Strong-field Breit-Wheeler pair production in QED$_{2+1}$

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The Breit-Wheeler pair production process in 2 + 1 dimensional spacetime is investigated. In the perturbative regime, non-vanishing rates at the energy threshold are found when odd numbers of photons take part on the reaction. This behaviour is understood as a direct consequence of the reduced dimensionality and resembles a corresponding prediction made in gapped graphene monolayers. In the non-perturbative strong field regime the effect of the dimensionality manifests itself in a different rate dependence on the quantum non-linearity parameter. The consequence of this deviation is discussed briefly in line with the applicability of perturbation theory. We argue that, in addition to large values of the quantum non-linearity parameter, the superrenormalisable character of quantum electrodynamics in 2 + 1 dimensional spacetime might give rise to a breakdown of perturbation theory within certain energy scales.

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I. INTRODUCTION

With the ongoing development of laser and accelerator infrastructures, the interest in producing electron-positron pairs from photon collisions has strongly grown over the last years as this process – despite of being one of the most fundamental predictions of quantum electrodynamics (QED$_{3+1}$) – has not been validated experimentally in full glory. After the pioneering work due to Breit and Wheeler on the pair production by the collision of two photons [1] (linear Breit-Wheeler process), further theoretical efforts revealed a phenomenology which manifests – among other issues – the non-linear feature of QED$_{3+1}$. Initially, several authors considered interaction of light quanta with a plane-wave electromagnetic background [2,4]. Special attention has been paid to the perturbative ($\xi \ll 1$) and non-perturbative ($\xi \gg 1$) regimes defined by the intensity parameter of the involved laser wave $\xi = -\epsilon E_0/(\omega m)$ with electron charge $\epsilon < 0$, its mass $m$, frequency $\omega$ and amplitude $E_0$ of the plane wave.¹ Besides, more elaborated treatments have been put forward in order to reach a more precise description of laser configurations [6,10] with particular emphasis on the highly non-linear non-perturbative regime due to important links to a wider range of elusive phenomena [17,22]. Of special interest has been the region in which the quantum non-linearity parameter $\chi = \xi k^2/m^2$ – involving photon momenta $k$ and $k^\prime$ – exceeds unity considerably as it might come accompanied with a breakdown of perturbation theory when $\alpha = e^2/4\pi \approx 1/137$ (or $\alpha = e^2/4\pi \approx 1/137$ is the fine structure constant) [23].

Despite the theoretical endeavour, experimental validation of Breit-Wheeler pair creation has so far been possible solely in the multiple-photon regime at $\xi \lesssim 1$ [31]. While this experiment provided important insights into this process, neither linear pair creation –for recent proposals see [32–37]– nor Breit-Wheeler pair creation at $\xi \gg 1$ has been observed by now. The reason why the former process is still elusive stems from the current limitations of reaching energetic photon beams with adequate intensities. Conversely, accessing to the non-perturbative regime demands peak intensities $I = E_0^2 \approx I_c \approx 4.6 \times 10^{29}$ Wcm$^{-2}$, corresponding to the Schwinger electric field scale $E_c \approx 1.3 \times 10^{16}$ Vcm$^{-1}$ in the relevant frame of reference. Reaching this condition requires ultrahigh laser intensities $I \sim (10^{21} - 10^{24})$ Wcm$^{-2}$ in the laboratory frame in conjunction with a beam of sufficiently energetic $\gamma$ photons, which represents a hard challenge to overcome (for recent proposals see [38,39]).

A low-energy test ground for yet unobserved QED$_{3+1}$ processes is provided by bandgapped graphene monolayers. In this two dimensional honeycomb of carbon atoms charge carriers near the degeneracy points possess a Dirac-like dispersion relation with the speed of light replaced by the Fermi velocity $v_F \approx 1/300$. Hence, their behaviour can be effectively described by a 2+1 dimensional Dirac model and the interband transition of electrons, i.e. the production of electron-hole pairs, can be exploited as a toy model for addressing QED$_{3+1}$ pair production-related questions. This idea has motivated investigations of analogues to the Schwinger mechanism [40,41] by delivering inherent insights caused by the low dimensional material: the power of the pre-exponential factor is changed from 2 in QED$_{3+1}$ to 3/2 in graphene. Likewise, bandgapped graphene has been presented as a suitable material to simulate the Breit-Wheeler process with two (linear case) and three photons [42]. In the latter case, as in the studies linked to the Schwinger-like mechanism, inherent properties due to the low dimensionality.
sional character of the medium came to a scene. Contrary to QED\textsubscript{3+1}, the rate linked to this process does not vanish at the energy threshold. This poses special feature questions whether its occurrence extends to perturbative Breit-Wheeler reactions in which more than three photons are absorbed and to which extent the Lorentz symmetry breaking caused by the Fermi velocity takes part in its realization. Addressing these questions is not an easy task within the Dirac model, mainly because the dispersive properties of the medium make the theoretical treatment difficult to handle.\(^2\)

Against this background, we study in the present paper the Breit-Wheeler pair production in 2+1 dimensional quantum electrodynamics (QED\textsubscript{2+1}) in various interaction regimes from weak to very strong fields. In particular we show that the threshold lifting persists in this Lorentz invariant framework when an odd number of photons are absorbed, confirming that this phenomenon results as an inherent consequence of the reduced spacetime dimensionality. Besides, we reveal characteristic influences of the dimensionality on the non-perturbative production rates for both small and large values of the quantum non-linearity parameter, and discuss the implications for a breakdown of perturbation theory in this scenario.

Considerations of low-dimensional scenarios have proven useful in various areas of physics. They revealed, for example, valuable insights on non-perturbative aspects of QCD (see e.g. \cite{43} and references therein) and even provided a solvable quantum field theory model for gravity \cite{44}. Investigations of the Chern-Simons term – intrinsically linked to QED\textsubscript{2+1} \cite{45} – have improved our understanding of the high temperature behaviour of QED\textsubscript{3+1}, leading in addition to an accurate description of the quantum Hall effect \cite{46}. Likewise, various research on the subject of Schwinger pair creation in arbitrary spacetime dimensions have been performed by considering different field configurations such as constant uniform electric and magnetic fields \cite{47} and electric fields of finite duration \cite{48}. Regarding Breit-Wheeler pair production, a recent study based on a model Hamiltonian in one spatial dimension has allowed for spacetime resolution of this process \cite{49}.

Another important area of applicability for lower dimensional theories is the field of quantum simulation. Gradually, this branch is gaining interest as it enables to simulate the behaviour of various many-body ensembles via ultracold atom systems placed in sophisticated optical lattices. Because of a complex technical implementation, these quantum simulators have been restricted so far to systems of lower dimensionality, so that the question arises, which differences occur as compared to 3+1 dimensions. In the realm of quantum electrodynamics, the progress in this field has led to an experimental implementation of the 1+1-dimensional Schwinger mechanism \cite{50,51}, which was investigated thoroughly by theoreticians \cite{52,53}, whereas current research efforts are spent towards QED\textsubscript{2+1} \cite{54,55}. Hence, with the present study of the non-perturbative Breit-Wheeler pair creation we provide a scenario which may be further explored through the branch of quantum simulation.

Our paper is organised as follows: in Sec. II we introduce Volkov states in QED\textsubscript{2+1} and apply them in Sec. III to obtain a general expression of the Breit-Wheeler pair production rate. Afterwards, in Sec. IV we evaluate the rate numerically in various intensity regimes and support this analysis by an asymptotic study. Our conclusions are given in Sec. V, while the two-photon pair production process is described in the lowest order of perturbation theory in Appendix A.

II. VOLKOV STATES IN 2+1 DIMENSIONS

In a 2+1 dimensional spacetime the time evolution of planar relativistic electrons interacting with an electromagnetic field \(a_\mu(x)\) is described by the Dirac equation

\[
(i\partial - m)\psi = 0,
\]

which manifests a SO(1,2) invariance \cite{59,60}. In this expression \(\psi(x)\) stands for a two component spinor wave function, whereas \(i\partial \equiv \gamma^\mu D_\mu\) and \(D_\mu = \partial_\mu + ie_\mu\) refers to the covariant derivative with \(e < 0\) denoting the electric charge in 2 + 1 dimensions. Observe that the latter notation differs from the one used in the introduction (e). The reason behind this change will be discussed below. The \(\gamma^\mu\)-matrices linked to this low dimensional scenario are determined by Pauli matrices \(\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1)\) and are also constricted by the Clifford-algebra \(\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{2\times 2}\) with \(\mu, \nu = 0, 1, 2\).

As in 3+1 dimensions the solvability of Eq. (1) is restricted to a certain class of electromagnetic fields. Here, we will focus on the solution resulting when \(a_\mu(x)\) is chosen as a polarized plane-wave-like three-potential

\[
a_\mu(kx) = a_0 e^{\mu} \cos(kx)
\]

characterised by the amplitude \(a_0\), the wave vector \(k^\mu = (k_0, \vec{k})\) and a transverse polarisation \(e^{\mu}\) \((e^\mu k_\mu = 0)\), which is normalised according to \(e^{\mu} e_\mu = -1\). We remark that in 2+1 dimensions the magnetic field provided by \(a_\mu(x)\) is a pseudo scalar, whereas the electric field is a two component vector. It is worth emphasising that the field in Eq. (2) is supposed to be a solution of a Maxwell equation without taking the Chern-Simons contribution into account, an assumption which applies whenever the energy-momentum transfer is much larger than the topological mass of the gauge field.

Despite of the inherent differences caused by the dimensionality the procedure for establishing the 2+1 dimensional Volkov-states does not differ much from the

\(^2\) As a consequence of the asymmetry introduced by the simultaneous appearance of the Fermi velocity and the speed of light, solutions to the Dirac equation of graphene in the field of a plane wave are much more difficult to obtain than in QED\textsubscript{3+1}.\footnote{As a consequence of the asymmetry introduced by the simultaneous appearance of the Fermi velocity and the speed of light, solutions to the Dirac equation of graphene in the field of a plane wave are much more difficult to obtain than in QED\textsubscript{3+1}.}
well-known approach in QED$_{3+1}$ (see for instance Refs. 3, 61, 62). In line, we find

$$\psi_p^-(x) = \sqrt{\frac{m}{q_0 A}} \left(1 + \frac{e}{2kq^{-}}k\hat{p}\right) u_p^- \times \exp\left[-iq^-x - i\frac{eaq^{-}\epsilon}{kq^{-}}\sin(kx) - i\frac{e^2a_q^2}{8kq^{-}}\sin(2kx)\right].$$

(3)

The state given above has been normalised in such a way that the time averaged electron density $\langle j^0 \rangle = \langle \psi_p^- \gamma^0 \psi_p^- \rangle$ with $\psi_p^- \equiv \psi_p^\dagger \gamma^0$ amounts to one particle in the normalisation area $A$. While $u_p^-$ refers to a free spinor $a_\mu = 0$ with positive energy, the quantity $q_\mu^- = p_\mu^- + \frac{e^2a_q^2}{4kq^-}k_\mu$ stands for an averaged effective electron momentum with $kp^- = kq^-$, $ep^- = eq^-$, where the electron momentum is denoted by $p^-$. Moreover, the Volkov solution for a positron $\psi_p^+(x)$ can be read off from Eq. (3) by carrying out the replacements: $p^- \rightarrow -p^+$ and $u_p^- \rightarrow v_p^+$, where $v_p^+$ is the free negative-energy solution. Both spinors $u_p^-$ and $v_p^+$ are normalised in accordance with the following rules

$$u_p^- u_{p^-} = \frac{p^- + m}{2m}, \quad v_p^+ v_{p^+} = \frac{p^+ - m}{2m}$$

(4)

and $\bar{u}_p^- \gamma_p^0 u_p^- = p^\mu_\mu / m$ for $\bar{u}_p^- \equiv u_p^\dagger \gamma^0$, $\bar{v}_p^+ \equiv v_p^+ \gamma^0$.

III. BREIT-WHEELER PAIR CREATION IN QED$_{2+1}$

A. General considerations

In QED$_{2+1}$, the reduction of space dimensions gives rise to changes in the physical dimensions of the involved fields $\psi$, $a_\mu$ and the electric charge $e$

$$[\psi] = \mathcal{E}, \quad [a] = \mathcal{E}^{1/2} \quad \text{and} \quad [e] = \mathcal{E}^{1/2},$$

(5)

where $\mathcal{E}$ stands for the dimension of energy. As the charge in the equation above has a positive energy dimension, QED$_{2+1}$ belongs to the class of super-renormalisable theories. This fact prevents us from identifying $e^2/4\pi$ with the traditional fine-structure constant $\alpha \approx 1/137$ and demands to look for an adequate dimensionless parameter from which a perturbative treatment can be justified.

In the scope of the strong-field Breit-Wheeler process two gauge fields are involved in the pair creation: a strong field $a_\mu$ and a low-intensity field $a'_\mu$. Hence, we can identify dimensionless intensity parameters $\eta = -ea_0^\dagger /m > 0$ and $\eta' = -ea'_0^\dagger /m > 0$, where $a_0$ and $a'_0$ stand for the field amplitudes, and perform the perturbative expansion in $\eta'$ assuming that $\eta' \ll 1$.

Accordingly, our starting point is the generic transition amplitude for the Breit-Wheeler process (see Fig. 1):

$$S_{fi} = -ie \int d^3 x \psi_{p^-}^-(x) \bar{\psi}_{p^+}^+(x),$$

(6)

where $\bar{\psi}_{p^+}^+(x) = \bar{a}^\mu e^{-ik'x}$ denotes the photon wave function, which refers to the amplitude $a'_0 \gamma^0$ rather than being normalised to one particle in the area, and the polarisation $\epsilon'_\mu$ satisfies the conditions $\epsilon'k' = 0, \epsilon' = -1$. In this expression $\psi_{p^-}^-(x)$ and $\psi_{p^+}^+(x)$ stand for the Volkov solutions as given in Sec. II.

However, we point out that the applicability of Eq. (6) also depends on the regime dictated by $\eta$, where additional specification of the expansion parameters may be in order. Indeed, as it happens in QED$_{3+1}$, large values of $\eta \gg 1$ might compensate the smallness of $\eta'$, making the effective coupling strength far from being perturbative. In this scenario the effect of radiative corrections should be included in the transition amplitude. Further discussion on this topic will follow at the end of Sec. IV.

To evaluate the integral in Eq. (6), we follow the usual procedure and introduce a Fourier expansion of an exponential

$$e^{-iz_- \sin(kx)} e^{iz_+ \sin(2kx)} = \sum_{n=-\infty}^{\infty} e^{-inkx} \tilde{J}_n(z_-,z_+),$$

(7)

in which the generalised Bessel function is defined as an infinite sum of a product of ordinary Bessel functions of one argument $3, 63, 64$.

$$\tilde{J}_n(z_-,z_+) = \sum_{m=-\infty}^{\infty} J_{n-2m}(z_-) J_m(z_-),$$

(8)

It is worth remarking that, in a minimally coupled framework, the combination $ea_0^\dagger$ always a dimension of energy regardless the number of space dimensions $d$: $[ea_0^\dagger] = \mathcal{E}^{(3-d)/2\mathcal{E}^{(d-1)/2}} = \mathcal{E}$.
and write \((l = 1, 2)\)

\[
\cos(kx)e^{-iz_{\text{ph}}(kx)-iz_{\text{ph}}(2kx)} = \sum_{n=-\infty}^{\infty} e^{-i nkx} \hat{\mathcal{J}}_n^l,
\]

\[
\hat{\mathcal{J}}_n^1 = \frac{1}{2} \left( \hat{J}_{n+1}(z_{\text{ph}}, z_{\text{ph}}) + \hat{J}_{n-1}(z_{\text{ph}}, z_{\text{ph}}) \right),
\]

\[
\hat{\mathcal{J}}_n^2 = \frac{1}{4} \left( \hat{J}_{n+2}(z_{\text{ph}}, z_{\text{ph}}) + 2\hat{J}_n(z_{\text{ph}}, z_{\text{ph}}) + \hat{J}_{n-2}(z_{\text{ph}}, z_{\text{ph}}) \right).
\]

By exploiting these definitions we can write Eq. (6) as

\[
S_{fi} = -\frac{e\alpha_0'}{2} \sqrt{\frac{m^2}{\hbar^2 \gamma_0 A^2}} \sum_{n=-\infty}^{\infty} (2\pi)^3 \delta_{nk+k', q^+ + q^-} \times \bar{u}_{p^+} M_n v_{p^+},
\]

where \(\delta_{x,y} \equiv \delta^3(x - y)\) and \(M_n\) is of the following form

\[
M_n = \frac{e\alpha_0'}{2} \sqrt{\frac{m^2}{\hbar^2 \gamma_0 A^2}} \left( \frac{\tilde{J}_n}{kq^-} - \frac{\tilde{J}_n}{kq^+} \right) \left( e^2 - \frac{e^2 a_0^2}{2kq^+ q^-} + e^p k_{\mu} \tilde{J}_n^2 \frac{k_{\mu}}{k^2} \right).
\]

Notice that energy-momentum conservation \(q^+ + q^- = nk + k'\), anticommutativity of gamma matrices as well as transversal condition have been employed and the arguments of generalised Bessel functions read \(z_{\text{ph}} = e\alpha_0(a_{\text{ph}} - \frac{k}{k^2}q_{\text{ph}})^{\frac{3}{2}}, z_{\text{ph}} = -\frac{e^2 a_0^2}{8 kq^+ q^-} - \frac{kk'}{kq^+ q^-}\). Next, we consider the differential rate of pair creation per area defined as

\[
dR = \frac{|S_{fi}|^2}{TA^2} \frac{d^2q^- d^2q^+}{(2\pi)^2 (2\pi)^2}.
\]

Here, the phase space is expressed in terms of the effective particle momenta, in accordance with the normalization chosen in Eq. (3). When inserting Eq. (10) into the equation above we end up with

\[
dR = \frac{e^2 a_0^2 m^2}{8\pi} \sum_{n=m_0}^{\infty} \delta_{nk+k', q^+ + q^-} \frac{d^2q^-}{q_0} \frac{d^2q^+}{q_0} \times \left( \frac{2 - (kk')^2}{4kq^+ q^-} \right) \left( \tilde{J}_n^2 - \tilde{J}_n \tilde{J}_n^2 \right).
\]

The fact that the summation index starts at \(n_0 = 2m_2/(kk')\) with \(m_2 = m^2(1 + \eta^2/2) = q^2\) referring to the effective electron mass is understood here as a consequence of the energy-momentum balance. Leaving aside the lower dimensionality of the delta functions and the involved integration measures, the main source of difference between \(dR\) and the unpolarized rate in 3+1 dimensions (see for instance Eq. (15) in [4]) lies in the precise structure of the squared amplitude (compare with Eq. (16)). It resembles the squared amplitude of the pair production by a photon, polarization of which is parallel to the electric field of the wave (as given in Eq. (35) in [4] or in Eq. (3) in [3]). This fact highlights the restriction

\[
\begin{align*}
\gamma_{\mu\nu}^{(2+1)} & = \gamma_{\mu\nu}^{(2+2)} - 6 \eta_{\mu\nu}, \\
\gamma_{\mu\nu}^{(2+2)} & = -2i\epsilon_{\mu\nu\rho\sigma} \gamma^{\rho\sigma}, \\
\gamma_{\mu\nu}^{(2+3)} & = 34 2x2, \\
\gamma_{\mu\nu}^{(2+4)} & = 2g_{\mu\nu}. \\
\end{align*}
\]

where \(\epsilon_{\mu\nu\rho\sigma} = 2(\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho})\) with Levi-Civita tensor \(\epsilon_{\mu\nu\rho\sigma}\) with \(\epsilon_{012} = 1\).
provided by a lower dimensionality: both photon wave vectors and polarisations are bound to a plane. Hence, when moving the system to a center-of-momentum frame, where photons are counterpropagating, their polarisations have to be parallel.

In order to highlight the consequences linked to the spacetime dimensionality further, we change the integration variables to a set referred to the center-of-momentum frame. In this case the remaining integral is expressed in polar coordinates

$$
R = \frac{e^2 a_0^2 m^2}{8\pi} \sum_{n \geq n_0} \int_0^{2\pi} d\phi \int_0^\infty dq \delta(q - q_n^*)
\times \left[ J_n^2 - \eta^2 \left( 1 - \frac{1}{1 - \frac{q^2}{2\pi} \cos^2(\phi)} \right) \left( J_n^* \right)^2 - J_n J_n^* \right],
$$

(19)

when setting $|\vec{q}| q_n^* \equiv q$, $kq^+ = \omega(\omega' - \omega \cos(\phi))$, $kq^{-} = \sqrt{\omega^2 - m^2}$, $\omega_n = \omega$. Additionally, we introduce a Mandelstam invariant $s = \sqrt{kq^+ \ell m^2}$ which allows us to write $q_n^* = \sqrt{\omega(\omega' - \omega \cos(\phi))}$. Requiring that the number of absorbed photons exceeds a critical number $n \geq (1 + \eta^2/2)/s^2$, we obtain for every value of $n$ a threshold energy $s^2 \geq (1 + \eta^2/2)/n = s_n^2$ which must be overpassed in order that the process takes place.

B. Threshold behaviour

Now, we are interested in investigating the behaviour of the rate at the energy threshold, where the particles are created with zero momentum ($q = 0$). It is noteworthy that at the threshold the integral over the phase space will not always vanish and is governed solely by the form of the amplitude. This fact represents a crucial difference to the 3+1 dimensional case where after integrating over $q^-$ in the center-of-momentum frame the remaining integral $\int \frac{d^3q}{(2\pi)^3} \delta(2q_0 - 2\omega') \propto q_n^*$ always goes to zero at the threshold. Therefore, we focus on the behaviour of the process amplitude. In this context, the arguments of the generalized Bessel functions in the center-of-momentum frame read

$$
z_- = -\frac{2\eta}{ms^2} \sqrt{1 - \frac{q^2}{2\pi} \cos^2(\phi)} \xrightarrow{s \to 0} 0,
$$

$$
z_+ = -\frac{\eta^2}{4s^2} \left[ 1 - \frac{q^2}{2\pi} \cos^2(\phi) \right] \xrightarrow{s \to 0} -\frac{\eta^2}{4s^2} \equiv z_+.
$$

(20)

and, consequently, the generalized Bessel function at the threshold can be written as

$$
\tilde{J}_n(0, z_+) = \begin{cases} 
J_{n/2}(z_+) & n \text{ even}, \\
0 & n \text{ odd}.
\end{cases}
$$

(22)

Hence, when an even number of strong field photons $n = 2\ell$ is absorbed, the rate (see Eq. (19)) at the threshold behaves

$$
R_{s \to s_n} = \frac{e^2 a_0^2 m}{8} \sum_{2\ell \geq n_0} J_{2\ell}^2 \left( \frac{-\eta^2 \ell}{2(1 + \eta^2/2)} \right).
$$

(23)

While this expression tends to vanish as $\eta$ grows ($\eta \gg 1$), the contribution for $\eta \ll 1$ approximates

$$
R_{q \ll 1} \approx \frac{e^2 a_0^2 m}{8} \sum_{2\ell \geq n_0} \eta^{4\ell} \left( \frac{2(\ell-1)}{2\ell} \right),
$$

(24)

where $\Gamma(x)$ stands for the gamma function. We see that the rate above does not vanish. This causes a lifting of the rate at each threshold when an even number of strong field photons is absorbed, which resembles the behaviour at low intensity of the Breit-Wheeler process and does not rely on the Lorentz invariance breakdown that is caused by the Fermi velocity of this medium.

IV. ASYMPTOTIC STUDY

This section is devoted to establishing asymptotic expressions of the Breit-Wheeler pair creation rate (see Eq. (18)) for different intensity parameters of the strong field $\eta$.

A. Behaviour for $\eta \ll 1$

For small laser parameter $\eta \ll 1$ the leading order contribution of Eq. (18) results, when energetically allowed, from the absorption of a single laser photon ($n = 1$). A representative Feynman diagram for this process is depicted in the upper panel of Fig. (2). In this limiting case the effective electron and positron momenta can be simply taken as $q^\pm = p^\pm$, so that the arguments of the generalized Bessel functions read $z_- = \eta m (p^+ - p^-)/p^\pm$, $z_+ = -\frac{\eta^2 m^2}{8} \frac{kk'}{kp^\pm k\prime}$. Their respective proportionality on $\eta$ and $\eta^2$ allows us to expand the Bessel functions (see Appendix C in [63]):

$$
\tilde{J}_n(\eta^\beta, \eta^2 \gamma) \approx \left( \frac{\eta^2 \gamma}{2} \right)^{\frac{\beta}{2}}
\times \left\{ \begin{array}{l}
\sum_{k=0}^{n/2} \frac{(\beta^2/2)^k}{(k!)^{n/2-k}} \\
\sum_{k=0}^{(n-1)/2} \frac{(\beta^2/2)^{k+1/2}}{(2k+1)!(n-1)/2-k)!
\end{array} \right\}
$$

(25)

for even $n$, for odd $n$, where the shorthand notation $\beta = m (p^+ - p^-)/p^\pm$, $\gamma = -\frac{m^2}{8} \frac{kk'}{kp^\pm k\prime}$ has been introduced. Hence, in the lowest
order in $\eta$ the squared amplitude can be written as

$$|M_{\eta=1}|^2 = \frac{\eta^2}{4} \left( 1 - \frac{(kk')^2}{4kp^+kp^-} \right)$$

(26)

The use of Eq. (15) in combination with the energy-momentum balance $p^+ + p^- = k + k'$ allows us to express the differential rate of the pair production in the following form

$$dR^{n=1}_{\eta=1} = \frac{e^2a_0^2m^2}{8\pi}\delta_{k+k',p^-+p^-}|M_{\eta=1}|^2 \frac{d^2p^-d^2p^+}{p_0^+p_0^-},$$

$$|M_{\eta=1}|^2 = \frac{\eta^2}{4} \left[ \frac{(kk')^2}{4kp^+kp^-} - 1 + \frac{2m^2kk'}{k^2p^+kp^-} - \frac{m^4(kk')^2}{(kp^+kp^-)^2} \right].$$

(27)

It is worth mentioning that this expression coincides with the outcome resulting from perturbation theory when the classical field in Eq. (1) is canonically quantised (see details in Appendix A).

Finally, we integrate the rate in Eq. (27) by adopting polar coordinates. To facilitate the integrations over the momenta we go over to the center-of-momentum frame, where $\vec{p} = \vec{p}' + = -\vec{p}^-$, $p_0^+ = p_0^-$, $\omega = \omega'$, $kk' = 2\omega^2$ and $kp = \omega(p_0 - |\vec{p}|\cos(\phi))$. Then, after integrating out $p^-$ and $p_0$ we arrive at

$$R^{n=1}_{\eta=1} = \frac{e^2a_0^2m^2\eta}{8\pi} \left[ -1 - \frac{4u^2}{1 - (1 - u)\cos^2(\phi)} + \frac{(1 + 4u)}{1 - (1 - u)\cos^2(\phi)} \right].$$

(28)

with $u = m^2/\omega^2$. The remaining integration can be performed analytically by using Eqs. (3.616.8) and (3.642.3) in Ref. [60].

$$R^{n=1}_{\eta=1} = \frac{e^2a_0^2m\eta^2}{8\pi} \left[ -1 - \frac{2(s^2 + 1)}{s^3} + \frac{(4 + s^2)}{s} \right],$$

(29)

where the result has been expressed in terms of the Mandelstam variable $s = \sqrt{kk'}/2m^2$. Observe that in this context the energy threshold translates into $s \geq 1$. We remark that for $s = 1$ the rate above vanishes. In the upper panel of Fig. 3 a comparison between the asymptotic expression in Eq. (29) (red dashed line) and the numerical evaluation for $n = 1$ in Eq. (18) (blue curve) is depicted. This assessment has been done by taking $\eta = 0.01$. For further comparison, Fig. 3 exhibits the behaviour of the corresponding process in 3+1 dimensions coloured in black. While the solid line results from an unpolarised photon beam, the dotted one refers to the case of parallel polarisation.

Usually, the derivation of the linear Breit-Wheeler rate is performed within the second order of perturbation theory when considering two quantised photons (see Appendix A). To bring Eq. (29) into the corresponding form

FIG. 2: Feynman diagrams of two (upper panel) and three (lower panel) photons pair creation processes. In both panels free fermions are represented by external solid lines, internal lines stand for the free fermion propagators, while wavy lines correspond to photons stemming from quantised and classical sources (the latter are marked by crossed circles).
the replacements $a_0^{(0)} = 2/\omega^{(0)} A$ need to be done. This way the quantity $\alpha_{2+1} = e^2/(4\pi m)$ appears in the rate, which is the counter part of the fine structure constant in QED, which is the counterpart of the fine structure constant $\alpha$ in physics. Preventing this way a mutual matching and so, a direct link with physical quantities. Next, we consider the case where two strong field photons are absorbed ($n = 2$) (see lower panel of Fig. 2) for the corresponding Feynman diagram. The leading order contribution to this process is described by the amplitude

$$|\mathcal{M}_{\eta<1}^{n=2}|^2 = \left( \frac{n^2 q}{2} + \frac{n^2 q^2}{8} \right)^2 - \frac{n^2}{4} \left( 1 - \frac{1}{4k^2} \right) \left( \frac{n^2 q^2}{8} - \frac{n^2 q^2}{2} \right),$$

(30)

where the involved Bessel functions were expanded with help of Eq. (25). Furthermore, we perform the integration in the center-of-momentum frame, which is defined similarly to the case $n = 1$ taking into account that $\omega' = 2\omega$. Consequently, the asymptote of the three-photon reaction reads

$$R_{\eta<1}^{n=2} = \frac{e^2a_0^2m}{8\pi} \frac{128s^8}{2} \left[ 10 - 30s^2 + 19s^4 + 12s^8 \right].$$

(31)

We note that at the threshold $s = s_2 = \sqrt{1/2}$ the expression above coincides with the outcome resulting from Eq. (24) for $n = 2$ providing a non-zero contribution. This effect can be seen in the lower panel of Fig. 2, where the asymptote given in the equation above is depicted in blue, whereas the red dashed curve results from evaluating Eq. (18) for $n = 2$. These outcomes are compared with the QED$_{3+1}$ rates for unpolarised photon (black solid) and photon polarised parallelly to the classical field (black dotted), which vanish at the threshold.

Finally, we remark that the fact that $\eta$ and $\eta'$ are dimensionless regardless of the spacetime dimension would allow us to transfer our predictions for the production rate from the 2+1 dimensional model to a real situation once it is assumed that they coincide with those defined from the physical charge and amplitude in 3+1 dimensions, i.e. if $\eta = \xi$. Observe that, identifying alternatively $\alpha_{2+1}$ with $\alpha \approx 1/137$, makes the rate a function depending explicitly on the field amplitude in 2+1 dimensions. However, the dimension of this quantity $|a_0| = \xi^{1/2}$ differs from the one of its 3+1 dimensional counterpart $\sim \xi$, preventing this way a mutual matching and so, a direct link with physical quantities.

B. Behaviour for $\eta \gg 1$

Contrary to the previous case of small laser parameters, for $\eta \gg 1$ a large number of summands in Eq (18) will provide non-negligible contributions to the rate. Indeed, a huge amount of photons is needed for the process to take place so that a transition to the continuum limit $\sum_{n} ... \to \int dn...$ can be developed. Afterwards, $n$ together with $\bar{q}^2$ are integrated out by using the delta functions. Consequently, Eq. (18) reduces to

$$R_{\eta>1}^{n=1} = \frac{e^2a_0^2m^2}{8\pi} \int \frac{d^2\bar{q}}{q_0} |M_{\eta>1}^{n=1}|^2.$$  

(32)

Here, $|M_{\eta>1}^{n=1}|^2$ is given in Eq. (16), where $n \rightarrow n^*$ and $n^* = q^2k^2/\omega(k_0^2 - k_0^2 - q_0^2 + q_2^2)$. We assume without loss of generality that the strong field $a^2$ is polarized in the $r_1$-direction. Next, let us introduce new variables

$$X = \frac{kq}{m^2} \eta, \quad X' = \frac{kq}{m^2} \eta, \quad \gamma = q_0 - q_2,$$

(33)

Observe that these definitions allow us to write $z_\pm$ (see below Eq. (7)) in the following form

$$z_+ = -\frac{n^2}{8XX'}, \quad z_- = \frac{n^2}{mXX'} (q^2 - q^2 \gamma X'),$$

(34)

where the quantum non-linearity parameter in 2+1 dimensions $\kappa = \eta k^2/m^2$ has been introduced. Further, we perform a substitution $q_2 \rightarrow \gamma$ which allows us to write the integral in Eq. (32) as

$$R_{\eta>1}^{n=1} = \frac{e^2a_0^2m^2}{8\pi} \frac{\eta}{m^2} \int_{-\infty}^{\infty} \frac{dq_-}{X} \int_0^\infty \frac{d\gamma}{\gamma} |M_{\eta>1}^{n=1}|^2,$$

(35)

where $\lambda = k_0^2 - k_2^2$ and

$$|M_{\eta>1}^{n=1}|^2 = J_0^2, \quad \eta^2 \left( 1 - \frac{\kappa^2}{4XX'} \right) \left[ (\tilde{J}_n^0)^2 - \tilde{J}_n z_+ \tilde{J}_n \right].$$

(36)

The matrix element above still contains generalized Bessel functions. In order to proceed, we have calculated their asymptotic expansions in the relevant limits, following the lines of Appendix B in Ref. [3] and paying special attention to modifications arising from the reduced dimensionality. It turns out that the latter introduces some minor changes but does not affect the derivations substantially. Having this in mind, we find that for $\eta \gg 1$ and $\eta > k^{1/3}$ the square of the matrix element approximates

$$|M_{\eta>1}^{n=1}|^2 \approx \frac{2}{\pi} \left( 1 - \frac{1}{4\eta^2 \sin^2(z_0)} \right)^{2/3} \times \left( \Phi^2(z) - \left( 1 - \frac{\kappa^2}{4XX'} \right) (\Phi^2(z) + \Phi^2(z)) \right).$$

(37)

---

5. Actually, the transition is made on the variable $s_\eta \equiv n/(1 + \eta/n^2) = 1/s_\eta^2$ for which the interval between two neighbour terms converges to zero $\Delta s_\eta = (1 + \eta^2/n^2)^{-1} \rightarrow ds$ as $\eta \rightarrow \infty$.

6. In order to exploit the large argument behaviour, the Bessel functions are written in their integral representation $J_n(z) = \frac{1}{2\pi j} \int_0^{2\pi} \text{d} \theta e^{j n \theta} f(\theta)$ with $f(\theta) = -i2\sin(\theta) - i2\sin(\theta) + i\theta$ and the integration is performed asymptotically via the method of steepest descent [27] by expanding $f(\theta)$. Since we are interested in deriving asymptotic formulas for both $\kappa < 1$ and $\kappa > 1$, an expansion up to the third order term is needed [28].
we further expand all involved functions in \( K \) with the modified Bessel functions of the second kind. Perform the remaining integrations, ending up with

\[
\frac{\kappa}{\sin(\omega x_0)} \approx \frac{e^{2a_0^2 m}}{8\pi} \frac{8}{22/3\pi^{3/2}} \int_0^{\pi/2} dx_0 \int_0^\infty \int_0^\infty \frac{d\theta}{\sqrt{1 + 2/3\sin^2(\theta)}} \Phi(t + 2^{3/2}z). \tag{39}
\]

For derivation of asymptotics we come back to Eq. (38) and start with considering \( \kappa \ll 1 \). Since the Airy functions decrease monotonically, the largest contribution to the integral in Eq. (38) comes from the region close to \( \bar{\theta} = 0 \) and \( \bar{x}_0 = \pi/2 \). Hence, we expand \( \Phi \) and \( \Phi' \) for large arguments as \( 1/\kappa \gg 1 \)

\[
\Phi^2(z) = \frac{z}{3\pi} K_{1/3}^2 \left( \frac{2}{3} z^{3/2} \right) \overset{z \rightarrow \infty}{\longrightarrow} \frac{z^{-1/2}}{4\pi} e^{-\frac{z^{3/2}}{2}}, \tag{40}
\]

\[
\Phi'^2(z) = \frac{z}{3\pi} K_{2/3}^2 \left( \frac{2}{3} z^{3/2} \right) \overset{z \rightarrow \infty}{\longrightarrow} \frac{z^{1/2}}{4\pi} e^{-\frac{z^{3/2}}{2}}, \tag{41}
\]

with the modified Bessel functions of the second kind \( K_{\nu}(x) \). Taking into account the expansions above we further expand all involved functions in \( \bar{\theta} \) and \( \bar{x}_0 \) and perform the remaining integrations, ending up with

\[
R_{\eta \gg 1, \kappa \ll 1} \approx \frac{e^{2a_0^2 m}}{8\pi} \frac{3\kappa}{8\sqrt{2}} e^{-\frac{4}{\sqrt{3}} \kappa}. \tag{42}
\]

Comparing the equation above with its 3+1 dimensional analogue (as given in Eq. (36) of Ref. [9]) we see that the power linked to the quantum non-linearity parameter \( \kappa \) in the pre-exponential factor is equal to 1 rather than 3/2. This difference comes from the reduced dimensionality of the phase space: Similarly as in QED\(_{3+1}\), the integrand in this limit depends on \( \kappa \) solely through the exponential function \( e^{-\frac{4}{\sqrt{3}}\kappa} \) and integration over each variable provides a factor \( \kappa^{1/2} \). Since in 2+1 dimensions \( z \) does not depend on a variable responsible for a dimension orthogonal to the plane spanned by photon polarization and propagation vectors, we obtain a power of 1/2 less in the quantum non-linearity parameter than in 3+1 dimensions.

Establishment of the asymptotic formula for \( \kappa \gg 1 \) demands a more elaborated procedure. To this end, we substitute \( \kappa \sin(\omega x_0) = p \) and \( \chi(\theta) = u \) in Eq. (38):

\[
R_{\kappa \gg 1, \kappa \gg 1} \approx \frac{e^{2a_0^2 m}}{8\pi} \frac{16}{3\pi^{3/2}} \int_0^\infty \frac{dp}{p\sqrt{r^2 - p^2}} \int_1^\infty \frac{du}{\sqrt{u^2 - 1}} \times \left[ u^2 K_{1/3}^2 \left( \frac{4u^2}{3p} \right) \right. \tag{43}
\]

and split the integration over \( p \) into two intervals: \( [0, p_0] \) and \( [p_0, \kappa] \) with \( p_0 \ll \kappa \). The contribution from the first interval will be negligible as it scales with \( \kappa^{-1} \):

\[
\int_0^{p_0} \frac{dp}{p\sqrt{r^2 - p^2}} \approx \kappa \int_0^{p_0} \frac{dp}{p} \sim \frac{4}{\kappa} \ll 1. \tag{44}
\]

For the extant integral we perform a transition to a new variable \( t = 4u^2/3p \), whereas \( p \) remains unchanged. As a consequence,

\[
R_{\kappa \gg 1, \kappa \gg 1} \approx \frac{e^{2a_0^2 m}}{8\pi} \frac{4}{3\pi^{3/2}} \int_0^p \frac{dp}{\sqrt{p\sqrt{r^2 - p^2}}} \int_1^\infty dt \times \left[ \frac{3}{4} \sqrt{\frac{t}{t - 1} K_{1/3}^2(t)} + \sqrt{\frac{3p - 1}{t} K_{2/3}^2(t)} \right]. \tag{45}
\]

At this point it is suited to split the integration over \( t \) into two sectors:

\[
\int_1^{t_0} dt \cdots = \int_{t_0}^{t_0} dt \cdots + \int_{t_0}^\infty dt \cdots. \tag{46}
\]

Here, the parameter \( t_0 \) satisfies the conditions \( 4/3\kappa \ll 4/3p_0 \ll t_0 \ll 1 \). Let us consider firstly the contribution defined over \( [t_0, \infty] \). In this region \( 3pt/4 \gg 1 \) holds and we obtain approximately

\[
\int_{t_0}^\infty dt \cdots \approx \int_{t_0}^\infty dt \sqrt{\frac{3p}{4} \left( K_{1/3}^2(t) + K_{2/3}^2(t) \right)} \approx \sqrt{\frac{3p}{4} \Gamma(\frac{\kappa}{2})/\kappa^{1/3}} \tag{47}
\]

where the smallness of \( t_0 \) has been used \( t_0 \ll 1 \) and \( \Gamma(x) \) denotes the gamma function [55].

Now, let us turn our attention to the contribution in Eq. (46) defined over \( [4/3p, t_0] \). As in this interval \( t \ll 1 \), we can use the small argument behaviour of the modified
Bessel functions $K_\nu(t) \approx \Gamma(\nu)/2(t/2)^\nu$ and, consequently, perform the integration over $t$. Taking advantage of the fact that $4/3pt_0 \ll 1$ and $t_0 \ll 1$ we arrive at

$$\int_{4/3p}^{t_0} dt... = -\sqrt{\frac{3p}{4}} \frac{3!^2 (\frac{t_0}{p})^{3/2}}{2^{2/3} t_0} \left( \frac{3^{5/6} \sqrt{\pi} \Gamma^2(\frac{3}{2}) \Gamma(\frac{1}{3})}{2^{10/3} \Gamma(\frac{11}{6})} \right) p^{5/6} + \frac{3^{1/6} \sqrt{\pi} \Gamma^2(\frac{1}{3}) \Gamma(\frac{1}{2})}{\Gamma(\frac{5}{6})} p^{1/6}. \quad (48)$$

We combine both Eqs. (17) and (48) in Eq. (46). As a consequence, $t_0$ drops out as it should and the resulting expression is inserted into Eq. (45). Hence, to the leading order in $\kappa$ we obtain

$$R_{0 \gg 1, \kappa \gg 1} \approx \frac{e^2 d^2 m}{8\pi} \frac{3^{11/6}}{5^{21/3} \pi} \Gamma^2 \left( \frac{2}{3} \right) \kappa^{1/3}. \quad (49)$$

As for the case $\kappa \ll 1$, a deviation with respect to QED$_{3+1}$ occurs. Indeed, the presence of the $\kappa^{1/3}$ dependence is in sharp contrast to the well established $\chi^{2/3}$ behavior that the rate in 3+1 dimensions manifests (for comparison see Eq. (36)) in Ref. [3]. It provides the first evidence that the proper expansion parameter for QED$_{2+1}$ in the strong field (see Eq. (1)) could be $g \sim 0.21 \kappa^{1/3}$, whenever $g \ll 1$. However, at this point, this statement must be considered as a conjecture; its confirmation will require a more elaborated investigation of consecutive terms within the perturbative expansion [27][30]. While this analysis is beyond the scope of the present study, some interesting consequences can be anticipated owing to the superrenormalisable feature of the theory. Firstly, upon identifying $\kappa \sim \chi \gg 1$, the presence of the $1/3$ exponent would induce a breakdown of perturbation theory softer than in strong-field QED$_{3+1}$ if $a_{2+1} \sim \alpha$. However, if $a_{2+1} \gg \alpha \kappa^{1/3}$, i.e. for a mass scale of the electric charge $e^2/4\pi \gg m a \kappa^{1/3}$, this break down in QED$_{2+1}$ could be stronger than in QED$_{3+1}$.

The result of this section is summarised in Fig. 4 which shows the dependence of $R$ on the quantum non-linearity parameter $\kappa$ [blue solid line]. This figure depicts in addition the asymptotes for small $\kappa$ [purple dotted line] and large $\kappa$ [red dashed line] given in Eqs. (42) and (49) correspondingly.

V. CONCLUSION

In the present work we have investigated the Breit-Wheeler pair creation in 2+1 dimensional Minkowski space. Special attention was paid to the domains of small and large intensity parameter $\eta$. We have shown that the dimensionality plays a crucial role when considering the process rates leading to surprising differences when comparing them to the 3+1 dimensional case. We have found that for $\eta \ll 1$ the low dimensionality causes non-vanishing contributions to the rate at the threshold when an even number of strong field photons is absorbed. This outcome supports and extends the results obtained in a recent study on bandgaped graphene [42]. As this feature persists in a Lorentz invariant vacuum, we conclude that in a honeycomb of carbon atoms it occurs independently of the medium.

In the high intensity regime ($\eta \gg 1$), the 2+1 dimensionality induces changes in the pair production rate which are manifested through a dependence on the quantum non-linearity parameter $\kappa$ different from the one appearing within the pure QED$_{3+1}$ context. On the basis of this finding we have argued that the so far accepted criteria within the strong field QED$_{3+1}$ community on the applicability of perturbation theory might be subject to modifications as the spacetime dimensionality changes. Indeed, we have seen that the super-renormalisable character of QED$_{2+1}$ constitutes an aspect to take into account in this regard because, once the electron mass becomes smaller than the energy scale linked to the square of the electric charge, the effective fine-structure constant is no longer a small parameter and a break down of perturbation theory could take place even for quantum non-linearity parameter of the order of unity.

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Appendix A: Pair creation by two photons in 2+1 dimensions

In 2+1 dimensions the S-matrix element which describes the electron positron pair creation out of two photons looks similarly to its counterpart in QED

\[ S_{fi} = -\frac{i^2 e^2}{A^2} \sqrt{\frac{m^2}{p_0^2}} \sqrt{\frac{1}{2\omega \omega'}} (2\pi)^3 \delta_{k'+p',p+q'} M_p \]  \hspace{1cm} (A1)

with

\[ M_p = \bar{u}_p \left( \frac{\gamma k' - \gamma p' + m + i0}{k' - p' + m + i0} + \frac{\gamma k - \gamma p + m + i0}{k - p + m + i0} \right) u_{p'}. \]  \hspace{1cm} (A2)

Here, the fermion propagator in 2+1 dimensions reads

\[ S_F(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip(x - y)}}{p - m + i0}. \]

The probability rate associated with the perturbative pair creation results from Eq. (12) when substituting \( q^- \to p^- \) and \( q^+ \to p^+ \). Explicitly,

\[ dR_{\gamma\gamma'} = \frac{e^4 m^2}{A^2} \frac{2}{8\pi \omega \omega'} \delta_{k' + k^+ + p'} |M_p|^2 \frac{d^2p_-}{p_0^2} \frac{d^2p_+}{p_0^2}, \]  \hspace{1cm} (A3)

where the square of the S-matrix element is

\[ |M_p|^2 = \frac{1}{16m^2} Tr \left[ \left( \frac{\gamma k' - \gamma p' + m}{k'p'} + \frac{\gamma k - \gamma p + m}{kp} \right) \left( \gamma p' + m \right) \right. \times \left. \left( \frac{\gamma k' - \gamma p' + m}{k'p'} + \frac{\gamma k - \gamma p + m}{kp} \right) \left( \gamma p + m \right) \right]. \]  \hspace{1cm} (A4)

Using the properties of the gamma matrices (see footnote 4) and the energy-momentum conservation, the expression above reduces to

\[ |M_p|^2 = \frac{1}{m^2} \left[ \frac{(kk')^2}{4kp^2k} - 1 + \frac{2m^2kk'}{kp^2k} - \frac{m^4(kk')^2}{(kp^2k)^2} \right]. \]  \hspace{1cm} (A5)

We note that the form of \( |M_p|^2 \) differs from its 3+1 dimensional counterpart (see [70]). When inserting Eq. (A5) into Eq. (A3) we obtain the differential rate

\[ dR_{\gamma\gamma'} = \frac{e^4 a_0^2 A^2}{32\pi} \delta_{k' + k^+ + p'} \frac{d^2p_-}{p_0^2} \frac{d^2p_+}{p_0^2} \times \left[ \frac{(kk')^2}{4kp^2k} - 1 + \frac{2m^2kk'}{kp^2k} - \frac{m^4(kk')^2}{(kp^2k)^2} \right]. \]  \hspace{1cm} (A6)

where the relation \( a_0^{(r)2} \equiv 2/\omega^{(r)} A \) has been used. This expression coincides with Eq. (27) provided the intensity parameter \( \eta = ea_0/m \) is identified.

[1] G. Breit and J.A. Wheeler, *Collision of Two Light Quanta*, Phys. Rev. 46, 1087 (1934).
[2] H. R. Reiss, *Absorption of Light by Light*, J. Math. Phys. 3, 50 (1962).
[3] A. I. Nikishov and V. I. Ritus, *Quantum processes in the field of a plane electromagnetic wave and in a constant field*, J. J. Exptl. Theor. Phys. 46, 776 (1964).
[4] A. I. Nikishov and V. I. Ritus, *Pair production by a photon and photon emission by an electron in the field of an intense electromagnetic wave and in a constant field*, J. J. Exptl. Theor. Phys. 52, 1707 (1967).
[5] V. I. Ritus, *Quantum effects of the interaction of elementary particles with an intense electromagnetic field*, (1985).
[6] F. Ehlotzky, K. Krajewska and J. Z. Kamiński, *Fundamental processes of quantum electrodynamics in laser fields of relativistic power*, Rep. Prog. Phys. 72, 046401 (2009).
[7] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan and C. H. Keitel, *Extremely high-intensity laser interactions with fundamental quantum systems*, Rev. Mod. Phys. 84, 1177 (2012).
[8] N. B. Narozhny and A. M. Fedotov, *Extreme light physics*, Contemp. Phys. 56, 249 (2015).
[9] T. Heinzl, A. Ilderton and M. Marklund, *Finite size effects in stimulated laser pair production*, Phys. Lett. B 692, 250 (2010).
[10] K. Krajewska and J. Z. Kamiński, *Breit-Wheeler process in intense short laser pulses*, Phys. Rev. A 86, 052104 (2012).
[11] A. I. Titov, H. Takabe, B. Kämpfer and A. Hosaka, *Enhanced subthreshold e+e- production in short laser pulses*, Phys. Rev. Lett. 108, 240406 (2012).
[12] A. I. Titov, B. Kämpfer, A. Hosaka, T. Nousch and
D. Seipt, Determination of the carrier envelope phase for short, circularly polarized laser pulses, Phys. Rev. D 93, 045010 (2016).

[13] M. J. A. Jansen and C. Müller, Strong-field Breit-Wheeler pair production in short laser pulses: Identifying multiphoton interference and carrier-envelope-phase effects, Phys. Rev. D 93, 053011 (2016).

[14] A. Di Piazza, Nonlinear Breit-Wheeler Pair Production in a Tightly Focused Laser Beam, Phys. Rev. Lett. 117, 213201 (2016).

[15] Q. Z. Lv, S. Dong, Y. T. Li, Z. M. Sheng, Q. Su and R. Grobe, Role of the spatial inhomogeneity on the laser-induced vacuum decay, Phys. Rev. A 97, 022515 (2018).

[16] A. Hartin, A. Ringwald and N. Tapia, Measuring the boiling point of the vacuum of quantum electrodynamics, Phys. Rev. D 99, 036008 (2019).

[17] S. Villalba-Chávez and C. Müller, Photo-production of scalar particles in the field of a circularly polarized laser beam, Phys. Lett. B 718, 992 (2013).

[18] M. J. A. Jansen, and C. Müller, Strongly enhanced pair production in combined high- and low-frequency laser fields, Phys. Rev. A 88, 052125 (2013).

[19] K. Krajeńska and J. Z. Kamiński, Coherent combs of antimatter from non-linear electron-positron-pair creation, Phys. Rev. A 90, 052108 (2014).

[20] S. Meuren, C. H. Keitel and A. Di Piazza, Semiclassical picture for electron-positron photocreation in strong laser fields, Phys. Rev. D 93, 085028 (2016).

[21] T. G. Blackburn and M. Marklund, Nonlinear Breit-Wheeler pair creation with bremsstrahlung γ rays, Plasma Phys. Control. Fusion 60, 054009 (2018).

[22] A. I. Titov and B. Kämpfer, Non-linear Breit-Wheeler process with linearly polarized beams, Eur. Phys. J. D 74, 218 (2020).

[23] C. Baumann and A. Pukhov, Laser-solid interaction and its potential for probing radiative corrections in strong-field quantum electrodynamics, Plasma Phys. Controlled Fusion 61, 074019 (2019).

[24] T. G. Blackburn, A. Ilderton, M. Marklund and C. P. Ridgers, Reaching supercritical field strengths with intense lasers, New J. Phys. 21, 053040 (2019).

[25] A. Di Piazza et al., Testing Strong Field QED Close to the Fully Nonperturbative Regime using Aligned Crystals, Phys. Rev. Lett. 124, 044801 (2020).

[26] V. Yakimenko et al., Prospect of studying nonperturbative QED with beam-beam collisions, Phys. Rev. Lett. 122, 190404 (2019).

[27] V. I. Ritus, Radiative effects and their enhancement in an intense electromagnetic field, J. Exptl. Theoret. Phys. 57, 2176 (1969).

[28] A. A. Mironov, S. Meuren and A. M. Fedotov, Resummation of QED radiative corrections in a strong constant crossed field, Phys. Rev. D 102, 053005 (2020).

[29] T. Podszus and A. Di Piazza, High-energy behavior of strong-field QED in an intense plane wave, Phys. Rev. D 99, 076004 (2019).

[30] A. Ilderton, Note on the conjectured breakdown of QED perturbation theory in strong fields, Phys. Rev. D 99, 085002 (2019).

[31] D. L. Burke et al., Positron Production in Multiphoton Light-by-Light Scattering, Phys. Rev. Lett. 79, 1626 (1997).

[32] B. King, H. Gies and A. Di Piazza, Pair production in a plane wave by thermal background photons, Phys. Rev. D 86, 125007 (2012).

[33] O. J. Pike et al., A photon-photon collider in a vacuum hohlraum, Nature Photonics 8, 434 (2014).

[34] X. Ribeyre et al., Pair creation in collision of γ-ray beams produced with high-intensity lasers, Phys. Rev. E 93, 013201 (2016).

[35] I. Drebot et al., Matter from light-light scattering via Breit-Wheeler events produced by two interacting Coulomb sources, Phys. Rev. Accel. Beams 20, 043402 (2017).

[36] T. Wang et al., Power Scaling for Collimated γ-Ray Beams Generated by Structured Laser-Irradiated Targets and Its Application to Two-Photon Pair Production, Phys. Rev. Applied 13, 054024 (2020).

[37] A. Golub, S. Villalba-Chávez, H. Ruhl and C. Müller, Linear Breit-Wheeler pair production by high-energy bremsstrahlung photons colliding with an intense X-ray laser pulse, Phys. Rev. D 103, 016009 (2021).

[38] H. Abramowicz et al., Letter of Intent for the LUXE Experiment, arXiv:1909.00860.

[39] The E-320 experiment of the FACET-II initiative at SLAC aims to measure the non-perturbative Breit-Wheeler process; https://portal.slac.stanford.edu/.

[40] For Schwinger effect in bandgapped graphene, see I. Akal, R. Egger, C. Müller and S. Villalba-Chávez, Low-dimensional approach to pair production in an oscillating electric field: Application to bandgap graphene layers, Phys. Rev. D 93, 116006 (2016).

[41] I. Akal, R. Egger, C. Müller and S. Villalba-Chávez, Simulating dynamically assisted production of Dirac pairs in gapped graphene monolayers, Phys. Rev. D 99, 016025 (2019).

[42] For Schwinger effect and other pair production-related phenomena in gapless graphene, we refer to D. Allor, T. D. Cohen and D. A. McGady, Schwinger mechanism and graphene, Phys. Rev. D 78, 096009 (2008).

[43] G. N. Klimchitskaya and V. M. Mostepanenko, Creation of quasiparticles in graphene by a time-dependent electric field, Phys. Rev. D 87, 125011 (2013).

[44] F. Fillion-Gourdeau and S. MacLean, Time-dependent pair creation and the Schwinger mechanism in graphen, Phys. Rev. B 92, 035401 (2015).

[45] F. Fillion-Gourdeau, D. Gagnon, C. Lefebvre and S. MacLean, Time-domain quantum interference in graphene, Phys. Rev. B 94, 125423 (2016).

[46] S. P. Gavrilov, D. M. Gitman and N. Yokomizo, Dirac fermions in strong electric field and quantum transport in graphene, Phys. Rev. D 86, 125022 (2012).

[47] H. K. Avetissian, A. K. Avetissian, G. F. Mrktchian and K. V. Sedrakian, Multiphoton resonant excitation of Fermi-Dirac sea in graphene at the interaction with strong laser fields, J. of Nanophotonics 6(1), 061702 (2012).

[48] M. C. Suster et al., Dynamical Mass Generation in Graphene by Bicircular Laser Fields, J. Phys. : Conf. Ser. 1508, 012004 (2020).

[49] A. Golub, R. Egger, C. Müller and S. Villalba-Chávez, Dimensionality-Driven Photoproduction of Massive Dirac Pairs near Threshold in Gapped Graphene Monolayers, Phys. Rev. Lett. 124, 110403 (2020).

[50] L. Fister, R. Alkofer and K. Schwenzer, On the Infrared Behavior of Landau Gauge Yang-Mills Theory with a Fundamentally Charged Scalar Field, Phys. Lett. B 688, 808 (2009).
237 (2010).

[44] T. Banks and M. O’Loughlin, *Two-dimensional quantum gravity in Minkowski space*, Nucl. Phys. B 362, 649 (1991).

[45] S. Desser, R. Jakiw and S. Templeton, *Topologically massive gauge theories*, Ann. of Phys. 140, 372 (1982).

[46] R. Prange and S. Girvin, *The quantum Hall effect*, Springer-Verlag, New York (1990).

[47] Q. Lin, *Electron-positron pair creation in a vacuum by an electromagnetic field in 3+1 and lower dimensions*, J. Phys. G: Nucl. Part. Phys. 25, 17 (1999).

[48] S. P. Gavrilov and D. M. Gitman, *Vacuum instability in external fields*, Phys. Rev. D 53, 7162 (1996).

[49] Y. Lu, N. Christensen, Q. Su and R. Grobe, *Space-time-resolved Breit-Wheeler process for a model system*, Phys. Rev. A 101, 022503 (2020).

[50] E. A. Martinez et al., *Real-time dynamics of lattice gauge theories with a few-qubit quantum computer*, Nature 543, 516 (2016).

[51] A. N. Pineiro, D. Genkina, M. Lu and I. B. Spielman, *Sauter-Schwinger effect with a quantum gas*, New J. Phys. 21, 083035 (2019).

[52] N. Szpak and R. Schützhold, *Quantum simulator for the Schwinger effect with atoms in bichromatic optical lattices*, Phys. Rev. A 84, 050101(R) (2011).

[53] N. Szpak and R. Schützhold, *Optical lattice quantum simulator for quantum electrodynamics in strong external fields: spontaneous pair creation and the Sauter–Schwinger effect*, New J. Phys. 14, 035001 (2012).

[54] T. Pichler, M. Dalmonte, E. Rico, P. Zoller and S. Montangero, *Real-Time Dynamics in U(1) Lattice Gauge Theories with Tensor Networks*, Phys. Rev. X 6, 011023 (2016).

[55] E. Zohar et al., *Digital Quantum Simulation of Z_{2} Lattice Gauge Theories with Dynamical Fermionic Matter*, Phys. Rev. Lett. 118, 070501 (2017).

[56] T. V. Zache et al., *Quantum simulation of lattice gauge theories using Wilson fermions*, Quantum Sci. Technol. 3, 034010 (2018).

[57] L. Klar, N. Szpak and R. Schützhold, *Quantum simulation of spontaneous pair creation in 2D optical lattices*, arXiv:1901.09880.

[58] R. Ott, T. V. Zache, N. Mueller and J. Berges, *Non-cancellation of the parity anomaly in the strong-field regime of QED_{2+1}*, Phys. Lett. B 805, 135459 (2020).

[59] J. M. Guilarte and M. de la Torre Mayado, *QED_{2+1}: the Compton effect*, Anales de Fisica (Spain) 6, 237 (2002).

[60] S. Bellucci, A. A. Saharian and V. M. Bardeghyan *Induced fermionic current in toroidally compactified spacetimes with applications to cylindrical and toroidal nanotubes*, Phys. Rev. D 82, 065011 (2010).

[61] D. M. Volovk, *On a class of solutions of the Dirac equation*, Z. Phys. 94, 250 (1935).

[62] V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, *Relativistic quantum theory* Pergamon Press Ltd. (1971).

[63] H. R. Reiss, *Effect of an intense electromagnetic field on a weakly bound system*, Phys. Rev. A 22, 1786 (1980).

[64] V. P. Krainov, H. R. Reiss and B. M. Smirnov, *Radiative processes in atomic physics* John Wiley and Sons (1997).

[65] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press (2010).

[66] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Sixth Edition), Academic Press (2000).

[67] M. J. Ablowitz and A. S. Fokas, *Complex Variables: Introduction and Applications*, Cambridge University Press (2003).

[68] C. Leubner, *Uniform asymptotic expansion of a class of generalized Bessel functions occurring in the study of fundamental scattering processes in intense laser fields*, Phys. Rev. A 23, 2877 (1981).

[69] D. E. Aspnes *Electric-field effects on optical absorption near threshold in solids*, Phys. Rev. 147, 554 (1966).

[70] W. Greiner and J. Reinhardt, *Quantum electrodynamics*, Springer-Verlag Berlin Heidelberg (2003).