On a necessary and sufficient identification condition of optimal treatment regimes with an instrumental variable

Yifan Cui and Eric Tchetgen Tchetgen
National University of Singapore, University of Pennsylvania

Abstract

Unmeasured confounding is a threat to causal inference and individualized decision making. Similar to Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020); Han (2020), we consider the problem of identification of optimal individualized treatment regimes with a valid instrumental variable. Han (2020) provided an alternative identifying condition of optimal treatment regimes using the conditional Wald estimand of Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020) when treatment assignment is subject to endogeneity and a valid binary instrumental variable is available. In this note, we provide a necessary and sufficient condition for identification of optimal treatment regimes using the conditional Wald estimand. Our novel condition is necessarily implied by those of Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020); Han (2020) and may continue to hold in a variety of potential settings not covered by prior results.

keywords: Individualized decision making, policy making, optimal treatment regimes, unmeasured confounding, sign identification, conditional average treatment effect
1 Introduction

Estimating optimal treatment regimes is a central task for precision medicine. In the health sciences and medicine, an individualized treatment regime provides a personalized treatment strategy for each patient in the population based on individual characteristics. A prevailing strand of work has been devoted to estimating optimal treatment regimes (Robins et al. (2000); Murphy (2003); Qian and Murphy (2011); Zhao et al. (2012); Zhang et al. (2012a) and many others), we refer to Chakraborty and Moodie (2013); Kosorok and Laber (2019); Tsiatis et al. (2019) for an up-to-date literature review on dynamic treatment regimes.

Recently, there has been a fast-growing literature on estimating individualized treatment regimes based on observational studies subject to potential unmeasured confounding (Kallus and Zhou (2018); Yadlowsky et al. (2018); Kallus et al. (2019); Han (2019); Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020); Han (2020)). In particular, Cui and Tchetgen Tchetgen (2020) and Qiu et al. (2020) tackled the problem of individualized decision making/estimating individualized treatment regimes by leveraging a valid instrumental variable (IV) to account for potential unmeasured confounding. Han (2020) provided an alternative identifying condition under which the conditional Wald estimand on which identification is based in Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020) continues to identify optimal treatment regimes.

In this paper, we further relax sufficient identifying conditions considered by Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020); Han (2020), and introduce the concept of identifying optimal treatment regimes by only identifying the sign of conditional average treatment effect (CATE). We propose a necessary and sufficient identification condition based on the possibility of identifying the sign of the CATE from the conditional Wald estimand without necessarily being able to identify the CATE nor the population value function for any given treatment regime. We illustrate the result by exploring realistic scenarios in which conditional optimal treatment regime can be identified in settings in
which identifying conditions of Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020); Han (2020) are not met. It is notable that our Theorem 3.1 allows for identification of optimal treatment regimes even when an unmeasured confounding factor is an effect modifier of both the first stage association between the IV and the endogenous treatment, and of the causal effect of the endogenous treatment on the outcome; a possibility first recognized by Han (2020) which we have hereby significantly expanded upon.

To conclude this section, we briefly introduce notation used throughout the paper. Let \( Y \) denote the outcome of interest and \( A \in \{-1, 1\} \) be a binary treatment indicator. Throughout it is assumed without loss of generality that larger values of \( Y \) are more desirable. Suppose that \( U \) is an unmeasured confounder of the effect of \( A \) on \( Y \). Suppose also that one has observed a pre-treatment binary IV \( Z \in \{-1, 1\} \). Let \( L \) denote a set of fully observed pre-IV covariates. Throughout we assume the complete data are independent and identically distributed realizations of \((Y, L, A, Z, U)\); thus the observed data are \((Y, L, A, Z)\).

## 2 IV approaches to optimal treatment regimes

We wish to identify an optimal treatment regime \( \mathcal{D}^* \), which is a mapping from the patient-level covariate space to the treatment space \( \{-1, 1\} \) that maximizes the corresponding expected potential outcome for the entire population, i.e.,

\[
\mathcal{D}^* = \arg\max_{\mathcal{D}} E_L \left[ E_{Y_\mathcal{D}}[Y_{\mathcal{D}(L)}|L] \right],
\]

where \( Y_a \) is a person’s potential outcome under an intervention that sets the treatment to value \( a \), and \( Y_{\mathcal{D}(L)} \) is the potential outcome under a hypothetical intervention that assigns treatment according to the regime \( \mathcal{D} \), i.e.,

\[
Y_{\mathcal{D}(L)} \equiv Y_1 I\{\mathcal{D}(L) = 1\} + Y_{-1} I\{\mathcal{D}(L) = -1\},
\]
where $I\{\cdot\}$ is the indicator function. Optimal individualized treatment regimes can alternatively be written as

$$\mathcal{D}^*(L) = \text{sign}\{\gamma(L)\},$$

(1)

where $\gamma(L) \equiv E(Y_1 - Y_{-1}|L) \neq 0$. Throughout the paper, we make the standard consistency and positivity assumptions as in Cui and Tchetgen Tchetgen (2020).

A significant amount of work has been devoted to estimating optimal treatment regimes relying on the following unconfoundedness assumption:

**Assumption 1.** (Unconfoundedness) $Y_a \perp \perp A|L$ for $a = \pm 1$.

The assumption essentially rules out the existence of an unmeasured factor $U$ that confounds the effect of $A$ on $Y$ upon conditioning on $L$. It is straightforward to verify that under Assumption 1, one can identify the value function (Qian and Murphy, 2011) $E[Y_D(L)]$ for a given treatment regime $D$. Furthermore, optimal treatment regimes in Equation (1) are identified from the observed data

$$\mathcal{D}^*(L) = \text{sign}\{E(Y|A = 1, L) - E(Y|A = -1, L)\}.$$

As established by Qian and Murphy (2011), learning optimal treatment regimes under Assumption 1 can be formulated as

$$\mathcal{D}^* = \arg \max_D E\left[\frac{YI\{D(L) = A\}}{f(A|L)}\right].$$

Zhang et al. (2012b) proposed to directly maximize the value function over a parametrized set of functions. Rather than maximizing the above value function, Zhao et al. (2012); Zhang et al. (2012a); Rubin and van der Laan (2012) transformed the above problem into a weighted classification approach, which was shown to have appealing robustness properties, particularly in a randomized study where no model assumption on $Y$ is needed.
Instead of relying on Assumption 1, we allow for unmeasured confounding. Let \( Y_{z,a} \) denote the potential outcome had, possibly contrary to fact, a person’s IV and treatment value been set to \( z \) and \( a \), respectively. Suppose that the following assumption holds.

**Assumption 2.** (Latent unconfoundedness) \( Y_{z,a} \perp \perp (Z,A)|L,U \) for \( z,a = \pm 1 \).

This assumption essentially states that together \( U \) and \( L \) would in principle suffice to account for any confounding bias. Because \( U \) is not observed, we propose to account for it when a valid IV \( Z \) is available that satisfies the following standard IV assumptions (Angrist et al., 1996):

**Assumption 3.** (IV relevance) \( Z \not\perp \perp A|L \).

**Assumption 4.** (Exclusion restriction) \( Y_{z,a} = Y_a \) for \( z,a = \pm 1 \) almost surely.

**Assumption 5.** (IV independence) \( Z \perp \perp U|L \).

**Assumption 6.** (IV positivity) \( 0 < f(Z = 1|L) < 1 \) almost surely.

Assumptions 3-5 are well-known IV conditions, while Assumption 6 is needed for nonparametric identification (Greenland, 2000; Hernan and Robins, 2006). Assumption 3 requires that the IV is associated with the treatment conditional on \( L \). Note that Assumption 3 does not rule out confounding of the \( Z-A \) association by an unmeasured factor, however, if present, such factor must be independent of \( U \). Assumption 4 states that there can be no direct causal effect of \( Z \) on \( Y \) not mediated by \( A \). Assumption 5 states that the direct causal effect of \( Z \) on \( Y \) would be identified conditional on \( L \) if one could intervene on \( A = a \). Hereinafter, we refer to Assumptions 3-6 as core IV assumptions. Figure 1 provides a graphical representation of Assumptions 4 and 5.

These four core IV assumptions together do not suffice for point identification of the counterfactual mean and average treatment effect. Cui and Tchetgen Tchetgen (2020) showed that \( D^* \) is nonparametrically identified by

\[
\arg\max_{D} E \left[ \frac{ZAYI\{A = D(L)\}}{\delta(L)f(Z|L)} \right],
\]
Figure 1: A causal graph with unmeasured confounding. The bi-directed arrow between $Z$ and $A$ indicates the possibility that there may be unmeasured common causes confounding their association.

and

\[
\begin{align*}
\arg\max_D E \left[ \frac{Y I\{Z = D(L)\}}{\delta(L)f(Z|L)} \right],
\end{align*}
\]

under no unmeasured common effect modifier assumption, i.e.,

\[
\text{Cov} \left\{ \tilde{\delta}(L,U), \tilde{\gamma}(L,U) \mid L \right\} = 0,
\]

almost surely, where $\tilde{\delta}(L,U) \equiv \Pr(A = 1|Z = 1, L, U) - \Pr(A = 1|Z = -1, L, U)$ and $\tilde{\gamma}(L,U) \equiv E(Y_1 - Y_{-1}|L,U)$, respectively, or independent compliance type assumption, i.e.,

\[
\delta(L) \equiv \Pr(A = 1|Z = 1, L) - \Pr(A = 1|Z = -1, L) = \tilde{\delta}(L,U) \text{ almost surely.}
\]

\text{Han (2020)} proposed the following alternative identifying assumption for (2) and (3) to identify optimal treatment regimes given for a causal IV: The following two conditions hold given $L$,

(a) either $\tilde{\gamma}(L,U) \geq 0$ or $\tilde{\gamma}(L,U) \leq 0$ almost surely; and
(b) either $\tilde{\delta}(L,U) \geq 0$ or $\tilde{\delta}(L,U) \leq 0$ almost surely.
3 Individualized treatment regimes: Identifying the sign of CATE

We note that in order to identify optimal treatment regimes, one only needs to identify the sign of CATE. Therefore, in the following theorem, we provide sufficient and necessary conditions for Equations (2) and (3) identifying $D^*$, or equivalently, the conditional Wald estimand $C(L)/\delta(L)$ in [Wald (1940); Wang and Tchetgen Tchetgen (2018)] having the same sign as $\gamma(L)$, where $C(L) \equiv E(Y|Z = 1, L) - E(Y|Z = -1, L)$.

**Theorem 3.1.** Under Assumptions 2-6, the following condition is necessary and sufficient for Equation (2) or (3) identifying $D^*(L)$,

$$E\left(\frac{\tilde{\gamma}(L,U)}{\gamma(L)} \times \frac{\tilde{\delta}(L,U)}{\delta(L)} \Bigg| L\right) > 0,$$

or equivalently,

$$\text{Cov}\left(\frac{\tilde{\gamma}(L,U)}{\gamma(L)}, \frac{\tilde{\delta}(L,U)}{\delta(L)} \Bigg| L\right) > -1. \quad (5)$$

The necessary and sufficient identifying condition clearly holds if

$$\text{Cov}\left(\frac{\tilde{\gamma}(L,U)}{\gamma(L)}, \frac{\tilde{\delta}(L,U)}{\delta(L)} \Bigg| L\right) \geq 0, \quad (6)$$

which essentially states that $\frac{\tilde{\gamma}(L,U)}{\gamma(L)}$ and $\frac{\tilde{\delta}(L,U)}{\delta(L)}$ are positively correlated conditional on $L$.

This would hold for instance if both functions are either non-increasing or non-decreasing as functions of $U$ whenever $U$ is scalar. Importantly, this assumption may hold if either $\tilde{\gamma}(L,U)$ or $\tilde{\delta}(L,U)$ is positive for some values of $U$ and negative for other values conditional on $L$ such that the sign of say $\tilde{\delta}(L,U)$ does not always agree with that of $\delta(L)$ therefore invalidating the assumption of Han (2020) (even if a causal IV is assumed in Han (2020)).

The theorem establishes that optimal treatment regimes are identified by the sign
of the conditional Wald estimand (Wald, 1940; Wang and Tchetgen Tchetgen, 2018) in the setting where individuals’ decision to uptake the intervention is concordant with an anticipated benefit from the intervention. Specifically, consider the following definition of concordant treatment uptake with anticipated benefit from the intervention in the special case where \( U \) can be taken as univariate: a) \( E(Y_1 - Y_{-1}|L,U) \) and \( E(A|Z = 1, L,U) - E(A|Z = -1, L,U) \) are both non-decreasing or non-increasing in \( U \) so that as anticipated expected treatment benefit increases with \( U \), expected treatment uptake likewise increases in \( U \); b) \( \text{sign}\{E(Y_1 - Y_{-1}|L)\} = \text{sign}\{E(A|Z = 1, L) - E(A|Z = -1, L)\} \) so that conditional on \( L \), patients’ expected decision to uptake the intervention matches their expected treatment benefit. We refer to subjects with an ability to fulfill both a) and b) as rational agents with perfect anticipation. It is then clear according to the theorem that if all subjects are rational agents with perfect anticipation, optimal treatment regimes can be identified on the basis of the sign of the conditional Wald estimand. Importantly, the theorem allows for some portion of the population of subjects failing to be rational possibly due to imperfect anticipation of expected treatment benefit; provided that they do not offset the contribution to the conditional Wald estimand from rational agents with perfect anticipation of expected treatment benefit.

Our necessary and sufficient condition is also clearly implied by those of Cui and Tchetgen Tchetgen (2020); Qiu et al. (2020); Han (2020). The assumption of Cui and Tchetgen Tchetgen (2020) corresponds to the case \( \text{Cov}\{\tilde{\gamma}(L,U), \tilde{\gamma}(L,U)|L\} = 0 \); while that of Han (2020) given for a causal IV: Conditional on \( L \),

(a) either \( \tilde{\gamma}(L,U) \geq 0 \) or \( \tilde{\gamma}(L,U) \leq 0 \) almost surely; and
(b) either \( \tilde{\delta}(L,U) \geq 0 \) or \( \tilde{\delta}(L,U) \leq 0 \) almost surely;

imply Equation (4) as it implies the sign of \( \tilde{\gamma}(L,U) \) agrees with the sign of \( \gamma(L) \) almost surely and likewise the sign of \( \tilde{\delta}(L,U) \) agrees with that of \( \delta(L) \) almost surely.
4 Three levels of assumptions for identifying optimal treatment regimes

In this section, as a summary, one may categorize various assumptions for identifying optimal treatment regimes into three levels, namely depending on whether one can identify the value function, the CATE, or only the sign of CATE in Table 1.

In particular, as for identifying the CATE, Assumption A5b(1b) of Qiu et al. (2020) is a special case of Assumption 7 of Cui and Tchetgen Tchetgen (2020), while Assumption A5b(1a) of Qiu et al. (2020) relaxes IV independence to uncorrelated IV. As for identifying the sign of CATE, while both imply Equation (4) or (5), Assumption A of Han (2020) and Equation (6) do not necessarily imply each other. Moreover, nonparametric IV bounds such as Balke-Pearl bounds (Balke and Pearl, 1997) can be used to identify the sign of CATE when not covering zero (Cui and Tchetgen Tchetgen, 2020).

Interestingly, we point out that while, on the left side of Table 1, the quantity that is identified is stronger from bottom to top, identifying assumptions in the top do not necessarily imply those in the bottom on the right side. For instance, Assumption 7 of Cui and Tchetgen Tchetgen (2020) and Assumption A of Han (2020) do not necessarily imply each other because no heterogeneity in $U$ of the compliance type or the average additive treatment effect on the outcome implies Assumption 7 of Cui and Tchetgen Tchetgen (2020) without implying Assumption A of Han (2020); and the latter assumption does not necessarily imply the no common unmeasured effect modifier condition.

| Quantity Identified | Identifying Assumptions |
|---------------------|-------------------------|
| the value function  | Assumption 8 of Cui and Tchetgen Tchetgen (2020), Assumption A5b(2) of Qiu et al. (2020) |
| the CATE            | Assumption 7 of Cui and Tchetgen Tchetgen (2020), Assumption A5b(1) of Qiu et al. (2020) |
| the sign of CATE    | Assumption A of Han (2020), Equation (4) or (5) or (6), IV CATE bounds when not covering 0 |

Table 1: A categorization of various identifying assumptions in Cui and Tchetgen Tchetgen (2020), Qiu et al. (2020), and Han (2020) for identifying the value function, CATE, and sign of CATE.
Appendix

A Proof of Theorem 3.1

Proof. From the proof of Theorems 2.1 and 2.2 in Cui and Tchetgen Tchetgen (2020), we have

\[
E \left[ \frac{ZAYI\{A = \mathcal{D}(L)\}}{\delta(L)f(Z|L)} \right] \\
= E \left[ \frac{[\Pr(A = 1|Z = 1, L, U) - \Pr(A = 1|Z = -1, L, U)] I\{\mathcal{D}(L) = 1\} E[Y_1|L, U]}{\delta(L)} \right] \\
+ E \left[ \frac{[\Pr(A = 1|Z = 1, L, U) - \Pr(A = 1|Z = -1, L, U)] I\{\mathcal{D}(L) = -1\} E[Y_{-1}|L, U]}{\delta(L)} \right],
\]

and

\[
E \left[ \frac{YI\{Z = \mathcal{D}(L)\}}{\delta(L)f(Z|L)} \right] \\
= E \left[ \frac{[\Pr(A = 1|Z = 1, L, U) - \Pr(A = 1|Z = -1, L, U)] I\{\mathcal{D}(L) = 1\} E[Y_1|L, U]}{\delta(L)} \right] \\
+ E \left[ \frac{[\Pr(A = 1|Z = 1, L, U) - \Pr(A = 1|Z = -1, L, U)] I\{\mathcal{D}(L) = -1\} E[Y_{-1}|L, U]}{\delta(L)} \right] \\
+ E \left[ \frac{\Pr(A = 1|Z = -1, L, U) E[Y_1|L, U]}{\delta(L)} \right] \\
+ E \left[ \frac{\Pr(A = -1|Z = -1, L, U) E[Y_{-1}|L, U]}{\delta(L)} \right].
\]

It is easy to see that

\[
\arg\max_{\mathcal{D}} E \left[ \frac{ZAYI\{A = \mathcal{D}(L)\}}{\delta(L)f(Z|L)} \right],
\]

and

\[
\arg\max_{\mathcal{D}} E \left[ \frac{YI\{Z = \mathcal{D}(L)\}}{\delta(L)f(Z|L)} \right],
\]

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equal to

$$\arg\max_{\mathcal{D}} E \left[ \frac{\tilde{\gamma}(L,U)\tilde{\delta}(L,U)}{\delta(L)} \mathbb{I}\{D(L) = 1\} \right],$$

(7)

which is denoted by $\tilde{D}$. This completes the proof as $\tilde{D}(L)$ necessarily agrees with the sign of $\gamma(L)$ whenever Equation (4) holds.

The proof can also be conducted from the perspective of conditional Wald estimand. Recall that

$$C(L) = E\left\{ Y \mid Z = 1, L, U \right\} - E\left\{ Y \mid Z = -1, L, U \right\} | L \right\}$$

$$= E\left\{ Y_1 \frac{1 + A}{2} \mid Z = 1, L, U \right\} + E\left\{ Y_{-1} \frac{1 - A}{2} \mid Z = 1, L, U \right\} | L \right\}$$

$$- E\left\{ Y_1 \frac{1 + A}{2} \mid Z = -1, L, U \right\} + E\left\{ Y_{-1} \frac{1 - A}{2} \mid Z = -1, L, U \right\} | L \right\}$$

$$= E\left( E\left\{ Y_1 - Y_{-1} \mid L, U \right\} \left\{ \Pr[A = 1 \mid Z = 1, L, U] - \Pr[A = 1 \mid Z = -1, L, U] \right\} \mid L \right)$$

$$= E\left( \tilde{\gamma}(L,U)\tilde{\delta}(L,U) \mid L \right).$$

Thus, we have that

$$\frac{C(L)}{\delta(L)} = E\left( \frac{\tilde{\gamma}(L,U)\tilde{\delta}(L,U)}{\delta(L)} \right| L).$$

Subsequently,

$$\frac{C(L)}{\delta(L)}/\gamma(L) = E\left( \frac{\tilde{\gamma}(L,U)}{\gamma(L)} \times \frac{\tilde{\delta}(L,U)}{\delta(L)} \right| L).$$

Also note that

$$Cov\left( \frac{\tilde{\gamma}(L,U)}{\gamma(L)}, \frac{\tilde{\delta}(L,U)}{\delta(L)} \mid L \right) + 1$$

$$= E\left( \frac{\tilde{\gamma}(L,U)}{\gamma(L)} \times \frac{\tilde{\delta}(L,U)}{\delta(L)} \mid L \right).$$

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which completes the proof as the sign of $\gamma(L)$ will necessarily agree with that of the conditional Wald estimand whenever the expression in above display is positive.

\[\square\]

\section*{B Identifying assumptions in Table 1}

For easy referencing, we provide identifying assumptions appeared in Table 1:

- Assumption 7 of \textit{Cui and Tchetgen Tchetgen (2020)} \textit{(No unmeasured common effect modifier)}: $\text{Cov}\left\{\tilde{\delta}(L,U), \tilde{\gamma}(L,U)|L\right\} = 0$ almost surely.

- Assumption 8 of \textit{Cui and Tchetgen Tchetgen (2020)} \textit{(Independent compliance type)}: $\delta(L) = \tilde{\delta}(L,U)$ almost surely.

- Assumption A5b(1) of \textit{Qiu et al. (2020)}: Both conditions below hold:
  
  (a) (Uncorrelated IV) $\text{Cov}(Y_1, Z|L) = 0$ almost surely;
  (b) (No unmeasured treatment-outcome effect modification) $E[Y_1 - Y_{-1}|L,U] = E[Y_1 - Y_{-1}|L]$ almost surely.

- Assumption A5b(2) of \textit{Qiu et al. (2020)}: Both conditions below hold:
  
  (a) (Independent IV) $Z$ and $U$ are independent given $L$;
  (b) (Independent compliance) $E[AZ|Z = 1, L,U] - E[AZ|Z = -1, L,U] = E[A|Z = 1, L] - E[A|Z = -1, L]$ almost surely.

- Assumption A of \textit{Han (2020)}: The following two conditions hold given $L$,

  (a) either $E[Y_1|L,U] \geq E[Y_{-1}|L,U]$ or $E[Y_1|L,U] \leq E[Y_{-1}|L,U]$ almost surely; and
  (b) either $E[A_1|L,U] \geq E[A_{-1}|L,U]$ or $E[A_1|L,U] \leq E[A_{-1}|L,U]$ almost surely.

\section*{References}

Angrist, J. D., Imbens, G. W., and Rubin, D. B. (1996), “Identification of Causal Effects Using Instrumental Variables,” \textit{Journal of the American Statistical Association}, 91,
Balke, A. and Pearl, J. (1997), “Bounds on Treatment Effects from Studies with Imperfect Compliance,” *Journal of the American Statistical Association*, 92, 1171–1176.

Chakraborty, B. and Moodie, E. (2013), *Statistical methods for dynamic treatment regimes*, Springer.

Cui, Y. and Tchetgen Tchetgen, E. (2020), “A Semiparametric Instrumental Variable Approach to Optimal Treatment Regimes Under Endogeneity,” *Journal of the American Statistical Association*, 0, 1–12.

Greenland, S. (2000), “An introduction to instrumental variables for epidemiologists,” *International Journal of Epidemiology*, 29, 722–729.

Han, S. (2019), “Optimal Dynamic Treatment Regimes and Partial Welfare Ordering,” *arXiv preprint arXiv:1912.10014*.

— (2020), “Comment: Individualized Treatment Rules Under Endogeneity,” *Journal of the American Statistical Association*.

Hernan, M. and Robins, J. (2006), “Instruments for Causal Inference: An Epidemiologist’s Dream?” *Epidemiology (Cambridge, Mass.)*, 17, 360–72.

Kallus, N., Mao, X., and Zhou, A. (2019), “Interval Estimation of Individual-Level Causal Effects Under Unobserved Confounding,” in *Proceedings of Machine Learning Research*, eds. Chaudhuri, K. and Sugiyama, M., PMLR, vol. 89 of *Proceedings of Machine Learning Research*, pp. 2281–2290.

Kallus, N. and Zhou, A. (2018), “Confounding-robust policy improvement,” in *Advances in neural information processing systems*, pp. 9269–9279.

Kosorok, M. R. and Laber, E. B. (2019), “Precision Medicine,” *Annual Review of Statistics and Its Application*, 6, 263–286.
Murphy, S. A. (2003), “Optimal dynamic treatment regimes,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65, 331–355.

Qian, M. and Murphy, S. A. (2011), “Performance guarantees for individualized treatment rules,” *Annals of statistics*, 39, 1180.

Qiu, H., Carone, M., Sadikova, E., Petukhova, M., Kessler, R. C., and Luedtke, A. (2020), “Optimal Individualized Decision Rules Using Instrumental Variable Methods,” *Journal of the American Statistical Association*, 0, 1–18.

Robins, J. M., Hernán, M. A., and Brumback, B. A. (2000), “Marginal structural models and causal inference in epidemiology.” *Epidemiology*, 11 5, 550–60.

Rubin, D. B. and van der Laan, M. J. (2012), “Statistical issues and limitations in personalized medicine research with clinical trials,” *The international journal of biostatistics*, 8, 18.

Tsiatis, A. A., Davidian, M., Holloway, S. T., and Laber, E. B. (2019), *Dynamic Treatment Regimes: Statistical Methods for Precision Medicine*, CRC Press.

Wald, A. (1940), “The fitting of straight lines if both variables are subject to error,” *The annals of mathematical statistics*, 11, 284–300.

Wang, L. and Tchetgen Tchetgen, E. (2018), “Bounded, efficient and multiply robust estimation of average treatment effects using instrumental variables,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80, 531–550.

Yadlowsky, S., Namkoong, H., Basu, S., Duchi, J., and Tian, L. (2018), “Bounds on the conditional and average treatment effect with unobserved confounding factors,” *arXiv preprint arXiv:1808.09521*.

Zhang, B., Tsiatis, A. A., Davidian, M., Zhang, M., and Laber, E. (2012a), “Estimating optimal treatment regimes from a classification perspective,” *Stat*, 1, 103–114.
Zhang, B., Tsiatis, A. A., Laber, E. B., and Davidian, M. (2012b), “A robust method for estimating optimal treatment regimes,” *Biometrics*, 68, 1010–1018.

Zhao, Y., Zeng, D., Rush, A. J., and Kosorok, M. R. (2012), “Estimating individualized treatment rules using outcome weighted learning,” *Journal of the American Statistical Association*, 107, 1106–1118.