UNIFORM GROWTH, ACTIONS ON TREES AND $GL_2$

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1. Exponential Growth

Choose a finite generating set $S = \{s_1, \cdots, s_p\}$ for the group $\Gamma$; define the $S$-length of an element as $\lambda_S(g) = \min \{ n \mid g = s_1 \cdots s_n, s_i \in S \cup S^{-1} \}$. The growth function $\beta_n(S, \Gamma) = |\{ g \mid \lambda_S(g) \leq n \}|$ depends on the chosen generating set. A group has exponential growth if the growth rate, $\beta(S, \Gamma) = \lim_{n \to \infty} \beta_n(S, \Gamma)^{\frac{1}{n}}$ is strictly greater than 1.

In fact, for another finite generating set $T = \{t_1, \cdots, t_q\}$ for $\Gamma$, if both $\max_j \lambda_S(t_j) \leq L$ and $\max_i \lambda_T(s_i) \leq L$, then $\beta_n(S, \Gamma) \leq \beta_n(L, \Gamma)$ and also the symmetric inequality. It then follows that $\beta(S, \Gamma)^L \leq \beta(T, \Gamma)$ and $\beta(T, \Gamma)^L \leq \beta(S, \Gamma)$. Using these remarks, Milnor showed that exponential growth is independent of the generating set.

For a group with exponential growth we consider

$$\beta(\Gamma) = \inf_S \beta(S, \Gamma).$$

If $\beta(\Gamma) > 1$ then $\Gamma$ is said to have uniform exponential growth.

Gromov has asked if there is a group of exponential growth which is not of uniform exponential growth. Indications are that such a group will be hard to find. Recently, [1, 9], it has been shown that all solvable groups which have exponential growth are of uniform exponential growth.

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2. Generalities

The following uses the fact that given a generating set $S$ for $\Gamma$ the set of elements of the subgroup $\mathcal{H}$, a subgroup of index $d$, which are words in $S$ of length at most $2d - 1$ give a generating set for $\mathcal{H}$.

**Proposition 2.1** ([10]). If a group $\Gamma$ has a subgroup $\mathcal{H}$ of finite index $d$, then $\beta(\Gamma) \geq \beta(\mathcal{H})^{2d-1}$.

The next proposition is elementary, but also a very useful result.

**Proposition 2.2.** If a group $\Gamma$ has homomorphic image which is of uniform exponential growth then $\Gamma$ has uniform exponential growth.

A group is called large if it has a homomorphism onto a free non-abelian group. It is important to realize that a free group of rank $n$ has uniform exponential growth of rate $\beta = 2n - 1$.

**Corollary 2.3.** If the group $\Gamma$ is virtually large then $\Gamma$ has uniform exponential growth.

There has been some interesting recent work on linear groups which are large, [7], [8], in a sense building on some ideas from [11].

We say that a group $\Gamma$ has the UF-property (uniformly contains a free nonabelian semigroup) if there is a constant $n_\Gamma \geq 1$ such that for every generating set $S$ of $\Gamma$ there exist two elements (depending on $S$) in $\Gamma$ of word length $\leq n_\Gamma$ and freely generating a free semigroup of rank 2.

**Proposition 2.4.** If a group $\Gamma$ has the UF-property then $\Gamma$ is of uniform exponential growth.

**Proof.** Let $S$ be an arbitrary finite generating set for $\Gamma$ and let $S_0 = \{g, h\}$ be the pair of words of $S$ – length less than or equal to $n_\Gamma$ and freely generating a free semigroup $\Gamma_0$ of rank 2. We have

$$\beta(S, \Gamma)^{nr} = \beta(S \cup S_0, \Gamma) \geq \beta(S_0, \Gamma_0) \geq 2$$

and the proof is complete. ■
3. Action on Trees

We use a variant on the usual ping-pong lemma to obtain free semigroups.

**Lemma 3.1** ([4]). Let $\Gamma_0$ be a group of isometries of a tree $X$ with the generating set $\{g_1, g_2\}$ where $g_1, g_2$ are hyperbolic isometries with distinct axes $A_1, A_2$. Then one of the four pairs $\{g_1^{\pm 1}, g_2^{\pm 1}\}$ freely generate a free semigroup of rank two.

**Proof.** If $A_1, A_2$ are disjoint then let $[a_1, a_2]$ be the shortest geodesic joining them. Let $A_1^+, A_2^+$ be the rays, starting at $a_1, a_2$, such that $g_1, g_2$ translate towards the ends of these rays. Let $R_1^+, R_2^+$ be the rays $A_1^+, A_2^+$ with the first unit length segments removed. Let $p_1, p_2$ be the geodesic projection maps of $X$ onto $R_1^+, R_2^+$ respectively. Set $X_i = p_i^{-1}(R_i^+)$, $i = 1, 2$. Clearly $X_1 \cap X_2 = \emptyset$ and $g_i^n(X_1 \cup X_2) \subset X_i$, $n \geq 1, i = 1, 2$. We assert now that $g_1, g_2$ freely generate a free semigroup of rank 2. Indeed let $w_1, w_2$ be (positive) words in $g_1, g_2$ and suppose $w_1 = w_2$ in $\Gamma_0$. If the words start with the same letter, the final segments are also equal in $\Gamma_0$; hence by induction we get that $w_1 = w_2$ as formal words. Thus, we may assume that the words $w_1, w_2$ start with different letters, say $g_1, g_2$ respectively. Then the image of $X_1 \cup X_2$ under $w_1, w_2$ belongs to $X_1, X_2$ respectively; hence a contradiction.  

**Theorem 3.2.** Suppose that $\Gamma$ is a finitely generated subgroup of $\text{GL}_2(K)$ over the field $K$.

(char $\neq 0$) If $K$ has nonzero characteristic and $\Gamma$ has exponential growth then $\Gamma$ satisfies the UF-property and consequently has uniform exponential growth.

(char $= 0$) If $K$ has characteristic zero and $\Gamma$ has exponential growth then either $\Gamma$ has uniform exponential growth or $\Gamma$ is conjugate to a subgroup of $\text{GL}_2(\mathcal{O})$, for $\mathcal{O}$ a ring of integers in an algebraic number field.

**Proof.** Since $\Gamma$ is finitely generated we may assume that $K$ is finitely generated. For any discrete rank one valuation $v$ of $K$ let $X_v$ be the corresponding Bruhat-Tits tree. We consider the action of $\Gamma$ (or a
subgroup of index 2, acting without inversions) on each of the Bruhat-Tits trees $X_v$ for each of these valuations and split the proof of the theorem into cases.

1. For any valuation $v$, $\Gamma$ has a fixed point on $X_v$. In this case, the ring $A$ generated by the traces of elements of $\Gamma$ lies in every valuation ring $A_v$ of $K$. In non-zero characteristic this ring is finite, [11] I.6.2, generated over the prime field by roots of unity, so in fact there are only finitely many traces and it follows that $\Gamma$ is finite, [12] 1.20, (hence of eventually constant growth), if the group acts irreducibly, or it is solvable if the group acts reducibly and hence also of uniform exponential growth, [9]. In characteristic zero, the ring of traces is the ring of algebraic integers $O$ in the algebraic closure of $\mathbb{Q}$ in $K$; as in [3], one can find a suitable module so that the group is conjugated into $\text{GL}_2(O)$, for a somewhat larger ring of integers $\overline{O}$.

2. There is a valuation such that $\Gamma$ has an invariant line on $X_v$. Then by [11] II.1.3, $\Gamma$ is contained in a Cartan subgroup, which is an extension of diagonal subgroup by cyclic of order 2. Hence $\Gamma$ is virtually abelian and has polynomial growth.

3. For some valuation $v$ there is neither a fixed point nor an invariant line on the Bruhat-Tits tree $X_v$. We prove that the image of $\Gamma$ (that is modulo scalar matrices) satisfies the UF-property with a constant $n_{\Gamma} = 6$ and hence so does the original group.

Let $S$ be a finite generating set of $\Gamma$. The subcases are as follows.

(a) $S$ contains a hyperbolic isometry, say $g$ and any $s \in S$ leaves the axis $A_g$ invariant. Then we are in case 2.

(b) $S$ contains a hyperbolic isometry, say $g$ and there is $s \in S$ such that $sA_g \neq A_g$. The isometry $h = sg s^{-1}$ is hyperbolic with the axis $sA_g \neq A_g$, hence by Lemma 3.1 one of the four pairs $\{g^{\pm 1}, h^{\pm 1}\}$ freely generate a free semigroup of rank 2 and the length of $g^{\pm 1}$ and $h^{\pm 1}$ is at most 3.

(c) $S$ does not contain a hyperbolic isometry, that is all isometries are elliptic, thus any $s \in S$ has a fixed vertex in $X_v$. Then there exists $g \in S \cup S^2$ which is hyperbolic, [11], I.6.4. Again we may assume there is $s \in S \cup S^2$ such that $sA_g \neq A_g$. The argument used for the previous case shows now that one of the four pairs $\{g^{\pm 1}, sg^{\pm 1}s^{-1}\}$ freely generate a free semigroup of rank 2 and the length of $g^{\pm 1}$, respectively, $sg^{\pm 1}s^{-1}$ is at most 6 and the proof is complete. ■
4. Incidentals

In the characteristic zero case, we have shown that the finitely generated group $\Gamma$ of exponential growth is of uniform exponential growth or it is conjugate to a subgroup of $\text{GL}_2(\mathcal{O})$, for $\mathcal{O}$ a ring of integers in an algebraic number field. This however is not a dichotomy. Certainly, there are many cases of discrete subgroups of $\text{GL}_2(\mathbb{C})$ which are of uniform exponential growth; these arise as the fundamental group of a hyperbolic manifold [6], [10].

For further analysis we consider the case when every element of $\Gamma$ is elliptic (considered as a group of Mobius transformations). By dint of a theorem of Lyndon-Ullman [3] the group is conjugate (via stereographic projection) to a group of rotations of the sphere. However, for a linear group which has compact closure, every eigenvalue of every element must be on the unit circle. If we are also in the case where $\Gamma$ is a subgroup of $\text{GL}_2(\mathcal{O})$, then under every embedding $\sigma : \mathcal{O} \rightarrow \mathbb{C}$ the above property of eigenvalues holds for $\sigma(\Gamma)$, since ellipticity is preserved. Hence every eigenvalue is a root of unity by Dirchlet’s Theorem. By passing to a subgroup of finite index (using compactness) we can assume the determinant of the matrices in this group are all unity. Thus the trace of the matrices are in fact a sum of a root of unity and its complex conjugate. But now there are only finitely many traces since the fraction field of $\mathcal{O}$ has finite dimension over $\mathbb{Q}$ and only finitely many roots of unity can belong to a number field of given degree. It now follows [2], that $\Gamma$ is finite since we can pass to a subgroup of finite index missing these finitely many possible traces $\neq 2$. But this subgroup of finite index has elements with all eigenvalues which are 1; thus every element is unipotent and so must trivial by compactness.

We summarize these remarks in the following statement.

**Corollary 4.1.** Suppose that $\Gamma$ is a finitely generated subgroup of $\text{GL}_2(\mathbb{C})$ of exponential growth and not of uniform exponential growth then $\Gamma$ can not consist entirely of elliptic elements.

5. Related Open Problems

**Question 1.** Prove uniform exponential growth for linear groups of exponential growth. For example show uniform exponential growth for $\text{SL}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of a real quadratic number field.
Question 2. Prove that uniform exponential growth is a quasi-isometry invariant.

Question 3. Classify or otherwise characterize the groups which have the property that every generating set has two elements which generate a free semigroup. (if torsion-free are they of finite cohomological dimension?)

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