This paper proposes a new logic RoCTL* to model robustness in concurrent systems. RoCTL* extends CTL* with the addition of Obligatory and Robustly operators, which quantify over failure-free paths and paths with one more failure respectively. We present a number of examples of problems to which RoCTL* can be applied.

The core result of this paper is to show that RoCTL* is expressively equivalent to CTL* but is non-elementarily more succinct. We present a translation from RoCTL* into CTL* that preserves truth but may result in non-elementary growth in the length of the translated formula as each nested Robustly operator may result in an extra exponential blowup. However, we show that this translation is optimal in the sense that any equivalence preserving translation will require an extra exponential growth per nested Robustly. We also compare RoCTL* to Quantified CTL* (QCTL*) and hybrid logics.

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1. INTRODUCTION

We introduce the Robust Full Computation Tree Logic (RoCTL*) as an extension of the branching time temporal logic CTL* to represent issues relating to robustness and reliability in systems. It does this by adding an Obligatory operator and a Robustly operator. The Obligatory operator specifies how the systems should ideally behave by quantifying over paths in which no failures occur. The Robustly operator specifies that something must be true on the current path and similar paths that “deviate” from the current path, having at most one more failure occurring. This notation allows phrases such as “even with \( n \) additional failures” to be built up by chaining \( n \) simple unary Robustly operators together.

RoCTL* is a particular combination of temporal and deontic logics allowing reasoning about how requirements on behaviour are progressed and change with time, and the unfolding of actual events. The RoCTL* Obligatory operator is similar to the Ob-
ligatory operator in Standard Deontic Logic (SDL), although in RoCTL* the operator quantifies over paths rather than worlds. However, it is the Robustly operator which gives RoCTL* many advantages over a simple combination of temporal logic and deontic logic as in [van der Torre and Tan 1998]. SDL has many paradoxes and some of these, such as the “Gentle Murderer” paradox (“if you murder, you must murder gently” [Forrester 1984]), spring from the inadequacy of SDL to deal with obligations caused by acting contrary to duty. Contrary-to-Duty (CtD) obligations are important for modeling a robust system, as it is often important to state that the system should achieve some goal and also that, if it fails, then it should act to mitigate or in some way recover from the failure.

RoCTL* can represent CtD obligations by specifying that the agent must ensure that the CtD obligation is met even if a failure occurs. SDL is able to distinguish what ought to be true from what is true, but is unable to specify obligations that come into force only when we behave incorrectly. Addition of temporal operators to deontic logic allows us to specify correct responses to failures that have occurred in the past [van der Torre and Tan 1998]. However, this approach alone is not sufficient [van der Torre and Tan 1998] to represent obligations such as “You must assist your neighbour, and you must warn them iff you will not assist them”. In RoCTL* these obligations can be represented if the obligation to warn your neighbour is robust but the obligation to assist them is not.

A number of other extensions of temporal logics have been proposed to deal with deontic or robustness issues [Broersen et al. 2004; Long et al. 2000; Hansson and Jonsson 1994; Aldewereld et al. 2005; Rodrigo and Eduardo 2005]. Each of these logics are substantially different from RoCTL*. Some of these logics are designed specifically to deal with deadlines [Broersen et al. 2004; Hansson and Jonsson 1994]. The Agent Communication Language was formed by adding deontic and other modal operators to CTL [Rodrigo and Eduardo 2005]; this language does not explicitly deal with robustness or failures. Hansson and Jonsson [1994] proposed an extension of CTL to deal with reliability. However, as well as being intended to deal with deadlines, their logic reasons about reliability using probabilities rather than numbers of failures, and their paper does not contain any discussion of the relationship of their logic to deontic logics. Like our embedding into QCTL*, [Aldewereld et al. 2005] uses a Viol atom to represent failure. However, their logic also uses probability instead of failure counts and is thus suited to a different class of problems than RoCTL*. Another formalisation of robustness is representing the robustness of Metric Temporal Logic (MTL) formulas to perturbations in timings [Bouyer et al. 2008]. None of these logics appear to have an operator that is substantially similar to the Robustly operator of RoCTL*.

In the last few years there has been considerable interest in logics for reasoning about systems that are robust to partial non-compliance with the norms. One approach has been to define robustness in terms of the ability of a multi-agent system to deal with having some subset of agents that are unwilling or unable to comply with the norms [van der Hoek et al. 2008; Agotnes et al. 2010]. Like RoCTL* they consider socially acceptable behaviours to be a subset of physically possible behaviours. A logic that like can discuss numbers of faults was suggested by [Faella et al. 2010], though this logic extended ATL instead of CTL* and defined fault-tolerance in terms of numbers of winning strategies. More recently the Deontic Computation Tree Logic (dCTL) was proposed [Castro et al. 2011]. Like RoCTL* the logic divides states into normal and abnormal states, but avoids capturing the full expressivity of CTL* to allow the model checking property to be polynomial like the simpler CTL logic. There is a restriction of RoCTL* that can be easily translated into CTL [McCabe-Dansted and Dixon 2010], allowing this restriction to be reasoned about as efficiently as CTL; how-
ever, dCTL is more expressive than CTL [Castro et al. 2011]. Finally, a Propositional Deontic Logic was proposed by [Acosta et al. 2012] than divided events into allowable and non-allowable depending on the current state.

Diagnosis problems in control theory [Jéron et al. 2006; Arnold et al. 2003] also deals with failures of systems. Diagnosis is in some sense the dual of the purpose of the RoCTL* logic, as diagnosis requires that failure cause something (detection of the failure) whereas robustness involves showing that failure will not cause something.

This paper provides some examples of robust systems that can be effectively represented in RoCTL*. It is easy to solve the coordinated attack problem if our protocol is allowed to assume that only \( n \) messages will be lost. The logic may also be useful to represent the resilience of some economy to temporary failures to acquire or send some resource. For example, a remote mining colony may have interacting requirements for communications, food, electricity and fuel. RoCTL* may be more suitable than Resource Logics (see for example [de Weerdt et al. 2003]) for representing systems where a failure may cause a resource to become temporarily unavailable. This paper presents a simple example where the only requirement is to provide a cat with food when it is hungry.

The Obligatory operator, as well as some uses of the Robustly operator, are easy to translate into CTL* [McCabe-Dansted 2008] but a general way to achieve a translation to CTL* is not obvious. The first translation in our paper is of RoCTL* into the tree semantics of Quantified CTL* (QCTL*). We note that a similar translation can be made into a fragment of Hybrid temporal logic. Although QCTL* is strictly more expressive than CTL* the translation of RoCTL* into QCTL* will be given for two reasons. Firstly the translation into QCTL* is very simple and thus is well suited as an introduction to reasoning with RoCTL*. Secondly, even this weak result is sufficient to demonstrate that RoCTL* is decidable. Finally, the translation into QCTL* is linear, while it will be shown that any translation to CTL* must be non-elementary in the worst case.

We then give a translation of RoCTL* formulas into CTL*. This results in a formula that is satisfied on a model iff the original formula is satisfied on the same model. This means that we can use all the CTL* model checkers, decision procedures and so forth for RoCTL*. Unfortunately, the translation can be quite long. We show that although all RoCTL* formulas can be translated into CTL*, the length of the CTL* formula is not elementary in the length of the RoCTL* formula. Hence some properties can be represented much more succinctly in RoCTL* than CTL*. This translation requires roughly one extra exponential per nested robustly operator. We will show that no translation can do better than this, so although RoCTL* is no more expressive than CTL it is very succinct in the sense that any translation of RoCTL* into either CTL* or tree automata will result in a non-elementary blowup in the length of the formula.

We can summarise the contributions of this paper as follows. Firstly, it defines a new intuitive and expressive logic, RoCTL*, for specifying robustness in systems. The logic seems to combine temporal and deontic notions in a way that captures the important contrary-to-duty obligations without the usual paradoxes. Secondly, it provides a proof that the logic can be translated in a truth-preserving manner into the existing CTL* logic. Thirdly, it provides a proof that RoCTL* is non-elementarily more succinct than CTL* for specifying some properties.

This paper extends results from the conference papers [French et al. 2007; McCabe-Dansted et al. 2009; McCabe-Dansted 2011a]. There is further discussion and more details in the thesis [McCabe-Dansted 2011b].

The structure of the paper is as follows. RoCTL* is introduced in the next section before we show that the new logic can be applied across a wide variety of examples, practical, theoretical and philosophical. In section 4 we revise a large collection of existing machinery that we will need in the subsequent expressivity and succinctness
proofs. In section 5, we show that RoCTL* is preserved under bisimulations: needed for some unwinding proofs, but also interesting to have. In section 6, we show the fairly straightforward translation of RoCTL* into QCTL*. Section 7 presents some useful conversions between automata. Section 8 contains the translation of RoCTL* into CTL*. In section 9, we show that this translation is optimal.

2. ROCTL*

In this section we define the RoCTL* logic. We first provide some basic definitions, starting with our set of variables.

**Definition 2.1.** We let \( V \) be our set of variables. The set \( V \) contains a special variable \( v \). A valuation \( g \) is a map from a set of worlds \( S \) to the power set of the variables. The statement \( p \in g(w) \) means roughly “the variable \( p \) is true at world \( w \)”.

The \( v \) atom will be used to define failing transitions. Informally it may be possible to enter a state labelled with \( v \), but it is forbidden to do so; entering such a state will be considered a failure.

As is normal we say a binary relation is serial if every element has a successor.

**Definition 2.2.** We say that a binary relation \( R \) on \( S \) is serial (total) if for every \( a \) in \( S \) there exists \( b \) in \( S \) such that \( aRb \).

We now provide a definition of a structure.

**Definition 2.3.** A structure \( M = (S, R, g) \) is a 3-tuple containing a set of worlds \( S \), a serial binary relation \( R \) on \( S \), a valuation \( g \) on the set of worlds \( S \).

While in some logics the truth of formulas depends solely on the current world, the truth of CTL* (and hence QCTL* and RoCTL*) may depend on which future eventuates. These futures are represented as infinitely long (full) paths through the structure. For this reason, we provide a formal definition of fullpaths.

**Definition 2.4.** We call an \( \omega \)-sequence \( \sigma = \langle w_0, w_1, \ldots \rangle \) of worlds a fullpath iff for all non-negative integers \( i \) we have \( w_iRw_{i+1} \). For all \( i \) in \( \mathbb{N} \) we define \( \sigma_{\geq i} \) to be the fullpath \( \langle w_i, w_{i+1}, \ldots \rangle \), we define \( \sigma_i \) to be \( w_i \) and we define \( \sigma_{\leq i} \) to be the sequence \( \langle w_0, w_1, \ldots, w_i \rangle \).

We now define the property of failure-freeness. This means that, in the future, no failing transitions are taken. Informally, a failure-free fullpath represents a perfect future. Whereas the Obligatory operator in SDL quantifies over acceptable worlds, the Obligatory operator we will define quantifies over failure-free fullpaths.

**Definition 2.5.** We say that a fullpath \( \sigma \) is failure-free iff for all \( i > 0 \) there is no \( v \) in \( \sigma \) with \( v \notin g(\sigma_i) \). We define \( ap(w) \) to be the set of all fullpaths starting with world \( w \) and \( S(w) \) to be the set of all failure-free fullpaths starting with \( w \). We call a structure a RoCTL-structure iff \( S(w) \) is non-empty for every \( w \) in \( S \).

We will now define deviations. Informally, these represent the possibility of adding an additional failure to some step \( i \) along a path. After \( i \) we follow a different path,

\(^1\)A variant of RoCTL* was presented in [French et al. 2007], which had two accessibility relations, a success and failure transition and thus did not need the special atom \( v \). The definition we use here was presented in McCabe-Dansted [2008]. These definitions are equivalent if we disallow the RoCTL* formulas from directly accessing the \( v \) atom [McCabe-Dansted 2011b]. All the known results on RoCTL* apply equally well using either definition, and no advantage is known to the definition in [French et al. 2007]. Using the definition of the structures for RoCTL* that have a single accessibility relation allows us to define both CTL* and RoCTL* structures in the same way, greatly simplifying the definition of the translations.
and we allow only a single failure not on the existing path so no failures occur after $i + 1$. Deviations are intended to represent possible failures we may wish to be able to recover from, and if our system is robust to failures we also want it to be robust in the face of correct transitions. For this reason we allow the new transition added at step $i$ to be a success as well as a failure.

**Definition 2.6.** For two fullpaths $\sigma$ and $\pi$ we say that $\pi$ is an $i$-deviation from $\sigma$ iff $\sigma_{\leq i} = \pi_{\leq i}$ and $\pi_{i+1} \in S(\pi_{i+1})$. We say that $\pi$ is a deviation from $\sigma$ if there exists a non-negative integer $i$ such that $\pi$ is an $i$-deviation from $\sigma$. We define a function $\delta$ from fullpaths to sets of fullpaths such that where $\sigma$ and $\pi$ are fullpaths, $\pi$ is a member of $\delta(\sigma)$ iff $\pi$ is a deviation from $\sigma$.

We see that $S(\sigma_0) \subseteq \delta(\sigma) \subseteq ap(\sigma_0)$. Where $p$ varies over $\mathcal{V}$, we define RoCTL* formulas according to the following abstract syntax

$$\phi := p | \neg \phi | (\phi \land \phi) | (\phi U \phi) | N \phi | A \phi | O \phi | \square \phi .$$

A formula that begins with $A$, $\neg A$, $O$, $\neg O$, $p$ or $\neg p$ is called a state formula. For consistency with [French et al. 2007], we do not consider a formula that explicitly contains $v$ to be a RoCTL* formula, although our translation into CTL* works equally well for such formulas. The $\top$, $\bot$, $\land$, $\lor$, $U$ and $A$ are the familiar “true”, “not”, “and”, “next”, “until” and “all paths” operators from CTL. The abbreviations $\bot$, $\lor$, $F$, $G$, $W$, $E$ and $\leftrightarrow$ are defined as in CTL* logic. As with Standard Deontic Logic (SDL) logic, we define $P \equiv \neg O \neg$. Finally, we define the dual $\triangle$ of $\square$ as the abbreviation $\triangle \equiv \neg \square \neg$. We call the $O$, $P$, $\square$, $\triangle$ operators Obligatory, Permissible, Robustly and Prone respectively.

We define truth of a RoCTL* formula $\phi$ on a fullpath $\sigma = \langle w_1, w_1, \ldots \rangle$ in a RoCTL-structure $M$ recursively as follows:

$$M, \sigma \models N \phi \iff M, \sigma_{\geq 1} \models \phi$$
$$M, \sigma \models \phi U \psi \iff \exists i \in \mathbb{N} \text{s.t. } M, \sigma_{\geq 1} \models \psi \text{ and } \forall j \in \mathbb{N} j < i \implies M, \sigma_{\geq j} \models \phi$$
$$M, \sigma \models A \phi \iff \forall \pi \in ap(\sigma_0) M, \pi \models \phi$$
$$M, \sigma \models O \phi \iff \forall \pi \in S(\sigma_0) M, \pi \models \phi$$
$$M, \sigma \models \square \phi \iff \forall \pi \in \delta(\sigma) M, \pi \models \phi \text{ and } M, \sigma \models \phi$$

The definition for $\top$, $p$, $\neg$ and $\land$ is as we would expect from classical logic. The intuition behind the $\square$ operator is that it quantifies over paths that could result if a single error was introduced; the deviations only have at most one failure not on the original path, and they are identical to the original path until this failure occurs.

**Definition 2.7.** We say that a function $\tau$ from formulas to formulas is truth-preserving iff for all $M, \sigma$ and $\phi$ it is the case that $M, \sigma \models \phi \iff M, \tau(\phi) \models \phi$.

Given that traditional modal logics define truth at worlds, instead of over paths, many important properties of modal logics assume such a definition of truth. When dealing with those properties we can use the following definition of truth of RoCTL* formulas at worlds.

**Definition 2.8.** A RoCTL* formula is true at a world if it is true on any path leading from that world, or more formally:

$$M, w \models \phi \iff \exists \pi \text{ s.t. } \pi_0 = w : M, \pi \models \phi.$$
3. EXAMPLES

In this section a number of examples are presented. These examples will demonstrate how combinations of RoCTL* operators can be used, and contrast the meaning of apparently similar combinations. A number of problem domains will be touched on briefly.

The first example will show how a variant of Chisholm’s paradox can be represented in RoCTL*. Example 3.2 examines the difference between the formula NO\(\phi\) and the formula ON\(\phi\), and shows how this combination of operators can be used to represent a contrary-to-duty obligation that is triggered by a failure in the past. Example 3.3 shows how RoCTL* may be used to specify a robust network protocol, in this case relating to the coordinated attack problem. Example 3.4 uses the feeding of a cat to show how we can reason about consequences of policies in RoCTL*. These examples frequently use the ▲/△ operator to form the pair O▲; Example 3.5 exhibits the simple formula O(△Fe → Fw) which nests ▲/△ in a less trivial way. Example 3.6 also nests ▲ in a less trivial way, as it is used to compare the meaning of ▲G with the meaning of G▲.

In each of these examples, an informal English requirement will be listed with formal specification as a RoCTL* formula. The informal requirements will have flavor and explanation that may not be expressed in the formal specification, and thus should not be interpreted as simple translations from RoCTL* to English.

EXAMPLE 3.1. We may represent a variant of Chisholm’s paradox [Chisholm 1963, p34–5] as follows:

OFh: You must help your neighbour (eventually)

O▲(¬Fh ↔ Fw): You must warn your neighbour that you will not help them iff you will not help them, even if a single failure occurs.

Note that O(¬Fh → Fw) would be redundant given OFh, as all failure-free paths would satisfy Fh and thus O(¬Fh → Fw) would be vacuously true. However, O▲(¬Fh → Fw) is not redundant, as this indicates that even if a single failure occurs. As with similar defeasible representations of this problem, the obligation to warn the neighbour is meaningful as it is stronger than the obligation to help the neighbour.

It may seem that the obligation to eventually help your neighbour is vacuous, as one could always claim that they will help their neighbour sometime later. In RoCTL* the obligation is not vacuous, as following a path where you never help the neighbour violates the norm. A common sense interpretation of this is, if you plan to never help your neighbour, then lying about that plan does not satisfy the first obligation, rather it also violates the second. RoCTL* focuses on modelling and verifying systems. It is reasonable and meaningful to state that a task must complete in finite time without specifying a deadline. Additionally, we note that if we have had multiple perfect opportunities to help our neighbour, and did not do so, the neighbour may become rightfully suspicious that we do not plan to help them; however, diagnosing systems on the basis of behaviour is outside the scope of this paper.

EXAMPLE 3.2. Here is an example of a simple Contrary-to-Duty obligation. This provides a counter example to both ON\(\phi\) → NO\(\phi\) and NO\(\phi\) → ON\(\phi\).

In some case decisiveness may be more important than making the right decision. For example, when avoiding collision with an object we may have the choice of veering right or left. In this case it may be more efficient to veer to the right, and so we should make this decision. However, changing our mind could cause a collision, so it is best to stay with the inferior decision once chosen. We show how we may formalise such a decision to demonstrate the difference in the meaning of ON and NO in RoCTL*.
ON(Gp). You should commit to the proper decision. (It is obligatory that by the next step, you will always “act according to proper decision” [p])
NO(G¬p ∨ Gp). Once you have made your decision, you should stick with it. (at the next step it is obligatory that you will always not p or always p)

It is logically consistent with both the above that we do not make the proper decision (N¬p), as the above only specifies what should happen not what actually will happen. Once we have made the wrong decision we cannot satisfy Gp, so we should stick with the wrong decision G¬p. Hence, in this case, both ON(Gp) and NO(G¬p) are true. Likewise ON(G¬p) and NO(Gp) are false. This demonstrates how obligations can change with time in RoCTL*.

We will now give an example of a structure \( M = (S, R, g) \) that satisfies these formulas:

\[
S = \{u, v, w, w'\}, \\
R = \{(u, v), (v, v), (u, w'), (w', w), (w, w)\}, \\
g(v) = \{p\}, \quad g(w) = g(u) = \emptyset, \quad g(w') = \{v\}.
\]

Let \( \sigma \) be the fullpath \( \langle u, w, w', w', \ldots \rangle \) corresponding to making the wrong decision. We see that \( M, \sigma \models \neg p \), so for every failure-free path \( \pi \) starting at \( \sigma_1 \) we have \( M, \sigma \not\models \neg p \) and hence \( M, \sigma_1 \models O \neg p \land \neg Op \). Thus \( M, \sigma \models NO \neg p \land N \neg Op \). As \( N \) is its own dual it follows that \( M, \sigma \models \neg NOp \).

Let \( \pi = \langle v, v, \ldots \rangle \). We see that \( M, \pi \models p \). We see that \( sp(u) = \{\langle u, v, v, \ldots \rangle\} \). Hence \( M, \sigma \models ONp \) and it follows that \( M, \sigma \models \neg O \neg p \) and so \( M, \sigma \not\models ON \neg p \).

Hence \( M, \sigma \models \langle ONp \land \neg NOp \rangle \) and so \( M, \sigma \not\models \langle ON \neg p \to NO \phi \rangle \) where \( \phi = p \). Likewise \( M, \sigma \models \langle NO \neg p \land ON \neg p \rangle \), so \( M, \sigma \not\models \langle NO \phi \to ON \phi \rangle \) where \( \phi = \neg p \).

It is well known that simple combinations of deontic and temporal logics can represent contrary-to-duty obligations of the form “If you have previously done \( \phi \), you should do \( \psi \)”. We now give an example of a contrary-to-duty obligation RoCTL* can express where time is not central to the obligation.

**Example 3.3.** In the coordinated attack problem we have two generals \( X \) and \( Y \). General \( X \) wants to organise an attack with \( Y \). A communication protocol will be presented such that a coordinated attack will occur if no more than one message is lost.

The coordinated attack problem requires that the both generals know that the other will attack despite the possibility that any message could be lost. This is known to be impossible. We will show how we can specify a policy on RoCTL* that specifies a weaker variant of the coordinated attack problem where we can achieve a coordinated attack provided no more than one message is intercepted (and both generals are willing to assume that no more than one message will be lost).

\( AG(s_X \to ONr_Y) \): If \( X \) sends a message, \( Y \) should (in an ideal world) receive it at the next step. Note that it may not actually be the case that the message arrives as it may be intercepted.
\( AG(\neg s_X \to \neg N r_Y) \): If \( X \) does not send a message now, \( Y \) will not receive a message at the next step.
\( AG(f_X \to AG f_X) \): If \( X \) commits to an attack, \( X \) cannot withdraw.
\( AG(f_X \to \neg s_X) \): If \( X \) has committed to an attack, it is too late to send messages.
\( A(\neg f_X W r_X) \): \( X \) cannot commit to an attack until \( X \) has received a message (which would contain plans from \( Y \)).
A \neg_X W_Y: X will not receive a message until Y sends one.

Similar constraints to the above also apply to Y. Below we add a constraint requiring X to be the general planning the attack

A \neg_Y W_Y: General Y will not send a message until Y has received a message.

No protocol exists to satisfy the original coordination problem, since an unbounded number of messages can be lost. Here we only attempt to ensure correct behaviour if one or fewer messages are lost.

A (s_X U r_X): General X will send plans until a response is received.

AG (r_X \rightarrow f_X): Once general X receives a response, X will commit to an attack.

A (\neg_Y W (r_Y \land (s_Y \land Ns_Y \land NNf_Y))): Once general Y receives plans, Y will send two messages to X and then commit to an attack.

Having the formal statement of the policy above and the semantics of RoCTL* we may prove that the policy \hat{\phi} is consistent and that it implies correct behaviour even if a single failure occurs:

\hat{\phi} \rightarrow O \triangle F (f_X \land f_Y).

Indeed, we will shown that such issues can be decided in finite time in Section 6.

For a more thorough specification of the Coordinated Attack problem, see for example [Halpern and Moses 1990].

EXAMPLE 3.4. We have a cat that does not eat the hour after it has eaten. If the cat bowl is empty we might forget to fill it. We must ensure that the cat never goes hungry, even if we forget to fill the cat bowl one hour. At the beginning of the first hour, the cat bowl is full. We have the following atoms:

b. “The cat bowl is full at the beginning of this hour”

d. “This hour is feeding time”

We can translate the statements above into RoCTL* statements:

(1) AG (d \rightarrow \neg Nd): If this hour is feeding time, the next is not.

(2) AG ((d \lor \neg b) \rightarrow \triangle N \neg b): If it is feeding time or the cat bowl was empty, a single failure may result in an empty bowl at the next step.

(3) AG ((\neg d \land b) \rightarrow Nb): If the bowl is full and it is not feeding time, the bowl will be full at the beginning of the next hour.

(4) O \triangle G (d \rightarrow b): It is obligatory that, even if a single failure occurs, it is always the case that the bowl must be full at feeding time.

(5) b: The cat bowl starts full.

Having formalised the specification it can be proven that the specification is consistent and that the policy implies O \triangle G \neg b, indicating that the bowl must be filled at every step (in case we forget at the next step), unless we have already failed twice. The formula \neg G \neg b \rightarrow O \triangle G (d \rightarrow b) can also be derived, indicating that following a policy requiring us to always attempt to fill the cat bowl ensures that we will not starve the cat even if we make a single mistake. Thus following this simpler policy is sufficient to discharge our original obligation.

EXAMPLE 3.5. Say that a bit ought to flip at every step, but might fail to flip at any particular step. This may be represented with the RoCTL* statement

AGO (b \leftrightarrow \neg Nb) \land AG \triangle (b \leftrightarrow Nb),

which is satisfied by the following model:
Then we may derive the following statements:

\[ O A ((b \land Nb) \rightarrow NG (b \leftrightarrow \neg Nb)) \]

If a single failure occurs, and the bit fails to flip at the next step, it will flip continuously from then on.

\[ O A FG (b \leftrightarrow \neg Nb) \]

Even if a single failure occurs, there will be time at which the bit will flip correctly from then on.

However, we will not be able to derive \( OF A G (b \leftrightarrow \neg Nb) \), as this would mean that there was a time at which a failure could not cause the bit to miss a step.

**Example 3.6.** Say a system has a battery that can sustain the system for a single step, even if a failure occurs (the fuse blows). Let \( \phi \) represent “the system has power now and at the next step”. Then, even if a single failure occurs, it will always be the case that even if a deviating event occurs the system will have power now and at the next step (\( OG A \phi \)). It would not follow that even if a single failure occurred the system would always have power (\( O A G \phi \)); the battery power would only last one step after the fuse blew. If we also specified that the fuse was an electronic fuse that automatically reset, then if a single failure occurs, the system would only have to rely on battery power for one step. Then, if the fuse only blows once then system will always have power (\( A G \phi \)).

As with the \( A \) operator in CTL*, \( A G \phi \rightarrow G A \phi \) is valid in RoCTL* but \( G A \phi \rightarrow A G \phi \) is not.

### 4. Technical Preliminaries

In this section we will provide definitions and background results that will be used in this paper. In **Section 4.1** we will define CTL* and its syntactic extension QCTL*.

In **Section 4.2** we will define various forms of Automata. In **Section 4.3** we will define Bisimulations. We will discuss expressive equivalences **Section 4.4** in particular between LTL and automata.

**4.1. Trees, LTL, CTL* and QCTL**

In this paper we will also briefly consider Linear Temporal Logic (LTL), CTL* and an extension QCTL* of CTL*. For the purposes of this paper we will define CTL* to be a syntactic restriction of RoCTL* excluding the \( O \) and \( A \) operator.

**Definition 4.1.** Where \( p \) varies over \( V \), we define CTL* formulas according to the following abstract syntax

\[ \phi := p \mid \neg \phi \mid (\phi \land \phi) \mid (\phi U \phi) \mid N \phi \mid A \phi . \]

We likewise define LTL to be the restriction of CTL* without the \( A \) operator.

**Definition 4.2.** Where \( p \) varies over \( V \), we define LTL formulas according to the following abstract syntax

\[ \phi := p \mid \neg \phi \mid (\phi \land \phi) \mid (\phi U \phi) \mid N \phi . \]

In turn we define QCTL* as an extension of CTL* with a \( \forall \) operator.
DEFINITION 4.3. A QCTL* formula has the following syntax:

\[ \phi ::= p | \neg \phi | (\phi \land \phi) | (\phi U \phi) | N \phi | A \phi \lor \forall p \phi. \]

The semantics of \( p, \neg, \land, U, N, \) and \( A \) are the same as in CTL* and RoCTL*. Before defining the Kripke semantics for QCTL* we need to define the concept of a \( p \)-variant. Informally a \( p \)-variant \( M' \) of a structure \( M \) is a structure that identical except in the valuation of the \( p \) atom.

DEFINITION 4.4. Given some CTL-structure \( M = (S, R, g) \) and some \( p \in \mathcal{V} \), a \( p \)-variant of \( M \) is some structure \( M' = (S, R, g') \) where \( g'(w) - \{ p \} = g(w) - \{ p \} \) for all \( w \in S \).

Under the Kripke semantics for QCTL*, \( \forall p \phi \) is defined as

\[ M, b \models \forall p \alpha \iff \text{For every } p \text{-variant } M' \text{ of } M \]

we have \( M', b \models \alpha \).

In this paper we will use the tree-semantics for QCTL*. These semantics are the same as the Kripke semantics except that, whereas the Kripke semantics evaluates satisfiability over the class \( C \) of CTL-structures, the tree semantics evaluate satisfiability over the class \( C_{tree} \) of CTL-structures which are trees (see Definition 4.6 below for a formal definition of trees). This changes which formulas are satisfiable in the logic as, unlike CTL* [Emerson 1983], QCTL* is sensitive to unwinding into a tree structure [Kupferman 1995]. Note that the atom \( p \) in \( \forall p \alpha \) is often called a variable.

THEOREM 4.5. The tree-semantics for QCTL* are decidable. [French 2006]

We now provide the formal definition of a tree.

DEFINITION 4.6. We say \( T = (S, R, g) \) is a \( \mathcal{V} \)-labelled tree, for some set \( \mathcal{V} \), iff

1. \( S \) is a non-empty set of nodes
2. for all \( x, y, z \in S \) if \( (x, z) \in R \) and \( (y, z) \in R \) then \( x = y \)
3. there does not exist any cycle \( x_0Rx_1 \cdots Rx_0 \) through \( R \)
4. there exists a unique node \( x \) such that for all \( y \in S \), if \( y \neq x \) there exists a sequence \( xRx_1 \cdots Ry \) through \( R \). We call the node \( x \) the root of the tree \( T \)
5. the valuation \( g \) (or labelling) is a function from \( S \) to \( 2^\mathcal{V} \), that is for each \( w \in S \), \( g(w) \subseteq \mathcal{V} \)

DEFINITION 4.7. We define the height of a finite tree \( T = (S, R, g) \) as follows: we let \( \text{root}(T) \) be the root of the tree \( T \). We let \( \text{height}(T) = \text{height}_R(\text{root}(T)) \) where \( \text{height}_R \) is a function from \( S \) to \( \mathbb{N} \) such that for all \( x \in S \), we let \( \text{height}_R(x) \) be the smallest non-negative integer such that \( \text{height}_R(x) > \text{height}_R(y) \) for all \( y \) such that \( (x, y) \in R \).

For example, a leaf node has a height of 0 since 0 is the smallest non-negative integer.

DEFINITION 4.8. We say that \( v \) is reachable from \( w \), with respect to an accessibility relation \( R \), iff there is a path through \( R \) from \( w \) to \( v \).

DEFINITION 4.9. We say that a binary relation \( R' \) is the fragment of another binary relation \( R \) on a set \( X \) iff

\[ \forall x, y : xR'y \iff x, y \in X \land xRy. \]

We say that a function \( g' \) is the fragment of another function \( g \) on a set \( X \) iff

\[ \text{range}(g) = X \subseteq \text{range}(g') \text{ and } g(x) = g'(x) \text{ for all } x \in X. \]
DEFINITION 4.10. We say $C = \langle S_C, R_C, g_C \rangle$ is a subtree of $T = (S, R, g)$ iff there exists $w \in S$ such that $S_C$ is the subset of $S$ reachable from $w$ and $R_C$ and $g_C$ are the fragments of $R$ and $g$ on $S_C$ respectively. We say $C$ is a direct subtree of $T = (S, R, g)$ if $C$ is a subtree of $T$ and $(\text{root}(T), \text{root}(C)) \in R$.

4.2. Automata

In this section we will define some basic terms and properties of automata that will be used later in this paper. We focus on showing that we can translate between counter-free automata and LTL formulas.

DEFINITION 4.11. A Finite State Automaton (FSA) $A = (\Sigma, S, Q_0, \delta, F)$ contains

- $\Sigma$: set of symbols (alphabet)
- $S$: finite set of automaton states
- $Q_0$: set of initial states $\subseteq S$
- $\delta$: a transition relation $\subseteq (S \times \Sigma \times S)$
- $F$: A set of accepting states $\subseteq S$

We call the members of $\Sigma^*$ words. Each transition of a path through an automaton is labelled with an element $e \in \Sigma$. We say $s_0 \xrightarrow{e_0} s_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{n-1}} s_n$ is a path of $A$ iff for all $0 \leq i < n$ the tuple $(s_i, e_i, s_{i+1})$ is in $\delta$. The label of the path is the word $(e_0, e_1, \ldots, e_n)$. We say that a path through an automaton is a run iff $s_0 \in Q_0$. A run of an FSA is called accepting if it ends in an accepting state. We define the language $L(A)$ recognised by an automaton to be the set of all words for which there is an accepting run.

DEFINITION 4.12. We let $L_{p,q}(A)$ be the set of all labels of paths through $A$ from $p$ to $q$.

Of particular importance to this paper are counter-free automata. As will be discussed later we can translate LTL formulas to and from counter-free automata.

DEFINITION 4.13. A counter-free automaton is an automaton such that for all positive integers $m$, states $s \in S$ and words $u \in \Sigma^*$, if $u^m \in L_{s,s}$ then $u \in L_{s,s}$ [Diekert and Gastin 2008].

DEFINITION 4.14. We define a Deterministic Finite Automaton (DFA) to be an FSA $A = (\Sigma, S, Q_0, \delta, F)$ where $|Q_0| = 1$ and for every $s \in S$ and $e \in \Sigma$ there exists exactly one $t \in S$ such that $(s, e, t) \in \delta$.

Having given the obvious definition of DFAs as a special case of FSAs, we will now define a standard determinisation for FSAs.

DEFINITION 4.15. Given an FSA $A = (\Sigma, S, Q_0, \delta, F)$, we define the determinisation of $A$ to be the DFA $\hat{A} = (\Sigma, \hat{S}, \{Q_0\}, \hat{\delta}, F)$ with:

- $\hat{S} = 2^S$. Each $\hat{s} \in \hat{S}$ represents the set of states of $A$ that $A$ could be in now.
- For each $\hat{s}, \hat{t} \in \hat{S}$, the tuple $(\hat{s}, e, \hat{t})$ is in $\hat{\delta}$ iff $\hat{t}$ is the maximal subset of $S$ satisfying $\forall t \in \hat{t} : \exists s \in \hat{s} : (s, e, t) \in \delta$.
- $\hat{s} \in F$ iff there is an $s \in \hat{s}$ such that $s \in F$.

The reason for presenting the above determinisation is to so that we can show that we can determinise FSA while preserving counter-free automata. While this intuitive, it is important to this paper so we will provide a formal proof.

LEMMA 4.16. If $A$ is counter-free then the determinisation $\hat{A}$ produced by the above procedure is counter-free.
PROOF. Say that \( \hat{A} \) is not counter-free. Thus there exists \( u, m \) and \( \hat{s} \) such that \( u^m \in L_{\hat{s},\hat{s}} \) but \( u \notin L_{\hat{s},\hat{s}} \).

Note that we have a cycle such that the word \( u \) takes us from \( \hat{s}_0 = \hat{s} \) to \( \hat{s}_1 \), from \( \hat{s}_1 \) to \( \hat{s}_2 \) and so on back to \( \hat{s}_0 = \hat{s} \), or more formally: \( u \in \bigcap_{i \leq m} L_{\hat{s}_i,\hat{s}_{i+1}} \) and \( u \in L_{\hat{s}_{m-1},\hat{s}_{0}} \).

Note also that \( \hat{s} \subseteq S \), and we see that \( u^m \in L_{s,s} \) for all \( s \in \hat{s} \). As \( \hat{A} \) is counter-free it is also the case that \( u \in L_{s,s} \) for all \( s \in \hat{s} \). As \( u \in L_{s,s} \) and \( s \in \hat{s}_0 \) it follows that \( s \in \hat{s}_1 \); we may repeat this argument to show that as \( s \in \hat{s}_1 \) it must also be the case that \( s \in \hat{s}_2 \) and so on. Thus \( \hat{s}_0 \subseteq \hat{s}_1 \subseteq \cdots \subseteq \hat{s}_0 \) and so \( \hat{s}_0 = \hat{s}_1 = \cdots = \hat{s}_0 \). We see \( L_{\hat{s}_0,\hat{s}_1} = L_{\hat{s},\hat{s}} \) and since \( u \in L_{\hat{s}_0,\hat{s}_1} \) it follows that \( u \in L_{\hat{s},\hat{s}} \), but we have assumed that \( u \notin L_{\hat{s},\hat{s}} \). Hence by contradiction, \( \hat{A} \) is counter-free. \( \Box \)

We will use the fact that the determinisation is counter-free to generalise the following theorem to non-deterministic automata.

**Theorem 4.17.** Translating a counter-free DFA into an LTL formula results in a formula of length at most \( m2^{O(n \log n)} \) where \( m \) is the size of the alphabet and \( n \) is the number of states [Wilke 1999].

One minor note is that [Wilke 1999] uses stutter-free operators so their \((\alpha U \beta)\) is equivalent to our \( N(\alpha U / \beta)\); however, this is trivial to translate.

As the determinisation from Definition 4.14 has \( 2^n \) states where \( n \) is the number of states in the original FSA, Corollary 4.18 below follows from Lemma 4.16 and Theorem 4.17.

**Corollary 4.18.** Translating a counter-free FSA into an LTL formula results in a formula of length at most \( m2^{O(2^{\log n})} \) where \( m \) is the size of the alphabet and \( n \) is the number of states.

We now define shorthand for discussing a variant of an automaton starting at a different state.

**Definition 4.19.** Given an automaton \( A = (\Sigma, S, Q_0, \delta, F) \), we use \( A^s \) as shorthand for \((\Sigma, S, \{s\}, \delta, F)\) where \( s \in S \). We say that an automaton \( A \) accepts a word from state \( s \) if the automaton \( A^s \) accepts the word.

**4.2.1. Automata on Infinite Words.** In this paper we use automata as an alternate representation of temporal logic formulas. LTL is interpreted over infinitely long paths, and so we are particularly interested in automata that are similarly interpreted over infinitely long runs. We will define an infinite run now.

**Definition 4.20.** We call the members of \( \Sigma^\omega \) infinite words. We say \( s_0 \xrightarrow{e_0} s_1 \xrightarrow{e_1} \cdots \) is an infinite path of \( A \) iff for all \( i \geq 0 \) the tuple \( \langle s_i, e_i, s_{i+1} \rangle \) is in \( \delta \). The label of the path is \( \langle e_0, e_1, \ldots \rangle \). An infinite run \( \rho \) of \( A \) is a path starting at a state in \( Q_0 \).

There are a number of different types of automata that can be interpreted over infinite runs. These are largely similar to FSA, but have different accepting conditions. Büchi automata are extensions of finite state automata to infinite worlds. A Büchi automaton is similar to an FSA, but we say an infinite run is accepting if a state in \( F \) occurs infinitely often in the run.

**Definition 4.21.** For a fixed structure \( M \), a fullpath \( \sigma \) through \( M \), and a set of state formulas \( \phi \) we let \( g_\sigma(\sigma_{\leq n}) = (w_0, w_1, \ldots, w_n) \) and \( g_\phi(\sigma_{\leq n}) = (w_0, w_1, \ldots) \) where \( w_i = \{ \phi : \phi \in \Phi \land M, \sigma_i \models \phi \} \) for each non-negative integer \( i \).
We are interested in counter-free automata because it is known that a language \( L \) is definable in LTL iff \( L \) is accepted by some counter-free Büchi automaton \( \) (Diekert and Gastin 2008) (see Theorem 4.30).

It is well known that we can represent a CTL* formula as an LTL formula over a path, where that path includes state formula as atoms; this is commonly used in model checking [Clarke et al. 1999; Emerson and Lei 1985; Clarke et al. 1986]. Recall that Theorem 4.30 states that a language \( L \) is definable in LTL iff \( L \) is accepted by some counter-free Büchi automaton \( \); thus we can also express this LTL formula as a counter-free Büchi automaton.

Formally, for any CTL* formula \( \phi \) there exists a set of state formulas \( \Phi \) and a counter-free automaton \( A = (2^{\Phi}, Q, Q_0, \delta, F) \) such that \( A \) accepts \( g_\Phi(\sigma) \) iff \( M, \sigma \models \phi \).

**DEFINITION 4.22.** We say an automaton \( A = (2^{\Phi}, Q, Q_0, \delta, F) \) is equivalent to a formula \( \phi \) iff for all structures \( M \) and fullpaths \( \sigma \) we have:

\[
(\forall M, \sigma : M, \sigma \models \phi) \iff (A \text{ accepts } g_\Phi(\sigma)).
\]

### 4.2.2. Alternating Tree Automata.

Our succinctness proof in Section 9 uses results that show CTL* can be translated to tree automata.

We will define a type of tree automata called symmetric alternating automata (SAA) (see for example [Kupferman and Vardi 2000]), these are a subclass of alternating automata, and can also be referred to as just alternating automata (see for example [Dam 1994]).

Every node, in the run of an SAA on an input structure \( M \), represents a world of \( M \). However, a world \( w \) in the input structure \( M \) may occur many times in a run. Where a non-deterministic automata would non-deterministically pick a next state, an SAA non-deterministically picks a conjunction of elements of the form \((\Box, q)\) and \((\Diamond, q)\); alternatively we may define SAA as deterministically picking a Boolean combination of requirements of this form, see for example [Kupferman and Vardi 2000]. Alternating automata can also be thought of as a type of parity game, see for example [Grädel et al. 2002]. An element of the form \((\Box, q)/(\Diamond, q)\) indicates for every/some child \( u \) of the current world \( w \) of the input structure \( M \), a run on \( M \) must have a branch which follows \( u \) and where \( q \) is the next state. Before defining SAA we will first define parity acceptance conditions.

**DEFINITION 4.23.** A **parity acceptance condition** \( F \) of an automaton \( (\Sigma, S, Q_0, \delta, F) \) is a map from \( S \) to \( \mathbb{N} \). We say that a path satisfies the parity condition \( F \) iff the largest integer \( n \), such that \( F(q) = n \) for some \( q \) that occurs infinitely often on the path, is even.

We can now define SAA.

**DEFINITION 4.24.** A symmetric alternating automata (SAA) is a tuple

\[
(\Sigma, S, Q_0, \delta, F)
\]

where \( \Sigma, S \) and \( S_0 \) are defined as in Büchi automata, and

\( \delta \) : a transition function \( \subseteq (S \times \Sigma \times 2^{(\Box, \Diamond)} \times S) \)

We define the acceptance condition \( F \) of an SAA to be a parity acceptance condition, but note that we can express Büchi parity conditions as parity acceptance conditions. The SAA accepts a run iff every infinite path through the run satisfies \( F \).

A run \( L = (S_L, R_L, g_L) \) of the SAA on a \( \mathcal{Y} \)-labelled pointed value structure \( (S, R, g)_w \) is an \( S \times S \)-labelled tree structure satisfying the following properties. Where \( g_L(\text{root}(L)) = (w, q) \), it is the case that \( q \in S_0 \) and \( w = a \). For every \( w_L \) in \( S_L \), where
calculus formula. Hence the translation into alternating automata is linear. The states in the resulting automata are subformulas of \( \mu \) calculus. The nodes are sets of formulas, so this is a singly exponential translation.

There are a number of translations of \( \mu \)-calculus into alternating automata. Wilke gives a simple translation that does not assume that the tree has any particular structure \([\text{Wilke} \ 2001]\). The states in the resulting automata are subformulas of the \( \mu \)-calculus formula. Hence the translation into alternating automata is linear. \( \Box \)

The translation via \( \mu \)-calculus above is sufficient for this paper. There are translations that result in more optimised model checking and decision procedure results \([\text{Kupferman} \ \text{and} \ \text{Vardi} \ 2000]\).

4.3. Bisimulations

An important concept relating to structures is bisimilarity, as two bisimilar structures satisfy the same set of modal formulas. We credit \([\text{Milner} \ 1980]\) and \([\text{Park} \ 1981]\) for developing the concept of bisimulation.

**Definition 4.26.** Where \( M = (S, R, g) \) is a structure and \( a \in S \), we say that \( M_a \) is a Pointed Valued Structure (PVS).

We now provide a definition of a bisimulation.

**Definition 4.27.** Given a PVS \( (S, R, g)_w \) and a PVS \( (S, R, g)_{\hat{w}} \) we say that a relation \( \mathcal{B} \) from \( S \) to \( \hat{S} \) is a bisimulation from \( (S, R, g)_w \) to \( (S, R, g)_{\hat{w}} \) iff

1. \((w, \hat{w}) \in \mathcal{B}\)
2. for all \((u, \hat{u}) \in \mathcal{B}\) we have \( g(u) = \hat{g}(\hat{u}) \).
3. for all \((u, \hat{u}) \in \mathcal{B}\) and \( v \in uR \) there is some \( \hat{v} \in \hat{u}\hat{R} \) such that \((v, \hat{v}) \in \mathcal{B}\).
4. for all \((u, \hat{u}) \in \mathcal{B}\) and \( \hat{v} \in \hat{u}\hat{R} \) there is some \( v \in uR \) such that \((v, \hat{v}) \in \mathcal{B}\).

Bisimulations can be used to define bisimilarity.

**Definition 4.28.** We say that \( (S, R, g)_w \) and \( (S, R, g)_{\hat{w}} \) are bisimilar iff there exists a bisimulation from \( (S, R, g)_w \) to \( (S, R, g)_{\hat{w}} \).

**Definition 4.29.** We say that a formula \( \phi \) of some logic \( L \) is bisimulation invariant iff for every bisimilar pair of PVS’s \( (M, w) \) and \( (M, \hat{w}) \) where \( M \) and \( M \) are structures that \( L \) is interpreted over, we have \( M, w \vdash \phi \) iff \( M, \hat{w} \vdash \phi \). We say the logic \( L \) is bisimulation invariant iff every formula \( \phi \) of \( L \) is bisimulation invariant.

Knowing that a logic is bisimulation invariant is useful because we can take the tree-unwinding of a structure without changing the set of formulas that it satisfies.

4.4. Expressive Equivalences

While this paper focuses on temporal logic, there are many ways of defining the languages expressive by LTL. This is very useful, as it provides us with many ways to
model the expressivity of temporal logics. We are particularly interested in the expressive equivalence of LTL with counter-free Büchi automata.

In Section 4.5, we will outline some important results relating to expressive equivalences, focusing on those presented in [Diekert and Gastin 2008]. There are a number of reasons we present these here. Firstly, by showing the many results that [Diekert and Gastin 2008] builds upon we hope to give the reader a feel for the complexity of attempting to follow approach of [Diekert and Gastin 2008] in proving that LTL and counter-free Büchi automata have the same expressive power. Secondly, since [Diekert and Gastin 2008] uses many results, having a map of those results and where to find them in the paper can be of assistance in following the work of [Diekert and Gastin 2008].

In Section 4.6, we outline the proof of [Wilke 1999] that any language recognised by a finite counter-free DFA can be represented in LTL. We note that this result is much weaker than the theorem of [Diekert and Gastin 2008]. However, this result is simple and constructive. This allows us to get an idea as to what the formulas translated from DFAs might look like, as well as an indication of the length of the translated formulas.

4.5. First-Order Definable Languages

We here present a summary of some significant results in first order definable languages. We focus on the survey paper of [Diekert and Gastin 2008], which provides a very powerful equivalence theorem.

**Theorem 4.30.** For any language L, the following statements are all equivalent. [Diekert and Gastin 2008]

- (1) L is first-order definable
- (2) L is star-free
- (3) L is aperiodic
- (4) L is definable in LTL
- (5) L is first-order definable with at most 3 names for variables
- (6) L is accepted by a counter-free Büchi automata
- (7) L is accepted by some aperiodic automata
- (8) L is accepted by some very weak automata

Below we summarise the results that provide the basis for this theorem. Given that the proofs are numerous and frequently complex we will not reproduce them here. Further, since we are only interested in counter-free Büchi automata and LTL we do not define the other terms used in the theorem. Readers are invited to read [Diekert and Gastin 2008] if they are interested in this detail.
[1] \implies [4]: This is in essence Kamp’s Theorem [Kamp 1968]. Note that Kamp focuses on translating into a temporal logic with past-time operators; however, this can be translated back into LTL [Gabbay et al. 1980, 1994].

[1] \iff [2]: [Diekert and Gastin 2008] cites [Perrin and Pin 1986], and presents a proof in their Section 4, as well as an alternate proof of [1] \implies [2] in their section 10.2.

[2] \iff [3]: [Perrin 1984], and [Diekert and Gastin 2008, Section 6]

[3] \implies [4]: This is one of the more complex proofs of this paper [Diekert and Gastin 2008, Section 8]. It serves a similar purpose to Kamp’s theorem.

[3] \implies [6] \implies [7] \implies [3]: This is their Proposition 34, [Diekert and Gastin 2008, p27]. This builds on a number of results discussed in the paper. For example, [6] \implies [7] is trivial because since any counter-free Büchi automaton is periodic, which is Lemma 29 of [Diekert and Gastin 2008, p25].

[4] \implies [8] \implies [7]: This is mentioned at the top of page 4. [4] \implies [8] is Proposition 41 of [Diekert and Gastin 2008, p35]. The proof takes LTL formulas in positive normal form and provides a simple construction of the corresponding weak alternating automata. [8] \implies [7] does not appear to be explicitly stated in the text, but a translation into Büchi automata is given and in the proof of Proposition 43 [Diekert and Gastin 2008, p36], it is mentioned that the automata has an aperiodic transition monoid, and so by definition is an aperiodic automata.

[4] \implies [5]: [Diekert and Gastin 2008] describes this as trivial and presents a simple proof (Section 7 p12–13).

[5] \implies [1]: Obvious as [5] is a restriction of [1].

[8] \implies [3]: Proposition 43 of [Diekert and Gastin 2008, p36].

We now present a brief outline of the path from counter-free automata to LTL, and where they are found in [Diekert and Gastin 2008]. First it is shown that counter-free automata are aperiodic [p25, lemma 29]. Translating aperiodic automata into aperiodic monoids is discussed [p28]. The most substantial part of the proof is the translation from aperiodic monoids (or homomorphisms). The set of words and the concatenation operator can be considered an infinite monoid [p13]. We can choose a homomorphism from that infinite monoid to a finite monoid. They present a factorisation of the words, and we can factorise words of a language to produce a simplified language. The translation into LTL has two major steps, translating the simplified language into LTL, and showing that the existence of an LTL formula for the simplified language demonstrates the existence of an LTL formula for the original language.

Translating LTL to counter-free Büchi automata would seem significantly more simple. The obvious powerset construction is counter-free, though it has a Streett acceptance condition rather than Büchi. Note that [Diekert and Gastin 2008] is used in this paper only for an existence result, and so the details are not important to this paper; following Figure [1] counter-clockwise from [4] to [6] is sufficient, even though this is presumably not cleanest or simplest route possible.

4.6. Finite Counter-free DFAs to LTL

We here outline the proof of [Wilke 1999], showing how we may translate a counter-free DFA into an LTL formula.

For any automaton (or pre-automaton) \( A \), word \( u \) and state \( q \). We use \( u^A(q) \) to represent the current state of the automaton after starting at state \( q \), and reading the word \( u \). For any function \( \alpha: Q \rightarrow Q \), we let the language \( L^A_\alpha \) be the set of words \( u \) such that \( u^A = \alpha \). For any set \( S \), we let \( u^A[S] = \{ u^A(q) : q \in S \} \).

**Theorem 4.31.** The language recognised by any counter-free DFA \( A \) can be expressed in LTL. [Wilke 1999]
Due to the importance of this result to Section 8.2, we will briefly outline their proof. They prove that for all words $u$ the language $L^A_u$ can be expressed in LTL. It is then clear that the language recognised by $A$ can be expressed by the LTL formula:

$$ \bigvee_{\alpha \in \Sigma} \text{LTL}(L^A_{\bar{\alpha}}) \text{,} $$

where LTL($L^A_{\bar{\alpha}}$) is the LTL formula that defines the language $L^A_{\bar{\alpha}}$.

The proof that $L^A_{\bar{\alpha}}$ can be expressed in LTL works by induction, either reducing the state space at the expense of increasing the alphabet, or shrinking the alphabet without increasing the state space.

They note that, since $A$ is counter-free, if $w^A[Q] = Q$ then $w^A$ is the identity (that is $w^A(q) = q$ for all $q \in Q$). Hence if $w^A[Q] = Q$ for all $w$ then it is trivial to express $L^A_{\bar{\alpha}}$ in LTL. Otherwise there is some input symbol $b$ such that $b^A[Q]$ is a strict subset of $Q$.

They then define three languages based on $b$; $L_0$ the restriction of $L^A_{\bar{\alpha}}$ where $b$ does not occur; $L_1$ the restriction of $L^A_{\bar{\alpha}}$ where $b$ occurs precisely once; and $L_2$ the restriction where $b$ occurs at least twice. Let $B$ be the obvious restriction of $A$ such that $b$ is removed from the input language, and let $L^B_0$ be $L^B_0 \cup \{\epsilon\}$. They also define $C$ such that the language recognised by $C$ is similar to that of $A$ except that the input symbols of $C$ are in essence words that end in $b$, and so we can restrict the states of $C$ to be $b^A[Q]$. Recall that $b^A[Q]$ is a strict subset of $Q$ and so we have reduced the number of states. They define a function $h$ to translate the words of $A$ into words of $C$, and likewise $h^{-1}$ translates the words of $C$ into words of $A$. They provide the following equalities:

$$ L_0 = L^B_0, \quad L_1 = \bigcup_{\alpha = \beta \cdot \gamma \cdot \beta'} L^B_{\beta \cdot \gamma \cdot \beta'}, \quad L_2 = \bigcup_{\alpha = \beta \cdot \gamma \cdot \beta'} L^B_{\beta \cdot \gamma \cdot \beta'} $$

They let $\Gamma = \Sigma - \{b\}$, and note that

$$ L_{\beta \cdot \beta'} = \bar{\Delta}_{\beta} b^* \cap \Gamma^* b_{\beta'}, \quad L_{\beta \cdot \gamma \cdot \beta'} = \Gamma^* b_{\beta'} \cap \Gamma^* b h^{-1}(L_{\gamma}^C) \cap \bar{\Delta}_{\beta} b^* \Gamma^* \cap \bar{\Delta}_{\beta} b^* \Gamma^* $$

Since $B$ has a smaller input language, and $C$ has a smaller state space, we can assume by way of induction that $L^B_0$, $\bar{\Delta}_{\beta}$, $\bar{\Delta}_{\beta'}$ and $L^C_0$ can be expressed in LTL. It follows that $L^A_{\bar{\alpha}}$ can be expressed in LTL. The result then follows from induction.

**Corollary 4.32.** Translating a counter-free DFA into an LTL formula results in a formula of length at most $m^2 2^{O(n \log n)}$ where $m$ is the size of the alphabet and $n$ is the number of states. [Wilke 1999]

One minor note is that [Wilke 1999] uses stutter-free operators so their $(\alpha U \beta)$ is equivalent to our $N(\alpha U \beta)$; however, this is trivial to translate.

5. **BISIMULATION INVARIANCE**

Recall that bisimulation invariance was defined in Definition 4.29. We shall now begin to prove some basic lemmas necessary to show that RoCTL* is bisimulation invariant. First we will prove that RoCTL* is bisimulation invariant, and define bisimulations on RoCTL-structures. Before reading the following definition recall the definition of a PVS, or pointed valued structure, from Definition 4.26.

**Definition 5.1.** Let $B$ be any bisimulation from some PVS $M_w$ to another PVS $M_w^\prime$. We define $B^\omega$ to be a binary relation from full paths through $M$ to full paths through $M$ such that $(\sigma, \tilde{\sigma}) \in B^\omega$ iff $(\sigma_i, \tilde{\sigma}_i) \in B$ for all $i \in \mathbb{N}$. We say that a PVS $M_w$ is a RoCTL-model iff $M$ is a RoCTL-structure.
It is important that for a path \( \sigma \) though \( M \) we can find a similar path \( \hat{\sigma} \) through \( \hat{M} \). We will now show that this is the case.

**Lemma 5.2.** Let \( \mathcal{B} \) be any bisimulation from some RoCTL-model \( M_w \) to another RoCTL-model \( \hat{M}_w \). For any fullpath \( \sigma \) where \( \sigma_0 = w \) through \( M \) there exists a fullpath \( \hat{\sigma} \) through \( \hat{M} \) such that \((\sigma, \hat{\sigma}) \in \mathcal{B}^\omega \) and \( \hat{\sigma}_0 = \hat{w} \); likewise for any fullpath \( \hat{\sigma} \) where \( \hat{\sigma}_0 = \hat{w} \) through \( \hat{M} \) there exists a fullpath \( \sigma \) through \( M \) such that \((\sigma, \hat{\sigma}) \in \mathcal{B}^\omega \) and \( \sigma_0 = w \).

**Proof.** We construct \( \hat{\sigma} \) from \( \sigma \) as follows: let \( \hat{\sigma}_0 = \hat{w} \). Once we have chosen \( \hat{\sigma} \), we choose \( \hat{\sigma}_{i+1} \) as follows: since \((\sigma_i, \hat{\sigma}_i) \in \mathcal{B} \) and \( \sigma_{i+1} \in \sigma_i \) there is some \( \hat{v} \in \hat{\sigma}_i \hat{R} \) such that \((\sigma_{i+1}, \hat{v}) \in \mathcal{B} \); we let \( \hat{\sigma}_{i+1} = \hat{v} \). We may construct \( \sigma \) from \( \hat{\sigma} \) likewise. \( \square \)

The following lemma is similar; however, we are specifically attempting to construct deviations.

**Lemma 5.3.** Let \( \mathcal{B} \) be a bisimulation from some RoCTL-model \( M_w \) to another RoCTL-model \( \hat{M}_w \). Let \((\sigma, \hat{\sigma}) \in \mathcal{B}^\omega \). Given a deviation \( \hat{\pi} \) from \( \hat{\sigma} \) we can construct a fullpath \( \pi \) such that \( \pi \) is a deviation from \( \sigma \) and \((\pi, \hat{\pi}) \in \mathcal{B}^\omega \).

**Proof.** As \( \hat{\pi} \) is a deviation from \( \hat{\sigma} \), it is the case that \( \hat{\pi} \) is an \( i \)-deviation from \( \hat{\sigma} \) for some non-negative integer \( i \). Since \((\sigma_i, \hat{\sigma}_i) \in \mathcal{B} \) we can construct a fullpath \( \tau \) such that \((\tau, \hat{\pi}_{\geq i}) \in \mathcal{B}^\omega \) and \( \tau_0 = \sigma_i \). We see that \( \sigma_{\leq i-1} \cdot \tau \) is a fullpath through \( M \), we call this fullpath \( \pi \). Since \( \hat{\pi}_{\geq i+1} \) is failure-free \( \tau_{\geq 1} \) is failure-free and thus \( \pi_{\geq i+1} \) is failure-free. Thus \( \pi \) is a deviation from \( \sigma \). \( \square \)

We will now state and prove the truth lemma.

**Lemma 5.4.** Let \( M_w \) and \( \hat{M}_w \) be a pair of arbitrary RoCTL-models and let \( \mathcal{B} \) be a bisimulation from \( M_w \) to \( \hat{M}_w \). Then for any \((\sigma, \hat{\sigma}) \in \mathcal{B}^\omega \), and for any formula \( \phi \) it is the case that \( M, \sigma \models \phi \iff M, \hat{\sigma} \models \phi \).

**Proof.** We construct \( \hat{\pi} \) from \( \pi \) as follows: let \( \hat{\pi}_0 = \hat{w} \). Once we have chosen \( \hat{\sigma} \), we choose \( \hat{\pi}_{i+1} \) as follows: since \((\sigma_i, \hat{\sigma}_i) \in \mathcal{B} \) and \( \sigma_{i+1} \in \sigma_i \) there is some \( \hat{v} \in \hat{\sigma}_i \hat{R} \) such that \((\sigma_{i+1}, \hat{v}) \in \mathcal{B} \); we let \( \hat{\pi}_{i+1} = \hat{v} \). We may construct \( \sigma \) from \( \hat{\sigma} \) likewise. \( \square \)
that from Lemma 5.3 we know there is a deviation \( \pi \) from \( \sigma \) such that \((\pi, \hat{\pi}) \in \mathcal{B}^w\). We see that \( M, \pi \not\models \psi \) and thus \( M, \sigma \not\models \Box \psi \).

By contradiction we know that no such \( \phi \) exists. \( \square \)

**Lemma 5.5.** \( \text{RoCTL}^* \) is bisimulation invariant.

**Proof.** Consider any \( \text{RoCTL}^* \) formula \( \phi \). Let \( \mathcal{B} \) be a bisimulation from some pair of PVS’s \((M, w)\) and \((\hat{M}, \hat{w})\), and say that \( M, \hat{w} \models \phi \) but \( M, w \not\models \phi \). Recall that under \( \text{RoCTL}^* \) we define truth at a world as follows:

\[
M, w \models \phi \iff \exists \pi \text{ s.t. } \pi_0 = w : M, \pi \models \phi .
\]

From Lemma 5.4 we know that there exists a fullpath \( \hat{\pi} \) through \( \hat{M} \) such that \( \hat{\pi} = \hat{w} \) and \( M, \hat{w} \models \phi \). Hence \( M, \hat{w} \models \phi \). Thus we see that for any bisimilar pair of PVS’s \((M, w)\) and \((\hat{M}, \hat{w})\) we have

\[
(M, w) \models \phi \iff (\hat{M}, \hat{w}) \models \phi .
\]

By definition we see that \( \phi \) is bisimulation invariant. Since \( \phi \) is an arbitrary \( \text{RoCTL}^* \) formula, we see that \( \text{RoCTL}^* \) is bisimulation invariant. \( \square \)

**6. Reduction into QCTL**

In this section we will present a translation of \( \text{RoCTL}^* \) (and \( \text{RoCTL}^* \)) formulas into QCTL* such that the formulas are satisfiable in the tree semantics of QCTL* if they are satisfiable in \( \text{RoCTL}^* \). As we have shown that \( \text{RoCTL}^* \) is bisimulation invariant in Lemma 5.5 in this section we will assume that all structures are tree structures. We will use \( \models^* \) to indicate \( \models \) is being interpreted according to the semantics of tree QCTL*.

**Definition 6.1.** We define a translation function \( \tau^O \) from QCTL* formulas to QCTL* formulas such that for any formula \( \phi^* \)

\[
\tau^O(\phi^*) = A(NG\neg\nu \rightarrow \phi^*)
\]

**Lemma 6.2.** Say that \( \phi \) is a \( \text{RoCTL}^* \) formula and \( \phi^* \) is a QCTL* formula such that for all \( M \) and \( \sigma \) it is the case that \( M, \sigma \models \phi \) iff \( M, \sigma \models^* \phi^* \). Then, for all \( M \) and \( \sigma \) it is the case that \( M, \sigma \models O\phi \) iff \( M, \sigma \models^* \tau^O(\phi^*) \).

**Proof.** (\( \Rightarrow \)) Say that \( M, \sigma \models O\phi \). Then for all failure-free paths \( \pi \) starting at \( \sigma_0 \), \( M, \pi \models \phi \) and so \( M, \pi \models^* \phi^* \). By definition, a path is failure-free iff for all \( i > 0 \) we have \( \nu \notin g(\pi_i) \). Since every path that satisfies \( NG\neg\nu \) is failure-free we see that every path that starts at \( \sigma_0 \) satisfies \( NG\neg\nu \rightarrow \phi^* \). Hence \( M, \sigma \models^* A(NG\neg\nu \rightarrow \phi^*) \).

(\( \Leftarrow \)) Say that \( M, \sigma \models^* A(NG\neg\nu \rightarrow \phi^*) \). Then every path starting at \( \sigma_0 \) satisfies \( NG\neg\nu \rightarrow \phi^* \). A path that satisfies \( NG\neg\nu \) is failure-free, so every failure-free path starting at \( \sigma_0 \) satisfies \( \phi^* \), and hence \( \phi \). Thus \( M, \sigma_0 \models O\phi \). \( \square \)

We let \( \gamma \) be the (Q)CTL formula \( NNG\neg\nu \). Thus \( \gamma \) does not specify whether the previous or next transitions are failures, but requires that all transitions after the next one be successes. The \( \gamma \) formula is used to represent the requirement that all transitions after a deviation must be successes.

We define a translation function \( \tau^A \) from QCTL* formulas to QCTL* formulas such that for any formula \( \phi^* \) and for some atom \( y \) not in \( \phi^* \):

\[
\tau^A(\phi^*) = \forall y[Gy \rightarrow E(\{Gy \lor F(y \land \gamma)\} \land \phi^*)] .
\]

Note that for \( \tau^A(\phi^*) \) to hold, \( E(\{Gy \lor F(y \land \gamma)\} \land \phi^*) \) must hold for all possible atoms \( y \) that satisfy \( Gy \), including the case where \( y \) is true only along the current fullpath \( \sigma \). The diagram below shows a fullpath \( \pi \) that satisfies \( F(y \land \gamma) \) for all such \( y \).
Recall from Definition 4.4 that a $p$-variant of a structure $M$ is a structure $M^p$ which values the atom $p$ differently but is otherwise similar.

**Lemma 6.3.** Say that $\phi$ is a RoCTL* formula and $\phi^*$ is a QCTL* formula such that for all $M$ and $\sigma$ it is the case that $M, \sigma \models \phi$ iff $M, \sigma \models \phi^*$. Then, for all $M$ and $\sigma$ it is the case that $M, \sigma \models \triangle \phi$ iff $M, \sigma \models \tau^\triangle (\phi^*)$.

**Proof.** ($\Rightarrow$) Say that $M, \sigma \models \triangle \phi$. Then $M, \sigma \models \phi$ or there exists a deviation $\pi$ from $\sigma$ such that $M, \pi \models \phi$. If $M, \sigma \models \phi$ then $M, \sigma \models \phi^*$ and so

$$M, \sigma \models \forall y \left[ G_y \rightarrow E \left[ G_y \land \phi^* \right] \right],$$

thus $M, \sigma \models \tau^\triangle (\phi^*)$.

On the other hand, if $M, \sigma \not\models \triangle \phi$ then, for some $i$, there exists an $i$-deviation $\pi$ from $\sigma$ such that $M, \pi \models \phi$. If $G_y$ holds along $\sigma$ then $y$ holds at $\pi_i = \sigma_i$. As $\pi$ is an $i$-deviation, all transitions following $\pi_{i+1}$ are success transitions, so $M, \pi_{i+1} \models \gamma$ and $M, \pi \models F (y \land \gamma) \land \phi^*$ from which it follows that $M, \sigma \models \tau^\triangle (\phi^*)$.

($\Leftarrow$) Say that $M, \sigma \models \tau^\triangle (\phi^*)$. Then

$$M^y, \sigma \models \left[ G_y \rightarrow E \left[ \left( G_y \lor F (y \land \gamma) \right) \land \phi^* \right] \right],$$

where $M^y$ is any $y$-variant of $M$. Consider an $M^y$ for which $y$ is true at a state $w \in \sigma$. Then $M^y, \sigma \models E \left[ \left( G_y \lor F (y \land \gamma) \right) \land \phi^* \right]$. Thus there exists some fullpath $\pi$ such that $\pi_0 = \sigma_0$ and $M^\pi, \pi \models F (y \land \gamma) \land \phi^*$ or $M^\pi, \pi \models G_y \land \phi^*$.

If $M^y, \sigma^y \models G_y \land \phi^*$ then $\pi = \sigma$, so $M, \sigma \models \phi^*$ and $M, \sigma \models \phi$. If $M^y, \pi \models F (y \land \gamma) \land \phi^*$ then there exists a non-negative integer $i$ such that $M^y, \pi_{i+1} \models y \land \gamma$. Since $y$ only occurs on the current path $\pi_{i+1} = \sigma_{i+1}$ and recall that the formula $\gamma$ indicates that we deviate here. Thus $\pi$ is an $i$-deviation from $\sigma$ and so $M, \sigma \not\models \triangle \phi$. \qed

We will now combine $\tau^O$ and $\tau^\triangle$ to provide a translation of RoCTL* into QCTL*.

**Definition 6.4.** We let $\tau$ be a function from formulas to formulas defined recursively as follows:

$$\begin{align*}
\tau (p) &= p \\
\tau (\neg \phi) &= \neg \tau (\phi) \\
\tau (\phi \land \psi) &= \tau (\phi) \land \tau (\psi) \\
\tau (\phi U \psi) &= \tau (\phi) U \tau (\psi) \\
\tau (N \phi) &= N \tau (\phi) \\
\tau (A \phi) &= A \tau (\phi) \\
\tau (O \phi) &= \tau^O (\tau (\phi)) \\
\tau (\boxdot \phi) &= \neg \tau^\triangle (\neg \tau (\phi)) .
\end{align*}$$
THEOREM 6.5. For any RoCTL* formula $\phi$ of length $n$ we can produce a QCTL* formula $\phi^\star$ of length $O(n)$ by simple recursive translation such that for any tree RoCTL-structure $M$ and fullpath $\sigma$ though $M$ we have $M, \sigma \models \phi$ iff $M, \sigma \models^\star \phi^\star$.

PROOF. From Lemma 6.2 and Lemma 6.3 above we see that $M, \sigma \models^\star \tau (\phi)$ iff $M, \sigma \models \phi$ where $\tau$ is the translation function from RoCTL* formulas to QCTL* formulas from Definition 6.4. □

We will also use the above translation to show that it is possible to decide the satisfiability of RoCTL* formulas.

LEMMA 6.6. Each RoCTL* formula $\phi$ is satisfiable in RoCTL* iff $\text{AGEN} \neg \nu \land \tau (\phi)$ is satisfiable in the tree semantics of QCTL*.

PROOF. Recall that a valued structure is a RoCTL-structure iff $\text{sp}(w)$ is non-empty for each world $w$ in the valued structure. The subformula $\text{AGEN} \neg \nu$ ensures that the translated formula is satisfiable on a path $\sigma$ through $M$ only if $\text{sp}(w)$ is non-empty on all worlds $w$ reachable from $\sigma_0$. It is trivial to show that removing all worlds not reachable from $\sigma_0$ from $M$ does not affect whether $M, \sigma \models \phi$. As such this result follows simply from Theorem 6.5. □

THEOREM 6.7. RoCTL* are decidable.

PROOF. Recall that every RoCTL* formula is a RoCTL* formula. As RoCTL* is bisimulation-invariant (Lemma 5.5) we can limit our selves to tree-structures without affecting the set of valid formulas. When we limit ourselves to tree-structures RoCTL* operates over the same structures as QCTL* and we see that for each such structure $M$, and from the previous lemma for every path $\sigma$ through $M$ we have $M, \sigma \models \phi$ iff $M, \sigma \models^\star \tau (\phi)$. Thus $\phi$ is satisfiable iff $\tau (\phi)$ is satisfiable.

As the tree semantics for QCTL* are decidable [Emerson and Sistla 1984; French 2001], it is obvious from Theorem 6.5 that RoCTL* is decidable. □

We can show that the above translation is also truth-preserving when using the amorphous semantics for QCTL*. The argument is similar to above, the $\forall$ operator in the amorphous semantics quantifies over all bisimulations, and some bisimulations are tree unwindings. These tree-unwindings will have a $y$-variant where $y$ is true only along the current path $\sigma$ as so the $(\leq)$ direction of the proof in Lemma 6.3 works similarly for the amorphous semantics of QCTL*. In the $(\Rightarrow)$ direction we have to consider arbitrarily bisimulations under the amorphous semantics; however, since RoCTL* is bisimulation invariant this does not cause problems.

The amorphous semantics provide a model-checking procedure for RoCTL*. Note that since the models are serial, all tree models have an infinite number of worlds. On the other hand the amorphous semantics can be model-checked; for example, by reduction to amorphous automata [French 2003].

We will not present the full proof that the above translation is also truth preserving when the amorphous semantics are used. Firstly it would be repetitive. The proof for the amorphous semantics is notationally more complex as it requires bisimulations, but this merely obfuscates the ideas central to the translation without introducing new fundamental ideas. Secondly we will get the model checking result for free when we introduce the translation into CTL* presented in Theorem 8.10.

6.1. A Comment on Hybrid Logic

Even the tree-semantics of QCTL* is non-elementary to decide and no translation into CTL* is elementary in length. For this reason we also investigated other logics to translate RoCTL* into. We know that we can represent RoCTL* with a single variable
A fragment of a hybrid extension of CTL*, by translating $\Delta \phi$ into a formula such as the following:

$$\phi \lor \exists x. (Fx \land E (\phi \land F (x \land NNG \neg v)))$$

where $\exists x. \psi$ is the hybrid logic formula indicating that the exists a valuation of $x$ such that $x$ is true at exactly one node on the tree and $\psi$ is true. This is still a way away from producing a decision procedure for RoCTL*. There has been considerable research into single variable fragments of Hybrid Logic recently (see for example [Kara et al. 2009] for a good overview of the results in this area). However, these fragments do not contain the $\exists$ operator as a base operator. Although $\exists x. \psi$ can be defined as an abbreviation, this requires two variables. Even adding a single variable hybrid logic to CTL* leads to a non-elementary decision procedure (see for example [Kara et al. 2009]), and adding two variables to an otherwise quite weak temporal logic again gives a non-elementary decision procedure [Schwentick and Weber 2007]. A potential avenue of research is investigating the complexity of deciding the fragment of Hybrid CTL* (HCTL*) where the hybrid component consists solely of the $\exists$ operator over a single variable, as the translation of RoCTL* into HCTL* falls inside this fragment. Although we have also given a linear translation into the tree-semantics of QCTL* logic, this single variable fragment of HCTL* seems much more restricted than QCTL*.

### 7. ALTL

Here we define a possible extension of LTL allowing automata to be used as operators, and briefly show to convert an ALTL formula $\phi$ into an automaton $A_\phi$.

**Definition 7.1.** We define ALTL formulas recursively according to the following BNF notation,

$$\phi ::= p \mid \neg \phi \mid (\phi \land \phi) \mid (\phi U \phi) \mid N\phi \mid A,$$

where $p$ varies over $\mathcal{V}$ and $A$ can be any counter-free FSA that accepts $2^\mathcal{V}$ as input, that is $\Sigma = 2^\mathcal{V}$. Recall from Definition 4.21 that $g_V$ is a simple conversion from fullpaths to words of an automaton. In this section we will assume that the special atoms required for the translation are members of $\mathcal{V}$, and so will use $\mathcal{V}$ as $\Phi$. The semantics of ALTL are defined similarly to LTL, with the addition that $M, \sigma \models A$ iff the automata $A$ accepts $g_V(\sigma)$, or in other words.

$$M, \sigma \models A \iff \exists \ i \ s.t. \ g_V(\sigma_{\leq i}) \in L(A)$$

Note that since automata can be ALTL formulas, the following definition also gives us a definition of equivalence between formulas and automata.

**Definition 7.2.** We say that a pair of formulas $\phi$, $\psi$ are equivalent ($\phi \equiv \psi$) iff for all structures $M$ and paths $\sigma$ through $M$:

$$M, \sigma \models \phi \iff M, \sigma \models \psi.$$
Definition 7.3. We define the length of an ALTL formula recursively as follows:

\[ |\phi \land \psi| = |\phi U \psi| = |\phi| + |\psi| \]
\[ |\neg \phi| = |\lnot \phi| = |\phi| + 1 \]
\[ |\alpha| = 1 \]
\[ |(\Sigma, Q_0, d, F)| = |S| \]

In some translations we encode state-formulas (e.g. \(A\psi\)) into atoms (labelled \(p_{A\psi}\)). We define the complexity \(|\phi|\) of an ALTL formula \(\phi\) similarly, except that we define the complexity \(|p_{\psi}|\) of an atom labelled \(p_{\psi}\) to be \(|\psi|\).

Lemma 7.4. The satisfiability problem for ALTL is decidable.

Proof. From Corollary 4.18 we can replace each automata in an ALTL formula \(\phi\) with an equivalent LTL formula. This will result in an LTL formula \(\phi'\) equivalent to \(\phi\). We can then use any decision procedure for LTL to decide \(\phi\). \(\square\)

7.1. A partial translation from ALTL into automata

Here we define a translation of an ALTL formula \(\phi\) into an automaton \(A_\phi\). However, we do not define a traditional acceptance condition as this is not required when constructing \(A_{A_\phi}\) from \(A_\phi\). In this section we will use \(s_\phi\) and \(t_\phi\) to represent arbitrary states of \(A_\phi\); we use \(x\) and \(y\) to represent arbitrary states of automata in \(\phi\).

Definition 7.5. The closure \(\mathcal{cl}\phi\) of the formula \(\phi\) is defined as the smallest set that satisfies the four following requirements:

1. \(\phi \in \mathcal{cl}\phi\)
2. For all \(\psi \in \mathcal{cl}\phi\), if \(\delta \models \psi\) then \(\delta \in \mathcal{cl}\phi\).
3. For all \(\psi \in \mathcal{cl}\phi\), \(\neg \psi \in \mathcal{cl}\phi\) or there exists \(\delta\) such that \(\psi = \neg \delta\) and \(\delta \in \mathcal{cl}\phi\).
4. If \(A \in \mathcal{cl}\phi\) then \(A^\tau \in \mathcal{cl}\phi\) for all states \(x\) of \(A\).

The states of \(A_\phi\) are sets of formulas that could hold along a single fullpath.

Proposition 7.6. The size of the closure set is linear in \(|\phi|\).

Definition 7.7. [MPC] We say that \(s_\phi \subseteq \mathcal{cl}\phi\) is Maximally Propositionally Consistent (MPC) iff for all \(\alpha, \beta \in s_\phi\):

- (M1) if \(\beta = \neg \alpha\) then \(\beta \in s_\phi\) iff \(\alpha \notin s_\phi\),
- (M2) if \(\alpha \land \beta \in \mathcal{cl}\phi\) then \((\alpha \land \beta) \in s_\phi \iff (\alpha \in s_\phi \text{ and } \beta \in s_\phi)\)

Definition 7.8. The set of states \(S_\phi\) is the set of all subsets \(s_\phi \subseteq \mathcal{cl}\phi\) satisfying:

- (S1) \(s_\phi\) is MPC
- (S2) if \(\alpha U \beta \in s_\phi\) then \(\alpha \in s_\phi\) or \(\beta \in s_\phi\)
- (S3) if \(\neg (\alpha U \beta) \in s_\phi\) then \(\beta \notin s_\phi\)
- (S4) \(s_\phi\) is non-contradictory, i.e. \(\land\) \(s_\phi\) is satisfiable.

Note that ALTL is decidable, so we can compute whether \(s_\phi\) is contradictory. We now define a standard temporal successor relation for LTL formula.

Definition 7.9. \([r_N]\) The temporal successor \(r_N\) relation on states is defined as follows: for all states \(s_\phi, t_\phi\) put \((s_\phi, t_\phi)\) in \(r_N\) iff the following conditions are satisfied:

- (R1) \(N\alpha \in s_\phi\) implies \(\alpha \in t_\phi\)
- (R2) \(\neg N\alpha \in s_\phi\) implies \(\alpha \notin t_\phi\)
- (R3) \(\alpha U \beta \in s_\phi\) and \(\beta \notin s_\phi\) implies \(\alpha U \beta \in t_\phi\)
- (R4) \(\neg (\alpha U \beta) \in s_\phi\) and \(\alpha \in s_\phi\) implies \(\neg (\alpha U \beta) \in t_\phi\)
DEFINITION 7.10. We define the transition relation $\delta_\phi \subseteq S_\phi \times \Sigma \times S_\phi$ as follows: a member $(s_\phi, e, t_\phi)$ of $S_\phi \times \Sigma \times S_\phi$ is a member of $\delta_\phi$ iff

\[(T1). \ (s_\phi, t_\phi) \in r_N \]

\[(T2). \ For \ each \ p \in V, \ it \ is \ the \ case \ that \ p \in e \ iff \ p \in s_\phi \]

\[(T3). \ If \ A^e \in s_\phi, \ and \ x \ is \ not \ an \ accepting \ state \ of \ A^e, \ then \ there \ must \ exist \ a \ state \ y \ of \ A^e \ such \ that \ A^v \in t_\phi \ and \ \langle x, e, y \rangle \ is \ a \ transition \ of \ A^e. \]

\[(T4). \ If \ \neg A^e \in s_\phi, \ then \ for \ each \ state \ y \ of \ A^e \ such \ that \ \langle x, e, y \rangle \ is \ a \ transition \ of \ A^e \ it \ must \ be \ the \ case \ that \ A^v \notin t_\phi. \]

\[
\text{LEMMA 7.13. For any fullpath } \pi, \text{ integer } j, \text{ pair of states } s_\phi, t_\phi \text{ such that} \\
\langle s_\phi, g_\psi (\pi_j), t_\phi \rangle \in \delta_\phi
\]

we have $\pi_{\geq j+1} \models t_\phi \implies \pi_{\geq j} \models s_\phi$.

PROOF. For contradiction assume that this lemma is false. Then $\pi_{\geq j+1} \models t_\phi$ and $\pi_{\geq j} \not\models s_\phi$. Since $\pi_{\geq j} \not\models s_\phi$, then there exists some $\psi \in s_\phi$ such that $\pi_{\geq j} \not\models \psi$. We assume without loss of generality that $\psi$ is the shortest such formula. We now consider each possible form of $\psi$, in each case recall that $\psi \in s_\phi$, $\pi_{\geq j+1} \models t_\phi$ and $(s_\phi, g_\psi (\pi_j), t_\phi) \in \delta_\phi$.

\[
\psi = \neg \alpha: \text{ From M1 and } \psi \in s_\phi \text{ we get } \alpha \in s_\phi \text{ and since } \alpha \text{ is shorter than } \neg \alpha \text{ it follows that } \pi_{\geq j} \models \alpha \text{ and so } \pi_{\geq j} \models \neg \alpha. \text{ However, by assumption } \pi_{\geq j} \not\models \psi. \\
\psi = p: \text{ From T2 we know that as } p \in s_\phi, \text{ we have } p \in g_\psi (\pi_j) \text{ and so } \pi_{\geq j} \models p. \text{ But by assumption } \pi_{\geq j} \not\models \psi. \\
\psi = \neg p: \text{ From M1 we know that } p \notin s_\phi, \text{ and from T2 we have } p \notin g_\psi (\pi_j) \text{ and so } \pi_{\geq j} \models \neg p. \\
\psi = \alpha \land \beta: \text{ As } s_\phi \text{ is MPC we see that } \alpha, \beta \in s_\phi. \text{ As we have assumed that } \psi \text{ is the shortest formula that provides a counterexample we see that } \pi_{\geq j} \models \alpha \text{ and } \pi_{\geq j} \not\models \beta. \text{ Hence } \pi_{\geq j} \not\models \alpha \land \beta. \\
\psi = \neg (\alpha \land \beta): \text{ As } s_\phi \text{ is MPC we see that } \alpha, \beta \not\in s_\phi. \text{ It follows that } \alpha \not\in s_\phi \text{ or } \beta \not\in s_\phi. \text{ Without loss of generality, assume } \alpha \not\in s_\phi. \text{ Thus } \pi_{\geq j} \not\models \alpha \text{ and } \pi_{\geq j} \not\models \neg (\alpha \land \beta). \text{ Hence } \pi_{\geq j} \not\models \neg (\alpha \land \beta). \\
\psi = N\theta: \text{ We see that if } \pi_{\geq j} \not\models N\theta \text{ then } \pi_{\geq j+1} \not\models \theta, \text{ but we see that from R1 that } \theta \in t_\phi. \text{ By contradiction } \theta \text{ cannot be of the form } N\theta. \\
\psi = \neg N\theta: \text{ We see that if } \pi_{\geq j} \not\models \neg N\theta \text{ then } \pi_{\geq j+1} \not\models \theta, \text{ but we see that from R2 that } \theta \notin t_\phi. \\
\psi = \alpha U \beta: \text{ We see that if } \alpha U \beta \in s_\phi \text{ then from S2 either } \alpha \in s_\phi \text{ or } \beta \in s_\phi. \text{ Since } \pi_{\geq j} \not\models \alpha U \beta \text{ it follows that } \pi_{\geq j} \models \neg \beta. \text{ As } \neg \beta \text{ is shorter than } \psi \text{ we have } \neg \beta \in s_\phi \text{ and so } \beta \not\in s_\phi. \text{ Since } \beta \not\in s_\phi, \text{ from R3 we have } \alpha U \beta \in t_\phi \text{ and so } \pi_{\geq j+1} \models \alpha U \beta. \text{ As } \alpha \in s_\phi \text{ and }
$\alpha$ is shorter than $\psi$ we see that $\pi_{\geq j} \models \alpha$. As $\pi_{\geq j} \models \alpha$ and $\pi_{\geq j+1} \models \alpha U \beta$ we see that $\pi_{\geq j} \models \alpha U \beta$.

$\psi = \neg (\alpha U \beta)$ We see that if $\neg (\alpha U \beta) \in s_\phi$ then from S3 we have $\beta \notin s_\phi$ and thus $\beta \in s_\phi$. As $\neg \beta$ is shorter than $\psi$ we have $\pi_{\geq j} \models \neg \beta$. Since $\pi_{\leq j} \not\models \neg (\alpha U \beta)$ we have $\pi_{\geq j} \models \alpha U \beta$; as $\pi_{\geq j} \models \neg \beta$ it follows that $\pi_{\geq j} \models \alpha$. Thus $\alpha \in s_\phi$, and from R4 we know $\neg (\alpha U \beta) \in t_\phi$ and hence $\pi_{\geq j+1} \models \neg (\alpha U \beta)$. As $\pi_{\geq j} \models \neg \beta$ it follows that $\pi_{\geq j} \models \neg (\alpha U \beta)$. By contradiction, $\psi$ cannot be of the form $\neg (\alpha U \beta)$.

$\psi = \mathcal{A}^x$: If $x$ is an accepting state of $\mathcal{A}^x$, then we see that $\mathcal{A}^x$ is satisfied on all fullpaths through $M$, including $\pi_{\geq j}$ and so $x$ is not an accepting state. We see from T3 that there exists a state $y$ of $\mathcal{A}^x$ such that $\mathcal{A}^y \in t_\phi$ and $\langle x, g_\mathcal{A}^y (\pi_j), y \rangle$ is a transition of $\mathcal{A}^x$. As $\pi_{\geq j+1} \models \bigwedge_{\phi} \pi_{\geq j+1} \models \mathcal{A}^y$. We can prepend the state $x$ and the symbol $g_\mathcal{A}^y (\pi_j)$ to the accepting path for $\mathcal{A}^y$ to construct an accepting path for $\mathcal{A}^x$, so we see that $\pi_{\geq j} \models \mathcal{A}^x$.

$\psi = \neg \mathcal{A}^x$: Since $\pi_{\geq j} \not\models \psi$ we see $\pi_{\geq j} \models \mathcal{A}^x$. Thus there must exist a state $y$ of $\mathcal{A}^x$ such that $\langle x, g_\mathcal{A}^y (\pi_j), y \rangle$ is a transition of $\mathcal{A}^x$. As $\pi_{\geq j+1} \models \bigwedge_{\phi} \pi_{\geq j+1} \models \mathcal{A}^y$. However, from T4 and M1, we see that $\neg \mathcal{A}^y \in t_\phi$, and since $\pi_{\geq j+1} \models \bigwedge_{\phi} \pi_{\geq j+1} \models \neg \mathcal{A}^y$.

We have considered all possible forms of $\psi$ and in each case got a contradiction. By contradiction this lemma must be true. \qed
Say that $\pi \not\models \phi$, but that the automata $A_\phi$ accepts the pair $(\pi, j + 1)$. Then there exists a path through $A_\phi$ labelled $g_\phi (\pi \leq j)$ ending at a state $t_\phi$ such that $\pi_{\geq j+1} \models \bigwedge t_\phi$; let $s_\phi$ be the state immediately preceding $t_\phi$ along that path. Since $\pi \not\models \phi$ and the lemma holds for $i = j$ we see that $\pi_{\geq j} \not\models \bigwedge s_\phi$. From Lemma 7.13 we get a contradiction. □

DEFINITION 7.15. We say that an ALTL formula is counter-free if all automata contained in the formula are counter-free.

Although we know that every LTL formula is equivalent to some counter-free automata in that they accept precisely the same strings/paths [Diekert and Gastin 2008], note that it is not the case that no non-counter free automata is equivalent to an LTL formula. For example, the following automata accepts the same paths that satisfy $G p$, yet it is not counter free as $pp \in L_{a,a}$ but $p \notin L_{a,a}$.

![Diagram](image.png)

We cannot assume that $A_\phi$ is counter free simply because $\phi$ is equivalent to an LTL formula. We will now prove that $A_\phi$ is counter-free. Although we have not defined a traditional acceptance condition for $A_\phi$, for the purposes of the next lemma we will say that the automata accepts a word $g_\psi (\pi)$ iff $A_\phi$ accepts $(\pi, i)$ for all $i \geq 0$.

LEMMA 7.16. If $\phi$ is counter-free then the automata $A_\phi$ is counter-free.

PROOF. Each state is a set of ALTL formula, by taking the conjunction of these formulas we get an ALTL formula $\psi$. Each automata $A^2$ in $\psi$ comes from some automata $A^1$ in $\phi$, and $A^1$ differs from $A^2$ only in the initial states. Since $A^1$ is counter-free we see that $A^2$ is counter-free. Since each automata in ALTL is counter-free we can find an equivalent LTL formula, and so $\psi$ is equivalent to some LTL formula $\psi'$.

If $A_\phi$ is not counter-free then there exists a positive integer $m$, state $s_\phi \in S_\phi$ and word $u$ in $\Sigma^*$ such that $u^m \in L_{s_\phi,s_\phi}$ and $u \notin L_{s_\phi,s_\phi}$. Since the states are non-contradictory we know that $A_\phi$ accepts some word $w$. For any state $t_\phi$ there exists some formula $\theta$ such that $\neg \theta \in s_\phi$ and $\theta \in t_\phi$ or visa-versa. As such $A_{\phi}^\theta$ does not accept the word $w$. Since $u \notin L_{s_\phi,s_\phi}$ and $u^m \in L_{s_\phi,s_\phi}$ we see that $A_{\phi}^u$ does not accept $u \cdot w$ but it does accept $u^m \cdot w$. By induction we discover that for all non-negative $i$ the automaton $A_{\phi}^u$ does not accept $u^{m+1} \cdot w$ but it does accept $u^m \cdot w$. We see that any automaton that accepts this language must have a counter, yet $A_{\phi}^\theta$ is equivalent to an LTL formula and so the language must be accepted by some counter-free automata. By contradiction we know that $A_\phi$ is counter-free. □

8. TRANSLATION INTO CTL*

We now present a translation from RoCTL* into CTL*. Note that $A$ indicates that $\phi$ holds on the current path or a deviation. As a convenience we use a pseudo-operator $\Delta$ which indicates that $\phi$ holds on a deviation. In Section 7.1 we presented a translation from ALTL into an automaton $A_\phi$; in Section 8.1 we will show how to construct an automaton $A_{\phi}^A$ which accepts iff $A_\phi$ would accept on a deviation from the current path, and then translate $A_{\phi}^A$ into $\phi \lor A_{\phi}^A$. In Section 8.2 we combine these translations to provide a translation of RoCTL* into ALTL and then into CTL*.
8.1. $A_\phi$ to $A_{A_\phi}$

In this section we will show how to construct an automaton $A_{A_\phi}$ from $A_\phi$. Where $A_\phi$ is equivalent to $\phi$, the automaton $A_{A_\phi}$ is equivalent to $A_\phi$. Note that the remainder of input from the current path is irrelevant once the deviation has occurred. Thus we may define $A_{A_\phi}$ as accepting finite words terminated by a state formula indicating that a deviation has occurred, and hence define $A_{A_\phi}$ as a finite automaton.

**Definition 8.1.** Where $A_\phi = \langle 2^V, S, Q_0, \delta, F \rangle$ is a counter-free automaton for $\phi$, we create a finite automaton $A_{A_\phi} = \langle 2^V, S_A, Q_0, \delta_A, F_A \rangle$ for $A_\phi$, where

1. $\Psi = \{ \psi_s : s \in S \}$, where $\psi_s$ is the following state formula:
   
   $$ E \left( \bigwedge s \land NNG \neg v \right) $$

   $\psi_s$ is roughly equivalent to saying "if we are in state $s$, we can deviate here".

2. We add a state $s_F$ indicating that there existed an accepting deviation from this path and so we shall accept regardless of further input. This input relates to the original path rather than the deviation and is thus irrelevant. As such, $S_A = S \cup s_F$ and $F_A = \{ s_F \}$.

3. $\delta_A$ is the relation that includes $\delta$ but at each state also gives the option to branch into $s_F$ when a deviation is possible and remain in that state regardless of the input along the current path. That is, $\delta_A$ is the minimal relation satisfying:
   
   (a) If for every tuple $(s, e, t) \in \delta$ the tuple $(s, e, t) \in \delta_A$. This is to ensure that wherever $g_V(\sigma)$ is a run of $A_\phi$, it is also the case that $g_V(\sigma)$ is a run of $A_{A_\phi}$.

   (b) For each $s \in S$ and each $e_A \in 2^V$ such that $p_{\psi_s} \in e_A$ we have $(s, e_A, s_F)$ in $\delta_A$.

   (c) For each $e_A \in 2^V$ we have $(s_F, e_A, s_F)$ in $\delta_A$.

The translation above is broadly similar to the translation presented in [McCabe-Dansted et al. 2009], but we translate the $\Lambda$ operator instead of the $\Delta$ operator so that we can use finite automata.

We fix $M$ to be some structure such that for all worlds $w$, formulas $\psi$ and all atoms-labelled $p_{E\psi}$, we have $M, w \models p_{E\psi}$ if and only if there exists a path $\sigma$ starting at $w$ such that $M, w \models \psi$. Recall that $A_\phi = \langle 2^V, S, Q_0, \delta, F \rangle$ is the translation of $\phi$ into an automaton, and $A_{A_\phi} = \langle 2^V, S_A, Q_0, \delta_A, F_A \rangle$ is the automaton constructed from $A_{A_\phi}$.

Here we present a lemma demonstrating that the translation of $\Lambda$ is correct.

**Lemma 8.2.** For any fullpath $\sigma$ and ATL$^L$ formula $\phi$ it is the case that $M, \sigma \models A_{A_\phi}$ iff there exists a deviation $\pi$ from $\sigma$ such that $M, \pi \models \phi$.

**Proof.**

(\(\Leftarrow\)) Say that there exists a deviation $\pi$ from $\sigma$ such that $M, \pi \models \phi$; then there exists an integer $i$ such that $\sigma_{\leq i} = \sigma_{\leq i}$ and $\pi_{\geq i+1}$ is failure-free. Since $\sigma \models \phi$ we know from Lemma 7.14 that $A_\phi$ accepts $(\pi, i)$, ending in some state $s$. As $\pi_{\geq i} = \bigwedge s$ and $\pi_{\geq i+1}$ is failure-free we see that $\pi_{\geq i} = \bigwedge s \land NNG \neg v$, and hence $p_{\psi_s} \in g_V(\pi_i)$ and so $(s, g_V(\pi_i), s_F) \in \delta_A$. Thus $M, \sigma \models A_{A_\phi}$.

(\(\Rightarrow\)) Say that $M, \sigma \models A_{A_\phi}$. Thus there is an accepting run $s_0 \xrightarrow{g_V(\sigma_0)} s_1 \xrightarrow{g_V(\sigma_1)} \cdots \xrightarrow{g_V(\sigma_i)} s_F$ for $A_{A_\phi}$.

We know from the construction of $A_{A_\phi}$ above that $p_{\psi_s} \in g_V(\sigma_i)$. Thus $\sigma_{\geq i} \models p_{\psi_s}$ and so there exists a fullpath $\pi$ such that $\pi_{\leq i} = \sigma_{\leq i}$ and $\pi_{\geq i} \models \bigwedge s \land NNG \neg v$. Hence $\pi_{\geq i+1}$ is failure-free and so $\pi$ is an $i$-deviation from $\sigma$. Since $\pi_{\geq i} = \bigwedge s_i$ and $s_{\geq 0} \xrightarrow{g_V(\sigma_0)} s_1 \xrightarrow{\cdots} s_{i-1}$ is a path of $A_\phi$ we see that $A_\phi$ accepts $(\pi, i)$. From Lemma 7.14 we know $\pi \models \phi$. \qed
8.2. RoCTL* to ALTL and CTL*

Here we define a translation $\varrho$ from RoCTL* into ALTL. It is well known that we can express a CTL* formula as an LTL formula over a path, where that path includes state formulas as atoms; this is commonly used in model checking, see for example [Clarke et al. 1999; Emerson and Lei 1985; Clarke et al. 1986]. This translation likewise replaces state formulas with atoms. It uses the standard translation of the $O$ operator found in [French et al. 2007], and the $f_\triangle$ translation from Definition 8.4. The translation $\varrho$ is defined recursively as follows:

$$
\varrho(\phi \land \psi) = \varrho(\phi) \land \varrho(\psi)
$$
$$
\varrho(\neg \phi) = \neg \varrho(\phi)
$$
$$
\varrho(A\phi) = p_{A\varrho}(\phi)
$$
$$
\varrho(O\phi) = p_{A(NGv\varrho(\phi))}
$$
$$
\varrho(\triangle \phi) = \neg f_\triangle (\neg \varrho(\phi))
$$
$$
\varrho(\square \phi) = N\varrho(\phi)
$$
$$
\varrho(\phi U \psi) = \varrho(\phi) U \varrho(\psi)
$$

**Definition 8.4.** For any ALTL formula $\phi$, we define $f_\triangle(\phi)$ to be $\phi \lor A_{\Lambda\phi}$.

**Theorem 8.5.** The translation $\varrho$ of RoCTL* into ALTL is truth-preserving if the atoms of the form $p_{A\varrho}$ are assumed to hold precisely at those worlds where $A\varrho$ holds.

**Proof.** It is easy to see from Lemma 8.2 that $\sigma \models f_\triangle(\phi)$ iff $\sigma \models \triangle \phi$. It is clear that $\sigma \models O\phi$ iff $\sigma \models A(NGv\varrho(\phi))$ as $NGv\varrho(\phi)$ is precisely satisfied on the failure-free paths, this was proven more formally in [French et al. 2007; McCabe-Dansted 2011b]. From these facts it is easy to see that $\varrho$ is truth-preserving. □

**Lemma 8.6.** The complexity of $f_\triangle(\phi)$ is singly exponential in $|\phi|$.

**Proof.** We see from Definition 7.8 that the translation of $\phi$ into $A_{\varrho}$ results in an automaton that has a number of states singly exponential in $|\phi|$. The automaton $A_{\Lambda\varrho}$ has exactly one more state than the automata $A_{\varrho}$, and so the number of states in $A_{\Lambda\varrho}$ is also singly exponential in $|\phi|$. From Definition 7.3 the length of the ALTL formula $A_{\Lambda\varrho}$ is the number of states in $A_{\Lambda\varrho}$, and so $|A_{\Lambda\varrho}|$ is also singly exponential in $|\phi|$. As $f_\triangle(\phi) = A_{\Lambda\varrho} \lor \phi$ we see that $|f_\triangle(\phi)|^2$ is singly exponential in $|\phi|$. □

**Corollary 8.7.** The translation into ALTL is at most $i$-exponential in length, for formulas with at most $i$ nested $\triangle$ operators.

**Definition 8.8.** We define a translation $\Delta$ from RoCTL* into CTL* such that for each RoCTL* formula $\phi$ we let $\Delta(\phi)$ be the ALTL formula $\varrho(\phi)$ with each atom of the form $p_{A\varrho}$ replaced with $A\varrho$, and each automata in $\varrho(\phi)$ replaced with the translation into an equivalent LTL formula referenced in Corollary 4.32.

The following theorem follows from Theorem 8.5.

**Lemma 8.9.** Where $\tau(\phi)$ is a truth-preserving translation from RoCTL* to CTL*, $\Gamma(\phi)$ is both truth and satisfiability preserving, where $\Gamma(\phi) \equiv \tau(\phi) \land AG\varnothing \neg v$.
PROOF. Consider some RoCTL-structure \( M \). Since \( \text{sp}(w) \) is non-empty for any world \( w \) of \( M \), there exists some fullpath \( \sigma \in \text{ap}(w) \) such that \( M, \sigma \models \neg \psi \). Hence \( M, w \models \neg EN \psi \). Since this is true for any arbitrary \( w \) we also see that \( M, w \models \neg \text{AGEN} \psi \). Thus for all fullpaths \( \pi \) we have \( M, \pi \models \tau(\phi) \iff M, \pi \models \Gamma(\phi) \), and so \( \Gamma \) is truth-preserving.

If \( \phi \) is satisfiable we see that there exists a RoCTL-structure \( M \) and fullpath \( \sigma \) through \( M \) such that \( M, \sigma \models \phi \). Hence \( M, \sigma \models \tau(\phi) \), and as before \( M, \sigma \models \Gamma(\phi) \). Thus \( \Gamma(\phi) \) is satisfiable.

Say \( \Gamma(\phi) \) is satisfiable in CTL*. Then there exists some CTL-structure \( M \) and fullpath \( \sigma \) through \( M \) such that \( M, \sigma \models \Gamma(\phi) \). We can assume without loss of generality that all worlds in \( M \) are reachable from \( \sigma_0 \), and so for every world \( w \) in we have \( M, w \models \neg EN \psi \). Thus for every world \( w \) we can pick a fullpath \( \sigma \) starting at \( w \) such that \( \sigma \models \text{AGEN} \psi \), and so \( \text{sp}(w) \) is non-empty. By definition \( M \) is a RoCTL-structure, and as \( M, \sigma \models \Gamma(\phi) \) we have \( M, \sigma \models \tau(\phi) \). Finally, \( M, \sigma \models \phi \), and so \( \phi \) is satisfiable in RoCTL*. □

THEOREM 8.10. The translation \( \text{RC} \) into CTL* is truth-preserving.

As the RoCTL-structures are precisely those structures where \( \text{sp}(w) \) is non-empty for each world \( w \) (see Lemma 8.9 for more detail), we have the following corollary.

COROLLARY 8.11. The translation \( \text{RC}_{\text{SAT}} \) is satisfaction preserving (and truth preserving) where \( \text{RC}_{\text{SAT}}(\phi) \equiv \text{RC}(\phi) \land \text{AGEN} \neg \psi \).

THEOREM 8.12. The translation \( \text{RC} \) is at most \((i+3)\)-exponential in the length, for formulas with at most \( i \) nested \( \Box \) operators.

PROOF. From Lemma 8.6 we see that there is at most a singly exponential blowup per \( \Box \) operator. Once we have translated the whole formula into an ALT formula \( \psi \), we know from Corollary 4.15 that we can translate the automata into LTL formulas with a 3-exponential blowup.

The automata are translated into LTL recursively, but the blowup remains 3-exponential. Say \( \phi \) is the formula being translated. We see that the number states in each automaton is no more than the complexity \( |\phi(\phi)|^\ast \) of \( \phi(\phi) \). Thus with each recursion we multiply the length of the translated formula by a number 3-exponential in \( |\phi(\phi)|^\ast \) which together still results in a 3-exponential blowup (note, for example the formula \( \left( \frac{2^n}{n!} \right) \) is singly exponential in \( n \), not \( i \)-exponential in \( n \)). □

9. OPTIMALITY OF REDUCTION INTO CTL*

In the previous section we showed that a satisfaction preserving translation from RoCTL* to CTL* exists. In this section we will show that any satisfaction preserving translation is non-elementary in the length of the formulas.

We will do this by taking a class of labelled trees which we will call \((h,l)\)-utrees, where \( h \) represents the height \( h \) and \( l \) is the number of bits per label. We will show that the number \#\((h,l)\)-utrees, of pairwise non-isomorphic \((h,l)\)-utrees, is non-elementary in \( h \). We will then present “suffix” and “prefix” encodings of utrees into RoCTL-structures, and for each pair of utrees will define \( u(T, T') \) to be the structure that results when the prefix encoding of \( T \) is joined/ followed by the suffix encoding of \( T' \). For each positive \( h \) and \( l \) we define a RoCTL* formula \( f(h,l) \) such that for any pair of utrees \( T \) and \( T' \) of height \( h \) it is the case that \( u(T, T') \) satisfies \( f(h,l) \) iff \( T, T' \) are isomorphic. For an automaton that accepts the tree-unwinding of \( u(T, T') \) iff \( T \) and \( T' \) are isomorphic, once the automaton has read the prefix encoding, the state of the automaton must give us enough information to determine which of \#\((h,l)\)-isomorphic equivalence classes
The definition of utrees where \( n \#(h,l) \) is non-elementary in \( h \), the number of states in the automata must also be non-elementary in \( h \). Since there are elementary translations of \( \text{CTL}^* \) into automata, we will conclude that there is no elementary translation of \( \text{RoCTL}^* \) into \( \text{CTL}^* \).

**Definition 9.1.** We define isomorphism on finite labelled trees recursively. We say that \( T = (S, R, g) \) and \( T' = (S', R', g') \) are isomorphic if \( g(\text{root}(T)) = g'(\text{root}(T')) \) and there exist orderings \( C = (C_1, \ldots, C_{|C|}) \) and \( C' = (C'_1, \ldots, C'_{|C'|}) \) of the direct subtrees of \( T \) and \( T' \) respectively such that \( C_i \) and \( C'_i \) are isomorphic for all \( i \in [1, |C|] \).

We define utrees below such that all \( (h,l) \)-utrees have the same number of direct subtrees, which are pairwise non-isomorphic. For any pair \( T, T' \) of \( (h,l) \)-utrees, this ensures that if there is a direct subtree of \( T \) that is not isomorphic to any subtree of \( T' \), there must also be a direct subtree of \( T' \) that is not isomorphic to any subtree of \( T \). This makes it easier to test whether a pair of utrees are isomorphic.

**Definition 9.2.** We define the concept of a utree recursively. We fix an infinite enumerated set \( V_0 = \{b_1, b_2, \ldots \} \). A tree \( T = (S, R, g) \) consisting of a single node \( n \) is a \((0, l)\)-utree iff \( g(n) \subseteq V_l \) where \( V_l = \{b_1, b_2, \ldots, b_l \} \). We let \( \#(h,l) \) be the number of pairwise non-isomorphic \((h,l)\)-utrees; then a tree \( T \) is a \((h+l, l)\)-utree iff \( g(\text{root}(T)) = \emptyset \) and \( T \) has \( \#(h, l)/2 \) direct subtrees, which are pairwise non-isomorphic \((h,l)\)-utrees.

**Example 9.3.** Here is an example \((1, 2)\)-utree. We use “11” as shorthand for \( b_1, b_2 \) and “01” as shorthand for \( b_2 \).

![Diagram of a (1, 2)-utree]

**Lemma 9.4.** The function \( \#(h, l) \) is at least \((h + 1)\)-exponential in \( l \).

**Proof.** We see that the number of pairwise non-isomorphic \((0, l)\)-utrees is \(2^l\). From the definition of utrees where \( n = 2 \lceil \#(h, l)/2 \rceil \),

\[
\#(h + 1, l) \geq nC \left( \frac{n}{2} \right) \\
= \frac{n!}{2^{\frac{n(n-1)}{2}}} \\
= \frac{n(n-1)\ldots \left( \frac{n}{2} \right) \ldots 2 \ldots 1}{\left( \frac{n}{2} \right) \ldots 2 \ldots 1} \\
= \frac{n(n-1)\ldots \left( \frac{n}{2} + 1 \right)}{\left( \frac{n}{2} \right) \ldots 2 \ldots 1} \\
\geq 2^{\left( \frac{n}{2} \right)}.
\]

Thus when \( \#(h, l) \) is \( j \)-exponential in \( l \), it is the case that \( \#(h + 1, l) \) is \((j + 1)\)-exponential in \( l \). As \( \#(0, l) \) is singly exponential in \( l \) it follows from induction that \( \#(h, l) \) is at least \((h + 1)\)-exponential in \( l \). \( \square \)

It is well known that we can describe the structure of a tree using a string of ‘{’ and ‘}’ characters. For example, “{1}” represents a tree with a single node, and “{{0}}” represents a tree where the root has two root nodes as successors. Algorithms 1 and 2 for
outputting the **prefix encoding** \( T \) of \( T \) use this principle. The function **prefix** is from utrees to labelled trees where each node has degree of at most one; essentially converting the utree into a linear string of symbols. In addition to the atoms used to label the input tree, the prefix encoding also uses the following atoms as labels, where \( h \) is the height of the tree and \( k \in [0, h] \).

\[ I_1 \] This atom indicates that we begin the description of a direct subtree of the tree we were describing. The current world also encodes the label of this subtree.

\[ I_f \] This atom indicates that we are ending the description of some tree.

\[ t_C \] This indicates that the description of the subtree \( C \) starts here. This is not used in function \( f \) below. It is only included to allow sections of the encoding to be easily and unambiguously referenced in the proof of correctness.

\[ H_k \] The current input character describes the start of a tree of height \( k \), we are at a node of height \( k \). Thus \( I_1 \land H_3 \) means we are beginning the definition of a tree of height 3 and \( I_f \land H_3 \) means we are ending the definition of a tree of height 3.

The final world in the prefix encoding is \( w_z \); the prefix encoding is not a transition structure as \( w_z \) has no successor.

**Example 9.5.** Below we present the prefix encoding of the utree \( T \) from Example 9.3.

\[ w_0[I_1, H_1, t_T] \]
\[ w_1[I_1, H_0, 01, t_{(n_2, \emptyset, \{n_2 \mapsto 01\})}] \]
\[ w_2[I_1, H_0] \]
\[ w_3[I_1, H_0, 11, t_{(n_3, \emptyset, \{n_3 \mapsto 11\})}] \]
\[ w_4[I_1, H_0] \]
\[ w_5[I_1, H_1] \]
\[ w_z \]

**Algorithm 1** \( T_2\text{prefix}(T) \)

1. \((g, i) := T_2g(T, \emptyset, 0)\)
2. \( S := \text{domain}(g) \cup \{w_z\} \)
3. \( \rightarrow := \{(w_{j-1}, w_j) : j \in [1, i]\} \cup (w_i, w_z) \)
4. \( \text{return}(S, \rightarrow, g) \)

**Algorithm 2** \( T_2g(T, g, i) \)

1. \((S^T, R^T, g^T) := T \)
2. \( g[w_i] := \{I_1, H_{\text{height}(T)}, t_T\} \cup g^T(\text{root}(T)); i := i + 1 \)
3. for each direct subtree \( C \) of \( T \): \((g, i) := T_2g(C, g, i)\)
4. \( g[w_i] := \{I_1, H_{\text{height}(T)}\}; i := i + 1 \)
5. \( \text{return}(g, i) \)
Strictly speaking, to be an algorithm, the **for each** in Algorithm 2 must iterate over the subtrees in some order, but the ordering chosen is unimportant and will not be defined here.

We now define the **suffix encoding** of a tree $T = (S^T, R^T, g^T)$. In addition to the atoms used in the labeling of the input tree $T$, the suffix encoding uses: the violation atom $v$ from RoCTL*; and $H_k^F$ for $k$ in $[0, h]$ which is used to indicate the height of the current node in the tree, much like $H_k$ is used in the prefix encoding. Let $N = \{n_1, \ldots, n_{|N|}\}$ be the set of nodes in the tree $T$. Let $N'$ be a numbered set such that $|N| = |N'|$; that is $N' = \{n_1', \ldots, n_{|N|}'\}$. Then for all trees $T$, if $(S, R, g) = \text{suffix}(T)$ we have

1. $S = N \cup N' \cup \{n_Z\}$
2. $R$ is the minimal relation satisfying: $R \supseteq R^T$, and
   \[
   \{(n, n_i'), (n_i', n_Z), (n_Z, n_Z)\} \subseteq R, \\
   \text{‘for all } i \in [1, |N'|].
   
3. the valuation $g$ is the valuation satisfying $g(n_i) = \{v\}; g(n_Z) = \emptyset$ and
   \[
   g(n_i') = g^T(n_i) \cup \{H_{\text{height}_{R^T}(n_i)}^F\}.
   
**Example 9.6.** Below we present the suffix encoding of the utree from Example 9.3

![Example Utree with Suffix Encoding](image)

**Definition 9.7.** We let $u(T, T')$ be the model that results when we join the prefix encoding of $T$ to the suffix encoding of $T'$ by adding $(w_Z, \text{root}(T'))$ to $R$. Formally, where $(S^P, R^P, g^P)$ is the prefix encoding of $T$ and $(S^S, R^S, g^S)$ is the suffix encoding of $T'$, it is the case that $u(T, T') = (S, R, g)$ where $S = S^P \cup S^S$, $g(w) = g^S(w)$ if $w \in S^S$, $g(w) = g^P(w)$ if $w \in S^P$, $R = R^S \cup R^P \cup \{(w_Z, \text{root}(T'))\}$.

**Definition 9.8.** Let us define a function $f$ as follows from pairs of natural numbers to RoCTL* formulas:

\[
\begin{align*}
    f(0, l) &= \bigwedge_{i \in [1, l]} (b_i \rightarrow F(H_0^F \land b_i)) \land \\
    &\bigwedge_{i \in [1, l]} (\neg b_i \rightarrow F(H_0^F \land \neg b_i)) \\
    f(k, l) &= ((I_k \land H_{k-1}) \rightarrow \triangle f(k - 1, l)) \lor (I_l \land H_k) \land F H_k^F \land (I_l \land H_k)
\end{align*}
\]
Recall that $F\phi$ is shorthand for $(TU\phi)$, and as such $M, \sigma \models F\phi \iff \exists \delta M, \sigma_{\geq \delta} \models \phi$.

The intuition behind $f$ is that a path $\sigma$ through $u(T, T') = (S, R, g)$ can correspond to both a subtree of $T$ and a subtree $T'$; if $t_C \in g(S_0)$ then $\sigma$ starts at the beginning of the prefix encoding of some subtree $C$ of $T$, and if $n_C'$ is in $\sigma$ then $\sigma$ corresponds to some subtree $C'$ of $T'$. The formula $f(0, l)$ is satisfied if the labels of $C$ and $C'$ match, so $f(0, l)$ is satisfied iff $C$ and $C'$ are isomorphic leaves. A deviation from the current path can only have one additional failure, and hence only one additional edge. So, where $D$ has a direct subtree isomorphic to $C'$, it is the case that $D'$ and $C$ are isomorphic leaves. A deviation from the current path can only have one additional failure, and hence only one additional edge. So, where $n_C'$, in $\sigma$, then for each subtree $D'$ of $T'$ satisfying height $(D') = \text{height}(C) - 1$ there exists a deviation from $\sigma$ containing $n_{C'}'$, iff $D'$ is a direct subtree of $C'$. As such, $\triangle f(0, l)$ is satisfied exactly on those paths that correspond to subtrees $C$ and $D'$ such that $C$ has a direct subtree isomorphic to $D'$. We use this intuition and recursion to prove the following lemma.

**Lemma 9.9.** For any integers $u$ and $l$, if $T$ and $T'$ are $(u, l)$-utrees then $u(T, T')$ satisfies $f(u, l)$ iff $T$ and $T'$ are isomorphic.

**Proof.** For each subtree $C$ of $T$, let $w_C$ be the world that is the beginning of the suffix encoding of $C$, or more formally the world where $t_C$ is true. For any path, $\sigma$ we define $\sigma_{\geq C}$ such that $\sigma_{\geq C} = \sigma_{\geq x}$, where $\sigma_x = w_C$.

$(\implies)$ Say that $u(T, T'), \sigma^T \models f(u, l)$ for some $\sigma^T$. We see that $\sigma_C^T = w_0$ as $f(u, l) \models I_1 \land H_u$. We define $\sigma_C$ recursively for each subtree $C$ of $T$. Say we have defined the path $\sigma_C$ for some subtree $C$ such that $u(T, T'), \sigma^C \models f(k, l)$ where $k$ is the height of $C$. Then for each direct subtree $D$ of $C$, we see that $\sigma_{D}^C \models \triangle f(k - 1, l)$ and thus there must exist a deviation from $\sigma_{D}^C$ satisfying $f(k - 1, l)$, we call this deviation $\sigma_D$.

We see that for each $C$ there is a unique $C'$ such that $n_{C'}$ is in the path $\sigma_C$. In the following paragraph we will show that for each subtree $C$ and direct subtree $D$ of $C$, we can produce $\sigma_D$ from $\sigma_{D}^C$ by replacing $n_{C'}$ with $n_{D'}$ and $n_{D''}$, and hence that $D'$ is a direct subtree of $C'$.

Consider where $\sigma_D$ deviates from $\sigma_{D}^C$. Say $n_{D'}$ is the first world in $\sigma_D$ not in $\sigma_C$, and that $n_{Z}$ is the last world in both $\sigma_C$ and $\sigma_D$. From the definition of deviations we see that $\sigma_{D}^C_{n_{D'}}$ is failure-free and so the next world on $\sigma_D$ must be $n_{D''}$. Since $\sigma_D = FH_k^k$, where $k$ is the height of $D$ it follows that $H^k_{k+1} \in g(n_{D''})$; from the structure of the suffix encoding it is clear that $B$ is a direct subtree of $A$, and $\text{height}(A) = k + 1$ and thus $H^k_{k+1}$ in $g(n_A)$). As each parent has a height greater than that of its direct subtrees, it follows that $n_{C'}$ is the only world in $\sigma_C$ such that $H^k_{k+1} \in n_{C'}$, and hence it follows that $n_{D} = n_{D'}$.

Consider $D$ of height 0. The path $\sigma_D$ is of the form

$$\langle w_D, \ldots, w_Z, n_T, \ldots, n_{C'}, n_{D'}, n_{D''}, n_{Z}, n_Z, \ldots \rangle$$

It is easy to show that $D$ and $D'$ are isomorphic. For each $C$, we choose $C'$ such that $n_{C'}$ is in the full path $\sigma_C$. Say that for every $D$ of height $k$ it is the case that $D'$ and $D$ are isomorphic. Consider $C$ of height $k + 1$. We have shown that for each direct subtree $D$ of $C$, it is the case that $D'$ is a direct subtree of $C'$, as $C'$ must have the same height as $C$ (otherwise the requirement that $\sigma \models FH_{k+1}^k$ would not be satisfied), $C'$ and $C$ have the same number of direct subtrees, each of height $k$. We have show previously that for each direct subtree $D$ of $C$, it is also the case that $D'$ is a direct subtree of $C'$. By assumption, each pair $D, D'$ are isomorphic, and so $C, C'$ are isomorphic. By induction $T$ and $T'$ are isomorphic.

$(\iff)$ Say that $T'$ and $T$ are isomorphic. Clearly suffix encodings of $T'$ and $T$ will also be isomorphic, and so $u(T, T')$ satisfies $f(u, l)$ iff $u(T, T)$ does. Thus we can assume without loss of generality that $T = T' = (S^T, R^T, g^T)$. 

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Likewise let $n_C$ be the node that is the root of the subtree $C$. We define $\sigma^C$ recursively as follows: let $\sigma^T$ be the fullpath starting at $w_0$ that passes through $n'_0$; that is, $\sigma^T = \langle w_0, \ldots, w_Z, n_{R}, n_{T}, n_Z, n_Z, \ldots \rangle$. Say that $D$ is the direct subtree of $C$, then where

$$\sigma^C = \langle w_C, \ldots, w_D, w_Z, n_{T}, \ldots, n_C, n'_C, n_Z, n_Z, \ldots \rangle$$

we let

$$\sigma^D = \langle w_D, \ldots, w_Z, n_T, \ldots, n_C, n_D, n'_D, n_Z, n_Z, \ldots \rangle.$$ 

In other words, we produce $\sigma^C$ from $\sigma^D$ by pruning everything prior to $w_D$, and $\sigma^D$ from $\sigma^C_D$ by replacing $n'_C$ with $n_D, n'_D$. This remains a full path, since $D$ is a direct subtree of $C$, and so $n_D$ is a child of $n_C$. Note also that $\sigma^D$ is a deviation from $\sigma^D_D$.

If $\text{height}(C) = 0$ it is easy to verify that $\sigma^C \models f(0, l)$, as $g(\langle w_C \rangle \cup \{H_0^F\} = g(\langle n_C \rangle \cup \{H_0, l_C, I_1\}$ so it is clear that $(-) b_i \rightarrow F (H_0^F \land (-) b_i)$. For $C$ of height $k$, it is likewise easy to see that $\sigma^C \models f(k - 1, l)$ for all $C$ of height $k - 1$. Now consider $C$ of height $k$. It is easy to show that

$$\sigma^C \models \left( (I_1 \land H_{k-1}) \rightarrow \bigvee_D t_D \right) \lor (I_1 \land H_k).$$

By assumption $\sigma^D \models f(k - 1, l)$, and $\sigma^D$ is a deviation from $\sigma^C_D$, so $\sigma^C_D \models \Delta f(k - 1, l)$. Thus

$$\sigma^C \models \left( (I_1 \land H_{k-1}) \rightarrow \Delta f(k - 1) \right) \lor (I_1 \land H_k).$$

Thus $\sigma^C \models f(k, l)$. By induction $u(T, T'), \sigma^T \models f(u, l)$. □

**Example 9.10.** In **Lemma 9.9** above, we proved that $u(T, T'), \sigma^T \models f(u, l)$ for some $\sigma^T$ iff $T$ and $T'$ are isomorphic. Using $T$ as the tree in **Example 9.3** let

$$\sigma^0 = \langle w_0, \ldots, w_Z, n_1, n_1', n_Z, \ldots \rangle$$

$$\sigma^1 = \langle w_1, \ldots, w_Z, n_1, n_2, n_2', n_Z, \ldots \rangle$$

$$\sigma^2 = \langle w_3, \ldots, w_Z, n_1, n_3, n_3', n_Z, \ldots \rangle$$

be paths through $u(T, T)$. We see that $\sigma^1$ and $\sigma^2$ satisfy $f(0, 2)$. As $\sigma^1$ and $\sigma^2$ are deviations from $\sigma^0_{2.1}$ and $\sigma^0_{2.3}$ respectively, it is the case that $\sigma^0_{2.1}$ and $\sigma^0_{2.3}$ satisfy $\Delta f(0, 2)$. Thus wherever $I_1 \land H_0$ is true, it is also the case that $\Delta f(0, 2)$ is true; hence $u(T, T), \sigma^0 \models f(1, 2)$.

**Definition 9.11.** We say an automaton $A$ accepts a structure $M$ iff the tree unwinding of $M$ is a member of $\mathcal{L}(A)$.

**Lemma 9.12.** For any arbitrary $h, l \in \mathbb{N}$, let $A = (\Sigma, Q, Q_0, \delta, F)$ be an SAA such that for any pair $T, T'$ of $(h, l)$-utrees $A$ accepts $u(T, T')$ iff $T$ and $T'$ are isomorphic; then $2^{|Q|} \geq \#(h, l)$.

**Proof.** Let $\{T_1, T_2, \ldots, T_{\#(h,l)}\}$ be a set of pairwise non-isomorphic $(h, l)$-utrees. For each $i$, let $\mathcal{R}_i = (S_{\mathcal{R}_i}, R_{\mathcal{R}_i}, g_{\mathcal{R}_i})$ be an accepting run of $A$ on $u(T_i, T_i)$; let $Q_i$ be the set of all states that the automata is in after reading the prefix encoding of $T_i$; formally let $Q_i \subseteq Q$ be the set of states such that for all $q \in Q$ we have $q \in Q_i$ iff there exists $w_{R} \in S_{\mathcal{R}_i}$ such that $(\text{root}(T_i), g) \in g_{\mathcal{R}_i}(w_{R})$. Recall that $\text{root}(T_i)$ is the beginning of the suffix encoding of $u(T_i, T_i)$.
Say that \( Q_i = Q_j \) for some \( i \neq j \). Recall from Definition 4.19 that \( A^q \) is shorthand for \( (X, Q, \{ q \}, \delta, F) \). In the next paragraph we will define a run \( R_{i}^{j} \) with the prefix from the run \( R_{i} \) and the suffix from \( R_{j} \).

Since all infinite paths of the run \( R_i \) are accepting, we see that for each \( q \in Q_i \), the relevant subtree \( R_{i,q}^{j} \) of \( R_i \) is an accepting run for \( A^q \) on the suffix encoding of \( T_i \). Let \( R_{i,j} \) be the tree that results when we replace the subtree beginning at \( w_{R_i} \) with \( R_{i,q}^{j} \), for each \( q \in Q_i = Q_j \) and \( w_{R_i} \in S_{R_i} \) satisfying \( g_{R_i} \). It is easy to show that \( R_{i,j} \) is an accepting run of \( A \) on \( u(T_j, T_i) \). However, we have assumed that \( T_i \) is not isomorphic to \( T_j \), and so \( A \) does not accept \( u(T_j, T_i) \). By contradiction \( Q_i \neq Q_j \) for any \( i,j \in [1, \#(h,l)] \) such that \( i \neq j \). As each \( Q_i \in 2^{Q} \), we can conclude from the pigeon hole principle that \( 2^{|Q|} \geq \#(h,l) \). \( \square \)

**Lemma 9.13.** For all fixed \( h \geq 1 \), there is no function \( e \) which is less than \((h-1)\)-exponential, such that the length \(|\phi_1|\) of the shortest CTL* formula \( \phi_1 \equiv f(h,l) \) satisfies \(|\phi_1| < e(l)\) for all \( l \).

**Proof.** Say \( e \) exists. Since \( \phi_1 \equiv f(h,l) \) then there exists a fullpath \( \sigma^T \) starting at \( w_0 \) through \( u(T,T') \) such that \( u(T,T'), \sigma^T \models \phi_1 \) iff \( T \) and \( T' \) are isomorphic. As \( e \) is less than \((h-1)\)-exponential, from Theorem 4.25 the size of the SAA is less than \( h \)-exponential in \( l \).

From Lemma 9.12 we have \( 2^n \geq \#(h,l) \) where \( n \) is the size of the automata, and from Lemma 9.4 we know that \( \#(h,l) \) is \((h+1)\)-exponential in \( l \). Hence \( 2^n \) is at least \((h+1)\)-exponential in \( l \), and so \( n \) is at least \( h \)-exponential in \( l \). By contradiction no such \( e \) exists. \( \square \)

**Lemma 9.14.** For all fixed \( h \geq 2 \), there is no function \( e \) which is less than \((h-2)\)-exponential such that for all RoCTL* formulas \( \phi \) with at most \( h \) nested \( \triangle \) (or \( \Box \)), the length \(|\psi|\) of the shortest CTL* formula \( \psi \) equivalent to \( \phi \) is no more than \( e(|\phi|) \).

**Proof.** This follows from the above lemma, and the fact that \( f(h,l) \) has at most \( h \) nested \( \triangle \) and \(|f(h,l)| \in O(h+l)\). \( \square \)

We can now state the main succinctness result.

**Theorem 9.15.** There is no truth preserving translation from RoCTL* to CTL* that is elementary in the length of the formula.

It is easy to prove this theorem from the lemma above. We only need to note that if there were an \( i \)-exponential translation of RoCTL* into CTL* for any \( i \in \mathbb{N} \) there would be an \( i \)-exponential translation of RoCTL* formulas with \( i+3 \) nested \( \triangle \) operators.

We see that the only non-classical operators in \( f(h,l) \) are positively occurring \( \triangle \), \( U \) and \( F \). Since \( F \psi \) is short hand for \( \top U \psi \) we see that alternations between positively occurring \( U \) and \( \triangle \) are sufficient to produce non-elementary blowup. By slightly modifying \( f \), we can similarly demonstrate that alternation between positively occurring \( \Box \) and \( U \) are also sufficient to produce non-elementary blowup. For example, the following \( f' \) contains only operators equivalent to negatively occurring \( U \), where \( W \) is the

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weak until operator and $H^F \approx \bigvee_i H^F_i$:

$$f'(0,l) = \bigwedge_{i \in [1,l]} (b_i \rightarrow G (H^F \rightarrow (H^F_0 \land b_i))) \land$$
$$\bigwedge_{i \in [1,l]} (\neg b_i \rightarrow G (H^F \rightarrow (H^F_0 \land \neg b_i)))$$

$$f'(k,l) = ((I_1 \land H_{k-1}) \rightarrow \triangle f'(k-1,l)) W (I_1 \land H_k)$$
$$\land F H^F_k \land (I_1 \land H_k)$$

Since there is no elementary translation of $f$ and $f'$ into CTL*, there is also no elementary translation of $\neg f$ and $\neg f'$ into CTL*.

9.1. Easily Translatable Fragments of RoCTL*

Although the translation is non-elementary in the worst case we note that real world formulas often fall into an easily translatable fragment of RoCTL*. The most common use for nested Robustly operators is to directly chain $n$ Robustly operators together to express the statement “Even with $n$ additional failures”. We also note that when describing the behaviour of a system, the specification of the system takes the form of a number of clauses each of which are reasonably short, see Example 3.3. We will now describing the behaviour of a system, the specification of the system takes the form of a number of clauses each of which are reasonably short, see Example 3.3. We will now translate these formulas into CTL*, and that it is easy to use CTL* decision procedures on such formulas.

It is easy to represent the statement “$v$ occurs at most $n$ times in future worlds” in LTL, we will call this statement $\gamma^n$. So for example, $\gamma^0 \equiv NG \neg v$, $\gamma^1 \equiv N(\neg v U G \neg v)$, and so forth. Note that $|\gamma^n| \in O(n)$. We see that translating $\Lambda^v \phi$ is no more complex than translating $\Lambda \phi$; we can translate $\Lambda^v \phi$ the same way as we translated $\Lambda \phi$ as above, but we replace $\psi_i$ with

$$E \left( \bigwedge s_i \land N \gamma^{n-1} \right).$$

We see that $\triangle \phi$ means $\phi$ holds on the original fullpath or a deviation, $\triangle \triangle \phi$ means that $\phi$ holds on the original path or a deviation, or a deviation from a deviation. In general $\triangle^n \phi$ means that $\phi$ holds on some path at most $n$ deviations from the current path. Thus:

$$\triangle^n \phi \equiv \phi \lor \Lambda \phi \lor \cdots \lor \Lambda^n \phi.$$ 

Thus we see that the length of the translation of $\triangle^n \phi$ is linear in $n$, and thus has no overall effect on the order of complexity. Note that $\Lambda^n \phi \equiv \neg \triangle^n \neg \phi$, so $\Lambda^n$ is also no harder to translate than a single $\Lambda$ operator. This is significant because one of the motivations of RoCTL* was to be able to express the statements of the form “If less than $n$ additional failures occur then $\phi$”. The related statement “If $n$ failures occur then $\phi$” is ever easier to translate into CTL* as $O \Lambda^n \phi \equiv A (\gamma^n \rightarrow \phi)$.

Let the $\triangle$-complexity of a formula $\phi$ be defined as follows:

$$|\phi|_\triangle = \max_{\Lambda^n \phi \leq \phi} |\psi|.$$ 

It is clear that there exists some function $f$ such that for all RoCTL* formula $\phi$ of length $n$ the translation of $\phi$ into CTL* is of length $f(n)$ or less. As the translation of $\triangle$ does not look inside state formulas it is clear that $|c(\phi)| \in O(\max_{\Lambda^n \phi} f(|\phi|_\triangle)|\phi|)$. In other words, for any fragment of RoCTL* where the length $|\phi|_\triangle$ of path-formulas contained within a $\Lambda$ operator is bounded there is a linear translation from this fragment to CTL*. As a result the complexity properties of RoCTL* formulas with bounded $|\phi|_\triangle$. 

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are similar to CTL*; we can decide the satisfiability problem in doubly exponential time and the model checking problem in time singly exponential in the length of the formula and linear in the size of the model, see [Clarke et al. 1999] for an example of a model checker for CTL*.

We can refine both above results by noting that the construction of $A_{\Lambda}$ does not look inside state formulas. Thus a fragment of RoCTL* which has a bounded number of $\Lambda$ nested within a path-formula (unbroken by $A$ or $O$) has an elementary translation into CTL*.

In [McCabe-Dansted and Dixon 2010], we discussed a fragment of RoCTL* called State-RoCTL. This fragment could naturally express many interesting robustness properties, but had a linear satisfaction preserving translation into CTL. The truth-preserving translation of State-RoCTL into RoCTL* was technically exponential, but had a linear number of unique sub-formulas and so has a natural and efficient compressed format; for example, the truth-preserving translation provided a polynomial-time model checking procedure.

10. CONCLUSION

We have defined a new, interesting, intuitive and expressive logic, RoCTL*, for specifying robustness in systems. The logic combines temporal and deontic notions in a way that captures the important contrary-to-duty obligations and seems free of the usual paradoxes.

We have shown that all RoCTL* formulas can be expressed as an equivalent CTL* formula. This translation can also be used to translate RoBCTL* [McCabe-Dansted 2008] formulas into BCTL* formulas. Once translated into CTL* formula we can use any of the standard methods for model checking, so this result provides us with a model checking procedure for RoCTL*. As with CTL*, the model checking problem for RoCTL* is linear with respect to the size of the model [Clarke et al. 1999]. Classes of RoCTL* formulas with bounded $\Lambda$-complexity have linear translations into CTL*.

Thus as with CTL* the model checking problem is also singly exponential [Clarke et al. 1999] with respect to the length of these formulas, and satisfiability is doubly exponential. Multiple nestings of $\Lambda$ (or $\Delta$) without any form of alternation can also be translated to CTL* without increasing the complexity of the translation over a single $\Lambda$ operator.

We have shown that RoCTL* is non-elementarily more succinct than CTL* for specifying some properties but we have not shown the exact complexity of the translation. However, asymptotically there is additionally one single exponential blowup per nested $\Lambda$ operator; never-the-less we expect model checking to be practical for some useful sub-classes of RoCTL* formulas. To verify this empirically we would need to implement the model checking procedure as a computer program. However, we have shown by hand that the given examples have translations into CTL* of reasonable length. Although a human translator can give better results than a naive computer translation, a practical model checking algorithm has the advantage that it can avoid translating automata back into CTL* and instead directly use the automaton to model check. While in other logics non-elementary blowup is frequently the result of unbounded alternations between positive and negative occurrences of the same operator, we do not need to alternate between $\Delta$ and $\Lambda$ to demonstrate non-elementary blowup. Indeed, the only non-classical operators in the function $f$ were positively occurring $U$ and $\Delta$. We may modify $f$ slightly so that it only contains positively occurring $U$ and $\Lambda$.

RoCTL* is known to be decidable, but without a known elementary upper bound. Our succinctness result shows that a full translation into CTL* or Tree Automata cannot result in elementary decision procedures. The question still remains as to whether some other elementary decision procedure can be found for RoCTL*. The discovery of
such a procedure would be interesting, as this would be the first modal logic which was elementary to decide but had only non-elementary translations into tree automata.

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