Remark on Weil’s conjectures

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Abstract

We introduce a cohomology theory for projective varieties over a finite field coming from the canonical trace on a $C^*$-algebra attached to the variety. Using the cohomology, we prove the rationality, functional equation and the Betti numbers conjectures for the zeta function of the variety.

Key words and phrases: Weil conjectures, Serre $C^*$-algebras

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1 Introduction

The aim of present note is a cohomology theory for projective varieties over the field with $q = p^r$ elements. Such a cohomology comes from the canonical trace on a $C^*$-algebra attached to the variety $V$; this theory will be called a trace cohomology and denoted by $H^*_tr(V)$. The trace cohomology is much parallel to the $\ell$-adic cohomology $H^*_{et}(V; \mathbb{Q}_\ell)$, see [Grothendieck 1968] [3] and [Hartshorne 1977] [4], pp. 453-457. Unlike the $\ell$-adic cohomology, it does not depend on a prime $\ell$ and the endomorphisms of $H^*_tr(V)$ always have an integer trace. While the $\ell$-adic cohomology counts isolated fixed points of the Frobenius endomorphism geometrically, the trace cohomology does it algebraically, i.e. taking into account the index $\pm 1$ of a fixed point. Moreover, the eigenvalues of Frobenius endomorphism on $H^*_{et}(V; \mathbb{Q}_\ell)$ are (complex) algebraic numbers of the absolute value $q^\frac{i}{2}$, yet such eigenvalues are real algebraic on the trace cohomology $H^*_tr(V)$. The cohomology groups $H^*_{tr}(V)$ are

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truly concrete and simple; they can be found explicitly in many important special cases, e.g. when $V$ is an algebraic curve, see Section 4. We shall pass to a detailed construction.

Denote by $V_C$ an $n$-dimensional projective variety defined by the equations of $V := V(F_q)$ over the field of complex numbers; in other words, $V$ is the reduction of $V_C$ modulo the prime ideal over $p$. Let $B(V_C, \mathcal{L}, \sigma)$ be the twisted homogeneous coordinate ring of projective variety $V_C$, where $\mathcal{L}$ is the invertible sheaf of linear forms on $V_C$ and $\sigma$ an automorphism of $V_C$, see [Stafford & van den Bergh 2001] [13], p. 180 for the details. The norm-closure of a self-adjoint representation of the ring $B(V_C, \mathcal{L}, \sigma)$ by the bounded linear operators on a Hilbert space $\mathcal{H}$ is a $C^*$-algebra, see e.g. [Murphy 1990] [6] for an introduction; we call it a Serre $C^*$-algebra of $V_C$ and denote by $A_V$. Let $K$ be the $C^*$-algebra of all compact operators on $\mathcal{H}$. We shall write $\tau : A_V \otimes K \to \mathbb{R}$ to denote the canonical normalized trace on $A_V \otimes K$, i.e.

![Image]

a positive linear functional of norm 1 such that $\tau(xy) = \tau(xy)$ for all $x, y \in A_V \otimes K$, see [Blackadar 1986] [2], p. 31. Because $A_V$ is a crossed product $C^*$-algebra of the form $A_V \cong C(V_C) \rtimes \mathbb{Z}$, one can use the Pimsner-Voiculescu six term exact sequence for the crossed products, see e.g. [Blackadar 1986] [2], p. 83 for the details. Thus one gets the short exact sequence of the algebraic $K$-groups: $0 \to K_0(C(V_C)) \overset{i_*}{\to} K_0(A_V) \to K_1(C(V_C)) \to 0$, where map $i_*$ is induced by the natural embedding of $C(V_C)$ into $A_V$. We have $K_0(C(V_C)) \cong K^0(V_C)$ and $K_1(C(V_C)) \cong K^{-1}(V_C)$, where $K^0$ and $K^{-1}$ are the topological $K$-groups of $V_C$, see [Blackadar 1986] [2], p. 80. By the Chern character formula, one gets $K^0(V_C) \otimes \mathbb{Q} \cong H^{even}(V_C; \mathbb{Q})$ and $K^{-1}(V_C) \otimes \mathbb{Q} \cong H^{odd}(V_C; \mathbb{Q})$, where $H^{even}$ ($H^{odd}$) is the direct sum of even (odd, resp.) cohomology groups of $V_C$. Notice that $K_0(A_V \otimes K) \cong K_0(A_V)$ because of a stability of the $K_0$-group with respect to tensor products by the algebra $K$, see e.g. [Blackadar 1986] [2], p. 32. One gets the commutative diagram in Fig. 1, where $\tau_*$ denotes a homomorphism induced on $K_0$ by the canonical trace $\tau$ on the $C^*$-algebra $A_V \otimes K$. Since $H^{even}(V_C) := \bigoplus_{i=0}^{n} H^{2i}(V_C)$ and $H^{odd}(V) := \bigoplus_{i=1}^{n} H^{2i-1}(V_C)$, one gets for each $0 \leq i \leq 2n$ an injective homomorphism $\tau_* : H^i(V_C) \to \mathbb{R}$.

**Definition 1** By an $i$-th trace cohomology group $H^i_{tr}(V)$ of variety $V$ one understands the abelian subgroup of $\mathbb{R}$ defined by the map $\tau_*$. Notice that each endomorphism of $H^i_{tr}(V)$ is given by a real number $\omega$, such that $\omega H^i_{tr}(V) \subseteq H^i_{tr}(V)$; thus the ring $\text{End} (H^i_{tr}(V))$ of all endomorphisms of $H^i_{tr}(V)$ is commutative. The $\text{End} (H^i_{tr}(V))$ is a commutative subring of the
Our main results are as follows.

**Theorem 1** The cardinality of variety $V(\mathbb{F}_q)$ is given by the formula:

$$|V(\mathbb{F}_q)| = 1 + q^n + \sum_{i=1}^{2n-1} (-1)^i \, \text{tr} \, (\omega_i),$$

(1)

where $\omega_i \in \text{End} \left( H^i_{tr}(V) \right)$ is generated by the Frobenius map of $V(\mathbb{F}_q)$.

**Theorem 2** The zeta function $Z_V(t) := \exp \left( \sum_{r=1}^{\infty} \frac{|V(\mathbb{F}_q^r)|}{r} t^r \right)$ of $V(\mathbb{F}_q)$ has the following properties:

(i) $Z_V(t) = \frac{P_1(t) \cdots P_{2n-1}(t)}{P_0(t) \cdots P_{2n}(t)}$ is a rational function;

(ii) $Z_V(t)$ satisfies the functional equation $Z_V \left( \frac{1}{q^n} t \right) = \pm q^n \chi(V_C) Z_V(t)$, where $\chi(V_C)$ is the Euler-Poincaré characteristic of $V_C$;

(iii) $\deg P_i(t) = \dim H^i(V_C)$.

**Remark 1** Roughly speaking, theorem 2 says that the standard properties of the trace cohomology imply all Weil’s conjectures, except for an analog of the Riemann hypothesis $|\alpha_{ij}| = q^{2i}$ for the roots $\alpha_{ij}$ of polynomials $P_i(t)$, see [Weil 1949] [14], p. 507; the latter property is proved in a separate note [9].
The article is organized as follows. Section 2 contains some useful definitions and notation. Theorems 1 and 2 are proved in Section 3. We calculate the trace cohomology for the algebraic (and elliptic, in particular) curves in Section 4.

2 Preliminaries

In this section we briefly review the twisted homogeneous coordinate rings and the Serre $C^*$-algebras associated to projective varieties, see [Artin & van den Bergh 1990] [1] and [Stafford & van den Bergh 2001] [13] for a detailed account. The $C^*$-algebras and their $K$-theory are covered in [Murphy 1990] [6] and [Blackadar 1986] [2], respectively. The Serre $C^*$-algebras were introduced in [8].

2.1 Twisted homogeneous coordinate rings

Let $V$ be a projective scheme over a field $k$, and let $\mathcal{L}$ be the invertible sheaf $\mathcal{O}_V(1)$ of linear forms on $V$. Recall, that the homogeneous coordinate ring of $V$ is a graded $k$-algebra, which is isomorphic to the algebra

$$B(V, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(V, \mathcal{L}^\otimes n).$$

(2)

Denote by $\text{Coh}$ the category of quasi-coherent sheaves on a scheme $V$ and by $\text{Mod}$ the category of graded left modules over a graded ring $B$. If $M = \oplus M_n$ and $M_n = 0$ for $n >> 0$, then the graded module $M$ is called right bounded. The direct limit $M = \lim M_\alpha$ is called a torsion, if each $M_\alpha$ is a right bounded graded module. Denote by $\text{Tors}$ the full subcategory of $\text{Mod}$ of the torsion modules. The following result is basic about the graded ring $B = B(V, \mathcal{L})$.

**Lemma 1** ([Serre 1955] [10]) $\text{Mod} (B) / \text{Tors} \cong \text{Coh} (V)$.

Let $\sigma$ be an automorphism of $V$. The pullback of sheaf $\mathcal{L}$ along $\sigma$ will be denoted by $\mathcal{L}^\sigma$, i.e. $\mathcal{L}^\sigma(U) := \mathcal{L}(\sigma U)$ for every $U \subset V$. The graded $k$-algebra

$$B(V, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(V, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \ldots \otimes \mathcal{L}^\sigma^{n-1}).$$

(3)

is called a **twisted homogeneous coordinate ring** of scheme $V$; notice that such a ring is non-commutative, unless $\sigma$ is the trivial automorphism. The
multiplication of sections is defined by the rule \( ab = a \otimes b^\sigma \), whenever \( a \in B_m \) and \( b \in B_n \). Given a pair \((V, \sigma)\) consisting of a Noetherian scheme \( V \) and an automorphism \( \sigma \) of \( V \), an invertible sheaf \( L \) on \( V \) is called \( \sigma \)-ample, if for every coherent sheaf \( F \) on \( V \), the cohomology group \( H^q(V, L \otimes L^{\sigma} \otimes \ldots \otimes L^{\sigma^{n-1}} \otimes F) \) vanishes for \( q > 0 \) and \( n \gg 0 \). Notice, that if \( \sigma \) is trivial, this definition is equivalent to the usual definition of ample invertible sheaf [Serre 1955] [10].

A non-commutative generalization of the Serre theorem is as follows.

**Lemma 2 ([Artin & van den Bergh 1990] [1])** Let \( \sigma : V \to V \) be an automorphism of a projective scheme \( V \) over \( k \) and let \( L \) be a \( \sigma \)-ample invertible sheaf on \( V \). If \( B(V, L, \sigma) \) is the ring (3), then

\[
\text{Mod } (B(V, L, \sigma)) / \text{Tors} \cong \text{Coh } (V). \tag{4}
\]

### 2.2 Serre \( C^* \)-algebras

Let \( V \) be a projective scheme and \( B(V, L, \sigma) \) its twisted homogeneous coordinate ring. Let \( R \) be a commutative graded ring, such that \( V = \text{Spec } (R) \). Denote by \( R[t, t^{-1}; \sigma] \) the ring of skew Laurent polynomials defined by the commutation relation \( b^\sigma t = tb \) for all \( b \in R \), where \( b^\sigma \) is the image of \( b \) under automorphism \( \sigma : V \to V \).

**Lemma 3 ([Artin & van den Bergh 1990] [1])** \( R[t, t^{-1}; \sigma] \cong B(V, L, \sigma) \).

Let \( \mathcal{H} \) be a Hilbert space and \( B(\mathcal{H}) \) the algebra of all bounded linear operators on \( \mathcal{H} \). For a ring of skew Laurent polynomials \( R[t, t^{-1}; \sigma] \), we shall consider a homomorphism

\[
\rho : R[t, t^{-1}; \sigma] \longrightarrow B(\mathcal{H}). \tag{5}
\]

Recall that algebra \( B(\mathcal{H}) \) is endowed with a \( * \)-involution; the involution comes from the scalar product on the Hilbert space \( \mathcal{H} \). We shall call representation (5) \( * \)-coherent, if (i) \( \rho(t) \) and \( \rho(t^{-1}) \) are unitary operators, such that \( \rho^*(t) = \rho(t^{-1}) \) and (ii) for all \( b \in R \) it holds \( (\rho^*(b))^{\sigma(\rho)} = \rho^*(b^\sigma) \), where \( \sigma(\rho) \) is an automorphism of \( \rho(R) \) induced by \( \sigma \). Whenever \( B = R[t, t^{-1}; \sigma] \) admits a \( * \)-coherent representation, \( \rho(B) \) is a \( * \)-algebra; the norm-closure of \( \rho(B) \) yields a \( C^* \)-algebra, see e.g. [Murphy 1990] [6], Section 2.1. We shall refer to such as the Serre \( C^* \)-algebra and denote it by \( A_V \).

Recall that if \( A \) is a \( C^* \)-algebra and \( \sigma : G \to \text{Aut } (A) \) is a continuous homomorphism of the locally compact group \( G \) group, then the triple \((A, G, \sigma)\) defines a \( C^* \)-algebra called a crossed product and denoted by \( A \rtimes_\sigma G \); we refer
the reader to [Williams 2007], pp 47-54 for the details. It is not hard to see, that $A_V$ is a crossed product $C^*$-algebra of the form $A_V \cong C(V) \rtimes_\sigma \mathbb{Z}$, where $C(V)$ is the $C^*$-algebra of all continuous complex-valued functions on $V$ and $\sigma$ is a $*$-coherent automorphism of $V$.

3 Proofs

3.1 Proof of theorem [1]

We shall prove a stronger result contained in the following lemma.

**Lemma 4** The Lefschetz number of the Frobenius map $f_C : V_C \to V_C$ is given by the formula:

$$L(f_C) = 1 - q^n + \sum_{i=1}^{2n-1} (-1)^i \text{tr} (\omega_i).$$  \hfill (6)

**Proof.** Recall that the Lefschetz number of a continuous map $g_C : V_C \to V_C$ is defined as

$$L(g_C) = \sum_{i=0}^{2n} (-1)^i \text{tr} (g_i^*),$$  \hfill (7)

where $g_i^* : H^i(V_C) \to H^i(V_C)$ is an induced linear map of the cohomology. Because $f_i^* \equiv \omega_i \in \text{End} (H^i_V (V))$, one gets

$$L(f_C) = \sum_{i=0}^{2n} (-1)^i \text{tr} (\omega_i).$$  \hfill (8)

We shall write equation (8) in the form

$$L(f_C) = \text{tr} (\omega_0) + \text{tr} (\omega_{2n}) + \sum_{i=1}^{2n-1} (-1)^i \text{tr} (\omega_i).$$  \hfill (9)

It is known, that $H^0(V_C) \cong \mathbb{Z}$ and $\omega_0 = 1$ is the trivial endomorphism; thus $\text{tr} (\omega_0) = 1$. Likewise, $H^{2n} \cong \mathbb{Z}$, but

$$\omega_{2n} = \text{sgn} [N(\omega_1)] q^n;$$  \hfill (10)
where \( N(\bullet) \) is the norm of an algebraic number. It is known, that the endomorphism \( \omega_1 \in \text{End} (H^1_{tr}(V)) \) has the following matrix form
\[
q^{1/2} \begin{pmatrix} A & I \\ I & 0 \end{pmatrix},
\]
where \( A \) is a positive symmetric and \( I \) is the identity matrix, see \((\text{[8]}, \text{Lemma 3})\). Thus
\[
\text{sgn} \ [N(\omega_1)] = \text{sgn} \ \det \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} = \text{sgn} \ [-\det (I^2)] = -\text{sgn} \ \det (I) = -1.
\]
Therefore, from \((\text{10})\) one obtains \( \omega_{2n} = -q^n \); in other words, the Frobenius endomorphism acts on \( H^2_{tr}(V) \approx \mathbb{Z} \) by multiplication on the negative integer \(-q^n\). Clearly, \( \text{tr} (\omega_{2n}) = -q^n \) and the substitution of these data in \((\text{9})\) gives us
\[
L(f_C) = 1 - q^n + \sum_{i=1}^{2n-1} (-1)^i \text{tr} (\omega_i).
\]
Lemma \( \text{4} \) follows. \( \Box \)

**Corollary 1** The total number of the index \(-1\) fixed points of the Frobenius map \( f_C \) is equal to \( q^n \).

**Proof.** It is known, that
\[
|V(F_q)| = 1 + q^n + \sum_{i=1}^{2n-1} (-1)^i \text{tr} (Fr_i^*),
\]
where \( Fr_i^* : H^i_{et}(V; \mathbb{Q}_\ell) \rightarrow H^i_{et}(V; \mathbb{Q}_\ell) \) is a linear map on the \( i \)-th \( \ell \)-adic cohomology induced by the Frobenius endomorphism of \( V \), see \([\text{Hartshorne 1977}]\) \([4]\), pp. 453-457. But according to \((\text{[8]}, \text{Lemma 4})\), it holds
\[
\text{tr} (Fr_i^*) = \text{tr} (\omega_i), \quad 1 \leq i \leq 2n - 1.
\]
Since \( |\text{Fix} (f_C)| = |V(F_q)| \), one concludes from lemma \( \text{4} \) that the algebraic count \( L(f_C) \) of the fixed points of \( f_C \) differs from its geometric count \( |\text{Fix} (f_C)| \) by exactly \( q^n \) points of the index \(-1\). Corollary \( \text{4} \) is proved. \( \Box \)

Theorem \( \text{4} \) follows formally from the equations \((\text{14})\) and \((\text{15})\). \( \Box \)
3.2 Proof of theorem 2

For the sake of clarity, let us outline the main idea. Since the trace cohomology accounts for the fixed points of the Frobenius map $f_C$ algebraically (see corollary 1), we shall deal with the corresponding Lefschetz zeta function

$$Z^L_V(t) := \exp \left( \sum_{r=1}^{\infty} \frac{L(f^r_C)}{r} t^r \right)$$

(16)

and prove items (i)-(iii) for the $Z^L_V(t)$. Because $f_C : V_C \to V_C$ is the Anosov-type map, one can use Smale’s formulas linking $Z^L_V(t)$ and $Z^V(t)$, see [Smale 1967] [12], Proposition 4.14; it will follow that items (i)-(iii) are true for the function $Z^V(t)$ as well. We shall pass to a detailed argument; the following general lemma will be helpful.

**Lemma 5** If $f : V \to V$ is a regular map, then all eigenvalues $\lambda_{ij}$ of the corresponding endomorphisms $\omega_i \in \text{End} \ (H^i_{tr}(V))$ of the trace cohomology are real algebraic numbers.

**Proof.** Since the endomorphisms of $H^i_{tr}(V)$ commute with each other, there exists a basis of $H^i_{tr}(V)$, such that each endomorphism is given in this basis by a symmetric integer matrix [8]. But the spectrum of a real symmetric matrix is known to be totally real and the eigenvalues of an integer matrix are algebraic numbers. Lemma 5 follows. □

(i) Let us prove rationality of the function $Z^L_V(t)$ given by formula (16). Using lemma 4 one gets

$$\log Z^L_V(t) = \sum_{r=1}^{\infty} \left[ 1 + (-q^n)^r + \sum_{i=1}^{2n-1} (-1)^i tr (\omega^r_i) \right] t^r = \sum_{r=1}^{\infty} t^r / r + \sum_{r=1}^{\infty} \frac{(-q^n)^r}{r} + \sum_{r=1}^{\infty} \left( \sum_{i=1}^{2n-1} (-1)^i tr (\omega^r_i) \right) t^r / r.$$ (17)

Taking into account the well-known summation formulas $\sum_{r=1}^{\infty} t^r / r = -\log(1-t)$ and $\sum_{r=1}^{\infty} \frac{(-q^n)^r}{r} = -\log(1+q^n t)$, one can bring equation (17) to the form

$$\log Z^L_V(t) = -\log(1-t)(1 + q^n t) + \sum_{i=1}^{2n-1} (-1)^i \sum_{r=1}^{\infty} tr (\omega^r_i) t^r / r.$$ (18)

On the other hand, it easy to see that

$$tr (\omega^r_i) = \lambda_i^r + \ldots + \lambda_b^r,$$ (19)
where $\lambda_j$ are the eigenvalues of the Frobenius endomorphism $\omega_i \in \text{End} \left( H^1_{tr}(V) \right)$ and $b_i$ is the $i$-th Betti number of $V_C$. Thus one can bring (18) to the form

$$\log Z^L_V(t) = - \log(1 - t)(1 + q^n t) + \sum_{i=1}^{2n-1} (-1)^i \sum_{r=1}^{\infty} \left[ \frac{(\lambda_1 t)^r}{r} + \ldots + \frac{(\lambda_{b_i} t)^r}{r} \right].$$

(20)

Using the summation formula $\sum_{r=1}^{\infty} \frac{(\lambda_j t)^r}{r} = - \log(1 - \lambda_j t)$, one gets from (20)

$$\log Z^L_V(t) = - \log(1 - t)(1 + q^n t) + \sum_{i=1}^{2n-1} (-1)^{i+1} \log [(1 - \lambda_1 t) \ldots (1 - \lambda_{b_i} t)].$$

(21)

Notice that the product $(1 - \lambda_1 t) \ldots (1 - \lambda_{b_i} t)$ is nothing but the characteristic polynomial $P_i(t)$ of the Frobenius endomorphism on the trace cohomology $H^1_{tr}(V)$; thus one can write (21) in the form

$$\log Z^L_V(t) = \log \frac{P_1(t) \ldots P_{2n-1}(t)}{(1 - t)P_2(t) \ldots P_{2n-2}(t)(1 + q^n t)}.$$ 

(22)

Taking exponents in the last equation, one obtains

$$Z^L_V(t) = \frac{P_1(t) \ldots P_{2n-1}(t)}{P_0(t) \ldots P_{2n}(t)},$$

(23)

where $P_0(t) = 1 - t$ and $P_{2n}(t) = 1 + q^n t$. Thus $Z^L_V(t)$ is a rational function.

To prove rationality of $Z_V(t)$, recall that a map $f_C : V_C \rightarrow V_C$ is called Anosov-type, if there exist a (possibly singular) pair of orthogonal foliations $F_u$ and $F_s$ of $V_C$ preserved by $f_C$. (Note that our definition is more general than the standard and includes all continuous maps $f_C$.) Consider the trace cohomology $H^1_{tr}(V)$ endowed with the Frobenius endomorphism $\omega_1 \in \text{End} \left( H^1_{tr}(V) \right)$. Let $F_s$ be a foliation of $V_C$, whose holonomy (Plante) group is isomorphic to $H^1_{tr}(V)$. Because $\omega_1 H^1_{tr}(V) \subset H^1_{tr}(V)$, one concludes that $F_s$ is an invariant stable foliation of the map $f_C$. The unstable foliation $F_u$ can be constructed likewise. Thus $f_C$ is the Anosov-type map of the manifold $V_C$. One can apply now (an extension of) [Smale 1967] [12], Proposition...
4.14, which says that one of the following formulas must hold:

\[
\begin{align*}
Z_V(t) &= \frac{1}{Z_V(t)}, \\
Z_V(t) &= Z_V(-t), \\
Z_V(t) &= \frac{1}{Z_V(-t)}.
\end{align*}
\]  

Since \(Z_V(t)\) is known to be a rational function (23), it follows from Smale’s formulas (24) that \(Z_V(t)\) is rational as well. Item (i) is proved.

(ii) Recall that the cohomology \(H^*(V_C)\) satisfies the Poincaré duality; the duality can be given by a pairing

\[
H^i(V_C) \times H^{2n-i}(V_C) \longrightarrow H^{2n}(V_C)
\]  

obtained from the cup-product on \(H^*(V_C)\).

Let \(f : V \rightarrow V\) be the Frobenius endomorphism and \(f_C : V_C \rightarrow V_C\) the corresponding algebraic map of \(V_C\) and consider the action \((f_C^{2n})^*\) on the pairing \(\langle \cdot, \cdot \rangle\) given by (25). Since \(H^{2n}(V_C) \cong \mathbb{Z}\) and the linear map \((f_C^{2n})^*\) multiplies \(H^{2n}(V_C)\) by the constant \(q^n\), one gets

\[
\langle (f_C^i)^* x, (f_C^{2n-i})^* y \rangle = q^n \langle x, y \rangle,
\]  

for all \(x \in H^i(V_C)\) and all \(y \in H^{2n-i}(V_C)\). Recall the linear algebra identities, given e.g. in [Hartshorne 1977] [4], Lemma 4.3, p. 456; then (26) implies the following formulas

\[
\begin{align*}
\det \left( (f_C^i)^* t \right) &= \frac{(-1)^b_i (q^{n^i b_i})}{\det \left( (f_C^{2n-i})^* \right)} \det \left[ I - \frac{1}{q^{n^i b_i}} (f_C^{2n-i})^* \right] \\
\det \left( f_C^i \right)^* &= \frac{\det \left( I - (f_C^{2n-i})^* \right)}{\det \left( (f_C^{2n-i})^* \right)}.
\end{align*}
\]  

(27)

where \(b_i = \dim H^i(V_C)\) are the \(i\)-th Betti numbers. But \(\det \left( I - (f_C^i)^* t \right) := P_i(t)\) and \(\det \left[ I - \frac{1}{q^{n^i b_i}} (f_C^{2n-i})^* \right] := P_{2n-i} \left( \frac{1}{q^{n^i b_i}} \right)\); therefore, the first equation of (27) yields the identity

\[
P_i(t) = \frac{(-1)^b_i (q^{n^i b_i})}{\det \left( f_C^{2n-i} \right)} t^{b_i} P_{2n-i} \left( \frac{1}{q^{n^i b_i}} \right).
\]  

(28)
Let us calculate \( Z_L^V (\frac{1}{q^n t}) \) using (28); one gets the following expression

\[
Z_L^V \left( \frac{1}{q^n t} \right) = \frac{P_1(\frac{1}{q^n t}) \ldots P_{2n-1}(\frac{1}{q^n t})}{P_0(\frac{1}{q^n t}) \ldots P_{2n}(\frac{1}{q^n t})} = \ldots
\]

\[
= \frac{P_1(t) \ldots P_{2n-1}(t)}{P_0(t) \ldots P_{2n}(t)} t^{(b_0 - b_1 + \ldots)} \ (1 - P_1(t) \ldots P_{2n-1}(t)) \det (f_1^* \ldots f_n^*) \det (f_{2n-1}^*) \ldots \det (f_n^*) \ldots \det (f_{2n-1}^*).
\]

(29)

Note that \( b_0 - b_1 + \ldots = \chi(V_C) \) is the Euler-Poincaré characteristic of \( V_C \). From the second equation of (27) one obtains the identity \( \det (f_1^*) \det (f_{2n-1}^*) = (q^n)^b_1 \). Thus (29) can be written in the form

\[
Z_L^V \left( \frac{1}{q^n t} \right) = t^{\chi(V_C)} (-1)^{\chi(V_C)} (q^n)^{\frac{1}{2}(b_0 + \ldots + b_{2n})} Z_V^V(t) =
\]

\[
= t^{\chi(V_C)} (-1)^{\chi(V_C)} (q^n)^{\frac{1}{2} \chi(V_C)} Z_V^V(t).
\]

(30)

Taking into account \((-1)^{-\chi(V_C)} = \pm 1\), one gets a functional equation for \( Z_L^V(t) \). We encourage the reader to verify using formulas (24), that the same equation holds for the function \( Z_V(t) \). Item (ii) of theorem 2 is proved.

(iii) To prove the Betti numbers conjecture, notice that equality (19) implies that \( \deg P_i(t) = \dim H^i_{tr}(V) \). But \( \dim H^i_{tr}(V) = \dim H^i(V_C) \) by the definition of trace cohomology; thus \( \deg P_i(t) = \dim H^i(V_C) \) for the polynomials \( P_i(t) \) in formula (23). Again, the reader can verify using (24), that the same relationship holds for the polynomials representing the rational function \( Z_V(t) \). Item (iii) is proved.

This argument completes the proof of theorem 2 \( \square \)

4 Examples

The groups \( H^i_{tr}(V) \) are truly concrete and simple; in this section we calculate the trace cohomology for \( n = 1 \), i.e. when \( V \) is a smooth algebraic curve. In particular, we find the cardinality of the set \( \mathcal{E}(\mathbb{F}_q) \) obtained by the reduction modulo \( q \) of an elliptic curve with complex multiplication.
Example 1  The trace cohomology of smooth algebraic curve $C(\mathbb{F}_q)$ of genus $g \geq 1$ is given by the formulas:

$$
\begin{align*}
H^{0}_{tr}(C) & \cong \mathbb{Z}, \\
H^{1}_{tr}(C) & \cong \mathbb{Z} + \mathbb{Z}\theta_1 + \ldots + \mathbb{Z}\theta_{2g-1}, \\
H^{2}_{tr}(C) & \cong \mathbb{Z},
\end{align*}
$$

(31)

where $\theta_i \in \mathbb{R}$ are algebraically independent integers of a number field of degree $2g$.

Proof. It is known that the Serre $C^*$-algebra of the (generic) complex algebraic curve $C$ is isomorphic to a toric AF-algebra $A_{\theta}$, see [7] for the notation and details. Moreover, up to a scaling constant $\mu > 0$, it holds

$$
\tau_*(K_0(A_{\theta} \otimes K)) = \begin{cases} \\
\mathbb{Z} + \mathbb{Z}\theta_1 + \ldots + \mathbb{Z}\theta_{6g-7} & \text{if } g > 1, \\
\mathbb{Z} + \mathbb{Z}\theta_1 + \ldots + \mathbb{Z}\theta_{2g-1} & \text{if } g = 1
\end{cases}
$$

(32)

where constants $\theta_i \in \mathbb{R}$ parametrize the moduli (Teichmüller) space of curve $C$, ibid. If $C$ is defined over a number field $k$, then each $\theta_i$ is algebraic and their total number is equal to $2g - 1$. (Indeed, since $Gal(\bar{k} \mid k)$ acts on the torsion points of $C(k)$, it is easy to see that the endomorphism ring of $C(k)$ is non-trivial. Because such a ring is isomorphic to the endomorphism ring of jacobian $Jac\ C$ and $dim_C Jac\ C = g$, one concludes that $End\ C(k)$ is a $\mathbb{Z}$-module of rank $2g$ and each $\theta_i$ is an algebraic number.) After scaling by a constant $\mu > 0$, one gets

$$
H^{1}_{tr}(C) := \tau_*(K_0(A_{\theta} \otimes K)) = \mathbb{Z} + \mathbb{Z}\theta_1 + \ldots + \mathbb{Z}\theta_{2g-1}
$$

(33)

Because $H^{0}(C) \cong H^{2}(C) \cong \mathbb{Z}$, one obtains the rest of formulas (31). □

Remark 2 Using theorem [1] one gets the formula

$$
|C(\mathbb{F}_q)| = 1 + q - tr\ (\omega) = 1 + q - \sum_{i=1}^{2g} \lambda_i,
$$

(34)

where $\lambda_i$ are real eigenvalues of the Frobenius endomorphism $\omega \in End\ (H^{1}_{tr}(C))$. Note that

$$
\lambda_1 + \ldots + \lambda_{2g} = \alpha_1 + \ldots + \alpha_{2g},
$$

(35)
where \( \alpha_i \) are the eigenvalues of the Frobenius endomorphism of \( H^1_{et}(\mathcal{C}; \mathbb{Q}_\ell) \).

However, there is no trace cohomology analog of the classical formula

\[
|\mathcal{C}(\mathbb{F}_{q^r})| = 1 + q^r - \sum_{i=1}^{2g} \alpha_i^r, \quad (36)
\]

unless \( r = 1 \); this difference is due to an algebraic count of the fixed points by the trace cohomology.

**Example 2**

The case \( g = 1 \) is particularly instructive; for the sake of clarity, we shall consider elliptic curves having complex multiplication. Let \( \mathcal{E}(\mathbb{F}_q) \) be the reduction modulo \( q \) of an elliptic with complex multiplication by the ring of integers of an imaginary quadratic field \( \mathbb{Q}(\sqrt{-d}) \), see e.g. [Silverman 1994] [11], Chapter 2. It is known, that in this case the trace cohomology formulas (31) take the form

\[
\begin{align*}
H^0_{tr}(\mathcal{E}(\mathbb{F}_q)) & \cong \mathbb{Z}, \\
H^1_{tr}(\mathcal{E}(\mathbb{F}_q)) & \cong \mathbb{Z} + \mathbb{Z}\sqrt{d}, \\
H^2_{tr}(\mathcal{E}(\mathbb{F}_q)) & \cong \mathbb{Z}.
\end{align*}
\]

(37)

We shall denote by \( \psi(\mathfrak{p}) \in \mathbb{Q}(\sqrt{-d}) \) the Grössencharacter of the prime ideal \( \mathfrak{p} \) over \( p \), see [Silverman 1994] [11], p. 174. It is easy to see, that in this case the Frobenius endomorphism \( \omega \in \text{End} \ (H^1_{tr}(\mathcal{E}(\mathbb{F}_q))) \) is given by the formula

\[
\omega = \frac{1}{2} \left[ \psi(\mathfrak{p}) + \overline{\psi(\mathfrak{p})} \right] + \frac{1}{2} \sqrt{ \left( \psi(\mathfrak{p}) + \overline{\psi(\mathfrak{p})} \right)^2 + 4q },
\]

(38)

and the corresponding eigenvalues

\[
\begin{align*}
\lambda_1 &= \omega = \frac{1}{2} \left[ \psi(\mathfrak{p}) + \overline{\psi(\mathfrak{p})} \right] + \frac{1}{2} \sqrt{ \left( \psi(\mathfrak{p}) + \overline{\psi(\mathfrak{p})} \right)^2 + 4q }, \\
\lambda_2 &= \bar{\omega} = \frac{1}{2} \left[ \psi(\mathfrak{p}) + \overline{\psi(\mathfrak{p})} \right] - \frac{1}{2} \sqrt{ \left( \psi(\mathfrak{p}) + \overline{\psi(\mathfrak{p})} \right)^2 + 4q }.
\end{align*}
\]

(39)

Using formula (34), one gets the following equation

\[
|\mathcal{E}(\mathbb{F}_q)| = 1 - (\lambda_1 + \lambda_2) + q = 1 - \psi(\mathfrak{p}) - \overline{\psi(\mathfrak{p})} + q,
\]

(40)

which coincides with the well-known expression for \( |\mathcal{E}(\mathbb{F}_q)| \) in terms of the Grössencharacter, see e.g. [Silverman 1994] [11], p. 175.
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