Rainbow independent sets on dense graph classes

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Abstract

Given a family $\mathcal{I}$ of independent sets in a graph, a rainbow independent set is an independent set whose vertices are contained in distinct independent sets in $\mathcal{I}$. Aharoni, Briggs, J. Kim, and M. Kim [Rainbow independent sets in certain classes of graphs. arXiv:1909.13143] determined for various graph classes $\mathcal{C}$ whether $\mathcal{C}$ satisfies a property that for every $n$, there exists $N = N(\mathcal{C}, n)$ such that every family of $N$ independent sets of size $n$ in a graph in $\mathcal{C}$ contains a rainbow independent set of size $n$. In this paper, we add two dense graph classes satisfying this property, namely, the class of graphs of bounded neighborhood diversity and the class of $r$-powers of graphs in a bounded expansion class.

1 Introduction

Given a family $\mathcal{M}$ of matchings in a graph, a rainbow matching is a matching whose edges are contained in distinct matchings in $\mathcal{M}$. Rainbow matchings in bipartite graphs have been attracted because of their connection to transversals of Latin squares. Drisko [6] showed that every family of $2n - 1$
matchings of size $n$ in a bipartite graph where one side has size $n$, and later Aharoni and Berger \cite{AharoniBerger1} showed that this is true for all bipartite graphs. A well-known conjecture of Ryser \cite{Ryser} states that for odd $n$, every family of edge-disjoint $n$ perfect matchings of $K_{n,n}$ has a rainbow matching of size $n$. For general graphs, Aharoni, Berger, Chudnovsky, Howard and Seymour \cite{AharoniBergerChudnovskyHowardSeymour} showed that every family of $3n - 2$ matchings of size $n$ contains a rainbow matching of size $n$. They conjectured that if $n$ is even, then $2n$ matchings of size $n$ have a rainbow matching of size $n$, and if $n$ is odd, then $2n - 1$ matchings have a rainbow matching of size $n$.

In this paper, we consider the rainbow version of independent sets. Given a family $\mathcal{I}$ of independent sets in a graph, a rainbow independent set is an independent set whose vertices are contained in distinct independent sets in $\mathcal{I}$. For the line graph $H$ of a graph $G$, rainbow matchings in $G$ correspond to rainbow independent sets in $H$. Thus, the result of Aharoni and Berger \cite{AharoniBerger1} implies that every family of $2n - 1$ independent sets of size $n$ in the line graph of a bipartite graph contains a rainbow independent set of size $n$.

Motivated from this fact, Aharoni, Briggs, J. Kim, and M. Kim \cite{AharoniBriggsKimKim} and Kim and Lew \cite{KimLew} considered the same type of problems for rainbow independent sets in various graph classes. They observed that contrary to the matching case, such a bound does not exist for all graphs. A simple example is the complete multipartite graph where each part has size $n$ and the number of parts is $k$. Clearly, this example has $k$ distinct independent sets of size $n$ which has no rainbow independent set. Thus, for the class of all complete multipartite graphs, there is no integer $N$ satisfying that the existence of $N$ independent sets of size $n$ guarantees the existence of a rainbow independent set of size $n$.

Aharoni et al. \cite{AharoniBriggsKimKim} suggested the following parameters. For a graph $G$ and positive integers $m$ and $n$ with $n \geq m$, let $f_{G}(n, m)$ be the minimum $k$ such that every family of $k$ independent sets of size $n$ has a rainbow independent set of size $m$. For a class $\mathcal{C}$ of graphs and positive integers $m$ and $n$ with $n \geq m$, let $f_{\mathcal{C}}(n, m) = \sup\{f_{G}(n, m) : G \in \mathcal{C}\}$. As discussed, it can be $\infty$. Note that $f_{\mathcal{C}}(n, m) \leq f_{\mathcal{C}}(n, n)$ for all $n \geq m$ because every rainbow independent set of size $n$ has a rainbow independent set of size $m$. We additionally introduce the following. A class $\mathcal{C}$ of graphs has the rainbow property (for independent sets) if for every positive integer $n$, $f_{\mathcal{C}}(n, n) < \infty$.

Aharoni et al. \cite{AharoniBriggsKimKim} established finiteness or infiniteness of $f_{\mathcal{C}}(n, m)$ for many graph classes. Among several results, the following results motivate our work. They showed that the class of $K_t$-induced subgraph free graphs and the class of chordal graphs have the rainbow property. On the other
hand, for every positive integer \( n > 1 \), \( f_{\mathcal{CM}}(n, n) = \infty \) for the class \( \mathcal{CM} \) of complete multipartite graphs, and \( f_{\mathcal{C}_n}(n, n) = \infty \) for the set \( \mathcal{C}_n \) of \((t - 1)\)-powers of cycles on \( tn \) vertices. For a graph \( G \) and a positive integer \( r \), the \( r \)-power \( G^r \) of \( G \) is the graph obtained from \( G \) by adding an edge between two vertices of distance at most \( r \) in \( G \). Thus, \( \mathcal{CM} \) and \( \mathcal{C}_n \) do not have the rainbow property.

For well-known sparse graph classes \( \mathcal{C} \) such as the class of planar graphs, we may observe that \( \mathcal{C} \) satisfies the rainbow property. Nešetřil and Ossona de Mendez [12] introduced the notion of bounded expansion classes of graphs, generalizing the classes with excluding minors and the classes of bounded degree. The same authors introduced more general sparse graph classes called nowhere dense classes of graphs [13], which further include the classes with locally excluding a minor. We refer to [14] for more extensive study on sparse graph classes. As an other extension, a bounded expansion class has bounded degeneracy, and classes of bounded degeneracy are incomparable with nowhere dense classes. It is not difficult to see that every nowhere dense class or every class of bounded degeneracy has the rainbow property because they are \( K_t \)-induced subgraph free for some \( t \) depending only on the class.

What about well-behaved dense graph classes? Graphs of bounded clique-width [5] (equivalently, rank-width [16]) are the graphs that can be decomposed into a tree-like structure where each vertex partition \((A, B)\) represented by an edge of the tree-structure has the property that \( A \) has a bounded number of neighborhood types to \( B \) and vice versa. As an example, complete graphs or complete multipartite graphs have small clique-width. But, because of complete multipartite graphs, generally, the class of graphs of bounded clique-width (even restricted linear clique-width, shrub-depth [8], or modular-width [7]) does not have the rainbow property.

As a much more restricted dense graph classes, we consider the class of graphs of bounded neighborhood diversity [11], and show that it has the rainbow property. We say that two vertices \( v \) and \( w \) in a graph \( G \) are twins if \( v \) and \( w \) have the same neighborhood in \( G \) outside \( v \) and \( w \). The neighborhood diversity of a graph \( G \) is the minimum \( t \) such that there is a partition of the vertex set of \( G \) into at most \( t \) sets, each of which is a set of pairwise twins.

**Theorem 1.1.** Let \( k \) and \( n \) be positive integers and \( \mathcal{N}_k \) be the class of graphs of neighborhood diversity at most \( k \). Then \( f_{\mathcal{N}_k}(n, n) \leq 2^k(n - 1) + 1 \).

A main idea of proving Theorem 1.1 is to consider the number of intersections of a given independent set with each maximal set of pairwise
twins. Since the number of such parts is at most \( k \) and the number of given independent sets is sufficiently large, we can find a large collection of the given set of independent sets such that the selected independent sets use exactly same parts. As there are no edges between the parts having vertices of chosen independent sets, we can reduce to find a rainbow independent set in a disjoint union of cliques.

We explore to use this idea for other classes. A map graph is a graph that can be obtained from a plane graph by making a vertex for each face, and adding an edge between two vertices, if the corresponding faces share a vertex. Map graphs may have large cliques and long induced cycles. Thus, it does not follow from the previous results whether the class of map graphs has the rainbow property.

Chen, Grigni, and Papadimitriou [4] observed that every map graph is an induced subgraph of the square of a planar graph. So, if the class of the squares of planar graphs has the rainbow property, then it is true for map graphs as well. We in fact, show a general statement that for every fixed integer \( r \) and every class \( C \) of bounded expansion, the class of \( r \)-powers of graphs in \( C \) has the rainbow property.

**Theorem 1.2.** Let \( C \) be a bounded expansion class and \( r \) be a positive integer. The class \( D = \{ G^r : G \in C \} \) has the rainbow property.

Contrary to Theorem 1.2, the class \( D^* \) of all possible graph powers of graphs in a bounded expansion class may have infinite \( f_{D^*}(n, n) \), because if we consider the class of all possible powers of planar graphs, then it contains \( C_n \), which is the set of \((t - 1)\)-powers of cycles on \( tn \) vertices.

To prove Theorem 1.2 we follow a similar approach used by Kwon, Pilipczuk, and Siebertz [10] to show that such class \( D \) admits low rank-width colorings. When they proved the result, in the last step, they used the fact that \( r \)-powers of graphs of bounded tree-depth have bounded rank-width. However, this does not help to show Theorem 1.2 because complete multipartite graphs have rank-width at most 1 while they do not have the rainbow property. Instead, we strongly show that the class of \( d \)-powers of graphs of bounded tree-depth has the rainbow property.

We remark that Theorem 1.2 cannot be extended to \( r \)-powers of graphs of bounded degeneracy, because every graph can be obtained as an induced subgraph of the 2-power of its 1-subdivision, where this 1-subdivision has degeneracy at most 2. We remain as an open question whether or not the class of \( r \)-powers of graphs in a nowhere dense class has the rainbow property.

The paper is organized as follows. In Section 2 we introduce necessary notations. In Section 3 we show that the class of bounded neighborhood
diversity has the rainbow property. In Section 4, we show that the class of \( r \)-powers of graphs of bounded tree-depth has the rainbow property, and based on this result, in Section 5, we prove that the class of \( r \)-powers of graphs in a bounded expansion class has the rainbow property. Finally, in Section 6 we conclude the paper by giving some problems.

2 Preliminaries

All graphs in this paper are finite, undirected graphs without loops or parallel edges. Let \( \mathbb{N} \) be the set of integers. Let \( G \) be a graph. We write \( V(G) \) for the vertex set of a graph \( G \) and \( E(G) \) for its edge set. For a vertex set \( S \) of \( G \), we denote by \( G - S \) the graph obtained from \( G \) by removing vertices in \( S \) and all incident edges, and denote by \( G[S] \) the subgraph of \( G \) induced by \( S \). A graph \( H \) is an induced subgraph of \( G \) if \( H \subseteq G[S] \) for some \( S \subseteq V(G) \).

The length of a path is the number of edges in the path. The distance between vertices \( u \) and \( v \) in \( G \), denoted \( \text{dist}_G(u, v) \), is the length of a shortest path between \( u \) and \( v \) in \( G \), or \( \infty \) if no such path exists. The \( r \)-power of a graph \( G \) is the graph \( G^r \) with vertex set \( V(G) \), where there is an edge between two vertices \( u \) and \( v \) in \( G^r \) if and only if their distance in \( G \) is at most \( r \). For a class \( \mathcal{C} \) of graphs, we define \( \mathcal{C}^r := \{ G^r : G \in \mathcal{C} \} \).

A set \( I \) of vertices in \( G \) is an independent set if no two vertices in \( I \) are adjacent. A set \( K \) of vertices in \( G \) is a clique if for every pair of distinct vertices \( v \) and \( w \) in \( K \), \( v \) is adjacent to \( w \). A set \( M \) of edges in \( G \) is a matching if no two edges in \( M \) share a vertex. A set \( S \) of vertices in \( G \) is a vertex cover if \( G - S \) has no edges.

We say that two vertices \( v \) and \( w \) in a graph \( G \) are twins if \( v \) and \( w \) have the same neighborhood in \( G \) outside \( v \) and \( w \). The neighborhood diversity of a graph \( G \) is the minimum \( t \) such that there is a partition of the vertex set of \( G \) into at most \( t \) sets, each of which is a set of pairwise twins.

We define the tree-depth of a graph \([14]\). A rooted forest is a forest in which every connected component has a specified node called a root. A vertex \( v \) is an ancestor of a vertex \( u \) in \( T \) if the path from \( u \) to the root in \( T \) contains \( v \). The closure of a rooted forest \( T \), denoted by \( \text{clos}(T) \), is the graph obtained from \( T \) by adding an edge between every vertex and all its ancestors. The height of a rooted forest is the number of vertices in a longest path from a root to a leaf. The tree-depth of a graph \( G \), denoted by \( \text{td}(G) \), is the minimum height of a rooted forest whose closure contains \( G \) as a subgraph.

A bounded expansion class was originally defined in terms of \( t \)-shallow
minors [12], but it turns out that low tree-depth colorings and weak colorings give alternative characterizations. As we use the two characterizations in the proof, we give them here.

A vertex coloring of a graph $G$ with colors from $S$ is a mapping $c: V(G) \rightarrow S$. For each $v \in V(G)$, we call $c(v)$ the color of $v$. A class $\mathcal{C}$ of graphs admits low tree-depth colorings if there exists a function $g: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $p \in \mathbb{N}$, every graph $G \in \mathcal{C}$ can be vertex colored with at most $g(p)$ colors such that the union of any $i \leq p$ color classes induces a subgraph of tree-depth at most $i - 1$.

For a graph $G$, we denote by $\Pi(G)$ the set of all linear orders of $V(G)$. For $u, v \in V(G)$ and an integer $r \geq 0$, we say that $u$ is weakly $r$-reachable from $v$ with respect to a linear order $L$, if there is a path $P$ of length at most $r$ between $u$ and $v$ such that $u$ is the smallest among the vertices of $P$ with respect to $L$. We denote by $WReach_r[G, L, v]$ the set of vertices that are weakly $r$-reachable from $v$ with respect to $L$. The weak $r$-coloring number $wcol_r(G)$ of $G$ is defined as

$$wcol_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} |WReach_r[G, L, v]|.$$

**Theorem 2.1** (Nešetřil and Ossona de Mendez [12]). A class of graphs has bounded expansion if and only if it admits low tree-depth colorings.

**Theorem 2.2** (Zhu [18]). A class $\mathcal{C}$ of graphs has bounded expansion if and only if for all integers $r \geq 0$, there is an integer $f(r)$ such that for all $G \in \mathcal{C}$, we have $wcol_r(G) \leq f(r)$.

Bounded expansion classes include classes of $H$-minor free graphs, classes of graphs of bounded degree, classes of graphs of bounded stack number, and classes of graphs of bounded queue number. We refer to [15] for examples of bounded expansion classes.

### 3 Graphs of bounded neighborhood diversity

In this section, we prove that the class of graphs of bounded neighborhood diversity has the rainbow property. We use the known result for chordal graphs. A chordal graph is a graph that does not contain an induced cycle of length at least 4.

**Theorem 3.1** (Aharoni, Briggs, J. Kim, and M. Kim [3]). Let $\mathcal{C}$ be the set of chordal graphs. For every positive integer $n$, $f_\mathcal{C}(n, n) = n$. 

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Theorem 1.1. Let \( k \) and \( n \) be positive integers and \( N_k \) be the class of graphs of neighborhood diversity at most \( k \). Then \( f_{N_k}(n, n) \leq 2^k(n - 1) + 1 \).

Proof. Let \( G \) be a graph of neighborhood diversity at most \( k \), and let \( C_1, C_2, \ldots, C_x \) be the partition of the vertex set of \( G \) such that \( x \leq k \) and each \( C_i \) is a set of pairwise twins. Note that each part \( C_i \) is either a clique or an independent set.

Let \( F = \{I_1, I_2, \ldots, I_t\} \) be a set of independent sets of size \( n \) in \( G \) where \( t \geq 2^k(n - 1) + 1 \). For each \( j \in \{1, 2, \ldots, t\} \), let \( U^j = (u^j_1, u^j_2, \ldots, u^j_k) \) be the vector such that

\[
 u^j_i = \begin{cases} 
 1 & \text{if } |I_j \cap C_i| \geq 1, \\
 0 & \text{if } |I_j \cap C_i| = 0.
\end{cases}
\]

As \( t \geq 2^k(n - 1) + 1 \), there is a subset \( F_1 \subseteq F \) of size \( n \) such that all independent sets \( I_j \) in \( F_1 \) have the same vectors, say \( U = (u_1, u_2, \ldots, u_k) \).

Now, observe that if \( u_{i_1} = 1 \) and \( u_{i_2} = 1 \) for some \( i_1 \neq i_2 \), then there is no edge between \( C_{i_1} \) and \( C_{i_2} \); if there is an edge between the two parts, then an independent set cannot lie in both parts. Thus, the subgraph \( G' \) of \( G \) induced by the union of sets \( C_i \) for which \( u_i = 1 \), is the disjoint union of cliques, which in particular is a chordal graph. By Theorem 3.1, \( F_1 \) contains a rainbow independent set of size \( n \), as required.

4 \( r \)-powers of graphs of bounded tree-depth

In Section 5, we show that the class of \( r \)-powers of graphs in a bounded expansion has the rainbow property. As a core, we prove that the class of \( r \)-powers of graphs of bounded tree-depth has the rainbow property. Let \( T_d \) be the class of graphs of tree-depth at most \( d \).

Theorem 4.1. Let \( d, r, n \) be positive integers. Then \( f_{T_d}(n, n) < \infty \).

For a positive integer \( d \), let \( F_d \) be the class of rooted trees \( F \) of height \( d \), and \( T_d^* \) be the set of all subgraphs of graphs in \( \{\text{clo}(F) : F \in F_d\} \). Note that \( T_d \subseteq T_{d+1}^* \). To prove Theorem 4.1 we will prove for \( T_d^* \) for all \( d \).

Let \( F \in F_d \) and let \( \mathcal{I} \) be a family of vertex sets of \( F \). We define \( t_F(\mathcal{I}) \in \{1, 2, \ldots, d\} \) as the minimum integer \( t \) satisfying the following: there is a node \( v \) of distance \( t - 1 \) from the root such that there are at least two subtrees, rooted at children of \( v \) in \( F \), meeting some vertex sets in \( \mathcal{I} \). If there is no such \( t \), then we define \( t_F(\mathcal{I}) = d \).

We use the tight value for graphs whose chromatic number is at most \( k \).
Claim 1. Let $I$ be a partition of the vertex set of $G$. We classify vertices of $G$ by the distance to each vertex in $\{v_1, v_2, \ldots, v_q\}$. For each vertex $v$ of $G$, let $d(v)$ be the vector $(d_1, d_2, \ldots, d_q)$ such that $d_i = \text{dist}_G(v, v_i)$ if $\text{dist}_G(v, v_i) \leq r$ and $d_i = r + 1$ otherwise. Note that $\text{dist}_G(v, v_i) = \infty$ if $v$ and $v_i$ are not contained in the same connected component of $G$. For each $A \in \{0, 1, \ldots, r + 1\}^q$, let $U_A := \{v \in V(G) : d(v) = A\}$. This is a partition of the vertex set of $G$ into $(r + 2)^q$ sets.

We use the following property of the sets $U_A$.

Claim 1. Let $v, w$ be two vertices of $G$ such that

- $v \in U_{(a_1, a_2, \ldots, a_q)}$ and $w \in U_{(b_1, b_2, \ldots, b_q)}$ for some vectors $(a_1, \ldots, a_q)$ and $(b_1, \ldots, b_q)$ in $\{0, 1, \ldots, r + 1\}^q$.
If $a_i + b_i \leq r$ for some $i \in \{1, 2, \ldots, q\}$, then $v$ is adjacent to $w$ in $G^r$. On the other hand, if $v$ is contained in some subtree of $H$, say $Q$, and $w$ is contained in $V(G)\setminus V(Q)$, and $a_i + b_i > r$ for all $i \in \{1, 2, \ldots, q\}$, then $v$ is not adjacent to $w$ in $G^r$.

**Proof.** Suppose that $a_i + b_i \leq r$ for some $i$. Then there is a path of length at most $r$ in $G$ from $v$ to $w$ through $v_i$. Thus it is obvious that $v$ is adjacent to $w$ in $G^r$.

Now, suppose that $v$ is contained in some subtree of $H$, say $Q$, and $w$ is contained in $V(G)\setminus V(Q)$, and $v$ is adjacent to $w$ in $G^r$. By definition, there is a path $P$ of length at most $r$ from $v$ to $w$ in $G$. Because in $G - \{v_1, v_2, \ldots, v_q\}$, there is no path from $V(Q)$ to $V(G)\setminus V(Q)$, the path $P$ meets $\{v_1, v_2, \ldots, v_q\}$. Then we have that $a_i + b_i \leq r$ for some $i \in \{1, 2, \ldots, q\}$.

As $|\mathcal{I}| = M = 2^{(r+2)^q(M_1 - 1)} + 1$, by the pigeonhole principle, we can choose a subset $\mathcal{I}_1$ of size at least $M_1$ such that

(*) for every $I_1, I_2 \in \mathcal{I}_1$ and every vector $A \in \{0, 1, \ldots, r+1\}^q$, $I_1 \cap U_A = \emptyset$ if and only if $I_2 \cap U_A = \emptyset$.

Now, for each vector $A \in \{0, 1, \ldots, r+1\}^q$, we construct an auxiliary bipartite graph $B_A$ on the bipartition $(\mathcal{H}, \mathcal{I}_1)$ such that for $H \in \mathcal{H}$ and $I \in \mathcal{I}_1$, $H$ is adjacent to $I$ in $B_A$ if and only if $I \cap U_A$ contains a vertex of $H$. We divide cases depending on whether $B_A$ contains a matching of size $n$ for some vector $A$ or not.

First suppose that there exists a matching of size $n$ in $B_A$ for some vector $A = (a_1, a_2, \ldots, a_q)$, say $\{F_1 I_1, F_2 I_2, \ldots, F_n I_n\}$ where $F_1, \ldots, F_n \in \mathcal{H}$ and $I_1, \ldots, I_n \in \mathcal{I}_1$. We choose a vertex $x_j$ in $I_j \cap V(F_j) \cap U_A$ for each $j$. If $2a_i > r$ for all $i \in \{1, 2, \ldots, q\}$, then by Claim 1, $x_i$ and $x_j$ are not adjacent for any $i \neq j$ since they are contained in distinct $V(F_j)$. Thus $\{x_1, x_2, \ldots, x_n\}$ is a rainbow independent set, so we are done. Therefore, we may assume that $2a_i \leq r$ for some $i \in \{1, 2, \ldots, q\}$. It implies that $U_A$ is a clique in $G^r$, because any two vertices in $U_A$ can be linked by a path of length at most $r$ through $v_i$.

Because of the existence of a matching in $B_A$, we know that for every $I \in \mathcal{I}_1$, $I$ contains exactly one vertex of $U_A$. We define that

$$\mathcal{I}_1^* := \{I \setminus U_A : I \in \mathcal{I}_1 \setminus \{I_1, I_2, \ldots, I_n\}\}.$$ 

Then $|\mathcal{I}_1^*| = M_1 - n \geq M_2 - n \geq M(d, n-1, p, r)$ and each independent set of $\mathcal{I}_1^*$ has size exactly $n - 1$. As $|\mathcal{I}_1^*| \geq M(d, n-1, p, r)$, $\mathcal{I}_1^*$ contains a
rainbow independent set $J$ of size $n - 1$. As $J$ has size $n - 1$, there exists $i \in \{1, 2, \ldots, n\}$ such that $J$ does not meet $F_i$. Without loss of generality, we assume that $J$ does not contain a vertex of $F_1$.

We claim that $x_1$ has no neighbor in $J$ in $G^r$, which implies that $\{x_1\} \cup J$ is a rainbow independent set in $G^r$. Take any vertex $w$ in $J$. Assume that $w$ is contained in $U_B$ for some vector $B = (b_1, b_2, \ldots, b_q) \in \{0, 1, \ldots, r + 1\}^q$. As $w$ is contained in $U_B$, by (*), $I_1$ also contains a vertex in $U_B$. Since $I_1$ is independent, by Claim II $a_i + b_i > r$ for all $i \in \{1, 2, \ldots, q\}$. On the other hand, since $w$ is not in $F_1$, by Claim I, $x_1$ is not adjacent to $w$ in $G^r$. So, we conclude that $\{x_1\} \cup J$ is a rainbow independent set, as required.

Now, we may assume that

- $B_A$ contains no matching of size $n$ for all vectors $A \in \{0, 1, \ldots, r + 1\}^q$.

By Kőnig’s Theorem, each $B_A$ contains a vertex cover $S_A$ of size at most $n - 1$. It means that all the independent sets that are contained in $I_1 \setminus S_A$ lie in the union of $\{v_1, v_2, \ldots, v_q\}$ and the vertex sets of the subtrees in $S_A \cap \mathcal{H}$. We define that

$$S := \bigcup_{A \in \{0, 1, \ldots, r + 1\}^q} S_A.$$  

Collecting the information of each $S_A$, we can observe that

- $|I_1 \setminus S| \geq M_1 - (n - 1)(r + 2)^q$ and $|S \cap \mathcal{H}| \leq (n - 1)(r + 2)^q$,

- every independent set in $I_1 \setminus S$ lies in the union of $\{v_1, v_2, \ldots, v_q\}$ and the vertex sets of the subtrees in $S \cap \mathcal{H}$.

Now, we want to take a subset of $I_1$ so that the selected independent sets have the same number of vertices in each set $V(H) \cap U_A$ for $H \in S \cap \mathcal{H}$ and $A \in \{0, 1, \ldots, r + 1\}^q$. Note that

$$M_1 - (n - 1)(r + 2)^q \geq M_2 \left( (n + 1)^{(n-1)(r+2)^q} - 1 \right) + 1.$$  

So, by the two properties, there exists a subset $I_2 \subseteq I_1$ of size at least $M_2$ such that

(**) all independent sets in $I_2$ have the same number of intersections on $V(H) \cap U_A$ for each subtree $H$ in $S \cap \mathcal{H}$ and each vector $A$ in $\{0, 1, \ldots, r + 1\}^q$.

Assume that an independent set in $I_2$ meets at least two subtrees in $S \cap \mathcal{H}$. Let $Q$ be a subtree in $S \cap \mathcal{H}$ that meets an independent set in $I_2$,
and \( y \) be the number of intersections of each independent set in \( Q \). Let 
\[ \mathcal{I}_2^* := \{ I \cap V(Q) : I \in \mathcal{I}_2 \} \]
Observe that
\[
M(d, n - 1, p, r) \geq M(d, i, j, r)
\]
for all \( i \in \{1, 2, \ldots, n - 1\} \) and \( j \in \{0, 1, \ldots, p\} \). Since
\[
|\mathcal{I}_2^*| = |\mathcal{I}_2| > M(d, n - 1, p, r),
\]
by the induction hypothesis and (1), \( \mathcal{I}_2^* \) contains a rainbow independent set 
\( R_1 = \{ z_{i_1}, z_{i_2}, \ldots, z_{i_q} \} \) of size \( y < n \) such that \( z_{i_j} \in I_{i_j} \in \mathcal{I}_2^* \). Let
\[
\mathcal{I}_2^+ := \{ I \cap V(Q) : I \in \mathcal{I}_2 \setminus \{ I_{i_1}, I_{i_2}, \ldots, I_{i_q} \} \}
\]
Note that each set in \( \mathcal{I}_2^+ \) has size \( n - y \). Since
\[
|\mathcal{I}_2^+| = |\mathcal{I}_2| - y \geq M(d, n - 1, p, r) + n - y > M(d, n - 1, p, r),
\]
by the induction hypothesis and (1), \( \mathcal{I}_2^+ \) also contains a rainbow independent set 
\( R_2 \) of size \( n - y \). In particular, \( R_1 \cup R_2 \) is a rainbow set for \( \mathcal{I}_2 \).

We claim that \( R_1 \cup R_2 \) is a rainbow independent set of size \( n \). Let 
\( w_1 \in R_1 \) and \( w_2 \in R_2 \). Assume that \( w_1 \in U_{B_1}, w_2 \in U_{B_2} \) for some vectors 
\( B_1 = (b_1^1, b_1^2, \ldots, b_q^1) \in \{0, 1, \ldots, r+1\}^q \) and \( B_2 = (b_1^2, b_2^2, \ldots, b_q^2) \in \{0, 1, \ldots, r+1\}^q \). Let \( I' \) be the independent set of \( \mathcal{I}_2 \) containing \( w_1 \). If \( w_2 = v_j \) for some 
\( j \in \{1, \ldots, q\} \), then \( b_j^2 = 0 \), and \( I' \) has to contain \( w_2 \) as well, because of
the property of \( \mathcal{I}_1 \). It means that \( w_1 \) is not adjacent to \( w_2 \). Thus, we may assume that \( w_2 \) is contained in some subtree of \( \mathcal{H} \) other than \( Q \), say \( Q' \).

As \( w_2 \) is contained in \( V(Q') \cap U_{B_2} \), by the property of \( \mathcal{I}_2 \), \( I' \) also contains
a vertex in \( V(Q') \cap U_{B_2} \). Since \( I' \) is independent, by Claim 1 \( b_i^1 + b_i^2 > r \)
for all \( i \in \{1, 2, \ldots, q\} \). On the other hand, since \( w_1 \) and \( w_2 \) are contained in
distinct subtrees of \( \mathcal{H} \), by Claim 1 \( w_1 \) is not adjacent to \( w_2 \) in \( G' \), as
required. We conclude that \( R_1 \cup R_2 \) is a rainbow independent set of size \( n \).

Lastly, we can assume that all the independent sets in \( \mathcal{I}_2 \) lie in the union of 
\( \{v_1, \ldots, v_q\} \) and exactly one subtree of \( S \cap \mathcal{H} \). It means that 
\( t_F(\mathcal{I}_2) > t_F(\mathcal{I}) \). As \( |\mathcal{I}_2| \geq M(d, n, p - 1, r) \geq M(d, n, j, r) \) for all \( 0 \leq j \leq p - 1 \), \( \mathcal{I}_2 \)
contains a rainbow independent set of size \( n \).

\[\square\]

**Proof of Theorem** Let \( M \) be the function in Proposition 4.3. We claim
that for every positive integer \( d \), we have \( f_{T_d^\mathcal{H}}(n, n) \leq M(d + 1, n, d + 1, r) \).
Let \( G \) be a tree-depth at most \( d \) and \( T \) be the rooted forest whose closure
contains \( G \) as a subgraph. Let \( F \) be a rooted tree obtained from \( T \) by
adding an isolated vertex, which is a new root, and adding edges between
the new vertex and original roots in the components of $T$. Then clearly, $G$ is a subgraph of $\text{clos}(F)$ and $F \in \mathcal{F}_{d+1}$. Let $\mathcal{I}$ be a set of $M(d+1,n,d+1,r)$ independent sets of size $n$ in $G^r$. Then by Proposition 4.3, $\mathcal{I}$ contains a rainbow independent set of size $n$.

5 $r$-powers of graphs of bounded expansion

In this section, we prove that the class of $r$-powers of graphs in a bounded expansion class has the rainbow property.

**Theorem 1.2.** Let $\mathcal{C}$ be a bounded expansion class and $r$ be a positive integer. The set $\mathcal{D} = \{G^r : G \in \mathcal{C}\}$ has the rainbow property.

We can start with low tree-depth colorings of a graph $G$ in $\mathcal{C}$ whose existence is guaranteed by Theorem 2.1. Note that the size $n$ of an independent set we are dealing is given, we will take a low tree-depth coloring where the union $S$ of $n$ color classes have small tree-depth. But when we take the $r$-power of $G$, $G^r[S]$ is not necessarily the same as $G[S]^r$, because the two vertices in $S$ can be adjacent in $G^r$ because of a path going through outside $S$. In fact, the similar problem happens in the result of Kwon, Pilipczuk, and Siebertz [10], and they resolve this problem by introducing a notion of $r$-shortest path closures.

For a graph $G$, $X \subseteq V(G)$, and a positive integer $r$, a superset $X'$ of $X$ is called an $r$-shortest path closure of $X$ if for each $u, v \in X$ with $\text{dist}_G(u, v) = \ell \leq r$, $G[X']$ contains a path of length $\ell$ between $u$ and $v$. For a graph $G$, a coloring $c$ of $G$, and integers $r \geq 2$, $d \geq 1$, a coloring $c'$ is a $(d, r)$-excellent refinement of $c$ if for every vertex set $X \subseteq V(G)$, there exists an $r$-shortest path closure $X'$ of $X$ such that if $X$ receives at most $p$ colors in $c'$, then $X'$ receives at most $d \cdot p$ colors in $c$.

**Lemma 5.1** (Kwon, Pilipczuk, and Siebertz [10]). Let $G$ be a graph, let $k \geq 1$, $r \geq 2$ be integers, and let $d_r := \prod_{2 \leq \ell \leq r} 2^{\text{wcol}_\ell(G)}$. Then every coloring $c$ of $G$ using at most $k$ colors has a $(d_r, r)$-excellent refinement coloring using at most $k^{d_r}$ colors.

**Proof of Theorem 1.2.** Let $n$ be a positive integer. We have to show that there exists $N$ such that for every graph $G$ in $\mathcal{C}$, $f_{G^r}(n,n) \leq N$.

By Theorem 2.2, for each $\ell$, $\text{wcol}_\ell(G)$ is bounded by a constant, say $W_\ell$, only depending on $\mathcal{C}$. Also, by Theorem 2.1, there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that for all $p \in \mathbb{N}$, every graph $G \in \mathcal{C}$ can be vertex colored with at most $g(p)$ colors such that the union of any $i \leq p$ color classes
induces a subgraph of tree-depth at most \( i - 1 \). Let \( d_r := \prod_{2 \leq \ell \leq r} 2W_\ell \), and \( L := f_{T_{d_r,n}}(n,n) \) which is finite by Theorem 4.1. Finally, we set \( N := g(d_r \cdot n)^{d_r \cdot n} \cdot L \).

Let \( G \in C \), and \( I \) be a set of independent sets of \( G^r \) of size \( n \) such that \( |I| \geq N \).

Let \( c \) be a \((d_r \cdot n)\)-tree-depth coloring with \( g(d_r \cdot n) \) colors. We take a \((d_r,r)\)-excellent refinement \( c' \) of \( c \) with at most \( g(d_r \cdot n)^{d_r \cdot n} \) colors by Lemma 5.1.

Since \( |I| \geq g(d_r \cdot n)^{d_r \cdot n} \cdot L \), there exist \( n \) color classes \( X_1, X_2, \ldots, X_n \) of \( c' \) and a subset \( I' \subseteq I \) such that \( |I'| \geq L \) and every independent set of \( I' \) is contained in \( X_1 \cup X_2 \cup \cdots \cup X_n \). As \( c' \) is a \((d_r,r)\)-excellent refinement of \( c \), there exists an \( r \)-shortest path closure \( X' \) of \( X_1 \cup X_2 \cup \cdots \cup X_n \) that uses at most \( d_r \cdot n \) colors of \( c \). Thus, the graph induced by \( X_1 \cup X_2 \cup \cdots \cup X_n \) in \( G^r \) is the same as the graph induced by \( X' \) in \( G[X'] \). Since \( G[X'] \) has tree-depth at most \( d_r \cdot n \) and \( |I'| \geq L = f_{T_{d_r,n}}(n,n) \), by Theorem 4.1 \( I' \) contains a rainbow independent set.

We discuss some applications of Theorem 1.2. It is not difficult to see that for every integer \( n > 1 \), the class \( C \) of complete multipartite graphs whose parts have size exactly \( n \) has infinite \( f_C(n,n) \). Thus, for every \( n > 1 \), the class of \( r \)-powers of graphs in a bounded expansion class does not contain all of complete multipartite graphs whose parts have size exactly \( n \).

**Corollary 5.2.** Let \( C \) be a bounded expansion class and \( r \) be a positive integer. For every integer \( n > 1 \), the class \( D = \{ G^r : G \in C \} \) does not contain all of complete multipartite graphs whose parts have size exactly \( n \).

As we discussed in the introduction, map graphs can be obtained as induced subgraphs of 2-powers of planar graphs. Thus, by Theorem 1.2, the class of map graphs has the rainbow property.

**Corollary 5.3.** The class of map graphs has the rainbow property.

We may also observe that induced matchings in a bounded expansion class have the rainbow property. A matching \( M \) is induced if for distinct edges \( a_1b_1, a_2b_2 \in M \), there are no edges between \( \{a_1, b_1\} \) and \( \{a_2, b_2\} \).

**Corollary 5.4.** Every bounded expansion class satisfies the rainbow property for induced matchings.

**Proof.** Let \( C \) be a bounded expansion class, \( n \) be an integer. We obtain \( D \) by taking 1-subdivisions of all graphs in \( C \). It is well known that \( D \) also has bounded expansion. Take \( M := f_{D^2}(n,n) \).
Let $G$ be a graph in $\mathcal{C}$, and $I$ be a family of $M$ induced matchings of size $n$ in $G$. Let $H$ be the 1-subdivision of $G$. For each induced matching in $I$, we correspond an edge $e$ of the matching with the subdivided vertex from $e$, and obtain an independent set in $H$. Let $I^*$ be the family of resulting independent sets.

We observe that for $I \in I^*$ and two distinct vertices $v, w \in I$, $\text{dist}_H(v, w) \geq 6$ as they came from an induced matching of $G$. It means that $I^*$ is a family of independent sets in the 5-power $H^5$ of $H$ as well. As $|I^*| \geq M = f_{D^5}(n, n)$, $I^*$ contains a rainbow independent set $X$ of size $n$ in $H^5$. Now, we obtain an edge set $Y$ in $G$ by taking the original edge from the subdivided vertices contained in $X$. Since $X$ is an independent set of $H^5$, $Y$ is again an induced matching of $G$.

\[\square\]

6 Conclusion

We mainly show that the class of $r$-powers of graphs in a bounded expansion class has the rainbow property for independent sets. As an essential part of the proof, we prove that the class of $r$-powers of graphs of bounded tree-depth has the rainbow property.

As we discussed in the introduction, in general, the class of graphs of bounded clique-width or rank-width has no rainbow property, because of complete multipartite graphs. A natural question is to determine whether $f_C(n, n)$ is finite or not if we further exclude some complete multipartite graph whose all parts have size at most $n$. To clarify the question, we define clique-width, which was introduced by Courcelle and Olariu [5].

A $k$-labeled graph is a pair $(G, \text{lab}_G)$ of a graph $G$ and a function $\text{lab}_G$ from $V(G)$ to $\{1, \ldots, k\}$. Clique-width is based on the following operations:

- creating a graph with a single vertex $x$ labeled with $i \in \mathbb{N}$;
- for a labeled graph $G$ and distinct labels $i, j \in \{1, \ldots, k\}$, relabeling the vertices of $G$ with label $i$ to $j$;
- for a labeled graph $G$ and distinct labels $i, j \in \{1, \ldots, k\}$, adding all the non-existent edges between vertices with label $i$ and vertices with label $j$;
- taking the disjoint union of two labeled graphs $G$ and $H$.

The clique-width of a graph $G$ is the minimum integer $k$ such that $G$ can be constructed by the above four operations using only $k$-labeled graphs. A
A graph class $C$ is hereditary if every graph isomorphic to an induced subgraph of a graph in $C$ is in $C$.

**Question 1.** Let $n \geq 2$ be an integer. Let $C$ be a hereditary class of graphs of bounded clique-width that does not contain all of complete multipartite graphs whose parts have size at most $n$. Is $f_C(n,n)$ finite?

This question is already interesting for cographs, which are the graphs of clique-width at most 2.

We repeat the question for nowhere dense classes in the introduction.

**Question 2.** For every positive integer $r$ and a nowhere dense class $C$, does $C^r$ have the rainbow property?

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