Stochastic resonance in the presence of slowly varying control parameters

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Abstract. The kinetics of transitions between states in a noisy system is studied in the simultaneous presence of a periodic forcing and a ramp. It is shown that the interaction between stochastic resonance and the action of the ramp may give rise to a new method for the control of the transition rates.

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1. Introduction

Many physical systems are described by evolution equations of the form [1, 2]

\[
\frac{dx}{dt} = v(x, \lambda) + F(t),
\]
where \( \mathbf{x} = (x_1, \ldots, x_n) \) denotes the state vector, \( \mathbf{v} = (v_1, \ldots, v_n) \) the evolution law, \( \lambda \) a set of control parameters and \( \mathbf{F}(t) = (F_1(t), \ldots, F_n(t)) \) a set of stochastic forcings accounting for the fluctuations generated spontaneously within the system or for the perturbations impinging on the system from the external world.

Ordinarily, the control parameters in (1) are assumed to remain constant. There are, however, situations in which this constitutes an oversimplification. As an example, \( \lambda \) may describe the effect of a constraint that is gradually switched on at some stage of the evolution, a climatic parameter drifting systematically e.g. as a result of anthropogenic effects, or a periodic forcing accounting for the action of external fields varying on a much slower time scale than the system itself.

It is by now established that when the system undergoes nonlinear dynamics leading to several coexisting invariant states, each of these effects taken individually can have significant consequences on the evolution. In particular:

(i) In a system possessing a stable steady state separated from another stable regime by an intermediate unstable steady state, stochastic forcings \( \mathbf{F}(t) \) in the form of Gaussian Markov noises induce transitions across the unstable state, tending to deplete a probability mass initially centred entirely on the stable state [3, 4].

(ii) Under the simultaneous presence of a stochastic forcing \( \mathbf{F}(t) \) and a slowly varying periodic forcing, the distribution of probability masses on the two sides of the unstable state may be deeply affected. In particular, the response to the periodic forcing can be substantially enhanced by the noise, a phenomenon referred as stochastic resonance [5]–[7].

(iii) Under the simultaneous presence of a stochastic forcing \( \mathbf{F}(t) \) and a slow increase of a control parameter in the form of a ramp,

\[
\lambda = \lambda_0 + \varepsilon t, \quad 0 < \varepsilon \ll 1, \tag{2}
\]

the system can be frozen on a preferred state by practically quenching the transitions across the unstable state [8, 9].

In the present paper, we report results on the response of a nonlinear system subjected simultaneously to noise, a ramp, and a slowly varying periodic forcing. We are especially interested in systems possessing two simultaneously stable steady states arising in the vicinity of a supercritical pitchfork bifurcation, for which the dynamics reduces to a universal (normal) form displaying a single order parameter. Under these conditions, equations (1) can be cast in the form of a Langevin equation [3, 4, 10]

\[
\frac{dz}{dt} = (\lambda_0 + \varepsilon t + \gamma \cos(\omega t + \phi))z - z^3 + \mathbf{F}(t), \tag{3}
\]

where \( t \) is a normalized time, \( z \) a linear combination of original variables \( \{x_i\} \) and the periodic forcing has been taken to act in a multiplicative fashion. In what follows, the stochastic forcing will be assimilated to a Gaussian white noise

\[
\langle \mathbf{F}(t) \rangle = 0, \quad \langle \mathbf{F}(t) \mathbf{F}(t') \rangle = q^2 \delta(t - t'). \tag{4}
\]

Notice that equation (3) can also be written as

\[
\frac{dz}{dt} = -\frac{\partial U(z, t)}{\partial z} + \mathbf{F}(t), \tag{5a}
\]
where $U$ is the kinetic potential, given by

$$U = -(\lambda_0 + \epsilon t + \gamma \cos(\omega t + \phi)) \frac{z^2}{2} + \frac{z^4}{4}. \quad (5b)$$

Equations (3)–(5) are encountered in a wide range of problems in physical, engineering, environmental and life sciences. Nonlinearity is a ubiquitous feature of such systems and bistability constitutes one of its principal manifestations. On the other hand, the environment in which these systems are embedded is as a rule a complex system, communicating to them perturbations that, for many purposes, can be assimilated to uncorrelated noise processes. Finally, the periodic forcing and the ramp may either constitute systematic signals superimposed on this otherwise random background, or be introduced deliberately in order to control the behaviour in a certain preassigned way, e.g. optimizing a regulatory or a cognitive process at the cellular or supercellular level, channel a reaction pathway in order to increase the selectivity of a desired product, and so forth.

In the following sections, the antagonistic effects arising from the noise, the ramp and the periodic forcing are combined to obtain the overall response of the system. In section 2, the deterministic dynamics of the noise-free system is first considered. The analytical formulation of stochastic dynamics is given in section 3 and is compared to the results of numerical simulations in section 4. The main conclusions are summarized in section 5.

2. The noise-free system: deterministic dynamics in the presence of a periodic forcing and a ramp

In the absence of noise, one deals with the evolution equation

$$\frac{dz}{dt} = (\lambda_0 + \epsilon t + \gamma \cos(\omega t + \phi))z - z^3. \quad (6)$$

This equation is of the Bernoulli type. It can be integrated exactly by first switching to the variable $u = z^{-1/2}$, yielding, after a standard calculation,

$$z(t) = \frac{z(0) \exp(\phi(t))}{1 + 2z^2(0) \int_0^t \exp(2\phi(t')) \, dt'}. \quad (7)$$

where

$$\phi(t) = \lambda_0 t + \frac{\epsilon^2}{2} + \frac{\gamma}{\omega} \sin(\omega t + \phi) \quad (8)$$

and $z(0)$ stands for the initial condition. We notice that $z = 0$ is a fixed-point solution, unstable as long as $\lambda_0$ and $\epsilon$ are positive. Furthermore, the solution $z(t)$ corresponding to a non-vanishing initial $z(0)$ keeps the sign of the initial condition forever, entailing that crossing of the $z = 0$ state is prohibited. This is due to the multiplicative character of the periodic forcing.

As it stands, solution (7) is rather formal. It simplifies considerably in the regime of the linear response to the periodic forcing ($\gamma \ll 1$), in which limit $z(t)$ can be expressed in terms of imaginary error functions involving the arguments $\epsilon^{1/2} t + [(\lambda_0 \pm i\omega)/\epsilon^{1/2}]$.
and \([(\lambda_0 + i\omega/2)/\varepsilon^{1/2}]\). Actually, it is more instructive to study linear response through the approximate solution of equation (6) obtained in the \textit{adiabatic approximation}. This amounts to introducing the slow time scale \(\tau = \varepsilon t\) and rewriting (6) as

\[
\varepsilon \frac{dz}{d\tau} = \left(\lambda_0 + \tau + \gamma \cos \left(\frac{\omega}{\varepsilon} \tau + \phi\right)\right) z - z^3.
\]  

To the dominant order in \(\varepsilon\) and as long as \(\omega/\varepsilon = \omega_1\) is of \(O(1)\), this equation admits the solutions

\[
z_0 = 0, \quad z_\pm = \pm [\lambda_0 + \tau + \gamma \cos(\omega_1 \tau + \phi)]^{1/2},
\]  

except for an initial time layer, if the system is started far from \(\pm \lambda_0^{1/2}\).

We notice that in the absence of both the ramp and the periodic forcing, equation (10) reproduces the two stable steady states \(z_{\text{ref}} = \pm \lambda_0^{1/2}\) of the reference system

\[
\frac{dz}{dt} = \lambda_0 z - z^3.
\]  

When the ramp is applied but the periodic forcing is absent, equation (10) provides the solutions \(z^{(0)}_\pm = \pm (\lambda_0 + \tau)^{1/2}\) of the equation

\[
\frac{dz}{dt} = (\lambda_0 + \tau) z - z^3
\]  

in the adiabatic approximation. As time increases, the distance of either \(z^{(0)}_+\) or \(z^{(0)}_-\) from the unstable state \(z_0 = 0\) increases monotonically. In this sense, the ramp has a stabilizing effect on the reference, completely unperturbed system.

We turn now to the type of effect that one may expect from the presence of the forcing. Expanding (10) in powers of \(\gamma\) and keeping the first non-trivial terms, we obtain (switching back to the original variables and parameters)

\[
z_\pm = \pm \left[ (\lambda_0 + \varepsilon t)^{1/2} + \frac{\gamma}{2(\lambda_0 + \varepsilon t)^{1/2}} \cos(\omega t + \phi) - \frac{\gamma^2}{4(\lambda_0 + \varepsilon t)^{3/2}} \cos^2(\omega t + \phi) \right].
\]  

A first measure of the overall effect of the forcing is provided by the phase-averaged response,

\[
\bar{z}_\pm = \frac{1}{2\pi} \int_0^{2\pi} d\phi z_\pm(t, \phi).
\]  

We obtain, using (11b) and (12),

\[
\bar{z}_+ = (\lambda_0 + \varepsilon t)^{1/2} - \frac{\gamma^2}{8(\lambda_0 + \varepsilon t)^{3/2}} \equiv z_+(0) - \frac{\gamma^2}{8(\lambda_0 + \varepsilon t)^{3/2}}
\]  

implying that \(\bar{z}_+ < z_+(0)\). This result can be interpreted as a ‘destabilizing’ trend induced by the periodic forcing, in the sense that the distance of the \(z_+^{(0)}\) state of the forcing-free system from the unstable state \(z_0 = 0\) tends to be decreased (and likewise for \(\bar{z}_-\) versus \(z_-^{(0)}\)). This trend becomes less pronounced as \(t\) increases and finally disappears for \(t \gg \varepsilon^{-1}\). Notice that, as pointed out in
connection with equation (7), the crossing of $z_0 = 0$ is impossible whatever the forcing amplitude $\gamma$ might be.

A finer measure of the effect of the periodic forcing can be obtained by keeping the phase dependence, in which case the last term in equation (12) can be neglected:

$$z_+(t) = (\lambda_0 + \epsilon t)^{1/2} + \frac{\gamma}{2(\lambda_0 + \epsilon t)^{1/2}} \cos(\omega t + \phi) \equiv z_+^{(0)}(t) + \gamma z_+^{(1)}(t)$$

(15)

and similarly for $z_-(t)$.

Since the contribution $z_+^{(1)}(t)$ of the forcing to the response $z_+$ is penalized by an inverse $t^{1/2}$ factor and given the systematic growth of $z_+^{(0)}$ in time, a relevant question to be asked pertains to the sign of $z_+^{(1)}$ during the time period up to its first extremum. The forcing will have a maximal destabilizing effect if initially $z_+^{(1)}$ and its time derivative are both negative and a maximal stabilizing effect if they are both positive. In the first case, one is led to the conditions

$$\cos \phi < 0 \quad \text{and} \quad \tan \phi > -\frac{\epsilon}{2\omega\lambda_0}.$$  \hspace{1cm} (16a)

The first inequality implies that $\phi$ must be limited in the interval $(\pi/2, 3\pi/2)$ and the second one that $\phi$ must exceed a crossover value, $\phi^*$, given by

$$\phi^* = \tan^{-1} \left( -\frac{\epsilon}{2\omega\lambda_0} \right).$$  \hspace{1cm} (16b)

Similar conditions can be derived in the opposite case where the forcing has a stabilizing effect.

In figure 1, the solution $z_+^{(0)}$ of the unforced system in the adiabatic approximation (solid line) is compared to the numerically computed solution of the full equation (6) for $z(0) = \lambda^{1/2}$. 

**Figure 1.** Time evolution as deduced from the full equation (6) (broken line) and from the adiabatic approximation equation (10) (solid line) with parameter values $\lambda_0 = 1$, $\epsilon = 0.1$, $\gamma = 0.5$, $\omega = 0.01$ and $\phi = 6\pi/5$. 

*New Journal of Physics 7* (2005) 8 ([http://www.njp.org/](http://www.njp.org/))
Under the conditions of relations (16a), the destabilizing action of the forcing as reflected by a transient trend to approach $z_0 = 0$ is clearly seen (broken line). A similar agreement with the analytical predictions is also found in the case opposite to (16a), the trend being now a stabilizing one.

3. Stochastic resonance-like behaviour in the presence of a ramp: analytical formulation

We have seen that, in the absence of fluctuations, the ramp has always a stabilizing effect, whereas under appropriate conditions the periodic forcing may have a destabilizing effect. We turn now to the dynamics of the fluctuations in the presence of both the ramp and the periodic forcing (equations (5)). Typically, an extra destabilizing trend will then be switched on in the form of a leakage of a probability mass initially placed around $z_{ref} = \pm \lambda_0^{1/2}$, following the crossing of the unstable state $z_0 = 0$. We shall analyse this crossing (which is here allowed owing to the additive character of the stochastic forcing) by adopting once again the adiabatic approximation, now suitably extended to account for stochastic effects.

The starting point is the Fokker–Planck equation associated with (5a),

$$\frac{\partial \rho(z, t)}{\partial t} = -\frac{\partial}{\partial z} \left( \frac{\partial U}{\partial z} \rho \right) + \frac{q^2}{2} \frac{\partial^2 \rho}{\partial z^2}, \quad (17)$$

where $\rho(z, t)$ denotes the probability density. The potential $U$ (equation (5b)) possesses three extrema, just as in the case $\varepsilon = 0, \gamma = 0$, given by $z_0$ and $z_\pm$ in equation (10). The values corresponding to these extrema are

$$U(0) = 0 \quad \text{(maximum)}, \quad U_\pm(t) = -\frac{1}{4}(\lambda_0 + \varepsilon t + \gamma \cos(\omega t + \phi))^2 \quad \text{(minima)}, \quad (18)$$

leading to a potential barrier

$$\Delta U_\pm(t) = \frac{1}{4}(\lambda_0 + \varepsilon t + \gamma \cos(\omega t + \phi))^2. \quad (19)$$

We observe that, as time grows, the minima tend to undergo a drift away from $z_0 = 0$ and to become deeper, but are also subject at the same time to a periodic modulation affecting momentarily their distance from $z_0 = 0$. If the noise strength $q^2$ is reasonably small, one may expect that in the long time regime, $\rho(z, t)$ will essentially be given by two Gaussians localized on $z_\pm(t)$, whose weights $N_\pm(t)$ will be varying slowly as a result of transitions across the barrier. We are thus led to the stochastic version of the adiabatic approximation [4, 5]

$$\rho(z, t) = N_- f_-(z, t) + N_+ f_+(z, t). \quad (20)$$

The form of $f_\pm$ is obtained by expanding $U(z, t)$ around $z_\pm$, keeping only the first non-trivial (here quadratic) terms

$$f_\pm = \sqrt{\frac{\lambda_0 + \varepsilon t + \gamma \cos(\omega t + \phi)}{\pi q^2/2}} \exp \left[ -\frac{(\lambda_0 + \varepsilon t + \gamma \cos(\omega t + \phi))(z - z_\pm)^2}{q^2/2} \right]. \quad (21)$$
We next turn to the dynamics of the weights \( N_\pm(t) \), which carry interesting information concerning the transitions across the barrier. Since the potential \( U(z, t) \) is symmetric around \( z = 0 \), it will be convenient if the transient behaviour from an initial state favouring \( z_+ \) or \( z_- \) is disentangled from the equipartition case, in which the two quasi-attraction basins of \( z_+ \) and \( z_- \) are given the same probability mass and hence there is no further evolution. We therefore set

\[
N_\pm(t) = \frac{1}{2} \pm \delta N(t). \tag{22}
\]

The excess probability mass \( \delta N(t) \) obeys then, in the adiabatic approximation, a rate equation similar to that appearing in the classical Kramers theory [4, 11]

\[
\frac{d\delta N}{dt} = -\frac{1}{\tau} \delta N \tag{23}
\]

in which the mean transition time \( \tau \) now depends on the ramp and the periodic forcing—and hence on time—through the potential \( U(z, t) \),

\[
\tau^{-1} = \frac{1}{2\pi} \left[ -U''(z_0, t)U''(z_\pm, t) \right]^{1/2} \exp \left[ -\Delta U_\pm(t) \frac{q^2}{2} \right] \tag{24}
\]

or, using equations (18) and (19),

\[
\tau^{-1} = \frac{\sqrt{2}}{\pi} (\lambda_0 + \epsilon t + \gamma \cos(\omega t + \phi)) \exp \left[ -\frac{1}{2q^2} (\lambda_0 + \epsilon t + \gamma \cos(\omega t + \phi))^2 \right]. \tag{25}
\]

Equation (23) can be integrated exactly, yielding

\[
\delta N(t) = \delta N(0) e^{V(t)}, \tag{26a}
\]

where

\[
V(t) = -\int_0^t dt' \frac{1}{\tau(t')} \tag{26b}
\]

As in section 2, we focus on the linear response to the forcing. Expanding \( V(t) \) in \( \gamma \) and keeping the first non-trivial terms, one obtains

\[
V(t) = V_0(t) + \gamma V_1(t) \tag{27}
\]

with

\[
V_0(t) = -\frac{\sqrt{2} q^2}{\pi \epsilon} \left[ \exp \left( -\frac{\lambda_0^2}{2q^2} \right) - \exp \left( -\frac{1}{2q^2} (\lambda_0 + \epsilon t)^2 \right) \right], \tag{28a}
\]

\[
V_1(t) = -\frac{\sqrt{2}}{\pi} \int_0^t dt' \left\{ \left[ -\frac{1}{q^2} (\lambda_0 + \epsilon t)^2 \exp \left( -\frac{(\lambda_0 + \epsilon t)^2}{2q^2} \right) + \exp \left( -\frac{(\lambda_0 + \epsilon t)^2}{2q^2} \right) \right] \right. \\
\times \cos \omega t \cos \phi \left[ -\frac{1}{q^2} (\lambda_0 + \epsilon t)^2 \exp \left( -\frac{(\lambda_0 + \epsilon t)^2}{2q^2} \right) \right. \\
+ \left. \exp \left( -\frac{(\lambda_0 + \epsilon t)^2}{2q^2} \right) \right] \sin \omega t \sin \phi \right\}. \tag{28b}
\]
The part in $V_0(t)$ arising from the presence of the ramp in the absence of the periodic forcing was analysed in detail by us [8]. It is responsible for a qualitatively new effect whereby, in the long-time limit $t \gg 1/\varepsilon$, equipartition (translated by $\delta N_\infty = 0$) is not always achieved: the system may remain blocked in one of the quasi-attraction basins (translated by $\delta N_\infty, 0$), depending on its initial preparation.

Our objective here is to test whether the term in $V_1(t)$ can counteract this blocking and facilitate, in the spirit of classical stochastic resonance, the transitions across the unstable state. We first observe that, using classic relations from the theory of special functions, this part can be expressed in terms of parabolic cylinder functions or, more conveniently for our purposes, in terms of error functions of complex argument:

$$-V_1(t) = \exp\left(-\frac{\lambda_0^2}{2q^2}\right)\left(\frac{-\lambda_0}{\varepsilon}\cos \phi + \frac{i\omega q^2}{\varepsilon^2} \sin \phi\right)$$

$$+ \exp\left(-\frac{(\lambda_0 + \varepsilon t)^2}{2q^2}\right)\left[\left(\frac{\lambda_0}{\varepsilon} + t\right)\cos(\omega t + \phi) - \frac{i\omega q^2}{\varepsilon^2} \sin(\omega t + \phi)\right]$$

$$+ \frac{q^3}{\varepsilon^3} \frac{\sqrt{\pi} \omega^2}{2\sqrt{2}} \exp\left(-\frac{\omega^2 q^2}{2\varepsilon^2}\right) (A(t) \cos \phi + B(t) \sin \phi),$$

where

$$A(t) = \exp\left(-\frac{i\omega \lambda_0}{\varepsilon}\right) \left[\text{erf}\left(\frac{\lambda_0 + \varepsilon t}{\sqrt{2}q}\right) - \text{erf}\left(\frac{\lambda_0}{\sqrt{2}q} - \frac{i\omega q}{\sqrt{2}\varepsilon}\right)\right] + \text{c.c.},$$

$$B(t) = i \exp\left(-\frac{i\omega \lambda_0}{\varepsilon}\right) \left[\text{erf}\left(\frac{\lambda_0 + \varepsilon t}{\sqrt{2}q}\right) - \text{erf}\left(\frac{\lambda_0}{\sqrt{2}q} - \frac{i\omega q}{\sqrt{2}\varepsilon}\right)\right] + \text{c.c.}$$

As we are interested primarily in the long-time behaviour (in the spirit of the adiabatic approximation) and/or in the weak-noise case, we are entitled to consider the limiting form of equations (29) and (30), in which the absolute value of the argument of the error functions goes to infinity. Performing the same limit in $V_0(t)$ as well, we obtain

$$\delta N_\infty = \delta N(0) \exp(V_\infty),$$

where

$$V_\infty = -\frac{\sqrt{2} q^2}{\pi} \exp\left(-\frac{\lambda_0^2}{2q^2}\right) + \frac{\sqrt{2}}{\pi} \gamma \exp\left(-\frac{\lambda_0^2}{2q^2}\right)$$

$$\times \frac{\lambda_0^2 \varepsilon}{\lambda_0^2 \varepsilon^2 + \omega^2 q^4} \left(\lambda_0 \cos \phi - \frac{q^2 \omega}{\varepsilon} \sin \phi\right) \equiv V^{(0)}_\infty + \gamma V^{(1)}_\infty.$$  

The second term in equation (32) contains the main features of the response of a noisy system subjected to a ramp to a superimposed periodic forcing. If this term is negative, the periodic forcing counteracts the action of the ramp, in the sense that the non-ergodic behaviour associated with freezing the system at some finite $\delta N_\infty$ tends to be relaxed. If, in contrast, this
term is positive, non-ergodicity is enhanced. In this connection, the following points are worth stressing.

(i) For a given $\lambda_0$, $\varepsilon$ and $\omega$, the regimes of enhanced ergodicity or non-ergodicity are delimited by a crossover value of the phase determined by cancelling the last term in parentheses in equation (32):
\[
\tan \phi^* = \frac{\varepsilon \lambda_0}{q^2 \omega}.
\] (33)

This relation is the probabilistic extension of equation (16b). Introducing $\phi^*$ into (32), one may express the linear response in the form
\[
\gamma V^{(1)}_\infty = -\gamma \frac{\sqrt{2}}{\pi} \exp \left( -\frac{\lambda_0^2}{2q^2} \right) \frac{\lambda_0^2 \varepsilon}{\sqrt{\lambda_0^2 \varepsilon^2 + q^4 \omega^2}} \sin(\phi - \phi^*).
\] (34)

The existence of a crossover value, $\phi^*$, can be understood qualitatively from inspection of equation (3). Taking $\phi = 0$, one sees that the periodic forcing tends to reinforce the action of the ramp during its first quarter period. This happens to be the most crucial one, since by the time the forcing will become negative the value of the barrier will be prohibitively large for the noise to induce transitions across the unstable state with appreciable probability. This is in good agreement with the fact that the $V^{(1)}_\infty$ term in equation (32) is positive in this case. Taking on the contrary $\phi = \pi$ amounts to counteracting the ramp momentarily during this same crucial period, again in good agreement with the fact that the $V^{(1)}_\infty$ term in equation (32) is now negative.

(ii) There is an optimal frequency extremizing the response
\[
\omega^* = \frac{\varepsilon \lambda_0 \cos \phi \pm 1}{q^2} \frac{\sin \phi}{\overline{\sin \phi}}
\] (35)
as long as the right-hand side is positive and finite. Taking, for instance, $\phi = -\pi/2$, corresponding to a forcing of the form $\sin \omega t$ in equation (3), one finds $\omega^* = \lambda_0 \varepsilon / q^2$.

(iii) There is a non-trivial dependence of the response to the forcing on $q^2$. $V^{(1)}_\infty$ tends to zero for very small and very large values of $q^2$ and possesses an extremum at some intermediate value, $q^*$. The value of this extremum depends strongly on the phase $\phi$ and on the parameters $\lambda_0$, $\varepsilon$ and $\omega$. Figure 2(a) provides a concrete illustration. Note that, in the range of small $q^2$’s both the forcing-free result $V^{(0)}_\infty$ and the linear response $V^{(1)}_\infty$ are very small under these particular conditions. In order to disentangle the role of the forcing, we depict in figure 2(b) the ratio $\delta N^{(0)}_\infty / \delta N^{(0)}_\infty$ of the full response (equation (26a)) to the solution of the rate equation (23) of the forcing-free system as a function of $q^2$. As can be seen, the effect of the forcing manifests itself in a significant way only beyond some finite value of the variance. One finds here one of the characteristic attributes of stochastic resonance.

4. Numerical solutions

The analytical results of the preceding section will now be confronted and complemented by those of the numerical solution of the Fokker–Planck equation (17) using a variable mesh method [12], and of the stochastic simulation of the Langevin equation (3).

We first address the role of the phase $\phi$ of the forcing. In figure 3, the asymptotic probability mass $N_\ast$ on the quasi-attraction basin $z_\ast$ as deduced from the solution of equation (17) is
Figure 2. (a) Sensitivity of the asymptotic exponents $V^{(0)}_{\infty}$ (solid line) and $\gamma V^{(1)}_{\infty}$ (broken line) scaled by $\exp(-\lambda_0^2/2q^2)$ on $q^2$ as predicted by the analytical evaluation, equation (32). (b) As in (a), except for the asymptotic value of the ratio of excess probability masses of the perturbed system, i.e. periodically forced system, and of the unperturbed system. Parameter values were $\lambda_0 = 1$, $\varepsilon = 10^{-4}$, $\gamma = 0.1$, $\phi = \pi/2$ and $\omega = 0.01$.

Figure 3. Asymptotic probability mass $N_+$ versus the phase $\phi$ as obtained from the numerical solution of the Fokker–Planck equation (17) with initial condition $N_+(0) = 1$. Parameter values were $\lambda_0 = 1$, $\varepsilon = 10^{-4}$, $\gamma = 0.1$, $q^2 = 0.08$ and $\omega = 10^{-3}$. $\phi^* = 0.896$ is the critical value predicted by the analytical evaluation (equation (33)) for which the response is insensitive to the periodic forcing.
Figure 4. Asymptotic $N_+$ versus $\omega$ as obtained from numerical solution of the Fokker–Planck equation (17) with initial condition $N_+ (0) = 1$. Parameters $\lambda_0, \varepsilon$, $\gamma$ and $q^2$ were as in figure 3 and $\phi = \pi/2$. The minimum value of $N_+$obtained is in accord with the analytical estimation $\omega^* = \lambda_0 \varepsilon / q^2 \approx 0.00125$.

represented as a function of $\phi$ (empty circles). The parameters are chosen so that in the absence of the forcing (broken line), $N_+$ shows a markedly non-ergodic behaviour, very far from the equipartition value $N_+ = 0.5$. As recalled in section 3, this is due entirely to the presence of the ramp. In the presence of the forcing, the situation is changing considerably, with the exception of a certain value $\phi$ (modulo $\pi$) for which the effect is vanishing. The particular value deduced from the simulation agrees reasonably well with the theoretical estimate $\phi^*$ of equation (33), considering the approximations involved. For $\phi > \bar{\phi}$, the forcing tends to equilibrate the distribution of probability masses across the unstable state, thereby enhancing ergodicity, with an optimum at $\phi = \hat{\phi} + (\pi/2)$. For $\phi < \bar{\phi}$, an opposite trend is observed. These features are all in good agreement with the analysis of section 3.

We next come to the dependence of the response on the forcing frequency $\omega$. In figure 4, $N_+$, deduced again from the solution of equation (17), is now represented as a function of $\omega$. The value of the phase is taken to be $\phi = \pi/2$, so that the $\omega = 0$ limit brings us automatically to the forcing-free case. The other parameters are so chosen that the system is again unable to reach equipartition and remains frozen around $z_+$ with high probability, owing to the action of the ramp. As can be seen from the figure, when the forcing is switched on, there is a tendency to equilibrate the probability masses and approach equipartition, thereby favouring ergodicity and the loss of memory of the initial condition. Since the forcing is weak (to comply with the analytical approximations), ergodicity is never strictly achieved, but the system comes as close to it as it can at some value $\omega^* \approx 0.00125$, which is practically indistinguishable from the theoretical estimate of equation (35).
Figure 5. As in figure 2(b), except that closed circles shown here stand for the results obtained from numerical integration of the Fokker–Planck equation.

Figure 5 summarizes the main results on the dependence of the response on $q^2$. The filled circles are obtained by the numerical solution of equation (17) and the solid line, taken from the analytical estimates of section 3, is identical to figure 2(b). The role of noise as an amplifier of the response to the forcing is again confirmed. Although the theoretical estimate is not fully matching the numerical values in the range of high $q$’s, the qualitative trend remains quite satisfactory.

In order to see how the variable $z(t)$ itself evolves in time, we perform a simulation of the full Langevin equation (3). Figure 6 depicts the stochastic trajectories obtained in the absence (a) and in the presence (b) of the forcing. As before, we place ourselves deliberately under the conditions that the action of the ramp tends to freeze the system in one of the quasi-attraction basins, here $z_+^{(0)}$. In figure 6, this is reflected by a stochastic trajectory exhibiting a small-scale variability around the instantaneous, slowly varying value of $z_+^{(0)}$. This trajectory is unable to cross the unstable state $z_0 = 0$ during the (long) simulation time. The situation changes radically in the presence of the forcing. The system can now perform one or several (as in figure 6(a)) transitions across $z_0 = 0$ during the simulation time window, before being eventually captured in the upper quasi-attraction basin or even in some percentage of the stochastic realizations in the lower one (as happened in figure 6(b)).

5. Conclusions

In this paper, a method for enhancing and controlling the sensitivity of nonlinear systems subjected to noise towards environmental constraints has been put forward. A typical setting where the method applies is that of systems possessing at least two simultaneously stable
Figure 6. Stochastic trajectory obtained numerically from equation (3), for the unperturbed system (a) and the perturbed one (b). Parameter values $\lambda_0 = 1$, $\epsilon = 10^{-4}$, $\gamma = 0.3$, $\omega = 10^{-3}$ and $q^2 = 0.06$.

Invariant states separated by an intermediate unstable one, in the presence of a ramp in the parameter controlling the bifurcation and of a multiplicative periodic forcing. It consists, then, in selectively freezing the system on a preferred state or, in contrast, in enabling it to perform transitions across the unstable state, by adequately tuning the values of parameters such as the ramp rate, amplitude, phase and period of the forcing and the noise strength. A number of situations have been identified in which the forcing counteracts the effect of the ramp with an optimal efficiency. Furthermore, as it turns out, the effect of the forcing manifests itself in a significant way only beyond some characteristic value of the noise strength.

These features are reminiscent of the phenomenon of stochastic resonance where, in the presence of noise, the response to a periodic forcing is amplified and, conversely, the presence of the periodic forcing organizes the distribution of the probability masses across the unstable state. The new element here is the presence of an antagonistic process provided by the ramp. This confers a transient character to the transition across the unstable state, in the sense that at some stage the transitions will tend to be quenched for increasingly long time windows. This entails that the facilitating role that the forcing can exert in the transition process is operational preferentially during the early stages of the evolution, prior to attaining a 'point of no return' due to a prohibitively high barrier that would form in the meantime as a result of the ramp. This interplay could be of interest in nonlinear optics and electronic circuit-related problems where, owing to the short time scales involved, the switching process may interfere in a non-trivial way with the dynamics. Another area of interest is atmospheric and climate dynamics, where periodic forcings associated with e.g. solar activity or with deep ocean circulation coexist with ramps arising from volcanic eruptions or from anthropogenic effects. More generally, the possibility to control a system by freezing it in a particular state in the course of the evolution could provide a mechanism for generic selection of certain modes of behaviour versus the full spectrum of possibilities available.

One direction along which the present work can be extended is to apply nonlinear ramps in the bifurcation parameter, such as ramps inducing a transition between two different levels of this parameter. Regarding the forcing, non-sinusoidal or non-periodic cases, e.g. a square
pulse or a quasi-periodic function, are frequently occurring in many—if not in most—physically relevant situations. Finally, spatially extended systems are expected to display responses with novel properties associated, for instance, with spatial or spatio-temporal pattern selection.

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References

[1] Nicolis G and Prigogine I 1977 *Self-organization in Nonequilibrium Systems* (New York: Wiley)
[2] Haken H 1977 *Synergetics* (Berlin: Springer)
[3] Van Kampen N 1981 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)
[4] Gardiner C 1983 *Handbook of Stochastic Methods* (Berlin: Springer)
[5] Nicolis C 1982 *Tellus* **34** 1
[6] Benzi R, Parisi G, Sutera A and Vulpiani A 1982 *Tellus* **34** 11
[7] Gammanitoni L, Hänggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
[8] Nicolis C and Nicolis G 2000 *Phys. Rev. E* **62** 197
[9] Nicolis C and Nicolis G 2004 *Europhys. Lett.* **66** 185
[10] Nicolis G 1995 *Introduction to Nonlinear Science* (Cambridge: Cambridge University Press)
[11] Talkner P and Luczka J 2004 *Phys. Rev. E* **69** 046109
[12] Chang J and Cooper G 1970 *J. Comput. Phys.* **6** 1

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