Non-Commutative Moduli Spaces, Dielectric Tori and T-duality

David Berenstein,\textsuperscript{*}Vishnu Jejjala,\textsuperscript{†}

\textit{Department of Physics, University of Illinois, Urbana, IL 61801}

and

Robert G. Leigh\textsuperscript{‡}

\textit{Enrico Fermi Institute, University of Chicago, Chicago, IL 60637}

\textit{and Department of Physics, University of Illinois, Urbana, IL 61801}

September 7, 2018

Abstract

We review and extend recent work on the application of the non-commutative geometric framework to an interpretation of the moduli space of vacua of certain deformations of \( \mathcal{N} = 4 \) super Yang-Mills theories. We present a simple worldsheet calculation that reproduces the field theory results and sheds some light on the dynamics of the D-brane bubbles. Different regions of moduli space are associated with \( D5 \)-branes of various topologies; singularities in the moduli space are associated with topology change. \( T \)-duality on toroidal topologies maps between mirror string realizations of the field theory.

\textsuperscript{*}e-mail: berenste@pobox.hep.uiuc.edu
\textsuperscript{†}e-mail: vishnu@pobox.hep.uiuc.edu
\textsuperscript{‡}e-mail: rgleigh@uiuc.edu
1 Introduction

The $N = 4$ super Yang-Mills theories in four dimensions possess exactly marginal deformations

$$W = \text{tr} \left( \phi_1 \phi_2 \phi_3 - q \phi_2 \phi_1 \phi_3 \right) + \lambda \text{tr} \left( \phi_1^3 + \phi_2^3 + \phi_3^3 \right)$$

(1)

The $q$-deformation is of particular interest. In its presence, we find $F$-term constraints

$$[\phi_1, \phi_2]_q = 0, \quad [\phi_2, \phi_3]_q = 0, \quad [\phi_3, \phi_1]_q = 0$$

(2)

The solutions of these equations determine the moduli space of supersymmetric vacua of the theory. Eqs. (2) give relations in the algebra of $N \times N$ matrices. This algebra is non-commutative, and thus the moduli space has a non-commutative geometric interpretation. Points in this non-commutative geometry are defined to be irreducible representations of the algebra (2), up to equivalences (amounting to gauge transformations on the brane world-volume). This definition is in accord with the usual definitions in non-commutative geometry. This structure persists in general, for the superpotential (1), but also for arbitrary relevant single-trace perturbations as well.

In Ref. [2], we presented a discussion of non-commutative moduli space for a variety of deformations. Moduli space is built of direct sums of irreducible representations of the algebra, and given a complete classification of the irreducible representations, it is possible to describe the moduli space as a non-commutative version of the symmetric product.

As well, we noted that these theories may be obtained from two distinct classes of string geometries. First, one may obtain these models via $\mathbb{Z}_m \times \mathbb{Z}_m$ orbifolds with discrete torsion [3, 4, 5], which is related to $q$, for $q^m = 1$. Second, the models are obtained through deformations of the near-horizon geometry, $AdS_5 \times S^5$, dual to the $N = 4$ field theory. These two models are related by mirror symmetry, which may be traced to a T-duality on torus fibrations of the 5-sphere of the near-horizon geometry. The significance of this torus manifests itself when one considers $D3$-branes in background $NS$ and $RR$ fields. We will describe these phenomena in detail below.

The principal effect here is that $D3$-branes in these backgrounds become $D5$-branes of non-trivial topology. This phenomenon has appeared in several guises over the past few years. First, it appeared in Ref. [6], wherein

In particular, each irreducible representation $\mathcal{A} \xrightarrow{\mu} \text{Mat}_N(\mathbb{C}) \rightarrow 0$ provides a maximal ideal, $\text{ker } \mu$. Maximal ideals, as in algebraic geometry, correspond to points.
membranes of Matrix theory were understood to be built from $D0$-branes. Spherical membranes in Matrix theory were considered in Ref. [7, 8]. More recently, Myers [9] has shown that $Dp$-branes in RR backgrounds in string theory carry a $D(p + 2)$-brane dipole moment, and thus may be thought of as $D(p + 2)$-branes with topology $\mathbb{R}^p \times S^2$. This effect comes into play, for example, for relevant deformations [10] within the AdS/CFT correspondence. One of the difficult issues is to understand this phenomenon for a single $Dp$-brane, as then the non-commutative geometry that one obtains is trivial, and one can just as well regard the brane as pointlike, as it seems to have no structure. Moreover, in this regime the $\alpha'$ corrections are not suppressed by $1/N$, and the DBI action is also unreliable.

One purpose of this letter is to try to understand this phenomenon of brane bubbling into spheres for a single $Dp$-brane by using worldsheet methods. We begin with a detailed account of the non-commutative moduli space of vacua. We then consider the AdS/CFT realization of these field theories. Here we are interested in a probe calculation, and so small deformations in the geometry can be taken into account systematically using worldsheet methods. This fills gaps in the discussion of Ref. [2]: the existence of certain massless modes can be proven and is important for the realization of the non-commutative moduli space. $Dp$-branes may be thought of as $D(p + 2)$-branes of various topologies. There is a T-dual orbifold description; singularities in the latter description, such as the fractionation of branes, corresponds directly to the degenerations of these topologies.

2 General Features of the Moduli Space

Let us begin with a description of the moduli space for the $q$-deformed theory, where $q^n = 1$ for some integer $n$. When $q$ is such a root of unity, there are branches of moduli space which do not otherwise exist. First, we note that the center of the algebra is generated by the elements $x = \phi_1^n$, $y = \phi_2^n$, $z = \phi_3^n$, and $w = \phi_1 \phi_2 \phi_3$, and that these satisfy the matrix relation

\[ (-w)^n + xyz = 0 \]  

Thus the center of the algebra reproduces the commutative moduli space, the orbifold upon which we expect point-like $D$-branes to propagate. We will use orbifold language to describe the moduli space here, but there is also a mirror description which we will give details of in the next section. The
orbifold space has singularities along complex lines where two of the \(x, y, z\) vanish.

![Singularity Diagram](image)

Figure 1: Moduli space of \(q\)-deformed theory.

Next, we look for representations of the algebra (2). In the bulk of the orbifold, where at least two of the \(x, y, z\) do not vanish, we find an \(n\)-dimensional\(^3\) representation\(^2\) \(G(\alpha, \beta, \gamma)\)

\[
\begin{align*}
\phi_1 &= aQ \\
\phi_2 &= bQ^{-1}P^{-1} \\
\phi_3 &= cP
\end{align*}
\]

where \(a, b, c\) are arbitrary complex numbers and \(P, Q\) satisfy \([P, Q]_q = 0\). An explicit representation is

\[
P = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & q & 0 & \ldots & 0 \\
0 & 0 & q^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & q^{n-1}
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

The individual (diagonal) elements of \(P\) represent fractional branes, as can be seen by going to the singularities. Indeed, at for example \(x = y = 0\), we find \(n\) one dimensional representations \(R(0, 0, c), R(0, 0, qc) \ldots, R(0, 0, q^{n-1}c)\) which are all distinct. As one approaches the singularity from

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\(^2\)We assume that \(n \leq N\).

\(^3\)This representation has been chosen so that \(\phi_3\) is diagonal at the singularity \(x = y = 0\). At other singularities, one may parameterize in a different way. This may be thought of in terms of patches.
the bulk, the bulk representation $R(a, b, c)$ becomes reducible, and decomposes into the direct sum

$$\lim_{a,b \to 0} R(a, b, c) = R(0, 0, c) \oplus R(0, 0, qc) \oplus \ldots R(0, 0, q^{n-1}c) \quad (8)$$

For $n < N$, we can build representations by composing these $n$-dimensional representations, together perhaps with one-dimensional representations associated to the fractional branes. When $q$ is not a root of unity, the representations that we have given do not exist, and the moduli space is reduced considerably, so only the fractional branes survive.

The decomposition (8) may be represented graphically as in Fig. 3. Note that this diagram is precisely the quiver diagram for the local singularity. Each node in the quiver represents one of the one-dimensional representations, and the lines connecting them represent the vevs of fields which vanish at the singularity.

![Figure 2: Quiver diagram for reducible representations.](image)

### 2.1 Relevant Deformations

It is interesting to see what happens to this moduli space when relevant deformations are added. We consider first the case of a rank one mass term $W = \frac{1}{2}m\phi_3^2$. When $q = 1$, it is clear what happens: at scales below $m$, we integrate out the field $\phi_3$, and the dimension of the moduli space is reduced from $3N$ to $2N$. This is also true for $q = -1$. For $q^n = 1, n > 2$, the mass term does not lift the moduli space in this way. Instead, it deforms it to

$$(-w)^n + xyz = -t^n z^2 \quad (9)$$

4The Casimir $w$ is shifted to $w = \phi_1\phi_2\phi_3 + t\phi_3^2$. 

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where \( t = m/(1-q^2) \). The representation \( R(a, b, c) \) is also suitably deformed to

\[
\begin{align*}
\phi_1 &= aQ \\
\phi_2 &= bQ^{-1}P^{-1} + dQ^{-1}P \\
\phi_3 &= cP
\end{align*}
\]

where \( ad(q^2 - 1) = mc \). Thus, although \( \phi_3 \) is determined, there is a third independent complex parameter – the moduli space has not decreased in dimension by the addition of the mass term. Note that one can solve for \( \phi_3 \) by the \( F \)-term equations and thus “integrate” it out of the superpotential. However, this procedure is misleading in this case, because \( \phi_1, \phi_2 \) have more degrees of freedom than one naively expects.

The mass term does resolve one of the singular complex lines; indeed, the variety \( \text{(9)} \) is singular at \( x = w = z = 0 \) and \( y = w = z = 0 \). Along the line \( x = y = 0 \), the representation is no longer reducible, as it was for \( m = 0 \).

Figure 3: Moduli space of \( q \)-deformed theory, with rank one mass.

One may also consider higher rank mass terms. One such rank three mass term may be written (when \( q \neq 1 \), through a holomorphic field redefinition, as

\[
W = \text{tr} \left( \phi_1 \phi_2 \phi_3 - q\phi_2 \phi_1 \phi_3 + \frac{m}{2} \phi_3^2 + \zeta_3 \phi_3 \right)
\]

This further deforms the Casimir \( w \), resulting in a deformed commutative geometry

\[
xyz + (-w)^n = -t^n z^2 + t_\zeta^n z
\]
where \( t_\zeta = \frac{\zeta^2}{q-1} \). Again, the irreducible representation has three independent complex parameters

\[
\begin{align*}
\phi_1 &= aQ \\
\phi_2 &= bQ^{-1}P^{-1} + dQ^{-1}P + eQ^{-1} \\
\phi_3 &= cP
\end{align*}
\] (15) (16) (17)

where \( ad(q^2 - 1) = mc \) and \( ae(q - 1) = \zeta_3 \). The two complex lines of singularities that were present for \( \zeta_3 = 0 \) are smoothed out to the hyperboloid

\[ xy = t_\zeta^n, \quad z = w = 0 \] (18)

Along this singularity, one can take the \( n \)th root of eq. (18), and obtain one-dimensional representations (corresponding to fractional branes).

Figure 4: Moduli space of \( q \)-deformed theory, with rank three mass.

In addition, there are isolated representations that are not covered by the above solution. For example, there are representations for which \( \phi_1^k = 0 \) non-trivially. This can happen for nilpotent \( \phi_1 \) where \( \phi_1^k = 0 \) for any \( 1 \leq k \leq n \). For \( k < n \), these representations may be thought of as \( q \)-deformed \( SL(2) \) representations, as we may bring the algebra to the form (for \( q \neq 0, 1 \)).

\[
\begin{align*}
[A_+, A_-]_q &= 2A_0, \\
[A_0, A_+]_q &= A_+, \\
[A_-, A_0]_q &= A_-
\end{align*}
\] (19) (20) (21)

Each of these \( n - 1 \) special branches occur at a fixed value of \( z \) along \( x = y = 0 \).

The rank three mass deformation that we have considered here is special, in that it is equivalent to mass eigenvalues \((m, -m, m')\). Other rank three
mass terms lead to algebras which are different from (19)–(21) as one cannot diagonalize the mass terms without altering the form of the $q$-commutator. Nevertheless, the physics is very similar\cite{2} to the case discussed here.

3 Spheres and Tori

Next, we consider the moduli space of these field theories by looking at the dual near-horizon string theory. The orbifold field theory has a description in terms of the near-horizon geometry $AdS_5 \times S_5/\Gamma$ with discrete torsion encoded in the boundary conditions of massless twisted states. Here we are interested instead with the deformation of the $N = 4$ theory. In the AdS/CFT correspondence\cite{11, 12, 13} we are instructed to interpret deformations of a superpotential in terms of non-normalizable modes of supergravity states. The $q$-deformation is induced\cite{14} by elements of the $45$ of $SU(4)_R$ which appears as the second harmonic of the anti-symmetric tensor modes, $G_{(3)} = \tilde{F}_{(3)} - \tau H_{(3)}$. As the deformation preserves conformal invariance (indeed, $G_{(3)}$ is independent of the $AdS_5$ radial coordinate), the geometry is of the form $AdS_5 \times N$ for some $N$.

As we have reviewed earlier, $D$-branes in background $RR$ fields attain multipole moments of branes of higher dimension. Fractional $D3$-branes should be thought of as $D5$-branes of topology $\mathbb{R}^4 \times S^2$, oriented in a certain way. There are actually two effects in the present case. First, the 5-brane is electrically charged with respect to $F_{(7)} \sim *_4 \tilde{F}_{(3)}$, which has support on $\mathbb{R}^4 \times D^3$, where $D^3$ is bounded by the $S^2$. This field is responsible for the dielectric effect. Secondly, there is a background $H_{(3)}^{NS}$. This is an $AdS_5$ scalar, and so has all three indices along $N$. Thus, $H_{(3)}$ contributes to the energy of a 5-brane if the $D^3$ has components along $N$. Taking both fields into account, we then expect that the radius of the disc $D^3$ is oriented partially in the $AdS_5$ direction, and partially along $N$. As shown in Ref. \cite{2}, energy considerations from the DBI action suggest, when the deformation is small, that the 2-sphere is of size

$$\langle r \rangle \sim (c_{NS} + g_{str} c_R)/(4\pi^2\alpha'\)^2$$

where $\int_{D^3} H_{(3)} = c_{NS} r^3$, $\int_{D^3} \tilde{F}_{(3)} = c_R r^3$. Thus the 2-spheres are expected to be of string size, and one should be hesitant in accepting the geometric picture described here. Nevertheless, we will continue with this way of speaking, as it provides useful intuition for the applications studied here.
We are interested in the new branches of moduli space which appear near the singularities. Near such a singularity, when $q^n = 1$, the background which corresponds to the $q$-deformation consists of $H_{(3)}$ only. As a result, the di-electric effect vanishes due to the orientation of the background field.

Small $D5$-branes with spherical topology correspond to fractional branes in the orbifold dual. If the spheres intersect each other, then semi-classically we expect that there are massless string modes stretching between them. However, the radii of the 2-spheres are small in string units, and thus this intuition may not necessarily be reliable. Moreover, two spheres coming together have locally opposite orientations, and thus correspond to a brane-antibrane system where one expects complicated dynamics. In the following subsection, we will show using a worldsheet computation that massless modes are present if we tune the separation of $D3$-branes appropriately. This occurs at finite separation in the presence of a non-zero $G_{(3)}$ field. Finally, in Section 4, we will argue that the existence of these massless modes gives full consistency with field theory expectations.

### 3.1 Massless States

We will demonstrate the existence of massless states between 5-brane spheres by returning to the description in terms of $D3$-branes in weak backgrounds. In particular, we will show that massless states arise when the $D3$-branes are held at fixed non-zero separation. This will be done at lowest order in the background expansion, but it is clear that the massless states persist in stronger backgrounds.

Note that, from the field theory perspective, we expect to see new massless modes when the branes are at special locations. Indeed, from the $q$-deformed superpotential, there are off-diagonal masses (for $\phi_3 = \text{diag}(a_1, a_2)$) proportional to $a_1 - qa_2$ and $a_2 - qa_1$. Thus, massless states are present for $|a_1| = |a_2|$ if we have $|q| = 1$. It is only when $q$ is a root of unity that the new branch in moduli space can open up, but the massless states between two $D$-branes persist.

In Ref. [3], we simply assumed that these massless states are present because of field theory considerations. There are several ways in which this effect may be seen from worldsheet computations. Here, we will consider the equation of motion of an open-string fermion. As is well-known, the Dirac equation is obtained by integrating the BRST current around the vertex operator. In this case, the background makes its presence felt through contact...
terms with the BRST operator. At lowest order in the RR background, the contact terms have been worked out in Ref. [15, 16]: the NSNS contact terms are standard. We will denote the separation of the branes by a vector $a^\mu$, of length $a$ and direction orthogonal to the branes. There is of course a linear contribution in $a^\mu$ to the mass of the stretched string state. The Dirac equation then takes the form

$$\left( i \nabla \cdot \Gamma + \frac{1}{2\pi \alpha'} a \cdot \Gamma + H_{\mu \nu \rho} \Gamma^{\mu \nu \rho} + g F_{\mu \nu \rho} \Gamma^{\mu \nu \rho} \right) \psi = 0 \quad (23)$$

The fields $H_{(3)}$ and $F_{(3)}$ have identical orientations in the background corresponding to the $q$-deformation. In an appropriate coordinate system near the singularity $x = y = 0$, $G_{(3)}$ is oriented along $d\theta \wedge (d\bar{x} \wedge dx - d\bar{y} \wedge dy)$ and $H_{(3)}/g (F_{(3)})$ is the imaginary (real) part. It is then easy to see that the $\Gamma$-matrix structure in eq. (23) has half of its eigenvalues equal to zero when $a = 0$. Also, by aligning $a^\mu$ appropriately, here along $z$, there are zero eigenvalues for a particular value of $|a|$ proportional to the field strength. This corresponds to a chiral fermion in four dimensions. The value of $|a|$ at which this occurs agrees with the Born-Infeld calculation presented above (up to vertex normalizations that we have not considered carefully).

### 3.2 Bulk branes and Tori

Given the presence of massless modes at the intersections of 2-spheres, it is natural to ask under what conditions one can turn on vevs for these modes. These vevs would have an interpretation in terms of smoothing the singularity of joined spheres. However, by consideration of appropriate string diagrams...
(or the equivalent field theory superpotential), we see that there is a potential preventing this condensation in general, consistent with supersymmetry. There is however a special configuration, corresponding to a linear combination of blowup modes, which evades this potential. This occurs when we have $n$ 2-spheres joined and wrapping around the 5-sphere, as is suggested by Fig. 6. This should be compared to the quiver diagram of Fig. 3. Turning on the vev referred to above corresponds to smoothing the structure in Fig. 6 out to a 2-torus, and is equivalent, in the orbifold language, to the process of moving a brane off of the singularity into the bulk. Thus a bulk 3-brane should be thought of as a $D5$-brane with topology $\mathbb{R}^4 \times T^2$. The space $N$ (the deformed $S^5$) should then be thought of topologically in terms of a torus fibration over $S^3$. Large tori are governed by the Born-Infeld action and are extrema of this action. Indeed, one can show that near the degeneration, the Born-Infeld action is independent of the size of the torus; to see this, one needs only the first-order analysis.

This description is related to the orbifold theory by T-duality. The T-duality extends in a natural way to non-commutative geometry. On one hand, we are instructed to think of an irreducible representation of the non-commutative algebra as a point in a non-commutative (moduli) space[[2]]. On the other hand, we often think of this very same algebra as that of the non-commutative torus, as in[[3]]. There is no tension between these two concepts; they are just mirror to each other.

### 3.3 Degenerations

To further explore the duality between the two descriptions, let us consider from a geometric standpoint the various degenerations of the toroidal 5-
branes, and how these processes are related to singularities in the orbifold moduli space. In the massless case, as we have stated, the degenerations in the moduli space correspond directly to brane fractionation.

With a rank one mass term, one of the singular lines is removed; algebraically, the representation of the algebra is no longer reducible there. Topologically, the torus pinches at those points in the moduli space, but only once, so no moduli are available to separate the surface. On a torus fibration, this fiber is singular and becomes a sphere with two points identified.

With a rank three mass term, the singularity is a hyperboloid, and again, along this locus, the torus degenerates to a collection of spheres. When a brane approaches the locus of the special isolated branches (at fixed $z, x = y = 0$), it may be able to split into different localized branes, although we have not been able to check this assertion precisely. In this case, we would get a degeneration into a collection of spheres, but no new moduli.

One can in fact interpret the matrix representations topologically. Diagonal entries, such as in the matrix $P$, correspond to spheres, while non-zero off-diagonal elements can be thought of in terms of a tube joining different spheres. The genus of the resulting Riemann surface is encoded in the number of off-diagonal elements of the generators. Degenerations may be understood in this language in terms of the vanishing of matrix elements.

4 Final remarks and outlook

In this letter we have seen various features of the interactions between $D$-branes and weakly deformed backgrounds. The semi-classical configuration of a $D(p+2)$-brane wrapping an $S^2$ provides us with useful intuition and is consistent with worldsheet results. In a sense however, the branes are truly pointlike, in that the spheres may pass through each other without deformation.

The effect of the background on the world-sheet is quantum-mechanical, and presumably may be thought of in terms of a shift in the zero point energy and modings of stretched strings.

From a dual perspective, the masslessness of these states arises because the fractional branes are coincident. The relation between these two different points of view, when the geometry can be well understood, corresponds to $T$-duality. Such dualities are expected to persist to more general situations.
Acknowledgments: We wish to thank M. Strassler and A. Hashimoto for discussions. RGL thanks the Enrico Fermi Institute at the University of Chicago for hospitality while much of this work was carried out. Work supported in part by U.S. Department of Energy, grant DE-FG02-91ER40677 and an Outstanding Junior Investigator Award.

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