Moduli Spaces in the Four-Dimensional Topological Half-Flat Gravity

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ABSTRACT

Abstract

We classify the moduli spaces of the four-dimensional topological half-flat gravity models by using the canonical bundle. For a $K3$-surface or $T^4$, they describe an equivalent class of a trio of the Einstein-Kähler forms (the hyperkähler forms). We calculate the dimensions of these moduli spaces by using the Atiyah-Singer Index theorem. We mention the partition function and the possibility of the observables in the Witten-type topological half-flat gravity model case.
I. Introduction

Recently, Witten gave some gravitational versions of topological quantum field theories [1]. These theories are important as the effective theories of the ordinal gravity theories. For example, he pointed out the relation between the two-dimensional topological gravity models and the string theory [1]. He also conjectured that $N = 4$ topological twisted supersymmetric Yang-Mills theory on four-dimensional manifold satisfies S-duality and has a link with the bosonic string or two-dimensional rational conformal field theories [2]. They seem to give the new light on the non-perturbative effect of the string theories and the gravity theories.

Since the work of Witten, there have been several attempts to construct four-dimensional topological gravity theories over different kind of the gravitational moduli spaces [3]-[5].

There two types of models have been proposed for the four-dimensional half-flat 2-form topological gravity. (A) Witten-type topological gravity model, which was given by Kunitomo [5] and (B) Schwarz-type topological gravity model which we proposed [6, 7]. The base of their formalism are given by ref. [8, 9]. The interesting relation between the half-flat gravity and 2-dim. conformal field theory is investigated by Park [10]. In the previous paper we showed that by taking the suitable gauge fixing condition and the limit of the coupling constant for (B), the bosonic part of the moduli spaces of (B) coincides with that of (A). These moduli spaces are those of the Einstein Kählerian manifolds with vanishing real first Chern class.

The purpose of this letter is to calculate the dimensions of the moduli spaces by using Atiyah-Singer index theorem and discuss about the possibility of the topological invariants such as the partition function and observables. We concentrate our attention for $K3$-surface and $T^4$ cases mainly in this paper. Their moduli spaces are identified with the deformation of a trio of the Einstein-Kähler forms (the hyperkähler forms) which is related to the Plebansky’s heavenly equations [11]. We also discuss the partition function for the Witten-type model case and mension the the possibility of the observables.

The extension of the algebraic curve with Einstein metric to the four dimensional case may be the algebraic surfaces with Einstein metrics. $T^4$ and $K3$ surface belong to the algebraic surfaces and we regard these models as the simple examples which treat the algebraic surfaces.

As another aspect, there have been discovered rich kind of non-compact gravitational instantons (i.e. ALE [12] or ALF [13]) which satisfy these equations. In this paper, we will treat the compact manifolds only. In the near future, we will extend our investigation to non-compact case.
II. The Four-Dimensional Half-Flat Topological Gravity Models

We derive the moduli spaces of the four-dimensional half-flat topological gravity models, which are partially contained in ref.\[6\]. There are two types of models: (A) Witten-type action given by Kunitomo \[3\] and (B) Schwarz-type action proposed by us \[16\]. We follow the notations of ref. \[5, 7\]. Fundamental fields in these models are a trio of \(su(2)\) valued 2-form \(\Sigma^k = \Sigma^k_{\mu\nu} dx^\mu \wedge dx^\nu\) and a trio of \(su(2)\) valued 1-form \(\omega^k = \omega^k_\mu dx^\mu\). The curvature tensor \(F^k = F^k_{\mu\nu} dx^\mu \wedge dx^\nu \equiv d\omega^k + (\omega \times \omega)^k = d\omega^k + f^{ijk} \omega^i \wedge \omega^j\) \((f^{ijk}\) is a structure constant of \(SU(2)\) and it will be \(\epsilon^{ijk}\) in later). Varying the actions of (A) or (B) with respect to \(\omega^k\) and \(\Sigma^k\) fields, we obtain the following equations\[5, 7\];

\[
F^k = 0, \quad D\Sigma^k = 0, \tag{1}
\]

\[
\iota_i \iota_j \Sigma^i \wedge \Sigma^j \equiv \Sigma^i \wedge \Sigma^j - \frac{1}{3} \delta_{ij} \Sigma^k \wedge \Sigma_k = 0, \tag{2}
\]

\(M_4\) is supposed to be a four-dimensional compact Kähler manifold with its real first Chern class \(c_1(M_4)_R = 0\), which is the sufficient condition for the existence of the moduli spaces \[7\]. From the Bogomolov decomposition theorem \[15\], complex tori \(T^4\) or \(K3\) are possible as the covering space of \(M_4\). The field \(\omega^k = \omega^k_\mu dx^\mu\) is restricted to a self-dual chiral part of a frame connection \((+)\omega^{ab}\) i.e., we consider only the principal bundle of oriented orthonormal frames \(P_{SO(4)}\) over \(M_4\) with structure group \(SO(4)\). From the assumption for \(M_4\), at least the reduction of the structure group \(G = SO(4) \rightarrow U(2)\) and of the bundle \(P_{SO(4)} \rightarrow P_{U(2)}\) are possible \[14\]. Thus, only \(U(1)\) component of \(\omega^k\) exists as the self-dual part.

\[
(+)\omega^{ab} = \bar{\eta}^{ab}_k \omega^k, \quad \omega^k \in su(2) \times \Lambda^1 \rightarrow \omega^1 \in u(1) \times \Lambda^1, \quad \omega^2 = \omega^3 = 0, \tag{3}
\]

where \(\bar{\eta}^{ij}_{ab}\) is the t’Hooft’s \(\eta\)-symbol \[17\].

Similarly, we get the following reductions \[4\] for \(\Sigma^k\), which supposed to be self-dual part of \(SO(4)\):

\[
\Sigma^i \in su(2) \otimes \Lambda^2 \rightarrow \Sigma^1 \in 1 \otimes \Lambda^2, \quad \Sigma^2 \in \Sigma^3 \in u(1) \otimes \Lambda^2. \tag{4}
\]

From eq. (2), \(\Sigma^k\) comes from a vierbein \(e^a = e^a_\mu dx^\mu\) \[5\]:

\[
\Sigma^k(e) = -\bar{\eta}^{kk}_a e^a \wedge e^b \propto g_{a\bar{\beta}} j^{k\bar{\beta}} dz^a \wedge d\bar{z}^\beta. \tag{5}
\]

\(\{J^k\}\) represents a complex structure or two almost complex structures which satisfy the quaternionic relations and \(g_{a\beta}\) is a hermite symmetric metric. The self-dual part of the Riemannian tensors is related to the curvature tensor \(F^k\) \[5, 7\]. It becomes zero (i.e., Riemannian half-flat) when it satisfies the eq. (1) and (2) \[5, 7\];

\[
(+)R_{\mu\nu\rho\tau} = 0. \tag{6}
\]

From the Hitchin’s theorem \[18\], \((M_4, g)\) is covered by a flat 4-torus or Kähler-Einstein K3-surface or Kähler-Einstein K3-surface \(/Z_2\) or Kähler-Einstein K3-surface \(/Z_2 \times Z_2\).

Before we proceed, it will be useful to introduce general spin bundles \(\Omega^{m,n}\) \[18\] and the canonical bundle \(K\) \[19\]. A \(U(1)\) part deformation of the frame connection is denoted by \(\omega^1_f\). It belongs to \(K^* \otimes \Lambda^1\).

\[
\omega^1_f \in \Omega^{0,2} \otimes \Lambda^1 \rightarrow \omega^1_f \in P_{U(1)} \times Ad_u(1) \otimes \Lambda^1 \cong K^* \otimes \Lambda^1, \tag{7}
\]

\(^1\)In the previous paper \[6\], we are setting \(\Sigma^k\) to be represented by \(su(2) \otimes \Lambda^2\). It will be more convenient to use the reduction forms \(1 \times \Lambda^2\) or \(u(1) \otimes \Lambda^2\) for them to see the dimensions of the each cohomology group which will appear in the later.
which comes from the following isomorphism \[7\]:

\[ P_{SO(4)} \times Ad_{so(4)} \cong M_4 \times so(4)/Ad \equiv M_4 \times \wedge^2 R^4/Ad \cong \wedge^2 TM_4^* \cong \Omega^{2,0} \oplus \Omega^{0,2} \quad (8) \]

Similarly, \( \{ \Sigma^k \} \) the deformations of \( \Sigma^k \) are represented by \( \Sigma^1 \in RS \otimes \Lambda^2, \Sigma^2 \) and \( \Sigma^3 \in K^* \otimes \Lambda^2 \), where \( S \) is a certain parallel section of \( \Omega^{0,2} \). \[22\].

\[ \Sigma^1_f, \Sigma^2_f, \Sigma^3_f \in \Omega^{0,2} \oplus \Lambda^2 \to \Sigma^1, \Sigma^2, \Sigma^3 \in (RS \oplus 2K)^* \otimes \Lambda^2. \quad (9) \]

The moduli spaces of (A) coincide with those of the bosonic part of (B) and are classified into two cases by using the property of the canonical bundle \( K \).[7].

Case(1) when the canonical bundles are trivial: \( M_4 \) is a K3-surface or \( T^4 \). On these two manifolds, the reductions of \( P_{U(2)} \) are possible due to the fact that \( K \) and \( P_{U(1)} \) are trivial when they have Einstein-Kahler metrics. These manifolds are called hyper-Kählerian.

The moduli space is the equivalent class of a trio of the Einstein-Kähler forms (the hyperkähler forms) \( \{ \Sigma^k(e) \} \) \[7\]:

\[ \mathcal{M}(\Sigma) = \{ \Sigma^k \mid \Sigma^1 \in 1 \otimes \wedge^2, \Sigma^2, \Sigma^3 \in u(1) \otimes \wedge^2, t.f. \Sigma^i \wedge \Sigma^j = 0, d\Sigma^k = 0 \}/diffgeo. \quad (10) \]

Case(2) when the canonical bundles are not trivial: \( M_4 \) is \( K3/Z_2 \), or \( K3/Z_2 \times Z_2 \), or \( T^4/\Gamma \) (\( \Gamma \) is some discrete group).

As \( K \) is not trivial on these manifolds, the reduction of \( P_{U(2)} \rightarrow P_{SU(2)}_R \) is not possible. These manifolds are called as the locally hyperkählerian. Their moduli spaces are given by

\[ \mathcal{M}(\Sigma, \omega) = \{ \Sigma^k, \omega^1 \mid \omega^1 \in u(1) \otimes \Lambda^1, F^1_{\mu \nu} = 0, \Sigma^1 \in 1 \otimes \wedge^2, \Sigma^2, \Sigma^3 \in u(1) \otimes \wedge^2, t.f. \Sigma^i \wedge \Sigma^j = 0, d\Sigma^1 = 0, D\Sigma^2 = D\Sigma^3 = 0 \}/diffgeo \times U(1). \quad (11) \]

These moduli spaces have the bundle structure such as the base is \( \mathcal{M}(\omega^k) \) with a fibre \( \mathcal{M}(\Sigma) \).

\[ \mathcal{M}(\Sigma) = \{ \Sigma^k \mid \Sigma^1 \in 1 \otimes \wedge^2, \Sigma^2, \Sigma^3 \in u(1) \otimes \wedge^2, t.f. \Sigma^i \wedge \Sigma^j = 0, d\Sigma^1 = 0, D\Sigma^2 = D\Sigma^3 = 0 \}/diffgeo, \quad \mathcal{M}(\omega) = \{ \omega^1 \mid \omega^1 \in 1 \otimes \Lambda^1, F^1_{\mu \nu} = 0 \}/U(1). \quad (12) \]

III. BRST quantization in the Witten-type model on \( T^4 \) or \( K3 \)

We discuss about the partition function of the Witten-type model on K3 or \( T^4 \), whose BRST symmetry and the action is given by \[22\]. (We will report the partition function for the Schwarz-type model in the next paper \[22\].) The action of the Witten-type model reduces to

\[ S_0^{\text{red}} = \int_{M_4} (\pi_i \wedge d\Sigma^i + \frac{1}{2} \pi_{ij} t.f. \Sigma^i \wedge \Sigma^j - \chi_i \wedge d\Psi^i - \frac{1}{2} \chi_{ij} t.f. \Sigma^i \wedge \Psi^j) \]

\[ = \int_{M_4} (\pi_i \oplus \pi_{ij}) \wedge D_1 \Sigma^j - (\chi_i \oplus \chi_{ij}) \wedge D_1 \Psi^j, \quad (13) \]

where \( D_1 \Sigma^j \equiv (\hat{D}_1 \oplus \hat{D}_1) \Sigma^j \equiv (d \oplus t.f. \Sigma_0^i \wedge) \Sigma^j \). This action is invariant under the diffeomorphism transformations and the supersymmetry. These transformations are invariant under the redundant diffeomorphism transformations of them. The symmetries of this model is interpretable as \( \text{diffgeo} \circ \text{super. diffeo} \). red. diffgeo.
We use the decomposition $\Sigma^k = \Sigma_0^k + \Sigma_j^k$ of the $\Sigma^k$ field to calculate of the partition function where $\Sigma_0^k$ is a back ground solution of the equations of motions. Gauge fixing conditions which we set are $D_0^* \Sigma_0^k = 0$ for the diffeomorphism and $D_0^* \Psi^k = 0$ for the redundant diffeomorphism and $D_1 \Sigma^k = 0$ for the super symmetry (except for the redundant diffeo symmetry). $D_0$ is defined by $D_0 \delta t \equiv \mathcal{L}_0 \Sigma^k$. $\ast$ denotes the Hodge star dual operation and $\mathcal{O}^* \equiv - * \mathcal{O}*$ is the adjoint operator of $\mathcal{O}$. The dimensional countings of the fundamental fields are given in the Table 1. In this model, there are the redundant symmetries such as $\delta \ast \pi_k = D_2^* \lambda_k$ and $\delta \ast \chi_k = D_2^* \eta_k$ due to $D_2 D_1 \Sigma^k \equiv D_2^2 \Sigma^k = 0$ and $D_2 D_1 \Psi_k = D_2^2 \Psi^k = 0$ [22]. $\lambda_i$ and $\lambda^k (\eta_k$ and $\eta^k)$ are a ghost and an anti-ghost of these redundant symmetries with $(2K \oplus \mathcal{R}S)^* \otimes \Lambda^2$ representations. To fix these symmetries, we set $D_2 \ast \pi_k = 0$ and $D_2 \ast \chi_k = 0$. Thus the quantum action $S_q$ is given by

$$S_q = S_q^{\text{red}} + S_{\text{g.f.}},$$

$$S_{\text{g.f.}} = \int_{M_4} \delta_B \{ *b \wedge D_0 \Sigma_j^k + *\lambda^k \wedge D_2 \ast \pi_k - *\beta \wedge D_0^* \Psi^k - *\eta^k \wedge D_2 \ast \chi_k \}. \quad (15)$$

$\delta_B$ denotes the BRST-transformation. We are now ready to evaluate the partition function:

$$Z = \int \mathcal{D}X \exp( - S_q), \quad (16)$$

where $\mathcal{D}X$ represents the path integral over the fields $\Sigma_j^k$, ghosts, anti-ghosts and N-L fields. In general, these fields contains zero modes and non-zero modes. We introduce the following the deformation complex [2]. The zero modes are the elements of the cohomology groups of this complex.

$$V_0 = \Omega^{1,1}, \quad V_1 = (2K \oplus \mathcal{R}S)^* \otimes \Lambda^2, \quad (17)$$

$$V_2 = (2K \oplus 2K^{\otimes 2} \oplus \mathcal{R}S)^* \otimes \Lambda^4 \oplus (2K \oplus \mathcal{R}S)^* \otimes \Lambda_3, \quad V_3 = 0 \oplus (2K \oplus \mathcal{R}S)^* \otimes \Lambda^4.$$

$$0 \overset{D_{-1}}{\longrightarrow} C^\infty(V_0) \overset{D_0}{\longrightarrow} C^\infty(V_1) \overset{D_1}{\longrightarrow} C^\infty(V_2) \overset{D_2}{\longrightarrow} C^\infty(V_3) \overset{D_3}{\longrightarrow} 0. \quad (18)$$

$D_{-1}$ and $D_3$ are identically zero operators. We can easily check the ellipticity of the deformation complex. We may then define the cohomology group.

$$H^i \equiv \text{Ker} D_i / \text{Im} D_{i-1} = \text{Ker} \Delta_i, \quad (19)$$

where $\Delta_i = D_{i-1} D_{i-1}^* + D_i^* D_i$ and $D_i D_i = \hat{D}_i^* \hat{D}_i + \hat{D}_i \hat{D}_i$. The dimensions of $H^i$ are finite and represented by $h_i$. $H_1$ is exactly identical with $T(\mathcal{M}(\Sigma))$ the tangent space of $\mathcal{M}(\Sigma)$:

$$T(\mathcal{M}(\Sigma)) = \{ \Sigma_j^k | \Sigma_j^k \in (2K \oplus \mathcal{R}S)^* \otimes \wedge^2, D_1 \Sigma_j^k = 0 \}/\text{diffeo.} \quad (20)$$

When expanded out by using the properties of $\delta_B$ [3], the quantum action is given by

$$S_q = \ast b \wedge TB \ast \ast F \wedge TF^t$$

$$+ \ast \beta \wedge \Delta_0^2 \gamma - *b \wedge \Delta_0 c$$

$$+ \ast \eta^k \wedge \Delta_3 \eta_k - *\lambda^k \wedge \Delta_3 \lambda_k + \text{other higher order terms}, \quad (21)$$

\[^2\text{We thank T. Ueno for pointing out the possibility of using the deformation complex and the Index theorem in these models \cite{23}. Before us, he composed the similar deformation complex and calculated the index for these models.}\]
with some field redefinitions.

\[
T = \begin{pmatrix}
0 & -D_0^* & 0 & 0 \\
-D_0 & 0 & -(\tilde{D}_1^* + \tilde{D}_1) & 0 \\
0 & -(\tilde{D}_1^* + \tilde{D}_1) & 0 & -D_2^* \\
0 & 0 & -D_2 & 0
\end{pmatrix}.
\]

We integrate over non-zero modes. The Gaussian integrals over the commuting \( \beta - \gamma \) and \( \lambda^k - \bar{\lambda}^k \) sets of fields cancel with the contribution coming from the anti-commuting sets of \( b - c \) and \( \bar{\pi}^k - \eta_k \). We integrate over the remaining \( B = \{ \pi_c, \Sigma^k, \star(\pi^k \oplus \pi_{ij}) \} \), \( \{ \pi^k, \pi_{ij} \} \)-system and \( F = \{ \chi_\gamma, \varphi^k, \star(\chi_i \oplus \chi_{ij}), \chi^i \} \)-system by taking \( \det T = \det^2(T^*T) \) and using the nilpotency \( D_i D_{i+1} = 0 \) \[24\]. Their contributions cancelled out each other. If the dimension of moduli space is zero, then there may exist the non-trivial partition with projecting out of the other zero modes.

\[
Z = \Sigma'_{\text{ instanton}} \frac{\{(\det \Delta_0)(\det \Delta_1)(\det \Delta_2)(\det \Delta_3)\}^{\frac{1}{4}}}{\{(\det \Delta_0)(\det \Delta_1)(\det \Delta_2)(\det \Delta_3)\}^{\frac{1}{4}}} \cdot \frac{\{(\det \Delta_0)(\det \Delta_3)\}}{\{(\det \Delta_0)(\det \Delta_3)\}} = \Sigma'_{\text{ instanton}} \pm 1
\]

(22)

Later, however, we show that \( h^1 \) is non-zero on \( K3 \) and on \( T^4 \). Thus, there arise fermionic zero-modes and bosonic zero-modes whose numbers are equal to \( h_1 \) in this model. The path integral over the fermionic (bosonic) zero modes makes the partition function trivial (divergent).

Some appropriate combinations of the following three ways can avoid these situations:

1) projecting out i.e., gauge fixing of the global symmetries caused by the zero modes
2) evaluation of the interaction terms of the quantum action which contains the appropriate zero modes
3) introducing some functional \( \mathcal{O} \) called ‘observable’ and calculating the vacuum expectation value of it \[\mathcal{O}\]. The path-integration over them may provide non-trivial information to distinguish differential structures on these manifolds.

Myers et al. have already proposed some ‘observable’ in their gravity model\[4\]. Their moduli space is the equivalent class of the Riemannian half-flat metric with respect to the diffeomorphism. Their ‘observable’ is composed of the BRST-extension of the Riemannian curvature form on \( M_4 \) and it might express the topological invariants made of curvatures on the tangent moduli space. The BRST symmetries of Myers et. al.’s model can be realized by using the composite fields of \( \Sigma^k \) and its super partner \( \Psi^k \) on \( K3 \) or \( T^4 \) in Kunitomo’s model since a metric can be made of \( \Sigma^k \) \[3\]. Thus the observables given by Myers et. al are also adopted to the Kunitomo’s model. The evaluation of them in the Kunitomo’s model will give some topological information.

**VI. The dimensions of the Moduli Spaces**

Let \( K(g) \) be the moduli space of Einstein-Kähler forms, \( e(g) \) be the moduli space of Einstein metrics and \( C(g) \) be the moduli space of complex structures, respectively. All of them are the equivalent classes with respect to the diffeomorphism. At first, we quote the result about the dimension of \( K(g) \) when \( M_4 \) is Kählerian manifold with \( c_1(M)_{R} = 0 \), which is given by \[13\]. Then we show the dimension of \( \mathcal{M}(\Sigma) \) or \( \mathcal{M}(\Sigma, \omega) \) by using the Atiyah-Singer Index theorem and clarify the difference between \( \mathcal{M}(\Sigma) \) and \( K(g) \), which is partially contained in ref. \[7\].

When \( c_1(M)_{R} = 0 \), the deformation of the Kähler class with a fixed complex structure induces a deformation of a Einstein metric from the Calabi-Yau theorem \[25\]. The deformation of Einstein-Kähler forms \( \{ \Sigma \} \) consists of that of Einstein metrics \( \{ g \} \) and complex structures \( \{ J \} \) and needs a careful examination of its degenerated part;
\[ \delta \Sigma = \delta g \circ J + g \circ \delta J. \] The dimensions of \( K(g) \), \( \epsilon(g) \) and \( C(g) \) for the Kählerian manifolds with \( c_1(M)_R = 0 \) are given by

\[ \text{dim. } C(K) = 2 \dim H^1_C(M, \Theta), \]  \tag{23}

\[ \text{dim. } \epsilon(K) = \dim H^1_R(M, J) - 1 + 2 \dim H^1_C(M, \Theta) - 2 \dim H^2_C(M, J), \]  \tag{24}

\[ \text{dim. } K = \dim H^1_R(M, J) - 1 + 2 \dim H^1_C(M, \Theta). \]  \tag{25}

where \( \Theta = \mathcal{O}(TM_z) \) i.e. the sheaf of the germs of holomorphic vector fields. \( TM_C = TM_A \otimes C = TM_L \oplus TM_{\bar{L}} \) with formal splittings \( \mathcal{2} \) : \( TM_z = L_1 + L_2, TM_{\bar{L}} = L_1 + L_2. \) \( L \) and \( \bar{L} \) are the line bundles.

We show how to calculate the dimension of the moduli space of \( M(\Sigma) \). The calculation of the index for \( M(\Sigma) \) is common to both of case(1) and case (2) because the index is independent on the connections. By applying the Atiyah-Singer index theorem \( \mathcal{2} \), we obtain

\[ \text{Index} = \sum_{i=0}^{3} (-1)^i h^i \]  \tag{26}

\[ = \int_{M_4} \frac{\text{td}(TM_4 \otimes C)}{e(TM_4)} \cdot \text{ch} \{ \sum_{n=0}^{3} \oplus (-1)^n V_n \} \]

\[ = 2\chi + 7\tau \to 2\chi - 7 | \tau |, \]

where \( \text{ch}, \epsilon \) and \( \text{td} \) are the Chern character, Euler class and Todd class of the various vector bundles involved. The index is determined by the Euler number \( \chi = \int_{M_4} x_1 x_2 \) and Hirzebruch signature \( \tau = \int_{M_4} \frac{x_1^2 + x_2^2}{3} \). \( x_i \) denotes the first Chern classes of \( L_i \) or \( \bar{L}_i \). By changing \( \tau \to | \tau | \), this index can be also adopted to manifolds with the opposite orientation.

In the above calculation, we use the character of the general spin bundles \( \mathcal{2} \) and that of a canonical bundle : \( \text{ch}(\Omega^{0,1}) = A_l + 2B_l(y_+)^2 + \frac{2}{3}C_l(y_+)^4 + \cdots \), with \( A_l = l_1 + 1, B_l = \sum_{k=0}^{l_1} (k - \frac{1}{2})^2, C_l = \sum_{k=0}^{l_1} (k - \frac{1}{2})^1 \). \( y_+ \) is replaced by \( y \) for \( \text{ch}(\Omega^{0,0}) \). \( y \) is given in terms of \( x_i \) due to the relation between the fundamental spinor representation and the adjoint representation of the Unitary group \( \mathcal{2} \) : \( \text{ch}(L_i) = \exp x_i, x_i = c_1(L_i) = -c_1(\bar{L}_i) \) and \( y_{\pm} = \frac{1}{2}(x_1 \pm x_2) \). \( \text{ch}(K^*) = \text{ch}(L^*_1 \otimes L^*_2) = \exp \{-x_1 + x_2 \} \). In terms of \( x_i \), \( \text{td}(TM_C) = 1 - \frac{x_1^2 + x_2^2}{12} + \) higher order terms, \( \epsilon(TM_A) |_{M_4} = x_1 x_2 \). It is easy to see that \( H^0 \) is equivalent to the space of the Killing vectors (Isometry), \( H^2 = 2H^2(M_4, \mathcal{O}(K^*)) + H^2(M_4, R) \) and \( H^3 = 2H^3(M_4, \mathcal{O}(K^*)) + H^4(M_4, R) \). We use the notation \( \dim H^i(M_4, \mathcal{O}(K^*)) = b_i(K^*) \) for the dimension of the cohomology group of the twisted de Rham complex:

\[ \ldots \to C^\infty(K^* \otimes \wedge^2 TM^*) \to C^\infty(K^* \otimes \wedge^3 TM^*) \to C^\infty(K^* \otimes \wedge^4 TM^*) \to 0. \]  \tag{27}

\[ \dim M(\Sigma) = h^1 = -2\chi + 7 | \tau | + \dim \text{of the Isometry} + 2b_3(K^*) + b_3 - 2b_4(K^*) - b_4. \]  \tag{28}

To know \( b_i(K^*) \), we use the following equations and \( b_{n-i}(K^*) = b_i(K^*) \).

\[ (A) b_1(K^*) = \dim H^0_0(M_4, \mathcal{O}(TM^*_z \otimes \wedge^2 TM^*_z)) + \dim H^1_0(M_4, \mathcal{O}(\wedge^2 TM^*_z)) \]  \tag{29}

\[ = \dim H^0_0(M_4, \Theta) + b_{1,2}, \]

\[ (B) b_2(K^*) = \dim H^0_0(M_4, \mathcal{O}(\wedge^2 TM^*_z)) + \dim H^0_0(M_4, \mathcal{O}(\wedge^2 TM^*_z \otimes \wedge^2 TM^*_z)) \]

\[ + \dim H^1_0(M_4, \mathcal{O}(TM^*_z)) \]

\[ = \dim H^{2,1}_0(M_4, \mathcal{O}(TM^*_z)) + \dim H^0_0(M_4, \mathcal{O}(\wedge^2 TM^*_z \otimes \wedge^2 TM^*_z)) + b_{2,2} \]

\[ = H^0_0(M_4, \Theta) + p^2 + b_0, \]
(C) $b_3(K^*) = \dim H^3_0(M, \mathcal{O}(\wedge^2 TM^*_\mathbb{C} \otimes \wedge^2 TM^*_\mathbb{C}))$
+ $\dim H^3_0(M, \mathcal{O}(TM^*_\mathbb{C} \otimes \wedge^2 TM^*_\mathbb{C}))$
\[\quad = \dim H^0_0(M, \Theta) + \left(\frac{7}{2} \tau + \frac{3}{2} \chi\right) + \dim H^1_0(M, \Theta) - b_{1,2}\]
\[\quad = \dim H^0_0(M, \Theta) + \left(-\frac{7}{2} |\tau| + \frac{3}{2} \chi\right) + \dim H^1_0(M, \Theta) - b_{1,2},\]

\[\quad = \text{Index of eq.(27)} - \sum_{i=0}^3 (-1)^i b_i(K^*),\] (E) $b_0(K^*) = b_{0,2}$.

$p^2$ is the 2-th plurigenus [19]. They are derived by the Serre duality [19] and the index theorem applied to the twisted Dolbeault complex [20] or the twisted de Rham complex [26].

\[
\dim H^p,q_0(M, \mathcal{O}(F)) \equiv \dim H^{2-p,2-q}_0(M, \mathcal{O}(F^*)) ,
\]
\[\quad = H^p,q_0(M, \mathcal{O}(F \otimes \wedge^p TM^*_\mathbb{C})),\]
\[\quad = \dim H^p_0(M, \mathcal{O}(\wedge^p TM^*_\mathbb{C})) = \dim H^p,q(M, C),\]

where $F$ is a holomorphic bundle. We used the result of ref. [21] to know the number of $p^2$ and $b_{2,0}$ for $K^3/Z_2 \times Z_2$ (see Table 2).

We introduce the notation $\tilde{\mathcal{M}}(\Sigma) \equiv \dim \mathcal{M}(\Sigma) - 1$ to remove a scale factor. For the case (1), $\dim \tilde{\mathcal{M}}(\Sigma)$ which we derived and $\dim K(g), \dim \epsilon(g)$ and $\dim C(g)$ given by [18] are summarized in

|        | $\dim \mathcal{M}(\Sigma)$ | $\dim K(g)$ | $\dim \epsilon(G)$ | $\dim C(G)$ |
|--------|-----------------------------|-------------|--------------------|-------------|
| $K3$   | 60                          | 59          | 57                 | 40          |
| $T^4$  | 12                          | 11          | 9                  | 8           |

The difference between $\tilde{\mathcal{M}}(\Sigma)$ of the case(1) and $K(g)$ is as follows ; $\tilde{\mathcal{M}}(\Sigma)$ represents the moduli space of hyperkähler forms i.e., the definition of $\tilde{\mathcal{M}}(\Sigma)$ describes a set of $(g, J^1, J^2, J^3)$ or equivalently $(\Sigma^1, \Sigma^2, \Sigma^3)$, which takes into account the degrees of freedoms of how one can choose $g$, a trio of $g$-orthogonal complex structures.

On the other hand, $K(g)$ designates $(g, J^1)$ or equivalently $(\Sigma^1)$ only. The degrees of a trio of $g$-orthogonal complex structures which satisfy the quaternionic relations for a fixed $g$ is 3. Namely, for a fixed $g$.

\[
\{ \text{$g$ - orthogonal quaternionic almost complex structures $J$} \} \quad \cong \quad SO(4)/U(2) \cong S^2 \cong \text{Im} \mathbf{H} |_{x_1^2 + x_2^2 + x_3^2 = 1},
\]

\[
\text{Im} \mathbf{H} \equiv \{ J = \sum_{i=0}^3 x_i \bar{j}^i \mid \bar{j}^i \bar{j}^j = -\bar{j}^j \bar{j}^i = J^k (i, j, k \text{ cyclic}), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, (\bar{j}^i)^2 = (\bar{j}^j)^2 = (\bar{j}^k)^2 = -1 \}.
\]

For the hyperkählerian manifold, a trio of almost complex structures reduces to a trio of complex structures. The degrees of freedom of how one can choose $J^1$ for a fixed $g$ is given by $\dim S^2 = 2$. The degrees of freedom of how one can obtain $J^2$ which is orthogonal to $J^1$ for a fixed pair $(g, J^1)$ is given by $S^1$ over $S^2 = 2, J^3$ is automatically arranged after $(g, J^1, J^2)$ are fixed.

The moduli space $\mathcal{M}(\Sigma)$ has a bundle structure which has the fiber $(J^1, J^2, J^3)$ over the base manifold $\epsilon(g)$.

\[
\dim \mathcal{M}(\Sigma) = \dim K(g) + \dim S^1 \quad \cong \quad \dim \epsilon(g) + 2\dim H^0_C(M, J) + \dim S^1 \quad \cong \quad \dim \epsilon(g) + \dim S^2 + \dim S^1.
\]
For the case (2), The dimension of the moduli space is given by

\[ \dim \mathcal{M}(\Sigma, \omega) = \dim \mathcal{M}(\Sigma) + \dim \mathcal{M}(\omega), \quad \dim \mathcal{M}(\omega) = b_1(K^*). \] (36)

\( \mathcal{M}(\Sigma) \) describes one Kähler form and two almost Kähler forms which satisfy the quaternionic relations, whose dimension is given by eq.(28). \( \dim \mathcal{M}(\Sigma) \) derived by us and \( \dim \mathcal{M}(\omega) \) and \( \dim K(g) \), \( \dim \epsilon(g) \) given by [21, 18] are summarized in

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & \dim \mathcal{M}(\omega) & \dim \mathcal{M}(\Sigma) & \dim K(g) & \dim \epsilon(g) & \dim C(G) \\
\hline
K3/Z_2 & 0 & 30 & 29 & 29 & 20 \\
K3/Z_2 \times Z_2 & 15 & 14 & 15 & 10 \\
\hline
\end{array}
\]

The difference between \( \dim K(g) \) and \( \dim \tilde{\mathcal{M}}(\Sigma) \) corresponds to \( \dim S^1 = 1 \) as before but the difference between \( \dim K(g) \) and \( \dim \epsilon(g) \) is no longer represented by \( \dim S^2 \) since \( J^1 \neq x_1 J^1 + x_2 J^2 + x_3 J^3 \) for a complex structure \( J^1 \).

\[
\begin{align*}
\dim \tilde{\mathcal{M}}(\Sigma) &= \dim K(g) + \dim S^1 \\
&= \dim \epsilon(g) + 2\dim H^2_C(M, J) + \dim S^1 \\
&\neq \dim \epsilon(g) + \dim S^2 + \dim S^1.
\end{align*}
\] (37)

We mention the Teichmüller space \( N \) on K3-surface [18] or \( T^4 \) [4], which is defined by the quotient of Kähler-Einstein metrics of volume one modulo the diffeomorphisms which induce the identity on the cohomology group \( H^2(M, Z) \);

\[ K3 : N \cong SO(3, 19)/SO(3) \times SO(19), \quad T^4 : N \cong SO(4) \setminus GL(4)/PSL(4, Z). \] (38)

The relations between the moduli spaces of \( K(g) \) and \( \epsilon(g) \) on K3 was already derived [18]. They are some manifolds with holes or singularities [24, 18]. There is a natural compactification of marked \( \epsilon(g) \) on K3 by the method of Satake, Baily-Borel and Mumford [29]. These mathematical results will be important for the further investigation into \( \mathcal{M}(\Sigma) \) and the evaluation of some observables.

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Table 1. A Dimension-Counting of Fundamental Variables in Witten-type model

| $\Sigma^k$ | $\Psi^k$ |
|------------|----------|
| (3 x 6 = 18) | (3 x 6 = 18) |
| diffeo. gauge fix. condi. | red. diffeo. gauge fix. condi. |
| $D^0_0 \Sigma^k = 0$ | $D^0_0 \Psi^k = 0$ |
| super. / red. gauge fix. condi. | eqs. of motion |
| $D_1 \Sigma^k = 0 : \{d \Sigma^k = 0\}/\{d^2 \Sigma^k = 0\}$ | $D_1 \Psi^k = 0 : \{d \Psi^k = 0\}/\{d^2 \Psi^k = 0\}$ |
| t.f. $\Sigma^i \wedge \Sigma^j = 0$ | t.f. $\Sigma^i \wedge \Psi^j = 0$ |




|                  | $T^4$ | K3   | $K3Z_2$ | $K3Z_2\otimes Z_2$ |
|------------------|-------|------|--------|-------------------|
| $\chi = c_2$     | 0     | 24   | 12     | 6                 |
| $\tau$           | 0     | -16  | -8     | -4                |
| $c_1$            | 0     | 0    | 0      | 0                 |
| $b_0(K^*)$       | 1     | 1    | 0      | $-\frac{1}{2}$    |
| $b_1(K^*)$       | 4     | 0    | 0      | 0                 |
| $b_2(K^*)$       | 6     | 22   | 12     | 7                 |
| $b_3(K^*)$       | 4     | 0    | 0      | 0                 |
| $b_4(K^*)$       | 1     | 1    | 0      | $-\frac{1}{2}$    |
| $h^0(M, \Theta)$ | 2     | 0    | 0      | 0                 |
| $h^1(M, \Theta)$ | 4     | 20   | 10     | 5                 |
| $h^2(M, \Theta)$ | 2     | 0    | 0      | 0                 |
| $b_0$            | 1     | 1    | 1      | 1                 |
| $b_1$            | 4     | 0    | 0      | 0                 |
| $b_2$            | 6     | 22   | 10     | 4                 |
| $b_{1,1}$        | 4     | 20   | 10     | 5                 |
| $b_{2,0}$        | 1     | 1    | 0      | $\frac{1}{2}$    |
| dim. Isometry    | 4     | 0    | 0      | 0                 |
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