An Efficient and Flexible Engine for Computing Fixed Points

Hai-Feng Guo
University of Nebraska at Omaha
and
Gopal Gupta
University of Texas at Dallas

An efficient and flexible engine for computing fixed-points is critical for many practical applications. In this paper, first, we present a goal-directed fixed-point computation strategy for the logic programming paradigm. Our strategy adopts tabled resolution (or memoized resolution) to mimic efficient semi-naive bottom-up computation. Its main idea is to dynamically identify and record those clauses that will lead to recursive variant calls, and then repetitively apply those alternatives incrementally until the fixed-point is reached. Second, we consider those situations in which a fixed-point contains a large number or even an infinite number of solutions. In these cases, a fixed-point computation engine may not be efficient enough or feasible at all. We present a mode-declaration scheme which provides the capabilities to reduce a fixed-point from a large set of solutions to a preferred smaller one, or from an infinite set that is infeasible to compute to a finite one. The mode declaration scheme can be characterized as a meta-level operation over the original fixed-point. We show the correctness of the mode declaration scheme. Third, the mode-declaration scheme provides a new declarative method for dynamic programming, which is typically used for solving optimization problems. Using the mode declaration scheme, there is no need to define the value of an optimal solution recursively, instead, defining a general solution suffices. The optimal value as well as its corresponding concrete solution can be derived implicitly and automatically using a mode-directed fixed-point computation engine. Finally, this fixed-point computation engine has been successfully implemented in a commercial Prolog system. Experimental results are shown to indicate that adopting the mode declaration scheme improves both time and space performances in solving dynamic programming problems.

Categories and Subject Descriptors: D.1.6 [Logic Programming]: Operational semantics; F.3.2 [Semantics of Programming Languages]: Operational semantics; F.4.1 [Computational logic, Logic programming]: Resolution; I.2.3 [Deduction and Theorem Proving]: Dynamic Programming

General Terms: Programming, Languages

Additional Key Words and Phrases: Tabled resolution, logic programming, dynamic programming, fixed-point

Author’s address: Hai-Feng Guo, Department of Computer Science, University of Nebraska at Omaha, 6001 Dodge St., Omaha, NE 68182-0500, USA. Email: haifengguo@mail.unomaha.edu
Gopal Gupta, Department of Computer Science, University of Texas at Dallas, Richardson, TX 75083-0688, USA. Email: gupta@utdallas.edu

Permission to make digital/hard copy of all or part of this material without fee for personal or classroom use provided that the copies are not made or distributed for profit or commercial advantage, the ACM copyright/server notice, the title of the publication, and its date appear, and notice is given that copying is by permission of the ACM, Inc. To copy otherwise, to republish, to post on servers, or to redistribute to lists requires prior specific permission and/or a fee.
© 1999 ACM 0164-0225/99/0100-0111 $00.75
1. INTRODUCTION

Due to their highly declarative nature and efficiency, Tabled Logic Programming (TLP) systems [Chen and Warren 1996; Zhou et al. 2000; Guo and Gupta 2001; Rocha et al. 2001] have been put to many innovative uses, such as model checking [Ramakrishnan et al. 1997] and non-monotonic reasoning [Swift 1999]. A tabled logic programming system can be thought of as an engine for efficiently computing fixed-points, which is critical for many practical applications. A TLP system is essential for extending traditional LP system (e.g., Prolog) with tabled resolution (or memoized resolution). The main advantages of tabled resolution are that it terminates more often by computing fixed-points, avoids redundant computation by memoing the computed answers, and keeps the declarative and procedural semantics of pure logic programs with bounded-size terms consistent.

The main idea of tabled resolution is never to compute the same call twice. Answers to certain calls are recorded in a global memo table (heretofore referred to as a table), so that whenever the same call is encountered later, the tabled answers are retrieved and used instead of being recomputed. This avoidance of recomputation not only gains better efficiency, more importantly, it also gets rid of many infinite loops, which often occur due to static computation strategies (e.g., SLD resolution [Lloyd 1987]) adopted in traditional logic programming systems.

In this paper, we present a novel tabled resolution strategy for computing fixed-points in the logic programming paradigm. We focus on definite logic programs. The strategy applies a memoized recursive algorithm to mimic efficient semi-naive bottom-up computation. Its main idea is to dynamically identify and record those clauses that will lead to recursive variant calls, and then repetitively apply these alternatives incrementally until the fixed-point is reached. Our tabled resolution scheme allows query evaluation to be performed with a single computation tree similar to traditional SLD resolution. As a result, this novel tabled resolution scheme can be easily incorporated into an existing pure Prolog system without involving any major changes to it.

Example 1. Consider the following two programs defining the reachability relation. The predicate reach(X,Y) in the program (a) checks whether a node Y is reachable from node X, while the predicate reach(X,Y,E) in the program (b) performs the same task but additionally returns the path from X to Y as an explanation (i.e., why Y is reachable from X).

```
:- table reach/2.
reach(X,Y) :- reach(X,Z), arc(Z,Y).
reach(X,Y) :- arc(X,Y).
arc(a,b).  arc(a,c).  arc(b,a).
:- reach(a,X).
```

(a) A Fixed-Point with Finite Number of Solutions

---

1We say two Prolog calls \( C_1 \) and \( C_2 \) are variant if there exist substitutions \( \phi \) and \( \sigma \) such that \( C_1 = C_2 \phi \) and \( C_2 = C_1 \sigma \).
:- table reach/3.
reach(X,Y,E) :-
    reach(X,Z,E1), arc(Z,Y,E2), append(E1,E2,E).
reach(X,Y,E) :- arc(X,Y,E).
arc(a,b,[(a,b)]). arc(a,c,[(a,c)]). arc(b,a,[(b,a)]).
:- reach(a,X,P).

(b) A Fixed-Point with Infinite Number of Solutions

The program in example 1(a), for checking the existence of reachability, does not work properly in a traditional Prolog system. With the declaration of a tabled predicate reach/2 in a tabled Prolog system, it can successfully find the complete set of solutions due to the fixed-point computation strategy. However, there are many situations in which a fixed-point contains a large number or even infinite number of solutions, which in turn affects the efficiency or completion of the computation.

Consider the reachability program shown in example 1(b), where append/3 is a standard predicate used to append one list to another. An extra argument is added to the predicate reach/3 to collect the path that connects its first argument to its second argument. However, this extra argument results in the fixed-point of the computation becoming infinite. Tabled resolution now becomes nonterminating, since now there are infinite number of paths from a to any node due to the cycle between a and b. Similar problems on evidence construction have been studied on justification in [Roychoudhury et al. 2000; Pemmasani et al. 2004]. One reasonable solution is presented in [Pemmasani et al. 2004] by asserting the first evidence into a dynamic database for each tabled answer. However, the evidence has to be organized as segments indexed by each tabled answer. That is, an extra procedure is required to construct the full evidence.

To avoid problems of inefficiency or nontermination, due to the fixed-point containing a large number or an infinite number of solutions respectively, it is often necessary to change the original problem so that there is only a small finite-sized solution set. For the reachability example, it is actually enough to find a single simple path to show the evidence of reachability. However, generally, it is not only difficult to alter the predicate definition of reach/3 to avoid nontermination in order to have a single simple path for each pair of reachable nodes, it also sacrifices the clarity of the original relation. In these cases, a fixed-point computation engine may not be efficient enough or feasible at all. We present a declarative method to get around this problem by introducing a new mode declaration scheme [Guo and Gupta 2004].

In this paper, we present a mode-declaration scheme [Guo and Gupta 2004] in a tabled Prolog paradigm which provides the user the capability to reduce the fixed-point of a definite logic program from a large- or infinite-sized solution set to a preferred finite-sized one. The method introduces a new mode declaration for tabled predicates. The mode declaration classifies arguments of a tabled predicate as indexed or non-indexed. Each non-indexed argument can be thought of as a function value uniquely determined by the indexed arguments. The tabled Prolog system is optimized to perform variant checking based only on the indexed argu-
ments during the computation of fixed-points. The mode declaration can further extend one of the non-indexed arguments to be an aggregated value, e.g., the minimum function, so that the global table will record answers with the value of that argument appropriately aggregated. Thus, in the case of the minimum function, a tabled answer can be dynamically replaced by a new one with a smaller value during the computation. This new declaration for tabled predicates and modified procedure for variant checking make it easier for the meta-level manipulation during computation of fixed-points.

Semantically, the mode declaration scheme can be characterized as a meta-level operation over the fixed-point of the original program. The semantics of a tabled Prolog program is formalized based on the Herbrand model [van Emden and Kowalski 1976; Lloyd 1987] and fixed-point theory, whereas the semantics of declared modes is defined as a strict partial order relation \(^2\) among the solutions. The mode declarations essentially provide a selection mechanism among the alternative solutions, thus making fixed-point computation more flexible. We formally present the semantics of mode declaration in a tabled Prolog program, and further show the correctness of its operational semantics in tabled resolution.

The new mode-declaration scheme, coupled with recursion, provides an attractive platform for making dynamic programming simpler: there is no need to define the value of an optimal solution recursively, instead, defining the value of a general solution suffices. The optimal value, as well as its associated solution, will be computed implicitly and automatically in a tabled Prolog system that uses our new mode declaration scheme and modified variant checking. Thus, dynamic programming problems are solved more elegantly and more declaratively.

We have successfully implemented our tabled resolution as well as the mode declaration scheme in the ALS Prolog, a commercial Prolog system. To implement the mode declaration scheme, no change is required to the tabled resolution mechanism; therefore, the same idea can also be applied to other tabled Prolog systems. Our experimental results show that the mode declaration scheme improves both time and space performance while solving dynamic programming problems.

The rest of the paper is organized as follows: Section 2 introduces tabled logic programming and our new tabled resolution scheme, dynamic reordering of alternatives (DRA) [Guo and Gupta 2001], for efficient fixed-point computation. Section 3 presents a mode declaration scheme for tabled predicates, shows how it affects fixed-point computation, and describes its implementation. Section 4 gives a detailed demonstration of how dynamic programming can benefit from this new scheme. Section 5 presents the performance results w.r.t. some dynamic programming benchmarks. Finally, section 6 gives our conclusions.

2. COMPUTING FIXED POINTS VIA TABLED RESOLUTION

2.1 Tabled Logic Programming (TLP)

Traditional logic programming systems (e.g., Prolog) use SLD resolution [Lloyd 1987] with the following computation strategy: subgoals of a resolvent are solved from left to right and clauses that match a subgoal are applied in the textual

\(^2\)A strict partial order relation is irreflexive and transitive.
order they appear in the program. It is well known that SLD resolution may lead to non-termination for certain programs, even though an answer may exist via the declarative semantics. That is, given any static computation strategy, one can always produce a program in which no answers can be found due to non-termination even though some answers may logically follow from the program. In case of Prolog, programs containing certain types of left-recursive clauses are examples of such programs.

Tabled logic programming [Tamaki and Sato 1986; Chen and Warren 1996] eliminates such infinite loops by extending logic programming with tabled resolution. The main idea of tabled resolution is to memoize the answers to some calls and use the memoized answers to resolve subsequent variant calls. Tabled resolution adopts a dynamic computation strategy while resolving subgoals in the current resolvent against matched program clauses or tabled answers. It keeps track of the nature and type of the subgoals; if the subgoal in the current resolvent is a variant of a former tabled call, tabled answers are used to resolve the subgoal; otherwise, program clauses are used following SLD resolution.

The main advantages of tabled resolution are that a TLP system terminates more often by computing fixed-points, avoids redundant computation by memoing the computed answers, and keeps the declarative and procedural semantics consistent for pure logic programs with bounded-size terms. A tabled logic programming system can be thought of as an engine for efficiently computing fixed-points, which is critical for many practical applications. Tabled logic programming (TLP) systems have been put to many innovative uses, such as model checking [Ramakrishnan et al. 1997] and non-monotonic reasoning [Swift 1999], due to their highly declarative nature and efficiency.

In a tabled logic programming system, only tabled predicates are resolved using tabled resolution. Tabled predicates are explicitly declared as follows:

\[- table \ p/n.\]

where \(p\) is a predicate name and \(n\) is its arity. A global data structure \textit{table} is introduced to memoize the answers of any subgoals to tabled predicates, and to avoid any recomputation.

### 2.2 Related Works

The first tabled resolution scheme, called OLDT [Tamaki and Sato 1986], was proposed in 1986 by Tamaki and Sato for avoiding some of the non-termination problems during evaluation of definite logic programs. The basic idea of OLDT is to maintain a global \textit{table} data structure to remember the queries seen so far and their corresponding answers produced so far. Such a predicate call that has been recorded in the table is referred to as a \textit{tabled call}. Any answers to a tabled call will be recorded in the table, and referred to as \textit{tabled answers}. OLDT resolution usually maintains multiple computation trees, in parallel, for computing the fixed-point, where each computation tree uniquely corresponding to one tabled call. Subsequently, this work has been extended to SLG resolution [Swift and Warren 1994; Chen et al. 1995; Chen and Warren 1996] for general logic programs with negation. The XSB system [Chen and Warren 1996] is an implementation of SLG resolution which supports well-founded semantics [Chen and Warren 1993; Chen et al. 1995]. The XSB system has bee implemented atop the Warren Abstract Ma-
chine (WAM) [Ait-Kaci 1991]. As is well known, compiling to the WAM instruction set is a standard way of implementing high performance logic programming systems. The implementation of the XSB system involved many non-trivial changes to the WAM, and was a very substantial engineering effort [Sagonas and Swift 1998].

The huge implementation effort needed for implementing OLDT and SLG can be avoided by choosing alternative methods for tabled resolutions that maintain a single computation tree similar to traditional SLD resolution, rather than maintaining a forest of SLD trees. SLDT resolution [Shen et al. 2001; Zhou et al. 2000] was the first attempt in this direction. The main idea behind SLDT is to steal the backtracking point—using the terminology in [Shen et al. 2001; Zhou et al. 2000]—of the previous tabled call when a variant call is found, to avoid exploring the current recursive clause which may lead to non-termination. However, because the variant call avoids applying the same recursive clause as the previous call, the computation may be incomplete. Thus, repeated computation of tabled calls is required to make up for the lost answers and to make sure that the fixed-point is complete. SLDT does not propose a complete theory regarding when a tabled call is completely evaluated, rather it relies on blindly recomputing the tabled calls to ensure completeness. SLDT resolution was implemented in early versions of B-Prolog system. However, recently this resolution strategy has been discarded, instead, a variant of DRA resolution [Guo and Gupta 2001] has been adopted in the latest version of B-Prolog system [Zhou and Sato 2003].

2.3 Dynamic Reordering of Alternatives (DRA)

The DRA resolution [Guo and Gupta 2001] computes a fixed-point in a very similar way as a goal-directed bottom-up execution of logic programs [Lloyd 1987]. Its main idea is to dynamically identify looping alternatives from the program clauses, and then repetitively apply those alternatives until no more answers can be found. A looping alternative refers to a clause that matches a tabled call and will lead to a resolvent containing a recursive variant call. The DRA resolution requires that not only the answers to variant calls are tabled, the alternatives leading to variant calls are also memorized in the table, which is essentially different from all the previous tabled resolution strategies.

As shown in Figure 1, the computation of a tabled call (e.g., reach(a,X)) is divided into three stages: normal, looping and complete. The purpose of the normal stage is to find all the looping alternatives (clause (1) leading to a variant subgoal reach(a,Z)) and record all the answers generated from the non-looping alternatives (clause (2)) into the table. The new-answer label indicates that the new answer generated from that successful path should be added into the table. Then, in the looping stage only the looping alternative (clause (1)) is applied repeatedly to consume new tabled answers until a fixed-point is reached, that is, no more answers for reach(a,X) can be found. Afterwards, the complete stage is reached. As a result, the query :- reach(a,X) returns a complete answer set X=b, X=c and X=a, albeit the predicate is defined left-recursively.

The DRA tabled resolution adopts a dynamic computation strategy by keeping track of the type of the current resolvent. If the current resolvent is a non-tabled call (e.g., all the calls to the predicate arc/2), then the traditional SLD resolution
is applied. Otherwise, if it is a tabled call, then the first occurrence of a tabled call in the computation is referred to as the *master* call; subsequent variant calls are called *slave* calls in DRA resolution. The master tabled call (e.g., \( \text{reach}(a,X) \)) is responsible for exploring the matched clauses, manipulating execution states, and repeatedly applying its corresponding looping alternatives in order to collect the tabled answers, whereas the slave tabled call (e.g., \( \text{reach}(a,Z) \)) only consumes tabled answers if there are any in the table.

In DRA resolution, the procedure of computing fixed-points of a definite logic program mimics the semi-naive bottom-up computation strategy [Balbin and Ramamohanarao 1987], whenever a looping alternative is applied again, the variant tabled calls only consume the incremental part of solutions in the table.

**Example 2.** Consider the following tabled Prolog program with multiple looping alternatives:

\[
\begin{align*}
: &- \\
&\text{table } r/2.
\end{align*}
\]

\[
\begin{align*}
r(X, Y) &::= r(X, Z), p(Z, Y). & \text{(1)} \\
r(X, Y) &::= p(X, Y). & \text{(2)} \\
r(X, Y) &::= r(X, Z), q(Z, Y). & \text{(3)}
\end{align*}
\]

\[
\begin{align*}
p(a, b). & \quad p(b, c). & \quad q(c, d).
\end{align*}
\]

\[
:- r(a, Y).
\]

Figure 2 gives the computation tree produced with the DRA scheme for example 2 (note that the labels on the branch refer to the clause used for creating that branch). Both clause (1) and clause (3) need to be tabled as looping alternatives for the tabled call \( r(a,Y) \). The second alternative is a non-looping alternative that produces an answer for the call \( r(a,Y) \) which is recorded in the table (denoted by the operation `new_answer` in the figure). The query call \( r(a,Y) \) is a master tabled call (since it is the first occurrence), while all the occurrences of \( r(a,Z) \) are slave tabled calls (since they are calls to variant of \( r(a,Y) \)). When the call \( r(a,Y) \) enters its looping state, it keeps applying the looping alternatives repeatedly until the answer set does
not change any more, i.e., until $r(a,Y)$ is completely evaluated. Note that if we added two more facts: $p(d,e)$ and $q(e,f)$, then we will have to go through the two looping alternatives one more time to produce the answers $r(a,e)$ and $r(a,f)$. Each time a looping alternative is invoked, only incremental tabled solutions are consumed.

| Tabled Subgoals | Answers | Looping Alternatives |
|----------------|---------|----------------------|
| $r(a,Y)$       | $r(a,b)$ | (1)                  |
|                | $r(a,c)$ |                     |
|                | $r(a,d)$ | (3)                  |

Fig. 2. DRA Resolution for Example 2

2.4 A Semi-naive Algorithm for DRA

We next give a meta-interpreter to describe how DRA works operationally. To keep the presentation and the meta-interpreter simple and clear, we assume that there are no nested tabled calls dependent on each other and we ignore any further optimizations except the incremental consumption for the semi-naive algorithm. We also assume that the table exists as a global data structure.

As shown in Figure 3, the solve/1 predicate is the entry point to the meta-interpreter and takes as input the goal to be executed. The goal clause(A,Cl) nondeterministically finds the matching clause, Cl, for the goal A, and then setCurrentAlt(A,Cl) records the current alternative Cl associated with A in a temporary global database. The goal tabled(A) checks whether A has been declared as tabled or not, state(A,X) checks to see whether A’s execution status is X (one of normal, looping, or complete), while the goal setState(A,X) changes the execution state of the goal A to the state X. The goal addTableAns(A) adds an answer for goal A to the table (if it is not already present), while the goal look_up_table(A) looks up answers for A that have been recorded so far in the table. The goal addLoopAlt(A,Cl) finds the current alternative of A (which was previously recorded by setCurrentAlt(A,Cl)), and then tables it as a looping alternative for A. The goal getLoopAlt(A,La) nondeterministically finds the next looping alternative, La, from the list of A’s looping alternatives.

The meta-interpreter in Figure 3 is pretty self-explanatory. A non-tabled goal is computed as in standard Prolog, with leftmost-first selection rule (line 2) and a depth-first search rule (line 27). A tabled call, on the other hand, is resolved
(1)  solve(true).
(2)  solve((A,B)) :- solve(A), solve(B).
(3)  solve(A) :-
(4)    (tabled(A) ->
(5)      (state(A, normal) -> %% A in normal state
(6)        (type(A, master) -> %% A is a master call
(7)          (clause(A,Cl) ->
(8)            setCurrentAlt(A,Cl), solve(Cl), addTableAns(A)
(9)          ; setState(A, looping), solve(A)
(10)        )
(11)      ; addLoopAlt(A), %% A is a slave call
(12)      look_up_table(A)
(13)    )
(14)    ; (state(A, looping) -> %% A in looping state
(15)      (type(A, master) -> %% A is a master call
(16)        (getLoopAlt(A, La) ->
(17)          solve(La), addTableAns(A)
(18)        ; (isNewSolFnd(A) ->
(19)          solve(A)
(20)        ; setState(A, complete)
(21)      )
(22)    )
(23)    ; look_up_table(A) %% A is a slave call
(24)  )
(25)  ; look_up_table(A) %% A in complete state
(26)  )
(27)  ; clause(A, Cl), solve(Cl) %% non-tabled call
(28)  ).

Fig. 3. An algorithm for DRA resolution

in a different manner by the DRA scheme based on its execution state (normal, looping or complete) and its master-slave mode. Consider a tabled goal A (line 4). If A's execution state is normal (line 5) and it is a master call (line 6), then a matching clause Cl for A is nondeterministically found, and the subgoals in the body of Cl recursively solved (line 7-8). An answer for A will be added into the table if its matching clause Cl is successfully solved (line 8). The matching clause Cl is identified as a looping alternative if a slave call of A is found (line 11), and the slave call can only be resolved against tabled answers (line 12). After all matching clauses have been seen, A's execution state is set to looping (line 9). During the looping state (line 14), the master call of A retrieves and executes the collected looping alternatives (line 15-17); whereas the slave call of A can only consume the tabled answers. If no new answer is found while applying one cycle of all the looping alternatives w.r.t. the master call of A, the execution state is set to complete (line 20); otherwise goal A continues to be executed (line 19) during its looping state until no new answer can be found. Once the tabled goal A reaches its complete state, all answers would have been recorded in the table, and thus only table look-ups are
used for resolving subsequent variant calls to $A$.

To make the algorithm efficient, we introduce an incremental consumption strategy for the predicate $\text{look\_up\_table/1}$ to mimic semi-naive evaluation of logic programs [Balbin and Ramamohanarao 1987]. Incremental consumption refers to avoiding re-consuming older tabled answers which have already been consumed earlier by the same slave tabled calls. Each looping alternative maintains two pointers, called $\text{begin}$ and $\text{end}$, which divide the tabled answer list into three parts: old answers, current answers, and new answers. Old answers are those tabled answers that were consumed during the previous round of exploring the looping alternative; current answers can be consumed during the current exploration; and new answers are kept for the next round of consumption. The answers between the pointers $\text{begin}$ and $\text{end}$ constitute the incremental answer set to be consumed in the current round. The incremental answer set is updated dynamically whenever a looping alternative is picked up for exploration. Therefore, the predicate $\text{look\_up\_table/1}$ only consumes the incremental answer set.

2.5 Experiments

The DRA resolution builds the computation tree as in normal Prolog execution based on SLD, however, when a variant tabled call is encountered, the branch that leads to that variant call is tabled as a looping alternative, and later applied again during the looping state. This has the same effect as shifting branches with variant calls to the right of the SLD tree to avoid exploring potentially non-terminating branches. Because of this simplicity, the DRA resolution can be incorporated very easily and without sacrificing efficiency in an existing Prolog system. This can have important consequences, given that tabling is so important for many serious applications of logic programming (e.g., model checking [Ramakrishnan et al. 1997]). The simplicity of our scheme guarantees that execution is not inordinately slowed down (e.g., in the previous B-Prolog tabled system [Zhou et al. 2000], a tabled call may have to be re-executed several times to ensure that all answers are found), nor considerable amount of memory used (e.g., in the XSB tabled system [Chen and Warren 1996] a large number of stacks/heaps may be frozen at any given time), rather, the raw speed of the Prolog’s WAM engine is available to execute even those programs that contain variant calls. The new tabling scheme allows one to incorporate tabling in an existing logic programming system with very little effort. Using the DRA scheme we were able to incorporate tabling in the commercial ALS Prolog system [ALS] in a few man-months of work. The time efficiency of our tabled ALS (called TALS) system is comparable to that of the XSB system. The space efficiency of our system is better than that of XSB system.

A preliminary implementation of DRA in the ALS Prolog system has been completed. Table I gives the comparison of running times (in seconds) between XSB and TALS for various benchmarks. These benchmarks are distributed with XSB and most of them table multiple predicates, many of whom manipulate structures. In general, the time performance of TALS is worse than that of XSB. That is mainly because SLG resolution has been implemented by a combination of computation suspension via stack freezing and maintaining a forest of SLD trees. Due to the computation suspensions and resumption, XSB avoids reconstructing execution environment for applying looping alternatives, which is typically recomputed.
### Table I. Running-time(Seconds)/Space-usage(KB)

| Benchmarks | TALS   | XSB    |
|------------|--------|--------|
| cs_r       | 0.37/29.8 | 0.20/58.3 |
| disj       | 0.25/18.7 | 0.05/54.2 |
| kalah      | 0.20/37.0 | 0.05/97.1 |
| peep       | 0.47/24.0 | 0.18/376.3 |
| pg         | 0.29/23.9 | 0.05/150.5 |
| read       | 0.62/35.8 | 0.23/616.7 |
| sg         | 0.03/27.4 | 0.08/48.3 |

Table I. Running-time(Seconds)/Space-usage(KB)

in TALS since looping alternatives have to be applied again to ensure the completion of fixed-point computation. On the other hand, due to this maintenance of forest of SLD trees and freezing of stacks, XSB cannot be implemented in the same way as SLD resolution using a simple stack-based memory structure. Consequently, the freezing of stacks results in significant space overheads. It should also be mentioned that DRA has been implemented on top of ALS system without modifying the compiler. XSB includes tabling in compiling stage, which may have significant impact on performance.

Tables I also compares the space usage between TALS and XSB systems. The space includes total stack and heap space used as well as space overhead to support tabling. The space overhead to support tabling in case of TALS includes table space, the extra space needed to record looping alternatives and extra fields for keeping track of the types of tabled calls. In case of XSB, the figure includes the table space and space used for suspension in SLG-WAM [Swift and Warren 1994]. As can be noticed from Table I, the space performance of TALS is significantly better than that of XSB (for some benchmarks, e.g., peep, pg and read, it is orders of magnitude better).

### 3. FLEXIBLE FIXED POINT COMPUTATION VIA MODES

There are many situations in which a fixed-point contains a large number or even infinite number of solutions (for example 1(b)). In these cases, a fixed-point computation engine may not be efficient enough or feasible at all. In this section, we present a mode-declaration scheme [Guo and Gupta 2004] which provides the users the capability to reduce a fixed-point from a large solution set to a preferred small one, or from an infeasible infinite set to a finite one. The mode declaration scheme can be characterized as a meta-level mapping operation over the original fixed-point.

#### 3.1 Mode Declarations

The fixed-point reduction can be achieved by a mode declaration for tabled predicates, which is described in the form of

\[ \text{:- table } q(m_1, ..., m_n), \]

where \( q/n \) is a tabled predicate name, \( n \geq 0 \), and each \( m_i \) has one of the forms as defined in Table II.

In order to find out how modes will affect fixed-point computations, we have to better understand the function of variant checking in tabled resolution. Variant
checking is a crucial operation for tabled resolution as it leads to avoidance of non-termination. It is used to differentiate between the various tabled calls as well as their answers. While computing the answers to a tabled goal \( p \) with tabled resolution, if another tabled subgoal \( q \) is encountered, the decision regarding whether to consume tabled answers or to try program clauses depends on the result of variant checking. If \( q \) is a variant of \( p \), the variant subgoal \( q \) will be resolved by unifying it with tabled answers, otherwise, traditional Prolog resolution is adopted for \( q \). Additionally, when an answer to a tabled goal is generated, variant checking is used to check whether the generated answer is variant of an answer that is already recorded in the table. If so, the table is not changed; this step is crucial in ensuring that a fixed-point is reached.

The main purpose of mode declaration is to classify the predicate arguments into two types: \textit{indexed} and \textit{non-indexed}. Only indexed arguments are used for variant checking while collecting answers for the table; for each tabled call, any answer generated later for the same value of the indexed arguments is discarded because it is a variant, w.r.t. the indexed arguments, of a previously tabled answer. Consider again the reachability program in Example 1(b). Suppose we declare the mode as "\:- table reach(+,+,-)"; this means that only the first two arguments of the predicate \textit{reach/3} are used for variant checking. The new computation of the query \textit{reach(a,Y,E)} is shown in Figure 4. Since only the first two arguments of \textit{reach/3} are used for variant checking, the last two answers "\(Y=b, E=\{(a,b),(b,a),(a,b)\}\)" and "\(Y=c, E=\{(a,b),(b,a),(a,c)\}\)"; shown on the rightmost two sub-branches, are variant answers to "\(Y=b, E=\{(a,b)\}\)" and "\(Y=c, E=\{(a,c)\}\)" respectively. Therefore, no new answers are added into the table at those points. The computation is then terminated properly with three answers. As a result, each reachable node from \( a \) has a simple path as an explanation.

The mode directive \texttt{table} makes it very easy and efficient to extract explanation for tabled predicates. In fact, our strategy of ignoring the explanation argument during variant checking results in only the first explanation for each tabled answer being recorded. Subsequent explanations are filtered by our modified variant checking scheme. This feature ensures that those generated explanations are concise and that cyclic explanations are guaranteed to be absent. For the reachability instance shown in Figure 4, each returned path is simple so that all arcs are distinct.

Essentially, if we regard a tabled predicate as a function, then all the non-indexed arguments are uniquely defined by the instances of indexed arguments. For the previous example, the third argument of \textit{reach/3} returns a single path depending on the first two arguments. Therefore, variant checking should be done w.r.t. only indexed arguments during tabled resolution. From this viewpoint, the mode declaration makes tabled resolution more efficient and flexible. More importantly,
this declaration scheme is especially useful in reducing an infinite set of solutions to a finite one for some practical uses, or to reduce a large finite set of solutions to an optimized smaller set as shown next.

3.2 Declaration of Aggregates

The mode directive table can be further extended to associate a non-indexed argument of a tabled predicate with some aggregate constraint. With the mode ‘-’, a non-indexed argument for each tabled answer only records the very first instance. This “very first” property can actually be generalized to support other preferences, e.g., the minimum value with mode min (or the maximum with mode max), in which case the global table will record answers with the value of that argument as small (or great) as possible. That is, a tabled answer can be dynamically replaced by a new one with smaller (or greater) value during the computation. Note that we only enumerate two typical aggregates as examples, other aggregates, such as sum, average, or even a user-defined one, can be used as well.

The aggregates, min and max, are specified via mode declarations as shown in Table II. Both modes imply that the declared arguments are non-indexed. The aggregates can be used to specify optimization problems more elegantly. For instance, in the code shown in example 3 for searching for shortest paths, instead of defining the shortest path directly, we only need to specify what is the definition for a general path. Clauses (2) to (5) make up the core program defining the path relation and a directed graph with a set of edges; Clause (1) specifies the predicate path/4 to be optimized and gives the criteria regarding how it should be optimized. The mode declaration path(+,+,min,-) means that only the first two arguments (pair of nodes) are used for variant checking when an answer is generated, and a minimum value (the shortest path) is expected for the third argument. Arguments with different modes are tested in the following order during variant checking of a recently generated answer: (i) the indexed argument with ‘+’ mode has the highest priority to be checked to identify whether it is a new answer. If that is the case, a
new tabled entry is required to record the answer; otherwise a tabled answer with the same indexed arguments is found. (ii) This tabled answer is then compared with the recently generated one w.r.t the argument with the optimum mode \textit{'min'}; if the new answer has a smaller value for this argument, then a replacement of the tabled answer is required such that the tabled answer keeps the minimum value as expected for this argument. (iii) The last argument with mode \textit{'}-\textit{'} will not be used for variant checking; if a replacement of a tabled answer happens, then the argument will be replaced as well; otherwise, the recently generated answer as well as its fourth argument are discarded.

**Example 3.** Consider the following program searching for a shortest path, where \texttt{path(X,Y,D,L)} denotes a path from \texttt{X} to \texttt{Y} with the distance \texttt{D} and the path route \texttt{L}.

\begin{verbatim}
:- table path(+, +, min, -). (1)
path(X, X, 0, []). (2)
path(X, Y, D, [e(X, Y)]) :- edge(X, Y, D). (3)
path(X, Y, D, [e(X, Z) | P]) :-
  edge(X, Z, D1), path(Z, Y, D2, P),
  D is D1 + D2. (4)
edge(a,b,4). edge(b,a,3). edge(b,c,2). (5)
:- path(a, X, D, P). (6)
\end{verbatim}

Example 3 shows that even though the core program defines a general path, with mode declaration, the predicate \texttt{path/4} can be easily upgraded to an optimization predicate. As long as the tabled Prolog engine is set to compute the fixed-point semantics for logic programs, the shortest path under consideration will always be found. Intuitively, given a tabled call \texttt{C}, the DRA resolution first finds all the answers for \texttt{C} using clauses not containing variant calls. Once this set of answers is computed and tabled, it is treated as a set of facts, and used for computing rest of the answers from the clauses leading to variant calls (looping alternatives). Whenever an answer to \texttt{C} is generated, it will be selectively added to the table either as a new entry or as a replacement based on the defined mode of the corresponding predicate. The process stops when no new answers can be computed via the looping alternatives, i.e., a fixed point is reached. In this regard, with the assistance of mode declaration and tabled resolution, the computation of program clauses only defining general solutions will still produce the optimal solution.

### 3.3 Operational Semantics

The operational semantics of a tabled program is dependent on tabled resolution [Chen and Warren 1993; Zhou et al. 2000; Guo and Gupta 2001], which can be formalized based on the Herbrand model [van Emde Boas and Kowalski 1976; Lloyd 1987] and fixed-point theory. In spite of having different tabled resolution, a tabled Prolog can be thought of as an engine for efficiently computing the least fixed-points. The procedure of computing fixed-points of a definite logic program in DRA resolution mimics the bottom-up computation strategy as follows. For the consideration of clarity and simplicity, we ignore any optimization used in DRA resolution (e.g. incremental consumption for simulating semi-naive bottom-up computation).
We use the following notational conventions: $P$ is used to denote a tabled logic program, $B_P$ to denote the Herbrand base of $P$, $2^B_P$ to denote the set of all Herbrand interpretations of $P$, a ground instance (e.g., a ground atom, a ground instance of a clause) to denote an instance without involving any variable. Note that $\omega$ is the first infinite ordinal, and $(F \uparrow n)(x)$ to denote applying the mapping $F$ $n$ times as $F(F(\cdots F(x)\cdots))$.

**Definition 1.** Let $P$ be a logic program and $B_P$ its Herbrand base. Let $\emptyset$ denote the empty set. We define a meta-level procedure $T_P : 2^B_P \rightarrow 2^B_P$. Given a Herbrand interpretation $I$, $T_P(I)$ performs:

1. $I_0 \leftarrow \emptyset$;
2. for each ground instance $A :\neg A_1, \cdots, A_m$ of a clause in $P$
   where $\{A_1, \cdots, A_n\} \subseteq I$
   do $I_0 \leftarrow I_0 \cup \{A\}$;
3. return $I_0$.

The fixed-point semantics of $P$ can be described as $T_P \uparrow \omega(\emptyset)$ [Lloyd 1987].

We next show that how mode declaration affects the fixed-point semantics of a logic program. One key ingredient for the mode declaration scheme to be applicable is the optimal-substructure property \(^3\), that is, the optimal solution to a tabled call contains optimal solutions to its tabled sub-calls. Typical examples of such problems are those for dynamic programming. For simplicity, we assume that for any tabled predicate, there is at most one optimization mode, ’min’ or ’max’, in the mode declaration. Tabled predicates whose multiple arguments have an optimization mode can be handled by combining them into one via program clause transformation. Additionally, we assign non-indexed modes different priorities, i.e., ‘min’ and ‘max’ have higher priorities than ’-‘.

Note that a tabled predicate without explicit mode declaration has a default one with all indexed modes ’+’. Although mode declarations are only allowed for tabled predicates, non-tabled predicates can also be simply treated as having implicitly declared indexed mode ’+’ for all arguments. Thus, in the rest of this subsection we will not distinguish tabled predicates from non-tabled ones, and each predicate defined in a tabled logic program is associated with a mode declaration.

**Definition 2.** Let $q/n$ be a predicate with a mode declaration $q(m_1, m_2, \cdots, m_n)$ in a tabled logic program $P$. Let the arguments $m_1, m_2, \cdots, m_k$ ($0 \leq k \leq n$) have the mode ‘+’ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$; let $m_j$ be the argument with the highest priority non-indexed mode if there are non-zero arguments in $q/n$ with non-indexed modes. We define two functions $K_{q/n}$ and $O_{q/n}$ as follows: given a ground atom $q(a_1, a_2, \cdots, a_n)$,

$$K_{q/n}(q(a_1, a_2, \cdots, a_n)) = (a_{i_1}, a_{i_2}, \cdots, a_{i_k});$$

$$O_{q/n}(q(a_1, a_2, \cdots, a_n)) = a_j, \quad \text{if } m_j \text{ exists.}$$

\(^3\)A problem has optimal-substructure property if its optimal solution can be expressed in terms of optimal solutions of its subproblems.
We say two ground atoms, \( t_1 \) and \( t_2 \), of \( q/n \) comparable if and only if \( K_{q/n}(t_1) = K_{q/n}(t_2) \).

The function \( K_{q/n} \) is used to return a sequence of indexed arguments in a left-to-right order, whereas \( O_{q/n} \) returns a non-indexed argument with the highest priority. We say two ground atoms of \( q/n \) are comparable if and only if these two atoms have the same indexed arguments. We abbreviate \( K_{q/n} \) and \( O_{q/n} \) to \( K \) and \( O \), respectively, whenever the predicate is obvious from the context. Thus, we have a preference relation defined as follows.

**Definition 3.** Let \( P \) be a logic program. A preference relation in \( P \) is a strict partial order \( \prec_P \) s.t. for any two ground atoms \( A_1 \) and \( A_2 \) of a predicate \( q/n \) in \( P \), \( A_1 \prec_P A_2 \) if both of the followings are true:

- \( K(A_1) = K(A_2) \);
- *cases the non-indexed mode with the highest priority in \( q/n \) of* 
  \[ \begin{align*}
  \text{min: } & O(A_1) > O(A_2) ; \\
  \text{max: } & O(A_1) < O(A_2) ; \\
  \text{−: } & A_2 \text{ generated earlier than } A_1 \text{ during tabled resolution.}
  \end{align*} \]

We abbreviate \( \prec_P \) to \( \prec \) whenever the tabled logic program is obvious from the context. It should be mentioned that the semantics of mode ‘−’ is heavily dependent on the order in which the answers are generated, which is decided by the tabled resolution procedure used. For Example 3, the preference relation \( \prec \) is the set

\[
\{ \text{path(a, a, 7, \_)} \prec \text{path(a, a, 0, \_)} , \ 
\text{path(a, a, 14, \_)} \prec \text{path(a, a, 0, \_)} , \ 
\text{path(a, a, 14, \_)} \prec \text{path(a, a, 7, \_)} , \\
\cdots \\
\text{path(a, b, 11, \_)} \prec \text{path(a, b, 4, \_)} , \ 
\text{path(a, b, 18, \_)} \prec \text{path(a, b, 4, \_)} , \ 
\text{path(a, b, 18, \_)} \prec \text{path(a, b, 11, \_)} , \\
\cdots 
\}
\]

where the numbers 0, 7, 11, … are the possible distances for their corresponding pair of nodes, and ‘\_’ means any ground term from the Herbrand universe. Note that no atoms of the non-tabled predicate \( \text{edge/3} \) are in the preference relation. This is because each non-tabled predicate is implicitly declared to have the index mode ‘+’ for all its arguments; therefore, none such predicate can satisfy the second condition of Definition 3. Similarly, all ground atoms of non-tabled predicates are optimized according to the following definition.

**Definition 4.** Let \( P \) be a logic program and \( I \) be one of its Herbrand interpretations; We say that \( A \) is an optimized ground atom, abbreviated as an optimized atom, in \( I \) if there does not exist any other ground atom \( A_1 \in I \) s.t. \( A \prec A_1 \).

---

\(^4\)A strict partial order relation is irreflexive and transitive.
**Definition 5.** Let \( P \) and \( B_P \) be a logic program and its Herbrand base. We define a meta-level procedure \( T'_P : 2^{B_P} \rightarrow 2^{B_P} \). Given a Herbrand interpretation \( I \), \( T'_P(I) \) performs:

1. \( I_0 \leftarrow \emptyset; \)
2. for each ground instance \( A : A_1, \ldots, A_m \) of a clause in \( P \) where \( \{A_1, \ldots, A_n\} \subseteq I \), do
   2a. \( I_0 \leftarrow I_0 \cup \{A\}; \)
   2b. \( I_0 \leftarrow I_0 - \{a_1 \in I_0 : \exists a_2 \in I_0 \text{ s.t. } a_1 \prec a_2\} \)
3. return \( I_0 \).

Thus, the fixed-point semantics of \( P \) can be described as \( T'_P \uparrow \omega(\emptyset) \).

Def. 5 gives the fixed-point semantics for a tabled logic program with mode declaration. The statement (*) shows how non-indexed modes affect the procedural semantics of the core program through the preference relation \( \prec \).

**Proposition 1.** Let \( P \) be a tabled logic program. For any atom \( A \in T_p \uparrow n(\emptyset) \), where \( n \geq 0 \), \( A \) is an optimized atom in \( T'_P \uparrow n(\emptyset) \).

**Proof:** This can be easily shown by mathematical induction on \( n \), mainly using the result of step 2b in Definition 5:

\[
I_0 \leftarrow I_0 - \{a_1 \in I_0 : \exists a_2 \in I_0 \text{ s.t. } a_1 \prec a_2\},
\]

so that any atom \( A \) in the resulting \( I_0 \) is an optimized atom according to Definition 4.

**Proposition 2.** Let \( P \) be a tabled logic program. If \( A \) is an optimized atom in \( T_P \uparrow n(\emptyset) \), then we have \( A \in T'_P \uparrow n(\emptyset) \), for any \( n \geq 0 \).

**Proof:** The proof is based on a mathematical induction on \( n \).

**Base case:** Consider \( n = 0 \). Since \( T_P \uparrow 0(\emptyset) \) is an empty set, the proposition is vacuously true.

**Inductive Case:** Assume that the proposition is true for some \( i \geq 0 \). We consider an optimized atom \( A \in T_P \uparrow (i+1)(\emptyset) \). \( A \) is obviously an optimized atom in \( T_P \uparrow (i+1)(\emptyset) \) as well due to the fact that \( T_P \uparrow (i+1)(\emptyset) \subseteq T_P \uparrow (i+1)(\emptyset) \). Next, we complete the proof by showing \( A \in T'_P \uparrow (i+1)(\emptyset) \). According to Definition 1, there exists a ground instance \( A : A_1, \ldots, A_m \) (for some \( m \geq 0 \)) of a clause in \( P \) where \( \{A_1, \ldots, A_m\} \subseteq T_P \uparrow i(\emptyset) \). Based on the optimal-substructure property, \( A_1, \ldots, A_m \) must be optimized atoms in \( T_P \uparrow i(\emptyset) \). Following the induction assumption, we have \( \{A_1, \ldots, A_m\} \in T'_P \uparrow i(\emptyset) \). Therefore, \( A \) satisfies the conditions specified in Definition 5; we have \( A \in T'_P \uparrow (i+1)(\emptyset) \).

**Proposition 3.** Let \( P \) be a tabled logic program. If \( A \in T'_P \uparrow n(\emptyset) \), then we have that \( A \) is an optimized atom in \( T_P \uparrow n(\emptyset) \), for any \( n \geq 0 \).

**Proof:** We can easily get \( A \in T_P \uparrow n(\emptyset) \) due to the fact that \( T'_P \uparrow n(\emptyset) \) is a subset of \( T_P \uparrow n(\emptyset) \). Assume that \( A' \) is an optimized atom in \( T_P \uparrow n(\emptyset) \) and \( A \prec A' \). Based on Proposition 2, we have \( A' \in T'_P \uparrow n(\emptyset) \), which is a contradiction with the fact that \( A \in T'_P \uparrow n(\emptyset) \) since \( A \prec A' \). Therefore, \( A \) must be an optimized atom in \( T_P \uparrow n(\emptyset) \).

Thus, we have the following main result showing the correctness of our mode declaration scheme for solving optimization problems with optimal-substructure property.
Theorem 4. Let $P$ be a tabled logic program. $A \in T_P \uparrow \omega(\emptyset)$ if and only if $A$ is an optimized atom in $T_P \uparrow \omega(\emptyset)$.

Proof: The proof follows trivially from Proposition 2 and Proposition 3. \qed

3.4 Implementation

The mode declaration scheme has been implemented in the authors’ TALS system, a tabled Prolog system implemented on the top of the WAM engine of the commercial ALS Prolog engine [ALS]. No change is required to the DRA resolution mechanism; therefore, the same idea can also be applied to other tabled Prolog systems such as XSB and B-Prolog.

In the TALS system, the global data structure $table$ is efficiently implemented thanks to the trie data structure [Rao et al. 1999]. The organization of the table can be abstractly described as a two-level hierarchical structure. The first level is used to organize different tabled calls indexed by their corresponding predicates names and arities; whereas the second level is used to organize tabled answers indexed by the tabled calls. Variant checking is the main operation used to locate the correct table position to access the table.

Two major changes to the global data structure $table$ are needed to support mode declarations. First, each table predicate is associated with a new item $mode$, which is represented as a bit string. The default mode for each argument in a table predicate is ‘+’. Second, the answers to a tabled call are selectively recorded depending on its mode declaration. The declared modes essentially specify the user preferences or selection constraints among the answers. When a new answer to a tabled goal is generated, variant checking on indexed arguments is invoked to determine whether the answer is variant to a previously tabled one. If that is the case, declared modes on non-indexed arguments are used to select a better answer to table; otherwise, a new table entry is added to record the answer. In fact, if an indexed argument is instantiated in advance before a tabled goal is called, variant checking on this indexed argument can be avoided since its value is same for all the answers; furthermore, it is not necessary to record the pre-instantiated value with each tabled answer because the same value has already been stored in the tabled call entry. This optimization leads to improvements on both time and space system performance.

Another important implementation issue is the replacement of tabled answers. In the current TALS system, if the tabled subgoal only involves numerals as arguments, then the tabled answer will be completely replaced if necessary. If the arguments involve structures, however, then the answer will be updated by a link to the new answer. Space taken up by the old answer has to be recovered by garbage collection (the ALS Prolog’s garbage collector has not yet been extended by us to include table space garbage recovery). As a result, if arguments of tabled predicates are bound to structures, more table space may be used up.

4. A DECLARATIVE METHOD FOR DYNAMIC PROGRAMMING

In the dynamic programming paradigm the value of an optimal solution is recursively defined in terms of optimal solutions to subproblems. Such dynamic programming definitions can be very tricky and error-prone to specify due to the involvement
of both optimization and recursion. In this section, we present a novel, elegant method based on tabled Prolog programming that simplifies the specification of such dynamic programming solutions. With the mode-declaration scheme, there is no need to define the value of an optimal solution recursively; instead, defining a general solution suffices for dynamic programming. The optimal value as well as its corresponding concrete solution can be derived implicitly and automatically using tabled Prolog systems.

4.1 Dynamic Programming with TLP

Dynamic programming algorithms are particularly appropriate for implementation with tabled logic programming [Warren 2004]. Dynamic programming is typically used for solving optimization problems. It is a general recursive strategy in which optimal solution to a problem is defined in terms of optimal solutions to its subproblems. Dynamic programming, thus, recursively reduces the solution to a problem to repetitively solving its subproblems. Therefore, for computational efficiency it is essential that a given subproblem is solved only once instead of multiple times. From this standpoint, tabled logic programming dynamically incorporates the dynamic programming strategy [Warren 2004] in the logic programming paradigm. TLP systems provide implicit tabulation scheme for dynamic programming, ensuring that subproblems are evaluated only once.

In spite of the assistance of tabled resolution, solving practical problems with dynamic programming is still not a trivial task. The main step in the dynamic programming paradigm is to define the value of an optimal solution recursively in terms of the optimal solutions to the subproblems. This definition could be very tricky and error-prone due to the interleaving of optimization and recursion. As the most widely used TLP system, XSB provides table aggregate predicates [XSB; Swift 1999], such as bagMin/2 and bagMax/2, to find the minimal or maximal value from tabled answers respectively. Those predicates are helpful in finding the optimal solutions, and therefore in implementing dynamic programming algorithms. However, users still have to define optimal solutions explicitly, that is, specify how the optimal value of a problem is recursively defined in terms of the optimal values of its subproblems. Furthermore, the aggregate predicates require the TLP system to collect all possible values, whether optimal or non-optimal, into the memo table, which could dramatically increase the amount of table space needed.

We use the matrix-chain multiplication problem [Cormen et al. 2001] as an example to illustrate how tabled logic programming can be adopted for solving dynamic programming problems. A product of matrices is fully parenthesized if it is either a single matrix or the product of two fully parenthesized matrix products, surrounded by parentheses. Thus, the matrix-chain multiplication problem can be stated as follows (detailed description of this problem can be found in any major algorithm textbook covering dynamic programming):

**Problem 1.** Given a chain \( (A_1, A_2, ..., A_n) \) of \( n \) matrices, where for \( i = 1, 2, ..., n \), matrix \( A_i \) has dimension \( p_{i-1} \times p_i \), fully parenthesize the product \( A_1A_2...A_n \) in a way that minimizes the number of scalar multiplications.

To solve this problem via dynamic programming, we need to define the cost of an optimal solution recursively in terms of the optimal solutions to subproblems. Let
\( m[i,j] \) be the minimum number of scalar multiplications needed to compute the matrix \( A_{i..j} \), which denotes a sub-chain of matrices \( A_i A_{i+1} \ldots A_j \) for \( 1 \leq i \leq j \leq n \). Thus, our recursive definition for the minimum cost of parenthesizing the product \( A_{i..j} \) becomes

\[
m[i,j] = \begin{cases} 0 & \text{if } i = j, \\ \min_{i \leq k < j} \{ m[i,k] + m[k+1,j] + p_{i-1} p_k p_j \} & \text{if } i < j. \\
\end{cases}
\]

As shown in Example 4, a tabled Prolog coding is given to solve the matrix-chain multiplication problem. The predicate \texttt{scalar\_cost}([P1, P2], 0, P1, P2) is tabled, where P0 and Pn are given by the user to represent the dimension sequence \([p_0, p_1, \ldots, p_n]\), the first dimension \(p_0\) and the last dimension \(p_n\), respectively, and \(V\) is the minimum cost of scalar multiplications to multiply \( A_{1..n} \); the built-in predicate \texttt{findall(X,G,L)} is used to find all the instances of \(X\) as a list \(L\) such that each instance satisfies the goal \(G\); the predicate \texttt{break(PL, PL1, PL2, Pk)} is used to split the dimension sequence at the point of \(P_k\) into two parts to simulate the parenthesization; and the predicate \texttt{list\_min(L, V)} finds a minimum number \(V\) from a given list \(L\).

**Example 4.** A tabled logic program for matrix-chain multiplication problems:

\[
\text{:- } \text{table scalar\_cost/4.}
\]

\[
\text{scalar\_cost([P1, P2], 0, P1, P2).
}\n\text{scalar\_cost([P1, P2, P3 | Pr], V, P1, Pn) :-
}\n\text{findall(V, ( break([P1, P2, P3 | Pr], PL1, PL2, Pk),
}\n\text{scalar\_cost(PL1, V1, P1, Pk),
}\n\text{scalar\_cost(PL2, V2, Pk, Pn),
}\n\text{V is V1 + V2 + P1 * Pk * Pn ), VL),
}\n\text{list\_min(VL, V).
}\n\text{break([P1, P2, P3], [P1, P2, P3], [P2, P3], [P1, P2, P3]).
}\n\text{break([P1, P2, P3, P4|Pr], [P1, P2], [P2, P3, P4|Pr], P2).
}\n\text{break([P1, P2, P3, P4|Pr], [P1|L1], L2, Pk) :-
}\n\text{break([P2, P3, P4|Pr], L1, L2, Pk).
}\n\]

Consider the problem for a chain \(\langle A_1, A_2, A_3 \rangle\) of three matrices. Suppose that the dimensions of the matrices are 10 x 100, 100 x 5, and 5 x 50, respectively. We can give a query :- scalar\_cost([10, 100, 5, 50], V, 10, 50) to find the minimum value of its scalar multiplications. As a result, \(V\) is instantiated to 7500, corresponding to the optimal parenthesization \((A_1 A_2) A_3\).

Program 4 shows that the programmer has to find the optimal value by comparing all possible multiplication costs explicitly. In fact, for a general optimization problem, the definition of an optimal solution could be quite complicated due to heterogeneous solution construction. Then, comparing all possible solutions explicitly to find the optimal one can be complicated.

### 4.2 A Declarative Method based on Modes

We present a declarative method based on the mode declaration scheme to separate the task of finding the optimal solution from the task of specifying the general
dynamic programming formulation. Using our method, the programmer is only required to define what a general solution is, while searching for the optimal solution is left to the TLP system.

The mode declaration can be used to make control of execution implicit during dynamic programming, making the specification of dynamic programming problems more declarative and elegant. For the matrix-chain multiplication, instead of defining the cost of an optimal solution, we only need to specify what the cost for a general solution is. Let \( m[i,j] \) be the number of scalar multiplications needed to compute the matrix \( A_{i..j} \) for \( 1 \leq i \leq j \leq n \), where \( n \) is the total number of matrices. The recursive definition for the cost of parenthesizing \( A_{i..j} \) becomes

\[
m[i,j] = \begin{cases} 
0 & \text{if } i = j, \\
m[i,k] + m[k+1,j] + p_{i-1}p_kp_j & \text{if } i < j,
\end{cases}
\]

where any \( k \in [i,j) \). Thus, we have the following program shown in Example 5.

**Example 5.** A tabled logic program with optimum mode declaration for matrix-chain multiplication problems:

\[
\text{:- table scalar\_cost(+, min, -, -).} \\
\text{scalar\_cost([P1, P2], 0, P1, P2).} \\
\text{scalar\_cost([P1, P2, P3 | Pr], V, P1, Pn) :-} \\
\text{break([P1, P2, P3 | Pr], PL1, PL2, Pk),} \\
\text{scalar\_cost(PL1, V1, P1, Pk),} \\
\text{scalar\_cost(PL2, V2, Pk, Pn),} \\
\text{V is V1 + V2 + P1 * Pk + Pn.}
\]

The mode declaration \text{scalar\_cost(+,min,-,-)} means that only the first argument (the list of matrix dimensions) is used for variant checking when an answer is generated, and a minimum value is expected from the second argument (the cost of scalar multiplication). Figure 5 shows a skeleton of the recursion tree produced by the query

\[
\text{:- scalar\_cost([10,100,5,50],V,10,50).}
\]

It's first tabled answer has \( V=75000 \). However, when the second answer \( V=7500 \) is computed, it will automatically replace the previous answer following the declared optimum mode. Therefore, there is at most one answer for the tabled call \text{scalar\_cost([10,100,5,50],V,10,50)} that exists in the table at any point in time, and it represents the optimal value computed up to that point.

To make the matrix-chain multiplication problem complete, we need to construct an optimal parenthesization solution corresponding to the minimal cost of scalar multiplication. This construction can be achieved with the strategy of introducing an extra non-indexed argument whose instantiation becomes the solution. The complete tabled logic program is shown below:

**Example 6.** A tabled logic program for the complete matrix-chain multiplication problem:

\[
\text{:- table scalar\_cost\_evid/5.} \\
\text{:- table mode scalar\_cost\_evid(+, min, -, -, -).} \\
\text{scalar\_cost\_evid([P1, P2], 0, P1, P2, (P1,P2)).} \\
\text{scalar\_cost\_evid([P1, P2, P3 | Pr], V, P1, Pn, (E1*E2)) :-}
\]
break([P1, P2, P3 | Pr], PL1, PL2, Pk),
scalar_cost_evid(PL1, V1, P1, Pk, E1),
scalar_cost_evid(PL2, V2, Pk, Pn, E2),
V is V1 + V2 + P1 * Pk * Pn.

5. EXPERIMENTAL RESULTS

Our experimental benchmarks include five typical dynamic programming examples. matrix is the matrix-chain multiplication problem; lcs is longest common subsequence problem; obst finds an optimal binary search tree; apsp finds the shortest paths for all pairs of nodes; and knap is the knapsack problem. All tests were performed in TALS system on an Intel Pentium 4 CPU 2.0GHz machine with 512M RAM running RedHat Linux 9.0.

| Benchmarks | without modes | with modes |
|------------|---------------|------------|
| matrix     | 1.99 (1.0)    | 1.01 (0.51) |
| lcs        | 0.80 (1.0)    | 0.37 (0.46) |
| obst       | 0.81 (1.0)    | 0.29 (0.36) |
| apsp       | 3.73 (1.0)    | 2.71 (0.73) |
| knap       | 51.79 (1.0)   | 35.04 (0.68) |

(a) Without evidence construction

| Benchmarks | without modes | with modes |
|------------|---------------|------------|
| matrix     | 2.74 (1.0)    | 1.97 (0.72) |
| lcs        | 0.86 (1.0)    | 0.55 (0.63) |
| obst       | 10.58 (1.0)   | 6.63 (0.63) |
| apsp       | 6.95 (1.0)    | 2.85 (0.47) |
| knap       | 126.25 (1.0)  | 38.56 (0.31) |

(b) With evidence construction

Table III. Running time performance comparison in Seconds (Ratio)

Table III compares the running time performance between the programs with and without mode declaration. The first group of benchmarks consists of programs that only seek the optimal value without evidence construction, while the second group...
consists of programs for the same dynamic programming problems with evidence construction. The experimental data indicates, based on the ratios in Table III, that the programs with mode declaration consume only 6% to 73% of the running time compared to corresponding programs without mode declaration.

![Graphs showing time performance of matrix-chain multiplication](image)

Figure 6 shows the timing information against different input sizes for matrix-chain multiplication problems. Notice that the numbers on X-axis represent the total number of matrices to be multiplied, and the numbers on Y-axis represent the running time with seconds. Whether without evidence construction (Figure 6(a)) or with evidence construction (Figure 6(b)), the graphs indicate that the timings of the programs with mode are consistently better than those without mode declaration.

The efficiency of these programs is mainly credited to two factors. First, tabled Prolog systems with mode declaration provides a concise but easy-to-use interface for dynamic programming, and it does not introduce any major overhead; mode declarations are flexible and powerful means of supporting meta-level manipulation of fixed-points; and, mode functionality is implemented at the system level rather than at the Prolog programming level. Second, tabled answers can be more efficiently organized due to the mode declaration. Indeed, if an indexed argument is instantiated in advance before a tabled goal is called, variant checking on this indexed argument can be avoided since its value is the same for all the answers; furthermore, it is not necessary to record the pre-instantiated value with each tabled answer because the same value has already been stored in the tabled call entry. These optimizations lead to considerable improvement in run-time performance.

The main disadvantage of our scheme w.r.t. efficiency is the need for frequent retrieval or replacement of tabled answers. This is because the optimized answer is dynamically selected by comparing it with old tabled answers according to the modes. The retrieval of a tabled answer for comparison incurs time overhead since information about each argument of the answer needs to be retrieved from the table.
For replacing a tabled answer in the current TALS system, if a tabled subgoal only involves numerals as arguments, then the tabled answer will be completely replaced if necessary. If the arguments involve structures, however, then the answer will be updated by a link to the new answer. Space taken up by the old answer has to be recovered by garbage collection later. As a result, the retrieval and replacement of tabled answers incurs performance overhead, especially space overhead as explained further below. The overhead will be minimal if the first tabled answer for each tabled call is optimal.

We compare the running space performance between the programs with and without mode declaration in Table IV. For benchmarks without evidence construction, our experiments indicate that with mode declaration, programs consumes only 7% to 73% of the space compared to those without mode declaration. With evidence construction included, space performance can be better or worse depending on the problem. For the matrix and obst problems that try to find the optimal binary tree structure, the programs without mode declaration generate all possible answers and then only table the optimal one, while the programs with mode declaration and implicit aggregation generate all possible answers and selectively table the better answers until the optimal one is found. In the latter case, some non-optimal answers may be replaced in the table during the computation, however, the space taken by those old answers, including tree structures, cannot be recovered immediately. If the optimal answer happens to be the first tabled answer, then no other un-optimal answers will be recorded. This is the reason why the benchmarks matrix and obst (with evidence construction) with mode declaration take more space than those without mode, as shown in Table IV.

| Benchmarks | without modes | with modes |
|------------|---------------|------------|
| matrix     | 4.98 (1.0)    | 0.93 (0.19) |
| lcs        | 78.75 (1.0)   | 23.57 (0.30) |
| obst       | 2.65 (1.0)    | 0.63 (0.24)  |
| apsp       | 20.17 (1.0)   | 14.66 (0.73) |
| knap       | 222.65 (1.0)  | 16.29 (0.07) |

(a) Without evidence construction

| Benchmarks | without modes | with modes |
|------------|---------------|------------|
| matrix     | 9.22 (1.0)    | 12.01 (1.30) |
| lcs        | 92.99 (1.0)   | 44.55 (0.48) |
| obst       | 4.44 (1.0)    | 19.37 (4.36) |
| apsp       | 29.90 (1.0)   | 21.60 (0.72) |
| knap       | 399.95 (1.0)  | 306.56 (0.77) |

(b) With evidence construction

Table IV. Running space comparison: Megabytes (Ratio)

6. CONCLUSIONS

We presented a new tabled resolution scheme based on dynamic reordering of alternatives (DRA) for efficient fixed-point computation. It works with a single SLD-
similar tree without suspension of any goals, and its operational semantics mimics
the semi-naive bottom-up computation. Our scheme can be easily implemented on
top of an existing Prolog system without modifying the kernel of the WAM engine
in any major way. We were able to implement it on top of an existing Prolog engine
(ALS Prolog) in a few man-months of work. Performance evaluation of our imple-
mentation shows that it is comparable in performance to well-engineered tabled
systems such as XSB, yet it is considerably easier to implement.

A mode declaration scheme for tabled predicates was also introduced in TLP
systems to aggregate information dynamically into the table. These modes provide
a declarative method to reduce a fixed-point from a large solution set to a preferred
small one, or from an infeasible infinite solution set to a finite one. The mode decla-
ration classifies arguments of tabled predicates as either indexed or non-indexed. As
a result, (i) a tabled predicate can be regarded as a function in which non-indexed
arguments (outputs) are uniquely defined by the indexed arguments (inputs); (ii)
concise explanation for tabled answers can be easily constructed in non-indexed
(output) arguments; (iii) the efficiency of tabled resolution can be improved since
only indexed arguments are involved in variant checking; and (iv) the non-indexed
arguments of a tabled predicate can be further qualified with an aggregate mode
such that an optimal value can be sought without explicit coding of the comparison.

This new mode declaration scheme, coupled with recursion, provides an elegant
method for specifying dynamic programming problems: there is no need to define
the value of an optimal solution recursively, instead, defining the value of a general
solution is enough. The optimal value, as well as its associated solution, is obtained
automatically by the TLP systems. This new scheme has been implemented in the
authors’ TALS system with very encouraging results.

Acknowledgments

We are grateful to David S. Warren, C. R. Ramakrishnan, Bart Demoen, Kostis
Sagonas and Neng-Fa Zhou for general discussions about tabled logic programming
and to Peter Stuckey for discussion on work presented in this paper.

REFERENCES

Ait-Kaci, H. 1991. Warren Abstract Machine: A Tutorial. MIT Press.
ALS. Applied logic systems, inc., http://www.als.com.
Balbin, I. and Ramamohanarao, K. 1987. A generalization of the differential approach to
recursive query evaluation. Journal of Logic Programming 4, 3, 259–262.
Chen, W., Swift, T., and Warren, D. 1995. Efficient top-down computation of queries under
the well-founded semantics. Journal of Logic Programming 24, 3 (September), 161–199.
Chen, W. and Warren, D. S. 1993. Query evaluation under the well-founded semantics. In ACM
Symposium on Principles of Database Systems. 168–179.
Chen, W. and Warren, D. S. 1996. Tabled evaluation with delaying for general logic programs.
Journal of the ACM 43, 1 (January), 20–74.
Cormen, T., Leiserson, C., and Rivest, R. 2001. Introduction to Algorithms. The MIT Press.
Guo, H.-F. and Gupta, G. 2001. A simple scheme for implementing tabled logic programming
systems based on dynamic reordering of alternatives. In Proceedings of International Confer-
ence on Logic Programming. 181–196.
Guo, H.-F. and Gupta, G. 2004. Simplifying dynamic programming via tabling. In Practical
Aspects of Declarative Languages. 163–177.
Lloyd, J. 1987. Foundations of Logic Programming. Springer-Verlag.
Pemmasani, G., Guo, H.-F., Dong, Y., Ramakrishnan, C., and Ramakrishnan, I. 2004. Online justification for tabled logic programs. In *International Symposium of Functional and Logic Programming*.

Ramakrishnan, Y., Ramakrishnan, C., Ramakrishnan, I., Smolka, S., Swift, T., and Warren, D. 1997. Efficient model checking using tabled resolution. In *Proceedings of Computer Aided Verification*. 143–154.

Rao, P., Ramakrishnan, I. V., Sagonas, K., Swift, T., and Warren, D. 1999. Efficient table access mechanisms for logic programs. *38*, 1 (January), 31–54.

Rocha, R., Silva, F., and Costa, V. S. 2001. On a tabling engine that can exploit or-parallelism. In *Proceedings of International Conference on Logic Programming*. 43–58.

Roychoudhury, A., Ramakrishnan, C., and Ramakrishnan, I. 2000. Justifying proofs using memo tables. In *Second International ACM SIGPLAN conference on Principles and Practice of Declarative Programming*. 178–189.

Sagonas, K. and Swift, T. An abstract machine for tabled execution of fixed-order stratified logic programs. ACM TOPLAS, *20*(3):586 - 635, May 1998.

Shen, Y.-D., Yuan, L.-Y., You, J.-H., and Zhou, N.-F. 2001. Linear tabulated resolution based on prolog control strategy. *Theory and Practice of Logic Programming 1*, 1, 71–103.

Swift, T. 1999. Tabling for non-monotonic programming. *Annals of Mathematics and Artificial Intelligence 25*, 3-4, 201–240.

Swift, T. and Warren, D. S. 1994. An abstract machine for slg resolution: Definite programs. In *Symposium on Logic Programming*. 633–652.

Tamaki, H. and Sato, T. 1986. Old resolution with tabulation. In *International Conference on Logic Programming (ICLP)*. 84–98.

van Emden, M. and Kowalski, R. 1976. The semantics of predicate logic as a programming language. *Journal of AMC 23*, 4, 733–742.

Warren, D. S. *Programming in Tabled Prolog (Draft Book)*. www.cs.sunysb.edu/~warren.

XSB. http://xsb.sourceforge.net.

Zhou, N.-F. and Sato, T. 2003. Efficient fixpoint computation in linear tabling. In *Proceedings of 5th International Conference on Principles and Practice of Declarative Programming*. 275–283.

Zhou, N.-F., Shen, Y., Yuan, L., and You, J. 2000. Implementation of a linear tabling mechanism. In *Proceedings of Practical Aspects of Declarative Languages*. 