Multiple summation inequalities and their application
to stability analysis of discrete-time delay systems

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Abstract

This paper is devoted to stability analysis of discrete-time delay systems based on a set of Lyapunov-Krasovskii functionals. New multiple summation inequalities are derived that involve the famous discrete Jensen’s and Wirtinger’s inequalities, as well as the recently presented inequalities for single and double summation in [15]. The present paper aims at showing that the proposed set of sufficient stability conditions can be arranged into a bidirectional hierarchy of LMIs establishing a rigorous theoretical basis for comparison of conservatism of the investigated methods. Numerical examples illustrate the efficiency of the method.

Keywords: Summation inequalities, stability analysis, discrete-time delay systems, hierarchy of LMIs

1 Introduction

Time delays are frequently encountered in various real life phenomena, e.g. in physical, industrial, and engineering systems. Since time delays may result in instability and poor performance of the systems, the stability of systems with time delays has received much attention during the past few decades: a comprehensive review can be found e.g. in [1], [2], [23]. (See also the references therein.)

This paper investigates the stability issue of linear discrete-time delay systems described by

\[
\begin{align*}
x(t + 1) &= Ax(t) + A_d x(t - \tau), \quad t = 0, 1, \ldots \\
x(s) &= x_0(s), \quad s = -\tau, -\tau + 1, \ldots, 0,
\end{align*}
\]

where \( x(t) \in \mathbb{R}^{n_x} \) is the state, \( A \) and \( A_d \) are given constant matrices of appropriate size, the time delay \( \tau \) is a known positive integer and \( x_0(.) \) is the initial function. It is well-known that a necessary and sufficient condition can be derived for the stability of (1) by the so called lifting technique in the form of the discrete Lyapunov inequality. This approach, however, suffers from the curse of dimensionality, since the number of decision variables in the LMI to be solved is \((\tau + 1)n_x((\tau + 1)n_x + 1)/2\), which may be too large, if \( \tau \) is large. Therefore, much effort has been devoted to obtain sufficient conditions that

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are computationally less intensive. The Lyapunov-Krasovskii functional (LKF) approach offers a fruitful alternative: during the past decades numerous LKFs have been proposed, and simultaneously, a large number of different techniques has been developed to get better estimations for the forward difference of the functional. Jensen’s inequality (a general form and historical comments can be found in [9], its role in delayed systems is apparent from references [1], [2], [5], [7], [10]-[23], [25]) and Wirtinger’s inequality (see [6] for original version and [4], [7], [10]-[12], [14]-[21], [25], [26] for generalizations) play an outstanding role in this respect. Recently, efficient inequalities for single and double summation have been published in [15] that involves the previous two inequalities. In this way, the derived sufficient conditions yield better and better results in respect of the tolerated delay bound. (See e.g. [7], [10]-[21], [24]-[26] and the references therein.)

The excellent idea of a hierarchy of sufficient LMI stability conditions has been introduced in [19, 20] that gives - among the others - a rigorous basis for comparison.

The aim of the present work is twofold. Firstly, new inequalities will be derived to the case of arbitrary number of summation both for functions and differences. Secondly, a set of LKFs will be proposed, and a hierarchy table of the obtained sufficient stability conditions will be demonstrated. We shall show by some benchmark examples that the proposed methods give better upper bounds for the tolerable time delay than the best ones that we could find in the previously published literature.

The notations applied in the paper are very standard, therefore we mention only a few of them. $P_K$ denotes the space of polynomials having degree not higher than $K$. Symbol $A \otimes B$ denotes the Kronecker-product of matrices $A, B$, while $S_n^+$ is the set of positive definite symmetric matrices of size $n \times n$.

## 2 Multiple summation inequalities

### 2.1 Discrete orthogonal polynomials

Suppose that $m$ and $N$ are given positive integers, and consider the support points $s_i = i$, $(i = 0, 1, \ldots, N - 1)$. For functions $f, g : \mathbb{Z} \rightarrow \mathbb{R}$, define a scalar product by

$$\langle f, g \rangle_m = \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} \ldots \sum_{i_m=0}^{i_{m-1}} f(i_m)g(i_m) = \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} \ldots \sum_{i_m=0}^{i_{m-1}} f(i_m)g(i_m),$$

and denote the corresponding norm by $\|f\|_m$. Since

$$\sum_{j_1=0}^{K} \sum_{j_2=0}^{j_1} \ldots \sum_{j_{m+1}=0}^{j_m} 1 = \binom{K+m+1}{m+1},$$

the scalar product in (2) can equivalently be written as

$$\langle f, g \rangle_m = \sum_{i_m=0}^{N-1} f(i_m)g(i_m) \sum_{i_1=i_m}^{N-1} \sum_{i_2=i_1}^{i_1} \ldots \sum_{i_m-1=i_{m-1}}^{i_{m-2}} 1 = \frac{1}{(m-1)!} \sum_{i=0}^{N-1} r_{N,m-1}(i) f(i)g(i).$$

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where \( r_{N,m}(i) = m! \binom{N-1+m-i}{m} \), \( i = 0, 1, ..., N-1 \). For convenience, we introduce another scalar product by
\[
<f, g>_{m} = \sum_{i=0}^{N-1} r_{N,m-1}(i) f(i) g(i),
\]
and denote the corresponding norm by \( \|f\|_{m} \). Obviously,
\[
\|f\|_{m}^2 = (m-1)! \|f\|_{m}^2.
\]
Let \( p_{ml}(.) \) denote the discrete orthogonal polynomials on \( \mathbb{Z} \cap [0, N-1] \) with respect to the scalar product (2)(or (4)), and with the exact degree \( l = 0, ..., N-1 \). It is well-known that these polynomials can be generated by applying either the Gram-Schmidt orthogonalization process or a three term recurrence relation (see e.g. [3]). If \( m = 1 \), the corresponding orthogonal polynomials are called discrete Chebysh ev polynomials, and they are given by the three term recurrence relation together with their norms e.g. in [3]. However, we are not aware of similar published formulas for \( m > 1 \). Therefore the polynomials \( p_{mj} \) have been generated for \( m > 1 \) using Wolfram Mathematica and the results can be found in [8].

### 2.2 Summation inequalities for functions

Let \( R \in \mathbb{S}_n^+ \) be given. For any \( f : \mathbb{Z} \rightarrow \mathbb{R}^n \), consider the functional
\[
J_{m}(f) = \sum_{i_{1}=0}^{N-1} \sum_{i_{2}=0}^{i_{1}-1} \cdots \sum_{i_{m}=0}^{i_{m-1}-1} f^{T}(i_{m}) R f(i_{m}).
\]

Our aim in this subsection is to derive lower bounds for functionals of this type.

Let \( \nu_{m} \) be given nonnegative integers satisfying condition \( \nu_{m} < \nu_{1} < N \). Set \( \bar{w}_{mj} = < f, p_{mj} >_m \) and \( w_{mj} = < f, p_{mj} >_m \), where \( j = 0, ..., \nu_{m} \), and the scalar product is taken componentwise. Clearly, \( w_{mj} = (m-1)! \bar{w}_{mj} \). In what follows, \( w_{1j} \) will play a key role, therefore we shall denote it specially
\[
\phi_{j} = w_{1j}, \quad j = 0, 1, ..., \nu_{1}.
\]
Suppose that \( \nu_{m} + m - 1 \leq \nu_{1} \) and set \( q_{N,m+j-1}(i) = r_{N,m-1}(i)p_{mj}(i) \). Then \( q_{N,m+j-1} \in \mathbb{P}_{\nu_{1}} \) with degree \( m + j - 1 \leq \nu_{m} + m - 1 < \nu_{1} \). Since the polynomials \( p_{l}, l = 0, ..., \nu_{1} \) form a basis of \( \mathbb{P}_{\nu_{1}} \), we have
\[
q_{N,m+j-1}(i) = \sum_{l=0}^{m+j-1} \xi_{j,l}^{m} p_{l}(i).
\]

Introduce now the row vector of length \( \nu_{1} + 1 \)
\[
\xi_{j}^{m} = [\xi_{j,0}^{m} \xi_{j,1}^{m} \cdots \xi_{j,\nu_{1}}^{m}]
\]
the first \( j + m \) entries of which equal to the coefficients in (7), while \( \xi_{j,l}^{m} = 0 \) if \( j + m - 1 < l \leq \nu_{1} \).
Then we have
\[ w_{mj} = \sum_{l=0}^{\nu_1} \xi^m_{j,l} \sum_{i=0}^{N-1} p_l(i) f(i) = \sum_{l=0}^{\nu_1} \xi^m_{j,l} \phi_l. \] (8)

Introduce the notations:
\[ \Xi_m = \begin{pmatrix} \xi^m_{0,0} & \cdots & \xi^m_{0,\nu_1} \\ \vdots & \ddots & \vdots \\ \xi^m_{\nu_m,0} & \cdots & \xi^m_{\nu_m,\nu_1} \end{pmatrix} \in \mathbb{R}^{(\nu_m+1) \times (\nu_1+1)}, \quad \Phi = \text{col} \{ \phi_0, \ldots, \phi_{\nu_1} \}, \] (9)
\[ R_m = \text{diag} \{ \chi_{0,R}, \ldots, \chi_{\nu_m,R} \}, \quad \chi_{m,j} = \frac{1}{\|p_{mj}\|^2_m}. \] (10)

**Theorem 1.** Let \( m, \nu_1, \nu_m, N \) be given integers satisfying conditions \( m \geq 1 \) and \( \nu_m < \nu_1 < N \). Let matrix \( R \in S^n_+ \) and function \( f: \mathbb{Z} \to \mathbb{R}^n \) be given. Then the following inequality holds:
\[ J_m(f) \geq \frac{1}{(m-1)!} \Phi^T (\Xi_m \otimes I)^T R_m (\Xi_m \otimes I) \Phi. \] (11)

**Proof.** Let \( \mu_j \) denote arbitrary constants and set
\[ z(i) = f(i) - \sum_{j=0}^{\nu_m} \mu_j \tilde{w}_{mj} p_{mj}(i). \]

Then we obtain
\[ 0 \leq J_m(z) = J_m(f) - \sum_{j=0}^{\nu_m} (2\mu_j - \mu_j^2 \|p_{mk}\|^2_m) \tilde{w}_{mj}^T R \tilde{w}_{mj}. \] (12)

The term \( 2\mu_j - \mu_j^2 \|p_{mk}\|^2_m \) takes its maximum at \( \mu_j^* = \frac{1}{\|p_{mk}\|^2_m} \). By rearranging (12) and by substituting \( \mu_j^* \) one obtains that
\[ J_m(f) \geq \sum_{j=0}^{\nu_m} \frac{1}{\|p_{mk}\|^2_m} \tilde{w}_{mj}^T R \tilde{w}_{mj}. \] (13)

Now, one has only to take into account (5), (8)-(10) and the relation between \( \tilde{w}_{mj} \) and \( w_{mj} \) to obtain (11).

From Theorem 1, one can easily derive several known summation inequalities such as the discrete Jensen inequality (see e.g. [5], [7], [9], [11], [14], [15], [21], [25]), the discrete Wirtinger inequality ([7], [11], [14], [15]), and the inequalities of Nam, Trinh and Pathirana [15]. As it is usual in the literature of stability analysis (see e.g. [7]), the lower estimations of type (11) are given by the single, double, triple, etc. summation of the state variable. We follow this line too, when formulating the corollaries of Theorem 1.

**Corollary 1.** If \( m = 1 \), and \( f(i) = x(i) \), (11) implies
- for \( \nu_1 = 0 \) the Jensen inequality
  \[ J_1(x) \geq \frac{1}{N} \Omega_{10}^T R \Omega_{10} \quad \text{with} \quad \Omega_{10} = \sum_{i_1=0}^{N-1} x(i_1); \] (14)
• for $\nu_1 = 1$, $N > 1$, the Wirtinger inequality

$$J_1(x) \geq \frac{1}{N} \left\{ \Omega_{10}^T R \Omega_{10} + 3 \frac{N + 1}{N - 1} \Omega_{11}^T R \Omega_{11} \right\}$$

with

$$\Omega_{11} = \sum_{i_1=0}^{N-1} x(i_1) - \frac{2}{N + 1} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} x(i_2); \quad (15)$$

• for $\nu_1 = 2$, $N > 1$, the inequality

$$J_1(x) \geq \frac{1}{N} \left\{ \Omega_{10}^T R \Omega_{10} + 3 \frac{N + 1}{N - 1} \Omega_{11}^T R \Omega_{11} + 5 \frac{(N + 1)(N + 2)}{(N - 1)(N - 2)} \Omega_{12}^T R \Omega_{12} \right\}$$

with

$$\Omega_{12} = \sum_{i_1=0}^{N-1} x(i_1) - \frac{6}{N + 1} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} x(i_2) + \frac{12}{(N + 1)(N + 2)} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} x(i_3). \quad (17)$$

Proof. Since the first three discrete Chebyshev polynomials and their norms are known to be

$$p_{10}(x) \equiv 1, \quad \|p_{10}\|_1^2 = N, \quad (19)$$

$$p_{11}(x) = 2x + 1 - N, \quad \|p_{11}\|_1^2 = \frac{N(N^2 - 1)}{3}, \quad (20)$$

$$p_{12}(x) = 6x^2 - 6(N - 1)x + (N - 1)(N - 2), \quad \|p_{12}\|_1^2 = \frac{(N^2 - 4)(N^2 - 1)N}{5}, \quad (21)$$

(see e.g. [3]), the proof consists of some straightforward but lengthy computations the details of which are omitted. □

Remark 1. Estimation (17)-(18) is less conservative than that in Lemma 1 of [7], and it is identical with that of [15], equation (27).

Corollary 2. If $m = 2$, and $f(i) = x(i)$, (11) implies

• for $\nu_2 = 0$, inequality

$$J_2(x) \geq \frac{2}{N(N + 1)} \Omega_{20}^T R \Omega_{20} \quad \text{with} \quad \Omega_{20} = \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} x(i_2); \quad (22)$$

• for $\nu_2 = 1$, $N > 1$, inequality

$$J_2(x) \geq \frac{2}{N(N + 1)} \left\{ \Omega_{20}^T R \Omega_{0} + 8 \frac{N + 2}{N - 1} \Omega_{21}^T R \Omega_{21} \right\}$$

with

$$\Omega_{21} = \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} x(i_2) - \frac{3}{N + 2} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} x(i_3). \quad (24)$$
Applying the first two orthogonal polynomials and their norms

\[
p_{20}(x) \equiv 1, \quad \|p_{20}\|_2^2 = \frac{N(N + 1)}{2},
\]
\[
p_{21}(x) = x + \frac{1 - N}{3}, \quad \|p_{21}\|_2^2 = \frac{(N - 1)N(N + 1)(N + 2)}{36},
\]
the proof consists of some straightforward but lengthy computations the details of which are omitted. □

Remark 2. Estimation (23) is equivalent to that of Lemma 2 in [7].

2.3 Summation inequalities for differences

Based on Theorem 1, we want to derive lower estimations for the functional \( J_m \) applied to the forward difference of a function. In doing so, let \( \rho \) be given by function \( f : \mathbb{Z} \to \mathbb{R}^n \) as

\[
\rho : \mathbb{Z} \to \mathbb{R}^n, \quad \rho(i) = f(i + 1) - f(i), \quad i = 0, 1, \ldots, N - 1.
\]

**Theorem 2.** Let \( m, \nu_1, \nu_m, N \) be given integers satisfying condition \( \nu_m + m - 1 \leq \nu_1 < N \). Let \( R \in \mathbb{S}_n^+ \) be given and let function \( \rho \) defined by (25). Then the following inequality holds:

\[
J_m(\rho) \geq \frac{1}{(m - 1)!} \Phi^T (\mathbb{Z}_m \otimes I)^T R_m (\mathbb{Z}_m \otimes I) \Phi,
\]

where \( \Phi = \text{col} \{ f(N), f(0), \phi_0, \ldots, \phi_{\nu_1-1} \} \), \( R_m \) is defined by (10) and \( \mathbb{Z}_m \) is given by (31) below.

**Proof.** Introduce the notation \( \theta_{mj} = \langle \rho, p_{mj} \rangle_m, \quad j = 0, \ldots, m \), where the scalar product is taken componentwise. From (12), it follows immediately that

\[
J_m(\rho) \geq \frac{1}{(m - 1)!} \sum_{j=0}^{\nu_m} \frac{1}{\|p_{mj}\|_2^2} \theta_{mj}^T R\theta_{mj}.
\]

The vectors \( \theta_{mj} \) have to be expressed by vectors \( \phi_l = w_{1l} \).

\[
\theta_{mj} = \langle \rho, p_{mj} \rangle_m = \sum_{i=0}^{N-1} r_{N,m-1}(i)p_{mj}(i)f(i + 1) - \sum_{i=0}^{N-1} r_{N,m-1}(i)p_{mj}(i)f(i)
\]
\[
= r_{N,m-1}(N - 1)p_{mj}(N - 1)f(N) - r_{N,m-1}(1)p_{mj}(1)f(0)
\]
\[
+ \sum_{i=0}^{N-1} [r_{N,m-1}(i - 1)p_{mj}(i - 1) - r_{N,m-1}(i)p_{mj}(i)] f(i).
\]

Apply now the notation

\[
\tilde{q}_{N,m+j-2}(i) = r_{N,m-1}(i - 1)p_{mj}(i - 1) - r_{N,m-1}(i)p_{mj}(i).
\]

The degree of polynomial (28) is exactly \( m + j - 2 \), and \( m + j - 2 \leq m + \nu_1 - 2 \leq \nu_1 - 1 \), therefore there exist coefficients \( \zeta_{j,l}^m \) such that

\[
\tilde{q}_{N,m+j-2}(i) = \sum_{l=0}^{m+j-2} \zeta_{j,l}^m p_{1l}(i).
\]
Set \( c_{m,j,0} = -r_{N,m-1}(-1)p_{mj}(-1) \) and \( c_{m,j,1} = r_{N,m-1}(N-1)p_{mj}(N-1) \). Then we have

\[
\theta_{m,j} = c_{m,j,1} f(N) + c_{m,j,0} f(0) + \sum_{l=0}^{m+j-2} \zeta_{j,l}^m \phi_l. \tag{30}
\]

Introduce now the row vector of length \( \nu_1 + 2 \)

\[
\zeta_j^m = \begin{bmatrix} c_{m,j,1} & c_{m,j,0} & \zeta_{j,0} & \cdots & \zeta_{j,\nu_1-1} \end{bmatrix},
\]

where entries \( \zeta_{j,l}^m \) equal to the corresponding coefficients of (29) for \( 0 \leq l \leq m+j-2 \), and \( \zeta_{j,l}^m = 0 \) is taken for \( m+j-2 < l \leq \nu_1 - 1 \).

We note that \( \zeta_{\nu_m,m+\nu_m-2}^m \neq 0 \), since the exact degree of \( \tilde{q}_{N,m+\nu_m-2} \) is \( m+\nu_m-2 \), and \( \zeta_{\nu_m,m+\nu_m-2}^m \) is the coefficient of the basis element \( p_{1,m+\nu_m-2} \). Set

\[
Z_m = \begin{pmatrix} \zeta_0^m \\ \vdots \\ \zeta_{\nu_m}^m \end{pmatrix} \in \mathbb{R}^{(\nu_m+1) \times (\nu_1+2)}. \tag{31}
\]

Then (26) immediately follows from (27), (30) and (31).

**Corollary 3.** If \( m = 1 \), and \( f(i) = x(i) \), (26) implies

- for \( \nu_1 = 0 \) the discrete Jensen inequality for differences

\[
J_1(\rho) \geq \frac{1}{N} \tilde{\Omega}_{10}^T R \tilde{\Omega}_{10} \quad \text{with} \quad \tilde{\Omega}_{10} = x(N) - x(0); \tag{32}
\]

- for \( \nu_1 = 1, N > 1 \), the discrete Wirtinger inequality for differences

\[
J_1(\rho) \geq \frac{1}{N} \left\{ \tilde{\Omega}_{10}^T R \tilde{\Omega}_{10} + 3 \frac{N+1}{N-1} \tilde{\Omega}_{11}^T R \tilde{\Omega}_{11} \right\}, \tag{33}
\]

with

\[
\tilde{\Omega}_{11} = x(N) + x(0) - \frac{2}{N+1} \sum_{i_1=0}^{N} x(i_1). \tag{34}
\]

**Proof.** We calculate with the polynomials (19)-(20) again: (32) is evident, while (33)-(34) follows by taking into account that

\[
\theta_{1,1} = (N-1)x(N) + (N+1)x(0) - 2 \sum_{i_1=0}^{N-1} x(i_1). \tag{30}
\]

**Remark 3.** The discrete Jensen inequality has a long history, it has been applied in a huge number of works. In contrast, the discrete Wirtinger inequality for differences has been developed very recently: it has been published independently by several authors in slightly different forms in works [7], [11], [14], [15], [21]; inequality (33)-(34) is identical with that of [21], and it is equivalent to all others.
3 Stability analysis of discrete delayed systems

The stability of the discrete delayed systems will be analyzed in this section by a set of quadratic Lyapunov-Krasovskii functionals (LKFs) applying estimations (27). In the past decades, numerous different LKFs have been proposed and various techniques have been applied to reduce the conservatism of the results. The effectiveness of the different LKFs are often tested by benchmark examples. Our purpose is to evaluate systematically the reduction of conservativeness when applying estimations (27). Similarly to the useful approach of [20], we will establish a hierarchy of linear matrix inequality (LMI) stability conditions based on estimation of the forward difference of the LKF. These LMIs depend both on the parameter \( m \) of the scalar product (2) and on the highest degree \( \nu_m \) of the approximating orthogonal polynomials. The different cases corresponding to pairs \( (m, \nu_m) \) will be theoretically compared.

Consider the linear discrete time-delay system described by (1).
Let us choose a positive integer \( m \) and nonnegative integers \( \nu_1 > \nu_2 > \ldots > \nu_m \geq 0 \). Introduce the notation

\[
x_{l-\tau}(i) = x(t - \tau + i), \quad \text{for} \quad i = 0, 1, \ldots, \tau - 1,
\]

and set \( \phi_j(t) = \sum_{i=0}^{\tau-1} p_{ij}(i)x_{l-\tau}(i) \). For \( \nu_1 \geq 1 \), consider the extended and the augmented state variables

\[
\bar{x}(t) = \begin{bmatrix} x(t), \phi_0(t), \ldots, \phi_{\nu_1-1}(t) \end{bmatrix} \in \mathbb{R}^{n_x(\nu_1+1)},
\]

\[
\bar{\Phi}(t) = \begin{bmatrix} x(t), x(t - \tau), 1 \tau \phi_0(t), \ldots, 1 \tau \phi_{\nu_1-1}(t) \end{bmatrix} \in \mathbb{R}^{n_x(\nu_1+2)},
\]

respectively, and for \( \nu_1 = 0 \), set

\[
\bar{x}(t) = x(t), \quad \bar{\Phi}(t) = \begin{bmatrix} x(t), x(t - \tau) \end{bmatrix}.
\]

Several further notations are needed for deriving the stability result. Set

\[
e_i = \begin{bmatrix} 0_{n_x \times n_x(i-1)}, I_{n_x}, 0_{n_x \times n_x(\nu_1+2-i)} \end{bmatrix}, \quad i = 1, 2,
\]

\[
A = Ae_1 + A_d e_2,
\]

\[
\Gamma_{\nu_1} = \left( \text{diag} \{ [1, 0], I_{\nu_1} \} T_{\nu_1} \right) \otimes I_{n_x}, \quad \bar{Z}_k = Z_k T_{\nu_1}, \quad k = 1, \ldots, m
\]

\[
\Lambda_l = \left( [c_{1,l,0} \lambda_{l,0}, \ldots, \lambda_{l,\nu_1-1}] T_{\nu_1} \right) \otimes I_{n_x},
\]

where \( Z_k \) is defined by (31), \( c_{1,l,0} = -p_{1l}(-1), c_{1,l,1} = p_{1l}(\tau - 1) \) as in the previous section, while \( \lambda_{l,s} \) is defined by the relation

\[
p_{1l}(i-1) = \sum_{s=0}^{l} \lambda_{l,s} p_{1s}(i),
\]

if \( 0 \leq s \leq l \) and \( \lambda_{l,s} = 0 \), if \( l < s \leq \nu_1 \). Finally set

\[
\Lambda_{\nu_1} = \left( A^T, \Lambda_{0}^T, \ldots, \Lambda_{\nu_1-1}^T \right)^T.
\]

**Theorem 3.** System (1) is asymptotically stable, if there are matrices \( P \in \mathbb{S}^+_{n_x(\nu_1+1)}, Q, R_j \in \mathbb{S}^+_{n_x}, j = 1, \ldots, m \) such that the LMI

\[
\Psi_{\nu_1}^1 + \Psi_{m,\nu_1}^2 - \Psi_{m,\nu_1,\ldots,\nu_m}^3 < 0,
\]

(37)
Let us introduce
Proof.

Furthermore, for terms (42)-(44) by \( \Delta \)

\[
\Psi_{m_1,\ldots,m_n} = \sum_{k=1}^{m} \left( \frac{1}{(k-1)!} \right) \left( \tilde{z}_k \otimes I \right)^T R_k \left( \tilde{z}_k \otimes I \right)
\]

with \( Q_{\nu_1} = \text{diag}\{Q, -Q, 0_{\nu_1 \times \nu_1}\} \) and \( R_k \) given by (31).

**Proof.** Let us introduce

\[
\rho_{t-\tau}(s) = x_{t+1-\tau}(s) - x_{t-\tau}(s) = x(t + 1 - \tau + s) - x(t - \tau + s),
\]

for \( s = 0, 1, \ldots, \tau - 1 \). Consider the Lyapunov-Krasovskii functional candidate

\[
V(x_{t-\tau}, \rho_{t-\tau}) = V_{00}(x_{t-\tau}) + V_{01}(x_{t-\tau}) + \sum_{j=1}^{m} V_{ij}(\rho_{t-\tau}),
\]

where

\[
V_{00}(x_{t-\tau}) = \tilde{x}^T(t) P \tilde{x}(t),
\]

\[
V_{01}(x_{t-\tau}) = \sum_{s=0}^{\tau-1} x_{t-\tau}^T(s) Q x_{t-\tau}(s),
\]

\[
V_{1,j}(\rho_{t-\tau}) = \sum_{i_1=0}^{\tau-1} \ldots \sum_{i_j=0}^{\tau-1} \rho_{t-\tau}^T(s) R_j \rho_{t-\tau}(s).
\]

Clearly, \( V \) is positive definite. Compute the forward difference of the LKF (41) along the solution of (1), denoting the forward difference of \( V \) by \( \Delta v \) and the forward differences of terms (42)-(44) by \( \Delta v_{1,j} \). It can easily be verified that

\[
\tilde{x}(t) = \Gamma_{\nu_1} \Phi(t),
\]

\[
\tilde{x}(t + 1) = \Delta_{\nu_1} \Phi(t),
\]

therefore one obtains

\[
\Delta v_{00}(t) = \tilde{x}^T(t + 1) P \tilde{x}(t + 1) - \tilde{x}^T(t) P \tilde{x}(t) = \Phi^T(t) \left[ \Delta_{\nu_1}^T P \Delta_{\nu_1} - \Gamma_{\nu_1}^T P \Gamma_{\nu_1} \right] \Phi(t).
\]

Furthermore,

\[
\Delta v_{01}(t) = x^T(t) Q x(t) - x^T(t - \tau) Q x(t - \tau) = \Phi^T(t) Q_{\nu_1} \Phi(t),
\]

and

\[
\Delta v_{1k}(t) = \sum_{i_1=0}^{\tau-1} \ldots \sum_{i_k=0}^{\tau-1} \left( \rho_{t-\tau}^T(\tau) R_k \rho_{t-\tau}(\tau) - \rho_{t-\tau}^T(i_k) R_k \rho_{t-\tau}(i_k) \right) =
\]

\[
= \rho_{t-\tau}^T(\tau) R_k \rho_{t-\tau}(\tau) \sum_{i_1=0}^{\tau-1} \ldots \sum_{i_k=0}^{\tau-1} 1 - \sum_{i_1=0}^{\tau-1} \ldots \sum_{i_k=0}^{\tau-1} \rho_{t-\tau}^T(i_k) R_k \rho_{t-\tau}(i_k) =
\]

\[
= \left( \frac{\tau - 1 + k}{k} \right) \Phi^T(t) (A - \epsilon_1)^T R_k (A - \epsilon_1) \Phi - J_k(\rho_{t-\tau}).
\]
For the estimation of $J_k(\rho_{t-\tau})$ we apply Theorem 2. By taking into account (36) and the definition of $\tilde{Z}_k$, one obtains that

$$J_k(\rho_{t-\tau}) \geq \frac{1}{(k-1)!} \tilde{\Phi}^T(t) \left( \tilde{Z}_k \otimes I \right)^T \mathcal{R}_k \left( \tilde{Z}_k \otimes I \right) \tilde{\Phi}(t).$$

Therefore,

$$\Delta v(t) \leq \tilde{\Phi}^T(t) \left\{ \Psi_{v_1} + \Psi_{m,v_1} - \Psi_{m,v_2,...,v_m} \right\} \tilde{\Phi}(t),$$

i.e. if (37) holds true, there exist a constant $\varepsilon > 0$ such that $\Delta v(t) \leq -\varepsilon \|x(t)\|^2$, thus the statement of the theorem follows. □

**Remark 4.** Matrices $A_{v_1}$, $Z_k$, together with the generating Wolfram Mathematica programs are given in [8] for several $v_1$, $k$ and $v_k$.

## 4 Hierarchy of the LMI stability conditions

This section is devoted to proving that the stability conditions (37) obtained for different choices of $m$ and $v_1 > v_2 > \ldots > v_m \geq 0$ can be arranged into a hierarchy table.

**Definition 1.** Let $\mathcal{L}^{(i)}$ denote LMI (37) obtained with the choice of $m^{(i)}, v_1^{(i)}, v_2^{(i)}, \ldots, v_m^{(i)}$, $(i = 1, 2)$. LMI $\mathcal{L}^{(2)}$ outperform LMI $\mathcal{L}^{(1)}$ if and only if for any $\tau$, for which $\mathcal{L}^{(1)}$ has a feasible solution, LMI $\mathcal{L}^{(2)}$ has it, as well. This relation will be denoted by $\mathcal{L}^{(1)} < \mathcal{L}^{(2)}$.

Let symbol $\mathcal{L}^{m}_{v_1,...,v_m}$ denote LMI (37). Let us arrange these LMIs into a hierarchy table as follows:

$$\begin{array}{cccc}
\mathcal{L}_0^1 & \mathcal{L}_1^1 & \mathcal{L}_2^1 & \ldots \\
\mathcal{L}_{1,0}^1 & \mathcal{L}_2^2 & \ldots \\
\mathcal{L}_{2,1,0}^2 & \mathcal{L}_{v_1,v_1-1,1}^3 & \ldots \\
\ldots & \ldots & \ldots \\
\mathcal{L}_{v_1,v_1-1,...,v_1-(m-1)}^m
\end{array}$$

Note that this table is finite for any given $\tau$ both to the right and to the bottom, since $\tau$ determines the maximal degree of the orthogonal polynomials that can be considered.

We want to show that a certain LMI in this table outperforms any other LMI, which is situated above and/or to the left of it.

**Theorem 4.** Let the integers $1 \leq m$ and $m < v_1^*$ be given. Then

- for $1 \leq \ell \leq m$ and $\ell - 1 \leq v_1 < v_1^*$,

$$\mathcal{L}_{v_1,v_1-1,...,v_1+1-\ell}^\ell < \mathcal{L}_{v_1+1,v_1,...,v_1+2-\ell}^\ell ;$$

- for $1 \leq \ell < m$ and $\ell - 1 \leq v_1 \leq v_1^*$,

$$\mathcal{L}_{v_1,v_1-1,...,v_1+1-\ell}^\ell < \mathcal{L}_{v_1,v_1-1,...,v_1+1-\ell,v_1-\ell}^{\ell+1}.$$  

**Proof.** Part I. First we show that (47) is valid. Suppose that $\mathcal{L}_{v_1,v_1-1,...,v_1+1-\ell}^\ell$ has the feasible solution $P^{(1)}$, $Q^{(1)}$, $R_1^{(1)}, \ldots, R_\ell^{(1)}$ for a given $\tau$. We show that, for the same value of $\tau$, there exists a feasible solution $P^{(2)}$, $Q^{(2)}$, $R_1^{(2)}, \ldots, R_\ell^{(2)}$ of $\mathcal{L}_{v_1,v_1-1,...,v_1+2-\ell}^\ell$. In what
follows, we shall use the upper index "(1)" and "(2)" to distinguish the matrices occurring in the first and in the second LMI, respectively.

Let us seek $P^{(2)}$ in the form of $P^{(2)} = \text{diag} \{ P^{(1)}, \varepsilon I_{n_x} \}$, where $\varepsilon$ is some positive constant, while we set $Q^{(2)} = Q^{(1)}$, $R^{(2)}_1 = R^{(1)}_1$, ..., $R^{(2)}_\ell = R^{(1)}_\ell$. Calculate first matrix $\Psi^{(2)}_{\nu_1+1}$. Observe that

$$Q^{(2)}_{\nu_1+1} = \begin{bmatrix} Q^{(1)} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma^{(2)}_{\nu_1+1} = \begin{bmatrix} \Gamma^{(1)} & 0 \\ 0 & \tau I_{n_x} \end{bmatrix}, \quad \Lambda^{(2)}_{\nu_1+1} = \begin{bmatrix} \Lambda^{(1)}_{\nu_1+1,1} & 0 \\ \Lambda^{(1)}_{\nu_1+1,2} & \Lambda^{(2)}_{\nu_1+1,2} \end{bmatrix},$$

(49)

where

$$\Lambda^{(1)}_{\nu_1+1,1} = (c_{1,\nu_1,1}, c_{1,\nu_1,0}, \lambda_{\nu_1,0}, \ldots, \lambda_{\nu_1-1}) T_{\nu_1} \otimes I, \quad \text{and} \quad \Lambda^{(1)}_{\nu_1+1,2} = \tau I_{n_x}.$$

By straightforward calculation we obtain

$$\Delta T^{(2)}_{\nu_1+1} P^{(2)}_{\nu_1+1} = \Delta^{(1)}_{\nu_1} P^{(1)}_{\nu_1} \Lambda^{(1)}_{\nu_1+1} + \varepsilon \begin{bmatrix} 0 & (\Lambda^{(2)}_{\nu_1+1})^T \\ 0 & \Lambda^{(1)}_{\nu_1+1} \Lambda^{(2)}_{\nu_1+1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \Lambda^{(1)}_{\nu_1+1} & \Lambda^{(2)}_{\nu_1+1} \end{bmatrix},$$

(50)

and

$$\Gamma^{(2)}_{\nu_1+1} P^{(2)}_{\nu_1+1} = \begin{bmatrix} \Gamma^{(1)}_{\nu_1} P^{(1)}_{\nu_1} & 0 \\ 0 & \varepsilon \tau I \end{bmatrix},$$

(51)

As far as $\Psi^2_{\ell,\nu_1+1}$ is concerned observe that

$$\Psi^2_{\ell,\nu_1+1} = \begin{bmatrix} \Psi^{(2)}_{\ell,\nu_1} & 0 \\ 0 & 0 \end{bmatrix},$$

(52)

Finally consider matrix $\Psi^3_{\ell,\nu_1+1,1,\ldots,\nu_1+2-\ell}$:

$$\Psi^3_{\ell,\nu_1+1,1,\ldots,\nu_1+2-\ell} = \sum_{k=1}^\ell \frac{1}{(k-1)!} \left( \tilde{Z}^{(2)}_k \otimes I \right)^T \mathcal{R}^{(2)}_k \left( \tilde{Z}^{(2)}_k \otimes I \right),$$

where

$$\mathcal{R}^{(2)}_k = \begin{bmatrix} \mathcal{R}^{(1)}_k & 0 \\ 0 & \chi_{k,\nu_1-k+2} R_k \end{bmatrix}, \quad \tilde{Z}^{(2)}_k = \begin{bmatrix} \tilde{Z}^{(1)}_k \\ \tilde{\zeta}^{k}_{\nu_1-k+2,1} \tilde{\zeta}^{k}_{\nu_1-k+2,2} \end{bmatrix}$$

with

$$\tilde{\zeta}^{k}_{\nu_1-k+2,1} = (c_{k,\nu_1-k+2,1}, c_{k,\nu_1-k+2,0}, \zeta^{k}_{\nu_1-k+2,0}, \ldots, \zeta^{k}_{\nu_1-k+2,\nu_1-1}) T_{\nu_1}$$

and

$$\tilde{\zeta}^{k}_{\nu_1-k+2,2} = \zeta^{k}_{\nu_1-k+2,\nu_1}. $$

We recall that $\zeta^{k}_{\nu_1-k+2,\nu_1} \neq 0$, (see the comment above (31)). By straightforward calculation we obtain

$$(\tilde{Z}^{(2)}_k \otimes I)^T \mathcal{R}^{(2)}_k (\tilde{Z}^{(2)}_k \otimes I) = \begin{bmatrix} (\tilde{Z}^{(1)}_k \otimes I)^T \mathcal{R}^{(1)}_k (\tilde{Z}^{(1)}_k \otimes I) & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \chi_{k,\nu_1-k+2} R_k \tilde{\zeta}^{k}_{\nu_1-k+2,1} \\ \tilde{\zeta}^{k}_{\nu_1-k+2,2} \end{bmatrix}^\top \chi_{k,\nu_1-k+2} R_k \begin{bmatrix} \tilde{\zeta}^{k}_{\nu_1-k+2,1} \\ \tilde{\zeta}^{k}_{\nu_1-k+2,2} \end{bmatrix}.$$
Since \( \chi_{k+1} R_k > 0 \), the second term on the right hand side of (53) is nonnegative. Therefore, matrix \( \Psi_{\ell,\nu_1+1,\nu_1+1-\ell} \) can be estimated from below by keeping only one of the second terms at summation, e.g. when \( k = 1 \). One obtains

\[
\Psi_{\ell,\nu_1+1,\nu_1+1-\ell} \geq \begin{bmatrix}
\Psi_{\ell,\nu_1,\nu_1-1,...,\nu_1+1-\ell} \\
0 \\
0
\end{bmatrix} + \chi_{1,\nu_1+1} R_1 \begin{bmatrix}
\tilde{\zeta}_{\nu_1+1,1} \\
\tilde{\zeta}_{\nu_1+1,2}
\end{bmatrix} \end{bmatrix}.
\]

To be short, let us use the notations

\[
\mathcal{M}^{(1)} = \Psi_{1,\nu_1} + \Psi_{\ell,\nu_1} - \Psi_{\ell,\nu_1,\nu_1-1,...,\nu_1+1-\ell}
\]

and

\[
\mathcal{M}^{(2)} = \Psi_{\ell,\nu_1+1} + \Psi_{\ell,\nu_1+1-1,...,\nu_1+1-\ell}.
\]

Employing (50)-(54) we obtain for \( \mathcal{L}_{\nu_1+1,\nu_1+1-1,...,\nu_1+2} \) that

\[
\mathcal{M}^{(2)} \leq \begin{bmatrix}
I & (\tilde{\zeta}_{\nu_1+1,1} \otimes I)^T \\
0 & \tilde{\zeta}_{\nu_1+1,2}
\end{bmatrix} \begin{bmatrix}
\mathcal{M}^{(1)} & 0 \\
0 & -\chi_{1,\nu_1+1} R_1
\end{bmatrix} \begin{bmatrix}
\tilde{\zeta}_{\nu_1+1,1} \\
\tilde{\zeta}_{\nu_1+1,2}
\end{bmatrix} + \varepsilon \begin{bmatrix}
0 & \left( \Lambda_{\nu_1+1}^{(1)} \right)^T \\
0 & \Lambda_{\nu_1+1}^{(2)}
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
\Lambda_{\nu_1+1}^{(1)} & \Lambda_{\nu_1+1}^{(2)}
\end{bmatrix}.
\]

Since \( \tilde{\zeta}_{\nu_1+1,2} \neq 0 \), the two extreme multiplier matrices in the first term on the right hand side of (55) are invertible. Therefore, if \( \mathcal{L}_{\nu_1+1,\nu_1+1-1,...,\nu_1+2-\ell} \) has a feasible solution, there exists a \( \mu_1 > 0 \) such that the first term on the right hand side of (55) is less than \( -\mu_1 I \).

On the other hand, there exists a constant \( \lambda^* \) such that the matrix product on the right hand side of (55) can be estimated as

\[
\begin{bmatrix}
0 & \left( \Lambda_{\nu_1+1}^{(1)} \right)^T \\
0 & \Lambda_{\nu_1+1}^{(2)}
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
\Lambda_{\nu_1+1}^{(1)} & \Lambda_{\nu_1+1}^{(2)}
\end{bmatrix} \leq \lambda^* I,
\]

therefore \( \mathcal{M}^{(2)} < 0 \) is satisfied, if \( \varepsilon \lambda^* < \mu_1 \). This proofs that (47) is true.

Part II. Next we show that one can move downwards in the hierarchy, too, if \( \ell \) is increased with fixed \( \nu_1 \), i.e. \( \mathcal{L}_{\nu_1,\nu_1-1,...,\nu_1+1-\ell} < \mathcal{L}_{\nu_1,\nu_1-1,...,\nu_1+1-\ell,\nu_1-\ell} \). We recalculate the terms (38)-(40) again in this case. We obtain

\[
\Psi_{1,\nu_1+1} = \Psi_{1,\nu_1},
\]

\[
\Psi_{2,\nu_1+1} = \Psi_{2,\nu_1} + \left( \tau - 1 + \ell + 1 \right) \left(A - e_1\right)^T R_{\ell+1} \left(A - e_1\right),
\]

\[
\Psi_{3,\nu_1,\nu_1-1,...,\nu_1+1-\ell} = \Psi_{3,\nu_1,\nu_1-1,...,\nu_1+1-\ell} + \frac{1}{\ell!} \left( \tilde{\zeta}_{\ell+1} \otimes I \right)^T R_{\ell+1} \left( \tilde{\zeta}_{\ell+1} \otimes I \right).
\]

Analogously to Part I, we shall use the brief notation

\[
\mathcal{N}^{(1)} = \Psi_{1,\nu_1} + \Psi_{2,\nu_1} - \Psi_{3,\nu_1,\nu_1-1,...,\nu_1+1-\ell}
\]

and

\[
\mathcal{N}^{(2)} = \Psi_{1,\nu_1} + \Psi_{2,\nu_1} - \Psi_{3,\nu_1,\nu_1-1,...,\nu_1+1-\ell}.
\]
Then \( \mathcal{N}^{(2)} \) can be expressed as \( \mathcal{N}^{(2)} = \mathcal{N}^{(1)} + \Upsilon \), where

\[
\Upsilon = \left( \frac{\tau - 1 + \ell + 1}{\ell + 1} \right) (A - e_1)^T R_{m+1} (A - e_1) \\
- \frac{1}{\ell!} \left( \tilde{Z}_{\ell+1} \otimes I \right)^T \mathcal{R}_{\ell+1} \left( \tilde{Z}_{\ell+1} \otimes I \right).
\]

Let us seek \( R_{\ell+1} \) in the form of \( R_{\ell+1} = \varepsilon I \). Then there is a \( \lambda^{**} \) such that \( \Upsilon \) can be estimated as

\[
\Upsilon = \varepsilon \left\{ \left( \frac{\tau - 1 + m + 1}{m + 1} \right) (A - e_1)^T (A - e_1) \\
- \frac{1}{\ell!} \left( \tilde{Z}_{\ell+1} \otimes I \right)^T \text{diag} \{\chi_{\ell,0} I, \ldots, \chi_{\ell+1,\nu_1+1-\ell} I\} \left( \tilde{Z}_{\ell+1} \otimes I \right) \right\} \leq \varepsilon \lambda^{**} I.
\]

If \( \mathcal{L}^{\ell}_{\nu_1,\nu_1-1,\ldots,\nu_1+1-\ell} \) has a feasible solution, then there exists a \( \mu_2 > 0 \) such that

\[ \mathcal{N}^{(1)} < -\mu_2 I, \]

therefore \( \mathcal{N}^{(2)} < 0 \) is satisfied if \( \varepsilon \lambda^{**} < \mu_2 \). This proofs that (48) is valid. □

### 5 Numerical examples

In this section, we apply the proposed method to two benchmark examples that have been extensively used in the literature to compare the results. The third example is a slight modification of Example 1 investigated in [5] in a different situation.

#### 5.1 Some remarks on the implementation

The computations of the data matrices for the application of Theorem 3 have been performed by using Wolfram Mathematica. First the monic orthogonal polynomials have been computed, then the polynomials of indeces \((k,j)\) have been normalized with a multiplier \(\pi_j\) so that \(p_{1,j}(-1) = (-1)^j\) is satisfied. In this way, it has been achieved that the elements of all matrices have values of reasonable order of magnitude, and the computations are numerically stable. As a result, we have e.g. the norm-squares \(\|p_{1,0}\|^2 = \tau\) and \(\|p_{1,j}\|^2 = \frac{\tau}{2j+1} \prod_{i=1}^{j} \frac{\tau+2}{\tau+i} \), if \(j = 1, \ldots, \nu_1\), while matrix \(\Lambda_5\) is as follows:

\[
\Lambda_5 = \begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 \\
1 & -\frac{2}{\tau+1} & \frac{6}{\tau+2} & 1 & 0 & 0 \\
1 & \frac{2}{\tau+1} & \frac{20(\tau+2)^2+11}{\tau+3} & \frac{30}{\tau+3} & 1 & 0 \\
1 & -\frac{2(\tau+1)^2+274}{\tau+3} & \frac{84(\tau+1)^2+11}{\tau+4} & \frac{126}{\tau+4} & 1 & -\frac{18}{\tau+5}
\end{bmatrix}
\]

The norm-squares of the other polynomials, the matrices \(\mathcal{Z}_k\) as well as the details of computations can be found in [8].

The LMIs have been solved by using MATLAB LMI Toolbox. The computations have been performed for \(m = 1, \ldots, 4\) and \(\nu_4 = 0, \ldots, 5\), but tables below include only the most informative results.
5.2 Numerical experiments

Example 1. Consider system (1) with
\[
A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.91 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}.
\]

The analytical range of the delay that retains stability of the system is \([0, 58] \cap \mathbb{Z}\). The number of decision variables is 7021, if the discrete Lyapunov inequality is used to determine the analytical bound. The results obtained by methods proposed in one of the most recent references [25] and by Theorem 3 for different values of \(m\) and \(\nu_1\) are given in Table 1. Further comparisons with results of several recent references is given in [25]. For \(m > 1\), the values of \(\nu_j\) are set as \(\nu_j = \nu_1 - (j - 1)\).

| Method          | \(m\) | \(\nu_1\) | \(\tau_M\) | NoDVs | \(m\) | \(\nu_1\) | \(\tau_M\) | NoDVs |
|-----------------|------|---------|--------|------|------|---------|--------|------|
| Zhang et al. [25] | 1    | 1       | 57     | 16   | 1    | 1       | 151    | 16   |
| Nam et al. [15]  | 2    | 2       | 168    | (*)  |      |         |        |      |
| Theorem 3        | 1    | 0       | 42     | 9    | 1    | 1       | 151    | 16   |
|                 | 1    | 1       | 57     | 16   | 1    | 2       | 168    | 27   |
|                 | 2    | 1       | 57     | 19   | 2    | 2       | 168    | 30   |
|                 | 1    | 2       | 58     | 27   | 1    | 4       | 169    | 61   |
|                 | 2    | 2       | 58     | 30   | 2    | 4       | 169    | 64   |

(*) not available

Example 2. Consider system (1) with
\[
A = \begin{bmatrix} 1 & 0.01 \\ -0.02 & 1.001 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 0.01 & 0 \end{bmatrix}.
\]

The analytical range of the delay that retains stability of the system is \([12, 169] \cap \mathbb{Z}\). The number of decision variables is 57970, if the discrete Lyapunov inequality is used to determine the analytical bound. The results obtained by methods proposed in most recent references [25] and [15] and by Theorem 3 for different values of \(m\) and \(\nu_1\) are given in Table 1. Similarly to the previous example, for \(m > 1\), the values of \(\nu_j\) are set as \(\nu_j = \nu_1 - (j - 1)\). Further comparisons with results of several recent references is given in [15]. The LMIs of Theorem 3 were feasible for \(\tau = 12\) in all cases.

Example 3. Consider system (1) with
\[
A = \begin{bmatrix} 0.12 & 0 & -0.12 \\ 0.06 & 0.36 & 0 \\ 0 & 0.24 & 0.72 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.4 & 0 & 0 \\ 0 & -0.2 & 0.2 \\ 0 & 0 & -0.4 \end{bmatrix}.
\]

The analytical range of the delay that retains stability of the system is \([0, 56] \cap \mathbb{Z}\). The number of decision variables is 14706, if the discrete Lyapunov inequality is used to determine the analytical bound. Similarly to the previous example, for \(m > 1\), the values of \(\nu_j\) are set as \(\nu_j = \nu_1 - (j - 1)\). The results obtained by methods proposed in [25] and by Theorem 3 for \(m = 1\) and for different values of \(\nu_1\) are given in Table 2. In cases of \(m = 2, 3, 4, 5\), the delay bounds were found to be the same as for \(m = 1\) applying the same value of \(\nu_1\).
| Method                        | m | $\nu_1$ | $\tau_M$ | NoDVs |
|------------------------------|---|---------|----------|-------|
| Zhang et al. 2015 [25]       | 1 | 1       | 50       | 33    |
| Theorem 3                    | 1 | 0       | 34       | 18    |
|                              | 1 | 1       | 50       | 33    |
|                              | 1 | 2       | 52       | 57    |
|                              | 1 | 3       | 52       | 90    |
|                              | 1 | 4       | 55       | 132   |
|                              | 1 | 5       | 56       | 183   |

### 5.3 Discussion

The numerical examples show that the application of the discrete Wirtinger inequality (case $m = 1$ and $\nu_1 = 1$) reduces the conservativeness of results obtained by the application of Jensen’s inequality (case $m = 1$ and $\nu_1 = 0$). They also show that further improvement can be achieved by the application of the inequalities of Theorem 2 both via increasing the number of multiple summation terms in the LKF (case $m > 1$) and the improvement of the lower estimations via increasing the degree of the orthogonal polynomials (i.e. $\nu_1 > 1, \ldots, \nu_m \geq 0$). Beside the above examples, we tested our approach for several other examples from the literature with the experience as follows. The increase of the complexity of the LKF (i.e. the increase of $m$) did not resulted in a better delay bound than the LKF with $m = 1$, if the same $\nu_1$ was applied. This means that the improvement is primarily due to the increase of the dimension of the extended state variable. This does not contradict to the reported improvements in the case the application of triple, etc. summation terms in the LKF, since - on the one hand - the applied lower estimations lead to introduction of some extended state variables with increased dimension. On the other hand, several authors apply in their developments not only a more complex LKF, but other methods as well (e.g. adding ’zero equality’, see e.g. [15], relaxation of the requirement of positive definiteness of certain matrices in the LKF, see e.g. [22], etc.). We have not applied these latter methods, since we wanted only to investigate the effect of the improvement of the lower estimation and the effect of the application of multiple summation terms in the LKF. In sum, the increase of the dimension of the extended state variable plays the basic role in the improvement.

We emphasize that analytical delay bounds could be achieved in all examples, if a sufficiently tight lower estimation is applied. Apparently, the necessary number of decision variables is dramatically lower than that under the application of the necessary and sufficient condition of stability.

### 6 Conclusion

In this paper, multiple summation inequalities were presented in the case of arbitrary number of summation both for functions and differences. The new inequalities involve the discrete Jensen’s and Wirtinger’s inequalities, as well as the recently presented inequalities for single and double summation in [15]. Applying the obtained inequalities, a set of sufficient LMI stability conditions for linear discrete-time delay systems are derived.
It was proven that these LMI conditions could be arranged into a bidirectional hierarchy establishing a rigorous theoretical basis for comparison of conservatism of the investigated methods. It was shown by some benchmark examples that the proposed methods give better upper bounds for the tolerable time delay than the best ones that we could find in the previously published literature. Several numerical examples showed that the improvement of the lower estimations can result in as much improvements as the application of more complex LKFs.

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