Entanglement-Enhanced Classical Communication on a Noisy Quantum Channel

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Abstract

We consider the problem of trying to send a single classical bit through a noisy quantum channel when two transmissions through the channel are available as a resource. Classically, two transmissions add nothing to the receiver’s capability of inferring the bit. In the quantum world, however, one has the possible further advantage of entangling the two transmissions. We demonstrate that, for certain noisy channels, such entangled transmissions enhance the receiver’s capability of a correct inference.

1 Introduction

Much of the growing field of quantum information theory is founded upon the asking of a single question: for a given task, can it help to use an entangled pair of particles as a resource? In this contribution, we ask just this question again, but in the context of a simple classical communication problem. The task is to use a noisy quantum channel as effectively as possible for the communication of a single bit, 0 or 1, given that only two transmissions through the channel are allowed. Alice, the transmitter, may use any means allowed by the laws of quantum mechanics for encoding the bit in the two transmissions; Bob, the receiver, may use any means allowed by quantum mechanics for measuring the output of the channel in an attempt to infer the actual bit. The question is, can the use of an encoding that takes advantage of entanglement between the transmissions increase Bob’s capability of inference?

As an example, the channel in question could be a somewhat depolarizing fiber optic cable. The two transmissions would then be represented by two separate photons being sent through the cable. In this case, the bit is to be encoded somehow in the polarization degrees of freedom of the photons, each represented by a two-dimensional Hilbert space \( \mathcal{H}_2 \). For this, our question boils down to whether the optimal encoding for such a channel will be in terms of product states—say by some \( |\psi_0\rangle|\psi_0\rangle \) and \( |\psi_1\rangle|\psi_1\rangle \) for 0 and 1 respectively—or rather by some entangled states \( |\Psi_0\rangle \) and \( |\Psi_1\rangle \) on \( \mathcal{H}_2 \otimes \mathcal{H}_2 \).

\(^1\)Information theoretically, this means that Alice sends each message with equal probability. The methods described here apply, with slight modification, to the case of unequal probabilities as well. We focus on the case of a full bit because the significant quantum behavior reported herein can be illustrated without such complications.
Classically, any attempt to protect the bit from noise by redundancy is always stalemated when only two transmissions are available. This is because Bob cannot perform any sort of “majority vote” error correction on two transmissions. In the quantum version of the problem, as posed above, however, things become much more interesting. Extensive numerical work shows that there are indeed noise models for which entangled transmissions are more reliable for carrying the bit from sender to receiver than would be the case otherwise. In fact, such noise models seem to be the rule rather than the exception.

This paper is devoted to demonstrating the existence of this effect—i.e., entanglement-enhanced classical communication—for a particularly simple noisy quantum channel, the “two-Pauli channel.” In Section 2 below, we develop the formalism required to give the general problem a precise statement. In Section 3, we introduce the two-Pauli channel and analyze what can be done with it upon one transmission and, alternatively, upon two transmissions but with product-state inputs. In Section 4, we find the optimal (entangled) encoding for two transmissions through the channel. We close the paper in Section 5 with a brief discussion of another channel of interest. Also we pose the deeper information-theoretic question of whether entanglement can be used to increase the classical information carrying capacity of a noisy quantum channel.

2 Noisy Channel Preliminaries

The general description of the problem is the following. The action of a noisy channel on the physical system transmitted through it is described by a mapping \( \phi \) from input density operators to output density operators on that system. In all cases, the mapping is assumed to come about via an interaction between the system of interest and an independently prepared environment, to which neither Alice nor Bob has access. Thus a noisy channel is captured formally by a mapping of the form

\[
\rho \rightarrow \phi(\rho) = \text{tr}_E \left( U (\rho \otimes \sigma) U^\dagger \right),
\]

where \( \sigma \) is the initial state of the environment, \( U \) is the unitary interaction between the system and environment, and \( \text{tr}_E \) denotes a partial trace over the environment’s Hilbert space. For specificity, we assume \( \rho \) is a density operator on a \( d \)-dimensional Hilbert space \( \mathcal{H}_d \).

A convenient means of representing all possible noise models is given by the Kraus representation theorem \( [1, 2] \). This states that Eq. (1) can be written in the form

\[
\phi(\rho) = \sum_i A_i \rho A_i^\dagger,
\]

where the \( A_i \) satisfy the completeness relation

\[
\sum_i A_i^\dagger A_i = I.
\]

Conversely, any set of operators \( A_i \) satisfying Eq. (3) can be used in Eq. (2) to give rise to a valid noisy channel in the sense of Eq. (1).

In the language of the Kraus theorem, the situation of two transmissions through the channel is described by the mapping

\[
R \rightarrow \Phi(R) = \sum_{i,j} (A_i \otimes A_j) R (A_i \otimes A_j)^\dagger,
\]

where \( R \) denotes a density operator on the \( d^2 \)-dimensional Hilbert space \( \mathcal{H}_d \otimes \mathcal{H}_d \).
In our particular problem, Alice encodes the bit she wishes to transmit to Bob by preparing a quantum system in one of two states \( R_0 \) or \( R_1 \) on \( \mathcal{H}_d \otimes \mathcal{H}_d \). The action of the channel via Eq. (4) leads to one of two possible output density operators for Bob at the receiving end, say either \( \tilde{R}_0 \) or \( \tilde{R}_1 \) respectively.

For the problem of distinguishing the two density operators \( \tilde{R}_0 \) and \( \tilde{R}_1 \), we imagine Bob performing a general quantum mechanical measurement, or positive operator-valued measure (POVM) \( \{E_b\} \), and then using the acquired data to venture a guess about the identity of the density operator. Depending upon which bit \( s \) Alice has sent, the probability of Bob’s measurement outcomes will be given by \( \text{tr}(\tilde{R}_s E_b) \). Clearly the best strategy for Bob in identifying the bit—upon finding an outcome \( b \)—is to guess the value of \( s \) for which \( \text{tr}(\tilde{R}_s E_b) \) is the largest. Since each input is equally likely, this gives rise to an average probability of error given by

\[
P_e(\{E_b\}) = \frac{1}{2} \sum_b \min\{\text{tr}(\tilde{R}_0 E_b), \text{tr}(\tilde{R}_1 E_b)\}.
\]

This makes it clear that the best measurement on Bob’s part is to choose the one that minimizes this expression. It turns out that this measurement can be described by a standard von Neumann measurement of the Hermitian operator \( \Gamma = \tilde{R}_1 - \tilde{R}_0 \). Moreover, with this measurement, Eq. (5) reduces to

\[
P_e = \frac{1}{2} - \frac{1}{4} \text{tr}\left|\tilde{R}_1 - \tilde{R}_0\right|.
\]

Here \( \text{tr}|A| \), for any Hermitian operator \( A \), should be interpreted as the sum of the absolute value of \( A \)'s eigenvalues.

The question we ask in this paper can now be stated in a precise manner. For any two orthogonal pure state inputs \( |0\rangle \) and \( |1\rangle \) on \( \mathcal{H}_d \otimes \mathcal{H}_d \), what is the smallest possible value that

\[
P_e = \frac{1}{2} - \frac{1}{4} \text{tr}\left|\Phi(|1\rangle\langle 1|) - \Phi(|0\rangle\langle 0|)\right|
\]

\[
= \frac{1}{2} - \frac{1}{4} \text{tr}\left|\Phi(|1\rangle\langle 1|) - |0\rangle\langle 0|\right|
\]

can take? And, more importantly, is it ever the case that the smallest value can be achieved only by entangled states?\(^2\)

### 3 The Two-Pauli Channel

The **two-Pauli channel** is a noisy quantum channel on a single qubit, \( \mathcal{H}_2 \), described by three Kraussian \( A_i \) operators:

\[
A_1 = \sqrt{x} I, \quad A_2 = \sqrt{\frac{1}{2}(1-x)} \sigma_1, \quad A_3 = -i \sqrt{\frac{1}{2}(1-x)} \sigma_2,
\]

where \( I \) is the identity operator and \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are the standard Pauli matrices, i.e.,

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
This channel has a simple interpretation: with probability \( x \), it leaves the qubit alone; with probability \( 1 - x \) it randomly applies one of the two Pauli rotations to the qubit.

Note that, because there is no \( A_i \) in Eq. (8) corresponding to the Pauli matrix \( \sigma_3 \), the two-Pauli channel cannot be thought of as a simple depolarizing channel. Nor can it be thought of as a simple dephasing channel, where only one Pauli rotation \( \sigma_j \) acts on the qubit. A classical bit can be sent perfectly through a dephasing channel by choosing \(|0\rangle\) and \(|1\rangle\) to be eigenstates of \( \sigma_j \). Moreover, numerical work demonstrates that there is no benefit from using entangled transmissions for the depolarizing channel. It turns out that the asymmetry of the two-Pauli channel is just right for seeing the entanglement enhancement effect in a particularly clean way.

Let us gain some intuition about the two-Pauli channel by first considering only one transmission through it. Suppose we consider two possible (commuting) inputs, \( \rho_+ \) and \( \rho_- \), given in Bloch sphere representation by

\[
\rho_\pm = \frac{1}{2} \left( I \pm \vec{a} \cdot \vec{\sigma} \right).
\]

(10)

Here, \( \vec{a} = (a_1, a_2, a_3) \) is any real vector of length 1 or less, and \( \vec{\sigma} \) is the vector of Pauli matrices.

One can easily verify that the action of the channel on these two density operators is:

\[
\rho_\pm \rightarrow \phi(\rho_\pm) = \frac{1}{2} \left( I \pm \vec{b} \cdot \vec{\sigma} \right),
\]

(11)

where

\[
\vec{b} = \left( a_1 x, a_2 x, a_3 (2x - 1) \right).
\]

(12)

Note that one transmission through the channel takes commuting density operators to commuting density operators. We shall see that this feature is not necessarily true for two transmissions.

If we now specifically consider the channel for communication purposes, we should make the final states as distinguishable as possible. This means we should pick the vector \( \vec{a} \) so that \( \phi(\rho_+) \) and \( \phi(\rho_-) \) are as pure as possible. How to do this will depend upon the value of the parameter \( x \), since the eigenvalues of both \( \rho_+ \) and \( \rho_- \) are

\[
\frac{1}{2} \pm \frac{1}{2} \sqrt{(a_1^2 + a_2^2) x^2 + a_3^2 (2x - 1)^2}.
\]

(13)

If \( x \geq \frac{1}{3} \), then \(|x| \geq |2x - 1|\). Hence, clearly, the optimal inputs will be pure states with \( a_3 = 0 \). If, on the other hand, \( x \leq \frac{1}{3} \), then \(|x| \leq |2x - 1|\). The optimal inputs in this case will again be pure states but with \( a_1 = a_2 = 0 \).

The probability of error in guessing the identity of the states works out easily enough. It is just

\[
P_e = \begin{cases} 
  x & \text{if } x \leq \frac{1}{3} \\
  \frac{1}{2} - \frac{1}{2} x & \text{if } x \geq \frac{1}{3} 
\end{cases}.
\]

(14)

With this much of an introduction to the two-Pauli channel, let us now briefly consider two transmissions through the channel, but with those transmissions restricted to be product states. There is not much to be said here. If we assume two inputs of the form

\[
R_0 = \rho_0 \otimes \rho_0 \quad \text{and} \quad R_1 = \rho_1 \otimes \rho_1,
\]

(15)

then the error probability remains the same as above. This can be corroborated easily both from classical principles and quantum principles. Since \( \hat{R}_0 \) and \( \hat{R}_1 \) themselves commute when the inputs are orthogonal and pure, working out Eq. (11) in this case is hardly more difficult than for the single transmission case. The best possible error probability remains that listed in Eq. (14).
4 Entangled Transmissions

In this Section, we turn to the problem of sending entangled transmissions down the channel. A convenient basis with which to write the input states for two transmissions through the two-Pauli channel is the Bell operator basis, where the four orthonormal basis vectors are

\[ |\Phi\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle \pm |\downarrow\rangle|\downarrow\rangle) \quad \text{and} \quad |\Psi\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle \pm |\downarrow\rangle|\uparrow\rangle). \]  

(16)

We use the usual notation that

\[ |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  

(17)

With respect to this basis, an arbitrary set of two input states can be written as

\[ |0\rangle = a_0|\Phi^+\rangle + b_0|\Phi^-\rangle + c_0|\Psi^+\rangle + d_0|\Psi^-\rangle \]  

(18)

and

\[ |1\rangle = a_1|\Phi^+\rangle + b_1|\Phi^-\rangle + c_1|\Psi^+\rangle + d_1|\Psi^-\rangle. \]  

(19)

Making the effort to work through Eq. (14), we find that the output density operator \( \tilde{R}_0 = \Phi(|0\rangle\langle 0|) \)

expressed in the basis of Eq. (17) is

\[
\begin{bmatrix}
    ea_0^2 + fb_0^2 + gc_0^2 + gd_0^2 & ha_0b_0 & xa_0c_0 & ka_0d_0 \\
    ha_0b_0 & fa_0^2 + eb_0^2 + gc_0^2 + gd_0^2 & kb_0c_0 & xb_0d_0 \\
    xa_0c_0 & kb_0c_0 & ga_0^2 + gb_0^2 + ec_0^2 + fd_0^2 & hc_0d_0 \\
    ka_0d_0 & xb_0d_0 & hc_0d_0 & ga_0^2 + gb_0^2 + ec_0^2 + fd_0^2
\end{bmatrix}
\]

where

\[ e = \frac{1}{2}(1 - 2x + 3x^2) \]  

(21)

\[ f = \frac{1}{2}(1 - x)^2 \]  

(22)

\[ g = x(1 - x) \]  

(23)

\[ h = 2x - 1 \]  

(24)

\[ k = x(2x - 1) \]  

(25)

A similar expression holds for \( \Phi(|1\rangle\langle 1|) \) but with 1 exchanged for 0 everywhere.

Before tackling the problem at hand, it is worthwhile exploring a few features of this noise model. For instance, it would be convenient if it worked out that, as with the product state, whenever \( |0\rangle \) and \( |1\rangle \) are orthogonal, \( \Phi(|0\rangle\langle 0|) \) and \( \Phi(|1\rangle\langle 1|) \) were assured to commute. This, unfortunately, is not the case. A simple counterexample suffices to show this. Simply take

\[ |0\rangle = \begin{pmatrix} -0.459506 \\ -0.870791 \\ 0.127295 \\ 0.119889 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} -0.578111 \\ 0.163069 \\ -0.770549 \\ -0.213192 \end{pmatrix}. \]  

(26)
Then \( \langle 0|1 \rangle = 0 \), but
\[
\left[ \Phi(\langle 1|1 \rangle), \Phi(\langle 0|0 \rangle) \right] \neq 0 .
\] (27)

Interestingly enough, however, there are special cases where the commutativity of the two outputs is assured. For instance, take
\[
|0 \rangle = \cos \alpha |B_1 \rangle + \sin \alpha |B_2 \rangle
\] (28)
and
\[
|1 \rangle = -\sin \alpha |B_1 \rangle + \cos \alpha |B_2 \rangle,
\] (29)
where \(|B_1 \rangle\) and \(|B_2 \rangle\) are any two Bell states. Then it is easily checked that \( \Phi(\langle 0|0 \rangle) \) and \( \Phi(\langle 1|1 \rangle) \) do indeed commute in this case.

As an alternate example, let \(|0 \rangle\) be any non-Bell state vector in the plane spanned by \(|\Phi^+ \rangle\) and \(|\Psi^+ \rangle\), and let \(|1 \rangle\) be any non-Bell state vector in the plane spanned by \(|\Phi^- \rangle\) and \(|\Psi^- \rangle\). It turns out that the outputs of the two-Pauli channel due to these inputs never commute except in the case that the channel parameter \( x \) equals either 0, 1, or 1/3.

These examples give some hint that the two-Pauli channel is channel fairly rich in structure. So, with this, let us return to the question of the optimal input states for two transmissions. Numerical work demonstrates that for channel parameter values \( x \leq 1/3 \), the optimal inputs are product states of the form given by Eq. (15). However, for channel parameters \( 1/3 < x < 1 \), entangled inputs give the minimal error probabilities. Moreover, within the latter regime, though there appear to be many equivalent optimal entangled signals, two inputs can always be taken to be of the form
\[
|0 \rangle = \cos \alpha |\Phi^+ \rangle + \sin \alpha |\Psi^+ \rangle,
\] (30)and
\[
|1 \rangle = -\sin \alpha |\Phi^+ \rangle + \cos \alpha |\Psi^+ \rangle
\] (31)without any loss of performance.

The remainder of this Section is devoted to fleshing out the consequences of taking Eqs. (30) and (31) as an ansatz in our problem. The best probability of error in Bob’s inference of the signal, in accordance with Eq. (7), follows after some algebra:
\[
P_e(\alpha) = \frac{1}{2} - \frac{1}{2} \left( \frac{1}{4} \left( 1 - 4x + 5x^2 \right)^2 \cos^2 2\alpha + x^2 \sin^2 2\alpha \right)^{1/2} + \frac{1}{2} (1 - x) \left( 1 - 3x \cos 2\alpha \right) .
\] (32)

If we define
\[
F = \frac{1}{4} (1 - x)(1 - 5x)(1 - 2x + 5x^2) \quad \text{(33)}
\]
\[
G = \frac{1}{2} (1 - x)(1 - 3x) \quad \text{(34)}
\]
\[
Z = \cos 2\alpha, \quad \text{(35)}
\]
the error probability as a function of the ansatz can be written more compactly as
\[
P_e(Z) = \frac{1}{2} - \frac{1}{2} \left( \sqrt{FZ^2 + x^2 + |GZ|} \right) .
\] (36)

Our task now reduces to optimizing the ansatz in order to find the two inputs that lead to the most distinguishable outputs. This is done by extremizing Eq. (36):
This variational equation will have a solution for an optimal $Z$ as long as $x$ is such that $Z^2$ remains within the range set by Eq. (35), i.e., between 0 and 1. This occurs for

$$0 \leq \frac{G^2 x^2}{F(F - G)} \leq 1,$$

(38)

which, in turn, requires that

$$x \geq \frac{4}{15} - \frac{41}{30} \left(15\sqrt{330} - 73\right)^{-1/3} + \frac{1}{30} \left(15\sqrt{330} - 73\right)^{1/3} \approx 0.227539$$

(39)

Since we only need solutions for $x \geq 1/3$, this implies that our ansatz at least remains valid within the range of interest.

For a given $x$, the optimal $Z^2$ works out to be given by

$$Z^2 = \frac{(1 - 3x)^2}{4x(5x - 1)(1 - 2x + 5x^2)}.$$  

(40)

Hence the optimal version of Eq. (32) reduces to

$$P_e = \frac{1}{2} - 2\sqrt{x^5 \left(\frac{1}{(5x - 1)(1 - 2x + 5x^2)}\right)}.$$  

(41)

This demonstrates our point: as long as $1/3 < x < 1$, entangled transmissions through the two-Pauli channel are more effective at disabling noise than product state transmissions. To convey a feeling for the effectiveness of entangled transmissions, we tabulate a few representative points below.

| $x$  | $P_e$ (product states) | $P_e$ (entangled states) |
|------|------------------------|--------------------------|
| .50  | 0.250000               | 0.241801                 |
| .60  | 0.200000               | 0.188231                 |
| .70  | 0.150000               | 0.137817                 |
| .80  | 0.010000               | 0.090072                 |
| .90  | 0.050000               | 0.044319                 |
| .95  | 0.025000               | 0.022009                 |

5 Discussion

This paper has been largely devoted to developing a formal framework for tackling the question of entanglement-enhanced classical communication and demonstrating the existence of this effect for a particular noisy channel, the two-Pauli channel. However, computer simulations further corroborate that this example is not in any way isolated: it may be a property of most noisy channels. For instance, another example where entangled transmissions are effective is a “amplitude damping channel,” where the qubit arises from either one or no photons in a mode. The noise in this channel is due to the possibility of a photon leaking off to infinity. The Kraussian $A_i$ operators for this channel are

$$A_1 = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{1-x} & 0 \end{pmatrix}.$$  

(42)
and an analysis similar to the preceding one for the two-Pauli channel can be carried out in like manner.

Finally, let us emphasize that what we have shown here is that, by increasing our resources from one to two transmissions and allowing those transmissions to be entangled, we can make a classical bit more resilient to noise. This has no analog in classical information theory. A deeper question arises from comparing the increase in resources to the overall information transmittable by those resources. This is the question of whether the classical information capacity of a noisy quantum channel can be increased by entangling transmissions \[7\]. Let us close the paper by making this question precise.

If \( n \) possible inputs to a channel—used with prior probabilities \( \pi_1, \pi_2, \ldots, \pi_n \)—lead to \( n \) distinct density operators \( \rho_1, \rho_2, \ldots, \rho_n \) at the output (with like prior probabilities), then for a fixed POVM \( \{E_b\} \), the mutual information recoverable about the identity of the input is

\[
I \left( \{\pi_i\}, \{E_b\} \right) = -\sum_b \text{tr}(\rho E_b) \log \text{tr}(\rho E_b) + \sum_{i=1}^n \pi_i \sum_b \text{tr}(\rho_i E_b) \log \text{tr}(\rho_i E_b),
\]

where

\[
\rho = \sum_{i=1}^n \pi_i \rho_i.
\]

This is the Shannon information of the output symbols minus the average Shannon information of the output symbols conditioned on the input. The channel capacity for the given set of inputs is found by optimizing the prior probabilities of the inputs and optimizing the quantum measurement used at the output:

\[
C(\rho_1, \rho_2, \ldots, \rho_n) = \max_{\{\pi_i\}} \max_{\{E_b\}} I \left( \{\pi_i\}, \{E_b\} \right).
\]

This defines the ultimate information carrying capacity for a single transmission through the channel as a function of the particular input quantum states.

What we are in search of is a comparison of the best possible capacity (as a function of the inputs) for one transmission versus the same for two transmissions. That is to say, under completely general uses of the channel, what is the best possible channel capacity

\[
C \left( \Phi(|1\rangle\langle 1|), \Phi(|2\rangle\langle 2|), \ldots, \Phi(|n\rangle\langle n|) \right),
\]

where an optimization must also encompass the number of inputs \( n \)? Moreover, how does this number compare to a similarly defined capacity \( C_1 \) for a single transmission through the channel? If Eq. \[46\] turns out to be more than twice \( C_1 \), then we would have that classical information capacities are “super-additive” with respect to multiple uses of the same channel. The existence of this phenomena would be yet another surprising quantum effect indeed.

6 Appendix: Why Orthogonal Signal States?

Suppose we have any quantum channel whatsoever and that its action on density operators is given (in standard form) by a trace-preserving completely positive map

\[
\rho \rightarrow \mathcal{E}(\rho) = \sum_i B_i \rho B_i^\dagger,
\]

where

\[
\sum_i B_i^\dagger B_i = I.
\]
For the problem of finding the two inputs that lead to two maximally distinguishable outputs (as in these notes), how do we know that the optimal inputs should pure states rather than mixed? Given that the inputs are pure states, how do we know that the optimal ones must be orthogonal?

If we grant two standard facts from linear algebra \[6\], then we can answer both these questions quite readily. The first fact is that, for any operator \(A\),

\[
\max_U \Re \text{tr}(UA) = \text{tr}|A|
\]

(49)

where the maximum is taken over all unitary operators \(U\). The second fact is that, for any two \(n \times n\) Hermitian operators \(A\) and \(B\),

\[
\sum_{i=1}^{n} \lambda_{n-i+1}(A) \lambda_i(B) \leq \text{tr}(AB) \leq \sum_{i=1}^{n} \lambda_i(A) \lambda_i(B) ,
\]

(50)

where \(\lambda_i(X)\) denotes the \(i\)th eigenvalue of \(X\) when enumerated in nonincreasing order.

Now, in order to minimize the probability of error in carrying one bit across the given channel, we must find two states \(\rho_1\) and \(\rho_0\) such that

\[
\text{tr}(|\mathcal{E}(\rho_1) - \mathcal{E}(\rho_0)|)
\]

is maximized. Therefore, let us focus on this expression. Define the “conjugate” mapping \(\mathcal{E}^*\) to \(\mathcal{E}\) by the action

\[
X \rightarrow \mathcal{E}^*(X) = \sum_i B_i X B_i .
\]

(52)

Then

\[
\text{tr}(|\mathcal{E}(\rho_1) - \mathcal{E}(\rho_0)|) = \text{tr}(|\mathcal{E}(\rho_1 - \rho_0)|) = \max_U \Re \text{tr}\left(U \mathcal{E}(\rho_1 - \rho_0)\right)
\]

\[
= \max_U \frac{1}{2} \text{tr}\left((U + U^\dagger) \mathcal{E}(\rho_1 - \rho_0)\right)
\]

\[
= \max_U \frac{1}{2} \text{tr}\left((\rho_1 - \rho_0) \mathcal{E}^*(U + U^\dagger)\right)
\]

\[
= \max_U \frac{1}{2} \left[\text{tr}(\rho_1 \mathcal{E}^*(U + U^\dagger)) - \text{tr}(\rho_0 \mathcal{E}^*(U + U^\dagger))\right] .
\]

(53)

(For the second to last step in this, we used the cyclic property of the trace.) Note that, because \(U + U^\dagger\) is an Hermitian operator, the operator \(\mathcal{E}^*(U + U^\dagger)\) is also Hermitian.

Let us focus, for the moment, on any particular unitary operator \(U\) in the maximization procedure above. Using Eq. (50), we have

\[
\text{tr}(\rho_1 \mathcal{E}^*(U + U^\dagger)) \leq \sum_{i=1}^{n} \lambda_i(\rho_1) \lambda_i\left(\mathcal{E}^*(U + U^\dagger)\right) \leq \lambda_1\left(\mathcal{E}^*(U + U^\dagger)\right)
\]

(54)

and

\[
-\text{tr}(\rho_0 \mathcal{E}^*(U + U^\dagger)) \leq -\sum_{i=1}^{n} \lambda_i(\rho_0) \lambda_{n-i+1}\left(\mathcal{E}^*(U + U^\dagger)\right) \leq -\lambda_n\left(\mathcal{E}^*(U + U^\dagger)\right) .
\]

(55)

However, if \(\rho_1\) and \(\rho_0\) are chosen to be the eigenprojectors of \(\mathcal{E}^*(U + U^\dagger)\), then equality will be achieved throughout these equations. Thus for any particular \(U\),

\[
\max_{\rho_0,\rho_1} \text{tr}\left((\rho_1 - \rho_0) \mathcal{E}^*(U + U^\dagger)\right) = \lambda_1\left(\mathcal{E}^*(U + U^\dagger)\right) - \lambda_n\left(\mathcal{E}^*(U + U^\dagger)\right)
\]

(56)
and $\rho_1$ and $\rho_0$ must be orthogonal pure states to achieve this. Therefore, it follows that the input states optimal for leading to the maximal distinguishability of the associated outputs will be orthogonal pure states.

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