Maximal analytic extensions of the Emparan-Reall black ring

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Abstract. The Emparan-Reall black rings are a two-parameter family of black hole solutions of the vacuum Einstein’s equation in dimension 4+1, the word “ring” refers to the event horizon, which has topology $S^2 \times S^1 \times \mathbb{R}$. We construct an analytic extension, whose global structure is similar to the Kruskal-Szekeres extension of the Schwarzschild space-time, and show that it is maximal, globally hyperbolic, and unique within a natural class of extensions.

1. Introduction

The Emparan-Reall [6] metrics form a remarkable class of vacuum black hole solutions of Einstein equations in dimension 4 + 1. Some aspects of their global properties have been studied in [6], where it was shown that the solution contains a Killing horizon with $S^2 \times S^1 \times \mathbb{R}$ topology. In fact, the event horizon coincides with the Killing horizon, and therefore also has this topology (see [4]). We show how to construct an analytic extension with a bifurcate Killing horizon; and we establish some global properties of the extended space-time. The extension resembles closely the Kruskal-Szekeres extension of the Schwarzschild space-time. We show the maximality of this extension, as well as global hyperbolicity, establish uniqueness of our extension within a natural class, and verify existence of a conformal completion at null infinity.

2. The Emparan-Reall space-time

In local coordinates $(t, \psi, z, x, \varphi)$, the Emparan-Reall metric can be written in the form

$$g = \frac{F(x)}{F(z)} \left( dt + \sqrt{\frac{\nu}{\xi_F} \frac{\xi_1 - z}{A}} d\psi \right)^2 + \frac{F(z)}{A^2(x - z)^2} \times$$

$$\left[ -F(x) \left( \frac{dz^2}{G(z)} + \frac{G(z)}{F(z)} d\psi^2 \right) + F(z) \left( \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 \right) \right],$$

where $A > 0$, $\nu$ et $\xi_F$ are constants, and

$$F(\xi) = 1 - \frac{\xi}{\xi_F},$$

$$G(\xi) = \nu \xi^3 - \xi^2 + 1 = \nu(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3),$$

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are polynomials, with \( \nu \) chosen so that \( \xi_1 < 0 < \xi_2 < \xi_3 \). The study of the coordinate singularities at \( x = \xi_1 \) and \( x = \xi_2 \) leads to the determination of \( \xi_F \) as:

\[
\xi_F = \frac{\xi_1 \xi_2 - \xi_3^2}{\xi_1 + \xi_2 - 2 \xi_3} \in (\xi_2, \xi_3).
\]

(4)

Emparan and Reall have established the asymptotically flat character of (1), as well as existence of an analytic extension across an analytic Killing horizon at \( z = \xi_3 \). The extension given in [6] is somewhat similar of the extension of the Schwarzschild metric that one obtains by going to Eddington-Finkelstein coordinates, and is not maximal. Now, existence of an analytic extension with a bifurcate horizon, \( \text{à la} \) Kruskal-Szekeres, is guaranteed from this by an analytic version of the analysis of Rácz and Wald in [9]. But the global properties of an extension so constructed are not clear. It is therefore of interest to present an explicit extension with good properties. This extension is outlined in Section 3, and its global properties are studied in Section 4.

The coordinate \( z \) can be seen as a radial coordinate, and runs in \( [\xi_3, +\infty[ \cup ] - \infty, \xi_1[ \), whereas \( \psi \) and \( \varphi \) are angular coordinates, and \( t \) is a time coordinate define in \( \mathbb{R} \). As in [6] we assume throughout that

\[
\xi_1 \leq x \leq \xi_2.
\]

(5)

As discussed in [6], the extremities correspond to a north and south pole of \( S^2 \), with a function \( \theta \) defined by \( d\theta = dx/\sqrt{G(x)} \) providing a latitude on \( S^2 \), except for the limit \( x - z \to 0, x \to \xi_1 \), which corresponds to an asymptotically flat region, see [6]; a detailed proof of asymptotic flatness can be found in [3]. The coordinate \( \varphi \) provides the corresponding azimuth on \( S^2 \). The surface \( \{z = \infty)\} \) can be identified with \( \{z = -\infty\} \) by introducing a coordinate \( Y = -1/z \), with the metric extending analytically across \( \{Y = 0\} \), see [6] for details. The symmetries are given by the Killing vector fields \( \partial_t, \partial_\psi \) and \( \partial_\varphi \), and one can check that \( \partial_t \) is timelike in the asymptotic flat region, and becomes null at \( \{Y = 0\} \). This gives the location of the ergosurface, and the space-time is stationary.

3. The extension

We start by working in the range \( z \in (\xi_3, \infty) \); there we define new coordinates \( w, v \) by the formulae

\[
dv = dt + \frac{bdz}{(z - \xi_3)(z - \xi_2)},
\]

(6)

\[
dw = dt - \frac{bdz}{(z - \xi_3)(z - \xi_2)},
\]

(7)

where \( b \) is a constant. (Our coordinates \( v \) and \( w \) are closely related to, but not identical, to the coordinates \( v \) and \( w \) used in [6] when extending the metric through the Killing horizon \( z = \xi_3 \)). Similarly to the construction of the extension of the Kerr metric in [1,2], we define a new angular coordinate \( \hat{\psi} \) by:

\[
d\hat{\psi} = d\psi - adt,
\]

(8)

where \( a \) is a constant. Then, similarly to what is done in [1,2], and to remove a signature degeneracy at \( z = \xi_3 \), one introduces

\[
\dot{v} = \exp(cv), \quad \dot{w} = - \exp(-cw),
\]

(9)

where \( c \) is some constant to be chosen. With a suitable choice of the constants \( a, b \) and \( c \) (see [4] for details), one may finally construct an analytic extension \( (\hat{\mathcal{M}}, \hat{g}) \) of the original Emparan-Reall spacetime \( (\mathcal{M}, g) \) (see Figure 1), with a bifurcate Killing horizon \( \{\dot{w}\dot{v} = 0\} \) extended from the Killing horizon \( \{z = \xi_3\} \).
4. Global properties

4.1. Maximal

The space-time \( \tilde{M}, \tilde{g} \) just constructed is a maximal analytic extension of the original Emparan-Reall space-time \( (M_1, g) \). One way to show this is to prove that \( (\tilde{M}, \tilde{g}) \) is timelike s-complete: by this we mean that any timelike, maximally extended geodesic of the extended space-time is either complete, or reaches a curvature singularity in finite affine time. In fact, the metric is singular on the set \( \{ z = \xi_3 \} \), which follows from the asymptotic expansion, derived by Alfonso García-Parrado and José María Martín García using the symbolic algebra package XAct [8], for the Kretschmann scalar:

\[
R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{12 A^4 \xi_F^4 G(\xi_F)^2 (x-z)^4 (1 + O(z-\xi_F))}{(\xi_F-x)^2 (z-\xi_F)^6}.
\]

This shows that the Emparan-Reall metric is \( C^2 \)-inextendible across \( \{ z = \xi_F \} \). Then, to prove the s-timelike completeness of \( (\tilde{M}, \tilde{g}) \), we study first the timelike geodesic segments of finite affine length which lie in the domain of outer communication \( \langle \langle M_{\text{ext}} \rangle \rangle \). To do so, we use various constants of motion, such as \( g(\dot{\gamma}, \dot{\gamma}) \) and \( g(\dot{\gamma}, X) \) where \( X \) is a Killing vector field, to study the evolution equations for \( x \) and \( z \) obtained from the Lagrangian \( L = g(\dot{\gamma}, \dot{\gamma}) \). One finds that the geodesics which are maximally extended within \( \langle \langle M_{\text{ext}} \rangle \rangle \) are either complete, or can be smoothly extended through the bifurcate Killing horizon \( \{ \dot{w} = 0 \} \). Then, one shows that causal geodesics in the region \( \{ \xi_F < z < \xi_3 \} \) reach the horizon \( \{ \dot{w} = 0 \} \) and are smoothly extendible there in one time direction, and reach the singular set \( \{ z = \xi_F \} \) in finite affine time in the other direction. A detailed proof of these facts is provided in [4]. Note that restricted classes of geodesics in the Emparan-Reall metrics have been analysed by different methods in Durkee, using constants of motion as well as the separability of the Hamilton-Jacobi equation for these geodesics (see [5]).

4.2. Global hyperbolicity

The maximal extension \( (\tilde{M}, \tilde{g}) \) is globally hyperbolic. Indeed, the surface \( S := \{ \dot{v} = 0 \} \) is a Cauchy surface of \( \tilde{M} \); note that \( S \) is a smooth extension of the hypersurface \( \{ t = 0 \} \) \( (= S \cap \{ \dot{v} > 0 \} \) of \( M_1 \) across the bifurcation surface \( \{ \dot{w} = \dot{v} = 0 \} \), to its image in \( M_{III} \).
under the map \((\hat{w}, \hat{v}, \hat{\psi}, x, \varphi) \mapsto (-\hat{w}, -\hat{v}, -\hat{\psi}, x, -\varphi)\). The proof of this uses the \(s\)-geodesic completeness of the metric.

### 4.3. A uniqueness theorem

We have the following uniqueness result:

**Theorem 4.1** The space-time \((\hat{M}, \hat{g})\) is unique within the class of simply connected analytic extensions of the Emparan-Reall space-time \((M_1, g)\) which have the property that all maximally extended causal geodesics on which \(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\) is bounded are \(s\)-complete.

Uniqueness is understood up to isometry. Theorem 4.1 follows from the analysis of geodesics just described and from Theorem 4.2, which is a Lorentzian version of [7, Theorem 6.3, p. 255] where geodesic completeness is weakened to timelike \(s\)-completeness (see [4] for details).

**Theorem 4.2** Let \((M, g), (M', g')\) be analytic Lorentzian manifolds of dimension \(n + 1, n \geq 1\), with \(M\) connected and simply connected, and \(M'\) timelike \(s\)-complete. Then every isometric immersion \(f_U : U \subset M \hookrightarrow M'\), where \(U\) is an open subset of \(M\), extends uniquely to an isometric immersion \(f : M \hookrightarrow M'\).

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