Geometric Phase Atom Optics and Interferometry

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We illustrate how geometric gauge forces and topological phase effects emerge in quantum systems without employing assumptions that rely on adiabaticity. We show how geometric magnetism may be harnessed to engineer novel quantum devices including a velocity sieve, a component in mass spectrometers, for neutral atoms or neutrons. We outline a possible experimental setup that demonstrates topological interferometry for neutral spin 1/2 systems.

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INTRODUCTION

Berry’s phase,[1] a realization of a non-integrable phase factor[2, 3], plays an important role in describing adiabatic quantum evolution in a semi-classical setting. It is also known[4, 5, 6, 7] that effective gauge potentials, that give rise to such phases, emerge in fully quantal systems in which true adiabaticity is ill-defined. In addition to their role in generating phase holonomies,[4] it was shown that such potentials may lead to effective Lorentz-like forces,[8, 9] acting on the quantum system.

Today, applications of geometric gauge forces in cold many-body systems[10, 11] is an active and topical area of research. Dressing atoms using lasers,[12] researchers[13, 14] have been able to engineer Lorentz-like forces, an effect sometimes called geometric[9, 15], synthetic[11] or artificial magnetism[10], in ensembles of cold atoms. It is hoped that the latter may allow realization of novel quantum Hall physics in a quantum degenerate gas.

Another, possible application of geometric magnetism is in the manipulation of individual neutral atoms or neutrons, e.g. realization of “magnetic lens” for neutral matter[15, 16, 17]. It is that application that we focus on in this Letter. Below we provide two striking illustrations of the latter. In order to demonstrate the phenomena we employ a two-level quantum system and using exact solutions for the coupled Schroedinger equations we show, first, how geometric magnetism allows realization of a velocity sieve, a component in mass spectrometers, for beams of neutral particles. In another example we consider a quantum mechanical analog of a field theoretical model, first introduced by March-Russell, Preskill and Wilczek[15, 17, 18], to demonstrate topological quantum interferometry for neutral spin-1/2 systems. It’s laboratory realization could have applications in topological quantum computing protocols.

Almost all previous theoretical studies rely on some form of the adiabatic approximation, i.e. the Born-Oppenheimer (BO) approximation. In the adiabatic picture the gauge potentials are explicit but it also know[19, 20] that those effective (non-Abelian) gauge potentials describe a pure gauge[3] which, at first sight, should not give rise to observable effects. Are non-trivial gauge forces an artifact of the adiabatic approximation?[21], are singularities in the adiabatic Hamiltonian solely responsible for the emergence of non-trivial topological phases? Several interpretations[22, 23] for the origins of geometric gauge forces have been advanced. However, compelling examples that offer fully quantum solutions to systems in which such forces arise in the adiabatic limit have largely been unavailable, and therefore, predictions are limited by the validity of assumptions based on adiabaticity.

In order to address questions and deficiencies in theoretical approaches that assume adiabaticity, we[15, 16] introduced a wave packet propagation scheme that does not insist on the assumptions and restrictions imposed by it. The resulting time dependent solutions for the systems (discussed below) are exact, within the bounds imposed by numerical error. Our solutions are therefore not compromised by issues relating to the robustness of the adiabatic approximation. This allows us to make definitive verdicts on the fidelity of predictions informed by the adiabatic approximation, and the gauge theory interpretation that follows from it.

Theory

Consider a Hamiltonian

\[ H = -\frac{\hbar^2}{2m} \nabla R^2 + H_{ad}(R) \]

\[ H_{ad}(R) = U(R) H_{BO}(R) U^\dagger(R) \]  \hspace{1cm} (1)

where \( H_{BO}(R) \) is an n-dimensional diagonal matrix with eigenvalues \( \epsilon_i(R) \), and \( U(R) \) is a unitary matrix. In this discussion we consider the case where \( n = 2 \), and \( R = (x, y) \). In solving the Schroedinger equation \( i\hbar \partial \psi / \partial t = H \psi \) it is useful to expand \( \psi \) in the basis of the adiabatic eigenstates of \( H_{ad} \). In that description we arrive at the set of coupled, Schroedinger-like, equations for the multichannel amplitudes that are minimally coupled to a pure, non-Abelian, gauge potential

as well as the diagonal scalar potential matrix \( H_{BO} \) whose entries are labeled by \( \epsilon_i(R) \) and correspond to the
Born-Oppenheimer energies of $H_{ad}$. If one of the eigenvalues, e.g. $\epsilon_1(R) \gg E > \epsilon_2(R)$, where $E$ is the total (collision) energy, we can use the Born-Oppenheimer (BO) approximation and project the coupled equations unto, open, ground-state BO amplitude $F(R)$. We get,

\[
-\frac{\hbar^2}{2m} \left( \nabla - iA_P \right)^2 F(R) + \tilde{V}_{BO}(R)F(R) = EF(R).
\]

(2)

Here, $A_P$ is a vector potential and is obtained by projecting the non-Abelian gauge potential so that $A_P = PA_P$, $P \equiv |g\rangle\langle g|$, where $|g\rangle$ is the ground eigenstate of $H_{BO}$ with eigenvalue $\epsilon_2(R)$.

Though $A$ describes a pure non-Abelian gauge and so has vanishing curvature, $A_P$ may have a non-trivial curvature, which for a non-degenerate ground state discussed here has the Abelian form $H \equiv \nabla \times A_P$. $\tilde{V}_{BO}(R)$ is a scalar potential that is a sum of the BO eigenvalue, $\epsilon_2(R)$ and a correction term, $b(R)$, that is inversely proportional to the mass $m$ of the atom/system.

The classical limit for Eq. (2) corresponds to a situation in which the motion of the atom is governed by

\[
m\frac{d^2x}{dt^2} = v \times H - \nabla \tilde{V}_{BO}
\]

(3)

where $R(t)$ is the atom position coordinate.

In addition to the conventional scalar gradient force $-\nabla \tilde{V}_{BO}(R)$, (sometimes called the Helmann-Feynman force), the atom experiences an effective velocity dependent Lorentz force. We argued\[15, 16\] that such forces could be exploited to construct effective “magnetic lenses” for neutral atoms or neutrons. Here we underscore that observation by demonstrating that both the induced scalar and vector forces could be used in conjunction to develop novel capabilities for the manipulation and control of neutral particle beams. In particular, we describe a neutral particle velocity sieve that exploits both the velocity dependent force arising from geometric magnetism as well as the Hellman-Feynman force.

It is well known that for charged particles a velocity selector can be realized by exploiting a suitable geometry in which a uniform magnetic induction, $H$, induces a Lorentz force that cancels the gradient force produced by an electric field, $E$, so that for a particle of charge $q$ and velocity $v$, $qvH = qE$. Pursuing this analogy we choose a Hamiltonian of the form $H$ where the entries of the BO eigenenergies are given by,

\[
H_{BO} = \sigma_3 (\epsilon(R) + \Delta) - b(R)
\]

\[
\epsilon(R) = v_0 y \tilde{H}(x) \quad \tilde{H}(x) \equiv |H|
\]

(4)

where $\Delta$ is chosen to be a sufficiently large energy gap so that the BO projection approximation into the ground state is justified. With this choice, Eq. (3) becomes

\[
m\frac{d^2x}{dt^2} = \frac{dy}{dt} H(x(t)) + y(t)v_0 \partial_x H(x(t))
\]

\[
m\frac{d^2y}{dt^2} = -\frac{dx}{dt} H(x(t)) + v_0 H(x(t)).
\]

(5)

In the asymptotic region $x \to \infty$ all forces vanish and if we take the initial condition $\dot{y} = 0, \dot{x} = v_0$, $x(t) = v_0t + x_0, y(t) = 0$ is a solution to Eqs. (5). For other impact parameters and velocities numerical solution of Eq. (5) predict trajectories in which the incoming particle is scattered. This behavior is illustrated in Figure 1 where dashed lines represent classical trajectories superimposed on the BO potential surface (ignoring the energy gap $\Delta$). The black line represents the solution described above in which an atom with initial velocity $v_0$ propagates at constant velocity unimpeded. The blue and red lines correspond to initial velocities slightly larger and smaller, respectively, than $v_0$. Atoms with these velocities are clearly scattered by the effective Lorentz and gradient forces.

Our goal is to demonstrate that this behavior is shared by the quantum evolution of wave packets for the Hamiltonian given in Eq. (1). We achieve this by propagating quantum mechanical wave packets, in which the atom is initially in its ground state in the asymptotic region, using the time dependent method described in detail in Refs. [13] [16], and discussed briefly in the enclosed supplementary material.

In Figure 1 we plot the expectation values $\langle x(t) \rangle, \langle y(t) \rangle$ for these wave packets, and which are illustrated by the colored spheres. All packets have an initial starting position outside the interaction region with a null impact parameter for the wave packet centers. The black spheres represent propagation of packets with initial velocity $v_0$. The centers of the packets track closely the classical tra-
jectory. The blue and red spheres have initial velocities slightly greater and smaller, respectively, than $v_0$. The quantum simulation illustrated in Figure 1 clearly shows the proposed velocity selection effect for this system. Though Hamiltonian (1) is a simple, time independent, two-state (or qubit) system, it’s experimental realization pose challenges. Below we introduce another, more familiar, two-state system a neutral spin 1/2 system (eg. atom, neutron) subjected to a static external magnetic field. In the geometry discussed below we show how geometric phase induced, Aharonov-Bohm like, interference can be realized by it.

**Geometric phase, Aharonov-Bohm, interferometry**

![Image](image-url)

**FIG. 2:** (Color online) Time lapse illustration of a pair of coherent Gaussian wave packets initially in the ground state. At $\tau = 0$ the pair of coherent packets are shown in the upper right and left sides of the figure. At a later time, the packets coalesce thus creating the interference pattern.

Consider the external magnetic field

$$B = B(\rho) \hat{\phi} + B_0 \hat{k}$$

where $\phi, \rho$ are the polar and radial coordinates in a cylindrical coordinate system, and $B_0$ is a constant. If $B(\rho) = \lambda/\rho$ then Eq. (6) describes the field generated by a wire with current along the $z$-axis axis superimposed with that of a homogeneous magnetic induction $B_0 \hat{k}$. In this field the Hamiltonian of a neutral spin-1/2 atom is

$$H = -\frac{\hbar^2}{2m} \nabla^2 + \mu_B \mathbf{\sigma} \cdot \mathbf{B}$$

where $m$ is the mass of the atom and $\mu_B$ is the Bohr magneton. Here we focus on a special case, that in which $B(\rho)$ is a constant $B_\rho$. Such a system could be realized by replacing the current carrying wire with a ferromagnetic material whose magnetization stays constant as a function of $\rho$. We find that $H = 0$, and because $V_{BO}(\rho)$ is also constant, and ignoring $b(\rho)$ (which is very small in the region $\rho \neq 0$ traversed by the packets), the atom does not experience either a velocity dependent Lorentz, or scalar force. However, the induced vector potential $A_P$ is not trivial, indeed we note the similarity with the Aharonov-Bohm vector potential

$$A_P = \Phi \frac{\hat{\phi}}{2\pi \rho} = A_{AB}$$

where, in that theory, $\Phi$ represents the flux enclosed by an AB-flux tube. According to the discussion in the supplement

$$\Phi = \pi(1 - \cos \Omega) = \pi(1 - \frac{B_0}{\sqrt{B_0^2 + B_\rho^2}}).$$

In order to demonstrate the proposed thesis we propagate two identical, coherent wave packets as shown in Figure 2. The packets are displaced from the origin and are allowed to propagate, having been given initial velocities that allow them to coalesce, at time $\tau$, and form the interference pattern shown in that figure. In our simulation we have first set $B(\rho) = 0$ so that the packets propagate freely. In that case the horizontal line that passes through the center where the two packets meet, the wave function has the analytic form (see the supplement)

$$\frac{8a^2 k^2 \exp(-\frac{2ak^2 \eta^2}{4a^2k^2 + \eta^2})}{4a^4k^2 \pi + \pi \eta_0^2} \cos^2(k\eta)$$

where $\eta$ is the horizontal coordinate, $k$ is a scaled wavenumber, $\eta_0$ are the initial displacements of the packets from the origin and $a$ is the initial width of each packet. This function is plotted by the red line in Figure 3. In panel (a) we superimpose the values, shown by the blue circles, obtained in our numerical simulation. We find excellent agreement with the analytic result Eq. (10), and validates the numerical procedure used in this study. In panel (b) we plot the correspond interference pattern when $B_0, B_\rho$ has the value so that $\Omega = \pi/2$. We notice a distinct shift in the calculated interference pattern from that given by Eq. (10). However, a fit with the replacement in Eq. (10)

$$\cos^2(k\eta) \rightarrow \cos^2(k \eta + \Phi/4)$$

provides an excellent approximation to the calculated data given by the simulation.

We now consider the propagation of the two initial packets, at $t = 0$, having their vertical velocities reversed so that they propagate into the upper half plane and coalesce at a point that is the reflection of the coalescence point shown in Figure 2. The resulting interference pattern is illustrated in Figure 4. At the point $d$ the pattern is again well described by Eq. (10) but with the replacement

$$\cos^2(k\eta) \rightarrow \cos^2(k \eta - \Phi/4)$$

Because Hamiltonian (7) is not invariant under reflection about the $\eta$ (horizontal) axis it is not surprising that the interference pattern at point $d$ differs from that at
point b. It might not be as obvious that the difference is topological in nature. According to the discussion above the locations on the η axis where local minima occur is given by

\[ \eta_m = \frac{m\pi}{2k} \pm \frac{\Phi}{4k} \]  

(13)

where the ± sign identifies the points on the horizontal lines passing through the points b and d respectively and m is an integer. Therefore there is a displacement

\[ \Delta \eta_m = \frac{\Phi}{2k} \]  

(14)

shown by the right hand panel of Figure 4, in the location of the relative minima between the upper and lower fringe patterns. It depends on the quantity Φ, which according to AB theory is given by

\[ \int_C \, d\mathbf{r} \cdot \mathbf{A}_{AB} \]  

(15)

where C is a contour that encircles the path adcb in Figure 4 and \( \mathbf{A}_{AB} \) is given by Eq. [5]. The connection with AB theory, and the topological nature of the fringe shift, becomes evident when we shift the packet paths, so that a displaced closed circuit \( a'd'c'b'a' \) no longer includes the fictitious flux tube located at the origin. In that case our simulations show that the difference in the fringes at the corresponding locations of \( b', d' \) disappear, in harmony with the predictions of the gauge theory analysis.

Our discussion addressed AB interferometry for setups in which interference patterns are compared following two independent open-loop measurements\[24\]. We also analyzed the standard single closed circuit (see supplement) AB setup. In it our calculations demonstrate that, in the adiabatic limit, solutions generated by the time evolution operator Eq. [7] with the external field configuration Eq. [6] reproduces the standard AB fringe shift. In the limit \( \Delta/k \rightarrow 0 \), non-adiabatic transitions conspire to wash out topological AB fringes. This conclusion is consistent with that given in Ref. [17]. Nevertheless, we find here that if spin-state dependent measurements are made during traversal of the circuit, topological shifts persist. This counter-intuitive observation will be discussed in more detail elsewhere\[20\].

Dirac\[2\] noted that if a solution \( \psi \) to the Schroedinger equation is multiplied by a phase factor so that \( \psi(\mathbf{R}) \rightarrow \psi'(\mathbf{R}) = \exp(-i\alpha(\mathbf{R}))\psi(\mathbf{R}) \), and since we can always represent \( \alpha(\mathbf{R}) = \int_{\mathbf{r}}^{\mathbf{R}} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r}) \), the Schroedinger equation for \( \psi' \) results in the replacement \( \nabla\psi' \rightarrow \nabla\psi - i\mathbf{A}\psi \). However, conventional quantum mechanics demands that \( \psi(\mathbf{R}) \) be single-valued at all \( \mathbf{R} \), and this condition constrains the gauge potential \( \mathbf{A} \) to a trivial, pure, gauge which does not lead to new physics. Dirac suggested that minimal coupling of a charged particle with a non-trivial gauge field follows from a non-integrable factor \( \alpha(\mathbf{R}) \), a functional that depends on the path C. Though explorations in that direction have been attempted\[27\], difficulties in enforcing single-valuedness has limited the utility of this approach.

The examples provided above illustrate how an aspect of Dirac’s vision is realized. Let \( \psi \) be a multi-component amplitude and consider the non-Abelian version of Dirac’s substitution, i.e. \( \psi \rightarrow \psi' = U(\mathbf{R})\psi \), where \( U(\mathbf{R}) \) is a single-valued, differentiable unitary matrix operator. In [15] we argued that \( U \) can be represented by a non-Abelian, pure, gauge potential \( \mathbf{A} \). Because \( \mathbf{A} \) is a pure gauge, issues involving multi-valuedness in \( U \), and hence \( \psi' \) do not arise. Nevertheless, if the gauge symmetry is broken by energy gaps, (i.e. non-degeneracy of \( H_{BO} \)) non-trivial gauge fields that lead to gauge forces and/or topological holonomies can emerge in low energy solutions to the Schroedinger equation.
Geometric Phase Atom Optics and Interferometry: Supplementary Material

MODEL HAMILTONIANS

Case I: Velocity sieve

We require $U(R)$ to be single-valued and express

$$ U = \exp(-i\sigma_3\phi/2) \exp(-i\sigma_2\Omega/2) \exp(i\sigma_3\phi/2) \quad (S1) $$

where $\sigma_i$ are the Pauli matrices, and $\phi, \Omega$ are single-valued functions of the planar coordinates $(x, y)$.

We choose, as in Ref. [S1],

$$ \Omega(x, y) = \frac{\pi}{2} (1 + \tanh(\beta x)) 
\phi(x, y) = LB_0 y \quad (S2) $$

where $L, B_0, \beta$ are constants. With this parameterization we obtain, for the ground adiabatic state, an effective curvature

$$ H = \nabla \times \hbar A_P = 
\hat{k} \frac{\hbar}{4} \pi B_0 L \beta \text{sech}^2(\beta x) \cos(\frac{\pi}{2} \tanh(\beta x)). \quad (S3) $$

For the diagonal Hamiltonian $H_{BO}$ we posit

$$ H_{BO} = \sigma_3 (\epsilon(R) + \Delta) - b(R) 
\epsilon(R) = v_0 y H(x) \quad H(x) \equiv |H| \quad (S4) $$

where $\Delta$ is chosen to be a sufficiently large energy gap so that the BO projection approximation into the ground state is justified. We have also included the counter term $b(R)$ in order to cancel the higher order induced scalar potential that arises in the adiabatic picture.

The wave packet solutions are described by the amplitude,

$$ \psi = \begin{pmatrix} f \\ y \end{pmatrix} $$
whose components obey the coupled Schroedinger equations

\[
\begin{align*}
\imath \hbar \frac{\partial f}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 f + (V - b(\mathbf{R})) f + V_{12} g \\
\imath \hbar \frac{\partial g}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 g + V^*_{12} f - (V + b(\mathbf{R})) g
\end{align*}
\]  

(S5)

where

\[
\begin{align*}
V &= (\Delta + \epsilon(\mathbf{R})) \cos(\Omega(x)) \\
V_{12} &= \exp(-i\phi(y)) (\Delta + \epsilon(\mathbf{R})) \sin(\Omega(x)).
\end{align*}
\]  

(S6)

We construct a wave packet that describes the system in the remote past and, in which, it is found in its instantaneous ground adiabatic state. The packet is allowed to propagate according to Eq. (S5) and the expectation values \(\langle x(t) \rangle\), \(\langle y(t) \rangle\) are evaluated. Their centers are plotted by the solid spheres shown in Figure (1) in the text.

**Case II: Neutral spin 1/2 particle in a magnetic field**

We can re-express the internal Hamiltonian \(H_{ad} = \mu_B \mathbf{\sigma} \cdot \mathbf{B}\) as

\[
H_{ad} = \mu_B \begin{pmatrix} B_0 & -i \exp(-i\phi)B(\rho) \\ i \exp(i\phi)B(\rho) & -B_0 \end{pmatrix} = U H_{BO} U^\dagger
\]  

(S7)

where

\[
H_{BO} = \mu_B \begin{pmatrix} \sqrt{B_0^2 + B^2(\rho)} & 0 \\ 0 & -\sqrt{B_0^2 + B^2(\rho)} \end{pmatrix}.
\]  

(S8)

\(U\) is given by Eq. (S1) with \(\phi, \rho\), the azimuthal angle and radial distance, in a cylindrical coordinate system, respectively and \(\tan(\Omega) = \frac{B(\rho)}{B_0}\). Thus,

\[
A(R) \equiv i U^\dagger(R) \nabla U(R) = \hat{\phi} A_\phi + \hat{\rho} A_\rho,
\]

\[
A_\phi = \frac{1}{2\rho} \begin{pmatrix} \cos \Omega(\rho) & -i \exp(-i\phi) \sin \Omega(\rho) \\ -i \exp(i\phi) \sin \Omega(\rho) & \cos \Omega(\rho) \end{pmatrix}
\]

\[
A_\rho = -\frac{\Omega'(\rho)}{2} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}
\]  

(S9)

and so,

\[
A_P = PA_P = \frac{1 - \cos \Omega(\rho)}{2\rho}
\]

\[
\tilde{V}_{BO} = -\mu_B \sqrt{B_0^2 + B^2(\rho)^2} + b(\rho)
\]

\[
b(\rho) = \frac{\hbar^2}{} \left( \frac{\sin^2 \Omega(\rho)}{2\rho^2} + \frac{\Omega'(\rho)}{4} \right).
\]  

(S10)

The effective curvature for the ground adiabatic state is,

\[
H \equiv \nabla \times \hbar A_P = \frac{k}{2\rho} \sin(\Omega(\rho)) \Omega'(\rho).
\]  

(S11)

The above analysis is relevant in studies of the motion of cold atoms, that have a magnetic dipole moment, in the vicinity of current carrying wire (or nanotube). Here we focus on a special case, that in which \(B(\rho)\) is a constant \(B_\rho\). For this case, according to Eq. (S11), \(H = 0\), and because \(V_{BO}(\rho)\) is also constant, and ignoring \(b(\rho)\) (which is very
small in the region $\rho \neq 0$ traversed by the packets), the atom does not experience either a velocity dependent Lorentz, or scalar force. However, the induced vector potential $A_P$ is not trivial and is given by

$$A_P = \hat{\phi} \frac{\Phi}{2\pi \rho}, \quad (S12)$$

where

$$\Phi = \pi (1 - \cos \Omega(\rho)) = \pi \left(1 - \frac{B_0}{\sqrt{B_0^2 + B_\rho^2}}\right). \quad (S13)$$

Packet propagation is again described as in Eq. $(S5)$ but now

$$V = \mu_B B_0$$
$$V_{12} = -i \mu_B \exp(-i\phi) B(\rho). \quad (S14)$$

**PACKET DYNAMICS**

Free particle propagation

At lower collision energies, in which the excited Zeeman level is closed, we consider the propagation of a coherent wave packet that is initially localized, as shown in Figure (2) in the text. Using the dimensionless coordinates defined in $[S1]$, $\xi = x/L, \eta = y/L, \tau = \frac{\hbar}{2mL^2} t$, we take the free particle wave packet (normalized to the value 2)

$$\psi(\xi, \eta, \tau) = \psi_1(\xi, \eta, \tau) + \psi_2(\xi, \eta, \tau)$$

$$\psi_i = \int dk_1 dk_2 \Phi(k_1 - k_{x_i}) \Phi(k_2 - k_{y_i}) \times \exp(-i\tau(k_1^2 + k_2^2)) \exp(ik_1(\xi - \xi_i)) \exp(ik_2(\eta - \eta_i)). \quad (S15)$$

With the choice

$$\Phi(k) = \frac{\sqrt{a}}{(2\pi^3)^{\frac{3}{4}}} \exp(-a^2 k^2) \quad (S16)$$

$\psi_i(\xi, \eta, \tau)$ is a solution to the equation

$$i \frac{\partial \psi_i}{\partial \tau} = -\frac{\partial^2 \psi_i}{\partial \xi^2} - \frac{\partial^2 \psi_i}{\partial \eta^2} + \tilde{\Delta} \psi_i \quad (S17)$$

and describes, at $\tau = 0$, a Gaussian wavepacket of width $a$ localized at $(\xi_i, \eta_i)$ with momenta $k_{x_i}, k_{y_i}$. $\tilde{\Delta}$ is the energy defect $\Delta$ in units of $\frac{2mL^2}{\hbar^2}$. By linearity, the coherent sum $\psi(\xi, \eta, \tau)$ is also a possible packet. Here we chose $\xi_1 = \xi_2 = 0, \eta_1 = -\eta_0, \eta_2 = \eta_0$, and $k_{x_1} = k_{x_2} = k, k_{y_1} = -k_{y_2} = k$. At time $\tau_c = \eta_0/2k$ the two packets coalesce and form the interference pattern illustrated in Figure (2) of the text. The center of the merged packets at $\tau_c$ passes through the line $\xi = \eta_0$. On it

$$\psi(\eta_0, \eta, \tau_c) = \int dk_1 dk_2 \Phi(k_1 - k) \Phi(k_2 - k) \times \exp(-i\tau_c(k_1^2 + k_2^2)) \exp(ik_1\eta_0) \left\{ \exp(ik_2(\eta + \eta_0)) + \exp(-ik_2(\eta - \eta_0)) \right\}. \quad (S18)$$

Integration of Eq. $(S18)$ yields

$$|\psi(\eta_0, \eta, \tau_c)|^2 = \frac{8a^2 k^2 \exp\left(-\frac{2a^2 k^2 \eta^2}{4a^4 k^2 + \pi \eta_0^2}\right)}{4a^4 k^2 + \pi \eta_0^2} \cos^2(\eta). \quad (S19)$$
Semi-classical description of packet propagation in an Abelian gauge potential

Consider the wave packet $\psi_1(\xi, \eta, \tau)$ defined by Eq. (S15) and whose center, at $\tau = 0$, is located at point $a$ in Figure (4) in the text. We need to predict the packet that grows out of it and whose evolution is determined by the coupled Schrödinger Eq. (S5). Our calculation show that, under the adiabatic condition $\Delta/k^2 \gg 1$, $\Delta \equiv \mu_B \sqrt{B_0^2 + B_2^2}$, and in the adiabatic gauge, a good approximation for it at the time its center arrives at $b$ is

$$U(a, b)\psi_1(b)$$  \hspace{1cm} (S20)

where $\psi_1$ is the free particle packet and the unitary operator $U(a, b)$ is given by

$$U(a, b) \approx \exp(i \int_{C_1} dr \cdot A_P).$$  \hspace{1cm} (S21)

Here $C_1$ represent a path integral along segment $a - b$ that starts at $a$ and ends at $b$ and $A_P$ is the gauge potential given by expression $\text{[S12]}$.

Similarly, a packet initially centered at $c$ at $\tau = 0$, translates along path $c - b$ and arrives at $b$ at $\tau_c$. It can be expressed

$$U(c, b)\psi_2(b)$$  \hspace{1cm} (S22)

where $\psi_2$ is defined in Eq. (S15). The coherent sum of these amplitudes at $\tau = \tau_c$ is then given by the expression

$$\psi(b) = U(a, b)\psi_1(b) + U(c, b)\psi_2(b) =$$

$$U(c, b)(U^{-1}(c, b)U(a, b)\psi_1(b) + \psi_2(b)) =$$

$$U(c, b)(U(b, c)U(a, b)\psi_1(b) + \psi_2(b)) =$$

$$U(c, b)(U(a, c)\psi_1 + \psi_2(b))$$  \hspace{1cm} (S23)

where we made use of the unitary property of $U$ and the relation $U(a, c) = U(b, c)U(a, b)$. Therefore

$$|\psi(b)|^2 = |\psi_2(b) + U(a, c)\psi_1(b)|^2$$  \hspace{1cm} (S24)

Evaluating

$$U(a, c) = \exp(i \int_{abc} dr \cdot A_P) = \exp(-i \frac{\Phi}{2})$$  \hspace{1cm} (S25)

and inserting this into Eq. (S18) we obtain the expression given in the main text.

We obtain an analogous relation for the case where the momenta, along the $\xi$ direction, of the initial wave packets at $a, c$ are reversed so that at time $\tau_c$ the packets meet at point $d$ in Figure 4. Following the steps outlined above we find

$$|\psi'(d)|^2 = |\psi_2'(d) + U'(a, c)\psi_1'(d)|^2$$  \hspace{1cm} (S26)

where $\psi_i'$ are the corresponding free-particle packets whose $\xi$ momenta are reversed, and

$$U'(a, c) = \exp(i \int_{abc} dr \cdot A_P) = U_W U(a, c)$$  \hspace{1cm} (S27)

where $U_W$ is a Wilson loop integral

$$U_W \equiv \exp(i \oint d\mathbf{r} \cdot A_P) = \exp(i\Phi)$$  \hspace{1cm} (S28)

and the closed, counterclockwise, circuit encloses the origin.

According to Eq. (S18) $\psi_i'(\xi, \eta, \tau) = \psi_i(-\xi, \eta, \tau)$ and therefore,

$$|\psi'(d)|^2 = |\psi_2(b) + U_W U(a, c)\psi_1(b)|^2.$$  \hspace{1cm} (S29)

So the interference pattern $|\psi'(d)|^2$ at the top panel in Figure (4) in the text, differs from that at the lower panel by a, gauge invariant, phase determined by the Wilson loop $U_W$, in harmony with the results obtained by the, fully quantal, numerical simulation.
FIG. S1: (Color online) Interference fringes for energy defect $\Delta/k^2 = 0.1$ and $\Omega = 2\pi/3$. Panels (a),(b) are fringes (blue points) at locations b,d (bottom,top) in Figure (4) of text. Panels (c),(d) show fringes for (excited) state selected measurements at the latter locations. The green vertical line is a reference line to aid the eye in comparing fringes between the top an bottom panels. The dashed red lines correspond to fringes predicted by the Abelian semiclassical theory.

FIG. S2: (Color online) Circuit $C$ which encloses fictitious flux tube (red disk), is translated to new circuit $C'$ that does not enclose it.

Semi-classical description of packet propagation in a non-Abelian gauge potential

If the collision energy, $\frac{\hbar^2 k^2}{2m}$, is much larger than the energy defect $2\Delta$, between the Zeeman split spin states, non-adiabatic transitions between those states can occur. Thus if the initial localized wavepacket, say at point $a$ in Figure (4) of the text, describes a particle in the ground Zeeman level, it will not necessarily stay in that level as the packet evolves in time.

In the discussion above we considered the adiabatic limit in which the ratio of energy defect to collision energy is large i.e. $\Delta/k^2 >> 1$. In that limit spin flipping transitions between ground and excited Zeeman levels are suppressed. We showed that, in this limit, the single channel (or Abelian) Schroedinger equation with gauge potential Eq. (S12) and $V_{BO} = 0$ accurately predicts wavepacket dynamics and topological AB features. As the collision energy is cranked up so that $2\Delta/k^2 \leq 1$ we anticipate that the single channel (Abelian) description breaks down and non-Abelian features arise.

In Figure (S1), panels (a), (b) we plot the interference patterns, corresponding to the top and bottom regions shown in Figure (4) in the text for the collision energy corresponding to $\Delta/k^2 = 0.1$. Though the collision energy is sufficient to cause Zeeman level transitions, our results suggest that many of the Abelian features persist. First we note that...
there is a phase shift between the interference patterns, for the top and bottom regions respectively, that is nearly, but not exactly, predicted by the Abelian AB theory (which are shown in red in that figure). Panels (c),(d) show the fringes, for the top and bottom regions respectively for state-dependent probabilities, in this case for excitation into the upper Zeeman level. A distinct phase shift in the fringe patterns for excitation is also seen, though its structure is not predicted by the Abelian theory.

In order to investigate whether these features are topological we translate the loop $C$, shown in Figure (S2), into the loop $C'$ and repeat the calculations described above. Because loop $C'$ does not enclose the fictitious flux tube (shown by the red disk) classical AB theory suggests that the difference in fringe shifts, evident in Figure (S1), is null. Indeed, this is the case. In panels (a),(b), of Figure (S3), the probability interference are shown. The patterns for top and bottom regions (points d,b respectively) line up and no fringe shifts are evident. Panels (b),(d) of that figure show the corresponding interference patterns for the excited Zeeman level probabilities. Though fringe differences are negligible, we note a strong suppression of the latter (when compared to that for loop $C$ shown in panels (b),(d) in Figure (S1)). The suppression of excitation for loop $C'$ is clearly a non-Abelian (or multichannel) feature.

Finally, we consider the extreme non-adiabatic regime. In it the ratio $\Delta/k^2 \rightarrow 0$ and we can again employ semiclassical methods to predict the fringe patterns that are generated by our fully quantal simulations and which are shown below. In Figure (S3) we repeat the calculations for propagation along loop $C$ in Figure (S2) for the values $\Delta/k^2 = 0.007$ and $\Omega = 2\pi/3$. Unlike the case for the adiabatic and near adiabatic regimes, in which topological fringe shifts arise, panels (a), (b) of Figure (S4) clearly demonstrate absence of the topological fringe shift. This behavior can be explained using semiclassical methods. For, in that description the total probability amplitude at point (b) in Figure (4) of the text, that grows out of wave packet $\psi_1(a)$ is approximated by the expression

$$P \exp(i \int_{ab} dr \cdot A) \psi_1(b)$$  \hspace{1cm} (S30)

where $P$ is a path-ordered integral, or Wilson line, along segment $a - b$, and $A$ is the non-Abelian, pure, gauge potential. Repeating the argument outlined above for the fringe shift in the adiabatic limit, we now find that the shift depends on the Wilson loop integral

$$U_W \equiv P \exp(-i \int dr \cdot A)$$  \hspace{1cm} (S31)

where $A$ is given by Eq. (S8). Because $A$ describes a pure gauge we find that $U_W = 1$ and so, unlike the case in the adiabatic regime, a fringe shift between the interference patterns at $b,d$ in Figure (4) of the text does not manifest. Surprisingly, this is no longer true if we perform state dependent measurements at locations $b$, $d$ in that figure. In

**FIG. S3:** (Color online) Interference fringes for energy defect $\Delta/k^2 = 0.1$ and $\Omega = 2\pi/3$ for loop diagram $C'$ in Figure [S2]. Panels (a),(b) show fringes (blue points) at locations $b,d$ (bottom,top) in Figure (S2) respectively. Panels (c),(d) show fringes for (excited) state selected measurements at points $b,d$ (bottom,top) in Figure (S2) respectively. The green vertical line is a reference line to aid the eye in comparing fringes between the top and bottom panels.
FIG. S4: (Color online) Interference fringes for energy defect $\Delta/k^2 = 0.007$ and $\Omega = 2\pi/3$. Panels (a,b) fringes (blue points) at locations c,a (bottom,top) in Figure (4) of the text respectively. Panels (c,d) fringes for (ground) state selected measurements at points c,a (bottom,top) in Figure (4) of the text respectively. The green vertical line is a reference line to aid the eye in comparing fringes between the top and bottom panels. The dashed red lines correspond to fringes predicted by the semiclassical Abelian theory.

panels (c), (d) of Figure (4) in the text we plot the calculated interference patterns at those points for a state selective measurement (in this case, the ground state). Those fringe shifts are, again, accurately predicted by the Abelian theory as illustrated by the red lines in those panels. Furthermore, this shift is also topological, in that the shifts vanish for loops that do not enclose the fictitious flux tube. A detailed discussion of the origin and implications of this observation will be presented elsewhere [S2].

FIG. S5: (Color online) Standard, single loop, AB interferometry setup. Two coherent packets at origin $a$ split and propagate to the mirrors at points $b$, $d$. The mirrors deflect the packets so they recombine at point $c$ where the measurements are made.
AB interferometry for a single loop

Our discussions addressed AB interferometry for setups in which interference patterns are compared following two independent open-loop measurements. In the classical single loop AB setup, an interference pattern is observed at a single screen (point $c$) as shown in Figure (S5). In it, a wavepacket coherently splits at the origin, point $a$, and propagates toward the mirrors at points $b, d$ respectively, the packets are deflected and allowed to recombine at point $c$ where a measurement is taken. In the calculations described above, and in the adiabatic limit $\Delta/k^2 \gg 1$, we found that (i) the wave packets propagate as a free particle, (ii) in the journey along curve $C$ the packets acquire, in addition to the standard dynamical the phase factor, the phase

$$\exp(i \int_C dr \cdot A_P).$$

(S32)

Using Eq. (S32) for the paths $a - b - c$ and $a - d - c$ we find that measurements at the screen will be a function of the path integral, along the closed loop,

$$\exp(i \oint dr \cdot A_P).$$

(S33)

If that loop encloses the fictitious flux tube then its value is $\exp(i\Phi)$, otherwise it has unit value. Therefore this setup exhibits topological properties consistent with AB theory. In the extreme non-adiabatic limit $\Delta/k^2 \ll 1$ our calculations again demonstrate the validity of properties (i), (ii), with the exception that “free particle” evolution is that of a two-component wavefunction and the phase factor multiplying it is a multi-channel unitary matrix Eq. (S30). Therefore, repeating the analysis given above we find measurements at screen $c$ are now proportional to the loop integral

$$\exp(i \oint dr \cdot A) = 1.$$

(S34)

That is, regardless of the loop topological fringe patterns do not arise, and there is no topological AB shift.

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[1] B. Zygelman, Phys Rev A 86, 042704 (2012).
[2] B. Zygelman, in preparation (2015).