Robust entangled qutrit states in atmospheric turbulence

Tobias Brünner and Filippus S Roux

1 Institute of Physics, Albert-Ludwigs University of Freiburg, Hermann-Herder Strasse 3, D-79104 Freiburg, Germany
2 CSIR National Laser Centre, PO Box 395, Pretoria 0001, South Africa
E-mail: fsroux@csir.co.za

Abstract. The entangled quantum state of a photon pair propagating through atmospheric turbulence suffers decay of entanglement due to the scintillation it experiences. In this paper, we investigate the robustness against this decay for different qutrit states. An infinitesimal propagation equation is used to obtain the density matrix as a function of the propagation distance and the tangle is used to quantify the entanglement between a pair of qutrits. We consider the evolution of various initial states as they propagate through turbulence. Using optimization of the parameters that define the initial state, we obtain expressions for bipartite qutrit states that retain their initial entanglement longer than the initially maximally entangled states.
1. Introduction

Free-space quantum communication is one of the major components in the new quantum information technology revolution. A challenge that confronts this technology is the decay of entanglement that entangled photons experience due to scintillation while propagating through a turbulent atmosphere. The approaches to overcome this challenge are based either on methods to correct the output optical field, such as using adaptive optics [1] or on choosing an input state that is to some extent robust against entanglement decay in atmospheric turbulence. Here we consider the latter approach. In other words, we investigate to what extent a quantum state can be optimized so that it will retain as much of its initial entanglement as possible, while propagating through turbulence. We will refer to such an optimized quantum state as a robust state.

In this paper we consider bipartite qutrit states in the Laguerre–Gaussian (LG) modal basis, which we restrict to the three elements that have a radial modal index of zero \( p = 0 \) and azimuthal modal indices of \( \ell = 1, 0, -1 \). This gives us a three-dimensional Hilbert space per photon \( \mathcal{H}_3 \). The orbital angular momentum (OAM) associated with an LG mode is proportional to \( \ell \). If we used larger values of \( \ell \) instead of choosing \( |\ell| \leq 1 \), the entanglement would have lasted longer [2]. This follows from the observation that the coupling from one OAM mode to other is stronger the closer the two OAM values are to each other [3, 4].

We use the infinitesimal propagation equation (IPE) [5] to calculate the propagation distance-dependent density matrix for the quantum state of the photon pair as it propagates through the turbulence. The two photons are assumed to propagate through different uncorrelated regions of atmospheric turbulence. The turbulence is modelled by the Kolmogorov power spectral density [6].

References

[1] adaptive optics
[2] longer
[3] closer
[4] stronger
[5] infinitesimal propagation equation
[6] Kolmogorov power spectral density
The entanglement between the two photons is quantified by the tangle $\tau$ [7, 8] as a function of propagation distance $z$. The tangle is equal to the square of the concurrence for pure states and gives a lower bound for the square of the concurrence for mixed quantum states. To find the most robust qutrit states, we optimize the tangle at a given propagation distance $z > 0$. The result is expressed in terms of the parameters for the initial pure state that will give the maximum tangle at that propagation distance.

It has been shown that if only one party of a bipartite (or multipartite) state passes through a dissipative channel, the final amount of entanglement is proportional to the initial entanglement of the state [9–12]. The entanglement of the quantum state decreases gradually as the photon propagates along the channel, and the shape of the curve is independent of the initial entanglement of the state. If the photon propagates far enough through the dissipative channel, the entanglement would eventually decay to zero. The point where it reaches zero would be the same regardless of the initial entanglement of the state. Owing to the initially maximally entangled states, therefore, retaining the maximal possible entanglement at any propagation distance, they are the so-called most robust entangled states. This argument has been extended to the case where all parties of an entangled state pass through dissipative channels, provided that these channels are trace preserving [13]. However, if both parties of an entangled bipartite state pass through uncorrelated dissipative channels that are not trace preserving, only an upper bound exists for the evolution of the entanglement [9–11]. Here we’ll show in particular that for two photons propagating through a turbulent atmosphere, corresponding to two qutrit states passing through uncorrelated dissipative channels that are not trace preserving, the initially maximally entangled state is not the most robust state.

2. Background

2.1. Turbulent atmosphere

While turbulence is a phenomenon found in fluids, causing random fluctuations in the refractive index of the fluid, scintillation is what happens to light propagating through a turbulent fluid due to the phase modulations induced by the random fluctuations in the refractive index, together with the subsequent diffraction. These fluctuations vary as a function of both time and space. Hence, one can regard the refractive index of the fluid as a random three-dimensional function that gradually changes with time. Although the primary effect of turbulence on the traversing optical beam is a pure phase modulation, after some subsequent propagation, the diffraction induced by the random phase modulations causes fluctuations in the intensity of the beam. Such intensity fluctuations can, for instance, be observed as the twinkling of stars in the night sky. The random phase modulation is done continuously along the propagation path, causing an accumulated effect that, together with the diffraction of the beam, progressively aggravates the distortion of the beam with increasing propagation distance. This combined effect is the scintillation process.

Since our interest with respect to the turbulent medium is related to how it affects the refractive index of the medium, the turbulence model is specified in terms of a power spectral density for the refractive index fluctuations. Here we use the Kolmogorov power spectral

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density [6], given by

\[ \Phi(k) = 0.033 \, C_n^2 k^{-11/3}, \]  

where \( k \) is the magnitude of the propagation vector and \( C_n^2 \) is the refractive index structure constant (units: \( \text{m}^{-3/2} \)), which quantifies the strength of the turbulence.

Strong turbulence implies that the fluctuations in the refractive index are relatively large. However, it is important to remember that strong turbulence does not necessarily imply strong scintillation. The amount of scintillation that an optical beam experiences does not only depend on the strength of the turbulence but also on the distance of propagation. Scintillation increases with distance of propagation and it increases more rapidly in strong turbulence. However, one can have the situation that scintillation is still weak even if the turbulence is strong, provided that the light propagated for only a short distance. The opposite is also true. If the turbulence is weak the scintillation of the beam can be strong if it propagated far enough.

When a classical optical beam with a particular beam profile propagates through a turbulent atmosphere, the combined effect of the random phase modulations and the subsequent diffraction causes the distortion of the intensity profile of the beam. One can view this as a scattering process in the space of transverse spatial modes, which contains all the beam profiles that can be expressed as a linear combination of an orthogonal basis, such as the LG modes, for example. Assume, for instance, that the initial beam profile is a particular modal function—one of the elements of the orthogonal basis. The distortion caused by the particular random phase modulation that the classical beam experiences implies that its spatial mode changes into a linear combination of basis functions. In other words, the initial basis function is scattered into other basis functions. In the process new basis functions are excited which do not exist in the initial beam profile.

A similar process occurs for photonic quantum states. Here the initial beam can be an entangled pair of photons. The photon pair is assumed to be entangled in its spatial degrees of freedom, which means that one can express it as a linear combination of pairs of spatial modes. One can use an orthogonal basis, such as the LG modes, for these spatial modes in the same way it is done for the classical case. When the entangled pair of photons propagates through the turbulent medium, the spatial modes will suffer the same distortion that a classical beam does. As a result the linear combination of pairs of spatial modes changes. In fact, the individual basis elements experience the same scattering as in the classical case. Although one can model a single pass of the quantum state through the scattering medium as a unitary operation, the ignorance about the details of the medium requires that one performs ensemble averaging, which then leads to a quantum mixture. Owing to the medium gradually changing during the experiment, we are effectively averaging over different random media. This scattering process eventually destroys the entanglement. This is seen as a gradual decay in the entanglement of the quantum state of the photon pair as a function of the propagation distance.

One can calculate the decay of entanglement for an initial photonic quantum state, using the IPE [5]. It is a set of coupled first-order differential equations that describes the evolution of the density matrix for a biphoton in the OAM basis as a function of the propagation distance \( z \) through a turbulent medium. It represents a multiple-phase screen approach, as opposed to the single-phase screen approach [3]. As a result it can simulate both phase and intensity fluctuations, whereas the single-phase screen approach can only simulate phase fluctuations. In the solution of the density matrix, obtained from the IPE, the strength of the turbulence is...
contained in a dimensionless parameter defined by

\[ \sigma = \frac{\pi^{3/2} C_n^2 w_0^{11/3}}{6 \Gamma(2/3) \lambda^3}, \]  

(2)

where \( w_0 \) is the radius of the Gaussian envelope at the waist of the beam and \( \lambda \) is the wavelength of the photons. Note that \( \sigma \) is not an indication of the scintillation strength, but rather of turbulence strength, because it does not contain the propagation distance. We also define a dimensionless normalized propagation distance

\[ t = \frac{z}{z_R}, \]  

(3)

where \( z_R \) is the Rayleigh range, defined by \( z_R = \pi w_0^2 / \lambda \). In the discussions below, the curves are shown as functions of \( t \). Henceforth, the ‘propagation distance’ always means the normalized propagation distance \( t \).

The evolution of entanglement as a function of propagation distance is governed by the strength of the turbulence, as given by \( \sigma \), and also by the shape of particular \( t \)-dependent functions produced by the IPE, which are discussed in appendix A. Due to the shape of these latter functions, the evolution of entanglement changes as a function of the strength of turbulence. This in turn allows for approximations (single-phase screen approximation [3], quadratic structure function approximation [2], etc) to be made under weak turbulence conditions. As a result, one may ask whether any conclusions that are to be drawn under certain turbulence conditions would be valid under all turbulence conditions. For this reason, in all the discussions below, we show curves for both weak and strong turbulence conditions and thus demonstrate that our conclusions are valid under all turbulence conditions.

2.2. Truncation

The spatial modal basis of the quantum state of an entangled photon pair that propagates through the atmosphere is infinite dimensional. This is true regardless of the fact that the initial quantum state contained nonzero coefficients only for a few basis elements—in our case only for elements in a \( H_3 \otimes H_3 \) subspace of the infinite-dimensional Hilbert space. As such, the propagation process represents a trace preserving quantum operation. However, in a practical quantum information system the information would be encoded in a finite-dimensional subspace of the infinite-dimensional Hilbert space. For this reason only the information that is contained in this finite subspace will be extracted during the measurement process after the photon pair propagated through the turbulence. As a result, the quantum state is truncated to this subspace, in which the trace is not preserved during propagation anymore. The truncation can be viewed as a post-selection process [14] that is performed on the measurement results. In other words, projective measurements are performed and only those results that are within the desired subspace are retained. This of course, inevitably means that some photons will be unusable for the communication protocol, resulting in a loss of photons. To some extent, the loss of photons can be compensated for in the experiment by increasing the intensity of the laser beam.

Although the IPE describes the evolution of the full infinite-dimensional density matrix that characterizes the biphoton state, i.e. a trace preserving quantum process, one cannot in practice solve an infinite number of coupled first-order differential equations. Therefore, one can only keep track of a finite number of density matrix elements and thus can only
calculate the evolution of a truncated density matrix. The truncated IPE does keep track of the scattering to density matrix elements that are outside the subspace, but the backward scattering from outside the subspace to elements in the subspace is lost. However, because the finite subspace is so much smaller than the rest of the infinite-dimensional Hilbert space, the backward scattering is expected to be negligible and therefore of minor importance to the evolution of the truncated density matrix. The scattering to elements outside the subspace is seen from a decay in the magnitudes of the density matrix elements inside the subspace. A consequence of this is a reduction in the trace of the truncated density matrix. The latter therefore requires a renormalization to be a valid density matrix, in the same way that the density matrix resulting from post-selective measurements needs to be renormalized. Physically, the number of discarded photons is proportional to the reduction of the trace of the truncated density matrix. The trace of the truncated density matrix thus provides useful information that is of practical importance in the experiment. We consider the decay of the trace in section 3 for this purpose.

2.3. Initial state

To find an initial quantum state that retains its entanglement for as far as possible, we start with the most general pure bipartite qutrit state that can be defined in terms of our Hilbert space $\mathcal{H}_3 \otimes \mathcal{H}_3$. Such a state can be expressed by

$$|\psi\rangle = \sum_{m,n} c_{m,n} |m\rangle |n\rangle,$$

where $c_{m,n}$ represents complex coefficients and $|m\rangle$ and $|n\rangle$ are OAM eigenstates (LG modes) in the respective subsystems, with $m, n \in \{1, 0, -1\}$. The normalization of the initial state implies that

$$\sum_{m,n} |c_{m,n}|^2 = 1.$$  \hspace{1cm} (5)

In the actual calculations we express the coefficients in terms of parameters that implicitly obey the normalization condition in (5) and removes an unimportant global phase factor. These parameters accommodate the relative phases and amplitudes between the basis states $|m\rangle |n\rangle$ and they also incorporate the symmetry associated with an interchange of the $\ell = 1$ and $-1$ states. The explicit expressions for this parameterization can be found in appendix B.

We use the IPE to calculate the density matrix $\rho(t)$ for the quantum state after the input state propagated a distance $t$ through the turbulent atmosphere. The input state is given by $\rho(0) = |\psi\rangle \langle \psi|$, where $|\psi\rangle$ is given in (4) in terms of the coefficients defined in (B.1). Although we assume that the initial state (4) is a pure state, the scintillation process causes it to become mixed during propagation, which necessitates a density matrix approach.

The resulting density matrix $\rho(z)$ is a $9 \times 9$ matrix, expressed in terms of the general parameters, provided in appendix B. We then perform a numerical optimization procedure to maximize the entanglement at a given value of $t$. From the results of the optimization procedure, we found that one can simplify the parameterization of the candidate quantum states significantly. The simpler parameterizations for the initial state, which are discussed in section 4.3, capture the essence of what is required to retain the maximum amount of entanglement.
Figure 1. Illustration of nearest-neighbour couplings between the basis elements inside the $\mathcal{H}_3 \otimes \mathcal{H}_3$ Hilbert space (green background) and the coupling to basis elements outside (white background). Vertical arrows represent nearest neighbour couplings for the A-subsystem and horizontal arrows represent nearest-neighbour couplings for the B-subsystem.

2.4. Scattering

As mentioned before, the scintillation process causes scattering of the spatial modes — different basis elements are generated due to the distortion of the initial basis elements in the initial state. The probability amplitude for a particular basis element to be generated when another basis element is scattered through the scintillation process is referred to as the coupling strength or just the coupling between these two basis elements. It has been shown [3] that the coupling between adjacent OAM modes (those with the smallest difference in OAM) is stronger than the coupling between OAM modes that are further apart.

The dynamics of the evolution of quantum states in a scintillating environment can in part be understood in terms of the dominant nearest-neighbour couplings. To visualize the dynamics, we provided a diagram in figure 1. It represents the nearest neighbour couplings between the basis elements in the central region of the infinite-dimensional Hilbert space of the bipartite photonic quantum state. The shaded (green) area represents the restricted $\mathcal{H}_3 \otimes \mathcal{H}_3$ Hilbert space. These basis states are included in the eventual quantum-state tomography measurements that are used to determine the entanglement. The diagram also shows the nearest basis elements outside the Hilbert space with a white background. The dominant nearest-neighbour couplings are represented by bidirectional arrows. The couplings are shown separately for the respective subsystems where vertical couplings are for the one subsystem (the A-subsystem) and horizontal couplings are for the other subsystem (the B-subsystem). Here the subsystems refer to the two respective photons in the entangled photon pair.

Two distinct processes can be explored with the help of the diagram in figure 1(a) the loss of photons due to couplings between elements from the shaded area to elements outside it (see section 3) and (b) the potential destruction of entanglement by couplings between elements inside the shaded area only (see section 4).
2.5. Measure of entanglement

For pure two-dimensional bipartite systems the concurrence [15] is a suitable measure of entanglement, but for mixed states in higher dimensions it is computationally demanding to calculate the concurrence directly through the construction of a convex roof [16]. Instead, we use the tangle $\tau\{\rho\}$, which is equal to the square of the concurrence for pure states and gives a lower bound for that of mixed quantum states [7]. The tangle is calculated via the purities of the (reduced) density matrices

$$\tau\{\rho\} = 2 \text{tr}(\rho^2) - \text{tr}(\rho_A^2) - \text{tr}(\rho_B^2),$$

(6)

where $\rho_A$ and $\rho_B$ are the respective reduced density matrices of the two subsystems. For a maximally entangled state $\tau = \tau_{\text{max}} = 2(d - 1)/d$, where $d$ is the dimension of the subsystems’ Hilbert spaces, and for a separable state $\tau = \tau_{\text{min}} = 0$.

3. Loss of photons

The first process we investigate is the loss of photons that is observed due to the post-selection, which is introduced by the projective measurements in the truncated Hilbert space. As explained above, this loss of photons is directly given by the reduction in the trace of the unnormalized truncated density matrix. One finds that this trace decreases as a function of the propagation distance, as can be seen in figure 2.

The decay of the trace as a function of the propagation distance depends on the initial state, because the different basis elements do not couple equally strongly to basis elements outside the restricted subspace. Due to the symmetries among the nine OAM basis elements from which the initial state can be composed, the basis elements are naturally divided into three groups:

- **Group 1** consists of only $|0, 0\rangle$. In the diagram in figure 1 we see that all the neighbouring states of $|0, 0\rangle$ lie within the shaded area of the truncated Hilbert space.


**Group 2** consists of \{\ket{0, 1}, \ket{1, 0}, \ket{0, -1}, \ket{-1, 0}\}. We see from the diagram in figure 1 that these elements each have one neighbouring state outside the truncated Hilbert space and three neighbouring states inside the truncated Hilbert space.

**Group 3** consists of \{\ket{1, 1}, \ket{1, -1}, \ket{-1, 1}, \ket{-1, -1}\}. According to the diagram in figure 1, these elements each have two neighbouring states outside and two neighbouring states inside the truncated Hilbert space.

Based on these considerations, groups 1–3 have progressively larger tendencies to couple to basis elements beyond the truncated Hilbert space. Hence, the rate of photon loss depends on how the initial state is composed out of the elements from these three respective groups. Initial states that are composed exclusively of elements from groups 1–3, respectively, are expected to show progressively higher rates of photon loss.

To illustrate how the photon loss depends on the composition of the initial state, we plot the trace as a function of the normalized propagation distance \(t\) in figure 2 for three cases when the initial state is composed of elements from the three respective groups. In the case of group 1, the initial state is just \(\ket{0, 0}\). In the case of groups 2 and 3, the initial state consists of a linear combination of the state from the respective groups, with arbitrary coefficients, provided that the state is normalized. Regardless of these coefficients, the evolution of the initial state from a particular group is always the same. The decay of the trace for these three cases is shown for strong turbulence (\(\sigma = 25\)) in figure 2(a) and for weak turbulence (\(\sigma = 0.25\)) in figure 2(b).

The plots in figure 2 confirm our expectations, as explained above. The difference in coupling 0 to states beyond the truncated Hilbert space manifests as a difference in the slope of the curve as it moves away from the point representing the initial state at \(t = 0\). Note that the state \(\ket{0, 0}\), which has the best performance of the three cases, is not an entangled state and therefore is not useful for applications that require entanglement.

### 4. Entanglement evolution

We investigate the evolution of entanglement, which will allow us to construct robust states, i.e. states that retain the highest amount of entanglement at a given propagation distance. This investigation starts by considering the evolution of all the Bell states that can be formed from the two-dimensional subspaces of our three-dimensional Hilbert space. Then we consider the evolution of the maximally entangled qutrit states. These investigations help us to obtain the most robust states.

#### 4.1. Bell states

For a two-dimensional Hilbert space, composed from OAM states with the azimuthal indices \(\ell = \pm 1\), one can show that the entanglement of the four maximally entangled Bell states, quantified by the concurrence, all decay equally [5]. This can be understood as a result of the fact that the scattering is symmetric with respect to \(\ell = 1\) and \(-1\). In the three-dimensional case, there is a third possible azimuthal index, which does not share this symmetry. In our case, we chose the third index to be \(\ell = 0\), and we quantify the entanglement by the tangle. As a result, we find that the 12 Bell states in the two-dimensional subspaces of our three-dimensional Hilbert space form three sets based on their decay curves. These sets are

\[
\text{Set 1} = \left\{ |\Phi_{0,1}^+\rangle, |\Phi_{0,-1}^-\rangle, |\Phi_{1,1}^+\rangle, |\Phi_{-1,1}^-\rangle \right\},
\]

\(\text{(7)}\)
Figure 3. Comparison of the tangle evolution of the Bell states in sets 1, 2 and 3 for strong turbulence (a) and for weak turbulence (b).

Set 2 = \{ |\Psi_{0,1}^+\rangle, |\Psi_{0,-1}^+\rangle, |\Psi_{0,1}^-\rangle, |\Psi_{0,-1}^-\rangle\}, \quad (8)

Set 3 = \{ |\Phi_{1,-1}^+\rangle, |\Phi_{1,-1}^-\rangle, |\Psi_{1,1}^+\rangle, |\Psi_{1,-1}^-\rangle\}, \quad (9)

where

$$|\Phi_{r,s}^\pm\rangle = \frac{1}{\sqrt{2}} (|r, r\rangle \pm |s, s\rangle), \quad (10)$$

$$|\Psi_{r,s}^\pm\rangle = \frac{1}{\sqrt{2}} (|r, s\rangle \pm |s, r\rangle), \quad (11)$$

the Bell states of the two-dimensional subspaces spanned by \( r \) and \( s \). The three sets of subspace Bell states give three different decay curves, as shown in figure 3.

We see from figure 3 that, although the initial amount of entanglement is the same for all three sets, the subsequent evolution of the three sets is different. Hence, the initial amount of entanglement does not uniquely determine the remaining entanglement after a propagation distance \( t \), as was found for the two-dimensional situation [5]. The Bell states of set 3 maintain a nonzero entanglement for longer than the states of sets 1 and 2. This can be understood by considering the diagram in figure 1. The basis states that are used to compose the states in set 1 (\(|1, 1\rangle, |0, 0\rangle\) or \(|-1, -1\rangle, |0, 0\rangle\)) have as their nearest-neighbour basis states (\(|1, 0\rangle, |0, 1\rangle\) or \(|-1, 0\rangle, |0, -1\rangle\)) that can be combined with the original basis states to form separable states. The same situation is true for the states in set 2. On the other hand, the nearest neighbour basis states for the states in set 3 cannot be combined with the original basis states in set 3 to form separable states. Hence, one can argue that states from set 3 have a slower entanglement decay than states from sets 1 and 2. This is confirmed by the difference in slope of the curve for set 3 leaving the point at \( t = 0 \) compared to those of sets 1 and 2.

The difference between the decay rates of sets 1 and 2 for larger \( t \) is not so easy to understand. At larger \( t \) higher order couplings are expected to start playing an important role, making the dynamics difficult to understand in terms of diagrams such as the one in figure 1.
4.2. Initially maximally entangled states

There are several maximally entangled qutrit states that can be composed in terms of the basis states of the truncated Hilbert space. These maximally entangled qutrit states are divided into two groups according to the curves of the decay of their entanglement. Examples of maximally entangled qutrit states from these two groups are given by

\[
\text{State 1} = \frac{1}{\sqrt{3}} \left[ |1, 1\rangle + |-1, -1\rangle + \exp(i\phi_3) |0, 0\rangle \right], \tag{12}
\]

\[
\text{State 2} = \frac{1}{\sqrt{3}} \left[ |0, -1\rangle + |-1, 0\rangle + |1, 1\rangle \right], \tag{13}
\]

respectively. Other examples of maximally entangled qutrit states in the respective groups can be obtained by performing local rotations between the $|1\rangle$ and $|-1\rangle$ basis states on either of the subsystems. To be maximally entangled, the magnitudes of the coefficients for the different terms in these states must all be equal to $1/\sqrt{3}$, but this still leaves some freedom with respect to the relative phases of these coefficients. However, we found that the only phase that affects the entanglement evolution in these states is the relative phase between the first two terms and the last term in state 1, as denoted by $\phi_3$. For this reason we ignore all the other phases in these states.

The optimal value of the phase $\phi_3$ depends on the propagation distance where the entanglement is to be maximized. Here we denote this propagation distance by $t_0$. The expression for $\phi_3$ that maximizes the tangle at $t = t_0$ is given by

\[
\phi_3 = \phi_{\text{opt}}(t_0) = \arctan \left( \frac{2 H_r(t_0) H_i(t_0)}{H_i(t_0)^2 - H_r(t_0)^2} \right), \tag{14}
\]

where the functions $H_r(t)$ and $H_i(t)$ are discussed in appendix A. The phase that minimizes the tangle is given by $\phi_3 = \phi_{\text{opt}}(t_0) - \pi$. A plot of $\phi_{\text{opt}}(t_0)$ is shown in figure 4. It starts from zero at $t = 0$ and then drops, approaching $-\pi$ as the propagation distance (where maximal entanglement is required) increases.

Figure 4. The optimal phase for $\phi_3$, as given in (14), that maximizes the tangle as a function of propagation distance.
Figure 5. Tangle evolution of the initially maximally entangled state in strong turbulence (a) and in weak turbulence (b), both showing three curves: states 1a, 1b and 2. In strong turbulence states 1a and 1b are obtained for $\phi = \pi$ and 0, respectively, and in weak turbulence states 1a and 1b are obtained for $\phi = 0.4\pi$ and $-0.6\pi$, respectively.

When we want to maximize the entanglement at small propagation distances $t_0 \lesssim 1/3$, the optimal phase that maximizes the tangle is $\phi_3 \approx 0$ and thus for $\phi_3 \approx \pi$ the tangle is minimized. Note that if we want to maximize the entanglement at larger propagation distances, $t_0 \gtrsim 10$ the reverse is true. In the case of strong turbulence the entanglement decays to zero before $t = 1/3$. Therefore, we will use this approximation for small propagation distances in the case of strong turbulence. We will denote the cases with minimum and maximum tangle as states 1a and 1b, respectively. The decay curves of states 1a, 1b and 2 are shown in figure 5, both in weak and strong turbulence. One finds that states represented by state 2 are more robust than those represented by state 1. At first the entanglement decay for all states is nearly identical and only starts to deviate as the entanglement decreases.

In this case, it is not so easy to understand in terms of the diagram in figure 1 why these states behave differently. The exact behaviour is not yet fully understood. Higher order couplings are expected to play an important role in this situation. Moreover, the interesting contribution of the optimal phase $\phi_3$ that influences the entanglement decay of state 1 is also not yet understood.

4.3. Most robust entangled state

Upon comparing the curves for initially maximally entangled qutrit states in figure 5 to those for the Bell states of the two-dimensional subsystems in figure 3, we find that the maximally entangled qutrit states are not the most robust qutrit states. The Bell states of set 3 maintain their entanglement longer than any of the initially maximally entangled qutrit states in (12) and (13), even though the initial entanglement of the Bell states is smaller. As a result, the most robust entangled state for a specific $t$ is not necessarily to be found among those states that are initially maximally entangled.

To find the most robust states, we start by performing numerical optimization of the tangle at a particular point $t = t_0$, for the most general initial pure state (4), using the parameterization

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Figure 6. Curves of optimized parameters $\mu_1(t_0)$ for state 1 and $\mu_2(t_0)$ for state 2 that gives the most robust entangled states $|\mu_1\rangle$ and $|\mu_2\rangle$ as a function of the normalized propagation distance $t_0$ where optimization is done, in strong turbulence (a) and weak turbulence (b).

from (B.1). This optimization process is similar to that in [8]. From the optimization results, expressed in terms of the general parameters, we found that the most robust states tend to have certain forms. The forms of these robust states are similar to the maximally entangled states shown in (12) and (13), but the Bell state (the first two terms in (12) and (13)) and the third term are weighted differently from those in a maximally entangled state. Following these observations, we introduce a simplified parameterization given by

$$|\mu_1\rangle = \cos(\mu_1) \frac{1}{\sqrt{2}} \left[ |1, 1\rangle + |-1, -1\rangle \right] + \sin(\mu_1) \exp(i\phi_3) |0, 0\rangle,$$

(15)

$$|\mu_2\rangle = \cos(\mu_2) \frac{1}{\sqrt{2}} \left[ |0, -1\rangle + |-1, 0\rangle \right] + \sin(\mu_2) |1, 1\rangle,$$

(16)

where $\mu_1$ and $\mu_2$ represent the optimization angular parameters for the two states, respectively, such that the sine and cosine of these parameters control the relative weighting of the Bell states and the third term. These parameters replace those given in (B.1) for the sake of simplicity. We still include the relative phase $\phi_3$ and equate it to the optimal phase value (14) in the subsequent discussion. Note that the values of $\mu_1$ must be restricted to the interval $[0, \pi]$ otherwise a shift of $\pi$ in the value of $\phi_3$ would be induced. The values of $\mu_2$ need not be restricted, because the optimization of state $|\mu_2\rangle$ is insensitive to the phase.

The simpler parameterization of (15) and (16), also simplifies the optimization process. From this optimization process we obtain the optimal values of $\mu_1$ and $\mu_2$ for every propagation distance. The curves of the optimized parameters $\mu_1(t_0)$ and $\mu_2(t_0)$ are shown as a function of the propagation distance in figure 6 for both weak and strong turbulence. We see that $\mu_1$ is indeed restricted to values in the interval $[0, \pi]$. The fact that the parameters are not independent of $t_0$ means that there does not exist one single state that is the most robust state for all propagation distances. Instead there are different states that have the highest remaining entanglement at each specific propagation distance $t = t_0$. For $t_0 = 0$ the optimized states $|\mu_1(0)\rangle$ and $|\mu_2(0)\rangle$ are the maximally entangled states given in (12) and (13), respectively, as expected.
Figure 7. Comparison of the tangle evolution of initially maximally entangled states (states 1 and 2) to those of the optimized states $|\mu_1\rangle$ and $|\mu_2\rangle$, (a) for strong turbulence, optimized at $t_0 = 0.03$ and (b) for weak turbulence, optimized at $t_0 = 1.6$.

Substituting (14) and the optimal values for $\mu_1$ and $\mu_2$ at $t_0 = 0.03$ ($t_0 = 1.6$) for strong (weak) turbulence into (15) and (16), we obtain states that maintain a nonzero entanglement much longer than the initially maximally entangled states of (12) and (13) or the Bell states in (7)–(9). The corresponding curves of the tangle are shown in figure 7. Note that state 1, which is less robust than state 2, can be optimized to produce state $|\mu_1\rangle$, which is more robust for large propagation distances than the optimized state $|\mu_2\rangle$. We do not yet understand the reason why this is the case. Again, we believe that higher order couplings and the non-trivial contribution of the phase $\phi_3$ could contribute to the observed effects. However, further investigation is necessary.

The optimized state $|\mu_1\rangle$ has an additional benefit: because all the terms have the same net OAM ($\ell = 0$), it can be prepared experimentally in spontaneous parametric down-conversion (SPDC). The down-converted bi-photon states that are produced in SPDC generally have different weights for different OAM basis states [17, 18]. Moreover, this OAM spectrum can be changed (made broader or narrower) by adjusting the appropriate parameters (e.g. beam sizes) in the experiment. In this way one can control the value of $\mu_1$. In case the range of weights that are thus produced does not include those that are required for the most robust state, one can introduce mode sorting [19]—an interferometer that can separate the OAM modes according to their OAM indices—to separate the different modes so that they can be given different weights and phases (such as $\phi_3$) before being recombined to form the required initial state. It would be harder to produce the optimized state $|\mu_2\rangle$ because the terms in this state do not have the same net OAM and would therefore not be directly produced in an SPDC process. However, using mode sorting [19] one may be able to change the OAM and the weights in the individual terms of some state that can be obtained from SPDC, such that the result is the required optimized state $|\mu_2\rangle$.

5. Summary and conclusions

We have shown that for three-dimensional systems, the initial amount of entanglement does not uniquely define the amount of entanglement left after propagation through a turbulent...
medium. This highlights the fact that the upper bound for the entanglement evolution [10, 11] has to be treated with caution. The reason is that in our three-dimensional Hilbert space, the coupling strength between differing magnitudes of azimuthal indices is different from the coupling strength between identical magnitudes. This follows from our choice of Hilbert space and the fact that the coupling strength decreases as the difference between the azimuthal modal indices increases.

By optimizing the tangle, starting from the most general initial pure state, we found that neither the initially maximally entangled states, nor one of the Bell states of the two-dimensional subsystems are the most robust states. Instead it is a coherent superposition of both. For each propagation distance, one can find a specific initial entangled state that retains the most entanglement up to that propagation distance. Thus, there is not one single most robust state for all distances.

The effect of the optimization of the robustness of the qutrit states is still rather small (it only increases the propagation distance by a few per cent). However, based on the difference going from two dimensions to three dimensions, it is reasonable to expect that the effect of such an optimization would become more significant for higher dimensional Hilbert spaces, making it worth doing the optimization to find robust quantum states in higher dimensional Hilbert spaces and thereby improving the performance of quantum communication systems. As a result, it is reasonable to expect that the optimization of the robustness of higher dimensional quantum states will have a significant impact on practical free-space quantum communication systems.

The optimized robust states can be prepared in experimental setups, using spontaneous parametric down-conversion, by exploiting the OAM spectrum of the down-converted state, as determined by the experimental parameters. In addition, mode sorting can be used to manipulate the individual terms of the quantum state.

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Appendix A. Propagation distance-dependent functions

After solving the differential equations of the IPE to obtain the density matrix as a function of the propagation distance, one finds that its dependence on the propagation distance is governed by two functions that are defined by the following integrals,

\[
Z(t) \equiv \int_0^t (1 + \tau^2)^{5/6} \, d\tau,
\]

and

\[
H(t) \equiv \int_0^t (1 + \tau^2)^{5/6} \left( \frac{1 + i\tau}{1 - i\tau} \right) \, d\tau,
\]

where \( t \) is the normalized propagation distance equation (3). For convenience we separate \( H(t) \) into its real and imaginary parts \( H(t) = H_r(t) + iH_i(t) \). These integrals can be solved to give
expressions in terms of hypergeometric functions

\[
Z(t) = t \, _2F_1 \left( \begin{bmatrix} -\frac{5}{6}, & \frac{1}{2} \\ \frac{5}{2} \end{bmatrix}, \ -t^2 \right),
\]

(A.3)

\[
H_i(t) = \frac{11}{8} \, t \, _2F_1 \left( \begin{bmatrix} \frac{1}{6}, & \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \ -t^2 \right) - \frac{2}{8} \left( 1 + t^2 \right)^{5/6},
\]

(A.4)

\[
H_i(t) = \frac{6}{5} \left[ (1 + t^2)^{5/6} - 1 \right],
\]

(A.5)

where \(_2F_1([\cdot], [\cdot], \cdot)\) represents Barnes’s extended hypergeometric functions [20].

Appendix B. Parameterization

The nine complex coefficients in (4) represent 18 real degrees of freedom. The normalization condition of (5) removes one of these real degrees of freedom and the overall phase factor that can be ignored implies the removal of another real degree of freedom. As a result one needs 16 real degrees of freedom to specify an arbitrary initial quantum state. These are divided into eight parameters for the magnitudes of the coefficients and eight parameters for the complex phase factors of the coefficients. Due to the form of the normalization condition in (5), it is natural to represent the eight parameters for the magnitudes of the coefficients in terms of sine and cosine functions. Therefore, the parameterization for the complex coefficients \(c_{m,n}\) in (4) is given by

\[
c_{1,1} = \sin(k_a \sin(k_b \sin(k_d) \sin(k_h) \exp[i(q_a + q_b + q_d + q_h)]),
\]

\[
c_{1,-1} = \cos(k_a \sin(k_b \sin(k_d) \sin(k_h) \exp[i(-q_a + q_b + q_d + q_h)]),
\]

\[
c_{-1,1} = \sin(k_c \cos(k_b \sin(k_d) \sin(k_h) \exp[i(q_c - q_b + q_d + q_h)]),
\]

\[
c_{-1,-1} = \cos(k_c \cos(k_b \sin(k_d) \sin(k_h) \exp[i(-q_c - q_b + q_d + q_h)]),
\]

\[
c_{0,0} = \cos(k_h) \exp(-iq_h),
\]

\[
c_{1,0} = \sin(k_c \sin(k_i \cos(k_d) \sin(k_h) \exp[i(q_e + q_t - q_d + q_h)]),
\]

\[
c_{-1,0} = \cos(k_c \sin(k_i \cos(k_d) \sin(k_h) \exp[i(-q_e + q_t - q_d + q_h)]),
\]

\[
c_{0,1} = \sin(k_g \cos(k_i \cos(k_d) \sin(k_h) \exp[i(q_g - q_t - q_d + q_h)]),
\]

\[
c_{0,-1} = \cos(k_g \cos(k_i \cos(k_d) \sin(k_h) \exp[i(-q_g - q_t - q_d + q_h)]).
\]

These parameters were chosen to follow the pattern imposed by the symmetries found in the couplings, as represented in the diagram in figure 1. In particular, note how the parameter \(k_h\) governs the relative weighting between \(|0, 0\) and the rest of the basis element (in other words, between group 1 on the one side and groups 2 and 3 on the other side, as introduced in section 3). Note also that \(k_d\) governs the relative weighting between groups 2 and 3.

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