THE ITALIAN BONDAGE AND REINFORCEMENT NUMBERS OF DIGRAPHS

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Abstract. An Italian dominating function on a digraph $D$ with vertex set $V(D)$ is defined as a function $f : V(D) \rightarrow \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under $f$ or one in-neighbor $w$ with $f(w) = 2$. The weight of an Italian dominating function $f$ is the value $\omega(f) = f(V(D)) = \sum_{v \in V(D)} f(v)$. The Italian domination number $\gamma(I)(D)$, is the minimum taken over the weights of all Italian dominating functions on $D$. The Italian bondage number of a digraph $D$, denoted by $b_I(D)$, is the minimum number of arcs whose addition to $D$ results in a digraph $D'$ with $\gamma_I(D') > \gamma_I(D)$. The Italian reinforcement number of a digraph $D$, denoted by $r_I(D)$, is the minimum number of extra arcs whose addition to $D$ results in a digraph $D'$ with $\gamma_I(D') < \gamma_I(D)$. In this paper, we initiate the study of Italian bondage and reinforcement numbers in digraphs and present some bounds for $b_I(D)$ and $r_I(D)$. We also determine the Italian bondage and reinforcement numbers of some classes of digraphs.

Key words: Italian domination number, Italian bondage number, Italian reinforcement number

1. Introduction

Let $D = (V, A)$ be a finite simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$, the order $n(D)$ of a digraph is the size of $V(D)$. For an arc $uv \in A(D)$, we say that $v$ is an out-neighbor of $u$ and $u$ is an in-neighbor of $v$. We denote the set of in-neighbors and out-neighbors of $v$ by $N_D^+(v)$ and $N_D^-(v)$, respectively. We write $deg_D(v)$ and $deg_D^-(v)$ for the size of $N_D^-(v)$ and $N_D^+(v)$, respectively. Let $N_D^-(v) = N_D^+(v) \cup \{v\}$ and $N_D^+(v) = N_D^-(v) \cup \{v\}$. For a subset $S$ of $V(D)$, we define $N^+(S) = \bigcup_{v \in S} N_D^+(v)$ and $N^+(S) = \bigcup_{v \in S} N_D^+(v)$. The maximum out-degree and maximum in-degree of a digraph $D$ are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively.

For a digraph $D$, a subset $S$ of $V(D)$ is a dominating set if $\bigcup_{v \in S} N_D^-(v) = V(D)$. The domination number $\gamma(D)$ is the minimum cardinality of a dominating set of $D$. The concept of the domination number of a digraph was introduced in [2].

The bondage number $b(D)$ of a digraph $D$ is the minimum number of arcs of $A(D)$ whose removal in $D$ results in a digraph $D'$ with $\gamma(D') > \gamma(D)$. The concept of the bondage number of a digraph was proposed in [1]. The reinforcement number $r(D)$ of a digraph $D$ is the minimum number of extra arcs whose addition to $D$ results in a digraph $D'$ with $\gamma(D') < \gamma(D)$. The concept of the reinforcement number of a digraph was introduced in [3].

Among the variations of domination, so called Italian domination of graphs is introduced in [3]. The authors of [3] present bounds relating the Italian domination number to some other domination parameters. The authors of [3] characterize the trees $T$ for which $\gamma(T) + 1 = \gamma(T)$ and also characterize the trees $T$ for which
\(\gamma_I(T) = 2\gamma(T)\). After that, there are many studies on Italian domination of graphs in \([2, 8, 9, 11, 13]\). Recently, the author of \([14]\) initiated the study of the Italian domination number in digraphs. Related results was given in \([12, 15]\). Our aim in this paper is to initiate the study of Italian bondage and reinforcement numbers for digraphs.

An **Italian dominating function** (IDF) on a digraph \(D\) with vertex set \(V(D)\) is defined as a function \(f : V(D) \to \{0, 1, 2\}\) such that every vertex \(v \in V(D)\) with \(f(v) = 0\) has at least two in-neighbors assigned 1 under \(f\) or one in-neighbor \(w\) with \(f(w) = 2\). An Italian dominating function \(f : V(D) \to \{0, 1, 2\}\) gives an ordered partition \((V_0, V_1, V_2)\) (or \((V_0^f, V_1^f, V_2^f)\) to refer to \(f\)) of \(V(D)\), where \(V_i := \{x \in V(D) \mid f(x) = i\}\). The weight of an Italian dominating function \(f\) is the value \(\omega(f) = f(V(D)) = \sum_{u \in V(D)} f(u)\). The **Italian domination number** of a digraph \(D\), denoted by \(\gamma_I(D)\), is the minimum taken over the weights of all Italian dominating functions on \(D\). A \(\gamma_I(D)\)-function is an Italian dominating function on \(D\) with weight \(\gamma_I(D)\).

The **Italian bondage number** of a digraph \(D\), denoted by \(b_I(D)\), is the minimum number of arcs of \(A(D)\) whose removal in \(D\) results in a digraph \(D'\) with \(\gamma_I(D') > \gamma_I(D)\).

The **Italian reinforcement number** of a digraph \(D\), denoted by \(r_I(D)\), is the minimum number of extra arcs whose addition to \(D\) results in a digraph \(D'\) with \(\gamma_I(D') < \gamma_I(D)\). The Italian reinforcement number of a digraph \(D\) is defined to be 0 if \(\gamma_I(D) \leq 2\). A subset \(R\) of \(A(D)\) is called an **Italian reinforcement set** (IRS) of \(D\) if \(\gamma_I(D + R) < \gamma_I(D)\). An \(r_I(D)\)-set is an IRS of \(D\) with size \(r_I(D)\).

This paper is organized as follows. In Section 2, we prepare basic results on the Italian domination number. In Section 3, we give some bounds of the Italian bondage number and determine the exact values of Italian bondage numbers of some classes of digraphs. In Section 4, we characterize all digraphs \(D\) with \(r_I(D) = 1\). We give some bounds of the Italian reinforcement number and also determine the exact values of Italian reinforcement numbers of compositions of digraphs.

## 2. The Italian Domination Numbers

In this paper, we make use of the following results.

**Observation 2.1.** For a digraph \(D\), \(\gamma_I(D) \leq n - \Delta^+(D) + 1\).

**Proof.** Let \(D\) be a digraph, and let \(v\) be a vertex with \(\deg_D^-(v) = \Delta^+(D)\). Define a function \(f : V(D) \to \{0, 1, 2\}\) by \(f(v) = 2\), \(f(x) = 0\) if \(x \in N^+(v)\), and \(f(x) = 1\) otherwise. It is easy to see that \(f\) is an IDF of \(D\). \(\square\)

The following result is the exact value of Italian domination number of a complete bipartite graph (see \([3]\) for the definition of Italian dominating function and domination number on a graph).

**Lemma 2.2 (\([3]\)).** For a complete bipartite graph \(K_{m,n}\) with \(1 \leq m \leq n\) and \(n \geq 2\),

\[
\gamma_I(K_{m,n}) = \begin{cases} 2 & \text{if } m \leq 2; \\ 3 & \text{if } m = 3; \\ 4 & \text{if } m \geq 4. \end{cases}
\]

**Theorem 2.3 (\([14]\)).** Let \(D\) be a digraph of order \(n\). Then \(\gamma_I(D) \geq \left\lceil \frac{2n}{\Delta^+(D)} \right\rceil\).

**Theorem 2.4.** Let \(D\) be a digraph of order \(n \geq 3\). Then \(\gamma_I(D) = 2\) if and only if \(\Delta^+(D) = n - 1\) or there exist two distinct vertices \(u\) and \(v\) such that \(V(D) \setminus \{u, v\} \subseteq N_D^+(u)\) and \(V(D) \setminus \{u, v\} \subseteq N_D^+(v)\).
Theorem 3.2. Let $w$ be an IDF of $D$ and $N$ in-neighbors of $D$. Moreover, if $yx, yz \in N$, then it is easy to see that $\gamma_1(D) = 2$.

Assume that $\gamma_1(D) = 2$. Let $(V_0, V_1, V_2)$ be a $\gamma_1(D)$-function. Then $\gamma_1(D) = 2 = |V_1| + 2|V_2|$ and $|V_2| \leq 1$. If $|V_2| = 1$, then $|V_1| = 0$ and hence $\Delta^+(D) = n - 1$. If $|V_2| = 0$, then $|V_1| = 2$ and, by the definition of IDF, there exist two distinct vertices $u$ and $v$ such that $V(D) \setminus \{u, v\} \subseteq N_D^+(u)$ and $V(D) \setminus \{u, v\} \subseteq N_D^+(v)$.

Theorem 2.5 ([4]). Let $D$ be a digraph of order $n \geq 3$. Then $\gamma_1(D) < n$ if and only if $\Delta^+(D) \geq 2$ or $\Delta^-(D) \geq 2$.

Corollary 2.6. If $D$ is a directed path or cycle of order $n$, then $\gamma_1(D) = n$.

3. The Italian bondage numbers

3.1. Bounds of the Italian bondage numbers. The underlying graph $G[D]$ of a digraph $D$ is the graph obtained by replacing each arc $uv$ by an edge $uv$. Note that $G[D]$ has two parallel edges $uv$ when $D$ contains the arc $uv$ and $vu$. A digraph $D$ is connected if the underlying graph $G[D]$ is connected. For a graph $G$, we denote the degree of $v \in V(G)$ by $deg_G(v)$. In particular, $\Delta(G)$ means the maximum degree in $G$.

Theorem 3.1. If $D$ is a digraph, and $xyz$ a path of length 2 in $G[D]$ such that $xy, yz \in A(D)$, then

$$b_1(D) \leq deg_{G[D]}(x) + deg_{G[D]}(y) + deg_{G[D]}(z) - |N^-(x) \cap N^-(y) \cap N^-(z)|.$$  

Moreover, if $x$ and $z$ are adjacent in $G[D]$, then

$$b_1(D) \leq deg_{G[D]}(x) + deg_{G[D]}(y) + deg_{G[D]}(z) - 1 - |N^-(x) \cap N^-(y) \cap N^-(z)|.$$  

Proof. Let $B$ be the set of all arcs incident with $x$ or $z$ and all arcs terminating at $y$ with the exception of all arcs from $N^-(x) \cap N^-(z)$ to $y$. Then

$$|B| \leq deg_{G[D]}(x) + deg_{G[D]}(y) + deg_{G[D]}(z) - |N^-(x) \cap N^-(y) \cap N^-(z)|$$  

and

$$|B| \leq deg_{G[D]}(x) + deg_{G[D]}(y) + deg_{G[D]}(z) - 1 - |N^-(x) \cap N^-(y) \cap N^-(z)|.$$  

when $x$ and $z$ are adjacent.

Let $D' = D - B$. In $D'$, $x$ and $z$ are isolated, and all in-neighbors of $y$ in $D'$, if any, lie in $N^-(x) \cap N^-(z)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_1(D')$-function. Then $f(x) = f(z) = 1$. If $f(y) = 2$, then

$$(V_0 \cup \{x, z\}, V_1 \setminus \{x, z\}, V_2)$$  

is an IDF of $D$ with weight less than $\omega(f)$. If $f(y) = 1$, then

$$(V_0 \cup \{x, z\}, V_1 \setminus \{x, y, z\}, V_2 \cup \{y\})$$  

is an IDF of $D$ with weight less than $\omega(f)$. However, if $f(y) = 0$, then there exists $w \in N^-(x) \cap N^-(y) \cap N^-(z)$ such that $f(w) = 2$ or there exist $w_1, w_2 \in N^-(x) \cap N^-(y) \cap N^-(z)$ such that $f(w_1) = f(w_2) = 1$. Since $w, w_1$ and $w_2$ are in-neighbors of $x$ and $z$ in $D$,

$$(V_0 \cup \{x, z\}, V_1 \setminus \{x, z\}, V_2)$$  

is an IDF of $D$ with weight less than $\omega(f)$. This completes the proof.

Theorem 3.2. Let $D$ be a digraph of order $n \geq 3$. If $G[D]$ is connected, then

$$b_1(D) \leq (\gamma_1(D) - 1)\Delta(G[D]).$$  


Proof. We proceed by induction on \( \gamma_f(D) \). Assume that \( \gamma_f(D) = 2 \). For a vertex \( u \in V_1 \cup V_2 \), let \( B_u \) be the set of arcs incident with \( u \). Since \( \gamma_f(D - u) \geq 2 \) by \( n \geq 3 \), we have

\[
\gamma_f(D - B_u) = \gamma_f(D - u) + 1 \geq 3.
\]

This implies that \( b_f(D) \leq |B_u| \) for \( u \in V_1 \cup V_2 \). Thus, \( b_f(D) \leq \Delta(G[D]) \).

Assume that the result is true for every digraph with the Italian domination number \( k \geq 3 \). Let \( D \) be a digraph with \( \gamma_f(D) = k + 1 \). Suppose to the contrary that \( b_f(D) > (\gamma_f(D) - 1)\Delta(G[D]) \). Let \( u \) be an arbitrary vertex of \( D \), and let \( B_u \) be the set of arcs incident with \( u \). Then we have \( \gamma_f(D) = \gamma_f(D - B_u) \). Let \( f \) be a \( \gamma_f(D - B_u) \)-function. Then \( f(u) = 1 \) and the function \( f \) restricted to \( D - u \) is also a \( \gamma_f(D - u) \)-function. This implies that \( \gamma_f(D - u) = \gamma_f(D) - 1 \). So, \( b_f(D) \leq b_f(D - u) + \text{deg}_{G[D]}(u) \). By the induction hypothesis, we have

\[
b_f(D) \leq b_f(D - u) + \text{deg}_{G[D]}(u) \\
\leq (\gamma_f(D - u) - 1)\Delta(G[D]) + \text{deg}_{G[D]}(u) \\
\leq (\gamma_f(D) - 1)\Delta(G[D]) + \Delta(G[D]) \\
= \gamma_f(D)\Delta(G[D]) \\
= (\gamma_f(D) - 1)\Delta(G[D]).
\]

This is a contradiction. \( \square \)

3.2. The Italian bondage numbers of some classes of digraphs. For a graph \( G \), the associated digraph \( G^* \) is the digraph obtained from \( G \) by replacing each edge of \( G \) by two oppositely oriented arcs. Note that \( \gamma_f(G) = \gamma_f(G^*) \) for any graph \( G \).

Theorem 3.3. Let \( K^*_n \) be the complete digraph of order \( n \geq 3 \). Then \( b_f(K^*_n) = n \).

Proof. Note that \( \gamma_f(K^*_n) = 2 \). Let \( B \) be an ar set of \( K^*_n \). Define \( D := K^*_n - B \). If \( D \) contain a vertex \( u \) such that \( \text{deg}_{G^*}(u) = n - 1 \), then it follows from Observation \( \square \) that \( \gamma_f(D) = 2 \). This implies that \( b_f(K^*_n) \geq n \).

Let \( \{x_1, x_2, \ldots, x_n\} \) be the vertex set of \( K^*_n \), and let \( B := \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\} \) be the arc set of a directed cycle in \( K^*_n \). Define \( D := K^*_n - B \). Then one can observe that there do not exist two distinct vertices \( u \) and \( v \) in \( D \) such that \( V(D) \setminus \{u, v\} \subseteq N_D^-(u) \) and \( V(D) \setminus \{u, v\} \subseteq N_D^-(v) \). It follows from Theorem \( \square \) that \( \gamma_f(D) \geq 3 \). This completes the proof. \( \square \)

The following result follows from the definition of associated digraph and Lemma \( \square \).

For a complete bipartite digraph \( K^*_{m,n} \) with \( 1 \leq m \leq n \),

\[
\gamma_f(K^*_{m,n}) = \begin{cases} 
2 & \text{if } m \leq 2; \\
3 & \text{if } m = 3; \\
4 & \text{if } m \geq 4.
\end{cases}
\]

Theorem 3.4. Let \( K^*_{m,n} \) be the complete bipartite digraph such that \( 1 \leq m < n \). Then

\[
b_f(K^*_{m,n}) = \begin{cases} 
1 & \text{if } m \leq 2; \\
2 & \text{if } m = 3; \\
2m + 2 & \text{if } m \geq 4.
\end{cases}
\]

Proof. We denote \( K^*_{m,n} \) by \( D \). Let \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \) be the partite sets of \( D \). The result is clear for \( m \leq 2 \).

Assume that \( m = 3 \). It follows from \( \square \) that \( \gamma_f(D) = 3 \). If we remove two arcs terminating at some vertex \( y_i \in Y \), then the Italian domination number of resulting digraph increases. So, \( b_f(D) \leq 2 \). For any arc \( e \) of \( A(D) \), there exist two vertices \( x_i \) and \( x_j \) such that \( N_{D^*}^+(x_i) = n \) and \( N_{D^*}^+(x_j) = n \). Thus, we have \( b_f(D) = 2 \).

Assume that \( m \geq 4 \). It follows from \( \square \) that \( \gamma_f(D) = 4 \). Let \( B = \{x_iy_i \mid 1 \leq i \leq m\} \cup \{y_1x_1, y_1x_2\} \). It is easy to see that \( \gamma_f(D - B) \geq 5 \). So, \( b_f(D) \leq m + 2 \).
Next, we show that \( b_I(D) \geq m + 2 \). Let \( B' \) be a subset of \( A(D) \) such that \( |B'| = m + 1 \), and let \( D' = D - B' \). Then \( D' \) has at least \( n - 1 \) vertices whose outdegree are equal in \( D \) and \( D' \). Let \( E = \{ v \in V(D) \mid d^+(v) = d^+_{D'}(v) \} \). If \( E \cap X \neq \emptyset \neq E \cap Y \), then clearly \( \gamma_I(D') = 4 \). Henceforth, we assume that \( E \cap X = \emptyset \) or \( E \cap Y = \emptyset \). Without loss of generality, assume that \( E \cap X = \emptyset \). Then \( E \subseteq Y \) and \( B' \) contains one outgoing arc for each \( x_i \in X \). Since \( |B'| = m + 1 < 2m \), \( B' \) contains exactly one outgoing arc for some \( x_i \in X \). Without loss of generality, assume that \( i = 1 \) and \( x_1y_1 \in B' \). If \( E = Y \), then

\[
(V(D') \setminus \{ x_1, y_1 \}, \emptyset, \{ x_1, y_1 \})
\]

is an IDF of \( D' \) with weight \( 4 \). Let \( E \subset Y \). We may assume that \( E \subseteq \{ y_1, y_2, \ldots, y_{n-1} \} \). Thus, \( B' \) contains one outgoing arc from \( y_n \), say \( y_nx_m \). Since \( |B'| = m + 1 \), \( B' \) contains exactly one outgoing arc for each \( x_i \in X \) and one outgoing arc from \( y_n \). If \( x_iy_j \in B' \) for some \( 1 \leq i \leq m \) and \( j < n \), then

\[
(V(D') \setminus \{ x_i, y_j \}, \emptyset, \{ x_i, y_j \})
\]

is an IDF of \( D' \) with weight \( 4 \). Thus, we assume that \( x_iy_n \in B' \) for each \( 1 \leq i \leq m \). But,

\[
(V(D') \setminus \{ x_m, y_n \}, \emptyset, \{ x_m, y_n \})
\]

is an IDF of \( D' \) with weight \( 4 \). Thus, we have \( b_I(D) \geq m + 2 \). \( \square \)

4. The Italian reinforcement numbers

4.1. Digraphs with \( r_I(D) = 1 \).

**Lemma 4.1.** Let \( D \) be a digraph with \( \gamma_I(D) \geq 3 \). Let \( F \) be an \( r_I(D) \)-set, and let \( g \) be a \( \gamma_I(D) \)-function of \( D + F \). Then the following hold:

(i) For each arc \( v_1v_2 \in F \), \( g(v_1) \neq 0 \) and \( g(v_2) = 0 \).

(ii) \( \gamma_I(D + F) = \gamma_I(D) - 1 \).

**Proof.** If there exists an arc \( v_1v_2 \in F \) such that either \( g(v_1) \geq 1 \) for each \( i \in \{1, 2\} \) or \( g(v_1) = g(v_2) = 0 \), then \( g \) is also an IDF of \( D + (F \setminus \{v_1v_2\}) \), and hence \( F \setminus \{v_1v_2\} \) is an IRS of \( D \), which contradicts the definition of \( F \). Thus, (i) holds.

By the definition of \( F \), we have \( \gamma_I(D + F) \leq \gamma_I(D) - 1 \). Suppose that \( \gamma_I(D + F) \leq \gamma_I(D) - 2 \). Let \( v_1v_2 \in F \). By (i), \( g(v_1) \neq 0 \) and \( g(v_2) = 0 \). Then the function \( g' : V(D + (F \setminus \{v_1v_2\})) \rightarrow \{0, 1, 2\} \) with

\[
g'(x) = \begin{cases} 1 & \text{if } x = v_2; \\ g(x) & \text{otherwise} \end{cases}
\]

is an IDF of \( D + (F \setminus \{v_1v_2\}) \) such that \( \omega(g') = \omega(g) + 1 \leq \gamma_I(D) - 1 \). This implies that \( F \setminus \{v_1v_2\} \) is an IRS of \( D \), which contradicts the definition of \( F \). Thus, (ii) holds. \( \square \)

**Lemma 4.2.** Let \( D \) be a digraph of order \( n \geq 3 \), \( \Delta^+(D) \geq 1 \) and \( \gamma_I(D) = n \). Then \( r_I(D) = 1 \).

**Proof.** It follows from Theorem 2.5 that \( \Delta^+(D) = 1 \). Since \( \sum_{v \in V(D)} \deg^+(v) = \sum_{v \in V(D)} \deg^-(v) \), we have \( \Delta^-(D) \geq 1 \). It also follows from Theorem 2.5 that \( \Delta^-(D) = 1 \). Thus, \( D \) is disjoint union of directed paths, cycles or isolated vertices. Let \( uv \in A(D) \) and \( w \in V(D) \setminus \{u, v\} \). It is easy to see that

\[
\{u, v, w\} \cup V(D) \setminus \{u, v, w\}, \{u\}
\]

is an IDF of \( D + uw \) with weight \( n - 1 \). Thus, we have \( r_I(D) = 1 \). \( \square \)

**Theorem 4.3.** Let \( D \) be a digraph with \( \gamma_I(D) \geq 3 \). Then \( r_I(D) = 1 \) if and only if there exist a \( \gamma_I(D) \)-function \( f = (V_0, V_1, V_2) \) of \( D \) and a vertex \( v \in V_1 \) satisfying one of the following conditions:
(i) \( f(N^-(v)) = 1 \) and \( f(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \cap V_0 \).

(ii) \( f(N^-(v)) = 0 \), \( f(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \), and \( V_2 \neq \emptyset \).

**Proof.** First, assume that (i) holds. Then it follows from \( f(N^-(v)) = 1 \) that there exists \( u \in V_1 \cap N^-(v) \). Since \( \gamma_I(D) \geq 3 \), there exists \( w \in (V_1 \cup V_2) \setminus \{v, u\} \). Since \( uw \in A(D) \) and \( f(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \cap V_0 \), \( V_0 \cup \{v\}, V_1 \setminus \{v\}, V_2 \) is an IDF of \( D + wv \) with weight \( \gamma_I(D) - 1 \). Thus, we have \( r_I(D) = 1 \).

Next, assume that (ii) holds. Let \( w \in V_2 \). Then it follows from \( f(N^-(v)) = 0 \) that \( uw \not\in A(D) \). Since \( f(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \cap V_0 \), \( V_0 \cup \{v\}, V_1 \setminus \{v\}, V_2 \) is an IDF of \( D + wv \) with weight \( \gamma_I(D) - 1 \). Thus, we have \( r_I(D) = 1 \).

Conversely, assume that \( r_I(D) = 1 \), and let \( uw \) be an arc of \( D \) with \( \gamma_I(D + uv) < \gamma_I(D) \). Let \( g \) be a \( \gamma_I(D + wv) \)-function. Then \( g(u) \neq 0 \) and \( g(v) = 0 \) by Lemma 4.1(i). The function \( f : V(D) \rightarrow \{0, 1, 2\} \) with

\[
 f(x) = \begin{cases} 1 & \text{if } x = v; \\ g(x) & \text{otherwise} \end{cases}
\]

is an IDF of \( D \). It follows from Lemma 4.1(ii) that \( f \) is a \( \gamma_I(D) \)-function.

Suppose that \( f(N^-(v)) \geq 2 \). Then \( g(N^-(v)) \geq 2 \). So, \( g \) is an IDF of \( D \). This means that \( \gamma_I(D) \leq \omega(g) = \gamma_I(D + uv) \), a contradiction. Thus, we have \( f(N^-(v)) \leq 1 \).

Note that \( f(N^-(x) \setminus \{v\}) = h(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \cap V'_I \), since \( g \) is a \( \gamma_I(D + uv) \)-function with \( g(v) = 0 \). If \( f(N^-(v)) = 1 \), then (i) holds. Now assume that \( f(N^-(v)) = 0 \). Then we have \( h(u) = h(u) = 2 \), since \( g(v) = 0 \) and \( u \) is an in-neighbor of \( v \) in \( D + uv \). As \( V'_I \neq \emptyset \), (ii) holds. □

### 4.2. Bounds of the Italian reinforcement numbers.

**Theorem 4.4.** If \( D \) is a digraph of order \( n \) with \( \gamma_I(D) \geq 3 \), then

\[
r_I(D) \leq n - \Delta^+(D) - \gamma_I(D) + 2.
\]

**Proof.** Since \( \gamma_I(D) \geq 3 \), it follows from Theorem 2.3 that \( \Delta^+(D) \leq n - 2 \). Let \( u \) be a vertex with \( \deg_D^+(u) = \Delta^+(D) \) and let \( R := \{uw \mid v \in V(D) \setminus N^+[u]\} \). Then \( (V(D) \setminus \{u\}, \emptyset, \{u\}) \) is an IDF of \( D + R \). Thus,

\[
r_I(D) \leq n - \Delta^+(D) - 1.
\]

There exist \( r_I(D) - 1 \) vertices \( v_1, v_2, \ldots, v_{r_I(D)-1} \) in \( V(D) \setminus N^+[u] \).

Let \( D' \) be a digraph obtained from \( D \) by adding \( r_I(D) - 1 \) arcs \( uw_i \). Then, by the definition of \( r_I(D) \) and Observation 2.1,

\[
\gamma_I(D) = \gamma_I(D') \leq n - \Delta^+(D') + 1.
\]

Since \( \Delta^+(D') = \Delta^+(D) + r_I(D) - 1 \), we have \( r_I(D) \leq n - \Delta^+(D) - \gamma_I(D) + 2 \). □

**Theorem 4.5.** If \( D \) is a digraph such that \( \gamma_I(D) = 3 \) and \( \gamma(D) = 2 \), then \( r(D) \leq r_I(D) + 1 \).

**Proof.** Let \( R \) be a \( r_I(D) \)-set. Then \( \gamma_I(D + R) = 2 \). If \( r \in R \), then clearly \( r_I(D + (R \setminus \{r\})) = 1 \).

By Theorem 4.3, there exist a \( \gamma_I(D + (R \setminus \{r\})) \)-function \( f = (V_0, V_1, V_2) \) of \( D + (R \setminus \{r\}) \) and a vertex \( v \in V_1 \) satisfying one of the following conditions:

(i) \( f(N^-(v)) = 1 \) and \( f(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \cap V_0 \).

(ii) \( f(N^-(v)) = 0 \), \( f(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \), and \( V_2 \neq \emptyset \).

Suppose that (i) holds. Since \( \gamma_I(D + (R \setminus \{r\}) \cap (R \setminus \{r\})) = 3 \), it follows from \( f(N^-(v)) = 1 \) that there exists \( u \in V_1 \) such that \( u \not\in N^-(v) \). Let \( w \in V_1 \cap N^-(v) \). Since \( f(N^-(x) \setminus \{v\}) \geq 2 \) for each \( x \in N^+(v) \cap V_0 \), we have \( u, xw \in A(D + (R \setminus \{r\})) \) for each \( x \in N^+(v) \cap V_0 \). Since \( \gamma_I(D + (R \setminus \{r\})) = 3 \), we have \( u, w \in N^+(v) \) for
Thus, \( \{u\} \) is a dominating set of \( D + ((R \setminus \{r\}) \cup \{uv, uw\}) \). This implies that \( r(D) \leq r_I(D) + 1 \).

Suppose that (ii) holds. Let \( V_2 = \{u\} \). Then we have \( V_0 \subseteq N^+(u) \). Thus, \( \{u\} \) is a dominating set of \( D + ((R \setminus \{r\}) \cup \{uv\}) \). This implies that \( r(D) \leq r_I(D) \). □

4.3. The Italian reinforcement numbers of compositions of digraphs. For two digraphs \( G \) and \( H \), two kinds of joins \( G \rightarrow H \) and \( G \leftrightarrow H \) were defined in [6]. The digraph \( G \rightarrow H \) consists of \( G \) and \( H \) with extra arcs from each vertex of \( G \) to every vertex of \( H \). The digraph \( G \leftrightarrow H \) can be obtained from \( G \rightarrow H \) by adding arcs from each vertex of \( H \) to every vertex of \( G \).

**Theorem 4.6.** Let \( G \) and \( H \) be two digraphs such that \( \Delta^+(G) \geq 1 \) and \( \Delta^+(H) \geq 1 \). Then

(i) \( \gamma_I(G \rightarrow H) = \gamma_I(G) \),
(ii) \( r_I(G \rightarrow H) = r_I(G) \).

**Proof.** (i) Let \( f \) be a \( \gamma_I(G) \)-function. Then it follows from the definition of IDF that \( f \) is extended to an IDF of \( G \rightarrow H \) by assigning 0 to every vertex of \( H \). Thus, \( \gamma_I(G \rightarrow H) \leq \gamma_I(G) \). On the other hand, if \( g = (V_0, V_1, V_2) \) is a \( \gamma_I(G \rightarrow H) \)-function, then clearly \( g_G := (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G)) \) is an IDF of \( G \). Thus, \( \gamma_I(G) \leq \gamma_I(G \rightarrow H) \).

(ii) If \( \gamma_I(G) = 2 \), then it follows from (i) that \( \gamma_I(G \rightarrow H) = 2 \). So, \( r_I(G \rightarrow H) = r_I(G) \). From now on, we assume \( \gamma_I(G) \geq 3 \). Let \( R \) be a \( r_I(G) \)-set. Then

\[
\gamma_I(G \rightarrow H) + R = \gamma_I(G + R \rightarrow H) = \gamma_I(G + R) < \gamma_I(G) = \gamma_I(G \rightarrow H).
\]

Thus, \( r_I(G \rightarrow H) \leq r_I(G) \).

Now we claim that \( r_I(G) \leq r_I(G \rightarrow H) \). Let \( R_1 \) be a \( r_I(G \rightarrow H) \)-set. Suppose that \( R_2 \) is a subset of \( R_1 \) such that two ends of arcs in \( R_2 \) lie in \( V(G) \). Let \( f = (V_0^f, V_1^f, V_2^f) \) be a \( \gamma_I(G \rightarrow H) + R_1 \)-function, and let \( g = f | G \). We divide our consideration into the following two cases.

**Case 1.** \( g \) is an IDF of \( G + R_2 \).

Then we have

\[
\gamma_I((G \rightarrow H) + R_1) = \omega(f) \\
\geq \omega(g) \\
\geq \gamma_I(G + R_2) \\
= \gamma_I((G + R_2) \rightarrow H) \\
= \gamma_I((G \rightarrow H) + R_2) \\
\geq \gamma_I((G \rightarrow H) + R_1).
\]

Since \( R_2 \subseteq R_1 \) and \( R_1 \) is a \( r_I(G \rightarrow H) \)-set, we have \( R_1 = R_2 \). So, \( \gamma_I(G + R_2) \leq \omega(g) = \gamma_I((G \rightarrow H) + R_2) < \gamma_I(G \rightarrow H) = \gamma_I(G) \). Thus, \( r_I(G) \leq |R_2| = |R_1| = r_I(G \rightarrow H) \).

**Case 2.** \( g \) is not an IDF of \( G + R_2 \).

Then some vertex \( u \in V_0^f \cap V(G) \) has an in-neighbor \( w \in V(H) \) such that \( uw \in R_1 \). Fix \( v \in V(G) \), and let \( R_3 = \{vu \mid u \in N\} \), where \( N = \{u \in V_0^f \cap V(G) \mid u \) does not dominated by the vertices of \( G \) under \( f \} \). Then clearly \( |R_2 \cup R_3| \leq |R_1| \). It is easy to see that the function \( h : V(G) \rightarrow \{0, 1, 2\} \) defined by \( h(v) = \max\{f(v), \max\{f(N^-(u) \cap V(H)) \mid u \in N\}\} \) and \( h(x) = f(x) \) otherwise, is an IDF
of \( G + (R_2 \cup R_3) \) with weight at most \( \omega(f) \). Now we have
\[
\gamma_I(G + (R_2 \cup R_3)) \leq \omega(h) \\
\leq \omega(f) = \gamma_I((G \to H) + R_1) < \gamma_I(G \to H) = \gamma_I(G).
\]
Thus, \( r_I(G) \leq |R_2 \cup R_3| \leq |R_1| = r_I(G \to H). \)

The corona \( G \bowtie H \) of two digraphs \( G \) and \( H \) is formed from one copy of \( G \) and \( n(G) \) copies of \( H \) by joining \( v_i \) to every vertex of \( H_i \), where \( v_i \) is the \( i \)th vertex of \( G \) and \( H_i \) is the \( i \)th copy of \( H \).

**Theorem 4.7.** Let \( G \) and \( H \) be two digraphs with \( n(H) \geq 2 \). Then

(i) \( \gamma_I(G \bowtie H) = 2n(G) \),

(ii) \( r_I(G \bowtie H) = \begin{cases} 0 & \text{if } n(G) = 1; \\ n(H) & \text{if } G \text{ is the empty digraph and } n(G) \geq 2; \\ n(H) - 1 & \text{otherwise.} \end{cases} \)

**Proof.** (i) If \( n(G) = 1 \), then clearly \( \gamma_I(G \bowtie H) = 2 \). Assume that \( n(G) \geq 2 \). It is easy to see that \( (V(G \bowtie H) \setminus V(G), 0, V(G)) \) is an IDF of \( G \bowtie H \). So, \( \gamma_I(G \bowtie H) \leq 2n(G) \).

Let \( f \) be a \( \gamma_I(G \bowtie H) \)-function. To dominate the vertices of \( H_i \), we must have \( \sum_{x \in V(H_i) \cup \{v_i\}} f(x) \geq 2 \). Since a single vertex of \( G \) does not dominate vertices in different copies of \( H \), we have \( \gamma_I(G \bowtie H) \geq 2n(G) \).

(ii) If \( n(G) = 1 \), then clearly \( r_I(G \bowtie H) = 0 \). Assume that \( n(G) \geq 2 \). We divide our consideration into the following two cases.

**Case 1.** \( A(G) = \emptyset \).

Let \( R = \{v_1u \mid u \in V(H_{n(G)})\} \). Then it is easy to see that
\[
(\bigcup_{i=1}^{n(G)}V(H_i) \cup \{v_{n(G)}\}, V(G) \setminus \{v_{n(G)}\})
\]
is an IDF of \( (G \bowtie H) + R \) with weight \( 2n(G) - 1 \). Thus, \( r_I(G \bowtie H) \leq n(H) \).

Let \( F \) be a \( r_I(G \bowtie H) \)-set. By Lemma 4.1(ii), \( \gamma_I((G \bowtie H) + F) = \gamma_I(G \bowtie H) - 1 \).

Let \( U_i = \{v_i\} \cup V(H_i) \) for \( 1 \leq i \leq n(G) \), and let \( f = (V_0, V_1, V_2) \) be a \( \gamma_I((G \bowtie H) + F) \)-function. Then \( \sum_{x \in U_i} f(x) \leq 1 \) for some \( i \), say \( i = n(G) \). To dominate the vertices in \( U_{n(G)} \), \( F \) must contain at least \( n(H) \) arcs which go from some vertices in \( (V_1 \cup V_2) \cap (\bigcup_{i=1}^{n(G)-1} U_i) \) to vertices in \( U_{n(G)} \). Thus, \( |F| \geq n(H) \) and so \( r_I(G \bowtie H) \geq n(H) \).

**Case 2.** \( A(G) \neq \emptyset \).

Without loss of generality, we assume that \( v_1v_{n(G)} \in A(G) \). Let \( V(H_{n(G)}) = \{w_1, \ldots, w_{n(H)}\} \), and let \( R = \{v_1w_j \mid w_j \in V(H_{n(G)}) \setminus \{w_1\}\} \). Then
\[
(\bigcup_{i=1}^{n(G)} V(H_i) \cup \{v_{n(G)}\}, \{w_1\}, \{v_1, \ldots, v_{n(G)-1}\})
\]
is an IDF of \( (G \bowtie H) + R \) with weight \( 2n(G) - 1 \). Thus, \( r_I(G \bowtie H) \leq n(H) - 1 \). By using the same argument given in Case 1, one can show that \( r_I(G \bowtie H) \geq n(H) - 1 \). \( \square \)
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