Pricing and Ordering Strategies of Supply Chain with Selling Gift Cards

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(Communicated by Xiaoqiang Cai)

Abstract. Gift cards is frequently used to replace traditional gift cash and gift products, especially when gift givers do not know gift receivers’ performances. Basing on this phenomenon, we analyze the supplier’s and the retailer’s strategies with selling gift cards. First, we develop a Stackelberg model without selling gift cards. Next, we develop two models with selling gift cards when unredeemed gift card balances become the retailer’s property and the state’s property, respectively. We present the optimal solutions and examine the impacts of parameters on the optimal decisions and the supply chain performance. When the retailer sells gift cards, the optimal order quantity is smaller than that without selling gift cards. The optimal wholesale price with selling gift cards is related to the treatment of unredeemed gift card balances. When unredeemed gift card balances become the retailer’s property, the wholesale price is lower than that without selling gift cards. However, when unredeemed gift card balances become the state’s property, the wholesale price is lower than that without selling gift cards in some conditions. And with selling gift cards, the optimal expected profits of retailer and supply chain are better off, but the optimal expected profit of supplier is worse off.

1. Introduction. Giving and receiving gifts are long-standing traditions on holiday season (Principe and Eisenhauer [15]). We often see that, in the pre-holiday period, customers (referred to as gift givers) buy gift products from the retailer, and give them to their family members, friends and so on (we call them gift receivers) in the holiday period. As the demands of gift products are much greater than those in normal days, the retailer have to order more gift products from the supplier in advance.

Gift products, gift cash and gift cards are three common gifts in holidays. They have different characteristics and utilities for both gift givers and receivers. Buying gift products is a good choice, but it costs gift givers much time and energies to carefully select the appropriate gifts for the gift receivers with different performances. Due to the fact that gift givers cannot know well the preference of gift receivers, the selected gifts may not be the ones those gift receivers really like and even may be the ones receivers already have. These unmatched behaviors between gift givers and gift receivers can lead to social risk (Austin and Huang [4]). To avoid wastage of the

2010 Mathematics Subject Classification. Primary: 90B05, 90B60; Secondary: 91B42.
Key words and phrases. Gift cards, Stackelberg model, supply chain, pricing, ordering.
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received gifts, gift receivers can return gift products where gift givers bought them. Generally, the retailer has no-hassle return policy and full return policy. Although gift receivers get full refunds from the retailer, gift receivers waste much time and money to return, they feel less satisfied with the retailer and produce deadweight loss in this process (Waldfogel [23] [24]). In order to make use of the holiday and seize the business opportunities, the retailer orders gift products in advance, such as roses in Valentine’s Day and Christmas trees in Christmas, these kinds of gift products are all sold in one period. If gift receivers return gift products, the retailer have to pay collecting cost, packing cost and extra cost.

Some gift givers refer to gift cash as gifts. Giving cash rather than product is efficient (Kaplan and Ruffle [9]). Gift cash is easier to classified to liquid asset in gift receiver’s subconscious (White [28]). But some gift receivers who buy things by cash feel guiltier than other ways they shop (Valentin and Allred [22]). Meanwhile, in western cultures, gift cash is not an appropriate gift between non-relatives (Webley, Lea and Portalsky [26]), and it is not a good manner in eastern countries, either.

A gift card with certain face value and expiry date is a better way to solve the aforementioned problems of gift products and gift cash. Gift cards not only greatly reduce gift givers’ searching time, but also avoid the gift receivers’ embarrassment for receiving unsuitable gifts and increase gift receivers’ satisfaction through redeeming products which suit their personal preferences. On the other hand, selling gift cards is a powerful tool for retailers to increase their profits (Shugan and Xie [17]). Gift cards can decrease the return rate of unsuitable gift products and the loss rate of consumers. Compared to taking cash, using gift cards has lower consumption guilty. Furthermore, gift cards can bring potential customers to the retailer. When gift receivers redeem gift cards in the retails, they are likely to spend more than gift card’s face value (White [29]). By the statistics, 65% of gift card users consume beyond 35% of gift card’s face value ([2]). Even some customers try to use mental budgeting to control overspending problems, however, they do not always succeed (Wertenbroch [27], Stilley, Inman and Wakefield [18][19]). As for supply chain, gift cards reduce the inventory of gift products and bring new sales of gift cards. According to the statistics, there are 93% of American customers have ever purchased or received gift cards ([1]). The sales of gift cards were about 60 billion dollars in 2005 (Offenberg [14]), it was up to 124 billion dollars in 2014 and the estimated sales of gift cards in 2016 will up to 140 billion dollars ([2]). So, gift cards are beneficial to the consumers, the retailers and the supply chain.

However, there are still some limitations of using gift cards in practice. Gift cards need to be redeemed to products from the retailer. Some gift cards require to be redeemed within one month of purchase (Tkacik [20]). Usually, some gift card receivers forget to redeem gift cards because of many reasons, such as beyond the period of validity and losing. According to statistics, the probability of unredeemed gift cards is 19% (Horne [8]). It is possible that many gift receivers have all kinds of reasons to redeem part or none of gift card’s face value (Thomas and Dillenbeck [21]), this phenomenon causes a discussion about the ownership of unredeemed gift cards balances. By the year of 2008, at least 39 states in U.S. issued laws to deal with these balances (Feinson [6]). Some states regulated that part or all of unredeemed balances are owned by the state (Alini [3]). At present, there are two usual ways to deal with unredeemed gift cards balances: the balances are owned by the state or by the retailer. In this paper, we consider these two ways into our models.
Although gift cards have a great potential development, there are few researches address the optimal pricing and ordering strategies of gift products with selling gift cards. The existing researches on gift cards mainly focus on following three aspects. Firstly, most papers study the “free” gift cards, which are offered by retailers or suppliers free to customers to attract additional purchase. Khouja et al. [10] developed a model to derive optimal purchase amount thresholds and gift card values, in which the retailer offers gift cards “free” to consumers who spend above specified thresholds in a single purchase. Khouja, Park and Zhou [12] proposed a newsvendor model to analyze the optimal strategies when the retailer gives “free” gift cards to customers who purchase a regularly priced product at the end of the selling season. Secondly, for the “no-free” gift cards, most scholars do qualitative research to investigate the impacts of gift cards on gift card users’ behaviors. Waldfogel [25] designed behavior experiments to prove that compared with receiving gifts, bounded rational customers prefer to buying gifts personally. It is because that a lot of cost are consumed when people give gifts, and giving gifts is not an effective way to allocate social resource. Yao and Chen [30] compared gift cards with gift cash in terms of the effects in customer’s information processing and product evaluating. Thirdly, few researchers do quantitative research on selling gift cards. Khouja, Pan and Zhou [11] studied the optimal ordering and discount of seasonal products when the retailers selling gift cards within the newsvendor model framework, but they did not consider the problem of supply chain. Although the literature about the decision of pricing and ordering with gift cards is limited, the research on the decision of pricing and ordering in supply chain without gift cards has received significant attention in the past two decades (Ghoreishi et al.[7], Sadigh, Chaharsooghi and Sheikhmohammady [16]).

In this paper, we do analysis of “no-free” gift cards in supply chain environment. When the retailer sells gift cards, gift cards can offset part of gift product return and(or) bring additional profit from redeeming gift cards. Meanwhile, the demand of gift product decreases because customers buy gift cards and redeem for non-gift products, the order quantity of gift product from the supplier decrease eventually. So it is important for the supplier and the retailer to make decisions to maximize their profits with selling gift cards, respectively. We answer the following three questions: (1) does the use of gift cards benefit to the retailer? (2) what are the optimal pricing and ordering strategies of the supplier and the retailer when the retailer sells gift cards, respectively? (3) how does the sale of gift cards impact supply chain’s performance?

The rest of this paper is organized as follows. In Section 2, we describe the problem with some reasonable assumptions. In Section 3, we present two Stackelberg models without and with selling gift cards, and give the optimal solutions, respectively. We further discuss the optimal strategies when demand is uniformly distributed in Section 4. Section 5 presents numerical analysis. Finally, we make conclusions in Section 6.

2. Basic assumptions. This paper analyzes the optimal pricing and ordering strategies of selling gift cards in supply chain with Stackelberg model. We suppose the demand of gift product is random by all customers. Let $f(x)$, $F(x)$ and $F(x)$ be the probability density function, the cumulative distribution function and the complementary cumulative distribution function, respectively. We define the failure rate of $x$ as $h(x) = \frac{f(x)}{F(x)}$ and assume that $x$ has an increasing failure rate.
(IFR). This assumption is not restrictive as it is satisfied by a large range of probability distributions, including but not limited to the uniform, Weibull, normal, and exponential distributions, and their truncated versions (Chua and Liu [5]). We further define the generalized failure rate of $x$ as $h(x) = \frac{f(x)}{F(x)}$. Distributions with an increasing failure rate (IFR) are clearly increasing generalized failure rate IGFR (Lariviere and Porteus [13]).

The supplier, as the Stackelberg leader, provides a single kind of gift product with marginal cost $c$ and wholesale price $w$. Meanwhile, the retailer orders the gift product from the supplier and sells it to customers. The order quantity is $q$ and sale price is $p$, and we assume that $p$ is exogenous in all conditions. In the pre-holiday period, gift givers purchase gifts (gift products/gift cards) from the retailer, then send the gifts to gift receivers on the holiday. In the post-holiday period, the unsold gift product is sold at a discount and the salvage value is $v$. In the event of a stock out, unmet demand is lost, resulting in the margin being lost, but without any additional stock-out penalty (Lariviere and Porteus[13]). Let $p > w > c > v$. A summary of notation is given in Table 1. We use $\pi$ to respect the profit. Superscripts $R$ and $S$ denote the retailer and the supplier, respectively. And subscripts NG, RG and SG denote the three different conditions that without selling gift cards, with selling gift cards when unredeemed gift cards balances become the retailer’s property and with selling gift cards when unredeemed gift cards balances become the state’s property, respectively. In our research, we do not consider the time value of money from unredeemed gift cards. Notice that the transaction can only proceed when the supplier and the retailer can get profits from the deal (Lariviere and Porteus [13]).

| Decision Variables | Description |
|--------------------|-------------|
| $q$                | order quantity of gift product of the retailer |
| $w$                | wholesale price of gift product of the supplier |

| Parameters          | Description |
|---------------------|-------------|
| $x$                 | demand for gift product/gift cards from gift givers before the holiday |
| $f(x), F(x)$        | the pdf and cdf of demand from gift givers before the holiday |
| $p$                 | sale price of unit gift product |
| $c$                 | cost of unit gift product |
| $v$                 | salvage value of unit gift product |
| $\theta$            | return rate of gift product/probability of a gift giving consumer buying gift card before the holiday |
| $\beta$             | probability of gift product buyers buying gift cards when gift product is stock-out before the holiday |
| $\alpha$            | average redemption rate of gift cards after the holiday |
| $m$                 | profit margin of non-gift products |

**Table 1. Notation**

3. Model formulation and solution. In this section, to get the optimal order quantities and wholesale prices of gift product with and without gift cards in supply chain, we introduce three models under three different conditions, namely NG, RG and SG, respectively.

3.1. Without selling gift cards (NG). When the retailer does not sell gift cards, decision behaviors among the supplier, the retailer, gift givers and gift receivers in decentralized supply chain are shown in Figure 1. First, in pre-holiday period, the supplier determines the wholesale price of the gift product, then the retailer determines the order quantity of the gift product from the supplier. Next, the gift giver purchases the gift product from the retailer. Finally, in holiday period , the
gift receiver gets the gift product from the gift giver. If the gift receiver does not like the gift product, she returns it to the retailer in post-holiday period. Suppose the return rate of the gift product is $\theta (0 \leq \theta \leq 1)$ and the gift receiver can return the gift product for full cash refund\(^1\).

![Figure 1. Decision behaviors in supply chain without gift cards](image)

The retailer’s profit function is

$$
\pi_{\text{NG}}^R = p(1 - \theta) \min\{q, x\} + v \theta \min\{q, x\} + v[q - \min\{q, x\}] - wq.
$$

(1)

In Equation (1), the first term is income of the selling gift product, the second term is salvage value of the returned gift product, the third term is salvage value of the unsold gift product and the fourth term is purchasing cost of the gift product. Let $A = (1 - \theta)p + v\theta$, then the retailer’s expected profit is

$$
E\pi_{\text{NG}}^R = \int_0^q [Ax + v(q - x)]f(x)dx + \int_q^{+\infty} Aqf(x)dx - wq.
$$

(2)

Accordingly, the supplier’s expected profit is

$$
E\pi_{\text{NG}}^S = (w - c)q.
$$

(3)

According to solving method of Stackelberg model, we can get Theorem 3.1.

**Theorem 3.1.** Without selling gift cards, the supplier’s optimal wholesale price of the gift product is $w_{\text{NG}}^* = (\frac{v - w}{\theta q_{\text{NG}}^*}) + v$, the retailer’s optimal order quantity of the gift product is $q_{\text{NG}}^* = \frac{F^{-1}(\frac{A - w}{A - v})}{A - v}$.

**Proof.** Taking the first and second derivations of $E\pi_{\text{NG}}^R$ in Equation (2) with respect to $q$, we have

$$
\frac{\partial E\pi_{\text{NG}}^R}{\partial q} = (v - A)F(q) + A - w,
$$

(4)

$$
\frac{\partial^2 E\pi_{\text{NG}}^R}{\partial q^2} = (v - A)f(q).
$$

(5)

According to Equation (5), since $v - A = (1 - \theta)(v - p)$, then $\frac{\partial^2 E\pi_{\text{NG}}^R}{\partial q^2} < 0$. So $E\pi_{\text{NG}}^R$ is concave in $q$. Hence, from the Equation (4), the retailer’s optimal order quantity of gift product $q_{\text{NG}}^*$ is set such that

$$
F(q_{\text{NG}}^*) = \frac{A - w}{A - v}.
$$

(6)

\(^1\)The process of return generates return cost, which mainly includes collecting and packing costs. To make the model simplify, the return cost is set for zero.
According to Equation (6), we know that the wholesale price $w$ and the retailer’s optimal order quantity of gift product $q^*_{NG}$ have a one-to-one correspondence. Hence, let $w(q^*_{NG})$ be the unique wholesale price that induces the retailer to order $q^*_{NG}$, then
\[ w(q^*_{NG}) = (A - v)\bar{F}(q^*_{NG}) + v. \] (7)

Substituting Equation (7) into Equation (3), the supplier’s expected profit is written as
\[ E\pi^S_{NG}(w(q^*_{NG})) = [(A - v)\bar{F}(q^*_{NG}) + v - c]q^*_{NG}. \]

So, we have
\[ \frac{\partial E\pi^S_{NG}(w(q^*_{NG}))}{\partial q^*_{NG}} = (A - v)\bar{F}(q^*_{NG})\left[1 - \frac{q^*_{NG}f(q^*_{NG})}{F(q^*_{NG})}\right] - (c - v) \]
\[ = (A - v)\bar{F}(q^*_{NG})\left[1 - g(q^*_{NG})\right] - (c - v). \] (8)

Since $F(x)$ is IGFR, then $g(q^*_{NG})$ is increasing in $q^*_{NG}$. So $\frac{\partial E\pi^S_{NG}(w(q^*_{NG}))}{\partial q^*_{NG}}$ is decreasing in $q^*_{NG}$. Thus, the supplier’s expected profit function $E\pi^S_{NG}(w(q^*_{NG}))$ is unimodal, and the optimal order quantity $q^*_{NG}$ is determined by the unique solution to the first-order condition. According to Equation (8), the first-order condition can be written as
\[ (A - v)\bar{F}(q^*_{NG})\left[1 - g(q^*_{NG})\right] = c - v. \] (9)

From Equation (7), the optimal wholesale price $w^*_{NG} = (A - v)\bar{F}(q^*_{NG}) + v$. Then we can use Equation (9) and (6) to yield $w^*_{NG} = \frac{c - v}{1 - g(q^*_{NG})} + v$ and $q^*_{NG} = F^{-1}\left(\frac{A - w^*_{NG}}{A - v}\right)$.

According to Theorem 3.1, when $w^*_{NG} = c$, the optimal stocking level of the integrated supply chain is $q^*_{NG} = F^{-1}\left(\frac{A - c}{A - v}\right)$.

### 3.2 Unredeemed gift cards balances become the retailer’s property (RG)

When the retailer sells gift cards, decision behaviors among the supplier, the retailer, gift givers and gift receivers in decentralized supply chain are depicted in Figure 2. Firstly, in pre-holiday period, the supplier determines the wholesale price of gift product, the retailer determines the order quantity of gift product, then the retailer offers the gift product and gift cards to customers. Next, the gift giver purchases the gift product or gift cards from the retailer. If the supply of gift product is sufficient, the gift giver who find the gift product is satisfied with the preference of gift receiver will purchase it, otherwise the gift giver will turn to buy gift cards instead of the gift product. Suppose that the probability of gift giver buying gift cards is $\theta(0 \leq \theta \leq 1)$

2 We assumed the probability of gift giver buying gift cards is same as return rate of gift product when the retailer does not sell gift cards, and the expense of gift giver buying gift cards is equal to the price of gift product.

3 For the sake of simplicity, we assume the gift receivers who receive gift cards are not interested in gift products, so they only redeem non-gift products from the retailer.
gift cards or missing redemption time. Suppose the average redemption rate of gift cards is \( \alpha (0 \leq \alpha \leq 1) \), and the profit margin of non-gift product is \( m (0 < m < 1) \).

In this subsection, we suppose that the unredeemed gift cards balances become the retailer’s property, then the retailer’s profit function is

\[
\pi_{RG}^R = p \min \{ q, (1 - \theta) x \} + m \alpha p \theta x + \beta (\max \{ q, (1 - \theta) x \} - q) \\
+ (1 - \alpha) p \theta x + \beta (\max \{ q, (1 - \theta) x \} - q) \\
+ v [q - \min \{ q, (1 - \theta) x \}] - w q. 
\]

In Equation (10), the first term is income of selling gift product, the second term is income of redeemed gift cards, the third term is income of unredeemed gift cards, the fourth term is salvage value of unsold gift product and the fifth term is purchasing cost of gift product. Let \( B = (1 - \alpha \theta + m \alpha \theta) p \) and \( C = (1 - \beta + \alpha \beta - m \alpha \beta) p \), then the retailer’s expected profit is

\[
E_{\pi_{RG}} = \int_0^q B x + v (q + \theta x - x) f(x) dx \\
+ \int_{q}^{+\infty} [B x - C (1 - \theta) x + C q] f(x) dx - w q. 
\]

And the supplier’s expected profit is

\[
E_{\pi_{RG}}^S = (w - c) q. 
\]

According to solving method of Stackelberg model, we get Theorem 3.2.

**Theorem 3.2.** when unredeemed gift cards balances become the retailer’s property, if \( (1 - \beta + \alpha \beta - m \alpha \beta) p - v > 0 \), the supplier’s optimal wholesale price of the gift product is \( w_{RG}^* = \frac{c - v}{1 - q (\frac{w_{RG}^*}{c - v})} + v \), the retailer’s optimal order quantity of the gift product is \( q_{RG}^* = (1 - \theta) F^{-1} (\frac{c - w_{RG}^*}{c - v}) \).

**Proof.** Taking the first and second derivations of \( E_{\pi_{RG}}^R \) in Equation (11) with respect to \( q \), we have

\[
\frac{\partial E_{\pi_{RG}}^R}{\partial q} = (v - C) F(\frac{q}{1 - \theta}) + C - w, \\
\frac{\partial^2 E_{\pi_{RG}}^R}{\partial q^2} = (\frac{v - C}{1 - \theta}) f(\frac{q}{1 - \theta}). 
\]
According to Equation (14), if \( v - C < 0 \), i.e., \((1 - \beta + \alpha \beta - m\alpha \beta)p - v > 0\), we have \( \frac{\partial^2 E\pi_{RG}^S}{\partial q_{RG}^2} < 0 \). So \( E\pi_{RG}^S \) is concave in \( q \). Hence, from the Equation (13), the retailer’s optimal order quantity of gift product \( q_{RG}^* \) is set such that

\[
F\left(\frac{q_{RG}^*}{1 - \theta}\right) = \frac{C - w}{C - v}.
\]

According to Equation (15), we know that the wholesale price \( w \) and the retailer’s optimal order quantity of gift product \( q_{RG}^* \) have a one-to-one correspondence. Hence, let \( w(q_{RG}^*) \) be the unique wholesale price that induces the retailer to order \( q_{RG}^* \), then

\[
w(q_{RG}^*) = (C - v)\bar{F}\left(\frac{q_{RG}^*}{1 - \theta}\right) + v.
\]

Substituting Equation (16) into Equation (12), the supplier’s expected profit is written as

\[
E\pi_{RG}^S(w(q_{RG}^*)) = [(C - v)\bar{F}\left(\frac{q_{RG}^*}{1 - \theta}\right) + v - c]q_{RG}^*.
\]

So, we have

\[
\frac{\partial E\pi_{RG}^S(w(q_{RG}^*))}{\partial q_{RG}^*} = (C - v)\bar{F}\left(\frac{q_{RG}^*}{1 - \theta}\right)[1 - \frac{\frac{q_{RG}^*}{1 - \theta} f\left(\frac{q_{RG}^*}{1 - \theta}\right)}{\bar{F}\left(\frac{q_{RG}^*}{1 - \theta}\right)}] - (c - v)
\]

\[
= (C - v)\bar{F}\left(\frac{q_{RG}^*}{1 - \theta}\right)[1 - g\left(\frac{q_{RG}^*}{1 - \theta}\right)] - (c - v).
\]

Since \( F(x) \) is IGFR, then \( g\left(\frac{q_{RG}^*}{1 - \theta}\right) \) is increasing in \( q_{RG}^* \). So \( \frac{\partial E\pi_{RG}^S(w(q_{RG}^*))}{\partial q_{RG}^*} \) is decreasing in \( q_{RG}^* \). Thus, the supplier’s expected profit function \( E\pi_{RG}^S(w(q_{RG}^*)) \) is unimodal, and the optimal order quantity \( q_{RG}^* \) is determined by the unique solution to the first-order condition. According to Equation (17), the first-order condition can be written as

\[
(C - v)\bar{F}\left(\frac{q_{RG}^*}{1 - \theta}\right)[1 - g\left(\frac{q_{RG}^*}{1 - \theta}\right)] = c - v.
\]

From Equation (16), the optimal wholesale price \( w_{RG}^* = (C - v)\bar{F}\left(\frac{q_{RG}^*}{1 - \theta}\right) + v \). Then we can use Equation (18) and (15) to yield \( w_{RG}^* = \frac{c - v}{1 - g\left(\frac{q_{RG}^*}{1 - \theta}\right)} + v \) and \( q_{RG}^* = (1 - \theta)F^{-1}\left(\frac{C - w_{RG}^*}{C - v}\right) \).

3.3. Unredeemed gift cards balances become the state’s property (SG).

Next, we consider a model in which the unredeemed gift cards balances become the state’s property. The retailer’s profit function is

\[
\pi_{SG}^R = p \min\{q, (1 - \theta)x\} + m\alpha p\theta x + \beta \max\{q, (1 - \theta)x\} - q]\]

\[
+ v[q - \min\{q, (1 - \theta)x\}] - wq.
\]

In Equation (19), the first term is income of selling gift product, the second term is income of redeemed gift cards, the third term is salvage value of unsold gift product and the fourth part is purchasing cost of gift product. Let \( D = (1 - \theta + m\alpha \theta)p \) and
\begin{align*}
K = (1 - m\alpha\beta)p, \text{ then the retailer's expected profit is} \\
E\pi^R_{SG} &= \int_0^\infty [Dx + v(q + \theta x - x)]f(x)dx \\
&\quad + \int_{-\infty}^\infty [Dx - K(1 - \theta)x + Kq]f(x)dx - wq.
\end{align*}

The supplier's expected profit is

\begin{equation}
E\pi^S_{SG} = (w - c)q.
\end{equation}

According to solving method of Stackelberg model, we get Theorem 3.3.

\textbf{Theorem 3.3.} \textit{when unredeemed gift cards balances become the state's property, if} \((1 - m\alpha\beta)p - v > 0\), \textit{the supplier's optimal wholesale price of the gift product is} \(w^*_{SG} = \frac{-v}{1 - g(SG^*)} + v\), \textit{the retailer's optimal order quantity of the gift product is} \(q^*_{SG} = (1 - \theta)F^{-1}(K - w^*_{SG})\).

\textbf{Proof.} Taking the first and second derivations of \(E\pi^R_{SG}\) in Equation (20) with respect to \(q\), we have

\begin{align*}
&\frac{\partial E\pi^R_{SG}}{\partial q} = (v - K)F\left(\frac{q}{1 - \theta}\right) + K - w, \\
&\frac{\partial^2 E\pi^R_{SG}}{\partial q^2} = \left(\frac{v - K}{1 - \theta}\right)f\left(\frac{q}{1 - \theta}\right).
\end{align*}

According to Equation (23), if \(v - K < 0\), i.e., \((1 - m\alpha\beta)p - v > 0\), we have \(\frac{\partial^2 E\pi^R_{SG}}{\partial q^2} < 0\). So \(E\pi^R_{SG}\) is concave in \(q\). Hence, from the Equation (22), the retailer's optimal order quantity of gift product \(q^*_{SG}\) is set such that

\begin{equation}
F\left(\frac{q^*_{SG}}{1 - \theta}\right) = \frac{K - w}{K - v}.
\end{equation}

According to Equation (24), we know that the wholesale price \(w\) and the retailer's optimal order quantity of gift product \(q^*_{SG}\) have a one-to-one correspondence. Hence, let \(w(q^*_{SG})\) be the unique wholesale price that induces the retailer to order \(q^*_{SG}\), then

\begin{equation}
w(q^*_{SG}) = (K - v)\bar{F}\left(\frac{q^*_{SG}}{1 - \theta}\right) + v.
\end{equation}

Substituting Equation (25) into Equation (21), the supplier's expected profit is written as

\begin{equation}
E\pi^S_{SG}(w(q^*_{SG})) = [(K - v)\bar{F}\left(\frac{q^*_{SG}}{1 - \theta}\right) + v - c]q^*_{SG}.
\end{equation}

So, we have

\begin{equation}
\frac{\partial E\pi^S_{SG}(w(q^*_{SG}))}{\partial q^*_{SG}} = (K - v)\bar{F}\left(\frac{q^*_{SG}}{1 - \theta}\right)[1 - g(SG^*)\frac{q^*_{SG}}{1 - \theta}] - (c - v)
\end{equation}

\begin{align*}
&= (K - v)\bar{F}\left(\frac{q^*_{SG}}{1 - \theta}\right)[1 - g(SG^*)\frac{q^*_{SG}}{1 - \theta}] - (c - v).
\end{align*}

Since \(F(x)\) is IGFR, \(g(SG^*)\) is increasing in \(q^*_{SG}\). So \(\frac{\partial E\pi^S_{SG}(w(q^*_{SG}))}{\partial q^*_{SG}}\) is decreasing in \(q^*_{SG}\). Thus, the supplier's expected profit function \(E\pi^S_{SG}(w(q^*_{SG}))\) is unimodal, and the optimal order quantity \(q^*_{SG}\) is determined by the unique solution
to the first-order condition. According to Equation (26), the first-order condition can be written as

\[(K - v)\tilde{F}\left(\frac{\tilde{q}^*_G}{1 - \theta}\right)[1 - g(\frac{\tilde{q}^*_G}{1 - \theta})] = c - v. \tag{27}\]

From Equation (25), the optimal wholesale price \(w^*_G = (K - v)\tilde{F}(\frac{\tilde{q}^*_G}{1 - \theta}) + v\). Then we can use Equation (27) and (24) to yield \(\tilde{w}^*_G = \frac{c - v}{1 - g(\frac{\tilde{q}^*_G}{1 - \theta})} + v\) and \(q^*_G = (1 - \theta)F^{-1}(\frac{C - w^*_G}{K - v})\).

3.4. Comparative static analysis. Based on the above analysis, we know that the optimal wholesale price and the optimal order quantity are related to many factors. We investigate the effects of some parameters on the optimal wholesale price and the optimal order quantity and the results are shown in Theorem 3.4 and Theorem 3.5.

**Theorem 3.4.** For the optimal order quantity of gift product, (i) under condition \(NG\), the optimal order quantity \(q^*_G\) is decreasing in the return rate \(\theta\); (ii) under condition \(RG\), the optimal order quantity \(q^*_G\) is increasing in the average redemption rate of gift cards \(\alpha\), but is decreasing in the probability of a gift product buyer buying gift cards when gift product is stock-out \(\beta\) and the profit margin of non-gift products \(\gamma\); and (iii) under condition \(SG\), the optimal order quantity \(q^*_G\) is decreasing in the average redemption rate of gift cards \(\alpha\), the probability of a gift product buyer buying gift cards when gift product is stock-out \(\beta\) and the profit margin of non-gift products \(\gamma\).

**Theorem 3.5.** For the optimal wholesale price of gift product, (i) under condition \(NG\), the optimal wholesale price \(w^*_G\) is decreasing in the return rate \(\theta\); (ii) under condition \(RG\), the optimal wholesale price \(w^*_G\) is increasing in the average redemption rate of gift cards \(\alpha\), but is decreasing in the probability of a gift product buyer buying gift cards when gift product is stock-out \(\beta\) and the profit margin of non-gift products \(\gamma\); and (iii) under condition \(SG\), the optimal wholesale price \(w^*_G\) is decreasing in the average redemption rate of gift cards \(\alpha\), the probability of a gift product buyer buying gift cards when gift product is stock-out \(\beta\) and the profit margin of non-gift products \(\gamma\).

**Proof of Theorem 3.4 and 3.5.** (i) According to Theorem 3.1, let

\[H_1 = \frac{c - v}{1 - g(q^*_G)} + v - w^*_G, \tag{28}\]

\[H_2 = \frac{A - w^*_G}{A - v} - F(q^*_G), \tag{29}\]

where \(A = (1 - \theta)p + v\theta\). Taking the first derivations of \(H_1\) and \(H_2\) in Equation (28) and (29) with respect to \(q^*_G\): \(w^*_G\) and \(\theta\), respectively. We have \(\frac{\partial H_1}{\partial q^*_G} = \frac{(c - v)g'(q^*_G)}{[1 - g(q^*_G)]^2} > 0\), \(\frac{\partial H_1}{\partial w^*_G} = -1\), \(\frac{\partial H_1}{\partial \theta} = 0\), \(\frac{\partial H_2}{\partial q^*_G} = -f(q^*_G) < 0\), \(\frac{\partial H_2}{\partial w^*_G} = -\frac{1}{A - v} < 0\) and \(\frac{\partial H_2}{\partial \theta} = \frac{(v - p)(w^*_G - v)}{(A - v)^2} < 0\).

Considering the impacts of \(\theta\) on \(q^*_G\) and \(w^*_G\), we have

\[
\begin{align*}
\frac{\partial H_1}{\partial q^*_G} + \frac{\partial H_1}{\partial w^*_G} + \frac{\partial H_1}{\partial \theta} = 0, \\
\frac{\partial H_2}{\partial q^*_G} + \frac{\partial H_2}{\partial w^*_G} + \frac{\partial H_2}{\partial \theta} = 0.
\end{align*}
\]
So we get
\[
\frac{\partial q_{NG}^*}{\partial \theta} = \frac{\partial H_1}{\partial \theta} \frac{\partial H_2}{\partial \theta} - \frac{\partial H_1}{\partial \theta} \frac{\partial H_2}{\partial \theta} < 0
\]
and
\[
\frac{\partial w_{NG}^*}{\partial \theta} = \frac{\partial H_1}{\partial \theta} \frac{\partial H_2}{\partial \theta} - \frac{\partial H_1}{\partial \theta} \frac{\partial H_2}{\partial \theta} < 0
\]
(ii) According to Theorem 3.2, let
\[
H_3 = \frac{C - v}{1 - \theta} + v - w_{RG}^*, \quad (30)
\]
\[
H_4 = \frac{C - w_{RG}^*}{C - v} - F\left(\frac{q_{RG}^*}{1 - \theta}\right), \quad (31)
\]
where \(C = (1 - \beta - \alpha \beta - ma \beta)p\) and \(C - v > 0\). Taking the first derivations of \(H_3\) and \(H_4\) in Equation (30) and (31) with respect to \(q_{RG}^*, w_{RG}^*, \alpha, \beta\) and \(m\), respectively.
We have
\[
\frac{\partial H_3}{\partial q_{RG}^*} = \frac{(c-v)^2}{\alpha \beta (1-\theta)} > 0, \quad \frac{\partial H_4}{\partial q_{RG}^*} = -1, \quad \frac{\partial H_3}{\partial \alpha} = 0, \quad \frac{\partial H_3}{\partial \alpha} = 0, \quad \frac{\partial H_3}{\partial m} = 0, \quad \frac{\partial H_4}{\partial m} = 0,
\]
\[
\frac{\partial H_3}{\partial \beta} = \frac{1}{(1-\theta)^2} \frac{q_{RG}^*}{1-\theta} < 0, \quad \frac{\partial H_4}{\partial \beta} = -\frac{1}{C-v} < 0, \quad \frac{\partial H_3}{\partial \alpha} = \frac{A(p(w_{RG}^*-v)}{(C-v)^2} > 0,
\]
\[
\frac{\partial H_4}{\partial \alpha} = \frac{(a-\alpha \beta - 1)p(w_{RG}^*-v)}{(C-v)^2} < 0 \text{ and } \frac{\partial H_4}{\partial c} = \frac{1}{(C-v)^2} < 0.
\]
Considering the impacts of \(\alpha\) on \(q_{RG}^*\) and \(w_{RG}^*\), we have
\[
\begin{align*}
\frac{\partial H_3}{\partial q_{RG}^*} &+ \frac{\partial H_3}{\partial w_{RG}^*} = 0, \\
\frac{\partial H_4}{\partial q_{RG}^*} &+ \frac{\partial H_4}{\partial w_{RG}^*} = 0.
\end{align*}
\]
So we get
\[
\frac{\partial q_{RG}^*}{\partial \alpha} = \frac{\partial H_3}{\partial w_{RG}^*} > 0
\]
and
\[
\frac{\partial w_{RG}^*}{\partial \alpha} = \frac{\partial H_3}{\partial q_{RG}^*} > 0.
\]
Considering the impacts of \(\beta\) on \(q_{RG}^*\) and \(w_{RG}^*\), we have
\[
\begin{align*}
\frac{\partial H_3}{\partial q_{RG}^*} &+ \frac{\partial H_3}{\partial w_{RG}^*} = 0, \\
\frac{\partial H_4}{\partial q_{RG}^*} &+ \frac{\partial H_4}{\partial w_{RG}^*} = 0.
\end{align*}
\]
So we get
\[
\frac{\partial q_{RG}^*}{\partial \beta} = \frac{\partial H_3}{\partial w_{RG}^*} < 0
\]
and
\[
\frac{\partial w_{RG}^*}{\partial \beta} = \frac{\partial H_3}{\partial q_{RG}^*} < 0.
\]
Considering the impacts of \( m \) on \( q^*_{RG} \) and \( w^*_{RG} \), we have

\[
\begin{cases}
\frac{\partial H_3}{\partial q^*_{RG}} + \frac{\partial H_3}{\partial w^*_{RG}} + \frac{\partial H_3}{\partial m} = 0, \\
\frac{\partial H_4}{\partial q^*_{RG}} + \frac{\partial H_4}{\partial w^*_{RG}} + \frac{\partial H_4}{\partial m} = 0.
\end{cases}
\]

So we get

\[
\frac{\partial q^*_{RG}}{\partial m} = \frac{\partial H_3}{\partial w^*_{RG}} \frac{\partial H_4}{\partial m} - \frac{\partial H_3}{\partial m} \frac{\partial H_4}{\partial w^*_{RG}} < 0
\]

and

\[
\frac{\partial w^*_{RG}}{\partial m} = \frac{\partial H_3}{\partial q^*_{RG}} \frac{\partial H_4}{\partial m} - \frac{\partial H_3}{\partial m} \frac{\partial H_4}{\partial q^*_{RG}} < 0.
\]

(iii) According to Theorem 3.3, let

\[
H_5 = \frac{c - v}{1 - g(\frac{q^*_{SG}}{1 - \theta})} + v - w^*_SG, \quad (32)
\]

\[
H_6 = \frac{K - w^*_SG}{K - v} - F(\frac{q^*_{SG}}{1 - \theta}). \quad (33)
\]

Where \( K = (1 - ma\beta)p \) and \( K - v > 0 \). Taking the first derivations of \( H_5 \) and \( H_6 \) in Equation (32) and (33) with respect to \( q^*_{SG}, w^*_SG, \alpha, \beta \) and \( m \), respectively.

We have \( \frac{\partial H_3}{\partial q^*_{SG}} = \frac{(c-v)g'(\frac{q^*_{SG}}{1-\theta})}{[1-\theta \frac{q^*_{SG}}{1-\theta}]^2(1-\theta)} > 0, \frac{\partial H_3}{\partial w^*_{SG}} = -1, \frac{\partial H_4}{\partial \alpha} = 0, \frac{\partial H_4}{\partial \beta} = 0, \frac{\partial H_4}{\partial m} = 0, \frac{\partial H_5}{\partial q^*_{SG}} = \frac{(c-v)g'(\frac{q^*_{SG}}{1-\theta})}{[1-\theta \frac{q^*_{SG}}{1-\theta}]^2(1-\theta)} > 0, \frac{\partial H_5}{\partial w^*_{SG}} = \frac{1}{(K-v)^2} < 0, \frac{\partial H_6}{\partial \alpha} = -\frac{\alpha m\beta(w^*_SG-v)}{(K-v)^2} < 0, \frac{\partial H_6}{\partial \beta} = \frac{1}{(K-v)^2} < 0, \frac{\partial H_6}{\partial m} = \frac{1}{(K-v)^2} < 0.

Considering the impacts of \( \alpha \) on \( q^*_{SG} \) and \( w^*_SG \), we have

\[
\begin{cases}
\frac{\partial H_5}{\partial q^*_{SG}} + \frac{\partial H_5}{\partial m} + \frac{\partial H_5}{\partial \alpha} = 0, \\
\frac{\partial H_6}{\partial q^*_{SG}} + \frac{\partial H_6}{\partial m} + \frac{\partial H_6}{\partial \alpha} = 0,
\end{cases}
\]

So we get

\[
\frac{\partial q^*_{SG}}{\partial \alpha} = \frac{\partial H_5}{\partial w^*_{SG}} \frac{\partial H_6}{\partial \alpha} - \frac{\partial H_5}{\partial \alpha} \frac{\partial H_6}{\partial w^*_{SG}} < 0
\]

and

\[
\frac{\partial w^*_{SG}}{\partial \alpha} = \frac{\partial H_5}{\partial q^*_{SG}} \frac{\partial H_6}{\partial \alpha} - \frac{\partial H_5}{\partial \alpha} \frac{\partial H_6}{\partial q^*_{SG}} < 0.
\]

Considering the impacts of \( \beta \) on \( q^*_{SG} \) and \( w^*_SG \), we have

\[
\begin{cases}
\frac{\partial H_5}{\partial q^*_{SG}} + \frac{\partial H_5}{\partial m} + \frac{\partial H_5}{\partial \beta} = 0, \\
\frac{\partial H_6}{\partial q^*_{SG}} + \frac{\partial H_6}{\partial m} + \frac{\partial H_6}{\partial \beta} = 0,
\end{cases}
\]

So we get

\[
\frac{\partial q^*_{SG}}{\partial \beta} = \frac{\partial H_5}{\partial w^*_{SG}} \frac{\partial H_6}{\partial \beta} - \frac{\partial H_5}{\partial \beta} \frac{\partial H_6}{\partial w^*_{SG}} < 0
\]
and
\[
\frac{\partial w_{SG}^*}{\partial \beta} = \frac{\partial H_5 \partial H_6}{\partial w_{SG}} \frac{\partial H_5}{\partial q_{SG}^*} - \frac{\partial H_5 \partial H_6}{\partial q_{SG}^*} < 0.
\]

Considering the impacts of \( m \) on \( q_{SG}^* \) and \( w_{SG}^* \), we have
\[
\begin{align*}
\frac{\partial H_5}{\partial q_{SG}^*} \frac{\partial q_{SG}^*}{\partial m} + \frac{\partial H_5}{\partial w_{SG}} \frac{\partial w_{SG}}{\partial m} + \frac{\partial H_5}{\partial m} &= 0, \\
\frac{\partial H_6}{\partial q_{SG}^*} \frac{\partial q_{SG}^*}{\partial m} + \frac{\partial H_6}{\partial w_{SG}} \frac{\partial w_{SG}}{\partial m} + \frac{\partial H_6}{\partial m} &= 0.
\end{align*}
\]

So we get
\[
\frac{\partial q_{SG}^*}{\partial m} = \frac{\partial H_5}{\partial w_{SG}} \frac{\partial H_6}{\partial q_{SG}^*} \frac{\partial H_5}{\partial m} - \frac{\partial H_5}{\partial m} \frac{\partial H_6}{\partial w_{SG}} \frac{\partial q_{SG}^*}{\partial m} < 0
\]
and
\[
\frac{\partial w_{SG}^*}{\partial m} = \frac{\partial H_5}{\partial q_{SG}^*} \frac{\partial H_6}{\partial w_{SG}} \frac{\partial H_5}{\partial m} - \frac{\partial H_5}{\partial m} \frac{\partial H_6}{\partial q_{SG}^*} \frac{\partial w_{SG}}{\partial m} < 0.
\]

Theorem 3.4 and 3.5 show that without selling gift cards, the optimal order quantity and the optimal wholesale price are decreasing with the return rate of gift product. This is because that larger return rate of gift product will increase the retailer’s return cost, then the retailer will decrease the order quantity of gift product. So, the supplier need to decrease the wholesale price to encourage the retailer to order more gift products. From Theorem 3.4 and 3.5, we know that with increasing of the proportion of buying gift cards from unmet gift product buyers, the retailer decreases the optimal order quantity and the supplier decreases the optimal wholesale price. It is because that the gift card, as a substitute for the gift product, can decline the demand of gift product. It also means that selling gift cards will diminish the supplier’s power in supply chain. Similarly, with increasing of the profit margin of non-gift products, the retailer decreases the optimal order quantity and the supplier decreases the optimal wholesale price too. If the profit margin of non-gift product is larger, then the retailer’s profit from gift card redemption is larger, so the retailer is more willing to sell gift cards. Hence, it is reasonable for the retailer to decline the order quantity of gift product. This imply that the profit margin of non-gift product will further decrease the profitability of the supplier. Under condition \( RG \), since the unredeemed gift cards balances become the retailer’s property, then the retailer can get more profit from lower redemption rate. So, if the redemption rate is lower, the retailer has no incentive to order more gift product, hence the supplier needs to decrease the wholesale price. Conversely, if the redemption rate is larger, the retailer’s optimal order quantity is larger, then the supplier can set a higher wholesale price. Under condition \( SG \), since the unredeemed gift cards balances become the state’s property, then the retailer cannot get benefit from unredeemed gift cards. So, with decreasing of the redemption rate, the retailer’s profit obtained from non-gift products will decrease, then the retailer will order more gift products. Consequently, the supplier may increase the optimal wholesale price.
4. Uniformly distributed demand. According to Theorem 3.1 – Theorem 3.3, we can get the optimal solutions for uniformly distributed demand $x \sim U(0, b)$, listed in Table 2.

| Optimal wholesale price | Optimal order quantity | Conditions |
|-------------------------|-----------------------|------------|
| $w^*_{NG} = \frac{1}{2} \left[ c + (\theta - \alpha)p + \theta v \right]$ | $q^*_{w} = \frac{t((1 - \theta)p + \theta v - c)}{2(1 - \theta)p - v}$ | -- |
| $w^*_{NG} = \frac{1}{2} \left[ c + (\beta - ma\beta + a\beta)p \right]$ | $q^*_{w} = \frac{t((1 - \theta)(1 - \beta - ma\beta + a\beta)p - c)}{2(1 - \beta - ma\beta + a\beta)p - v}$ | $(1 - \beta - ma\beta + a\beta)p - v \geq 0$ |
| $w^*_{SG} = \frac{1}{2} \left[ c + (\alpha - ma\alpha)p \right]$ | $q^*_{w} = \frac{t((1 - \theta)(1 - \alpha)p - c)}{2(1 - \alpha)p - v}$ | $(1 - \alpha)p - v \geq 0$ |

**Table 2. Optimal solutions for uniformly distributed demand**

Based on the optimal policies in Table 2, Theorem 4.1 compares without selling gift cards, unredeemed gift cards balances become the retailer’s property and unredeemed gift cards balances become the state’s property.

**Theorem 4.1.** When the customer demand is uniformly distributed, comparing $NG$, $RG$ and $SG$, we have: (i) for the optimal wholesale price, if $\theta \leq \frac{ma\beta \beta}{p - v}$, then $w^*_{NG} \geq w^*_{SG} \geq w^*_{RG}$; if $\theta \geq \frac{(\beta + ma\beta - a\beta)p}{p - v}$, then $w^*_{NG} \geq w^*_{SG} \geq w^*_{RG}$; if $\theta \geq \frac{(\beta + ma\beta - a\beta)p}{p - v}$, then $w^*_{SG} \geq w^*_{RG} \geq w^*_{NG}$; (ii) for the optimal order quantity, $q^*_{NG} \geq q^*_{SG} \geq q^*_{RG}$.

**Proof.** From Table 2, we have

$$w^*_{SG} - w^*_{RG} = \frac{1}{2}(1 - \alpha)\beta p \geq 0,$$

$$w^*_{RG} - w^*_{NG} = \frac{1}{2}(\theta - \alpha - ma\beta - \beta)p - \theta v,$$

$$w^*_{SG} - w^*_{NG} = \frac{1}{2}(\theta - ma\beta)p - \theta v.$$

If $\theta \leq \frac{(\beta + ma\beta - a\beta)p}{p - v}$, then $w^*_{RG} - w^*_{NG} \leq 0$; if $\theta \geq \frac{(\beta + ma\beta - a\beta)p}{p - v}$, then $w^*_{RG} - w^*_{NG} \geq 0$; if $\theta \leq \frac{ma\beta \beta}{p - v}$, then $w^*_{SG} - w^*_{NG} \leq 0$; if $\theta \geq \frac{ma\beta \beta}{p - v}$, then $w^*_{SG} - w^*_{NG} \geq 0$.

Since $\frac{ma\beta \beta}{p - v} \leq \frac{(\beta + ma\beta - a\beta)p}{p - v}$, then we have the results of Theorem 4.1.

From Table 2, we also have

$$q^*_{SG} - q^*_{RG} = \frac{bp(\beta - \beta)(1 - \alpha)}{2(1 - \beta - ma\beta + a\beta)p - v} \geq 0,$$

$$q^*_{NG} - q^*_{SG}|_{\beta = 0} = \frac{bp(1 - \theta) + v - c(2 - \theta)}{2(1 - \theta)(p - v)} \geq 0.$$

According to Theorem 3.4, $q^*_{SG}$ is decreasing in $\beta$, so $q^*_{NG} \geq q^*_{SG}$ for all $0 \leq \beta \leq 1$. Thus, for the optimal order quantity, we have $q^*_{NG} \geq q^*_{SG} \geq q^*_{RG}$.

**Theorem 4.1** shows that the optimal wholesale price in $SG$ is always larger than that in $RG$. Note that $\theta$ represents the return rate without selling gift cards and the proportion of gift card sales with selling gift cards. We know that the optimal wholesale price in $NG$ is decreasing in $\theta$ from Theorem 3.5. Theorem 4.1 also shows that, the optimal wholesale price in $SG$ is always larger than that in $RG$; however, with the increasing of $\theta$, the optimal wholesale price in $NG$ will change from above the optimal wholesale price in $SG$ to below the optimal wholesale price in $RG$. 


As for the optimal order quantity, the optimal order quantity in $NG$ is always larger than those in $RG$ and in $SG$. It means that gift cards can effectively reduce the retailer’s loss that due to uncertain demand. Meanwhile, when unredeemed gift card balances stay with the retailer, the retailer prefers to reduce the order quantity of gift product.

5. **Numerical analysis.** In this section, we do some numerical analyses to exam the influences of different parameters on the optimal strategies and the supply chain performance with normally distribution demand. We also consider the condition of integrated supply chain without selling gift cards (denotes it by $ING$) in this section. Suppose that the mean value $\mu = 100$, the sale price of gift product $p = 10$ and the salvage value of gift product $v = 0.5c$.

5.1. **Coefficient of variance.** The impacts of $CV$ (coefficient of variance, $\sigma/\mu$) on the optimal wholesale price, the optimal order quantity and the expected profits are shown in Figure 3, respectively. In Figure 3(a), the optimal wholesale prices in $NG$, $RG$ and $SG$ are decreasing in $CV$. With the increase of $CV$, the demand will be more and more instable, the supplier has to reduce wholesale price to encourage the retailer to order more gift products. The optimal wholesale price without selling gift cards is higher than that with selling gift cards, and the optimal wholesale price in $SG$ is higher than that in $RG$. Figure 3(b) shows that the optimal order quantities with and without selling gift cards decrease in small $CV$, but increase in big $CV$. The optimal order quantity without selling gift cards is larger than that with selling gift cards, meanwhile, the optimal order quantity in $SG$ is larger than that in $RG$. As for the retailer’s performance, Figure 3(c) shows that the retailer has the largest optimal expected profit in $RG$, however the retailer’s optimal expected profit in $NG$ is the lowest. All these profits are increasing in $CV$. Figure 3(d) shows that the supplier’s optimal expected profit without selling gift cards is larger than that with selling gift cards, and the optimal wholesale price in $SG$ is lower than that in $NG$ just for bigger $c/p$. The optimal order quantity without selling gift cards is larger than that with selling gift cards, meanwhile, the optimal order quantity in $SG$ is larger than that in $RG$. Figure 3(d) also shows that the supplier’s optimal expected profits in three conditions decrease as $CV$ increase. Figure 3(e) illustrates that the performances of supply chain in three conditions are also decreasing in $CV$. According to Figure 3(c), 3(d) and 3(e), we can know that selling gift cards is benefit to the retailer, but it is not benefit to the supplier. While, the supply chain’s optimal expected profit with gift cards is larger than that without gift cards in integrated supply chain. Hence, selling gift cards gives much rise to the supply chain’s optimal expected profit.

5.2. **Unit cost of gift product/sale price of gift product.** The impacts of $c/p$ (unit cost of gift product/sale price of gift product) on the optimal wholesale price, the optimal order quantity and the optimal expected profit are shown in Figure 4, respectively. Figure 4(a) shows that the optimal wholesale price in $RG$ is lower than those in $NG$ and $SG$, but the optimal wholesale price in $SG$ is lower than that in $NG$ just for bigger $c/p$. The optimal wholesale prices in three conditions are all increasing in $c/p$. Figure 4(b) shows the optimal order quantity without selling gift cards is larger than that with selling gift cards, and the optimal order quantity in $SG$ is larger than that in $RG$. The optimal order quantities in three conditions are all decreasing in $c/p$. From Figure 4(c), 4(d) and 4(e), we know that the optimal expected profits of the retailer, the supplier and the supply chains are all decreasing in $c/p$ because of larger manufacturing cost margin of gift product. Selling gift cards
Figure 3. $w^*$, $q^*$ and $\pi^*$ vs. $CV$ (Note: $c = 5, \theta = 0.15, \alpha = 0.8, \beta = 0.5$ and $m = 0.3$)

is benefit to the retailer, but it is not benefit to the supplier. Although the supply chain’s optimal expected profit with selling gift cards is not always larger than the integrated supply chain’s optimal expected profit without selling gift cards, selling gift cards is also benefit to the supply chain.

Figure 4. $w^*$, $q^*$ and $\pi^*$ vs. $c/p$ (Note: $\sigma = 40, \theta = 0.15, \alpha = 0.8, \beta = 0.5$ and $m = 0.3$)
5.3. **Return rate of gift products (purchase rate of gift cards).** The impacts of return rate of gift product (or purchase rate of gift cards, $\theta$) on the optimal wholesale price, the optimal order quantity and the optimal expected profit are shown in Figure 5, respectively. Figure 5(a) shows that the optimal wholesale price of gift product without selling gift cards is decreasing in $\theta$, however the optimal wholesale price of gift product with selling gift card is constant in $\theta$. The optimal wholesale price in $NG$ is higher than the optimal wholesale prices in $SG$ and $RG$ for small $\theta$. Figure 5(b) shows that the optimal order quantities in three conditions are all decreasing in $\theta$. From Figure 5(c), 5(d) and 5(e), we get gift products return is harmful to the supplier, the retailer and the supply chain under without selling gift cards condition, however, selling gift cards can improve their profits because gift cards can offset return loss in a certain extent.

![Figure 5](image)

5.4. **Average redemption rate of gift cards.** The impacts of average redemption rate of gift cards ($\alpha$) on the optimal wholesale price, the optimal order quantity and the optimal expected profit are shown in Figure 6, respectively. Figure 6(a) and 6(b) show that the optimal wholesale price and the optimal order quantity in $SG$ are larger than those in $RG$. When $\alpha$ increases, the optimal wholesale price and the optimal order quantity in $RG$ increase, while the optimal wholesale price and the optimal order quantity in $SG$ decrease. When $\alpha$ is smaller, the optimal wholesale price in $SG$ is higher than that in $NG$. While, the optimal wholesale price in $SG$ is smaller than that in $NG$ for bigger $\alpha$. The more gift cards are redeemed, the less unredeemed gift cards balances belong to the retailer. Figure 6(c) shows that the retailer’s optimal expected profit in $RG$ is decreasing in $\alpha$, and the retailer’s optimal expected profit in $SG$ is increasing in $\alpha$. Figure 6(d) shows that the supplier’s optimal expected profit in $RG$ is increasing in $\alpha$, and the supplier’s optimal
expected profit in SG is decreasing in $\alpha$. From Figure 6(e), we get that selling gift cards makes the supply chain’s optimal expected profit increase significantly.

![Figure 6](image)

**Figure 6.** $w^*$, $q^*$ and $\pi^*$ vs. $\alpha$ (Note: $\sigma = 40, c = 5, \theta = 0.15, \beta = 0.5$ and $m = 0.3$)

5.5. Proportion of gift cards sales from unmet gift product buyers. The impacts of the proportion of a gift product buyer buying gift cards when gift product is stock-out ($\beta$) on the optimal wholesale price, the optimal order quantity and the optimal expected profit are shown in Figure 7, respectively. Figure 7(a) shows that the optimal wholesale price with selling gift cards is lower than that without selling gift cards for bigger $\beta$, and the optimal wholesale price with selling gift cards is decreasing in $\beta$. It means that with the increase of the proportion of gift cards sales from unmet gift product buyers, the supplier should reduce the wholesale price of gift product. Figure 7(b) shows that the optimal order quantity with selling gift cards is decreasing in $\beta$. It means that with the increase of $\beta$, the retailer will decrease the order quantity of gift product. The impacts of on profits are shown in Figure 7(c), 7(d) and 7(e). The retailer’s and supply chain’s optimal expected profits with selling gift cards are increasing in $\beta$, while the supplier’s optimal expected profit with selling gift cards is decreasing in $\beta$. For smaller $\beta$, selling gift cards is benefit to the supplier.

5.6. Profit margin of non-gift products. The impacts of profit margin of non-gift products ($m$) on the optimal wholesale price, the optimal order quantity and the optimal expected profit are shown in Figure 8, respectively. If the profit margin of non-gift products is higher, the retailer prefers to sell gift cards. Figure 8(a) shows that the optimal wholesale prices in RG and SG are decreasing in $m$. It means that with the increase of profit margin of non-gift products, the optimal wholesale price with selling gift cards becomes lower and lower. Figure 8(b) shows that the optimal order quantity of gift product with selling gift cards is decreasing in $m$. Figure 8(c),
**Figure 7.** $w^*$, $q^*$ and $\pi^*$ vs. $\beta$ (Note: $\sigma = 40, c = 5, \theta = 0.15, \alpha = 0.8$ and $m = 0.3$)

8(d) and 8(e) show that the optimal expected profits of retailer and supply chain increase as $m$ increases, however the supplier's optimal expected profit decreases as $m$ increases.

**Figure 8.** $w^*$, $q^*$ and $\pi^*$ vs. $m$ (Note: $\sigma = 40, c = 5, \theta = 0.15, \alpha = 0.8$ and $\beta = 0.5$)
Conclusion. In this paper, we develop Stackelberg models in three conditions to analyze the optimal strategies of the supply chain. Based on our theoretical and numerical analysis, we conclude our work as follows.

For the retailer, selling gift cards can improve her profit dramatically, especially for unredeemed gift cards balances belong to her property. So the retailer has an incentive to encourage selling gift cards, but the retailer does not have an incentive to encourage gift cards redemption when the retailer keeps unredeemed gift cards balances. Meanwhile, higher proportion of gift cards sales from unmet gift product buyers and higher profit margin of non-gift products increase the profitability of the retailer.

For the supplier, his expected profit with selling gift cards is lower than that without gift cards in most conditions. So, in general, selling gift cards is harmful to the supplier. The reason is that the gift card, as a substitute for the gift product, can make the retailer order less gift products from the supplier. Hence, the supplier has to reduce the wholesale price to encourage the retailer to order more gift products. Especially, when the unredeemed gift cards balances belong to the retailer’s property, the supplier set the wholesale price at a lower level. Therefore, the supplier prefers the state government taking the unredeemed gift cards balances or a higher redemption rate of gift cards.

For the supply chain, selling gift card increases the total expected profit of supply chain. The additional profit mainly comes from gift cards redemption for non-gift products. Meanwhile, selling gift cards will make the supply chain decrease the shortage cost and overstock cost. We also find that the decentralized supply chain’s expected profit with selling gift cards may be larger than the integrated supply chain’s expected profit without selling gift cards. It provides the possibility of developing cooperation between the supplier and the retailer to make the supply chain be better off.

There are several limitations in this paper. Firstly, we did not consider the consumer behavior of buying gift cards. In this paper, we only consider the benefits from reduced product return. In fact, there are many factors which affect the gift giver’s buying decision. Secondly, the price of gift product was exogenous in our model. The extension of this paper may consider the price as a decision variable. Thirdly, we did not consider the supplier’s participation constraint and the coordination of supply chain.

Acknowledgments. The authors are much grateful to the Editor and two anonymous referees for their constructive comments which improved the presentation of the paper. This work is supported by the Natural Sciences Foundation of China under Grant No. 71272127 and 71531003, and the Soft Science Research Project of Sichuan Province under Grant No. 2016ZR0118.

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Received April 2016; revised May 2016.

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