Robustness of Bayesian Neural Networks to Gradient-Based Attacks

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Abstract

Vulnerability to adversarial attacks is one of the principal hurdles to the adoption of deep learning in safety-critical applications. Despite significant efforts, both practical and theoretical, the problem remains open. In this paper, we analyse the geometry of adversarial attacks in the large-data, overparametrized limit for Bayesian Neural Networks (BNNs). We show that, in the limit, vulnerability to gradient-based attacks arises as a result of degeneracy in the data distribution, i.e., when the data lies on a lower-dimensional submanifold of the ambient space. As a direct consequence, we demonstrate that in the limit BNN posteriors are robust to gradient-based adversarial attacks. Experimental results on the MNIST and Fashion MNIST datasets with BNNs trained with Hamiltonian Monte Carlo and Variational Inference support this line of argument, showing that BNNs can display both high accuracy and robustness to gradient based adversarial attacks.

1. Introduction

Adversarial attacks are small, potentially imperceptible perturbations of test inputs that can lead to catastrophic misclassifications in high-dimensional classifiers such as deep neural networks (NN). Since the seminal work of Szegedy et al. (2013), adversarial attacks have been intensively studied, and even state-of-the-art deep learning models, trained on very large data sets, have been shown to be susceptible to such attacks (Goodfellow et al., 2014). In the absence of effective defenses, the widespread existence of adversarial examples has raised serious concerns about the security and robustness of models learned from data (Biggio & Roli, 2018). As a consequence, the development of machine learning models that are robust to adversarial perturbations is an essential pre-condition for their application in safety-critical scenarios, where model failures have already led to fatal accidents (Yadron & Tynan, 2016).

Many attack strategies are based on identifying directions of high variability in the loss function by evaluating gradients w.r.t. input points (see, e.g., Goodfellow et al. (2014); Madry et al. (2017)). Since such variability can be intuitively linked to uncertainty in the prediction, Bayesian Neural Networks (BNNs) (Neal, 2012) have been recently suggested as a more robust deep learning paradigm, a claim that has found some empirical support (Feinman et al., 2017; Gal & Smith, 2018; Bekasov & Murray, 2018; Liu et al., 2018). However, neither the source of this robustness, nor its general applicability are well understood mathematically.

In this paper we show a remarkable property of BNNs: in a suitably defined large data limit, we prove that the gradients of the expected loss function of a BNN w.r.t. the input points vanish. As a consequence, in the limit BNNs are provably immune to gradient-based adversarial attacks.

We verify our theoretical findings on various BNN architectures trained with both Hamiltonian Monte Carlo (HMC) and Variational Inference (VI) on both MNIST and Fashion MNIST data sets, empirically showing that the magnitude of the gradients indeed decreases when more posterior samples are taken. We also show that two popular, highly effective gradient-based attack strategies are unsuccessful on BNNs. Finally, we conduct a large-scale experiment on thousands of different networks, showing that for BNNs high accuracy correlates with high robustness to gradient-based adversarial attacks, contrary to what observed for networks trained via...
standard Stochastic Gradient Descent (SGD) (Zhang et al., 2019).

In summary, this paper makes the following contributions:

- We provide a theoretical framework to analyse adversarial robustness of BNNs in the large data limit.
- We show that, in this limit, the posterior average of the gradients of the loss function vanish, providing robustness against gradient-based attacks.
- We substantiate empirically our arguments on a large-scale experiment, showing empirically that BNNs are immune from the well known accuracy-robustness trade-off.

1.1. Related Work

The robustess of BNNs to adversarial examples has been already observed by Gal & Smith (2018); Bekasov & Murray (2018). In particular, in (Bekasov & Murray, 2018) the authors define Bayesian adversarial spheres and empirically show that, for BNNs trained with HMC, adversarial examples tend to have high uncertainty, while in (Gal & Smith, 2018) sufficient conditions for idealised BNNs to avoid adversarial examples are derived. However, it is unclear how such conditions could be checked in practice, as it would require one to check that the BNN architecture is invariant under all the symmetries of the data.

Empirical methods to detect adversarial examples for BNNs that utilise pointwise uncertainty have been introduced in (Li & Gal, 2017; Feinman et al., 2017; Rawat et al., 2017). However, these approaches have largely relied on Monte Carlo dropout as a posterior inference approximation which can be fooled by attacks that generate adversarial examples with small uncertainty (Carlini & Wagner, 2017). We discuss reasons these methods are fooled under our framework in Sections 5 and 6. Statistical techniques for the quantification of adversarial robustness of BNNs have been introduced by Cardelli et al. (2019a) and employed in (Michelmore et al., 2019) to detect erroneous behaviours in the context of autonomous driving. Furthermore, in (Ye & Zhu, 2018) a Bayesian approach has been considered in the context of adversarial training, where the authors showed improved performances with respect to other, non-Bayesian, adversarial training approaches.

2. Gradient Based Adversarial Attacks

Gradient-based attacks are among the most employed techniques for fast testing of NNs in adversarial settings. The basic principle is to perform gradient ascent on the loss to identify candidate adversarial perturbations. Briefly, in the adversarial setting, instead of minimizing the loss function $L$ w.r.t. the NN weights on a given set (as in the training phase), the objective of the gradient method becomes loss maximisation w.r.t. the input coordinates, while predictions of the network are performed with fixed weights. In other words, the gradient descent on the weights of the network becomes a local gradient ascent on the network input.

Specifically, let $f(x, w)$ be a NN with input $x$ and network parameters (weights) $w$, and denote by $L(f)$ the associated loss function. Given an input point $x^*$, and a strength (i.e. maximum perturbation magnitude) $\epsilon > 0$ of the attack, the worst-case adversarial perturbation can be defined as the point around $x^*$ that maximises the loss:

$$\hat{x} := \arg\max_{x:||x-x^*||_p \leq \epsilon} L(f(\hat{x}, w)).$$

If the the network prediction on $\hat{x}$ differs from the original prediction on $x^*$, then $\hat{x}$ is called an adversarial example. As $f(x, w)$ is non-linear, this poses a non-linear optimisation problem for which several approximate solution methods have been proposed (Biggio & Roli, 2018). While the results discussed in Section 3 hold for any gradient-based methods, in the experiments reported in Section 6 we directly look at the Fast Gradient Sign Method (FGSM) (Goodfellow et al., 2014) and the Projected Gradient Descent method (PGD) (Madry et al., 2017). The former, FGSM, works by approximating $\hat{x}$ by taking an $\epsilon$-step in the direction of the sign of the gradient in $x$. In the case of $l_{\infty}$ perturbations, that is:

$$\hat{x} \approx x^* + \epsilon \cdot \text{sgn}(\nabla_x L(f(x^*, w))).$$

PGD is based on a iterative generalisation of FGSM (Madry et al., 2017). It starts from a random perturbation in an $l_p$-ball of radius $\epsilon$ around the input sample $x^*$, then performs a gradient step in the direction of the greatest loss, projecting the so obtained point in the $\epsilon$-$l_p$-ball centered in $x^*$, and iterates the gradient step to improve approximations $\hat{x}^t$:

$$\hat{x}^{t+1} = \text{Proj}_{l_p}(x^* - \epsilon \cdot \text{sgn}(\nabla_x L(f(x^*, w)))) \cdot \hat{x}^t + \epsilon \cdot \text{sgn}(\nabla_x L(f(\hat{x}^t, w))).$$

We stress that also attacks that do not rely directly on the gradient of the network under consideration have been developed (Papernot et al., 2017; Ilyas et al., 2018; Wicker et al., 2018).

3. Bayesian Neural Networks and Adversarial Attacks

Bayesian modelling aims to capture the intrinsic epistemic uncertainty of data models by defining ensembles of predictors (see e.g. (Barber, 2012)); it does so by turning algorithm parameters (and consequently also predictions) into random variables. In a NNs scenario (Neal, 2012), one starts with a prior measure over the network weights $p(w)$. The fit of the
network with weights $w$ to the data $D$ is assessed through the likelihood $p(D|w)$ (Bishop, 2006). Bayesian inference then combines likelihood and prior via Bayes theorem to obtain a posterior measure on the space of weights

$$p(w|D) = \frac{1}{Z} p(D|w) p(w).$$

(1)

Maximising the likelihood w.r.t. the weights $w$ is in general equivalent to minimising the loss function in standard NNs; indeed, standard training of NNs can be viewed as an approximation to Bayesian inference which replaces the posterior distribution with a delta function at its mode.

It should be noted that obtaining the posterior distribution exactly is impossible for non-linear/ non-conjugate models such as NNs. Asymptotically exact samples from the posterior distribution can be obtained via procedures such as Hamiltonian Monte Carlo (HMC); approximate samples can be obtained more cheaply via Variational Inference (VI).

Irrespective of the posterior inference method of choice, Bayesian predictions at a new input $x^*$ are obtained from an ensemble of NNs, each with its individual weights drawn from the posterior distribution $p(w|D)$

$$f(x^*|D) = \langle f(x^*, w) \rangle_{p(w|D)}$$

$$\approx \frac{1}{n} \sum_{i=1}^{n} f(x^*, w_i) \quad w_i \sim p(w|D)$$

(2)

where $\langle \cdot \rangle_p$ denotes expectation w.r.t. the distribution $p$. The ensemble of NNs is the predictive distribution of the BNN.

Attacks against a BNN are attacks against the predictive distribution (2). The FGSM attack then becomes

$$\hat{x} = x + \epsilon \text{sgn} \left( \langle \nabla_x L(x, w) \rangle_{p(w|D)} \right)$$

(3)

$$\approx x + \epsilon \text{sgn} \left( \sum_{i=1}^{n} \nabla_x L(x, w_i) \right)$$

(4)

where the final expression is a Monte Carlo approximation with samples $w_i$ drawn from the posterior $p(w|D)$. Expression for the PGD or other gradient-based attacks are analogous.

4. Adversarial robustness of Bayesian predictive distributions

Equation (3) suggests a possible explanation for the observed robustness of BNNs to adversarial attacks: the averaging under the posterior might lead to cancellations in the final expectation. It turns out that this averaging property is intimately related to the geometry of the so called data manifold $M_D \subset \mathbb{R}^d$, i.e. the support of the data generating distribution $p(D)$. The key result that we leverage is a recent breakthrough (Du et al., 2018; Rotskoff & Vanden-Eijnden, 2018; Mei et al., 2018) which proved that global convergence of (stochastic) gradient descent (at the distributional level) in the overparametrised, large data limit. We refer to the original publications for precise definitions; we define as a fully trained, overparametrized BNN an ensemble of NNs satisfying the conditions in (Rotskoff & Vanden-Eijnden, 2018) and at full convergence of the training algorithm.

We now prove our main result:

**Theorem 1.** Let $f(x, w)$ be a fully trained overparametrized BNN on a prediction problem with data manifold $M_D \subset \mathbb{R}^d$ and posterior weight distribution $p(w|D)$. Assuming $M_D \subset C^\infty$ almost everywhere, in the large data limit we have a.e. on $M_D$

$$\langle \nabla_x L(x, w) \rangle_{p(w|D)} = 0$$

(5)

By the definition of the FGSM attack in Equation (3) and other gradient-based attacks, Theorem 1 directly implies that any gradient-based attack will be ineffective against a BNN. The proof of Theorem 1 is instructive; we divide it in two parts in the following.

**Dimensionality of the data manifold** An important result proved in (Du et al., 2018; Rotskoff & Vanden-Eijnden, 2018; Mei et al., 2018) is that, at convergence, overparametrised NNs provably achieve zero loss on the whole data manifold $M_D$ in the infinite data limit. An immediate consequence of this result is the following

**Lemma 1.** Let $f(x, w)$ be a fully trained overparametrized NN on a prediction problem with data manifold $M_D \subset \mathbb{R}^d$. Let $x^* \in M_D$ s.t. $B_d(x^*, \epsilon) \subset M_D$, with $B_d(x^*, \epsilon)$ the d-dimensional ball centred at $x^*$ of radius $\epsilon$ for some $\epsilon > 0$. Then $f(x, w)$ is robust to gradient-based attacks at $x^*$ in the large data limit.

This is a trivial consequence of the network achieving zero loss on the data manifold (Du et al., 2018; Rotskoff & Vanden-Eijnden, 2018; Mei et al., 2018), since the function would be locally constant at $x^*$.

A corollary of Lemma 1 is

**Corollary 1.** Let $f(x, w)$ be a fully trained overparametrized NN on a prediction problem with data manifold $M_D \subset \mathbb{R}^d$ smooth a.e. If $f$ is vulnerable to gradient-based attacks almost everywhere in $M_D$ in the infinite data limit, then $\dim(M_D) < d$.

This corollary confirms the widely held conjecture that adversarial attacks originate from degeneracies of the data manifold (Goodfellow et al., 2014; Fawzi et al., 2018). In
A consequence of Corollary 1 is that $M$ on a prediction problem with data manifold. The proof of this lemma rests on the observation that finding a suitable function is equivalent to solving the Cauchy boundary value problem specified by zero loss on the data manifold and normal gradient field $-\nabla w(x)$. Since we are in the overparameterized, large data limit, any such function will be realisable as a NN with suitable weights choice $w'$.

### 5. Consequences and limitations

Theorem 1 has the natural consequence of protecting BNNs against all gradient-based attacks, due to the vanishing average of the expectation of the gradients in the limit. Its proof also sheds light on a number of observations made in recent years. Before moving on to empirically validating the theorem, it is worth reflecting on some of its implications and limitations:

- Theorem 1 holds in a specific thermodynamic limit, however we expect the averaging effect of BNN gradients to still provide considerable protection in conditions where the network architecture and the data amount lead to high accuracy and strong expressivity. In practice, high accuracy might be a good indicator of robustness.

- Theorem 1 holds when the ensemble is drawn from the true posterior distribution; nevertheless it is not obvious (and likely not true) that the posterior distribution is the sole ensemble with the zero averaging property of the gradients. Cheaper approximate Bayesian inference methods which retain ensemble predictions such as VI may in practice provide good protection.

- Theorem 1 is proven under the assumption of uniform priors; in practice, (vague) Gaussian priors are more frequently used for computational reasons. Once again, unless the priors are too informative, we do not expect a major deviation from the idealised case.

- Gaussian Processes (Williams & Rasmussen, 2006) are equivalent to infinitely wide BNNs with a single hidden layer, and therefore constitute overparameterized BNNs by definition (although scaling their training to the large data limit might be problematic). Theorem 1 provides theoretical backing to recent empirical observations of their adversarial robustness (Blaas et al., 2019; Cardelli et al., 2019b).

- While the Bayesian posterior ensemble may not be the only randomization to provide protection, it is clear that some simpler randomizations such as bootstrap will be ineffective, as noted empirically in (Bekasov & Murray, 2018). This is because bootstrap resampling introduces variability along the data manifold, rather than randomising in orthogonal directions. In this sense, the often repeated mantra that bootstrap is an approximation to Bayesian inference is strikingly
Figure 1. The expected loss gradients of BNNs exhibit a vanishing behaviour when increasing the number of samples from the posterior predictive distribution. Above, we show example images from MNIST (top row) and Fashion MNIST (bottom row) and their expected loss gradients wrt networks trained with HMC (left) and VI (right). To the right of the images we plot a heat map of gradient values. The set of figures to the left demonstrate the vanishing behavior of gradients wrt a posterior distribution approximated with HMC; whereas the figures on right demonstrate the vanishing behavior of gradients wrt a posterior distribution approximated with VI.

inaccurate when the data distribution has zero measure support. Similarly, we don’t necessarily expect gradient smoothing approaches to be successful (Athalye et al., 2018), since the type of smoothing performed by Bayesian inference is specifically informed by the geometry of the data manifold.

6. Empirical Results

In this section we investigate the applicability and relevance of our theoretical results on different BNN architectures. We train a variety of BNNs on the MNIST and Fashion MNIST (Xiao et al., 2017) datasets, and evaluate posterior distributions using HMC (asymptotically exact but more expensive), as well as cheaper approximations through VI. We explicitly verify the zero-averaging property of gradients implied by Theorem 1 (Section 6.1), and show (Section 6.2) empirically that FGSM and PGD attacks on BNNs fail to perform better than random attack. Finally, in Section 6.3 we analyse how robustness and accuracy are correlated on thousands of different neural networks architectures trained with HMC, VI and standard Stochastic Gradient Descent (SGD).

6.1. Evaluation of the Gradient of the Loss for BNNs

To investigate the practical relevance of Theorem 1 in the finite data setting, we consider two large fully-connected BNNs with Gaussian priors placed over all of the networks weights and biases, evaluated over the popular MNIST and Fashion MNIST benchmarks. We train a two hidden layers (with 1024 neurons per layer, and approximately a total of 1.8 million parameters) network with HMC and a three hidden layers networks (512 neurons per layer, i.e., almost 1 million parameters) with VI. The BNNs achieved approximately 95% test set accuracy on MNIST and 89% on Fashion MNIST when trained with HMC; as well as 95% and 92% when trained with VI (using KL minimization) on MNIST and Fashion MNIST, respectively. The numbers of parameters and high accuracies suggest that this practical scenario is close to the thermodynamic limit of Theorem 1.

First, we examine the behaviour of the component-wise expectation of the loss gradient as more samples from the posterior distribution, \( p(w|D) \), are used in the expectation computation. In line with Theorem 1, we expect gradient expectations to shrink the more posterior samples we use. Figure 1 shows anecdotal evidence of this trend on four example images from both MNIST and Fashion MNIST, for BNNs trained with HMC (left half of the figure) and VI (right half of the figure). The heatmaps show the components of the expected gradient loss, when 1, 10 and 100 posterior samples are used in the computation, demonstrating a clear decrease component-wise of the gradients as we increase the number of samples and, hence, better approximate the posterior expectation.

We then systematically examine convergence of all the components of the expected loss gradient in Figure 2 for both HMC (top row) and VI (bottom row). Each dot represents one of the \( 28 \times 28 \) components of the expected loss gradients computed on 1000 test images, for a total of 784000 gradients components used to shown their empirical distribution for each number of samples used to approximate the expectation (x-axis of the plots).

For both HMC and VI the magnitude of the gradient components drop quickly and tend to stabilize around the zero mean at about 100 samples. The residual variance is to be expected, since we are conducting our experiments on a finite approximation of the limiting regime.

6.2. Gradient-Based Attacks for BNNs

Showing that gradient cancellation occurs does not directly imply that the network predictions are robust to gradient-
Figure 2. The components values of the expected loss gradients approaches zero as the number of samples from the posterior distribution increases. For each fixed number of samples, the figure shows 784 gradient components for 1000 different test images, from both the MNIST and Fashion MNIST datasets. The gradients are computed on HMC (a) and VI (b) trained BNNs. We inset a plot with a more suitable y-axis range in the case of HMC/MNIST in order to better visualize the trend of convergence.

based attacks in the finite case. For example, FGSM attacks are crafted such that the direction of the manipulation is given only by the sign of expectation of the loss gradient and not the magnitude. Thus, even if the entries of the expectation drop to an infinitesimal magnitude, but maintain the correct sign, then FGSM will produce the same attack direction.

In order to test the implications of vanishing gradients on the robustness of the posterior predictive distribution against gradient-based attacks, we compare FGSM and PGD to a random attack. The random attack simply draws a random value from the set \{-\epsilon, \epsilon\} for each input dimension and treats this as the resulting perturbation on a test image.

In Figure 3 we compare the effectiveness of FGSM, PGD and of the random attack as the number of samples drawn from the posterior distribution increases. For the evaluation we consider a notion of robustness, which we call softmax difference. Given \( n \) posterior samples from a BNN, let \( f_n(x, w) := \sum_{i=1}^{n} f(w_i, x) \) be the estimator of the expected output of the BNN computed over \( n \) samples drawn from the BNN posterior \( p(w|D) \), then the softmax difference is defined as:

\[
\frac{1}{N} \sum_{j=1}^{N} |f_n(x_j, w) - f_n(\tilde{x}_j, w)|_\infty.
\]

That is, the softmax difference computes the average distance over \( N \) test points of the variation of the softmax layer caused by the adversarial attack of a network whose posterior distribution is estimated with \( n \) weights. In Figure 3 we consider the cases where FGSM and PGD attacks are evaluated over the full posterior or only with respect to the weights used to craft the attack (‘fixed FGSM’ and ‘fixed PGD’). In other words, in ‘fixed’ attacks the attacker has access to all posterior samples used to evaluate expectations (in particular, the case of a single sample is equivalent to a deterministic NN). The fixed sample case converges to the full inference case as the number of samples \( n \) goes to \( \infty \).

In the ideal case, we expect, via Theorem 1, that FGSM and PGD would perform identically to a random attack. As the number of samples increases, if expectation of the loss gradient vanishes so does any potentially detected attack direction, thus any small residual noise in the loss gradient might be tantamount to random noise.
6.3. Robustness Accuracy Analysis in Deterministic and Bayesian Neural Networks

In the discussion of Section 5, we suggested that as a consequence of Theorem 1, high accuracy might be related to high robustness to gradient based attacks in Bayesian settings, this would run counter to what has been commonly observed for deterministic neural networks trained with SGD (Zhang et al., 2019). In this Section, we look at an array of 1500 different BNN architectures trained with HMC and VI on both the MNIST and Fashion-MNIST datasets and experimentally evaluate their accuracy/robustness trade-off on FGSM attacks as compared to that obtained with comparable deterministic NNs trained via SGD based methods.

The results of this analyses are plotted in Figures 4 and 5 for MNIST and Fashion MNIST respectively. Each dot in the scatter plots represent the results obtained for each specific network architecture trained with SGD (blue dots), HMC (pink dots in plots (a)) and VI (pink dots in plots (b)). As already reported in the literature (Zhang et al., 2019) we observe a marked trade-off between accuracy and robustness (i.e., 1 - softmax difference) for deterministic networks, with network having high-accuracy being also fragile to FGSM, and vice-versa. Interestingly, this trend is fully reversed for BNNs trained with HMC (plots (a) in Figures 4-5), where we find that as networks become more accurate, they additionally become more robust to FGSM attacks.

To further examine this trend we inspect how the number of parameters affects the robustness of the BNNs. In the case of HMC, we plot (boxplots in plots (a) of Figures 4-5) the effect of the number of neurons in the network versus the robustness of the resulting posterior and find that there exists an increasing trend in robustness as we increase the number of neurons in the network. Similarly, for VI we observe that there is some trend dealing with the size of the model, but we only observe this in the case of VI trained on MNIST where it can be seen that model robustness may increase as the width of the layers increases, but this can also lead to poor robustness as well (which may be indicative of mode collapse). This is in line with what we observed in the previous two sections: as the network approaches the over-parametrised limits the conditions for Theorem 1 are approximately met and the network is protected against gradients attack.
Figure 4. Robustness-Accuracy trade-off on the MNIST dataset for network trained with HMC (left), VI (right) and SGD. While a marked trade-off between accuracy and robustness occur for deterministic network trained with SGD, for HMC and VI the experiments show a positive correlation between accuracy and robustness. The boxplots show the correlation between model capacity and robustness.

Figure 5. Robustness-Accuracy trade-off on the Fashion MNIST dataset for network trained with HMC (left), VI (right) and SGD. The boxplots show the correlation between model capacity and robustness. Notice that different attack strength (\(\epsilon\)) are used for the three methods accordingly to their average robustness.

As expected, the trade-off behaviours are less obvious for the BNNs trained with VI and on Fashion-MNIST, that is, with a more approximate inference method on a more complex dataset. In particular, in plot (b) of Figure 5 we find that, similarly to the deterministic case, also for BNNs, robustness seems to have a negative correlation with accuracy. However, interestingly, we should note that for this particular case VI was the most robust in terms of its ability to withstand attacks with high magnitude (\(\epsilon = 0.15\)). So, while we report that robustness is not positively correlated with accuracy in this case, we do find that the tested networks trained with VI obtain greater robustness to gradient-based attacks.

7. Conclusions

The quest for robust, data-driven models is an essential component towards the construction of AI-based technologies. In this respect, we believe that the fact that Bayesian ensembles of NNs can evade a broad class of adversarial attacks will be of great relevance.

While promising, this result comes with some significant limitations. First and foremost, performing Bayesian inference in large non-linear models is extremely challenging. While in our hands cheaper approximations such as VI also enjoyed a degree of adversarial robustness, albeit reduced, there are no guarantees that this will hold in general. To this end, we hope that this result will spark renewed interest in the pursuit of efficient Bayesian inference algorithms.

Secondly, our theoretical results hold in a thermodynamic limit which is never realised in practice. More worryingly, we currently have no rigorous diagnostics to understand how near we are to the limit case, and can only reason about this empirically. We notice here that several other studies (Bekasov & Murray, 2018; Li & Gal, 2017; Feinman et al., 2017; Rawat et al., 2017) have focused on pointwise uncertainty to detect adversarial behaviour; while this does not appear relevant in the limit scenario, it might be a promising indicator of robustness in finite data conditions.

Thirdly, we have focused on two attack strategies which directly utilise gradients in our empirical evaluation. More complex gradient-based attacks, such as (Carlini & Wagner, 2016; Papernot et al., 2017; Moosavi-Dezfooli et al., 2016),
as well as non-gradient based/ query-based attacks, also exist (Ilyas et al., 2018; Wicker et al., 2018). Evaluating the robustness of BNNs against these attacks would also be interesting.

Finally, the proof of our main result highlighted a profound connection between adversarial vulnerability and the geometry of data manifolds; it was this connection that enabled us to show that randomisation might be an effective way to provide robustness in the high dimensional context. We hope that this connection will inspire novel algorithmic strategies which can offer adversarial protection at a cheaper computational cost.
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