Boundary weak Harnack estimates and regularity for elliptic PDE in divergence form

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Abstract. We obtain a global extension of the classical weak Harnack inequality which extends and quantifies the Hopf-Oleinik boundary-point lemma, for uniformly elliptic equations in divergence form. Among the consequences is a boundary gradient estimate, due to Krylov and well-studied for non-divergence form equations, but completely novel in the divergence framework. Another consequence is a new more general version of the Hopf-Oleinik lemma.

1 Introduction

We study boundary estimates and global extensions of the weak Harnack inequality for PDE driven by a general linear uniformly elliptic second order operator in divergence form

\[ L_D[u] := -\text{div}(A(x)Du + \beta(x)u) + b(x)Du + c(x)u. \] (1)

We assume that the matrix \( A \) is bounded and uniformly positive; the lower-order coefficients belong to Lebesgue spaces which make possible for weak solutions to satisfy the maximum principle and the Harnack inequality:

(H1) \( A(x) \in L^\infty(\Omega), \quad \lambda I \leq A(x) \leq \Lambda I \) for some \( 0 < \lambda \leq \Lambda \),

(H2) \( \beta, b, g \in L^q_{\text{loc}}(\Omega) \) for some \( q > n \), \( c, f \in L^p_{\text{loc}}(\Omega) \) for some \( p > n/2 \),

for some bounded \( \Omega \subset \mathbb{R}^n \); and consider (in)equalities in the form

\[ L_D[u] \leq \geq \leq f + \text{div}(g) \] in \( \Omega \) (2)

satisfied in the usual weak Sobolev sense (see [12, Chapter 8]) by \( u \in H^1_{\text{loc}}(\Omega) \).

We recall the De Giorgi-Moser “weak Harnack inequality” (WHI), a fundamental result in the theory of elliptic PDE. In its classical form it states that for any nonnegative supersolution of (2) and \( B_{2R} = B_{2R}(x_0) \subset \Omega \),

\[ \left( \int_{B_R} u^\epsilon \, dx \right)^{1/\epsilon} \leq C_0 \left( \inf_{B_R} u + \| f \|_{L^p(B_{2R})} + \| g \|_{L^q(B_{2R})} \right), \] (3)

where \( \epsilon < n/(n-2)_+ \), \( C_0 \) depends on \( n, \lambda, \Lambda, p, q, R, \epsilon \), and the above Lebesgue norms of the coefficients \( \beta, b, c \) in \( B_{2R} \) (see [12, Th. 8.18] and [37]). The WHI
has a wide range of applications, the best-known being the local H"older regularity and bounds for solutions of $\mathcal{L}_D[u] = f + \text{div}(g)$. As for global bounds, essential in the study of boundary value problems, it is known that the WHI applied to $\Omega = \mathbb{R}^n$ and the supersolution $u_m(x) = \min\{u(x), m\}$ if $x \in \Omega$, $u_m(x) = m$ if $x \not\in \Omega$, with $m = \inf_{\partial \Omega} u$, is sufficient to obtain global Hölder estimates if $\Omega$ has for instance the exterior cone property - see [12, Section 8.10, Theorems 8.26, 9.27], as well as [15, 2], for variants of this "boundary weak Harnack inequality" (bWHI).

Note this bWHI is void if $u$ vanishes on the boundary ($m = 0$). One may wonder whether there is a way to quantify the positivity of the supersolution $u$ close to $\partial \Omega$ in the same way as the WHI quantifies the positivity of $u$ in the interior. Such results have appeared only recently (a review is given below), for equations in non-divergence form. Our first statement deals with this question in the previously unstudied divergence setting of (1).

We set $d(x) = \text{dist}(x, \partial \Omega)$, $\Omega_{d_0} = \{x \in \Omega : d(x) < d_0\}$, and assume that (H3) the boundary of $\Omega$ is $C^{1,\text{Dini}}$, the coefficients $A, \beta, g$ have Dini mean oscillation in $\Omega_{d_0}$, and $b, c, f \in L^q(\Omega_{d_0})$, for some $q > n$, $d_0 > 0$.

Below we recall and discuss the regularity notions in (H3). For $B_R = B_R(x_0)$, $x_0 \in \partial \Omega$, $R \leq d_0/2$, we denote $B_R^+ = B_R \cap \Omega$, $B_R^- = B_R \cap \partial \Omega$, and set $k_{2R} = \|f\|_{L^q(B_R^+)} + \|g\|_{L^\infty(B_R^+)} + \mathcal{M}_g(B_{2R}^+)$ (here $\mathcal{M}_g$ quantifies the Dini mean oscillation of $g$ through the function $\varrho_{mD,g}$ defined below). If $g \in C^\alpha$ we can take $\mathcal{M}_g$ to be the standard $\alpha$-Hölder bracket of $g$. Since all hypotheses are preserved under $C^{1,\text{Dini}}$-changes of variable, we can assume $B_R^0$ is flat.

**Theorem 1.1** 1) Assume (H1)-(H3) and $\mathcal{L}_D[u] \geq f + \text{div}(g)$, $u \geq 0$ in $B_R^2$.

Then for $\varepsilon > 0$ depending on $n, \lambda, \Lambda, q, \varrho_{mD,A}$, and $C > 0$ depending on $n, \lambda, \Lambda, q, \varrho_{mD,A}$, and $R$, $\|b\|_{L^q(B_{2R}^+)}$, $\|c\|_{L^q(B_{2R}^+)}$, $\|\beta\|_{L^\infty(B_{2R}^+)}$, $\mathcal{M}_\beta(B_{2R}^+)$,

$$
\left(\frac{\int_{B_{2R}^+} \left(\frac{u}{d}\right)^\varepsilon}{\varepsilon}\right)^{1/\varepsilon} \leq C \left(\inf_{B_{2R}^+} \frac{u}{d} + k_{2R}\right).
$$

2) Assume $\mathcal{L}_D^{(1)}$, $\mathcal{L}_D^{(2)}$ are operators in the form (1) under (H1) - (H3) and $\mathcal{L}_D^{(1)}[u] \leq f^{(1)} + \text{div}(g^{(1)})$, $\mathcal{L}_D^{(2)}[u] \geq f^{(2)} + \text{div}(g^{(2)})$, $u \geq 0$ in $B_R^2$, $u = 0$ on $B_R^0$. There exists $C > 0$ depending on $n, \lambda, \Lambda, q, \varrho_{mD,A}^{(i)}$, and $R$, $\|b^{(i)}\|_{L^q(B_{2R}^+)}$, $\|c^{(i)}\|_{L^q(B_{2R}^+)}$, $\|\beta^{(i)}\|_{L^\infty(B_{2R}^+)}$, $\mathcal{M}_{\beta^{(i)}}(B_{2R}^+)$, $i = 1, 2$, such that

$$
\sup_{B_{2R}^+} \frac{u}{d} \leq C \left(\inf_{B_{2R}^+} \frac{u}{d} + k_{2R}^{(1)} + k_{2R}^{(2)}\right).
$$

3) All $B^+$ (resp. $B_0^+$) can be replaced by $\Omega$ (resp. $\partial \Omega$) in 1) and 2), with $C$ depending also on $p$ and $\Omega$. 

2
Remark. The functions $g$ are defined in the next section. For a coefficient 
$\xi \in C^\alpha$, $\alpha \in (0, 1)$, the dependence of $\varepsilon, C$ in $\theta_{mD, \xi}$ (resp. in $M_\xi$) reduces to 
dependence in $\alpha$ and the $C^\alpha$-norm of the coefficient.

The classical and fundamental Zaremba-Hopf-Oleinik lemma, or “boundary point principle” (BPP), states that a nontrivial nonnegative supersolution of $L[u] \geq 0$ is indeed “strictly positive” close to $\partial \Omega$, in the sense that
$\inf_{\Omega}(u/d) > 0$, for a sufficiently smooth $\partial \Omega$. A lot of work has been dedicated to getting optimal conditions for the validity of BPP, in terms of the regularity or the geometry of the domain, or of the nature of the coefficients of the elliptic operator. We refer to [1], [3], [4], [8], [11], [20], [24], [28], [29] for such conditions, as well as historical reviews and more references. The survey [4] is very complete and up-to-date.

Note that (4) with $f = g = 0$ ($k_{2R} = 0$) implies the BPP, and quantifies it in the following sense: if $L_D[u] \geq 0$ in $B^+_2$, $u \geq c_0d$ in a subset $\omega \subset B^+_1$ of positive measure, then $u \geq \kappa c_0d$ in the whole $B^+_1$, for some $\kappa > 0$ depending only on $|\omega|$ and the data – this can be thought of as a boundary variant of a “growth lemma”. To our knowledge, there are no previous results which quantify the BPP in such a way for any type of divergence form equations which cannot be related to non-divergence ones. In particular, (4) is new for inequalities such as $-\text{div}(A(x)Du) \geq 0$ or $-\Delta u \geq \text{div}(g)$, with $A, g \in C^\alpha$.

Furthermore, even the BPP itself implied by Theorem 1.1 with $f = g = 0$ appears to be new for non-Dini continuous leading coefficients. The best available hypothesis under which the BPP was proved for $-\text{div}(A(x)Du) \geq 0$, was Dini continuity of $A$, see [4]. Previous works on the BPP also make hypotheses on the distributional sign (or absence) of $\text{div}(\beta) - c$.

The importance of such a quantification of the BPP was recognized only recently, but already a number of applications have appeared. The uniform up-to-the boundary inequality (4) for non-divergence form equations was proved in [31]. In the non-divergence framework we also refer to [8] Lemma 1.6], [6], [20], for estimates like (4) in which the left-hand side contains an integral on a interior subset, and the constant $C$ degenerates if this subset approaches the boundary. The best constant $\varepsilon$ for which (4) holds was specified in [33], for operators which are both in divergence and non-divergence form (the optimal value of $\varepsilon$ is an open problem for more general operators). The results in [31], [33] have been instrumental in a new method for a priori bounds for positive solutions of nonlinear elliptic equations – see [32] and the references there. Another application has just appeared in [13]. We will use Theorem 1.1 in the boundary regularity Theorem 1.2 below, as well as in the forthcoming works [25] on solvability of equations having quadratic dependence in the gradient and [35] on the Landis conjecture and elliptic.
estimates with optimized constants.

As often happens in elliptic theory, the statement of Theorem 1.1 is similar to that of the non-divergence case [31, Theorem 1.2]; however, the main point of the proof (the boundary growth lemma) requires a different approach. Here we use the classical idea of [11] to compare \( u \) with a solution of a “frozen coefficients” equation in a sufficiently small annulus which touches the boundary; however, we combine this comparison with direct use of elliptic estimates, in particular the Stampacchia maximum principle and the global \( C^1 \)-estimates from [10], thus avoiding the use of Green functions which has been frequent in proofs of the BPP in the divergence framework.

Our second main result is an application of Theorem 1.1 to boundary regularity theory. It concerns the following classical property: given two elliptic operators such that the solutions of the Dirichlet problem in \( \Omega \) for each of them have uniformly continuous gradient in \( \partial \Omega \); and a function which is only a subsolution and a supersolution of two different equations involving these operators, then this function may not even be differentiable in \( \Omega \) but still has a uniformly continuous gradient at \( \partial \Omega \). This is a fundamental result in the non-divergence theory, which goes back to Krylov and his proof of solvability and regularity of the Dirichlet problem [16]. Krylov’s property has been studied, extended and used over the years by many authors, see [27], [18], [21], [30], [19], and [5] for very general results and a large discussion, as well as the references in these works. However, this fact has never been proven for pure divergence-form equations, even in the simplest cases.

We show that Krylov’s property is valid in the divergence framework.

**Theorem 1.2** Assume \( \partial \Omega \) is in \( C^{1,\overline{\alpha}} \), \( \mathcal{L}_D^{(1)} \cdot \mathcal{L}_D^{(2)} \) are operators in the form (1) under (H1), whose coefficients \( A^{(i)}, \beta^{(i)}, g^{(i)} \in C^{\overline{\alpha}}(B_1^+), B^{(i)}, c^{(i)}, f^{(i)} \in L^q(B_1^+) \), for some \( \overline{\alpha} > 0, q > n, i = 1, 2 \). Assume \( u \in H^1(B_1^+) \) is such that

\[
\mathcal{L}_D^{(1)}[u] \leq f^{(1)}+\text{div}(g^{(1)}), \quad \mathcal{L}_D^{(2)}[u] \geq f^{(2)}+\text{div}(g^{(2)}) \text{ in } B_1^+, \quad u|_{B_0^+} \in C^{1,\overline{\alpha}}(B_0^+).
\]

Then there exists \( G \in C^{\alpha}(B_{1/2}^+, \mathbb{R}^n) \) (the “gradient” of \( u \) on \( B_{1/2}^+ \)), such that

\[
\|G\|_{C^{\alpha}(B_{1/2}^+)} \leq CW,
\]

and for every \( x \in B_{1/2}^+ \) and every \( x_0 \in B_{1/2}^0 \) we have

\[
|u(x) - u(x_0) - G(x_0) \cdot (x - x_0)| \leq CW|x - x_0|^{1+\alpha}, \quad \text{where} \quad W := \|u\|_{L^\infty(B_1^+)} + \|u\|_{C^{1,\alpha}(B_1^+)} + L, \quad L = \sum_i (\|f^{(i)}\|_{L^q(B_1^+)} + \|g^{(i)}\|_{C^{\overline{\alpha}}(B_1^+)}) .
\]

Here \( \alpha, C > 0 \) depend on \( n, \lambda, \Lambda, q, \overline{\alpha}, \|A^{(i)}\|_{C^{\overline{\alpha}}(B_1^+)} \); \( C \) also depends on the Hölder, resp. Lebesgue, norms of the lower-order coefficients and \( \partial \Omega \).
In Theorem 1.2 we strengthened the regularity assumptions on the coefficients to the most important and often encountered Hölder continuity. This permits us to ease technicalities and present the result as a consequence from Theorem 1.1 and the method developed in [30] for the non-divergence case.

In the next section we give some more comments on our hypotheses and framework. The last section is devoted to the proofs of the theorems.

2 Further comments

The distinction “divergence” vs. “non-divergence” is particularly relevant and delicate with regard to the BPP and its ramifications. For nondivergence type inequalities, say $\text{tr}(A(x)D^2u) \leq 0$, the BPP is true for any $A(x) \in L^\infty(\Omega)$. On the other hand, for inequalities in divergence form, say $\text{div}(A(x)Du) \leq 0$, the BPP may fail even for $A(x) \in C(\Omega)$ (see [24], [4], for counterexamples and more references). However, the BPP is true for that inequality if $A$ is Dini continuous (see [3], [4]), and as we now know by Theorem 1.1 even if $A$ has Dini mean oscillation. Furthermore, it is rather remarkable that the standard boundary Harnack inequality (in which two positive solutions are compared close to the boundary, as opposed to one solution and the distance function) is valid for $\text{div}(A(x)Du) = 0$ with $A \in L^\infty$ (see [7]), but fails for $\text{div}(A(x)Du) = f$, $f \neq 0$; however for the latter it is true if $A$ is only continuous, as was recently shown in [26]. Another example of how delicate the role of the regularity assumptions on the coefficients may be are the recent deep works on “propagation of smallness” (see [22]) for solutions of $\text{div}(A(x)Du) = 0$, which are valid for a symmetric Lipschitz $A$, but fail for $A \in C^\alpha$, $\alpha < 1$. In a certain sense, Theorem 1.1 above is a ”propagation of smallness” of $u/d$ from the boundary to the whole of the domain.

Next we comment on the regularity and integrability assumptions we make on the coefficients of the elliptic operators. We crucially use that the standard Dirichlet problem associated to the operator has global $C^1$-estimates. The Dini mean oscillation assumption on the leading coefficients is currently the most general available hypothesis under which a $C^1$-estimate up to the boundary is known, while mere continuity is not sufficient for such an estimate; we believe our method is sufficiently versatile to adapt to other situations, if global $C^1$-estimates are proved in the future under even more general assumptions.

We have assumed that the lower-order coefficients in $\mathcal{L}_D$ belong to $L^q$ with $q > n$, which is certainly the optimal Lebesgue integrability for Theorem 1.1.
(and even for the BPP, which is known to fail for instance for $b \in L^n$, see [28, Example 4.1]). On the other hand, the BPP is known under finer restrictions on the lower-order coefficients, such as intermediate spaces between $L^q$ for $q > n$, and $L^n$, see [3], [4], and the references there. Theorem 1.1 should be true under such assumptions too; however, since it is new even for operators without lower order coefficients, and to avoid technical complications, we do not study such extensions here. Our assumption permits us to directly quote the $C^1$-estimate in [10, Theorem 1.3] and concentrate on its use.

Similarly, while Theorem 1.2 is proved in large generality (and is new in the simplest cases such as equations without lower-order coefficients and with zero right-hand side), we expect and conjecture that it is true for even more general coefficients and operators. It should be possible to replace the H"{o}lder by Dini mean continuity in the assumptions on the leading coefficients; however this would render the rescaling argument which is in the core of the proof considerably more delicate. Furthermore, the result should be true for quasi-linear operators whose associated Dirichlet problem has $C^1$ estimates, such as operators considered in [17]. For instance, we expect Theorem 1.2 to be valid for operators with quadratic growth in the gradient as in [5], replacing the Pucci operators there by divergence form operators with coefficients which have Dini mean oscillation.

3 Proofs

3.1 Preliminaries

We start by recalling the $C^1$ estimate from [10]. Following that paper, a function $\varrho : [0, 1] \to [0, \infty]$ is a Dini function (we write $\varrho \in \mathcal{D}$) if $\varrho(0) = 0$, $0 < c_1 \rho(t) \leq \rho(s) \leq c_2 \rho(t)$ for $0 < t/2 \leq s \leq t$ and $\int_0(\varrho(s)/s) \ ds$ converges.

We note it is possible to assume without restricting the generality that $\varrho(s)$ is non-decreasing and continuously differentiable for $s > 0$, and $\varrho(s)/s$ is non-increasing, see [4] (so we can take $c_1 = 1/2$, $c_2 = 1$).

A function $h$ is Dini continuous on $\Omega$ (we write $h \in C^D(\Omega)$) if

$$\varrho_{D,h}(r) = \sup_{x \in \Omega} \sup_{y', y'' \in B^*_r(x)} |h(y') - h(y'')| \in \mathcal{D}.$$ 

We say that $\Omega$ is in $C^{1,D}$ if each point on $\partial \Omega$ has a neighborhood in which $\partial \Omega$ is the graph of a continuously differentiable function whose derivatives are in $C^D$. 

6
A function $h$ has Dini mean oscillation on $\Omega$ (we write $h \in C^{mD}(\Omega)$) if

$$
\varrho_{mD,h}(r) = \sup_{x \in \Omega} \int_{B^*_r(x)} \left| h(y) - \int_{B^*_r(x)} h(y) \, dy \right| \, dy \in D.
$$

Here as usual $f_G = \frac{1}{|G|} \int_G$. Note that $\varrho_{mD,h}(r) \leq \varrho_{D,h}(r)$, so Dini mean oscillation is a weaker hypothesis than Dini continuity. A standard example of non-Dini continuous function which has Dini mean oscillation is $h(x) = |\log |x||^{-\gamma}$, $\gamma \in (0,1]$ (for enlightening examples on the difference between Dini and mean Dini conditions, see [9], [23]). For $k > 0$, $t \in (0,1)$, under the rescaling

$$
\tilde{h}(y) = kh(ty), \text{ we have } \varrho_{D,h}(r) = k \varrho_{D,h}(tr), \quad \varrho_{mD,h}(r) = k \varrho_{mD,h}(tr). \quad (8)
$$

It is also true that in a $C^1$-domain $\Omega$, if $h \in C^{mD}(\Omega)$ then $h$ is uniformly continuous in $\Omega$, with a modulus of continuity $\omega_h(r)$ dominated by $\int_0^r (\varrho_{mD,h}(s)/s) \, ds$ (see [13] Lemma A.1).

**Theorem 3.1** (Dong-Escauriaza-Kim, [14]) Assume (H1) and (H3) hold in $\Omega$, $\partial \Omega \in C^{1,D}$, $\text{diam}(\Omega) \leq 1$. If $u \in H^1_0(\Omega)$ solves $L_D[u] = f + \text{div}(g)$ in $\Omega$, then $u \in C^1(\Omega)$. In addition,

$$
\|u\|_{C^1(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} + \|g\|_{L^\infty(\Omega)} + \mathcal{M}_q(\Omega)), \quad (9)
$$

where $\mathcal{M}_q(\Omega)$ is a quantity which describes the Dini mean oscillation of $g$ and is defined through the values of $\varrho_{mD,g}(r)$ (in particular of $\int_0^r (\varrho_{mD,g}(s)/s) \, ds$, $r \in (0,1]$). The constant $C$ is bounded above in terms of $n, \lambda, \Lambda, q$, the $C^{1,D}$-norm of $\partial \Omega$, upper bounds on the $L^q$-norms of $b, c, f$, the $L^\infty$-norm of $\beta$, $\mathcal{M}_A(\Omega)$, $\mathcal{M}_\beta(\Omega)$.

Furthermore, there exists a modulus of continuity $\sigma$ determined by $n, \lambda, \Lambda, q$, the $L^q$-norms of $b, c, f$, the functions $\varrho_{mD}$, $\varrho_{D}$ corresponding to $A, \beta, g$ and $\partial \Omega$, and by $\|u\|_{L^2(\Omega)}$, such that

$$
|Du(x) - Du(y)| \leq \sigma(|x - y|).
$$

This statement can be inferred from [10] Theorem 1.3 and its proof. See in particular inequalities (2.31) and (2.36) in [10]. Note the term $\|Du\|_{L^1(\Omega)}$ which appears in (2.31) is bounded by the right-hand side of (9) by the standard Sobolev bounds for weak solutions (see for instance [36] Theorem 3.2)

$$
\|u\|_{H^1(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} + \|g\|_{L^\infty(\Omega)}). \quad (10)
$$

**Remark 3.1.** We have that $\mathcal{M}_{tg}(\Omega) = t\mathcal{M}_g(\Omega)$ for $t > 0$ and for each $\epsilon > 0$ there is $\delta > 0$ such that $\mathcal{M}_g(\Omega) < \epsilon$ if $\int_0^1 (\varrho_{mD,g}(s)/s) \, ds < \delta$.

**Remark 3.2.** If $g \in C^\alpha$ we can take $\mathcal{M}_g$ to be the usual $\alpha$-Hölder bracket of $g$. 

7
3.2 Proof of Theorem 1.1

We observe that all hypotheses on the operator $\mathcal{L}_D$ are preserved under a $C^{1,\varphi}$-regular change of variables (that Dini mean oscillation is preserved follows from [10, Lemma 2.1]). So from now on we will assume that the boundary of $\Omega$ is locally flat, included in $\{x : x_n = 0\}$. In the following $Q_\rho = Q_\rho(\rho e)$ denotes the cube with center $\rho e$ and side $\rho$, where $e = (0, \ldots, 0, 1/2)$. To avoid confusion, the reader’s attention is brought to the fact that $Q_\rho$ is not centered at the origin but has its bottom on $\{x_n = 0\}$.

We first establish the following growth lemma. We assume that in $Q_2$ all coefficients have the regularity given in (H3), and set

$$W = \|f\|_{L^q(Q_2)} + \|g\|_{L^\infty(Q_2)} + \mathcal{M}_g(Q_2).$$

Lemma 3.1 Given $\nu > 0$, there exist $a, k \in (0, 1)$ depending on $n, \lambda, \Lambda, q, \theta_{mD,A}, \nu$, such that if $u \in H^1(Q_2)$ is a weak solution of

$$\mathcal{L}_D[u] \geq f + \text{div}(g), \quad u \geq 0 \text{ in } Q_2,$$

$$\|b\|_{L^q(Q_2)} \leq 1, \quad \|c\|_{L^q(Q_2)} \leq 1, \quad \|\beta\|_{L^\infty(Q_2)} + \mathcal{M}_\beta(Q_2) \leq 1, \quad W \leq a,$$

and we have

$$|\{u > x_n\} \cap Q_1| \geq \nu, \quad (11)$$

then $u > kx_n$ in $Q_1$.

Proof. For all $\rho \in (0, 1]$ set $x_\rho = (0, \ldots, 0, \rho)$, $A_\rho = B_\rho(x_\rho) \setminus B_{\rho/2}(x_\rho)$. For $y \in A_1$, $v \in H^1(A_1)$ introduce the operator

$$\mathcal{L}_\rho[v] := -\text{div}_y(A(y)Dv(y) + \beta_\rho(y)v(y)) + b_\rho(y)D_yv(y) + c_\rho(y)v(y),$$

where $\beta_\rho(y) = \rho \beta(\rho y)$, $b_\rho(y) = \rho b(\rho y)$, $c_\rho(y) = \rho^2 c(\rho y)$.

Note $\mathcal{L}_D[u] \leq f + \text{div}(g)$ in $A_\rho$ is equivalent to

$$\mathcal{L}_\rho[u_\rho] \leq f_\rho(y) + \text{div}_y(g_\rho(y)) \quad \text{in } A_1,$$

if we set

$$y = x/\rho, \quad u_\rho(y) = u(x), \quad f_\rho(y) = \rho^2 f(\rho y), \quad g_\rho(y) = \rho g(\rho y).$$

Then

$$\|\beta_\rho\|_{L^q(A_1)} = \rho^{1-n/q}\|\beta\|_{L^q(A_\rho)} \leq \rho^{1-n/q}\|\beta\|_{L^q(Q_2)} \leq 2^n \rho^{1-n/q}, \quad (12)$$

and
and similarly for the other coefficients with subscript $\rho$. So all these coefficients have $L^q$ norms in $A_1$ that tend to zero as $\rho \to 0$, since $1 - n/q > 0$ for $n < q \leq \infty$. Also by the definition of $\g_mD$, (8) and Remark 3.1 we have

$$\mathcal{M}_{g_\rho}(A_1) \leq \rho \mathcal{M}_g(Q_2) \leq \rho \quad \text{and} \quad \mathcal{M}_{\beta_\rho}(A_1) \leq \rho \mathcal{M}_\beta(Q_2) \leq \rho. \quad (13)$$

Fix a smooth function $\psi$ such that $\psi = 1$ in $B_{1/2}$, $\psi = 0$ outside $B_1$, $\|\psi\|_{C^1} = C(n)$. Set

$$\bar{f}_\rho = f_\rho - b.D\psi - c\psi, \quad \bar{g}_\rho = g_\rho + AD\psi + \beta\psi.$$ 

It follows from classical solvability results (see Theorems 3.1 and 3.3 of [36]) that for some $\rho_0 \in (0, 1/2)$ depending only on $n, \lambda, \Lambda, q$, and for all $\rho \in (0, \rho_0)$ there is a unique function

$$w_\rho \in H^1_0(A_1) \quad \text{such that} \quad \mathcal{L}_\rho[w_\rho] = \bar{f}_\rho + \text{div}(\bar{g}_\rho) \quad \text{in} \quad A_1.$$ 

Hence $v_\rho = w_\rho + \psi \in H^1(A_1)$ solves

$$\begin{cases} 
\mathcal{L}_\rho[v_\rho] = f_\rho + \text{div}(g_\rho) & \text{in} \quad A_1 \\
v_\rho = 1 & \text{on} \quad \partial B_{1/2} \\
v_\rho = 0 & \text{on} \quad \partial B_1.
\end{cases} \quad (14)$$

We set $\mathcal{L}_0[\cdot] = -\text{div}(A(0)D\cdot)$ (note $\mathcal{L}_0$ has constant coefficients and is also in non-divergence form) and let $v_0$ be the solution of

$$\begin{cases} 
\mathcal{L}_0[v_0] = 0 & \text{in} \quad A_1 \\
v_0 = 1 & \text{on} \quad \partial B_{1/2} \\
v_0 = 0 & \text{on} \quad \partial B_1.
\end{cases} \quad (15)$$

By the maximum principle $0 < v_0 < 1$ in $A_1$. Since $\mathcal{L}_0[1] = 0$, if we extend $v_0 = 1$ in $B_{1/2}$ we obtain a supersolution, $\mathcal{L}_0[v_0] \geq 0$ in $B_1$. By theorem 4.1.2 in [31] (or Theorem 1.2 in [33]) we have

$$v_0(y) \geq c_0 \text{dist}(y, \partial B_1) \quad (16)$$

for all $y \in A_1$, and some $c_0 > 0$ depending only on $n, \lambda, \Lambda$. Also, by standard elliptic estimates for equations with constant coefficients

$$\|v_0\|_{C^2(A_1)} \leq C_0 = C_0(n, \lambda, \Lambda). \quad (17)$$

Set $z_\rho = v_\rho - v_0$. We have

$$\mathcal{L}_\rho[z_\rho] = f_\rho + \text{div}(g_\rho) + (\mathcal{L}_0 - \mathcal{L}_\rho)[v_0] = \bar{f}_\rho + \text{div}(\bar{g}_\rho).$$

9
in $A_1$, where

$$f_\rho = f_\rho - b_\rho Dv_0 - c_\rho v_0$$

$$\tilde{g}_\rho = (A(\rho y) - A(0)) Dv_0 + \beta_\rho v_0 + g_\rho.$$ 

Clearly by (12) and (17) $\alpha = 1 - n/q$ and $\omega_A$ is the uniform modulus of continuity of $A$ given by [13, Lemma A.1]). Further, by writing the last equation in the form

$$\tilde{\mathcal{L}}_\rho[z_\rho] = \tilde{f}_\rho - c_\rho(y)z_\rho(y) + \text{div}(\tilde{g}_\rho - \beta_\rho(y)z_\rho(y))$$

where

$$\tilde{\mathcal{L}}_\rho[z_\rho] = -\text{div}(A(\rho y) Dz_\rho(y)) + b_\rho(y) \cdot Dz_\rho(y)$$

and by applying Stampacchia generalized maximum principle ([12, Theorem 8.16]), since $z_\rho = 0$ on $\partial A_1$ we get

$$\|z_\rho\|_{L^\infty(A_1)} \leq C(\|\tilde{f}_\rho\|_{L^q(A_1)} + \|\tilde{g}_\rho\|_{L^q(A_1)})$$

so setting $\rho_1 > 0$ such that $C_2 \rho_1^\alpha = 1/2$ we have for $\rho < \rho_2 = \min\{\rho_0, \rho_1\}$

$$\|z_\rho\|_{L^\infty(A_1)} \leq C(\|\tilde{f}_\rho\|_{L^q(A_1)} + \|\tilde{g}_\rho\|_{L^q(A_1)}) \leq C(\omega_A(\rho) + \rho^\alpha).$$

The first of these inequalities is independent of the form of $\tilde{f}_\rho, \tilde{g}_\rho$, and means that the generalized maximum principle holds for the equation $\mathcal{L}_\rho[z_\rho] = \tilde{f}_\rho + \text{div}(\tilde{g}_\rho)$ in $A_1$, and the comparison principle is valid for $\mathcal{L}_\rho$ in $A_1$. By standard Sobolev estimates for weak solutions $z_\rho \in H^1_0(A_1)$ of this equation

$$\|z_\rho\|_{H^1(A_1)} \leq C(\|z_\rho\|_{L^2(A_1)} + \|\tilde{f}_\rho\|_{L^q(A_1)} + \|\tilde{g}_\rho\|_{L^q(A_1)}) \leq C(\omega_A(\rho) + \rho^\alpha).$$

We now apply Theorem 3.1 to the equation $\mathcal{L}_\rho[z_\rho] = \tilde{f}_\rho + \text{div}(\tilde{g}_\rho)$. Note that

$$\|\tilde{g}_\rho\|_{L^\infty(A_1)} + M_{\tilde{g}_\rho}(A_1) \to 0 \text{ as } \rho \to 0$$

by the homogeneity property in [8], Remark 3.1, [12], [13]. Therefore, Theorem 3.1 implies that there exists $\sigma_\rho$ with $\sigma_\rho \to 0$ as $\rho \to 0$, such that

$$\|z_\rho\|_{C^1(A_1)} \leq \sigma_\rho.$$ 

By the mean value theorem, for each $y = (0, \ldots, 0, y_n) \in A_1$

$$\left| \frac{z_\rho(y)}{y_n} - \frac{z_\rho(0)}{y_n} \right| \leq \sigma_\rho.$$
Finally, we get by (10)
\[
v_{\rho}(y) = v_0(y) + z_{\rho}(y) \geq c_0 - \sigma_{\rho} \geq c_0/2, \quad y = (0, \ldots, 0, y_n) \in A_1.
\]
provided \( \rho \in (0, \rho_3) \), for \( \rho_3 > 0 \) fixed so that \( \sigma_{\rho} \leq c_0/2 \) for \( \rho \in (0, \rho_3) \).

By (11) there is \( \rho_4 > 0 \) depending on \( \nu \) such that
\[
|\{ u > \rho_4 \} \cap Q_1 \cap \{ x_n \geq \rho_4 \}| \geq \nu/2.
\]
Set \( \bar{\rho} = \min_{0 \leq i \leq 4} \rho_i \). Then by the interior weak Harnack inequality applied in \( \{ x_n \geq \bar{\rho}/2 \} \cap Q_1 \) to the inequality satisfied by \( u \) we get
\[
u \geq c \bar{\rho} \nu - C' W \geq c' - C'a
\]
in \( \{ x_n \geq \rho_4/2 \} \cap Q_1 \) and hence by choosing \( a \) sufficiently small \( u \geq c \bar{\rho} > 0 \) in that set. This implies that \( u_{\rho} \geq c \bar{\rho} v_{\rho} \) on \( \partial A_1 \), and since \( L_{\rho}[u_{\rho}] \geq L_{\rho}[v_{\rho}] \) in \( A_1 \), by the comparison principle, \( v_{\rho} \geq c_0/2 \) in \( A_1 \) and \( x_n = \bar{\rho} y_n \) we get
\[
u \geq \min \{ c_0 c_{\bar{\rho}}/(2 \bar{\rho}), c \bar{\rho} \},
\]
provided \( x = (0, \ldots, 0, x_n) \in Q_1 \). We can shift the origin along \( \{ x_n = 0 \} \) soLemma 3.1 is proved.

**Proof of Theorem 1.1** It is enough to prove (4) for \( R = 1 \), \( x_0 = 0 \) (the general case follows by scaling and translation). We can also assume \( d_0 = 2 \).

We set
\[
r_0 = (4 + \| b \|_{L_q(B_2^+)}^{1-n/q} + \| c \|_{L_q(B_2^+)}^{1-n/q} + \| \beta \|_{L_\infty(B_2^+)} + M_\beta(B_2^+))^{-1}.
\]

By the same change of variables as in the beginning of the proof of Lemma 3.1 precisely for \( \rho = r_0 \), replacing \( L_D \) by \( L_{r_0} \) and \( u \) by \( u_{r_0} \) we obtain an equation in a set containing \( B_1^{t_1} \), whose lower order coefficients \( \beta, b, c \) have the bounds required in Lemma 3.1. Note that if Theorem 1.1 is proved for \( L_{r_0} \) in a half-ball of unit size, scaling back we obtain (1) for \( L_D, u, \) and balls with radius \( R = r_0 \), with a constant \( C \) depending also on \( r_0 \). We can then cover \( B_1^{t_1} \) with overlapping balls and semi-balls of size \( r_0 \), use that (1) holds in each of these balls and a Harnack chain argument, to deduce (1) in \( B_1^{t_1} \) (see the proof of Theorem 2.1 in [34] for such a Harnack chain argument).

To prove Theorem 1.1 for \( L_{r_0} \) in \( B_1^{t_1} \) we use Lemma 3.1 which we proved above. Actually, once Lemma 3.1 is available, Theorem 1.1 follows as in (the non-divergence case) [31], the proof becomes essentially independent
of the nature of the PDE. We sketch the argument, for completeness and convenience.

The inequality (4) in Theorem 1.1 follows from Lemma 3.1 in exactly the same way as [31, Theorem 1.2] follows from [31, Lemma 4.1]. We may repeat almost verbatim the argument on pages 7475-7478 in that paper. We note this argument uses only standard analysis, cube decomposition, the interior Harnack inequality and Lemma 3.1, so is independent of the nature (divergence or non-divergence) of the PDE.

To prove (5) in Theorem 1.1 we first observe that $u$ is Hölder continuous in $B^{+}_{3/2}$ (see Proposition 3.1 below). Further, we have the following Lipschitz estimate: if $u \in H^{1}(B^{+}_{2}) \cap C(B^{+}_{2})$ is a weak solution of $\mathcal{L}_{D}[u] \leq f + \text{div}(g)$ in $B^{+}_{2}$ with $u \leq 0$ on $\partial B^{+}_{2}$ then

$$u(x) \leq C(\sup_{B^{+}_{3/2}} + \|f\|_{L^{q}(B^{+}_{2})} + \|g\|_{L^{\infty}(B^{+}_{2})} + \mathcal{M}_{g}(B^{+}_{2})) \cdot x, \quad x \in B^{+}_{r}. \quad (18)$$

This can be proved in our setting in exactly the same way as [31, Theorem 2.3], by replacing the ABP inequality in the proof there by the Stampacchia generalized maximum principle ([12, Theorem 8.15-8.16]) here, and by using the $C^{1}$-estimate in Theorem 3.1 together with the solvability results we already quoted ([36, Theorems 3.1 and 3.3]) in a sufficiently small neighborhood of the boundary.

Then, thanks to (18) we can repeat the proof of [31, Theorem 1.3] (see page 7464 in [31]) in order to show that, in our divergence setting, each weak solution of $\mathcal{L}_{D}[u] \leq f + \text{div}(g)$ in $B^{+}_{2}$ with $u \leq 0$ on $\partial B^{+}_{2}$ is such that for each $r > 0$

$$\sup_{B_{r}} \left( \frac{u^{+}}{x_n} \right) \leq C \left( \left( \int_{B^{+}_{3/2}} (u^{+})^{r} \right)^{1/r} + \|f\|_{L^{q}(B^{+}_{2})} + \|g\|_{L^{\infty}(B^{+}_{2})} + \mathcal{M}_{g}(B^{+}_{2}) \right).$$

This, together with (4), gives (5).

The third statement in Theorem 1.1 follows from (4), (5), the interior (weak) Harnack inequality, and the same covering/Harnack chain argument as above.

### 3.3 Proof of Theorem 1.2

Theorem 1.2 follows from Theorem 1.1 and the method developed in [30]. More precisely, we will see that Theorem 1.2 is to the divergence case what [30, Theorem 1.1] is to the non-divergence one. For the reader’s convenience
we are going to fully review the proof and make explicit all parallels between
the two works, giving all details at points where differences appear.

We recall we can assume that the boundary of \( \Omega \) is locally flat, included
in \( \{ x : x_n = 0 \} \). We first observe that global Hölder estimates are available
for functions which satisfy the hypothesis of Theorem 1.2 (this will replace
the use of \[30, Proposition 2.6\] in our case).

**Proposition 3.1** Assume \( \mathcal{L}_D^{(1)} : \mathcal{L}_D^{(2)} \) are operators in the form \[ (1) \] which satis-
ify \((H1)-(H2)\). Assume \( u \in H^1(B_1^+) \) is such that
\[
\mathcal{L}_D^{(1)}[u] \leq f^{(1)} + \text{div}(g^{(1)}), \quad \mathcal{L}_D^{(2)}[u] \geq f^{(2)} + \text{div}(g^{(2)}) \quad \text{in } B_1^+, \quad u|_{B_0^+} \in C^{\overline{\pi}}(B_1^+),
\]
for some \( \overline{\pi} > 0 \). Then \( u \in C^\alpha(B_3/4^+) \) and
\[
\|u\|_{C^\alpha(B_3/4^+)} \leq C(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|g\|_{L^q(B_1^+)}),
\]
for \( \alpha, C > 0 \) depending on \( n, \lambda, \Lambda, p, q, \overline{\pi} \) and upper bounds on the Lebesgue
norms of \( \beta, b, c \) from \((H2)\) in \( B_1^+ \).

**Proof.** This is \[12, Theorem 8.29\] together with the remark at the end of
Section 8.10 of \[12\] or \[37, Corollary 6.1\]. Note these results were stated for
solutions of equations, but their proofs were actually done for any function \( u \)
which is a supersolution and a subsolution of two different equations - since
the proof is essentially an application of the WHI and bWHI to functions
in the form \( \sup_B u - u \) and \( u - \inf_B u \) for well chosen balls and half-balls \( B \).

Therefore from now on we can assume without restricting the generality
that \( u \in H^1(B_1^+) \cap C^\alpha(B_1^+) \) as well as \( e^{(i)} = \beta^{(i)} = 0 \) in Theorem 1.2
(by replacing \( f^{(i)} \) by \( f^{(i)} - c^{(i)} u \), \( g^{(i)} \) by \( g^{(i)} + \beta^{(i)} u \)).

To parallel the notations in \[30\] we set
\[
K = \sum_i \|b^{(i)}\|_{L^q(B_1^+)}, \quad L = \sum_i (\|f^{(i)}\|_{L^p(B_1^+)} + \|g^{(i)}\|_{C^{\overline{\pi}}(B_1^+)}).
\]

The inequality \( (5) \) plays a pivotal role in the proof of Theorem 1.2
providing the same bound as in \[30, Proposition 2.5\] (we will take \( \varepsilon_0 = 1 \) in that
proposition). As before \( e = (0, \ldots, 0, 1/2) \).

**Proposition 3.2** Assume \( u \) is as in Theorem 1.2, with \( u \geq 0 \) in \( B_1^+ \), \( u = 0 \)
on \( B_0^1 \), and \( K \leq 1 \). Then
\[
u(x) \geq c_0(u(e) - C_0 L) x_n, \quad x \in B_{3/4}^+,
\]
for some \( c_0, C_0 > 0 \) depend on \( n, \lambda, \Lambda, \overline{\pi}, \|A^{(i)}\|_{C^{\overline{\pi}}(B_1^+)} \).

13
Proof. This follows directly from (5). □

Below we will also use the Lipschitz estimate (18) which replaces in our setting Proposition 2.4 in [30].

Proposition 3.3 Assume $u$ is as in Theorem 1.2 with $u = 0$ on $B_0^1$. Then

$$|u(x)| \leq \bar{C}(\|u\|_{L^\infty(B_1^+)} + L) x_n, \quad x \in B_{3/4}^+, \quad (21)$$

where $\bar{C} > 0$ depends on $n, \lambda, \Lambda, q, \alpha, \parallel A^{(i)} \parallel_{C^1(B_1^+)}$, $K$.

Proof. This follows directly from (18) applied to $u$ and $-u$. □

Proof of Theorem 1.2. The first (and key) step in the proof is the particular case when $u$ vanishes on the boundary, $u|_{B_0^1} = 0$.

Proposition 3.4 Theorem 1.2 is valid under the additional assumption $u|_{B_0^1} = 0$.

Proof. Substituting $u$ by $u/W$, we can assume $W \leq 1$, that is, $|u| + L \leq 1$.

First, we observe that it is enough to establish the existence of a “boundary gradient” at one point, for instance, it is enough to prove that there is a constant $G_0 \in \mathbb{R}$, $|G_0| \leq C$, such that (17) holds for $x_0 = 0$, $G(0) = (0, \ldots, 0, G_0)$. Once this is proved, Proposition 3.4 follows from a simple translation and analysis argument, described in the proof of [30, Theorem 3.2] (note that argument does not depend on the PDE).

So let us prove that such a constant $G_0$ exists. The proof uses the same idea as the one of [30, Lemma 3.1], which is the corresponding result for non-divergence form equations (setting $A = G_0$ there). We can repeat the beginning of that proof, defining the sequences $r_k = 2^{-k}$, and $U_k, V_k$, to be built in such a way that $U_1 = \bar{C}$, $V_1 = -\bar{C}$ ($\bar{C} \geq 1$ is the constant from the Lipschitz estimate (21)), $U_k$ is decreasing, $V_k$ is increasing,

$$V_k x_n \leq u(x) \leq U_k x_n \quad \text{in} \quad B_{r_k}, \quad U_k - V_k \leq M r_k^\alpha,$$

which guarantees that the limit $G_0 = \lim_{k \to \infty} V_k = \lim_{k \to \infty} U_k$ is such that $|G_0| \leq \bar{C}$ and has the desired property

$$|u(x) - G_0 x_n| \leq 2^\alpha M |x|^{1+\alpha}, \quad x \in B_{1/2}^+.$$

The constants $\alpha, M$ are defined by

$$\alpha = \frac{1}{2} \min\{1 - n/q, \tilde{\alpha}, \tilde{\alpha}\}, \quad \text{with} \quad 2^{-\tilde{\alpha}} = 1 - c_0/8,$$
where \( c_0 \leq 1 \) is the constant from \([20]\) and
\[
M = 2\tilde{C}r_k^{-\alpha} = \tilde{C}2^{1+\alpha k_0}, \quad \text{with } k_0 \text{ chosen so that}
\]
\[
(\tilde{C}K + 1)r_k^{1-\alpha-n/q} < 1/(16C_0), \quad (\tilde{C}\|A^{(i)}\|_{C^0(B_1^+)} + 1)r_k^{\alpha-\alpha} < 1/(16C_0),
\]
where \( C_0 \geq 1 \) is the constant from \([20]\).

Exactly as in \([30, \text{Lemma 3.1}]\) we define \( U_1 = \ldots = U_k, V_1 = \ldots = V_k \) and construct iteratively \( U_k, V_k \) for \( k > k_0 \). The iterative procedure is the same, up to the introduction of the rescaled function
\[
v_k(x) = r_k^{1-\alpha}(u(r_kx) - V_kr_kx), \quad x \in B_1^+.
\]

where the only real difference with the proof of \([30, \text{Lemma 3.1}]\) appears: in our setting the new function \( v_k \) satisfies in \( B_1^+ \)
\[
\frac{1}{r_k} \text{div}(A(r_kx)(r_k^\alpha Du_k(x) + V_ke_n)) + b(r_kx)(r_k^\alpha Du_k(x) + V_ke_n)) \\
\leq (\geq) f(r_kx) + \frac{1}{r_k} \text{div}(g(r_kx))
\]
(we do not write the indices \( i = 1, 2 \) for display convenience), which can be rewritten as
\[
-\text{div}(A(r_kx)Du_k(x)) + r_k b(r_kx)Du_k(x) \leq (\geq) r_k^{1-\alpha}(V_kb(r_kx)e_n + f(r_kx)) \\
+ \text{div}\left(V_k A(r_kx) - A(0) \right) + \frac{1}{r_k^\alpha} \text{div}\left( g(r_kx) - g(0) \right).
\]

We observe that
\[
\|A(r_kx)\|_{C^0(B_1^+)} \leq \|A(x)\|_{C^0(B_1^+)},
\]
\[
r_k \|b(r_kx)\|_{L^1(B_1^+)} \leq r_k^{1-n/q} \|b\|_{L^1(B_1^+)} \leq K r_k^{1-n/q},
\]
\[
r_k^{1-\alpha} \|V_k b(r_kx)\|_{L^q(B_1^+)} \leq C r_k^{1-\alpha-n/q} \|b\|_{L^q(B_1^+)} \leq C r_k^{1-\alpha-n/q},
\]
\[
r_k^{1-\alpha} \|f(r_kx)\|_{L^q(B_1^+)} \leq r_k^{1-\alpha-n/q} \|f\|_{L^q(B_1^+)} \leq r_k^{1-\alpha-n/q}, \quad (L \leq 1),
\]
\[
\left\| V_k A(r_kx) - A(0) \right\|_{C^0(B_1^+)} \leq C \|A\|_{C^0(B_1^+)} r_k^{\alpha-\alpha},
\]
\[
\left\| g(r_kx) - g(0) \right\|_{C^0(B_1^+)} \leq \|g\|_{C^0(B_1^+)} r_k^{\alpha-\alpha} \leq r_k^{\alpha-\alpha}, \quad (L \leq 1),
\]

Hence the last differential inequality is in the form
\[
-\text{div}(A(r_kx)Du_k(x)) + b_k Du_k \leq (\geq) f_k + \text{div}(\hat{g}_k)
\]
where \( \hat{f}_k, \hat{g}_k \) are such that the corresponding quantities \( \hat{K}, \hat{L} \) satisfy \( \hat{K} < 1 \) and \( \hat{L} < 1/(8C_0) \), thanks to the choice of \( k_0 \) and \( k \geq k_0 \). So we can apply Proposition 3.2 getting

\[
v_k(x) \geq c_0(v_k(\epsilon) - C_0\hat{L})x_n, \quad x \in B^+_1/2.
\]

This inequality replaces the inequality (3.4) in [30] and the rest of the proof of Lemma 3.1 there is repeated without any change, the PDE is no longer used.

Thus Proposition 3.4 is established.

To extend Proposition 3.4 to arbitrary boundary data, we can just remove a \( C^1,\bar{\alpha} \)-function from \( u \). For instance, for \( x = (x', x_n) \in B_1 \) we set

\[
\psi(x) = u(x', 0) \quad \text{and} \quad v = u - \psi.
\]

Then \( v \) vanishes on the flat boundary and solves the same equation as \( u \), with \( f \) replaced by \( f - bD\psi \) and \( g \) replaced by \( g + A\psi \). Since obviously \( \|\psi\|_{C^1,\bar{\alpha}(B^+_1)} = \|u\|_{C^1,\bar{\alpha}(B^+_1)} \), it is trivial to see that Proposition 3.4 applied to \( v \) implies the statement of Theorem 1.2.

Funding acknowledgements. B. Sirakov is supported by CNPq grant 310989/2018-3 and FAPERJ grant E-26/201.205/2021. M. Soares is supported by a post-doctoral grant DGAPA–UNAM.

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\(^1\)We note a small misprint in [30], one sets there \( \varepsilon_1 = 1/(8C_0) \).
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