SAHA AND THE DYON

A.P. Balachandran

Department of Physics, Syracuse University,
Syracuse, NY 13244-1130

Abstract

Meghnad Saha occupies a special role in the history of Indian science, having been a pioneer in its organization already from the oppressive colonial period and having left important legacies to post-colonial India like the Saha Institute of Nuclear Physics. He is famous for his research in astrophysics, and has also made important, but less well-known contributions to magnetic monopole theory. This article attempts an elementary survey of this theory and its implications with special emphasis on Saha’s work and argues that the latter has had wide ranging consequences for fundamental physics.

To be published in the Commemoration Volume celebrating the Birth Centenary of Professor M.N. Saha.
The interactions of electrically charged matter and the electromagnetic field are described by the Maxwell-Lorentz theory which has the Lorentz force and Maxwell equations as its principle dynamical equations. This theory is immensely successful in describing a wide range of phenomena already in its classical form. When it is appropriately first and second quantized, it becomes capable of accounting for observations all the way from chemistry and condensed matter physics to Lamb shift and electron-positron ($e^- - e^+$) pair production. The typical energies involved at one end of this spectrum are a few electron volts whereas it costs a minimum of 1 MeV to pair produce $e^- - e^+$. It is thus a theory which works well for at least six decades in energy. Its range of validity is in fact greater than is indicated by these examples. It is also a theory accurate to an extent never before encountered in science. The agreement between theory and experiment for the anomaly in the electron gyromagnetic ratio is about one part in $10^7$. Quantum electrodynamics is one of the best scientific models about nature ever constructed by humans.

It was apparently felt already in the last century by Poincaré, Thomson and perhaps others as well that there was room for improvement of electrodynamics despite its empirical success. Such a speculation becomes understandable when it is realized that the equations of electrodynamics contain an intrinsic asymmetry between electricity and magnetism, allowing as they do for electric monopoles, but not for magnetic monopoles. It can therefore be proposed with good reason that just as there are electrically charged particles which act as sources for the electric field $\vec{E}$, so too must there exist magnetically charged particles which are sources for the magnetic induction $\vec{B}$. In the presence of a particle of charge $e$ at point $\vec{z}$, Gauss’s law is $\nabla \cdot \vec{E}(\vec{x}) = 4\pi e \delta^3(x-z)$. In the same way, the
equation for $\vec{\nabla} \cdot \vec{B}$ will read $\vec{\nabla} \cdot \vec{B}(x) = 4\pi g \delta^3(x - \vec{z})$ when there is a particle of magnetic charge $g$ at $\vec{z}$. The relativistically covariant version of this equation is $\partial^\lambda * F_{\lambda \nu} = 4\pi j^M_\nu$ where $* F_{\lambda \nu}$ is the dual of the Maxwell tensor field $F_{\lambda \nu}$, $j^M_\nu$ is the conserved magnetic four vector current and we use units where the speed of light $c$ is 1. In the work of Poincaré, the dynamics of an electrically and a magnetically charged particle in interaction were also studied in the nonrelativistic approximation. There were no doubt early speculations as well about the physical implications of the existence of a magnetic monopole.

There is one striking consequence of the existence of a magnetic monopole already for classical electromagnetic theory. It is that with a magnetic monopole around, it is impossible to introduce a smooth vector potential $\vec{A}$ for $\vec{B}$. This is because $\vec{B} = \vec{\nabla} \times \vec{A}$ implies that the net flux of $\vec{B}$ over any closed surface is zero by Stokes’s theorem whereas it must be $4\pi g$ for a sphere enclosing magnetic charge $g$. We may now remember that the existence of a vector potential $A_\mu$ ($\mu = 0, 1, 2, 3$) is used in a significant manner for writing a Lagrangian for conventional electrodynamics. An electrodynamic theory which has magnetic charges as well does not therefore admit a simple Lagrangian description. [But see however ref.3.]

The implications of this remark are not serious for classical physics. It is possible to do a great deal of the latter without ever talking about Lagrangians and Hamiltonians. Lagrangian and Hamiltonian formalisms are largely superstructures on the edifice of classical theory. Perhaps the most serious exception to the decorative role they play occurs in classical statistical mechanics which uses the phase space in a basic manner. But electric and magnetic monopoles in interaction do admit a Hamiltonian description so that the lack of a decent Lagrangian of a conventional sort for this system is not a serious matter.
for its classical physics.

But that however is far from being the case in quantum theory, a profound fact discovered by Dirac in a paper published in 1931.\(^4\) One way to sketch Dirac’s reasoning about the electric charge-magnetic monopole (or charge-monopole) system is as follows. Since a Coulomb magnetic induction does not admit a vector potential, let us modify it and instead consider

\[
\vec{B} = \vec{B}_C + \vec{B}_S,
\]

\[
\vec{B}_C = \frac{g \vec{x}}{r^3}, \quad r = |\vec{x}|,
\]

\[
\vec{B}_S = -\vec{n} \ 4\pi g \ \theta(x_3) \delta(x_1) \delta(x_2), \quad \vec{n} = (0, 0, 1),
\]

(1)

\(\theta\) being the standard step function. In this expression, the Coulomb field \(\vec{B}_C\) of a magnetic monopole (or monopole for short) at the origin has been augmented by a field \(\vec{B}_S\) concentrated on the nonnegative third axis \(x_3 \geq 0, \ x_1 = x_2 = 0\). This new \(\vec{B}\) is the field of an infinitely thin solenoid on the nonnegative third axis and is divergenceless. It can thus be written as \(\vec{\nabla} \times \vec{A}\). The dynamics of a charge in this field can also be described using a Lagrangian.

The field \(\vec{B}\) is identical to the Coulomb field \(\vec{B}_C\) away from the third axis. But that is not the case on the third axis so that without further qualifications or physical inputs, it can not be regarded as the field of a monopole, but rather must be thought of as the field of a solenoid.

A brilliant contribution of Dirac to the charge-monopole theory was to establish that \(\vec{B}_S\) is entirely unobservable if the quantization condition

\[
eg g = n \frac{\hbar}{2}, \quad n = 0, \pm 1, \pm 2, ...
\]

(2)

is fulfilled. The location \(L\) of the solenoid has been chosen to be along the nonnegative third axis in (1). But it is obvious that a vector potential \(\vec{A}\) exists even if this location is
along any line from origin to infinity. The line of location $L$ of the solenoid is known as the Dirac string. The result of Dirac is that the Dirac string can not be observed if the quantization condition (2) is fulfilled.

This result can be established by semiclassical reasoning as follows. Let us for definiteness assume that $L$ is along the third axis as in (1) and imagine an Aharonov-Bohm experiment\(^5\) with the flux on $L$ serving as the flux line. In this experiment, a coherent beam of particles of charge $e$ and mass $m$ (regarded for simplicity as spinless) are injected through a slit and detected at a point $\vec{x}_0$ on a screen as in Fig. 1. The wave function $\psi$

\[
\psi(\vec{x}_0) = \text{Overall factor} \times \left[ e^{iS_0/\hbar} + e^{iS_1/\hbar} \right]. \tag{3}
\]
If $P_i$ are the classical paths from slit to $\vec{x}_0$, one behind and the other before the Dirac string as in Fig. 1, then $S_i$ is the classical action for path $P_i$:

$$S_i = \int_{P_i} \left\{ \frac{1}{2}m \left( \frac{dx_i(t)}{dt} \right)^2 - V(\vec{x}(t)) + eA_i(\vec{x}(t)) \frac{dx_i(t)}{dt} \right\} .$$

(4)

Here we have included a possible potential term (the integral of $-V$) in the action which may be required to realise the classical solutions with paths $P_i$. [Instead of a potential, we can also use mirrors (or any other suitable device) to reflect the split beams from the slit so as to realise a path $P_1$ behind and a path $P_2$ before $L$.] The field on the string does not affect the classical trajectories. It can therefore be detected only by its contribution to the phase difference of the terms in (3). This contribution is contained in

$$\exp i \left\{ \frac{e}{\hbar} \left( \int_{P_1} - \int_{P_2} \right) A_i(\vec{x}(t)) \frac{dx_i(t)}{dt} \right\}$$

(5)

which by Stokes’s theorem is

$$\exp i \left\{ \frac{e}{\hbar} \times \text{Magnetic flux through loop } C \right\}$$

(6)

where $C$ circles around $L$ as shown in Fig. 1. Besides the monopole field contribution, this flux contains also the string contribution $-4\pi g$, the shift in the phase difference caused by the flux supported on $L$ being

$$\exp i \{-4\pi g/e\} .$$

(7)

If $-4\pi g/e$ is $2\pi \times$ an integer $n$, then (7) is 1 and the Dirac string can not be detected, the answer for the phase difference being the same whether or not there is the flux supported on $L$. In other words, according to Dirac, an electric charge coupled to a potential $\vec{A}$ with $\vec{B}$ as its curl leads to a theory of charges and monopoles (and not of charges and solenoids) provided.

$$eg = \frac{nh}{2}, \ n \in \text{the set } \mathbb{Z} \text{ of integers} .$$

(8)
It is thus the case that we have the quantization rule (8) for electric and magnetic charges.

A very significant implication of (8) is that electric charges must be quantized if there are magnetic monopoles. Thus suppose that the least nonzero value of $|g|$ occurs when $g = g_0$. Then all electric charges are integral multiples of $\hbar/(2g_0)$ by (8). Magnetic charges too must come in quantized units for a similar reason. Since experiments indicate that electric charges are quantized, this prediction of (8) was for a long time considered as a strong argument in favour of the existence of magnetic monopoles.

There is an interesting alternative approach to the derivation of (8) which is due to Saha. Let us consider the motion of a particle of electric charge $e$ in the Coulomb field of a particle of magnetic charge $g$. For nonrelativistic kinematics, if $\vec{z}$ is the relative coordinate and $\mu$ is the reduced mass, the Lorentz force equation is

$$\mu \ddot{\vec{z}} = e g \frac{\vec{z} \times \dot{\vec{z}}}{|\vec{z}|^3},$$

the dots denoting time derivatives. The usual angular momentum $\mu \vec{z} \times \dot{\vec{z}}$ is not a constant of motion for (9). Instead, as is well known,

$$\bar{J} = \mu \vec{z} \times \dot{\vec{z}} + e g \hat{z}, \quad \hat{z} := \frac{\vec{z}}{|\vec{z}|}$$

is a constant of motion for (9) and should be regarded as the angular momentum of the charge-monopole composite. There is thus a new term in the angular momentum pointing along the line joining the charge and the monopole:

$$\bar{J} \cdot \hat{z} = e g.$$

The familiar quantization condition for components of angular momentum is now seen to require the condition (8).
Saha also emphasized that the term $eg \hat{z}$ can be obtained from the integral

$$\frac{1}{4\pi} \int \left\{ \vec{x} \times \left[ \vec{E}(\vec{x}) \times \vec{B}(\vec{x}) \right] \right\} d^3x$$

(12)

which is the contribution of the electromagnetic field to angular momentum. This result is originally due to Thomson\(^2\). It can be derived by substituting the Coulomb fields of a charge and a monopole for $\vec{E}$ and $\vec{B}$ in (12) and performing the integral.

The derivation of the quantization condition (8) from the properties of angular momentum in quantum theory reveals certain notable properties of the dyon or the Saha dyon as the charge-monopole composite is often called. We can explain them, and bring out their novelty as well, by first considering a system of two electrically charged particles with no magnetic charge. Let us assume furthermore that the particles are spinless. In that case, the angular momentum or spins of the composite are integral. In other words, we can not form a composite of spin $2S$ with $2S$ an odd integer (or spin “half-integral”) out of spinless constituents. This remark is correct for most familiar interactions between particles and also if both particles have integral or half-integral spin. There is an obvious modification of this observation if one of the particles is of integer spin and the second of half-integer spin.

The story is strikingly different for a Saha dyon. For a dyon, if $eg$ is an odd multiple of $\hbar/2$, then the component of angular momentum in the $\hat{z}$ direction, and hence the spin of the dyon, is half-integral. This is so even though both constituents are spinless (or for that matter, of any integral or half-integral spin). It is thus possible to contemplate half-integral spin or “spinorial” composites made up of integral spin or “tensorial” constituents. Saha’s derivation of (8) in this manner also emphasizes a very interesting physical principle.

It is not a priori obvious that such spinorial composites also obey Fermi-Dirac statistics and hence are fermions. As dyons have several unusual properties, one could suspect that their statistics too is unusual. They may for example violate the spin-statistics theorem
or obey a relatively unfamiliar parastatistics. It is known however that the spin-statistics
correlation for dyons is normal and that tensorial (spinorial) dyons obey Bose-Einstein
(Fermi-Dirac) statistics and hence are bosons (fermions).

Dirac himself and Tamm soon after him, had studied the Schrödinger equation of
the charge-monopole system with spinless constituents in polar coordinates and thereby
found half-integral angular momenta for suitable eg for this composite. The merit of
Saha’s approach in contrast to that of these authors is that it is very simple and reveals
the result in an easy manner. It has since been generalized and is frequently used in
research in elementary particle theory.

Summing up, the work of the above authors and later developments have revealed a
very important physical possibility: It is conceivable to have spinorial, fermionic com-
posites made up of tensorial, bosonic constituents. Later on, we will have more to say on
this remarkable fact originating in dyon theory.

In elementary particle physics, the so-called standard model nicely accounts for prop-
erties of strong, electromagnetic and weak interactions. It is very successful for energies
up to 1 TeV, and may in fact be good for energies all the way up to about $10^{15}$ or $10^{16}$
GeV.

The standard model however is unsatisfactory in at least one respect: whereas it
successfully unifies electromagnetic and weak interactions, explaining their origin from a
common source, it does not unify all the preceding three interactions in a similar way.
Many particle physicists therefore suspect that it will break down at $10^{15} - 10^{16}$ GeV,
the unity of the three interactions becoming manifest at such energies. [The reasons for
arriving at the “grand unification” scale of $10^{15} - 10^{16}$ GeV are rather technical.] According
To these physicists, there must exist a unified theory of all these interactions superseding the standard model which should be used for a satisfactory description of nature at these energies. There are at present several proposed models attempting this unification and collectively known as Grand Unified Theories (GUT’s) or Supersymmetric Grand Unified Theories (SUSY-GUT’s). There is no single GUT or SUSY-GUT which we can now say is closer to reality than others, but a common expectation is that a unique theory will eventually emerge as the winner from among the competing GUT’s and SUSY-GUT’s.

It is not necessary to distinguish GUT’s from SUSY-GUT’s for the purposes of this article. We will therefore refer only to GUT’s.

One feature of GUT’s is of particular interest for monopole theory. It is that GUT’s generically predict the existence of monopoles (and dyons). [For reviews of GUT monopoles and dyons, see for example ref. 9.] The prediction of monopoles by field theories which share the pertinent features of GUT’s was first pointed out by ‘t Hooft and Polyakov.9 There are differences in detail between GUT monopoles and those of Dirac, but there are enough points of similarity that a particle physicist does not always bother to distinguish these monopoles. We will accept this practice here.

Monopoles predicted by GUT’s are extremely massive, with masses of the order of the grand unification scale. They are impossible to create in accelerators. They could however have been produced during cosmic history. In that case, they may be detectable by their flux on earth from outer space, or by astrophysical observations which look for their effects on magnetic fields in galaxies or neutron stars. Experiments along these lines have been done and are in progress, but they have yet to detect a monopole. The discovery of a monopole will be of great significance for fundamental theory.

GUT monopoles have many exotic properties. Just as the Saha dyon, they can be spinorial fermions, even though their constituent fields are of integer spin. They also have an astonishing ability to catalyze fast baryon decays,10 a feature which they share with
certain GUT “cosmic strings” and which will now be briefly discussed.

GUT’s generically contain interactions which lead to nonconservation of baryon (and lepton) number. In the absence of monopoles, these interactions predict the decay of protons into leptons plus other particles like mesons and photons. The life-time for proton decay is typically predicted to be $10^{32}$ years or longer. Experiments have not yet detected proton decay. Its observation will have as much an impact as the discovery of a monopole on fundamental theory.

Monopoles in GUT’s are made of fields which can mediate baryon decays. It was discovered in the ’80’s that as a consequence, a baryon scattering off a monopole can turn into a lepton plus possible debris like mesons, leaving the monopole intact. The existence of such catalytic processes in itself is not a surprise. What is a surprise is that their cross-sections are comparable to strong interaction cross-sections because of certain powerful attractive interactions caused by monopole topology. The upshot is that proton decay catalyzed by monopoles has a life time of only about $10^{-20}$ seconds (give or take a few orders of magnitude) and not $10^{32}$ years. Thus life as we know it will certainly not exist in an ambience with an abundance of GUT monopoles.

Monopoles also have implications for the “standard model” of the history of the universe. It is based on the “big bang” alleged to be the beginning of all history. In its original version, it predicted the presence of too many monopoles in outer space leading to contradictions with observations. This was among the reasons leading to revisions of this version and to proposals of the “inflationary” scenario and its variants. The interested reader should consult a suitable semipopular or technical review of cosmology for further details.

We can conclude this section as follows: whereas for electrodynamics, a monopole was a postulate grafted on to existing theory, that is not the case for GUT’s which generically predict them as stable states. Their experimental discovery will therefore have far reaching
The possibility of creating spinorial composites out of tensorial constituents is a very basic physical result. It could have been adequately understood by any physicist after the work of Dirac\(^4\) and Saha\(^6\). But that is not what happened, and it was only in the late ’60’s and the ’70’s that this result began to be widely known in the theoretical physics community. There was at that time a pervasive feeling too that it hinged on rare topological features and was not likely to find too many applications.

But all that is changed today especially because of the discoveries of Skyrme beginning from the late ’50’s. [See for example ref. 3 or 11 for a review of Skyrme’s work and related developments.] He showed in his studies that spin half excitations can exist in a theory of pions. He thus proved that spinorial states can emerge from a tensorial field theory (a variant of the result for the Saha dyon) and went on to propose that nucleons are twisted topological lumps in pion fields. Skyrme’s work was revived in the ’80’s by Pak and Tze and by the Syracuse group and found universal acceptance after fundamental contributions by Witten. This model of the nucleon is called the Skyrmion.

As the Skyrmion is widely known these days at least by name, one supposes that many physicists also appreciate now that Lagrangians built from tensorial variables can nevertheless possess spinorial states.

“Geometric Quantization”\(^{12}\) is a particular approach to quantization of classical theories based on powerful mathematical and in particular topological ideas. The relatively few physicists actively pursuing this approach have long appreciated that the Dirac-Tamm-Saha ideas are hardly unique to the charge-monopole system. They had in fact found several exceptionally elementary systems many years ago which we can regard as illus-
trating these ideas. One such example will be mentioned here. [Refs. 3 and 12 can be consulted for details and more citations.] Suppose that one wishes to describe a particle of fixed spin $j$ by a Lagrangian. There is then such a Lagrangian based on elements of the rotation group SO(3) or on oriented orthonormal frames which are both tensorial variables. This is so for $2j$ even or $2j$ odd, the states of course being spinorial in the latter case. Particle of fixed half-integral spin thus illustrate how Lagrangians with tensorial variables can nevertheless have spinorial states.

There are also many examples of this sort in molecular physics in the Born-Oppenheimer approximation$^{13}$ and in the collective model approach to nuclei.$^{14}$ In the former approximation for example, a molecule can be described by a Lagrangian based on SO(3). This description is not unlike the one commonly used for a rigid rotor. Now it is easy to demonstrate that the latter admits quantization with spinorial states [or spinorial quantization]. [Cf. ref. 3, Chapter 13.4.] For related reasons, the Born-Oppenheimer Lagrangian too admits spinorial quantization for a great many molecules, $^{15}$ the spinorial states admitting an interpretation as well in terms of intrinsic spins of nuclei and electrons. Molecular and nuclear physicists routinely use such quantization, but apparently without emphasizing the remarkable topological and conceptual foundations of quantum physics which allow this possibility. If they had paid careful attention to these features, they would have enriched and significantly extended the Dirac-Tamm-Saha studies and contributed to the foundations of important research in elementary particle theory such as that on the Skyrmion. [Skyrme of course did pay attention to topological and conceptual features, but apparently not in the context of research on the collective model approach.
Meghnad Saha is a major figure in the history of recent Indian science, being a pioneer in its organization in the modern era. He was a talented physicist particularly well known for his work in astrophysics. His “Treatise on Heat” is a splendid text book which has trained generations of Indian students including the author. In this short essay, an attempt has been made to survey magnetic monopole theory with particular attention to Saha’s work and to point out that it is as important as his better known astrophysical research. Indeed it has had wide ranging repercussions which continue to be influential particularly in elementary particle theory.

Acknowledgments

I am most grateful to Arshad Momen, Carl Rosenzweig, V.V. Sreedhar, Paul Souder, Ajit Mohan Srivastava and Paulo Teotonio for generous help in preparing this article. Special thanks go to Paulo for drawing the figure. This work was supported by the Department of Energy under contract number DEFG02-85ER40231.

References

The references below are not meant to be exhaustive and are very limited in number.

1. H. Poincaré, Compt. Rend. 123, 530 (1896).

2. J.J. Thomson, “Elements of the Mathematical Theory of Electricity and Magnetism” [Cambridge University Press, 1904].
3. A.P. Balachandran, G. Marmo, B.S. Skagerstam and A. Stern, “Classical Topology and Quantum States” [World Scientific, 1991].

4. P.A.M. Dirac, *Proc. Roy. Soc.* (London) **A133**, 60 (1931).

5. Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).

6. M.N. Saha, *Ind. J. Phys.* **10**, 145 (1936). Saha’s derivation of (8) was rediscovered by Wilson. See H.A. Wilson, *Phys. Rev.* **75**, 308 (1949) and M.N. Saha, *Phys. Rev.* **75**, 1968 (1949). See also M. Fierz, *Helv. Phys. Acta* **17**, 27 (1944) for a closely related work.

7. A.S. Goldhaber, *Phys. Rev.* **36**, 1122 (1976); R.D. Sorkin, *Phys. Rev.* **D27**, 1787 (1983).

8. I. Tamm, *Z. Physik* **71**, 141 (1931).

9. P. Goddard and D. Olive, *Rep. Prog. Phys.* **41**, 1357 (1978); S. Coleman, “The Magnetic Monopole Fifty Years Later” in “The Unity of Fundamental Interactions”, edited by A. Zichichi [Plenum Press, 1983].

10. V.A. Rubakov, *Pis’ma Zh. Eksp. Teor. Fiz.* **33**, 659 (1981) [JETP Lett. **33**, 644 (1981)]; *Nucl. Phys.* **B203**, 311 (1982); C.G. Callan, *Phys. Rev.* **D25**, 2141 (1982); **26**, 2058 (1982).

11. V.G. Makhankov, Yu. P. Rybakov and V.I. Sanyuk, “The Skyrme Model, Fundamentals, Methods, Applications” [Springer-Verlag (in press)]. For a semi-popular article on the role of topology in physics, see A.P. Balachandran, “Topology in Physics-A Perspective”, an article written in honor of Fritz Rohrlich, Syracuse University preprint SU-4228-533 (1993), to be published in *Foundations in Physics*. 
12. Cf. D.J. Simms and N. Woodhouse, “Geometric Quantization”, Lecture Notes in Physics 53 [Springer-Verlag, 1976]; N. Woodhouse, “Geometric Quantization”, Oxford Mathematical Monographs [Clarendon, 1980]; J. Sniatycki, “Geometric Quantization and Quantum Mechanics” [Springer-Verlag, 1980].

13. Cf. L.D. Landau and M. Lifshitz, “Quantum Mechanics, Non-Relativistic Theory” [Pergamon, 1977].

14. Cf. A. Bohr and B.R. Mottleson, “Nuclear Structure, Volume II: Nuclear Deformations” [W.A. Benjamin, Inc., 1975]; P. Ring and P. Schuck, “The Nuclear Many-Body Problem” [Springer-Verlag, 1980].

15. A.P. Balachandran, A. Simoni and D.M. Witt, *Int. J. Mod. Phys.* **A7**, 2087 (1992).
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9303312v1