Research Article

The Generalized Burnside Theorem in Noncommutative Deformation Theory*

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Abstract Let $A$ be an associative algebra over a field $k$, and let $\mathcal{M}$ be a finite family of right $A$-modules. A study of the noncommutative deformation functor $\operatorname{Def}_{\mathcal{M}}$ of the family $\mathcal{M}$ leads to the construction of the algebra $O^A(\mathcal{M})$ of observables and the generalized Burnside theorem, due to Laudal (2002). In this paper, we give an overview of aspects of noncommutative deformations closely connected to the generalized Burnside theorem.

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1 Introduction

Let $k$ be a field and let $A$ be an associative $k$-algebra. For any right $A$-module $M$, there is a commutative deformation functor $\operatorname{Def}_{A}: \mathbb{L} \to \operatorname{Sets}$ defined on the category $\mathbb{L}$ of local Artinian commutative $k$-algebras with residue field $k$. We recall that for an algebra $R$ in $\mathbb{L}$, a deformation of $R$ is a pair $(M_R, \tau)$, where $M_R$ is an $R$-$A$-bimodule (on which $k$ acts centrally) that is $R$-flat, and $\tau: k \otimes_R M_R \to M$ is an isomorphism of right $A$-modules.

Let $\mathcal{M}$ be the category of $R$-pointed Artinian $k$-algebras for $r \geq 1$, the natural noncommutative generalization of $\mathbb{L}$. We recall that an algebra $R$ in $\mathcal{M}$ is an Artinian ring, together with a pair of structural ring homomorphisms $f: k^r \to R$ and $g: R \to k^r$ with $g \circ f = \text{id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. Any algebra $R$ in $\mathcal{M}$ has $r$ simple right modules of dimension one, the natural projections $\{k_1, \ldots, k_r\}$ of $k^r$.

In [2], a noncommutative deformation functor $\operatorname{Def}_{\mathcal{M}}: \mathcal{M} \to \operatorname{Sets}$ of a finite family $\mathcal{M} = \{M_1, \ldots, M_r\}$ of right $A$-modules was introduced, as a generalization of the commutative deformation functor $\operatorname{Def}_{A}: \mathbb{L} \to \operatorname{Sets}$ of a right $A$-module $M$. In the case $r = 1$, this generalization is completely natural, and can be defined word for word as in the commutative case. The generalization to the case $r > 1$ is less obvious and has further-reaching consequences, but is still very natural. A deformation of $\mathcal{M}$ to $R$ is defined to be a pair $(M_R, \{\tau_i\}_{1 \leq i \leq r})$, where $M_R$ is an $R$-$A$-bimodule (on which $k$ acts centrally) that is $R$-flat, and $\tau_i: k_i \otimes_R M_R \to M_i$ is an isomorphism of right $A$-modules for $1 \leq i \leq r$. We remark that $M_R$ is $R$-flat if and only if

$$M_R \cong (R_{ij} \otimes_k M_j) = \left( \begin{array}{cccc} R_{11} \otimes_k M_1 & R_{12} \otimes_k M_2 & \cdots & R_{1r} \otimes_k M_r \\ R_{21} \otimes_k M_1 & R_{22} \otimes_k M_2 & \cdots & R_{2r} \otimes_k M_r \\ \vdots & \vdots & \ddots & \vdots \\ R_{r1} \otimes_k M_1 & R_{r2} \otimes_k M_2 & \cdots & R_{rr} \otimes_k M_r \end{array} \right),$$

considered as a left $R$-module, and that a deformation in $\operatorname{Def}_{\mathcal{M}}(R)$ may be thought of as a right multiplication $A \to \operatorname{End}_R(M_R)$ of $A$ on the left $R$-module $M_R$ that lifts the multiplication $\rho: A \to \bigoplus_{1 \leq i \leq r} \operatorname{End}_k(M_i)$ of $A$ on the family $\mathcal{M}$.

There is an obstruction theory for $\operatorname{Def}_{\mathcal{M}}$, generalizing the obstruction theory for the commutative deformation functor. Hence there exists a formal moduli $(H_M, M_H)$ for $\operatorname{Def}_{\mathcal{M}}$ (assuming a mild condition on $\mathcal{M}$). We consider the algebra of observables $O^A(\mathcal{M}) = \operatorname{End}_H(M_H) \cong \langle H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j) \rangle$ and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & O^A(\mathcal{M}) \\
\rho \downarrow & & \downarrow \pi \\
\bigoplus_{1 \leq i \leq r} \operatorname{End}_k(M_i) & & &
\end{array}$$

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given by the versal family $M_H \in \text{Def}_{\mathcal{A}}(H)$. The algebra $B = \mathcal{O}^A(M)$ has an induced right action on the family $\mathcal{M}$ extending the action of $A$, and we may consider $\mathcal{M}$ as a family of right $B$-modules. In fact, $\mathcal{M}$ is the family of simple $B$-modules since $\pi$ can be identified with the quotient morphism $B \rightarrow B/\text{rad } B$.

When $A$ is an algebra of finite dimension over an algebraically closed field $k$ and $\mathcal{M}$ is the family of simple right $A$-modules, Laudal proved the generalized Burnside theorem in [2], generalizing the structure theorem for semi-simple algebras and the classical Burnside theorem. Laudal’s result is stated in the following form.

**Theorem** (The generalized Burnside theorem). Let $A$ be a finite-dimensional algebra over a field $k$, and let $\mathcal{M} = \{M_1, M_2, \ldots , M_r\}$ be the family of simple right $A$-modules. If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\eta : A \rightarrow \mathcal{O}^A(M)$ is an isomorphism. In particular, $\eta$ is an isomorphism when $k$ is algebraically closed.

Let $A$ be an algebra of finite dimension over an algebraically closed field $k$ and let $\mathcal{M}$ be any family of right $A$-modules of finite dimension over $k$. Then the algebra $B = \mathcal{O}^A(M)$ has the property that $\eta_B : B \rightarrow \mathcal{O}^B(M)$ is an isomorphism, or equivalently, that the assignment $(A, M) \mapsto (B, M)$ is a closure operation. This means that the family $\mathcal{M}$ has exactly the same module-theoretic properties, in terms of (higher) extensions and Massey products, considered as a family of modules over $B$ as over $A$.

## 2 Noncommutative deformations of modules

Let $k$ be a field. For any integer $r \geq 1$, we consider the category $a_r$ of $r$-pointed Artinian $k$-algebras. We recall that an object in $a_r$ is an Artinian ring $R$, together with a pair of structural ring homomorphisms $f : k^r \rightarrow R$ and $g : R \rightarrow k^r$ with $g \circ f = \text{id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. The morphisms of $a_r$ are the ring homomorphisms that commute with the structural morphisms. It follows from this definition that $I(R)$ is the Jacobson radical of $R$, and therefore that the simple right $R$-modules are the projections $\{k_1, \ldots , k_r\}$ of $k^r$.

Let $A$ be an associative $k$-algebra. For any family $\mathcal{M} = \{M_1, \ldots , M_r\}$ of right $A$-modules, there is a noncommutative deformation functor $\text{Def}_A : a_r \rightarrow \text{Sets}$, introduced by Laudal [2]; see also Eriksen [1]. For an algebra $R$ in $a_r$, we recall that a deformation of $\mathcal{M}$ over $R$ is a pair $(M_R, \tau_1)_{1 \leq i \leq r}$, where $M_R$ is an $R$-$A$ bimodule (on which $k$ acts centrally) that is $R$-flat, and $\tau_i : k_i \otimes_R M_R \rightarrow M_i$ is an isomorphism of right $A$-modules for $1 \leq i \leq r$. Moreover, $(M_R, \{\tau_i\}) \sim (M_R', \{\tau_i'\})$ are equivalent deformations over $R$ if there is an isomorphism $\eta : M_R \rightarrow M'_R$ of $R$-$A$ bimodules such that $\tau_i \approx \tau_i' \circ (1 \otimes \eta)$ for $1 \leq i \leq r$. We may prove that $M_R$ is $R$-flat if and only if

$$
M_R \cong (R_{ij} \otimes_k M_j) = \begin{pmatrix}
R_{11} \otimes_k M_1 & R_{12} \otimes_k M_2 & \cdots & R_{1r} \otimes_k M_r \\
R_{21} \otimes_k M_1 & R_{22} \otimes_k M_2 & \cdots & R_{2r} \otimes_k M_r \\
\vdots & \vdots & \ddots & \vdots \\
R_{r1} \otimes_k M_1 & R_{r2} \otimes_k M_2 & \cdots & R_{rr} \otimes_k M_r
\end{pmatrix},
$$

considered as a left $R$-module, and a deformation in $\text{Def}_A(R)$ may be thought of as a right multiplication $A \rightarrow \text{End}_R(M_R)$ of $A$ on the left $R$-module $M_R$ that lifts the multiplication $\rho : A \otimes_k \text{End}_k(M_i)$ of $A$ on the family $\mathcal{M}$.

Let us assume that $\mathcal{M}$ is a swarm, that is, $\text{Ext}_A^1(M_i, M_j)$ has finite dimension over $k$ for $1 \leq i, j \leq r$. Then $\text{Def}_A$ has a pro-representing hull or a formal moduli $(H, M_H)$; see Laudal [2, Theorem 3.1]. This means that $H$ is a complete $r$-pointed $k$-algebra in the pro-category $a_r$, and that $M_H \in \text{Def}_A(H)$ is a family defined over $H$ with the following versal property: for any algebra $R$ in $a_r$ and any deformation $M_R \in \text{Def}_A(R)$, there is a homomorphism $\phi : H \rightarrow R$ such that $\text{Def}_A(\phi)(M_H) = M_R$. The formal moduli $(H, M_H)$ is unique up to non-canonical isomorphism. However, the morphism $\phi$ is not uniquely determined by $(R, M_R)$.

When $\mathcal{M}$ is a swarm with formal moduli $(H, M_H)$, right multiplication on the $H$-$A$ bimodule $M_H$ by elements in $A$ determines an algebra homomorphism

$$
\eta : A \rightarrow \text{End}_H(M_H).
$$

We write $\mathcal{O}^A(M) = \text{End}_H(M_H)$ and call it the algebra of observables. Since $M_H$ is $H$-flat, we have that $\text{End}_H(M_H) \cong (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j))$, and it follows that $\mathcal{O}^A(M)$ is explicitly given as the matrix algebra

$$
\begin{pmatrix}
H_{11} \otimes_k \text{End}_k(M_1) & H_{12} \otimes_k \text{Hom}_k(M_1, M_2) & \cdots & H_{1r} \otimes_k \text{Hom}_k(M_1, M_r) \\
H_{21} \otimes_k \text{Hom}_k(M_2, M_1) & H_{22} \otimes_k \text{End}_k(M_2) & \cdots & H_{2r} \otimes_k \text{Hom}_k(M_2, M_r) \\
\vdots & \vdots & \ddots & \vdots \\
H_{r1} \otimes_k \text{Hom}_k(M_r, M_1) & H_{r2} \otimes_k \text{Hom}_k(M_r, M_2) & \cdots & H_{rr} \otimes_k \text{End}_k(M_r)
\end{pmatrix}.
$$
Let us write \( \rho_i : A \rightarrow \text{End}_k(M_i) \) for the structural algebra homomorphism defining the right \( A \)-module structure on \( M_i \) for \( 1 \leq i \leq r \), and

\[
\rho : A \rightarrow \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i)
\]

for their direct sum. Since \( H \) is a complete \( r \)-pointed algebra in \( \mathcal{A}_r \), there is a natural morphism \( H \rightarrow k^r \), inducing an algebra homomorphism

\[
\pi : \mathcal{O}^A(M) \rightarrow \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i).
\]

By construction, there is a right action of \( \mathcal{O}^A(M) \) on the family \( M \) extending the right action of \( A \), in the sense that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & \mathcal{O}^A(M) \\
\downarrow{\rho} & & \downarrow{\pi} \\
\bigoplus_{1 \leq i \leq r} \text{End}_k(M_i) & & \\
\end{array}
\]

commutes. This makes it reasonable to call \( \mathcal{O}^A(M) \) the algebra of observables.

3 The generalized Burnside theorem

Let \( k \) be a field and let \( A \) be a finite-dimensional associative \( k \)-algebra. Then the simple right modules over \( A \) are the simple right modules over the semi-simple quotient algebra \( A / \text{rad}(A) \), where \( \text{rad}(A) \) is the Jacobson radical of \( A \). By the classification theory for semi-simple algebras, it follows that there are finitely many non-isomorphic simple right \( A \)-modules.

We consider the noncommutative deformation functor \( \text{Def}_M : \mathcal{A}_r \rightarrow \text{Sets} \) of the family \( M = \{ M_1, M_2, \ldots, M_r \} \) of simple right \( A \)-modules. Clearly, \( M \) is a swarm, hence \( \text{Def}_M \) has a formal moduli \( (H, M_H) \), and we consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & \mathcal{O}^A(M) \\
\downarrow{\rho} & & \downarrow{\pi} \\
\bigoplus_{1 \leq i \leq r} \text{End}_k(M_i) & & \\
\end{array}
\]

By a classical result, due to Burnside, the algebra homomorphism \( \rho \) is surjective when \( k \) is algebraically closed. This result is conveniently stated in the following form.

**Theorem 1** (Burnside’s theorem). If \( \text{End}_A(M_i) = k \) for \( 1 \leq i \leq r \), then \( \rho \) is surjective. In particular, \( \rho \) is surjective when \( k \) is algebraically closed.

**Proof.** There is a factorization \( A \rightarrow A / \text{rad}(A) \rightarrow \bigoplus_i \text{End}_k(M_i) \) of \( \rho \). If \( \text{End}_A(M_i) = k \) for \( 1 \leq i \leq r \), then \( A / \text{rad}(A) \rightarrow \bigoplus_i \text{End}_k(M_i) \) is an isomorphism by the classification theory for semi-simple algebras. Since \( \text{End}_A(M_i) \) is a division ring of finite dimension over \( k \), it is clear that \( \text{End}_A(M_i) = k \) whenever \( k \) is algebraically closed. \( \square \)

Let us write \( \overline{\eta} : A / \text{rad}(A) \rightarrow \bigoplus_i \text{End}_k(M_i) \) for the algebra homomorphism induced by \( \rho \). We observe that \( \rho \) is surjective if and only if \( \overline{\eta} \) is an isomorphism. Moreover, let us write \( J = \text{rad}(\mathcal{O}^A(M)) \) for the Jacobson radical of \( \mathcal{O}^A(M) \). Then we see that

\[
J = (\text{rad}(H) \otimes_k \text{Hom}_k(M_i, M_j)) = \ker(\pi).
\]

Since \( \rho(\text{rad} A) = 0 \) by definition, it follows that \( \eta(\text{rad} A) \subseteq J \). Hence there are induced morphisms

\[
\text{gr}(\eta)_q : \text{rad}(A)^q / \text{rad}(A)^{q+1} \rightarrow J^q / J^{q+1}
\]

for all \( q \geq 0 \). We may identify \( \text{gr}(\eta)_0 \) with \( \overline{\eta} \), since \( \mathcal{O}^A(M) / J \cong \bigoplus_i \text{End}_k(M_i) \). The conclusion in Burnside’s theorem is therefore equivalent to the statement that \( \text{gr}(\eta)_0 \) is an isomorphism.
Theorem 2 (The generalized Burnside theorem). Let $A$ be a finite-dimensional algebra over a field $k$, and let $\mathcal{M} = \{M_1, M_2, \ldots, M_r\}$ be the family of simple right $A$-modules. If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\eta : A \to O^A(\mathcal{M})$ is an isomorphism. In particular, $\eta$ is an isomorphism when $k$ is algebraically closed.

Proof. It is enough to prove that $\eta$ is injective and that $\text{gr}(\eta)_q$ is an isomorphism for $q = 0$ and $q = 1$, since $A$ and $O^A(\mathcal{M})$ are complete in the $\text{rad}(A)$-adic and $J$-adic topologies. By Burnside’s theorem, we know that $\text{gr}(\eta)_0$ is an isomorphism. To prove that $\eta$ is injective, let us consider the kernel $\ker(\eta) \subseteq A$. It is determined by the obstruction calculus of $\text{Def}_{\mathcal{M}}$; see Laudal [2, Theorem 3.2] for details. When $A$ is finite-dimensional, the right regular $A$-module $A_A$ has a decomposition series

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = A_A$$

with $F_p/F_{p-1}$ a simple right $A$-module for $1 \leq p \leq n$. Namely, $A_A$ is an iterated extension of the modules in $\mathcal{M}$. This implies that $\eta$ is injective; see Laudal [2, Corollary 3.1]. Finally, we must prove that $\text{gr}(\eta)_1 : \text{rad}(A)/\text{rad}(A)^2 \to J/J^2$ is an isomorphism. This follows from the Wedderburn-Malcev theorem; see Laudal [2, Theorem 3.4], for details.

4 Properties of the algebra of observables

Let $A$ be a finite-dimensional algebra over a field $k$, and let $\mathcal{M} = \{M_1, \ldots, M_r\}$ be any family of right $A$-modules of finite dimension over $k$. Then $\mathcal{M}$ is a swarm, and we denote the algebra of observables by $B = O^A(\mathcal{M})$. It is clear that

$$B/\text{rad}(B) \cong \bigoplus_i \text{End}_k(M_i)$$

is semi-simple, and it follows that $\mathcal{M}$ is the family of simple right $B$-modules. In fact, we may show that $\mathcal{M}$ is a swarm of $B$-modules, since $B$ is complete and $B/(\text{rad}B)^n$ has finite dimension over $k$ for all positive integers $n$.

Proposition 3. If $k$ is an algebraically closed field, then $\eta_B : B \to O^B(\mathcal{M})$ is an algebra isomorphism.

Proof. Since $\mathcal{M}$ is a swarm of $A$-modules and of $B$-modules, we may consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta^A} & B = O^A(\mathcal{M}) & \xrightarrow{\eta^B} & C = O^A(\mathcal{M}) \\ \rho \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i). & & \end{array}$$

The algebra homomorphism $\eta^B$ induces maps $B/\text{rad}(B)^n \to C/\text{rad}(C)^n$ for all $n \geq 1$. Since $k$ is algebraically closed and $B/\text{rad}(B)^n$ has finite dimension over $k$, it follows from the generalized Burnside theorem that $B/\text{rad}(B)^n \to C/\text{rad}(C)^n$ is an isomorphism for all $n \geq 1$. Hence $\eta^B$ is an isomorphism.

In particular, the proposition implies that the assignment $(A, \mathcal{M}) \mapsto (B, \mathcal{M})$ is a closure operation when $k$ is algebraically closed. In other words, the algebra $B = O^A(\mathcal{M})$ has the following properties:

1. the family $\mathcal{M}$ is the family of the simple $B$-modules;
2. the family $\mathcal{M}$ has exactly the same module-theoretic properties, in terms of (higher) extensions and Massey products, considered as a family of modules over $B$ as over $A$.

Moreover, these properties characterize the algebra $B = O^A(\mathcal{M})$ of observables.

5 Examples: representations of ordered sets

Let $k$ be an algebraically closed field, and let $A$ be a finite ordered set. Then the algebra $A = k[\Lambda]$ is an associative algebra of finite dimension over $k$. The category of right $A$-modules is equivalent to the category of presheaves of vector spaces on $A$, and the simple $A$-modules correspond to the presheaves $\{M_\lambda : \lambda \in A\}$ defined by $M_\lambda(\lambda) = k$ and $M_\lambda(\lambda') = 0$ for $\lambda' \neq \lambda$. The following results are well known:

1. if $\lambda > \lambda'$ in $A$ and $\{\gamma \in A : \lambda > \gamma > \lambda'\} = \emptyset$, then $\text{Ext}_A^1(M_\lambda, M_{\lambda'}) \cong k$;
2. if $\{\gamma \in A : \lambda > \gamma \geq \lambda'\}$ is a simple loop in $A$, then $\text{Ext}_A^1(M_\lambda, M_{\lambda'}) \cong k$;
3. in all other cases, $\text{Ext}_A^1(M_\lambda, M_{\lambda'}) = \text{Ext}_A^2(M_\lambda, M_{\lambda'}) = 0$. 


5.1 A hereditary example

Let us first consider the following ordered set. We label the elements by natural numbers, and write $i \rightarrow j$ when $i > j$:

\[
\begin{array}{ccc}
& 1 & \\
\downarrow & & \downarrow \\
& 2 & \rightarrow \\
& & \downarrow \\
& 3 & \rightarrow \\
& & \downarrow \\
& 4 & \\
\end{array}
\]

In this case, the simple modules are given by $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$, and we can easily compute the algebra $O^A(\mathcal{M})$ of observables since $\text{Ext}_A^2(M_i, M_j) = 0$ for all $1 \leq i, j \leq 4$. We obtain

\[
O^A(\mathcal{M}) = (H_{ij} \otimes_k \text{Hom}_k (M_i, M_j)) \cong H \cong \begin{pmatrix} k & 0 & 0 & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}.
\]

It follows from the generalized Burnside theorem that $\eta : A \rightarrow O^A(\mathcal{M})$ is an isomorphism. Hence we recover the algebra $A \cong O^A(\mathcal{M}) \cong H$.

5.2 The diamond

Let us also consider the following ordered set, called the diamond. We label the elements by natural numbers, and write $i \rightarrow j$ when $i > j$:

\[
\begin{array}{ccc}
& 1 & \\
\downarrow & & \downarrow \\
& 2 & \rightarrow \\
& & \downarrow \\
& 3 & \rightarrow \\
& & \downarrow \\
& 4 & \\
\end{array}
\]

In this case, the simple modules are given by $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$. Since $\text{Ext}_A^2(M_1, M_4) \cong k$, we must compute the cup-products

\[
\begin{align*}
\text{Ext}_A^1(M_1, M_2) \cup \text{Ext}_A^1(M_2, M_4) & \rightarrow \text{Ext}_A^2(M_1, M_4), \\
\text{Ext}_A^1(M_1, M_3) \cup \text{Ext}_A^1(M_3, M_4) & \rightarrow \text{Ext}_A^2(M_1, M_4)
\end{align*}
\]

in order to compute $H$. These cup-products are non-trivial; see Laudal [2, Remark 3.2] for details. Hence we obtain

\[
O^A(\mathcal{M}) = (H_{ij} \otimes_k \text{Hom}_k (M_i, M_j)) \cong H \cong \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix}.
\]

Note that $H_{14}$ is two-dimensional at the tangent level and has a relation. Also in this case, it follows from the generalized Burnside theorem that $\eta : A \rightarrow O^A(\mathcal{M})$ is an isomorphism. Hence we recover the algebra $A \cong O^A(\mathcal{M}) \cong H$.

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