Semi-Hyers–Ulam–Rassias Stability of a Volterra Integro-Differential Equation of Order I with a Convolution Type Kernel via Laplace Transform

Daniela Inoan † and Daniela Marian *

Department of Mathematics, Technical University of Cluj-Napoca, 28 Memorandumului Street, 400114 Cluj-Napoca, Romania; Daniela.Inoan@math.utcluj.ro
* Correspondence: daniela.marian@math.utcluj.ro
† These authors contributed equally to this work.

Abstract: In this paper, we investigate the semi-Hyers–Ulam–Rassias stability of a Volterra integro-differential equation of order I with a convolution type kernel. To this purpose the Laplace transform is used. The results obtained show that the stability holds for problems formulated with various functions: exponential and polynomial functions. An important aspect that appears in the form of the studied equation is the symmetry of the convolution product.

Keywords: Laplace transform; semi-Hyers–Ulam–Rassias stability

MSC: 47H10; 45G10; 47N20

1. Introduction

A famous question concerning the stability of homomorphisms was formulated by Ulam in 1940 [1]. In 1941 Hyers, [2] gave an answer, in the case of the additive Cauchy equation in Banach spaces, to the problem posed by Ulam [1]. Many mathematicians posed and solved similar problems by replacing functional equations with differential equations, partial differential equations or integral equations. The first result for Hyers–Ulam stability of differential equations was given by Obloza [3]. Alsina and Ger [4] investigated the stability of the differential equation \( y' = y \). In the papers [5–10], the stability of first-order linear differential equations and linear differential equations of higher order was studied.

Several results for the Hyers–Ulam stability of integral equations were obtained in [11–14]. In [11], a class of nonlinear integral equations was studied; in [12], an integral equation with supremum; in [13], a class of fractional integro-differential equations; and in [14], a class of Volterra–Hammerstein integral equations with modified arguments.

The first result regarding the Hyers–Ulam stability of partial differential equations was given by A. Prastaro and Th.M. Rassias in [15]. Ulam–Hyers stability of partial differential equations was also studied in [16–21]. In [22], Brzdek, Popa, Rasa and Xu presented a unified and systematic approach to the field. Laplace transform was used recently to investigate the stability of linear differential equations in the work of Rezaei–Jung–Rassias [23]. The idea was extended in other papers, such as [24–26]. This method was also used in [27] where the stability of Laguerre and Bessel differential equations was studied. Replacing Laplace transform with z-transform, a similar method was used in [28] to prove the stability of linear difference equations with constant coefficients.

Laplace transform is an effective tool for solving several types of differential and integral equations. For further properties of Laplace-type integral transforms; see for instance [29]. There are numerous applications of Laplace transform in various domains: civil engineering [30], electrical engineering [31], finance [32], geology [33] and medical applications [34]. Laplace transform was also used for fractional partial differential equations.
We remark that the term containing convolution product is symmetric relative to
well known that the Laplace transform is linear and one-to-one if the involved functions
we recall first some properties of the Laplace transform used in the paper and formulate
were obtained. The system of governing equations in many fractional models is best
(see Shah et al. [35]), where analytical solutions for temperature and velocity functions
and some special functions (Hajizadeh et al. [36]). For a review of applications of Laplace transformations in various
fields, see [37].

In [38], Volterra integro-differential equation of order $p$, with a convolution type kernel
is defined as follows.

**Definition 1** ([38]). The equation

$$a_p(t)y^{(p)}(t) + a_{p-1}(t)y^{(p-1)}(t) + \cdots + a_0(t)y(t) = f(t) + \lambda \int_0^t g(t-u)y(u)du,$$

$$y(0) = y_0, \ y'(0) = y_1, \ldots, y^{(p-1)}(0) = y_{p-1},$$

(1)

is called Volterra integro-differential equation of convolution type of order $p$.

This kind of equation appear in models from many problems in mechanics and physics,
biology, engineering and astronomy, for example in heat and mass transfer, diffusion
process, growth of cells and describing the motion of satellites. Inspired by the method of
Rezaei–Jung–Rassias [23], we establish in this paper the semi-Hyers–Ulam–Rassias stability
of a Volterra integro-differential equation of convolution type of order $p$.

**Main Results**

In what follows, we denote by $\mathbb{F}$ the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $f, g, y : (0, \infty) \rightarrow \mathbb{F}$ be functions of exponential order and continuous. We write $f(0)$, $g(0)$, $y(0)$ instead the lateral limits $f(0^+)$, $g(0^+)$, $y(0^+)$, respectively. $\Re(\sigma)$ and $\Im(\sigma)$ stand for the
real part and the imaginary part, respectively, of the complex number $\sigma$.

We denote by $\mathcal{L}(f)$ the Laplace transform of the function $f$, defined by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)e^{-st}dt,$$

on the open half plane $\{s \in \mathbb{C} \mid \Re(s) > \sigma_f\}$, where $\sigma_f$ is the abscissa of convergence. It is well
known that the Laplace transform is linear and one-to-one if the involved functions
are continuous. The inverse Laplace transform will be denoted by $\mathcal{L}^{-1}(F)$. Two of the
properties used in the paper are

$$\mathcal{L}(f^{(n)})(s) = s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

$$\mathcal{L}(f \ast g)(s) = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s),$$

where $(f \ast g)(t) = \int_0^t f(t-u)g(u)du$ is the convolution product of $f$ and $g$.

In what follows, we consider the equation

$$y'(t) + \int_0^t y(u)g(t-u)du - f(t) = 0, \quad t \in (0, \infty).$$

(2)

We remark that the term containing convolution product is symmetric relative to $y$ and $g$. 
Let $\varepsilon > 0$. We also consider the inequality
\[
\left| y'(t) + \int_0^t y(u)g(t-u)du - f(t) \right| \leq \varepsilon, \quad t \in (0, \infty).
\] (3)
According to [39], we give the following definition:

**Definition 2.** Equation (2) is called semi-Hyers–Ulam–Rassias stable if there exists a function $k : (0, \infty) \to (0, \infty)$ such that for each solution $y$ of the inequality (3), there exists a solution $y_0$ of Equation (2) with
\[
|y(t) - y_0(t)| \leq k(t), \quad \forall t \in (0, \infty).
\] (4)

**Remark 1.** A function $y : (0, \infty) \to \mathbb{F}$ is a solution of (3) if and only if there exists a function $h : (0, \infty) \to \mathbb{F}$ such that
1) $|h(t)| \leq \varepsilon$, $\forall t \in (0, \infty)$,
2) $y'(t) + \int_0^t y(u)g(t-u)du - f(t) = h(t)$, $\forall t \in (0, \infty)$.

We suppose that the inverse $L^{-1}\left(\frac{1}{s+L(g)}\right)$ exists and $L^{-1}\left(\frac{1}{s+L(g)}\right)(0) = 1$.

**Theorem 1.** If a function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality (3), then there exists a solution $y_0 : (0, \infty) \to \mathbb{F}$ of (2) such that
\[
|y(t) - y_0(t)| \leq \varepsilon \int_0^t \left| L^{-1}\left(\frac{1}{s+L(g)}\right)(t-u) \right| du, \quad \forall t \in (0, \infty),
\]
that is, Equation (2) is semi-Hyers–Ulam–Rassias-stable.

**Proof.** Let $h : (0, \infty) \to \mathbb{F}$,
\[
h(t) = y'(t) + \int_0^t y(u)g(t-u)du - f(t), \quad t \in (0, \infty).
\] (5)
We have
\[
L(h) = sL(y) - y(0) + L(y) \cdot L(g) - L(f),
\]
hence
\[
L(y) = \frac{L(h)}{s+L(g)} + \frac{y(0) + L(f)}{s+L(g)}.
\]
Let
\[
y_0(t) = y(0) \cdot L^{-1}\left(\frac{1}{s+L(g)}\right)(t) + L^{-1}\left(\frac{L(f)}{s+L(g)}\right)(t), \quad \forall t \in (0, \infty).
\]
It can be seen from the condition $L^{-1}\left(\frac{1}{s+L(g)}\right)(0) = 1$ and the definition of convolution product that $y_0(0) = y(0)$.

Hence, we get
\[
\begin{align*}
& L \left[ y_0'(t) + \int_0^t y_0(u)g(t-u)du - f(t) \right] = sL(y_0) - y_0(0) + L(y_0) \cdot L(g) - L(f) \\
& = s \left[ y(0) \cdot \frac{1}{s+L(g)} + \frac{L(f)}{s+L(g)} \right] - y_0(0) \\
& \quad + \left[ y(0) \cdot \frac{1}{s+L(g)} + \frac{L(f)}{s+L(g)} \right] \cdot L(g) - L(f) = 0.
\end{align*}
\]
Since $L$ is one-to-one, it follows that
\[
y_0(t) + \int_0^t y_0(u)g(t-u)du - f(t) = 0,
\]
Theorem 2. Let \( g \) from Theorem 1, we have
\[
\mathcal{L}(y) - \mathcal{L}(y_0) = \frac{\mathcal{L}(h)}{s + \mathcal{L}(g)},
\]
hence
\[
|y(t) - y_0(t)| = \left| \mathcal{L}^{-1}\left( \frac{\mathcal{L}(h)}{s + \mathcal{L}(g)} \right) \right| = \left| \mathcal{L}^{-1}(\mathcal{L}(h)) \ast \mathcal{L}^{-1}\left( \frac{1}{s + \mathcal{L}(g)} \right) \right|
\]
so
\[
|y(t) - y_0(t)| = \left| h \ast \mathcal{L}^{-1}\left( \frac{1}{s + \mathcal{L}(g)} \right) \right| = \left| \int_0^t h(u) \cdot \mathcal{L}^{-1}\left( \frac{1}{s + \mathcal{L}(g)} \right)(t-u)du \right|
\]
\[
\leq \int_0^t |h(u)| \cdot \left| \mathcal{L}^{-1}\left( \frac{1}{s + \mathcal{L}(g)} \right)(t-u) \right| du \leq \varepsilon \int_0^t \mathcal{L}^{-1}\left( \frac{1}{s + \mathcal{L}(g)} \right)(t-u) du.
\]

Next we consider different functions \( g \). For the case where \( g \) is the exponential function, we will use an auxiliary result:

**Lemma 1.** Let \( \alpha \in \mathbb{C}^*, \alpha \neq 2 \) and let \( \sigma_1, \sigma_2 \) be the roots of the equation \( s^2 - as + 1 = 0 \). The following statements are equivalent:

(i) \( \Re(\alpha) = 0 \),

(ii) \( \Re(\sigma_1) = 0 \),

(iii) \( \Re(\sigma_2) = 0 \).

**Proof.** We have \( \sigma_1 \sigma_2 = 1 \) and \( \sigma_1 + \sigma_2 = \alpha \). This gives
\[
\sigma_2 = \frac{1}{\sigma_1} = \frac{1}{\Re(\sigma_1) + i\Im(\sigma_1)} = \frac{\Re(\sigma_1) - i\Im(\sigma_1)}{\Re(\sigma_1)^2 + (\Im(\sigma_1))^2}.
\]
Since \( \Re(\sigma_2) = \frac{\Re(\sigma_1)}{(\Re(\sigma_1)^2 + (\Im(\sigma_1))^2)} \), it follows that (ii) and (iii) are equivalent.

Now \( \alpha = \sigma_1 + \sigma_2 \) implies that \( \Re(\alpha) = \Re(\sigma_1) \left( 1 + \frac{1}{(\Re(\sigma_1))^2 + (\Im(\sigma_1))^2} \right) \), so \( \Re(\alpha) = 0 \) iff \( \Re(\sigma_1) = \Re(\sigma_2) = 0 \). \( \square \)

**Theorem 2.** Let \( g : (0, \infty) \to \mathbb{F}, g(t) = e^{\alpha t}, \) with \( \alpha \in \mathbb{C}^*, \alpha \neq 2 \).

If a function \( y : (0, \infty) \to \mathbb{F} \) satisfies the inequality (3), then there exists a solution \( y_0 : (0, \infty) \to \mathbb{F} \) of (2) such that
\[
|y(t) - y_0(t)| \leq \left\{ \begin{array}{ll}
\varepsilon \left( \frac{|A_1|}{\Re(\sigma_1)} \left( e^{\Re(\sigma_1)t} - 1 \right) + \frac{|A_2|}{\Re(\sigma_2)} \left( e^{\Re(\sigma_2)t} - 1 \right) \right), & \text{if } \Re(\alpha) \neq 0,
\varepsilon t(|A_1| + |A_2|), & \text{if } \Re(\alpha) = 0,
\end{array} \right.
\]
for any \( t \in (0, \infty) \), where \( \sigma_1, \sigma_2 \) are the roots of \( s^2 - as + 1 = 0 \) and \( A_1 = \frac{\sigma_1 - \alpha}{\sigma_1 - \sigma_2}, \ A_2 = \frac{\sigma_2 - \alpha}{\sigma_2 - \sigma_1} \).

**Proof.** From Theorem 1, we have
\[
|y(t) - y_0(t)| \leq \varepsilon \int_0^t \left| \mathcal{L}^{-1}\left( \frac{1}{s + \mathcal{L}(g)} \right)(t-u) \right| du, \quad \forall t \in (0, \infty).
\]
For \( g(t) = e^{\alpha t} \), we get
\[
\mathcal{L}^{-1}\left( \frac{1}{s + \mathcal{L}(g)} \right)(t) = \mathcal{L}^{-1}\left( \frac{s - \alpha}{s^2 - as + 1} \right)(t).
\]
Since \( \alpha \neq 2 \), the equation \( s^2 - \alpha s + 1 = 0 \) admits two distinct roots, denoted by \( \sigma_1 \) and \( \sigma_2 \). It follows that \( \frac{s-a}{s^2 - \alpha s + 1} = \frac{A_1}{s - \sigma_1} + \frac{A_2}{s - \sigma_2} \), with \( A_1 = \frac{\sigma_1 - a}{\sigma_1 - \sigma_2} \) and \( A_2 = \frac{\sigma_2 - a}{\sigma_2 - \sigma_1} \). Subsequently,

\[
\mathcal{L}^{-1}\left(\frac{s-a}{s^2 - \alpha s + 1}\right)(t) = A_1 e^{\sigma_1 t} + A_2 e^{\sigma_2 t}.
\]

Obviously, \( \mathcal{L}^{-1}\left(\frac{s-a}{s^2 - \alpha s + 1}\right)(0) = 1 \). We have

\[
|y(t) - y_0(t)| \leq \varepsilon \int_0^t \left| A_1 e^{\sigma_1 (t-u)} + A_2 e^{\sigma_2 (t-u)} \right| du \\
\leq \varepsilon \left( |A_1| e^{\Re(\sigma_1) 0} + |A_2| e^{\Re(\sigma_2) 0} \right) \int_0^t e^{-\Re(\sigma_2) u} du.
\]

If \( \Re(\alpha) \neq 0 \), then \( \Re(\sigma_1) \neq 0 \) and \( \Re(\sigma_2) \neq 0 \), so

\[
|y(t) - y_0(t)| \leq \varepsilon \left( |A_1| |e^{\Re(\sigma_1) t} - 1| + |A_2| |e^{\Re(\sigma_2) t} - 1| \right).
\]

If \( \Re(\alpha) = 0 \), then \( \Re(\sigma_1) = 0 \) and \( \Re(\sigma_2) = 0 \), so

\[
|y(t) - y_0(t)| \leq \varepsilon (|A_1| + |A_2|).
\]

In the case \( \alpha = 2 \), the roots \( \sigma_1 \) and \( \sigma_2 \) are equal.

**Theorem 3.** Let \( g : (0, \infty) \rightarrow \mathbb{F} \), \( g(t) = e^{2t} \).

If a function \( y : (0, \infty) \rightarrow \mathbb{F} \) satisfies the inequality (3), then there exists a solution \( y_0 : (0, \infty) \rightarrow \mathbb{F} \) of (2) such that

\[
|y(t) - y_0(t)| \leq \left\{ \begin{array}{ll}
\varepsilon (2e^t - t e^t - 2), & \forall t \in (0, 1) \\
\varepsilon (t e^t - 2 e^t + 2 e - 2), & \forall t \in [1, \infty).
\end{array} \right.
\]

**Proof.** To apply Theorem 1, we determine

\[
\mathcal{L}^{-1}\left(\frac{1}{s+\mathcal{L}(g)}\right)(t) = \mathcal{L}^{-1}\left(\frac{s-2}{s^2 - 2s + 1}\right)(t) = \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{(s-1)^2}\right)(t) = e^t (1 - t).
\]

Then

\[
|y(t) - y_0(t)| \leq \varepsilon \int_0^t |e^{\sigma_1 (t-u)}(1 - t + u)| du \\
\leq \varepsilon (2e^t - t e^t - 2), \quad \forall t \in (0, 1) \\
\leq (t e^t - 2 e^t + 2 e - 2), \quad \forall t \in [1, \infty).
\]

We now consider the case where \( g : (0, \infty) \rightarrow \mathbb{F}, g(t) = t^n, \ n \in \mathbb{N} \).

**Theorem 4.** Let \( g : (0, \infty) \rightarrow \mathbb{F}, g(t) = t^n, \ n \in \mathbb{N} \). Let \( \sigma_1, \sigma_2, \ldots, \sigma_{n+2} \) be the roots of the equation \( s^{n+2} + n! = 0 \) and \( A_1, A_2, \ldots, A_{n+2} \in \mathbb{F} \) such that

\[
\frac{s^{n+1}}{s^{n+2} + n!} = \sum_{k=1}^{n+2} \frac{A_k}{s - \sigma_k}.
\]
If a function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality (3), then there exists a solution $y_0 : (0, \infty) \to \mathbb{F}$ of (2) such that

$$|y(t) - y_0(t)| \leq \left\{ \begin{array}{ll}
\varepsilon \sum_{k=1}^{n+2} \frac{|A_k|}{\Re(\sigma_k)} \left( e^{\Re(\sigma_k) t} - 1 \right), & \text{if } \Re(\sigma_k) \neq 0, \forall k = 1, n+2 \\
\varepsilon \left( t \sum_{k=1}^{l} |A_k| + \sum_{k=l+1}^{n+2} \frac{|A_k|}{\Re(\sigma_k)} \left( e^{\Re(\sigma_k) t} - 1 \right) \right), & \text{if } \exists l \in \{1, 2, \ldots, n+1\} \text{such that } \Re(\sigma_k) = 0, \forall k = 1, l, \\
\varepsilon t \sum_{k=1}^{n+2} |A_k|, & \text{if } \Re(\sigma_k) = 0, \forall k = l+1, n+2
\end{array} \right. $$

Proof. We remark that the roots $\sigma_1, \sigma_2, \ldots, \sigma_{n+2}$ are distinct and $L^{-1} \left( \frac{1}{s + L(g)} \right) (0) = 1$. Indeed,

$$L^{-1} \left( \frac{1}{s + L(g)} \right) (0) = L^{-1} \left( \frac{1}{s + \frac{n+2}{s+1}} \right) (0) = L^{-1} \left( \frac{sn+1}{s+n+1} \right) (0) = \sum_{k=1}^{n+2} A_k = 1.$$

We also have

$$L^{-1} \left( \frac{1}{s + L(g)} \right) (t) = L^{-1} \left( \frac{sn+1}{s+n+2} \right) (t) = L^{-1} \left( \sum_{k=1}^{n+2} \frac{A_k}{s - \sigma_k} \right) (t) = \sum_{k=1}^{n+2} A_k e^{\sigma_k t}.$$

If $\Re(\sigma_k) \neq 0, \forall k = 1, n+2$, we apply Theorem 1 and we get

$$|y(t) - y_0(t)| \leq \varepsilon \int_0^t \left| L^{-1} \left( \frac{1}{s + L(g)} \right) (t-u) \right| du \leq \varepsilon \int_0^t |\sum_{k=1}^{n+2} A_k e^{\sigma_k (t-u)}| du = \varepsilon \sum_{k=1}^{n+2} \frac{|A_k|}{\Re(\sigma_k)} \left( e^{\Re(\sigma_k) t} - 1 \right).$$

If $\exists l \in \{1, 2, \ldots, n+1\}$ such that $\Re(\sigma_k) = 0, \forall k = 1, l$ and $\Re(\sigma_k) \neq 0, \forall k = l+1, n+2$, we apply Theorem 1 and we get

$$|y(t) - y_0(t)| \leq \varepsilon \int_0^t \left| L^{-1} \left( \frac{1}{s + L(g)} \right) (t-u) \right| du \leq \varepsilon \int_0^t |\sum_{k=1}^{n+2} A_k e^{\sigma_k (t-u)}| du = \varepsilon \left( \sum_{k=1}^{l} |A_k| t + \sum_{k=l+1}^{n+2} \frac{|A_k|}{\Re(\sigma_k)} \left( e^{\Re(\sigma_k) t} - 1 \right) \right).$$
Analogously, if $\Re(\sigma_k) = 0, \forall k = 1, n + 2$, we have
\[
|y(t) - y_0(t)| \leq \varepsilon \sum_{k=1}^{n+2} |A_k| e^{\Re(\sigma_k)t} \int_0^t e^{-\Re(\sigma_k)u} \, du = \varepsilon \sum_{k=1}^{n+2} |A_k| e^{\Re(\sigma_k)t} t \int_0^t e^{-\Re(\sigma_k)u} \, du = \varepsilon t \sum_{k=1}^{n+2} |A_k|.
\]

\[\square\]

**Example 1.** For $n = 0$ in Theorem 4, we have $g : (0, \infty) \to \mathbb{F}$, $g(t) = 1$. Then
\[
\mathcal{L}^{-1}\left(\frac{1}{s + \mathcal{L}(g)}\right)(t) = \mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{s}}\right)(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right)(t)
\]
\[= \mathcal{L}^{-1}\left(\frac{1}{2} \left(\frac{1}{s - i} + \frac{1}{s + i}\right)\right) = \frac{1}{2} (e^{it} + e^{-it}).\]

Here, the real parts of both roots of $s^2 + 1 = 0$ are zero. We apply Theorem 4 and we have
\[
|y(t) - y_0(t)| \leq \varepsilon t.
\]

**Example 2.** For $n = 1$ in Theorem 4, we have $g : (0, \infty) \to \mathbb{F}$, $g(t) = t$. Let $s_1, s_2$ be the roots of the equation $s^2 - s + 1 = 0$. Then
\[
\mathcal{L}^{-1}\left(\frac{1}{s + \mathcal{L}(g)}\right)(t) = \mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{s^2}}\right)(t) = \mathcal{L}^{-1}\left(\frac{s^2}{s^2 + 1}\right)(t)
\]
\[= \mathcal{L}^{-1}\left(\frac{1}{3} \left(\frac{1}{s + 1} + \frac{1}{s - s_1} + \frac{1}{s - s_2}\right)\right) = \frac{1}{3} (e^{-t} + e^{s_1 t} + e^{s_2 t}).\]

Applying Theorem 4, it follows that
\[
|y(t) - y_0(t)| \leq \varepsilon \frac{1}{3} \left(-e^{-t} + 4e^{\frac{t}{2}} - 3\right).
\]

### 3. Conclusions

In this paper, we studied semi-Hyers–Ulam–Rassias stability of a Volterra integro-differential Equation (2), with a convolution type kernel, of order I, using the Laplace transform. The aim of the paper was achieved by proving first the semi-Hyers–Ulam–Rassias stability in the general case (Theorem 1). In Theorems 2–4, various functions $g$ that appear in the equation were considered: exponential and polynomial functions. Some examples were given. We intend to continue the study for Volterra integro-differential Equation (1) with a convolution type kernel, of an arbitrary order.

**Author Contributions:** Conceptualization, writing, methodology: D.I. and D.M. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

### References
1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, 27, 222–224. [CrossRef]
3. Obloza, M. Hyers stability of the linear differential equation. *Rocznik Nauk-Dydakt. Prace Mat.* **1993**, 13, 259–270.
4. Alsina, C.; Ger, R. On some inequalities and stability results related to exponential function. *J. Inequal. Appl.* **1998**, *2*, 373–380. [CrossRef]

5. Takahasi, S.E.; Takagi, H.; Miura, T.; Miyajima, S. The Hyers-Ulam stability constant of first order linear differential operators. *J. Math. Anal. Appl.* **2004**, *296*, 403–409. [CrossRef]

6. Jung, S.-M. Hyers-Ulam stability of linear differential equations of first order, III. *J. Math. Anal. Appl.* **2005**, *311*, 139–146. [CrossRef]

7. Cimpean, D.S.; Popa, D. On the stability of the linear differential equation of higher order with constant coefficients. *Appl. Math. Comput.* **2010**, *217*, 4141–4146. [CrossRef]

8. Popa, D.; Rasa, I. Hyers-Ulam stability of the linear differential operator with non-constant coefficients. *Appl. Math. Comput.* **2012**, *219*, 1562–1568.

9. Otrocol, D. Ulam stabilities of differential equation with abstract Volterra operator in a Banach space. *Nonlinear Funct. Anal. Appl.* **2010**, *15*, 613–619.

10. Novac, A.; Otrocol, D.; Popa, D. Ulam stability of a linear difference equation in locally convex spaces. *Results Math.* **2021**, *76*, 1–13. [CrossRef]

11. Cadariu, L. The generalized Hyers-Ulam stability for a class of the Volterra nonlinear integral equations. *Sci. Bull. Politehnica Univ. Timis. Trans. Math. Phys.* **2011**, *56*, 30–38.

12. Ilea, V.; Otrocol, D. Existence and Uniqueness of the Solution for an Integral Equation with Supremum, via w-Distances. *Symmetry* **2020**, *12*, 1554. [CrossRef]

13. Oliveira, E.C.; Sousa, J. Ulam–Hyers–Rassias Stability for a Class of Fractional Integro-Differential Equations. *Results Math.* **2018**, *73*, 111. [CrossRef]

14. Marian, D.; Ciplea, S.A.; Lungu, N. On a functional integral equation. *Symmetry* **2021**, *13*, 1321. [CrossRef]

15. Prastaro, A.; Rassias, T.M. Ulam stability in geometry of PDE’s. *Nonlinear Funct. Anal. Appl.* **2003**, *8*, 259–278.

16. Jung, S.-M. Hyers-Ulam stability of linear partial differential equations of first order. *Appl. Math. Lett.* **2009**, *22*, 70–74. [CrossRef]

17. Jung, S.-M.; Lee, K.-S. Hyers-Ulam-Rassias stability of first order linear partial differential equations with constant coefficients. *Math. Inequal. Appl.* **2007**, *10*, 261–266. [CrossRef]

18. Lungu, N.; Ciplea, S. Ulam-Hyers-Rassias stability of pseudoparabolic partial differential equations. *Carpathian J. Math.* **2015**, *31*, 233–240. [CrossRef]

19. Lungu, N.; Marian, D. Ulam-Hyers-Rassias stability of some quasilinear partial differential equations of first order. *Carpathian J. Math.* **2019**, *35*, 165–170. [CrossRef]

20. Lungu, N.; Popa, D. Hyers-Ulam stability of a first order partial differential equation. *J. Math. Anal. Appl.* **2012**, *385*, 86–91. [CrossRef]

21. Marian, D.; Ciplea, S.A.; Lungu, N. Ulam-Hyers stability of Darboux-Ionescu problem. *Carpathian J. Math.* **2021**, *37*, 211–216. [CrossRef]

22. Brzdek, J.; Popa, D.; Rasa, I.; Xu, B. *Ulam Stability of Operators*; Elsevier: Amsterdam, The Netherlands, 2018.

23. Rezaei, J.; Jung, S.-M.; Rassias, T. Laplace transform and Hyers-Ulam stability of linear differential equations. *J. Math. Anal. Appl.* **2013**, *403*, 244–251. [CrossRef]

24. Alqifiary, Q.; Jung, S.-M. Laplace transform and generalized Hyers-Ulam stability of linear differential equations. *Electron. J. Differ. Equ.* **2014**, *2014*, 1–11.

25. Murali, R.; Ponmana Selvan, A. Mittag–Leffler-Hyers-Ulam stability of a linear differential equation of first order using Laplace transforms. *Can. J. Appl. Math.* **2020**, *2*, 47–59.

26. Shen, Y.; Chen, W. Laplace Transform Method for the Ulam Stability of Linear Fractional Differential Equations with Constant Coefficients. *Medit. J. Math.* **2017**, *14*, 1–17. [CrossRef]

27. Bicer, E.; Tunç, C. On the Hyers-Ulam Stability of Laguerre and Bessel Equations by Laplace Transform Method. *Nonlinear Dyn. Syst. Syst.* **2017**, *17*, 340–346.

28. Shen, Y.; Li, Y. The z-transform method for the Ulam stability of linear difference equations with constant coefficients. *Adv. Differ. Equ.* **2018**, *396*, 1–16. [CrossRef]

29. Sattaso, S.; Nonlaopon, K.; Kim, H. Further properties of Laplace-type integral transforms. *Dyn. Syst. Appl.* **2019**, *28*, 195–215.

30. Iwinski, T. *Theory of Beams. The Applications of the Laplace Transformation Method to Engineering Problems*; Pergamon Press: Oxford, UK; London, UK, 1967.

31. Grasso, F.; Manetti, S.; Piccirilli, M.C.; Reatti, A. A Laplace transform approach to the simulation of DC-DC converters. *Int. J. Numer. Model.* **2019**, *32*, e2618. [CrossRef]

32. Daci, A.; Tola, S. Applications of Laplace transform in finance. *Int. Sci. J. Math. Model.* **2020**, *2*, 130–133.

33. Lumentat, M.F. Analytical techniques for broadband multi electrochemical piezoelectric bimorph beams with multifrequency power harvesting. *IEEE Trans. Ultrason. Ferroelectr.Freq. Control* **2012**, *59*, 2555–2568. [CrossRef]

34. Hodasaleh, E.A.; Ebaid, A. Medical applications for the flow of carbon nano tubes suspended nano fluids in the presence of convective condition using Laplace transform. *J. Assoc. Arab. Univ. Basic Appl. Sci.* **2017**, *24*, 206–212.

35. Shah, N.A.; Elnaqeeb, T.; Animasaun, I.L.; Mahsud, Y. Insight into the natural convection flow through a vertical cylinder using Caputo time-fractional derivatives. *Int. J. Appl. Comput. Math.* **2018**, *4*, 80. [CrossRef]
36. Hajizadeh, A.; Shah, N.A.; Shah, S.I.A.; Animasaun, I.L.; Rahimi-Gorji, M.; Alarifi, I.M. Free convection flow of nanofluids between two vertical plates with damped thermal flux. *J. Mol. Liq.* 2019, 289, 110964. [CrossRef]
37. Reddy, K.J.P.; Kumar, K.; Satish, J.; Vaithyasubramanian, S. A review on applications of Laplace transformations in various fields. *J. Adv. Res. Dyn. Control Syst.* 2017, 9, 14–24.
38. Babolian, E.; Salimi Shamloo, A. Numerical solution of Volterra integral and integro-differential equations of convolution type by using operational matrices of piecewise constant orthogonal functions. *J. Comput. Appl. Math.* 2008, 214, 495–508. [CrossRef]
39. Castro, L.P.; Simões, A.M. Different Types of Hyers-Ulam-Rassias Stabilities for a Class of Integro-Differential Equations. *Filomat* 2017, 31, 5379–5390. [CrossRef]