**TOPOLOGICAL MATTER**

Choreographed entanglement dances: Topological states of quantum matter

Xiao-Gang Wen

**BACKGROUND:** Our world is very rich. One aspect of its richness is reflected in the existence of many different phases of matter. More than half a century ago, Landau developed a theory to describe phases of matter on the basis of symmetry breaking. He pointed out that the distinction between different phases stems from the way their constituent particles are organized (ordered); different phases correspond to different symmetries of the particles' ordering. For many years, it was widely believed that the symmetry-breaking theory described all phases and all phase transitions.

**ADVANCES:** However, the study of chiral spin liquids and quantum Hall (QH) liquids eventually revealed phases of matter and organizations of particles not described by the symmetry-breaking theory. This new kind of order was referred to as topological order, because it is closely related to the topological quantum field theory introduced by Witten in 1989.

It took researchers 20 years to realize that topological order is nothing but the patterns of quantum entanglement in many-body systems, which can be intuitively understood via two analogies to dancing: (i) Dance of particles (or step dance). Particles move in a spiral fashion and take a fixed number of steps to dance around each other. (ii) Dance of strings. The local degrees of freedom form strings that join in a particular way (see the figure). The strings can dance by moving around and reconnecting freely.

The first type of dance, the step dance, describes topological order in chiral spin liquids and QH liquids, whereas the string dance describes topological order in other spin liquids. The QH liquids have been realized by electron systems at the interface of semiconductors and by graphene, under strong magnetic fields. The topological order categorized by the string dance may be realized by electron spins in certain materials, such as herbertsmithite and RuCl₃.

In the string-net liquids described by the string dance, the strings can be viewed as the "electric" flux of a gauge theory, and the string density wave give rise to an emergent (non-Abelian) gauge field. The ends of the strings are topological excitations that may carry fractional charges, fractional (non-Abelian) statistics in two-dimensional (2D) systems, and Fermi statistics in 3D systems. The QH liquids categorized by the step dance also have emergent gauge theory—the Chern-Simons gauge theory. This type of dance leads to indestructible perfect conducting boundaries, as well as indestructible qubits (units of quantum information). Topologically ordered states are materials with intriguing properties, which may be useful in electronic devices and topological quantum computation.

**OUTLOOK:** The emergence of topological phases of matter from the patterns of many-body quantum entanglement is a truly new phenomenon. New mathematics is needed to describe and classify topological orders. Recent studies have revealed that a unitary modular tensor category is required to classify 2D bosonic topological orders, and unitary braided fusion categories are necessary to classify 2D fermionic topological orders. To classify 2D topological orders with symmetry G, a G-cross unitary modular tensor category (for bosons) or a unitary braided fusion categories over Rep(G) (for bosons and fermions) is needed. However, the mathematical theory, including higher-category theory, to classify topological orders in three dimensions and beyond is still evolving.

Many-body entanglement is not only the origin of many new states of quantum matter (such as topological orders), it is also the origin of emergent gauge fields, as well as emergent Fermi or fractional statistics, from the simple bosonic qubits that form the system. Recent work has indicated that our empty space itself might be a system formed by many qubits—a qubit ocean. In other words, the space itself is formed by entangled qubits; if there is no qubit, there is no sense of space. The entanglement of the qubits provides a sense of neighborhood and dimension of the space. If the quantum entanglement of the qubits in the ocean is described by a particular string-net dance, then the string density waves in the string-net liquid create electromagnetic waves that satisfy the Maxwell equation and gluon waves that satisfy the Yang-Mills equation. String ends produce electrons and quarks that carry Fermi statistics and satisfy the Dirac equation. Those emergent gauge fields and fermions are the elementary particles in the standard model. Such an emergence picture based on string-nets represents a unification of matter and information (see the figure).

Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA.
Email: xwen@mit.edu

Cite this article as X.-G. Wen et al., Science 363, eaal3099 (2019). DOI: 10.1126/science.aal3099
Choreographed entanglement dances: Topological states of quantum matter

Xiao-Gang Wen

It has long been thought that all different phases of matter arise from symmetry breaking. Without symmetry breaking, there would be no pattern, and matter would be featureless. However, it is now clear that for quantum matter at zero temperature, even symmetric disordered liquids can have features, giving rise to topological phases of quantum matter. Some of the topological phases are highly entangled (that is, have topological order), whereas others are weakly entangled (that is, have symmetry-protected trivial order). This Review provides a brief summary of these zero-temperature states of matter and their emergent properties, as well as their importance in unifying some of the most basic concepts in nature.

Since the 1980s, the study of topological phases of quantum matter has been steadily becoming more active and is now a mainstream topic in condensed matter physics. But what are topological phases? Why are people interested in them? Quantum matter refers to states of matter at zero temperature, at which the quantum nature of constituent particles becomes important. Topological quantum phases are a subclass of the phases of matter at zero temperature (which are also called quantum matter). They appear to be featureless and have nonzero energy gaps. The energy gaps imply that, like inert gases, topological phases hardly respond to any external perturbations. So they are featureless and inert—which, on the surface, sounds quite boring!

In fact, topological phases of quantum matter represent an unexplored world in condensed matter physics. Many notable phenomena, such as emergent gauge interaction and emergent Fermi and non-Abelian statistics, have been discovered. Of these phenomena, previously thought to be impossible, are having a deep impact on different fields of physics and mathematics, such as condensed matter physics, quantum information science, high-energy particle physics, algebraic topology, and higher category theory. The discovery of topological phases of quantum matter and the associated many-body entanglement arguably represent a second quantum revolution in physics.

The era of broken symmetry

Many different phases of matter exist in nature, and several of these phases can be understood from the point of view of symmetry. For example, a liquid has randomly distributed atoms. It preserves all spatial symmetries because it remains the same, on average, after it has been displaced and/or rotated arbitrarily. Because liquids have all spatial symmetries, they are featureless. In contrast, crystals do not have all symmetries, and they remain unchanged only when displaced by a particular set of distances (integers of lattice constant). So the translational symmetry of a crystal is discrete. The phase transition between a liquid and a crystal is thus a transition that reduces the continuous translational symmetry to a discrete symmetry. This change in symmetry is known as spontaneous symmetry breaking (Fig. 1). The complex crystal structures actually come from the partial breaking of the translation and rotation symmetries.

Landau theory generalizes the above picture to describe all phases and all phase transitions (1). Within this theory, the symmetry of the ordering of constituent particles distinguishes one phase from another. As a material changes phase, the symmetry of the organization of the particles changes as well. Such a comprehensive theory that describes all phases of matter led many to believe that condensed matter theory reached maturity beyond which only incremental progress could be expected.

New world beyond symmetry breaking

Disordered liquids are not featureless

According to the Landau theory, patterns in phases of matter actually emerge from symmetry breaking. Without symmetry breaking, as is the case in liquids, there would be no patterns and the state would be featureless. But, in the late 1980s, it became clear that even disordered liquids can have features. In an effort to describe the exotic properties of superconductors at high critical temperatures, a two-dimensional (2D) disordered spin liquid—i.e., a chiral spin liquid—was discovered (2, 3). A chiral spin liquid is characterized by the absence of any spin order (Fig. 1C) and perfect edge heat conduction (because all edge excitations move in the same direction). It was quickly realized that there can be many different chiral spin liquids with exactly the same symmetry (4) but different numbers of heat-conducting edge modes (5). Therefore, symmetry alone is not enough to characterize and distinguish different chiral spin liquids. This means that chiral spin liquids must exhibit another kind of order beyond the usual symmetry description. The proposed new kind of order was named “topological order” (6) (the term “topological” was motivated by the low-energy effective theory of the chiral spin liquids (3), which is a topological quantum field theory (6)).

Unfortunately, chiral spin liquids have not yet been realized in experiments (7). However, the related fractional quantum Hall (FQH) phases have been experimentally observed under strong magnetic fields. In FQH liquids, an electric field induces a current density in the transverse direction: \( j_y = \sigma_{xy} E_y \). The Hall conductance \( \sigma_{xy} \) is precisely quantized as a rational number \( v = \frac{q}{p} \) (where \( p \) and \( q \) are any integers) if measured in terms of \( c \): \( \sigma_{xy} = \frac{e^2}{h} c \) (e, elementary charge; \( h \), Planck’s constant) (8, 9). Different values of quantized \( \sigma_{xy} \) correspond to different FQH phases. Just like the chiral spin liquids, those different FQH phases all have the same symmetry and cannot be distinguished by symmetry breaking. This suggests that FQH liquids may also contain the new topological orders.

What is the essence of FQH liquids?

C. N. Yang once pointed out that the Bardeen-Cooper-Schrieffer (BCS) theory of fermionic superfluids and superconductors captures their essence. But what is this essence? To address this question, he developed the theory of off-diagonal long-range order (10), which reveals that the long-range correlation of a local order parameter is the essence of any symmetry-breaking order. Similarly, Laughlin’s theory (11) captures the essence of the FQH effect, but what is this essence? It turns out that the essence is not long-range order. Actually, the essence hidden in chiral spin liquids and the FQH effect is so new that it does not even have a name. So we are free to call it topological order. Over the past three decades, the theory of topological order was developed toward understanding the nature of this hidden essence. Our exploration is a journey into a surprisingly rich unknown world.

What is topological order?

In the above discussion, I only refer to topological order by what it is not: symmetry-breaking order. But this is not enough; topological order must be defined by what it is. A new concept in physics must be defined via physical properties that can be measured in experiments and/or numerical calculations. Furthermore, those measurable properties must be robust against any small perturbations that can break any symmetries, in order to define new orders beyond symmetry. In comparison, superfluid order is characterized by zero...
viscosity and vortex quantization, both of which are robust against any small perturbations that do not break the particle-number-conservation symmetry. Thus, superfluid order is protected by particle-number-conservation symmetry.

How do we define and characterize topological order? Influenced by Landau symmetry-breaking theory, many researchers tried to use order parameters and long-range correlations to characterize the essence of FQH liquids (12–16), which resulted in the Ginzburg-Landau-Chern-Simons effective theory of quantum Hall states. However, the order parameter and long-range correlation used in the characterization were not physical. In fact, all local physical operators have short-ranged correlations in FQH liquids. So to reveal the essence of FQH liquids, we must think in a new direction.

Motivated by research in chiral spin states (4), I identified new physical properties that reveal the essence of FQH liquids: (i) the ground-state degeneracy (4, 15) on a torus (and other spaces with nontrivial topology), (ii) the non-Abelian geometric phase (16) of those degenerate ground states (4) which form a representation of modular group $SL(2, \mathbb{Z})$, and (iii) the gapless edge modes (17, 18). All of these properties have proven resistant to any small perturbations (even those that break all symmetries) (15, 18).

Thus, these robust topological properties play the role of an order parameter and therefore enable a macroscopic definition of topological order.

The notion of topological order can also be defined microscopically. First, recall that a product state is a many-body state formed by a fixed pattern of local quantum states on each site, such as the antiferromagnetic state: $\left|\uparrow\uparrow\downarrow\uparrow\downarrow\right\rangle$. Such states are unentangled states. A spin flip $\left|\uparrow\uparrow\downarrow\uparrow\downarrow\right\rangle$ represents an excitation above the antiferromagnetic ground state. If all such excitations cost a finite energy, then the ground state is said to have a finite energy gap. The notion of product state is opposite to that of topological order in the sense that a gapped product state has no topological order. Moreover, a topologically ordered state is a gapped ground state that cannot be continuously deformed into a product state without closing the energy gap (i.e., without a phase transition). Two topologically ordered states belong to the same phase (i.e., have the same topological order) if they can deform into each other smoothly without a phase transition. From the above definition, one can show that all states with no topological order actually belong to the same phase, because the deformation mentioned above can deform any product state into any other product state. A deformation of a topologically ordered state is caused by a deformation of the Hamiltonian.

Deformations that do not close the energy gap can modify the entanglement only over a short distance. Thus, states with no topological order are called short-range entangled states, and states with topological order are referred to as long-range entangled states (19). Hence, topological order is nothing but a pattern of long-range entanglement. Long-range entanglement is the essence of topological order, as well as the essence of FQH liquids and chiral spin liquids.

**Topological orders as choreographed quantum dances**

Topological order is a concept that describes quantum entanglement in many-body systems. To provide insight into this difficult-to-understand concept, I describe topological order pictorially. Note that the pictures are meant to promote an intuitive understanding rather than serve as an exact analogy.

**Static pattern versus dancing pattern**

The product states (which correspond to symmetry-breaking states) are characterized by fixed static patterns, such as $\left|\uparrow\uparrow\uparrow\uparrow\uparrow\right\rangle$ (Fig. 2, A and B). In contrast, topologically ordered ground states are superpositions of many different configurations, such as different spin configurations $\left|\uparrow\uparrow\uparrow\uparrow\uparrow\right\rangle + \left|\downarrow\downarrow\downarrow\downarrow\downarrow\right\rangle + \cdots$. Such states are also referred as having strong quantum fluctuations. I will now use a dancing analogy to provide an intuitive picture of topological order (i.e., pattern of quantum fluctuations or quantum entanglement).

A topological order is described by a global dance (Fig. 2, C and D), in which every particle (or spin) is dancing with every other particle (or spin) in a very organized way. All spins or particles dance according to a set of local dancing “rules” that aim to lower the energy of a local Hamiltonian. The local dancing “rules” enforce a global dancing pattern, which corresponds to the topological order (i.e., long-range entanglement).

For example, in FQH liquids, the electrons dance according to the following local rules: (i) Electrons always dance antidiagonally, which implies that the electron wave function only depends on the electron coordinates $(x, y)$ via $z = x + i y$ (where $i^2 = -1$). (ii) Each electron always takes exactly “three steps” to dance around any other electron, indicating that the phase of the wave function changes by $6\pi$ as one electron moves around another.

**Fig. 1. Symmetry breaking.** (A, C, and E) Disordered liquid states that do not break any symmetry. (B, D, and F) Ordered states that spontaneously break some symmetries. For example, the energy function $e_{\phi}(\phi)$ has a symmetry $\phi \rightarrow -\phi$: $e_{\phi}(\phi) = e_{\phi}(-\phi)$. However, as the parameter $g$ (e.g., magnetic field) changes, the minimal energy state (the ground state) sometimes respects the symmetry [(A), (C), and (E)] and other times must settle into a state that does not respect the symmetry [(B), (D), and (F)]. A ferromagnetic spin order, which breaks rotational symmetry, is depicted in (D); a crystal order, which breaks continuous translational and rotational symmetries, is shown in (F). A, symmetric state; B and $B'$, symmetry-breaking states.

**Fig. 2. Pictorial representations of order in quantum matter.** (A and B) Static patterns for the symmetry-breaking orders (i.e., product states). (C and D) Dancing patterns for the topological orders: globally correlated group dances.
These two local dancing rules enforce a global dancing pattern that corresponds to the Laughlin wave function $|\Psi_{\text{FQH}}\rangle = \prod_{i<j} (z_i - z_j)^3$. Such collective movement gives rise to topological order (or long-range entanglement) in the filling fraction $\nu = 1/3$ FQH liquid (i.e., with Hall conductance $\sigma_{xy} = e^2/h$).

In addition to FQH liquids, some spin liquids (20) also contain topological orders (21, 22) in which the spins dance according to the following local rules: (i) Down spins form closed strings with no open ends (Fig. 3), (ii) Strings can otherwise move and reconnect freely.

The global dance formed by the spins in accordance with the above dancing rules yields a quantum spin liquid that is a superposition of all closed-string configurations: $\langle \Psi_{\text{String}} \rangle = \sum_{\text{all string patterns}} |X\rangle$. Such a state is a vector, with those vectors spanning a lattice.

First, let us view each column of the $K$-matrix that describes the dancing pattern (the pattern of long-range entanglement). As a result, the fractionalization is determined by the $K$-matrix, which encodes the quantum dimension of a particle. But for a particle with non-Abelian statistics, $d$ may not even be an integer, as the internal degrees of freedom of the particle are described by a vector space with a non-integer dimension $d$. In this case, the particle is said to have fractional degrees of freedom.

For $N$ identical non-Abelian particles, exchanging two of them leads to a $D_N$ by $D_N$ unitary matrix acting on the degenerate states (36, 37). In contrast to some descriptions of non-Abelian statistics, the exchange unitary matrix is not a property of the two exchanging particles. Rather, it depends on all particles in the system.

Non-Abelian statistics—in particular, its fractional degrees of freedom—is so unconventional that it is difficult to imagine that such a thing can appear in condensed matter systems formed by simple electrons and atoms. But the quantum entanglement between electrons is a very powerful “creator”; it can even make such an impossibility occur.

Two concrete realizations of non-Abelian statistics in FQH liquids were proposed in the same year (38, 39). One of the proposed FQH liquids at $\nu = 2/3$ is given by the wave function $|\Psi_{\text{FQH}}\rangle = \prod_{i<j} (z_i - z_j)^3$, where $\chi_{K}(\{z_i\})$ is the many-electron wave function with $n$ filled Landau levels (38). It has a type-SU(3)$_3$ non-Abelian particle (a Fibonacci anyon) with quantum dimension $d = \sqrt{5}/2 = 0.694$ qubits. Such SU(3)$_3$ non-Abelian statistics also appear in $Z_3$ parafermion FQH state (40).

The degeneracy from non-integer quantum dimension cannot be viewed as local degrees of freedom associated with each particle, although I will call it “the internal degrees of freedom” of the particle. In fact, those degrees of freedom are...
distributed between well-separated non-Abelian particles and are intrinsically nonlocal. As a result, the quantum information carried by those degenerate states is immune to the decoherence caused by the environment, which interacts locally with each particle. The degeneracy [called topological degeneracy (4)] is robust against any local perturbations, including any perturbations acting on the particles. Thus, non-Abelian topological order may be used to perform topological quantum computation (23, 35, 41, 42). In particular, SU(3)2 non-Abelian topological order can perform universal topological quantum computation (42).

A simpler non-Abelian FQH liquid at \( v = \frac{1}{2} \) is given by \( \Psi_{v \to \frac{1}{2}}(\{z_i\}) = \mathcal{Z}_2(\{z_i\})|\mathcal{X}_2(\{z_i\})|^2 \) or by \( \Psi_{v \to \frac{1}{2}} = \mathcal{A}\left(\frac{1}{\pi \sin \frac{\pi}{2m}}\right) \prod_{i<j} (z_i - z_j)^2 \). Both have type-SU(2)2 non-Abelian statistics, with quantum dimension \( d = \sqrt{3} \) (a half qubit). The experimentally realized \( v = \frac{1}{2} \) FQH state is likely to be such an SU(2)2 non-Abelian FQH state (43). Unfortunately, however, SU(2)2 non-Abelian topological order cannot be used to perform universal topological quantum computation (42).

### String liquid in spin liquid: Emergence of gauge theory

FQH liquids and chiral spin liquids do not have the simplest topological order. The simplest one is the \( Z_2 \) topological order defined by emergent \( Z_2 \) gauge theory at low energy. The \( Z_2 \) topological order (and the emergent \( Z_2 \) gauge theory) was first proposed to exist in a frustrated 2+1D spin liquid (21, 22). Later, a numerical calculation confirmed its existence in a closely related quantum dimer model on a triangular lattice (44). If the spin rotation symmetry is broken, the \( Z_2 \) topological order can be realized by an exactly soluble model—the toric code model (23). The toric code model reveals that the \( Z_2 \) topological order happens to be the topological order in the string dance of nonoriented loops described in the “Static pattern versus dancing pattern” section above. Unlike FQH states whose boundary is always gapless, the \( Z_2 \) topological order can have boundaries that are gapped (45).

In addition to the above theoretical realizations, the \( Z_2 \) topological order (21, 22) may be realized in herbertsmithite (spin-½ on a kagome lattice) (46), as indicated by experimental evidence (47, 48). The early numerical calculation of Yan et al. (49) suggested that the spin-½ Heisenberg model on a kagome lattice is gapped, but the details of the results are inconsistent with \( Z_2 \) topological order, which led researchers to suspect that the model is instead gapless. A more recent numerical calculation indicates that the model has \( Z_2 \) topological order with a long correlation length of \( \sim 10 \) unit cells (50), whereas several other calculations suggest gapless U(1) spin-liquid ground states (51–53).

The string dancing picture for the \( Z_2 \) topological order and emergent \( Z_2 \) gauge theory can be generalized by allowing strings to have more types and by allowing three strings to join in a certain way (Fig. 4). This gives rise to a string-net–condensed state (24), which can lead to non-Abelian topological orders and emergent non-Abelian gauge theory. In fact, the strings behave like the electric flux of the gauge theory, and the string density wave corresponds to a non-Abelian gauge field that generates gauge bosons. The ends of strings correspond to gauge charges, which may carry non-Abelian statistics in 2+1D or Fermi statistics in 3+1D.

Such a string-net construction is also very powerful and comprehensive: It can produce all 2+1D topological orders with gappable boundaries (24, 54). Such a construction has a mathematical basis in unitary fusion category theory (55) and is closely related to the Turaev-Viro invariant of 3D manifolds (56). To be more precise, unitary fusion categories classify 2+1D topological orders with gappable boundaries.

### Even disordered product states are not featureless

For a long time, symmetry breaking in product states was thought to describe all possible phases of matter. But the existence of long-range entangled states, which leads to rich topologically ordered phases, is now recognized. Considering only product states and assuming no symmetry breaking, the corresponding symmetric product states are expected to be trivial. However, this speculation turns out to be incorrect. If the Hamiltonian has symmetry, even when its ground state is a product state that does not spontaneously break any symmetry, such a ground state can still be nontrivial. This is the most “trivial” nontrivial state of quantum matter.

### Haldane phase for integer-spin chain

Owing to quantum fluctuations, the ground state of antiferromagnetic spin-½ chain does not break the SO(3) spin-rotation symmetry. However, it almost breaks the symmetry in the sense that spin-spin correlation has an exponential decay (similar to nondecaying long-range order) and the chain is gapless (as if having a Goldstone mode). For a spin-½ chain with \( S > ½ \), the quantum fluctuations are weaker than the spin-½ chain. For that reason, it was believed that a spin-½ chain for \( S > ½ \) has the same properties as a spin-½ chain.

However, in 1983, Haldane pointed out (57) that the chain spin is actually gapped and the spin-spin correlation has an exponential decay when the spin has an integer value. The gapped ground state of the integer-spin chain is called the Haldane phase. At that time, researchers did not distinguish even-integer-spin chains from those with odd-integer spins and believed the Haldane phase for both even- and odd-integer-spin chains to be a trivial disordered phase, just like the product state formed by spin-0 on each site.

### Even-integer Haldane phase is trivial; odd-integer Haldane phase is nontrivial

Only even-integer Haldane phases behave like the product states formed by spin-0 on each site. The odd-integer Haldane phases actually correspond to a nontrivial phase!
What is the essence of the gapped ground state of an odd-integer-spin chain? In (58), the spin-1 chain was studied using a tensor network renormalization approach. It was found that the tensor network, describing the space-time spin fluctuations, flows to a so-called corner-double-line tensor under the coarse graining of the network (Fig. 5A). To understand the physical meaning of the corner-double-line structure, it is necessary to describe the coarse-graining process from the point of view of the ground-state wave function.

The ground state of the spin-1 chain (Fig. 5B) is formed by spins of value 1 on each site. A number of sites may be grouped into an effective site. When the effective site is large enough, the direct entanglement between two spins of value 1 (represented by the blue curves) can only appear between the neighboring effective sites. Then a unitary transformation acting within an effective site may be used to simplify the entanglement within each effective site (Fig. 5C). After removing the degrees of freedom that are entangled only within each effective site, a simplified coarse-grained wave function is obtained (Fig. 5D) that corresponds to the corner-double-line structure. In the coarse-grained wave function, each effective site has four states: spin-0, spin-1, spin-½, and spin-singlet. Additionally, the coarse-grained wave function is a product state of singlets (Fig. 5A and D), thus confirming that the Haldane phase is a trivial product state formed by spins of value 0.

However, the coarse-grained wave function is not the product state formed by spin-singlets on each effective site; rather, it is the product state formed by spin-singlets between the neighboring effective sites (Fig. 5). Furthermore, the spin-singlets between sites are formed by spins of value ½ (58), which are not representations of SO(3) but instead are projective representations of SO(3) (59, 60). This entanglement pattern and the associated corner-double-line structure is robust against any perturbations that preserve SO(3) symmetry. In other words, the product state formed by spin-singlets on each site and the product state formed by spin-singlets between sites belong to two different phases, provided that the spin-singlets are formed by projective representations of SO(3) (i.e., half-integer spins). In this case, one state cannot be deformed into the other without encountering a gap-closing phase transition, if the deformation preserves the SO(3) symmetry. Thus, the ground state of a spin-1 chain is actually a new phase of matter that is robust against any symmetry-preserving perturbations despite being a product state that does not break any symmetry. This new phase was named the SPT phase (which stands for symmetry-protected trivial phase) (61), and a theory for the boundary of SPT phases was developed by generalizing the Wess-Zumino-Witten model (66, 67). It was found that SPT phases cannot have a trivial gapped boundary (68, 69). The boundary must be gapless, symmetry breaking, or topologically ordered (if the boundary is 2D or higher). Many new phases of matter were discovered (68, 70, 71), including the first SPT state beyond 1+1D—the CZX state (68).

A more systematic theory of bosonic SPT phases in all d dimensions was developed on the basis of group cohomology \( H^{d+1}(G, R/Z) \) (65), and a theory for the boundary of SPT phases was developed by generalizing the Wess-Zumino-Witten model (66, 67). It was found that SPT phases cannot have a trivial gapped boundary (68, 69): The boundary must be gapless, symmetry breaking, or topologically ordered (if the boundary is 2D or higher). Many new phases of matter were discovered (68, 70, 71), including the first SPT state beyond 1+1D—the CZX state (68).

Two years later, a classification of all bosonic SPT phases in d dimensions was obtained on the basis of group cohomology \( H^{d+1}(G × SO_n, R/Z) \) (72) and cobordism (73).}

This perspective on the SPT phases (stressing their lack of entanglement) led to rapid development of the field. Indeed, only 1 year later, a classification (60–62) of all 1+1D SPT phases protected by symmetry group \( G \) was found in terms of the projective representations of \( G \) (59). This discovery facilitated a complete classification of all 1+1D gapped phases of matter. In particular, the classification predicts that all 1+1D SPT states have symmetry-protected boundary degeneracy described by the projective representations of the symmetry. Such a measurable character of SPT states agrees with an earlier result (63) that spin-1 Haldane phase has a degenerate spin-½ degree of freedom at the chain end. However, in the same study, Affleck et al. also predict degenerate spin-1 degrees of freedom at a boundary of the spin-2 Haldane phase, which seems to indicate that the spin-2 Haldane phase is also a nontrivial SPT phase. Now it is evident that such spin-1 boundary degrees of freedom are not protected by symmetry because they can be gapped out by adding several spin-2 degrees of freedom to the boundary without breaking the SO(3) symmetry. Thus, the spin-2 Haldane phase is a trivial SPT phase (64).

The essence of the gapped state of the spin-1 chain is the pattern of entanglement (Fig. 5D) that spin-1 Haldane phase has a degenerate spin-½ degree of freedom at the chain end. However, in the same study, Affleck et al. also predict degenerate spin-1 degrees of freedom at a boundary of the spin-2 Haldane phase, which seems to indicate that the spin-2 Haldane phase is also a nontrivial SPT phase. Now it is evident that such spin-1 boundary degrees of freedom are not protected by symmetry because they can be gapped out by adding several spin-2 degrees of freedom to the boundary without breaking the SO(3) symmetry. Thus, the spin-2 Haldane phase is a trivial SPT phase (64).

Two years later, a systematic theory of bosonic SPT phases in all d dimensions was developed on the basis of group cohomology \( H^{d+1}(G, R/Z) \) (65), and a theory for the boundary of SPT phases was developed by generalizing the Wess-Zumino-Witten model (66, 67). It was found that SPT phases cannot have a trivial gapped boundary (68, 69): The boundary must be gapless, symmetry breaking, or topologically ordered (if the boundary is 2D or higher). Many new phases of matter were discovered (68, 70, 71), including the first SPT state beyond 1+1D—the CZX state (68).

Shortly afterward, a classification of all bosonic SPT phases in d dimensions was obtained on the basis of group cohomology \( H^{d+1}(G × SO_n, R/Z) \) (72) and cobordism (73). SPT phases can appear in both bosonic and fermionic systems. Besides the bosonic gapped spin-1 phase that has been realized experimentally (74), the other well-known SPT phase is the fermionic topological insulator protected by \( G' = [U(1) × Z_2^t] / Z_2 \) symmetry (75–80) that has also been experimentally realized (81, 82). Here, \( Z_2^t \) is the symmetry group generated by time-reversal symmetry, \( U(1) \) is the symmetry group for charge conservation. The early theories of topological insulators and topological superconductors (83) are based on non-interacting fermions, in terms of K-theory (84) or replica theory (85). But many topological insulators and topological superconductors obtained in non-interacting theory are actually not topological insulators and topological superconductors in the presence of interactions (86). With interaction, topological insulators and topological superconductors are not described by K-theory or replica theory. In this case, they should be viewed as fermionic SPT phases protected by a symmetry \( G' \), which are described by group-supercohomology theory of \( G' \) (87, 88), spin cobordism theory (89, 90), and/or, if in 2+1D, by modular extensions.
of $s\text{Rep}(G')$—the symmetric fusion category formed by $Z_2$-graded representations of $G'$.  

### Four revolutions in physics and the second quantum revolution

Several times in the history of physics, new mathematics has been required to describe a new phenomenon. In fact, the appearance of new mathematics is a sign of a truly new discovery in physics. For example, Newton's theory of mechanics required calculus to describe the phenomena of curved motion of particles that were believed to form all matter. Maxwell's theory of electromagnetism revealed a new form of matter—wave-like matter (electromagnetic waves and light)—which necessitated the mathematical theory of fiber bundles. Einstein's theory of general relativity revealed yet another form of wave-like matter: gravitational waves, which required Riemannian geometry. Quantum theory unified particle-like matter and wave-like matter, as explained via linear algebra. Those are mathematical theories required: Group cohomology ($\mathbb{H}, \mathbb{C}$) or cobordism ($\mathbb{H}, \mathbb{C}$, $\mathbb{C}$) is needed to classify SPT orders, a unitary modular tensor category ($\mathbb{H}, \mathbb{C}, \mathbb{C}$) is required to classify 2+1D bosonic topological orders, and unitary braided fusion categories over $s\text{Rep}(Z^2)$ ($\mathbb{H}$) are needed to classify 2+1D fermionic topological orders. The mathematical theory to describe topological orders in 3+1D and beyond is still evolving (580).

Many-body entanglement gives rise to new topological states of quantum matter, enables topological quantum computation, and provides a framework for the unification of matter and information and potentially the origin of matter (and even space) from entangled qubits—it may well inspire a new mathematical foundation of nature. Thus, many-body entanglement (i.e., topological order) arguably represents a second quantum revolution. Condensed matter physics is not at the beginning of its end; it is at the end of its beginning.

### REFERENCES AND NOTES

1. L. D. Landau, E. M. Lifschitz. *Statistical Physics - Course of Theoretical Physics*, vol. 5 (Pergamon, 1958).

2. V. Kalmykov, R. B. Laughlin. Equivalence of the resonating-valence-bond and fractional quantum Hall states. *Phys. Rev. Lett.* 59, 2095–2098 (1987). doi: 10.1103/PhysRevLett.59.2095; pmid: 1003548

3. X.-G. Wen, F. Wilczek, A. Zee. Chiral spin states and superconductivity. *Phys. Rev. B* 39, 11433–11423 (1989). doi: 10.1103/PhysRevB.39.11413; pmid: 9947970

4. X.-G. Wen. Topological orders in rigid states. *Int. J. Mod. Phys. B* 4, 239–271 (1990). doi: 10.1142/S021797929000139

5. X.-G. Wen, Gapless boundary excitations in the quantum Hall states and in the chiral spin states. *Phys. Rev. B* 43, 11025–11036 (1991). doi: 10.1103/PhysRevB.43.11025; pmid: 9995836

6. E. Witten. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* 121, 351–399 (1989). doi: 10.1007/BF01217730

7. The chiral spin liquid is realized numerically in the Hessenberg model on a kagome lattice with $J_1-J_2-J_3$ coupling (99, 100).

8. K. von Klitzing, G. Dorda, M. Pepper. New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance. *Phys. Rev. Lett.* 45, 494–497 (1980). doi: 10.1103/PhysRevLett.45.494

9. D. C. Tsui, H. L. Stormer, A. C. Gossard. Two-Dimensional Magnetotransport in the Extreme Quantum Limit. *Phys. Rev. Lett.* 48, 1559–1562 (1982). doi: 10.1103/PhysRevLett.48.1559

10. C. N. Yang. Concept of Off-Diagonal Long-Range Order and the Quantum Phases of Liquid He and of Superconductors. *Rev. Mod. Phys.* 34, 694–704 (1962). doi: 10.1103/RevModPhys.34.694

11. R. B. Laughlin. Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations. *Phys. Rev. Lett.* 50, 1395–1398 (1983). doi: 10.1103/PhysRevLett.50.1395

12. S. M. Girvin, A. H. MacDonald. Off-diagonal long-range order, oblique confinement, and the fractional quantum Hall effect. *Phys. Rev. Lett.* 58, 1252–1255 (1987). doi: 10.1103/PhysRevLett.58.1252; pmid: 10034381

13. S. C. Zhang, T. H. Hansson, S. Kivelson. Effective-field-theory model for the fractional quantum Hall effect. *Phys. Rev. Lett.* 62, 82–85 (1989). doi: 10.1103/PhysRevLett.62.82; pmid: 10039554

14. N. Read. Order parameter and Ginsburg-Landau theory for the fractional quantum Hall effect. *Phys. Rev. Lett.* 62, 86–89 (1989). doi: 10.1103/PhysRevLett.62.86; pmid: 10039555

15. X.-G. Wen, Q. Niu. Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces. *Phys. Rev. B* 41, 9377–9396 (1990). doi: 10.1103/PhysRevB.41.9377; pmid: 9993283

16. For an explanation of non-Abelian geometric phase, see (102).

17. B. I. Halperin, Quantized Hall conductance, current-carrying edge states, and the existence of extended states.
18. M. Freedman, C. Nayak, K. Shtengel, K. Walker, Z. Wang, Symmetry protected topological orders and the group cohomology of their abelian subgroups. Phys. Rev. B 87, 155114 (2013). doi:10.1103/PhysRevB.87.155114

19. J. Weiss, B. Zurnino. Consequences of anomalous ward identities. Phys. Lett. B 37, 95–97 (1971). doi:10.1016/0370-2693(71)90531-4

20. Spin liquids refer to ground states of quantum spin systems that do not spontaneously break any symmetry.

21. N. Read, S. Sachdev. Large-N expansion for frustrated quantum antiferromagnets. Phys. Rev. Lett. 66, 1773–1776 (1991). doi:10.1103/PhysRevLett.66.1773; pmid:10453748

22. B. I. Halperin, Statistics of Quasiparticles and the e/3 Fractionally Charged Laughlin Quasiparticle. Phys. Rev. Lett. 52, 1583 (1984). doi:10.1103/PhysRevLett.52.1583

23. J. S. Halton et al., Spin dynamics of the spin-1/2 kagome lattice antiferromagnet ZnCu3(OH)6Cl2. Phys. Rev. B 87, 144420 (2013). doi:10.1103/PhysRevB.87.144420

24. X.-G. Wen. Mean-field theory of spin-ladder states with finite energy gap and topological orders. Phys. Rev. B 44, 2664–2672 (1991). doi:10.1103/PhysRevB.44.2664; pmid:999836

25. M. Levine, X.-G. Wen. String-net condensation: A physical mechanism for topological phases. Phys. Rev. B 71, 045110 (2005). doi:10.1103/PhysRevB.71.045110

26. D. Arovas, J. R. Schrieffer, W. Wilczek, Fractional Statistics and the Quantum Hall Effect. Phys. Rev. Lett. 53, 722–725 (1984). doi:10.1103/PhysRevLett.53.722

27. B. I. Halperin, Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States. Phys. Rev. Lett. 52, 1583–1586 (1984). doi:10.1103/PhysRevLett.52.1583

28. M. Freedman, C. Nayak, K. Shtengel, K. Walker, Z. Wang, Symmetry protected topological orders and the group cohomology of their abelian subgroups. Phys. Rev. B 87, 155114 (2013). doi:10.1103/PhysRevB.87.155114

29. J. Weiss, B. Zurnino. Consequences of anomalous ward identities. Phys. Lett. B 37, 95–97 (1971). doi:10.1016/0370-2693(71)90531-4

30. Spin liquids refer to ground states of quantum spin systems that do not spontaneously break any symmetry.

31. N. Read, S. Sachdev. Large-N expansion for frustrated quantum antiferromagnets. Phys. Rev. Lett. 66, 1773–1776 (1991). doi:10.1103/PhysRevLett.66.1773; pmid:10453748

32. B. I. Halperin, Statistics of Quasiparticles and the e/3 Fractionally Charged Laughlin Quasiparticle. Phys. Rev. Lett. 52, 1583 (1984). doi:10.1103/PhysRevLett.52.1583

33. J. S. Halton et al., Spin dynamics of the spin-1/2 kagome lattice antiferromagnet ZnCu3(OH)6Cl2. Phys. Rev. B 87, 144420 (2013). doi:10.1103/PhysRevB.87.144420

34. X.-G. Wen. Mean-field theory of spin-ladder states with finite energy gap and topological orders. Phys. Rev. B 44, 2664–2672 (1991). doi:10.1103/PhysRevB.44.2664; pmid:999836

35. M. A. Kitaev. Fault-tolerant quantum computation by anyons. Ann. Phys. 303, 2–30 (2000). doi:10.1006/aphy.2000.6198

36. Y.-S. Wu, General theory for quantum statistics in two dimensions. Phys. Rev. B 41, 2383–2384 (1990). doi:10.1103/PhysRevB.41.2383; pmid:993578

37. X. Chen, Z.-C. Gu, X.-G. Wen. Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order. Phys. Rev. B 82, 235138 (2010). doi:10.1103/PhysRevB.82.235138

38. Spin liquids refer to ground states of quantum spin systems that do not spontaneously break any symmetry.

39. G. Moore, N. Read. Nonabelions in the fractional quantum hall effect. Nucl. Phys. B 360, 362–396 (1991). doi:10.1016/0550-3213(91)90407-O

40. N. Read, E. Rezayi. Beyond paired quantum Hall states: Paratemperatures and incompressible states in the first excited Landau level. Phys. Rev. B 59, 8084–8092 (1999). doi:10.1103/PhysRevB.59.8084

41. M. Freedman, Quantum computation and the localization of modular fucntors. arXiv:quant-ph/0003032 (28 March 2000).
90. D. S. Freed, M. J. Hopkins, Reflection positivity and invertible topological phases. arXiv:1604.06527 [hep-th] (22 April 2016).
91. T. Lan, L. Kong, X.-G. Wen, Classification of (2+1)-dimensional topological order and symmetry-protected topological order for bosonic and fermionic systems with on-site symmetries. Phys. Rev. B 95, 235140 (2017).

ACKNOWLEDGMENTS
I thank M. DeMarco for comments. Funding: This work was supported by NSF grants DMR-1506475 and DMS-1664412. Competing interests: None declared.

10.1126/science.aal3099