Universal rings arising in geometry and group theory

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Abstract. Various algebraic structures in geometry and group theory have appeared to be governed by certain universal rings. Examples include: the cohomology rings of Hilbert schemes of points on projective surfaces and quasi-projective surfaces; the Chen-Ruan orbifold cohomology rings of the symmetric products; the class algebras of wreath products, as well as their associated graded algebras with respect to a suitable filtration. We review these examples, and further provide a new elementary construction and explanation in the case of symmetric products. We in addition show that the Jucys-Murphy elements can be used to clarify the Macdonald’s isomorphism between the FH-ring for the symmetric groups and the ring of symmetric functions.

1. Introduction

1.1. The Hilbert scheme $X^{[n]}$ of $n$ points on a complex (quasi-)projective surface $X$ has provided a natural setup for an interplay among geometry, representation theory and combinatorics. The starting point is the construction of a Heisenberg algebra in terms of incidence varieties which acts irreducibly on the direct sum over all $n$ of the cohomology groups $H^*(X^{[n]})$ with $\mathbb{C}$-coefficients $[38]$ (cf. $[19]$ for another construction and $[15, 45]$ for motivations). In a nutshell, the Heisenberg algebra reveals the geometric structures of the Hilbert schemes in a way similar to the cycle structures for the symmetric groups. This has recently led to a new approach toward the cohomology rings of Hilbert schemes.

Being closely related yet complementary to each other, the results on these cohomology rings can be roughly divided as follows: the connections with vertex operators and $W$ algebras $[27, 30, 33]$; the ring generators (and relations) for the rings $H^*(X^{[n]})$ $[30, 31, 34]$ (also cf. $[10, 3, 36]$ for a classical approach for some special surfaces); the constructions of the universal rings which govern the rings $H^*(X^{[n]})$ depending on whether or not $X$ is compact $[32, 34]$; the intertwining relations with the study of Chen-Ruan orbifold cohomology rings $[6]$ of the symmetric products $[28, 29, 40, 42, 44, 24, 17]$ as well as the class algebras of the wreath products $[47, 14, 11, 26, 49, 50]$. We remark that there has been also remarkable connections of Hilbert schemes with combinatorics $[20]$ and interesting work on the motives of Hilbert schemes $[7, 16]$.

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Much of the advances in these directions has been made after an earlier review was written. The main purpose of the present paper is to review some of these recent progress complementary to loc. cit., with an emphasis on the appearance of certain universal rings which govern various algebraic structures. Also it is interesting to compare with where the axiomatic nature of the vertex operator approach developed for Hilbert schemes was formulated explicitly and similar framework was established for the Chen-Ruan orbifold cohomology rings of the symmetric products. The Sections 4 and 5.5 contain new constructions and results.

1.2. The Heisenberg algebra provides a distinguished linear basis for $H^\ast(X^{[n]})$ parameterized by multi-partitions (this basis will be referred to as the Heisenberg monomial basis). It is natural to ask how the structure constants of the cup product among these basis elements (up to some suitable scalings) behave. It turns out that the answer depends essentially on whether $X$ is projective or quasi-projective.

When $X$ is projective, a stability result formulated in states that the modified structure constants with respect to a suitably chosen set of elements in $H^\ast(X^{[n]})$ (which include the Heisenberg monomial basis elements) can be uniquely taken to be independent of $n$. This has led to a universal ring (termed as the Hilbert ring associated to $X$) which governs the cohomology rings $H^\ast(X^{[n]})$ for all $n$. These are explained in Sect. The Hilbert ring actually depends not on the projective surface $X$, but on the ring $H^\ast(X)$ and the canonical class of $X$. While all these statements are described in the classical language, the machinery used to establish them replies heavily on the connections with vertex operators.

The stability for the cohomology rings of $H^\ast(X^{[n]})$ for $X$ quasi-projective requires a quite different formulation. It is shown that the structure constants with respect to the Heisenberg monomial basis of $H^\ast(X^{[n]})$ are independent of $n$, for a large class of quasi-projective surfaces (conjecturally for all quasi-projective surfaces). This has led to a universal ring (called the FH-ring in ) which governs the cohomology rings $H^\ast(X^{[n]})$ for all $n$. This stability result can in turn be interpreted as giving rise to a surjective ring homomorphism from $H^\ast(X^{[n]})$ to $H^\ast(X^{[n-1]})$ for each $n$. This phenomenon has been rather unusual, since there is no natural algebraic embedding of $X^{[n-1]}$ into $X^{[n]}$ for $m<n$. In fact, an incidence variety relating $X^{[n]}$ and $X^{[n-1]}$ induces such a ring homomorphism. This will be explained in Sect. 3.

1.3. Recall that the Hilbert-Chow morphism $\pi_n : X^{[n]} \to X^n/S_n$ from the Hilbert schemes to the symmetric products is a crepant resolution of singularities, where $S_n$ is the $n$-th symmetric group. A general principle states that the geometry of an orbifold is “equivalent” to the geometry of its crepant resolutions. Originally, the word “geometry” here was expressed in terms of (orbifold) Betti numbers. Over the years, the “equivalence” extends to (orbifold) Hodge numbers, (orbifold) elliptic genera, motivic integration, and (orbifold) cohomology rings etc, (see, for example, and the references therein). In particular, the Chen-Ruan cohomology ring $H_{CR}(X^n/S_n)$ of the symmetric products has attracted much attention, as this is related to the cohomology rings of Hilbert schemes and vertex operators. This provides nice examples of testing and verifying Ruan’s conjectures. We remark

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1 The use of the terminology “quasi-projective” excludes “projective” in the present paper.
that there has been some further reformulation and development of the orbifold cohomology rings \([12, 44]\).

It developed that the structures of the rings \(H^*_{\text{CR}}(X^n/S_n)\) for a compact complex manifold \(X\) are very similar to the cohomology rings of Hilbert schemes of points on projective surfaces. A stable ring introduced in \([40]\), which are analogous to the Hilbert ring mentioned above for Hilbert schemes, governs the Chen-Ruan cohomology rings \(H^*_{\text{CR}}(X^n/S_n)\). The proof therein relies on an axiomatic vertex operator approach which are parallel to the developments in Hilbert schemes. An additional bonus of such an approach is to provide a more conceptual proof of Ruan’s conjecture on the ring isomorphism between \(H^*(X^{[n]})\) and \(H^*_{\text{CR}}(X^n/S_n)\) when \(X\) is projective and has a numerically trivial canonical class (which was also established in \([12, 44]\) using \([29]\)).

However, the orbifold side is simpler than the crepant resolution side, and often group-theoretic techniques are applicable. A nice example is provided by the Jucys-Murphy elements of symmetric groups \([22, 37]\) which feature significantly in the study of the rings \(H^*_{\text{CR}}(X^n/S_n)\) \([40]\). In Sect. 1, we shall provide a new and direct construction (without using vertex operators) which explains the stability for the rings \(H^*_{\text{CR}}(X^n/S_n)\). The idea here is to enlarge further the noncommutative orbifold cohomology rings \([12]\) (also cf. \([29]\)) of the symmetric products.

### 1.4. The Hilbert schemes are also resolutions of singularities of wreath product orbifolds (which are generalized symmetric products) \([47]\), where the wreath product \(\Gamma_n := \Gamma^n \rtimes S_n\) is the semidirect product of the product group \(\Gamma^n\) and \(S_n\). The study of the class algebras \(R(\Gamma_n)\) of the wreath products \(\Gamma_n\) (which can be regarded as the Chen-Ruan cohomology ring of the symmetric product of the orbifold \(\text{pt}/\Gamma\)) reveals a stability similar to the Hilbert schemes for projective surfaces \([49]\). We can further introduce a filtration on \(R(\Gamma_n)\) and study its associated graded algebras \(G_{\Gamma}(n)\) \([50]\) (also compare \([11, 13, 28, 46]\)). It is shown \([50]\) that the structure constants of these graded algebras \(G_{\Gamma}(n)\) with respect to the conjugacy classes (which are the analog of the Heisenberg monomial basis) are independent of \(n\). This motivated and corresponds to the stability explained above of the cohomology rings of Hilbert schemes when the surface is quasi-projective. This allows one to introduce a universal ring \(G(\Gamma)\) (called the FH-ring) which governs the algebras \(G_{\Gamma}(n)\) for all \(n\). Both the Chen-Ruan cohomology ring of the symmetric product and the class algebra of wreath product are nontrivial generalizations in two directions of the class algebra of the symmetric group, yet they can be further unified under the common roof of the wreath product orbifolds.

Macdonald \([35]\) has earlier constructed an isomorphism between the FH-ring \(G\) \([13]\) of the symmetric groups (i.e. when \(\Gamma = 1\)) and the ring of symmetric functions. (Actually, the universal ring in each of the different setups above has been shown to be a polynomial ring which is isomorphic to a tensor product of several copies of the ring of symmetric functions.) In addition, he describes explicitly the symmetric functions corresponding to the stable conjugacy classes in \(G\). In Sect. 5 we will show how the Jucys-Murphy elements can be used to provide further explicit correspondences under Macdonald’s isomorphism.

A better way of reading the present paper is to read first Sect. 4 by setting \(X\) to be a point and Sect. 5 by setting \(\Gamma\) to be trivial, where the results and notations would be significantly simplified. In this way the formulations of the stability in Sections 2 and 3 may become more transparent.
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2. The Hilbert ring of Hilbert schemes for projective surfaces

2.1. Generalities on Hilbert schemes. Let $X$ be a (quasi-)projective complex surface, and $X^{[n]}$ be the Hilbert scheme of $n$ points on $X$. An element in the Hilbert scheme $X^{[n]}$ is represented by a length $n$ 0-dimensional closed subscheme of $X$. According to Fogarty, $X^{[n]}$ is smooth. We denote by $H^*(X^{[n]})$ the cohomology group/ring of $X^{[n]}$ with complex coefficient. We denote

$$
\mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]}).
$$

The element $1$ in $H^0(X^{[0]}) = \mathbb{C}$ is called the vacuum vector and denoted by $|0\rangle$. A non-degenerate super-symmetric bilinear form $(\cdot,\cdot)$ on $\mathbb{H}_X$ is induced from the standard one on $H^*(X^{[n]})$ defined by $(\alpha,\beta) = \int_{X^{[n]}} \alpha \beta$ for $\alpha,\beta \in H^*(X^{[n]})$. For $f \in \text{End}(\mathbb{H}_X)$ we denote by $f^\dagger \in \text{End}(\mathbb{H}_X)$ its adjoint operator.

The Betti numbers of Hilbert schemes for an arbitrary (quasi-)projective surface was computed in \cite{Lehn2005}. As a corollary, one obtains the graded dimension of $\mathbb{H}_X$ to be

$$
\text{dim}_q \mathbb{H}_X := \sum_{n=0}^{\infty} \text{dim} H^*(X^{[n]}) q^n = \prod_{r=1}^{\infty} \frac{(1 + q^r)^{h_{\text{odd}}(X)} - (1 - q^r)^{h_{\text{ev}}(X)}}{r}
$$

where $h_{\text{odd}}(X)$ and $h_{\text{ev}}(X)$ denote the dimensions of the cohomology group of $X$ of odd and respectively even degrees. Based on this formula, it is suggested \cite{Lehn2005} that the space $\mathbb{H}_X$ can be identified as the Fock space of a Heisenberg algebra associated to the lattice $H^*(X, \mathbb{Z})/\text{tor}$. This has been established firmly in terms of incidence varieties \cite{Nakajima1999}. Similar results were obtained in \cite{Lehn2005}.

We recall the construction \cite{Nakajima1999, Nakajima2001} when $X$ is projective. For $n > 0$ and $\ell \geq 0$, we define $Q^{[n+\ell]} X^{[n]} \subseteq X^{[n+\ell]} \times X^{[n]}$ to be the closed subset

$$
\{(\xi,\eta) \in X^{[n+\ell]} \times X^{[n]} \mid \xi \supset \eta \text{ and } \text{Supp}(I_\eta / I_\xi) = \{x\}\}.
$$

The linear operator $a_{-n}(\alpha) \in \text{End}(\mathbb{H}_X)$ (called the creation operator) with $\alpha \in H^*(X)$ is defined by

$$
a_{-n}(\alpha)(a) = pr_1^\ast ([Q^{[m+n,\ell]}] \cdot \rho^\ast \alpha \cdot pr_2^\ast a).
$$

for all $a \in H^*(X^{[m]})$, where $pr_1, \rho, pr_2$ are the projections of $X^{[m+n]} \times X^{[m]}$ to $X^{[m+n]}, X^{[m]}$ respectively. Here and below we use $\gamma \cdot \beta$ or $\gamma \beta$ to denote the cup product of two cohomology classes $\gamma, \beta$, and omit the Poincaré duality used to switch between a homology class and a cohomology class. The annihilation operator $a_n(\alpha)$ can be defined to be $a_n(\alpha) = (-1)^n \cdot a_{-n}(\alpha)^\dagger$. Also set $a_0(\alpha) = 0$. Nakajima’s theorem \cite{Nakajima2001} says that

$$
[a_n(\alpha), a_m(\beta)] = -n \cdot \delta_{n,-m} \cdot \int_X (\alpha \beta) \cdot \text{Id}.
$$
Moreover, the space $\mathbb{H}_X$ is an irreducible representation of the Heisenberg algebra.

**Remark 2.1.** The above construction remain to be valid when $X$ is quasi-projective after some suitable modification, cf. [33]. That is, we have the creation operators $a_n(\alpha)$ ($n > 0$) associated to a cohomology class $\alpha$ of $X$ as above. On the other hand, the annihilation operators $a_n(\gamma)$ ($n > 0$) are associated to a cohomology class $\gamma$ with compact support of $X$.

**2.2. The Hilbert ring.** Given a $\mathbb{Z}_2$-graded finite set $S = S_0 \cup S_1$, we denote by $\mathcal{P}(S)$ the set of partition-valued functions $\rho = (\rho(c))_{c \in S}$ on $S$ such that for every $c \in S_1$, the partition $\rho(c)$ is required to be strict in the sense that $\rho(c) = (1^{m_1(c)})2^{m_2(c)}\ldots$ with $m_r(c) = 0$ or 1 for all $r \geq 1$.

Now let us take a linear basis $S = S_0 \cup S_1$ of $H^*(X)$ such that $1_X \in S_0$, $S_0 \subset H^{even}(X)$ and $S_1 \subset H^{odd}(X)$. If we write a partition-valued function $\rho = (\rho(c))_{c \in S}$, where $\rho(c) = (r^{m_r(c)})_{r \geq 1} = (1^{m_1(c)})2^{m_2(c)}\ldots$, then we define

$$
\ell(\rho) = \sum_{c \in S} \ell(\rho(c)) = \sum_{c \in S, r \geq 1} m_r(c)
$$

$$
\|\rho\| = \sum_{c \in S} |\rho(c)| = \sum_{c \in S, r \geq 1} r \cdot m_r(c)
$$

$$
\mathcal{P}_n(S) = \{ \rho \in \mathcal{P}(S) \mid \|\rho\| = n \}.
$$

Given $\rho = (\rho(c))_{c \in S} = (r^{m_r(c)})_{c \in S, r \geq 1}$, we define

$$
a_{-\rho(c)}(c) = \prod_{r \geq 1} a_{-r(c)}^{m_r(c)}
$$

$$
a_\rho(n) = 1_{-(n-\|\rho\|)} \prod_{c \in S} a_{-\rho(c)}(c) \cdot |0\rangle \in H^*(X^{[n]})
$$

where we fix the order of the elements $c \in S_1$ appearing in $\prod_{c \in S}$ once and for all, and $1_{-k}$ ($k \geq 1$) denotes $a_{-1}(1_X)^k/k!$. By definition, $a_\rho(n) = 0$ for $n < \|\rho\|$.

**Theorem 2.2.** [33] Let $X$ be a projective surface, and $S$ be a linear basis of $H^*(X)$ which contains $1_X$. Let $\rho, \sigma \in \mathcal{P}(S)$. Then the cup product $a_\rho(n) \cdot a_\sigma(n)$ in $H^*(X^{[n]})$ can be expressed uniquely as a linear combination

$$
a_\rho(n) \cdot a_\sigma(n) = \sum_{\nu \in \mathcal{P}(S)} A^\nu_{\rho\sigma}a_\nu(n)
$$

where $\|\nu\| \leq \|\rho\| + \|\sigma\|$ and the structure coefficients $A^\nu_{\rho\sigma}$ are independent of $n$.

Note that the cohomology classes $a_\nu(n)$ with $\|\nu\| \leq n$ span $H^*(X^{[n]})$ but are not linearly independent for $n > 1$.

**Remark 2.3.** Although it is not manifest in the formulation of the statement, the proof of Theorem 2.2 used the connections between the cup products in $H^*(X^{[n]})$ and vertex operators developed in [27, 30] in an essential way.

**Definition 2.4.** The **Hilbert ring** associated to a projective surface $X$, denoted by $\mathfrak{H}_X$, is defined to be the ring with a linear basis formed by the symbols $a_\rho$, $\rho \in \mathcal{P}(S)$ and with the multiplication defined by $a_\rho \cdot a_\sigma = \sum_\nu A^\nu_{\rho\sigma}a_\nu$ where the structure constants $A^\nu_{\rho\sigma}$ are from (2.2).
Note that the Hilbert ring does not depend on the choice of a linear basis $S$ of $H^*(X)$ containing $1_X$ since the operator $a_n(a)$ depends on the cohomology class $a \in H^*(X)$ linearly. The Hilbert ring $\mathcal{H}_X$ captures all the information of the cohomology rings of $X^{[n]}$ for all $n$, as we easily recover the relations (2.2) from the ring $\mathcal{H}_X$. These observations can be summarized into the following.

**Theorem 2.5.** \[^{[32]}\] For a given projective surface $X$, the cohomology rings $H^*(X^{[n]})$, $n \geq 1$ give rise to a Hilbert ring $\mathcal{H}_X$ which completely encodes the cohomology ring structure of $H^*(X^{[n]})$ for each $n$.

We shall see in Sect. \[^{[33]}\] that analogous results hold in the symmetric product setup which afford simple proofs and explanations.

3. The FH-ring of Hilbert schemes for quasi-projective surfaces

3.1. $n$-independence of the structure constants. Let $X$ be a smooth quasi-projective surface. As before, let us take a linear basis $S = S_0 \cup S_1$ of $H^*(X)$ such that $1_X \in S_0$, $S_0 \subset H^{\text{even}}(X)$ and $S_1 \subset H^{\text{odd}}(X)$.

Fix $n \geq 1$. For a given $\rho \in \mathcal{P}(S)$, we set $b_{\rho}(n) = 0 \in H^*(X^{[n]})$ if $n < \|\rho\| + \ell(\rho(1_X))$. If $n \geq \|\rho\| + \ell(\rho(1_X))$, we define $\hat{\rho} \in \mathcal{P}(S)$ by putting $m_r(\hat{\rho}(c)) = m_r(\rho(c))$ for $c \in S - \{1_X\}$, $m_1(\hat{\rho}(1_X)) = n - \|\rho\| - \ell(\rho(1_X))$, and $m_r(\hat{\rho}(1_X)) = m_{r-1}(\rho(1_X))$ for $r \geq 2$. Note that $\|\hat{\rho}\| = n$. We define $b_{\rho}(n) \in H^*(X^{[n]})$ by

$$b_{\rho}(n) = \frac{1}{\prod_{r \geq 2} (r^{m_r(\hat{\rho}(1_X))} m_r(\hat{\rho}(1_X)))!} \left( \prod_{\substack{c \in S, r \geq 1 \backslash \{1_X\} \atop r \geq 1}} a_{\rho}(c)^{m_r(\hat{\rho}(c))} \right) |0|,$$

where we fix the order of the elements $c \in S_1$ appearing in the product $\prod_{c \in S}$ once and for all. We remark that the only part in the definition of $b_{\rho}(n)$ involving $n$ is the factor $1 - (n - \|\rho\| - \ell(\rho(1_X)))$.

As a corollary to the construction of Heisenberg algebra \[^{[33]}\], $H^*(X^{[n]})$ has a linear basis consisting of the classes $b_{\rho}(n)$ where $\rho \in \mathcal{P}(S)$ and $\|\rho\| + \ell(\rho(1_X)) \leq n$. Fix a positive integer $n$ and $\rho, \sigma \in \mathcal{P}(S)$ satisfying $\|\rho\| + \ell(\rho(1_X)) \leq n$ and $\|\sigma\| + \ell(\sigma(1_X)) \leq n$. Then we can write the cup product $b_{\rho}(n) \cdot b_{\sigma}(n)$ as

$$(3.1) \quad b_{\rho}(n) \cdot b_{\sigma}(n) = \sum_{\nu \in \mathcal{P}(S)} B^\nu_{\rho\sigma}(n) b_{\nu}(n)$$

with structure constants $B^\nu_{\rho\sigma}(n)$, where $\|\nu\| + \ell(\nu(1_X)) \leq n$.

We shall need the following technical definition.

**Definition 3.1.** A smooth quasi-projective surface $X$ is said to satisfy the $S$-property if it can be embedded in a smooth projective surface $\overline{X}$ such that the induced ring homomorphism $H^*(\overline{X}) \to H^*(X)$ is surjective.

Let $X$ be a smooth quasi-projective surface with the $S$-property. This class of quasi-projective surfaces is very large, including the minimal resolution of a simple singularity, the cotangent bundle of a smooth projective curve, and the surface obtained from a smooth projective surface by deleting a point, etc.
Theorem 3.2. Let \( X \) be a smooth quasi-projective surface satisfying the S-property. Then all the structure constants \( B^\nu_{\rho\sigma}(n) \) given in (3.1) are independent of \( n \).

Remark 3.3. The statement in Theorem 3.2 was conjectured \( ^{50} \) to be valid for every quasi-projective surface. The proof of Theorem 3.2 given in \( ^{34} \) used in an essential way relations with Hilbert schemes of points on projective surfaces \( ^{32, 33} \). Some new approach will be needed toward the general conjecture.

Remark 3.4. The definitions of the elements \( a^\rho(n) \) in Sect. 2 and \( b^\rho(n) \) above (with different parametrization and scaling) are made in order to formulate two different types of stability depending on whether or not the surface \( X \) is projective. For example, Theorem 3.2 is invalid if \( X \) is projective.

3.2. The FH-ring of Hilbert schemes.

Recall from Remark 2.1 that the annihilation operators \( a^\gamma(n) \) \( (n > 0) \) are associated to a cohomology class \( \gamma \) with compact support of \( X \), when \( X \) is quasi-projective. Set \( x^c \in H^4_c(X) \) to be the Poincaré dual of the homology class in \( H_0(X) \) represented by a point in \( X \). Then \( \mathfrak{A} = -a_1([x^c]) \) is induced from an incidence variety in \( X^{[n-1]} \times X^{[n]} \). The following theorem is an equivalent formulation of Theorem 3.2.

Theorem 3.5. Let \( X \) be a smooth quasi-projective surface satisfying the S-property. Then the linear map \( \mathfrak{A} : H^*(X^{[n]}) \rightarrow H^*(X^{[n-1]}) \) is a surjective ring homomorphism. More explicitly, it sends \( b^\rho(n) \) to \( b^\rho(n-1) \) for all \( \rho \).

Remark 3.6. In the case when \( X \) is the affine plane, the above theorem was also contained in \( ^{28} \). The work of Farahat-Higman is very relevant here \( ^{13, 50} \) (also cf. Sect. 5 below).

Definition 3.7. Let \( X \) be a smooth quasi-projective surface satisfying the S-property. We define the FH-ring \( G_X \) associated to \( X \) to be the ring with a linear basis given by the symbols \( b^\rho \), \( \rho \in P(S) \), with the product given by

\[
b^\rho \cdot b^\sigma = \sum_{\nu \in P(S)} B^\nu_{\rho\sigma} b^\nu,
\]

where the structure constants \( B^\nu_{\rho\sigma} \) come from (3.1).

The above results have led to the following.

Theorem 3.8. For a quasi-projective surface \( X \) which satisfies the S-property, the cohomology rings \( H^*(X^{[n]}) \), \( n \geq 1 \) give rise to the FH-ring \( G_X \) which completely encodes the cohomology ring structure of \( H^*(X^{[n]}) \) for each \( n \).

4. The stable ring of the symmetric products \( X^n/S_n \) for \( X \) compact

4.1. Generalities on Chen-Ruan cohomology rings. Let \( M \) be a complex manifold of complex dimension \( d \) with a finite group \( G \) action. Introduce

\[
M \circ G = \{(g, x) \in G \times M \mid gx = x \} = \bigsqcup_{g \in G} M^g,
\]

with a \( G \)-action given by \( h \cdot (g, x) = (hgh^{-1}, hx) \). As a vector space, we define \( H^*(M, G) \) to be the cohomology group of \( M \circ G \) with \( \mathbb{C} \)-coefficient, or equivalently,
\[ H^* (M, G) = \bigoplus_{g \in G} H^* (M^g). \]

The space \( H^* (M, G) \) has a natural induced \( G \) action, which is denoted by \( \text{ad} \ h : H^* (M^g) \to H^* (M^{gh^{-1}}) \). As a vector space, the orbifold cohomology group \( H^*_{CR} (M/G) \) is the \( G \)-invariant part of \( H^* (M, G) \), which is isomorphic to

\[ \bigoplus_{[g] \in G_*} H^* (M^g / Z(g)) \]

where \( G_* \) denotes the set of conjugacy classes of \( G \) and \( Z(g) \) denotes the centralizer of \( g \) in \( G \).

For \( g \in G \) and \( x \in M^g \), write the eigenvalues of the action of \( g \) on the complex tangent space \( TM_x \) to be \( \mu_k = e^{2\pi i r_k} \), where \( 0 \leq r_k < 1 \). The degree shift number or age is the rational number \( F_2^g = \sum_{k=1}^{\text{odd}} r_k \), cf. \([32]\). It depends only on the connected component \( Z \) which contains \( x \), so we can denote it by \( F_2^g \). Then associated to a cohomology class in \( H^* (Z) \), we assign the corresponding element in \( H^* (M, G) \) (and thus in \( H^*_{CR} (M/G) \)) a degree of \( r + 2F_2^g \).

A graded ring structure on \( H^*_{CR} (M/G) \) was introduced by Chen and Ruan \([4]\). This was subsequently reformulated in \([12]\) by introducing a ring structure on \( H^* (M, G) \) first and then passing to \( H^*_{CR} (M/G) \) by restriction. We shall use \( \circ \) to denote this product. The ring has the following property: \( \alpha \circ \beta \) lies in \( H^* (M^g) \) for \( \alpha \in H^* (M^g) \) and \( \beta \in H^* (M^h) \).

### 4.2. Enlarging the ring \( H^* (X^n, S_n) \)

Let \( X \) be a compact complex manifold of dimension \( d \). Our main objects here are the Chen-Ruan cohomology ring \( H^*_{CR} (X^n / S_n) \). Let us fix a linear basis \( S = S_0 \sqcup S_1 \) of \( H^* (X) \) such that \( S_0 \subset H^{ev} (X) \), \( 1_X \in S_0 \), and \( S_1 \subset H^{odd} (X) \).

If \( U \) is a finite set, we denote by \( S_U \) the symmetric group of permutations on \( U \). We denote by \( n \) the set \( \{1, 2, \ldots, n\} \), so \( S_n \) is just our usual symmetric group \( S_n \). For \( U \subset n \), we regard \( S_U \) as a subgroup of \( S_n \). We define \( X^U = \{ f : U \to X \} \) with a natural action by \( S_U \). In particular, \( X^\emptyset = X^n \).

Following \([21]\), we introduce the semigroup of “partial permutations” \( PS_n \) as follows. A partial permutation of the set \( n \) is a pair \((\sigma, U)\) which consists of a finite subset \( U \subset n \) and an element \( \sigma \in S_U \). Denote by \( PS_n \) the set of all partial permutations of \( n \). The finite set \( PS_n \) is endowed with a natural semigroup structure by letting the product of two elements \((\sigma, U)\) and \((\tau, V)\) in \( PS_n \) to be \((\sigma \tau, U \cup V)\). We denote by \( \mathbb{Z}[PS_n] \) the semigroup algebra over \( \mathbb{Z} \). The \( S_n \) acts on \( PS_n \) and thus on \( \mathbb{Z}[PS_n] \) by \( \text{ad} \ g : (\sigma, U) \mapsto (g \sigma g^{-1}, gU) \).

We introduce

\[ PH^* (X^n, S_n) = \bigoplus_{(\sigma, U) \in PS_n} H^{*-2F_2} ((X^U)^\sigma) \]

and define a product \( \bullet \) on \( PH^* (X^n, S_n) \) as follows. Let \((\sigma, U)\) and \((\tau, V)\) be in \( PS_n \). Given \( \alpha \in H^{*-2F_2} ((X^U)^\sigma) \) and \( \beta \in H^{*-2F_2} ((X^V)^\tau) \), we may identify \( \alpha \) as an element in \( H^{*-2F_2} ((X^{U \cup V})^\tau) \) via the natural embedding

\[ H^{*-2F_2} ((X^U)^\sigma) \subset H^{*-2F_2} ((X^{U \cup V})^\sigma) \]

where \( \sigma \in S_U \) is regarded as a permutation in \( S_{U \cup V} \) by fixing \( V \setminus U \cap V \) point-wise. Similarly, we may regard \( \beta \) as an element in \( H^{*-2F_2} ((X^{U \cup V})^\tau) \). We define
the product \( \alpha \cdot \beta \) to be \( \alpha \circ \beta \in H^{*-2F_{\infty}}((X^{U}U)^{\sigma}) \). Then the associativity of \( H^*(X^n, S_n) \) (together with the trivial associativity of the union operation of sets) implies that \( PH^*(X^n, S_n) \) thus defined is an associative algebra.

Note that \( \text{ad } g : H^*((X^U)^\sigma) \to H^*((X^{U})^{g\sigma g^{-1}}) \) for \( g \in S_n \) and \((\sigma, U) \in PS_n\) defines an action of \( S_n \) on \( PH^*(X^n, S_n) \). The product structure on \( PH^*(X^n, S_n) \) is clearly compatible with the \( S_n \)-action. We denote by \((H_{X,n} \cdot \cdot)\) the \( S_n \)-invariant subalgebra of \( PH^*(X^n, S_n) \).

### 4.3. The stability of the rings \( H^*_{\text{Gr}}(X^n/S_n) \)

By associating a partition of \( |U| \) (the cardinality of \( U \)) to an element \((\sigma, U) \in PS_n\), we have a natural parametrization of the \( S_n \)-orbits of \( PS_n \) by partitions \( \lambda \) with \( ||\lambda|| \leq n \). Just as \( H^*_{\text{Gr}}(X^n/S_n) \) has a linear basis given by \( P_n(S) \) (cf. e.g. \([40]\)), we see that \( H_{X,n} \) affords a linear basis parameterized by \( \rho \in P(S) \) with \( ||\rho|| \leq n \) as follows. Given \( \rho = (\rho(s))_{s \in S} \in P(S) \) with \( ||\rho|| \leq n \), define a partition \( \bar{\rho} \) of \( ||\rho|| \) to be \( \cup_{s \in S} \rho(s) \); that is, \( \bar{\rho} \) is obtained from \( \rho \in P(S) \) by forgetting the indices \( s \in S \) and then rearranging the parts in descending order. Take an element \((\sigma, U) \in PS_n \) with \( |U| = ||\rho|| \) and \( \sigma \) of cycle type \( \bar{\rho} \), and take an \( x \in H^*((X^U)^\sigma) \cong H^*(X)^{\otimes (\bar{\rho})} \) with \( (\rho(s)) \) factors equal to \( s \) for each \( s \in S \) such that the cycles of \( \sigma \) associated to \( s \) correspond to the partition \( \rho(s) \). Denote by \( c_{\rho}(n) \in H_{X,n} \) the sum of the \( S_n \)-orbit of \( x \). It is easy to see that \( c_{\rho}(n) \) does not depend on the choice of \((\sigma, U) \) and \( x \), but only on \( \rho \) and \( n \). The \( c_{\rho}(n) \), where \( \rho \in P(S) \) and \( ||\rho|| \leq n \), form a linear basis for \( H_{X,n} \).

Let us write

\[ c_{\rho}(n) \cdot c_{\sigma}(n) = \sum_{\nu \in P(S)} C_{\rho \sigma}^{\nu}(n) c_{\nu}(n) \]

summed over \( \nu \) with \( ||\nu|| \leq n \), where \( C_{\rho \sigma}^{\nu}(n) \) denotes the structure constants of the ring \( H_{X,n} \).

Keeping \([4.1]\) in mind, we define a linear map (for \( m \leq n \))

\[ \theta_{n,m} : PH^*(X^n, S_n) \to PH^*(X^m, S_m) \]

to be the identity map on the component \( H^{*-2F_{\infty}}((X^U)^\sigma) \) associated to \((\sigma, U) \in PS_n \) if \( U \subset m \) and 0 if \( U \not\subset m \). It is clear that \( \theta_{n,m} \) is a ring homomorphism and it is surjective. Since the homomorphisms \( \theta_{n,m} \) for all \( m \leq n \) are compatible, we can define an inverse limit

\[ PH^*(X^\infty, S_{\infty}) := \lim_{\nu} PH^*(X^n, S_n) \]

which has an induced algebra structure and an induced action by the group \( S_{\infty} = \cup_n S_n \). Denote by \( H_{X,\infty} \) the algebra of \( S_{\infty} \)-invariants in \( PH^*(X^\infty, S_{\infty}) \). One can construct a linear basis \( c_{\rho} \) with \( \rho \in P(S) \) of \( H_{X,\infty} \), in the same way as constructing the basis \( c_{\rho}(n) \)'s for \( H_{X,n} \). Write

\[ c_{\rho} \cdot c_{\sigma} = \sum_{\nu} C_{\rho \sigma}^{\nu}(n) c_{\nu}. \]

The homomorphisms \( \theta_{n,m} \) induces surjective ring homomorphisms, which will be also denoted by \( \theta_{n,m} \), from \( H_{X,n} \) to \( H_{X,m} \). This in turn is compatible with a surjective ring homomorphism \( \theta_{n} : H_{X,\infty} \to H_{X,n} \) by letting \( \theta_{n}(c_{\rho}) = c_{\rho}(n) \) if \( ||\rho|| \leq n \) and \( \theta_{n}(c_{\rho}) = 0 \) otherwise. It follows that \( C_{\rho \sigma}^{\nu}(n) = C_{\rho \sigma}^{\nu} \) for \( \rho, \sigma, \nu \) such that \( ||\rho|| \leq n, ||\sigma|| \leq n \) and \( ||\nu|| \leq n \). We have established the following.
Theorem 4.1. The structure constants $C^\nu_{\rho \sigma}(n)$ are independent of $n$.

The ‘forgetful’ map $f_n : PH^*(X^n, S_n) \to H^*(X^n, S_n)$ which, when restricted to the component $H^{-2F_\sigma}((X^U)^\sigma)$ for $(\sigma, U) \in PS_n$, is given by the inclusion map $H^{-2F_\sigma}((X^U)^\sigma) \subset H^{-2F_\rho}((X^U)^\sigma)$ where $\sigma$ in the latter is regarded as an element of $S_n$ by fixing pointwise $U \backslash U$. Clearly, $f_n$ is indeed a surjective $S_n$-equivariant ring homomorphism, and thus induces a surjective ring homomorphism from $H_{X,n} \to H_{CR}^*(X^n/S_n)$ which will be again denoted by $f_n$. We define $p_{\rho}(n) = f_n(c_\rho(n))$. It follows from (4.2) that

\begin{equation}
 p_{\rho}(n) \circ p_{\sigma}(n) = \sum C^\nu_{\rho \sigma} p_\nu(n).
\end{equation}

Remark 4.2. It is quite straightforward to define $p_\rho(n)$ directly, just as we defined $c_\rho(n)$. We denote $F_X = \oplus_{n=0}^\infty H^{CR}_*(X^n/S_n)$. It is well known that one can construct a Heisenberg algebra associated to the lattice $H^*(X)/tor$ acting irreducibly on $F_X$ (cf. e.g. [40]). One can show that the $p_\rho(n)$ in the present paper coincides with the one defined in [40], Sect. 4.4, in terms of the Heisenberg monomials (up to a scale multiple independent of $n$). The $p_\rho(n)$’s span $H^*_{CR}(X^n/S_n)$ but are not linearly independent when $n > 1$.

We summarize the above into a diagram of surjective ring homomorphisms:

\[
\begin{array}{ccccccc}
H_{X,\infty} & = & \lim_{n \to \infty} H_{X,n} & \overset{\theta_{n,\infty}}{\longrightarrow} & H_{X,n-1} & \overset{\theta_{n,n-1}}{\longrightarrow} & \ldots \\
\downarrow f_n & & \downarrow f_{n-1} & & \downarrow & & \\
H_{CR}^*(X^n/S_n) & & H_{CR}^*(X^{n-1}/S_{n-1}) & & & &
\end{array}
\]

In this way, the algebra $H_{X,\infty}$ with the multiplication (1.2) governs the structures of the rings $H_{CR}^*(X^n/S_n)$. The identity (1.3) with structure constants $C^\nu_{\rho \sigma}$ shows that the algebra $H_{X,\infty}$ can be identified with the stable ring $R_X$ associated to a compact complex manifold $X$ introduced in [40]. Thus, we have established the following theorem which first appeared in [40] with the assumption that the complex dimension of $X$ is even. The proof in loc. cit. was highly nontrivial and used a vertex operator method developed therein for the rings $H_{CR}^*(X^n/S_n)$.

Theorem 4.3. For a compact complex manifold $X$, the Chen-Ruan cohomology rings $H_{CR}^*(X^n/S_n)$, $n \geq 1$ give rise to a stable ring $H_{X,\infty}$ which completely encodes the cohomology ring structure of $H_{CR}^*(X^n/S_n)$ for each $n$.

Remark 4.4. In the case when $X$ is a point, our constructions of $H_{X,n}$ etc simplify greatly and specialize to that of [21], which were used to explain the stability in the class algebras of symmetric groups discovered in [25] (also see [19] for the extension to the wreath product setup.)

Remark 4.5. Assume that $H^*(X, Z)$ has no torsion. We can refine the definition of $H_{CR}^*(X^n/S_n)$ to define the orbifold cohomology ring of the symmetric product with integer coefficient, which has no torsion. If we further choose $S$ to be an integral linear basis for $H^*(X, Z)$, then the structure constants $C^\nu_{\rho \sigma}$’s above can be shown to be integers. Also compare with [1] where the orbifold cohomology rings of a general orbifold can be defined over integers.
5. The FH-ring of wreath products

5.1. Preliminaries on the wreath products. Let $\Gamma$ be a finite group, and $\Gamma_\ast$ be the set of conjugacy classes of $\Gamma$. We will denote the identity of $\Gamma$ by 1 and the identity conjugacy class in $\Gamma$ by $c^0$. The symmetric group $S_n$ acts on the product group $\Gamma^n = \Gamma \times \cdots \times \Gamma$ by permutations: $\sigma(g_1, \cdots, g_n) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(n)})$. The wreath product of $\Gamma$ with $S_n$ with the multiplication $(g, \sigma) \cdot (h, \tau) = (g \sigma(h), \sigma \tau)$. The $i$-th factor subgroup of the product group $\Gamma^n$ will be denoted by $\Gamma^{(i)}$. The wreath product $\Gamma_n$ embeds in $\Gamma_{n+1}$ as the subgroup $\Gamma_n \times 1$, and the union $\Gamma_\infty = \bigcup_{n \geq 1} \Gamma_n$ carries a natural group structure. When $\Gamma$ is trivial, $\Gamma_\infty$ reduces to $S_\infty = \bigcup_{n \geq 1} S_n$.

The space $R_2(\Gamma)$ of the class functions of a finite group $\Gamma$ (which is often called the class algebra of $\Gamma$) is closed under the convolution. In this way, $R_2(\Gamma)$ is also identified with the center of the group algebra $Z[\Gamma]$. We will denote by $R(\Gamma_n)$ (resp. $R_2(\Gamma_n)$) the class algebra of complex (resp. integer) class functions on $\Gamma_n$ endowed with the convolution product.

The conjugacy classes of $\Gamma_n$ can be described in the following way (cf. [35, 53]). Let $x = (g, \sigma) \in \Gamma_n$, where $g = (g_1, \ldots, g_n) \in \Gamma^n$, $\sigma \in S_n$. The permutation $\sigma$ is written as a product of disjoint cycles. For each such cycle $y = (i_1 i_2 \cdots i_k)$, the element $p_y = g_{i_k} g_{i_{k-1}} \cdots g_{i_1} \in \Gamma$ is determined up to conjugacy in $\Gamma$ by $g$ and $y$, and will be called the cycle-product of $x$ corresponding to the cycle $y$. For any conjugacy class $c \in \Gamma_\ast$ and each integer $i \geq 1$, the number of $i$-cycles in $\sigma$ whose cycle-product lies in $c$ will be denoted by $m_i(\rho(c))$, and $\rho(c)$ denotes the partition $(\substack{1^{m_1(c)} 2^{m_2(c)} \cdots})$. Then each element $x = (g, \sigma) \in \Gamma_n$ gives rise to a partition-valued function $\rho = (\rho(c))_{c \in \Gamma_\ast} \in \mathcal{P}(\Gamma_\ast)$ such that $\|\rho\| := \sum_{c \in \Gamma_\ast} r m_r(\rho(c)) = n$. The $\rho$ is called the type of $x$. It is known that any two elements of $\Gamma_n$ are conjugate in $\Gamma_n$ if and only if they have the same type.

5.2. The modified types. Let $x$ be an element of $\Gamma_n$ of type $\rho = (\rho(c))_{c \in \Gamma_\ast} \in \mathcal{P}_n(\Gamma_\ast)$. If we regard it as an element in $\Gamma_{n+k}$ by the natural inclusion $\Gamma_n \leq \Gamma_{n+k}$, then $x$ has the type $\rho \cup (1^k) \in \mathcal{P}_{n+k}(\Gamma_\ast)$, where $(\rho \cup (1^k))(c) = \rho(c)$ for $c \neq c^0$ and $(\rho \cup (1^k))(c^0) = (\rho(c^0), 1, \ldots, 1) = \rho(c^0) \cup (1^k)$. It is convenient to introduce the modified type of $x$ to be $\rho \in \mathcal{P}_{n-1}(\Gamma_\ast)$, where $r = l(\rho(c^0))$, as follows: $\tilde{\rho}(c) = \rho(c)$ for $c \neq c^0$ and $\tilde{\rho}(c^0) = (\rho_1 - 1, \ldots, \rho_r - 1)$ if we write the partition $\rho(c^0) = (\rho_1, \ldots, \rho_r)$. Two elements in $\Gamma_\infty$ are conjugate if and only if their modified types coincide.

Given $\mu \in \mathcal{P}(\Gamma_\ast)$, we denote by $K_\mu$ the conjugacy class in $\Gamma_\infty$ of elements whose modified type is $\mu$. For each $n \geq 0$ and each $\mu \in \mathcal{P}(\Gamma_\ast)$, let $K_\mu(n)$ be the characteristic function of the conjugacy class in $\Gamma_n$ whose modified type is $\mu$, i.e. the sum of all $\sigma \in \Gamma_n \cap K_\mu$. The nonzero $K_\mu(n)$’s form a $\mathbb{Z}$-basis for $R_2(\Gamma_n)$. If $x \in K_\mu$, the degree $\|x\|$ of $x$ is defined to be the degree $\|\mu\|$ of its modified type.

Given $g \in \Gamma$ and $1 \leq i \neq j$, we denote by $(i \xrightarrow{g^j} j)$ the element $((g_1, g_2, \ldots), (i, j))$ in $\Gamma_\infty$, where $(i, j) \in S_\infty$ is a transposition, $g_j = g$, $g_i = g^{-1}$, and $g_k = 1$ for $k \neq i, j$. Note that $(i \xrightarrow{g^j} j) = (j \xrightarrow{g^{-1}} i)$ and it is of order 2. The elements in $\Gamma_\infty$ of the form $(i \xrightarrow{g} j)$, where $g$ runs over $\Gamma$ and $(i, j)$ (where $i < j$) runs over all transpositions of $S_\infty$, form the single conjugacy class whose cycle-products are all $c^0$ and the partition corresponding to $c^0$ is $(2, 1, 1, \ldots)$. Clearly, any element $x$ in $\Gamma_\infty$ can be written as a product of elements in $K_{(1, 0)}$ and elements of the form $h^{(i)} \in \Gamma^{(i)}$, $i \geq 1$. Such a
product is called a reduced expression for \( x \) if \( x \) cannot be written as a product of fewer such elements. For a general element in \( \Gamma_\infty \), a reduced expression can be constructed cycle-by-cycle.

The number of elements appearing in a reduced expression of \( x \) is shown in loc. cit. to be \( \|\lambda\| \) for a given element \( x \in \Gamma_\infty \) of modified type \( \lambda \). It follows that \( \|xy\| \leq \|x\| + \|y\| \), and \( \| \cdot \| \) defines an algebra filtration on \( R_\mathbb{Z}(\Gamma_n) \). The associated graded ring will be denoted by \( G_{\Gamma}(n) \).

5.3. The FH-ring of wreath products. Given \( \lambda, \mu \in \mathcal{P}(\Gamma_\ast) \), we write the convolution product \( K_\lambda(n)K_\mu(n) \) in \( R_\mathbb{Z}(\Gamma_n) \) as a linear combination of \( K_\nu(n) \):

\[
K_\lambda(n)K_\mu(n) = \sum_\nu a_{\lambda\mu}^\nu K_\nu(n)
\]

where the structure constants \( a_{\lambda\mu}^\nu \) are nonnegative integers independent of \( n \).

The results above specialize to the results of Farahat-Higman \([13]\) (also cf. \([35]\)) in the symmetric group case. Thus, we make the following definition.

Definition 5.3. The Farahat-Higman ring (or FH-ring) of the wreath products associated to \( \Gamma \), denoted by \( G_{\Gamma}(n) \), is a \( \mathbb{Z} \)-ring with a \( \mathbb{Z} \)-basis \( K_\lambda \), where \( \lambda \in \mathcal{P}(\Gamma_\ast) \), and a multiplication (called the FH-product)

\[
(5.1) \quad K_\lambda K_\mu = \sum_{\|\nu\| = \|\lambda\| + \|\mu\|} a_{\lambda\mu}^\nu K_\nu.
\]

There is a natural surjective ring homomorphism \( \mathcal{G}_\Gamma \to \mathcal{G}_\Gamma(n) \) for each \( n \) which is compatible with the surjective ring homomorphism \( \text{res}_n : \mathcal{G}_\Gamma(n) \to \mathcal{G}_\Gamma(n-1) \) by restriction. The restriction map \( \text{res}_n \) clearly sends \( K_\lambda(n) \) to \( K_\lambda(n-1) \) for all \( \lambda \). The ring \( \mathcal{G}_\Gamma \) can be regarded as the inverse limit of the family of rings \( \{\mathcal{G}_\Gamma(n)\}_{n \geq 1} \) under the restriction maps. Theorem 5.2 is equivalent to the statement that the restriction maps are ring homomorphisms. We summarize the sequence of surjective ring homomorphisms above into the following diagram:

\[
\mathcal{G}_\Gamma = \lim_{\rightarrow n} \mathcal{G}_\Gamma(n) \quad \ldots \quad \mathcal{G}_\Gamma(n) \xrightarrow{\text{res}_n} \mathcal{G}_\Gamma(n-1) \to \ldots
\]
5.4. Macdonald’s interpretation of the FH-ring. We restrict ourselves to the symmetric group case (i.e. when $\Gamma = 1$) in the remainder of this section. In this case, we shall omit the subscript $\Gamma$ from the notations for the rings $\mathcal{G}_\Gamma, \mathcal{G}_\Gamma(n)$.

Denote by $\Lambda = \oplus_k \Lambda^k$ the $\mathbb{Z}$-ring of symmetric functions (in infinitely many variables), and $\Lambda^k$ denotes the subspace of symmetric functions of degree $k$. Denote by $e_k, h_k, p_k \in \Lambda^k$ respectively the $k$-th elementary, complete, and power-sum symmetric functions. There is a standard bilinear form $\langle - , - \rangle$ on $\Lambda$ and the Schur functions form an orthonormal $\mathbb{Z}$-basis of $\Lambda$. Denote by $\lambda'$ the transpose of a partition $\lambda$. An involution $\omega : \Lambda \to \Lambda$ is defined by $\omega(s_\lambda) = s_{\lambda'}$ for all $\lambda$, and it can also be characterized as the ring isomorphism of $\Lambda$ by switching $e_n$ and $h_n$ for all $n$.

$\Lambda$ (less standard) involution on the ring $\Lambda$ is defined as follows (Macdonald) [35]. Let

$$u = t + h_1t^2 + h_2t^3 + \ldots.$$  

Then $t$ can be expressed as a power series in $u$, say

$$t = u + h_1^u u^2 + h_2^u u^3 + \ldots,$$

with coefficients $h_k^u \in \Lambda^k$ ($k \geq 1$). The ring homomorphism $\psi : \Lambda \to \Lambda$ defined by $\psi(h_k) = h_k^u$, $k \geq 1$, is an involution. We denote $f^* = \psi(f)$. For example, $h_1^u = h_1^u h_2^u \ldots$ for each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, and the $h_k^u$’s form a $\mathbb{Z}$-basis of $\Lambda$. We denote by $(g_\lambda)$ the dual basis, i.e. $\langle g_\lambda, h_n^u \rangle = \delta_{\lambda,\mu}$.

The following theorem is due to Macdonald [35], Ex. 25, pp. 132 (also cf. Goulden-Jackson [18] for another proof).

**Theorem 5.4.** (Macdonald) The linear map $\varphi : \Lambda \to \mathcal{G}$ defined by $\varphi(g_\lambda) = K_\lambda$ for all partitions $\lambda$ is a ring isomorphism.

5.5. Jucys-Murphy elements and Macdonald’s isomorphism. Recall [22, 37] that the Jucys-Murphy elements $\xi_{j,n}$ of the symmetric group $S_n$ are defined to be the sums of transpositions:

$$\xi_{j,n} = \sum_{i < j} (i, j), \quad j = 1, \ldots, n.$$  

In particular, we have $\xi_{1,n} = 0$. When it is clear from the text, we may simply write $\xi_{j,n}$ as $\xi_j$. An important property of the JM elements is that any symmetrization of $\{\xi_1, \ldots, \xi_n\}$ lies in $R_{\mathbb{Z}}(S_n)$. It is further known that the $k$-th elementary symmetric function $e_k(\Xi_n)$ is equal to the sum of all permutations in $S_n$ having exactly $(n - k)$ cycles. For example, $e_1(\Xi_n)$ is exactly the characteristic function $K_{(1)}(n)$ of the conjugacy classes of transpositions in $S_n$. We will use $*$ to denote the product in the FH-ring $\mathcal{G}(n)$. We define

$$(-\Xi)^* \epsilon_k(n) = \sum_{j=1}^n (-1)^{j} \xi_{j,n} \ast \ldots \ast \xi_{j,n}.$$  

**Proposition 5.5.** In the ring $\mathcal{G}(n)$, we have the equality:

$$(-\Xi)^* \epsilon_k(n) = \begin{cases} (-1)^k K_{(k)}(n), & \text{if } k < n \\ 0, & \text{if } k \geq n. \end{cases}$$
Proof. Since we have $\|x\| \leq n-1$ for every element $x$ in $R_\mathbb{C}(S_n)$, the statement is clear for $k \geq n$.

Now assume $k < n$. We first note that $(i \ j) \ast (i \ j) = 0$ for any transposition $(i \ j)$, since $(i \ j)^2 = 1$, $\|(i \ j)\| = 1$, and $\|1\| = 0$. It follows from the definition of the Jucys-Murphy elements $\xi_{j:n}$ that the only permutations which survive in $\xi_{j:n} \ast \ldots \ast \xi_{j:n}$ is the product (under the ordinary group multiplication) of $k$ distinct transpositions and any such a product is necessarily a $(k+1)$-cycle. Since $(-\Xi)^k(n)$ lies in $R_\mathbb{C}(S_n)$, it is a multiple of $K_{(k)}(n)$. It remains to show that the multiple is indeed 1.

The total number of such $(k+1)$-cycles in $\xi_{j:n} \ast \ldots \ast \xi_{j:n}$ is $k! \cdot \binom{j-1}{k}$, and thus the total number of such products in $(-\Xi)^k(n)$ is $\sum_{j=2}^{n} k! \cdot \binom{j-1}{k}$. The number of $(k+1)$-cycles appearing in $K_{(k)}(n)$ is $\frac{n!}{(n-k-1)! \cdot (k+1)!}$. We conclude that $(-\Xi)^k(n) = K_{(k)}(n)$ from the identity (which can be proved easily by induction on $n$)

$$\sum_{j=2}^{n} k! \cdot \binom{j-1}{k} = \frac{n!}{(n-k-1)! \cdot (k+1)!}.$$ 

□

Now, the elements $(-\Xi)^k(n)$, $n \geq 1$, give rise to an element $(-\Xi)^k$ in $\mathcal{G}$. Let us denote by $\phi : \mathcal{G} \to \Lambda$ the isomorphism inverse to $\varphi$.

Proposition 5.6. We have $\phi(K_{(k)}) = -p_k$, and $\omega \phi((-\Xi)^k) = p_k$.

Proof. The first part was established in [35], Ex. 25 (b), pp. 133. The second part now follows from the first part, Proposition 5.3, and the fact that $\omega(p_k) = (-1)^{k-1} p_k$. □

Denote by $e_k(-\Xi)$, $h_k(-\Xi)$ and $s_{\lambda}(-\Xi)$ respectively the $k$-th elementary, complete, and Schur symmetric function in the variables $-\xi_1, \ldots, -\xi_n$ (using the FH-product). We denote by $e_k^*(\Xi)$, $h_k^*(\Xi)$ and $s_{\lambda}^*(\Xi)$ the corresponding elements in $\mathcal{G}$.

Theorem 5.7. We have the following correspondence under the isomorphisms $\phi$ and $\omega \phi$ from $\mathcal{G}$ to $\Lambda$:

1. $\phi(K_{\lambda}) = g_{\lambda}$.
2. $\omega \phi((-\Xi)^k) = p_k$.
3. $\omega \phi(e_k^*(\Xi)) = e_k$.
4. $\omega \phi(h_k^*(\Xi)) = h_k$.
5. $\omega \phi(s_{\lambda}^*(\Xi)) = s_{\lambda}$.

Proof. Part (1) is Theorem 5.4 and part (2) is Proposition 5.6. The rest follows from part (2) and the definitions of $e_k^*(\Xi)$, $h_k^*(\Xi)$ and $s_{\lambda}^*(\Xi)$. □

Remark 5.8. The Jucys-Murphy elements have been vital in understanding the rings $H_{\text{CR}}(X^n/S_n)$ for compact $X$ [40], and they are intimately related to vertex operators, cf. [26, 49, 40]. The natural role of the Jucys-Murphy elements in $\mathcal{G}$ indicates that they should play also an important role in the study of the rings $H_{\text{CR}}^*(X^n/S_n)$ for noncompact $X$ even without vertex operators.
Remark 5.9. For an arbitrary finite group $\Gamma$, it has been shown that the FH-ring $G_\Gamma$ is isomorphic to a tensor product of several copies (parametrized by $\Gamma^*$) of the ring $\Lambda$ of symmetric functions. It will be very interesting to generalize Macdonald’s symmetric function interpretation of $G$ to the general FH-ring $G_\Gamma$.

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