MEAN-VARIANCE INVESTMENT AND CONTRIBUTION
DECISIONS FOR DEFINED BENEFIT PENSION PLANS IN A
STOCHASTIC FRAMEWORK

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ABSTRACT. In this paper we investigate the management of a defined benefit pension plan under a model with random coefficients. The objective of the pension sponsor is to minimize the solvency risk, contribution risk and the expected terminal value of the unfunded actuarial liability. By measuring the solvency risk in terms of the variance of the terminal unfunded actuarial liability, we formulate the problem as a mean-variance problem with an additional running cost. With the help of a system of backward stochastic differential equations, we derive a time-consistent equilibrium strategy towards investment and contribution rate. The obtained equilibrium strategy turns out to be a good candidate for a stable contribution plan. When the interest rate is given by the Vasicek model and all other coefficients are deterministic, we obtain closed-form solutions of the equilibrium strategy and efficient frontier.

1. Introduction. As one of the most popular pension schemes, defined benefit (DB, for short) pension schemes have attracted tremendous attention in academic research. In a DB plan, retirement benefits are determined in advance by a formula (usually based on earnings history and years of service), and in the funding phase, contributions are paid in accordance with an actuarial scheme. The fund associated

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with such a plan makes the benefit payments and receives income from the contributions and investment returns. It is widely acknowledged that the DB pension plan is confronted with two main types of risks: (i) the solvency risk which is related to the security of the plan and, (ii) the contribution risk which is associated with the stability of the contribution. Therefore, the management of a DB pension fund is mostly formulated as a risk minimization problem, in which the control variables include the investment strategy and the contribution rate.

Since the work of [16], the risk minimization problems for DB pension plans are usually modeled as linear-quadratic (LQ, for short) control problems, where the solvency risk and the contribution rate risk are defined as the quadratic deviations of the fund and contributions from the benefit and normal cost, respectively. See, e.g., [6], [7], [19, 20]. Recently, there is a growing interest in studying DB pension plans under the mean-variance (MV, for short) framework. [8] and [18] consider static MV type problems for DB pension plans. [10] studies the MV investment and contribution strategies in a pre-retirement accumulation phase of a DB pension plan. Under a model with stochastic benefits outgo, [21] defines the solvency risk as the variance of the terminal unfunded actuarial liability and formulate the optimal management of the DB pension plan as a non-standard MV problem. They obtain the (pre-commitment) optimal contribution and investment strategies by applying the embedding method proposed by [24] and [34].

Since the variance is a non-linear function of the expectation, the mean-variance optimization problem is time-inconsistent in the sense that it does not admit a Bellman optimality principle. In other words, the optimal strategy obtained for the initial time may fail to be optimal for some later time. However, finding a time-consistent strategy is highly relevant, especially for the long-term decision-making problems in which the decision-maker may not be able to pre-commit their future behavior. For example, the decision-maker may be changed and the successor may lack the incentive to follow the strategy executed by the predecessor. To obtain a time-consistent strategy, [30] proposes a game theoretic method. He views the original time-inconsistent control problem as a non-cooperative game with one player at each time $t$, and then seeks a subgame perfect Nash equilibrium strategy.

In recent years, there have been considerable attempts in the literature to deal with the time-inconsistent LQ and/or MV problems in continuous-time settings. [1] derives a time-consistent strategy for the mean-variance asset allocation problem in an incomplete-market setting. [17] considers a time-inconsistent LQ control problem with random coefficients. As an application, they solve the MV problem with state-dependent risk aversion and random coefficients (the interest rate in their model is deterministic). [9] investigates the time-consistent solution for the mean-variance problem in a general semi-martingale framework. [4] studies the MV portfolio selection problem with state-dependent risk aversion. [3] establishes the extended HJB equation and the verification theorem for the time-inconsistent control problem in a general Markovian framework and solve a time-inconsistent LQ problem as an example. [31] obtains a time-consistent equilibrium strategy for the MV asset-liability management problem with all coefficients (including the interest rate) being random. [32] studies the open-loop and closed-loop equilibrium strategies for a time-inconsistent LQ problem for mean-field stochastic differential equations. There are also many papers considering the time-inconsistent portfolio selection problems with non-exponential discounting. See, e.g., [12], [26], [11], [33] and references therein.
In this paper, we consider a risk minimization problem for a DB pension plan. Similar to [21], the cost functional consists of a standard MV cost and a running cost measuring the contribution risk. Since the management of DB pensions usually spans over a long planning horizon, we aim to find time-consistent equilibrium investment and contribution strategies. Furthermore, to capture the randomnesses of the market coefficients in the long term, we assume that all the parameters in the model are stochastic processes. Following a method similar to [31], we obtain an equilibrium strategy by using the solution to a system of backward stochastic differential equations (BSDEs, for short). As an example, we discuss the case where the interest rate is given by the Vasicek model and all the other coefficients are deterministic functions. In this case, the time-consistent equilibrium strategy is given in closed-form in terms of the solution to a system of partial differential equations (PDEs, for short).

In practice, finding stable contribution plans is an important theme for pension scheme sponsors (see, e.g., [23], [27], [14] and [15]). A stable contribution plan is one that would not be significantly disturbed by the investment performance. It is interesting to see that the time-consistent equilibrium contribution strategy obtained in this paper is only determined by the normal cost, the risk aversion and the coefficients in the model, and is independent of the fund and the actuarial liability. To our best knowledge, the optimal contribution strategies in most of the existing literature depend on the value of the fund (see, e.g., [6], [19, 20, 21] and [29]). In this regard, our time-consistent equilibrium contribution strategy could serve as a candidate for the stable contribution plan.

The paper is organized as follows. Section 2 introduces the DB pension scheme, the financial market and the risk minimization problem. Section 3 presents the main results. Section 4 discusses an example with stochastic interest rate. Section 5 concludes the paper and the appendices collect some lengthy proofs.

2. The model. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a fixed complete probability space on which an \(m\)-dimensional and an \(n\)-dimensional standard Brownian motions, namely \(W(\cdot) \equiv (W_1(\cdot), \ldots, W_m(\cdot))^\top\) and \(\bar{W}(\cdot) \equiv (\bar{W}_1(\cdot), \ldots, \bar{W}_n(\cdot))^\top\), are defined. Suppose that \(W(\cdot)\) and \(\bar{W}(\cdot)\) are independent. Let \(T > 0\) be the fixed finite time horizon (e.g., the date of the end of the pension plan), \(\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0,T]}\) be the augmented filtration generated by \((W(\cdot)^\top, \bar{W}(\cdot)^\top)^\top\) and \(\mathbb{F}^W := \{\mathcal{F}_t^W\}_{t \in [0,T]}\) be the augmented filtration generated by \(W(\cdot)\).

For \(p \geq 1\), \(H := \mathbb{R}^n, \mathbb{R}^{n \times m}\), etc. and \(0 \leq s \leq t \leq T\), define

\[
L^p_{\mathbb{F}}(\Omega; \mathcal{H}) := \left\{ \mathbf{X} : \Omega \rightarrow \mathcal{H} \mid \mathbf{X}(\cdot) \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}[|\mathbf{X}|^p] < \infty \right\},
\]

\[
L^p_{\mathbb{F}}(s, t; \mathcal{H}) := \left\{ \mathbf{X} : [s, t] \times \Omega \rightarrow \mathcal{H} \mid \mathbf{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted, and } \mathbb{E} \left[ \int_s^t |\mathbf{X}(v)|^p \, dv \right] < \infty \right\},
\]

\[
L^p_{\mathbb{F}}(\Omega; L^2(s, t; \mathcal{H})) := \left\{ \mathbf{X} : [s, t] \times \Omega \rightarrow \mathcal{H} \mid \mathbf{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted, and } \mathbb{E} \left[ \left( \int_s^t |\mathbf{X}(v)|^2 \, dv \right)^p \right] < \infty \right\},
\]

\[
L^p_{\mathbb{F}}(\Omega; C([s, t]; \mathcal{H})) := \left\{ \mathbf{X} : [s, t] \times \Omega \rightarrow \mathcal{H} \mid \mathbf{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted, and } \mathbb{E} \left[ \left( \int_s^t |\mathbf{X}(v)| \, dv \right)^p \right] < \infty \right\}.
\]
has continuous paths and $\mathbb{E}\left[ \sup_{v \in [s,t]} |X(v)|^p \right] < \infty$.

In the rest of the paper, unless specified otherwise, we denote matrices and vectors by bold-face letters and the transpose of a matrix or vector by the superscript $\top$.

2.1. The DB pension plan. We consider a stochastic DB pension model which is an extension of the ones in [20, 21, 22] by considering random coefficients. This model is very flexible and general, and has the capacity to model various changing factors in economic and market conditions, such as stochastic interest rate and stochastic volatility. The modelling framework also allows for path-dependent factors, which is an important phenomenon in the market, but seldom considered in the context of pension planning.

The benefit promised to the participants (or the liability of the sponsor) at time $t$, denoted by $L(t)$, is modelled by a geometric Brownian motion model:

$$
\begin{aligned}
\begin{cases}
    dL(s) = L(s) \left[ \kappa(s) ds + \xi^\top(s) dW(s) + \bar{\xi}^\top(s) d\bar{W}(s) \right], & s \in [0, T], \\
    L(0) = L_0 > 0,
\end{cases}
\end{aligned}
$$

(1)

where $L_0$ is a constant representing the initial liability, $\kappa(\cdot) \in \mathbb{R}$, $\xi(\cdot) \in \mathbb{R}^m$ and $\bar{\xi}(\cdot) \in \mathbb{R}^n$ are bounded $\mathbb{F}$-adapted stochastic processes.

Assume that all the members join the plan at age $a$ and retire at age $d$. Denote by $M(x)$ the distribution function representing the percentage of actuarial value of future benefits accumulated until age $x$, and $m(x)$ the associated density function. Then the actuarial liability $AL(\cdot)$ and the normal cost $NC(\cdot)$ are defined as

$$AL(t) = \mathbb{E}_t \left[ \int_a^d e^{-\int_t^{t+d-x} \delta(s) ds} M(x)L(t+d-x)dx \right],$$

$$NC(t) = \mathbb{E}_t \left[ \int_a^d e^{-\int_t^{t+d-x} \delta(s) ds} m(x)L(t+d-x)dx \right],$$

where $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$ is the expectation conditioned on $\mathcal{F}_t$ and the technical rate of actualization $\delta(\cdot) \in \mathbb{R}$ is assumed to be a bounded $\mathbb{F}$-adapted stochastic process. To derive the equations satisfied by $AL(\cdot)$ and $NC(\cdot)$, we make the following assumption.

**Assumption 2.1.** It holds that

$$\kappa(\cdot) = \delta(\cdot) + \gamma_1(\cdot),$$

(2)

where $\gamma_1(\cdot) \in \mathbb{R}$ is a bounded deterministic function.

Since we shall consider random market coefficients, it is reasonable to assume that the technical rate $\delta(\cdot)$, which is closely related to the interest rate, is also random. Assumption 2.1 means that the discounted liability value $e^{-\int_t^s \delta(s) ds} L(t)$ increases/decreases on average at a deterministic exponential rate. This is similar to the common assumption that the difference between the appreciation rate of the stock and the interest rate, i.e., the risk premium, is deterministic in the portfolio selection problem with stochastic interest rate.

To simplify our presentation, we define

$$\psi_{AL}(t) := \int_a^d e^{\int_t^{t+d-x} \gamma_1(s) ds} M(x)dx, \quad \psi_{NC}(t) := \int_a^d e^{\int_t^{t+d-x} \gamma_1(s) ds} m(x)dx,$$
\[ \zeta_{AL}(t) := \int_a^d e^{\int_a^t d\gamma_1(t) + m(x)dx} \gamma_1(t + d - x)M(x)dx - \gamma_1(t)\psi_{AL}(t), \]
\[ \zeta_{NC}(t) := \int_a^d e^{\int_a^t d\gamma_1(t) + m(x)dx} \gamma_1(t + d - x)m(x)dx - \gamma_1(t)\psi_{NC}(t), \]
which are all deterministic functions.

**Proposition 2.2.** Given Assumption 2.1, it holds that for all \( t \geq 0 \),
\[ AL(t) = \psi_{AL}(t)L(t), \quad NC(t) = \psi_{NC}(t)L(t) \]
and
\[ NC(t) - L(t) = \left( \frac{\zeta_{AL}(t)}{\psi_{AL}(t)} + \gamma_1(t) \right) AL(t). \]  

Moreover, the actuarial liability \( AL(\cdot) \) and the normal cost \( NC(\cdot) \) satisfy
\[ \left\{ \begin{array}{l}
\mathrm{d}AL(s) = AL(s) \left[ \left( \frac{\zeta_{AL}(s)}{\psi_{AL}(s)} + \kappa(s) \right) ds + \xi^T(s) d\bar{W}(s) \right], \quad s \in [0, T], \\
AL(0) := AL_0 = \psi_{AL}(0)L_0
\end{array} \right. \]
and
\[ \left\{ \begin{array}{l}
\mathrm{d}NC(s) = NC(s) \left[ \left( \frac{\zeta_{NC}(s)}{\psi_{NC}(s)} + \kappa(s) \right) ds + \xi^T(s) d\bar{W}(s) \right], \quad s \in [0, T], \\
NC(0) := NC_0 = \psi_{NC}(0)L_0
\end{array} \right. \]
respectively.

The proof of the above proposition is similar to [22, Proposition 2.1] and it is omitted here. We should point out that without Assumption 2.1, Proposition 2.1 in [22] does not hold. Since the coefficients are bounded, it is easy to see that \( AL(\cdot), NC(\cdot) \in L^p_\mathbb{F}(\Omega; C([0, T]; \mathbb{R})) \) for any \( p > 1 \).

2.2. The optimization problem. In this subsection, we formulate the risk minimization problem of the sponsor. Consider a financial market consisting of a bank account and \( m \) risky assets traded within the time horizon \([0, T]\). The price process of the bank account \( S_0(\cdot) \) is governed by
\[ \left\{ \begin{array}{l}
\mathrm{d}S_0(s) = r(s)S_0(s) ds, \quad s \in [0, T], \\
S_0(0) = s_0 > 0
\end{array} \right. \]
where \( r(\cdot) \) is the risk-free interest rate.

For \( i = 1, 2, \cdots, m \), the price of the \( i \)-th risky asset \( S_i(\cdot) \) is given by
\[ \left\{ \begin{array}{l}
\mathrm{d}S_i(s) = S_i(s) \left[ \mu_i(s) ds + \sum_{j=1}^m \sigma_{ij}(s) dW_j(s) \right], \quad s \in [0, T], \\
S_i(0) = s_i > 0
\end{array} \right. \]
where \( \mu_i(\cdot) \) is the expected return rate of the \( i \)-th risky asset and \( \sigma_{ij}(\cdot), \quad j = 1, \cdots, m \), are volatility rates.

In this paper, we assume that the interest rate \( r(\cdot) \) is a bounded, continuous and \( \mathbb{F}^{W} \)-adapted process, which implies that the interest rate risk can be fully hedged in the financial market (e.g., by trading zero-coupon bonds). Furthermore, we assume that \( \mu(\cdot) := (\mu_1(\cdot), \cdots, \mu_m(\cdot))^T \) and \( \sigma(\cdot) := (\sigma_{ij}(\cdot))_{1 \leq i, j \leq m} \) are bounded, continuous and \( \mathbb{F} \)-adapted processes such that \( \mu_i(\cdot) > r(\cdot) \) for all \( i = 1, \cdots, m \) and \( \sigma(\cdot) \sigma^T(\cdot) \geq \varrho I_{m \times m} \) for some \( \varrho > 0 \), where \( I_{m \times m} \) denotes the \( m \times m \) identity matrix. Unlike the interest rate, risks from \( \mu(\cdot) \) and \( \sigma(\cdot) \) can only be partially hedged.
Let $u_i(t)$ be the amount of the fund invested in the $i$-th risky asset and $C(t)$ be the contribution rate at time $t$. Then the value of the fund asset, denoted by $F(\cdot)$, is governed by
\[
\begin{aligned}
dF(s) &= [r(s)F(s) + C(s) - L(s) + \theta^T(s)u(s)] \, ds + u^T(s)\sigma(s) dW(s),
F(0) = F_0,
\end{aligned}
\]
where
\[
\theta(\cdot) := (\mu_1(\cdot) - r(\cdot), \mu_2(\cdot) - r(\cdot), \cdots, \mu_m(\cdot) - r(\cdot))^T
\]
and $u(\cdot) := (u_1(\cdot), \cdots, u_m(\cdot))^T$.

Denote by $UAL(\cdot) := AL(\cdot) - F(\cdot)$ the unfunded actuarial liability and $SC(\cdot) := C(\cdot) - NC(\cdot)$ the supplementary cost. Let $X(\cdot) := -UAL(\cdot) = F(\cdot) - AL(\cdot)$. By (2-4) and (8), we have
\[
\begin{aligned}
dX(s) &= [r(s)X(s) + (r(s) - \delta(s)) AL(s) + SC(s) + \theta^T(s)u(s)] \, ds
+ (u^T(s)\sigma(s) - AL(s)\xi^T(s)) \, dW(s) - AL(s)\xi^T(s) \, dW(s),
X(0) = x_0 = F_0 - AL_0.
\end{aligned}
\]

For simplicity, in the following we take the pair $(u(\cdot), SC(\cdot))$, instead of $(u(\cdot), C(\cdot))$, as the control variable. Similar to [21], we assume that the objective of the pension fund sponsor is to minimize the expectation and the variance of the terminal unfunded actuarial liability $UAL(T)$ and the contribution risk during the planning horizon. Therefore, for any initial state $(t, X(t), AL(t))$, we consider the cost functional
\[
J(t, X(t), AL(t); u(\cdot), SC(\cdot)) := \frac{1}{2} \mathbb{E}_t \left[ \int_t^T SC^2(s) \, ds \right] + \frac{1}{2} \text{Var}_t [X(T)] - \lambda \mathbb{E}_t [X(T)],
\]
where $\text{Var}_t [\cdot]$ denotes the variance conditioned on $\mathcal{F}_t$, and $\lambda > 0$ is a given constant balancing the relative importance of minimizing the risks and minimizing the debt (we can also interpret $1/(2\lambda)$ as a risk aversion parameter). Since there is an additional control variable $SC(\cdot)$ in the state process (9) and an additional running cost in (10), the optimization problem considered in this paper is different from the one in [31].

The definition of admissible strategies is given in the following.

**Definition 2.3.** A pair $(u(\cdot), SC(\cdot))$ is said to be admissible if it is in $L^2_\mathbb{F}(0, T; \mathbb{R}^m) \times L^2_\mathbb{F}(0, T; \mathbb{R})$ such that (9) admits a unique solution $X(\cdot) \in L^2_\mathbb{F}(\Omega; C ([0, T]; \mathbb{R}))$.

The sponsor aims to solve the following optimization problem.

**Problem.** At any time $t \in [0, T]$ with $X(t)$ and actuarial liability $AL(t)$, specify an admissible strategy $(u(\cdot), SC(\cdot)) \in L^2_\mathbb{F}(0, T; \mathbb{R}^m) \times L^2_\mathbb{F}(0, T; \mathbb{R})$ that minimizes cost functional (10).

It is well known that the mean-variance optimization problem is time-inconsistent. That is the optimal strategy that minimizes (10) for the initial state $(t, X(t), AL(t))$ may be suboptimal at some later time. In this paper, we seek the so-called time-consistent equilibrium strategy for Problem 2.2. The following definition can be found in [17].

**Definition 2.4.** Let $(\hat{u}(\cdot), SC(\cdot)) \in L^2_\mathbb{F}(0, T; \mathbb{R}^m) \times L^2_\mathbb{F}(0, T; \mathbb{R})$ be a given strategy and $\hat{X}(\cdot)$ be the corresponding state process. The strategy $(\hat{u}(\cdot), SC(\cdot))$ is
called a time-consistent equilibrium strategy if for any \( t \in [0, T) \) and \((\nu_1, \nu_2) \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^m) \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R})\),
\[
\liminf_{\varepsilon \downarrow 0} \frac{J(t, \tilde{X}(t), AL(t); u^{t, \varepsilon, \nu_1}(-), SC^{t, \varepsilon, \nu_2}(-)) - J(t, \tilde{X}(t), AL(t); \tilde{u}(-), \tilde{SC}(-))}{\varepsilon} \geq 0,
\]
where
\[
\begin{align*}
(u^{t, \varepsilon, \nu_1}(s) := \tilde{u}(s) + \nu_1 I_{[t, t+\varepsilon]}(s), & \quad \text{if } s \in [t, T], \\
SC^{t, \varepsilon, \nu_2}(s) := \tilde{SC}(s) + \nu_2 I_{[t, t+\varepsilon]}(s), & \quad \text{if } s \in [t, T].
\end{align*}
\]

The equilibrium value function is defined by
\[
V(t, \tilde{X}(t), AL(t)) := J(t, \tilde{X}(t), AL(t); \tilde{u}(-), \tilde{SC}(-)).
\]

Note that the above time-consistent equilibrium strategy is defined within the class of open-loop controls, while it is defined within the class of deterministic feedback controls in \([12], [11]\) and \([3]\). However, the intuitions behind these two types of definitions are the same, i.e., the time-consistent equilibrium strategies are the strategies such that, given that they will be implemented in the future, it is optimal to implement them right now.

3. Main results. In this section, we give a time-consistent equilibrium investment-contribution strategy and the corresponding efficient frontier. Since we are working in a non-Markovian framework, we shall use BSDEs to construct the equilibrium strategy.

To begin with, let us consider the following BSDEs:
\[
\begin{align*}
&dP_0(s) = -r(s)P_0(s)ds + Q_0^\top(s)dW(s), \quad s \in [0, T], \\
&P_0(T) = 1,
\end{align*}
\]
and
\[
\begin{align*}
&dP_i(s) = -f_i(s, P_i(s), Q_i(s), \bar{Q}_i(s))ds + Q_i^\top(s)dW(s) + \bar{Q}_i^\top(s)d\bar{W}(s), \quad s \in [0, T], \\
&P_i(T) = 1, \quad \text{if } i = 1, 4, \text{ and } P_i(T) = 0, \quad \text{if } i = 2, 3,
\end{align*}
\]
where (suppressing the variable \( s \))
\[
\begin{align*}
f_1(s, P_1, Q_1, \bar{Q}_1) &= rP_1 - \theta^\top \left( \sigma^\top \right)^{-1} Q_1 - \frac{1}{P_1} |Q_1|^2, \\
f_2(s, P_2, Q_2, \bar{Q}_2) &= (r - \delta)P_1 + P_2 \left( \frac{\xi_{AL}}{\psi_{AL}} + \kappa \right) + \xi^\top (Q_2 - Q_1) + \xi^\top (\bar{Q}_2 - \bar{Q}_1) \\
&\quad - (P_2 \xi - P_1 \xi + \bar{Q}_2)^\top \left( \sigma^{-1} \theta + \frac{Q_1}{P_1} \right), \\
f_3(s, P_3, Q_3, \bar{Q}_3) &= P_0 P_1 (P_3 - P_4) \\
&\quad - \left\{ Q_4 - (P_3 - P_4) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) \right\}^\top \left( \sigma^{-1} \theta + \frac{Q_1}{P_1} \right), \\
f_4(s, P_4, Q_4, \bar{Q}_4) &= f_3(s, P_3, Q_3, \bar{Q}_3) + (P_3 - P_4) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right)^\top \frac{Q_0}{P_0}.
\end{align*}
\]

Since \( r(\cdot) \) is bounded, we know that \((14)\) admits a unique solution \((P_0(\cdot), Q_0(\cdot))\) such that for all \( t \in [0, T) \), \( P_0(t) = \mathbb{E}_t \left[ e^{r(T-t)}d\bar{r} \right] \) is bounded almost surely and \( \int_0^T Q_0^\top(s)dW(s) \) is a bounded mean oscillation (BMO, for short) martingale (see,
Remark 3.2. By [25, Section 5], we know that there exist constants $0 < \delta_1 < \delta_2$ such that

Theorem 3.3. Let $\delta_1 < P_1(\cdot) < \delta_2$. Thus ($P_0(\cdot), Q_0(\cdot), \bar{Q}_1(\cdot), i = 1, 2, 3, 4$. The proof is the same as [31, Appendix B] and omitted here.

Proposition 3.1. The system (15) admits a unique solution $(P_1(\cdot), Q_1(\cdot), \bar{Q}_i(\cdot), i = 1, 2, 3, 4$), in $L^p(\Omega; C([0, T]; \mathbb{R})) \times L^p(\Omega; L^2(0, T; \mathbb{R}^m))$, for all $p > 1$.

By [25, Section 5], we know that there exist constants $0 < \delta_1 < \delta_2$ such that $\delta_1 < P_1(\cdot) < \delta_2$.

Theorem 3.3. Let

$$\bar{u}(\cdot) = (\sigma^T(\cdot))^{-1} \left[ \Theta_1(\cdot) \bar{X}(\cdot) + \Theta_2(\cdot) \bar{A}(\cdot) + \Theta_3(\cdot) \right],$$

$$\bar{X}(\cdot) = -\lambda P_0(\cdot) [P_0(\cdot) - P_1(\cdot)],$$

$$\Theta_1(\cdot) = -\frac{Q_2(\cdot)}{P_1(\cdot)} - \frac{1}{P_1(\cdot)} \left[ (P_1(\cdot) - P_1(\cdot)) \xi(\cdot) + Q_2(\cdot) \right],$$

$$\Theta_3(\cdot) = -\frac{\lambda}{P_1(\cdot)} \left[ (P_3(\cdot) - P_4(\cdot)) \left( \sigma^{-1}(\cdot) \theta(\cdot) + \frac{Q_0(\cdot)}{P_0(\cdot)} \right) - Q_3(\cdot) \right].$$

If it holds that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |Q_0(t)|^2 \right] < \infty,$$

then $\left( \bar{u}(\cdot), \bar{SC}(\cdot) \right)$ is an equilibrium strategy for Problem 2.2.

The proof of the above theorem is given in Appendix A. The technical condition (18) is only a sufficient condition ensuring that $\left( \bar{u}(\cdot), \bar{SC}(\cdot) \right)$ is an equilibrium strategy (see the end of Appendix A for more details). If $r(\cdot)$ is deterministic, then $Q_0(\cdot) \equiv 0$ which implies (18) immediately. In the general case, (18) may be ensured by imposing proper assumptions on the Malliavin derivative of $r(\cdot)$ (see, e.g., [13, Section 5.2]).

From (16), it is easy to see that the equilibrium contribution strategy is only determined by $\lambda$, the coefficients in the model and the normal cost, and is independent of the value of the fund and the liability. This is a quite interesting and useful result, since the sponsor of the pension scheme always attempts to find a stable contribution plan that is not significantly disturbed by the investment performance (see e.g. [23], [14] and [15]). Here the equilibrium contribution strategy we find can serve as a candidate of stable contribution strategies. In contrast, neither the pre-commitment optimal contribution strategy obtained in [21] nor the optimal strategies under quadratic, power or exponential cost functionals (see e.g. [6], [19, 20] and [29]) have this appealing feature.

Furthermore, letting

$$(P_5(\cdot), Q_5(\cdot), \bar{Q}_5(\cdot)) := (P_5(\cdot) - P_4(\cdot), Q_3(\cdot) - Q_4(\cdot), Q_3(\cdot) - \bar{Q}_4(\cdot)),$$
we have
\[
\left\{ \begin{array}{l}
\frac{dP_5(s)}{P_5(s)} = P_5 \left[ \sigma^{-1} \left( \theta + \sigma \frac{Q_6(s)}{P_5(s)} \right) \right] + Q_5(dW(s)) + Q_6(s) dW(s), \\ P_5(T) = -1.
\end{array} \right. \tag{19}
\]

It is easy to see that \( P_5(.) < 0 \) a.s. Thus, the equilibrium supplementary cost \((16)\) is always positive, i.e., the amortization rate is always higher than the normal cost.

The equilibrium investment strategy \((17)\) is in a linear feedback form of the unfunded actuarial liability and the liability. Since
\[
\Theta_3(.): = \frac{SC(.)}{P_{1,0}P_0(.)} \left( \sigma^{-1}(.\theta(.) + \frac{Q_3(.)}{P_0(.)}\right) + \lambda \frac{Q_4(.)}{P_1(.)},
\]
once can easily identify a linear relationship between the equilibrium supplementary cost and the investment strategy. Similar to \([21]\), this can be viewed as a “rule of thumb” for the manager. If all the coefficients are deterministic, then
\[
\hat{u}(.) = - \left( \sigma^T(.) \right) ^{-1} \left[ \frac{1}{P_{1,0}(.)} \left( P_2(.) - P_1(.) \right) \xi(.) AL(.) + \lambda \frac{P_3(.)}{P_1(.)} \sigma^{-1}(.\theta(.)\right].
\]

Obviously, in this case, if \( \lambda \) increases, then the control of variance becomes relatively less important and the manager invests more fund into the risky assets to earn a higher expected return. One can also see that the equilibrium strategy \( \hat{u}(.)\) depends on the growth rate of the benefits through \( P_2(.)\), which is different from the investment strategy in \([21]\).

Next, we are going to find the efficient frontier. To this end, we introduce the following BSDE:
\[
\left\{ \begin{array}{l}
\frac{dP_6(s)}{P_6(s)} = - \left[ P_5(s) \left( \sigma^{-1}(s) \theta(s) + \frac{Q_6(s)}{P_5(s)} \right) \right] + Q_5(s) dW(s) \\
\quad + Q_6(s) dW(s), \\
\quad s \in [0, T], \\
P_6(T) = 0.
\end{array} \right. \tag{20}
\]

By Theorem 10 in \([5]\) and Proposition 3.1, we know that \((20)\) admits a unique solution \((P_6(.), Q_6(.), Q_6(.))\) in the space \(L^p(\Omega;C([0, T]; \mathbb{R})) \times L^p(\Omega;L^2(0, T; \mathbb{R}^m)) \times L^p(\Omega;L^2(0, T; \mathbb{R}^n))\) for any \( p > 1 \). Furthermore,
\[
P_6(t) = E_t \left[ \int_t^T P_5(s) \left( \sigma^{-1}(s) \theta(s) + \frac{Q_6(s)}{P_5(s)} \right) + Q_5(s) \right] ds \geq 0.
\]

Theorem 3.4. If \( \mu(.) \) and \( \sigma(.) \) are \( \mathbb{F}^W \)-adapted, then the corresponding efficient frontier is given by
\[
\text{Var}_t \left[ \tilde{X}(T) \right] = \frac{P_6(t)}{P_7(t)} \left\{ P_1(t) \tilde{X}(t) + P_2(t) AL(t) - E_t \left[ \tilde{X}(T) \right] \right\} ^2 \\
\quad + E_t \left[ \int_t^T \left( \frac{1}{P_6(s)} \left( P_3(s) - P_1(s) \right) \xi(s) - P_3(s) \xi(s) \right) ^2 AL^2(s) ds \right], \tag{21}
\]
for \((t, \omega)\) such that \( P_3(t, \omega) \neq 0 \).

Proof. Since \( \mu(.) \) and \( \sigma(.) \) are only \( \mathbb{F}^W \)-adapted, we have \( \bar{Q}_1(.) = \bar{Q}_3(.) \equiv \bar{0} \). By the derivation of \((43)\), it is easy to check that
\[
d \left( P_1(s) \tilde{X}(s) + P_2(s) AL(s) - \lambda P_3(s) \right)
\]
\[
= -\lambda \left[ Q_0 + P_0 \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) \right]^T dW(s) + AL \left( Q_2 + P_2 \xi - P_1 \xi \right)^T dW(s).
\]

Then
\[
\hat{X}(T) = P_1(0)\hat{X}(T) + P_2(0)AL(T) - \lambda P_3(T) \\
= P_1(t)\hat{X}(t) + P_2(t)AL(t) - \lambda P_3(t) \\
- \lambda \int_t^T \left\{ P_5 \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) + Q_5 \right\}^T dW(s) \\
+ \int_t^T AL \left( Q_2 + P_2 \xi - P_1 \xi \right)^T dW(s),
\]

and
\[
\hat{X}^2(T) = \left( P_1(t)\hat{X}(t) + P_2(t)AL(t) - \lambda P_3(t) \right)^2 \\
+ \int_t^T \left\{ \lambda^2 \left| P_5 \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) + Q_5 \right|^2 + \left| Q_2 + P_2 \xi - P_1 \xi \right|^2 AL^2 \right\} ds \\
- \int_t^T 2\lambda \left( P_1 \hat{X} + P_2 AL - \lambda P_3 \right) \left\{ P_5 \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) + Q_5 \right\}^T dW(s) \\
+ \int_t^T 2 \left( P_1 \hat{X} + P_2 AL - \lambda P_3 \right) \left( Q_2 + P_2 \xi - P_1 \xi \right)^T AL dW(s).
\]

Consequently,
\[
\mathbb{E}_t \left[ \hat{X}(T) \right] = P_1(t)\hat{X}(t) + P_2(t)AL(t) - \lambda P_3(t), \quad (22)
\]
\[
\mathbb{E}_t \left[ \hat{X}^2(T) \right] = \left( \mathbb{E}_t \left[ \hat{X}(T) \right] \right)^2 + \lambda^2 P_0(t) \\
+ \mathbb{E}_t \left[ \int_t^T \left| Q_2(s) + P_2 \xi(s) - P_1(s) \xi(s) \right|^2 AL^2(s) ds \right]. \quad (23)
\]

By (22), we have
\[
\lambda = \frac{1}{P_3(t)} \left\{ P_1(t)\hat{X}(t) + P_2(t)AL(t) - \mathbb{E}_t \left[ \hat{X}(T) \right] \right\} \quad (24)
\]
on the set \( \{ (t, \omega) : P_3(t, \omega) \neq 0 \} \). Inserting (24) into (23) leads to the (21). \( \square \)

**Remark 3.5.** In the general case with \( \mu(\cdot) \) and \( \sigma(\cdot) \) being \( \mathbb{F} \)-adapted, one can also obtain an expression of the efficient frontier by using some BSDEs, for which we are not clear how to show the solvability at the moment.

Since there is unhedgeable risk \( \hat{W}(\cdot) \) in the market, the efficient frontier (21) is not a perfect square. If the market is complete, i.e. \( \xi(\cdot) \equiv 0 \) and all the coefficients are \( \mathbb{F}^W \)-adapted, then we have \( \hat{Q}_2(\cdot) \equiv 0 \). Hence, in the complete market case, the efficient frontier can be characterized by a mean-standard deviation plane:
\[
\mathbb{E}_t \left[ \hat{X}(T) \right] = \sqrt{\frac{P_3^2(t)}{P_0(t)}} \sigma_t \left[ \hat{X}(T) \right] + P_1(t)\hat{X}(t) + P_2(t)AL(t), \quad (25)
\]
for \( (t, \omega) \) such that \( P_0(t, \omega) \neq 0 \), where \( \sigma_t [\cdot] = \sqrt{\text{Var}_t [\cdot]} \) is the standard deviation. Note that in our setting the current actuarial liability appears in (25), which is different from the result in [21].
4. An example: Stochastic interest rate. In this section, we consider the case where the interest rate is stochastic and the other coefficients in the model are deterministic. We refer the reader to [2] and [22] for the studies of pension plans with stochastic interest rates. For simplicity, we assume that \( m = 2 \) and \( n = 1 \), i.e., the Brownian motions \( W(\cdot) = (W_1(\cdot), W_2(\cdot))^\top \) and \( \bar{W}(\cdot) \) are two-dimensional and one-dimensional, respectively. One can easily extend the results to the general case with \( m > 2 \) and \( n > 1 \).

We assume that the instantaneous interest rate \( r(\cdot) \) satisfies the Vasicek model:

\[
\begin{aligned}
\begin{cases}
    dr(t) = \alpha (\beta - r(t)) \, dt + \rho dW_1(t), & t \in [0, T], \\
    r(0) = r_0,
\end{cases}
\end{aligned}
\]  

(26)

where \( \alpha, \beta > 0 \) and \( \rho \) are all constants. Obviously, the interest rate \( r(\cdot) \) given by (26) does not satisfy the boundedness assumption imposed in the previous sections. Note that this assumption is only used to show the existence and uniqueness of the solutions to BSDEs (14) and (15). In this section, we shall derive the solutions explicitly, and thus we can use Theorems 3.3 and 3.4, even though \( r(\cdot) \) is not bounded almost surely.

Clearly, the Vasicek model can lead to negative interest rates, on which we have to compromise when taking advantage of the tractability of this model. Though being criticized periodically, the Vasicek model may still be practically meaningful nowadays. Recent policies of central banks around the world seem to resurrect the Vasicek model. Since the 2007-2008 global financial crisis, ultra low interest rates policies have been deployed to revive global economy; some central banks (e.g., European Central Bank and Bank of Japan) have gone even further and pushed official interest rates below zero. This provides us empirical evidence to support the use of the Vasicek model.

Given \( r(\cdot) \), we assume that a zero-coupon bond with a fixed maturity \( T_1 > T \) is available in the market. Let \( \phi \) be the market price of interest rate risk, which is assumed to be a real-valued constant. Then the price of a zero coupon bond is given by (see, e.g. [2] and [22]):

\[
B(t, T_1) = e^{c(t, T_1) - b(t, T_1) r(t)},
\]

where

\[
c(t, T_1) = -R(\infty)(T_1 - t) + b(t, T_1) \left( R(\infty) - \frac{\rho^2}{2\alpha^2} \right) + \frac{\rho^2}{4\alpha^2} \left( 1 - e^{-2\alpha(T_1 - t)} \right),
\]

\[
b(t, T_1) = \frac{1}{\alpha} \left( 1 - e^{-\alpha(T_1 - t)} \right), \quad R(\infty) = \beta + \rho \phi \alpha - \frac{\rho^2}{2\alpha^2}.
\]

Applying Itô’s formula, the price process of the bond satisfies the following stochastic differential equation (SDE):

\[
\begin{aligned}
\begin{cases}
    dB(t, T_1) = B(t, T_1) \left[ (r(t) + \rho \phi b(t, T_1)) \, dt - \rho b(t, T_1) \, dW_1(t) \right], & t \in [0, T_1], \\
    B(T_1, T_1) = 1.
\end{cases}
\end{aligned}
\]

(27)

To simplify the notation, let \( S_1(\cdot) = B(\cdot, T_1) \). Then it holds that

\[
\begin{aligned}
\begin{cases}
    dS_1(s) = S_1(s) \left[ \mu_1(s) \, ds + \sigma_{11}(s) \, dW_1(s) \right], & s \in [0, T], \\
    S_1(0) = B(0, T_1) > 0,
\end{cases}
\end{aligned}
\]

(28)

where \( \mu_1(\cdot) := r(\cdot) + \rho \phi b(\cdot, T_1) \) and \( \sigma_{11}(\cdot) := -\rho b(\cdot, T_1) \).
Besides the zero-coupon bond, a stock is traded in the financial market, whose price $S_2(t)$ follows
\[
\begin{cases}
dS_2(t) = S_2(t) [\mu_2(t) dt + \sigma_2(t) dW_1(t) + \sigma_2(t) dW_2(t)], & s \in [0, T], \\
S_2(0) = s_2 > 0.
\end{cases}
\]  

(29)

The dynamics of the liability in this example is given by
\[
\begin{cases}
dL(s) = L(s) [\kappa(s) dt + \xi_1(s) dW_1(s) + \xi_2(s) dW_2(s) + \xi^T(s) dW(s)], & s \in [0, T], \\
L(0) = L_0 > 0.
\end{cases}
\]

In this setting, we have $\sigma(t) = \begin{pmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix}$, $\xi(t) = (\xi_1(t), \xi_2(t))^T$ and $\theta(t) = (\theta_1(t), \theta_2(t))^T$, where $\theta_1(t) = \rho \delta(t, T)$ and $\theta_2(t) = \mu_2(t) - r(t)$ is the expected excess return from investing in the stock. Obviously, the functions $\theta_1(t)$ and $\sigma_{11}(t)$ are bounded. We further assume that $\sigma_{21}(t), \sigma_{22}(t), \xi_1(t), \xi_2(t)$ and $\xi(t)$ are bounded deterministic functions, and $\sigma(t)$ is invertible for any $t \in [0, T]$.

To obtain an explicit solution, we assume that the technical interest rate is chosen as the interest rate plus an additional term.

**Assumption 4.1.** The technical rate of actualization satisfies $\delta(t) = r(t) + \gamma_2(t)$, where $\gamma_2(t)$ is a bounded deterministic function.

[20, 21, 22] make similar assumptions in which $\gamma_2(t)$ is specified to be related with the market risk of the liabilities, and they justify the assumptions from the viewpoint of risk-neutral valuation of the liabilities. By Assumptions 2.1 and 4.1, it is easy to see that $\kappa(t) = r(t) + \gamma(t)$, where $\gamma(t) := \gamma_1(t) + \gamma_2(t)$.

Given Assumptions 2.1 and 4.1, since $r(t)$ is adapted to the filtration generated by $W_1(t)$ and the other parameters in BSDE (15) are deterministic, we know that $Q_i(t) = (Q_{i1}(t), 0)^T$, $i = 0, \cdots, 4$ and $\bar{Q}_i(t) \equiv 0$, $i = 1, \cdots, 4$. With a little abuse of notation, we write $f_i(t, P_i(t), Q_i(t)) = f_i(t, P_i(t), Q_i(t), 0)$, $i = 1, \cdots, 4$. By the generalized Feynman–Kac formulation (see, e.g. [13, Proposition 4.3]), we consider the following partial differential equations (PDEs, for short):

\[
\begin{cases}
F_{i, t}(t, r) + F_{i, r}(t, r) \alpha (\beta - r) + \frac{1}{2} F_{i, rr}(t, r) \rho^2 + f_i(t, F_i(t, r), (F_i, r(t, r) \rho, 0)^T) = 0, & t \in [0, T], \\
F_i(T, r) = P_i(T), & i = 0, \cdots, 4.
\end{cases}
\]

(30)

Here $F_{i, t}$ is the first-order derivative of $F_i(t, r)$ with respect to $t$, $F_{i, r}$ and $F_{i, rr}$ are the first-order and second-order derivatives of $F_i(t, r)$ with respect to $r$, respectively.

If (30) admits a unique classical solution and
\[
(F_i(\cdot, r(\cdot)), F_{i, r}(\cdot, r(\cdot)) \rho) \in L^p_b(\Omega; C([0, T]; \mathbb{R})) \times L^p_b(\Omega; L^2(0, T; \mathbb{R}))
\]
for all $p > 1, i = 0, \cdots, 4$, then by Itô’s formula we know that
\[
(P_i(\cdot), Q_i(\cdot)) := \left( F_i(\cdot, r(\cdot)), (F_i, r(t, r) \rho, 0)^T \right)
\]
(31)
satisfies BSDEs (14) and (15).

The proof of the following proposition is given in Appendix B.

**Proposition 4.2.** Let
\[
g(t) = \frac{1}{\alpha} \left( 1 - e^{-\alpha (T-t)} \right), \quad t \in [0, T].
\]
Proposition 4.3. Suppose the interest rate \( r(\cdot) \) is given by (26). The equilibrium supplementary cost and investment strategy given by (16) and (17) become

\[
\hat{S}C(\cdot) = -\lambda G_0(\cdot)G_5(\cdot)e^{\alpha r(\cdot)},
\]

\[
\hat{u}(\cdot) = (\sigma^T(\cdot))^{-1} \left[ \Theta_1(\cdot)\tilde{X}(\cdot) + \Theta_2(\cdot)AL(\cdot) + \Theta_3(\cdot) \right],
\]

where

\[
\Theta_1(\cdot) = (-\sigma (\cdot), 0)^T, \quad \Theta_2(\cdot) = -\left( \xi_1(\cdot) + \rho g(\cdot) \left( 1 + \frac{G_2(\cdot)}{G_1(\cdot)} \right) \right)^T \xi_2(\cdot),
\]

Then, we have

\[
F_0(t, r) = G_0(t)e^{\alpha r}, \quad F_1(t, r) = G_1(t)e^{\alpha r},
\]

\[
F_2(t, r) = (G_1(t) + G_2(t))e^{\alpha r}, \quad F_3(t, r) = F_3(t, r) - G_5(t),
\]

where

\[
G_0(t) = \exp \left\{ \int_t^T g(s) \left( \alpha + \frac{1}{2}g^2(s) \rho^2 \right) ds \right\},
\]

\[
G_1(t) = \exp \left\{ \int_t^T g(s) \left[ \alpha - \frac{1}{2}g^2(s) \rho^2 + \phi \rho \right] ds \right\},
\]

\[
G_2(t) = -G_1(t)e^{\int_t^T \Xi(s)ds} \left[ 1 - \int_t^T e^{-\int_t^s \Xi(u)du} \left( \frac{\zeta_{AL}(s)}{\psi_{AL}(s)} + \gamma_1(s) \right) ds \right],
\]

\[
G_5(t) = -\exp \left\{ -\int_t^T [g(s)\rho - \phi] g(s) \rho ds \right\},
\]

\[
\Xi(t) = \frac{\zeta_{AL}(t)}{\psi_{AL}(t)} + \gamma(t) + \phi \xi_1(t) - \frac{\phi \sigma_2(t) + \theta_2(t)}{\sigma_2(t)} \xi_2(t)
\]

and

\[
F_3(t, r) = \int_t^T G_5(s) \left[ (g(s)\rho - \phi)^2 + \left( \frac{\beta_2(s) + \phi \sigma_2(s)}{\alpha s_2(s)} \right)^2 \right] ds
\]

\[
+ \int_t^T G_0(s)G_1(s)G_5(s) \exp \left\{ 2r \left[ g(t) - \frac{1}{\alpha} \left( 1 - e^{-\alpha(s-t)} \right) \right] \right\} \times \exp \left\{ 2\alpha \beta \int_t^s \left[ g(z) - \frac{1}{\alpha} \left( 1 - e^{-\alpha(s-z)} \right) \right] dz \right\}
\]

\[
\times \exp \left\{ 2\rho^2 \int_t^s \left[ g(z) - \frac{1}{\alpha} \left( 1 - e^{-\alpha(s-z)} \right) \right]^2 dz \right\}
\]

\[
\times \exp \left\{ -2\rho \int_t^s (g(z)\rho - \phi) \left[ g(z) - \frac{1}{\alpha} \left( 1 - e^{-\alpha(s-z)} \right) \right] dz \right\} ds.
\]

By Lemma 4.1 in [31], it holds that \( \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Q_0(t)|^2 \right] < \infty \) for any constant \( p \geq 0 \). It is easy to see that \( (P_i(\cdot), Q_i(\cdot)), i = 0, \cdots, 4 \) given by (31) are in the space \( L_p^p(\Omega; C([0, T]; \mathbb{R})) \times L_p^p(\Omega; L^2(0, T; \mathbb{R}^2)) \) for any \( p > 1 \). Furthermore, we have

\[\mathbb{E} \left[ \sup_{t \in [0, T]} |Q_0(t)|^2 \right] = \mathbb{E} \left[ \sup_{t \in [0, T]} G_0^2(t)e^{2\alpha r(\cdot)} \right] < \infty,\]

i.e., the condition (18). Thus, we have the following result.
\[
\Theta_3(\cdot) = -\frac{\lambda}{G_1(\cdot)e^{\theta(r(\cdot))}} \left( G_5(\cdot)(g(\cdot)\rho - \phi) - F_{3,r}(\cdot, r(\cdot))(G_5(\cdot)\frac{\phi \sigma_{21}(\cdot) + \theta_2(\cdot)}{\sigma_{22}(\cdot)}) \right)^T.
\]

Clearly, the randomness of the equilibrium supplementary cost only comes from the stochastic interest rate, and the investment strategy is a feedback of the fund, actuarial liability and the interest rate. By (24), (33), (4) and (5), the expected discounted cumulative equilibrium supplementary cost from 0 to \( T \) is given by

\[
\overline{SC} := \mathbb{E} \left[ \int_0^T e^{-f^*_0 r(u)du} \overline{SC}(s)ds \right]
\]

\[
= -\mathbb{E} \left[ \int_0^T x e^{-f^*_0 r(u)du} G_0(s)G_5(s)ds \right]
\]

\[
= \frac{1}{F_3(0, r_0)} \left\{ G_1(0)e^{\theta(r(0))}X_0 + (G_1(0) + G_2(0))e^{\theta(r(0))}AL_0 - \mathbb{E} \left[ \tilde{X}(T) \right] \right\}
\]

\[
\times \int_0^T \exp \left\{ r_0 g(0) - \frac{2}{\alpha} (1 - e^{-\alpha z}) + \int_0^z \beta [g(z) - 2(1 - e^{-\alpha z})]dz \right\}
\]

\[
\times \exp \left\{ \int_0^z \tilde{\rho}^2 \left[ g(z) - \frac{2}{\alpha} (1 - e^{-\alpha(s-z)}) \right]^2 dz \right\} G_0(s)G_5(s)ds.
\]

It follows from (5) that the total expected contribution is given by

\[
\overline{C} := \mathbb{E} \left[ \int_0^T e^{-f^*_0 r(u)du} NC(s)ds \right] + \overline{SC}
\]

\[
= NC_0 \int_0^T e^{f^*_0 (\frac{\gamma(NC(s))}{\gamma(NC(s))} + \gamma(v))} dv + \overline{SC}.
\]

In contrast to [21], the total expected contribution \( \overline{C} \) depends on the diffusion parameters of the benefits through \( G_2(0) \) in this paper.

Now, we are going to derive the efficient frontier. Let us study BSDE (20) which becomes

\[
\begin{cases}
    dP_0(s) = -G_2^2(s) \left| \sigma^{-1}(s)\theta(s) + \frac{Q_0(s)}{Q_0(t)} \right|^2 ds + Q_0^T(s)dW(s), & s \in [0, T], \\
    P_0(T) = 0.
\end{cases}
\]

It is easy to check that

\[
P_0(t) = \int_t^T G_2^2(s) \left[ (g(s)\rho - \phi)^2 + \left( \frac{\phi \sigma_{21}(s) + \theta_2(s)}{\sigma_{22}(s)} \right)^2 \right] ds, \quad (35)
\]

and \( Q_0(\cdot) \equiv (0, 0)^T \).

**Proposition 4.4.** Let \( r(\cdot) \) be given by (26). The efficient frontier (21) becomes

\[
\text{Var} \left[ \tilde{X}(T) \right]
\]

\[
= \frac{P_0(t)}{F_3^2(t, r(t))} \left\{ G_1(t)e^{\theta(r(t))}\tilde{X}(t) + (G_1(t) + G_2(t))e^{\theta(r(t))}AL(t) - \mathbb{E} \left[ \tilde{X}(T) \right] \right\}^2
\]

\[
+ e^{2\theta(r(t))}AL^2(t) \int_t^T \tilde{\xi}^2(s)G_2^2(s) \exp \left\{ \int_t^s \left[ \alpha \beta + 2\rho \xi_1(v) + \rho^2 g(v) \right] 2g(v)dv \right\}
\]
\begin{align*}
\times \exp \left\{ \int_t^s \left[ 2 \left( \frac{\zeta_{AL}(v)}{\psi_{AL}(v)} + \gamma(v) \right) + \xi^2(v) + \xi^2(v) \right] dv \right\} \, ds. \quad (36)
\end{align*}

**Proof.** Since \( \bar{Q}_2(\cdot) \equiv 0 \), the efficient frontier (21) can be rewritten as

\begin{align*}
\Var_t \left[ X(T) \right] &= \frac{P_0(t)}{P^2_2(t)} \left\{ P_1(t) \bar{X}(t) + P_2(t) AL(t) - E_t \left[ \bar{X}(T) \right] \right\}^2 \\
&\quad + E_t \left[ \int_t^T \xi^2(s) (P_2(s) - P_1(s))^2 AL^2(s) ds \right]. \quad (37)
\end{align*}

It suffices to calculate the second term on the right-hand-side of (37). Noting that

\begin{align*}
AL(s) &= AL(t) \exp \left\{ \int_t^s \left[ \frac{\zeta_{AL}(v)}{\psi_{AL}(v)} + r(v) + \gamma(v) - \frac{1}{2} \left( |\xi|^2(v) + \xi^2(v) \right) \right] dv \right\} \\
&\quad + \int_t^s \xi^T(v) dW(v) + \int_t^s \xi(v) d\tilde{W}(v)
\end{align*}

by (4) and (5), we obtain

\begin{align*}
\xi^2(s) (P_2(s) - P_1(s))^2 AL^2(s) &= e^{2g(t-r(t))} AL^2(t) \xi^2(s) G^2_2(s) \\
&\times \exp \left\{ 2 \int_t^s \left[ \alpha \beta g(v) + \frac{\zeta_{AL}(v)}{\psi_{AL}(v)} + \gamma(v) - \frac{1}{2} \left( |\xi|^2(v) + \xi^2(v) \right) \right] dv \right\} \\
&\times \exp \left\{ 2 \int_t^s (\rho g(v) + \xi_1(v)) dW_1(v) + 2 \int_t^s \xi_2(v) dW_2(v) + 2 \int_t^s \xi(v) d\tilde{W}(v) \right\}.
\end{align*}

Therefore,

\begin{align*}
E_t \left[ \int_t^T \xi^2(s) (P_2(s) - P_1(s))^2 AL^2(s) ds \right] \\
&= e^{2g(t-r(t))} AL^2(t) \int_t^T \xi^2(s) G^2_2(s) \exp \left\{ 2 \int_t^s [\alpha \beta + 2 \rho \xi_1(v) + \rho^2 g(v)] g(v) dv \right\} \\
&\times \exp \left\{ \int_t^s \left[ 2 \left( \frac{\zeta_{AL}(v)}{\psi_{AL}(v)} + \gamma(v) \right) + |\xi|^2(v) + \xi^2(v) \right] dv \right\} \, ds.
\end{align*}

The proof is completed. \( \square \)

It is well known that a pension fund is intended to cover the retirement benefits rather than produce any excessive assets. This is usually referred as the sufficiency principle of the pension funding (see [27]). Letting \( E_t \left[ \bar{X}(T) \right] = 0 \) in (36), one can easily get the variance of the terminal unfunded actuarial liability for the sponsor who expects that the terminal value of the fund exactly meets the terminal benefit.

**Example 4.5.** Assume that the participants join the pension plan at the age of \( a = 30 \) and retire at the age of \( d = 60 \) and, the benefits are accumulated uniformly, i.e., \( M(x) = (x-a)/(d-a) \) (see [22]). The planning horizon is \( T = 5 \). The values of parameters of the benefits and the financial market are \( \xi_1 = 0.1, \xi_2 = 0.2, \xi = 0.1, \theta_2 = 0.01, \sigma_2 = 0.3, \sigma_{22} = 0.2, \alpha = 0.2, \beta = 0.05, T_1 = 10, \phi = 0.15, \gamma_1 = 0.01 \) and \( \gamma_2 = 0.02 \). The initial values are \( AL_0 = 1, F_0 = 0.8 \) and \( \gamma_0 = 0.05 \).

Let \( E_0 \left[ \bar{X}(T) \right] = 0 \). Figure 1 (A) shows that the variance \( \Var_0 \left[ \bar{X}(T) \right] \) (as a function of \( \rho \)) attains its minimum approximately at \( \rho = -0.055 \). In this case, the
interest rate and the stock price are negatively correlated and thus the risk of the portfolio is reduced. However, the portfolio with minimum risk does not lead to a minimum total contribution simultaneously. Figure 1 (B) shows that the minimum of \( \rho \) is attained at about \( \rho = 0.1 \), in which case the interest rate and the stock price are positively correlated. This is because the riskier portfolio with \( \rho = 0.1 \) has a higher expected return than the one with \( \rho = -0.055 \). This result is in line with the one in [21], that is, the contribution under risky investment is less than that under safety investment.

5. Concluding remarks. In this paper we have studied the management of a DB pension plan in a continuous-time model with random coefficients. The objective is to find a time-consistent equilibrium investment-contributions strategy that minimizes the solvency risk, the contribution risk and the expected terminal value of the unfunded actuarial liability. We have obtained an equilibrium strategy by using the solution of a system of BSDEs. It is shown that the equilibrium contribution rate is stable in the sense that it is independent of the fund and actuarial liability. We have also discussed an example with the interest rate given by the Vasicek model and obtained a time-consistent equilibrium strategy in closed form.

In further research, we may make some interesting extensions by considering regulatory constraints on the strategies, such as short-selling and borrowing restrictions.

Appendix A. Proof of Theorem 3.3. We first give a sufficient condition for the equilibrium strategy. Since the interest rate is deterministic in [17] and there is no running cost in the cost functional of [31], we cannot directly apply their results.

**Theorem A.1.** A strategy \( (\bar{u}(\cdot), \bar{SC}(\cdot)) \) \( \in L^2_\mathbb{F} (0, T; \mathbb{R}^m) \times L^2_\mathbb{F} (0, T; \mathbb{R}) \) is an equilibrium strategy if for any \( t \in [0, T] : 

1. there exist \((Y(\cdot; t), Z(\cdot; t), U(\cdot; t)) \in L^2_\mathbb{F} (\Omega; C([t, T]; \mathbb{R})) \times L^2_\mathbb{F} (\Omega; L^2(t, T; \mathbb{R}^m)) \times L^2_\mathbb{F} (\Omega; L^2(t, T; \mathbb{R}^m)) \) and \( \bar{X}(\cdot) \in L^2_\mathbb{F} (\Omega; C([0, T]; \mathbb{R})) \) that solve the following system of forward-backward stochastic differential equations

\[
\begin{align*}
    \frac{\mathrm{d} \bar{X}(s)}{\mathrm{d}s} &= \left[ r(s) \bar{X}(s) + (r(s) - \delta(s)) \ AL(s) + \bar{SC}(s) + \theta(s) \bar{u}(s) \right] \ \mathrm{d}s \\
    &\quad + \left( u^\top(s) \sigma(s) - AL(s) \xi^\top(s) \right) \ \mathrm{d}W(s) - AL(s) \xi^\top(s) \ \mathrm{d}W(s), \quad s \in [t, T], \\
    \frac{\mathrm{d} Y(s)}{\mathrm{d}s} &= -r(s) Y(s; t) \ \mathrm{d}s + Z^\top(s; t) \ \mathrm{d}W(s) + U^\top(s; t) \ \mathrm{d}W(s), \quad s \in [t, T], \\
    \bar{X}(0) &= x_0, \quad Y(T; t) = \bar{X}(T) - E_x \left[ \bar{X}(T) \right] - \lambda;
\end{align*}
\]

(38)
2. the process \( \{Y(t; t)\}_{t \in [0, T]} \) has continuous paths and it holds that
\[
\overline{S}C(t) = -Y(t; t), \quad \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_t \left[ \int_t^{t+\varepsilon} \Lambda(s; t)ds \right] = 0, \quad a.s.,
\]
where \( \Lambda(s; t) = Y(s; t)\theta(s) + \sigma(s)Z(s; t), s \in [t, T] \).

Proof. Let \( (\hat{u}(\cdot), \overline{S}C(\cdot)) \) be a strategy satisfying the conditions (i) and (ii). Consider the strategy \( (u^{t,\varepsilon,\nu_1}(\cdot), SC^{t,\varepsilon,\nu_2}(\cdot)) \) given by (12) and define the process \( X_1^{t,\varepsilon,\nu}(\cdot) := X^{t,\varepsilon,\nu}(\cdot) - \hat{X}(\cdot) \), where \( \hat{X}(\cdot) \) and \( X^{t,\varepsilon,\nu}(\cdot) \) are the state processes associated with \( (\hat{u}(\cdot), \overline{S}C(\cdot)) \) and \( (u^{t,\varepsilon,\nu_1}(\cdot), SC^{t,\varepsilon,\nu_2}(\cdot)) \), respectively. Obviously, we have
\[
\begin{align*}
\{dX_1^{t,\varepsilon,\nu}(s) &= [r(s)X_1^{t,\varepsilon,\nu}(s) + (\nu_2 + \theta^T(s)\nu_1) I_{[t, t+\varepsilon]}(s)] ds \\
+ &\nu_1^T \sigma(s) I_{[t, t+\varepsilon]}(s) dW(s), \quad s \in [t, T],
\end{align*}
\]
\( X_1^{t,\varepsilon,\nu}(t) = 0 \).

By the standard SDE theory, one can see that \( X_1^{t,\varepsilon,\nu}(\cdot) \in L^2(\Omega; C(t, T; \mathbb{R})) \).

It is easy to check that
\[
J \left( t, \hat{X}(t), AL(t); u^{t,\varepsilon,\nu_1}(\cdot), SC^{t,\varepsilon,\nu_2}(\cdot) \right) - J \left( t, \hat{X}(t), AL(t); \hat{u}(\cdot), \overline{S}C(\cdot) \right)
= \frac{1}{2} \sigma^2 \varepsilon + \mathbb{E} \left[ \int_t^{t+\varepsilon} \overline{S}C(s)\nu_2 ds \right] + J_1(t) + J_2(t),
\]
where
\[
J_1(t) := \frac{1}{2} \mathbb{E}_t \left[ (X_1^{t,\varepsilon,\nu}(T) - \mathbb{E}_t [X_1^{t,\varepsilon,\nu}(T)]) X_1^{t,\varepsilon,\nu}(T) \right],
\]
and
\[
J_2(t) := \mathbb{E}_t \left[ \left( \hat{X}(T) - \mathbb{E}_t [\hat{X}(T)] - \lambda \right) X_1^{t,\varepsilon,\nu}(T) \right].
\]

Note that \( J_1(t) = \frac{1}{2} \text{Var}_t [X_1^{t,\varepsilon,\nu}(T)] \geq 0 \). Thus, it is sufficient to show that
\[
\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ \mathbb{E} \left[ \int_t^{t+\varepsilon} \overline{S}C(s)\nu_2 ds \right] + J_2(t) \right\} = 0.
\]
Applying Itô’s formula to \( Y(\cdot; t)X_1^{t,\varepsilon,\nu}(\cdot) \) gives
\[
d(Y(s; t)X_1^{t,\varepsilon,\nu}(s)) = \Lambda^T(s; t)\nu_1 I_{[t, t+\varepsilon]}(s)ds + Y(s; t)\nu_2 I_{[t, t+\varepsilon]}(s)ds
\]
\[
+ (Y(s; t)\nu_1^T \sigma(s) I_{[t, t+\varepsilon]}(s) + X_1^{t,\varepsilon,\nu}(s)Z^T(s; t)) dW(s)
\]
\[
+ X_1^{t,\varepsilon,\nu}(s)U^T(t; s)dW(s).
\]

Integrating (40) from \( t \) to \( T \), taking conditional expectation and recalling that \( X_1^{t,\varepsilon,\nu}(t) = 0 \), we have
\[
J_2(t) = \mathbb{E}_t \left[ \int_t^{t+\varepsilon} \Lambda^T(s; t)\nu_1 ds + \int_t^{t+\varepsilon} Y(s; t)\nu_2 ds \right].
\]

By Condition (ii), we have
\[
\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ \mathbb{E}_t \left[ \int_t^{t+\varepsilon} \overline{S}C(s)\nu_2 ds \right] + J_2(t) \right\}
= \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_t \left[ \int_t^{t+\varepsilon} \Lambda^T(s; t)\nu_1 ds + \overline{S}C(t) + Y(t; t) \right] \nu_2 = 0.
\]

The proof is completed. \( \square \)
Now we are going to verify that \((\mathbf{u}(\cdot), \mathbf{SC}(\cdot))\) given by (16) and (17) is indeed a strategy such that the conditions in Theorem A.1 are satisfied. First, let us consider the existence of the solution to the system (38). Inserting (16) and (17) into the forward equation in (38), we obtain the following linear SDE:

\[
\begin{align*}
d\bar{X}(s) &= \left\{ r(s) + \theta^T(s) (\sigma^T(s))^{-1} \Theta_1(s) \right\} \bar{X}(s) \\
& \quad + \left\{ r(s) - \delta(s) + \theta^T(s) (\sigma^T(s))^{-1} \Theta_2(s) \right\} AL(s) \\
& \quad + \left( \Theta_1(s) \bar{X}(s) + \Theta_2(s)AL(s) - AL(s)\xi(s) + \Theta_3(s) \right)^T dW(s) \\
& \quad - AL(s)\xi^T(s)dW(s). \numberthis \tag{41}
\end{align*}
\]

By Proposition 3.1 and following the same method of [31, Appendix C], one can show (41) admits a unique solution \(\bar{X}(\cdot) \in L_p^p(\Omega; C([0,T];\mathbb{R}))\), for all \(p > 1\). Consequently, it is easy to see from (16), (17) and Proposition 3.1 that \((\mathbf{u}(\cdot), \mathbf{SC}(\cdot)) \in L_p^p(0,T;\mathbb{R}^m) \times L_p^p(0,T;\mathbb{R})\) for all \(p > 1\).

To show the existence of the solution to the backward equation in (38), we make the following ansatz:

\[
Y(s;t) = P_0(s) \left\{ P_1(s)\bar{X}(s) - \mathbb{E}_t \left[ P_1(s)\bar{X}(s) \right] \right\} + P_2(s)AL(s) \\
- \mathbb{E}_t \left[ P_2(s)AL(s) + \lambda \mathbb{E}_t \left[ P_2(s) \right] - \lambda P_2(s) \right], \numberthis \tag{42}
\]

for \(0 \leq t \leq s \leq T\).

By Itô’s formula, we have (suppressing the variable \(s\))

\[
\begin{align*}
d\left( P_1 \bar{X} \right) &= \left\{ P_1 \left[ \left( r + \theta^T (\sigma^T)^{-1} \Theta_1 \right) \bar{X} + \left( r - \delta + \theta^T (\sigma^T)^{-1} \Theta_2 \right) AL \right. \right. \\
& \quad + \tilde{SC} + \theta^T (\sigma^T)^{-1} \Theta_3 \right\} - \bar{X} f_1 (s, P_1, Q_1, \bar{Q}_1) \\
& \quad + \left[ \Theta_1 \bar{X} + \Theta_2 AL - AL\xi + \Theta_3 \right]^T Q_1 - AL\xi^T Q_1 \right\} ds \\
& \quad + \left[ \bar{X} Q_1^T + P_1 \left[ \Theta_1 \bar{X} + \Theta_2 AL - AL\xi + \Theta_3 \right]^T \right] dW(s) \\
& \quad + \left( \bar{X} Q_1^T - P_1 AL\xi^T \right) dW(s) \\
& = \left\{ P_1 \left[ \left( r - \delta + \theta^T (\sigma^T)^{-1} \Theta_2 \right) AL + \tilde{SC} + \theta^T (\sigma^T)^{-1} \Theta_3 \right] \right. \\
& \quad + \left[ \Theta_2 AL - AL\xi + \Theta_3 \right]^T Q_1 - AL\xi^T Q_1 \right\} ds \\
& \quad + P_1 \left[ \Theta_2 AL - AL\xi + \Theta_3 \right]^T dW(s) + \left( \bar{X} Q_1^T - P_1 AL\xi^T \right) dW(s),
\end{align*}
\]

and

\[
\begin{align*}
d\left( P_2 AL \right) &= AL \left[ P_2 \left( \frac{\zeta_{AL}}{\psi_{AL}} + \kappa \right) - f_2 (s, P_2, Q_2) + \xi^T Q_2 + \xi^T \bar{Q}_2 \right] ds \\
&\quad + AL \left( P_2 \xi^T + Q_2^T \right) dW(s) + AL \left( \bar{Q}_2^T + P_2 \xi^T \right) dW(s).
\end{align*}
\]
Thus, taking expectations conditioned on $\mathcal{F}_t$ leads to
\[
d\mathbb{E}_t \left[ P_1 \tilde{X} \right] = \mathbb{E}_t \left[ P_1 \left( \left( r - \delta + \theta^\top (\sigma^\top)^{-1} \Theta_2 \right) AL + \overline{SC} + \theta^\top (\sigma^\top)^{-1} \Theta_3 \right) + \left[ \Theta_2 AL - AL \xi + \Theta_3 \right]^\top Q_1 - AL \xi^\top \tilde{Q}_1 \right] ds
\]
and
\[
d\mathbb{E}_t [P_2 AL] = \mathbb{E}_t AL \left[ P_2 \left( \frac{\zeta_{AL}}{\psi_{AL}} + \kappa \right) - f_2(s, P_2, Q_2, \tilde{Q}_2) + \xi^\top Q_2 + \xi^\top \tilde{Q}_2 \right] ds.
\]
Hence,
\[
d \left\{ P_1 \tilde{X} - \mathbb{E}_t \left[ P_1 \tilde{X} \right] + P_2 AL - \mathbb{E}_t [P_2 AL] + \lambda \mathbb{E}_t [P_3] - \lambda P_4 \right\}
\]
\[
= \left\{ P_1 \left( \left( r - \delta + \theta^\top (\sigma^\top)^{-1} \Theta_2 \right) AL + \overline{SC} + \theta^\top (\sigma^\top)^{-1} \Theta_3 \right) + \left[ \Theta_2 AL - AL \xi + \Theta_3 \right]^\top Q_1 - AL \xi^\top \tilde{Q}_1 \right\} ds
\]
\[
+ P_1 \left[ \Theta_2 AL - AL \xi + \Theta_3 \right]^\top dW(s) + \left( \tilde{X} \tilde{Q}_1 - P_1 AL \xi^\top \right) d\tilde{W}(s)
\]
\[
+ AL \left[ P_2 \left( \frac{\zeta_{AL}}{\psi_{AL}} + \kappa \right) - f_2(s, P_2, Q_2, \tilde{Q}_2) + \xi^\top Q_2 + \xi^\top \tilde{Q}_2 \right] ds
\]
\[
+ AL \left( P_2 \xi^\top + Q_2^\top \right) dW(s) + AL \left( Q_2^\top + P_2 \xi^\top \right) d\tilde{W}(s)
\]
\[
+ \lambda \left[ f_4(s, P_4, Q_4, \tilde{Q}_4) ds - Q_4^\top dW(s) - \tilde{Q}_4^\top d\tilde{W}(s) \right]
\]
\[
- \mathbb{E}_t \left\{ P_1 \left( \left( r - \delta + \theta^\top (\sigma^\top)^{-1} \Theta_2 \right) AL + \overline{SC} + \theta^\top (\sigma^\top)^{-1} \Theta_3 \right) + \left[ \Theta_2 AL - AL \xi + \Theta_3 \right]^\top Q_1 - AL \xi^\top \tilde{Q}_1 \right\} ds
\]
\[
- \mathbb{E}_t \left\{ AL \left[ P_2 \left( \frac{\zeta_{AL}}{\psi_{AL}} + \kappa \right) - f_2(s, P_2, Q_2, \tilde{Q}_2) + \xi^\top Q_2 + \xi^\top \tilde{Q}_2 \right] \right\} ds
\]
\[
- \lambda \mathbb{E}_t \left[ f_3(s, P_3, Q_3, \tilde{Q}_3) \right] ds
\]
\[
= \lambda (P_3 - P_4) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right)^\top Q_0 ds - \lambda \mathbb{E}_t \left[ \left( P_3 - P_4 \right) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) \right]^\top dW(s)
\]
\[
+ \left[ \left( \tilde{X} \tilde{Q}_1^\top - P_1 AL \xi^\top \right) + AL \left( Q_2^\top + P_2 \xi^\top \right) - \lambda Q_4^\top \right] d\tilde{W}(s).
\]
By Itô’s formula again, we have
\[
dY(s; t)
\]
\[
= P_0 d \left\{ P_1 \tilde{X} - \mathbb{E}_t \left[ P_1 \tilde{X} \right] + P_2 AL - \mathbb{E}_t [P_2 AL] + \lambda \mathbb{E}_t [P_3] - \lambda P_4 \right\}
\]
\[
\]
\[
+ \left\{ P_1 \tilde{X} - \mathbb{E}_t \left[ P_1 \tilde{X} \right] + P_2 AL - \mathbb{E}_t [P_2 AL] + \lambda \mathbb{E}_t [P_3] - \lambda P_4 \right\} dP_0(s)
\]
\[
+ d \left\{ P_1 \tilde{X} - \mathbb{E}_t \left[ P_1 \tilde{X} \right] + P_2 AL - \mathbb{E}_t [P_2 AL] + \lambda \mathbb{E}_t [P_3] - \lambda P_4, P_0 \right\}(s)
\]
\[
= \lambda (P_3 - P_4) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right)^\top Q_0 ds - \lambda P_0 \left[ \left( P_3 - P_4 \right) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) \right]^\top dW(s)
\]
\[
+ P_0 \left[ \left( \tilde{X} \tilde{Q}_1^\top - P_1 AL \xi^\top \right) + AL \left( Q_2^\top + P_2 \xi^\top \right) - \lambda Q_4^\top \right] d\tilde{W}(s)
\]
\[
- rP_0 \left\{ P_1 \tilde{X} - \mathbb{E}_t \left[ P_1 \tilde{X} \right] + P_2 AL - \mathbb{E}_t [P_2 AL] + \lambda \mathbb{E}_t [P_3] - \lambda P_4 \right\} ds
\]
\( - \lambda \left[ (P_3 - P_4) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right) \right]^T Q_0 ds \)
\( + \left\{ P_1 \tilde{X} - E_t \left[ P_1 \tilde{X} \right] + P_2 AL - E_t [P_2 AL] + \lambda E_t [P_3] - \lambda P_4 \right\} Q_0^T dW(s) \)
\( = - r Y(s; t) ds - \lambda P_0 (P_3 - P_4) \left( \sigma^{-1} \theta + \frac{Q_0}{P_0} \right)^T dW(s) \)
\( + \left\{ P_1 \tilde{X} - E_t \left[ P_1 \tilde{X} \right] + P_2 AL - E_t [P_2 AL] + \lambda E_t [P_3] - \lambda P_4 \right\} Q_0^T dW(s) \)
\( + P_0 \left[ \tilde{X} \tilde{Q}_1 + AL (\tilde{Q}_2 + P_2 \xi - P_1 \bar{\xi}) - \lambda \tilde{Q}_4 \right]^T d\tilde{W}(s). \)

By taking
\[ Z(s; t) = - \lambda P_0 (s) (P_3(s) - P_4(s)) \left[ \sigma^{-1} \theta(s) + \frac{Q_0(s)}{P_0(s)} \right] \]
\[ + \left\{ P_1(s) \tilde{X}(s) - E_t \left[ P_1(s) \tilde{X}(s) \right] + P_2(s) AL(s) - E_t [P_2(s) AL(s)] + \lambda E_t [P_3(s)] - \lambda P_4(s) \right\} Q_0(s), \]
\[ U(s; t) = P_0(s) \left[ \tilde{X}(s) \tilde{Q}_1(s) + AL(s) \left( \tilde{Q}_2(s) + P_2(s) \xi(s) - P_1(s) \bar{\xi}(s) \right) - \lambda \tilde{Q}_4(s) \right], \]
we know that \( (Y(\cdot; t), Z(\cdot; t), U(\cdot; t)) \) given by (42), (45) and (46) satisfies the backward equation in (38). Since \( \tilde{X}(\cdot), AL(\cdot) \in L^p_\Omega (\Omega; C([0, T]; \mathbb{R})) \) for all \( p > 1 \), it is easy to see from Proposition 3.1 that \( (Y(\cdot; t), Z(\cdot; t), U(\cdot; t)) \) is in \( L^2_\Omega (\Omega; C([t, T]; \mathbb{R}) \times L^2_\Omega (\Omega; L^2(t; \mathbb{R}^m) \times L^2_\Omega (\Omega; L^2(t; \mathbb{R}^n))) \).

Finally, we verify the condition (ii). It is easy to see from (42) that the process \( \{Y(t; t)\}_{t \in [0, T]} \) has continuous paths. The first equality of (39) directly follows from (16) and (42). To show the second relation in (39), let us define
\[ \Gamma(s; t) := P_1(s) \tilde{X}(s) - E_t \left[ P_1(s) \tilde{X}(s) \right] + P_2(s) AL(s) - E_t [P_2(s) AL(s)] + \lambda E_t [P_3(s)] - \lambda P_4(s). \]

It can be shown easily that
\[ \Lambda(s; t) = \Gamma(s; t) [P_0(s) \theta(s) + \sigma(s) Q_0(s)]. \]

Since \( \Gamma(\cdot; t) \in L^2_\Omega (\Omega; C([t, T]; \mathbb{R})), P_0(\cdot) \in L^2_\Omega (\Omega; C([0, T]; \mathbb{R})) \) and \( \Gamma(t; t) = 0 \), it follows from the Lebesgue differentiation theorem that
\[ \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E_t \left[ \int_t^{t + \varepsilon} |\Gamma(s; t) P_0(s) \theta(s)| ds \right] = \Gamma(t; t) P_0(t) \theta(t) = 0. \]

Furthermore, by Hölder's inequality and (18), we have
\[ \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E_t \left[ \int_t^{t + \varepsilon} |\Gamma(s; t) \sigma(s) Q_0(s)| ds \right] \]
\[ \leq \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ E_t \left[ \int_t^{t + \varepsilon} |\Gamma(s; t) \sigma(s)|^2 ds \right] E_t \left[ \int_t^{t + \varepsilon} |Q_0(s)|^2 ds \right] \right\}^{\frac{1}{2}} \]
\[ \leq \liminf_{\varepsilon \downarrow 0} \left\{ \frac{1}{\varepsilon} E_t \left[ \int_t^{t + \varepsilon} |\Gamma(s; t) \sigma(s)|^2 ds \right] \right\}^{\frac{1}{2}} \left\{ E_t \left[ \sup_{s \in [0, T]} |Q_0(s)|^2 \right] \right\}^{\frac{1}{2}} \]
\[ = 0. \]
We point out that without proper regularity condition on the path of $Q_0(\cdot)$, we cannot directly apply Lebesgue differentiation theorem to get
\[
\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_t \left[ \int_t^{t+\varepsilon} |\Gamma(s; t)\sigma(s)Q_0(s)| \, ds \right] = 0.
\]

This is why we impose the condition (18).

**Appendix B. Proof of Proposition 4.2.** Note that $g(\cdot)$ satisfies the ordinary differential equation
\[
g_t(t) - \alpha g(t) + 1 = 0, \quad t \in [0, T],
\]
with $g(T) = 0$.

Inserting the ansatz $F_0(t, r) = G_0(t)e^{g(t)r}$ into (30), by (1) it is easy to obtain that
\[
G_{0, t}(t) + G_0(t)g(t) \left( \alpha\beta + \frac{1}{2}g(t)^2 \rho^2 \right) = 0, \quad t \in [0, T],
\]
and $G_0(T) = 1$. Thus
\[
G_0(t) = \exp \left\{ \int_t^T g(s) \left( \alpha\beta + \frac{1}{2}g(s)^2 \rho^2 \right) \, ds \right\}.
\]

Similarly, consider the ansatz $F_1(t, r) = G_1(t)e^{g(t)r}$. Putting it back into (30) we have for $t \in [0, T)$,
\[
G_{1, t}(t)e^{g(t)r} + rg_t(t)G_1(t)e^{g(t)r} + G_1(t) \left( g(t)G_1(t) + \alpha \beta - r \right) + \frac{1}{2}g^2(t)G_1(t)e^{g(t)r} \rho^2 + rG_1(t)e^{g(t)r} + \phi g(t)G_1(t)e^{g(t)r} - \frac{1}{2}G_1(t)e^{g(t)r} \rho^2 = 0,
\]
and $G_1(T) = 1$, which imply that
\[
G_1(t) = \exp \left\{ \int_t^T g(s) \left( \alpha\beta + \frac{1}{2}g(s)^2 + \phi \rho \right) \, ds \right\}.
\]

We look for $F_2(\cdot, \cdot)$ in the form of $F_2(t, r) = (G_1(t) + G_2(t))e^{g(t)r}$. Then by (30), (1) and (2), it holds that
\[
\left[ G_{1, t}(t) + G_{2, t}(t) \right] e^{g(t)r} + rg_t(t) \left[ G_1(t) + G_2(t) \right] e^{g(t)r} + g(t) \left[ G_1(t) + G_2(t) \right] e^{g(t)r} \alpha (\beta - r) + \frac{1}{2}g^2(t) \left[ G_1(t) + G_2(t) \right] e^{g(t)r} \rho^2 + (r - \delta(t))G_1(t)e^{g(t)r} + (r - \delta(t))G_2(t)e^{g(t)r}
\]
\[
+ \left[ \frac{\zeta_{AL}(t)}{\psi_{AL}(t)} + \kappa(t) \right] (G_1(t) + G_2(t)) e^{g(t)r} + \xi_1(t)g(t)G_2(t)e^{g(t)r} \rho
\]
\[
- \left[ G_2(t)e^{g(t)r} \xi_1(t) + g(t) (G_1(t) + G_2(t)) e^{g(t)r} \rho \right] (g(t)\rho - \phi)
\]
\[
- G_2(t)e^{g(t)r} \xi_2(t) \frac{\phi \sigma_2(t) + \theta_2(t)}{\sigma_2(t)}
\]
\[
e^{g(t)r} \left\{ \frac{\zeta_{AL}(t)}{\psi_{AL}(t)} + \gamma_1(t) \right\} G_1(t) + G_{2, t}(t).
\]
which is an exponential martingale. It follows from Proposition 2.2 in [13] that

\[
+ G_2(t) \left[ g(t) \alpha \beta + \frac{1}{2} g^2(t) \rho^2 + \left( \frac{\zeta_{AL}(t)}{\psi_{AL}} + \gamma(t) \right) \right] \\
+ \xi_1(t) g(t) \rho - (\xi_1(t) + g(t) \rho) (g(t) \rho - \phi) - \xi_2(t) \frac{\phi \sigma_{21}(t) + \theta_2(t)}{\sigma_{22}(t)} \right) = 0,
\]

and \( G_2(T) = -1 \). It is easy to see that

\[
G_2(t) = -G_1(t) e^{\int_t^T \Xi(s) ds} \left[ 1 - \int_t^T e^{-\int_u^T \Xi(v) dv} \left( \frac{\zeta_{AL}(s)}{\psi_{AL}(s)} + \gamma_1(s) \right) ds \right].
\]

To get \( F_3(\cdot, \cdot) \) and \( F_4(\cdot, \cdot) \), we first solve \( (P_5(\cdot), Q_5(\cdot)) \) (note that \( Q_5(\cdot) = 0 \)), which satisfies

\[
\begin{aligned}
\left\{ \right. &dP_5(s) = P_5 \left[ \sigma^{-1} \left( \theta + \sigma \frac{Q_0}{P_0} \right) \right]^T Q_0 / P_0 \left[ \sigma^{-1} \left( \theta + \sigma \frac{Q_0}{P_0} \right) \right] \left. \right]\ \\
&+ \left. G_0(s) G_1(s) G_5(s) e^{2g(s)r(s) - (g(s) \rho - \phi)} \right] ds + Q_3(s) dW(s), \quad s \in [0, T], \\
P_5(T) & = -1.
\end{aligned}
\]

Clearly, (30) and (31) still hold for \( i = 5 \) with

\[
f_5(s, P_5, Q_5) = -P_5 \left[ \sigma^{-1} \left( \theta + \sigma \frac{Q_0}{P_0} \right) \right]^T Q_0 / P_0.
\]

Let us consider the ansatz \( F_5(t, r) = G_5(t) \). Then

\[
F_{5, t}(t, r) + F_{5, r}(t, r) \alpha (\beta - r) + \frac{1}{2} F_{5, rr}(t, r) \rho^2 = F_5(t, r) \left[ g(t) \rho - \phi \right] g(t) \rho
\]

\[
= G_5, t(t) - G_5(t) \left[ g(t) \rho - \phi \right] g(t) \rho = 0,
\]

which, together with \( F_5(T, r) = -1 \), implies

\[
G_5(t) = -\exp \left\{ -\int_t^T [g(s) \rho - \phi] g(s) \rho ds \right\}.
\]

In the following, we solve \( (P_3(\cdot), Q_3(\cdot)) \) by using the probability method. Since \( P_3(T) = 0 \) and \( (P_5(\cdot), Q_5(\cdot)) = (G_5(\cdot), 0) \), we have for \( s \in [t, T] \),

\[
dP_3(s) = - \left\{ \right. G_3(s) \left[ (g(s) \rho - \phi)^2 + \left( \frac{\theta_2(s) + \phi \sigma_{21}(s)}{\sigma_{22}(s)} \right)^2 \right] \right. \\
+ G_0(s) G_1(s) G_3(s) e^{2g(s)r(s) - (g(s) \rho - \phi)} \left. Q_3(s) \right] ds + Q_3(s) dW_1(s).
\]

Let

\[
\Upsilon(s; t) = \exp \left\{ -\frac{1}{2} \int_t^s (g(z) \rho - \phi)^2 dz - \int_t^s (g(z) \rho - \phi) dW_1(z) \right\},
\]

which is an exponential martingale. It follows from Proposition 2.2 in [13] that

\[
P_3(t) = E_t \int_t^T \Upsilon(s; t) G_3(s) \left\{ (g(s) \rho - \phi)^2 + \left( \frac{\theta_2(s) + \phi \sigma_{21}(s)}{\sigma_{22}(s)} \right)^2 \right\} ds
\]

\[
+ G_0(s) G_1(s) e^{2g(s)r(s)} \left\} ds
\]

\[
= \int_t^T G_3(s) \left\{ (g(s) \rho - \phi)^2 + \left( \frac{\theta_2(s) + \phi \sigma_{21}(s)}{\sigma_{22}(s)} \right)^2 \right\}
\]

\[
+ G_0(s) G_1(s) E_t \left[ \Upsilon(s; t) e^{2g(s)r(s)} \right] ds.
\]
Noting that for \( s \in [t, T] \),
\[
r(s) = e^{-\alpha(s-t)}r(t) + \beta \left(1 - e^{-\alpha(s-t)}\right) + \int_t^s e^{-\alpha(s-v)} \rho dW_1(v),
\]
we get
\[
\int_t^s r(z)dz = \int_t^s \left[e^{-\alpha(z-t)}r(t) + \beta \left(1 - e^{-\alpha(z-t)}\right) + \int_t^z e^{-\alpha(z-v)} \rho dW_1(v)\right] dz
\]
\[
= r(t) \frac{1}{\alpha} \left(1 - e^{-\alpha(s-t)}\right) + \int_t^s \beta \left(1 - e^{-\alpha(z-t)}\right) dz + \int_t^s \frac{\rho}{\alpha} \left(1 - e^{-\alpha(s-v)}\right) dW_1(v).
\]
Applying Itô’s formula yields
\[
d(g(t)r(t)) = (g(t)\alpha \beta - r(t)) dt + g(t)\rho dW_1(t).
\]
Thus
\[
g(s)r(s) = g(t)r(t) + \int_t^s \left(g(z)\alpha \beta - r(z)\right) ds + \int_t^s g(z)\rho dW_1(z)
\]
\[
= r(t) \left[g(t) - \frac{1}{\alpha} \left(1 - e^{-\alpha(s-t)}\right)\right] + \int_t^s \left[g(z)\alpha - 1 + e^{-\alpha(z-t)}\right] \beta dz - \int_t^s \left[\frac{1}{\alpha} \left(1 - e^{-\alpha(s-z)}\right) - g(z)\right] \rho dW_1(z),
\]
and
\[
E_t \left[Y(s; t) e^{2g(s)r(s)}\right] = \exp \left\{2r(t) \left[g(t) - \int_t^s e^{-\alpha(z-t)} dz\right]
\right.
\]
\[
+ \int_t^s \left[-\frac{1}{2} (g(z)\rho - \phi)^2 + 2 \left(g(z)\alpha - 1 + e^{-\alpha(z-t)}\right) \beta\right] dz\right\}
\times E_t \left[\exp \left\{ - \int_t^s \left\{ \left(g(z)\rho + \frac{\rho}{\alpha} \right) \right\} dz\right\}
\right.
\]
\[
+ 2 \left[\frac{1}{\alpha} \left(1 - e^{-\alpha(s-z)}\right) - g(z)\right] \rho \right\} dW_1(z)\left]\right]\right\}
\times \exp \left\{2r(t) \left[g(t) - \frac{1}{\alpha} \left(1 - e^{-\alpha(s-t)}\right)\right]
\right.
\]
\[
+ \int_t^s \left[-\frac{1}{2} (g(z)\rho - \phi)^2 + 2 \left(g(z)\alpha - 1 + e^{-\alpha(z-t)}\right) \beta\right] dz\right\}
\times \exp \left\{\frac{1}{2} \int_t^s \left\{ \left(g(z)\rho - \phi\right) + 2 \left[\frac{1}{\alpha} \left(1 - e^{-\alpha(s-z)}\right) - g(z)\right] \rho\right\}^2 dz\right\}. \quad (6)
\]
It follows from (3) and (6) that \( P_3(t) = F_3(t, r(t)) \) with \( F_3(t, r) \) given by (32). Consequently, \( Q_{31}(t) = F_{3, r}(t, r)\rho \) and \( F_4(t, r) = F_3(t, r) - G_5(t) \). The proof is completed.

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