Abstract. We prove that, if $\Delta_1$ is the Hodge Laplacian acting on differential 1-forms on the $(2n+1)$-dimensional Heisenberg group, and if $m$ is a Mihlin-Hörmander multiplier on the positive half-line, with $L^2$-order of smoothness greater than $n + \frac{1}{2}$, then $m(\Delta_1)$ is $L^p$-bounded for $1 < p < \infty$. Our approach leads to an explicit description of the spectral decomposition of $\Delta_1$ on the space of $L^2$-forms in terms of the spectral analysis of the sub-Laplacian $L$ and the central derivative $T$, acting on scalar-valued functions.

Introduction

The $(2n+1)$-dimensional Heisenberg group $H_n$ has a (unique modulo dilations) left-invariant Riemannian structure which is invariant under the action of the unitary group $U(n)$ by automorphisms (i.e. the natural action on the $\mathbb{C}^n$-component, when $H_n$ is realized as $\mathbb{C}^n \times \mathbb{R}$). Various differential-geometric aspects of this structure have been analyzed in the literature [DT, L, R1, R2].

On the contrary, from an analytic point of view, most of the attention has been given to the operators related to the CR-structure on $H_n$, or to its sub-Riemannian structure (the sub-Laplacian and the Kohn Laplacians), leaving only a marginal rôle to the “Riemannian” operators. Our interest here is in the Hodge Laplacians $\Delta_k = dd^* + d^*d$ acting on differential $k$-forms on $H_n$, a family of operators that naturally arise in the Riemannian setting, and in their $L^p$-functional calculus. For $k \geq 1$, $\Delta_k$ is far from being diagonal (in contrast with the Kohn Laplacians for the $\bar{\partial}_b$-complex) in any reasonable basis of forms. This makes its analysis quite involved, with a level of complexity that increases with $k$ (as long as $k \leq n$; it goes without saying that we are dispensed from treating higher values of $k$ by Hodge duality). For this reason our results are limited to the case $k = 1$ (together with the “scalar” case $k = 0$), and we believe that investigating Laplacians on higher-order forms would require a more sophisticated understanding of the decomposition of the space of $L^2$-forms under the action of $\Delta_k$.

Our main result is Theorem 6.8, proving that, if $m$ is a Mihlin-Hörmander multiplier on the positive half-line with a sufficiently high order of smoothness, then $m(\Delta_1)$ is bounded on 1-forms in $L^p$, for $1 < p < \infty$. The order of smoothness is measured in terms of “scale-invariant” local Sobolev norms (called $L^2_{\tau, \text{sloc}}$), and $\tau$
is required to be strictly larger than \( n + \frac{1}{2} \), i.e. half of the dimension of \( H_n \) as a manifold.

As a preliminary result, the same statement is proved for the Laplace-Beltrami operator \( \Delta_0 \) acting on functions (Theorem 6.4). That the critical value for \( \tau \) is \( n + \frac{1}{2} \) in this case is not surprising, because \( \Delta_0 \) locally behaves like the ordinary Laplacian on \( \mathbb{R}^{2n+1} \), and at infinity like the sub-Laplacian \( L \), and it is known that \( n + \frac{1}{2} \) is critical for both these operators (see [MS] for what concerns \( L \)). To be more specific, if we scale on \( H_n \) isotropically by a parameter tending to zero, we produce a deformation of \( \Delta_0 \) which in the limit gives the Laplacian; on the other hand, if we scale by a parameter tending to infinity in the automorphic (non-isotropic) way, the resulting deformation of \( \Delta_0 \) tends to \( L \) [NRS]. As observed in [R2], this doubly asymptotic picture has no analogue for forms of order \( k \geq 1 \). The fact that \( n + \frac{1}{2} \) remains the critical value for \( \tau \) also when \( k = 1 \) turns out to be a consequence of the fact that the space of \( L^2 \)-1-forms decomposes as the orthogonal sum of five subspaces such that on each of them the action of \( \Delta_1 \) is unitarily equivalent (possibly modulo an intertwining operator) to the action of a “scalar” differential or pseudo-differential operator related to \( \Delta_0 \). Precisely, we find

1. the space \( V_1 \) of exact forms, where the action of \( \Delta_1 \) is unitarily equivalent to \( \Delta_0 \) acting on scalar functions;
2. the space \( V_2^+ \) of \( \partial^*_b \)-closed (1, 0)-forms, where \( \Delta_1 \) acts as \( \Delta_0 - iT \) componentwise;
3. the space \( V_2^- \) of \( \bar{\partial}^*_b \)-closed (0, 1)-forms, where \( \Delta_1 \) acts as \( \Delta_0 + iT \) componentwise;
4. two other subspaces, \( V_3^\pm \), where the action of \( \Delta_1 \) is unitarily equivalent to that of \( \Delta_0 + \frac{n}{2} \pm \sqrt{\Delta_0 + \frac{n^2}{4}} \) on scalar functions.

Whereas \( V_1, V_2^+, V_2^- \) can be detected by a simple inspection, the last two subspaces are not so visible, and their description involves a rather delicate formalism. The presence of \( V_3^\pm \) had been detected before in [L] for \( H_1 \). We thank Michael Christ for bringing this reference to our attention.

Once this is established, the task is to prove first that the decomposition of the space of 1-forms into these five subspaces also makes sense in \( L^p \) for \( p \neq 2 \) in the range \( 1 < p < \infty \) (i.e. to prove that the corresponding orthogonal projections are \( L^p \)-bounded), and then to prove that Mihlin-Hörmander multipliers with order of smoothness \( \tau > n + \frac{1}{2} \) give bounded operators on \( L^p \) when applied to the five operators above. In doing so, we heavily rely on the results in [MRS1, MRS2].

1. The Hodge Laplacians

Let \( H_n \) be the \((2n+1)\)-dimensional Heisenberg group with coordinates \((x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\), and with a basis of left-invariant vector fields given by

\[
X_j = \partial_{x_j} - \frac{y_j}{2} \partial_t , \quad Y_j = \partial_{y_j} + \frac{x_j}{2} \partial_t , \quad T = \partial_t ,
\]

for \( 1 \leq j \leq n \). The dual basis of 1-forms is given by the \( 2n \) elementary forms \( dx_j, dy_j \) and by the contact form

\[
\theta = dt - \frac{1}{2} \sum_{j=1}^n (x_j dy_j - y_j dx_j) .
\]
We denote by $\Lambda^k = \Lambda^k(h_n^*)$ the $k$-th exterior product of the dual of the Lie algebra $h_n$ of $H_n$ (also identifiable with the space of left-invariant $k$-forms on $H_n$). We call

$$\mathcal{D} \Lambda^k(H_n) = \mathcal{D}(H_n) \otimes \Lambda^k$$

the space of smooth $k$-forms on $H_n$ with compact support. This notation will be consistently adapted to function spaces other than $\mathcal{D}(H_n)$ or to subspaces of $\Lambda^k$.

We shall often meet differential (or pseudo-differential) operators which act separately on each scalar component of a given form. In these cases we will denote by the same symbol the operator, call it $D$, acting on scalar-valued functions, and the corresponding operator acting on forms, which should be more correctly denoted by $D \otimes I$.

It will be convenient for us to work with different bases of complex vector fields and forms. We then set

$$B_j = \frac{1}{\sqrt{2}}(X_j - iY_j), \quad \bar{B}_j = \frac{1}{\sqrt{2}}(X_j + iY_j),$$

and

$$\beta_j = \frac{1}{\sqrt{2}}(dx_j + idy_j), \quad \bar{\beta}_j = \frac{1}{\sqrt{2}}(dx_j - idy_j).$$

The relevant commutation relation is

$$[B_j, \bar{B}_j] = iT.$$

The differential $df$ of a smooth function is then given by

$$df = \sum_{j=1}^{n} (X_j f dx_j + Y_j f dy_j) + Tf \theta = \sum_{j=1}^{n} (B_j f \beta_j + \bar{B}_j f \bar{\beta}_j) + Tf \theta,$$

and similarly for exterior derivatives of differential forms. Observe that, in particular,

$$d\theta = -\sum_{j=1}^{n} dx_j \wedge dy_j = -i \sum_{j=1}^{n} \beta_j \wedge \bar{\beta}_j.$$

A $k$-form $\omega$ decomposes uniquely as

$$\omega = \omega_1 + \theta \wedge \omega_2,$$

with

$$\omega_1 = \sum_{|I|+|I'|=k} f_{I,I'} \beta^I \wedge \bar{\beta}^{I'},$$

$$\omega_2 = \sum_{|I|+|I'|=k-1} g_{J,J'} \beta^J \wedge \bar{\beta}^{J'},$$

where we have followed the usual convention that, if $I = \{i_1, \ldots, i_p\}$ is a finite subset of $\{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_p$, then

$$\beta^I = \beta_{i_1} \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_p}.$$
and similarly for $\bar{\beta}'$.

Clearly, $\omega_2 = 0$ if $k = 0$ and $\omega_1 = 0$ for $k = 2n + 1$.

We say that $\omega$ is horizontal if $\omega_2 = 0$, and we call horizontal differential of a smooth function $f$ the horizontal form

$$d_H f = \sum_{j=1}^{n} (B_j f \beta_j + \bar{B}_j f \bar{\beta}_j).$$

We denote by $\Lambda^k_{H}$ (resp. $\mathcal{D}(\Lambda^k_{H}(H_n))$) the subspace of $\Lambda^k$ consisting of horizontal $k$-forms which are left-invariant (resp. with compact support).

The notion of “horizontal form” presents serious problems, that are treated in a systematic way in [R1]. For instance, the natural extension to horizontal forms of the operator $d_H$ in (1.6) does not define a complex, because $d_H^2 \neq 0$. However we shall not use any such property, and on the other hand (1.6) provides a convenient notation. For instance, w.r. to the decomposition (1.4), we have

$$d(\omega_1 + \theta \wedge \omega_2) = \left( d_H \omega_1 + (d\theta) \wedge \omega_2 \right) + \theta \wedge (T \omega_1 - d_H \omega_2).$$

Identifying $\omega$ with the pair $\left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right)$ in (1.4), the operator

$$d = d_k : \mathcal{D}(\Lambda^k(H_n)) \to \mathcal{D}(\Lambda^{k+1}(H_n))$$

is then represented by the matrix

$$(1.8) \qquad d = \begin{pmatrix} d_H & e(d\theta) \\ T & -d_H \end{pmatrix},$$

where $e$ denotes exterior multiplication, i.e. $e(d\theta)\omega = (d\theta) \wedge \omega$.

We introduce on $H_n$ the left-invariant Riemannian metric that makes the basis (1.1) orthonormal at each point. W.r. to the induced inner product on $\Lambda^k$, the elements

$$\beta^I \wedge \bar{\beta}^I' , \quad \theta \wedge \beta^J \wedge \bar{\beta}^J'$$

(with $|I| + |I'| = k$, $|J| + |J'| = k - 1$) also form an orthonormal basis. Let $\omega, \omega'$ be two $k$-forms, with

$$\omega = \omega_1 + \theta \wedge \omega_2 , \quad \omega' = \omega'_1 + \theta \wedge \omega'_2 ,$$

and let $f_{I,I'}, g_{J,J'}$ be the coefficients of $\omega$ as in (1.5), and $f'_{I,I'}, g'_{J,J'}$ the corresponding coefficients of $\omega'$. The inner product in $L^2 \Lambda^k(H_n) = L^2(H_n) \otimes \Lambda^k$ is such that

$$\langle \omega, \omega' \rangle_k = \sum_{I,I'} \langle f_{I,I'}, f'_{I,I'} \rangle + \sum_{J,J'} \langle g_{J,J'}, g'_{J,J'} \rangle$$

$$= \langle \omega_1, \omega'_1 \rangle_k + \langle \omega_2, \omega'_2 \rangle_{k-1},$$

where the inner products of the coefficients are taken in $L^2(H_n)$. In particular the decomposition (1.4) is orthogonal. The formal adjoint of $d_{k-1}$,

$$d^* = d^*_{k-1} : \mathcal{D}(\Lambda^k(H_n)) \to \mathcal{D}(\Lambda^{k-1}(H_n)),$$
is represented by the adjoint matrix of (1.8), i.e.

\[
(1.9) \quad d^* = \begin{pmatrix} d_H^* & -T \\ i(d\theta) & -d_H^* \end{pmatrix},
\]

where \(i(d\theta) = e(d\theta)^*\) is the interior multiplication operator

\[
i(d\theta)\omega = i \sum_{j=1}^{n} i(\beta_j)i(\beta_j)\omega = i \sum_{j=1}^{n} \beta_j \iota(\beta_j\omega).
\]

It follows that the Hodge Laplacian on \(k\)-forms

\[
\Delta_k = dd^* + d^*d
\]

is expressed by the matrix

\[
\Delta_k = \begin{pmatrix} d_H & e(d\theta) \\ T & -d_H \end{pmatrix} \begin{pmatrix} d_H^* & -T \\ i(d\theta) & -d_H^* \end{pmatrix} + \begin{pmatrix} d_H^* & -T \\ i(d\theta) & -d_H^* \end{pmatrix} \begin{pmatrix} d_H & e(d\theta) \\ T & -d_H \end{pmatrix}
\]

\[
(1.10) \quad = \begin{pmatrix} \Delta_H - T^2 + e(d\theta)i(d\theta) \\ [i(d\theta), d_H] \end{pmatrix} \begin{pmatrix} [d_H^*, e(d\theta)] \\ \Delta_H - T^2 + i(d\theta)e(d\theta) \end{pmatrix},
\]

where

\[
\Delta_H = d_Hd_H^* + d_H^*d_H.
\]

In particular, for \(k = 0\) we simply have

\[
(1.11) \quad \Delta_0 = d^*d = -\sum_{j=1}^{n} (B_j\bar{B}_j + \bar{B}_jB_j) - T^2,
\]

acting on scalar-valued functions.

2. The CR-structure

It is possible to simplify various terms in (1.10) and get a better understanding of that formula by appropriate decompositions of the space of horizontal forms. In order to do so, we must refer to the standard CR-structure on \(H_n\). The operators

\[
\partial_b f = \sum_{j=1}^{n} B_j f \beta_j, \quad \bar{\partial}_b f = \sum_{j=1}^{n} \bar{B}_j f \bar{\beta}_j,
\]

initially defined on functions, are naturally extended to forms. They satisfy the following identities:

\[
(2.1) \quad \partial_b^2 = \bar{\partial}_b^2 = \partial_b \bar{\partial}_b^* + \bar{\partial}_b^* \partial_b = \bar{\partial}_b \partial_b^* + \partial_b^* \bar{\partial}_b = 0,
\]

as well as

\[
(2.2) \quad d = \partial_b + \bar{\partial}_b.
\]
Observe that, by (1.8),
\[ d_H^2 = \partial_b \bar{\partial}_b + \bar{\partial}_b \partial_b = -Te(d\theta) . \]

Setting
\[ \square = \partial_b \partial_b^* + \partial_b^* \partial_b , \quad \bar{\square} = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b , \]
we obtain that
\[ (2.3) \quad \Delta_H = \square + \bar{\square} . \]

Then \( \square \) is the Kohn Laplacian and \( \square \) its complex conjugate. A \( (p, q) \)-form, \( p, q \leq n \), is a horizontal form
\[ \omega = \sum_{|I|=p, |I'|=q} f_{I, I'} \beta^I \wedge \bar{\beta}^{I'} . \]

Clearly, the decomposition (1.4) can be further refined, by decomposing \( \omega_1 \) as a sum of \( (p, k - p) \)-forms and \( \omega_2 \) as a sum of \( (p, k - 1 - p) \)-forms. The notation \( \Lambda^{p, q}, \mathcal{D}\Lambda^{p, q}(H_n) \), etc. refers to \( (p, q) \)-forms. It is well known [FS] that \( \square \) and \( \bar{\square} \) act as scalar operators on \( (p, q) \)-forms (we shall write \( \square_{p, q} \) and \( \bar{\square}_{p, q} \) when appropriate). If
\[ L = -\sum_{j=1}^{n} (B_j \bar{B}_j + \bar{B}_j B_j) = -\sum_{j=1}^{n} (X_j^2 + Y_j^2) \]
is the sub-Laplacian, then
\[ (2.4) \quad \square_{p, q} = \frac{1}{2} L + i \left( \frac{n}{2} - p \right) T ; \]
similarly,
\[ (2.5) \quad \bar{\square}_{p, q} = \frac{1}{2} L - i \left( \frac{n}{2} - q \right) T . \]

It follows from (2.3) that
\[ (2.6) \quad \Delta_H = L + i(q - p) T \]
on \( (p, q) \)-forms.

We next describe the structure of the remaining diagonal terms in (1.10), i.e. \( e(d\theta)i(d\theta) \) and its transpose \( i(d\theta)e(d\theta) \). Since these operators do not involve any differentiation, their action can be analyzed on exterior forms. Many of the formulas below are also stated in [R1,2] and are derived from the formulas for the Lefshetz decomposition on Kähler manifolds in [W]. For completeness, we give some explicit proofs, and we allow forms of any order, even though we shall later restrict ourselves to 1-forms.
Proposition 2.1. Consider the following subspaces of $\Lambda^{p,q}$,

\[ V_{j}^{p,q} = e(d\theta)j \text{ ker}_{\Lambda^{p-j,q-j}} i(d\theta), \]
\[ W_{\ell}^{p,q} = i(d\theta)\ell \text{ ker}_{\Lambda^{p+\ell,q+\ell}} e(d\theta). \]

Then $V_{j}^{p,q}$ is non-trivial if and only if $\max\{0, k - n\} \leq j \leq \min\{p, q\}$, $W_{\ell}^{p,q}$ is non-trivial if and only if $\max\{0, n - k\} \leq \ell \leq \min\{n - p, n - q\}$, and we have the equality

\[ V_{j}^{p,q} = W_{\ell}^{p,q}, \quad \text{for } \ell = j + n - k = \ell(j). \]

Moreover, $\Lambda^{p,q}$ is the orthogonal sum of the non-trivial $V_{j}^{p,q}$, and

\[ e(d\theta)i(d\theta) = j(j + 1 + n - k) = (\ell(j) + 1)(\ell(j) + k - n) \]
\[ i(d\theta)e(d\theta) = (j + 1)(j + n - k) = \ell(j)(\ell(j) + 1 + k - n) \]

on $V_{j}^{p,q}$.

Proof. Because $\text{ker} i(d\theta) = (e(d\theta)\Lambda^{p-1,q-1})^\perp$ inside $\Lambda^{p,q}$, every $(p, q)$-form $\omega$ can be uniquely decomposed into the orthogonal sum

\[ \omega = \omega_0 + e(d\theta)\alpha \]

with $i(d\theta)\omega_0 = 0$. Next, we decompose $\alpha$ as

\[ \alpha = \omega_1 + e(d\theta)\alpha', \]

with $i(d\theta)\omega_1 = 0$. The resulting decomposition

\[ \omega = \omega_0 + e(d\theta)\omega_1 + e(d\theta)^2\alpha' \]

is also orthogonal, because

\[ \langle e(d\theta)\omega_1, e(d\theta)^2\alpha' \rangle = \langle i(d\theta)e(d\theta)\omega_1, e(d\theta)\alpha' \rangle = (n - k + 2)\langle \omega_1, e(d\theta)\alpha' \rangle = 0. \]

Iterating this procedure, we end up with writing

\[ \omega = \sum_{j=0}^{\min\{p,q\}} e(d\theta)^j \omega_j, \]

with $\omega_j \in \Lambda^{p-j,q-j}$ and $i(d\theta)\omega_j = 0$. A direct computation shows that, when applied to horizontal $k$-forms,

\[ [i(d\theta), e(d\theta)] = (n - k)I. \]
where the summands are non-trivial and mutually orthogonal. Then
\[
(i(d\theta)e(d\theta)^j)\omega_j = [i(d\theta), e(d\theta)^j]\omega_j \\
= \sum_{i=0}^{j-1} e(d\theta)^i [i(d\theta), e(d\theta)] e(d\theta)^{j-1-i}\omega_j
\]
(2.8)
\[
= \sum_{i=0}^{j-1} (n - k + 2 + 2i)e(d\theta)^{j-1-i}\omega_j \\
= j(n - k + j + 1)e(d\theta)^{j-1}\omega_j .
\]

Hence,
\[
e(d\theta)i(d\theta)\omega = \sum_{j=0}^{\min\{p,q\}} j(n - k + j + 1)e(d\theta)^j\omega_j ,
\]
showing that \(e(d\theta)i(d\theta)\) diagonalizes w.r. to the decomposition
\[
\Lambda^{p,q} = \sum_{j=0}^{\min\{p,q\}} V_j^{p,q} .
\]

By (2.7), \(i(d\theta)e(d\theta)\) also diagonalizes w.r. to the same decomposition, and
\[
i(d\theta)e(d\theta)\omega = e(d\theta)i(d\theta)\omega + (n - k)\omega
\]
(2.10)
\[
= \sum_{j=0}^{\min\{p,q\}} (j + 1)(n - k + j)e(d\theta)^j\omega_j .
\]

But \(i(d\theta)e(d\theta)\) is positive semidefinite, so that \(\omega_j\) must be 0 for \(j < k - n\). Therefore \(V_j^{p,q}\) can be non-trivial only if \(\max\{0, k - n\} \leq j \leq \min\{p, q\}\). In order to see that this condition is also sufficient, observe that for \(j\) in this range, \(0 \leq p + q - 2j \leq k - 2\max\{0, k - n\} = \min\{k, 2n - k\} \leq n\). Then
\[
\omega = \beta_1 \wedge \cdots \wedge \beta_{p-j} \wedge \bar{\beta}_{p-j+1} \wedge \cdots \wedge \bar{\beta}_{p+q-2j}
\]
is a non-zero element of \(\Lambda^{p-j,q-j}\) satisfying \(i(d\theta)\omega = 0\). That \(e(d\theta)^j\omega\) is non-zero is trivial for \(j = 0\) and it follows by induction from (2.8). In conclusion,
\[
\Lambda^{p,q} = \sum_{j=\max\{0,k-n\}}^{\min\{p,q\}} V_j^{p,q} ,
\]
where the summands are non-trivial and mutually orthogonal.

A repetition of the same arguments with the rôles of \(e(d\theta)\) and \(i(d\theta)\) interchanged shows that
\[
\Lambda^{p,q} = \sum_{\ell=\max\{0,n-k\}}^{\min\{n-p,n-q\}} W_{\ell}^{p,q} ,
\]
and that \(i(d\theta)e(d\theta) = \ell(\ell + k - n + 1)I\) on \(W_{\ell}^{p,q}\).

A comparison with the eigenvalues in (2.10) provides the identification of \(V_j^{p,q}\) with \(W_{\ell(j)}^{p,q}\).

Consider now the off-diagonal terms
\[
[i(d\theta), d_H] , \quad [d_H, e(d\theta)]
\]
in (1.10). They can be simplified using the following identities:
Proposition 2.2. We have

\[
\begin{align*}
[i(d\theta), \partial_b] &= -i\bar{\partial}_b^* , \\
[\partial_b^*, e(d\theta)] &= i\bar{\partial}_b , \\
[d\theta, \partial_b] &= -i\bar{\partial}_b^* , \\
[\partial_b^*, e(d\theta)] &= -i\partial_b .
\end{align*}
\]

In particular,

\[
\begin{align*}
[i(d\theta), d_H] &= i\partial_b^* - i\bar{\partial}_b^* , \\
[d_H, e(d\theta)] &= i\bar{\partial}_b - i\partial_b .
\end{align*}
\]

Proof. Given \( j \in \{1, \ldots, n\} \) and \( I, J \subseteq \{1, \ldots, n\} \), define \( \varepsilon^I_{j,I} \) as 0 unless \( j \notin I \) and \( \{j\} \cup I = J \), in which case

\[
\varepsilon^I_{j,I} = \prod_{i \in I} \text{sgn} \left( i - j \right) ,
\]

i.e. the signature of the permutation that moves \( j \) from the left of \( I \) to its correct position w.r. to the natural ordering of \( J \). Let \( \omega = f \beta^I \wedge \bar{\beta}^{I'} \). Then

\[
\begin{align*}
(i(\beta_j)\partial_b + \partial_b i(\beta_j))\omega &= i(\beta_j) \sum_{\ell,J} \varepsilon^J_{\ell,I} B_{\ell f} \beta^J \wedge \bar{\beta}^{I'} \\
&\quad + \partial_b \sum_M \varepsilon^I_{J,M} f \beta^M \wedge \bar{\beta}^{I'} \\
&= \sum_{\ell,J,L} \varepsilon^J_{\ell,I} \varepsilon^L_{J,L} B_{\ell f} \beta^L \wedge \bar{\beta}^{I'} \\
&\quad + \sum_{\ell,M,L} \varepsilon^I_{J,M} \varepsilon^L_{\ell,M} B_{\ell f} \beta^L \wedge \bar{\beta}^{I'} \\
&= \sum_{\ell,L} \left( \sum_J \varepsilon^J_{\ell,I} \varepsilon^L_{J,L} + \sum_M \varepsilon^I_{J,M} \varepsilon^L_{\ell,M} \right) B_{\ell f} \beta^L \wedge \bar{\beta}^{I'} .
\end{align*}
\]

Consider the expression

\[
\sum_J \varepsilon^J_{\ell,I} \varepsilon^L_{J,L} + \sum_M \varepsilon^I_{J,M} \varepsilon^L_{\ell,M}
\]

for fixed \( \ell, L \). Assume first that \( \ell \neq j \). The first sum does not vanish only in one case: \( \ell \notin I, j \in I, L = I \cup \{\ell\} \setminus \{j\} \), with the only non-vanishing term in the sum corresponding to \( J = \{\ell\} \cup I \). But this is also the only case when the second sum has a non-vanishing term, the one corresponding to \( M = I \cap L \). It takes a few moments to verify that, if this is the case, the two terms have opposite signs, so that the total expression is always 0 for \( \ell \neq j \).

Assume now that \( \ell = j \in I \). The first term is 0, and the second term is also 0 unless \( M = I \setminus \{j\} \) and \( L = I \). In this case the total expression gives 1. Finally, if \( \ell = j \notin I \), the first term is 1 and the second is 0. The conclusion is that the expression under consideration equals 1 if \( \ell = j \) and \( L = I \) and 0 otherwise. Hence

\[
(i(\beta_j)\partial_b + \partial_b i(\beta_j))\omega = B_{\ell f} \beta^L \wedge \bar{\beta}^{I'} .
\]
A similar computation shows that
\[ (i\overline{\beta}_j)\partial_b + \partial_b i(\overline{\beta}_j))\omega = 0. \]

Putting these identities together, we find that
\[ i(d\theta)\partial_b \omega = i\sum_{j=1}^n i(\overline{\beta}_j)i(\beta_j)\partial_b \omega = i\sum_{j=1}^n B_j f i(\overline{\beta}_j)\beta^I \wedge \overline{\beta}^I - i\sum_{j=1}^n i(\overline{\beta}_j)\partial_b i(\beta_j)\omega = i\sum_{j=1}^n B_j f i(\overline{\beta}_j)\beta^I \wedge \overline{\beta}^I + i\sum_{j=1}^n \partial_b i(\overline{\beta}_j)i(\beta_j)\omega = -i\partial_b^* \omega + \partial_b i(d\theta)\omega. \]

This gives the first identity in the statement. Taking complex conjugates and transposes, the other three follow. □

In combination with the formula preceding (2.3), this immediately gives

**Corollary 2.3.** We have
\[ \square \partial_b = \partial_b \square - iT \partial_b, \]
\[ \Box \partial_b = \partial_b \Box + iT \partial_b, \]
hence, by duality,
\[ \partial_b^* \Box = \Box \partial_b^* - iT \partial_b^*, \]
\[ \partial_b^* \square = \square \partial_b^* + iT \partial_b^*. \]

3. Spectral multipliers of \( i^{-1}T \) and \( L \)

The operators \( i^{-1}T \) and \( L \) admit commuting self-adjoint extensions on \( L^2(H_n) \), and their joint spectrum is the *Heisenberg fan* \( F_n \subset \mathbb{R}^2 \). If
\[ \ell_m = \{(\lambda, \xi) : \xi = (n + 2m)|\lambda|, \lambda \in \mathbb{R}\}, \]
then
\[ F_n = \bigcup_{m \in \mathbb{N}} \ell_m. \]

The variable \( \lambda \) corresponds to \( i^{-1}T \) and \( \xi \) to \( L \), i.e., calling \( dE(\lambda, \xi) \) the spectral measure on \( F_n \),
\[ i^{-1}T = \int_{F_n} \lambda dE(\lambda, \xi), \quad L = \int_{F_n} \xi dE(\lambda, \xi). \]

It follows from the Plancherel formula that the spectral measure of the vertical half-line \( \{(0, \xi) : \xi \geq 0\} \subset F_n \) is zero. A spectral multiplier is therefore a function \( \mu(\lambda, \xi) \) on \( F_n \) whose restriction to each \( \ell_m \) is measurable w.r. to \( d\lambda \). Later on we shall use results from [MRS1,2] concerning \( L^p \)-boundedness of spectral multipliers. For the moment, we use these facts to discuss \( L^2 \)-boundedness of certain operators that will appear in the next Section, together with some \( L^p - L^q \)-estimates for unbounded multipliers.
Lemma 3.1. The operators
\[ L^r(\Delta_0 + i\alpha T)^{-r}, \quad T^{2r}(\Delta_0 + i\alpha T)^{-r} \]
are bounded on \( L^2(H_n) \) for \( |\alpha| < n \) and \( r > 0 \).

**Proof.** By (1.11), \( \Delta_0 = L - T^2 \). Hence we just need to observe that the multipliers
\[ \mu_1(\lambda, \xi) = \frac{\xi^r}{(\xi + \lambda^2 - \alpha \lambda)^r}, \quad \mu_2(\lambda, \xi) = \frac{\lambda^{2r}}{(\xi + \lambda^2 - \alpha \lambda)^r} \]
are bounded on \( F_n \). \( \square \)

The Cauchy-Szegő projection \( C \) is the orthogonal projection of \( L^2(H_n) \) onto the Hardy space \( H^2(H_n) \), consisting of the \( L^2 \)-functions \( f \) such that \( \partial_b \bar{f} = 0 \). It is a well-known fact (see [S, Ch. XIII]) that \( H^2(H_n) \) is also the null-space of
\[ L - inT = 2 \Box_{0,0} = 2 \partial_b^* \partial_b = -2 \sum_{j=1}^n B_j \bar{B}_j. \]

What is relevant for us at this stage is that \( C = \mu(i^{-1}T, L) \), where \( \mu \) is equal to 1 on the half-line \( \xi = -n\lambda \), with \( \lambda < 0 \), and 0 elsewhere. In the same way, the complex conjugate \( \bar{C} \) of \( C \) projects \( L^2(H_n) \) onto the null space of \( \partial_b \), which is the same as the null space of \( L + inT = 2 \Box_{0,0} \), and its multiplier equals 1 on the half-line \( \xi = n\lambda \), with \( \lambda > 0 \), and 0 elsewhere. The next result follows easily.

**Lemma 3.2.** The operators
\[ L^r(L - inT)^{-r}(I - C), \quad L^r(L + inT)^{-r}(I - \bar{C}) \]
are bounded on \( L^2 \).

We pass now to the \( L^p - L^q \)-estimates.

**Lemma 3.3.** Let \( \mu(\lambda, \xi) \) be a smooth function defined on an angle \( D_\delta = \{(\lambda, \xi) : \xi > (n-\delta)|\lambda|\} \), with \( \delta > 0 \), and homogeneous of degree \(-d\), with \( 0 < d < n+1 \). Then \( \mu(i^{-1}T, L) \) is well-defined and bounded from \( L^p(H_n) \) to \( L^q(H_n) \) for \( 1 < p < q < \infty \) and \( \frac{1}{p} - \frac{1}{q} = \frac{d}{n+1} \).

**Proof.** It follows from [G, AD] that \( \mu(i^{-1}T, L)f = f * K \), where \( K \) is smooth away from the origin and homogeneous of degree \(-(2n+2-2d)\). The conclusion follows from the generalized Young inequality. \( \square \)

4. **Decomposition of \( L^2\Lambda^1(H_n) \) under the action of \( \Delta_1 \)**

For \( k = 1 \), the conclusions of Sections 1 and 2 lead us to write the generic 1-form \( \omega \) as
\[ \omega = \omega_+ + \omega_- + h\theta, \]
where \( \omega_+ \) is a \((1,0)\)-form and \( \omega_- \) is a \((0,1)\)-form. Then
\[
\begin{pmatrix}
\omega_+ \\
\omega_-
\end{pmatrix}
= \begin{pmatrix}
\Delta_0 - iT & 0 & -i\partial_b \\
0 & \Delta_0 + iT & i\partial_b \\
i\partial^*_b & -i\partial^*_b & \Lambda + n
\end{pmatrix}
\begin{pmatrix}
\omega_+ \\
\omega_-
\end{pmatrix},
\]
(4.1)
Obviously, $\Delta_1$, initially defined on $D\Lambda^1(H_n)$, is essentially self-adjoint, and the domain of its self-adjoint (Friedrichs) extension is

$$\text{dom} \Delta_1 = \{ \omega \in L^2\Lambda^1(H_n) : \Delta_1 \omega \in L^2\Lambda^1(H_n) \} ,$$

where $\Delta_1 \omega$ is meant in the sense of distributions.

If $\omega \in D\Lambda^1(H_n)$ is exact, say $\omega = d\varphi$, then

$$\Delta_1 \omega = dd^*d\varphi = d\Delta_0 \varphi ,$$

i.e. $d$ intertwines the action of $\Delta_1$ on $\omega$ with the action of $\Delta_0$ on $\varphi$. We shall show that a similar statement holds for exact $L^2$-forms, with $d$ replaced by a modified intertwining operator which is $L^2$-bounded. Before doing so, we must make some preliminary remarks.

**Lemma 4.1.** The operator $R = d\Delta_0^{-\frac{1}{2}}$ is isometric from $L^2(H_n)$ to its image in $L^2\Lambda^1(H_n)$.

**Proof.** By Lemma 3.1, $L^\frac{1}{2} \Delta_0^{-\frac{1}{2}}$ and $T \Delta_0^{-\frac{1}{2}}$ are bounded on $L^2(H_n)$. We recall that the Riesz transforms $B_j L^{-\frac{1}{2}}, B_j \Delta_0^{-\frac{1}{2}}$ are also bounded on $L^2$. Since

$$B_j \Delta_0^{\frac{1}{2}} = (B_j L^{-\frac{1}{2}})(L^{\frac{1}{2}} \Delta_0^{\frac{1}{2}}) ,$$

it follows that $B_j \Delta_0^{\frac{1}{2}}$ is $L^2$-bounded, and similarly for $\bar{B}_j \Delta_0^{-1}$.

Hence, for $\varphi \in D(H_n)$,

$$\|R\varphi\|_2^2 = \|d\Delta_0^{-\frac{1}{2}}\varphi\|_2^2 = \langle \Delta_0^{-\frac{1}{2}} d^* d \Delta_0^{-\frac{1}{2}} \varphi, \varphi \rangle = \|\varphi\|_2^2 . \quad \Box$$

We say that $\omega \in L^2\Lambda^1(H_n)$ is *exact* if there exists $u \in D'(H_n)$ such that $\omega = du$ in the sense of distributions (componentwise). In the same sense we shall talk later on of $\partial_b$-exact forms or of $\bar{\partial}_b$-exact forms.

**Lemma 4.2.** Let $r$ be such that $\frac{1}{2} - \frac{1}{r} = \frac{1}{2n+2}$. If $\omega \in L^2\Lambda^1(H_n)$ is exact, then $\omega = dv$, in the sense of distributions, for some $v \in L^r(H_n)$.

**Proof.** By definition, there is $u \in D'(H_n)$ such that $\omega = du$. Define

$$v = \Delta_0^{-1} d^* \omega = L^{-\frac{1}{2}} (\Delta_0^{-\frac{1}{2}} L^{\frac{1}{2}}) R^* \omega ,$$

where $R^* = \Delta_0^{\frac{1}{2}} d^*$ is the adjoint of the operator $R$ in Lemma 4.1. Then $R^*$ is $L^2$-bounded, and $\Delta_0^{-\frac{1}{2}} L^{\frac{1}{2}}$ is too, by the spectral theorem. Finally, $L^{-\frac{1}{2}}$ is bounded from $L^2$ to $L^r$, e.g. by Lemma 3.3. Hence $v \in L^r(H_n)$. Moreover,

$$dv = RR^* \omega \in L^2\Lambda^1(H_n) ,$$

and

$$\Delta_0 v = d^* \omega = \Delta_0 u .$$

Observe now that

$$\Delta_1 (\omega - dv) = \Delta_1 d(u - v) = d\Delta_0 (u - v) = 0 .$$

The conclusion will follow from the next lemma. $\Box$
Lemma 4.3. The Hodge Laplacian $\Delta$ is injective on $L^2 \Lambda^1(H_n)$.

Proof. Assume that $\omega = \omega_+ + \omega_- + h\theta$ satisfies $\Delta \omega = 0$ in the sense of distributions. By (4.1), this means that

$$
\begin{align*}
(\Delta - iT)\omega_+ &= i\partial_h h , \\
(\Delta + iT)\omega_- &= -i\bar{\partial}_h h , \\
(\Delta + n)h &= -i\bar{\partial}^*_h \omega_+ + i\partial^*_h \omega_- .
\end{align*}
$$

We multiply the first equation in (4.2) by $(\Delta - iT)\partial_h^*$, and the second equation by $(\Delta + iT)\bar{\partial}_h^*$. Using the identities

$$
\partial_h^*(\Delta - iT) = (\Delta + iT)\partial_h^* , \\
\bar{\partial}_h^*(\Delta + iT) = (\Delta - iT)\bar{\partial}_h^*
$$

– easily deduced from (2.4) and (2.5) –, and performing some simple computations, we obtain that

$$
(\Delta^2 + T^2)(\Delta + n)h = (\Delta^2 + T^2(\Delta + n))h ,
$$

i.e.

$$
\Delta^2(\Delta + n - 1)h = 0 .
$$

Since the zero set of the multiplier corresponding to the operator on the left-hand side is the origin, and it has measure zero in the Heisenberg fan, this implies that $h = 0$. □

Proposition 4.4. The operator $P_1 = RR^*$ on $L^2 \Lambda^1(H_n)$ is the orthogonal projection onto the subspace of exact $L^2$-forms. In particular, this subspace is closed. Moreover, $P_1$ maps $\text{dom } \Delta$ into itself.

Proof. Clearly, $P_1$ is self-adjoint. Assume that $\omega \in L^2 \Lambda^1(H_n)$ is exact. By Lemma 4.2, there is $v \in L'(H_n)$ such that $\omega = dv$. Let $\chi$ be a non-negative, smooth function on $H_n$ with compact support, equal to 1 on a neighborhood of the origin, and define $\chi_j(z, t) = \chi(z/j, t/j^2)$. Let also $\{\varphi_j\}_{j \in \mathbb{N}}$ be an approximate identity in $\mathcal{D}(H_n)$. If

$$
v_j = \varphi_j * (\chi_j v) ,
$$

then $v_j \to v$ in $L'(H_n)$. Moreover,

$$
dv_j = \varphi_j * (\chi_j \omega) + \varphi_j * (vd\chi_j) ,
$$

if we interpret the convolution $\varphi_j * \alpha$ of $\varphi_j$ with a 1-form $\alpha$ componentwise.

If $| \cdot |$ denotes a homogeneous norm on $H_n$,

$$
\|\varphi_j * (vd\chi_j)\|_2 \leq \|vd\chi_j\|_2
$$

$$
\leq C \frac{j}{j} \left( \int_{|x|<j} |v|^2 \right)^\frac{1}{2}
$$

$$
\leq C \frac{j}{j} (2n+2)^{\frac{r-2}{2}} \left( \int_{|x|<j} |v|^r \right)^\frac{1}{2}
$$

$$
= C \left( \int |v|^r \right)^\frac{1}{2} ,
$$
and it tends to zero as \( j \) tends to infinity. Hence \( dv_j \to dv \) in \( L^2 \Lambda^1(H_n) \).

Given \( \sigma \in \mathcal{D} \Lambda^1(H_n) \), we then have

\[
\langle P_1\omega, \sigma \rangle = \langle dv, P_1\sigma \rangle = \lim_{j \to \infty} \langle dv_j, P_1\sigma \rangle = \lim_{j \to \infty} \langle v_j, d^* \sigma \rangle = \langle v, d^* \sigma \rangle = \langle dv, \sigma \rangle,
\]

showing that \( P_1\omega = \omega \).

On the other hand, if \( \omega = P_1\omega' \), let \( v = \Delta^{-1} d^* \omega' \in L^r(H_n) \), as in the proof of Lemma 4.2. If \( \sigma \in \mathcal{D} \Lambda^1(H_n) \),

\[
\langle dv, \sigma \rangle = \langle \Delta^{-1} d^* \omega', d^* \sigma \rangle = \langle \omega', P_1\sigma \rangle,
\]

so that \( dv = P_1\omega' = \omega \). To prove the last part of the statement, take again \( \sigma \in \mathcal{D} \Lambda^1(H_n) \). Then

\[
\Delta_1 P_1 \sigma = \Delta_1 (d \Delta^{-1} d^* \sigma) = dd^* d \Delta^{-1} d^* \sigma = dd^* \sigma,
\]

and

\[
P_1 \Delta_1 \sigma = d \Delta^{-1} d^* \Delta_1 \sigma = d \Delta^{-1} d^* d \Delta^{-1} \sigma = dd^* \sigma.
\]

Therefore \( \Delta_1 P_1 = P_1 \Delta_1 \) on \( \mathcal{D} \Lambda^1(H_n) \). For a general \( \sigma \in \text{dom} \Delta_1 \), we take a sequence of forms \( \sigma_j \in \mathcal{D} \Lambda^1(H_n) \) such that \( \sigma_j \to \sigma \) and \( \Delta_1 \sigma_j \to \Delta_1 \sigma \) in the \( L^2 \)-norm. Then \( P_1 \sigma_j \to P_1 \sigma \), and

\[
P_1 \Delta_1 \sigma = \lim_{j \to \infty} P_1 \Delta_1 \sigma_j = \lim_{j \to \infty} \Delta_1 P_1 \sigma_j.
\]

Since \( \Delta_1 \) is closed, \( P_1 \sigma \in \text{dom} \Delta_1 \), and \( P_1 \Delta_1 \sigma = \Delta_1 P_1 \sigma \). \( \square \)

**Proposition 4.5.** Let \( V_1 \) be the range of \( P_1 \) in \( L^2 \Lambda^1(H_n) \), i.e. the space of exact \( L^2 \)-forms. Then \( R \) maps \( \text{dom} \Delta_0 \) onto \( (\text{dom} \Delta_1) \cap V_1 \), and intertwines the action of \( \Delta_0 \) with that of \( \Delta_1 \), i.e.

\[
R \Delta_0 = \Delta_1 R,
\]
on \( \text{dom} \Delta_0 \).

**Proof.** If \( \varphi \in \mathcal{D}(H_n) \),

\[
\Delta_1 R \varphi = (dd^* + d^* d) d \Delta_0^{-\frac{1}{2}} \varphi
\]

\[
= d (d^* d) \Delta_0^{-\frac{1}{2}} \varphi
\]

\[
= d \Delta_0^{-\frac{1}{2}} \Delta_0 \varphi
\]

\[
= R \Delta_0 \varphi.
\]
An adaptation of the proof of Proposition 4.4 shows that \( R(\text{dom } \Delta_0) \subseteq \text{dom } \Delta_1 \), and that \( R \Delta_0 = \Delta_1 R \) on \( \text{dom } \Delta_0 \).

Conversely, take \( \omega \in (\text{dom } \Delta_1) \cap V_1 \) and \( \varphi \in \mathcal{D}(H_n) \). Since \( \mathcal{D}A^1(H_n) \) is a core for \( \Delta_1 \), we find a sequence \( \{ \omega_j \}_j \) in this space such that \( \omega = \lim \omega_j \) and \( \Delta_1 \omega = \lim \Delta_1 \omega_j \) in \( L^2 \). Moreover, \( \Delta_1 R \varphi = R \Delta_0 \varphi \in \mathcal{L}^2 \mathcal{A}^1(H_n) \), and thus

\[
\langle R^* \Delta_1 \omega, \varphi \rangle = \langle \Delta_1 \omega, R \varphi \rangle
\]

\[
= \lim_{j \to \infty} \langle \Delta_1 \omega_j, R \varphi \rangle = \lim_{j \to \infty} \langle \omega_j, \Delta_1 R \varphi \rangle
\]

\[
= \lim_{j \to \infty} \langle \omega_j, R \Delta_0 \varphi \rangle
\]

\[
= \langle \omega, R \Delta_0 \varphi \rangle
\]

\[
= \langle \Delta_0 R^* \omega, \varphi \rangle,
\]

showing that \( \Delta_0 (R^* \omega) \), defined in the sense of distributions, is equal to \( R^* \Delta_1 \omega \). In particular, \( R^* \omega \in \text{dom } \Delta_0 \).

Since \( \omega = P_1 \omega = R(R^* \omega) \), it follows that \( \omega \in R(\text{dom } \Delta_0) \). □

We are so led to study \( \Delta_1 \) on \( V_1^+ \), the orthogonal complement of the exact \( L^2 \)-forms. This is the space of co-closed \( L^2 \)-forms, i.e. the forms \( \omega \) such that \( d^* \omega = 0 \). We denote by \( V_2^+ \) (resp. \( V_2^- \)) the space of co-closed \((1,0)\) forms (resp. \((0,1)\) forms).

**Proposition 4.6.** For \( \omega \in V_2^\pm \), \( \Delta_1 \omega = (\Delta_0 \mp iT) \omega \) in the sense of distributions.

**Proof.** If \( \omega \in V_2^+ \), then \( \partial_b^* \omega = d^* \omega = 0 \). The conclusion follows from (4.1), and similarly for \( V_2^- \). □

Observe that, on \( H_1 \), \( V_2^+ \) consists of the \((1,0)\)-forms \( f \beta \) with \( \partial_b^* (f \beta) = -B f = 0 \). Therefore, on \( V_2^+ \),

\[
\Delta_0 - iT = -(2BB^* + T^2) = -T^2.
\]

In the same way, \( \Delta_0 + iT = -T^2 \) on \( V_2^- \).

We want to describe now the orthogonal projections \( P_2^\pm \) from \( \mathcal{L}^2 \mathcal{A}^1(H_n) \) onto \( V_2^\pm \). We look at \( P_2^+ \) as the composition of the orthogonal projection \( Q^+ \) from \( \mathcal{L}^2 \mathcal{A}^1(H_n) \) onto \( \mathcal{L}^2 \mathcal{A}^{1,0}(H_n) \) followed by the orthogonal projection from \( \mathcal{L}^2 \mathcal{A}^{1,0}(H_n) \) onto \( V_2^+ \) (and similarly for \( P_2^- \)). It is immediate to verify that

\[
Q^\pm (\omega_+ + \omega_- + h \theta) = \omega_\pm.
\]

In order to describe the second factor in the decomposition of \( P_2^+ \), it is preferable to consider its complementary projection, from \( \mathcal{L}^2 \mathcal{A}^{1,0}(H_n) \) onto the orthogonal complement \( (V_2^+)^\perp \). Since \( V_2^+ \) is the null space of \( \partial_b^* \), \( (V_2^+)^\perp \) is the closure of the space of \( \partial_b^* \)-exact \( L^2 \)-\((1,0)\)-forms. In the same way, \( (V_2^-)^\perp \) is the closure of the space of \( \partial_b^* \)-exact \( L^2 \)-\((0,1)\)-forms.

These projections involve the operators \( \Box_{0,0}, \Box_{0,0} \) in (2.4) and (2.5),

\[
\Box_{0,0} = \partial_b^* \partial_b = \frac{1}{2}(L + iT), \quad \Box_{0,0} = \partial_b^* \partial_b = \frac{1}{2}(L - iT).
\]

As there will be no confusion from now on, we drop the double subscript and simply write \( \Box \) and \( \Box \). As we have observed already,

\[
(4.3) \quad \Box u = 0 \iff \partial_b^* u = 0, \quad \Box u = 0 \iff \partial_b^* u = 0.
\]
Consequently, the image in $L^2(H_n)$ of $\partial_b^*$ is contained in $(\ker \Box)^\perp$ and the image in $L^2(H_n)$ of $\bar{\partial}_b^*$ is contained in $(\ker \Box)^\perp$.

In particular,

\begin{equation}
\partial_b^* = (I - \bar{C})\partial_b^* , \quad \bar{\partial}_b^* = (I - C)\bar{\partial}_b^* .
\end{equation}

It follows from Lemma 3.2 and boundedness of the Riesz transforms that

\begin{align*}
\Box^{-\frac{1}{2}}\partial_b^* &= \sqrt{2} \left( L^\frac{1}{2} (L + i\mu)^{-\frac{1}{2}} (I - \bar{C}) \right) \left( L^{-\frac{1}{2}} \partial_b^* \right), \\
\Box^{-\frac{1}{2}}\bar{\partial}_b^* &= \sqrt{2} \left( L^\frac{1}{2} (L - i\mu)^{-\frac{1}{2}} (I - C) \right) \left( L^{-\frac{1}{2}} \bar{\partial}_b^* \right)
\end{align*}

are well defined and bounded from $L^2 \Lambda^{1,0}(H_n)$ (resp. $L^2 \Lambda^{0,1}(H_n)$) to $L^2(H_n)$. If the factors $I - C$ and $I - \bar{C}$ are superfluous in the above formulas because of (4.4), the same is not true for the adjoint operators, $\partial_b \Box^{-\frac{1}{2}}(I - \bar{C})$ and $\bar{\partial}_b \Box^{-\frac{1}{2}}(I - C)$.

We conclude that the four operators we will be dealing with,

\begin{equation}
\mathcal{R} = \partial_b \Box^{-\frac{1}{2}}(I - \bar{C}) , \quad \mathcal{R} = \bar{\partial}_b \Box^{-\frac{1}{2}}(I - C) , \\
\mathcal{R}^* = \Box^{-\frac{1}{2}}\partial_b^* , \quad \mathcal{R}^* = \Box^{-\frac{1}{2}}\bar{\partial}_b^* ,
\end{equation}

are $L^2$-bounded.

**Proposition 4.7.** The operator $\mathcal{R} \mathcal{R}^*$ is the orthogonal projection from $L^2 \Lambda^{1,0}(H_n)$ onto the subspace of $\partial_b$-exact forms, and $\mathcal{R}^* \mathcal{R}$ is the orthogonal projection from $L^2 \Lambda^{0,1}(H_n)$ onto the subspace of $\bar{\partial}_b$-exact forms. In particular, these two subspaces are closed. Moreover, $\mathcal{R}^* \mathcal{R} = I - \bar{C}$, $\mathcal{R} \mathcal{R}^* = I - C$.

**Proof.** The argument is the same as in the proof of Proposition 4.4. The only substantial difference is that we must write

$$\Box^{-1}\partial_b^* = \Box^{-\frac{1}{2}}(I - \bar{C})\Box^{-\frac{1}{2}}\partial_b^* ,$$

and notice that Lemma 3.3 can be applied to the factor $\Box^{-\frac{1}{2}}(I - \bar{C})$. In fact this operator can be realized as $\mu(i^{-1}T, L)$, if $\mu$ is an appropriately chosen smooth function on some angle $D_{\delta}$, homogeneous of degree $-1/2$, equal to $(\xi - n\lambda)^{-\frac{1}{2}}$ on $F_{\delta}$ except for the half-line $\xi = n\lambda$, $\lambda > 0$, where it is set equal to 0. $\Box$

**Corollary 4.8.** The orthogonal projections $P_2^\pm$ from $L^2 \Lambda^1(H_n)$ onto $V_2^\pm$ are given by

$$P_2^+ = (I - \mathcal{R} \mathcal{R}^*)Q^+ , \quad P_2^- = (I - \mathcal{R}^* \mathcal{R})Q^- .$$

They map $\text{dom} \Delta_1$ into itself.

5. **Decomposition of the action of $\Delta_1$ on $V_3$**

It remains to describe the action of $\Delta_1$ on the orthogonal complement $V_3$ of $V_1 \oplus V_2^2 \oplus V_2^- \in L^2 \Lambda^1(H_n)$. Notice that $\Delta_1(V_3 \cap \text{dom} \Delta_1) \subset V_3$. It follows from Proposition 4.4 and Corollary 4.8 that $V_3 \cap \text{dom} \Delta_1$ is dense in $V_3$. 
In order to describe $V_3$ we take a detour that has the advantage of making this space somewhat more explicit. We forget for a moment that $V_1$ has been analyzed already, and we look at the full orthogonal complement of $V_2$,

$$V_2^\perp = \{ \omega : \omega = \omega_+ + \omega_- + h\theta, \omega_+ \text{ is } \partial_b - \text{exact}, \omega_- \text{ is } \bar{\partial}_b - \text{exact} \}.$$ 

Since $\omega_+$ is $\partial_b$-exact, let $u = R^*\omega_+$. Then $u \in L^2$ and $\bar{C}u = 0$. Moreover, we can recover $\omega_+$ from $u$, since $\omega_+ = R\omega$, by Prop. 4.7. Analogously, we set $v = \bar{R}^*\omega_-$ so that $v \in L^2$, $Cv = 0$ and $\omega_- = \bar{R}v$. Thus, we are lead to consider the closed subspace of $(L^2)^3$

$$W = \{ (u, v, h) \in (L^2)^3 : \bar{C}u = Cv = 0 \}.$$ 

**Lemma 5.1.** Define $\Gamma : W \to V_2^\perp$ by setting

$$\Gamma(u, v, h) = R\omega + \bar{R}v + h\theta.$$ 

Then $\Gamma$ is unitary and its inverse is given by

$$\Gamma^*(\omega) = (R^*\omega_+, \bar{R}^*\omega_-, h).$$

**Proof.** By definition of $R$ and $\bar{R}$ it is clear that $\Gamma$ maps $W$ into $V_2^\perp$. Next, by Proposition 4.7,

$$\langle \Gamma(u, v, h), \Gamma(u', v', h') \rangle = \langle Ru, Ru' \rangle + \langle \bar{R}v, \bar{R}v' \rangle + \langle h, h' \rangle$$

$$= \langle R^*Ru, u' \rangle + \langle \bar{R}^*\bar{R}v, v' \rangle + \langle h, h' \rangle$$

$$= \langle (u, v, h), (u', v', h') \rangle,$$

which shows that $\Gamma$ preserves the inner product. The previous discussion shows that $\Gamma^*\Gamma = \text{Id}_W$, and furthermore $\Gamma$ is onto since $\Gamma^* = \text{Id}_{V_2^\perp}$. 

We now set

$$D_1 = \Gamma^*\Delta_1\Gamma,$$

being $\text{dom}(D_1) = \Gamma^*(\text{dom}(\Delta_1) \cap V_2^\perp)$. We compute $D_1$ explicitly. Writing $\Gamma(u, v, h) = \omega(u, v, h)$ and recalling that $\Delta_1$ is given by (4.1), we have

$$\Delta_1\omega(u, v, h) = \omega(u', v', h'),$$

where

$$\begin{cases}
R u' = (\Delta_0 - iT)Ru - i\partial_b h \\
\bar{R} v' = (\Delta_0 + iT)Rv + i\bar{\partial}_b h \\
h' = i\partial^*_b R u - i\bar{\partial}^*_b \bar{R} v + (\Delta_0 + n) h.
\end{cases}$$

By applying $R^*$ to the first equation and $\bar{R}^*$ to the second one and using the commutation relations from Corollary 2.3 we obtain

$$\begin{cases}
u' = (\Delta_0 + iT)u - i\square^1 h \\
v' = (\Delta_0 - iT)v + i\square^2 h \\
h' = i\square_b^1 u - i\bar{\partial}^2_b v + (\Delta_0 + n) h.
\end{cases}$$
Therefore,
\[
D_1 = \begin{pmatrix}
\Delta_0 + iT & 0 & -i\Box^1 \\
0 & \Delta_0 - iT & i\Box^1 \\
i\Box^1 & -i\Box^1 & \Delta_0 + n
\end{pmatrix}.
\]

Consider the corresponding matrix of spectral multipliers
\[
d_1 = \begin{pmatrix}
\xi + \lambda^2 - \lambda & 0 & -i\sqrt{\frac{1}{2}(\xi - n\lambda)} \\
0 & \xi + \lambda^2 + \lambda & i\sqrt{\frac{1}{2}(\xi + n\lambda)} \\
i\sqrt{\frac{1}{2}(\xi - n\lambda)} & -i\sqrt{\frac{1}{2}(\xi + n\lambda)} & \xi + \lambda^2 + n
\end{pmatrix}
= (\xi + \lambda^2)I + \begin{pmatrix}
-\lambda & 0 & -i\sqrt{\frac{1}{2}(\xi - n\lambda)} \\
0 & \lambda & i\sqrt{\frac{1}{2}(\xi + n\lambda)} \\
i\sqrt{\frac{1}{2}(\xi - n\lambda)} & -i\sqrt{\frac{1}{2}(\xi + n\lambda)} & n
\end{pmatrix}.
\]

Diagonalization of \(d_1\) will have the following implication. Assume that
\[
v = \begin{pmatrix}
a_1(\lambda, \xi) \\
a_2(\lambda, \xi) \\
a_3(\lambda, \xi)
\end{pmatrix}
\]
is a unit eigenvector of \(d_1\) of eigenvalue \(\mu(\lambda, \xi)\). If we take a scalar function \(f \in L^2(H_n)\) such that
\[
F = \begin{pmatrix}
a_1(i^{-1}T, L)f \\
a_2(i^{-1}T, L)f \\
a_3(i^{-1}T, L)f
\end{pmatrix} \in W,
\]
then \(D_1F = \mu(i^{-1}T, L)F\).

**Lemma 5.2.** The eigenvalues of \(d_1\) are \(\xi + \lambda^2\) and \(\xi + \lambda^2 + \frac{n^2}{4} \pm \sqrt{\xi + \lambda^2 + \frac{n^2}{4}}\). The matrix entries of the orthogonal projections to the eigenspaces of \(d_1\) are functions of \((\lambda, \xi)\) which are bounded on the Heisenberg fan \(F_n\).

**Proof.** We compute the eigenvalues of \(m_1 = d_1 - (\xi + \lambda^2)I\):
\[
\det(m_1 - \mu I) = -\mu^3 + n\mu^2 + (\xi + \lambda^2)\mu,
\]
so that \(m_1\) has eigenvalues
\[
\mu = 0, \quad \mu_\pm = \frac{n}{2} \pm \sqrt{\xi + \lambda^2 + \frac{n^2}{4}}.
\]

Next we determine the eigenvectors and the orthogonal projections onto the eigenspaces of \(m_1\).

A unit eigenvector for \(\mu = 0\) is
\[
v_0 = \frac{1}{\sqrt{\xi + \lambda^2}} \begin{pmatrix}
\sqrt{\frac{1}{2}(\xi - n\lambda)} \\
\sqrt{\frac{1}{2}(\xi + n\lambda)} \\
i\lambda
\end{pmatrix}.
\]
In order to describe the eigenvectors corresponding to the eigenvalues $\mu_\pm$, we set

$$a = a(\lambda, \xi) = \sqrt{\xi + \lambda^2 + \frac{n^2}{4}}$$

$$q_\delta^\pm = q_\delta^\pm(\lambda, \xi) = a + \varepsilon \frac{n}{2} + \delta \lambda,$$

where $\varepsilon, \delta = \pm 1$. Notice that the following identities hold:

$$q_+^+ q_-^- = \xi - n \lambda$$

$$q_+^+ q_-^- = \xi + n \lambda$$

$$q_+^+ + q_-^- = q_-^- + q_+^+ = 2a$$

$$q_+^+ q_+^+ = (a + \lambda)^2 - \frac{n^2}{4}$$

$$q_+^- q_-^- = (a - \lambda)^2 - \frac{n^2}{4}$$

$$q_+^+ q_-^- = (a + \frac{n}{2})^2 - \lambda^2$$

$$q_-^- q_-^- = (a - \frac{n}{2})^2 - \lambda^2.$$  

Since

$$m_1 - \mu_\pm I =
\begin{pmatrix}
-\lambda - \frac{n}{2} \mp \sqrt{\xi + \lambda^2 + \frac{n^2}{4}} & 0 & -i \sqrt{\frac{1}{2} (\xi - n \lambda)} \\
0 & \lambda - \frac{n}{2} \mp \sqrt{\xi + \lambda^2 + \frac{n^2}{4}} & \sqrt{\frac{1}{2} (\xi + n \lambda)} \\
i \sqrt{\frac{1}{2} (\xi - n \lambda)} & \sqrt{\frac{1}{2} (\xi + n \lambda)} & \frac{n}{2} \mp \sqrt{\xi + \lambda^2 + \frac{n^2}{4}}
\end{pmatrix},$$

a unit eigenvector relative to $\mu_\pm$ is

$$v_+ = \frac{1}{\sqrt{2a(a + \frac{n}{2})}} \begin{pmatrix} -i \sqrt{\frac{1}{2} (q_+^- q_-^-)} \\ i \sqrt{\frac{1}{2} (q_+^+ q_+^+)} \\ \sqrt{q_+^+ q_-^-} \end{pmatrix},$$

where we have used the identities (5.3) to obtain the normalizing factor.

Similar computations show that a unit eigenvector relative to $\mu_-$ is

$$v_- = \frac{1}{\sqrt{2a(a - \frac{n}{2})}} \begin{pmatrix} i \sqrt{\frac{1}{2} (q_+^+ q_+^+)} \\ -i \sqrt{\frac{1}{2} (q_+^- q_-^-)} \\ \sqrt{q_-^- q_-^-} \end{pmatrix}.$$
Next, we wish to decompose \( W \) as the direct sum of subspaces in such a way that \( D_1 \) acts as a scalar operator on any of these subspaces. Recalling the definition (5.2) of \( a \) and \( q_0^\xi \), we set

\[
\mathcal{A} = \sqrt{\Delta_0 + \frac{n^2}{4}} = a(i^{-1}T, L), \quad Q_0^\xi = \mathcal{A} + \varepsilon \frac{n}{2} - \delta iT = q_0^\xi(i^{-1}T, L),
\]

where \( \varepsilon, \delta = \pm 1 \). By (5.3) and (2.4), (2.5) we then have the following identities

\[
Q_+^\xi Q_-^\xi = 2 \square \quad Q_-^\xi Q_+^\xi = 2 \square.
\]

**Proposition 5.3.** Define \( S_0, S_\pm \) resp. to be the operators from \( L^2(H_n) \) to \( L^2(H_n)^3 \) having \( v_0, v_\pm \) resp. as spectral multipliers. Then \( S_0 \) and \( S_\pm \) map \( L^2(H_n) \) isometrically into \( W \), and \( S_- \) maps \( L_0^2(H_n) = \{ f \in L^2(H_n) : Cf = \overline{Cf} = 0 \} \) isometrically into \( W \).

Moreover, \( W \) is the orthogonal sum of \( W_0 = S_0L^2(H_n), W_+ = S_+L^2(H_n), W_- = S_-L_0^2(H_n) \). More precisely, every \( (u, v, h) \in W \) decomposes uniquely as

\[
(u, v, h) = S_0 f_0 + S_+ f_+ + S_- f_-,
\]

with \( f_0 = S_0^* (u, v, h) \in L^2(H_n), f_+ = S_+^* (u, v, h) \in L^2(H_n), \) and \( f_- = S_-^* (u, v, h) \in L_0^2(H_n) \). Finally, the operators \( P_0 = S_0S_0^* \), \( P_\pm = S_\pm S_\pm^* \) on \( W \) whose spectral multipliers are \( p_0, p_\pm \) resp., are the orthogonal projections onto \( W_0, W_\pm \) respectively.

**Proof.** We know that, for every fixed \((\lambda, \xi) \in F_n\),

\[
I = p_0 + p_+ + p_- = v_0 v_0^* + v_+ v_+^* + v_- v_-^* \quad \text{on} \quad \mathbb{C}^3,
\]

where \( p_0, p_+, p_- \) are pairwise orthogonal projections. By the spectral theorem, this implies

\[
I = P_0 + P_+ + P_- = S_0 S_0^* + S_+ S_+^* + S_- S_-^* \quad \text{on} \quad L^2(H_n)^3,
\]

where \( P_0, P_+ \) and \( P_- \) are pairwise orthogonal projections. Moreover, since the spectral multiplier for \( S_0^* S_0 \) is \( v_0^* v_0 = ||v_0||^2 = 1 \), \( S_0 \) is isometric, and the same is true for \( S_+, S_- \), by similar reasoning.

Thus, every \((u, v, h) \in L^2(H_n)^3\) uniquely decomposes as the orthogonal sum

\[
(u, v, h) = S_0 f_0 + S_+ f_+ + S_- f_-,
\]

with \( f_0 = S_0^* (u, v, h), f_+ = S_+^* (u, v, h), f_- = S_-^* (u, v, h) \in L^2(H_n) \). There remains to prove that the mapping

\[
\mathcal{T} : (f_0, f_+, f_-) \mapsto S_0 f_0 + S_+ f_+ + S_- f_-,
\]

when restricted to the subspace \( \Omega = L^2(H_n) \times L^2(H_n) \times L_0^2(H_n) \), maps into and onto \( W \). To this end, notice that the first components in \( v_0 \) and \( v_+ \) vanish for \( \xi = n\lambda \).

Together with the fact that the spectral multiplier of \( \tilde{C} \) is the characteristic function of the set where \( \xi = n\lambda \), this implies that, if \((u, v, h)\) equals \( S_0 f_0 \) or \( S_+ f_+ \), then \( \tilde{C} u = 0 \). A similar argument shows that \( \tilde{C} v = 0 \). The same conclusion holds for \( \tilde{C} h \).
\((u, v, h) = S_- f_-\) if we impose that \(Cf_- = Cf_- = 0\), i.e. \(f_- \in L^2_0(H_n)\). Thus \(\mathcal{T}(\Omega) \subset W\).

Conversely, given \((u, v, h) \in W\), define \(f_0, f_\pm \in L^2(H_n)\) as in the statement. In particular,

\[
f_- = \frac{1}{\sqrt{2A(A - \frac{n^2}{4})}} \left( i \sqrt{\frac{1}{2} Q^+ Q^- u} - i \sqrt{\frac{1}{2} Q^+ Q^- v} + \sqrt{Q^+ Q^-} h \right).
\]

From the identities 
\(\bar{C}u = 0\), \(Cv = 0\), \(\bar{C}Q^- = 0\), \(CQ^+ = 0\),
we conclude that \(\bar{C}f_- = Cf_- = 0\), hence \((f_0, f_+, f_-) \in \Omega\). \(\square\)

**Remark.** It can be proved that it is possible to give another description of the three subspaces of \(W\) as

\[
W_0 = \{(u, v, h) \in W : Tu = \frac{1}{2} h, \ Tv = \frac{1}{2} h\}
\]

\[
W_+ = \{(u, v, h) \in W : Q^+ u = -i \frac{1}{2} h, \ Q^+ v = i \frac{1}{2} h\}
\]

\[
W_- = \{(u, v, h) \in W : Q^- u = i \frac{1}{2} h, \ Q^- v = -i \frac{1}{2} h\}.
\]

Composing with \(\Gamma\), this decomposition of \(W\) gives rise to an orthogonal decomposition of \(V_2^\perp\). Notice that, if \((u, v, h) \in W_0\), i.e.

\[
u = \Delta_0^{-\frac{1}{2}} \Box \frac{1}{2} f, \hspace{1cm} v = \Delta_0^{-\frac{1}{2}} \Box \frac{1}{2} f, \hspace{1cm} h = T \Delta_0^{-\frac{1}{2}} f,
\]

for some \(f \in L^2(H_n)\), then

\[
\Gamma(u, v, h) = R \Delta_0^{-\frac{1}{2}} \Box \frac{1}{2} f + R \Delta_0^{-\frac{1}{2}} \Box \frac{1}{2} f + T \Delta_0^{-\frac{1}{2}} f = R f,
\]

so that \(\Gamma W_0 = V_1\), the space of exact forms.

Define

\[
V_3^\pm = \Gamma(W_\pm).
\]

**Proposition 5.4.** The orthogonal complement of \(V_1 \oplus V_2\) in \(L^2 \Lambda_1\) is the subspace \(V_3 = V_3^+ \oplus V_3^-\). The operators \(\Gamma S_+\) and \(\Gamma S_-\) are unitary respectively from \(L^2(H_n)\) onto \(V_2^\perp\) and from \(L^2_0(H_n)\) onto \(V_3^-\). The orthogonal projections from \(V_2^\perp\) onto \(V_3^-\) and \(V_3^-\) are

\[
\Pi_\pm = \Gamma S_\pm S_\pm^* \Gamma^*.
\]

Moreover,

\[
\Gamma S_\pm \left( \Delta_0 + \frac{n}{2} \pm \sqrt{\Delta_0 + \frac{n^2}{4}} \right) = \Delta_1 \Gamma S_\pm.
\]

**Proof.** The first part of the statement is obvious. What concerns the action of \(\Delta_1\) follows from the fact that, since \(D_1\) commutes with \(P_0, P_\pm\),

\[
\text{dom}(D_1) = (\text{dom}(D_1) \cap W_0) \oplus (\text{dom}(D_1) \cap W_+) \oplus (\text{dom}(D_1) \cap W_-).
\] \(\square\)
6. $L^p$-boundedness of spectral multipliers of $\Delta_1$

On the basis of the previous analysis, we can say that

$$\Delta_1 = R\Delta_0 R^{-1} = R\Delta_0 R^*$$

on $V_1$,

$$\Delta_1 = \Delta_0 \mp iT$$

on $V_2^\pm$, and, by Proposition 5.4,

$$\Delta_1 = \Gamma S_\pm \left( \Delta_0 + \frac{n}{2} \pm \sqrt{\Delta_0 + \frac{n^2}{4}} \right) S^*_\pm \Gamma^*$$

on $V_3^\pm$.

This implies that, given a bounded Borel function $m$ on $(0, +\infty) = \mathbb{R}_+$, the operator $T_m = m(\Delta_1)$ equals

$$Rm(\Delta_0)R^*, \quad m(\Delta_0 \mp iT), \quad \Gamma S_\pm m \left( \Delta_0 + \frac{n}{2} \pm \sqrt{\Delta_0 + \frac{n^2}{4}} \right) S^*_\pm \Gamma^*$$

on the corresponding subspaces. Denoting by $P_3 = I - P_1 - P^+_2 + P^-_2$ the orthogonal projection from $L^2 A^1(H_n)$ onto $V_3$, we obviously have

$$m(\Delta_1) = m(\Delta_1)P_1 + m(\Delta_1)P^+_2 + m(\Delta_1)P^-_2 + m(\Delta_1)\Pi_+ P_3 + m(\Delta_1)\Pi_- P_3 .$$

Observe that, since $R^* R = I$, we have $R^* P_1 = R^* R R^* = R^*$. Similarly,

$$S^*_\pm \Gamma^* \Pi_\pm = S^*_\mp \Gamma^* .$$

We then have

$$m(\Delta_1) = Rm(\Delta_0)R^* + m(\Delta_0 - iT)P^+_2 + m(\Delta_0 + iT)P^-_2$$

$$\quad + \Gamma S_\pm m \left( \Delta_0 + \frac{n}{2} \pm \sqrt{\Delta_0 + \frac{n^2}{4}} \right) S^*_\pm \Gamma^* P_3$$

$$\quad + \Gamma S_\mp m \left( \Delta_0 + \frac{n}{2} - \sqrt{\Delta_0 + \frac{n^2}{4}} \right) S^*_\mp \Gamma^* P_3 .$$

(6.1)

We are so led to discuss $L^p$ boundedness of each of the operators appearing in (6.1). For this purpose, we recall the following result, taken from [MRS2, Cor.2.4], and concerning Marcinkiewicz multipliers of $i^{-1}T$ and $L$. We shall present a series of technical lemmas in a rather self-contained fashion. We do not claim full originality for every single statement. In particular, various overappings with arguments in [MS] are present. Given $\rho, \sigma > 0$, we say that a function $f(\lambda, \xi)$ is in the mixed Sobolev space $L^2_{\rho, \sigma} = L^2_{\rho, \sigma}(\mathbb{R}^2)$ if

$$\|f\|^2_{L^2_{\rho, \sigma}} = \int_{\mathbb{R}^2} \left( (1 + |\xi'|)^{2\rho} (1 + |\lambda'|)^{2\sigma} |\hat{f}(\lambda', \xi')|^2 d\lambda' d\xi' \right. \left. + \rho(1 + |\lambda'|)^{2\rho} (1 + |\xi'|)^{2\sigma} |\hat{f}(\lambda', \xi')|^2 d\lambda' d\xi' \right)^{\frac{1}{2}} .$$

(6.2)
is finite. When \( \rho \) and \( \sigma \) are integers, this condition means that the derivatives \( \partial^i_\lambda \partial^j_\xi f \) are in \( L^2 \) for \( i \leq \sigma \) and \( i + j \leq \rho + \sigma \). We shall make use of this characterization, together with the fact that the \( L^2_{\rho, \sigma} \) form an interpolation family.

Let \( \eta \in \mathcal{D}((\mathbb{R}^*_+)^2) \) be a non-trivial, non-negative, smooth function (briefly, a bump function). We say that a bounded function \( \mu(\lambda, \xi) \) defined on \( (\mathbb{R}^*_+)^2 \) is in \( L^2_{\rho, \sigma, \text{sloc}}((\mathbb{R}^*_+)^2) \) if for every \( r = (r_1, r_2) \in (\mathbb{R}^*_+)^2 \), the function \( \mu^r(\lambda, \xi) = \mu(r_1 \lambda, r_2 \xi) \eta(\lambda, \xi) \) is in \( L^2_{\rho, \sigma} \) and

\[
(6.3) \quad \| \mu \|_{L^2_{\rho, \sigma, \text{sloc}}} = \sup_r \| \mu^r \|_{L^2_{\rho, \sigma}}
\]
is finite. We extend this definition to functions \( \mu \) defined on \( \mathbb{R} \times \mathbb{R}^*_+ \) by requiring that both \( \mu(\lambda, \xi) \) and \( \mu(-\lambda, \xi) \) are in \( L^2_{\rho, \sigma, \text{sloc}}((\mathbb{R}^*_+)^2) \).

If \( \rho \) and \( \sigma \) are integers, to require that \( \mu \in L^2_{\rho, \sigma, \text{sloc}}((\mathbb{R}^*_+)^2) \) is equivalent to requiring that

\[
(6.4) \quad \sup_{r_1, r_2 > 0} r_1^{-1+2i} r_2^{-1+2j} \int_{r_1 < |\lambda| < 2r_1 \cdot r_2 < \xi < 2r_2} |\partial^i_\lambda \partial^j_\xi \mu(\lambda, \xi)|^2 d\lambda d\xi < +\infty,
\]
for all \( i, j \) such that \( i \leq \sigma, i + j \leq \rho + \sigma \). In particular, the definition of \( L^2_{\rho, \sigma, \text{sloc}} \) is independent of the choice of \( \eta \). The same is true for every \( \rho, \sigma > 0 \), as the following lemma shows.

**Lemma 6.1.** Given two bump functions \( \eta_1 \) and \( \eta_2 \) on \( (\mathbb{R}^*_+)^2 \), the norms (6.3) that they define are equivalent. Let \( \Omega \) be a family of bump functions, such that all the \( \eta \in \Omega \) are supported on the same compact subset of \( (\mathbb{R}^*_+)^2 \), and that their \( C^k \)-norms are uniformly bounded for some \( k \geq \rho + \sigma \). Given another bump function \( \eta_0 \), the norms (6.3) defined by the \( \eta \in \Omega \) are controlled uniformly by the norm defined by \( \eta_0 \).

**Proof.** If \( \varphi \in \mathcal{D}(\mathbb{R}^2) \), the operation of multiplication by \( \varphi \) is continuous on \( L^2_{\rho, \sigma} \), with a norm controlled by the \( C^k \)-norm of \( \varphi \), if \( k \geq \rho + \sigma \). This is trivial if \( \rho \) and \( \sigma \) are integers, and it follows by interpolation in the general case.

Given \( \eta_1 \) and \( \eta_2 \) as above, there are \( r^{(1)}, \ldots, r^{(k)} \in (\mathbb{R}^*_+)^2 \) such that

\[
\psi(\lambda, \xi) = \sum_{j=1}^{k} \eta_1(r^{(j)}_1 \lambda, r^{(j)}_2 \xi) \geq \delta > 0
\]
on the support of \( \eta_2 \). Hence \( \eta_2 = \varphi \psi \) for some \( \varphi \in \mathcal{D}(\mathbb{R}^2) \). Then

\[
(6.5) \quad \| \mu(r \cdot) \eta_2 \|_{L^2_{\rho, \sigma}} \leq C \| \mu(r \cdot) \psi \|_{L^2_{\rho, \sigma}}
\]

\[
\leq C \sum_{j=1}^{k} \| \mu(r \cdot) \eta_1(r^{(j)}) \|_{L^2_{\rho, \sigma}}
\]

\[
\leq C' \sum_{j=1}^{k} \| \mu(r(r^{(j)})^{-1}) \eta_1 \|_{L^2_{\rho, \sigma}},
\]
and this implies the first part of the statement.

Given a family \( \Omega \) of bump functions as above, the same \( \psi \) can be used for all the \( \eta \in \Omega \), because of the condition on the supports. It follows that the set \( \{ \varphi = \eta/\psi : \eta \in \Omega \} \) is bounded in \( C^k \) for every \( k \). Hence the constant \( C' \) appearing in (6.5), with \( \eta_2 = \eta \) and \( \eta_1 = \eta_0 \), can be taken independently of \( \eta \).  \( \square \)
Theorem 6.2 [MRS2]. Let $\mu$ be a bounded function in $L^2_{\rho,\sigma,\text{sloc}}(\mathbb{R} \times \mathbb{R}^*_+)$ for some $\rho > n$ and $\sigma > \frac{1}{2}$. Then $\mu(i^{-1}T,L)$ is bounded on $L^p(H_n)$ for $1 < p < \infty$, with norms controlled by $\|\mu\|_{L^2_{\rho,\sigma,\text{sloc}}}$.

From this statement we shall derive a result concerning spectral multipliers of $\Delta_0 + i\alpha T$ for $|\alpha| < n$. Observe that, if $m$ is a bounded function on $\mathbb{R}^*_+$, then $m(\Delta_0 + i\alpha T) = \mu(i^{-1}T,L)$, with

$$\mu(\lambda, \xi) = m(\lambda^2 + \xi - \alpha\lambda).$$

If $\tau > 0$, we say that $m \in L^2_{\tau,\text{sloc}}(\mathbb{R}^*_+)$ (or that $m$ is a Mihlin-Hörmander multiplier of order $\tau$) if

$$\|m\|_{L^2_{\tau,\text{sloc}}} = \sup_{r > 0} \|m(r\cdot)\varphi\|_{L^2_{\tau}}$$

is finite, where $\varphi$ is a non-trivial, non-negative, smooth bump function on $\mathbb{R}^*_+$ and the $L^2_{\tau}$-norm is the ordinary Sobolev norm on $\mathbb{R}$. It will be useful to observe that $m \in L^2_{\tau,\text{sloc}}(\mathbb{R}^*_+)$ if and only if $\mu(\lambda, \xi) = m(\xi)$ is in $L^2_{\tau,\sigma,\text{sloc}}((\mathbb{R}^*_+)^2)$ for any $\sigma$.

In particular, the analogue of Lemma 6.1 can be formulated, with the obvious modifications, for $L^2_{\tau,\text{sloc}}(\mathbb{R}^*_+)$. One important technical aspect of our argument is the following.

Proposition 6.3. Let $\rho, \sigma > 0$, $\alpha \in (-n,n)$, and let $m$ be a Mihlin-Hörmander multiplier of order $\tau = \rho + \sigma$. Then $\mu(\lambda, \xi) = m(\lambda^2 + \xi - \alpha\lambda)$ coincides on $F_n$ with a function in $L^2_{\rho,\sigma,\text{loc}}$.

This and Theorem 6.2 imply the following result.

Theorem 6.4. If $m$ is a Mihlin-Hörmander multiplier of order $\tau > n + \frac{1}{2}$, then $m(\Delta_0 + i\alpha T)$ is a bounded operator on $L^p(H_n)$ for $|\alpha| < n$ and $1 < p < \infty$.

The proof of Proposition 6.3 requires a few lemmas.

Lemma 6.5. If $\mu(\lambda, \xi)$ is in $L^2_{\rho,\sigma,\text{sloc}}((\mathbb{R}^*_+)^2)$, then $\mu(\lambda^2, \xi)$ is in $L^2_{\rho,\sigma,\text{sloc}}(\mathbb{R} \times \mathbb{R}^*_+)$, with equivalence of norms.

Proof. Let $K$ be a compact subset of $(\mathbb{R}^*_+)^2$. If $\rho$ and $\sigma$ are integers, it is quite clear that a function $f(\lambda, \xi)$ supported on $K$ is in $L^2_{\rho,\sigma}$ if and only if $f(\lambda^2, \xi)$ is in $L^2_{\rho,\sigma}$. By complex interpolation, the same holds for all $\rho, \sigma > 0$.

In order to prove the Lemma, it is sufficient to consider the restriction $\tilde{\mu}(\lambda, \xi)$ of $\mu(\lambda^2, \xi)$ to $(\mathbb{R}^*_+)^2$. If $\eta$ is a bump function, the $L^2_{\rho,\sigma}$-norm of

$$\tilde{\mu}(r_1\lambda, r_2\xi)\eta(\lambda, \xi) = \mu(r_1^2\lambda^2, r_2\xi)\eta(\lambda, \xi)$$

is controlled by the $L^2_{\rho,\sigma}$-norm of $\mu(r_1^2\lambda, r_2\xi)\eta(\sqrt{\lambda}, \xi)$. The conclusion follows easily from Lemma 6.1. \qed

Lemma 6.6. Let $\mu \in L^2_{\rho,\sigma,\text{sloc}}((\mathbb{R}^*_+)^2)$, and $\delta > 0$. Let also $\psi$ be smooth on $\mathbb{R} \times \mathbb{R}^*_+$, homogeneous of degree zero, and supported on the angle $D_\delta = \{(\lambda, \xi) : \xi \geq (n-\delta)|\lambda|\}$. If $\alpha < n - \delta$, then

$$\mu'(\lambda, \xi) = \mu(\lambda, \xi - \alpha\lambda)\psi(\lambda, \xi)$$

is also in $L^2_{\rho,\sigma,\text{sloc}}((\mathbb{R}^*_+)^2)$.\qed
Proof. If \( \gamma \in \mathbb{R} \), the linear change of variables \((\lambda, \xi) \mapsto (\lambda, \xi + \gamma \lambda)\) induces an isomorphism of \(L^2_{\rho, \sigma}(\mathbb{R}^2)\) onto itself, with constants controlled by \(\gamma\). This follows easily from \((6.2)\). Therefore, if \(\eta_0\) is a bump function on \((\mathbb{R}^*_+)^2\) and \(r_1, r_2 > 0\), the \(L^2_{\rho, \sigma}\)-norm of
\[
\mu'(r_1 \lambda, r_2 \xi) \eta(\lambda, \xi) = \mu(r_1 \lambda, r_2 \xi - \alpha r_1 \lambda) \psi(r_1 \lambda, r_2 \xi) \eta_0(\lambda, \xi)
\]
is equivalent to the \(L^2_{\rho, \sigma}\)-norm of
\[
\mu(r_1 \lambda, r_2 \xi') \psi(r_1 \lambda, r_2 \xi' + \alpha r_1 \lambda) \eta_0(\lambda, \xi' + \alpha \frac{r_1}{r_2} \lambda),
\]
with constants controlled by the ratio \(r_1/r_2\). Let
\[
\eta_r(\lambda, \xi') = \psi(r_1 \lambda, r_2 \xi' + \alpha r_1 \lambda) \eta_0(\lambda, \xi' + \alpha \frac{r_1}{r_2} \lambda) = \psi(\lambda, \frac{r_2}{r_1} \xi' + \alpha \frac{r_1}{r_2} \lambda) \eta_0(\lambda, \xi' + \alpha \frac{r_1}{r_2} \lambda).
\]
The conclusion follows if we prove that, for an appropriate choice of \(\eta_0\), the set \(\Omega = \{\eta_r : r \in (\mathbb{R}^*_+)^2\}\) satisfies the assumptions of Lemma 6.1, and that \(\eta_r \neq 0\) only if the ratio \(r_1/r_2\) is bounded.

Assume that the support of \(\eta_0\) is contained in the square \([1 - \varepsilon, 1 + \varepsilon]^2\), with \(\varepsilon \in (0, 1)\) to be determined. A necessary condition for having \(\eta_r \neq 0\) is that there exists \((\lambda, \xi')\) such that the conditions
\[
(6.6) \quad \left( \lambda, \xi' + \alpha \frac{r_1}{r_2} \lambda \right) \in [1 - \varepsilon, 1 + \varepsilon]^2, \quad \left( \lambda, \frac{r_2}{r_1} \xi' + \alpha \lambda \right) \in D_\delta,
\]
are satisfied, or, otherwise stated, that
\[
\left( [1 - \varepsilon, 1 + \varepsilon] \times \left[ \frac{r_2}{r_1} (1 - \varepsilon), \frac{r_2}{r_1} (1 + \varepsilon) \right] \right) \cap D_\delta \neq \emptyset.
\]

This occurs if and only if the point \(\left( 1 - \varepsilon, \frac{r_2}{r_1} (1 + \varepsilon) \right) \in D_\delta\), i.e. if and only if
\[
(6.7) \quad \frac{r_2}{r_1} > (n - \delta) \frac{1 - \varepsilon}{1 + \varepsilon}.
\]
The requirement about the boundedness of the ratios \(r_1/r_2\) is then fulfilled.

Once \((6.7)\) is satisfied, we check that the supports of the \(\eta_r\) are contained in a common compact subset of \((\mathbb{R}^*_+)^2\). Clearly, if \((\lambda, \xi') \in \text{supp} \eta_r\), then \(\lambda \in [1 - \varepsilon, 1 + \varepsilon]\). As to \(\xi'\), we impose, for all the \((r_1, r_2)\) satisfying \((6.7)\), the condition
\[
1 - \varepsilon \leq \xi' + \alpha \frac{r_1}{r_2} \lambda \leq 1 + \varepsilon,
\]
taken from the first of \((6.6)\). The existence of an upper bound for \(\xi'\) independent of \(r\) follows from the fact that the ratios \(r_1/r_2\) are bounded. For the lower bound, there is no problem if \(\alpha \leq 0\). If \(0 < \alpha < n - \delta\), taking into account \((6.7)\) and that \(\lambda < 1 + \varepsilon\), we are done if
\[
(1 - \varepsilon) - \alpha \frac{1 + \varepsilon}{1 - \varepsilon} \frac{1}{1 + \varepsilon} (1 + \varepsilon) = (1 - \varepsilon) \left( 1 - \frac{\alpha}{\delta} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^2 \right) > 0.
\]
This can be obtained by choosing \( \varepsilon \) small enough. A simple verification shows that the derivatives of the \( \eta_r \) are uniformly bounded, so that the conclusion follows from Lemma 6.1. \( \square \)

We can now prove Proposition 6.3.

**Proof of Proposition 6.3.** Given a Mihlin-Hörmander multiplier \( m \) of order \( \tau \) on the positive half-line, consider

\[
\mu_1(\lambda, \xi) = m(\lambda + \xi)
\]
on \( (\mathbb{R}_+^*)^2 \). Applying Lemma 6.6 to \( \mu_0(\lambda, \xi) = m(\xi) \), with \( \delta = n \) and \( \alpha = -1 \), we obtain that \( \mu_1 \in L^2_{\rho,\sigma,\text{sloc}}((\mathbb{R}_+^*)^2) \).

By Lemma 6.5, \( \mu_2(\lambda, \xi) = m(\lambda^2 + \xi) \) is in \( L^2_{\rho,\sigma,\text{sloc}}(\mathbb{R} \times \mathbb{R}_+^*) \).

Given \( \alpha \in \mathbb{R} \) with \( |\alpha| < n \), take \( \delta > 0 \), \( \delta < n - |\alpha| \) and construct \( \psi \) smooth, homogeneous of degree 0 supported on \( D_\delta \) and equal to 1 on \( F_n \). Applying Lemma 6.6 to \( m_2(\pm \lambda, \xi) \) restricted to \( (\mathbb{R}_+^*)^2 \), we conclude that also \( \mu_3(\lambda, \xi) = m(\lambda^2 + \xi - \alpha \lambda) \psi(\lambda, \xi) \) is in \( L^2_{\rho,\sigma,\text{sloc}}(\mathbb{R} \times \mathbb{R}_+^*) \). \( \square \)

**Proposition 6.7.** The operators \( S_+ \) and \( S_- \) are bounded from \( L^p(H_n) \) to \( L^p(H_n)^3 \) for \( 1 < p < \infty \).

**Proof.** The components of the operators \( S_+ \) and \( S_- \) are spectral multiplier operators whose multipliers are the components of \( v_+ \) and \( v_- \).

It turns out that the half-lines \( \xi = n \lambda \) and \( \xi = -n \lambda \) play a special role here, which is why we discuss them separately. We restrict ourselves to the half-line \( \xi = n \lambda \); the other half-line can be treated in a similar way.

On the former half-line, we have \( a = \lambda + \frac{n}{2} \), and, using (5.3), one finds that

\[
v_+ = \left( \begin{array}{c} i \sqrt{\frac{0}{\lambda + n}} \\
\sqrt{\frac{n}{\lambda + n}} \end{array} \right), \quad v_- = \left( \begin{array}{c} i \\
0 \\
0 \end{array} \right).
\]

The components of \( v_+ \) and \( v_- \) are Mihlin-Hörmander multipliers as functions of \( \lambda > 0 \), and since the operator \( \bar{C} \), which corresponds to the restriction to the spectral half-line \( \xi = n \lambda \), is \( L^p \)-bounded, we see that the components of \( v_+, v_- \), when restricted to this half-line, give rise to \( L^p \)-bounded operators for \( 1 < p < \infty \).

In view of the definition of the Heiseberg fan \( F_n \), it thus suffices to consider the domain where \( \xi > (n + 1)|\lambda| \). Notice that the components of \( v_\pm \) are all products of multipliers of the form

\[

v_0 = (q_\delta^+)^{\frac{1}{2}} a^{-\frac{1}{2}}, \quad v_+ = \left( q_\delta^+ \right)^{\frac{1}{2}} (a + \frac{n}{2})^{-\frac{1}{2}}, \quad v_- = \left( q_\delta^- \right)^{\frac{1}{2}} (a - \frac{n}{2})^{-\frac{1}{2}}.
\]

We show that for \( \xi > (n + 1)|\lambda| \) they satisfy the pointwise estimates

\[
(6.8) \quad |\partial^i_\lambda \partial^j_\xi \nu(\lambda, \xi)| \leq C_{i,j} |\lambda|^{-i} \xi^{-j}
\]

for \( i = 0, 1 \) and \( j \) arbitrary. This implies that they can be appropriately extended to the upper half-plane so that (6.4) holds with \( \sigma = 1 \) and \( \rho \) arbitrary, so that the proposition is a consequence of Theorem 6.2.
Using the identity $\partial_\xi a = \partial_\xi q_\delta^\epsilon = \frac{1}{2a}$, we see by induction that

$$\partial_\xi^j \nu_0 = \sum_{k=0}^j c_{jk}(q_\delta^\epsilon)^{\frac{3}{2}-k}a^{-\frac{3}{2}+k-2j} = \nu_0 \sum_{k=0}^j c_{jk}(q_\delta^\epsilon)^{-k}a^{k-2j}.$$ 

To prove (6.8) we shall use the following elementary relation

$$\sqrt{\alpha + \beta} - \sqrt{\alpha} \simeq \begin{cases} \alpha, & \text{if } \alpha \gtrsim \beta, \\ \beta, & \text{if } \alpha \lesssim \beta, \end{cases}$$

valid for every $\alpha, \beta \geq 0$.

We first consider the case $i = 0$ in (6.8). Clearly, $\nu_0$ is bounded. Moreover,

$$a \simeq \begin{cases} |\lambda| + \frac{n}{2}, & \text{if } |\lambda| + \frac{n}{2} \gtrsim \sqrt{\xi}, \\ \sqrt{\xi}, & \text{if } |\lambda| + \frac{n}{2} \lesssim \sqrt{\xi}, \end{cases}$$

and, by (6.9), since

$$q_\delta^\epsilon \geq \sqrt{\xi + \lambda^2 + \frac{n^2}{4}} - |\lambda| - \frac{n}{2} = \sqrt{(|\lambda| + \frac{n}{2})^2 + (\xi - n|\lambda|)} - \sqrt{(|\lambda| + \frac{n}{2})^2},$$

we have

$$q_\delta^\epsilon \gtrsim \begin{cases} \frac{\xi - n|\lambda|}{|\lambda| + \frac{n}{2}}, & \text{if } |\lambda| + \frac{n}{2} \gtrsim \sqrt{\xi - n|\lambda|}, \\ \sqrt{\xi - n|\lambda|}, & \text{if } |\lambda| + \frac{n}{2} \lesssim \sqrt{\xi - n|\lambda|}. \end{cases}$$

Notice that $\xi - n|\lambda| \simeq \xi$, since we assume $\xi \geq (n + 1)|\lambda|$, and thus

$$q_\delta^\epsilon \gtrsim \begin{cases} \frac{\xi}{|\lambda| + \frac{n}{2}}, & \text{if } |\lambda| + \frac{n}{2} \gtrsim \sqrt{\xi}, \\ \sqrt{\xi}, & \text{if } |\lambda| + \frac{n}{2} \lesssim \sqrt{\xi}. \end{cases}$$

For simplicity of notation, let us assume that $\lambda > 0$. Then, by (6.10), (6.11),

$$(q_\delta^\epsilon)^{-k}a^{k-2j} \lesssim \begin{cases} (\lambda + \frac{n}{2})^{2k-2j}\xi^{-k}, & \text{if } \lambda + \frac{n}{2} \gtrsim \sqrt{\xi}, \\ (\sqrt{\xi})^{-2j}, & \text{if } \lambda + \frac{n}{2} \lesssim \sqrt{\xi}. \end{cases}$$

This shows that (6.8) holds for $i = 0$.

Consider now the case $i = 1$. To control $\partial_\lambda \nu_0$, we write $\nu_0 = \psi^\frac{1}{2}$, where

$$\psi^\epsilon = \xi^\epsilon a^{-1} = 1 + (\xi^\epsilon + \delta)\psi^{-1}.$$
Then
\[
\partial \lambda \psi = \delta a^{-1} - (\varepsilon \frac{n^2}{4} + \delta \lambda) \lambda a^{-3}
\]
\[
= (\delta (\xi + \frac{n^2}{4}) - \varepsilon \frac{n^2}{2} \lambda) a^{-3},
\]
hence
\[
\partial \lambda \nu_0 = \frac{1}{2} (\delta \xi + \delta \frac{n^2}{4} - \varepsilon \frac{n^2}{2} \lambda)(q_{\delta}^{\xi})^{-\frac{1}{2}} a^{\frac{1}{2} - 3}.
\]
By induction, one then finds that
\[
\partial_j \xi \partial \lambda \nu_0 = \nu_0 \sum_{k=0}^{j-1} c_{jk}(q_{\delta}^{\xi})^{-k-1} a^{k-2j} + \nu_0 \sum_{k=0}^{j} d_{jk}(\delta \xi + \delta \frac{n^2}{4} - \varepsilon \frac{n^2}{2} \lambda)(q_{\delta}^{\xi})^{-k-1} a^{k-2(j+1)}.
\]
By (6.10), (6.11) (assuming again that \(\lambda > 0\)) we see that
\[
(q_{\delta}^{\xi})^{-k-1} a^{k-2j} \lesssim \begin{cases} 
\frac{1}{\lambda + \frac{n^2}{2}} (\lambda + \frac{n^2}{2})^{2k-2(j-1)} \xi^{-k-1}, & \text{if } \lambda + \frac{n^2}{2} \gtrsim \sqrt{\xi}, \\
(\sqrt{\xi})^{-2j-1}, & \text{if } \lambda + \frac{n^2}{2} \lesssim \sqrt{\xi}.
\end{cases}
\]
Since \(k \leq j - 1\), these terms are of order \(O((\lambda + \frac{n^2}{2})^{-1} \xi^{-j})\).
Noticing that \(|\delta \xi + \delta \frac{n^2}{4} - \varepsilon \frac{n^2}{2} \lambda| \lesssim 1 + \xi\), when \(\xi > (n+1) \lambda\), one finds in a similar way that the terms in the second sum are of the order
\[
(1 + \xi)(\lambda + \frac{n^2}{2})^{-1} \xi^{-(j+1)} \lesssim \lambda^{-1} \xi^{-j},
\]
so that (6.8) also holds for \(i = 1\).
Next,
\[
\nu_+ = \nu_0 \sqrt{\frac{a}{a + \frac{n^2}{2}}},
\]
where the square root only depends on \(\xi + \lambda^2\), and is a Mihlin-Hörmander multiplier in this variable. So Theorem 6.4 applies to this factor.
There remain the multipliers of the form
\[
\nu_- = (q_{\delta}^{-})^{\frac{1}{2}} (a - \frac{n^2}{2})^{-\frac{1}{2}}.
\]
We begin with \(i = 0\) in (6.8), assuming again for simplicity that \(\lambda > 0\). By induction, we here see that
\[
\partial_j \xi \nu_- = \nu_- \sum_{k+l \leq j} c_{jkl}(q_{\delta}^{-})^{-k}(a - \frac{n^2}{2})^{-l} a^{k+l-2j}.
\]
Moreover, by (6.9),
\[
a - \frac{n^2}{2} = \sqrt{\frac{n^2}{4} + (\xi + \lambda^2)} - \sqrt{\frac{n^2}{4}}
\]
\[
\approx \begin{cases} 
\frac{\xi + \lambda^2}{n/2}, & \text{if } \frac{n^2}{4} \gtrsim \xi + \lambda^2, \\
\sqrt{\xi + \lambda^2}, & \text{if } \frac{n^2}{4} \lesssim \xi + \lambda^2.
\end{cases}
\]
Assume first that $\xi \ls 1$. Since $\xi > (n+1)\lambda$, then

$$a - \frac{n}{2} \approx \xi, \quad a \approx 1 \quad \text{and} \quad q_\delta^- \gtrsim \xi.$$ 

Hence

$$(q_\delta^-)^{-k}(a - \frac{n}{2})^{-l}a^{k+l-2j} \ls \xi^{-k-l} \ls \xi^{-j},$$

so that (6.8) holds.

Let next $\xi \gs 1$. Then $a - \frac{n}{2} \approx \sqrt{\xi + \lambda^2}$, hence

$$a - \frac{n}{2} \approx \left\{ \begin{array}{ll}
\lambda, & \text{if } \lambda \gs \sqrt{\xi}, \\
\sqrt{\xi}, & \text{if } \lambda \ls \sqrt{\xi}.
\end{array} \right.$$ 

In combination with (6.10), (6.11), this gives

$$(q_\delta^-)^{-k}(a - \frac{n}{2})^{-l}a^{k+l-2j} \ls \left\{ \begin{array}{ll}
\left(\frac{\xi}{\lambda}\right)^{-k}\lambda^{-l}\lambda^{k+l-2j}, & \text{if } \lambda \gs \sqrt{\xi}, \\
\sqrt{\xi}^{-k}\sqrt{\xi}^{-l}\sqrt{\xi}^{k+l-2j}, & \text{if } \lambda \ls \sqrt{\xi} 
\end{array} \right.$$ 

so that (6.8) holds for $i = 0$.

Let next $i = 1$. Arguing similarly as for $\nu_0$, we here put

$$\psi = q_\delta^- (a - \frac{n}{2})^{-1} = 1 + \delta \lambda (a - \frac{n}{2})^{-1},$$

so that

$$\partial_\lambda \nu_- = \frac{\delta}{2} \left( \xi a^{-1} (a - \frac{n}{2})^{-1} - \frac{n}{2} a^{-1} \right) (q_\delta^-)^{-\frac{1}{2}} (a - \frac{n}{2})^{-\frac{1}{2}}$$

consists of terms

$$\mu_1 = \xi (q_\delta^-)^{-\frac{1}{2}} (a - \frac{n}{2})^{-\frac{1}{2}} a^{-1}, \quad \mu_2 = (q_\delta^-)^{-\frac{1}{2}} (a - \frac{n}{2})^{-\frac{1}{2}} a^{-1}.$$

Then

$$\partial^j_\xi \mu_1 = \nu_- \sum_{k+l \leq j-1} c_{jkl} (q_\delta^-)^{-k-1} (a - \frac{n}{2})^{-l-1} a^{-1+k+l-2(j-1)}$$

$$+ \nu_- \sum_{k+l \leq j} d_{jkl} \xi (q_\delta^-)^{-k-1} (a - \frac{n}{2})^{-l-1} a^{-1+k+l-2j}$$

and

$$\partial^j_\xi \mu_2 = \nu_- \sum_{k+l \leq j} b_{jkl} (q_\delta^-)^{-k-1} (a - \frac{n}{2})^{-l-1} a^{-1+k+l-2j}.$$ 

If $\xi \ls 1$, in view of the previous discussion one easily finds that each term arising in these sums is of order $O(\xi^{-1-j})$, so that $|\partial^j_\xi \mu_{1/2}| \ls \xi^{-1-j} \ls \lambda^{-1} \xi^{-j}$.

Similarly, if $\xi \gs 1$, then, e.g.

$$\xi (q_\delta^-)^{-k-1} (a - \frac{n}{2})^{-l-1} a^{-1+k+l-2j} \ls \left\{ \begin{array}{ll}
\xi (\frac{\xi}{\lambda})^{-k-1} \lambda^{-l-1} \lambda^{-1+k+l-2j}, & \text{if } \lambda \gs \sqrt{\xi}, \\
\xi \sqrt{\xi}^{-k-1} \sqrt{\xi}^{-l-1} \sqrt{\xi}^{-1+k+l-2j}, & \text{if } \lambda \ls \sqrt{\xi}
\end{array} \right.$$ 

$$\ls \lambda^{-1} \xi^{-j}$$

if $k + l \leq j$, and the other terms can be estimated in a similar way.

We thus see that (6.8) also holds for $i = 1$. \(\square\)

We can now prove our main result.
Theorem 6.8. If $m$ is a Mihlin-Hörmander multiplier of order $\tau > n + \frac{1}{2}$, then $m(\Delta_0)$ is a bounded operator on $L^p \Lambda^1(H_n)$ for $1 < p < \infty$.

Proof. We show that, if $m$ is as stated, then each individual operator appearing in (6.1) is $L^p$-bounded. We begin with the orthogonal projections and the intertwining operators. As in the proof of Lemma 4.1, we write the components of $R = d\Delta_0^{-\frac{3}{2}}$ as

$$(B_j L^{-\frac{3}{2}})(L^\frac{3}{2} \Delta_0^{-\frac{3}{2}}), \quad (\bar{B}_j L^{-\frac{3}{2}})(L^\frac{3}{2} \Delta_0^{-\frac{3}{2}}), \quad T \Delta_0^{-\frac{3}{2}}.$$  

The Riesz transforms $B_j L^{-\frac{3}{2}}$, $\bar{B}_j L^{-\frac{3}{2}}$ are bounded on $L^p$, being homogeneous singular integral operators with smooth kernels away from the origin. The operators $L^\frac{3}{2} \Delta_0^{-\frac{3}{2}}$ and $T \Delta_0^{-\frac{3}{2}}$ are also bounded on $L^p$ by Theorem 6.2. In fact their spectral multipliers satisfy the stronger pointwise condition (6.8) for every $i, j$. By duality, $R^*$ is also $L^p$-bounded. By Corollary 4.8, $L^p$-boundedness of $P^+_2$ reduces to $L^p$-boundedness of $\mathcal{R}$, i.e. of each operator

$$B_j \square^{-\frac{3}{2}}(I - \bar{C}) = (B_j L^{-\frac{3}{2}}) L^\frac{3}{2} \square^{-\frac{3}{2}}(I - \bar{C}).$$

Being homogeneous of degree zero, the spectral multiplier of $L^\frac{3}{2} \square^{-\frac{3}{2}}(I - \bar{C})$ satisfies (6.8), and we can apply again Theorem 6.2. The argument is completely analogous for $P^-_2$.

It remains to discuss the last two terms. Since $\Gamma$ and $\Gamma^*$ only contain $\mathcal{R}, \bar{\mathcal{R}}$ and their adjoints, we can pass directly to $S_\pm$ and $S^*_\pm$, and these operators are $L^p$-bounded by Proposition 6.7.

We finally consider the terms containing the multiplier $m$. By Theorem 6.4, $m(\Delta_0)$ is bounded on $L^p$, and the same is true for $m(\Delta_0 \pm iT)$ as long as $n \geq 2$. On $H_1$, the restriction of $m(\Delta_0 \mp iT)$ to $V^\pm_2$ equals $m(-T^2)$, as we have already observed in Section 4. Hence this case is even simpler, the conclusion following by transference from $\mathbb{R}$ to $H_1$ (or by Theorem 6.2). Finally, once we have observed that $U$ is $L^p$-bounded, it remains to consider $m\left(\Delta_0 + \frac{n}{2} \pm \sqrt{\Delta_0 + \frac{n^2}{4}}\right)$. The $L^p$-boundedness of these operators follows from the fact that also

$$m_\pm(s) = m\left(s + \frac{n}{2} \pm \sqrt{s + \frac{n^2}{4}}\right)$$

satisfy a Mihlin-Hörmander condition of order $\tau$, as a consequence of the following last two lemmas. □

Lemma 6.9. Let $m$ be a Mihlin-Hörmander multiplier of order $\tau$ on $\mathbb{R}^*_+$, and let $\varphi : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ be a smooth increasing function with the following properties

(i) there exist exponents $\gamma$ and $\gamma'$ such that, if $k$ is the smallest integer greater than or equal to $\tau$ and $j \leq k$, then $|\varphi^{(j)}(s)| \leq Ms^{\gamma-j}$ for $s$ close to 0 and $|\varphi^{(j)}(s)| \leq Ms^{\gamma'-j}$ for $s$ close to $+\infty$;

(ii) there is $\delta > 0$ such that, for $j = 0, 1$, $\varphi^{(j)}(s) \geq \delta s^{\gamma-j}$ for $s$ close to 0, and $\varphi^{(j)}(s) \geq \delta s^{\gamma'-j}$ for $s$ close to $+\infty$.

Then $m \circ \varphi$ is also a Mihlin-Hörmander multiplier of order $\tau$.

Proof. Let $I, J$ be compact intervals contained in $\mathbb{R}^*_+$, let $\psi : I \to J$ be a $C^k$-map with never vanishing derivative, and let $f \in L^2_p$ be supported on $J$. If $\tau \leq k$ is an integer, then $f \circ \psi \in L^p_r$, and

$$\|f \circ \psi\|_p \leq C\|f\|_{L^p_p}.$$
with $C$ depending on the $C^k$-norm of $\psi$ and on the infimum of $|\psi'|$.

By complex interpolation, the same is true for every $\tau \leq k$. Take now $m \in L^2_{\tau,\text{sloc}}(\mathbb{R}_+^*)$, and consider, for $r > 0$,

$$m_r(s) = m \circ \varphi(rs) \eta_0(s),$$

with $\eta_0$ a bump function supported in $I = [1,2]$. Define, for $r \leq 1$,

$$\psi_r(s) = r^{-\gamma} \varphi(rs).$$

By (i) and (ii), $\psi_r(I) \subseteq [\delta, M2\gamma] = J$. The $C^k$-norms of the $\psi_r$ are uniformly bounded by (i), and the derivatives $\psi'_r$ are uniformly bounded from below by (ii).

By (6.12),

$$\|m_r\|_{L^2_r} \leq C\|m(r^{\gamma})\eta_0 \circ \psi_r^{-1}\|_{L^2_r}.$$  

Consider the set $\Omega$ consisting of the bump functions

$$\eta_r(u) = \eta_0 \circ \psi_r^{-1}(u) = \eta_0\left(r^{-1} \varphi^{-1}(r\gamma u)\right),$$

supported on $J$. It follows from (i) and (ii) that

$$\left|\left(\varphi^{-1}\right)^{(j)}(u)\right| \leq M'u_{\gamma}^{j-1},$$

for $j \leq k$, and

$$(\varphi^{-1})^{(j)}(u) \geq \delta' u_{\gamma}^{k-j},$$

for $j = 0,1$. These inequalities imply that the $\eta_r$ have uniformly bounded $C^k$-norms. Applying now Lemma 6.1, we obtain that

$$\sup_{r \leq 1} \|m_r\|_{L^2_r} \leq C\|m\|_{L^2_{\tau,\text{sloc}}}.$$  

The same argument works for $r > 1$, replacing $\gamma$ with $\gamma'$. $\square$

**Lemma 6.10.** If $m(s)$ is a Mihlin-Hörmander multiplier on $\mathbb{R}_+^*$ of order $\tau > \frac{1}{2}$, the same is true for $m(s + a)$, for every $a > 0$.

**Proof.** By scale-invariance, we can assume that $a = 1$. Take a bump function $\eta_0$ with sufficiently small support, and consider first $r$ large. By translation-invariance, the $L^2_r$-norm of $\tilde{m}(rs)\eta_0(s)$ equals the $L^2_r$-norm of $m(rs)\eta_0(s-r^{-1})$. The functions $\eta_r(s) = \eta_0(s-r^{-1})$ are supported on the same compact subset of $\mathbb{R}_+^*$, so that we can apply Lemma 6.1 to conclude that

$$\sup_{r \geq 1} \|\tilde{m}(r\cdot)\eta_0\|_{L^2_r} \leq C\|m\|_{L^2_{\tau,\text{sloc}}}.$$  

If we now restrict our attention to $r$ small, we can replace $m$ by $m\chi$, where $\chi$ is smooth and supported on some interval $[1-\delta,1+M]$. Hence we can assume that $m \in L^2_r$, so that $\tilde{m}$ is the restriction to $\mathbb{R}_+^*$ of a function in $L^2_r$, supported on $[-\delta,M]$. We prove that, for $r$ small,

$$(6.13) \quad \|\tilde{m}(r\cdot)\eta\|_{L^2_r} \leq C\|\tilde{m}\|_{L^2_r},$$

with $C$ independent of $r$.

If $\tau = k$ is an integer, it follows from Leibniz’s rule that the left-hand side is controlled by the $L^2$-norms of $r^j\tilde{m}^{(j)}(rs)$ over the support of $\eta$. For $j = 0$, such norms are uniformly bounded by the boundedness of $\tilde{m}$, and for $j \geq 1$ by change of variable in the $L^2$-integral. For general $\tau$, (6.13) follows by complex interpolation. $\square$
References

[AD] F. Astengo, B. Di Blasio, *The Gelfand transform of homogeneous distributions on Heisenberg (-type) groups*, preprint.

[DT] P. de Bartolomeis, A. Tomassini, *On formality of some symplectic manifolds*, Int. Math. Res. Notices 24 (2001), 1287-1314.

[FS] G. Folland, E. M. Stein, *Estimates for the $\bar{\partial}$-complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. 27 (1974), 429-522.

[G] D. Geller, *Local solvability and homogeneous distributions on the Heisenberg group*, Comm. PDE 5 (1980), 475-560.

[L] J. Lott, *Heat kernels on covering spaces and topological invariants*, J. Diff. Geom. 35 (1992), 471-510.

[MRS1] D. Müller, F. Ricci, E. M. Stein, *Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, I*, Inv. Math. 119 (1995), 199-233.

[MRS2] D. Müller, F. Ricci, E. M. Stein, *Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, II*, Math. Z. 221 (1996), 267-291.

[MS] D. Müller, E. M. Stein, *On spectral multipliers for Heisenberg and related groups*, J. Math. Pures Appl. 73 (1994), 413-440.

[NRS] A. Nagel, F. Ricci, E. M. Stein, *Harmonic analysis and fundamental solutions on nilpotent Lie groups*, In: Analysis and Partial Differential Equations (C. Sadosky ed.). Lecture Notes in Pure Appl. Math. M. Dekker NY 122 (1990), 249-275.

[R1] M. Rumin, *Formes différentielles sur les variétés de contact*, J. Diff. Geom. 39 (1994), 281-330.

[R2] M. Rumin, *Sub-Riemannian limit of the differential form spectrum of contact manifolds*, GAFA 10 (2000), 407-452.

[S] E. M. Stein, *Harmonic Analysis. Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993.

[W] A. Weil, *Introduction à l’étude des variétés kählériennes*, Hermann, 1958.

Christian-Albrechts-Universität zu Kiel, Mathematisches Seminar, Ludewig-Meyn-Str. 4, D-24098 Kiel, Germany

E-mail address: mueller@math.uni-kiel.de
URL: http://analysis.math.uni-kiel.de/mueller

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

E-mail address: peloso@calvino.polito.it
URL: http://calvino.polito.it/~peloso/

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

E-mail address: fricci@sns.it
URL: http://www.math.sns.it/HomePages/Ricci/