On B-type open-closed Landau-Ginzburg theories defined on Calabi-Yau Stein manifolds

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Abstract: We consider the bulk algebra and topological D-brane category arising from the differential model of the open-closed B-type topological Landau-Ginzburg theory defined by a pair \((X,W)\), where \(X\) is a non-compact Calabi-Yau manifold and \(W\) has compact critical set. When \(X\) is a Stein manifold (but not restricted to be a domain of holomorphy) we extract equivalent descriptions of the bulk algebra and of the category of topological D-branes which are constructed using only the analytic space associated to \(X\). In particular, we show that the D-brane category is described by projective factorizations defined over the ring of holomorphic functions of \(X\). We also discuss simplifications of the analytic models which arise when \(X\) is holomorphically parallelizable and illustrate these analytic models in a few classes of examples.

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1. Introduction

Quantum oriented open-closed B-type Landau-Ginzburg theories are non-anomalous quantum field theories conjecturally associated to pairs \((X,W)\), where \(X\) is a non-compact Calabi-Yau manifold (non-compact Kählerian manifold with trivial canonical line bundle) and \(W : X \to \mathbb{C}\) is a non-constant holomorphic function. Such theories should be obtained by quantization of the classical models constructed in [1,2]. A successful quantization must in particular associate to each pair \((X,W)\) an oriented two-dimensional open-closed TFT (topological field theory) in the sense axiomatized in [3,4,5], which in turn is equivalent with an algebraic structure known as a “TFT datum”. When the critical set of \(W\) is compact, non-rigorous path integral arguments
imply [2] that the TFT datum of such theories can be recovered from the cohomology of any member of a family of differential models built using a certain dg (differential graded) algebra \((PV(X), \delta_W)\) and a certain dg-category \(DF(X, W)\) of bundle-valued differential forms defined on \(X\) (see [6] for a rigorous construction of such models).

As already pointed out in [2], the differential models of the TFT datum associated to \((X, W)\) are quasi-isomorphic to each other and their cohomology admits an equivalent analytic description which is controlled by certain spectral sequences whose limits depend markedly on the geometry of \(X\) and on the nature of the critical locus of \(W\). For example, when \(X = \mathbb{C}^n\) and \(W\) has finite critical set, it was argued in [2] that such spectral sequences provide an equivalent analytic model of the TFT datum which is realized in terms of residues [7,8], a model which was studied rigorously in [9,10,11]. When \(X\) is a domain of holomorphy in \(\mathbb{C}^n\), some aspects of the differential and analytic models were studied in [12,13].

In this paper, we consider the problem of constructing equivalent analytic models for the TFT datum in the more general setting when \(X\) is an arbitrary Calabi-Yau Stein manifold (not restricted to be a domain of holomorphy). We first show that, without any restrictions on \(X\), the cohomological algebra \(HPV(X, W) \overset{\text{def}}{=} H(PV(X), \delta_W)\) admits an isomorphic analytic model constructed as the hypercohomology of the Koszul complex of the holomorphic 1-form \(-i\partial W\). When \(X\) is Stein and \(W\) has isolated critical points, the latter reduces to the global Jacobi algebra of \(W\), whose description simplifies further when \(X\) is holomorphically parallelizable.

When \(X\) is Stein, we also show that the cohomological category \(HDF(X, W) \overset{\text{def}}{=} H(DF(X, W))\) is equivalent with a simpler category \(HF(X, W) \overset{\text{def}}{=} H(F(X, W))\), where \(F(X, W)\) is another dg-category introduced in [6], which is defined using only analytic data. Through the Serre-Swan correspondence for Stein manifolds [14,15], the category \(HF(X, W)\) is itself equivalent with a category of projective factorizations. A projective factorization is a pair \((P, D)\), where \(P\) is a finitely-generated \(\mathbb{Z}_2\)-graded module over the algebra \(O(X)\) of complex-valued holomorphic functions defined on \(X\) and \(D\) is an odd endomorphism of \(P\) which squares to \(\text{Id}_P\). In the Stein case, we also obtain analytic models of the cohomological disk algebra \(HPV(X, a) \overset{\text{def}}{=} H(PV(X, a), \Delta_a)\) of a holomorphic factorization \(a\) of \(W\), which is defined [6] as the cohomology of a certain dg-algebra \((PV(X, a), \Delta_a)\). Finally, we illustrate these analytic models in a few classes of examples.

The paper is organized as follows. Section 2 describes some basic analytic objects associated to a pair \((X, W)\). Section 3 recalls the definition [6] of the dg-algebra \((PV(X), \delta_W)\) and of the dg-categories \(DF(X, W)\) and \(F(X, W)\), as well as of the differential graded disk algebra \((PV(X, a), \Delta_a)\). Sections 4, 5 and 6 discuss the equivalent analytic models of \(HPV(X, W)\), \(HDF(X, W)\) and \(HPV(X, a)\). Section 7 illustrates these analytic models in a few classes of examples. Appendix A summarizes some relevant properties of Stein manifolds.

1.1. Notations and conventions. We use the notations and conventions of [6, Subsection 1.1]. Given a commutative ring \(R\), let \(\text{Mod}_R\) denote the category of \(R\)-modules and \(\text{mod}_R\) denote the full sub-category of finitely-generated \(R\)-modules. Let \(\text{Proj}_R\) denote the full subcategory of \(\text{Mod}_R\) consisting of projective \(R\)-modules and \(\text{proj}_R\) denote the full subcategory of \(\text{mod}_R\) consisting of finitely-generated projective \(R\)-modules. All manifolds considered are smooth, paracompact, connected and of non-zero dimension and all vector bundles considered are smooth.

2. Landau-Ginzburg pairs

Definition 2.1 A Landau-Ginzburg (LG) pair of dimension \(d\) is a pair \((X, W)\), where:
A. \( X \) is a non-compact Kählerian manifold of complex dimension \( d \) which is Calabi-Yau in the sense that the canonical line bundle \( K_X \) is holomorphically trivial.

B. \( W : X \to \mathbb{C} \) is a non-constant complex-valued holomorphic function defined on \( X \).

The signature \( \mu(X, W) \) is the mod 2 reduction of \( d \):

\[
\mu(X, W) \overset{\text{def}}{=} d \in \mathbb{Z}_2 .
\]

Let \( (X, W) \) be a Landau-Ginzburg pair. Let \( \mathcal{O}_X \) denote the sheaf of locally-defined complex-valued holomorphic functions and \( \mathcal{O}(X) = \Gamma(X, \mathcal{O}_X) = H^0(\mathcal{O}_X) \) denote the ring of globally-defined holomorphic functions from \( X \) to \( \mathbb{C} \). Let:

\[
\iota_W \overset{\text{def}}{=} -i(\partial W) : TX \to \mathcal{O}_X
\]

denote the morphism of sheaves of \( \mathcal{O}_X \)-modules given by left contraction with \(-i\partial W\), where we identify the holomorphic tangent bundle \( TX \) with its locally-free sheaf of holomorphic sections.

**Definition 2.2** The critical set of \( W \) is defined through:

\[
Z_W \overset{\text{def}}{=} \{ p \in X | (\partial W)(p) = 0 \} .
\]

The critical sheaf of \( W \) is the ideal sheaf:

\[
\mathcal{J}_W \overset{\text{def}}{=} \text{im}(\iota_W : TX \to \mathcal{O}_X) \subset \mathcal{O}_X .
\]

The critical ideal of \( (X, W) \) is the following ideal of the commutative ring \( \mathcal{O}(X) \):

\[
J(X, W) \overset{\text{def}}{=} \mathcal{J}_W(X) = \iota_W(\Gamma(X, TX)) \subset \mathcal{O}(X) .
\]

Notice that \( i\partial W \in \Gamma(X, T^*X) \) is a holomorphic section of the holomorphic cotangent bundle \( T^*X \) and that \( Z_W \) is the vanishing locus of this section. For any open subset \( U \subset X \) supporting local complex coordinates \( z^1, \ldots, z^d \), the ideal \( \mathcal{J}_W(U) \subset \mathcal{O}_X(U) \) is generated by the partial derivatives \( \frac{\partial W}{\partial z^1}, \ldots, \frac{\partial W}{\partial z^d} \in \mathcal{O}_X(U) \).

**Definition 2.3** The Jacobi sheaf of \( W \) is the sheaf of commutative \( \mathcal{O}_X \)-algebras defined through:

\[
\text{Jac}_W \overset{\text{def}}{=} \mathcal{O}_X / \mathcal{J}_W .
\]

The Jacobi algebra \( \text{Jac}(X, W) \) of the LG pair \( (X, W) \) is the commutative \( \mathcal{O}(X) \)-algebra of globally-defined sections of the Jacobi sheaf:

\[
\text{Jac}(X, W) \overset{\text{def}}{=} \text{Jac}_W(X) = H^0(\text{Jac}_W) .
\]

Let \( \mathcal{O}_X \to \text{Jac}_W \) denote the projection map. The Jacobi sheaf is supported on the critical set \( Z_W \) and the restriction \( \mathcal{O}_{Z_W} \overset{\text{def}}{=} \text{Jac}_W|_{Z_W} \) makes \( Z_W \) into an analytic subspace \( (Z_W, \mathcal{O}_{Z_W}) \) of the analytic space \( (X, \mathcal{O}_X) \), which we call the Jacobi space of \( W \).
Definition 2.4 The sheaf Koszul complex of $W$ is the following complex of locally-free sheaves of $\mathcal{O}_X$-modules:

$$(\mathcal{K}_W): \ 0 \to \wedge^d TX \overset{\iota_W}{\to} \wedge^{d-1} TX \overset{\iota_W}{\to} \ldots \overset{\iota_W}{\to} \mathcal{O}_X \to 0,$$  

(2.2)

where $\mathcal{O}_X$ sits in degree zero and we identify the exterior power $\wedge^k TX$ with its locally-free sheaf of holomorphic sections.

With our convention, $\mathcal{K}_W$ is concentrated in non-positive degrees.

Proposition 2.5 Assume that the critical set of $W$ is finite, i.e. $\dim_\mathbb{C} Z_W = 0$. Then the sequence:

$$0 \to \wedge^d TX \overset{\iota_W}{\to} \wedge^{d-1} TX \overset{\iota_W}{\to} \ldots \overset{\iota_W}{\to} TX \overset{\iota_W}{\to} \mathcal{O}_X \to \text{Jac}_W \to 0 \ (2.3)$$

is exact and thus provides a resolution of $\text{Jac}_W$ through locally free sheaves of $\mathcal{O}_X$-modules. In particular, the sheaf Koszul complex (2.2) is exact except at the last term.

Proof. Since $\dim_\mathbb{C} Z_W = 0$, we have $\text{codim}_\mathbb{C} Z = d = \dim_\mathbb{C} T^*X$, which means that $-i\partial W$ is a regular section of $T^*X$. Hence the sheaf sequence:

$$0 \to \wedge^d TX \overset{\iota_W}{\to} \wedge^{d-1} TX \overset{\iota_W}{\to} \ldots \overset{\iota_W}{\to} TX \overset{\iota_W}{\to} \mathcal{J}_W \to 0$$

is exact in the Abelian category $\text{Coh}(X)$ of coherent sheaves of $\mathcal{O}_X$-modules, being a resolution of the ideal sheaf $\mathcal{J}_W$ of $(Z_W, \mathcal{O}_{Z_W})$ (see [16, page 5]). Since $\mathcal{J}_W = \ker(\mathcal{O}_X \to \text{Jac}_W)$, this implies the conclusion. $\square$

Remark 2.1. Assume that $X$ is Stein. Then the critical set $Z_W$ is compact iff it is a finite set. Indeed, $Z_W$ is a subvariety of $X$ and a Stein manifold does not admit compact subvarieties of positive dimension (see Theorem 3 on page 152 in [24, Chap V.4]).

3. Some structures of the differential model

Let $(X, W)$ be a Landau-Ginzburg pair of dimension $d$. Using path integral arguments, it was argued in [2] that, when the critical set $Z_W$ is compact, the TFT datum [6] of the B-type open-closed topological Landau-Ginzburg theory defined by $(X, W)$ admits a family of cochain-level realizations. The differential models proposed in [2] were studied from a mathematical perspective in [6], to which we refer the reader for details. They involve a certain $\mathcal{O}(X)$-linear dg-algebra $(\text{PV}(X), \delta_W)$ and a certain dg-category $\text{DF}(X, W)$, as well as cochain-level realizations of the bulk and boundary traces and bulk-boundary and boundary-bulk maps of the TFT datum. In this section, we briefly recall the definition of these structures and of the differential graded disk algebra of [6]. We also recall the definition of a dg-category $\text{F}(X, W)$ discussed in loc. cit, which will arise in later sections. Notice that some of these structures can be defined without assuming compactness of $Z_W$ (and in this paper they are considered in this more general situation), though their Physics interpretation is less clear unless one adds that assumption.

3.1. The differential graded bulk algebra. The $C^\infty(X)$-module:

$$\text{PV}(X) \overset{\text{def.}}{=} \mathcal{A}(X, \wedge TX) \simeq \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, \wedge TX)$$

carries a natural multiplication which makes it into a unital and associative $C^\infty(X)$-algebra. For any $i = -d, \ldots, 0$ and $j = 0, \ldots, d$, let:

$$\text{PV}^{i,j}(X) \overset{\text{def.}}{=} \mathcal{A}^i(X, \wedge^{|i|} TX)$$
and set $PV^{i,j}(X) = 0$ for $i \not\in \{-d, \ldots, 0\}$ or $j \not\in \{0, \ldots, d\}$. Then the decomposition:

$$PV(X) \overset{\text{def}}{=} \bigoplus_{i=-d}^{d} \bigoplus_{j=0}^{d} PV^{i,j}(X)$$

makes $PV(X)$ into a unital associative $\mathbb{Z} \times \mathbb{Z}$-graded $C^\infty(X)$-algebra, whose grading is concentrated in bidegrees $(i,j)$ satisfying $i \in \{-d, \ldots, 0\}$ and $j \in \{0, \ldots, d\}$. The canonical $\mathbb{Z}$-grading of $PV(X)$ is the total grading of this bigrading:

$$PV^k(X) \overset{\text{def}}{=} \bigoplus_{i+j=k} PV^{i,j}(X) \quad (k \in \mathbb{Z}) ,$$

while the canonical $\mathbb{Z}_2$-grading is the mod 2 reduction of the former:

$$PV^0(X) \overset{\text{def}}{=} \bigoplus_{k=\text{ev}} PV^k(X) ,$$

$$PV^1(X) \overset{\text{def}}{=} \bigoplus_{k=\text{odd}} PV^k(X) .$$

We trivially extend $\iota_W$ to a map from $PV(X)$ to $PV(X)$ denoted by the same symbol. Let $\overline{\partial} := \overline{\partial}_{TX} : PV(X) \to PV(X)$ be the Dolbeault differential of the holomorphic vector bundle $\wedge TX$. Then $(PV(X), \iota_W, \overline{\partial})$ is a bicomplex. By definition, the twisted differential $\delta_W : PV(X) \to PV(X)$ is the total differential of this bicomplex:

$$\delta_W \overset{\text{def}}{=} \overline{\partial} + \iota_W .$$

**Definition 3.1** The twisted Dolbeault algebra of polyvector-valued forms of the LG pair $(X, W)$ is the supercommutative $\mathbb{Z}$-graded $O(X)$-linear dg-algebra $(PV(X), \delta_W)$, where $PV(X)$ is endowed with the canonical $\mathbb{Z}$-grading. The cohomological twisted Dolbeault algebra of $(X, W)$ is the total cohomology algebra:

$$HPV(X, W) \overset{\text{def}}{=} H(PV(X), \delta_W) .$$

### 3.2. Differential graded categories of holomorphic factorizations.

**Definition 3.2** A holomorphic factorization of $W$ is a pair $a = (E, D)$, where $E = E^0 \oplus E^1$ is a holomorphic vector superbundle on $X$ and $D \in \Gamma(X, \text{End}^1(E))$ is an odd holomorphic section of $\text{End}(E)$ which satisfies $D^2 = \text{Wid}_E$.

Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$ of $W$, the space $\mathcal{A}(X, Hom(E_1, E_2)) \simeq \mathcal{A}(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, Hom(E_1, E_2))$ is $\mathbb{Z} \times \mathbb{Z}_2$-graded and carries two differentials, namely the Dolbeault differential $\overline{\partial}_{a_1,a_2} := \overline{\partial}_{\text{Hom}(E_1,E_2)}$ and the defect differential $\partial_{a_1,a_2}$. The latter is the $C^\infty(X)$-linear endomorphism of $\mathcal{A}(X, Hom(E_1, E_2))$ determined by the condition:

$$\partial_{a_1,a_2}(\rho \otimes f) = (-1)^{rk\rho} \rho \otimes (D_2 \circ f) - (-1)^{rk\rho + \sigma(f)} \rho \otimes (f \circ D_1)$$

for all pure rank forms $\rho \in \mathcal{A}(X)$ and all pure $\mathbb{Z}_2$-degree elements $f \in \Gamma_\infty(X, Hom(E_1, E_2))$, where $\sigma(f)$ denotes the $\mathbb{Z}_2$-degree of $f$. The Dolbeault and defect differentials square to zero.
and anticommut e, making $A(X, \text{Hom}(E_1, E_2))$ into a $\mathbb{Z} \times \mathbb{Z}_2$-graded bicomplex. The twisted differential $\delta_{a_1,a_2}$ is defined through:

$$\delta_{a_1,a_2} \overset{\text{def.}}{=} \partial_{a_1,a_2} + d_{a_1,a_2}.$$ 

The total $\mathbb{Z}_2$-grading of $A(X, \text{Hom}(E_1, E_2))$ is given by the sum of the $\mathbb{Z}_2$-degree with the mod 2 reduction of the $\mathbb{Z}$-degree. Notice that $\delta_{a_1,a_2}$ is odd with respect to this $\mathbb{Z}_2$-grading.

**Definition 3.3** The twisted Dolbeault category of holomorphic factorizations of $W$ is the $\mathbb{Z}_2$-graded $O(X)$-linear dg-category $DF(X,W)$ defined as follows:

- The objects are the holomorphic factorizations of $W$.
- Given two objects $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$, the module of morphisms from $a_1$ to $a_2$ is:

$$\text{Hom}_{DF(X,W)}(a_1,a_2) \overset{\text{def.}}{=} A(X, \text{Hom}(E_1, E_2)),$$

endowed with the total $\mathbb{Z}_2$-grading and with the twisted differential $\delta_{a_1,a_2}$.
- The composition of morphisms is given by the wedge product of bundle-valued forms.

Let:

$$HF(X,W) \overset{\text{def.}}{=} H(DF(X,W))$$

denote the total cohomology category of $DF(X,W)$. We will show in Section 5 that $HF(X,W)$ admits a simpler description when $X$ is a Stein manifold, being equivalent with the category $HF(X,W)$ defined below.

**Definition 3.4** The holomorphic dg-category of holomorphic factorizations is the $\mathbb{Z}_2$-graded $O(X)$-linear dg-category $F(X,W)$ defined as follows:

- The objects are the holomorphic factorizations of $W$.
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$ of $W$, the module of morphisms from $a_1$ to $a_2$ is:

$$\text{Hom}_{F(X,W)}(a_1,a_2) \overset{\text{def.}}{=} \Gamma(X, \text{Hom}(E_1, E_2)),$$

endowed with the $\mathbb{Z}_2$-grading, with homogeneous components:

$$\text{Hom}_{F(X,W)}^\kappa(a_1,a_2) \overset{\text{def.}}{=} \Gamma(X, \text{Hom}^\kappa(E_1, E_2)),$$

and with the differentials $\partial_{a_1,a_2}$ determined uniquely by the condition:

$$\partial_{a_1,a_2}(f) \overset{\text{def.}}{=} D_2 \circ f - (-1)^\kappa f \circ D_1, \quad \forall f \in \Gamma(X, \text{Hom}^\kappa(E_1, E_2)),$$

$\forall \kappa \in \mathbb{Z}_2$.
- The composition of morphisms is the obvious one.

Let:

$$HF(X,W) \overset{\text{def.}}{=} H(F(X,W))$$

denote the total cohomology category of $F(X,W)$, viewed as a $\mathbb{Z}_2$-graded $O(X)$-linear category.

**Remark 3.1.** When the critical set $Z_W$ is compact, the path integral arguments of [2] imply [6] that the cohomological category $HF(X,W)$ can be identified with the category of topological D-branes of the corresponding open-closed topological field theory. In this case, the cohomology algebra $HPV(X,W)$ can be identified with the bulk algebra of the same theory.
3.3. The disk algebra of a holomorphic factorization. Fix a holomorphic factorization \( a = (E, D) \) of \( W \) and set \( \overline{\delta}_a := \overline{\delta}_{a,a} = \delta_{End(E)} \), \( a_a := [D, \cdot] \) and \( \delta_a := \delta_{a,a} = \overline{\delta}_a + a_a \). Consider the \( \mathbb{Z}_2 \)-graded unital associative \( C^\infty(X) \)-algebra:

\[
PV(X, \text{End}(E)) \overset{\text{def}}{=} A(X, \wedge TX \otimes \text{End}(E)) \simeq PV(X) \otimes_{C^\infty(X)} \Gamma_\infty(X, \text{End}(E)) ,
\]

where \( \otimes_{C^\infty(X)} \) denotes the graded tensor product and \( PV(X) \) is endowed with the canonical \( \mathbb{Z}_2 \)-grading. The twisted disk differential \( \Delta_a \) is the odd \( O(X) \)-linear differential on \( PV(X, \text{End}(E)) \) defined through:

\[
\Delta_a \overset{\text{def}}{=} \delta_W \otimes A(X) id_{A(X,\text{End}(E))} + \text{id}_{PV(X)} \otimes A(X) \delta_a .
\]

**Definition 3.5** The differential graded disk algebra of a holomorphic factorization \( a = (E, D) \) is the \( O(X) \)-linear \( \mathbb{Z}_2 \)-graded unital dg-algebra \( (PV(X, \text{End}(E)), \Delta_a) \). The cohomological disk algebra \( HPV(X, a) \) of \( a \) is the total cohomology algebra:

\[
HPV(X, a) \overset{\text{def}}{=} H(PV(X, \text{End}(E)), \Delta_a) .
\]

Let \( \overline{\delta} := \overline{\delta}_{\wedge TX \otimes \text{End}(E)} : PV(X, \text{End}(E)) \to PV(X, \text{End}(E)) \) be the Dolbeault differential\(^1\) of the holomorphic vector bundle \( \wedge TX \otimes \text{End}(E) \). Then:

\[
\Delta_a = \overline{\delta} + \iota_W + a_a ,
\]

where we trivially extended \( \iota_W \) and \( a_a \) to mutually anti-commuting differentials in \( PV(X, \text{End}(E)) \)[6]. Notice that \( \overline{\delta} \) anticommutes with \( \iota_W \) and \( a_a \).

4. Analytic models for the cohomological twisted Dolbeault algebra \( HPV(X, W) \)

Let \( (X, W) \) be a Landau-Ginzburg pair of dimension \( d \).

4.1. The generally-valid analytic model. Let \( \mathbb{H}(\mathcal{K}_W) \) denote the hypercohomology of the sheaf Koszul complex (2.2), viewed as a finite complex of sheaves of \( O_X \)-modules concentrated in non-positive degrees.

**Proposition 4.1** There exists a natural isomorphism of \( \mathbb{Z} \)-graded \( O(X) \)-modules:

\[
HPV(X, W) \simeq_{O(X)} \mathbb{H}(\mathcal{K}_W) ,
\]

where \( HPV(X, W) \) is endowed with the canonical \( \mathbb{Z} \)-grading. Thus:

\[
H^k(PV(X), \delta_W) \simeq_{O(X)} \mathbb{H}^k(\mathcal{K}_W) , \quad \forall k \in \{-d, \ldots, d\} .
\]  
(4.1)

Moreover, we have:

\[
\mathbb{H}^k(\mathcal{K}_W) = \bigoplus_{i+j=k} E_{i,j}^\infty ,
\]  
(4.2)

where \( E_{i,j}^\infty \) is the limit of the spectral sequence \( E := (E_{i,j}, d_r)_{r \geq 0} \) which starts with:

\[
E_{i,j}^0 \overset{\text{def}}{=} PV^{i,j}(X) = A^j(X, \wedge [i] TX) , \quad d_0 \overset{\text{def}}{=} \overline{\delta} := \overline{\delta}_{\wedge TX} , \quad (i = -d, \ldots, 0 , \quad j = 0, \ldots, d) .
\]  
(4.3)

The zeroth page of this sequence is shown in Diagram 1.

\(^1\) Note that symbol \( \overline{\delta} \) will be used to simplify notation in three different cases: \( \overline{\delta} := \overline{\delta}_{\wedge TX \otimes \text{End}(E)} \) for the disk algebra, \( \overline{\delta} := \overline{\delta}_{\wedge TX} \) in the algebra of polyvector-valued forms and \( \overline{\delta} := \overline{\delta}_{a_1, a_2} := \overline{\delta}_{\text{End}(E_1, E_2)} \) in Section 5.
Diagram 1

Proof. To compute the hypercohomology of $K_W$, we can use the Dolbeault resolutions of the sheaves $\wedge^k TX$:

\[ 0 \to \wedge^k TX \xrightarrow{\delta} A^1 \otimes \wedge^k TX \xrightarrow{\delta} \ldots \xrightarrow{\delta} A^d \otimes \wedge^k TX \to 0 \]

(4.4)

where $A^i$ are the sheaves of smooth $(0,j)$-forms on $X$ (viewed as sheaves of $\mathcal{O}_X$-modules by restriction of scalars) and the tensor product is taken over $\mathcal{O}_X$. Notice that $(PV(X), \iota_W, \partial)$ coincides with the $\mathbb{Z} \times \mathbb{Z}$-graded bicomplex whose node $(i,j)$ is given by $A^i \otimes \wedge^j TX$ and whose horizontal and vertical differentials are given respectively by $\iota_W$ and $\partial$. This provides an acyclic resolution of the sheaf Koszul complex $K_W$ defined in (2.2). The hypercohomology of $K_W$ coincides with the total cohomology of this bicomplex, which in turn coincides with HPV($X, W$). Thus (4.1) holds. The spectral sequence $(E^{i,j}_r, d_r)_{r \geq 0}$ is the spectral sequence determined by the bicomplex described above, showing that (4.2) holds. □

4.2. Analytic models of HPV($X, W$) for the Stein case.

Theorem 4.2 Suppose that $X$ is Stein. Then the spectral sequence $E$ defined above collapses at the second page $E_2$ and HPV($X, W$) is concentrated in non-positive degrees. For all $k = -d, \ldots, 0$, the $\mathcal{O}(X)$-module $HPV^k(X, W) \defeq H^k(PV(X), \delta_W)$ is isomorphic with the cohomology at position $k$ of the following sequence of finitely-generated projective $\mathcal{O}(X)$-modules:

\[ (K_W): \quad 0 \to H^0(\wedge^d TX) \xrightarrow{\iota_W} H^0(\wedge^{d-1} TX) \xrightarrow{\iota_W} \ldots \xrightarrow{\iota_W} H^0(TX) \xrightarrow{\iota_W} \mathcal{O}(X) \to 0 \]

(4.5)

where $\mathcal{O}(X)$ sits in position zero.

Proof. Since $X$ is Stein, Cartan’s theorem B implies $E^{i,j}_1 \defeq H^j_{\partial}(A(X, \wedge^i TX)) = 0$ for $j > 0$ and all $i = -d, \ldots, 0$. Thus the only non-trivial row of the page $E_1$ of the spectral sequence is the bottom row $E_{1}^{0,0}$ with differential $d_1 \defeq \iota_W$, whose nodes are given by:

\[ E_{1}^{0,0} = H^0_{\partial}(A(X, \wedge^i TX)) = H^0_{\partial}(PV^{i,0}(X)) = \Gamma(X, \wedge^i TX) = H^0(\wedge^i TX) \]

This is because the sheaf $A^k \otimes E$ is fine (and hence soft and acyclic) for any holomorphic vector bundle $E$ defined on $X$, where we identify $E$ with its locally-free sheaf of holomorphic sections.
for all \( i = -d, \ldots, 0 \). Thus page \( E_1 \) reduces to:

\[
\begin{array}{cccccc}
E_i^{d,d} & \longrightarrow & E_i^{d+1,d} & \longrightarrow & E_i^{d+2,d} & \longrightarrow & E_i^{d,0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_i^{d-1,1} & \longrightarrow & E_i^{d+1,1} & \longrightarrow & E_i^{d+2,1} & \longrightarrow & E_i^{d,1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_i^{d-2,2} & \longrightarrow & E_i^{d+1,2} & \longrightarrow & E_i^{d+2,2} & \longrightarrow & E_i^{d,2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_i^{d-3,3} & \longrightarrow & E_i^{d+1,3} & \longrightarrow & E_i^{d+2,3} & \longrightarrow & E_i^{d,3} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\end{array}
\]

Diagram 2

We note that all differentials \( d_2 \) of \( E_\bullet^\bullet \) will be trivial, since they are maps between different rows. Hence the spectral sequence collapses at \( E_2 \) and we have \( E_\infty^k = E_2^k = H^k(K_W) \) for all \( k = -d, \ldots, 0 \). Since \( \wedge^k TX \) are vector bundles, the Serre-Swan theorem for Stein manifolds \([14,15]\) implies that (4.5) is a sequence of finitely-generated projective \( O(X) \)-modules. ☐

**Proposition 4.3** Suppose that \( X \) is Stein and \( \dim C Z_W = 0 \). Then \( HPV^k(X,W) = 0 \) for \( k \neq 0 \) and there exists a natural isomorphism of \( O(X) \)-modules:

\[
HPV^0(X,W) \simeq_{O(X)} H^0(JacW) = Jac(X,W) .
\]

Moreover, we have an isomorphism of \( O(X) \)-algebras:

\[
Jac(X,W) \simeq O(X)/J(X,W) . \tag{4.6}
\]

**Proof.** Lemma 2.5 implies that the hypercohomology \( H^k(K_W) \) of the Koszul complex (2.2) coincides with sheaf cohomology of the Jacobi sheaf \( JacW \):

\[
H^k(K_W) = H^k(JacW) . \tag{4.7}
\]

Combining this with (4.1) gives:

\[
HPV^k(X,W) \simeq_{O(X)} H^k(JacW) , \quad \forall k \in Z . \tag{4.8}
\]

Thus \( HPV^k(X,W) = 0 \) for \( k \neq 0 \) by Cartan’s Theorem B and \( HPV^0(X,W) \simeq_{O(X)} H^0(JacW) = Jac(X,W) \). Since \( X \) is Stein, Theorem 4.2 shows that \( HPV^0(X,W) \) also coincides with the cohomology of the sequence (4.5) at position zero, which equals \( O(X)/\text{im}(\iota_W : H^0(TX) \to O(X)) = O(X)/J(X,W) \). This shows that (4.6) holds. ☐

**Remark 4.1.** In applications to Physics, the set \( Z_W \) is usually assumed to be compact, since this condition implies \([6]\) finite-dimensionality of \( HPV(X,W) \) as well as Hom-finiteness of the category \( \text{HDF}(X,W) \) over \( C \). In such applications, the condition \( \dim C Z_W = 0 \) is automatically satisfied when \( X \) is Stein, provided that the critical set is compact (see Remark 2.1).
4.3. Analytic model of HPV(X,W) when X is Stein and holomorphically parallelizable with \( \dim_{\mathbb{C}} Z_W = 0 \). Recall that X is called holomorphically parallelizable if its holomorphic tangent bundle TX is holomorphically trivial. This happens if X admits global holomorphic frames\(^3\).

**Lemma 4.4** Suppose that X is holomorphically parallelizable and let \( u_1, \ldots, u_d \) be any holomorphic frame of X. Then the critical sheaf \( \mathcal{J}_W \) is generated on every open subset of X by the restrictions of the holomorphic functions \( u_1(W), \ldots, u_d(W) \in O(X) \). In particular, the critical ideal \( J(X,W) \) is generated by these holomorphic functions:

\[
J(X,W) = \langle u_1(W), \ldots, u_d(W) \rangle.
\]

**Proof.** The holomorphic frame \( u_1, \ldots, u_d \) of X induces an isomorphism of locally-free sheaves:

\[
\psi : \mathcal{O}_{\bar{X}}^{\oplus d} \xrightarrow{\sim} TX
\]

given on open subsets \( U \subset X \) by:

\[
\psi_U(f_1, \ldots, f_d) \overset{\text{def}}{=} \sum_{i=1}^{d} f_i u_i \in \Gamma(U,TX), \quad \forall f_1, \ldots, f_d \in \mathcal{O}_X(U).
\]

Since \( \imath_W(u_i) = -iu_i(W) \in O(X) \), the morphism of sheaves \( \imath_W : TX \to O_X \) can be identified with the morphism \( \tau : \mathcal{O}_{\bar{X}}^{\oplus d} \to O_X \) which is given on open sets \( U \) by:

\[
\tau_U(f_1, \ldots, f_d) = -\sum_{i=1}^{d} f_i [u_i(W)]|_{U} \in \mathcal{O}_X(U), \quad \forall f_1, \ldots, f_d \in \mathcal{O}_X(U).
\]

This implies \( \mathcal{J}_W = \imath \tau = \sum_{i=1}^{d} \mathcal{O}_X u_i(W) \) and in particular \( J(X,W) = \mathcal{J}_W(X) = \sum_{i=1}^{d} O(X) u_i(W) \). Thus \( J(X,W) \) coincides with the ideal \( \langle u_1(W), \ldots, u_d(W) \rangle \) generated inside \( O(X) \) by the holomorphic functions \( u_i(W) \). \( \square \)

**Proposition 4.5** Let \( (X,W) \) be a Landau-Ginzburg pair such that X is Stein and holomorphically parallelizable and such that \( \dim_{\mathbb{C}} Z_W = 0 \). Let \( u_1, \ldots, u_d \) be any holomorphic frame of X. Then \( \text{HPV}(X,W) = \text{HPV}^0(X,W) \simeq_{O(X)} \text{Jac}(X,W) \) and we have:

\[
\text{Jac}(X,W) = O(X)/\langle u_1(W), \ldots, u_d(W) \rangle.
\]

**Proof.** Follows immediately from Lemma 4.4 and Proposition 4.3. \( \square \)

5. Analytic models for the category HDF(X,W)

Let \( (X,W) \) be a Landau-Ginzburg pair of dimension \( d \).

5.1. A generally-valid analytic model. Let \( a_1 \overset{\text{def}}{=} (E_1,D_1) \) and \( a_2 \overset{\text{def}}{=} (E_2,D_2) \) be two holomorphic factorizations of W. Let \( \mathcal{F} := \mathcal{F}_{a_1,a_2} \) and \( \mathcal{D} := \mathcal{D}_{a_1,a_2} \) be the Dolbeault and defect differentials on \( \text{Hom}_{\text{HDF}(X,W)}(a_1,a_2) = A(X,Hom(E_1,E_2)) \). Let \( \delta := \delta_{a_1,a_2} \) be the twisted differential. Consider the complex \( \text{Hom}_{\text{HDF}(X,W)}(a_1,a_2,\delta) \) (endowed with the total \( \mathbb{Z}_2 \)-grading), whose total cohomology equals the \( \mathbb{Z}_2 \)-graded \( O(X) \)-module \( \text{Hom}_{\text{HDF}(X,W)}(a_1,a_2) \).

\(^3\) A global holomorphic frame of X is a family of globally-defined holomorphic vector fields \( u_1, \ldots, u_d \in \Gamma(X,TX) \) such that the vectors \( u_1(x), \ldots, u_d(x) \) form a basis of \( T_x X \) for any point \( x \in X \).
**Definition 5.1** The unwinding of the $\mathbb{Z} \times \mathbb{Z}_2$-graded bicomplex $(A(X, \text{Hom}(E_1, E_2)), \mathfrak{g}, \mathfrak{d})$ is the $\mathbb{Z} \times \mathbb{Z}$-graded complex $C^{\bullet, \bullet}$ of $O(X)$-modules with homogeneous components:

$$C^{i,j} \overset{\text{def}}{=} A^j(X, \text{Hom}^i(E_1, E_2)),$$ 

whose non-trivial horizontal and vertical differentials are $\mathfrak{d}$ and $\nabla$, both of which have degree $+1$ (see Diagram 3). This bicomplex is 2-periodic in the horizontal direction and concentrated in degrees $0, \ldots, d$ in the vertical direction, thus $C^{i,j} = 0$ for $j \not\in \{0, \ldots, d\}$ and all $i$.

Diagram 3

Consider the total complex $C^\bullet$ of the bicomplex $C^{\bullet, \bullet}$. Its homogeneous components are:

$$C^k \overset{\text{def}}{=} \bigoplus_{i+j=k} C^{i,j}, \quad \forall k \in \mathbb{Z}$$

and its differential equals $\delta = \nabla + \mathfrak{d}$. Note that the sum on the right hand side has only a finite number of nonzero terms. The complex $C^\bullet$ is 2-periodic:

$$C^k = C^{k+2}, \quad \forall k \in \mathbb{Z}$$

and the same holds for its cohomology $H^k(C^\bullet)$:

$$H^k(C^\bullet) = H^{k+2}(C^\bullet), \quad \forall k \in \mathbb{Z}.$$  

We have:

$$H^k(C) = H^k(A(X, \text{Hom}(E_1, E_2)), \delta) = \text{Hom}^k_{\text{HDF}(X,W)}(a_1, a_2), \quad \forall k \in \mathbb{Z}. \quad (5.1)$$

Consider the spectral sequence $E = (E^{r, \bullet}_r, d_r)_{r \geq 0}$ whose zeroth page is given by:

$$E^{0,j}_r = C^{i,j} = A^j(X, \text{Hom}^i(E_1, E_2))$$

dowered with the vertical differential $d_0 \overset{\text{def}}{=} \nabla$ in Diagram 3.

**Proposition 5.2** The spectral sequence $E$ defined above converges. Moreover, we have a natural isomorphism of $O(X)$-modules:

$$\bigoplus_{i+j=t} E^{i,j}_{d+2} \simeq_{O(X)} \text{Hom}_{\text{HDF}(X,W)}^i(a_1, a_2), \quad \forall t \in \mathbb{Z}.$$
The only nonzero row of \( E \). Hence all differentials \( d \) rows. Thus the spectral sequence degenerates at \( E \).

Since \( X \) is Stein, Cartan’s Theorem B yields:

\[
\text{Combining this with (5.1) gives the conclusion.} \quad \Box
\]

Remark 5.1. For a pair of holomorphic factorizations \((a_1, a_2)\) the limit of the spectral sequence \( E \) defined above looks like the ”hypercohomology” of the following 2-periodic complex of locally-free sheaves:

\[
(Q_{a_1, a_2}) : \ldots \to Hom^{1}(E_1, E_2) \xrightarrow{\partial} Hom^{0}(E_1, E_2) \xrightarrow{\partial} Hom^{1}(E_1, E_2) \to \ldots ,
\]

where \( Hom^{0}(E_1, E_2) \) sits in even positions. Indeed, the columns give (2-periodic) bounded acyclic (Dolbeault) resolutions of nodes \( Hom^{0}(E_1, E_2) \) and \( Hom^{1}(E_1, E_2) \), while \( Hom_{HDF(X,W)}(a_1, a_2) \) is isomorphic to the total cohomology of such bicomplex. However, the notion of hypercohomology for unbounded complexes is ambiguous (see [18]).

5.2. An analytic model of \( HDF(X, W) \) when \( X \) is Stein. Recall the category \( HF(X, W) \) defined in Subsection 3.2.

**Lemma 5.3** Suppose that \( X \) is Stein. Then the spectral sequence \( E \) defined in (5.1) degenerates at \( E_2 \) and we have an isomorphism of \( \mathbb{Z}_2 \)-graded \( O(X) \)-modules:

\[
\text{Hom}_{HDF(X,W)}(a_1, a_2) \cong_{O(X)} \text{Hom}_{HF(X,W)}(a_1, a_2) .
\]

**Proof.** The first page of the spectral sequence is given by:

\[
E_1^{i,j} \text{ def } \text{H}(E_0^{i,j}, \overline{\partial}) = \text{H}(\mathcal{A}^{i}(X, Hom^{i}(E_1, E_2)), \overline{\partial}) = H^{i}_{\overline{\partial}}(Hom^{i}(E_1, E_2)) .
\]

Since \( X \) is Stein, Cartan’s Theorem B yields:

\[
E_1^{i,j} = 0 \quad \text{for } j > 0 \text{ and } i \in \mathbb{Z} .
\]

Hence all differentials \( d \) of the next page \( E_2 \) must vanish, since they are maps between different rows. Thus the spectral sequence degenerates at \( E_2 \) and we have:

\[
\text{Hom}_{HDF(X,W)}^{i}(a_1, a_2) \cong_{O(X)} \bigoplus_{i+j=t} E_2^{i,j} = \bigoplus_{i+j=t} \text{H}(E_1^{i,j}, d_1) , \quad \forall t \in \mathbb{Z} .
\]

The only nonzero row of \( E_1 \) is the bottom row \( E_1^{i,0} \), which is a 2-periodic sequence with nodes \( E_1^{i,0} = H^{i}_{\overline{\partial}}(Hom^{i}(E_1, E_2)) = \Gamma(X, Hom^{i}(E_1, E_2)) \) and differential \( d_1 \) def. \( \partial \):

\[
\ldots \to E_1^{i-1,0} \xrightarrow{\partial} E_1^{i,0} \xrightarrow{\partial} E_1^{i+1,0} \to \ldots
\]
The cohomology of this sequence at node $i$ equals $H^i_{\mathcal{B}}(\Gamma(X, \text{Hom}^i(E_1, E_2))) = \text{Hom}_{\text{HF}(X,W)}^i(a_1, a_2)$. Hence (5.4) reduces to:

$$\text{Hom}_{\text{HDF}(X,W)}^i(a_1, a_2) \simeq_{O(X)} H(E_1^t, \mathcal{D}) = H^i_{\mathcal{B}}(\Gamma(X, \text{Hom}(E_1, E_2))) = \text{Hom}_{\text{HF}(X,W)}^i(a_1, a_2),$$

for all $t \in \mathbb{Z}$, which gives (5.3). □

**Theorem 5.4** Suppose that $X$ is Stein. Then $\text{HDF}(X,W)$ and $\text{HF}(X,W)$ are equivalent as $\mathbb{Z}_{2}$-graded $O(X)$-linear categories.

**Proof.** It is easy to see that the isomorphism of Lemma 5.3 is natural with respect to $a_1$ and $a_2$ and that it preserves units. □

5.3. **Relation to projective analytic factorizations in the Stein case.** Recall that $O(X) = O_X(X)$ denotes the commutative ring of complex-valued holomorphic functions defined on $X$.

**Definition 5.5** An $O(X)$-supermodule is a $\mathbb{Z}_2$-graded $O(X)$-module $M$ endowed with a direct sum decomposition $M = M^0 \oplus M^1$ into submodules.

Notice that $O(X)$-supermodules form an $O(X)$-linear $\mathbb{Z}_2$-graded category $\text{Mod}_{O(X)}^s$ if we define the Hom space $\text{Hom}_{O(X)}(M_1, M_2)$ from a supermodule $M_1$ to a supermodule $M_2$ to be the $\mathbb{Z}_2$-graded $O(X)$-module with homogeneous components:

$$\text{Hom}_{O(X)}^0(M_1, M_2) \overset{\text{def}}{=} \text{Hom}_{O(X)}(M_1^0, M_2^0) \oplus \text{Hom}_{O(X)}(M_1^1, M_2^1),$$

$$\text{Hom}_{O(X)}^1(M_1, M_2) \overset{\text{def}}{=} \text{Hom}_{O(X)}(M_1^0, M_2^1) \oplus \text{Hom}_{O(X)}(M_1^1, M_2^0). \quad (5.5)$$

The composition is defined in the obvious manner. Given an $O(X)$-supermodule $M$, let

$$\text{End}_{O(X)}(M) \overset{\text{def}}{=} \text{Hom}_{O(X)}(M, M).$$

**Definition 5.6** An $O(X)$-supermodule $M = M^0 \oplus M^1$ is called finitely-generated if both of its $\mathbb{Z}_2$-homogeneous components $M^0$ and $M^1$ are finitely-generated over $O(X)$. It is called projective if both $M^0$ and $M^1$ are projective $O(X)$-modules.

Finitely-generated $O(X)$-supermodules form a full $\mathbb{Z}_2$-graded $O(X)$-linear subcategory $\text{mod}^s_{O(X)}$ of $\text{Mod}^s_{O(X)}$, while projective and finitely-generated $O(X)$-supermodules form a full $\mathbb{Z}_2$-graded $O(X)$-linear subcategory $\text{proj}^s_{O(X)}$ of $\text{mod}^s_{O(X)}$.

**Definition 5.7** A projective analytic factorization of $W$ is a pair $(P, D)$, where $P$ is a finitely-generated projective $O(X)$-supermodule and $D \in \text{End}_{O(X)}^1(P)$ is an odd endomorphism of $P$ such that $D^2 = \text{Wid}P$.

**Definition 5.8** The dg-category $\text{PF}(X,W)$ of projective analytic factorizations of $W$ is the $\mathbb{Z}_2$-graded $O(X)$-linear dg-category defined as follows:

- The objects are the projective analytic factorizations of $W$. 


Given two projective analytic factorizations \((P_1, D_1)\) and \((P_2, D_2)\) of \(W\), we set:

\[
\text{Hom}_{\text{PF}(X,W)}((P_1, D_1), (P_2, D_2)) \overset{\text{def}}{=} \text{Hom}_{\text{O}(X)}(P_1, P_2),
\]

endowed with the \(\mathbb{Z}_2\)-grading \((5.5)\) inherited from \(\text{mod}^\mathbb{Z}_2\text{O}(X)\) and with the \(\text{O}(X)\)-linear odd differential \(\mathfrak{d} := \mathfrak{d}_{(P_1, D_1), (P_2, D_2)}\) determined uniquely by the condition:

\[
\mathfrak{d}(f) \overset{\text{def}}{=} D_2 \circ f - (-1)^{\deg f} f \circ D_1
\]

for all elements \(f \in \text{Hom}_{\text{O}(X)}(P_1, P_2)\) which have pure \(\mathbb{Z}_2\)-degree.

The composition of morphisms is inherited from \(\text{mod}^\mathbb{Z}_2\text{O}(X)\).

The cohomological category \(\text{HPF}(X,W)\) of analytic projective factorizations of \(W\) is the total cohomology category:

\[
\text{HPF}(X,W) \overset{\text{def}}{=} \text{H}(\text{PF}(X,W))
\]

which is a \(\mathbb{Z}_2\)-graded \(\text{O}(X)\)-linear category.

Let us assume that \(X\) is Stein. Then the Serre-Swan theorem for Stein manifolds \([14,15]\) states that the functor \(\Gamma_X \overset{\text{def}}{=} \Gamma(X, \cdot)\) of taking global holomorphic sections gives an equivalence of \(\text{O}(X)\)-linear categories:

\[
\Gamma_X : \text{VB}(X) \overset{\sim}{\rightarrow} \text{proj}_{\text{O}(X)},
\]

where \(\text{proj}_{\text{O}(X)}\) is the category of finitely-generated projective \(\text{O}(X)\)-modules. This induces a \((\text{O}(X)\)-linear, degree zero) dg-functor \(\Gamma_X : \text{F}(X,W) \rightarrow \text{PF}(X,W)\) which sends a holomorphic factorization \((E, D)\) of \(W\) into the projective factorization \(\Gamma_X(E, D) \overset{\text{def}}{=} (\Gamma(X, E), D)\), where \(D \in \Gamma(X, \text{End}^1(E)) \cong_{\text{O}(X)} \text{End}^1_{\text{O}(X)}(\Gamma(X, E))\).

The proof of the following statement is immediate:

**Proposition 5.9** Assume that \(X\) is Stein. Then the dg-functor \(\Gamma_X\) is an equivalence of \(\mathbb{Z}_2\)-graded \(\text{O}(X)\)-linear dg-categories between \(\text{F}(X,W)\) and \(\text{PF}(X,W)\). In particular, the \(\mathbb{Z}_2\)-graded \(\text{O}(X)\)-linear cohomological categories \(\text{HF}(X,W)\) and \(\text{HPF}(X,W)\) are equivalent.

Theorem 5.4 and Proposition 5.9 imply that the categories \(\text{HDF}(X,W)\) and \(\text{HPF}(X,W)\) are equivalent when \(X\) is Stein.

### 5.4. Free holomorphic factorizations and analytic matrix factorizations.

**Definition 5.10** A holomorphic vector bundle \(E\) on \(X\) with \(\text{rk}_{\mathbb{C}} E = r\) is called holomorphically trivial if it is isomorphic (as a holomorphic vector bundle) with the trivial holomorphic vector bundle \(\mathcal{O}_X^r\). A holomorphic vector superbundle \(E = E^0 \oplus E^1\) on \(X\) is called holomorphically trivial if both its even and odd sub-bundles \(E^0\) and \(E^1\) are holomorphically trivial.

**Remark 5.2.** Two holomorphic vector bundles \(E\) and \(F\) defined on \(X\) are isomorphic in the category \(\text{VB}(X)\) iff they are isomorphic in the usual category of holomorphic vector bundles defined on \(X\). Indeed, \(\text{VB}(X)\) is equivalent with the full subcategory of \(\text{Coh}(X)\) consisting...
of locally-free sheaves of finite rank. Also, an isomorphism of sheaves between the sheaves of holomorphic sections $E$ and $F$ of $E$ and $F$ has trivial kernel and image equal to $F$, which means that the corresponding isomorphism in the category $VB(X)$ is an ordinary isomorphism of vector bundles (since its kernel and image are sub-bundles of $E$ and $F$, respectively). In particular, a vector bundle $E$ is holomorphically trivial iff it is isomorphic with a trivial vector bundle in the category $VB(X)$.

Let $VB_{triv}(X)$ denote the full subcategory of $VB(X)$ whose objects are the holomorphically trivial holomorphic vector bundles defined on $X$ and $VB^s_{triv}(X)$ denote the full subcategory of $VB^s(X)$ whose objects are the holomorphically trivial holomorphic vector superbundles defined on $X$.

**Remark 5.3.** Suppose that $X$ is Stein. Then the Oka-Grauert principle [19,20] implies that a holomorphic vector bundle $E$ is holomorphically trivial iff it is topologically trivial, i.e. isomorphic with a trivial vector bundle in the category of complex vector bundles defined on $X$.

**Definition 5.11** A holomorphic factorization $(E,D)$ of $W$ is called free if the holomorphic vector superbundle $E$ is holomorphically trivial.

**Definition 5.12** The category $F_{\text{free}}(X,W)$ of free holomorphic factorizations of $W$ is the full $\mathbb{Z}_2$-graded $O(X)$-linear dg-subcategory of the category $F(X,W)$ whose objects are the free holomorphic factorizations of $W$. The cohomological category $HF_{\text{free}}(X,W)$ of free holomorphic factorizations of $W$ is the $\mathbb{Z}_2$-graded $O(X)$-linear category defined as the total cohomology category of $F_{\text{free}}(X,W)$:

$$HF_{\text{free}}(X,W) \overset{\text{def}}{=} \text{H}(F_{\text{free}}(X,W))$$

**Definition 5.13** An $O(X)$ supermodule $M = M^0 \oplus M^1$ is called free if its even and odd submodules $M^0$ and $M^1$ are free $O(X)$-modules.

Let $\text{free}_{O(X)}$ denote the full subcategory of $\text{proj}_{O(X)}$ consisting of those finitely-generated projective $O(X)$-modules which are free. Let $\text{free}_{O(X)}^s$ denote the full subcategory of $\text{proj}_{O(X)}^s$ consisting of those finitely-generated projective $O(X)$-supermodules which are free.

**Definition 5.14** An analytic matrix factorization of $W$ is a pair $(M,D)$, where $M$ is a free and finitely-generated projective $O(X)$-supermodule and $D \in \text{End}_{O(X)}(M)$ is an odd endomorphism of $M$ such that $D^2 = \text{Wid}_M$.

**Definition 5.15** The dg-category $MF(X,W)$ of analytic matrix factorizations of $W$ is the full $\mathbb{Z}_2$-graded $O(X)$-linear dg-subcategory of $PF(X,W)$ whose objects are the analytic matrix factorizations of $W$. The cohomological category $HMF(X,W)$ of analytic matrix factorizations is the $\mathbb{Z}_2$-graded $O(X)$-linear category defined as the total cohomology category of $MF(X,W)$:

$$HMF(X,W) \overset{\text{def}}{=} \text{H}(MF(X,W))$$

When $X$ is Stein, the Serre-Swan equivalence $\Gamma_X : VB(X) \xrightarrow{\sim} \text{proj}_{O(X)}$ restricts to an equivalence of $O(X)$-linear categories between $VB_{triv}(X)$ and $\text{free}_{O(X)}$. This implies:

**Proposition 5.16** Assume that $X$ is Stein. Then the equivalence of categories $\Gamma_X : F(X,W) \rightarrow PF(X,W)$ restricts to an equivalence of $\mathbb{Z}_2$-graded $O(X)$-linear categories between $F_{\text{free}}(X,W)$ and $MF(X,W)$. In particular, the $\mathbb{Z}_2$-graded $O(X)$-linear categories $HF_{\text{free}}(X,W)$ and $HMF(X,W)$ are equivalent.
Corollary 5.17 Assume that $X$ is Stein and that any holomorphic vector bundle defined on $X$ is topologically trivial. Then $F(X,W) = F_{\text{free}}(X,W)$ and the $\mathbb{Z}_2$-graded $O(X)$-linear dg-categories $F(X,W)$ and $MF(X,W)$ are equivalent. In particular, the $\mathbb{Z}_2$-graded $O(X)$-linear categories $HF(X,W)$ and $HMF(X,W)$ are equivalent.

Proof. By the Oka-Grauert principle [19,20], topological triviality of a holomorphic vector bundle $E$ implies holomorphic triviality of $E$. Thus the hypothesis implies $F(X,W) = F_{\text{free}}(X,W)$. The remaining statements follow immediately from the results above. □

Remark 5.4. In general, the category $HF(X,W)$ has many more objects than the category $HF_{\text{free}}(X,W)$, since a generic Calabi-Yau Stein manifold $X$ has many holomorphic vector bundles which are not topologically trivial.

6. Analytic models for the cohomological disk algebra $HPV(X,a)$

Let $(X,W)$ be a Landau-Ginzburg pair of dimension $d$. Fix a holomorphic factorization $a = (E,D)$ of $W$ and set $\overline{\delta}_a := \overline{\delta}_{a,a} = \overline{\delta}_{\text{End}(E)}$, $\delta_a := \delta_{a,a} = [D,\cdot]$ and $\delta_a := \delta_{a,a} = \overline{\delta}_a + \delta_a$. Recall from Subsection 3.3 that we have:

$$\Delta_a = \overline{\delta} + \vartheta_a ,$$

where $\overline{\delta} := \overline{\delta}_{TX \otimes \text{End}(E)}$ and

$$\vartheta_a \text{ def.} = t_W + \delta_a .$$

Notice that $(\vartheta_a)^2 = 0$ and that $\vartheta_a$ anticommutes with $\overline{\delta}$. Consider the 2-periodic complex of holomorphic vector bundles:

$$(P_a) : \ldots \xrightarrow{\vartheta_a} \bigoplus_{s+t=k} \wedge^{|t|} TX \otimes \text{End}^k(E) \xrightarrow{\vartheta_a} \bigoplus_{s+t=k+1} \wedge^{|t|} TX \otimes \text{End}^k(E) \xrightarrow{\vartheta_a} \ldots$$

and the 2-periodic sequence of projective $O(X)$-modules:

$$(P_a) : \ldots \xrightarrow{\vartheta_a} H^0( \bigoplus_{s+t=k} \wedge^{|t|} TX \otimes \text{End}^k(E) ) \xrightarrow{\vartheta_a} H^0( \bigoplus_{s+t=k+1} \wedge^{|t|} TX \otimes \text{End}^k(E) ) \xrightarrow{\vartheta_a} \ldots ,$$

where $P^k_a \text{ def.} = \bigoplus_{s+t=k} \wedge^{|t|} TX \otimes \text{End}^k(E)$ and $P^k_a = H^0(P^k_a)$ sit in position $k$ and $s \in \mathbb{Z}$, $t \in \mathbb{Z}_{\leq 0}$.

Proposition 6.1 Suppose that $X$ is Stein. Then for each $k$, the $O(X)$-module $HPV^k(X,a)$ is naturally isomorphic with the cohomology at position $k$ of the sequence of projective $O(X)$-modules $(6.1)$.

Proof. The bicomplex $(PV(X,End(E)), \vartheta_a, \overline{\delta})$ can unwind to a horizontally 2-periodic bicomplex $^{1}C^{\bullet\bullet}$ with vertical differential $\overline{\delta}$ and horizontal differential $\vartheta_a$, where:

$$^{1}C^{i,j} \text{ def.} = \bigoplus_{t+s=i} A^j(X, \wedge^{|t|} TX \otimes \text{End}^k(E)) , \quad \forall s,j \in \mathbb{Z} , \quad \forall t \in \mathbb{Z}_{\leq 0} .$$

We have $^{1}C^{i,j} = 0$ unless $j \in \{0,\ldots,d\}$, so this complex is vertically bounded. Its associated spectral sequence $^{1}E \text{ def.} = (^{1}E^r_{i,j}, ^{1}d_r)_{r \geq 0}$ has zeroth page given by:

$$^{1}E^0_{i,j} \text{ def.} = ^{1}C^{i,j} , \quad ^{1}d_0 \text{ def.} = \overline{\delta} .$$

(6.2)
The columns are the Dolbeault resolutions of $P_a^i = \bigoplus_{t+s=i} \wedge |t| TX \otimes \text{End}^s(E)$. This spectral sequence converges since the bicomplex $^{1\mathcal{C}\mathcal{C\bullet}}$ is vertically bounded. Since $X$ is Stein, Cartan’s Theorem B implies $H^j_{\mathcal{D}}(\wedge |s| TX \otimes \text{End}^s(E)) = 0$ for $j > 0$. Thus on page 1 the spectral sequence is concentrated at the zeroth row ($j = 0$) and has differential $1 \mathcal{D}_1 \overset{\text{def}}{=} \vartheta_a$. It follows that $1 \mathcal{D}_2 = 0$, since these differentials are maps between different rows. Thus $^{1\mathcal{E}}$ degenerates at the second page and:

$$H^k_{\Delta_a}(PV(X, \text{End}(E))) = H^k_{\vartheta_a}(({^{1\mathcal{E}}}_0^1)) = H^k(P_a) \ . \ (6.3)$$

$\square$

Even without the assumption that $X$ be Stein, the cohomology of the complex (6.1) can itself be computed using another spectral sequence. Indeed, the decomposition $\vartheta_a = \iota_W + \mathfrak{d}_a$ implies:

$$P_a^i = \bigoplus_{s+t=i} 2\mathcal{C}^s,t \ ,$$

where $^{2\mathcal{C}\mathcal{C\bullet}}$ is the bicomplex of $O(X)$-modules defined through:

$$^{2\mathcal{C}}_{s,t} \overset{\text{def}}{=} H^0(\wedge |s| TX \otimes \text{End}^t(E)) \ , \ \forall s \in \mathbb{Z} \ , \ \forall t \in \mathbb{Z}_{\leq 0} \ (6.4)$$

with vertical differential $\iota_W$ and horizontal differential $\mathfrak{d}_a$. Consider the following spectral sequence $^{2\mathcal{E}} = (^{2\mathcal{E}}_{s,t}^r, 2\mathcal{D}_r)_{r \geq 0}$ with zeroth page defined by:

$$^{2\mathcal{E}}_{0,t} \overset{\text{def}}{=} ^{2\mathcal{C}}_{s,t} \ , \ \ 2\mathcal{D}_0 \overset{\text{def}}{=} \iota_W \ , \ (6.5)$$

with $^{2\mathcal{C}\mathcal{C\bullet}}$ as in (6.4).

Lemma 6.2 The spectral sequence $^{2\mathcal{E}}$ defined above degenerates at page at most $d + 2$ and converges to the cohomology of the complex of projective $O(X)$-modules (6.1).

Proof. The complex $^{2\mathcal{C}\mathcal{C\bullet}}$ is horizontally 2-periodic and vertically bounded, since $^{2\mathcal{E}}_{0,t}^r$ vanishes for $t \not\in \{-d, \ldots, 0\}$. The differentials $2\mathcal{D}_r : {^{2\mathcal{E}}_{s,t}^r} \to {^{2\mathcal{E}}_{s+r,t-r+1}}$ vanish for $r = d + 2$ and all $s$ and $t$ since $t$ and $t - r + 1$ cannot both lie in $\{-d, \ldots, 0\}$. Hence, the spectral sequence is convergent and it degenerates at page at most $d + 2$. By construction, the limit of $^{2\mathcal{E}}$ equals the cohomology of $P_a$. $\square$

Proposition 6.3 Suppose that $X$ is Stein. Then the spectral sequence $^{2\mathcal{E}}$ defined above degenerates at the second page and $^{2\mathcal{E}}_2$ has nodes given by:

$$^{2\mathcal{E}}_{2,t} \simeq_{O(X)} \text{HPV}^t(X, W) \otimes_{O(X)} \text{End}^t_{\text{HF}(X,W)}(a) \ . \ (6.6)$$

Proof. Recall the sequence $K_W$ of projective $O(X)$-modules defined in (4.5). Since $X$ is Stein, we have $H^0(\wedge |t| TX \otimes \text{End}^t(E)) \simeq_{O(X)} H^0(\wedge |s| TX) \otimes_{O(X)} H^0(\text{End}^s(E))$ (see [14, page 403]. Hence (6.5) reduces to:

$$^{2\mathcal{E}}_{0,t}^0 = H^0(\wedge |t| TX) \otimes_{O(X)} H^0(\text{End}^t(E)) = K_W \otimes_{O(X)} H^0(\text{End}^t(E)) \ ,$$

with differential $2\mathcal{D}_0 \overset{\text{def}}{=} \iota_W$. Thus:

$$^{2\mathcal{E}}_{1,t}^0 = H^1(K_W) \otimes_{O(X)} H^0(\text{End}^t(E)) = H^1(K_W) \otimes_{O(X)} \Gamma(X, \text{End}^t(E))$$

with first page differentials $2\mathcal{D}_1 \overset{\text{def}}{=} \mathfrak{d}_a$. This implies that the second page has nodes given by:

$$^{2\mathcal{E}}_{2,t}^0 = H^1(K_W) \otimes_{O(X)} \text{End}^t_{\text{HF}(X,W)}(a) \ . \ (6.7)$$

Since $X$ is Stein, Theorem 4.2 gives $H^1(K_W) \simeq_{O(X)} \text{HPV}^t(X, W)$, so (6.7) reduces to (6.6). $\square$
Proposition 6.4 Suppose that $X$ is Stein and $\dim_{\mathbb{C}} Z_{W} = 0$. Then the spectral sequence $^{2}\mathbb{E}$ degenerates at the second page and there exists a natural isomorphism of $\mathbb{Z}_{2}$-graded $O(X)$-modules:

$$
HPV(X,a) \simeq_{O(X)} H_{0,a}(H_{2W}(^{2}\mathbb{C}^{\bullet,\bullet})) \simeq_{O(X)} \text{Jac}(X,W) \otimes_{O(X)} \text{End}_{HF(X,W)}(a) .
$$

(6.8)

Proof. Since $X$ is Stein and $\dim_{\mathbb{C}} Z_{W} = 0$, Proposition 4.3 gives $HPV(X,W) \simeq \text{Jac}(X,W)$, where the right hand side is concentrated in degree 0. Now the statement follows easily from Proposition 6.3. \qed

7. Some examples

7.1. Domains of holomorphy in $\mathbb{C}^{d}$. Let $X = U \subseteq \mathbb{C}^{d}$ be a domain of holomorphy$^{4}$. In this case, $U$ is Stein and holomorphically parallelizable (and hence Calabi-Yau). Moreover, $U$ admits integrable holomorphic frames given by $u_{i} = \partial_{i}$, where $\partial_{i} := \frac{\partial}{\partial z_{i}}$ and $z_{1}, \ldots, z_{d}$ is a system of globally-defined complex coordinates on $U$. Assume that $W \in O(U)$ has isolated critical points. Then Proposition 4.5 gives:

$$
HPV(U,W) = HPV^{0}(U,W) \simeq \text{Jac}(U,W) = O(U)/\langle \partial_{1}W, \ldots, \partial_{d}W \rangle ,
$$

thereby recovering a result of [12]. Let us further assume that $U$ is contractible. Then any finitely-generated projective $O(U)$-module is free$^{5}$. In this case, projective analytic factorizations coincide with analytic matrix factorizations and $HDF(U,W)$ is equivalent with the $\mathbb{Z}_{2}$-graded cohomological category $HMF(U,W)$ of analytic matrix factorizations of $W$ by Theorem 5.4 and Corollary 5.17. Proposition 6.4 gives $HPV(U,a) \simeq \text{Jac}(U,W) \otimes_{O(U)} \text{End}_{HF(U,W)}(a)$ for any holomorphic factorization $a$. The simplest situation is obtained for $U = \mathbb{C}^{d}$. In that case, $O(U) = O(\mathbb{C}^{d})$ is the ring of entire functions of $d$ variables, which is already very rich [21]. For $d = 1$, we have $O(U) = O(\mathbb{C})$, which is much larger than the ring $O_{alg}(\mathbb{C}) = \mathbb{C}[z]$ of univariate polynomials with complex coefficients. Indeed, Liouville’s theorem implies that an entire function $f \in O(\mathbb{C})$ is a polynomial iff it has at most a pole singularity at infinity. Thus $O(\mathbb{C}) \setminus O_{alg}(\mathbb{C})$ consists of all entire functions with an essential singularity at infinity (a simple example of which is the exponential function $f(z) = e^{z}$). There appears to exist no good reason (apart from mere convenience) to require $W$ to be a polynomial.

7.2. Non-compact Riemann surfaces. Let $X$ be a non-compact Riemann surface. As recalled in Appendix A, every such surface is Stein by a result of Behnke and Stein [26]. Moreover, any holomorphic vector bundle on $X$ is holomorphically trivial [27, Theorem 30.3]. In particular, $X$ is holomorphically parallelizable and hence Calabi-Yau. Since $X$ is complex one-dimensional, holomorphic parallelizability implies existence of a global complex coordinate $z$ on $X$.

Proposition 7.1 Let $W \in O(X)$ be a non-constant holomorphic function. Then:

$$
HPV(X,W) = HPV^{0}(X,W) \simeq \text{Jac}(X,W) = O(X)/\langle \frac{dW}{dz} \rangle .
$$

Moreover, there exist equivalences of $O(X)$-linear $\mathbb{Z}_{2}$-graded categories:

$$
HDF(X,W) \simeq HF(X,W) \simeq HF_{free}(X,W) \simeq HMF(X,W) .
$$

---

$^{4}$ By the Cartan-Thullen theorem, a domain $U \subseteq \mathbb{C}^{d}$ is Stein iff it is holomorphically convex, i.e. iff it is a domain of holomorphy. Notice that domains of holomorphy are very special cases of Stein manifolds.

$^{5}$ In this case, any complex vector bundle defined on $U$ is topologically trivial so the Oka-Grauert principle [19, 20] implies that any holomorphic vector bundle defined on $U$ is holomorphically trivial.
For any holomorphic factorization $a$ of $W$, we have:

$$\text{HPV}(X, a) \simeq_{\text{O}(X)} \text{Jac}(X, W) \otimes \text{End}_{\text{HF}(X, W)}(a).$$

**Proof.** Notice that $\dim_{\mathbb{C}} Z_W = 0$ since $W$ is non-constant. The conclusion now follows immediately from Proposition 4.5, Theorem 5.4, Corollary 5.17 and Proposition 6.4 upon using the observations made above. \(\square\)

### 7.3. Analytic complete intersections

Let $X \subset \mathbb{C}^N$ be an analytic complete intersection of complex dimension $d$, defined by the regular sequence $f_1, \ldots, f_{N-d} \in \text{O}(X)$. Then $X$ is holomorphically parallelizable by results of [22] (see Theorem A.14). In particular, $X$ is Calabi-Yau. Let $u_1, \ldots, u_d$ be a holomorphic frame of $TX$.

**Proposition 7.2** Let $W \in \text{O}(X)$ be a holomorphic function such that $\dim_{\mathbb{C}} Z_W = 0$. Then:

$$\text{HPV}(X, W) = \text{HPV}^0(X, W) \simeq \text{Jac}(X, W) = \text{O}(X)/\langle u_1(W), \ldots, u_d(W) \rangle.$$ 

Moreover, we have equivalences of $\text{O}(X)$-linear $\mathbb{Z}_2$-graded categories:

$$\text{HDF}(X, W) \simeq \text{HF}(X, W) \simeq \text{HPF}(X, W).$$

For any holomorphic factorization $a$ of $W$, we have:

$$\text{HPV}(X, a) \simeq \text{Jac}(X, W) \otimes \text{End}_{\text{HF}(X, W)}(a).$$

**Proof.** Follows immediately from Proposition 4.5, Theorem 5.4 and Proposition 6.4 using the observations made above. \(\square\)

### A. Stein manifolds

In this Appendix, we recall the definition and some properties of Stein manifolds. For more information on Stein manifolds and Stein spaces we refer the reader to [19,20,22,23,24,25,26].

Let $X$ be a complex manifold. We say that holomorphic functions separate points of $X$ if for every pair of distinct points $x \neq y$ in $X$, there exists a holomorphic function $f \in \text{O}(X)$ such that $f(x) \neq f(y)$. The holomorphic (or analytic) hull of a compact subset $K \subset X$ is defined through:

$$\hat{K}_{\text{O}(X)} \overset{\text{def.}}{=} \{ x \in X | \forall f \in \text{O}(X) : |f(x)| \leq \max_{y \in K} |f(y)| \},$$

where $\text{O}(X) = \text{O}_X(X)$ is the $\mathbb{C}$-algebra of complex-valued holomorphic functions defined on $X$.

**Definition A.1** [23] A complex manifold $X$ is called holomorphically convex if $\hat{K}_{\text{O}(X)}$ is compact for any compact subset $K \subset X$.

**Definition A.2** [23] Let $X$ be a complex manifold with $\dim_{\mathbb{C}} X = d$. We say that $X$ is Stein if the following three conditions are satisfied:

1. Holomorphic functions separate points of $X$.
2. $X$ is holomorphically convex.
3. For every point $x \in X$ there exist globally-defined holomorphic functions $f_1, \ldots, f_d \in \text{O}(X)$ whose differentials $\partial f_j$ are linearly independent at $x$ over $\mathbb{C}$. 

On B-type open-closed Landau-Ginzburg theories defined on Calabi-Yau Stein manifolds
**Definition A.3** A complex manifold $X$ is called $K$-complete if for every point $x \in X$ there exists a holomorphic map $f : X \to \mathbb{C}^N$ for some $N$ such that $x$ is an isolated point of the fiber $f^{-1}(f(x))$.

**Theorem A.4** A complex manifold $X$ is Stein iff it satisfies the following properties:

1. Holomorphic functions separate points on $X$.
2. $X$ is holomorphically convex.
3. $X$ is $K$-complete.

**Theorem A.5** A complex manifold is Stein iff it is biholomorphic to a closed complex submanifold of $\mathbb{C}^N$ for some $N$.

**Theorem A.6** [23, page 337] Every Stein manifold of dimension $d > 1$ admits a proper holomorphic embedding into $\mathbb{C}^{Nd}$ for $N_d = \left[\frac{3d}{2}\right] + 1$ and a proper holomorphic immersion into $\mathbb{C}^{Md}$ for $M_d = \left[\frac{3d+1}{2}\right]$.

**Remark A.1.** The $\mathbb{C}$-algebra $O(X)$ of complex-valued holomorphic functions defined on a Stein manifold $X$ need not be finitely-generated. It is known [25] that $O(X)$ is finitely-generated if $X$ can be embedded as a closed polynomially convex subset of $\mathbb{C}^N$ for some $N$.

A.1. Cartan-Serre theorems and the Oka-Grauert principle.

**Theorem A.7 (Cartan)** [24, page 124] For every coherent analytic sheaf $F$ on a Stein manifold $X$, the following statements hold:

A. For any $x \in X$, the stalk $F_x$ is generated as an $O_{X,x}$-module by global sections of $F$.
B. $H^i(X,F) = 0$ for all $i > 0$.

Since $H^{i,j}(X) = H^j(X,\wedge^iTX)$, this implies:

**Corollary A.8** On any Stein manifold $X$ the Dolbeault cohomology groups $H^{i,j}(X)$ vanish for all $i \geq 0$ and $j \geq 1$.

The following result is known as the “Oka-Grauert principle”:

**Theorem A.9** [19] The holomorphic and topological classifications of holomorphic vector bundles over a Stein manifold coincide.

**Corollary A.10** Let $X$ be a Stein manifold. Then $X$ is Calabi-Yau iff $c_1(TX) = 0$.

**Proof.** $X$ is Calabi-Yau iff its canonical line bundle $K_X$ is holomorphically trivial. Since $X$ is Stein, the Oka-Grauert principle shows that $K_X$ is holomorphically trivial iff it is topologically trivial, i.e. iff $c_1(K_X) = 0$. Since $c_1(K_X) = -c_1(TX)$, this amounts to the condition $c_1(TX) = 0$.

A.2. Holomorphically parallelizable Stein manifolds. A special class of Stein manifolds consists of those complex manifolds which can be embedded as analytic complete intersections in some complex affine space. Such Stein manifolds are always Calabi-Yau. In fact, they coincide with the class of holomorphically parallelizable Stein manifolds, by the results of [22] recalled below.
**Definition A.11** A complex manifold $X$ with $\dim_{\mathbb{C}} X = d$ is called **holomorphically parallelizable** if the holomorphic tangent bundle $TX$ is holomorphically trivial, i.e. it is isomorphic with the trivial holomorphic vector bundle of rank $d$ in the category of holomorphic vector bundles over $X$.

It is a consequence of the Oka-Grauert principle [19,20] that a Stein manifold $X$ is holomorphically parallelizable iff $TX$ is topologically trivial (see [22]). Notice that any holomorphically parallelizable manifold is Calabi-Yau. The following results were established in [22]:

A $d$-dimensional analytic submanifold $X$ of $\mathbb{C}^N$ is called an **analytic complete intersection** in $\mathbb{C}^N$ if the ideal $I_N(X) \subset O(\mathbb{C}^N)$ of all holomorphic functions defined on $\mathbb{C}^N$ and which vanish identically on $X$ can be generated by $N-d$ independent elements, i.e. if there exist $N-d$ holomorphic functions $f_1, \ldots, f_{N-d}: \mathbb{C}^N \to \mathbb{C}$ such that:

$$X = \{x \in \mathbb{C}^N \mid f_1(x) = \ldots = f_{N-d}(x) = 0\}$$

and such that the rank of the matrix $\left(\frac{\partial f_i(x)}{\partial x_j}\right) \in \text{Mat}(N-d, N, \mathbb{C})$ equals $N-d$ at every point $x \in X$.

**Theorem A.12** A Stein manifold is holomorphically parallelizable iff it is biholomorphic with an analytic complete intersection in $\mathbb{C}^N$ for some $N$.

**Lemma A.13** Let $X$ be a complex-analytic submanifold of $\mathbb{C}^N$ with $\dim_{\mathbb{C}} X = d$. Then the following statements hold:

(i) If the normal bundle of $X$ is trivial, then $X$ is holomorphically parallelizable.

(ii) If $X$ is holomorphically parallelizable and $N \geq \frac{3d}{2}$, then the normal bundle of $X$ is trivial.

**Theorem A.14** Let $X$ be an analytic submanifold of $\mathbb{C}^N$ with $\dim_{\mathbb{C}} X = d$.

(i) If $X$ is an analytic complete intersection in $\mathbb{C}^N$, then it is holomorphically parallelizable.

(ii) If $X$ is holomorphically parallelizable and $N \geq \frac{3d}{2}+1$, then $X$ is a complete intersection in $\mathbb{C}^N$.

**Corollary A.15** Let $X$ be an analytic submanifold of $\mathbb{C}^N$ with $\dim_{\mathbb{C}} X = N-2$. Then $X$ is holomorphically parallelizable iff the first Chern class $c_1(X)$ vanishes.

**Corollary A.16** Let $N \leq 7$. Then an analytic submanifold $X$ of $\mathbb{C}^N$ is a complete intersection in $\mathbb{C}^N$ iff $X$ is holomorphically parallelizable.

Since for a $d$-dimensional Stein manifold we have $H^i(X, \mathbb{Z}) = 0$ for all $i > d$, one can express holomorphic parallelizability of low-dimensional Stein manifolds completely in terms of Chern classes. For example:

**Proposition A.17** Let $X$ be a Stein manifold with $\dim_{\mathbb{C}} X \leq 5$. Then $X$ is holomorphically parallelizable iff $c_1(X) = c_2(X) = 0$.

**Theorem A.18** [24, page 126] Every $d$-dimensional Stein manifold can be biholomorphically mapped onto a closed complex submanifold of $\mathbb{C}^{2d+1}$.

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6 It is not assumed that $X$ is an analytic complete intersection inside this $\mathbb{C}^N$. 

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