SECTIONS OF CONVEX BODIES IN JOHN'S AND MINIMAL SURFACE AREA POSITION

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Abstract. We prove several estimates for the volume, mean width, and the value of the Wills functional of sections of convex bodies in John’s position, as well as for their polar bodies. These estimates extend some well-known results for convex bodies in John’s position to the case of lower-dimensional sections, which had mainly been studied for the cube and the regular simplex. Some estimates for centrally symmetric convex bodies in minimal surface area position are also obtained.

1. Introduction and notation

Ball showed in [3] that if $B_{\infty}^n$ denotes the $n$-dimensional cube and $F \in G_{n,k}$ is a $k$-dimensional linear subspace of $\mathbb{R}^n$ then $|B_{\infty}^n \cap F| \leq 2^{n+k}/2^n$. He also obtained the bound

$$|B_{\infty}^n \cap F|^{1/k} \leq \sqrt{n} |B_{\infty}^k|^{1/k},$$

which is optimal if $k | n$.

It follows from results of Ball [3] that the $k$-dimensional sections of a regular simplex with largest volume are exactly its $k$-dimensional faces. Let us also mention that Webb showed in [26] that if $S_n$ denotes the regular simplex with inradius $r(S_n) = 1$ then, for every hyperplane through the origin $F \in G_{n,n-1}$,

$$|S_n \cap F|^{\frac{1}{n-1}} \leq \frac{1}{n(n+1)^{\frac{n-1}{2}}} \sqrt{n(n+1)} |S_{n-1}|^{\frac{1}{n-1}}.$$

There is equality for the sections passing through the origin that contain $n-1$ of the vertices.

Dirksen proved in [9, Theorem 6.1], the following estimate for the volume of $k$-dimensional sections of the regular simplex through the center of mass. If $S_n$ denotes the centered regular $n$-dimensional simplex with inradius $r(S_n) = 1$ then for every $F \in G_{n,k}$ we have that

$$|S_n \cap F|^{1/k} \leq \frac{1}{(k+1)^{n+1}} \sqrt{n(n+1)} |S_k|^{1/k}.$$

Besides, this estimate is asymptotically sharp.

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Passing to the general case, we observe that it is not possible to obtain an upper bound for the volume of sections of a general centered convex body \( K \) (i.e., a compact convex set with non-empty interior) without any additional assumption since, considering different positions of \( K \) (i.e., affine images of \( K \)), we can obtain sections with volume as large as desired. For that matter we consider convex bodies in a particular position. In this paper we consider convex bodies in John’s position and in minimal surface area position.

A convex body \( K \subseteq \mathbb{R}^n \) is said to be in John’s position if the Euclidean unit ball \( B^n_2 \) is contained in \( K \) and for every non-degenerate linear map \( T \in \text{GL}(n) \) such that \( T(B^n_2) \subseteq K \) we have that \( |T(B^n_2)| \leq |B^n_2| \), where \(| \cdot |\) denotes the volume of a convex body. In other words, \( K \) is in John’s position if the Euclidean unit ball is the maximal volume ellipsoid contained in \( K \). This position is uniquely determined up to orthogonal transformations. Both the cube and the regular simplex with inradius 1 considered above are in John’s position.

In [4], Ball proved that among all convex bodies in John’s position, \( S_n \), the regular simplex with inradius \( r(S_n) = 1 \), has the largest volume and, among all the centrally symmetric convex bodies in John’s position, the cube \( B^n_\infty \) has the largest volume. In a recent article [10], Markessinis claimed to have obtained an upper bound for the volume of \( k \)-dimensional central sections of convex bodies in John’s position. However, although the estimate given for central sections of centrally symmetric convex bodies in John’s position is correct, the proof in the not necessarily symmetric case is not correct. In the following theorem we give an upper bound for the volume of central (and non-central) \( k \)-dimensional sections of an arbitrary convex body which is in John’s position.

**Theorem 1.1.** Let \( K \subseteq \mathbb{R}^n \) be a convex body in John’s position and \( F \in G_{n,k} \). Then

\[
|K \cap F|^{1/k} \leq \frac{1}{(k+1)^{n-k+1} k^{n-k+1}} \left( \frac{n(n+1)}{k(k+1)} \right)^{1/k} |S_k|^{1/k}.
\]

Furthermore, if \( K \) is centrally symmetric

\[
|K \cap F|^{1/k} \leq \sqrt{\frac{n}{k}} |B^k_\infty|^{1/k}.
\]

Moreover, if \( F \) is a \( k \)-dimensional affine subspace at distance \( d \) from the origin and \( K \) is a convex body in John’s position then

\[
|K \cap F|^{1/k} \leq \sqrt{\frac{n(n+1)}{k(k+1)^{1+1/k}}} \left( \frac{n}{n+d^2} \right)^{1/k} |S_k|^{1/k}.
\]

**Remark.** The proof of the symmetric case is the same as the one given by Markessinis. Nevertheless, we will reproduce it for the sake of completeness. We can also obtain it as a direct consequence of Theorem 1.5 as well as a consequence of Theorem 8.1 below (see Section 2.6). This estimate is a sharp generalization of Ball’s estimate (1.1) for the cube. Moreover, the case \( k = 1 \) gives one more proof of John’s theorem in the symmetric case: if \( K \) is a centrally symmetric convex body in \( \mathbb{R}^n \) whose maximal volume ellipsoid is \( B^n_2 \) then \( K \subseteq \sqrt{n} B^n_2 \).

**Remark.** Notice that we recover the estimate in (1.3), which is asymptotically sharp for the simplex. Besides, if we take non-central sections by \( k \)-dimensional subspaces
a distance \(d = \sqrt{\frac{n(n-k)}{k+1}}\) from the origin, which is the distance from the origin to any \(k\)-dimensional face of \(S_n\) we obtain that
\[
|K \cap F|^{1/k} \leq \sqrt{\frac{n(n+1)}{k(k+1)}} |S_k|^{1/k},
\]
which is exactly the volume of the \(k\)-dimensional faces of \(S_n\). The estimate for general affine subspaces can also be obtained as a direct consequence of Theorem 8.1 below.

Dual to this position is the so called L"owner’s position. A convex body is said to be in L"owner’s position if the minimal volume ellipsoid containing it is the Euclidean unit ball. A convex body \(K \subseteq \mathbb{R}^n\) is in John’s position if and only if \(K^\circ\) is in L"owner’s position, where \(K^\circ\) denotes the polar body of \(K\), defined as
\[
K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in K\}.
\]
In [3], Ball also showed that among all convex bodies in L"owner’s position, \(\tilde{S}_n\), the regular simplex with circumradius \(R(\tilde{S}_n) = 1\), has the smallest volume and among all centrally symmetric convex bodies in L"owner’s position, \(B_\infty\), the \(\ell_\infty^n\)-ball has the smallest volume.

Concerning the volume of polar bodies of sections of convex bodies in John’s position (i.e, projections of convex bodies in L"owner’s position), it was proved by Barthe in his PhD thesis (see also [1]) that in the case of the \(\ell_p^n\)-balls, if \(1 \leq p \leq 2\) and \(F \in G_{n,k}\) then
\[
|P_F(B_p^n)|^{1/k} \geq \left(\frac{k}{n}\right)^{\frac{1}{p} - \frac{1}{2}} |B_p^k|^{1/k}.
\]
In particular, we have the following estimate for the projections of \(B_1^n\): For every \(F \in G_{n,k}\)
\[
|P_F(B_1^n)|^{1/k} \geq \sqrt{\frac{k}{n}} |B_1^k|^{1/k}.
\]
We can also obtain a lower bound for the volume of \(k\)-dimensional projections of convex bodies in L"owner’s position.

**Theorem 1.2.** Let \(K \subseteq \mathbb{R}^n\) be a convex body in L"owner’s position and \(F \in G_{n,k}\). Then
\[
|P_F(K)|^{1/k} \geq \sqrt{\frac{k}{n}} |S_k|^{1/k}
\]
Furthermore, if \(K\) is centrally symmetric
\[
|P_F(K)|^{1/k} \geq \sqrt{\frac{k}{n}} |B_1^k|^{1/k}.
\]

The mean width of a convex body \(K \subseteq \mathbb{R}^n\) is defined as
\[
w(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta),
\]
where, for every \(x \in \mathbb{R}^n\), \(h_K(x) = \sup\{\langle x, y \rangle : y \in K\}\) is the support function of \(K\) at \(x\) and \(d\sigma\) denotes the uniform probability measure on \(S^{n-1}\). In [23], the authors proved that among all centrally symmetric convex bodies in John’s position in \(\mathbb{R}^n\), \(w(K)\) is maximized if \(K = B_\infty^n\). The not necessarily symmetric case was treated in [6], where it was proved that among all convex bodies in John’s position in \(\mathbb{R}^n\), \(w(K)\) is maximized if \(K = S_n\), where \(S_n\) denotes the regular simplex in John’s
position. If we pass to them mean width of sections, then a direct consequence of [7, Theorem 10], is that for any $k$-dimensional linear subspace $F \in G_{n,k}$

$$w(B_{\infty}^n \cap F) \leq \sqrt{\frac{n}{k}} w(B_{k}^1)$$

and this estimate is sharp when $k \mid n$.

Furthermore, it was proved in [23] that among all centrally symmetric convex bodies in Löwner’s position in $\mathbb{R}^n$, $w(K)$ is minimized if $K = B_{1}^n$ and it was proved in [24] that among all convex bodies in Löwner’s position in $\mathbb{R}^n$, $w(K)$ is minimized if $K = \tilde{S}_n$. We will prove the following results on the mean width of sections of convex bodies in John’s position

**Theorem 1.3.** Let $K \subseteq \mathbb{R}^n$ be a convex body in John’s position and $F \in G_{n,k}$. Then

$$w(K \cap F) \leq C \sqrt{\frac{n \log n}{k \log k}} w(S_k),$$

where $C$ is an absolute constant. Furthermore, if $K$ is centrally symmetric then

$$w(K \cap F) \leq \sqrt{\frac{n}{k}} w(B_{\infty}^k).$$

We shall also prove the following result on the mean width of projections of convex bodies in Löwner’s position.

**Theorem 1.4.** Let $K \subseteq \mathbb{R}^n$ be a convex body in Löwner’s position. Then, for any $k$-dimensional linear subspace $F \in G_{n,k}$,

$$w(P_F(K)) \geq C \sqrt{\frac{k}{n}} w(\tilde{S}_k).$$

Furthermore, if $K$ is centrally symmetric then

$$w(P_F(K)) \geq \sqrt{\frac{k}{n}} w(B_1^k).$$

For any compact convex set $K \subseteq \mathbb{R}^n$, by Steiner’s formula (see [24, Equation (4.1)]) the volume of $K + tB_2^n$ can be expressed as a polynomial in the variable $t$

$$|K + tB_2^n| = \sum_{i=0}^{n} \binom{n}{i} |W_i(K)| t^i, \quad \forall t \geq 0,$$

where the numbers $W_i(K)$ are the so-called quermaßintegrals of $K$. We have that $W_0(K) = |K|$ is the volume of $K$, $nW_1(K) = |\partial K|$ is the surface area of $K$, and $W_{n-1} = |B_2^n| w(K)$, where $w(K)$ is the mean width of $K$. If $K$ is contained in a $k$-dimensional subspace $F \in G_{n,k}$, we can compute its quermaßintegrals in $\mathbb{R}^n$, but also its quermaßintegrals with respect to the subspace $F$, which we identify with $\mathbb{R}^k$. If we denote these quermaßintegrals by $W_i^{(k)}(K)$, for $i = 0, \ldots, k$, we have that (see e.g. [22, Property 3.1])

$$W_i^{(k)}(K) = \binom{n}{k-1} \frac{|B_2^n|}{|B_2^{n-k+i}|} |W_{n-k+i}(K)|, \quad \forall 0 \leq i \leq k,$$
while $W_i(K) = 0$ for all $0 \leq i < n - k$. In order to avoid the issue that quermaßintegrals depend on the space where the convex body is embedded, McMullen \[18\] defined the intrinsic volumes of a compact convex set $K \subseteq \mathbb{R}^n$ as

$$V_i(K) = \frac{n!}{|B_2^{n-i}|} W_{n-i}(K), \quad \forall 0 \leq i \leq n.$$  

In \[27\] Wills introduced and studied the functional

$$W(K) = \sum_{i=0}^{n} V_i(K)$$  

because of its possible relation with the so-called lattice-point enumerator $G(K) = \#(K \cap \mathbb{Z}^n)$. In \[2\], it was proved that among symmetric convex bodies in John’s position, $W(K)$ is maximized if $K = B_\infty^n$. Here, we prove the following:

**Theorem 1.5.** Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body in John’s position. Then, for any $F \in G_{n,k}$ and every $\lambda \geq 0$,

$$W(\lambda(K \cap F)) \leq W\left(\lambda \sqrt{\frac{n}{k}} B_\infty^k\right).$$

This theorem will give, as direct consequences, the symmetric cases of Theorem 1.1 and Theorem 1.3. We also prove the following estimate for the Wills functional of projections of convex bodies in Löwner’s position.

**Theorem 1.6.** Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body in Löwner’s position. Then, for any $F \in G_{n,k}$,

$$W(P_F(K)) \geq \frac{1}{k^{k/2}}.$$  

The main tool used to obtain most of the estimates above is the fact that a decomposition of the identity operator is associated to any convex body in John’s position, and that this decomposition allows the use of Brascamp-Lieb inequality (see 2.1 below). When $K$ is a polytope in minimal surface area, then there is again a decomposition of the identity associated to $K$ (see Section 2.7). A similar use of Brascamp-Lieb inequality, together with an approximation by polytopes, will lead to similar estimates for sections of convex bodies in minimal surface area position. Namely, we can prove the following

**Theorem 1.7.** Let $K \subseteq \mathbb{R}^n$ be a convex body in minimal surface area position. Then, for any $k$-dimensional linear subspace $F \in G_{n,k}$ we have

(a) $|\Pi^* K \cap F| \leq \frac{4^k n^k}{k!} \frac{1}{\partial(K)^k}$,

(b) $|P_F(\Pi K)| \geq \left(\frac{\partial(K)}{n}\right)^k$.

Furthermore, if $K$ is centrally symmetric, then for any $k$-dimensional linear subspace $F \in G_{n,k}$ we have

(i) $W(K \cap F) \leq W\left(\frac{n^2}{k} \frac{|K|}{\partial(K)} B_\infty^k\right)$,

(ii) $|K \cap F|^{1/k} \leq \frac{n^2}{k} \frac{|K|}{\partial(K)} |B_\infty^k|^{1/k}$,

(iii) $w(K \cap F) \leq \frac{n^2}{k} \frac{|K|}{\partial(K)} w(B_\infty^k)$,

(iv) $|(K \cap F)\circ|^{1/k} \geq \frac{k}{n^2} \frac{|K|}{\partial(K)} |(B_\infty^k)\circ|^{1/k}$,
\( w((K \cap F)') \geq \frac{k}{n^2} \frac{\partial(K)}{|\partial(K)|} w((B_k^n)') \).

**Remark.** Notice that if \( k = n \) then (a) recovers the right-hand side of (2.3), (b) recovers the left-hand side of (2.4), (ii) recovers the estimate given by Ball’s reverse isoperimetric inequality in [4] and (iii) recovers the estimate given in [17, Theorem 7.1].

2. Preliminaries

2.1. John’s position. As mentioned in the introduction, a convex body is said to be in John’s position if the maximal volume ellipsoid contained in it is the Euclidean unit ball. A classical theorem of John [15] (see also [5]) states that

\[ \text{John’s position} \iff \text{the maximal volume ellipsoid contained in } K \text{ is the Euclidean unit ball}. \]

Recall that a convex body \( K \subseteq \mathbb{R}^n \) is said to be in John’s position if the maximal volume ellipsoid contained in it is the Euclidean unit ball. A classical theorem of John [15] (see also [5]) states that a convex body \( K \subseteq \mathbb{R}^n \) is in John’s position if and only if the maximal volume ellipsoid contained in \( K \) is the Euclidean unit ball. A classical theorem of John [15] (see also [5]) states that a convex body \( K \subseteq \mathbb{R}^n \) is in John’s position if and only if

\[ \text{the maximal volume ellipsoid contained in } K \text{ is the Euclidean unit ball}. \]

Remark. Notice that for any such decomposition of the identity, we have that for every \( 1 \leq k \leq m \)

\[ 1 = |u_k|^2 = \sum_{j=1}^{m} c_j \langle u_k, u_j \rangle \geq c_k \langle u_k, u_k \rangle = c_k, \]

and so all the numbers \( \{c_j\}_{j=1}^m \) are in the interval \([0, 1]\).

2.2. Brascamp-Lieb inequality. We will make use of the Brascamp-Lieb inequality and the reverse Brascamp-Lieb inequality in the following form.

**Theorem 2.1.** Let \( m \geq n \), \( \{u_j\}_{j=1}^{m} \subseteq S^{n-1} \) and \( \{c_j\}_{j=1}^{m} \subseteq (0, \infty) \) be such that \( I_n = \sum_{j=1}^{m} c_j u_j \otimes u_j \). Then, for any integrable functions \( \{f_j\}_{j=1}^{m} : \mathbb{R} \to [0, \infty) \) we have that

\[ \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^j (\langle x, u_j \rangle) dx \leq \prod_{j=1}^{m} \left( \int_{\mathbb{R}} f_j(t) dt \right)^{c_j}. \]

Besides, for any integrable functions \( \{h_j\}_{j=1}^{m} : \mathbb{R} \to [0, \infty) \) verifying that

\[ h \left( \sum_{j=1}^{m} \theta_j c_j u_j \right) \geq \prod_{j=1}^{m} h_j^j (\theta_j) \text{ for every } \{\theta_j\}_{j=1}^{m} \subseteq \mathbb{R}, \]

we have that

\[ \int_{\mathbb{R}^n} h(x) dx \geq \prod_{j=1}^{m} \left( \int_{\mathbb{R}} h_j(t) dt \right)^{c_j}. \]

2.3. The regular simplex. Let \( \Delta_k \) denote the \( k \)-dimensional regular simplex

\[ \Delta_k = \text{conv} \{e_1, \ldots, e_{k+1} \} \subseteq H_0, \]

where \( H_0 = \{ x \in \mathbb{R}^{k+1} : \sum_{i=1}^{k+1} x_i = 1 \} \) is identified with \( \mathbb{R}^k \) and \( \left( \frac{1}{k+1}, \ldots, \frac{1}{k+1} \right) \) is identified with the origin. It is well known that
\[ |\Delta_k| = \frac{\sqrt{k+1}}{k}, \]
\[ r(\Delta_k) = \frac{1}{\sqrt{k(k+1)}}, \]
\[ R(\Delta_k) = \frac{k}{k+1}, \]
\[ \Delta_k^\circ = -(k+1)\Delta_k, \]
\[ w(\Delta_k) \simeq \sqrt{\log k}, \]

where \( a \simeq b \) denotes the fact that there exist two positive absolute constants \( c_1, c_2 \) such that \( c_1 a \leq b \leq c_2 a \). Thus, \( \frac{1}{r(\Delta_k)} \Delta_k \) is in John’s position and \( \frac{1}{R(\Delta_k)} \Delta_k \) is in Löwner’s position. Then, if \( S_k \) denotes the \( k \)-dimensional simplex in John’s position and \( \tilde{S}_k \) denotes the \( k \)-dimensional simplex in Löwner’s position, we have that

\[ S_k = \sqrt{k(k+1)} \Delta_k \quad \text{and} \quad \tilde{S}_k = \frac{k+1}{k} \Delta_k. \]

Therefore,

\[ |S_k|^{1/k} = \frac{\sqrt{k(k+1)^{1+\frac{1}{k}}}}{(k!)^{1/k}} \quad \text{and} \quad |\tilde{S}_k|^{1/k} = \frac{1}{(k!)^{1/k}} \sqrt{(k+1)^{1+\frac{1}{k}}}. \]

and also

\[ w(S_k) \simeq \sqrt{k \log k} \quad \text{and} \quad w(\tilde{S}_k) \simeq \sqrt{\frac{\log k}{k}}. \]

### 2.4. Mean width.

Let \( K \subseteq \mathbb{R}^n \) be a convex body. The mean width of \( K \) is defined as

\[ w(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta), \]

where, for every \( \theta \in S^{n-1} \), \( h_K(\theta) \) is the support function of \( K \) at \( \theta \) and \( d\sigma \) denotes the uniform probability measure on \( S^{n-1} \). If we also assume that \( K \) contains the origin in its interior, then \( h_K \) is homogeneous of degree 1 and, integrating in polar coordinates, we have that if \( G \) is a standard Gaussian random vector in \( \mathbb{R}^n \) then

\[ E h_K(G) = \int_{\mathbb{R}^n} h_K(x) e^{-\frac{|x|^2}{2}} dx = n |B_2^n| \int_0^\infty r^n e^{-\frac{r^2}{2}} \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) \]

\[ = c_n \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) = c_n w(K), \]

where \( c_n = \frac{n |B_2^n| \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2\pi} \Gamma\left(\frac{n}{2}\right)} = \sqrt{\pi \left(\frac{n+1}{2}\right)}. \) Likewise, since for any convex body containing the origin in its interior the support function of \( K^\circ \) is \( h_{K^\circ} = \| \cdot \|_K \), where \( \| \cdot \|_K \) is the Minkowski gauge function of \( K \), given by

\[ \|x\|_K := \inf\{ \lambda > 0 : x \in \lambda K \} \]

for all \( x \in \mathbb{R}^n \), we have that if \( G \) is a standard Gaussian random vector in \( \mathbb{R}^n \)

\[ E\|G\|_K = c_n w(K^\circ). \]
2.5. **Log-concave functions.** A function \( f: \mathbb{R}^n \to [0, \infty) \) is called log-concave if \( f(x) = e^{-v(x)} \) where \( v: \mathbb{R}^n \to (-\infty, \infty) \) is a convex function. It is well-known that any integrable log-concave function \( f: \mathbb{R}^n \to [0, \infty) \) is bounded and has moments of all orders. If \( K \subseteq \mathbb{R}^n \) is a convex body then its indicator function \( \chi_K \) is integrable and log-concave, with integral \(|K|\) and if \( K \) is a convex body containing the origin, then \( e^{-\|x\|} \) is integrable and log-concave, with integral \( n!!|K|\).

Given a log-concave function \( f = e^{-x} \), where \( v: \mathbb{R}^n \to (-\infty, \infty) \) is a convex function, its polar function is the function \( f^\circ: \mathbb{R}^n \to [0, \infty) \) given by
\[
f^\circ(x) = e^{-L(v)(x)},
\]
where \( L(v) \) denotes the Legendre transform
\[
L(v)(x) = \sup_{y \in \mathbb{R}^n} \langle x, y \rangle - v(y), \quad x \in \mathbb{R}^n.
\]

For more information on log-concave functions we refer the reader to [8, Chapter 2].

2.6. **The Wills functional.** Let us recall that for any \( n \)-dimensional convex body \( K \), its Wills functional is defined as
\[
W(K) = \sum_{i=0}^{n} V_i(K),
\]
where \( V_i(K) \) denotes the \( i \)-th intrinsic volume of \( K \). Many properties of the Wills functional can be found in [25], [14], [19], or [2]. For our purposes we emphasize the following two:

1. (Hadwiger, see [14 (1.3)]) For any convex body \( K \subseteq \mathbb{R}^n \)
\[
W(K) = \int_{\mathbb{R}^n} e^{-\pi d(x,K)^2} \, dx,
\]
where \( d(x,K) \) denotes the Euclidean distance from \( x \) to \( K \).

2. (Hadwiger, see [14 (2.3)]) If \( E \) is a linear subspace of \( \mathbb{R}^n \), \( K_1 \subseteq E \) and \( K_2 \subseteq E^\perp \), then
\[
W(K_1 \times K_2) = W(K_1)W(K_2).
\]
In particular, if \( K = [-a,a] \subseteq \mathbb{R} \) we have that
\[
W([-a,a]) = 2a + 2 \int_{a}^{\infty} e^{-\pi(x-a)^2} \, dx = 2a + 1
\]
and if \( K = aB_\infty^n \subseteq \mathbb{R}^n \) then \( W(aB_\infty^n) = (1 + 2a)^n \).

Let us point out that for any \( \lambda > 0 \)
\[
W(\lambda K) = \sum_{i=0}^{n} V_i(\lambda K) = 1 + \sum_{i=1}^{n} \lambda^i V_i(K).
\]
Therefore, if two convex bodies \( K, L \subseteq \mathbb{R}^n \) verify that \( W(\lambda K) \leq W(\lambda L) \) for every \( \lambda \geq 0 \), then one immediately obtains that \( V_n(K) \leq V_n(L) \) and \( V_1(K) \leq V_1(L) \) or, equivalently, \(|K| \leq |L|\) and \( w(K) \leq w(L)\).

The first property above shows that for any convex body, its Wills functional is the integral of the log-concave function \( f_K: \mathbb{R}^n \to [0, \infty) \) given by
\[
f_K(x) = e^{-\pi d(x,K)^2}.
\]
Lemma 2.1. Let \( A \subseteq \mathbb{R}^n \) be two convex bodies containing the origin in their interiors. Assume that there exist two numbers \( A > 0 \) such that, for any \( K \subseteq (0, \lambda) \),

\[
\int_{\mathbb{R}^n} f_{\lambda K}^\circ(x) dx \leq A \int_{\mathbb{R}^n} f_{\lambda L}^\circ(x) dx.
\]

Then \( |K| \leq A|L| \).

Proof. Notice that for any convex body \( K \subseteq \mathbb{R}^n \) containing the origin in its interior and any \( \lambda > 0 \)

\[
\int_{\mathbb{R}^n} f_{\lambda K}^\circ(x) dx = \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx = \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx = \lambda^n \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx.
\]

Therefore, we have that for every \( \lambda \in (0, \lambda_0) \)

\[
\int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx \leq A \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx
\]

and, taking the limit as \( \lambda \) tends to 0 we obtain that

\[
n!|K| = \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx \leq A \int_{\mathbb{R}^n} e^{-||x|| L} dx = n!|L|.
\]

\[\square\]

2.7. Convex bodies in minimal surface area position. A convex body \( K \subseteq \mathbb{R}^n \) is said to be in minimal surface area position if it has minimal surface area among all of its volume preserving affine images. That is, if

\( \partial(K) = \min \{ \partial(TK) : T \in SL(n) \} \),

where \( SL(n) \) denotes the set of non-degenerate linear maps \( T \in GL(n) \) with \( |\det(T)| = 1 \). The surface area measure of a convex body \( K \) is the measure on the sphere defined by

\( \sigma_K(A) := \nu \{ x \in \partial K : \nu_K(x) \in A \} \quad \forall A \) Borel set in \( S^{n-1} \),

where \( \nu \) denotes the Hausdorff measure on \( \partial K \) and \( \nu_K(x) \) is the outer normal vector to \( K \) at \( x \), which is defined except for a set of measure \( \nu \) equal to 0.

The projection body \( \Pi K \) and its polar, the polar projection body \( \Pi^* K \), of a convex body \( K \) are the centrally symmetric convex bodies defined by

\[
h_{\Pi K}(x) = ||x||_{\Pi^*(K)} = ||x|| P_{x,K} = \frac{1}{2} \int_{S^{n-1}} |\langle x, \theta \rangle| d\sigma_K(\theta),
\]

Using a double polarity (both in the convex body and in the family of log-concave functions) we define for any convex body \( K \subseteq \mathbb{R}^n \) containing the origin in its interior, the log-concave function \( f_{\lambda K}^\circ \). It was proved in \cite{2} Lemma 3.1] that for every \( x \in \mathbb{R}^n \)

\[
f_{\lambda K}^\circ(x) = e^{-\frac{||x||^2}{\lambda^2} - ||x|| \lambda}.
\]

The following lemma shows that if, for every \( \lambda > 0 \), the integral of \( f_{\lambda K}^\circ(x) \) is bounded by the integral of \( f_{\lambda L}^\circ(x) \), then \( |K| \leq |L| \).

Lemma 2.1. Let \( K, L \subseteq \mathbb{R}^n \) be two convex bodies containing the origin in their interiors. Assume that there exist two numbers \( A > 0 \) such that, for any \( \lambda \in (0, \lambda_0) \),

\[
\int_{\mathbb{R}^n} f_{\lambda K}^\circ(x) dx \leq A \int_{\mathbb{R}^n} f_{\lambda L}^\circ(x) dx.
\]

Then \( |K| \leq A|L| \).

Proof. Notice that for any convex body \( K \subseteq \mathbb{R}^n \) containing the origin in its interior and any \( \lambda > 0 \)

\[
\int_{\mathbb{R}^n} f_{\lambda K}^\circ(x) dx = \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx = \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx = \lambda^n \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx.
\]

Therefore, we have that for every \( \lambda \in (0, \lambda_0) \)

\[
\int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx \leq A \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx
\]

and, taking the limit as \( \lambda \) tends to 0 we obtain that

\[
n!|K| = \int_{\mathbb{R}^n} e^{-\frac{||x||^2}{\lambda^2}} e^{-\frac{||x||\lambda}{} dx \leq A \int_{\mathbb{R}^n} e^{-||x|| L} dx = n!A|L|.
\]

\[\square\]
where for any $x \neq 0$, $|P_{\perp} K|$ denotes the $(n - 1)$-dimensional volume of the projection of $K$ onto the hyperplane orthogonal to $x$ and the last equality is the well-known Cauchy’s formula (see, for instance, [23, Equation (5.80)]).

It was proved by Petty [20] (see also [10]) that $K$ is in minimal surface area position if and only if $\sigma_K$ is isotropic, i.e., if

$$I_n = \frac{n}{\partial K} \int_{S^{n-1}} u \otimes ud\sigma_K(u),$$

and it was observed in [11] that the latter happens if and only if $\Pi K$ is in minimal mean width position, i.e.,

$$w(\Pi K) = \min\{w(T\Pi K) : T \in SL(n)\}.$$

Notice that if $K$ is a polytope with facets $\{F_j\}_{j=1}^m$ with outer normal vectors $\{u_j\}_{j=1}^m$ then the surface area measure of $K$ is $\sigma_K = \sum_{j=1}^m |F_j| \delta_{u_j}$, where $\delta_{u_j}$ denotes the Dirac delta measure on $u_j$ and $K$ is in minimal surface area position if and only if

$$I_n = \sum_{j=1}^m n |F_j| u_j \otimes u_j.$$

In particular, if $K$ is a polytope with facets $\{F_j\}_{j=1}^m$ and outer normal vectors $\{u_j\}_{j=1}^m$ then for every $x \in \mathbb{R}^n$

$$h_{\Pi K}(x) = \|x\|_{\Pi^* K} = \frac{1}{2} \sum_{j=1}^m |F_j| \langle x, u_j \rangle.$$

It was proved in [11] that, as a consequence of a lemma obtained from the Brascamp-Lieb inequality (see [2]), if $K$ is a convex body in minimal surface area position then

$$|B_n^2| \left( \frac{n |B_2^n|}{|B_2^n|} \right)^n \frac{1}{\partial(K)^n} \leq |\Pi K| \leq \frac{4^n n^n}{n!} \frac{1}{\partial(K)^n}$$

and, as a consequence of the Blaschke-Santaló inequality and its exact reverse for zonoids (see [12] and [21]), or as a direct consequence of the reverse form of Brascamp-Lieb inequality (see [11]),

$$\left( \frac{\partial(K)}{n} \right)^n \leq |\Pi K| \leq \left( \frac{|B_2^n|}{|B_2^n|} \right)^n \frac{1}{\partial(K)^n}.$$

3. General setting

In this section we introduce the notation for a setting that will be used in several of our proofs. Let $K \subseteq \mathbb{R}^n$ be a (not necessarily symmetric) convex body. Then there exist $\{u_j\}_{j=1}^m \subseteq \partial K \cap S^{n-1}$ and $\{c_j\}_{j=1}^m \subseteq (0, \infty)$ such that

$$I_n = \sum_{j=1}^m c_j u_j \otimes u_j, \quad \sum_{j=1}^m c_j u_j = 0 \quad \text{and} \quad \sum_{j=1}^m c_j = n.$$
where the third equality is obtained from the first one by taking traces. We will denote by $C \subseteq \mathbb{R}^n$ the convex body

$$C = \{ x \in \mathbb{R}^n : \langle x, u_j \rangle \leq 1, \forall 1 \leq j \leq m \}.$$  

It is easily verified that $K \subseteq C$. We will denote, for every $1 \leq j \leq m$,

- $v_j = \frac{n}{n+1} (-u_j, \frac{1}{\sqrt{n}}) \in \mathbb{S}^n$, and
- $\delta_j = \frac{n}{n+1} c_j \in (0,1]$.

These vectors satisfy

$$I_{n+1} = \sum_{j=1}^{m} \delta_j v_j \otimes v_j, \quad \sum_{j=1}^{m} \delta_j v_j = (0, \sqrt{n+1}) \quad \text{and} \quad \sum_{j=1}^{m} \delta_j = n + 1.$$ 

We will denote by $L$ the cone

$$L = \{ y = (x, r) \in \mathbb{R}^{n+1} : \langle y, v_j \rangle \geq 0, \forall 1 \leq j \leq m \}.$$ 

The next lemma relates $L$ and $C$.

**Lemma 3.1.** Let $K \subseteq \mathbb{R}^n$ be a convex body in John’s position and let $L$ be defined as above. Then

$$L = \{ (x, r) \in \mathbb{R}^{n+1} : r \geq 0, x \in \frac{r}{\sqrt{n}} C \}.$$ 

**Proof.** Let $y = (x, r) \in L$. By the definition of $v_j$ we have that for each $1 \leq j \leq m$

$$\langle y, v_j \rangle = -\sqrt{\frac{n}{n+1}} \langle x, u_j \rangle + \frac{r}{\sqrt{n+1}}.$$ 

Assume that $r < 0$. Then, since $\langle y, v_j \rangle \geq 0$ for every $1 \leq j \leq m$ we have that

$$-\sqrt{\frac{n}{n+1}} \langle x, u_j \rangle + \frac{r}{\sqrt{n+1}} \geq 0 \quad \forall 1 \leq j \leq m$$

and then $\langle x, u_j \rangle < 0$ for every $1 \leq j \leq m$. As a consequence, since $\{c_j\}_{j=1}^{m} \subseteq (0, \infty)$,

$$\sum_{j \in J} c_j \langle x, u_j \rangle < 0,$$

which contradicts the fact that $\sum_{j=1}^{m} c_j u_j = 0$. Therefore, if $y = (x, r) \in L$ then $r \geq 0$ and for any $r \geq 0$ we have $\langle y, v_j \rangle \geq 0$ for every $1 \leq j \leq m$ if and only if $\langle x, u_j \rangle \leq \frac{r}{\sqrt{n}}$ for every $1 \leq j \leq m$. The latter condition is true if and only if $x \in \frac{r}{\sqrt{n}} C$.

Conversely, if $y = (x, r)$ verifies that $r \geq 0$ and $x \in \frac{r}{\sqrt{n}} C$, which happens if and only if $\langle x, u_j \rangle \leq \frac{r}{\sqrt{n}}$ for every $1 \leq j \leq m$, then for every $1 \leq j \leq m$

$$\langle y, v_j \rangle = -\sqrt{\frac{n}{n+1}} \langle x, u_j \rangle + \frac{r}{\sqrt{n+1}} \geq 0$$

and then $y \in L$. \hfill $\square$

Given any $k$-dimensional affine subspace $F$ in $\mathbb{R}^n$, we will consider the linear $(k+1)$-dimensional subspace in $\mathbb{R}^{n+1}$

$$H = \text{span}\{ (x, \sqrt{n}) : x \in F \}.$$ 

Notice that if $F \in G_{n,k}$ is a linear subspace then $H = F \times \mathbb{R}$. We set $J = \{ 1 \leq j \leq m : P_H v_j \neq 0 \}$ and, for every $j \in J$, we define
\[ w_j = \frac{P_H v_j}{\|P_H v_j\|_2}, \]
\[ \kappa_j = \delta_j \|P_H v_j\|_2^2 = \frac{n+1}{n} c_j \|P_H v_j\|_2^2. \]

Then, we have that
\[ I_H = \sum_{j \in J} \kappa_j w_j \otimes w_j \quad \text{and} \quad \sum_{j \in J} \kappa_j = k + 1, \]
where \( I_H \) denotes the identity in the linear subspace \( H \). Furthermore, if for every \( j \in J \) we denote by \( s_j = \frac{1}{\|w_j\|_2} \) we have that for every \( y = (x, r) \in H \subseteq \mathbb{R}^{n+1} \)
\[ \sum_{j \in J} \kappa_j s_j \langle y, w_j \rangle = \sum_{j \in J} \delta_j \langle y, P_H v_j \rangle = \sum_{j=1}^m \delta_j \langle y, v_j \rangle = \sum_{j=1}^m \delta_j \langle y, v_j \rangle = r \sqrt{n+1}. \]

The following lemma shows that if \( F \in G_{n,k} \) is a linear subspace, and if \( H = F \times \mathbb{R} \), then we have that \( J = \{1, \ldots, m\} \) and there is a lower bound for the Euclidean norm of \( P_H v_j \) for every \( 1 \leq j \leq m \).

**Lemma 3.2.** Let \( \{u_j\}_{j=1}^m \subseteq S^{n-1}, \{c_j\}_{j=1}^m \) be such that (2.1) holds, \( F \in G_{n,k}, H = F \times \mathbb{R} \in G_{n+1,k+1} \), and \( \{v_j\}_{j=1}^m \subseteq S^n \) be defined as above. Then, for every \( 1 \leq j \leq m \) we have
\[ \frac{1}{n + 1} \leq \|P_H v_j\|_2^2 \leq 1. \]

**Proof.** Let \( c = \left(0, \frac{1}{\sqrt{n+1}}\right) \in H \) and notice that for every \( 1 \leq j \leq m \)
\[ \langle P_H (v_j - c), c \rangle = \langle v_j - c, c \rangle = \frac{1}{n+1} - \frac{1}{n+1} = 0 \]
and then, since \( c \in H \),
\[ \|P_H v_j\|_2^2 = \|c + (P_H (v_j - c))\|_2^2 = \|c\|_2^2 + \|P_H (v_j - c)\|_2^2 \geq \|c\|_2^2 = \frac{1}{n + 1}. \]

Thus, for every \( 1 \leq j \leq m \) we have that
\[ \frac{1}{n + 1} \leq \|P_H v_j\|_2^2 \leq 1. \]

\[ \square \]

If \( K \subseteq \mathbb{R}^n \) is a centrally symmetric convex body in John’s position then we will also denote by \( C_0 \) the symmetric convex body
\[ C_0 = \{x \in \mathbb{R}^n : |\langle x, u_j \rangle| \leq 1, \forall 1 \leq j \leq m\}, \]
which verifies that \( K \subseteq C_0 \). If \( F \in G_{n,k} \) is a linear subspace, we set \( J_0 = \{1 \leq j \leq m : F_{F u_j} \neq 0\} \) and for every \( j \in J_0 \), we define
\[ v^0_j = \frac{F_{F u_j}}{\|F_{F u_j}\|_2}, \]
\[ \delta^0_j = c_j \|F_{F u_j}\|_2^2. \]

Then, we have that
\[ I_F = \sum_{j \in J_0} \delta^0_j v^0_j \otimes v^0_j \quad \text{and} \quad \sum_{j \in J_0} \delta^0_j = k, \]
where \( I_F \) denotes the identity operator in \( F \), and also
\[ K \cap F \subseteq C_0 \cap F = \{x \in F : |\langle x, u_j \rangle| \leq 1, \forall 1 \leq j \leq m\}. \]
where for every \( j \in J_0 \) we have set \( t_j = \frac{1}{\|P_F u_j\|_2} = \left( \frac{c_j}{\delta_j} \right)^{1/2} \). Furthermore,

\[
(K \cap F)^\circ \supseteq (C_0 \cap F)^\circ = P_F(C_0^\circ) = P_F \{ \pm u_j : 1 \leq j \leq m \}
= \text{conv} \{ \pm P_F u_j : 1 \leq j \leq m \}
= \text{conv} \{ \pm P_F u_j : j \in J_0 \}.
\]

Let \( K \) be a (not necessarily centrally symmetric) polytope in minimal surface area position with facets \( \{ F_j \}_{j=1}^m \) and outer normal vectors \( \{ u_j \}_{j=1}^m \), and let \( F \in G_{n,k} \) be a \( k \)-dimensional linear subspace. Then,

\[
K = \{ x \in \mathbb{R}^n : \langle x, u_j \rangle \leq h_K(u_j), \forall 1 \leq j \leq m \}
\]
and

\[
I_n = \sum_{j=1}^m \frac{n|F_j|}{\partial(K)} u_j = \sum_{j=1}^m c_j u_j \otimes u_j,
\]
where \( c_j = \frac{n|F_j|}{\partial(K)} \) for every \( 1 \leq j \leq m \). Besides (see, for instance, [13, Theorem 18.2])

\[
\sum_{j=1}^m c_j u_j = \frac{n}{\partial(K)} \sum_{j=1}^m |F_j| u_j = 0.
\]

and

\[
(3.2) \quad \sum_{j=1}^m c_j h_K(u_j) = \sum_{j=1}^m \frac{n|F_j|}{\partial(K)} h_K(u_j) = \frac{n^2|K|}{\partial(K)}.
\]

Note also that if \( K \) is a centrally symmetric polytope in minimal surface area position, with facets \( \{ F_j \}_{j=1}^m \) and outer normal vectors \( \{ u_j \}_{j=1}^m \), and if \( F \in G_{n,k} \) is a \( k \)-dimensional linear subspace, then

\[
K = \{ x \in \mathbb{R}^n : |\langle x, u_j \rangle| \leq h_K(u_j), \forall 1 \leq j \leq m \}.
\]

As in the case where the decomposition of the identity comes from a centrally symmetric convex body in John’s position, we set \( J_0 = \{ 1 \leq j \leq m : P_F u_j \neq 0 \} \) and, for every \( j \in J_0 \), we define

- \( v_j^0 = \frac{P_F u_j}{\|P_F u_j\|_2} \),
- \( \delta_j^0 = c_j \|P_F u_j\|_2^2 = \frac{n|F_j|\|P_F u_j\|_2^2}{\partial(K)} \).

We have that

\[
I_F = \sum_{j=1}^m c_j P_F u_j \otimes P_F u_j = \sum_{j \in J_0} \delta_j^0 v_j^0 \otimes v_j^0.
\]

Besides, if we denote \( t_j = \frac{1}{\|P_F u_j\|_2} = \left( \frac{c_j}{\delta_j} \right)^{1/2} \) for every \( j \in J_0 \), then

\[
K \cap F = \{ x \in F : |\langle x, v_j^0 \rangle| \leq t_j h_K(u_j), \forall j \in J_0 \}
\]
and

\[
(K \cap F)^\circ = \text{conv} \left\{ \pm \frac{P_F(u_j)}{h_K(u_j)} : 1 \leq j \leq m \right\}.
\]
where for every $k$-dimensional subspace. Following the notation in Section 3 we have that $K \cap H = \{ x \in F : |\langle x, v_j^0 \rangle | \leq t_j, \forall j \in J_0 \}$, where for every $j \in J_0$ we have denoted $t_j = \frac{1}{\| P_F v_j \|_2} = \left( \frac{c_j}{\delta_j^0} \right)^{1/2}$. Therefore, by the Brascamp-Lieb inequality,

$$|K \cap F| \leq |C_0 \cap F| = \int_F \prod_{j \in J_0} \chi_{\{ -t_j, t_j \}}(\langle x, v_j^0 \rangle)dx = \int_F \prod_{j \in J_0} \chi_{\{ -t_j, t_j \}}(\langle x, v_j^0 \rangle)dx \quad \leq \prod_{j \in J_0} \left( \int \chi_{\{ -t_j, t_j \}}(t) \right)^{\delta_j^0} = \prod_{j \in J_0} (2t_j)^{\delta_j^0} = 2^k \prod_{j \in J_0} \left( \frac{c_j}{\delta_j^0} \right)^{\delta_j^0}.$$  

By the arithmetic-geometric mean inequality we get

$$\prod_{j \in J_0} \left( \frac{c_j}{\delta_j^0} \right)^{\delta_j^0} \leq \sum_{j \in J_0} \frac{\delta_j^0}{k} \frac{c_j}{\delta_j^0} = \frac{1}{k} \sum_{j \in J_0} c_j \leq \frac{1}{k} \sum_{j=1}^m c_j = \frac{n}{k},$$

and hence,

$$|K \cap F|^{1/k} \leq 2 \sqrt{\frac{n}{k}} = \sqrt{\frac{n}{k}} |B_k^k|^{1/k}.$$  

Assume now that $K \subseteq \mathbb{R}^n$ is a (not necessarily symmetric) convex body in John’s position and $F \in G_{n,k}$ a linear $k$-dimensional subspace. Following the notation introduced in Section 3 we have that if we take $H = F \times \mathbb{R} \in G_{n+1,k+1}$ then

$$L \cap H = \left\{ (x,r) \in F \times \mathbb{R} : r \geq 0, x \in \frac{r}{\sqrt{n}}(C \cap F) \right\}.$$  

Denoting $s_j = \frac{1}{\| P_{F \times v_j} \|_2}$ we have, by the Brascamp-Lieb inequality, that

$$\int_{L \cap H} e^{-\sum_{j=1}^m \kappa_j s_j(y,w_j) t} \ dy \ = \ \int_{H} \prod_{j=1}^m \left( \chi_{[0,\infty)}(\langle y, v_j \rangle) e^{-s_j(y,w_j)} \right)^{\kappa_j} \ dy \ = \ \int_{H} \prod_{j=1}^m \left( \chi_{[0,\infty)}(\langle y, P_H v_j \rangle) e^{-s_j(y,w_j)} \right)^{\kappa_j} \ dy \ = \ \int_{H} \prod_{j=1}^m \left( \chi_{[0,\infty)}(\langle y, w_j \rangle) e^{-s_j(y,w_j)} \right)^{\kappa_j} \ dy \ \leq \ \prod_{j=1}^m \left( \int_0^\infty e^{-s_j t} dt \right)^{\kappa_j}.$$
On the other hand, taking into account (3.1) we see that
\[
\int_{L \cap H} e^{-\sum_{j=1}^m \kappa_j s_j(y, w_j)} dy = \int_0^\infty \int_{(C \cap F)} e^{-r\sqrt{n+1}} dx dr
\]
\[
= \int_0^\infty \frac{k^k}{n^{k}(n+1)^{k+1}} |C \cap F|
\]
\[
= \frac{k^k}{n^{k}(n+1)^{k+1}} |C \cap F|.
\]

Let us maximize \(\prod_{j=1}^m \|P_{H v_j}\|_2^2 \|P_{H v_j}\|_2^2\) under the constraints
\[
\begin{align*}
&\frac{1}{n+1} \leq \|P_{H v_j}\|_2^2 \leq 1 \forall 1 \leq j \leq m, \\
&\sum_{j=1}^m \delta_j \|P_{H v_j}\|_2^2 = k + 1, \\
&\sum_{j=1}^m \delta_j = n + 1, \\
&0 \leq \delta_j \leq 1.
\end{align*}
\]
Equivalently, let us maximize \(F(x, \delta) = \frac{1}{2} \sum_{j=1}^m \delta_j x_j \log x_j\) under the constraints
\[
\begin{align*}
&\frac{1}{n+1} \leq x_j \leq 1 \forall 1 \leq j \leq m, \\
&\sum_{j=1}^m \delta_j x_j = k + 1, \\
&\sum_{j=1}^m \delta_j = n + 1, \\
&0 \leq \delta_j \leq 1.
\end{align*}
\]
First notice that the function \(F(x, \delta)\) is continuous on the compact domain in \(M \subseteq \mathbb{R}^{2m}\) given by the constraints and, therefore, it attains its maximum. For every \(x = (x_1, \ldots, x_m)\) with \(\frac{1}{n+1} \leq x_j \leq 1\) with \(1 \leq j \leq m\), for all \(1 \leq j \leq m\), let \(F_x(\delta)\) be the function
\[
F_x(\delta) = \frac{1}{2} \sum_{j=1}^m \delta_j x_j \log x_j.
\]
Notice that \(F_x\) is a convex function and then, since the set
\[
A = \{ \delta \in \mathbb{R}^m : \sum_{j=1}^m \delta_j x_j = k + 1, \sum_{j=1}^m \delta_j = n + 1, 0 \leq \delta_j \leq 1 \forall 1 \leq j \leq m \}
\]
is a compact convex set, \(F_x\) attains its maximum on some extreme point of \(A\). These are the points of intersection of the 2-dimensional faces of the cube \(\{ \delta \in \mathbb{R}^m : 0 \leq \delta_j \leq 1 \forall 1 \leq j \leq m \}\) with the \((m-2)\)-dimensional affine subspace
\[
\{ \delta \in \mathbb{R}^m : \sum_{j=1}^m \delta_j x_j = k + 1, \sum_{j=1}^m \delta_j = n + 1 \}.
\]
Therefore, a maximizer of the function \(F_x\) has to be a point of the form
\[
\delta_\lambda = (\underbrace{1, 1, \ldots, 1}_n, \underbrace{\lambda, 1 - \lambda, 0, \ldots, 0}_{m-n-2}).
\]
for some $\frac{1}{2} \leq \lambda \leq 1$ (or a permutation of it) such that $\sum_{j=1}^{m} \delta_j x_j = k + 1$ is satisfied. Let us find, for every $\delta_\lambda$ with $\frac{1}{2} \leq \lambda \leq 1$, the maximizer of the function

$$F_{\delta_\lambda}(x) = \frac{1}{n} \sum_{j=1}^{m} \delta_j x_j \log x_j$$

on the compact convex set

$$B_\lambda = \left\{ x \in \mathbb{R}^m : \sum_{j=1}^{m} \delta_{\lambda,j} x_j = k + 1, \frac{1}{n+1} \leq x_j \leq 1 \ \forall 1 \leq j \leq m \right\}.$$  

We can assume, without loss of generality, that $\delta_\lambda = \delta_\lambda^*$, where $\delta_\lambda^*$ is the decreasing rearrangement of $\delta_\lambda$. Let

$$C = \frac{k + 1}{n + 1}$$

and

$$\tilde{x} = \left( 1, 1, \ldots, 1, C, \frac{1}{n+1}, \ldots, \frac{1}{n+1} \right)_{m-k-1}.$$  

We check that $\frac{1}{n+1} \leq C \leq 1$. Moreover,

$$\sum_{j=1}^{m} \delta_{\lambda,j} \tilde{x}_j = k + C + \frac{n - k}{n + 1} = k + 1$$

and for every $x = (x_1, \ldots, x_m) \in B_\lambda$, since the first $k + 1$ coordinates of $\tilde{x}$ are as large as they can, we have $\tilde{x} \succ (x_1, \ldots, x_m)$, i.e.,

- $\sum_{j=1}^{m} \delta_{\lambda,j} \tilde{x}_j = \sum_{j=1}^{m} \delta_{\lambda,j} x_j = k + 1$,
- $\sum_{j=1}^{m} \delta_{\lambda,j} \tilde{x}_j \geq \sum_{j=1}^{m} \delta_{\lambda,j} x_j \ \forall 1 \leq l \leq m$.

Therefore, by the weighted Karamata’s inequality we have that, for every $x \in B_\lambda$,

$$F_{\delta_\lambda}(x) \leq F_{\delta_\lambda}(\tilde{x}) \leq \max_{(\delta, x) \in M} F(\delta, x)$$

and then, since

$$\max_{(\delta, x) \in M} F(\delta, x) = \max_{\lambda \in \left[ \frac{1}{2}, 1 \right]} F(\delta_\lambda, x) = \max_{\lambda \in \left[ \frac{1}{2}, 1 \right]} \left\{ \frac{1}{2} C \log C + \frac{n - k}{2(n + 1)} \log \left( \frac{1}{n + 1} \right) \right\}$$

we see that

$$\max_{(\delta, x) \in M} F(\delta, x) = \frac{1}{2} C \log C - \frac{n - k}{n + 1} \log(n + 1).$$

Thus,

$$\prod_{j=1}^{m} \left\| P_H v_j \right\|_2^2 \leq e^{\frac{1}{2} C \log C - \frac{n - k}{2(n + 1)}} \frac{n + 1}{(n + 1)^{2(n + 1)}} = \left( \frac{k + 1}{n + 1} \right)^{\frac{n + 1}{2(n + 1)}}.$$
Therefore, since \(|K \cap F| \leq |C \cap F|\) we have that

\[
|K \cap F|^{1/k} \leq \frac{1}{(k + 1)^{n/2(n+1)}} \frac{n(n+1)}{k(k+1)} |S_k|^{1/k}.
\]

Finally, assume now that \(K \subseteq \mathbb{R}^n\) is a (not necessarily symmetric) convex body in John’s position and \(F\) is a \(k\)-dimensional affine subspace at distance \(d\) from 0. Following the notation introduced in Section 3, given the \(k\)-dimensional affine subspace \(F\), if we take the linear subspace \(H = \text{span}\{(x, \sqrt{n}) : x \in F_1\} \in G_{n+1, k+1}\) we have that

\[
L \cap H = \left\{ (x, r) \in \mathbb{R}^{n+1} : r \geq 0, x \in \frac{r}{\sqrt{n}}(C \cap F) \right\}.
\]

Setting \(J = \{1 \leq j \leq m : P_Hv_j \neq 0\}\) and \(s_j = \frac{1}{\|P_Hv_j\|_2}\) we have, by the Brascamp-Lieb inequality, that

\[
\int_{L \cap H} e^{-\sum_{j \in J} \kappa_j s_j \langle y, w_j \rangle} \, dy = \int_H \prod_{j \in J} \left( \chi_{[0, \infty)}(\langle y, v_j \rangle) e^{-s_j \langle y, w_j \rangle} \right)^{\kappa_j} \, dy
\]

\[
= \int_H \prod_{j \in J} \left( \chi_{[0, \infty)}(\langle y, P_Hv_j \rangle) e^{-s_j \langle y, w_j \rangle} \right)^{\kappa_j} \, dy
\]

\[
= \int_H \prod_{j \in J} \left( \chi_{[0, \infty)}(\langle y, w_j \rangle) e^{-s_j \langle y, w_j \rangle} \right)^{\kappa_j} \, dy
\]

\[
\leq \prod_{j \in J} \left( \int_0^\infty e^{-s_j t} \, dt \right)^{\kappa_j} = \prod_{j \in J} \|P_Hv_j\|_2^{\delta_j \|P_Hv_j\|_2^2}
\]

\[
\leq 1.
\]

Taking into account (3.1) we have that

\[
\int_{L \cap H} e^{-\sum_{j \in J} \kappa_j s_j \langle y, w_j \rangle} \, dy = \int_0^\infty \int_{\mathbb{R}^{n+1}(C \cap F)} e^{-\sqrt{n+1} r} \, dx \sqrt{\frac{n + d^2}{n}} \, dr
\]

\[
= \int_0^\infty \left( \frac{\sqrt{n}}{\sqrt{n+1}} \right)^{k/2} |C \cap F| e^{-\sqrt{n+1} r} \, dr
\]

\[
= \frac{(n + d^2)^{\frac{k}{2}} |C \cap F|}{n^{\frac{k}{2}} (n + 1)^{\frac{k}{2} + 1}}.
\]

Since \(|K \cap F| \leq |C \cap F|\), we get

\[
|K \cap F| \leq \frac{n^k (n + 1)^{\frac{k+1}{2}}}{k!} \sqrt{\frac{n}{n + d^2}}
\]

or, equivalently,

\[
|K \cap F|^{1/k} \leq \sqrt{\frac{n(n + 1) \frac{1}{2} + \frac{k}{2}}{k(k+1)} \left( \frac{n}{n + d^2} \right)^{\frac{1}{2}}} |S_k|^{1/k}.
\]

\(\square\)
5. Volume of projections of convex bodies in Löffner’s position

In this section we will give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let us start with the symmetric case. Assume that $K$ is a centrally symmetric convex body in John’s position. Following the notation in Section 3 we have that $K \subseteq C_0$, therefore $K \cap F \subseteq C_0 \cap F$. This gives that

$$(K \cap F)^0 \supseteq (C_0 \cap F)^0 = \text{conv}\{\pm P_F u_j : j \in J_0\}.$$ 

Therefore, for every $x \in F$ we have that

$$h_{K \cap F}(x) \leq h_{C_0 \cap F}(x) = \|x\|_{(C_0 \cap F)^0} = \inf \left\{ \sum_{j \in J_0} |\alpha_j| : x = \sum_{j \in J_0} \alpha_j P_F u_j \right\}$$

$$= \inf \left\{ \sum_{j \in J_0} |\alpha_j| : x = \sum_{j \in J_0} \alpha_j \|P_F u_j\|_{2^0} \right\}$$

$$= \inf \left\{ \sum_{j \in J_0} |\beta_j| : x = \sum_{j \in J_0} \beta_j v_j^0 \right\}$$

$$= \inf \left\{ \sum_{j \in J_0} \delta_j^0 \theta_j t_j : x = \sum_{j \in J_0} \delta_j^0 \theta_j v_j^0 \right\},$$

where we have set, for every $j \in J_0$, $t_j = \frac{1}{\|P_F u_j\|_{2^0}} = \left(\frac{c_j}{n}\right)^{\frac{1}{2}}$. For every $j \in J_0$, we set

$$f_j(t) := e^{-|t| t_j}, \quad t \in \mathbb{R}.$$ 

Then, if $x = \sum_{j \in J_0} \delta_j^0 \theta_j v_j^0$ for some $\{\theta_j\}_{j \in J_0} \subseteq \mathbb{R}$, we have

$$\prod_{j \in J_0} f_j^{\delta_j^0}(\theta_j) = e^{-\sum_{j \in J_0} \delta_j^0 |\theta_j| t_j} \leq e^{-h_{K \cap F}(x)}.$$ 

Therefore, by the reverse Brascamp-Lieb inequality we obtain

$$k! |(K \cap F)^0| = \int_F e^{-h_{K \cap F}(x)} dx \geq \prod_{j \in J} \left( \int_{\mathbb{R}} e^{-|t| t_j} dt \right)^{\delta_j}$$

$$= \frac{2^k}{\prod_{j \in J_0} \frac{c_j}{n}} \leq \frac{2^k}{\prod_{j \in J_0} \frac{c_j}{n}}.$$ 

As we have seen in the proof of Theorem 1.1

$$\prod_{j \in J_0} \left( \frac{c_j}{n} \right)^{\frac{1}{2^k}} \leq \sum_{j \in J_0} \frac{\delta_j^0 c_j}{k \delta_j} = \frac{1}{k} \sum_{j \in J_0} c_j \leq \frac{1}{k} \sum_{j=1}^m c_j = \frac{n}{k}.$$ 

Taking into account that $|(B^k_\infty)^0| = |B^k_1| = \frac{2^k}{k!}$, we obtain

$$|(K \cap F)^0|^{1/k} \geq \sqrt[k]{\frac{k}{n}} |(B^k_\infty)^0|^{1/k}.$$
Assume now that \( K \subseteq \mathbb{R}^n \) is a (not necessarily symmetric) convex body in John’s position and \( F \in G_{n,k} \) is a \( k \)-dimensional linear subspace. Following the notation in Section 3 we have that

\[
(C \cap F)^c = P_F(C^c) = P_F(\text{conv}\{u_j : 1 \leq j \leq m\}) = \text{conv}\{P_Fu_j : 1 \leq j \leq m\}.
\]

Let us denote, for any \( y = (x, r) \in H = F \times \mathbb{R} \)

\[
N(y) = \inf \left\{ \sum_{j=1}^m \frac{\kappa_j \theta_j}{\sqrt{n\|P_Fu_j\|^2 + 1}} : \theta_j \geq 0, y = \sum_{j=1}^m \kappa_j \theta_j w_j \right\},
\]

where the latter infimum is understood as \( \infty \) if there do not exist \( \{\theta_j\}_{j=1}^m \) with \( \theta_j \geq 0 \) such that \( y = \sum_{j=1}^m \kappa_j \theta_j w_j \). Notice that, for any \( \{\theta_j\}_{j=1}^m \subseteq \mathbb{R} \),

\[
y = \sum_{j \in J} \kappa_j \theta_j w_j \iff (x, r) = \left( -\sum_{j=1}^m \frac{\kappa_j \theta_j P_F(u_j)}{\sqrt{n\|P_Fu_j\|^2 + 1}}, \sum_{j=1}^m \frac{\kappa_j \theta_j}{\sqrt{n\|P_Fu_j\|^2 + 1}} \right) \]

\[
\iff (x, r) = \left( -r \sqrt{n} \sum_{j=1}^m \frac{\kappa_j \theta_j P_F(u_j)}{\sqrt{n\|P_Fu_j\|^2 + 1}}, \sum_{j=1}^m \frac{\kappa_j \theta_j}{\sqrt{n\|P_Fu_j\|^2 + 1}} \right)
\]

and then there exist \( \{\theta_j\}_{j=1}^m \subseteq \mathbb{R} \) with \( \theta_j \geq 0 \) for every \( 1 \leq j \leq m \) such that the latter equality holds if and only if

\[
(x, r) \in L_1 := \left\{ (x, r) \in F \times \mathbb{R} : r \geq 0 : x \in -r \sqrt{n}(C \cap F)^c \right\},
\]

and for all such \( y = (x, r) \in L_1 \) we have that \( N(y) = r \). Therefore, for every \( y \in H \)

\[
\sup_{y = \sum_{j=1}^m \kappa_j \theta_j w_j} \prod_{j=1}^m \left( \chi_{[0,\infty)}(\theta_j) e^{-\frac{\kappa_j \theta_j}{\sqrt{n\|P_Fu_j\|^2 + 1}}} \right)^{\kappa_j} = e^{-N(y)}.
\]

Thus, by the reverse Brascamp-Lieb inequality,

\[
\int_H e^{-N(y)} dy \geq \prod_{j=1}^m \left( \int_0^\infty e^{-\frac{\kappa_j \theta_j}{\sqrt{n\|P_Fu_j\|^2 + 1}}} dt \right)^{\kappa_j}.
\]

On the one hand,

\[
\int_H e^{-N(y)} dy = \int_0^\infty e^{-r} \left| -r \sqrt{n}(C \cap F)^c \right| dr = k! n^{k/2} |(C \cap F)^c|.
\]

On the other hand, for every \( 1 \leq j \leq m \)

\[
\int_0^\infty e^{-\frac{r \sqrt{n\|P_Fu_j\|^2 + 1}}{\sqrt{n\|P_Fu_j\|^2 + 1}}} dr = \sqrt{n\|P_Fu_j\|^2 + 1} = \sqrt{n + 1}\|P_Fv_j\|_2.
\]

Therefore, since \( (K \cap F)^c \geq (C \cap F)^c \), we obtain

\[
|(K \cap F)^c| \geq \left( \frac{n + 1}{k + 1} \right)^{k \over 2} \prod_{j=1}^m \|P_Fv_j\|_2^{\delta_j} \|P_Hv_j\|_2^2 \\
= \left( \frac{n + 1}{k + 1} \right)^{k^2 \over 2} \left( \frac{k}{n} \right)^{k^2 \over 2} \prod_{j=1}^m \|P_Hv_j\|_2^{\delta_j} \|P_Hv_j\|_2^2 |S_k^0|.
\]
Since the function \( f(x) = x \log x \) is convex in \((0, \infty)\) we have, by Jensen’s inequality, that
\[
\sum_{j=1}^{m} \frac{\delta_j}{n+1} \|P_H v_j\|^2 \log \|P_H v_j\|^2 \geq f \left( \sum_{j=1}^{m} \frac{\delta_j}{n+1} \|P_H v_j\|^2 \right) = f \left( \frac{k+1}{n+1} \right)
\]
and then
\[
\prod_{j=1}^{m} \|P_H v_j\|^2_2 = e^{\frac{n+1}{2} \sum_{j=1}^{m} \frac{\delta_j}{n+1} \log \|P_H v_j\|^2} \geq \left( \frac{k+1}{n+1} \right)^{\frac{k+1}{2}}.
\]
Therefore,
\[
|(K \cap F)^\circ|_{c} \geq \sqrt{n} S_k^{\frac{1}{n}}.
\]

6. MEAN WIDTH OF SECTIONS OF CONVEX BODIES IN JOHN’S POSITION

In this section we will prove Theorem 1.3.

**Proof of Theorem 1.3.** Let us start with the symmetric case. Assume that \( K \) is a centrally symmetric convex body in John’s position and \( F \in G_{n,k} \) is a \( k \)-dimensional linear subspace. Following the notation in Section 3 we have that \( K \subseteq C_0 \), \( K \cap F \subseteq C_0 \cap F \), and
\[
(K \cap F)^\circ \supseteq (C_0 \cap F)^\circ = \text{conv}\{ \pm P_F u_j : j \in J_0 \}.
\]
Thus, for every \( x \in F \),
\[
h_{K \cap F}(x) \leq h_{C_0 \cap F}(x) = \inf \left\{ \sum_{j \in J_0} |\alpha_j| : x = \sum_{j \in J} \alpha_j P_F u_j \right\}
\]
\[
= \inf \left\{ \sum_{j \in J_0} |\alpha_j| : x = \sum_{j \in J_0} \alpha_j \|P_F u_j\|_2^2 v_j^0 \right\}
\]
\[
= \inf \left\{ \sum_{j \in J_0} |\beta_j| t_j : x = \sum_{j \in J_0} \beta_j v_j^0 \right\},
\]
where for every \( j \in J_0 \) we have set \( t_j = \frac{1}{\|P_F u_j\|_2^2} \). Since, for every \( x \in F \), we have that
\[
x = \sum_{j \in J_0} \delta_j^0 \langle x, v_j^0 \rangle v_j^0,
\]
then, for every \( \theta \in S_F \),
\[
h_{K \cap F}(x) \leq \sum_{j \in J_0} \delta_j^0 t_j \langle x, v_j^0 \rangle.
\]
Therefore, if \( G_1 \) is a standard Gaussian random vector in \( F \) and \( G_2 \) is a standard Gaussian random vector on \( \mathbb{R}^k \),
\[
\mathbb{E} h_{K \cap F}(G_1) \leq \sum_{j \in J_0} \delta_j^0 t_j \mathbb{E} \|G_1, v_j\| = \mathbb{E} \|G_2, e_1\| \sum_{j \in J_0} \delta_j^0 t_j
\]
\[
= \frac{1}{k} \mathbb{E} \|G_2\| \sum_{j \in J_0} \delta_j^0 t_j = \frac{1}{k} \sum_{j \in J_0} \delta_j^0 t_j \mathbb{E} h_{B_k^0}(G_2).
\]
Since, by Hölder’s inequality,
\[
\frac{1}{k} \sum_{j \in J_0} \delta_j^0 t_j = \frac{1}{k} \sum_{j \in J_0} c_j \|P_F u_j\|_2 \leq \frac{1}{k} \left( \sum_{j \in J_0} c_j \right)^\frac{1}{p} \left( \sum_{j \in J_0} c_j \|P_F u_j\|_2^2 \right)^\frac{1}{q} \leq \frac{1}{k} \left( \sum_{j \in J_0} c_j \right)^\frac{1}{p} \left( \frac{1}{k} \sum_{j \in J_0} \delta_j^0 t_j \right)^\frac{1}{q} \leq \frac{\sqrt{nk}}{k} = \sqrt{n/k},
\]
we obtain
\[
\text{E} h_{K \cap F}(G_1) \leq \sqrt{\frac{n}{k}} \text{E} h_{B^k}(G_2).
\]
Equivalently
\[
w(K \cap F) \leq \sqrt{\frac{n}{k}} w(B^k).
\]
Let us now assume that $K$ is a not necessarily symmetric convex body in John’s position and let $F \in G_{n,k}$. Following the notation in Section 3, we have that for every $x \in F$
\[
h_{K \cap F}(x) \leq h_{C \cap F}(x) = \inf \left\{ \sum_{j=1}^m a_j : x = \sum_{j=1}^m a_j P_F u_j, a_j \geq 0 \right\}.
\]
Let $\theta \in S_F$. Since $\sum_{j=1}^m c_j P_F u_j = 0$ we may write
\[
\theta = \sum_{j=1}^m c_j \|P_F u_j\|_2 \left( \frac{\theta}{\|P_F u_j\|_2} - \min_{1 \leq k \leq m} \left\{ \frac{\theta}{\|P_F u_k\|_2} \right\} \right) P_F u_j
\]
and then, setting (like in the symmetric case before) $J_0 = \{1 \leq j \leq m : P_F u_j \neq 0\}$ and $v_j = \frac{P_F u_j}{\|P_F u_j\|_2}$ for $j \in J_0$, we get
\[
w(K \cap F) \leq w(C \cap F) = \int_{S_F} h_{C \cap F}(\theta) d\sigma(\theta)
\]
\[
\leq \int_{S_F} \sum_{j=1}^m c_j \|P_F u_j\|_2 \left( \langle \theta, v_j^0 \rangle - \min_{1 \leq k \leq m} \langle \theta, v_k^0 \rangle \right) d\sigma(\theta)
\]
\[
= \int_{S_F} \sum_{j=1}^m c_j \|P_F u_j\|_2 \max_{1 \leq k \leq m} \left| \langle \theta, v_k^0 \rangle \right| d\sigma(\theta)
\]
\[
= \sum_{j=1}^m c_j \|P_F u_j\|_2 \int_{S_F} \max_{1 \leq k \leq m} \left| \langle \theta, v_k^0 \rangle \right| d\sigma(\theta).
\]
Since by Cauchy-Schwarz inequality
\[
\sum_{j=1}^m c_j \|P_F u_j\|_2 \leq \left( \sum_{j=1}^m c_j \right)^\frac{1}{p} \left( \sum_{j=1}^m c_j \|P_F u_j\|_2^2 \right)^\frac{1}{q} = \sqrt{nk}
\]
and since there exists an absolute constant \( c > 0 \) such that
\[
\int_{S'} \max_{1 \leq i \leq m} |\langle \theta, v_i^f \rangle| \, d\sigma(\theta) \leq c \sqrt{\log m/k},
\]
taking into account that \( m = O(n^2) \) and that \( w(S_k) \approx \sqrt{k \log k} \) we obtain that there exists an absolute constant \( C > 0 \) such that
\[
w(K \cap F) \leq C \sqrt{\frac{n \log n}{k \log k}} w(S_k).
\]

\[ \square \]

7. MEAN WIDTH OF PROJECTIONS OF CONVEX BODIES IN LÖWNER’S POSITION

In this section we will prove Theorem 1.4. We will make use of the following lemma.

**Lemma 7.1.** Let \( K \subseteq \mathbb{R}^n \) be a (not necessarily symmetric) convex body, let \( F \) be a \( k \)-dimensional affine subspace at distance \( d \) from the origin and let \( \alpha, \beta \in \mathbb{R} \), \( \beta \leq 0 \). Let us identify \( F \) with \( \mathbb{R}^k \) with the origin at the closest point in \( F \) to \( 0 \) and let \( \gamma_k \) be the \( k \)-dimensional Gaussian measure on \( F \). Then,
\[
\sqrt{\frac{n + d^2}{n}} \int_0^\infty e^{-\frac{(r-\alpha \sqrt{n+1})^2}{2 \pi}} e^{\beta \sqrt{n+1} \gamma_k \left( \frac{r}{\sqrt{n}} (C \cap F) \right)} \, dr \leq \int_0^\infty e^{-\frac{(r-\alpha d_1)^2}{2 \pi}} e^{\beta \sqrt{k+1} \gamma_k \left( r \sqrt{k+1} \Delta_k \right)} \, dr
\]
where \( d_1 = \frac{1}{\sqrt{k+1}} \sum_j \delta_j \| P_H v_j \|_2 \), and \( H, J, v_j \), and \( \delta_j \) are defined as in Section 3.

**Proof.** Following the notation introduced in Section 3 we have that if \( H \in G_{n+1,k+1} \) is the \((k+1)\)-dimensional linear subspace \( H = \text{span}\{(x, \sqrt{n}) : x \in F\} \) then
\[
L \cap H = \left\{(x, r) : r \geq 0, x \in \frac{r}{\sqrt{n}}(C \cap F)\right\}.
\]
For any \( \alpha, \beta \in \mathbb{R} \), let \( \mu_{\alpha, \beta} \) be the measure on \( H \) whose density with respect to the Lebesgue measure at a point \( y = (x, r) \) is
\[
d\mu_{\alpha, \beta}(r) = e^{-\frac{\|y\|^2}{2}} e^{(\alpha + \beta) \sqrt{n+1} r} dy.
\]
Denoting by \( \gamma_k \) the \( k \)-dimensional Gaussian measure on \( F \), we have that, on the one-hand, for any \( \alpha, \beta \in \mathbb{R} \)
\[
\mu_{\alpha, \beta}(L \cap H) = \int_0^\infty e^{-\frac{r^2}{2 \pi}} e^{\alpha r \sqrt{n+1}} e^{\beta \sqrt{n+1} r} \gamma_k \left( \frac{r}{\sqrt{n}} (C \cap F) \right) \sqrt{\frac{n + d^2}{n}} \, dr
\]
\[
= e^{\alpha^2(n+1)/2} \int_0^\infty e^{-\frac{(r-\alpha \sqrt{n+1})^2}{2 \pi}} e^{\beta \sqrt{n+1} r} \gamma_k \left( \frac{r}{\sqrt{n}} (C \cap F) \right) \sqrt{\frac{n + d^2}{n}} \, dr.
\]
On the other hand, setting $J = \{1 \leq j \leq m : P_Hv_j \neq 0\}$ and $s_j = \frac{1}{\|P_Hv_j\|_2}$ for every $j \in J$, and denoting $y = (x, r) \in H$ we have, by the Brascamp-Lieb inequality,

$$
\mu_{\alpha, \beta}(L \cap H) = \int_H e^{\frac{\|y\|^2}{2}} e^{\alpha \sqrt{n+1}r} e^{\beta \sqrt{n+1}r} \prod_{j \in J} \chi_{[0, \infty)}((y, v_j))dy
$$

$$
= \int_H \left( \frac{e^{-\|y\|^2/2}}{2\pi} \right)^{(k+1)/2} \prod_{j \in J} \chi_{[0, \infty)}((y, v_j))dy
$$

$$
= \int_H \prod_{j \in J} \left( \frac{e^{-\|y\|^2/2}}{\sqrt{2\pi}} \right)^{\kappa_j} \prod_{j \in J} \chi_{[0, \infty)}((y, v_j))dy
$$

$$
\leq \prod_{j \in J} \left( \int_0^\infty e^{-\frac{(t-a_j)^2}{2}} e^{\beta s_j t} dt \right)^{\kappa_j}
$$

$$
= \prod_{j \in J} \left( \int_0^\infty e^{-\frac{(t-a_j)^2}{2}} e^{\beta s_j t} dt \right)^{\kappa_j}
$$

Therefore, for any $\alpha, \beta \in \mathbb{R}$,

$$
\sqrt{n + d^2/n} \int_0^\infty e^{-\frac{(r-a\sqrt{n+1})^2}{2\pi}} e^{\beta r \sqrt{n+1}} \chi_k \left( \frac{r}{\sqrt{n}} (C \cap F) \right) dr
$$

$$
\leq \prod_{j \in J} \left( \int_0^\infty e^{-\frac{(t-a_j)^2}{2}} e^{\beta s_j t} dt \right)^{\kappa_j}
$$

and, by the Prékopa-Leindler inequality, setting $d_1 = \frac{1}{\sqrt{k+1}} \sum_{j=1}^m \kappa_j s_j$ we have that if $\beta \leq 0$ then (taking into account that $s_j = \frac{1}{\|P_Hv_j\|_2} \geq 1$)

$$
\sqrt{n + d^2/n} \int_0^\infty e^{-\frac{(r-a\sqrt{n+1})^2}{2\pi}} e^{\beta r \sqrt{n+1}} \chi_k \left( \frac{r}{\sqrt{n}} (C \cap F) \right) dr
$$

$$
\leq \prod_{j \in J} \left( \int_0^\infty e^{-\frac{(t-a_j)^2}{2}} e^{\beta s_j t} dt \right)^{\kappa_j}
$$

$$
\leq \left( \int_0^\infty e^{-\frac{(t-a\sqrt{n+1})^2}{2\pi}} e^{\beta t} dt \right)^{k+1}
$$

$$
= \int_{(0, \infty)^{k+1}} \prod_{i=1}^{k+1} \frac{(t_i - a_i \sqrt{n+1})^2}{\sqrt{2\pi}} e^{\beta t_i} dt
$$

$$
= \int_{(-a \sqrt{n+1}, \infty)^{k+1}} \prod_{i=1}^{k+1} \frac{t_i^2}{\sqrt{2\pi}} e^{\beta t_i} dt,
$$
where \( v_0 = \left( \frac{1}{\sqrt{k+1}}, \ldots, \frac{1}{\sqrt{k+1}} \right) \). Therefore, for any \( \alpha \in \mathbb{R} \) and any \( \beta \leq 0 \),

\[
\sqrt{n + d^2/n} \int_0^\infty e^{-\frac{(r - \alpha \sqrt{n+1})^2}{2}} e^{\beta r \sqrt{n+1}} \gamma_k \left( \frac{r}{\sqrt{n}}(C \cap F) \right) \, dr 
\leq \int_{\left[ \frac{-\alpha d}{\sqrt{k+1}}, \infty \right)} e^\frac{1}{2} e^{\beta \sqrt{k+1} (t, v_0)} e^{\beta \alpha d_1 \sqrt{k+1}} \, dt 
= \int_{-\alpha d_1}^\infty e^{-\frac{(r - \alpha d_1)^2}{2}} e^{\beta \sqrt{k+1} (t, v_0)} e^{\beta \alpha d_1 \sqrt{k+1}} \gamma_k \left( (t + \alpha d_1) \sqrt{k + 1} \Delta_k \right) \, dt 
= \int_0^\infty e^{-\frac{(r - \alpha d_1)^2}{2}} e^{\beta \sqrt{k+1} (t, v_0)} e^{\beta \alpha d_1 \sqrt{k+1}} \gamma_k \left( r \sqrt{k + 1} \Delta_k \right) \, dr.
\]

Proof of Theorem 1.4. Let us start with the symmetric case. Assume that \( K \) is a centrally symmetric convex body in John’s position and \( F \in G_{n,k} \) is a \( k \)-dimensional linear subspace. We want to prove that

\[
w((K \cap F)^o) \geq w \left( \left( \sqrt[n]{B_k} \right)^\circ \right),
\]

which is equivalent to

\[
E\|G_1\|_{K \cap F} \geq E\|G_2\|_{\sqrt[n]{B_k}},
\]

where \( G_1 \) is a standard Gaussian random vector on \( F \) and \( G_2 \) is a standard Gaussian random vector on \( \mathbb{R}^k \). Since for any convex body \( L \subseteq \mathbb{R}^k \) containing the origin in its interior we have, by Fubini’s theorem, that if \( G \) is a standard Gaussian random vector then

\[
E\|G\|_L = \int_0^\infty P(\|G\|_L \geq t) \, dt = \int_0^\infty \gamma_k(\mathbb{R}^n \setminus tL) \, dt
\]

where \( \gamma_k(A) \) denotes the Gaussian measure of the \( k \)-dimensional set \( A \), we obtain that the statement we want to prove is equivalent to

\[
\int_0^\infty \gamma_k(F \setminus t(K \cap F)) \, dt \geq \int_0^\infty \gamma_k(\mathbb{R}^n \setminus t\sqrt[n]{B_k}) \, dt
\]

or, equivalently,

\[
\int_0^\infty (1 - \gamma_k(t(K \cap F))) \, dt \geq \int_0^\infty \left( 1 - \gamma_k \left( t\sqrt[n]{B_k} \right) \right) \, dt.
\]

We are going to prove that for any \( t \geq 0 \)

\[
\gamma_k(t(K \cap F)) \leq \gamma_k \left( t\sqrt[n]{B_k} \right),
\]

which implies the latter inequality.

Following the notation in Section 3, since \( K \) is a centrally symmetric convex body in John’s position we have that

\[
K \cap F \subseteq C_0 \cap F = \left\{ x \in F : \langle x, v_j^0 \rangle \leq t_j, \forall j \in J_0 \right\},
\]
where, for every $j \in J_0$ we have set $t_j = \frac{1}{\| F_{-t_j} \|_2} = \left( \frac{c_j}{\delta_j} \right)^{1/2}$. Therefore, for every $t \geq 0$,

$$t(K \cap F) \subseteq t(C_0 \cap F) = \{ x \in F : |\langle x, v_j^0 \rangle | \leq tt_j, \forall j \in J \},$$

and then, by the Brascamp-Lieb inequality

$$\gamma_k(t(K \cap F)) \leq \gamma_k(t(C_0 \cap F)) = \int_F \left( \prod_{j \in J} \chi_{[-tt_j, tt_j]}(\langle x, v_j^0 \rangle) \right) e^{-\| x \|^2 / (2\pi)^{k/2}} dx \leq \prod_{j \in J_0} \left( \int_{-tt_j}^{tt_j} e^{-t^2 / 2} dt \right)^{\delta_j^0} = \left( \prod_{j \in J_0} \gamma_1(tt_j|e_1, e_1)\frac{s_j^0}{\pi} \right)^k.$$

Since $\gamma_1$ is log-concave, we have that

$$\gamma_k(t(K \cap F)) \leq \gamma_1 \left( \left( \frac{t}{\frac{\sum_{j \in J_0} t_j^0}{k}} \right)^k \right)$$

Since, by Hölder’s inequality

$$\sum_{j \in J_0} \frac{t_j^0}{k} \leq \sum_{j \in J_0} \frac{\sqrt{c_j^0}}{k} \leq \frac{1}{k} \left( \sum_{j \in J_0} c_j \right)^{1/2} \left( \sum_{j \in J_0} \delta_j^0 \right)^{1/2} \leq \frac{1}{k} \left( \sum_{j=1}^m c_j \right)^{1/2} \left( \sum_{j=1}^m \delta_j^0 \right)^{1/2} = \sqrt{\frac{n}{k}},$$

we obtain that for every $t \geq 0$

$$\gamma_k(t(K \cap F)) \leq \gamma_k \left( t \sqrt{\frac{n}{k}} B_k^\infty \right).$$

Assume now that $K \subseteq \mathbb{R}^n$ is a (not necessarily symmetric) convex body in John’s position and $F \in G_{n,k}$ is a linear subspace. Following the notation introduced in Section 3 we have that if $H = F \times \mathbb{R}$ then

$$L \cap H = \left\{ (x, r) \in F \times \mathbb{R} : r \geq 0, x \in \frac{r}{\sqrt{n}} (C \cap F) \right\}.$$

By Lemma 7.1 with $\beta = 0$ and taking into account that $F \in G_{n,k}$ is a linear subspace, for any $\alpha \in \mathbb{R}$ we have that

$$\int_0^\infty e^{-\frac{(r - \sqrt{n}(C \cap F))^2}{2\sqrt{2\pi}}} \gamma_k \left( \frac{r}{\sqrt{n}} (C \cap F) \right) dr \leq \int_0^\infty e^{-\frac{(r - \sqrt{k} + \Delta_k)}{2\sqrt{2\pi}}} \gamma_k \left( r \sqrt{k} + 1 \Delta_k \right) dr,$$
where \( d_1 = \frac{1}{\sqrt{k+1}} \sum_{j \in J} \delta_j \| P_H v_j \|_2 \). Applying the latter inequality to \(-\alpha\), we also get

\[
\int_0^\infty e^{-\frac{(r-\alpha+1) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( \frac{r}{\sqrt{n}} (C \cap F) \right) dr \leq \int_0^\infty e^{-\frac{(r-\alpha) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( r\sqrt{k+1} \Delta_k \right) dr
\]
or, equivalently,

\[
\int_{-\infty}^0 e^{-\frac{(r-\alpha+1) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( \frac{|r|}{\sqrt{n}} (C \cap F) \right) dr \leq \int_{-\infty}^0 e^{-\frac{(r-\alpha) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( |r|\sqrt{k+1} \Delta_k \right) dr.
\]

Therefore, for any \( \alpha \in \mathbb{R} \),

\[
\int_{-\infty}^\infty e^{-\frac{(r-\alpha+1) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( \frac{|r|}{\sqrt{n}} (C \cap F) \right) dr \leq \int_{-\infty}^\infty e^{-\frac{(r-\alpha) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( |r|\sqrt{k+1} \Delta_k \right) dr
\]

d and hence,

\[
\int_{-\infty}^\infty e^{-\frac{(r-\alpha+1) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( F \setminus \left( \frac{|r|}{\sqrt{n}} (C \cap F) \right) \right) dr \geq \int_{-\infty}^\infty e^{-\frac{(r-\alpha) \gamma_k}{\sqrt{2}\pi}} \gamma_k \left( \mathbb{R}^k \setminus (|r|\sqrt{k+1} \Delta_k) \right) dr.
\]

Integrating in polar coordinates, and taking into account that \( K \subseteq C \) we obtain

\[
\frac{1}{\sqrt{n+1}} \int_0^\infty \gamma_k \left( F \setminus \left( \frac{r}{\sqrt{n}} (C \cap F) \right) \right) dr \geq \frac{1}{d_1} \int_0^\infty \gamma_k \left( \mathbb{R}^k \setminus (r\sqrt{k+1} \Delta_k) \right) dr.
\]

Equivalently,

\[
\frac{1}{\sqrt{n+1}} \int_0^\infty \gamma_k \left( F \setminus \left( \frac{r}{\sqrt{n}} (C \cap F) \right) \right) dr \geq \frac{1}{d_1} \int_0^\infty \gamma_k \left( \mathbb{R}^k \setminus (r\sqrt{k+1} \Delta_k) \right) dr,
\]
or

\[
\sqrt{\frac{n}{n+1}} \int_0^\infty \gamma_k \left( F \setminus (r(C \cap F)) \right) dr \geq \frac{1}{d_1 \sqrt{k+1}} \int_0^\infty \gamma_k \left( \mathbb{R}^k \setminus (r\Delta_k) \right) dr.
\]

Integrating in polar coordinates, and taking into account that \( K \subseteq C \) we obtain

\[
w((K \cap F)^\circ) \geq \frac{1}{d_1} \sqrt{\frac{n+1}{n(k+1)}} w((\Delta_k)^\circ).
\]

Since

\[
\sqrt{k(k+1)\Delta_k} = S_k,
\]

where \( S_k \) denotes the \( k \)-dimensional regular simplex in John’s position, we have that, for any \( k \)-dimensional subspace \( F \),

\[
w((K \cap F)^\circ) \geq \frac{1}{d_1} \sqrt{\frac{k(n+1)}{n}} w((S_k)^\circ).
\]

Thus, for any \( k \)-dimensional subspace \( F \),

\[
w((K \cap F)^\circ) \geq \sqrt{\frac{k(k+1)(n+1)}{n}} \frac{1}{d_1} w((S_k)^\circ).
\]

Since

\[
d_1 = \sum_{j=1}^m \delta_j \| P_H v_j \|_2 \leq \left( \sum_{j=1}^m \delta_j \right)^{\frac{1}{2}} \left( \sum_{j=1}^m \delta_j \| P_H v_j \|_2^2 \right)^{\frac{1}{2}} = \sqrt{(n+1)(k+1)},
\]
we have that
\[ w((K \cap F)^\circ) \geq \sqrt{\frac{k}{n}} w(S_k^\circ). \]

□

8. The Wills functional of sections of convex bodies in John’s position

In this section we will give the proof of Theorem 1.5.

Proof of Theorem 1.5. Let \( K \) be a centrally symmetric convex body in John’s position and \( F \in G_{n,k} \) a \( k \)-dimensional linear subspace. Following the notation in Section 3 we have that, for every \( \lambda \geq 0 \),
\[ \lambda(K \cap F) \subseteq \lambda(C_0 \cap F) = \{ x \in F : |\langle x, v_0^j \rangle| \leq \lambda t_j, \forall j \in J_0 \}, \]
where, for every \( j \in J_0 \), we denote \( t_j = \frac{1}{\|P_{F}u_j\|_2} = \left( \frac{c_j}{\lambda t_j} \right)^{1/2} \). Let, for every \( j \in J_0 \),
\[ f_j : \mathbb{R} \to [0, \infty) \]
be the function
\[ f_j(t) = e^{-\pi d((tv_0^j, P_{v_0^j}(\lambda(C_0 \cap F)))^2)} \forall t \in \mathbb{R}, \]
where \( \langle v_0^j \rangle \) denotes the 1-dimensional subspace spanned by \( v_0^j \). Notice that, for every \( j \in J_0 \),
\[ \int_{\mathbb{R}} f_j(t) dt = \int_{\mathbb{R}} e^{-\pi d((tv_0^j, P_{v_0^j}(\lambda(C_0 \cap F)))^2)} dt = \mathcal{W}(P_{v_0^j}(\lambda(C_0 \cap F))), \]
and, since for every \( j \in J_0 \) we have that
\[ P_{v_0^j}(\lambda(C_0 \cap F)) \subseteq [-\lambda t_j, \lambda t_j] v_j, \]
we see that, for every \( j \in J_0 \),
\[ \int_{\mathbb{R}} f_j(t) dt \leq \mathcal{W}([-\lambda t_j, \lambda t_j] v_j) = (1 + 2\lambda t_j). \]

Therefore, by the Brascamp-Lieb inequality,
\[ \int_{F} e^{-\pi \sum_{j \in J_0} \delta_0^0 d((x, v_0^j), P_{v_0^j}(\lambda(C_0 \cap F)))^2} = \prod_{j \in J_0} f_j^0(\langle x, v_0^j \rangle) dx \leq \prod_{j \in J_0} \left( \int_{\mathbb{R}} f_j(t) dt \right)^{\delta_0^0} \leq \prod_{j \in J_0} (1 + 2\lambda t_j)^{\delta_0^0}. \]

By the arithmetic-geometric mean inequality we have
\[ \prod_{j \in J_0} (1 + 2\lambda t_j)^{\delta_0^0} \leq \sum_{j \in J_0} \frac{\delta_0^0}{k} (1 + 2\lambda t_j) \leq 1 + \frac{2\lambda}{k} \sum_{j \in J_0} c_j \|P_{F}u_j\|_2 \]
\[ \leq 1 + \frac{2\lambda}{k} \left( \sum_{j \in J_0} c_j \right)^{1/2} \left( \sum_{j \in J_0} c_j \|P_{F}u_j\|_2^2 \right)^{1/2} \]
\[
1 + \frac{2\lambda}{k} \left( \sum_{j=1}^{m} c_j \right)^{\frac{1}{2}} \left( \sum_{j=1}^{m} \|P_{F_j}u_j\|_2^2 \right)^{\frac{1}{2}} \\
= 1 + 2\lambda \sqrt{\frac{n}{k}},
\]
and then
\[
\prod_{j \in J_0} (1 + 2\lambda s_j) s_j^{\delta_j} \leq \left( 1 + 2\lambda \sqrt{\frac{n}{k}} \right)^k = \mathcal{W} \left( \lambda \sqrt{\frac{n}{k}B^k_\infty} \right).
\]
On the other hand, for every \( x_0 \in \lambda(C_0 \cap F) \) we have that for every \( x \in F \) and every \( j \in J_0 \)
\[
d \left( \langle x, v_j^0 \rangle v_j^0, P_{(v_j^0)}(\lambda(C_0 \cap F)) \right)^2 \leq d \left( \langle x, v_j^0 \rangle v_j^0, (x_0, v_j^0) \right)^2 = \langle x - x_0, v_j \rangle^2.
\]
Thus, for every \( x_0 \in \lambda(C_0 \cap F) \) and every \( x \in F \),
\[
\sum_{j \in J_0} \delta_j^{\lambda} d \left( \langle x, v_j^0 \rangle v_j^0, P_{(v_j^0)}(\lambda(C_0 \cap F)) \right)^2 \leq \sum_{j \in J_0} \delta_j^{\lambda} |x - x_0|^2 = |x - x_0|^2,
\]
and hence, for every \( x \in F \),
\[
\sum_{j \in J_0} \delta_j^{\lambda} d \left( \langle x, v_j^0 \rangle v_j^0, P_{(v_j^0)}(\lambda(C_0 \cap F)) \right)^2 \leq d(x, \lambda(C_0 \cap F))^2.
\]
Consequently,
\[
\mathcal{W}(\lambda(C_0 \cap F)) = \int_F e^{-\pi d(x,\lambda(C_0 \cap F))^2} dx \leq \int_F e^{-\pi \sum_{j \in J_0} \delta_j^{\lambda} d \left( \langle x, v_j^0 \rangle v_j^0, P_{(v_j^0)}(\lambda(C_0 \cap F)) \right)^2} dx \leq \mathcal{W} \left( \lambda \sqrt{\frac{n}{k}B^k_\infty} \right).
\]
Therefore, since \( K \cap F \subseteq C_0 \cap F \), by the monotonicity of the Wills functional we get
\[
\mathcal{W}(\lambda(K \cap F)) \leq \mathcal{W}(\lambda(C_0 \cap F)) \leq \mathcal{W} \left( \lambda \sqrt{\frac{n}{k}B^k_\infty} \right).
\]

The following result gives a similar upper bound for a quantity defined via a double polarity, both on the convex body and on the log-concave function. We will denote, for any \( k \) dimensional linear subspace \( F \in G_{n,k} \) and any convex body \( K \subseteq F \),
\[
f_K(x) = e^{-\pi d(x,K)} \quad \forall x \in F.
\]

\textbf{Theorem 8.1.} Let \( K \subseteq \mathbb{R}^n \) be a convex body in John’s position and let \( F \) be a \( k \)-dimensional affine subspace at distance \( d \) from \( 0 \). Then, for every \( \lambda > 0 \),
\[
\int_F f_{\lambda(K \cap F)}^\circ(x) dx \leq \frac{n+1}{k+1} \sqrt{\frac{n}{n+d^2}} \int_{\mathbb{R}^k} f_{\lambda \sqrt{\frac{n(n+1)}{k+1}S_\lambda}}^\circ(x) dx,
\]
where the polarity is taken with respect to the closest point in \( F \) to the origin if it belongs to the relative interior to \( K \cap F \).
Furthermore, if $K$ is centrally symmetric and $F \in G_{n,k}$ is a $k$-dimensional linear subspace then, for every $\lambda > 0$,
\[
\int_{F} f_{(\lambda(K \cap F))^}\circ(x)dx \leq \int_{\mathbb{R}^{k}} f_{(\lambda \sqrt{\mathbb{B}_{k}^{\perp}})^}\circ(x)dx.
\]

Proof. Let $K \subseteq \mathbb{R}^{n}$ be a centrally symmetric convex body in John’s position and let $F \in G_{n,k}$ be a $k$-dimensional linear subspace. From the definition of $f_{(\lambda(K \cap F))^}\circ$ and \(2.2\) we have that, for every $\lambda > 0$,
\[
\int_{F} f_{(\lambda(K \cap F))^}\circ(x)dx = \int_{F} e^{-\frac{1+i\frac{1}{2}t}{n}} e^{-\|x\|_{(\lambda(K \cap F))}}dx = \int_{F} e^{-\frac{1+i\frac{1}{2}t}{n}} \int_{0}^{\infty} e^{-t}dt dx
\]
\[
= \int_{0}^{\infty} e^{-t} \int_{(\lambda(K \cap F))} e^{-\frac{1+i\frac{1}{2}t}{n}} dt dx
\]
\[
= (2\pi)^{k} \int_{0}^{\infty} e^{-t} \int_{(\lambda(K \cap F))} e^{-\|x\|_{(\lambda(K \cap F))}} (\sqrt{2\pi})^{k} dt dx
\]
\[
= (2\pi)^{k} \int_{0}^{\infty} e^{-t} \gamma_{k} \left( \frac{t\lambda}{\sqrt{2\pi}} (K \cap F) \right) dt.
\]

Similarly, for every $\lambda > 0$,
\[
\int_{\mathbb{R}^{k}} f_{(\lambda \sqrt{\mathbb{B}_{k}^{\perp}})^}\circ(x)dx = (2\pi)^{k} \int_{0}^{\infty} e^{-t} \gamma_{k} \left( \frac{\lambda}{\sqrt{2\pi}} \sqrt{n} B_{k}^{\perp} \right) dt.
\]

As we have seen in the proof of Theorem 1.24, for every $t \geq 0$ and every $\lambda > 0$,
\[
\gamma_{k} \left( \frac{t\lambda}{\sqrt{2\pi}} (K \cap F) \right) \leq \gamma_{k} \left( \frac{\lambda}{\sqrt{2\pi}} \sqrt{n} B_{k}^{\perp} \right).
\]

Therefore, for every $\lambda > 0$,
\[
\int_{F} f_{(\lambda(K \cap F))^}\circ(x)dx \leq \int_{\mathbb{R}^{k}} f_{(\lambda \sqrt{\mathbb{B}_{k}^{\perp}})^}\circ(x)dx.
\]

Assume now that $K \subseteq \mathbb{R}^{n}$ is a (not necessarily symmetric) convex body in John’s position and $F$ is a $k$-dimensional affine subspace at distance $d$ from the origin. As before, we have that, for every $\lambda > 0$,
\[
\int_{F} f_{(\lambda(K \cap F))^}\circ(x)dx = (2\pi)^{k} \int_{0}^{\infty} e^{-t} \gamma_{k} \left( \frac{t\lambda}{\sqrt{2\pi}} (K \cap F) \right) dt,
\]
where the identity holds identifying the point with respect to which the polar body is taken with the origin in the affine subspace. Following the notation in Section 3 by Lemma 7.1 we have that, for any $\alpha \in \mathbb{R}$ and any $\beta \leq 0$,
\[
\sqrt{\frac{n+2}{n}} \int_{0}^{\infty} e^{-\frac{r-n\alpha r}{n}} e^{\beta r \sqrt{n} \gamma_k} \left( \frac{r}{\sqrt{n}} (C \cap F) \right) dr \leq \int_{0}^{\infty} e^{-\frac{r-n\alpha r}{n}} e^{\beta r \sqrt{n} \gamma_k} \left( r \sqrt{k+1} \Delta_k \right) dr,
\]
where $d_{1} = \frac{1}{\sqrt{k+1}} \sum_{j \in J} \delta_{j} \| P_{H} v_{j} \|_{2}$. Integrating with respect to $\alpha \in \mathbb{R}$ we see that, for any $\beta \leq 0$,
\[
\sqrt{\frac{n+2}{n(n+1)}} \int_{0}^{\infty} e^{\beta r \sqrt{n} \gamma_k} \left( \frac{r}{\sqrt{n}} (C \cap F) \right) dr \leq \frac{1}{d_{1}} \int_{0}^{\infty} e^{\beta \sqrt{n} \gamma_k} \left( r \sqrt{k+1} \Delta_k \right) dr,
\]
or, equivalently, for any $\beta \leq 0$,
\[
\sqrt{\frac{n + d^2}{n + 1}} \int_0^\infty e^{-\frac{u}{\sqrt{2\pi}}} \gamma_k \left(\frac{u}{\sqrt{2\pi}}(C \cap F)\right) du \leq \frac{1}{d_1 \sqrt{k + 1}} \int_0^\infty e^{\frac{\beta}{\sqrt{2\pi}}} \gamma_k \left(\frac{u}{\sqrt{2\pi}} \Delta_k\right) du.
\]
Taking, for any $\lambda > 0$, $\beta = -\frac{1}{k} \sqrt{\frac{2\pi}{n(n+1)}}$ we obtain
\[
\sqrt{\frac{n + d^2}{n + 1}} \int_0^\infty e^{-\frac{u}{\sqrt{2\pi}}} \gamma_k \left(\frac{u}{\sqrt{2\pi}}(C \cap F)\right) du \leq \frac{1}{d_1 \sqrt{k + 1}} \int_0^\infty e^{-\frac{\beta}{\sqrt{2\pi}}} \gamma_k \left(\frac{u}{\sqrt{2\pi}} \Delta_k\right) du,
\]
or, equivalently,
\[
\sqrt{\frac{n + d^2}{n + 1}} \int_0^\infty e^{-\frac{u}{\sqrt{2\pi}}} \gamma_k \left(\frac{u\lambda}{\sqrt{2\pi}}(C \cap F)\right) du \leq \frac{\sqrt{n(n+1)}}{d_1 \sqrt{k + 1}} \int_0^\infty e^{-\frac{u}{\sqrt{2\pi}}} \gamma_k \left(\frac{u\lambda}{\sqrt{2\pi}} \Delta_k\right) du.
\]
Since $\sqrt{k(k+1)} \Delta_k = S_k$ we see that, for every $\lambda > 0$,
\[
\int_0^\infty e^{-u} \gamma_k \left(\frac{u\lambda}{\sqrt{2\pi}}(C \cap F)\right) du \leq \frac{(n+1)^\sqrt{n}}{d_1 \sqrt{(k+1)(n+d^2)}} \int_0^\infty e^{-u} \gamma_k \left(\frac{u\lambda}{\sqrt{2\pi}} \pi \Delta_k\right) du.
\]
Consequently, for any $\lambda > 0$, taking polar with respect to the closest point in $F$ to the origin which belongs to the relative interior of $K \cap F$,
\[
\int_F f_{\lambda(K\cap F)^\circ}(x) dx \leq \int_F f_{\lambda(C\cap F)^\circ}(x) dx \leq d_1 \frac{\pi \sqrt{n(n+1)}}{(k+1)(n+1)} \int_{\mathbb{R}^k} f_{\lambda \sqrt{\frac{n(n+1)}{k(k+1) S_k}}} (x) dx.
\]
Since
\[
d_1 \sqrt{k+1} = \sum_{j=1}^m \kappa_s j \geq \sum_{j=1}^m \kappa_s j = k + 1,
\]
we have that, for every $\lambda > 0$,
\[
\int_F f_{\lambda(K\cap F)^\circ}(x) dx \leq \frac{n + 1}{k} \sqrt{\frac{n}{n+d^2}} \int_{\mathbb{R}^k} f_{\lambda \sqrt{\frac{n(n+1)}{k(k+1) S_k}}} (x) dx.
\]
\[\square\]

9. The Wills functional of projections of convex bodies in Löwner’s position

In this section we will give the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $K$ be a centrally symmetric convex body in John’s position and let $F \in G_{n,k}$ be a $k$-dimensional linear subspace. Following the notation in Section 3 we have that $K \subseteq C_0$, $K \cap F \subseteq C_0 \cap F$, and $(K \cap F)^\circ \supseteq (C_0 \cap F)^\circ = \text{conv}\{\pm \|P_F u_j\|_2 \nu j, j \in J_0\}$.

Therefore, since the function $d((C_0 \cap F)^\circ)^2$ is convex, for any $x \in F$ and any $\{\theta_j\}_{j \in J_0} \subseteq \mathbb{R}$ such that $x = \sum_{j \in J_0} \delta_0^0 \theta_j v_j^0$, we have that
\[
d(x, (K \cap F)^\circ)^2 \leq d(x, (C_0 \cap F)^\circ)^2 = d\left(\sum_{j \in J_0} \delta_0^0 k \theta_j v_j^0, (C_0 \cap F)^\circ\right)^2
\]
is a centrally symmetric polytope in minimal surface area position and the notation introduced in Section 3. 

By an approximation argument the inequalities obtained will also be true for any centrally symmetric convex body in minimal surface area position. We will follow 

Therefore, by the reverse Brascamp-Lieb inequality, 

and then setting, for every \( j \in J_0 \), 

we have that, for any \( x \in F \) and any \( \{ \theta_j \}_{j \in J_0} \subseteq \mathbb{R} \) such that \( x = \sum_{j \in J_0} \delta_j \theta_j v_j^0 \), 

\[
\prod_{j \in J_0} f_j^0(\theta_j) \leq e^{-\pi d(x,(K \cap F)^c)^2}.
\]

Therefore, by the reverse Brascamp-Lieb inequality, 

\[
\mathcal{W}((K \cap F)^c) = \int_F e^{-\pi d(x,(K \cap F)^c)^2} dx \geq \prod_{j \in J_0} \left( \int_{\mathbb{R}} f_j(t) dt \right)^{\delta_j} \\
= \prod_{j \in J_0} \left( \frac{1}{\sqrt{k}} \int_{\mathbb{R}} e^{-\pi d(\sqrt{k}t v_j^0, [-\frac{\|P_F u_j\|_2 v_j^0, \frac{P_F u_j}{\|P_F u_j\|_2} v_j^0]}^2) dt \right)^{\delta_j} \\
= \frac{1}{k^{k/2}} \prod_{j \in J_0} \mathcal{W}\left( [-\frac{\|P_F u_j\|_2 v_j^0, \frac{P_F u_j}{\|P_F u_j\|_2} v_j^0]} \right)^{\delta_j} \\
= \frac{1}{k^{k/2}} \prod_{j \in J_0} \left( 1 + \frac{2\|P_F u_j\|_2}{\|P_F u_j\|_2} \right)^{\delta_j} \geq \frac{1}{k^{k/2}}.
\]

\[\square\]

10. SECTIONS OF CONVEX BODIES IN MINIMAL SURFACE AREA POSITION

In this section we are going to prove Theorem 1.7. Let us start assuming that \( K \) is a centrally symmetric polytope in minimal surface area position and \( F \in G_{n,k} \). By an approximation argument the inequalities obtained will also be true for any centrally symmetric convex body in minimal surface area position. We will follow the notation introduced in Section 3.

Let, for every \( j \in J_0 \), \( f_j : \mathbb{R} \rightarrow (0, \infty) \) be the function 

\[ f_j(t) = e^{-\pi d(t v_j^0, P_{\langle v_j^0 \rangle}(K \cap F))^2} \forall t \in \mathbb{R}, \]

where \( \langle v_j^0 \rangle \) denotes the 1-dimensional subspace spanned by \( v_j^0 \). Notice that, for every \( j \in J_0 \), 

\[ \int_{\mathbb{R}} f_j(t) dt = \int_{\mathbb{R}} e^{-\pi d(t v_j^0, P_{\langle v_j^0 \rangle}(K \cap F))^2} dt = \mathcal{W}(P_{\langle v_j \rangle}(K \cap F)), \]

and, since for every \( j \in J_0 \) we have that 

\[ P_{\langle v_j \rangle}(K \cap F) \subseteq [-t_j h_K(u_j), t_j h_K(u_j)] v_j^0, \]
we have that, for every \( j \in J_0 \),
\[
\int_{\mathbb{R}} f_j(t) dt \leq W([-t_j h_K(u_j), t_j h_K(u_j)] v_j) = (1 + 2t_j h_K(u_j)).
\]
Therefore, by the Brascamp-Lieb inequality
\[
\int_{F} e^{-\pi \sum_{j \in J_0} \delta_j^0 d \left( (x, v_j^0) v_j^0, P_{(v_j^0)} (C_0 \cap F) \right)^2} = \int_{F} \prod_{j \in J_0} f_j^\delta \left( (x, v_j^0) \right) dx 
\leq \prod_{j \in J_0} \left( \int_{\mathbb{R}} f_j(t) dt \right)^{\delta_j^0} \leq \prod_{j \in J_0} (1 + 2t_j h_K(u_j))^{\delta_j^0}.
\]
By the arithmetic-geometric mean inequality we have
\[
\prod_{j \in J_0} (1 + 2t_j h_K(u_j))^{\delta_j^0} \leq \sum_{j \in J_0} \frac{\delta_j^0}{k} (1 + 2t_j h_K(u_j)) = 1 + \frac{2}{k} \sum_{j \in J_0} \frac{n |F_j| \|P_{F_j} u_j\| h_K(u_j)}{\partial(K)} 
\leq 1 + \frac{2n}{k \partial(K)} \sum_{j \in J_0} |F_j| h_K(u_j) 
\leq 1 + \frac{2n}{k \partial(K)} \sum_{j=1}^m |F_j| h_K(u_j) = 1 + \frac{n^2 |K|}{k \partial(K)},
\]
and then,
\[
\prod_{j \in J_0} (1 + 2t_j h_K(u_j))^{\delta_j^0} \leq \left( 1 + \frac{n^2 |K|}{k \partial(K)} \right)^k = W \left( \frac{n^2 |K|}{k \partial(K)} B_{\infty} \right).
\]
On the other hand, for every \( x_0 \in K \cap F \) we have that, for every \( x \in F \) and every \( j \in J_0 \),
\[
d \left( (x, v_j^0) v_j^0, P_{(v_j^0)} (K \cap F) \right)^2 \leq d \left( (x, v_j^0) v_j^0, (x_0, v_j^0) v_j^0 \right)^2 = (x - x_0, v_j)^2.
\]
Thus, for every \( x_0 \in C_0 \cap F \) and every \( x \in F \),
\[
\sum_{j \in J_0} \delta_j^0 d \left( (x, v_j^0) v_j^0, P_{(v_j^0)} (K \cap F) \right)^2 \leq \sum_{j \in J_0} \delta_j^0 (x - x_0, v_j^0)^2 = |x - x_0|^2,
\]
and hence, for every \( x \in F \),
\[
\sum_{j \in J_0} \delta_j^0 d \left( (x, v_j^0) v_j^0, P_{(v_j^0)} (K \cap F) \right)^2 \leq d(x, K \cap F)^2.
\]
Consequently,
\[
W(K \cap F) = \int_{F} e^{-\pi d(x, K \cap F)^2} dx \leq \int_{F} e^{-\pi \sum_{j \in J_0} \delta_j^0 d \left( (x, v_j^0) v_j^0, P_{(v_j^0)} (K \cap F) \right)^2} dx 
\leq W \left( \frac{n^2 |K|}{k \partial(K)} B_{\infty} \right),
\]
which proves (i).
Therefore, by the reverse Brascamp-Lieb inequality,

$$\mathcal{W}(\lambda(K \cap F)) \leq \mathcal{W}\left(\lambda \frac{n^2 |K|}{\partial(K)} B_{k}^k\right).$$

Therefore, as explained in Section 2.6, we obtain that $V'(K \cap F) \leq V_1 \left(\lambda \frac{n^2 |K|}{\partial(K)} B_{k}^k\right)$ and $V_n((K \cap F) \leq V_n \left(\lambda \frac{n^2 |K|}{\partial(K)} B_{k}^k\right)$, which is equivalent to (ii) and (iii).

Now, observe that for every $x \in F$ we have that

$$h_{K \cap F}(x) = \inf \left\{ \sum_{j \in J_0} |\alpha_j| : x = \sum_{j \in J_0} \frac{\alpha_j}{h_{K}(u_j)} P_{F} u_j \right\}$$

$$= \inf \left\{ \sum_{j \in J_0} |\beta_j| t_j h_{K}(u_j) : x = \sum_{j \in J_0} \beta_j v_j^0 \right\},$$

where we have set, for every $j \in J_0$, $t_j = \frac{1}{\| P_{F} u_j \|_2} = \left(\frac{\alpha_j}{\| u_j \|}\right)^{\frac{1}{k}}$. For every $j \in J_0$, we define

$$f_j(t) := e^{-t|t_j h_{K}(u_j)|}, \quad t \in \mathbb{R}.$$  

Then, if $x = \sum_{j \in J_0} \delta_j \theta_j v_j^0$ for some $\{\theta_j\}_{j \in J_0} \subseteq \mathbb{R}$, we have

$$\prod_{j \in J_0} f_j^{\delta_j}(u_j) = e^{-\sum_{j \in J_0} \delta_j |t_j h_{K}(u_j)|} \leq e^{-h_{K \cap F}(x)}.$$  

Therefore, by the reverse Brascamp-Lieb inequality,

$$k! |(K \cap F)^\circ| = \int_F e^{-h_{K \cap F}(x)} \, dx \geq \prod_{j \in J} \left(\int_{\mathbb{R}} e^{-|t_j h_{K}(u_j)|} \, dt\right)^{\delta_j} = \frac{2^k}{\prod_{j \in J_0} (t_j h_{K}(u_j))^{\delta_j}}.$$  

By the arithmetic-geometric mean inequality,

$$\prod_{j \in J_0} (t_j h_{K}(u_j))^{\frac{\delta_j}{k}} \leq \sum_{j \in J_0} \frac{\delta_j}{k} t_j h_{K}(u_j) = \sum_{j \in J_0} \frac{n |F_j| \| P_{F} u_j \|_2 h_{K}(u_j)}{k \partial(K)} \leq \frac{n^2 |K|}{k \partial(K)} \sum_{j=1}^m |F_j| h_{K}(u_j) = \frac{n^2 |K|}{k \partial(K)}.$$  

Taking into account that $|B_{k}^k| = |B_{k}^k| = 2^k$, we obtain

$$|(K \cap F)^\circ|^{1/k} \geq \frac{k |\partial(K)|}{n^2 |K|} |(B_{k}^k)^\circ|^{1/k},$$

which gives us (iv).

Finally, observe that for every $t \geq 0$

$$t(K \cap F) = \{ x \in F : |\{x, v_j^0\}| \leq tt_j h_{K}(u_j), \forall j \in J_0 \}.$$
By the Brascamp-Lieb inequality he have that
\[
\gamma_k(t(K \cap F)) = \int_{\mathbb{R}^n} \prod_{j \in J_0} \chi_{[-tt_j h_K(u_j), tt_j h_K(u_j)]}(|x, v_0^j|) e^{-\sum_{j \in J_0} \frac{\delta_j^0 |x, v_0^j|^2}{2}} \frac{dx}{(2\pi)^{k/2}} \\
\leq \prod_{j \in J_0} \left( \int_{-tt_j h_K(u_j)}^{tt_j h_K(u_j)} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt \right) \\
= \prod_{j \in J_0} \gamma_1 ([tt_j h_K(u_j), tt_j h_K(u_j)])^{\delta_j^0}.
\]

Since the 1-dimensional Gaussian measure is log-concave, we have that
\[
\prod_{j \in J_0} \gamma_1 ([tt_j h_K(u_j), tt_j h_K(u_j)])^{\delta_j^0} \leq \gamma_1 \left( \sum_{j \in J_0} \frac{\delta_j^0 tt_j h_K(u_j)}{k} \right)^k [-e_1, e_1] \\
= \gamma_k \left( t \left( \sum_{j \in J_0} \frac{\delta_j^0 tt_j h_K(u_j)}{k} \right) B^k_{\infty} \right).
\]

Since
\[
\sum_{j \in J_0} \frac{\delta_j^0 tt_j h_K(u_j)}{k} = \sum_{j \in J_0} \frac{n|F_j||P_{F_j}u_j|_2 h_K(u_j)}{k \partial(K)} \leq \sum_{j \in J_0} \frac{n|F_j|h_K(u_j)}{k \partial(K)} = \sum_{j=1}^m \frac{n|F_j|h_K(u_j)}{k \partial(K)} = \frac{n^2 |K|}{k \partial(K)},
\]
we have that, for any \( t \geq 0 \),
\[
\gamma_k(t(K \cap F)) \leq \gamma_k \left( \left( \frac{n^2 |K|}{k \partial(K)} \right) B^k_{\infty} \right).
\]

Therefore,
\[
w((K \cap F)^o) \geq w \left( \left( \frac{n^2 |K|}{k \partial(K)} \right) B^k_{\infty} \right) = \frac{k \partial(K)}{n^2 |K|} w((B^k_{\infty})^o),
\]
and we obtain (v).

Let us now assume that \( K \) is a (not necessarily symmetric) polytope in minimal surface area position and \( F \in G_{n,k} \). Again, by approximation the inequalities obtained will be true for any convex body. Following the notation in Section 8 we have that, for any \( x \in F \),
\[
\|x\|_{\Pi^* K \cap F} = \frac{1}{2} \sum_{j=1}^m |F_j| \|\langle x, u_j \rangle\| = \frac{1}{2} \sum_{j \in J_0} |F_j| \|P_{F_j}u_j\|_2 \|\langle x, v_0^j \rangle\| \\
= \sum_{j \in J_0} \frac{\partial(K) \delta_j^0 t_j}{2n} \|\langle x, v_0^j \rangle\|.
\]

Therefore, by the Brascamp-Lieb inequality
\[
k! \|\Pi^* K \cap F\| = \int_F e^{-\|x\|_{\Pi^* K \cap F}} dx = \int_F e^{-\sum_{j \in J_0} \frac{\partial(K) \delta_j^0 t_j}{2n} |\langle x, v_0^j \rangle|} dx \\
= \int_F \prod_{j \in J_0} \left( e^{-\frac{\partial(K) \delta_j^0}{2n} |\langle x, v_0^j \rangle|} \right)^{\delta_j^0} dx.
\]
and hence, by the reverse Brascamp-Lieb inequality
\[
\|x\|_{P_F\Pi K} = \inf \left\{ \max_{j \in J_0} |\tau_j| : x = \sum_{j \in J_0} \frac{\partial(K)\delta_j^0 t_j}{2n} v_j^0 \right\}.
\]
On the other hand, we have that \(P_F\Pi K = (\Pi^* K \cap F)^o\) and then, for every \(x \in F\)
\[
\|x\|_{P_F\Pi K} = \inf \left\{ \max_{j \in J_0} |\tau_j| : x = \sum_{j \in J_0} \frac{\partial(K)\delta_j^0 t_j}{2n} v_j^0 \right\}.
\]
Therefore, taking for every \(j \in J_0\) the function \(h_j = \chi \left[ -\frac{\partial(K)\delta_j^0 t_j}{2n}, \frac{\partial(K)\delta_j^0 t_j}{2n} \right] \) we have that any decomposition of \(x\) of the form \(x = \sum_{j \in J_0} \delta_j^0 \theta_j v_j^0\) with \(|\theta_j| \leq \frac{\partial(K)\delta_j^0 t_j}{2n}\) gives a decomposition of \(x\) of the form
\[
x = \sum_{j \in J_0} \frac{\partial(K)\delta_j^0 t_j}{2n} \theta_j v_j^0 \quad \text{with} \quad \tau_j = \frac{2n\theta_j}{\partial(K)\delta_j^0 t_j},
\]
and then \(\max_{j \in J_0} |\tau_j| \leq 1\). Therefore, the function \(h = \chi_{P_F\Pi K}\) has the property that, for every \(\{\theta_j\}_{j \in J_0} \subseteq \mathbb{R}\),
\[
h \left( \sum_{j \in J_0} \delta_j^0 \theta_j v_j^0 \right) \geq \prod_{j \in J_0} h_j^\delta (\theta_j),
\]
and hence, by the reverse Brascamp-Lieb inequality
\[
|P_F\Pi K| = \int_F h(x) dx \geq \prod_{j \in J_0} \left( \int_{\mathbb{R}} h_j(t) dt \right) \delta_j^0 = \prod_{j \in J_0} \left( \frac{\partial(K)\delta_j^0 t_j}{n} \right) \delta_j^0
\]
\[
= \left( \frac{\partial(K)}{n} \right) \prod_{j \in J_0} \left( \frac{1}{\|P_Fu_j\|} \right) \delta_j^0 \geq \left( \frac{\partial(K)}{n} \right)^k.
\]

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