Unification of quantum resources in distributed scenarios

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Quantum resources, such as coherence, discord, and entanglement, play a key role for demonstrating advantage in many computation and communication tasks. In order to find the nature behind these resources, tremendous efforts have been made to explore the connections between them. In this work, we extend the single party coherence resource framework to the distributed scenario and relate it to basis-dependent discord. We show the operational meaning of basis-dependent discord in quantum key distribution. By formulating a framework of basis-dependent discord, we connect these quantum resources, coherence, discord, and entanglement, quantitatively, which leads to a unification of measures of different quantum resources.

I. INTRODUCTION

Coherence, discord, and entanglement are fundamental resources in many tasks that cannot be achieved by classical physics. Coherence characterizes the superpositions 1, 2, serving as a resource of quantum randomness generation 3, 4, quantum metrology 6, 7, quantum computation 10, 13, and quantum thermodynamics 14–19. As one of the most widely used quantum resources, entanglement 20–27 plays a key role in quantum teleportation 28, quantum key distribution 29, 30, and dense coding 31, and also interprets the violation of Bell inequalities. Discord characterizes quantum correlations beyond entanglement 32, 33. It is the resource for remote state preparation 37, and might explain the acceleration in discrete quantum computation with one qubit and other quantum computation circuits 39.

Although these quantum resources play different roles in different tasks, the nature behind the resources might be the same. To find out such a non-classical nature, a natural idea is to build a unification framework of these quantum resources. Recently some researches have made progress for this goal 40–53. Early researches in this field focus on the transformation between distillable entanglement and discord 44, 45. Since the framework of coherence is proposed 2, there have been substantial attempts for unifying coherence and entanglement resource theory by designing protocols where these two resources can be converted into each other 16, 50. One example is that a single partite state with non-zero coherence is shown to be able to generate entanglement with bipartite incoherent operations 46. Similar results are extended to discord and generalized to multipartite systems in 12, where it is shown that the quantum discord created by multipartite incoherent operations is bounded by the quantum coherence consumed in its subsystems. Another connection between coherence and entanglement lies in quantum state merging 51. A standard quantum state merging can lead to a gain of entanglement, while the incoherent quantum state merging 52 where one of the parties is restricted with local incoherent operations only, shows that entanglement and coherence cannot be gained at the same time.

All the works above are trying to connect part of these resources. Recently a unification of all the three resources based on an interferometric scenario is proposed in 53. Considering a phase encoding process of an input state, the interferometry power, i.e., how much phase information can be obtained is determined by the quantum resource contained in the input state. In such an interferometric framework, different quantum resources corresponds to the interferometry power in different scenarios. Although coherence, discord, and entanglement are qualitatively unified in the interferometric framework, a quantitative unification is still an open problem.

In this work, we construct such a quantitative unification of the three resources. We first review the general definitions of resource frameworks and summarize the corresponding definitions for coherence, discord, and entanglement. Then, we extend the single party coherence resource framework to the bipartite distributed scenario in several different ways. It turns out that one of the definitions is identical to basis-dependent (BD) discord 12, 13. We construct the resource framework of BD-discord, where we propose its operational meaning in quantum key distribution (QKD) and give some examples of BD-discord measures. With the help of BD-discord, measures of coherence, discord, and entanglement can be naturally defined and unified. We believe our unified framework of quantum resources can make a substantial progress in understanding the quantum nature.

II. PRELIMINARIES

In this section, we first review the definitions of a general resource framework. Then, we briefly summarize the coherence framework and refer the reader to Appendix A.
for a detailed review of discord and entanglement frameworks.

A. Resource framework

A general resource framework [2, 13, 54, 62] consists of the definition of free state, free operation, and resource quantifiers.

Free state is a set of states \( \mathcal{F} \) that contain no resource while a state \( \rho \notin \mathcal{F} \) contains resource.

Free operations are physical realizable operations characterized by completely positive and trace preserving (CPTP) maps. They should at least transform free states only into free states, i.e., \( \Lambda_{\text{CPTP}}(\rho) \in \mathcal{F}, \forall \rho \in \mathcal{F} \) which can be rewritten as \( \sum_n K_n \rho K_n^\dagger \in \mathcal{F}, \forall \rho \in \mathcal{F} \) in Kraus presentation. Here \( \{K_n\} \) is the set of Kraus operators satisfying \( \sum_n K_n^\dagger K_n = I \). Different other free operations can be defined based on different extra requirements.

Quantifiers are real-valued functions \( f \) mapping states to non-negative real numbers. The free states should be mapped to zero, i.e., \( f(\rho) = 0, \forall \rho \in \mathcal{F} \). And for an arbitrary state, the function value should not increase under free operations, i.e., \( f(\rho) \geq f(\Lambda_{\text{CPTP}}(\rho)) \). Other principles are required for different resources and different tasks.

B. Framework of coherence

The general resource framework reduces to a specific one when we consider coherence, discord, and entanglement as the resource. We briefly review the coherence framework introduced in [2, 61], focusing on quantum states in a \( d \)-dimensional Hilbert space.

Incoherent and maximally coherent states. Given a classical computational basis \( J = \{j\}, (j = 1, 2, \ldots, d) \), an incoherent state refers to a state without superposition on the basis, which can be described by

\[
\sigma = \sum_{j=1}^d p_j |j\rangle\langle j|,
\]

where \( p_{ja} \in [0, 1], \forall j \) and \( \sum_j p_{ja} = 1 \). At the meantime, maximally coherent states can be expressed as:

\[
|\Psi_d\rangle = \frac{1}{d} \sum_{j=1}^d e^{i\phi_j} |j\rangle,
\]

where \( \phi_j \in [0, 2\pi) \).

Incoherent operations. Incoherent operations map an incoherent state only to an incoherent state. That is, \( \sum_n K_n \rho K_n^\dagger \in \mathcal{C}, \forall \rho \in \mathcal{C} \), where \( \mathcal{C} \) is the set of incoherent states, \( \{K_n\} \) is a series of Kraus operators satisfying \( \sum_n K_n^\dagger K_n = I \).

Coherence measures. A coherence measure \( C(\rho) \) is defined by a function that maps a quantum states \( \rho \) to a real non-negative number, which satisfies the following conditions in Table 1.

| Table 1: Properties of a coherence quantifier. |
|------------------------------------------------|
| (C1) \( C(\sigma) = 0 \) when \( \sigma \) is an incoherent state. A stronger condition is (C1') \( C(\sigma) = 0 \) if and only if \( \sigma \) is an incoherent state; |
| (C2) Monotonicity: Coherence should not increase under incoherent operations, that is, (C2a) \( C(\rho) \geq C[\Phi_{\text{CPTP}}(\rho)] \), (C2b) \( C(\rho) \geq \sum_n p_n C(\rho_n) \), where \( \rho_n = K_n \rho K_n^\dagger / \text{tr}(K_n \rho K_n^\dagger) \); |
| (C3) Convexity: Coherence cannot increase under mixing, that is, \( \sum_{e} p_e C(\rho_e) \geq C(\sum_{e} p_e \rho_e) \). |

We leave the framework of the other two quantum resources, discord and entanglement in Appendix A.

III. EXTENDING COHERENCE TO THE DISTRIBUTED SCENARIO

Quantum coherence is defined in the single party scenario while discord and entanglement are defined for at least two parties. Therefore, to unify the three measures, we should generalize coherence to multiple parties. In this section, we consider three approaches to generalize coherence to the bipartite distributed scenario, where we begin with three possible generalized definitions of the incoherent state.

A. Incoherent-incoherent bipartite coherence

A natural extension is the bipartite coherence proposed in [46], which considers the joint basis \( J_A J_B = \{|j_A\rangle |j_B\rangle\} \) \( (j_A = 1, 2, \ldots, d_A, j_B = 1, 2, \ldots, d_B) \) with \( d_A \) and \( d_B \) being dimensions of the local Hilbert spaces of system \( A \) and \( B \), respectively. The bipartite incoherent state in can be rewritten as

\[
\sigma_{\text{II}}^{AB} = \sum_{j_A, j_B} p_{j_A j_B} |j_A\rangle\langle j_A| \otimes |j_B\rangle\langle j_B|.
\]

It is not hard to see that the bipartite incoherent state defined above is a specific type of classical-classical state \( \sigma_{\text{II}}^{AB} = \sum_{m,n} p_{mn} |m\rangle\langle m| \otimes |n\rangle\langle n| \) with certain local bases. Here we call Eq. (3) as incoherent-incoherent (II) state. A bipartite state contains bipartite coherence if it is not an incoherent-incoherent state.
B. Incoherent-classical bipartite coherence

When focusing the coherence in a local basis of system \( A \) (say \( J_A \)) and ignore the local basis of system \( B \), we define the incoherent-classical (IC) state as

\[
\sigma_{AB}^{IC} = \sum_{j_A,n} p_{j_A n} |j_A\rangle\langle j_A| \otimes |n\rangle\langle n|
\]

which is still a classical-classical state. Although \( |j_A\rangle \) is still from the \( J_A \) basis, \( |n\rangle \) can be from an arbitrary basis of the system \( B \). Note that any incoherent-classical state can be obtained by applying a local unitary operation on system \( B \) to a incoherent-incoherent state, i.e., \( \sigma_{AB}^{IC} = U_B \sigma_{AB}^{IQ} U_B^\dagger \). Therefore, the set of incoherent-classical states is larger than the set of incoherent-incoherent states. A bipartite state contains incoherent-classical bipartite coherence if it is not an incoherent-classical state.

C. Incoherent-quantum bipartite coherence

In the above generalization, we still consider the incoherent state as a classical-classical state. If we only focus on the coherence in a local basis of system \( A \) (say \( J_A \)), and totally ignore the other party \( (B) \), we can generalize coherence to be incoherent-quantum (IQ) coherence when studying discord. Here we formulate its framework.

\[
\sigma_{AB}^{IQ} = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_B^{j_A}.
\]

Equivalently, it can be written as

\[
\sigma_{AB}^{IQ} = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \left( \sum_{i_{j_A}} t_{j_A} |i_{j_A}\rangle\langle i_{j_A}| \right),
\]

with a spectral decomposition in party \( B \). It is not hard to see that incoherent-quantum state can be obtained by mixing incoherent-classical states. Or we can regard the set of incoherent-quantum state as the convex hull of the set of the incoherent-classical states. A bipartite state contains incoherent-quantum bipartite coherence if it is not an incoherent-quantum state.

We generalize the bipartite coherence in distributed scenarios with the track of \( \sigma_{AB}^{IQ} \rightarrow \sigma_{AB}^{IC} \rightarrow \sigma_{AB}^{IQ} \). The incoherent-incoherent state is a subset of incoherent-classical state which is further a subset of incoherent-quantum state. We illustrate the relationship of these states in Fig. 1. The incoherent-quantum bipartite coherence is actually identical to the basis-dependent (BD) discord, which is the key resource for our unification framework.

IV. BASIS-DEPENDENT DISCORD

A. Framework of basis-dependent discord

The concept of BD-discord has been proposed in 12, 13 when studying discord. Here we formulate its resource framework, beginning with definitions of free states for BD-discord given a local computational basis \( J_A = \{|j_A\rangle \{j = 1, 2, \ldots, d_A \} \} \) on system \( A \).

Definition 1. A zero basis-dependent discord state in \( J_A \) is an incoherent-quantum state in Eq. (5).

Second we define free operations for BD-discord, which map incoherent-quantum states to incoherent-quantum states.

Definition 2. The free operations for BD-discord are separable-quantum-incoherent (SQI) operations 64

\[
\Lambda_{SQI}(\sigma_{AB}^{IQ}) = \sum_{n} \hat{A}_n \otimes \hat{B}_n \sigma_{AB}^{IQ} \hat{A}_n^\dagger \otimes \hat{B}_n^\dagger \subset \delta_{IQ},
\]

where \( \delta_{IQ} \) is the set of incoherent-quantum states, \( \hat{A}_n \otimes \hat{B}_n \) is a series of Kraus operators satisfying the completeness condition \( \sum_n \hat{A}_n^\dagger \hat{A}_n = I \), and \( \{ \hat{A}_n \} \) is a set of incoherent operations on \( A \).

Finally we define the measures of BD-discord, \( BD_{J_A}(\rho_{AB}) \), which map a bipartite quantum states \( \rho_{AB} \) to a real non-negative number, satisfying the conditions in Table 2.

B. Examples of basis-dependent discord measures

Here we give two categories of BD-discord measures that fulfill the conditions in Table 2. One is the distance-based measure. The BD-discord equals to the distance...
from IQ states, which is expressed as

$$BD_{J_A}(\rho_{AB}) = \min_{\sigma_{AB}^{IQ}} \sum_{j_A} p_{j_A} R(|\Psi_{AB}\rangle_e) \tag{9}$$

Specifically, the distance can be various of measures given in Table 3, where the superscript in $\rho_{AB}^{Adiag}$ means a local dephasing operation on $A$. Actually these measures are widely used in entanglement, discord and coherence.

The other is the convex roof of local randomness,

$$BD_{J_A}(\rho_{AB}) = \min_{\rho_{AB}} \sum_{j_A} p_{j_A} R(|\Psi_{AB}\rangle_e)$$

C. Operational meaning of the basis-dependent discord

In this section, we consider the operational meaning of BD-discord, which is the local randomness of the raw key in QKD. In the QKD security analysis, the communication partners, Alice and Bob, share a bipartite state $\rho_{AB}$, while the adversary Eve, is assumed to hold a purification $|\Psi_{ABE}\rangle$ of Alice’s and Bob’s system $AB$, which enables her to obtain the most information. The Devetak-Winter formula [65] gives an asymptotic key rate with one-way direct reconciliation. When $\rho_{AB}$ is known to Alice and Bob, the formula is expressed as

$$K = S(Z_A|E) - S(Z_A|Z_B) \tag{11}$$

Proposition 1. The local randomness in QKD, i.e., the conditional entropy $S(Z_A|E)$ in the Devetak-Winter formula, is a BD-discord measure.

Proof. The conditional entropy $S(Z_A|E)$ can be expressed as

$$S(Z_A|E) = S(\rho_{AE}^{Adiag}) - S(\rho_{E}) \tag{12}$$

Suppose the tripartite state after Alice’s local measurement is $\rho_{ABE}^{Adiag} = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_{BE}^{j_A}$, where $\rho_{BE}^{j_A}$ is a pure state since $|\Psi_{ABE}\rangle$ is a pure state, then

$$\rho_{AB}^{Adiag} = \text{tr}_E(\rho_{ABE}^{Adiag}) = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_{BE}^{j_A} \tag{13}$$

and $\rho_{AE}^{Adiag}$ has a similar expression of $\rho_{AE}^{Adiag} = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_{E}^{j_A}$. Consider the von Neumann entropy of a classical-quantum state,

$$H(\{p_{j_A}\}) - \sum_{j_A} p_{j_A} S(\rho_{E}^{j_A})$$

where $H(\cdot)$ is the Shannon entropy function and the second equality uses the fact that $S(\rho_{E}^{j_A})=S(\rho_{E})$ when $\rho_{BE}$ is a pure state, then Eq. [12] becomes

$$S(Z_A|E) = S(\rho_{AB}^{Adiag}) - S(\rho_{E}) \tag{15}$$

where the last equation is the relative entropy measure of basis-dependent discord given in Table 3.
V. UNIFYING MEASURES OF QUANTUM RESOURCES

With the help of the framework of BD-discord, now we are ready to unify the measures of different quantum resources.

A. BD-discord to coherence

In previous section, BD-discord is extended from bipartite coherence. And now we redefine the original single partite coherence \(2\) with BD-discord.

**Theorem 1.** The BD-discord measure of a tensor product state \(\rho_A \otimes \rho_B\) is a coherence monotone of \(\rho_A\), i.e.,

\[
C_{JA}(\rho_A) = BD_{JA}(\rho_A \otimes \rho_B)
\]

For simplicity, we can calculate the coherence of \(\rho_A\) by \(BD_{JA}(\rho_A \otimes I_B)\), where \(I_B\) is an identity matrix of \(B\). If \(BD_{JA}(\rho_A \otimes \rho_B)\) is further convex over \(\rho_A \otimes \rho_B\), \(C_{JA}(\rho_A)\) becomes a coherence measure.

**Proof.** First, for an incoherent state \(\sigma_A = \sum_j \rho_{JA}(|j_A\rangle\langle j_A|)\), \(\sigma_A \otimes \rho_B = \sum_j \rho_{JA}(|j_A\rangle\langle j_A|) \otimes \rho_B\) is an IQ state. Then the rhs of Eq. (10) equals to zero, which means \(C_{JA}(\sigma_A) = 0\) for an incoherent state \(\sigma_A\).

Second, according to the contractivity of a BD-discord measure under \(A_{SQI}\), \(BD_{JA}(A_{SQI}(\rho_A \otimes \rho_B)) = BD_{JA}(\Lambda_{IO}(\rho_A \otimes \Phi(\rho_B))) \leq BD_{JA}(\rho_A \otimes \rho_B)\), where \(\Lambda_{IO}\) is an incoherent operation and \(\Phi\) is an arbitrary operation. Then \(C_{JA}(\Lambda_{IO}(\rho_A)) \leq C_{JA}(\rho_A)\), which means \(C_{JA}(\rho_A)\) is contractive under \(\Lambda_{IO}\).

Finally, if \(BD_{JA}(\rho_A \otimes \rho_B)\) is convex over \(\rho_A \otimes \rho_B\), i.e., \(BD_{JA}(\rho_A \otimes \rho_B) \leq \sum \rho_n BD_{JA}(\rho^n_A \otimes \rho^n_B)\), where \(\rho_A \otimes \rho_B = \sum \rho_n \rho^n_A \otimes \rho^n_B\). Then \(C_{JA}(\rho_A) \leq \sum \rho_n C_{JA}(\rho^n_A)\), which shows the convexity of \(C_{JA}(\rho_A)\). A coherence monotone with convexity is a coherence measure. \(\square\)

B. BD-discord to discord

Furthermore, we can define a discord measure from any BD-discord measure. The free state for discord we consider here is the classical-quantum state, i.e.,

\[
\sigma_{CA}^{CQ} = \sum_n p_n |n\rangle \langle n| \otimes \rho^n_B,
\]

where \(|n\rangle\) is orthogonal for different \(n\), \(p_n \in [0,1]\), \(\forall n\) and \(\sum_n p_n = 1\). As the set of classical-quantum states contains all the incoherent-quantum states in different local bases, one can regard discord as a basis-independent version of BD-discord. Based on such an intuition, we can define a discord measure by Theorem \(2\)

\[
D(\rho_{AB}) = \min_{U_A} BD_{JA}(U_A \otimes I \rho_{AB} U_A^\dagger \otimes I)
\]

**Theorem 2.** A discord measure is a minimization of BD-discord measure over local bases, i.e.,

\[
D(\rho_{AB}) = \min_{U_A} BD_{JA}(U_A \otimes I \rho_{AB} U_A^\dagger \otimes I)
\]

We leave the proof in Appendix \(C\)

C. BD-discord to entanglement

To define entanglement measures from BD-discord measures, we consider the strong adversary scenario in \(53\). For a given input state \(\rho_{AB}\), some phase information is encoded in the local basis \(J_A\), i.e., by a local operation of \(U_A = \sum_{jA} e^{i\phi_{jA}} |j_A\rangle \langle j_A|\). After the phase encoding, a joint measurement is performed on both \(A\) and \(B\) to extract the phase information. It turns out that the interferometry power, i.e., how much phase information can be extracted corresponds to the BD-discord of the input state \(\rho_{AB}\). A strong adversary holds a purification of \(\rho_{AB}\) with \(\rho_{AB} = tr_E(|\Psi\rangle \langle \Psi|_{ABE})\). In order to let the extracted phase information as little as possible, the adversary will choose an optimal measurement on her local quantum system \(E\) and rotate the phase-encoding basis according to the measurement results. In this case the interferometry power corresponds to entanglement. Since the local measurement on \(E\) will effectively make the remaining system be with a certain decomposition, \(\rho_{AB} = \sum p_\epsilon |\psi_{\epsilon AB}\rangle \langle \psi_{\epsilon AB}|\) and the basis rotation operation depends on \(\epsilon\), the interferometry power will be minimized over all kinds of decompositions and the local unitary operations on \(A\). Therefore we have the following theorem.

**Theorem 3.** An entanglement measure is a convex roof of a discord measure, i.e.,

\[
E(\rho_{AB}) = \min_{p_\epsilon, |\psi_{\epsilon AB}\rangle} \sum p_\epsilon \min_{U_A} BD_{JA}(U_A \otimes I |\psi_{\epsilon AB}\rangle \langle \psi_{\epsilon AB}| \epsilon U_A^\dagger \otimes I)
\]

where the minimization is over all possible decompositions of \(\rho_{AB} = \sum p_\epsilon |\psi_{\epsilon AB}\rangle \langle \psi_{\epsilon AB}|\) and \(|\psi_{\epsilon AB}\rangle\) is a pure state.

We leave the proof in Appendix \(D\)

D. Example with distance-based measures

In this section we show an example of the measure unification of different quantum resources, the distance-based measures. Given distance-based BD-discord in Eq. (8), the distance-based coherence, discord and entanglement measures are given by Theorem \(1\) and Theorem \(2\) and Theorem \(3\).
quantified by relative entropy.

We note that the unification results will also be applied for other measures. And the operational meanings of each resource will be consistent in our unification framework. For example, the relative entropy measure of BD-discord will be transformed into distillable coherence, discord and entanglement by \[20\] which are also quantified by relative entropy.

VI. DISCUSSION AND CONCLUSION

In this work, we propose a unification framework on coherence, basis-dependent discord, discord and entanglement. We begin with constructing a resource framework of basis-dependent discord. As a bridge, basis-dependent discord connects coherence for their basis-dependence nature. On the other hand, it relates discord and entanglement since they all characterize bipartite quantum correlations. A unification framework of these quantum resources is established with the help of BD-discord. Moreover, we give the operational meanings of basis-dependent discord in QKD, which correspond to the local randomness of keys.

For future work, it is interesting to generalize these results to continuous variable cases, especially for Gaussian states. Discord and entanglement for Gaussian states have been well defined based on covariance matrix presentations \[35, 60\], however, the quantum coherence or a coherence-like basis-dependent quantity is still missing. This work can provide an inspiration to complete the unifications of quantum resources for continuous variables. And this will also help us understand the quantum resource behind the secure keys in continuous variable QKD.

Acknowledgement. We acknowledge Y. Zhou and X. Zhang for the insightful discussions. This work was supported by the National Natural Science Foundation of China Grants No. 11674193, the National Key R&D Program of China (2017YFA0303900, 2017YFA0304004), the National Research Foundation (NRF), NRF-Fellowship (Reference No: NRF-NRFF2016-02), BP plc and the EPSRC National Quantum Technology Hub in Networked Quantum Information Technology (EP/M013243/1).

H.Z. and X.Y. contributed equally to this work.

\[
C_J(\rho_A) = \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_A \otimes \rho_B || \sigma_{AB}^{IQ})
\]
\[
D(\rho_{AB}) = \min_{U_A} \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A \otimes I \rho_{AB} U_A^\dagger \otimes I || \sigma_{AB}^{IQ})
\]
\[
E(\rho_{AB}) = \min_{p_e, \{\psi_{AB}\}_e} \sum_e p_e \min_{U_A} \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^e \otimes I |\psi_{AB}\rangle_e \langle \psi_{AB}| e U_A^{\dagger e} \otimes I || \sigma_{AB}^{IQ}).
\]

Appendix A: Framework of discord and entanglement

1. Discord

In this part, we briefly review the framework for quantum discord \[35, 67, 68\] in a bipartite system \(AB\).

Definition of classical state. A state is classical for discord when it is a classical-quantum state, i.e.,

\[
\sigma_{AB}^{CQ} = \sum_n p_n |n\rangle \langle n| \otimes \rho_B^n,
\]

where \(|n\rangle\) is orthogonal for different \(n\), \(p_n \in [0, 1]\), \(\forall n\) and \(\sum_n p_n = 1\).

Definition of classical operation. The classical operation for discord is defined by local operations on \(B\), i.e., \(I_A \otimes \Phi_B\).

Discord measure. A discord measure \(D(\rho_{AB})\) is defined by a function that maps a quantum states \(\rho\) to a real non-negative number, which satisfies the following conditions in Table 4.

Table 4: Discord properties.

| Condition | Description |
|-----------|-------------|
| (D1) \(D(\sigma_{AB})\) vanishes for classical-quantum states, \(\sigma_{AB} = \sum_n p_n |n\rangle \langle n| \otimes \rho_B^n\). |
| (D2) Monotonicity: \(D(\rho_{AB})\) cannot increase under local operations, \(D(I_A \otimes \Phi_B(\rho_{AB})) \leq D(\rho_{AB})\). |
| (D3) \(D(\rho_{AB})\) is invariant under all local unitary operations, \(D(\rho_{AB}) = D(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger)\). |

2. Entanglement

In this part, we summarize the framework for entanglement \[24, 69, 70\] in a bipartite system \(AB\).

Definition of classical state. A state is classical for entanglement when it is separable, i.e.,

\[
\sigma_{AB}^{sep} = \sum_n p_n \rho_{A}^n \otimes \rho_B^n,
\]

where \(p_n \in [0, 1]\), \(\forall n\) and \(\sum_n p_n = 1\).
Definition of classical operation. The classical operation for entanglement is defined by local operation and classical communication (LOCC). In the following, we denote LOCC operations by $\Lambda_{\text{LOCC}}$.

Entanglement measure. An entanglement measure $D(\rho_{AB})$ is defined by a function that maps a quantum states $\rho$ to a real non-negative number, which satisfies the following conditions in Table 5.

| Table 5: Entanglement properties. |
|------------------------------------|
| (E1) $E(\rho_{AB})$ vanishes when $\rho_{AB}$ is separable. |
| (E2) Monotonicity: $E(\rho_{AB})$ cannot increase under LOCC operation, that is, (E2a) $E[\Lambda_{\text{LOCC}}(\rho_{AB})] \leq E(\rho_{AB})$. This condition is often replaced by another stronger one. (E2b) $E(\rho_{AB})$ should not increase on average under LOCC operations which map $\rho_{AB}$ to $\rho_{AB}^k$ with probability $p_k$, then $\sum_k p_k E(\rho_{AB}^k) \leq E(\rho_{AB})$. |
| (E3) Convexity: $E(\rho_{AB})$ decreases under mixing, $E(\sum_k p_k \rho_{AB}^k) \leq \sum_k p_k E(\rho_{AB}^k)$. |
| (E4) $E(\rho_{AB})$ is invariant under all local unitary operations, that is, $E(\rho_{AB}) = E(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger)$. |

Lemma 3. The coherence of an arbitrary state is invariant under incoherent unitary operations.

Proof. Consider an incoherent unitary operation $U$. Its reverse operation $U^\dagger$ is also a unitary operation $U$ according to Lemma 2. Then $C(\rho) = C(U^\dagger U \rho U^\dagger U) \leq C(U \rho U^\dagger)$. On the other hand, $C(U \rho U^\dagger) \leq C(\rho)$ since $U$ is an incoherent operation, which leads to $C(\rho) = C(U \rho U^\dagger)$. □

With the lemmas above, we can first prove that the distance-based measure in Eq. 8 satisfy all the conditions of a BD-discord measure.

Proof of (BD1). It is straightforward that $BD_{I_A}(\sigma_{AB}^{IQ}) = 0$ according to the definition.

Proof of (BD2). We have such relations

$$BD_{I_A}[\Lambda_{\text{SQI}}(\rho_{AB})] = \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\Lambda_{\text{SQI}}(\rho_{AB}))||\sigma_{AB}^{IQ}$$

$$= \min_{\rho_{AB}^{IQ} \in \delta_{IQ}} d(\Lambda_{\text{SQI}}(\rho_{AB}))||\Lambda_{\text{SQI}}(\sigma_{AB})$$

$$\leq \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} \min_{\rho_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB}||\sigma_{AB}^{IQ})$$

where the second equality is because $\Lambda_{\text{SQI}}(\sigma_{AB}^{IQ}) \in \delta_{IQ}$ and the inequality is due to the contractive nature of a distance measure, i.e., the distance will not increase under a completely positive and trace preserving (CPTP) map.

Proof of (BD3). Note the local incoherent unitary operation on $A$ and a unitary operation on $B$ as $U_A^\dagger \otimes U_B$, which is a SQI operation, then

$$\min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^\dagger \otimes U_B \rho_{AB} U_A \otimes U_B^\dagger||\sigma_{AB}^{IQ})$$

$$\leq \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB}||\sigma_{AB}^{IQ})$$

On the other hand, the reverse operation $U_A \otimes U_B^\dagger$ is also a SQI operation since $U_A^\dagger$ is an incoherent operation according to Lemma 2 then

$$\min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB}||\sigma_{AB}^{IQ})$$

$$= \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^\dagger U_B \otimes U_B \rho_{AB} U_A \otimes U_B^\dagger||\sigma_{AB}^{IQ})$$

$$\leq \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^\dagger \otimes U_B \rho_{AB} U_A \otimes U_B^\dagger||\sigma_{AB}^{IQ})$$

Thus we conclude that

$$\min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB}||\sigma_{AB}^{IQ})$$

$$= \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^\dagger \otimes U_B \rho_{AB} U_A \otimes U_B^\dagger||\sigma_{AB}^{IQ})$$

□

Appendix B: Proofs for BD-discord measures

In order to formulate the conditions of BD-discord measures, we investigate the properties of incoherent unitary operations.

Lemma 1. The Kraus operator of a unitary operation is unique.

Proof. Consider a unitary operation $U$, one possible Kraus operator representation can be written as $UU^\dagger$ which is rank 1. All of its other Kraus operator representations are $E_i = \sum_j u_{ij} U_j$, where $u_{ij}$ is a unitary matrix. Since $U$ is rank 1, the matrix $u_{ij}$ reduces to 1 and $E_i = U$. □

Lemma 2. If a unitary operation is an incoherent operation, its reverse operation is also an incoherent operation.

Proof. Consider a unitary operation $U$, it has unique Kraus operator representation $UU^\dagger$ according to Lemma 1. If it is an incoherent operation, we have

$$C(U \rho U^\dagger) \leq C(\rho)$$

for an arbitrary state $\rho$. Assume that its reverse operation, $U^{-1} = U^\dagger$, is not an incoherent operation, then $C(\rho) = C(U^\dagger U \rho U^\dagger U) > C(U \rho U^\dagger)$, which leads to a contradiction with Eq. (B1). □
Next we prove that the convex roof measure Eq. (3) also satisfies all conditions of a BD-discord measure.

Proof. Proof of (BD1). Consider the spectral decomposition of ρ_B | in Eq. (5), an IQ state can be rewritten as

\[ \sigma_{AB}^{IQ} = \sum_{jA} p_{ja} |jA\rangle \langle jA| \]  

(B6)

For each pure state component |jA\rangle, the local randomness is zero according to Eq. (11). And such a decomposition is an optimal decomposition due to the non-negativity of a BD-discord measure.

Proof of (BD2). Suppose the optimal decomposition is ρ_{AB} = \sum_e p_e |\psi_{AB}\rangle \langle \psi_{AB}|_e. For an arbitrary component |\psi_{AB}\rangle_ e, the local randomness is

\[ R(|\psi_{AB}\rangle_ e) = S(\sum_j |jA\rangle \left| tr_B |\psi_{AB}\rangle_ e \right|^2 |jA\rangle \langle jA| ) \]  

(B7)

We notice that Eq. (B7) is equal to the relative entropy of BD-discord measure of |\psi_{AB}\rangle, which is a distance-based measure and contractive under Λ_{SQI}. The convex roof is a mixture of the local randomness for each pure state component, and the mixture is also contractive under Λ_{SQI}.

Proof of (BD3). Same as the proof for distance-based measure.

Appendix C: Proof of Theorem 2

Proof. Proof of (D1). For a classical-quantum state in Eq. (11), we set U_A |n\rangle = |jA\rangle for n = 1, 2, · · · d_A, then

\[ BD_{J_A} \left[ U_A \otimes I \left( \sum_n p_n |n\rangle \langle n| \otimes \rho_B^n \right) U_A^\dagger \otimes I \right] \]

= BD_{J_A} \left[ \sum_n p_n (U_A |n\rangle \langle n| U_A^\dagger) \otimes \rho_B^n \right] \]

= BD_{J_A} \left( \sum_{jA} p_{ja} |jA\rangle \langle jA| \otimes \rho_B^n \right) \]

= 0.

We can see that such a U_A is optimal, which realizes a minimization of D(ρ_{AB}) due to the non-negativity of a basis-dependent discord measure.

Proof of (D2). Since I ⊗ Φ_B ⊂ Λ_{SQI}, from (BD2) we have

\[ BD_{J_A} [I \otimes \Phi_B (\rho_{AB})] \leq BD_{J_A} (\rho_{AB}) \]  

(C2)

and their minimization on the local basis also satisfies

\[ \min_{U_A} BD_{J_A} [I \otimes \Phi_B (U_A \otimes I) \rho_{AB} U_A^\dagger \otimes I] \]

\leq \min_{U_A} BD_{J_A} (U_A \otimes I) \rho_{AB} U_A^\dagger \otimes I \]

(C3)

Proof of (D3). Our target is to prove

\[ \min_{U_A} BD_{J_A} [(U_A \otimes U_B) (U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I) (U_B^\dagger \otimes U_B)] \]

= \min_{U_A} BD_{J_A} [(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

(C4)

Note that, in our definition of discord, the minimization is over all local basis, it is equal to prove that

\[ \min_{U_A} BD_{J_A} [(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

\leq \min_{U_A} BD_{J_A} [(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

(C5)

Since I ⊗ U_B ⊂ I ⊗ Φ_B, according to (D2) we have

\[ \min_{U_A} BD_{J_A} [(I \otimes U_B) (U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

\leq \min_{U_A} BD_{J_A} [(I \otimes U_B) (U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

(C6)

Apply local operation I ⊗ U_B on both sides,

\[ \min_{U_A} BD_{J_A} [(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

\leq \min_{U_A} BD_{J_A} [(I \otimes U_B) (U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

\leq \min_{U_A} BD_{J_A} [(I \otimes U_B) (U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

(C7)

As the local operation I ⊗ U_B ⊂ I ⊗ Φ_B, we also have

\[ \min_{U_A} BD_{J_A} [(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

\leq \min_{U_A} BD_{J_A} [(I \otimes U_B) (U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \]

(C8)

Thus we prove Eq. (C4).

Appendix D: Proof of Theorem 3

Proof. Since condition (E2a) can be derived with (E2b) and (E3),

\[ E(Λ_{LOCC}(ρ_{AB})) = E(\sum_n p_n \rho_{AB}^n) \]

\[ \leq \sum_n p_n E(\rho_{AB}^n) \]

\[ \leq E(ρ_{AB}), \]

(D1)

where \rho_{AB}^n = K_n \rho_{AB} K_n^\dagger / p_n and p_n = Tr(K_n \rho_{AB} K_n^\dagger), we only need to prove (E1), (E2b), (E3) and (E4).

Proof of (E1). Since the set of separable states is convex and closed, a separable state \sigma_{AB} = \sum_j p_j \rho_A^j ⊗ \rho_B^j can always be expressed as a mixture of pure separable states, i.e., product states.

\[ \sigma_{AB} = \sum_j p_j \rho_A^j ⊗ \rho_B^j = \sum_e p_e |\psi_A\rangle_ e \langle ψ_A|_e |\psi_B\rangle_ e \langle ψ_B|_e \]

(D2)

Substitute Eq. (D2) into Eq. (19), for each pure state component |\psi_A\rangle_ e |\psi_B\rangle_ e, we set a certain U_A such that U_A |\psi_A\rangle = |jA\rangle, then
where the non-negativity of a basis-dependent discord measure.

Similarly we assume the optimal decomposition for $\rho_{AB} = \sum_i p_i \rho_{AB}^i$, and
\begin{equation}
\sum_l p_l E(\rho_{AB}^l) = \sum_l p_l \min_{\rho_{AB}^l} \sum_e p_e |\psi_{AB}^l\rangle_e \langle \psi_{AB}^l| U_{AB}^t \otimes I = 0 \tag{D3}
\end{equation}
where we simplify the subscript of minimizing decomposition $p_e, |\psi_{AB}^l\rangle_e$ to $p_e$. Suppose for each component $\rho_{AB}^l$, the optimal decomposition is $\rho_{AB} = \sum_e |\psi_{AB}^l\rangle_e \langle \psi_{AB}^l| U_{AB}^t \otimes I$.

Similarly we assume the optimal decomposition for $\rho_{AB}$ is $\rho_{AB} = \sum_e p_e |\psi_{AB}\rangle_e \langle \psi_{AB}|$.
\begin{equation}
E(\rho_{AB}) = \sum_e p_e \min_{U_{AB}^t} \sum_e p_e |\psi_{AB}\rangle_e \langle \psi_{AB}| U_{AB}^t \otimes I \tag{D6}
\end{equation}

}\begin{equation}\end{equation}

\begin{equation}
\rho_{AB}^n = \frac{\hat{K}_n \rho_{AB} \hat{K}_n^\dagger}{p_n} = \sum_{p_{en}} \frac{p_{en}}{p_n} \hat{K}_n \rho_{AB} \hat{K}_n^\dagger \tag{D8}
\end{equation}

where the first inequality is due to the selective monotonicity of distance-based BD-discord measure,
\begin{equation}
BD_J(\rho_{AB}) \geq \sum_n \rho_{en} BD_J(\rho_{AB}^n) \tag{D10}
\end{equation}
the second inequality is due to the minimization over local basis according to each component after channel $U_{An}$ is more powerful than an entire minimization $U_{A}$. Proof of (E4). Local unitary operations $U_A \otimes U_B$ belong to LOCC. Then according to (E2a),
\begin{equation}
E(\rho_{AB}) \geq E(U_A \otimes U_B) \tag{D11}
\end{equation}
Apply $U_A^\dagger \otimes U_B^\dagger$ to the last equation,
\begin{equation}
E(U_A^\dagger \otimes U_B^\dagger) \geq E(\rho_{AB}) \tag{D12}
\end{equation}
On the other hand, operations $U_A^\dagger \otimes U_B^\dagger$ also belong to LOCC.
\begin{equation}
E(U_A^\dagger \otimes U_B^\dagger) \leq E(\rho_{AB}) \tag{D13}
\end{equation}
Therefore we have
\begin{equation}
E(\rho_{AB}) = E(U_A \otimes U_B) \tag{D14}
\end{equation}
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