Viscous Cosmology, Entropy, and the Cardy-Verlinde Formula

Iver Brevik

March 24, 2022

Abstract

The holographic principle in a radiation dominated universe, as discussed first by Verlinde [hep-th/008140], is extended so as to incorporate the case of a bulk-viscous cosmic fluid. This corresponds to a non-conformally invariant theory. Generalization of the Cardy-Verlinde entropy formula to the viscous case appears to be formally possible, although on physical grounds one may question some elements in this type of theory, especially the manner in which the Casimir energy is evaluated. Also, we consider the observation made by Youm [Phys. Lett. B531, 276 (2002)], namely that the entropy of the universe is no longer expressible in the conventional Cardy-Verlinde form if one relaxes the radiation dominance equation of state for the fluid and instead merely assumes that the pressure is proportional to the energy density. We show that Youm’s generalized entropy formula remains valid when the cosmic fluid is no longer ideal, but endowed with a constant bulk viscosity. In the introductory part of this article, we take a rather general point of view and survey the essence of cosmological theory applied to a fluid containing both a constant shear viscosity and a constant bulk viscosity.

1 Introduction

Perfect fluid models have for a long time been used in cosmological theory. The introduction of the viscosity concept came later, and there has in fact been an open question as to whether the viscosity concept is needed for an explanation of the observed quantities in the universe. Misner [1] was probably the first to introduce viscosity in cosmology in connection with his study of how initial anisotropies in the early universe became relaxed. Cosmological model with viscosity have later now and then been discussed in the literature, from various points of view. A useful review of the subject, with many references to the earlier literature up to 1990, was given by Gron [2].

*Department of Energy and Process Engineering, Norwegian University of Science and Technology, N-7491 Trondheim, Norway. E-mail address: iver.h.brevik@ntnu.no
From a physical point of view it would in our opinion be almost surprising if the viscosity concept were not of importance in cosmology. An essential ingredient of the Friedmannian model of the universe is after all to borrow the energy-momentum tensor from non-viscous fluid dynamics and insert it into the Einstein equations. From fluid dynamics we know that in many situations the non-viscous theory is inadequate, in particular, if anisotropies or turbulence effects are involved. Use of the molecular shear and bulk viscosities means in effect an expansion of the theory to first order in the deviations from thermal equilibrium. Moreover, once turbulence occurs in the velocity field it is the Reynolds stresses, rather than the stresses caused by molecular viscosities, that govern the behavior of the fluid. In view of these known facts from ordinary fluid dynamics it is natural to be somewhat reluctant to regard cosmology as an exceptional case for which the viscosity concepts are of no use. One main reason why the viscous cosmological theory has so far not gained large attention is undoubtedly that the viscosity-generated entropy $\sigma$ per baryon is observed to be very large, $\sigma \sim 10^9$, and previous investigations have shown that a straightforward use of the kinematically derived bulk viscosity in the isotropic and homogeneous universe is unable to explain this large magnitude [3, 4, 5].

The fluidlike behavior of the universe during its early epochs is in all probability very complicated. One particular facet of the problem is the existence of one or more phase transitions that may have led to sudden changes whose description is naturally given in terms of the phenomenological viscosity coefficients. An eclatant example of this sort is the transition from the de Sitter universe back to the FRW universe at the end of the inflationary era, at $t \sim t^{-33}$ s. We may call the effective viscosity operative during such a phase transition (bulk viscosity in case of an isotropic universe) an "impulsive" viscosity. Examples of this kind of theory were examined in Refs. [6, 7].

Another interesting aspect of viscous cosmology that has arisen recently is the influence from shear viscosity. The shear viscosity comes into play in connection with anisotropy. Thus Weinberg [8] derived an analytic formula for the traceless part of the anisotropic stress tensor due to freely streaming neutrinos. Another recent example is the suggestion of Kovtun et al. [9] and Karch [10] (cf. also the discussion in [11]) that there exists in cosmology a universal lower bound on the ratio $\eta/s$, where $\eta$ is the shear viscosity and $s$ the entropy per unit volume.

The main topic to be dealt with in the present article is the incorporation of the bulk viscosity $\zeta$ in the so-called holographic principle for the early universe. In the case of a radiation dominant ideal-fluid universe, Verlinde [12] put forward the idea that there exists a bound on the subextensive entropy associated with the Casimir energy. This bound is called the holographic bound. When this bound is saturated, there exists a formal coincidence between the Friedmann equation for $H^2 \equiv (\dot{a}/a)^2$ and the Cardy entropy formula known from conformal field theory [13, 14]. The question that naturally arises is whether this merging between the holographic principle, the entropy formula from conformal field theory, and the Friedmann equation from cosmology, is of deeper physical significance and thus
reproducible under more general conditions, or if it is just a formal coincidence. As one would expect, the Verlinde proposal has been the subject of study in cases where more general effects have been accounted for. For instance, Wang et al. \cite{15} have considered universes having a cosmological constant, and Nojiri et al. \cite{16} have considered quantum bounds for the Cardy-Verlinde formula. There are several papers in this area of Padmanabhan et al. \cite{17}, and of Cai et al. \cite{18}; the paper of Youm \cite{19} contains an extensive list of references. We also mention the considerable interest that has arisen in connection with entropy and energy as following from quantum and thermal fluctuations in conformal field theories \cite{20}.

We begin in the next section by giving a survey of cosmological theory applied to a fluid that is general enough to possess both a constant shear viscosity \( \eta \) and a constant bulk viscosity \( \zeta \). Thereafter, in Sect. 3 we consider the Verlinde setting when the fluid is taken to possess a bulk viscosity only. In Sect. 4 we carry out an analogous reasoning for the case that the bulk-viscosity fluid is not necessarily radiation dominated but instead required to obey the equation of state

\[
p = (\gamma - 1)\rho,
\]

with \( \gamma \) a constant. Sections 3 and 4 are based upon material given previously in Refs. \cite{21, 22, 23}. We supply the analysis with some numerical estimates; thus we exploit the smallness of \( \zeta \) in the plasma era to calculate the viscosity-induced correction to the scale factor \( a(t) \), assuming for simplicity that the spatial curvature \( k \) is zero. In Sect. 5 we test the proposed holographic entropy bound on an example, taken from the plasma era in the early universe.

We mostly use natural units, with \( \hbar = c = k_B = 1 \).

### 2 Survey of viscous cosmology

We use the convention in which the Minkowski metric is \((-+++\)). Let \( U^\mu = (U^0, U^i) \) be the four-velocity of the cosmic fluid. In comoving coordinates, \( U^0 = 1 \), \( U^i = 0 \).

Let \( g_{\mu\nu} \) be the general metric. Using the projection tensor

\[
h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu,
\]

we define the rotation tensor as

\[
\omega_{\mu\nu} = h_\alpha^\mu h_\beta^\nu U_{[\alpha;\beta]} = \frac{1}{2}(U_{\mu;\alpha} h_\alpha^\nu - U_{\nu;\alpha} h_\alpha^\mu)
\]

and the expansion tensor as

\[
\theta_{\mu\nu} = h_\alpha^\mu h_\beta^\nu U_{(\alpha;\beta)} = \frac{1}{2}(U_{\mu;\alpha} h_\alpha^\nu + U_{\nu;\alpha} h_\alpha^\mu).
\]

The scalar expansion is \( \theta \equiv \theta_\mu^\mu = U^\mu_{\mu,\mu} \). The shear tensor, as defined by

\[
\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu} \theta,
\]
is traceless, $\sigma^\mu_\nu = 0$. The following decomposition of the covariant derivative is often useful:

$$U_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}h_{\mu\nu}\theta - A_\mu U_\nu,$$  \hspace{1cm} (6)

where $A_\mu \equiv \dot{U}_\mu = U^\nu U_{\mu;\nu}$ is the four-acceleration of the fluid.

We write down the expression for the energy-momentum tensor $T_{\mu\nu}$ of the viscous fluid, taking into account also the conduction of heat. As already mentioned, $\eta$ and $\zeta$ are respectively the shear and the bulk viscosities; if moreover $\kappa$ is the thermal conductivity (all quantities taken in accordance with their nonrelativistic definitions), then

$$Q^\mu = -\kappa h^\mu_\nu (T_{\nu,\mu} + TA_\nu)$$  \hspace{1cm} (7)

is the spacelike heat flux density four-vector, and

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - \zeta \theta)h_{\mu\nu} - 2\eta \sigma_{\mu\nu} + Q_\mu U_\nu + Q_\nu U_\mu.$$  \hspace{1cm} (8)

Here $\rho$ is the mass-energy density and $p$ is the isotropic pressure, both taken in the local rest inertial frame. The last term in Eq. (7), containing $TA_\nu$, is of relativistic origin. If one ignores this term, one is left with $Q^\mu = -\kappa h^\mu_\nu T_{\nu,\mu}$. This expression is defined such that in a local rest inertial frame (designated with a "hat") $Q^0 = 0$, whereas $Q^i = -\kappa T^i_\nu$ is the heat energy per unit time crossing a unit surface orthogonal to the unit vector $e^i$.

Consider next the production of entropy. It is here instructive to start from nonrelativistic theory. If $u_i$ are the nonrelativistic velocity components, and $\sigma$ the nonrelativistic entropy per particle (baryon), then the ordinary entropy per unit volume is $S = n\sigma$, $n$ being the baryon number density. From nonrelativistic theory we have [24]

$$\frac{dS}{dt} = 2\eta T \left( \theta_{ik} - \frac{1}{3} \delta_{ik} \nabla \cdot u \right)^2 + \frac{\zeta}{T} (\nabla \cdot u)^2 + \frac{\kappa}{T^2} (\nabla T)^2,$$  \hspace{1cm} (9)

where $\theta = u_{(i,k)}$. The transition to relativistic theory may be made via the effective substitutions

$$\theta_{ik} \rightarrow \theta_{\mu\nu}, \quad \delta_{ik} \rightarrow h_{\mu\nu}, \quad \nabla \cdot u \rightarrow \theta, \quad -\kappa T^i_\nu \rightarrow Q^i_\mu,$$  \hspace{1cm} (10)

from which we obtain

$$S^\mu_{\nu,\mu} = \frac{2\eta}{T} \sigma_{\mu\nu} \sigma^\mu_\nu + \frac{\zeta}{T} \theta^2 + \frac{1}{\kappa T^2} Q_\mu Q^\mu.$$  \hspace{1cm} (11)

Here, $S^\mu$ is the entropy four-vector

$$S^\mu = n\sigma U^\mu + \frac{1}{T} Q^\mu.$$  \hspace{1cm} (12)

Our treatment above follows Ref. [6]. The same result is obtained from a more careful analysis taking into account the relativistic thermodynamic equations [3] [25].
In the case of thermal equilibrium, \( Q_\mu = 0 \). Moreover, in accordance with usual practice we omit the shear viscosity in view of the assumed complete isotropy of the cosmic fluid, although we have to remark that this is actually a nontrivial point. The reason is that the shear viscosity is usually so much greater than the bulk viscosity. Typically, after termination of the plasma era at the time of recombination \((T \simeq 4000 \text{ K})\), the ratio \( \eta/\zeta \) as calculated from kinetic theory is as large as about \( 10^{12} \). Thus, even a slight anisotropy in the fluid would easily outweigh the effect of the effect of the very small bulk viscosity.

3 Cardy-Verlinde formula in a bulk-viscous radiation dominated universe

Assume now that the metric is of the Friedmann-Robertson-Walker (FRW) type,

\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),
\]

\( k = -1, 0, 1 \) being the curvature parameter. In comoving coordinates, \( U_{\mu\nu} = \Gamma^0_{\mu\nu} \), so that in view of the standard relations for the Christoffel symbols we get \( U^\mu_{\mu\nu} = h^\mu_\mu \dot{a}/a \), with \( \dot{a} = da/dt \). From these equations we see that the rotation and shear tensors both vanish,

\[
\omega_{\mu\nu} = \sigma_{\mu\nu} = 0,
\]

whereas the scalar expansion is

\[
\theta = 3\dot{a}/a = 3H,
\]

\( H \) being the Hubble parameter. The four-acceleration is zero, \( A_\mu = 0 \). The energy-momentum tensor becomes

\[
T_{00} = \rho, \quad T_{0k} = 0, \quad T_{ik} = (p - \zeta \theta) g_{ik}.
\]

There is thus no conduction of heat in the FRW space, which is a homogeneous space. The entire effect of the bulk viscosity is to reduce the pressure by an amount \( \zeta \theta \), so that the effective pressure becomes \( \tilde{p} = p - \zeta \theta \).

When applied to the FRW space, Eqs. (11) and (12) yield

\[
S^\mu_{\cdots \mu} = \frac{\zeta}{T} \theta^2, \quad S^0 = n\sigma, \quad S^i = 0.
\]

Consider now the Einstein equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.
\]
from which we obtain the first Friedmann equation (the ”initial value equation”)

\[ H^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} - \frac{k}{a^2}, \quad (19) \]

\( \rho = E/V \) being the energy density. This equation contains no viscous term. The second Friedmann equation (the ”dynamic equation”), when combined with Eq. (19), yields

\[ \dot{H} = -4\pi G(\rho + \tilde{p}) + \frac{k}{a^2}, \quad (20) \]
in which the presence of viscosity is explicit.

The differential conservation for energy, \( T_{\alpha\nu;\nu} = 0 \), yields

\[ \dot{\rho} + (\rho + p)\dot{\theta} = \zeta \theta^2. \quad (21) \]
The conservation equation for baryon particle number is

\[ (nU^\mu)_{;\mu} = 0, \quad (22) \]

which means that \( na^3 = \) constant in the comoving frame. Then from Eq. (17) we obtain, when substituting \( S^\mu \) from Eq. (12) and observing that \( \dot{Q}^0 = 0 \) in the comoving frame,

\[ n\dot{\sigma} = \frac{\zeta}{T} \theta^2. \quad (23) \]

We now recall that the entropy of a (1+1) dimensional CFT is given by the Cardy formula \[13, 14\]

\[ S = 2\pi \sqrt{c/6 \left( L_0 - c/24 \right)}, \quad (24) \]

where \( c \) is the central charge and \( L_0 \) the lowest Virasoro generator.

Let us assume that the universe is closed, and has a vanishing cosmological constant, \( k = +1, \quad \Lambda = 0 \). This is the case considered by Verlinde \[12\] (in his formalism the number \( n \) of space dimensions is set equal to 3). The Friedmann equation (19) is seen to agree with the CFT equation (24) if we perform the substitutions

\[ L_0 \to \frac{1}{3} Ea, \quad c \to \frac{3}{\pi G a}, \quad S \to \frac{HV}{2G}. \quad (25) \]

These substitutions are the same as in Ref.\[12\]. We see thus that Verlinde’s argument remains valid, even if the fluid possesses a bulk viscosity. Note that no assumptions have so far been made about the equation of state for the fluid. At this point the following question thus naturally arises: is there a deeper connection between the laws of general relativity and those of quantum field theory?

Continuing this kind of reasoning, let us consider the three actual entropy definitions.
• First, there is the Bekenstein entropy \[ S_B = \frac{2\pi}{3} E_a. \] (26)

The arguments for deriving this expression seem to be of a general nature; in accordance with Verlinde we find it likely that the Bekenstein bound \( S \leq S_B \) is universal. We shall accept this expression for \( S_B \) in the following, even when the fluid is viscous.

• The next kind of entropy is the Bekenstein-Hawking expression \( S_{BH} \), which is supposed to hold for systems with limited self-gravity:

\[ S_{BH} = \frac{V}{2Ga}. \] (27)

Again, this expression relies upon the viscous-insensitive member (19) of Friedmann’s equations. Namely, when \( \Lambda = 0 \) this equation yields

\[ S_B < S_{BH} \quad \text{when} \quad Ha < 1, \]
\[ S_B > S_{BH} \quad \text{when} \quad Ha > 1. \] (28)

The borderline case between a weakly and a strongly gravitating system is thus at \( Ha = 1 \). It is reasonable to identify \( S_{BH} \) with the holographic entropy of a black hole with the size of the universe.

• The third entropy concept is the Hubble entropy \( S_H \). It can be introduced by starting from the conventional formula \( A/4G \) for the entropy of a black hole. The horizon area \( A \) is approximately \( H^{-2} \), so that

\[ S_H \sim \frac{H^{-2}}{4G} \sim \frac{HV}{4G}, \] (29)

since \( V \sim H^{-3} \). Arguments have been given by Easther and Lowe [27], Veneziano [28], Bak and Rey [29], and Kaloper and Linde [30] for assuming the maximum entropy inside the universe to be produced by black holes of the size of the Hubble radius (cf. also [31]). According to Verlinde the FSB prescription (see [12] for a closer discussion) one can determine the prefactor:

\[ S_H = \frac{HV}{2G}. \] (30)

It is seen to agree with Eq. (25).

One may now choose (see below) to define the Casimir energy \( E_C \) as the violation of the Euler identity:

\[ E_C = 3(E + pV - TS) \] (31)
where, from scaling, the total energy $E$ can be decomposed as ($E_E$ is the extensive part) $E(S,V) = E_E(S,V) + \frac{1}{2}E_C(S,V)$. Due to conformal invariance the products $E_E a$ and $E_C a$ are independent of the volume $V$, and a function of the entropy $S$ only. From the known extensive behaviour of $E_E$ and the sub-extensive behaviour of $E_C$ one may write (for CFT)

$$E_E = \frac{\alpha}{4\pi a} S^{4/3}, \quad E_C = \frac{\beta}{2\pi a} S^{2/3}, \quad (32)$$

where $\alpha, \beta$ are constants whose product for CFTs is known: $\sqrt{\alpha \beta} = n = 3$ (this follows from the AdS/CFT correspondence, cf. [12]). From these expressions it follows that

$$S = \frac{2\pi a}{3} \sqrt{E_C(2E - E_C)}. \quad (33)$$

This is the Cardy-Verlinde formula for the radiation dominated universe. Identifying $E_E a$ with $L_0$ and $E_C a$ with $c/12$ we see that Eq. (33) becomes the same as Eq. (24), except from a numerical prefactor which is related to our assumption about $n = 3$ space dimensions instead of $n = 1$ as assumed in the Cardy formula.

The question is now: can the above line of arguments be carried over to the case of a viscous fluid? The most delicate point here appears to be the assumed pure entropy dependence of the product $E_E a$. As we mentioned above, this property was derived from conformal invariance, a property that is absent in the case under discussion. To examine whether the property still holds when the fluid is viscous (and conformal invariance is lost), we can start from the Friedmann equations (19) and (20), in the case $k = 1$, $\Lambda = 0$, and derive the "energy equation", which can be transformed to

$$\frac{d}{da} \left( \rho a^4 \right) = (\rho - 3\tilde{p}) a^3. \quad (34)$$

Thus, for a radiation dominated universe, $p = \rho/3$, it follows that

$$\frac{d}{dt} \left( \rho a^4 \right) = \zeta \theta^2 a^4. \quad (35)$$

Let us compare this expression, which is essentially the time derivative of the volume density of the quantity $E_E a$ under discussion, with our earlier expression (23): both the two time derivatives are seen to be proportional to $\zeta$. Since $\zeta$ is small, we can insert for $a = a(t)$ the expression pertinent for a non-viscous, closed universe: $a(t) = a_* \sin \eta$, where $\eta$ here denotes conformal time, and $a_*$ is the constant

$$a_* = \sqrt{(8\pi G/3) \rho_{in} a_{in}^4}. \quad (36)$$

The subscript "in" designates the initial instant of the onset of viscosity. Imagine now that Eqs. (35) and (23) are integrated with respect to time. Then, since the densities $\zeta^{-1} \rho a^4$ and $\zeta^{-1} n_\sigma$ can be drawn as functions of $t$, it follows that $\rho a^4$ can be considered as a function of $n_\sigma$, or, equivalently, that $E_E a$ can be considered as a
function of $S$. We conclude that this property, previously derived on the basis of CFT, really appears to carry over to the viscous case.

The following point ought to be commented upon. The specific entropy $\sigma$ in Eq. (23) is the usual thermodynamic entropy per particle. The identification of $S$ with $HV/(2G)$, as made in Eq. (25), is however something different, since it is derived from a comparison with the Cardy formula (24). Since this entropy is the same as the Hubble entropy $S_H$ we can write the equation as $n \sigma_H = H/(2G)$, where $\sigma_H$ is the Hubble entropy per particle. This quantity is different from $\sigma$, since it does not follow from thermodynamics plus Friedmann equations alone, but from the holographic principle. The situation is actually not peculiar to viscous cosmology. It occurs if $\zeta = 0$ also. The latter case is easy to analyze analytically, if we focus attention on the case $t \to 0$. Then, for any value of $k$, we have $a \propto t^{1/2}$, implying that $H = 1/(2t)$. Moreover, the equation of continuity $(nU^\mu)_{\mu} = 0$ implies, as we have seen, that $na^3 = \text{constant}$ for a FRW universe, so that $n \propto t^{-3/2}$. The above equation for $\sigma_H$ then yields $\sigma_H \propto t^{1/2}$. This is obviously different from the result for the thermodynamic entropy $\sigma$: from Eq. (28) we simply get $\sigma = \text{constant}$ when $\zeta = 0$. The two specific entropies are thus different even in this case.

### 3.1 Remarks on the physical interpretation

Let us make three remarks on the interpretation of the above formalism. They are based on physical, rather than mathematical, considerations.

- First, one may wonder about the legitimacy of defining the Casimir energy such as in Eq. (31). Usually, within the Green function approach, in a spherical geometry the Casimir energy is calculated indirectly, by integration of the Casimir surface force density $f = -(1/4\pi a^2) \partial E/\partial a$. The force $f$ in turn is calculated by first subtracting off the volume-dependent parts of the two scalar Green functions; this is in agreement with the physical requirement that $f \to 0$ at $r \to \infty$. (The typical example of this configuration is that of a conducting shell; cf., for instance, Ref. [32].) That this kind of procedure should lead to the same result as Eq. (31), which merely expresses a violation of the thermodynamic Euler identity, is in our opinion not evident.

- Our second remark is about the physical meaning of taking the Casimir energy $E_C$ to be positive. Verlinde assumes that $E_C$ is bounded by the total energy $E$: $E_C \leq E$. This may be a realistic bound for some of the CFTs. However, in general cases, it is not true. For a realistic dielectric material it is known that the full Casimir energy is not positive; the dominant terms in $E_C$ are definitely negative. From a statistical mechanical point of view this follows immediately from the fact that the Casimir force is the integrated effect of the attractive van der Waals force between the molecules. Now the case of a singular conducting shell is complicated - there are two limits involved, namely
the infinitesimal thickness of the shell and also the infinite conductivity (or infinite permittivity) - and a microscopical treatment of such a configuration has to our knowledge not been given. What is known, is the microscopical theory for a dielectric ball. Let us write down, for illustration, the expression derived by Barton \[33\] for a dilute ball:

\[
E_C = \frac{3\gamma V}{2\pi^2 \lambda^4} + \gamma^2 \left( -\frac{3}{128\pi^2 \lambda^4} + \frac{7}{360\pi^3 \lambda^3} - \frac{1}{20\pi^2 \lambda} + \frac{23}{1536\pi a} \right),
\]

where \(\gamma = (\epsilon - 1)/\epsilon\), \(A\) is the surface area, and \(\lambda\) is a cutoff parameter. This expression, derived from quantum mechanical perturbation theory, agrees with the statistical mechanical calculation in Ref. \[34\], and also essentially with Ref. \[35\] (there are some numerical factors different in the cutoff-dependent terms). It is evident from this expression that the dominant, cutoff, dependent volume terms, are negative.

We see that there remains one single, cutoff independent, term in Eq. (37). This term is in fact positive. It can be derived from macroscopic electrodynamics also, by using either dimensional continuation or zeta-function regularization, as has been done in Ref. \[36\]. In the present context the following question becomes however natural: how can a positive, small, cutoff dependent term in the Casimir energy play a major role in cosmology? In another words, why should the matter necessarily be conformal? Of course, our universe is different from a dielectric ball, and we are not simply stating that Verlinde’s method is incorrect. Our aim is merely to stress the need of caution, when results from one field in physics are applied to another field. In any case, all this suggests that the consideration of non-conformally invariant situations should significantly change the dynamical entropy bounds and bounds for Casimir energy.

- Our discussion on the generalized Verlinde formula in the present section was based upon the set of cosmological assumptions \(\{p = \rho/3, k = +1, \Lambda = 0\}\). The recent development of Wang et al. \[15\] is interesting, since it allows for a nonvanishing cosmological constant (still assuming a closed model). One of the scenarios treated in \[15\] is that of a de Sitter universe (\(\Lambda > 0\)) occupied by a universe-sized black hole. A black hole in de Sitter space has the metric

\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2,
\]

where \(f(r) = 1 - 2MG/r - \Lambda r^2/3\). The region of physical interest is that lying between the inner black hole horizon and the outer cosmological horizon, the latter being determined by the magnitude of \(\Lambda\).
Although we do not enter into any detail about this theory, we make the following observations: the above metric is static; there is no time-dependent scale factor involved, and the influence from viscosity will not turn up in the line element. Moreover, Wang et al. make use of only the member of Friedmann’s equations which, as we have noticed, is formally independent of viscosity.

Does this imply that viscosity is without any importance for the present kind of theory? The answer in our opinion is no, since the theory operates implicitly with the concept of the maximum scale factor $a_{\text{max}}$ in the closed Friedmann universe. In order to calculate $a_{\text{max}}$, one has to solve the Friedmann equation also, which contains the viscosity through the modified pressure $\tilde{p}$. Thus, viscosity comes into play after all, though in an indirect way.

4 Cardy-Verlinde formula in the presence of a general equation of state

The paper of Youm mentioned above is interesting, since it shows that the entropy of the universe can no longer be expressed in the conventional form of Eq. (33) if one relaxes the radiation dominance state equation and instead assumes the more general equation (1). And this brings us naturally to the problem of how the presence of a bulk viscosity, together with Eq. (1), influences the entropy formula. Actually, it will turn out that the modified entropy formula found by Youm still persists, even when a constant bulk viscosity is allowed for. The central formula is Eq. (42) below.

The main part of the formalism has been given already. Thus Friedmann’s equations take the same form (19) and (20) as before (with $k = +1, \Lambda = 0$), the conservation equation for energy is still Eq. (21), and also the production rate for entropy is as in Eq. (23). However, Eq. (35) gets replaced by

$$\frac{d}{dt}(\rho a^{3\gamma}) = \zeta \theta^2 a^{3\gamma}. \quad (39)$$

We can now carry out the same kind of reasoning as above: Since $\zeta$ is assumed small, we can use for $a = a(t)$ the same expression as for a nonviscous closed universe, namely $a(t) = a_\ast \sin \eta$ with $a_\ast$ given by Eq. (36) (although this approximate expression strictly speaking assumes that the universe is radiation dominated). Imagine that Eqs. (39) and (20) are integrated with respect to time. Since $\zeta^{-1} \rho a^{3\gamma}$ and $\zeta^{-1} n a^{3\gamma}$ can be drawn as functions of $t$, it follows that $\rho a^{3\gamma}$ can be considered as a function of $n a$. Then, since the total energy is $E \sim \rho a^3$ and the total entropy is $S \sim n a^3$, it follows that $E a^{3(\gamma - 1)}$ is independent of the volume $V$ and is a function of $S$ only. This generalizes the pure entropy dependence of the product $E a$, found by Verlinde in the case of a nonviscous radiation dominated universe. And it is
A Sample Document

noteworthy that the derived property of $Ea^{3(\gamma-1)}$ formally agrees exactly with the property found by Youm [19] when $\zeta = 0$.

Let us carry out the analysis a bit further, and write the total energy $E$ as a sum of an extensive part $E_E$ and a subextensive part $E_C$, as in the previous section. Under a scale transformation $S \rightarrow \lambda S$ and $V \rightarrow \lambda V$ with constant $\lambda$, $E_E$ scales linearly with $\lambda$. But the term $E_C$ scales with a power of $\lambda$ that is less than one: as $E_C$ is the volume integral over a local energy density expressed in the metric and its derivatives, each of which scales as $\lambda^{-1/3}$, and as the derivatives occur in pairs, the power in $\lambda$ has to be $1-2/3= 1/3$. Thus we have

$$E_E(\lambda S, \lambda V) = \lambda E_E(S, V), \quad E_C(\lambda S, \lambda V) = \lambda^{1/3} E_C(S, V),$$

(40)

which implies

$$E_E = \frac{\alpha}{4\pi a^{3(\gamma-1)}} S^\gamma, \quad E_C = \frac{\beta}{2\pi a^{3(\gamma-1)}} S^{\gamma-2/3},$$

(41)

Together with the decomposition of $E(S, V)$, this leads to

$$S = \left[ \frac{2\pi a^{3(\gamma-1)}}{\sqrt{\alpha\beta}} \sqrt{E_C(2E - E_C)} \right]^{\frac{1}{3\gamma-1}}.$$  

(42)

This is the generalized Cardy-Verlinde formula, in agreement with Eq. (20) in Youm’s paper, reducing to the standard formula (with square root) in the case of a radiation dominated universe. In conclusion, we have extended the basis of Eq. (12) so as to include the presence of a constant bulk viscosity in the cosmic fluid.

4.1 Numerical estimates for a spatially flat universe

It is of physical interest to supplement the above considerations with some simple numerical estimates, showing, in particular, the order of magnitude of the viscosity terms. Let us assume, as mentioned above, that the effect of the bulk viscosity effectively sets in at some initial instant $t = t_{\text{in}}$ and that $\zeta$ is thereafter constant for the times that we consider. For definiteness we choose

$$t_{\text{in}} = 1000 \text{ s}$$

(43)

after the big bang. The universe is then in the plasma (or radiation) era; it consists of ionized H and He in equilibrium with radiation. The particle density is $n_{\text{in}} \simeq 10^{19} \text{ cm}^{-3}$, and the temperature is $T_{\text{in}} \simeq 4 \times 10^8 \text{ K}$. The advantage of considering this relatively late stage of the universe’s history is that the magnitude of $\zeta$ can be calculated using conventional kinetic theory. At $t = t_{\text{in}}$ one finds

$$\zeta = 7.0 \times 10^{-3} \text{ g cm}^{-1} \text{ s}^{-1}$$

(44)

(cf. [6] and further references therein).
For our estimate purposes it appears reasonable to assume that the influence from the spatial curvature is not very important. For simplicity let us put $k = 0$. This implies that there exists the following simple differential equation for the scalar expansion [6]:

$$\dot{\theta}(t) + \frac{1}{2} \gamma \theta^2(t) - 12\pi G \zeta \theta(t) = 0,$$

(45)

which can be solved to give the expression for the scale factor:

$$a(t) = a_{\text{in}} \left[ 1 + \frac{1}{2} \gamma \theta_{\text{in}} t_c \left( e^{(t-t_{\text{in}})/t_c} - 1 \right) \right]^{2/(3\gamma)},$$

(46)

where

$$t_c = (12\pi G \zeta)^{-1}.$$  

(47)

The corresponding expression when viscosity is absent, is

$$a(t, \zeta = 0) = a_{\text{in}} \left[ 1 + \frac{1}{2} \gamma \theta_{\text{in}} (t - t_{\text{in}}) \right]^{2/(3\gamma)}.$$  

(48)

The ratio between the expressions (46) and (48) reduces to unity in the limit when $(t - t_{\text{in}})/t_c \ll 1$. This is what we would expect. The influence from viscosity generally turns up only in the factor $t_c$, and the effect becomes strengthened when $t_c$ becomes smaller, i.e., when $\zeta$ becomes larger. The numerical values given above for the instant $t_{\text{in}} = 1000$ s correspond to

$$\theta_{\text{in}} = 1.5 \times 10^{-3} \text{ s}^{-1}, \quad t_c = 5.1 \times 10^{28} \text{ s}.$$  

(49)

### 4.2 Perturbative expansion for a radiation dominated closed universe

The smallness of $\zeta$ makes it natural, as an alternative to the approach of the previous subsection, to make a Stokes expansion in $\zeta$. For simplicity we now assume that $\gamma = 4/3$, i.e., that the universe is radiation dominated. We put $k = 1$. Let subscript zero refer to quantities in the nonviscous case. We write the first order expansions as

$$a = a_0 (1 + \zeta a_1), \quad \rho = \rho_0 (1 + \zeta \rho_1),$$  

(50)

where the functions $a_1$ and $\rho_1$ are of zeroth order in $\zeta$; they are regarded as functions of $t$ or alternatively as functions of the conformal time $\eta$. Correspondingly, the scalar expansion $\theta = 3\dot{a}/a$ is written as

$$\theta = \theta_0 (1 + \zeta \theta_1).$$  

(51)

The Friedmann equation (19), and Eq. (39), now yield to the zeroth order

$$\theta_0^2 = 24\pi G \rho_0 - \frac{9}{a_0^2},$$  

(52)
\[ \rho_i a_i^4 = \rho_0 a_0^4, \]  

and to the first order

\[ \frac{\theta_0^2}{\theta_1} = 12\pi G \rho_0 \rho_1 + \frac{9a_1}{a_0^2}, \]  

\[ \rho_i a_i^4 (\dot{\rho}_1 + 4\dot{a}_1) = \theta_0^2 a_0^4. \]  

The solutions of Eqs. (52) and (53) are

\[ a_0 = a_* \sin \eta, \quad \rho_0 = \frac{3}{8\pi G} \frac{1}{a_*^2} \frac{1}{\sin^4 \eta}, \]  

\[ \theta_0 = \frac{3 \cos \eta}{a_* \sin^2 \eta}, \]  

where \( a_* \) as before is given by Eq. (36). From Eq. (55) we now have

\[ \frac{d\rho_1}{d\eta} + 4 \frac{da_1}{d\eta} = 24\pi G a_* \sin \eta \cos^2 \eta, \]  

which can be integrated from \( \eta = \eta_\text{in} \) onwards:

\[ \rho_1 + 4a_1 = 8\pi G a_* (\cos^3 \eta_\text{in} - \cos^3 \eta). \]  

We have here assumed that \( a_1 = \rho_1 = 0 \) at the initial instant \( t = t_\text{in} \). With

\[ \theta_1 = \frac{da_1}{d\eta} \tan \eta \]  

(cf. Eq. 59), we obtain from Eq. 54 a first order differential equation for the scale factor correction:

\[ \sin 2\eta \frac{da_1}{d\eta} + 2(1 + \cos^2 \eta)a_1 + 8\pi G a_* \cos^3 \eta = 8\pi G a_* \cos^3 \eta_\text{in}. \]  

After some calculation we find the following solution, again observing the initial conditions at \( t = t_\text{in} \):

\[ a_1(\eta) = \frac{4\pi G a_*}{\sin^2 \eta} \left[ \cos^3 \eta_\text{in} + \left( \frac{1}{4} \cos 2\eta \cos \eta - \frac{1}{4} \cos 2\eta_\text{in} \right) \cos \eta \right]. \]  

Once the scale factor correction \( a_1 \) is known, the corresponding density correction \( \rho_1 \) follows from Eq. (58).

From Eq. 61 it is apparent that the relative correction \( \zeta a_1 \) to the scale factor is of order \( 4\pi G \zeta a_* \) or, in dimensional units, \( 4\pi G \zeta a_* c^2 \). We here note that \( 4\pi G \zeta /c^2 = 1/(3t_c) \), thus about \( 6.5 \times 10^{-30} \text{ s}^{-1} \) according to Eq. 17, which in dimensional form reads \( t_c^{-1} = 12\pi G \zeta /c^2 \). The quantity \( a_* \), according to Eq. (56), is the maximum value of the scale factor in a nonviscous \( k = 1 \) theory. Let us put \( a_* \) equal to the commonly accepted value of the radius of the universe, \( i.e., a_* = 10^{28} \text{ cm} \). Then we
obtain $4\pi G \zeta a_*/c^3 = 2 \times 10^{-12}$. The relative correction to the scale factor is thus in this case very small. Physically, this is due to the fact that we are considering an instant relatively late in the history of the universe. If the bulk viscosity $\zeta$ were higher at earlier times, or, if there were an anisotropic stage present in the early universe at which the enormously higher shear viscosity would come into play [6], then the effect would be significantly enhanced.

5 Remarks on shear viscosity and the holographic entropy bound

We round off this article by making some remarks on the influence from shear viscosity in cosmology. The shear viscosity concept, as mentioned in the Introduction, has received considerable attention recently [8, 9, 10, 11]. What we shall focus attention on here, is the suggestion of Kovtun et al. [9] that there exists a cosmological universal lower bound on the ratio $\eta/s$, $\eta$ denoting as usual the shear viscosity and $s$ the entropy per unit volume. The mentioned authors are concerned with the infrared properties of theories whose gravity duals contain a black brane with a nonvanishing Hawking temperature, the point being that the infrared behavior is governed by hydrodynamical laws. If we for definiteness consider a stack of $N$ non-extremal D3 branes in type IIB supergravity, the metric near the horizon is given by

$$ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + dx^2 + dy^2 + dz^2] + \frac{R^2}{r^2f(r)}dr^2 + R^2d\Omega_5^2,$$

(62)

where $R \propto N^{1/4}$ is a constant, and $f(r) = 1 - r_0^4/r^4$ with $r_0$ being the horizon. The Hawking temperature of this metric is $T = r_0/\pi R^2$, and $\eta$ and $s$ are given by

$$\eta = \frac{1}{8\pi N^2 T^3}, \quad s = \frac{1}{2\pi^2 N^2 T^3}.$$  
(63)

Thus, when using hereafter dimensional notation,

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} = 6.08 \times 10^{-13} \text{ Ks}.$$

(64)

The conjecture of Kovtun et al. is that the value in Eq. (63) is a lower bound for $\eta/s$. Since this bound does not involve the speed of light, the authors even conjecture that this bound exists for all systems, including non-relativistic ones. The idea has recently been further elaborated in [10], arguing that the bound follows from the generalized covariant entropy bound.

Let us check the proposed bound, by considering an example taken from the plasma era in the early universe. We choose the same instant as in subsection 4.1, namely $t_{in} = 1000$ s, corresponding to $n \simeq 10^{19}$ cm$^{-3}$ (subscript “in” here omitted), $T \simeq 4 \times 10^8$ K, whereas the energy density is $\rho c^2 = a_r T^4$, where $a_r =$
\[ \pi^2 k_B^4 / (15 \hbar^3 c^3) = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4} \] is the radiation constant. The viscosity coefficients - whose existence is due to the imperfectness of thermal equilibrium - can be calculated from relativistic kinetic theory. Let \( x = m_e c^2 / k_B T \) be the ratio between electron rest mass and thermal energy; when \( x \gg 1 \) it is convenient to use the polynomial approximations \( \eta \) and \( \zeta \) for the evaluation of \( \eta \) and \( \zeta \):

\begin{align*}
\eta &= \frac{5 m_e^6 c^8 \zeta(3)}{9 \pi^2 \hbar^3 e^4 n} x^{-4}, \quad \zeta = \frac{\pi c^2 \hbar^3 n}{256 e^4 \zeta(3)} x^3, \\
\zeta(3) &= 1.202
\end{align*}

\( \zeta(3) = 1.202 \) being the Riemann zeta function. At \( T = 4 \times 10^8 \text{ K} \) one has \( x = 14.8 \), leading to

\[ \eta = 2.8 \times 10^{14} \text{ g cm}^{-1} \text{ s}^{-1}, \tag{66} \]

which is enormously greater than \( \zeta \) given by Eq. (44). We note that both \( \eta \) and \( \zeta \) contain \( \hbar \). The entropy density, in view of the radiation dominance, is given by

\[ s = \frac{4}{3} a_r T^3 = 6.45 \times 10^{11} \text{ erg cm}^{-3} \text{ K}^{-1}, \tag{67} \]

and so

\[ \frac{\eta}{s} = 435 \text{ K s}. \tag{68} \]

When the state equation is instead taken as \( p = (\gamma - 1) \rho c^2 \), we obtain analogously

\[ \frac{\eta}{s} = \frac{578}{\gamma} \text{ K s}, \tag{69} \]

agreeing with Eq. (68) when \( \gamma = 4/3 \).

Other cases analyzed in [11], taken under widely different physical conditions, turned out to give values of \( \eta/s \) of roughly the same order of magnitude as above.

**Acknowledgment**

I thank Professor Sergei Odintsov for valuable information.

**References**

[1] C. W. Misner, *Astrophys. J.* 151, 431 (1968).

[2] Ø. Grøn, *Astrophys. Space Sci.* 173, 191 (1990).
[3] S. Weinberg, *Astrophys. J* 168, 175 (1971).

[4] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York (1972).

[5] V. B. Johri and R. Sudarshan, *Phys. Lett.* 132A, 316 (1988).

[6] I. Brevik and L. T. Heen, *Astrophys. Space Sci.* 219, 99 (1994).

[7] I. Brevik and G. Stokkan, *Astrophys. Space Sci.* 239, 89 (1996).

[8] S. Weinberg, *Phys. Rev. D* 69, 023503 (2004).

[9] P. Kovtun, D. T. Son, and A. O. Starinets, hep-th/0309213.

[10] A. Karch, hep-th/0311116.

[11] I. Brevik, S. Nojiri, S. D. Odintsov, and L. Vanzo, hep-th/0401073.

[12] E. Verlinde, hep-th/0008140.

[13] J. L. Cardy, *Nucl. Phys.* B270, 186 (1986); *ibid.* B275, 200 (1986).

[14] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, *Phys. Rev. Lett.* 56, 742 (1986).

[15] B. Wang, E. Abdalla, and R.-K. Su, *Phys. Lett.* B503, 394 (2001).

[16] S. Nojiri, O. Obregón, S. D. Odintsov, H. Quevedo, and M. P. Ryan, *Mod. Phys. Lett. A* 16, 1181 (2001); S. Nojiri and S. D. Odintsov, *Class. Quant. Grav.* 18, 5227 (2001).

[17] T. Padmanabhan and S. M. Chitre, *Phys. Lett. A* 120, 433 (1987). Cf. also the recent review articles of T. Padmanabhan, *Phys. Rep.* 380, 235 (2003); gr-qc/0311036 (to appear in *Rev. Mod. Phys*), with further references therein.

[18] R. G. Cai, *Phys. Rev. D* 63, 124018 (2001); *Phys. Lett. B* 525, 331 (2002); *Nucl. Phys.* B628, 375 (2002); R. G. Cai, Y. S. Myung, and N. Ohta, *Class. Quant. Grav.* 18, 5429 (2001); R. G. Cai and Y. Z. Zhang, *Phys. Rev. D* 64, 104015 (2001); R. G. Cai and Y. S. Myung, *Phys. Rev. D* 67, 124021 (2003).

[19] D. Youm, *Phys. Lett.* B531, 276 (2002).

[20] D. Kutasov and F. Larsen, *J. High Energy Phys.* 01, 001 (2001); D. Klemm, A. C. Petkou, and G. Siopsis, *Nucl. Phys.* B 601, 380 (2001); S. Nojiri and S. D. Odintsov, *Int. J. Mod. Phys.* A 16, 3273 (2001); I. Brevik, K. A. Milton, and S. D. Odintsov, *Ann. Phys. (N.Y.)* 302, 120 (2003), with further references therein.

[21] I. Brevik and S. D. Odintsov, *Phys. Rev. D* 65, 067302 (2002).
[22] I. Brevik, *Phys. Rev. D* **65**, 127302 (2002).

[23] I. Brevik, *Int. J. Mod. Phys. A* **18**, 2145 (2003).

[24] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed., Pergamon Press, Oxford (1987), Sect. 49.

[25] A. H. Taub, *Annual Rev. Fluid Mech.* **10**, 301 (1978).

[26] J. D. Bekenstein, *Phys. Rev. D* **23**, 287 (1981); *Int. J. Theor. Phys.* **28**, 967 (1989).

[27] R. Easther and D. Lowe, *Phys. Rev. Lett.* **82**, 4967 (1999).

[28] G. Veneziano, *Phys. Lett.* **B454**, 22 (1999).

[29] D. Bak and S.-J. Rey, *Class. Quant. Grav.* **17**, L83 (2000).

[30] N. Kaloper and A. Linde, *Phys. Rev. D* **60**, 103509 (1999).

[31] W. Fischler and L. Susskind, [hep-th/9806039](http://arxiv.org/abs/hep-th/9806039).

[32] K. A. Milton, L. L. DeRaad, Jr. and J. Schwinger, *Ann. Phys. (NY)* **115**, 388 (1978); K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero-Point Energy*, World Scientific, Singapore (2001).

[33] G. Barton, *J. Phys. A* **32**, 525 (1999).

[34] J. S. Høye and I. Brevik, *J. Stat. Phys.* **100**, 223 (2000).

[35] M. Bordag, K. Kirsten and D. Vassilevich, *Phys. Rev. D* **59**, 085011 (1999).

[36] I. Brevik, V. N. Marachevsky and K. A. Milton, *Phys. Rev. Lett.* **82**, 3948 (1999).

[37] N. Caderni and R. Fabbri, *Phys. Lett.* **69B**, 508 (1977).