On representing the positive semidefinite cone using the second-order cone

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Abstract The second-order cone plays an important role in convex optimization and has strong expressive abilities despite its apparent simplicity. Second-order cone formulations can also be solved more efficiently than semidefinite programming problems in general. We consider the following question, posed by Lewis and Glineur, Parrilo, Saunderson: is it possible to express the general positive semidefinite cone using second-order cones? We provide a negative answer to this question and show that the $3 \times 3$ positive semidefinite cone does not admit any second-order cone representation. In fact we show that the slice consisting of $3 \times 3$ positive semidefinite Hankel matrices does not admit a second-order cone representation. Our proof relies on exhibiting a sequence of submatrices of the slack matrix of the $3 \times 3$ positive semidefinite cone whose “second-order cone rank” grows to infinity.

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1 Introduction

Let $\mathcal{Q} \subseteq \mathbb{R}^3$ denote the three-dimensional second-order cone (also known as the “ice-cream” cone or the Lorentz cone):

$$\mathcal{Q} = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : \|x\| \leq t\}.$$
It is known that $Q$ is linearly isomorphic to the cone of $2 \times 2$ real symmetric positive semidefinite matrices. Indeed we have:

$$(x_1, x_2, t) \in Q \iff \begin{bmatrix} t - x_1 & x_2 \\ x_2 & t + x_1 \end{bmatrix} \succeq 0.$$  

Despite its apparent simplicity the second-order cone $Q$ has strong expressive abilities and allows us to represent various convex constraints that go beyond “simple quadratic constraints”. For example it can be used to express geometric means ($x \mapsto \prod_{i=1}^{n} x_i^{p_i}$ where $p_i \geq 0$, rational, and $\sum_{i=1}^{n} p_i = 1$), $\ell_p$-norm constraints, multifocal ellipses (see e.g., [11, Equation (3.5)]), robust counterparts of linear programs, etc. We refer the reader to [4, Section 3.3] for more details.

Given this strong expressive ability one may wonder whether the general positive semidefinite cone can be represented using $Q$. This question was posed in particular by Adrian Lewis (personal communication) and Glineur, Parrilo and Saunderson [7]. In this paper we show that this is not possible, even for the $3 \times 3$ positive semidefinite cone. To make things precise we use the language of lifts (or extended formulations), see [8]. We denote by $Q^k$ the Cartesian product of $k$ copies of $Q$:

$$Q^k = Q \times \cdots \times Q \ (k \text{ copies}).$$

A linear slice of $Q^k$ is an intersection of $Q^k$ with a linear subspace. We say that a convex cone $K \subset \mathbb{R}^m$ has a second-order cone lift of size $k$ (or simply $Q^k$-lift) if it can be written as the projection of a slice of $Q^k$, i.e.:

$$K = \pi \left( Q^k \cap L \right)$$

where $\pi : \mathbb{R}^{3k} \to \mathbb{R}^m$ is a linear map and $L$ is a linear subspace of $\mathbb{R}^{3k}$. Let $S_+^n$ be the cone of $n \times n$ real symmetric positive semidefinite matrices. In this paper we prove:

**Theorem 1** The cone $S_+^3$ does not admit any $Q^k$-lift for any finite $k$.

Actually our proof allows us to show that the slice of $S_+^3$ consisting of Hankel matrices does not admit any second-order representation (see Sect. 4 for details). Note that higher-dimensional second order cones of the form

$$\{(x, t) \in \mathbb{R}^n \times t : \|x\| \leq t\}$$

where $n \geq 3$ can be represented using the three-dimensional cone $Q$, see e.g., [5, Section 2]. Thus Theorem 1 also rules out any representation of $S_+^3$ using the higher-dimensional second-order cones. Moreover since $S_+^3$ appears as a slice of higher-order positive semidefinite cones Theorem 1 also shows that one cannot represent $S_+^n$, for $n \geq 3$ using second-order cones.
2 Preliminaries

The paper [8] introduced a general methodology to prove existence or nonexistence of lifts in terms of the slack matrix of a cone. In this section we review some of the definitions and results from this paper, and introduce the notion of a second-order cone factorization and the second-order cone rank.

Let $E$ be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and let $K \subseteq E$ be a cone. The dual cone $K^*$ is defined as:

$$K^* = \{ x \in E : \langle x, y \rangle \geq 0 \ \forall \ y \in K \}.$$

We also denote by $\text{ext}(K)$ the extreme rays of a cone $K$. The notion of slack matrix plays a fundamental role in the study of lifts.

**Definition 1 (Slack matrix)** The slack matrix of a cone $K$, denoted $S_K$, is a (potentially infinite) matrix where columns are indexed by extreme rays of $K$, and rows are indexed by extreme rays of $K^*$ (the dual of $K$) and where the $(x, y)$ entry is given by:

$$S_K[x, y] = \langle x, y \rangle \ \forall (x, y) \in \text{ext}(K^*) \times \text{ext}(K).$$  \hfill (3)

Note that, by definition of dual cone, all the entries of $S_K$ are nonnegative. Also note that an element $x \in \text{ext}(K^*)$ (and similarly $y \in \text{ext}(K)$) is only defined up to a positive multiple. Any choice of scaling gives a valid slack matrix of $K$ and the properties of $S_K$ that we are interested in will be independent of the scaling chosen.

The existence/nonexistence of a second-order cone lift for a convex cone $K$ will depend on whether $S_K$ admits a certain second-order cone factorization which we now define.

**Definition 2 ($Q^k$-factorization and second-order cone rank)** Let $S \in \mathbb{R}^{|I| \times |J|}$ be a matrix with nonnegative entries. We say that $S$ has a $Q^k$-factorization if there exist vectors $a_i \in Q^k$ for $i \in I$ and $b_j \in Q^k$ for $j \in J$ such that $S[i, j] = \langle a_i, b_j \rangle$ for all $i \in I$ and $j \in J$. The smallest $k$ for which such a factorization exists will be denoted $\text{rank}_{soc}(S)$.

**Remark 1** Recall that for any $a, b \in Q$ we have $\langle a, b \rangle \geq 0$. This means that any matrix with a second-order cone factorization is elementwise nonnegative.

**Remark 2** It is important to note that the second-order cone rank of any matrix $S$ can be equivalently expressed as the smallest $k$ such that $S$ admits a decomposition

$$S = M_1 + \cdots + M_k$$  \hfill (4)

where $\text{rank}_{soc}(M_l) = 1$ for each $l = 1, \ldots, k$ (i.e., each $M_l$ has a factorization $M_l[i, j] = \langle a_i, b_j \rangle$ where $a_i, b_j \in Q$). This simply follows from the fact that $Q^k$ is the Cartesian product of $k$ copies of $Q$.

We now state the result from [8] that we will need.
Theorem 2 (Existence of a lift, special case of [8]) Let $K$ be a convex cone. If $K$ has a $Q^k$-lift then its slack matrix $S_K$ has a $Q^k$-factorization.

This theorem can actually be turned into an if and only if condition under mild conditions on $K$ (e.g., $K$ is proper), see [8], but we have only stated here the direction that we will need.

The cone $S^3_+$. In this paper we are interested in the cone $K = S^3_+$ of real symmetric $3 \times 3$ positive semidefinite matrices. The extreme rays of $S^3_+$ are rank-one matrices of the form $xx^T$ where $x \in \mathbb{R}^3$. Also $S^3_+$ is self-dual, i.e., $(S^3_+)^* = S^3_+$. The slack matrix of $S^3_+$ thus has its rows and columns indexed by three-dimensional vectors and

$$S_{S^3_+}[x, y] = \langle xx^T, yy^T \rangle = \left(x^T y\right)^2 \quad \forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3. \quad (5)$$

In order to prove that $S^3_+$ does not admit a second-order representation, we will show that its slack matrix does not admit any $Q^k$-factorization for any finite $k$. In fact we will exhibit a sequence $(A_n)$ of submatrices of $S_{S^3_+}$ where $\text{rank}_{\text{soc}}(A_n)$ grows to $+\infty$ as $n \to +\infty$.

Before introducing this sequence of matrices we record the following simple (known) proposition concerning orthogonal vectors in the cone $Q$ which will be useful later.

Proposition 1 Let $a, b_1, b_2 \in Q$ nonzero and assume that $\langle a, b_1 \rangle = \langle a, b_2 \rangle = 0$. Then $b_1$ and $b_2$ are collinear.

Proof This is easy to see geometrically by visualizing the “ice cream” cone. We include a proof for completeness: let $a = (a', t) \in \mathbb{R}^2 \times \mathbb{R}$ and $b_i = (b'_i, s_i) \in \mathbb{R}^2 \times \mathbb{R}$ where $\lVert a' \rVert \leq t$ and $\lVert b'_i \rVert \leq s_i$. Note that for $i = 1, 2$ we have $0 = \langle a, b_i \rangle = \langle a', b'_i \rangle + ts_i \geq -\lVert a' \rVert \lVert b'_i \rVert + ts_i \geq 0$ where in the first inequality we used Cauchy-Schwarz and in the second inequality we used the definition of the second-order cone. It thus follows that all the inequalities must be equalities: by the equality case in Cauchy-Schwarz we must have that $b'_i = \alpha_i a'$ for some constant $\alpha_i < 0$ and we must also have $t = \lVert a' \rVert$ and $s_i = \lVert b'_i \rVert$. Thus we get that $b_i = (\alpha_i a', |\alpha_i| \lVert a' \rVert) = |\alpha_i|(-a', \lVert a' \rVert)$. This shows that $b_1$ and $b_2$ are both collinear to the same vector $(-a', \lVert a' \rVert)$ and thus completes the proof. \hfill \qed

3 Proof of Theorem 1

A sequence of matrices We now define our sequence $A_n$ of submatrices of the slack matrix of $S^3_+$. For any integer $i$ define the vector

$$v_i = (1, i, i^2) \in \mathbb{R}^3. \quad (6)$$

Note that this sequence of vectors satisfies the following:

For all distinct integers $i_1, i_2, i_3$ $\det(v_{i_1}, v_{i_2}, v_{i_3}) \neq 0$. \quad (7)
Our matrix $A_n$ has size $\binom{n}{2} \times n$ and is defined as follows (rows are indexed by 2-subsets of $[n]$ and columns are indexed by $[n]$):

$$A_n[[i_1, i_2], j] := \left( (v_{i_1} \times v_{i_2})^T v_j \right)^2 = \det(v_{i_1}, v_{i_2}, v_j)^2 \quad \forall [i_1, i_2] \in \binom{[n]}{2}, \forall j \in [n] \quad (8)$$

where $\times$ denotes the cross-product of three-dimensional vectors. It is clear from the definition of $A_n$ that it is a submatrix of the slack matrix of $S^3_+$. Note that the sparsity pattern of $A_n$ satisfies the following:

$$A_n[e, j] = 0 \quad \text{if } j \in e$$
$$A_n[e, j] > 0 \quad \text{otherwise} \quad e \in \binom{[n]}{2}, \quad j \in [n]. \quad (9)$$

Also note that $A_n$ satisfies the following important recursive property: for any subset $C$ of $[n]$ of size $n_0$ the submatrix $A_n[[C], C]$ has the same sparsity pattern as $A_{n_0}$ (up to relabeling of rows and columns). In our main theorem we will show that the second-order cone rank of $A_n$ grows to infinity with $n$.

**Remark 3** (Geometric interpretation of (9)) The property (9) of the matrices $A_n$ will be the key to prove a lower bound on their second-order cone rank. Geometrically, the property (9) reflects a certain 2-neighborliness property of the extreme rays $\text{ext}(S^3_+)$ of $S^3_+$: for any two distinct extreme rays $xx^T$ and $yy^T$ of $S^3_+$, there is a supporting hyperplane $H$ to $S^3_+$ that touches $\text{ext}(S^3_+)$ precisely at $xx^T$ and $yy^T$. This 2-neighborliness property turns out to be the key geometric obstruction for the existence of second-order cone lifts for $S^3_+$.

**Covering numbers** Our analysis of the matrix $A_n$ will only rely on its sparsity pattern. Given two matrices $A$ and $B$ of the same size we write $A \supseteq B$ if $A$ and $B$ have the same support (i.e., $A_{ij} = 0$ if and only if $B_{ij} = 0$ for all $i, j$). We now define a combinatorial analogue of the second-order cone rank:

**Definition 3** Given a nonnegative matrix $A$, we define the $\text{soc}$-covering number of $A$, denoted $\text{cov}_{\text{soc}}(A)$ to be the smallest number $k$ of matrices $M_1, \ldots, M_k$ with $\text{rank}_{\text{soc}}(M_l) = 1$ for $l = 1, \ldots, k$ that are needed to cover the nonzero entries of $A$, i.e., such that

$$A \supseteq M_1 + \cdots + M_k. \quad (10)$$

**Proposition 2** For any nonnegative matrix $A$ we have $\text{rank}_{\text{soc}}(A) \geq \text{cov}_{\text{soc}}(A)$.

**Proof** This follows immediately from Remark 2 concerning $\text{rank}_{\text{soc}}$ and the definition of $\text{cov}_{\text{soc}}$. \qed

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In fact here we only work with the extreme rays $\{v_n v_n^T : n \in \mathbb{N}\}$, see Sect. 4 for the implication of this.
A simple but crucial fact concerning soc-coverings that we will use is the following: in any soc-covering of $A$ of the form (10), each matrix $M_i$ must satisfy $M_i[i, j] = 0$ whenever $A[i, j] = 0$. This is because the matrices $M_1, \ldots, M_k$ are all entrywise nonnegative.

We are now ready to state our main result.

**Theorem 3** Consider a sequence $(A_n)$ of matrices of sparsity pattern given in (9). Then for any $n_0 \geq 2$ we have $\covsoc(A_{3n_0^2}) \geq \covsoc(A_{n_0}) + 1$. As a consequence $\covsoc(A_n) \to +\infty$ when $n \to +\infty$.

The proof of our theorem rests on a key lemma concerning the sparsity pattern of any term in a soc-covering of $A_n$.

**Lemma 1** (Main) Let $n$ be such that $n \geq 3n_0^2$ for some $n_0 \geq 2$. Assume $M \in \mathbb{R}^{(2) \times n}$ satisfies $\rank(M) = 1$ and $M[e, j] = 0$ for all $e \in (2)$ and $j \in [n]$ such that $j \in e$. Then there is a subset $C$ of $[n]$ of size at least $n_0$ such that the submatrix $M[\binom{C}{2}, C]$ is identically zero.

Before proving this lemma, we show how this lemma can be used to easily prove Theorem 3.

**Proof of Theorem 3** Let $n = 3n_0^2$ and consider a soc-covering of $A_n$ $\supp M_1 + \cdots + M_r$ of size $r = \covsoc(A_n)$ (note that we have of course $r \geq 1$ since $A_n$ is not identically zero). By Lemma 1 there is a subset $C$ of $[n]$ of size $n_0$ such that $M_1[\binom{C}{2}, C] = 0$. It thus follows that we have $A_n[\binom{C}{2}, C] \supp M_2[\binom{C}{2}, C] + \cdots + M_r[\binom{C}{2}, C]$. Also note that $A_n[\binom{C}{2}, C] \supp A_{n_0}$. It thus follows that $A_{n_0}$ has a soc-covering of size $r - 1$ and thus $\covsoc(A_{n_0}) \leq \covsoc(A_{3n_0^2}) - 1$. This completes the proof.

For completeness we show how Theorem 1 follows directly from Theorem 3.

**Proof of Theorem 1** Since for any $n \geq 1$, $A_n$ is a submatrix of the slack matrix of $S^3_+$, Theorem 3 shows that the slack matrix of $S^3_+$ does not admit any $Q^k$-factorization for finite $k$. This shows, via Theorem 2, that $S^3_+$ does not have a $Q^k$-lift for any finite $k$.

The rest of the section is devoted to the proof of Lemma 1.

**Proof of Lemma 1** Let $M \in \mathbb{R}^{(2) \times n}$ and assume that $M$ has a factorization $M_{e,j} = \langle a_e, b_j \rangle$ where $a_e, b_j \in Q$ for all $e \in \binom{[n]}{2}$ and $j \in [n]$, and that $M_{e,j} = 0$ whenever $j \in e$.

Let $E_0 := \{e \in \binom{[n]}{2} : a_e = 0\}$ be the set of rows of $M$ that are identically zero and let $E_1 = \binom{[n]}{2} \setminus E_0$. Similarly for the columns we let $S_0 := \{j \in [n] : b_j = 0\}$ and $S_1 = [n] \setminus S_0$.

In the next lemma we use the sparsity pattern of $A_n$ together with Proposition 1 to infer additional properties on the sparsity pattern of $M$.

**Lemma 2** Let $C$ be a connected component of the graph with vertex set $S_1$ and edge set $E_1(S_1)$ (where $E_1(S_1)$ consists of elements in $E_1$ that connect only elements of $S_1$). Then necessarily $M[\binom{C}{2}, C] = 0$. 

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Proof We first show using Proposition 1 that all the vectors \( \{b_j\}_{j \in C} \) are necessarily collinear. Let \( j_1, j_2 \in S_1 \) such that \( e = \{j_1, j_2\} \in E_1 \). Note that since \( M_{e,j_1} = M_{e,j_2} = 0 \) then we have, by Proposition 1 that \( b_{j_1} \) and \( b_{j_2} \) are collinear. It is easy to see thus now that if \( j_1 \) and \( j_2 \) are connected by a path in the graph \((S_1, E_1(S_1))\) then \( b_{j_1} \) and \( b_{j_2} \) must be collinear.

We thus get that all the columns of \( M \) indexed by \( C \) must be proportional to each other, and so they must have the same sparsity pattern. Now let \( e \in \binom{C}{2} \). If \( a_e = 0 \) then \( M(e, C) = 0 \) since the entire row indexed by \( e \) is zero. Otherwise if \( a_e \neq 0 \) let \( e = \{j_1, j_2\} \) with \( j_1, j_2 \in C \). Since, by assumption, \( M_{e,j_1} = 0 \) it follows that for any \( j \in C \) we must have \( M_{e,j} = 0 \), i.e., \( M(e, C) = 0 \). This is true for any \( e \in \binom{C}{2} \) thus we get that \( M(\binom{C}{2}), C) = 0 \).

To complete the proof of Lemma 1 assume that \( n \geq 3n_0^2 \) for some \( n_0 \geq 2 \). We need to show that there is a subset \( C \) of \([n]\) of size at least \( n_0 \) such that \( M(\binom{C}{2}), C) \neq 0 \).

First note that if the graph \((S_1, E_1(S_1))\) has a connected component of size at least \( n_0 \) then we are done by Lemma 2. Also note that if \( S_0 \) has size at least \( n_0 \) we are also done because all the columns indexed by \( S_0 \) are identically zero by definition.

In the rest of the proof we will thus assume that \(|S_0| < n_0\) and that the connected components of \((S_1, E_1(S_1))\) all have size < \( n_0 \). We will show in this case that \( E_0 \) necessarily contains a clique of size at least \( n_0 \) (i.e., a subset of the form \( \binom{C}{2} \) where \(|C| \geq n_0\)) and this will prove our claim since all the rows in \( E_0 \) are identically zero by definition. The intuition is as follows: the assumption that \(|S_0| < n_0\) and that the connected components of \((S_1, E_1(S_1))\) have size < \( n_0 \) mean that the graph \((S_1, E_1(S_1))\) is very sparse. In particular this means that \( E_1 \) has to be small which means that \( E_0 = E_1^c \) must be large and thus it must contain a large clique.

More precisely, to show that \( E_1 \) is small note that it consists of those edges that are either in \( E_1(S_1) \) or, otherwise, they must have at least one node in \( S_1^c = S_0 \). Thus we get that

\[
|E_1| \leq |E_1(S_1)| + |S_0|(n - 1) \leq |E_1(S_1)| + (n_0 - 1)(n - 1).
\]

where in the second inequality we used the fact that \(|S_0| < n_0\). Also since the connected components of \((S_1, E_1(S_1))\) all have size < \( n_0 \) it is not difficult to show that \(|E_1(S_1)| < n_0 n / 2\) (indeed if we let \( x_1, \ldots, x_k \) be the size of each connected component we have \(|E_1(S_1)| \leq \frac{1}{2} \sum_{i=1}^{k} x_i^2 < \frac{1}{2} \sum_{i=1}^{k} n_0 x_i \leq \frac{n_0}{2} n_0 n \)). Thus we get that

\[
|E_1| \leq \frac{n_0 n}{2} + (n_0 - 1)(n - 1) \leq \left( \frac{3}{2} n_0 - 1 \right) n
\]

Thus this means, since \( E_0 = \binom{n}{2} \setminus E_1 \):

\[
|E_0| \geq \left( \frac{n}{2} \right) - \left( \frac{3}{2} n_0 - 1 \right) n > \frac{n^2}{2} - \frac{3}{2} n_0 n
\]

We now invoke a result of Turán to show that \( E_0 \) must contain a clique of size at least \( n_0 \):

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Theorem 4 (Turán, see e.g., [2]) Any graph on \( n \) vertices with more than \((1 - \frac{1}{k}) \frac{n^2}{2}\) edges contains a clique of size \( k + 1 \).

By taking \( k = n_0 - 1 \) we see that \( E_0 \) contains a clique of size \( n_0 \) if

\[
\frac{n^2}{2} - 3 \frac{n^2}{2} - 2 \frac{n^2}{2} - 2 \frac{n^2}{2} > \left(1 - \frac{1}{n_0 - 1} \right) \frac{n^2}{2}
\]

This simplifies into

\[
n > 3n_0(n_0 - 1)
\]

which is true for \( n \geq 3n_0^2 \).

\( \square \)

4 Slices of the 3 × 3 positive semidefinite cone

Hankel slice The proof given in the previous section actually shows the following more general statement.

Theorem 5 Let \( \mathcal{H} \) denote the cone of 3 × 3 positive semidefinite Hankel matrices:

\[
\mathcal{H} = \{ X \in S_+^3 : X_{13} = X_{22} \}.
\]

Assume \( K \) is a convex cone that is “sandwiched” between \( \mathcal{H} \) and \( S_+^3 \), i.e., \( \mathcal{H} \subseteq K \subseteq S_+^3 \). Then \( K \) does not have a second-order cone representation.

Proof The proof follows from the observation that the matrices \( A_n \) considered in Sect. 3 (see Eq. (8)) are actually submatrices of the generalized slack matrix of the pair of nested cones \((\mathcal{H}, S_+^3)\), the definition of which we now recall (see e.g., [9, Definition 6]): The generalized slack matrix of a pair of convex cones \((K_1, K_2)\) with \( K_1 \subseteq K_2 \) is a matrix whose rows are indexed by \( \text{ext}(K_2^*) \) (the valid linear inequalities of \( K_2 \)) and its columns indexed by \( \text{ext}(K_1) \) and is defined by

\[
S_{K_1, K_2}[x, y] = \langle x, y \rangle \quad x \in \text{ext}(K_2^*), \ y \in \text{ext}(K_1).
\]

When \( K_1 = K_2 \) this is precisely the slack matrix of \( K_1 = K_2 \). The following theorem is a generalization of Theorem 2 and can be proved using very similar arguments (see e.g., [9, Proposition 7]).

Theorem 6 (Generalization of Theorem 2 to nested cones) Let \( K_1, K_2 \) be two convex cones with \( K_1 \subseteq K_2 \), and assume there exists a convex cone \( K \) with a \( Q^k \)-lift such that \( K_1 \subseteq K \subseteq K_2 \). Then \( S_{K_1, K_2} \) has a \( Q^k \)-factorization.

The main observation is to see that the vector \( v_i \) defined in (6) satisfies \( v_i v_i^T \in \mathcal{H} \), and so this shows that \( A_n \) defined in Equation (8) is a submatrix of the generalized slack matrix of the pair \((\mathcal{H}, S_+^3)\). Since \( \text{rank}_{\text{soc}}(A_n) \) grows to infinity with \( n \), this gives the desired result.

\( \square \)
The dual cone of $\mathcal{H}$ is the cone of nonnegative quartic polynomials on the real line (see e.g., [3, Section 3.5]). It thus follows that the latter is also not second-order cone representable using the second-order cone. More generally we can prove:

**Corollary 1** Let $\Sigma_{n,2d}$ be the cone of polynomials in $n$ variables of degree at most $2d$ that are sums of squares. Then $\Sigma_{n,2d}$ is not second-order cone representable except in the case $(n,2d) = (1,2)$.

For the proof we recall that in the cases $n = 1$ (univariate polynomials) and $2d = 2$ (quadratic polynomials), nonnegative polynomials are sums of squares.

**Proof** For $2d = 2$, $\Sigma_{n,2d}$ is the cone of nonnegative quadratic polynomials in $n$ variables. By homogenization, this cone is linearly isomorphic to $S_{n+1}^{n+1}$, the cone of nonnegative quadratic forms in $n + 1$ variables. By Theorem 1, this shows that $\Sigma_{n,2}$ is not second-order cone representable for $n \geq 2$. The case $(n, 2d) = (1,2)$ is clearly second-order cone representable because $S_2^2$ is linearly isomorphic to the second-order cone.

If $2d \geq 4$ then the cone of nonnegative quartic polynomials on the real line can be obtained as a section of $\Sigma_{n,2d}$ by setting to zero the coefficients of some appropriate monomials. This shows that $\Sigma_{n,2d}$ is not second-order cone representable when $2d \geq 4$.

**Other slices of $S_+^3$ that are second-order cone representable** There are certain slices of $S_+^3$ of codimension 1 that are, on the other hand, known to admit a second-order cone representation. For example the following second-order cone representation of the slice $\{X \in S_+^3 : X_{11} = X_{22}\}$ appears in [7]:

$$
\begin{bmatrix}
t & a & b \\
a & t & c \\
b & c & s
\end{bmatrix} \succeq 0 \iff \exists u, v \in \mathbb{R} \text{ s.t. } \begin{bmatrix} t + a & b + c \\ b + c & u \end{bmatrix} \succeq 0, \begin{bmatrix} t - a & b - c \\ b - c & v \end{bmatrix} \succeq 0, \quad u + v = 2s. \tag{11}
$$

(The $2 \times 2$ positive semidefinite constraints can be converted to second-order cone constraints using (1)). To see why (11) holds note that by applying a congruence transformation by $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ on the $3 \times 3$ matrix on the left-hand side of (11) we get that

$$
\begin{bmatrix}
t & a & b \\
a & t & c \\
b & c & s
\end{bmatrix} \succeq 0 \iff \begin{bmatrix} t + a & 0 & b + c \\ 0 & t - a & b - c \\ b + c & b - c & 2s \end{bmatrix} \succeq 0.
$$

The latter matrix has an arrow structure and thus using results on the decomposition of matrices with chordal sparsity pattern [1,6,10] we get the decomposition (11).
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