Normal form of bimeromorphically contractible holomorphic Lagrangian submanifolds

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Abstract
Let $M$ be a holomorphically symplectic complex manifold, not necessarily compact or quasiprojective, and $X \subset M$ a compact Lagrangian submanifold. We construct a deformation to the normal cone, showing that a neighbourhood of $X$ can be deformed to its neighbourhood in $T^*X$. This is used to study Lagrangian submanifolds which can be bimeromorphically contracted to a point. We prove that such submanifolds are biholomorphic to $\mathbb{C}P^n$, and show that a certain neighbourhood of $X$ is symplectically biholomorphic to a neighbourhood of the zero section of its cotangent bundle. This gives a holomorphic version of the Weinstein’s normal neighbourhood theorem.

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1 Both authors acknowledge support of HSE University basic research program; also partially supported by ANR (France) project FANOHK
2 Partially supported by FAPERJ E-26/202.912/2018 and CNPq - Process 310952/2021-2.

Keywords: hyperkähler manifold, Lagrangian submanifold, normal form, holomorphically symplectic manifold

2010 Mathematics Subject Classification: 53C26
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1 Introduction

1.1 Holomorphically symplectic Moser’s lemma and the deformation to the normal cone

A smooth complex manifold $M$ is holomorphically symplectic if it carries a closed holomorphic 2-form $\Omega$, non-degenerate at each point. Clearly, in this case $M$ is of even dimension $2n$ and the canonical bundle of $M$ is trivial.

A subvariety $X \subset M$ is called Lagrangian if the restriction of $\Omega$ to $X$ is zero and $\dim(X) = n$, in other words, $X$ is isotropic of maximal possible dimension.

It is not hard to see that the holomorphically symplectic form defines the complex structure in a unique way (Theorem 3.4). This gives a way to describe the holomorphically symplectic structures symplectically, without referring to the complex structure. We define a C-symplectic structure (Definition 3.3) on a real manifold of dimension $4n$ as a closed complex-valued form $\Omega$ such that $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^2$ is non-degenerate. Treated in this fashion, the holomorphic symplectic structure gains the features we could expect from the symplectic form. In [SV], a C-symplectic version of Moser’s lemma was proven. Just like in case of Moser’s lemma, it was shown that a smooth family $\Omega_t$ of cohomologous C-symplectic structures is trivialized by an appropriate flow of diffeomorphisms. However, a significant caveat applies: Moser’s lemma fails unless one requires the vanishing of the cohomology group $H^1(O_{M,I_t})$, where $I_t$ the family of complex structures on $M$ induced by the C-symplectic forms $\Omega_t$, and $O_{M,I}$ is the sheaf of holomorphic functions on $M$ with the complex structure $I_t$.

Weinstein normal neighbourhood theorem is one of the most immediate and important applications of Moser’s lemma. This theorem, stated in the traditional symplectic context, identifies a tubular neighbourhood of a Lagrangian submanifold with its neighbourhood in its cotangent space, equipped with the standard symplectic structure.
This paper is an attempt to apply this reasoning to C-symplectic manifolds. However, the condition $H^1(\mathcal{O}_M, i_*) = 0$ is quite restrictive. It is not always true, and can otherwise be hard to establish, that $H^1(\mathcal{O}_M, i_*) = 0$ for a neighbourhood of a holomorphic Lagrangian submanifold. One exception is when the holomorphic Lagrangian submanifold $S \subset M$ is bimeromorphically contractible, that is, there exists a holomorphic, bimeromorphic map $M \to M_0$, bijective on $M \setminus S$, which contracts $S$ to a point $s$. As follows from Grauert-Riemenschneider theorem, in this situation the preimage $V := \pi^{-1}(U) \subset M$ of a Stein neighbourhood $U$ of $s \in M_0$ satisfies $H^1(\mathcal{O}_V) = 0$ (Corollary 2.16).

To establish the existence of a normal form, we prove that the neighbourhood $V \supset S$ can be deformed to a neighbourhood of $S$ in its cotangent bundle. For complex manifolds this construction, called “deformation to the normal cone”, is quite standard. In Theorem 4.3, we show that the deformation to the normal cone can be performed in the holomorphically symplectic category. We construct a smooth family of holomorphic symplectic manifolds connecting a neighbourhood of a holomorphic Lagrangian submanifold $S \subset M$ to its neighbourhood in $T^*S$.

The content of this paper is split between Sections 3 and 4, where the applications of C-symplectic Moser’s lemma are discussed and the deformation to the normal cone is constructed, and Section 2, where we show that any bimeromorphically contractible Lagrangian submanifold is biholomorphic to the complex projective space.

1.2 Holomorphic Weinstein theorem

Recently, Lagrangian subvarieties have attracted considerable attention in algebraic geometry of holomorphic symplectic manifolds. One of the reasons is their relation to the birational geometry: Lagrangian subvarieties appear as exceptional loci of birational contractions of certain type. These are well-known to be isomorphic to the projective space whenever smooth (a brief sketch of proof is given in the beginning of Section 2). However, the general, non-algebraic and non-compact case, apparently has not been studied up to now. In this situation, the exceptional subvariety of a contraction is defined as follows.

**Definition 1.1:** Let $X \subset Y$ be a complex subvariety in a complex manifold $Y$. We say that $X$ can be **bimeromorphically contracted** if there exists a proper morphism of complex varieties $Y \to Y_0$ mapping $X$ to a point, which is an isomorphism outside of $X$. 
Our main result is an analogue of Weinstein tubular neighbourhood theorem for bimeromorphically contractible Lagrangian subvarieties, as follows.

**Theorem 1.2:** Let \((M, I, \Omega)\) be a holomorphically symplectic manifold (not necessarily compact or quasi-projective), and \(E \subset (M, I)\) a compact holomorphic Lagrangian submanifold. Assume that \(E\) can be bimeromorphically contracted to a point. Then \(E\) is isomorphic to \(\mathbb{C}P^n\). Moreover, \(E\) has a neighbourhood which is biholomorphically symplectomorphic to a neighbourhood of the zero section in \(T^*\mathbb{C}P^n\).

The first statement shall be proved in the next section, and the proof of the second part shall be given in Subsection 4.2, after recalling Moser’s isotopy lemma and its holomorphic version in Section 3.

**Remark 1.3:** Weinstein tubular neighbourhood theorem fails when \(E\) is a fiber of a holomorphic Lagrangian fibration on a hyperkähler manifold (say, on an elliptic K3 surface). Indeed, the normal bundle \(NE\) is trivial, but the elliptic curve in the elliptic family varies, hence its neighbourhood cannot be isomorphic to a neighbourhood of the zero section in the total space of \(T^*E = E \times \mathbb{C}\).

## 2 Contractible Lagrangian submanifolds

The main result of this section is the following theorem.

**Theorem 2.1:** Let \(X\) be a bimeromorphically contractible compact Lagrangian submanifold of a holomorphic symplectic manifold \(M\) of dimension \(2n\), not necessarily compact or quasiprojective. Then \(X\) is isomorphic to \(\mathbb{C}P^n\).

This section is devoted to the proof, in fact we shall provide two versions of the argument. These versions have a common beginning which proves that \(X\) is Moishezon. Then one may proceed either by adaptation of log-MMP\(^1\) following [Kaw], [N], or by using the positivity properties of the cotangent bundle restricted to a movable curve (see [CP], [CPT]).

\(^1\)This part has been added after the first version of this paper has been written, and is suggested by the subsequent discussions with C. Shramov and Yu. Prokhorov.
To begin with, we recall that in the projective setting, this result is well-known, see e.g. [CMS], chapter 8. The main steps of the proof in the projective case (that is, assuming the contraction morphism \( f : M \to M_0 \) projective, as well as \( M \) and \( M_0 \)) are briefly sketched as follows.

- **Step 1:** The contraction morphism is a log-extremal contraction with respect to a certain boundary divisor \( \Delta \) on \( M \). More precisely, one takes \( H \) ample on \( M_0 \), then \( f^*H \) is big, and writes \( f^*H = A + E \) where \( A \) is ample on \( M \) and \( E \) effective on \( M \). Then \((M, \epsilon E)\) is a klt-pair for a small rational number \( \epsilon \), and the contraction is log-extremal for this pair, since \( E \) is \( f \)-negative. By the log Minimal Model theory, the contraction locus \( X \) is uniruled, i.e. covered by rational curves (see e.g. [Kaw]).

- **Step 2:** According to Z. Ran [Ra], Corollary 5.1, any rational curve in a holomorphic symplectic manifold \( M \) (not necessarily algebraic or compact) deforms in a family of dimension at least \( 2n - 2 \).

- **Step 3:** If \( C \subset X \) is a rational curve, it is contracted by \( f \), so that any deformation of \( C \) inside \( M \) must be contracted as well and therefore lies in \( X \). Hence any rational curve in \( X \) deforms in a family of dimension at least \( 2n - 2 \). In particular this is true for a covering family of minimal rational curves\(^2\). Now the results by S. Kebekus [Ke] imply that this is possible only when \( X \cong \mathbb{P}^n \).\(^3\)

In this section we give a proof of Theorem 2.1 which does not require any projectivity assumption. We start with a contraction criterion by Ancona and Vo Van Tan [AnVo], and obtain in particular that \( X \) is Moishezon (i.e. bimeromorphic to projective). Then the results by Campana–Peternell–Toma and Campana–Paun [CP] (formulated in the projective context in [CP], but remarked by the authors to be of bimeromorphic nature) imply uniruledness. Applying Step 2, we obtain a \( 2n - 2 \)-dimensional family of...\(^2\)I.e. such that all of its members passing through a general point are irreducible. Informally, \( X \) should not covered by rational curves splitting off some curves in this family.

\(^3\)See [CMS], Theorem 4.2. Since [CMS] is known to be not always accurate, we prefer to rely on [Ke], who proves in Proposition 3.1 that the base \( H_x \) of a family of minimal rational curves passing through the general point \( x \) and covering the variety \( X \) is \( \mathbb{P}^{n-1} \), and in the following subsection 3.2 computes that the total space \( U_x \) of this family is \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) \), so that the evaluation map contracts the exceptional section onto \( \mathbb{P}^n \).
minimal rational curves on $X$. It remains to remark that Kebekus’ proof is also valid when $X$ is Moishezon.

Alternatively (we thank Yuri Prokhorov and Costya Shramov for many indications on this argument), once we know that $X$ is Moishezon we can remark that either $X$ has rational curves, or $X$ is projective (or both) ([Sh]). Then in the first case we repeat Step 2 and Step 3, and in the second case we make a small adjustment to reduce to the projective case.

**Remark 2.2:** As soon as we know that $X \cong \mathbb{P}^n$, we can apply Grauert’s cohomological criterion ([G], §4, Satz 7, Corollar) to show that a neighbourhood of $X$ in $M$ is biholomorphic to a neighbourhood of the zero section in the cotangent bundle of $\mathbb{P}^n$: see [CMS], Example 8.1 and Lemma 8.2.

Later on in this paper we also prove the Weinstein normal form theorem for the contractible Lagrangian submanifold (Subsection 4.2), i.e., that there exists a biholomorphism taking the symplectic form on $M$ into the standard symplectic form on the cotangent bundle.

### 2.1 Vo Van Tan’s characterization of contractible subvarieties

Here we recall the main results about the bimeromorphically contractible subvarieties, following [G] [Vo], [AnVo]. All varieties and analytic spaces are assumed to be connected.

**Definition 2.3:** A complex analytic space $X$ is called 1-convex if it admits a proper holomorphic, bimeromorphic map $p : X \to Y$ to a Stein space $Y$, and $p_*\mathcal{O}_X = \mathcal{O}_Y$.

**Remark 2.4:** By Remmert Reduction theorem ([Rem]), a complex variety is holomorphically convex if and only if it admits a proper, holomorphic map with connected fibers to a Stein variety. The 1-convexity is a stronger property which requires this map to be bimeromorphic.

**Definition 2.5:** Given a coherent sheaf $A$ on $X$, denote by $L(A)$ the complex analytic space obtained as the relative spectrum of the sheaf of rings $\bigoplus \text{Sym}^k(A)$ over $X$.

**Remark 2.6:** When $A$ is a vector bundle, $L(A)$ is the total space of the dual bundle $A^*$. 
Definition 2.7: A coherent sheaf $A$ on a compact complex analytic space $X$ is called **ample** if for any coherent sheaf $\mathcal{F}$ on $X$ there exists $k > 0$ such that $\text{Sym}^k(A) \otimes \mathcal{F}$ is globally generated. It is called **cohomologically positive** if there exists $k > 0$ such that $H^i(X, \text{Sym}^k(A) \otimes \mathcal{F}) = 0$ for all $i > 0$, and **weakly positive** if the zero section of $L(A)$ admits a 1-convex neighbourhood.

Claim 2.8: Suppose that $A$ is a torsion-free coherent sheaf on $X$. Then $A$ is weakly positive if and only if the zero section $X \subset L(A)$ is bimeromorphically contractible.

**Proof. Step 1:** The group $\mathbb{C}^*$ acts on $L(A)$ by taking any compact subvariety which is not contained in the zero section to a non-compact family of compact subvarieties. Since 1-convexity implies that all positive-dimensional compact subvarieties of $L(A)$ are contained in a compact subset, this is impossible, and any bimeromorphically contractible subset of $L(A)$ is contained in the zero section.

**Step 2:** The 1-convexity means that $L(A)$ is equipped with a bimeromorphic contraction $f$ to a Stein variety. Since the zero section $X \subset L(A)$ is compact, and a Stein variety has no compact complex subvarieties, the image $f(X)$ is zero-dimensional, hence it is contracted. As follows from Step 2, nothing else is contracted. ■

Weak positivity has been introduced by Grauert in [G], who proved that for line bundles on compact complex manifolds weak positivity is equivalent to positivity, i.e. ampleness, and that a compact complex space carrying a weakly positive vector bundle is projective algebraic ([G], §3, Satz 1 and 2). These results were subsequently extended by Vo Van Tan, [Vo], as follows.

**Theorem 2.9:** ([Vo, Theorem 1]) Let $A$ be a coherent sheaf on a compact irreducible complex space $X$. Then the three conditions defined above (ampleness, weak positivity, cohomological positivity) are equivalent. ■

**Theorem 2.10:** ([Vo, Corollary 6]) A compact irreducible complex space carries a torsion-free weakly positive coherent sheaf if and only if it is Moishezon. ■
Remark 2.11: Grauert proved in [G, §3, Satz 8], that a compact submanifold $X$ in an analytic space $M$ with weakly positive conormal bundle is bimeromorphically contractible, but observed that the converse is not true, that is, the conormal bundle of a contractible submanifold does not have to be weakly positive. He asked whether the contractibility of $X$ implies that the conormal sheaf of some analytic space structure on $X$ is weakly positive. This question has been answered affirmatively by Vo Van Tan and Ancona.

More precisely, there is the following contractibility criterion:

Theorem 2.12: ([Vo, Corollary 7], [AnVo, Corollary 3]) Let $M$ be a complex analytic space and $X \subset M$ a compact subvariety. Then $X$ is bimeromorphically contractible if and only if there exists an ideal sheaf $J \subset \mathcal{O}_M$ such that $\text{Supp}(\mathcal{O}_M/J) = X$ and the conormal sheaf, which is defined as $J/J^2$, is ample on $X$.

Remark 2.13: To say that $X$ carries an ample coherent torsion-free sheaf is equivalent to saying that there is an ample coherent sheaf with sheaf-theoretic support equal to $X$.

Remark 2.14: In particular, it follows that a bimeromorphically contractible subvariety is Moishezon.

2.2 Contractible subvarieties are isotropic

Recall the following classical theorem of complex geometry by Grauert and Riemenschneider [GR], Satz 2.4 (see also [T], Corollary I).

Theorem 2.15: Let $f: U \rightarrow V$ be a bimeromorphic contraction of complex spaces with $U$ smooth. Then $R^i f_*(K_U) = 0$ for $i > 0$, where $K_U$ is the canonical bundle of $U$.

Corollary 2.16: Let $X \subset M$ be a bimeromorphically contractible subvariety of a manifold with trivial canonical bundle. Then any open neighbourhood of $X$ in $M$ contains a tubular neighbourhood $U \supset X$ such that $H^i(\mathcal{O}_U) = 0$ for any $i > 0$.

Proof: Let $f : M \rightarrow M_0$ be the bimeromorphic contraction, mapping $X$ to a point $x \in M_0$. Consider a Stein neighbourhood $V \ni x$, and let $U := f^{-1}(V)$. By assumption, $K_M = \mathcal{O}_M$. Grauert-Riemenschneider theorem (Theorem 2.15) implies that $R^i f_*(\mathcal{O}_U) = 0$. The Grothendieck spec-
tral sequence with $E_2$-table $H^j(R^if_*(O_U))$ converges to $H^{i+j}(O_U)$, giving $H^k(O_U) = H^k(f_*O_U) = H^k(O_V) = 0$ because $V$ is Stein. 

The fact that a contractible subvariety is Moishezon, together with Grauert-Riemenschneider theorem, yields the following result which is well-known in the projective case (see e. g. [Ka]):

**Proposition 2.17:** Let $M$ be a holomorphic symplectic manifold and $X$ a contractible subvariety. Then $X$ is isotropic, that is, the restriction of the holomorphic symplectic form $\Omega$ to $X$ is zero.

**Proof:** We may assume that $M$ is a neighbourhood of $X$ as above, so that $H^2(O_M) = 0$. Let $h: X' \to X \subset M$ be a resolution of singularities of $X$ with $X'$ projective (it exists because $X$ is Moishezon, Remark 2.14). The Dolbeault cohomology class of the $(2,0)$-form $h^*\Omega$ is zero, because the Dolbeault cohomology class of $\Omega$ itself is zero on $M$. Since $X'$ is smooth projective, by Hodge theory $h^*\Omega = 0$, so that $\Omega$ must be zero on $X$. 

### 2.3 An MMP-style proof of Theorem 2.1

Let $f: M \to M_0$ be our bimeromorphic contraction. Without loss of generality, we replace $M_0$ by a small Stein neighbourhood of $x = f(X)$. In particular, we may assume that $M$ retracts on $X$. We have established that $X$ is Moishezon. It is known, see [P] in dimension 3, [Sh], [VP] in arbitrary dimension, that all Moishezon varieties either contain rational curves, or are projective, or both.

**Case 1:** $X$ contains a rational curve. Since $X$ is isotropic, we know that $\dim(X) \leq n$. By [Ra], any rational curve in $M$ deforms in a family of dimension at least $2n - 2$. Since $M_0$ is Stein, all rational curves in $M$ are contained in the contraction locus $X$. We claim they cover $X$ and the dimension of $X$ is $n$. Indeed without loss of generality we may consider a family of "minimal", or "generically unsplit" curves: this means that all curves through a general point $z$ of the subvariety $Z \subset X$ which they cover, are irreducible. Then by bend-and-break there is only a finite number of minimal curves through $z$ and any other point of $Z$. Dimension count gives $\dim(Z) \geq n$ and $Z = X$. One proceeds to the proof of Theorem 2.25 to finish the argument.

**Case 2:** $X$ is projective.
Claim 2.18: Let $M$ be a complex manifold with trivial canonical bundle admitting a holomorphic birational contraction $f : M \to M_0$, contracting a submanifold $X \subset M$ to a point $x$. If $X$ is projective, then some ample line bundle $L$ on $X$ extends, together with its sections, to a line bundle $L'$ on $f^{-1}(U)$, where $U$ is a suitable neighbourhood of $x$.

Proof: We choose $U$ in such a way that it is Stein and $X$ is a deformation retract of $f^{-1}(U)$. To simplify the notation, we may assume that $U = M_0$, and $f^{-1}(U) = M$. Consider the exponential exact sequence $0 \to \mathbb{Z}_M \to \mathcal{O}_M \to \mathcal{O}^*_M \to 0$. From this exact sequence (and a similar sequence on $X$) we obtain the following diagram, with all rows exact.

\[
\begin{array}{cccc}
H^1(M, \mathcal{O}_M) & \to & H^1(M, \mathcal{O}^*_M) & \to & H^2(M, \mathbb{Z}) & \to & H^2(M, \mathcal{O}_M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(X, \mathcal{O}_X) & \to & H^1(X, \mathcal{O}^*_X) & \to & H^2(X, \mathbb{Z}) & \to & H^2(X, \mathcal{O}_X)
\end{array}
\]

The vertical arrow $H^2(M, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ in this diagram is an isomorphism, and $H^2(M, \mathcal{O}_M) = 0$ by Grauert-Riemenschneider (Theorem 2.15). This implies that any $\eta \in H^2(X, \mathbb{Z})$ which belongs to the image of $H^1(X, \mathcal{O}^*_X)$ can be lifted to an element of $H^2(M, \mathbb{Z})$ which belongs to the image of $H^1(M, \mathcal{O}^*_M)$. Therefore, the restriction map is surjective on the set of connected components of $H^1(X, \mathcal{O}^*_X)$. Since ampleness is a numerical condition, a connected component of Pic($X$) containing an ample line bundle consists of ample line bundles.

Therefore some ample $L$ on $X$ extends to a line bundle $L'$ on $M$. The proof of [N], Proposition 1.4, shows that for a sufficiently large $k$, the restriction map $(L')^\otimes k \to L^\otimes k$ is surjective on the global sections.

Hence the contraction $f$ is projective. Now we obtain the uniruledness by the proof of [Kaw], Theorem 2. Indeed, this theorem is a consequence of [Kaw], Lemma 2, where the crucial step is the vanishing of higher direct images from [N], Theorem 3.6, valid in the complex-analytic context. Once the uniruledness is established, we again proceed to Theorem 2.25.

Remark 2.19: As noted in the proof of [N], Theorem 3.6, we are in fact dealing with a version of log-MMP here. Namely, to apply log-MMP one needs a boundary divisor negative on $X$. In the projective setting, it is given by decomposing the big divisor $f^*H$, $H$ ample, in the sum of ample
and effective. In the complex-analytic setting one remarks that $(L')^{-1}$ is effective, indeed $f_*((L')^{-1})$ has sections because $M_0$ is Stein. This gives the boundary divisor negative on $X$.

In the rest of this section, we give another proof of Theorem 2.1, where the uniruledness of $X$ is obtained directly from Ancona-Vo Van Tan criterion with the help of [CP].

### 2.4 Lagrangian contractible submanifolds are uniruled

Here we apply Vo Van Tan’s results to a bimeromorphically contractible Lagrangian submanifold $X$ and prove that it is uniruled (that is, covered by rational curves).

**Lemma 2.20:** i) A quotient of an ample sheaf is ample;

ii) A tensor power of an ample sheaf is again ample;

iii) If $\text{Supp}(\mathcal{A}) = X$, the restriction of an ample sheaf $\mathcal{A}$ to a curve on $X$, which is not contained in the singular set of $\mathcal{A}$, is of strictly positive degree.

**Proof:** The first assertion is true because the $k$-th symmetric power of a sheaf surjects onto the $k$-th symmetric power of its quotient, the second is proved using the representation theory, exactly as in the vector bundle case treated in [H], and the third results from the fact that a globally generated sheaf of positive rank on a curve has strictly positive degree, unless it is trivial.

**Lemma 2.21:** Let $X \subset M$ be a smooth submanifold, and $\mathcal{J} \subset \mathcal{O}_M$ an ideal sheaf defining an analytic subspace of $M$ with support $X$ (that is, $\text{Supp}(\mathcal{O}_M/\mathcal{J}) = X$). Denote by $J_X$ the ideal of $X$. Then there exists a non-zero map from $\mathcal{J}/\mathcal{J}^2$ to $\text{Sym}^r(J_X/J_X^2) = J_X^r/J_X^{r+1}$ for some $r$.

**Proof:** We have $\mathcal{J} \subset J_X$. Choose $r$ such that $\mathcal{J} \subset J_X^r$ and $\mathcal{J} \subsetneq J_X^{r+1}$. Then the natural map $\mathcal{J} \rightarrow J_X^r/J_X^{r+1}$ is non-zero. However, since $\mathcal{J} \subset J_X$, this map vanishes on $\mathcal{J}^2$, defining a non-zero sheaf morphism $\mathcal{J}/\mathcal{J}^2 \rightarrow J_X^r/J_X^{r+1}$.

**Corollary 2.22:** If $X$ is a contractible Lagrangian submanifold in a holomorphic symplectic $M$, then for some $r$, $\text{Sym}^r TM$ contains an ample subsheaf.
Proof: For such an $X$, the holomorphic symplectic form provides an isomorphism between the conormal bundle and the tangent bundle. Take $J$ as in Theorem 2.12 and apply the above lemma.

In the sequel, we will need the following theorem, implicit in the work of Campana and Paun [CP] and the preceding work of Campana, Peternell and Toma [CPT]. Define a movable curve $C$ on $X$ as a member of a dominating family of irreducible curves.

Theorem 2.23: Let $X$ be a Moishezon manifold. Assume that for some $r > 0$, $\text{Sym}^r TX$ has a subsheaf $\mathcal{E}$ such that $\deg(\mathcal{E}|_C) > 0$ for a movable curve $C$. Then $X$ is uniruled.

Proof: Let $\alpha$ be the cohomology class of $C$, then, in the terminology of [CP], $\mu_{\alpha, \text{max}}(\text{Sym}^r TX) > 0$. Then also $\mu_{\alpha, \text{max}}(TX) > 0$ ([CP], Theorem 2.9), and the maximal destabilizing sheaf $F$ of $TX$ has positive $\alpha$-slope (together with all its quotients). If $X$ was projective, we would conclude immediately that $X$ is uniruled, by [CP], Theorem 1.1. Otherwise, we let $\pi : X' \to X$ be a projective modification. The class $\pi^* \alpha$ is a movable class on $X'$. Indeed, by [BDPP] a movable class $\beta$ on a projective manifold is characterized by $D\beta \geq 0$ for all effective divisor classes $D$. Setting $\beta = \pi^* \alpha$, we get that this condition is satisfied by the projection formula: indeed $\alpha \pi_* D = 0$ when $D$ is contracted by $\pi$, and is non-negative otherwise. Now in the same way as in Lemma 2.12 of [CP], the subsheaf $\mathcal{F}' \subset TX'$ defined by $\mathcal{F}' = \pi^* F \cap TX'$ is of positive slope with respect to the movable class $\pi^* \alpha$, and we conclude by the main results of [CP] that $X'$, and hence also $X$, is uniruled.

Corollary 2.24: A contractible Lagrangian submanifold of a holomorphic symplectic manifold is uniruled.

2.5 Uniruled Lagrangian submanifolds

Following [CMS, HY, Ke], we use the uniruledness (Theorem 2.23) to prove the following theorem.

Theorem 2.25: Let $X \subset M$ be a bimeromorphically contractible Lagrangian submanifold of a holomorphically symplectic $2n$-dimensional manifold $M$. Then $X$ is biholomorphic to $\mathbb{C}P^n$. 
Proof: By [Ra], Corollary 5.1, any rational curve $C$ on $M$ deforms in a family of dimension at least $2n - 2$. When $C \subset X$, any deformation of $C$ also lies in $X$, because $X$ is contractible. We conclude that any family of rational curves in $X$, $\dim(X) = n$, has dimension at least $2n - 2$. Such families exist because $X$ is uniruled.

We apply this to a complete family of minimal rational curves covering $X$. “Minimal” in this context means that all curves from the family passing through a sufficiently general point $x$ of $X$ are irreducible. It is easy to see by noetherian induction that such minimal rational curves exist. We obtain a family of irreducible curves through $x$ of dimension at least $n - 1$. These curves must cover $X$, by bend-and-break (yielding that there are only finitely many curves connecting $x$ to another point $y$) and by compacity of families of cycles on Moishezon manifolds. Now we are in the situation studied by Kebekus for projective manifolds, see [Ke]. The arguments carry over verbatim to the Moishezon case and prove that $X$ is isomorphic to the projective space.

3 Moser’s lemma and normal form of Lagrangian submanifolds

3.1 Moser’s lemma for holomorphic symplectic and C-symplectic structures

The complex structure on a holomorphically symplectic manifold is uniquely determined by the holomorphic symplectic form. This was the starting point of [SV]. In [SV] the authors obtain a version of holomorphically symplectic Moser’s lemma which accounts for the deformations of complex structure as well as for the deformations of the holomorphic symplectic form.

Moser’s lemma is the following fundamental result of symplectic geometry.

Theorem 3.1: Let $\omega_t$, $t \in [0, 1]$ be a smooth family of symplectic structures on a compact manifold $M$. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in $t$. Then there exists a smooth family $\Psi_t \in \text{Diff}_0(M)$ of diffeomorphisms such that $\Psi_t^* \omega_0 = \omega_t$.

Proof: [McDS].

Remark 3.2: The proof proceeds by constructing a vector field and in-
integrating it to a flow of diffeomorphisms. It is for integration that one needs compacity. In particular, one can obtain a similar statement replacing a compact manifold $M$ by a neighbourhood of a compact submanifold $X \subset M$.

For a holomorphically symplectic version of this lemma, we recall the notion of C-symplectic structures. A C-symplectic structure is a holomorphically symplectic form understood abstractly, that is, without fixing a complex structure (which is uniquely determined by the holomorphically symplectic form nevertheless).

**Definition 3.3:** ([BDV]) Let $M$ be a smooth $4n$-dimensional manifold. A closed complex-valued form $\Omega$ on $M$ is called **C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form.

**Theorem 3.4:** Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T_{\Omega}^{0,1}(M) := \ker \Omega$, where
\[ \ker \Omega := \{ v \in TM \otimes \mathbb{C} \mid \Omega \wedge v = 0 \}. \]
Then $T_{\Omega}^{0,1}(M) \oplus T_{\Omega}^{0,1}(M) = TM \otimes \mathbb{C}$, hence the sub-bundle $T_{\Omega}^{0,1}(M)$ defines an almost complex structure $I_\Omega$ on $M$. Moreover, $I_\Omega$ is integrable, and $\Omega$ is holomorphically symplectic on $(M, I_\Omega)$.

**Proof:** [BDV, Proposition 2.12], [Ve, Theorem 3.5].

Now we can state the C-symplectic version of Moser’s lemma.

**Theorem 3.5:** Let $(M, I_t, \Omega_t)$, $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold, with corresponding complex structures $I_t$. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}(M, I_t))$ is the first cohomology of the sheaf of holomorphic functions. Then there exists a smooth family of diffeomorphisms $V_t \in \text{Diff}_0(M)$, such that $V_t^*\Omega_0 = \Omega_t$.

**Proof:** [SV, Theorem 2.5 (2)].

For non-compact manifolds, a slightly more cumbersome version of Moser’s lemma can be stated.

**Theorem 3.6:** Let $\pi: \mathcal{X} \to \Delta$ be a smooth family of holomorphic symplectic manifolds (not necessarily compact) over the unit disc, locally trivial as a family of $C^\infty$ manifolds. Denote by $\mathcal{X}_t = \pi^{-1}(t)$ its fiber, and let $\Omega_t \in$
\[ H^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t}) \] be its holomorphic symplectic form, smoothly depending on \( t \). Using the \( C^\infty \) trivialization to identify cohomology groups of the fibres, assume that the cohomology class of \( \Omega_t \) does not depend on \( t \in \Delta \), and \( H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0 \). Let \( K \subset \mathcal{X}_{t_0} \) be a compact subset. Then there exists an open neighbourhood \( U \subset \Delta \) of \( t_0 \in \Delta \), and an open subset \( \tilde{U} \subset \pi^{-1}(U) \), with \( K \subset \tilde{U} \), with the following property. The set \( \tilde{U} \) is locally trivially fibered over \( U \), with all fibres \( \tilde{U} \cap \pi^{-1}(t) \), \( t \in U \) isomorphic as holomorphically symplectic manifolds.

**Proof:** [SV, Theorem 2.5 (1)]. \( \blacksquare \)

We will use an equivalent version of this result.

**Theorem 3.7:** Let \( \pi: \mathcal{X} \to \Delta \) be a smooth family of holomorphic symplectic manifolds (not necessarily compact) over the unit disc, trivial as a family of \( C^\infty \) manifolds. Denote by \( \mathcal{X}_t = \pi^{-1}(t) \) its fiber, and let \( \Omega_t \in H^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t}) \) be its holomorphic symplectic form, smoothly depending on \( t \). Using the \( C^\infty \) trivialization to identify cohomology groups of the fibres, assume that the cohomology class of \( \Omega_t \) does not depend on \( t \in \Delta \), and \( H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0 \). Let \( K_t \subset \mathcal{X}_t \) be a locally trivial family of compact subsets, and \( K \subset \mathcal{X} \) their union. Then there exists an open neighbourhood \( U \subset \Delta \) of \( t_0 \in \Delta \), and an open subset \( \tilde{U} \subset \pi^{-1}(U) \), with \( K_t \subset \tilde{U} \), with the following property. The set \( \tilde{U} \) is locally trivially fibered over \( U \), with all fibres \( \tilde{U} \cap \pi^{-1}(t) \), \( t \in U \) isomorphic as holomorphically symplectic manifolds.

**Proof:** Chose \( U \subset \Delta \) and \( \tilde{U} \subset \mathcal{X} \) as in Theorem 3.6. Denote by \( \Delta_\lambda \) the disc of radius \( \lambda \) centered in 0. We need to show that \( \mathcal{K} \cap \pi^{-1}(\Delta_\lambda) \subset \tilde{U} \) for \( \lambda \) sufficiently small. Introduce a Riemannian metric on \( \mathcal{X} \). Since \( \tilde{U} \subset \mathcal{X} \) is open, it contains an \( \varepsilon \)-neighbourhood of \( K_0 \), for a sufficiently small \( \varepsilon > 0 \). Choose \( \lambda \) in such a way that \( K_t \) belongs to an \( \varepsilon \)-neighbourhood \( K_0(\varepsilon) \) of \( K_0 \) for all \( t \in \Delta_\lambda \); this is possible to do, because the Hausdorff distance \( d_H(K_t, K_0) \) converges to 0 as \( t \) converges to 0. For such \( \lambda \), clearly, one has

\[ \mathcal{K} \cap \pi^{-1}(\Delta_\lambda) \subset K_0(\varepsilon) \subset \tilde{U}. \]

\( \blacksquare \)

### 3.2 Moser’s lemma for a neighbourhood of a holomorphic Lagrangian submanifold

We will apply Theorem 3.7 when \( K \) is bimeromorphically contractible Lagrangian submanifolds. We need the following version of Moser’s lemma.
**Lemma 3.8:** Let \((M, I_t, \Omega_t), t \in [0, 1]\) be a smooth family of C-symplectic manifolds (not necessarily compact), with all \(\Omega_t\) exact, and \(E_t \subset (M, I_t)\) be a family of compact holomorphic Lagrangian subvarieties. Assume that \(H^{0, 1}(M, I_t) = 0\). Then \(E_t\) have open neighbourhoods \(U_t\) in \(M\) such that \((U_t, I_t, \Omega_t, E_t)\) is trivialized by a flow of diffeomorphisms.

**Proof:** Follows immediately from Theorem 3.7. ■

In the situation described by Corollary 2.16, any deformation of a neighbourhood of a holomorphic Lagrangian subvariety is locally trivial by Lemma 3.8, because the relevant cohomology groups vanish. This brings

**Corollary 3.9:** Let \((M, I, \Omega)\) be a holomorphically symplectic manifold (not necessarily compact), and \(E \subset (M, I)\) a compact holomorphic Lagrangian subvariety biholomorphic to \(\mathbb{C}P^n\). Assume that a neighbourhood of \(E\) can be smoothly deformed to a neighbourhood of the zero section in \(T^*E\) as a C-symplectic manifold. Then \(E\) has a neighbourhood which is isomorphic to a neighbourhood of \(E\) in \(T^*E\) as a holomorphically symplectic manifold.

**Proof:** We apply Lemma 3.8 to the deformation \((U_t, I_t, \Omega_t, E_t)\) connecting a neighbourhood \(U_0\) of \(E\) in \(M\) to a neighbourhood \(U_1\) of \(E\) in \(T^*E\). Note that since \(E \cong \mathbb{P}^n\), all \(E_t\) are isomorphic to \(\mathbb{P}^n\) as well. Therefore they are all bimeromorphically contractible inside \(U_t\): indeed \(N^*E_t \cong T E_t\) is isomorphic to the cotangent bundle of \(\mathbb{P}^n\) and hence ample, and we conclude by Remark 2.11. Hence \(H^i(\mathcal{O}_{U_t}), i > 0\), of a tubular neighbourhood of \(E_t\) in \((M, I_t)\) vanish by Corollary 2.16, and Lemma 3.8 applies. ■

**Remark 3.10:** The exactness condition on the symplectic forms is satisfied automatically in a sufficiently small neighbourhood \(U_t\) of \(E_t\), because \(U_t\) retracts on \(E_t\).

### 4 Deformation to the normal cone of a Lagrangian submanifold

To apply Corollary 3.9, we need to deform a neighbourhood of a contractible Lagrangian submanifold to its neighbourhood in its cotangent space. This
Deformation to the normal cone construction is well known in algebraic geometry under the name “deformation to the normal cone”; however, we have to make sure it works in the holomorphic symplectic category.

4.1 Deformation to the normal cone: an introduction

Let $X \subset M$ be a complex submanifold in a complex manifold. **Deformation to the normal cone** is a holomorphic deformation of a neighbourhood of $X \subset M$ over the disk such that its central fiber is the total space of the normal bundle $N_X$, and the rest of the fibers are $M$. It is obtained as follows.

Let $X \subset M$ be a complex subvariety. Consider a product $M_1 := M \times \Delta$ of $M$ with the disk $\Delta$, and let $\tilde{M}_1$ be the blow-up of $M_1$ in $X \times \{0\}$. Denote by $\tilde{\pi}_1 : \tilde{\mathcal{N}} \to \Delta$ the blow-down composed with the projection. The preimage $\tilde{\pi}_1^{-1}(0)$ is a union of two irreducible components, the proper preimage of $M \times \{0\}$, denoted $D_1$, and the blow-up divisor, denoted $D_2$.

**Definition 4.1**: The **deformation to the normal cone** is the complement $\tilde{M} := \tilde{M}_1 \setminus D_1$, considered as a fibration over the disk $\Delta$.

Clearly, the central fiber of the natural projection $\tilde{M} \to \Delta$ is $D_2 \setminus (D_1 \cap D_2)$.

**Claim 4.2**: When $X$ is smooth, and $M$ is a tubular neighbourhood of $X$ in $M$, the complement $D_2 \setminus (D_1 \cap D_2)$ is naturally isomorphic to $N_X$, and the “deformation to the normal cone” family $\tilde{\pi} : \tilde{M} \to \Delta$ is locally trivial in the smooth category.

**Proof. Step 1**: The blow-up divisor $E = \mathbb{P}N_{M_1}X = \mathbb{P}(N_X \oplus O_X)$, and its intersection with $D_1$ is the set of all $l \in \mathbb{P}(N_X \oplus O_X)$ tangent to $M \times \{0\}$. We identify this intersection with $\mathbb{P}(N_X)$. This gives an isomorphism $D_2 \setminus (D_1 \cap D_2) = \mathbb{P}(N_X \oplus O_X) \setminus \mathbb{P}(N_X) = \text{Tot}(N_X)$.

**Step 2**: Now, the tubular neighbourhood of $X \subset M$ is diffeomorphic to $\text{Tot}(N_X)$, hence all fibers of $\tilde{\pi} : \tilde{M} \to \Delta$ are diffeomorphic. ■
4.2 Deformation to the normal cone in holomorphic symplectic category

The following theorem, together with Corollary 3.9, implies the holomorphic version of the Weinstein normal neighbourhood theorem (Theorem 1.2).

**Theorem 4.3:** Let \((M, \Omega)\) be a holomorphically symplectic manifold, and \(X \subset M\) a holomorphically Lagrangian submanifold. Then there exists a smooth, holomorphic deformation of a neighbourhood of \(X\) in \(M\) over the disk \(\Delta\), such that its central fiber is biholomorphic to a neighbourhood of \(X\) in \(T^*X\), the rest of the fibers are biholomorphic to a neighbourhood of \(X\) in \(M\), and the holomorphic symplectic form on \(T^*X\) can be smoothly extended to the holomorphic symplectic form on the rest of the fibers.

**Remark 4.4:** Note that this deformation in complex analytic category is already constructed: it is the “deformation to the normal cone” family. However, to apply the C-symplectic Moser lemma, we need to have a smooth family of holomorphically symplectic forms on its fibers.

**Proof of Theorem 4.3. Step 1:** Let \(\tilde{M} \xrightarrow{\pi} \Delta\) be the deformation to the normal cone family, and \(t\) the coordinate on \(\Delta\). Locally in \(X\) we can write \(X\) by a system of holomorphic equations \(q_1 = q_2 = \ldots = q_n = 0\), and the holomorphically symplectic form as \(\Omega = \sum_{i=1}^n dp_i \wedge dq_i\). We are going to prove that \(t^{-1}\Omega\) is extended to a non-degenerate form on the central fiber of \(\pi\).

Since this extension is unique, if would suffice to prove that \(t^{-1}\Omega\) can be extended to the central fiber locally in \(X\), and to check that it is non-degenerate.

**Step 2:** The coordinates on the central fiber of the deformation to the normal cone family \(\tilde{M} \xrightarrow{\tilde{\pi}} \Delta\) are given by \(p_1, \ldots, p_n, \tilde{q}_1, \ldots, \tilde{q}_n\). Trivializing the neighbourhood of \(x \times \Delta \in X \times \Delta\) along \(\Delta\) in the usual way, we write \(\tilde{q}_i = t^{-1}q_i\): this is the standard way to write coordinates on the blow-up.

Locally, we can always find a \(\theta \in \Omega^1 M\) such that \(d\theta = \Omega\) and \(\theta \big|_X = 0\). For example, we can take \(\theta = \sum_i q_i dp_i\) in the above coordinates. Writing \(\theta\) in the coordinates \(\tilde{q}_i, p_i\), we get \(\tilde{\theta} = t \sum_i \tilde{q}_i dp_i\). Then \(d(t^{-1}\theta) = \sum_i d\tilde{q}_i \wedge dp_i\) is a holomorphic form on \(\tilde{M}\). Restricted to the fibers \(\pi^{-1}(u)\) of the projection \(\tilde{M} \xrightarrow{\tilde{\pi}} \Delta\), this form is equal to \(t^{-1}\Omega + d(t^{-1}) \wedge \theta\); the second term restricted to \(\pi^{-1}(u)\) vanishes, hence the restrictions of \(d(t^{-1}\theta)\) extend
Step 3: It remains to show that $d(t^{-1}\theta)$ is non-degenerate on the central fiber of $\tilde{M} \rightarrow \Delta$. Writing $\Omega = \sum_{i=1}^{n} dp_{i} \wedge dq_{i}$ as above and passing to the coordinates $q_{i} = t\tilde{q}_{i}$, we obtain $\Omega = \sum_{i=1}^{n} tdp_{i} \wedge d\tilde{q}_{i} + \sum_{i=1}^{n} dp_{i} \wedge \tilde{q}_{i}dt$. Since the last term vanishes on the fibers, the form $t^{-1}\Omega = \sum_{i=1}^{n} dp_{i} \wedge d\tilde{q}_{i}$ is smooth, non-degenerate on the central fiber, and equal to $t^{-1}\Omega$ on the general fibers. This proves Theorem 4.3.

Acknowledgements: We are grateful to Frédéric Campana, Andreas Höring, Dmitry Kaledin, Yuri Prokhorov, Costya Shramov and Andrey Soldatenkov for their insightful discussions, and to Arnaud Beauville for his valuable email communication.

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