ADAPTIVE DAMPING AND MEAN REMOVAL FOR THE GENERALIZED APPROXIMATE MESSAGE PASSING ALGORITHM

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ABSTRACT

The generalized approximate message passing (GAMP) algorithm is an efficient method of MAP or approximate-MMSE estimation of \( x \) observed from a noisy version of the transform coefficients \( z = Ax \). In fact, for large zero-mean i.i.d sub-Gaussian \( A \), GAMP is characterized by a state evolution whose fixed points, when unique, are optimal. For generic \( A \), however, GAMP may diverge. In this paper, we propose adaptive-damping and mean-removal strategies that aim to prevent divergence. Numerical results demonstrate significantly enhanced robustness to non-zero-mean, rank-deficient, column-correlated, and ill-conditioned \( A \).

Index Terms—Approximate message passing, belief propagation, compressed sensing.

1. INTRODUCTION

Consider estimating a realization \( x \in \mathbb{R}^N \) of a random vector \( x \) with statistically independent components \( x_n \sim p_{x_n} \) from observations \( y \in \mathbb{R}^M \) that are conditionally independent given the transform outputs

\[
z = Ax
\]

for some known matrix \( A = \{a_{nm}\} \in \mathbb{R}^{M \times N} \). Here, the likelihood function can be written as \( p_{y|x}(y|Ax) \) with separable \( p_{y|x} \), i.e., \( p_{y|x}(y|z) = \prod_{m=1}^{M} p_{y|m|x}(y_m|z_m) \). Such problems arise in a range of applications including statistical regression, inverse problems, and compressive sensing. Note that, for clarity, we use sans-serif fonts (e.g., \( x, x_n \)) to denote random quantities and serif fonts (e.g., \( x, x_n \)) to denote deterministic ones.

Assuming knowledge of the prior \( p_x(x) = \prod_{n=1}^{N} p_{x_n}(x_n) \) and likelihood \( p_{y|x}(y|x) \), typical estimation goals are to compute the minimum mean-squared error (MMSE) estimate \( \hat{x}_{\text{MMSE}} = \arg \min_x p_{y|x}(y|x) \) or the maximum a posteriori (MAP) estimate \( \hat{x}_{\text{MAP}} = \arg \max_x p_{x|y}(x|y) = \arg \max_x J_{\text{MAP}}(x) \) for the MAP cost

\[
J_{\text{MAP}}(\hat{x}) \triangleq - \ln p_{y|x}(y|Ax) - \ln p_x(\hat{x}).
\]

Recently, the generalized approximate message passing (GAMP) algorithm [1] has been proposed as a means of tackling these two problems in the case that \( M \) and \( N \) are large. Essentially, GAMP uses a high-dimensional approximation of loopy belief propagation to convert the MMSE or MAP inference problems into a sequence of tractable scalar inference problems.

GAMP is well motivated in the case that \( A \) is a realization of a large random matrix with i.i.d zero-mean sub-Gaussian entries. For such \( A \), the large-system limit (i.e., \( M, N \to \infty \) for fixed \( M/N \in \mathbb{R}_+ \)), GAMP is characterized by a state evolution whose fixed points, when unique, are MMSE or MAP optimal [1][3]. Furthermore, for generic \( A \), it has been shown [4] that MAP-GAMP’s fixed points coincide with those of a Bethe free entropy [5], as discussed in detail in Section 2.2.

For generic \( A \), however, GAMP may not reach its fixed points, i.e., it may diverge (e.g., [6]). The convergence of GAMP has been fully characterized in [7] for the simple case that \( p_{x_n} \) and \( p_{y|m|x} \) are Gaussian. There, it was shown that Gaussian-GAMP converges if and only if the peak-to-average ratio of the squared singular values of \( A \) is sufficiently small. A damping modification was then proposed in [7] that guarantees the convergence of Gaussian-GAMP with arbitrary \( A \), at the expense of a slower convergence rate. For strictly log-concave \( p_{x_n} \) and \( p_{y|m|x} \), the local convergence of GAMP was also characterized in [7]. However, the global convergence of GAMP under generic \( A \), \( p_{x_n} \), and \( p_{y|m|x} \) is not yet understood.

Because of its practical importance, prior work has attempted to robustify the convergence of GAMP in the face of “difficult” \( A \) (e.g., high peak-to-average singular values) for generic \( p_{x_n} \) and \( p_{y|m|x} \). For example, “swept” GAMP (SwAMP) [8] updates the estimates of \( \{x_n\}_{n=1}^{N} \) and \( \{z_m\}_{m=1}^{M} \) sequentially, in contrast to GAMP, which updates them in parallel. Relative to GAMP, experiments in [8] show that SwAMP is much more robust to difficult \( A \), but it is slower and cannot facilitate fast implementations of \( A \) like an FFT. As another example, the public-domain GAMPrematlab implementation [9] has included “adaptive damping” and “mean removal” mechanisms for some time, but they have never been described in the literature.

In this paper, we detail the most recent versions of GAMPmatlab’s adaptive damping and mean-removal mechanisms, and we experimentally characterize their performance on non-zero-mean, rank-deficient, column-correlated, and ill-conditioned \( A \) matrices. Our results show improved robustness relative to SwAMP and enhanced convergence speed.

2. ADAPTIVELY DAMPED GAMP

Damping is commonly used in loopy belief propagation to “slow down” the updates in an effort to promote convergence. (See, e.g., [10] for damping applied to the sum-product algorithm and [7][11] for damping applied to GAMP.) However, since not enough damping...
is already at its minimum allowed value \( \beta_{\text{min}} \); else it fails. If the iteration passes, \((R14)-(R15)\) implement a stopping condition, \((R16)\) increases \( \beta(t) \) by a factor \( G_{\text{pass}} \geq 1 \) (up to the maximum value \( \beta_{\text{max}} \)), and \((R17)\) increments the counter \( t \). If the iteration fails, \((R18)\) decreases \( \beta(t) \) by a factor \( G_{\text{fail}} < 1 \) (down to the minimum value \( \beta_{\text{min}} \)) and the counter \( t \) is not advanced, causing AD-GAMP to re-try the \( t \)th iteration with the new value of \( \beta(t) \).

In the MAP case, line \((R12)\) simply computes the cost \( J_{\text{MAP}}(\tilde{t}(t+1)) = J_{\text{MAP}}(\tilde{x}(t+1)) \) for \( J_{\text{MAP}} \) from \( \tilde{t} \). The MMSE case, which is more involved, will be described next.

2.2. MMSE-GAMP Cost Evaluation

As proven in \((R4)\) and interpreted in the context of Bethe free entropy in \((R5)\), the fixed points of MMSE-GAMP are critical points of the optimization problem

\[
(f_x, f_z) = \arg \min_{f_x, f_z} J_{\text{Bethe}}(f_x, f_z) \text{ s.t. } E[f_x|b_z] = A E[X|b_x]
\]

\[
J_{\text{Bethe}}(f_x, f_z) = D(b_x||p_x) + D(b_z||p_yf_z|Z) + H(b_z|\nu^p)
\]

\[
H(b_z|\nu^p) \triangleq \frac{1}{2} \sum_{m=1}^{M} \left( \frac{\nu_m b_z, d_m}{\nu^p_m} + \ln 2\pi \nu^p_m \right)
\]

where \( b_x \) and \( b_z \) are separable pdfs, \( Z \) is an upper bound on the entropy of \( b_z \) that is tight when \( b_z \) is independent Gaussian with variances in \( \nu^p \). In other words, the pdfs \( f_x(x; \tilde{r}, \nu^p) = \int_{\nu^p} f_x(x; \nu^p) f_{\tilde{r}}(\nu^p) d\nu^p \) and \( f_z(x; \tilde{r}, \nu^p) = \int_{\nu^p} f_z(x; \nu^p) f_{\tilde{r}}(\nu^p) d\nu^p \) given in lines \((D1)\) and \((D2)\) of Table 1 are critical points of \((R4)\) for fixed-point versions of \( \tilde{r}, \nu^p, \tilde{p}, \nu^p \).

Since \( f_x \) and \( f_z \) are functions of \( \tilde{r}, \nu^p, \tilde{p}, \nu^p \), the cost \( J_{\text{Bethe}} \) can be written in terms of these quantities as well. For this, we first note

\[
D(f_x||p_x) = \int f_x(x; \tilde{r}, \nu^p) \ln \frac{p_x(x)N(x; \tilde{r}, \nu^p)}{p_x(x)C_{\nu^p}(\tilde{r}, \nu^p)} dx
\]

\[
= - \ln C_{\nu^p}(\tilde{r}, \nu^p) - \frac{\ln 2\pi \nu^p_m}{2} - \frac{\nu_m^2}{2\nu^p_m} + \frac{\|\tilde{r} - \tilde{r}\|^2}{2\nu^p_m}
\]

\[
D(f_z||p_yf_z|Z) = \sum_{m=1}^{M} D(f_z||p_z|Z_m)
\]

\[
= - \ln B_{\nu^p_m}(\tilde{p}_m, \nu^p_m) - \frac{\ln 2\pi \nu^p_m}{2} - \frac{\|\tilde{r}_m - \tilde{p}_m\|^2 + \nu^p_m}{2\nu^p_m}
\]

where \( \tilde{x}_m \) and \( \nu^p_m \) are the mean and variance of \( f_x(x; \tilde{r}_m, \nu^p_m) \) from \((R9)\) and \((R8)\). Following a similar procedure,

\[
D(f_x||p_yf_z|Z) = \sum_{m=1}^{M} D(f_x||p_z|Z_m)
\]

\[
= - \ln B_{\nu^p_m}(\tilde{p}_m, \nu^p_m) - \frac{\ln 2\pi \nu^p_m}{2} - \frac{\|\tilde{r}_m - \tilde{p}_m\|^2 + \nu^p_m}{2\nu^p_m}
\]

\[
= - \ln C_{\nu^p_m}(\tilde{r}_m, \nu^p_m) - \frac{\ln 2\pi \nu^p_m}{2} - \frac{\|\tilde{r}_m - \tilde{r}_m\|^2}{2\nu^p_m}
\]

\[
J_{\text{Bethe}}(f_x, f_z) = - \frac{1}{2} \sum_{m=1}^{M} \left( \ln B_{\nu^p_m}(\tilde{p}_m, \nu^p_m) + \frac{\|\tilde{r}_m - \tilde{p}_m\|^2}{2\nu^p_m} \right)
\]

\[
- \frac{\ln 2\pi \nu^p_m}{2} - \frac{\|\tilde{r}_m - \tilde{r}_m\|^2}{2\nu^p_m} + \text{const}.
\]

where we have written \( J_{\text{Bethe}}(f_x, f_z) \) as \( J_{\text{Bethe}}(\tilde{r}, \nu^p, \tilde{p}, \nu^p) \) to make the \( \tilde{r}, \nu^p, \tilde{p}, \nu^p \)-dependence clear, and where \text{const} collects terms invariant to \( \tilde{r}, \nu^p, \tilde{p}, \nu^p \).
The advantage of (14) is that the rows and columns of \( \mathbf{A} \) are approximately zero-mean. This can be seen by first verifying, via the definitions above, that \( e^\top_{0}1_{N} = 0 \), \( \mathbf{A}1_{N} = 0 \), and \( 1_{M}\mathbf{A} = 0_{M} \), which implies that the elements in every row and column of \( \mathbf{A} \) are zero-mean. Thus, for large \( N \) and \( M \), the elements in all but a vanishing fraction of the rows and columns of \( \mathbf{A} \) will also be zero-mean. The mean-square coefficient size in the last two rows and columns of \( \mathbf{A} \) can be made to match that in \( \mathbf{A} \) via choice of \( b_{12}, b_{13}, b_{21}, b_{31} \).

To understand the construction of (13), note that (15) implies

\[
\bar{z} = \mathbf{A}x = \bar{\mathbf{A}}x + b_{12}1_{N}\bar{x}/b_{12} + b_{13}1_{M}\bar{e}/b_{13},
\]

which explains the first \( M \) rows of (14). To satisfy the definitions in (19), we then require that \( z_{M+1} = 0 \) and \( z_{M+2} = 0 \) in (14), which can be ensured through the Dirac-delta likelihood

\[
p_{ym}(y_m | z_m) = \delta(z_m) \quad \text{for} \quad m \in \{M+1, M+2\}.
\]

Meanwhile, we make no assumption about the newly added elements \( x_{N+1} \) and \( x_{N+2} \), and thus adopt the improper uniform prior

\[
p_{xm}(x_n) \propto 1 \quad \text{for} \quad n \in \{N+1, N+2\}.
\]

In summary, the mean-removal approach suggested here runs GAMP or AD-GAMP (as in Table 1) with \( \mathbf{A} \) in place of \( \mathbf{A} \) and with the likelihoods and priors augmented by (20) and (21). It is important to note that, if multiplication by \( \mathbf{A} \) and \( \mathbf{A}^\top \) can be implemented using a fast transform (e.g., FFT), then multiplication by \( \mathbf{A}^\top \) and \( \mathbf{A} \) can too; for details, see the GAMP/mATlab implementation [3].

3. NUMERICAL RESULTS

We numerically studied the recovery NMSE \( \triangleq \| \bar{z} - \bar{x} \|^2/\| x \|^2 \) of SwAMP [8] and the MMSE version of the original GAMP from (11) relative to the proposed mean-removed (M-GAMP) and adaptively damped (AD-GAMP) modifications, as well as their combination (MAD-GAMP). In all experiments, the signal \( x \) was drawn Bernoulli-Gaussian (BG) with sparsity rate \( \tau \) and length \( N = 1000 \), and performance was averaged over 100 realizations. Average NMSE was clipped to 0 dB for plotting purposes. The matrix \( \mathbf{A} \) was drawn in one of four ways:

(a) Non-zero mean: \( \text{i.i.d} a_{mn} \sim \mathcal{N}(\mu, \frac{1}{\nu}) \) for a specified \( \mu \neq 0 \).

(b) Low-rank product: \( \mathbf{A} = \frac{1}{\sqrt{N}} \mathbf{U} \mathbf{V}^\top \) with \( \mathbf{U} \in \mathbb{R}^{M \times R}, \mathbf{V} \in \mathbb{R}^{R \times N} \), and i.i.d. \( \mathbf{u}_{nr} \sim \mathcal{N}(0, 1) \), for a specified \( R \). Note \( \mathbf{A} \) is rank deficient when \( R < \min\{M, N\} \).

(c) Column-correlated: the rows of \( \mathbf{A} \) are independent zero-mean stationary Gaussian-Markov processes with a specified correlation coefficient \( \rho = E\{a_{mn}a_{mn+1}\} / E\{a_{mn}^2\} \).

(d) Ill-conditioned: \( \mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^\top \) where \( \mathbf{U} \) and \( \mathbf{V} \) are the left and right singular vector matrices of an i.i.d. \( N(0, 1) \) matrix and \( \Sigma \) is a singular value matrix such that \( \|\Sigma\|_{\text{ii}}/\|\Sigma\|_{\text{ii}, 1+1+i} = (\kappa)^{1/\min(M, N)} \) for \( i = 1, \ldots, \min\{M, N\} - 1 \), with a specified condition number \( \kappa > 1 \).

For all algorithms, we used \( T_{\max} = 1000 \) and \( \epsilon = 10^{-5} \). Unless otherwise noted, for adaptive damping, we used \( T_B = 0 \), \( G_{\text{pass}} = 1.1 \), \( G_{\text{fail}} = 0.5 \), \( \beta_{\max} = 1 \), and \( \beta_{\min} = 0.01 \). For SwAMP, we used the authors’ publicly available code [12].

First we experiment with compressive sensing (CS) in AWGN at \( \text{SNR} \triangleq E\{\|z\|^2\} / E\{\|y - z\|^2\} = 60 \) dB. For this, we used
Next, we tried “robust” CS by repeating the previous experiment with non-zero-mean matrices, Fig. 1(a) shows that the proposed M-GAMP and MAD-GAMP provided near-genie performance for all tested means \( \mu \). In contrast, GAMP only worked with zero-mean matrices. AD-GAMP has a fast convergence speed of MAD-GAMP to SwAMP, Fig. 3(a) shows that A-GAMP and SwAMP to converge for each matrix type. Table 3 shows that the whole, MAD-GAMP ran several times faster than SwAMP but used more iterations. Thus, it may be possible to reduce SwAMP’s runtime to below that of MAD-GAMP. We proposed adaptive damping and mean-removal modifications of GAMP that help prevent divergence in the case of “difficult” \( \mathbf{A} \) matrices. We then numerically demonstrated that the resulting modifications significantly increase GAMP’s robustness to non-zero-mean, low-rank product, column-correlated, and ill-conditioned \( \mathbf{A} \) matrices. Moreover, they provide robustness similar to the recently proposed SwAMP algorithm, while running faster than the current SwAMP implementation. For future work, we note that the sequential update of SwAMP could in principle be combined with the proposed mean-removal and/or adaptive damping to perhaps achieve a level of robustness greater than either SwAMP or (M)AD-GAMP.

**4. CONCLUSIONS**

Finally, we experiment with noiseless 1-bit CS [13], where \( \mathbf{y} = \text{sgn}(\mathbf{A}\mathbf{x}) \), using \( M = 3000 \) measurements and sparsity ratio \( \tau = 0.125 \). In each realization, the empirical mean was subtracted from the non-zero entries of \( \mathbf{x} \) to prevent \( \mathbf{y}_m = 1 \) \( \forall m \). For (M)AD-GAMP, we used \( \beta_{\max} = 0.5 \). For SwAMP, we increased the stop-
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