MAGNETOSTATIC SPIN WAVES AND MAGNETIC-WAVE CHAOS IN FERROMAGNETIC FILMS.

I. THEORY OF MAGNETOSTATIC WAVES IN PLATES UNDER ARBITRARY ANISOTROPY AND EXTERNAL FIELDS

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Abstract. General phenomenological theory of magnetic spin waves in ferromagnetic media is originally reformulated and applied to analysis of magnetostatic waves in films and plates with arbitrary anisotropy under arbitrary external field. Exact expressions are derived for propagator of linear waves between antennae (inductors) and mutual impedances of antennae, and exact unified dispersion equation is obtained which describes all types of magnetostatic wave eigen-modes. Characteristic frequencies (spectra) and some other important properties of main modes are analytically considered and graphically illustrated. Besides, several aspects of non-linear excitations and magnetic-wave chaos are discussed, including two-dimensional “non-linear Schrödinger equations” for magnetostatic wave packets and envelope solitons.

1. Introduction

This preprint contains some results of research performed in Department of kinetic properties of disordered and nonlinear systems of DonPTI NASU, between 2001 and 2003, and devoted to excitation and synchronization of chaotic magnetostatic spin waves (MSW) in ferromagnetic films. In fact, this is a part of report written at that time. It reflects our theoretical MSW’s considerations which, in our opinion, may be useful for concerned readers. Next preprints will reflect our numeric simulations of MSW and magnetic-wave chaos in films and corresponding real experiments.

Key words and phrases. Phenomenology of ferromagnetic media and structures, Landau-Lifshitz-Gilbert torque equation, Dipole-dipole interactions, Magnetostatic excitations and waves, Magnetostatic wave propagator in films and plates, Impedances of wire inductors near ferromagnetic film, Dispersion equation and classification of magnetostatic spin waves in films and plates, Anisotropy’s influence onto spectra and propagation of magnetostatic waves, Non-linear magnetostatic waves, Parametric magnetostatic-wave interactions, Two-dimensional non-linear Schrödinger equations and magnetostatic-wave solitons.
It should be noticed that a portion of the following Sections 2 and 3 already was presented (in a modified form) by arXiv preprint [Yu. Kuzovlev, N. Mezin, and G. Yarosh, cond-mat/0405640], but we keep it here for better reader’s accommodation.

2. Basic properties and vocabulary of magnetic waves

2.1. MAGNETIC TORQUE EQUATION.

In the classical phenomenology of ferro- and ferri-magnetic solids [1-6], their magnetization is characterized by 3-dimensional vector field, $M(r,t)$, representing density of elementary magnetic moments. The important property of this vector is that its length is constant: $|M(r,t)| = M_s$, with $M_s = \text{const}$ termed saturation magnetization. Hence, it is natural to measure magnetization and magnetic fields in $M_s$ units and introduce the unit-length “spin” vector $S(r,t)$ by means of $M = M_s S$, $|S| = 1$. Then natural time unit will be $\tau_0 = (2\pi g M_s)^{-1}$, where $g$ is gyromagnetic ratio ($g \approx 2.8$ MHz/Oe) [1-3]. By this definition, $2\pi \tau_0$ is period of precession of a separate spin at field equal to $M_s$. The rotation of $S(r,t)$ is subject to the Landau-Lifshitz torque equation [1-6] which in these units reads

$$\frac{dS}{dt} = [F, S] + \gamma \{F - S \langle S, F \rangle \}, \quad F = -\frac{\delta E}{\delta S},$$

(2.1)

with $E$ being magnetic energy functional and $F$ full effective local magnetic field acting on spins (called also local thermodynamic force). Here and below $\langle a, b \rangle$, $[a, b]$ and $a \otimes b$ denote scalar product, vector product and tensor product of two vectors respectively ($c = a \otimes b$ is matrix with elements $c_{\alpha\beta} = a_\alpha b_\beta$). After transition to dimensionless time, $E$ is factual energy divided by $M_s^2$, hence, its dimensionality is spatial volume, while $F$ becomes dimensionless. The term containing $\gamma$ represents “friction”, i.e. dissipative interaction between magnetization and other microscopic degrees of freedom (thermostat). It can be written in several equivalent
forms in view of identities

\[ F - S\langle S, F \rangle = [S, [F, S]] = (1 - S \otimes S) F \]

In principle, the Eq.1 is the only possible law of magnetic dynamics compatible simultaneously with the constancy \(|S| = 1\), energy conservation under spin rotation itself and total energy balance with thermostat. In non-principally complicated models the friction may depend on spatial gradients of magnetization, \(\nabla S(r, t)\).

2.2. FERROMAGNETIC ENERGY.

In any magnet, its energy functional \(E\) includes

(I) anisotropy energy

\[ E_a\{S\} = \int A(S) \, dr \]  \hspace{1cm} (2.2)

which dictates locally preferred magnetization orientation (\(dr\) is volume differential);

(II) exchange interaction energy, which ensures local spatial uniformity of \(S(r)\) pattern,

\[ E_e\{S\} = \frac{1}{2} \int \langle \nabla_\alpha S, \nabla_\alpha S \rangle \, dr ; \]  \hspace{1cm} (2.3)

it is invariant with respect to \(S\) rotation; in general, it may be anisotropic with respect to gradient;

(III) dipole interaction energy which causes long-range non-uniformity (demagnetization) of \(S(r)\) pattern and initializes formation of magnetic domains:

\[ E_d\{S\} = \frac{1}{2} \int \langle S, \hat{G} S \rangle \, dr , \]  \hspace{1cm} (2.4)

with \(\hat{G}\) being the integral operator,

\[ \hat{G} S(r) \equiv \int G(r - rt) S(rt) \, dr , \]  \hspace{1cm} (2.5)
and \( G(r) \) the 3 \( \times \) 3 -matrix function

\[
G_{\alpha \beta}(r) = \frac{\delta_{\alpha \beta}}{|r|^3} - \frac{3r_\alpha r_\beta}{|r|^5} = -\nabla_\alpha \nabla_\beta \frac{1}{|r|}; 
\]

(IV) interaction with an external magnetic field \( H_e(r,t) \),

\[
E_h = -\int \langle H_e S \rangle \, dr
\]

In summary,

\[
E = E\{S, H_e\} = E_a + E_e + E_d + E_h + ...
\]

The dots replace additional interactions for instance, with distortions and defects of atomic lattice. As a rule, their contribution to long magnetic waves related phenomena (to be under our interest) can be either neglected or reduced to renormalization of \( M_s \), \( \gamma \) or other material parameters [1,2,5]. From experiments it is known that especial surface anisotropy may cause pinning of magnetization on the sample surface. This effect can be described in terms of boundary conditions for the spin field \( S(r,t) \) [8].

For example, in good quality yttrium-iron garnet (YIG) ferrites [4,5], \( M_s \approx 140 \) Oe, correspondingly \( \tau_0 \approx 0.4 \) ns the exchange interaction radius \( r_0 \approx 5 \cdot 10^{-6} \) cm, characteristic anisotropy field is less than \( M_s \) (that is \( |\partial A(S)/\partial S| \lesssim 1 \)), and the dimensionless friction coefficient \( \gamma \) is rather small, \( \gamma \approx 10^{-4} \).

2.3. DIPOLE INTERACTION.

The dipole contribution (4) is nothing but approximate form of complete magnetization interaction with self-induced magnetic field, \( H_S(r,t) \). Usually, characteristic spatial scales (sample size, wavelength) and group velocity of magnetic waves are much less than length and velocity, respectively, of free-space electromagnetic wave with the same frequency. Therefore, \( H_S(r,t) \) can be obtained as solution to quasi-static
Maxwell equations

\[ \text{div} \ (H_S + 4\pi S) = 0, \ \text{curl} \ H_S = 0, \quad (2.7) \]
even in high-frequency (microwave) band. Besides if there are no metal surfaces in the vicinity of magnetic sample (waveguide walls or other), one can suppose \( H_S \to 0 \) at infinity and come to (4). Oppositely, Eqs.7 should be solved under actual boundary conditions at surroundings and expression (4) be generalized as \( E_d = -\int \langle H_S S \rangle \, dr/2 \).

The principal peculiarity of dipole interaction is its specific long-range character: with increasing distance \( |r| \) it decreases exactly as the volume marked by this distance grows \( G(r) \propto r^{-3} \). Consequently, the integral dipole force (5) is indifferent to absolute spatial scales of magnetization pattern and depends on their ratios only. In particular, for uniform magnetization it depends on a shape of magnetic sample but not on its size.

2.4. STATIC MAGNETIZATION.

Let \( S_0(r) \) stands for a static (equilibrium, metastable or unstable) magnetization configuration in a constant external field, \( H_e = H_0(r) \). According to Eq.1, the essence of static state is that everywhere spins are oriented in parallel to thermodynamic force, \( F \parallel S \), that is

\[ -\frac{\delta E\{S_0 H_0\}}{\delta S_0} = W_0(r) \, S_0, \quad (2.8) \]

where scalar function \( W_0(r) \) (complete internal magnetic field) is determined by the requirement \( |S_0| = 1 \). Generally speaking, this equation possesses many solutions (various domain structures) whose physical realization depends on history of the system. But if \( H_0 \) is sufficiently strong then an unique solution must stay only \[1\], even in spite of the dipole interaction, since the dipole force (5) is bounded, \( |\hat{G} S| \leq 4\pi \), at any nonsingular magnetization distribution.
2.5. MAGNETIC EXCITATIONS.

Consider perturbation of static state, \( s = S - S_0 \), for instance, caused by additional time-varying external field, \( h(r, t) \), when \( H_e = H_0 + h \). Under a perturbation, the energy divides into two parts \( E = E_\parallel + E_\perp \), the ground

\[
E_\parallel = -\frac{1}{2} \int \{ W_0 + \langle H_0 S_0 \rangle \} \, dr - \int \langle h, S_0 \rangle \, dr ,
\]

and excess energy \( E_\perp \) which vanishes at \( s = 0 \) and in its turn consists of two parts

\[
E_\perp \{ sh \} = \frac{1}{2} \int W_0 s^2 \, dr + \tilde{E} \{ sh \} ,
\]

\[
\tilde{E} \{ sh \} = \int \tilde{A}(s) \, dr + E_e \{ s \} + E_d \{ s \} - \int \langle h, s \rangle \, dr .
\]

The first contribution to (9) arises from interaction between \( s(r, t) \) and static magnetization (lowering static magnetic order by magnetic excitations when \( W_0(r) = \langle S_0 F \{ S_0 \} \rangle > 0 \), or, in opposite, maintaining it, when \( W_0(r) < 0 \)). The second, \( \tilde{E} \{ sh \} \), represents \( s(r, t) \) interaction with itself. Its construction is quite similar to \( E \{ S, H \} \), except that anisotropy term is modified:

\[
\tilde{A}(s) = A(S_0 + s) - A(S_0) - \langle s A'(S_0) \rangle , \quad A(S) \equiv \frac{\partial A(S)}{\partial S} .
\]

This self-interaction induces the response operator \( \hat{L} \), as defined by

\[
\hat{L} s \equiv \frac{\delta \tilde{E}}{\delta s} + h = A(S_0 + s) - A(S_0) + \{ \hat{G} - r_0^2 \nabla^2 \} s .
\]

Evidently, in case of easy axis or easy plane anisotropy, when \( A(S) \) and hence \( \tilde{A}(s) \) are quadratic forms \( \hat{L} \) is purely linear operator. In general, it is reasonable (and commonly used) approximation if possible dependence of \( \hat{L} \) on \( S_0 \) is took into account while its nonlinearity (dependence on \( s \)) neglected, thus replacing \( \hat{L} \) by linear operator

\[
\hat{L} = \hat{A} - r_0^2 \nabla^2 + \hat{G} , \quad \text{where } (\hat{A} s)_{\alpha} = \frac{\partial^2 A(S_0)}{\partial S_{0\alpha} \partial S_{0\beta}} s_{\beta} .
\]
2.6. HAMILTONIAN FORMULATION.

Due to $|S_0 + s| = 1$, only two of three components of the excitation $s(r,t)$ must be considered as independent variables namely, the components perpendicular to $S_0(r)$, since weak excitation is always perpendicular to $S_0(r)$. Thus define the two-dimensional vector

$$ S_\perp = \widehat{\Pi} S , \quad \widehat{\Pi} \equiv 1 - S_0 \otimes S_0 , $$

where matrix $\widehat{\Pi} = \widehat{\Pi}(r)$ performs projection onto the plane, $\Pi(r)$, perpendicular to $S_0(r)$. Evidently,

$$ s = S_\perp + (S_\parallel - 1) S_0 , \quad S_\parallel \equiv \langle S_0, S \rangle , \quad (2.12) $$

where scalar $S_\parallel$ represents projection of $S$ onto the static magnetization direction, and

$$ S_\parallel^2 + |S_\perp|^2 = 1 , \quad |s|^2 = 2(1 - S_\parallel) \quad (2.13) $$

In terms of $S_\perp$ the torque equation (1) transforms into

$$ \frac{dS_\perp}{dt} = S_\parallel [F_\perp S_0] + \gamma(1 - S_\perp \otimes S_\perp) F_\perp , \quad F_\perp = -\frac{\delta E_\perp\{sh\}}{\delta S_\perp} \quad (2.14) $$

Here $E_\perp\{sh\}$ should be expressed, with the help of (12) and (13), as a functional of $S_\perp$, yielding

$$ F_\perp = -\frac{S_\perp}{S_\parallel} \left( W_0 - \left\langle S_0 \frac{\delta E}{\delta s} \right\rangle \right) - S_\parallel \widehat{\Pi} \frac{\delta E}{\delta s} \quad (2.15) $$

Notice that frictionless version of the Eq.14 (for $\gamma = 0$) follows from the variational principle

$$ \delta \int \left( \int \left\langle S_0 \left[ \frac{dS_\perp}{dt}, S_\perp \right] \right\rangle \frac{dr}{1 + S_\parallel} + E_\perp \{s, h\} \right) dt = 0 \quad (2.16) $$

It takes canonical Hamiltonian form after the change of variables
where vector $Q$ is also situated in the plane $\Pi(r)$ [6-7]. Simultaneously such a choice of variables ensures unambiguous parametrization of $S_\parallel$ even in case of spin flipping when $S_\parallel$ can become negative.

2.7. DYNAMIC EQUATIONS FOR MAGNETIZATION.

It is convenient to introduce the rotation (precession) operator, $\hat{R}$, and damp operator, $\hat{\Gamma}$, by means of

$$
\hat{R}V = [S_0 V], \quad \hat{\Gamma}V \equiv \hat{\Gamma}(S_\perp)V = \frac{\gamma}{S_\parallel} \Pi(1 - S_\perp \otimes S_\perp)V, \quad (2.18)
$$

where $V$ is arbitrary vector. Then the Eqs.14-15 takes more pragmatic form:

$$
\frac{dS_\perp}{dt} = \{\hat{R} - \hat{\Gamma}(S_\perp)\} \{W_0 + (S_0 f)\} S_\perp - S_\parallel f, \quad f \equiv h - \hat{L}\{S_\perp + (S_\parallel - 1) S_0\}, \quad (2.19)
$$

to be supplemented by relations (11-13). Note that $\hat{\Pi}^2 = \hat{\Pi}$, $\hat{R}\hat{\Pi} = \hat{\Pi}\hat{R} = \hat{R}$, $\hat{R}^2 = -\hat{\Pi}$, and the latter equality is equivalent to $\hat{R}^2 = -1$ in planes $\Pi(r)$.

In generalized version of Eq.19, direct dipole interaction is replaced, in analogy with (7), by self-induced field of the excitation, $h_S(r,t)$, according to the rule $h_S \leftrightarrow -\hat{G}s$. In such an approach

$$
f = h + h_S - (\hat{A} - r_0^2 \nabla^2) s, \quad (2.20)
$$

and Eq.19 should be solved together with magnetostatic equations

$$
\text{div} \ (h_S + 4\pi s) = 0, \quad \text{curl} \ h_S = 0, \quad (2.21)
$$

under classical electromagnetic boundary conditions at the sample surface and surrounding bodies.
2.8. SMALL-AMPLITUDE DYNAMICS.

If a static state $S_0$ is stable, i.e. represents (global or local) energy minimum in the space of magnetization configurations then $E_{\perp}\{s_{h}\}$, for small $s(r,t)$, is positive quadratic form. Hence, we have rights to speak about small-amplitude magnetic oscillations and waves. Linearization of Eq.19 results in

$$\frac{dS_{\perp}}{dt} = (\hat{R} - \gamma \hat{\Pi})(\hat{W} S_{\perp} - h), \quad \hat{W} \equiv W_0 + \hat{A} - r_0^2 \nabla^2 + \hat{G},$$

(2.22)

or, in the self-induced field representation,

$$\frac{dS_{\perp}}{dt} = (\hat{R} - \gamma \hat{\Pi})\{(W_0 + \hat{A} - r_0^2 \nabla^2) S_{\perp} - h - \hat{h} S\}$$

(2.23)

Let the perturbation and hence linear response vary harmonically: $h(r,t) = h(r) \exp(-i\omega t)$, and so on ($i \equiv \sqrt{-1}$). Then Eq.23 formally yields

$$S_{\perp} = \hat{\chi}(h + \hat{h} S), \quad \hat{\chi} = \hat{\chi}(\omega) \equiv \{i\omega + (\hat{R} - \gamma \hat{\Pi})(W_0 + \hat{A} - r_0^2 \nabla^2)\}^{-1}(\hat{R} - \gamma \hat{\Pi})$$

(2.24)

Substituting this expression to Eqs.21, we transform the problem to solving partial differential equations

$$\text{div} \; \hat{\mu}(h + \hat{h} S) = 0, \quad \hat{\mu}(\omega) \equiv 1 + 4\pi \hat{\chi}(\omega), \quad \text{curl} \; \hat{h} S = 0$$

(2.25)

Here $\hat{\chi}$ and $\hat{\mu}$ are play the role of linear polarizability and susceptibility operators [2]. With respect to polarization, $\hat{\chi}$ acts as $2 \times 2$-matrix performing projection onto planes $\Pi(r)$. It may be thought local spin precession response to given local magnetic field, but, from the rigorous mathematical point of view, presence of the Laplasian $\nabla^2$ makes it integral (nonlocal) operator.

2.9. LINEAR FREE WAVES.
Omitting in (22) friction and external pump, one can find eigenmodes and eigenfrequencies of free magnetic waves (MW):

\[ S_\perp \equiv V e^{-i\omega t}, \quad -i\omega V = \hat{R} \hat{W} V \]  

(2.26)

It is sufficient to consider positive frequencies only, \( \omega > 0 \), since opposite sign means complex conjugating the same mode. Let different modes be enumerated by a set of indexes \( k \). Since operator \( \hat{W} \) is positively defined, we can rewrite (26) as

\[ \omega_k \bar{V}_k = i \hat{W}^{1/2} \hat{R} \hat{W}^{1/2} \bar{V}_k, \quad \bar{V}_k \equiv \hat{W}^{1/2} V_k \]  

(2.27)

Operator in the left equation is Hermitian because \( i\hat{R} \) is Hermitian, hence, the solutions can be chosen mutually orthogonal and normalized to \( \int \langle \bar{V}_m \bar{V}_k \rangle dr = \omega_k \delta_{mk} \) (with Kroneker symbol on right-hand side). Returning to (27) and (26) shows that

\[ i \int \langle V_m^* \hat{R} V_k \rangle dr = i \int \langle S_0 [V_k V_m^*] \rangle dr = \delta_{mk} \]  

(2.28)

This gives the orthogonality rule for the eigenwaves.

The vectors \( V \) always can be represented in the form

\[ V(r) = e^{i\vartheta(r)} \{a(r) + ib(r)\}, \quad a \perp b, \quad a \perp S_0, b \perp S_0, \]  

(2.29)

where both \( a \) and \( b \) are real-valued vectors situated in plane \( \Pi(r) \) and perpendicular one to another. This means merely that spin vector draws elliptic trajectory whose main axes are just \( a \) and \( b \).

2.10. PLANE WAVES AND POLARIZATION DECOMPOSITION.

Consider free MW in an infinite-size uniformly magnetized sample (\( S_0 = const \)). In this envisioned situation, eigenmodes (29) are plane waves that is \( \vartheta(r) = \langle k, r \rangle \), with \( k \) being wave vector, and \( a \) and \( b \) are constants. One wave only corresponds to
every $k$, because magnetization precession is definitely clockwise. Putting on $V_k(r) = V_k \exp(i \langle k, r \rangle)$, $V_k = a_k + ib_k$ in the Eq.26, one easy obtains

$$-\omega_k [S_0 b_k] + i\omega_k [S_0 a_k] = \tilde{W} a_k + i\tilde{W} b_k, \quad \tilde{W} \equiv e^{-i\langle k, r \rangle} \Pi \tilde{W} \Pi e^{i\langle k, r \rangle}$$ (2.30)

The vectors $a$ and $b$ always can be such ordered that $\langle S_0 [a, b] \rangle > 0$, then (30) yields

$$\omega_k p_k a_k + i\omega_k \frac{b_k}{p_k} = \tilde{W} a_k + i\tilde{W} b_k, \quad p_k \equiv \frac{|b_k|}{|a_k|}$$ (2.31)

Since operator $\tilde{W}$, by its definition, is real-valued symmetric operator, this equation clearly shows that

$$\tilde{W} a_k = \alpha_k a_k, \quad \tilde{W} b_k = \beta_k b_k$$ (2.32)

Hence, the main axes of MW polarization ellipse, $a_k$ and $b_k$, are nothing but eigenvectors of operator (matrix) $\tilde{W}$. Besides two corresponding eigenvalues determine frequency and eccentricity, $p_k$, of MW:

$$p_k = \sqrt{\alpha_k / \beta_k}, \quad \omega_k^2 = \alpha_k \beta_k = \det \tilde{W}$$ (2.33)

(of course, symbol $\det$ relates to 2-dimensional projection subspace, while the third eigenvalue is zero).

If the rotation operator $\hat{R}$ was discarded from Eq.22, instead of $-\gamma \Pi$, this equation would describe monotonic decay instead of oscillations with two relaxation rates $\gamma \alpha_k$, $\gamma \beta_k$. In this sense, $V_k = a_k + ib_k$ is decomposition of MW polarization to two relaxation modes.

2.11. SPIN WAVES AND MAGNETOSTATIC WAVES.

According to (22) and (30),
\[ \mathcal{W} = \hat{\Pi} \{ W_0 + \hat{A} + r_0^2 k^2 + \tilde{G}(k) \} \hat{\Pi}, \]  

\[ \tilde{G}(k) \equiv e^{-i(k,r)} \tilde{G} e^{i(k,r)} = 4\pi \frac{k \otimes k}{k^2}, \]

where \( \tilde{G}(k) \) is Fourier transform of dipole interaction, and \( k^2 \equiv |k|^2 \). To express the frequency more or less evidently, let us introduce the wave vector projection onto the plain \( \Pi \), \( k_{\perp} \equiv \hat{\Pi} k \), the projected anisotropy matrix \( \hat{A}_{\perp} \equiv \hat{\Pi} \hat{A} \hat{\Pi} \) and two its eigenvalues \( A_1, A_2 \). Then

\[ \omega_k^2 = (W_0 + r_0^2 k^2 + A_1)(W_0 + r_0^2 k^2 + A_2) + 4\pi \frac{k \otimes k}{k^2} \left\{ \frac{k_{\perp}^2}{k^2} \left( W_0 + r_0^2 k^2 + A_1 + A_2 \right) - \left\langle k, \hat{\Pi} \hat{A} \hat{\Pi} k \right\rangle \right\} \]  

\[ \alpha_k + \beta_k = \text{Tr} \mathcal{W}, \quad \text{Tr} \mathcal{W} = W_0 + r_0^2 k^2 + A_1 + A_2 + 4\pi k_{\perp}^2 / k^2 \]

The latter equality as combined with (32) determines the eccentricity. Notice that \( k_{\perp}^2 = [S_0^2 k]^2 \).

Alternatively, the frequency can be found from the Eqs.25, if exclude friction and external field and represent the magnetization field in potential form, \( h_s \propto k \exp(i \langle k, r \rangle) \). This leads to the dispersion equation,

\[ \langle k, \hat{\mu}(\omega) k \rangle = k^2 + 4\pi \left\langle k, \{ i\omega + \hat{R}(W_0 + \hat{A} + r_0^2 k^2) \}^{-1} \hat{R} k \right\rangle = 0, \]  

whose solution \( \omega = \omega_k \) coincides with (36), while wave polarization follows from Eqs.24.

This consideration fully neglects actual size and shape of a ferromagnet sample. Dipole interaction in the envisioned boundless MW finds no spatial scales to compare with its wavelength. That is why dipole contribution to the dispersion law (36) is
insensible to wave length being function of wave direction only. Perhaps the waves whose length is ten or more times less than least from the sample dimensions may be treated in such a way. This MW are called spin waves (SW). Neither dispersion nor polarization of SW depend on sample geometry.

However, in relatively long wave dipole interaction inevitably compares their length with sample dimensions. As the result, dispersion law acquires the ratio of these two scales:

$$\omega_k = \omega(D|k|, k/|k|, r_0^2k^2), \quad (2.38)$$

where $D$ is some characteristic sample size. Waves which for this correction is essential are called magnetostatic waves (MSW). In most of modern applications just MSW are directly excited and detected while SW generated from them in nonlinear wave processes.

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3. **Linear waves in films and plates: propagator, excitation, and dispersion equation**

In compactly designed microwave devices current carrying conductors (wires) are rather suitable inductors and antennas for magnetic waves (MW) in ferro- and ferrimagnetic films. Therefore, it is reasonable to unify consideration of the waves and consideration of electromagnetic impedance of conductors interacting with film. We will assume characteristic sizes of a circuit be less than $2\pi c/\omega$, with $\omega$ being characteristic operating frequency and $c$ speed of light. This allows to use quasi-static Maxwell equations.

At the same time, real processes in thin films (thin in the relative sense that thickness $D$, is much less than length and width) can be successfully analyzed in the formally infinite film approximation [2,3]. In particular, domainless static magnetization of a thin film is uniform everywhere except narrow regions adjoining its edges with width of order of film thickness. Concretely, evaluation of dipole force (Eq.2.5, that is Eq.5 in Sec.2) in finite-size plate geometry shows that demagnetizing field stipulated by film edges decreases at least as $2D/d$ where $d$ is distance from a nearest edge. Thus one can surely suppose $S_0 = const$.

Being relatively thin a film may be thick in the absolute sense that $D >> r_0$ ($r_0$ is exchange radius). We will consider small-amplitude (linear) MW excitation in such a film, by wires whose radius much exceeds $r_0$. So thick wires in linear regime can directly excite long magnetostatic waves (MSW) only. Therefore, it is reasonable to
simplify mathematics by neglecting exchange contribution to the polarizability $\hat{\chi}$ (see Eq.2.24). This transforms $\hat{\chi}$ from integral operator into matrix which locally connects spin precession with magnetic field.

Under these conditions general formula (2.38) for MSW frequency takes the form

$$\omega_k = \omega_N(D|k|, k/|k|),$$

(3.1)

where $k$ is in-plane wave vector and integer $N$ enumerates wave branches different with respect to thickness. Importantly, it shows that (at least at given wave direction) the group velocity of MSW, $v_g = \partial \omega_k / \partial k \propto D$, is as small as film thickness.

3.1. WAVE EXCITATION BY EXTERNAL CURRENTS.

The external field $h(r, t)$ (see Sec.2) produced by currents is solution to the equations

$$h(r, t) = \sum h_m(r) I_m(t), \quad \text{div } h_n = 0, \quad \text{curl } h_n = \frac{4\pi}{c} J_n(r),$$

(3.2)

where $I_n$, $J_n$ and $h_n J_n$ are total current, its density distribution (normalized to unit) and its field for $n$-th wire, respectively. Clearly, so defined $h_n$ are nothing but Green functions which determine simultaneously electric moving force (EMF) induced in conductors by time varying film magnetization. The expression for EMF, $\varepsilon_n$, follows merely from Eq.2.14 as the consequence of energy balance and reads

$$\varepsilon_n = \int \langle h_n \frac{dS}{dt} \rangle dr$$

(3.3)

This formula is valid for arbitrary strong nonlinear excitation. For weak excitation to be considered, EMF is linear function of currents and relation

$$\varepsilon_n = \sum \hat{Z}_{nm} I_m$$

(3.4)
defines mutual impedances of the wires $\hat{Z}_{nm}$.

In linear regime we may treat dipole interaction by means of Eqs.2.23-2.25. Both external and magnetization induced field can be represented in potential form,

$$h_n = -\nabla U_n, \quad h_s = \sum \hat{h}_{Sm} I_m \hat{h}_{Sn} = -\nabla \tilde{U}_{Sn}$$

excluding (for external field) places occupied by currents. Here $\tilde{U}_{Sn}$ is contribution obliged to $n$-th inductor. The hat means that it is time convolution operator. Suppose film situated in region $-D/2 < z < D/2$ in parallel to $xy$-plane, and expand variables and patterns into Fourier series:

$$I_n(t) = \int e^{-i\omega t} \tilde{I}_n(\omega) \frac{d\omega}{2\pi}, \quad U_n(r) = \int \exp(ik_x x + ik_y y)\tilde{U}_n(k, z) \, dk$$

and so on, where $k = \{k_x, k_y\}$ is in-plane wave vector. The potentials strictly determine magnetization:

$$\tilde{S}_\perp(\omega, k, z) = -\hat{\chi}(\omega) \nabla \sum \tilde{I}_m(\omega)[\tilde{U}_{Sm}(\omega, k, z) + \tilde{U}_m(k, z)]$$

where, after Fourier transform is made, $\nabla = \{ik_x, ik_y, \nabla_z\}$.

3.2. SOURCE FORM-FACTORS.

In the film interior $\nabla^2 U_n = 0$, therefore external potentials possess simple exponential behavior:

$$\tilde{U}_n(k, z) = \Phi_n(k) \exp\{|k|(\sigma_n z - D/2)|, \quad |k| \equiv \sqrt{k_x^2 + k_y^2}$$ (3.5)

where $\sigma_n = 1$ (or $-1$) if $n$-th inductor is placed above (or under) film, and the form-factor, $\Phi_n(k)$, reflects distribution of $n$-th current. In particular, let $n$-th wire has round cross-section and oriented strictly along $y$-axis at position $x_n$, and distance between its center line and film top or bottom equals to $\rho_n$, then
\[
\Phi_n(k) = \frac{4\pi^2}{i\kappa_x} \exp(-|k|\rho_n - ik_{x}x_{n})\delta(k_{y}) ,
\] (3.6)

with \(\delta()\) being Dirac delta-function.

3.3. IMPEDANCE TO MAGNETIC POTENTIAL RELATION.

Besides the equality \(\nabla^2 U_n = 0\) as combined with usual boundary conditions (continuity of normal magnetic inductance and tangential filed) allows to express the impedances through the potentials taken at film top or film bottom:

\[
Z_{nm} \equiv e^{i\omega t}\tilde{Z}_{nm}e^{-i\omega t} = \frac{i\omega}{2\pi} \int |k|\tilde{U}_{Sm}(\omega,k,\sigma_{n}D/2)\tilde{U}_{n}(-k,\sigma_{n}D/2) \, dk
\] (3.7)

(we omit details of calculation). Hence, potentials are taken at the surface most close to receiving (\(n\)-th) conductor. More exactly, in view of the definitions (3) and (4), Eq.8 describes the film contribution to full impedance (which, of course, has also contribution from direct magnetic interaction between conductors).

3.4. WAVE POTENTIAL AND NORMAL WAVE NUMBERS.

According to Eqs.2.25, we have to solve the equation

\[
\langle \nabla, \hat{\mu} \nabla \rangle (\tilde{U}_{S} + \tilde{U}) = 0 \, , \, \hat{\mu} \equiv 1 + 4\pi\hat{\chi}(\omega) ,
\] (3.8)

addressed to either potential of particular current or total one. Of course, the unit here (as well as in Eqs.2.21) means nothing but the unit matrix. To carefully consider this equation, we need in introducing two 3-dimensional unit vectors

\[
\nu \equiv \{k_{x}/|k|, k_{y}/|k|, 0\} \, , \, \bar{\nu} \equiv \{0, 0, 1\} ,
\]
and \(2 \times 2\) matrix

\[
M = \begin{bmatrix}
\mu_{\nu\nu} & \mu_{\nu z} \\
\mu_{z\nu} & \mu_{zz}
\end{bmatrix}
\equiv \begin{bmatrix}
\langle \nu, \hat{\mu} \nu \rangle & \langle \nu, \hat{\mu} \bar{\nu} \rangle \\
\langle \bar{\nu}, \hat{\mu} \nu \rangle & \langle \bar{\nu}, \hat{\mu} \bar{\nu} \rangle
\end{bmatrix}
\]
The solution to Eq.9 is

$$\tilde{U}_S + \tilde{U} = u_+ \exp(\lambda_+ |k|z) + u_- \exp(\lambda_- |k|z), \quad (3.9)$$

where $\lambda_\pm = \lambda_\pm(\omega)$ are two roots of the dispersion equation which follows from Eq.9:

$$\mu_{zz} \lambda^2 + i(\mu_{\nu z} + \mu_{z\nu}) \lambda - \mu_{\nu \nu} = 0, \quad (3.10)$$

$$\lambda = \lambda_\pm \equiv \lambda_0 \pm \Lambda, \quad \lambda_0 = -\frac{i(\mu_{\nu z} + \mu_{z\nu})}{2\mu_{zz}}, \quad \Lambda = \left\{ \frac{\mu_{\nu \nu}}{\mu_{zz}} - \left( \frac{\mu_{\nu z} + \mu_{z\nu}}{2\mu_{zz}} \right)^2 \right\}^{1/2} (3.11)$$

In general $\lambda_\pm$ can be either real or imaginary or complex.

Obviously, matrix $M$ and hence $\lambda_\pm$ are sensitive to direction of the in-plane wave vector $k$ but indifferent to its absolute value, $|k|$. Therefore, as the Eq.9 shows potentials and magnetization (5) vary in $z$-direction as quickly or slowly as in plane. This is characteristic property of magnetostatic waves (see Sec.2).

To find the coefficients $u_j$ and then boundary values of potentials $\tilde{U}_S n(\omega, k, \pm D/2)$, for substitution into Eq.8, we must once again use standard boundary conditions. The final result for the boundary potential, at the side closest to a given inductor, is presented by formulas

$$\tilde{U}_S n(\omega, k, \sigma_n D/2) = \Phi_n(k) F(\omega, k), \quad (3.12)$$

$$F(\omega, k) \equiv \frac{[1 - \Delta - i(\mu_{z\nu} - \mu_{\nu z})] \sinh(\Lambda |k|D)}{(1 + \Delta) \sinh(\Lambda |k|D) + 2\mu_{zz} \Lambda \cosh(\Lambda |k|D)}, \quad (3.13)$$

$$\Delta \equiv \det M = \mu_{\nu \nu} \mu_{zz} - \mu_{\nu z} \mu_{z\nu} \quad (3.14)$$

Let us pay attention to the term $i(\mu_{z\nu} - \mu_{\nu z})$ in the nominator of (14). It changes sign when $k$ turns in opposite direction, $-k$. This is example of unreciprocity
inherent to phenomena under static magnetization, which leads to difference between $Z_{nm}$ and $Z_{mn}$.

3.5. POLARIZABILITY MATRIX.

After excluding exchange from the polarizability $\hat{\chi}$ (see Eq.2.24), the inversion of denominator operator in Eq.2.24 reduces to algebraic manipulations. The result reads

$$\hat{\chi}(\omega) = \frac{(\bar{W}_0 + A_1 + A_2)\bar{\Pi} - \hat{A}_\perp + i\bar{\omega}\bar{R}}{(\bar{W}_0 + A_1)(\bar{W}_0 + A_2) - \bar{\omega}^2}, \quad \bar{W}_0 \equiv W_0 - i\gamma\bar{\omega}, \bar{\omega} \equiv \frac{\omega}{1 + \gamma^2}, \quad (3.15)$$

with $A_{1,2}$ being two nonzero eigenvalues of the projected anisotropy matrix $\hat{A}_\perp = \bar{\Pi}\hat{A}\bar{\Pi}$ (see Sec.2). Additional simplification is implied by the inequality $\gamma << 1$ which is always hoped in practice and, indeed, it is very well satisfied in ferrites ($\gamma = 10^{-4} \div 10^{-3}$). By this reason one can discard all the terms terms containing $\gamma^2$ (all the more higher powers), in particular, make no difference between $\bar{\omega}$ and $\omega$.

3.6. SUSCEPTIBILITY MATRIX.

To detail the susceptibility matrix $M$, a few definitions are necessary. Let $\theta$ be the angle between static magnetization vector $S_0$ and $z$-axis orthogonal to film, and $\varphi$ the angle clockwise counted between $y$-axis and projection of $S_0$ onto the film plane $x$-$y$ (so that $\theta = \pi/2$ and $\varphi = 0$ correspond to in-plane magnetization strictly parallel to $y$-axis). Next, involve to consideration two unit-length eigenvectors of $\hat{A}_\perp$, $\overline{A}_{1,2}$ (of course, mutually orthogonal), which lie in the plane $\Pi \perp S_0$ and correspond to the eigenvalues $A_{1,2}$. We always can order them so that $[\overline{A}_1 \overline{A}_2] = S_0$. Then consider the plane, $z$-$S_0$, what passes through both $z$-axis and $S_0$, and define one more angle, $\psi$, be the angle clockwise counted between this plane and $\overline{A}_1$. Thus $\psi = 0$ means that vector $\overline{A}_2$ lies in plane $x$-$y$. Also we need in the quantities

$$\nu\parallel \equiv \nu_x \sin \varphi + \nu_y \cos \varphi = (k_x \sin \varphi + k_y \cos \varphi)/|k| = \cos \phi, \quad (3.16)$$
\[ \nu_\perp \equiv \nu_x \cos \varphi - \nu_y \sin \varphi = (k_x \cos \varphi - k_y \sin \varphi)/|k| = \sin \phi, \]

which represent cosine and sine, respectively, of the angle, \( \phi \), between the in-plane wave vector \( k \) and projection of \( S_0 \) onto plane \( x-y \) (i.e. between \( k \) and \( z-S_0 \) plane).

In addition, it is convenient to use the notations

\[ A_+ \equiv (A_1 + A_2)/2 \ , \ A_- \equiv (A_2 - A_1)/2 \ , \ W_{1,2} \equiv W_0 + A_{1,2} \ , \ \omega_0^2 \equiv \omega_1 \omega_2, \quad (3.17) \]

\[ \hat{\Omega} = \hat{\Omega}(\omega) \equiv (W_0 + A_1 + A_2)\hat{\Pi} - \hat{A}_\perp + i\omega \hat{R} \quad \text{(3.18)} \]

Recall that at given external field, the vectors \( S_0, \ A_{1,2} \) and scalars \( W_0, \ A_{1,2} \) are completely determined by solution to the static magnetization equation (2.8). In these notations \( A_1 = A_+ - A_- \ , \ A_2 = A_+ + A_- \), and

\[ \mu_{zz} = 1 + \frac{4\pi \Omega_{zz}}{\omega_0^2 - \omega^2}, \ \mu_{\nu \nu} = 1 + \frac{4\pi \Omega_{\nu \nu}}{\omega_0^2 - \omega^2}, \ \mu_{z \nu} = \frac{4\pi \Omega_{z \nu}}{\omega_0^2 - \omega^2}, \ \mu_{\nu z} = \frac{4\pi \Omega_{\nu z}}{\omega_0^2 - \omega^2}, \quad (3.19) \]

where the four matrix elements of \( \hat{\Omega} \) are defined quite similarly to elements of matrix \( M \),

\[ \Omega_{zz} \equiv \langle z, \hat{\Omega} z \rangle, \ \Omega_{\nu \nu} \equiv \langle \nu, \hat{\Omega} \nu \rangle, \ \Omega_{z \nu} \equiv \langle z, \hat{\Omega} \nu \rangle, \ \Omega_{\nu z} \equiv \langle \nu, \hat{\Omega} z \rangle \quad \text{(3.20)} \]

After a lot of algebra one can find:

\[ \Omega_{zz} = (W_0 + A_+ + A_- \cos 2\psi) \sin^2 \theta, \quad (3.21) \]
\[ \Omega_{\nu\nu} = (\overline{W_0} + A_+)(\nu_\perp^2 + \nu_\parallel^2 \cos^2 \theta) + A_- \{ (\nu_\parallel^2 \cos^2 \theta - \nu_\perp^2) \cos 2\psi - 2\nu_\parallel \nu_\perp \sin 2\psi \cos \theta \}, \]  
(3.22)

\[ \Omega_{z\nu} = \Omega_x - i\omega \nu_\perp \sin \theta, \quad \Omega_{\nu z} = \Omega_x + i\omega \nu_\perp \sin \theta, \]  
(3.23)

\[ \Omega_x \equiv \{ A_- \nu_\perp \sin 2\psi - \nu_\parallel (\overline{W_0} + A_+ + A_- \cos 2\psi) \cos \theta \} \sin \theta \]  
(3.24)

Importantly, these complicated expressions always compensate one another so that the determinant (15) has the simple pole only, see below.

3.7. RESPONSE FUNCTION.

To comfortably express the determinant (15), the roots (12) of dispersion equation (11), and the whole function (14) which connects by Eq.15 the form-factor of current distribution, \( \Phi_n(k) \), and magnetic potential of self-induced field, let us introduce the characteristic frequencies \( \omega_u, \omega_1, \omega_2, \) and \( \omega_3 \), as follows:

\[ \omega_u^2 \equiv W_1W_2 + 4\pi(\overline{W_0} + A_+ + A_- \cos 2\psi) \sin^2 \theta, \]  
(3.25)

\[ \omega_{1,2}^2 \equiv W_1W_2 + 2\pi(\Omega_{zz} + \Omega_{\nu\nu}) \pm 2\pi \sqrt{(\Omega_{zz} + \Omega_{\nu\nu})^2 - (2\nu_\parallel \sin \theta)^2W_1W_2}, \]  
(3.26)

\[ \omega_3^2 \equiv W_1W_2 + 2\pi(\Omega_{zz} + \Omega_{\nu\nu}) + (4\pi \nu_\perp \sin \theta)^2 / 2 = \{ \omega_1^2 + \omega_2^2 + (4\pi \nu_\perp \sin \theta)^2 \} / 2 \]  
(3.27)

After one more portion of algebra, eventually we obtain:

\[ \Delta = \frac{2\omega_3^2 - \omega_0^2 - \omega_3^2}{\omega_0^2 - \omega_3^2}, \quad \mu_{zz} = \frac{\omega_u^2 - \omega_0^2}{\omega_0^2 - \omega_3^2}, \quad \mu_{z\nu} - \mu_{\nu z} = -\frac{8\pi i\omega \nu_\perp \sin \theta}{\omega_0^2 - \omega_3^2}, \]  
(3.28)
\[ \lambda_0 = \frac{4\pi \Omega_x}{\omega^2 - \omega_u^2}, \quad \Lambda = \Lambda(\omega) \equiv \frac{\sqrt{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}}{\omega^2 - \omega_u^2} \] (3.29)

In this designations the response function (14) takes the form

\[ F(\omega, k) \equiv \frac{(\omega_0^2 - \omega_3^2 - 4\pi \omega \nu_1 \sin \theta) \sinh [\Lambda(\omega)|k|D] + (\omega_u^2 - \omega^2)\Lambda(\omega) \cosh [\Lambda(\omega)|k|D]}{(\omega_0^2 - \omega^2) \sinh [\Lambda(\omega)|k|D] + (\omega_u^2 - \omega^2)\Lambda(\omega)} \] (3.30)

Notice that all the above defined frequencies \( \omega_0 \), \( \omega_u \), \( \omega_1 \), \( \omega_2 \) and \( \omega_3 \) include complex factor \( W_0 \equiv W_0 - i\gamma \omega \), hence, they themselves are complex functions of \( \omega \) although with small imaginary parts (and thus weakly depending on \( \omega \)). At \( \gamma \to 0 \) all they become real values which characterize spectrum of free MSW (eigenwaves). In particular (see Sec.4), \( \omega_u \) is the frequency of spatially uniform spin precession in film.

3.8. FILM INDUCED IMPEDANCE OF CONDUCTORS.

As the consequence of Eqs.8 and 13, mutual impedance of two conductors situated on one and the same hand from film, is given by

\[ Z_{nm} = \frac{i\omega}{2\pi} \int |k|\Phi_n(-k)\Phi_m(k)F(\omega, k) \, dk, \] (3.31)

where, as well as in (8), integration is performed over all the two-dimensional in-plane wave vectors while \( F(\omega, k) \) is determined by Eq.14 and previous listing of matrix \( M \).

In all the above formulas the letter \( \omega \) (as well as all the frequency related designations \( \omega_u \), \( \omega_k \), \( W \), \( W_0 \), etc.) means dimensionless angular frequency which equals to actual dimensional angular frequency \( 2\pi f \) (or \( 2\pi f_u \), etc.) expressed in units of \( 1/\tau_0 = 2\pi gM_s \) (see Sec.2). Hence, correspondence between quantities like \( \omega \) and \( f \) is established by the rule

\[ \omega = f \text{[GHz]} / f_0 \text{[GHz]}, \quad f_0 \text{[GHz]} \equiv g \text{[GHz/kOe]} M_s \text{kOe}, \] (3.32)
where \( f \) is dimensional frequency, \( g \approx 2.8 \text{ GHz} / \text{kOe} \) is gyromagnetic ratio, \( M_s \) is saturation magnetization, and square brackets enclose physical unit names. For YIG, \( f_0 \approx 0.39 \text{ [GHz]} \).

3.9. IMPEDANCE OF STRAIGHT WIRES AND LOOPS.

Substituting (7) in (32), we obtain mutual impedance for a pair of cylindrical wires which are directed along \( y \)-axis in parallel one to another and to film and lying on the same hand from it (again we omit rather tremendous manipulations). To express the result in pleasant form, let us measure distances and sizes in centimeters and impedances in Ohm. Then

\[
\frac{Z_{nm}[\text{Ohm}]}{w[\text{cm}]f[\text{GHz}]} = 4\pi i \int_{0}^{\infty} e^{-q(\rho_n+\rho_m)} \left[ \frac{(1 - \Delta) \cos(qx) + (\mu_{zz} - \mu_{zv}) \sin(qx)}{(1 + \Delta) \sinh(\Lambda qD) + 2\mu_{zz} \cosh(\Lambda qD)} \right] dq
\]

(3.33) Here \( x = x_n - x_m \) is distance between the wires and \( w \) is their length (in centimeters). The integral is taken over \( q \equiv k_x > 0 \), \( k_y = 0 \), that is in all \( \mu \)-related parameters of the integrand it should be put on \( \nu = \{1,0,0\}, \nu_{\perp} = \cos \varphi, \nu_{\parallel} = \sin \varphi \), while the frequency \( \omega \) should be mentioned in accordance with (33).

For the case when two parallel wires are situated on the opposite parties from the film, evaluation of corresponding boundary potentials yields (in the same units):

\[
\frac{Z_{nm}}{wf} = 2\pi i \int_{-\infty}^{\infty} e^{-|q|(\rho_n+\rho_m) + iqx} \left\{ e^{-|q|D} - \frac{2\mu_{zz} \Lambda \exp(\lambda_0|q|D)}{(1 + \Delta) \sinh(\Lambda|q|D) + 2\mu_{zz} \Lambda \cosh(\Lambda|q|D)} \right\} \frac{dq}{|q|}
\]

(3.34) (to be accompanied by definitions (12) and (15)). Of course, here \( \nu = \{ \text{sign}(q),0,0\} \) in the integrand. Due to the definition (8), \( Z = R - 2\pi i f L \), where \( R = \text{Re} Z \) and \( L = -\text{Im} \frac{Z}{2\pi f} \) play roles of resistance and inductance.

In fact, what we made is evaluation of impedances per unit length for long line inductors with not taking into account disturbance of their parallelity and edge effects.
Formulas (34) and (35) can be obviously generalized to strip-shaped wires and to the case when transmitting or/and receiving inductor consist of two parallel wires which form loop (or many wires which form antennae lattice) and can perform more or less wave selection. What is for more complicated configurations to investigate them one should return to formulas (13)-(15) and (32).

By its definition, the distance between any wire and film, $\rho_n$, can not be less than the wire radius. Closely looking on integrands in (34) and (35) we see, firstly, that only such the waves are excited (and contributing to the impedance) whose wavelength notably exceeds $\rho_n+\rho_m$. Secondly, impedance depends on the ratio $\delta \equiv \frac{2D}{(\rho_n+\rho_m)}$ but not on $D$ or $\rho_n$ separately.

3.10. RESISTANCE OF PARALLEL WIRE.

Exact analytical integration in (34) is impossible. But in the important special case when conductors are lasting along external field, $H_0$, which lies in the film plane, there exists satisfactory analytical approximation. In particular, for magnetic contribution to resistance of a single wire (at $x=0$), under neglecting anisotropy, we found the estimate

\[
\frac{R_{11}^{\text{wire}}[\text{Ohm}]}{w[\text{cm}]} \approx \frac{2\pi (2\pi + H_0) fX(f)}{[1 - X^2(f)] \arctanh X(f)} \exp \left\{ -\frac{\rho}{D} \arctanh X(f) \right\}, \quad \text{if } f_u < f < f_0(H_0 + 2\pi),
\]

(3.35)

\[
X(f) \equiv \frac{f^2_0 - f_u^2}{(2\pi f_0)^2}, \quad f_u \equiv f_0\sqrt{H_0(H_0 + 4\pi)}, \quad \arctanh X = \frac{1}{2} \ln \frac{1 + X}{1 - X}
\]

In these formulas $f$ is frequency in GHz, $H_0$ expressed in units of $M_s$, and $f_u$ is the uniform spin precession frequency of in-plane magnetized film (with no anisotropy, see next Section). Outside of the marked frequency interval, resistance turns into zero.
Corresponding wire inductance can be simply estimated at $\delta \lesssim 0.5$ only, and in this case

$$\frac{L_{\text{wire}}[^{\text{nH}}]}{w[^{\text{cm}}]} \approx -(2 + H_0/\pi) e^{-X(f)} \text{Ei}(X(f)),$$

with $\text{Ei}(\xi)$ being exponential integral function [4].

Notice that at in-plane magnetization $W_0 = H_0$ because in-plane demagnetization factors are equal to zero.

3.11. NUMERICAL EXAMPLES AND DAMON-ESHBACH WAVES.

In general, the integrals can be obtained by numerically. For this purpose, formula (14) is better than (31) because numerical procedure deals directly with matrix (16). Some results are illustrated by Figs.1a-1d, 2a-2b and 3a-3b. Since we are most interested in YIG samples the value $M_s \approx 140$ Oe was substituted.

The Fig.1a relates to the same case as the Eq.36. To show characteristic influence by anisotropy, two pairs of curves are presented, for $A_{1,2} = 0$ and $A_2 = -A_1 = H_a = 0.5$, supposing its main axes are $x$- and $z$-axes at $y$-directed field (i.e. $\psi = 0$). As typically, anisotropy increases frequencies of MSW running across the field. Unfortunately, real anisotropy in YIG (function $A(S)$ in Eq.2.2) involves essential complications. For the present, in next examples it was neglected at all. Inductors lie on one and the same top side of film and last along $y$-axis while $H_0$ is oriented either along $y$- or $x$-axis.

The Fig.1b presents impedance for the loop consisting of two parallel wires which continue one another and carry the same current but in opposite directions (see inset in Fig.1d). Any inductor parallel to magnetization naturally generates so-called Damon-Eshbach magnetostatic waves first discovered in [5]. They run perpendicular to $S_0$. Uniquely, their dispersion law can be found (if neglect anisotropy) in simple analytical
Fig. 1a. Film obliged impedance (per unit length) of cylindric wire lying on film in parallel to in–plane static magnetization, at external field $H_0 = 3M_s$, anisotropy $H_a = 0$ or $H_a = 0.5M_s$ (left or right–hand curves), film thickness $D=10 \, \mu m$ and wire radius $\rho=20 \, \mu m$ ($\delta = D/\rho=0.5$).

Fig. 1b. Film obliged impedance of two–element loop with current parallel to static magnetization, at the wires separation $l=0.05 \, \text{cm}$, film thickness $D=10 \, \mu m$, $H_0 = 3M_s$, $H_a = 0$, $\rho=20 \, \mu m$. 

Active Impedance
Reactive Impedance
Fig. 1c. Mutual impedances $Z_{21}$ and $Z_{12}$ of cylindric wires parallel to in-plane film magnetization and distanced by $x=0.1$ cm from each other, at $H_0=3M_s$, $D=10\mu$, $\rho=20\mu$. Wire 2 is on the right of wire 1 if look in the field direction.

Fig. 1d. Mutual impedance $Z_{21}$ of two identical loops parallel to film magnetization and separated by $x=2l=0.1$ cm (with $l=0.05$ cm being inter-wire distance in loops), at $H_0=3M_s$, $D/\rho=0.5$. Thick and thin curves are Re $Z_{21}$ and Im $Z_{21}$, respectively. Loop 2 is on the right of loop 1 if look in the field direction.
Maximums of the loop resistance on Fig.1b exactly correspond to Damon-Eshbach (DE) waves with lengths $2ml$, where $l$ is the width of the loop (i.e. inter-wire distance) and $m = 1, 3, ...$ odd integers just as one could expect. Notice that all the spectrum of DE waves lies above the uniform precession frequency, $\omega_u = \sqrt{H_0 (H_0 + 4\pi)}$. In dimensional form, at $H_0/M_s = 3$ chosen for these examples $f_u \approx 2.68$ GHz.

Figs.1c and 1d show mutual impedances of two wires and two loops respectively, under the same orientation. Both ”from left to right” impedance and ”from right to left” are presented in Fig.1c. The latter clearly demonstrates violation of the reciprocity: every inductor placed above the film top better excites waves going clockwise from $H_0$ and $S_0$ than inverse waves. To change preferred direction one must remove inductors under film. The half of the sum of these two impedances by its magnitude, equals approximately to impedance on Fig.1a. One may find also that maximums of absolute value of impedance in Fig.1d and maximums of resistance in Fig.1b take place at the same frequencies and have equal amplitude ratios. In other words loops separation strongly influences phase of mutual impedance but slightly its frequency filtering characteristics.

Figs.2a and 2b relate to wires and loops which are oriented perpendicular to magnetization (see inset in Fig.2b) and hence excite MSW running along it. The essential difference from previous case is that these MSW are irradiated symmetrically (reciprocity takes place), and their spectrum lies below $\omega_u$. Since this highest frequency responds to least wave number, the group velocity of these waves is negative, i.e. directed in opposite to phase velocity.
Fig. 2a. Impedances of wire and of loop with \( l = 0.05 \) cm perpendicular to in–plane static magnetization, at \( H_0 = 3M_s \), \( D/\rho = 0.5 \), \( M_s = 140 \) Oe.

Fig. 2b. Mutual impedance of two loops with inter–wire distance \( l = 0.05 \) cm separated by \( x = 2l = 0.1 \) cm and perpendicular to in–plane magnetization, at \( H_0 = 3M_s \), \( D/\rho = 0.5 \).
3.12. ROLE OF FILM THICKNESS.

Let us return to wire parallel to magnetization, and consider Fig.3a which contains the series of resistance via frequency curves for different thickness values. Naively, one would predict nearly linear dependence $R(D)$ if suppose the EMF (voltage) be proportional to time-varying magnetization, $S_\perp$, and to a number of contributing spins (that is to thickness), $\varepsilon \propto D|S_\perp|$, while $S_\perp$ proportional to exciting current, $|S_\perp| \propto I$. But Figs.3a,3b and formula (36) show that, at better excited lower part of the spectrum, resistance is almost independent on $D$ (higher frequencies are rejected merely by the exponential wire form-factor, see Eq.7).

What is the matter? The answer comes from Eqs.1 and 38: the smaller thickness the smaller group velocity of irradiated waves $v_g$, therefore, the smaller is energy outflow from the inductor. Assume this outflow be proportional to product $Dv_g|S_\perp|^2$, equate it to the pumped power $RI^2$ and combine the resulting relation with $\varepsilon \propto D|S_\perp|$. These reasoning yield $R \propto D/v_g$. Taking into account that the ratio $v_g/D$ is a function of product the $D|k|$ or, equivalently, of the frequency only, we get the explanation of approximate constancy of resistance.

From Fig.3a and Eq.36 it is evident that at $D \gtrsim 1.5\rho$ the form-factor becomes unimportant and resistance almost independent on $D$ in all the frequency region. The strong rise of resistance at frequencies close to $f_u + 2\pi f_0$ reflects fast falling of group velocity in this region. Indeed, the Eq.38 implies for DE waves

$$v_g = D\{(H_0 + 2\pi)^2 - \omega^2\}/\omega$$ (3.38)

Hence, the assumption $R \propto D/v_g$ implies $R \propto 1/(H_0 + 2\pi - f/f_0)$ what is qualitatively confirmed by both Eq.36 and numerical results. This rise becomes better
clear if notice that less group velocity means greater density of (excited) states (what
is highlighted by right-hand peaks condensation in Fig.1b).

Incidentally, we can conclude that $|S_\perp| \propto I/D$, i.e. at given current the more thin
is film the stronger swing of magnetization and thus the closer nonlinear excitation
regime. But EMF voltage signal remains approximately the same, even if $|S_\perp|$ is
comparable with unit. Therefore, to get greater voltage signal in nonlinear regime, one
is enforced to make film thicker.

3.13. ROLE OF FRICTION.

According to Eqs.16 the factual dimensionless friction coefficient is not $\gamma$ itself but
product $\Gamma \equiv \gamma \omega$ (also small quantity). In dimensional form,

$$\Gamma = \gamma \omega / \tau_0 = 2\pi \gamma f \lesssim 2\pi \gamma f_0 (H_0 + 2\pi)$$

(3.39)

The latter estimate relates to in-plane magnetized film where, as we could conclude,
$f_0(H_0 + 2\pi)$ is the upper bound of MSW spectrum (for small anisotropy and long
MSW as compared with $r_0$; see Sec.4).

In infinite-size film under above formal consideration any external source generates
continuous wave spectrum, regardless of concrete $\Gamma$ value, i.e. non-resonant excitation
takes place. That is why $\gamma$ in no way manifests itself in (36). But real finite-area
film has discrete MSW spectrum. If characteristic frequency separation of excitable
eigenwaves $\delta f$, essentially exceeds their spectral broadening, $\delta f \gg \Gamma / 2\pi$, then
it is principally possible to resonantly distinguish them.

Let us allow that we select only one-dimensional set of DE modes running in $x$ -
direction and uniform in $y$ -direction ($k_y = 0$ ). Their separation by wavenumber is
on order of $\approx 2\pi / d$, where $d$ stands for wire length. Hence, frequency separation is
$\delta f \approx f_0 (\partial \omega_k / \partial k) : (2\pi / d) = v_g / d$ (with $v_g$ being dimensional group velocity),
Fig. 3a. Film obliged wire resistance, $\Re Z_{11}$, via frequency at different dimensionless thickness $\delta = D/\rho : 0.025, 0.05, 0.1, 0.25, 0.5, 0.75, 1, 1.5, 2, 3.5$. Wire is parallel to in-plane static magnetization, $H_0 = 3M_s (M_s = 140 \text{ Oe})$.

Fig. 3b. Dependence of wire resistance on dimensionless film thickness $\delta = D/\rho$ at different frequencies (2700:100:3600 MHz) and the same conditions as in Fig. 3a.
and the resonance is possible if \( \Gamma d \lesssim v_g \). Expressing \( v_g \) from Eq.38, for example, at \( H_0 \sim 3 \) and \( D \sim 10 [\mu] \), we obtain \( v_g \approx 4\pi^2 D/\tau_0 \omega_u \sim 10^7 [\text{cm/s}] \). For \( \gamma \sim 3 \cdot 10^{-4} \) and \( D \sim 10 [\mu] \), any length \( d \ll 10 [\text{cm}] \) occurs sufficiently small!

In such the case is the theory applicable to real films with \( d \sim 0.5 [\text{cm}] \)? Yes if wave selection, under realistic source form-factor, is not so perfect as was assumed. Even if a few modes only with nonzero \( k_y \sim 2\pi/w \) are excited in addition to \( k_y = 0 \), the sufficiently small length easy fails down to 0.1 [cm] or less. Besides in real finite-amplitude process directly excited modes transmit their energy to other modes by means of non-linear wave interactions. Thus the latters effectively increase friction and approach situation to the idealized model.

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4. LINEAR WAVES IN FILMS AND PLATES: EIGEN-MODES AND DISPERSION LAWS

4.1. IN-PLANE WAVE REPRESENTATION.

Consider magnetic field created by the magnetization wave

\[ S_\perp \equiv Ve^{-i\omega t}, \quad V \equiv V(z) \exp\{i(k_xx + k_yy)\}, \]
where \( k = \{k_x, k_y\} \) is in-plane wave vector. For any function \( G(r) \) whose 3-dimensional Fourier transform is known be \( \widetilde{G}(K) \), with \( K = \{k_x, k_y, k_z\} \), and any function \( f(z) \), the relations take place as follows

\[
\int G(r-r')f(z')e^{i(k_x x + k_y y)}
\]

\[
\int G(k_z) dz', G(k, z) \equiv \int e^{ik_z z} \widetilde{G}(K) \frac{dk_z}{2\pi}
\]

If applying this theorem to dipole interaction matrix function defined by (2.5) and (2.6), in accordance with (2.35) we have

\[
G(k, z) = 2\pi \left[ \frac{(k \otimes k)/|k|}{i \left[ k_x \quad k_y \right] \text{sign}(z)} \right] e^{-|k||z|},
\]

where \( \delta(z) \) is Dirac delta-function. Expression (1) is dipole interaction kernel in \((k, z)\)-representation. In this representation, operator \( \hat{G} \), the linear dynamical operator \( \hat{W} \) defined in Eq.2.22, and the self-induced field take the form

\[
\hat{G}f \equiv \int_{-D/2}^{D/2} G(k, z - z')f(z')dz' \hat{W} = W_0 + A + r_0^2(|k|^2 - \nabla_z^2) + \hat{G}, h_S = -\hat{G}V
\]

SINGULARITY OF DIPOLE INTERACTION.

From Eq.1 we see that in the long wave limit, when \( |k|D \to 0 \) , all the matrix elements of \( G(k, z) \) turn into zero except the only singular element, \( G_{zz} \to 4\pi\delta(z) \), which performs purely local connection in \( z \)-direction:

\[
\hat{W} \Rightarrow W_0 + A - r_0^2\nabla_z^2 + 4\pi \vec{z} \otimes \vec{z}, \vec{z} \equiv \{0, 0, 1\}
\]

In this limit, \( h_S(z) \Rightarrow -4\pi \vec{z} V_z(z) \), that is the field vanishes everywhere outside film while in its interior in any layer \( z = \text{const} \) it fully reduces to magnetization of that layer.

This trivial fact of magnetostatics means with respect to spin dynamics that if being uniformly magnetized any separate flat layer acts on itself only. Therefore, spin
oscillations in different layers can behave independently, resulting in strange property of MSW: at $|k|D << 1$ many MSW branches with different normal wave numbers have almost the same frequencies very close to the frequency of uniform spin precession, $\omega_u$.

Hence, $\omega_u$ is essentially peculiar degenerated point of MSW spectrum. Although exchange interaction forbids too arbitrary $z$-distributions and removes exact degeneracy, the latter remains important in absolutely thick plates hindering resonant excitation of too long MSW.

4.3. UNIFORM PRECESSION.

Under uniform precession, all spins in the sample rotate with exactly the same phase, that is $V = const$, and $\vartheta(r) = const$ in Eq.2.29. According to (3), in this situation, as in case of boundless MW considered in Sec.2.10, operator $\hat{W}$ transforms to algebraic one. As the consequence, clearly, again the polarization decomposition is possible. Therefore, we can write

$$\omega_u^2 = \det \hat{W}, \hat{W} \equiv \hat{\Pi}(W_0 + \hat{A} + 4\pi \hat{z} \otimes \hat{z})\hat{\Pi}$$

Since matrix $\hat{W}$ has exactly the same structure as matrix in Eq.2.34, with $\hat{\tau}$ in place of $k/|k|$ and zero in place of $r_0^2k^2$, in fact the answer is already presented by Eq.2.36, namely,

$$\omega_u^2 \equiv (W_0 + A_1)(W_0 + A_2) + 4\pi(W_0 + A_1 \sin^2 \psi + A_2 \cos^2 \psi) \sin^2 \theta$$

The angles $\theta$ and $\psi$ were introduced in Sec.3.6. This expression is equivalent to Eq.3.26 obtained when considering MW excitation.
The polarization and eccentricity also can be obtained with recipes of Sec.2.10 and 2.11. Of course, in absence of anisotropy the main axes look as $a \parallel z$ and $b \parallel [S_0 \times z]$. In this case,

$$p = |b|/|a| = \sqrt{1 + \frac{4\pi}{W_0} \sin^2 \theta} , \quad (A = 0) ,$$

that is polarization ellipse of uniform precession is always in-plane stretched (spins do not like piercing the plane).

For further, let us agree that $A = 0$ and $\psi = 0$ will mark neglected (or effectively negligible) anisotropy or specially oriented anisotropy (see Sec.3.6), respectively.

4.4. DISPERSION EQUATION.

In fact, the results of Sec.3 are sufficient to analyze all the variety of free eigenwaves in (absolutely thick) film geometry. As usually, their frequencies are nothing but poles of the response function, $F(\omega, k)$ (see Eqs.3.14, 3.31), at vanishing friction. Hence, to get the dispersion equation, we must equate denominator of (3.31) to zero. Elimination of friction from formulas of Sec.3 is achieved merely by returning real quantity $W_0$ (static effective field) in place of complex one, $\overline{W}_0$ introduced in Eq.3.16. Thus we come to the dispersion equation as follows

$$\coth [\Lambda(\omega)|k|D] = \frac{(\omega_3^2 - \omega^2) \text{sign} (\omega^2 - \omega_u^2)}{\sqrt{(\omega_3^2 - \omega^2)(\omega_u^2 - \omega^2)}} , \quad \Lambda(\omega) = \frac{\sqrt{(\omega_3^2 - \omega^2)(\omega_u^2 - \omega^2)}}{|\omega^2 - \omega_u^2|}$$

In view of Eqs.3.22-3.28 and Eq.4, here $\omega_3 = \omega_3(k/|k|)$ is depending on orientation of in-plane wave vector while $\omega_u^2$ is constant.

At given in-plane wave vector, Eq.7 has either one or infinitely many real roots with respect to $\omega^2$, determining different MSW branches. For any of roots $\omega^2 = \omega_N^2(k)$, we can obtain also corresponding pair of out-plane wave numbers defined in accordance with Eqs.3.10-3.12 and 3.30:
\[ i q_\pm(k) = \lambda_\pm(\omega)|k| = ik_0(\omega) \pm \Lambda(\omega)|k|, \quad k_0(\omega) \equiv \lambda_0(\omega)|k|/i = \frac{4\pi\Omega_x|k|}{\omega^2 - \omega_n^2}, \quad (4.8) \]

with \( \Omega_x = \Omega_x(k/|k|) \) being expressed by Eq.3.15, and \( \omega = \omega_N(k) \). Notice that \( \Omega_x \) and consequently \( k_0(\omega) \) always are real-valued.

4.5. TWO TYPES OF WAVES.

In a bulk wave, by its definition, magnetization harmonically oscillates along \( z \)-axis i.e. \( \Lambda(\omega) \) takes some imaginary value. Clearly, this is the case if a root of Eqs.7 belongs to the interval

\[ \omega_1^2 < \omega^2 < \omega_2^2 \quad (4.9) \]

Otherwise, \( \Lambda \) is real and magnetization varies exponentially responding to what is usually termed surface wave. However, due to the common scaleless nature of MW governed by dipole interaction, characteristic exponents \( \pm \Lambda(\omega)|k| \), are of order of in-plane wave number. At \( D|k| >> 1 \), such a wave is indeed concentrated in the vicinity of film surfaces. But in practically important case, when in-plane wavelength is greater than thickness this wave is indistinguishable from bulk one. The Figs.1a-1d and 3a-3b relate just to such ”surface-bulk” MSW (see below).

At arbitrary wave vector \( k \), Eq.7 has infinitely many solutions in the interval (9) but no more than one solution outside this interval, i.e. there are infinitely many branches of bulk MSW but unique branch of surface MSW.

4.6. PHASE VELOCITY VECTOR.

It should be underlined that generally neither bulk nor surface waves are standing waves with respect to \( z \)-coordinate. The matter is that \( \Omega_x \) and thus \( k_0(\omega) \) differ from zero, and actual wave phase is \( \vartheta(r) = k_xx + k_yy + k_0z \). In other words phase velocity vector in MSW is not in-plane oriented but has also non-zero out-plane
component. This fact means that usual attempts to find eigenwaves assuming
the
\[ q_- = -q_+ \]
are wrong. According to Eq.3.25, this equality takes place at
special orientation of \( k \) or static magnetization only. In particular, for \( A = 0 \), it
is true if \( \nu \parallel \cos \theta \sin \theta = 0 \), i.e. if static magnetization \( S_0 \) vector is either strictly
orthogonal or strictly parallel to film plane or if \( k \) is strictly parallel to \( S_0 \) projection
onto this plane.

4.7. CHARACTERISTIC FREQUENCIES.

Let us discuss characteristic frequencies defined by Eqs.3.26-28. It can be proved,
firstly, that the sum \( \Omega_{zz} + \Omega_{\nu\nu} \) is always positive and, secondly, the expression under
square root in Eq.3.27 is always non-negative. Hence, indeed \( \omega_1^2 \) and \( \omega_2^2 \) are positive
values and both the frequencies \( \omega_1 \) and \( \omega_2 \) do exist. Thirdly, the inequalities

\[
\omega_1^2 \leq \omega_u^2 \leq \omega_2^2
\]  (4.10)

take place, that is the uniform precession frequency is either immersed into bulk waves
spectrum or coincides with its edge. This is manifestation of above mentioned fact that
arbitrary internal layer can undergo autonomous precession.

Next, notice that \( \omega_3 \geq \omega_1 \) (see Eq.3.28). In view of this inequality as combined
with (10), the dispersion equation (7) can not be satisfied at \( \omega < \omega_1 \). At the same
time, it is easy to see that each frequency from the bulk waves interval (9) can be
solution to Eq.7 at some appropriate \( |k| \). Consequently, \( \omega_1 \) is nothing but lower
bound of total MSW spectrum.

It is necessary to remember that all the characteristic frequencies except \( \omega_u \) only,
are flowing in the sense of their essential dependence on wave direction, even at \( A = 0 \).
This is specific anisotropy dictated by flat geometry of dipole interaction.
From previous reasonings we must conclude that the surface eigenmodes if they exist at all, possess frequencies higher than any bulk wave, with $\omega_2$ being lower bound of surface wave spectrum. But, evidently, the Eq.7 does not have roots at $\omega > \max(\omega_2, \omega_3)$. Hence, surface waves do exist at those directions $\nu = \frac{k}{|k|}$ which satisfy the condition $\omega_3^2 > \omega_2^2$. Then Eq.7 has roots at $\kappa(\nu) < |k| < \infty$ with lowest wave number, $\kappa(\nu)$, determined by equality $\omega_3^2 = \omega_2^2$:

$$D\kappa(\nu) = (\omega_2^2 - \omega_3^2)/(\omega_3^2 - \omega_2^2)$$ (4.11)

4.8. SURFACE WAVES.

From previous reasonings we can conclude that the surface eigenmodes if they exist at all, must possess eigen-frequencies higher than bulk waves with $\omega_2$ being lower bound of their spectrum. But, evidently, the Eq.7 does not have roots at $\omega > \max(\omega_2, \omega_3)$. Consider more carefully the condition for surface modes to exist, $\omega_3^2 > \omega_2^2$. With the help of Eq.3.28, this condition takes the form

$$(4\pi \sin \phi \sin \theta)^2 > \omega_3^2 - \omega_1^2 = 4\pi \sqrt{(\Omega_{zz} + \Omega_{\nu\nu})^2 - (2 \sin \phi \sin \theta)^2\{(W_0 + A_+)^2 - A_-^2\}}$$ (4.12)

($\phi$ was defined in Eqs.3.17). The Eqs.3.22-23 imply

$$\Omega_{zz} + \Omega_{\nu\nu} = (W_0 + A_+)(1 + \sin^2 \phi \sin^2 \theta) +$$

$$+ A_-[(\sin^2 \phi \sin^2 \theta + \cos 2\phi) \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi]$$

Analyzing these formulas one can see that the requirement (12) can be most easy satisfied for nearly in-plane static magnetization $S_0$ and then for waves propagating nearly perpendicular to $S_0$. Any surface modes with parallel propagation or in
normally magnetized film are clearly forbidden. From the other hand, taking exactly in-plane \( S_0 \) and \( \nu_2^2 = 1 \), i.e. strictly perpendicular waves we come from (12) to inequality

\[
2\pi > |A_\perp|,
\]  

which is practically always true.

Hence, if anisotropy is not extremely strong (in the sense of (14)), then surface waves definitely exist in some region surrounding the point \( \theta = \phi = \pi/2 \). For given \( \theta \) and \( \phi \) in this region, let \( \omega_h \) be their highest (upper) frequency (thus \( \omega_h \) is also upper bound of MSW spectrum). Naturally, \( \omega_h \) is achieved at \( |k| \to \infty \) when \( \cot |k|D \to 1 \), therefore it follows from Eq.7 that

\[
\omega_h^2 = \frac{\omega_3^4 - \omega_1^2 \omega_2^2}{(4\pi \sin \phi \sin \theta)^2} = \left( \frac{\Omega_{zz} + \Omega_{\nu\nu}}{2 \sin \phi \sin \theta} + 2\pi \sin \phi \sin \theta \right)^2
\]  

(4.15)

Particularly, for strictly in-plane magnetization and perpendicular propagation, \( k \perp S_0 \), this expression reduces to

\[
\omega_h(\theta = \pi/2, \phi = \pi/2) = W_0 + A_+ + 2\pi
\]  

(4.16)

We can expect this is maximum \( \omega_h \).

4.9. ISOTROPIC CASE.

All the algebra becomes much more visual if anisotropy contribution disappears. This does not necessarily mean that anisotropy is absent at all. For example, in the easy axis or easy plane case, anisotropy energy (2.2) and matrix \( \hat{A} \) (see Sec.2, Eq.2.11) take the form

\[
A(S) = A_0 \langle \hat{u} S \rangle^2 / 2, \quad \hat{A} = A_0 \hat{u} \otimes \hat{u}
\]
where \( \mathbf{u} \) is unit vector showing easy (or heavy) axis. Hence, at special static magnetization, when \( S_0 \parallel \mathbf{u} \), the projected anisotropy matrix exactly turns into zero: 
\[ \hat{\Pi} \hat{A} \hat{\Pi} = 0 \.
\]
In absence of anisotropy contribution, the inequality (10) reduces to
\[
\omega^2 = W_0^2 + 4\pi W_0 \nu^2 \sin^2 \theta \leq \omega_u^2 = W_0^2 + 4\pi W_0 \sin^2 \theta \leq \omega_2^2 = W_0^2 + 4\pi W_0 \ , \ (A = 0)
\]
(4.17)
We see that under in-plane magnetization, i.e. at \( \theta = \pi/2 \), (i) \( \omega_2 = \omega_u \), that is all the bulk wave spectrum lies under uniform precession frequency, and (ii) there are no bulk waves propagating perpendicular to static magnetization, because the equality \( \omega_1 = \omega_2 \) takes place at \( \nu^2 \parallel = \sin^2 \phi = 1 \).

The condition (12) determining the surface modes region now reads
\[
\sin^2 \phi \sin^2 \theta > \sin^2 \theta_0 \equiv \frac{W_0}{W_0 + 4\pi} \ , \ (A = 0) ,
\]
while their upper frequency is presented by
\[
\omega_h^2 = W_0(W_0 + 4\pi)\frac{1}{4} \left( \frac{\sin \theta_0}{\sin \phi \sin \theta} + \frac{\sin \phi \sin \theta}{\sin \theta_0} \right)^2 , \ (A = 0)
\]
In accordance with (16), its absolute maximum is \( \max \omega_h = W_0 + 2\pi \).

The lowest wave number of surface modes is achieved at \( k \) strictly perpendicular to \( z - S_0 \)-plane. From Eq.11 we obtain
\[
\min D\kappa(\nu) = D\kappa(\nu \perp S_0) = \frac{2W_0}{4\pi \tan^2 \theta - W_0}
\]
(4.19)
Naturally, it turns into infinity at \( \theta = \theta_0 \) when surface modes disappear. For in-plane magnetization it turns into zero, and then surface waves occupy all the sector.
\[
\frac{\nu_\parallel}{\nu_\perp} = \left| \frac{k_\parallel}{k_\perp} \right| < \sqrt{\frac{4\pi}{W_0}} \quad (4.20)
\]

It should be added that under in-plane external field \( \theta = \pi/2 \), and the static internal field, \( W_0 \), trivially reduces to the external one, \( W_0 = H_0 \), because of zero demagnetization.

4.10. DAMON-ESHBACH WAVES.

As was mentioned, at \( A = 0 \), in-plane magnetization ( \( \theta = \pi/2 \) ) and \( \sin^2 \phi = 1 \) all three frequencies present in (10) coincide one with another. Hence, in this specific case identically \( \Lambda(\omega) = 1 \), and with accounting for (17) the Eq.7 becomes linear equation:

\[
\coth (|k|D) = (\omega_3^2 - \omega^2)/(\omega^2 - \omega_u^2), \quad \omega_3^2 = \omega_u^2 + 8\pi^2, \quad \omega_u^2 = H_0(H_0 + 4\pi)
\]

Its solution is given by the classical Damon-Eshbach dispersion law (3.38). Otherwise, unfortunately, Eq.7 can not be solved in such an evident form. But in wide sense all the surface modes can be called Damon-Eshbach waves.

4.11. MSW CLASSIFICATION.

In general, dispersion law must be obtained numerically. But also we may treat Eq.7 be evident expression for \( |k| \) as a function of wave direction and wave frequency:

\[
|k|D = \text{Re} \frac{|\omega^2 - \omega_u^2|}{\sqrt{(\omega^2 - \omega_1^2)(\omega_2^2 - \omega^2)}} \left\{ \pi N + \arctan \frac{\text{sign}(\omega^2 - \omega_u^2)\sqrt{(\omega^2 - \omega_1^2)(\omega_2^2 - \omega^2)}}{\omega_3^2 - \omega^2} \right\}, \quad (4.21)
\]

where \( N \) is non-negative integer, the square root should be chosen in upper half-plane, while \( \arctan \) in right-hand half-plane, and the requirement \( |k|D \geq 0 \) serves as selection rule for permissible frequencies. Under these conditions \( N \) plays no role outside of interval (9) while inside it \( N \geq 0 \) enumerates branches of bulk waves with
Fig. 4a. Equi-frequency lines of lowest bulk MSW mode (left plot) and surface MSW (right-hand plot), for in-plane magnetized film ($\theta=90^\circ$), at $S_0 \parallel Y$, $H_x=3$ and $H_a=0$. The fat line marks edge of surface wave region, $|k_y|/|k_x|=(4\pi H_0)^{1/2}$. The arrows look to higher frequencies.

Fig. 4b. Equi-frequency lines of lowest bulk MSW mode with frequency less (on the left) and greater (dashed line on the right) than $\omega_u$ (uniform precession frequency) and of surface MSW (right-hand plot), for film magnetized at angle $\theta=45^\circ$, with $S_0 \parallel YZ$-plane, $W_0=3$ and $H_a=0$. The fat curve marks the edge of surface waves.
different normal wavenumbers $|k|D|\Lambda(\omega)|$. In fact at any $N$ two modes can be distinguished, with $\omega < \omega_u$ and $\omega > \omega_u$, to be enumerated as $N+$ and $N-$, respectively.

4.12. MAIN MODES.

Fig. 4a shows equal-frequency lines calculated from Eq. 22 for the lowest bulk mode $0-$ and surface mode under in-plane magnetization and in absence of anisotropy (for details see captures). That are just the two sorts of MSW discussed in Sec. 3 as contributors to impedances at Fig. 2a-2b and 1a-1d, respectively.

Wonderfully, their frequency spectra, although lying on opposite hands from $\omega_u$, at $|k|D \to 0$ are sewed together in the directions (21), i.e. along the edge of surface wave region. Therefore, with respect to sufficiently long waves both the sorts can be effectively unified into single main mode. Corresponding expansion of Eq. 7 or Eq. 22 gives its dispersion law as follows:

$$\omega = \omega_0(kD), \quad \omega_0(kD) \approx \omega_u + \frac{\omega_u^2}{2\omega_u}|k|D = \omega_u + \frac{\pi D(4\pi k_x^2 - H_0 k_y^2)}{|k|\sqrt{H_0(H_0 + 4\pi)}}, \quad \text{(4.22)}$$

This equation extends the Damon-Eshbach formula (3.38) to arbitrary propagation angles although at small wave numbers. The inequality $|k|D \lesssim 0.2$ is quite sufficient to apply Eq. 23.

Fig. 4b shows what does occur if static magnetization is put out from the film plane. We see that now main bulk mode $0-$ becomes strongly separated from surface wave but instead the latter well merges with mode $0+$. 

4.13. EFFECTS OF ANISOTROPY.

To feel principal influence by anisotropy, let us confine ourselves by special case $\psi = 0$, when one of main axes of $\hat{\Pi}\hat{A}\hat{\Pi}$, namely $\overline{A}_1$, lies in $z-S_0$-plane while $\overline{A}_2$ in $x-y$-plane. Then
Fig. 5a. Equi–frequency lines of main bulk MSW mode with frequency less (on the left) and greater (dotted line on the right) than $\omega_u$ and of surface MSW (right–hand plot), for nearly in–plane magnetized anisotropic film, at $\theta=85^\circ$, $H_0=3$, $H_a=1.5$ and $\psi=0$. The arrows look to higher frequencies.

Fig. 5b. Equi–frequency lines of main bulk MSW mode with frequency less (on the left) and greater (dotted line on the right) than $\omega_u$ and of surface MSW (right–hand plot), for nearly in–plane magnetized anisotropic film, at $\theta=88^\circ$, $H_0=3$, $H_a=-1.5$ and $\psi=0$. The fat curve marks the edge of surface wave region.
\[ \omega_\alpha^2 = W_2(W_1 + 4\pi \sin^2 \theta), \ (\psi = 0), \quad (4.23) \]

\[ \omega_{1,2}^2 = W_1W_2 + 2\pi[W_1 \sin^2 \phi + W_2(1 - \cos^2 \theta \sin^2 \phi)] \mp 2\pi[(W_1 \sin^2 \phi - W_2 \sin^2 \theta)^2 + (4.24) \]

\[ + W_2^2 \cos^2 \theta \cos^2 \phi (1 + \sin^2 \theta - \cos^2 \theta \sin^2 \phi) + 2W_1W_2 \cos^2 \theta \sin^2 \phi \cos^2 \phi]^{1/2}, \]

where \( W_{1,2} = W_0 + A_{1,2} = W_0 + A_+ \mp A_- \). Instead of (20) one can obtain

\[
\min D\kappa(\nu) = D\kappa(\nu \perp S_0) = \frac{X + |X|}{4\pi \sin^2 \theta - |X|}, \quad X \equiv W_1 - W_2 \sin^2 \theta, \ (\psi = 0) \quad (4.25)
\]

Clearly, the role of \( A_+ \) is merely shift of the internal field, and non-trivial effects may come from the effective anisotropy field \( H_a \equiv A_- \) only.

For example, consider nearly in-plane magnetized film (\( \theta = 85^\circ \)) with such an anisotropy. From Eq.26 it is seen that at \( H_a > 0 \) (\( W_1 < W_2 \)) and \( \theta \) close to \( 90^\circ \) the beginning of the surface wave sector remains staying at \( k = 0 \). That is illustrated by Fig.5a. It shows also that interface between surface mode and \( 0^+ \) bulk mode is rather sharp, so that the \( 0^- \) mode (left plot) seems be better continuation of the surface sector. Indeed, at \( \theta \to 90^\circ \) the picture becomes very similar to Fig.4a.

Hence, positive anisotropy field, \( H_a > 0 \), (i) produces no qualitative change in relation between the two sorts of waves. However, it (ii) allows \( 0^- \) mode to propagate in \( x \)-direction, (iii) in accordance with Eq.24, it narrows down the surface wave sector, and (iv) rises \( \omega_\alpha \) and thus MSW frequencies (as reflected by Fig.1a too).

In contrast, Fig.5b demonstrates that negative anisotropy, \( H_a < 0 \) (\( W_1 > W_2 \)), (i) causes essential change in relative disposition of bulk and surface modes on \( k \)-plane. Namely, the \( 0^+ \) mode becomes captured in the bubble immersed into the
surface sector (the right-hand edge of this bubble is determined by Eq.26). The matter is that at $\theta \approx 90^\circ$, according to Eqs.24 and 25,

$$\omega_u^2 = \omega_1^2 \text{ at } \sin^2 \phi > W_2/W_1, \quad \omega_u^2 = \omega_2^2 \text{ at } \sin^2 \phi < W_2/W_1$$

This results also in (ii) precise angular separation of 0− (left plot) and 0+ modes which facilitates to treat, at $|k|D << 1$, all three modes as single anisotropic mode. Besides negative anisotropy (iii) obviously expands the surface sector and (iv) decreases $\omega_u$ and MSW frequencies.

4.14. LONG-WAVE ASYMPTOTICS AND EXCHANGE CONTRIBUTION.

In general (at $\sin^2 \theta < 1$), the dispersion law for long MSW, as compared with film thickness can approximated by linear function of $|k|D$, like in Eq.23. The Eq.7 yields

$$\omega^2_{N \pm} - \omega_u^2 \approx \frac{|k|D(\omega_3^2 - \omega_u^2)X}{\pi N + \arctan X + \pi(1 - \text{sign } X)/2}, \quad X \equiv \pm \sqrt{\frac{(\omega_u^2 - \omega_1^2)(\omega_2^2 - \omega_u^2)}{\omega_3^2 - \omega_u^2}}$$

(4.26)

Hence, MSW modes with greater $N$ are more strongly pressed to the uniform precession frequency. This paradoxical fact is the consequence of above discussed singularity of dipole interaction.

Under strictly in-plane magnetization ( $\sin^2 \theta = 1$), at $N = 0$ this expression reduces to Eq.23 which unify main modes (may be with the surface one instead of 0+). But for $N > 0$ dispersion becomes quadratic leading to even stronger frequency compression:

$$\omega^2_{N \pm} - \omega_u^2 \approx \pm(|k|D)^2(\omega_u^2 - \omega_1^2)/(\pi N)^2, \quad (N > 0)$$

(4.27)
However, at sufficiently large $N$ the exchange interaction enters the game and increases $\omega_{N\pm}$. At least under in-plane magnetization, the exchange contribution can be described by the replacement

$$W_0 \Rightarrow W_0 + r_0^2(|k|^2 + q_{N\pm}^2) \approx W_0 + r_0^2(\pi N/D)^2,$$

(4.28)

with $q_{N\pm}$ being out-plane wave numbers $q_{\pm}$ (see Eq.8), for $N$-th modes in exact analogy with how exchange interaction contributes to frequency of boundless wave (Sec.2.11). In the latter equality, we took into account that for higher order long-wave modes $r_0|k|$ is negligibly small while their normal wave numbers $q_{\pm} = \pm|k|\Lambda(\omega)$ are close to $\pi N/D$.

5. Some aspects of non-linear phenomena and chaos

5.1. Nonlinear Processes.

The unique peculiarity of spin waves (SW) and especially magnetostatic waves (MSW) in ferrimagnets (e.g. YIG) is that their relaxation rate, $\Gamma \sim 5 \cdot \mu \text{s}^{-1}$ or even less is very small as compared with other wave excitations in solids in the same (microwave) frequency region. So low decay ensures effective generation and nonlinear transformations of MSW at small pumping power [1-5].

At the same time, usually swing of spin precession remains far from spin flipping, i.e. $|S_\perp|^2 << 1$ in Eq.2.19, therefore three-wave and four-wave processes only are of great importance. In the firsts either (P) some already excited mode with frequency $\omega_0$ serves as parametric pump for two other modes whose frequencies satisfy the condition $\omega_1 + \omega_2 = \omega_0$ or, in opposite, (G) two modes mix up one another being the source for $\omega_0$ mode. Here the bracketed letters G and P, abbreviate generation and parametric excitation. Among fourth-order processes most important one is the combined G-P-process satisfying $\omega_3 + \omega_4 = \omega_1 + \omega_2$. 
If accounting for these processes only, the Eq.2.19 transforms into the approximate equation,

$$\frac{dS_\perp}{dt} = \left[ S_0 (W_0 + \hat{L}) S_\perp - h \right] - \gamma \hat{\Pi} (W_0 + \hat{L}) S_\perp -$$

$$- \left\langle S_0 \hat{L} S_\perp - h \right\rangle [S_0 S_\perp] - [S_0 \hat{L}(|S_\perp|^2 S_0)] / 2 -$$

$$- |S_\perp|^2 [S_0 \hat{L} S_\perp] / 2 + \left\langle S_0 \hat{L}(|S_\perp|^2 S_0) \right\rangle [S_0 S_\perp] / 2,$$

where three rows contain linear, quadratic and cubic terms respectively. Higher-order terms and all the nonlinear contributions to friction (as well as to anisotropy, see Sec.2.5) are neglected, and most important entries of $h(r, t)$ are kept only.

Clearly, the external field, $h(r, t)$, also can act as either additive source (G-process) or parametric pump (P-process). The first variant is more effective if realizes by way of ferromagnetic resonance (FMR). In best real YIG samples the power consumption of order of tens microwatt may be sufficient to initiate nonlinear processes [2]. At greater pump, one can observe rich variety of nonlinear phenomena including formation of envelope solitons [2,3-9], parametric amplification [10-12], magnetization reversal [13], self-focusing of MW beams[14], generation of harmonics subharmonics and ultra-short pulses [15], non-linear short electromagnetic waves [16] (the alternate to MSW high-frequency branch of mutual magnetization and EM-field hybridization).

But most interesting phenomenon is magnetic chaos (chaotic oscillations of magnetization pattern) produced if external pump exceeds certain critical level [2,17-19].

5.2. NONLINEAR WAVES.

In special class of nonlinear phenomena qualified as weakly nonlinear magnetic waves a narrow region of total MSW frequency band (all the more of whole MW spectrum) is involved only and, hence, third-order processes (quadratic terms in second row of
Eq.1) are not at business. In sufficiently long waves their dispersion (spatial derivatives) also is weak and therefore naturally separates from nonlinearity, so that approximate wave equation turns into sum of spatially non-local (differential) linear terms and local nonlinear (cubic) terms [3].

If speak about films the long magnetostatic nonlinear waves are of special interest composed by the main branch of linear MSW (most homogeneous with respect to normal $z$-coordinate). In this case the exchange part of operator $\hat{L}$ can be neglected, and dispersion is completely determined by dipole interaction (see Sec.4). But since the singular part of dipole interaction (Sec.4.1) is factually local, its product with cubic nonlinearity should be kept. Then the Eq.1 (as combined with Eq.4.1) reduces to

$$\frac{dS_\perp}{dt} = [S_0 (W_0 + \hat{A} + \hat{G}) S_\perp - h] - \gamma \hat{H} (W_0 + \hat{A} + 4\pi \mathbf{z} \otimes \mathbf{z}) S_\perp + (5.2)$$

$$+ \frac{1}{2} |S_\perp|^2 [S_0 (A_\parallel + 4\pi S_0^2 - \hat{A} - 4\pi \mathbf{z} \otimes \mathbf{z}) S_\perp]$$

where characteristic frequency of $h$ is supposed the same as carrying frequency of $S_\perp$. Of course, still this is formal storage only for more correct equation which must be free of third-order harmonics and concern $S_\perp$'s envelope. Such the equation can be deduced, as usually [20], from variational formulation of Eq.1, or by means of time averaging over the carrier period.

5.3. NONLINEAR WAVE EQUATION.

In accordance with Sec.2.9 and Sec.4.3, the main (as well as any other) branch of eigenwave modes looks as

$$S_\perp \propto V_k(r) \exp\{-i\omega_0(kD)t\}, \quad V_k(r) = \{a_k(z) + ib_k(z)\} \exp\{i \langle k, \rho \rangle\}$$
Here its dispersion law is written in the form $\omega = \omega_0(kD)$, $k = \{k_x, k_y\}$ is in-plane wave vector, $\rho \equiv \{x, y\}$, and $a_k(z)$ and $b_k(z)$ are mutually orthogonal real-valued vectors. Arbitrary non-autonomous (externally influenced) wave composed by these modes can be expanded into Fourier integral

$$S_\perp = \Re \int \{a_k(z) + ib_k(z)\}e^{i(k,\rho)C(k)\tilde{\Psi}(k, t)} \, dk,$$

where function $\tilde{\Psi}(k, t)$ contains one-signed (e.g. positive) frequencies only, that is represents an analytical signal. The $C(k)$ in (3) being real positive factor serves for suitable normalization of the eigenmodes. If it is fixed then, instead of (3), one can equivalently consider the “wave function”

$$\Psi(x, y, t) \equiv e^{i(k, \rho)\tilde{\Psi}(k, t)} \, dk$$

Correspondingly to (3), it useful to introduce analytical signal, $\tilde{h}$, for the external pump too:

$$h(r, t) = \Re \tilde{h}(r, t), \int e^{i\omega t}\tilde{h}(r, t) \, dt \equiv 0 \text{ at } \omega \leq 0$$

For the wave function, the Eq.2 implies the equation as follows (we omit its derivation):

$$\frac{\partial \Psi}{\partial t} + i\omega_0(-iD\nabla)\Psi = -i|x|\Psi^2\Psi - \Gamma\Psi + \eta$$

Here $\nabla = \{\partial/\partial x, \partial/\partial y\}$ and operator $\omega_0(-iD\nabla)$ (formally differential) is determined by the dispersion law. Let $a, b$ and $\alpha, \beta$ be the pair of eigenvectors and related eigenvalues of the uniform precession operator, $\tilde{W} \equiv \hat{\Pi}(W_0 + \hat{A} + 4\pi z \otimes z)\hat{\Pi}$, considered in Sec.4.3, and $p$ eccentricity of uniform precession. Besides for any two vectors $u$ and $v$, let $u_v$ means $u$’s projection onto $v$, i.e. $u_v \equiv \langle v, u \rangle/|v|$. 
Then the parameters of Eq.5, friction coefficient $\Gamma$, nonlinearity scale $\kappa$ and pump, $\eta$, read

$$\Gamma = (\alpha + \beta) \gamma / 2, \quad (5.6)$$

$$\eta = \frac{1}{2D} \int \left\{ \sqrt{p} h_a - i h_b / \sqrt{p} \right\} dz , \quad p = \sqrt{\frac{\alpha}{\beta}}, \quad (5.7)$$

$$\kappa = \frac{1}{8} \left\{ (W_0 + A_\parallel + 4\pi S_0^2) \left[ \frac{3}{2} \left( \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right) + 1 \right] - 2(\alpha + \beta) \right\}, \quad (5.8)$$

while approximate connection between the wave function and magnetization is established by

$$\Psi = \sqrt{p} S_{\perp a} - i S_{\perp b} / \sqrt{p}, \quad |\Psi|^2 \approx ||S_{\perp a}|| ||S_{\perp b}|| \quad (5.9)$$

Here $||...||$ denotes envelope (amplitude) of an oscillating variable.

In particular case of tangential magnetization and not strong anisotropy, vector $a$ is nearly parallel to normal $z$-axis vector $b$ lies in the film plane, and formulas (6-8) are simplified to

$$\Gamma \approx (H_0 + 2\pi) \gamma, \quad p \approx \sqrt{1 + 4\pi H_0}, \quad \kappa \approx -\frac{\pi(H_0 + \pi)}{H_0 + 4\pi} \quad (5.10)$$

5.4. NONLINEAR SHRÖDINGER EQUATION.

For waves and wave packets formed by a narrow set of in-plane wavevectors concentrated about some $k_0$, the Eq.5 reduces to the nonlinear Shrödinger equation (NLS),

$$\frac{\partial \psi}{\partial t} + \langle v_g \nabla \rangle \psi = i \left\langle \nabla, \nabla \right\rangle \psi - i \kappa|\psi|^2 \psi - \Gamma \psi + \tilde{\eta}, \quad (5.11)$$

$$\psi \equiv \exp\{i\omega_0(Dk_0)t - i \left\langle k_0 \rho \right\rangle\} \Psi , \quad \tilde{\eta} \equiv \exp\{i\omega_0(Dk_0)t - i \left\langle k_0 \rho \right\rangle\} \eta ,$$
Here \( v_g \) and \( \hat{D} \) are group velocity vector and diffusivity tensor, respectively, and \( \psi \) plays the role of envelope of wave function \( \Psi \).

Evident analytical expressions for the latter quantities can be obtained in a few special cases only, particularly, for Damon-Eshbach waves (see Sec.3 and Sec.4) in exactly in-plane magnetized film with zero (or weak) anisotropy. In this case, if magnetizing field \( H_0 \) is oriented along \( y \)-axis then for surface waves nearly parallel to \( x \)-axis \( (k_y^2 \ll |k|^2) \) the Eq.4.7 yields (in the dimensionless time units):

\[
\omega_0^2(Dk) \approx \omega_u^2 + \pi \{1 - \exp(-2D|k|)}(4\pi k_x^2 - H_0 k_y^2)/|k|^2
\]

Hence, wave packet running along \( x \)-axis with \( k_{0y} = 0 \), has group velocity and diffusivity as follow:

\[
v_{yy} = 0, \quad v_{yx} = 4\pi^2 D \exp(-2D|k_0|)\text{sign}(k_0)/\omega_0, \quad \omega_0 \equiv \omega_{DE}(D|k_0|), \tag{5.14}
\]

with function \( \omega_{DE}(D|k|) \) given by Eq.3.38, and

\[
\hat{D}_{xx} = -4\pi^2 D^2 \{(H_0 + 2\pi)^2 - 2\pi^2 \exp(-2D|k_0|)} \exp(-2D|k_0|)/\omega_0^3, \tag{5.15}
\]

\[
\hat{D}_{xy} = 0, \quad \hat{D}_{yy} = -\pi (H_0 + 4\pi)\{1 - \exp(-2D|k_0|)}/2|k_0|^2 \tag{5.16}
\]

In contrary to this specific case, generally propagation direction of envelope of the wave packet differs from its carrier wave direction, \( k_0 \). Clearly, the group velocity is perpendicular to equi-frequency curves shown at Fig.4a-b and Fig.5a-b (see Sec.4). These figures (as well as formulas of Sec.4) show that wave packets which are formed
by surface MSW and have non-zero $k_{0y}$ comparable with $k_{0x}$ must prefer directions characterized by

$$v_{gy} \approx \pm v_{gx}\sqrt{H_0/4\pi} \quad (5.17)$$

If the carrier wave is not long, that is the value $D|k_0|$ is comparable with unit, then the main axes of polarization ellipse, $a, b$, its eccentricity, $p$, and the eigenvalues $\alpha, \beta$ in Eqs.6-10 should be calculated just for the $k_0$ mode (instead of uniform one), i.e. mentioned as $a_{k_0}, b_{k_0}$, and so on, in the sense of Sec.2.9-10.

5.5. NON-ISOCRONITIVITY AND INSTABILITY OF MAGNETIC WAVES.

Consider autonomous waves i.e. in absence of pump and dissipation. Cubic nonlinear terms in Eq.5 and Eq.11 involve fundamental non-isochronity property of nonlinear MW: their frequencies depend on their amplitudes. Indeed, for a plane autonomous wave with amplitude $A$ the Eq.5 gives

$$\Psi = A \exp\{-i[\omega_0(Dk) + \kappa A^2]t + i \langle k, \rho \rangle\} \quad (5.18)$$

According to Eq.10, in tangentially magnetized film intensification of wave leads to lowering its frequency.

What does occur if the amplitude is not uniform but slightly spatially modulated? As in general [20], result depends on concurrence between nonlinearity and dispersion which in our case is described by diffusional term in Eq.11. To see the result, let us search for evolution of the wave envelope in the form

$$\psi(\rho, t) = [A + \chi(\rho - v_g t, t)]\exp(-i\kappa A^2 t), \quad \chi = \chi_1 + i\chi_2, \quad A = const, \quad (5.19)$$

with $\chi$ being (infinitely) small non-uniform perturbation. It is easy to derive from Eq.11 the linearized equations for $\chi_1(\rho, t)$ and $\chi_2(\rho, t)$ as follows
\[
\frac{\partial}{\partial t} \begin{pmatrix}
\chi_1 \\
\chi_2 
\end{pmatrix} = \begin{pmatrix}
0 & -\langle \nabla, \hat{D} \nabla \rangle \\
-\langle \nabla, \hat{D} \nabla \rangle & 2A^2 
\end{pmatrix} \begin{pmatrix}
\chi_1 \\
\chi_2 
\end{pmatrix}
\] (5.20)

Let initially the amplitude was periodically modulated with some wave vector \( q \), for instance, \( \chi_1(\rho, 0) \propto \cos \langle k, \rho \rangle \), \( \chi_2(\rho, 0) = 0 \). Then solution to (20) consists of two definitely weighted exponents \( \exp(\pm \lambda t) \), where \( \pm \lambda \) are eigenvalues of the right-hand matrix operator,

\[
\lambda^2 = -\langle q, \hat{D}q \rangle \left( \langle q, \hat{D}q \rangle + 2\kappa A^2 \right)
\] (5.21)

Here \( \langle q, \hat{D}q \rangle \) represents deviation of wave frequency coming from the dispersion. If it is of the same sign as the deviation due to non-isochronity then \( \lambda \) has imaginary value. Hence, in this case initial non-uniformity results in small amplitude and phase oscillations which in reality decay due to dissipation. But if

\[\kappa \langle q, \hat{D}q \rangle < 0 \text{ and } 2|\kappa|A^2 > |\langle q, \hat{D}q \rangle| + \Gamma^2 |\langle q, \hat{D}q \rangle|^{-1}, \] (5.22)

then the sufficiently intensive wave occurs unstable with respect to small amplitude disturbance. The latter grows and the wave inevitably breaks into a chain of energy slots (solitons).

According to the instability conditions (22), if friction was absent then sufficiently smooth spatial modulation always would unstable. Due to friction, however, both long and short modulations always are stable (taking into account that in reality \( A < 1 \), because of relations (9)). The instability starts from moderate modulation scales \( q_c \), and after exceeding at least minimum threshold amplitude value, \( A_{\text{min}} \), as determined by (22),

\[|\langle q_c \hat{D} q_c \rangle| \approx \Gamma, \ A_{\text{min}} \approx \sqrt{\frac{\Gamma}{|\kappa|}} \] (5.23)
5.6. SOLITONS.

Magnetic envelope soliton is a single spatially local wave packet stabilized (protected from diffusional bleed) by non-linearity and described by Eq.5 or Eq.11. Its existence is implied by the same instability (they must satisfy the first of the inequalities (22)), but to created it one should use suitably localized external pump, instead of a spreaded wave. These solitons are called also “bright solitons”.

The envelope of one-dimensional (flat) autonomous (at no pump at no friction) bright soliton, moving in direction of some unit-length vector \( \vec{n} \), is determined by the equations

\[
\psi(\rho, t) = e^{-i\Omega t} F(\xi) , \: \xi \equiv \langle \vec{n} \rho - v_g t \rangle , \: \delta \frac{d^2 F}{d\xi^2} - i\kappa F^3 + \Omega F = 0 , \: \delta \equiv \langle \vec{n} \hat{D} \vec{n} \rangle ,
\]

which directly follow from Eq.11. The solution to (24) is

\[
F(\xi) = A/\cosh(A\xi|\kappa|/2\delta|^{1/2}) , \: \Omega = \kappa \delta A^2 / 2 ,
\]

with the magnitude \( A \) being free parameter.

Alternatively, so-called “black” (dark) envelope solitons can exist representing “holes” (dips) in amplitude of spreaded (plane) wave. From the point of view of above consideration, these objects formally correspond to imaginary modulation wave vector, \( q \to iq \). Thus for them the first of the instability conditions (22) turns into opposite, but the second remains valid. Hence, sufficiently intensive MW inevitably loses stability and produces some soliton structure by either one or another way.

The envelope of black soliton satisfies the same equations (24), but with non-zero boundary values at infinity, and has the form

\[
F(\xi) = A \tanh(A\xi|\kappa|/2\delta|^{1/2}) , \: \Omega = \kappa A^2
\]

(5.26)
Clearly, both the types of solitons are as much narrow (wide) as strong (weak). The peculiarity of the black soliton is phase slip in its center by $\pi$. For a given direction $\nabla$, either bright (if $\delta x < 0$) or black (if $\delta x > 0$) solitons exist only. In reality, two-dimensional solitons are under use [3-9], but their analytical investigation is much more hard task.

5.7. MAGNETIC CHAOS.

The comprehend reviews of experimental data on magnetic chaos its theoretical interpretation and numerical reproduction are presented in [2,17,18,19]. The important conclusion from both theory and numerical simulations is that even two MW modes (i.e. four variables: two amplitudes and two phases) are sufficient to realize chaotic behavior.

In standard scenario, the mechanism of chaos is dependence of frequencies of the modes on amplitudes because of their nonlinear self-interaction and mutual parametric interaction. At small amplitudes they are coherently (resonantly or parametrically) excited by external field and one by another. At large amplitudes the coherence destroys and dissipative damping prevails which restores coherent interaction and returns to beginning of the cycle. Under sufficiently strong pump, this cycle becomes unstable with respect to infinitely small perturbation and thus chaotic.

Hence, the same property (non-isochronity) of magnetization oscillations at the same degree of nonlinearity is responsible for both chaos in only two-mode model and for regular soliton structures consisting of very many MW modes (let us recollect that autonomous NSE dynamics is integrable and thus can not produce chaos [20]). This fact demonstrates that principal origin of magnetic chaos is nothing but energy transfer through magnetic system (from external source to thermostat). In other words this is dissipative chaos characterized by phase volume contraction and dissipative strange (zero Lebesgue measure) attractors [21], although (due to small friction) possessing
many properties of Hamiltonian chaos [21]. Then, it is not surprising that just the power absorption (energy consumption by a ferromagnet sample per unit time) mostly highlights magnetic chaos [2,17,18] and can be used as control variable for its identification and synchronization [19].

In principle, in presence of periodic perturbation accompanied by dissipation even an individual spin (magnetic moment with fixed length and thus two independent variables only) can undergo chaotic behavior [22-24]. Nevertheless the two MW modes are too few to adequately imitate real magnetic chaos since they are forced to incur roles of other modes. Therefore a variety of many-mode models was suggested for numerical investigation [2] which are able to reproduce (i) typically observed spikes in the power time series (ii) their intermittency, (iii) their fractal properties and, moreover, (iv) characteristic frequencies of chaotic power oscillations usually in the interval from 0.1 MHz to 10 MHz.

5.8. FRACTAL DIMENSION AND CONTROL OF CHAOS.

Physically, the peculiarity of chaos (in contrary to noise) is that very many degrees of freedom are governed by a few independent variables only. What anybody needs in when describing chaos is adequate choice of such the relevant variables (which may differ from some particular modes). At least the number of relevant variables \( d_{rel} \), can be determined if estimate so-called fractal dimension, \( d_{frac} \), of time series under observation.

The quantity \( d_{frac} \) characterizes dimension of a manifold (attractor) filled by trajectories of the relevant variables. It is obvious that \( d_{frac} < d_{rel} \). At the same time, \( d_{frac} > d_{rel} - 1 \), since the opposite case would mean that one of variables is somehow dependent on others. For example, if the attractor was periodic (limit) cycle whose dimension \( d_{frac} = 1 \) it would be described by single variable (its phase). For chaotic (strange) attractor, its dimensionality \( d_{frac} \) is inevitably non-integer.
This means that its intersection with (almost any) one-dimensional line (in \( d_{rel} \)-dimensional embedding space) represents so-called Cantor set. The latter is infinitely rarefied (nowhere dense) set of uncountably many points [21]. Roughly speaking, if line contains \( \aleph \) points then a Cantor set on it contains \( \aleph^\delta \) points with \( \delta < 1 \). Then
\[
\frac{d}{d_{rel}} = \frac{d_{rel} - 1 + \delta}{d_{rel} - 1}.
\]

If \( d_{frac} \) is known then the number, \( d_{rel} \), of variables which are governing chaotic dynamics can be found as the integer number exceeding \( d_{frac} \) but most close to it. The cases when \( d_{frac} \geq 3 \) (and thus \( d_{rel} \geq 4 \)) are called hyperchaos. For the same purpose of \( d_{rel} \) determination, the correlation dimension can be used.

The correlation dimension [25], \( d_{cor} \), characterizes statistics of distances between points of the attractor taken at discrete time moments \( t_n = t_0 + n\tau \), with some reasonable time interval \( \tau \) and \( n = 1..N \), at \( N \to \infty \). Let \( X_d(t) \), \( d = 1..d_{rel} \), be relevant variables under consideration. Then the set of distances
\[
R_{ij} = \left\{ \sum_d (X_d(t_i) - X_d(t_j))^2 \right\}^{1/2}
\]
is investigated as characterized by the so-called correlation sum,
\[
\sigma(R) \equiv \frac{2}{N(N - 1)} \sum_{1 \leq i < j \leq N} \{R_{ij} < R\}, \text{ where } \{R_{ij} < R\} \equiv \begin{cases} 1 & \text{if } R_{ij} < R \\ 0 & \text{if } R_{ij} \geq R \end{cases}
\]
Due to fractal (scale-invariant) structure of Cantor sets it can be expected that at small distances and large number of points the power law takes place:
\[
\sigma(R) \to \Omega \left( \frac{R}{R_{max}} \right)^{d_{cor}} \text{ at } N \to \infty, \quad \frac{R}{R_{max}} \to 0, \quad R_{max} \equiv \max_{ij} R_{ij}, \quad (5.27)
\]
where \( \Omega \) is some constant, and the right-hand limit of the correlation sum presents definition of \( d_{cor} \). Naturally, under rather general assumptions \( d_{cor} = d_{frac} \) [25-28].
There are two important statements. First, $d_{cor}$ is insensible to smooth transformations of attractor variables including (not too long) time delays. Therefore, equivalently one may analyze discrete sequence, $x(t_0 + n\tau)$, of any available variable, $x(t)$ (of course, well connected to attractor), considering the subsequences $\{x(t_0 + n\tau), x(t_0 + n\tau + \tau), ..., x(t_0 + n\tau + d\tau)\}$, with $d \geq d_{rel}$, quite like the attractor points above. Second, in principle, estimate of $d_{cor}$ is insensible to $d$ (called embedding dimension) if only $d \geq d_{rel}$. But it is sensible to noise, either external parasitic one (errors of measurements etc.) or ham noise produced by a real system itself. Hence, a factual $d_{cor}$’s dependence on $d$ can inform about quality of data under analysis.

From the other hand, in general structure of chaos (strange attractor) may be better characterized by a spectrum of fractal dimensions instead of a single one [27,28]. Then different variables may give more or less different correlation dimensions.

The essence of chaotic motion is its exponential instability, that is exponential growth of response to arbitrarily small disturbance. Nevertheless this motion obey deterministic law. Hence, if its current state is controlled with accuracy up to $n$ binary digits then its future can be somehow predicted for a time, $n \cdot t_{inf}$, proportional to $n$. Then $h_{ch} = \ln 2/t_{inf}$ is called entropy of chaotic attractor, while the sense of $t_{inf}$ is lifetime of information bits. The latter approximately coincides with characteristic correlation time of chaotic variables [21,33].

Let there are two identical chaotic generators initially delivered in the same state to some extent of precision. To keep the same equality of states in future and thus synchronize one generator by another, we should send from one to another at least one bit of information per time $t_{inf}$. A representative chaotic variable carry just such the amount of information and hence can be used for the synchronization [29,30]. The real example of magnetic chaos synchronization was reported in [19].
However, to make this minimum necessary information to be also factually sufficient, it should be chosen and applied in adequate way. Concretely, one must take into account the topology of attractors i.e. graph of transitions between its Cantor subsets. For example [34], the discrete-time chaotic evolution described by the tent map, \( x(t + 1) = 1 - |2x(t) - 1| \), with \( 0 < x < 1 \), has \( h_{ch} = \ln 2 \), i.e. one bit of information per time step is sufficient for synchronization. But this principal possibility turns into reality if only the bit is chosen be 0 at \( x < x_0 \) and 1 at \( x > x_0 \), with certainly \( x_0 = 1/2 \). Any other rule (or other \( x_0 \)) either leads to errors or requires additional information. Generally, determination of adequate rule (termed generating partition of phase space) and corresponding most meaningful information sequences (so-called symbolic dynamics) is very non-trivial task [34,35], even if dynamic law of chaos is known, all the more if it is under question. From this point of view, the results of [19] seem extremely interesting.

In principle, an adequate rule allows to synchronize non-identical attractors too if they have similar topologies and equal entropies. At more simplified approaches to synchronization (but instead practically applied ones see [31,32] and references therein), rather small non-identity of “master” and “slave” chaotic systems can forbid it, even in spite of quantitative excess of information.

Let \( X(t) \) and \( Y(t) \) are vector (\( d_{rel} \)-dimensional) variables of two chaotic systems which obey the same dynamic equations but the second is influenced by the first as follows:

\[
dX/dt = F(X) \quad dY/dt = F(Y) - g \cdot (Y - X)
\]

(5.28)

Here \( g \) is positive matrix, hence, it introduces additional damping. Let the latter is so strong that suppresses exponential instability of the slave system. Then it easy to
see that after some time the only possible solution for $Y(t)$ will exactly reproduce $X(t)$, and thus one can say that the slave system is ideally synchronized by the master system.

Perhaps however, it would be more correct to name this copying chaos. Indeed, the slave factually loses its autonomy (since at $X = 0$ it would produce neither chaos nor any other motion instead tending to a stable state), and behaves as passive repeater of external signal.

In more general and fine variant of such kind of synchronization,

$$
\frac{dX}{dt} = F(X_1 X), \quad \frac{dY}{dt} = F(X_1 Y),
$$

(5.29)

where $X_1$ is some (say first) of $d_{rel}$ attractor variables and again function $F(X_1 Y)$ of two arguments is arranged so that first equation produces chaos while solution of the second falls into stable point solution as $X_1 = 0$.

The copying of chaos is rather sensible to non-identity of the slave and master systems to adding external noise or any distortion of master signal in transmission channel (see, for instance, [36] and references therein). Relative error of the reproduction occurs be at least the same as relative difference of master and slave parameters plus noise to signal ratio and plus relative distortions.

In practical applications more reasonable approach may be to surely recognize and reproduce some particular characteristics of chaotic signal only, instead of its literal but erroneous copying. The example is mutual phase synchronization of chaotic oscillators (for instance, famous Rossler systems) which does not need in simultaneous amplitude synchronization and therefore is possible for non-identical oscillators in presence of noise. More general possibility is so-called event synchronization where events mean definite well characterizable fragments of chaotic trajectory.
Then the natural step is an artificial creation of events in master system which can serve for encoding information and then its decoding in similar slave system. In particular, this may be switching between different trajectories on the same attractor.

When artificially manipulating trajectory of a chaotic system, one needs in a set (alphabet) of easy creatable and identifiable “events”. Such the possibility is ensured by unstable periodic orbits (UPO), i.e. periodic trajectories which always take place on chaotic attractors. Moreover, from practical point of view one may treat a strange attractor merely as a collections of periodic orbits with different length, from some minimum period up to infinity. The instability of finite-length orbits means that their measure (relative number of attractor points belonging them) is zero, therefore, almost even insignificant deviation from short periodic orbit for certain injects to very long one (chaotic).

But, remarkably, UPO’s can be stabilized and thus practically installed into master’s chaotic trajectory by means of specially programmed feedback (see, for example, [37]). Then similar feedback in slave system helps to unambiguously recognize an UPO’s installation although it looks quite as typical fragment of transmitted signal. Such the discrete chaotic encryption of information can well protect it from noise and signal distortions.

During recent decade many ideas of chaos application to secure communication were suggested. One of schemes successfully realized in [31] is based on introducing communication signal, \(s(t)\), into Eqs.29:

\[
\begin{align*}
\frac{dX}{dt} &= F(X_1 + s(t), X) , \\
\frac{dY}{dt} &= F(X_1 + s(t), Y)
\end{align*}
\]  

(5.30)

Thus the master (in [31] it is chaotic Chua’s generator) produces chaos influenced by the signal. The factually transmitted information is the sum \(s_{trans}(t) = X_1(t) + s(t)\). If
the communication \( s(t) \) was absent \( Y(t) \) would be precise copy of \( X(t) \). Therefore the communication can be restored as \( s(t) = s_{\text{trans}}(t) - Y_1(t) \).

Analogously, discrete chaos (chaotic maps) can be used. For simplest example, let \( x(t+1) = F(x(t)) \) be some one-dimensional chaotic map, and we introduce discrete-time information \( s(t) \) by means of \( x(t+1) = F(x(t) + s(t)) \). If \( s_{\text{trans}}(t) \equiv x(t) + s(t) \) is sent, then in identical receiving system the information can be recovered merely as \( s(t) = s_{\text{trans}}(t) - F(s_{\text{trans}}(t-1)) \). This possibility was suggested in [38] for information encoding in chaotic impulse communication.

Principally similar ideas were experimentally realized for communication with optical chaos [39]. The peculiarity of the latter is an essential time delay in a feedback part of optical (laser) chaotic generators. Correspondingly, their dynamics undergo difference-differential nonlinear equations which can produce chaos whose fractal dimensionality, \( d_{\text{frac}} \), exceeds formal number of variables (number of equations).

Such the schemes (in which an information is either masked by chaos or modulates it) possess all the potential defects of chaos copying. Besides they do not allow for multiple-access chaotic communication (many senders) in the same time-frequency domain. The matter is that chaotic system can not recognize even its own signal if is mixed with a signal from other system.

Still chaotic extension of modern digital code-division multiple access (CDMA) communication is under discussion. Although CDMA also uses chaotic signals (pseudo-random coding sequences) but these are discrete exactly predictable (periodic) signals only whose entropy is zero. The interesting scheme of multiplexed chaotic communication based on analog chaotic signals was suggested in [40,41]. It shows how the unrecognizability of mixed chaotic messages can be overcame. The idea is that all the users simultaneously take part in creating chaos which thus becomes common for all the network and therefore recognizable by any participating chaotic generator. At
present form, however, this scheme needs in temporal separation of users and other limitations [40,41].

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**Conclusion**

We hope that at least some parts of the aforesaid material can be useful supplement to existing literature on magnetic waves. Anyway, the presented approach, - based on first principles only, - well helps to understand and interpret results of numerical simulation of linear and non-linear magnetostatic waves and magnetic chaos. This will be subject of continuation of this manuscript.

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