STRONG CONVERGENCE OF FULL-DISCRETE NONLINEARITY-TRUNCATED ACCELERATED EXPONENTIAL EULER-TYPE APPROXIMATIONS FOR STOCHASTIC KURAMOTO-SIVASHINSKY EQUATIONS

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Abstract. This article introduces and analyzes a new explicit, easily implementable, and full-discrete accelerated exponential Euler-type approximation scheme for additive space-time white noise driven stochastic partial differential equations (SPDEs) with possibly non-globally monotone nonlinearities such as stochastic Kuramoto-Sivashinsky equations. The main result of this article proves that the proposed approximation scheme converges strongly and numerically weakly to the solution process of such an SPDE. Key ingredients in the proof of our convergence result are a suitable generalized coercivity-type condition, the specific design of the accelerated exponential Euler-type approximation scheme, and an application of Fernique’s theorem.

Keywords. Stochastic differential equation; strong convergence; numerical approximation; stochastic Kuramoto-Sivashinsky equations; coercivity-type condition; accelerated exponential Euler approximations.

AMS subject classifications. 60H35; 65C30.

1. Introduction

For strong $L^2$-convergence of a sequence of approximations to a square integrable limit, it is necessary that the $L^2$-norms of the approximations are uniformly bounded. In the case of finite-dimensional stochastic differential equations (SDEs), this can be achieved by the well-known coercivity-type condition that there exists $c \in \mathbb{R}$ such that for all $x \in \mathbb{R}^d$ it holds that

$$\langle x, \mu(x) \rangle_{\mathbb{R}^d} + \frac{1}{2} \|\sigma(x)\|_{\text{HS}(d,d)}^2 \leq c(1 + \|x\|_{\mathbb{R}^d}^2),$$

where $\mu : \mathbb{R}^d \to \mathbb{R}^d$ is the drift coefficient of the SDE under consideration, where $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is the diffusion coefficient of the SDE under consideration, and where $d \in \mathbb{N}$ is the dimension of the SDE under consideration. In the case of an infinite-dimensional separable Hilbert space $H$, this coercivity-type condition, in particular, forces the diffusion coefficient to satisfy that $\|\sigma(0)\|_{\text{HS}(H,H)} < \infty$. So, the infinite-dimensional version of the well-known coercivity-type condition (1.1) is not satisfied in the important case of additive space-time white noise where the diffusion coefficient is constantly equal to the identity operator or a non-zero multiple hereof (note for every $d \in \mathbb{N}$ that the Hilbert-Schmidt norm of the identity operator $I_{\mathbb{R}^d}$ on the $\mathbb{R}^d$ is equal to $\|I_{\mathbb{R}^d}\|_{\text{HS}(\mathbb{R}^d,\mathbb{R}^d)} = \sqrt{d}$). This is one central reason why almost all temporal strong convergence results in the literature (see the discussion in the next paragraph) apply only to trace-class noise. In particular, to the best of our knowledge, there exists no strong approximation result for stochastic Kuramoto-Sivashinsky (K-S) equations with space-time white noise in the

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scientific literature. The key contribution of this work is to impose an appropriately
generalized coercivity-type condition in which the coercivity constant (that is, the constant $c \in \mathbb{R}$ in (1.1)) may depend on the noise process (cf. (1.3), (1.8), Theorem 4.1, and Corollary 5.2 below) and to introduce a suitable new explicit approximation scheme which is, roughly speaking, designed in a way so that it respects this generalized coercivity-type condition (see (1.8)-(1.10) and Proposition 2.1 below). This new coercivity-type condition allows us to analyse a number of additive space-time white noise driven stochastic evolution equations (SEEs) with superlinearly growing nonlinearities which could not be handled before. In particular, it enables us to prove strong convergence of the proposed scheme in the case of stochastic K-S equations (see Theorem 4.1 and Corollary 5.2 below). The analysis of further SEEs is subject to future research.

Next, we review the literature on strongly converging approximations of additive noise-driven SEEs with superlinearly growing nonlinearities. It was shown that the explicit Euler scheme and the linear-implicit Euler scheme do, in general, not converge strongly and numerically weakly in the case of such SEEs; cf., e.g., Theorem 2.1 in [22], Theorem 2.1 in [24], and Section 5.1 in Kurniawan [32]. Fully drift-implicit Euler methods, by contrast, converge strongly for some SEEs with superlinearly growing nonlinearities; see, e.g., Theorem 2.4 in Hu [18], Theorem 2.10 in Gyöngy & Millet [13], Theorem 7.1 in Brzeźniak [5], and Theorem 1.1 in Kovács et al. [31]. However, to implement these methods, a nonlinear equation has to be solved in each time step approximatively and this results in an additional computational cost (especially, when the state space of the considered SEE is high dimensional, see, e.g., Figure 4 in [23]). Moreover, it is not yet known whether such approximate implementations of fully drift-implicit Euler schemes do converge strongly. Recently, a series of appropriately modified versions of the explicit Euler scheme have been proposed and shown to converge strongly for some SEEs with superlinearly growing nonlinearities; cf., e.g., Hutzenthaler et al. [23], Wang & Gan [42], Hutzenthaler & Jentzen [21], Tretyakov & Zhang [41], and Sabanis [37,38] in the case of finite dimensional SEEs and cf., e.g., Gyöngy et al. [14], Kurniawan [32], Jentzen & Pušnik [29], and Becker & Jentzen [2] in the case of infinite dimensional SEEs. These methods are explicit, easily realizable, and somehow tame/truncate superlinearly growing nonlinearities to prevent from strong divergence. However, except of Becker & Jentzen [2], each of the above mentioned temporal strong convergence results for implicit (see, e.g., [5,13,18,31]) or explicit (see, e.g., [2,14,21,23,29,32,37,38,41,42]) schemes applies merely to trace-class noise driven SEEs and excludes the important case of the more irregular space-time white noise. In Becker & Jentzen [2] a coercivity/Lyapunov-type condition has been imposed and used to establish strong convergence rates in the case of stochastic Ginzburg-Landau equations with additive space-time white noise; cf. (85) in [2], Lemma 6.2 in [2], and Corollaries 6.16–6.17 in [2]. However, the machinery in [2] does not exploit the powerful negativity of the linear operator (cf. (85) in [2] with (1.8) below where the $H_{1/2}$-norm appears on the right-hand side) and thereby applies merely to stochastic Ginzburg-Landau equations but excludes most of the challenging additive space-time white noise driven SEEs with superlinearly growing nonlinearities such as stochastic K-S equations. To be more specific, [2] employs, roughly speaking, a coercivity-type condition of the form that for a given Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ there exist a real number $c \in \mathbb{R}$ and a suitable measurable function $\Phi : H \to [0, \infty)$ such that for all $v, w \in H$ it holds that

$$\langle v, F(v + w) \rangle_H \leq c\|v\|_H^2 + \Phi(w),$$

where $F$ is the nonlinearity of the SEE under consideration (cf., e.g., in [2, (85) and Lemma 6.2]). In this work, we employ a coercivity-type condition of the form that there
exist suitable measurable functions $\phi, \Phi: D(A) \to [0, \infty)$ and a real number $\varphi \in [0,1)$ such that for all $v, w \in D(A)$ it holds that
\[ \langle v, \varphi A v + F(v+w) \rangle_H \leq \phi(w) \|v\|_H^2 + \Phi(w), \tag{1.3} \]
where $A$ is the dominant linear operator of the SEE under consideration. In the case of stochastic K-S equations the linear operator $A$ is the square of the Laplace operator with suitable boundary conditions plus appropriate lower order terms. This operator is highly negative and, thereby, helps us to control the $F(\cdot)$-term in the inner product in (1.3). In particular, the coefficients of the stochastic K-S equations fail to satisfy (1.2) but they do satisfy (1.3) (see Lemma 5.2 below).

In the following, we illustrate the main result of this article (see Theorem 4.1 in Section 4 below) by means of an application of this result in the case of stochastic K-S equations (see Corollary 5.2 in Section 5 below). More formally, let $H^1_p((0,1),\mathbb{R})$ be the Sobolev space of weakly differentiable periodic functions with square integrable derivatives, let $T \in (0,\infty)$, $\xi \in H^1_p((0,1),\mathbb{R})$, $H = L^2((0,1),\mathbb{R})$, let $F: L^4((0,1);\mathbb{R}) \to H^{-1}((0,1),\mathbb{R})$ be the function which satisfies for all $v \in L^4((0,1);\mathbb{R})$ that $F(v) = v - 1/2(v^2)'$, let $A: D(A) \subseteq H \to H$ be the Laplacian with periodic boundary conditions on $H$, let $A: D(A) \subseteq H \to H$ be the linear operator which satisfies for all $v \in D(A)$ that $D(A) = D(A^2)$ and $Av = -A^2v - Av - v$, let $B \in L(\mathcal{T},H^{-1}((0,1),\mathbb{R}))$ be the linear operator with the property that for all $v \in H$ it holds that $Bv = v'$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, and let $(W_t)_{t \in [0,T]}$ be an $\mathbb{R}^n$-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process. The above assumptions ensure that there exists an up to indistinguishability unique $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic process $X: [0,T] \times \Omega \to L^4((0,1);\mathbb{R})$ with continuous sample paths which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that
\[ X_t = e^{-tA} \xi + \int_0^t e^{(t-s)A} F(X_s) \, ds + \int_0^t e^{(t-s)A} B \, dW_s \tag{1.4} \]
(cf., e.g., Duan & Ervin [10]). The stochastic process $X$ is thus a mild solution of the stochastic K-S equation
\[ \frac{\partial}{\partial t} X_t(x) = -\frac{\partial^4}{\partial x^4} X_t(x) - \frac{\partial^2}{\partial x^2} X_t(x) - X_t(x) \cdot \frac{\partial}{\partial x} X_t(x) + \frac{\partial^2}{\partial x^2} W_t(x) \tag{1.5} \]
with $X_t(0) = X_t(1)$, $X'_t(0) = X'_t(1)$, $X''_t(0) = X''_t(1)$, $X'''_t(0) = X'''_t(1)$, and $X_0(x) = \xi(x)$ for $x \in (0,1)$, $t \in [0,T]$. Note that the noise in (1.4) and (1.5) is quite rough in the sense that $\frac{\partial^2}{\partial x^2} W_t(x)$, $x \in (0,1)$, $t \in [0,T]$, is the distributional space derivative of the space-time white noise $\frac{\partial}{\partial t} W_t(x)$, $x \in (0,1)$, $t \in [0,T]$. In this article we introduce the following nonlinearly-truncated accelerated exponential Euler-type scheme to approximate the solution process $X$ of the SPDE (1.5). Let $(\epsilon_n)_{n \in \mathbb{N}} \subseteq H$, $(\rho_n)_{n \in \mathbb{N}} \subseteq L(H)$, $(\chi_n)_{n \in \mathbb{N}} \subseteq (0,\infty)$, $\varphi \in (1/6,3/2]$, $\chi \in (0,\varphi/2 - 1/32]$ satisfy for all $n \in \mathbb{N}$, $v \in H$ that $\epsilon_0 = 1$, $\epsilon_n(\cdot) = \sqrt{2} \cos(2n\pi(\cdot))$, $e_n(\cdot) = \sqrt{2} \sin(2n\pi(\cdot))$, $P_n(v) = \sum_{k=-n}^{n} \langle e_k, v \rangle_H e_k$, $\limsup_{k \to \infty} h_k = 0$, let $[\cdot]_H: \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, be the mappings which satisfy for all $h \in (0,\infty)$, $t \in \mathbb{R}$ that $[t]_H = \max((\infty, t] \cap \{0, h-h, 2h-h, 2h, 3h-h, \ldots\})$, and let $O^n: [0,T] \times \Omega \to P_n(H)$, $n \in \mathbb{N}$, and $X^n: [0,T] \times \Omega \to P_n(H)$, $n \in \mathbb{N}$, be stochastic processes which satisfy that for all $n \in \mathbb{N}$, $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $O^n_t = \int_0^t P_n e^{(t-s)A} B \, dW_s$ and
\[ X^n_t = P_n e^{-tA} \xi + O^n_t \]
\[ + \int_0^t P_n e^{(t-s)A} 1_{\{\|(-A)^{\varphi} X^n_{\cdot[H]}(\cdot)\|_H + \|(-A)^{\varphi} [O^n_{s[H]} + P_n e^{(s-H)A} \xi]\|_H \leq h_n s \}} F(X^n_{s[H]}(\cdot)) \, ds. \tag{1.6} \]
In Corollary 5.2 in Section 5 below, we demonstrate that the approximation scheme (1.6) converges strongly to the solution of the SPDE (1.5). More precisely, Corollary 5.2 (with 
\[ \beta = \frac{3}{16}, \eta = \kappa = 1, \theta = \varrho, b_k = 0, b_k = 2k\pi, \chi^n = \chi^\alpha, O^n = O^\alpha, X = X \text{ for } k \in \mathbb{Z}, n \in \mathbb{N} \] 
in the notation of Corollary 5.2) proves that for all \( p \in (0, \infty) \) it holds that
\[
\limsup_{n \to \infty} \sup_{t \in [0, T]} \mathbb{E} \left[ ||X_t - \chi^n_t||^p_H \right] = 0. \tag{1.7}
\]
Corollary 5.2 follows from an application of Theorem 4.1 below, which is the main result of this paper. Theorem 4.1 establishes strong convergence for a more general class of SPDEs as well as for a more general type of approximation schemes.

We now add a few comments on the approximation scheme (1.6) and on key ideas in the proof of Corollary 5.2 and Theorem 4.1, respectively. First, we note that the approximation scheme (1.6) does not temporally discretize the semigroup \((e^{tA})_{t \in [0, \infty)}\) appearing in (1.4) and is thus an appropriate modification of the accelerated exponential Euler scheme in Section 3 in Jentzen & Kloeden [26] (cf., e.g., also Section 4 in Jentzen & Kloeden [25] for an overview and e.g., Lord & Tanneum [33] and Wang & Qi [43] for further results on accelerated exponential Euler approximations). This lack of discretization of the semigroup in the stochastic integral (1.4) has been proposed in Jentzen & Kloeden [26] to obtain an approximation scheme which converges under suitable assumptions with a significant higher convergence rate than previously analyzed approximation schemes such as the linear implicit Euler scheme or the exponential Euler scheme (cf., e.g., Theorem 3.1 in Jentzen & Kloeden [26], Theorem 1 in [27], Theorem 3.1 in Wang & Qi [43], and Theorem 3.1 in Qi & Wang [36]). In this article, the lack of discretization of the semigroup in the non-stochastic integral in (1.4) is employed for a different purpose, that is, here this lack of discretization is used to obtain a scheme that inherits an appropriate a priori estimate from the exact solution process of the SPDE (1.5). More specifically, we observe that the nonlinearity \( F: L^4((0,1);\mathbb{R}) \to H^{-1}((0,1),\mathbb{R}) \) appearing in (1.4) satisfies that there exist suitable measurable functions \( \phi, \Phi: C([0,1],\mathbb{R}) \to [0,\infty) \) and a real number \( \varphi \in [0,1) \) such that for all \( v, w \in H_1 \) it holds that
\[
\langle v, F(v + w) \rangle_H \leq \phi(w)\|v\|_H^2 + \varphi\|v\|_{H^{1/2}}^2 + \Phi(w) \tag{1.8}
\]
(see Lemma 5.2 for the proof of (1.8) and see also the proof of Corollary 5.2 for the specific choice of \( \phi, \Phi, \) and \( \varphi \)). Inequality (1.8), in turn, ensures that for every continuous stochastic process \( O: [0,T] \times \Omega \to C([0,1],\mathbb{R}) \) with \( \forall u \in [0,T]: \mathbb{P}(O_u = \int_0^u e^{(u-s)A}BdW_s) = 1 \) and every \( t \in [0,T] \) it holds \( \mathbb{P} \)-a.s. that
\[
\|X_t\|_H \leq \|O_t\|_H + \sqrt{\int_0^t 2\phi(O_u)ds \|\xi\|_H^2 + 2 \int_0^t e^{\int_0^s 2\phi(O_u)du} \Phi(O_s)ds}. \tag{1.9}
\]
Note that (1.8) is an appropriate generalized coercivity-type condition for the SPDE under consideration (cf., e.g., Chapter 4 in Prévôt & Röckner [35]). A key contribution of this paper is to reveal that the approximation scheme (1.6) inherits (1.9) in the sense that there exists \( \theta \in (0, \infty) \) such that for all \( t \in [0,T], n \in \mathbb{N} \) it holds \( \mathbb{P} \)-a.s. that
\[
\|X^n_t\|_H \leq \|O^n_t\|_H + \sqrt{\int_0^t 2\phi(O^n_{[s]n_1})ds \|\xi\|_H^2 + 2 \int_0^t e^{\int_0^s 2\phi(O^n_{[s]n_1})du} \left[ \Phi(O^n_{[s]n_1} + \theta|h_n|^{1/\theta}) \right] ds}. \tag{1.10}
\]
(see Proposition 2.1 for the proof of (1.10)). Strong convergence results for explicit (see, e.g., [2, 14, 21, 23, 29, 32, 37, 38, 41, 42]) and implicit (see, e.g., [5, 13, 18, 31]) numerical approximation schemes for SEEs in the literature impose coercivity-type assumptions in which $\phi$ and $\Phi$ are constants (cf., e.g., Assumption (B)' in Hu [18], (C2) in Gyöngy & Millet [13], Section 1 in Hutzenthaler et al. [23], Assumption 2.1 in Wang & Gan [42], (2.11) in Hutzenthaler & Jentzen [21], Assumption 2.1 in Tretyakov & Zhang [41], Section 7 in Brzeźniak [5], (A-1) in Sabanis [37], (A-4) in Sabanis [38], Assumption 1 in Gyöngy et al. [14], (4.79) in Kurniawan [32], Section 7.4 in Jentzen & Pušnik [29], Section 3.1 in Kovács et al. [31], and (85) in Becker & Jentzen [2]). Such a coercivity-type condition is not fulfilled in the case of a number of nonlinear SPDEs such as (1.5). In particular, none of the above mentioned references applies to the stochastic K-S equation (1.5) and Theorem 4.1 and Corollary 5.2 below, respectively, are, to the best of our knowledge, the first strong approximation results for the stochastic K-S equation (1.5). We would also like to add that in the above mentioned articles on accelerated exponential Euler approximations, it was crucial to avoid the discretization of the semigroup in the stochastic integral while our analysis exploits the fact that the semigroup in the non-stochastic integral in (1.4) is not discretized but allows discretizations of the semigroup in the stochastic integral (cf. Theorem 4.1 in Section 4).

Next, we observe that the approximation scheme (1.6) can be easily realized on a computer. More formally, note that for all $n \in \mathbb{N}$, $k \in \mathbb{N} \cap (-\infty, T/n - 1]$ it holds $\mathbb{P}$-a.s. that

\[
\begin{align*}
O_{(k+1)h_n}^n &= e^{h_n A} O_{kh_n}^n + \int_{kh_n}^{(k+1)h_n} P_n e^{((k+1)h_n-s)A} B dW_s, \\
X_{(k+1)h_n}^n &= e^{h_n A} X_{kh_n}^n + O_{(k+1)h_n}^n - e^{h_n A} O_{kh_n}^n \\
&\quad + P_n A^{-1} (e^{h_n A} - \text{Id}_H) \mathbb{1}_{\{\|((A)e^{h_n A}X_{kh_n}^n\|_H + \|(A)e^{(A)e^{h_n A}A}\|_H \leq |h_n| - \chi\}} F(A_{kh_n}^n)
\end{align*}
\]

and (1.11) can be used directly in an implementation. We illustrate this in Figures 1.1 and 1.2 where three realizations of $X_T(\omega)$, $\omega \in \Omega$, are calculated approximatively with the numerical approximation method (1.6) for the case where $T = 1$, $n = 10000$, $h_n = 1/\sqrt{n}$, $\varrho = 5/64$, $\chi = 1/128$, and $\xi = 0$. The MATLAB code used to generate Figure 1.1 can be found in Figure 1.2 below. The approximation scheme (1.6) is thus an easily implementable strongly convergent approximation method for the SPDE (1.5). In particular, to the best of our knowledge, the scheme (1.6) is the first approximation method in the scientific literature that has been shown to converge strongly to the solution of the stochastic K-S equation (1.5).

In addition, we would like to point out that although the main result of this paper proves strong convergence, it does not provide any information on the speed of convergence. Moreover, we note that in the general framework of our main result in Theorem 4.1, it is not possible to prove strong convergence rates (cf., e.g., [28] and [11, 15, 16, 22, 24, 34, 44, 45]). Nonetheless, it is an interesting subject of future research to provide further conditions besides the hypotheses that we use which are sufficient to ensure that a numerical scheme converges strongly to the solution of a stochastic K-S equation with a strictly positive polynomial rate of convergence. The difficulty which arises in achieving a strictly positive polynomial strong rate of convergence in the case of stochastic K-S equations and other equations with similar nonlinearities is that these equations fail to satisfy a global monotonicity property under which many
results and techniques are available and the counterexamples in [28] and [11] lie within the region where the global monotonicity condition is not satisfied. There are also some results in the literature which prove strong convergence rates in the case of multidimensional SDEs and SPDEs with non-globally monotone nonlinearities (see, for example, Theorem 1.3 and Theorem 1.4 in [20] and Corollary 3.2 in Dörsek [9]). The approach developed in [20] employs a suitable perturbation theory together with suitable exponential integrability properties of the numerical approximation. It is a subject of future research to establish whether this approach can be extended to stochastic K-S equations, and, in particular, to study whether suitable exponential integrability properties can be established for appropriate numerical approximations in the case of the stochastic K-S equation (1.5).

The remainder of this article is organized as follows. In Section 2 the required a priori moment bounds for the nonlinearity-truncated approximation schemes are established and in Section 3 the error analysis is performed in the pathwise sense under the hypothesis of suitable a priori bounds for the approximation processes. Section 4 combines the results of Section 2 and Section 3 and thereby establishes strong convergence in Theorem 4.1, which is the main result of this article. The analysis in Sections 2–4 is carried out for abstract stochastic evolution equations on separable Banach and Hilbert spaces, respectively. Section 5 then verifies that the assumptions of Theorem 4.1 in Section 4 are satisfied in the case of concrete K-S equations of the type (1.5) and, in particular, establishes Corollary 5.2.

1.1. Notation. Throughout this article, the following notation is used. We denote by \( \mathbb{N} = \{1, 2, 3, \ldots\} \) the set of all natural numbers. For two sets \( A \) and \( B \) we denote by \( \mathcal{M}(A, B) \) the set of all mappings from \( A \) to \( B \). For a set \( A \) we denote by \( \mathcal{P}(A) \) the power set of \( A \), we denote by \( \#_A : \mathcal{P}(A) \to [0, \infty] \) the counting measure on \( A \), and we denote by \( \mathcal{P}_0(A) \) the set given by \( \mathcal{P}_0(A) = \{B \in \mathcal{P}(A) : \#_A(B)<\infty\} \). For two measurable spaces \( (A, \mathcal{A}) \) and \( (B, \mathcal{B}) \) we denote by \( \mathcal{M}(A, B) \) the set of all \( \mathcal{A}/\mathcal{B} \)-measurable mappings. Let \( \Gamma : (0, \infty) \to (0, \infty) \) be the function with the property that for all \( x \in (0, \infty) \) it holds that \( \Gamma(x) = \int_0^\infty t^{(x-1)} e^{-t} \, dt \) (Gamma function). Let \( E_r : [0, \infty) \to [0, \infty), \ r \in (0, \infty), \) be the functions with the property that for all \( r \in (0, \infty), \ x \in [0, \infty) \) it holds that \( E_r[x] = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) (cf. Chapter 7 in Henry [17]). For a topological space \( (X, \tau) \) we denote by \( \mathcal{B}(X) \) the Borel sigma-algebra of \( (X, \tau) \). For a set \( A \) we denote by \( \text{Id}_A : A \to A \) the mapping with the property that for all \( a \in A \) it holds that \( \text{Id}_A(a) = a \) (identity mapping on \( A \)). For a set \( A \in \mathcal{B}(\mathbb{R}) \) we denote by \( \lambda_A : \mathcal{B}(A) \to [0, \infty] \)
the Lebesgue-Borel measure on $A$. For a measure space $(\Omega,\mathcal{F},\mu)$, a measurable space $(S,S)$, a set $R \subseteq S$, and a function $f: \Omega \to R$ we denote by $[f]_{\mu,S}$ the set given by $[f]_{\mu,S} = \{g \in \mathcal{M}(\mathcal{F},S): (\exists A \in \mathcal{F}: \mu(A) = 0 \text{ and } \{w \in \Omega: f(w) \neq g(w)\} \subseteq A\}$. We denote by $\lfloor \cdot \rfloor_h: \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, and $\lceil \cdot \rceil_h: \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, the mappings with the property that for all $t \in \mathbb{R}$, $h \in (0,\infty)$ it holds that $\lfloor t \rfloor_h = \max((-\infty,t] \cap \{0,h,-h,2h,-2h,\ldots\})$ and $\lceil t \rceil_h = \min([t,\infty),\{0,h,-h,2h,-2h,\ldots\})$. For real numbers $p \in [1,\infty)$, $\theta \in (0,1)$ and a $\mathcal{B}((0,1))/\mathcal{B}(\mathbb{R})$-measurable function $v: (0,1) \to \mathbb{R}$ we denote by $\|v\|_{W^{\theta,p}((0,1),\mathbb{R})}$ the extended real number given by

$$
\|v\|_{W^{\theta,p}((0,1),\mathbb{R})} = \left( \int_0^1 |v(x)|^p dx + \int_0^1 \int_0^1 \frac{|v(x) - v(y)|^p}{|x-y|^{1+\theta p}} dx dy \right)^{\frac{1}{p}}.
$$

2. A priori bounds

In this section we accomplish in Proposition 2.1 and Corollary 2.1 below appropriate a priori bounds for our approximation scheme. Before we establish Proposition 2.1 and Corollary 2.1 below, we present in Lemma 2.1, Lemma 2.2, Lemma 2.3, and Lemma 2.4 a few elementary auxiliary results for Proposition 2.1 and Corollary 2.1.

2.1. Regularity of the numerical approximations.

The following elementary and well-known lemma is a slight generalization of Lemma 3.3 in Becker & Jentzen [2]. In particular, the proof of Lemma 2.1 is a slight adaptation of the proof of Lemma 3.3 in Becker & Jentzen [2].

**Lemma 2.1.** Let $(V,\|\cdot\|_V)$ be an $\mathbb{R}$-Banach space, let $A: D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with spectrum($A$) $\subseteq \{z \in \mathbb{C}: \text{Re}(z) < 0\}$, and let $T,h \in (0,\infty)$, $Y,Z \in \mathcal{M}([0,T),V)$ satisfy for all $t \in [0,T]$ that $Y_t = \int_0^t e^{(t-s)A}Z_{[s],h}ds$. 

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1. rng ('default');
2. N = 10000; h = 1/sqrt(N); chi = 1/128; varrho = 5/64;
3. A = -(-N:N).*4*pi*16*pi+(N:N).*2.*pi*1-1;
4. S = sqrt((exp(2*h*A)-1)./A/2).*2*pi.*(-N:N); X = zeros(3,2*N+1);
5. for m = 1:3
6. Y = zeros(1,2*N+1); O_old = zeros(1,2*N+1);
7. for n = 1:sqrt(N)
8. O_new = exp(A*h).*O_old+S.*randn(1,2*N+1);
9. y = [Y(N+1),11*Y(N-1:1):sqrt(2)+Y(N+2:2:end)]
10. y = real(fftv(y)); y1 = ifft(y.*2);
11. y2 = [imag(y1(N+1-1:2))];
12. FY = (Y-pi*flipl(y2).*(-N:N))...
13. *(norm((-A).*varrho)+norm((-A).*varrho.*Y))<=(h^(-chi));
14. Y = exp(A*h).*Y+A.*(-1).*(exp(A*h)-1).*FY+O_new-exp(A*h).*O_old;
15. O_old = O_new;
16. end
17. X(m,:) = [Y(N+1),11*Y(N-1:1):sqrt(2)+Y(N+2:2:end)]
18. X(m,:) = real(fftv(X(m,:)));
19. end
20. figure(1); subplot(1,3,1);
21. plot((1:2*N+1)/(2*N+2),X(1,:),'k','LineWidth',0.3); ylim([1 -1]);
22. subplot(1,3,2);
23. plot((1:2*N+1)/(2*N+2),X(2,:),'k','LineWidth',0.3); ylim([1 -1]);
24. subplot(1,3,3);
25. plot((1:2*N+1)/(2*N+2),X(3,:),'k','LineWidth',0.3); ylim([1 -1]);

**Fig. 1.2. MATLAB code for Figure 1.1.**
In the following we prove (2.3) by induction on 

\( Y \) 

probability space, let 

The proof of Lemma 2.2 is thus completed.

Let \( Y \) follows from the fact that 

Lemma 2.3.

(ii) it holds that the function \([0,T] \ni t \mapsto Y_t \in D(A)\) is continuous,

(iii) it holds that the function \([0,T] \ni t \mapsto Y_t \in V\) is Lipschitz continuous,

(iv) it holds that the function \([0,T] \setminus \{0, 2h, \ldots\} \ni t \mapsto Y_t \in V\) is continuously differentiable,

(v) it holds for all \( t \in [0,T]\) that 

\( \frac{dY_t}{dt} = AY_t + Z_{[t]h}, \)

and

(vi) it holds for all \( t \in [0,T]\) that 

\( Y_t = \int_0^t \left[ AY_s + Z_{[s]h} \right] ds. \)

Then it follows from (2.1) that 

\( \limsup_{h \searrow 0} \|Y_{t+h} - Y_t\|_V = 0. \)

The proof of Lemma 2.2 is thus completed.

\[ \int_0^T \left( e^{(t+h)A} - e^{(t-s)A} \right) Z_s \|_V ds + \int_t^T \left( e^{(t-h+s)A} \right) Z_s \|_V ds \]

Combining Lebesgue's theorem of dominated convergence with the assumption that 

\( A: D(A) \subseteq V \rightarrow V\) is a generator of a strongly continuous semigroup and the assumption that 

\( \sup_{s \in [0,T]} \| Z_s \|_V < \infty \) hence yields that for all \( t \in [0,T]\) it holds that

\[ \limsup_{h \searrow 0} \|Y_{t+h} - Y_t\|_V = 0. \]

The proof of Lemma 2.2 is thus completed.

\[ \int_0^T \left( e^{(t+h)A} - e^{(t-s)A} \right) Z_s \|_V ds + \int_t^T \left( e^{(t-h+s)A} \right) Z_s \|_V ds + h \left( \sup_{s \in [0,T]} \| e^{sA} \|_{L(V)} \right) \left( \sup_{s \in [0,T]} \| Z_s \|_V \right). \]

Lemma 2.3.

Let \( (V, \| \cdot \|_V) \) be a separable \( \mathbb{R}\)-Banach space, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, let \( A \in D(A) \subseteq V \rightarrow V \) be a generator of a strongly continuous semigroup, let \( O: [0,T] \times \Omega \rightarrow V \) be a stochastic process, and let \( T, h \in (0, \infty) \), \( Y \in \mathcal{M}(\mathcal{B}([0,T], \mathcal{F}(V)), \mathcal{M}(\mathcal{B}(\mathcal{V}^2), \mathcal{B}(V))\) satisfy for all \( t \in [0,T]\) that 

\( Y_t = \int_0^t e^{(t-s)A} F(Y_{[s]h}, O_{[s]h}) ds + O_t. \)

Then it holds that 

\( Y: [0,T] \times \Omega \rightarrow V\) is a stochastic process and it holds that 

\( Y - O: [0,T] \times \Omega \rightarrow V\) is a stochastic process with right-continuous sample paths.

\[ \forall t \in [0, \min\{T, kh\}] : Y_t \in \mathcal{M}(\mathcal{F}, \mathcal{B}(V)). \]

In the following we prove (2.3) by induction on \( k \in \mathbb{N}_0\). The base case \( k = 0 \) follows from the fact that 

\( Y_0 = O_0 \in \mathcal{M}(\mathcal{F}, \mathcal{B}(V)). \) Next, observe that the fact that 

\( \forall t \in [0,T] : O_t \in \mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \) shows that for all \( k \in \mathbb{N}_0\), \( t \in [\min\{T, kh\}, \min\{T, (k+1)h\}] \) with \( \forall s \in [0, \min\{T, kh\}] : Y_s \in \mathcal{M}(\mathcal{F}, \mathcal{B}(V)) \) it holds that

\[ Y_t = e^{(t-\min\{T, kh\})A} Y_{\min\{T, kh\}} + \int_{\min\{T, kh\}}^t e^{(t-s)A} F(Y_{[s]h}, O_{[s]h}) ds \]
\[+O_t - e^{(t-\min\{T,kh\})A}O_{\min\{T,kh\}}\]
\[= e^{(t-\min\{T,kh\})A}Y_{\min\{T,kh\}} + \int_{\min\{T,kh\}}^t e^{(t-s)A}F(Y_{\min\{\tau,s\},k})O_{\min\{\tau,s\},k})\,ds\]
\[+ O_t - e^{(t-\min\{T,kh\})A}O_{\min\{T,kh\}}\in M(\mathcal{F},\mathcal{B}(V)).\] (2.4)

The induction step \(\mathbb{N}_0 \ni k \to k+1 \in \mathbb{N}\) follows from (2.4) and the induction hypothesis. Induction hence proves (2.3). In the next step we observe that \(O\colon [0,T] \times \Omega \to V\) is a stochastic process with right-continuous sample paths. The proof of Lemma 2.3 is thus completed.

2.2. Semi-globally Lipschitz continuous functions.

**Lemma 2.4.** Let \((V,\|\cdot\|_V), (\mathcal{V},\|\cdot\|_\mathcal{V}), (W,\|\cdot\|_W)\) be normed \(\mathbb{R}\)-vector spaces with \(\mathcal{V} \subseteq \mathcal{V}\) continuously, \(W \subseteq \mathcal{W}\) continuously, and \(V \neq \{0\}\) and let \(\epsilon, \theta \in [0,\infty), \epsilon, \theta \in (0,\infty), F \in \mathcal{M}(V,W)\) satisfy for all \(v,w \in V\) that \(\|F(v) - F(w)\|_W \leq \epsilon (1 + \|v\|_V + \|w\|_V)\|v - w\|_V\), \(\theta = 2\epsilon\), and

\[
\theta = \max\left\{3\epsilon^2 \left[ \sup_{u \in W \setminus \{0\}} \frac{\|u\|_W}{\|u\|_W} \right] \left[ 1 + \sup_{u \in V \setminus \{0\}} \frac{\|u\|_V}{\|u\|_V} \right] (1 + 2^{2\max\{2\epsilon-1,0\}}), (8\epsilon^2 + 2\|F(0)\|_W^2) \max\left\{1, \sup_{u \in V \setminus \{0\}} \frac{\|u\|_V^{2+\epsilon}}{\|u\|_V}\right\} \right\}. (2.5)
\]

Then it holds for all \(v,w \in V\) that \(\|F(v)\|_W^2 \leq \theta \max\{1,\|v\|_V^{2+\theta}\}\) and

\[
\|F(v) - F(w)\|_W^2 \leq \theta \max\{1,\|v\|_V^{2+\theta}\} \|v - w\|_V^2 + \theta \|v - w\|_V^{2+\theta}. (2.6)
\]

**Proof.** Combining the fact that \(\forall a,b,c \in \mathbb{R}\colon (a+b+c)^2 \leq 3(a^2 + b^2 + c^2)\) and the fact that \(\forall a,b,x \in [0,\infty)\colon (a+b)^x \leq 2^{\max\{x-1,0\}}(a^x + b^x)\) with the triangle inequality proves that for all \(v,w \in V\) it holds that

\[
\|F(v) - F(w)\|_W^2 \leq \left[ \sup_{u \in W \setminus \{0\}} \frac{\|u\|_W}{\|u\|_W} \right]^2 \|F(v) - F(w)\|_W^2
\]
\[\leq 3\epsilon^2 \left[ \sup_{u \in W \setminus \{0\}} \frac{\|u\|_W}{\|u\|_W} \right]^2 \left(1 + \|v\|_V^{2\epsilon} + \|w\|_V^{2\epsilon}\right) \|v - w\|_V^2
\]
\[\leq 3\epsilon^2 \left[ \sup_{u \in W \setminus \{0\}} \frac{\|u\|_W}{\|u\|_W} \right]^2 \left(1 + (1 + 2^{2\max\{2\epsilon-1,0\}})\|v\|_V^{2\epsilon} + 2^{2\max\{2\epsilon-1,0\}}\|v - w\|_V^2\right) \|v - w\|_V^2
\]
\[\leq 3\epsilon^2 \left[ \sup_{u \in W \setminus \{0\}} \frac{\|u\|_W}{\|u\|_W} \right]^2 \left(1 + \left[ \sup_{u \in V \setminus \{0\}} \frac{\|u\|_V}{\|u\|_V} \right]^{2\epsilon} \right) \left(1 + 2^{2\max\{2\epsilon-1,0\}}\right) \|v - w\|_V^2
\]
\[\leq 3\epsilon^2 \left[ \sup_{u \in W \setminus \{0\}} \frac{\|u\|_W}{\|u\|_W} \right]^2 \left(1 + \left[ \sup_{u \in V \setminus \{0\}} \frac{\|u\|_V}{\|u\|_V} \right]^{2\epsilon} \right) \left(1 + 2^{2\max\{2\epsilon-1,0\}}\right) \|v - w\|_V^2
\]
therefore ensures for all Lemma 2.1 hence proves that \( \bar{\theta} \). Moreover, the triangle inequality implies that for all \( v \in V \) it holds that

\[
\|F(v)\|_W^2 \leq (\|F(v) - F(0)\|_W + \|F(0)\|_W)^2
\]

\[
\leq 2(\epsilon^2 (1 + \|v\|^2) \|v\|_V^2 + \|F(0)\|_W^2)
\]

\[
\leq (8\epsilon^2 + 2\|F(0)\|_W^2) \max\{1, \|v\|_V^{2+2\epsilon}\}
\]

\[
\leq \theta \max\{1, \|v\|_V^{2+\theta}\}.
\]

(2.7)

This and (2.7) complete the proof of Lemma 2.4.

\[\square\]

2.3. A priori bounds.

Proposition 2.1 (A priori bounds). Let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a separable \( \mathbb{R} \)-Hilbert space, let \((\Omega, F, \mathbb{P})\) be a probability space, let \( A \in L(H) \) be a diagonal linear operator with \( \sup(\sigma_p(A)) < 0 \), let \((H_t, \langle \cdot, \cdot \rangle_{H_t}, \| \cdot \|_{H_t})\), \( \tau \in \mathbb{R} \), be a family of interpolation spaces associated to \( -A \) (cf., e.g., [39, Section 3.7]), let \( Y, O : [0, T] \times \Omega \to H \) be stochastic processes, and let \( F \in C(H, H), \phi, \Phi \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}([0, \infty))) \), \( \varphi \in [0, 1] \), \( \alpha \in [0, 1/2] \), \( \rho \in [0, 1 - \alpha] \), \( \theta, \delta, \chi \in [0, \infty] \), \( T \in (0, \infty) \), \( h \in (0, 1] \) satisfy for all \( v, w, \varphi, H_t, t \in [0, T] \) that \( \langle v, F(v + w) \rangle_H \leq \frac{1}{2} \varphi(w) \|v\|_H^2 + \varphi\|v\|_{H_{t/2}}^2 + \frac{1}{2} \Phi(w), \|F(v) - F(w)\|_{H_{t/2}}^2 \leq \theta \max\{1, \|v\|_{H_{t/2}}^{2+\theta}\} \), and

\[
Y_t = e^{tA}(Y_0 - O_0) + \int_0^t e^{(t-s)A} \mathbb{1}\{\|Y_{s\wedge}\|_{H_{t/2}} + \|O_{s\wedge}\|_{H_{t/2}} \leq h^{-\chi}\} F(Y_{s\wedge}) \, ds + O_t.
\]

(2.9)

Then it holds for all \( t \in [0, T] \) that

\[
\|Y_t - O_t\|_H^2 + (1 - \varphi) \int_0^t e^{\int_0^s \Phi(O_{u\wedge})} \|Y_s - O_s\|_{H_{t/2}}^2 \, ds
\]

\[
\leq e^{\int_0^t \Phi(O_{u\wedge})} \|Y_0 - O_0\|_H^2 + \int_0^t e^{\int_0^s \Phi(O_{u\wedge})} \left[ \Phi(O_{s\wedge})
\right.
\]

\[
\left. + \theta (1 + \sqrt{\varphi}) h^{2+\theta} \min((1+\varphi/2)(1-\alpha-(1+\varphi/2)\chi), \varphi - (1+\varphi/2)\chi, (1-\alpha-(1+\varphi/2)\chi)
\right]
\]

\[
\right) \, ds.
\]

(2.10)

Proof. Throughout this proof let \( \Omega_t \subseteq \Omega \), \( t \in [0, T] \), be the sets with the property that for all \( t \in [0, T] \) it holds that \( \Omega_t = \{\|Y_{t\wedge}\|_{H_{t/2}} + \|O_{t\wedge}\|_{H_{t/2}} \leq h^{-\chi}\} \) and let \( \tilde{Y} \in \mathcal{M}([0, T], H) \) be the function with the property that for all \( t \in [0, T] \) it holds that \( \tilde{Y}_t = Y_t - O_t \). Next, note that for all \( t \in [0, T] \) it holds that

\[
\tilde{Y}_t = e^{tA}\tilde{Y}_0 + \int_0^t e^{(t-s)A} \mathbb{1}_{\Omega_s} F(Y_{s\wedge}) \, ds.
\]

(2.11)

Lemma 2.1 hence proves that \( \tilde{Y} \) has continuous sample paths and that for all \( t \in [0, T] \) it holds that \( \tilde{Y}_t = \tilde{Y}_0 + \int_0^t A\tilde{Y}_s + \mathbb{1}_{\Omega_s} F(Y_{s\wedge}) \, ds \). The fundamental theorem of calculus therefore ensures for all \( t \in [0, T] \) that

\[
eq \int_0^t \Phi(O_{s\wedge}) \, ds \|\tilde{Y}\|_H^2.
\]
\[
\begin{align*}
\|\bar{Y}_0\|_H + \int_0^t 2 e^{-f_0^s \phi(O_{[s],h})} du \left\langle \bar{Y}_s, A\bar{Y}_s + 1_{\Omega_s} F\left(Y_{[s],h}\right) \right\rangle_H ds \\
- \int_0^t \phi(O_{[s],h}) e^{-f_0^s \phi(O_{[s],h})} du \|\bar{Y}_s\|^2_H ds
\end{align*}
\]

\[= \|\bar{Y}_0\|_H + 2 \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left\langle \bar{Y}_s, A\bar{Y}_s + 1_{\Omega_s} F\left(Y_{[s],h}\right) \right\rangle_H ds
\]

\[+ 2 \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left[ 1_{\Omega_s}\left( -A \right)^{1/2} \bar{Y}_s \left( -A \right)^{-1/2} \left[ F\left(Y_{[s],h}\right) - F\left(\bar{Y}_s + O_{[s],h}\right) \right] \right]_H ds
\]

\[- \int_0^t \phi(O_{[s],h}) e^{-f_0^s \phi(O_{[u],h})} du \|\bar{Y}_s\|^2_H ds.
\]

Next, observe that for all \( s \in [0,T] \) it holds that

\[\left\langle \bar{Y}_s, A\bar{Y}_s \right\rangle_H + 1_{\Omega_s} \left( \frac{1-\varphi}{2} \right) \|\bar{Y}_s\|^2_{H_{1/2}} \leq \left\langle \bar{Y}_s, A\bar{Y}_s \right\rangle_H + \left( \frac{1-\varphi}{2} \right) \|\bar{Y}_s\|^2_{H_{1/2}}
\]

\[= \left\langle \bar{Y}_s, A\bar{Y}_s \right\rangle_H - \left( \frac{1-\varphi}{2} \right) \left\langle \bar{Y}_s, A\bar{Y}_s \right\rangle_H
\]

\[= \left\langle \bar{Y}_s, \left[ 1 - \left( \frac{1-\varphi}{2} \right) A \right] \bar{Y}_s \right\rangle_H
\]

\[= \left\langle \bar{Y}_s, \left( \frac{1+\varphi}{2} \right) A\bar{Y}_s \right\rangle_H.
\]

This, (2.12), and the Cauchy-Schwartz inequality show for all \( t \in [0,T] \) that

\[e^{-f_0^t \phi(O_{[t],h})} ds \|\bar{Y}_t\|^2_H
\]

\[= \|\bar{Y}_0\|^2_H + 2 \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left\langle \bar{Y}_s, A\bar{Y}_s + 1_{\Omega_s} F\left(Y_{[s],h}\right) \right\rangle_H ds
\]

\[+ 2 \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left[ 1_{\Omega_s}\left( -A \right)^{1/2} \bar{Y}_s \left( -A \right)^{-1/2} \left[ F\left(Y_{[s],h}\right) - F\left(\bar{Y}_s + O_{[s],h}\right) \right] \right]_H ds
\]

\[- \int_0^t \phi(O_{[s],h}) e^{-f_0^s \phi(O_{[u],h})} du \|\bar{Y}_s\|^2_H ds
\]

\[\leq \|\bar{Y}_0\|^2_H + 2 \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left\langle \bar{Y}_s, \left( \frac{1+\varphi}{2} \right) A\bar{Y}_s + 1_{\Omega_s} F\left(Y_{[s],h}\right) \right\rangle_H ds
\]

\[+ 2 \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left[ \|\bar{Y}_s\|^2_{H_{1/2}} \left| F\left(Y_{[s],h}\right) - F\left(\bar{Y}_s + O_{[s],h}\right) \right\|_{H_{-1/2}} \right] ds
\]

\[- \left( \frac{1-\varphi}{2} \right) \|\bar{Y}_s\|^2_{H_{1/2}} \right] ds - \int_0^t \phi(O_{[s],h}) e^{-f_0^s \phi(O_{[u],h})} du \|\bar{Y}_s\|^2_H ds.
\]

The fact that \( \forall a, b \in \mathbb{R}, c \in (0, \infty): ab \leq c a^2 + \frac{b^2}{4c} \) therefore proves for all \( t \in [0,T] \) that

\[e^{-f_0^t \phi(O_{[t],h})} ds \|\bar{Y}_t\|^2_H
\]

\[\leq \|\bar{Y}_0\|^2_H + \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left\langle \bar{Y}_s, (1+\varphi) A\bar{Y}_s + 21_{\Omega_s} F\left(\bar{Y}_s + O_{[s],h}\right) \right\rangle_H ds
\]

\[+ \left( \frac{1}{1-\varphi} \right) \int_0^t e^{-f_0^s \phi(O_{[u],h})} du \left[ \|F\left(Y_{[s],h}\right) - F\left(\bar{Y}_s + O_{[s],h}\right)\|_{H_{-1/2}} \right]^2 ds
\]

\[- \int_0^t \phi(O_{[s],h}) e^{-f_0^s \phi(O_{[u],h})} du \|\bar{Y}_s\|^2_H ds.
\]
In the next step we use the fact that \( \forall v, w \in H_1 = H : \|F(v) - F(w)\|_{H^{-1/2}}^2 \leq \theta \max\{1, \|v\|_{H^2}^2\} \|v - w\|_{H^2}^2 + \theta \|v - w\|_{H^2}^2 \) to obtain for all \( v, w \in H_1 = H, s \in [0, T] \) that
\[
1_{\Omega_s} \|F(Y_{[s],h}) - F(\tilde{Y}_s + O_{[s],h})\|_{H^{-1/2}}^2 \\
\leq 1_{\Omega_s} \theta \max\{1, \|Y_{[s],h}\|_{H^2}^2\} \|\tilde{Y}_s - Y_{[s],h}\|_{H^2}^2 + 1_{\Omega_s} \theta \|\tilde{Y}_{[s],h} - \tilde{Y}_s\|_{H^2}^2 \\
\leq 1_{\Omega_s} \theta \|\tilde{Y}_{[s],h} - \tilde{Y}_s\|_{H^2} \left( h^{-\vartheta} + \|\tilde{Y}_{[s],h} - \tilde{Y}_s\|_{H^2}^2 \right).
\] (2.16)

Moreover, observe that the fact that \( \forall q \in [0, 1], t \in (0, \infty) : \|(-A)^{-q}(e^{tA} - \Id_H)\|_{L(H)} \leq t^q \) and the fact that \( \forall q \in [0, 1], t \in (0, \infty) : \|(-A)^q e^{tA}\|_{L(H)} \leq t^{-q} \) imply that for all \( s \in [0, T] \) it holds that
\[
1_{\Omega_s} \|\tilde{Y}_{[s],h} - \tilde{Y}_s\|_{H^2} = 1_{\Omega_s} \left\| \left( e^{(s-\lfloor s \rfloor)A} - \Id_H \right) \tilde{Y}_{[s],h} + \int_{\lfloor s \rfloor}^s (e^{(s-u)A} F(Y_{[s],h}) \, du \right\|_{H^2} \\
\leq 1_{\Omega_s} \left| s - \lfloor s \rfloor \right| h^{1-\varrho} \|\tilde{Y}_{[s],h}\|_{H^2} + \int_{\lfloor s \rfloor}^s \left( s - u \right)^{-(\alpha+\rho)} \|F(Y_{[s],h})\|_{H^2} \, du \right\|_{H^2} \\
\leq 1_{\Omega_s} \left| s - \lfloor s \rfloor \right| h^{1-\varrho} \|\tilde{Y}_{[s],h}\|_{H^2} + \int_{\lfloor s \rfloor}^s \left( s - u \right)^{-(\alpha+\rho)} \theta \max\{1, \|Y_{[s],h}\|_{H^2}^{1+\varrho/2}\} \right\|_{H^2} \\
\leq \frac{1_{\Omega_s}}{1-\alpha-\rho} \left[ h^{1-\varrho-\chi + \sqrt{\vartheta} h^{1-\alpha-\rho-\chi/2}} \right] \leq \frac{1}{(1-\alpha-\rho)} \frac{\frac{1}{(1-\alpha-\rho)^2}}{1-\alpha-\rho}. 
\] (2.17)

Putting (2.17) into (2.16) shows for all \( s \in [0, T] \) that
\[
1_{\Omega_s} \|F(Y_{[s],h}) - F(\tilde{Y}_s + O_{[s],h})\|_{H^{-1/2}}^2 \leq \frac{\theta (1 + \sqrt{\vartheta})^2 h^{2\min\{\rho-\chi, 1-\alpha-\rho-(1+\vartheta/2)\} + \vartheta}}{1-\alpha-\rho} \cdot \frac{h^{\vartheta\min\{\rho-\chi, 1-\alpha-\rho-(1+\vartheta/2)\} + \vartheta}}{1-\alpha-\rho} \\
\leq \frac{2\theta (1 + \sqrt{\vartheta})^2 h^{2\min\{\rho-\chi, 1-\alpha-\rho-(1+\vartheta/2)\} + \vartheta}}{1-\alpha-\rho} \\
= \frac{2\theta (1 + \sqrt{\vartheta})^2 h^{2\min\{1+\vartheta/2(1-\alpha-\rho-(1+\vartheta/2))\} + \vartheta}}{1-\alpha-\rho}.
\] (2.18)

In the next step we put (2.18) into (2.15) to obtain that for all \( t \in [0, T] \) it holds that
\[
e^{-\int_0^s \phi(O_{[u],h}) \, du} \|\tilde{Y}_t\|_{H^2}^2 \\
\leq \|\tilde{Y}_0\|_{H^2}^2 + \int_0^t e^{-\int_0^s \phi(O_{[u],h}) \, du} \left[ -(1 + \varphi) \|\tilde{Y}_s\|_{H^2}^2 + 21_{\Omega_s} \langle \tilde{Y}_s, F(\tilde{Y}_s + O_{[s],h}) \rangle \right] \, ds \\
+ \frac{\theta (1 + \sqrt{\vartheta})^2 h^{2\min\{1+\vartheta/2(1-\alpha-\rho-(1+\vartheta/2))\} + \vartheta}}{1-\alpha-\rho} \int_0^t e^{-\int_0^s \phi(O_{[u],h}) \, du} \, ds \\
- \int_0^t \phi(O_{[s],h}) e^{-\int_0^s \phi(O_{[u],h}) \, du} \|\tilde{Y}_s\|_{H^2}^2 \, ds.
\] (2.19)
The assumption that $\forall v,w \in H_1$: $\langle v,F(v+w) \rangle_H \leq \frac{1}{2} \phi(w)\|v\|_H^2 + \varphi\|v\|_{H_{1/2}}^2 + \frac{1}{2} \Phi(w)$ hence proves that for all $t \in [0,T]$ it holds that

$$e^{-\int_0^t \phi(O_{[s]})} ds \|\bar{Y}_t\|_H^2 \leq \|\bar{Y}_0\|_H^2 + \int_0^t e^{-\int_s^t \phi(O_{[u]})} ds \left\{ (2\varphi - (1+\varphi))\|\bar{Y}_s\|_{H_{1/2}}^2 + \Phi(O_{[s]}) \right\} ds$$

$$+ \theta(1+\sqrt{\theta})^{2+\theta} h^{2+\varphi} e^{-\int_0^t \phi(O_{[u]})} ds \left\{ (1+\varphi)(1-\alpha-\rho-(1+\varphi)\chi,1-\alpha-\rho-(1+\varphi)\chi) \right\} ds.$$

This assures for all $t \in [0,T]$ that

$$\|\bar{Y}_t\|_{H_{1/2}}^2 + (1-\varphi) \int_0^t e^{\int_s^t \phi(O_{[u]})} ds \|\bar{Y}_s\|_{H_{1/2}}^2 ds$$

$$\leq \int_0^t e^{\int_s^t \phi(O_{[u]})} ds \left\{ (2\varphi - (1+\varphi))\|\bar{Y}_s\|_{H_{1/2}}^2 + \Phi(O_{[s]}) \right\} ds$$

$$+ \theta(1+\sqrt{\theta})^{2+\theta} h^{2+\varphi} e^{-\int_0^t \phi(O_{[u]})} ds \left\{ (1+\varphi)(1-\alpha-\rho-(1+\varphi)\chi,1-\alpha-\rho-(1+\varphi)\chi) \right\} ds.$$

The proof of Proposition 2.1 is thus completed. 

**Corollary 2.1 (A priori moment bounds).** Let $(H,\langle \cdot,\cdot \rangle_H,\|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $A \in H$ be a diagonal linear operator with $\sup(\sigma_p(A)) < 0$, let $(H_r,\langle \cdot,\cdot \rangle_{H_r},\|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [39, Section 3.7]), let $O: [0,T] \times \Omega \to H$ be a stochastic process, and let $Y \in \mathcal{M}([0,T] \times \Omega, H), \quad F \in C(H,H)$, $\phi,\Phi \in \mathcal{M}(\mathcal{B}(H),\mathcal{B}([0,\infty])), \quad \varphi \in [0,1], \quad \alpha \in [0,1/2], \quad \rho \in [0,1-\alpha], \quad \varrho \in [\rho,\rho + 1], \quad \theta,\vartheta \in [0,\infty), \quad \chi \in [0,(2-2\alpha - 2\rho)/(2+\vartheta)], \quad T \in (0,\infty)$, $\ h \in (0,1)$, $\ p \in [2,\infty)$ satisfy for all $v,w \in H_1$, $t \in [0,T]$ that $\langle v,F(v+w) \rangle_{H} \leq \frac{1}{2} \phi(w)\|v\|_H^2 + \varphi\|v\|_{H_{1/2}}^2 + \frac{1}{2} \Phi(w), \quad \|F(v)-F(w)\|_{H_{-1/2}}^2 \leq \theta \max\{1,\varrho\|v\|_{H_1}^2 \} \|v-w\|_{H_1}^2 + \theta \|v-w\|_{H_{1/2}}^2 + \frac{1}{2} \Phi(w), \quad \|F(v)\|_{H_{-\alpha}}^2 \leq \theta \max\{1,\varrho\|v\|_{H_1}^2 \} \|v-w\|_{H_1}^2.$

Then it holds that $Y - O: [0,T] \times \Omega \to H$ is a stochastic process with continuous sample paths and it holds that

$$\|\sup_{t \in [0,T]} \|Y_t - O_t\|_H\|_{L^p(\mathbb{P};\mathbb{R})} \leq \theta \max\{1,\varrho\|v\|_{H_1}^2 \} \|v-w\|_{H_1}^2 + \theta \|v-w\|_{H_{1/2}}^2 + \frac{1}{2} \Phi(w), \quad \|F(v)\|_{H_{-\alpha}}^2 \leq \theta \max\{1,\varrho\|v\|_{H_1}^2 \} \|v-w\|_{H_1}^2.$$

(2.23)
\begin{align*}
&\leq \sqrt{\int_0^T \left\| e^{\int_0^T \phi(O_{[s\wedge t]}h) \, du} \left[ \Phi(O_{[s]}h) + \frac{\theta(1+\sqrt{\theta})^2 + \theta_h \min(2p, 2-2\alpha - \vartheta \chi) - 2\varrho - 2(2+\varrho)\chi}{(1/2-\varphi/2)(1-\alpha-\rho)^{2+\varrho}} \right] \right\|_{\mathcal{L}^{p/2}(\mathbb{P}; \mathbb{R})}} \, dt \\
&\leq \sqrt{\frac{1 + \frac{\theta(1+\sqrt{\theta})^2 + \theta_h \min(2p, 2-2\alpha - \vartheta \chi) - 2\varrho - 2(2+\varrho)\chi}{(1/2-\varphi/2)(1-\alpha-\rho)^{2+\varrho}}}{\sqrt{\mathbb{E}}} \left[ e^{\int_0^T \phi(O_{[s\wedge t]}h) \, du} \max\{1, \left| \Phi(O_{[s]}h) \right|^{\eta/2} \} \right]^{2/p}} \, dt.
\end{align*}

(2.24)

**Proof.** Note that the assumption that \( O : [0, T] \times \Omega \to H \) is a stochastic process and Lemma 2.3 yield that \( Y : [0, T] \times \Omega \to H \) is also a stochastic process. Combining Proposition 2.1 with the assumption that \( \chi \in [0, (2-2\alpha - 2\varrho)/(2+\varrho)] \) hence yields that

\[
\sup_{t \in [0, T]} \left\| Y_t - O_t \right\|_H^2
\leq \sup_{t \in [0, T]} \left( \int_0^t e^{\int_s^t \phi(O_{[u\wedge t]}h) \, du} \left[ \Phi(O_{[s]}h) + \frac{\theta(1+\sqrt{\theta})^2 + \theta_h \min(2p, 2-2\alpha - \vartheta \chi) - 2\varrho - 2(2+\varrho)\chi}{(1/2-\varphi/2)(1-\alpha-\rho)^{2+\varrho}} \right] \, ds \right) \\
= \sup_{t \in [0, T]} \left( \int_0^t e^{\int_s^t \phi(O_{[u\wedge t]}h) \, du} \left[ \Phi(O_{[s]}h) + \frac{\theta(1+\sqrt{\theta})^2 + \theta_h \min(2p, 2-2\alpha - \vartheta \chi) - 2\varrho - 2(2+\varrho)\chi}{(1/2-\varphi/2)(1-\alpha-\rho)^{2+\varrho}} \right] \, ds \right) \\
= \int_0^T e^{\int_s^T \phi(O_{[u\wedge t]}h) \, du} \left[ \Phi(O_{[s]}h) + \frac{\theta(1+\sqrt{\theta})^2 + \theta_h \min(2p, 2-2\alpha - \vartheta \chi) - 2\varrho - 2(2+\varrho)\chi}{(1/2-\varphi/2)(1-\alpha-\rho)^{2+\varrho}} \right] \, ds.
\]

(2.25)

Moreover, the fact that \( Y : [0, T] \times \Omega \to H \) is a stochastic process, the assumption that \( O : [0, T] \times \Omega \to H \) is a stochastic process, (2.23), and Lemma 2.1 prove that \( Y - O : [0, T] \times \Omega \to H \) is a stochastic process with continuous sample paths. Hence, we obtain that \( (\Omega \ni \omega \mapsto \sup_{t \in [0, T]} \| Y_t(\omega) - O_t(\omega) \|_H \in \mathbb{R}) \in \mathcal{M}(F, B(\mathbb{R})) \). This, (2.25), Minkowski’s integral inequality, and the assumption that \( p \geq 2 \) show that

\[
\left\| \sup_{t \in [0, T]} \left\| Y_t - O_t \right\|_H \right\|_{\mathcal{L}^{p/2}(\mathbb{P}; \mathbb{R})} \leq \sqrt{\int_0^T \left\| e^{\int_s^T \phi(O_{[u\wedge t]}h) \, du} \left[ \Phi(O_{[s]}h) + \frac{\theta(1+\sqrt{\theta})^2 + \theta_h \min(2p, 2-2\alpha - \vartheta \chi) - 2\varrho - 2(2+\varrho)\chi}{(1/2-\varphi/2)(1-\alpha-\rho)^{2+\varrho}} \right] \right\|_{\mathcal{L}^{p/2}(\mathbb{P}; \mathbb{R})}} \, ds.
\]

(2.26)

The proof of Corollary 2.1 is thus completed. \[ \square \]

3. Pathwise convergence

3.1. Setting. Let \((V, \| \cdot \|_V)\) and \((W, \| \cdot \|_W)\) be \( \mathbb{R} \)-Banach spaces and let \( T, \chi \in (0, \infty), \quad \mathcal{T} \subset \mathbb{R}, \quad \alpha \in (0, 1), \quad (P_n)_{n \in \mathbb{N}} \in \mathcal{M}(\mathbb{N}, L(V)), \quad (h_n)_{n \in \mathbb{N}} \in \mathcal{M}(\mathbb{N}, (0, \infty)), \quad F \in \mathcal{C}(V, W), \quad \Psi \in \mathcal{M}([0, \infty), [0, \infty)), \quad X, O : \mathcal{C}([0, T], V), \quad S \in \mathcal{M}(\mathcal{B}([0, T]), \mathcal{B}(L(W, V))), \quad (X^n)_{n \in \mathbb{N}}, \quad (O^n)_{n \in \mathbb{N}} \in \mathcal{M}(\mathbb{N}, \mathcal{M}([0, T], V)) \) satisfy for all \( r \in [0, \infty), \quad t \in [0, T], \quad n \in \mathbb{N} \) that \( \limsup_{m \to \infty} h_m = 0, \quad \Psi([0, \infty)) \subseteq \mathbb{R}, \quad \mathcal{T} = \sup_{t \in [0, T]} (t^\alpha \| S_t \|_{L(W, V)}), \quad \Psi(r) = \sup \{ 0 \cup \{ \| F(v) - F(w) \|_W \} : v, w \in V, v \neq w, \max \{ \| v \|_V, \| w \|_V \} \leq r \}, \quad X_t = \int_0^t S_{t-s} F(X_s) \, ds + O_t, \) and

\[
X^n_t = \int_0^t P_n S_{t-s} 1_{[0, |h_n| \wedge \chi]} (\| X^n_{[s]} h_n \|_V + \| O^n_{[s]} h_n \|_V) F(\mathcal{A}^n_{[s], h_n}) \, ds + O^n_t.
\]

(3.1)
3.2. Auxiliary results.

**Lemma 3.1.** Assume the setting in Section 3.1 and let $n \in \mathbb{N}$, $t \in [0,T]$. Then

\[
\|X_t - \mathcal{X}^n_t\|_V \leq \|O_t - \mathcal{O}^n_t\|_V + \|\mathcal{Y}^n_t\|_V \Psi \left( \sup_{s \in [0,t]} \|X_s\|_V \right) \left[ \sup_{s \in [0,t]} \|X_s - X_{\lfloor s \rfloor_{h_n}}\|_V \right] \\
\leq \frac{\mathcal{Y} |h_n| \mathcal{Y}^{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,t]} \|F(X_s)\|_W \right) \left( \sup_{s \in [0,t]} \|X_s\|_V + \|\mathcal{O}^n_s\|_V \right) \\
\leq \frac{\mathcal{Y} |h_n| \mathcal{Y}^{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,t]} \|F(X_s)\|_W \right) \int_0^t \|\text{Id}_V - P_n\|_{L(W,V)} ds \\
+ \mathcal{Y} \left[ \|P_n\|_{L(W)} \Psi \left( \sup_{s \in [0,\max\{0,\lfloor t \rfloor_{h_n} - h_n\}] \max\{ \|X_s\|_V, \|\mathcal{X}^n_s\|_V \} \right) \right] \\
+ |h_n|^\chi \left( \sup_{s \in [0,t]} \|F(X_s)\|_W \right) \int_0^t (t-s)^{-\alpha} \|X_{\lfloor s \rfloor_{h_n}} - \mathcal{X}^n_{\lfloor s \rfloor_{h_n}}\|_V ds. \tag{3.2}
\]

**Proof.** Observe that the triangle inequality proves that

\[
\|X_t - \mathcal{X}^n_t\|_V \leq \|O_t - \mathcal{O}^n_t\|_V + \int_0^t \|S_{t-s}[F(X_s) - F(X_{\lfloor s \rfloor_{h_n}})]\|_V ds \\
\leq \int_0^t \|S_{t-s}[F(X_s) - F(X_{\lfloor s \rfloor_{h_n}})]\|_V ds \\
\leq \mathcal{Y} \Psi \left( \sup_{s \in [0,t]} \|X_s\|_V \right) \int_0^t (t-s)^{-\alpha} \|X_s - X_{\lfloor s \rfloor_{h_n}}\|_V ds \\
\leq \frac{\mathcal{Y} |h_n| \mathcal{Y}^{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,t]} \|X_s\|_V \right) \left[ \sup_{s \in [0,t]} \|X_s - X_{\lfloor s \rfloor_{h_n}}\|_V \right]. \tag{3.3}
\]

Next, note that

\[
\int_0^t \|S_{t-s}[F(X_s) - F(X_{\lfloor s \rfloor_{h_n}})]\|_V ds \\
\leq \int_0^t \|S_{t-s}\|_{L(W,V)} \|F(X_s) - F(X_{\lfloor s \rfloor_{h_n}})\|_W ds \\
\leq \mathcal{Y} \Psi \left( \sup_{s \in [0,t]} \|X_s\|_V \right) \int_0^t (t-s)^{-\alpha} \|X_s - X_{\lfloor s \rfloor_{h_n}}\|_V ds \\
\leq \frac{\mathcal{Y} |h_n| \mathcal{Y}^{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,t]} \|X_s\|_V \right) \left[ \sup_{s \in [0,t]} \|X_s - X_{\lfloor s \rfloor_{h_n}}\|_V \right]. \tag{3.4}
\]

Moreover, observe that

\[
\int_0^t \|S_{t-s}[F(X_{\lfloor s \rfloor_{h_n}}) - \mathbb{1}_{0,\lfloor h_n \rfloor - \chi}}\|_V \|F(X_{\lfloor s \rfloor_{h_n}})\|_W ds \\
= \int_0^t \|S_{t-s}\|_{L(W,V)} \|F(X_{\lfloor s \rfloor_{h_n}})\|_W ds \\
\leq \int_0^t \|S_{t-s}\|_{L(W,V)} \|F(X_{\lfloor s \rfloor_{h_n}})\|_W ds \\
\leq \mathcal{Y} \left( \sup_{s \in [0,t]} \|F(X_s)\|_W \right) \int_0^t (t-s)^{-\alpha} \|X_{\lfloor s \rfloor_{h_n}}\|_V \|\mathcal{X}^n_{\lfloor s \rfloor_{h_n}}\|_V ds \\
\leq \mathcal{Y} \left( \sup_{s \in [0,t]} \|F(X_s)\|_W \right) \int_0^t (t-s)^{-\alpha} \|X_{\lfloor s \rfloor_{h_n}}\|_V \|\mathcal{X}^n_{\lfloor s \rfloor_{h_n}}\|_V ds \\
\leq \mathcal{Y} \left( \sup_{s \in [0,t]} \|F(X_s)\|_W \right) \int_0^t (t-s)^{-\alpha} \|X_{\lfloor s \rfloor_{h_n}}\|_V \|\mathcal{X}^n_{\lfloor s \rfloor_{h_n}}\|_V ds.
\]
and

\begin{align*}
&\int_0^t \| [S_{t-s} - P_n S_{t-s}] \mathbb{I}_{[0,[h_n]-\chi]} (\|X^n_{[s]}\|_V + \|O^n_{[s]}\|_V) F(X^n_{[s]}) \|_V \, ds \\
&\leq \int_0^t \| (\text{Id}_V - P_n) S_{t-s} \|_{L(W,V)} \| F(X^n_{[s]}) \|_V \, ds \\
&\leq \left( \sup_{s \in [0,t]} \| F(X_s) \|_W \right) \int_0^t \| (\text{Id}_V - P_n) S_{t-s} \|_{L(W,V)} \, ds \\
&= \left( \sup_{s \in [0,t]} \| F(X_s) \|_W \right) \int_0^t \| \text{Id}_V - P_n \|_{L(W,V)} \, ds. \tag{3.6}
\end{align*}

Furthermore, note that

\begin{align*}
&\int_0^t \| P_n S_{t-s} \mathbb{I}_{[0,[h_n]-\chi]} (\|X^n_{[s]}\|_V + \|O^n_{[s]}\|_V) [F(X^n_{[s]}) - F(X^n_{[s]})] \|_V \, ds \\
&\leq \int_0^t \| P_n \|_{L(V)} \| S_{t-s} \|_{L(W,V)} \| F(X^n_{[s]}) - F(X^n_{[s]}) \|_V \, ds \\
&\leq \gamma \| P_n \|_{L(V)} \int_0^t (t-s)^{-\alpha} \| X^n_{[s]} - X^n_{[s]} \|_V \Psi(\max\{\|X^n_{[s]}\|_V, \|X^n_{[s]}\|_V\}) \max\{\|X_s\|_V, \|X^n_{[s]}\|_V\}) \\
&\cdot \int_0^t (t-s)^{-\alpha} \| X^n_{[s]} - X^n_{[s]} \|_V \, ds. \tag{3.7}
\end{align*}

Combining (3.3)-(3.7) completes the proof of Lemma 3.1.

**Corollary 3.1.** Assume the setting in Section 3.1, let \( n \in \mathbb{N} \), and assume that \( \sup_{t \in [0,T]} \| O_t^n \|_V < \infty \). Then it holds for all \( t \in [0,T] \) that

\[ \sup_{s \in [0,\max\{0,[t]_h - h_n\}]} \max\{\|X_s\|_V, \|X^n_{[s]}\|_V\} < \infty \text{ and} \]

\[ \sup_{s \in [0,T]} \| X_s - X^n_{[s]} \|_V \]

\[ \leq \left[ \sup_{s \in [0,T]} \| O_s - O^n_s \|_V + \frac{\gamma T^{1-\alpha}}{(1-\alpha)} \Psi\left( \sup_{s \in [0,T]} \| X_s \|_V \right) \left( \sup_{s \in [0,T]} \| X_s - X^n_{[s]} \|_V \right) \\
+ \frac{\gamma \| h_n \|_V^{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,T]} \| F(X_s) \|_W \right) \left( \sup_{s \in [0,T]} \| X_s \|_V + \| O^n_s \|_V \right) \\
+ \left( \sup_{s \in [0,T]} \| F(X_s) \|_W \right) T \int_0^t \| (\text{Id}_V - P_n) S_s \|_{L(W,V)} \, ds \right) \\
\cdot E_{1-\alpha} \left[ t \left(1-\alpha \right)^{1/(1-\alpha)} \right] \left( \frac{\gamma T^{1-\alpha}}{(1-\alpha)} \right) \left( \sup_{s \in [0,\max\{0,[t]_h - h_n\}]} \max\{\|X_s\|_V, \|X^n_{[s]}\|_V\} \right) \]
This yields that
\[ + |h_n|^\chi \left( \sup_{s \in [0,T]} \| F(X_s) \|_W \right) \left( 1/(1-\alpha) \right) < \infty. \tag{3.8} \]

**Proof.** Note that Lemma 3.1 implies for all \( t \in [0,T] \) that
\[
\sup_{s \in [0,t]} \| X_s - \mathcal{X}_s^n \|_V \\
\leq \sup_{s \in [0,T]} \| O_s - \mathcal{O}_s^n \|_V + \frac{T^{1-\alpha}}{(1-\alpha)} \Psi \left( \sup_{s \in [0,T]} \| X_s \|_V \left( \sup_{s \in [0,T]} \| X_s - X_{\max h_n} \|_V \right) + \frac{T}{(1-\alpha)} \left( \sup_{s \in [0,T]} \| F(X_s) \|_W \right) \right) \left( \sup_{s \in [0,T]} \| X_s \|_V + \| \mathcal{O}_s^n \|_V \right) \\
+ \left( \sup_{s \in [0,T]} \| F(X_s) \|_W \right) \int_0^T \| (\text{Id}_V - P_n) S_s \|_{L(W,V)} ds \\
+ \Psi \left( \sup_{s \in [0,T]} \| X_s \|_V \left( \sup_{s \in [0,T]} \| X_s - \mathcal{X}_s^n \|_V \right) \right) \\
+ \frac{T}{(1-\alpha)} \left( \sup_{s \in [0,T]} \| X_s - \mathcal{X}_s^n \|_V \right). \tag{3.9} \]

Moreover, note that the assumption that \( \sup_{t \in [0,T]} \| \mathcal{O}_t^n \|_V < \infty \) and the assumption that \( O \in \mathcal{C}([0,T],V) \) imply that
\[
\sup_{t \in [0,T]} \| O_t - \mathcal{O}_t^n \|_V \leq \sup_{t \in [0,T]} \| O_t \|_V + \sup_{t \in [0,T]} \| \mathcal{O}_t^n \|_V < \infty. \tag{3.10} \]

This yields that
\[
\sup_{t \in [0,T]} \| \mathcal{X}_t^n \|_V \leq \sup_{t \in [0,T]} \| \mathcal{O}_t^n \|_V \\
+ \| P_n \|_{L(V)} \left( \max_{s \in [0,2h_n,...]} \right) \sup_{t \in [0,T]} \int_0^T \| S_{t-s} \|_{L(W,V)} ds \]
\[
\leq \sup_{t \in [0,T]} \| \mathcal{O}_t^n \|_V \\
+ \Psi \left( \sup_{t \in [0,T]} \| P_n \|_{L(V)} \left( \max_{s \in [0,2h_n,...]} \right) \sup_{t \in [0,T]} \int_0^T (t-s)^{-\alpha} ds \right) \]
\[
= \sup_{t \in [0,T]} \| \mathcal{O}_t^n \|_V + \frac{T^{1-\alpha}}{(1-\alpha)} \Psi \left( \sup_{t \in [0,T]} \| P_n \|_{L(V)} \left( \max_{s \in [0,2h_n,...]} \right) \sup_{t \in [0,T]} \int_0^T (t-s)^{-\alpha} ds \right) < \infty. \tag{3.11} \]

The assumption that \( X \in \mathcal{C}([0,T],V) \) hence yields that
\[
\sup_{t \in [0,T]} \| X_t - \mathcal{X}_t^n \|_V \leq \sup_{t \in [0,T]} \| X_t \|_V + \sup_{t \in [0,T]} \| \mathcal{X}_t^n \|_V < \infty. \tag{3.12} \]

Moreover, note that
\[
\int_0^T \| (\text{Id}_V - P_n) S_s \|_{L(W,V)} ds \leq \left( \| \text{Id}_V \|_{L(V)} + \| P_n \|_{L(V)} \right) \int_0^T \| S_s \|_{L(W,V)} ds \\
\leq \Psi \left( 1 + \| P_n \|_{L(V)} \right) \int_0^T s^{-\alpha} ds = \frac{\Psi (1 + \| P_n \|_{L(V)}) T^{1-\alpha}}{(1-\alpha)} < \infty. \tag{3.13} \]

In the next step we combine (3.9)-(3.13) with the generalized Gronwall lemma in Chapter 7 in Henry [17] (cf., e.g., Lemma 2.6 in Andersson et al. [1]) to obtain that for all \( t \in [0,T] \) it holds that
\[
\sup_{s \in [0,t]} \| X_s - \mathcal{X}_s^n \|_V 

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The proof of Corollary 3.1 is thus completed.

3.3. Pathwise convergence.

Proposition 3.1. Assume the setting in Section 3.1 and assume that limsup_{n→∞}\left(\int_0^T \| (Id_V - P_n) S_s \|_{L(W,V)} ds + \sup_{s ∈ [0,T]} \| O_s - O^n_s \|_V \right) = 0 and
linsup_{n→∞} \| P_n \|_{L(V)} < ∞. Then

(i) it holds that limsup_{n→∞} \sup_{s ∈ [0,T]} \| X_s - X^n_s \|_V = 0 and

(ii) it holds that there exists a real number \( C \in (0,∞) \) such that for all \( n ∈ N \) it holds that

\[
\sup_{s ∈ [0,T]} \| X_s - X^n_s \|_V \leq C \left[ \sup_{s ∈ [0,T]} \| O_s - O^n_s \|_V + \| h_n \|^\chi + \sup_{s ∈ [0,T]} \| X_s - X_{[s] h_n} \|_V \\
+ \int_0^T \| (Id_V - P_n) S_s \|_{L(W,V)} ds \right].
\]

(3.15)

Proof. Note that the assumption that limsup_{n→∞} \sup_{s ∈ [0,T]} \| O_s - O^n_s \|_V = 0 and the assumption that \( O ∈ C([0,T],V) \) imply that

\[
limsup_{n→∞} \sup_{s ∈ [0,T]} \| O^n_s \|_V \leq \sup_{s ∈ [0,T]} \| O_s \|_V + \limsup_{n→∞} \sup_{s ∈ [0,T]} \| O_s - O^n_s \|_V < ∞.
\]

(3.16)

This and the assumption that
\[
limsup_{n→∞} \left(\int_0^T \| (Id_V - P_n) S_s \|_{L(W,V)} ds + \sup_{s ∈ [0,T]} \| O_s - O^n_s \|_V \right) = 0
\]
yield that
\[
\limsup_{n→∞} \left[ \frac{YT_{1-α}}{(1-α)} \Psi \left( \sup_{s ∈ [0,T]} \| X_s \|_V \right) \left( \sup_{s ∈ [0,T]} \| X_s - X_{[s] h_n} \|_V \right) \\
+ \frac{\| h_n \|^\chi YT_{1-α}}{(1-α)} \left( \sup_{s ∈ [0,T]} \| F(X_s) \|_W \right) \left( \sup_{s ∈ [0,T]} \| X_s \|_V + \| O^n_s \|_V \right) \\
+ \sup_{s ∈ [0,T]} \| O_s - O^n_s \|_V + \left( \sup_{s ∈ [0,T]} \| F(X_s) \|_W \right) \int_0^T \| (Id_V - P_n) S_s \|_{L(W,V)} ds \right] = 0.
\]

(3.17)

Combining this with (3.16) and the fact that \( \sup_{n ∈ N} \left[ \| P_n \|_{L(V)} + \| h_n \|^\chi \right] < ∞ \) ensures that there exists a natural number \( N ∈ N \) such that

\[
\sup_{n ∈ \{N,N+1,...\}} \sup_{s ∈ [0,T]} \| O^n_s \|_V < ∞ 
\]

(3.18)
This proves (3.22) for the base case $n = 0$. The induction step

\[ \sup_{n \in \{N, N + 1, \ldots \}} \left( \left( \frac{\gamma T^{1-\alpha}}{1-\alpha} \Psi \left( \sup_{s \in [0,T]} \|X_s\|_V \right) \sup_{s \in [0,T]} \|X_s - X_{[s]_{h_n}}\|_V \right) + \frac{\gamma |h_n|^\frac{\gamma T^{1-\alpha}}{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,T]} \|F(X_s)\|_W \right) \left( \sup_{s \in [0,T]} \|X_s\|_V + \|\mathcal{O}_s^n\|_V \right) \right) \]

\[ + \left( \sup_{s \in [0,T]} \|F(X_s)\|_W \right) f \left(\|\text{Id}_V - P_n\|S\|_{L(W,V)} \right) ds \]

\[ \cdot E_{1-\alpha} \left[ T |(1-\alpha) \gamma |^{1/(1-\alpha)} \left( \|P_n\|_{L(V)} \Psi \left( \sup_{s \in [0,T]} \|X_s\|_V + \sup_{s \in [0,k_{h_n}]} \|X_s - X^n_s\|_V \right) \right) \right] ^{1/(1-\alpha)} \]

\[\leq 1 < \infty. \quad (3.19)\]

Moreover, observe that the triangle inequality shows for all $t \in [0,T]$, $n \in \mathbb{N}$ that

\[ \sup_{s \in [0,t]} \max \left\{ \|X_s\|_V, \|X^n_s\|_V \right\} \leq \sup_{s \in [0,t]} \max \left\{ \|X_s\|_V, \|X_s + \|X_s - X^n_s\|_V \right\} \]

\[ = \sup_{s \in [0,t]} \left( \|X_s\|_V + \|X_s - X^n_s\|_V \right) \leq \sup_{s \in [0,T]} \|X_s\|_V + \sup_{s \in [0,T]} \|X_s - X^n_s\|_V \]

\[ \leq \sup_{s \in [0,T]} \|X_s\|_V + 2 \sup_{s \in [0,t]} \max \left\{ \|X_s\|_V, \|X^n_s\|_V \right\}. \quad (3.20)\]

Combining Corollary 3.1 with (3.18) and the fact that $\Psi$ is non-decreasing hence proves for all $n \in \{N, N + 1, \ldots \}$, $k \in \mathbb{N}_0 \cap (-\infty, T/h_n - 1]$ that $\sup_{s \in [0,k_{h_n}]} \|X_s - X^n_s\|_V < \infty$ and

\[ \sup_{s \in [0,(k+1)h_n]} \|X_s - X^n_s\|_V \leq \sup_{s \in [0,T]} \|O_s - \mathcal{O}_s^n\|_V \]

\[ + \frac{\gamma T^{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,T]} \|X_s\|_V \right) \sup_{s \in [0,T]} \|X_s - X_{[s]_{h_n}}\|_V \]

\[ + \frac{\gamma |h_n|^\frac{\gamma T^{1-\alpha}}{1-\alpha}}{(1-\alpha)} \left( \sup_{s \in [0,T]} \|F(X_s)\|_W \right) \left( \sup_{s \in [0,T]} \|X_s\|_V + \|\mathcal{O}_s^n\|_V \right) \]

\[ + \left( \sup_{s \in [0,T]} \|F(X_s)\|_W \right) f \left(\|\text{Id}_V - P_n\|S\|_{L(W,V)} \right) ds \]

\[ \cdot E_{1-\alpha} \left[ T |(1-\alpha) \gamma |^{1/(1-\alpha)} \left( \|P_n\|_{L(V)} \Psi \left( \sup_{s \in [0,T]} \|X_s\|_V + \sup_{s \in [0,k_{h_n}]} \|X_s - X^n_s\|_V \right) \right) \right] ^{1/(1-\alpha)} \]

\[\leq 1 < \infty. \quad (3.21)\]

Next, let $n \in \{N, N + 1, \ldots \}$. We then claim that for all $k \in \mathbb{N}_0 \cap [0,T/h_n]$ it holds that

\[ \sup_{s \in [0,k_{h_n}]} \|X_s - X^n_s\|_V \leq 1. \quad (3.22)\]

We prove (3.22) by induction on $k \in \mathbb{N}_0 \cap [0,T/h_n]$. Combining (3.19) and the fact that $n \in \{N, N + 1, \ldots \}$ with the fact that $\forall x \in [0,\infty): E_{1-\alpha}[x] \geq 1$ shows that

\[ \sup_{s \in \{0\}} \|X_s - X^n_s\|_V = \|X_0 - X^n_0\|_V = \|O_0 - \mathcal{O}_0^n\|_V \leq \sup_{s \in [0,T]} \|O_s - \mathcal{O}_s^n\|_V \leq 1. \quad (3.23)\]

This proves (3.22) for the base case $k = 0$. The induction step $\mathbb{N}_0 \cap (-\infty, T/h_n - 1] \ni k \mapsto k + 1 \in \mathbb{N} \cap (-\infty, T/h_n]$ is an immediate consequence of (3.19), (3.21), and the induction
hypothesis. Induction hence proves (3.22). Inequality (3.22), in particular, shows that for all \( n \in \{N, N+1, \ldots \} \) it holds that

\[
\sup_{s \in [0, \max\{0, [T]_{n} - h_{n}\}]} \|X_{s} - \mathcal{X}_{s}^{n}\|_{V} \leq 1. \tag{3.24}
\]

In the next step we combine (3.24) and the fact that \( \forall n \in \{N, N+1, \ldots \} : \sup_{s \in [0, T]} \|O_{s}^{n}\|_{V} < \infty \) with Corollary 3.1 and (3.20) to obtain that for all \( n \in \{N, N+1, \ldots \} , t \in [0, T] \) it holds that

\[
\sup_{s \in [0, t]} \|X_{s} - \mathcal{X}_{s}^{n}\|_{V} \leq \left[ \sup_{s \in [0, T]} \|O_{s} - \mathcal{O}_{s}^{n}\|_{V} + \frac{\Upsilon T}{(1 - \alpha)} \Psi \left( \sup_{s \in [0, T]} \|X_{s}\|_{V} \right) \right] \sup_{s \in [0, T]} \|X_{s} - X_{[s]}_{h_{n}}\|_{V} + \frac{\Upsilon |h_{n}| T}{(1 - \alpha)} \left( \sup_{s \in [0, T]} \|F(X_{s})\|_{W} \right) \left( \sup_{s \in [0, T]} \|X_{s}\|_{V} + \|\mathcal{O}_{s}^{n}\|_{V} \right) + \left( \sup_{s \in [0, T]} \|F(X_{s})\|_{W} \right) \frac{T}{0} \left( \|\text{Id}_{V} - P_{n}\| \right) S_{s} \|L_{V} \|_{V} \|ds
\]

\[
\cdot E_{1 - \alpha} \left[ T \Gamma(1 - \alpha) Y \right]^{(1(1 - \alpha))} \left[ \|P_{n}\|_{L(V)} \Psi \left( \sup_{s \in [0, \max\{0, [T]_{n} - h_{n}\}]} \max \{ \|X_{s}\|_{V}, \|\mathcal{X}_{s}^{n}\|_{V} \} \right) \right] + |h_{n}| \left( \sup_{s \in [0, T]} \|F(X_{s})\|_{W} \right) \left[ (1(1 - \alpha)) \right] < \infty. \tag{3.25}
\]

This and the fact that \( \sup_{n \in \mathbb{N}} \left[ \|P_{n}\|_{L(V)} + |h_{n}| \right] < \infty \) imply (3.15). Moreover, (3.25), the fact that \( \sup_{n \in \mathbb{N}} \left[ \|P_{n}\|_{L(V)} + |h_{n}| \right] < \infty \), and (3.17) prove

\[
\limsup_{n \to \infty} \sup_{s \in [0, T]} \|X_{s} - \mathcal{X}_{s}^{n}\|_{V} = 0.
\]

The proof of Proposition 3.1 is thus completed. \( \square \)

4. Strong convergence

In this section, we accomplish in Theorem 4.1 strong convergence for our approximation scheme. Before we establish Theorem 4.1, we present in Lemma 4.1, Lemma 4.2, Lemma 4.3, Corollary 4.1, and Proposition 4.1 a few elementary results on an appropriate convergence concept for random fields. We employ Corollary 4.1 and Proposition 4.1 in the proof of Theorem 4.1.

4.1. Weakly uniform convergence in probability

**Lemma 4.1.** Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, let \( \mathbb{P}^{*} : \mathcal{P}(\Omega) \to [0, \infty] \) be the mapping with the property that for all \( A \in \mathcal{P}(\Omega) \) it holds that \( \mathbb{P}^{*}(A) = \inf \{ \mathbb{P}(B) : B \in \mathcal{F} \text{ and } A \subseteq B \} \), let \( \mathcal{F} \subseteq \{ A : r \in \mathcal{P}(\Omega) \} : \mathbb{P}^{*}(A) = 1 \}, \) and let \( X_{n} : \Omega \to \mathbb{R} \cup \{ -\infty, -\infty \} , \) \( n \in \mathbb{N} , \) be mappings which satisfy for all \( \omega \in \Omega \) that \( \limsup_{n \to \infty} |X_{n}(\omega)| = 0 \). Then it holds for all \( \varepsilon \in (0, \infty) \) that \( \liminf_{n \to \infty} \mathbb{P}^{*}(\{ |X_{n}| \leq \varepsilon \}) = 1 \).

**Proof.** Throughout this proof let \( Y_{n} : \Omega \to [0, \infty] , \) \( n \in \mathbb{N} , \) be the mappings with the property that for all \( n \in \mathbb{N} \) it holds that

\[
Y_{n} = \sup_{m \in \mathbb{N} \cap [n, \infty]} |X_{m}|. \tag{4.1}
\]

Note that the fact that \( \forall n \in \mathbb{N} : Y_{n+1} \leq Y_{n} \) ensures that for all \( n \in \mathbb{N} , \varepsilon \in (0, \infty) \) it holds that \( \{ Y_{n} \leq \varepsilon \} \subseteq \{ Y_{n+1} \leq \varepsilon \} \). Proposition 1.5.12 in Bogachev [4] and the fact that \( \mathbb{P}^{*} : \mathcal{P}(\Omega) \to [0, \infty] \) is non-decreasing hence prove for all \( \varepsilon \in (0, \infty) \) that

\[
\mathbb{P}^{*}(\cup_{n \in \mathbb{N}} \{ Y_{n} \leq \varepsilon \}) = \liminf_{n \to \infty} \mathbb{P}^{*}(Y_{n} \leq \varepsilon) \leq \liminf_{n \to \infty} \mathbb{P}^{*}(|X_{n}| \leq \varepsilon). \tag{4.2}
\]
Moreover, again the fact that $\mathbb{P}^*: \mathcal{P}(\Omega) \to [0, \infty]$ is non-decreasing shows that for all $\varepsilon \in (0, \infty)$ it holds that

$$\mathbb{P}^*(\cup_{n \in \mathbb{N}} \{ Y_n \leq \varepsilon \}) = \mathbb{P}^*(\{ \exists n \in \mathbb{N} : Y_n \leq \varepsilon \}) = \mathbb{P}^*(\{ \exists n \in \mathbb{N} : (\forall m \in \mathbb{N} \cap [n, \infty) : X_m \leq \varepsilon) \}) \geq \mathbb{P}^*(\Omega) = 1.$$  

(4.3)

Combining this with (4.2), the fact that $\mathbb{P}^* : \mathcal{P}(\Omega) \to [0, \infty]$ is non-decreasing, and the fact that $\mathbb{P}^*|_{\mathcal{F}} = \mathbb{P}$ ensures that for all $\varepsilon \in (0, \infty)$ it holds that

$$1 \leq \mathbb{P}^*(\cup_{n \in \mathbb{N}} \{ Y_n \leq \varepsilon \}) \leq \liminf_{n \to \infty} \mathbb{P}^*(\{ |X_n| \leq \varepsilon \}) \leq \mathbb{P}^*(\Omega) = \mathbb{P}(\Omega) = 1.$$  

(4.4)

This completes proof of Lemma 4.1.

**Lemma 4.2.** Let $I$ be a non-empty set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $c \in (0, \infty)$, and let $X^n : I \times \Omega \to \mathbb{R} \cup \{ \infty, -\infty \}$, $n \in \mathbb{N}$, be random fields. Then the following three statements are equivalent:

(i) It holds for all $\varepsilon \in (0, \infty)$ that $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{P}(|X^n_i| \geq \varepsilon) = 0$.

(ii) It holds for all $\varepsilon \in (0, \infty)$ that $\liminf_{n \to \infty} \inf_{i \in I} \mathbb{P}(|X^n_i| \leq \varepsilon) = 1$.

(iii) It holds that $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E}[\min\{c, |X^n_i|\}] = 0$.

**Proof.** First, note that Markov’s inequality proves for all $\varepsilon \in (0, c)$, $n \in \mathbb{N}$, $i \in I$ that

$$\mathbb{P}(|X^n_i| \geq \varepsilon) = \mathbb{P}(\min\{c, |X^n_i|\} \geq \varepsilon) \leq \frac{\mathbb{E}[\min\{c, |X^n_i|\}]}{\varepsilon}.$$  

(4.5)

This shows that $((iii) \Rightarrow (i))$. In the next step observe for all $\varepsilon \in (0, \infty)$ that

$$\limsup_{n \to \infty} \sup_{i \in I} \mathbb{P}(|X^n_i| \geq \varepsilon) = \limsup_{n \to \infty} \sup_{i \in I} \left[ 1 - \mathbb{P}(|X^n_i| < \varepsilon) \right] = \limsup_{n \to \infty} \left[ 1 - \inf_{i \in I} \mathbb{P}(|X^n_i| < \varepsilon) \right]$$

$$= 1 - \liminf_{n \to \infty} \inf_{i \in I} \mathbb{P}(|X^n_i| < \varepsilon).$$  

(4.6)

This ensures that $((i) \Leftrightarrow (ii))$. It thus remains to prove that $((i) \Rightarrow (iii))$. Note that for all $\varepsilon \in (0, \infty)$ it holds that

$$\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E}[\min\{c, |X^n_i|\}]$$

$$\leq \limsup_{n \to \infty} \sup_{i \in I} \left[ \mathbb{E}[\mathbb{1}_{\{|X^n_i| \geq \varepsilon\}} \min\{c, |X^n_i|\}] + \limsup_{n \to \infty} \sup_{i \in I} \mathbb{E}[\mathbb{1}_{\{|X^n_i| < \varepsilon\}} \min\{c, |X^n_i|\}] \right]$$

$$\leq \limsup_{n \to \infty} \mathbb{P}(|X^n_i| \geq \varepsilon) + \varepsilon.$$  

(4.7)

This shows that $((i) \Rightarrow (iii))$. The proof of Lemma 4.2 is thus completed.

**Lemma 4.3.** Let $\Omega$ and $I$ be non-empty sets, let $\mu : \mathcal{P}(\Omega) \to [0, \infty]$ be a non-decreasing mapping, and let $X^n : I \times \Omega \to \mathbb{R} \cup \{ \infty, -\infty \}$, $n \in \mathbb{N}$, be mappings. Then it holds for all $\varepsilon \in (0, \infty)$, $n \in \mathbb{N}$ that $\inf_{i \in I} \mu(|X^n_i| \leq \varepsilon) \geq \mu(\sup_{i \in I} |X^n_i| \leq \varepsilon)$ and $\liminf_{n \to \infty} \inf_{i \in I} \mu(|X^n_i| \leq \varepsilon) \geq \liminf_{m \to \infty} \inf_{i \in I} \mu(|X^m_i| \leq \varepsilon)$.

**Proof.** Note that the fact that $\mu : \mathcal{P}(\Omega) \to [0, \infty]$ is non-decreasing ensures that for all $n \in \mathbb{N}$, $j \in I$, $\varepsilon \in (0, \infty)$ it holds that

$$\mu(|X^n_j| \leq \varepsilon) \geq \mu(\sup_{i \in I} |X^n_i| \leq \varepsilon).$$  

(4.8)
This yields for all $n \in \mathbb{N}$, $\varepsilon \in (0, \infty)$ that $\inf_{i \in I} \mu(|X^n_i| \leq \varepsilon) \geq \mu(\sup_{i \in I} |X^n_i| \leq \varepsilon)$. This completes the proof of Lemma 4.3.

Informally speaking, the following corollary, Corollary 4.1, shows that convergence uniformly in an index set $I$ on a measurable set of probability 1 implies convergence in probability uniformly in the index set. This statement is nontrivial since arbitrary suprema over random variables are, in general, not random variables.

**Corollary 4.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\Omega = \{ A \in \mathcal{F}: \mathbb{P}(A) = 1 \}$, let $I$ be a non-empty set, and let $X^n : I \times \Omega \to \mathbb{R} \cup \{ \infty, -\infty \}$, $n \in \mathbb{N}$, be random fields which satisfy for all $\omega \in \Omega$ that $\limsup_{n \to \infty} \sup_{i \in I} |X^n_i(\omega)| = 0$. Then it holds for all $\varepsilon \in (0, \infty)$ that $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{P}(|X^n_i| \geq \varepsilon) = 0$.

**Proof.** Throughout this proof let $\mathbb{P}^* : \mathcal{P}(\Omega) \to [0, \infty]$ be the mapping with the property that for all $A \in \mathbb{P}(\Omega)$ it holds that $\mathbb{P}^*(A) = \inf(\{ \mathbb{P}(B) \in [0, 1]: B \in \mathcal{F} \text{ and } A \subseteq B \})$ and let $Y_n : \Omega \to \mathbb{R} \cup \{ \infty, -\infty \}$, $n \in \mathbb{N}$, be the mappings with the property that for all $n \in \mathbb{N}$, $\omega \in \Omega$ it holds that $Y_n(\omega) = \sup_{i \in I} |X^n_i(\omega)|$. Next, note that $\mathbb{P}^*(\Omega) = 1$. Combining Lemma 4.1 with the fact that $\forall \omega \in \Omega: \limsup_{n \to \infty} |Y_n(\omega)| = 0$ hence proves for all $\varepsilon \in (0, \infty)$ that $\liminf_{n \to \infty} \mathbb{P}^*(|Y_n| \leq \varepsilon) = 1$. This implies for all $\varepsilon \in (0, \infty)$ that $\liminf_{n \to \infty} \mathbb{P}^*(\sup_{i \in I} |X^n_i| \leq \varepsilon) = 1$. The fact that $\mathbb{P}^*|_\mathcal{F} = \mathbb{P}$ and Lemma 4.3 therefore prove that

$$\liminf_{n \to \infty} \inf_{i \in I} \mathbb{P}(|X^n_i| \leq \varepsilon) = \liminf_{n \to \infty} \inf_{i \in I} \mathbb{P}^*(|X^n_i| \leq \varepsilon) \geq \liminf_{n \to \infty} \mathbb{P}^*(|X^n_i| \leq \varepsilon) = 1.$$  (4.9)

Hence, it holds for all $\varepsilon \in (0, \infty)$ that $\liminf_{n \to \infty} \inf_{i \in I} \mathbb{P}(|X^n_i| \leq \varepsilon) = 1$. Combining this with Lemma 4.2 shows that for all $\varepsilon \in (0, \infty)$ it holds that $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{P}(|X^n_i| \geq \varepsilon) = 0$. The proof of Corollary 4.1 is thus completed.

Informally speaking, the following proposition, Proposition 4.1, proves for all $p \in (0, \infty)$ that convergence in probability uniformly in an index set $I$ together with uniform moment bounds of the approximations implies for every $q \in (0, p)$ $L^q$-convergence uniformly in $I$. In applications to stochastic processes the index set $I$ can be a time interval.

**Proposition 4.1.** Let $I$ be a non-empty set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $p \in (0, \infty)$, let $(V, \| \cdot \|_V)$ be a separable normed $\mathbb{R}$-vector space, and let $X^n : I \times \Omega \to V$, $n \in \mathbb{N}_0$, be random fields which satisfy for all $\varepsilon \in (0, \infty)$ that $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \|X^n_i - X^n_i\|_V < \infty$ and $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \|X^n_i - X^n_i\|_V \geq \varepsilon = 0$. Then it holds for all $q \in (0, p)$ that $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \|X^n_i - X^n_i\|_V = 0$ and $\sup_{i \in I} \mathbb{E} \|X^n_i\|_V < \infty$.

**Proof.** Observe that, e.g., Lemma 3.10 in [21], the assumption that $\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \|X^n_i\|_V < \infty$, and the assumption that $\forall \varepsilon \in (0, \infty): \limsup_{n \to \infty} \sup_{i \in I} \mathbb{P}(\|X^n_i - X^n_i\|_V \geq \varepsilon) = 0$ yield that

$$\sup_{i \in I} \mathbb{E} \|X^n_i\|_V \leq \sup \liminf_{n \to \infty} \mathbb{E} \|X^n_i\|_V \leq \limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \|X^n_i\|_V < \infty.$$  (4.10)

Next, note that Hölder’s inequality ensures for all $q \in (0, p)$, $n \in \mathbb{N}$ that

$$\sup_{i \in I} \mathbb{E} \|X^n_i - X^n_i\|_V^q = \sup_{i \in I} \left( \mathbb{E} \left[ \mathbb{1}_{\{\|X^n_i - X^n_i\|_V \geq 1\}} \|X^n_i - X^n_i\|_V \right] + \mathbb{E} \left[ \mathbb{1}_{\{\|X^n_i - X^n_i\|_V < 1\}} \|X^n_i - X^n_i\|_V \right] \right)$$
The fact that for all \( q \in (0, p), n \in \mathbb{N} \)
that
\[
\sup_{i \in I} \mathbb{E} \left[ \| X_i^n - X_i^m \|_{Y_0}^q \right] \leq 2^q \sup_{i \in I} \left( \mathbb{P} \left( \| X_i^n - X_i^m \|_{V_0} \geq 1 \right) \left( \mathbb{E} \left[ \| X_i^n \|_{V_0}^p \right] \right)^{\frac{q}{p}} \right)
+ 2^q \sup_{i \in I} \left( \mathbb{P} \left( \| X_i^n - X_i^m \|_{V_0} \geq 1 \right) \left( \mathbb{E} \left[ \| X_i^n \|_{V_0}^p \right] \right)^{\frac{q}{p}} \right) + \sup_{i \in I} \mathbb{E} \left[ \min \left\{ 1, \| X_i^n - X_i^m \|_{V_0} \right\} \right].
\]

Moreover, observe that Lemma 4.2 and the assumption that
\[
\forall \varepsilon \in (0, \infty) : \limsup_{n \to \infty} \inf_{i \in I} \mathbb{P} \left( \| X_i^n - X_i^m \|_{V_0} \geq \varepsilon \right) = 0
\]
prove that for all \( q \in (0, p) \) it holds that
\[
\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \left[ \min \left\{ 1, \| X_i^n - X_i^m \|_{Y_0} \right\} \right] = 0.
\]
This, (4.12), (4.10), the assumption that \( \limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \left[ \| X_i^n \|_{Y_0} \right] < \infty \), and the fact that \( \limsup_{n \to \infty} \sup_{i \in I} \mathbb{P} \left( \| X_i^n - X_i^m \|_{V_0} \geq 1 \right) = 0 \) yield that for all \( q \in (0, p) \) it holds that
\[
\limsup_{n \to \infty} \sup_{i \in I} \mathbb{E} \left[ \| X_i^n - X_i^m \|_{Y_0} \right] = 0.
\]

Combining this with (4.10) completes the proof of Proposition 4.1.
This together with the assumption that $\liminf_{n\to\infty}\inf\{\lambda_b : b \in \mathbb{H} \setminus \mathbb{H}_n \cup \{\infty\}\} = \infty$ ensures that

$$
\limsup_{n \to \infty} \sup_{t \in [0,T]} \left( \int_0^t \|(\text{Id}_H - P_n)e^{sA}\|_{L(H_{-\alpha,H_\alpha})} ds \right) = 0.
$$

\(\square\)
In addition, the assumption that $\forall \omega \in \Omega: \limsup_{n \to \infty} \sup_{t \in [0,T]} \|O_t(\omega) - \tilde{O}^n_t(\omega)\|_{H^\rho} = 0$ and the fact that $\hat{\Omega} \subseteq \overline{\Omega}$ imply that $\limsup_{n \to \infty} \sup_{t \in [0,T]} \|\tilde{O}_t - \tilde{O}^n_t\|_{H^\rho} = 0$. Combining this with (4.17), (4.19), the fact that $\forall t \in [0,T]: \tilde{X}_t = \int_0^t e^{(t-s)A} F(\tilde{X}_s) ds + \tilde{O}_t$, and the fact that $\limsup_{n \to \infty} \|P_n\|_{L(H^\rho)} = 1 < \infty$ allows us to apply Proposition 3.1 (with $V = H_0$, $W = H_{-\alpha}$, $T = T$, $\chi = \chi$, $\alpha = \alpha + \alpha$, $P_n = (H_0 \ni v \mapsto P_n(v) \in H_0)$, $h_n = h_n$, $F = F|_{H_0} \in C(H_0,H_{-\alpha})$, $S_l = (H_{-\alpha} \ni v \mapsto e^{tA}v \in H_0)$ for $t \in (0,T)$, $n \in \mathbb{N}$ in the notation of Proposition 3.1) to obtain that $\limsup_{n \to \infty} \sup_{t \in [0,T]} \|\tilde{X}_t - \tilde{X}^n_t\|_{H^\rho} = 0$. This together with the fact that $\forall \omega \in \hat{\Omega}$, $t \in [0,T]$, $n \in \mathbb{N}$: $\tilde{X}_t(\omega) - \tilde{X}^n_t(\omega) = X_t(\omega) - X^n_t(\omega)$ proves (ii).

In the next step note that Lemma 2.1 yields for every $n \in \mathbb{N}$ that $X^n - \tilde{O}^n : [0,T] \times \Omega \to H_0$ is a stochastic process with continuous sample paths. This establishes (iii). Next, observe that Lemma 2.4 implies that for all $v,w \in H_0$ it holds that

$$\|F(v) - F(w)\|_{H_0}^{1/2} \leq \tilde{\theta} \max\{1,\|v\|_{H_0}^2\} \|v - w\|_{H_0}^{1/2}$$

and

$$\|F(v)\|_{H_{-\alpha}}^2 \leq \tilde{\theta} \max\{1,\|v\|_{H_0}^{2+\tilde{\theta}}\}. \tag{4.21}$$

In addition, observe that the assumption that $\chi \in (0,(1-\alpha-\rho)/(1+2\rho)]$, in particular, assures that $\chi \in (0,(2-2\alpha-2\rho)/(2+\tilde{\theta})]$. Combining this with (4.20) and (4.21) enables us to apply Corollary 2.1 (with $H = P_n(H)$, $(\Omega,F,P) = (\Omega,F,P)$, $A = (P_n(H) \ni v \mapsto Av \in P_n(H)) \in L(P_n(H))$, $O = ([0,T] \times \Omega \ni (t,\omega) \mapsto \tilde{O}^n_t(\omega) \in P_n(H))$, $Y = ([0,T] \times \Omega \ni (t,\omega) \mapsto \tilde{X}^n_t(\omega) \in P_n(H))$, $F = (P_n(H) \ni v \mapsto P_n F(v) \in P_n(H)) \in C(P_n(H),P_n(H))$, $\phi = \phi_n$, $\Phi = \Phi_n$, $\varphi = \varphi$, $\alpha = \alpha$, $\rho = \rho$, $\vartheta = \vartheta$, $\theta = \theta$, $\vartheta = \vartheta$, $\chi = \chi$. $T = T$, $h = h_n$, $p = p$ for $n \in \{m \in \mathbb{N} : h_m \leq 1\}$ in the notation of Corollary 2.1) to obtain that for all $n \in \mathbb{N}$ with $h_n \leq 1$ it holds that

$$\sup_{t \in [0,T]} \|\tilde{X}^n_t - \tilde{O}^n_t\|_{H^\rho} \leq \sqrt{\left[1 + \frac{\tilde{\theta}(1+\tilde{\theta})^{1/2}\tilde{\theta}}{h_n \min(2\varphi,2-2\alpha-\rho)}\right]^{\frac{1}{2}} \|\tilde{\Phi}(\tilde{X}^n_{[t]}|_{h_n})|^{\rho/2}\| ds}$$

and

$$\sqrt{\left[1 + \frac{\tilde{\theta}(1+\tilde{\theta})^{1/2}\tilde{\theta}}{h_n \min(2\varphi,2-2\alpha-\rho)}\right]^{\frac{1}{2}} \|\Phi(\tilde{X}^n_{[t]}|_{h_n})|^{\rho/2}\| ds}. \tag{4.22}$$

Next, observe that the assumption that $\chi \in (0,(1-\alpha-\rho)/(1+2\rho)] \cap (0,\vartheta/(1+2\rho)]$ ensures that

$$(1-\alpha-\rho) - (1+2\vartheta)\chi \geq 0 \quad \text{and} \quad (\rho-\alpha) - (1+\vartheta)\chi \geq 0. \tag{4.23}$$

This, in turn, proves that

$$\min\{(\rho,1-\alpha-\vartheta\chi) - \rho - (1+\vartheta)\chi = \min\{(\rho-\alpha) - (1+\vartheta)\chi,(1-\alpha-\rho) - (1+2\vartheta)\chi\} \geq 0. \tag{4.24}$$
Hence, we obtain for all \( n \in \mathbb{N} \) with \( h_n \leq 1 \) that 
\[ |h_n|^{2\min\{\alpha, 1-\alpha\}} - \rho - (1+\vartheta)\chi \leq 1. \]
Combining this with (4.22) proves that for all \( n \in \mathbb{N} \) with \( h_n \leq 1 \) it holds that
\[
\left\| \sup_{t \in [0,T]} \| \tilde{X}_t^n - \tilde{\sigma}_t^n \|_H \right\|_{L^p(P;\mathbb{R})} 
\leq \sqrt{2 \left( 1 + \frac{\vartheta (1+|\vartheta|)^{2+2\vartheta}}{(1-\vartheta/2)(1-\alpha-\rho)^{2+2\vartheta}} \right) \int_0^T \mathbb{E} \left[ e^{p\int_0^T \phi(\tilde{O}^{n}_{[s],h_n}) du} \max \{1, |\Phi(\tilde{O}^{n}_{[s],h_n})|^{p/2} \} \right] } ds.
\] (4.25)

The fact that \( \forall \omega \in \tilde{\Omega}, \ t \in [0,T], \ n \in \mathbb{N} \): \( \tilde{X}_t^n(\omega) = X_t^n(\omega) \), \( \tilde{\sigma}_t^n(\omega) = \sigma_t^n(\omega) \), (i), and (iii) hence yield that for all \( n \in \mathbb{N} \) with \( h_n \leq 1 \) it holds that
\[
\left\| \sup_{t \in [0,T]} \| X_t^n - \sigma_t^n \|_H \right\|_{L^p(P;\mathbb{R})} 
\leq \sqrt{2 \left( 1 + \frac{\vartheta (1+|\vartheta|)^{2+2\vartheta}}{(1-\vartheta/2)(1-\alpha-\rho)^{2+2\vartheta}} \right) \int_0^T \mathbb{E} \left[ e^{p\int_0^T \phi(\sigma^{n}_{[s],h_n}) du} \max \{1, |\Phi(\sigma^{n}_{[s],h_n})|^{p/2} \} \right] } ds
\] \leq \sqrt{2 \left( 1 + \frac{\vartheta (1+|\vartheta|)^{2+2\vartheta}}{(1-\vartheta/2)(1-\alpha-\rho)^{2+2\vartheta}} \right) T + \mathbb{E} \left[ \int_0^T e^{p\int_0^T \phi(\sigma^{n}_{[s],h_n}) du} \max \{1, |\Phi(\sigma^{n}_{[s],h_n})|^{p/2} \} \right] } ds. \] (4.26)

Combining this with the assumption that \( \limsup_{n \to \infty} h_n = 0 \) implies that
\[
\limsup_{n \to \infty} \left\| \sup_{t \in [0,T]} \| X_t^n - \sigma_t^n \|_H \right\|_{L^p(P;\mathbb{R})} < \infty. \] (4.27)
This, the assumption that \( \limsup_{n \to \infty} \sup_{s \in [0,T]} \mathbb{E} \left[ \| \sigma_s^n \|_H^p \right] < \infty, \) and the triangle inequality assure that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E} \left[ \| X_t^n \|_H^p \right] < \infty. \] (4.28)

Next, note that (ii) and the fact \( H_{\varrho} \subseteq H \) continuously ensure for all \( \omega \in \tilde{\Omega} \) that
\( \limsup_{n \to \infty} \sup_{t \in [0,T]} \| X_t(\omega) - X_t^n(\omega) \|_H = 0 \). Combining this with (i) allows us to apply Corollary 4.1 to obtain that for all \( \epsilon \in (0, \infty) \) it holds that \( \limsup_{n \to \infty} \sup_{t \in [0,T]} \mathbb{P} \left( \| X_t - X_t^n \|_H \geq \epsilon \right) = 0 \). Proposition 4.1 together with (4.28) hence ensures that for all \( q \in (0, p) \) it holds that \( \sup_{t \in [0,T]} \mathbb{E} \left[ \| X_t \|_H^q \right] < \infty \) and \( \limsup_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E} \left[ \| X_t - X_t^n \|_H^q \right] = 0 \). Combining this with (4.27), (4.28), and the assumption that \( \forall n \in \mathbb{N}, \ t \in [0,T]: \mathbb{P}(\sigma_s^n = X_s^n) = 1 \) establishes (iv) and (v). The proof of Theorem 4.1 is thus completed.

5. Stochastic Kuramoto-Sivashinsky equations

In this section we establish a few elementary results which, in particular, demonstrate that Theorem 4.1 can be applied to the stochastic K-S equation (1.5).

5.1. Setting. Let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H) = (L^2(\lambda_0; \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\lambda_0; \mathbb{R})}, \| \cdot \|_{L^2(\lambda_0; \mathbb{R})})\), \( \beta \in (1/8, 1/2), \) \( T, \eta \in (0, \infty), \) \( \kappa \in \mathbb{R}, \) \( \varrho \in (1/16, \beta/2), \) \( \chi \in (0, \varrho/2 - 1/32), \) \( \xi \in H_{1/4}, \) \( e_k, b_k \in \mathbb{M}(\mathbb{Z}, H), \) \( \lambda_k \subseteq \mathbb{Z}, \) \( b_k \subseteq \mathbb{Z}, \) \( \tilde{b}_k \subseteq \mathbb{M}(\mathbb{Z}, H) \) satisfy for all \( n \in \mathbb{N}, \) \( k \in \mathbb{Z} \)
\[ e_0 = \left[ \frac{\chi}{1} \right]_{x \in (0,1)}, B(\mathbb{R}), \] \( e_n = \left[ \frac{2\cos(2\pi x)}{x \in (0,1)} \right]_{x \in (0,1)}, B(\mathbb{R}), \] \( \lambda_k = 16\kappa \pi^4 - 4\kappa^2 \pi^2 + \eta, \) \( \sum_{m \in \mathbb{Z}} \| b_m \|^2 + |\tilde{b}_m|^2 < \infty, \) let \( A: D(A) \subseteq H \to H \) be the linear operator such that \( D(A) = \{ v \in H : \sum_{k \in \mathbb{Z}} \| \lambda_k (e_k, v) \|^2 < \infty \} \) and such that for all \( v \in D(A) \) it holds...
that \(Av = \sum_{k \in \mathbb{Z}} -\lambda_k \langle e_k, v \rangle_H e_k\), let \((H_r, \langle \cdot, \cdot \rangle_H, \|\cdot\|_{H_r})\), \(r \in \mathbb{R}\), be a family of interpolation spaces associated to \(-A\) (cf., e.g., [39, Section 3.7]), let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a normal filtration \((\mathcal{F}_t)_{t \in [0,T]}\), let \(F \in \mathcal{M}(H_{1/16}, H_{-1/4})\), \(B \in \mathcal{M}(H,H)\), \((h_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}(\mathbb{N}, L(H))\) satisfy for all \(v \in H_{1/16}\), \(n \in \mathbb{N}\) that \(F(v) = \eta v - \frac{\kappa}{2} (\eta^2)\), \(Bv = \sum_{k \in \mathbb{Z}} (\langle e_k, v \rangle_H + \tilde{b}_k \langle e_{-k}, v \rangle_H) e_k\), \(\limsup_{k \to \infty} h_k = 0\), \(P_n(v) = \sum_{k=-n}^n \langle e_k, v \rangle_H e_k\), for every \(w \in \{v\}_{\lambda_{(0,1),B}(\mathbb{R})} \subseteq L^0(\lambda_{(0,1),\mathbb{R}}): v \in \mathcal{C}((0,1), \mathbb{R})\)

\[\mathcal{O} \left(\mathcal{O}_\ell, \lambda_{(0,1),B}(\mathbb{R}) = w, \text{ let } (W_t)_{t \in [0,T]} \text{ be an } \text{Id}_{H^c}\text{-cylindrical } (\mathcal{F}_t)_{t \in [0,T]}\text{-Wiener process, let } \mathcal{X}^n: [0,T] \times \Omega \to P_n(H), n \in \mathbb{N}, \text{ and } \mathcal{O}^n: [0,T] \times \Omega \to P_n(H), n \in \mathbb{N}, \text{ be stochastic processes, and assume for all } n \in \mathbb{N}, \text{ that } [\mathcal{O}_\ell]_{n, P_n(H)} = \int_0^t P_n e^{(t-s)A} B dW_s \text{ and} \]

\[
\mathbb{P} \left( \mathcal{X}_t^n = \mathcal{O}_t^n + \int_0^t P_n e^{(t-s)A} \frac{1}{\| \mathcal{X}_{\langle s \rangle_{h_n}}^n \|_{H_\theta}} + \| \mathcal{O}_{\langle s \rangle_{h_n}} + P_n e^{(s-t)A} \|_{H_\theta} \leq \| h_n \|_{H_\theta} \right) = 1. \tag{5.1}
\]

5.2. Properties of the nonlinearity. In Lemma 5.2 and Lemma 5.3 below, we demonstrate that the function \(F\) in Section 5.1 fulfills the hypotheses of Theorem 4.1 above. Our proofs of Lemma 5.2 and Lemma 5.3 use the following well-known lemma.

**Lemma 5.1.** Assume the setting in Section 5.1 and let \(v \in H_1\). Then \(\|v\|_{H_1} \leq 2^{1/4}\|v\|_{H_{1/4}}\|v\|_{H_{1/4}} \leq \|v\|_{H_1} \|v\|_{H_{1/4}}\).

**Proof.** Note that Parseval’s identity and integration by parts prove that

\[
\|v\|_{H_1}^2 = \sum_{k \in \mathbb{Z}} |\langle e_k, v \rangle_H|^2 = \sum_{k \in \mathbb{Z}} \int_0^1 e_k(x)(v)'(x) dx = \sum_{k \in \mathbb{Z}} \int_0^1 (e_k)'(x)v(x) dx \int_0^1 \int_0^1 2k\pi \int \langle e_{-k}, v \rangle_H dx d^2 = 4k^2 \pi^2 |\langle e_{-k}, v \rangle_H| \sum_{k \in \mathbb{Z}} 4k^2 \pi^2 |\langle e_k, v \rangle_H|^2 \leq \sqrt{2} \left(16k^4 \pi^4 - 4k^2 \pi^2 + \eta \right)^{1/2} |\langle e_k, v \rangle_H|^2 = \sqrt{2} \sum_{k \in \mathbb{Z}} \lambda_k^{1/2} |\langle e_k, v \rangle_H|^2 \leq 2 \|v\|_{H_{1/4}}^2. \tag{5.2}
\]

Moreover, Hölder’s inequality shows that

\[
\|v\|_{H_{1/4}}^2 = \sum_{k \in \mathbb{Z}} \lambda_k^{1/2} |\langle e_k, v \rangle_H|^2 \leq \sqrt{\sum_{k \in \mathbb{Z}} |\langle e_k, v \rangle_H|^2} \sqrt{\sum_{k \in \mathbb{Z}} \lambda_k |\langle e_k, v \rangle_H|^2} = \|v\|_{H_1} \|v\|_{H_{1/4}}. \tag{5.3}
\]

Combining this and (5.2) completes the proof of Lemma 5.1. \(\blacksquare\)

The next simple lemma is a slight modification of Lemma 5.7 in Blömker & Jentzen [3].

**Lemma 5.2.** Assume the setting in Section 5.1 and let \(v, w \in H_1, \varepsilon \in (0, \infty)\). Then

\[
\langle v, F(v + w) \rangle_H \leq \|v\|_{H_1}^2 \left( \frac{3\eta}{2} + \frac{\kappa^4}{16} \left[1 + 1/\sqrt{\varepsilon} \right]^2 + \varepsilon \left[\sup_{x \in (0,1)} |w(x)|^2 \right]\right) + \frac{1}{2} \|v\|_{H_{1/2}}^2 + \frac{\kappa}{2} \|w\|_{H_1} + \frac{\kappa}{4} \|w\|_{H_1}^2. \tag{5.4}
\]
Proof. Note that integration by parts yields that
\[
\langle v, F(v + w) \rangle_H = \langle v, \eta (v + w) - \frac{\eta}{2} ((v + w)^2) \rangle_H \\
= \eta \|v\|_H^2 + \eta \langle v, w \rangle_H - \frac{\kappa}{2} \int_0^1 \eta'(x) \langle |v + w|^2 \rangle'(x) dx \\
= \eta \|v\|_H^2 + \eta \langle v, w \rangle_H + \frac{\kappa}{2} \int_0^1 (v)'(x) [v(x) + w(x)]^2 dx \\
= \eta \|v\|_H^2 + \eta \langle v, w \rangle_H + \frac{\kappa}{2} \int_0^1 (v)'(x) w(x) dx + \frac{\kappa}{2} \int_0^1 (v)'(x) |w(x)|^2 dx.
\]
(5.5)

The Cauchy-Schwartz inequality and the fact that \( v, x, r \in (0, \infty) : xy \leq \frac{x^2}{2r} + \frac{ry^2}{2} \) therefore prove that
\[
\langle v, F(v + w) \rangle_H \leq \eta \|v\|_H^2 + \eta \|v\|_H \|w\|_H + |\kappa| \|v'\|_H \|v\|_H \|w\|_H + |\kappa| \|v'\|_H \|w^2\|_H \\
\leq \eta \|v\|_H^2 + \eta \|v\|_H \|w\|_H + \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|v'\|_H^2 + \epsilon \|v\|_H \|w^2\|_H + \frac{\kappa^2}{4} \|v'\|_H \|w^2\|_H \\
\leq \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|w\|_H^2 + (\frac{\kappa^2}{4} + \frac{\kappa^2}{4}) \|v\|_H^2 + \frac{\kappa^2}{4} \|v'\|_H^2 + \frac{\kappa^2}{4} \|w^2\|_H \\
\leq \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|w\|_H^2 + \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|v'\|_H^2 + \frac{\kappa^2}{4} \|w^2\|_H \\
= \|v\|_H^2 \left( \frac{\kappa^2}{4} + \frac{\kappa^2}{4} \right) + \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|v'\|_H^2 + \frac{\kappa^2}{4} \|w^2\|_H.
\]
(5.6)

Lemma 5.1 and again the fact that \( v, x, r \in (0, \infty) : xy \leq \frac{x^2}{2r} + \frac{ry^2}{2} \) hence show that
\[
\langle v, F(v + w) \rangle_H \\
\leq \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|w\|_H^2 + \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|v'\|_H^2 + \frac{\kappa^2}{4} \|w^2\|_H \\
+ \epsilon \|v\|_H^2 \left( \sup_{x \in (0,1)} |w(x)|^2 \right) + \frac{\kappa^2}{4} \|w^2\|_H \\
\leq \|v\|_H^2 \left( \frac{\kappa^2}{4} + \frac{\kappa^2}{4} \right) + \frac{\kappa^2}{4} \|v\|_H^2 + \frac{\kappa^2}{4} \|v'\|_H^2 + \frac{\kappa^2}{4} \|w^2\|_H.
\]
(5.7)

The proof of Lemma 5.2 is thus completed.

**Lemma 5.3.** Assume the setting in Section 5.1 and let \( v, w \in H_{1/16} \). Then
\[
\|F(v) - F(w)\|_{H^{-1/4}} \leq \left( \eta^{1/16} + |\kappa| \sup_{u \in H_{1/16} \setminus \{0\}} \frac{\|u\|_{L_{1/16}}^2}{\|u\|_{H_{1/16}}^2} \right) \cdot \left( 1 + \|v\|_{H_{1/16}} + \|w\|_{H_{1/16}} \right) \|v - w\|_{H_{1/16}} < \infty.
\]
(5.8)

**Proof.** First, note that
\[
\|v - w\|_{H^{-1/4}} \leq \|(A)_{-5/16}\|_{L(H)} \|v - w\|_{H_{1/16}} = \eta^{-5/16} \|v - w\|_{H_{1/16}} \\
\leq \eta^{-5/16} \left( 1 + \|v\|_{H_{1/16}} + \|w\|_{H_{1/16}} \right) \|v - w\|_{H_{1/16}}.
\]
(5.9)

Next, observe that for all \( u \in H \) it holds that
\[
(-A)^{-1/4}(u') = ((-A)^{-1/4}u)'.
\]
(5.10)
This and Lemma 5.1 prove that

\[ \|(v^2)' - (w^2)'\|_{H^{-1/4}} = \|(-A)^{-1/4}((v^2)' - (w^2)')\|_H = \|((-A)^{-1/4}[v^2 - w^2])'\|_H \]

\[ \leq 2^{1/4}\|(-A)^{-1/4}[v^2 - w^2]\|_{H^{1/4}} = 2^{1/4}\|v^2 - w^2\|_H \]

\[ \leq 2^{1/4}\sup_{u \in H^{1/16}\setminus\{0\}} \frac{\|u\|_{L^4(\lambda_{(0,1)}; \mathbb{R})}}{\|u\|_{H^{1/16}}} \left(\|v + w\|_{H^{1/16}} \|v - w\|_{H^{1/16}} \right) \]

\[ \leq 2 \left( \sup_{u \in H^{1/16}\setminus\{0\}} \frac{\|u\|_{L^4(\lambda_{(0,1)}; \mathbb{R})}}{\|u\|_{H^{1/16}}} \right)^2 \left(1 + \|v\|_{H^{1/16}} + \|w\|_{H^{1/16}} \right) \|v - w\|_{H^{1/16}}. \]  

(5.11)

This together with (5.9) shows that

\[ \|F(v) - F(w)\|_{H^{-1/4}} = \|\eta(v - w) - \frac{k}{2}((v^2)' - (w^2)')\|_{H^{-1/4}} \]

\[ \leq \|v - w\|_{H^{-1/4}} + \frac{|k|}{2}\|((v^2)' - (w^2)')\|_{H^{-1/4}} \]

\[ \leq \left( \eta^{1/16} + |k| \right) \left( \sup_{u \in H^{1/16}\setminus\{0\}} \frac{\|u\|_{L^4(\lambda_{(0,1)}; \mathbb{R})}}{\|u\|_{H^{1/16}}} \right)^2 \left(1 + \|v\|_{H^{1/16}} + \|w\|_{H^{1/16}} \right) \|v - w\|_{H^{1/16}}. \]  

(5.12)

Next, observe that the Sobolev embedding theorem ensures that

\[ \sup_{u \in H^{1/16}\setminus\{0\}} \frac{\|u\|_{L^4(\lambda_{(0,1)}; \mathbb{R})}}{\|u\|_{H^{1/16}}} < \infty. \]  

(5.13)

Combining this with (5.12) completes the proof of Lemma 5.3.

\[ \Box \]

5.3. Fernique’s theorem. In the next result, Lemma 5.4 below, we recall a well-known version of Fernique’s theorem (cf., for example, Stroock [40, Theorem 8.2.1]). Lemma 5.4 will be used in the proof of Corollary 5.1 to establish integrability properties for the stochastic convolution processes in (5.32) below. Corollary 5.1 will be employed in the proof of Corollary 5.2 in Section 5.5 to establish strong convergence in (5.51) below.

**Lemma 5.4.** Let \((V, \|\cdot\|_V)\) be a separable \(\mathbb{R}\)-Banach space, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \(\mathcal{X}: \Omega \rightarrow V\) be a mapping which satisfies that for every \(\varphi \in V’\) it holds that \(\varphi \circ \mathcal{X}: \Omega \rightarrow \mathbb{R}\) is a centered Gaussian random variable, and let \(r \in (0, \infty)\) satisfy that \(\mathbb{P}(\|\mathcal{X}\|_V^2 > r) \leq 1/10\). Then

\[ \mathbb{E}\left[ \exp\left( \frac{\|\mathcal{X}\|_V^2}{18r} \right) \right] \leq \sqrt{e} + \sum_{k=0}^{\infty} \left[ \frac{e}{3} \right]^{2^k} < 13. \]  

(5.14)

**Proof.** Note that (5.14) is an immediate consequence of the fact that \(\mathbb{P}(\|\mathcal{X}\|_V^2 \leq r) \geq 9/10\) and of Fernique’s theorem (see, e.g., Stroock [40, Theorem 8.2.1]). The proof of Lemma 5.4 is thus completed.  

\[ \Box \]
5.4. Properties of the stochastic convolution process.

**Lemma 5.5.** Assume the setting in Section 5.1 and let \((a_k)_{k \in \mathbb{Z}} \in \mathcal{M}(\mathbb{Z}, \mathbb{R}), S \in \mathcal{P}_0(\mathbb{Z})\). Then

\[
\mathbb{E}\left[ \sum_{k \in S} a_k \int_0^t e^{-\lambda_k (t-s)} \left( b_k d(e_k, W_s)_H + \tilde{b}_k d(e_{-k}, W_s)_H \right)^2 \right] \leq \sum_{k \in S} \frac{|a_k b_k|^2 + |a_k \tilde{b}_k|^2}{\lambda_k}.
\]

(5.15)

**Proof.** Throughout this proof let \(S \in \mathcal{P}_0(\mathbb{Z})\) be the set given by \(S = \{-k \colon k \in S\}\).

Next, note that Itô’s isometry proves that

\[
\mathbb{E}\left[ \sum_{k \in S} a_k \int_0^t e^{-\lambda_k (t-s)} \left( b_k d(e_k, W_s)_H + \tilde{b}_k d(e_{-k}, W_s)_H \right)^2 \right] = \mathbb{E}\left[ \sum_{k \in S} a_k b_k \int_0^t e^{-\lambda_k (t-s)} d(e_k, W_s)_H^2 \right] + \mathbb{E}\left[ \sum_{k \in S} a_k \tilde{b}_k \int_0^t e^{-\lambda_k (t-s)} d(e_{-k}, W_s)_H^2 \right]
\]

\[
= \sum_{k \in S} |a_k b_k + a_{-k} \tilde{b}_{-k}| \int_0^t e^{-2\lambda_k (t-s)} ds + \sum_{k \in S} |a_k \tilde{b}_{-k}|^2 \int_0^t e^{-2\lambda_k (t-s)} ds.
\]

(5.16)

The fact that \(\forall x, y \in \mathbb{R}: |x + y|^2 \leq 2x^2 + 2y^2\) hence ensures that

\[
\mathbb{E}\left[ \sum_{k \in S} a_k \int_0^t e^{-\lambda_k (t-s)} \left( b_k d(e_k, W_s)_H + \tilde{b}_k d(e_{-k}, W_s)_H \right)^2 \right] \leq \sum_{k \in S} \frac{|a_k b_k + a_{-k} \tilde{b}_{-k}|^2}{2\lambda_k} + \sum_{k \in S} \frac{|a_k b_k|^2}{2\lambda_k} + \sum_{k \in S} \frac{|a_{-k} \tilde{b}_{-k}|^2}{2\lambda_k}
\]

\[
\leq \sum_{k \in S} \frac{|a_k b_k|^2 + |a_{-k} \tilde{b}_{-k}|^2}{\lambda_k} + \sum_{k \in S \setminus S} \frac{|a_k b_k|^2}{\lambda_k} + \sum_{k \in S \setminus S} \frac{|a_{-k} \tilde{b}_{-k}|^2}{\lambda_k}.
\]

(5.17)

This yields that

\[
\mathbb{E}\left[ \sum_{k \in S} a_k \int_0^t e^{-\lambda_k (t-s)} \left( b_k d(e_k, W_s)_H + \tilde{b}_k d(e_{-k}, W_s)_H \right)^2 \right] \leq \sum_{k \in S \cap S} \frac{|a_k b_k|^2}{\lambda_k} + \sum_{k \in S \setminus S} \frac{|a_k \tilde{b}_k|^2}{\lambda_k} + \sum_{k \in S} \frac{|a_k b_k|^2}{\lambda_k} + \sum_{k \in S \setminus S} \frac{|a_{-k} \tilde{b}_{-k}|^2}{\lambda_k}
\]

\[
= \sum_{k \in S} \frac{|a_k b_k|^2}{\lambda_k} + \sum_{k \in S} \frac{|a_k \tilde{b}_k|^2}{\lambda_k} + \sum_{k \in S \setminus S} \frac{|a_k b_k|^2}{\lambda_k} + \sum_{k \in S \setminus S} \frac{|a_{-k} \tilde{b}_{-k}|^2}{\lambda_k} = \sum_{k \in S} \frac{|a_k b_k|^2 + |a_k \tilde{b}_k|^2}{\lambda_k}.
\]

(5.18)
The proof of Lemma 5.5 is thus completed.

**Lemma 5.6.** Assume the setting in Section 5.1, let \( p \in (\frac{1}{\beta}, \infty) \), \( t \in [0, T] \), \( n \in \mathbb{N} \), and let \( Y : \Omega \to \mathbb{R} \) be a standard normal random variable. Then

\[
\left( \mathbb{E} \left[ \sup_{x \in (0, 1)} |\mathcal{O}^n_t(x)|^2 \right] \right)^{1/2} \leq \sqrt{10} \left( \mathbb{E} \left[ |Y|^p \right] \right)^{1/p} \sum_{k=-n}^{n} \max \{ |k|^{4\beta}, 1 \} \left( |b_k|^2 + |\tilde{b}_k|^2 \right) \left( \frac{\lambda_k}{\lambda} \right)^{1/2} \cdot \sup \left( \sup_{x \in (0, 1)} |v(x)| : [v \in C((0, 1), \mathbb{R}) \text{ and } \|v\|_{W^{\beta,p}((0, 1), \mathbb{R})} \leq 1] \right) < \infty. \tag{5.19}
\]

**Proof.** First, note that Jensen’s inequality proves that

\[
\mathbb{E} \left[ \sup_{x \in (0, 1)} |\mathcal{O}^n_t(x)|^2 \right] \leq \left( \mathbb{E} \left[ \left\| \mathcal{O}^n_t \right\|_{W^{\beta,p}((0, 1), \mathbb{R})}^2 \right] \right)^{1/2} \mathbb{E} \left[ \left\| \mathcal{O}^n_t \right\|_{W^{\beta,p}((0, 1), \mathbb{R})}^p \right] \leq \left( \mathbb{E} \left[ \left\| \mathcal{O}^n_t \right\|_{W^{\beta,p}((0, 1), \mathbb{R})}^p \right] \right)^{1/2} \mathbb{E} \left[ \left\| \mathcal{O}^n_t \right\|_{W^{\beta,p}((0, 1), \mathbb{R})}^p \right]. \tag{5.20}
\]

Moreover, observe that

\[
\mathbb{E} \left[ \left\| \mathcal{O}^n_t \right\|_{W^{\beta,p}((0, 1), \mathbb{R})}^p \right] = \mathbb{E} \left[ \int_0^1 |\mathcal{O}^n_t(x)|^p dx + \int_0^1 \int_0^1 \frac{|\mathcal{O}^n_t(x) - \mathcal{O}^n_t(y)|^p}{|x-y|^{1+\beta p}} dx dy \right] = \mathbb{E} \left[ |Y|^p \right] \int_0^1 \left( \mathbb{E} \left[ |\mathcal{O}^n_t(x)|^2 \right] \right)^{p/2} dx + \mathbb{E} \left[ |Y|^p \right] \int_0^1 \left( \mathbb{E} \left[ \left| \mathcal{O}^n_t(x) - \mathcal{O}^n_t(y) \right|^2 \right] \right)^{p/2} \frac{dx}{|x-y|^{1+\beta p}} dy. \tag{5.21}
\]

Next, note that Lemma 5.5 ensures that for all \( x \in (0, 1) \) it holds that

\[
\mathbb{E} \left[ |\mathcal{O}^n_t(x)|^2 \right] = \mathbb{E} \left[ \sum_{k=-n}^{n} e_k(x) \int_0^t e^{-\lambda_k(t-s)} \left( b_k d(e_k, W_s)_H + \tilde{b}_k d(e_{-k}, W_s)_H \right) ds \right]^2 \leq \sum_{k=-n}^{n} \frac{|e_k(x)|^2 |b_k|^2 + |\tilde{b}_k|^2}{\lambda_k} \leq 2 \sum_{k=-n}^{n} \frac{|b_k|^2 + |\tilde{b}_k|^2}{\lambda_k}. \tag{5.22}
\]

This yields that

\[
\int_0^1 \left( \mathbb{E} \left[ |\mathcal{O}^n_t(x)|^2 \right] \right)^{p/2} dx \leq 2^{p/2} \left( \sum_{k=-n}^{n} \frac{|b_k|^2 + |\tilde{b}_k|^2}{\lambda_k} \right)^{p/2}. \tag{5.23}
\]

Moreover, Lemma 5.5 proves that for all \( x, y \in (0, 1) \) it holds that

\[
\mathbb{E} \left[ |\mathcal{O}^n_t(x) - \mathcal{O}^n_t(y)|^2 \right] \leq \mathbb{E} \left[ |\mathcal{O}^n_t(x)|^2 \right] + \mathbb{E} \left[ |\mathcal{O}^n_t(y)|^2 \right].
\]
\[
= \mathbb{E} \left[ \left| \sum_{k=-n}^{n} \left[ e_k(x) - e_k(y) \right] \int_{0}^{t} e^{-\lambda_k(t-s)} \left( b_k d\langle e_k, W_s \rangle_H + \tilde{b}_k d\langle e_{-k}, W_s \rangle_H \right) \right|^2 \right] \\
\leq \sum_{k=-n}^{n} \frac{|e_k(x) - e_k(y)|^2 |b_k|^2 + |e_k(x) - e_k(y)|^2 |\tilde{b}_k|^2}{\lambda_k}. \quad (5.24)
\]

In addition, the assumption that \( \beta < 1/2 \) and the fact that \( \forall x, y \in \mathbb{R}: \max\{|\sin(x) - \sin(y)|, |\cos(x) - \cos(y)|\} \leq |x - y| \) ensure for all \( x, y \in (0, 1) \), \( k \in \mathbb{Z} \) that
\[
|e_k(x) - e_k(y)|^2 \leq 2 \max\{|\sin(2k\pi x) - \sin(2k\pi y)|^2, |\cos(2k\pi x) - \cos(2k\pi y)|^2\} \\
\leq 2 \beta \max\{|\sin(2k\pi x) - \sin(2k\pi y)|^{4\beta}, |\cos(2k\pi x) - \cos(2k\pi y)|^{4\beta}\} \\
\leq 2^3 |k\pi|^{4\beta} |x-y|^{4\beta}. \quad (5.25)
\]

Combining this with (5.24) proves for all \( x, y \in (0, 1) \) that
\[
\mathbb{E}\left[ |\mathcal{O}_n^\alpha(x) - \mathcal{O}_n^\alpha(y)|^2 \right] \leq 2^2 |x-y|^{4\beta} \sum_{k=-n}^{n} \frac{|k\pi|^{4\beta} (|b_k|^2 + |\tilde{b}_k|^2)}{\lambda_k}. \quad (5.26)
\]

This and the assumption that \( \beta p > 1 \) yield that
\[
\int_{0}^{1} \int_{0}^{1} \left( \mathbb{E}\left[ |\mathcal{O}_n^\alpha(x) - \mathcal{O}_n^\alpha(y)|^2 \right] \right)^{\eta/2} dx dy \\
\leq 2^{3\eta/2} \left[ \sum_{k=-n}^{n} \frac{|k\pi|^{4\beta} (|b_k|^2 + |\tilde{b}_k|^2)}{\lambda_k} \right]^{\eta/2} \int_{0}^{1} \int_{0}^{1} |x-y|^{\beta p - 1} dx dy \\
\leq 2^{3\eta/2} \left[ \sum_{k=-n}^{n} \frac{|k\pi|^{4\beta} (|b_k|^2 + |\tilde{b}_k|^2)}{\lambda_k} \right]^{\eta/2}. \quad (5.27)
\]

Combining (5.21), (5.23), and the fact that \( \forall x, y \in \mathbb{R}: |x+y|^{2/p} \leq |x|^{2/p} + |y|^{2/p} \) hence shows that
\[
\left( \mathbb{E}\left[ \left\| \mathcal{O}_n^\alpha \right\|_{W^{\beta,p}((0,1),\mathbb{R})}^p \right] \right)^{1/p} \\
\leq \left( \mathbb{E}\left[ |Y|^p \right] \right)^{1/p} \left\{ 2^{\eta/2} \left[ \sum_{k=-n}^{n} \frac{|b_k|^2 + |\tilde{b}_k|^2}{\lambda_k} \right]^{\eta/2} + 2^{3\eta/2} \left[ \sum_{k=-n}^{n} \frac{|k\pi|^{4\beta} (|b_k|^2 + |\tilde{b}_k|^2)}{\lambda_k} \right]^{\eta/2} \right\}^{1/p} \\
\leq \sqrt{10} \left( \mathbb{E}\left[ |Y|^p \right] \right)^{1/p} \left[ \sum_{k=-n}^{n} \frac{\max\{|k\pi|^{4\beta}, 1\} (|b_k|^2 + |\tilde{b}_k|^2)}{\lambda_k} \right]^{1/2}. \quad (5.28)
\]

Next, observe that the Sobolev embedding theorem and the assumption that \( \beta p > 1 \) ensure that
\[
\sup\left\{ \left\{ \sup_{x \in (0,1)} |v(x)| : [v \in C((0,1),\mathbb{R}) \text{ and } \|v\|_{W^{\beta,p}((0,1),\mathbb{R})} \leq 1] \right\} \right\} < \infty. \quad (5.29)
\]

Combining this with (5.20) and (5.28) establishes (5.19). The proof of Lemma 5.6 is thus completed. \( \square \)
Lemma 5.7. Let $a \in (0, \infty)$, $x, r \in [0, \infty)$. Then $x^r \leq a^{-r}([r]_1 + 1)! e^{ax}$.

Proof. Note that

$$e^{ax} \geq 1 + \frac{|ax|^{[r]_1 + 1}}{([r]_1 + 1)!} \geq 1 + \frac{|ax|^{[r]_1 + 1}}{([r]_1 + 1)!} \geq \frac{|ax|^r}{([r]_1 + 1)!},$$

(5.30)

The proof of Lemma 5.7 is thus completed.

Corollary 5.1. Assume the setting in Section 5.1 and let $\phi, \Phi \in \mathcal{M}(\mathcal{B}(H_1), \mathcal{B}([0, \infty)))$, $p \in (1/\beta, \infty)$, $\varepsilon \in (0, \infty)$ satisfy for all $v \in H_1$ that $\phi(v) = \frac{3n}{2} + \frac{3}{16} [1 + 1/\varepsilon]^2 + \varepsilon \left[ \sup_{x \in (0,1)} |v(x)|^2 \right]$, $\Phi(v) = \frac{\sqrt{n}}{2} ||v||^2_H + \frac{1}{4} ||v||^2_H$, and

$$\varepsilon \leq \frac{1}{72pT \varepsilon} \left[ \max \left\{ 1, \sum_{k \in \mathbb{Z}} \frac{\max\{kB_{1/4}, 1\}}{\lambda_k} \right\} \right]^{-1} \cdot \left[ \sup\left\{ \sup_{x \in (0,1)} |v(x)| : \left[ v \in C((0,1), \mathbb{R}) \text{ and } ||v||_{W^{\beta,p}(0,1), \mathbb{R}} \leq 1 \right] \right\} \right]^2. \tag{5.31}

Then it holds that $\limsup_{m \to \infty} \sup_{s \in [0,T]} \mathbb{E} [\|O^m_s + P_m e^{sA} \xi \|_H^p] < \infty$ and

$$\limsup_{m \to \infty} \mathbb{E} \left[ \int_0^T \exp \left( \int_0^r p \phi(O^m_{[u]_m} + P_m e^{[u]_m A} \xi) \, du \right) \right] \cdot \max \left\{ 1, \left| \Phi(O^m_{[r]_m} + P_m e^{[r]_m A} \xi) \right|^{p/2} \right\} \, dr < \infty. \tag{5.32}

Proof. First, note that Markov’s inequality, e.g., Lemma 4.7 in [19], Lemma 5.6, and (5.31) imply for all $m \in \mathbb{N}$, $t \in [0, T]$ that

$$\mathbb{P} \left( \sup_{x \in (0,1)} |O^m_t(x)|^2 \geq \frac{1}{72pT \varepsilon} \right) \leq 72pT \varepsilon \mathbb{E} \left[ \sup_{x \in (0,1)} |O^m_t(x)|^2 \right] \leq 720p^3T \varepsilon \left[ \sup \left\{ \left. \left( \sup_{x \in (0,1)} |v(x)| : \left[ v \in C((0,1), \mathbb{R}) \text{ and } ||v||_{W^{\beta,p}(0,1), \mathbb{R}} \leq 1 \right] \right) \right\} \right]^2 \cdot \left[ \sum_{k=-m}^m \frac{\max\{kB_{1/4}, 1\}}{\lambda_k^2} \right] \leq 10^{-1}. \tag{5.33}

Lemma 5.4 hence shows that for all $m \in \mathbb{N}$, $t \in [0, T]$ it holds that

$$\mathbb{E} \left[ \exp \left( 4pT \varepsilon \left\{ \sup_{x \in (0,1)} |O^m_t(x)|^2 \right\} \right) \right] \leq 13. \tag{5.34}

Moreover, Hölder’s inequality ensures for all $r \in [0, T]$, $m \in \mathbb{N}$ that

$$\mathbb{E} \left[ \exp \left( \int_0^r p \phi(O^m_{[u]_m} + P_m e^{[u]_m A} \xi) \, du \right) \right] \cdot \max \left\{ 1, \left| \Phi(O^m_{[r]_m} + P_m e^{[r]_m A} \xi) \right|^{p/2} \right\} \leq \mathbb{E} \left[ \int_0^r p \phi(O^m_{[u]_m} + P_m e^{[u]_m A} \xi) \, du \right] \cdot \max \left\{ 1, \left| \Phi(O^m_{[r]_m} + P_m e^{[r]_m A} \xi) \right|^{p/2} \right\} \leq \sqrt{\mathbb{E} \left[ \int_0^r 2p \phi(O^m_{[u]_m} + P_m e^{[u]_m A} \xi) \, du \right]} \cdot \mathbb{E} \left[ 1 + \left| \Phi(O^m_{[r]_m} + P_m e^{[r]_m A} \xi) \right|^p \right]. \tag{5.35}
Next, note that the fact that \( \forall x, y \in \mathbb{R}: (x + y)^2 \leq 2x^2 + 2y^2 \) yields that for all \( m \in \mathbb{N} \) it holds that
\[
\mathbb{E} \left[ \exp \left( \frac{T}{2} \eta^2 \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) du \right] 
\leq \exp \left( 3pT \eta^2 + \frac{pT}{16} \left( 1 + 1/\varepsilon \right) \right) \quad \text{and} \quad \mathbb{E} \left[ \exp \left( \frac{T}{4} \eta^2 \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) du \right] 
\leq \exp \left( 3pT \eta + \frac{pT}{16} \eta \right) \left( 1 + 1/\varepsilon \right) \quad \text{and} \quad \mathbb{E} \left[ \exp \left( \frac{T}{4} \eta^2 \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) du \right].
\]

In addition, observe that the triangle inequality and the fact that \( \forall x, y \in \mathbb{R}, a \in [1, \infty): |x + y|^a \leq 2^{a-1} |x|^a + 2^{a-1} |y|^a \) show for all \( r \in [0, T], m \in \mathbb{N} \) that
\[
\mathbb{E} \left[ | \Phi \left( \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) \right]^p 
= \mathbb{E} \left[ \frac{p}{2} \left\| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right\|^2_H + \frac{1}{4} \left\| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right\|_H^2 \right]^p 
\leq \mathbb{E} \left[ \frac{p}{2} \left\| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right\|^2_H + \frac{1}{4} \left\| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right\|_H^2 \right]^p 
\leq \mathbb{E} \left[ \left\{ \sup_{x \in (0,1)} \left| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right| \right\}^{2p} \right] 
+ \frac{1}{4} \left\{ \sup_{x \in (0,1)} \left| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right| \right\}^{4p} \right].
\]

Lemma 5.7 hence proves for all \( r \in [0, T], m \in \mathbb{N} \) that
\[
\mathbb{E} \left[ | \Phi \left( \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) \right]^p 
\leq 2^{2p-2} \eta^p \left\{ \sup_{x \in (0,1)} \left| P_{m} \mathcal{e}_{u}^{m} |A| \xi \right| \left| \left| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right| \right| \right\}^{2p} 
+ 2^{2p-2} \left\{ \sup_{x \in (0,1)} \left| P_{m} \mathcal{e}_{u}^{m} |A| \xi \right| \left| \left| \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right| \right| \right\}^{4p} \right].
\]

Combining this with (5.35), (5.36), and the fact that \( \forall x, y \in [0, \infty): \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \) ensures that for all \( m \in \mathbb{N} \) it holds that
\[
\mathbb{E} \left[ \int_{0}^{T} \exp \left( \frac{T}{r} \phi ( \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi ) du \right) \right] \max \left\{ 1, \left| \Phi \left( \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) \right|^{r/2} \right\} dr 
\leq \exp \left( \frac{3pT \eta^2}{2} + \frac{pT \eta}{16} \left( 1 + 1/\varepsilon \right) \right) \quad \text{and} \quad \mathbb{E} \left[ \exp \left( \frac{T}{4} \eta^2 \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) du \right] 
\leq \exp \left( 3pT \eta + \frac{pT}{16} \eta \right) \left( 1 + 1/\varepsilon \right) \quad \text{and} \quad \mathbb{E} \left[ \exp \left( \frac{T}{4} \eta^2 \mathcal{O}_{u}^{m} + P_{m} \mathcal{e}_{u}^{m} |A| \xi \right) du \right].
\]

Moreover, the Sobolev embedding theorem implies that

\[ + \int_0^T \sqrt{\left( \frac{\eta^p(|r|+1)!}{4pT|\xi|^p} + \frac{(2p^2p^2+1)!}{2p^2p^2|\xi|^p} \right) \mathbb{E} \left[ \exp \left( 4pT \varepsilon \left( \sup_{x \in (0,1)} |O_{m(h_m,x)}^m(x)|^2 \right) \right) \right] } \, dr \]

Next, note that Jensen’s inequality (cf., e.g., Lemma 2.22 in Cox et al. [6]) and (5.34) show that for all \( m \in \mathbb{N} \) it holds that

\[ \mathbb{E} \left[ \exp \left( \int_0^T 4pT \varepsilon \left( \sup_{x \in (0,1)} |O_{m(h_m,x)}^m(x)|^2 \right) \, du \right) \right] \leq \frac{1}{T} \int_0^T \mathbb{E} \left[ \exp \left( 4pT \varepsilon \left( \sup_{x \in (0,1)} |O_{m(h_m,x)}^m(x)|^2 \right) \right) \right] \, du \leq 13. \]  

(5.40)

Combining (5.39) with (5.34) hence shows for all \( m \in \mathbb{N} \) that

\[ \mathbb{E} \left[ \int_0^T \exp \left( \int_0^T p \phi(O_{m(h_m,x)}^m + P_m e^{[r]_{h_m,A}^m A} \xi) \, du \right) \max \left\{ 1, |\Phi(O_{m(h_m,x)}^m + P_m e^{[r]_{h_m,A}^m A} \xi)|^{n/2} \right\} \, dr \right] \]

\[ \leq \sqrt{13} \exp \left\{ 3pT \eta + pT^2 \frac{1}{16} [1 + 1/\varepsilon]^2 + 2p \int_0^T \left\{ \sup_{x \in (0,1)} |P_m e^{[r]_{h_m,A}^m A} \xi(x)|^2 \right\} \, du \right\} \]

\[ \cdot \left( T + \int_0^T 2^{p-1} \eta^{p/2} \left\{ \sup_{x \in (0,1)} |P_m e^{[r]_{h_m,A}^m A} \xi(x)|^p \right\} \, dr \right) \]

\[ + \int_0^T 2^{2p-1} \left\{ \sup_{x \in (0,1)} |P_m e^{[r]_{h_m,A}^m A} \xi(x)|^{2p} \right\} \, dr + T \sqrt{13 \left( \frac{\eta^p(|r|+1)!}{4pT|\xi|^p} + \frac{(2p^2p^2+1)!}{2p^2p^2|\xi|^p} \right) } . \]

(5.41)

Moreover, the Sobolev embedding theorem implies that

\[ \sup \left\{ \left\{ \sup_{x \in (0,1)} |v(x)| : [v \in H_{1/4} \text{ and } \|v\|_{H_{1/4}} \leq 1] \right\} \right\} < \infty. \]  

(5.42)

This yields for all \( s \in [0,T], m \in \mathbb{N} \) that

\[ \sup_{x \in (0,1)} |P_m e^{sA} \xi(x)| \]

\[ \leq \left[ \sup \left\{ \left\{ \sup_{x \in (0,1)} |v(x)| : [v \in H_{1/4} \text{ and } \|v\|_{H_{1/4}} \leq 1] \right\} \right\} \right] \|P_m e^{sA} \xi\|_{H_{1/4}} \]

\[ \leq \left[ \sup \left\{ \left\{ \sup_{x \in (0,1)} |v(x)| : [v \in H_{1/4} \text{ and } \|v\|_{H_{1/4}} \leq 1] \right\} \right\} \right] \|\xi\|_{H_{1/4}} < \infty. \]  

(5.43)

Combining this with (5.41) implies that

\[ \limsup_{m \to \infty} \mathbb{E} \left[ \int_0^T \exp \left( \int_0^T p \phi(O_{m(h_m,x)}^m + P_m e^{[r]_{h_m,A}^m A} \xi) \, du \right) \right. \]

\[ \cdot \left. \max \left\{ 1, |\Phi(O_{m(h_m,x)}^m + P_m e^{[r]_{h_m,A}^m A} \xi)|^{n/2} \right\} \, dr \right] < \infty. \]  

(5.44)

In the next step observe that Lemma 5.7 and (5.34) prove that for all \( m \in \mathbb{N}, s \in [0,T] \) it holds that

\[ \mathbb{E} \left[ \|O_s^m\|_H^2 \right] \leq \mathbb{E} \left[ \sup_{x \in (0,1)} |O_s^m(x)|^2 \right] \leq \mathbb{E} \left[ \exp \left( 4pT \varepsilon \left( \sup_{x \in (0,1)} |O_s^m(x)|^2 \right) \right) \right] \]

\[ \leq \mathbb{E} \left[ \exp \left( 4pT \varepsilon \left( \sup_{x \in (0,1)} |O_s^m(x)|^2 \right) \right) \right] . \]
The triangle inequality and the fact that $\forall x, y \in \mathbb{R}: |x + y|^p \leq 2^{p-1}|x|^p + 2^{p-1}|y|^p$ hence show that

$$\limsup_{m \to \infty} \sup_{s \in [0, T]} \mathbb{E}\left[\|O^m_s + P_m e^{sA}\xi\|^p_H\right] \leq 2^{p-1} \limsup_{m \to \infty} \left( \sup_{s \in [0, T]} \mathbb{E}\left[\|O^m_s\|^p_H\right] + \sup_{s \in [0, T]} \|P_m e^{sA}\xi\|^p_H\right) \leq \frac{13((p/2)+1)!}{2^{p(p/2)+p/2}} + 2^{p-1}\|\xi\|^p_H < \infty.$$  

(5.46)

Combining this with (5.44) completes the proof of Corollary 5.1.

**Lemma 5.8.** Assume the setting in Section 5.1, let $p \in [2, \infty)$, $n \in \mathbb{N}$, $\varepsilon \in [0, \beta/2 - \varrho)$, let $O: [0, T] \times \Omega \to \mathcal{H}$ be a stochastic process, and assume for all $t \in [0, T]$ that $|O_t|_{\mathbb{P}, \mathcal{H}} = \int_0^t e^{(t-s)A} B dW_s$. Then

$$\sup_{t \in [0, T]} \|O_t - O^n_t\|_{\mathbb{P}, \mathcal{H}} \leq \left[ \frac{p(p-1)T(\beta-2\varrho-2\varepsilon)}{2(\beta-2\varrho-2\varepsilon)} \right]^{1/2} \|B\|_{HS(H, H(\beta-1)/2)} n^{-\varrho \varepsilon}.$$  

(5.47)

**Proof.** First, observe that the Burkholder-Davis-Gundy-type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] implies that for all $t \in [0, T]$ it holds that

$$\|O_t - O^n_t\|_{\mathbb{P}, \mathcal{H}} \leq \left[ \int_0^t \| (\text{Id}_H - P_n) e^{(t-s)A} B dW_s \|_{\mathbb{P}, \mathcal{H}} \right]^{1/2}.$$  

(5.48)

Next, note that the fact that $\forall q \in [0, 1], t \in (0, \infty): \|(\text{Id}_H - P_n) e^{tA}\|_{L(H)} \leq t^{-q}$ proves for all $t \in [0, T]$ that

$$\int_0^t \| (\text{Id}_H - P_n) e^{(t-s)A}\|_{L(H(\beta-1)/2, \mathcal{H})}^2 ds$$

$$\leq \int_0^t \| (\text{Id}_H - P_n) e^{(t-s)A}\|^2_{L(H(\beta-1)/2, \mathcal{H})} ds$$

$$\leq \| (\text{Id}_H - P_n)\|^2_{L(H(\beta-1)/2, \mathcal{H})} \int_0^t \| (\text{Id}_H - P_n) e^{(t-s)A}\|^2_{L(H(\beta-1)/2, \mathcal{H})} ds$$

$$\leq (\lambda n + 1)^{-2\varepsilon} \int_0^t s^{-2(\varrho + \varepsilon + (1-\beta)/2)} ds \leq \frac{n^{-8\varepsilon} t^{(\beta-2\varrho-2\varepsilon)}}{(\beta-2\varrho-2\varepsilon)}.$$  

(5.49)

This together with (5.48) yields that for all $t \in [0, T]$ it holds that

$$\|O_t - O^n_t\|_{\mathbb{P}, \mathcal{H}} \leq \left[ \frac{p(p-1)T(\beta-2\varrho-2\varepsilon)}{2(\beta-2\varrho-2\varepsilon)} \right]^{1/2} \|B\|_{HS(H, H(\beta-1)/2)} n^{-\varrho \varepsilon}.$$  

(5.50)

The proof of Lemma 5.8 is thus completed. 

□
5.5. Strong convergence.

COROLLARY 5.2. Assume the setting in Section 5.1 and let $X: [0,T] \times \Omega \rightarrow H_0$ be a stochastic process with continuous sample paths which satisfies for all $t \in [0,T]$ that $[X_t]_{p,B(H)} = [e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s) \, ds]_{p,B(H)} + \int_0^t e^{(t-s)A} B dW_s$. Then it holds for all $p \in (0, \infty)$ that

$$\limsup_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E} \left[ \| X_t - X^n_t \|^p_H \right] = 0. \quad (5.51)$$

Proof. Throughout this proof let $\phi, \Phi \in \mathcal{M}(\mathcal{B}(H_1), \mathcal{B}([0,\infty)))$, $\varepsilon \in (0, \beta/2 - q)$, $p, r, q \in (0, \infty)$, $q \in (\max\{p,1/\beta, 1/4\}, \infty)$ satisfy for all $v \in H_1$ that $\phi(v) = \frac{3q}{2} + \frac{3}{16}[1 + \gamma] + \frac{1}{2}\|v\|^2_H$, and

$$\gamma \leq \frac{1}{7200q^2} \left[ \max\left\{ 1, \sum_{k \in \mathbb{Z}} \frac{\max\{k!^{4\beta}, 1\}(|k|_2^2 + |k|_4^2)}{\lambda_k} \right\} \right]^{-1} \cdot \left[ \sup\left\{ \left\{ \sup_{x \in (0,1)} |v(x)| : v \in C((0,1), \mathbb{R}) \text{ and } \|v\|_{W^{\beta,q}((0,1), \mathbb{R})} \leq 1 \right\} \right\} \right]^{-2}. \quad (5.52)$$

Next, note that Lemma 5.2 implies that for all $v, w \in H_1$ it holds that

$$\left( \langle v, F(v+w) \rangle - \phi(w)v + \frac{1}{2}\|v\|^2_H + \Phi(w) \right). \quad (5.53)$$

Moreover, Lemma 5.3 proves that for all $v, w \in H_1$ it holds that $F \in C(H_{1/16}, H_{-1/4})$ and

$$\| F(v) - F(w) \|_{H_{-1/4}} \leq \left( \frac{\gamma_{1/16}}{1 + \|v\|_{H_{1/16}} + \|w\|_{H_{1/16}}} \right) \|v - w\|_{H_{1/16}} < \infty. \quad (5.54)$$

In the next step observe that the Burkholder-Davis-Gundy-type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] shows that for all $n \in \mathbb{N}$, $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ it holds that

$$\left\| \int_0^{t_1} P_n e^{(t_1-s)A} B dW_s - \int_0^{t_2} P_n e^{(t_2-s)A} B dW_s \right\|_{L^p(\mathbb{P}; H_0)}^2 \leq q(q-1) \int_{t_1}^{t_2} \| e^{(t_2-s)A} B \|_{H^S(H, H_0)}^2 ds \leq q(q-1) \int_{t_1}^{t_2} \| (e^{(t_2-s)A} - e^{(t_2-s)A}) B \|_{H^S(H, H_0)}^2 ds \leq q(q-1) \|B\|_{H^S(H, H_{(\beta-1)/2})}^2 \cdot \left[ \int_{t_1}^{t_2} \| e^{(t_2-s)A} \|_{L(H_{(\beta-1)/2}, H_0)}^2 ds + \int_0^{t_1} \| e^{(t_1-s)A} (\text{Id}_H - e^{(t_2-t_1)A}) \|_{L(H_{(\beta-1)/2}, H_0)}^2 ds \right]. \quad (5.55)$$

The fact that $\forall r \in [0, 1], t \in (0, \infty): \| (A)^{-r}(e^{tA} - \text{Id}_H) \|_{L(H)} \leq t^r$ and the fact that $\forall r \in [0, 1], t \in (0, \infty): \| (A)^r e^{tA} \|_{L(H)} \leq t^{-r}$ therefore imply that for all $n \in \mathbb{N}$, $t_1, t_2 \in [0, T]$
with \( t_1 < t_2 \) it holds that
\[
\left\| \int_0^{t_1} P_n e^{(t_1-s)A} B dW_s - \int_0^{t_2} P_n e^{(t_2-s)A} B dW_s \right\|_{L^n(P; H_\epsilon)}^2 \\
+ \left\| \int_0^{t_1} e^{(t_1-s)A} B dW_s - \int_0^{t_2} e^{(t_2-s)A} B dW_s \right\|_{L^n(P; H_\epsilon)}^2 \\
\leq q(q-1) \| B \|_{HS(H,H, (\beta-1)/2)}^2 \left[ \int_{t_1}^{t_2} \| (A)^{(t_2-s)_/2} e^{(t_2-s)A} \|_{L(H)}^2 ds \\
+ \int_{t_1}^{t_2} \| (A)^{-(t_2-s)_{/2}} e^{(t_2-s)A} \|_{L(H)}^2 \| (A)_{/2} (t_2-t_1) \|^2_{L(H)} ds \right] \\
\leq q(q-1) \| B \|_{HS(H,H, (\beta-1)/2)}^2 \left[ (t_2-t_1)_{/2} (\beta-2_2) \right] \\
\cdot \left[ \int_{t_1}^{t_2} (t_2-s)_{/2} (\beta-2_2) ds + \int_{t_1}^{t_2} (t_2-s)_{/2} (\beta-2_2) ds \right] \\
\leq \frac{2q(q-1) \| B \|_{HS(H,H, (\beta-1)/2)}^2 T_{/2} (\beta-2_2) T_{/2} (t_2-t_1)^{\beta-2_2}}{\left( \beta-2_2 \right)}.
\] (5.56)

Combining this with the Kolmogorov-Chentsov theorem and the fact that \( q \epsilon > 1 \) yields that there exist stochastic processes \( O: [0, T] \times \Omega \rightarrow H_\epsilon \) and \( \tilde{O}^n: [0, T] \times \Omega \rightarrow P_n(H), \) \( n \in \mathbb{N}, \) with continuous sample paths which satisfy for all \( n \in \mathbb{N}, t \in [0, T] \) that \( [O_t]_{\mathbb{P}; B(H)} = \int_\epsilon^t e^{(t-s)A} B dW_s \) and \( [\tilde{O}^n_t]_{\mathbb{P}, B(H)} = \int_\epsilon^t P_n e^{(t-s)A} B dW_s. \) Next, observe that Lemma 5.8 proves that for all \( n \in \mathbb{N} \) it holds that
\[
\sup_{t \in [0, T]} \left\| O_t - \tilde{O}^n_t \right\|_{L^n(P; H_\epsilon)} \leq \left[ \frac{\left( q(q-1) T_{/2} (\beta-2_2) \right)^{1/2}}{\left( \beta-2_2 \right)} \right] \| B \|_{HS(H,H, (\beta-1)/2)} n^{-\epsilon}.
\] (5.57)

This, the fact that \( O: [0, T] \times \Omega \rightarrow H_\epsilon \) and \( \tilde{O}^n: [0, T] \times \Omega \rightarrow P_n(H), \) \( n \in \mathbb{N}, \) are stochastic processes with continuous sample paths, (5.56), and Corollary 2.11 in Cox et al. [7] (with \( T = T, \ p = q, \ \beta = \epsilon, \ \theta^N = \{ k \}_{k \in \mathbb{N}} \in [0, \infty): k \in \mathbb{N} \cap [0, N] \}, \ (E, \ ||\cdot||_E) = (H_\epsilon, \ ||\cdot||_{H_\epsilon}), \ Y^N = ([0, T] \times \Omega \ni (t, \omega) \rightarrow \tilde{O}^n_t (\omega) \in H_\epsilon), \ Y^0 = O, \ \alpha = 0, \ \epsilon = \epsilon/2 \) for \( N \in \mathbb{N} \) in the notation of Corollary 2.11 in Cox et al. [7]) ensure that
\[
\sup_{n \in \mathbb{N}} \left( \sup_{t \in [0, T]} \left\| O_t - \tilde{O}^n_t \right\|_{H_\epsilon} \right) \leq \left\| B \right\|_{HS(H,H, (\beta-1)/2)} n^{-\epsilon}.
\] (5.58)

Lemma 3.21 in [21] (cf., e.g., Theorem 7.12 in Graham & Talay [12] and Lemma 2.1 in Kloeden & Neuenkirch [30]) together with the fact that \( \epsilon/2 - 1/q > 1/q \) hence yields that
\[
\mathbb{P} \left( \limsup_{n \to \infty} \sup_{s \in [0, T]} \left\| O_s - \tilde{O}^n_s \right\|_{H_\epsilon} = 0 \right) = 1.
\] (5.59)

In the next step observe that for all \( n \in \mathbb{N}, t \in [0, T] \) it holds that
\[
\| (\text{Id}_H - P_n) e^{tA} \|_{H_\epsilon} \leq \| (A)^{-1/4} (\text{Id}_H - P_n) \|_{L(H)} \| \xi \|_{H_1/4} \\
= \left| \lambda_{n+1} \right|^{1/4} \| \xi \|_{H_1/4} \leq n^{1/4} \| \xi \|_{H_1/4}.
\] (5.60)
Combining this with (5.59) proves that
\[
\mathbb{P}\left(\limsup_{n \to \infty} \sup_{s \in [0,T]} \| (O_{s} + e^{sA} \xi) - (\tilde{O}_{s}^{n} + P_{n} e^{sA} \xi) \|_{H_{\varphi}} = 0 \right) = 1. \tag{5.61}
\]
Moreover, note that the fact that \( \forall n \in \mathbb{N}, t \in [0,T] : \mathbb{P}(O_{t}^{n} = \tilde{O}_{t}^{n}) = 1 \) and (5.1) ensure that for all \( n \in \mathbb{N}, t \in [0,T] \) it holds that
\[
\mathbb{P}\left( X_{t}^{n} = P_{n} e^{tA} \xi + \tilde{O}_{t}^{n} + \int_{0}^{t} e^{(t-s)A} \mathbb{1}_{\{ \|= X_{[s]}^{n} \|_{H_{\varphi}} + \| \tilde{O}_{[s]}^{n} + P_{n} e^{sA} \xi \|_{H_{\varphi}} \leq |h_{n}| - \varepsilon \}} F(\tilde{X}_{[s]}^{n}) ds \right) = 1. \tag{5.62}
\]
In addition, Corollary 5.1, (5.52), and again the fact that \( \forall n \in \mathbb{N}, t \in [0,T] : \mathbb{P}(O_{t}^{n} = \tilde{O}_{t}^{n}) = 1 \) show that
\[
\limsup_{m \to \infty} \mathbb{E} \left[ \int_{0}^{T} \exp \left( \int_{T}^{t} \Phi(\tilde{O}_{[r]}^{m} + P_{m} e^{[r]A} \xi) \right) dr \right] + \limsup_{m \to \infty} \sup_{s \in [0,T]} \mathbb{E} \left[ \| \tilde{O}_{s}^{m} + P_{m} e^{sA} \xi \|_{H_{\varphi}}^{q} \right] < \infty. \tag{5.63}
\]
Combining (5.53)-(5.54), (5.61)-(5.63), the fact that \( p \in (0,q) \), the fact that \( \mathbb{P}(X_{t} = \int_{0}^{t} e^{(t-s)A} F(X_{s}) ds + O_{t} + e^{tA} \xi) = 1 \), and Item (v) in Theorem 4.1 (with \( \mathbb{H} = \{ e_{k} \in \mathbb{H} : k \in \mathbb{Z} \} \), \( \alpha = 1/4 \), \( \varphi = 1/2 \), \( \rho = 1/16 \), \( \varrho = \varphi \), \( \theta = \eta^{1/16} + |\kappa| \sup_{u \in H_{1/16} \setminus \{0\}} \| u \|_{L^{2}(\omega,0,1;B)/\| u \|_{H_{1/16}}} \), \( \vartheta = 1 \), \( p = q \), \( F = F \), \( \phi = \phi \), \( \Phi = \Phi \), \( H_{n} = \{ e_{k} \in \mathbb{H} : k \in \{ -n, -n+1, \ldots, n-1, n \} \} \), \( h_{n} = h_{n} \), \( X_{n} = \{ [0,T] \times \Omega \ni (t,\omega) \mapsto X_{t}^{n}(\omega) \in H_{\varphi} \} \), \( O_{n} = \{ [0,T] \times \Omega \ni (t,\omega) \mapsto (\tilde{O}_{t}^{n}(\omega) + P_{n} e^{tA} \xi) \in H_{\varphi} \} \), \( X = X \), \( O = \{ [0,T] \times \Omega \ni (t,\omega) \mapsto (O_{t}(\omega) + e^{tA} \xi) \in H_{\varphi} \} \), \( q = p \) for \( n \in \mathbb{N} \) in the notation of Theorem 4.1) completes the proof of Corollary 5.2.

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