AFFINE LINES IN THE COMPLEMENT OF A SMOOTH PLANE CONIC

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Abstract. We classify closed curves isomorphic to the affine line in the complement of a smooth rational projective plane conic $Q$. Over a field of characteristic zero, we show that up to the action of the subgroup of the Cremona group of the plane consisting of birational endomorphisms restricting to biregular automorphisms outside $Q$, there are exactly two such lines: the restriction of a smooth conic osculating at a rational point and the restriction of the tangent line to $Q$ at a rational point. In contrast, we give examples illustrating the fact that over fields of positive characteristic, there exist exotic closed embeddings of the affine line in the complement of $Q$. We also determine an explicit set of birational endomorphisms of the plane whose restrictions generates the automorphism group of the complement of $Q$ over a field of arbitrary characteristic.

Introduction

A famous theorem of Abhyankar and Moh [1] asserts that over a field $k$ of characteristic zero, all closed embeddings of the affine line $A_k^1$ into the affine plane $A_k^2$ are equivalent under the action of the group $\text{Aut}_k(A_k^1)$ of algebraic $k$-automorphisms of $A_k^2$: for any two such closed embeddings with images $A$ and $A'$, there exists $\Psi \in \text{Aut}_k(A_k^2)$ such that $A' = \Psi(A)$. In this article, we consider the classification of equivalence classes of closed embeddings of the affine line into another smooth affine surface very similar to the affine plane: the complement of a smooth $k$-rational conic $Q$ in the projective plane $\mathbb{P}_k^2$. In the complex case, such a smooth affine surface $S = \mathbb{P}_k^2 \setminus Q$ has divisor class group $\text{Cl}(S) = \mathbb{Z}_2$, integral homology groups $H_0(S; \mathbb{Z}) = \mathbb{Z}$, $H_1(S; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(S; \mathbb{Z}) = 0$ for every $i \geq 2$, and its logarithmic Kodaira dimension $\kappa(S) = \kappa(\mathbb{P}^2, K_{\mathbb{P}^2} + Q)$ (see [10]) is equal to $-\infty$. It is thus very close to the affine plane from both algebraic and topological points of view. It also contains many closed curves isomorphic to the affine line $A_k^1$. For instance, for every point $p \in Q$, $Q$ and twice its tangent line $T_pQ$ at $Q$ generate a pencil $\mathcal{P}_p \subset |O_{\mathbb{P}_k^2}(2)|$ whose members, except for the one $2T_pQ$, are smooth conics intersecting $Q$ with multiplicity 4 at $p$. The intersections with $S$ of all members of $\mathcal{P}_p$ except $Q$ are thus isomorphic to $A_k^1$. The subgroup $\text{Aut}(\mathbb{P}_k^2, Q)$ of $\text{Aut}(\mathbb{P}_k^2)$ consisting of automorphisms preserving $Q$ acts transitively on $Q$, and for a given point $p_0 \in Q$, the action on $\mathcal{P}_{p_0} \setminus Q$ of the subgroup $\text{Aut}(\mathbb{P}_k^2, Q, p_0)$ of $\text{Aut}(\mathbb{P}_k^2, Q)$ consisting of automorphisms fixing $p_0$ has exactly two orbits: a fixed point $T_{p_0}Q$ and its complement $\mathcal{P}_{p_0} \setminus (Q \cup T_{p_0}Q)$. Viewing $\text{Aut}(\mathbb{P}_k^2, Q)$ as a subgroup of $\text{Aut}(S)$ via the natural restriction homomorphism, it follows in particular that $\text{Aut}(S)$ acts on the set of so-defined affine lines in $S$ with at most two orbits: the one of $T_{p_0}Q \cap S$ and the one of $Q_1 \cap S$ for a fixed member $Q_1$ of $\mathcal{P}_{p_0} \setminus (Q \cup T_{p_0}Q)$. But since $\text{Cl}(S) \setminus (S \cap T_{p_0}Q_1))$ is trivial while $\text{Cl}(S) \setminus (S \cap Q_1)) \simeq \mathbb{Z}_2$, it follows that $T_{p_0}Q \cap S$ and $Q_1 \cap S$ cannot belong to a same orbit of the action of $\text{Aut}(S)$ on the set of closed curves in $S$ isomorphic to $A_k^1$. So in contrast with the case of $A_k^2$, the best we can hope for is that the action of $\text{Aut}(S)$ on the set of such closed curves has precisely two orbits. Our main result just below implies that this is exactly the case:

Theorem 1. Let $k$ be a field of characteristic zero, let $Q \subset \mathbb{P}_k^2$ be a smooth conic and let $S = \mathbb{P}_k^2 \setminus Q$. Suppose that $A \subset S$ is a closed curve isomorphic to $A_k^1$ and let $\overline{A} \subset \mathbb{P}_k^2$ be its closure. Then $Q$ is $k$-rational and for every given $k$-rational point $p_0 \in Q$ and every smooth $k$-rational member $Q_1 \neq Q$ of the pencil $\mathcal{P}_{p_0}$ generated by $Q$ and $2T_{p_0}Q$, there exists a birational map $\Phi : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ defined over $k$, restricting to an automorphism of $S$, such that

$$\Phi_*(\overline{A}) = \begin{cases} Q_1 & \text{if } \deg \overline{A} \text{ is even} \\ T_{p_0}Q & \text{if } \deg \overline{A} \text{ is odd}. \end{cases}$$

In particular, there are precisely two classes of closed curves isomorphic to $A_k^1$ in $S$ up to the action of $\text{Aut}_k(S)$. 

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Note that the dichotomy depending on the degree of $\overline{\mathcal{A}}$ follows from the fact that the divisor classes of $Q$ and $\overline{\mathcal{A}}$ either generate $\text{Cl}(\mathbb{P}^2_k)$ when $\deg \overline{\mathcal{A}}$ is odd, or generate a proper subgroup of index 2 when $\deg \overline{\mathcal{A}}$ is even, so that $\text{Cl}(\mathbb{P}^6 \setminus \mathcal{A}) = \{0\}$ or $\mathbb{Z}_2$ according to $\deg \overline{\mathcal{A}}$ is odd or even.

Recall that by a theorem of Jung and van der Kulk \[11\] \[15\], the automorphism group of the affine plane $k^2 = \text{Spec}(k[x,y])$ over an arbitrary field $k$ is the free product of the group of affine automorphisms and of the group of automorphisms of the form $(x,y) \mapsto (ax + b, cy + s(x))$, where $a, c \in k^*$, $b \in k$ and $s \in k[t]$, amalgamated over their intersection. Viewing $k^2$ as the complement of the line at infinity $L = \{z = 0\}$ in $\mathbb{P}^2_k = \text{Proj}(k[x,y,z])$, these two subgroups coincide respectively with the restriction to $k^2$ of the group $\text{Aut}(\mathbb{P}^2_k, L)$ and with the group $\text{Aut}(k^2, \text{pr}_1)$ of automorphisms preserving globally the $k^1$-fibration $\text{pr}_1 : k^2 \to k^1$ induced by the restriction of the pencil of lines through the point $[0 : 1 : 0]$. Our second result consists of an analogous presentation of the automorphism group of the complement of a smooth $k$-rational conic in $\mathbb{P}^2_k$, providing in particular a complete description of the birational maps $\Phi : \mathbb{P}^2_k \dasharrow \mathbb{P}^2_k$ which can occur in Theorem 1.

**Theorem 2.** Let $k$ be a field of arbitrary characteristic and let $S$ be the complement of a smooth conic $Q \subset \mathbb{P}^2_k$ with a $k$-rational point $p$. Let $\text{Aut}(S, \rho_p)$ denote the subgroup of $\text{Aut}(\mathbb{P}^2_k)$ consisting of automorphisms which preserve globally the $k^1$-fibration $\rho_p : S \to k^1$ induced by restriction of the pencil $\mathcal{P}_p \subset \mathcal{O}_{\mathbb{P}^2_k}(2)$ generated by $Q$ and twice its tangent line $T_pQ$ at $p$. Then $\text{Aut}(\mathbb{P}^2_k)$ is the free product of $\text{Aut}(\mathbb{P}^2_k, Q)|_S$ and $\text{Aut}(S, \rho_p)$ amalgamated along their intersection.

The scheme of the article is the following: in the first section, we review standard material on projective completions of smooth quasi-projective surfaces and certain rational fibrations on them. Section two is devoted to the proof of Theorem 1 which proceeds through the analysis of the structure of the total transform of the divisor $Q \cup \overline{\mathcal{A}}$ in a minimal log-resolution of the pair $(\mathbb{P}^2_k, Q \cup \overline{\mathcal{A}})$. Our argument, inspired by a recent alternative proof of the Abhyankar-Moh and Lin-Zaïdenberg theorems due to Palka \[14\], uses techniques and classification results from the theory of $Q$-acyclic complex surfaces, that is, normal complex surfaces with trivial reduced rational homology groups. A standard reference for most of these results is \[13\], to which we refer the reader for a more complete picture of the theory of non complete algebraic surfaces. Theorem 2 is proved in the third section, in which we give in addition an explicit set of generators of $\text{Aut}(\mathbb{P}^2_k)$ for a suitably chosen model of $Q$ up to projective equivalence. We also derive from this description examples illustrating that similarly to the situation for the affine plane, Theorem 1 does not hold over fields of positive characteristic.

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1. **Preliminaries and notations**

In what follows, the term $k$-variety refers to a geometrically integral scheme $X$ of finite type over a base field $k$ of arbitrary characteristic. A morphism of $k$-varieties is a morphism of $k$-schemes. A surface $V$ is a $k$-variety of dimension 2, and by a curve on a surface, we mean a geometrically reduced closed sub-scheme $C \subset V$ of pure codimension 1 defined over $k$.

1.1. **SNC divisors and smooth completions.**

(i) An *SNC divisor* on a smooth projective surface $X$ is a curve $B$ on $X$ with smooth irreducible components and ordinary double points only as singularities. Equivalently, for every closed point $p \in B$, the local equations of the irreducible components of $B$ passing through $p$ form a part of regular sequence in the maximal ideal $\mathfrak{m}_{S_p}$ of the local ring $\mathcal{O}_{S_p}$ of $S$ at $p$.

An SNC divisor $B$ on $X$ is said to be *SNC-minimal* if there does not exist any strictly birational projective morphism $\tau : X \to X'$ onto a smooth projective surface $X'$ with exceptional locus contained in $B$ such that $\tau_*(B)$ is SNC. If $k$ is algebraically closed, then this property is equivalent to the fact that any $(-1)$-curve $E$ contained in $B$ is branching, i.e. meets at least three other irreducible components of $B$.

(ii) A *smooth completion* of a smooth quasi-projective surface $V$ is a pair $(X, B)$ consisting of a smooth projective surface $X$ and an SNC divisor $B \subset V$ such that $X \setminus B \simeq V$. 

1.2. Rational trees and rational chains.

(i) A geometrically rational tree $B$ on a smooth projective surface $X$ is an SNC divisor whose irreducible components are geometrically rational curves and such that the dual graph of the base extension $B_E$ of $B$ to an algebraic closure $\overline{k}$ of $k$ is a tree. A geometrically rational chain $B$ is a geometrically rational tree such that the dual graph of $B_T$ is a chain. A rational tree (resp. rational chain) is a geometrically rational tree (resp. geometrically rational chain) whose irreducible components are all $k$-rational.

The irreducible components $B_0, \ldots, B_r$ of a rational chain $B$ can be ordered in such a way that $B_i \cdot B_j = 1$ if $|i - j| = 1$ and 0 otherwise. A rational chain $B$ with such an ordering on the set of its irreducible components is said to be oriented. The components $B_0$ and $B_r$ are called respectively the left and right boundaries of $B$, and we say by extension that an irreducible component $B_i$ of $B$ is on the left of another one $B_j$ when $i < j$.

The sequence of self-intersections $\left|B_0^2, \ldots, B_r^2\right|$ is called the type of the oriented rational chain $B$. An oriented subchain of an oriented rational chain $B$ is a rational chain $Z$ whose support is contained in that of $B$. We say that an oriented rational chain $B$ is composed of subchains $B_1, \ldots, B_s$, and we write $B = D_1 \sqsupset \cdots \sqsupset D_s$, if the $D_i$ are oriented subchains of $B$ whose union is $B$ and the irreducible components of $D_i$ precede those of $D_j$ for $i < j$.

(ii) An oriented rational chain $F \sqsupset C \sqsupset E$ where $F$ and $C$ are irreducible and $E$ is an oriented subchain, possibly empty, is said to be $m$-standard, $m \in \mathbb{Z}$, if it is of type $[0, -m]$ or $[0, -m, -a_1, \ldots, -a_r]$ where $a_i \geq 2$ for every $i = 1, \ldots, r$. It is an elementary exercise (see e.g. [3]) to check that every chain with non negative definite intersection matrix can be transformed by a sequence of blow-ups and blow-downs whose centers are contained in the successive total transforms of $B$ either into a 0-curve, or into a chain of type $[0, 0, 0]$, or into an $m$-standard chain for every $m \in \mathbb{Z}$.

(iii) In particular, every affine surface $S$ non isomorphic to $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$ admitting a smooth completion $(X_0, B_0)$ for which $B_0$ is a rational chain, admits a smooth completion $(X, B)$ for which $B = F \sqsupset C \sqsupset E$ is $m$-standard chain (see e.g. [3] Lemma 2.7). Furthermore, it follows from a result of Danilov and Gizatullin [3 Corollary 2] that the type of the subchain $E$ is an invariant of $S$, in the sense that if $(X', B')$ is another smooth completion of $S$ by an $m'$-standard chain $B' = F' \sqsupset C' \sqsupset E'$ then the type of $E'$ is either equal to that of $E$ or to that of $E$ equipped with the reversed orientation. For instance, if $S = \mathbb{P}^2 \setminus Q$ is the complement of a smooth $k$-rational conic $Q$ in $\mathbb{P}^2$, then for every smooth completion $(X, B)$ of $S$ by a $m$-standard chain $B = F \sqsupset C \sqsupset E$, the subchain $E$ has type $[-2, -2, -2]$.

1.3. Recollection on $\mathbb{P}^1$, $\mathbb{A}^1$ and $\mathbb{A}^1_\kappa$-fibrations. We review some basic properties of $\mathbb{P}^1$-fibrations on smooth projective surfaces and their restrictions to certain of their open subsets, see e.g. [3] Chapter 3 for more details.

(i) By a $\mathbb{P}^1$-fibration on a smooth projective surface $X$, we mean a surjective morphism $\overline{\eta} : X \to \overline{\mathbb{Z}}$ onto a smooth projective curve $\overline{Z}$ whose generic fiber is isomorphic to the projective line over the function field of $\overline{Z}$. It is well known that every $\mathbb{P}^1$-fibration $\eta : X \to \mathbb{Z}$ is obtained from a Zariski locally trivial $\mathbb{P}^1$-bundle over $\overline{\mathbb{Z}}$ by a finite sequence of blow-ups of points. In particular, every such $\mathbb{P}^1$-fibration has a section, and its singular fibers are supported by geometrically rational trees on $X$. If $X$ is $k$-rational, then it follows from the Riemann-Roch Theorem, that for every smooth $k$-rational curve $F$ with self-intersection 0, the complete linear system $|F|$ defines a $\mathbb{P}^1$-fibration $\overline{\eta}_F : X \to \mathbb{P}^1$ having $F$ as a smooth fiber.

(ii) An $\mathbb{A}^1$-fibration on a smooth quasi-projective surface $V$ is a surjective morphism $\rho : V \to Z$ onto a smooth curve $Z$ whose generic fiber is isomorphic to the affine line over the function field of $Z$. Every $\mathbb{A}^1$-fibration is the restriction of a $\mathbb{P}^1$-fibration $\overline{\eta} : X \to \overline{\mathbb{Z}}$ over the smooth projective model $\overline{Z}$ of $Z$ on a smooth completion $(X, B)$ of $\overline{\eta}$. Furthermore, one can always find such a smooth completion for $B$ has the form $B = \bigcup_{z \in \overline{Z} \setminus Z} F_z \cup C \cup \bigcup_{z \in Z} H_z$, where, $F_z = \overline{\eta}^{-1}(z) \cong \mathbb{P}^1_{k(z)}$ for every $z \in \overline{Z} \setminus Z$, $C$ is a section of $\overline{\eta}$, and where for every $z \in Z$, $H_z$ is an SNC-minimal geometrically rational subtree of $\overline{\eta}^{-1}(z)$, possibly empty, the support of the fiber $\overline{\eta}^{-1}(z)$ being equal to the union of $H_z$ and of the closure in $X$ of the support of $\rho^{-1}(z)$.

If in addition $V$ is affine, then every nonempty $H_z$ contains a $\kappa(z)$-rational irreducible component intersecting $C$, the closure in $X$ of every irreducible component of $\rho^{-1}(z)$ is isomorphic to the projective line over a finite extension $\kappa'$ of $\kappa(z)$, and it intersects $H_z$ transversally in a unique $\kappa'$-rational point. A scheme theoretic closed fiber $\rho^{-1}(z)$ of $\rho : V \to Z$ which is not isomorphic to $\mathbb{A}^1_{k(z)}$ is said to be degenerate.

(iii) An $\mathbb{A}^1_\kappa$-fibration on smooth quasi-projective surface $V$ is a surjective morphism $\xi : V \to Z$ onto a smooth affine curve $\overline{Z}$ whose geometric generic fiber is isomorphic to the punctured affine line $\mathbb{A}^1_\kappa = \mathbb{A}^1 \setminus \{0\}$ over an algebraic closure of the function field of $Z$. We say that $\xi$ is twisted if the generic fiber of $\xi$ is a nontrivial form of $\mathbb{A}^1_\kappa$ over the function field of $Z$, and untwisted otherwise.
2. Affine lines in $\mathbb{P}^2 \setminus Q$

This section is devoted to the proof of Theorem 1. A field $k$ of characteristic zero being fixed throughout, we let $S$ be the complement of a smooth conic $Q$ in $\mathbb{P}^2_k$, we let $A \subset S$ be a closed curve isomorphic to $\mathbb{A}^1_k$ and we let $\overline{A}$ be its closure in $\mathbb{P}^2$. Let $\nu : C \to \overline{A}$ be the normalization of $\overline{A}$. Since $A \cong \mathbb{A}^1_k$, $C$ is isomorphic to $\mathbb{P}^1_k$ and $C \setminus \nu^{-1}(A)$ consists of a unique point. So $\overline{A} \setminus A$ consists of a unique point $p = Q \cap \overline{A}$, which is thus necessarily $k$-rational, at which $\overline{A}$ has a unique local analytic branch. In particular, $Q$ is $k$-rational. Since $\text{Aut}(\mathbb{P}^2_k)$ acts transitively on the set of pairs $(Q,p)$ consisting of a smooth $k$-rational conic and a $k$-rational point on it and since the stabilizer $\text{Aut}(\mathbb{P}^2_k,Q,p) \subset \text{Aut}(\mathbb{P}^2_k)$ of a given pair $(Q,p)$ acts transitively on $Q$, we are reduced to establish the following:

**Proposition 3.** There exists a smooth $k$-rational conic $Q' \subset \mathbb{P}^2_k$ and a birational map $\Psi : \mathbb{P}^2_k \dasharrow \mathbb{P}^2_k$ restricting to an isomorphism $\psi : S = \mathbb{P}^2_k \setminus Q \xrightarrow{\sim} \mathbb{P}^2_k \setminus Q'$ mapping $\overline{A}$ either to a smooth $k$-rational conic intersecting $Q'$ in a unique $k$-rational point $p'$ or to the tangent line $T_{p'}Q'$ of $Q'$ at a $k$-rational point $p'$.

The proof of this proposition is given in §2.1 and §2.2 below.

2.1. **Log-resolution setup and preliminary observations.** Let $\sigma : (X',D') \to (\mathbb{P}^2_k,Q \cup \overline{A})$ be the minimal log-resolution of the pair $(\mathbb{P}^2_k,Q \cup \overline{A})$. Recall that by definition, $X'$ is smooth, $\sigma$ is a projective birational morphism restricting to an isomorphism over $\mathbb{P}^2_k \setminus Q \cup \overline{A}$, and minimal for the property that $D' = \sigma^{-1}(Q \cup \overline{A})_{\text{red}}$ is an SNC divisor. Note that if $p = Q \cap \overline{A}$ is a singular point of $\overline{A}$ then $\sigma$ is in particular a log-resolution of the singularity of $\overline{A}$. Since $\overline{A} \cdot Q \geq 2$ and $p$ is $k$-rational, $\sigma$ consists of the blow-up of $p$ followed by a sequence of blow-ups of $k$-rational points supported on the successive total transforms of $Q \cup \overline{A}$. Since $\overline{A}$ has a unique analytic branch at $p$, $D'$ is a rational tree of the form $\overline{A} \cup D'_1 \cup E' \cup D'_2$, where $D'_1 \cup E' \cup D'_2 = \sigma^{-1}(Q)_{\text{red}}$ consists of a rational tree $D'_1$ containing the proper transform of $Q$, a nonempty SNC-minimal rational chain $D'_2$ with negative definite intersection matrix and a $(-1)$-curve $E'$ such that $D'_1 \cap E'$, $D'_2 \cap E'$ and $\overline{A} \cap E'$ all consist of a unique point.

![Diagram](image)

**Figure 2.1.** Structure of the divisor $D'$ in the case where $\overline{A}$ is a line, a rational conic, and a general curve respectively. The gray dotted lines represent rational subtrees of $D'_1$.

The proper transform of $Q$ is the unique possible non-branching $(-1)$-curve in $D'_1$ and we let $\sigma' : (X',D') \to (X,D)$ be the map consisting of the contraction all successive non-branching $(-1)$-curves in $D'_1$. The image of $D'$ by $\sigma'$ is again a rational tree $D = \tilde{A} \cup D_1 \cup E \cup D_2$ where $\tilde{A} = \sigma_*(\overline{A}) \cong \mathbb{P}^1_k$, $D_1 = \sigma'_*(D'_1)$ is an SNC-minimal rational tree, $D_2 = \sigma'_*(D'_2)$ is a rational chain isomorphic to $D_2$, and $E = \sigma'_*(E')$. By construction $S = \mathbb{P}^2_k \setminus Q$ is isomorphic to $X \setminus (D_1 \cup E \cup D_2)$.

We now establish two crucial auxiliary results which will serve for the analysis of the case where the self-intersection of the proper transform $\tilde{A}$ of $\overline{A}$ in $X$ is negative.
Lemma 4. If \( \hat{A}^2 < 0 \) then the following hold:

a) The rational tree \( D_1 \) is not empty, and its intersection matrix is not negative definite.

b) Every closed irreducible curve \( C \) in \( X \) distinct from \( \hat{A} \) or an irreducible component of \( D_2 \) intersects \( D_1 \).

Proof. These properties are invariant under extension and restriction of the base field \( k \). So by first replacing \( k \) by a subfield \( k_0 \subset k \) of finite transcendence degree over \( \mathbb{Q} \) over which \( X \) and \( D \) are defined and extension to \( k \) via the choice of an embedding \( k_0 \hookrightarrow k \), we may assume from the beginning that \( k = \mathbb{C} \). Then since \( S \approx X \setminus (D_1 \cup E \cup D_2) \approx \mathbb{P}^2 \setminus \mathbb{Q} \) is \( \mathbb{Q} \)-acyclic, the classes \( E \) and of the irreducible components of \( D_1 \) and \( D_2 \) form a basis of \( \text{Cl}(X) \otimes \mathbb{Q} \). Indeed, by Lemma 4.2.1 in [13, Lemma 4.2.1]. If \( \text{Cl}(X) \otimes \mathbb{Q} \), we may thus write \( \hat{A} \equiv (\hat{A}^2)E + R \), where \( R \) is the class of a \( \mathbb{Q} \)-divisor supported on \( D_1 \cup D_2 \) and since \( \hat{A}^2 \neq 0 \), it follows that the classes of \( \hat{A} \) and of the irreducible components of \( D_1 \) and \( D_2 \) also form a basis of \( \text{Cl}(X) \otimes \mathbb{Q} \). Let \( \tau : X \to \hat{X} \) be the birational morphism onto a normal surface with at most cyclic quotient singularities obtained by contracting \( \hat{A} \) and the negative definite rational chain \( D_2 \). The irreducible components of \( \tau(D_1) \) then form a basis of \( \text{Cl}(X) \otimes \mathbb{Q} \), and since \( \tau(D_1) \) is connected, it follows that \( \hat{X} \setminus \tau(D_1) \) is a normal \( \mathbb{Q} \)-acyclic surface, hence an affine surface by virtue [12]. So by [8], \( \tau(D_1) \) is the support of an ample divisor on \( \hat{X} \); in particular, \( \tau(D_1) \) is not empty, its intersection matrix is not negative definite and it intersects every proper curve in \( \hat{X} \). Because \( D_1 \) is disjoint from \( D_2 \) and \( \hat{A} \), \( \tau \) restricts to an isomorphism in an open neighborhood of \( D_1 \), and so the assertion follows.

\[ \square \]

Lemma 5. If \( E^2 = -1 \) and \( \hat{A}^2 < 0 \) then \( \hat{A}^2 = -1 \).

Proof. Similarly as in the proof of the previous lemma, the assertion is invariant under restriction to a subfield \( k_0 \subset k \) of finite transcendence degree over \( \mathbb{Q} \) over which \( X \) and \( D \) are defined and extension to \( \mathbb{C} \) via the choice of an embedding \( k_0 \hookrightarrow \mathbb{C} \). So we may again assume that \( k = \mathbb{C} \). The following argument is inspired from [14]. Suppose for contradiction \( \hat{A}^2 < -1 \). Then \( \Delta = D_1 \cup \hat{A} \) is an SNC-minimal divisor on \( X \) with three connected components, whose complement is a smooth quasi-projective surface \( V \) such that \( V \setminus (E \cap V) \approx S \setminus A \). The theory of minimal models of log-surfaces (see [13, Chapter 3] for a detailed account) asserts the existence of a sequence of projective birational morphisms

\[ f = f_n \circ \cdots \circ f_1 : (X, \Delta) = (X_0, \Delta_0) \to (X_1, \Delta_1) \to \cdots \to (X_n, \Delta_n) \]

with the following properties:

a) For every \( i = 1, \ldots, n \), \( X_i \) is a smooth projective surface and \( \Delta_i \) is an SNC-minimal reduced divisor

b) For every \( i = 1, \ldots, n \), \( f_i : (X_{i-1}, \Delta_{i-1}) \to (X_i, \Delta_i) \) is the contraction of a \((-1)\)-curve \( \ell_i \not\subset \Delta_{i-1} \) intersecting \( \Delta_{i-1} \) transversally in at most two smooth points and each connected component of \( \Delta_{i-1} \) at most once and such that \( \pi(X_i, \Delta_i) = \pi(X, \Delta) \cup \ell_i \), followed by the SNC-minimalization \( \Delta_i \) of the push-forward \( f_i \circ \Delta_{i-1} \) of \( \Delta_i \). Furthermore, if \( \ell_i \) intersects precisely two connected components of \( \Delta_{i-1} \) then one of these components is rational chain with negative definite intersection matrix.

c) The pair \( (X_n, \Delta_n) \) is almost-minimal, meaning that every log Minimal Model Program ran from \( (X_n, \Delta_n) \) terminates with a log terminal projective surface \( (X_{n+1}, \Delta_{n+1}) \) and the induced birational morphism \( f_{n+1} : (X_n, \Delta_n) \to (X_{n+1}, \Delta_{n+1}) = (f_{n+1})_*(\Delta_n) \) contracts only irreducible components of \( \Delta_n \) onto quotient singularities of \( X_{n+1} \). Equivalently, \( (X_{n+1}, \Delta_{n+1}) \) is a log-terminal pair which does not contain any proper curve \( \ell \) such that \( \ell^2 < 0 \) and \( \ell \cdot (K_{X_{n+1}} + \Delta_{n+1}) < 0 \) and \( f_{n+1} : (X_n, \Delta_n) \to (X_{n+1}, \Delta_{n+1}) \) is its minimal log-resolution.

Since \( \Delta_0 = D_1 \cup \hat{A} \cup D_2 \) has three connected components, \( \Delta_n \) has at most three connected components, and since by Lemma 4.1a) the intersection matrix of \( D_1 \) is not negative definite, the connected component of \( \Delta_n \) containing the image of \( D_1 \) is not contracted to a point by \( f_{n+1} \). So the exceptional locus \( \text{Exc}(f_{n+1}) \) consists of at most two connected components of \( \Delta_n \), and since \( \Delta_n \) is SNC-minimal, \( f_{n+1}(\text{Exc}(f_{n+1})) \) consists of singular points of \( X_{n+1} \). In particular, the local fundamental group \( G_p \) of \( p \in f_{n+1}(\text{Exc}(f_{n+1})) \) has order at least 2. An elementary calculation shows that the topological Euler characteristic of the surface \( X_{i-1} \setminus \Delta_{i-1} \) increases at a step if and only if the curve \( \ell_i \) contracted by \( f_i \) intersects two connected components of \( \Delta_{i-1} \) and the union of \( \ell_i \) with these components is contracted by \( f_i \) to a smooth point of \( X_i \). If such a curve existed, then by Lemma 4.1b), one of these connected components would necessarily be the one containing the image of \( D_1 \), which would imply in turn that the intersection matrix of \( D_1 \) is definite negative, a contradiction to Lemma 4.1a). So for every \( i = 1, \ldots, n \),

\[ \chi(X_i \setminus \Delta_i) \leq \chi(V) = \chi(S \setminus A) + \chi(E \cap V) = -1. \]

Now suppose that \( \overline{X} \setminus \Delta \) has non-negative logarithmic Kodaira dimension \( \pi(X, \Delta) \geq 0 \). Then \( \pi(X_n, \Delta_n) \geq 0 \) and since \( (X_n, \Delta_n) \) is almost-minimal, it follows from the logarithmic Bogomolov-Miyaoka-Yau inequality.
that
\[ 0 \leq \chi(X_n \setminus \Delta_n) + \sum_{p \in f_{n+1}(\text{Exc}(f_{n+1}))} \frac{1}{G_p} \leq \chi(X_n \setminus \Delta_n) + \frac{1}{2} \pi_0(\text{Exc}(f_{n+1})). \]

The only possibility is thus that $\chi(X_n \setminus \Delta_n) = \chi(V) = -1$ and that $f_{n+1}(\text{Exc}(f_{n+1}))$ consists of two points, with local fundamental group $\mathbb{Z}_2$. The corresponding connected components of $\Delta_n$ are thus simply $(-2)$-curves, and we deduce in turn from Lemma 12 that $D_2$ and $\hat{A}$ are $(-2)$-curves themselves. Since $E^2 = -1$ by hypothesis, the complete linear system $|D_2 + 2E + \hat{A}|$ on $X$ defines a $\mathbb{P}^1$-fibration $\overline{\xi} : X \to \mathbb{P}^1$ having the irreducible component $D_{1,E}$ of $D_1$ intersecting $E$ as a $2$-section. The restriction of $\xi$ to $S = X \setminus (D_1 \cup E \cup D_2)$ is thus a twisted $\mathbb{A}_n^1$-fibration $\xi : S \to Z$ over an open subset $Z$ of $\mathbb{P}^1$. Since $S$ is a $\mathbb{Q}$-acyclic, it follows from 13 Lemma 4.5.1 that $Z = \mathbb{A}^1$. The fiber $\overline{\xi}$ over the point $\infty = \mathbb{P}^1 \setminus Z$ is supported by a connected component $F_{\infty}$ of $D_1 - D_{1,E}$, and since $D_1$ is a rational tree, $D_{1,E}$ intersects $F_{\infty}$ transversally in a unique point. Since $D_1$ is SNC-minimal, we infer that $F_{\infty} = F_{\infty,1} \triangleright L \triangleright F_{\infty,2}$ is a chain of type $[-2, -1, -2]$ intersecting $D_{1,E}$ along $L$. Since $S = \mathbb{P}^2 \setminus Q$ admits a smooth completion by a rational curve, it follows from 4 Theorem 2.16 that by contracting successively $E$, $D_2$ and then all successive non-branching $(-1)$-curves in $D_1$, the image of $D$ in the corresponding smooth projective surface $Y$ is an SNC-minimal chain $B$ such that $Y \setminus B \simeq S$. Since the irreducible components $F_{\infty,1}$ and $F_{\infty,2}$ are untouched during this process, $B$ must be equal to the image of $F_{\infty}$, which is a chain of type $[-2, a, -2]$ for some $a \geq 0$. But one checks that no such chain can be transformed into one of type $[0, -1, -2, -2, -2]$, a contradiction to §12 (iii).

So $\pi(X, \Delta) = -\infty$, and hence $\pi(X_n, \Delta_n) = -\infty$. Since $\chi(X_n \setminus \Delta_n) \leq -1$ and the connected component of $\Delta_n$ containing the image of $D_1$ is not contracted by $f_{n+1}$, it follows from Theorem 3.15.1 and Theorem 5.1.2 in 13 that $X_n \setminus \Delta_n$ is $\mathbb{A}_n^1$-ruled, i.e., contains a Zariski open subset of the form $C \times \mathbb{A}^1$ for a certain smooth rational curve $C$. It follows in turn that $V$ is $\mathbb{A}_n^2$-ruled, and we let $q : V \dashrightarrow \mathbb{P}^1$ and $\overline{q} : X \dashrightarrow \mathbb{P}^1$ be the rational maps induced by the projection $p_C$. By virtue of Lemma 4 b), the closure $F$ in $X$ of a general fiber of $q$ intersects $D_1$. So $q : V \to Z$ is a well defined $\mathbb{A}^1$-fibration over an open subset $Z$ of $\mathbb{P}^1$, and if $\overline{q}$ is not regular, then its unique proper base point is supported on $D_1$. Furthermore, $D_2$ and $\hat{A}$ are necessarily contained in fibers of $\overline{q} : X \dashrightarrow \mathbb{P}^1$. So $\overline{q}$ is well defined on $S$, restricting to an $\mathbb{A}^1$-fibration $\rho : S \to \mathbb{A}^1$ containing $\hat{A}$ in one of its fibers. Let $\delta : Y \to X$ be the minimal resolution of the indeterminacies of $\overline{q}$, so that $\overline{q} = \overline{q} \circ \delta : Y \to \mathbb{P}^1$ is an everywhere defined $\mathbb{P}^1$-fibration, say with section $C \subset \delta^{-1}(D_1)$. Since the closures in $X$ of the general fibers of $q$ intersect $D_1$, the proper transform in $Y$ of $D_2 \cup E \cup \hat{A}$ is contained in $\overline{q}^{-1}(\rho(A))$ and since by 13 Theorem 4.3.1], all fibers of $\rho : S \to \mathbb{A}^1$ are irreducible, $\overline{q}^{-1}(\rho(A))$ is the union of the proper transform of $D_2 \cup E \cup \hat{A}$, the proper transform of a subset of irreducible components of $D_1$, possibly empty, and a subset of exceptional divisors of $\sigma$, again possibly empty. Since $D_2$ is a nonempty chain with negative definite intersection matrix and $\hat{A}^2 < -1$, at the step of the contraction of $\overline{q}^{-1}(\rho(A))$ onto a smooth fiber of a $\mathbb{P}^1$-fibration, the image of $E$ would have to become a $(-1)$-curve intersecting the images of $D_2$ and $\hat{A}$, and either an irreducible component of the image of $\overline{q}^{-1}(\rho(A))$ or the image of the section $C$, which is impossible. This is absurd, and so $\hat{A}^2 = -1$ necessarily.

2.2. Proof of Proposition 3

Proposition 3 and hence Theorem 1 are now consequences of the following lemma:

**Lemma 6.** With the notation of 27 above, the following alternative holds:

1) Either $\hat{A}^2 = 0$ and then there exists a birational map $\psi : X \dashrightarrow \mathbb{P}^2$ restricting to an isomorphism between $S = X \setminus D$ and the complement of a smooth conic $Q'$ and mapping $\hat{A}$ to a smooth conic intersecting $Q'$ with multiplicity $4$ at a single $k$-rational point $p'$.

2) Or $\hat{A}^2 = -1$ and then there exists a birational map $\psi : X \dashrightarrow \mathbb{P}^2$ restricting to an isomorphism between $S = X \setminus D$ and the complement of a smooth conic $Q'$ and mapping $\hat{A}$ to the tangent line to $Q'$ at a $k$-rational point $p'$.

**Proof.** We consider the two cases $\hat{A}^2 \geq 0$ and $\hat{A}^2 < 0$ separately.

Case 1). $\hat{A}^2 \geq 0$. Suppose first that $a = \hat{A}^2 > 0$. Then by performing a minimal sequence of blow-ups of $k$-rational points supported on the successive proper transforms of $\hat{A}$, starting with that of $E \cap A$ and then continuing with those of the intersection point of the proper transform of $\hat{A}$ with the previous exceptional divisor produced, we obtain a surface $\beta : Y \to X$ in which the proper transform of $\hat{A}$ has self-intersection $0$, while its reduced total transform is a rational chain $\hat{A} \triangleright H \triangleright D_3$ where $H$ is a $(-1)$-curve and $D_3$ is a chain of $a - 1$ curves with self-intersection $-2$ connecting $H$ to $E$. The complete linear system $|\hat{A}|$ then defines a $\mathbb{P}^1$-fibration $\overline{q} : Y \to \mathbb{P}^1_k$ having $\hat{A}$ as a smooth fiber and $H$ as a section.

Case 2). $\hat{A}^2 < 0$. Suppose now that $a = \hat{A}^2 < 0$. Then $\hat{A}$ is not a rational curve, and if $\hat{A}$ is contained in a fiber of $\overline{q}$, then $\overline{q}^{-1}(\rho(A))$ contains a smooth curve $\overline{q}^{-1}(\rho(A))$, contradicting the fact that $\overline{q}$ is a $\mathbb{P}^1$-fibration.
The surface $Y$, the rational chain $\tilde{A} \supset H \supset D_3$ and the $\mathbb{P}^1$-fibration are all defined over a subfield $k_0 \subset k$ of finite transcendence degree over $\mathbb{Q}$, and applying [13, Theorem 4.3.1] to the complex surface obtained by base change via an embedding $k_0 \hookrightarrow \mathbb{C}$, we conclude that the restriction of $\tilde{\sigma}$ to $S \simeq Y \setminus \beta^{-1}(D)$ is an $A^1$-fibration $q : S \to \mathbb{A}^1_k$ with a unique degenerate fiber. Since the union of the proper transform of $D_1 \cup E \cup D_2$ with $D_3$ is connected, it would be fully contained in a unique degenerate fiber of $\tilde{\sigma} : Y \to \mathbb{P}^1_k$, hence equal to it, and $q$ would be an $A^1$-fibration without any degenerate fiber, a contradiction.

So $\tilde{A}^2 = 0$, and the $\mathbb{P}^1$-fibration $\tilde{\sigma} : X \to \mathbb{P}^1_k$ defined by the complete linear system $|A|$ restricts to an $A^1$-fibration $\rho : S \to \mathbb{A}^1_k$ having $\tilde{A} \cap S \simeq A$ as a fiber. Since $E$ is a section of $\tilde{\sigma}$, for the same reason as before, either $D_1$ or $D_2$ supports a full fiber $F$ of $\tilde{\sigma} : X \to \mathbb{P}^1_k$, and since the intersection matrix of $D_2$ is negative definite and $D_1$ is SNC minimal, it must be that $F = D_1$ is a $(0)$-curve. So $D = D_1 \supset E \supset D_2$ is a $(-E^2)$-standard chain, and by [13, Theorem 4.6.1 (ii)], $D_2$ is thus a chain of type $[-2, -2, -2]$. After performing elementary transformations with center on $E$ if necessary to reach a smooth completion $(X', B')$ of $S$ by a rational chain $F_\infty \supset E \supset D_2$ of type $[0, -1, -2, -2, -2]$, the images of $F_\infty$ and $\tilde{A}$ by the contraction $\tau : X' \to X''$ of $E \cup D_2$ are curves of self-intersection 4 intersecting each other with multiplicity 4 in a single $k$-rational point. Since $X'' \setminus \tau(F_\infty \setminus E) \simeq S$ and $\text{Cl}(S) \simeq \mathbb{Z}_2$, $\text{Cl}(X'') \otimes \mathbb{Q}$ is freely generated by the class of $\tau(F_\infty)$, and since $X''$ is a smooth $k$-rational surface, we conclude that $X'' \simeq \mathbb{P}^2_k$ and that $\tau(F_\infty)$ and $\tau(\tilde{A})$ are smooth $k$-rational conics.

Case 2). $\tilde{A}^2 < 0$. By Lemma [4, a), $D_1$ is not empty. We consider two subcases according to the self-intersection of $E$.

Subcase 1). $E^2 = -1$. By virtue of Lemma [5], $\tilde{A}^2 = -1$. It follows that the complete linear system $|E + \tilde{A}|$ on $X$ defines a $\mathbb{P}^1$-fibration $\tilde{\sigma} : X \to \mathbb{P}^1_k$ having the irreducible components $D_{1,E}$ and $D_{2,E}$ of $D_1$ and $D_2$ intersecting $E$ as disjoint sections. The restriction of $\tilde{\sigma}$ to $S = X \setminus (D_1 \cup E \cup D_2)$ is thus an untwisted $A^1$-fibration $\xi : S \to Z$ over a smooth curve $Z \subset \mathbb{P}^1_k$ having $A = \tilde{A} \cap S$ as a degenerated fiber of multiplicity 1. Let again $k_0 \subset k$ be a subfield of finite transcendence degree over $\mathbb{Q}$ over which $X$, $D$, and $\tilde{\sigma}$ are defined, denote by $X_C$, $D_C$ and $\tilde{\sigma}_C$ the corresponding surface, divisor and morphism obtained by base extension via an embedding $k_0 \hookrightarrow \mathbb{C}$, and let $S_C = X_C \setminus (D_{1,C} \cup D_{2,C} \cup E_C)$ isomorphic to $\mathbb{P}^2_{\mathbb{C}} \setminus \mathbb{Q}_C$. It follows from Lemma 4.5.1 and Theorem 4.6.2 in [13] applied to $S_C$ that $Z_C \simeq \mathbb{P}^2_{\mathbb{C}}$ and that $\xi_C = \xi |_{S_C}$ has at most a second degenerate fiber, whose support $F$ is isomorphic to $A^1_k$. Since $(D_{2,E})^2_C = D^2_{2,E} \leq -2$ and $H_1(S_C; \mathbb{Z}) = \mathbb{Z}_2$ we deduce from § 4.5.2 (5) and Theorem 4.6.1 (2) in [13] that $\xi_C^{-1}(\xi_C(E_C \cup A))$ is actually the unique degenerate fiber of $\tilde{\sigma}_C$. It follows in turn that $D_{1,C} = (D_{1,E})_C$ and $D_{2,C} = (D_{2,E})_C$ and hence that $D_1 = D_{1,E}$ and $D_2 = D_{2,E}$. This $D$ is a chain $D_1 \supset E \supset D_2$ of type $[D^2_1, -1, D^2_2]$, where $D^2_2 \leq -2$ and where, by virtue of Lemma [4, a), $D^2_1 \geq 0$ because the intersection matrix of $D_1$ is not negative definite. Such a chain has a 1-standard form of type $[0, -1, -2, -2, -2]$ if and only if $D^2_1 = 2$ and $D^2_2 = -2$, and then the images of $D_1$ and $\tilde{A}$ by the contraction $\tau : X \to \mathbb{P}^2_k$ of $E$ and $D_2$ are respectively a smooth $k$-rational conic $Q'$ and its tangent line $T_{p'}Q'$ at the $k$-rational point $p' = \tau(E \cup D_2)$.

Subcase 2). $E^2 \geq 0$. Since $D_1$ is SNC-minimal, $D$ is SNC minimal, and since the boundary of every SNC-minimal completion of $S$ is a rational chain by virtue of [4, Theorem 2.16], $D$ is a rational chain. So $D_1$ is an SNC-minimal rational chain with non negative definite intersection matrix, and hence it contains an irreducible component with non negative self-intersection. If $D_1$ is a $(0)$-curve, then the $\mathbb{P}^1$-fibration $\tilde{\sigma} : X \to \mathbb{P}^1_k$ defined by $|D_1|$ has $E$ as a section, hence restricts to an $A^1$-fibration on $S$ containing $A = \tilde{A} \cap S$. 

**Figure 2.2.** The $\mathbb{P}^1$-fibrations $\tilde{\sigma} : Y \to \mathbb{P}^1_k$ and $\tilde{\sigma} : X \to \mathbb{P}^1_k$ respectively.
in one of its fibers. Since $D_2$ and $\tilde{A}$ both intersect $E$, they are contained in two different fibers of $\mathfrak{F}$. But since $\tilde{A}^2 < 0$, $\tilde{A}$ would be properly contained in a degenerate fiber of $\mathfrak{F}$, which is impossible by virtue of §1.3 (ii). So $D_1$ is either reducible or irreducible with positive self-intersection. By elementary birational transformations whose centers blown-up and curves contracted are $k$-rational and supported on $D_1$ and its successive images, we obtain a smooth completion $W$ of $S$ for which the reduced total transform $\tilde{D} = W \setminus S$ of $D$ is a rational chain $\tilde{D}_1 \supset E \supset D_2$, where the reduced total transform $\tilde{D}_1 = F_\infty \supset C \supset D_3$ of $D_1$ is a 1-standard chain. The complete linear system $|F_\infty|$ on $Y$ defines a $\mathbb{P}^1$-fibration $\mathfrak{F} : W \to \mathbb{P}^1_k$ with section $C$, containing $D_3 \cup E \cup D_2 \cup A$ in one of its fibers.

\[ F_\infty \quad 0 \quad \tilde{D}_3 \quad \tilde{D}_2 \quad \tilde{A} \]

\[ \mathfrak{F} \]

\[ \mathbb{P}^1_k \]

**Figure 2.3.** The $\mathbb{P}^1$-fibration $\mathfrak{F} : W \to \mathbb{P}^1_k$.

So $E^2 \leq -1$ and since $E$ intersects $D_2$, $\tilde{A}$ and either an irreducible component of $D_3$ if $D_1$ is not empty or $C$ otherwise, we have $E^2 \leq -2$ necessarily. The chain $D_3 \supset E \supset D_2$ is thus SNC-minimal, hence of type $[-2, -2, -2]$ by [12] (iii). By applying [13] Theorem 4.3.1 to the surface $S_C$ defined in a similar way as in the previous subcase, we deduce that $A_C$ must be the support of the unique degenerate fiber of the restriction $\rho_C : S_C \to A_C^1$ of $\mathfrak{F}$. So $\mathfrak{F}_C^{-1}(\rho_C(A_C)) \mathfrak{F}_3 = D_3,C \cup D_2,C \cup A_C$ implying that $A_C^2 = -1$. Thus $\tilde{A}^2 = -1$ and the images of $F_\infty$ and $\tilde{A}$ by the contraction of $C \cup D_3 \cup E \cup D_2$ to a k-rational point $p'$ are then respectively a smooth k-rational conic $Q'$ in $\mathbb{P}^1_k$ and its tangent line $T_pQ'$ at $p'$. □

As a consequence of Proposition 6 of the fact that $\text{Aut}(\mathbb{P}_k^2)$ acts transitively on the set of pairs $(Q, p)$ consisting of a smooth k-rational conic and a k-rational point on it, we obtain the following:

**Corollary 7.** Let $k$ be a field of characteristic 0 and let $S = \mathbb{P}_k^2 \setminus Q$ be the complement of a smooth k-rational conic $Q \subset \mathbb{P}_k^2$. Then the following hold:

1. Every closed curves $A \simeq A^1_k$ is equal to the support of a fiber of an $A^1$-fibration $\rho : S \to A^1_k$. More precisely, there exists a smooth completion $(\mathbb{P}_k^2, Q')$ of $S$ by a smooth k-rational conic $Q'$ such that $A$ is the support of a fiber of the $A^1$-fibration induced by the restriction of the pencil generated by $Q$ and twice its tangent line at a k-rational point $p$.

2. There exists a unique equivalence class of $A^1$-fibrations $\rho : S \to A^1_k$ on $S$ up to automorphisms, in the sense that every two such $A^1$-fibrations $\rho : S \to A^1_k$ and $\rho' : S \to A^1_k$ fit into a commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\psi} & S \\
\downarrow{\rho} & & \downarrow{\rho'} \\
A^1_k & \xrightarrow{\psi} & A^1_k \\
\end{array}
\]

for some automorphisms $\Psi$ and $\psi$ of $S$ and $A^1_k$ respectively.

**Remark 8.** In the complex case, it was established more generally in [9] Theorem 2.1 that on a smooth $\mathbb{Q}$-acyclic surface $S$ admitting a smooth completion $(X, B)$ by a chain of rational curves, every closed curve isomorphic to $A^1_k$ is the support of a fiber of an $A^1$-fibration $\rho : S \to A^1_k$. But every such $\mathbb{Q}$-acyclic surface different from $\mathbb{P}^2_k$ or $\mathbb{P}^2_k \setminus Q$ turns out to have more than one equivalence class of $A^1$-fibrations up to action of its automorphism groups. Indeed, by [5] Theorem 5.6, a $\mathbb{Q}$-acyclic surface as above different from $A^2_k$ is isomorphic to the quotient $S_{m,0}$ of a smooth surface $S_m = \{xz = y^m - 1\} \subset A^2_k$, $m \geq 2$, by a free action of
the group $\mu_m$ of complex $m$-th roots of unity of the form $(x, y, z) \mapsto (\varepsilon x, \varepsilon^r y, \varepsilon^{-1} z)$ where $q \in \{1, \ldots, m - 1\}$ and $\gcd(m, q) = 1$. Note that for every $m$ and every such $q$, the involution $(x, y, z) \mapsto (z, y, x)$ of $S_m$ descends to an isomorphism between $S_{m,q}$ and $S_{m,m-q}$. The $A^1$-fibration $pr_x : S_m \to A^1_k$ having support $S_m$ descends to an $A^1$-fibration $\rho_{m,q} : S_{m,q} \to A^1_k$, of multiplicity $m$, whose support $F_{m,q}$ generates the divisor class group $\text{Cl}(S_{m,q}) \cong \mathbb{Z}/m$ of $S_{m,q}$. Furthermore, the $\mu_m$-equivariant regular 2-form $x^{m-q}_m \in \Omega^2_k$ on $S_m$ descends to a regular 2-form vanishing at order $m - q$ along $F_{m,q}$ and nowhere else, implying that $K_{S_{m,q}} \cong (m - q)F_{m,q}$. It follows that if $m \geq 3$, then the $A^1$-fibrations $\rho_{m,q}$ and $\rho_{m,m-q}$ on $S_{m,q} \cong S_{m,m-q}$ are not equivalent under the action of $\text{Aut}(S_{m,q})$. Indeed, otherwise there would exist an isomorphism $\Psi : S_{m,q} \cong S_{m,m-q}$ and an automorphism $\psi$ of $A^1_k$, fixing the origin such that $\rho_{m,m-q} \circ \Psi \circ \rho_{m,q}$, and we would have the relation $(m - q)F_{m,q} = qF_{m,q}$ in $\text{Cl}(S_{m,q})$, in contradiction with the fact that $F_{m,q}$ has order $m$ in $\text{Cl}(S_{m,q})$.

3. AUTOMORPHISMS OF $\mathbb{P}^2 \setminus Q$ AND EXOTIC AFFINE LINES

In this section, we fix a base field $k$ of arbitrary characteristic $p \geq 0$. Recall that a smooth $k$-rational conic $Q \subset \mathbb{P}^2_k$ is projectively equivalent to that that $Q_0 \subset \mathbb{P}^2_k$ defined by the equation $q_0 = x^2 + y^2 = 0$ and that the induced action on $Q_0$ of the stabilizer $\text{Aut}(\mathbb{P}^2_k, Q_0)$ of $Q_0$ in $\text{Aut}(\mathbb{P}^2_k)$ is transitive on the set of $k$-rational points of $Q_0$. We let $S_0 = \mathbb{P}^2 \setminus Q_0$, $p_0 = [0:0:1]$ and we denote by

$$\rho_0 : S_0 \to A^1_k = \text{Spec}(k[t]), \quad [x : y : z] \mapsto \frac{x^2}{q_0},$$

the $A^1$-fibration induced by the restriction to $S_0$ of the rational map $\mathcal{P}_{p_0} : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k$ defined by the pencil $\mathcal{P}_{p_0} \subset \{q_2(2)\}$ generated by $Q_0$ and twice its tangent line $T_{p_0}Q_0$ at $p_0$. We denote by $\text{Aut}(S_0, \rho_0)$ the group of $k$-automorphisms of $S_0$ preserving $\rho_0$ globally, that is, automorphisms $\Psi \in \text{Aut}_k(S_0)$ for which there exists $\psi_{p_0} \in \text{Aut}(A^1_k)$ such that $\rho_0 \circ \Psi = \psi_{p_0} \circ \rho_0$.

3.1. AUTOMORPHISMS OF $S_0$. This subsection is devoted to the proof of the following more precise version of Theorem [2]

Proposition 9. With the notation above, the following hold:

1) The group $\text{Aut}_k(S_0)$ is isomorphic to the free product of $\text{Aut}(\mathbb{P}^2_k, Q_0)|_{S_0}$ and $\text{Aut}(S_0, \rho_0)$ amalgamated along their intersection.

2) The group $\text{Aut}(\mathbb{P}^2_k, Q_0) \subset \text{PGL}(3,k)$ is isomorphic to $\text{PGL}(2,k)$, generated by the following automorphisms:

a) $[x : y : z] \mapsto [x : y + bx : z - 2by - b^2x], b \in k,$

b) $[x : y : z] \mapsto [ax : y : a^{-1}z], a \in k^*,$

c) $[x : y : z] \mapsto [z : -y : x].$

3) The group $\text{Aut}(S_0, \rho_0)$ is generated by the restrictions to $S_0$ of birational endomorphisms of $\mathbb{P}^2_k$ of the form

$$[x : y : z] \mapsto \left[ x : y + s(\frac{x^2}{q_0})x : z - 2ys(\frac{x^2}{q_0}) - xs(\frac{x^2}{s_0})^2 \right], \quad s \in k[t],$$

and of the elements of the subgroup $\text{Aut}(\mathbb{P}^2, Q_0, p_0) \subset \text{Aut}(\mathbb{P}^2, Q_0)$ consisting of automorphisms fixing the point $p_0 \in Q_0$.

Proof. 1) The assertion already appeared in [3] §4.1.3 in the case $k = \mathbb{C}$. It extends readily to the case of an arbitrary base field $k$ thanks to the techniques developed in [2], so we just sketch the argument for the convenience of the reader. Every automorphism $\varphi$ of $S_0$ uniquely extends to a birational map $\varphi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$. If $\varphi$ is biregular then it is an automorphism preserving $S_0$, hence its complement $Q_0$, and so $\varphi \in \text{Aut}(\mathbb{P}^2_k, Q_0)$. Otherwise if $\varphi$ is strictly birational, it lifts in a unique way to a strictly birational endomorphism of the 1-standard completion $(X_0, B_0)$ of $S_0$ by a chain $Q_0 \supset C \supset E$ of type $[0, -1, -2, -2, -2]$ obtained by taking the minimal resolution of $\mathcal{P}_{p_0} : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$. By virtue of Theorem 3.0.2 in [2] this lift factors in a unique way into a sequence of two particular types of birational maps between 1-standard completions $(X_1, B_1 = Q_1 \supset C_1 \supset E_1)$ of $S_0$, called fibered modifications and reversions. In our case, a fibered modification $(X_{i-1}, B_{i-1}) \dashrightarrow (X_i, B_i)$ descends through the contractions of the rational sub-chains $C_{i-1} \supset E_{i-1}$ and $C_i \supset E_i$ in $X_{i-1}$ and $X_i$ onto $k$-rational points $x_{i-1} \in Q_{i-1}$ and $y_i \in Q_i$ to a birational endomorphism $\varphi_i : (\mathbb{P}^2_k, Q_{i-1}) \dashrightarrow (\mathbb{P}^2_k, Q_i)$ with the following properties:

a) $x_{i-1}$ and $y_i$ are the unique proper base points of $\varphi_i$ and $\varphi_i^{-1}$ respectively,
b) $\varphi_i$ maps the pencil $P_{x_{i-1}} \subset O_{2k}^*(2)$ generated by the smooth $k$-rational conic $Q_{i-1}$ and $2T_{x_{i-1}}, Q_{i-1}$ onto the pencil $P_y \subset O_{2k}^*(2)$ generated by the smooth $k$-rational conic $Q_i$ and $2T_x, Q_i$.

c) $\varphi_i$ restricts to an isomorphism between $\mathbb{P}^2_k \setminus Q_{i-1}$ and $\mathbb{P}^2_k \setminus Q_i$.

On the other hand, the definition of a reversion $(X_i, B_{i-1}) \mapsto (X_i, B_i)$ (see [2] Definition 2.3.1) implies that such a map descends via the same contractions to an isomorphism of pairs $(\mathbb{P}^2_k, Q_{i-1}) \to (\mathbb{P}^2_k, Q_i)$. As a consequence, every strictly birational endomorphism $\varphi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ restricting to an automorphism of $S_0 = \mathbb{P}^2_k \setminus Q_0$ admits a decomposition into a finite sequence of strictly birational maps of pairs

$$\varphi = \varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1 : (\mathbb{P}^2_k, Q_0) \to (\mathbb{P}^2_k, Q_1) \to \cdots \to (\mathbb{P}^2_k, Q_n) = (\mathbb{P}^2_k, Q_0)$$

satisfying properties a), b) and c) above. Now for every $i = 1, \ldots, n$, there exists an automorphism $\alpha_i : (\mathbb{P}^2_k, Q_0) \to (\mathbb{P}^2_k, Q_{i-1})$ of $\mathbb{P}^2_k$ mapping $Q_0$ onto $Q_{i-1}$ and $p_0$ onto the proper base point $x_{i-1}$ of $\varphi_i$ and an automorphism $\beta_i : (\mathbb{P}^2_k, Q_{i-1}) \to (\mathbb{P}^2_k, Q_i)$ mapping $Q_{i-1}$ onto $Q_i$ and $p_0$ onto the proper base point $y_i$ of $\varphi_i^{-1}$.

$$\varphi_i = \beta_i^{-1} \circ \varphi_i \circ \alpha_i$$

is then a birational map of pairs $(\mathbb{P}^2_k, Q_0) \to (\mathbb{P}^2_k, Q_0)$ mapping the pencil $P_{p_0}$ onto itself, and restricting to an automorphism $\psi_i$ of $S_0 = \mathbb{P}^2_k \setminus Q_0$ preserving the $\mathbb{A}^1$-fibration $\rho_0 : S_0 \to \mathbb{A}^1$ globally. Writing

$$\varphi = \varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1 = (\beta_n \circ \psi_n \circ \alpha_n^{-1}) \circ \cdots \circ (\beta_2 \circ \psi_2 \circ \alpha_2^{-1}) \circ (\beta_1 \circ \psi_1 \circ \alpha_1^{-1})$$

we obtain a decomposition of $\varphi$ into an alternating sequence of automorphisms $\beta_n, (\alpha_i^{-1} \circ \beta_i)_{i=1, \ldots, n-1}, \alpha_1^{-1}$ of the pair $(\mathbb{P}^2_k, Q_0)$ and birational endomorphisms $\psi_i$ of $\mathbb{P}^2_k$ restricting to elements of the group $\text{Aut}(S_0, \rho_0)$. This shows that $\text{Aut}(S_0)$ is generated by the subgroups $\text{Aut}(\mathbb{P}^2_k, Q_0)|S_0$ and $\text{Aut}(S_0, \rho_0)$. The existence of an amalgamated product structure follows from general properties of the above decompositions into birational maps, see [2] §3, Proposition 16[ and [2] Lemma 3.2.4).

2) The description of the generators of $\text{Aut}(\mathbb{P}^2_k, Q_0)$ follows from the classical faithful representation $\text{Aut}(Q_0) = \text{PGL}_2(k)$ as the special orthogonal group $SO_3(q_0) \subset \text{GL}_3(k)$ of the quadratic form $q_0 = xz + y^2$, defined by the action $\sigma : \text{PGL}_2(k) \times T \to T$ of $\text{PGL}_2(k)$ by conjugation on the space $T \cong \mathbb{R}^3$ of $2 \times 2$ matrices of trace zero. Explicitly, the representation $\gamma : \text{PGL}_2(k) \to SO_3(q_0)$ is given by

$$\text{PGL}_2(k) \cong \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a^2 & -2ab & -b^2 \\ -a(c+2d) & ad + bc & bd \\ -c^2 & 2cd & d^2 \end{bmatrix} \in SO_3(q_0),$$

and the listed generators of $\text{Aut}(\mathbb{P}^2_k, Q_0)$ coincide with the respective images in $\text{PGL}_3(k)$ of the generators

$$\begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}, b \in k, \quad \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, a \in k^* \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

of $\text{PGL}_2(k)$ by $\gamma$.

3) The generators of $\text{Aut}(S_0, \rho_0)$ can be determined as follows. The correspondence which maps every $\Psi \in \text{Aut}(S_0, \rho_0)$ to the unique element $\psi_0, \rho_0, \psi_0 \in \text{Aut}(\mathbb{P}^2_k)$ such that $\rho_0 \circ \Psi = \psi_0 \circ \rho_0$ defines a group homomorphism $d : \text{Aut}(S_0, \rho_0) \to \text{Aut}(\mathbb{P}^2_k)$. Since $\rho_0^{-1}(0)$ is the unique degenerate fiber of $\rho_0, \psi_0$ necessarily fixes the origin, hence belongs to the sub-torus $\mathbb{G}_{m, k} \times \{0\}$ of $\text{Aut}(\mathbb{P}^2_k) \cong \mathbb{G}_{m, k} \times \mathbb{G}_{a, k}$. Conversely, the existence of the homomorphism $\mathbb{G}_{m, k} \to \text{Aut}(S_0, \rho_0) \cap \text{Aut}(\mathbb{P}^2_k, Q_0), a \mapsto [ax : y : a^{-1}z]$ implies that we have a split exact sequence

$$0 \to \text{Aut}_0(S_0, \rho_0) \to \text{Aut}(S_0, \rho_0) \to \mathbb{G}_{m, k} \to 0,$$

and it remains to describe the elements of the group $\text{Aut}_0(S_0, \rho_0)$ of automorphisms of $S_0$ preserving $\rho_0$ fiber wise. Every such automorphism $\Psi$ restricts to an automorphism of the complement of $\rho_0^{-1}(0)$ in $S_0$. Under
the isomorphisms

$$S_0 \setminus \rho_0^{-1}(0) \simeq \mathbb{P}^2 \setminus (Q_0 \cup T_{\rho_0}Q_0)$$

$$\simeq \text{Spec}(k[Y, Z]) \setminus \{Z + Y^2 = 0\}$$

$$\simeq \text{Spec}(k[Y, t^{\pm 1}])$$

where $Y = y/x$, $Z = z/x$ and $t = x^2q_0 = (Z + Y^2)^{-1}$, $\Psi|_{S_0 \setminus \rho_0^{-1}(0)}$ coincides with a $\text{Spec}(k[t^{\pm 1}])$-automorphism of $\text{Spec}(k[Y, t^{\pm 1}])$ which is thus of the form $(t, Y) \mapsto (t, \lambda t^nY + s(t))$ for some $\lambda \in k^*, n \in \mathbb{Z}$ and $s(t) \in k[t^{\pm 1}]$. It follows that $\Psi$ is induced by the restriction of a birational endomorphism $\overline{\Psi}$ of $\mathbb{P}^2_k$ of the form

$$[x : y : z] \mapsto [x : (x^2q_0)y + s(x^2q_0)x : z + (1 - \lambda^2(x^2q_0)^2y^2) - 2\lambda(x^2q_0)^n s(x^2q_0)y - s(x^2q_0)^2 x].$$

If $(1 - \lambda^2(x^2q_0)^2n) \neq 0$ or $s(t) \in k[t^{\pm 1}] \setminus k[t]$ then such a birational endomorphism $\overline{\Psi}$ contracts the tangent line $T_{\rho_0}Q_0 = \{x = 0\}$ to the point $[0 : 0 : 1]$, hence is not the extension of any automorphism of $S_0$. So $n = 0$, $\lambda = \pm 1$, $s(t) \in k[t]$ necessarily. Conversely, every $\overline{\Psi}$ of the form

$$[x : y : z] \mapsto [x : \lambda y + s(x^2q_0)x : z - 2\lambda y s(x^2q_0) - x z(x^2q_0)^2]$$

where $\lambda = \pm 1$ and $s \in k[t]$ is the composition of an element $[x : y : z] \mapsto [x : \pm y : z]$ of $\text{Aut}(\mathbb{P}^2_0, Q_0, \rho_0)$ and of a birational endomorphism of the desired type, which indeed restricts to an automorphism of $S_0 = \mathbb{P}^2_k$. $\square$

3.2. Exotic affine lines in positive characteristic. A well known consequence of the structure of $\text{Aut}(A^2_k)$ is that if an embedded affine line $A \simeq A^1_k$ in $A^2_k$ with parametrization $t \mapsto (xt, yt)$ belongs to the $\text{Aut}(A^2_k)$-orbit of the coordinate line $\{x = 0\}$, then either $\deg_x(x(t))$ divides $\deg_y(y(t))$ or $\deg_y(y(t))$ divides $\deg_x(x(t))$. Letting $L_0 = \{x = 0\}$ and $L_1 = \{x^2 - q_0 = 0\}$ be the reduced fibers of the $A_k$-fibration

$$\rho_0 : S_0 \rightarrow A^1_k = \text{Spec}(k[t]), \quad [x : y : z] \mapsto \frac{x^2}{q_0}$$

over the closed points 0 and 1 of $A^1_k$ respectively, the description of $\text{Aut}(S_0)$ given in Proposition 9 leads to the following analogue for closed embeddings of $A^1_k$ in $S_0$:

**Lemma 10.** Let $j : A^1_k \hookrightarrow S_0$, $t \mapsto [x(t) : y(t) : z(t)]$ a closed embedding with image $A$. If $A$ belongs to the $\text{Aut}(S_0)$-orbit of $L_0$ or $L_1$, then up to composition by the involution $[x : y : z] \mapsto [z : y : x]$, the following hold:

a) $\deg_x(x(t)) < \deg_y(y(t)) < \deg_x(z(t))$.

b) If $\deg_x(x(t)) \neq -\infty$, then it divides $\deg_y(y(t))$ and $\deg_x(z(t))$.

**Proof.** This holds for the parametrizations $t \mapsto [0 : 1 : t]$ and $t \mapsto [1 : t : 1 - t^2]$ of $L_0$ and $L_1$ respectively, and both properties are preserved under the application of any of the generator of $\text{Aut}(S_0)$ listed in Proposition 9.

As a consequence of the above lemma, we obtain in every characteristic $p \geq 3$ the existence of closed embeddings $j : A^1_k \hookrightarrow S_0$ whose image does not belong to the $\text{Aut}(S_0)$-orbit of $L_0$ or $L_1$, a phenomenon similar to the failure of the Abhyankar-Moh Theorem in positive characteristic. Namely, we have the following family of examples:

**Proposition 11.** Let $k$ be a field of characteristic $p \geq 3$. Then the morphism

$$j : A^1_k \hookrightarrow S_0, \quad t \mapsto \left[t^{p^2} : t^{p^2} (p^2 + p + t + 1 : -t^{p^2} (p^2 + p + t + 2) - 2(t^{p^2 + p + t})ight]$$

is a closed embedding whose image does not belong to the $\text{Aut}(S_0)$-orbit of $L_0$ or $L_1$.

**Proof.** Once we show that $j$ is indeed a closed embedding, the conclusion follows immediately from Lemma 10 above. Letting $\tilde{S}_0 \subset A^2_k = \text{Spec}(k[x, y, z])$ be the smooth affine surface with equation $xz + y^2 - 1 = 0$, $j$ is the composition of the morphism

$$\tilde{j} : A^1_k \hookrightarrow \tilde{S}_0, \quad t \mapsto (t^{p^2}, t^{p^2} (p^2 + p + t + 1 : -t^{p^2} (p^2 + p + t + 2) - 2(t^{p^2 + p + t}))$$

with the étale Galois double cover $\pi : \tilde{S}_0 \rightarrow S_0$, $(x, y, z) \mapsto [x : y : z]$. Letting $\tilde{A}$ be the image of $\tilde{j}$, $\pi^{-1}(\pi(\tilde{A}))$ is the disjoint union of $\tilde{A}$ with its image by the action $(x, y, z) \mapsto (-x, -y, -z)$ of the Galois group of $\pi$. So $\pi$ induces an isomorphism between $\tilde{A}$ and the image of $j$. Since $\tilde{S}_0$ is affine, every embedding of $A^1_k$ into it is necessarily closed, and hence, it now suffices to show that $j$ is an embedding. Noting that $\tilde{A}$ is contained...
in the complement $V$ of the curve with equation \( \{x = y + 1 = 0\} \subset \tilde{S}_0 \) and that $V$ is isomorphic to $\mathbb{A}^2_k$ via the restriction of the rational map

\[
\alpha : \tilde{S}_0 \to \mathbb{A}^2_k = \text{Spec}(k[x, v]), \quad (x, y, z) \mapsto (x, x^{-1}(1 - y) = (y + 1)^{-1}z),
\]

we are reduced to check that $\alpha \circ \tilde{j} : \mathbb{A}^1_k \to \mathbb{A}^2_k$, $t \mapsto (x(t), v(t)) = (t^{p^2}, t^{p^2+p} + t)$ is an embedding. This follows from the identity $t^{p(p+1)} = (v(t)p - x(t)p+1)^{p+1}$ which implies that the inclusion $k[x(t), v(t)] \subset k[t]$ is an equality.

\[\square\]

Remark 12. The morphism $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$, $t \mapsto (t^{p^2}, t^{p^2+p} + t)$ used in the proof of the proposition above is a typical example of closed embedding of the line in $\mathbb{A}^2_k$ whose image, as a consequence of the Jung and van der Kulk Theorem, does not belong to the $\operatorname{Aut}(\mathbb{A}^2_k)$-orbit of the coordinate line $\{x = 0\}$.

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