The Fractal Nature of Drought: Power Laws and Fractal Complexity of Arizona Drought

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ABSTRACT. In this study, we explore the possibility that the Drought Monitor database belongs to class of fractal process which can be characterized using a single scaling exponent. The Drought Monitor map identifies areas of drought and labels them by intensity: $D_0$ abnormally dry, $D_1$ moderate drought, $D_2$ severe drought, $D_3$ extreme drought, and $D_4$ exceptional drought. The vibration analysis using power spectral densities (PSD) method has been carried out to discover whether some type of power-law scaling exists for various statistical moments at different scales of this database. We perform multi-fractal analysis to estimate the multi-fractal spectrum of each group. We apply Higuchi algorithm to find the fractal complexity of each group and then compare them for different time intervals. Our findings reveal that we have a wide range of exponents for $D_0-D_4$. Therefore, $D_0-D_4$ belong to class of multi-fractal process for which a large number of scaling exponents are required to characterize the scaling structure.

1. Introduction

Drought is defined as a moisture deficit bad enough to have social, environmental or economic effects. Drought is a recurring feature of nearly every climate on the planet [1–5]. In many parts of the world, including North America, we have little ability to predict exactly when drought will happen next. But if we look at history and climate data, we can be sure that drought will happen again at some point. In the United States, a well-developed economy and agricultural system generally protect citizens from the most critical effects of drought such as shortages of food and water. However, drought still causes extreme hardship for farm and ranch families, and individual wells may run dry. Besides affecting municipal water suppliers, drought affects businesses and environmental interests that are reliant on adequate and timely amounts of precipitation and water, such as habitat for fish and wildlife, outdoor recreation outfitters, and landscaping and car
wash services [5–14]. The Drought Monitor map identifies areas of drought and labels them by intensity. $D_1$ is the least intense level and $D_4$ the most intense. $D_0$ areas are not in drought, but are experiencing abnormally dry conditions that could turn into drought or are recovering from drought but are not yet back to normal.

There are different indices which have been used to assess drought severity and impacts in different time-scales. The normalized difference vegetation index (NDVI) is one of the most widely utilized drought indices to determine different drought levels [15–17]. Satellite databases have been extensively used to record and quantify the changes may happen in vegetation coverage due to changing climate conditions. The NDVI is estimated using visible and near-infrared (NIR) bands from Advanced Very-High-Resolution Radiometer (AVHRR), Terra Moderate Resolution Imaging Spectroradiometer (MODIS), and Landsat sensors. In general, positive NDVI values demonstrate vegetated areas, zero and negative values are associated with bare soil and water bodies [15]. The time series of the average NDVI for Arizona (Arizona includes regions with moderate to exceptional drought $D_1$–$D_4$) shows the highest and lowest NDVI values during 10 years (2010–2020) (for the month of Jan–Dec each year) (see figure (1)). The NDVI data selected from the Google Earth Enterprise open source which is derived using Terra Moderate Resolution Imaging Spectroradiometer (MODIS) and would be useful to forecast the future changes in vegetation in Arizona. Each state experiences different sets of impacts during a drought. We have also displayed the table of reported impacts during past droughts in Arizona for each level of drought on the U.S. Drought Monitor in table (2) (Source(s): NDMC, NOAA, USDA). When we study real world time series data, depending on scale and higher order moments, we may confront with data that display nonlinear power-law behaviours. For these type data, we need to apply multifractal analysis. In multifractal analysis we discover whether some type of power-law scaling exists for various statistical moments at different scales. A process called mono-fractal, if it can be characterized using a single scaling exponent, or this process is a linear function of the moments. Likewise, a process called multi-fractal, if we see the scaling behavior follows a function which is non-linear in the moments. When we study scale invariant time series data, or data with different scaling behavior, we are not able to use the classical time series analysis and we need to perform fractal analysis.

In this study, we use fractal geometry to classify drought severity from 2000 to 2021 in Arizona. We perform multifractal analysis to discover whether some type of power-law scaling exists for various statistical moments at different scales of these data sets. We plot the multifractal spectra to compare the width of the scaling exponent for each spectrum. A quantitative analysis commonly known as the Fractal Dimension (FD) using Higuchi algorithm.

2. Materials, Methods and results

2.1. Data. Here, data has been collected using U.S. Drought Monitor for each week of the selected time period (January 2000 to Nov 2021) and location (Arizona, USA), see figures (3) and (4). The
Normalized Difference Vegetation Index (NDVI) Arizona between 2010-2020; Google Earth Enterprise Open Source

U.S. Drought Monitor which started from 1999, is a partnership between the National Drought Mitigation Center (NDMC) at the University of Nebraska-Lincoln, the United States Department of Agriculture (USDA), and the National Oceanic and Atmospheric Administration (NOAA). Each Thursday, the U.S. Drought Monitor (USDM) will be updated to demonstrate the location and intensity of drought across the country. Using the experts’ assessments, drought categories display conditions related to dryness and drought such as observations of how much water is available in streams, lakes, and soils compared to usual time of year (Source(s): NDMC, NOAA, USDA) [18].

2.2. Time–Frequency Analysis and Continuous Wavelet Transform (CWT). Continuous Wavelet Transform (CWT) provides a linear time-frequency representation of non-stationary signals called scalogram by breaking the data into scales by preserving time shifts and time scales. Therefore, the wavelet transform makes the analysis of the data in different frequency ranges easier and we can...
To compute the scalogram of data which is function of time and frequency, at first we split the time series data into overlapping segments, then we need to compute the absolute value of the continuous wavelet transform coefficients of each segment and finally, plot it. We have displayed the scalogram plots of Drought Monitor Categories Arizona database (2000 - present) in figures (5)-(6).

2.3. **Vibration frequency analysis using Power spectral densities (PSD).** The fast Fourier transform (FFT) has been used widely to analysis of vibration frequency in computing discrete Fourier transform (DFT). However, FFT only works accurately if we have a finite number of dominant frequency components. To overcome this problem, we use the power spectral densities (PSD) which
Arizona Percent Area in U.S. Drought Monitor Categories database (2000 - present); National Drought Mitigation Center (NDMC), the U.S. Department of Agriculture (USDA), and the National Oceanic and Atmospheric Administration (NOAA)

Histogram of Arizona Percent Area in U.S. Drought Monitor Categories (2010-2020)

Figure 3. Arizona Percent Area in U.S. Drought Monitor Categories database (2000 - present); National Drought Mitigation Center (NDMC), the U.S. Department of Agriculture (USDA), and the National Oceanic and Atmospheric Administration (NOAA)

Histogram of Arizona Percent Area in U.S. Drought Monitor Categories (2010-2020)

Figure 4. Histogram of Arizona Percent Area in U.S. Drought Monitor Categories database (2000 - present); National Drought Mitigation Center (NDMC), the U.S. Department of Agriculture (USDA), and the National Oceanic and Atmospheric Administration (NOAA)
Time-frequency representations of Drought Monitor Categories database (2000 - present) using Continuous Wavelet Transform (CWT) in two dimensional Time-Frequency space.

![CWT graphs](image)

**Figure 5.** Time-frequency representations of Drought Monitor Categories database (2000 - present) using Continuous Wavelet Transform (CWT) in two dimensional Time-Frequency space.

Time-frequency representations of Drought Monitor Categories database (2000 - present) using Continuous Wavelet Transform (CWT) in three dimensional Time-Frequency-Magnitude space.

![Scalogram graphs](image)

**Figure 6.** Time-frequency representations of Drought Monitor Categories database (2000 - present) using Continuous Wavelet Transform (CWT) in three dimensional Time-Frequency-Magnitude space.

is applied to characterize random vibration in time series data. To compute PSD, we multiply each frequency bin of FFT by its complex conjugate to get a real spectrum and then normalize the results.
to frequency bin width. Here, we have applied the (PSD) method for our database and then we fit
the logarithm power spectral densities to their frequencies in log format using least squares ap-
proximation method. Finally, we calculate the slope for each regression line captures the linearity
of data. In figure (7), we can see the fitted least squares approximation to the logarithm of power
spectral density of Arizona drought database.

Moreover, we have plotted the scaling exponent graphs for Arizona drought database in figure (8).

![Figure 7](image.png)

**Figure 7.** Fitted least squares approximation to the logarithm of power spectral
density of Arizona drought database (2000 – present) obtained by wavelet tech-
niques

### 2.4. Multifractal Analysis and Discrete Wavelet Transform (DWT).

Fractal dimension is one of the
most often used algorithm to describe the complexity of a fractal object by measuring the changes
of coverings relative to the scaling factor [20–25]. It also specifies the space filling capacity of a
fractal object with respect to its scaling properties in the space [26–29]. The relationship between
scaling and covering is often hard to be characterized. The variation in the number of coverings,
\(N(\epsilon)\), with respect to the scaling factor \(\epsilon\), can be written as

\[ N(\epsilon) \propto \epsilon^{-D} \tag{1} \]

where \(D\) is the fractal dimension. The relation (1) is called scaling law that is used to demonstrate
the size distribution of many objects in nature. The box counting formula which has been widely
applied to approximate the fractal dimension of an irregular object is defined as

\[
D_B = \lim_{a \to 0} \frac{\ln(N(a))}{\ln(1/a)}
\]  

(2)

However, this monofractal dimension is not able to fully characterize complex scaling behaviors of many irregular objects in the real world. That’s why to study irregular objects we need to apply the multifractal algorithm. The multifractal analysis utilizes a spectrum of singularity exponents to provide a detailed and local description of complex scaling behaviors. In order to quantify local densities of the fractal set, we approximate the mass probability using the following formula

\[
P_i(a) = \frac{N_i(a)}{N}
\]

(3)

where \(N_i(a)\) is the number of mass in the \(i\)th subset of measure \(a\), \(N\) is the total mass of the set. When we scale the mass probability \(P_i(a)\) with measure \(a\) of a multifractal set, it also demonstrates the power law behavior:

\[
P_i(a) \propto a^{\alpha_i}
\]

(4)

where \(\alpha_i\) is the singularity exponent characterizing the local scaling in the \(i\)th subset. The multifractal spectrum \(f(\alpha)\) provides a statistical distribution of singularity exponents \(\alpha_i\). In general,
$f(\alpha)$ may be estimated using the Legendre transformation

$$f(\alpha) = q\alpha - \tau(q)$$

$$\alpha(q) = \frac{d\tau(q)}{dq}$$

where $q$ is the moment and $\tau(q)$ is the mass exponent of the $q$th order moment. In addition, the multifractal measures may be specified by scaling of $q$th moments of $P_i(a)$ as

$$\sum_{i=1}^{N(a)} P_i^q(a) \propto a^{\tau(q)} = a^{(q-1)D_q}$$

where $D_q = \frac{\tau(q)}{(q-1)}$ is the generalized fractal dimension. For $q = 0$ equation (2.4) becomes

$$N(a) \propto a^{-D_0}$$

which is similar to formula (1).

From multifractal analysis results of Arizona drought database (see figure (9)), we can easily see that we have a wide range of exponents for $D_0$-$D_4$, which indicates they have multifractal structure. The drought database needs to be indexed by different exponents as we decompose them into different subsets. Therefore, $D_0$-$D_4$ require much more exponents to characterize their scaling properties.

**Figure 9.** The multi-fractal spectrum analysis of Arizona Drought Monitor Categories database (2000 - present) shows the occurrence of multi-fractality with a broad range of exponents in data structure of $D_0$-$D_4$. 
2.5. **Higuchi Fractal Dimension Algorithm.** When we use box counting method, we compute the fractal dimension and or the complexity of a fractal process in two dimensional space [30]. However, when we are working with many real world time series data, it fails to recognize the sudden changes happen in data [31]. To solve this problem, there are different methods such as Higuchi algorithm, power spectrum analysis, and Katz algorithm that help to analyze the complexity of irregular data [32–34]. Here we use Higuchi Algorithm. We start with a finite time series \( x_1, x_2, x_3, \ldots, x_N \). Then, we create \( k \) new time series \( x^k_m \) of the form

\[ x_m, x_{m+k}, x_{m+2k}, \ldots, x_{[m+A]k} \]

where \( A = (N - m)/k \). For each time interval \( k \) and the initial time \( m \) such that \( m = 1, 2, \ldots, k \), we calculate the length of \( x^k_m \) using

\[ L^k_m = \sum_{i=1}^{[A]} |x_{m+ik} - x_{m+(i-1)k}| \]

where \( R = (N - 1)/[A]k \) is the curve length normalization factor. To compute the average of curve length for each \( k \), we calculate the mean of \( L^k_m \) for \( m = 1, 2, \ldots, k \) and take the average for \( k = 1, \ldots, k_{max} \). Next, we plot \( \log(L^k_m) \) versus \( \log(1/k) \) for different time interval \( k \). Finally, we calculate the slope of regressed line which is obtained by the least-squares approximation as the Higuchi fractal dimension for time interval \( k = 500 \). We have estimated the fractal dimension of the Arizona drought database and plotted their regression models for each data in figure (10).

![Plots of log(L^k_m) versus log(1/k) for time interval k = 500](image)

**Figure 10.** Plots of \( \log(L^k_m) \) versus \( \log(1/k) \) for time interval \( k = 500 \), the logarithmic scale and the corresponding slope of fitted regression line (the Higuchi fractal dimension) for Arizona Drought Monitor Categories database (2000 - present).
3. Discussion

Drought as a slowly progressed and hidden disaster takes place in normal cycles of climate and it affects adversely environment and economic the same as other disasters. Therefore, characterizing the complexity of its nature will help to predict and recognize its different stages before it damages our societies. In U.S., the National Integrated Drought Information System (NIDIS) which is a multi-agency partnership, works on facilitating drought recording, predicting, risk management and planning at different national levels. Because of obvious impacts of drought on agriculture, water supply, energy production, public health, and wildlife, we decided to find analytical and computational techniques to characterize the complexity of different levels of drought from moderate drought $D_1$ to exceptional drought $D_4$ using the Arizona Drought Monitor databases. We applied the time-frequency analysis using Continuous Wavelet Transform (CWT) to visualize the non-linearity in the structure of five different groups of drought levels in the frequency domain of data vibration. Moreover, we carried out the vibration analysis using the power-law exponent and (PSD) to discover the power-law and self-similarity behaviors in the structure of drought database. we performed the multi-fractal analysis in studying the multi-fractal properties of our time series data. This analysis revealed the presence of a wide range of scaling exponent for $D_0$-$D_4$ and multi-fractal structure of the drought database. We continued our study by measuring the fractal complexity in drought time series data using Higuchi algorithm. This analysis helped to compare the self-similarity of different drought levels. Although these methods helped to characterize the complexity in the nature of our database, however it requires further studies to find an appropriate mathematical model (deterministic or stochastic) governing the complex dynamics of drought time series data.

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