Local well-posedness of compressible-incompressible two-phase flows with phase transitions

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Abstract

This paper is concerned with the basic model for compressible and incompressible two-phase flows with phase transitions. The flows are separated by nearly flat interface represented as a graph over the $N-1$ dimensional Euclidean space $\mathbb{R}^{N-1}$ ($N \geq 2$). The local well-posedness is proved by the Banach fixed point theorem based on the maximal $L_p$-$L_q$ regularity theorem for the linearized problem.

1 Introduction

The three states of matter are solids, liquids and gases. The flow consisting of two phases, which are mixed and interacting each other, is called the two phase flow. To analyze the two phase flow is the important problem in the field of the fluid machine. For example, nowadays it is known that cavitation noise and the damage of hard material for turbo-machines and ship propellers are induced by impulsive pressures that are caused by the collapse of cloud of bubbles in the water. In fact, K. Yamamoto [20] investigated that the water jets injected from submerged nozzle with narrow orifice were observed by a high-speed video camera and he found that the several times rebound of the cloud of bubbles creates very strong pressure pulses which cause the cavitation noise and the damage of hard material. Moreover, D. Rosseinelli et al [9] showed the simulation of cloud cavitation collapse by the high performance computer. Thus, the study of the cavitation, which is described by the compressible and incompressible fluid flow mathematically, has new development in the experimental fluid mechanics and computational fluid mechanics, rather recently. On the other hand, the mathematical approach to two phase problem with liquids and bubbles is rare, even when they are described by the compressible and incompressible viscous fluid flow with sharp interface. The author knows only results due to Denisova [2] and Kubo, Shibata and Soga [4]. In this paper, we start with the modeling of the two phase problem which can be found in the usual engineering text books without any mathematical proof and we prove the local well-posedness in the phase transition case, which has not been yet treated in any mathematical literature as far as the author knows.

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2 Modeling.

Following the Prüss idea in [6], we discuss the modeling. Let Ω be a domain in the $N$ dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) with boundary $\Gamma_0$. Let $\Omega_-$ be a subdomain of $\Omega$ with boundary $\Gamma$. We assume that $\Gamma = \partial \Omega_+ \subset \Omega$ and that $\Gamma_0 \cap \Gamma = \emptyset$. Set $\Omega_+ = \Omega - \overline{\Omega_-}$. Let $\varphi = \varphi(\xi, t) = (\varphi_1(\xi, t), \ldots, \varphi_N(\xi, t))$ be a function defined on the closure of $\Omega$ for each time variable $t \in (0, T)$, $\xi = (\xi_1, \ldots, \xi_N)$ being the reference coordinate system. We assume that the map $\xi \mapsto \varphi(\xi, t)$ is one to one for each $t \in (0, T)$.

Set $(\partial_\xi \varphi)(\xi, t) = \mathbf{v}(x, t)$ with $x = \varphi(\xi, t)$, where $\mathbf{v} = (v_1, \ldots, v_N)$ is the velocity field, and $\mathbf{v}_\alpha$ is the mass field.

Moreover, we know that the well-known Reynolds transport theorem:

$$
\frac{d}{dt} \int_{\Omega(t)} f \, dx = \int_{\Omega(t)} \partial_t f \, dx + \int_{\Gamma(t)} [\mathbf{v} \cdot \mathbf{n}_{\Gamma(t)}] \, dv + \int_{\Gamma_0} f \mathbf{v} \cdot \mathbf{n}_{\Gamma_0} \, dv,
$$

where $dv$ represents the surface element not only of $\Gamma(t)$ but also of $\Gamma_0$. In this orientation, we know that

$$
\frac{d}{dt} |\Gamma(t)| = -\int_{\Gamma(t)} H_\mathbf{v} \cdot \mathbf{n}_{\Gamma} \, dv \quad (2.1)
$$

Here and in the following, we use bold small letters to denote $N$-vector valued functions and the bold capital letters to denote $N \times N$ matrix valued functions, respectively. For $\mathbf{v} = (v_1, \ldots, v_N)$ and $\mathbf{w} = (w_1, \ldots, w_N)$, we set $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^{N} v_j w_j$, which is the usual inner product of $\mathbb{R}^N$.

In the following, we use the following notation:

- $\rho : \hat{\Omega}(t) \to \mathbb{R}_+$ is the mass field,
- $\mathbf{u} : \hat{\Omega}(t) \to \mathbb{R}^N$ the velocity field,
- $\pi : \hat{\Omega}(t) \to \mathbb{R}$ the pressure field,
- $T : \hat{\Omega}(t) \to \{ \mathbf{A} \in GL_N(\mathbb{R}) \mid \tilde{T} \mathbf{A} = \mathbf{A} \}$ the stress tensor field,
- $\theta : \hat{\Omega}(t) \to \mathbb{R}_+$ the thermal field,
- $e : \hat{\Omega}(t) \to \mathbb{R}$ the internal energy,
- $q : \hat{\Omega}(t) \to \mathbb{R}^N$ the heat flux.

Since this subsection is concerned with the modeling, we do not care the regularity of boundary and the map $\varphi$. Moreover, we do not mention any integrability of functions rigorously. These are formulated mathematically in sections 2.
\( \eta : \Omega(t) \to \mathbb{R} \) the entropy, where, we have set \( \mathbb{R}_+ = (0, \infty) \). For the modeling, we use the following Navier-Stokes-Fourier system of equations: for \( x \in \Omega(t) \) ad \( t > 0 \)

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho \mathbf{u}) &= 0; \\
\partial_t (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) - \text{div} \mathbf{T} &= 0; \\
\partial_t (\frac{\rho}{2} \mathbf{u}^2 + \rho e) + \text{div} (\frac{\rho}{2} \mathbf{u}^2 + \rho e) \mathbf{u} - \text{div} \left( \mathbf{T} \mathbf{u} - \mathbf{q} \right) &= 0.
\end{align*}
\]  

(2.2)

(2.3)

(2.4)

Here, for any \( \mathbf{u} = (u_1, \ldots, u_N) \), \( \mathbf{u} \otimes \mathbf{u} \) is the \( N \times N \) matrix whose \((i,j)\) component is \( u_i u_j \), and for any \( \mathbf{w} = (w_1, \ldots, w_N) \) and \( \mathbf{S} = (S_{ij}) \) the divergence forms \( \text{div} \mathbf{w} \) and \( \text{div} \mathbf{S} \) are defined by

\[
\text{div} \mathbf{w} = \sum_{j=1}^{N} \partial_j w_j, \quad \text{div} \mathbf{S} = \left( \sum_{j=1}^{N} \partial_j S_{1j}, \ldots, \sum_{j=1}^{N} \partial_j S_{Nj} \right).
\]

During our discussion of the jump condition on \( \Gamma(t) \) and boundary condition on \( \Gamma_0 \), we assume that \( \mathbf{v} \neq \mathbf{u} \) on \( \Gamma(t) \), but \( \mathbf{v} = \mathbf{u} \) on \( \Gamma_0 \).

First, we consider the mass conservation:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho \, dx = 0. \tag{2.5}
\]

By (2.2) and the Reynolds transport theorem, we have

\[
\frac{d}{dt} \int_{\Omega(t)} \rho \, dx = \int_{\Omega(t)} \partial_t \rho \, dx + \int_{\Gamma(t)} \left[ \rho \right] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \rho \mathbf{u} \cdot \mathbf{n}_{\Gamma_0} \, d\nu \\
= -\int_{\Omega(t)} \text{div} (\rho \mathbf{u}) \, dx + \int_{\Gamma(t)} \left[ \rho \right] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \rho \mathbf{u} \cdot \mathbf{n}_{\Gamma_0} \, d\nu \\
= -\int_{\Gamma(t)} \left[ \rho (\mathbf{u} - \mathbf{v}) \right] \cdot \mathbf{n}_{\Gamma(t)} \, d\nu.
\]

Thus, to obtain (2.6), it is sufficient to assume that

\[ \left[ \rho (\mathbf{u} - \mathbf{v}) \right] \cdot \mathbf{n}_{\Gamma(t)} = 0 \quad \text{on} \ \Gamma(t). \tag{2.6} \]

In this case, \( \rho_+ (\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} = \rho_- (\mathbf{u}_- - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} \) on \( \Gamma(t) \), so that the phase flux \( j \) is defined by

\[ j = \rho_+ (\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} = \rho_- (\mathbf{u}_- - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)}. \tag{2.7} \]

- When \( j = 0 \), we have \( \left[ \left[ \mathbf{u} \right] \right] \cdot \mathbf{n}_{\Gamma} = 0. \)

- When \( j \neq 0 \) and \( \left[ \left[ \rho \right] \right] \neq 0 \), we have

\[ j = \frac{\left[ \left[ \mathbf{u} \right] \right] \cdot \mathbf{n}_{\Gamma(t)}}{\left[ \left[ 1/\rho \right] \right]} \tag{2.8}. \]

- When \( j \neq 0 \) and \( \left[ \left[ \rho \right] \right] = 0 \), \( j \) can not be decided by the velocity field \( \mathbf{u} \).
The case where \( j = 0 \) is called without phase transition and the case where \( j \neq 0 \) with phase transition.

Next, we consider the conservation of momentum:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = 0. \tag{2.9}
\]

By (2.3) and the Reynolds transport theorem, we have

\[
\frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = - \int_{\Omega(t)} \text{div} \left( \rho u \otimes u \right) \, dx + \int_{\Omega(t)} \text{div} \, T \, dx
\]

\[
= - \int_{\Gamma(t)} \left( \left[ \rho u \otimes (u - v) - \left[ T \right] \right] n_{\Gamma(t)} \right) \, d\nu + \int_{\Gamma_0} T n_{\Gamma_0} \, d\nu.
\]

Thus, in order that \( \frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = 0 \) holds, it is sufficient to assume that

\[
\begin{cases}
\left[ \rho u \otimes (u - v) - T \right] n_{\Gamma(t)} = \text{div}_T T_{\Gamma} & \text{on } \Gamma(t), \\
T n_{\Gamma_0} = 0 & \text{on } \Gamma_0,
\end{cases} \tag{2.10}
\]

Here, \( T_{\Gamma} \) is the stress tensor field on \( \Gamma(t) \). Note that \( \int_{\Gamma(t)} \text{div}_T T_{\Gamma} \, d\nu = 0 \). We assume that \( \text{div}_T T_{\Gamma} = -\sigma H_{\Gamma} n_{\Gamma(t)} \), where \( \sigma \) is a non-negative constant describing the coefficient of surface tension.

We represent the interface condition (2.10) with the help of the phase flux \( j \) as follows:

\[
\left[ \rho u \otimes (u - v) - T \right] n_{\Gamma(t)} = \text{div}_T T_{\Gamma} \quad \text{on } \Gamma(t) \quad \text{and} \quad T n_{\Gamma_0} = 0 \quad \text{on } \Gamma_0.
\]

Moreover, by (2.2) we rewrite (2.3) as follows:

\[
\partial_t (\rho u) + \text{div} (\rho u \otimes u) = \rho(\partial_t u + u \cdot \nabla u) = \rho(\partial_t u + u \cdot \nabla u).
\]

Summing up, we have obtained

\[
\begin{cases}
\rho(\partial_t u + u \cdot \nabla u) - \text{div} \, T = 0 & \text{in } \Omega(t), \\
j[[u]] - [[T n_{\Gamma(t)}]] = -\sigma H_{\Gamma} n_{\Gamma(t)} & \text{on } \Gamma(t), \\
T n_{\Gamma_0} = 0 & \text{on } \Gamma_0.
\end{cases} \tag{2.11}
\]

Here and in the following, for any \( N \)-vector functions \( w = (w_1, \ldots, w_N) \), \( z = (z_1, \ldots, z_N) \) and scalar function \( f \), we set \( w \cdot \nabla f = \sum_{j=1}^N w_j \partial_j f \), and \( w \cdot \nabla z \) is an \( N \)-vector function whose \( i \) th component is \( w \cdot \nabla z_i \).

Next, we consider the balance of energy. We look for a sufficient condition to obtain the conservation of energy:

\[
\frac{d}{dt} \left( \int_{\Omega(t)} \left( \frac{\rho}{2} |u|^2 + \rho e \right) \, dx + \sigma |\Gamma(t)| \right) = 0. \tag{2.12}
\]

By (2.4) and the Reynolds transport theorem, we have

\[
\frac{d}{dt} \int_{\Omega(t)} \left( \frac{\rho}{2} |u|^2 + \rho e \right) \, dx
\]

\[
= - \int_{\Gamma(t)} \left( \left[ \frac{\rho}{2} |u|^2 + \rho e \right] (u - v) - \left( T u - q \right) \right) \cdot n_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \left( T u - q \right) \cdot n_{\Gamma_0} \, d\nu.
\]
Moreover, using (2.2) and (2.3), we rewrite (2.4) as follows:

\[
\frac{d}{dt}(\int_{\Omega(t)} \left( \frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) \, dx + \sigma |\Gamma(t)|) = \int_{\Gamma_0} (\mathbf{T}\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_{\Gamma_0} \, d\nu \\
- \int_{\Gamma(t)} (\langle \rho \mathbf{u}^2 + \rho e \mathbf{u} - (\mathbf{T}\mathbf{u} - \mathbf{q}) \rangle \cdot \mathbf{n}_{\Gamma(t)} + \sigma \mathbf{H}_t \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)}) \, d\nu.
\]

Thus, in order to obtain (2.12), it is sufficient to assume that

\[
\int_{\Gamma(t)} (\langle \rho \mathbf{u}^2 + \rho e \mathbf{u} - (\mathbf{T}\mathbf{u} - \mathbf{q}) \rangle \cdot \mathbf{n}_{\Gamma(t)} + \sigma \mathbf{H}_t \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)}) \, d\nu = 0
\]

Since \( \mathbf{T}\mathbf{n}_{\Gamma_0} = 0 \) on \( \Gamma_0 \), we assume that \( \mathbf{q} \cdot \mathbf{n}_{\Gamma_0} = 0 \) on \( \Gamma_0 \). By (2.7) and (2.11),

\[
\langle \rho \mathbf{u}^2 + \rho e \mathbf{u} - (\mathbf{T}\mathbf{u} - \mathbf{q}) \rangle \cdot \mathbf{n}_{\Gamma(t)} + \sigma \mathbf{H}_t \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = 0
\]

Since \( \langle \rho (\mathbf{u} - \mathbf{v}) \rangle \cdot \mathbf{n}_{\Gamma(t)} = j[\mathbf{e}] \), the first equation of (2.13) becomes:

\[
\frac{1}{2} \langle |\mathbf{u} - \mathbf{v}|^2 \rangle + j[\mathbf{e}] - \langle \mathbf{T}(\mathbf{u} - \mathbf{v}) \rangle \cdot \mathbf{n}_{\Gamma(t)} + \langle \mathbf{q} \rangle \cdot \mathbf{n}_{\Gamma(t)} = 0.
\]

Moreover, using (2.2) and (2.3), we rewrite (2.4) as follows:

\[
\frac{d}{dt}(\int_{\Omega(t)} \left( \frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) \, dx + \sigma |\Gamma(t)|) = \int_{\Gamma_0} (\mathbf{T}\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_{\Gamma_0} \, d\nu \\
- \int_{\Gamma(t)} (\langle \rho \mathbf{u}^2 + \rho e \mathbf{u} - (\mathbf{T}\mathbf{u} - \mathbf{q}) \rangle \cdot \mathbf{n}_{\Gamma(t)} + \sigma \mathbf{H}_t \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)}) \, d\nu.
\]

Here, we have set \( \mathbf{T} : \nabla \mathbf{u} = \sum_{i,j=1}^{N} T_{ij} \partial_i u_j \). Thus, we have

\[
\rho (\partial_t \mathbf{e} + \mathbf{u} \cdot \nabla \mathbf{e}) + \text{div} \mathbf{q} - \mathbf{T} : \nabla \mathbf{u} = 0.
\]

Summing up, we have obtained

\[
\begin{cases}
\rho (\partial_t \mathbf{e} + \mathbf{u} \cdot \nabla \mathbf{e}) + \text{div} \mathbf{q} - \mathbf{T} : \nabla \mathbf{u} = 0 \quad \text{in } \Omega(t), \\
\frac{1}{2} \langle |\mathbf{u} - \mathbf{v}|^2 \rangle + j[\mathbf{e}] - \langle \mathbf{T}(\mathbf{u} - \mathbf{v}) \rangle \cdot \mathbf{n}_{\Gamma(t)} + \langle \mathbf{q} \rangle \cdot \mathbf{n}_{\Gamma(t)} = 0 \quad \text{on } \Gamma(t), \\
\mathbf{q} \cdot \mathbf{n}_{\Gamma_0} = 0 \quad \text{on } \Gamma_0.
\end{cases}
\]

The number of interface conditions is so far not enough. To find one more condition, we consider the law of entropy increase:

\[
\frac{d}{dt} \int_{\Omega(t)} \rho \eta \, dx \geq 0.
\]
For this purpose, we introduce the constitutive laws in the phases. According to the Newton law, the stress tensor $T$ is given by

$$T = 2\mu D(u) + (\lambda - \mu)\text{div} u I - \pi I.$$  

Here, $D(u) = \frac{1}{2}(T \nabla u + \nabla u)$ is the strain tensor field, $I$ the $N \times N$ identity matrix, $\mu$ and $\lambda$ are the first and second viscosity coefficients satisfying the condition:

$$\mu > 0, \quad \lambda > \frac{N - 2}{N} \mu.$$  

(2.16)

To prove local well-posedness, it suffices to assume that $\mu > 0$ and $\lambda > 0$. According to the Fourier law, the heat flux $q$ is given by

$$q = -d \nabla \theta.$$  

(2.17)

with thermal conductivity $d$ satisfying the condition: $d > 0$. Moreover, the first law of thermodynamics tells us that the internal energy $e$, the entropy $\eta$, and the pressure term $\pi$ have the relation:

$$de = \theta d\eta + \frac{\pi}{\rho^2} d\rho.$$  

(2.18)

We define the free energy $\psi$ for the unit mass and the specific heat $\kappa$ by

$$\psi = e - \theta \eta, \quad \kappa = \frac{\partial e}{\partial \theta},$$  

(2.19)

respectively. We assume that $\kappa > 0$. Since $\frac{\partial e}{\partial \eta} = \theta$ and $\frac{\partial e}{\partial \rho} = \frac{\pi}{\rho^2}$ as follows from (2.18), by (2.2)

$$\partial_t e + u \cdot \nabla e = \frac{\partial e}{\partial \eta} (\partial_t \eta + u \cdot \nabla \eta) + \frac{\partial e}{\partial \rho} (\partial_t \rho + u \cdot \nabla \rho)$$

$$= \theta (\partial_t \eta + u \cdot \nabla \eta) - \frac{\pi}{\rho} \text{div} u.$$  

(2.20)

In addition,

$$T : \nabla u = 2\mu |D(u)|^2 + (\lambda - \mu)(\text{div} u)^2 - \pi \text{div} u,$$  

(2.21)

which, combined with the first equation of (2.14), (2.20), and (2.21), furnishes

$$\rho \theta (\partial_t \eta + u \cdot \nabla \eta) - \text{div} (d \nabla \theta) - (2\mu |D(u)|^2 + (\lambda - \mu)(\text{div} u)^2) = 0.$$  

(2.22)

On the other hand, we have

$$\partial_t (\rho \eta) + \text{div} (\rho \eta u) = \eta (\partial_t \rho + \text{div} (\rho u)) + \rho (\partial_t \eta + u \cdot \nabla \eta) = \rho (\partial_t \eta + u \cdot \nabla \eta).$$  

(2.23)

In the following, we assume that

$$\theta > 0, \quad [[\theta]] = 0.$$  

(2.24)

Since $\theta$ represents the absolute temperature, $\theta > 0$ is natural assumption. While phase transition happens, the temperature does not change, so that $[[\theta]] = 0$ is also natural assumption. By (2.22) and (2.23)

$$\partial_t (\rho \eta) + \text{div} (\rho \eta u) = \frac{1}{\theta} \{ \text{div} (d \nabla \theta) + 2\mu |D(u)|^2 + (\lambda - \mu)(\text{div} u)^2 \}.$$  

(2.25)
By the Reynolds transport theorem, \[(2.25)\], and the divergence theorem of Gauss,

\[
\frac{d}{dt} \int_{\Omega(t)} \rho \eta \, dx = - \int_{\Omega(t)} \text{div} (\rho \eta \mathbf{u}) \, dx + \int_{\Omega(t)} \frac{1}{\theta} (\text{div} (d\nabla \theta) + (2\mu |\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu) (\text{div} \mathbf{u})^2) \, dx \\
+ \int_{\Gamma(t)} [(\rho \eta)] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \rho \eta \mathbf{u} \cdot \mathbf{n}_{\Gamma_0} \, d\nu
\]

Moreover, by \[(2.27)\], we have

\[
\frac{d}{dt} \int_{\Omega(t)} \rho \eta \, dx = - \int_{\Omega(t)} \text{div} (\rho \eta \mathbf{u}) \, dx + \int_{\Gamma(t)} [(\rho \eta)] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \rho \eta \mathbf{u} \cdot \mathbf{n}_{\Gamma_0} \, d\nu
+ \int_{\Gamma(t)} \frac{d}{\theta^2} \left[\frac{\theta}{2} \nabla \theta \cdot \mathbf{n}_{\Gamma(t)} \right] \, d\nu.
\]

Since \(2\mu |\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu) (\text{div} \mathbf{u})^2 \geq 0\) as follows from \[(2.16)\], to obtain \[(2.15)\] it is sufficient to assume that

\[
[[\rho \eta]](\mathbf{u} - \mathbf{v}) - \frac{d}{\theta} \nabla \theta \big|_{\Gamma(t)} \mathbf{n}_{\Gamma(t)} = 0 \quad \text{on} \quad \Gamma(t),
\]

\[
d\nabla \theta \cdot \mathbf{n}_{\Gamma_0} = 0 \quad \text{on} \quad \Gamma_0.
\]

Moreover, by \[\theta = 0\] and \[(2.7)\], the first equation of \[(2.26)\] becomes

\[
\mathbf{j}[[\theta]] - [[d\nabla \theta]] \cdot \mathbf{n}_{\Gamma(t)} = 0 \quad \text{on} \quad \Gamma(t),
\]

which is called the Stefan law. In fact, by \[(2.7)\] and \[\theta = 0\],

\[
0 = [[\rho \eta]](\mathbf{u} - \mathbf{v}) - \frac{d}{\theta} \nabla \theta \big|_{\Gamma(t)} \mathbf{n}_{\Gamma(t)}
\]

\[
= (\rho - \rho_\eta)(\mathbf{u} - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} - (\rho + \rho_\eta)(\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} - \frac{(d_+ \nabla \theta_+ - d_- \nabla \theta_-) \cdot \mathbf{n}_{\Gamma(t)}}{\theta}
\]

\[
= \frac{1}{\theta} \left[ [[\theta]] - [[d\nabla \theta]] \cdot \mathbf{n}_{\Gamma(t)} \right].
\]

Note that the Stefan law becomes the usual jump condition: \[[[d\nabla \theta]] \cdot \mathbf{n}_{\Gamma(t)} = 0 \quad \text{on} \quad \Gamma(t)\) provided that \(j = 0\).

Next, assuming that \(j \neq 0\) and \[\rho \neq 0\] and using \[(2.27)\], we rewrite \[(2.11)\]. Given \(\mathbf{w}\), we set \(\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w} = \mathbf{w} - (\mathbf{w} \cdot \mathbf{n}_{\Gamma(t)}) \mathbf{n}_{\Gamma(t)}\), which is the tangential part of \(\mathbf{w}\) along \(\mathbf{n}_{\Gamma(t)}\). Since \(\mathbf{w} = (\mathbf{w} \cdot \mathbf{n}_{\Gamma(t)}) \mathbf{n}_{\Gamma(t)} + \mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w}\) and \(\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w} \cdot \mathbf{n}_{\Gamma(t)} = 0,\) we have

\[
|\mathbf{w}|^2 = |\mathbf{w} \cdot \mathbf{n}_{\Gamma(t)}|^2 + |\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w}|^2.
\]

In the following, we assume that

\[
\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} [u - v] = 0.
\]

Especially, we have

\[
|\mathbf{u}| = (|\mathbf{u}| \cdot \mathbf{n}_{\Gamma(t)}) \mathbf{n}_{\Gamma(t)}.
\]

Since \(|\mathbf{w}|^2 - |\mathbf{v}|^2 = (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} + \mathbf{v})\) for any \(\mathbf{w}\) and \(\mathbf{v}\), by \[(2.29)\] [[\(\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} (\mathbf{u} - \mathbf{v})^2\)] = 0, so that by \[(2.7)\] and \[(2.28)\]

\[
\frac{1}{2} |u - v|^2 = \frac{1}{2} [[(u - v) \cdot \mathbf{n}_{\Gamma(t)}|^2] = \frac{j^2}{2} [\frac{1}{\rho^2}].
\]
To derive (2.32), we assume that \( \rho \neq (2.14) \). By (2.29), combined with (2.29), furnishes that \( u \neq (2.32) \).

When \( \gamma = 0 \), it follows from (2.7) and (2.29) that 

\[
\psi = \frac{1}{2\rho^2} \left( u - v \right) \cdot n_{\Gamma(t)} + j([\psi]).
\]

Moreover, noting that \( T_\pm \) are symmetric matrices, we have

\[
\left[ [T(u - v)] \cdot n_{\Gamma(t)} \right] = \left[ [(u - v) \cdot Tn_{\Gamma(t)}] \right] = \left[ [(u - v) \cdot n_{\Gamma(t)}] n_{\Gamma(t)} \cdot Tn_{\Gamma(t)} \right] + \left[ [Tn_{\Gamma(t)}(u - v) \cdot Tn_{\Gamma(t)}] \right].
\]

By (2.7),

\[
\left[ [(u - v) \cdot n_{\Gamma(t)}] n_{\Gamma(t)} \cdot Tn_{\Gamma(t)} \right] = j\left[ \frac{1}{\rho} n_{\Gamma(t)} \cdot Tn_{\Gamma(t)} \right].
\]

On the other hand, by (2.31) and (2.29) and (2.30),

\[
\left[ [Tn_{\Gamma(t)}(u - v) \cdot Tn_{\Gamma(t)}] \right] = Tn_{\Gamma(t)}(u - v) \cdot [Tn_{\Gamma(t)}] = Tn_{\Gamma(t)}(u - v) \cdot (j[\rho]) \cdot n_{\Gamma(t)} + \sigma H\Gamma n_{\Gamma(t)} = 0.
\]

Thus, we have obtained \( 0 = \gamma([\psi]) + j^2\left[ \frac{1}{2\rho^2} \right] - \left[ \frac{1}{\rho} n_{\Gamma(t)} \cdot Tn_{\Gamma(t)} \right] \). Since \( j \neq 0 \), finally we arrive at the condition:

\[
\left[ [\psi] \right] + j^2\left[ \frac{1}{2\rho^2} \right] - \left[ \frac{1}{\rho} n_{\Gamma(t)} \cdot Tn_{\Gamma(t)} \right] = 0 \quad \text{on } \Gamma(t), \tag{2.32}
\]

which is called the generalized Gibbs-Thomson law.

Finally, we calculate \( V_\Gamma := v \cdot n_{\Gamma(t)} \). By (2.7), we have \( v \cdot n_{\Gamma(t)} = u - n_{\Gamma(t)} - \frac{1}{\rho} \).

When \( j = 0 \), it follows from (2.7) and (2.29) that \( [u] = 0 \), so that \( v \cdot n_{\Gamma(t)} = u \cdot n_{\Gamma(t)} \).

When \( j \neq 0 \) and \( [\rho] \neq 0 \), by (2.32), we have \( [\rho] = \left[ \frac{1}{\rho} \right] \), so that

\[
v \cdot n_{\Gamma(t)} = u \cdot n_{\Gamma(t)} - \frac{1}{\rho} = \frac{[\rho u] \cdot n_{\Gamma(t)}}{[\rho]}.
\]

Summing up, we have obtained

\[
V_\Gamma := v \cdot n_{\Gamma(t)} = u \cdot n_{\Gamma(t)} \quad (j = 0),
\]

\[
V_\Gamma := v \cdot n_{\Gamma(t)} = \frac{[\rho u] \cdot n_{\Gamma(t)}}{[\rho]} \quad (j \neq 0 \text{ and } [\rho] \neq 0). \tag{2.33}
\]

Next, we consider the case where \( j \neq 0 \) and \( [\rho] = 0 \). In this case, \( [u] \cdot n_{\Gamma(t)} = 0 \), which, combined with (2.29), furnishes that \( [u] = 0 \), so that (2.11) becomes

\[
[Tn_{\Gamma(t)}] = \sigma H\Gamma n_{\Gamma(t)} \quad \text{on } \Gamma(t). \tag{2.34}
\]

To derive (2.32), we assume that \( [\rho] \neq 0 \), so that we reconsider the second condition of (2.14). By \( [u] = 0 \), \( [u - v]^2 = 0 \). By (2.14) and (2.21), \( j([v]) + [q] \cdot n_{\Gamma(t)} = 8\).
\[ j[[\psi]]. \] In addition, by (2.4), (2.5), and the symmetricity of \( T_\pm \)

\[ [T(u - v)] \cdot n_{\Gamma(t)} = (u_- - v) \cdot T_\pm n_{\Gamma(t)} - (u_+ - v) \cdot T_\pm n_{\Gamma(t)} \]

\[ = (u_- - v) \cdot [Tn_{\Gamma(t)}] = (u_- - v) \cdot \sigma Hn_{\Gamma(t)} \]

Since \( j \neq 0 \), the second equation of (2.14) becomes

\[ ||\psi|| - \frac{\sigma}{\rho_-} H = 0. \quad \text{on } \Gamma(t). \quad (2.35) \]

Noting that \( \partial_t \rho + u \cdot \nabla \rho = -\rho \text{div } u \) as follows from (2.2) and recalling the formulas: \( \frac{\partial c}{\partial \theta} = \kappa \) and \( \frac{\partial c}{\partial \rho} = \frac{\pi}{\rho^2} \) (cf. (2.18) and (2.19)), we have

\[ \rho(\partial_t e + u \cdot \nabla e) = \rho(\frac{\partial c}{\partial \theta} \partial_t \theta + \frac{\partial c}{\partial \rho} \partial_t \rho + \frac{\partial e}{\partial \theta} u \cdot \nabla \theta + \frac{\partial e}{\partial \rho} u \cdot \nabla \rho) \]

\[ = \rho \kappa (\partial_t \theta + u \cdot \nabla \theta) - \frac{\pi}{\rho} \text{div } u, \]

which, combined with the first equation in (2.14) and (2.21), furnishes that

\[ \rho \kappa (\partial_t \theta + u \cdot \nabla \theta) - \text{div} (\text{div } \theta) = (2 \mu |D(\theta)|^2 + (\lambda - \mu)(\text{div } u)^2) + \pi (1 - \frac{1}{\rho}) \text{div } u = 0. \]

Summing up, we have obtained the equations: for \( x \in \Omega(t) \) and \( t > 0 \)

\[ \partial_t \rho + \text{div} (\rho u) = 0, \]

\[ \rho(\partial_t u + u \cdot \nabla u) - \text{div } T = 0, \quad (2.36) \]

\[ \rho \kappa (\partial_t \theta + u \cdot \nabla \theta) - \text{div} (\text{div } \theta) = (2 \mu |D(\theta)|^2 + (\lambda - \mu)(\text{div } u)^2) - \pi (1 - \frac{1}{\rho}) \text{div } u, \]

subject to the boundary condition: for \( x \in \Gamma_0 \) and \( t > 0 \):

\[ Tn_{\Gamma_0} = 0, \quad d\nabla \theta \cdot n_{\Gamma_0} = 0 \quad \text{on } \Gamma_0, \quad (2.37) \]

and one of the following interface conditions: for \( x \in \Gamma(t) \) and \( t > 0 \)

(1) When \( j = 0 \),

\[ [[u]] = 0, \quad ||[Tn_{\Gamma(t)}]] = \sigma Hn_{\Gamma(t)}, \quad ||[\theta]] = 0, \]

\[ ||[d\nabla \theta \cdot n_{\Gamma(t)}]] = 0, \quad v \cdot n_{\Gamma(t)} = u \cdot n_{\Gamma(t)}. \quad (2.38) \]

(2) When \( j \neq 0 \) and \( ||[\rho]] \neq 0, \)

\[ T_{\Gamma(t)}[[u]] = 0, \quad j[[u]] - ||[Tn_{\Gamma(t)}]] = -\sigma Hn_{\Gamma(t)}, \quad ||[\theta]] = 0, \]

\[ j[[\theta]] - ||[d\nabla \theta \cdot n_{\Gamma(t)}]] = 0, \quad ||[\psi]] + \frac{1}{2} [[2|D(\theta)|^2]] - [[\frac{1}{\rho} n_{\Gamma(t)}]Tn_{\Gamma(t)}] = 0, \]

\[ v \cdot n_{\Gamma(t)} = \frac{[[\rho u]] \cdot n_{\Gamma(t)}}{[[\rho]]}, \quad J = \frac{[[u]] \cdot n_{\Gamma(t)}}{[[1/\rho]]}. \quad (2.39) \]

(3) When \( j \neq 0 \) and \( \rho = \rho_- = \rho_+ \) (constants),

\[ [[u]] = 0, \quad ||[Tn_{\Gamma(t)}]] = \sigma Hn_{\Gamma(t)}, \quad ||[\theta]] = 0, \quad j[[\theta]] - ||[d\nabla \theta \cdot n_{\Gamma(t)}]] = 0, \]

\[ \rho[[\psi]] - \sigma H = 0, \quad v \cdot n_{\Gamma(t)} = u \cdot n_{\Gamma(t)} - \frac{1}{\rho}. \quad (2.40) \]
Remark 1. Assuming that $\Omega_- = \Omega$ and $\Omega_+ = \emptyset$, we have the one phase problem.
In this case, as boundary condition on $\Gamma_0$, we have
\[ T_n\Gamma_0 = \sigma H_{\Gamma_0} n_{\Gamma_0}, \quad d\nabla \theta \cdot n_{\Gamma_0} = 0 \quad \text{on } \Gamma_0. \]

3 Problem

The problem of this paper is concerned with the compressible and incompressible two phase flow separated by a nearly flat interface with phase transition. Let $h_0(x')$ be a given function with respect to $x' = (x_1, \ldots, x_{N-1})$ and we set
\[ \Omega_\pm = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \pm (x_N - h_0(x')) > 0 \quad \text{for } x' \in \mathbb{R}^{N-1} \}, \]
\[ \Gamma = \{ x \in \mathbb{R}^N \mid x_N = h(x') \quad \text{for } x' \in \mathbb{R}^{N-1} \}. \]
In this case, $\Omega = \mathbb{R}^N$ and $\Gamma_0 = \emptyset$. Let $h(x', t)$ be a unknown function and we assume that the time evolutions of domains $\Omega_\pm$ and the surface $\Gamma$ are given by
\[ \Omega_\pm(t) = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \pm (x_N - h(x', t)) > 0 \quad \text{for } x' \in \mathbb{R}^{N-1} \}, \]
\[ \Gamma(t) = \{ x \in \mathbb{R}^N \mid x_N = h(x', t) \quad \text{for } x' \in \mathbb{R}^{N-1} \}. \] (3.1)
In this case, $n_{\Gamma(t)} = (-\nabla' h, 1)/\sqrt{1 + |\nabla' h|^2}$, with $\nabla' h = (\partial_1 h, \ldots, \partial_{N-1} h)$ ($\partial_j = \partial/\partial x_j$). Moreover, $\varphi(x, t) = (x', x_N + h(x', t))$, so that $\nu \cdot n_{\Gamma(t)} = \partial_t \varphi \cdot n_{\Gamma(t)} = \partial_t h/\sqrt{1 + |\nabla' h|^2}$.

In view of (2.21), (2.36), (2.37) and (2.39), our problem is given as follows:

For $x \in \Omega_+(t)$, $t > 0$,
\[ \begin{align*}
\rho_+ (\partial_t u_+ + u_+ \cdot \nabla u_+) - \text{Div } T_+ &= 0, \quad \partial_t \rho_+ + \text{div } (\rho_+ u_+) = 0, \\
\rho_+ \kappa_+ (\partial_t \theta_+ + u_+ \cdot \nabla \theta_+) - \text{div } (d_+ \nabla \theta_+ - T_+ : \nabla u_+) - \frac{x}{\rho} \text{div } u_+ &= 0 \quad \text{(3.2)}
\end{align*} \]
and, for $x \in \Omega_-(t)$, $t > 0$,
\[ \begin{align*}
\rho_- (\partial_t u_- + u_- \cdot \nabla u_-) - \text{Div } T_- &= 0, \quad \text{div } u_- = 0, \\
\rho_- \kappa_- (\partial_t \theta_- + u_- \cdot \nabla \theta_-) - \text{div } (d_- \nabla \theta_- - T_- : \nabla u_-) &= 0
\end{align*} \]
subject to the jump conditions: for $x \in \Gamma(t)$ and $t > 0$,
\[ \begin{align*}
&\left[ \left( \frac{1}{\rho} \right) \right] n_{\Gamma} = \left[ [T] n_{\Gamma} \right] = -\sigma H_{\Gamma} n_{\Gamma}, \quad T_{\Gamma(t)}([\rho]) = 0, \\
&[\partial \theta] - [\text{div } (\nabla \theta)] = 0, \quad [\theta] = 0, \\
&[\psi] + \frac{[\psi]}{[\rho]} = 0, \quad \partial_t h = \frac{[\rho u] \cdot (-\nabla' h, 1)}{[\rho]} \quad \text{(3.3)}
\end{align*} \]
and the initial conditions:
\[ \begin{align*}
(u_+, \theta_+)_{|t=0} &= (u_{0+}, \theta_{0+}) \quad \text{in } \Omega_+, \\
(u_-, \theta_-)_{|t=0} &= (u_{0-}, \theta_{0+}) \quad \text{in } \Omega_-, \quad h_{|t=0} = h_0 \quad \text{on } \Gamma. \quad \text{(3.4)}
\end{align*} \]
Here, \( \rho_{\pm}, \theta_{\ast} \) and \( \sigma \) are positive constants describing the reference mass densities of \( \Omega_{\pm} \), the reference temperature of \( \Omega_{\pm} \) and the coefficient of the surface tension, respectively. Moreover, \( T_{\pm} = S_{\pm} - \pi_{\pm} I \) with

\[
S_{\pm} = S_{\pm}(u_{\pm}, \rho_{\pm}, \theta_{\ast}) = \mu_{\pm} D(u_{\pm}) + (\lambda_{\pm} - \mu_{\pm}) \text{div} u_{\pm},
\]

\[
S_{-} = S_{-}(u_{-}, \theta_{\ast}) = \mu_{-} D(u_{-}).
\]

Here, \( d_{\pm} = d_{\pm}(\rho, \theta), \mu_{\pm} = \mu_{\pm}(\rho, \theta), \lambda_{\pm} = \lambda_{\pm}(\rho, \theta), \kappa_{\pm} = \kappa_{\pm}(\rho, \theta) \) are positive \( C^{\infty} \) functions with respect to \( (\rho, \theta) \in (0, \infty) \times (0, \infty) \), and \( \psi_{+}(\theta, \rho) \) and \( \eta_{+}(\theta, \rho) \) are real valued \( C^{\infty} \) functions with respect to \( (\rho, \theta) \in (0, \infty) \times (0, \infty) \), while \( d_{-} = d_{-}(\theta), \mu_{-} = \mu_{-}(\theta), \kappa_{-} = \kappa_{-}(\theta) \) are positive \( C^{\infty} \) functions with respect to \( \theta \in (0, \infty) \), and \( \psi_{-}(\theta) \) and \( \eta_{-}(\theta) \) are real valued \( C^{\infty} \) functions with respect to \( \theta \in (0, \infty) \). Moreover, we also assume that \( \pi_{\pm} \) is given by \( \pi_{\pm} = P_{\pm}(\rho, \theta) \), where \( P_{\pm} \) is some \( C^{\infty} \) function with respect to \( (\rho, \theta) \in (0, \infty) \times (0, \infty) \) such that \( \frac{\partial P_{\pm}}{\partial \rho} > 0 \) for any \( (\rho, \theta) \in (0, \infty) \times (0, \infty) \).

The main purpose of this paper is to show the local wellposedness of problem (3.2), (3.3) and (3.4) in the maximal \( L_{p}-L_{q} \) regularity class under the assumption that \( \rho_{\pm} \) and \( \theta_{\ast} \) satisfy the condition:

\[
\rho_{\ast} - \rho_{\pm}, \quad \psi(\theta_{\ast}) - \psi(\rho_{\ast}, \theta_{\ast}) + \left( \frac{1}{\rho_{\ast}} - \frac{1}{\rho_{\pm}} \right) P(\rho_{\ast}, \theta_{\ast}) = 0. \tag{3.5}
\]

To state our main result, we transform \( \Gamma(t) \) to the flat interface. Set

\[
\mathbb{R}^{N}_{\pm} = \{ x = (x_{1}, \ldots, x_{N}) \in \mathbb{R}^{N} \mid \pm x_{N} > 0 \}, \quad \mathbb{R}^{N}_{0} = \{ x \in \mathbb{R}^{N} \mid x_{N} = 0 \}.
\]

We transfer the problem given in domains \( \Omega_{\pm}(t) \) to that in \( \mathbb{R}^{N}_{\pm} \cup \mathbb{R}^{N}_{0} \) with interface \( \partial \mathbb{R}^{N}_{0} \). Let \( h(x', t) \) be a function appearing in the definition of \( \Gamma(t) \) in (3.1).

Let \( H(x, t) \) be a solution to the equations: (1 - \( \Delta \)) \( H = 0 \) in \( \mathbb{R}^{N}_{\pm} \) with \( H|_{x_{N}=0} = h(x', t) \), where \( \Delta H = \sum_{j=1}^{N} \frac{\partial^{2}}{\partial x_{j}^{2}} H \). To prove the local well-posedness, we assume that \( h_{0} \) is small enough, so that we may assume that

\[
1 + \frac{\partial}{\partial x_{N}} H(x, t, \xi) \geq \frac{1}{2} \quad \text{for any } x \in \mathbb{R}^{N}_{\pm} \text{ and } t \in (0, T). \tag{3.6}
\]

If we consider the transformation:

\[
y_{N} = x_{N} + H(x, t), \quad y_{j} = x_{j} \quad (j = 1, \ldots, N - 1),
\]

then by (3.6) \( \Omega_{\pm}(t) \) and \( \Gamma(t) \) are transformed to \( \mathbb{R}^{N}_{\pm} \) and \( \mathbb{R}^{N}_{0} \), respectively, because \( y_{N} = h(y', t) \) when \( x_{N} = 0 \) and

\[
\frac{\partial y_{N}}{\partial x_{N}} = 1 + \frac{\partial H}{\partial x_{N}}(x, t) \geq \frac{1}{2}.
\]

Let \( \mathbf{u}_{\pm}, \rho_{\ast}, \pi_{\ast} \) and \( \theta_{\ast} \) satisfy problem (3.2), (3.3) and (3.4). Set

\[
\hat{\mathbf{u}}_{\pm}(x, t) = \mathbf{u}_{\pm}(x', x_{N} + H(x, t), t), \quad \hat{\rho}_{\pm}(x, t) = \hat{\rho}_{\ast}(x', x_{N} + H(x, t), t),
\]

\[
\hat{\theta}(x, t) = \theta_{\ast}(x', x_{N} + H(x, t), t), \quad \hat{\mu}_{+} = \mu_{+}(\rho_{\ast}, \theta_{\ast}), \quad \hat{\mu}_{-} = \mu_{-}(\theta_{\ast}), \quad \hat{\kappa}_{+} = \kappa_{+}(\rho_{\ast}, \theta_{\ast}),
\]

\[
d_{\pm} = d_{\pm}(\rho_{\ast}, \theta_{\ast}), \quad \hat{d}_{\pm} = d_{\pm}(\rho_{\ast}, \theta_{\ast}), \quad \tilde{d}_{\pm} = \tilde{d}_{\pm}(\rho_{\ast}, \theta_{\ast}), \quad \hat{d}_{\pm} = \hat{d}_{\pm} - d_{\pm}. \tag{3.7}
\]
Setting $H_0 = \partial_t H$, $H_j = \partial_j H$ ($j = 1, \ldots, N$), we have

$$
(\partial_t f)(x', x_N + H(x, t), t) = \partial_t \hat{f}(x, t) = \frac{H_0}{1 + H_N} \partial_N \hat{f}(x, t),
$$

$$
(\partial_j f)(x', x_N + H(x, t), t) = \partial_j \hat{f}(x, t) = \frac{H_j}{1 + H_N} \partial_N \hat{f}(x, t) \quad (j = 1, \ldots, N).
$$

(3.8)

In the following, we set

$$
K_j = \frac{H_j}{1 + H_N} \quad (j = 0, 1, \ldots, N), \quad K = (K_1, \ldots, K_N), \quad K_0 = (K_0, K).
$$

By (3.8) we have

$$
\nabla \pi_{-} = Q \nabla \hat{\pi}_{-} = \begin{pmatrix} 1 & 0 & \cdots & 0 & K_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & K_{N-1} \\ 0 & 0 & \cdots & 0 & 1 + H_N \end{pmatrix} \begin{pmatrix} \partial_1 \hat{\pi}_{-} \\ \vdots \\ \partial_N \hat{\pi}_{-} \end{pmatrix},
$$

and $Q^{-1}$ is given by

$$
Q^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -H_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -H_{N-1} \\ 0 & 0 & \cdots & 0 & 1 + H_N \end{pmatrix} = I + Q_1 \quad \text{with} \quad Q_1 = \begin{pmatrix} 0 & \cdots & 0 & -H_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -H_{N-1} \end{pmatrix}.
$$

By (3.8) we have

$$
\text{div } u_{\pm} = \text{div } \hat{u}_{\pm} + V_{\text{div }}(\hat{u}_{\pm}, H)
$$

$$
= \frac{1}{1 + H_N} \{ \text{div } \hat{u}_{\pm} - f_{-}(\hat{u}_{\pm}, H) \}
$$

$$
= \frac{1}{1 + H_N} \{ \text{div } \hat{u}_{\pm} - \text{div } f_{-}(\hat{u}_{\pm}, H) \}
$$

(3.9)

with

$$
V_{\text{div }}(\hat{u}_{\pm}, H) = -\sum_{j=1}^{N} K_j \partial_N \hat{u}_{\pm,j}, \quad f_{-}(\hat{u}_{\pm}, H) = \sum_{j=1}^{N-1} (H_N \partial_j \hat{u}_{\pm,j} - H_j \partial_N \hat{u}_{\pm,j}),
$$

$$
f_{-}(\hat{u}_{\pm}, H) = -(H_N \hat{u}_{\pm,1}, \ldots, H_N \hat{u}_{\pm,N-1}, - \sum_{j=1}^{N-1} H_j \hat{u}_{\pm,j}).
$$

For any $N \times N$ matrix of functions $G = (g_1, \ldots, g_N)$, where $^T$M denotes the transposed M, with $^T g_j = (g_{i1}, \ldots, g_{iN})$, by (3.9) we have

$$
\text{Div } G = \text{Div } \hat{G} + V_{\text{Div }}(G, H)
$$

(3.10)

with $V_{\text{Div }}(G, H) = ^T(V_{\text{div }}(\hat{g}_1, H), \ldots, V_{\text{div }}(\hat{g}_N, H))$. Moreover, we set

$$
D_{ij}(u_{\pm}) = D_{ij}(\hat{u}_{\pm}) + V_{D_{ij}}(\hat{u}_{\pm}, H), \quad D(u_{\pm}) = D(\hat{u}_{\pm}) + V_D(\hat{u}_{\pm}, H),
$$

(3.11)
where $V_{D_{ij}}(\tilde{u}_\pm, H) = -(K_i\partial_N\tilde{u}_{\pm j} + K_j\partial_N\tilde{u}_{\pm i})$ and $V_D(\tilde{u}_\pm, H)$ is the $N \times N$ matrix whose $(i, j)$ component is $V_{D_{ij}}(\tilde{u}_\pm, H)$.

Under these preparations, we see easily that problem (3.22), (3.3) and (3.4) is transformed to the following problem:

\[
\begin{cases}
\partial_t \rho_+ + v_+ \cdot \nabla \rho_+ + \rho_+(\text{div} u_+ + V_{\text{div}}(\tilde{u}_+, H)) = 0 \\
\rho_+ \partial_t \tilde{u}_+ - \text{Div} S_+(\tilde{u}_+) = F_+ \\
\rho_+ \kappa_+ \partial_t \hat{\theta}_+ - d_+ \Delta \hat{\theta}_+ = F_{\theta+} \\
\rho_- \partial_t \tilde{u}_- - \text{Div} S_-(\tilde{u}_- + \nabla \tilde{\pi}_-) = F_- \\
\text{div} \tilde{u}_- = f_- = \text{div} f \\
\rho_- \kappa_- \partial_t \hat{\theta}_- - d_- \Delta \hat{\theta}_- = F_{\theta-} \\
\mu_+ D_N(\hat{\tilde{u}}_+)\nabla H + G_+ \\
T_- - T_+ = \sigma \Delta H + G_N, \quad \rho_+ \Delta T_- - \rho_- \Delta T_+ = G_{N+1} \\
\hat{u}_{-i} - \hat{u}_{+i} = K_i \\
\hat{\theta}_{-i} - \hat{\theta}_{+i} = 0, \quad d_- \partial_N \hat{\theta}_- - d_+ \partial_N \hat{\theta}_+ = G_\theta \\
\partial_t H - \left( \frac{\rho_+ - \rho_{\ast}}{\rho_{\ast} + \rho_{+}} \hat{u}_{-N} - \frac{\rho_- - \rho_+}{\rho_+ - \rho_-} \hat{u}_{+N} \right) = G_h \\
[\hat{\rho}_+, \hat{u}_+, \hat{\theta}_+]|_{t=0} = (\rho_0, \tilde{u}_0, \tilde{\theta}_0) \quad \text{in } \mathbb{R}^N_+, \quad H|_{t=0} = H_0 \quad \text{on } \mathbb{R}^N_0 \quad (3.12)
\end{cases}
\]

where $i = 1, \ldots, N - 1$, and we have set

\[
\begin{align*}
T_- = (\mu_+ D_N(\hat{u}_-\nabla H) - \tilde{\pi}_-)|_{-}, \\
T_+ = (\mu_+ D_N(\hat{u}_+) + (\lambda_+ - \mu_+)\text{div} \hat{u}_+), \\
v_+ = (\hat{u}_{+1}, \ldots, \hat{u}_{+N-1}, \hat{u}_{+N} - K_0 \sum_{j=1}^N K_j \hat{u}_{+j}), \\
S_+(u) = \mu_+ D(u) + (\lambda_+ - \mu_+)\text{div} u, \\
S_-(u) = \mu_- D(u), \\
\rho_0(x) = \rho_0(x', x_N + H_0(x)), \quad u_{0\pm}(x') = u_{0\pm}(x', x_N + H_0(x)), \\
\tilde{\theta}_0(x) = \theta_0(x', x_N + H_0(x)).
\end{align*}
\]

Here, $f|_{x=0} = \lim_{r \to 0} f(x)$ for $x_0 \in \mathbb{R}^N_0$. Moreover, $H_0$ is a function satisfying the equations $(1 - \Delta)H_0 = 0$ in $\mathbb{R}^N_0$, and $H_0|_{x_N=0} = h_0$, and the right-hand sides in (3.12) are defined by the following formulas:

\[
\begin{align*}
F_+ &= F_+(\tilde{u}_+, \hat{u}_+, H) \\
&= -\tilde{\rho}_+ \partial_t \hat{u}_+ - \tilde{\rho}_+ \{ K_0 \partial_N \hat{u}_+ - \hat{u}_+ \cdot \nabla \hat{u}_+ + (\hat{u}_+ \cdot K) \partial_N \hat{u}_+ \} \\
&+ \text{Div} (\tilde{\rho}_+ D(\hat{u}_+) + \tilde{\rho}_+ V_D(\tilde{u}_+, H)) + V_D(\tilde{\rho}_+ (D(\hat{u}_+) + V_D(\tilde{u}_+, H))) \\
&+ \nabla \{ (\hat{\lambda}_+ - \tilde{\mu}_+) \text{div} \hat{u}_+ + (\hat{\lambda}_+ - \tilde{\mu}_+) V_{\text{div}}(\tilde{u}_+, H) \} \\
&+ V_{\text{Div}} ((\hat{\lambda}_+ - \tilde{\mu}_+) (\text{div} \hat{u}_+ + V_{\text{div}}(\tilde{u}_+, H)) \mathbf{I} - Q V P_+ (\tilde{\rho}_+, \tilde{\theta}_+) \\
F_{\theta+} &= F_{\theta+}(\tilde{u}_+, \hat{u}_+, \hat{\theta}_+, H) = -d_+ \sum_{j=1}^N \partial_j (K_j \partial_N \hat{\theta}_+ )
\end{align*}
\]
\[ + \sum_{j=1}^{N} \partial_j(\hat{d}_+ (\partial_j \hat{\theta}_+ - K_j \partial_N \hat{\theta}_+)) + \sum_{j=1}^{N} K_j \partial_N (\hat{d}_+ (\partial_j \hat{\theta}_+ - K_j \partial_N \hat{\theta}_+)) \]

\[- (\hat{\rho}_+ \hat{r}_+ - \rho_+ \kappa_+) \partial_j \hat{\theta}_+ + \hat{\rho}_+ \hat{r}_+ (K_0 \partial_N \hat{\theta}_+ - \hat{u}_+ \cdot \nabla \hat{\theta}_+ + (\hat{u}_+ \cdot K) \partial_N \hat{\theta}_+) \]

\[+ 2 \hat{\mu}_+ |D(\hat{u}_+)| + V_D(\hat{u}_+, H)|^2 \]

\[+ \lambda_+ - \hat{\mu}_+) (\text{div} \hat{u}_+ + V_{\text{div}}(\hat{u}_+, H)) \]

\[+ P_+(\hat{\rho}_+ \hat{\theta}_+) (1 - \frac{1}{\hat{\rho}_+}) (\text{div} \hat{u}_+ + V_{\text{div}}(\hat{u}_+, H)) \]

\[ \mathbf{F}_- = \mathbf{F}_- (\hat{u}_-, H) = -Q_1 (\rho_- \partial_j \hat{u}_- - \mu_- \partial_j \text{div} \mathbf{D}(\hat{\mathbf{u}}_-)) \]

\[- (I + Q_1) \{ \rho_- (K_0 \partial_N \hat{u}_- - \hat{u}_- \cdot \nabla \hat{u}_- + (\hat{u}_- \cdot K) \partial_N \hat{u}_-) \]

\[+ \text{div} (\hat{\mu}_- \mathbf{D}(\hat{u}_-) + \hat{\mu}_- \mathbf{V}_D(\hat{u}_-, H)) + V_{\text{div}} (\hat{\mu}_- \text{div} \mathbf{D}(\hat{u}_-) + V_D(\hat{u}_-, H)) \}, \]

\[ f_+ f_- (\hat{u}_-, H) = \sum_{j=1}^{N-1} \{ H_N \partial_j \hat{u}_{-j} - H_j \partial_N \hat{u}_{-j} \}, \]

\[ f_- f_- (\hat{u}_-, H) = - (H_N \hat{u}_{-1}, \ldots, H_N \hat{u}_{-N-1}, - \sum_{j=1}^{N-1} H_j \hat{u}_{-j}) \]

\[ F_{\theta_-} = F_{\theta_-} (\hat{u}_-, \hat{\theta}_-, H) = - \rho_- \hat{\theta}_- \partial_j \hat{\theta}_- - \hat{d}_- \sum_{j=1}^{N} \partial_j (K_j \partial_N \hat{\theta}_-) \]

\[+ \rho_- \hat{\theta}_- (K_0 \partial_N \hat{\theta}_- - \hat{u}_- \cdot \nabla \hat{\theta}_- + (\hat{u}_- \cdot K) \partial_N \hat{\theta}_-) + \sum_{j=1}^{N} \partial_j (\hat{d}_+ (\partial_j \hat{\theta}_- - K_j \partial_N \hat{\theta}_-)) \]

\[+ \sum_{j=1}^{N} K_j \partial_N (\hat{d}_- (\partial_j \hat{\theta}_- - K_j \partial_N \hat{\theta}_-)) + 2 \hat{\mu}_- |D(\hat{u}_-)| + V_D(\hat{u}_-, H)|^2 \]

\[ G_i = G_i (\hat{\rho}_+, \hat{u}_+, H) = \]

\[- \{ \{ \hat{\mu}_- D_{\mathbf{D}_N} (\hat{\mathbf{u}}_-) + \hat{\mu}_- \mathbf{V}_{\mathbf{D}_N}(\hat{\mathbf{u}}_-, H) \}|_+ - (\hat{\mu}_+ D_{\mathbf{D}_N} (\hat{\mathbf{u}}_+) + \hat{\mu}_+ \mathbf{V}_{\mathbf{D}_N}(\hat{\mathbf{u}}_+, H)) \}|_+ \]

\[+ \sum_{j=1}^{N-1} \{ (\partial_j H) \{ \hat{\mu}_- (D_{j \mathbf{D}} (\hat{\mathbf{u}}_-) + V_{D_{j \mathbf{D}}}(\hat{\mathbf{u}}_-, H))|_+ - \hat{\mu}_+ (D_{j \mathbf{D}} (\hat{\mathbf{u}}_+) + V_{D_{j \mathbf{D}}}(\hat{\mathbf{u}}_+, H)) \}|_+ \}

\[- (\partial_+ H) \{ \sum_{j=1}^{N} (\partial_j H) \{ \hat{\mu}_- (D_{j \mathbf{D}} (\hat{\mathbf{u}}_-) + V_{D_{j \mathbf{D}}}(\hat{\mathbf{u}}_-, H))|_+ - \hat{\mu}_+ (D_{j \mathbf{D}} (\hat{\mathbf{u}}_+) + V_{D_{j \mathbf{D}}}(\hat{\mathbf{u}}_+, H)) \}|_+ \}

\[- \{ \hat{\mu}_- (D_{\mathbf{D}_N} (\hat{\mathbf{u}}_-) + V_{\mathbf{D}_N}(\hat{\mathbf{u}}_-, H))|_+ - \hat{\mu}_+ (D_{\mathbf{D}_N} (\hat{\mathbf{u}}_+) + V_{\mathbf{D}_N}(\hat{\mathbf{u}}_+, H)) \}|_+ \}, \]

\[ G_N = G_N (\hat{\rho}_+, \hat{u}_+, H) = \]

\[- (\hat{\mu}_- D_{\mathbf{D}_N} (\hat{\mathbf{u}}_-) + \hat{\mu}_- \mathbf{V}_{\mathbf{D}_N}(\hat{\mathbf{u}}_-, H))|_+ - (\hat{\mu}_+ D_{\mathbf{D}_N} (\hat{\mathbf{u}}_+) + \hat{\mu}_+ \mathbf{V}_{\mathbf{D}_N}(\hat{\mathbf{u}}_+, H)) \}|_+ \]

\[+ \{ (\lambda_+ - \hat{\mu}_+) \text{div} \hat{u}_+ + (\lambda_+ - \hat{\mu}_+) V_{\text{div}}(\hat{u}_+, H) - (P_+(\hat{\rho}_+ \hat{\theta}_+) - P_+ (\rho_+ \theta_+)) \}|_+ \]

\[+ \sum_{j=1}^{N-1} (\partial_j H) \{ \hat{\mu}_- (D_{j \mathbf{D}} (\hat{\mathbf{u}}_-) + V_{D_{j \mathbf{D}}}(\hat{\mathbf{u}}_-, H))|_+ - \hat{\mu}_+ (D_{j \mathbf{D}} (\hat{\mathbf{u}}_+) + V_{D_{j \mathbf{D}}}(\hat{\mathbf{u}}_+, H)) \}|_+ \}

\[+ \sigma \{ (1 - \frac{1}{\sqrt{1 + |\nabla H|^2}})^2 \frac{\Delta' H}{\sum_{j=1}^{N-1} (\partial_j H)(\partial_j \partial_+ H)} \}

\[+ \frac{1}{\hat{\rho}_+ |_+} (\hat{\theta}_- N |_+ - \hat{\theta}_+ N |_+)^2 (1 + |\nabla H|^2) \].
\[ G_{N+1} = G_{N+1}(\tilde{\rho}_+, \tilde{u}_\pm, \tilde{\theta}_\pm, H) = \]
\[ -\frac{1}{\rho_{+ \pm}}(P_+(\tilde{\rho}_+, \tilde{\theta}_+) - P_+(\rho_{+ \pm}, \theta_{+ \pm})) - \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right)P_+(\tilde{\rho}_+, \tilde{\theta}_+) \]
\[ + (\psi_-(\tilde{\theta}_-) - \psi_-(\theta_-))|_{-} - (\psi_+(\tilde{\rho}_+, \tilde{\theta}_+) - \psi_+(\rho_{+ \pm}, \theta_{+ \pm}))|_{+} \]
\[ + (\psi_-(\tilde{\theta}_-) - \psi_-(\theta_-))|_{-} - \psi_+(\tilde{\rho}_+, \tilde{\theta}_+) - \psi_+(\rho_{+ \pm}, \theta_{+ \pm}))|_{+} \]
\[ + \left(\psi_-(\tilde{\theta}_-) - \psi_-(\theta_-))|_{-} - \psi_+(\tilde{\rho}_+, \tilde{\theta}_+) - \psi_+(\rho_{+ \pm}, \theta_{+ \pm}))|_{+}\right)|_{+} \]
\[ + \frac{1}{2}\left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right)\right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left\{ \frac{1}{\rho_{+ \pm}}\hat{\mu}_{- \pm} D_{N+1}(\tilde{u}_-|_{-} - \frac{1}{\rho_{+ \pm}}(\hat{\mu}_+ D_{N+1}(\tilde{u}_+|_{+} + (\hat{\lambda}_- - \hat{\mu}_+)\text{div}u_+)|_{+}) \right\} \]
\[ + \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right)\{\hat{\mu}_+(D_{N+1}(\tilde{u}_+) + V_{D_{N+1}}(\tilde{u}_+, H)) \]\n\[ + (\hat{\lambda}_- - \hat{\mu}_+)\text{div}u_+ + V_{\text{div}}(\tilde{u}_+, H)\}) \right\} \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
\[ \left(\frac{1}{\rho_{+ \pm}} - \frac{1}{\rho_{+ \pm}}\right) \right)^{-1}(u_{- \pm} - u_{+ \pm})^2 (1 + |\nabla' H|^2)^2 \]
The phase flux $j$ is eliminated by using the formula:

$$j = (\tilde{u}_N | - \tilde{u}_N | +) \left( \frac{1}{\rho_+} - \frac{1}{\rho_+ | +} + \rho_+ | + \right)^{-1} \sqrt{1 + |\nabla H|^2}$$

on $\mathbb{R}_0^N \times (0, T)$.

Moreover, we have used the formulas: for $x \in \Gamma(t)$ and $t > 0$

$$\pi_+ - P_+ + (\lambda_+ - \mu_+) \text{div } u_+$$

$$= -\sigma H_\Gamma - \left[ \frac{[\mu \hat{u}]}{[\rho]} \right] - \sum_{j=1}^{N-1} [[\mu D_N j]] (\partial_j H) + [[\mu D_N N]]$$

$$H_\Gamma u_\Gamma(t) = \left\{ \text{div}\left( \frac{\nabla H}{1 + |\nabla H|^2} \right) \right\} \left( -\nabla' H, 1 \right)/\sqrt{1 + |\nabla' H|^2}$$

where $\nabla' H = (\partial_1 H, \ldots, \partial_{N-1} H)$ and $\text{div}' v' = \sum_{j=1}^{N-1} \partial_j v_j$ for $v' = (v_1, \ldots, v_{N-1})$.

Since $T_- - T_+ = \sigma \Delta' H + G_N$ and $\rho_{-1}^{-1} T_- + \rho_{-1} T_+ = G_{N+1}$ are equivalent to

$$T_\pm = \frac{\rho_{\pm} \sigma}{\rho_{-} - \rho_{+}} \Delta' H + \frac{\rho_{+} - \rho_{+}}{\rho_{-} - \rho_{+}} (\rho_{\pm}^{-1} G_N - G_{N+1}),$$

the compatibility condition for problem (3.12) is

$$\text{div } \hat{u}_0 = f_\pm (\hat{u}_0, H_0) = \text{div} f_\pm (\hat{u}_0, H_0) \text{ in } \mathbb{R}_0^N$$

$$\mu_+ - D_N (\hat{u}_0 | +) - \mu_+ - D_N (\hat{u}_0 | +) = G_i (0, u_0, \hat{u}_0, H_0) \quad (i = 1, \ldots, N - 1),$$

$$\hat{u}_0 | - \hat{u}_0 | + = 0,$$

$$d_+ - d_+ - d_+ + d_+ \partial_0 \hat{0}_0 | + = G_0 (0, u_0, \hat{0}_0, H_0),$$

$$(\mu_+ + D_N (\hat{u}_0 | +) + \lambda_+ - \mu_+) \text{div } \hat{u}_0 | + = \frac{\rho_{\pm} \sigma}{\rho_{-} - \rho_{+}} \Delta' h_0$$

$$+ \frac{\rho_{+} - \rho_{+}}{\rho_{-} - \rho_{+}} (\rho_{\pm}^{-1} G_N (0, u_0, \hat{0}_0, H_0) - G_{N+1}(0, u_0, \hat{0}_0, H_0)).$$

The following theorem is the main result of this paper concerning the local well-posedness of problem (3.12).

**Theorem 3.1.** Let $1 < p, q < \infty$ with $2/p + N/q < 1$. Assume that $\rho_{\pm}$ and $\theta_\star$ satisfy the condition $3.5$. Then, given any positive time $T$, there exists an $\epsilon > 0$ such that problem (3.12) admits unique solutions $\hat{u}_0, \hat{u}_0, \hat{\theta}_0$ with $\hat{u}_\pm, \hat{\theta}_\pm$ such that

$$\hat{\rho}_0 \in W^1_p((0, T), L_q(\mathbb{R}^N_\pm)) \cap L_p((0, T), W^1_q(\mathbb{R}^N_\pm))$$

$$(\hat{u}_\pm, \hat{\theta}_\pm) \in W^1_p((0, T), L_q(\mathbb{R}^N_\pm)) \cap L_p((0, T), W^1_q(\mathbb{R}^N_\pm))$$

$$H \in W^1_p((0, T), L_q^2(\mathbb{R}^N_\pm)) \cap L_p((0, T), W^1_q(\mathbb{R}^N_\pm))$$

for any initial data

$$\hat{\rho}_0 \in W^1_q(\mathbb{R}^N_+), \quad (\hat{u}_0, \hat{\theta}_0) \in B^2(1/p)(\mathbb{R}^N_\pm),$$

$satisfying$ the smallness condition:

$$\|\hat{\rho}_0\|_{W^1_q(\mathbb{R}^N_+)} + \sum_{\ell = \pm} (\|\hat{u}_{0, \ell} \|_{B^2(1/p)(\mathbb{R}^N_\pm)} + \|H_0\|_{B^2_p(\mathbb{R}^N_\pm)} \leq \epsilon$$

and compatibility condition $3.10$. 

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Here and in the following, \( L_q(\mathbb{R}^N_+) \) and \( W^m_q(\mathbb{R}^N_+) \) denote the usual Lebesgue space and Sobolev space of order \( m \) in the \( L_q(\mathbb{R}^N_+) \) sense, while \( \| \cdot \|_{L_q(\mathbb{R}^N_+)} \) and \( \| \cdot \|_{W^m_q(\mathbb{R}^N_+)} \) denote their norms, respectively. For the Banach space \( X \), \( L_p((0, T), X) \) and \( W^m_p((0, T), X) \) denote the Lebesgue space and the Sobolev space with values in \( X \), while \( \| \cdot \|_{L_p((0, T), X)} \) and \( \| \cdot \|_{W^m_p((0, T), X)} \) denote their norms, respectively. \( \mathcal{B}^{(1-\theta)\alpha+\theta b}_{p, \theta}(\mathbb{R}^N_+) \) denotes the real interpolation space \( (W^a_q(\mathbb{R}^N_+), W^b_q(\mathbb{R}^N_+))_{\theta, p} \) with real interpolation functor \((\cdot, \cdot)_{\theta, p}\) and \( 0 < \theta < 1 \).

**Remark 2.** (1) The mathematical study of the compressible and incompressible two phase problem is quite rare as far as the author knows. First Denisova \[2\] studied the evolution of the compressible and incompressible two phase flow with tension and without phase transition. This paper is the first manuscript to treat the compressible and incompressible two phase problem with phase transition. The incompressible and incompressible two phase problem with phase transition was studied by J. Prüss, et al. \[6, 7, 8\].

**4 Maximal \( L_p - L_q \) regularity**

In the following, we assume that \( N < q < \infty \) in view of the Sobolev imbedding theorem: \( \| v \|_{L_{\infty}(\Omega)} \leq C \| v \|_{W^2_q(\Omega)} \) with \( \Omega = \mathbb{R}^N_+ \) and \( \Omega = \mathbb{R}^N \). To solve problem \[3.12\], we use the maximal \( L_p - L_q \) regularity for the parabolic equations. From this point of view, we represent \( \hat{\rho}_+ \) by the integration along the characteristic curve generated by \( v_+ \) to eliminate \( \hat{\rho}_+ \) from the first equation of \[3.12\], which is the hyperbolic equation for \( \hat{\rho}_+ \).

Given function \( f \) defined on \( \mathbb{R}^N_+ \), the Lions extension \( \text{Ext}[f] \) of \( f \) is defined by

\[
\text{Ext}[f](x, t) = \begin{cases} f(x, t) & \text{for } x_N > 0, \\ 3f(x', -x_N, t) - 2f(x', -2x_N, t) & \text{for } x_N < 0, \end{cases}
\]

Set \( \hat{w}_+ = \text{Ext}[\hat{u}_+] \) and \( v = (\hat{w}_+, \ldots, \hat{w}_{N-1}, \hat{w}_N - K_0 - \sum_{j=1}^N K_j \hat{w}_j) \). Note that \( v = v_+ \) on \( \mathbb{R}^N_+ \). We assume that

\[
\int_0^T \| \nabla v(\cdot, t) \|_{L_{\infty}(\mathbb{R}^N)} \, dt \leq \epsilon_1 \tag{4.1}
\]

with some small positive constant \( \epsilon_1 > 0 \). We use the usual fixed point argument to solve the nonlinear problem and in this argument we keep the situation where \( \hat{u}_+ \) and \( H \) satisfy \((4.1)\).

---

1. Modeling and the main results of this paper were announced in the abstract of 39th Sapporo symposium on PDE at Hokkaido University (cf. \[13\]).
2. Tani \[17\] represented the mass density with the help of the velocity field to prove the local well-posedness of the Navier-Stokes equations describing the compressible viscous fluid flow (cf. also \[16\], \[18\]). It was also suggested by J. Prüss to the author to represent \( \hat{\rho}_+ \) by \( \hat{u}_+ \) and \( H \) with the help of the equation of balance of mass when the author visited Halle university in the early of April, 2014.
Let \( \hat{\xi} \) be the solution to the Cauchy problem:

\[
\frac{d}{dt}\hat{\xi}(\eta, t) = \nu(\hat{\xi}(\eta, t), t), \quad \hat{\xi}(\eta, 0) = \eta \in \mathbb{R}^N.
\]

According to Ströhmer [15], we choose an \( \epsilon_1 > 0 \) so small that the map: \( \eta \mapsto \xi \) is bijective on \( \mathbb{R}^N \) for any \( t \in [0, T] \). We denote its inverse map by \( \tilde{\eta} = \tilde{\eta}(\xi, t) \). We look for \( \hat{\rho}_+ \) satisfying the equation:

\[
\partial_t \hat{\rho}_+ + \nu \cdot \nabla \hat{\rho}_+ + \hat{\rho}_+(\text{div} \hat{\omega}_+ + V_{\text{div}}(\hat{\omega}_+, H)) = 0 \quad \text{in} \ \mathbb{R}^N \times (0, T). \tag{4.2}
\]

Since

\[
\partial_t \hat{\rho}_+(\xi(\eta, t), t) = (\partial_t \hat{\rho}_+ + \nu \cdot \nabla \hat{\rho}_+)(\xi(\eta, t), t) = g(\xi(\eta, t), t)\rho_+(\xi(\eta, t), t),
\]

with \( g = -(\text{div} \hat{\omega}_+ + V_{\text{div}}(\hat{\omega}_+, H)) \), defining \( \hat{\rho}_+(\xi, t) \) by

\[
\hat{\rho}_+(\xi, t) = (\rho_+ + \hat{\rho}_0(\eta))e^{-\int_0^t(\text{div} \hat{\omega}_+ + V_{\text{div}}(\hat{\omega}_+, H))(\xi(\eta, s), \eta) \, ds} \tag{4.3}
\]

with \( \eta = \tilde{\eta}(\xi, t) \), where \( \hat{\rho}_0(\eta) = \text{Ext}[\hat{\rho}_0+] \) to \( \mathbb{R}^N \), we see that \( \hat{\rho}_+ \) is a required function satisfying (4.2) with \( \hat{\rho}_+(\xi, 0) = \rho_+ + \hat{\rho}_0(\xi) \) in \( \mathbb{R}^N \).

Inserting the formula of \( \hat{\rho}_+ \) given in (4.3) into the right-hand sides: \( F_+ = F_+(\hat{\rho}_+, \hat{u}_+, H), F_{\theta+} = F_{\theta+}(\hat{\rho}_+, \hat{u}_+, \hat{\theta}_+, H), G_j = G_j(\hat{\rho}_+, \hat{u}_+, H) \) \((j = 1, \ldots, N + 1)\) and \( G_{\theta} = G_{\theta}(\hat{\rho}_+, \hat{u}_+, \hat{\theta}_+, H) \) in (3.12), we have the interface problem of the final form, which is a quasilinear parabolic equation. As the linearized problem, we have the decoupled two systems. One is the Stokes equation with interface condition:

\[
\begin{cases}
\rho_+ \partial_t u_+ - \text{Div} S_+(u_+) = f_+ & \text{in} \ \mathbb{R}^N_+ \times (0, T) \\
\rho_- \partial_t u_- - \text{Div} S_-(u_-) + \nabla \pi_- = f_- & \text{in} \ \mathbb{R}^N_- \times (0, T) \\
\text{div} u_+ = f_{\text{div}} & \text{in} \ \mathbb{R}^N_+ \times (0, T)
\end{cases} \tag{4.4}
\]

subject to the interface condition: for \( x \in \mathbb{R}^N_0 \) and \( t \in (0, T) \)

\[
\begin{align*}
\mu_{\pm} D_{iN}(u_-)|_{-} - \mu_{\pm} D_{iN}(u_+)|_{+} &= g_i \quad (i = 1, \ldots, N - 1), \\
(\mu_{\pm} + D_{NN}(u_-) - \pi_-)|_{-} &= \sigma_- \Delta H + g_N \\
(\mu_{\pm} + D_{NN}(u_+)) + (\lambda_{\pm} - \mu_{\pm})\text{div} u_+)|_{+} &= \sigma_+ \Delta H + g_{N+1}, \\
\hat{u}_-|_{-} - \hat{u}_+|_{+} &= h_i \quad (i = 1, \ldots, N - 1), \\
\partial_t H - \left( \frac{\rho_-}{\rho_+ - \rho_-} u_{-N} - \frac{\rho_+}{\rho_+ - \rho_-} u_{+N} \right) &= d
\end{align*} \tag{4.5}
\]

and the initial condition:

\[
u(u_\pm)|_{t=0} = u_{0\pm} \quad \text{in} \ \mathbb{R}^N_\pm, \quad H|_{t=0} = H_0 \quad \text{in} \ \mathbb{R}^N_0, \tag{4.6}
\]

where we have set \( \sigma_\pm = \rho_\pm \sigma(\rho_- - \rho_+)^{-1} \) and we have used the equivalent relations (3.15). Another is the heat equations with interface condition:

\[
\begin{align*}
\rho_+ \kappa_+ \partial_t \theta_+ - d_+ \Delta \theta_+ &= \hat{f}_+ & \text{in} \ \mathbb{R}^N_+ \times (0, T) \\
\rho_- \kappa_- \partial_t \theta_- - d_- \Delta \theta_- &= \hat{f}_- & \text{in} \ \mathbb{R}^N_- \times (0, T)
\end{align*} \tag{4.7}
\]
subject to the interface condition: for \( x \in \mathbb{R}^N_0 \) and \( t \in (0, T) \)
\[
\theta_-|_t - \theta_+|_t = 0, \quad d_\nu \partial_N \theta_-|_t - d_\nu \partial_N \theta_+|_t = \tilde{g} \quad (4.8)
\]
and the initial condition:
\[
\theta_{\pm}|_{t=0} = \theta_{0, \pm} \quad \text{on } \mathbb{R}^N_\pm. \quad (4.9)
\]

We have the following theorem about the maximal \( L_p - L_q \) regularity for problem (4.4), (4.5), (4.6).

**Theorem 4.1.** Let \( 1 < p, q < \infty \) and \( 0 < T < \infty \). Assume that \( \rho_- \neq \rho_+ \). Then, for any initial data \( u_{0, \pm} \in B_{q,p}^1((\mathbb{R}^N_\pm)) \) and \( H_0 \in B_{q,p}^{-1}((\mathbb{R}^N)) \), and right-hand sides of (4.4) and (4.5)

\[
f_\pm \in L_p((0, T), L_q((\mathbb{R}^N_\pm))), \quad f_{\text{div}} \in L_p((0, T), W^1_q((\mathbb{R}^N))), \quad f_\div \in W^1_p((0, T), L_q((\mathbb{R}^N)))
\]
\[
d \in L_p((0, T), W^2_q((\mathbb{R}^N))), \quad g_i \in L_p((0, T), W^1_q((\mathbb{R}^N)) \cap W^3_q((0, T), W^{-1}_q((\mathbb{R}^N)))
\]
\[
h_j \in L_p((0, T), W^2_q((\mathbb{R}^N)) \cap W^3_p((0, T), L_p((\mathbb{R}^N)))
\]

for \( i = 1, \ldots, N+1 \) and \( j = 1, \ldots, N-1 \), satisfying the compatibility conditions:

\[
\text{div } u_{0, -} = f_-|_{t=0} = \text{div } f_{\text{div}}|_{t=0} \quad \text{in } \mathbb{R}^N_-, \quad \mu_- D_{iN}(u_{0, -})_t - \mu_+ D_{iN}(u_{0, +})_t = g_i|_{t=0} \quad (i = 1, \ldots, N-1) \quad \text{on } \mathbb{R}^N_-, \quad (\mu_+ D_{iN}(u_{0, +}) + (\lambda_+ - \mu_+) \text{div } u_{0, +})_t
\]
\[
= \sigma_+ H_0 + g_{N+1}|_{t=0} \quad \text{on } \mathbb{R}^N_0, \quad u_{0, -}|_t - u_{0, +}|_t = h_t \quad (i = 1, \ldots, N-1) \quad \text{on } \mathbb{R}^N_0.
\]

then, problem (4.4), (4.5), (4.6) admits unique solutions \( u_{\pm} \) and \( H \) with

\[
u_{\pm} \in L_p((0, T), W^2_q((\mathbb{R}^N_\pm))) \cap W^3_p((0, T), L_q((\mathbb{R}^N))), \quad H \in L_p((0, T), W^3_q((\mathbb{R}^N)) \cap W^3_p((0, T), L^2_q((\mathbb{R}^N)))
\]

possessing the estimates:

\[
\sum_{\ell = \pm} \left\{ \| u_{\ell} \|_{L_p((0, t), W^2_q((\mathbb{R}^N_\ell)))} + \| \partial_t u_{\ell} \|_{L_p((0, t), L_q((\mathbb{R}^N_\ell)))} \right\}
\]
\[
+ \| \partial_t H \|_{L_p((0, t), W^1_q((\mathbb{R}^N)))} + \| H \|_{L_p((0, t), W^2_q((\mathbb{R}^N)))}
\]
\[
\leq C \gamma T \left\{ \sum_{\ell = \pm} \left\{ \| u_{\ell} \|_{L_p((0, t), W^2_q((\mathbb{R}^N_\ell)))} + \| \partial_t u_{\ell} \|_{L_p((0, t), L_q((\mathbb{R}^N_\ell)))} \right\} + \| f_{\text{div}} \|_{L_p((0, t), W^1_q((\mathbb{R}^N)))}
\]
\[
+ \| f_{\div} \|_{L_p((0, t), L_q((\mathbb{R}^N)))} + \sum_{i=1}^{N+1} \left( \| g_i \|_{L_p((0, t), W^1_q((\mathbb{R}^N)))} + \| \partial_t g_i \|_{L_p((0, t), W^{-1}_q((\mathbb{R}^N)))} \right)
\]
\[
+ \sum_{j=1}^{N-1} \left( \| h_j \|_{L_p((0, t), W^2_q((\mathbb{R}^N)))} + \| \partial_t h_j \|_{L_p((0, t), L_q((\mathbb{R}^N)))} \right) + \| d \|_{L_p((0, t), W^2_q((\mathbb{R}^N)))}
\]

for any \( t \in (0, T) \) with some positive constants \( C \) and \( \gamma \) independent of \( t \) and \( T \).

And also, we have the following theorem about the maximal \( L_p - L_q \) regularity for problem (4.7), (4.8), (4.9).
**Theorem 4.2.** Let $1 < p, q < \infty$ and $0 < T < \infty$. Then, for any initial data $\theta_0 \in B^{2(1-1/q)}_{q,p}(\mathbb{R}^N)$ and right-hand sides

$$\bar{f}_\pm \in L_p((0,T),L_q(\mathbb{R}^N)), \quad \bar{g} \in L_p((0,T),W^1_q(\mathbb{R}^N)) \cap W^1_p((0,T),W^{-1}_q(\mathbb{R}^N))$$

satisfying the compatibility condition:

$$[[\theta_0]] = 0, \quad d_+\partial_N\theta_0-|_- - d_-\partial_N\theta_0+|_+ = \bar{g}|_{t=0} \text{ on } \mathbb{R}^N_0,$$

problem (4.7) and (4.8) admits unique solutions $\theta_\pm$ with

$$\theta_\pm \in L_p((0,T),W^2_q(\mathbb{R}^N)) \cap W^1_p((0,T),L_q(\mathbb{R}^N))$$

satisfying the estimate:

$$\sum_{\ell=\pm} \{\|\theta_\ell\|_{L_p((0,t),W^2_q(\mathbb{R}^N))} + \|\partial_t \theta_\ell\|_{L_p((0,t),L_q(\mathbb{R}^N))}\}$$

$$\leq C^\gamma t \left(\sum_{\ell=\pm} \{\|\theta_\ell\|_{B^{2(1-1/q)}_{q,p}(\mathbb{R}^N)} + \|\bar{f}\|_{L_p((0,t),L_q(\mathbb{R}^N))}\} + \|\bar{g}\|_{L_p((0,t),W^1_q(\mathbb{R}^N))} + \|\partial_t \bar{g}\|_{L_p((0,t),W^{-1}_q(\mathbb{R}^N))}\}\right)$$

for any $t \in (0,T)$ with some positive constants $C$ and $\gamma$ independent of $t$ and $T$.

**Remark 3.** (1) The proof of Theorem 4.1 is given in [14]. The proof of Theorem 4.2 is found in [3], but it can be proved by using the same argument as in the proof of Theorem 4.1 in [14].

(2) Theorem 4.1 is proved with the help of Theorem 4.1 and Theorem 4.2, the Banach fixed point argument and, some bootstrap arguments. The argument is quite standard, so that we may omit the proof of Theorem 4.1 (cf. Prüss [5]).

## 5 $\mathcal{R}$-bounded solution operators

To prove Theorem 4.1 we consider the following generalized resolvent problem:

$$\rho_{\pm}\lambda u_\pm - \text{Div } S_{\pm}(u_\pm) = f_\pm \quad \text{in } \mathbb{R}^N_\pm$$

$$\rho_{\pm}\lambda u_\pm - \text{Div } S_{\pm}(u_\pm) + \nabla \pi_\pm = f_\mp \quad \text{div } u_\mp = f_{\text{div}} = f_\text{div} \quad \text{in } \mathbb{R}^N_\mp \quad (5.1)$$

subject to the interface condition: for $x \in \mathbb{R}^N_0$

$$\mu_{\pm}\text{Div } \mathbf{N}(u_\pm)|_- - \mu_{\pm}\text{Div } \mathbf{N}(u_\pm)|_+ = g_i \quad (i = 1, \ldots, N-1),$$

$$\rho_{\pm}\mu_{\pm}\text{Div } \mathbf{N}(u_\pm)|_- - \rho_{\pm}\mu_{\pm}\text{Div } \mathbf{N}(u_\pm)|_+ = \sigma_+\Delta' H + g_{N},$$

$$\mu_{\pm}\text{Div } \mathbf{N}(u_\pm) + (\lambda_{\pm\pm} - \mu_{\pm\pm})\text{div } u_\pm|_| = \sigma_+\Delta' H + g_{N+1},$$

$$\u_\pm|_- - \u_\pm|_+ = h_i \quad (i = 1, \ldots, N-1),$$

$$\lambda H = \left(\frac{\rho_{\pm\pm}}{\rho_{\pm\pm} - \rho_{\pm\pm}} \u_\pm - \frac{\rho_{\pm\pm}}{\rho_{\pm\pm} - \rho_{\pm\pm}} \u_\pm + \lambda N\right) = d,$$

which is corresponding to the time dependent problem (4.1), (4.5), (4.6).

Before stating the main result of this section, we first introduce the definition of $\mathcal{R}$-boundedness and the operator valued Fourier multiplier theorem due to Weis [19].
Definition 5.1. Let $X$ and $Y$ be two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$ there holds the inequality:

$$\left\{ \left\| \sum_{j=1}^n r_j(u)T_jf_j \right\|_Y^p \right\}^{1/p} \leq C \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u)f_j \right\|_X^p \, du \right\}^{1/p}.$$

The smallest such $C$ is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$. Here, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from $X$ into $Y$.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all $X$ valued $C^\infty$ functions having compact supports and the Schwartz space of rapidly decreasing $X$ valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\text{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M \phi = \mathcal{F}^{-1}[M \mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)), \quad (5.3)$$

The following theorem is obtained by Weis [19].

Theorem 5.2. Let $X$ and $Y$ be two UMD Banach spaces and $1 < p < \infty$. Let $M$ be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X,Y))$ such that

$$\mathcal{R}_{\mathcal{L}(X,Y)}\left(\{\frac{d}{d\tau} M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}\right) \leq \kappa < \infty \quad (\ell = 0, 1)$$

with some constant $\kappa$. Then, the operator $T_M$ defined in (5.3) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by $T_M$, we have

$$\|T_M\|_{\mathcal{L}(L_p(X), L_p(Y))} \leq C \kappa$$

for some positive constant $C$ depending on $p$, $X$ and $Y$.

Remark 4. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W^m_q(\Omega)$ are both UMD spaces.

Theorem 5.3. Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Set $\Sigma_{\epsilon, \lambda_0} = \{ \lambda \in \Sigma_\epsilon \mid |\lambda| \geq \lambda_0 \}$ ($\lambda_0 > 0$) with $\Sigma_\epsilon = \{ \lambda = \gamma + i\tau \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon \}$, and

$$X_q = \{ (f_+, f_-, f_{\text{div}}, f_{\text{div}}, g, h, d) \mid f_+ \in L_q(\mathbb{R}_+^N), f_- \in L_q(\mathbb{R}_-^N), d \in W_2^q(\mathbb{R}^N), f_{\text{div}} \in W_2^q(\mathbb{R}^N), g = (g_1, \ldots, g_{N+1}) \in W_2^q(\mathbb{R}^N), h = (h_1, \ldots, h_N \pm 1) \in W_2^q(\mathbb{R}^N) \},$$

$$X_q = \{ F = (F_+^1, F_-^1, F_-^2, \ldots, F_-^4, F_+, F_1, F_2, F_3, F_4, F_5, F_6) \mid F_{\pm 1} \in L_q(\mathbb{R}_+^N), \quad F_1, F_2, F_3, F_4, F_5 \in L_q(\mathbb{R}_-^N), \quad F_6 \in W_2^q(\mathbb{R}_-^N) \}.$$

Then, there exist a constant $\lambda_0 > 0$ and operator families

$$A_\pm(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, W_2^q(\mathbb{R}^N_\pm))), \quad \mathcal{P}_- \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, W_2^q(\mathbb{R}^N_-))),$$

$$\mathcal{H}(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, W_2^q(\mathbb{R}^N)))$$

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such that $u_\pm = A_\pm(\lambda)F_\lambda$, $\pi_- = \mathcal{P}_-(\lambda)F_\lambda$, and $H = \mathcal{H}(\lambda)F_\lambda$ are unique solutions of problem (5.1) and (5.2) for any $\lambda \in \Sigma_{e,\lambda_0}$ and $F = (f_\uparrow, f_\downarrow, f_\text{div}, g, h, d) \in X_q$, and we have

\[
\mathcal{R}_c(\lambda, L_q(\mathbb{R}^N))\left(\{\tau \partial_\tau\}^\ell G_\pm^1(\lambda) \mid \lambda \in \Sigma_{e,\lambda_0}\right) \leq c \quad (\ell = 0, 1), \\
\mathcal{R}_c(\lambda, L_q(\mathbb{R}^N))\left(\{\tau \partial_\tau\}^\ell \nabla \mathcal{P}_-(\lambda) \mid \lambda \in \Sigma_{e,\lambda_0}\right) \leq c \quad (\ell = 0, 1), \\
\mathcal{R}_c(\lambda, W^2_q(\mathbb{R}^N))\left(\{\tau \partial_\tau\}^\ell \mathcal{H}(\lambda) \mid \lambda \in \Sigma_{e,\lambda_0}\right) \leq c \quad (\ell = 0, 1)
\]

with some constant $c$. Here,

\[
G_\pm^1(\lambda) = (\lambda A_\pm(\lambda), \lambda^{1/2} \nabla A_\pm(\lambda), \nabla^2 A_\pm(\lambda)), \quad G_\pm^2(\lambda) = (\lambda \mathcal{H}(\lambda), \nabla \mathcal{H}(\lambda)), \\
F_\lambda = (f_\uparrow, f_\downarrow, \lambda^{1/2} f_\text{div}, \nabla f_\text{div}, f_\text{div}, \lambda^{1/2} g, \nabla g, \lambda h, \lambda^{1/2} \nabla h, \nabla^2 h, d), \\
\nabla H^1_q(\mathbb{R}^N) = \{\pi_- \in L_{q,\text{loc}}(\mathbb{R}^N) \mid \nabla \pi_- \in L_q(\mathbb{R}^N)\},
\]

Hol$(U, X)$ denotes the set of all holomorphic functions defined on $U$ with their values in $X$, $\nabla = (\partial_1, \ldots, \partial_N)$ and $\nabla^2 = (\partial_i \partial_j \mid i, j = 1, \ldots, N)$.

**Remark 5.** (1) $F_{\pm 1}, F_{-2}, F_{-3}, F_{-4}, F_1, F_2, F_3, F_4, F_5$ and $F_6$ are corresponding variables to $f_\pm$, $\lambda^{1/2} f_\text{div}$, $\nabla f_\text{div}$, $\partial_\text{div}$, $\lambda^{1/2} g$, $\nabla g$, $\lambda h$, $\lambda^{1/2} \nabla h$, $\nabla^2 h$, and $d$, respectively.

(2) The proof of Theorem 5.3 is given in [14].

**References**

[1] H. Amann, *Linear and Quasilinear Parabolic Problems, Vol. I*, Birkhäuser, Basel, 1995.

[2] I. V. Denisova, Evolution of compressible and incompressible fluids separated by a closed interface, *Interface Free Bound* **2** (3) (2000), 283–313.

[3] J. Escher, J. Prüss, and G. Simonett, Analytic solutions for a Stefan problem with Gibbs-Thomson correction, *J. reine angew. Math.*, **563** (1) (2003), 1–52.

[4] T. Kubo, Y. Shibata, and K. Soga, On the $\mathcal{R}$-boundedness for the two phase problem: compressible-incompressible model problem, to appear in *Boundary Value Problem*.

[5] J. Prüss, Maximal regularity for evolution equations in $L_p$-spaces, *Conf. Semin. Mat. Univ. Bari*, **285** (2003), 1–39.

[6] J. Prüss, Y. Shibata, S. Shimizu, and G. Simonett, On well-posedness of incompressible two-phase flows with phase transitions: the case of equal densities, *Evolution Equations and Control Theory* **1** (2012), 917–941.

[7] J. Prüss and S. Shimizu, On well-posedness of incompressible two-phase flows with phase transitions: the case of non-equal densities, *J. Evol. Equ.* **12** (2012), 917–941.

[8] J. Prüss, S. Shimizu, and M. Wilke, Qualitative behaviour of incompressible two-phase flows with phase transitions: the case of non-equal densities, *Comm. Partial Differential Equations* **39** (7) (2014), 1236–1283.
[9] D. Rossinelli, B. Hejazialhosseini, P. Hadjidoukas, C. Bekas, A. Curioni, A. Bertsch, S. Futral, S. Schmidt, N. Adams, P. Koumoutsakos, Simulations of cloud cavitation collapse, *SC’13 Proc. International Conf. on High Performance Computing, Networking, Storage and Analysis*, ACM New York, NY, USA 2013, Doi:10.1145/2503210.2504565

[10] Y. Shibata, Generalized resolvent estimates of the Stokes equations with first order boundary condition in a general domain, *J. Math. Fluid Mech.*, 15 (1) (2013), 1–40.

[11] Y. Shibata, On the R-boundedness of solution operators for the Stokes equations with free boundary condition, *Diff. Integr. Equ.*, 27 (2014), 313–368.

[12] Y. Shibata On some free boundary problem for the Navier-Stokes equations in the maximal $L_p$-$L_q$ regularity class, submitted on Feb.4.2014.

[13] Y. Shibata, On the 2 phase problem including the phase transition, *Abstract for the 39th Sapporo symposium on PDE at Hokkaido University*, 2014, [http://www.math.sci.hokudai.ac.jp/sympo/sapporo/](http://www.math.sci.hokudai.ac.jp/sympo/sapporo/)

[14] Y. Shibata On the R boundedness for the two phase problem with phase transition: compressible-incompressible model problem. preprint.

[15] G. Ströhmer, About a certain class of parabolic-hyperbolic system of differential equations, *Analysis* 9 (1989), 1–39.

[16] V. A. Solonnikov and A. Tani, Evolution free boundary problem for equations of motion of viscous compressible barotropic liquid, in *The Navier-Stokes equations II - theory and numerical methods (Oberwolfach, 1991)*, Lecture Notes in Math., 1530, Springer, Berlag, (1992), 30–55

[17] A. Tani, On the free boundary problem for compressible viscous fluid motion, *J. Math. Kyoto Univ.*, 21 (1981), 839–859.

[18] A. Tani, Two phase free boundary problem for compressible viscous fluid motion, *J. Math. Kyoto Univ.*, 24 (1984), 243–267.

[19] L. Weis, Operator-valued Fourier multiplier theorems and maximal $L_p$-regularity. *Math. Ann.*, 319 (2001), 735–758.

[20] K. Yamamoto, On the collapse behavior of bubble clouds in cavitation jets, *Proceeding of 10th Pacific Rim International Conference on Water Jet technology*, April 2013, Jeju, Korea.