Deformed quantum mechanics and $q$-Hermitian operators

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Abstract
Starting on the basis of the non-commutative $q$-differential calculus, we introduce a generalized $q$-deformed Schrödinger equation. It can be viewed as the quantum stochastic counterpart of a generalized classical kinetic equation, which reproduces at the equilibrium the well-known $q$-deformed exponential stationary distribution. In this framework, $q$-deformed adjoint of an operator and $q$-Hermitian operator properties occur in a natural way in order to satisfy the basic quantum mechanics assumptions.

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1. Introduction
In the recent past, there has been a great deal of interest in the study of quantum algebra and quantum groups in connection with several physical fields [1]. From the seminal work of Biedenharn [2] and Macfarlane [3], it was clear that the $q$-calculus, originally introduced in the study of the basic hypergeometric series [4–6], plays a central role in the representation of the quantum groups with a deep physical meaning and not merely a mathematical exercise. Many physical applications have been investigated on the basis of the $q$-deformation of the Heisenberg algebra [7–11]. In [12, 13], it was shown that a natural realization of quantum thermostatistics of $q$-deformed bosons and fermions can be built on the formalism of the $q$-calculus. In [14], a $q$-deformed Poisson bracket, invariant under the action of the $q$-symplectic group, has been derived and a classical $q$-deformed thermostatistics has been proposed in [15]. Furthermore, it is remarkable to observe that the $q$-calculus is very well suited to describe fractal and multifractal systems. As soon as the system exhibits a discrete-scale invariance, the natural tool is provided by Jackson $q$-derivative and $q$-integral, which constitute the natural generalization of the regular derivative and integral for discretely self-similar systems [16].

In the past, the study of generalized linear and nonlinear Schrödinger equations has attracted a lot of interest because many collective effects in quantum many-body models can be
described by means of effective theories with a generalized one-particle Schrödinger equation [17–20]. On the other hand, it is relevant to mention that in recent years many investigations in literature have been devoted to non-Hermitian and pseudo-Hermitian quantum mechanics [21–25, 27].

In the framework of the $q$-Heisenberg algebra, $q$-deformed Schrödinger equations have been proposed [28, 29]. Although the proposed quantum dynamics is based on the noncommutative differential structure on configuration space, we believe that a fully consistent $q$-deformed formalism of the quantum dynamics, based on the properties of the $q$-calculus, has been still lacking.

In this paper, starting on a generalized classical kinetic equation reproducing as stationary distribution of the well-known $q$-exponential function, we study a generalization of the quantum dynamics consistently with the prescriptions of the $q$-differential calculus. At this scope, we introduce a $q$-deformed Schrödinger equation with a deformed Hamiltonian which is a non-Hermitian operator with respect to the standard (undeformed) operators properties but its dynamics satisfies the basic assumptions of the quantum mechanics under generalized operators properties, such as the definition of $q$-adjoint and $q$-Hermitian operators.

2. Noncommutative differential calculus

We shall briefly review the main features of the noncommutative differential $q$-calculus for real numbers. It is based on the following $q$-commutative relation among the operators $\hat{x}$ and $\hat{\partial}_x$:

$$\hat{\partial}_x \hat{x} = 1 + q \hat{x} \hat{\partial}_x,$$

with $q$ a real and positive parameter.

A realization of the above algebra in terms of ordinary real numbers can be accomplished by the replacement [14, 30]

$$\hat{x} \rightarrow x,$$
$$\hat{\partial}_x \rightarrow \mathcal{D}_x^{(q)},$$

where $\mathcal{D}_x^{(q)}$ is the Jackson derivative [4] defined as

$$\mathcal{D}_x^{(q)} = \frac{D(q)}{x} - \frac{1}{(q - 1)x},$$

where

$$D(q) = q^{x^0},$$

is the dilatation operator. Its action on an arbitrary real function $f(x)$ is given by

$$\mathcal{D}_x^{(q)} f(x) = \frac{f(qx) - f(x)}{(q - 1)x}. $$

The Jackson derivative satisfies some simple properties which will be useful in the following. For instance, its action on a monomial $f(x) = x^n$ is given by

$$\mathcal{D}_x^{(q)} x^n = [n]_q x^{n-1}$$

and

$$\mathcal{D}_x^{(q)} x^{-n} = -\frac{[n]_q}{q^n} x^{n+1},$$

where $n \geq 0$ and

$$[n]_q = \frac{q^n - 1}{q - 1}.$$
are the so-called basic numbers. Moreover, we can easily verify the following $q$-version of the Leibnitz rule:

$$D_{x}^{(q)}(f(x)g(x)) = D_{x}^{(q)} f(x)g(x) + f(qx)D_{x}^{(q)} g(x),$$

$$= D_{x}^{(q)} f(x)g(x) + f(x)D_{x}^{(q)} g(x).$$  \(10\)

A relevant role in the $q$-algebra, as developed by Jackson, is given by the basic binomial series defined by

$$(x + y)^{(n)} = (x + y)(x + qy)(x + q^2y) \cdots (x + q^{n-1}y)$$

$$\equiv \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right] q^{r(n-r)/2} x^{n-r} y^r.$$  \(11\)

where

$$\left[ \begin{array}{c} n \\ r \end{array} \right] q = \frac{[n]_q!}{[r]_q! [n-r]_q!}.$$  \(12\)

is known as the $q$-binomial coefficient which reduces to the ordinary binomial coefficient in the $q \to 1$ limit [6]. We should remark that equation (12) holds for $0 \leq r \leq n$, while it is assumed to vanish otherwise and we have defined $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$. Remarkably, a $q$-analogue of the Taylor expansion has been introduced in [4] by means of a basic binomial

$$f(x) = f(a) + \frac{(x-a)^{(1)}}{[1]!} D_{x}^{(q)} f(x) \bigg\vert_{x=a} + \frac{(x-a)^{(2)}}{[2]!} D_{x}^{(q)} f(x) \bigg\vert_{x=a} + \cdots,$$  \(13\)

where $D_{x}^{(q)^2} \equiv D_{x}^{(q)} D_{x}^{(q)}$ and so on.

Consistently with the $q$-calculus, we also introduce the basic integration

$$\int_{0}^{x} f(y) \, d_{q}y = \sum_{n=0}^{\infty} \Delta_{q}^{n} f(\lambda_{n}),$$  \(14\)

where $\Delta_{q}^{n} = \lambda_{n} - \lambda_{n+1}$ and $\lambda_{n} = \lambda_{0} q^{n}$ for $0 < q < 1$ whilst $\Delta_{q}^{n} = \lambda_{n-1} - \lambda_{n}$ and $\lambda_{n} = \lambda_{0} q^{-n-1}$ for $q > 1$ [5, 6, 15, 16]. Clearly, equation (14) is reminiscent of the Riemann quadrature formula performed now in a $q$-nonuniform hierarchical lattice with a variable step $\Delta_{q}^{n}$. It is trivial to verify that

$$D_{x}^{(q)} \int_{0}^{x} f(y) \, d_{q}y = f(x),$$  \(15\)

for any $q > 0$.

Let us now introduce the following $q$-deformed exponential function defined by the series:

$$E_{q}(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = 1 + x + \frac{x^2}{[2]_q!} + \frac{x^3}{[3]_q!} + \cdots,$$  \(16\)

which will play an important role in the framework we are introducing. The function (16) defines the basic exponential, well known in the literature since a long-time ago, originally introduced in the study of basic hypergeometric series [5, 6]. In this context, let us observe that definition (16) is fully consistent with its Taylor expansion, as given by equation (13).

The basic exponential is a monotonically increasing function, $d E_{q}(x) / dx > 0$, convex, $d^2 E_{q}(x) / dx^2 > 0$, with $E_{q}(0) = 1$ and reducing to the ordinary exponential in the $q \to 1$ limit: $E_{1}(x) \equiv \exp(x)$. An important property satisfied by the $q$-exponential can be written formally as [6]

$$E_{q}(x + y) = E_{q}(x) E_{q^{-1}}(y),$$  \(17\)
where the left-hand side of equation (17) must be considered by means of its series expansion in terms of basic binomials,

\[ E_q(x + y) = \sum_{k=0}^{\infty} \frac{(x + y)^{(k)}}{[k]!}. \]  

By observing that \((x - x)^{(k)} = 0\) for any \(k > 0\), since \((x - x)^{(0)} = 1\), from equation (17) we can see that [15]

\[ E_q(x)E_q(-x) = 1. \]  

The above property will be crucial in the following introduction to a consistent q-deformed quantum mechanics.

Among many properties, it is important to recall the following relation [6]:

\[ D^q_x(E_q(ax)) = aE_q(ax), \]  

and its dual

\[ \int_0^x E_q(ay) \, dqy = \frac{1}{a}[E_q(ax) - 1]. \]  

Finally, it should be pointed out that equations (20) and (21) are two important properties of the basic exponential which turns out to be not true if we employ the ordinary derivative or integral.

3. Classical q-deformed kinetic equation

Starting from the realization of the q-algebra, defined in equations (2)–(3), we can write for a homogeneous system the following q-deformed Fokker–Planck equation [31]:

\[ \frac{\partial f(x, t)}{\partial t} = D^{q}_x \left[ -J^{q}_1(x) + J^{q}_2 D^{q}_x \right] f(x, t), \]  

where \(J^{q}_1(x)\) and \(J^{q}_2\) are the drift and diffusion coefficients, respectively.

The above equation has stationary solution \(f^{q}_{st}(x)\) that can be written as

\[ f^{q}_{st}(x) = N_q E_q[-\Phi^{q}_1(x)]. \]  

where \(N_q\) is a normalization constant, \(E_q[x]\) is the q-deformed exponential function defined in equation (16) and we have defined\(^1\)

\[ \Phi^{q}_1(x) = -\frac{1}{J^{q}_2} \int_0^x J^{q}_1(y) \, dqy. \]  

If we postulate a generalized Brownian motion in a q-deformed classical dynamics by mean the following definition of the drift and diffusion coefficients:

\[ J^{q}_1(x) = -\gamma x (q D^{q}_x + 1), \quad J^{q}_2 = \gamma/\alpha, \]  

where \(\gamma\) is the friction constant, \(\alpha\) is a constant depending on the system and \(D^{q}_x\) is the dilatation operator (5), the stationary solution \(f^{q}_{st}(x)\) of the above Fokker–Planck equation can be obtained as solution of the following stationary q-differential equation:

\[ D^{q}_x f^{q}_{st}(x) = -\alpha x [q f^{q}_{st}(qx) + f^{q}_{st}(x)]. \]  

It easy to show that the solution of the above equation can be written as

\[ f^{q}_{st}(x) = N_q E_q[-\alpha x^2]. \]  

\(^1\) In the following, for simplicity, we limit ourselves to consider the drift coefficient as a monomial function of \(x\).
4. \( q \)-deformed Schrödinger equation

We are now able to derive a \( q \)-deformed Schrödinger equation by means of a stochastic quantization method \([32]\).

Starting from the following transformation of the probability density:

\[
\rho_q(x,t) = \mathcal{E}_q \left[ \frac{1}{\Phi_1^2(x)} \right] \psi_q(x,t), \tag{28}
\]

where \( \Phi_q(x) \) is the function defined in equation (24), the \( q \)-deformed Fokker–Planck equation (22) can be written as

\[
\frac{\partial \psi_q(x,t)}{\partial t} = J_2^{(q)} D_x^{(q)} \psi_q(x,t) - V_q(x) \psi_q(x,t), \tag{29}
\]

where

\[
V_q(x) = \frac{1}{2} D_x^{(q)} J_1^{(q)}(x) + \frac{\left[J_1^{(q)}(x)\right]^2}{4 J_2^{(q)}}. \tag{30}
\]

The above equation has the same structure of the time-dependent Schrödinger equation. In fact, the stochastic quantization of equation (22) can be realized with the transformations

\[
t \rightarrow t - i \hbar, \tag{31}
\]

\[
J_2^{(q)} \rightarrow \frac{\hbar^2}{2m}, \tag{32}
\]

getting the \( q \)-generalized Schrödinger equation

\[
i\hbar \frac{\partial \psi_q(x,t)}{\partial t} = H_q \psi_q(x,t), \tag{33}
\]

where

\[
H_q = -\frac{\hbar^2}{2m} D_x^{(q)} + V_q(x) \tag{34}
\]

is the \( q \)-deformed Hamiltonian. Let us note that the Hamiltonian (34) is a not-Hermitian operator with respect to the standard definition based on the ordinary (undeformed) scalar product of square-integrable functions \([9, 14]\). In the following section, we will see as this aspect can be overridden by means the introduction of a \( q \)-deformed scalar product and generalized properties of operators inspired to the \( q \)-calculus.

The above equation admits factorized solution

\[
\psi_q(x,t) = \phi(t) \varphi_q(x), \tag{35}
\]

where \( \phi(t) \) satisfies to the equation

\[
i\hbar \frac{d\phi(t)}{dt} = E \phi(t), \tag{36}
\]

with the standard (undeformed) solution

\[
\phi(t) = \exp \left( -\frac{i}{\hbar} Et \right), \tag{37}
\]

while \( \varphi_q(x) \) is the solution of time-independent \( q \)-Schrödinger equation

\[
H_q \varphi_q(x) = E \varphi_q(x). \tag{38}
\]

In one-dimensional case, for a free particle \( (V_q = 0) \) described by the wavefunction \( \varphi_q(x) \), equation (37) becomes

\[
D_x^{(q)} \varphi_q(x) + k^2 \varphi_q(x) = 0, \tag{39}
\]

where \( k \) is the wavenumber.
where \( k = \sqrt{2mE/\hbar^2} \). The solution of the previous equation can be written as

\[
\psi_f^q(x) = NE_q(ikx).
\]  

(39)

The above equation generalizes the plane wavefunction in the framework of the \( q \)-calculus.

**5. \( q \)-deformed products and \( q \)-Hermitian operators**

In order to develop a consistent deformed quantum dynamics, we have to generalize the products between functions and properties of the operators in the framework of the \( q \)-calculus.

Let us start on the basis of equation (19), which implies

\[
E_q(ix)(E_q^{-1}(ix))^* = 1,
\]  

(40)

\[
E_q(ix)^* = (E_q^{-1}(ix))^{-1},
\]  

(41)

and in terms of the \( q \)-plane wave (39)

\[
\psi_{f -1}^q(x)^*\psi_f^q(x) = N^2.
\]  

(42)

Inspired to the above equation, it appears natural to introduce the complex \( q \)-conjugation of a function as

\[
\psi_{q^-1}^*(x) = \psi_{q^-1}^*(x),
\]  

(43)

and, consequently, the probability density of a single particle in a finite space as

\[
\rho_q(x, t) = |\psi_q(x, t)|^2_q = \psi_{q^-1}^*(x)\psi_q(x, t) \equiv \psi_{q^-1}^*(x, t)\psi_q(x, t).
\]  

(44)

Thus, the wavefunctions must be \( q \)-square-integrable functions of configuration space, that is to say the functions \( \psi_q(x) \) such that the integral

\[
\int \int \rho_q(x, t)^2_q \, dq \, dx
\]  

(45)

converges.

The function space defined above is a linear space. If \( \psi_q \) and \( \varphi_q \) are two \( q \)-square-integrable functions, any linear combinations \( \alpha\psi_q + \beta\varphi_q \), where \( \alpha \) and \( \beta \) are arbitrarily chosen complex numbers, are also \( q \)-square-integrable functions.

Following this line, it is possible to define a \( q \)-scalar product of the function \( \psi \) by the function \( \varphi \) as

\[
\langle \psi, \varphi \rangle_q = \int \psi_{q^-1}^*(x)\varphi_q(x) \, dq \, dx \equiv \int \psi_{q^-1}^*(x)\varphi_q(x) \, dq \, dx.
\]  

(46)

This is linear with respect to \( \psi \), the norm of a function \( \psi_q \) is a real, non-negative number, \( \langle \psi, \psi \rangle_q \geq 0 \) and

\[
\langle \psi, \varphi \rangle_q = \langle \varphi, \psi \rangle_q^*.
\]  

(47)

Analogously to the undeformed case, it is easy to see that from the above properties of the \( q \)-scalar product follows the \( q \)-Schwarz inequality

\[
|\langle \psi, \varphi \rangle_q^2_q| \leq \langle \psi, \psi \rangle_q \langle \varphi, \varphi \rangle_q.
\]  

(48)

Consistently with the above definitions, the \( q \)-adjoint of an operator \( A_q \) is defined by means of the relation

\[
\langle \psi, A_q^\dagger \psi \rangle_q = \langle \psi, A_q \psi \rangle_q^*.
\]  

(49)
and, by definition, a linear operator is $q$-Hermitian if it is its own $q$-adjoint. More explicitly, an operator $A_q$ is $q$-Hermitian if for any two states $\psi_q$ and $\psi_q$ we have

$$\langle \psi, A_q \psi \rangle_q = \langle A_q \psi, \psi \rangle_q.$$  

(50)

First of all, the above properties are crucial to have a consistent conservation in time of the probability densities, defined in equation (44). In fact, by taking the complex $q$-conjugation of equation (33), summing and integrating term by term the two equations, we get

$$i\hbar \frac{\partial}{\partial t} \int \psi^*_q \psi_q dx = \int \left[ \psi^{*^{-1}}_q (H_q \psi_q) - (H_q^{-1} \psi^{*^{-1}}_q) \psi_q \right] dq x = 0,$$

(51)

where the last equivalence follows from the fact that the operator Hamiltonian is $q$-Hermitian.

In this context, it is relevant to observe that it is possible to verify the above property by using the time-spatial factorization solution $\psi_q(x, t) = \phi(t) \psi_q(x)$ of the $q$-Schrödinger equation. In fact, we have

$$i\hbar \frac{\partial}{\partial t} \int \psi^*_q \psi_q dx = \int \phi^* \phi \left[ \psi^{*^{-1}}_q (H_q \psi_q) - (H_q^{-1} \psi^{*^{-1}}_q) \psi_q \right] dq x.$$  

(52)

From the stationary Schrödinger equation (37) and its complex $q$-conjugation we have directly

$$\psi^{*^{-1}}_q (H_q \psi_q) = (H_q^{-1} \psi^{*^{-1}}_q) \psi_q,$$

(53)

and the terms in the square bracket of equation (52) go to zero.

6. Observables in $q$-deformed quantum mechanics

On the basis of the above properties, we have the recipe to generalize the definition of observables in the framework of $q$-deformed theory by postulating that:

- with the dynamical variable $A(x, p)$ associate the linear operator $A_q(x, -i\hbar D_x^{(q)})$;
- the mean value of this dynamical variable, when the system is in the dynamical (normalized) state $q$, is

$$\langle A \rangle_q = \int \psi^*_q A_q \psi_q dq x = \int \psi^{*^{-1}}_q A_q \psi_q dq x.$$  

(54)

Observables are real quantities, hence the expectation value (54) must be real for any state $q$, while

$$\int \psi^*_q A_q \psi_q dq x = \int (A_q \psi_q) \psi_q dq x,$$

(55)

therefore, on the basis of equation (50), observables must be represented by $q$-Hermitian operators.

If we require there is a state $q$ for which the result of measuring the observable $A$ is unique, in other words that the fluctuations

$$(\Delta A_q)^2 = \int \psi^*_q (A_q - \langle A_q \rangle_q)^2 \psi_q dq x$$

(56)

must vanish, we obtain the following $q$-eigenvalue equation of a $q$-Hermitian operator $A_q$ with eigenvalue $a$:

$$A_q \psi_q = a \psi_q.$$  

(57)

As a consequence, the eigenvalues of a $q$-Hermitian operator are real because $\langle A \rangle_q$ is real for any state; in particular, for an eigenstate with the eigenvalue $a$ for which $\langle A \rangle_q = a$.  

(57)
Furthermore, as in the undeformed case, two eigenfunctions $\psi_{q,1}$ and $\psi_{q,2}$ of the $q$-Hermitian operator $A_q$, corresponding to different eigenvalues $a_1$ and $a_2$, are orthogonal. We can always normalize the eigenfunction; therefore, we can choose all the eigenvalues of a $q$-Hermitian operator orthonormal, i.e.,

$$\int \psi_{q,n}^\dagger \psi_{q,m} d_q x = \delta_{n,m}. \quad (58)$$

Consequently, two eigenfunctions $\psi_{q,1}$ and $\psi_{q,2}$ belonging to different eigenvalues are linearly independent.

It is easy to see that, adapting step by step the undeformed case to the introduced $q$-deformed framework, the totality of the linearly independent eigenfunctions $\{\psi_{q,n}\}$ of $q$-Hermitian operator $A_q$ form a complete (orthonormal) set in the space of the previously introduced $q$-square-integrable functions. In other words, if $\psi_q$ is any state of a system, then it can be expanded in terms of the eigenfunctions (with a discrete spectrum) of the corresponding $q$-Hermitian operator $A_q$ associate with the observable

$$\psi_q = \sum_n c_{q,n} \psi_{q,n}, \quad (59)$$

where

$$c_{q,n} = \int \psi_{q,n}^\dagger \psi_q d_q x. \quad (60)$$

The above expansion allows us, as usual, to write the expectation value of $A_q$ in the normed state $\psi_q$ as

$$\langle A \rangle_q = \int \psi_q^\dagger A_q \psi_q d_q x = \sum_n |c_{q,n}|_q^2 a_n. \quad (61)$$

where $\{a_n\}$ are the set of eigenvalues (assumed, for simplicity, discrete and non-degenerate) and the normalization condition of the wavefunction can be written in the form

$$\sum_n |c_{q,n}|_q^2 = 1. \quad (62)$$

7. Conclusions

On the basis of the stochastic quantization procedure and on the $q$-differential calculus, we have obtained a generalized linear Schrödinger equation which involves a $q$-deformed Hamiltonian that is non-Hermitian with respect to the standard (undeformed) definition. However, under an appropriate generalization of the operators properties and the introduction of a $q$-deformed scalar product in the space of $q$-square-integrable wavefunctions, such equation of motion satisfies the basic quantum mechanics assumptions.

Although a complete physical and mathematical description of the introduced quantum-dynamical equations lies out the scope of this paper, we think that the results derived here appear to provide a deeper insight into a full consistent $q$-deformed quantum mechanics in the framework of the $q$-calculus and may be a relevant starting point for future investigations.

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