THE PLANE WAVE EXPANSION, INFINITE INTEGRALS AND IDENTITIES INVOLVING SPHERICAL BESSEL FUNCTIONS

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ABSTRACT

This paper shows that the plane wave expansion can be a useful tool in obtaining analytical solutions to infinite integrals over spherical Bessel functions and the derivation of identities for these functions. The integrals are often used in nuclear scattering calculations, where an analytical result can provide an insight into the reaction mechanism. A technique is developed whereby an integral over several special functions which cannot be found in any standard integral table can be broken down into integrals that have existing analytical solutions.

Keywords

Plane Wave Expansion-Infinite Integrals Over Spherical Bessel Functions-Spherical Bessel Function Identities-Special Functions-Nuclear Reactions Theory

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1. Introduction

The plane wave expansion, or the Rayleigh equation, is commonly used in nuclear physics; in particular it can be found in any standard text on scattering theory [1]. In the plane wave approximation, a particle incident on a nucleus has a wavefunction approximated by a plane wave of the form $e^{i\vec{k} \cdot \vec{r}}$, where $\vec{k}$ is the particle’s wavenumber and $\vec{r}$ is its position. This approximation is often used to get an insight into the reaction mechanism of the process studied, without the complications arising from using the more accurate distorted waves.

Using the plane wave approximation will result in radial integrals that involve spherical Bessel functions. On the other hand, there are many other applications in nuclear physics that encounter these types of integrals amongst which are distorted wave calculations [2] and nuclear response function calculations [3]. These types of integrals have in the past been calculated analytically (many references exist, see for example references [2] and [4-11]), or numerically using complex-plane methods (see for example references [12-14]). In this paper, an additional advantage to the plane wave expansion is presented, whereby the expansion itself is used to derive analytical solutions to integrals over spherical Bessel functions and identities involving them. The technique itself is not new [11], but the author is not aware of any prior paper that has highlighted its versatility. In section 2, the plane wave expansion is introduced from which the integral representation of the spherical Bessel function is derived. Section 3 builds on that by deriving integrals involving one spherical Bessel function. Section 4 details integrals involving two spherical Bessel functions, amongst which is the Closure Relation for spherical Bessel functions. Section 5 deals with integrals involving three spherical Bessel functions. Section 6 extends the previous results to a larger number of any special or elementary function. Here the Closure relation is used to simplify an integral that cannot be found in any standard integral tables [15]. This can be done,
for example, by breaking an integral over 4 special functions into an integral over two integrals of 3 special functions each. Concluding remarks then follow.

2. The Plane Wave Expansion

The plane wave expansion, also known as the Rayleigh equation, is given by

\[ e^{i \vec{k} \cdot \vec{r}} = 4\pi \sum_{L=0}^{\infty} \sum_{m=-L}^{L} i^L Y_L^M(\hat{r}) Y_L^{M*}(\hat{k}) j_L(kr) \]

(2.1)

where \( Y_L^M(\hat{r}) \) is the spherical harmonic function for the unit vector \( \hat{r} \), and \( j_L(kr) \) is the spherical Bessel function for \( k \geq 0 \), which is assumed for the rest of the paper along with \( k_n \geq 0 \), for integer \( n \). An alternative formula is

\[ e^{i \vec{k} \cdot \vec{r}} = \sum_{L=0}^{\infty} (i)^L (2L + 1) P_L(\cos \theta) j_L(kr), \]

(2.2)

using the Vector Addition Theorem given by

\[ P_L(\cos \theta) = \frac{4\pi}{2L + 1} \sum_{M=-L}^{M=L} Y_L^M(\hat{r}) Y_L^{M*}(\hat{k}), \]

(2.3)

where \( P_L(\cos \theta) \) is the Legendre polynomial and \( \theta \) is the angle between \( \vec{k} \) and \( \vec{r} \).

To limit the sum over \( L \) in eq. (2.1) or eq. (2.2) to a specific value, one needs to use the orthogonality of the spherical harmonics

\[ \int Y_L^M(\hat{k}) Y_{L'}^{M*}(\hat{k}) d\Omega_{\hat{k}} = \delta_{L,L'} \delta_{M,M'}, \]

(2.4)

where \( \Omega_{\hat{k}} \) is the solid angle subtended by \( \hat{k} \). Alternatively, we can use the orthogonality
of the Legendre polynomials

\[ \int_{-1}^{1} P_L(\cos \theta) P_{L'}(\cos \theta) \, d(\cos \theta) = \frac{2}{2L + 1} \delta_{L,L'} \]  

(2.5)

to arrive at the integral representation of the spherical Bessel function

\[ j_L(kr) = \frac{(-i)^L}{2} \int_{-1}^{1} P_L(\cos \theta) e^{ikr \cos \theta} \, d(\cos \theta). \]  

(2.6)

Setting \( k = 0 \), one arrives at the identity

\[ j_L(0) = \delta_{L,0} \]  

(2.7)

3. Infinite Integrals Over One Spherical Bessel Function

Integrating eq. (2.6) over \( r \) and interchanging the order of integration we obtain

\[ \int_{-\infty}^{\infty} j_L(kr) \, dr = \frac{(-i)^L}{2} \int_{-1}^{1} P_L(\cos \theta) \int_{-\infty}^{\infty} e^{ikr \cos \theta} \, dr \, d(\cos \theta). \]  

(3.1)

However,

\[ \int_{-\infty}^{\infty} e^{ikr} \, dr = 2\pi \delta(k), \]  

(3.2)

resulting in

\[ \int_{-\infty}^{\infty} j_L(kr) \, dr = \frac{(-i)^L}{k} \pi P_L(0). \]  

(3.3)

Another technique is to integrate the plane wave, eq. (2.1) over all space and using

\[ \int d^3 r \, e^{i\vec{k} \cdot \vec{r}} = (2\pi)^3 \delta^3(\vec{k}), \]  

(3.4)
one reaches

\[ \int_0^\infty r^2 j_0(kr) \, dr = 2\pi^2 \delta^3(\vec{k}), \]  

(3.5)

which shows the spherical symmetry of \( \delta^3(\vec{k}) \), since there is no angular dependence on the left hand side. Also, since

\[ \int \frac{e^{i\vec{k} \cdot \vec{r}}}{r^2} \, d^3r = \frac{2\pi^2}{k}, \]  

(3.6)

we can reach, using eqs. (2.1) and (2.4),

\[ \int_0^\infty j_0(kr) \, dr = \frac{\pi}{2k}. \]  

(3.7)

4. Infinite Integrals Over Two Spherical Bessel Functions

Again the plane wave expansion gives

\[ e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}} = (4\pi)^2 \sum_{L_1,M_1} \sum_{L_2,M_2} (i)^{L_1+L_2} Y_{L_1}^{M_1}(\hat{k}_1) Y_{L_2}^{M_2}(\hat{k}_2) \]
\[ \times Y_{L_1}^{M_1*}(\hat{r}) Y_{L_2}^{M_2*}(\hat{r}) j_{L_1}(k_1r) j_{L_2}(k_2r). \]  

(4.1)

Integrating over all space results in fixing \( L_2 = L_1 \equiv L \) and \( M_2 = M_1 \equiv M \)

\[ (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2) = (4\pi)^2 \sum_{L,M} (-1)^L Y_L^M(\hat{k}_1) Y_L^{M*}(\hat{k}_2) \]
\[ \times \int_0^\infty r^2 j_L(k_1r) j_L(k_2r) \, dr. \]  

(4.2)

Using

\[ \delta^3(\vec{k}_1 + \vec{k}_2) = \frac{\delta(k_1 - k_2)}{k_1^2} \delta(\Omega - \vec{k}_1 - \vec{k}_2), \]  

(4.3)

multiplying both sides of eq. (4.2) by \( Y_L^{M*}(\hat{k}_1) Y_L^M(\hat{k}_2) \), followed by integrating over
solid angles $\hat{k}_1$ and $\hat{k}_2$ and using

$$Y_L^M(\hat{k}) = (-1)^L Y_L^M(\hat{k}), \quad (4.4)$$

results in

$$\int_0^\infty r^2 j_L(k_1 r) j_L(k_2 r) \, dr = \frac{\pi}{2 k_1^2} \delta(k_1 - k_2), \quad (4.5)$$

which is known as the *Closure Relation* for spherical Bessel Functions.

Another relation that can be derived using eqs. (2.6) and (3.2) is

$$\int_{-\infty}^\infty j_L(x) j_{L'}(x) \, dx = \frac{(-i)^{L+L'}}{2} \pi \int_{-1}^1 P_L(y) P_{L'}(-y) \, dy. \quad (4.6)$$

Now

$$P_L(-y) = (-1)^L P_L(y), \quad (4.7)$$

just as

$$j_L(-x) = (-1)^L j_L(x), \quad (4.8)$$

which reduces eq. (4.6) to

$$\int_{-\infty}^\infty j_L(x) j_{L'}(x) \, dx = \frac{\pi}{2L + 1} \delta_{L,L'}, \quad (4.9)$$

known as the *Orthogonality Relation* for the spherical Bessel functions. Using this equation together with the plane wave expansion, eq. (2.2) leads to

$$P_L(x) = \frac{(-i)^L}{\pi} \int_{-\infty}^\infty e^{ixy} j_L(y) \, dy, \quad (4.10)$$

which is an integral representation for the Legendre polynomial valid for $-1 < x < 1$. 

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Also, dividing eq. (4.1) by $r^2$ and integrating over all space results in

$$\frac{2\pi^2}{|\vec{k}_1 + \vec{k}_2|} = (4\pi)^2 \sum_{L,M} (-1)^L Y_L^M(\hat{k}_1) Y_{L^*}^M(\hat{k}_2)$$

\[ \times \int_0^\infty j_L(k_1r) j_L(k_2r) \, dr. \tag{4.11} \]

Now

$$\frac{1}{|\vec{k}_1 + \vec{k}_2|} = 4\pi \sum_{L,M} \frac{(-1)^L}{2L+1} \frac{(k_<)^L}{(k_>)^{L+1}} Y_L^M(\hat{k}_1) Y_{L^*}^M(\hat{k}_2), \tag{4.12}$$

where $k_<$ and $k_>$ are the smaller and larger of $k_1$ and $k_2$, respectively. So substituting eq. (4.12) into eq. (4.11), multiplying by $Y_{L^*}^M(\hat{k}_1) Y_L^M(\hat{k}_2)$ and integrating over solid angles $\hat{k}_1$ and $\hat{k}_2$ results in

$$\int_0^\infty j_L(k_1r) j_L(k_2r) \, dr = \frac{\pi}{2(2L+1)} \frac{(k_<)^L}{(k_>)^{L+1}}, \tag{4.13}$$

which agrees with reference [4], equation 1, page 405.

An identity can also be derived by equating the expansion in eq. (4.1) to the alternative expansion in terms of $j_L(|\vec{k}_1 + \vec{k}_2|r)$ resulting in

$$\langle i \rangle^L Y_L^M(\overrightarrow{k_1} + \overrightarrow{k_2}) j_L(\overrightarrow{k_1} + \overrightarrow{k_2}|r) = \sqrt{4\pi} \sum_{L_1,M_1} \sum_{L_2,M_2} \langle i \rangle^{L_1+L_2} \sqrt{\frac{(2L_1+1)(2L_2+1)}{(2L+1)}}$$

\[ \times < L_1 L_2 0 0 | L 0 > < L_1 L_2 M_1 - M_2 | L M > Y_{L_1}^{M_1}(\hat{k}_1) Y_{L_2}^{M_2}(\hat{k}_2) \]

\[ \times j_{L_1}(k_1r) j_{L_2}(k_2r), \tag{4.14} \]

using

$$\int d\Omega_\hat{r} \, Y_{l_1}^{m_1}(\hat{r}) Y_{l_2}^{m_2}(\hat{r}) Y_{l}^{m*}(\hat{r}) = \frac{(2l_1+1)(2l_2+1)}{4\pi (2l+1)}$$

\[ \times < l_1 l_2 0 0 | l 0 > < l_1 l_2 m_1 m_2 | l m >, \tag{4.15} \]
where \( < l_1 l_2 m_1 m_2 | l m > \) is a Clebsch-Gordan coefficient, coupling angular momenta \( l_1 \) and \( l_2 \) to angular momentum \( l \) such that \( m_1 + m_2 = m \).

If \( \vec{k}_1 \) and \( \vec{k}_2 \) point in the same direction, i.e \( \hat{k}_1 = \hat{k}_2 \equiv \hat{k} \), then multiplying eq. (4.14) by \( Y^M_L(\hat{k}) \), integrating over solid angle \( \hat{k} \) then summing over \( M_1 \) and \( M_2 \) using the unitarity of the Clebsch-Gordan coefficients

\[
\sum_{M_1, M_2} < L_1 L_2 M_1 M_2 | L M > < L_1 L_2 M_1 M_2 | L' M' > = \delta_{L, L'} \delta_{M, M'},
\]

(4.16)
equation (4.14) reduces to the identity

\[
j_L[(k_1 + k_2)r] = \sum_{L_1, L_2} (i)^{L_1 + L_2 - L} \frac{(2L_1 + 1)(2L_2 + 1)}{(2L + 1)} < L_1 L_2 0 0 | L 0 >^2 \times j_{L_1}(k_1r) j_{L_2}(k_2r).
\]

(4.17)

Similarly,

\[
j_L[(k_1 - k_2)r] = \sum_{L_1, L_2} (i)^{L_1 - L_2 - L} \frac{(2L_1 + 1)(2L_2 + 1)}{(2L + 1)} < L_1 L_2 0 0 | L 0 >^2 \times j_{L_1}(k_1r) j_{L_2}(k_2r).
\]

(4.18)

Redefining \( x \equiv k_1r \) and \( y \equiv k_2r \), multiplying eq. (4.18) by \( j_l(x) \) and integrating over \( x \) would result in an equation of the form

\[
\int_{-\infty}^{\infty} j_L(x - y) j_l(x) dx = \pi \sum_{L_2} (i)^{L_1 - L_2 - L} < L_1 0 0 | L_2 0 >^2 j_{L_2}(y),
\]

(4.19)

using eq. (4.9) and the symmetry properties of the Clebsch-Gordan coefficients.
5. Integrals Over Three Spherical Bessel Functions

Using the same procedure as before, integrating over the expansion of $e^{i(k_1 + k_2 + k_3) \cdot \hat{r}}$ and using eq. (4.15) amounts to

\[
\delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) = \frac{4}{\sqrt{\pi}} \sum_{L_1, M_1} \sum_{L_2, M_2} \sum_{L_3, M_3} (i)^{L_1 + L_2 + L_3} \times \sqrt{(2L_1 + 1)(2L_2 + 1)} \frac{(2L_3 + 1)}{2L_3 + 1} \times \int_0^\infty r^2 j_{L_1}(k_1 r) j_{L_2}(k_2 r) j_{L_3}(k_3 r) \, dr,
\]

where

\[
K(L_1, L_2, L_3, \hat{k}_1, \hat{k}_2, \hat{k}_3) \equiv \sum_{M_1} \sum_{M_2} \sum_{M_3} < L_1 L_2 M_1 M_2 | L_3 M_3 > \times Y^{M_1 *}_{L_1}(\hat{k}_1) Y^{M_2 *}_{L_2}(\hat{k}_2) Y^{M_3}_{L_3}(\hat{k}_3).
\]

Multiplying both sides of eq. (5.1) by $K^*(L'_1, L'_2, L'_3, \hat{k}_1, \hat{k}_2, \hat{k}_3)$ and using the orthogonality of the spherical harmonics

\[
\int d\Omega_{\hat{k}_1} d\Omega_{\hat{k}_2} d\Omega_{\hat{k}_3} K^*(L'_1, L'_2, L'_3, \hat{k}_1, \hat{k}_2, \hat{k}_3) K(L_1, L_2, L_3, \hat{k}_1, \hat{k}_2, \hat{k}_3) = (2L_3 + 1) \delta_{L'_1, L_1} \delta_{L'_2, L_2} \delta_{L'_3, L_3},
\]

results in

\[
< L_1 L_2 0 0 | L 0 > = \frac{\sqrt{\pi}}{4} \frac{(-i)^{L_1 + L_2 + L_3}}{\sqrt{(2L_1 + 1)(2L_2 + 2)(2L_3 + 1)}} \times \int d\Omega_{\hat{k}_1} d\Omega_{\hat{k}_2} d\Omega_{\hat{k}_3} K^*(L_1, L_2, L_3, \hat{k}_1, \hat{k}_2, \hat{k}_3) \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3).
\]

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Now
\[
\delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) = \delta(k_3 - |\vec{k}_1 + \vec{k}_2|) \frac{\delta(\Omega_{\hat{k}_3} - \Omega_{\hat{k}_1 + \hat{k}_2})}{k_3^2}.
\] (5.5)

Using
\[
\delta(k_3 - |\vec{k}_1 + \vec{k}_2|) = \frac{k_3}{k_1 k_2} \delta(\cos \theta - \frac{k_1^2 + k_2^2 - k_3^2}{2k_1 k_2}),
\] (5.6)

where \(\theta\) is the angle between \(\hat{k}_1\) and \(\hat{k}_2\), and the completeness relation for the Legendre polynomials
\[
\delta(x - y) = \sum_L \frac{2L+1}{2} P_L(x) P_L(y),
\] (5.7)

gives
\[
\delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) = \frac{\beta(\Delta)}{2k_1 k_2 k_3} \delta(\Omega_{\hat{k}_3} - \Omega_{\hat{k}_1 + \hat{k}_2}) \sum_L (2L+1) P_L(\Delta) P_L(\cos \theta),
\] (5.8)

where
\[
\Delta = \frac{k_1^2 + k_2^2 - k_3^2}{2k_1 k_2},
\] (5.9)

and the factor \(\beta(\Delta)\) is given by [11]
\[
\beta(\Delta) = \frac{1}{2}, \quad \Delta = \pm 1
\]
\[
= 1, \quad -1 < \Delta < 1
\]
\[
= 0, \quad \text{otherwise}.
\] (5.10)

Equation (5.4) then becomes
\[
< L_1 L_2 0 0 | L 0 > \int_0^\infty r^2 j_{L_1}(k_1 r) j_{L_2}(k_2 r) j_{L_3}(k_3 r) \, dr
= \sqrt{\pi} \beta(\Delta) \frac{(-i)^{L_1+L_2+L_3}}{8k_1 k_2 k_3} \frac{(-1)^{L_1+L_2+L_3}}{\sqrt{(2L_1 + 1)(2L_2 + 2)(2L_3 + 1)}}
\times \int d\Omega_{\hat{k}_1} d\Omega_{\hat{k}_2} dK^*(L_1, L_2, L_3, \hat{k}_1, \hat{k}_2, -(\vec{k}_1 + \vec{k}_2)) \sum_L (2L+1) P_L(\Delta) P_L(\cos(\theta)).
\] (5.11)
The integral in eq. (5.11) can be easily evaluated using eq. (2.3), eq. (4.4) and the Solid-Harmonic Addition Theorem for spherical harmonics [16]

\[ Y_{L_3}^{M_3}(\mathbf{k}_1 + \mathbf{k}_2) = \left( \frac{k_1}{k_3} \right)^{L_3} \sum_{L=0}^{L_3} \sqrt{\frac{4\pi}{2L+1}} \left( \frac{2L_3+1}{2L} \right)^{1/2} \left( \frac{k_2}{k_1} \right)^L \sum_M < (L_3 - L) L (M_3 - M) M | L_3 M_3 > Y_{L_3-L}^{M-M}(\mathbf{k}_1) Y_L^M(\mathbf{k}_2), \]

where the binomial coefficient is defined by

\[ \binom{n}{m} \equiv \frac{n!}{(n-m)!m!}. \]

The result of the integration is then the formula for the integral over three spherical Bessel functions

\[ < L_1 L_2 0 0|L_3 0 > = \int_0^\infty r^2 j_{L_1}(k_1 r) j_{L_2}(k_2 r) j_{L_3}(k_3 r) \, dr \]

\[ = \frac{\pi \beta(\Delta)}{4k_1 k_2 k_3} (i)^{L_1+L_2+L_3} (2L_3 + 1) \left( \frac{k_1}{k_3} \right)^{L_3} \sum_{L=0}^{L_3} \left( \frac{2L_3+1}{2L} \right)^{1/2} \left( \frac{k_2}{k_1} \right)^L \]

\[ \times \sum_l < L_1 (L_3 - L) 0 0 | l 0 > < L_2 L 0 0 | l 0 > \left\{ \begin{array}{ccc} L_1 & L_2 & L_3 \\ L & L_3 - L & l \end{array} \right\} P_l(\Delta), \]

where \( \left\{ j_1 j_2 j_3 \right\} \) is a 6-j symbol defined in any standard angular momentum text [17]. An interesting special case of the above equation is when \( L_3 = 0 \), which imposes \( L_1 = L_2 \equiv \lambda \) leading to

\[ \int_0^\infty r^2 j_\lambda(k_1 r) j_\lambda(k_2 r) j_0(k_3 r) \, dr = \frac{\pi \beta(\Delta)}{4k_1 k_2 k_3} P_\lambda \left( \frac{k_2^2 + k_3^2 - k_1^2}{2k_1 k_2} \right), \]

and the inverse equation

\[ j_\lambda(k_1 r) j_\lambda(k_2 r) = \frac{1}{2k_1 k_2} \int_{|k_2 - k_1|}^{k_1+k_2} k_3 j_0(k_3 r) P_\lambda \left( \frac{k_2^2 + k_3^2 - k_1^2}{2k_1 k_2} \right) \, dk_3. \]
6. Generalisation

It is possible to extend the results of the last section to finding analytical solutions to infinite integrals over several special or elementary functions (provided the integral exists) if the integral over a fewer number of these functions combined with a spherical Bessel function is known. One can show, using the *Closure Relation* for spherical Bessel functions eq. (4.5), that

\[
\int_0^\infty r^2 \chi_{L_1}(k_1, r) \chi_{L_2}(k_2, r) \chi_{L_3}(k_3, r) \chi_{L_4}(k_4, r) dr = \frac{2}{\pi} \int_0^\infty k^2 dk \times \left( \int_0^\infty r^2 \chi_{L_1}(k_1, r) \chi_{L_2}(k_2, r) j_L(kr) dr \right) \left( \int_0^\infty r'^2 \chi_{L_3}(k_3, r') \chi_{L_4}(k_4, r') j_L(kr') dr' \right),
\]

(6.1)

where \( \chi_L(k, r) \) is a special function of order \( L \) that, in general, depends on \( k \) and \( r \). So, in effect, the integral over four special has been broken up into two integrals over 3 special functions. The final integral over \( k \) may be easier to handle than the initial integral, or in the least is an alternative approach to solving the integral analytically. The following example illustrates the advantage:

Example- Evaluate the following integral over 4 spherical Bessel functions analytically

\[
I \equiv \int_0^\infty r^2 j_1(kr) j_1(kr) j_2(kr) j_2(kr) dr.
\]

(6.2)

Solution- The integral can be rewritten as

\[
I = \frac{2}{\pi} \int_0^\infty q^2 dq \left( \int_0^\infty r^2 j_1(kr) j_1(kr) j_0(qr) dr \right) \left( \int_0^\infty r'^2 j_2(kr') j_2(kr') j_0(qr') dr' \right).
\]

(6.3)
Using eq. (5.16) this reduces to

\[ I = \frac{\pi}{8k^4} \int_0^{2k} P_1 \left( \frac{2k^2 - q^2}{2k^2} \right) P_2 \left( \frac{2k^2 - q^2}{2k^2} \right) dq. \] (6.4)

This can be easily evaluated using \( P_1(x) = x \) and \( P_2(x) = \frac{1}{2} (3x^2 - 1) \) to give

\[ I = \frac{23\pi}{420k^3}. \] (6.5)
7. Conclusions

The plane wave expansion was utilised to derive integrals over one, two, and three spherical Bessel functions. The generalisation to a larger number of special functions is exhibited through the use of the Closure Relation for spherical Bessel functions, which is also derived here using the plane wave expansion. In addition, several identities involving spherical Bessel functions were also derived using the same techniques.

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