REALIZATION OF $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ AT THE CRITICAL LEVEL

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Abstract. An explicit realization of the affine Lie algebra $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ at the critical level is constructed using a mixture of bosons and parafermions. Subsequently a representation of the associated Lepowsky-Wilson $\mathcal{Z}$-algebra is given on a space of the tensor product of bosonic fields and certain semi-infinite wedge products.

1. Introduction

Affine Lie algebras form an important class of Kac-Moody Lie algebras (cf. [9]). The first nontrivial construction of a representation of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ was discovered in terms of certain differential operators [14] called vertex operators acting on a Fock Space. This explicit realization of affine Lie algebras by vertex operators initiated a flurry of activities which led to many important connections of affine Lie algebra representations with other areas of mathematics and physics. It also led to the discovery of new algebraic structures such as vertex operator algebras (cf. [1, 7, 12]).

In [15, 16] Lepowsky and Wilson introduced a certain nonassociative algebra called a $\mathcal{Z}$-algebra associated with an integrable highest-weight module of an affine Lie algebra and a suitable infinite-dimensional Heisenberg subalgebra. In particular, the $\mathcal{Z}$-algebra is generated by certain operators called $\mathcal{Z}$-operators centralizing the action of the Heisenberg subalgebra on the representation space, hence acting on a special subspace called the vacuum space. It was shown in [16] that the representation space is irreducible if and only if the vacuum space is irreducible as a $\mathcal{Z}$-algebra module. In [13], Lepowsky and Primc used the $\mathcal{Z}$-algebra theory to give explicit constructions of all integrable highest weight $\widehat{\mathfrak{sl}_2(\mathbb{C})}$-modules at positive integral level in the homogeneous gradation. Such $\mathcal{Z}$-algebras are essentially equivalent to parafermion algebras as explained in [2]. The construction was extended to that of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2(\mathbb{C})})$ in [8].

On the other hand, Wakimoto [18] gave a free field realization of the affine Lie algebra $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ at an arbitrary level. Subsequently, this construction was extended to other affine Lie algebras by Feigin and Frenkel [4, 5]. Geometric realizations of the Wakimoto representation at the critical level were given in [6]. For a generic weight $\lambda$ at the critical level $-2$ the graded dimension of the simple $\widehat{\mathfrak{sl}_2(\mathbb{C})}$-module $L(\lambda)$ is known (cf. [10], [3], [11], [17]).

Motivated by explicit constructions in [13, 8] at the positive levels, we present an explicit realization of the affine Lie algebra $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ at the critical level $-2$. The corresponding representation involves an interesting Clifford-type (or parafermion) algebra and is simpler than the Wakimoto module. It would be interesting to compare our new construction with the level 2 $\widehat{\mathfrak{sl}_2(\mathbb{C})}$-modules constructed in [8]. Finally, we give a representation of the associated Lepowsky-Wilson $\mathcal{Z}$-algebra and determine its graded dimension. It is worthwhile
to note that this $\mathbb{Z}$-algebra realization at the critical level $-2$ is different from the general construction of [13] for positive levels.

2. Preliminaries

The simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by $X$, $Y$, and $H$, where

\[
X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

The roots of $\mathfrak{sl}_2(\mathbb{C})$ with Cartan subalgebra $CH$ are $\{\alpha, -\alpha\}$ with $\alpha(H) = 2$.

The affine Lie algebra $\widehat{\mathfrak{sl}}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $c$ is the central element and $d = 1 \otimes \frac{d}{dt}$ is the degree derivation. For any $a \in \mathfrak{sl}_2(\mathbb{C})$ and $m \in \mathbb{Z}$, we denote $a(m) = a \otimes t^m \in \widehat{\mathfrak{sl}}_2(\mathbb{C})$. For any $a, b \in \mathfrak{sl}_2(\mathbb{C})$ and $m, n \in \mathbb{Z}$, the commutation relations in $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ are given by:

\[
\begin{cases}
[a(m), b(n)] = [a, b](m + n) + \text{Tr}(ab)mc\delta_{m,-n}, \\
[c, \widehat{\mathfrak{sl}}_2(\mathbb{C})] = 0, \\
[d, a(m)] = ma(m).
\end{cases}
\]

The affine Lie algebra $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ is generated by $h_0, h_1, e_0, e_1, f_0, f_1$ and $d$ where:

\[
\begin{align*}
e_0 &= Y(1), & e_1 &= X(0), \\
f_0 &= X(-1), & f_1 &= Y(0), \\
h_0 &= -H(0) + c, & h_1 &= H(0).
\end{align*}
\]

Let $\mathfrak{h} = \left( \bigoplus_{n \in \mathbb{Z}} CH(n) \right) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\mathfrak{h}_0 = CH(0) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the affine Cartan subalgebra. Then, for $\lambda \in \mathfrak{h}_0^\ast$, $\lambda(c)$ is called the level of $\lambda$, and for $\widehat{\mathfrak{sl}}_2(\mathbb{C})$, $\lambda(c) = -2$ is the critical level. We consider the Heisenberg subalgebra $\mathfrak{h}' = \left( \bigoplus_{n \neq 0} CH(n) \right) \oplus \mathbb{C}c$ of $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ and set $\mathfrak{h}'^\pm = \bigoplus_{n > 0} CH(\pm n)$.

It follows by (2.1) that $[H(m), H(n)] = 2mc\delta_{m,-n}$. Throughout this paper, we use the notations and settings of [7] and [12]. We define the action of the Heisenberg subalgebra $\mathfrak{h}'$ on the symmetric algebra $S(\mathfrak{h}'^-)$ as follows:

\[
c \cdot v = -2v,
\]

\[
H(-m) \cdot v = H(-m)v,
\]

\[
H(m) \cdot v = -4m\partial_{H(-m)}(v),
\]

for all $v \in S(\mathfrak{h}'^-), m > 0$, where $\partial_{H(-m)}$ denotes the formal partial derivative with respect to $H(-m)$.

Let $a(z) = \sum_{m \in \mathbb{Z}} a(m)z^{-m}$, for all $a \in \mathfrak{sl}_2(\mathbb{C})$. We call $a(m)$ the $m^{th}$ component of $a(z)$.

Let $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ be the formal delta function. Then using (2.1) we have:
\begin{align}
(2.2) \quad [H(z), X(w)] &= 2X(w)\delta\left(\frac{w}{z}\right), \\
(2.3) \quad [H(z), Y(w)] &= -2Y(w)\delta\left(\frac{w}{z}\right), \\
(2.4) \quad [X(z), Y(w)] &= H(w)\delta\left(\frac{w}{z}\right) + zw\partial_w\delta\left(\frac{w}{z}\right).
\end{align}

Now let \( V \) be any \( \widehat{\mathfrak{sl}_2}(\mathbb{C}) \)-module induced from that of the Heisenberg subalgebra \( \mathfrak{h}' \), we define the exponential operators on \( V \):

\[
E^\pm_+(z) = \exp\left(\sum_{n>0} \frac{H(-n)}{2n} z^n\right) \quad \text{and} \quad E^\pm_-(z) = \exp\left(\pm\sum_{n>0} \frac{H(n)}{2n} z^{-n}\right).
\]

The following identities are straightforward.

\textbf{Lemma 2.1.} \textit{(cf. [16])}

\[
E^\pm_+(z)E^\mp_+(z) = 1, \\
E^\pm_+(z)E^\mp_+(w) = E^\pm_+(w)E^\mp_+(z) \left(1 - \frac{w}{z}\right)^{-1}, \\
E^+_+(z)E^-_+(w) = E^-_+(w)E^+_+(z) \left(1 - \frac{w}{z}\right)^{}, \\
E^+_-(z)E^+_+(w) = E^+_+(w)E^+_-(z), \\
E^+_-(z)E^-_+(w) = E^-_+(w)E^-_-(z), \\
E^-_-(z)E^-_+(w) = E^-_+(w)E^-_-(z), \\

\partial_z \left( E^\pm_+(z)E^\pm_-(z) \right) = E^\pm_+(z) \left( \pm\sum_{n\neq 0} \frac{H(n)}{2n} z^{-n-1} \right) E^\pm_-(z).
\]

\textit{Proof.} The proof of all properties except the last one are similar to the corresponding results in [16]. We include the proof of the last property below.

\[
\partial_z \left( E^\pm_+(z)E^\pm_-(z) \right) = E^\pm_+(z)\partial_z \left[ \pm\sum_{n>0} \frac{H(-n)}{2n} z^n \right] E^\pm_+(z) + E^\pm_+(z)\partial_z \left[ \pm\sum_{n>0} \frac{H(n)}{2n} z^{-n} \right] E^\pm_-(z) \\
= E^\pm_+(z) \left[ \pm\sum_{n>0} \frac{H(-n)}{2} z^{n-1} \right] E^\pm_+(z) + E^\pm_+(z) \left[ \pm\sum_{n>0} \frac{H(n)}{2} z^{-n-1} \right] E^\pm_-(z) \\
= E^\pm_+(z) \left[ \pm\sum_{n\neq 0} \frac{H(n)}{2} z^{-n-1} \right] E^\pm_-(z)
\]
Now we define the $\mathbb{Z}$-operators on $V$ as follows:

\[ Z^+(z) = Z(\alpha, z) = E^+_+(z)X(z)E^+_-(z), \]
\[ Z^-(z) = Z(-\alpha, z) = E^+_+(z)Y(z)E^+_-(z). \]

Let $Z^\pm(z) = \sum_{m \in \mathbb{Z}} Z^\pm(m)z^{-m}$. The $\mathbb{Z}$-algebra $\mathbb{Z}$ is generated by the homogeneous components $Z^\pm(m)$, for all $m \in \mathbb{Z}$. As shown in [16], $V$ has the following tensor product decomposition:

\[ V \simeq S(\mathfrak{h}^-) \otimes \Omega(V), \]

where $\Omega(V) = \{ v \in V| \mathfrak{h}^+ \cdot v = 0 \}$ is called the vacuum space. The following lemma can be proved using standard techniques, as in [16].

**Lemma 2.2.** (cf. [16]) The $\mathbb{Z}$-operators $Z^\pm(m)$ commutes with the $H(n)$'s and hence with the Heisenberg algebra $\mathfrak{h}^\prime$.

As a consequence of Lemma 2.2, we have

\[ [E^\pm_+(z), Z^\pm(w)] = 0 = [E^\pm_-(z), Z^\pm(w)] \]
\[ [E^\pm_+(z), Z^-(w)] = 0 = [E^\pm_-(z), Z^+(w)]. \]

By Lemma 2.2, the $\mathbb{Z}$-operators centralize the action of the Heisenberg subalgebra and hence act on the vacuum space $\Omega(V)$. Thus $\Omega(V)$ is a $\mathbb{Z}$-module. As shown in [16], $V$ is irreducible if and only if $\Omega(V)$ is irreducible as a $\mathbb{Z}$-module.

For $\phi_1, \phi_2 = \pm \alpha$, we define the generalized commutator bracket as follows:

\[ (2.5) \quad [[Z(\phi_1, z), Z(\phi_2, w)]] = Z(\phi_1, z)Z(\phi_2, w) \left( 1 - \frac{w}{z} \right)^{(\phi_1, \phi_2)} - Z(\phi_2, w)Z(\phi_1, z) \left( 1 - \frac{z}{w} \right)^{(\phi_1, \phi_2)} \cdot \]

The following Lemma follows easily using Lemma 2.2 as in [16].

**Lemma 2.3.** (cf. [16]) The following relations hold:

\[ (2.6) \quad \begin{align*}
[[[Z^\pm(z), Z^\pm(w)]]] &= 0, \\
[[[Z^\pm(z), Z^- (w)]]] &= H(0)\delta \left( \frac{w}{z} \right) - 2w\partial_w \delta \left( \frac{w}{z} \right). 
\end{align*} \]

3. **Representation of $\widehat{sl}_2(\mathbb{C})$ at the Critical Level**

We define the Clifford-like algebra $\mathcal{A}$ generated by $\{ A(m), A^*(m) | m \in \mathbb{Z} + \frac{1}{2} \}$ subject to the relations:

\[ (3.1) \quad \{ A(m), A^*(n) \} = - \left( m^2 - \frac{1}{4} \right) \delta_{m+n,0}, \]
\[ (3.2) \quad \{ A(m), A(n) \} = 0 = \{ A^*(m), A^*(n) \}, \]

where $\{ , \}$ denotes the anti-commutator bracket.

For $a \in \{ A, A^* \}$ we define the formal generating series $a(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} a(n)z^{-n-\frac{1}{2}}$. 

Definition 3.1. For \( m, n \in \mathbb{Z} + \frac{1}{2}, a, b \in \{ A, A^* \} \) we define the normally ordered products
\[
: a(m)b(n) : = \begin{cases} 
  a(m)b(n), & m < 0 \\
  -b(n)a(m), & m > 0
\end{cases}
\]

The normally ordered products for the generating series are given by
\[
: a(z)b(w) : = \sum_{m,n \in \mathbb{Z} + \frac{1}{2}} : a(m)b(n) : z^{-m-\frac{1}{2}}w^{-n-\frac{1}{2}}
\]
for \( a, b \in \{ A, A^* \} \) and we have \( : a(z)b(w) : = - : b(w)a(z) : \). Therefore, \( : a(w)a(w) : = 0 \) for \( a \in \{ A, A^* \} \).

For \( a, b \in \{ A, A^* \} \) we define the contraction function \( : a(z)b(w) = a(z)b(w) - : a(z)b(w) : \).

Then by (3.1) and (3.2) we have
\[
\begin{align*}
A(z)A(w) &= 0 = A^*(z)A^*(w), \\
A(z)A^*(w) &= \frac{-2zw}{(z-w)^2} = A^*(z)A(w)
\end{align*}
\]

We consider a natural representation of the algebra \( \mathcal{A} \) as follows. The exterior algebra \( \Lambda(V) \) of the infinite-dimensional vector space \( V = \oplus_{i \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}_{i} \) is spanned by the finite wedge products \( u = i_1 \wedge \cdots \wedge i_k \), where \( i_1 < \cdots < i_k \), \( i_j \in \mathbb{Z} + \frac{1}{2} \). The set of indices appearing in \( u \) is called the support of \( u \), i.e., \( \text{supp}(u) = \{ i_1, \ldots, i_k \} \). We now consider the semi-infinite wedge space \( \Lambda := \Lambda^{\infty/2}(V) \), which is an infinite-dimensional \( \Lambda(V) \)-module spanned by the semi-infinite wedge products (of weight \( r \))
\[
v = i_1 \wedge i_2 \wedge \cdots \wedge i_k \wedge \cdots,
\]
where the support \( \text{supp}(v) = \{ i_1, i_2, \cdots \} \) satisfy the condition that \( i_1 < i_2 < \cdots < i_k < \cdots \) and \( i_k = k + r + 1/2 \) for a fixed \( r \in \mathbb{Z} \) and sufficiently large integer \( k \). Here \( \Lambda(V) \) acts on \( \Lambda^{\infty/2}(V) \) by concatenation. A general homogeneous element \( v \) in \( \Lambda \) is characterized by the condition that both \( \text{supp}(v) \cap (\mathbb{Z} + \frac{1}{2}) \) and \( (\mathbb{Z} + \frac{1}{2}) - \text{supp}(v) \) are finite subsets. Here \( \mathbb{Z}_+ \) (or \( \mathbb{Z}_- \)) denotes the set of non-negative (or negative) integers respectively.

The semi-infinite wedge products satisfy similar properties as finite wedge products such as adjacent factors anti-commute with each other:
\[
\tilde{i} \wedge \tilde{i} \wedge \cdots = -\tilde{i} \wedge \tilde{i} \wedge \cdots.
\]

It follows that \( \tilde{i} \wedge \tilde{i} \wedge \cdots = 0 \), for any \( \tilde{i} \). Also, it follows that the factors of \( v \) or any similar semi-infinite wedge product can always be placed into ascending numerical order, as described above in the definition of \( v \) (see Eq. (3.4)).

Since, for \( n_j \in \mathbb{Z}_+ + \frac{1}{2} \), only finitely many \( n_j \) are not in \( \text{supp}(v) \), it may then be convenient to use the notation \( \tilde{i} \) to imply that \( \tilde{i} \) is not a factor of \( v \). For each \( i \in \mathbb{Z} + \frac{1}{2} \) we define the operator \( \partial_i \) acting on \( \Lambda \) as follows. For any semi-infinite wedge product \( v \) as in Eq. (3.4), if \( i \notin \text{supp}(v) \) then \( \partial_i v = 0 \). Otherwise, if \( i_k = i \)
\[
\partial_i (i_1 \wedge \cdots \wedge i_k \wedge \cdots) = (-1)^{k+1}i_k \wedge i_{k+1} \wedge \cdots \wedge i_{k+2} \wedge \cdots.
\]
The action of $\partial_2$ is well-defined. In fact, without loss of generality suppose we rewrite the semi-infinite wedge product $v = i_1 \wedge \cdots \wedge i_k \wedge \cdots = -i_1 \wedge \cdots \wedge i_{k+1} \wedge i_{k+2} \wedge \cdots$. Since $i_k = i$ is now in the $(k+1)$th position, our rule gives that

$$\partial_2(v) = \partial_2(-i_1 \wedge \cdots \wedge i_{k+1} \wedge i_k \wedge \cdots) = -(-1)^{k+2}i_1 \wedge i_2 \wedge \cdots \wedge i_k \wedge \cdots$$

which is the same as the right-hand side of (3.6).

An example of homogeneous semi-infinite wedge vector $v$ is

$$v = -9/2 \wedge -3/2 \wedge -1/2 \wedge 1/2 \wedge 5/2 \wedge 7/2 \wedge 9/2 \wedge \ldots,$$

which has $\text{supp}(v) = \{-9/2, -3/2, -1/2\} \cup (Z_+ + 1/2) - \{3/2\}$.

Equivalently, we could write this example as

$$v = -9/2 \wedge -7/2 \wedge -5/2 \wedge -3/2 \wedge -1/2 \wedge 1/2 \wedge 3/2 \wedge 5/2 \wedge 7/2 \wedge 9/2 \wedge \ldots,$$

or

$$v = -\partial_{-5/2}(-9/2 \wedge -5/2 \wedge -3/2 \wedge -1/2 \wedge 1/2 \wedge 5/2 \wedge 7/2 \wedge 9/2 \wedge \ldots).$$

Now, define $\Lambda_{-1/2}$ to be the subspace of $\Lambda$ spanned by the vectors of the form

$$v = m_r \wedge \cdots \wedge m_1 \wedge -1/2 \wedge 1/2 \wedge n_1 \wedge n_2 \wedge \ldots.$$

In particular, we stress that for these vectors $v$, $-1/2 \in \text{supp}(v)$ and $1/2 \notin \text{supp}(v)$. Using the notation described above to highlight the factors omitted from $\text{supp}(v)$, we can write any vector in $\Lambda_{-1/2}$ as a linear combination of vectors of the form:

$$v = m_r \wedge \cdots \wedge m_1 \wedge -1/2 \wedge 1/2 \wedge n_1 \wedge n_2 \wedge \ldots,$$

where $m_r < \ldots < m_1 < \frac{-1}{2} < n_1 < \ldots < n_s$, for some positive integers $r$ and $s$. Note especially that the important characteristic of this $v$ is that the negative half-integers $m'_r$s are included in $\text{supp}(v)$ and the positive half-integers $n'_s$s are not included in $\text{supp}(v)$. Now for a homogeneous element $v \in \Lambda_{-1/2}$, as in (3.7), we define the degree of $v$ by

$$\deg(v) = \sum_{i=1}^r \left(-m_i - \frac{1}{2}\right) + \sum_{j=1}^s \left(n_j - \frac{1}{2}\right).$$

For example, let

$$w = -11/2 \wedge -5/2 \wedge -3/2 \wedge -1/2 \wedge 3/2 \wedge 7/2 \wedge 11/2 \wedge 15/2 \wedge 17/2 \wedge \ldots$$

Then,

$$\deg(w) = \left(\frac{11}{2} - \frac{1}{2}\right) + \left(\frac{5}{2} - \frac{1}{2}\right) + \left(\frac{3}{2} - \frac{1}{2}\right) + \left(\frac{5}{2} - \frac{1}{2}\right) + \left(\frac{9}{2} - \frac{1}{2}\right) + \left(\frac{13}{2} - \frac{1}{2}\right) = 20.$$
Define the action of the algebra $\mathcal{A}$ on $\Lambda_{\frac{1}{\hat{\gamma}}}$ as follows:

\begin{align}
A(m) \cdot v &= \left(m - \frac{1}{2}\right) m \wedge v, \\
A^*(m) \cdot v &= \left(m - \frac{1}{2}\right) \partial m(v),
\end{align}

where $v = m_1 \wedge \cdots \wedge m_k \wedge \frac{1}{2} \wedge \cdots \wedge \frac{1}{2} \wedge \cdots \wedge \frac{1}{2} \wedge \cdots$ in $\Lambda_{\frac{1}{\hat{\gamma}}}$, and $A(m), A^*(m) \in \mathcal{A}$. This action is extended by linearity to all of $\Lambda_{\frac{1}{\hat{\gamma}}}$.

**Theorem 3.2.** Under the above action $\Lambda_{\frac{1}{\hat{\gamma}}}$ is an $\mathcal{A}$-module.

**Proof.** It is sufficient to show that the action preserves relations (3.1) and (3.2) for $v = m_1 \wedge \cdots \wedge m_k \wedge \frac{1}{2} \wedge \cdots \wedge \frac{1}{2} \wedge \cdots \wedge \frac{1}{2} \wedge \cdots \in \Lambda_{\frac{1}{\hat{\gamma}}}$. First, we prove that (3.1) holds:

$$\{A(m), A^*(n)\} \cdot v = -\left(m^2 - \frac{1}{4}\right) \delta_{m-n,0} v.$$

For $m, n \in \mathbb{Z} + \frac{1}{2}$, there are four cases to consider. Namely,

(i) $m, -n \in \text{supp}(v)$,  
(ii) $m \notin \text{supp}(v), -n \in \text{supp}(v)$,  
(iii) $m \in \text{supp}(v), -n \notin \text{supp}(v)$, and  
(iv) $m, -n \notin \text{supp}(v)$.

We only prove that (3.1) holds in case (i) below since the other three cases are similar. In this case, we have

$$\{A(m), A^*(n)\} \cdot v = A(m) \cdot \left((n - \frac{1}{2}) \partial m(v)\right) + A^*(n) \cdot \left((m - \frac{1}{2}) m \wedge v\right)$$

$$= A(m) \cdot \left((n - \frac{1}{2}) \partial m(v)\right)$$

$$= \left(m - \frac{1}{2}\right) \left(n - \frac{1}{2}\right) m \wedge (\partial m(v)).$$

Clearly, $m \wedge (\partial m(v)) \neq 0$ if and only if $m = -n$. Hence, when $m, -n \in \text{supp}(v)$, we have $\{A(m), A^*(n)\} \cdot v = -(m^2 - \frac{1}{4}) v\delta_{m,-n}$ proving (3.1).

Now to show that $\{A(m), A(n)\} \cdot v = 0$ holds, we observe that

$$\{A(m), A(n)\} \cdot v = A(m)A(n) \cdot v + A(n)A(m) \cdot v$$

$$= A(m) \cdot \left((n - \frac{1}{2}) m \wedge v\right) + A(n) \cdot \left((m - \frac{1}{2}) m \wedge v\right)$$

$$= \left(m - \frac{1}{2}\right) \left(n - \frac{1}{2}\right) m \wedge m \wedge v + \left(n - \frac{1}{2}\right) \left(m - \frac{1}{2}\right) m \wedge m \wedge v$$

$$= 0,$$

since $m \wedge n = -n \wedge m$.

It can be shown similarly using relation (3.6) that $\{A^*(m), A^*(n)\} \cdot v = 0$ also holds. □
Consider the group algebra $\mathbb{C}[\mathbb{Z}a]$ where, for all $\beta, \lambda \in \mathbb{Z}a$, we define the actions

\[
\begin{aligned}
&\begin{cases}
e^\beta \cdot e^\lambda = e^{\beta+\lambda},
d \cdot e^\lambda = -\frac{1}{n}(\lambda, \lambda)e^\lambda.
\end{cases}
\end{aligned}
\]

(3.11)

For any coweight $\beta(0)$ ($\beta \in \mathbb{Z}a$) we define $z^{\beta(0)} \cdot e^\lambda = z^{(\beta, \lambda)}e^\lambda$.

and we identify $\alpha(0)$ with $H(0)$.

**Theorem 3.3.** Let $V = S(\mathfrak{h}^-) \otimes \Lambda_{\frac{1}{2}, \frac{1}{2}} \otimes \mathbb{C}[\mathbb{Z}a]$. Define the map $\pi : \hat{\mathfrak{sl}}_2(\mathbb{C}) \rightarrow gl(V)$ by

\[
\begin{aligned}
X(z) &\mapsto E_+^+(z)E_+^-(z) \otimes A(z) \otimes e^\alpha z^{-\frac{\alpha(0)}{2}}, \\
Y(z) &\mapsto E_+^+(z)E_+^-(z) \otimes A^*(z) \otimes e^{-\alpha} z^{\frac{\alpha(0)}{2}}, \\
H(z) &\mapsto H^*(z) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes H(0), \\
c &\mapsto -2, \\
d &\mapsto d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d,
\end{aligned}
\]

where $H^*(z) = H(z) - H(0)$. Then $\pi$ is a representation of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ on $V$.

**Proof.** To see that $\pi$ defines a representation of $\hat{\mathfrak{sl}}_2(\mathbb{C})$, we simply need to show that the map $\pi$ preserves relations (2.2) – (2.4). For example, using (3.3) we have

\[
\begin{aligned}
\pi(X(z))\pi(Y(w)) &= E_+^+(z)E_+^-(z)E_+^-(w)E_--(w) \otimes A(z)e^\alpha z^{-\frac{\alpha(0)}{2}}A^*(w)e^{-\alpha}w^{\frac{\alpha(0)}{2}} \\
&= E_+^+(z)E_+^-(w)E_+^+(z)E_--(w) \otimes A(z)A^*(w) : (z-w)z^{\frac{\alpha(0)}{2}}w^{\frac{\alpha(0)}{2}} \\
&\quad + E_+^+(z)E_+^-(w)E_+^+(z)E_--(w) \otimes \left(\frac{-2zw}{(z-w)^2}\right)w^{\frac{\alpha(0)}{2}},
\end{aligned}
\]

and

\[
\begin{aligned}
\{\pi(X(z)), \pi(Y(w))\} &= -2E_+^+(z)E_+^-(w)E_+^+(z)E_--(w) \left(\frac{w}{z}\right)^{\frac{\alpha(0)}{2}} \left[ w\partial_w \delta \left(\frac{w}{z}\right) \right] \\
&= -2w\partial_w \left[ E_+^+(z)E_+^-(w)E_+^+(z)E_--(w) \left(\frac{w}{z}\right)^{\frac{\alpha(0)}{2}} \delta \left(\frac{w}{z}\right) \right] \\
&\quad + 2w\partial_w \left[ E_+^+(z)E_+^-(w)E_+^+(z)E_--(w) \left(\frac{w}{z}\right)^{\frac{\alpha(0)}{2}} \delta \left(\frac{w}{z}\right) \right] \\
&\quad + 2wE_+^+(z)E_+^-(w)E_+^+(z)E_--(w)\partial_w \left[ \left(\frac{w}{z}\right)^{\frac{\alpha(0)}{2}} \delta \left(\frac{w}{z}\right) \right] \\
&= -2w\partial_w \delta \left(\frac{w}{z}\right) + \sum_{n \neq 0} H(n)w^{-n} \delta \left(\frac{w}{z}\right) + \alpha(0) \delta \left(\frac{w}{z}\right) \\
&= \sum_{n \neq 0} H(n)w^{-n} \delta \left(\frac{w}{z}\right) + H(0) \delta \left(\frac{w}{z}\right) - 2w\partial_w \delta \left(\frac{w}{z}\right)
\end{aligned}
\]
Consider Theorem 3.6. It is known (cf. [10, 11, 17]) that the graded $\delta$-dimension of an irreducible module is $\dim_q(V) = \prod_{n \geq 0} (1 + q^n)^2 \sum_{\nu \in \mathbb{Z}^2} q^\nu$. Therefore, $\pi$ is a representation of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ on $V$.

Now, with the degree defined in (3.8, 3.11), the following result is immediate.

**Corollary 3.4.** The graded $q$-dimension of $V$ is 
$$\dim_q(V) = \prod_{n \geq 0} (1 + q^n)^2 \sum_{\nu \in \mathbb{Z}^2} q^\nu.$$  

**Remark 3.5.** It is known (cf. [10, 11, 17]) that the graded $q$-dimension of an irreducible $\hat{\mathfrak{sl}}_2(\mathbb{C})$ highest weight module $L(\lambda)$ of level $-2$ is $\dim_q(L(\lambda)) = \prod_{n \geq 0} (1 + q^n)^2$.

**Theorem 3.6.** Consider $v_0 = 1 \otimes -1/2 \wedge 3/2 \wedge 5/2 \wedge \cdots \otimes 1$, and $v_1 = 1 \otimes -1/2 \wedge 3/2 \wedge 5/2 \wedge \cdots \otimes e^{-\alpha}$ in $V$. Then, $v_0$ and $v_1$ are highest weight vectors with weights $\lambda_0 = -2\Lambda_0$ and $\lambda_1 = -2\Lambda_1 - \frac{1}{2}\delta$, respectively.

**Proof.** Recall that $e_0 = Y(1)$ and $e_1 = X(0)$. Then,

$$Y(z) \cdot v_0 = E_+(z) \cdot 1 \otimes \sum_{n \in \mathbb{Z} + \frac{1}{2}} A^\ast(n) \cdot (-1/2 \wedge 3/2 \wedge 5/2 \wedge \ldots) \cdot z^{-n/2} \otimes e^{-\alpha} z^\frac{n}{2} \cdot 1$$

$$= E_+(z) \otimes \sum_{n \leq \frac{1}{2}} \left( n - \frac{1}{2} \right) \partial_n (-1/2 \wedge 3/2 \wedge 5/2 \wedge \ldots) \otimes e^{-\alpha} z^{-n - \frac{1}{2}}.$$ 

We observe that as $Y(z)$ acts on $v_0$, only the components $Y(n)$ with $n < 0$ have non-zero action. Hence $e_0 = Y(1)$ and $f_1 = Y(0)$ annihilate $v_0$.

Similarly,

$$X(z) \cdot v_0 = E_-(z) \cdot 1 \otimes \sum_{n \in \mathbb{Z} + \frac{1}{2}} A(n) \cdot (-1/2 \wedge 3/2 \wedge 5/2 \wedge \ldots) \cdot z^{-n/2} \otimes e^{\alpha} z^{-\frac{n}{2}} \cdot 1$$

$$= E_-(z) \otimes \sum_{n \leq \frac{1}{2}} \left( n - \frac{1}{2} \right) n \wedge (-1/2 \wedge 3/2 \wedge 5/2 \wedge \ldots) \otimes e^{\alpha} z^{-n - \frac{1}{2}}.$$ 

Thus, $e_1 \cdot v_0 = X(0) \cdot v_0 = 0$. Furthermore,

$$f_0 \cdot v_0 = -2 \left( 1 \otimes -3/2 \wedge -1/2 \wedge 3/2 \wedge 5/2 \wedge \ldots \otimes e^{\alpha} \right).$$

Letting $Y(z)$ act on $f_0 \cdot v_0$, we easily see that $e_0 \cdot (f_0 \cdot v_0) = -2v_0$. Therefore, we have $h_0 \cdot v_0 = (e_0 f_0 - f_0 e_0) \cdot v_0 = e_0 \cdot (f_0 \cdot v_0) = -2v_0$, and $h_1 \cdot v_0 = (e_1 f_1 - f_1 e_1) \cdot v_0 = 0$. Using (3.8) and (3.11), we observe that $d \cdot v_0 = 0$. Hence, the weight of $v_0$ is $\lambda_0 = -2\Lambda_0$.

Also, $Y(z) \cdot v_1 = E_+(z) \otimes \sum_{n \leq \frac{3}{2}} (n - \frac{1}{2}) \partial_n (-1/2 \wedge 3/2 \wedge 5/2 \wedge \ldots) \otimes e^{-2\alpha} z^{-n - \frac{3}{2}}$, and

$$X(z) \cdot v_1 = E_-(z) \cdot 1 \otimes \sum_{n \leq -\frac{3}{2}} (n - \frac{1}{2}) n \wedge (-1/2 \wedge 3/2 \wedge 5/2 \wedge \ldots) \otimes 1 \cdot z^{-n + \frac{3}{2}}.$$
4. Representation of the \( \mathbb{Z} \)-algebra

Recall that the \( \mathbb{Z} \)-algebra \( \mathcal{Z} \) associated with \( \hat{\mathfrak{sl}}_2(\mathbb{C}) \) is generated by the operators \( \{ \mathcal{Z}^\pm(m) \mid m \in \mathbb{Z} \} \) which satisfy the relations:

\[
\begin{align*}
(4.1) & \quad \left[ \left[ \mathcal{Z}^\pm(z), \mathcal{Z}^\pm(w) \right] \right] = 0, \\
(4.2) & \quad \left[ \left[ \mathcal{Z}^+(z), \mathcal{Z}^-(w) \right] \right] = H(0)\delta \left( \frac{w}{z} \right) - 2w\partial_w \delta \left( \frac{w}{z} \right),
\end{align*}
\]

where \( \mathcal{Z}^\pm(z) = \sum_{m \in \mathbb{Z}} \mathcal{Z}^\pm(m)z^{-m} \) and \( \left[ \left[ , \right] \right] \) is the generalized commutator defined in (2.5).

We define a representation of the \( \mathbb{Z} \)-algebra \( \mathcal{Z} \) on the vacuum space \( \Omega(V) = \Lambda_{\frac{1}{T}} \otimes \mathbb{C} [z^\alpha] \).

**Theorem 4.1.** Define the map \( \pi_\Omega : \mathcal{Z} \rightarrow \text{End}(\Omega(V)) \) by

\[
\begin{align*}
\mathcal{Z}^+(z) & \mapsto A(z)e^\alpha z^{-\frac{\alpha}{2}}, \\
\mathcal{Z}^-(-z) & \mapsto A^*(z)e^{-\alpha} z^{-\frac{\alpha(0)}{2}}.
\end{align*}
\]

The map \( \pi_\Omega \) is a representation of \( \mathcal{Z} \) on \( \Omega(V) \).

**Proof.** It suffices to show that the map \( \pi_\Omega \) preserves the relations (4.1) and (4.2).

First, by (3.3) we have \( \pi_\Omega(\mathcal{Z}^+(z))\pi_\Omega(\mathcal{Z}^+(w)) = :A(z)A(w): e^{2\alpha(z^{-1}w-\frac{\alpha(0)}{2})} \). Hence

\[
\begin{align*}
\left[ \left[ \pi_\Omega(\mathcal{Z}^+(z)), \pi_\Omega(\mathcal{Z}^+(w)) \right] \right] &= :A(z)A(w): e^{2\alpha(z^{-1}w-\frac{\alpha(0)}{2})} \delta \left( \frac{w}{z} \right) \\
&= 0 = \pi_\Omega \left( \left[ \left[ \mathcal{Z}^+(z), \mathcal{Z}^+(w) \right] \right] \right).
\end{align*}
\]

Similarly, \( \left[ \left[ \pi_\Omega(\mathcal{Z}^-(z)), \pi_\Omega(\mathcal{Z}^-(w)) \right] \right] = 0 = \pi_\Omega \left( \left[ \left[ \mathcal{Z}^-(z), \mathcal{Z}^-(w) \right] \right] \right). \)

Now using (3.3) we have,

\[
\begin{align*}
\pi_\Omega(\mathcal{Z}^+(z))\pi_\Omega(\mathcal{Z}^-(w)) &= A(z)e^\alpha z^{-\frac{\alpha}{2}} A^*(w)e^{-\alpha} w^{-\frac{\alpha(0)}{2}} = \left( :A(z)A^*(w): - \frac{2zw}{(z-w)^2} \right) z \left( \frac{w}{z} \right)^{\frac{\alpha}{2}}.
\end{align*}
\]
Hence,
\[
\left[ [\pi_{\Omega}(Z^+(z)), \pi_{\Omega}(Z^-(w))] \right] = A(z)A^*(w): ((z-w) + (w-z)) \left( \begin{array}{c} w \\ z \end{array} \right) \frac{2}{\frac{w}{z}}
\]
\[
= -2 \left( \frac{zw}{(z-w)^2} - \frac{zw}{(w-z)^2} \right) \left( \begin{array}{c} w \\ z \end{array} \right) \frac{2}{\frac{w}{z}}
\]
\[
= -2w\partial_w \left[ \delta \left( \frac{w}{z} \right) \right] \left( \begin{array}{c} w \\ z \end{array} \right) \frac{2}{\frac{w}{z}}
\]
\[
= 2w\partial_w \left[ \left( \begin{array}{c} w \\ z \end{array} \right) \frac{2}{\frac{w}{z}} \delta \left( \frac{w}{z} \right) \right] - 2w\partial_w \left[ \left( \begin{array}{c} w \\ z \end{array} \right) \frac{2}{\frac{w}{z}} \delta \left( \frac{w}{z} \right) \right]
\]
\[
= H(0)\delta \left( \frac{w}{z} \right) - 2w\partial_w \delta \left( \frac{w}{z} \right) = \pi_{\Omega} (\left[ [Z^+(z), Z^-(w)] \right] ) .
\]

Thus, \( \pi_{\Omega} \) is a representation of the algebra \( Z \) on \( \Omega(V) \).

\[\text{Remark 4.2.}\] We note that Theorem 4.1 gives another proof that \( \pi \) is a representation of \( \widehat{sl}_2(\mathbb{C}) \) on \( V \) by the general theory of \( Z \)-algebras.

\[\text{Remark 4.3.}\] It follows from Corollary 3.4 and Theorem 4.1 that
\[
\dim_{q}(\Omega(V)) = \prod_{m \geq 0} (1 + q^m)^2 \sum_{p \in \mathbb{Z}} q^{\frac{p}{2}} .
\]

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