On resistance matrices of weighted balanced digraphs

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\textbf{ABSTRACT}

Let $G$ be a connected graph with $V(G) = \{1, \ldots, n\}$. Then the resistance distance between any two vertices $i$ and $j$ is given by $r_{ij} := l_{ii}^\dagger + l_{jj}^\dagger - 2l_{ij}^\dagger$, where $l_{ij}^\dagger$ is the $(i,j)$th entry of the Moore-Penrose inverse of the Laplacian matrix of $G$. For the resistance matrix $R := [r_{ij}]$, there is an elegant formula to compute the inverse of $R$. This says that

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau^\dagger R \tau} \tau \tau^\dagger,$$

where

$$\tau := (\tau_1, \ldots, \tau_n)' \quad \text{and} \quad \tau_i := 2 - \sum_{j \sim i} r_{ij} \quad i = 1, \ldots, n.$$

A far reaching generalization of this result that gives an inverse formula for a generalized resistance matrix of a strongly connected and matrix weighted balanced directed graph is obtained in this paper. When the weights are scalars, it is shown that the generalized resistance is a non-negative real number. We also obtain a perturbation result involving resistance matrices of connected graphs and Laplacians of digraphs.

\section{1. Introduction}

Let $G$ be a simple connected graph. Suppose $x$ and $y$ are any two vertices of $G$. The length of the shortest path connecting $x$ and $y$ in $G$ is the natural way to define the distance between $x$ and $y$. This classical distance has certain limitations. For instance, consider two graphs $G_1$ and $G_2$ such that

(i) $V(G_1) = V(G_2) = \{1, \ldots, n\}$.
(ii) $i$ and $j$ are adjacent in both $G_1$ and $G_2$.
(iii) There is only path between $i$ and $j$ in $G_1$ and there are multiple paths connecting $i$ and $j$ in $G_2$. 
Then the shortest distance between $i$ and $j$ in both $G_1$ and $G_2$ is one. However, since there are multiple paths connecting $i$ and $j$ in $G_2$, the communication between $i$ and $j$ in $G_2$ is better than in $G_1$. This significance is not reflected in the shortest distance. In several applications, this limitation needs to be overcome. Instead of the classical distance, the so-called resistance distance is used widely in many situations like in electrical networks, chemistry and random walks: see for example [1] and [2]. If there are multiple paths between two vertices, then the resistance distance is less than the shortest distance. The resistance matrix is now the matrix with $(i,j)$th entry equal to the resistance distance between $i$ and $j$. Resistance matrices are non-singular and the inverse is given by an elegant formula that can be computed directly from the graph. The main purpose of this paper is to deduce a formula for the inverse of a generalized resistance matrix of a simple digraph with some special properties. This new formula generalizes the following known results.

### 1.1. Inverse of the resistance matrix of a connected graph

Let $G$ be a connected graph with $V(G) = \{1, \ldots, n\}$. Let $\delta_i$ denote the degree of the vertex $i$ and $A$ be the adjacency matrix of $G$. Then the Laplacian matrix of $G$ is $L = \text{Diag} (\delta_1, \ldots, \delta_n) - A$. Now the resistance between any two vertices $i$ and $j$ in $G$ is

$$r_{ij} := \hat{l}_{ii}^i + \hat{l}_{jj}^j - 2\hat{l}_{ij}^i,$$  \hspace{1cm} (1)

where $\hat{l}_{ij}^i$ is the $(i,j)$th entry of the Moore Penrose inverse of $L$. Define $R := [r_{ij}]$. Then the inverse of $R$ is given by

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R \tau} \tau \tau',$$  \hspace{1cm} (2)

where

$$\tau := (\tau_1, \ldots, \tau_n)' \text{ and } \tau_i := 2 - \sum_{j \sim i} r_{ij} \quad i = 1, \ldots, n.$$  

The proof of (2) is given in Theorem 9.1.2 in [1].

### 1.2. Inverse of the distance matrix of a tree

Let $T$ be a tree with $V(T) = \{1, \ldots, n\}$ and $r_{ij}$ (defined in (1)) be the resistance distance between any two vertices $i$ and $j$. If $d_{ij}$ is the length of the shortest path connecting $i$ and $j$ in $T$, then by an induction argument, it can be shown that $d_{ij} = r_{ij}$. Define $D := [d_{ij}]$. Specializing formula (2) to $T$ gives

$$D^{-1} = -\frac{1}{2}L + \frac{(2 - \delta_1, \ldots, 2 - \delta_n)'(2 - \delta_1, \ldots, 2 - \delta_n)}{2(n - 1)},$$  \hspace{1cm} (3)

where $\delta_i$ is the degree of the vertex $i$ and $L$ is the Laplacian matrix of $T$. This formula is obtained by Graham and Lovász in [3].

### 1.3. Inverse of the distance matrix of a weighted tree

Formula (3) can be generalized to weighted trees. We first need to define the Laplacian matrix of a weighted tree. Consider a tree $G = (V, \Omega)$ with $V = \{1, \ldots, n\}$. To an edge
(i, j) ∈ Ω, we assign a positive real number \( w_{ij} \). Define

\[
l_{ij} := \begin{cases} 
    -\frac{1}{w_{ij}} & (i, j) \in \Omega \\
    0 & i \neq j \text{ and } (i, j) \notin \Omega \\
    \sum_{\{k : (i, k) \in \Omega\}} \frac{1}{w_{ik}} & i = j.
\end{cases}
\]

Then the Laplacian matrix of \( G \) is \( L := [l_{ij}] \). The distance matrix of \( G \) is the symmetric matrix \( D \) with \((i, j)\)th entry equal to sum of all the weights that lie in the path connecting \( i \) and \( j \). In this case, by an induction argument, it can be shown that \( L D L + 2L = 0 \) and from this identity it is easy to show that \( d_{ij} = l_{ii}^T + l_{jj}^T - 2l_{ij}^T \), where \( l_{ij}^T \) is the \((i, j)\)th entry of the Moore Penrose inverse of \( L \). Let \( \delta_i \) be the degree of the vertex \( i \). In this setting, the following inverse formula is obtained in [4]:

\[
D^{-1} = \frac{1}{2} \left( L - \frac{\tau \tau'}{\sum_{i,j} w_{ij}} \right),
\]

where \( \tau \) is the vector \((2 - \delta_1, \ldots, 2 - \delta_n)'\).

### 1.4. Inverse of the distance matrix of a tree with matrix weights

Formula (4) can be generalized. Consider a tree on \( n \) vertices with vertex set \( V(T) = \{1, \ldots, n\} \) and edge set \( E(T) \). To an edge \((i, j)\) in \( T \), assign a positive definite matrix \( W_{ij} \) of some fixed order \( s \). Define

\[
L_{ij} := \begin{cases} 
    -W_{ij}^{-1} & (i, j) \in E(T) \\
    O_s & i \neq j \text{ and } (i, j) \notin E(T) \\
    \sum_{\{k : (i, k) \in E(T)\}} W_{ik}^{-1} & i = j.
\end{cases}
\]

(Here \( O_s \) is the \( s \times s \) matrix with all entries equal to zero.) The Laplacian matrix \( L \) of \( T \) is then the \( ns \times ns \) matrix with \((i, j)\)th block equal to \( L_{ij} \). The distance between any two vertices \( i \) and \( j \) in \( T \) is the sum of all positive definite matrices that lie in the path connecting \( i \) and \( j \). Let the \((i, j)\)th block of the Moore-Penrose inverse of \( L \) be given by \( M_{ij} \). Then, by induction it can be shown that

\[
D_{ij} = M_{ii} + M_{jj} - 2M_{ij}.
\]

The inverse of \( D \) is

\[
D^{-1} = -\frac{1}{2} \left( L - FS^{-1}F' \right),
\]

where

\[
S = \sum_i \sum_j W_{ij} \quad \text{and} \quad F = (2 - \delta_1, \ldots, 2 - \delta_n)' \otimes I_s.
\]

Formula (5) is obtained in [5].
1.5. Inverse of the resistance matrix of a directed graph

Let $G = (V, E)$ be a digraph with vertex set $V = \{1, \ldots, n\}$. A directed edge from a vertex $i$ to a vertex $j$ in $G$ will be denoted by $(i, j)$. Recall that $G$ is said to be strongly connected if there is a directed path between any two vertices $i$ and $j$. A vertex $i$ is said to be balanced if the outdegree of $i$ and the indegree of $i$ are equal. If all the vertices are balanced, then we say that $G$ is balanced. We now assume that $G$ is a strongly connected and balanced digraph.

If $i$ and $j$ are any two vertices, define

$$a_{ij} := \begin{cases} 1 & (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian matrix of $G$ is the matrix $L = [l_{ij}]$ such that

$$l_{ij} := \begin{cases} -a_{ij} & i \neq j \\ \sum_{\{k: k \neq i\}} a_{ik} & i = j. \end{cases}$$

Now, let $L^\dagger := [l_{ij}^\dagger]$ be the Moore-Penrose inverse of $L$. Then the resistance between any two vertices $i$ and $j$ in $G$ is given by

$$r_{ij} := l_{ii}^\dagger + l_{jj}^\dagger - 2l_{ij}^\dagger.$$

For the resistance matrix $R = [r_{ij}]$ of $G$, we have the following inverse formula from [6]:

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R \tau} (\tau (\tau' + 1' \text{diag}(L^\dagger)) M),$$

where

$$M := L - L', \quad \tau := 2 - \sum_{\{j: (i, j) \in E\}} r_{ij},$$

and $1$ is the vector of all ones in $\mathbb{R}^n$.

We now ask if there is a formula that unifies (2), (3), (4), (5) and (6).

1.6. Results obtained

We consider a simple digraph $G = (V, E)$ with $V = \{1, \ldots, n\}$. An element in $E$ will be denoted by $(i, j)$. Precisely, $(i, j) \in E$ means that there is a directed edge from a vertex $i$ to a vertex $j$ in $G$. All edges are assigned a positive definite matrix of some fixed order $s$. These positive definite matrices will be called weights. Let $W_{ij}$ be the weight of the edge $(i, j)$. In this setup, we define the following.
Laplacian of $G$: If $i$ and $j$ are any two distinct vertices in $G$, define

$$L_{ij} := \begin{cases} -W_{ij}^{-1} & (i,j) \in E \\ 0_s & \text{otherwise}. \end{cases}$$

(Here, $O_s$ is the $s \times s$ matrix with all entries equal to zero.) The Laplacian of $G$ is then the $ns \times ns$ matrix

$$L(G) := \begin{bmatrix} - \sum_{\{j:j \neq 1\}} L_{1j} & L_{12} & \ldots & L_{1n} \\ L_{21} & - \sum_{\{j:j \neq 2\}} L_{2j} & \ldots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \ldots & - \sum_{\{j:j \neq n\}} L_{nj} \end{bmatrix}.$$ We shall say that $L_{ij}$ is the $(i,j)$th block of $L(G)$. We note that since $G$ is a digraph, $L(G)$ is not symmetric in general.

Resistance matrix of $G$: Let $i$ and $j$ be any two vertices in $G$. Fix $a, b > 0$. Then, a generalized resistance distance between $i$ and $j$ is the $s \times s$ matrix

$$R_{ij} = a^2 K_{ii} + b^2 K_{jj} - 2ab K_{ij},$$

where $K_{ij}$ is the $(i,j)$th block of the Moore Penrose inverse of $L$. The resistance matrix corresponding to $a$ and $b$ is then the $ns \times ns$ matrix with $(i,j)$th block equal to $R_{ij}$.

Balanced vertices: We say that a vertex $j \in V$ is balanced if

$$\sum_{\{i \in V : (i,j) \in E\}} W_{ij}^{-1} = \sum_{\{i \in V : (j,i) \in E\}} W_{ji}^{-1}.$$ We obtain the following result in this paper.

**Theorem 1.1:** Let $G = (V, E)$ be a weighted, balanced and strongly connected digraph, where $V = \{1, \ldots, n\}$. Let $W_{ij}$ be the weight of the edge $(i,j)$. Fix $a, b > 0$. If $R$ is the resistance matrix of $G$, $L$ is the Laplacian of $G$, and $L^\dagger$ is the Moore-Penrose inverse of $G$, then

$$R^{-1} = -\frac{1}{2ab} L + \tau (\tau' R \tau)^{-1} (\tau' - a^2 U' \Delta(L^\dagger) L' + b^2 U' \Delta(L^\dagger) L),$$

where $\tau = (\tau_1, \ldots, \tau_n)'$ is given by

$$\tau_i := 2ab I_s + L_{ii} R_{ii} - \sum_{\{j : (i,j) \in E\}} W_{ij}^{-1} R_{ji}.$$

$U = [I_s, \ldots, I_s]$ and $\Delta(L^\dagger) = \text{Diag}(K_{11}, \ldots, K_{nn})$, with $K_{ii}$ being the $i$th diagonal block of $L^\dagger$. 

(iv) Balanced digraphs: If every vertex of $G$ is balanced, then we shall say that $G$ is balanced.
To illustrate, we give an example.

**Example 1.2:** Consider the following graph $G$ on four vertices. Let

\[ W_{14} = W_{21} = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ -4 & 7 \end{bmatrix}, \quad W_{43} = W_{32} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \]

and

\[ W_{42} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}. \]

The graph $G$ is balanced with the above matrix weights. The Laplacian of $G$ is

\[
L = [L_{ij}] = \begin{bmatrix}
7 & 4 & 0 & 0 & 0 & 0 & -7 & -4 \\
4 & 3 & 0 & 0 & 0 & 0 & -4 & -3 \\
-7 & -4 & 7 & 4 & 0 & 0 & 0 & 0 \\
-4 & -3 & 4 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 & 2 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & -5 & -3 & -2 & -1 & 7 & 4 \\
0 & 0 & -3 & -2 & -1 & -1 & 4 & 3
\end{bmatrix}.
\]

The Moore-Penrose inverse of $L$ is the matrix

\[
L^\dagger = [K_{ij}] = \frac{1}{80} \begin{bmatrix}
20 & -25 & -16 & 23 & -12 & 11 & 8 & -9 \\
-25 & 45 & 23 & -39 & 11 & -23 & -9 & 17 \\
8 & -9 & 20 & -25 & -24 & 27 & -4 & 7 \\
-9 & 17 & -25 & 45 & 27 & -51 & 7 & -11 \\
-12 & 11 & 0 & -5 & 36 & -33 & -24 & 27 \\
11 & -23 & -5 & 5 & -33 & 69 & 27 & -51 \\
-16 & 23 & -4 & 7 & 0 & -5 & 20 & -25 \\
23 & -39 & 7 & -11 & -5 & 5 & -25 & 45
\end{bmatrix}.
\]

Suppose $a = 2$ and $b = 3$. The resistance matrix of $G$ is

\[
R = [4K_{ij} + 9K_{ij} - 12K_{ij}]
\]

\[
= \frac{1}{80} \begin{bmatrix}
20 & -25 & 452 & -601 & 548 & -529 & 164 & -217 \\
-25 & 45 & -601 & 1053 & -529 & 1077 & -217 & 381 \\
164 & -217 & 20 & -25 & 692 & -721 & 308 & -409 \\
-217 & 381 & -25 & 45 & -721 & 1413 & -409 & 717 \\
468 & -489 & 324 & -297 & 36 & -33 & 612 & -681 \\
-489 & 957 & -297 & 621 & -33 & 69 & -681 & 1293 \\
452 & -601 & 308 & -409 & 404 & -337 & 20 & -25 \\
-601 & 1053 & -409 & 717 & -337 & 741 & -25 & 45
\end{bmatrix}.
\]

Next, we compute $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)'$. Recall that

\[ \tau_i := 2abI_s + L_{ii}R_{ii} - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1}R_{ji}. \]
Thus,

\[
\tau = \frac{1}{5} \begin{bmatrix}
15 & 0 & 15 & 0 & 21 & 2 & 9 & -2 \\
0 & 15 & 0 & 15 & 2 & 19 & -2 & 11
\end{bmatrix}',
\]

\[
\tau'R\tau = \frac{1}{5} \begin{bmatrix}
2916 & -3159 \\
-3159 & 6075
\end{bmatrix},
\]

and

\[
\tau' - 4U'\Delta(L^\dagger)L' + 9U'\Delta(L^\dagger)L = \frac{1}{10} \begin{bmatrix}
30 & 0 & 3 & -9 & 57 & 9 & 30 & 0 \\
0 & 30 & -9 & 12 & 48 & 9 & 0 & 30
\end{bmatrix}.
\]

Hence, we have

\[
-\frac{1}{2ab}L + \tau' R\tau^{-1}(\tau' - 4U'\Delta(L^\dagger)L' + 9U'\Delta(L^\dagger)L)
\]

\[
= \frac{1}{42444} \begin{bmatrix}
-23259 & -13368 & -84 & -138 & 3084 & 1698 & 26259 & 14928 \\
-13368 & -9891 & -138 & 54 & 1698 & 1386 & 14928 & 11331 \\
26259 & 14928 & -24843 & -14286 & 3084 & 1698 & 1500 & 780 \\
14928 & 11331 & -14286 & -10557 & 1698 & 1386 & 780 & 720 \\
2204 & 1188 & 6938 & 3351 & -2530 & -975 & 2204 & 1188 \\
1188 & 1016 & 3351 & 3587 & -975 & -1555 & 1188 & 1016 \\
796 & 372 & 17653 & 10521 & 8698 & 4371 & -23963 & -13776 \\
372 & 424 & 10521 & 7132 & 4371 & 4327 & -13776 & -10187
\end{bmatrix}
\]

which is equal to \( R^{-1} \).

### 1.7. Other results

We obtain the following two results after proving Theorem 1.1.

- By numerical computations, we observe that for any \( a, b > 0 \),
  \[
  R_{ij} = a^2K_{ii} + b^2K_{jj} - 2abK_{ij}
  \]
  is positive semidefinite. We do not know how to prove this result in general. However, when the weights in \( G \) of Theorem 1 are positive scalars, we show that \( R_{ij} \) is always a non-negative real number.

- Let \( T \) be a tree on \( n \) vertices. Suppose \( D \) and \( L \) are the distance and Laplacian matrices of \( T \). Then, from (3) it can be deduced that
  \[
  (D^{-1} - L)^{-1} = \frac{1}{3}D + \frac{1}{3}\left( \sum_{ij} w_{ij} \right) 11'.
  \]

In particular, this equation says that every entry in \( (D^{-1} - L)^{-1} \) is non-negative. Suppose \( M \) is the Laplacian matrix of an arbitrary tree on \( n \) vertices. It can be shown that \( D^{-1} - M \) is non-singular. We now say that \( (D^{-1} - M)^{-1} \) is a perturbation of the distance matrix \( D \). In [4], it is shown that all perturbations of \( D \) are non-negative matrices. We now assume that \( R \) is the resistance matrix of a connected graph (defined in
Section 1.1) on \( n \) vertices. Now consider the Laplacian matrix \( L \) of \( G \) in Theorem 1.1.

Suppose all the weights in \( G \) are positive scalars. It can be shown that \((R^{-1} - L)\) is always non-singular. We now say that \((R^{-1} - L)^{-1}\) is a perturbation of \( R \). By performing certain numerical experiments, we observed that similar to the result in [4], all perturbations of \( R \) are non-negative matrices. Since \( G \) is a digraph, \( L \) is not symmetric in general and hence all perturbations of \( R \) are not symmetric matrices. Despite this difficulty, by using an argument different from [4], we show that all perturbations of \( R \) are non-negative. This result is proved in the final part of this paper.

2. Preliminaries

We mention the notation and some basic results that will be used in the paper.

(i) We reserve \( G \) to denote a simple, strongly connected, weighted, and balanced digraph with vertex set \( V = \{1, \ldots, n\} \). A directed edge from \( i \) to \( j \) in \( G \) will be denoted by \((i, j)\). We use \( E \) to denote the edge set of \( G \). The weight of an edge \((i, j)\) will be denoted by \( W_{ij} \). All weights will be symmetric positive definite matrices and have fixed order \( s \).

(ii) Let \( B_{ns} \) be the set of all real \( ns \times ns \) matrices. A matrix \( A \) in \( B_{ns} \) will be denoted by \([A_{ij}]\), where \( A_{ij} \) is an \( s \times s \) matrix. We shall say that \( A_{ij} \) is the \((i, j)\)th block of \( A \). There are \( n \) blocks in \( A \). The null-space of a matrix \( A \) is denoted by \( \text{null}(A) \) and the column space by \( \text{col}(A) \).

(iii) The vector of all ones in \( \mathbb{R}^n \) will be denoted by \( 1 \). The matrix \( 1' \otimes I_s \) will be denoted by \( U' \), i.e.

\[
U := [I_s, \ldots, I_s]',
\]

where \( I_s \) appears \( n \) times. We use \( J \) to denote the matrix in \( B_{ns} \) with all blocks equal to \( I_s \). Note that \( J = UU' \).

(iv) The Laplacian matrix of \( G \) will be denoted by \( L \) and its Moore-Penrose by \( L^\dagger \). We note that \( L \) and \( L^\dagger \) belong to \( B_{ns} \). The \((i, j)\)th block of \( L^\dagger \) will be denoted by \( K_{ij} \). We use \( \Delta(L^\dagger) \) to denote the block diagonal matrix

\[
\text{Diag}(K_{11}, \ldots, K_{nn}).
\]

Let \( G' \) be the digraph such that \( V(G') := \{1, \ldots, n\} \) and

\[
E(G') := \{(j, i) : (i, j) \in E(G)\}
\]

To an edge \((i, j)\) of \( G' \), we assign the weight \( W_{ji} \). Again \( G' \) will be strongly connected, and balanced. The Laplacian of \( G' \) is clearly \( L' \).

(v) Fix \( a, b > 0 \). The generalized resistance matrix of \( G \) corresponding to \( a \) and \( b \) will be denoted by \( R_{a,b} \). Thus, \( R_{a,b} \) is an element in \( B_{ns} \) with \((i, j)\)th block equal to

\[
a^2K_{ii} + b^2K_{jj} - 2abK_{ij}.
\]

(vi) Let \( A \) be an \( m \times m \) matrix.

(a) We say that \( A \) is positive semidefinite if \( x'Ax \geq 0 \) for all \( x \in \mathbb{R}^m \) and positive definite if \( x'Ax > 0 \) for all \( 0 \neq x \in \mathbb{R}^m \). (To define positive semidefiniteness, we do not assume that \( A \) is symmetric.)
(b) Following [7], we say that $A$ is almost positive definite if for each $x \in \mathbb{R}^m$, either $x'Ax > 0$ or $Ax = 0$. Suppose $A$ is almost positive definite. Then the Moore-Penrose inverse of $A$ is also almost positive definite: see Corollary 2 in [7].

(vii) Let $B = [A_1, \ldots, A_n]$, where each $A_j$ is an $s \times s$ matrix. As before, we say that $A_j$ is the $j$th block of $B$. Let $\text{Diag}(B)$ be the $ns \times ns$ block matrix

$$< p > < p > \text{Diag}(A_1, \ldots, A_n).$$

(viii) We use $[n]$ for $\{1, \ldots, n\}$. The zero matrix of order $s \times s$ ($s \geq 2$) will be denoted by $O_s$. The identity matrix of order $k$ will be denoted by $I_k$.

(ix) Let $A = [a_{ij}]$ be an $m \times m$ matrix. We say that $A$ is row diagonally dominant if

$$|a_{ii}| \geq \sum_{\{j: j \neq i\}} |a_{ij}|.$$ We shall use the following well known result on diagonally dominant matrices (see Theorem 2.5.12 in [8]).

Theorem 2.1: Let $A$ be row diagonally dominant. Suppose $A$ is non-singular. Let $B := A^{-1}$ and $B = [b_{ij}]$. Then,

$$|b_{ii}| \geq |b_{ji}| \forall j.$$ 

3. Results

To prove our main result, we need to show that the Laplacian of $G$ has certain properties.

3.1. Properties of the Laplacian

From the digraph $G = (V, E)$, we define a simple undirected graph $\tilde{G}$ as follows.

Definition 3.1: Let $V(\tilde{G}) := \{1, \ldots, n\}$. We say that any two vertices $i, j \in V(\tilde{G})$ are adjacent in $\tilde{G}$ if and only if either $(i, j) \in E$ or $(j, i) \in E$.

Definition 3.2: To an edge $(i, j)$ of $\tilde{G}$, define

$$\tilde{W}_{ij} := \begin{cases} (W_{ij}^{-1} + W_{ji}^{-1})^{-1} & (i, j) \in E \text{ and } (j, i) \in E \\ W_{ij} & (i, j) \in E \text{ and } (j, i) \notin E \\ W_{ji} & (i, j) \notin E \text{ and } (j, i) \in E. \end{cases}$$

We now have the weighted graph $\tilde{G}$. Let $E$ be the set of all edges of $\tilde{G}$.

Definition 3.3: The Laplacian of $\tilde{G}$ is the matrix $L(\tilde{G}) = [M_{ij}] \in B^{ns}$, where the $(i, j)$th block is defined as follows:

$$M_{ij} := \begin{cases} -\tilde{W}_{ij}^{-1} & (i, j) \in E \\ O_s & i \neq j \text{ and } (i, j) \notin E \\ \sum_{\{k: k \neq i\}} \tilde{W}_{ik}^{-1} & i = j. \end{cases}$$
We now have the following proposition. Recall that $L$ is the Laplacian of $G$.

**Proposition 3.4:** $L(\tilde{G}) = L + L'$.

**Proof:** We shall write

$$A = L + L' \quad \text{and} \quad M = L(\tilde{G}).$$

Let the $(i,j)$th block of $A$ be $A_{ij}$ and $L$ be $L_{ij}$. We need to show that

$$A_{ij} = M_{ij} \quad \text{for all } i,j \in V,$$

where $M_{ij}$ is defined in (8). Note that $A_{ij} = L_{ij} + L_{ji}$. Partition the set of all edges $E$ of $\tilde{G}$ as follows:

$$\begin{align*}
S_1 & := \{(i,j) \in E : (i,j) \notin E \text{ and } (j,i) \notin E \} \\
S_2 & := \{(i,j) \in E : (i,j) \notin E \text{ and } (j,i) \in E \} \\
S_3 & := \{(i,j) \in E : (i,j) \in E \text{ and } (j,i) \notin E \}.
\end{align*}$$

Fix $i$ and $j$ in $\{1, \ldots, n\}$. We consider the following cases.

**Case 1:** Suppose $i$ and $j$ are not adjacent in $\tilde{G}$. This means $(i,j) \notin E$ and $(j,i) \notin E$. So, $L_{ij} = O_S$ and $L_{ji} = O_S$. By (8), $M_{ij} = O_S$. Therefore,

$$A_{ij} = L_{ij} + L_{ji} = O_S = M_{ij}.$$

**Case 2:** Suppose $i$ and $j$ are adjacent in $\tilde{G}$. We consider three possible sub-cases.

**Case (i):** Let $(i,j) \in S_1$. Then, by (7), $\tilde{W}_{ij} = W_{ij}$. So, $M_{ij} = -W_{ij}^{-1}$. Because the weight of the edge $(i,j)$ in $G$ is $W_{ij}$, $L_{ij} = -W_{ij}^{-1}$. Since $(j,i) \notin E$, $L_{ji} = O_S$ and therefore,

$$A_{ij} = L_{ij} + L_{ji} = -W_{ij}^{-1} = M_{ij}. \quad (9)$$

**Case (ii):** Let $(i,j) \in S_2$. Then, by (7), $\tilde{W}_{ij} = W_{ji}$. So, $M_{ij} = -W_{ji}^{-1}$. As $(i,j) \notin E$, $L_{ij} = O_S$. The weight of the edge $(j,i)$ in $G$ is $W_{ji}^{-1}$. So, $L_{ji} = -W_{ji}^{-1}$; and hence

$$A_{ij} = L_{ij} + L_{ji} = -W_{ji}^{-1} = M_{ij}. \quad (10)$$

**Case 3:** Let $(i,j) \in S_3$. Then,

$$\tilde{W}_{ij} = (W_{ij}^{-1} + W_{ji}^{-1})^{-1}.$$

So,

$$M_{ij} = -\tilde{W}_{ij}^{-1} = -(W_{ij}^{-1} + W_{ji}^{-1}).$$

The weights of the edges $(i,j)$ and $(j,i)$ in $G$ are respectively, $W_{ij}$ and $W_{ji}$. So,

$$L_{ij} = -W_{ij}^{-1} \quad \text{and} \quad L_{ji} = -W_{ji}^{-1}.$$

Thus,

$$A_{ij} = L_{ij} + L_{ji} = -(W_{ij}^{-1} + W_{ji}^{-1}) = M_{ij}. \quad (11)$$

Since

$$(L + L')U = MU = O_S,$$

it follows that $A_{ii} = M_{ii}$ for each $i = 1, \ldots, n$. The proof is complete. \qed
We now deduce some properties of the Laplacian matrix $L$ of $G$.

**Proposition 3.5:** The following are true.

(i) $L$ is positive semidefinite.
(ii) $\text{null}(L) = \text{null}(L') = \text{col}(J)$.
(iii) $L^\dagger$ is almost positive definite.
(iv) $LL^\dagger = L^\dagger L = I_{ns} - \frac{1}{n}$.

**Proof:** Consider the undirected graph $\tilde{G} = (V, E)$ in Definition 3.1. Put $\tilde{L} = L(\tilde{G})$ and $S_{ij} = \tilde{W}_{ij}^{-1}$, where $\tilde{W}_{ij}$ are defined in (7). Corresponding to an edge $(p, q)$ in $\tilde{G}$, we now define $\$ (p, q) $ \in B_{ns}$ with $(i, j)$th block given by

$$
\$ (p, q)_{ij} := \begin{cases} 
-S_{pq} & \text{(i,j) = (p, q) or (i,j) = (q, p)} \\
S_{pq} & i = j = p \text{ or } i = j = q \\
O_s & \text{else}.
\end{cases}
$$

Now,

$$
\tilde{L} = \sum_{(p,q) \in E} \$ (p, q).
$$

Let $x \in \mathbb{R}^{ns}$. Write

$$
x = (x^1, \ldots, x^j, \ldots, x^n)', \text{ where each } x^j \in \mathbb{R}^s.
$$

By an easy verification, we find that, if $(p, q) \in E$, then

$$
x' \$ (p, q)x = \langle S_{pq}(x^p - x^q), x^p - x^q \rangle.
$$

Thus,

$$
x' \tilde{L}x = \sum_{(i,j) \in E} \langle S_{ij}(x^i - x^j), x^i - x^j \rangle. \tag{12}
$$

Each $S_{ij}$ is a positive definite matrix. So, $x' \tilde{L}x \geq 0$. By Proposition 3.4, $\tilde{L} = L + L'$. Therefore, $x' \tilde{L}x = 2x' L' x$. So, $x' L' x \geq 0$. This proves (i).

Let $x \in (L)$. As, $\tilde{L} = L + L'$, we see that $x' \tilde{L}x = 0$. By (12), if $(i, j) \in E$, then $x^i = x^j$. Because $G$ is strongly connected, $\tilde{G}$ is connected. So, $x^i = x^j$ for all $i, j$. Thus,

$$
x \in \text{span}\{(w, \ldots, w)' \in \mathbb{R}^{ns} : w \in \mathbb{R}^s\}.
$$

Since, $\text{col}(J) = \text{span}\{(w, \ldots, w) : w \in \mathbb{R}^s\}$, we see that $x = Jp$ for some $w \in \mathbb{R}^s$. So, $(L) = \text{col}(J)$. Now, $\tilde{G}'$ is a strongly connected, and balanced digraph. Because $L'$ is the Laplacian of $\tilde{G}'$, we see that $(L') = \text{col}(J)$. This proves (ii).

We now prove (iii). Let $y \in \mathbb{R}^{ns}$. Since $L$ is positive semidefinite, $y' Ly \geq 0$. Suppose $y' Ly = 0$. Then, $y' \tilde{L} y = 0$. By Equation (12), it follows that $y \in \text{col}(J)$. Since $\text{col}(J) = (L)$, we have $Ly = 0$. Thus, either $y' Ly > 0$ or $Ly = 0$. So, $L$ is almost positive definite. By item (vi) in Section 2, $L^\dagger$ is almost positive definite as well. This proves (iii).
To prove (iv), we show that
\[ LL^\dagger v = v \quad \text{for all } v \in (J). \]

Let \( v \in (J) \). Suppose \( LL^\dagger v = w \). Then,
\[ Jw = 0 \quad \text{and} \quad L^\dagger LL^\dagger v = L^\dagger w. \]

Since \( L^\dagger LL^\dagger = L^\dagger \), we get \( L^\dagger v = L^\dagger w \) and hence \( v = w \in (L^\dagger) \). As \( (L^\dagger) = \text{col}(J) \), we get
\[ v - w \in \text{col}(J). \]

Since \( JLL^\dagger v = Jw \), and \( JL = O_{ns} \), \( Jw = 0 \). As \( v \in (J) \), \( Jv = 0 \). So, \( J(v - w) = 0 \) and hence
\[ v - w \in (J). \]

We now have \( v - w \in (J) \cap \text{col}(J) \). So, \( v = w \). The proof is complete. ■

**Proposition 3.6:** \( \Delta(L^\dagger) \) is a positive definite matrix.

**Proof:** We recall that \( \Delta(L^\dagger) = \text{Diag}(K_{11}, \ldots, K_{nn}) \). Fix \( i \in \{1, \ldots, n\} \). We show that \( K_{ii} \) is positive definite. Let \( y \in \mathbb{R}^s \). Define \( q := (q^1, \ldots, q^n)' \in \mathbb{R}^{ns} \) by
\[
q_j := \begin{cases} 
y & j = i \\
0 & \text{else.}
\end{cases}
\]

In view of previous lemma, \( L^\dagger \) is almost positive definite. So, \( q' L^\dagger q > 0 \) or \( L^\dagger q = 0 \). We note that \( q' L^\dagger q = y' K_{ii} y \). Hence, if \( q' L^\dagger q > 0 \), then \( y' K_{ii} y > 0 \). Suppose \( L^\dagger q = 0 \). Since \( \text{null}(L^\dagger) = \text{null}(L') \) and \( \text{null}(L') = \text{col}(J) \), \( q = 0 \). This means \( y \) is zero. So, \( K_{ii} \) is positive definite. The proof is complete. ■

**3.2. Inverse formula**

Recall that the generalized resistance matrix of \( G \) corresponding to two positive real numbers \( a \) and \( b \) is
\[
R_{a,b} := [R_{ij}] = [a^2 L_{ii}^\dagger + b^2 L_{jj}^\dagger + 2ab L_{ij}^\dagger].
\]

Define
\[
\tau_i := 2abI_s + L_{ii}R_{ii} - \sum_{\{j: (i,j) \in E\}} W_{ij}^{-1} R_{ji} \quad \text{for all } i = 1, \ldots, n. \quad (13)
\]

Now, let \( \tau \) be the \( ns \times s \) matrix \( (\tau_1, \ldots, \tau_n)' \). The inverse formula will be proved by using the following lemma.

**Lemma 3.7:** The following are true.

(i) \( \tau = a^2 L \Delta(L^\dagger) U + \frac{2ab}{n} U \).

(ii) \( \tau' + a^2 U' \Delta(L^\dagger)(L - L') = a^2 U' \Delta(L^\dagger)L + \frac{2ab}{n} U' \).
(iii) $LR_{a,b} + 2abI_{ns} = \tau U'$.
(iv) $R_{a,b}L + 2abl_{ns} = U\tau' - a^2UU'\Delta(L^\dagger)L' + b^2UU'\Delta(L^\dagger)L$.
(v) $U'\tau = 2abI_s$.
(vi) $\tau'R_{a,b}\tau = 2a^3b^3\tilde{X}'L\tilde{X} + \frac{1}{n}4a^2b^2(a^2 + b^2)\sum_{i=1}^n K_{ii}$, where $\tilde{X} := \Delta(L^\dagger)U$.
(vii) $\tau'R_{a,b}\tau$ is a positive definite matrix.

**Proof:** Fix $i \in [n]$. For simplicity, we shall use $R$ for $R_{a,b}$. Since $L = [L_{ij}]$ and $L^\dagger = [K_{ij}]$, the $(i,j)$th block of $LL^\dagger$ is the $s \times s$ matrix

$$L_{ii}K_{ii} - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} K_{ji}.$$ 

The $(i,j)$th block of $I_{ns} - \frac{1}{n}I_s$ is $(1 - \frac{1}{n})I_s$. Since $LL^\dagger = I_{ns} - \frac{1}{n}J$, we see that

$$L_{ii}K_{ii} - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} K_{ji} = \left(1 - \frac{1}{n}\right)I_s.$$ 

Rewriting the above equation, we have

$$\sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} K_{ji} = L_{ii}K_{ii} - \left(1 - \frac{1}{n}\right)I_s. \quad (14)$$

By definition,

$$\tau_i = 2abI_s + L_{ii}R_{ii} - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} R_{ji}.$$ 

Because

$$R_{ji} = a^2K_{jj} + b^2K_{ii} - 2abK_{ji} \quad \text{and} \quad R_{ii} = (a - b)^2K_{ii},$$

we have

$$\tau_i = 2abI_s + (a - b)^2 L_{ii}K_{ii} - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} (a^2K_{jj} + b^2K_{ii} - 2abK_{ji}). \quad (15)$$

We recall that

$$L_{ii} = \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1}.$$ 

So,

$$\sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} K_{ii} = L_{ii}K_{ii}. \quad (16)$$

Substituting (16) in (15),

$$\tau_i = 2abI_s + (a - b)^2 L_{ii}K_{ii} - b^2L_{ii}K_{ii} - a^2 \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} K_{jj} + 2ab \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} K_{ji}. $$
Using (14) in the above equation, we get
\[ \tau_i = 2abI_s + (a - b)^2L_{ii}K_{ii} - b^2L_{ii}K_{ii} - a^2\sum_{\{j : (i,j) \in E\}}W_{ij}^{-1}K_{jj} + 2abL_{ii}K_{ii} - 2ab(1 - \frac{1}{n})I_s. \]

After simplification,
\[ \tau_i = a^2L_{ii}K_{ii} - a^2\sum_{\{j : (i,j) \in E\}}W_{ij}^{-1}K_{jj} + 2abL_{ii}K_{ii} - 2ab\frac{n}{n}I_s. \] (17)

Let \( A := \text{Diag}(L) - L \). Write \( A = [A_{ij}] \). Then,
\[ \sum_{\{j : (i,j) \in E\}}W_{ij}^{-1}K_{jj} = \sum_{j=1}^{n}A_{ij}K_{jj} = (A\Delta(L^\dagger)U)_i. \] (18)

We now compute \( AU \). Because
\[
A = \begin{bmatrix}
O_s & -L_{12} & -L_{13} & \ldots & -L_{1n} \\
-L_{21} & O_s & -L_{23} & \ldots & -L_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-L_{n1} & -L_{n2} & -L_{n3} & \ldots & O_s
\end{bmatrix},
\]
it follows that
\[ (AU)_i = -\sum_{j \in [n] \setminus \{i\}}L_{ij}. \]

Put
\[ P := \Delta(L^\dagger)U. \]

Then,
\[ P_i = K_{ii}. \]

Thus,
\[ (\text{Diag}(AU)P)_i = -\sum_{j \in [n] \setminus \{i\}}L_{ij}K_{ji}. \]

As \( \sum_{j=1}^{n}L_{ij} = O_s \), we get
\[ (\text{Diag}(AU)P)_i = -\sum_{j \in [n] \setminus \{i\}}L_{ij}K_{ji} = L_{ii}K_{ii}. \] (19)

By (17), (18) and (19),
\[ \tau_i = a^2(\text{Diag}(AU)P - AP)_i + \frac{2ab}{n}I_s. \] (20)
Put
\[ \tilde{A} := \text{Diag}(AU) - A. \]

In view of (20),
\[ \tau_i = a^2 \tilde{A} P_i + \frac{2ab}{n} I_s. \]  \hspace{1cm} (21)

But a direct verification tells that \( \tilde{A} = L \).

Therefore by (21),
\[ \tau = a^2 \Delta(L^\dagger) U + \frac{2ab}{n} U. \]

This proves (i).

We now prove (ii). Put \( M := L - L' \). Then by (i),
\[ a^2 U' \Delta(L^\dagger) M + \tau' = a^2 U' \Delta(L^\dagger) L - a^2 U' \Delta(L^\dagger) L' + a^2 U' \Delta(L^\dagger) L + \frac{2ab}{n} U' \]
\[ = a^2 U' \Delta(L^\dagger) L + \frac{2ab}{n} U'. \]  \hspace{1cm} (22)

The proof of (ii) is complete.

We now prove (iii). Since
\[ R_{ij} = a^2 L_{ii}^\dagger + b^2 L_{jj}^\dagger - 2ab L_{ij}^\dagger, \]
and \( R = [R_{ij}] \), it is easy to see that
\[ R = a^2 \Delta(L^\dagger) U U' + b^2 U U' \Delta(L^\dagger) - 2ab L^\dagger. \]

As \( LL^\dagger = I_{ns} - \frac{1}{n} U U' \) and \( LU = O_s \), we get
\[ LR = a^2 \Delta(L^\dagger) U U' - 2ab LL^\dagger \]
\[ = a^2 \Delta(L^\dagger) U U' + \frac{2ab}{n} U U' - 2ab I_{ns} \]
\[ = \left( a^2 \Delta(L^\dagger) U + \frac{2ab}{n} \right) U - 2ab I_{ns}. \]  \hspace{1cm} (23)

By (i),
\[ \tau = a^2 \Delta(L^\dagger) U + \frac{2ab}{n} U. \]

Hence,
\[ LR = \tau U' - 2ab I_{ns}. \]

This completes the proof of (iii).
To prove (iv), first we observe that
\[ RL = b^2 UU' \Delta(L^\dagger)L - 2abL^\dagger L. \]
Since \( L^\dagger L = I_{ns} - \frac{1}{n} UU' \), we get
\[ RL + 2abI_{ns} = b^2 UU' \Delta(L^\dagger)L + \frac{2ab}{n} UU'. \]  
(24)

By (i),
\[ U^\tau' = a^2 UU' \Delta(L^\dagger)L' + \frac{2ab}{n} UU'. \]  
(25)

From (24) and (25),
\[ RL + 2abI_{ns} = U^\tau' - a^2 UU' \Delta(L^\dagger)L' + b^2 UU' \Delta(L^\dagger)L. \]

The proof of (iv) is complete.

By item (i),
\[ U^\tau' = a^2 U' \Delta(L^\dagger)U + \frac{2ab}{n} U' U. \]

As \( U' U = I_{ns} \) and \( U' L = O_s \), it follows that
\[ U^\tau' = 2abI_{ns}. \]

This proves (v).

Put \( M = L - L' \). By (ii),
\[ \tau' R^\tau = (a^2 U' \Delta(L^\dagger)L + \frac{2ab}{n} U') - a^2 U' \Delta(L^\dagger)M)R^\tau. \]  
(26)

Because \( M = L - L' \),
\[ a^2 U' \Delta(L^\dagger)L + \frac{2ab}{n} U' - a^2 U' \Delta(L^\dagger)L'M = a^2 U' \Delta(L^\dagger)L' + \frac{2ab}{n} U'. \]

Substituting \( \tau' \) from (i) in (26), we get
\[ \tau' R^\tau = \left(a^2 U' \Delta(L^\dagger)L' + \frac{2ab}{n} U'\right) R \left(a^2 L \Delta(L^\dagger)U + \frac{2ab}{n} U\right). \]

Therefore,
\[ \tau' R^\tau = a^4 U' \Delta(L^\dagger)L' R L \Delta(L^\dagger)U + \frac{2a^3 b}{n} U' \Delta(L^\dagger)L' RU + \frac{2a^3 b}{n} U' RL \Delta(L^\dagger)U \]  
\[ + \frac{4a^2 b^2}{n^2} U' RU. \]  
(27)

As \( LU = L' U = O_s \) and \( R = a^2 \Delta(L^\dagger)UU' + b^2 UU' \Delta(L^\dagger) - 2abL^\dagger \),
we have

\[ U' \Delta(L^\dagger)L' R L \Delta(L^\dagger) U = -2abU' \Delta(L^\dagger)L' L R \Delta(L^\dagger) U. \]

Since \( L^\dagger L = I_{ns} - \frac{1}{n} U U' \) and \( L' U = O_s \),

\[ U' \Delta(L^\dagger)L' R L \Delta(L^\dagger) U = -2abU' \Delta(L^\dagger)L' \left( I_{ns} - \frac{1}{n} U U' \right) \Delta(L^\dagger) U = -2abU' \Delta(L^\dagger)L' \Delta(L^\dagger) U. \]  

(28)

Define

\[ \tilde{X} := \Delta(L^\dagger) U. \]

Then,

\[ \tilde{X}' L \tilde{X} = U' \Delta(L^\dagger)L \Delta(L^\dagger) U. \]

(29)

By (28),

\[ U' \Delta(L^\dagger)L' R L \Delta(L^\dagger) U = -2ab \tilde{X}' L \tilde{X}. \]

In view of (iv) and (i), we have

\[ RL + 2abI_{ns} = U \tau' - a^2 U U' \Delta(L^\dagger)L' + b^2 U U' \Delta(L^\dagger)L; \]

\[ U \tau' = a^2 U U' \Delta(L^\dagger)L' + \frac{2ab}{n} U U'. \]

These two equations imply

\[ RL = b^2 U U' \Delta(L^\dagger)L + \frac{2ab}{n} U U' - 2abI_{ns}. \]

Hence,

\[ U' R L \Delta(L^\dagger) U = U' \left( b^2 U U' \Delta(L^\dagger)L + \frac{2ab}{n} U U' - 2abI_{ns} \right) \Delta(L^\dagger) U = (b^2 n) U' \Delta(L^\dagger)L \Delta(L^\dagger) U. \]

By (29),

\[ U' R L \Delta(L^\dagger) U = b^2 n \tilde{X}' L \tilde{X}. \]

(30)

We also note that

\[ U' \Delta(L^\dagger)L' R U = U' \Delta(L^\dagger)L' \left( a^2 \Delta(L^\dagger)L U U' + b^2 U U' \Delta(L^\dagger) - 2ab L^\dagger \right) U. \]

\[ = a^2 n U' \Delta(L^\dagger)L' \Delta(L^\dagger) U \]

\[ = a^2 n \tilde{X}' L \tilde{X}, \]

(31)

where the second equality follows from \( L' U = L^\dagger U = O_s \) and the last one from (29). Since

\[ R = a^2 \Delta(L^\dagger)L U U' + b^2 U U' \Delta(L^\dagger) - 2ab L^\dagger, \]
we see that

\[ U'RU = n(a^2 + b^2) \sum_{i=1}^{n} K_{ii}. \quad (32) \]

Substituting (28), (30), (31) and (32) in (27), we get

\[ \tau'R\tau = 2a^3b^3\tilde{X'}L\tilde{X} + \frac{4a^2b^2(a^2 + b^2)}{n} \sum_{i=1}^{n} K_{ii}. \]

The proof of (vi) is complete.

Since \( L \) is positive semidefinite, \( \tilde{X'}L\tilde{X} \) is positive semidefinite. By Proposition (3.6), each \( K_{ii} \) is positive definite. So, \( \tau'R\tau \) is positive definite. This proves (vii). The proof is complete. \( \blacksquare \)

Next, we prove the inverse formula in Theorem 1.1.

**Theorem 3.8:**

\[ R_{a,b}^{-1} = -\frac{1}{2ab} L + \tau(\tau'R\tau)^{-1}(\tau' - a^2U'\Delta(L^\dagger)L' + b^2U'\Delta(L^\dagger)L). \]

**Proof:** Again, as in the proof of above lemma, we shall use \( R \) for \( R_{a,b} \). By item (iii) of Lemma 3.7,

\[ LR + 2abi_{ns} = \tau U'. \]

In view of item (v) of the previous Lemma, \( U'\tau = 2abi_s \). So,

\[ LR\tau + 2abi\tau = \tau U'\tau = 2abi. \]

This implies

\[ LR\tau = O_s. \]

We know that

\[ (L) = \text{span}\{(p, \ldots, p)': p \in \mathbb{R}^s\}. \]

So,

\[ R\tau = UC, \]

where \( C \) is a \( s \times s \) matrix. Since \( \tau'R\tau \) is a positive definite matrix, \( R\tau \) cannot be zero. Hence, \( C \neq O_s \). As \( \tau'U = 2abi_s \), we get

\[ C = \frac{1}{2ab} \tau'R\tau. \]

Therefore,

\[ R\tau = \frac{1}{2ab} U(\tau'R\tau). \quad (33) \]
Since \( L'U = O_s \), from item (iv) of Lemma 3.7, we deduce that
\[
\left( \tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E'RL + 2abI_{ns} \right)
= 2ab(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E')L.
\]
Simplifying the above equation, we get
\[
(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E')RL = O_s. \tag{34}
\]
We now claim that
\[
(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E')R \neq O_s.
\]
If not, then
\[
\tau'R \tau - a^2 U'E(L^\dagger)E' \tau + b^2 U'E(L^\dagger)E'R \tau = O_s. \tag{35}
\]
By (33),
\[
R \tau = \frac{1}{2ab} U \tau'R \tau.
\]
So,
\[
LR \tau = O_s \quad \text{and} \quad L' \tau = O_s.
\]
Hence (35) leads to \( \tau'R \tau = O_s \). This contradicts that \( \tau'R \tau \) is positive definite. Hence,
\[
(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E')R \neq O_s.
\]
Since nullity of \( L' \) is \( s \) and \( L'U = O_s \), by (34), there exists an \( s \times s \) matrix \( \tilde{C} \) such that
\[
(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E')R = \tilde{C}U'.
\]
We know that \( U' \tau = 2abI_s \). So, from the previous equation,
\[
\tilde{C} = \frac{1}{2ab} \tau'R \tau.
\]
Thus,
\[
(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E')R = \frac{\tau'R \tau}{2ab} - U'. \tag{36}
\]
We now have
\[
Q := \left( -\frac{1}{2ab} L + \tau(\tau'R \tau)^{-1}(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E') \right)R
\]
\[
= -\frac{1}{2ab} LR + \tau(\tau'R \tau)^{-1}(\tau' - a^2 U'E(L^\dagger)E' + b^2 U'E(L^\dagger)E')R.
\]
By (36), we have
\[
Q = -\frac{1}{2ab} (LR - \tau U'). \tag{37}
\]
Item (iii) of Lemma 3.7 says that
\[
LR + 2abI_{ns} = \tau U'.
\]
Substituting back in (37), we get \( Q = I_{ns} \). The proof is complete. \( \blacksquare \)
3.3. Special cases

(i) Suppose all the weights in $G$ are equal to 1. Choose $a = b = 1$. We shall denote $R_{ij}$ by $r_{ij}$ and define $R := [r_{ij}]$. Now by (13),

$$\tau_i = 2 - \sum_{\langle i,j \rangle \in E} r_{ji}.$$  

We note that $r_{ii} = 0$ and $U = 1$. Hence, by our formula in Theorem 1.1,

$$R^{-1} = -\frac{1}{2}L + (\tau' R \tau)^{-1}(\tau' - 1') \Delta(L^\dagger)L' + 1' \Delta(L^\dagger)L$$

$$= -\frac{1}{2}L + \frac{\tau'}{\tau' R \tau}(\tau' - 1') \Delta(L^\dagger)(L - L').$$

Thus we get (6).

(ii) Suppose $T$ is a tree with $V(T) = \{1, \ldots, n\}$. To denote an edge in $T$, we shall use the notation $ij$. Let the weight of an edge $ij$ be $W_{ij}$. Assume that all weights are positive definite matrices of order $s$. Now, define a directed graph $\tilde{T}$ as follows. Let $V(\tilde{T}) = \{1, \ldots, n\}$. We use the notation $(i, j)$ to denote a directed edge from $i$ to $j$. Now, we define $E(\tilde{T}) := \{(i, j), (j, i) : ij \in E(T)\}$. Now we assign the weight $W_{pq}$ to an edge $(p, q)$ in $\tilde{T}$. It is clear that $\tilde{T}$ is strongly connected, weighted and balanced digraph. Now define the Laplacian matrix of $T$, say, $L(T)$ as given in 1.4 and the Laplacian of $\tilde{T}$, say, $L(\tilde{T})$ as given in item (i) of 1.6. We note that $L(T) = L(\tilde{T})$. Fix $a = b = 1$. Let $R_{ij}$ be the resistance between $i$ and $j$. Then,

$$R_{ij} = M_{ii} + M_{jj} - 2M_{ij},$$

where $M_{ij}$ is the $(i, j)$th block of the Moore Penrose inverse of $L(\tilde{T})$. If $D_{ij}$ is the shortest distance between $i$ and $j$ in $T$, then by the argument mentioned in 1.4, $D_{ij} = R_{ij}$. Define $D := [D_{ij}]$. Because $D_{ii} = O_s$, by Theorem 1.1, we have

$$\tau_i := 2I_s - \sum_{\langle i,j \rangle \in E} W_{ij}^{-1} R_{ji}$$

$$= 2I_s - \sum_{\langle i,j \rangle \in E} W_{ij}^{-1} W_{ij}$$

$$= (2 - \delta_i)I_s,$$

where $\delta_i$ is the out-degree of the vertex $i$. We now define

$$\tau := (2 - \delta_1, \ldots, 2 - \delta_n)' \otimes I_s.$$  

By an induction argument it follows that if $S$ is the sum of all the weights in $T$, then

$$D \tau = [S, \ldots, S]'$$.
Because $T$ is a tree, $\sum_{i=1}^{n} \delta_i = 2(n-1)$. Thus, $\sum_{i=1}^{n} \tau_i = 2I_x$ and hence $\tau' D \tau = 2S$. By our formula in Theorem 1.1,

$$D^{-1} = -\frac{1}{2} L(T) + \tau (\tau' D \tau)^{-1} \tau'$$

$$= -\frac{1}{2} L(T) + \frac{1}{2} \tau S \tau'.$$

(38)

This is formula (5).

In a similar manner, we get (2), (3), and (4).

### 3.4. Non-negativity of the resistance

From numerical computations, we observe that $R_{ij} = a^2 l_{ii} + b^2 l_{jj} - 2ab l_{ij}$ is always positive semi-definite. But at this stage, we do not know how to prove this. However, when all the weights in $G$ are positive scalars, we now show that the resistance is always non-negative. We need the following lemma.

**Lemma 3.9:** Let $A = [a_{ij}]$ be an $n \times n$ matrix with the following properties.

(i) All the off-diagonal entries are non-positive.

(ii) $A1 = A'1 = 0$.

(iii) $\text{rank}(A + A') = n - 1$.

(iv) $A$ is positive semidefinite.

Let $A^\dagger := [p_{ij}]$ be the Moore-Penrose inverse of $A$. Then,

$$p_{ii} \geq p_{ij} \quad \text{and} \quad p_{ii} \geq p_{ji} \quad \forall \ j.$$

**Proof:** By a permutation similarity argument, without loss of generality, we may assume that $i = 1$ and $j = n$. We now show that $p_{11} \geq p_{1n}$ and $p_{11} \geq p_{n1}$. By symmetry of our assumptions, it is sufficient to show that $p_{11} \geq p_{1n}$.

Let $1_{n-1}$ be the vector of all ones in $\mathbb{R}^{n-1}$. By (ii) we can partition $A$ as follows:

$$A = \begin{bmatrix} B & -B1_{n-1} \\ -1_{n-1}'B & 1_{n-1}'B1_{n-1} \end{bmatrix}.$$  

All the row sums of $A + A'$ are equal to zero. So, all the cofactors of $A + A'$ are equal. As $\text{rank}(A + A') = n - 1$, we now deduce that the common cofactor of $A + A'$ is non-zero. In particular, $\det(B + B') \neq 0$. Since $A$ is positive semidefinite, $B + B'$ is positive semidefinite. Because $B + B'$ is non-singular, $B + B'$ is positive definite. So, $B$ is positive definite. All the off-diagonal entries of $B$ are non-positive. By a well-known theorem on $\mathbb{Z}$-matrices, $B$ is non-singular and all entries of $B^{-1}$ are non-negative. By a direct verification,

$$A^\dagger = \begin{bmatrix} B^{-1} - \frac{1}{n} 1_{n-1}1_{n-1}'B^{-1} - \frac{1}{n} 1_{n-1}B^{-1}1_{n-1}' - \frac{1}{n} B^{-1}1_{n-1} & 1_{n-1}B^{-1}1_{n-1} \\ -\frac{1}{n} 1_{n-1}'B^{-1} & 0 \end{bmatrix} + \frac{1_{n-1}'B^{-1}1_{n-1}}{n^2} (11').$$

(39)
Put 

\[ C = [c_{ij}] := B^{-1} \quad \text{and} \quad \delta := \frac{1}{n^2} 1_{n-1}^t B^{-1} 1_{n-1}. \]

Then, \( c_{ij} \geq 0 \forall i, j \) and

\[
\begin{align*}
p_{11} &= c_{11} - \frac{1}{n} \sum_{j=1}^{n-1} c_{1j} - \frac{1}{n} \sum_{j=1}^{n-1} c_{1j} + \delta, \\
p_{1n} &= -\frac{1}{n} \sum_{j=1}^{n-1} c_{1j} + \delta.
\end{align*}
\]

Now,

\[
p_{11} - p_{1n} = c_{11} - \frac{1}{n} \sum_{i=1}^{n-1} c_{i1}.
\]

By Theorem 2.1,

\[ c_{11} \geq c_{j1} \quad \forall j = 1, \ldots, n - 1. \]

So,

\[
-\frac{1}{n} \sum_{i=1}^{n-1} c_{i1} \geq - \left( \frac{n-1}{n} \right) c_{11}.
\]

Hence,

\[
c_{11} - \frac{1}{n} \sum_{i=1}^{n-1} c_{i1} \geq c_{11} - \frac{n-1}{n} c_{11} = \frac{1}{n} c_{11}.
\]

Since \( c_{11} \geq 0 \), we conclude that

\[ p_{11} - p_{1n} \geq 0. \]

The proof is complete. \( \blacksquare \)

Now it can be easily shown that any generalized resistance is non-negative.

**Theorem 3.10:** Suppose all the weights in \( G \) are positive scalars. Let \( a, b > 0 \). Let \( L^\dagger = [k_{ij}] \) be the Moore-Penrose inverse of the Laplacian of \( G \). Then,

\[ r_{ij} := a^2 k_{ii} + b^2 k_{jj} - 2ab k_{ij} \geq 0. \]

**Proof:** We note that the Laplacian matrix \( L \) of \( G \) satisfies all the conditions of the previous lemma. Moreover, by Proposition 3.6, \( k_{ii} \) and \( k_{jj} \) are positive. As a consequence of Lemma 3.9, we deduce that

\[ \min(k_{ii}, k_{jj}) \geq \max(k_{ij}, k_{ji}). \]

Now

\[ a^2 k_{ii} + b^2 k_{jj} - 2ab k_{ij} \geq 0, \]

follows from the arithmetic mean and geometric mean inequality. \( \blacksquare \)
3.5. A perturbation result

We now show that if $R$ is the resistance matrix of a connected graph with $n$ vertices, and if $L$ is the Laplacian matrix of $G$ with positive scalar weights, then $(R^{-1} - L)^{-1}$ has all entries non-negative.

**Theorem 3.11:** Let $H$ be a simple (undirected) connected graph with $n$ vertices and $R$ be the resistance matrix of $H$. Assume that all the weights in $G$ are positive scalars. Then, $R^{-1} - L$ is non-singular and every entry in $(R^{-1} - L)^{-1}$ is non-negative.

**Proof:** Let $M = [m_{ij}]$ be the Moore-Penrose inverse of the Laplacian matrix of $H$. Then the $(i, j)$th entry $r_{ij}$ of $R$ is given by

$$r_{ij} = m_{ii} + m_{jj} - 2m_{ij}.$$

Fix $\alpha \geq 0$. Define $S := \alpha L$. We complete the proof by using the following claims.

**Claim 1:** $R^{-1} - S$ is non-singular.

To prove this claim, we can assume that $S = \alpha L$, where $\alpha > 0$. By Proposition 3.5, rank$(S) = n - 1$, $S + S'$ is positive semidefinite and $S'1 = S1 = 0$. Let $x \in \mathbb{R}^n$ be such that $(R^{-1} - S)(x) = 0$. (40)

Put $u := R^{-1}x$. Assuming $u \neq 0$, we now get a contradiction. As $1'S = 0$, it follows that $1'R^{-1}x = 0$ and hence $u \in 1^\perp$. Writing

$$R = \text{Diag}(M)11' + 11'\text{Diag}(M) - 2M,$$

we get

$$u'Ru = u'(\text{Diag}(M)11' + 11'\text{Diag}(M) - 2M)u$$

$$= -2u'Mu.$$

Since $(M) = \text{span}\{1\}$, $M$ is positive definite on $1^\perp$. So, $u'Mu > 0$. Hence,

$$u'Ru < 0. \quad (41)$$

It is easy to see that

$$u'Ru = x'R^{-1}x \quad (42)$$

By (40),

$$x'R^{-1}x = x'Sx.$$

Since $S + S'$ is positive semidefinite, $x'Sx \geq 0$. So, $x'R^{-1}x \geq 0$. Hence by (42),

$$u'Ru \geq 0. \quad (43)$$

Thus, we get a contradiction from (41) and (43). Therefore, $u = R^{-1}x = 0$. This implies $x = 0$. So, $R^{-1} - S$ is non-singular. The claim is proved.
**Claim 2:** If $C$ is a $k \times k$ proper principal submatrix of $S - R^{-1}$, then
\[ q^T C q > 0 \quad \text{for all } 0 \neq q \in \mathbb{R}^k. \]

If $A$ is an $n \times n$ matrix, we shall use the notation $A[i]$ to denote the principal submatrix of $A$ obtained by deleting the $i$th row and $i$th column of $A$. Fix $1 \leq i \leq n$ and define
\[ B := S[i] - R^{-1}[i]. \]

Since $R$ is negative definite on $1^{\perp}$ and the diagonal entries are zero, $R$ has exactly one positive eigenvalue. By an application of interlacing theorem [10], we see that $-R^{-1}[i]$ is positive semidefinite. Hence
\[ -p^T R^{-1}[i] p \geq 0 \quad \text{for all } w \in \mathbb{R}^{n-1}. \] (44)

As the row sums and the column sums of $S$ are equal to zero, it follows that all the cofactors of $S$ are equal. Because $S + S'$ is positive semidefinite and has rank $n-1$, it follows that every proper principal submatrix of $S + S'$ is positive definite. So, we have
\[ p^T S[i] p > 0 \quad \text{for all } 0 \neq p \in \mathbb{R}^{n-1}. \] (45)

By (44) and (45),
\[ p^T B p > 0 \quad \text{for all } 0 \neq p \in \mathbb{R}^{n-1}. \]

The claim is proved.

In particular, we note that all principal minors of $S - R^{-1}$ with order less than $n$ are positive.

**Claim 3:** $\det(S - R^{-1}) < 0$.

Because $\gamma L - R^{-1}$ is non-singular for every $\gamma \geq 0$,
\[ \text{sgn } \det(\gamma L - R^{-1}) = \text{sgn } \det(-R^{-1}). \]

Since $-R$ has exactly one negative eigenvalue, $\det(-R^{-1}) < 0$. So,
\[ \det(\gamma L - R^{-1}) < 0 \quad \forall \gamma \geq 0. \]

This proves the claim.

**Claim 4:** All principal minors of $(S - R^{-1})^{-1}$ are negative.

Put
\[ G := (S - R^{-1})^{-1} \quad \text{and} \quad H := S - R^{-1}. \]

Let $s < n$, and let $\hat{G}$ be a $s \times s$ principal submatrix of $G$. Suppose $\hat{H}$ is the complementary submatrix of $\hat{G}$ in $H$. By Jacobi identity,
\[ \det(\hat{G}) = \frac{\det(\hat{H})}{\det(H)}. \]

By Claim 2 and 3,
\[ \text{sgn}(\det\hat{H}) > 0 \quad \text{and} \quad \text{sgn}(\det(H)) < 0. \]

So, $\det(\hat{G}) < 0$. The claim is proved.
Figure 1. Directed Graph \( G \).

Figure 2. (a) Graph \( H \), and (b) Graph \( G \).

We now complete the proof of the theorem. Given \( \beta \geq 0 \), let

\[
\begin{bmatrix}
  y_{ii}(\beta) & y_{ij}(\beta) \\
  y_{ji}(\beta) & y_{jj}(\beta)
\end{bmatrix}
\]

denote a \( 2 \times 2 \) principal submatrix of \((\beta L - R^{-1})^{-1}\). By Claim 4,

\[ y_{ii}(\beta) < 0 \quad \text{and} \quad y_{jj}(\beta) < 0 \quad \text{for all} \quad \beta \geq 0. \]  

(46)

We now show that \( y_{ij}(\beta) < 0 \) for all \( \beta \geq 0 \). Since \( y_{ij}(0) = -r_{ij}, y_{ij}(0) < 0 \). If \( y_{ij}(\alpha) > 0 \) for some \( \alpha > 0 \), then by continuity, \( y_{ij}(\delta) = 0 \) for some \( \delta > 0 \). Hence by (46),

\[
\det\begin{bmatrix}
  y_{ii}(\delta) & y_{ij}(\delta) \\
  y_{ji}(\delta) & y_{jj}(\delta)
\end{bmatrix} = y_{ii}(\delta)y_{jj}(\delta) > 0.
\]

However by Claim 4,

\[
\det\begin{bmatrix}
  y_{ii}(\delta) & y_{ij}(\delta) \\
  y_{ji}(\delta) & y_{jj}(\delta)
\end{bmatrix} < 0.
\]

Thus, we have a contradiction. So, \( y_{ij}(\alpha) \leq 0 \) for all \( \alpha \geq 0 \). We now conclude that every entry in \((L - R^{-1})^{-1}\) is negative. The proof is complete.

We illustrate the above result with an example.

Example 3.12: Consider the graphs \( H \) and \( G \) given in Figure 2.
Let the positive scalar weights $w_{ij}$ assigned to each edge $(i, j)$ of $G$ be

$$w_{14} = w_{21} = \frac{10}{7}, \quad w_{32} = w_{43} = 5 \quad \text{and} \quad w_{42} = 2.$$

The Laplacian matrix of $G$ is

$$L = \frac{1}{10} \begin{bmatrix} 7 & 0 & 0 & -7 \\ -7 & 7 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -5 & -2 & 7 \end{bmatrix}.$$

The resistance matrix $R$ of $H$ is

$$R = \frac{1}{4} \begin{bmatrix} 0 & 3 & 4 & 3 \\ 3 & 0 & 3 & 4 \\ 4 & 3 & 0 & 3 \\ 3 & 4 & 3 & 0 \end{bmatrix}.$$

Now,

$$(R^{-1} - L)^{-1} = \frac{1}{8612} \begin{bmatrix} 2335 & 6555 & 7515 & 5125 \\ 5125 & 2585 & 6905 & 6915 \\ 7515 & 5975 & 1135 & 6905 \\ 6555 & 6415 & 5975 & 2585 \end{bmatrix},$$

which is a non-negative matrix.

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