How to obtain a covariant Breit type equation from relativistic Constraint Theory

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Abstract

It is shown that, by an appropriate modification of the structure of the interaction potential, the Breit equation can be incorporated into a set of two compatible manifestly covariant wave equations, derived from the general rules of Constraint Theory. The complementary equation to the covariant Breit type equation determines the evolution law in the relative time variable. The interaction potential can be systematically calculated in perturbation theory from Feynman diagrams. The normalization condition of the Breit wave function is determined. The wave equation is reduced, for general classes of potential, to a single Pauli-Schrödinger type equation. As an application of the covariant Breit type equation, we exhibit massless pseudoscalar bound state solutions, corresponding to a particular class of confining potentials.

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1 Introduction

Historically, The Breit equation [1] represents the first attempt to describe the relativistic dynamics of two interacting fermion systems. It consists in summing the free Dirac hamiltonians of the two fermions and adding mutual and, eventually, external potentials. This equation, when applied to QED, with one-photon exchange diagram considered in the Coulomb gauge, and solved in the linearized approximation, provides the correct spectra to order $\alpha^4$ [1, 2] for various bound state problems.

However, attempts to improve the predictivity of the equation, by developing from it a systematic perturbation theory or by solving it exactly, have failed, due to its inability to incorporate the whole effects of the interaction hamiltonian of QED [2]. Also, this equation does not satisfy global charge conjugation symmetry [3]. Finally, the Breit equation, as it stands, is not relativistically invariant, although one might consider it valid in the c.m. frame.

Nevertheless, despite these drawbacks, the Breit equation has remained popular. The main reason for this is due to the fact that it is a differential equation in $x$-space and thus it permits the study of effective local potentials with standard techniques of quantum mechanics. Improvements of this equation usually transform it into integral equations in momentum space, because of the presence of projection operators [3, 4].

The purpose of this article is to derive, from relativistic Constraint Theory [5], a covariant Breit type equation, where the free part is the sum of individual Dirac hamiltonians. The latter framework ensures the relativistic invariance of the equations describing two-particle systems with mutual interactions [1, 6] and establishes the connection with Quantum Field Theory and the Bethe-Salpeter equation by means of a Lippmann-Schwinger-Quasipotential type equation relating the potential to the off-mass shell scattering amplitude [8].

The fact that a covariant Breit type equation can be obtained from Constraint Theory was already shown by Crater and Van Alstine [9]. In the present paper we show that, for general classes of interaction, the covariant Breit type equation is equivalent to the Constarint Theory wave equations, provided it is supplemented with a second equation which explicitly eliminates the relative energy variable and at the same time ensures Poincaré invariance of the theory. The potential that appears in the main equation has a c.m. energy dependence that also ensures the global charge conjugation symmetry of the
The paper is organized as follows. In Sec. 2, we derive the covariant Breit type equation from the relativistic Constraint Theory wave equations. In Sec. 3, we determine the normalization condition of the wave function. In Sec. 4, we reduce the wave equation, relative to a sixteen-component wave function, to a Pauli-Schrödinger type equation, relative to a four-component wave function. As an application of the covariant Breit equation, we exhibit, in Sec. 5, massless pseudoscalar bound state solutions, corresponding to a particular class of confining potentials, involving mainly pseudoscalar and spacelike vector interactions. Conclusion follows in Sec. 6.

2 The covariant Breit equation

We begin with the Constraint Theory wave equations describing a system of two spin-1/2 particles composed of a fermion of mass \( m_1 \) and an antifermion of mass \( m_2 \), in mutual interaction [7]:

\[
\begin{align*}
(\gamma_1 \cdot p_1 - m_1) \tilde{\Psi} &= (-\gamma_2 \cdot p_2 + m_2) \tilde{V} \tilde{\Psi}, \\
(-\gamma_2 \cdot p_2 - m_2) \tilde{\Psi} &= (\gamma_1 \cdot p_1 + m_1) \tilde{V} \tilde{\Psi}.
\end{align*}
\]

Here, \( \tilde{\Psi} \) is a sixteen-component spinor wave function of rank two and is represented as a \( 4 \times 4 \) matrix:

\[
\tilde{\Psi} = \tilde{\Psi}_{\alpha_1 \alpha_2}(x_1, x_2) \quad (\alpha_1, \alpha_2 = 1, \ldots, 4),
\]

where \( \alpha_1(\alpha_2) \) refers to the spinor index of particle 1(2). \( \gamma_1 \) is the Dirac matrix \( \gamma_1 \) acting in the subspace of the spinor of particle 1 (index \( \alpha_1 \)); it acts on \( \tilde{\Psi} \) from the left. \( \gamma_2 \) is the Dirac matrix acting in the subspace of the spinor of particle 2 (index \( \alpha_2 \)); it acts on \( \tilde{\Psi} \) from the right; this is also the case of products of \( \gamma_2 \) matrices, which act on \( \tilde{\Psi} \) from the right in the reverse order:

\[
\begin{align*}
\gamma_1 \mu \tilde{\Psi} &\equiv (\gamma_\mu)_{\alpha_1 \beta_1} \tilde{\Psi}_{\beta_1 \alpha_2}, \\
\gamma_2 \mu \tilde{\Psi} &\equiv \tilde{\Psi}_{\alpha_1 \beta_2}(\gamma_\mu)_{\beta_2 \alpha_2}, \\
\gamma_2 \nu \gamma_2 \nu \tilde{\Psi} &\equiv \tilde{\Psi}_{\alpha_1 \beta_2}(\gamma_\nu \gamma_\mu)_{\beta_2 \alpha_2},
\end{align*}
\]

\[
\sigma_{a \alpha \beta} = \frac{1}{2i} \left[ \gamma_{aa}, \gamma_{a \beta} \right] \quad (a = 1, 2).
\]

In Eqs. (2.1) \( p_1 \) and \( p_2 \) represent the momentum operators of particles 1 and 2, respectively. \( \tilde{V} \) is a Poincaré invariant potential.
The compatibility (integrability) condition of the two equations (2.1) imposes conditions on the wave function and the potential. For the wave function, one finds the constraint:

\[
\left( p_1^2 - p_2^2 \right) - \left( m_1^2 - m_2^2 \right) \tilde{\Psi} = 0 ,
\]

which allows one to eliminate the relative energy variable in a covariant form. For eigenfunctions of the total momentum operator \( P \), the solution of Eq. (2.4) is:

\[
\tilde{\Psi} = e^{-iP.X} e^{-i(m_1^2 - m_2^2)P.x/(2P^2)} \tilde{\psi}(x^T),
\]

where we have used notations from the following definitions:

\[
P = p_1 + p_2 , \quad p = \frac{1}{2}(p_1 - p_2) , \quad M = m_1 + m_2 ,
\]

\[
X = \frac{1}{2}(x_1 + x_2) , \quad x = x_1 - x_2.
\]

We also define transverse and longitudinal components of four-vectors with respect to the total momentum \( P \):

\[
q^T_\mu = q_\mu - \frac{(q.P)}{P^2} P_\mu , \quad q^L_\mu = (q.\hat{P})\hat{P}_\mu , \quad \hat{P}_\mu = P_\mu/\sqrt{P^2} ,
\]

\[
q_L = q.\hat{P} , \quad P_L = \sqrt{P^2}.
\]

This decomposition is manifestly covariant. In the c.m. frame the transverse components reduce to the three spacelike components, while the longitudinal component reduces to the timelike component of the corresponding four-vector. (Note that \( x^{T^2} = -x^2 \) in the c.m. frame.) Also notice that, with the definition of the longitudinal components, \( P_L \), which is the positive square root of \( P^2 \), does not change sign for negative energy states (under the change \( P \to -P \)); in this case, it is the longitudinal components, \( q_L \), of those four-vectors which are independent of \( P \) that change sign, since these are linear functions of \( \hat{P} \).

For the potential, one finds the constraint:

\[
\left[ p_1^2 - p_2^2 , \tilde{V} \right] \tilde{\Psi} = 0 ,
\]

which means that \( \tilde{V} \) is independent of the relative longitudinal coordinate \( x_L \):

\[
\tilde{V} = \tilde{V}(x^T, P_L, p^T, \gamma_1, \gamma_2).
\]
Equations (2.5) and (2.9) show that the internal dynamics of the system is three-dimensional, besides the spin degrees of freedom, described by the three-dimensional transverse coordinate $x^T$.

The relationship between the potential $\tilde{V}$ and Feynman diagrams is summarized by the following Lippmann-Schwinger-Quasipotential type \[10, 11, 12, 13, 14, 15, 16, 17, 8, 18\] equation:

$$\tilde{V} - \tilde{T} - \tilde{V}G_0\tilde{T} = 0 ,$$

(2.10)

$$\tilde{T}(P,p^T,p'^T) \equiv \frac{i}{2P_L} \left[ T(P,p,p') \right]_{C(p),C(p')} ,$$

where:

i) $T$ is the off-mass shell fermion-antifermion scattering amplitude;

ii) $C$ is the constraint (2.4):

$$C(p) \equiv (p_1^2 - p_2^2) - (m_1^2 - m_2^2) = 2P_LP_L - (m_1^2 - m_2^2) \approx 0 ;$$

(2.11)

in Eq. (2.10) the external momenta of the amplitude $T$ are submitted to the constraint $C$;

iii) $G_0$ is defined as:

$$G_0(p_1, p_2) = \tilde{S}_1(p_1) \tilde{S}_2(-p_2) H_0 ,$$

(2.12)

where $\tilde{S}_1$ and $\tilde{S}_2$ are the propagators of the two fermions, respectively, in the presence of the constraint (2.11), and $H_0$ is the Klein-Gordon operator, also in the presence of the constraint (2.11):

$$H_0 = (p_1^2 - m_1^2) \bigg|_C = (p_2^2 - m_2^2) \bigg|_C = \frac{P^2}{4} - \frac{1}{2} (m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4P^2} + p'^2 .$$

(2.13)

In order to obtain the covariant Breit equation, we define the covariant Dirac “hamiltonians”:

$$\mathcal{H}_1 = m_1\gamma_1L - \gamma_1L\gamma_1^T P_L^T ,$$

(2.14a)

$$\mathcal{H}_2 = -m_2\gamma_2L - \gamma_2L\gamma_2^T P_L^T .$$

(2.14b)

We then multiply Eq. (2.1a) by $\gamma_1L$ and Eq. (2.1b) by $\gamma_2L$, respectively. After subtracting the two equations from each other, we obtain the equation:

$$(P_L - (\mathcal{H}_1 + \mathcal{H}_2)) \frac{\partial \Psi}{\partial T} = - (P_L + (\mathcal{H}_1 + \mathcal{H}_2)) (\gamma_1L\gamma_2L\tilde{V}) \frac{\partial \Psi}{\partial \tilde{V}} ,$$

(2.15)
which can be rewritten as:

\[
\left[ P_L(1 + \gamma_{1L}\gamma_{2L}\bar{V}) - (H_1 + H_2)(1 - \gamma_{1L}\gamma_{2L}\bar{V}) \right] \bar{\Psi} = 0 .
\] (2.16)

Addition of the two equations to each other leads to the equation:

\[
(2p_L - (H_1 - H_2)) \bar{\Psi} = (2p_L + (H_1 - H_2))(\gamma_{1L}\gamma_{2L}\bar{V})\bar{\Psi} ,
\] (2.17)

which can be rewritten as:

\[
\left[ 2p_L(1 - \gamma_{1L}\gamma_{2L}\bar{V}) - (H_1 - H_2)(1 + \gamma_{1L}\gamma_{2L}\bar{V}) \right] \bar{\Psi} = 0 .
\] (2.18)

Upon multiplying this equation by \((H_1 + H_2)\) and noticing that \(H_1^2 - H_2^2 = m_1^2 - m_2^2\), it becomes, after using Eq. (2.16):

\[
\left[ 2p_L P_L - (m_1^2 - m_2^2) \right] (1 + \gamma_{1L}\gamma_{2L}\bar{V})\bar{\Psi} = 0 .
\] (2.19)

We now define the Breit wave function \(\Psi_B\) by:

\[
\Psi_B = (1 - \gamma_{1L}\gamma_{2L}\bar{V})\bar{\Psi} .
\] (2.20)

Then, Eq. (2.16) takes the form:

\[
\left[ P_L(1 + \gamma_{1L}\gamma_{2L}\bar{V})(1 - \gamma_{1L}\gamma_{2L}\bar{V})^{-1} - (H_1 + H_2) \right] \Psi_B = 0 ,
\] (2.21)

while Eq. (2.19) yields, after factorizing the term \((1 + \gamma_{1L}\gamma_{2L}\bar{V})(1 - \gamma_{1L}\gamma_{2L}\bar{V})^{-1}\):

\[
\left[ 2p_L P_L - (m_1^2 - m_2^2) \right] \Psi_B = 0 .
\] (2.22)

Equations (2.21) and (2.22) are the two wave equations satisfied by the Breit wave function \(\Psi_B\). As far as the wave function transformation (2.20) is nonsingular, they are equivalent to the initial two wave equations (2.1) of Constraint Theory.

Equation (2.21) is the obvious generalization of the Breit equation. Its interaction dependent part has an explicit c.m. energy \((P_L)\) dependence which restores the global charge conjugation symmetry that was lacking in the Breit equation. For each solution of Eq. (2.21) with total momentum \(P\), there will correspond, for charge conjugation invariant interactions, a charge conjugated solution with momentum \(-P\).

Equation (2.22) determines the relative time evolution law of the wave function, as in Eq. (2.3), and ensures the relativistic invariance of the theory. While Eq. (2.21) might
be considered alone in the c.m. frame, Eq. (2.22) indicates the way of passing to other reference frames.

In the c.m. frame, with the standard definitions \( \beta = \gamma_0 \) and \( \alpha = \gamma_0 \gamma \), Eq. (2.21) becomes:

\[
[P_0(1 + \beta_1 \beta_2 \tilde{V})(1 - \beta_1 \beta_2 \tilde{V})^{-1} - (m_1 \beta_1 + \alpha_1 \cdot p - m_2 \beta_2 - \alpha_2 \cdot p)] \Psi_B = 0.
\] (2.23)

In perturbation theory, \( \tilde{V} \) has, in lowest order, according to Eq. (2.10), the structure:

\[
\tilde{V} = -\frac{1}{2\sqrt{P^2}} U(x^T, \gamma_1, \gamma_2),
\] (2.24)

where \( U \) is the three-dimensionally reduced form of the propagator of the exchanged particle, including the couplings at the vertices. To this order, Eq. (2.23) takes the form:

\[
[P_0 - \beta_1 \beta_2 \epsilon(P_0)U - (m_1 \beta_1 + \alpha_1 \cdot p - m_2 \beta_2 - \alpha_2 \cdot p)] \Psi_B = 0.
\] (2.25)

We notice here, in distinction from the Breit equation, the presence of the energy sign factor in front of the potential \( U \); it is this factor which ensures the global charge conjugation symmetry of the equation.

Finally, in the limit when \( m_2 \) tends to infinity, Eq. (2.23) yields the Dirac equation of particle 1, with the potential \( \beta_1 U \) (\( \beta_2 \) is replaced by \(-1\) for the antifermion and \( \epsilon(P_0) \) by \(+1\) in this limit).

Equations (2.1), or equivalently (2.21) and (2.22), were analyzed, in Ref. [18], in the nonrelativistic limit, to order \( 1/c^2 \), in particular for the electromagnetic interaction case. For an arbitrary covariant gauge of the photon propagator, the corresponding Hamiltonian receives contributions (among others) from quadratic terms generated by the one-photon exchange diagram as well as from the two-photon exchange diagrams. However, it turns out that in the Coulomb gauge (and also in the Landau gauge to that order) the two-photon contribution cancels the quadratic terms arising from the one-photon exchange diagram and one then is left with the Breit Hamiltonian [2, 13]. This explains why the Breit equation in its linearized approximation provides a correct result to order \( \alpha^4 \). However, in other gauges than the Coulomb and Landau gauges, it is necessary to take into account the quadratic terms as well as the two-photon exchange contribution to obtain a correct result.
3 Normalization condition

The normalization condition of the wave function $\tilde{\Psi}$ can be determined either from the construction of tensor currents of rank two, satisfying two independent conservation laws, with respect to $x_1$ and $x_2$ [6], or from the integral equation of the corresponding Green's function [14, 17]. One finds for the norm of $\tilde{\psi}$ [Eq. (2.5)] the formula (in the c.m. frame and for local potentials in $x^T$):

$$\int d^3x \, Tr \left\{ \tilde{\psi}^\dagger \left[ 1 - \tilde{V}^\dagger \tilde{V} + 4\gamma_{10}\gamma_{20}P_0^2 \frac{\partial \tilde{V}}{\partial P^2} \right] \tilde{\psi} \right\} = 2P_0 \ ,$$

where $\tilde{V}$ satisfies the hermiticity condition:

$$\tilde{V}^\dagger = \gamma_{10}\gamma_{20} \tilde{V} \gamma_{10}\gamma_{20} \ .$$

(3.2)

For energy independent potentials (in the c.m. frame) the norm of $\tilde{\psi}$ is not positive definite for arbitrary $\tilde{V}$. In order to ensure positivity, it is sufficient that the potential $\tilde{V}$ satisfy the inequality

$$\frac{1}{4} Tr(\tilde{V}^\dagger \tilde{V}) < 1 \ .$$

(3.3)

In this case one is allowed to make the wave function transformation

$$\tilde{\Psi} = \left[ 1 - \tilde{V}^\dagger \tilde{V} \right]^{-\frac{1}{2}} \tilde{\psi} \ ,$$

(3.4)

and to reach a representation where the norm for c.m. energy independent potentials is the free norm.

In this respect, the parametrization suggested by Crater and Van Alstine [20], for potentials that commute with $\gamma_{1L}\gamma_{2L}$ (and hence $\tilde{V}^\dagger = \tilde{V}$),

$$\tilde{V} = \tanh V \ ,$$

(3.5)

satisfies condition (3.3) and allows one to bring the equations satisfied by $\Psi$ [Eq. (3.4)] into forms analogous to the Dirac equation, where each particle appears as placed in the external potential created by the other particle, the latter potential having the same tensor nature as potential $V$ of Eq. (3.3).

We shall henceforth adopt the above parametrization (3.5). For more general potentials that do not commute with $\gamma_{1L}\gamma_{2L}$, the natural extension of parametrization (3.5) is:

$$\gamma_{1L}\gamma_{2L} \tilde{V} = \tanh(\gamma_{1L}\gamma_{2L} V) \ .$$

(3.6)
According to Eqs. (3.2) and (3.4), we shall introduce the wave function transformation:

\[ \tilde{\Psi} = \cosh(\gamma_1 L \gamma_2 V) \Psi. \]  \hspace{1cm} (3.7)

The norm of the new wave function \( \Psi \) then becomes (in the c.m. frame):

\[ \int d^3x \text{Tr} \left\{ \psi^\dagger \left[ 1 + 2P_0^2 \left( e^{-\gamma_1 \gamma_2 V} \frac{\partial}{\partial P^2} e^{\gamma_1 \gamma_2 V} - e^{\gamma_1 \gamma_2 V} \frac{\partial}{\partial P^2} e^{-\gamma_1 \gamma_2 V} \right) \right] \psi \right\} = 2P_0. \]  \hspace{1cm} (3.8)

(The relationship between \( \Psi \) and \( \psi \) is the same as in Eq. (2.5).)

Equations (2.1) then take the form:

\[ (\gamma_1 p_1 - m_1) \cosh(\gamma_1 L \gamma_2 V) \Psi = (-\gamma_2 p_2 + m_2)\gamma_1 L \gamma_2 V \sinh(\gamma_1 L \gamma_2 V) \Psi, \]  \hspace{1cm} (3.9a)

\[ (-\gamma_2 p_2 - m_2) \cosh(\gamma_1 L \gamma_2 V) \Psi = (\gamma_1 p_1 + m_1)\gamma_1 L \gamma_2 V \sinh(\gamma_1 L \gamma_2 V) \Psi. \]  \hspace{1cm} (3.9b)

In order to determine the normalization condition of the Breit wave function \( \Psi_B \), we first define from \( V \) a potential \( V_B \) as:

\[ V_B = \gamma_1 L \gamma_2 V. \]  \hspace{1cm} (3.10)

(Notice that because of Eq. (3.2) \( V_B^\dagger = V_B \) in the c.m. frame.) With this potential, the relationship (2.20) takes the form:

\[ \Psi_B = e^{-V_B} \tilde{\Psi}, \]  \hspace{1cm} (3.11)

while the relationship between \( \Psi_B \) and \( \Psi \) [Eq. (3.7)] is:

\[ \Psi_B = e^{-V_B} \Psi. \]  \hspace{1cm} (3.12)

The Breit type equation (2.21) becomes:

\[ \left[ P_L e^{2V_B} - (\mathcal{H}_1 + \mathcal{H}_2) \right] \Psi_B = 0. \]  \hspace{1cm} (3.13)

The normalization condition of \( \psi_B \) (defined from \( \Psi_B \) as in Eq. (2.5)) is, in the c.m. frame:

\[ \int d^3x \text{Tr} \left\{ \psi_B^\dagger \left[ e^{2V_B} + 2P_0^2 \frac{\partial}{\partial P^2} e^{2V_B} \right] \psi_B \right\} = 2P_0. \]  \hspace{1cm} (3.14)

We therefore end up with three different representations for the two-particle wave function. The first one, \( \tilde{\Psi} \) [Eqs. (2.1)], corresponds to the framework where Constraint
Theory conditions, as well as connection with Quantum Field Theory and the Bethe-Salpeter equation, are most easily established. The second one, $\Psi$ [Eqs. (3.7) and (3.9)], corresponds to the “canonical” representation, for which the norm, for c.m. energy independent potentials, is the free one. This representation is also the one used by Crater and Van Alstine [20]. The third one, $\Psi_B$ [Eqs. (3.11), (3.12), (3.13) and (2.22)], corresponds to the Breit representation.

4 Resolution of the Breit type equation

By decomposing the wave function $\Psi$ along $2 \times 2$ matrix components, Eqs. (3.9) can be solved with respect to one of these components and transformed, for the case of potentials commuting with $\gamma_1L\gamma_2L$, into a second order differential equation of the Pauli-Schrödinger type [18]. A similar reduction is also possible starting from the Breit type equation (3.13).

The relative time dependence of the wave function being determined by Eq. (2.22), with a solution of the form (2.5), one decomposes the internal $4 \times 4$ matrix wave function $\psi_B$ on the basis of the matrices $1$, $\gamma_L$, $\gamma_5$ and $\gamma_L\gamma_5$ by defining $2 \times 2$ matrix components:

$$\psi_B = \psi_{B1} + \gamma_L\psi_{B2} + \gamma_5\psi_{B3} + \gamma_L\gamma_5\psi_{B4} \equiv \sum_{i=1}^{4} \Gamma_i\psi_{Bi}.$$ (4.1)

We consider the case of potentials that are local in $x^T$ (but having eventually a c.m. energy dependence) and that are functions of products of $\gamma_1$ and $\gamma_2$ matrices in equal number (general vertex corrections do not satisfy the latter property); then $V$ commutes with $\gamma_1\gamma_2L$:

$$\gamma_1\gamma_2L V = V\gamma_1\gamma_2L.$$ (4.2)

We introduce projection matrices for the above $2 \times 2$ component subspaces:

$$P_1 = \frac{1}{4}(1 + \gamma_1\gamma_2L)(1 + \gamma_1\gamma_2L), \quad P_2 = \frac{1}{4}(1 + \gamma_1\gamma_2L)(1 - \gamma_1\gamma_2L),$$

$$P_3 = \frac{1}{4}(1 - \gamma_1\gamma_2L)(1 + \gamma_1\gamma_2L), \quad P_4 = \frac{1}{4}(1 - \gamma_1\gamma_2L)(1 - \gamma_1\gamma_2L).$$ (4.3)

They satisfy the relations:

$$P_iP_j = \delta_{ij}P_j, \quad P_i\Gamma_j = \delta_{ij}\Gamma_j \quad (i, j = 1, \ldots, 4).$$ (4.4)
(The Γ’s are defined in Eq. (4.1).)

Then, the most general (parity and time reversal invariant) potential \( V \) [Eqs. (3.5) and (3.10)] we may consider has the decomposition on the basis (4.3) :

\[
V = \sum_{i=1}^{4} a_i P_i .
\]  

(4.5)

The potentials \( a_i \) themselves may still have spin dependences. The spin operators, which act in the \( 2 \times 2 \) component subspaces, are defined by means of the Pauli-Lubanski operators:

\[
W_{1SF} = -\frac{h}{4} \epsilon_{\alpha\beta\mu\nu} P^\beta \sigma_1^{\mu\nu} , \quad W_{2SF} = -\frac{h}{4} \epsilon_{\alpha\beta\mu\nu} P^\beta \sigma_2^{\mu\nu} \quad (\epsilon_{0123} = +1) ,
\]

\[
W_{1S} = W_{2S} = -\frac{3}{4} h^2 P^2 , \quad W_S = W_{1S} + W_{2S} .
\]  

(4.6)

They also satisfy the relations :

\[
\gamma_{1L} W_{1SF} = \frac{h P_L}{2} \gamma_{1T} \gamma_{15} , \quad \gamma_{2L} W_{2SF} = \frac{h P_L}{2} \gamma_{2T} \gamma_{25} .
\]  

(4.7)

We introduce the operators :

\[
w = \left( \frac{2}{h P_L} \right)^2 W_{1S} W_{2S} , \quad w_{12} = \left( \frac{2}{h P_L} \right)^2 \frac{W_{1S} x^T W_{2S} x^T}{x^T x^T} ;
\]  

(4.8)

then, the potentials \( a_i \) [Eq. (4.5)] can be decomposed as :

\[
a_i = A_i + w B_i + w_{12} C_i \quad (i = 1, \ldots, 4) ,
\]  

(4.9)

where the potentials \( A_i, B_i, C_i \) are functions of \( x^T x^T \) and eventually of \( P^2 \).

The projectors (4.3)-(4.5) satisfy the simple property :

\[
\exp \left( \sum_{i=1}^{4} a_i P_i \right) = \sum_{i=1}^{4} P_i e^{a_i} .
\]  

(4.10)

The Breit potential \( V_B \) [Eq. (3.10)] has also a decomposition like (4.5) :

\[
V_B = \sum_{i=1}^{4} a_{Bi} P_i ,
\]  

(4.11)

with the following relations with the \( a_i \)’s :

\[
a_{B1} = a_1 , \quad a_{B2} = a_2 , \quad a_{B3} = -a_3 , \quad a_{B4} = -a_4 .
\]  

(4.12)
The relationship (3.12) between $\psi_B$ and $\psi$ can be rewritten for their $2 \times 2$ components as well:

$$\psi_{Bi} = e^{-a_{Bi}} \psi_i \quad (i = 1, \ldots, 4).$$

(4.13)

[$\psi$ is defined from a decomposition of $\psi$ as in Eq. (4.1).]

The Breit type equation (3.13) is now easily decomposed into four equations for the four components $\psi_{Bi} (i = 1, \ldots, 4)$:

$$P_L e^{2a_1} \psi_{B1} - (m_1 - m_2) \psi_{B2} + \frac{2}{h P_L} (W_{1S} - W_{2S})_p \psi_{B3} = 0,$$

(4.14a)

$$P_L e^{2a_2} \psi_{B2} - (m_1 - m_2) \psi_{B1} - \frac{2}{h P_L} W_{S,p} \psi_{B4} = 0,$$

(4.14b)

$$P_L e^{-2a_3} \psi_{B3} = M \psi_{B4} + \frac{2}{h P_L} (W_{1S} - W_{2S})_p \psi_{B1} = 0,$$

(4.14c)

$$P_L e^{-2a_4} \psi_{B4} = M \psi_{B3} - \frac{2}{h P_L} W_{S,p} \psi_{B2} = 0.$$

(4.14d)

These equations allow one to eliminate the components $\psi_{B1}, \psi_{B2}$ and $\psi_{B4}$ in terms of $\psi_{B3}$, which is a surviving component in the nonrelativistic limit. Upon defining

$$e^{2h} = 1 - \frac{(m_1^2 - m_2^2)^2}{M^2 P^2} e^{-2(a_1 + a_2)},$$

(4.15)

one finds for $\psi_{B1}$ and $\psi_{B2}$ the relations:

$$P_L \psi_{B1} = e^{-2(a_1 + a_2 + h)} \left( \frac{2}{h P_L} \right) \left\{ - e^{2a_2} (W_{1S} - W_{2S})_p \right. \right.$$  

$$\left. + \frac{(m_1^2 - m_2^2)}{M^2} W_{S,p} e^{-2a_3} \right\} \psi_{B3},$$

(4.16)

$$M \psi_{B2} = e^{-2(a_1 + a_2 + h)} \left( \frac{2}{h P_L} \right) \left\{ + e^{2a_1} W_{S,p} e^{-2a_3} \right. \right.$$  

$$\left. - \frac{(m_1^2 - m_2^2)}{P^2} (W_{1S} - W_{2S})_p \right\} \psi_{B3}. \quad (4.17)$$

One then obtains two independent equations for $\psi_{B3}$ and $\psi_{B4}$:

$$M P_L e^{-2a_4} \psi_{B4} = M^2 \psi_{B3} + \left( \frac{2}{h P_L} \right)^2 W_{S,p} e^{-2(a_1 + a_2 + h)}$$  

$$\times \left\{ + e^{2a_1} W_{S,p} e^{-2a_3} \right. \right.$$  

$$\left. - \frac{(m_1^2 - m_2^2)}{P^2} (W_{1S} - W_{2S})_p \right\} \psi_{B3},$$

(4.18a)
\[ MP_L \psi_{B4} = \frac{p^2}{2} e^{-2a_3} \psi_{B3} + \left( \frac{2}{\hbar P_L} \right)^2 (W_{1S} - W_{2S}) \cdot p e^{-2(a_1 + a_2 + h)} \times \left\{ -e^{2a_2} (W_{1S} - W_{2S}) \cdot p \right. \]
\[ + \left. \frac{(m_1^2 - m_2^2)}{M^2} W_{S \cdot p} e^{-2a_3} \right\} \psi_{B3} . \quad (4.18b) \]

Elimination of \( \psi_{B4} \) leads to the eigenvalue equation for \( \psi_{B3} \):
\[ \left[ \frac{p^2}{2} e^{-2(a_3 + a_4)} - M^2 \right] \psi_{B3} \]
\[ - \left( \frac{2}{\hbar P_L} \right)^2 W_{S \cdot p} e^{-2(a_1 + a_2 + h)} \left\{ e^{2a_1} W_{S \cdot p} e^{-2a_3} \right. \]
\[ - \left. \frac{(m_1^2 - m_2^2)}{P^2} (W_{1S} - W_{2S}) \cdot p \right\} \psi_{B3} \]
\[ - \left( \frac{2}{\hbar P_L} \right)^2 e^{-2a_4} (W_{1S} - W_{2S}) \cdot p e^{-2(a_1 + a_2 + h)} \left\{ e^{2a_2} (W_{1S} - W_{2S}) \cdot p \right. \]
\[ - \left. \frac{(m_1^2 - m_2^2)}{M^2} W_{S \cdot p} e^{-2a_3} \right\} \psi_{B3} = 0 . \quad (4.19) \]

Equation (4.19) is a second order differential equation for the component \( \psi_{B3} \). Usually, by wave function transformations one can simplify the structure of the differential operators in it. For the general potential (4.5), the second order differential operator will still exhibit a spin dependence. However, for simpler types of potential, the spin dependence of the second order differential operator also disappears. This is the case of the potential composed of general combinations of scalar, pseudoscalar and vector potentials. It has the following structure:
\[ V = V_1 + \gamma_{15} \gamma_{25} V_3 + \gamma_1 \gamma_2 \left( g_{\mu \nu} V_2 + g_{\mu \nu} U_4 + \frac{x_T^{\mu} x_T^{\nu}}{x_T^2} T_4 \right) , \quad (4.20) \]
and the decomposition of the potentials \( a_i \) [Eq. (4.5)] along these potentials is given by the relations:
\[ a_1 = V_1 + V_2 + V_3 + wU_4 + w_{12} T_4 , \]
\[ a_2 = V_1 + V_2 - V_3 - wU_4 - w_{12} T_4 , \]
\[ a_3 = V_1 - V_2 + V_3 - wU_4 - w_{12} T_4 , \]
\[ a_4 = V_1 - V_2 - V_3 + wU_4 + w_{12} T_4 . \quad (4.21) \]

With potentials of the type (4.20) and after using the wave function transformation
\[ \psi_{B3} = eV_1 - V_2 + V_3 + 2U_4 + 2T_4 + h \]
\[
\times \left\{ - \left(2 + \frac{W^2}{\hbar^2 P^2}\right) e^{2V_2} - U_4 - T_4 \sinh(2U_4) \\
+ \frac{1}{2}(1 - w_{12})e^{2V_1} + U_4 + T_4 + \frac{1}{2}(1 + w_{12})e^{2V_2} + U_4 - T_4 \right\} \phi_3 \quad (4.22)
\]

\(w_{12}\) defined in Eq. (4.8), Eq. (4.19) reduces to a Pauli-Schrödinger type equation, where the radial differential operators are those of the Laplace operator. This equation, which is also obtained from a wave function transformation in the equation satisfied by \(\psi_3\) [Eq. (4.13)], was presented in Ref. [18].

Equations (4.14) could also have been solved with respect to \(\psi_{B4}\) instead of \(\psi_{B3}\).

## 5 Zero mass solutions

As a straightforward application of the covariant Breit equation, with the class of potentials considered in Sec. 4, we shall exhibit, in this section, a class of solutions which correspond to massless pseudoscalar bound states in the limit when the masses of the constituent particles tend to zero.

The key observation is that, because of the presence of the kernel \(e^{2V_B}\) in the normalization condition (3.14), one is allowed to search for solutions in which some of the components \(\psi_{Bi}\) are constants, provided the kernel \(e^{2V_B}\) is rapidly decreasing at infinity.

The quantum numbers of the state are determined by those of the components \(\psi_{B3}\) and \(\psi_{B4}\), which are the surviving components in the nonrelativistic limit. For the ground state they have the quantum numbers \(s = 0\) (for the total spin operator defined in Eq. (4.6)), \(\ell = 0\) (for the orbital angular momentum operator) and \(j = 0\) (for the total angular momentum operator); these quantum numbers are those of a pseudoscalar state. We shall restrict the search by demanding that the components \(\psi_{B1}\) and \(\psi_{B2}\) be zero for the ground state solution.

Inspection of Eqs. (4.14a) and (4.14b) shows that \(\psi_{B3}\) must be a constant:

\[\psi_{B3} = \phi_0 = \text{const.} \quad (5.1)\]

(The vanishing of the components \(\psi_{B1}\) and \(\psi_{B2}\) can then also be checked directly in Eqs. (4.16) and (4.17).)

One is left with the two equations (4.14c) and (4.14d), which become simple algebraic equations:

\[P_L e^{-2a_3} \phi_0 - M \psi_{B4} = 0 \quad (5.2a)\]
\[ P_L e^{-2a_4 \psi_{B4}} - M \phi_0 = 0. \]  

These equations have a nontrivial solution only if \( a_3 + a_4 \) is a constant:

\[ a_3 + a_4 = C = \text{const.} \]  

Then:

\[ \psi_{B4} = \frac{M}{P_L} e^{2a_4 \phi_0} = \frac{P_L}{M} e^{-2a_3 \phi_0}, \]

\[ P_L = M e^C. \]

We now check the normalizability of the solution thus found. For simplicity, we shall consider potentials that are independent of \( P^2 \) in the c.m. frame; the corresponding conclusions are not much affected by an eventual smooth \( P^2 \) dependence of the potentials.

The normalization condition (3.14) becomes:

\[ 4 \int d^3x \left[ \psi_{B3}^\dagger e^{-2a_3 \psi_{B3}} + \psi_{B4}^\dagger e^{-2a_4 \psi_{B4}} \right] = 2P_0, \]

\[ 4|\phi_0|^2 \int d^3x \left[ e^{-2a_3} + e^{2a_4} e^{-2C} \right] = 2P_0, \]

\[ 8|\phi_0|^2 \int d^3x e^{-2a_3} = 2P_0, \]

where Eqs. (5.1) and (5.3)-(5.5) were used; furthermore, the spin 0 projection must be taken in the potentials. Therefore, \( e^{-2a_3} \) must be a rapidly decreasing function when \( |x| \to \infty \), or, equivalently, \( a_3 \) must be an increasing function of \( |x| \) at infinity, indicating the confining nature of the potential.

To have a more explicit representation of the potentials satisfying the above conditions, let us consider again the class of potentials composed of general combinations of scalar, pseudoscalar and vector potentials [Eqs. (4.20) and (4.21)]. Condition (5.3) means that

\[ 2(V_1 - V_2) = C. \]

Thus, the scalar and timelike vector potentials cannot be chosen independently from each other.

The normalizability condition (5.6) implies:

\[ \lim_{|x| \to \infty} (V_3 - 3U_4 - T_4) = \infty. \]
(The spin 0 projection of the potentials has been taken.) This combination of the pseudoscalar and spacelike vector potentials must therefore be of the confining type.

The component $\psi_{B4}$ [Eqs. (5.4) and (5.9)] of the wave function then becomes:

$$
\psi_{B4} = e^{-2(V_3 - 3U_4 - T_4)}\phi_0 .
$$

Equation (5.9) shows that when the masses of the constituent particles vanish, then the mass of the bound state also vanishes. [The fact that the right-hand side of Eq. (5.6) vanishes in this limit should not lead one to the immediate conclusion that this state disappears from the spectrum. It is its coupling to the axial vector current which is important on physical grounds, and this coupling involves the relationship of the wave function to the Bethe-Salpeter wave function through nonlocal operators, where also $P_L$ is involved [8].]

As far as the potentials do not have singularities at finite distances, the function $\psi_{B4}$ [Eqs. (5.4) and (5.9)] does not vanish at finite distances and the corresponding wave function $\psi_B$ does not have nodes; it is then a candidate for the ground state of the spectrum. To conclude that this is actually the case necessitates a detailed study of the various potentials in all sectors of quantum numbers. Conditions (5.8) and (5.7) are not sufficient to guarantee confinement in general. There are cases of potentials satisfying these conditions, for which confinement does not occur in a particular sector of quantum numbers or for which some solutions become unnormalizable. However, there are also cases for which the above solution is the ground state of the spectrum; in particular, when the confining potential is represented by the pseudoscalar potential, Eq. (4.19) can easily be analyzed; in this case all solutions other than the one found above remain massive in the limit of vanishing constituent masses.

Equations (4.14) also have solutions for which $\psi_{B3} = \psi_{B4} = 0$ and $\psi_{B1}$ and $\psi_{B2}$ are nonzero. In this case one finds the solution $P_L = |m_1 - m_2|e^{-(a_1 + a_2)}$, with $(a_1 + a_2)$ equal to a constant. This solution is, however, unphysical, since it belongs to one of the unphysical subspaces, where one of the longitudinal momenta, $p_{1L}$ or $p_{2L}$, calculated from Eqs. (2.6) and (2.22), may become negative [7].

Finally, the solution found above can also be expressed in the “canonical” representation. Taking into account the relationship (3.12), one finds:

$$
\psi = (1 + \gamma_L)\gamma_5 e^{-a_3}\phi_0 .
$$
The massless pseudoscalar bound state solution found in this section does not of course exhaust all possibly existing solutions. Furthermore, several types of mechanism may lead to the occurrence of massless pseudoscalar bound states, in connection with the spontaneous breakdown of chiral symmetry. Two such mechanisms are: i) the dynamical fermion mass generation, due to radiative corrections in the fermion self-energy part \[21\]; ii) the fall to the center phenomenon, due to short distance singularities \[22\]. Our solution differs from the above two in that it is a direct consequence of the particular confining nature of the interaction and therefore hinges on long distance forces, rather than on the short distance ones or on the radiative corrections. The solution corresponding to the pure pseudoscalar interaction case was studied in detail in Ref. \[23\].

6 Conclusion

We have shown that, by an appropriate modification of the structure of the interaction potential, the Breit equation can be incorporated into a set of two compatible manifestly covariant wave equations, derived from the general rules of Constraint Theory. The complementary equation to the covariant Breit type equation determines the evolution law of the system in the relative time variable and also determines its relative energy with respect to the other variables. Furthermore, in this covariant version of the Breit equation, the interaction potential can be systematically calculated in perturbation theory from Feynman diagrams by means of a Lippmann-Schwinger-Quasipotential type equation, relating it to the off-mass shell scattering amplitude.

The normalization condition of the Breit wave function indicates the presence of an interaction dependent kernel in it, which should be taken into account for consistent evaluations of physical quantities, like coupling constants, or for the selection of acceptable (normalizable) solutions to the wave equations. In this respect, we exhibited, as a straightforward application of the covariant Breit equation, massless pseudoscalar bound state solutions, corresponding to a class of confining potentials, essentially composed of pseudoscalar and spacelike vector potentials with eventually a particular combination of scalar and timelike vector potentials.

The covariant two-body Breit equation suggests several possibilities for its generalization to the \(N\)-body case \((N > 2)\) or for the incorporation of external potentials. However,
one meets here the known difficulty of the “continuum dissolution” problem [24, 25], which prevents the existence of normalizable states. Usually, this difficulty is circumvented by the introduction of projection operators, either in the potential [26, 27] or in the kinetic terms [4]. It is not yet known whether some local generalization of the Breit equation may avoid the above difficulty.
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