Ginzburg-Landau equations for superconducting quark matter in neutron stars

D. Blaschke
Institute for Nuclear Theory, University of Washington
Box 351550, Seattle, WA 98195
and
Fachbereich Physik, Universität Rostock, D-18051 Rostock, Germany
and
Bogoliubov Laboratory for Theoretical Physics, Joint Institute for Nuclear Research
14 19 80 Dubna, Russia

D. Sedrakian
Observatoire de Paris, F-92195 Meudon Cedex, France
and
Department of Physics, Yerevan State University, 375 025 Yerevan, Armenia

Abstract

We investigate magnetic properties of color superconducting quark matter within a Ginzburg-Landau approach. The simultaneous coupling of the quark fields to gluonic and electromagnetic gauge fields leads to rotated electromagnetism with a massive (Higgsed) and a massless photon-gluon field. We derive the Ginzburg-Landau equations for superconducting quark matter taking into account the rotated electromagnetism in the general case when the rotation angle $\alpha$ is an arbitrary function of the coordinates. We solve these equations for the isolated vortex in superconducting quark matter. We obtain a solution for the magnetic and gluomagnetic fields and expressions for the calculation of the penetration depth and the quantized magnetic flux in quark matter. For this case we have demonstrated that the occurrence of electric and color Meissner currents is a consequence of the color superconducting state of quark matter.

PACS number(s): 12.38.Mh, 26.60+c, 97.60.Jd
I. INTRODUCTION

Recently, the possible formation of diquark condensates in QCD at finite density has been reinvestigated in a series of papers following Refs. [1,2]. It has been shown that in chiral quark models with a nonperturbative 4-point interaction motivated from instantons [3] or nonperturbative gluon propagators [4,5] the anomalous quark pair amplitudes in the color antitriplet channel can be very large: of the order of 100 MeV. Therefore, in two-flavor QCD, one expects this diquark condensate to dominate the physics at densities beyond the deconfinement/chiral restoration transition and below the critical temperature ($\approx 50$ MeV) for the occurrence of this two-flavor color superconductivity (2SC) phase. In a three-flavor theory it has been found [6,7] that there can exist a color-flavor locked (CFL) phase for not too large strange quark masses [8] where color superconductivity is complete in the sense that diquark condensation produces a gap for quarks of all three colors and flavors, which is of the same order of magnitude as that in the two-flavor case.

The high-density phases of QCD at low temperatures are most relevant for the explanation of phenomena in rotating compact stars - pulsars. Conversely, the physical properties of these objects (as far as they are measured) could constrain our hypotheses about the state of matter at the extremes of densities. In contrast to the situation for the cooling behaviour of compact stars where the CFL phase is dramatically different from the 2SC phase [9], we don’t expect qualitative changes of the magnetic field structure between these two phases. Consequently, we will restrict ourselves here to the discussion of the simpler two-flavor theory first. The 2SC phase window is expected to occur at densities just above the deconfinement transition inside a compact star where the strange quark flavor is either still confined [10] or the self consistently determined strange quark masses are large enough to entail color-flavor unlocking [11].

According to Bailin and Love [12] the magnetic field of pulsars should be expelled from the superconducting interior of the star due to the Meissner effect and decay subsequently within $\approx 10^4$ years. For their estimate, they used a perturbative gluon propagator which yielded a very small pairing gap and they made the assumption of a homogeneous magnetic field. Since both assumptions seem not to be valid in general, we have performed in Ref. [13] a reinvestigation of the question with the result that 2SC quark matter is a type-II superconductor which can form vortices in response to the external magnetic field thus behaving similar to protons in the superfluid neutron phase of a neutron star [14,15]. The magnetic field in the quark core of the neutron star can then exist for periods much longer than the spin-down age of a pulsar.

The authors of Ref. [16] have considered this question taking into account the rotated electromagnetism. They came to the conclusion that the magnetic field will live in the quark core sufficiently long although it does not form the structure of quantum vortices because it obeys the force-free Maxwell equation.

In the present work we derive the Ginzburg-Landau equations with account of rotated electromagnetism and solve these equations for the example of an isolated quantum vortex in quark matter which demonstrates that there is always an electric and color Meissner current when the quark matter is superconducting. This conclusion holds also for the case of a homogeneous external magnetic field as we have shown in a separate paper [17].
II. GINZBURG-LANDAU EQUATIONS

In the paper [13] we have obtained the Ginzburg-Landau equations for relativistic superconducting quarks, supposing that the superconducting quark matter phase with \( ud \) diquark pairing has \( P_p = 0 \), where \( p \) is the color antitriplet index. We study the consequences of superconducting quark cores in neutron stars for the magnetic field of pulsars. We find that within recent nonperturbative approach to the effective quark interaction the diquark condensate forms a type II superconductor whereas previously quark matter was considered as a type I superconductor [12]. In both cases the magnetic field which is generated in the superconducting hadronic shell of superfluid neutrons and superconducting protons can penetrate into the quark matter core since it is concentrated in proton vortex clusters where the field strength exceeds the critical value. Recently, in the paper [16], discussing the Meissner effect for color superconducting quark matter, the authors introduce the \( \hat{Q} \)-charge generator

\[
\hat{Q} = Q + \eta P_8 ,
\]

where \( Q \) is the conventional electromagnetic charge generator and \( P_8 \) is associated with the one of the gluons in the representation of the quarks

\[
Q = \text{diag} \left( \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right) \text{ in flavor } u, d, s \text{ space } ,
\]

\[
P_8 = \frac{1}{\sqrt{3}} \text{diag} \left( 1, 1, -2 \right) \text{ in color } r, g, b \text{ space } .
\]

The condition they requested was that the \( \hat{Q} \)-charge of all Cooper pairs which form the condensate vanishes

\[
\hat{Q} d_p = 0 .
\]

Applying this condition (instead of \( P_8 = 0 \)) for the derivation of the Ginzburg-Landau equations, we will see that in the superconducting quark matter two different types of magnetic fields can exist simultaneously. The penetration depths \( \lambda_q \) for these two magnetic fields are completely different. For one of them the penetration depth is finite and much shorter than has been found in [13], but for the other one it becomes infinitely large.

The free energy density in the superconducting quark matter with \( ud \) diquark pairing can be written in the following form [3]

\[
f = f_n + ad_p d_p^* + \frac{1}{2} \beta (d_p d_p^*)^2 + \gamma |\hat{P} d_p|^2 + \frac{(\text{rot}
\vec{A})^2}{8\pi} + \frac{(\text{rot}
\vec{G}_8)^2}{8\pi} ,
\]

where \( a = dn/dE \ t , \beta = dn/dE \ 7\zeta(3)(\pi k_B T_c)^{-2}/8 , \gamma = p_F^2 \beta/(6\mu^2) \) and \( t = (T - T_c)/T_c \) with \( T_c \) being the critical temperature, \( p_F \) the quark Fermi momentum. In zeroth order with respect to the coupling constant, \( dn/dE = \mu p_F/\pi^2 \). \( |\hat{P} d_p|^2 \) is the kinetic term of the free energy density [3] which is given by

\[
|\hat{P} d_p|^2 = |(\vec{\nabla} + ie\vec{A} Q + ig\vec{G}_8 P_8) d_p|^2 .
\]
If we take into account the condition (4) and use (1) we will have
\[ Qd = -\eta P_8 d. \] (7)
Inserting (7) into (6) we will get
\[ |\hat{P}d_p|^2 = |(\vec{\nabla} + i\eta \vec{A} P_8 + ig \vec{G}_8 P_8)d_p|^2. \] (8)
Following the paper [16], let us introduce instead of the original gauge fields \( \vec{A} \) and \( \vec{G}_8 \) the linear combinations \( \vec{A}_x \) and \( \vec{A}_y \)
\[ \vec{A}_x = \frac{-\eta e \vec{A} + \sqrt{\eta^2 e^2 + g^2} \vec{G}_8}{\sqrt{\eta^2 e^2 + g^2}} = -\sin \alpha \vec{A} + \cos \alpha \vec{G}_8, \] (9)
\[ \vec{A}_y = \frac{g \vec{A} + \eta e \vec{G}_8}{\sqrt{\eta^2 e^2 + g^2}} = \cos \alpha \vec{A} + \sin \alpha \vec{G}_8. \] (10)
This corresponds to a rotation with the angle \( \alpha \) in the orthogonal basis of the fields \( \vec{A}_x \) and \( \vec{A}_y \), where
\[ \cos \alpha = \frac{g}{\sqrt{\eta^2 e^2 + g^2}}, \] (11)
while the normalization is preserved
\[ \vec{A}_x^2 + \vec{A}_y^2 = \vec{A}_x^2 + \vec{G}_8^2. \] (12)
At neutron star densities the gluons are strongly coupled (\( g^2/4\pi \approx 1 \)) and the photons are of course weakly coupled (\( e^2/4\pi \approx 1/137 \)), so that \( \alpha \approx \eta e/g \) is small. For the diquark condensate, where blue-green and green-blue \( ud \) quarks are paired, \( \eta = 1/\sqrt{3} \) and therefore \( \alpha \approx 1/20 \). Taking into account (9) and (10), the kinetic term (8) will become
\[ |\hat{P}d_p|^2 = (\vec{\nabla} - iq\eta \vec{A}_x)d_p^*(\vec{\nabla} + iq\eta \vec{A}_x)d_p, \] (13)
where
\[ q = \sqrt{\eta^2 e^2 + g^2 P_8}, \] (14)
and \( P_8 = 1/\sqrt{3} \). We see that the kinetic term has the conventional expression but instead of the charge of the \( ud \) diquark pair equal to \( e/3 \) we have the new charge \( q \) which is much larger than \( e/3 \) (about 20 times). The Ginzburg-Landau equations are obtained in the usual way. If we demand that the variation of the free energy with respect to the parameters \( d^* \), \( \vec{A}_x \) and \( \vec{A}_y \) have to be equal to zero, we will get
\[ 0 = ad_p + \beta (d_p d^*_p) d_p + \gamma (i\vec{\nabla} + q\vec{A}_x)^2 d_p, \] (15)
the equation of motion for the diquark condensate and for the gauge fields
0 = \sin \alpha \text{rot rot} \vec{A} - \cos \alpha \text{rot rot} \vec{G}_8 - 4\pi i q \gamma [\vec{d} \vec{\nabla} d^* - d^* \vec{\nabla} \vec{d}] - 8\pi q^2 \gamma |\vec{d}|^2 \vec{A}_x , \quad (16)

0 = \cos \alpha \text{rot rot} \vec{A} + \sin \alpha \text{rot rot} \vec{G}_8 . \quad (17)

These equations can be written in the following form

\[ \lambda^2 q \text{rot rot} \vec{A} + \sin^2 \alpha \vec{A} = i \frac{\sin \alpha (d_p \vec{\nabla} d_p^* - d_p^* \vec{\nabla} d_p)}{|d|^2} + \sin \alpha \cos \alpha \vec{G}_8 \quad (18) \]

\[ \lambda^2 q \text{rot rot} \vec{G}_8 + \cos^2 \alpha \vec{G}_8 = -i \frac{\cos \alpha (d_p \vec{\nabla} d_p^* - d_p^* \vec{\nabla} d_p)}{|d|^2} + \sin \alpha \cos \alpha \vec{A} , \quad (19) \]

where

\[ \lambda^{-1}_q = \sqrt{8\pi \gamma |d| q} = \sqrt{-\frac{4q^2 t^3_F}{3\pi \mu}} . \quad (20) \]

This is the system of Ginzburg-Landau equations for superconducting quark matter which takes into account the “rotated electromagnetism”. When we define in the equations (18) and (19) the order parameter in the following form

\[ d_p = |d_p| e^{i \phi} , \quad (21) \]

where \( \phi \) is the phase of the order parameter, then this system of equations can be written in the following form

\[ \lambda^2 q \text{rot rot} \vec{A} + \sin^2 \alpha \vec{A} = i \frac{\Phi_q \sin \alpha}{2\pi} \vec{\nabla} \phi + \sin \alpha \cos \alpha \vec{G}_8 \quad (22) \]

\[ \lambda^2 q \text{rot rot} \vec{G}_8 + \cos^2 \alpha \vec{G}_8 = -i \frac{\Phi_q \cos \alpha}{2\pi} \vec{\nabla} \phi + \sin \alpha \cos \alpha \vec{A} \quad (23) \]

Here \( \Phi_q \) has the well known form

\[ \Phi_q = \frac{2\pi \hbar c}{q} . \quad (24) \]

If we take the contour integral on both sides of equation (22) where the contour \( L \) is chosen as the equatorial ring which limits the quark matter core region and take into account that

\[ \oint_L \text{rot rot} \vec{A} \, d\vec{l} = \oint_L \text{rot} \vec{B} \, d\vec{l} = 0 , \quad (25) \]

\[ \oint_L \vec{G}_8 \, d\vec{l} = 0 , \quad (26) \]

so that the electrical currents and the vector potential of the gluon field on the surface of the quark core must vanish, then we get

\[ \oint_L \vec{A} \, d\vec{l} = \oint_S \text{rot} \vec{A} \, d\vec{S} = \oint_S \vec{B} \, d\vec{S} = \frac{\Phi_q}{\sin \alpha} N . \quad (27) \]
This means that the flux of the magnetic field through an arbitrary surface over the contour $L$ is equal to the sum of fluxes which are generated by $N$ vortices each of which has a flux $\Phi_q/\sin \alpha$.

When we calculate this flux we obtain

$$\Phi'_q = \frac{6\pi \hbar c}{e} = 6 \Phi_0 ,$$

(28)

where $\Phi_0 = 2 \times 10^{-7}$ G cm$^2$ is the magnetic flux quantum. In order to obtain this result we have used equation (22). It is easy to see that the same result can be obtained using equation (23), as to be expected.

**III. SOLUTION OF GINZBURG-LANDAU EQUATIONS FOR ISOLATED VORTEX IN QUARK MATTER**

We now consider the solution of the Ginzburg-Landau equations (22, 23) for an isolated vortex situated in homogeneous superconducting quark matter. To this end it is sufficient to find the solution of these equations without the vortex terms. The contribution of an isolated vortex is taken into account by the condition (27) where $N = 1$. Consequently we have to solve the following system of equations

$$\lambda_q^2 \text{rot rot} \vec{A} + \sin^2 \alpha \vec{A} = \sin \alpha \cos \alpha \vec{G}_8$$

(29)

$$\lambda_q^2 \text{rot rot} \vec{G}_8 + \cos^2 \alpha \vec{G}_8 = \sin \alpha \cos \alpha \vec{A}.$$  

(30)

From these equations it is easy to see that

$$\text{rot rot} \vec{G}_8 = -\cot \alpha \text{rot rot} \vec{A}.$$  

(31)

When we act on both sides of equation (29) with the operator rot rot and take into account that $\alpha = \text{const}$, we obtain

$$\lambda_q^2 \text{rot rot rot rot} \vec{A} + \sin^2 \alpha \text{rot rot} \vec{A} = \sin \alpha \cos \alpha \text{rot rot rot} \vec{G}_8.$$  

(32)

Inserting (31) into (32), we obtain the equation for the determination of the electromagnetic vector potential

$$\lambda_q^2 \text{rot rot rot rot} \vec{A} + \text{rot rot} \vec{A} = 0 .$$  

(33)

Let us denote

$$\text{rot rot} \vec{A} = \vec{M} .$$  

(34)

Then, equation (31) takes the form

$$\lambda_q^2 \text{rot rot} \vec{M} + \vec{M} = 0 .$$  

(35)

Consequently, instead of equation (33) we can solve the system of equations (34) and (35).
Let us consider the solution of equation (35). Since our problem has cylindric symmetry, the unknown function shall depend only on the coordinate $r$ denoting the distance to the vortex. It is easy to see that the vectors $\vec{M}$ and $\vec{A}$ have only azimuthal components $\vec{M}_\varphi(r)$ and $\vec{A}_\varphi(r)$. Then equation (35) takes the form

$$\frac{d^2 M_\varphi(r)}{dr^2} + \frac{1}{r} \frac{dM_\varphi(r)}{dr} - \left(\frac{1}{r^2} + \frac{1}{\lambda_q^2}\right) M_\varphi(r) = 0 .$$

The general solution of this equation which tends to zero at infinity is given by

$$M_\varphi(r) = c K_1(r/\lambda_q) ,$$

where $K_1$ is the modified Bessel function of first kind. Next we solve the equation (34). It is easy to see that a special solution of this equation is $\vec{A} = -\lambda_q^2 \vec{M}$, and the general solution which vanishes at $r \to \infty$ will be $c_2/r$. Finally for the vector potential we obtain

$$A_\varphi(r) = c_1 K_1(r/\lambda_q) + c_2/r .$$

The nonvanishing component of the magnetic field has $z$-direction and is determined by the formula

$$B_z(r) = \frac{1}{r} \frac{d}{dr} \left( r A_\varphi(r) \right) .$$

Inserting (38) into (39) we finally obtain

$$B_z(r) = c_1 K_0(r/\lambda_q) ,$$

where $K_0$ is the modified Bessel function of zero kind. The constant $c_1$ is determined from the equation (37) as

$$c_1 = \frac{\Phi'}{2\pi \lambda_q^2} .$$

Then we have

$$B_z(r) = \frac{\Phi'}{2\pi \lambda_q^2} K_0 \left( \frac{r}{\lambda_q} \right) .$$

If we introduce the gluomagnetic field $\vec{B}_8 = \text{rot} \vec{G}_8$, then we can determine it using equation (31). The condition that the magnetic fields $\vec{B}$ and $\vec{B}_8$ have to vanish at $r \to \infty$ immediately determine the solution for $\vec{B}_8$

$$\vec{B}_8 = -\vec{B} \cot \alpha ,$$

where $\vec{B}$ is given by equation (42).

As can be seen from the solution (42) the field of the quantum vortex in quark matter formally has the same form as the field of a vortex in an ordinary superconductor. Whereas in the case of the ordinary superconductor in the definition of the flux $\Phi$ and the penetration
depth \( \lambda \) enters the same charge of the Cooper pair (i.e. 2e, where e is the charge of the electron), in quark matter in the definition of \( \Phi_q \) enters the charge of the quark Cooper pair \( e/3 \), and therefore \( \Phi'_q = 6\Phi_0 \), and in the definition of the penetration depth \( \lambda_q \) enters the charge \( q \), which is about 20 times larger than \( e/3 \). Therefore, the penetration depth \( \lambda_q \) which is proportional to \( q^{-1} \) is 20 times smaller than that of hadronic matter. It is obvious from equation (43) that the magnetic quantum vortex is accompanied by a gluomagnetic quantum vortex the center of which coincides with that of the former, but its force lines have the opposite direction to the magnetic ones. The flux of the gluomagnetic vortex is greater than that of the magnetic vortex by a factor \( \cot \alpha \).

Let us consider the equations for the rotated magnetic fields \( \vec{B}_x = \text{rot}\vec{A}_x \) and \( \vec{B}_y = \text{rot}\vec{A}_y \), which we obtain from the equations (16), (17)

\[
\begin{align*}
\lambda_q^2 \text{rot} \vec{B}_x + \vec{B}_x &= \Phi_q \delta(\vec{r}) , \\
\text{rot} \vec{B}_y &= 0 .
\end{align*}
\]

The energy of isolated quark vortex is given by

\[
E = \frac{1}{8\pi} \int [\vec{B}_y^2 + \vec{B}_x^2 + (\lambda_q \text{rot} \vec{B}_x)^2] \, dV
\]

Note that from equations (11) and (13) follows that the magnetic field \( \vec{B}_y \) of the isolated quark vortex is equal to zero. The magnetic field \( \vec{B}_x \) will be found from the solution of equation (14), which coincides with the well known equation for the magnetic field of an isolated vortex in ordinary superconductors. We can use now these solutions and calculate the energy of an isolated quark vortex in the form

\[
E = \left( \frac{\Phi_q}{8\pi \lambda_q} \right)^2 \ln \frac{\lambda_q}{\xi_q}
\]

As we see from this expression the energy of quark vortex depends logarithmically on the renormalized charge \( q \).

In concluding this section let us add the following: If the quark matter occupies a spherical volume of radius \( a \), and the vortex passes through the center of this volume then the force lines of the magnetic vortex outside of this volume appear to have the dipole form, whereas the force lines for the gluomagnetic field have to be confined within this volume because of the confinement condition \( \hat{G}_8(a) = 0 \). Outside of the quark matter region we have only the ordinary magnetic field. We also note that we can use these results in the calculation of the distribution of magnetic and gluomagnetic fields in a neutron star if its core consists of superconducting quark matter.

**IV. CONCLUSION**

In this article we derived the Ginzburg-Landau equations with account for *rotated electromagnetism*. From the form of these equations written for the physical fields \( \vec{B} \) and \( \vec{B}_8 \) it is obvious that they are coupled, i.e. the appearance of one of them leads to the immediate
appearance of the other. We have solved these equations for an isolated quantum vortex in quark matter and have shown that the centers of electromagnetic and gluomagnetic vortices coincide. The distribution around the continuation of the center of the vortex in quark matter for the fields $\vec{B}$ and $\vec{B}_8$ has the same form as for ordinary superconductors. Nevertheless, the flux of the magnetic vortex is 6 times the magnetic flux quantum $\Phi_0$ and the penetration depth in quark matter is 20 times less than that in hadronic matter. Only the force lines of the magnetic field can leave the quark matter volume whereas the gluomagnetic field is confined therein. As we have demonstrated in this paper for the case of an isolated vortex in quark matter the occurrence of electric and color Meissner currents is a consequence of the color-superconducting state of quark matter.

ACKNOWLEDGEMENT

We thank M. Alford, K. Rajagopal, R. Rapp, E. Shuryak, D. Voskresensky and J. Wambach for discussions. We acknowledge financial support from the DAAD for the exchange program between the Universities of Rostock and Yerevan and from the ECT* Trento for our participation at the workshop: Physics of Neutron Star Interiors. D.S. is grateful to the Observatoire de Paris at Meudon for its hospitality during a research visit; D.B. thanks the Institute for Nuclear Theory at the University of Washington for its hospitality and the Department of Energy for partial support during the programs INT-00-1: QCD at Nonzero Baryon Density and INT-01-2: Neutron Stars.
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