ASYMPTOTIC BEHAVIOUR OF QUASI-ORTHOGONAL POLYNOMIALS

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Abstract

We obtain explicit upper and lower bounds on the norms of the spectral projections of the non-self-adjoint harmonic oscillator. Some of our results apply to a variety of other families of orthogonal polynomials.

1 Introduction

We consider polynomials $p_n$ which are orthogonal with respect to a complex weight $\sigma$ on $[0, \infty)$ in the following sense. We suppose that $p_n$ is of degree $n$ and

$$\int_0^\infty p_m(x)p_n(x)\sigma(x)^2 \, dx = \delta_{m,n}$$

for all non-negative integers $m, n$. (All of our statements and proofs can be rewritten with $(0, \infty)$ replaced by $\mathbb{R}$, and we will not keep repeating this point.) If $\sigma > 0$ and $p_m$ are real-valued, then they are orthonormal in $L^2((0, \infty), \sigma(x)^2 \, dx)$ in the usual sense, but for complex-valued $\sigma$ such an interpretation is not possible. Our goal is to obtain bounds on the quantities

$$N_n = \int_0^\infty |p_n(x)\sigma(x)|^2 \, dx$$

for all $n$.

This problem arose in the context of the non-self-adjoint harmonic oscillator

$$(Hf)(x) = -f''(x) + z^4 x^2 f(x)$$

(1)

acting in $L^2(\mathbb{R})$ for some complex $z$. In this situation the relevant weight is

$$\sigma(x) = e^{-z^2x^2/2}$$
and $N_n$ is the norm of the spectral projection $P_n$ of $H$ associated with its $n$th eigenvalue, $\lambda_n = z^2(2n + 1)$. In the numerical literature $N_n$ is called the condition number of the eigenvalue $\lambda_n$. Numerical calculations in [1] indicated that $\|P_n\|$ increases at an exponential rate as $n \to \infty$, and it was proved in [4] that there was no polynomial bound on $\|P_n\|$ for this and certain other Schrödinger operators. The super-polynomial rate of increase of the associated resolvent norms in the semi-classical limit was proved in [3] by a method which was greatly generalized in [6]. For certain classes of operators with analytic coefficients it was recently proved that the resolvent norms increase at an exponential rate in the semiclassical limit, [5]. However, the precise exponential constants have not been identified in any example.

A consequence of our theorems is that there exists a positive critical constant $t_z$ such that the ‘spectral expansion’

$$e^{-Ht} = \sum_{n=0}^{\infty} e^{-\lambda_n t} P_n$$

is norm convergent if $t > t_z$ and divergent if $0 \leq t < t_z$. Our method provides explicit upper and lower bounds on $t_z$ but not its precise value.

The problem may be reformulated as finding the norms of $\phi_n(x) = p_n(x)\sigma(x)$ in $L^2((0, \infty), dx)$, where $\phi_n$ are obtained by applying a modified Gram-Schmidt orthogonalization process to the functions $x^n\sigma(x)$. This procedure is modified in the sense that we require

$$\int_0^\infty \phi_m(x)\phi_n(x) \, dx = \delta_{m,n}$$

without any complex conjugates. This is equivalent to requiring that $\phi_m$ and $\phi_n^*(x) = \bar{\phi_n}(x)$ form a biorthogonal system in $L^2((0, \infty), dx)$ in the sense that

$$\langle \phi_m, \phi_n^* \rangle = \delta_{m,n}$$

for all $m, n$. If $P_n$ is the (non-orthogonal) projection

$$P_n f = \langle f, \phi_n^* \rangle \phi_n$$

then $P_m P_n = \delta_{m,n} P_n$ for all $m, n$ and it is easily seen that

$$\|P_n\| = N_n.$$

In order to make some progress with this problem, we make the following assumptions on the weight $\sigma$. We assume that $\sigma(z)$ is an analytic function of $z$ in the sector $S = \{z : |\arg(z)| < \alpha\}$, and that it is positive on the real axis. We also assume that

$$\int_0^\infty x^n|\sigma(e^{i\theta} x)|^2 \, dx < \infty$$
for all $n \geq 0$ and $|\theta| < \alpha$, in order that $p_n$ should be well-defined. Our most important condition is that

$$|\sigma(e^{i\theta}r)| \geq c_\theta \sigma(s_\theta r)$$

(2)

for all $|\theta| < \alpha$ and all $r > 0$, where $c_\theta > 0$ and $0 < s_\theta < 1$. Our main theorem provides a lower bound on $N_n$ for the weight $x \rightarrow \sigma(e^{i\theta}x)$ under these assumptions. Examples of such weights are given in Section 3. Finally in Section 4 we compare the bounds obtained with numerical evidence.

2 The Lower Bound

Let $\{p_n\}_{n=0}^\infty$ denote the standard orthonormal sequence of real-valued polynomials with respect to the positive weight $\sigma^2$ on $(0, \infty)$. We define

$$p_{n,z}(x) = z^{1/2}p_n(zx)$$

where $z \in S$ and $x > 0$. If $z > 0$ then

$$\int_0^\infty p_{m,z}(x)p_{n,z}(x)\sigma(zx)^2 \, dx = \delta_{m,n}$$

by making the change of variable $zx = u$. By analytic continuation the same holds for all complex $z \in S$. We are interested in obtaining a lower bound on the quantity

$$N_{n,z} = \int_0^\infty |p_{n,z}(x)\sigma(zx)|^2 \, dx$$

for complex $z \in S$. Note that $N_{n,z} = 1$ for all positive real $z$.

**Theorem 1** Under the assumption (2) we have

$$N_{n,z} \geq c_\theta^2 s_\theta^{-2n-1}$$

(3)

provided $z = re^{i\theta}$ and $|\theta| < \alpha$.

**Proof** We have

$$N_{n,z} = |z| \int_0^\infty |p_n(zx)\sigma(zx)|^2 \, dx$$

$$\geq c_\theta^2 r \int_0^\infty |p_n(zx)\sigma(s_\theta rx)|^2 \, dx$$

$$= c_\theta^2 s_\theta^{-1} \int_0^\infty |p_n(zx/s_\theta r)\sigma(x)|^2 \, dx.$$
for constants \( k_j \) which we need not evaluate. By the orthogonality of the polynomials, we have

\[
\int_0^\infty |p_n(zx/s_\theta r)\sigma(x)|^2 \, dx = s_\theta^{-2n} + \sum_{j=0}^{n-1} |k_j|^2 \geq s_\theta^{-2n}.
\]

The statement of the theorem follows.

We next consider the example

\[
\sigma(z) = z^{\gamma/2} e^{-z^\beta}
\]

where \( \gamma > -1 \) and \( \beta > 0 \). If \( r > 0 \) and \( |\theta| < \pi/(2\beta) \) then

\[
|\sigma(re^{i\theta})| = r^{\gamma/2} e^{-r^\beta \cos(\theta \beta)} = c_\theta \sigma(s_\theta r)
\]

where \( s_\theta = \{\cos(\theta \beta)\}^{1/\beta} \) and \( c_\theta = s_\theta^{-\gamma/2} \). After replacing \((0, \infty)\) by \((-\infty, \infty)\), the particular choice \( \gamma = 0 \) and \( \beta = 2 \) leads one to the study of the Hermite polynomials with a complex scaling, which is relevant to the non-self-adjoint harmonic oscillator. The choice \( \beta = 1 \) leads to the Laguerre polynomials \( L_\gamma^n \). As far as we know, all other choices lead to non-classical polynomials.

The following theorem provides a more general type of weight satisfying (2), and can itself easily be generalized.

**Theorem 2** If

\[
\sigma(x) = \exp\left\{-\sum_{j=1}^{n} c_j x^j\right\}
\]

for all \( x \in (0, \infty) \), where \( c_j \in \mathbb{R} \) for all \( j \) and \( c_n > 0 \), then \( \sigma \) satisfies (2) provided \( |\theta| < \pi/(2n) \).

**Proof** We have to find \( k_\theta > 0 \) and \( s_\theta \in (0, 1) \) such that

\[
\sum_{j=1}^{n} c_j \cos(j\theta) r^j \leq k_\theta + \sum_{j=1}^{n} c_j s_\theta^j r^j
\]

for all \( r > 0 \) and \( |\theta| < \pi/(2n) \). The validity of such an inequality depends upon the coefficient of \( r^n \). We achieve the required bound \( \cos(n\theta) < s_\theta^n < 1 \) by putting

\[
s_\theta = \{(1 + \cos(n\theta))/2\}^{1/n}.
\]

**Note** If \((0, \infty)\) is replaced by \( \mathbb{R} \) in the above theorem, we must also assume that \( n \) is even.
3 The Upper Bound

It is surprisingly difficult to obtain an upper bound on $N_n$, and we treat only two cases. We start with the orthonormal sequence of Laguerre polynomials, associated with the weight $\sigma(x) = e^{-x/2}$ on $(0, \infty)$. We have

$$p_n(x) = \frac{(-1)^n}{n!} e^x \frac{d^n}{dx^n} \left(x^n e^{-x}\right) = \sum_{r=0}^{n} b_{n,r} x^r$$

where

$$b_{n,r} = (-1)^{n-r} \frac{n!}{(r!)^2(n-r)!}$$

satisfies

$$|b_{n,r}| \leq \frac{2^n}{r!}$$

by virtue of the general inequality

$$(r+s)! \leq 2^{r+s}r!s!$$

The following theorem provides an upper bound on $N_{n,z}$ which complements the lower bound of Theorem 1.

**Theorem 3** If $\sigma(x) = e^{-x/2}$ and $z = e^{i\theta}$ then

$$N_{n,z} \leq s_\theta^{-2n-12n^2+2}$$

for all $n \geq 0$, provided $|\theta| < \pi/2$ and $s_\theta = \cos(\theta)$.

**Proof** We start with the equality

$$N_{n,z} = s_\theta^{-1} \int_{0}^{\infty} |p_n(e^{i\theta}x/s_\theta)\sigma(x)|^2 \, dx,$$

which is proved as in Theorem 1. We have $c_\theta = 1$ and $s_\theta = \cos(\theta)$ by (4). We deduce that

$$N_{n,z} \leq s_\theta^{-1} \int_{0}^{\infty} \sum_{r,s=0}^{n} |b_{n,r}b_{n,s}|s_\theta^{-r-s}x^{r+s}e^{-x} \, dx$$

$$\leq s_\theta^{-2n-12n^2} \int_{0}^{\infty} \sum_{r,s=0}^{n} \frac{x^{r+s}}{r!s!}e^{-x} \, dx$$

$$\leq s_\theta^{-2n-12n^2} \sum_{r,s=0}^{n} \frac{(r+s)!}{r!s!}$$

$$\leq s_\theta^{-2n-12n^2} \left( \sum_{r=0}^{n} 2^r \right)^2$$

$$\leq s_\theta^{-2n-12n^2+2}.$$
Note. This proof can be extended to more general weights provided suitable bounds on the coefficients $b_{n,r}$ can be obtained, but in general this is not easy.

We next consider the non-self-adjoint harmonic oscillator. The orthonormal sequence of polynomials corresponding to the weight $\sigma(x) = e^{-x^2/2}$ is given by $p_n(x) = k_n H_n(x)$, where

$$k_n = \pi^{-1/4}2^{-n/2}(n!)^{-1/2}$$

and $H_n$ are the Hermite polynomials

$$H_n(x) = (2x)^n - \frac{n!}{1!(n-2)!}(2x)^{n-2} + \frac{n!}{2!(n-4)!}(2x)^{n-4} - ...$$

We will need the following lemma.

**Lemma 4** If $r, s$ are non-negative integers then

$$\int_{-\infty}^{\infty} x^{2r+2s}e^{-x^2} \, dx \leq \pi^{1/2}2^{r+s}r!s!$$

**Proof** The left hand-side equals

$$\int_{0}^{\infty} u^{r+s}e^{-s} \, ds = \Gamma(r+s+1/2) \leq \frac{\pi^{1/2}\Gamma(r+s+1)}{\sqrt{\pi(2r+1)}} \leq \pi^{1/2}2^{r+s}r!s!$$

In the following theorem we restrict attention to the case of even integers; the treatment of odd integers is very similar.

**Theorem 5** Let $z = e^{i\theta}$ where $|\theta| < \pi/4$, and put $s_\theta = (\cos(2\theta))^{1/2}$. Then

$$N_{2n,z} \leq \pi(n+1)^{1/2}2^{4n+2}s_\theta^{-4n-1}.$$ for all non-negative integers $n$.

**Proof** We start with the identity

$$p_{2n}(x) = \sum_{r=0}^{n} b_{n,r} x^{2r}$$

where

$$b_{n,r} = \frac{(-1)^{n-r}2^{2r-n}\sqrt{(2n)!}}{\pi^{1/4}(n-r)!(2r)!}.$$ In the following chain of inequalities we will use

$$2^{-2r} \sqrt{r+1} \leq \frac{(r!)^2}{(2r)!} \leq 2^{-2r} \sqrt{\pi(r+1)}$$
for all non-negative integers \( r \); this is proved using induction and Stirling’s formula.

Following the method of Theorem 3 we have

\[
N_{2n,z} \leq s_{\theta}^{-4n-1} \sum_{r,s=0}^{n} |b_{n,r}b_{n,s}| s_{\theta}^{-2r-2s} \int_{-\infty}^{\infty} x^{2r+s} e^{-x^2} \, dx
\]

\[
\leq s_{\theta}^{-4n-1} \sum_{r,s=0}^{n} |b_{n,r}b_{n,s}| \pi^{1/2} 2^{r+s} r! s!
\]

\[
\leq s_{\theta}^{-4n-1} (n+1)^{-1/2} \left( \sum_{r=0}^{n} \frac{2^{3r} r! n!}{(n-r)! (r)! (2r)!} \right)^2
\]

\[
\leq s_{\theta}^{-4n-1} (n+1)^{-1/2} \left( \sum_{r=0}^{n} \frac{2^{3r} (r!)^2}{(n-r)! r! (2r)!} \right)^2
\]

\[
\leq s_{\theta}^{-4n-1} (n+1)^{-1/2} 2^{2n} \left( \sum_{r=0}^{n} \frac{2^r}{\sqrt{\pi (r+1)}} \right)^2
\]

\[
\leq s_{\theta}^{-4n-1} \pi (n+1)^{1/2} 2^{4n+2}.
\]

4 The Spectral Expansion

Let \( H \) denote the non-self-adjoint harmonic oscillator acting in \( L^2(\mathbb{R}) \), with eigenvalues \( \lambda_n = z^2(2n+1) \) and spectral projections \( P_n \). If the right-hand-side of the expansion

\[
e^{-Ht} = \sum_{n=0}^{\infty} e^{-\lambda_n t} P_n \tag{5}
\]

is norm convergent, then by comparing the action of the two sides on the eigenfunctions \( \phi_n \) we see that they coincide on a dense subspace, and hence on the whole of \( L^2(\mathbb{R}) \).

If we put

\[
s_z = \limsup_{n \to \infty} n^{-1} \log(\|P_n\|)
\]

then our theorems imply that \( 0 < s_z < \infty \) provided \( 0 < |\theta| < \pi/4 \). They also provide explicit upper and lower bounds on \( s_z \).

**Theorem 6** The spectral expansion \([5]\) is norm convergent if \( t > t_z = s_z/(2 \cos(2\theta)) \) and is norm divergent if \( 0 \leq t < t_z \).
For \( t > t_z \) the terms of the series decrease at an exponential rate, while for \( 0 \leq t < t_z \) they are not uniformly bounded in norm.

### 5 Numerical Results

The non-self-adjoint harmonic oscillator \( (1) \) has eigenvalues \( \lambda_n = z^2(2n + 1) \) and eigenfunctions

\[
\phi_n(x) = k_n e^{-z^2 x^2 / 2} H_n(z x)
\]

for \( n = 0, 1, \ldots \), where \( k_n \) are normalization constants, \( H_n \) are the Hermite polynomials, and \( |\arg(z)| < \pi/4 \).

**Theorem 7** If \( P_n \) is the \( n \)th spectral projection of \( H \) and \( z = r e^{i\theta} \) then

\[
\lim_{n \to \infty} \inf n^{-1} \log(\|P_n\|) \geq \log(\sec(2\theta)).
\]

**Proof** This follows directly from Theorem 1 upon observing that \( \|P_n\| = N_{n,z} \) and \( s_{\theta} = \cos(2\theta)^{1/2} \).

We have previously evaluated these norms numerically for \( z^4 = c = \sqrt{i} \), i.e. \( \theta = \pi/16 \). See \( \kappa_n^{(1)} \) in Table 4 of [1]. It appears from the computations there that

\[
\lim_{n \to \infty} n^{-1} \log(\|P_n\|) \sim 0.40
\]

which is considerably larger than the lower bound 0.079 of Theorem 7.

We now report on a more systematic numerical investigation of the spectral projections of \( (1) \). We evaluated \( \sigma_n(\theta) = \sqrt{\|P_n\| / \|P_{n-2}\|} \) for various \( n \) and \( \theta \) using Maple. (This was easier than evaluating \( \|P_n\| / \|P_{n-1}\| \) because different algorithms are needed for even and odd \( n \).) The method used was the same as that described in [1] sect. 4.3. We put \( Digits := 200 \), and included enough terms of the sequence determining the eigenvector to achieve stability. For each \( \theta \) it appeared that \( \sigma_n(\theta) \) was an increasing function of \( n \), so the limiting value is probably larger than the computed value. For \( \theta = 0 \) the operator \( H \) is self-adjoint, and the projections have norm 1. As stated earlier one must restrict \( \theta \) to the range \( |\theta| < \pi/4 \). The results are shown for \( n = 100 \) in Table 1. The second column lists the constants \( s_{\theta}^{-2} = \sec(2\theta) \) (rounded down) associated with the lower bound of Theorem 7. The fourth column lists the constants \( 4s_{\theta}^{-2} = 4 \sec(2\theta) \) (rounded up) associated with the upper bound of Theorem 7. The final column lists the values of \( \mu(\theta) = \exp(\tan(2\theta)) \), for reasons explained below.

The approximations \( \mu(\theta) \) were obtained by the following non-rigorous method. For even values of \( n \) the eigenfunction \( \phi_n \) of \( H \) is an even function of \( x \) which is concentrated around the points \( \pm x_0 \), where \( x_0 \) is defined below. On the positive
half-line the semi-classical analysis of [2, Sect. 2] suggests that for large enough \( \eta > 0 \)

\[
\phi(s + x_0) \sim e^{-\psi_1 s - \psi_2 s^2/2}
\]

is an approximate eigenvector of \( H \) with approximate eigenvalue \( \lambda \), where \( x_0 = \eta \), \( \psi_1 = i\eta \), \( \psi_2 = -iz^4 \), and \( \lambda = (1 + z^4)\eta^2 \); in the notation of [2] we are putting \( c = z^4 \) and \( \alpha = 1 \), and are ignoring the term involving \( \psi_3 \).

If \( n \) is a positive integer and we put \( \eta = \{n/ \cos(2\theta)\}^{1/2} \), then a direct calculation shows that \( \lambda = 2nz^2 \), which equals the \( n \)th eigenvalue of \( H \) to leading order as \( n \to \infty \). This suggests that

\[
\|P_n\| \sim \frac{\int_0^\infty |\phi(s + x_0)|^2 \, ds}{\int_0^\infty |\phi(s + x_0)|^2 \, ds}
= \frac{\int_{-\infty}^\infty e^{-2Re(\psi_1)s - Re(\psi_2)s^2} \, ds}{\int_{-\infty}^\infty e^{-2\psi_1 s - \psi_2 s^2} \, ds}
= \exp\{n \tan(2\theta)\}.
\]

In view of the crude character of the approximations above, the similarity of \( \sigma_{100}(\theta) \) and \( \mu(\theta) \) in Table 1 is interesting. We conjecture that a more detailed semiclassical analysis might yield the correct asymptotic constant. This also seems the best hope for treating more general non-self-adjoint Schrödinger operators.

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