LOCAL NULL CONTROLLABILITY FOR A PARABOLIC EQUATION WITH LOCAL AND NONLOCAL NONLINEARITIES IN MOVING DOMAINS

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ABSTRACT. In this paper, we establish a local null controllability result for a nonlinear parabolic PDE with local and nonlocal nonlinearities in a domain whose boundary moves in time by a control force with a multiplicative part acting on a prescribed subdomain. We prove that, if the initial data is sufficiently small and the linearized system at zero satisfies an appropriate condition, the equation can be driven to zero.

1. Introduction. In this paper we investigate the question of local null controllability of a nonlinear parabolic equation in domains which are moving in time. To make notations clear, let \( \Omega \subset \mathbb{R}^n \), \( n \geq 1 \) be a nonempty bounded connected open set, with boundary \( \Gamma = \partial \Omega \) of class \( C^2 \). For \( T > 0 \), we represent by \( Q = \Omega \times (0, T) \) of \( \mathbb{R}^{n+1} \), with lateral boundary \( \Sigma = \Gamma \times (0, T) \). Let us consider a family of functions \( \{ \tau_t \}_{0 \leq t \leq T} \), where for each \( t \), \( \tau_t \) is a deformation of \( \Omega \) into an open bounded set \( \Omega_t \) of \( \mathbb{R}^n \) defined by

\[
\Omega_t = \{ x \in \mathbb{R}^n ; x = \tau_t(y) \quad \text{for} \quad y \in \Omega \}.
\]

For \( t = 0 \), we identify \( \Omega_0 \) with \( \Omega \) so that \( \tau_0 \) is the identity mapping. For convenience of notation, for the points in the reference domain \( \Omega \), we will write \( y = (y_1, \cdots, y_n) \), while those in \( \Omega_t \) are denoted by \( x = (x_1, \cdots, x_n) \). The smooth boundary of \( \Omega_t \) is represented by \( \Gamma_t \). The noncylindrical domain \( \tilde{Q} \) and its lateral boundary \( \tilde{\Sigma} \) are defined by

\[
\tilde{Q} = \bigcup_{0 \leq t \leq T} \{ \Omega_t \times \{ t \} \} \quad \text{and} \quad \tilde{\Sigma} = \bigcup_{0 \leq t \leq T} \{ \Gamma_t \times \{ t \} \},
\]

respectively.

We assume the following regularity on the functions \( \tau_t \) for \( 0 \leq t \leq T \):

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(R1) $\tau_t$ is a $C^2$ diffeomorphism from $\Omega$ to $\Omega_t$.

(R2) $\tau_t$ lies in $C^1([0,T]; C^0(\Omega))$.

Thus we have a natural diffeomorphism $\tau : Q \to \hat{Q}$ defined by $(y,t) \in Q \to (x,t) \in \hat{Q}$, where $x = \tau_t(y)$.

To simplify the presentation, the reference domain $\Omega$ is assumed to be bounded and of class $C^2$. Nevertheless, we remark that most of the results we present here still hold when $\Omega$ is a Lipschitz-continuous and unbounded. The regularity assumptions on the diffeomorphism $\tau_t$ may also be weakened. However, the minimal assumptions on the reference domain $\Omega$ and the transformation $\tau_t$ will depend very much on the notion of solution and the type of control problem under consideration.

Concerning the class of domains $\hat{Q}$ which we are considering, it is important to point out that the assumptions above are not very restrictive. For instance, the condition (R2) that $\tau_t$ depends in a $C^1$ way on time (that, in practice, can often be replaced by a Lipschitz dependence) indicates that the domain does not evolve in time too roughly but allows all kinds of deformations on its shape. But, the conditions that $\Omega_t$ can be mapped into the reference domain $\Omega$ at every $t$ by means of a $C^2$ diffeomorphism impose that the topology of $\Omega_t$ does not change as time evolves. This is the main restriction that we impose on the geometry of the space-time domain $\hat{Q}$ under consideration. In particular, we do not adress here the problems in which holes appear or disappear in $\Omega_t$ as time increases. This type of situation requires a separate analysis since solutions may develop singularities at those points where the topology of $\Omega_t$ changes.

Our main goal is to establish the null controllability of the following nonlinear system:

$$
\begin{aligned}
&u_t - a\left(\int_{\Omega_t} u \, dx, \int_{\Omega_t} \nabla u \, dx\right) \Delta u + f(u) = h \hat{1}_{\hat{\omega}}(1 + u) \quad \text{in} \quad \hat{Q}, \\
u = 0 & \quad \text{on} \quad \hat{\Sigma}, \\
u(x,0) = u_0(x) & \quad \text{in} \quad \Omega,
\end{aligned}
$$

(1)

where $u = u(x,t)$ denotes the state and $h = h(x,t)$ is the control which acts on the system through an arbitrarilly small open set $\hat{\omega}$, where $\hat{\omega}$ is the image by $\tau_t$ of a non-empty open subset $\omega$ of $\Omega$.

Let us denote by $\hat{\omega}_1$, the image by $\tau_t$ of a non-empty open subset $\omega_1$ of $\Omega$, where $\omega_1 \subset \subset \omega$. Here, $\hat{1}_{\hat{\omega}} \in C_0^\infty(\hat{Q})$, $0 < \hat{1}_{\hat{\omega}} \leq 1$ in $\hat{\omega}$, $\hat{1}_{\hat{\omega}} = 1$ in $\hat{\omega}_1$ and $\hat{1}_{\hat{\omega}} = 0$ outside $\hat{\omega}$. The functions $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are of class $C^1$ with bounded derivatives, globally Lipschitz-continuous and satisfy

$$
0 < C_0 \leq a \leq C_1 \quad \text{and} \quad f(0) = 0.
$$

The controllability of linear and semilinear parabolic systems has been analyzed in several papers. Among them, let us mention Fursikov and Imanuvilov [17], Fernández-Cara and Zuazua [16] and Doubova et al. [12].

The control of PDEs equations and systems has been the subject of a lot of papers the last years. In particular, important progress has been made recently in the controllability analysis of semilinear parabolic equations. We refer to the works [9, 12, 13, 16, 17, 20, 22] and the references therein in the context of bounded domains, [3, 10] in the context of more general domains, and the works [7, 15] in the context of nonlocal terms.
On the other hand, noncylindrical problems similar to (1) have been motivated by different works. We will make the following chronological reference:

- **Approximated controllability and Null controllability:**
  The following boundary value problem for the heat equation in the non-
  cylindrical domain \( \hat{Q} \) has been considered by Límaco et al. in 2002; see for
  instance [25],
  \[
  \begin{cases}
  u' - \Delta u = h1_{\hat{Q}} & \text{in } \hat{Q}, \\
  u = 0 & \text{on } \hat{\Sigma}, \\
  u(x,0) = u_0(x) & \text{in } \Omega.
  \end{cases}
  \]

- **Finite approximated controllability:**
  In 2004, De Menezes et al. [11] studied properties of the finite approximated
  controllability for the following semilinear heat equation:
  \[
  \begin{cases}
  u' - \Delta u + f(u) = h1_{\hat{Q}} & \text{in } \hat{Q}, \\
  u = 0 & \text{on } \hat{\Sigma}, \\
  u(x,0) = u_0(x) & \text{in } \Omega,
  \end{cases}
  \]

- **Null controllability:**
  Límaco et al. [24] proved in 2016 the null controllability of the semilinear
  reaction-diffusion system with only one control force:
  \[
  \begin{cases}
  u' - \Delta u + f(u,v) = 0 & \text{in } \hat{Q}, \\
  v' - \Delta v + g(u,v) = h1_{\hat{Q}} & \text{in } \hat{Q}, \\
  u = v = 0 & \text{on } \hat{\Sigma}, \\
  u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega,
  \end{cases}
  \]
  where \( f \) is globally Lipschitz-continuous. The proof was based on Schauder’s
  Fixed Point Theorem.

In cylindrical domain, we will make the following reference:

- **Null controllability:**
  In 2012, Fernández-Cara et al. [15] proved by Kakutani’s Fixed Point
  Theorem the null controllability of the system:
  \[
  \begin{cases}
  u_t - A(t)u = h1_{\omega} & \text{in } Q, \\
  u = 0 & \text{on } \Sigma, \\
  u(x,0) = u_0(x) & \text{in } \Omega,
  \end{cases}
  \]
  where, \( A(t)u = \sum_{i,j=1}^{n} B_{ij}(u(.,t),t) \frac{\partial^2 u}{\partial x_i \partial x_j} \).

In 2013, Fernández-Cara et al. [7] proved by Liusternik’s Inverse Function
Theorem the null controllability of the system:

\[
\begin{cases}
  u_t - \alpha \left( \int_{\Omega} u \, dx, \int_{\Omega} v \, dx \right) \Delta u + f(u,v) = h1_{\omega} & \text{in } Q, \\
  v_t - \beta \left( \int_{\Omega} u \, dx, \int_{\Omega} v \, dx \right) \Delta v + g(u,v) = 0 & \text{in } Q, \\
  u = v = 0 & \text{on } \Sigma, \\
  u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega,
\end{cases}
\]
where $\alpha, \beta, f, g$ are of class $C^1$ with bounded derivatives and globally Lipschitz-
continuous.

Recently, in 2019, Prouvée and Límaco [33] proved the null controllability
(through a local inversion method) of the following parabolic-elliptic coupled
nonlinear systems:

$$
\begin{aligned}
\frac{du}{dt} - \beta_1 \left( \int_{\Omega} u \, dx, \int_{\Omega} v \, dx, \int_{\Omega} \nabla u \, dx, \int_{\Omega} \nabla v \, dx \right) \Delta u + f(u, v) &= h_1 \omega_{1,\omega} \quad \text{in } Q, \\
-\beta_2 \left( \int_{\Omega} u \, dx, \int_{\Omega} v \, dx, \int_{\Omega} \nabla u \, dx, \int_{\Omega} \nabla v \, dx \right) \Delta v + g(u, v) &= 0 \quad \text{in } Q, \\
u = v &= 0 \quad \text{on } \Sigma, \\
u(x, 0) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{du}{dt} - \beta_1 \left( \int_{\Omega} u \, dx, \int_{\Omega} v \, dx, \int_{\Omega} \nabla u \, dx, \int_{\Omega} \nabla v \, dx \right) \Delta u + f(u, v) &= 0 \quad \text{in } Q, \\
-\beta_2 \left( \int_{\Omega} u \, dx, \int_{\Omega} v \, dx, \int_{\Omega} \nabla u \, dx, \int_{\Omega} \nabla v \, dx \right) \Delta v + g(u, v) &= h_2 \omega_{1,\omega} \quad \text{in } Q, \\
u = v &= 0 \quad \text{on } \Sigma, \\
u(x, 0) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
$$

where $h_1$ is the control for the parabolic equation in (1), $h_2$ is the control
for the elliptic equation in (1) and $\beta_1, \beta_2, f, g$ are of class $C^1$ with bounded
derivatives and globally Lipschitz-continuous.

In this paper, we extended the results obtained in [15] and [33] considering non-
cylindrical domains and a multiplicative control $h_1 \omega_{1,\omega} u$.

In order, to obtain such multiplicative control, we strongly used the Proposition 3,
which gives us the regularity of control. It should be mentioned that in [33] it is
not possible to use parabolic regularity results to prove Proposition 3. Thus, with
the methodology used in [33], it is not possible to consider a multiplicative control.

The nonlocal terms in (1) appear naturally in some physical models. For exam-
ple, they can arise in heat conduction in materials with memory, nuclear reactors,
and population dynamics, for instance the bacteria in a container, the diffusion
coefficients may depend on the total amount of individuals; see for instance [6, 34].
We also mention that, in the context of elasticity theory, terms of the form

$$
a \left( \int_{\Omega} |u(x, t)|^2 \, dx \right) \quad \text{and} \quad a \left( \int_{\Omega} |\nabla u(x, t)|^2 \, dx \right)
$$

appear, respectively, in the Carrier and Kirchhoff equations. These equations arise
in nonlinear vibration theory; see for instance [29].

We also mention some basic references on the analysis on partial differential equa-
tions in noncylindrical domains. There is an extensive literature and the following
works are worth mentioning among many others: Lions [27], Cooper and Bardos [8],
Medeiros [28], Inoue [23], Nakao and Narazaki [32] for wave equations; Bernardi,
Bonfanti and Lutteroti [2], Miranda and Medeiros [31] for Schrödinger equations;
He and Hisano [19] for Euler equations; Miranda and Límaco [30] for Navier-Stokes
equations; Chen and Frid [4] for hyperbolic systems of the conservation law.
The main goal of this paper is to establish a null controllability property for the nonlinear system (1).

The system (1) is to be said null controllable at time $T$ if, for any $u_0 \in H_0^1(\Omega)$, there exist controls $h \in L^2(\tilde{\omega} \times (0,T))$ such that the associated solution to (1) satisfy

$$u(\cdot, T) = 0 \quad \text{in } \Omega_t.$$ (2)

Our main result is the following:

**Theorem 1.1.** Under the previous assumptions on $a$ and $f$, the nonlinear system (1) is locally null-controllable at any time $T > 0$. In other words, there exists $\varepsilon > 0$ such that, whenever $u_0 \in H_0^1(\Omega)$ and $\|u_0\|_{H_0^1(\Omega)} \leq \varepsilon$, there exist at least a control $h \in L^2(\tilde{\omega} \times (0,T))$ and associated state $u$ satisfying (2).

The proof of Theorem 1.1 relies on an application of the Liusternik’s Inverse Function Theorem in Banach spaces; see [1]. To this end, we will first use a suitable change of variables that transforms (1) in a parabolic problem in a fixed cylindrical domain. Then, we will introduce two appropriate Banach spaces $Y, Z$ and a mapping $H : B_r(0) \subset Y \mapsto Z$ (where $B_r(0)$ is a open ball of radius $r$), and we will rewrite (1) as an equation of the form

$$H(v, \tilde{h}) = (0, v_0), \quad (v, \tilde{h}) \in B_r(0) \subset Y.$$ (3)

Finally, we will check that the assumptions of Liusternik’s Theorem are satisfied by $H : B_r(0) \subset Y \mapsto Z$ and, consequently for any small initial data, (1) is solvable.

This paper is organized as follows. In Section 2, we give the details of the announced change of variables and we recall some previous results. In Section 3, we consider and solve a null controllability problem for a linear parabolic equation; this will be needed later to prove that the hypotheses of Liusternik’s Theorem are fulfilled. Section 4 deals with the proof of Theorem 1.1. Finally, some additional comments and questions are presented in Section 5.

### 2. Some previous results.

As already mentioned, the methodology in the present paper consists in turning the noncylindrical state equation (1) into a cylindrical one (see (5) below) by the diffeomorphism $\tau_t$.

### 2.1. Reduction to a fixed cylindrical domain.

To carry on this methodology, we first denote by $\psi_t(x)$ the inverse map of $\tau_t$, that is, $\psi_t = \tau_t^{-1}$. According the assumption (R1), $\psi_t$ is a $C^2$-map from $\tilde{\Omega}_t$ to $\Omega$, for all $0 \leq t \leq T$. We shall use the notation $\psi(x, t) = \psi_t(x)$. Thus the state on $Q$ is defined by

$$v(y, t) = u(\tau_t(y), t) = u(\tau_t(y), t) \quad \text{for all } y \in \Omega.$$ (4)

Equivalently in $\tilde{Q}$ we have

$$u(x, t) = v(\tau_t^{-1}(x), t) = v(\psi(x, t), t) \quad \text{for all } x \in \Gamma_t.$$ (5)

Therefore, the initial-boundary value problem (1) is equivalent to:

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
 v_t + a \left( \int_{\Omega} v |J| \, dy, \int_{\Omega} (\nabla v \cdot \nabla \psi) |J| \, dy \right) A(t)v + \vec{b} \cdot \nabla v + f(v) \\
 = \tilde{h}1_{\omega_t}(1 + v) & \quad \text{in } Q, \\
 v = 0 & \quad \text{on } \Gamma, \\
 v(y, 0) = v_0(y) & \quad \text{in } \Omega,
\end{array}
\right.
\end{align*}
\]
where,
\[
A(t)v(y, t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} \left( \alpha_{ij}(y, t) \frac{\partial v}{\partial y_j} \right),
\]

\[
\alpha_{ij} = \frac{\partial}{\partial x_j} (\tau_i(y), t) \frac{\partial}{\partial x_i} (\tau_j(y), t),
\]

\[
\tilde{b}(y, t) = (b_j(y, t))_{1 \leq j \leq n},
\]

\[
b_j(y, t) = \frac{\partial \psi}{\partial x_j} (\tau_i(y), t) + \sum_{i,j=1}^{n} \frac{\partial \alpha_{ij}}{\partial y_i} (y, t) - \Delta_x \psi_j (\tau_i(y), t),
\]

\[
\tilde{h}(y, t) = h(\tau_i(y), t), \quad \tilde{1}_{\omega}(y, t) = \tilde{1}_{\omega}(\tau_i(y), t)
\]

\[
v_0(y) = u_0(\tau_0(y)),
\]

\[
J(y, t) = \text{the Jacobian of the transformation } \Omega_t = \tau_t(\Omega).
\]

The system (5) is a variable coefficient parabolic equation in the cylindrical domain \(Q\). From the technical point of view, a new problem arises because the state equation (5) contains a uniformly coercive operator \(A(t)\).

The operator \(A(t)\) is associated with the following bilinear form:
\[
\alpha(v, u) = \sum_{i,j=1}^{n} \int_{\Omega} \alpha_{ij}(y, t) \frac{\partial v}{\partial y_j} \frac{\partial u}{\partial y_i} \, dy, \quad \forall v, u \in H_0^1(\Omega).
\]

This bilinear form is bounded because \(\psi(x, t) = \tau_t^{-1}(x)\) is a \(C^2\) diffeomorphism between \(\Omega_t\) and \(\Omega\). Then its matrix \(M = \left(\frac{\partial \psi}{\partial x_i}\right)_{1 \leq i, j \leq n}\) is invertible and for all \(\eta \in \mathbb{R}^n\) we have
\[
\|M^{-1} \eta\|_{\mathbb{R}^n} \leq \frac{1}{\alpha_0} \|\eta\|_{\mathbb{R}^n}, \quad \alpha_0 > 0.
\]

From the last inequality, we have the estimate:
\[
\alpha(v, v) \geq \alpha_0^2 \|v\|^2_{H_0^1(\Omega)}, \tag{6}
\]

proving the coercivity of \(\alpha\) in \(H_0^1(\Omega) \times H_0^1(\Omega)\).

Applying the method in [5], it follows that given \(v_0 \in H_0^1(\Omega)\) and \(\tilde{h} \in L^2(\tilde{\omega} \times (0, T))\), then (5) admits an unique strong solution \(v \in C([0, T]; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))\). Otherwise, if \(v_0 \in L^2(\Omega)\) and \(\tilde{h} \in L^2(\omega \times (0, T))\), then (5) admits an unique weak solution \(v \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))\).

Consequently, by using the diffeomorphism \((y, t) \rightarrow (x, t)\), from \(Q\) to \(\tilde{Q}\), we obtain a unique solution \(u\) to problem (1) with the regularity, namely:

- If \(u_0 \in H_0^1(\Omega)\) and \(h \in L^2(\tilde{\omega} \times (0, T))\), then
  \[
  u \in C([0, T]; H_0^1(\Omega_t)) \cap L^2(0, T; H^2(\Omega_t)) \cap H^1(0, T; L^2(\Omega_t)).
  \]

- If \(u_0 \in L^2(\Omega)\) and \(h \in L^2(\tilde{\omega} \times (0, T))\), then
  \[
  u \in C([0, T]; L^2(\Omega_t)) \cap L^2(0, T; H_0^1(\Omega_t)).
  \]

At this point we underline that, under assumptions (R1) – (R2), the transformation \(y \rightarrow x\) does indeed map the space of functions \(C([0, T]; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))\) into \(C([0, T]; H_0^1(\Omega_t)) \cap L^2(0, T; H^2(\Omega_t)) \cap H^1(0, T; L^2(\Omega_t))\).

2.2. Liusternik’s inverse function theorem. Let us consider the mapping
\[
H(v, \tilde{h}) = (H_1(v, \tilde{h}), v(\cdot, 0))
\]
where,
\[
H_1(v, \tilde{h}) = v + a \left( \int_{\Omega} v |J| \, dy + \int_{\Omega} (\nabla v \cdot \nabla \psi) |J| \, dy \right) A(t)v + \\
b \cdot \nabla v + f(v) - \tilde{h} h_1 \omega (1 + v).
\]

Later, our approach will be to locally invert this mapping, with the aim to ensure, for any small initial data \(v_0\), the existence of a pair \((v, \tilde{h})\) such that \(H(v, \tilde{h}) = (0, v_0)\).

To accomplish this goal, we will apply the following local inversion result in Banach spaces, which is a consequence of the so-called Liusternik-Graves Theorem (see for instance [1]):

**Theorem 2.1.** Let \(Y\) and \(Z\) be Banach spaces and let \(H : B_r(0) \subset Y \rightarrow Z\) be a \(C^1\) mapping. Let us assume that the derivative \(H'(0) : Y \rightarrow Z\) is onto and let us set \(\xi_0 = H(0)\). Then there exist \(\varepsilon > 0\), a mapping \(W : B_\varepsilon(\xi_0) \subset Z \rightarrow Y\) and a constant \(K > 0\) satisfying
\[
\{ \begin{array}{l}
W(z) \in B_\varepsilon(0) \text{ and } H(W(z)) = z, \forall z \in B_\varepsilon(\xi_0), \\
\|W(z)\|_Y \leq K \|z - H(0)\|_Z, \forall z \in B_\varepsilon(\xi_0).
\end{array} \]

Notice that, in this Theorem, \(W\) is the inverse-to-the-right of \(H\).

Once the inversion is performed, we take \((v, \tilde{h}) = H^{-1}(0, v_0)\) and, returning to the original coordinates, we see that the function \(u(x,t) = v(\psi(x,t), t)\) satisfies (1).

3. **Analysis of the controllability of the linearized system.** Let us consider the linearized system at zero
\[
\begin{cases}

v_t + a(0,0)A(t)v + \tilde{b} \cdot \nabla v + cv = \tilde{h} h_1 \omega + k & \text{in } Q, \\
v = 0 & \text{on } \Sigma, \\
v(y,0) = v_0(y) & \text{in } \Omega,
\end{cases}
\]
where the coefficient \(c\) is obtained from derivative of \(f\) at 0 and \(k \in L^2(Q)\).

As usual, the controllability of (7) is closely related to the properties of the solutions to the associated adjoint states. In this case, the adjoint of (7) is given by
\[
\begin{cases}

-\varphi_t + a(0,0)A^*(t)\varphi - \nabla \cdot (\tilde{b}\varphi) + c\varphi = F & \text{in } Q, \\
\varphi = 0 & \text{on } \Sigma, \\
\varphi(T) = \varphi_T & \text{in } \Omega,
\end{cases}
\]
where \(A^*(t)\) is the formal adjoint of the operator \(A(t), F \in L^2(Q)\) and \(\varphi_T \in L^2(\Omega)\).

Next we sketch the points used in the proof of the null controllability of the system (7) using suitable Carleman estimates. First, we use a global Carleman inequality satisfied by the solutions to (8). Second, this inequality allows us to establish an observability estimate. Third, we prove a new Carleman inequality with weights that do not vanish at \(t = 0\). Finally, we prove the null controllability of (7) by using the new Carleman estimate.

In this approach, the following technical result due to Fursikov and Imanuvilov [17], is fundamental.

**Lemma 3.1.** Let \(\omega_1 \subset \subset \omega\) a non-empty open subset. Then, there exists a function \(\beta_0 \in C^2(\Omega)\) satisfying:

- \(\beta_0(y) > 0 \ \forall y \in \Omega\),
\( \beta_0 = 0 \ \forall y \in \partial \Omega, \)
\( |\nabla \beta_0(y)| > 0 \ \forall y \in \bar{\Omega} - \omega_1. \)

Let us introduce the weight functions
\[
\alpha(y, t) := \frac{\alpha_1(y)}{\beta(t)}, \quad \phi(y, t) := e^{\lambda \beta_0(y)} \quad \text{with}
\]
\[
\alpha_1(y) := e^{R \lambda} - e^{\lambda \beta_0(y)}, \quad \beta(t) := t(T - t), \ 0 < t < T,
\]
where, \( R > \| \beta_0 \|_{L^\infty(\Omega)} + \ln 6 \) and \( \lambda > 0. \)

Also, let us set
\[
\alpha^*(t) := \max_{y \in \overline{\Omega}} \frac{\alpha_1(y)}{\beta(t)}, \quad \tilde{\beta}(t) := \min_{y \in \overline{\Omega}} \frac{\alpha_1(y)}{\beta(t)}
\]
\[
\phi^*(t) := \max_{y \in \overline{\Omega}} \phi(y, t), \quad \tilde{\phi}(t) := \min_{y \in \overline{\Omega}} \phi(y, t),
\]
\[
\tilde{\alpha} := 2\tilde{\alpha} - \alpha^*.
\]

The following global Carleman estimate hold for the solution to (8):

**Proposition 1.** There exist positive constants \( \lambda_0, s_0 \) and \( C_0 \) such that, for any \( \lambda \geq \lambda_0, s \geq s_0 \) and any \( \varphi_T \in L^2(\Omega) \) and \( F \in L^2(Q) \), the associated solution to (8) satisfies
\[
\int_Q e^{-2s\alpha}\left( (s\varphi)^{-1}\left( |\varphi_t|^2 + \sum_{i,j=1}^n \left| \frac{\partial^2 \varphi}{\partial y_i \partial y_j} \right|^2 \right) + \lambda^2 (s\varphi)|\nabla \varphi|^2 + \lambda^4 (s\varphi)^3 |\varphi|^2 \right) dy dt
\]
\[
\leq C_0 \left( \int_Q e^{-2s\alpha} |F|^2 dy dt + \int_{\omega_1 \times (0, T)} e^{-2s\tilde{\alpha}} \lambda^4 (s\phi^*)^3 |\varphi|^2 dy dt \right).
\]

Furthermore, \( C_0 \) and \( \lambda_0 \) only depend on \( \Omega \) and \( \omega \) and \( s_0 \) can be chosen of the form \( s_0 = C(\Omega, \omega)(T + T^2) \).

This result is essentially proved in [17] (in fact, similar Carleman inequalities were established there for more general linear parabolic equations); see also [14]. In fact, the coefficients of the principal part \( A^*(t) \), according to the assumptions (R1) and (R2), are of class \( C^1 \). Under these conditions, the Carleman inequalities presented in [17] or [14] guarantee (10). The explicit dependence on time of the constants is note given in [17]. We refer to [14], where the above formula for \( s_0 \) was obtained.

An important consequence of Proposition 1 is the following observability inequality:

**Corollary 1.** Suppose that \( F = 0 \) in (8). Then, there exists a positive constant \( C \) depending on \( T, \) and \( \lambda, \) such that, for any solution to (7), one has:
\[
|\varphi(0)|_{L^2(\Omega)} \leq C \int_{\omega_1 \times (0, T)} e^{-2s\tilde{\alpha}} (s\phi^*)^3 |\varphi|^2 dy dt,
\]
where \( s \) and \( \lambda \) are taken as in Proposition 1.

**Proof.** From the Carleman inequality in Proposition 1 with \( \lambda = \lambda_0 \) and \( s = s_0, \) we get:
\[
\int_Q e^{-2s\alpha} \lambda^4 (s\varphi)^3 |\varphi|^2 dy dt \leq C \int_{\omega_1 \times (0, T)} e^{-2s\tilde{\alpha}} \lambda^4 (s\phi^*)^3 |\varphi|^2 dy dt.
\]
We have
\[
\frac{1}{t^3(T-t)^3} \leq \phi^3(t) \leq \frac{C}{t^3(T-t)^3},
\]
(12)
because
\[
\phi(y,t) = e^{\lambda \beta_0(y)} \geq \frac{1}{t(T-t)} ,
\]
\[
\phi(y,t) \leq e^{\lambda t \beta_0 l\infty(\Omega)} \leq \frac{C}{t(T-t)}.
\]
From (11) and (12) we obtain
\[
\int_0^T e^{-2saT} d\tau \int_0^T \int_\Omega e^{-\frac{a}{2} \phi^3(s\phi)^3 |\varphi|^2} \, dy \, dt \leq C \int_0^T e^{-\frac{a}{2} \phi^3(s\phi)^3 |\varphi|^2} \, dy \, dt
\]
(13)
Let \( C_0 = \min_{\frac{T}{2} \leq t \leq \frac{3T}{4}} e^{-2saT} \phi^3 > 0 \). Whence
\[
C_0 \int_0^{\frac{3T}{4}} \int_\Omega |\varphi|^2 \, dy \, dt \leq C \int_0^{\frac{3T}{4}} \int_\Omega e^{-2saT} \phi^3 |\varphi|^2 \, dy \, dt
\]
(14)
Moreover,
\[
\int_0^{\frac{3T}{4}} \int_\Omega |\varphi|^2 \, dy \, dt \leq C \int_{\Omega} e^{-2saT} \phi^3 |\varphi|^2 \, dy \, dt
\]
(15)
Multiplying both sides of (8) (with \( F = 0 \)) by \( \varphi \) and integrating on \( \Omega \), we obtain
\[
-\frac{1}{2} \frac{d}{dt} |\varphi|^2 + a(0,\tilde{\varphi}) \int_\Omega \sum_{i,j=1}^n \alpha_{ij} \frac{\partial \varphi}{\partial y_i} \frac{\partial \varphi}{\partial y_j} \, dy = -\int_\Omega (\tilde{b}_0 y \nabla \varphi - c \varphi^2) \, dy.
\]
(16)
Now, using (6) and Young’s inequality, we rewrite (16) and obtain
\[
-\frac{1}{2} \frac{d}{dt} |\varphi|^2 \leq C |\nabla \varphi|^2 L_2(\Omega) \leq C |\varphi|^2 L_2(\Omega).
\]
(17)
Thus
\[
|\varphi(0)|^2_{L_2(\Omega)} \leq C |\varphi|^2_{L_2(\Omega)}.
\]
(18)
Employing (15) and (18) we finally obtain
\[
|\varphi(0)|^2_{L_2(\Omega)} = \frac{2}{T} \int_0^T \int_\Omega |\varphi(0)|^2 \, dy \, dt
\]
(19)
We will also need some Carleman inequalities for the solution to (8) with suitable weights that do not vanish at $t = 0$. To this end, let $m$ be a function satisfying

$$m \in C^\infty([0, T]), \ m(t) \geq \frac{T^2}{4} \quad \text{in} \ [0, T/2], \ m(t) = t(T - t) \quad \text{in} \ [T/2, T].$$

We consider

$$\theta(y, t) := \frac{e^{\lambda t_0(y)}}{m(t)}, \ \ A(y, t) := \frac{a_1(y)}{m(t)},$$

$$A^* := \max_{y \in \Omega} a_1(y), \ \ \hat{A} := \min_{y \in \Omega} a_1(y),$$

$$\theta^*(t) := \max_{y \in \Omega} \theta(y, t), \ \ \hat{\theta}(t) := \min_{y \in \Omega} \theta(y, t),$$

$$\bar{A} := 2\hat{A} - A^*.$$

**Remark 1.** From the definitions of $R, A^*$ and $\hat{A}$, it follows that

$$2\hat{A} - \frac{3}{2}A^* > 1 \quad \text{and} \quad 6\hat{A} - 5A^* > 1.$$  

First, we consider $A^* = e^{RL} - 1$ and $\hat{A} = e^{RL} - e^{\lambda\|\beta_0\|}$. Now, note that

$$2\hat{A} - \frac{3}{2}A^* > 1 \iff 4\hat{A} - 3A^* > 2$$

$$\iff 4(e^{RL} - e^{\lambda\|\beta_0\|}) - 3(e^{RL} - 1) > 2$$

$$\iff e^{RL} - 4e^{\lambda\|\beta_0\|} > -1.$$  

Since, $R > \|\beta_0\|_{L^\infty(\Omega)} + \ln 6$ one has

$$e^{RL} > e^{\lambda\|\beta_0\|} \cdot e^{\lambda \ln 6}$$

$$= 6e^{\lambda\|\beta_0\|} \cdot e^{\lambda}, \quad (\lambda \geq 1)$$

$$> 6e^{\lambda\|\beta_0\|}.$$  

Thus,

$$e^{RL} - 4e^{\lambda\|\beta_0\|} > 2e^{\lambda\|\beta_0\|} > -1.$$  

Analogously, we conclude that $6\hat{A} - 5A^* > 1$.

Now, we will use the following notation:

$$I(s, \lambda; \varphi) := \int_Q e^{-2sA}\left[(s\theta)^{-1}\left(|\varphi|^2 + \sum_{i,j=1}^n \left|\frac{\partial^2 \varphi}{\partial y_i \partial y_j}\right|^2\right)\right]$$

$$+ \lambda^2(s\theta)|\nabla \varphi|^2 + \lambda^4(s\theta)^3|\varphi|^2 \ dydt.$$  

One has the following:

**Proposition 2.** Under the assumptions of Proposition 1, there exist positive constants $\lambda_2, s_2$ such that, for any $\lambda \geq \lambda_2$ and $s \geq s_2$, there exists $C_2(s, \lambda)$ with the following property: for any $\varphi_T \in L^2(\Omega)$ the associated solution to (8) satisfies

$$I(s, \lambda; \varphi) \leq C_2(s, \lambda)\left(\int_Q e^{-2sA}|F|^2 \ dydt + \int_{\omega_1 \times (0, T)} e^{(-2sA)/m(\theta^*)^3}|\varphi|^2 \ dydt\right),$$

where $\lambda_2, s_2$ only depend on $\Omega, \omega, T, a(0, \bar{0}), |b|$ and $C_2(s, \lambda)$ only depend on these data $s$ and $\lambda$.  

$$\text{(20)}$$
Proof. We decompose the integrals in $I(s, \lambda; \varphi)$ in the form

$$\int_{Q} = \int_{\Omega \times (0,T/2)} + \int_{\Omega \times (T/2,T)}.$$

Let us gather together all the integrals in $\Omega \times (0, T/2)$ (resp. $\Omega \times (T/2, T)$) in $I_1(s, \lambda; \varphi)$ (resp. $I_2(s, \lambda; \varphi)$). Then,

$$I(s, \lambda; \varphi) = I_1(s, \lambda; \varphi) + I_2(s, \lambda; \varphi).$$

Let us start again from the Carleman inequality in Proposition 1, with $s_2 = s_0$, $\lambda_2 = \lambda_0$ such that $s \geq s_0$ and $\lambda \geq \lambda_0$. We have

$$I_2(s, \lambda; \varphi) = \int_{\Omega \times (T/2,T)} e^{-2sA} \left[ (s\theta)^{-1} \left( |\varphi_t|^2 + \sum_{i,j=1}^n |\frac{\partial^2 \varphi}{\partial y_i \partial y_j}|^2 \right) + \lambda^2(s\theta)|\nabla \varphi|^2 + \lambda^4(s\theta)^3 |\varphi|^2 \right] dydt$$

$$\leq \int_{Q} e^{-2s\alpha} \left[ (s\phi)^{-1} \left( |\varphi_t|^2 + \sum_{i,j=1}^n |\frac{\partial^2 \varphi}{\partial y_i \partial y_j}|^2 \right) + \lambda^2(s\phi)|\nabla \varphi|^2 + \lambda^4(s\phi)^3 |\varphi|^2 \right] dydt$$

$$\leq C \left( \int_{Q} e^{-2s\alpha}|F|^2 dydt + \int_{\omega_1 \times (0,T)} e^{-2s\alpha/3} \lambda^4(s\varphi)^3 |\varphi|^2 dydt \right)$$

$$\leq C \left( \int_{Q} e^{-2sA} |F|^2 dydt + \int_{\omega_1 \times (0,T)} e^{-(2sA)/m(s\phi)^3} |\varphi|^2 dydt \right).$$

Now, let us come back to the energy estimate for $\varphi$. We have the following for all $t \in (0, T/2)$:

$$-\frac{1}{2} \frac{d}{dt} |\varphi|^2_{L^2(\Omega)} + C |\nabla \varphi|^2_{L^2(\Omega)} \leq C (|\varphi|^2_{L^2(\Omega)} + |F|^2_{L^2(\Omega)}).$$

(22)

Multiplying (22) by $2e^{2Ct}$, we obtain

$$-\frac{d}{dt} (e^{2Ct} |\varphi|^2_{L^2(\Omega)}) + 2Ce^{2Ct} |\nabla \varphi|^2_{L^2(\Omega)} \leq 2Ce^{2Ct} |F|^2_{L^2(\Omega)}.$$

Integrating from $t_1$ to $t_2$, where $t_1 \in [0, T/2]$ and $t_2 \in [T/2, 3T/4]$, one has

$$|\varphi(t_1)|^2_{L^2(\Omega)} + \int_{t_1}^{t_2} |\nabla \varphi|^2_{L^2(\Omega)} dt \leq C \left( \int_{t_1}^{t_2} |F|^2_{L^2(\Omega)} dt + |\varphi(t_2)|^2_{L^2(\Omega)} \right).$$

So, it follows that

$$|\varphi(t_1)|^2_{L^2(\Omega)} \leq C \left( \int_{t_1}^{t_2} |F|^2_{L^2(\Omega)} dt + |\varphi(t_2)|^2_{L^2(\Omega)} \right)$$

(23)

and

$$\int_{t_1}^{t_2} |\nabla \varphi|^2_{L^2(\Omega)} dt \leq C \left( \int_{t_1}^{t_2} |F|^2_{L^2(\Omega)} dt + |\varphi(t_2)|^2_{L^2(\Omega)} \right).$$

(24)

Integrating (23) from 0 to $T/2$ with respect to $t_1$ and from $T/2$ to $3T/4$ with respect to $t_2$, we see that

$$\int_{0}^{T/2} |\varphi|^2_{L^2(\Omega)} dt \leq C \left( \int_{0}^{ST/4} |F|^2_{L^2(\Omega)} dt + \int_{T/2}^{3T/4} |\varphi|^2_{L^2(\Omega)} dt \right).$$

(25)
On the other hand, from (24) we deduce that
\[
\int_0^{T/2} |\nabla \varphi|^2_{L^2(\Omega)} dt \leq C \left( \int_0^{3T/4} |F|^2_{L^2(\Omega)} dt + \int_{T/2}^{3T/4} |\varphi|^2_{L^2(\Omega)} dt \right). \tag{26}
\]
From (25) and (26), we conclude that
\[
\int_{\Omega \times (0,T/2)} |\varphi|^2 + |\nabla \varphi|^2 \, dydt \leq C \left( \int_{\Omega \times (T/2,3T/4)} |\varphi|^2 + |\nabla \varphi|^2 \, dydt + \int_{\Omega \times (0,3T/4)} |F|^2 \, dydt \right). \tag{27}
\]
A second-order energy estimate can also be deduced for \( \varphi \):
\[
- \frac{1}{2} \frac{d}{dt} |\varphi|^2_{L^2(\Omega)} + \sum_{i,j=1}^n \left| \frac{\partial^2 \varphi}{\partial y_j \partial y_i} \right|^2 \leq C(|\varphi|^2_{L^2(\Omega)} + |F|^2_{L^2(\Omega)}), \tag{28}
\]
for all \( t \in (0,T/2) \). This leads to the following:
\[
\int_{\Omega \times (0,T/2)} \sum_{i,j=1}^n \left| \frac{\partial^2 \varphi}{\partial y_j \partial y_i} \right|^2 \, dydt \leq C \left( \int_{\Omega \times (T/2,3T/4)} (|\varphi|^2 + |\nabla \varphi|^2) \, dydt + \int_{\Omega \times (0,3T/4)} |F|^2 \, dydt \right). \tag{29}
\]
Finally, from PDE in (8), (27) and (29) yield:
\[
\int_{\Omega \times (0,T/2)} |\varphi|^2 \, dydt \leq C \left( \int_{\Omega \times (T/2,3T/4)} (|\varphi|^2 + |\nabla \varphi|^2) \, dydt + \int_{\Omega \times (0,3T/4)} |F|^2 \, dydt \right). \tag{30}
\]
From (27)-(29), we deduce that
\[
I_1(s, \lambda; \varphi) \leq C \left( \int_{\Omega \times (T/2,3T/4)} (|\varphi|^2 + \sum_{i,j=1}^n \left| \frac{\partial^2 \varphi}{\partial y_j \partial y_i} \right|^2 + |\nabla \varphi|^2 + |\varphi|^2) \, dydt + \int_{\Omega \times (0,3T/4)} |F|^2 \, dydt \right) \tag{31}
\]
whence
\[
I_1(s, \lambda; \varphi) \leq C(s, \lambda) \left( I(s, \lambda; \varphi) + \int_{\Omega \times (0,3T/4)} |F|^2 \, dydt \right) \leq C(s, \lambda) \left( \int_{\omega_1 \times (0,T)} e^{(-2sA)/m} (\theta^*)^3 |\varphi|^2 \, dydt + \int_{\Omega \times (0,3T/4)} |F|^2 \, dydt \right).
\]
Combining (21) with these inequalities, we obtain (20). \qed

In the sequel, we fix \( \lambda = \lambda_2, \ s = s_2 \) and we set
\[
\rho := e^{sA}, \quad \rho_0 := e^{sA} \theta^{-3/2}, \quad \rho_* := e^{(sA)/m} (\theta^*)^{-3/2},
\]
\[
\hat{\rho} := e^{(sA)/2m} e^{(sA)/m} (\theta^*)^{-3/2}, \quad \bar{\rho} := e^{s(4A - 3A^*)/m} (\theta^*)^{-5}.
\]
Then, we deduce from (20) that the solution to (8) satisfies:
\[
I(s, \lambda; \varphi) \leq C_2(s, \lambda) \left( \int_Q e^{-2sA} |F|^2 \, dydt + \int_{\omega_1 \times (0,T)} \rho_*^{-2} |\varphi|^2 \, dydt \right). \tag{32}
\]
Remark 2. From the definitions of the weights $\rho, \rho_0, \rho_*$ and $\hat{\rho}$, we verify
\[
\begin{align*}
&\rho_0 \leq \kappa_1 \rho, \quad \rho_* \leq \hat{\rho}, \\
&\hat{\rho} \leq \rho_0, \quad \hat{\rho}^2 \leq \frac{1}{4} \rho_*^2 + \varepsilon \rho_0^2, \\
&|\hat{\rho} \partial_t \hat{\rho}| \leq \kappa_2 \rho_0^2, \quad |\hat{\rho} \partial_y \hat{\rho}| \leq \kappa_3 \hat{\rho}^2, \\
&|\rho_*(\rho_*)_t| \leq \kappa_4 \rho_*^2,
\end{align*}
\]
where,
\[
\begin{align*}
\kappa_1 &= \max_{0 \leq t \leq T} m(t)^{3/2}, \\
\kappa_2 &= \left( \frac{s \Lambda}{2} \max_{0 \leq t \leq T} |m'| + \frac{3}{2} \max_{0 \leq t \leq T} |m'm| \right) e^{-(A^* - \tilde{A})} m^{-5/2}, \\
\kappa_3 &= \max_{y \in \Omega} s \lambda \epsilon \beta_0(y) \left| \frac{\partial \beta_0(y)}{\partial y} \right|, \\
\kappa_4 &= \left( \frac{s \Lambda}{2} \max_{0 \leq t \leq T} |m'| + \frac{3}{2} \max_{0 \leq t \leq T} |m'm| \right) e^{(s \Lambda - s \alpha_1(y))} m^{-5/2}.
\end{align*}
\]
In fact, let us denote
\[
\begin{align*}
\rho_0 &= e^{s \alpha_1(y)/m} \theta - \frac{3}{2} \\
&= e^{s \alpha_1(y)/m} \left( \frac{m}{e^{\Lambda \beta_0(y)}} \right)^{3/2} \\
&\leq \max_{0 \leq t \leq T} m^{3/2} e^{s \alpha_1(y)/m} \\
&= \kappa_1 \rho, \\
\rho_* &= e^{s \alpha_1(y)/m} \theta - \frac{3}{2} \\
&\leq e^{(s \alpha_1(y) + s \Lambda)/2m(\theta^*)} - \frac{3}{2}, \\
\hat{\rho} &= e^{(s \alpha_1(y) + s \Lambda)/2m(\theta^*)} - \frac{3}{2} \\
&= e^{s \alpha_1(y)/m} \left( \frac{m}{e^{\Lambda \beta_0(y)}} \right)^{3/2} \\
&\leq e^{s \alpha_1(y)/m} \theta - \frac{3}{2} = \rho_0, \\
\hat{\rho}^2 &\leq \frac{1}{4} e^{s \Lambda/m(\theta^*)} - \frac{3}{2} + e^{s \alpha_1(y)/m(\theta^*)} - \frac{3}{2} \\
&\leq \frac{1}{4} e^{2s \Lambda/m(\theta^*)} - \frac{3}{2} + e^{s \alpha_1(y)/m(\theta^*)} - \frac{3}{2} \\
&\leq \frac{1}{4} e^{2s \Lambda/m(\theta^*)} - \frac{3}{2} + e^{2s \alpha_1(y)/m(\theta^*)} - \frac{3}{2} = \frac{1}{4} \rho_*^2 + \varepsilon \rho_0^2,
\end{align*}
\]
Assume that the function $k$ for the right hand side $\rho$ satisfies
\[ \max_{0 \leq t \leq T} e^{\lambda_0(y) t} \leq \frac{1}{\max_{0 \leq t \leq T} e^{\lambda_0(y) t}} \]

Then (7) is null controllable. More precisely, for any $v_0 \in L^2(\Omega)$, there exist controls $\hat{h} \in L^2(\omega \times (0, T))$ such that the state-control $(v, \hat{h})$ satisfies
\[ \int_{\omega \times (0, T)} \rho_0^2 |\hat{h}|^2 dydt \leq C \left( |v_0|_{L^2(\Omega)}^2 + \int_{Q} \rho_0^2 |k|^2 dydt \right), \tag{33} \]

where, in particular, $v(y, T) = 0$ in $\Omega$.

**Proof.** Here we will use well known ideas from the work by Fursikov and Imanuvilov [17].

For each $n \geq 1$, let us introduce the functions
\[ A_n := \frac{A(T-t)}{(T-t) + 1/n}, \quad \theta_n := \frac{\theta(T-t)}{(T-t) + 1/n}, \quad \rho_n := e^{sA_n}, \quad \rho_0, n := \rho_0 \theta_n^{-3/2} \text{ and} \]

Thanks to Proposition 2, we will be able to prove the null controllability to (7) for the right hand side $k$ that decay fast to zero as $t \to T$. Indeed, one has the following:

**Theorem 3.2.** Assume that the function $k$ satisfy
\[ \int_{Q} \rho_0^2 |k|^2 dydt < +\infty. \]

Then (7) is null controllable. More precisely, for any $v_0 \in L^2(\Omega)$, there exist controls $\hat{h} \in L^2(\omega \times (0, T))$ such that the state-control $(v, \hat{h})$ satisfies
\[ \int_{Q} \rho_0^2 |\hat{h}|^2 dydt \leq C \left( |v_0|_{L^2(\Omega)}^2 + \int_{Q} \rho_0^2 |k|^2 dydt \right), \]

where, in particular, $v(y, T) = 0$ in $\Omega$. 

For each $n \geq 1$, let us introduce the functions
\[ A_n := \frac{A(T-t)}{(T-t) + 1/n}, \quad \theta_n := \frac{\theta(T-t)}{(T-t) + 1/n}, \quad \rho_n := e^{sA_n}, \quad \rho_0, n := \rho_0 \theta_n^{-3/2} \text{ and} \]
\( \rho_{v,n} := \rho_v \cdot m_n = e^{(s\overline{\lambda})/m} (\theta^*)^{-3/2} \cdot m_n \) where \( m_n = \begin{cases} 1 & \text{in } \omega \\ n & \text{in } \Omega - \overline{\omega} \end{cases} \)

and the functional \( J_n : L^2(Q) \times L^2(\omega \times (0,T)) \rightarrow \mathbb{R} \), with

\[
J_n(v, \tilde{h}) := \frac{1}{2} \int_Q \left( \rho_{0,n}^2 |v|^2 + \rho_{*n}^2 |\tilde{h}|^2 \right) \, dydt.
\]

Let us consider the following extremal problem:

\[
\begin{cases}
  v_{n,t} + a(0,0)A(t)v_n + \tilde{b} \cdot \nabla v_n + cv_n = \tilde{h}_n \mathbf{1}_\omega + k & \text{in } Q, \\
  v_n = 0 & \text{on } \Sigma, \\
  v_n(y,0) = v_0(y) & \text{in } \Omega, \\
  -p_{n,t} + a(0,0)A^*(t)p_n - \nabla \cdot (\tilde{b} p_n) + cp_n = -\rho_{0,n}^2 v_n & \text{in } Q, \\
  p_n = 0 & \text{on } \Sigma, \\
  p_n(y,T) = 0 & \text{in } \Omega,
\end{cases}
\]

\[ p_n \mathbf{1}_\omega = -\rho_{*n}^2 \tilde{h}_n \text{ in } Q. \tag{36} \]

Multiplying (35) by \( v_n \) and integrating in \( Q \), we obtain

\[
0 = \int_Q \left[ -p_{n,t} + a(0,0)A^*(t)p_n - \nabla \cdot (\tilde{b} p_n) + cp_n + \rho_{0,n}^2 v_n \right] v_n \, dydt.
\]

Integrating by parts, we see that

\[
\int_Q \rho_{0,n}^2 |v_n|^2 \, dydt = \int_Q \left[ v_{n,t} + a(0,0)A(t)v_n + \tilde{b} \cdot \nabla v_n + cv_n \right] p_n \, dydt
\]

\[ + \int_{\Omega} p_n(y,0) v_0(y) \, dy. \tag{37} \]

From (37), taking into account (36) and the PDE from (34), and recalling the definition of \( J_n \), we find that

\[
J_n(v_n, \tilde{h}_n) = -\frac{1}{2} \left( \int_Q p_n k \, dydt + \int_{\Omega} p_n(y,0) v_0(y) \, dy \right).
\]

Consequently,

\[
J_n(v_n, \tilde{h}_n) \leq \frac{1}{2} \left[ \left( \int_Q \rho^{-2}\theta^3 |p_n|^2 \, dydt \right)^{1/2} \left( \int_Q \rho_{0,n}^2 |k|^2 \, dydt \right)^{1/2}
\]

\[ + |p_n(y,0)|_{L^2(\Omega)} |v_0|_{L^2(\Omega)} \right]^2 \tag{38} \]

\[
\leq \frac{1}{2} \left[ |p_n(y,0)|_{L^2(\Omega)}^2 + \int_Q \rho^{-2}\theta^3 |p_n|^2 \, dydt \right]^{1/2}
\]

\[ \cdot \left[ |v_0|^2_{L^2(\Omega)} + \int_Q \rho_{0,n}^2 |k|^2 \, dydt \right]^{1/2}. \]

Let us apply the Carleman inequality (20) to the solution \( p_n \) to (35) and use the facts that
\[ \rho_n \leq \rho, \quad \rho_{0,n} \leq C\rho \quad \text{and} \quad \rho_{*,n} = \rho_* \cdot m_n = e^{(s\overline{\lambda})/m(\theta^*)^{-3/2}} \cdot m_n, \]
it follows that,

\[
\int_Q \rho^{-2} \theta^3 |p_n|^2 \, dydt + \int_Q \rho^{-2} \theta^{-1} |p_{n,t}|^2 \, dydt \\
\leq \tilde{C}(s, \lambda) \left( \int_Q \rho^{-2} \rho_{0,n}^d |v_n|^2 \, dydt + \int_{\omega_1 \times (0,T)} e^{-2(s\overline{\lambda})/m(\theta^*)^3} |p_n|^2 \, dydt \right) \\
\leq \tilde{C}(s, \lambda) \left( \int_Q \rho_0^2 \rho_{0,n}^d |v_n|^2 \, dydt + \int_{\omega \times (0,T)} \rho_*^{-2} \rho_*^d |\tilde{h}_n|^2 \, dydt \right) \\
\leq \tilde{C}(s, \lambda) \left( \int_Q \rho_0^2 |v_n|^2 \, dydt + \int_{\omega \times (0,T)} \rho_*^2 |\tilde{h}_n|^2 \, dydt \right) \\
\leq C J_n(v_n, \tilde{h}_n).
\]

We also have

\[
|p_n(\cdot, 0)|^2_{L^2(\Omega)} \leq C J_n(v_n, \tilde{h}_n). \tag{40}
\]

Indeed, since

\[
|(\rho^{-1} \theta^{-1/2} p_n)_t|_{L^2(Q)} \leq |(\rho^{-1} \theta^{-1/2} p_{n,t})|_{L^2(Q)} + C |(\rho^{-1} \theta^{3/2} p_n)|_{L^2(Q)},
\]
from (39) one has

\[
(\rho^{-1} \theta^{-1/2} p_n)_t, \quad \rho^{-1} \theta^{-1/2} p_n \in L^2(0, T; L^2(\Omega))
\]

and

\[
|(\rho^{-1} \theta^{-1/2} p_n)_t|_{L^2(\Omega)} + |\rho^{-1} \theta^{-1/2} p_n|_{L^2(\Omega)} \leq C \sqrt{J_n(v_n, \tilde{h}_n)}.
\]

Therefore,

\[
\rho^{-1} \theta^{-1/2} p_n \in C(0, T; L^2(\Omega)) \quad \text{and} \quad |(\rho^{-1} \theta^{-1/2} p_n)(t)|_{C(0, T; L^2(\Omega))} \leq C \sqrt{J_n(v_n, \tilde{h}_n)}.
\]

So, we obtain

\[
|(|\rho^{-1} \theta^{-1/2} p_n)(0)|_{L^2(\Omega)} \leq C \sqrt{J_n(v_n, \tilde{h}_n)}
\]

and, we conclude (40).

Then, from (39)-(40)

\[
|p_n(\cdot, 0)|^2_{L^2(\Omega)} + \int_Q \rho^{-2} \theta^3 |p_n|^2 \, dydt \leq C J_n(v_n, \tilde{h}_n). \tag{41}
\]

From (38) and (41) one has

\[
J_n(v_n, \tilde{h}_n) \leq C \left( |v_0|^2_{L^2(\Omega)} + \int_Q \rho_0^2 |k|^2 \, dydt \right).
\]

So, we have the following estimate

\[
\int_Q \rho_{0,n}^2 |v_n|^2 \, dydt + \int_{\omega \times (0,T)} \rho_{*,n}^2 |\tilde{h}_n|^2 \, dydt \\
\leq C \left( |v_0|^2_{L^2(\Omega)} + \int_Q \rho_0^2 |k|^2 \, dydt \right), \tag{42}
\]

whence we can extract subsequences indexed by \( n \) satisfying

\[
\begin{aligned}
\rho_{0,n} v_n \rightharpoonup v_1 \quad \text{in} \quad L^2(Q) \\
\rho_{*,n} \tilde{h}_n \rightharpoonup v_2 \quad \text{in} \quad L^2(Q).
\end{aligned}
\]
It is routine calculation to check that $\nu_1 = \rho_0 v$ and $\nu_2 = \rho_1 \hat{h}_1 \omega$ for some $v$ and $\hat{h}$. So, from (42), we can take limits in the linear system (34) and deduce that $v$ is the state associated to $\hat{h}$, i.e.

$$\begin{cases}
v_t + a(0, \vec{0})A(t)v + \vec{b} \cdot \nabla v + cv = \hat{h}_1 \omega + k & \text{in } Q,
v = 0 & \text{on } \Sigma, 
v(y, 0) = v_0(y) & \text{in } \Omega,
\end{cases}$$

and also, that

$$\int_Q \rho_0^2 |v|^2 \, dy \, dt \leq \liminf \int_Q \rho_{0,n}^2 |v_n|^2 \, dy \, dt \leq C \left( |v_0|^2_{L^2(\Omega)} + \int_Q |\rho_0^2|^2 |k|^2 \, dy \, dt \right),$$

$$\int_{\omega \times (0, T)} \rho_{0,n}^2 |\hat{h}_n|^2 \, dy \, dt \leq \liminf \int_{\omega \times (0, T)} \rho_n^2 |\hat{h}_n|^2 \, dy \, dt \leq C \left( |v_0|^2_{L^2(\Omega)} + \int_Q \rho_0^2 |k|^2 \, dy \, dt \right).$$

Thus, $v(y, T) = 0$ in $\Omega$. This ends the proof.

3.1. Some additional estimates. In this section, the results provide additional properties of the control and the state found in Theorem 3.2.

They will be needed below, in Section 4.

**Remark 3.** Let $\tilde{\rho} = e^{-(sA^* / m)(\theta^*)^{7/2}}$, we verify

- $\tilde{\rho} \leq C \rho_{0,n}^{-1}$,
- $\tilde{\rho}_t \leq C \tilde{\rho}_{0,n}^{-1}$.

In fact, note that

- $\rho_{0,n} = e^{sA_{n}} \theta_{n}^{-3/2}$
  $$\leq e^{sA_{n}} \left( \frac{(T - t) + 1/n}{\theta(t) (T - t)} \right)^{3/2} \leq Ce^{sA_{n} (\theta^*)^{7/2}} \leq Ce^{(sA^*) / m (\theta^*)^{7/2}} = C(\tilde{\rho})^{-1},$$
- $\tilde{\rho}_t = e^{-(sA^*) / m} m' \left( \frac{m}{e^{\lambda\|B_0\|}} \right)^{7/2} + \frac{1}{(e^{\lambda\|B_0\|})^{7/2}} \cdot \frac{7}{2} \cdot m' m^{5/2}$
  $$\leq Ce^{-sA_{n}} \left( \frac{m}{e^{\lambda\|B_0\|}} \right)^{3/2} \leq Ce^{-sA_{n}} \theta_{n}^{-3/2} = C \rho_{0,n}^{-1}.$$
Proposition 3. Let $\overline{p} = e^{s(4\overline{A} - 3A^4)/m}(\theta^*)^{-5}$. Then one has
\[
\overline{p}h \in V = L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \cap L^\infty(0, T; H^1_0(\Omega)) \quad \text{and} \quad |\overline{p}h|^r \leq C(|v_0|_{L^2(\Omega)} + |\rho_0k|_{L^2(Q)}) \tag{43}
\]
\[
|\overline{p}h|^p \leq C(|v_0|_{L^2(\Omega)} + |\rho_0k|_{L^2(Q)}) \tag{44}
\]

Proof. Arguing as the proof of Theorem 3.2 to $\overline{p}p_n$ in (35), we obtain:
\[
\begin{cases}
-\overline{p}p_n + a(0, 0)A^*(t)(\overline{p}p_n) - \nabla \cdot (\overline{p}p_n) + c\overline{p}p_n = 0 & \text{in } Q,
\overline{p}p_n = 0 & \text{on } \Sigma,
\overline{p}p_n(0, T, T) = 0 & \text{in } \Omega,
\end{cases}
\tag{45}
\]
From the Remark 3, we have
\[
|\overline{p}p_n|^2 \leq C|\rho_0p_n|.
\]
So, from (42) we obtain
\[
|\overline{p}p_n|^2 \in L^2(Q) \quad \text{and}
|\overline{p}p_n|^2|_{L^2(Q)} \leq C\left(|v_0|_{L^2(\Omega)} + \left(\int Q |\rho_0^2|k|^2 \, dydt\right)^{1/2}\right) \tag{46}
\]
Once again, using Remark 3 one has
\[
|\overline{p}p_n|^2 \leq |\rho_0^{-1}p_n|.
\]
Now, from Carleman estimate to (35) one has
\[
|\rho_0^{-1}p_n|^2 \leq C\left(\int Q |\rho_0|\rho_0^4|v_n|^2 \, dydt + \int \omega \times (0, T) \rho_0^2|p_n|^2 \, dydt\right)
\leq C\left(\int Q \rho_0^2|v_n|^2 \, dydt + \int \omega \times (0, T) \rho_0^2|\hat{p}_n|^2 \, dydt\right)
\leq C\left(\int Q \rho_0^2|v_n|^2 \, dydt + \int \omega \times (0, T) \rho_0^2|\hat{h}_n|^2 \, dydt\right).
\tag{47}
\]
Then, from (42)
\[
\rho_0^{-1}p_n \in L^2(Q) \quad \text{and}
|\rho_0^{-1}p_n|^2 \leq C\left(|v_0|_{L^2(\Omega)} + \left(\int Q |\rho_0^2|k|^2 \, dydt\right)^{1/2}\right) \tag{48}
\]
In view of (45), (46), (47) and parabolic regularity, we have
\[
\overline{p}p_n \in L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \cap L^\infty(0, T; H^1_0(\Omega))
\]
From (36) one has
\[
\overline{p}_n \hat{h}_n = \overline{p}p_n \overline{t}_n, \quad \text{where} \quad \overline{p}_n = \overline{p} \cdot m_n^2.
\]
Therefore,
\[
\overline{p}_n \hat{h}_n \in L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \cap L^\infty(0, T; H^1_0(\Omega)) \quad \text{and}
|\overline{p}_n \hat{h}_n|_{L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega))} + |\overline{p}_n \hat{h}_n|_{L^\infty(0, T; H^1_0(\Omega))} \leq C(|v_0|_{L^2(\Omega)} + |\rho_0k|_{L^2(Q)})
\]
So, we can extract subsequences indexed by $n$ satisfying
\[
\begin{cases}
\overline{p}_n \rightarrow \overline{p} \text{ in } L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega)),
\overline{p}_n \hat{h}_n \rightarrow \overline{p}h \text{ in } L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega)),
\overline{p}_n \hat{h}_n \rightarrow \overline{p}h \text{ in } L^\infty(0, T; H^1_0(\Omega))
\end{cases}
\]
and also,

\[ |\tilde{p}h|_{L^2(0,T;H_0^2 \cap H^2)} \leq \lim \inf |\tilde{p}_n h_n|_{L^2(0,T;H_0^2 \cap H^2)} \leq C \left( |v_0|_{L^2(\Omega)} + |\rho_0 k|_{L^2(Q)} \right), \]

\[ |\tilde{p}h|_{L^\infty(0,T;H_0^1)} \leq \lim \inf |\tilde{p}_n h_n|_{L^\infty(0,T;H_0^1)} \leq C \left( |v_0|_{L^2(\Omega)} + |\rho_0 k|_{L^2(Q)} \right). \]

**Proposition 4.** Let the hypotheses in Theorem 3.2 be satisfied and let \( v \) and \( \tilde{h} \) satisfy (7) and (33). Then one has

\[
\int_Q \tilde{\rho}^2 |\nabla v|^2 \, dy \, dt \leq C \int_Q \rho_0^2 |v|^2 \, dy \, dt + \int_{\omega \times (0,T)} \rho_0^2 |\tilde{h}|^2 \, dy \, dt \\
+ C |v_0|^2_{L^2(\Omega)} + C \int_Q \rho_0^2 |k|^2 \, dy \, dt .
\]

**Proof.** Let us multiply (7) by \( \tilde{\rho}^2 v \) and let us integrate in \( \Omega \). We obtain:

\[
\int_\Omega \tilde{\rho}^2 (v_t + a(0,\tilde{0}) A(t)v) \, dv = - \int_\Omega \tilde{\rho}^2 (\tilde{b} \cdot \nabla v + cv - \tilde{h}_1 \omega - k)v \, dv .
\]

From Remark 2 and (6), we obtain

\[
\left| \int_\Omega \tilde{\rho}^2 (b \cdot \nabla v + cv) v \, dv \right| \leq \frac{C(\tilde{b},c)}{4\varepsilon} \int_\Omega \tilde{\rho}^2 |v|^2 \, dy + \varepsilon \alpha_0^2 \int_\Omega \tilde{\rho}^2 |\nabla v|^2 \, dy ,
\]

\[
\int_\Omega \tilde{\rho}^2 v_t v \, dy = \frac{1}{2} \frac{d}{dt} \int_\Omega \tilde{\rho}^2 |v|^2 \, dy - \int_\Omega \tilde{\rho}^2 |\tilde{h}|^2 \, dy ,
\]

\[
\int_\Omega \tilde{\rho}^2 \tilde{h}_1 \omega v \, dy \leq \frac{1}{4\varepsilon} \int_{\omega} \rho_0^2 |\tilde{h}|^2 \, dy + \varepsilon \int_\Omega \rho_0^2 |v|^2 \, dy ,
\]

\[
\int_\Omega \tilde{\rho}^2 (a(0,\tilde{0}) (A(t)v)) \, dv = \int_\Omega \tilde{\rho}^2 ((a(0,\tilde{0})) \alpha_i j \frac{\partial v}{\partial y_i} \frac{\partial v}{\partial y_j}) \\
+ 2 \int_\Omega a(0,\tilde{0}) \tilde{\rho} \frac{\partial \tilde{\rho}}{\partial y_i} \left( \alpha_i j \frac{\partial v}{\partial y_i} \frac{\partial v}{\partial y_j} \right) \, dy
\]

\[
2 \int_\Omega a(0,\tilde{0}) \tilde{\rho} \frac{\partial \tilde{\rho}}{\partial y_i} \left( \alpha_i j \frac{\partial v}{\partial y_i} \frac{\partial v}{\partial y_j} \right) \, dy \leq \frac{C(A)}{4\varepsilon} \int_\Omega \tilde{\rho}^2 |v|^2 \, dy + \varepsilon \alpha_0^2 \int_\Omega \tilde{\rho}^2 |\nabla v|^2 \, dy ,
\]

and

\[
\int_\Omega \tilde{\rho}^2 k v \, dy \leq \frac{1}{2} C \int_\Omega \tilde{\rho}^2 |k|^2 \, dy + \frac{1}{2} \int_\Omega \tilde{\rho}^2 |v|^2 \, dy .
\]

Therefore, from (37) and taking \( 2\varepsilon = \frac{1}{2} \), the following is deduced:

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \tilde{\rho}^2 |v|^2 \, dy + \frac{\alpha_0^2}{2} \int_\Omega \tilde{\rho}^2 (a(0,\tilde{0}) |\nabla v|^2) \, dv \\
\leq C |v_0|^2_{L^2(\Omega)} + \frac{1}{2} C \int_\omega \rho_0^2 |k|^2 \, dy \\
+ C \int_\Omega \rho_0^2 |v|^2 \, dy + C \int_\omega \rho_0^2 |\tilde{h}|^2 \, dy .
\]

Integrating the last estimate in time and using Gronwall inequality, we get the desired result. \( \square \)
Proposition 5. Let the hypotheses in Theorem 3.2 be satisfied and let \( \hat{h} \) and \( v \) be the control and the associated state furnished by this result and let us assume that \( v_0 \in H^1_0(\Omega) \).

Then one has

\[
\begin{align*}
\int_Q \rho^2 (|v_t|^2 + \sum_{i,j=1}^n \left| \frac{\partial^2 v}{\partial y_j \partial y_i} \right|^2) \, dydt &+ \sup_{t \in [0,T]} \int_{\Omega} \rho^2 |\nabla v|^2 \, dy \\
\leq C \int_Q \rho_0^2 |v|^2 \, dydt &+ C \int_{\Omega \times (0,T)} \rho^2 |\hat{h}|^2 \, dydt \\
&+ C \|v_0\|^2_{H^1_0(\Omega)} + C \int_Q \rho_0^2 |k|^2 \, dydt.
\end{align*}
\]

(50)

Proof. Let us multiply (7) by \( \rho^2 v_t \) and let us integrate in \( \Omega \). The following holds:

\[
\begin{align*}
\frac{1}{2} \int_{\Omega} \rho^2 |v_t|^2 \, dy &+ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^2 a(0,\vec{0}) |\nabla v|^2 \, dy \\
\leq C \int_{\Omega} (\rho^2_{i,\omega}) |\nabla v|^2 \, dy &+ C \int_{\Omega} \rho^2 |v|^2 \, dy \\
&+ C \int_{\omega} \rho^2 |k|^2 \, dy.
\end{align*}
\]

(51)

From the Remark 2, we have that the function into parentheses in the first integral in the right hand side is bounded by \( C \rho^2 \). Consequently, one has

\[
\begin{align*}
\frac{1}{2} \int_{\Omega} \rho^2 |v_t|^2 \, dy &+ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^2 a(0,\vec{0}) |\nabla v|^2 \, dy \\
\leq C \int_{\Omega} \rho^2 |\nabla v|^2 \, dy &+ C \int_{\Omega} \rho^2 |v|^2 \, dy \\
&+ C \int_{\omega} \rho^2 |k|^2 \, dy.
\end{align*}
\]

Integrating in time and recalling (49) and (48), we get the desired estimate for \( |v_t|^2 \).

In order to prove the same estimate for \( \sum_{i,j=1}^n \left| \frac{\partial^2 v}{\partial y_j \partial y_i} \right|^2 \), let us multiply (7) by \( -\rho^2 \Delta v \). After integration in \( \Omega \), we have,

\[
\begin{align*}
\frac{1}{2} \int_{\Omega} \rho^2 a(0,\vec{0}) \sum_{i,j=1}^n \left| \frac{\partial^2 v}{\partial y_j \partial y_i} \right|^2 \, dy &+ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^2 |\nabla v|^2 \, dy \\
\leq C \int_{\Omega} (\rho^2_{i,\omega}) |\nabla v|^2 \, dy &+ C \int_{\Omega} \rho^2 |v|^2 \, dy \\
&+ C \int_{\omega} \rho^2 |k|^2 \, dy.
\end{align*}
\]

From the definitions of \( \hat{\rho} \) and \( \rho_* \), it is clear that the function between parentheses in the first integral in the right hand side is bounded by \( C \hat{\rho}^2 \). Consequently,

\[
\begin{align*}
\frac{1}{2} \int_{\Omega} \rho^2 a(0,\vec{0}) \sum_{i,j=1}^n \left| \frac{\partial^2 v}{\partial y_j \partial y_i} \right|^2 \, dy &+ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^2 |\nabla v|^2 \, dy \\
\leq C \int_{\Omega} (\hat{\rho}^2_{i,\omega}) |\nabla v|^2 \, dy &+ C \int_{\Omega} \hat{\rho}^2 |v|^2 \, dy \\
&+ C \int_{\omega} \hat{\rho}^2 |k|^2 \, dy.
\end{align*}
\]
\[
\leq C \int_{\Omega} \rho^2 |\nabla v|^2 \, dy + C \int_{\Omega} \rho^*_v |v|^2 \, dy \\
+ C \int_{\omega} \rho^2_* |\tilde{h}|^2 \, dy + C \int_{\Omega} \rho^2_0 |k|^2 \, dy.
\]

Integrating in time and recalling again (49) and (48), we get the desired estimates for
\[
\sum_{i,j=1}^{n} \left| \frac{\partial^2 v}{\partial y_j \partial y_i} \right|^2.
\]

4. The local null controllability of the nonlinear problem (1). The aim of this section is to prove Theorem 1.1. For this, we will use the Liusternik’s Inverse Function Theorem.

Let us start by defining the following function spaces:

\[
Y := \left\{(v, \tilde{h}) : \tilde{h} \in L^2(\omega \times (0,T)), \int_{\omega \times (0,T)} \rho^2_* |\tilde{h}|^2 \, dydt < +\infty, \right. \\
v, \partial_t v, v_t + a(0, \vec{0})A(t)v, v \in L^2(Q), \int_Q \rho^2_0 |v|^2 \, dydt < +\infty, \\
\left. \int_Q \rho^2_0 |v_t + a(0, \vec{0})A(t)v + \vec{b} \cdot \nabla v - \tilde{h}1_\omega|^2 \, dydt < +\infty, v(\cdot, 0) \in H^1_0(\Omega) \right\},
\]

\[
G := \left\{k \in L^2(Q) : \int_Q \rho^2_0 |k|^2 \, dydt < +\infty \right\}
\]

and

\[
Z := G \times H^1_0(\Omega).
\]

We consider these spaces with the following Hilbertian norms:

\[
\|(v, \tilde{h})\|_Y^2 := \int_Q \rho^2_0 |v|^2 \, dydt + \int_{\omega \times (0,T)} \rho^2_* |\tilde{h}|^2 \, dydt \\
+ \int_Q \rho^2_0 \left(|v_t + a(0, \vec{0})A(t)v + \vec{b} \cdot \nabla v - \tilde{h}1_\omega|^2 \right) \, dydt \\
+ \|(v(\cdot, 0))\|_{H^1_0(\Omega)}^2,
\]

\[
\|k\|_G^2 := \int_Q \rho^2_0 |k|^2 \, dydt
\]

and

\[
\|(k, v(\cdot, 0))\|_Z^2 := \|k\|_G^2 + \|v(\cdot, 0)\|_{H^1_0(\Omega)}^2.
\]

Let us consider the mapping \( H : Y \rightarrow Z \) with

\[
H(v, \tilde{h}) = (H_1, H_2)(v, \tilde{h}),
\]

\[
H_1(v, \tilde{h}) = v_t + a \left( \int_{\Omega} v |J| \, dy, \int_{\Omega} (\nabla v \cdot \nabla \psi) |J| \, dy \right) A(t)v + \\
+ \vec{b} \cdot \nabla v + f(v) - \tilde{h}1_\omega (1 + v), \\
H_2(v, \tilde{h}) := v(\cdot, 0). \tag{51}
\]

We will prove that there exists \( \varepsilon > 0 \) such that, if \((k, v_0) \in Z \) and \( \|(k, v_0)\|_Z \leq \varepsilon \), then the equation

\[
H(v, \tilde{h}) = (k, v_0), \quad (v, \tilde{h}) \in Y,
\]
possesses at least one solution.

In particular, this show that (1) is locally null controllable and, furthermore, the state-control pair can be chosen in \( Y \).

To show that Theorem 2.1 can be applied in this setting, we will use several lemmas.

First, let us prove that the definition of \( H \) is correct.

**Lemma 4.1.** Let \( H : Y \to Z \) be the mapping defined by (51) and (52). Then \( H \) is well defined and continuous.

**Proof.** For \( (v, \tilde{h}) \in Y \), let us see that \( H_1(v, \tilde{h}) \) make sense and belongs to \( G \), and also, that \( H_2(v, \tilde{h}) \) make sense and belongs to \( H^0_0(\Omega) \).

Since \( f \) is Lipschitz-continuous, for any \( (v, \tilde{h}) \in Y \), we have

\[
\int \int \rho^2_0 |H_1(v, \tilde{h})|^2 \, dy \, dt = \int \int \rho_0^2 |v_t + a \left( \int_{\Omega} |J| \, dy, \int_{\Omega} (\nabla v \cdot \nabla \psi) |J| \, dy \right) A(t) v \\
+ b \cdot \nabla v + f(v) - \tilde{h}1_\omega (1 + v)|^2 \, dy \, dt \\
\leq C \left[ \int \int \rho_0^2 |v_t + a(0, \tilde{0}) A(t) v + b \cdot \nabla v - \tilde{h}1_\omega|^2 \, dy \, dt \\
+ C \left[ \int \int \rho_0^2 |a| \left( \int_{\Omega} |J| \, dy, \int_{\Omega} (\nabla v \cdot \nabla \psi) |J| \, dy \right) \right. \\
\left. + a(0, \tilde{0}) \right|^2 |A(t)v|^2 \, dy \, dt \\
+ C \int \int \rho_0^2 |v|^2 \, dy \, dt + C \int \int \rho_0^2 v \, dy \, dt \right] \\
= A_1 + A_2 + A_3 + A_4.
\]

From the definition of the space \( Y \),

\[
A_1 = \int \int \rho_0^2 |v_t + a(0, \tilde{0}) A(t) v + b \cdot \nabla v - \tilde{h}1_\omega|^2 \, dy \, dt \leq C\|(v, \tilde{h})\|_Y^2 < +\infty
\]

and

\[
A_3 = \int \int \rho_0^2 |v|^2 \, dy \, dt \leq C\|(v, \tilde{h})\|_Y^2 < +\infty.
\]

Now, let us analyze \( A_2 \). Since \( a \) is \( C^1 \) and globally Lipschitz-continuous, one has:

\[
A_2 \leq \int \int \rho_0^2 \left[ \left( \int_{\Omega} |J| \, dy \right)^2 + \left( \int_{\Omega} (\nabla v \cdot \nabla \psi) |J| \, dy \right)^2 \right] |A(t)v|^2 \\
= J_1 + J_2.
\]

From Proposition 5, we know that

\[
\int \int \rho_*^2 \sum_{i,j=1}^n \left| \frac{\partial^2 v}{\partial y_i \partial y_j} \right|^2 \, dy \, dt < +\infty \quad \text{and} \quad \sup_{t \in [0, T]} \int_{\Omega} \rho_*^2 |\nabla v|^2 \, dy < +\infty,
\]

where \( \rho_* = \frac{C_0 e^{(\lambda T)/m} \lambda^{3/2}}{C_0 e^{(2sA^* - A^*)/m} \lambda^{3/2}} \), \( C_0 = e^{-(3/2)\lambda \|\beta_0\|_\infty} \).

Then, in order to prove \( H_1 \) is well defined, we must have to obtain that \( A_2 < +\infty \).

In this case, from (54), we just have to demonstrate that \( J_2 < +\infty \) (\( J_1 \) is similar).
which is true since, from Remark 1 one has

\[ - \text{ (as mentioned in Session 3.1).} \]

and for this objective, we must have

\[ \text{To achieve our goal, we have to prove that} \]

\[ I = e^{(2sA^*)/m}m^3e^{(-8s\tilde{A}+4sA^*)/m}m^{-6} \left( \int_{\Omega} \rho_2^2 |\nabla v|^2 dy \right) < +\infty \]

and for this objective, we must have

\[ e^{(2sA^*)/m}m^3e^{(-8s\tilde{A}+4sA^*)/m}m^{-6} \leq C, \]

which is true since, from Remark 1 one has \(-8s\tilde{A}+6sA^* < -1\).

Next, let us analyze \( A_4 \). This term involve the multiplicative control \( \tilde{h}v \) (as mentioned in Session 3.1).

So, from Proposition 3, Proposition 5 and Remark 1 one has

\[ A_4 \leq \int_0^T \int_{\Omega} \rho_2^2 \tilde{1}_\omega |\tilde{h}v|^2 dydt \]

\[ \leq \int_0^T \int_{\Omega} e^{2sA\theta^{-3}} \tilde{1}_\omega |\tilde{h}|^2 |v|^2 dydt \]

\[ \leq C \int_0^T \int_{\Omega} e^{(2sA^*)/m(\theta^*)^{-3}} \tilde{1}_\omega |\tilde{h}|^2 |v|^2 dydt \]

\[ \leq C \int_0^T \int_{\Omega} e^{(2sA^*)/m(\theta^*)^{-3}} |\tilde{h}|_L^2(\Omega) |v|_L^2(\Omega) dt \]

\[ \leq C \int_0^T \int_{\Omega} e^{(2sA^*)/m(\theta^*)^{-3}} |\nabla \tilde{h}|_L^2(\Omega) |\nabla v|_L^2(\Omega) dt \]

\[ \leq C \int_0^T \int_{\Omega} e^{(2sA^*)/m(\theta^*)^{-3}} |\rho_2^{-2}(\tilde{p})|^{-2} \left( \tilde{p}^2 |\nabla \tilde{h}|_L^2(\Omega) \right) \left( \rho_2^2 |v|^2 |L^2(\Omega) \right) dt \]

\[ \leq C \int_0^T \int_{\Omega} e^{(2sA^*)/m(\theta^*)^{-3}} e^{-2s(2\tilde{A}-A^*)/m(\theta^*)} \cdot e^{-2s(4\tilde{A}-3A^*)/m(\theta^*)} \left( \tilde{p}^2 |\nabla \tilde{h}|_L^2(\Omega) \right) \left( \rho_2^2 |v|^2 |L^2(\Omega) \right) dt \]
compute the G-derivative $H$

Let us first prove that $H$ is G-differentiable at any $(v, \hat{h}) \in Y$ and we will use the notation $A^\ast = H^\ast$. Then, that the two mappings $H_i$ are continuous is very easy to prove using similar arguments.

**Lemma 4.2.** The mapping $H : Y \to Z$ is continuously differentiable.

**Proof.** Let us first prove that $H$ is G-differentiable at any $(v, \hat{h}) \in Y$ and let us compute the G-derivative $H'(v, \hat{h})$.

Thus, let us fix $(v, \hat{h}) \in Y$ and let us take $(v', \hat{h}') \in Y$ and $\sigma > 0$. For simplicity, we will use the notation

$$a_\sigma := a \left( \int_\Omega (v + \sigma v') |J| \ dy, \int_\Omega (\nabla v + \sigma \nabla v') \cdot \nabla \psi |J| \ dy \right),$$

$$\bar{a} := a \left( \int_\Omega v |J| \ dy, \int_\Omega (\nabla v \cdot \nabla \psi) |J| \ dy \right),$$

$$a^\eta := a \left( \int_\Omega v^\eta |J| \ dy, \int_\Omega (\nabla v^\eta \cdot \nabla \psi) |J| \ dy \right),$$

$$\bar{a}_i := D_i a \left( \int_\Omega v |J| \ dy, \int_\Omega (\nabla v \cdot \nabla \psi) |J| \ dy \right), \quad i = 1, 2, ..., n + 1,$$

$$a^\eta_i := D_i a \left( \int_\Omega v^\eta |J| \ dy, \int_\Omega (\nabla v^\eta \cdot \nabla \psi) |J| \ dy \right), \quad i = 1, 2, ..., n + 1,$$

$$f_\sigma := f(v + \sigma v), \quad \bar{f} := f(v), \quad f^\eta := f(v^\eta),$$

$$\bar{f}^i := f^i(v), \quad f^\eta := f^\eta(v^\eta).$$

We have

$$\frac{1}{\sigma} [H_1((v, \hat{h}) + \sigma(v', \hat{h}')) - H_1(v, \hat{h})] = v'_\sigma + \frac{1}{\sigma} [a_\sigma - \bar{a}] A(t)v + a_\sigma A(t)v' + \bar{b} \cdot \nabla v'$$

$$+ \frac{1}{\sigma} [f_\sigma - \bar{f}] - \hat{h}' \bar{1}_\omega - (\hat{h}' v + \hat{h}v') \bar{1}_\omega.$$
\[ DH_2(v', \tilde{h}') = v'(\cdot, 0), \]  
(58)

where \( \Gamma_1 = (\bar{a}_2, \ldots, \bar{a}_{n+1}) \in \mathbb{R}^n. \)

For all \((v', \tilde{h}') \in Y, \) one has

\[
\frac{1}{\sigma}[H_1((v, \tilde{h}) + \sigma(v', \tilde{h}')) - H_1(v, \tilde{h})] \longrightarrow DH_1(v', \tilde{h}') \text{ strongly in } G \text{ as } \sigma \to 0. \quad (59)
\]

Indeed, we have:

\[
\frac{1}{\sigma} \left[ (a_\sigma - \bar{a}) \right] A(t)v' \|_G \\
\leq \| (a_\sigma - \bar{a}) A(t) v' \|_G \\
+ \left\| \left[ \frac{1}{\sigma} (a_\sigma - \bar{a}) \right] A(t)v' \right\|_G \\
+ \left\| \left[ \frac{1}{\sigma} (f_\sigma - \bar{f}) \right] A(t)v' \|_G + \| \sigma \bar{h}'v' \|_G \\
= B_1 + B_2 + B_3 + B_4.
\]

Arguing as the proof of (54) and using Proposition 5, we obtain the following result, as a consequence of Lebesgue’s Theorem:

\[ B_1^2 = \int_Q \rho_0^2 (a_\sigma - \bar{a})^2 |A(t)v'|^2 dydt \longrightarrow 0 \text{ as } \sigma \to 0. \]

Once again, arguing as the proof of (54) and using Proposition 5, one has from Lebesgue’s Theorem:

\[ B_2^2 = \int_Q \rho_0^2 \left[ \frac{1}{\sigma} (f_\sigma - \bar{f}) \right] A(t)v'^2 dydt \\
= \int_Q \rho_0^2 \left[ (D_1 a^* - \bar{a}_1) \right] A(t)v'^2 dydt \\
\longrightarrow 0, \]

as \( \sigma \to 0, \) where the \( D_1 a^* \) are the partial derivatives of \( a \) at some intermediate points, in particular \( D_1 a^* \in \mathbb{R} \) and \( D_2 a^* \in \mathbb{R}^n. \)

For \( B_3, \) the argument is very similar. Indeed, we have

\[ B_3^2 = \int_Q \rho_0^2 \left[ (f^* - \bar{f})v' \right]^2 dydt \\
= \int_Q \rho_0^2 \left[ (f^* - \bar{f})v' \right]^2 dydt \\
= \int_Q \rho_0^2 \left[ f'^2 - \bar{f}'^2 \right] dydt, \]

where the \( f'^* \) is the derivative of \( f \) at some intermediate points. As \( f \in C_b^1(\mathbb{R}), \) then, arguing as the proof of (54) and using Proposition 5 and Lebesgue’s Theorem, once more we also find that \( B_3 \longrightarrow 0. \)

Finally, we see that

\[ B_4^2 = \int_Q \rho_0^2 (\sigma \bar{h}'v')^2 dydt \longrightarrow 0 \text{ as } \sigma \to 0. \]

Taking into account the behavior of \( B_1, B_2, B_3 \) and \( B_4, \) we deduce that (59) is true.
Consequently,
\[
\lim_{\sigma \to 0} \frac{1}{\sigma} [H((v, \tilde{h} + \sigma(v, \tilde{h}')) - H(v, \tilde{h})) = DH(v, \tilde{h}') \text{ in } G,
\]
whence we have that \( H \) is G-differentiable at any \((v, \tilde{h}) \in Y, \) with a G-derivative given by \( DH. \)

As usual let us denote by \( H'(v, \tilde{h}) \) the linear mapping defined by (56)-(58). Now, we shall prove that the mapping \((v, \tilde{h}) \mapsto H'(v', \tilde{h}')\) is continuous from \( Y \) to \( \mathcal{L}(Y; Z). \)

In other words, we will prove that, whenever \((v^n, \tilde{h}^n) \to (v, \tilde{h}) \in Y, \) one has
\[
\| (DH(v^n, \tilde{h}^n) - DH(v, \tilde{h}))(v', \tilde{h}')\|_Z \leq \varepsilon_n \| (v', \tilde{h}')\|_Y \text{ for some } \varepsilon_n \to 0. \quad (61)
\]

Then, we have just to prove that
\[
\| (DH_1(v^n, \tilde{h}^n) - DH_1(v, \tilde{h}))(v', \tilde{h}')\|_G \leq \varepsilon_n \| (v', \tilde{h}')\|_Y \text{ for some } \varepsilon_n \to 0. \quad (62)
\]

In effect,
\[
\|(DH_1(v^n, \tilde{h}^n) - DH_1(v, \tilde{h}))(v', \tilde{h}')\|_G^2 \leq C \int_Q \rho^2 \left[ a^n \int_{\Omega} v' |J| \, d\gamma A(t) v^n - \bar{a} \int_{\Omega} v' |J| \, d\gamma A(t) v \right]^2 \, dydt
\]
\[
+ C \int_Q \rho^2 \left[ \left( \Gamma^n \cdot \int_{\Omega} \nabla v' \cdot \nabla \psi |J| \, d\gamma A(t) v^n \right) - \left( \Gamma_1 \cdot \int_{\Omega} \nabla v' \cdot \nabla \psi |J| \, d\gamma A(t) v \right) \right]^2 \, dydt
\]
\[
+ C \int_Q \rho^n (\tilde{h}'(v^n - v) + (\tilde{h}^n - \bar{h}) v')^2 \, dydt
\]
\[
= E_1 + E_2 + E_3 + E_4 + E_5.
\]

Now, we will check that each \( E_i \) can be bounded as in (61). For instance, we have
\[
E_1 \leq C \int_Q \rho^2 |a^n| - a_1|^2 \left( \int_{\Omega} v' \, d\gamma \right)^2 |A(t) v|^2 \, dydt
\]
\[
+ C \int_Q \rho^n |a^n|^2 \left( \int_{\Omega} v' \, d\gamma \right)^2 |A(t) v^n - A(t) v|^2 \, dydt.
\]

The first and second integrals in the right hand side can be bounded as follow:
\[
\int_Q \rho^2 |a^n| - a_1|^2 \left( \int_{\Omega} v' \, d\gamma \right)^2 |A(t) v|^2 \, dydt
\]
\[
\leq C \left( \int_Q \rho^2 |A(t) v|^2 |a^n| - a_1|^2 \, dydt \right) \| (v', \tilde{h}')\|_Y^3.
\]

and
\[
\int_Q \rho^n |a^n|^2 \left( \int_{\Omega} v' \, d\gamma \right)^2 |A(t) v^n - A(t) v|^2 \, dydt
\]
\[
\leq C \left( \int_Q \rho^n |A(t) v^n - A(t) v|^2 \, dydt \right) \| (v', \tilde{h}')\|_Y^3.
\]
Lemma 4.3. Let \( a \) give us the desired estimate for \( L \) satisfied and ends the proof.

Similarly, we obtain the same conclusion for the others \( E_i \)'s, with \( i = 2, \cdots, 4 \).

In the sequel, let us analyze \( E_5 \) (this is another term involving multiplicative control). Note that:

\[
\int_0^T \int_\Omega \rho_0^2 (\tilde{h}'(v^n - v) + (\tilde{h} - \hat{h})v')^2 \, dydt \leq 2 \int_0^T \int_\Omega \rho_0^2 (\tilde{h}'^2 (v^n - v)^2) \, dydt
\]

\[
= I_1 + I_2.
\]

Proceeding as in (4) and using Proposition 3, we see that

\[
I_1 = 2 \int_0^T \int_\Omega \rho_0^2 (\tilde{h}'^2 (v^n - v)^2) \, dydt
\]

\[
\leq C (|\tilde{h}'|_{L^2(0,T;H^1(\Omega))} |\rho_*(v^n - v)|_{L^\infty(0,T;H^1(\Omega))})
\]

\[
\leq C (\|v'(\cdot,0)\|_{L^2(\Omega)}^2 + |\rho_0 k'|_{L^2(\Omega)}) |\rho_*(v^n - v)|_{L^\infty(0,T;H^1(\Omega))}
\]

\[
\leq C (\|v'(\cdot,0)\|_{H^1(\Omega)}^2 + |\rho_0 k'|_{L^2(\Omega)}) |\rho_*(v^n - v)|_{L^\infty(0,T;H^1(\Omega))}
\]

\[
\leq C (\|v'(\cdot,0)\|_{H^1(\Omega)}^2 + |\rho_0 v'|_{L^2(\Omega)}) (\|v^n - v\|_{L^\infty(0,T;H^1(\Omega))})
\]

\[
\leq C (\|\rho_*(v^n - v)|_{L^\infty(0,T;H^1(\Omega))}) (\|\tilde{h}' - v'\|_{L^2}\frac{1}{2})
\]

and

\[
I_2 = 2 \int_0^T \int_\Omega \rho_0^2 \left(\tilde{h}' - \hat{h}\right)^2 \, dydt
\]

\[
\leq C (|\tilde{h}' - \hat{h}|_{L^\infty(0,T;H^1(\Omega))} |\rho_0 v'|_{L^\infty(0,T;H^1(\Omega))})
\]

\[
\leq C (|\tilde{h}' - \hat{h}|_{L^\infty(0,T;H^1(\Omega))} |\rho_0 v'|_{L^\infty(0,T;H^1(\Omega))})
\]

\[
\leq C (|\tilde{h}' - \hat{h}|_{L^\infty(0,T;H^1(\Omega))} (\|\tilde{h}' - v'\|_{L^2})
\]

From the estimates above for the \( E_i \)'s, \( i = 1, \cdots, 5 \) we conclude that (61) is satisfied and ends the proof.

\[ \square \]

**Lemma 4.3.** Let \( H \) be the mapping defined by (51) and (52). Then \( H'(0,0) \in \mathcal{L}(Y';Z) \) is onto.

**Proof.** First, notice that

\[
H'(0,0)(v', \tilde{h}') = (H_1'(0,0)(v', \tilde{h}'), H_2'(0,0)(v', \tilde{h}'))
\]

\[
= (K_1(v', \tilde{h}'), K_2(v', \tilde{h}'))
\]

where

\[
K_1(v', \tilde{h}') := v'_t + a(0,\tilde{0})A(t)v' + \tilde{b} \cdot \nabla v' + cv' - \tilde{h}' \tilde{1},
\]

\[
K_2(v', \tilde{h}') := v'(-, 0),
\]
for all \((v', \tilde{h}') \in Y\). Here, the coefficient \(c\) is given by \(c = f'(0)\).

Consequently \(H'(0,0)\) is onto if and only if for each \((k, v_0) \in Z\), there exists \((v, \tilde{h}) \in Y\) satisfying

\[
\begin{align*}
&v + a(0,0)A(t)v + b \cdot \nabla v + cv = \tilde{h}1_{\omega} + k \quad \text{in} \quad Q, \\
v = 0 \quad \text{on} \quad \Sigma, \\
v(y, 0) = v_0(y) \quad \text{in} \quad \Omega.
\end{align*}
\]

Hence, the existence of \((v, \tilde{h})\) with these properties is ensured by Theorem 3.2. This shows that \(H'(0,0)\) is surjective and this ends the proof.

In view of Lemmas 4.1, 4.2 and 4.3, we can apply Liusternik’s Theorem to the mapping \(H : Y \to Z\) and (5) is locally null controllable, with \((v, \tilde{h}) \in Y\).

Consequently, by using the diffeomorphism \((y, t) \to (x, t)\) from \(Q\) to \(\hat{Q}\), we obtain that (1) is null controllable.

5. Additional comments and open questions. As a first comment, an interest question is concerned with global null controllability to (1), which does not seem to be simple. Indeed, the smallness assumption on the data in Theorem 1.1 is clearly necessary if one tries to apply Theorem 2.1 or another result of the same kind. To prove a global result, we would have to make use a global inverse mapping theorem, but this needs much more complicate estimates, that do not seem affordable.

Other important topics arise from our current research:

- In the system (1), we can replace the local nonlinearity \(f(u)\) by \(f(u, \nabla u)\), in order to analyze whether it is possible to prove results about null controllability.

- The arguments in Sections 3 and 4 also lead to the local boundary null controllability of similar equations:

\[
\begin{align*}
&u_t - a\left(\int_{\Omega_t} u \, dx, \int_{\Omega_t} \nabla u \, dx\right)\Delta u + f(u) = 0 \quad \text{in} \quad \hat{Q}, \\
u = h1_{\hat{\Sigma}} \quad \text{on} \quad \hat{\Sigma}, \\
u(x, 0) = u_0(x) \quad \text{in} \quad \Omega,
\end{align*}
\]

In fact, it is not too difficult to deduce such control results from Theorem 1.1. It suffices to apply this result to an appropriately modified open set \(\Omega_t \cup D\) and, then, consider the restriction to

\[
\bigcup_{0 \leq t \leq T} \{\Omega_t \times \{t\}\}
\]

of the resulting state.

Let us now indicate some open questions that arise naturally in the context of the results in this paper:

- Under the assumptions of Theorem 1.1, is it possible to prove the local exact controllability to (smooth) trajectories?

In other words, let \(\overline{u}\) be given, with

\[
\begin{align*}
&\overline{u}_t - a\left(\int_{\Omega_t} \overline{u} \, dx, \int_{\Omega_t} \nabla \overline{u} \, dx\right)\Delta \overline{u} + f(\overline{u}) = 0 \quad \text{in} \quad \hat{Q}, \\
\overline{u} = 0 \quad \text{on} \quad \hat{\Sigma}, \\
\overline{u}(x, 0) = \overline{u}_0(x) \quad \text{in} \quad \Omega,
\end{align*}
\]
Assume that $\bar{u}$ is regular (in a sense to be precised). Is it true that, for $\delta$ sufficiently small and $|u_0 - \bar{u}_0| \leq \delta$, there exist controls $h$ and associated states $u$ such that $u(x, T) = \bar{u}(x, T) \in \Omega_t$?

Finally, can we deduce local controllability results for the wave equation with nonlocal nonlinear terms of the same kind? In particular, let us consider the system

$$
\begin{cases}
u_{tt} - a \left( \int_{\Omega_t} u \, dx, \int_{\Omega_t} \nabla u \, dx \right) \Delta u + f(u) = h1_{\bar{\Omega}} & \text{in } \bar{\Omega}, \\
u = 0 & \text{on } \bar{\Sigma}, \\
u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x) & \text{in } \Omega,
\end{cases}
$$

where $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$. Is it true that, for $\delta$ sufficiently small, $(u_0, u_1), (z_0, z_1) \in H^1_0(\Omega) \times L^2(\Omega)$ and $\|(u_0, u_1) - (z_0, z_1)\|_{H^1_0 \times L^2} \leq \delta$ there exist controls $h$ and associated states $u$ such that $u(x, T) = z_0(x), \; u_t(x, T) = z_1(x) \in \Omega_t$?

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REFERENCES

[1] V. M. Alekseev, V. M. Tikhomirov and S. V. Fomin, Optimal Control, Consultants Bureau, New York, 1987.
[2] M. L. Bernardi, G. Bonfanti and F. Lutteroti, Abstract Schrödinger type differential equations with variable domain, J. Math. Anal. and Appl., 211 (1997), 84–105.
[3] V. R. Cabanillas, S. B. De Menezes and E. Zuazua, Null controllability in unbounded domains for the semilinear heat equation with nonlinearities involving gradient terms, J. Optim. Theory Appl., 110 (2001), 245–264.
[4] G.-Q. Chen and H. Frid, Divergence-measure fields and hyperbolic conservation laws, Archive Rat. Mech. Anal., 147 (1999), 89–118.
[5] M. Chipot and B. Lovat, Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems, Advances in quenching, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 8 (2001), 35–51.
[6] M. Chipot, V. Valente and G. Caffarelli, Remarks on a nonlocal problem involving the Dirichlet energy, Rend. Semin. Mat. Univ. Padova, 110 (2003), 199–220.
[7] H. R. Clark, E. Fernández-Cara, J. Límaco and L. A. Medeiros, Theoretical and numerical local null controllability for a parabolic system with local and nonlocal nonlinearities, Appl Math and Comp, 223 (2013), 483–505.
[8] J. Cooper and C. Bardos, A nonlinear wave equation in a time dependent domain, J. Math. Anal. and Appl., 42 (1973), 29–60.
[9] J-M. Coron, Control and Nonlinearity, Mathematical Surveys and Monographs, 136. American Mathematical Society, Providence, RI, 2007.
[10] S. B. De Menezes, J. Límaco and L. A. Medeiros, Remarks on null controllability for semilinear heat equation in moving domains, Electronic J. of Qualitative Theory of Differential Equations, 16 (2003), No. 16, 32 pp.
[11] S. B. De Menezes, J. Límaco and L. A. Medeiros, Finite approximate controllability for semilinear heat equations in non-cylindrical domains, Annals of the Brazilian Academy of Sciences, 76 (2004), 475–487.
[12] A. Doubova, E. Fernández-Cara, M. González-Burgos and E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, SIAM, J. Control Optim., 41 (2002), 798–819.
[13] C. Fabre, J.-P. Puel and E. Zuazua, *Approximate controllability of the semilinear heat equation*, Proc. Royal Soc. Edinburgh, Ser. A, 125 (1995), 31–61.

[14] E. Fernández-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*, SIAM J. Control Optim., 45 (2006), 1395–1446.

[15] E. Fernández-Cara, J. Limaco and S. B De Menezes, *Null controllability for a parabolic equation with nonlocal nonlinearities*, Systems and Control Letters, 61 (2012), 107–111.

[16] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability of weakly blowing-up semilinear heat equations*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, 17 (2000), 583–616.

[17] A. V. Fursikov and O. Yu Imanuvilov, *Controllability of Evolution Equations*, Lectures Notes, Vol. 34, Seoul National University, Korea, 1996.

[18] M. González-Burgos and R. Pérez-García, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal., 46 (2006), 123–162.

[19] C. He and L. Hsiao, *Two dimensional Euler equations in a time dependent domain*, J. Diff. Equations, 163 (2000), 265–291.

[20] O. Yu Imanuvilov, *Controllability of parabolic equations (in Russian)*, Mat. Sbornik. Novaya Seriya, 186 (1995), 109–132.

[21] O. Yu Imanuvilov, *Remarks on exact controllability for the Navier-Stokes equations*, ESAIM Control Optim. Calc. Var., 6 (2001), 39–72.

[22] O. Yu Imanuvilov and M. Yamamoto, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Publ. Res. Inst. Math. Sci., 39 (2003), 227–274.

[23] A. Inoue, *Sur □u + u³ = f dans un domaine noncylindrique*, J. Math. Anal. and Appl., 46 (1974), 777–819.

[24] J. Limaco, M. Clark, A. Marinho, S. B. De Menezes and A. T. Louredo, *Null controllability of some reaction-diffusion systems with only one control force in moving domains*, Chin. Ann. Math. Ser. B, 37 (2016), 29–52.

[25] J. Limaco, L. A. Medeiros and E. Zuazua, *Existence, uniqueness and controllability for parabolic equations in non-cylindrical domains*, Matemática Contemporânea, 23 (2002), 49–70.

[26] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes Aux Limites Nonlinéaires*, Dunod, Paris, 1960.

[27] J.-L. Lions, *Une remarque sur les problèmes d’évolution nonlinéaires dans le domaines non-cylindriques* (In french), Rev. Roumaine Math. Pures Appl., 9 (1964), 11–18.

[28] L. A. Medeiros, *Nonlinear wave equations in domains with variable boundary*, Arch. Rational Mech. Anal., 47 (1972), 47–58.

[29] L. A. Medeiros, J. Limaco and S. B. Menezes, *Vibrations of elastic strings: Mathematical aspects - part one*, J. Comput. Anal. Appl., 4 (2002), 91–127.

[30] M. M. Miranda and J. L. Ferrel, *The Navier-Stokes equation in noncylindrical domain*, Comput. Appl. Math., 16 (1997), 247–265.

[31] M. M. Miranda and L. A. Medeiros, *Contrôlabilité exacte de l’équation de Schrödinger dans des domaines noncylindriques*, C. R. Acad. Sci. Paris, 319 (1994), 685–689.

[32] M. Nakao and T. Narazaki, *Existence and decay of some nonlinear wave equations in noncylindrical domains*, Math. Rep. Kyushu Univ., 11 (1978), 117–125.

[33] L. Prouvée and J. Limaco, *Local null controllability for a parabolic-elliptic system with local and nonlocal nonlinearities*, Electron. J. of Qualitative Theory of Differential Equations, 74 (2019), 1–31.

[34] S. Zheng and M. Chipot, *Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms*, Asymptot. Anal., 45 (2005), 301–312.

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