Stability of homogeneous bundles on $\mathbb{P}^3$

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Abstract

We study the stability of some homogeneous bundles on $\mathbb{P}^3$ by using their representations of the quiver associated to the homogeneous bundles on $\mathbb{P}^3$. In particular we show that homogeneous bundles on $\mathbb{P}^3$ whose support of the quiver representation is a parallelepiped are stable, for instance the bundles $E$ whose minimal free resolution is of the kind $0 \rightarrow S^{\lambda_1, \lambda_2, \lambda_3} V(t) \rightarrow S^{\lambda_1 + s, \lambda_2, \lambda_3} V(t + s) \rightarrow E \rightarrow 0$ are stable.

1 Introduction

In [O-R1] we studied homogeneous bundles on $\mathbb{P}^2$ and in particular their simplicity and stability from the standpoint of their minimal free resolutions and their representations of the quiver associated to homogeneous bundles on $\mathbb{P}^2$ introduced by Bondal and Kapranov in [B-K].

This point of view was born from the consideration that, homogeneous bundles on $\mathbb{P}^2 = SL(3)/P$ can be described by representations of the parabolic subgroup $P$, but since $P$ is not a reductive group, there is a lot of indecomposable reducible representations of $P$ and to classify homogeneous bundles on $\mathbb{P}^2$ and among them the simple ones, the stable ones, etc. by means of the study of the representations of the parabolic subgroup $P$ is difficult. On the other hand quivers allow us to handle well and “to make explicit” the homogeneous subbundles of a homogeneous bundle $E$ and Rohmfeld’s criterion (see [Rohm]) in this context is equivalent to saying that $E$ is semistable if and only if the slope of every subbundle associated to a subrepresentation of the quiver representation of $E$ is less or equal than the slope of $E$; so quivers and representations of quivers associated to homogeneous bundles are particularly suitable for the study of stability.

This paper is, in a certain sense, the continuation of [O-R1]; in fact here we try to study the same problems for $\mathbb{P}^3$. 

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**2000 Mathematical Subject Classification:** 14M17, 14F05, 16G20.

**Key-words:** homogeneous bundles, minimal resolutions, quivers
If $E$ is a homogeneous vector bundle on $\mathbb{P}^3 = \mathbb{P}(V)$ there exists a minimal free resolution of $E$
\[ 0 \to \bigoplus_q \mathcal{O}(-q) \otimes_{\mathcal{O}} A_q \to \bigoplus_q \mathcal{O}(-q) \otimes_{\mathcal{O}} B_q \to \bigoplus_q \mathcal{O}(-q) \otimes_{\mathcal{O}} C_q \to E \to 0 \]
with $A_q, B_q, C_q$ $SL(V)$-representations and maps $SL(V)$-invariant.

We prove that homogeneous bundles whose support of the quiver representation is a parallelepiped are stable (if they are not trivial) and also the bundles whose support of the quiver representation has the form of a particular staircase are stable. In terms of minimal free resolutions this can be restated respectively in the following way:

**Theorem 1** Let $E$ be a homogeneous bundle on $\mathbb{P}^3 = \mathbb{P}(V)$ whose minimal free resolution is one of the following for some $\lambda_1, \lambda_2, \lambda_3, s, t, r, l, k \in \mathbb{N}$, with $s \geq 1$, $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and maps $SL(V)$-invariant and with all the components nonzero:
\[ 0 \to S^{\lambda_1,\lambda_2,\lambda_3} V(t) \to S^{\lambda_1+s,\lambda_2,\lambda_3} V(t+s) \to E \to 0 \]
\[ 0 \to S^{\lambda_2+s-1,\lambda_2,\lambda_3} V(t-\lambda_2-s+1+\lambda_1) \to S^{\lambda_1,\lambda_2,\lambda_3} V(t) \to S^{\lambda_1,\lambda_2+s,\lambda_3} V(t+s) \to E \to 0 \]
\[ 0 \to S^{\lambda_1-l,\lambda_2-k,\lambda_3} V(t-k-l) \to S^{\lambda_1,\lambda_2-k,\lambda_3} V(t-k) \oplus S^{\lambda_1-l,\lambda_2,\lambda_3} V(t-l) \to \]
\[ \to S^{\lambda_1,\lambda_2,\lambda_3} V(t) \to E \to 0 \]

Then $E$ is stable.

**Theorem 2** Let $E$ be a homogeneous bundle on $\mathbb{P}^3 = \mathbb{P}(V)$ whose minimal free resolution is
\[ 0 \to \delta S^{\lambda_1+1-k,\lambda_2-r-k,\lambda_3} V(1-r-2k) \to \delta S^{\lambda_1+r-k,\lambda_2-r-k,\lambda_3} V(-2k) \oplus \bigoplus_{i=r,\ldots,1} S^{\lambda_1+i,\lambda_2-i,\lambda_3} V(1) \to E \to 0 \]
for some $\lambda_1, \lambda_2, \lambda_3, r \in \mathbb{N}$, $\epsilon, \delta \in \{0,1\}$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$, $SL(V)$-invariant maps, the only nonzero component of the second map is the first, the third map restricted to $S^{\lambda_1+i,\lambda_2-i,\lambda_3} V$ has only the components into $S^{\lambda_1+i+1,\lambda_2-i,\lambda_3} V(1)$ and $S^{\lambda_1+i,\lambda_2-i+1,\lambda_3} V(1)$ nonzero and the third map restricted to $S^{\lambda_1+r-k,\lambda_2-r-k,\lambda_3} V(-2k)$ has all the components nonzero.

Then $E$ is stable.

Besides, again by using the simplicity of certain staircases, we prove

**Theorem 3** Let $E$ be a homogeneous bundle on $\mathbb{P}^3$ such that there exist $\lambda_1, \lambda_2, \lambda_3, \overline{s}_1, c \in \mathbb{N}$ with $\overline{s}_1 \neq c$, $\lambda_1 \geq \lambda_2 \geq \lambda_3$ such that the minimal free resolution of $E$ is
\[ 0 \to S^{\lambda_1,\lambda_2,\lambda_3} V \xrightarrow{\alpha} \bigoplus_{(s_1,s_2,s_3,s_4) \in T-c} S^{\lambda_1+s_1,\lambda_2+s_2,\lambda_3+s_3} V(s_1+s_2+s_3+s_4) \to E \to 0 \]
where
\[ T = \{(s_1,s_2,s_3) \mid s_1, s_2, s_3 \in \mathbb{N}, s_i \leq \lambda_{i-1} - \lambda_i \text{ for } i = 2,3, s_4 \leq \lambda_3, s_1 + s_2 + s_3 + s_4 = c \} \]
\[ C = \{ (\overline{s}_1, s_2, s_3, 0) \mid s_2, s_3 \in \mathbb{N}, s_i \leq \lambda_{i-1} - \lambda_i \text{ for } i = 2,3, \overline{s}_1 + s_2 + s_3 = c \} \]

Then $E$ is simple.
2 Notation and some preliminary lemmas

We recall some facts on representation theory (see for instance [F-H]). Let $d$ be a natural number and let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $d$ with $\lambda_1 \geq \ldots \geq \lambda_k$. We can associate to $\lambda$ a Young diagram with $\lambda_i$ boxes in the $i$-th row, the rows lined up on the left. The conjugate partition $\lambda'$ is the partition of $d$ whose Young diagram is obtained from the Young diagram of $\lambda$ interchanging rows and columns.

A tableau with entries in $\{1, \ldots, n\}$ on the Young diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $d$ is a numbering of the boxes by the integers $1, \ldots, n$, allowing repetitions (we say also that it is a tableau on $\lambda$).

Definition 4 Let $V$ be a complex vector space of dimension $n$. Let $d \in \mathbb{N}$ and let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $d$, with $\lambda_1 \geq \ldots \geq \lambda_k$. Number the boxes of the Young diagram of $\lambda$ with the numbers $1, \ldots, d$ from left to right beginning from the top row. Let $\Sigma_d$ be the group of permutations on $d$ elements; let $R$ be the subgroup of $\Sigma_d$ given by the permutations preserving the rows and let $C$ be the subgroup of $\Sigma_d$ given by the permutations preserving the columns. We define

$$S^\lambda V := \text{Im}(\sum_{a \in C, s \in R} \text{sign}(a)s \circ a : \otimes^d V \to \otimes^d V)$$

The $S^\lambda V$ are called Schur representations.

The $S^\lambda V$ are irreducible $SL(V)$-representations and it is well-known that all the irreducible $SL(V)$-representations are of this form.

We recall that Pieri’s formula says that, if $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition of a natural number $d$ with $\lambda_1 \geq \lambda_2 \geq \ldots$ and $t$ is a natural number, then

$$S^\lambda V \otimes S^t V = \oplus_{\nu} S^\nu V$$

as $SL(V)$-representation, where the sum is performed on all the partitions $\nu = (\nu_1, \ldots)$ with $\nu_1 \geq \nu_2 \geq \ldots$ of $d + t$ whose Young diagrams are obtained from the Young diagram of $\lambda$ adding $t$ boxes not two in the same column.

Finally we observe that, if $V$ is a complex vector space of dimension $n$, then $S^{(\lambda_1,\ldots,\lambda_{n-1})} V$ is isomorphic, as $SL(V)$-representation, to $S^{(\lambda_1+r,\ldots,\lambda_{n-1+r},r)} V$ for all $r \in \mathbb{N}$. Besides $(S^{(\lambda_1,\ldots,\lambda_{n})} V)^\vee$ is isomorphic, as $SL(V)$-representation, to $S^{(\lambda_1-\lambda_n,\ldots,\lambda_1-\lambda_2)} V$. Moreover $(S^\lambda V)^\vee \simeq S^\lambda V^\vee$. 

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Notation 5 • In all the paper V will be a complex vector space of dimension 4 if not otherwise specified.
• If E is a vector bundle on P(V) then μ(E) will denote the slope of E, i.e. the first Chern class divided by the rank.

Lemma 6 Let E be a homogeneous vector bundle on P^3 = P(V). By Horrocks’ theorem [Hor] the bundle E has a minimal free resolution

\[ 0 \to \oplus_q O(-q) \otimes C D_q \to \oplus_q O(-q) \otimes C C_q \to \oplus_q O(-q) \otimes C B_q \to E \to 0 \]

(D_q, C_q, B_q C-vector spaces). Since E is homogeneous we can suppose the maps are SL(V)-invariant maps (D_q, C_q and B_q are SL(V)-representations).

Proof. We can argue as in the proof of Lemma 7 in [O-R1] to prove that there is an exact sequence

\[ 0 \to A \to B \to E \to 0 \]

where \( B = \oplus_q O(-q) \otimes C B_q \) (B_q SL(V)-representations) and \( H^1(A(t)) = 0 \forall t \). Repeating the same argument for \( A \) instead of \( E \) we get an exact sequence

\[ 0 \to D \to C \to A \to 0 \]

where \( C = \oplus_q O(-q) \otimes C C_q \) (C_q SL(V)-representations). We want to prove that also \( D \) splits: it is sufficient to prove that \( H^1(D(t)) = H^2(D(t)) = 0 \) and we can prove that by using the cohomology exact sequence associated to \( 0 \to D \to C \to A \to 0 \).

Remark 7 a) If U, W, V are three vector spaces, then \( \text{Hom}(U \otimes O_P(V)(-s), W) = \text{Hom}(U \otimes S^s V, W) \) (the isomorphism can be given by \( H^0(\cdot, \cdot) \)).

b) Let V be a vector space. For any \( \lambda, \mu \) partitions, \( s \in \mathbb{N} \), up to multiples there is a unique SL(V)-invariant map

\[ S^\lambda V \otimes O(-s) \to S^\mu V \otimes O \]

by Pieri’s formula, Schur’s lemma and part a of the remark.

Lemma 8 Let V be a complex vector space of dimension n. Let \( \lambda_1, \ldots, \lambda_{n-1}, s \in \mathbb{N} \) with \( \lambda_1 \geq \ldots \geq \lambda_{n-1} \). On \( P^{n-1} = P(V) \) any SL(V)-invariant nonzero map

\[ S^{\lambda_1, \ldots, \lambda_{n-1}} V(-s) \to S^{\lambda_1 + s, \lambda_2, \ldots, \lambda_{n-1}} V \]

is injective.

The above lemma is well known; for the proof see for instance [O-R1].
Lemma 9 Let $E$ and $E'$ be two homogeneous vector bundles on $\mathbb{P}^3$ and

$$0 \to R \xrightarrow{h} S \xrightarrow{f} T \xrightarrow{g} E \to 0 \quad \quad 0 \to R' \xrightarrow{h'} S' \xrightarrow{f'} T' \xrightarrow{g'} E' \to 0$$

be two minimal free resolutions. Any map $\eta : E \to E'$ induces maps $A$, $B$ and $C$ such that the following diagram commutes:

$$\begin{array}{c}
0 \to R \xrightarrow{h} S \xrightarrow{f} T \xrightarrow{g} E \to 0 \\
\downarrow A \quad \downarrow B \quad \downarrow C \quad \downarrow \eta \\
0 \to R' \xrightarrow{h'} S' \xrightarrow{f'} T' \xrightarrow{g'} E' \to 0
\end{array}$$

and given $A$, $B$ and $C$ such that the above diagram commutes we have a map $E \to E'$.

Given $\eta$, the map $C$ is unique if and only if $\text{Hom}(T, \ker g') = 0$. (In particular if $\text{Hom}(T, \ker g') = 0$ and we find $A$, $B$ and $C$ such that the diagram commutes and $C$ is not a multiple of the identity, we can conclude that $E$ is not simple.)

Proof. Let $K = \ker g$ and $K' = \ker g'$. By applying $\text{Hom}(T, \cdot)$ to $0 \to K' \to T' \to E' \to 0$ we get that $\eta \circ g$ can be lifted to a map $C : T \to T'$ if $H^1(T^\vee \otimes K') = 0$, and, by applying $\text{Hom}(T, \cdot)$ to $0 \to R' \to S' \to K' \to 0$, we get that $H^1(T^\vee \otimes K') = 0$. Obviously $C : T \to T'$ induces $C|_K : K \to K'$. By applying $\text{Hom}(S, \cdot)$ to $0 \to R' \to S' \to K' \to 0$ we get that $C|_K \circ f$ can be lifted to a map $B : S \to S'$ (since $H^1(R' \otimes S^\vee) = 0$).

\[ \square \]

3 Quivers

We recall now the main definitions and results on quivers and representations of quivers associated to homogeneous bundles introduced by Bondal and Kapranov in [B-K]. Quivers will allow us to handle well and “to make explicit” the homogeneous subbundles of a homogeneous bundle.

Namely we are recalling the definition of a quiver $Q$ such that the category of the homogeneous bundles on $\mathbb{P}^n$ is equivalent to the category of finite dimensional representations of $Q$ with some relations $R$. We refer to [B-K], [Hil1], [Hil2], [O-R2], for the proofs and for the definitions and the statements in a more general setting.

Definition 10 (See [Sim], [King], [Hil1], [G-R].) A quiver is an oriented graph $Q$ with the set $Q_0$ of vertices (or points) and the set $Q_1$ of arrows. A path in $Q$ is a formal composition of arrows $\beta_m \ldots \beta_1$ where the source of an arrow $\beta_i$ is the sink of the previous arrow $\beta_{i-1}$.

A relation in $Q$ is a linear form $\lambda_1 c_1 + \ldots + \lambda_r c_r$ where $c_i$ are paths in $Q$ with a common source and a common sink and $\lambda_i \in \mathbb{C}$.

A representation of a quiver $Q = (Q_0, Q_1)$, or $Q$-representation, is the couple of a set of vector spaces $\{X_i\}_{i \in Q_0}$ and of a set of linear maps $\{\varphi_\beta\}_{\beta \in Q_1}$ where $\varphi_\beta : X_i \to X_j$ if $\beta$ is an arrow from $i$ to $j$. 

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A representation of a quiver $Q$ with relations $R$ is a $Q$-representation such that

$$\sum_j \lambda_j \varphi_{\beta_m j} \cdots \varphi_{\beta_1} = 0$$

for every $\sum_j \lambda_j \beta_m \cdots \beta_1 \in R$.

Let $(X_i, \varphi_\beta)_{i \in Q_0, \beta \in Q_1}$ and $(Y_i, \psi_\beta)_{i \in Q_0, \beta \in Q_1}$ be two representations of a quiver $Q = (Q_0, Q_1)$. A morphism $f$ from $(X_i, \varphi_\beta)_{i \in Q_0, \beta \in Q_1}$ to $(Y_i, \psi_\beta)_{i \in Q_0, \beta \in Q_1}$ is a set of linear maps $f_i : X_i \to Y_i, i \in Q_0$ such that, for every $\beta \in Q_1$, $\beta$ arrow from $i$ to $j$, the following diagram is commutative:

$$\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_j \\
\varphi_\beta \downarrow & & \downarrow \psi_\beta \\
X_j & \xrightarrow{f_j} & Y_j
\end{array}$$

A morphism $f$ is injective if the $f_i$ are injective.

**Notation 11** We say that a representation $(X_i, \varphi_\beta)_{i \in Q_0, \beta \in Q_1}$ of a quiver $Q = (Q_0, Q_1)$ has multiplicity $m$ in a point $i$ of $Q$ if $\dim X_i = m$.

The support (with multiplicities) of a representation of a quiver $Q$ is the subgraph of $Q$ constituted by the points of multiplicity $\geq 1$ and the nonzero arrows (with the multiplicities associated to every point of the subgraph).

We introduce the following partial order on $Q_0$: we say that $A > B$ if there is a path from $B$ to $A$ (that is whose source is $B$ and whose sink is $A$).

First some notation. Let $P$ and $R$ be the following subgroup of $SL(n+1)$:

$$P = \{ A \in SL(n+1) \mid A_{i,1} = 0 \text{ for } i = 2, \ldots, n+1 \}$$

$$R = \{ A \in SL(n+1) \mid A_{1,i} = A_{1,1} = 0 \text{ for } i = 2, \ldots, n+1 \}$$

Observe that $R$ is reductive. We can see $P^n$ as

$$P^n = SL(n+1)/P$$

($P$ is the stabilizer of $[1 : 0 : \ldots : 0]$). Let $p$ and $r$ be the Lie algebras associated respectively to $P$ and $R$. Let $n$ be the Lie algebra

$$n = \{ A \in M(n+1 \times n+1) \mid A_{i,j} = 0 \text{ for } i = 2, \ldots, n+1, \quad j = 1, \ldots, n+1 \text{ and } A_{1,1} = 0 \}$$

Thus $p = r \oplus n$ (Levi decomposition).

We recall that the homogeneous bundles on $P^n = SL(n+1)/P$ are given by the representations of $P$ (this bijection is given by taking the fibre over $[1 : 0 : \ldots : 0]$ of a homogeneous vector bundle); by composing the projection from $P$ to $R$ with a representation of $R$ we get a representation of $P$ and the set of the homogenous bundles obtained in this way from the irreducible representations of $R$ are

$$\{ S^{l_1} \cdots S^{l_{n-1}} Q(t) \mid l_1, \ldots, l_{n-1} \in \mathbb{N}, \quad t \in \mathbb{Z} \}$$

where $Q = T_{P^n}(-1)$.
Definition 12 Let $Q$ be the following quiver:

- Let $Q_0 = \{ \text{irreducible representations of } R \} = \{ S^1, \ldots, S^{l_{n-1}}(t) \mid l_1, \ldots, l_{n-1} \in \mathbb{N}, t \in \mathbb{Z} \} = \{ \text{dominant weights of } r \}$

- Let $Q_1$ be defined in the following way: there is an arrow from $\lambda$ to $\mu$, $\lambda, \mu \in Q_0$, if and only if $n \otimes \Sigma_\lambda \supset \Sigma_\mu$, where $\Sigma_\lambda$ denotes the representation of $r$ with dominant weight $\lambda$.

Definition 13 Let $E$ be a homogeneous vector bundle on $\mathbb{P}^n$. The $Q$-representation associated to $E$ is the following (see [B-K], [Hil1], [Hil2], [O-R2]):

consider the fiber $E_{[1:0: \ldots : 0]}$ of $E$ on $[1 : 0 : \ldots : 0]$; it is a representation of $p$; as $r$-representation we have

$$E_{[1:0: \ldots : 0]} = \bigoplus_{\lambda \in Q_0} X_\lambda \otimes \Sigma_\lambda$$

for some vector spaces $X_\lambda$; we associate to $\lambda \in Q_0$ the vector space $X_\lambda$; for any $\lambda$ dominant weight vector we fix $v_\lambda \in \Sigma_\lambda$ and $\eta_0, \ldots, \eta_{n-1}$ eigenvectors of the $p$-representation $n$; let $V_0, \ldots, V_{n-1}$ be their weights respectively; we associate to the arrow $\lambda \rightarrow \mu$ a map $f : X_\lambda \rightarrow X_\mu$ defined in the following way: let $i$ be such that $\lambda + \psi_i = \mu$; consider the composition

$$\Sigma_\lambda \otimes n \otimes X_\lambda \rightarrow \Sigma_\mu \otimes X_\mu$$

given by the action of $n$ over $E_{[1:0: \ldots : 0]}$ followed by projection; it maps $v_\lambda \otimes \eta_i \otimes v$ to $v_\mu \otimes w$; we define $f(v) = w$ (it does not depend on the choice of the dominant weight vector).

Lemma 14 The adjoint representation of $p$ on $n$ corresponds to $(\wedge^{n-1} Q)(-2) = \Omega^1$,

(we recall that, by the Euler sequence, $\wedge^n Q = O(1)$, thus $\Omega^1 = \wedge^{n-1} Q(-2)$).

By the previous remark, if $\Sigma_\lambda$ is the representation corresponding to $S^1, \ldots, S^{l_{n-1}}(t)$ and $\Sigma_\mu$ is the representation corresponding to $S^{l'_1, \ldots, l'_{n-1}}(t')$, the condition

$$n \otimes \Sigma_\lambda \supset \Sigma_\mu$$

is equivalent to the fact $S^{l'_1, \ldots, l'_{n-1}}(t')$, a direct summand of $\wedge^{n-1} Q(-2) \otimes S^1, \ldots, S^{l_{n-1}}(t)$ and this is true if and only if

$(l'_1, \ldots, l'_{n-1}, t') = (l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_{n-1}, t - 1)$ for some $i$

or

$(l'_1, \ldots, l'_{n-1}, t') = (l_1 + 1, \ldots, l_{n-1} + 1, t - 2)$

Thus our quiver has $n$ connected components $Q^{(1)}, \ldots, Q^{(n)}$ (given by the congruence class modulo $(n + 1)/n$ of the slope of the homogeneous vector bundles corresponding
to the points of the connected component); we identify the points of every connected component \(Q^{(r)}\) of \(Q\) with a subset of \(\mathbb{Z}^n\) for convenience.

We define \(V_i\) the vector \(S_{l_1, \ldots, l_{i-1}} Q(t - 1) - S_{l_1, \ldots, l_n} Q(t)\) \(\forall i = 1, \ldots, n - 1\) and \(V_0\) the vector \(S_{l_1, \ldots, l_{n-1} + 1} Q(t - 2) - S_{l_1, \ldots, l_n} Q(t)\).

The figure shows one of the connected components in the case \(n = 3\).

Definition 15 Let \(\beta_{T, P}\) denote the arrow from \(P\) to \(T\). Let \(R\) be the set of relations on \(Q\) given by the commutativity of the squares i.e.

\[
\beta_{P + V_i, P} - \beta_{P + V_i, P + V_k, P + V_i, P} = \beta_{P + V_k, P + V_i, P} - \beta_{P + V_k, P + V_i, P + V_k, P}
\]

for any \(P \in Q^j\) for some \(j\) such that \(P + V_i, P + V_k, P + V_i + V_k \in Q_0\). and by

\[
\beta_{P + V_i, P + V_k, P} = \beta_{P + V_k, P + V_i, P}
\]

for any \(P \in Q^j\) for some \(j\) such that \(P + V_i \notin Q_0\).

(For instance for \(n = 3\) we have \(\beta_{P + V_0, V_1, P + V_0, P} \forall P \in \sigma = \{V_0, V_1 + V_2\}\) and \(\beta_{P + V_1, V_2, P + V_2, P} \forall P \in \sigma = \{V_0, V_1 + V_2\}\)).

Theorem 16 (Bondal, Kapranov, Hille) \([B-K], [Hil1], [Hil2], [O-R2]\). The category of the homogeneous bundles on \(\mathbb{P}^n\) is equivalent to the category of finite dimensional representation of the quiver \(Q\) with the relations \(R\).

Observe that in Def. 13 with respect to Bondal-Kapranov-Hille’s convention in \([B-K], [Hil1], [Hil2]\), we preferred to invert the arrows in order that an injective \(SL(V)\)-equivariant map of bundles corresponds to an injective morphism of \(Q\)-representations. For example \(\mathcal{O}\) injects in \(V(1)\) whose support is the arrow from \(Q(1)\) to \(\mathcal{O}\).
**Notation 17** • We will often speak of the \( Q \)-support of a homogeneous bundle \( E \) instead of the support with multiplicities of the \( Q \)-representation of \( E \) and we will denote it by \( \text{Q-su}pp(E) \).

• The word “parallelepiped” will denote the subgraph with multiplicities of \( Q \) given by the subgraph of \( Q \) included in a parallelepiped whose sides are parallel to \( \langle V_{i_1}, \ldots, V_{i_{n-1}} \rangle \) for some distinct \( i_1, \ldots, i_{n-1} \), with the multiplicities of all its points equal to 1.

• If \( A \) and \( B \) are two subgraphs of \( Q \), \( A \cap B \) is the subgraph of \( Q \) whose vertices and arrows are the vertices and arrows both of \( A \) and of \( B \); \( A - B \) is the subgraph of \( Q \) whose vertices are the vertices of \( A \) not in \( B \) and the arrows are the arrows of \( A \) joining two vertices of \( A - B \).

**Remark 18** ([B-K]) The \( Q \)-support of \( S^{p_1, \ldots, p_n} V(t) \) is a parallelepiped with vertex with maximum slope \( S^{p_1, \ldots, p_{n-1}, -p_n} Q(t + p_n) \) and the side of direction \( V_1 \) of length \( p_1 - p_2, \ldots, \), the side of direction \( V_{n-1} \) of length \( p_{n-1} - p_n \), the side of direction \( V_0 \) of length \( p_n \).

In fact: by the Euler sequence \( S^{p_1, \ldots, p_n} V(t) = S^{p_1, \ldots, p_n} (O(-1) \oplus Q)(t) \) as \( R \)-representation; by the formula of a Schur functor applied to a direct sum (see [F-H], Exercise 6.11) we get
\[
S^{p_1, \ldots, p_n} V = \bigoplus S^\lambda Q \otimes S^m O(-1)
\]
as \( R \)-representations, where the sum is performed on \( m \in \mathbb{N} \) and on \( \lambda \) Young diagram obtained from the Young diagram of \( (p_1, \ldots, p_n) \) by taking off \( m \) boxes not two in the same column; thus
\[
S^{p_1, \ldots, p_n} V = \bigoplus_{0 \leq m_1 \leq p_1 - p_2, \ldots, \begin{array}{c} 0 \leq m_{n-1} \leq p_{n-1} - p_n, \\ 0 \leq m_n \leq p_n \end{array}} S^{p_1, p_2, \ldots, p_n} O(-m_1 - \ldots - m_n + t) 
\]
\[
= \bigoplus_{0 \leq m_1 \leq p_1 - p_2, \ldots, \begin{array}{c} 0 \leq m_{n-1} \leq p_{n-1} - p_n, \\ 0 \leq m_n \leq p_n \end{array}} S^{p_1, p_2 - (m_1 - m_2), \ldots, p_{n-1} - p_n - (m_{n-1} - m_n)} Q(-m_1 - \ldots - m_n + t + p_n - m_n)
\]

Finally to show the maps associated to the arrows in the parallelepiped are nonzero we can consider on the set of the vertices of the rectangle the following equivalence relation: \( P \sim Q \) if and only if there exist two paths with \( P \) and \( Q \) respectively as sources and common sink such that the map associated to any arrow of the two paths is nonzero; if a map associated to an arrow (say from \( P_1 \) to \( P_2 \)) of the rectangle is zero then, by the “commutativity of the squares”, precisely by the relations in Definition 15, there would be at least two equivalence classes (the class of \( P_1 \) and the class of \( P_2 \)), but this is impossible by the irreducibility of \( S^{p_1, \ldots, p_n} V \).

**4 Some lemmas and notation**

In this section we collect some technical notation and lemmas, which will be useful in the next section to study homogeneous subbundles (in particular their slope) of homogeneous
bundles and then to study stability.

**Remark 19** i) The first Chern class of a homogeneous bundle \( E \) can be calculated as the sum of the first Chern classes of the irreducible bundles corresponding to the vertices of the \( Q \)-support of \( E \) multiplied by the multiplicities. The rank of \( E \) is the sum of the ranks of the irreducible bundles corresponding to such vertices multiplied by the multiplicities.

We will often speak of the slope (resp. first Chern class, rank) of a graph with multiplicities instead of the slope (resp. first Chern class, rank) of the vector bundle whose \( Q \)-support is that graph with multiplicities.

ii) Suppose the set of the vertices of the \( Q \)-support of \( E \) is the disjoint union of the vertices of the supports of two \( Q \)-representations \( A \) and \( B \); if \( \mu(A) = \mu(B) \) then \( \mu(E) = \mu(A) = \mu(B) \), if \( \mu(A) < \mu(B) \) then \( \mu(A) < \mu(E) < \mu(B) \).

iii) We recall that on \( \mathbb{P}^n \) we have

\[
\text{rk}(S^{l_1, \ldots, l_{n-1}} Q(t)) = \prod_{1 \leq i < j \leq n} \frac{l_i - l_j}{j - i}, \quad l_n := 0
\]

\[
c_1(S^{l_1, \ldots, l_{n-1}} Q(t)) = \prod_{1 \leq i < j \leq n} \frac{l_i - l_j}{j - i} \left( \frac{l_1 + \ldots + l_{n-1}}{n} + t \right)
\]

**Remark 20** On \( \mathbb{P}^n \) we have

\[
S^{l_1, \ldots, l_{n-1}} Q(t)^\vee = S^{l_1, l_2 - l_{n-1}, \ldots, l_2} Q(-t - l_1).
\]

In particular on \( \mathbb{P}^3 \) \( S^{l_1, l_2} Q(t)^\vee = S^{l_1, l_2 - l_1} Q(-t - l_1) \). Thus, by dualizing, the vector \( V_0 \) goes into \(-V_1 \) and \( V_2 \) into \(-V_2 \).

We introduce now particular \( Q \)-representations, called “staircases”. Their importance is due to the fact that they are the \( Q \)-supports of the homogeneous subbundles of the homogeneous bundles whose \( Q \)-supports are parallelepiped (in particular of the trivial homogeneous bundles).

**Remark 21** Let \( E \) be a homogeneous bundle on \( \mathbb{P}^n \) and \( F \) be a homogeneous subbundle.

Let \( S \) and \( S' \) be the \( Q \)-supports of \( E \) and \( F \) respectively. By Theorem 16 the \( Q \)-representation of \( F \) injects into the \( Q \)-representation of \( E \). If the multiplicities of \( S \) are all 1 and \( S' \) contains the source of an arrow \( \beta \) in \( S \) then \( S' \) contains \( \beta \).

**Definition 22** We say that a subgraph with multiplicities of \( Q \) is a staircase \( S \) in a parallelepiped \( R \) if all its multiplicities are 1 and the graph of \( S \) is a subgraph of \( R \) satisfying the following property: if \( V \) is a vertex of \( S \) then the arrows of \( R \) having \( V \) as source must be arrows of \( S \) (and then also their sinks must be vertices of \( S \)).

We say that a subgraph with multiplicities of \( Q \) is a staircase if it is a staircase in some parallelepiped.

By Remark 21 the \( Q \)-support of a homogeneous subbundle of a homogeneous bundle whose \( Q \)-support is a parallelepiped is a staircase in the parallelepiped.
**Notation 23** Given a staircase $S$ in a parallelepiped we define $V_S$ to be the set of the vertices of $S$ that are not sinks of any arrow of $S$. We call the elements of $V_S$ the vertices of the steps. We say that a staircase has $k$ steps if the cardinality of $V_S$ is $k$. Let $V \in V_S$. We define the sticking out part relative to $V$ as the part of $S$ whose vertices are exactly the points of $S$ greater than $V$ but not greater than any other element of $V_S$ and the arrows are all the arrows connecting any of these vertices (see Notation 11).

**Remark 24** Let $a, b, c, d, r, s \in \mathbb{R}$ with $r$ and $s$ positive. Suppose $\frac{a}{b} > \frac{c}{d}$. Then
\[
\frac{a + sc}{b + sd} > \frac{a + c}{b + d} \text{ if } s < r \quad \text{and} \quad \frac{a + sc}{b + sd} < \frac{a + c}{b + d} \text{ if } s > r.
\]

**Lemma 25** Let $c \in \mathbb{N}$. Let $E_{S}^{l_1, l_2}Q(t), c, V_i, V_k$ be the hypotenuse of the isosceles triangle whose vertices are:
\[
A = S^{l_1, l_2}Q(t), \quad A + cV_i, \quad A + cV_k
\]
for some $i, k \in \{0, 1, 2\}$. Let $R_{S}^{l_1, l_2}Q(t), c, V_i, V_k$ be the hypotenuse of the isosceles triangle whose vertices are:
\[
A = S^{l_1, l_2}Q(t), \quad A - cV_i, \quad A - cV_k
\]
Let $x = l_1 - l_2 + 1$ and $z = l_2 + 1$.

Let $e_{S}^{l_1, l_2}Q(t), c, V_i, V_k$ and $r_{S}^{l_1, l_2}Q(t), c, V_i, V_k$ be the sum of the ranks of the vector bundles corresponding to the points of $E_{S}^{l_1, l_2}Q(t), c, V_i, V_k$ and of $R_{S}^{l_1, l_2}Q(t), c, V_i, V_k$ respectively. We have:

1. $e_{S}^{l_1, l_2}Q(t), c, V_i, V_k = (c + 1)[cx(x + 2z + 1) + 2xz(x + z)]$
2. $r_{S}^{l_1, l_2}Q(t), c, V_i, V_k = (c + 1)[cx(x + 2z - 1) + 2xz(z + x)]$
3. $e_{S}^{l_1, l_2}Q(t), c, V_i, V_k = -c^2 + c(x + z)(x - z + 1) + 2xz(x + z)$

**Proof.**

1. 
\[
e_{S}^{l_1, l_2}Q(t), c, V_i, V_k = \sum_{s=0,\ldots,c} (x + 2s - c)(x + z - c + s)(z - s) = (c + 1)[cx(x + 2z + 1) + 2xz(x + z)]
\]

2. 
\[
r_{S}^{l_1, l_2}Q(t), c, V_i, V_k = \sum_{s=0,\ldots,c} (x - 2s + c)(x + z + c - s)(z + s) = (c + 1)[cx(x + 2z - 1) + 2xz(z + x)]
\]

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iii) 
\[ e_{\mathbb{N}^{1/2}(t),c,V_0,V_1} = \sum_{s=0,...,c} (x - s - c)(x + z - c - 2s)(z - s) = \]
\[ = -c^2 + c(x + z)(x - z + 1) + 2xz(x + z) \]

**Proposition 26** Let \( S \) be a rectangle in \( Q^{(r)} \) for some \( r \). Suppose the rectangle is parallel to \( \langle V_i, V_k \rangle \) for some \( i, k \in \{0, 1, 2\}, i \neq k \). Let \( S' \) be obtained by translating \( S \) in \( Q^{(r)} \subset \mathbb{Z}^3 \) by \( V_j - V_i \) or by \( V_j - V_k \) where \( j \in \{0, 1, 2\}, j \neq i, k \). Then \( \mu(S) > \mu(S') \) with only two exceptions:

a) \( \{i, k\} = \{1, 2\} \), \( S' \) is obtained by translating \( S \) by \( V_0 - V_1 \) and the length of the side of \( S \) with direction \( \langle V_2 \rangle \) is greater than the length of the side of \( S \) with direction \( \langle V_1 \rangle \).

b) \( \{i, k\} = \{1, 2\} \), \( S' \) is obtained by translating \( S \) by \( V_0 - V_2 \) and the length of the side of \( S \) with direction \( \langle V_1 \rangle \) is greater than the length of the side of \( S \) with direction \( \langle V_2 \rangle \).

**Proof.** To prove the statement, we will consider the quotient between the sum of the ranks of the bundles corresponding the points of \( S \) with a fixed \( \mu \) and the sum of the ranks of the bundles corresponding the points of \( S' \) with the same \( \mu \). We will show that this quotient is a decreasing function of \( \mu \). By Remark 24 this is sufficient to prove our statement.

Obviously the points of \( S \) corresponding to bundles with a fixed \( \mu \) form a segment forming an angle of 45 degrees with the sides of \( S \).

We have to consider three cases:

1) \( S \) is parallel to \( \langle V_1, V_2 \rangle \)
2) \( S \) is parallel to \( \langle V_0, V_1 \rangle \)
3) \( S \) is parallel to \( \langle V_0, V_2 \rangle \)

**1) \( S \) IS PARALLEL TO \( \langle V_1, V_2 \rangle \).**

Obviously we have to divide this case into two main subcases: the case where \( S' \) is obtained by translating \( S \) by \( V_0 - V_1 \) and the case where \( S' \) is obtained by translating \( S \) by \( V_0 - V_2 \).

Observe that if we consider the segment given by the points of \( S \) with a certain \( \mu \) and the segment given by the points of \( S' \) with the slope equal to \( \mu - \mu(\Omega^1) \), we have the four subsubcases a,b,c,d shown in the picture:
• Subcase: \( S' \) is obtained by translating \( S \) by \( V_0 - V_1 \)

Subcase A:
Let \( S^{h,b} Q(t) \) be the vector bundle corresponding to the vertex of \( S \) with greatest slope. Let \( x = l_1 - l_2 + 1 \) and \( z = l_2 + 1 \). Let

\[
\nu(c) = \frac{e^{S^{h,b} Q(t),c,V_1,V_2}}{e^{S^{h+2,b+1} Q(t-1),c,V_1,V_2}} = \frac{-c(x + 1)(x + 2z + 4) + 2(x + 1)(z + 1)(x + z + 2)}{-cx(x + 2z + 1) + 2xz(x + z)}
\]

We have to prove that \( \nu(c) \) is an increasing function of \( c \).
Obviously the derivative \( \nu'(c) \) has the same sign as its numerator, i.e.

\[
-(x + 2z + 4)(x + 1)(2xz)(x + z) - (-x)(x + 2z + 1)2(x + 1)(z + 1)(x + z + 2) =
\]

\[
= 2x(x + 1)[-3z(x + z) + (x + 2z + 1)(x + 3z + 2)]
\]

which is obviously positive.

Subcase B:
Let \( S^{h,b} Q(t) \) be the vector bundle corresponding to the vertex of \( S \) with least slope. Let \( x = l_1 - l_2 + 1 \) and \( z = l_2 + 1 \). Let

\[
\nu(c) = \frac{r^{S^{h,b} Q(t),c,V_1,V_2}}{r^{S^{h+2,b+1} Q(t-1),c,V_1,V_2}} = \frac{c(x + 1)((x + 1) + 2(z + 1) - 1] + 2(z + 1)(x + z + 2)}{cx(x + 2z - 1) + 2xz(x + z)}
\]

We have to prove that \( \nu(c) \) is a decreasing function of \( c \).
The numerator of the derivative \( \nu'(c) \) is

\[
2(x + 1)x[(x + 2z + 2)z(x + z) - (z + 1)(x + z + 2)(x + 2z - 1)] =
\]

\[
= 2(x + 1)x[3z(x + z) - (x + 3z + 2)(x + 2z - 1)]
\]

which is obviously negative.

Subcase D:
Let $S^{h, l}Q(t)$ be the vector bundle corresponding to the vertex of $S$ with greatest slope.

Let $x = l_1 - l_2 + 1$ and $z = l_2 + 1$ and $y = x + z$. Consider

$$\alpha(c, z, y) = \frac{e_{S^{h, l}Q(t-1), c, V_1, V_2} - e_{S^{h, l}Q(t), c, V_1, V_2}}{c(y - z)(y + z + 1) - 2(y - z)zy}$$

We have to show that $\frac{\partial \alpha}{\partial y}$ is negative. Let $l$ be the number such that $x = lz$ (thus $y = lz + z$). The numerator of $\frac{\partial \alpha}{\partial y}$ expressed in function of $z, c, l$ is

$$(-12z^3 + 2z^2 - 2z)l^2 + (8z^2 - 8cz - 4c^2 z^2 - 12z^2)l + 4c - 4c^2 z^2 + 2cz^2 - 8z^2 - 4z^2 + 8z^2 - 4cz - 12z - 4c^2$$

The coefficient of $l^2$ is obviously negative. The coefficient of $l$ is

$$-4z(2z^3 - 2c + c^2 z + 4z + 6z^2 - 7cz - 3z^2 c + 2c^2)$$

which is negative if and only if

$$2z^3 - 2c + c^2 z + 4z + 6z^2 - 7cz - 3z^2 c + 2c^2$$

is positive and this is true, in fact we can see this polynomial as sum of $2z^3 + c^2 z - cz - 3z^2 c, 4z - 2c$ and $6z^2 + 2c^2 - 6cz$, which are obviously positive.

The known term can be seen as the sum of $22cz^2 + 10c^2 z - 12z^3, 14cz - 8z^2 - 6c^2$ and $8z^3 - 4c^2 z^2 - 4z^4 + 4c + 2c^2$, which are negative (in order to prove that the last one is negative observe that

$$4z^2(2cz - c^2 - z^2) + 4c + 2c^2 = -4z^2(z - c)^2 + 4c + 2c^2$$

$$\leq -4(c + 1)^2 + 4c + 2c^2 \leq 0$$

**Subcase**: $S'$ is obtained by translating $S$ by $V_0 - V_2$

Subcase $A$:

Let $S^{h, l}Q(t)$ be the vector bundle corresponding to the vertex of $S$ with greatest slope.

Let $x = l_1 - l_2 + 1$ and $z = l_2 + 1$. Let

$$\nu(c) = \frac{e_{S^{h, l}Q(t-1), c, V_1, V_2} - e_{S^{h, l}Q(t), c, V_1, V_2}}{-c(x + 2z + 1 + 2x)(x + z)}$$

We have to prove that $\nu(c)$ is an increasing function of $c$.

The numerator of the derivative $\nu'(c)$ is

$$(x + 2z + 4)(-x + 1)(2xz)(x + z) + x(x + 2z + 1)(x + z)2(x - 1)(x + z)(x + z + 1) = 2x(x - 1)(-3z(x + z) + (x + 2z + 1)(x + 2x + 2z + 1))$$
which is obviously positive.

Subsubcase B:
Let \( S^{h, h} Q(t) \) be the vector bundle corresponding to the vertex of \( S \) with least slope.
Let \( x = h_1 - h_2 + 1 \) and \( z = h_2 + 1 \). Let

\[
\nu(c) = \frac{r_{S^{h, h} Q(t), c, V_1, V_2}}{r_{S^{h+1, h+2} Q(t-1), c, V_1, V_2}} = \frac{c(x - 1)(x + 2z + 2) + 2(x - 1)(z + 2)(x + z + 1)}{cx(x + 2z - 1) + 2xz(x + z)}
\]

We have to prove that \( \nu(c) \) is a decreasing function of \( c \).
The numerator of the derivative \( \nu'(c) \) is

\[
2(x - 1)x[(x + 2z + 2)z(x + z) - (z + 2)(x + z + 1)(x + 2z - 1)] =
\]

\[
= 2(x - 1)x[(3z(x + z) - (2x + 3z + 2)(x + 2z - 1)]
\]

which is obviously negative.

Subsubcase C:
Let \( S^{h, h} Q(t) \) be the vector bundle corresponding to the vertex of \( S \) with greatest slope.
Let \( x = h_1 - h_2 + 1 \) and \( z = h_2 + 1 \) and \( y = x + z \). Consider

\[
\alpha(c, z, y) = \frac{e_{S^{h+1, h+2} Q(1), c, V_1, V_2}}{e_{S^{h, h} Q(1), c, V_1, V_2}} = \frac{c(y - z - 1)(y + z + 4) - 2(y - z - 1)(z + 2)(y + 1)}{c(y - z)(y + z + 1) - 2(y - z)zy}
\]

We have to show that \( \frac{\partial \alpha}{\partial y} \) is negative. Let \( l \) be the number such that \( x = lz \) (thus \( y = lz + z \)). The numerator of \( \frac{\partial \alpha}{\partial y} \) expressed in function of \( z, c, l \) is

\[
-8z^4l^4 + (-16z^4 + 8z^3 c)l^3 + (-12z^4 + 8z^2 + 12z^3 c - 4c^2 z^2 - 2cz^2)l^2
\]

\[
+(-8z^4 - 12z^3 + 12cz^3 - 4c^2 z^2 + 16cz^2 - 2cz^2 z + 6cz)l
\]

\[
+4c - 4c^2 z^2 + 22cz^2 - 8z^2 - 4z^2 + 8z^3 c - 10c^2 z + 14cz - 12c^3 - 4c^2
\]

The coefficients of \( l^4 \) and \( l^3 \) are obviously negative. The coefficient of \( l^2 \) is negative too, in fact it can be seen as the sum of \(-12z^4 + 12z^3 c \) and \((-4c^2 - 2c + 8)z^2 \).
The first term is obviously negative, the second is negative for \( c \geq 2 \). In the case \( c = 1 \), observe that the coefficient of \( l^2 \) becomes \(-12z^4 + 12z^3 + 2z^2 \) which is negative for \( z \geq 2 \) (which is our case since \( c = 1 \) and \( c \leq z - 1 \))
The coefficient of \( l \) is obviously negative for \( z \geq 2 \). For \( z = 1 \), \( c \) must be 0 and thus the coefficient of \( l \) is still negative.
The known term is obviously negative for \( z \geq 2 \). For \( z = 1 \), \( c \) must be 0 and thus the coefficient of \( l \) is still negative.

2) \( S \) IS PARALLEL TO \( (V_0, V_1) \).
• Subcase: \( S' \) is obtained by translating \( S \) by \( V_2 - V_0 \)

Subsubcase A:
Let $S^{h,b}Q(t)$ be the vector bundle corresponding to the vertex of $S$ with greatest slope. Let $x = h_1 - l_2 + 1$ and $z = l_2 + 1$. Let

$$\nu(c) = \frac{e_{S^{h,b}Q(t),c,V_0,V_1}}{e_{S^{h-1,b-2}Q(t+1),c,V_0,V_1}} = \frac{-c^2 + c(x + z - 1)(x - z + 4) + 2(x + 1)(z - 2)(x + z - 1)}{-c^2 + c(x + z)(x - z + 1) + 2xz(x + z)}$$

We have to prove that $\nu(c)$ is an increasing function of $c$. Obviously the derivative $\nu'(c)$ has the same sign as its numerator. The coefficient of $c^2$ is

$$-(x + z)(x - z + 1) + (x + z - 1)(x - z + 4) = 2x + 4z - 4$$

The coefficient of $c$ is

$$-2xz(x + z) + 2(x + 1)(z - 2)(x + z - 1) = -4xz - 4x^2 + 2z^2 - 6z + 4$$

The term not depending on $c$ is

$$2xz(x + z)(x + z - 1)(x - z + 4) - 2(x + 1)(z - 2)(x + z - 1)(x + z)(x - z + 1) = 2(x + z)(x + z - 1)[2x^2 + 4x + z^2 - 3z + 2]$$

So the numerator of the derivative $\nu'(c)$ is obviously positive (observe that $c$ must be less or equal than $z$).

Subsubcase B:

It follows from subcase a) by duality (see Remark 20).

Subsubcase C:

Let $S^{h,b}Q(t)$ be the vector bundle corresponding to the vertex of $S$ with greatest slope. Let $x = h_1 - l_2 + 1$ and $z = l_2 + 1$ and $y = x + z$. Consider

$$\alpha(c, z, y) = \frac{e_{S^{h-1,b-2}Q(t-1),c,V_1,V_2}}{e_{S^{h,b}Q(t),c,V_1,V_2}} = \frac{-c^2 + c(y - 1)(y - z + 1)(y - 2z + 4) + 2(y - z + 1)(y - 2)(y - 1)}{-c^2 + cy(y - z)(y - 2z + 1) + 2(y - z)zy}$$

We have to show that $\frac{\partial \alpha}{\partial y}$ is negative. Let $l$ be the number such that $x = lz$ (thus $y = lz + z$). The numerator of $\frac{\partial \alpha}{\partial y}$ expressed in function of $z, c, l$ is

$$(-4z^2c - 2c^2z^2)^2 + [-8z^4 + (24 - 12c)z^3 + (-8c^2 + 20c - 16)z^2 + (-8c + 16c^2)z]l$$

$$-4z^4 + (12 - 4c)z^3 + (-22c^2 + 2c - 8)z^2 + (6c + 2c^2)z - 4c - 2c^3 + 4c^2$$

The known term is obviously negative if $z \geq 2$ and $c \geq 1$. One can easily check that also in the other cases, i.e. $z = 2, c = 0$ and $z = 1, c = 0$ (we recall that $c \leq z - 1$), it is negative. The coefficient of $l^2$ is always less or equal than 0. The coefficient of $l$ is
obviously negative if \( c \geq 2 \). If \( c = 1 \) then \( z \) must be greater or equal than 2 and also in this case we can easily check that the coefficient of \( l \) is negative.

Subsubcase D: It follows from subcase c) by duality (see Remark 20).

- **Subcase: \( S' \) is obtained by translating \( S \) by \( V_2 - V_1 \)**

Subsubcase A:
Let \( S^{t,b} Q(t) \) be the vector bundle corresponding to the vertex of \( S \) with greatest slope.
Let \( x = l_1 - l_2 + 1 \) and \( z = l_2 + 1 \). Let

\[
\nu(c) = \frac{e_{S^{t,b} Q(t),c, V_2, V_1}}{e_{S^{t,b-1} Q(t+1),c, V_0, V_1}} = \frac{-c^2 + c(x + z + 1)(x - z + 4) + 2(x + 2)(z - 1)(x + z + 1)}{-c^2 + c(x + z)(x - z + 1) + 2xz(x + z)}
\]

We have to prove that \( \nu(c) \) is an increasing function of \( c \).
The coefficient of \( c^2 \) of the numerator of the derivative \( \nu'(c) \) is

\[-(x + z)(x - z + 1) + (x + z + 1)(x - z + 4) = 4x + 2z + 4 \]

The coefficient of \( c \) is

\[-2xz(x + z) + 2(x + 2)(z - 1)(x + z + 1) = -2x^2 - 6x + 4xz + 4z^2 - 4 \]

The term not depending on \( c \) is

\[xz(x + z)(x + z + 1)(x - z + 4) - (x + 2)(z - 1)(x + z + 1)(x + z)(x - z + 1) = (x + z)(x + z + 1)[x^2 + 3x + 2x^2 - 4z + 2] \]

So the numerator of the derivative \( \nu'(c) \) is obviously positive (observe that \( c \) must be less or equal than \( z \)).

Subsubcase B: It follows from subcase a) by duality (see Remark 20).

Subsubcase C:
Let \( S^{t,b} Q(t) \) be the vector bundle corresponding to the vertex of \( S \) with greatest slope.
Let \( x = l_1 - l_2 + 1 \) and \( z = l_2 + 1 \) and \( y = x + z \). Consider

\[
\alpha(c, z, y) = \frac{e_{S^{t,b-1} Q(t-1),c, V_1, V_2}}{e_{S^{t,b} Q(t),c, V_1, V_2}} = \frac{-c^2 + c(y + 1)(y - z + 2)(y - 2z + 4) + 2(y - z + 2)(z - 1)(y + 1)}{-c^2 + cy(y - z)(y - 2z + 1) + 2(y - z)zy} \]

We have to show that \( \frac{\partial \alpha}{\partial y} \) is negative. Let \( l \) be the number such that \( x = lz \) (thus \( y = lz + z \)). The numerator of \( \frac{\partial \alpha}{\partial y} \) expressed in function of \( z, c, l \) is

\[-12z^4 + (12c+12)z^3 + (4c-4c^2)z^2 + [-16z^4 - 12z^3c + (16 + 4c^2 - 20c)z^2 + (8c - 4c^2)z]l \]
\[-8z^4 - 8z^3c + (8 - 2c - 4c^2)z^2 + (-2c^2 - 6c)z - 4c^3 + 2c^2 + 4c \]
One can easily see that every coefficient of the above polynomial in \( l \) is negative.

**Subsubcase D**: It follows from subcase c) by duality (see Remark 20).

3) **S IS PARALLEL TO \( \langle V_0, V_2 \rangle \)**.

It follows from case 1 by duality (see Remark 20).

**Corollary 27** Let \( S \) be a segment and let \( S' \) be obtained by translating \( S \) by \( V_i \) with \( i \in \{1, 2, 3\} \): \( S' = S + V_i \). Then \( \mu(S') < \mu(S) \).

**Proof.** i) If the direction of \( S \) is \( \langle V_i \rangle \), it is obvious

ii) Otherwise let the direction of \( S \) be \( \langle V_j \rangle \). From Proposition 26 and part i) respectively we have \( \mu(S) > \mu(S + V_i - V_j) > \mu(S + V_i) \).

**Corollary 28** Let \( U \) be a rectangle and let \( U' \) be obtained by translating \( U \) by \( V_i \) with \( i \in \{1, 2, 3\} \). Then \( \mu(U') < \mu(U) \).

**Proof.** i) When \( V_i \) is contained in the direction of \( U \) we get the statement by Corollary 27.

ii) Suppose \( V_i \) is not contained in the direction of \( U \). Let the direction of \( U \) be \( \langle V_j, V_k \rangle \) with \( i \neq j, k \) and let’s suppose the length of the side of \( U \) with direction \( \langle V_j \rangle \) greater than the length of the side of \( U \) with direction \( \langle V_k \rangle \). By Proposition 26 and part i) respectively we have \( \mu(U) > \mu(U + V_i - V_j) > \mu(S + V_i) \).

5 **Results on stability and simplicity**

**Definition 29** We say that a \( G \)-homogeneous bundle is multistable if it is the tensor product of a stable \( G \)-homogeneous bundle and an irreducible \( G \)-representation.

**Theorem 30 (Rohmfeld, Faini)** i) \([\text{Rohm}]\) A homogeneous bundle \( E \) is semistable if and only if \( \mu(F) \leq \mu(E) \) for any subbundle \( F \) of \( E \) induced by a subrepresentation of the \( P \)-representation inducing \( E \).

ii) \([\text{Fa}]\) A homogeneous bundle \( E \) is multistable if and only if \( \mu(F) < \mu(E) \) for any subbundle \( F \) of \( E \) induced by a subrepresentation of the \( P \)-representation inducing \( E \).

**Remark 31** The support of \( S^{h,b} Q(t) \otimes S^q V \) is the one shown in the figure, possibly cut off by planes; precisely if \( q > l_1 - l_2 \) it is cut off by a plane with direction \( \langle V_0, V_1 \rangle \) passing through the point \( \sigma \cap \{ S^{h,b} Q(t - q) + kV_2 | k \in \mathbb{R} \} \) and if \( q > l_2 \) it is cut off by a plane with direction \( \langle V_1, V_2 \rangle \) passing through the point \( \pi \cap \{ S^{h,b} Q(t - q) + kV_0 | k \in \mathbb{R} \} \). (In fact, as \( R \)-representation, \( S^{h,b} Q(t) \otimes S^q V \) is equal to \( S^{h,b} Q(t) \otimes (\oplus_{i=0,\ldots,q} S^{q-i} Q(-i)) \); apply Pieri’s formula.)

Analogously \( S^{h,b} Q(t) \otimes S^{q,q} V \) has the support shown in the figure.
Theorem 32 Let $E$ be the homogeneous vector bundle on $\mathbb{P}^3 = \mathbb{P}(V)$ whose $Q$-support is a parallelepiped. Then $E$ is stable if it is not trivial (in particular it is simple).

Proof. First we prove that, to show that $E$ is stable, it is sufficient to show that it is multistable.

If $E$ is the tensor product of a stable homogeneous vector bundle $E'$ with an $SL(V)$-representation $W \neq \mathbb{C}$, then we can suppose that $W$ is irreducible; let $W = S^{p,q,r}V$. Suppose first that $S^{p,q,r}V$, as $R$-representations, doesn’t contain any $O(t)$; then, as $R$-representations, the tensor product $E' \otimes S^{p,q,r}V$ is given by the tensor product of every summand of $E'$ equal to $O(t)$ for some $t$ with $S^{p,q,r}V$ (which is a parallelepiped) and by the tensor products of every summand of $E'$ different from $O(t) \forall t$ with every summand of $S^{p,q,r}V$ (which is a figure with more than one point and parallel to $(V_1 - V_0, V_2 - V_0)$).

Now suppose that $S^{p,q,r}V$, as $R$-representations, contains $O(t)$ for some $t$; then $q = r$. If $p \neq q \neq 0$ then $S^{p,q,r}V$ contains $S^{p-q-1}Q(-1)$ and $S^{p-q+1,1}Q(-1)$ and so, if $S^{l,m}Q(t)$ is not trivial, then $S^{l,m}Q(t) \otimes S^{p,q,r}V$ contains points with multiplicity 2. If $q = r = 0$ as $R$-representations, the tensor product $E' \otimes S^pV$ is given by the tensor product of all the summands of $E'$ equal to $O(t)$ for some $t$ with $S^pV$ (which is a parallelogram with sides parallel to $V_1$ and $V_0 + V_1 + V_2$) and by the tensor products of every summand of $E'$ different from $O(t) \forall t$ with $S^pV$ (figures shown in Remark 31) and a union of such figures can’t be a parallelepiped unless $p = q = r = 0$. Analogously the dual case $p = q = r$. 

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To show that $E$ is multistable we consider the $Q$-representation associated to $E$.

By Theorem 30, $E$ is multistable if $\mu(F) < \mu(E)$ for any subbundle $F$ of $E$ induced by a subrepresentation of the $P$-representation inducing $E$. Observe that, by Remark 21, the support of the $Q$-representation of any such subbundle $F$ must be a staircase $C$ in $R$ and vice versa any $Q$-representation whose support is a staircase $C$ in $R$ is the $Q$-representation of a subbundle $F$ of $E$ induced by a subrepresentation of the $P$-representation inducing $E$.

We will show by induction on the cardinality of $V_C$ that $\mu(C) < \mu(R)$ for any $C$ staircase in $R$.

$k = 1$. In this case $C$ is a subparallelepiped in the parallelepiped $R$. Thus this case follows from Corollary 28.

$k - 1 \Rightarrow k$. We will show that, given a staircase $C$ in $R$ with $k$ steps, there exists a staircase $C'$ in $R$ with $k - 1$ steps such that $\mu(C) \leq \mu(C')$. If we prove this, we conclude because $\mu(C) \leq \mu(C') < \mu(R)$, where the last inequality holds by induction hypothesis.

Let $C_1$ and $C_2$ be two staircases with $k - 1$ steps obtained from $C$ respectively “removing and adding” a parallelepiped $O$ and a unions of two parallelepiped $T$. ($O$ is a “sticking out part” of $C$ and $T$ is a nonempty union of parallelepiped adjacent to $O$ disjoint from $C$ such that the union of the points of $T$ with the point of $C$ gives a staircase with $k - 1$ steps).

If $\mu(C_1) \geq \mu(C)$ we conclude at once.

Thus we can suppose that $\mu(C_1) < \mu(C)$. We state that in this case $\mu(C_2) \geq \mu(C)$. In fact: let $\mu(C_1) = \frac{a}{b}$, $\mu(O) = \frac{c}{d}$ and $\mu(T) = \frac{e}{f}$, where the numerators are the first Chern classes and the denominators the ranks; since $\mu(C_1) < \mu(C)$, we have $\frac{a}{b} < \frac{a+c}{b+d}$, thus $\frac{a}{b} < \frac{c}{d}$; besides by Corollary 28 $\mu(O) < \mu(T)$, i.e. $\frac{c}{d} < \frac{e}{f}$; thus $\frac{a+c+e}{b+d+f} \geq \frac{a+c}{b+d}$ i.e. $\mu(C_2) \geq \mu(C)$. 

Definition 33 We say that a staircase contained in a plane parallell to $\langle V_i, V_j \rangle$, for some $i, j$, is completely regular if all the bundles corresponding the vertices of the steps (see Notation 23) are $P + I(V_i - V_j) I = 0, ..., r$ for some $P$ point of the quiver and $r \in \mathbb{N}$.

Theorem 34 Every staircase which is a cylinder on a completely regular staircase in a plane parallel to $\langle V_1, V_2 \rangle$ is multistable and it is stable unless it is a cylinder of height 0, i.e. it is a completely regular staircase in a plane parallel to $\langle V_1, V_2 \rangle$. 

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Proof. First we are proving the statement on multistability.

Fact 1. For any staircase which is a cylinder on a completely regular staircase in a plane parallel to \( \langle V_1, V_2 \rangle \), let’s order the vertices \( P_1, P_2, \ldots \) by going in the direction \( V_1 - V_2 \) and let \( R_i \) be the parallellepipded contained in the staircase whose vertices are exactly those greater than \( P_i \). Let \( H_i = R_i - R_{i-1}, R_0 = \emptyset \) (horizontal steps in the figure) and \( E_i = R_i - R_{i+1}, R_0 = \emptyset \) (vertical steps in the figure). We have

\[
\mu(H_i) > \mu(H_{i-1}) \quad \mu(E_i) > \mu(E_{i+1})
\]

for any \( i \).

Proof. It follows from Proposition [26] and Corollary [28].

Fact 2. Let \( S \) be a staircase which is a cylinder on a completely regular staircase in a plane parallel to \( \langle V_1, V_2 \rangle \). Then for every sticking out part \( O \) of \( S \) we have

\[
\mu(O) > \mu(S - O)
\]

Therefore

\[
\mu(S) > \mu(S - O)
\]

More generally let us define a “piece \( O \) of the staircase \( S \)” in the following way. Let \( P \) and \( Q \) be two vertices with \( Q = P + m(V_2 - V_1) \) and let us consider the triangle \( T \) with vertices \( P, Q, Q + mV_1 \) and let \( O \) be the part of the staircase cylinder on \( T \). If \( O_1, \ldots, O_k \) are pieces of the staircases \( S \), we have that \( \mu(O_i) > \mu(S - O_1 - \ldots - O_{i-1} - O_{i+1} - \ldots - O_k) \).

Proof. Let \( b \) be the plane on which the base of \( O \) is and let \( l \) be the plane on which the left side of \( O \) is. Let \( T_1 \) be the staircase whose vertices are the vertices of \( S \) that are either above \( b \) or on \( b \) and on the left of \( l \) (see the figure below, section for a plane parallel to \( \langle V_1, V_2 \rangle \)). Let \( T_2 \) be the staircase whose vertices are the vertices of \( S \) that are below \( b \) and either on the right of \( l \) or on \( l \).

Let \( K \) be the rectangle

\[
K = S - T_1 - T_2 - O
\]
By Corollary 28 $\mu(O) > \mu(K)$. Besides, by applying Fact 1 to the staircases $T_1 + O$ and $T_2 + O$ (where $T_i + O$ is the smallest staircase containing $T_i$ and $O$), we get

$$\mu(O) > \mu(T_1), \quad \mu(O) > \mu(T_2)$$

Hence $\mu(O) > \max\{\mu(K), \mu(T_1), \mu(T_2)\} \geq \mu(S - O)$ (see Remark 19).

Analogously for the second statement.

Now we are ready to prove that every bundle such that its $Q$-support is a staircase $S$ which is a cylinder on a completely regular staircase in a plane parallel to $\langle V_1, V_2 \rangle$ is multistable. Let $C$ be the support of a $Q$-representation subrepresentation of $S$ (thus again a staircase by Remark 21). We want to prove $\mu(C) < \mu(S)$ by induction on the number $k$ of steps of $C$.

$k = 1$. The statement follows from Corollary 28 and Fact 1.

$k - 1 \Rightarrow k$. To prove this implication we do induction on

$$- \text{area}(bd(C) \cap bd(S))$$

where $bd$ denotes the border and the border of a staircase is the border of the part of the space inside the staircase.

Let $C$ be a staircase with $k$ steps support of a subrepresentation of $S$.

- If $\mu(C - O) \geq \mu(C)$ for some sticking out part $O$ of $C$, we conclude at once because $C - O$ has $k - 1$ steps; thus by induction assumption $\mu(S) > \mu(C - O)$ and then $\mu(S) > \mu(C)$.
- Thus we can suppose $\mu(C) > \mu(C - O)$ for every sticking out part $O$ of $C$ i.e. $\mu(O) > \mu(C)$ for every sticking out part $O$ of $C$.

Suppose there exists a sticking out part $O$ of $C$ such that there exists $A$ parallelepiped or union of two parallelepipeds such that $A$ is disjoint from $C$, $A + C$ is a staircase with less steps (where $A + C$ is the smallest staircase containing $C$ and $A$), a side of $A$ is equal to a side of $O$ and $A' := A \cap S \neq \emptyset$.

Since $\mu(A') > \mu(O)$ by Corollary 28 and $\mu(O) > \mu(C)$ by assumption, we have $\mu(A') > \mu(C)$ and thus

$$\mu(C + A') > \mu(C) \quad (1)$$
Firstly observe that the supports of the bundles are parallelepipeds or staircases. In this section we investigate the minimal free resolutions of the bundles whose supports are parallelepipeds or staircases.

**A**

Let \( P \) be a parallelepiped with an edge on \( \sigma \) touching \( \sigma \) and \( \pi \) an edge on \( \sigma \) touching \( \sigma \). Thus the resolution of a bundle \( P \) whose support is a parallelepiped with an edge on \( \pi \) on \( (\sigma, V) \) be the parallelepiped containing \( P \), touching \( \sigma \) and \( \pi \). We can get the minimal free resolution of the bundle relative to \( P \) in the following way: let \( P' \) be the parallelepiped containing \( P \), touching \( \sigma \) and \( \pi \) and whose edges in the directions \( V_0 \) and \( V_2 \) have the same length of the corresponding edges of \( P \) (we get it by going in the direction \( V_1 \)). We get the minimal free resolution (by denoting the relative bundles with the same name of their supports)

\[
0 \to P' \to P \to P' \to P \to 0
\]

Thus the resolution of a bundle \( P \) whose support is a parallelepiped with an edge on \( \pi = \langle V_1, V_2 + V_0 \rangle \) and not touching \( \sigma = \langle V_0, V_1 + V_2 \rangle \) is

\[
0 \to S_{\lambda_1, \lambda_2, \lambda_3} V(t) \to S_{\lambda_1 + s, \lambda_2, \lambda_3} V(t + s) \to P \to 0
\]

(with \( \lambda_1, \lambda_2, \lambda_3, s \in \mathbb{N}, s \geq 1 \) and \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \)).

---

**6 Resolutions of parallelepipeds and staircases**

In this section we investigate the minimal free resolutions of the bundles whose supports are parallelepipeds or staircases.

Firstly observe that the supports of the bundles \( S_{\lambda_1, \lambda_2, \lambda_3} V(t) \) are parallelepipeds with an edge on \( \pi = \langle V_1, V_2 + V_0 \rangle \) and an edge on \( \sigma = \langle V_0, V_1 + V_2 \rangle \) (border planes of the quiver).

**Remark 35** Let \( P \) be a parallelepiped with an edge on on \( \pi = \langle V_1, V_2 + V_0 \rangle \) and not touching \( \sigma = \langle V_0, V_1 + V_2 \rangle \). We can get the minimal free resolution of the bundle relative to \( P \) in the following way: let \( P' \) be the parallelepiped containing \( P \), touching \( \sigma \) and \( \pi \) and whose edges in the directions \( V_0 \) and \( V_2 \) have the same length of the corresponding edges of \( P \) (we get it by going in the direction \( V_1 \)). We get the minimal free resolution (by denoting the relative bundles with the same name of their supports)

\[
0 \to P' \to P \to P' \to P \to 0
\]
Remark 36  Let $R$ be a parallelepiped with an edge on $\sigma = \langle V_0, V_1 + V_2 \rangle$ and not touching $\pi = \langle V_1, V_2 + V_0 \rangle$. We can get the minimal free resolution of the bundle relative to $R$ in the following way: let $R'$ be the parallelepiped containing $R$, touching $\sigma$ and $\pi$ and whose edges in the directions $V_1$ and $V_0$ have the same length of the corresponding edges of $R$ (we get it by going in the direction $V_2$). Let $P = R' - R$ and let $P'$ be constructed as in the remark above. We get the minimal free resolution (by denoting the relative bundles with the same name of their supports)

$$0 \to P' - P \to P' \to R' \to R \to 0$$

Thus the resolution of a bundle $R$ whose support is a parallelepiped with an edge on $\sigma = \langle V_0, V_1 + V_2 \rangle$ and not touching $\pi = \langle V_1, V_2 + V_0 \rangle$ is

$$0 \to S^{\lambda_2 + s-1, \lambda_2, \lambda_3} V(t - \lambda_2 - s + 1 + \lambda_1) \to S^{\lambda_1, \lambda_2, \lambda_3} V(t) \to S^{\lambda_1, \lambda_2 + s, \lambda_3} V(t + s) \to R \to 0$$

(with $\lambda_1, \lambda_2, \lambda_3, s, t \in \mathbb{N}$, $s \geq 1$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3$).

Remark 37  Let $R$ be a parallelepiped touching neither $\pi = \langle V_1, V_2 + V_0 \rangle$ nor $\sigma = \langle V_0, V_1 + V_2 \rangle$. We can get the minimal free resolution of the bundle relative to $R$ in the following way: let $R'$ be the parallelepiped containing $R$, touching $\sigma$ and $\pi$ and whose edge in the direction $V_0$ has the same length of the corresponding edge of $R$ (we get it by going in the directions $V_1$ and $V_2$). and let $T$ be the parallelepiped containing $R$,
touching \( \pi \) and whose edges in the directions \( V_1 \) and \( V_0 \) have the same length of the corresponding edges of \( R \) (we get it by going in the direction \( V_2 \)).

Let \( Q = R' - T \).

Let \( P = T - R \) and let \( P' \) be constructed as in the first remark (that is let \( P' \) be the parallelepiped containing \( P \), touching \( \sigma \) and \( \pi \) and whose edges in the directions \( V_0 \) and \( V_2 \) have the same length of the corresponding edges of \( P \) (we get it by going in the direction \( V_1 \))).

We get the minimal free resolution

\[
0 \to P' \to P' \oplus Q \to R' \to R \to 0
\]

(all the components of the maps nonzero). Thus the resolution of a bundle \( R \) whose support is a parallelepiped touching neither \( \pi \) nor \( \sigma \) is

\[
0 \to S^{\lambda_1 - l, \lambda_2 - k, \lambda_3} V(t - k - l) \to S^{\lambda_1, \lambda_2 - k, \lambda_3} V(t - k) \oplus S^{\lambda_1 - l, \lambda_2, \lambda_3} V(t - 1) \to \]

\[
S^{\lambda_1, \lambda_2, \lambda_3} V(t) \to R \to 0
\]

(with \( \lambda_1, \lambda_2, \lambda_3, t, k, l \in \mathbb{N}, k, l \geq 1 \) and \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \) and all the components of the maps nonzero).

The three above remarks show that Theorem 32 can be restated in Theorem 11.

Now we want to study the resolution of the bundle correspondent to a staircase which is a cylinder on a staircase in a plane parallel to \( \langle V_1, V_2 \rangle \). For any such staircase \( S \) let’s order the vertices \( P_1, P_2, \ldots \) by going in the direction \( V_1 - V_2 \) and let \( R_i \) be the parallelepiped contained in the staircase whose vertices are exactly those greater than \( P_i \). Let \( E_i = R_i - R_{i+1}, R_0 = \emptyset \) (vertical steps).

First let’s suppose that \( S \) touches \( \pi \). Let \( K_i \) be the parallelepiped touching \( \sigma \) and \( \pi \) containing \( E_i \) and with the length of the edges in the direction \( V_0 \) and \( V_2 \) equal to the corresponding edges of \( E_i \) (we get it by “going” in the direction of \( V_1 \)).

We get the minimal free resolution:

\[
0 \to \bigoplus_i(K_i - E_i) \to \bigoplus_i K_i \to S \to 0
\]

where the second map restricted to \( K_i - E_i \) has only the components \( K_i - E_i \to K_i \) and \( K_i - E_i \to K_{i+1} \) nonzero.
Now let’s suppose that $S$ doesn’t touch $\pi$. Let $E'_i$ be the parallelepiped containing $E_i$ and with the length of the edges in the direction $V_0$ and $V_1$ equal to the corresponding edges of $E_i$ (we get it by “going” in the direction of $V_2$). Let $K'_i$ be the parallelepiped touching $\sigma$ and $\pi$ containing $E'_i$ and with the length of the edges in the direction $V_0$ and $V_2$ equal to the corresponding edges of $E'_i$ (we get it by “going” in the direction of $V_1$). Let $Z$ be the parallelepiped touching $\sigma$ and $\pi$ containing $E'_i - E_1$ and with the length of the edges in the direction $V_0$ and $V_2$ equal to the corresponding edges of $E'_1 - E_1$ (we get it by “going” in the direction of $V_1$).

We get the minimal free resolution:

$$0 \to Z - \oplus_i E'_i \to Z \oplus \oplus_1(K'_i - E'_i) \to \oplus_1 K'_i \to S \to 0$$

where the third map restricted to $K'_i - E'_i$ has only the components $K'_i - E'_i \to K'_i$ and $K'_i - E'_i \to K'_{i+1}$ nonzero.

So we have that the resolution of the bundle $E$ correspondent to a staircase which is a cylinder on a staircase in a plane parallel to $\langle V_1, V_2 \rangle$ and whose vertices are lined up in a line parallel to $V_1 - V_2$ is, for some $\lambda_1, \lambda_2, \lambda_3, r, s_1, ..., s_r, t_1, ..., t_r \in \mathbb{N}$, $s_1 > ... > s_r$, $s_i = s_{i+1} + t_{i+1}$ for $i = 1, ..., r - 1$, $\epsilon, \delta \in \{0, 1\}$, $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $SL(V)$-invariant maps:

$$0 \to \delta S^{\lambda_1 + s_r - k, \lambda_2 - s_1 - k, \lambda_3} V(-2k + s_r - s_1) \to \delta S^{\lambda_1 + s_r - k, \lambda_2 - s_1 - k, \lambda_3} V(-2k) \oplus \oplus_{\epsilon=1}^{r-1} S^{\lambda_1 + s_r - s_\epsilon - k, \lambda_2 - s_\epsilon - k, \lambda_3} V \to$$
where the only nonzero component of the second map is the first, the third map restricted to \( S^{\lambda_1+s_i,\lambda_2-s_i,\lambda_3} V(t_i) \) has only the components into \( S^{\lambda_1+s_i+t_i,\lambda_2-s_i,\lambda_3} V(t_i) \) and \( S^{\lambda_1+s_i+1,\lambda_2-s_i+1,\lambda_3} V(t_{i+1}) \) nonzero and the third map restricted to \( S^{\lambda_1+s_i-k,\lambda_2-s_i-k,\lambda_3} V(-2k) \) has all the components nonzero. (\( \delta \) is 0 iff the staircase touches \( \pi \) \( \epsilon \) is 1 iff the staircase touches \( \sigma \)).

So Theorem 34 can be restated in Theorem 2.

7 Other applications

In this section we want to prove Theorem 3.

Notation 38 For every parallelepiped \( P \), we will denote the vertex with maximum slope by \( M_P \) and the vertex opposite to \( M_P \) in the side in the direction \( \langle V_1, V_2 \rangle \) by \( F_P \); if \( P \) is the support of \( S^{\lambda_1,\lambda_2,\lambda_3} V \) we have that \( F_P \) corresponds to \( S^{\lambda_1-\lambda_3,\lambda_2-\lambda_3} Q(-\lambda_1+2\lambda_3+t) \).

Lemma 39 Let \( \lambda_1, \ldots, \lambda_n, s \in \mathbb{N} \) with \( \lambda_1 \geq \ldots \geq \lambda_n \). Let \( H = \{(s_1, \ldots, s_{n+1}) \in \mathbb{N}^{n+1} \mid s_1 + \ldots + s_{n+1} = s, \ s_i \leq \lambda_i - \lambda_{i+1} \ i = 2, \ldots, n \ \ s_{n+1} \leq \lambda_n \} \). For every \( M \subset H \) let \( \mathcal{P}_M \) be the following statement: for every \( V \) complex vector space of dimension \( n+1 \), the commutativity of the diagram of bundles on \( P(V) \):

\[
\begin{array}{c}
S^{\lambda_1,\ldots,\lambda_n} V(-s) \\
\downarrow A
\end{array} \xrightarrow{\varphi} \bigoplus_{(s_1,\ldots,s_{n+1}) \in M} S^{\lambda_1+s_1,\ldots,\lambda_n+s_{n+1}} V
\]

(\( \text{where } A \text{ and } B \text{ are linear maps and the components of } \varphi \text{ are nonzero } SL(V)\)-invariant maps) implies \( A = \lambda I \) and \( B = \lambda I \) for some \( \lambda \in \mathbb{C} \).

Let \( H = R \cup T \) with \( R \cap T = \emptyset, \ R \neq \emptyset, \ T \neq \emptyset \). Then \( \mathcal{P}_R \) is true if and only if \( \mathcal{P}_T \) is true.
Proof. Completely analogous to the proof of Lemma 38 in [O-R1].

Proof of Theorem 3. First let us suppose that \( \lambda_3 \neq 0 \). Let \( A \) be an endomorphism of \( E \). It induces a commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & S^{\lambda_1,\lambda_2,\lambda_3} V & \xrightarrow{\alpha} & \bigoplus_{(s_1,s_2,s_3)\in C} S^{\lambda_1+s_1,\lambda_2+s_2,\lambda_3+s_3} V(s_1+s_2+s_3) & \rightarrow & E & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & S^{\lambda_1,\lambda_2,\lambda_3} V & \xrightarrow{\alpha} & \bigoplus_{(s_1,s_2,s_3)\in C} S^{\lambda_1+s_1,\lambda_2+s_2,\lambda_3+s_3} V(s_1+s_2+s_3) & \rightarrow & E & \rightarrow & 0
\end{array}
\]

and thus, by Lemma 39, to show our statement, it is sufficient to show that the vertical map of a diagram

\[
S^{\lambda_1,\lambda_2,\lambda_3} V \xrightarrow{\alpha} \bigoplus_{(s_1,s_2,s_3)\in C} S^{\lambda_1+s_1,\lambda_2+s_2,\lambda_3+s_3} V(s_1+s_2+s_3)
\]

(where the horizontal arrows are \( SL(V) \)-invariant) are the identity maps.

Observe that \( D|_{\text{Ker}(\varphi)} \subset \text{Ker}(\varphi) \). Thus we have a commutative diagram

\[
\begin{array}{ccc}
\text{Ker}(\varphi) & \rightarrow & S^{\lambda_1,\lambda_2,\lambda_3} V \\
\downarrow D|_{\text{Ker}(\varphi)} & & \downarrow D \\
\text{Ker}(\varphi) & \rightarrow & S^{\lambda_1,\lambda_2,\lambda_3} V
\end{array}
\]

Let

\[
0 \rightarrow R \rightarrow S \rightarrow T \rightarrow \text{Ker}(\varphi) \rightarrow 0
\]

be a minimal free resolution of \( \text{Ker}(\varphi) \). Since \( \text{Ker}(\varphi) \) is simple (it is a cylinder on a completely regular staircase in a plane with direction \( \langle V_1, V_2 \rangle \) with nonzero height since \( \lambda_3 \neq 0 \)) the map \( D|_{\text{Ker}(\varphi)} : \text{Ker}(\varphi) \rightarrow \text{Ker}(\varphi) \) is the identity.

Thus we get a commutative diagram

\[
\begin{array}{ccc}
S & \rightarrow & \text{Ker}(\varphi) \\
\downarrow I & & \downarrow D|_{\text{Ker}(\varphi)} = I \\
S & \rightarrow & \text{Ker}(\varphi)
\end{array}
\]

and then

\[
\begin{array}{ccc}
S_{\text{max}} & \rightarrow & \text{Ker}(\varphi) \\
\downarrow I & & \downarrow D|_{\text{Ker}(\varphi)} = I \\
S_{\text{max}} & \rightarrow & \text{Ker}(\varphi)
\end{array}
\]

and then

\[
\begin{array}{ccc}
S_{\text{max}} & \rightarrow & S^{\lambda_1,\lambda_2,\lambda_3} V \\
\downarrow I & & \downarrow D \\
S_{\text{max}} & \rightarrow & S^{\lambda_1,\lambda_2,\lambda_3} V
\end{array}
\]

Thus \( D = I \).

If \( \lambda_3 = 0 \) the proof is completely analogous to the one of Lemma 45 in [O-R1]. \( \Box \)
Remark 40  Obviously there are other cases when the support of the kernel of a \( SL(V) \)-invariant map \( \varphi: S^{\lambda_1, \lambda_2, \lambda_3} V \to \bigoplus_{(s_1, s_2, s_3, s_4) \in C} S^{\lambda_1 + s_1, \lambda_2 + s_2, \lambda_3 + s_3, \lambda_4} V(s_1 + s_2 + s_3 + s_4) \) with \( C \) such that there exists a constant \( c \) such that \( \forall (s_1, s_2, s_3, s_4) \in C \), \( s_1 + s_2 + s_3 + s_4 = c \) is a completely regular staircase cylinder on a staircase on a plane parallel to \( \langle V_1, V_2 \rangle \).
So we could deduce generalization of the above theorem. Since the statement would be rather complicated, we preferred to limit ourselves to the above statement.

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