Research Article

Upper and Lower Bounds for the Kirchhoff Index of the $n$-Dimensional Hypercube Network

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1. Introduction

Network is usually modelled by a connected graph $G = (V_G, E_G)$ with order $n$, labeled as $V_G = \{v_1, v_2, \ldots, v_n\}$ and $E_G = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A(G)$ of $G$ is a square matrix with $n$ vertices, in which elements $a_{ij}$ are 1 or 0, depending on whether there is an edge or not between vertices $i$ and $j$. The degree diagonal matrix of $G$ is denoted by $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$, where $d_1, d_2, \ldots, d_n$ are the degree of vertices $v_1, v_2, \ldots, v_n$, respectively. Together with the adjacency and degree matrix, one arrives at the Laplacian matrix, whose expression can be written as $L(G) = D(G) - A(G)$. For other notations and graph theoretical terminologies that not state here, we follow [1].

Various parameters are always used to characterize and describe the complex networks of which the fundamental one is named as the distance $d_{ij}$, concerned as the shortest path between the vertices $i$ and $j$ in networks. Similarly considering the distance $d_{ij}$, Klein and Randić in 1993 presented a novel distance function, named as resistance distance [2]. Denote $r_{ij}$ the resistance distance between two arbitrary vertices $i$ and $j$ in electrical networks by replacing every edge by a unit resistor [3–7]. The Kirchhoff index $\text{Kf}(G)$ of networks is defined as

$$\text{Kf}(G) = \sum_{i < j} r_{ij}(G).$$

The Kirchhoff index has attracted more and more attentions due to its practical applications in the fields of physical interpretations, electric circuit, and so on [8–11]. The Kirchhoff index of some product graphs, join graphs, and corona graphs were studied [5, 7]. The more results of the applications on the Kirchhoff index were explored in [12–14].

In what follows, the rest of the context is summarized. Section 2 proposes the main definition and preliminaries in our discussion. Some bounds on the Kirchhoff index of hypercubes $Q_n$ are deduced in Section 3. We conclude the paper in Section 4.

2. Definition and Preliminaries

In this section, we recall some basic definition in graph theory. The hypercube network $Q_n$ may be constructed from the family of subsets of a set with a binary string of length $n$, by making a vertex for each possible subset and joining two vertices by an edge whenever the corresponding subsets differ in a single binary string. The hypercube network $Q_n$ admits several definitions of which one is stated as below [15].

The hypercube network $Q_n$ is repeatedly constructed by making two copies of $Q_{n-1}$, written as $Q_{n-1}^0$ and $Q_{n-1}^1$, respectively. Meanwhile, adding repeatedly $2^{n-1}$ edges as below, let $V(Q_{n-1}^0) = \{0U = 0u_1u_2 \ldots u_n; \ u_i = 0 \text{ or } 1\}$ and $V(Q_{n-1}^1) = \{1V = 1v_1v_2 \ldots v_n; \ v_i = 0 \text{ or } 1\}$. A node $0U = 0u_1u_2 \ldots u_n$ of
\(Q_n^0\) is linked to another node \(1V = v_1v_2 \ldots v_n\) of \(Q_{n-1}\) if and only if \(u_i = v_i\) for each \(i, 2 \leq i \leq n\).

The hypercube network \(Q_n\) obtained more and more admirable concentrations due to its surprising properties, for instance, symmetry, regular structure, strong connectivity, small diameter, and so on [16, 17]. For more results on the hypercube network and its applications, see [18–21].

Next, we recall the formula for the Kirchhoff index in the hypercube \(Q_n\) with \(n \geq 2\).

**Theorem 1** (see [3]). For the hypercube network \(Q_n\) with \(n \geq 2\),

\[
Kf(Q_n) = 2^n \sum_{i=1}^{n} \binom{n}{i} \frac{1}{2^i}.
\]

(2)

where \(2i(i=1, \ldots, n)\) is the eigenvalue of the Laplacian matrix of the hypercube network and the binomial coefficients \(\binom{n}{i}\) are the multiplicities of the eigenvalues \(2i\).

**Theorem 2** (see [22]).

\[
\lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{n}{2^i (n-i)} = 2.
\]

(3)

The authors of [23] obtained a closed-form formula for the Kirchhoff index of the \(d\)-dimensional hypercube and found the asymptotic value \(2^{d/2}\) by using probabilistic tools. The result of Theorem 3 is obtained by directly calculating the eigenvalues of the Laplacian matrix of the hypercube network, which is different from the technique in [23].

### 3. Main Results

In this section, one will estimate the Kirchhoff index of \(n\)-dimensional hypercube, i.e., our goal is to estimate the quantity:

\[
2^n \sum_{i=1}^{n} \binom{n}{i} \frac{1}{2^i}.
\]

(4)

**Theorem 3.** For the hypercube network \(Q_n\) with \(n \geq 2\),

\[
\frac{4^n}{n} \left( \frac{n}{n+1} \right) - \frac{n(n+2)}{2^{n+1}} \leq Kf(Q_n).
\]

(5)

Consider that

\[
\sum_{i=1}^{n} \binom{n}{i} \frac{1}{2^i} = \sum_{i=0}^{n-1} \frac{n}{2^i (n-i)} + 1.
\]

\[
= \frac{1}{2} \binom{n}{2} + \frac{1}{3} \binom{n}{3} + \ldots + \frac{1}{n+1} \binom{n}{n+1}
\]

\[
= \frac{1}{2!} \frac{n!}{(n-1)!} + \frac{1}{3!} \frac{n!}{2!(n-2)!} + \ldots + \frac{1}{n+1} \frac{n!}{(n+1)!} + 1
\]

\[
= \sum_{i=1}^{n} \frac{C_{i+1}^{n}}{n+1}.
\]

(6)

By virtue of

\[
\sum_{i=1}^{n} \binom{n}{i} \frac{1}{i+1} = 2^{n+1} - n - 2.
\]

(7)

By means of calculating the right of equation (6), one can establish the following identity:

\[
\sum_{i=1}^{n} \binom{n}{i} \frac{1}{i+1} = 2^{n+1} - n - 2.
\]

(8)

Since

\[
\frac{2^{n+1} - n - 2}{n+1} \leq \sum_{i=1}^{n} \binom{n}{i} \frac{1}{i+1} = 2^{n+1} - n - 2,
\]

(9)

Hence,

\[
\frac{2^{n+1} - 2n - 2}{n+1} \leq \sum_{i=1}^{n} \binom{n}{i} \frac{1}{i+1} = Kf(Q_n).
\]

(10)

Simply, from the left of the above inequality, we obtain

\[
\frac{4^n}{n} \left( \frac{n}{n+1} \right) - \frac{n(n+2)}{2^{n+1}} \leq Kf(Q_n).
\]

(11)

Apparently, the left of the above inequality converges to the asymptotic value \(2^{d/2}\) for large enough \(n\). The proof of lower bound is completed.

For the upper bound, we have similar theorem to consider as follows.

**Theorem 4.** For the hypercube networks \(Q_n\) with \(n \geq 2\), then

\[
Kf(Q_n) \leq \frac{4^n}{n} \left( \frac{2n}{n+1} - \frac{n+2}{2^n (n+1)} \right).
\]

(12)

\[
\sum_{i=1}^{n} \frac{C_{i+1}^{n}}{n+1} \leq \sum_{i=1}^{n} \frac{1}{2^i} \frac{1}{n+1}
\]

\[
= \frac{1}{2} \frac{C_n}{n} + \frac{C_n}{3} + \ldots + \frac{C_n}{n+1}
\]

\[
= \frac{1}{2} \frac{1}{n+1} + \frac{1}{3} \frac{n!}{2!(n-2)!} + \ldots + \frac{1}{n+1}
\]

\[
= \frac{1}{n+1} + \frac{1}{3} \frac{(n+1)!}{n+1} + \frac{1}{3} \frac{(n+1)!}{3!(n-2)!} + \ldots + \frac{1}{n+1}
\]

\[
= \sum_{i=1}^{n} \frac{1}{n+1} \frac{C_{i+1}^{n}}{n+1}.
\]

(13)

Based on equation (8), we can obtain that

\[
2^n \sum_{i=1}^{n} \frac{C_{i+1}^{n}}{2^i} = Kf(Q_n) \leq 2^{n+1} - n - 2.
\]

(14)

Hence,
Theorem 5. For the hypercube network $Q_n$ with $n \geq 2$,

\[ Kf(Q_n) \leq 2^n \frac{2^n n - 2}{n + 1} = \frac{4^n}{n} \left( \frac{2n}{n + 1} - \frac{n + 2}{2^n (n + 1)} \right). \tag{15} \]

The above estimate looks a little complicated. The upper bound is roughly twice the asymptotic value. Hence, a new upper bound is explored as follows.

**Theorem 6.** For the hypercube network $Q_n$ with $n \geq 2$,

\[ Kf(Q_n) \leq \frac{4^n}{n}. \tag{16} \]

Following the identity which is obtained in [24],

\[ \sum_{i=1}^{n} x^i = \left( \frac{1}{n} + \frac{1}{n-1} + \cdots + 1 \right) + \sum_{i=1}^{n} \left( \frac{n}{i} \right) \frac{(x-1)^i}{i}. \tag{17} \]

Fixing $x = 2$, one arrives at

\[ \sum_{i=1}^{n} \left( \frac{n}{i} \right) \frac{1}{i} = \sum_{i=1}^{n-1} \left( \frac{2^n}{i} - \frac{1}{2^n} + \cdots + 1 \right). \tag{18} \]

Namely,

\[ \sum_{i=1}^{n} \left( \frac{n}{i} \right) \frac{1}{i} = \sum_{i=1}^{n-1} \frac{2^n - 1}{i}. \tag{19} \]

According to equation (19) and Theorem 2, one obtains

\[ Kf(Q_n) = 2^{n-1} \cdot \sum_{i=1}^{n-1} \frac{2^n - 1}{i}. \tag{20} \]

Using equation (20), one has

\[ Kf(Q_n) \leq 2^{n-1} \cdot \sum_{i=1}^{n-1} \frac{2^n - 1}{i}. \tag{21} \]

On the contrary,

\[ 2^{n-1} \cdot \sum_{i=1}^{n-1} \frac{2^n - 1}{i} = 1 \cdot 2^{n-1} \cdot \sum_{i=1}^{n-1} \frac{n}{2^n (n-i)} \tag{22} \]

Using Theorem 2 and substituting equations (22) to (21), one obtains the desired result:

\[ Kf(Q_n) \leq \frac{4^n}{n}. \tag{23} \]

This has completed the proof.

4. **Further Discussion**

We, at this place, try another way to estimate the Kirchhoff index of $n$-dimensional hypercubes.

**Theorem 6.** For the hypercube networks $Q_n$ with $n \geq 2$,

\[ Kf(Q_n) = 2^{n-1} \sum_{i=1}^{n-1} \frac{2^n - 1}{i}. \tag{24} \]

Let $S_n = \sum_{i=1}^{n} C^i_n / i$, then

\[ S_n - S_n-1 = \sum_{i=1}^{n-1} \frac{C^i_n}{i} - \sum_{i=1}^{n-1} \frac{C^{i-1}_{n-1}}{i} \]

\[ = \frac{1}{n} \left[ \sum_{i=1}^{n-1} \frac{C^i_n}{i} - \sum_{i=1}^{n-1} \frac{C^{i-1}_{n-1}}{i} \right] \]

\[ = \frac{1}{n} \left[ 1 + \sum_{i=1}^{n-1} \left( C^i_n - C^{i-1}_{n-1} \right) \right] \]

Consequently,

\[ n \cdot (S_n - S_{n-1}) = 1 + \sum_{i=1}^{n-1} \frac{n!}{i! (n-i)!} = 1 + \sum_{i=1}^{n-1} C^i_n = 2^n - 1. \tag{25} \]

One can easily check that $S_1 = 1$. Hence, $S_n - S_{n-1} = (2^n - 1)/n$.

By virtue of the above equality, we obtain

\[ S_n = \sum_{i=1}^{n} \frac{2^n - 1}{i}. \tag{26} \]

Therefore,

\[ Kf(Q_n) = 2^{n-1} \sum_{i=1}^{n-1} \frac{2^n - 1}{i}. \tag{27} \]

The proof of Theorem 6 is completed.

**Data Availability**

The data used to support the findings of this study are available within paper.

**Conflicts of Interest**

The authors declare no conflicts of interest.

**Authors’ Contributions**

Data curation was carried out by J-B.L.; J-B.L. and J.Cao helped with the methodology; J.Z., Z-Y.S., and F.E. Alsaaadi wrote the original draft. All authors read and approved the final manuscript.

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