REMARKS ON THE DAMPED NONLINEAR
SCHRÖDINGER EQUATION

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Abstract. It is the purpose of this note to investigate the initial value problem for a focusing semi-linear damped Schrödinger equation. Indeed, in the energy sub-critical regime, one obtains global well-posedness and scattering in the energy space, depending on the order of the fractional dissipation.

1. Introduction. Consider the Cauchy problem for a damped nonlinear Schrödinger equation

\[
\begin{cases}
  i\dot{u} + \Delta u + i\gamma(-\Delta)^s u + |u|^{p-1} u = 0; \\
  u|_{t=0} = u_0,
\end{cases}
\]

where \( u \) is a complex valued function of the variable \( (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \) for some \( N \geq 2 \). The damping coefficient is \( \gamma > 0 \) and the Laplacian operator with order \( s > 0 \) stands for

\[ (-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u). \]

In the particular case \( s = 0 \), the equation (1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics, it has been studied by many mathematicians and physicists [1, 2, 9, 10, 20, 21]. Indeed, it is known that the Cauchy problem is well-posed in the energy space and the solution is global for large damping \( \gamma \) (Theorem 1 in [16], see also [18, 19]). The case \( s = 1 \) corresponds to a complex Ginzburg-Landau equation studied in [17].

The Schrödinger problem (1) in the mass critical case \( p = 1 + \frac{4}{N} \) was treated in [7], where global and non global existence of solutions were investigated.

It is known [8] that for small data, the nonlinear focusing Schrödinger problem without any dissipation, i.e. (1) with \( \gamma = 0 \), is globally well-posed and scatters in the energy space. The damping term allows to overcome the restriction on the data size. Indeed, unconditional global well-posedness and scattering are obtained in this paper. In other words, the dissipative quantity is shown to stabilize the solution, in the sense that finite time blow-up is prevented.

The stabilization for the Schrödinger equation with a localized damping was treated by many authors, see for instance Cavalcanti et al. [3, 4, 5], and Natali [14, 15].

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It is the purpose of this paper to establish that the damping prevents finite time blow-up. Comparing with the previous work [7], we assume that $p = 1 + \frac{4s}{N} \leq 1 + \frac{4}{N}$, which may be larger than the mass critical nonlinearity $1 + \frac{4}{N}$, if $1 < s < \frac{N}{4}$.

The rest of the paper is organized as follows. The second section is devoted to give the main results and some tools needed in the sequel. The third one deals with proving unconditional global well-posedness and scattering of (1). Section four contains a proof of global well-posedness and scattering for small data. In the appendix, local well-posedness of (1) in the energy space is established.

Here and hereafter, $C$ is a constant which may vary from line to line, $A \lesssim B$ denotes an estimate of the form $A \leq CB$ for some absolute constant $C$. For easy notation $L^p := L^p(\mathbb{R}^N)$ is the Lebesgue space endowed with the norm $\| \cdot \|_p := \| \cdot \|_{L^p}$ and $\| \cdot \| := \| \cdot \|_2$. The classical Sobolev space is $W^{s,p} := (I - \Delta)^{-\frac{s}{2}}L^p$ and $H^s := H^{s,2}$ is the inhomogeneous Sobolev space endowed with the complete norm

$$
\| u \|_{H^s} := \left( \| u \|^2 + \| (-\Delta)^{\frac{s}{2}} u \|^2 \right)^{\frac{1}{2}}.
$$

Finally, if $T > 0$ and $X$ is an abstract functional space, denote $C_T(X) := C([0, T], X)$, $L^p_T(X) := L^p([0, T], X)$ and $X_{rd}$ the set of radial elements in $X$, moreover for an eventual solution to (1), $T^* := T_{\gamma,s} > 0$ denotes its lifespan.

2. Background material. This section is devoted to give the main results of this note and some technical tools needed in the sequel.

2.1. Preliminary. Let us start with recalling two relevant quantities in the study of solutions to the Schrödinger problem (1). First, The mass is non increasing.

$$
M(t) := M(u(t)) := \int_{\mathbb{R}^N} |u(t, x)|^2 \, dx = M(0) - 2\gamma \int_0^t \| (-\Delta)^{\frac{s}{2}} u(\tau, x) \|^2 \, d\tau.
$$

Second, the energy satisfies formally the following identity.

$$
E(t) := E(u(t)) := \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{1 + p} |u(t, x)|^{1+p} \right) \, dx
= E(0) - \gamma \int_0^t \| (-\Delta)^{\frac{s}{2}} u(\tau) \|^2 \, d\tau + \gamma \delta \left( \int_0^t \int_{\mathbb{R}^N} |(-\Delta)^{s} u| |u|^{p-1} u \, dx \, d\tau \right).
$$

Now, some useful properties of the free damping Schrödinger kernel are gathered in what follows.

**Proposition 2.1.** Denoting the free operator associated to the damped Schrödinger equation (1),

$$
T(t) \phi := T_{\gamma,s}(t) \phi := \mathcal{F}^{-1}(e^{-(i|\cdot|^2 + \gamma|\cdot|^{2s})} t) \ast \phi, \quad t > 0
$$

yields

1. $T_{\gamma,s}(t)u_0$ is the solution to the linear problem associated to (1);
2. $T_{\gamma,s}(t)u_0 + i \int_0^t T_{\gamma,s}(t-s) |u|^{p-1} u \, ds$ is the solution to the problem (1);
3. \( T_{\gamma,s}(t + t') = T_{\gamma,s}(t)T_{\gamma,s}(t') \);
4. \( \|T_{\gamma,s}(\cdot)\| \leq 1 \);
5. \( T_{0,s} \) is an isometry of \( L^2 \).

Here and hereafter, denote the real numbers

\[
q := \frac{4s(1 + p)}{N(p - 1)}, \quad \theta := \frac{2s(p^2 - 1)}{2s(1 + p) - N(p - 1)}, \quad \frac{1}{\theta} := \frac{N(\frac{1}{2} - \frac{1}{1 + p})}{s}.
\]

**Remark 2.2.** The following identities hold
1. \((q, 1 + p)\) is an admissible pair;
2. if \( p \leq \frac{N + 2s}{N - 2s} \), then \( q \geq 2 \);
3. \( \frac{1}{\theta} = \frac{1}{q} + \frac{p - 1}{s} \);
4. there exists \( p_0(N, s) \in (1, 1 + 4s) \) such that \( \theta > 1 \) for any \( p > p_0(N, s) \);
5. \( p\theta^\gamma = \theta \).

2.2. **Main results.** Results proved in this paper are listed in this subsection.

Strichartz estimate \([6]\) is a standard tool to control solutions of a Schrödinger equation in Lebesgue spaces.

**Definition 2.3.** A couple of real numbers \((q, r)\) such that \( q, r \geq 2 \) is said to be admissible if \((q, r, N) \neq (2, \infty, 2)\) and

\[
N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2s}{q}.
\]

**Proposition 2.4.** Let \( N \geq 2, s > 0 \) and an admissible pair \((q, r)\). Then,
1. \( \|u\|_{L^q_t(L^r)} \cap L^\infty_t(L^2) \lesssim N, s, q \|u_0\| + \|i\dot{u} + \Delta u + i\gamma(-\Delta)u\|_{L^q_t(L^r)} \);
2. \( \|u\|_{L^q_t(L^r)} \lesssim N, s, q \|u_0\| + \|i\dot{u} + \Delta u + i\gamma(-\Delta)u\|_{L^q_t(L^r)}, \)

whenever \( t > 0, \theta, \theta' > 1, r > 2 \) and

\[
\frac{1}{\theta} + \frac{1}{\theta'} = \frac{2}{q}.
\]

Local well-posed of the Cauchy problem (1) in the energy space holds using Strichartz estimate and a classical contraction method.

**Proposition 2.5.** Let \( N \geq 2, s > 0, u_0 \in H^1 \) and \( p_0(N, s) < p \leq 1 + \frac{4}{N^2 - 2} \). Then, there exist \( T^* > 0 \) and a unique \( u \in C_{T^*}(H^1) \) maximal solution to (1). Moreover,
1. \( u \in L^\infty_{loc}([0, T^*), W^{1,1+p}) \);
2. \( u \) satisfies (2)-(3) on \([0, T^*) \);
3. the energy is decreasing if \( s < 1, p = 1 + \frac{4}{N} \) and \( \|u_0\| \) is small enough.

**Remark 2.6.** For the reader convenience, we give a proof of the previous result in the appendix.

The main result of this manuscript deals with unconditional global well-posedness and scattering of the focusing problem (1).

**Theorem 2.7.** Let \( N \geq 2, 0 < s < \frac{N}{2}, p_0(N, s) < p \leq \min\{1 + \frac{4s}{N}, 1 + \frac{4}{N^2 - 2}\} \) and \( u_0 \in H^1 \). Then, there exists a unique global solution to (1),

\[
uu C(\mathbb{R}^+, H^1).
\]

Moreover, if \( s \leq \frac{N}{N-2} \) and \( p = 1 + \frac{4s}{N} \), then there exists \( u_+ \in H^1 \) such that

\[
\|(u - T_{\gamma,s}(\cdot)u_+)(t)\|_{H^1} \to 0, \quad \text{as} \quad t \to \infty
\] (4)
and the mapping
\[ S : H^1 \to H^1 \quad u_0 \mapsto u_+ \]
is continuous and one to one.

**Remark 2.8.** Since the case \( s = 1 \) in (1) corresponds to the complex Ginzburg-Landau equation, the previous Theorem improves the existing result \([11]\), about global existence of solutions in the energy space for small data.

For small data, the previous result remains true if the condition \( p \leq 1 + \frac{4s}{N} \) is relaxed.

**Proposition 2.9.** Let \( N \geq 2, 0 < s < \frac{N}{2} \), \( p_0(N, s) < p < \min\{1 + \frac{4s}{N-2}, 1 + \frac{4}{N-2}\} \)
and \( u_0 \in H^1 \). There exists \( \varepsilon > 0 \) such that if \( \|T_{\gamma, s}(\cdot)u_0\|_{L^p(\mathbb{R}_+, L^{1+p})} < \varepsilon \), then there is a unique global solution \( u \in C(\mathbb{R}_+, H^1) \) to (1). Moreover,
1. \( u \in L^q(\mathbb{R}_+, W^{1,1+p}) \),
2. \( u \) satisfies (2)-(3) on \( \mathbb{R}_+ \).

2.3. **Tools.** Let us collect some classical estimates needed forward this manuscript. The following Gagliardo-Nirenberg inequality holds \([13]\).

**Lemma 2.10.** Let \( 2 \leq p \leq \frac{2N}{N-2} \) and \( \mu = N\left(\frac{1}{2} - \frac{1}{p}\right) \). Then,
\[
\|u\|_p \lesssim \|u\|^{1-\mu}_p \|u\|_H^\mu.
\]

Sobolev injections \([12]\) give a meaning to the energy and several computations done in this note.

**Lemma 2.11.** Let \( N \geq 2 \) and \( 0 < s < \frac{N}{2} \). Then,
\[
H^s \hookrightarrow L^q \quad \text{for any} \quad q \in \left[2, \frac{2N}{N-2s}\right].
\]

Let us close this subsection with an abstract result.

**Lemma 2.12.** Let \( T > 0 \) and \( X \in C([0, T], \mathbb{R}_+) \) such that
\[
X \leq a + b X^\theta \quad \text{on} \quad [0, T],
\]
where \( a, b > 0, \theta > 1, a < \left(1 - \frac{1}{\theta}\right)(\theta b)^{\frac{1}{\theta-1}} \) and \( X(0) \leq \left(\theta b\right)^{\frac{1}{\theta-1}} \). Then,
\[
X \leq \frac{\theta}{\theta-1} a \quad \text{on} \quad [0, T].
\]

**Proof.** The function \( f(x) := bx^\theta - x + a \) is decreasing on \([0, (b\theta)^{\frac{1}{\theta-1}}]\) and increasing on \([(b\theta)^{\frac{1}{\theta-1}}, \infty)\). The assumptions imply that \( f((b\theta)^{\frac{1}{\theta-1}}) < 0 \) and \( f((\theta b)^{\frac{1}{\theta-1}}) \leq 0 \). As \( f(X(t)) \geq 0, f(0) > 0 \) and \( X(0) \leq (b\theta)^{\frac{1}{\theta-1}} \), we conclude the proof by a continuity argument.

3. **Proof of Proposition 2.4.** The proof of Strichartz estimate in Proposition 2.4 is based on several lemmas. In this section, denote \( \mathcal{F} \) the Fourier transform, an admissible pair \((q, r)\) and the functions
\[
\phi_{\gamma, s}(f)(t) = \int_0^t T_{\gamma, s}(t-s)f(s)\,ds, \quad \psi_{\gamma, s}(f)(t, \tau) = \int_0^t T_{\gamma, s}(\tau-s)f(s)\,ds.
\]

**Lemma 3.1.**
\[
\|T_{\gamma, s}(t)f\|_r \lesssim_{N, s} \left(\gamma t\right)^{-\frac{N}{2}(\frac{1}{r} - \frac{1}{r'})}\|f\|_{r'}.
\]
Proof. Thanks to Plancherel Theorem, write \( \|T_{\gamma,s}(t)f\| \leq \|e^{-\gamma t|x|^2} \mathcal{F}(f)\| \leq \|f\| \) and
\[
\|T_{\gamma,s}(t)f\|_{\infty} = \|\mathcal{F}^{-1}(e^{-it|x|^2-\gamma t|x|^2}) \ast f\|_{\infty} \\
\leq \|\mathcal{F}^{-1}(e^{-it|x|^2-\gamma t|x|^2})\|_{1} \|f\|_{1} \\
\leq \|e^{-\gamma t|x|^2}\|_{1} \|f\|_{1} \\
\lesssim_{N,s} (\gamma t)^{-\frac{N}{2}} \|f\|_{1}.
\]
The result follows with interpolation. □

Proposition 2.4 is a direct consequence of the next estimates.

Lemma 3.2. For any \( T > 0 \), hold
1. \( \|\phi_{\gamma,s}(f)\|_{L^{q}_{T}(L^{r})} \lesssim_{N,s,q} \|f\|_{L^{q}_{T}(L^{r})} \);  
2. \( \|\psi_{\gamma,s}(f)(t,\cdot)\|_{L^{q}_{T}(L^{r})} \lesssim_{N,s,q} \|f\|_{L^{q}_{T}(L^{r})} \);  
3. \( \|\phi_{\gamma,s}(f)\|_{L^{q}_{T}(L^{r})} \lesssim_{N,s,q} \|f\|_{L^{q}_{T}(L^{r})} \);  
4. \( \|\phi_{\gamma,s}(f)\|_{L^{q}_{T}(L^{r})} \lesssim_{N,s,q} \|f\|_{L^{q}_{T}(L^{r})} \).

Proof. 1. The first and second point follow similarly.
\[
\|\psi_{\gamma,s}(f)(t,\tau)\|_{r} \leq \int_{\tau}^{t} \|T_{\gamma,s}(t-t')f(t')\|_{r} dt' \\
\lesssim \int_{\tau}^{t} |t-t'|^{-\frac{N}{2} (\frac{1}{r} - \frac{1}{2})} \|f(t')\|_{r'} dt' \\
\lesssim \int_{\tau}^{t} |t-t'|^{-\frac{N}{2}} \|f(t')\|_{r'} dt'.
\]
The result follows with Riesz potential inequalities.
2. Write for \( t > 0 \),
\[
\|\phi_{\gamma,s}(f)(t)\|^{2} = \int_{0}^{t} T_{\gamma,s}(t-t')f(t') dt' , \int_{0}^{t} T_{\gamma,s}(t-\tau)f(\tau) d\tau > L^{2}_{t} \\
= \int_{0}^{t} \int_{0}^{t} T_{\gamma,s}(t-t')f(t'), T_{\gamma,s}(t-\tau)f(\tau) > L^{2}_{t} dt'd\tau \\
= \int_{0}^{t} \int_{0}^{t} e^{-\gamma(t-\tau):|t'|^{2}} f(t'), T_{\gamma,s}(t'-\tau)f(\tau) > L^{2}_{t} dt'd\tau.
\]
Then, thanks to the previous Lemma,
\[
\|\phi_{\gamma,s}(f)(t)\|^{2} \leq \int_{0}^{t} \|f(t')\|_{r'} \psi_{\gamma,s}(f)(t, t') dt' \\
\leq \|f\|_{L^{q}_{T}(L^{r})} \psi_{\gamma,s}(f)(t, \cdot)\|_{L^{q}_{T}(L^{r})} \\
\lesssim \|f\|_{L^{q}_{T}(L^{r})}^{2}.
\]
3. Using the first Lemma of this section, we get
\[
\|\phi_{\gamma,s}(f)(t)\|_{r} \lesssim_{N,s} \int_{0}^{t} (\gamma |t-t'|)^{-\frac{N}{2} (\frac{1}{r} - \frac{1}{2})} f(t') dt'.
\]
We conclude with Riesz potential inequality. □
4. Proof of Theorem 2.7. In this section, we prove Theorem 2.7 about global existence and scattering of solutions to (1) in the energy space. The proof is based on two auxiliary results. First, let us give a global existence criterion.

**Lemma 4.1.** Let $N \geq 2$, $s > 0$, $p_0(N, s) < p \leq 1 + \frac{4}{N-2}$ and $u_0 \in H^1$. Take the unique maximal solution $u \in C_T(H^1)$ to (1). If $u \in L^\infty_T_{\ast}(L^{1+p})$, then $T^\ast = \infty$.

**Proof.** Assume with contradiction that $T^\ast < \infty$ and $\|u\|_{L^\infty_T_{\ast}(L^{1+p})} < \infty$. Take $0 < t_0 < T^\ast$ and write

$$u(t_0 + \tau) = T_{\gamma,s}(\tau)u(t_0) + i \int_{t_0}^T T_{\gamma,s}(t - \tau)(|u|^{p-1}u)\,d\tau.$$ 

With Strichartz estimate

$$\|T_{\gamma,s}(\tau)u(t_0)\|_{L^p_T(t_0+\tau,L^{1+p})} \leq \|u\|_{L^p(t_0,T^\ast \cdot, L^{1+p})} + \|u\|_{L^p(t_0,T^\ast \cdot, L^{(1+p)'})}.$$ 

Taking $t_0$ next to $T^\ast$ such that $\|u\|_{L^p((t_0,T^\ast \cdot, L^{1+p})} + \|u\|_{L^p((t_0,T^\ast \cdot, L^{1+p})} < \varepsilon$ and applying Proposition 2.9, it follows that $u$ can be extended on $C([0,T^\ast],H^1)$ which a is contradiction and so $T^\ast = \infty$. □

Second, let us give some properties of the solution to (1), which are direct consequences of the mass decay (2).

**Lemma 4.2.** Let $N \geq 2$, $s > 0$, $p_0(N, s) < p \leq 1 + \frac{2N}{N-2}$ and $u_0 \in H^1$. Take $u \in C_T(H^1)$ be the unique maximal solution to (1). Then,

$$\|u\|_{L^\infty_T(L^2)} \leq \|u_0\|$$ and $$\|u\|_{L^\infty_T,H^s} \leq \frac{1}{\sqrt{2\gamma}}\|u_0\|.$$

4.1. **Global existence.** Using Lemma 2.10, write for

$$q := \frac{4s(1+p)}{N(p-1)}, \quad \theta := \frac{q(p-1)}{q-2} \quad \text{and} \quad \mu := \frac{N}{s}\left(\frac{1}{2} - \frac{1}{1+p}\right),$$

$$\|u(t)\|_{1+p} \leq \|u(t)\|^{1-\mu}\|u(t)\|_{H^s}^{\mu} \leq \|u_0\|^{1-\mu}\|u(t)\|_{H^s}^{\mu}.$$ 

Integrating in time, yields for $0 < T < T^\ast$,

$$\|u\|_{L^p_T(L^{1+p})} \leq \|u_0\|^{1-\mu}\|u\|_{L^p_T(\nu)^\ast}^{\mu} \leq \|u_0\|^{1-\mu}\|u\|_{L^p_T(H^s)}^{\mu} T^{\frac{\mu}{\nu} - \frac{1}{2}} \leq \|u_0\|^{1-\mu} - \frac{1}{2} T^{\frac{\mu}{\nu} - \frac{1}{2}}.$$ 

This implies that

$$\|u\|_{L^p_T,(1+p)} \leq \infty.$$ 

Then, thanks to Lemma 4.1, it follows that

$$T^\ast = \infty.$$ 

**Remark 4.3.** The condition $p \leq 1 + \frac{4}{N}$ is equivalent to $\mu \theta \leq 2$. In the particular case $p = 1 + \frac{4}{N}$ which corresponds to $\mu \theta = 2$, it follows that

$$u \in L^\theta(\mathbb{R}_+,L^{1+p}).$$
4.2. Scattering. Thanks to Proposition 2.1, it is sufficient to show that
\[ \|T_{\gamma,s}(-t)u - u_+\|_{H^1} \to 0, \quad \text{as} \quad t \to \infty. \] (5)

Taking account of Cauchy criteria and denoting \( v(t) := T_{\gamma,s}(-t)u(t) \), it is sufficient to prove that
\[ \|v(t) - v(t')\|_{H^1} \to 0, \quad \text{as} \quad t, t' \to \infty. \]

Taking account of Proposition 2.1, we have
\[ v(t) = u_0 - i \int_0^t T_{\gamma,s}(-s)(|u|^{p-1}u) \, ds. \]

Therefore, we are reduced to prove that
\[ \| \int_{t'}^t T_{\gamma,s}(-s)(|u|^{p-1}u) \, ds \|_{H^1} \to 0, \quad \text{as} \quad t, t' \to \infty. \]

Applying Strichartz estimate and arguing as previously, we get
\[ \| \int_{t'}^t T_{\gamma,s}(-s)(|u|^{p-1}u) \, ds \|_{H^1} \lesssim \| |u|^{p-1}u \|_{L^p((t', t), L^{1+\frac{4}{p}})} \]
\[ \lesssim \| (1 + \nabla) [|u|^{p-1}u] \|_{L^p((t', t), L^{1+\frac{4}{p}})} \]
\[ \lesssim \| |u|^{p-1} |u| L^{\frac{p}{2}}((t', t), L^{1+\frac{4}{p}}) \| u \| L^s((t', t), W^{1,1+p}). \]

Since by the previous remark and (8),
\[ u \in L^q((0, \infty), L^{1+p}) \cap L^q((0, \infty), W^{1,1+p}), \]
the proof of (5) is finished. Moreover, taking the limit when \( t \) goes to infinity, we get
\[ u_+ = \lim_{t \to \infty} v(t) = u_0 - i \int_0^\infty T_{\gamma,s}(-s)(|u|^{p-1}u) \, ds. \]

Now, we establish that \( S \) is one to one. Take \( u_+ \in H^1 \), we prove that there exists a unique \( u \in C(\mathbb{R}_+, H^1) \) solution to the first equation in (1), such that (4) is satisfied. We proceed with a fixed point argument at infinity. Take the function
\[ \phi(u)(t) := \tilde{u}(t) := T_{\gamma,s}(t)u_+ + i \int_t^\infty T_{\gamma,s}(t - s)(|u|^{p-1}u) \, ds. \]

For \( T > 0 \) and \( C_q \) a constant depending on \( q \), define the space
\[ X_T := \left\{ u \in C([T, \infty), H^1); \| u \|_{L^q((T, \infty), W^{1,1+p})} \leq 2C_q\| u_+ \|_{H^1}, \right. \\
\| u \|_{L^\infty((T, \infty), H^1)} \leq 2\| u_+ \|_{H^1,} \right\} \leq 2\| T_{\gamma,s}(.)u_+ \|_{L^q((T, \infty), L^{1+p})} \}
\]
endowed with the distance
\[ d(u, v) := \| u - v \|_{L^q((T, \infty), L^{1+p})}. \]

Using Hölder and Strichartz inequalities, it follows that
\[ \| \tilde{u} \|_{L^q((T, \infty), W^{1,1+p})} \leq C_q \left( \| u_+ \|_{H^1} + \| |u|^{p-1}u \|_{L^q((T, \infty), W^{1,1+p})} \right) \]
\[ \leq C_q \left( \| u_+ \|_{H^1} + \| (1 + \nabla) |u|^{p-1}u \|_{L^q((T, \infty), L^{1+\frac{4}{p}})} \right) \]
\[ \leq C_q \left( \| u_+ \|_{H^1} + C \| u \|_{L^p((T, \infty), L^r)} \| u \|_{L^q((T, \infty), W^{1,1+p})} \right). \]
Similarly, we get thanks to the mean value Theorem, we get This implies that So Now, by Strichartz estimate
\[
\parallel \tilde{u} \parallel_{L^{\infty}(T, \infty), W^{1,1+p}} \\
\leq C_q \left( \parallel u_+ \parallel_{H^1} + \parallel u \parallel_{L^q((T, \infty), L^{1+p})} \parallel u \parallel_{L^q((T, \infty), W^{1,1+p})} \right) \\
\leq C_q \left( \parallel u_+ \parallel_{H^1} + \parallel u \parallel_{L^q((T, \infty), L^{1+p})} \parallel u \parallel_{L^q((T, \infty), W^{1,1+p})} \right) \\
\leq C_q \left( \parallel u_+ \parallel_{H^1} + \parallel u \parallel_{L^q((T, \infty), L^{1+p})} \parallel u \parallel_{L^q((T, \infty), W^{1,1+p})} \right).
\]

Using the definition of the space $X_T$, yields
\[
\parallel \tilde{u} \parallel_{L^q((T, \infty), H^1)} \\
\leq C_q \left( \parallel u_+ \parallel_{H^1} + C \parallel T_{\gamma, s}()u_+ \parallel_{L^q((T, \infty), L^{1+p})} \parallel u_+ \parallel_{H^1} \right).
\]

Similarly, we get
\[
\parallel \tilde{u} \parallel_{L^q((T, \infty), L^r)} \\
\leq \parallel T_{\gamma, s}()u_+ \parallel_{L^q((T, \infty), L^{1+p})} + C \parallel T_{\gamma, s}()u_+ \parallel_{L^q((T, \infty), L^{1+p})} \parallel u_+ \parallel_{H^1}.
\]

Now, by Strichartz estimate
\[
T_{\gamma, s}()u_+ \in L^q((0, \infty), L^{1+p}).
\]

So
\[
\lim_{t \to \infty} \parallel T_{\gamma, s}()u_+ \parallel_{L^q((t, \infty), L^{1+p})} = 0. \tag{6}
\]

This implies that $\phi(X_T) \subset X_T$ for $T$ large enough. Moreover, for $u_1, u_2 \in X_T$, thanks to the mean value Theorem, we get
\[
\parallel \tilde{u}_1 - \tilde{u}_2 \parallel_{L^q((T, \infty), L^{1+p})} \\
\leq \parallel u_1 \parallel_{L^q((T, \infty), L^{1+p})} - \parallel u_2 \parallel_{L^q((T, \infty), L^{1+p})}
\leq \sum_{i=1}^{2} \parallel (u_1 - u_2) u_i^{p-1} \parallel_{L^q((T, \infty), L^{1+p})}
\leq \sum_{i=1}^{2} \parallel u_i \parallel_{L^q((T, \infty), L^{1+p})} \parallel u_1 - u_2 \parallel_{L^q((T, \infty), L^{1+p})}
\leq \sum_{i=1}^{2} \parallel u_i \parallel_{L^q((T, \infty), L^{1+p})} \parallel u_i \parallel_{L^q((T, \infty), L^{1+p})} \parallel u_1 - u_2 \parallel_{L^q((T, \infty), L^{1+p})}.
\]

By the definition of $X_T$, yields
\[
\parallel \tilde{u}_1 - \tilde{u}_2 \parallel_{L^q((T, \infty), L^{1+p})} \\
\leq \parallel T_{\gamma, s}()u_+ \parallel_{L^q((T, \infty), L^{1+p})} \parallel u_1 + \parallel u_2 \parallel_{H^1} \parallel u_1 - u_2 \parallel_{L^q((T, \infty), L^{1+p})}.
\]

Taking account of (6), $\phi$ is a contraction of $X_T$, for $T > 0$ large enough. Since, this space is complete [6], with classical Picard argument, $\phi$ has a fixed point satisfying for some large $T_0 > 0$, which implies that
\[
u(t) = T_{\gamma, s}(t) u_+ + i \int_{t}^{\infty} T_{\gamma, s}(t-s) \parallel u \parallel_{L^1} u \hspace{1em} ds, \hspace{1em} t \geq T_0.
\]
Denoting $\psi := T_{\gamma,s}(-T_0)u(T_0)$, yields
\[ u(t) = T_{\gamma,s}(t)\psi - i\int_{T_0}^t T_{\gamma,s}(t-s)(|u|^{p-1}) u \, ds + i\int_{T_0}^\infty T_{\gamma,s}(t-s)(|u|^{p-1}) u \, ds \]
\[ = T_{\gamma,s}(t)\psi - i\int_{T_0}^t T_{\gamma,s}(t-s)(|u|^{p-1}) u \, ds. \]

It follows that $u$ resolves the problem (1) for $t > T_0$, moreover by Theorem 2.7, $u$ is global. Now, when $t$ tends to infinity
\[ \|u(t) - T_{\gamma,s}(t)u_+\|_{H^1} = \| \int_{t}^{\infty} T_{\gamma,s}(t-s)(|u|^{p-1}) u \|_{L^\infty((t,\infty), H^1)} \]
\[ \lesssim \|u\|_{L^p((0,\infty), W^{1,1+p})} \|
\[ \lesssim \|u\|_{L^p((0,\infty), W^{1,1+p})}\|u\|_{L^q((0,\infty), W^{1,1+p})} \to 0 \]
because by (7)-(8),
\[ u \in L^q((0,\infty), L^{1+p}) \cap L^q((0,\infty), W^{1,1+p}). \]

For uniqueness of such $u$, take $u_1, u_2 \in C(\mathbb{R}^+, H^1)$ two solutions to the first equation in (1) such that
\[ \lim_{t \to \infty} \|(u_1 - T_{\gamma,s}(\cdot)u_+)(t)\|_{H^1} = 0. \]

With the integral formula of Duhamel, we get $\tilde{u}_i = u_i$ for $i \in \{1, 2\}$. Moreover, arguing as previously
\[ \|u_1 - u_2\|_{L^q((T,\infty), L^{1+p})} \lesssim \|T_{\gamma,s}(\cdot)u_+\|_{L^q((T,\infty), L^{1+p})}\|u_+\|_{H^1} \|u_1 - u_2\|_{L^q((T,\infty), L^{1+p})}. \]

Uniqueness follows because, for large time
\[ \|u_1 - u_2\|_{L^q((T,\infty), L^{1+p})} \leq \frac{1}{2}\|u_1 - u_2\|_{L^q((T,\infty), L^{1+p})}. \]

The continuity of $S$ is a consequence of previous computations.

5. **Proof of Proposition 2.9.** We proceed with a fixed point argument. Take the function
\[ \phi(u)(t) := \tilde{u}(t) := T_{\gamma,s}(t)u_0 + i\int_0^t T_{\gamma,s}(t-\tau)(|u|^{p-1}) u \, d\tau. \]

For $C_q$ a constant depending on $q$ and $R := 2C_q\|u_0\|_{H^1}$, define the space
\[ X_R := \left\{ u \in L^q(\mathbb{R}^+, W^{1,1+p}), \ |u|_{L^q(\mathbb{R}^+, W^{1,1+p})} \leq R \right\} \]
derived with the distance
\[ d(u, v) := \|u - v\|_{L^q(\mathbb{R}^+, L^{1+p})}. \]

For $u_1, u_2 \in X_R$, thanks to the mean value Theorem, we get
\[ d(\tilde{u}_1, \tilde{u}_2) \lesssim \|u_1 - u_2\|_{L^q(\mathbb{R}^+, L^{1+p})} \]
\[ \lesssim \sum_{i=1}^2 \|u_1 - u_2\|_{L^q(\mathbb{R}^+, L^{1+p})}. \]
\[
\begin{align*}
&\lesssim \sum_{i=1}^{2} \|u_i\|_{L^2_\gamma(L^{1}+p)}^{p-1} \|u_1 - u_2\|_{L^q(R_+, L^{1}+p)} \\
&\lesssim \sum_{i=1}^{2} \|u_i\|_{L^2_\gamma(L^{1}+p)}^{p-1} d(u_1, u_2).
\end{align*}
\]

Thanks to Strichartz estimate and Hölder inequality via Lemma 2.12, yields

\[
\begin{align*}
\|u_i\|_{L^p(R_+, L^{1}+p)} &\lesssim \|T_{\gamma,s}(\cdot)u_0\|_{L^p(R_+, L^{1}+p)} + \|u_i\|_{L^{p'}(R_+, L^{1}+\frac{q}{p})}^{p-1} \\
&\lesssim \|T_{\gamma,s}(\cdot)u_0\|_{L^p(R_+, L^{1}+p)} + \|u_i\|_{L^{p'}(R_+, L^{1}+p)}^{p} \\
&\lesssim \epsilon + \|u_i\|_{L^p(R_+, L^{1}+p)} \\
&\lesssim \epsilon.
\end{align*}
\]

Then,

\[
d(\tilde{u}_1, \tilde{u}_2) \lesssim \epsilon^{p-1} d(u_1, u_2).
\]

Moreover, using Hölder and Strichartz inequalities, it follows that

\[
\begin{align*}
\|\tilde{u}\|_{L^p(R_+, W^{1,1}+p)} &\leq C_q \left( \|u_0\|_{H^1} + \|u\|_{L^p(L^{1}+\frac{q}{p})}^{p-1} \right) \\
&\leq C_q \left( \|u_0\|_{H^1} + (1 + \nabla) \|u\|_{L^{p'}(R_+, L^{1}+\frac{q}{p})}^{p-1} \right) \\
&\leq C_q \left( \|u_0\|_{H^1} + C \|u\|_{L^{p'}(R_+, L^{1}+p)}^{p-1} \|u\|_{L^q(R_+, W^{1,1}+p)} \right) \\
&\leq C_q \left( \|u_0\|_{H^1} + CR\epsilon^{p-1} \right) \\
&\leq \frac{R}{2} + CR\epsilon^{p-1}.
\end{align*}
\]

Then, for small $\epsilon > 0$, $\phi$ is a contraction of $X_R$. Since, this space is complete, with classical Picard argument, $\phi$ has a fixed point satisfying (1) in $L^2_q(W^{1,1}+p)$. Moreover, with previous computations, $u \in C(R_+, H^1)$ and is unique.

**Appendix.** This section is devoted to prove Proposition 2.5 about local well-posedness of (1). We proceed with a fixed point argument. Take the function

\[
\phi(u)(t) := \tilde{u}(t) := T_{\gamma,s}(\cdot)u_0 + i \int_0^t T_{\gamma,s}(t - \tau)(|u|^{p-1}u) \, d\tau.
\]

For $C_q$ a constant depending on $q$ and $R := 2C_q\|u_0\|_{H^1}$, define the space

\[
X_{R,T} := \left\{ u \in L^2_q(W^{1,1}+p) \cap C_T(H^1), \quad \|u\|_{L^2_q(W^{1,1}+p) \cap L^p_T(H^1)} \leq R \right\}
\]

endowed with the distance

\[
d(u, v) := \|u - v\|_{L^2_q(L^{1}+p)}.
\]

For $u_1, u_2 \in X_{R,T}$, thanks to the mean value theorem and Strichartz estimate, we get

\[
\begin{align*}
\|u_1 - u_2\|_{L^2_q(L^{1}+p)} &\leq \sum_{i=1}^{2} \|u_i\|_{L^{p'}(L^{1}+\frac{q}{p})}^{p-1} d(u_1, u_2) \\
&\leq \sum_{i=1}^{2} \|u_i\|_{L^{p'}(L^{1}+\frac{q}{p})}^{p-1} \|u_i\|_{L^{p'}(L^{1}+\frac{q}{p})}
\end{align*}
\]
Then, for small $T > 0$, decreasing. Recall the identity (3), yields

Taking account of the estimates

Moreover, using Hölder and Strichartz inequalities, it follows that

Then, for small $T > 0$, $\phi$ is a contraction of $X_{R,T}$. Since, this space is complete, with classical Picard argument, $\phi$ has a fixed point satisfying (1) in $L^p_{T}(W^{1,1+p})\cap C_T(H^1)$. Moreover, with previous computations, this solution is unique.

Now, assuming that $s \in (0,1)$ and $p = 1 + \frac{1}{N}$, we prove that the energy is decreasing. Recall the identity (3),

Taking account of the estimates

yields

This finishes the proof.

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