FRACTIONAL OPERATORS WITH BOUNDARY POINTS DEPENDENT KERNELS AND INTEGRATION BY PARTS

THABET ABDELJAWAD
Department of Mathematics and General Sciences, Prince Sultan University
P. O. Box 66833, 11586 Riyadh, Saudi Arabia
Department of Medical Research, China Medical University
Taichung 40402, Taiwan
Department of Computer Science and Information Engineering, Asia University
Taichung, Taiwan

ABSTRACT. Recently, U. N. Katugampola presented some generalized fractional integrals and derivatives by iterating a $t^{\rho-1}$-weighted integral, $\rho > 0$. The case $\rho = 1$ produces Riemann and Caputo fractional derivatives and the limiting case $\rho \to 0^+$ results in Hadamard type fractional operators. In this article, we discuss the differences between a new class of nonlocal generalized fractional derivatives generated by iterating left and right type conformable integrals weighted by $(t-a)^{\rho-1}$ and $(b-t)^{\rho-1}$ and the ones introduced by Katugampola. In fact, we will present very different integration by parts formulas by presenting new mixed left and right generalized fractional operators with boundary points dependent kernels. The properties of this new class of mixed fractional operators are analyzed in newly defined function spaces as well.

1. Introduction and preliminaries. Fractional calculus which deals with integration and differentiation with respect to arbitrary order is of increase interest among researchers in the last two decades or so [39, 40, 34, 38, 24]. Up to three years ago, fractional calculus was depending on the classical fractional operators that are obtained by iterating integrals and then replacing them by a single integral in which the kernel is singular, having delay reflecting the non-locality, and does not depend on the boundary points $a$ and $b$. This approach produced nonlocal fractional derivatives such as Riemann-Liouville, Caputo and Hadamard types [35, 29, 23]. On the other hand the Grünwald-Letnikov fractional operators depends on iterating the usual derivatives [34]. Very recently and for the hope of obtaining better nonlocal fractional operators in describing complex models in many fields of science and engineering and without simulation difficulties when dealing with the kernel, some authors presented and studied new fractional operators with nonsingular kernels. The idea behind generating such operators was by imposing a nonsingular exponential or Mittag-Leffler kernel so that in the limiting case $\alpha \to 1$ the usual derivatives are reobtained by the help of the delta dirac functions [1, 2, 17, 21, 37, 22, 3, 4, 15, 16, 5, 7, 8, 9]. For further recent developed class of fractional differential operators with nonsingular kernels and their application to real world problems we refer the reader to [36, 18, 19].

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Very recently, the authors in [30], depending on the conformable integrals and derivatives developed in [10] and as an answer to the open problem presented there, generated new types of left and right fractional integrals and operators with kernels depending on the boundary points $a$ and $b$. Such new fractional operators brought a new trend in fractional calculus where researchers will be in need of new convolution theory and slightly different Laplace transforms. However, in [30], the authors were not able to formulate reasonable integration by parts under the existence left and right fractional operators. The trend of developing new fractional operators with kernels depending on the starting point $a$ very recently started to be studied in quantum fractional calculus as well [41].

Motivated by and supporting to what we have mentioned above, we shall define a new class of mixed left and right conformable fractional integrals and derivatives with kernels depending on the boundary points. The properties of the new presented operators will be studied in a newly defined function spaces, and will be used to formulate fractional integration by parts. We also shall make some comparisons to the generalized fractional operators in the sense of Katugampola and show that they are different. Our paper is organized as follows: In the next part of this section we give some basic necessary definitions and results about fractional calculus, generalized fractional operators in the sense of Katugampola and conformable fractional operators following [30, 10]. In Section 2 we make some comparisons between the generalized fractional operators in the sense of Katugampola and conformable fractional operators. In Section 3, we define and study the properties of new class of mixed left and right conformable fractional operators in both Riemann and Caputo settings. In section 4, we formulate and prove new different integration-by-parts formulas for the conformable and mixed conformable fractional operators. In section 5, we have conclusions.

1.1. Some general fractional calculus concepts. First, we recall some formulas from the classical fractional calculus [39, 40, 34].

The left Riemann-Liouville fractional of order $\alpha$, $\Re(\alpha) > 0$ is defined by

$$
(aI_\alpha^f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \quad (1.1)
$$

The right Riemann-Liouville fractional of order $\alpha$, $\Re(\alpha) > 0$ ending at $b > a$ is defined by

$$
(bI_\alpha^f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds. \quad (1.2)
$$

The left Riemann-Liouville fractional derivative of order $\alpha$, $\Re(\alpha) \geq 0$ is given as

$$
(aD_\alpha^f)(t) = \frac{d^n}{dt^n} (aI^{n-\alpha}_\alpha f)(t) = \frac{d^n}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad n = \lceil \Re(\alpha) \rceil + 1. \quad (1.3)
$$

The right Riemann-Liouville fractional derivative of order $\alpha$, $\Re(\alpha) \geq 0$ reads

$$
(bD_\alpha^f)(t) = (-1)^n \frac{d^n}{dt^n} (bI^{n-\alpha}_\alpha f)(t)
= (-1)^n \frac{d^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f(s) ds, \quad n = \lceil \Re(\alpha) \rceil + 1. \quad (1.4)
$$
The left Caputo fractional of order $\alpha$, $\Re(\alpha) \geq 0$ has the following form
\[
( C_a^\alpha D^n f)(x) = ( aI^{n-\alpha}f^{(n)})(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \ n = [\Re(\alpha)] + 1.
\] (1.5)

The right Caputo fractional derivative of order $\alpha$, $\Re(\alpha) \geq 0$ reads
\[
( C_b^\alpha D^n f)(t) = ( b^{-\alpha} f^{(n)})(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)ds, \ n = [\Re(\alpha)] + 1.
\] (1.6)

**Lemma 1.1.** [34] Let $\Re(\alpha) \geq 0$ and $n = [\Re(\alpha)] + 1$. If $f \in AC^n[a, b]$, where $0 < a < b < \infty$. Then,
\[
\left( C_a^\alpha D^n f\right)(t) = \left( aD^\alpha f\right)(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1)} (t-a)^{k-\alpha}, \quad (1.7)
\]
\[
\left( C_b^\alpha D^n f\right)(t) = \left( D_b^\alpha f\right)(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k+1)} (b-t)^{k-\alpha}. \quad (1.8)
\]

**Definition 1.1.** [34] The Mittag-Leffler function of one parameter is defined by
\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad (z \in \mathbb{C}; \ \Re(\alpha) > 0), \quad (1.9)
\]
and the one with two parameters $\alpha$ and $\beta$ by
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (z, \beta \in \mathbb{C}; \ \Re(\alpha) > 0), \quad (1.10)
\]

where $E_{\alpha,1}(z) = E_\alpha(z)$.

For Hadamard fractional operators, as operators of logarithmic kernels, we refer the reader to [35] and for their Caputo modifications we refer to [29, 23].

1.2. The generalized fractional operators in the sense of Katugampola.

For $a < b$, $c \in \mathbb{R}$ and $1 \leq p < \infty$, define the function space
\[
X_p^p(a, b) = \{ f : [a, b] \rightarrow \mathbb{R} : \| f \|_{X_p^p} = \left( \int_a^b |t^p f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty \}.
\]
For $p = \infty$, $\| f \|_{X_\infty^p} = ess sup_{a \leq t \leq b} |t^p f(t)|$. The generalized left and right fractional integrals of order $\alpha$, $\Re(\alpha) > 0$ are defined in [32, 33] as
\[
(aI_\alpha^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} f(s) \frac{ds}{s^{1-\rho}} \quad (1.11)
\]
and
\[
(bI_\alpha^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left( \frac{s^\rho - t^\rho}{\rho} \right)^{\alpha-1} f(s) \frac{ds}{s^{1-\rho}}, \quad (1.12)
\]
respectively. It can be easily noticed that when $\rho = 1$, the integrals in (1.11) and (1.12) reduces to the integrals in (1.1) and (1.2), respectively.
Lemma 1.2. [32, 33, 12, 31] For $\Re(\alpha) > 0$, $\Re(\mu) > 0$ and $\rho > 0$ we have
\[
a I_{a}^{\alpha, \rho} \left[ \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\mu-1} \right] (x) = \frac{\Gamma(\mu)}{\Gamma(\mu + \alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha + \mu - 1} \tag{1.13}
\]
\[
b I_{b}^{\alpha, \rho} \left[ \left( \frac{b^\rho - t^\rho}{\rho} \right)^{\mu-1} \right] (x) = \frac{\Gamma(\mu)}{\Gamma(\mu + \alpha)} \left( \frac{b^\rho - x^\rho}{\rho} \right)^{\alpha + \mu - 1} \tag{1.14}
\]
Proof. The proof of the first part is done by definition and by using the change of variable $u = \frac{t^\rho - a^\rho}{\rho}$. We give the details for the proof of the second part.
\[
b I_{b}^{\alpha, \rho} \left[ \left( \frac{b^\rho - t^\rho}{\rho} \right)^{\mu-1} \right] (x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \left( \frac{t^\rho - x^\rho}{\rho} \right)^{\alpha - 1} \left( \frac{b^\rho - t^\rho}{\rho} \right)^{\rho - 1} dt \tag{1.15}
\]
By using the change of variable $u = \frac{t^\rho - x^\rho}{\rho}$, we have
\[
b I_{b}^{\alpha, \rho} \left[ \left( \frac{b^\rho - t^\rho}{\rho} \right)^{\mu-1} \right] (x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{1} u^{\alpha - 1} (1 - u)^{\mu - 1} du, \tag{1.16}
\]
and hence (1.14) follows. \hfill \Box

The left and right generalized fractional derivatives of order $\alpha, \Re(\alpha) \geq 0$ are defined by (see [33])
\[
(aD^{\alpha, \rho} f)(t) = \gamma^n (aI^{n-\alpha, \rho} f)(t) = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_{a}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha - 1} f(s) \frac{ds}{s^{1-\rho}} \tag{1.17}
\]
and
\[
bD^{\alpha, \rho} f(t) = (-\gamma)^n (bI^{n-\alpha, \rho} f)(t) = \frac{(-\gamma)^n}{\Gamma(n-\alpha)} \int_{t}^{b} \left( \frac{s^\rho - t^\rho}{\rho} \right)^{n-\alpha - 1} f(s) \frac{ds}{s^{1-\rho}}, \tag{1.18}
\]
respectively, where $\gamma = t^{1-\rho} \frac{dT}{dt}$. Putting $\rho = 1$ in (1.17) and (1.18) one gets the Riemann-Liouville fractional derivatives (1.3) and (1.4) and letting $\rho$ tend to 0, one gets the Hadamard fractional derivatives.

For the functions in $AC^n_{\alpha} [a, b] = \left\{ f : [a, b] \rightarrow \mathbb{C} \text{ and } \gamma^{n-1} f \in AC[a, b], \gamma = t^{1-\rho} \frac{dT}{dt} \right\}$ and $C^n_{\alpha} [a, b] = \left\{ f : [a, b] \rightarrow \mathbb{C} \text{ and } \gamma^{n-1} f \in C[a, b], \gamma = t^{1-\rho} \frac{dT}{dt} \right\}$, the left and right generalized Caputo fractional derivatives of order $\alpha, \Re(\alpha) > 0$ are given respectively as in [31] by
\[
C_{a}D^{\alpha, \rho} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha - 1} \left( \frac{\gamma f}{s^{1-\rho}} \right) ds = aI^{n-\alpha, \rho} (\gamma^n f)(t) \tag{1.19}
\]
and
\[
C_{b}D^{\alpha, \rho} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{b} \left( \frac{s^\rho - t^\rho}{\rho} \right)^{n-\alpha - 1} \left( \frac{\gamma^n f}{s^{1-\rho}} \right) ds = bI^{n-\alpha, \rho} ((-\gamma)^n f)(t). \tag{1.20}
\]

It should be noticed that the derivative in (1.19) becomes the left Caputo derivative (1.5) once one replaces $\rho$ by 1. The same relation holds for (1.20) and (1.6).

Below we further present some formulas that will be used later in this work.

Theorem 1.1. [33] Let $\alpha > 0$, $1 \leq p \leq \infty$ and $c \in \mathbb{R}$. Then for $f \in X_{c}^{p}(a, b)$ where $a > 0$, $\rho > 0$, we have
\[
aD^{\alpha, \rho} aI^{\alpha, \rho} f = f \text{ and } bD_{\alpha, \rho} bI_{\alpha, \rho} f = f \tag{1.21}
\]
The proof of the following lemma can be executed using Definition 3.1 and Lemma 2.8 in [31].

**Lemma 1.3.** Let \( \Re(\alpha) \geq 0 \) and \( n = \lceil \Re(\alpha) \rceil + 1 \). If \( f \in AC^\alpha_\rho[a, b] \), where \( 0 < a < b < \infty \). Then,

\[
\left( C D^\alpha_\rho f \right)(t) = \left( D^\alpha_\rho f \right)(t) - \sum_{k=0}^{n-1} \frac{\gamma^k f(a)}{\Gamma(k - \alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k-\alpha}, \quad (1.22)
\]

\[
\left( C D^\alpha_\rho f \right)(t) = \left( D^\alpha_\rho f \right)(t) - \sum_{k=0}^{n-1} \frac{(-1)^{k+1} f(b)}{\Gamma(k - \alpha + 1)} \left( \frac{b^\rho - t^\rho}{\rho} \right)^{k-\alpha}. \quad (1.23)
\]

1.3. **The fractional operators generated by conformable integrals and derivatives.** In [10], the left-conformable derivative starting from \( a \) of a function \( f: [a, b] \to \Re \) of order \( 0 < \rho \leq 1 \) was defined by

\[
\left( aT^\rho f \right)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t - a)^{1-\rho}) - f(t)}{\epsilon}.
\]

If \( \left( aT^\rho f \right)(t) \) exists on \( (a, b) \) then \( aT^\rho f(a) = \lim_{t \to a^+} \left( aT^\rho f \right)(t) \), and the right conformable derivative ending at \( b \) was defined by

\[
\left( T^\rho_b f \right)(t) = - \lim_{\epsilon \to 0} \frac{f(t + \epsilon(b - t)^{1-\rho}) - f(t)}{\epsilon}.
\]

If \( \left( T^\rho_b f \right)(t) \) exists on \( (a, b) \) then \( T^\rho_b f(b) = \lim_{t \to b^-} \left( aT^\rho f \right)(t) \).

If \( f \) is differentiable in \( (a, b) \) then

\[
\left( aT^\rho f \right)(t) = (t - a)^{1-\rho} f'(t) \quad (1.24)
\]

and

\[
\left( T^\rho_b f \right)(t) = -(b - t)^{1-\rho} f'(t) \quad (1.25)
\]

We also recall from [10], the corresponding left and right conformable integrals for a real-valued function \( f \) defined on \( [a, b] \) of order \( \rho \in (0, 1] \) by

\[
a^\rho f(t) = \int_a^t f(s) d\rho(s, a) = \int_a^t f(s)(s - a)^{\rho-1} ds \quad (1.26)
\]

and

\[
^\rho_b f(t) = \int_t^b f(s) d\rho(b, s) = \int_t^b f(s)(b - s)^{\rho-1} ds. \quad (1.27)
\]

The extension to higher order was given in [10] as well. The authors in [30] used the conformable integrals of order \( \rho > 0 \) in (1.26) and (1.27) to generate a new class of left and right generalized fractional integrals and derivatives with memories. We shall call them conformable fractional integrals and derivatives. Indeed, we list the following operators

**Definition 1.2.** [30] The left-fractional conformable integral operator is defined by

\[
a^\alphaT^\rho f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \frac{(x - a)^\rho - (t - a)^\rho}{\rho} \right)^{\alpha-1} f(t) \frac{dt}{(t - a)^{1-\rho}}. \quad (1.28)
\]

where \( \alpha \in \mathbb{C}, \Re(\alpha) \geq 0, \rho > 0 \).
Definition 1.3. [30] The left-fractional conformable derivative of order $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) \geq 0$ in Riemann-Liouville setting is defined by
\[
{^aD}^{\alpha,\rho} f(x) = {^aT}^\rho \left( {^aD}^{\alpha-\rho} f(x) \right) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left( \frac{(x-a)^\rho - (x-t)^\rho}{\rho} \right)^{n-\alpha-1} f(t) \frac{dt}{(t-a)^{1-\rho}},
\]
(1.29)
and in the Caputo setting by
\[
{^C}D^{\alpha,\rho} f(x) = \left( {^C}T^{\alpha,\rho} f(x) \right)^{(n)} = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left( \frac{(x-a)^\rho - (x-t)^\rho}{\rho} \right)^{n-\alpha-1} {^aT}^\rho f(t) \frac{dt}{(t-a)^{1-\rho}},
\]
(1.30)
where $n = \lfloor \text{Re}(\alpha) \rfloor + 1$, $n {^aT}^\rho = {^aT}^\rho {^aT}^\rho \cdots {^aT}^\rho$ and $a T^\rho$ is the left conformable differential operator presented in (1.24).

Definition 1.4. [30] The right-fractional conformable integral operator is defined by
\[
{^bI}^{\alpha,\rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{\alpha-1} f(t) \frac{dt}{(b-t)^{1-\rho}},
\]
(1.31)
where $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) \geq 0$, $\rho > 0$.

The following is Lemma 2.2 in [30].

Lemma 1.4. [30] For $\text{Re}(\mu) > 0$, $\text{Re}(\alpha) > 0$ and $\rho > 0$ we have
\[
( {^aJ}^{\alpha,\rho} (t-a)^{\rho \mu - \alpha} ) (x) = \frac{\Gamma(\mu)}{\rho^\alpha \Gamma(\mu + \alpha)} (x-a)^{\rho(\alpha + \mu - 1)},
\]
(1.32)
\[
( {^bJ}^{\alpha,\rho} (b-t)^{\rho \mu - \alpha} ) (x) = \frac{\Gamma(\mu)}{\rho^\alpha \Gamma(\mu + \alpha)} (x-b)^{\rho(\alpha + \mu - 1)},
\]
(1.33)

Lemma 1.5. [30] For $\text{Re}(\mu) > 0$, $\text{Re}(\alpha) \geq 0$ and $\rho > 0$ we have
\[
( {^aD}^{\alpha,\rho} (t-a)^{\rho \mu - \alpha} ) (x) = \frac{\rho^\alpha \Gamma(\mu)}{\Gamma(\mu - \alpha)} (x-a)^{\rho(\mu - \alpha - 1)},
\]
(1.34)
\[
( {^bD}^{\alpha,\rho} (b-t)^{\rho \mu - \alpha} ) (x) = \frac{\rho^\alpha \Gamma(\mu)}{\Gamma(\mu - \alpha)} (b-x)^{\rho(\mu - \alpha - 1)},
\]
(1.35)

Definition 1.5. [30] The left-fractional conformable derivative of order $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) \geq 0$ in Riemann-Liouville setting is defined by
\[
{^aD}^{\alpha,\rho} f(x) = {^bT}^\rho ( {^bJ}^{\alpha,\rho} f(x) ) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \left( \frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{n-\alpha-1} f(t) \frac{dt}{(b-t)^{1-\rho}},
\]
(1.36)
and in the Caputo setting by
\[
{^C}D^{\alpha,\rho} f(x) = \left( {^bJ}^{\alpha,\rho} {^bT}^\rho f(x) \right)^{(n)} = \frac{1}{\Gamma(n-\alpha)} \int_x^b \left( \frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right)^{n-\alpha-1} {^bT}^\rho f(t) \frac{dt}{(b-t)^{1-\rho}},
\]
(1.37)
where \( n = \lfloor \text{Re}(\alpha) \rfloor + 1 \), \( ^nT_b^\rho = T_b^\rho T_b^\rho \cdots T_b^\rho \) and \( T_b^\rho \) is the right conformable differential operator presented in (1.25).

**Definition 1.6.** For \( a < b \), \( c \in \mathbb{R} \) and \( 1 \leq p < \infty \), define the function spaces

\[
\alpha X^{c,p}(a,b) = \{ f : [a,b] \to \mathbb{R} : \| f \|_{\alpha X^{c,p}} = \left( \int_a^b |(t-a)^c f(t)|^p \frac{dt}{(t-a)} \right)^{1/p} < \infty \}.
\]

For \( p = \infty \), \( \| f \|_{\alpha X^{c,p}} = \text{ess sup}_{a \leq t \leq b} |(t-a)^c f(t)| \).

\[
X^{c,p}_c(a,b) = \{ f : [a,b] \to \mathbb{R} : \| f \|_{X^{c,p}_c} = \left( \int_a^b |(b-t)^c f(t)|^p \frac{dt}{(b-t)} \right)^{1/p} < \infty \}.
\]

For \( p = \infty \), \( \| f \|_{X^{c,p}_c} = \text{ess sup}_{a \leq t \leq b} |(b-t)^c f(t)| \).

**Definition 1.7.** For \( \rho > 0 \) and \( c \geq \rho \) define the function spaces

\[
\alpha AC^\rho[a,b] = \left\{ f : [a,b] \to \mathbb{C} : f(x) = c_1 + \int_a^x \varphi(t)d\rho (t,a), \varphi \in \alpha X^{c,1} \right\}, \quad (1.38)
\]

\[
AC^\rho_b[a,b] = \left\{ f : [a,b] \to \mathbb{C} : f(x) = c_2 + \int_x^b \varphi(t)d\rho (b,t), \varphi \in X^{c,1}_b \right\}, \quad (1.39)
\]

\[
b AC^\rho[a,b] = \left\{ f : [a,b] \to \mathbb{C} : f(x) = c_3 + \int_a^x \varphi(t)d\rho (b,t), \varphi \in X^{c,1}_b \right\}, \quad (1.40)
\]

\[
a AC^\rho_b[a,b] = \left\{ f : [a,b] \to \mathbb{C} : f(x) = c_4 + \int_x^b \varphi(t)d\rho (t,a), \varphi \in \alpha X^{c,1} \right\}, \quad (1.41)
\]

**Definition 1.8.** For \( \rho > 0 \), \( c \geq \rho \) and \( n = 1, 2, ..., \) define the function spaces

\[
\alpha AC^{n,\rho}[a,b] = \{ f : [a,b] \to \mathbb{C} : \ n^{-1}T_b^\rho f(x) \in \alpha AC^\rho[a,b] \}, \quad (1.42)
\]

\[
AC^{n,\rho}_b[a,b] = \{ f : [a,b] \to \mathbb{C} : \ n^{-1}T_b^\rho f(x) \in AC^\rho_b[a,b] \}, \quad (1.43)
\]

\[
b AC^{n,\rho}[a,b] = \{ f : [a,b] \to \mathbb{C} : \ n^{-1}T_b^\rho f(x) \in \alpha AC^\rho[a,b] \}, \quad (1.44)
\]

\[
a AC^{n,\rho}_b[a,b] = \{ f : [a,b] \to \mathbb{C} : \ n^{-1}T_b^\rho f(x) \in \alpha AC^\rho_b[a,b] \}, \quad (1.45)
\]

where \( AC^{1,\rho} = AC^\rho \).

**Remark 1.1.** The spaces \( \alpha AC^{n,\rho}[a,b] \) and \( AC^{n,\rho}_b[a,b] \) defined above are the same as the spaces \( C_{\rho,a}^n[a,b] \) and \( C_{\rho,b}^n[a,b] \) given in Definition 3.1 in [30]. The function spaces listed in Definition 1.7 and Definition 1.8 will be used to present new fractional by integration parts in the setting of Caputo for conformable and mixed conformable fractional operators.

**Lemma 1.6.** Let \( \Re(\alpha) \geq 0 \) and \( n = \lfloor \Re(\alpha) \rfloor + 1 \). If \( f \in \alpha AC^{n,\rho}[a,b] \) (\( f \in \alpha AC^{n,\rho}_b[a,b] \)), where \( 0 < a < b < \infty \). Then,

\[
\left( C \alpha D^{\alpha,\rho} f \right)(t) = \left( \alpha D^{\alpha,\rho} f \right)(t) - \sum_{k=0}^{n-1} k^{\alpha-k-1} f^{(k)}(a) (t-a)^{\rho(k-\alpha)}, \quad (1.46)
\]

\[
\left( C \alpha D^{\alpha,\rho}_b f \right)(t) = \left( \alpha D^{\alpha,\rho}_b f \right)(t) - \sum_{k=0}^{n-1} k^{\alpha-k-1} f^{(k)}(b) (b-t)^{\rho(k-\alpha)} \quad (1.47)
\]

**Proof.** The proof follows from Definition 4.1 in [30] and Lemma 1.5 above. \( \square \)
2. Comparisons between Katugampola type fractional operators and conformable fractional operators. In this section, for the sake of comparison we list some properties of Katugampola type fractional operators and conformable fractional operators. There are many comparison aspects, however, we just mention integration by parts and the solutions for Caputo setting linear fractional initial value problems with constant coefficient. Other aspects can be also reported and we shall remark them at the end of this section.

First we recall the integration by parts for Katugampola type fractional operators as was presented in [11].

The authors in [11] presented the following function spaces: For \( p \geq 1 \) and \( \alpha > 0 \), we define

\[
aI^\alpha (X_p) = \{ f : f = aI^\alpha \varphi, \ \varphi \in X_p^p(a, b) \}. \tag{2.1}
\]

and

\[
bI^\alpha (X_p) = \{ f : f = bI^\alpha \psi, \ \psi \in X_p^p(a, b) \}. \tag{2.2}
\]

**Theorem 2.1.** [11] Let \( \alpha > 0 \), \( p \geq 1 \), \( q \geq 1 \), and \( \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha \) \((p \neq 1\) and \( q \neq 1 \) in the case \( \frac{1}{p} + \frac{1}{q} = 1 + \alpha \)). Then

- If \( \varphi(t) \in X_p^p(a, b) \) and \( \psi(t) \in X_p^q(a, b) \), then
  \[
  \int_a^b \psi(t)(aI^\alpha \varphi)(t) \frac{dt}{t^{1-\rho}} = \int_a^b \varphi(t)(bI^\alpha \psi)(t) \frac{dt}{t^{1-\rho}}.
  \]

- If \( f(t) \in bI^\alpha(X_p) \) and \( g(t) \in aI^\alpha(X_q) \), then
  \[
  \int_a^b f(t)(aD^\alpha g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b (bD^\alpha f)(t)g(t) \frac{dt}{t^{1-\rho}}.
  \]

Below we state integration by parts formulas for functions in the space \( AC_n^\alpha[a, b] \) or \( C_\gamma^n[a, b] \) as was proved in [11].

**Theorem 2.2.** [11]. Let \( \alpha > 0, \ n = [\alpha] + 1 \) and \( f, g \in AC_n^\alpha[a, b] \) or \( C_\gamma^n[a, b] \). Then

\[
\int_a^b f(t)(C D^\alpha g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t)(aD^\alpha f)(t) \frac{dt}{t^{1-\rho}} + \left( \sum_{k=0}^{n-1} (-1)^{n-k}(\gamma^{n-1-k}g(t)(aI^{n-k-\alpha}f)(t) \right)_{a}^{b} \tag{2.3}
\]

**Corollary 2.1.** [11] Let \( \alpha > 0, \ n = [\alpha] + 1 \) and \( f, g \in AC_n^\alpha[a, b] \) or \( C_\gamma^n[a, b] \). Then,

\[
\int_a^b f(t)(D^\alpha g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t)(aD^\alpha f)(t) \frac{dt}{t^{1-\rho}} \sum_{k=0}^{n-1} (-1)^{n-k}(\gamma^{n-1-k}g(t)(aI^{n-k-\alpha}f)(a)) \tag{2.4}
\]

**Theorem 2.3.** [11]. Let \( \alpha > 0, \ n = [\alpha] + 1 \) and \( f, g \in AC_n^\alpha[a, b] \) or \( C_\gamma^n[a, b] \). Then

\[
\int_a^b f(t)(C aD^\alpha g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t)(D^\alpha f)(t) \frac{dt}{t^{1-\rho}} + \left( \sum_{k=0}^{n-1} (\gamma^{n-1-k}g(t)(bI^{n-k-\alpha}f)(t) \right)_{a}^{b} \tag{2.5}
\]
Corollary 2.2. [11] Let $\alpha > 0$, $n = [\alpha] + 1$ and $f, g \in AC^n_\gamma[a, b]$ or $C^n_\gamma[a, b]$. Then,

$$
\int_a^b f(t)(aD^{\alpha, \rho}g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t)(D^{\alpha, \rho}f)(t) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} (\gamma^{n-1-k}g)(b)(I_{b}^{n-k-\alpha, \rho}f)(b^-) \tag{2.6}
$$

Remark 2.1. As we noticed above, the integration by parts formulas have been presented by just using left and right type fractional operators. However, in next section we shall present new mixed types of conformable fractional operators in order to prove convenient integration by parts. Therefore, Katugampola type and conformable fractional type operators are very different in the aspect of integration by parts. Also, we would like to mention the integration by parts given here and cited by [11] are more convenient than those presented in [14]. For more about the application of fractional integration by parts in variational calculus we refer for example to [13, 20, 25, 26, 27, 28].

Below we proceed to explain more differences between the two types.

Consider the initial value problem

$$
C^\alpha Dh(t) = \lambda x(t) + f(t), \quad t > a, \quad x(a) = x_0, \tag{2.7}
$$

where $\alpha \in (0, 1)$, $f$ real-valued function and $\rho > 0$. If we proceed by successive approximation, similar to the proof of Theorem 2.5 below, and by the help of Theorem 4.5 in [30] we can state the following representation.

Theorem 2.4. The solution of the Caputo initial value problem (2.7) is given by

$$
x(t) = E_\alpha \left( \frac{\lambda}{\rho^\alpha} (t - a)^\alpha \right) x_0 + \int_a^t \left( \frac{(t - a)^\alpha - (s - a)^\alpha}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left( \lambda \left( \frac{(t - a)^\alpha - (s - a)^\alpha}{\rho} \right)^\alpha \right) f(s) (s - a)^{\rho-1} ds. \tag{2.8}
$$

Next we give the proof of solution representation for Caputo type Katugumapola fractional linear nonhomogeneous equation by using the successive approximation through applying Theorem 3.6 in [31] and Lemma 1.4 in [12].

Theorem 2.5. The Cauchy problem

$$
C^\alpha Dh(t) - \lambda x(t) = f(t), \quad t > a, \quad 0 < \alpha \leq 1, \quad \lambda \in \mathbb{R},
$$

$x(a) = x_0, \quad b \in \mathbb{R}$

has the solution

$$
x(t) = x_0 E_\alpha \left( \frac{\lambda}{\rho^\alpha} (t - a)^\alpha \right) + \int_a^t \left( \frac{t^\rho - \tau^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left( \lambda \left( \frac{t^\rho - \tau^\rho}{\rho} \right)^\alpha \right) f(\tau) \frac{d\tau}{\tau^{1-\rho}}. \tag{2.10}
$$

Proof. Consider the successive approximation

$$
x_n(t) = x_0 + \lambda a^{\alpha, \rho} x_{n-1}(t) + a I^{\alpha, \rho} f(t), \quad n = 1, 2, ..., x_0(t) = x_0.
$$

Then, for $n = 1$ we have

$$
x_1(t) = x_0 + \lambda \frac{\Gamma(1)}{\Gamma(\alpha + 1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^\alpha x_0 + a I^{\alpha, \rho} f(t).
$$
Proceeding inductively and by making use of the semigroup property of the general-
ized fractional integrals we have
\[
x_n(t) = x_0 + \sum_{k=0}^{n-1} \lambda^k(t^\rho - a^\rho)^k\alpha \frac{t^{\rho\alpha}}{\Gamma(k\alpha + 1)} + \sum_{k=1}^{n} \lambda^{k-1} a^{k\alpha,\rho} f(t).
\]
(2.11)
Then, we reach our claim by expanding \( a^{k\alpha,\rho} \) in the second summation, shifting
the index \( k \), interchanging the order of the integral and summation, and letting
\( n \to \infty \).

**Remark 2.2.** It is clear that the solution representations (2.8) and (2.9) are dif-
ferent and coincide when \( a = 0 \). However, no coincidence if we consider the right
fractional initial value problem. If we also consider the Riemann-Liouville initial
value problems we can notice the difference in solution representations as well. All
the differences depend on the fact that in case of defining the conformable fractional
integrals we depend on iterating weighted integrals allowing the appearance of the
boundary points. Indeed, we iterate integrals weighted by \((t−a)^{\rho−1}\) in the left case
and by \((b−t)^{\rho−1}\) in the right case. However, in the Katugampola fractional we
iterate integrals with respect to \( t^{\rho−1} dt \). For the sake of more comparisons compare
Lemma 1.2 and Lemma 1.4.

3. **Mixed types conformable fractional operators.** The following definition
will be the key to present integration by parts formula for conformable fractional
operators.

**Definition 3.1.** Let \( f \) be a real-valued function defined on \([a, b]\) and \( \rho > 0 \). Then,
• The mixed left conformable integral of order \( \rho \) is defined by
\[
\mathcal{I}_a^\rho \left( \int_a^t f(s)(b−s)^{\rho−1}ds \right). \tag{3.1}
\]
• The mixed right conformable integral of order \( \rho \) is defined by
\[
\mathcal{I}_b^\rho f(t) = \int_t^b f(s)(s−a)^{\rho−1}ds. \tag{3.2}
\]

By the help of (1.24) and (1.25), it is straightforward to verify that \( −T^\rho \mathcal{I}_a^\rho f(t) = f(t) \) and \( −T^\rho \mathcal{I}_b^\rho f(t) = f(t) \). In the sequel, we may use the notations
\( d^\rho(t, a) = (t−a)^{\rho−1}dt \) and \( d^\rho(b, t) = (b−t)^{\rho−1}dt \).
Next, we iterate the integrals in (3.1) and (3.2) in order to generate mixed left
and mixed right fractional integrals. Let \( \mathcal{I}_a^\rho f(t) = \mathcal{I}_a^\rho f(t) \) and
\[
\mathcal{I}_a^\rho f(x) = \int_a^x \left( \int_a^t f(s)(b−s)^{\rho−1}ds \right)(b−t)^{\rho−1}dt.
\]
Interchanging the order of integrals and evaluating the inner integral, we conclude
that
\[
\mathcal{I}_a^\rho f(x) = \int_a^x f(s) \left( \frac{(b−s)^\rho − (b−x)^\rho}{\rho} \right)(b−s)^{\rho−1}ds.
\]
If we proceed inductively on \( n \), we reach the formula
\[
\mathcal{I}_a^\rho f(x) = \frac{1}{(n−1)!} \int_a^x f(s) \left( \frac{(b−s)^\rho − (b−x)^\rho}{\rho} \right)^{n−1}(b−s)^{\rho−1}ds. \tag{3.3}
\]
Similarly, if we iterate the mixed right conformable integral \( a^b J^1_{b, \rho} f(x) = a^b J^0_{b, \rho} f(x) \), we reach the formula

\[
a^b J^\alpha_{b, \rho} f(x) = \frac{1}{(n-1)!} \int_a^b f(s) \left( \frac{(s-a)^\rho - (x-a)^\rho}{\rho} \right)^{n-1} (s-a)^{\rho-1} ds.
\]  

(3.4) 

Upon (3.3) and (3.4) we can define the following new types of fractional integrals.

**Definition 3.2.** (Mixed left and right conformable fractional integrals) Let \( f \) be defined on \([a, b]\) and \( \alpha, \rho \in \mathbb{C}, \ Re(\alpha) > 0, \ Re(\rho) > 0 \). Then

- The mixed left conformable fractional integral of \( f \) is defined by
  \[
  b^a J^\alpha_{a, \rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(s) \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} \, ds.
  \]  

(3.5) 

- The mixed right conformable fractional integral of \( f \) is defined by
  \[
  a^b J^\alpha_{b, \rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(s) \left( \frac{(s-a)^\rho - (x-a)^\rho}{\rho} \right)^{\alpha-1} \, ds.
  \]  

(3.6) 

**Remark 3.1.** Recalling that the \( Q \)-operator is defined as \( Qf(x) = f(a + b - x) \), one easily can verify that

\[
Q a^b J^\alpha_{b, \rho} f(x) = b^a J^{\alpha, \rho} Qf(x),
\]  

(3.7) 

and hence, since \( Q^2 f(x) = f(x) \) we have \( Q a^b J^\alpha_{a, \rho} f(x) = a^b J^\alpha_{b, \rho} Qf(x) \). This is immediate by using definition and the change of variable \( u = a + b - s \).

**Lemma 3.1.** For \( Re(\alpha) > 0, \ Re(\mu) > 0 \) and \( \rho > 0 \) we have

\[
b^a J^\alpha_{a, \rho} ((b-a)^\rho - (b-s)^\rho)^{\mu-1} (x) = \frac{\Gamma(\mu)}{\Gamma(\mu + \rho)} (b-a)^\rho (b-x)^\rho \mu^{\mu+\rho-1},
\]  

(3.8) 

and

\[
a^b J^\alpha_{b, \rho} ((s-a)^\rho - (x-a)^\rho)^{\mu-1} (x) = \frac{\Gamma(\mu)}{\Gamma(\mu + \rho)} (b-a)^\rho (x-a)^\rho \mu^{\mu+\rho-1}.
\]  

(3.9) 

**Proof.** We prove (3.8) and the proof of (3.9) is achieved by the action of the \( Q \)-operator, mentioned in Remark 3.1 above, to (3.8). From definition we have

\[
b^a J^\alpha_{a, \rho} ((b-a)^\rho - (b-s)^\rho)^{\mu-1} (x) = \frac{1}{\Gamma(\alpha)} \int_a^x ((b-a)^\rho - (b-s)^\rho)^{\mu-1} \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} \, ds.
\]  

(3.10) 

By using the substitution \( u = \frac{(b-a)^\rho - (b-s)^\rho}{(b-a)^\rho - (b-x)^\rho} \), we see that

\[
b^a J^\alpha_{a, \rho} ((b-a)^\rho - (b-s)^\rho)^{\mu-1} (x) = \int_0^1 u^{\mu-1} (1-u)^{\alpha-1} \frac{[(b-a)^\rho - (b-x)^\rho]^{\mu+\rho-1}}{\Gamma(\alpha) \rho^\rho} \, du.
\]  

(3.11) 

and hence (3.8) is proved. \( \square \)

**Theorem 3.1.** For \( Re(\alpha) > 0, \ Re(\mu) > 0, \ Re(\rho) > 0 \) and \( f \) defined on \([a, b]\), we have

\[
b^a J^\alpha_{a, \rho} (b^a J^\mu_{a, \rho} f(x)) = b^a J^{\alpha+\mu, \rho} f(x).
\]  

(3.12)
Lemma 3.2. For \( a^{\alpha \rho} ( a^{\mu \rho}) f(x) = a^{\alpha + \mu \rho} f(x) \).

\( \text{Proof.} \) We prove only the first part. The second part is similar or it follows by the the first part and the action of the \( Q \)-operator in Remark 3.1. From definition, we have

\[
\int_a^x \left( \frac{(b-t)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} \int_a^t \left( \frac{(b-s)^\rho - (b-t)^\rho}{\rho} \right)^{\mu-1} f(s) d\rho(b,s) d\rho(b,t).
\]

By interchanging the order of the integrals in (3.14) we see that

\[
a^{\alpha \rho} ( a^{\mu \rho}) f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\mu)} \int_a^x f(s) \left( \int_s^x \left( \frac{(b-t)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} \left( \frac{(b-s)^\rho - (b-t)^\rho}{\rho} \right)^{\mu-1} d\rho(b,t) \right) d\rho(b,s).
\]

Using the change of variable \( z = \frac{(b-s)^\rho - (b-x)^\rho}{(b-s)^\rho - (b-x)^\rho} \), we conclude that

\[
\int_0^1 z^{\mu-1}(1-z)^{\alpha-1} dz = B(\alpha,\mu) = \frac{\Gamma(\alpha)\Gamma(\mu)}{\Gamma(\alpha+\mu)}.
\]

Upon Definition 3.2 above, we present the following new class of generalized fractional differential operators with kernels depending on the boundary points.

**Definition 3.3.** (Mixed left and right conformable fractional derivatives in Riemann-Liouville setting)

For a function \( f \) defined on \([a, b]\), \( Re(\alpha) > 0 \), \( \rho > 0 \), \( n = [Re(\alpha)] + 1 \), we have

- The mixed left \( RL \)-conformable fractional derivative is defined by
  \[
a^{\alpha \rho}_a D^{n \alpha \rho} f(x) = \frac{(-1)^n a^{T^\rho}}{\Gamma(n-\alpha)} \int_a^x \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{n-\alpha-1} f(s) d\rho(b,s).
\]

- The mixed right \( RL \)-conformable fractional derivative is defined by
  \[
  a^{\alpha \rho}_b D^{n \alpha \rho} f(x) = \frac{(-1)^n a^{T^\rho}}{\Gamma(n-\alpha)} \int_x^b \left( \frac{(s-a)^\rho - (x-a)^\rho}{\rho} \right)^{n-\alpha-1} f(s) d\rho(s,a).
\]

**Lemma 3.2.** For \( Re(\alpha) \geq 0 \), \( Re(\mu) > 0 \) and \( \rho > 0 \) we have

\[
b^{\alpha \rho}_a D^{(b-a)^\rho - (b-s)^\rho \mu - 1} f(x) = \frac{\rho^\mu \Gamma(\mu)}{\Gamma(\mu - \alpha)} \left( \frac{(b-a)^\rho - (b-x)^\rho}{b-a} \right)^{\mu - \alpha - 1},
\]

(3.17)
If we multiply both sides of (3.22) by \( \psi \) where \( \psi \) is a function, we can proceed inductively by multiplying by \( \psi \) if and only if \( f \) has the representation

\[
a_D^{\alpha,\rho} ((b-a)^\rho - (s-a)^\rho)^{\mu-1} (x) = \frac{\rho^n \Gamma(n)}{\Gamma(n-\alpha)} ((b-a)^\rho - (x-a)^\rho)^{\mu-\alpha-1}.
\]

(3.18)

**Proof.** We prove (3.17) for \( 0 < \alpha < 1 \) and the other details follow straightforwardly by induction. Indeed, from definition we have

\[
b_D^{\alpha,\rho} ((b-a)^\rho - (b-s)^\rho)^{\mu-1} (x)
\]

\[
= (-T_0^b) b_D^{1-\alpha,\rho} ((b-a)^\rho - (b-s)^\rho)^{\mu-1} (t)
\]

\[
= (b-t)^{1-\rho} \frac{d}{dt} b_D^{1-\alpha,\rho} ((b-a)^\rho - (b-s)^\rho)^{\mu-1} (t).
\]

(3.19)

Then, the proof is completed by applying (3.8) in Lemma 3.1 with \( 1 - \alpha \) instead of \( \alpha \) and then differentiating. \( \square \)

**Remark 3.2.** It can be shown that for any \( \alpha \) and \( \rho > 0 \) we have

\[
b_D^{\alpha,\rho} f = b_T^{\alpha,\rho} f, \quad a_D^{\alpha,\rho} f = a_T^{\alpha,\rho} f
\]

(3.20)

**Lemma 3.3.** Let \( \rho > 0 \) and \( n \in \mathbb{N} \). Then a function \( f \in a AC_b^{\alpha,\rho} [a,b] \) if and only if \( f \) has the representation

\[
f(x) = \sum_{k=0}^{n-1} (-1)^k b_T^{\rho} f(b) \frac{\rho^{n-\alpha-1} x^{\rho}}{k!} + a_D^{\alpha,\rho} \psi(x),
\]

where \( \psi(t) = -\frac{\alpha}{\rho} T^\rho(t) \).

Proof. Let \( f \in a AC_b^{\alpha,\rho} [a,b] \), then \( n^{-1} T^\rho f \in a AC_b [a,b] \) and hence

\[
n^{-1} T^\rho f(x) = c_4 + \int_x^b \psi(t) d_{\rho}(t,a), \quad c_4 = n^{-1} T^\rho f(b), \quad \psi \in a X^{\infty}.
\]

(3.22)

If we multiply both sides of (3.22) by \( (x-a)^{\rho-1} \) then we have

\[
\frac{d}{dx} n^{-2} T^\rho f(x) = c_4 (x-a)^{\rho-1} + (x-a)^{\rho-1} \int_x^b \psi(t) d_{\rho}(t,a).
\]

(3.23)

Integrating both sides of (3.23) with changing the order of integration will lead to

\[
a^{-2} T^\rho f(x) = a^{-2} T^\rho f(b) - c_4 \frac{(b-a)^{\rho} - (x-a)^{\rho}}{\rho} - \int_x^b \psi(t) \left( \frac{(t-a)^{\rho} - (x-a)^{\rho}}{\rho} \right) d_{\rho}(t,a).
\]

(3.24)

If we proceed inductively by multiplying by \( (x-a)^{\rho-1} \), integrating and changing the order of integration we obtain the representation in (3.21). Sufficiency can be proved by applying the operator \( a T^\rho \) to (3.21) \( n \) times. \( \square \)

**Lemma 3.4.** Let \( \rho > 0 \) and \( n \in \mathbb{N} \). Then a function \( f \in b AC_b^{\alpha,\rho} [a,b] \) if and only if \( f \) has the representation

\[
f(x) = \sum_{k=0}^{n-1} (-1)^k b_T^{\rho} f(a) \frac{\rho^{n-\alpha-1} x^{\rho}}{k!} + a_D^{\alpha,\rho} \psi(x),
\]

where \( \psi(t) = n^{-1} T^\rho(t) \).

(3.25)
Proof. If \( f \in \mathring{\mathcal{D}}_{a}^{\gamma,\alpha}[a,b] \) then
\[
 n^{-1}T_{b}^{\rho}f(x) = c_{3} + \int_{a}^{x} \psi(t) d_{\rho}(b,t), \ c_{3} = n^{-1}T_{b}^{\rho}f(a), \ \psi(t) \in X_{b}^{c,1}. \tag{3.26}
\]
Then, we proceed similar to the proof of Lemma 3.3 but by multiplying by \(-(b-x)^{\rho-1}\) and integrating from \(a\) to \(x\).

\[\blacksquare\]

**Remark 3.3.** The spaces \(\mathring{\mathcal{D}}_{a}^{\gamma,\alpha}[a,b\) and \(\mathring{\mathcal{D}}_{b}^{\gamma,\alpha}[a,b]\) were characterized in Lemma 3.1 and Lemma 3.2 in \([30]\), respectively.

**Definition 3.4.** (Mixed left and right conformable fractional derivatives in Caputo setting)

For a function \( f \) defined on \([a,b]\), \( Re(\alpha) > 0, \ \rho > 0, \ n = [Re(\alpha)] + 1 \), we have

- The mixed left Caputo conformable fractional derivative is defined by

\[
 C_{a}^{\cdot,\cdot}D_{a}^{\alpha,\rho}f(x) = b_{a}^{\gamma^{-\alpha,\rho}}(-T_{a}^{\rho})^{n}f(x)
\]

\[
 = \left(\frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{a}^{x} \left(\frac{b-s)\rho - (b-x)\rho x}{\rho}\right)^{n-\alpha-1} \right) \cdot nT_{b}^{\rho}f(s) d_{\rho}(b,s). \tag{3.29}
\]

- The mixed right Caputo conformable fractional derivative is defined by

\[
 C_{b}^{\cdot,\cdot}D_{b}^{\alpha,\rho}f(x) = a_{b}^{\gamma^{-\alpha,\rho}}(-aT^{\rho})^{n}f(x)
\]

\[
 = \left(\frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} \left(\frac{(s-a)\rho - (x-a)\rho}{\rho}\right)^{n-\alpha-1} \right) \cdot aT^{\rho}f(s) d_{\rho}(s,a). \tag{3.30}
\]

**Remark 3.4.** Using similar arguments to that in Theorem 4.1 in \([30]\), we have the following equivalent definitions to \( C_{a}^{\cdot,\cdot}D_{a}^{\alpha,\rho}f(x) \) and \( C_{b}^{\cdot,\cdot}D_{b}^{\alpha,\rho}f(x) \) for a function \( f \in \mathring{\mathcal{D}}_{a}^{\gamma,\alpha}[a,b] \) and \( f \in \mathring{\mathcal{D}}_{b}^{\gamma,\alpha}[a,b] \), respectively.

\[
 C_{a}^{\cdot,\cdot}D_{a}^{\alpha,\rho}f(x) = b_{a}^{\gamma^{-\alpha,\rho}} \left[ f(t) - \sum_{k=0}^{n-1} \left(\frac{(-1)^{k}kT_{a}^{\rho}f(a)}{k!}\right) \right] (x), \tag{3.27}
\]

and

\[
 C_{b}^{\cdot,\cdot}D_{b}^{\alpha,\rho}f(x) = a_{b}^{\gamma^{-\alpha,\rho}} \left[ f(t) - \sum_{k=0}^{n-1} \left(\frac{(-1)^{k}kT_{b}^{\rho}f(b)}{k!}\right) \right] (x). \tag{3.28}
\]

**Lemma 3.5.** For \( Re(\alpha) > 0, \ Re(\mu) > n = [Re(\alpha)] + 1 \) we have

\[
 C_{a}^{\cdot,\cdot}D_{a}^{\alpha,\rho} \left( (t-a)\rho^{(\alpha-1)} \right) (x) = \frac{\rho^{\alpha}\Gamma(\mu)}{\Gamma(\mu-\alpha)}(t-a)^{(\rho-\alpha-1)}, \tag{3.29}
\]

\[
 C_{b}^{\cdot,\cdot}D_{b}^{\alpha,\rho} \left( (b-t)\rho^{(\mu-1)} \right) (x) = \frac{\rho^{\alpha}\Gamma(\mu)}{\Gamma(\mu-\alpha)}(b-t)^{(\rho-\alpha-1)}, \tag{3.30}
\]

and for \( k = 0, 1, 2, ..., n-1 \) we have

\[
 C_{a}^{\cdot,\cdot}D_{a}^{\alpha,\rho} \left( (t-a)^{\rho k} \right) (x) = 0, \ C_{b}^{\cdot,\cdot}D_{b}^{\alpha,\rho} \left( (b-t)^{\rho k} \right) (x) = 0. \tag{3.31}
\]

In particular, \( C_{a}^{\cdot,\cdot}D_{a}^{\alpha,\rho}1(x) = (C_{b}^{\cdot,\cdot}D_{b}^{\alpha,\rho}1)(x) = 0. \)

**Proof.** The proof follows by definition and applying Lemma 2.2 in \([30]\) or Lemma 1.4 in this article. \hfill \square

Similarly, for the mixed Caputo type case we have the following.
Lemma 3.6. For $Re(\alpha) > 0$, $Re(\mu) > n = [Re(\alpha)] + 1$ we have
\[
C_a b \mathcal{D}_{\alpha,\rho} ( (b - a)\rho - (b - s)\rho )^{\mu - 1} (x) = \frac{\rho^\mu \Gamma(\mu)}{\Gamma(\mu - \alpha)} ( (b - a)^\rho - (b - x)^\rho )^{\mu - \alpha - 1},
\]
(3.32)
\[
C_a b \mathcal{D}_{\alpha,\rho} ( (b - a)\rho - (s - a)\rho )^{\mu - 1} (x) = \frac{\rho^\mu \Gamma(\mu)}{\Gamma(\mu - \alpha)} ( (b - a)^\rho - (x - a)^\rho )^{\mu - \alpha - 1},
\]
(3.33)
and for $k = 0, 1, 2, \ldots, n - 1$ we have
\[
C_a b \mathcal{D}_{\alpha,\rho} \left( ((b - a)\rho - (b - s)\rho)^k \right)(x) = 0, \quad C_a b \mathcal{D}_{\alpha,\rho} \left( ((b - a)\rho - (s - a)\rho)^k \right)(x) = 0.
\]
(3.34)
In particular, \((C_a b \mathcal{D}_{\alpha,\rho})^1(x) = (C_a b \mathcal{D}_{\alpha,\rho})^1(x) = 0\).

Proof. The proof follows from definition and by applying Lemma 3.1.

Remark 3.5. If we proceed, we can prove many other analogous properties for the mixed left and right conformable fractional integrals as it was proved for left and right ones in [30]. For example, see Theorem 3.2, Theorem 3.4, Corollary 3.5, Theorem 3.6, Lemma 4.2, Lemma 4.3, Theorem 4.4. We leave these details for researchers who are interested. We just below prove the mixed conformable fractional analogue of Theorem 3.7 and Theorem 4.5 from [30] that will be essential to mixed conformable fractional dynamical systems in the Riemann and Caputo settings.

Theorem 3.2. For $\rho > 0$, $Re(\alpha) > 0$, $n = [Re(\alpha)] + 1$ and for a function $f \in AC^{n,\rho}[a, b]$ \((f \in AC_b^{n,\rho}[a, b])\) respectively, we have
\[
b_a C \mathcal{D}_{\alpha,\rho} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k T_a^\rho f(a)}{k!} \left( \frac{(b - a)^\rho - (b - x)^\rho}{\rho} \right)^k, \quad (3.35)
\]
and
\[
a_b C \mathcal{D}_{\alpha,\rho} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k T_a^\rho f(b)}{k!} \left( \frac{(b - a)^\rho - (t - a)^\rho}{\rho} \right)^k. \quad (3.36)
\]

Proof. The proof follows by Theorem 3.1, the definition of the Caputo mixed conformable fractional derivatives and integration by parts. For example, to prove (3.35) we have
\[
b_a C \mathcal{D}_{\alpha,\rho} f(x) = b_a C \mathcal{D}_{\alpha,\rho} f(x) = b_a C \mathcal{D}_{\alpha,\rho} \mathcal{D}_{\alpha,\rho} (-1)^n T_a^\rho f(x) = b_a C \mathcal{D}_{\alpha,\rho} (-1)^n T_a^\rho f(x), \quad (3.37)
\]
and hence by integration by parts $n$ times (3.35) is proved. The proof of (3.36) is similar.

The following theorem relates Riemann and Caputo mixed conformable fractional derivatives.

Theorem 3.3. Let $\Re(\alpha) \geq 0$ and $n = [\Re(\alpha)] + 1$. If $f \in AC^{n,\rho}[a, b]$ \((f \in AC_b^{n,\rho}[a, b])\), where $0 < a < b < \infty$. Then,
\[
C_a b \mathcal{D}_{\alpha,\rho} f(x) = b_a C \mathcal{D}_{\alpha,\rho} f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k T_a^\rho f(a)}{\Gamma(k + 1 - \alpha)} \left( \frac{(b - a)^\rho - (b - x)^\rho}{\rho} \right)^{k-\alpha}, \quad (3.38)
\]
and
\[
C_{\alpha} D_b^{\alpha, \rho} f(x) = a D_b^{\alpha, \rho} f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k k^\rho f(b)}{\Gamma(k+1-\alpha)} \left( \frac{(b-a)^\rho - (t-a)^\rho}{\rho} \right)^{k-\alpha}.
\]

(3.39)

Proof. The proof follows by (3.27) and (3.28) and Lemma 3.2. Also, the proof can be achieved by Lemma 3.7 below.

Lemma 3.7. Let \( \zeta^\rho f(t) = -T^\rho f(t) = (b-t)^{1-\rho} f'(t) \) and \( n \zeta^\rho f(t) \) is the composition of \( \zeta^\rho \) \( n \)-times. Also, let \( \eta^\rho f(x) = -a T^\rho f(x) = -(x-a)^{1-\rho} f'(x) \) and \( n \eta^\rho f(t) \) is the composition of \( \eta^\rho \) \( n \)-times. Then, for any \( \rho > 0 \), and \( \alpha \in \mathbb{R} \) we have
\[
b a \gamma^\alpha, \rho n \zeta^\rho f(x) = n \zeta^\rho n \gamma^\alpha, \rho f(x) - \sum_{k=0}^{n-1} \frac{k \zeta^\rho f(a)}{\Gamma(\alpha + k - n + 1)} \left( \frac{(b-a)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-n+k}, \tag{3.40}
\]
and
\[
a \gamma^\alpha, \rho n \eta^\rho f(x) = n \eta^\rho a \gamma^\alpha, \rho f(x) - \sum_{k=0}^{n-1} \frac{k \eta^\rho f(b)}{\Gamma(\alpha + k - n + 1)} \left( \frac{(b-a)^\rho - (a-x)^\rho}{\rho} \right)^{\alpha-n+k}. \tag{3.41}
\]

Proof. For \( \alpha > 0 \) and \( n = 1 \) we first have
\[
b a \gamma^\alpha, \rho n \zeta^\rho f(x) = \frac{1}{\Gamma(\alpha)} \int_a^b \left[ \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} f(s) \right] ds.
\]

On the other hand, we have
\[
\zeta^\rho b a \gamma^\alpha, \rho f(x) = -\frac{1}{\Gamma(\alpha)} \int_a^b \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} f(s) ds.
\]

From which it follows that
\[
b a \gamma^\alpha, \rho n \zeta^\rho f(x) = \zeta^\rho b a \gamma^\alpha, \rho f(x) - \frac{1}{\Gamma(\alpha)} \left( \frac{(b-a)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} f(a). \tag{3.42}
\]

Then (3.40) follows by applying (3.42) recursively \( n \) times through making use of the observation that \( \zeta^\rho \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-1} = (\alpha-1) \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{\alpha-2} \). Finally, by the help of the identity
\[
n \zeta^\rho \left( \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \right)^{n-\alpha-1} = \frac{(b-s)^\rho - (b-x)^\rho}{\rho} \frac{1}{\Gamma(n-\alpha)},
\]
and Remark 3.2 we see that (3.40) is valid for any \( \alpha \in \mathbb{R} \). The \( \alpha = 0 \) case is trivial since dividing by a ball leads to zero. The proof of (3.41) is analogous. \( \square \)
Theorem 3.4. For $\text{Re}(\alpha) > 0$, $n = -[\text{Re}(\alpha)]$ and $f \in X^1_{b,c}$, $b\mathcal{J}_a^n - \alpha,\rho \in b\mathcal{AC}_{a,b}^n, \mathcal{J}_a^n - \alpha,\rho \in \mathcal{AC}_{a,b}^n$ [a, b] (f \in aX^{1,c}$, $a\mathcal{J}_b^n - \alpha,\rho \in a\mathcal{AC}_{b,a}^n$, $a\mathcal{J}_b^n - \alpha,\rho \in \mathcal{AC}_{b,a}^n$, $a\mathcal{J}_b^n - \alpha,\rho \in \mathcal{AC}_{b,a}^n$) we have

\[
b\mathcal{J}_a^n - \alpha,\rho f(x) = f(x) - \sum_{k=1}^{n-1} \frac{b\mathcal{J}_a^n - \alpha,\rho f(a_k^+) (b-a) - (b-x)^\rho}{k + \alpha - k + 1} \rho^{-1} \]

and

\[
a\mathcal{J}_b^n - \alpha,\rho f(x) = f(x) - \sum_{k=1}^{n-1} \frac{a\mathcal{J}_b^n - \alpha,\rho f(b_k^-) (b-a) - (b-x)^\rho}{k + \alpha - k + 1} \rho^{-1} \]

Proof. By the help of Lemma 3.7, and that $k\mathcal{J}_a^n - \alpha,\rho f(x) = b\mathcal{J}_a^n - \alpha,\rho f(x)$ we have

\[
b\mathcal{J}_a^n - \alpha,\rho a\mathcal{J}_b^n - \alpha,\rho f(x) = \]

\[
b\mathcal{J}_a^n - \alpha,\rho n\mathcal{J}_b^n - \alpha,\rho f(x) = \]

\[
f(x) - \sum_{k=1}^{n-1} \frac{k\mathcal{J}_a^n - \alpha,\rho f(a_k^+) (b-a) - (b-x)^\rho}{k + \alpha - k + 1} \rho^{-1} \]

\[
f(x) - \sum_{k=1}^{n-1} \frac{a\mathcal{J}_b^n - \alpha,\rho f(a_k^+) (b-a) - (b-x)^\rho}{k + \alpha - k + 1} \rho^{-1} \]

The proof of (3.44) is analogous. $\square$

4. Integration by parts for conformable fractional operators. The following integration by parts formula for left and right conformable derivatives was proved in [10].

Proposition 4.1. Let $f, g : [a, b] \to \mathbb{R}$ be functions and $0 < \rho \leq 1$. Then

\[
\int_a^b (a\mathcal{J}_b^n - \alpha,\rho f(t))g(t)d_\rho(t, b, t) = \int_a^b f(t)(a\mathcal{J}_b^n - \alpha,\rho g(t))d_\rho(t, a, b). \tag{4.1}
\]

By means of the mixed left and right fractional integrals we can prove the following integration by parts statements.

Proposition 4.2. Let $f, g : [a, b] \to \mathbb{R}$ be functions and $0 < \rho \leq 1$. Then

\[
\int_a^b g(t)(a\mathcal{J}_b^n - \alpha,\rho f(t))d_\rho(t, b, a) = \int_a^b f(t)(a\mathcal{J}_b^n - \alpha,\rho g(t))d_\rho(t, a, b). \tag{4.2}
\]

Proof. The proof is straightforward and follows by interchanging the order of integrals and using the definition of (mixed) left and (mixed) right conformable integrals.

Next, we shall generalize the integration by parts formulas given in Proposition 4.1 and Proposition 4.2 to (mixed) left and right conformable fractional integrals. In fact, we shall see that Proposition 4.2 is better generalized so that it will be used to prove integration by parts formula for the (mixed) conformable fractional derivatives.
Lemma 4.1. For a function $f$ defined on $[a, b]$ and $\alpha > 0$, $\rho > 0$ we have

$$
\frac{b}{a} \mathcal{G}_\alpha^\rho \mathcal{G}_\rho f(t) = \frac{1}{\Gamma(\alpha + 1)} \int_a^t \left( \frac{(b-s)^\rho - (b-t)^\rho}{\rho} \right)^\alpha f(s) d_\rho(s, a). \tag{4.4}
$$

Proof. From definition we have

$$
\frac{b}{a} \mathcal{G}_\alpha^\rho \mathcal{G}_\rho f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \int_a^t f(s) d_\rho(s, a) \right) \left[ \frac{(b-t)^\rho - (b-x)^\rho}{\rho} \right]^{\alpha-1} d_\rho(b, t). \tag{4.5}
$$

By interchanging the order of integrals in (4.5), we have

$$
\frac{b}{a} \mathcal{G}_\alpha^\rho \mathcal{G}_\rho f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x f(s) \left( \int_s^x \left[ \frac{(b-t)^\rho - (b-x)^\rho}{\rho} \right]^{\alpha-1} d_\rho(b, t) \right) d_\rho(s, a). \tag{4.6}
$$

Finally, (4.4) follows by evaluating the inner integral of (4.6) using the substitution $u = \frac{(b-t)^\rho - (b-x)^\rho}{\rho}$ and the proof is completed. \qed

Proposition 4.3. For $\rho > 0$ and $\alpha > 1$ we have

$$
\int_a^b g(x)(\mathcal{G}_\rho^\alpha f)(x) d_\rho(x, a) = \int_a^b f(x)(\mathcal{G}_\rho^\alpha f)(x) d_\rho(x, a). \tag{4.7}
$$

Proof. From definition we have

$$
\int_a^b g(x)(\mathcal{G}_\rho^\alpha f)(x) d_\rho(x, a) = \int_a^b g(x) \left( \int_a^x \left[ \frac{(b-t)^\rho - (b-x)^\rho}{\rho} \right]^{\alpha-1} f(t) d_\rho(b, t) \right) d_\rho(x, a). \tag{4.8}
$$

Then, the result follows by interchanging the order of the integrals and making use of Lemma 4.1. \qed

Remark 4.1. Note that if we let $\alpha \to 1^+$ in Proposition 4.3 then Proposition 4.1 is reobtained.

In order to prove an integration by part formula for the generalized type conformable fractional derivatives and integrals we introduce the following function spaces: For $p \geq 1$ and $\alpha > 0$, we define

$$
a \mathcal{G}^{\alpha, \rho}(X_p) = \{ f : f = a \mathcal{G}_\alpha^\rho \varphi, \quad \varphi \in a X^{c, p}(a, b) \}. \tag{4.9}
$$

and

$$
b \mathcal{G}^{\alpha, \rho}(X_p) = \{ f : f = b \mathcal{G}_\alpha^\rho \psi, \quad \psi \in X^{c, p}_b(a, b) \}. \tag{4.10}
$$

Theorem 4.1. (Integration by parts for mixed conformable fractional operators)

Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). Then

- If $\varphi(t) \in X^{c, p}_b(a, b)$ and $\psi(t) \in X^{c, q}_b(a, b)$, then

$$
\int_a^b \varphi(t)(\mathcal{G}_b^\alpha \psi)(t) d_\rho(b, t) = \int_a^b \psi(t)(\mathcal{G}_b^\alpha \varphi)(t) d_\rho(b, t).
$$
• If \( f(t) \in {}_a\mathcal{J}^{\alpha,\rho}(X_p) \) and \( g(t) \in {}_b\mathcal{J}^{\alpha,\rho}(X_q) \), then
\[
\int_a^b g(t) ( {}_a\mathcal{D}^{\alpha,\rho} f)(t) d\rho(b,t) = \int_a^b ( {}_b\mathcal{D}^{\alpha,\rho} g)(t) f(t) d\rho(b,t)
\]

**Proof.**  
- From the definition and by interchanging the order of integrals we have
\[
\int_a^b \varphi(x)(\mathcal{J}^{\alpha,\rho}_b \psi)(x) d\rho(b,x) = \int_a^b \left( \frac{1}{\Gamma(\alpha)} \int_x^b \left[ \frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right]^{\alpha-1} \psi(t) d\rho(b,t) \right) \times \varphi(x) d\rho(b,x)
\]
\[
= \int_a^b \left( \frac{1}{\Gamma(\alpha)} \int_a^t \left[ \frac{(b-x)^\rho - (b-t)^\rho}{\rho} \right]^{\alpha-1} \varphi(x) d\rho(b,x) \right) \times \psi(t) d\rho(b,t)
\]
\[
= \int_a^b \psi(t)(\mathcal{J}^{\alpha,\rho}_b \varphi)(t) d\rho(b,t).
\]

- From definition and the first part we have
\[
\int_a^b g(t)( {}_a\mathcal{D}^{\alpha,\rho} f)(t) d\rho(b,t) = \int_a^b {}_a\mathcal{D}^{\alpha,\rho} \mathcal{J}^{\alpha,\rho}_b \psi(t)(\mathcal{J}^{\alpha,\rho}_b \varphi)(t) d\rho(b,t)
\]
\[
= \int_a^b \varphi(t)(\mathcal{J}^{\alpha,\rho}_b \psi)(t) d\rho(b,t)
\]
\[
= \int_a^b ( {}_b\mathcal{D}^{\alpha,\rho} g)(t) f(t) d\rho(b,t).
\]

We have used the facts that \( {}_a\mathcal{D}^{\alpha,\rho} \mathcal{J}^{\alpha,\rho}_b \psi(t) = \psi(t) \) and \( {}_a\mathcal{D}^{\alpha,\rho} {}_b\mathcal{J}^{\alpha,\rho} \varphi(t) = \varphi(t) \).

\[\square\]

Analogously, by introducing the following function spaces
\[
\mathcal{J}^{\alpha,\rho}_b(X_p) = \{ f : f = \mathcal{J}^{\alpha,\rho}_b \varphi, \ \varphi \in \mathcal{X}^{\alpha,\rho}_c(a,b) \}, \quad (4.11)
\]
and
\[
\mathcal{J}^{\alpha,\rho}_b(X_p) = \{ f : f = \mathcal{J}^{\alpha,\rho}_b \psi, \ \psi \in \mathcal{X}^{\alpha,\rho}_c(a,b) \}, \quad (4.12)
\]
we can state the following integration by parts.

**Theorem 4.2.** Let \( \alpha > 0, p \geq 1, q \geq 1, \) and \( \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha \) (\( p \neq 1 \) and \( q \neq 1 \) in the case \( \frac{1}{p} + \frac{1}{q} = 1 + \alpha \) ). Then

• If \( \varphi(t) \in {}_a\mathcal{X}^{\alpha,\rho}(a,b) \) and \( \psi(t) \in {}_a\mathcal{X}^{\alpha,\rho}(a,b) \), then
\[
\int_a^b \varphi(t)( {}_a\mathcal{D}^{\alpha,\rho} \psi)(t) d\rho(t,a) = \int_a^b \psi(t)( {}_a\mathcal{D}^{\alpha,\rho} \varphi)(t) d\rho(t,a).
\]

• If \( f(t) \in \mathcal{J}^{\alpha,\rho}_b(X_p) \) and \( g(t) \in {}_a\mathcal{J}^{\alpha,\rho}_b(X_q) \), then
\[
\int_a^b g(t)(\mathcal{D}^{\alpha,\rho}_b f)(t) d\rho(t,a) = \int_a^b ( {}_a\mathcal{D}^{\alpha,\rho} g)(t) g(t) d\rho(t,a)
\]

Below we present integration by parts formula for functions in the spaces \( {}_a\mathcal{AC}^{\alpha,\rho}[a,b], \ {}_a\mathcal{AC}^{\alpha,\rho}[a,b], \ {}_a\mathcal{AC}^{\alpha,\rho}[a,b] \) and \( {}_a\mathcal{AC}^{\alpha,\rho}_b[a,b] \).
Theorem 4.3. Let $\alpha > 0, n = [\alpha] + 1$ and $f \in X_b^{c,p}(a,b)$ and $g \in AC_b^{n,\rho}[a,b]$. Then

$$
\int_a^b f(t)(C\mathcal{D}_b^{\alpha,\rho}g)(t) d\rho(b,t) = \int_a^b g(t)(\frac{b}{a}\mathcal{D}_b^{\alpha,\rho}f)(t) d\rho(b,t)
$$

\begin{equation}
- \left( \sum_{k=0}^{n-1} \alpha, \rho \right)^k T_k^\rho f(t) \left( \frac{b}{a} T_k^\rho f(t) \right) \right|_a^b.
\end{equation}

In particular, if $0 < \alpha < 1$ we have

$$
\int_a^b f(t)(C\mathcal{D}_b^{\alpha,\rho}g)(t) d\rho(b,t) = \int_a^b g(t)(\frac{b}{a}\mathcal{D}_b^{\alpha,\rho}f)(t) d\rho(b,t) - \left( \frac{b}{a} T_1^{1,\rho} f(t) g(t) \right) \bigg|_a^b.
$$

Proof. We prove (4.14) and the proof of (4.13) follows inductively. By definition and applying the first part of Theorem 4.1 and the ordinary integration by parts we have

$$
\int_a^b f(t)(C\mathcal{D}_b^{\alpha,\rho}g)(t) d\rho(b,t) = \int_a^b f(t).T_1^{1,\rho}(T_b^\rho g)(t) d\rho(b,t)
$$

$$
= \int_a^b (\frac{b}{a} T_1^{1,\rho} f(t)).g(t) d\rho(b,t)
$$

$$
= \int_a^b (\frac{b}{a} T_1^{1,\rho} f(t)).g'(t) dt
$$

$$
= \int_a^b (\frac{b}{a} T_1^{1,\rho} f(t)) g(t) \bigg|_a^b + \int_a^b g(t)(b - t)^{1,\rho} \frac{d}{dt} (\frac{b}{a} T_1^{1,\rho} f(t)) d\rho(b,t).
$$

$$
= \int_a^b g(t)(\frac{b}{a} \mathcal{D}_b^{\alpha,\rho} f)(t) d\rho(b,t) - \left( \frac{b}{a} T_1^{1,\rho} f(t) g(t) \right) \bigg|_a^b.
$$

Corollary 4.1. Let $\alpha > 0, n = [\alpha] + 1$ and $f \in X_b^{c,p}(a,b)$ and $g \in AC_b^{n,\rho}[a,b]$. Then

$$
\int_a^b f(t)(D_b^{\alpha,\rho}g)(t) d\rho(b,t) =
$$

$$
\int_a^b g(t)(\frac{b}{a} \mathcal{D}_b^{\alpha,\rho} f)(t) d\rho(b,t) + \sum_{k=0}^{n-1} b^{n-k-1} T_k^\rho g(a) \frac{b}{a} T_k^\rho f(a^+).
$$

In particular, if $0 < \alpha < 1$ we have

$$
\int_a^b f(t)(D_b^{\alpha,\rho}g)(t) d\rho(b,t) = \int_a^b g(t)(\frac{b}{a} \mathcal{D}_b^{\alpha,\rho} f)(t) d\rho(b,t) + g(a)(\frac{b}{a} T_1^{1,\rho} f(a^+))
$$

(4.16)

Proof. By (1.47) in Lemma 1.6 and Theorem 4.3 we have

$$
\int_a^b f(t)(D_b^{\alpha,\rho}g)(t) d\rho(b,t) = \int_a^b g(t)(\frac{b}{a} \mathcal{D}_b^{\alpha,\rho} f)(t) d\rho(b,t)
$$

$$
- \left( \sum_{k=0}^{n-1} b^{n-k-1} T_k^\rho g(t)(\frac{b}{a} T_k^\rho f(t)) \right) \bigg|_a^b.
$$

(4.17)
Let Corollary 4.2.

If we notice that
\[ \frac{1}{\Gamma(k - \alpha + 1)} \int_a^b \left( \frac{(b - t)^\alpha}{\rho} \right)^{k-\alpha} f(t) dt \]
then we conclude (4.15). In particular, if \( 0 < \alpha < 1 \) we have
\[ \int_a^b f(t) (D_b^{\alpha, \rho} g)(t) dt \rho(t, t) = \int_a^b g(t) (D_b^{\alpha, \rho} f)(t) dt \rho(t, a) + \int_a^b g(t)(\alpha^1 - \alpha, \rho)f(t) dt |_{t=a}^{t=b}, \]  
(4.18)
and hence (4.16) follows.

Theorem 4.4. Let \( \alpha > 0, n = [\alpha] + 1 \) and \( f \in _aX^{c, \rho}(a, b) \) and \( g \in _aAC^{n, \rho}[a, b] \). Then
\[ \int_a^b f(t) (C_b^{\alpha, \rho} g)(t) dt \rho(t, a) = \int_a^b g(t)(\alpha^1 - \alpha, \rho)f(t) dt \rho(t, a) + (\alpha^1 - \alpha, \rho)f(t) dt |_{t=a}^{t=b}, \]
(4.19)
In particular, if \( 0 < \alpha < 1 \) we have
\[ \int_a^b f(t) (C_b^{\alpha, \rho} g)(t) dt \rho(t, a) = \int_a^b g(t)(\alpha^1 - \alpha, \rho)f(t) dt \rho(t, a) + (\alpha^1 - \alpha, \rho)f(t) dt |_{t=a}^{t=b}, \]
(4.20)
Proof. We prove (4.20) and the proof of (4.19) follows inductively. By definition and applying the first part of Theorem 4.2 and the ordinary integration by parts we have
\[ \int_a^b f(t) (C_b^{\alpha, \rho} g)(t) dt \rho(t, a) = \int_a^b f(t)(\alpha^1 - \alpha, \rho)T^\rho g(t) dt \rho(t, a) \]
\[ = \int_a^b \alpha^1 - \alpha, \rho f(t)(\alpha^1 - \alpha, \rho)T^\rho g(t) dt \rho(t, a) \]
\[ = \int_a^b g(t)(\alpha^1 - \alpha, \rho)f(t) dt \rho(t, a) \]
\[ = \alpha^1 - \alpha, \rho f(t) g(t) |_{t=a}^{t=b} - \int_a^b g(t) \frac{d}{dt} \alpha^1 - \alpha, \rho f(t) dt \]
\[ = \alpha^1 - \alpha, \rho f(t) g(t) |_{t=a}^{t=b} + \int_a^b g(t)(\alpha^1 - \alpha, \rho)T^\rho f(t) dt \rho(t, a) \]
\[ = \int_a^b g(t)(\alpha^1 - \alpha, \rho)T^\rho f(t) dt \rho(t, a) + (\alpha^1 - \alpha, \rho)f(t) g(t) |_{t=a}^{t=b}. \]

Corollary 4.2. Let \( \alpha > 0, n = [\alpha] + 1 \) and \( f \in _aX^{c, \rho}(a, b) \) and \( g \in _aAC^{n, \rho}[a, b] \). Then
\[ \int_a^b f(t)(\alpha^1 - \alpha, \rho)g(t) dt \rho(t, a) = \]
Theorem 4.5. Let first parts of Theorem 4.1 and Theorem 4.2, and Theorem 3.3.
integration by parts without proofs. The proof can be done by making use of the

\[ \int_a^b g(t) \left( C_a^\alpha D_b^{\alpha-\rho} f(t) \right) d_{\rho}(t, a) + \sum_{k=0}^{n-1} \left( \frac{n-k-1}{a} \int_a^b (C_a^\alpha D_b^{\alpha-\rho} f(t)) \right) (\alpha^\gamma_{n-k-\alpha-\rho} f(b^-)) \] (4.21)

In particular, if \( 0 < \alpha < 1 \) we have

\[ \int_a^b f(t) \left( a D_b^{\alpha-\rho} g(t) \right) d_{\rho}(t, a) = 
\int_a^b g(t) \left( a D_b^{\alpha-\rho} f(t) \right) d_{\rho}(t, a) + g(b) \left( a D_b^{\alpha-\rho} f(b^-) \right). \] (4.22)

Proof. From (1.46) in Lemma 1.6 and Theorem 4.4 we have

\[ \int_a^b f(t) \left( a D_b^{\alpha-\rho} g(t) \right) d_{\rho}(t, a) = \int_a^b g(t) \left( a D_b^{\alpha-\rho} f(t) \right) d_{\rho}(t, a) 
+ \left( \sum_{k=0}^{n-1} \left( \frac{(n-k-1)}{a} T_a^{\rho} g(t) \right) \right) (\alpha^\gamma_{n-k-\alpha-\rho} f(t)) \big|_a^b \] (4.23)

\[ + \sum_{k=0}^{n-1} \frac{k T_a^{\rho} g(a)}{\Gamma(k - \alpha + 1)} \int_a^b \left( \frac{(t-a)}{\rho} \right)^{k-\alpha} f(t) d_{\rho}(t, a). \]

If we notice that

\[ \frac{1}{\Gamma(k - \alpha + 1)} \int_a^b \left( \frac{(t-a)^{\rho}}{\rho} \right)^{k-\alpha} f(t) d_{\rho}(t, a) = a D_b^{\alpha-\rho+1} f(a), \]
then we conclude (4.21). In particular, if \( 0 < \alpha < 1 \) we have

\[ \int_a^b g(t) \left( C_a^\alpha D_b^{\alpha-\rho} f(t) \right) d_{\rho}(b, t) = 
\int_a^b g(t) \left( a D_b^{\alpha-\rho} f(t) \right) d_{\rho}(b, t) + \left( a D_b^{\alpha-\rho} f(t) \right) \big|_a^b + g(b) \left( a D_b^{\alpha-\rho} f(b^-) \right) \big|_a^b \] (4.24)

and hence (4.22) follows. \( \square \)

Analogous to what we have proved above, we state the following versions of integration by parts without proofs. The proof can be done by making use of the first parts of Theorem 4.1 and Theorem 4.2, and Theorem 3.3.

**Theorem 4.5.** Let \( \alpha > 0, n = [\alpha] + 1 \) and \( f \in a X_{a}^{\alpha-\rho}(a,b) \) and \( g \in a AC_{a}^{\alpha-\rho}[a,b] \). Then

\[ \int_a^b f(t) \left( C_a^\alpha D_b^{\alpha-\rho} g(t) \right) d_{\rho}(t, a) = \int_a^b g(t) \left( a D_b^{\alpha-\rho} f(t) \right) d_{\rho}(t, a) 
+ \left( \sum_{k=0}^{n-1} (-1)^{n-k} \left( \frac{n-1-k}{a} T_a^{\rho} g(t) \right) \right) (\alpha^\gamma_{n-k-\alpha-\rho} f(t)) \big|_a^b. \] (4.25)

In particular, if \( 0 < \alpha < 1 \) we have

\[ \int_a^b f(t) \left( C_a^\alpha D_b^{\alpha-\rho} g(t) \right) d_{\rho}(t, a) = \int_a^b g(t) \left( a D_b^{\alpha-\rho} f(t) \right) d_{\rho}(t, a) - \left( a D_b^{\alpha-\rho} f(t) \right) \big|_a^b \] (4.26)
Corollary 4.3. Let \( \alpha > 0, n = [\alpha] + 1 \) and \( f \in \mathcal{X}^{\alpha,\rho}(a, b) \) and \( g \in \mathcal{A}^{n,\rho}_{a,b}[a, b] \). Then

\[
\int_{a}^{b} f(t)(\mathcal{A}^{\alpha,\rho}_{a,b}g)(t)d_{\rho}(t, a) = \\
\int_{a}^{b} g(t)(\mathcal{D}^{\alpha,\rho}_{a,b}f)(t)d_{\rho}(t, a) - \sum_{k=0}^{n-1} (-1)^{n-k} \int_{a}^{b} n^{k-1}T_{\rho}^{\alpha}(g)(a)\mathcal{D}^{n-k,\rho,\alpha}_{a,b}f(a^{+}).
\]

In particular, if \( 0 < \alpha < 1 \) we have

\[
\int_{a}^{b} f(t)(\mathcal{A}^{\alpha,\rho}_{a,b}g)(t)d_{\rho}(t, a) = \int_{a}^{b} g(t)(\mathcal{D}^{\alpha,\rho}_{a,b}f)(t)d_{\rho}(t, a) + g(a)(\mathcal{D}^{1-\alpha,\rho}_{a,b}f)(a^{+}).
\]

Theorem 4.6. Let \( \alpha > 0, n = [\alpha] + 1 \) and \( f \in \mathcal{X}^{\alpha,\rho}_{b}(a, b) \) and \( g \in \mathcal{A}^{n,\rho}_{a,b}[a, b] \). Then

\[
\int_{a}^{b} f(t)(\mathcal{C}^{\alpha,\rho}_{a,b}g)(t)d_{\rho}(b, t) = \\
\int_{a}^{b} g(t)(\mathcal{D}^{\alpha,\rho}_{a,b}f)(t)d_{\rho}(b, t) - \sum_{k=0}^{n-1} (-1)^{n-k} \int_{a}^{b} n^{k-1}T_{\rho}^{\alpha}(g)(t)\mathcal{D}^{n-k,\rho,\alpha}_{a,b}f(t))d_{\rho}(a, b). \tag{4.27}
\]

In particular, if \( 0 < \alpha < 1 \) we have

\[
\int_{a}^{b} f(t)(\mathcal{C}^{\alpha,\rho}_{a,b}g)(t)d_{\rho}(b, t) = \int_{a}^{b} g(t)(\mathcal{D}^{\alpha,\rho}_{a,b}f)(t)d_{\rho}(b, t) + \mathcal{D}^{1-\alpha,\rho}_{b,a}f(t)g(t)\bigg|_{a}^{b}. \tag{4.28}
\]

Corollary 4.4. Let \( \alpha > 0, n = [\alpha] + 1 \) and \( f \in \mathcal{X}^{\alpha,\rho}_{b}(a, b) \) and \( g \in \mathcal{A}^{n,\rho}_{a,b}[a, b] \). Then

\[
\int_{a}^{b} f(t)(\mathcal{C}^{\alpha,\rho}_{b,a}g)(t)d_{\rho}(b, t) = \\
\int_{a}^{b} g(t)(\mathcal{D}^{\alpha,\rho}_{b,a}f)(t)d_{\rho}(b, t) - \sum_{k=0}^{n-1} (-1)^{n-k} \int_{a}^{b} n^{k-1}T_{\rho}^{\alpha}(g)(b)\mathcal{D}^{n-k,\rho,\alpha}_{b,a}f(b^{-}) \tag{4.30}.
\]

In particular, if \( 0 < \alpha < 1 \) we have

\[
\int_{a}^{b} f(t)(\mathcal{C}^{\alpha,\rho}_{b,a}g)(t)d_{\rho}(b, t) = \\
\int_{a}^{b} g(t)(\mathcal{D}^{\alpha,\rho}_{b,a}f)(t)d_{\rho}(b, t) + g(b)\mathcal{D}^{1-\alpha,\rho}_{b,a}f(b^{-}). \tag{4.31}
\]

5. Conclusions. We have made comparisons between generalized fractional integrals and derivatives in the sense of Katugampola and fractional operators generated by conformable integrals weighted by \((t-a)^{\rho-1}\) and \((b-t)^{\rho-1}\). In fact, the generated conformable fractional operators have kernels depending on the boundary points \(a\) and \(b\) while Katugampola ones do not depend. This, brings new trend in fractional calculus whereas the kernels of the left and right fractional operators are boundary-point free. We have realized that the two types, in particular, are very different in the integration by parts. In order to formulate versions for the integration by parts in case of conformable fractional operators we needed to define and analyze new class of mixed left and right conformable fractional operators. We believe that left and right conformable fractional operators and the new defined mixed ones together with their integration by parts will play an important role in modelling and will
open the door to study new types of fractional variational problems and fractional Sturm-Liouville problems.

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*E-mail address: tabdeljawad@psu.edu.sa*