BOHR INEQUALITY FOR CERTAIN HARMONIC MAPPINGS

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Abstract. A function \( f \in C(\phi) \) if \( 1 + zf'''(z)/f'(z) \prec \phi(z) \) and \( f \in C_c(\phi) \) if \( 2(zf''(z)/f(z) + f'(z)/f(z)) \prec \phi(z) \) for \( z \in \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \). In this article, we consider the classes \( HC(\phi) \) and \( HC_c(\phi) \) consisting of harmonic mappings \( f = h + \overline{g} \) of the form

\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n
\]

in the unit disk \( \mathbb{D} \), where \( h \) belongs to \( C(\phi) \) and \( C_c(\phi) \) respectively, with the dilation \( g'(z) = \alpha h'(z) \) and \( |\alpha| < 1 \). Using Bohr phenomenon for subordination classes \([13, \text{Lemma } 1]\), we find a radius \( R_f < 1 \) such that Bohr inequality

\[
|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial \mathbb{D})
\]

holds for \( |z| = r \leq R_f \) for the classes \( HC(\phi) \) and \( HC_c(\phi) \).

1. Introduction and Preliminaries

In recent years, studying the Bohr inequality has become an interesting area of research, which (in the final form has been independently proved by Weiner, Riesz and Schur) states that if \( f \) be an analytic function in the unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) with the following Taylor series expansion

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

such that \( |f(z)| < 1 \) in \( \mathbb{D} \), then the majorant series \( M_f(r) \) associated with \( f \) satisfies the following inequality

\[
M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \quad \text{for} \quad |z| = r \leq 1/3,
\]

and the constant \( 1/3 \), known as Bohr radius, cannot be improved. Analytic functions \( f \) of the form (1.1) with \( |f(z)| < 1 \) satisfying the inequality (1.2) for \( |z| = r \leq 1/3 \), are sometimes said to satisfy the classical Bohr phenomenon. It is worth noting that the inequality (1.2), called Bohr inequality, can be written in the following form

\[
\sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |a_0| = d(f(0), \partial \mathbb{D})
\]

for \( |z| = r \leq 1/3 \) and the constant \( 1/3 \) is independent on the coefficients of the Taylor series (1.1), where \( d \) is the Euclidean distance. In a better way we can demonstrate this fact by saying that Bohr phenomenon occurs in the class of analytic self maps of the unit disk \( \mathbb{D} \). In view of the distance form (1.3), the notion of the Bohr phenomenon can be generalized to the class \( \mathcal{G} \) consisting of analytic functions \( f \) in...
$\mathbb{D}$ which take values in a given domain $D \subseteq \mathbb{C}$ such that $f(\mathbb{D}) \subseteq D$ and the class $\mathcal{G}$ is said to satisfy the Bohr phenomenon if there exists largest radius $r_D \in (0,1)$ such that the inequality \[ |f(z)| = r \leq r_D \] holds for all functions $f \in \mathcal{G}$. The largest radius $r_D$ is called the Bohr radius for the class $\mathcal{G}$.

Let $\mathcal{A}$ denote the class of analytic functions in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Each function $f \in \mathcal{A}$ has the following representation:

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the univalent functions. Let $\mathcal{S}^*$ (respectively $\mathcal{C}$) be the subclass of $\mathcal{S}$ consisting of starlike (respectively convex) functions in $\mathbb{D}$. Using the notion of subordination, Ma and Minda \[21\] have introduced more general subclasses of $\mathcal{S}^*$ and $\mathcal{C}$, denoted by $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$, consisting of functions in $\mathcal{S}$ for which $zf'(z)/f(z) < \phi(z)$ and $1 + zf''(z)/f'(z) < \phi(z)$ respectively. Here the function $\phi : \mathbb{D} \to \mathbb{C}$, called Ma-Minda function, is analytic and univalent in $\mathbb{D}$ such that $\phi(\mathbb{D})$ has positive real part, symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A Ma-Minda function has the series representation of the form $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$ ($B_1 > 0$). Similarly, it is natural to consider the function $\psi$, called non-Ma-Minda function, with the condition $\psi'(0) < 0$ and the other conditions on $\psi$ are the same as that of $\phi$. Note that $\psi$ can be obtained from $\phi$ by a rotation, namely, $z \to -z$. By similar fashion as the definition of $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$, we consider the classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$, where $\psi$ is non-Ma-Minda function. The extremal functions $K$ and $H$ respectively for the classes $\mathcal{C}(\phi)$ and $\mathcal{S}^*(\phi)$ are as follows:

\[ 1 + \frac{zK''(z)}{K'(z)} = \phi(z) \quad \text{and} \quad \frac{zH'(z)}{H(z)} = \phi(z) \]

with the normalizations $K(0) = K'(0) - 1 = 0$ and $H(0) = H'(0) - 1 = 0$. The functions $K$ and $H$ belong to the classes $\mathcal{C}(\phi)$ and $\mathcal{S}^*(\phi)$ and they play the role of the Koebe functions in the respective classes. We have the following subordination theorems and growth estimates for the class $\mathcal{C}(\phi)$ due to Ma-Minda \[21\].

**Lemma 1.6.** \[21\] Let $f \in \mathcal{S}^*(\phi)$. Then $zf'(z)/f(z) \prec zH'(z)/H(z)$ and $f(z)/z \prec H(z)/z$.

**Lemma 1.7.** \[21\] Let $f \in \mathcal{C}(\phi)$. Then $zf''(z)/f'(z) \prec zK''(z)/K'(z)$ and $f''(z) \prec K'(z)$.

**Lemma 1.8.** \[21\] Assume $f \in \mathcal{C}(\phi)$ and $|z| = r < 1$. Then

\[ K'(-r) \leq |f'(z)| \leq K'(r). \]

Equality holds for some $z \neq 0$ if, and only, if $f$ is a rotation of $K$.

In \[22\], Ravichandran has considered the classes $\mathcal{S}^*_c(\phi)$ and $\mathcal{C}_c(\phi)$, the classes of Ma-Minda type starlike functions with respect to the conjugate points and classes of Ma-Minda type convex functions with respect to the conjugate points respectively. A function $f \in \mathcal{S}$ is in the class $\mathcal{S}^*_c(\phi)$ if

\[ \frac{2zf'(z)}{f(z) + f(\bar{z})} < \psi(z) \quad \text{for} \quad z \in \mathbb{D} \]
and is in the class $C_c(\phi)$ if

$$\frac{2(zf'(z))'}{(f(z) + \overline{f(z)})^2} < \phi(z), \quad \text{for } z \in \mathbb{D}.$$ 

If $\phi(z) = (1 + z)/(1 - z)$, then $S_c(\phi)$ and $C_c(\phi)$ reduce to the classes of standard starlike and convex functions with respect to the conjugate points.

**Lemma 1.10.** Let $\min_{|z|=r} |\phi(z)| = \phi(-r)$, $\max_{|z|=r} |\phi(z)| = \phi(r)$, $|z| = r$. If $f \in C_c(\phi)$, then

(i) $K'(-r) \leq |f'(z)| \leq K'(r)$

(ii) $-K(-r) \leq |f(z)| \leq K(r)$

(iii) $f(\mathbb{D}) \supset \{w : |w| \leq -K(-1)\}$.

The results are sharp.

Recall that a complex-valued function $f$ in $\mathbb{D}$ is said to be harmonic if it satisfies the Laplace equation $\Delta f = 4f'' + \overline{f'} = 0$. Every harmonic mapping $f$ in $\mathbb{D}$ has a unique canonical decomposition $f = h + \overline{g}$, where $h$ and $g$ are analytic functions with $g(0) = 0$. We know that a harmonic mapping $f$ is locally univalent at $z_0$ if and only if, its Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2 \neq 0$ at $z_0$, and is sense-preserving if $J_f(z) > 0$ in $\mathbb{D}$ i.e., the dilation $\omega$ of $f$, given by $\omega(z) = g'(z)/h'(z)$, satisfies $|\omega(z)| < 1$ in $\mathbb{D}$. Let $\mathcal{H}$ be the class of normalized harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ of the form

(1.11) \[h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n\]

and $\mathcal{S}_\mathcal{H}$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving harmonic mappings in $\mathbb{D}$. It is proved that the class $\mathcal{S}_\mathcal{H}$ is normal but not compact. Observe that $\mathcal{S}_\mathcal{H}$ reduces to the class $\mathcal{S}$ if $g \equiv 0$ in $\mathbb{D}$. Now we consider the following new subclasses of $\mathcal{H}$ as follows:

**Definition 1.1.** For $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, let $\mathcal{HC}(\phi)$ and $\mathcal{HC}_c(\phi)$ denote the class of harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ of the form (1.11), whose analytic part $h$ belongs $C(\phi)$ and $C_c(\phi)$ respectively, with $h'(0) \neq 0$, along with the condition $g'(z) = \alpha z h'(z)$.

It is easy to see that the dilation of functions belongs to these classes are $\omega(z) = \alpha z$ and $|\omega(z)| < 1$. Hence these classes are sense-preserving in $\mathbb{D}$. For $\phi(z) = (1 + (1 - 2\beta)z)/(1 - z)$ with $-1/2 \leq \beta < 1$, the class $\mathcal{HC}(\phi)$ reduces to $\mathcal{M}(\alpha, \beta)$ consisting of harmonic mappings $f$ of the form (1.11), with $h'(0) \neq 0$, which satisfies $g'(z) = \alpha z h'(z)$ and $\Re (1 + zf''(z)/f'(z)) > \beta$. For $\alpha = 1$ and $\beta = -1/2$, $\mathcal{M}(\alpha, \beta)$ reduces to $\mathcal{M}(1, -1/2)$. The class $\mathcal{M}(1, -1/2)$ with $|\alpha| = 1$, has been extended to $\mathcal{M}(\alpha, -1/2)$.

In 2018, Bhownik and Das [13] proved the following interesting result for subordination classes.

**Lemma 1.12.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two analytic functions in $\mathbb{D}$ and $g < f$, then

(1.13) \[\sum_{n=0}^{\infty} |b_n| r^n \leq \sum_{n=0}^{\infty} |a_n| r^n\]

for $|z| = r \leq 1/3$. 
2. **Main Results**

First we prove the sharp growth estimate for the classes $\mathcal{HC}(\phi)$ and $\mathcal{HC}_c(\phi)$ which will be useful to prove our main results.

**Theorem 2.1.** Let $f \in \mathcal{HC}(\phi)$ (respectively $\mathcal{HC}_c(\phi)$). Then $f$ satisfies the following inequalities

\[(2.2)\]

\[L(r, \alpha) \leq |f(z)| \leq R(r, \alpha),\]

where

\[L(r, \alpha) = -K(-r) - |\alpha| \int_0^r tK'(-t) \, dt \quad \text{and} \quad R(r, \alpha) = K(r) + |\alpha| \int_0^r tK'(t) \, dt.\]

The bounds are sharp being the extremal function $f_\alpha = h_\alpha + \overline{g_\alpha}$ with $h_\alpha = K$ or its rotations.

Let $S_r$ be the area of the image $f(D_r)$, where $D_r := \{z \in \mathbb{D} : |z| = r < 1\}$.

**Theorem 2.3.** Let $f \in \mathcal{HC}(\phi)$. Then the following inequalities hold

\[(2.4)\]

\[2\pi \int_0^r t (1 - |\alpha|^2 t^2) (K'(-t))^2 \, dt \leq S_r \leq 2\pi \int_0^r t (1 - |\alpha|^2 t^2) (K'(t))^2 \, dt.\]

In the following theorem we obtain the Bohr inequality for the class $\mathcal{HC}(\phi)$

**Theorem 2.5.** Let $f \in \mathcal{HC}(\phi)$ be of the form (1.11). Then the majorant series of $f$ satisfies the following inequality

\[(2.6)\]

\[|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(D))\]

for $|z| = r \leq \min\{1/3, r_f\}$, where $r_f$ is the smallest positive root of $R(0) = L(1, \alpha)$. Here $R(0) = M_K(r) + |\alpha| \int_0^r tM_K(t) \, dt$ and $L(r, \alpha)$ is defined as in Theorem 2.1.

Next, we establish the Bohr phenomenon for the class $\mathcal{HC}_c(\phi)$

**Theorem 2.7.** Let $f \in \mathcal{HC}_c(\phi)$ be of the form (1.11). Then the majorant series of $f$ satisfies the following inequality

\[(2.8)\]

\[|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(D))\]

for $|z| = r \leq \min\{1/3, r_f\}$, where $r_f$ is the smallest positive root of $R(0) = L(1, \alpha)$. Here $R(0) = T(r) + |\alpha| \int_0^r tT_c(t) \, dt$ with

\[(2.9)\]

\[T_c(r) := \frac{1}{r} \int_0^r M_K(t) M_\phi(t) \, dt, \quad T(r) = \int_0^r T_c(t) \, dt\]

and $L(r, \alpha)$ is defined as in Theorem 2.1.
3. Proof of the main results

Proof of Theorem 2.1. Let \( f = h + \overline{g} \in H\mathcal{C}(\phi) \) (respectively \( H\mathcal{C}_e(\phi) \)). Then from definition, we have \( h \in \mathcal{C}(\phi) \) (respectively \( \mathcal{C}_e(\phi) \)) and the distortion Lemmas 1.8 and 1.10 assert that

\[
K'(r) \leq |h'(z)| \leq K'(r) \quad \text{for} \quad |z| = r.
\]

Let \( \gamma \) be the linear segment joining 0 to \( z \) in \( \mathbb{D} \). Then we have

\[
|f(z)| = \left| \int_\gamma \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \bar{\xi}} d\bar{\xi} \right|
\]

\[
\leq \int_\gamma (|h'(\xi)| + |g'(\xi)|) |d\xi|
\]

\[
= \int_\gamma (1 + |\alpha||\xi|) |h'(\xi)| |d\xi|.
\]

Hence by using (3.1) and (3.2), we obtain

\[
|f(z)| \leq \int_0^r (1 + |\alpha|t) K'(t) \, dt = K(r) + |\alpha| \int_0^r tK'(t) \, dt = R(r, \alpha).
\]

Let \( \Gamma \) be the preimage of the line segment joining 0 to \( f(z) \) under the function \( f \), then we have

\[
|f(z)| = \left| \int_\Gamma \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \bar{\xi}} d\bar{\xi} \right|
\]

\[
\geq \int_\Gamma (|h'(\xi)| - |g'(\xi)|) |d\xi|
\]

\[
= \int_\Gamma (1 - |\alpha||\xi|) |h'(\xi)| |d\xi|.
\]

In view of (3.1) and (3.4), we obtain

\[
|f(z)| \geq \int_0^r (1 - |\alpha|t) K'(-t) \, dt = -K(-r) - |\alpha| \int_0^r tK'(-t) \, dt = L(r, \alpha).
\]

From (3.3) and (3.5), we have

\[
L(r, \alpha) \leq |f(z)| \leq R(r, \alpha).
\]

To show the sharpness we consider the function \( f_\alpha = h_\alpha + \overline{g_\alpha} \) with \( h_\alpha = K \) or its rotations. It is easy to see that \( h_\alpha = K \in \mathcal{C}(\phi) \) and satisfies \( g_\alpha(z) = \alpha z h'_\alpha(z) \), which shows that \( f_\alpha \in H\mathcal{C}(\phi) \). The equalities hold in (3.1) for suitable rotations of \( K \). For \( 0 < \alpha < 1 \), it is easy to see that \( f_\alpha(r) = R(r, \alpha) \) and \( f_\alpha(-r) = -L(r, \alpha) \). Hence \( |f_\alpha(r)| = R(r, \alpha) \) and \( |f_\alpha(-r)| = L(r, \alpha) \). This completes the proof. \( \square \)

Proof of Theorem 2.3. Let \( f = h + \overline{g} \in H\mathcal{C}(\phi) \) and \( z = x + iy \). Then the area of image of \( \mathbb{D}_r \) under harmonic function \( f \) is given by

\[
S_r = \int_{\mathbb{D}_r} \left( |h'(z)|^2 - |g'(z)|^2 \right) \, dx \, dy = \int_{\mathbb{D}_r} \left( 1 - |\alpha|^2 |z|^2 \right) |h'(z)|^2 \, dx \, dy.
\]
Since \( h \in C(\phi) \), in view of inequalities (3.1) and (3.7), we obtain
\[
\int_0^r \int_0^{2\pi} t \left(1 - |\alpha|^2 t^2\right) (K'(-t))^2 \, d\theta \, dt \leq S_r \leq \int_0^r \int_0^{2\pi} t \left(1 - |\alpha|^2 t^2\right) (K'(t))^2 \, d\theta \, dt.
\]
i.e.,
\[
2\pi \int_0^r t \left(1 - |\alpha|^2 t^2\right) (K'(-t))^2 \, dt \leq S_r \leq 2\pi \int_0^r t \left(1 - |\alpha|^2 t^2\right) (K'(t))^2 \, dt.
\]
This completes the proof. \( \square \)

**Proof of Theorem 2.5** Let \( f = h + \mathcal{F} \in \mathcal{HC}(\phi) \). Since \( h \in C(\phi) \), from Lemma 1.7 we have
\[
(3.8) \quad h' < K'.
\]
Let \( K(z) = z + \sum_{n=2}^\infty k_n z^n \). In view of Lemma 1.12 and the inequality (3.8), we obtain
\[
(3.9) \quad 1 + \sum_{n=2}^\infty n|a_n|r^{n-1} = M_{h'}(r) \leq M_{K'}(r) = 1 + \sum_{n=2}^\infty n|k_n|r^{n-1}
\]
for \( |z| = r \leq 1/3 \). Integrating (3.9) with respect to \( r \) limit from 0 to \( r \), we obtain
\[
(3.10) \quad M_h(r) = r + \sum_{n=2}^\infty |a_n|r^{n} \leq r + \sum_{n=2}^\infty |k_n|r^{n} = M_K(r) \quad \text{for} \quad r \leq 1/3.
\]
From the definition of \( \mathcal{HC}(\phi) \), we have \( g'(z) = \alpha h'(z) \). This relation along with the inequality (3.9) yields that
\[
(3.11) \quad \sum_{n=2}^\infty n|b_n|r^{n-1} = M_{g'}(r) = |\alpha| r M_{h'}(r) \leq |\alpha| r M_{K'}(r) \quad \text{for} \quad r \leq 1/3.
\]
By integrating (3.11) with respect to \( r \) limit from 0 to \( r \), we obtain
\[
(3.12) \quad M_g(r) = \sum_{n=2}^\infty |b_n|r^n \leq |\alpha| \int_0^r t M_{K'}(t) \, dt, \quad r \leq 1/3.
\]
Therefore, for \( |z| = r \leq 1/3 \), the inequalities (3.10) and (3.12) yeild that
\[
(3.13) \quad M_f(r) = |z| + \sum_{n=2}^\infty (|a_n| + |b_n|)r^n \leq M_K(r) + |\alpha| \int_0^r t M_{K'}(t) \, dt = R_C(r).
\]
Now, from the inequality (2.2), it is evident that the Euclidean distance between \( f(0) \) and the boundary of \( f(\mathbb{D}) \) is given by
\[
(3.14) \quad d(f(0), \partial f(\mathbb{D})) = \liminf_{|z| \to 1} |f(z) - f(0)| \geq L(1, \alpha).
\]
We note that \( R_C(r) \leq L(1) \) whenever \( r \leq r_f \), where \( r_f \) is the smallest positive root of \( R_C(r) = L(1, \alpha) \) in \((0, 1)\). Let \( H_1(r) = R_C(r) - L(1, \alpha) \) then \( H_1(r) \) is a continuous
function in \([0,1]\). Since \(M_K(r) \geq K(r) > -K(-r)\), it follows that

\[
H_1(1) = R_C(1) - L(1, \alpha)
\]

\[
= M_K(1) + K(-1) + |\alpha| \int_0^r t (M_{K'}(t) + K'(t)) \, dt
\]

\[
\geq K(1) + K(-1) + |\alpha| \int_0^r t (M_{K'}(t) + K'(t)) \, dt > 0.
\]

On the other hand,

\[
H_1(0) = -L(1, \alpha) = -K(-1)(1 - |\alpha|) + |\alpha| \int_0^1 -K(-t) \, dt < 0.
\]

Therefore, \(H_1\) has a root in \((0,1)\). Let \(r_f\) be the smallest root of \(H_1\) in \((0,1)\). Then \(R_C(r) \leq L(1, \alpha)\) for \(r \leq r_f\). Now by combining the inequalities (3.13) and (3.14) with the fact that \(R_C(r) \leq L(1, \alpha)\) for \(r \leq r_f\), we obtain

\[
|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(\mathbb{D}))
\]

for \(|z| = r \leq \min\{1/3, r_f\}\). This completes the proof. \(\square\)

**Proof of Theorem 2.7.** Let \(f = h + \overline{g} \in \mathcal{HC}_c(\phi)\). Then \(h \in \mathcal{C}_c(\phi)\). Let \(g_c(z) := (h(z) + \overline{h}(\overline{z}))/2\). Since \(\phi\) is starlike and symmetric with respect to real axis, \(g_c \in \mathcal{C}(\phi)\). By the definition of \(\mathcal{C}_c(\phi)\), we have

\[
(\zeta h'(\zeta))' = g'_c(\zeta)\phi(\zeta),
\]

where \(\omega : \mathbb{D} \to \mathbb{D}\) is analytic with \(\omega(0) = 0\). A simplification of (3.15) gives

\[
h'(\zeta) = \frac{1}{\zeta} \int_0^\zeta g'_c(\xi)\phi(\omega(\xi)) \, d\xi.
\]

Since \(g_c \in \mathcal{C}(\phi)\), from Lemma 1.7 we have \(g'_c \prec K'\) and hence by Lemma 1.12 we obtain

\[
M_{g'_c}(r) \leq M_{K'}(r) \quad \text{for} \quad r \leq 1/3.
\]

Since \(\phi \circ \omega \prec \phi\), by Lemma 1.12 we have

\[
M_{\phi \circ \omega}(r) \leq M_{\phi}(r) \quad \text{for} \quad |z| = r \leq 1/3.
\]

In view of [12, Lemma 2.1] and by using (3.16), (3.17) and (3.18), we obtain

\[
M_{K'}(r) \leq \frac{1}{r} \int_0^r M_{g'_c}(t) M_{\phi \circ \omega}(t) \, dt
\]

\[
\leq \frac{1}{r} \int_0^r M_{K'}(t) M_{\phi}(t) \, dt
\]

\(=: T_c(r)\)
for \( r \leq 1/3 \). Integrating \((3.19)\) with respect to \( r \) from 0 to \( r \), we obtain

\[
M_h(r) \leq \int_0^r T_c(t) \, dt =: T(r) \quad \text{for} \quad r \leq 1/3.
\]

From the definition of \( HC_c(\phi) \), we have \( g'(z) = az'z(z) \). This relation along with the inequality \((3.19)\) asserts that

\[
\sum_{n=2}^\infty n|b_n|r^{n-1} = M'_g(r) = |\alpha|rM'_h(r) \leq |\alpha|rT_c(r) \quad \text{for} \quad r \leq 1/3.
\]

Integrating with respect to \( r \) from 0 to \( r \), we obtain

\[
\sum_{n=2}^\infty |b_n|r^n = M_g(r) \leq |\alpha| \int_0^r tT_c(t) \, dt \quad \text{for} \quad r \leq 1/3.
\]

Therefore, for \( |z| = r \leq 1/3 \), the inequalities \((3.20)\) and \((3.22)\) yield that

\[
M_f(r) = |z| + \sum_{n=2}^\infty (|a_n| + |b_n|)r^n \leq T(r) + |\alpha| \int_0^r tT_c(t) \, dt = R_{C_\alpha}(r).
\]

From the inequality \((2.2)\), it is evident that the Euclidean distance between \( f(0) \) and the boundary of \( f(\mathbb{D}) \) is given by

\[
d(f(0), \partial f(\mathbb{D})) = \lim_{|z| \to 1} \inf |f(z) - f(0)| \geq L(1, \alpha).
\]

We note that \( R_{C_\alpha}(r) \leq L(1, \alpha) \) whenever \( r \leq r_f \), where \( r_f \) is the smallest positive root of \( R_{C_\alpha}(r) = L(1, \alpha) \) in \((0, 1)\). Let \( H_2(r) = R_{C_\alpha}(r) - L(1, \alpha) \) then \( H_2(r) \) is a continuous function in \([0, 1]\). Clearly,

\[
H_2(1) = R_{C_\alpha}(1) - L(1, \alpha) = T(1) + K(-1) + |\alpha| \int_0^1 t(T'(t) + K'(-t)) \, dt.
\]

A simple observation shows that

\[
M_{K'}(r) \geq K'(r) \geq K'(-r), \quad M_\phi(r) \geq \phi(r) \quad \text{and} \quad K'(r) + rK''(r) = K'(r)\phi(r).
\]

Therefore, using \((3.26)\), we obtain

\[
T(1) + K(-1) = \int_0^1 \int_0^s M_{K'}(t)M_\phi(t) \, dt \, ds + K(-1)
\]

\[
\geq \int_0^1 \int_0^s K'(t)\phi(t) \, dt \, ds + K(-1)
\]

\[
= \int_0^1 \int_0^s (tK''(t) + K'(t)) \, dt \, ds + K(-1)
\]

\[
= \int_0^1 (sK'(s) - K(s) + K(s)) \, ds + K(-1)
\]

\[
= K(1) + K(-1) > 0.
\]
Similarly, using (3.26), we see that
\[
T'(r) + K'(-r) = \frac{1}{r} \int_0^r M_{K'}(t)M_\phi(t) \, dt + K'(-r)
\]
\[
\geq \frac{1}{r} \int_0^r K'(t)\phi(t) \, dt + K'(-r)
\]
\[
= \frac{1}{r} \int_0^r (K'(t) + tK''(t)) \, dt + K'(-r)
\]
\[
= K'(r) + K'(-r) > 0
\]
and hence
\[
(3.28) \quad \int_0^1 (T'(t) + K'(-t)) \, t \, dt > 0.
\]
Combining (3.27) and (3.28) with (3.25), we obtain \( H_2(1) > 0 \). Similarly, using (3.26), we obtain
\[
H_2(0) = -L(1, \alpha) = -K(-1)(1 - |\alpha|) + |\alpha| \int_0^1 -K(-t) \, dt < 0.
\]
Therefore \( H_2 \) has a root in \((0, 1)\). Let \( r_f \) be the smallest root of \( H_2 \) in \((0, 1)\). Then \( R_{f_c}(r) \leq L(1, \alpha) \) for \( r \leq r_f \). Combining the inequalities (3.13) and (3.14) with the fact that \( R_{f_c}(r) \leq L(1, \alpha) \) for \( r \leq r_f \), we obtain
\[
|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \leq d(f(0), \partial f(\mathbb{D}))
\]
for \(|z| = r \leq \min\{1/3, r_f\}\). This completes the proof. \( \square \)

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