Analytical approach to the transition to thermal hopping in the thin- and thick-wall approximations

Hatem Widyan*, A. Mukherjee†, N. Panchapakesan‡ and R. P. Saxena§

Department of Physics and Astrophysics,
University of Delhi, Delhi-110 007, India

Abstract

The nature of the transition from the quantum tunneling regime at low temperatures to the thermal hopping regime at high temperatures is investigated analytically in scalar field theory. An analytical bounce solution is presented, which reproduces the action in the thin-wall as well as thick-wall limits. The transition is first order for the case of a thin wall while for the thick wall case it is second order.

I. INTRODUCTION

The decay of metastable states is a basic phenomenon of great generality with numerous applications in a large number of contexts, ranging from the decay of the false vacuum [1], e.g., in cosmology [2], to the creep-type motion of topological defects in solids [3]. At a given temperature $T$ the decay rate of a metastable state can be written in the form

$$\Gamma = Ae^{-S_E(T)/\hbar},$$

with $S_E(T)$ being the Euclidean action of the saddle-point configuration (the bounce) and $A$ being the prefactor determined by the associated fluctuations. At zero temperature, the decay is determined by quantum effects. With increasing temperature, the nature of the decay changes from quantum to classical. The function $S_E(T)$ might either be a smooth function of temperature or exhibit a kink with a discontinuity in its derivative at some temperature $T_c$. In the former case, the transition from the quantum tunneling regime is said to be of second order while in the latter case it is said to be of first order. The word

*E-mail : h_widyan@maktoob.com
†E-mail : am@ducos.ernet.in
‡E-mail : panchu@vsnl.com
§E-mail : rps@ducos.ernet.in
transition is appropriate as the crossovers have all the features of mean-field phase transition upon identifying the Euclidean action with the free energy [1].

Affleck [5] studied the phase transition at nonzero temperature for a quantum mechanical problem. He argued that a second order transition from the quantum tunneling regime to the thermal hopping regime takes place at some critical temperature. Chudnovsky [6] and Garriga [7] have given criteria for determining when the transition is first-order and when it is second-order. There have been many studies on the order of the phase transition in the context of condensed matter; for a recent work see [4]. On the other hand, there have been very few studies in quantum field theoretic situations.

A field theoretic system, consisting of a single scalar field with a $\varphi$ symmetry breaking term, has been studied recently by Ferrera [8] and also by us. We have also investigated in detail the nature of the phase transition for single scalar field theory with $\varphi^3$ symmetry breaking [3]. Our numerical results show that, for small values of the symmetry-breaking coupling $f$, the transition from the quantum regime to the thermal hopping regime is first-order. We have argued that for large values of $f$ the transition is second order. We have also calculated the action analytically at zero temperature by assuming an appropriate ansatz solution for the bounce. The result is in good agreement with the exact numerical result in the thin-wall approximation (TWA).

It is convenient to use the following form of the potential proposed by Adams [10]

$$U(\varphi) = \frac{1}{4}\varphi^4 - \varphi^3 + \frac{\delta}{2}\varphi^2,$$

where $0 \leq \delta \leq 2$. Any quartic potential can be reduced to this form by shifting and rescaling of the field. In this paper we extend our calculation of the action to finite temperatures and study the nature of the transition. We propose a general ansatz at finite temperature in the thin wall ($\delta \to 2$) and thick wall ($\delta \to 0$) limits. We find that for a thin wall the transition is first-order while for thick wall it is second-order. This result is in agreement with Ferrera [8] who found that only for very large wall thickness (i.e., $\delta \sim 0.6$) a second order transition takes place, while for all other cases a first order transition occurs. We would like to point out that the analytical approach is very much easier than the numerical one and less time consuming [3].

In Sec. II we present our analytic calculation for the action at zero temperature and high temperature, including some earlier work which is presented for completeness. In Sec. III, we extend the calculations to finite temperatures and discuss the nature of the phase transition. Section IV contains our results for intermediate wall sizes. Section V contains our conclusions. The algebraic expressions for the integrals appearing in the analytic formalism are given in the appendices A and B.
II. ANALYTICAL SOLUTION FOR ZERO AND HIGH TEMPERATURES

We use the following potential (see [10] and [9]) to calculate the action analytically in two extreme limits: the thin-wall and thick-wall limits

\[ U(\varphi) = \frac{1}{4}\varphi^4 - \varphi^3 + \frac{\delta}{2}\varphi^2. \]  

(2)

A. Thin-wall limit at zero temperature: \( \delta \to 2 \)

We find that an analytic solution for the bounce of the form of a Fermi function:

\[ \varphi = \gamma \frac{e^{(\rho^2 - R^2)/\Lambda^2}/(e^\rho + 1)}{\sqrt{\rho^2 + \tau^2}}, \]  

(3)

where \( \rho = \sqrt{x^2 + \tau^2}, \) \( R \) is the radius of the bubble and \( \Lambda \) its width, acts like a bounce in the TWA and leads to the correct value for \( S_4, \) the action at zero temperature. Here the false minimum of the potential is at \( \varphi = 0 \) and the true minimum lies between 2 (for \( \delta = 2 \)) and 3 (for \( \delta = 0 \)). The parameter \( \gamma \) is approximately equal to the true minimum in the thin wall approximation. The bounce has values \( \varphi = \gamma \) at \( \rho = 0 \) and 0 at \( \rho \to \infty. \) These boundary conditions are satisfied by Eq. (3).

To evaluate \( \gamma, \) \( R, \) and \( \Lambda, \) we substitute the ansatz (3) in the equation of motion:

\[ \frac{d^2\varphi}{d\rho^2} + \frac{3}{\rho} \frac{d\varphi}{d\rho} = \varphi^3 - 3\varphi^2 + \delta \varphi. \]  

(4)

Then the left-hand side (L.H.S.) and the right-hand side (R.H.S.) are respectively

\[
\begin{align*}
L.H.S. & = \frac{8\gamma \rho^2 / \Lambda^4}{(e^{(\rho^2 - R^2)/\Lambda^2}/(e^\rho + 1))^3} + \frac{\gamma(-12\rho^2 / \Lambda^4 + 8 / \Lambda^2)}{(e^{(\rho^2 - R^2)/\Lambda^2}/(e^\rho + 1))^2} + \frac{\gamma(4\rho^2 / \Lambda^4 - 8 / \Lambda^2)}{(e^{(\rho^2 - R^2)/\Lambda^2}/(e^\rho + 1))^3} \\
R.H.S. & = \gamma \frac{\delta}{e^{(\rho^2 - R^2)/\Lambda^2}} + \frac{3\gamma^2}{e^{(\rho^2 - R^2)/\Lambda^2}/(e^\rho + 1)^2} + \frac{\gamma^3}{e^{(\rho^2 - R^2)/\Lambda^2}/(e^\rho + 1)^3}.
\end{align*}
\]  

(5)

In the TWA, the solution is constant except in a narrow region near the wall at \( \rho = R. \) So, we replace in Eq. (3)

\[ 8\rho^2 / \Lambda^4 \]  

by \( \frac{8R^2}{\Lambda^4}(1 - a\Lambda^2 / R^2) \) in the \( \frac{1}{e^{(\rho^2 - R^2)/\Lambda^2}/(e^\rho + 1)^3} \) term,

(7)
\[
\frac{8}{\Lambda^2} - 12 \frac{\rho^2}{\Lambda^4} \text{ by } -\frac{12R^2}{\Lambda^4} (1 - b\Lambda^2/R^2) \text{ in the term, } (8)
\]
\[
4\frac{\rho^2}{\Lambda^4} - 8/\Lambda^2 \text{ by } \frac{4R^2}{\Lambda^4} (1 - d\Lambda^2/R^2) \text{ in the term, } (9)
\]

where \(a\), \(b\) and \(d\) are parameters to be determined later.

Comparing Eq. (5) with Eq. (6) in the range
\[
R^2(1 - \frac{\Lambda^2}{R^2}) = R^2 - \Lambda^2 < \rho^2 < R^2 + \Lambda^2 = R^2(1 + \frac{\Lambda^2}{R^2})
\]
where \(\rho^2 \simeq R^2\) as \(\frac{\Lambda^2}{R^2} \ll 1\), we have:
\[
\frac{\gamma^2}{8} = \frac{R^2}{\Lambda^4} (1 - a\Lambda^2/R^2),
\]
\[
\frac{\gamma}{4} = \frac{R^2}{\Lambda^4} (1 - b\Lambda^2/R^2),
\]
\[
\frac{\delta}{4} = \frac{R^2}{\Lambda^4} (1 - d\Lambda^2/R^2),
\]

(10)

We can now evaluate the zero-temperature action \(S_4\):
\[
S_4 = 2\pi^2 \int_0^\infty d\rho \rho^3 \left[ \frac{1}{2} \left( \frac{d\phi}{d\rho} \right)^2 + U(\phi) \right].
\]

(11)

Substituting Eq. (3) in Eq. (11) and integrating we get
\[
S_4 = 2\pi^2 \gamma^2 R^4 \left[ \frac{1}{6\Lambda^2} \left( 1 + \left( \frac{\pi^2}{3} - 2 \right) \frac{\Lambda^4}{R^4} \right) + \frac{\delta}{8} \left( 1 - 2\Lambda^2/R^2 + \frac{\pi^2 \Lambda^4}{3 R^4} \right) \right.
\]
\[
+ \frac{\gamma}{4} \left( 1 - 3\Lambda^2/R^2 + \left( \frac{\pi^2}{3} + 1 \right) \frac{\Lambda^4}{R^4} \right)
\]
\[
+ \frac{\gamma^2}{16} \left( 1 - \frac{11\Lambda^2}{3 R^2} + \left( \frac{\pi^2}{3} + 2 \right) \frac{\Lambda^4}{R^4} \right) \left( \frac{\Lambda^4}{R^4} \right) \right].
\]

(12)

We now determine the parameters \(a\), \(b\), and \(d\) by demanding \(dS_4/dR^2 = dS_4/d\Lambda^2 = dS_4/d\gamma = 0\). Differentiating Eq. (12) and using Eq. (10), we find that, to leading order in \(\Lambda^2/R^2\),
\[
4b - 2a - 2d + 1 = 0,
\]
\[
3b - 2a - d = 0,
\]
\[
3b - 11a/6 - d = 0,
\]

which leads to \(a = 0\), \(b = 1/2\) and \(d = 3/2\). Using Eq. (10), we can rewrite Eq. (12) as:
\[
S_4 = 2\pi^2 \frac{8R^6}{\Lambda^6} \left[ \left( 1/3 - d/2 - a/2 + b \right) + \frac{\Lambda^2}{R^2} \left( d - 3b + 11a/6 \right) \right],
\]

(14)

where the coefficient of \(\frac{\Lambda^4}{R^4}\) evaluated by the usual methods of statistical mechanics for the Fermi function vanishes. This gives
\[
S_4 = \frac{4\pi^2 R^6}{3} \frac{\Lambda^4}{\Lambda^6} + O\left( \frac{\Lambda^6}{R^6} \right).
\]

(15)
The quantities \( \gamma, R \) and \( \Lambda \) are determined from Eq. (10) using the values of \( a, b, \) and \( d \). So we have

\[
\gamma^2 - 2\gamma \frac{d - a}{d - b} + 2\delta \frac{b - a}{d - b} = 0 ,
\]

(16)

which gives

\[
\Lambda^2 = \frac{8(b - a)}{\gamma^2 - 2\gamma} = \frac{4(d - b)}{\gamma - \delta} ,
\]

(17)

with \( \gamma \) given by Eq. (16). We have then, for \( \delta = 1.9, \gamma = 2.1 \), which implies that \( R^2/\Lambda^2 = \gamma^2b/(\gamma - \delta) = 11 \). Thus we have

\[
S_4 = \frac{4\pi^2}{3}(11)^3 ,
\]

(18)

while the action from the TWA formula is ( see [10]) \( S_{TW} = \frac{4\pi^2}{3}(10)^3 \) for \( \delta = 1.9 \). The departure from TWA, \( S_4/S_{TW} = 1.33 \), is in agreement with Ref. [10]. The expressions seem certainly valid for values of \( \delta \) in the range 2.0 to 1.8.

**B. Thin-wall limit at high temperature: \( \delta \to 2 \)**

The bounce takes the following form

\[
\varphi = \gamma \frac{e^{(r^2 - R^2)/\Lambda^2}}{1 + 1} ,
\]

(19)

where \( r^2 = \vec{x}^2 \) and the other parameters \( R \) and \( \Lambda \) have the same physical significance in three dimensions. The boundary conditions are \( \varphi = \gamma \) at \( r = 0 \) and \( \varphi = 0 \) at \( r \to \infty \), \( d\varphi/dr = 0 \) at \( r = 0 \). The equation of motion is

\[
\frac{d^2\varphi}{dr^2} + 2 \frac{d\varphi}{r} = \varphi^3 - 3\varphi^2 + \delta \varphi .
\]

(20)

As in the earlier subsection, we substitute the ansatz bounce in the equation of motion and assume the solution is constant except in a narrow region near the wall at \( r = R \). The resulting equations have a structure similar to that of Eqs. (3) to (10).

\[
S_3 = 4\pi \int_0^\infty dr r^2 \left[ \frac{1}{2} \left( \frac{d\varphi}{dr} \right)^2 + U(\varphi) \right] .
\]

(21)

After substituting the bounce Eq. (19) in the action and integrating, we get the following

\[
S_3 = \frac{4\pi}{3} \gamma^2 R^3 \left[ \frac{1}{2\Lambda^2} \left( 1 + \left( \frac{\pi^2}{8} - \frac{3}{4} \Lambda^4 \right) \right) R^4 \right] \\
+ \frac{\gamma^2}{4} \left( 1 - \frac{11}{4} \frac{\Lambda^2}{R^2} + \left( \frac{\pi^2}{8} + \frac{3}{4} \right) \frac{\Lambda^4}{R^4} \right) \\
- \gamma \left( 1 - \frac{9}{4} \frac{\Lambda^2}{R^2} + \left( \frac{\pi^2}{8} + \frac{3}{4} \right) \frac{\Lambda^4}{R^4} \right)
\]

5
\[
+ \frac{\delta}{2} \left( 1 - \frac{3}{2} \frac{\Lambda^2}{R^2} + \frac{\pi^2}{8} \frac{\Lambda^4}{R^4} \right). \tag{22}
\]

In terms of the parameters \(a, b\) and \(d\), the action takes the simpler form

\[
S_3 = \frac{32\pi}{3} \frac{R^5}{\Lambda^4} \left( 1 - 2a + 4b - 2d + \frac{\Lambda^2}{R^2} \left( \frac{11}{2} a - 9b + 3d \right) \right), \tag{23}
\]

where the relations between \(a, b\) and \(d\) to leading order in \(\Lambda^2/R^2\) are

\[
-2a + 4b - 2d + \frac{2}{3} = 0,
\]

\[
-\frac{11}{2}a + 9b - 3d = 0,
\]

\[
-2a + 3b - d = 0, \tag{24}
\]

this leads to \(a = 0, b = 1/3\) and \(d = 1\).

Hence the action in Eq. (23) is reduced to

\[
S_3 = \frac{32\pi}{9} \frac{R^5}{\Lambda^6} + O\left( \frac{\Lambda^6}{R^6} \right), \tag{25}
\]

which agrees with the TWA formula and the expression of Adams [10].

**C. Thick-wall limit at zero temperature: \(\delta \to 0\)**

The form of the bounce in Eq. (3) suggests that the thick wall limit, which would correspond to small values of \(R^2/\Lambda^2\), would be obtained by approximating the Fermi function by the Maxwell-Boltzmann function, which leads to a Gaussian:

\[
\varphi = \gamma e^{-\rho^2/\Lambda^2}. \tag{26}
\]

The action for this form of bounce is found to be

\[
S_4 = \pi^2\gamma^2 \Lambda^4 \left[ \frac{1}{2\Lambda^2} + \frac{\delta}{8} - \frac{\gamma}{9} + \frac{\gamma^2}{64} \right]. \tag{27}
\]

For small values of \(R^2/\Lambda^2\), the relation between the parameters in the bounce Eq. (26) and the constants \(a, b\) and \(d\) is given by

\[
\frac{\gamma^2}{8} = -\frac{a}{\Lambda^2}, \quad \frac{\gamma}{4} = -\frac{b}{\Lambda^2}, \quad \frac{\delta}{4} = -\frac{d}{\Lambda^2}. \tag{28}
\]

Note that in this case \(\gamma \ll 1\), so \(\gamma^2\) is negligible.
The values of $b$ and $d$ are again obtained by demanding $dS_4/d\Lambda^2 = dS_4/d\gamma = 0$. This gives $b = -9/8$, $d = -1/2$, giving

$$\Lambda^2 = \frac{2}{\delta}, \quad \gamma = \frac{9}{4}\delta.$$  \hspace{1cm} (29)

This yields the action

$$S_4 = \frac{4\pi^2}{3}(1.9)\delta \left(1 + O\left(\frac{R^2}{\Lambda^2}\right)\right).$$  \hspace{1cm} (30)

The ratio of the action to the TWA value is

$$R_4 = \frac{S_4}{S_{TW}} = 1.9\delta(2 - \delta)^3.$$  \hspace{1cm} (31)

For $\delta = 0.1$, $R_4 = 1.31$, which agrees with Adams’ result.

We find the ratio of the actions when we take the next order in $\delta$ in $S_4$ to be

$$R_4 \propto \delta \left(1 - 0.826 \delta - 0.150 \delta^2 + 0.320 \delta^3\right),$$  \hspace{1cm} (32)

as compared to Adams’

$$R_4 \propto \delta \left(1 - 0.8 \delta + 0.15 \delta^2\right).$$  \hspace{1cm} (33)

We find the two expressions agree for $0 < \delta < 0.5$.

\textbf{D. Thick-wall limit at high temperature: $\delta \to 0$}

At higher temperatures the bounce takes the form

$$\varphi = \gamma e^{-r^2/\Lambda^2},$$  \hspace{1cm} (34)

with the action

$$S_3 = 4\pi^{3/2}\gamma^2\Lambda^3 \left[ \frac{3}{16\sqrt{2}} \frac{1}{\Lambda^2} + \frac{\gamma^2}{128} - \frac{\gamma}{12\sqrt{3}} + \frac{\delta}{16\sqrt{2}} \right].$$  \hspace{1cm} (35)

Defining $\gamma/4 = -b/\Lambda^2$, $\delta/4 = -d/\Lambda^2$ and neglecting $\gamma^2$, we find $b$ and $d$ as before by demanding $dS_3/d\Lambda = dS_3/d\gamma = 0$. The relation between $b$ and $d$ is given by

$$3 + 12\left(\frac{2}{3}\right)^{3/2}b - 4d = 0,$$

$$\frac{3}{4} + 6\left(\frac{2}{3}\right)^{3/2}b - 3d = 0,$$  \hspace{1cm} (36)

which leads to $b = -\frac{3}{4}(3/2)^{1/2}$ and $d = -3/4$, giving $\Lambda^2 = 3/\delta$ and $\gamma = \sqrt{3/2}\delta$. The action can be simplified to

$$S_3 = \frac{3\sqrt{3}}{4\sqrt{2}}\pi^{3/2}\delta^{3/2}\left(1 + \frac{9\sqrt{2}}{32}\delta\right).$$  \hspace{1cm} (37)
Following Adams [10], we can calculate the ratio of this action to the thick-wall action and we find
\[ R_3 = 5.828\delta^{3/2}\left(1 - 0.602\delta - 0.148\delta^2 + 0.099\delta^3\right), \]  
(38) and this matches very well with Adams’ result
\[ R_3 = 5.864\delta^{3/2}\left(1 - 0.667\delta + 0.099\delta^2\right). \]  
(39) We find both compare well for values of \( \delta \) from 0 to 0.5.

Thus, the form of the bounce given by Eqs. (3) and (19) seems valid over the whole range of \( \delta \) (from 0 to 2), and in the two extreme limits is amenable to analytic calculations. This suggests that we look for an interpolating form valid at all temperatures that reduces to Eq. (3) at \( T = 0 \) and to Eq. (19) at high temperatures.

### III. RESULTS FOR INTERMEDIATE TEMPERATURES

#### A. Thin-wall limit : \( \delta \to 2 \)

The action at finite temperature of a single scalar field \( \varphi \) is given by the following formula
\[ S(T) = 4\pi \int_{-\beta/2}^{\beta/2} d\tau \int_0^\infty dr r^2 \left[ \frac{1}{2} \left( \frac{\partial \varphi}{\partial \tau} \right)^2 + \frac{1}{2} \left( \frac{\partial \varphi}{\partial r} \right)^2 + U(\varphi) \right]. \]  
(40) The equation of motion derived from the above action is given by the following expression
\[ \frac{\partial^2 \varphi}{\partial \tau^2} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{2}{r} \frac{\partial \varphi}{\partial r} = \frac{\partial U(\varphi, T)}{\partial \varphi}, \]  
(41) with boundary conditions
\[ \varphi \to \varphi_- \text{ as } r \to \infty, \quad \partial \varphi / \partial \tau = 0 \text{ at } \tau = \pm \beta/2, 0, \]  
(42) where \( \varphi_- \) is the false vacuum of the potential \( U \), \( \beta \) is the period of the solution and \( r = \sqrt{x^2} \).

We assume for the solution of the equation of motion the following ansatz
\[ \varphi(r, \tau) = \frac{\gamma}{e^{(r^2 + \frac{\beta^2}{4}\sin^2(\frac{\pi \tau}{\beta}) - R^2)/\Lambda^2} + 1}, \]  
(43) which is periodic in the interval \((-\beta/2, \beta/2)\) and satisfies the required boundary conditions (Eq. (42)), viz
\[ \frac{\partial \varphi}{\partial r} = 0 \text{ at } r = 0, \quad \frac{\partial \varphi}{\partial \tau} = 0 \text{ at } \tau = 0 \text{ and } \pm \beta/2, \text{ and } \varphi = 0 \text{ as } r \to \infty. \]  
(44) Note that for the potential given by Eq. (1), \( \varphi = 0 \) is always the false vacuum.
We evaluate the action for potential given by Eq. (1)

\[ U(\varphi) = \frac{1}{4} \varphi^4 - \varphi^3 + \frac{\delta}{2} \varphi^2 , 0 \leq \delta \leq 2 . \]  

(45)

After substituting the ansatz function Eq. (13) into the equation of motion Eq. (11), we have

\[ \frac{\gamma}{(e^{(r^2 + \frac{\beta^2}{4\pi^2}\sin^2(\frac{2\pi r}{\beta}) - R^2)/\Lambda^2} + 1)^3} \left[ \frac{8r^2}{\Lambda^4} + \frac{2\beta^2}{\pi^2 \Lambda^4} \sin^2(\frac{2\pi r}{\beta}) \right] \]

\[ + \frac{\gamma}{(e^{(r^2 + \frac{\beta^2}{4\pi^2}\sin^2(\frac{2\pi r}{\beta}) - R^2)/\Lambda^2} + 1)} \left[ - \frac{12r^2}{\Lambda^4} + \frac{6}{\Lambda^2} + \frac{2}{\Lambda^2} \cos(\frac{2\pi r}{\beta}) - \frac{3\beta^2}{\pi^2 \Lambda^4} \sin^2(\frac{2\pi r}{\beta}) \right] \]

\[ + \frac{\gamma}{e^{(r^2 + \frac{\beta^2}{4\pi^2}\sin^2(\frac{2\pi r}{\beta}) - R^2)/\Lambda^2} + 1} \left[ \frac{4r^2}{\Lambda^4} - \frac{6}{\Lambda^2} + \frac{\beta^2}{\pi^2 \Lambda^4} \sin^2(\frac{2\pi r}{\beta}) - \frac{2}{\Lambda^2} \cos(\frac{2\pi r}{\beta}) \right] \]

\[ = \frac{3\gamma^3}{(e^{(r^2 + \frac{\beta^2}{4\pi^2}\sin^2(\frac{2\pi r}{\beta}) - R^2)/\Lambda^2} + 1)^2} + \frac{\gamma \delta}{(e^{(r^2 + \frac{\beta^2}{4\pi^2}\sin^2(\frac{2\pi r}{\beta}) - R^2)/\Lambda^2} + 1)} . \]  

(46)

Equating terms with different powers of exponentials separately, we have with \(r^2 + \frac{\beta^2}{4\pi^2}\sin^2(\frac{2\pi r}{\beta}) \approx R^2\)

\[ \frac{\gamma^2}{8} = \frac{R^2}{\Lambda^4} \left[ 1 - a\frac{\Lambda^2}{R^2} \right] . \]

\[ \frac{\gamma}{4} = \frac{R^2}{\Lambda^4} \left[ 1 - b\frac{\Lambda^2}{R^2} \right] . \]

\[ \frac{\delta}{4} = \frac{R^2}{\Lambda^4} \left[ 1 - d\frac{\Lambda^2}{R^2} \right] . \]  

(47)

As in the last section, the parameters \(a\), \(b\) and \(d\) are found by the requirement that the variation of \(S(T)\) with respect to the parameters \(R\), \(\Lambda\) and \(\gamma\) in Eq. (13) vanish.

The integrals in the action are obtained in powers of \(\Lambda^2/R^2\) using the usual methods for evaluating integrals of the Fermi function (see e.g. Huang [11]). We get

\[ S(T) = \frac{8\pi}{3} R^4 \kappa E_3 \gamma \left[ \frac{1}{2\Lambda^2} \left( 1 + \kappa^2 \frac{E_T}{E_3} + \left( \frac{\pi^2}{8} - \frac{3}{4} \right) \frac{E_0}{E_3} \frac{\Lambda^4}{R^4} \right) \right] \]

\[ + \frac{\gamma^2}{4} \left( 1 - \frac{11}{4} \frac{E_1}{E_3} \frac{\Lambda^2}{R^2} + \left( \frac{\pi^2}{8} + \frac{3}{4} \right) \frac{E_0}{E_3} \frac{\Lambda^4}{R^4} \right) \]

\[ - \gamma \left( 1 - \frac{9}{4} \frac{E_1}{E_3} \frac{\Lambda^2}{R^2} + \left( \frac{\pi^2}{8} + \frac{3}{4} \right) \frac{E_0}{E_3} \frac{\Lambda^4}{R^4} \right) \]

\[ + \frac{\delta}{2} \left( 1 - \frac{2}{3} \frac{E_1}{E_3} \frac{\Lambda^2}{R^2} + \frac{\pi^2}{8} \frac{E_0}{E_3} \frac{\Lambda^4}{R^4} \right) \]  

(48)
where

\[ E_0 = \int_0^1 \frac{dt}{\sqrt{1 - t^2 \sqrt{1 - \kappa^2 t^2}}} , \]
\[ E_1 = \int_0^1 \frac{dt \sqrt{1 - \kappa^2 t^2}}{\sqrt{1 - t^2}} , \]
\[ E_3 = \int_0^1 \frac{dt (1 - \kappa^2 t^2)^{3/2}}{\sqrt{1 - t^2}} , \]
\[ E_T = \int_0^1 \frac{dt (1 - t^2) t^2 \sqrt{1 - \kappa^2 t^2}}{\sqrt{1 - t^2}} , \]

are complete elliptic integrals which can be represented in terms of the basic complete elliptic integrals \( E_0 \) and \( E_1 \) (see Appendix [A]), \( \kappa = \frac{\beta}{\pi R} \) and \( t = \sin \frac{\pi}{\beta} \tau \).

We now determine the parameters \( a, b \) and \( d \) by demanding the vanishing of \( dS(T)/dR^2 \), \( dS(T)/d\Lambda^2 \) and \( dS(T)/d\gamma \). Differentiating Eq. (48) and using Eq. (47), we find that to leading order in \( \Lambda^2/R^2 \),

\[-2a + 4b - 2d + \frac{1}{2} + \frac{1}{2} \frac{E_1 - 2 \kappa^2 E'_1}{2 E_3 - 2 \kappa^2 E'_3} + \frac{\kappa^2}{2} \frac{E_T - 2 \kappa^2 E'_T}{2 E_3 - 2 \kappa^2 E'_3} = 0 . \]

\[ a \left( 11g - 2 \gamma \epsilon_T E_3/E_1 \right) + b \left( \gamma^2 \epsilon_T E_3/E_1 - 18g \right) + 6gd = 0 . \]

\[-2a + 3b - d - \frac{1}{4} \epsilon_T = 0 , \quad \text{(49)} \]

where \( g = \gamma^2 - 2\gamma \), \( E'_1 \) is the derivative of \( E_1 \) with respect to \( \kappa^2 \) (and similarly for \( E'_T \) and \( E'_3 \)), and \( \epsilon_T = E_1/E_3 - \kappa^2 E_T/E_3 - 1 \).

By using Eq. (49), we can find a relation between the constants \( a, b \) and \( d \),

\[ d - a = \frac{3}{4} (1 + c) + \frac{\epsilon_T}{2} , \]
\[ d - b = \frac{1}{2} (1 + c) + \frac{\epsilon_T}{4} , \]
\[ b - a = \frac{1}{4} (1 + c + \epsilon_T) , \]
\[ a = \frac{\epsilon_T [\gamma^2(b - a) E_3/(E_1 g) - 1.5]}{(1 - \epsilon_T E_3/E_1)} , \quad \text{(50)} \]

where \( c \) is given by the following expression

\[ c = \frac{E_1 - 2 \kappa^2 E'_1}{3 E_3 - 2 \kappa^2 E'_3} + \frac{\kappa^2 (E_T - 2 \kappa^2 E'_T)}{3 E_3 - 2 \kappa^2 E'_3} . \quad \text{(51)} \]

Thus Eq. (48) can be expressed in terms of \( a, b \) and \( d \). It reads as

\[ S(T) = \frac{64}{3} \pi \kappa \left( \frac{R}{\Lambda} \right)^6 \left[ \left( 1 - \frac{\Lambda^2}{R^2} \right) \left( 1 - 2(b - a) \frac{E_3}{E_1} \right) - \frac{\Lambda^2}{R^2} \left( \frac{3}{4} \epsilon_T + \frac{a}{2} \right) \right] . \quad \text{(52)} \]
With these expressions we calculate the values of $a$, $b$ and $d$ and also for the action $S(T)$. In calculating $S(T)$ for $\beta \to \infty$, the integrals $E_{i4}$ are to be used (see Appendix A). In these cases $\kappa > 1$, and we restrict the upper limit of the elliptic integrals to $1/\kappa$ as they become complex for larger values of $\kappa$.

One observation is the occurrence of a singularity in $(b-a)$ (which is directly proportional to $R$) due to $E_0(\kappa)$ becoming singular at $\kappa = 1$ (i.e. at $\beta = \pi R$). The values of $S(T)$ are obtained and plotted for $\delta = 1.9$ and 1.85 as shown in Figs. 1 and 2 respectively. The temperature $T_*$ ($\beta_* = 1/T_*$) is defined by $\beta_* S_3/S_4$. The transition point $\beta_c$ can in principle be different from $\beta_*$, but in the TWA they are equal [4]. In our result (Figs. 1 and 2) we can readily determine $\beta_*$ by extending the horizontal part of the curve to the left. For Fig. 2, for example, this yields $\beta_* \approx 25$, which is close to the value of $\beta_c$ obtained numerically as well as analytically [4]. We conclude, therefore, that the singularity at $\beta \approx 45$ in Fig. 2 is an artifact of the method, and does not represent the transition point. The phase transition actually takes place at a much lower value of $\beta_c$, and is first-order. We suggest that the same prescription be used to determine the transition point for other values of $\delta$ within the TWA. This means that the phase transition is first order, which is consistent with the definitions available in the literature (see, for example, [11]).

It can be shown that in the limit of zero temperature (i.e. $\kappa \to \infty$), the action in Eq. (52) reduces to the action given by Eq. (14), while at high temperature (i.e. $\kappa \to 0$), it reduces to the one given by Eq. (23).

### B. Thick-wall limit: $\delta \to 0$

The form of the bounce in Eq. (13) suggests that the thick wall limit, which would correspond to small values of $R^2/\Lambda^2$, would be obtained by approximating the Fermi function by the Maxwell-Boltzmann function, which leads to a Gaussian:

$$\varphi(r, \tau) = \gamma e^{-(r^2 + \frac{\beta^2}{\pi^2} \sin^2(\frac{\pi \tau}{\beta}))}/\Lambda^2,$$

which satisfies the boundary conditions given by Eq. (12).

The action for this form of bounce is found to be

$$S(T) = 2\pi^2 \gamma^2 \Gamma(3/2) \Lambda x \left( \frac{\Lambda^2}{2} \right)^{3/2} e^{-x^2} I_0(x^2) \left[ \frac{3}{2\Lambda^2} \left( 1 + \frac{I_1(x^2)}{3 I_0(x^2)} \right) + \frac{1}{4} \left( \frac{3}{2} \right)^{3/2} \gamma^2 e^{-x^2} \frac{I_0(2x^2)}{I_0(x^2)} \right]$$

$$- \left( \frac{2}{3} \right)^{3/2} \gamma^2 e^{-x^2/2} \frac{I_0(\frac{2x^2}{x^2})}{I_0(x^2)} + \frac{\delta}{2},$$

where $x = \frac{\beta}{\pi \Lambda}$ and $I_\nu(x^2)$ are the modified Bessel functions.
Eq. (54) then reduces to
\[
\frac{\gamma^2}{8} = -\frac{a}{\Lambda^2}, \quad \frac{\gamma}{4} = -\frac{b}{\Lambda^2}, \quad \frac{\delta}{4} = -\frac{d}{\Lambda^2}.
\] (55)

Here again as in Sec. II, \( \gamma \ll 1 \), so \( \gamma^2 \) is negligible.

The values of \( b \) and \( d \) are again obtained by demanding \( dS(T)/d\gamma = dS(T)/d\Lambda = 0 \). This gives the following
\[
3\left(1 + \frac{1}{3} \frac{I_1(x^2)}{I_0(x^2)}\right) + 12\left(\frac{2}{3}\right)^{3/2}b e^{-x^2/2} I_0\left(\frac{3}{2}x^2\right) - 4d = 0.
\] (56)
\[
\frac{3}{4}e^{-x^2} I_0(x^2) + x^2 e^{-x^2} \left(I_0(x^2) - I_1(x^2)\right) + \frac{3}{4} e^{-x^2} I_1(x^2) + 3\left(\frac{2}{3}\right)^{3/2} F b e^{-\frac{3}{4}x^2} I_0\left(\frac{3}{2}x^2\right) - dE e^{-x^2} I_0(x^2) = 0.
\] (57)

where
\[
E = 3 + 2x^2 \left(1 - \frac{I_1(x^2)}{I_0(x^2)}\right),
\]
\[
F = 2 + 2x^2 \left(1 - \frac{I_1\left(\frac{3}{2}x^2\right)}{I_0\left(\frac{3}{2}x^2\right)}\right).
\] (58)

Using Eqs. (56) and (57), the values of \( b \) and \( d \) are given by
\[
b = \frac{E}{4} \left(3I_0(x^2) + I_1(x^2) - \frac{3}{4} e^{-x^2} \left(I_1(x^2) + I_0(x^2)\right) - x^2 e^{-x^2} \left(I_0(x^2) - I_1(x^2)\right)\right).
\] (59)
\[
d = \frac{E}{4} \left(3I_0(x^2) + I_1(x^2) - \frac{3}{4} e^{-x^2} \left(I_1(x^2) + I_0(x^2)\right) - x^2 e^{-x^2} \left(I_0(x^2) - I_1(x^2)\right)\right).
\] (60)

This yields the action
\[
S(T) = 32\pi^2 \left(\frac{1}{2}\right)^{3/2} \left(\frac{b}{\Lambda^2}\right)^2 \Gamma(3/2) \Lambda^2 x e^{-x^2} I_0(x^2) \left[3\left(1 + \frac{1}{3} \frac{I_1(x^2)}{I_0(x^2)}\right) + 4\left(\frac{2}{3}\right)^3 e^{-x^2} I_0\left(\frac{3}{2}x^2\right)\right] - 2d.
\] (61)

For a given value of temperature (i.e., \( x \)) we can calculate \( b \) and \( d \). Hence \( \gamma \) and \( \Lambda \) are determined. Thus we can calculate the action at different values of temperatures. Figures 3 and 4 show the value of the action at different values of inverse of temperatures for \( \delta = 0.3 \) and \( \delta = 0.1 \) respectively. As we can see from the figure, the action goes smoothly from the zero temperature regime to high temperature regime without any singularity at the transition point. This means that in the thick-wall limit the transition is second order.

It can be shown easily that in the limit of zero temperature (i.e., \( x \to \infty \)), the action in Eq. (54) reduces to that in Eq. (27). Also we can recover the values of \( b \) and \( d \), i.e., \( b = -9/8 \) and \( d = -1/2 \). In the limit of high temperatures (i.e., \( x \to 0 \)), Eq. (54) reduces to Eq. (55) and the values of \( b \) and \( d \) are recovered.
IV. RESULTS FOR INTERMEDIATE WALL SIZES

The results in Sec. II and III apply to situations where the bubble size $R$ is large (TWA) or the bubble has no radius but only a (exponentially) decreasing wall. In this section we consider expressions for the action with $y_{\Lambda} = R^2/\Lambda^2$ a small quantity.

A. zero temperature

For the zero temperature or O(4) invariant bounce, an exact expression for the action can be found and it can be approximated for large or small values of $y_{\Lambda}$. For small $y_{\Lambda}$ we have

\[
S_4 = 2\pi^2 \gamma^2 \Lambda^4 \left[ \frac{1}{\Lambda^2} \left( \frac{\pi^2}{36} - \frac{1}{6} + y_{\Lambda} \left( \frac{\ln 2}{3} - \frac{1}{12} \right) \right) + \frac{\delta}{4} \left( \frac{\pi^2}{12} - \ln 2 + y_{\Lambda} \left( \ln 2 - \frac{1}{2} \right) \right) \right. \\
- \gamma \left( \frac{\pi^2}{12} + \frac{1}{4} - \frac{3}{2} \ln 2 + y_{\Lambda} \left( \ln 2 - \frac{5}{8} \right) \right) \\
+ \frac{\gamma^2}{8} \left( \frac{\pi^2}{12} + \frac{11}{24} - \frac{11}{6} \ln 2 + y_{\Lambda} \left( \ln 2 - \frac{2}{3} \right) \right) \right].
\] (62)

Extremization with respect to $\gamma$ and $\Lambda$ leads to $a = -14.9924$, $b = -6.2335$ and $d = -1.5785$ with values of $\gamma$ given by the equation

\[
\gamma^2 - 5.76 \gamma + 3.76 d = 0, \quad 3.7 < \gamma < 5.8, \quad 0 < \delta < 2.
\] (63)

The value of $y_{\Lambda}$ for $\delta = 0$ is $y_{\Lambda} \approx -1.5$ though the expression for the action is not valid as now $|y_{\Lambda}| > 1$. Interestingly, $3e^{-y_{\Lambda}}$ corresponds to the value of $\gamma(\approx \delta)$ obtained in Sec. II. As we expect the value of $\gamma$ to be 3 (value of $\varphi$ at the true minimum) it appears that the limiting solution is of the form

\[
\varphi = 3 e^{-y_{\Lambda}} e^{-\rho^2/\Lambda^2}.
\] (64)

We see that $y - \Lambda \approx 0$ for $\delta \approx 1.2$. The action calculated agrees with the action calculated numerically by Adams [10] in the region $1.2 < \delta < 1.4$.

B. High temperature

In this case we start with the expression for small $y_{\Lambda}$ given by

\[
S_3 = 4\pi^{3/2} \gamma^2 \Lambda^3 \left[ \frac{1}{\Lambda^2} \frac{3}{16\sqrt{2}} \left( 1 + y_{\Lambda} \left( 2 - \frac{16\sqrt{2}}{9\sqrt{3}} \right) \right) + \frac{\delta}{16\sqrt{2}} \left( 1 + y_{\Lambda} \left( 2 - \frac{4\sqrt{2}}{3\sqrt{3}} \right) \right) \right. \\
- \gamma \left( \frac{1}{12\sqrt{3}} \left( 1 + y_{\Lambda} \left( 3 - \frac{9\sqrt{3}}{8} \right) \right) + \frac{\gamma^2}{128} \left( 1 + y_{\Lambda} \left( 4 - \frac{32\sqrt{5}}{25} \right) \right) \right) \right].
\] (65)
Minimization leads to $d = -1.588$, $b = -2.403$, and $a = -2.2857$. Again we find the values of $y_{\Lambda} \approx 0$ for $\delta = 1.25$ and the expressions are valid between $1.1 < \delta < 1.4$. Thus we have (semi) analytic expressions for the regions $0 < \delta < 0.5$, $1.1 < \delta < 1.4$ and $1.7 < \delta < 2.0$.

V. CONCLUSIONS

We now discuss the nature of the transition as we go from zero to high temperatures. In quantum mechanics, definitive criteria for the continuity or discontinuity (corresponding to second order and first order respectively) in the derivative of the action have been obtained by Chudnovsky [6] and Garriga [7]. It has further been shown that the lowest action at any temperature is possessed by either the zero temperature or the high temperature solutions.

In quantum field theory the situation seems to be different. Both Ferrera [8] and we [9] find that there is an interpolating solution which can be used to determine whether the transition is first order or second order (i.e. with or without a kink).

In Sec. III we find that for a thin wall ($\delta \approx 2$) the interpolating solution has a singularity at $\beta = \pi R$. The expansion in terms of $y_{\Lambda}$ breaks down at this point. So we do not expect a real singularity at this point. However our numerical solutions (as well as those of Ferrera) show that a kink is present in the TWA, showing that the transition is first order. For $\delta \approx 0$ (thick wall) we find that there is no kink and the transition is smooth (second order).

It seems, therefore, a reasonable conclusion that below $\delta < 1.2$ ($y_{\Lambda} \approx 0$) we have a second order transitions and the graph of the action against $\beta$ is smooth. For $\delta > 1.2$ we have a kink and the transition is first order. Thus for a potential with a $\varphi$ symmetry breaking and coupling $f = 0.75$ corresponding to $\delta = 0.65$, Ferrera finds a smooth transition. With a $\varphi^3$ symmetry breaking term and $f = 0.75$ corresponding to $\delta = 1.46$ (see table III in our paper [9]) we find a kink. So by transforming the potential to the Adams form and looking at the resulting $\delta$ we can predict whether there will be a first or a second order transition.

It is suggested that our method could be used to study in detail the nature of the phase transition in electroweak theory. Such a study could be of importance in models of electroweak baryogenesis and other phenomena in the early universe.
A EXPRESSIONS OF ELLIPTIC INTEGRAL IN TERMS
OF THE BASIC INTEGRALS $E_0$ AND $E_1$

For $\kappa < 1$

\[
\begin{align*}
E_0 &= \int_0^1 \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - \kappa^2 t^2}} \\
E_1 &= \int_0^1 \frac{dt\sqrt{1 - \kappa^2 t^2}}{\sqrt{1 - t^2}} \\
E_3 &= \int_0^1 \frac{dt(1 - \kappa^2 t^2)^{3/2}}{\sqrt{1 - t^2}} = E_1\left(\frac{4}{3} - \frac{2}{3}\kappa^2\right) + E_0\left(\frac{\kappa^2 - 1}{3}\right) \\
E_T &= \int_0^1 \frac{dt\sqrt{1 - \kappa^2 t^2 t^2(1 - t^2)}}{\sqrt{1 - t^2}} = \frac{2E_1}{15}\left(\frac{1}{\kappa^4 - \frac{1}{\kappa^2}} + 1\right) + \frac{E_0}{15}\left(-\frac{2}{\kappa^4} + \frac{3}{\kappa^2} - 1\right) \\
E'_1 &= \frac{E_1 - E_0}{2\kappa^2} \\
E'_3 &= \frac{E_0}{2}\left(1 - \frac{1}{\kappa^2}\right) + E_1\left(-1 + \frac{1}{2\kappa^2}\right) \\
E'_T &= \frac{E_1}{15}\left(-\frac{4}{\kappa^6} + \frac{3}{2\kappa^4} + \frac{1}{\kappa^2}\right) + \frac{E_0}{15}\left(\frac{4}{\kappa^6} - \frac{7}{2\kappa^4} - \frac{1}{2\kappa^2}\right)
\end{align*}
\]

For $\kappa > 1$

\[
\begin{align*}
E_0(1/\kappa^2) &= \int_0^1 \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - t^2/\kappa^2}} \\
E_1(1/\kappa^2) &= \int_0^1 \frac{dt\sqrt{1 - t^2/\kappa^2}}{\sqrt{1 - t^2}} \\
E_{04}(\kappa^2) &= \int_0^{1/\kappa} \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - \kappa^2 t^2}} \\
E_{14}(\kappa^2) &= \int_0^{1/\kappa} \frac{dt\sqrt{1 - \kappa^2 t^2}}{\sqrt{1 - t^2}} = \kappa E_1(1/\kappa^2) - \frac{\kappa^2 - 1}{\kappa} E_0(1/\kappa^2) \\
E_{34}(\kappa^2) &= \int_0^{1/\kappa} \frac{dt(1 - \kappa^2 t^2)^{3/2}}{\sqrt{1 - t^2}} = \frac{1}{\kappa}\left[E_1(1/\kappa^2)\left(\frac{4\kappa^2}{3} - \frac{2\kappa^4}{3}\right)
+ E_0(1/\kappa^2)\left(1 - \frac{5\kappa^2}{3} + \frac{2\kappa^4}{3}\right)\right] \\
E_{T4}(\kappa^2) &= \int_0^{1/\kappa} \frac{dt\sqrt{1 - \kappa^2 t^2 t^2(1 - t^2)}}{\sqrt{1 - t^2}} = \frac{1}{\kappa}\left[E_1(1/\kappa^2)\left(-\frac{2}{15} + \frac{2}{15\kappa^2} + \frac{2\kappa^2}{15}\right)
+ E_0(1/\kappa^2)\left(\frac{1}{5} - \frac{1}{15\kappa^2} - \frac{2\kappa^2}{15}\right)\right]
\end{align*}
\]
\[
\frac{dE_0(1/\kappa^2)}{d\kappa^2} = \frac{1}{2\kappa^2}E_0(1/\kappa^2) - \frac{E_1(1/\kappa^2)}{2(\kappa^2 - 1)} \tag{79}
\]

\[
\frac{dE_1(1/\kappa^2)}{d\kappa^2} = \frac{1}{2\kappa^2} \left( E_0(1/\kappa^2) - E_1(1/\kappa^2) \right) \tag{80}
\]

\[
\frac{dE_{14}(\kappa^2)}{d\kappa^2} = \frac{1}{2\kappa} \left( E_1(1/\kappa^2) - E_0(1/\kappa^2) \right) \tag{81}
\]

\[
\frac{dE_{34}(\kappa^2)}{d\kappa^2} = E(1/\kappa^2) \left( \frac{1}{2\kappa} - \kappa \right) + E_0(1/\kappa^2) \left( - \frac{1}{\kappa} + k \right) \tag{82}
\]

\[
\frac{dE_{T4}(\kappa^2)}{d\kappa^2} = E_0(1/\kappa^2) \left( \frac{2}{15\kappa^5} - \frac{1}{15\kappa^3} - \frac{1}{15\kappa} \right) + E_1(1/\kappa^2) \left( - \frac{4}{15\kappa^5} + \frac{1}{10\kappa^3} + \frac{1}{15\kappa} \right) \tag{83}
\]

**B EVALUATION OF INTEGRALS INVOLVING THE FERMI FUNCTION**

For large \( R^2/\Lambda^2 \) an asymptotic expansion may be obtained through a method due to Sommerfeld as follows. Let \( y = \rho^2/\Lambda^2 \) and \( y_0 = R^2/\Lambda^2 \), then

\[
\int_0^\infty dy \frac{y^{1/2}}{(e^y - y_0 + 1)^4} = \frac{2}{3} y_0^{3/2} - \frac{11}{6} y_0^{1/2} + \left( \frac{\pi^2}{12} + \frac{1}{2} \right) y_0^{-1/2} + O(y_0^{-3/2}) \tag{84}
\]

\[
\int_0^\infty dy \frac{y^{1/2}}{(e^y - y_0 + 1)^3} = \frac{2}{3} y_0^{3/2} - \frac{3}{2} y_0^{1/2} + \left( \frac{\pi^2}{12} + \frac{1}{4} \right) y_0^{-1/2} + O(y_0^{-3/2}) \tag{85}
\]

\[
\int_0^\infty dy \frac{y^{1/2}}{(e^y - y_0 + 1)^2} = \frac{2}{3} y_0^{3/2} - y_0^{1/2} + \frac{\pi^2}{12} y_0^{-1/2} + O(y_0^{-3/2}) \tag{86}
\]

\[
\int_0^\infty dy \frac{y^{1/2}}{e^y - y_0 + 1} = \frac{2}{3} y_0^{3/2} + \frac{\pi^2}{12} y_0^{-1/2} + O(y_0^{-3/2}) \tag{87}
\]

\[
\int_0^\infty dy \frac{y^{1/2}e^{2(y-y_0)}}{(e^y - y_0 + 1)^4} = \frac{1}{6} y_0^{1/2} + O(y_0^{-3/2}) \tag{88}
\]

\[
\int_0^\infty dy \frac{y^{3/2}e^{2(y-y_0)}}{(e^y - y_0 + 1)^4} = \frac{1}{6} y_0^{3/2} + \left( \frac{\pi^2}{48} - \frac{1}{8} \right) y_0^{-1/2} + O(y_0^{-3/2}) \tag{89}
\]
References

[1] S. Coleman, Phys. Rev. D 15, 2929 (1977).
    C. Callan and S. Coleman, Phys. Rev. D 16, 1762 (1977).
    For a review of instanton methods and vacuum decay at zero temperature, see, e.g.,
    S. Coleman, Aspects of Symmetry (Cambridge University Press, Cambridge, England
    1985).

[2] A. D. Linde, Nucl. Phys. B216, 421 (1983); Particle Physics and Inflationary Cosmology
    (Harwood Academic Publishers, Chur, Switzerland, 1990).

[3] For a review of quantum and classical creep of vortices in high-$T_c$ superconductors, see
    G. Blatter, M. N. Feigel'man, V. B. Geshkenbein, A. I. Larkin and V. M. Vinokur, Rev.
    Mod. Phys. 66, 1125 (1994).

[4] D. A. Gorokhov and G. Blatter, Phys. Rev. B 58, 5486 (1998).
    D. A. Gorokhov and G. Blatter, Phys. Rev. B 56, 3130 (1997).

[5] I. Affleck, Phys. Rev. Lett. 46, 388 (1981).

[6] E. M. Chudnovsky, Phys. Rev. A 46, 8011 (1992).

[7] J. Garriga, Phys. Rev. D 49, 5497 (1994).

[8] A. Ferrera, Phys. Rev. D 52, 6717 (1995).

[9] H. Widyan, A. Mukherjee, N. Panchapakesan and R. P. Saxena, Phys. Rev. D 59, 045003 (1999).

[10] F. C. Adams, Phys. Rev. D 48, 2800 (1993).

[11] K. Huang, Statistical Mechanics (John Wiley & Sons, New York, 1963)
Figure Caption

FIG. 1. Temperature dependence of the Euclidean action in thin-wall limit: $S(T)$ vs $\beta$ for $\delta = 1.90$.

FIG. 2. Temperature dependence of the Euclidean action in thin-wall limit: $S(T)$ vs $\beta$ for $\delta = 1.85$.

FIG. 3. Temperature dependence of the Euclidean action in thick-wall limit: $S(T)$ vs $\beta$ for $\delta = 0.30$.

FIG. 4. Temperature dependence of the Euclidean action in thick-wall limit: $S(T)$ vs $\beta$ for $\delta = 0.10$. 

Figure 1:
Figure 2:
Figure 3:
Figure 4: