QUANTUM CHAOS IN THE HEAVY QUARKONIA

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The binding energy spectra of the heavy quarkonia are calculated by solving the Schrödinger equations with Coulomb plus confining potentials. Statistical properties of the obtained spectra are studied by plotting nearest level spacing distribution histograms.

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I. INTRODUCTION

Level statistics, i.e. the statistical theory describing the fluctuation properties of the quantum spectra was originally developed in nuclear physics in early fifties. Pioneering work dates back to Wigner [1] who gave the first detailed theoretical analysis of the nuclear spectra from the viewpoint of random matrix theory. Further developments of this subject were carried out by Porter [2], Mehta [3] and Dyson [4]. In particular, it has been established that the spectra of the complex quantum many-body systems have a behaviour similar to that of the random matrices [5].

Later the development of the statistical spectroscopy was stimulated by quantum chaology which arose in early eighties as a quantum theory of classical chaotic systems. It was established that quantized spectra of classical chaotic system spectra have the same statistical properties as those of the random matrices. It is well known, that the classical chaotic dynamics is characterised by an exponential divergence of the neighboring trajectories in phase-space. How far the complexity of the classical motion is reflected in the wave dynamics of the corresponding quantum system (described by a linear wave equation) is an interesting aspect of the study of deterministic non-linear dynamics. Numerical evidence has shown that the statistical properties of the energy spectra of the quantum systems depend crucially on the degree of chaos in the underlying classical dynamics. Namely, the spectrum of the quantum operator, describing classically non-integrable systems, has the same statistical properties as those of random matrices. It has been established that certain distribution functions and fluctuation measures for a wide variety of complex nuclear spectra follow a universal behavior and that moreover they agree very well with the results obtained by ensemble averaging over certain matrix ensembles with randomly distributed matrix elements. In particular, for systems with time-reversal symmetry, good agreement was found with a Gaussian orthogonal ensemble (GOE) [6,7]. It has also been established, that for integrable systems the nearest neighbor spacing distribution is dominated by level clustering and is expected to be Poisson-like. Later, this has been confirmed in a number of studies of model systems like billiard balls, simple rotators etc. [6,8,9] as well as in realistic systems such as hydrogen atom in a magnetic field [10,11]. Details of the progress of level statistics and random matrix theory are given in a recent review [12].

Here we deal with the spectral statistics of the quark-antiquark systems (quarkonia). Quarkonium spectroscopy has been the subject of extensive experimental as well as theoretical studies [13-17]. Theoretical description of quarkonium properties can be done quantum mechanically as well as within the QCD approach. Quantum mechanical description of hadrons is reduced to solving wave equations with Coulomb plus a confining potential. Considerable progress has been made in the calculation of the spectra of quarkonium [14-16] and baryons [17]. At present a large amount of theoretical and experimental data is available. Recently energy fluctuations and quantum chaos in hadrons and QCD has become a subject of theoretical studies [18-24]. In particular, it has been found that QCD is governed by quantum chaos in both confinement and deconfinement phases [18-22]. The statistical analysis of the measured meson and baryon spectra shows that there is quantum chaos phenomenon in these systems [20]. The study of the charmonium spectral statistics and its dependence on color screening has established quantum chaotic behaviour of this system [24]. It was claimed that such a behaviour could be the reason for J/Ψ suppression [24].

In the present work we concentrate our attention on level fluctuations and quantum chaos in heavy quarkonia due to the confining forces. It is well known that many realistic and model confined dynamical systems such as an atom in a magnetic field, particle motion in resonators, quantum dots, billiards exhibit chaotic dynamics. Quarks in the hadrons are also confined systems, and to some extent are equivalent to motion in billiards (domain with hard walls where elastic scattering of the particle will occur) [23,24]. Depending on the shape of the billiard domain the motion can be regular or chaotic [24]. If one uses for quark-antiquark potential Coulomb plus harmonic oscillator potential, the sys-
system becomes equivalent to the atom in a magnetic field, which is also described by a Coulomb plus harmonic oscillator potential whose dynamics is chaotic in classical and quantum cases \[10,11\]. An analysis of the quarkonium would suggest that the dynamics of such a system could be chaotic in classical as well as in quantum cases. Therefore the present paper may be viewed as an attempt to use well-known technique to the quarkonium systems to establish their chaotic behaviour.

Charmonium and bottomonium spectral statistics based on the solution of the Schrödinger equation for this system is considered. Two types of potentials for quark-antiquark interaction are considered: Coulomb plus linear and Coulomb plus harmonic oscillator potentials. Solving the Schrödinger equation analytically using the scaling variational method we obtain the mass spectra of charmonium and bottomonium (for simplicity we consider quarks with the same energy levels of the quarkonium system. This paper is based on the solution of the Schrödinger equation for 

\[ H_0 \Psi_{nl} = \varepsilon_{nl} \Psi_{nl}, \]

satisfying the the normalization condition

\[ < \phi_{nl}|\phi_{nl'} > = \delta_{nn'}\delta_{ll'}. \]

and the eigenvalues are given as

\[ \varepsilon_{nl} = \frac{Z^2}{2n^2}. \]

Then the eigenvalues of eq. (1) can be calculated as

\[ E_{nl} = \varepsilon_{nl}(b_0), \]

where

\[ \varepsilon_{nl}(b) = b^3 < \phi(b, r)|H_0 + H'|\phi(b, r) >, \]

and \( b_0 \) is found by minimizing this energy, i.e.

\[ \frac{\partial \varepsilon}{\partial b} = 0. \] (7)

The quantity \( \varepsilon_{nl}(b) \) may be calculated as:

\[ \varepsilon_{nl}(b) = n^{-2} \left( \frac{b^2}{2} - Zb - \frac{q^{b-k}}{k} \right), \]

where

\[ q = k\lambda n^2 < \phi_{nl}|r^k|\phi_{nl} >. \]

These matrix elements are calculated analytically \[31\].

For \( k = 1 \) we have

\[ q = \lambda n^2 \frac{1}{2} (3n^2 - l(l + 1)). \]

where \( n \) and \( l \) are the principal and orbital quantum numbers, respectively. Then the extremum condition gives

\[ b_0 = \frac{Z}{3} + \frac{2^{\frac{1}{2}}Z^2}{3(27q + 2Z^3 + 3q(27q + 4Z^3))^{\frac{1}{2}}} + \frac{3(27q + 2Z^3 + 3q(27q + 4Z^3))^{\frac{1}{2}}}{2^{\frac{3}{2}}} \]

For \( k = 2 \) we have

\[ q = 5n^2 + 1 - 3l(l + 1). \]

The extremum condition \[17\] leads to an equation for defining \( b_0 \)

\[ b_0^4 - Zb_0^3 - q = 0. \]

Solving this equation we get \( b_0 \) that leads to the energy eigenvalues for the case \( k = 2 \). Thus the solutions of the Schrödinger equation for Coulomb plus confining potentials are obtained analytically using the scaling variational method.
III. SPECTRAL STATISTICS

Now an analysis of the statistical properties of the spectra is to be considered. One of the main characteristics of the statistical properties of the spectra is the level spacing distribution \( P(S) \) function. We calculate the nearest-neighbor level-spacing distribution for quarkonium spectra. The nearest neighbor level spacings are defined as \( S_i = E_{i+1} - E_i \), where \( E_i \) are the energies of the unfolded levels, which are obtained by the following way: The spectrum \( \{E_i\} \) is separated into smoothed average part and fluctuating parts. Then the number of the levels below \( E \) is counted and the following staircase function is defined:

\[
N(E) = N_{av}(E) + N_{fluct}(E).
\]

The unfolded spectrum is finally obtained with the mapping

\[
\tilde{E}_i = N_{av}(E_i).
\]

Then the nearest level spacing distribution function \( P(S) \) is defined as the probability of \( S \) lying within the infinitesimal interval \([S, S + dS] \).

For the quantum systems which are chaotic in the classical limit this distribution function is the same as that of the random matrices \([8, 12]\). For systems which are regular in the classical limit its behaviour is close to a Poissonian distribution function. This distribution is usually taken to be a Gaussian with a parameter \( d \):

\[
P(H) \sim \exp(-Tr\{HH^+\}/2d^2),
\]

and the random matrix ensemble corresponding to this distribution is called the Gaussian ensemble. For Hamiltonians invariant under rotational and time-reversal transformations the corresponding ensemble of matrices is called Gaussian orthogonal ensemble (GOE). It was established \([8, 8, 32]\) that GOE describes the statistical fluctuation properties of a quantum system whose classical analog is completely chaotic. The nearest neighbor level spacing distribution for GOE is described by Wigner distribution:

\[
P(S) = \frac{1}{2} \pi S \exp\left(-\frac{1}{4} \pi S^2\right).
\]

(8)

The common way to study of the level statistics is to compare the calculated nearest-neighbor level-spacing distribution histogram with the Wigner distribution.

For systems whose classical motion is neither regular nor fully chaotic (mixed dynamics) the level spacing distribution will be intermediate between the Poisson and GOE limits. Several empirical functional forms for the distribution were suggested for this case \([12]\). If the Hamiltonian is not time-reversal invariant, irrespective of its behavior under rotations, the Hamiltonian matrices are complex Hermitian and the corresponding ensemble is called Gaussian unitary ensemble (GUE). If the Hamiltonian of the systems is time-reversal invariant but not invariant under rotations, then the corresponding ensemble is called the Gaussian symplectic ensemble. Quarkonium system is time reversal and rotational invariant. Therefore in the case of chaotic motion its nearest-neighbor level spacing distribution is expected to be GOE-type. Now we carry out a detailed analysis of the spectra to establish the nearest-neighbor level spacing distribution function to determine the existence of quantum chaos in our system.

The following values of the \( c \) and \( b \) quark masses and potential parameters are chosen in these calculations:

For charmonium: \( m_c = 1.486 GeV \), \( \alpha_s = 0.32 \), \( \lambda = 0.2 GeV^2, 0.4 GeV^2, 0.6 GeV^2 \).

For bottomonium: \( m_b = 4.88 GeV \), \( \alpha_s = 0.22 \), \( \lambda = 0.4 GeV^2, 0.6 GeV^2, 0.8 GeV^2 \). First 1000 eigenvalues are used to plot each histogram. In our calculations we vary only the principal quantum number \( n \), keeping the orbital quantum number \( l \) constant.

Figs. 1 and 2 show the level spacing distribution for charmonium and bottomonium, respectively for Coulomb plus linear potential (\( k = 1 \) case, and in Figs. 3 and 4 the level spacing distributions are plotted for Coulomb plus harmonic oscillator potential (\( k = 2 \) case, at various values of the confining potential parameter. It is clear that for the smaller values of the parameter \( \lambda \) this distribution is close to GOE, that means the quantum system is becoming chaotic. It allows us to estimate the values of \( \lambda \) where a transition from regular to chaotic behaviour will occur. Furthermore we find that for the same values of \( \lambda \) the bottomonium motion is more regular than that of the charmonium. Comparison of the level spacing distribution for \( cc \) and \( bb \) systems shows us that the transition from regular to chaotic motion for these systems occurs at different values of \( \lambda \). So, for \( cc \) system (Figs.1 and 2) we have GOE-type distributions at \( \lambda = 0.2 GeV^2 \) and \( \lambda = 0.4 GeV^2 \) and Poissonian distribution at \( \lambda = 0.6 GeV^2 \), while the motion of \( bb \) system is still chaotic at \( \lambda = 0.6 GeV^2 \) (Figs. 3 and 4). At \( \lambda = 0.4 GeV^2 \) and \( \lambda = 0.8 GeV^2 \) the level spacing distribution for bottomonium is GOE and Poisson type, respectively. Thus it may be concluded from this comparison that the motion of \( bb \) system is more chaotic than that of the \( cc \) system.

Indeed, if one writes the Schrödinger equations for charmonium and bottomonium in the units \( m_q = \hbar = 1 \), the value of the confining potential for bottomonium is much smaller than that of charmonium. The chaotic nature of the motion in the quarkonium potential can be also understood from the following fact: formally the Hamiltonian for Coulomb plus harmonic oscillator potential is equivalent to the Hamiltonian of an atom in an uniform magnetic field, which is a nonintegrable system. It is well known, that the Hamiltonian of an atom in an uniform magnetic field can be reduced to the Hamilto-
nian of coupled nonlinear oscillators [10], which exhibits chaotic dynamics. With this analogy the change in the confining potential parameters to some extent is equivalent to the change of the shape of billiard domain.

IV. CONCLUSIONS

Summarizing, in this work we have applied the technique well-known in nuclear physics, atomic physics and other areas of physics to treat quantum chaos in heavy quarkonia, using for their description the Schrödinger equation with Coulomb plus confining potentials, a non-integrable system. It should be emphasized that this system is quite similar to the case of an atom in a constant magnetic field. It is well known that such a system is non-integrable and exhibits chaotic dynamics in both classical as well as quantum cases [11]. The analysis of the nearest level spacing distributions of these systems shows, that for some smaller values of the confining potential parameter the motion of the heavy quarkonium is chaotic while for larger values of $\lambda$ the level spacing distribution is a Poisson type, that implies that the motion is becoming regular. Previously [24] the quantum chaos is related to the color screening parameter, while the present work treats the transition from regular to a chaotic motion related to a confining potential parameters. The quarkonia can exhibit regular as well as chaotic dynamics depending on the value of the confining force. Study of the level statistics and quantum chaos in hadrons could play the same role in particle physics as it did in the case of nuclear physics. Therefore quantum chaosology of hadrons could become one of the exciting topics in particle physics. More comprehensive analysis of the statistical properties of the heavy quarkonia spectra performed by solving the Dirac equation with Coulomb plus a confining potential is needed. This would establish the role played by relativistic effects in the approach to a level-spacing distribution.

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