The Four Loop QCD Rapidity Anomalous Dimension

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ABSTRACT: The rapidity anomalous dimension controls the scaling of transverse momentum dependent observables in the Sudakov region. In a conformal theory it is equivalent to the soft anomalous dimension, but in QCD this relation is broken by anomalous terms proportional to the $\beta$-function. In this paper we first give a simple proof of this relation using two different representations of the energy-energy correlator observable. We then calculate the anomalous terms to three loops by computing the three-loop fully differential soft function to $\mathcal{O}(\epsilon)$. Combined with recent perturbative data from the study of on-shell form factors and splitting functions, this allows us to derive the four loop rapidity anomalous dimension in QCD.

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1 Introduction

Precision calculations in perturbative QCD play a crucial role in interpreting data in collider experiments. This is particularly true for key observables, such as QCD event shapes, which can be used for precision measurements of the strong coupling [1–4], or the Sudakov region of the Higgs transverse moment spectrum, where precision enables study of the coupling of the Higgs to light quarks [5, 6]. Due to progress in understanding the structure and universality of infrared evolution equations, as well as in fixed order calculations, these observables can be computed to next-to-next-to-next-to-leading logarithm (N^3LL) matched to next-to-next-to-leading order (NNLO) (see e.g. [2, 3, 7–21]).

In the last several years, there has been remarkable progress in extending calculations of a number of key gauge theory quantities to the four loop level. These include the β-function [22–25], the anomalous dimensions of twist-2 operators [26–29], the cusp anomalous dimension [26, 29–40], the quark and gluon collinear anomalous dimensions [31–34, 36, 37, 37, 40–42], and most recently the full form factors [43–45]. These anomalous dimensions govern the behavior of scattering amplitudes and cross sections in infrared limits, allowing the resummation of logarithms that appear in these limits, and which often invalidate a fixed order perturbative expansion, to N^4LL. They have recently seen their first phenomenological applications with the calculation of the four loop threshold corrections to Higgs production [46].
However, for a number of the most important phenomenological observables an additional anomalous dimension, the rapidity anomalous dimension, is required to describe the infrared dynamics. This is the case for the energy-energy-correlator (EEC) [47] in the Sudakov observable, a key observable for precision studies of QCD event shapes, as well as for the transverse momentum, $p_T$, spectrum of the Higgs/electroweak bosons at small $p_T$. More generally, the rapidity anomalous dimension controls the scaling evolution of transverse momentum dependent observables, as formulated in the Collins-Soper-Sterman approach [48–50] or in the rapidity renormalization group [51, 52] approach in SCET [53–57], and is therefore a universal anomalous dimension describing the infrared dynamics of QCD. The goal of this paper is to fill this gap, and compute the four loop rapidity anomalous dimension in QCD. For the EEC, this allows resummation at $N^4\text{LL}$ in the back-to-back region.\(^1\) For the case of the $p_T$ spectra of the Higgs or electroweak bosons, the complete four loop DGLAP evolution is also required. This is currently known for the non-singlet splitting functions [27], and will hopefully be known for the singlet combinations in the near future.

In this paper we compute the four loop rapidity anomalous dimension in QCD by exploiting its relation in a conformal field theory (CFT) to the soft anomalous dimension, $\gamma_S$. This relation was discovered through perturbative calculations to three loops in [58], and subsequently proven to all orders in [59, 60]. In a non-conformal theory this relation is broken by terms proportional to the $\beta$-function. The key advantage of exploiting this relation, is that since these corrections multiply $\beta$-functions, in perturbation theory they only need to be computed to one lower order. Therefore, we are able to compute the four loop anomalous dimension using known results for the soft function combined with a three loop calculation. Technically, we will obtain the anomalous terms by calculating the fully differential soft function [61] to $O(\epsilon)$ at three loops in dimensional regularization.\(^2\)

In addition to presenting the result for the four loop rapidity anomalous dimension in QCD, we also give an additional proof of the correspondence between the rapidity anomalous dimension and the soft anomalous dimension in a conformal theory using a recent derivation by Korchemsky of the back-to-back asymptotics of the EEC in a conformal field theory from a representation in terms of a Euclidean four point function [64]. The equivalence of this result with the timelike factorization derived in [65] provides a simple proof of this equivalence. We hope that the results of this paper clarify the relations between different anomalous dimensions used in the literature, and emphasizes the importance of understanding relations between them.

An outline of this paper is as follows. In Sec. 2 we summarize recent progress in the calculation of four loop anomalous dimensions which we will use in our calculation of the four loop anomalous dimension. In Sec. 3, we review the definition of the rapidity

\(^1\)The five loop cusp anomalous dimension is also required to perform resummation at $N^4\text{LL}$, however, its effect should be minuscule. A rough estimate of the value of the five loop cusp can be obtained from the calculation of the five loop non-singlet anomalous dimensions [28]. Only the four loop anomalous dimensions in this paper are required to predict the singular structure at $N^3\text{LO}$.

\(^2\)This approach, as well as the result for the four loop anomalous dimensions were presented by the authors at World SCET 2020 [62] and at an online QCD seminar hosted by Xiangdong Ji [63].
anomalous dimension as the anomalous dimension of a particular configuration of Wilson lines. In Sec. 4 we present a simple proof of the correspondence between soft and rapidity anomalous dimensions in a conformal field theory using two different representations of the Sudakov asymptotics of the energy-energy correlators. In Sec. 5 we describe our calculation of the N^3LO fully differential soft function to \mathcal{O}(\epsilon) . In Sec. 6 we compute the difference between the soft anomalous dimension \gamma_S and the rapidity anomalous dimension \gamma_R at four loops, and hence derive the four loop QCD rapidity anomalous dimension. We conclude in Sec. 7. Additional perturbative data is collected in the Appendix, as well as the attached ancillary files.

2 Summary of Perturbative Data for Anomalous Dimensions

In this section we summarize known results for QCD anomalous dimensions that we will use as perturbative data, and discuss some relations between these anomalous dimensions. This will also allow us to state our conventions and notation, since a number of anomalous dimensions are given different names in different contexts.

Apart from the \beta-function, the simplest and most well studied anomalous dimensions in gauge theories are the anomalous dimensions of twist-2 spin-J operators, or equivalently, the moments of the spacelike splitting functions. In the limit \( S \rightarrow \infty \), the twist-2 anomalous dimensions behave as (see e.g. [66, 67])

\[
\gamma(J, \alpha_s) = \Gamma^{\text{cusp}}(\alpha_s)(\log J + \gamma_E) - B_\delta(\alpha_s) + \mathcal{O}(1/J) .
\] (2.1)

This is equivalent to the limit \( z \rightarrow 1 \) in the splitting functions, where they behave as [66, 68, 69]

\[
P(z, \alpha_s) = \frac{\Gamma^{\text{cusp}}(\alpha_s)}{(1 - z)_+} + B_\delta(\alpha_s)\delta(1 - z) + \cdots .
\] (2.2)

These limits define two anomalous dimensions, the cusp anomalous dimension [70], \( \Gamma^{\text{cusp}} \), which describes the leading divergences for a cusped Wilson line, and the virtual anomalous dimension, \( B_\delta \).

Significant perturbative data is available for both of these anomalous dimensions. In planar \( \mathcal{N} = 4 \) super Yang-Mills, both \( \Gamma^{\text{cusp}} \) [71, 72] and \( B_\delta \) [73–75] are known to all loops from integrability. In QCD (and in non-planar \( \mathcal{N} = 4 \) SYM), the cusp anomalous dimension was recently obtained analytically to four loops [39, 40, 76] following much earlier work (see [26, 29–38] for the analytic calculation of the planar and matter dependent terms, and [27, 29, 77–79] for numerical extractions). Due to an extensive program calculating splitting functions in QCD (see e.g. [26–29] for recent results), \( B_\delta \) is fully known analytically at three loops [80, 81] for both quarks and gluons. All color structures in \( B_\delta \) are also known at four loops for both quarks and gluons, either numerically, or analytically. In particular, the planar result is known analytically [27], as are results for the \( n_f^2 \) and \( n_f^3 \) terms [82, 83]. Updated numerical results for all additional color structures were given in [41] for the quark case, and in [46] for the gluon case. For convenience, these anomalous dimensions are summarized in the Appendix.
Our second source of perturbative data will come from the study of on-shell form factors. The logarithms in the on-shell form factors are well studied (see e.g. [84–91]). Here we follow the notation of [92]. The all orders formula for a form factor with momentum transfer $Q^2$, evaluated at the scale $\mu^2 = Q^2$ can be written

$$F(1, \alpha_s(Q), \epsilon) = \exp \left[ \frac{1}{2} \int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} \left( G(1, \alpha_s(\lambda, \epsilon), \epsilon) - \Gamma^{\text{cusp}}(\alpha_s(\lambda, \epsilon)) \log \frac{Q^2}{\lambda^2} \right) \right], \quad (2.3)$$

where $\epsilon = 2 - d/2$ is the dimensional regulator, and $\alpha_s(\lambda, \epsilon)$ is the $d$-dimensional running coupling at scale $\lambda$. As with the twist-2 operators, the leading logarithms are governed by the cusp anomalous dimension. The subleading logarithms are governed by the so called “collinear anomalous dimension”, $\gamma_G$,

$$\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} G(1, \alpha_s(\lambda, \epsilon), \epsilon) = \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_G(\alpha_s(\lambda, \epsilon)) + O(\epsilon^0), \quad (2.4)$$

where it is understood that all the poles are absorbed by the integration with $\gamma_G$. The collinear anomalous dimension is known analytically to four loops in planar $\mathcal{N} = 4$ SYM [93] (for an earlier numerical calculation, see [94]), and has been computed numerically in non-planar $\mathcal{N} = 4$ SYM [78, 79]. In QCD there has been an extensive program computing the four loop form factors. In particular, it was computed in the planar limit in [31, 33], the $n_f^3$ terms were computed in [32], the $n_f^2$ terms were computed in [34, 37] and quartic color factor contributions were computed in [36, 37, 41]. Very recently all matter dependent terms were obtained in [40] (using previous results e.g. [95, 96]), and finally the full four loop form factors were computed [43–45].

The anomalous dimension that we are ultimately interested in, and which is closely related to the rapidity anomalous dimension is the “soft anomalous dimension”, $\gamma_S$, which can be expressed in terms of the anomalous dimensions described above as [91, 92, 97]

$$\gamma_G - 2B_\delta = f_{\text{eik}} = -2\gamma_S. \quad (2.5)$$

The relation connecting collinear, virtual, and soft anomalous dimension, (2.5), can be easily understood by considering threshold resummation using SCET, where all these three anomalous dimension appear, see e.g. [98–101]. The soft anomalous dimension is sometimes also referred to as $f_{\text{eik}}$, since it can be thought of as an eikonal version of the collinear anomalous dimension. For a recent discussion of this relation and other similar relations, see [92]. This anomalous dimension controls soft gluon radiation from Wilson lines, as we will describe in more detail in Sec. 3, and describes for example, Drell-Yan or gluon fusion at threshold, or dijet event shapes in the back-to-back limit, see e.g. [101–103].

Using the known perturbative data for $\gamma_G$, and $B_\delta$, the soft anomalous dimensions were assembled using Eq. (2.5) in [41, 46] (additional data was recently provided in [40] has been incorporated in our results). This provides the following result for the four loop soft anomalous dimension in QCD:

$$\gamma^S_3 = C_A^3 C_F \left( - \frac{b^{(4)}_{FA}}{24} \frac{9311591}{13122} + \frac{2189}{9} \zeta_3^2 + \left( \frac{1057}{3} \zeta_4 + \frac{414602}{243} \right) \zeta_3 - \frac{53467}{27} \zeta_5 \right)$$
where \( C_r = C_F \) for fundamental Wilson lines and \( C_r = C_A \) for adjoint Wilson lines, and \( N_r \) is the dimension of representation, \( N_r = N_c \) for fundamental and \( N_r = N_c^2 - 1 \) for adjoint. We have defined here and below a perturbative expansion for a generic anomalous dimension \( \gamma(\alpha_s) \) as

\[
\gamma(\alpha_s) = \sum_{i=0}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^{i+1} \gamma_i \equiv \sum_{i=0}^{\infty} a_s^{i+1} \gamma_i .
\]

(2.7)

The \( n_f C_F^2 C_r \) and \( n_f C_F C_A C_r \) and all other coefficients in the collinear anomalous dimension were computed analytically (or with high numerical precisions ) in \([40, 42]\), and by combing all the knowledges for the anomalous dimensions, especially from the study of \( B_4 \) in \([41, 46]\), it turns out that our results only need four of the numeric values collected in Table 1 of the work \([41]\). The extra numeric values are linearly dependent on those in Table 2, and they are numerically consistent as compared with the numbers collected in Table 1 of \([41]\).
The coefficients of these quartic Casimirs in both the cusp and eikonal anomalous dimensions exhibit a generalized Casimir scaling, namely that the contributions to the eikonal anomalous dimension from quartic Casimirs are determined by only two functions, instead of three \([29, 107]\). We have employed the generalized Casimir scaling to present the soft function anomalous dimension for generic SU\((N_c)\) representation \(r\) in \((2.6)\). We will also exploit this relation later in this paper to obtain the rapidity anomalous dimensions for both quarks and gluons.

### 3 The Rapidity Anomalous Dimension

The rapidity anomalous dimension governs the rapidity evolution of TMD quantities. Its calculation typically requires a an definition of rapidity regulator. Here we adopt the exponential regulator \([108]\), which has been successfully applied at three-loop order \([58, 109–112]\). Other definitions of rapidity regulator can be found in \([51, 52, 113–118]\). We can define the following matrix element of soft Wilson lines (in the following we use fundamental Wilson lines as explicit example, while our conclusion apply equally to adjoint Wilson lines.)

\[
S_{EEC}(\vec{b}_\perp, \mu, \nu) = \lim_{\nu \to +\infty} \frac{1}{N_c} \text{Tr} \left[ T \left[ S_n^\dagger(0) S_n(0) \right] T \left[ S_n^\dagger \left( y_\nu(\vec{b}_\perp) \right) S_n \left( y_\nu(\vec{b}_\perp) \right) \right] \right] |0\rangle ,
\]

where

\[
y_\nu(\vec{b}_\perp) = \left( i b_0 / \nu, i b_0 / \nu, \vec{b}_\perp \right) , \quad b_0 = 2e^{-\gamma_E} ,
\]

and the soft Wilson line is defined as

\[
S_n(x) = P \exp \left( ig_s \int_0^\infty ds \, n \cdot A_s(x + st) \right) .
\]
This configuration is shown in Fig. 1. This particular matrix element describes soft gluons in transverse momentum resummation, and certain $e^+e^-$ dijet event shapes in the back-to-back limit. Here we have chosen the orientation of the Wilson lines as appropriate for the case of an $e^+e^-$ event shape. For the case of $q_T$, the orientation of the Wilson lines is flipped, but this does not change the associated anomalous dimensions [119, 120], at least through to three loops.

This matrix element exhibits divergences that are regulated by dimensional regularization, leading to an RG equation in $\mu$, and divergences that are regulated by $\nu$, leading to a rapidity renormalization group in $\nu$. The RG equation in $\mu$ is given by

$$\frac{dS_{EEC}(\vec{b}_\perp, \mu, \nu)}{d\mu} = \left[2\Gamma_{cusp}(\alpha_s) \ln \frac{\mu^2}{b^2} - 4\gamma_S(\alpha_s)\right] S_{EEC}(\vec{b}_\perp, \mu, \nu),$$

and is governed by the cusp anomalous dimension and the soft anomalous dimension, $\gamma_s$ described above. The evolution in $\nu$ is given by [51, 52, 58]³

$$\frac{dS_{EEC}(\vec{b}_\perp, \mu, \nu)}{d\nu} = \left[-2 \int \frac{d\mu^2}{\mu^2} \Gamma_{cusp}(\alpha_s(\mu)) + 4\gamma_R(\alpha_s(b_0/|\vec{b}_\perp|))\right] S_{EEC}(\vec{b}_\perp, \mu, \nu),$$

which is governed by the cusp anomalous dimension, and the rapidity anomalous dimension $\gamma_R$. Identical evolution equations hold for the case of transverse momentum resummation. The goal of this paper will be to compute the rapidity anomalous dimension to four loops in QCD, which completes the scale understanding of this matrix element at four loops.

This soft function was computed to three loops in [58], and there it was noticed that in a conformal theory one has the relationship

$$\gamma_R = \gamma_S.$$  

Note that the rapidity anomalous dimension, begin a evolution equation in rapidity, contains higher order terms in $\epsilon$, which are neglected in (3.6). This relation was subsequently generalized to all orders in [59, 60], where the relation

$$\gamma_R(\epsilon^*) = \gamma_S$$

was obtained, with explicit dependence of $\epsilon$ replaced by $\epsilon^*$, which is determined by equation $\beta^*(\epsilon^*) \equiv \frac{1}{\alpha_s} \beta(\alpha_s) - 2\epsilon^* = 0$ order-by-order in QCD. In other words, it is the critical dimension where QCD is conformal order by order in perturbation theory. This relationship allows us to compute $\gamma_R$ at a given loop order by knowing $\gamma_S$ at that loop order, and $\gamma_R$ to higher orders in $\epsilon$ at lower loop orders. This approach has been utilized in [59] to reproduce the three-loop $\gamma_R$ from a two-loop calculation and knowledge of the three-loop $\gamma_S$. We will adopt this approach in this paper. For a recent formal study of the uses of conformal symmetry in $d = 4 - 2\epsilon$, see [121].

³Note that the convention for $\gamma_R$ is such that it differs from $\gamma_r$ in [58] by a normalization, $2\gamma_R = \gamma_r$. Similarly, the soft anomalous dimension also differed by a factor of 2 compared with $\gamma_s$ in [58], $2\gamma_S = \gamma_s$. 

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Figure 1. The configuration of Wilson lines used to define the rapidity anomalous dimension. This configuration appears in the description of the EEC in the back-to-back limit, and a related configuration appears in the description of the $p_T$ spectrum.

4 Proof of Soft-Rapidity Correspondence Using Energy Correlators

Since we will exploit heavily the correspondence between the rapidity anomalous dimension and the soft anomalous dimension in a CFT, in this section we give an independent proof of this fact using the energy-energy correlator (EEC) observable. This proof exploits the universality of factorization, combined with the fact that the EEC also admits a representation in terms of a Euclidean four point function. By comparing two distinct results for the EEC we are able to prove $\gamma_R = \gamma_S$ in a conformal theory.

The EEC admits a timelike factorization formula \[65\] derived in SCET \[53–57\]

\[
\frac{d\sigma}{dz} = \frac{1}{4} \int_0^\infty db b J_0(bQ\sqrt{1-z}) H(Q,\mu_h) j_{EEC}^q(b, b_0/b, Q) j_{EEC}^{\bar{q}}(b, b_0/b, Q) S_{EEC}(b, \mu_s, \nu_s) \cdot \left( \frac{Q^2}{\mu_s^2} \right)^{2\gamma_R(\alpha_s(b_0/b))} \exp \left[ \int_{\mu_h^2}^{\mu_b^2} \Gamma_{\text{cusp}}(\alpha_s(\bar{\mu})) \ln \frac{b^2}{b_0^2} \right]
\]

\[
+ \int_{\mu_h^2}^{\mu_b^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left( \Gamma_{\text{cusp}}(\alpha_s(\bar{\mu})) \ln \frac{b^2}{b_0^2} + \gamma_H(\alpha_s(\bar{\mu})) \right) - \int_{\mu_h^2}^{\mu_b^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \gamma_S(\alpha_s(\bar{\mu})) \right],
\]

where $\gamma_H = -\gamma_G = -2B_3 + 2\gamma_S$ is the collinear anomalous dimension. This factorization formula is identical in structure to that for $p_T$ resummation. The latter has been proven rigorously, including the cancellation of Glauber gluons \[122\]. It has been used to compute the EEC at $N^3LL$ \[123\] using results for the three-loop transverse momentum dependent fragmentation functions \[111, 112\]. For other studies of the EEC in the back-to-back limit, see \[124–127\].

This formula applies in both conformal and non-conformal gauge theories, however, to prove the equality of the soft and rapidity anomalous dimensions, we can consider this
formula in the conformal limit. After setting all scales equal to their canonical values such that all large logarithms are absorbed into the Sudakov exponent,

\[ \mu_h = Q, \quad \mu_s = \frac{b_0}{b}, \quad \nu_s = \frac{b_0}{b}, \]

and using the fact that the coupling constant does not run in a conformal theory, we have

\[ \frac{d\sigma}{dz} = \frac{1}{4} \int_0^\infty db J_0(bQ\sqrt{1-z}) H(Q, \mu_h) j_{EEC}(b, b_0/b, Q) j_{EEC}(b, b_0/b, Q) S_{EEC}(b, \mu_s, \nu_s) \]

\[ \exp \left[ -\frac{1}{2} \Gamma_{cusp} \log^2 \left( \frac{b^2 Q^2}{b_0^2} \right) + 2B_\delta \log \left( \frac{b^2 Q^2}{b_0^2} \right) + 2(\gamma_R - \gamma_S) \log \left( \frac{b^2 Q^2}{b_0^2} \right) \right]. \]  

(4.3)

The EEC is special in that it also admits a representation as a four point Wightman function [128–130], enabling it to be studied using a completely different set of techniques from conformal field theory. In particular, the EEC can be written as

\[ \text{EEC} \sim \int d^4xe^{i\vec{x} \cdot \vec{q}} \langle 0 | J_\mu(0) \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) J_\mu(x) | 0 \rangle, \]

(4.4)

where

\[ \mathcal{E}(\vec{n}) = \int_0^\infty dt \lim_{r \to \infty} r^2 n^i T_{0i}(t, r\vec{n}), \]

(4.5)

is an energy flow operator [128–134]. It is therefore expressible as an integral kernel acting on the four point function of stress tensors. Using an understanding of the behavior of the four point function as operators become lightlike separated [135, 136] and the high spin limit of the twist two operators Eq. (2.1), Korchemsky proved in [64] that the EEC in a conformal theory is governed by the anomalous dimensions \( \Gamma_{cusp} \) and \( B_\delta \) only, and is described by the formula \(^4\)

\[ \frac{d\sigma}{dz} = \frac{1}{4} \int_0^\infty db J_0(bQ\sqrt{1-z}) H(Q, \mu_h) j_{EEC}(b, b_0/b, Q) j_{EEC}(b, b_0/b, Q) S_{EEC}(b, \mu_s, \nu_s) \]

\[ \exp \left[ -\frac{1}{2} \Gamma_{cusp} \log^2 \left( \frac{b^2 Q^2}{b_0^2} \right) + 2B_\delta \log \left( \frac{b^2 Q^2}{b_0^2} \right) + 2(\gamma_R - \gamma_S) \log \left( \frac{b^2 Q^2}{b_0^2} \right) \right]. \]

(4.6)

By comparing this formula derived from the four point function with the timelike factorization formula in Eq. (4.3), we immediately prove \( \gamma_R = \gamma_S \) in a CFT. We therefore arrive at a transparent proof of the relation between soft and rapidity anomalous dimensions.\(^5\)

\(^4\)See e.g. Eq. (1.5) of [64]. We have converted it to the conventions used in this paper.

\(^5\)It is interesting to note that non-trivial relations between anomalous dimensions (in particular the Basso-Korchemsky reciprocity [137]) can also be derived by studying equivalent expressions for the EECs in the collinear limit. Ref. [138] provides an explicit example of such relation at three loops in QCD. On the one hand they can be expressed in terms of spacelike anomalous dimensions using the light-ray OPE [64, 128, 139], and on the other hand they can be expressed in terms of timelike anomalous dimensions using factorization formulas derived in SCET [140, 141]. These relations suggest much is still to be understood about Lorentzian dynamics.
Although this relation is sufficient for our purposes, it would be interesting to extend this to study the behavior of rapidity anomalous dimensions for soft functions that have Wilson lines in greater than two directions. Higher point energy correlators exhibit Sudakov behavior in particular kinematic limits, and this has interesting applications at hadron colliders [124]. These singular regions can be accessed also on the correlator side. However, unlike the more well studied case [135], for three-point correlators and higher the operators become lightlike separated, but not in a sequential limit.

5 The $N^3LO$ Fully Differential Soft Function to $\mathcal{O}(\epsilon)$

In this section we describe the methods used to compute the $N^3LO$ fully differential soft function to $\mathcal{O}(\epsilon)$ at three loops, from which we will extract the rapidity anomalous dimension, retaining higher-order dependence in $\epsilon$. The fully differential soft function was introduced in [61] for the study of transverse momentum resummation. It has been used in [142] as a deformation of the threshold soft function, and in [108] as a deformation of TMD soft function. It has also been exploited to calculate Sudakov radiator for jet observable [143].

It is mostly conveniently defined on the exponent,

$$F(q) = \int \prod_i \frac{d^d q_i}{(2\pi)^d} \frac{\delta_+(q_i^2)(2\pi)^d}{\delta(q - \sum_i q_i)} \mathcal{W}_{n\bar{n}}(\{q_i\}),$$

(5.1)

where additional symmetry factor for identical final-state particle is implicitly understood. The integrand $\mathcal{W}_{n\bar{n}}(\{q_i\})$ corresponds to single or multiple soft emissions web diagrams from two lightlike Wilson lines $S_n$ and $S_{\bar{n}}$, such that non-abelian exponentiation theorem is made manifest whenever the observable have that property. Definition of web diagrams can be found, e.g., in [144, 145]. The main task of the work here is to calculate the NNNLO fully differential soft function to higher orders in $\epsilon$. The TMD soft function [58] and the threshold soft function [103, 146–148] can then all be obtained as limits of the fully differential soft function.

To perform the calculation, Feynman diagrams were generated by Qgraph [149], color/Dirac algebra and integrand manipulations were performed in form [154]. With the help of reverse unitarity [155], the integrals can be reduced with Integration-By-Parts identities [156]. We use the publicly available Mathematica package LiteRed [157]. We obtained systems of master integrals to formulate differential equations with respect to $n$ and $\bar{n}$ rescaling invariant $x$,

$$x \equiv \frac{q^2 (n \cdot \bar{n})}{2(q \cdot n)(q \cdot \bar{n})} = \frac{q^2}{q^2 q^2},$$

(5.2)

which can be solved using the method of differential equation [158]. Note that the final-state total momentum satisfy $q^2 \geq 0$, $x \in [0, 1]$.

We note that several results for the soft web diagrams are available in the literature, namely the two-loop single soft current [147, 148], one-loop double soft current [119, 120], and tree-level triple soft current [150]. Furthermore, three-loop threshold function has also been obtained in [103, 146, 151–153]. We choose to generate the relevant soft currents using an independent code, therefore providing additional checks on the existing results.
The differential equations allow an $\epsilon$-form \[159\] with the help of CANONICA \[160\], after which we can derive the boundary constants by considering the limit $x \to 1$ and $x \to 0$.

To all orders in perturbation, the unnormalized results can be written as

$$F_b(q, \epsilon) = \sum_{j=0}^{\infty} (a_b^j)^{j+1} \frac{F_{T,b}^j(x, \epsilon)}{(q_\perp^2)^{2+j \epsilon}}, \quad \text{(5.3)}$$

where the variable $x$ is defined in \(5.2\) and $a_b^j = \alpha_b^j/(4\pi)$ is the bare coupling constant. We have retain the higher order in $\epsilon$ dependence in the definition of full differential soft function, in anticipating that further integration of $q$ can leads to poles in $\epsilon$. The bare perturbation results for $F_{T,b}^j(x, \epsilon)$, as well as the renormalized ones, are given in the ancillary file `fully.m` through to three loops.

### 6 Soft and Rapidity Anomalous Dimensions at $N^4$LO

In this section we present the main result of this paper, namely the four loop relation between the soft and rapidity anomalous dimensions in QCD obtained by computing the fully differential soft function at three loops to higher orders in $\epsilon$. We then combine this with the perturbative data summarized earlier to give the four loop rapidity anomalous dimension in QCD.

The bare TMD soft function with renormalized coupling is obtained by perform Fourier transformation and Laplace transformation to the fully differential soft function, and taking the rapidity regulator $\tau = 1/\nu \to 0$,

$$s^b(\vec{b}_\perp, \nu = 1/\tau, \mu) \equiv \int d^d q \exp \left(- (q \cdot n + q \cdot \vec{n}) \tau e^{-\gamma_E} - i\vec{b}_\perp \cdot \vec{q}_\perp \right) F_b(q, \epsilon)$$

$$= \sum_{j=0}^{\infty} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^{j+1} \int_0^1 dx \, P_j \left(x, |\vec{b}_\perp| \mu/b_0, \mu/\nu \right) F_{T,b}^j(x, \epsilon) (1-x)^{-2}, \quad \text{(6.1)}$$

where

$$P_j \left(x, |\vec{b}_\perp| \mu/b_0, \mu/\nu \right) \equiv \left( \frac{\tau^2 \mu^2}{b_0^2} \right)^{2+(j+1)\epsilon} \frac{\Gamma(-j+1)\epsilon^2}{\Gamma(1-\epsilon)^2} (1-x)^2$$

$$\times \, _2F_1 \left( -(j+1)\epsilon, -(j+1)\epsilon; 1-\epsilon; -\frac{\vec{b}_\perp^2 (1-x)}{\tau^2} \right)$$

$$= \frac{\Gamma(-j+1)\epsilon}{\Gamma(1+j\epsilon)} \left[ L_r + \ln(1-x) \right]$$

$$- H(-1 -(j+1)\epsilon) - H(j\epsilon), \quad \text{(6.2)}$$

and $L_r = \ln \left( \nu^2 \vec{b}_\perp^2 / b_0^2 \right)$ is the rapidity logarithm, $H$ are the harmonic numbers, and $F_{T,b}^j(x, \epsilon)$ are the expansion coefficients of the unrenormalized fully differential soft function. The gamma function $\Gamma(-(j+1)\epsilon)$ in \(6.2\) leads to at most double poles in $\epsilon$ for
\( s^b \), which should be renormalized in \( \overline{\text{MS}} \) scheme. After operator and coupling constant renormalization, the soft function still contains higher order in \( \epsilon \) dependence. Note that the soft function calculated above is really the logarithm of TMD soft function. The usual TMD soft function can be written as

\[
S(b, \nu, \mu, \epsilon) = \exp(s(b, \nu, \mu, \epsilon)).
\]  

(6.3)

With the renormalized, finite TMD soft function \( s(b_\perp, \nu, \mu, \epsilon) \), we obtain the rapidity anomalous from

\[
\gamma^R(\epsilon) \equiv \frac{1}{2} \left. \frac{\partial s(b_\perp, \nu, \mu, \epsilon)}{\partial L_r} \right|_{L_r \to 0}.
\]  

(6.4)

The explicit expression for \( \gamma^R(\epsilon) \) can be read off of the results given in App. A.3

\[
\gamma^R(\epsilon) = \frac{\alpha_s}{4\pi} \left\{ C_T \left( -\frac{9}{8} \zeta_4 \epsilon^3 - \frac{2}{3} \zeta_3 \epsilon^2 - \zeta_2 \epsilon \right) \right\} + \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_A C_T \left( \frac{46}{3} \zeta_3 \zeta_2 - \frac{404}{27} \zeta_2 - \frac{134}{27} \zeta_3 - \frac{55}{24} \zeta_4 + \frac{62}{9} \epsilon - \frac{14576}{243} \right) \epsilon^2 \right. \\
\left. + \left( -\frac{67}{9} \zeta_2 + 31 \zeta_4 - \frac{2428}{81} \right) \epsilon + 14 \zeta_3 - \frac{404}{27} \right\} + C_T N_f \left( \frac{56}{27} \zeta_2 + \frac{20}{27} \zeta_3 + \frac{5}{12} \zeta_4 + \frac{1952}{243} \zeta_2 + \frac{164}{81} \zeta_3 - \frac{172}{3} \epsilon - \frac{1433}{4} \zeta_4 - \frac{56}{3} \zeta_5 \right) + C_T N_f \left( \frac{716509}{4374} \epsilon - \frac{412}{81} \zeta_2 - \frac{452}{27} \zeta_3 + \frac{10}{3} \zeta_4 + \frac{31313}{729} \right) \\
+ C_T^2 \left( \frac{77}{3} \zeta_4 - 96 \zeta_5 - \frac{297029}{1458} \right) + C_T N_f \left( \frac{100}{27} \zeta_2 - \frac{160}{27} \zeta_3 - \frac{8}{3} \zeta_4 + \frac{11584}{2187} \right) \epsilon - \frac{16}{9} \zeta_3 - \frac{928}{729} \right. \\
\left. + C_T N_f \left( \frac{55}{6} \zeta_2 - \frac{1384}{27} \zeta_3 - \frac{76}{3} \zeta_4 - \frac{112}{3} \zeta_5 + \frac{42727}{324} \right) \epsilon \\
- \frac{152}{9} \zeta_3 - 8 \zeta_4 + \frac{1711}{54} \right\} \right\} + \mathcal{O}(\alpha_s^4). 
\]  

(6.5)

The \( \mathcal{O}(\epsilon^0) \) terms at \( \alpha_s^3 \) appeared in [58], the higher order terms in \( \epsilon \) are new.

Using this result, we are now able to extract the rapidity anomalous dimension in QCD using the relation [59, 60]

\[
\gamma^R(\epsilon^*) = \gamma^S,
\]  

(6.6)
where again $\epsilon^*$ is known as the critical number of space-time dimension, whose value is determined by equation $\beta^*(\epsilon^*) \equiv \frac{1}{\alpha_s} \beta(\alpha_s) - 2\epsilon^* = 0$ order-by-order in QCD. Explicitly,

$$\epsilon^* = -a_s \beta_0 - a_s^2 \beta_1 - a_s^3 \beta_2 - \cdots . \quad (6.7)$$

The relation helps to build a bridge between different regimes of physics, such that the soft anomalous dimension can be obtained from rapidity anomalous dimension by substituting the dimensional regulator at its critical point, namely the QCD $\beta$-function (perturbative results for the $\beta$-function are collected in App. A.1). This allows us to derive an $N^4$LO relation between the soft and rapidity anomalous dimensions

$$\gamma^S_3 - \gamma^R_3 = CACrN_f \beta_0 \left( \frac{8}{9} \zeta_3 \zeta_2 - \frac{3925}{243} \zeta_2 + \frac{1264}{81} \zeta_3 + \frac{172}{3} \zeta_4 + \frac{56}{3} \zeta_5 - \frac{716509}{4374} \right) + C^2 ACr \beta_0 \left( \frac{680}{3} \zeta_3^2 - \frac{440}{9} \zeta_2 \zeta_3 - \frac{41548}{81} \zeta_3 + \frac{8237}{486} \zeta_4 - \frac{1675}{3} \zeta_5 \right) - 308 \zeta_5 + \frac{4409}{9} \zeta_6 + \frac{7135981}{8748} \right) + CA \beta_0 \left( \frac{46}{3} \zeta_3 \zeta_2 - \frac{404}{27} \zeta_2 - \frac{134}{27} \zeta_3 - \frac{55}{24} \zeta_4 + 62 \zeta_5 - \frac{14576}{243} \right) + \frac{16}{9} \beta_0 \left( \frac{67}{9} \zeta_2 - 31 \zeta_4 + \frac{2428}{81} \right) \right) + CA \epsilon_0 \left( \frac{56}{27} \zeta_2 + \frac{20}{27} \zeta_4 + \frac{5}{12} \zeta_4 + \frac{1952}{243} \right) + \beta_1 \left( - \frac{10}{9} \zeta_2 - \frac{328}{81} \right) + CA \epsilon_0 \left( \frac{100}{81} \zeta_2 + \frac{160}{27} \zeta_4 + \frac{8}{3} \zeta_4 + \frac{11584}{2187} \right) + CA \epsilon_0 \left( \frac{8 \zeta_3 \zeta_2 - \frac{55}{6} \zeta_2 + \frac{1384}{27} \zeta_3 + \frac{76}{3} \zeta_4 + \frac{112}{3} \zeta_5 - \frac{42727}{324} \right) + \frac{9}{8} \beta_0 \left( \frac{4}{3} \beta_1 \zeta_4 + \beta_2 \zeta_2 \right) = C^2 FCrN_f \zeta_2 + CACrN_f \zeta_2, \quad (6.8)$$
and this is the primary new perturbative ingredient presented in this paper.

Using the perturbative data from the splitting function and form factor anomalous dimensions presented earlier, this relation allows us to derive the four loop anomalous dimension in QCD

\[
\gamma_3^R = \frac{d_A^{abcd} d_r^{abcd}}{N_r} \left( b(d_F^{(4)}) + \frac{1672\zeta_2^2}{3} + \left( 184\zeta_4 + \frac{3904}{9} \right) \zeta_3 - \frac{1738}{9} \zeta_6 \right) \\
- \frac{56}{3} \zeta_4 + \frac{920}{9} \zeta_5 + \zeta_2 \left( 896\zeta_3 - 512\zeta_5 + \frac{1088}{3} \right) - 1742\zeta_7 - 96 \right) + C_A^2 C_r \left( - \frac{b(d_F^{(4)})}{24} \right) \\
- \frac{28290079}{8748} - \frac{5291}{9} \zeta_2^3 + \left( \frac{1057}{3} \zeta_4 + \frac{300436}{81} \right) \zeta_3 - \frac{45481}{27} \zeta_5 - \frac{15895}{27} \zeta_6 \right) \\
+ \frac{2072}{9} \zeta_4 + \frac{11071}{12} \zeta_7 + \left( - \frac{6526}{9} \zeta_3 + \frac{688}{3} \zeta_5 + \frac{389083}{486} \right) \zeta_2 \right) \\
+ N_f C_A^2 C_r \left( - \frac{1}{2} b \left( N_f C_F C_A C_r \right) \right) - \frac{1}{4} b \left( N_f C_F^2 C_r \right) - \frac{b(d_F^{(4)})}{48} - \frac{2146\zeta_3^2}{9} - \frac{61913}{81} \zeta_3 \\
- \frac{4484}{27} \zeta_5 + \frac{791}{54} \zeta_6 + \frac{10906}{27} \zeta_4 + \left( \frac{520}{3} \zeta_3 - \frac{91067}{486} \right) \zeta_2 + \frac{10761379}{11664} \right) \\
+ N_f C_A C_F C_r \left( b \left( N_f C_F C_A C_r \right) \right) - \frac{1700\zeta_2^2}{3} - \frac{30554}{27} \zeta_4 - \frac{473}{9} \zeta_3 + \frac{5476}{9} \zeta_5 - \frac{2216}{9} \zeta_3 \zeta_2 \\
+ \frac{2561}{54} \zeta_2 - \frac{359\zeta_6 + 2149049}{1944} + N_f C_F^2 C_r \left( b \left( C_F^2 \right) N_f \right) - \frac{184\zeta_2^2}{9} \zeta_3 - \frac{1936}{3} \zeta_5 + \frac{560}{9} \zeta_3 \\
+ \frac{7334}{9} \zeta_6 + \left( \frac{256}{3} \zeta_3 - 162 \right) \zeta_2 + 167 \zeta_4 + \frac{27949}{216} \right) + N_f^2 C_A C_r \left( \frac{40}{3} \zeta_4 + \frac{1732}{27} \zeta_3 + \frac{40}{9} \zeta_4 \right) \\
+ \frac{368}{9} \zeta_5 + \frac{56}{9} \zeta_3 + \frac{1688}{243} \zeta_2 - \frac{898033}{11664} \right) + N_f^2 C_F C_r \left( \frac{40}{3} \zeta_4 + \frac{1732}{27} \zeta_3 + \frac{40}{9} \zeta_4 \right) \\
- \frac{2272}{9} \zeta_2 + 192 + C_r N_f^2 \left( - \frac{4}{9} \zeta_4 + \frac{40}{9} \zeta_3 + \frac{2608}{2187} \right). \quad (6.9)
\]

For fundamental Wilson lines with generic \( N_f \), the numeric results through to four loops are

\[
\gamma_R^a = a_s^2(7.46333 + 2.76543 N_f) + a_s^3(70.068 + 77.1286 N_f - 4.54662 N_f^2) \\
+ a_s^4(-350.8 + 2428 N_f - 378.3 N_f^2 + 8.072 N_f^3) + \mathcal{O}(a_s^5), \quad (6.10)
\]

while for adjoint Wilson lines it is given by

\[
\gamma_R^b = a_s^2(16.7925 + 6.22222 N_f) + a_s^3(157.653 + 173.539 N_f - 10.2299 N_f^2) \\
+ a_s^4(333.8 + 5506 N_f - 851.2 N_f^2 + 18.16 N_f^3) + \mathcal{O}(a_s^5), \quad (6.11)
\]

These are the primary results of our paper.
In Fig. 2, we plot the perturbative rapidity anomalous dimension through to four loops in QCD, using fundamental Wilson lines as an example. The four-loop corrections are non-negligible, and should be taken into account in future phenomenological studies. We also show the four-loop results without the quartic Casimir corrections, which illustrates that these corrections are small.

For the case of $q_T$ resummation rapidity renormalization group consistency implies that the rapidity anomalous dimension given in Eq. (6.9) also determines the beam function rapidity anomalous dimension to four loops.

7 Conclusions

In this paper we have computed the four loop rapidity anomalous dimension in QCD. This result was derived by exploiting a relation between the rapidity anomalous dimension and the soft anomalous dimension in a conformal theory, and computing the corrections to this relation arising from conformal symmetry breaking in QCD. At a technical level this was achieved by computing the fully differential soft function to $O(\epsilon)$ at three loops. We also described in some detail how this calculation was performed with the hope that it can be useful for performing calculations of other rapidity divergent soft functions.

In this paper we have also attempted to further clarify the nature of the rapidity anomalous dimension by presenting a simple proof relating the soft and rapidity anomalous dimensions using two equivalent representations of the the energy-energy correlator observable in the back-to-back limit. This relates the behavior of the four-point correla-

\footnote{For non-perturbative calculations of the rapidity anomalous dimension or TMD soft function from lattice QCD, see e.g. [161–166].}
tor [64], with the rapidity anomalous dimension appearing in the time like factorization formula of [65].

The four loop rapidity anomalous dimension enables the resummation of the EEC in the back-to-back region at N^4LL, which is the first event shape for which all ingredients are available to achieve this perturbative accuracy. It will be particularly interesting to study the convergence of the perturbative series, as well as the phenomenological implications for extractions of the strong coupling. For the case of p_T resummation for the Higgs or electroweak p_T spectra, the remaining ingredient required to achieve N^4LL resummation is the singlet four loop splitting functions.

Our approach to computing the four loop rapidity anomalous dimension by exploiting its relation to other anomalous dimensions enabling it to be obtained through a three loop calculation illustrates the importance of better understanding relations between different anomalous dimensions in gauge theories. Much recent progress has been made along these lines [92]. An interesting generalization of the case considered in this paper is to understand the relation of the rapidity anomalous dimension for observables with more than two colored lines, such as the transverse EEC as relevant for hadron colliders [124], and its relation to the soft anomalous dimension with multiple lines, which is known at three loops [167, 168]. In particular, it is interesting to understand whether the dipole conjecture holds for the rapidity anomalous dimension. Another rapidity type anomalous dimension that is difficult to compute directly is the Regge trajectory, which was recently computed in QCD to three loops [169–179]. With the wealth of four loop data for anomalous dimensions, we believe it is important to understand its relation to other anomalous dimensions to enable its calculation to higher orders, and to obtain a more complete understanding of the relations between infrared anomalous dimensions.

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Note Added: Simultaneously with this paper, the work of [180] appeared, which also presented a result for the anomalous dimension, we have verified that our result agrees.

A Anomalous Dimensions and TMD Soft Function

In this Appendix, we collect the ingredients that entered into our calculations. We also summarize the results for the beam, soft and rapidity anomalous dimension at NNNNNLO, and the NNNLO TMD soft function to higher orders in the \( \epsilon \) expansion.
A.1 QCD Beta Function

The QCD beta function is defined as

\[ \frac{d\alpha_s}{d\ln \mu} = \beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^{n+1} \beta_n, \]  

(A.1)

with [22]

\[ \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F N_f, \]

\[ \beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F N_f - 4 C_F T_F N_f, \]

\[ \beta_2 = \left( \frac{158 C_A}{27} + \frac{44 C_F}{9} \right) N_f^2 T_F^2 + \left( - \frac{205 C_A C_F}{9} - \frac{1415 C_A^2}{27} + 2 C_F^2 \right) N_f T_F + \frac{2857 C_A^3}{54}. \]

(A.2)

A.2 Anomalous Dimensions

We expand all anomalous dimensions in \( \alpha_s \) as

\[ \gamma(\alpha_s) = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^{n+1} \gamma_n. \]

(A.3)

The three-loop cusp, soft and rapidity anomalous dimension can be found in [58, 59, 81, 103]. The four loop cusp anomalous dimension in QCD can be found in [39, 40]. The numerical-analytical expressions for virtual and soft anomalous dimensions were obtained in [40–42, 44]. Based on these results, we obtain the full expressions for rapidity anomalous dimensions, albeit with some color coefficients a numerical value (see in Tab. 2).

The collected results for the coefficients up to \( \mathcal{O}(\alpha_s^4) \) are given by

\[ \Gamma_0^{\text{cusp}} = 4 C_r, \]

\[ \Gamma_1^{\text{cusp}} = \left( \frac{268}{9} - 8\zeta_2 \right) C_A C_r - \frac{40}{9} C_r N_f, \]

\[ \Gamma_2^{\text{cusp}} = \left( \frac{160\zeta_2}{9} - \frac{112\zeta_3}{3} - \frac{836}{27} \right) C_A C_r N_f + \left( 32\zeta_3 - \frac{110}{3} \right) C_F C_r N_f + \left( - \frac{1072\zeta_2}{9} + \frac{88\zeta_3}{3} + \frac{88\zeta_4}{4} + \frac{490}{3} \right) C_A^2 C_r - \frac{16}{27} C_r N_f^2, \]

\[ \Gamma_3^{\text{cusp}} = \frac{64}{27} \frac{\zeta_3 - \frac{32}{81}}{N_f^3 C_r} + \left( \frac{-224}{15} \zeta_2^2 + \frac{2240}{27} \zeta_2 \zeta_3 - \frac{608}{81} \zeta_2 + \frac{923}{81} \right) N_f^2 C_A C_r + \frac{20320}{81} \zeta_2 - \frac{24137}{81} \right) N_f C_A C_r + \left( 160\zeta_5 - 128\zeta_3 \zeta_2 - \frac{352}{5} \zeta_2^2 + \frac{3712}{9} \zeta_3 + \frac{44}{3} \right) C_A^2 C_r + \frac{-34066}{81} \zeta_2 - \frac{24137}{81} \right) \frac{N_f C_A C r}{N_r} + \left( - \frac{1280}{3} \zeta_5 - \frac{256}{3} \zeta_3 \right)

+ 256 \zeta_2 \right) \frac{N_f d_F^{abcd} d_r^{abcd}}{N_r} (A.4)
\[-8\zeta_4 + 1711 \frac{N_j}{54} C_F C_r N_f,\]
\[\gamma_S^S = \frac{\alpha_{d_{\text{em}} d_{\text{em}}}}{N_r} \left( b(d_{\text{em}}^{(4)}) + \frac{1672\zeta_3^2}{3} + \left( \frac{184\zeta_4 + 3904}{9} \right) \zeta_3 - \frac{1738}{9} \zeta_6 \right),\]
\[-\frac{56}{3} \zeta_4 + \frac{920}{9} \zeta_5 + \zeta_2 \left( 896\zeta_3 - 512\zeta_5 + \frac{1088}{3} \right) - 1742\zeta_7 - 96 \right) + C_A^3 C_r \left( - \frac{b(d_{\text{em}}^{(4)})}{24} \right),\]
\[-7405 \frac{N_j}{243} C_A^2 C_r \left( \frac{1000\zeta_3^2}{3} - \frac{26372}{27} \zeta_4 + \frac{11759}{81} \zeta_3 + \frac{2236}{3} \zeta_5 - \frac{1952}{9} \zeta_3 \zeta_2 \right),\]
\[-\frac{673}{54} \zeta_2 - 359\zeta_6 + 1092511 \frac{N_j}{1944} C_{C_F C_r} \left( b \left( C_{C_F N_f}^2 \right) - \frac{184\zeta_3^2}{3} - \frac{1936}{3} \zeta_5 - \frac{560}{9} \zeta_3 \right),\]
\[+ \frac{7334}{9} \zeta_6 + \frac{256}{3} \zeta_3 - 161 \zeta_2 + 167\zeta_4 - \frac{27949}{216} \right) + C_A^2 C_r \left( - \frac{16076}{243} \zeta_3 - \frac{194}{9} \zeta_4 \right),\]
\[+ \frac{112}{9} \zeta_3 \zeta_2 + \frac{15481}{1458} \zeta_2 + 56\zeta_5 - \frac{27875}{34992} \right) + C_F C_r N_f^2 \left( - \frac{152}{9} \zeta_5 - \frac{32}{9} \zeta_4 + \frac{2284}{81} \zeta_3 - \frac{16}{3} \zeta_3 \zeta_2 \right),\]
\[+ \frac{86}{9} \zeta_4 - \frac{16733}{972} + \frac{N_j}{972} d_{\text{em}} d_{\text{em}} \left( b(d_{\text{em}}^{(3)}) - \frac{592}{3} \zeta_6 + \frac{400}{3} \zeta_4 + \frac{2656}{9} \zeta_3 \right),\]
\[+ \frac{10880}{9} \zeta_5 + \zeta_2 \left( - \frac{64\zeta_3 - 2272}{3} \right) + 192 \right) + C_r N_f^3 \left( - \frac{64}{27} \zeta_4 + \frac{8}{81} \zeta_2 + \frac{200}{243} \zeta_3 + \frac{8080}{6561} \right).\]
anomalous dimensions can be trivially obtained by generalized Casimir scaling.

so above we only show the results for quark anomalous dimensions, the corresponding gluon anomalous dimensions satisfy generalized Casimir scaling, 

\[ \gamma_3^R = \frac{d_{ABC}^a d_{ABC}^d}{N_f} \left( b(d_{FA}^{(4)}) + \frac{1672\zeta_5^2}{3} + \left( 184\zeta_3 + \frac{300436}{31} \right) \zeta_4 - \frac{1738}{9} \zeta_6 \right) + 56\zeta_4 + \frac{920}{9} \zeta_5 + \zeta_2 \left( 896\zeta_3 - 512\zeta_5 + \frac{1088}{3} \right) - 1742\zeta_7 - \frac{96}{24} + C_F^3 N_f \left( - \frac{198}{27} \right) \left( \frac{92}{61913} \right) \zeta_3 \]

The cusp and soft and rapidity anomalous dimensions satisfy generalized Casimir scaling, so above we only show the results for quark anomalous dimensions, the corresponding gluon anomalous dimensions can be trivially obtained by generalized Casimir scaling.

The quark virtual anomalous dimensions are

\[ B_0^q = 3C_F, \]

\[ B_1^q = \left( \frac{3}{2} - 12\zeta_2 + 24\zeta_5 \right) C_F^2 + \left( \frac{17}{6} + \frac{44\zeta_2}{3} - 12\zeta_3 \right) C_A C_F + \left( - \frac{1}{3} - \frac{8\zeta_2}{3} \right) C_F N_f, \]

\[ B_2^q = \left( \frac{-1336\zeta_3}{27} + \frac{200\zeta_5}{9} + 2\zeta_4 + 20 \right) C_A C_F N_f + \left( \frac{20\zeta_3}{3} - \frac{136\zeta_5}{3} + \frac{116\zeta_4}{3} - 23 \right) C_F^2 N_f + \left( 16\zeta_3 \zeta_2 - \frac{410\zeta_4}{3} + \frac{844\zeta_5}{3} - \frac{494\zeta_4}{3} + 120\zeta_5 + \frac{151}{4} \right) C_A C_F^2 + \left( \frac{80\zeta_2}{27} - \frac{16\zeta_3}{9} - \frac{17}{9} \right) C_F N_f^2 \]

\[ \frac{-2272}{3} \zeta_2 + 192 \right) + C_F N_f^3 \left( - \frac{4}{3} \zeta_4 + \frac{10}{9} \zeta_3 + \frac{2608}{2187} \right). \]
The gluon virtual anomalous dimensions are

\[ B_3^g = (d^{(4)}_{FA})_A^{d\text{bed}} N_F^{d\text{bed}} C_F \left(- \frac{b(d^{(4)}_{FF})}{24} - \frac{371201}{648} + \frac{528}{3} \zeta_3 \right) \]

\[ + 144 \zeta_4 - 240 \zeta_5 \right) C_F^4, \]

\[ B_3^g = b(d^{(4)}_{FA})_A^{d\text{bed}} N_F^{d\text{bed}} C_F \left( - \frac{b(d^{(4)}_{FF})}{24} - \frac{371201}{648} + \frac{528}{3} \zeta_3 \right) \]

\[ + \left( \frac{8 \zeta_4 - 153670}{81} \right) - \frac{11194}{27} \zeta_3 - \frac{6046}{9} \zeta_6 + \frac{11372}{9} \zeta_5 + \frac{472}{3} \zeta_3 \zeta_2 + 504 \zeta_5 \zeta_2 \]

\[ + \frac{4582}{3} \zeta_2 - \frac{2870 \zeta_7}{9} \right) + N_F C_A^2 C_F \left( - \frac{1}{2} b \left( C_A C_F^2 N_f - \frac{1}{4} b \left( C_F^3 N_f - \frac{b(d^{(4)}_{FF})}{48} \right) \right) \right) \]

\[ + \frac{16 \zeta_3}{3} - \frac{248}{3} \zeta_5 - \frac{137}{9} \zeta_3 + 16186 \zeta_4 + \left( - \frac{584}{9} \zeta_3 - \frac{85175}{162} \zeta_2 - 144 \zeta_6 + \frac{353}{3} \right) \]

\[ + N_F C_A C_F^2 b \left( C_A C_F^2 N_f \right) + N_F C_F^2 b \left( C_F^3 N_f \right) + \left( - \frac{320}{9} \zeta_3 - \frac{88}{9} \zeta_5 - \frac{80}{9} \zeta_4 \right) \]

\[ + \left( \frac{80}{3} \zeta_3 + \frac{3170}{81} \right) \zeta_2 - \frac{193}{54} \right) N_F^2 C_A C_F + \left( - \frac{2104}{27} \zeta_4 + \frac{56}{27} \zeta_3 + \frac{328}{9} \zeta_5 - \frac{60}{9} \zeta_3 \zeta_2 \right) \]

\[ + \frac{1244}{27} \zeta_2 - \frac{188}{27} \right) N_F^2 C_F^2 + C_A C_F^2 \left( - \frac{2085}{4} + \frac{3220}{3} + \frac{128 \zeta_4 - 3260}{3} \right) \zeta_3 \]

\[ + \frac{7929}{18} \zeta_6 + 2167 \zeta_4 - 976 \zeta_5 + \zeta_2 \left( - \frac{1988}{3} \zeta_4 + 2064 \zeta_5 + 1167 \right) - 10920 \zeta_7 \]

\[ - \frac{5497}{2} \zeta_6 + \zeta_2 \left( \frac{2096}{9} \zeta_3 - 2104 \zeta_5 + \frac{46771}{27} \right) + 8610 \zeta_7 \right) + \left( - \frac{32}{27} \zeta_4 + \frac{32}{81} \zeta_2 + \frac{304}{81} \zeta_3 - \frac{131}{81} \right) \zeta_5 \]

\[ + C_A^2 N_f \left( \frac{29639}{36} - \frac{710 \zeta_3}{3} + \left( \frac{129662}{27} - 32 \zeta_4 \right) \zeta_3 + \frac{5354}{9} \zeta_5 - \frac{60850}{27} \right) \zeta_4 \]

\[ - \frac{5497}{2} \zeta_6 + \zeta_2 \left( 2096 \zeta_3 - 2104 \zeta_5 - \frac{46771}{27} \right) + 8610 \zeta_7 \right) + \left( - \frac{32}{27} \zeta_4 + \frac{32}{81} \zeta_2 + \frac{304}{81} \zeta_3 - \frac{131}{81} \right) \zeta_5 \]

\[ + C_A^2 N_f \left( - \frac{10}{3} \zeta_4 - \frac{80}{3} \zeta_3 - \frac{8}{3} \zeta_2 - \frac{233}{18} \right) + \frac{27}{18} C_A N_f^2 - \frac{241}{18} C_A C_F N_f \]

\[ + C_F^2 N_f + \frac{11}{9} C_F N_f^2, \]

\[ B_3^g = N_f \frac{d\text{bed}^{d\text{bed}}}{d\text{bed}^{d\text{bed}}} \left( b(d^{(4)}_{FF}) - \frac{1520}{3} \zeta_5 - \frac{1496}{9} \zeta_6 + \frac{1016}{3} \zeta_4 + \frac{1312}{3} \zeta_3 \right) \]

\[ + \zeta_2 \left( \frac{544 \zeta_3 - 2368}{3} + \frac{1952}{9} \right) + C_A \left( - \frac{b(d^{(4)}_{FA})}{24} + \frac{682 \zeta_3}{3} + \left( 168 \zeta_4 + \frac{40888}{27} \right) \zeta_3 \right) \]
allows an exponential form

\[
\begin{align*}
- \frac{14617}{9} \zeta_5 - \frac{19129}{54} \zeta_6 + \frac{8965}{54} \zeta_4 + \zeta_2 & \left( - \frac{3902}{9} \zeta_3 + \frac{80 \zeta_5 + 2098}{27} \right) + 700 \zeta_7 + \frac{50387}{486} \\
+ N_f C_A^3 & \left( - \frac{1}{2} b \left( C_A C_F^2 N_f \right) - \frac{1}{4} b \left( C_F^3 N_f \right) - \frac{b(d_F^{(4)})}{48} - \frac{1268 \zeta_3^2}{3} - \frac{22714}{27} \zeta_3 - \frac{1777}{54} \zeta_6 \right) \\
+ \frac{919}{9} \zeta_5 + \frac{7789}{18} \zeta_4 + \left( \frac{1874}{9} \zeta_3 - \frac{6155}{54} \right) \zeta_2 & \left( - \frac{8075}{108} \right) + N_f C_A C_F \left( b \left( C_A C_F^2 N_f \right) + \frac{1928}{3} \zeta_3^2 \\
- \frac{27269}{27} \zeta_4 - \frac{2879}{9} \zeta_6 + \frac{8854}{27} \zeta_3 + \frac{6712}{9} \zeta_5 & \left( - \frac{2744}{9} \zeta_3 + \frac{4198}{27} \zeta_2 + \frac{23566}{243} \right) \\
+ N_f C_A C_F^2 & \left( b \left( C_F^3 N_f \right) - 224 \zeta_2^2 + \frac{2948}{9} \zeta_3 + \frac{6434}{9} \zeta_6 + \left( \frac{256}{3} \zeta_3 - 162 \right) \zeta_2 + 204 \zeta_4 \\
- \frac{912 \zeta_5 - \frac{2723}{27}}{27} + 23 N_f C_F^3 & \left( \frac{160}{9} \zeta_3 + \frac{3910}{243} \right) N_f^2 C_A C_F \left( - \frac{8}{9} \zeta_5 + \frac{200}{27} \zeta_4 + \frac{289}{27} \zeta_3 \\
+ \left( - \frac{32}{9} \zeta_3 + \frac{37}{27} \right) \zeta_2 & + \frac{1352}{81} \right) N_f^2 C_A + \left( - \frac{176}{9} \zeta_3 + \frac{338}{27} \right) N_f^2 C_F^3 + \frac{5}{243} C_A N_f^3 \\
+ \frac{154}{243} C_F N_f^3 & + \frac{p^{\text{bed}}_{AB} p^{\text{bed}}_{AC}}{N_a} \left( b(d_F^{(4)}) - \frac{784}{3} \zeta_3 - \frac{508}{3} \zeta_4 + \frac{748}{9} \zeta_6 + \frac{760}{3} \zeta_5 + \zeta_2 \left( \frac{1184}{3} - 272 \zeta_3 \right) \\
- \frac{800}{9} \right) + \left( \frac{512}{3} \zeta_3 - \frac{704}{9} \right) \frac{N_f^2 p^{\text{bed}}_{EF} p^{\text{bed}}_{FA}}{N_a}. \tag{A.5}
\end{align*}
\]

A.3 TMD Soft Function

In addition to the finite terms that were obtained in Ref. [58], in this Appendix we report a result to higher order in dimensional regulator \( \epsilon \), using the exponential regulator [108].

Thanks to the Non-Abelian exponentiation theorem [144, 145], the TMD soft function allows an exponential form

\[
s_1 = C_r \left[ \epsilon^3 \left( \frac{L_b^2}{9} - \zeta_2 L_r - \frac{4 \zeta_3}{3} \right) + L_b \left( - \frac{4}{3} \zeta_3 L_r - \frac{27 \zeta_4}{4} \right) - \frac{1}{6} L_b^4 L_r - \frac{1}{3} \zeta_2 L_b^3 \\
+ \frac{L_b^5}{30} - \frac{9}{4} \zeta_3 L_r - \frac{8}{3} \zeta_2 \zeta_3 - \frac{16 \zeta_5}{5} \right] + \epsilon^2 \left( \frac{L_b}{3} \left( -2 \zeta_2 L_r + \frac{8 \zeta_3}{3} \right) - \frac{2}{3} \zeta_2 L_b^3 \right) \\
- \zeta_2 L_b^2 + \frac{L_b^4}{6} - \frac{4}{3} \zeta_3 L_r - \frac{27 \zeta_4}{4} \right] + \epsilon \left( -2 L_b^2 L_r - 2 \zeta_2 L_b + \frac{2 \zeta_3^2}{3} - 2 \zeta_2 L_r + \frac{8 \zeta_3}{3} \right) \\
- 4 L_b L_r + 2 \zeta_2 \right],
\]

\[
s_2 = C_r N_f \left[ - \frac{4 L_b^3}{9} + \left( \frac{4 L_r}{9} - \frac{20}{9} \right) L_b^2 + \left( \frac{40 L_r}{9} + \frac{8 \zeta_2}{3} - \frac{112}{27} \right) L_b + \frac{112 L_r}{27} + \frac{10 \zeta_2}{3} \\
+ \frac{28 \zeta_3}{9} + \epsilon \left( - \frac{L_b^4}{3} + \left( \frac{4 L_r}{3} - \frac{40}{27} \right) L_b^3 + \left( \frac{40 L_r}{9} + \frac{10 \zeta_2}{3} - \frac{112}{27} \right) \right) L_b^2 \\
+ \left( \frac{20 \zeta_2}{3} + L_r \left( \frac{4 \zeta_3}{3} + \frac{224}{27} \right) + \frac{8 \zeta_3}{3} - \frac{656}{81} \right) L_b + \frac{56 \zeta_2}{9} + L_r \left( \frac{20 \zeta_2}{9} + \frac{656}{81} \right) \\
+ \frac{220 \zeta_3}{27} + \frac{35 \zeta_4}{3} - \frac{1952}{243} \right] + \epsilon^2 \left( - \frac{7 L_b^5}{45} + \left( \frac{7 L_r}{9} - \frac{20}{27} \right) L_b^4 + \left( \frac{8 L_r}{27} + \frac{22 \zeta_2}{9} \right)
\right].
\]


\[- \frac{224}{81} L_b^3 + \left( \frac{20 \zeta_2}{3} + L_r \left( 2 \zeta_2 + \frac{224}{27} \right) + \frac{80 \zeta_3}{9} - \frac{656}{81} \right) L_b^2 + \left( \frac{112 \zeta_2}{9} + L_r \left( \frac{40 \zeta_2}{9} + \frac{8 \zeta_3}{9} + \frac{1312}{81} \right) + \frac{440 \zeta_3}{27} + \frac{167 \zeta_4}{6} - \frac{3904}{243} \right) L_b + \left( \frac{328 \zeta_2}{27} + \frac{44}{9} \zeta_2 \zeta_3 + \frac{1232 \zeta_3}{81} + L_r \left( \frac{112 \zeta_2}{27} + \frac{40 \zeta_3}{27} + \frac{5 \zeta_4}{6} + \frac{3904}{243} \right) + \frac{485 \zeta_4}{18} \right) + L_r \left( \frac{284 \zeta_5}{15} - \frac{11680}{729} \right) - \frac{328}{81} \right] + C_A C_r \left[ \frac{22 L_b^3}{9} + \left( - \frac{22 L_r}{3} - 4 \zeta_2 + \frac{134}{9} \right) \right] \]

\[
\left( - \frac{44 \zeta_2}{3} + L_r \left( 8 \zeta_2 - \frac{268}{9} \right) - 28 \zeta_3 + \frac{808}{27} \right) L_b - \frac{67 \zeta_2}{3} - 154 \zeta_4 \]

\[
+ L_r \left( \frac{28 \zeta_3 - 808}{27} \right) + 10 \zeta_4 + \epsilon \left( \frac{11 L_r^4}{6} + \left( - \frac{22 L_r}{3} - 8 \zeta_2 + \frac{268}{27} \right) \right]^3 \]

\[
+ \left( - \frac{55 \zeta_2}{3} + L_r \left( 8 \zeta_2 - \frac{268}{9} \right) - 28 \zeta_3 + \frac{808}{27} \right) L_b^2 + \left( - \frac{134 \zeta_2}{3} - 44 \zeta_3 \right) \]

\[- \frac{1474 \zeta_4}{27} - 385 \zeta_4 + 4856 \zeta_4 \left( \frac{172}{3} \zeta_3 \zeta_2 - \frac{808 \zeta_2}{9} - \frac{2948 \zeta_3}{27} - \frac{1837 \zeta_4}{243} \right) \]

\[
+ 20 \zeta_4 + \frac{4856}{81} \zeta_4 - \frac{2428 \zeta_2}{27} - \frac{242 \zeta_2 \zeta_3 - 8888 \zeta_3}{81} - \frac{6499 \zeta_4}{36} - \frac{1562 \zeta_5}{15} + L_r \left( \frac{92}{3} \zeta_3 \zeta_2 \right) \]

\[
- \frac{808 \zeta_2}{27} - \frac{268 \zeta_3}{27} - \frac{55 \zeta_4}{12} + 124 \zeta_5 - \frac{29152}{243} \right) + \frac{2009 \zeta_6}{8} + \frac{87472}{729} + \frac{2428}{81} \left( \right) \]

\[
s_3 = C_A C_r^2 \left[ \frac{121 L_r^4}{27} + \left( - \frac{484 L_r}{27} - \frac{88 \zeta_2}{9} + \frac{3560}{81} \right) L_b^3 + \left( - \frac{340 \zeta_2}{3} + L_r \left( \frac{88 \zeta_2}{3} \right) \right) \right] \]

\[
- \frac{3560}{27} - \frac{88 \zeta_4 + 44 \zeta_4 + 15503}{81} \right) L_b^2 + \left( \frac{176}{3} \zeta_3 \zeta_2 - \frac{27752 \zeta_2}{81} - \frac{15232 \zeta_3}{27} \right) \]

\[
+ L_r \left( \frac{1072 \zeta_2}{9} + 176 \zeta_3 - 88 \zeta_4 - \frac{31006}{81} \right) + \frac{748 \zeta_4}{3} + 192 \zeta_5 + \frac{297029}{729} \right) \]

\[
+ \frac{928 \zeta_4}{9} - \frac{297481 \zeta_2}{729} + \frac{1100 \zeta_2 \zeta_3 - 151132 \zeta_3}{243} + \frac{3649 \zeta_4}{27} \]

\[
+ L_r \left( - \frac{176}{3} \zeta_3 \zeta_2 + \frac{6392 \zeta_2}{81} + \frac{12328 \zeta_4}{27} + \frac{154 \zeta_4}{3} - \frac{192 \zeta_5 - 297029}{729} \right) \]

\[- 22 \]
\[ + \frac{1804 \zeta_5}{9} - \frac{3086 \zeta_6}{27} + \epsilon \left( \frac{242 L_5^5}{45} + \left( - \frac{242 L_r}{9} - \frac{110 \zeta_2}{9} + \frac{4297}{81} \right) L_b + \left( - \frac{4270 \zeta_2}{27} \right) \right) \]
\[ + L_r \left( \frac{440 \zeta_2}{9} - \frac{17188}{81} \right) - \frac{148 \zeta_3}{9} + \frac{44 \zeta_4}{3} + \frac{64285}{243} \right) L_b^3 + \left( \frac{88 \zeta_3 \zeta_2 - \frac{18604 \zeta_2}{27}}{27} \right) \]
\[ - \frac{906 \zeta_3}{9} + L_r \left( \frac{1366 \zeta_2}{9} + \frac{148 \zeta_3}{9} - \frac{132 \zeta_4}{3} + \frac{64285}{81} \right) + \frac{1342 \zeta_4}{3} + \frac{288 \zeta_5}{9} \]
\[ + \frac{403861}{486} \right) L_b^2 + \left( \frac{928 \zeta_2^2}{3} + \frac{5192}{9} \zeta_2 \zeta_3 - \frac{18600 \zeta_3}{81} - \frac{37747 \zeta_2}{243} - \frac{586 \zeta_4}{9} \right) \]
\[ + L_r \left( - \frac{176 \zeta_3 \zeta_2 + \frac{944 \zeta_2}{9} + \frac{1232 \zeta_3}{9} + \frac{1826 \zeta_4}{3} - \frac{576 \zeta_5}{3} - \frac{403861}{243} \right) \]
\[ + \frac{1760 \zeta_5}{3} - \frac{3086 \zeta_6}{9} + \frac{7135981}{4374} \right) L_b + \frac{18524 \zeta_2}{27} + \frac{6837355 \zeta_2}{4374} + \frac{20792}{27} \zeta_2 \zeta_3 \]
\[ + \frac{1708132 \zeta_3}{729} - \frac{2816 \zeta_3 \zeta_4}{9} - \frac{96557 \zeta_4}{81} - \frac{1448 \zeta_2 \zeta_4}{9} - \frac{135}{36} \zeta_2 \zeta_5 - \frac{3350 \zeta_4}{3} + \frac{616 \zeta_5}{5} \]
\[ + L_r \left( - \frac{1360 \zeta_3^2}{3} - \frac{880 \zeta_3 \zeta_2}{9} + \frac{83096 \zeta_3}{81} - \frac{8237 \zeta_2}{243} + \frac{3350 \zeta_4}{3} + \frac{416 \zeta_4}{27} + \frac{1960 \zeta_3}{27} - \frac{184 \zeta_4}{3} \right) \]
\[ + \frac{8818 \zeta_5}{9} - \frac{7135981}{4374} + \frac{121627 \zeta_6}{54} + \frac{10870951}{26244} + \frac{5211949}{13122} \right] \]
\[ + C_r C_A N_f \left[ - \frac{44 \zeta_2}{27} + \frac{176 L_r}{27} + \frac{16 \zeta_2}{9} - \frac{1156}{81} \right) L_b^3 + \left( L_r \left( \frac{1156}{27} - \frac{16 \zeta_2}{3} \right) \right) \]
\[ + \frac{256 \zeta_2}{9} - \frac{4102}{81} L_b + \left( L_r \left( \frac{804 \zeta_2}{81} - \frac{160 \zeta_2}{9} \right) + \frac{7760 \zeta_2}{81} + \frac{1960 \zeta_3}{27} - \frac{184 \zeta_4}{3} \right) \]
\[ - \frac{6266 \zeta_2}{729} L_b + \frac{74530 \zeta_3}{729} + \frac{40 \zeta_2 \zeta_3}{9} + \frac{8152 \zeta_3}{81} - \frac{416 \zeta_4}{27} + \frac{L_r}{81} \left( \frac{824 \zeta_2}{27} \right) \]
\[ - \frac{904 \zeta_3}{27} + \frac{20 \zeta_4}{3} + \frac{6266 \zeta_2}{729} \right) - \frac{184 \zeta_5}{3} + \epsilon \left( - \frac{88 L_b^5}{45} + \frac{88 L_r}{9} + \frac{20 \zeta_2}{9} - \frac{1400}{81} \right) L_b^4 \]
\[ + \left( L_r \left( \frac{5600}{81} - \frac{80 \zeta_2}{9} \right) + \frac{1208 \zeta_2}{27} + \frac{112 \zeta_3}{9} - \frac{18002}{243} \right) L_b^3 + \left( \frac{5434 \zeta_2}{27} \right) \]
\[ + L_r \left( - \frac{152 \zeta_2}{9} - \frac{112 \zeta_3}{3} + \frac{18002}{9} \right) + \frac{1508 \zeta_3}{9} - \frac{316 \zeta_4}{3} - \frac{48241}{243} \right) L_b^5 \]
\[ + \left( - \frac{224}{9} \zeta_3 \zeta_2 + \frac{10016 \zeta_2}{243} + \frac{35912 \zeta_2}{81} + L_r \left( \frac{332 \zeta_2}{27} - \frac{904 \zeta_3}{9} - \frac{188 \zeta_4}{3} \right) \right) \]
\[ + \frac{96482}{243} + \frac{1124 \zeta_4}{9} - \frac{544 \zeta_5}{3} - \frac{716509}{2187} \right) L_b - \frac{1160 \zeta_2^2}{27} + \frac{79951 \zeta_2}{2187} \]
\[ + \frac{452}{27} \zeta_2 \zeta_3 + \frac{33047 \zeta_2}{729} + \frac{33430 \zeta_3}{81} + L_r \left( - \frac{16}{9} \zeta_3 \zeta_2 + \frac{7850 \zeta_2}{243} - \frac{2528 \zeta_3}{81} \right) \]
\[ - \frac{344 \zeta_4}{3} - \frac{112 \zeta_5}{3} - \frac{716509}{2187} + \frac{15784 \zeta_5}{135} - \frac{10607 \zeta_6}{27} - \frac{375175}{1458} - \frac{412765}{651} \right] \]
\[ + C_r N_f^2 \left[ \frac{4 L_b^4}{27} + \left( \frac{80 L_r}{81} - \frac{16 L_r}{27} \right) L_b^3 + \left( - \frac{80 L_r}{27} - \frac{16 \zeta_2}{9} + \frac{200 \zeta_2}{81} \right) L_b^4 + \left( - \frac{400 L_r}{81} \right) \]
\[ - \frac{160 \zeta_2}{27} + \frac{1856}{729} L_b - \frac{136 \zeta_2}{27} + L_r \left( - \frac{32 \zeta_3}{9} - \frac{1856}{729} \right) - \frac{560 \zeta_3}{243} - \frac{44 \zeta_4}{27} \]
\[ - 23 \]
\[
+ \frac{\epsilon (8 \frac{L_5^2}{45} + \left( \frac{100}{81} - \frac{8L_r}{9} \right) L_6^4 + \left( - \frac{400L_r}{81} - \frac{88\zeta_2}{27} + \frac{1048}{243} \right) \frac{L_3^3}{9} + \left( \frac{8\zeta_2}{9} \frac{L_3^3}{9} \right)}{48} - \frac{1048}{81} - \frac{40\zeta_2}{3} + \frac{16\zeta_3}{3} + \frac{2240}{243} \right) L_2^2 + \left( - \frac{632\zeta_2}{27} + \frac{80\zeta_2}{27} - \frac{32\zeta_2}{3} \frac{81}{243} + \frac{3712\zeta_3}{3} \right) L_6 - \frac{12512\zeta_2}{2187} \frac{L_6}{729} \frac{32}{3} \frac{81}{135} + \frac{23168}{6561} \frac{L_6}{1561} \right) \] 
\[+ L_r \frac{-200\zeta_2}{81} \frac{320\zeta_2}{27} \frac{16\zeta_4}{3} \frac{23168}{2187} \frac{L_2}{2540\zeta_4} \frac{1744\zeta_5}{81} - \frac{32\zeta_3}{3} \frac{81}{243} - \frac{304\zeta_3}{9} \frac{16\zeta_4}{9} + \frac{1711}{27} \frac{L_2}{275\zeta_2} \frac{80}{3} \frac{\zeta_2\zeta_3}{3} + \frac{3488\zeta_3}{81} \frac{L_2}{-304\zeta_3} \frac{9}{16\zeta_4} \frac{152\zeta_4}{9} \frac{224\zeta_5}{9} \frac{\zeta_2}{3} + \frac{\epsilon (4\frac{L_r}{3} + \left( \frac{16\zeta_2}{3} + \frac{16\zeta_6}{3} \right) L_3 + \left( \frac{14\zeta_2}{3} \frac{16\zeta_2}{3} + \frac{16\zeta_6}{3} \right) L_3 + \left( - \frac{80\zeta_3}{3} + \frac{275\zeta_2}{3} \right) \frac{L_2}{3632\zeta_3} \frac{l}{27} + \frac{152\zeta_3}{9} \frac{24\zeta_4}{3} \frac{1711}{18} \frac{L_2}{152\zeta_4} \frac{224\zeta_5}{3} \frac{1711}{18} \frac{L_2}{-48\zeta_3} \frac{152\zeta_3}{3} + \frac{24\zeta_4}{3} \frac{1711}{18} \frac{L_2}{736\zeta_3} \frac{8555\zeta_2}{9} \frac{760}{3} \frac{\zeta_2\zeta_3}{3} + \frac{4769\zeta_2}{243} \frac{2536\zeta_4}{27} + \frac{224\zeta_5}{3} \frac{1711}{18} \frac{L_2}{-16\zeta_3} \frac{152\zeta_4}{3} + \frac{224\zeta_5}{3} \frac{1711}{18} \frac{L_2}{55\zeta_2} \frac{2768\zeta_3}{27} \frac{152\zeta_4}{3} + \frac{224\zeta_5}{3} \frac{1711}{18} \frac{L_2}{2128\zeta_3} \frac{50\zeta_6}{3} + \frac{951775}{2916} \frac{L_2}{42727} \frac{1711}{18} \frac{L_2}{-42727} \frac{1711}{18} \frac{L_2}{486} \right] .
\]

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