Dissipativity verification with guarantees for polynomial systems from noisy input-state data

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Abstract—In this paper, we investigate the verification of dissipativity properties for polynomial systems without explicit knowledge of a model but directly from noise-corrupted measurements. Contrary to most data-driven approaches for nonlinear systems, we determine dissipativity properties over infinite time horizon using input-state data. To this end, we propose two characterizations of the noise that affects the system and deduce from each characterization a data-based set-membership representation of the ground-truth system. Each representation then serves as a framework to derive computationally attractive conditions to verify dissipativity properties with rigorous guarantees from noise-corrupted data using SOS optimization.

I. INTRODUCTION

The standard approach to obtain a controller for nonlinear systems requires to retrieve a sufficiently precise model and then the application of nonlinear control design techniques [1]. However, the modelling and the identification of nonlinear systems are in general time consuming and difficult. Hence, the interest on data-driven controller design techniques, where the controller is deduced without knowledge of a model but directly from measured data of the system, rises. An overview of such approaches can be found in [2].

One well-elaborated theory for the controller design of nonlinear systems are dissipativity properties [3]. Since these system properties give insight to the system and facilitate a controller design without knowledge of the system, the verification of these properties from measured trajectories can be leveraged to a data-driven controller design with stability and performance guarantees. For LTI systems, [4] determines dissipativity properties over finite time horizon from a noise-free single input-output trajectory on the basis of [5]. By exploiting the set-membership representation of an unknown LTI system by noisy input-state samples from [6], [7] provides guaranteed dissipativity properties over infinite time horizon from noisy input-state measurements and from noise-free input-output data. For nonlinear systems, [8] is tailored to estimate certain dissipativity properties over finite time horizon, as the $L_2$-gain or conic relations [9], from a large number of input-output trajectories based on the Lipschitz constant of the system operator. To reduce the amount of required data, [10] proposes sequential experiments to improve iteratively the accuracy of a non-parametric data-based Lipschitz approximation of the system operator. Nevertheless, the amount of data might be still too large for a real application as also indicated by bounds on the sampling complexity in [11].

For that reason, we examine throughout this paper data-based determination of dissipativity properties over infinite time horizon of unknown nonlinear systems with polynomial dynamics. Contrary to [12], we consider polynomial systems in discrete time and measurements in presence of noise. By characterizing this noise by two distinct descriptions, we propose two data-based set-membership representations of the ground-truth system which constitute two frameworks to deduce computationally attractive conditions for verifying dissipativity properties using SOS optimization. The first noise description bounds the amplitude of the noise signal in each time step which is commonly assumed, e.g., in set-membership identification [13]. This characterization yields for the verification of dissipativity properties with polynomial supply rates a SOS optimization problem which can be solved by semi-definite programming using standard SOS techniques [14]. Since the complexity of the proposed SOS optimization problem increases for additional samples, the second ansatz characterizes the noise by a single cumulative property as the energy of the noise over the measured time horizon. This approach was first introduced in [6] and yields a feasibility condition of an LMI to verify $(Q,S,R)$-dissipativity.

The paper is organized as follows. First, we introduce some notations for SOS optimization and specify the problem setup for dissipativity verification of polynomial systems. Subsequent, we propose in Section [II] and Section [IV] each one noise description to deduce a data-based set-membership representation of the ground-truth system which yields a computationally tractable condition for verifying dissipativity. In Section [V] we compare both approaches and apply them on two numerical examples in Section [VII].

II. PRELIMINARIES

In this section, we introduce the notion of SOS polynomials and matrices and formulate the problem of verifying dissipativity properties for polynomial systems from noise-corrupted input-state data.

A. SOS optimization

For a vectorial index $\alpha \in \mathbb{N}_0^n$ and a vector $x = [x_1 \cdots x_n]^T \in \mathbb{R}^n$, we write $|\alpha| = a_1 + \cdots + a_n$, the monomial

$$x^\alpha = x_1^{a_1} \cdots x_n^{a_n},$$

and $\mathbb{R}[x]$ for the set of all polynomials $p$ in $x$, i.e.,

$$p(x) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq d} a_\alpha x^\alpha,$$

with real coefficients $a_\alpha \in \mathbb{R}$. $d$ corresponds to the degree of the polynomial if there is an $a_\alpha \neq 0$ with $|\alpha| = d$. Furthermore, we denote $\mathbb{R}[x]^m$ as the set of all $m$-dimensional...
vectors with entries in $\mathbb{R}[x]$ and $\mathbb{R}[x]^{r \times s}$ as the set of all $r \times s$-matrices with entries in $\mathbb{R}[x]$. The degree of a polynomial matrix is the largest degree of its elements.

**Definition 1 (SOS matrix).** A polynomial matrix $P \in \mathbb{R}[x]^{q \times q}$ with even degree is called a sum-of-square (SOS) matrix if there exists a matrix $Q \in \mathbb{R}[x]^{q \times q}$ such that

$$P(x) = Q(x)^T Q(x).$$

Moreover, let the set of all $q \times q$-SOS matrices be denoted by $\text{SOS}[x]^{q \times q}$. For $q = 1$, $P(x)$ is called SOS polynomial.

From a computational perspective, SOS matrices are interesting as we can verify whether a polynomial is a SOS matrix by an LMI feasibility condition which follows from the next proposition.

**Proposition 2 (SOS decomposition).** A polynomial matrix $P \in \mathbb{R}[x]^{p \times q}$ is a SOS matrix if and only if there exists a real matrix $X \succ 0$ such that

$$P(x) = [Z(x) \otimes I_q]^T X [Z(x) \otimes I_q],$$

where the vector $Z \in \mathbb{R}[x]^p$ contains monomials $x^\alpha$, $I_q$ denotes the $q \times q$-identity matrix, and $\otimes$ corresponds to the Kronecker product.

In our application of SOS optimization, we are confronted with the verification of the definiteness of a polynomial matrix $P(x)$ for all $x \in \{x \in \mathbb{R}^n : c_1(x) \geq 0, \ldots, c_l(x) \geq 0\}$. Since any SOS matrix is positive semidefinite together with a generalization of the S-procedure for SOS matrices, we can boil down this problem to an LMI feasibility condition using the following SOS relaxation from [15].

**Proposition 3 (SOS relaxation).** For a matrix $P \in \mathbb{R}[x]^{q \times q}$ and a set $\mathcal{X} = \{x \in \mathbb{R}^n : c_1(x) \geq 0, \ldots, c_l(x) \geq 0\}$ with $c_i \in \mathbb{R}[x]$, if there exist SOS matrices $T_i \in \text{SOS}[x]^{q \times q}, i = 1, \ldots, l$ (and $\epsilon > 0$) such that

$$P(-\epsilon I_q) - \sum_{i=1}^l T_i c_i \in \text{SOS}[x]^{q \times q}$$

then $P(x) \succeq 0(> 0)$ for all $x \in \mathcal{X}$.

Since not every positive definite polynomial matrix $P(x)$ is a SOS matrix, the SOS relaxation is in general not a tight description of positive definite polynomial matrices. However, the relaxation in Proposition 3 is indeed asymptotically exact in the sense that the SOS relaxation is tight if the degree of the SOS matrices $T_i(x)$ tends to infinity [15].

**B. Problem setup**

We consider the nonlinear discrete-time system with polynomial dynamics

$$x(t + 1) = f(x(t), u(t)), f \in \mathbb{R}[x, u]^n$$

(1)

and state-input constraints $(x, u) \in \mathcal{Z}$ with

$$\mathcal{Z} = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : p_i(x, u) \leq 0, p_i \in \mathbb{R}[x, u], i = 1, \ldots, c\}. $$

(2)

The goal of this paper is the derivation of computationally tractable conditions to check whether system (1) is dissipative without identifying a model but directly from input-state data. Since dissipativity properties are originally defined for continuous-time unconstrained systems in [3], we specify a suitable notion of dissipativity for discrete-time systems under constraints.

**Definition 4 (Dissipativity).** System (1) is dissipative on $\mathcal{Z} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ with respect to the given supply rate $s : \mathcal{Z} \rightarrow \mathbb{R}$ if there exists a continuous storage function $\lambda : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u), \quad \forall (x, u) \in \mathcal{Z},$$

(3)

where $\mathcal{X}$ denotes the projection of $\mathcal{Z}$ on the state-space $\mathbb{R}^n$. Moreover, if the system is dissipative with respect to the supply rate

$$s(x, u) = [x^T u^T]^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} [x^T u]^T$$

then the system is called $(Q, S, R)$-dissipative.

Definition 4 also differs from the definition in [3] as the supply rate originally depends on the input and output of the system. Due to the access on input-state data, however, the measured output corresponds to the state $x$. Moreover, Definition 4 incorporates as special case if the supply rate depends on the output $y = h(x)$ with known function $h : \mathcal{X} \rightarrow \mathbb{R}^k$.

While the verification of dissipativity inequality (3) for a (known) polynomial system (1) and polynomial supply rate using SOS optimization is well-investigated [3], dissipativity verification of an unidentified polynomial system from noisy data as formulated next hasn’t been analyzed yet, to the authors’ best knowledge.

Suppose an upper bound on the degree of $f$ is known while its coefficients are unidentified. Then the system dynamics (1) can be written as

$$f(x, u) = Az(x, u) = (I_n \otimes c(z(x, u)^T) a$$

(4)

where $z \in \mathbb{R}[x, u]^l$ contains monomials of $x$ and $u$ and $A \in \mathbb{R}^{n \times l}$ and $a \in \mathbb{R}^l$, respectively, the unidentified coefficients. Furthermore, we assume the access to input-state data of system (1)

$$\{(\tilde{x}_i^+, \tilde{x}_i, \tilde{u}_i)_{i=1, \ldots, D}\}$$

(5)

satisfying $\tilde{x}_i^+ = f(\tilde{x}_i, \tilde{u}_i) + \tilde{d}_i$. Here, $\tilde{d}_i \in \mathbb{R}^n$ summarizes process noise, measurement noise, input uncertainties, and unconsidered parts of the system dynamics which are potentially non-polynomial. As clarified in [7], we could also study noise that affects the system dynamics through a matrix $B$ to include addition knowledge on the influence of the noise.

In the sequel, we characterize the noise $\tilde{d}_i, i = 1, \ldots, D$ more precisely to derive data-based set-membership representations of the unidentified polynomial system (1) which constitute frameworks to verify whether system (1) is dissipative.

**III. DATA-DRIVEN DISSIPATIVITY VERIFICATION FOR SEPARATELY BOUNDED NOISE**

In this section, we develop a framework for dissipativity verification of polynomial system (1) from noise-corrupted data if the noise is bounded explicitly in each time step as specified in the following assumption.
Assumption 5 (Separately bounded noise). For the measured data \(\{(x_i, u_i)\}_{i=1}^D\), suppose for \(i = 1, \ldots, D\)

\[\|d_i\| \leq \delta_i\]  \hspace{1cm} (6)

with \(\delta_i \geq 0\).

By Assumption 5, the noise is unbiased and has bounded amplitudes \(\delta_i\) which is supposed, e.g., in system identification \([13]\). Moreover, this characterization incorporates noise that exhibits a fixed signal-to-noise-ratio \(\delta\), i.e., \(\|d_i\| \leq \delta \|x_i\|\).

To derive a data-based set-membership representation of the ground-truth system (1) which is the basis to verify dissipativity properties without an explicit model, we next define the set of all systems

\[x(t+1) = (I_n \otimes z(x,u)^T) a, \quad := Z(x,u)\]  \hspace{1cm} (7)

with unidentified coefficient vector \(a \in \mathbb{R}^{nl}\) and known vector \(z \in \mathbb{R}^{[x,u]^l}\), which explains the data (5).

**Definition 6 (Feasible system set).** The set of all systems (7) admissible with the measured data (5) for separately bounded noise (6) is given by the feasible system set \(\text{FSS}_a = \{Z(x,u) : a \in \Sigma_a\}\) with

\[\Sigma_a = \{a : \forall i \in \{1, \ldots, D\} \exists \tilde{d}_i \text{ satisfying (6)} \text{ and} \tilde{x}^t_i = Z(\tilde{x}_i, \tilde{u}_i) a + \tilde{d}_i\} \]

Since the ground-truth system \(f\) explains the data (5) for \(\tilde{d}_i, i = 0, 1, \ldots, D\) which satisfy Assumption 5, the ground-truth system (1) is an element of \(\text{FSS}_a\), i.e., \(f(x,u) \in \text{FSS}_a\). Thereby, \(\text{FSS}_a\) is a set-membership representation of the ground-truth system (1). Analogously to (5), we deduce in the following lemma a data-based and tight description of \(\text{FSS}_a\).

**Lemma 7.** The set of all coefficients \(\Sigma_a\) for which system (2) explains the measured data set (5) for separately bounded noise (6) is equivalent to

\[\{a : \begin{bmatrix} a \\ 1 \end{bmatrix} Q_i \begin{bmatrix} a \\ 1 \end{bmatrix} \leq 0, \forall i \in \{1, \ldots, D\}\} \]  \hspace{1cm} (8)

with the data-dependent matrices

\[Q_i = \begin{bmatrix} Z(\tilde{x}_i, \tilde{u}_i)^T Z(\tilde{x}_i, \tilde{u}_i) - Z(\tilde{x}_i, \tilde{u}_i)^T \tilde{x}^+ & -Z(\tilde{x}_i, \tilde{u}_i)^T \tilde{x}^+ \tilde{x}^T - \tilde{d}^2_i \\ -\tilde{x}^T \tilde{x}^T \tilde{x}^+ & \tilde{x}^T \tilde{x}^+ - \tilde{d}^2_i \end{bmatrix} \]

**Proof.** If \(a \in \Sigma_a\) then there exist realizations of the noise \(d_i, i = 1, \ldots, D\) such that \(\tilde{x}^+ = Z(\tilde{x}_i, \tilde{u}_i) a + \tilde{d}_i\) and \(\|d_i\| \leq \delta_i\). Combining both yields

\[(\tilde{x}^+ - Z(\tilde{x}_i, \tilde{u}_i) a)^T (\tilde{x}^+ - Z(\tilde{x}_i, \tilde{u}_i) a) \leq \tilde{d}^2_i\]

which is equivalent to (8).

To prove the converse, suppose \(a\) is an element of (8). Then construct \(\tilde{d}_i, i = 1, \ldots, D\) such that \(\tilde{x}^+ = Z(\tilde{x}_i, \tilde{u}_i) a + \tilde{d}_i\). Since \(a\) satisfies (8), \(d_i, i = 1, \ldots, D\) satisfy (6), and hence \(a \in \Sigma_a\).

Before continuing with the verification of dissipativity properties of the ground-truth system (1), we link Definition 6 and Lemma 7 to the feasible system set considered in the set-membership identification literature \([13]\).

**Remark 8.** Since the matrices \(Q_i, i = 1, \ldots, D\) can be calculated from the data set (5), \(\text{FSS}_a\) can be seen as a data-based set-membership model of the unknown system (1). A similar set-membership description for nonlinear systems has been examined for set-membership identification \([13]\). There, Lipschitz bounds on the system dynamics are considered in order to bound the variety of the system dynamics as otherwise there exist infinitely many systems that explain the data. Similarly, the variety of the system dynamics in Definition 6 is bounded by the assumption of a polynomial system with bounded degree.

Since \(\text{FSS}_a\) contains the ground-truth systems, (1) is dissipative if all systems of the feasible system set \(\text{FSS}_a\) are dissipative. Based on this idea, the following theorem provides a data-based SOS condition for the verification of dissipativity properties without an explicit model of (1).

**Theorem 9.** Let the data samples (5) satisfy Assumption 5. Then system (1) is dissipative on (2) with respect to the given supply rate \(s \in \mathbb{R}^{[x,u]}\) if there exist a storage function \(\lambda \in \text{SOS}[x]\) and polynomials \(s_i \in \text{SOS}[x,u,a], i = 1, \ldots, c\) and \(t_i \in \text{SOS}[x,u,a], i = 1, \ldots, D\) such that \(\psi \in \text{SOS}[x,u,a]\) with

\[\psi(x,u,a) = s(x,u) - \lambda(Z(x,u)a) + \lambda(x) + \ldots + \sum_{i=1}^c p_i(x) s_i(x,u,a) + \ldots\]

**Proof.** By Definition 6, all systems of the feasible system set \(\text{FSS}_a\), and hence system (1), are dissipative on (2) if there exists a continuous storage function \(\lambda : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}\) such that

\[s(x,u) - \lambda(Z(x,u)a) + \lambda(x) \geq 0, \forall (x,u) \in \mathbb{Z}, \forall a \in \Sigma_a\]

By Lemma 7 and Proposition 8, this holds if there exist SOS polynomials \(s_i \in \text{SOS}[x,u,a], i = 1, \ldots, c\) and \(t_i \in \text{SOS}[x,u,a], i = 1, \ldots, D\) such that \(\psi \in \text{SOS}[x,u,a]\).

Even though \(f(x,u) = Z(x,u)a\) is an unidentified polynomial vector in \(\mathbb{R}^{[x,u]}\), it is a known polynomial vector in \(\mathbb{R}^{[x,u]}\). For that reason, we can verify \(\psi \in \text{SOS}[x,u,a]\) as SOS problem with free variables \(x, u, a\) and \(y\) by applying standard SOS solvers, e.g., \([16]\). Nevertheless, we can exploit the quadratic structure of \(\Sigma_a\) from Lemma 7 to achieve a SOS condition independent of \(a\).

**Corollary 10.** Let the data samples (5) satisfy Assumption 5. Then system (1) is dissipative on (2) with respect to the supply rate \(s \in \mathbb{R}^{[x,u]}\) if there exist a storage function

\[\lambda(x) = \begin{bmatrix} x^T \\ 1 \end{bmatrix} P \begin{bmatrix} x \\ 1 \end{bmatrix}, P = \begin{bmatrix} P_{11} & P_{12} \\ P_{T1} & P_{T2} \end{bmatrix} \geq 0\]

and polynomials \(t_i \in \text{SOS}[x,u], i = 1, \ldots, D\) and

\[s_i(x,u,a) = \begin{bmatrix} a \\ 1 \end{bmatrix} S_i(x,u) \begin{bmatrix} a \\ 1 \end{bmatrix}, i = 1, \ldots, c,\]
with $S_i \in \text{SOS}[x,u]^{(n_l+1)\times(n_l+1)}$ such that $\Psi \in \text{SOS}[x,u]^{(n_l+1)\times(n_l+1)}$ with

\[
\Psi(x,u) = \sum_{i=1}^{c} p_i(x,u)S_i(x,u) + \sum_{i=1}^{D} Q_i t_i(x,u) + \ldots
\]

\[
\ldots + \begin{bmatrix} -Z(x,u)^TP_{11}Z(x,u) & -Z(x,u)^TP_{12} \\ -P_{12}Z(x,u) & s(x,u) + \frac{x^T}{1} \begin{bmatrix} P & \frac{x}{1} \end{bmatrix} \end{bmatrix}.
\]

Proof. Pursuing the proof of Theorem 9 and applying a generalized S-procedure for SOS matrices [14], system (1) is dissipative if there exist a $P \succeq 0$, $t_i \in \text{SOS}[x,u]$, $i = 1, \ldots, D$, and $S_i \in \text{SOS}[x,u]^{(n_l+1)\times(n_l+1)}$ such that

\[
\begin{bmatrix} a^T \\ a \end{bmatrix} \Psi(x,u) \begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0, \forall (x,u,a) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n.
\]

This is equivalent to $\Psi(x,u) \geq 0, \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m$, and thus is implied by $\Psi \in \text{SOS}[x,u]^{(n_l+1)\times(n_l+1)}$. \qed

Remark 11. If we have prior insight to the system dynamics (1), then we could consider instead of (7) the system dynamics

\[x(t + 1) = F(x(t), u(t))a + G(x(t), u(t)),\]

with unidentified coefficients $a \in \mathbb{R}^{n_l}$, known matrix $F \in \mathbb{R}^{[(x,u)] \times n_l}$, and known vector $G \in \mathbb{R}^{[(x,u)]}$. For this system dynamics, we can analogously derive $\text{FFS}_{\Sigma_a}$ and conditions for verifying dissipativity properties similar to Theorem 9 and Corollary 10.

Remark 12. Theorem 9 is based on the verification of dissipativity inequality (3) for all systems in the feasible system set $\text{FSS}_A$ using state-dependent storage functions. Hence, we actually employ Theorem 9 dissipativity properties for system (1) with time-varying coefficients $a(t) \in \Sigma_a$. Since the ground-truth system dynamics (1) are supposed to be polynomial with constant coefficients, state-dependent storage functions increase the conservatism of Theorem 9. To prevent time-varying coefficients and hence improve the accuracy of Theorem 9 and Corollary 10 respectively, we could consider parametric storage functions $\lambda \in \text{SOS}[x,a]$ or

\[\lambda(x,a) = \begin{bmatrix} x^T \\ a \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0,
\]

respectively, and the dissipativity inequality

\[\lambda(Z(x,u)a,a) - \lambda(x,a) \leq s(x,u), \forall (x,u) \in \mathbb{Z}, \forall a \in \Sigma_a.
\]

Since this dissipativity inequality corresponds to the dissipativity of

\[x(t + 1) = Z(x(t), u(t))a(t),
\]

\[a(t+1) = a(t),
\]

the coefficients $a \in \Sigma_a$ are constant over time.

IV. DATA-DRIVEN DISSIPATIVITY VERIFICATION FOR CUMULATIVELY BOUNDED NOISE

We again tackle the problem of verifying whether the unidentified polynomial system (1) is dissipative by means of noisy data. However, instead of bounding the noise separately as in the previous section, now the noise is characterized by one property that bounds cumulatively the noise realizations of data samples (5), which was first proposed in [6].

Assumption 13 (Cumulatively bounded noise). For the measured data (5), suppose $\tilde{d}_i, i = 1, \ldots, D$ satisfy

\[\begin{bmatrix} \tilde{D}^T \\ I \end{bmatrix} \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^T & \Delta_3 \end{bmatrix} \begin{bmatrix} \tilde{D}^T \\ I \end{bmatrix} < 0 \]

with $\tilde{D} = [\tilde{d}_1 \ldots \tilde{d}_D]$ and $\Delta_1 \geq 0$.

By Assumption 13 all noise realizations $\tilde{d}_1, \ldots, \tilde{d}_D$ are cumulatively bounded as $\Delta_1 \geq 0$ and $\tilde{D} \tilde{D}^T = \sum_{i=1}^{D} \tilde{d}_i \tilde{d}_i^T$. Exemplarily, (9) incorporates noise with (strictly) bounded energy $\sum_{i=1}^{D} \tilde{d}_i \tilde{d}_i^T < \delta_2^2 I_D$. Note that the strictness of (9) could be switched with $\Delta_1 \geq 0$ while the results of the remainder of this section could be adapted. For more details on the noise description in Assumption 13 we refer to [6] and to Section V for a comparison with Assumption 5.

Analogously to [6] and Section VII-A combining Assumption 13 data samples (5), and the knowledge on the structure of the system dynamics (9) explains the measured data set (5) for cumulatively bounded noise (9) is given by the feasible system set $\text{FSS}_{A} = \{Az(x,u) : A \in \Sigma_A\}$ with

\[\Sigma_A = \{ A : \forall i \in \{1, \ldots, D\} \exists \tilde{d}_i \text{ satisfying (9) and } \tilde{x}_i^+ = Az(\tilde{x}_i, \tilde{u}_i) + \tilde{d}_i \}.
\]

Lemma 15. The set of all coefficients $\Sigma_A$ for which system (10) admits the measured data set (5) for cumulatively bounded noise (9) is equivalent to

\[\{ A : \begin{bmatrix} A^T \\ I \end{bmatrix} \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^T & \Delta_3 \end{bmatrix} \begin{bmatrix} A^T \\ I \end{bmatrix} < 0 \}
\]

with the data-dependent matrices

\[
\tilde{Z} = [z(\tilde{x}_1, \tilde{u}_1) \ldots z(\tilde{x}_D, \tilde{u}_D)],
\]

\[
\tilde{X}^+ = [\tilde{x}_1^+ \ldots \tilde{x}_D^+],
\]

\[
\begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^T & \Delta_3 \end{bmatrix} = \begin{bmatrix} \tilde{Z} \Delta_1 \tilde{Z}^T \\ -\tilde{Z}(\Delta_1 \tilde{X}^+ + \Delta_2) \tilde{Z}^T \\ -(\tilde{X}^+ \Delta_1^T + \Delta_2^T) \tilde{Z}^T \end{bmatrix} \begin{bmatrix} (\tilde{X}^+)^T \\ \Delta_1 & \Delta_2 \\ \Delta_2^T & \Delta_3 \end{bmatrix} \begin{bmatrix} \tilde{X}^+^T \\ I \end{bmatrix}.
\]
Proof. The statement follows analogously to (Lemma 4) and the proof of Lemma 7 respectively. □

However, since the data-based description of Σ_A in Lemma 15 is not tractable for the verification of dissipativity inequality (15), we also introduce its dual version as in (7).

Lemma 16. Suppose Assumption 15 holds and the inverse
\[
\begin{bmatrix}
-\hat{\Delta}_1 & \hat{\Delta}_2 \\
\hat{\Delta}_1^T & -\hat{\Delta}_3
\end{bmatrix}^{-1} = \begin{bmatrix}
\Delta_1 & \Delta_2 \\
\Delta_1^T & \Delta_3
\end{bmatrix}
\]
exists. Then any matrix \( A \in \mathbb{R}^{n \times l} \) is an element of \( \Sigma_A \) if and only if
\[
A \in \Sigma_A = \{ A : [ I ]^T \begin{bmatrix}
\Delta_1 & \Delta_2 \\
\Delta_1^T & \Delta_3
\end{bmatrix} [ I ] < 0 \}. \quad (13)
\]

Proof. By Assumption 15, the coefficient matrix \( A_{gt} \) of the ground-truth system (1), i.e., \( f(x,u) = A_{gt} z(x,u) \), is an element of \( \Sigma_A \), and moreover \( \Delta_1 \succeq 0 \). Thus, the dualization lemma 15 implies that \( (A_{gt} \in \Sigma_A \) and \( \Delta_3 \succeq 0 \). Thereby, any matrix \( A \in \mathbb{R}^{n \times l} \) satisfies \( A \in \Sigma_A \) if and only if \( A \in \Sigma_A \) again by the dualization lemma. □

As implied by the proof of Lemma 16 and under Assumption 12 and existence of (12), the feasible system sets \( \text{FSS}_A \) and
\[
\text{FSS}_A = \{ Az(x,u) : A \in \Sigma_A \},
\]
are equivalent and contain the ground-truth system (1). Therefore, we can derive analogously to Section III-A a condition to verify dissipativity properties of polynomial system (1) without identifying a model but from noisy input-state measurement.

Theorem 17. Suppose the data samples (5) satisfy Assumption 13 and the inverse (12) exists, the state-input constraints (4) are specified by
\[
p_i(x,u) = \begin{bmatrix}
z(x,u) \\
1
\end{bmatrix}^T P_i \begin{bmatrix}
z(x,u) \\
1
\end{bmatrix} \quad i = 1, \ldots, c, \quad (14)
\]
and, without loss of generality, there exist matrices \( T_x \in \mathbb{R}^{n \times l} \) and \( T \in \mathbb{R}^{(n+m) \times l} \) such that \( x = T x_0 \) and \( z = T z \). Then system (1) is \( (Q,S,R) \)-dissipative on (2) with quadratic constraints (14) if there exist a storage function \( \lambda(x) = x^T P x, P \succeq 0 \), a non-negative constant \( \tau \), and polynomials \( \tau_i \in \text{SOS}[x,u], i = 1, \ldots, c \) with
\[
\tau_i(x,u) \begin{bmatrix}
z \\
1
\end{bmatrix}^T P_i \begin{bmatrix}
z \\
1
\end{bmatrix} = \begin{bmatrix}
z \\
1
\end{bmatrix}^T \tilde{P}_i \begin{bmatrix}
z \\
1
\end{bmatrix} \quad (15)
\]
satisfying the LMI condition (16).

Proof. Since the ground-truth system (1) is an element of \( \text{FSS}_A \), the result follows directly from Theorem 17. □

By the generalized S-procedure for polynomial matrices (14), this conditioned inequality holds if there exist a non-negative constants \( \tau \) and polynomials \( \tau_i \in \text{SOS}[x,u], i = 1, \ldots, c \) with (15) satisfying
\[
*^T \Theta \begin{bmatrix}
A z(x,u) \\
T z(x,u)
\end{bmatrix} \begin{bmatrix}
I \quad 0 \\
0 \quad T_x
\end{bmatrix} \begin{bmatrix}
z(x,u) \\
Az(x,u)
\end{bmatrix} \begin{bmatrix}
I \quad 0 \\
0 \quad I
\end{bmatrix} \begin{bmatrix}
z(x,u) \\
1
\end{bmatrix} \leq 0. \quad (17)
\]
for all \( (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \) and \( A \in \mathbb{R}^{n \times l} \). Here, \( * \) denotes the same vector as on the right-hand side of \( \Theta \). To attain a tractable LMI condition, we extract the matrix \( A \) and the nonlinear proportion \( z(x,u) \)
\[
*^T \Theta \begin{bmatrix}
A \quad 0 \\
0 \quad T_x
\end{bmatrix} \begin{bmatrix}
z(x,u) \\
1
\end{bmatrix} \leq 0. \quad (18)
\]
Obviously, (18) holds for all \( (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \) and all \( A \in \mathbb{R}^{n \times l} \) if (16) is satisfied. Moreover, since \( \tilde{P}_i \) of the quadratic decomposition (15) contains linearly the to-be-optimized coefficients of the SOS polynomial \( \tau_i(x,u) \), (16) is indeed an LMI.

In Theorem 17, dissipativity verification boils down to an LMI feasibility problem instead of a SOS problem as in Theorem 9 because the implication of inequality (18) by LMI (16) corresponds to a SOS relaxation. Moreover, note that we proceed similarly as for providing quadratic performance guarantees for linear fractional representations which are exploited in (7) to verify dissipativity properties for unknown linear systems.

In Theorem 17 we consider SOS polynomials \( \tau_i(x,u), i = 1, \ldots, c \) instead of non-negative constants as otherwise LMI (18) becomes indefinite if \( \tilde{P}_i \) contains a negative right lower element which is mostly the case, e.g., for \( x^T x \leq 1 \). Furthermore, note that the quadratic decomposition (15) is not unique but is spanned by a linear subspace which provides additional degrees of freedom to deteriorate the conservatism of condition (16).

We conclude this section by demonstrating the flexibility of this framework by employing prior system knowledge and appending additional nonlinearities and uncertainties.

Remark 18. First, we can generalize Theorem 17 for supply rates \( s(x,u) = z(x,u)^T Q z(x,u) \). Second, we can take prior knowledge of the system dynamics into account by considering
\[
x(t+1) = A z_1(x(t),u(t)) + [\bar{A}_1 \quad \bar{A}_2] \begin{bmatrix}
z_1(x(t),u(t)) \\
z_2(x(t),u(t))
\end{bmatrix}
\]
with unidentified matrix \( A \) and known matrices \( \bar{A}_1 \) and \( \bar{A}_2 \). The additional vector of monomials \( z_2(x,u) \) is beneficial if, for instance, \( g(x) \) of a (polynomial) control-affine system
$x(t+1) = f(x(t)) + g(x(t))u(t)$ is known from some insight to the system. Moreover, $\gamma(x, u)$ might be necessary for the quadratic decomposition (15).

Inspired by the extraction of nonlinearities in inequality (18), the third extension might be the consideration of quadratically bounded non-polynomial nonlinearities $g(x, u)$

$$x(t + 1) = A \begin{bmatrix} z(x(t), u(t)) \\ g(x(t), u(t)) \end{bmatrix}$$

where $z(x, u)$ still contains only monomials in $x$ and $u$. Potentially, the additional nonlinearities could be bounded dynamically using integral quadratic constraints in discrete time. However, note that non-polynomial nonlinearities lead to the loss of some properties of the SOS relaxation as the asymptotically exactness mentioned in Section VI-A. Moreover, this extension is also conceivable for the first framework for separately bounded noise.

V. COMPARISON OF BOTH FRAMEWORKS FOR SEPARATELY BOUNDED NOISE

Motivated by the frequently assumed separately bounded noise from Assumption 5 as non-probabilistic noise description, e.g., in system identification (13), we compare both previously proposed frameworks for data-driven dissipativity verification for this noise characterization.

As suggested in (6), the cumulatively bounded noise description (9) can incorporate the separately bounded noise (6) by $\dot{D}D^T = \sum_{i=1}^{D} \dot{d}_i d_i^T \preceq \sum_{i=1}^{D} \delta_i^2 I$. However, this characterization also facilitates, e.g., noise with bounded energy $\sum_{i=1}^{D} q_i^2 \dot{d}_i \preceq \sum_{i=1}^{D} \delta_i^2$ which includes more noise realizations than (6). Hence, Assumption 5 provides a more accurate description than Assumption 13 and therefore leads to tighter set-membership representation of the ground-truth system (1). For that reason, Theorem 9 provides a less conservative condition for dissipativity verification than Theorem 17 which is indeed the case in both examples in Section VI.

Furthermore, the feasible system set $FSS_{ss}$ cannot increase by considering additional data. Contrary to this, we show in Subsection VI-A that adding samples with high signal-to-noise-ratio might render the feasible system set $FSS_{ss}$ larger and LMI (16) infeasible. To some extend, this observation is important, e.g., if the data set includes outliers because for the cumulative noise description they influence negatively the whole data set if we don’t neglect them.

Further advantages of Theorem 9 compared to Theorem 17 are that its accuracy can be improved by parametrized storage functions as shown in Remark 12 and general polynomial state-input constraints and supply rates can be handled.

On the other hand, the framework of cumulative bounded noise is computationally more attractive. Since all data samples are cumulated into one condition, the verification condition in Theorem 17 doesn’t increase with the amount of samples and boils down to an LMI condition, while Theorem 9 requires one additional SOS polynomial multiplier for each sample which might yield to a non-tractable optimization problem. Although this problem could be circumvented by forcing $t_1(x, u) = \cdots = t_D(x, u)$ in Theorem 9, this simplification corresponds to a cumulative noise characterization such that Theorem 17 could be applied instead.

Furthermore, in our testing in Section VI, system description (10) is computationally more efficient to tackle systems (4) with a large number of unidentified coefficients.

To summarize this discussion, while the framework of separately bounded noise provides a data-efficient approach for the often used noise characterization (6), the framework of cumulatively bounded noise is computationally more attractive. For that reason, the latter framework should always be considered if the noise is characterized by some cumulative property.

VI. NUMERICAL EXAMPLES

To measure the conservatism of both frameworks for separately bounded noise, we apply Corollary 10 and Theorem 17 to find a guaranteed upper bound on the $\ell_2$-gain of two polynomial systems. The $\ell_2$-gain $\gamma \geq 0$ corresponds to the supply rate

$$s(x, u) = \gamma^2 u^T u - h(x)^T h(x)$$

with the measured output $h(x) = x$. The SOS problem of Corollary 10 and the LMI feasibility problem of Theorem 17 are extended by the minimization of $\gamma$ and are solved in Matlab using YALMIP (16) and the solver MOSEK.

A. Example 1

In the first example, we determine an upper bound on the $\ell_2$-gain of the locally asymptotically stable system

$$x(t + 1) = -0.8x(t) + 0.1x(t)^2 + u(t) + d(t)$$

with state constraint $p_1 = x^2 - 1 \leq 0$, input constraint $p_2 = u^2 - 0.1 \leq 0$, and (separately) bounded noise $|d(t)| \leq 0.02$. We receive the upper bound $\gamma \leq 100.5$ by SOS optimization.
expanding the system dynamics with \( d(t) = 0 \).

To apply our data-driven methods, we draw samples from a single trajectory with initial condition \( x(0) = 1 \), input \( u(t) = 0.1, t \geq 0 \), and a random sampled noise signal \( |d(t)| \leq 0.02, t \geq 0 \). For Corollary 10 we use a parametrized storage function, quadratic SOS polynomials \( s_i(x, u) \), \( i = 1, 2 \), and quartic SOS polynomials \( t_i(x, u) \), \( i = 1, \ldots, D \). In Theorem 17 quadratic SOS polynomials \( \tau_i(x, u) \), \( i = 1, 2 \) are considered. All optimization problems are solved in less than a second on a Lenovo i5 notebook.

Considering the first three data samples of the trajectory, we receive the upper bounds for the \( \ell_2 \)-gain \( \gamma_{SB} = 280.8 \) from Corollary 10 and \( \gamma_{CB} = 291.2 \) from Theorem 17.

As stated in Section V additional data always lead to a non-increase of \( \Sigma_0 \) but potentially to an increase of \( \Sigma_A \). Indeed, while the upper bound \( \gamma_{SB} \) decreases to \( \gamma_{TS} \approx 190 \) using the first 20 data samples, \( \gamma_{CB} \) increases to 5570 using the first 6 samples and LMI 16 even becomes infeasible for more samples. This observation is due to the high signal-to-noise-ratio of the measured trajectory for \( t \geq 5 \).

B. Example 2

In the second example, the \( \ell_2 \)-gain of the locally asymptotically stable system

\[
\begin{bmatrix}
    x_1(t + 1) \\
    x_2(t + 1)
\end{bmatrix} = \begin{bmatrix}
    -0.5x_1 + 0.3x_2^2 + 0.2x_1x_2 \\
    0.4x_2 + 0.1x_2^2 - 0.2x_1^2 + u(t)
\end{bmatrix}
\]

with \( x_1^2 \leq 1 \), \( x_2^2 \leq 1 \), and \( u^2 \leq 1 \) is examined. Given the ground-truth system, we determine 4.6 as upper bound of the \( \ell_2 \)-gain. We suppose that the noise exhibit constant signal-to-noise-ratio \( |d(t)| \leq 0.02 \). Furthermore, we apply the input signal \( u(t) = 0.7 \sin(0.06t^2 + 0.5t) \) such that the system is excited over the whole time horizon. Moreover, we assume that the unidentified system 4 contains 12 unknown coefficients with \( z(x, u) = [x_1^2 \ x_2^2 \ x_1x_2 \ x_2^3 \ u]^T \).

Using the first 20 samples of the input-state trajectory, we calculate the bounds \( \gamma_{SB} = 8.9 \) and \( \gamma_{CB} = 109.1 \). While the computation time to solve the optimization problem of Theorem 17 doesn’t increase remarkable compared to the first example, solving the SOS problem of Corollary 10 takes now about 45 seconds.

With 100 data samples available, we can reduce the upper bound \( \gamma_{CB} \) to 12.7. While the LMI 16 is infeasible when increasing the signal-to-noise-ratio to \( |d(t)| \leq 0.04 \), Theorem 9 still provides an upper bound of \( \gamma_{SB} = 80.7 \) for \( D = 20 \).

Note that both frameworks determine meaningful bounds on the \( \ell_2 \)-gain with less data samples than [8] at the cost of input-state measurements and a polynomial description of the system which requires more insight to the system. For example, [8] estimates the \( \ell_2 \)-gain of a similar complex system by approximately \( 10^4 \) data samples.

VII. CONCLUSIONS

We established two frameworks to check without a model but from noise-corrupted input-state measurements whether an unidentified polynomial system is dissipative. First, Theorem 9 and Corollary 10 provide a data-efficient but computationally expensive framework for separately bounded noise using standard SOS optimization. Second, a cumulatively bounded noise is considered to deduce a more computationally attractive LMI condition with SOS multipliers for dissipativity verification in Theorem 17 which corresponds to a generalization of Theorem 12 in [7] for polynomial systems. In numerical examples, we showed that both frameworks are more data-efficient than using Lipschitz approximations [8], [10] at the cost of input-state measurements and a polynomial description of the system which requires more insight to the system.

In a future work, we might extend the results for finding optimal dissipativity properties as conic relations [9] or nonlinearity measures [17]. Subject of another future research might be the extension of these frameworks for input-output measurements.

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