Universal quantum (semi)groups and Hopf envelopes: Erratum

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Abstract

In [F] there is a statement generalizing the results in [FG]. Unfortunately there is a mistake in a computation that affects the main result. I don’t know if the main result in [F] is true or not, but I propose an alternative statement (Theorem 3.7) that was actually the main motivation in [FG]. This statement answers in an affirmative way the question whether the localization of the FRT construction with respect to a quantum determinant is a Hopf algebra, in case the Nichols algebra associated to the braiding is finite dimensional.

1 Error in ”Universal quantum(semi)groups and Hopf envelopes”

For a bilinear form $b : V \times V \to k$ in a finite dimensional vector space $V$ over a field $k$ with basis $x_\mu$, defined by $b_\mu\nu := b(x_\mu, x_\nu)$

Dubois-Violette and Launer [DV-L] define a Hopf algebra with generators $t_\lambda^\mu (\lambda, \mu = 1, \ldots , \dim V)$ and relations (sum over repeated indexes)

$$b_\mu\nu t_\lambda^\mu t_\rho^\nu = b_\lambda\rho 1$$

$$b^\mu\nu t_\lambda^\mu t_\rho^\nu = b^\lambda\rho 1$$

In [F] there is a Lemma 2.1 saying that equation (2) is redundant. Unfortunately the proof is incorrect. I thank Hongdi Huang and her collaborators Padmini Veerapen, Van Nguyen, Charlotte Ure, Kent Washaw and Xingting Wang for pointing me up the error. A lot of important consequences in [F] are derived from this lemma, mainly Sections 2 and 3:

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• Corollary 2.2 in [F], saying that \( A(b) \), the universal bialgebra associated to a bilinear form, is a Hopf algebra.

• Theorem 2.4 in [F], saying that a universal bialgebra associated to a specific bilinear form and a quotient of it is a Hopf algebra.

• And the main result: Theorem 3.10 of [F], that says that \( A(c) = \text{the FRT construction associated to a solution } c : V \to V \to V \otimes V \text{ of the braid equation in } V \text{ admitting a weakly graded-Frobenius algebra (WGF), becomes a Hopf algebra when localizing with respect to the quantum determinant associated that WGF algebra.} \)

On the other hand, the general universal constructions of Section 1 is independent of Lemma 2.1 and the following parts are still safe:

• The bialgebraic nature of the construction (Theorem 1.1)

• Its universal property (Proposition 1.3).

• Example of computation 1.4 and Remark 1.5.

• Section 4: the locally finite graded case and comments on other related works.

2 The mistake, and alternatives to Lemma 2.1 in [F]

The mistake in the proof of Lemma 2.1 in [F] relies in the confusion of the matrix \( t \) with entries \((t)_{ij} = t^j_i\) between the inverse of \( t \) and the inverse of the transposed matrix of \( t \). Even though I do not have a concrete counter-example, I think Lemma 2.1 is false in its full generality. However, one can still view Dubois-Violette and Launer’s Hopf algebra as a universal bialgebra construction. Recall briefly the universal construction in [F]:

**Definition 2.1.** Let \( V \) be a finite dimensional vector space with basis \( \{x_i\}_{i=1}^{\dim V} \) and \( f : V^\otimes n_i \to V^\otimes n_i : i \in I \) be a linear map. Consider free generators \( t^j_i \) \((i,j = 1, \ldots, \dim V)\) and using multi-index notation

\[
x_I := x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_\ell} \in V^\otimes \ell
\]

\[
t_I^j := t^j_{i_1} t^j_{i_2} \cdots t^j_{i_\ell} \in k\{t^j_i : i, j = 1, \ldots, \dim V\}
\]

write \( f(x_I) = \sum_J f_J^I x_J \) and define the two-sided ideal \( I_f := \langle \sum_J (t_J^I f_J^I - f_J^I t_J^I) : \forall I, K \rangle \) and the algebra

\[
A(f) := k\{t^j_i : i, j = 1, \ldots, \dim V\}/I_f
\]

with comultiplication induced by

\[
\Delta t^j_i = \sum_{k=1}^{\dim V} t^j_k \otimes t^j_i
\]

If \( F = \{f_i : V^\otimes n_i \to V^\otimes n_i : i \in I\} \) is a family of linear maps indexed by a set \( I \), define \( I_F := \sum_{i \in I} I_{f_i} \) and \( A(F) := k\{t^j_i : i, j = 1, \ldots, \dim V\}/I_F \)
2.1 Dubois-Violette and Launer’s Hopf algebra as a universal bialgebra

If \( b : V \times V \to k \) is a non-degenerate bilinear form and using the notation \( b_{ij} = b(x_i, x_j) \), we consider two linear maps

\[
b : V^2 \to V = V^0
\]

\[
x_i \otimes x_j \mapsto b_{ij}
\]

and

\[
i_b : k \to V^2
\]

\[
1 \mapsto \sum_{i,j} b_{ij} x_i \otimes x_j
\]

where \( b_{ij} \) are the \( ij \)-entries of the inverse of the matrix \((B)_{ij} = b_{ij}\).

If we denote \( H_{DV-L}(b) \) the Dubois-Violette and Launer’s Hopf algebra, one tautologically has that

\[
H_{DV-L} = A(b, i_b)
\]

(but not \( A(b) \)).

For reasons that will be clear soon, let us write \( t_{\mu\nu} := t_{\nu\mu}' \). Let us denote \( B \) the matrix with indices \( (B)_{\mu\nu} = b_{\mu\nu} \), and keep the “up convention” for \( b_{\mu\nu} = (B^{-1})_{\mu\nu} \). Then the above equations are (sum over repeated indexes)

\[
\begin{align*}
b_{\mu\nu} t_{\lambda\mu} t_{\rho\nu} &= b_{\lambda\rho} 1 \\
b_{\mu\nu} t_{\mu\lambda} t_{\nu\rho} &= b_{\lambda\rho}' 1
\end{align*}
\]

We see that if \( t \) is the matrix with entries \( (t)_{ij} = t_{ij} \) then the equations are

\[
\begin{align*}
t B t'^r &= B \\
t'^r B^{-1} t &= B^{-1}
\end{align*}
\]

where \( t'^r \) denotes the transposed matrix.

Equation (5) says that \( t \) has a right inverse (and \( t'^r \) has a left inverse), but it is not obvious that this single equation implies that \( t \) has a left inverse (or that \( t'^r \) has a right inverse). But clearly equation (6) says that \( t \) has an inverse from the other side. The key point when proving both axioms of the antipode is to prove that a given matrix has both left and right inverse. We formalize the statement in the following lemma:

**Lemma 2.2.** Assume \( H \) is a bialgebra generated by some group-like elements and a set \( \{ t_{ij}, i, j = 1, \ldots, n \} \) with \( \Delta(t_{ij}) = \sum_{k=1}^n t_{ik} \otimes t_{kj} \) and \( \epsilon(t_{ij}) = \delta_{ij} \). Denote \( t \in M_n(H) \) the \( n \times n \) matrix with entries \( (t)_{ij} \). Then the bialgebra \( H \) is a Hopf algebra if and only there exist an anti-algebra morphism \( S : H \to H \) such that \( S(D) = D^{-1} \) for all group-like generators \( D \) and the matrix \( S(t) \) with coefficients \( (S(t))_{ij} = S(t_{ij}) \) is the inverse of the matrix \( t \) in \( M_n(H) \).

**Proof.** Assume \( H \) is a Hopf algebra. The antipode axiom says in particular

\[
m(\text{Id} \otimes S)\Delta(t_{ij}) = \epsilon(t_{ij}) = m(S \otimes \text{Id})\Delta(t_{ij})
\]
But because of the comultiplication and counit properties of the $t_{ij}$ these equations translate into

$$\sum_{k=1}^{n} t_{ik} S(t_{kj}) = \delta_{ij} = \sum_{k=1}^{n} S(t_{ik}) t_{kj}$$

Denoting $S(t)$ the matrix with entries $S(t)_{ij} = S(t_{ij})$, the above equation for all $ij$ is simply the entries of the single matrix equation

$$t \cdot S(t) = \text{Id}_{n \times n} = S(t) \cdot t$$

On the other hand, assume $H$ is a bialgebra generated by group-like elements and a set $\{t_{ij} : i, j = 1, \ldots, n\}$ where the elements $t_{ij}$ satisfy $\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$ and $\epsilon(t_{ij}) = \delta_{ij}$. If $S(t) = t^{-1}$ and $S(D) = D^{-1}$ for all group-like in the set of generators, then clearly $S$ satisfies the antipode axiom on generators. Hence, $S$ is the antipode for $H$ and $H$ is a Hopf algebra.

Recall the notation in [FRT]: $c: V^\otimes 2 \to V^\otimes 2$ is a solution of the braid equation and $A(c)$ is its universal bialgebra, that is the algebra with generators $t_{ij}$ and relations

$$\sum_{k,\ell} c_{ij}^{k\ell} t_{kr} t_{\ell s} = \sum_{k,\ell} t_{ik} t_{j\ell} c_{k\ell}^{rs} \quad \forall 1 \leq i, j, r, s \leq n.$$  

(7)

that coincides with the FRT construction [FRT]. Using the same idea as in the above lemma we have:

**Lemma 2.3.** Assume $D \in A(c)$ is a group-like element such that the matrix $t \in M_n(A(c)[D^{-1}])$ is invertible, then $A(c)[D^{-1}]$ is a Hopf algebra.

**Proof.** Assume $t$ is an invertible matrix, call $u$ its inverse and $u_{ij} := (u)_{ij}$. Let us prove that there exists a unique well-defined anti-algebra map

$$A(c) \to A(c)[D^{-1}]$$

$$t_{ij} \mapsto u_{ij}$$

Since $A(c)$ is freely generated by the $t_{ij}$ with relations

$$\sum_{k,\ell} c_{ij}^{k\ell} t_{kr} t_{\ell s} = \sum_{k,\ell} t_{ik} t_{j\ell} c_{k\ell}^{rs} \quad \forall 1 \leq i, j, r, s \leq n.$$  

(7)

one should check the opposite relation in $A(c)[D^{-1}]$

$$\sum_{k,\ell} c_{ij}^{k\ell} u_{\ell s} u_{kr} \overset{\Delta}{=} \sum_{k,\ell} u_{j\ell} u_{ik} c_{k\ell}^{rs} \quad \forall 1 \leq i, j, r, s \leq n.$$  

But because the matrix $t$ is invertible in $M_n(A[D^{-1}])$, we apply the operator

$$\sum_{r,s,i,j} t_{di} t_{cj} \left(-\right) t_{ra} t_{sb}$$
and we get the equivalent checking
\[
\sum_{k,\ell,r,s,i,j} t_{di} t_{cj} c_{ij}^{rs} t_{ra} t_{sb} = \sum_{k,\ell,r,s,i,j} t_{di} t_{cj} u_{ji} u_{ik} c_{kr}^{rs} t_{ra} t_{sb} \quad \forall \ 1 \leq i, j, r, s \leq n.
\]

Now using (on LHS) \(\sum_{r,s} u_{\ell s} u_{kr} t_{ra} t_{sb} = \delta_{\ell b} \delta_{ka}\) and (on RHS) \(\sum_{ij} t_{di} t_{cj} u_{j\ell} u_{ik} = \delta_{dk} \delta_{c\ell}\) we get
\[
\sum_{i,j} t_{di} t_{cj} c_{ij}^{ab} = \sum_{r,s} c_{kr}^{rs} t_{ra} t_{sb}
\]
and this is the same relation as \(7\), that is valid on \(A(c)\), hence, it is valid in \(A(c)[D^{-1}]\) as well.

Recall that the non-commutative localization \(A(c)[D^{-1}]\) is the algebra freely generated by \(A(c)\) and the symbol \(D^{-1}\) with (the same relations as in \(A(c)\) and)
\[
DD^{-1} = 1 = D^{-1} D
\]
Having defined an antialgebra map \(A(c) \to A(c)[D^{-1}]\) we extend to a map \(A(c)[D^{-1}] \to A(c)[D^{-1}]\) by sending \(D \mapsto D^{-1}\) and this define the desired map \(S\): the antipode axioms for \(S\) are easily checked on generators. \(\square\)

3 Nichols Algebras and an alternative to Theorem 3.10 of \([F]\)

We will use the following well-known facts from finite dimensional Nichols algebras. Assume \((V, c)\) a rigid solution of YBeq such that \(\mathcal{B}(V, c)\) is finite dimensional.

**Fact 3.1.** \(\mathcal{B}\) is a graded algebra and coalgebra. It is not a Hopf algebra in the usual sense, but it is a Hopf algebra in the category of Yetter-Drinfeld modules over some Hopf algebra \(H\).

For instance, \(H = H(c)\) the Hopf envelope of \(A(c)\) do the work.

**Fact 3.2.** Denoting \(\mathcal{B}^{top}\) the highest non-zero degree of \(\mathcal{B}\), one has \(\dim \mathcal{B}^{top} = 1\), say \(\mathcal{B}^{top} = kb\) for a choice of a non-zero element \(b \in \mathcal{B}^{top}\). The projection into the coefficient of \(b\)
\[
\mathcal{B} \ni \omega = \omega_0 + \omega_1 + \cdots + \omega_{top} = \omega_0 + \omega_1 + \cdots + \lambda b \quad \mapsto \quad \lambda \in k
\]
is an integral of the braided Hopf algebra \(\mathcal{B}\). Also, because of \(\dim \mathcal{B}^{top} = 1\), the \(A(c)\)-comodule structure gives a non trivial grouplike element \(D\) determined by
\[
\rho(b) = D \otimes b
\]

**Fact 3.3.** For each degree, the multiplication map induces a non-degenerate pairing
\[
m| : \mathcal{B}^p \otimes \mathcal{B}^{top-p} \to \mathcal{B}^{top}
\]
In particular, since \(\mathcal{B}^1 = V\), the restriction of the multiplication
\[
V \otimes \mathcal{B}^{top-1} \to \mathcal{B}^{top} = bk
\]
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gives a non-degenerate pairing. In terms of basis, if \( \omega^i \) is a basis of \( B^{\text{top}}_{-1} \) and \( x_i \) is a basis of \( V \), then write

\[
\omega^i x_j = m_{ij} b, \quad m_{ij} \in k
\]

and the matrix \((m_{ij})\) is invertible, and one can choose a "dual basis" \( \hat{\omega}^i \) such that

\[
x_i \hat{\omega}^j = \delta_{ij} b
\]

**Fact 3.4.** If \((V, c)\) is rigid then so is \((V^*, c^*)\), and \( B(V^*, c^*) \) is the graded dual (as algebra and coalgebra) of \( B(V, c) \).

**Fact 3.5.** By duality (using Fact 3.3), denoting \( \text{coev} \) the comultiplication composed with projection

\[
B^{\text{top}} \xrightarrow{\Delta} B^{\text{top} \otimes p} \otimes B^p \xrightarrow{\pi} B^{\text{top} - 1} \otimes V,
\]

it is non degenerate, in the sense that if \( x_i, \hat{\omega}^i \) are bases of \( V \) and \( B^{\text{top} - 1} \) respectively, and

\[
\text{coev}(b) = \sum_{ij} \text{coev}_{ij} \hat{\omega}^i \otimes x_j, \quad \text{coev}_{ij} \in k
\]

then the matrix \((\text{coev}_{ij})\) is invertible. In particular, there exists a basis \( \hat{\omega}^i \) such that

\[
\text{coev}(b) = \sum_i \hat{\omega}^i \otimes x_i
\]

**Fact 3.6.** The maps in 3.3 and 3.5 are \( A(c) \)-colinear.

Now denote \( \rho : B \to A(c) \otimes B \) the comodule structure map and write \( \rho(\omega^j) = T_{jk} \otimes \omega^k \) and \( \rho(\hat{\omega}^j) = \hat{T}_{jk} \otimes \hat{\omega}^k \), where \( \{\omega^j\} \) and \( \{\hat{\omega}^j\} \) are basis of \( B^{\text{top} - 1} \) as in 3.3 and 3.5. By \( A(c) \)-colinearity we have

\[
D \otimes \delta_{ij} b = \rho(\delta_{ij} b) = \rho(x_i \omega^j) = t_{ik} T_{jl} \otimes x_k \omega^l = t_{ik} T_{jl} \otimes \delta^l_k b = t_{ik} T_{jk} \otimes b
\]

\[
\Rightarrow t_{ik} T_{jk} = D \delta_{ij}
\]

\[
\iff t \cdot T^{\text{tr}} = D \text{id}
\]

Now using the \( \hat{\omega}^j \)'s:

\[
\rho(\sum_i \hat{\omega}^i \otimes x_i) = \rho(\text{coev}(b)) = (1 \otimes \text{coev}) \rho(b)
\]

\[
= (1 \otimes \text{coev})(D \otimes b) = D \otimes \sum_i (\hat{\omega}^i \otimes x_i)
\]

but also

\[
\rho(\sum_i \hat{\omega}^i \otimes x_i) = \sum_{i,j,k} \hat{T}_{ij} t_{ik} \otimes \hat{\omega}^j \otimes x_k
\]

This proves

\[
\hat{T}_{ij} t_{ik} = D \delta_{jk} \Rightarrow \hat{T}^{\text{tr}} \cdot t = D \text{id}
\]

hence, \( D^{-1} \hat{T}^{\text{tr}} \) is a left inverse of \( t \), that is, \( t \) is invertible in \( M_n(A(c)[D^{-1}]) \).

Using 2.3 and observing that \( A(c) = A(qc) \) for any \( 0 \neq q \in k \), one can conclude the following main result, that is an alternative to Theorem 3.10 in [1]:

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Theorem 3.7. Let \( V \) be a finite dimensional vector space, \( c : V^\otimes 2 \to V^\otimes 2 \) a rigid solution of the braid equation and assume there is a non-zero scalar \( 0 \neq q \in k \) such that \( \mathcal{B} := \mathcal{B}(V, cq) \) is finite dimensional. Denote \( D \) the associated group-like element in \( A(c) \) coming from \( \mathcal{B}^{\text{top}} \). Then \( A(c)[D^{-1}] \) is a Hopf algebra.

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