Dynamical system of relativistic particle under one dimensional harmonic oscillator potential

L A Sanjaya, T B Prayitno*, W Widyanirmala and I M Astra

Department of Physics, Faculty of Mathematics and Natural Science, Universitas Negeri Jakarta, Kampus A Jl. Rawamangun Muka, Jakarta Timur 13220, Indonesia

*Teguh-budi@unj.ac.id

Abstract. Dynamical system of a relativistic particle under harmonic oscillator potential was considered through its stability. We first construct the Hamiltonian of the system, which represents the dynamic motion of the relativistic particle. The stability of the system was then determined by the singular point. We found that the relativistic motion under the harmonic oscillator is a stable system where the phase curve forms an ellipse.

1. Introduction

The harmonic oscillator is an interesting discussion in physics both in the classical physics and in the quantum physics. In classical physics, the energy of the harmonic oscillator can be arbitrary while the energy of the quantum harmonic oscillator should be discrete. The model of harmonic oscillator can be found in the mathematic discussions [1-5] and in the physical discussions [6-9].

On the other side, the characteristic of the physical system is also interesting. This is related to the stable or unstable system. According to the dynamical system, a harmonic oscillator is stable by observing the phase curve, which forms an ellipse. This means that this system is robust under a disturbance. A next question is ‘does the stability still exist in relativistic version?’

This paper briefly discusses the stability of the relativistic particle under harmonic oscillator potential. To do that, we look for the singular point by means of the Hamiltonian equation in section 2. In the next section, we use the evolution matrix to investigate the stability of the system and draw the phase portrait. The final conclusions will be mentioned in the last section.

2. Methods

The purpose of this section is to evaluate the singular point by using the Hamiltonian equation. First, the Hamiltonian form of the relativistic particle under the harmonic oscillator potential can be written as

\[ H(x, p) = c \sqrt{m^2 c^2 + p^2} + \frac{1}{2} k x^2 \]  

(1)

with \( c \) is the velocity of light, \( m \) is the rest mass of particle, and \( k \) is a spring constant. For the comparison, the Hamiltonian for the non-relativistic particle (\( v \ll c \)) is given by

\[ H(x, p) = \frac{p^2}{2m} + \frac{1}{2} k x^2 \]  

(2)

Through Eq. (1), the dynamic motion of the system can be obtained by the following equation of motions
\[
\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{pc}{\sqrt{m^2c^2 + p^2}} \tag{3}
\]

\[
\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -kx. \tag{4}
\]

If we define \( f(x, p) = dx/dt \) and \( g(x, p) = dp/dt \), the singular points can be found by imposing the conditions \( f(x, p) = 0 \) and \( g(x, p) = 0 \), respectively.

Substituting Eq. (1) into Eq. (3) and Eq. (4), and imposing the above condition, we only obtain one singular point \((x, p) = (0,0)\). In addition, the phase trajectory can also be found by combining Eq. (3) and Eq. (4), which yields

\[
J(x, p) = \frac{1}{2} c \sqrt{m^2c^2 + p^2} + \frac{1}{2} kx^2 = 0. \tag{5}
\]

Observing Eq. (5), we can guess that the phase curve of our system should be an ellipse, which leads to a stable system.

3. Results and discussions

To see a stable system represented in Eq. (5), we can draw the phase trajectory as given in Fig. (1) below.

![Phase trajectory of relativistic particle motion under harmonic oscillator.](image)

\textbf{Figure 1.} Phase trajectory of relativistic particle motion under harmonic oscillator.

As immediately seen in Fig. 1, the phase trajectory forms an open-up parabola, which means that a relativistic mass particle will oscillate around \((0, 0)\) as a stable point, thus indicating a periodic motion. Consequently, if we employ a disturbance, the particle will tend to go its stable point.

To study a dynamical system, we need an evolution matrix at the singular point through Eq. (3) and Eq. (4), which can be written as

\[
\Gamma(x, y) = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial p}
\end{pmatrix}
\bigg|_{x=0, y=0}
\]

After substituting all the quantities, the evolution matrix for the relativistic harmonic oscillator at the singular point reads

\[
\Gamma(0,0) = \begin{pmatrix}
0 & 1/m \\
-k & 0
\end{pmatrix}
\]
Using Eq. (7), the eigenvalues of the matrix can be evaluated by solving the characteristic equation as
\[
\det\begin{pmatrix}
-\lambda & 1/m \\
-k & -\lambda
\end{pmatrix} = 0
\] (8)
with \(\lambda\) is the eigenvalue. After applying some algebra, we find two different roots, i.e.,
\(\lambda_1 = i\sqrt{k/m}\) dan \(\lambda_2 = -i\sqrt{k/m}\). Since \(k\) and \(m\) are real positive, then the relativistic particle under harmonic oscillator potential is a stable system with \((0,0)\) as a center. It also can be shown in Fig. (2) that the phase portrait forms an ellipse. The phase portrait is a set of phase curves which have the different energies.

Based on the results, we justify that both the non-relativistic particle and the relativistic particle under the harmonic oscillator potential are stable systems. This means that those two system will preserve their stability under a disturbance.

![Figure 2. Phase portrait of relativistic particle motion under harmonic oscillator.](image)

Note that our approach is a non-quantum formulation. For the comparison, the metastable state can occur in the relativistic oscillator. As reported in Giachetti and Grecchi [10] which discussed the one-dimensional Dirac equation under harmonic oscillator potential, the metastable state appears by solving the eigenvalue problem, the similar discussion can also be found in Schwinger [11]. For the similar situation with ours, Kowalski and Reimbielinski also found the periodic solution when discussing the relativistic massless particle under harmonic oscillator [12]. It seems that the relativistic massive and massless particles under harmonic oscillator have the same property.

4. Conclusions
We have proven that the dynamics of a relativistic particle motion with respect to the harmonic oscillator is a stable system, which is similar to the non-relativistic version. It can be seen that the phase trajectory forms an open-up parabola, which interprets a periodic motion around a stable point. We also show the phase curve, which relates the position and linear momentum of the relativistic particle, forms an ellipse where \((0,0)\) is a center. This means that this system is robust under a small perturbation.

References
[1] Cordero-Soto R, Lopez R M, Suazo E and Suslov S K 2008 Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields Letters in Mathematical Physics 84 159
[2] Cordero-Soto R, Suazo E and Suslov S K 2009 Models of damped oscillators in quantum mechanics Journal of Physical Mathematics 1 1
[3] Cordero-Soto R, Suazo E and Suslov S K 2010 Quantum integrals of motion for variable quadratic Hamiltonians Ann. Phys. 325 1884
[4] Achar B N N, Hanneken J W, Enck T and Clarke T 2001 Dynamics of the fractional oscillator
Physica A 297 361
[5] Um C, Yeon K and George T F 2002 The quantum damped harmonic oscillator Phys. Rep. 362
63
[6] Pérez-García V M, Michinel and Herrero H 1998 Bose-Einstein solitons in highly asymmetric
traps Phys. Rev. A 57 3837
[7] Kivshar Y S, Alexander T J and Turitsyn S K 2001 Nonlinear modes of a macroscopic quantum
oscillator Phys. Lett. A 278 225
[8] Prayitno T B 2014 Fixed conditions for achieving the real-valued partition function of one-
dimensional Gross-Pitaevskii equation coupled with time-dependent potential AIP Conf. Proc. 1589 87
[9] Prayitno T B, Budi E and Fahdiran R 2019 Ideal gas model of Bose-Einstein condensates confined
in the parabolic trap J. Phys.: Conf. Ser. 1402 044084
[10] Giachetti R and Grecchi V J 2011 PT-symmetric operators and metastable states of the 1D
relativistic oscillators J. Phys. A: Math. Theor. 44 095308
[11] Schwinger J 1951 On gauge invariance and vacuum polarization Phys. Rev. 82 664
[12] Kowalski K, Reimbieliński J 2010 Relativistic massless harmonic oscillator Phys. Rev. A 81
012118