SLANT RIEMANNIAN SUBMERSIONS FROM SASAKIAN MANIFOLDS

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Abstract. We introduce slant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We survey main results of slant Riemannian submersions defined on Sasakian manifolds. We also give an example of such slant submersions.

1. Introduction

Let $F$ be a $C^\infty$-submersion from a Riemannian manifold $(M, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then according to the conditions on the map $F : (M, g_M) \to (N, g_N)$, we have the following submersions:

- semi-Riemannian submersion and Lorentzian submersion [11],
- Riemannian submersion ([19], [12]), slant submersion ([9], [24]), almost Hermitian submersion [27], contact-complex submersion [16], quaternionic submersion [15], almost $h$-slant submersion and $h$-slant submersion [21], semi-invariant submersion [25], $h$-semi-invariant submersion [22], etc.

As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([6], [28]), Kaluza-Klein theory ([7], [13]), Supergravity and superstring theories ([14], [29]). In [23], Sahin introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. He also suggested to investigate anti invariant submersions from almost contact metric manifolds onto Riemannian manifolds in [20].

So the purpose of the present paper is to study similar problems for slant Riemannian submersions from Sasakian manifolds to Riemannian manifolds. We also want to carry anti-invariant submanifolds of Sasakian manifolds to anti-invariant Riemannian submersion theory and to prove dual results for submersions. For instance, a slant submanifold of a $K$-contact manifold is an anti invariant submanifold if and only if $\nabla Q = 0$ (see: Proposition 4.1 of [3]). We get similar result as Proposition 4. Thus, it will be worth the study area which is anti-invariant submersions from almost contact metric manifolds onto Riemannian manifolds.

The paper is organized as follows: In section 2, we present the basic information about Riemannian submersions needed for this paper. In section 3, we mention about Sasakian manifolds. In section 4, we give definition of slant Riemannian submersions and introduce slant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We survey main results of slant submersions defined on Sasakian manifolds. We also give an example of slant submersions such that characteristic vector field $\xi$ is vertical.

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2. Riemannian Submersions

In this section we recall several notions and results which will be needed throughout the paper.

Let \((M, g_M)\) be an \(m\)-dimensional Riemannian manifold, let \((N, g_N)\) be an \(n\)-dimensional Riemannian manifold. A Riemannian submersion is a smooth map \(F : M \to N\) which is onto and satisfies the following axioms:

\(S1\). \(F\) has maximal rank.

\(S2\). The differential \(F_*\) preserves the lengths of horizontal vectors.

The fundamental tensors of a submersion were defined by O’Neill \([19, 20]\). They are \((1, 2)\)-tensors on \(M\), given by the formula:

\begin{align*}
(2.1) \quad \mathcal{T}(E, F) &= \mathcal{T}_E F = \mathcal{H}\nabla_{VE}VF + \mathcal{V}\nabla_{VE}HF, \\
(2.2) \quad \mathcal{A}(E, F) &= \mathcal{A}_E F = \mathcal{V}\nabla_{HE}HF + \mathcal{H}\nabla_{HE}VF,
\end{align*}

for any vector field \(E\) and \(F\) on \(M\). Here \(\nabla\) denotes the Levi-Civita connection of \((M, g_M)\). These tensors are called integrability tensors for the Riemannian submersions. Note that we denote the projection morphism on the distributions \(\ker F_*\) and \((\ker F_*)^\perp\) by \(\mathcal{V}\) and \(\mathcal{H}\), respectively. The following Lemmas are well known \([19, 20]\).

**Lemma 1.** For any \(U, W\) vertical and \(X, Y\) horizontal vector fields, the tensor fields \(\mathcal{T}\) and \(\mathcal{A}\) satisfy:

\begin{align*}
(2.3) \quad i) & \quad \mathcal{T}_U W = \mathcal{T}_W U, \\
(2.4) \quad ii) & \quad \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2} \mathcal{V}[X, Y].
\end{align*}

It is easy to see that \(\mathcal{T}\) is vertical, \(\mathcal{T}_E = \mathcal{T}_{VE}\) and \(\mathcal{A}\) is horizontal, \(\mathcal{A} = \mathcal{A}_{HE}\).

For each \(q \in N\), \(F^{-1}(q)\) is an \((m-n)\)-dimensional submanifold of \(M\). The submanifolds \(F^{-1}(q)\), \(q \in N\), are called fibers. A vector field on \(M\) is called vertical if it is always tangent to fibers. A vector field on \(M\) is called horizontal if it is always orthogonal to fibers. A vector field \(X\) on \(M\) is called basic if \(X\) is horizontal and \(F\)-related to a vector field \(\hat{X}\) on \(N\), i.e., \(F_*X_p = \hat{X}_p\) for all \(p \in M\).

**Lemma 2.** Let \(F : (M, g_M) \to (N, g_N)\) be a Riemannian submersion. If \(X, Y\) are basic vector fields on \(M\), then:

\begin{itemize}
  \item[i)] \(g_M(X, Y) = g_N(X_*, Y_*) \circ F\),
  \item[ii)] \(\mathcal{H}[X, Y]\) is basic, \(F\)-related to \([X_*, Y_*]\),
  \item[iii)] \(\mathcal{H}(\nabla_X Y)\) is basic vector field corresponding to \(\nabla_{X_*} Y_*\) where \(\nabla^*\) is the connection on \(N\),
  \item[iv)] for any vertical vector field \(V\), \([X, V]\) is vertical.
\end{itemize}

Moreover, if \(X\) is basic and \(U\) is vertical then \(\mathcal{H}(\nabla_U X) = \mathcal{H}(\nabla_X U) = \mathcal{A}_X U\). On the other hand, from \((2.1)\) and \((2.2)\) we have

\begin{align*}
(2.5) \quad \nabla_V W &= \mathcal{T}_V W + \nabla_V W, \\
(2.6) \quad \nabla_V X &= \mathcal{H}\nabla_V X + \mathcal{T}_V X, \\
(2.7) \quad \nabla_X V &= \mathcal{A}_X V + \mathcal{V}\nabla_X V, \\
(2.8) \quad \nabla_X Y &= \mathcal{H}\nabla_X Y + \mathcal{A}_X Y
\end{align*}

for \(X, Y \in \Gamma((\ker F_*)^\perp)\) and \(V, W \in \Gamma(\ker F_*),\) where \(\nabla_V W = \mathcal{V}\nabla_V W\).

Notice that \(\mathcal{T}\) acts on the fibers as the second fundamental form of the submersion and restricted to vertical vector fields and it can be easily seen that \(\mathcal{T} = 0\) is equivalent
to the condition that the fibres are totally geodesic. A Riemannian submersion is called a Riemannian submersion with totally geodesic fiber if \( T \) vanishes identically. Let \( U_1, ..., U_{m-n} \) be an orthonormal frame of \( \Gamma(\ker F) \). Then the horizontal vector field \( H = \frac{1}{m-n} \sum_{j=1}^{m-n} T U_j U_j \) is called the mean curvature vector field of the fiber. If \( H = 0 \) the Riemannian submersion is said to be minimal. A Riemannian submersion is called a Riemannian submersion with totally umbilical fibers if

\[
T U W = g(M(U, W) H
\]

for \( U, W \in \Gamma(\ker F) \). For any \( E \in \Gamma(TM) \), \( T E \) and \( A E \) are skew-symmetric operators on \( (\Gamma(TM), g_M) \) reversing the horizontal and the vertical distributions. By Lemma 1 horizontally distribution \( \mathcal{H} \) is integrable if and only if \( A = 0 \). For any \( D, E, G \in \Gamma(TM) \) one has

\[
g(T D E, G) + g(T D G, E) = 0,
\]

(2.10)

\[
g(A D E, G) + g(A D G, E) = 0.
\]

We recall the notion of harmonic maps between Riemannian manifolds. Let \( (M, g_M) \) and \( (N, g_N) \) be Riemannian manifolds and suppose that \( \varphi : M \to N \) is a smooth map between them. Then the differential \( \varphi_* \) of \( \varphi \) can be viewed a section of the bundle \( \text{Hom}(TM, \varphi^{-1}TN) \to M \), where \( \varphi^{-1}TN \) is the pullback bundle which has fibres \( (\varphi^{-1}TN)_p = T_{\varphi(p)}N, p \in M \). \( \text{Hom}(TM, \varphi^{-1}TN) \) has a connection \( \nabla \) induced from the Levi-Civita connection \( \nabla \) and the pullback connection. Then the second fundamental form of \( \varphi \) is given by

\[
(\nabla \varphi_*)(X, Y) = \nabla^\varphi_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y)
\]

for \( X, Y \in \Gamma(TM) \), where \( \nabla^\varphi \) is the pullback connection. It is known that the second fundamental form is symmetric. If \( \varphi \) is a Riemannian submersion it can be easily prove that

\[
(\nabla \varphi_*)(X, Y) = 0
\]

for \( X, Y \in \Gamma((\ker F)^\perp) \). A smooth map \( \varphi : (M, g_M) \to (N, g_N) \) is said to be harmonic if \( \text{trace}(\nabla \varphi_*) = 0 \). On the other hand, the tension field of \( \varphi \) is the section \( \tau(\varphi) \) of \( \Gamma(\varphi^{-1}TN) \) defined by

\[
\tau(\varphi) = \text{div} \varphi_* = \sum_{i=1}^m (\nabla \varphi_*)(e_i, e_i),
\]

where \( \{e_1, ..., e_m\} \) is the orthonormal frame on \( M \). Then it follows that \( \varphi \) is harmonic if and only if \( \tau(\varphi) = 0 \), for details, [2].

3. Sasakian Manifolds

A \( n \)-dimensional differentiable manifold \( M \) is said to have an almost contact structure \( (\phi, \xi, \eta) \) if it carries a tensor field \( \phi \) of type \( (1, 1) \), a vector field \( \xi \) and 1-form \( \eta \) on \( M \) respectively such that

\[
\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1,
\]

where \( I \) denotes the identity tensor.
The almost contact structure is said to be normal if $N + d\eta \otimes \xi = 0$, where $N$ is the Nijenhuis tensor of $\phi$. Suppose that a Riemannian metric tensor $g$ is given in $M$ and satisfies the condition
\begin{equation}
(3.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi).
\end{equation}
Then $(\phi, \xi, \eta, g)$-structure is called an almost contact metric structure. Define a tensor field $\Phi$ of type $(0, 2)$ by $\Phi(X, Y) = g(\phi X, Y)$. If $d\eta = \Phi$ then an almost contact metric structure is said to be normal contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies
\begin{equation}
(3.3) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,
\end{equation}
where $\nabla$ denotes the Levi-Civita connection of $g$. For a Sasakian manifold $M = M^{2n+1}$, it is known that
\begin{align}
(3.4) & \quad R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \\
(3.5) & \quad S(X, \xi) = 2n\eta(X)
\end{align}
and
\begin{equation}
(3.6) \quad \nabla_X \xi = -\phi X.
\end{equation}

Now we will introduce a well known Sasakian manifold example on $\mathbb{R}^{2n+1}$.

**Example 1** ([4]). We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_i, y_i, z)$ ($i = 1, \ldots, n$) and its usual contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y_idx_i)$. The characteristic vector field $\xi$ is given by $2\partial/\partial z$ and its Riemannian metric $g$ and tensor field $\phi$ are given by
\begin{equation}
g = \frac{1}{4}(\eta \otimes \eta + \sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2)), \quad \phi = \begin{pmatrix}
0 & \delta_{ij} & 0 \\
-\delta_{ij} & 0 & 0 \\
0 & y_j & 0
\end{pmatrix}, \quad i, j = 1, \ldots, n
\end{equation}
This gives a contact metric structure on $\mathbb{R}^{2n+1}$. The vector fields $E_i = 2\partial/\partial y_i$, $E_{n+i} = 2\left(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}\right)$, $\xi$ form a $\phi$-basis for the contact metric structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\phi, \xi, \eta, g)$ is a Sasakian manifold.

4. Slant Riemannian submersions

**Definition 1.** Let $M(\phi, \xi, \eta, g_M)$ be a Sasakian manifold and $(N, g_N)$ be a Riemannian manifold. A Riemannian submersion $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ is said to be slant if for any non zero vector $X \in \Gamma(\text{ker} F_* ) - \{\xi\}$, the angle $\theta(X)$ between $\phi X$ and the space $\text{ker} F_*$ is a constant (which is independent of the choice of $p \in M$ and of $X \in \Gamma(\text{ker} F_* ) - \{\xi\}$). The angle $\theta$ is called the slant angle of the slant submersion. Invariant and anti-invariant submersions are slant submersions with $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submersion which is not invariant nor anti-invariant is called proper submersion.

Now we will introduce an example.
Example 2. \( \mathbb{R}^5 \) has got a Sasakian structure as in Example 1. Let \( F: \mathbb{R}^5 \to \mathbb{R}^2 \) be a map defined by \( F(x_1, x_2, y_1, y_2, z) = (x_1 - 2\sqrt{2}x_2 + y_1, 2x_1 - 2\sqrt{2}x_2 + y_1) \). Then, by direct calculations

\[
\ker F_* = \text{span}\{V_1 = 2E_1 + \frac{1}{\sqrt{2}}E_4, V_2 = E_2, V_3 = \xi = E_5\}
\]

and

\[
(\ker F_*)^\perp = \text{span}\{H_1 = 2E_1 - \frac{1}{\sqrt{2}}E_4, H_2 = E_3\}.
\]

Then it is easy to see that \( F \) is a Riemannian submersion. Moreover, \( \phi V_1 = 2E_3 - \frac{1}{\sqrt{2}}E_2 \) and \( \phi V_2 = E_4 \) imply that \(|g(\phi V_1, V_2)| = \frac{1}{\sqrt{2}}\). So \( F \) is a slant submersion with slant angle \( \theta = \frac{\pi}{4} \).

In Example 2, we note that the characteristic vector field \( \xi \) is vertical. If \( \xi \) is orthogonal to \( \ker F_* \) we will give following Theorem.

**Theorem 1.** Let \( F \) be a slant Riemannian submersion from a Sasakian manifold \( M(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N, g_N) \). If \( \xi \) is orthogonal to \( \ker F_* \), then \( F \) is anti-invariant.

**Proof.** By (3.6), (2.6), (2.10) and (2.3) we have

\[
g(\phi U, V) = -g(\nabla_U \xi, V) = -g(T_U \xi, V) = g(T_U V, \xi) = g(T_\psi U, \xi) = g(U, \phi V)
\]

for any \( U, V \in \Gamma(\ker F_*) \). Using skew symmetry property of \( \phi \) in the last relation we complete the proof of the Theorem. \( \square \)

**Remark 1.** We note Lotta [17] proved that if \( M_1 \) is a submanifold of contact metric manifold of \( \tilde{M}_1 \) and \( \xi \) is orthogonal to \( M_1 \), then \( M_1 \) is anti-invariant submanifold. So, our result can be seen as a submersion version of Lotta’s result.

Now, let \( F \) be a slant Riemannian submersion from a Sasakian manifold \( M(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N, g_N) \). Then for any \( U, V \in \Gamma(\ker F_*) \), we put

\[
(4.1) \quad \phi U = \psi U + \omega U,
\]

where \( \psi U \) and \( \omega U \) are vertical and horizontal components of \( \phi U \), respectively. Similarly, for any \( X \in \Gamma(\ker F_*)^\perp \), we have

\[
(4.2) \quad \phi X = BX + CX,
\]

where \( BX \) (resp. \( CX \)) is vertical part (resp. horizontal part) of \( \phi X \).

From (4.2), (4.1) and (4.2) we obtain

\[
(4.3) \quad g_M(\psi U, V) = -g_M(U, \psi V)
\]

and

\[
(4.4) \quad g_M(\omega U, Y) = -g_M(U, BY).
\]

for any \( U, V \in \Gamma(\ker F_*) \) and \( Y \in \Gamma((\ker F_*)^\perp) \).

Using (2.5), (4.1) and (3.6) we obtain

\[
(4.5) \quad T_U \xi = -\omega U, \quad \nabla_U \xi = -\psi U
\]

for any \( U \in \Gamma(\ker F_*) \).
Now we will give the following proposition for a Riemannian submersion with two dimensional fibers which is similar to Proposition 3.2. of [1].

**Proposition 1.** Let $F$ be a Riemannian submersion from almost contact manifold onto a Riemannian manifold. If $\text{dim}(\ker F^*) = 2$ and $\xi$ is vertical then fibers are anti-invariant.

As the proof of the following proposition is similar to slant submanifolds (see [8]) we don’t give its proof.

**Proposition 2.** Let $F$ be a Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $\xi \in \Gamma(\ker F^*)$. Then $F$ is anti-invariant submersion if and only if $D$ is integrable, where $D = \ker F^* - \{\xi\}$.

**Theorem 2.** Let $M(\phi, \xi, \eta, g_M)$ be a Sasakian manifold of dimension $2m+1$ and $(N, g_N)$ is a Riemannian manifold of dimension $n$. Let $F : M(\phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ be a slant Riemannian submersion. Then the fibers are not totally umbilical.

**Proof.** Using (2.5) and (3.6) we obtain

$$T_U \xi = -\omega U$$

for any $U \in \Gamma(\ker F_*)$. If the fibers are totally umbilical, then we have $T_U V = g_M(U, V)H$ for any vertical vector fields $U, V$ where $H$ is the mean curvature vector field of any fibre. Since $T_\xi \xi = 0$, we have $H = 0$, which shows that fibres are minimal. Hence the fibers are totally geodesic, which is a contradiction to the fact that $T_U \xi = -\omega U \neq 0$. □

By (2.5), (2.6), (4.1) and (4.2) we have

$$\nabla_U \omega V = CT_U V - T_U \psi V,$$

$$\nabla_U \phi V = BT_U V - T_U \omega V + R(\xi, U)V,$$

where

$$\nabla_U \omega V = H\nabla_U \omega V - \omega \tilde{\nabla}_U V,$$

$$\nabla_U \psi V = \tilde{\nabla}_U \psi V - \psi \tilde{\nabla}_U V,$$

for $U, V \in \Gamma(\ker F_*)$.

**Theorem 3.** Let $F$ be a Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $\xi \in \Gamma(\ker F_*)$. Then, $F$ is a slant Riemannian submersion if and only if there exist a constant $\lambda \in [0, 1]$ such that

$$\psi^2 = -\lambda(I - \eta \otimes \xi).$$

Furthermore, in such case, if $\theta$ is the slant angle of $F$, it satisfies that $\lambda = \cos^2 \theta$.

**Proof.** Firstly we suppose that $F$ is not an anti-invariant Riemannian submersion. Then, for $U \in \Gamma(\ker F_*)$,

$$\cos \theta = \frac{g_M(\phi U, \psi U)}{|\phi U| |\psi U|} = \frac{|\psi U|^2}{|\phi U| |\psi U|} = \frac{|\psi U|}{|\phi U|}.$$
From (4.10) and (4.11) we have

\[(4.12) \quad |\psi U|^2 = |\psi^2 U| |\phi U|\]

On the other hand, one can get following

\[(4.13) \quad g_M(\psi^2 \, U, \psi^2 \, U) = g_M(\phi \psi U, \psi U) = -g_M(\psi U, \psi U) = -|\psi U|^2.\]

Using (4.12) and (4.13) we get

\[(4.14) \quad g_M(\psi^2 \, U, \psi^2 \, U) = -|\psi^2 U| |\phi U|\]

Also, one can easily get

\[(4.15) \quad g_M(\psi^2 \, U, \phi^2 \, U) = -g_M(\psi^2 \, U, \psi^2 \, U).\]

So, by help (4.14) and (4.15) we obtain

\[g_M(\psi^2 \, U, \phi^2 \, U) = 0.\]

We denote the complementary orthogonal distribution to \(\omega(\ker F_*)\) in \((\ker F_*)^\perp\) by \(\mu\). Then we have

\[(4.18) \quad (\ker F_*)^\perp = \omega(\ker F_*) \oplus \mu.\]

Lemma 3. Let \(F\) be a slant Riemannian submersion from a Sasakian manifold \(M(\phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\) with slant angle \(\theta\). Then the following relations are valid

\[(4.16) \quad g_M(\psi U, \psi V) = \cos^2 \theta (g_M(U, V) - \eta(U)\eta(V)),\]

\[(4.17) \quad g_M(\omega U, \omega V) = \sin^2 \theta (g_M(U, V) - \eta(U)\eta(V))\]

for any \(U, V \in \Gamma(\ker F_*)\).

Lemma 4. Let \(F\) be a proper slant Riemannian submersion from a Sasakian manifold \(M(\phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\) then \(\mu\) is an invariant distribution of \((\ker F_*)^\perp\), under the endomorphism \(\phi\).

Proof. For \(X \in \Gamma(\mu)\), from (3.2) and (4.1), we obtain

\[g_M(\phi X, \omega V) = g_M(\phi X, \phi V) - g_M(\phi X, \psi V) = g_M(X, V) - \eta(X)\eta(V) - g_M(\psi X, \psi V) = -g_M(X, \phi^2 V).\]

Using (4.19) and (4.18) we have

\[g_M(\phi X, \omega V) = -\cos^2 \theta g_M(X, V - \eta(V)\xi) = g_M(X, \omega^2 V) = 0.\]
In a similar way, we have \( g_M(\phi X, U) = -g_M(X, \phi U) = 0 \) due to \( \phi U \in \Gamma((\ker F^*) \oplus \omega(\ker F_*)) \) for \( X \in \Gamma(\mu) \) and \( U \in \Gamma(\ker F_*) \). Thus the proof of the lemma is completed. \( \square \)

By help (4.17), we can give following Corollary 1.

**Corollary 1.** Let \( F \) be a proper slant Riemannian submersion from a Sasakian manifold \( M^{2m+1}(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N^n, g_N) \). Let
\[
\{e_1, e_2, \ldots, e_{2m-n}, \xi\}
\]
be a local orthonormal basis of \( (\ker F_*) \), then \( \{\csc \theta we_1, \csc \theta we_2, \ldots, \csc \theta we_{2m-n}\} \) is a local orthonormal basis of \( \omega(\ker F_*) \).

By using (4.18) and Corollary 1 one can easily prove the following Proposition.

**Proposition 3.** Let \( F \) be a proper slant Riemannian submersion from a Sasakian manifold \( M^{2m+1}(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N^n, g_N) \). Then \( \dim(\mu) = 2(n - m) \).

If \( \mu = \{0\} \), then \( n = m \).

By (4.3) and (4.16) we have

**Lemma 5.** Let \( F \) be a proper slant Riemannian submersion from a Sasakian manifold \( M^{2m+1}(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N^n, g_N) \). If \( e_1, e_2, \ldots, e_k, \xi \) are orthogonal unit vector fields in \( (\ker F_*) \), then
\[
\{e_1, \sec \theta \psi e_1, e_2, \sec \theta \psi e_2, \ldots, e_k, \sec \theta \psi e_k, \xi\}
\]
is a local orthonormal basis of \( (\ker F_*) \). Moreover \( \dim(\ker F_*) = 2m - n + 1 = 2k + 1 \) and \( \dim N = n = 2(m - k) \).

**Lemma 6.** Let \( F \) be a slant Riemannian submersion from a Sasakian manifold \( M(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N, g_N) \). If \( \omega \) is parallel then we have
\[
T_U^\psi \psi U = -\cos^2 \theta (T_U U + \eta(U) \omega U)
\]

**Proof.** If \( \omega \) is parallel, from (4.17), we obtain \( CT_U V = T_U \psi V \) for \( U, V \in \Gamma(\ker F_*) \). We interchange \( U \) and \( V \) and use (2.3) we get
\[
T_U \psi V = T_V \psi U.
\]
Substituting \( V \) by \( \psi U \) in the above equation and then using Theorem 3 we get the required formula. \( \square \)

We give a sufficient condition for a slant Riemannian submersion to be harmonic as an analogue of a slant Riemannian submersion from a Sasakian manifold onto a Riemannian manifold in [24].

**Theorem 4.** Let \( F \) be a slant Riemannian submersion from a Sasakian manifold \( M(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N, g_N) \) If \( \omega \) is parallel then \( F \) is a harmonic map.

**Proof.** From [10] we know that \( F \) is harmonic if and only if \( F \) has minimal fibres. Thus \( F \) is harmonic if and only if \( \sum_{i=1}^{n_1} T_{e_i} e_i = 0 \). Thus using the adapted frame for slant Riemannian submersion and by the help of (2.14) and Lemma 5 we can write
\[
\tau = -\sum_{i=1}^{m-n} F_*(T_{e_i} e_i + T_{\sec \theta \psi e_i} \sec \theta \psi e_i) - F_*(T_{\xi} \xi).
\]
Using $T_\xi \xi = 0$ we have
\[
\tau = - \sum_{i=1}^{m-2} F_*(T_\xi e_i + \sec^2 \theta T_\psi e_i)
\]
By virtue of (4.19) in the above equation, we obtain
\[
\tau = - \sum_{i=1}^{m-2} F_*(T_\xi e_i + \sec^2 \theta(T_\xi e_i + \eta(e_i) \omega e_i))
\]
\[
= - \sum_{i=1}^{m-2} F_*(T_\xi e_i - T_\xi e_i) = 0
\]
So we prove that $F$ is harmonic. \(\square\)

Now setting $Q = \psi^2$, we define $\nabla Q$ by
\[
(\nabla_U Q)V = \nabla_U QV - Q\hat{\nabla}_U V
\]
for any $U, V \in \Gamma(\ker F_*)$. We give a characterization for a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ by using the value of $\nabla Q$.

**Proposition 4.** Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then, $\nabla Q = 0$ if and only if $F$ is an anti-invariant submersion.

**Proof.** By using (4.9),
\[
Q\hat{\nabla}_U V = - \cos^2 \theta(\hat{\nabla}_U V - \eta(\hat{\nabla}_U V) \xi)
\]
for each $U, V \in \Gamma(\ker F_*)$, where $\theta$ is slant angle.

On the other hand,
\[
\nabla_U QV = - \cos^2 \theta(\hat{\nabla}_U V - \eta(\hat{\nabla}_U V) \xi + g(V, \psi U) \xi + \eta(V) \psi U).
\]
So, from (4.20) and $\nabla Q = 0$ if and only if $\cos^2 \theta(g(V, \psi U) \xi + \eta(V) \psi U) = 0$ which implies that $\psi U = 0$ or $\theta = \frac{\pi}{2}$. Both the cases verify that $F$ is an anti-invariant submersion. \(\square\)

We now investigate the geometry of leaves of $(\ker F_*)^\perp$ and $\ker F_*$. 

**Proposition 5.** Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on $M$ if and only if
\[
g_M(\mathcal{H} \nabla X Y, \omega \psi U) - \sin^2 \theta g_M(Y, \phi X) \eta(U) = g_M(AX BY, \omega U) + g_M(\mathcal{H} \nabla X CY, \omega U)
\]
for any $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

**Proof.** From (3.3) and (4.1) we have
\[
g_M(\nabla X Y, U) = -g_M(\phi \nabla X \phi Y, U) + g_M(Y, \phi X) \eta(U)
\]
\[
= g_M(\nabla X \phi Y, \phi U) + g_M(Y, \phi X) \eta(U)
\]
\[
= g_M(\nabla X \phi Y, \psi U) + g_M(\nabla X \phi Y, \omega U) + g_M(Y, \phi X) \eta(U).
\]
for any $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$. 

Using (3.3) and (4.1) in (4.22), we obtain
\[ g_M(\nabla_X Y, U) = -g_M(\nabla_X Y, \psi^2 U) - g_M(\nabla_X Y, \omega \psi U) + g_M(Y, \phi X) \eta(U) + g_M(\nabla_X \phi Y, \omega U). \]

By (4.2) and (4.9) we have
\[ g_M(\nabla_X Y, U) = \cos^2 \theta g_M(\nabla_X Y, U) - \cos^2 \theta \eta(U) \eta(\nabla_X Y) - g_M(\nabla_X Y, \psi U) - g_M(\nabla_X B Y, \omega U) + g_M(\nabla_X C Y, \omega U). \]

Using (2.7), (2.8) and (3.6) in the last equation we obtain
\[ \sin^2 \theta g_M(\nabla_X Y, U) = \sin^2 \theta g_M(Y, \phi X) \eta(U) - g_M(H \nabla_X Y, \omega \psi U) + g_M(A X B Y, \omega U) + g_M(H \nabla_X C Y, \omega U). \]

which prove the theorem. \( \square \)

**Proposition 6.** Let \( F \) be a slant Riemannian submersion from a Sasakian manifold \( M(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N, g_N) \). If the distribution \( \ker F^* \) defines a totally geodesic foliation on \( M \) then \( F \) is an invariant submersion.

**Proof.** By (4.25), if the distribution \( \ker F^* \) defines a totally geodesic foliation on \( M \) then we conclude that \( \omega U = 0 \) for any \( U \in \Gamma(\ker F^*) \) which shows that \( F \) is an invariant submersion. \( \square \)

**OpenProblem:**
Let \( F \) be a slant Riemannian submersion from a Sasakian manifold \( M(\phi, \xi, \eta, g_M) \) onto a Riemannian manifold \( (N, g_N) \). In [3], Barrera et.al. define and study the Maslov form of non-invariant slant submanifolds of \( S \)-space form \( \tilde{M} \). They find conditions for it to be closed. By similar discussion in [3] we can define Maslov form \( \Omega \) of \( M \) as the dual form of the vector field \( B \), that is,
\[ \Omega(U) = g_M(U, B) \]
for any \( U \in \Gamma(\ker F^*) \). So, it will be interesting for giving a characterization respect to \( \Omega \) for slant submersions, where \( H = \sum_{i=1}^{m-2} T_{e_i}, e_i + T_{\sec \theta e_i}, \sec \theta e_i \) and
\[ \{e_1, \sec \theta e_1, e_2, \sec \theta e_2, ... , e_k, \sec \theta e_k, \xi \} \]
is a local orthonormal basis of (\( \ker F^* \)).

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