EXISTENCE AT LEAST FOUR SOLUTIONS FOR A SCHRÖDINGER EQUATION WITH MAGNETIC POTENTIAL INVOLVING SIGN-CHANGING WEIGHT FUNCTION

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ABSTRACT

In this paper we consider the following class of elliptic problems

\[-\Delta_A u + u = a_\lambda(x)|u|^{q-2}u + b_\mu(x)|u|^{p-2}u,\]

for \(x \in \mathbb{R}^N, 1 < q < 2 < p < 2^* - 1 = \frac{2N}{N-2}, a_\lambda(x)\) is a sign-changing weight function, \(b_\mu(x)\) has some additional conditions, \(u \in H^1_A(\mathbb{R}^N)\) and \(A : \mathbb{R}^N \rightarrow \mathbb{R}^N\) is a magnetic potential. Exploring the Bahri Li argument and some preliminary results we will discuss the existence of four solution to the problem in question.

Keywords sign-changing weight functions · magnetic potential · Nehari Manifold · Fibering map

1 Introduction

In this work we are interested in studying the existence of a fourth solution for the following classes of concave-convex elliptical problem

\[
(P_1) \begin{cases}
-\Delta_A u + u = a_\lambda(x)|u|^{q-2}u + b_\mu(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\
u \in H^1_A(\mathbb{R}^N),
\end{cases}
\]

where \(N \geq 3, -\Delta_A = (-i\nabla + A)^2\), \(1 < q < 2 < p < 2^* = \frac{2N}{N-2}\), \(a_\lambda(x)\) is a family of functions that can change signal, \(b_\mu(x)\) is continuous and satisfies some additional conditions, \(u : \mathbb{R}^N \rightarrow \mathbb{C}\) with \(u \in H^1_A(\mathbb{R}^N)\) (such space will be defined later), \(\lambda > 0\) and \(\mu > 0\) are real parameters, \(u : \mathbb{R}^N \rightarrow \mathbb{C}\) and \(A : \mathbb{R}^N \rightarrow \mathbb{R}^N\) is a magnetic potential in \(L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)\).

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In [12] the authors show the existence of three solutions for this problem and also prove their regularity. In this case we will show the existence of fourth solution. Many works have been developed with the magnetic laplacian. Its importance in physics was discussed e. g. in Alves and Figueiredo [1] and in Arioli and Szulkin [2].

There are so many works in literature with similar problem to \((P_1)\) with \(A = 0\) like in Ambrosetti, Brezis and Cerami [2], where the following problem was considered

\[
\begin{cases}
-\Delta u + u = \lambda u^{q-1} + u^{p-1} \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \partial \Omega,
\end{cases}
\]

where \(\Omega\) is a bounded regular domain of \(\mathbb{R}^N\) \((N \geq 3)\), with smooth boundary and \(1 < q < 2 < p \leq 2^*\). Combining the method of sub and super-solutions with the variational method, the authors proved the existence of a certain \(\lambda_0 > 0\) such that there are two solutions when \(\lambda \in (0, \lambda_0)\), one solutions if \(\lambda = \lambda_0\) and no solutions if \(\lambda > \lambda_0\).

The concave-convex problem like

\[
-\Delta u + u = \lambda f(x)u^{q-1} + u^{p-1} \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \partial \Omega,
\]

with \(f \in C(\overline{\Omega})\) a sign changing function and \(1 < q < 2 < p \leq 2^*\), was studied by Wu in [24]. It proves that the problem has at least two positive solutions for values of \(\lambda\) small enough. Therefore, many studies have been devoted to the analysis of existence and multiplicity of concave-convex elliptic problems in bounded domains, for instance, we can cite Brown [8]; Brown and Wu [6]; Brown and Zhang [7]; Hsu [18]; Hsu and Lin [17] and references contained in these articles.

In an unbounded domain we can cite Chen [10], Huang, Wu and Wu [19], who have worked with a similar cases in \(\mathbb{R}^N\). In [25], Wu deals with the problem

\[
\begin{cases}
-\Delta u + u = f_\lambda(x)u^{q-1} + g_\mu u^{p-1} \quad \text{in } \mathbb{R}^N, \\
u \geq 0 \quad \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

with \(1 < q < 2 < p \leq 2^*, g_\mu \geq 0\) or \(f_\lambda\) being able to change of signal, among other additional hypotheses. It seeks to show the existence of at least four solutions to the problem when \(\lambda\) and \(\mu\) small enough. This result was extend in [12], investigating if it would be possible to obtain similar consequences when we replace the magnetic laplacian in the place of the usual Laplacian. In this work we will show the existence of the fourth solution for this problem.

The first results in non-linear Schrödinger equations, with \(A \neq 0\) can be atributed to Esteban and Lions [14] in which the existence of stationary solutions for equations of the type

\[
-\Delta_A + Vu = |u|^{p-2}u, u \neq 0, u \in L^2(\mathbb{R}^N),
\]

with \(V = 1\) and \(p \in (2, \infty)\), were obtained using minimization method with constant magnetic field and also for the general case.

Chabrowski and Szulkin [9] worked with this operator in the critical case and with the electric potential \(V\) being able to change the signal. Already Cingolani, Jeanjean and Secchi [11] consider the existence of mult-peak solutions in the subcritical case.

A problem of the type

\[
-\Delta_A u = \mu |u|^{q-2}u + |u|^{2^*-2}u, u \neq 0, \Omega \subset \mathbb{R}^N,
\]

with \(\mu > 0\) and \(2 \leq q < 2^*\), is treated by Alves and Figueiredo [1] in which the number of solutions with the topology of \(\Omega\) is related.

In [12] they deal with the non-zero \(A\) case with a weight function that changes sign in the concave-convex case, like the problem in this work. They prove the existence of three solutions for the problem and now, we would like to show the existence of the fourth solution. In [12] was used the Nehari manifold linked with the behavior of functions known as fibering map and Category theory.

In the sequence we will announce some preliminaries results and the result that we seek to show. Observe that

\[
J_{\lambda, \mu}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla A_{\lambda, \mu}u|^2 + u_{\lambda, \mu}^2)dx - \frac{1}{q} \int_{\mathbb{R}^N} a_{\lambda}(x)|u|^qdx - \frac{1}{p} \int_{\mathbb{R}^N} b_\mu(x)|u|^pdx,
\]

(1)
is the functional associated with the problem \((P_1)\) and is of class \(C^1\) in \(H^1_\lambda(\mathbb{R}^N)\) as can be seen in [22]. Also, the critical points of \(J_{\lambda,\mu}(u)\) are weak solutions of problem \((P_1)\). We will work with the hypotheses that we will enunciate next. Consider the function \(a(x) \in L^q(\mathbb{R}^N)\), \(q' = \frac{p}{p-q}\) and \(a_{\pm} = \pm \max\{\pm a(x),0\} \neq 0\). Let us assume

\[
a_{\lambda}(x) = \lambda a_+(x) + a_-(x).
\]

\((A)\) \(a(x) \in L^q(\mathbb{R}^N)\), \(q' = \frac{p}{p-q}\) and exists \(\hat{c} > 0\) and \(r_{a_-} > 0\), such that

\[
a_-(x) > -\hat{c} \exp(-r_{a_-}|x|) \quad \text{for all} \quad x \in \mathbb{R}^N.
\]

In addition to \((A)\), we will assume that \(b_\mu(x) = b_1(x) + \mu b_2(x)\), where

\((B_1)\) \(b_1(x) > 0\) in continuous in \(\mathbb{R}^N\), with \(b_1(x) \to 1\) as \(|x| \to \infty\) and exists \(r_{b_1} > 0\), such that

\[
1 \geq b_1(x) \geq 1 - c_0 \exp(-r_{b_1}|x|) \quad \text{for some} \quad c_0 < 1 \quad \text{and for all} \quad x \in \mathbb{R}^N.
\]

\((B_2)\) \(b_2(x) > 0\) is continuous in \(\mathbb{R}^N\), \(b_2(x) \to 0\) as \(|x| \to \infty\) and exists \(r_{b_2} > 0\), with \(r_{b_2} < \min\{r_{a_-}, r_{b_1}, q\}\) such that

\[
b_2(x) \geq d_0 \exp(-r_{b_2}|x|) \quad \text{for some} \quad d_0 < 1 \quad \text{and for all} \quad x \in \mathbb{R}^N.
\]

Those hypotheses were used in [12]. Consider

\[
\Upsilon_0 = (2-q)^{2-q} \left( \frac{p-2}{|a_+| q'} \right)^{p-2} \left( \frac{S_p}{p-q} \right)^{p-q}, \quad \text{where}
\]

\[
S_p = \inf_{u \in H^1_\lambda(\mathbb{R}^N \setminus \{0\})} \frac{\left( \int_{\mathbb{R}^N} |\nabla A u|^2 + u^2 dx \right)^{\frac{p}{2}}}{\left( \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{p}{2}}} > 0.
\]

In [12] the first result, assuming the hypotheses \((A)\), \((B_1)\) and \((B_2)\), and \(\Upsilon_0\) as defined above, it was proved that \((P_1)\) has at least one solution, provided that

\[
\lambda^{p-2}(1 + \mu ||b_2||_{\infty})^{2-q} < \left( \frac{q}{2} \right)^{p-2} \Upsilon_0
\]

(3)

holds for each \(\lambda > 0\) and \(\mu > 0\). Then, adding the hypothesis that the potential is asymptotic to a constant in infinity, they prove the existence of at least two solutions \(u_{\lambda,\mu}^+\) and \(u_{\lambda,\mu}^-\) with \(J_{\lambda,\mu}(u_{\lambda,\mu}^+) < 0 < J_{\lambda,\mu}(u_{\lambda,\mu}^-)\).

In the previous result, the existence is valid for all \(\lambda\) and \(\mu\) satisfying the inequality [3]. So, if we additionally set values of \(\lambda\) and \(\mu\) conveniently small we obtain the multiplicity result, that is, the existence of at least three solutions. Actually they showed the existence of \(\lambda_0 > 0\) and \(\mu_0 > 0\) with \(\lambda_0^{p-2}(1 + \mu_0 ||b_2||_{\infty})^{2-q} < \left( \frac{q}{2} \right)^{p-2} \Upsilon_0\), such that for all \(\lambda \in (0, \lambda_0)\) and \(\mu \in (0, \mu_0)\), the problem \((P_1)\) has at least three solutions.

Now, in this work, we observe that for the problem in question, the numbers \(\lambda_0\) and \(\mu_0\) as previously mentioned are independent of the value of \(a_-\). However, considering some additional hypotheses and taking values of \(||a_-||_{q'}\) sufficiently small we have the results getting another solution. Before enunciate this result we will present the following hypotheses:

\((C_1)\) \(b_1(x) < 1\) in \(\mathbb{R}^N\) in a positive measure set;
\((C_2)\) \(r_{b_\mu} > 2\).

**Theorem 1.1.** Suppose that the potential \(A \to d\) where \(d\) constant as \(|x| \to \infty\). Assuming the hypotheses \((A)\), \((B_1)\), \((B_2)\), \((C_1)\) and \((C_2)\) there are positive values of \(\lambda_0 \leq \lambda_0\), \(\mu_0 \leq \mu_0\) and \(v_0\) such that for \(\lambda \in (0, \lambda_0)\), \(\mu \in (0, \mu_0)\) and \(||a_-||_{q'} < v_0\), the problem \((P_1)\) has at least four solutions.

For these first three solutions results of this problem was used the Nehari method together with the category theory. We will continue to make use of variational methods to prove the above theorem. We will work under a few more assumptions to estimate different energy levels and will use the Bahri-Li min-max argument to show that for very small values of \(||a_-||_{q'}\), the problem has at least four distinct solutions.
2 Initial considerations

According to Tang [23], we denote by $H^q_A(\mathbb{R}^N)$ the Hilbert space obtained by the closing of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with following inner product:

$$< u, v >_A = Re \int_{\mathbb{R}^N} (\nabla u \overline{\nabla v} + u \overline{v}) dx,$$

where $\nabla_A u := (D_1 u, D_2 u, ..., D_N u)$ and $D_j := -i \partial_j - A_j(x)$, with $j = 1, 2, ..., N$, with $A(x) = (A_1(x), ..., A_N(x))$. The norm induced by this product is given by

$$||u||_A^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx.$$

It is proved by Esteban and Lions, [14] Section II that for all $u \in H^1_A(\mathbb{R}^N)$ it is worth diamagnetic inequality

$$|\nabla |u||_A^2 = |Re \left( \nabla u \overline{\nabla u} \right)| = |Re \left( \nabla u - iAu \right) \overline{\nabla u} | \leq |\nabla u(x)|$$

So, if $u \in H^1_A(\mathbb{R}^N)$ we have that $|u|$ belongs to the usual Sobolev space $H^1_0(\mathbb{R}^N)$.

2.1 Preliminary results

To obtain results of existence in this case, we introduced the Nehari manifold

$$M_{\lambda, \mu} = \{ u \in H^1_A(\mathbb{R}^N) \setminus \{0\} : \langle J_{\lambda, \mu}^A(u), u \rangle = 0 \},$$

where $\langle , \rangle$ denotes the usual duality between $H^1_A(\mathbb{R}^N)^*$ and $H^1_A(\mathbb{R}^N)$, where $H^1_A(\mathbb{R}^N)^*$ is the dual space to the corresponding $H^1_A(\mathbb{R}^N)$ space. The Nehari manifold is linked to the functions of the form $F_{u} : t \rightarrow J_{\lambda, \mu}(tu); \; (t > 0)$, called fibering map. Note that the fabering map it was defined and depends on $u, \lambda$ and $\mu$, so that proper notation would be $F_{u, \lambda, \mu}$, but in order to simplify the notation, we will denote by $F_u$. If $u \in H^1_A(\mathbb{R}^N)$, we have

$$F_u(t) = \frac{t^2}{2} ||u||_A^2 - \frac{t^q}{q} \int_{\mathbb{R}^N} a_\lambda(x)||u|^q dx - \frac{t^p}{p} \int_{\mathbb{R}^N} b_\mu(x)||u|^p dx, \quad (4)$$

$$F'_u(t) = t||u||_A^2 - t^{q-1} \int_{\mathbb{R}^N} a_\lambda(x)||u|^q dx - t^{p-1} \int_{\mathbb{R}^N} b_\mu(x)||u|^p dx, \quad (5)$$

$$F''_u(t) = ||u||_A^2 - (q-1)t^{q-2} \int_{\mathbb{R}^N} a_\lambda(x)||u|^q dx - (p-1)t^{p-2} \int_{\mathbb{R}^N} b_\mu(x)||u|^p dx. \quad (6)$$

The following remark relates the Nehari manifold and the Fibering map.

**Remark 2.1.** Let $F_u$ be the application defined above and $u \in H^1_A(\mathbb{R}^N)$, then:

(i) $u \in M_{\lambda, \mu}$ if, and only if, $F'_u(1) = 0$;

(ii) more generally $tu \in M_{\lambda, \mu}$ and only if, $F'_u(t) = 0$.

From the previous remark we can conclude that the elements in $M_{\lambda, \mu}$, correspond to the critical points of the Fibering map. Thus, as $F_u(t) \in C^2(\mathbb{R}^+, \mathbb{R})$, we can divide the Nehari manifold into three parts

$$M_{\lambda, \mu}^+ := \{ u \in M_{\lambda, \mu} : F''_u(1) > 0 \};$$

$$M_{\lambda, \mu}^- := \{ u \in M_{\lambda, \mu} : F''_u(1) < 0 \};$$

$$M_{\lambda, \mu}^0 := \{ u \in M_{\lambda, \mu} : F''_u(1) = 0 \}.$$

Lemma below shows us under some conditions the $M_{\lambda, \mu}^0$ is empty.

**Lemma 2.2.** Let $\mu \geq 0$ and $\lambda > 0$ such that

$$\lambda^{p-2}(1 + \mu||b_2||_\infty)^{2-q} < \Upsilon_0. \quad (7)$$

Then $M_{\lambda, \mu}^0 = \emptyset$.

**Proof.** The proof is similar to what was done in [6, Lemma 2.2].

\[ \square \]
In [12] they showed that under certain conditions on \( \lambda \) and \( \mu \), we have a minimizer in \( M_{\lambda,\mu}^+ \) and another in \( M_{\lambda,\mu}^- \), whose minimum levels of energy will be denoted respectively by

\[
m_{\lambda,\mu}^+ = \inf_{u \in M_{\lambda,\mu}^+} J_{\lambda,\mu}(u) \]

and

\[
m_{\lambda,\mu}^- = \inf_{u \in M_{\lambda,\mu}^-} J_{\lambda,\mu}(u). \]

To establish the existence of the first two solutions and compare with the energy level of the fourth solution, we will need the following result that was shown in [12].

**Lemma 2.3.** For each \( u \in H_A^1(\mathbb{R}^N) \setminus \{0\} \) and \( \mu > 0 \) we have

\((i)\) If \( \int_{\mathbb{R}^N} a_\lambda(x) |u|^q \, dx \leq 0 \), there is a single \( t^-(u) > t_{\max}(u) \) such that \( t^-(u) u \in M_{\lambda,\mu}^- \). Also, \( F_u(t) \) is increasing in \((0, t^-(u))\), decreasing in \((t^-(u), +\infty)\) and \( F_u(t) \to -\infty \) as \( t \to +\infty \).

\((ii)\) If \( \int_{\mathbb{R}^N} a_\lambda(x) |u|^q \, dx > 0 \) and \( \lambda \) is such that \( \lambda^{p-2} (1+\mu) |b_2|_\infty^{2-q} < \Upsilon_0 \), so there is \( 0 < t^+(u) < t_{\max}(u) < t^+(u) \) such that \( t^\pm(u) u \in M_{\lambda,\mu}^\pm \). Also, \( F_u(t) \) is decreasing in \((0, t^+(u))\), increasing in \((t^+(u), t^-(u))\) and decreasing in \((t^-(u), +\infty)\). Furthermore, \( F_u(t) \to -\infty \) as \( t \to +\infty \).

Our next result shows that these points are well defined.

**Lemma 2.4.** The functional \( J_{\lambda,\mu} \) is coercive and bounded from below in \( M_{\lambda,\mu} \).

**Proof.** The proof is similar to that made in [17] Lemma 2.1. \(\square\)

For the next results we will need some estimates about the values of the functions in \( m_{\lambda,\mu}^\pm \). To do this, from [7] we have

\[
|u|_A^2 < \frac{p-q}{p-2} \int_{\mathbb{R}^N} a_\lambda(x) |u|^q \, dx \leq \Upsilon_0^{1/(p-2)} \frac{p-q}{p-2} s_p^{-\frac{q}{p}} |a_+||L_{v'}||u||_{L^q}^q.
\]

Therefore

\[
|u|_A \leq \left( \Upsilon_0^{1/(p-2)} \frac{p-q}{p-2} s_p^{-\frac{q}{p}} |a_+||L_{v'}| \right)^{1/(2-q)} |u||_{L^q}^q \tag{8}
\]

for all \( u \in M_{\lambda,\mu}^\pm \). Also, if \( \lambda = 0 \), then (7) is satisfied, so that by Lemma 2.3(i), \( M_{\lambda,\mu}^+ = \emptyset \), and we have \( M_{\lambda,\mu} = M_{\lambda,\mu}^- \) for all \( \mu \geq 0 \). By has been seen, we will show the following results on the values of \( m_{\lambda,\mu} \).

**Lemma 2.5.** \((i)\) If \( \lambda^{p-2}(1+\mu) |b_2|_\infty^{2-q} < (\frac{\lambda}{\Upsilon})^{p-2} \Upsilon_0 \), then \( m_{\lambda,\mu}^- > 0 \);

\((ii)\) For \( \lambda > 0 \) and \( \mu \geq 0 \) with \( \lambda^{p-2}(1+\mu) |b_2|_\infty^{2-q} < \Upsilon_0 \), then \( m_{\lambda,\mu}^+ < 0 \). In particular, if \( \lambda^{p-2}(1+\mu) |b_2|_\infty^{2-q} < (\frac{\lambda}{\Upsilon})^{p-2} \Upsilon_0 \), then

\[
m_{\lambda,\mu}^+ = \inf_{M_{\lambda,\mu}} J_{\lambda,\mu}(u).
\]

**Proof.** The proof is similar to what was done in [25] Theorem 3.1. \(\square\)

By Lemma 2.5 we can conclude that for every \( u \in H_A^1(\mathbb{R}^N) \setminus \{0\} \)

\[
J_{\lambda,\mu}(t^-(u) u) = \max_{t \leq 0} J_{\lambda,\mu}(tu), \tag{9}
\]

whenever \( \lambda^{p-2}(1+\mu) |b_2|_\infty^{2-q} < (\frac{\lambda}{\Upsilon})^{p-2} \Upsilon_0 \), with \( \lambda \geq 0 \) and \( \mu > 0 \).
3 Existence of \( m_\infty \)

In this section we will define the energy level of the limit problem and make some energy estimates in relation to the energy levels of the solutions in the Nehary manifold. Therefore, we will have tools to show that the fourth solution to be found has a different level than other solutions already found. For this, consider the following semilinear elliptical problem

\[
(P_A) \quad \begin{cases} 
-\Delta A u + u = |u|^{p-2} u & \text{in } \mathbb{R}^N, \\
u \in H_A^1(\mathbb{R}^N).
\end{cases}
\]

Define \( J_\infty(u) = \frac{1}{2}||u||_A^2 - \frac{1}{p}||u||_p^p \), the functional associated with the problem \((P_A)\), then \( J_\infty \) is a functional \( C^2 \) in \( H_A^1(\mathbb{R}^N) \). The Nehary manifold associated with problem \((P_A)\) is given by

\[
M_\infty = \{ u \in H_A^1(\mathbb{R}^N) \setminus \{0\} : J'_\infty(u)u = 0 \}.
\]

In this problem we can observe that if \( u \in N_\infty \), then \( ||u||_A^2 = ||u||_p^p \). Now consider the following minimization problem

\[
m_\infty = \inf_{M_\infty} J_\infty(u). \quad (10)
\]

In [12] they prove that exists \( \bar{u} \in H_A^1(\mathbb{R}^N) \) such that \( m_\infty = \inf_{N_\infty} J_\infty(u) = J_\infty(\bar{u}) \). From these considerations we will show the following result that gives us a description of a sequence (PS) of \( J_{\lambda, \mu} \).

**Lemma 3.1.** Let \( \{u_n\} \subset M_{\lambda, \mu}^- \) be a sequence (PS) in \( H_A^1(\mathbb{R}^N) \) of \( J_{\lambda, \mu} \), this is, a sequence satisfying \( J_{\lambda, \mu}(u_n) = \beta + o_n(1) \) and \( J'_{\lambda, \mu}(u_n) = o_n(1) \) in \( H_A^{-1} \) as \( n \to \infty \), where

\[
m_{\lambda, \mu}^+ + m_\infty < \beta < m_{\lambda, \mu}^- + m_\infty,
\]

then there is a subsequence \( \{u_n\} \) and \( u_0 \in H_A^1(\mathbb{R}^N) \), with a non zero \( u_0 \), such that \( u_n = u_0 + o_n(1) \) strong in \( H_A^1(\mathbb{R}^N) \) and \( J_{\lambda, \mu}(u_0) = \beta \). Moreover, \( u_0 \) is a solution of \((P_1)\).

**Proof.** For \((A), (B_1)\) and \((B_2)\), we obtain by a standard argument that \( \{u_n\} \) is bounded sequence in \( H_A^1(\mathbb{R}^N) \). Then there is a subsequence \( \{u_n\} \) and \( u_0 \in H_A^1(\mathbb{R}^N) \) such that \( u_n \rightharpoonup u_0 \) weakly in \( H_A^1(\mathbb{R}^N) \) as \( n \to \infty \). Taking \( u_n = u_n - u_0 \), we have \( v_n \rightharpoonup 0 \) weak in \( H_A^1(\mathbb{R}^N) \) as \( n \to \infty \). Denoting by \( B(0, 1) \) the ball centered on the origin of radius 1, we have in \( B(0, 1) \) the strong convergence

\[
\int_{B(0, 1)} |u_n|^q \to \int_{B(0, 1)} |u_0|^q.
\]

By the Dominated Convergence Theorem we obtain

\[
\int_{B(0, 1)} a_\lambda ||u_n|^q - |u_0|^q| \to 0, \text{ when } n \to \infty.
\]

Then, by Hölder and the integrability of \( a_\lambda \) follows

\[
\left| \int a_\lambda(x)(|u_n|^q - |u_0|^q) \right| \leq o_n(1) + \int_{B(0, 1)} a_\lambda(x)||u_n|^q - |u_0|^q| \\
\leq o_n(1) + \left( \int_{B(0, 1)} a_\lambda(x)|v|^q \right)^{\frac{1}{q}}(||u_n||_p^p + ||u_0||_p^p) \\
\leq o_n(1) + C.
\]

As \( \epsilon > 0 \) it is arbitrary, we have

\[
\int a_\lambda(x)(|u_n|^q - |u_0|^q) = o_n(1).
\]

On the other hand, \((B_1)\) and \((B_2)\) and by Brezis-Lieb lemma (see [24]), we can conclude that \( \mu \int b_2(x)|v_n|^p = o_n(1) \), \( \int (1 - b_1(x))|v_n|^p = o_n(1) \) and \( \int b_1(x)(|u_n|^p - |v_n|^p - |u_0|^p) = o_n(1) \), which together with the above inequality gives us

\[
J_{\lambda, \mu}(u_n) = J_\infty(v_n) + J_{\lambda, \mu}(u_0) + o_n(1).
\]

In a similar way we obtain that \( J'_{\lambda, \mu}(v_n)v_n = J'_{\lambda, \mu}(u_0)v_n - J'_{\lambda, \mu}(u_0)u_0 + o_n(1) \). By hypothesis \( J'_{\lambda, \mu}(u_n) \to 0 \) strong in \( H_A^{-1}(\mathbb{R}^N) \) and \( u_n \rightharpoonup u_0 \) weak in \( H_A^1(\mathbb{R}^N) \) as \( n \to \infty \) and so we have \( J'_{\lambda, \mu}(u_0) = 0 \). Now, define \( \delta = \lim \sup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} |v_n|^p \). So we have two cases:
(i) $\delta > 0$, or
(ii) $\delta = 0$.

Suppose that (i) happen. Then there will be a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B(y_n, r)} |v_n|^p \geq \frac{\delta}{4}$ and for all $n \in \mathbb{N}$. Define $\tilde{v}_n(x) = v_n(x + y_n)$. We have that $\{\tilde{v}_n\}$ is bounded and $\tilde{v}_n \rightharpoonup v$ weak and almost everywhere. Making a change of variables we obtain

$$
\int_{B(0,1)} |\tilde{v}_n|^p \geq \frac{\delta}{4}.
$$

Then

$$
\int_{B(0,1)} |v|^p \geq \frac{\delta}{4},
$$

(11)
giving us $v \neq 0$. But, $v_n \to 0$ weakly, then

$$
\int_{\mathbb{R}^N} |v_n|^p \geq \int_{B(y_n,1)} |v_n|^p \geq \frac{\delta}{2} > 0.
$$

(12)

See that

$$
J_\infty(v_n) = \frac{1}{2} \int (|A\nabla v_n|^2 + v_n^2) dx - \frac{1}{p} \int |v_n|^p dx.
$$

Likewise,

$$
F_{v_n}(t) = J_\infty(t v_n) = \frac{t^2}{2} ||v_n||_A^2 - \frac{t^p}{p} ||v_n||^p.
$$

For each $n \in \mathbb{N}$, we can get $t_n$ such that $t_n v_n \in M_\infty$. So we build a sequence $\{t_n\} \subset \mathbb{R}^N$ with $t_n \to t_0$ as $n \to \infty$, such that $t_n v_n \in M_\infty$, that is, such that $J'(t_n v_n)t_n v_n = 0$. See also that

$$
J'_\infty(v_n)v_n = ||v_n||^2_A - ||v_n||^p = o_n(1)
$$

and

$$
F_{v_n}'(t) = J'_\infty(t v_n)v_n = t||v_n||^2_A - t^{p-1}||v_n||^p = o_n(1).\tag{13}
$$

With this

$$
(t_n - t_n^{p-1})||v_n||^2_A = t_n(1 - t_n^{p-2})||v_n||^2_A = o_n(1).\tag{14}
$$

For (11) we know that $||v_n||_A^2 \to 0$ (that is, $v_n$ does not converge to zero). Also note that $t_n^{2-p} = \int |v_n|^p \geq \frac{\delta}{2\delta}$. With that and by (14) we get that $(1 - t_n^{p-2}) \to 0$, giving us that $t_n \to 1$. Now, see that $v_n \to 0$ weak in $H^1_{A(\mathbb{R}^N)}$ as $n \to \infty$. With this and by the fact $t_n \to 1$, we can conclude that

$$
J_{\lambda,\mu}(u_n) = J_\infty(t_n v_n) + J_{\lambda,\mu}(u_0) + o_n(1) \geq m_\infty + J_{\lambda,\mu}(u_0).
$$

Note that by hypothesis $J_{\lambda,\mu}(u_0) = \beta + o_n(1)$ with $\beta < m_\infty + m^+_{\lambda,\mu}$. From there we obtain

$$
\beta + o_n(1) = J_{\lambda,\mu}(u_0) = J_\infty(t_n v_n) + J_{\lambda,\mu}(u_0) + o_n(1) \geq m_\infty + J_{\lambda,\mu}(u_0),
$$

giving us

$$
m_\infty + J_{\lambda,\mu}(u_0) \leq \beta + o_n(1) < m_\infty + m^+_{\lambda,\mu} + o_n(1),
$$

therefore

$$
J_{\lambda,\mu}(u_0) < m^+_{\lambda,\mu} + o_n(1).\tag{15}
$$

We have already seen that $J'_{\lambda,\mu}(u_0)$ converges strongly to zero, therefore we get $J'_{\lambda,\mu}(u_0) = 0$. Thus $u_0 \in M_{\lambda,\mu}$. Still, by Lemma 2.2, $M_{\lambda,\mu} = \emptyset$ and by Lemma 2.5 we conclude that $m^+ > 0$ and $m^- < 0$. Then,

$$
J_{\lambda,\mu}(u_0) \geq \inf_{M_{\lambda,\mu}} J_{\lambda,\mu}(u) = \inf_{M^+_{\lambda,\mu}} J_{\lambda,\mu}(u) = m^+,
$$

which contradicts what we have concluded in (15). We conclude that (ii) occurs. In this case, $\{v_n\}$ such that $\int |v_n|^p \to 0$ if $n \to \infty$.

As we already have $J_\infty(v_n)v_n = o_n(1)$ with $J'_\infty(v_n)v_n = ||v_n||^2_A - ||v_n||^p$ and $\int |v_n|^p \to 0$, we conclude that $||v_n||^2 \to 0$ giving us $u_n \to u_0$ strong in $H^1_{A(\mathbb{R}^N)}$. See also that $u_0 \neq 0$. In fact, note that if $u_0 = 0$ so $\tilde{v}_n = v_n = u_n$ and $\int_{B(0,1)} |u_n|^p \geq \frac{\delta}{4}$, which we have already seen to be an absurd.
To treat the existence of the second solution of the problem \( (P_1) \), we need to make some considerations. Note that equation
\[
-\Delta_A u + u = a_\lambda(x)|u|^{p-2}u + b_\mu(x)|u|^{p-2}u \quad (P_1)
\]
is such that \( a_\lambda(x) \to 0 \) and \( b_\mu(x) \to 1 \) as \( |x| \to \infty \). Adding the hypothesis of \( A \to d \) with \( d \) constant as \( |x| \to \infty \), the problem \( (P_1) \) converges at infinity for the problem
\[
-\Delta_d u + u = |u|^{p-2}u. \quad (P_\infty),
\]
where \( -\Delta_d = (-i\nabla + d)^2 \). Thus, by a result of Ding and Liu [13 Lemma 2.5], \( u \) is a solution of Problem \( (P_\infty) \) if and only if \( v(x) := |u(x)| \in H^1 \) it is a solution to the problem
\[
-\Delta v + v = v^{p-1}; \quad v > 0. \quad (E_\infty)
\]
Moreover, the equations \( (P_\infty) \) and \( (E_\infty) \) have the same energy level, that is
\[
J_\infty(u) = I_\infty(v) = m_\infty;
\]
on what \( J_\infty \) and \( I_\infty \) are the respective functional associated with the previous problems. According to Berestick, Lions [5] or Kwong [21], the equation \( (E_\infty) \) has a unique solution \( z_0 \) symmetrical, positive and radial. By [15 Theorem 2], for all \( \epsilon > 0 \), exists \( A_{\epsilon}, B_{\epsilon} \) and \( C_\epsilon \) positive such that
\[
A_{\epsilon} \exp(-(1 + \epsilon)|x|) \leq z_0(x) \leq B_{\epsilon} \exp(-|x|) \quad (16)
\]
and
\[
|\nabla z_0(x)| \leq C_\epsilon \exp(-(1 - \epsilon)|x|). \quad (17)
\]
According Kurata [20, Lemma 4], defining \( w_0 = z_0 e^{-i\epsilon x} \) we have \( w_0 \) is a solution of \( (P_\infty) \), unique, symmetrical, positive and radial. So we will have \( J_\infty(w_0) = m_\infty \). See also that \( z_0 = |w_0| \), which together with (16) gives us the following inequalities
\[
A_{\epsilon} \exp(-(1 + \epsilon)|x|) \leq |w_0(x)| \leq B_{\epsilon} \exp(-|x|) \quad (18)
\]
and
\[
|\nabla w_0(x)| \leq C_\epsilon \exp(-(1 - \epsilon)|x|). \quad (19)
\]
Next, we will make some estimates about the minimum energy levels in the Nehari Manifold to prove the existence of a second solution. In order to not overload the notation, we will denote \( u^{+}_\lambda,\mu := u^{+} \). Considering \( J(u^{+}) = m^{+} \), \( m^{-} = \inf_{u \in M^{-}} J_{\lambda,\mu}(u) \) and \( m_\infty = \inf_{u \in M^\infty} J_\infty(u) = J_\infty(w_0) \), we will make the following estimate for such energy levels.

**Proposition 3.2.** For all \( \lambda > 0 \) and \( \mu > 0 \) satisfying \( \lambda^p(1 + \mu||b_2||_\infty)^{2-q} \leq \gamma_0 \), we have \( m^{-} < m^{+} + m_\infty \).

**Proof.** The proof is similar to what was done in [12 Proposition 6.1].

\( \square \)

### 4 Third Solution

#### 4.1 Some considerations

To get the third solution of the \( (P_1) \) problem, we will need some results that is done next. For this, we highlight the set defined below for \( \lambda = 0 \) and \( \mu = 0 \)
\[
M_{a_0, b_0}^- = \{u \in H_1^1(\mathbb{R}^N) \setminus \{0\} : \langle J_{a_0, b_0}(u), u \rangle = 0 \}
\]
where
\[
J_{a_0, b_0} = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla_A u\lambda,\mu|^2 + u_{\lambda,\mu}^2\right) dx - \frac{1}{q} \int a_0(x)|u|^q dx - \frac{1}{p} \int b_0(x)|u|^p dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla_A u\lambda,\mu|^2 + u_{\lambda,\mu}^2\right) dx - \frac{1}{q} \int a_-(x)|u|^q dx - \frac{1}{p} \int b_1(x)|u|^p dx.
\]

**Lemma 4.1.** We have
\[
\inf_{u \in M_{a_0, b_0}} J_{a_0, b_0}(u) = \inf_{u \in M} J_\infty(u) = m_\infty.
\]
We also have to make sure that the problem has a unique solution. To prove our result, we will need this lemma that establishes values of $w$ of the problem.

Let $w_0$ be a solution of problem $(E_{\infty})$ and remembering that the functional associated with $(E_{\infty})$ is given by $I(u) = \frac{1}{2}||u||_{\lambda}^2 - \frac{1}{p}||u||_p^p$, and $I'(u) = ||u||_{\lambda}^2 - ||u||_p^p$ we have

$$I'(w_0)w_0 = ||w_0||_{\lambda}^2 - ||w_0||_p^p.$$

Therefore

$$m_\infty = I(w_0) = \frac{1}{2}||w_0||_{\lambda}^2 - \frac{1}{p}||w_0||_p^p =\frac{1}{2}||w_0||_{\lambda}^2 - \frac{1}{p}||w_0||_A = \frac{p-2}{2p}||w_0||_A.$$

Being $w_0$ solution of problem $(E_{\infty})$ follows that $w_k(x) = w_0(x + ke)$. With this and $I'(w_0)w_0 = 0$, we have $I'(w_k)w_k = 0$. So that

$$||w_k||_{\lambda}^2 = \int_{\mathbb{R}^N} ||w_k||_q^qdx = \frac{2p}{p-2}m_\infty \quad \text{for all } k \geq 0.$$

It is known that $w_n$ is bounded in $L^{r'}$ and $w_n \to 0$ a.e., by Theorem [16, Theorem 13.44] that $w_n \to 0$ weakly in $L^{r'}$.

By the condition $(A)$, $a_- \in (L^{r'})' = L^r$ we get

$$\int_{\mathbb{R}^N} a_-|w_k|^qdx \to 0 \text{ as } k \to \infty. \quad (22)$$

In addition, by $(B_1)$ and $(B_2)$ we get

$$\int_{\mathbb{R}^N} (1-b_1)|w_k|^qdx = \int_{B(0,R)} (1-b_1)|w_k|^qdx + \int_{B^\complement(0,R)} (1-b_1)|w_k|^qdx \to 0, \quad (23)$$

as $|w_k| \to \infty$. By $(20)$, $(22)$ and $(23)$ we have that $t^{-}(w_k) \to 1$ as $k \to \infty$. Likewise

$$\lim_{k \to \infty} J_{a_-,b_0}(t^{-}(w_k)) = \lim_{k \to \infty} J_{\infty}(t^{-}(w_k)) = m_\infty.$$

Thus

$$m_\infty = \inf_{u \in M^\infty} J_{\infty}(u) = \lim_{k \to \infty} J_{\infty}(t^{-}(w_k)) \geq \inf_{u \in M^\infty} J_{a_-,b_0}(u). \quad (24)$$

We also have to $u \in M^\infty_{a_-,b_0}$, by Lemma [2,3] (i), $J_{a_-,b_0}(u) = \sup_{t \geq 0} J_{a_-,b_0}(tu)$, and more, there is a single $t^\infty > 0$ such that $t^\infty u \in M^\infty$. So

$$J_{a_-,b_0}(t^\infty u) = \frac{1}{2}||t^\infty u||_{\lambda}^2 - \frac{(t^\infty)^q}{q} \int_{\mathbb{R}^N} a_-|u|^qdx - \frac{(t^\infty)^p}{p} \int_{\mathbb{R}^N} b_1|u|^pdx \geq \frac{1}{2}||t^\infty u||_{\lambda}^2 - \frac{(t^\infty)^p}{p} \int_{\mathbb{R}^N} |u|^pdx \geq J_{\infty}(t^\infty u) \geq m_\infty,$$

therefore

$$\inf_{u \in M^\infty_{a_-,b_0}} J_{a_-,b_0}(t^\infty u) \geq m_\infty. \quad (25)$$

By $(24)$ and $(25)$

$$\inf_{u \in M^\infty_{a_-,b_0}} J_{a_-,b_0}(u) = \inf_{u \in M^\infty} J_{\infty}(u) = m_\infty.$$

To prove our result, we will need this lemma that establishes values of $\lambda$ and $\mu$ suitable values to get the fourth solution of the problem.
Lemma 4.2. Exist } \lambda_0 > 0 \text{ and } \mu_0 > 0 \text{ with } \\
\lambda_0^{p-2}(1 + \mu_0||b_1||_\infty)^{2-q} < \left(\frac{q}{2}\right)^{p-2} \gamma_0, \\
such that for all } \lambda \in (0, \lambda_0) \text{ and } \mu \in (0, \mu_0), \text{ we have } \\
\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx \neq 0 \\
\text{for all } u \in M_{a_1,b_1}^- \text{ with } J_{\lambda,\mu}(u) < m_{a_1,b_1}^+ + m^\infty. \\

Proof. The proof is in accordance with what was done in [12] Lemma 7.6. \hfill \Box

5 Fourth Solution

We will work in this section with estimates of the energy levels of the functional associated with the main problem to prove the existence of a solution whose energy level satisfies the conditions of Proposition 3.1(ii), that is, a distinct solution of the three solutions already found in previous sections. For } \alpha > 0, \text{ we define } \\
J_{0,\alpha b_0}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_A u|^2 + u^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} \alpha b_0|u|^p dx, \\
M_{0,\alpha b_0} = \{u \in H^1_A(\mathbb{R}^N) \setminus \{0\}; \langle J_{0,\alpha b_0}(u), u \rangle = 0\}.

We now define the following subset of unitary ball \\
\mathcal{B} = \{u \in H^1_A(\mathbb{R}^N) \setminus \{0\}; \|u\|_A \geq 1\}.

Let us recall that for every } u \in H^1_A(\mathbb{R}^N) \setminus \{0\} \text{ there exists a unique } t^-(u) > 0 \text{ and } t_0(u) > 0 \text{ such that } t^-(u) \in M_{a_1,b_1}^- \text{ and } t_0(u) \in M_{0,b_0}. \text{ In order to apply the minimax argument of Bahri-Li we present the following result.}

Lemma 5.1. For each } u \in \mathcal{B} \text{ we will have } \\
(i) \text{ There is a single } t_0^u = t_0^u(u) > 0 \text{ such that } t_0^u u \in M_{0,\alpha b_0} \text{ and } \\
\sup_{t \geq 0} J_{0,\alpha b_0}(tu) = J_{0,\alpha b_0}(t_0^u u) = \frac{p-2}{2p} \left(\int_{\mathbb{R}^N} \alpha b_0|u|^p dx\right)^{\frac{2}{p-2}}.

(ii) \text{ For } \rho \in (0,1), \\
J_{\alpha,\rho b_0}(t^-(u)u) \geq \frac{(1 - \rho)^{\frac{2}{p-2}}}{(1 + \mu||b_2/b_1||_\infty)^{\frac{2}{p-2}}} J_{0,\rho b_0}(t_0(u)u) - \frac{2 - q}{2q} (\rho S_p)^{\frac{2}{q-2}} (\lambda ||a_+||_q^* + ||a_-||_q^*)^{\frac{2}{q-2}} \\
\text{and } \\
J_{\alpha,\rho b_0}(t^-(u)u) \leq \frac{1 + \rho}{2} J_{0,\rho b_0}(t_0(u)u) + \frac{2 - q}{2q} (\rho S_p)^{\frac{2}{q-2}} (\lambda ||a_+||_q^* + ||a_-||_q^*)^{\frac{2}{q-2}}.

Proof. (i) For each } u \in \mathcal{B}, \text{ consider } \\
K_u(t) = J_{0,\alpha b_0}(tu) = \frac{1}{2} t^2 - \frac{1}{2} \int_{\mathbb{R}^N} \alpha b_0|u|^p dx, \\
so } K_u(t) \to -\infty \text{ as } t \to \infty \text{ and } \\
K'_u(t) = t - t^{p-1} \int_{\mathbb{R}^N} \alpha b_0|u|^p dx.

Thus, } K'_u(t_0^u) = 0, \text{ and } t_0^u u \in M_{0,\alpha b_0}.$
Thus, \( t_0^\alpha = t_0^\alpha(u) = \left( \int_{\mathbb{R}^N} \alpha b_0 |u|^p \, dx \right)^{\frac{1}{p}} > 0 \)

Moreover, \( K''_u(t) = 1 - (p - 1)t^{p-2} \int_{\mathbb{R}^N} \alpha b_0 |u|^p \, dx \). So, in \( t_0^\alpha(u) \) we have

\[ K''_u(t_0^\alpha) = 2 - p < 0, \]

that is, \( t_0^\alpha \) is a maximum point of \( K_u \). Then, there exists a unique \( t_0^\alpha = t_0^\alpha(u) > 0 \) such that \( t_0^\alpha u \in M_{0, \alpha b_0} \) and also by definition \( K_u(t) = J(tu) \) we get

\[ \sup_{t \geq 0} J_{0, \alpha b_0}(tu) = J_{0, \alpha b_0}(t_0^\alpha u) = \frac{p - 2}{2p} \left( \int_{\mathbb{R}^N} \alpha b_0 |u|^p \, dx \right)^{\frac{2}{p}}. \]

\( (ii) \) Consider \( \alpha = (1 + \mu ||b_2/b_1||_\infty)/(1 - \rho) \). Then, for each \( u \in \mathcal{B} \) and \( \rho \in (0, 1) \), we have

\[
\int_{\mathbb{R}^N} a_\lambda |t_0^\alpha u|^q \, dx \leq \lambda S_p^{\frac{2q}{q - 2}} ||a_+||_{q'} ||t_0^\alpha u||_A^q
\]

\[
\leq \frac{2 - q}{2} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} \right)^{\frac{2q}{q - 2}} + \frac{q}{2} \left( \rho \right)^{\frac{2q}{q - 2}} ||t_0^\alpha u||_A
\]

\[
= \frac{2 - q}{2} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} \right)^{\frac{2q}{q - 2}} + \frac{q \rho}{2} ||t_0^\alpha u||_A^q. \tag{26}
\]

Then, for the part (i) and by (26),

\[
\sup_{t \geq 0} J_{a_\lambda, b_\mu}(tu) \geq J_{a_\lambda, b_\mu}(t_0^\alpha u)
\]

\[
\geq \frac{1 - \rho}{2} \left( ||t_0^\alpha u||_A^2 \right) - \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} \right)^{\frac{2q}{q - 2}}
\]

\[
- \frac{p}{1 + \mu ||b_2/b_1||_\infty} \int_{\mathbb{R}^N} b_0 |t_0^\alpha u|^p \, dx
\]

\[
= \left( 1 - \rho \right) J_{0, \alpha b_0}(t_0^\alpha u) - \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} \right)^{\frac{2q}{q - 2}}
\]

\[
= \left( 1 - \rho \right) \left( \frac{p - 2}{2p} \right) \left( 1 + \mu ||b_2/b_1||_\infty \right) \int_{\mathbb{R}^N} b_0 |u|^p \, dx^{\frac{2}{p - 2}} - \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} \right)^{\frac{2q}{q - 2}}
\]

\[
= \left( 1 - \rho \right) \left( \frac{p - 2}{2p} \right) \left( 1 + \mu ||b_2/b_1||_\infty \right) \int_{\mathbb{R}^N} b_0 |u|^p \, dx^{\frac{2}{p - 2}} J_{0, \alpha b_0}(t_0(u)u) - \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} \right)^{\frac{2q}{q - 2}}.
\]

Still, by Lemma 4.3 and by Theorem 2.5

\[
\sup_{t \geq 0} J_{a_\lambda, b_\mu}(tu) = J_{a_\lambda, b_\mu}(t^-(u)u).
\]

Thus,

\[
J_{a_\lambda, b_\mu}(t^-(u)u) \geq \frac{1 - \rho}{2} \left( \frac{p - 2}{2p} \right) \left( 1 + \mu ||b_2/b_1||_\infty \right) \int_{\mathbb{R}^N} b_0 |t_0(u)u|^p \, dx^{\frac{2}{p - 2}} J_{0, \alpha b_0}(t_0(u)u) - \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} \right)^{\frac{2q}{q - 2}}.
\]

Further, by Hölder, Sobolev and Young’s inequalities

\[
\left| \int_{\mathbb{R}^N} a_\lambda |tu|^q \, dx \right| \leq \int_{\mathbb{R}^N} a_\lambda |tu|^q \, dx \leq \left( \lambda ||a_+||_{q'} + ||a_-||_{q'} \right) S_p^{\frac{2q}{q - 2}} ||tu||_A^q
\]

\[
\leq \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} + ||a_-||_{q'} \right)^{\frac{2q}{q - 2}} + \frac{q \rho}{2} ||tu||_A^q.
\]

Also,

\[
J_{a_\lambda, b_\mu}(tu) \leq \frac{(1 + \rho)^{\frac{2q}{q - 2}}}{2} J_{0, b_0}(t_0(u)u) + \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} + ||a_-||_{q'} \right)^{\frac{2q}{q - 2}} - \frac{1}{p} \int_{\mathbb{R}^N} b_0 |tu|^p \, dx
\]

\[
\leq \frac{(1 + \rho)^{\frac{2q}{q - 2}}}{2} J_{0, b_0}(t_0(u)u) + \frac{2 - q}{2q} \left( \rho S_p \right)^{\frac{2q}{q - 2}} \left( \lambda ||a_+||_{q'} + ||a_-||_{q'} \right)^{\frac{2q}{q - 2}}.
\]
Then,
\[ J_{a, b_0}(t^-(u)u) \leq \frac{(1 + \rho)^{\frac{p}{p^-}}}{2} J_{0, b_0}(t_0(u)u) + \frac{2 - q}{2q} (\rho S_p)^{\frac{q}{q^-}} (\lambda ||a_+||_{q^*} + ||a_-||_{q^*})^{\frac{2}{q^*}}. \]

As we wanted to prove.

Note that as \( m_{a, b_0}^- > 0 \) for all \( \lambda \in (0, \lambda_0) \) and \( \mu \in (0, \mu_0) \), we define
\[ I_{a, b_0}(u) = \sup_{t \geq 0} J_{a, b_0}(tu) = J_{a, b_0}(t^-(u)u) > 0, \]

where \( t^-(u)u \in M_{a, b_0}^- \). We can see that if \( \lambda, \mu \) and \( ||a_-||_{q^*} \) are sufficiently small, we can use the minimax Bahri-Li’s argument [4] for our functional \( J_{a, b_0} \). Let
\[ \Gamma_{a, b_0} = \{ \gamma \in C(B(N(0, k), \mathbb{B}); \gamma|_{\partial B(N(0, k))} = w_k/||w_k||_A \} \]

be for values of \( l \) large enough.

We define
\[ n_{a, b_0} = \inf_{\gamma \in \Gamma_{a, b_0}} \sup_{x \in \mathbb{R}^N} I_{a, b_0}(\gamma(x)) \]

By Lemma [5.1 ii], for \( 0 < \rho < 1 \), we have
\[ n_{a, b_0} \geq \frac{(1 - \rho)^{\frac{p}{p^-}}}{(1 + \mu) ||b_2/b_1||_\infty^{p^-} m_{a, b_0}} - \frac{2 - q}{2q} (\rho S_p)^{\frac{q}{q^-}} (\lambda ||a_+||_{q^*} + ||a_-||_{q^*})^{\frac{2}{q^*}} \]
\[ n_{a, b_0} \leq (1 + \rho)^{\frac{p}{p^-}} m_{a, b_0} + \frac{2 - q}{2q} (\rho S_p)^{\frac{q}{q^-}} (\lambda ||a_+||_{q^*} + ||a_-||_{q^*})^{\frac{2}{q^*}}. \]

We will need estimates of energy levels as follows.

**Lemma 5.2.** \( m^\infty < n_{a, b_0} < 2m^\infty. \)

**Proof.** From the results of Bahri and Li [4] we have that the equation \( (E_{0, b_0}) \) admits at least one solution \( u_0 \) with \( J_{0, b_0}(u_0) = n_{a, b_0} < 2m^\infty. \) In addition, by the condition \( (C_1) \), the equation \( (E_{0, b_0}) \) does not have a minimum energy solution. Like this, \( m^\infty < n_{a, b_0} < 2m^\infty. \)

**Theorem 5.3.** Let \( \lambda_0 \) and \( \mu_0 \) be as in **Lemma 2.2** Then there will be positive values \( \lambda_0 \leq \lambda_0, \mu_0 \leq \mu_0 \) and \( \nu_0 \leq \nu_0 \) such that for \( \lambda \in (0, \lambda_0), \mu \in (0, \mu_0) \) and \( ||a_-||_{q^*} < \nu_0 \), we have
\[ m_{a, b_0}^+ + m^\infty < n_{a, b_0} < m_{a, b_0}^- + m^\infty. \]

In addition, \( (P_1) \) admits a solution \( v_{a, b_0} \) with
\[ J_{a, b_0}(v_{a, b_0}) = n_{a, b_0}. \]

**Proof.** By Lemma [5.1 ii], we have for \( 0 < \rho < 1 \)
\[ m_{a, b_0}^- \geq \frac{(1 - \rho)^{\frac{p}{p^-}}}{(1 + \mu) ||b_2/b_1||_\infty^{p^-}} m^\infty - \frac{2 - q}{2q} (\rho S_p)^{\frac{q}{q^-}} (\lambda ||a_+||_{q^*} + ||a_-||_{q^*})^{\frac{2}{q^*}} \]
and
\[ m_{a, b_0}^- \leq (1 + \rho)^{\frac{p}{p^-}} m^\infty + \frac{2 - q}{2q} (\rho S_p)^{\frac{q}{q^-}} (\lambda ||a_+||_{q^*} + ||a_-||_{q^*})^{\frac{2}{q^*}}. \]

For each \( \epsilon > 0 \) there are positive values \( \lambda_1 \leq \lambda_0, \mu_1 \leq \mu_0 \) and \( \nu_1 \) such that \( \lambda \in (0, \lambda_1), \mu \in (0, \mu_1) \) and \( ||a_-||_{q^*} < \nu_1 \), we have
\[ m^\infty - \epsilon < n_{a, b_0} < m^\infty + \epsilon. \]
Then,

\[ 2m^\infty - \epsilon < n_{a,\lambda, b, \mu} + m^\infty < 2m^\infty + \epsilon. \]

Using (27) and (28) for all \( \delta > 0 \), there will be positive values \( \tilde{\lambda}_2 \leq \lambda_0, \tilde{\mu}_2 \leq \mu_0 \) and \( \nu_2 \) such that for \( \lambda \in (0, \tilde{\lambda}_2), \mu \in (0, \tilde{\mu}_2) \), and \( ||a_-||_{q^*} < \nu_2 \), we have

\[ n_{\alpha, b, \mu} - \delta < n_{\alpha, \lambda, b, \mu} < n_{\alpha, b, \mu} + \delta. \]

Fixing small values of \( 0 < \epsilon < (2m^\infty - n_{\alpha, b, \mu})/2, \) and being \( m^\infty < n_{\alpha, b, \mu} < 2m^\infty \), and choosing \( \delta > 0 \) so that for \( \lambda < \lambda_0 = \min\{\lambda_1, \lambda_2\}, \mu < \mu_0 = \min\{\mu_1, \mu_2\} \) and \( ||a_-||_{q^*} < \nu_0 = \min\{\nu_1, \nu_2\} \), we will have

\[ m_{a, \lambda, b, \mu}^+ + 2m^\infty < n_{a, \lambda, b, \mu} < 2m^\infty - \epsilon < m_{a, \lambda, b, \mu}^-. \]

Thus, by Proposition 3.1(ii), we obtain that the problem \((P_1)\) has a solution \(v_{a, \lambda, b, \mu}\) with

\[ J_{a, \lambda, b, \mu}(v_{a, \lambda, b, \mu}) = n_{a, \lambda, b, \mu}. \]

Proof of Theorem 1.1 With the result of theorem 3.3 we can complete the proof of theorem 1.1. For \( \lambda \in (0, \tilde{\lambda}_0), \mu \in (0, \tilde{\mu}_0) \), and \( ||a_-||_{q^*} < \epsilon < n_{\alpha, b, \mu} \), also using the results presented in the introduction about the existence of the first three solutions and 3.3 we obtain that the equation \((P_1)\) admits at least four solutions.

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