Graph Neural Networks with Low-rank Learnable Local Filters

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Abstract

For the classification of graph data consisting of features sampled on an irregular coarse mesh like landmark points on face and human body, graph neural network (gnn) models based on global graph Laplacians may lack expressiveness to capture local features on graph. The current paper introduces a new gnn layer type with learnable low-rank local graph filters, which significantly reduces the complexity of traditional locally connected gnn. The architecture provides a unified framework for both spectral and spatial convolutional gnn constructions. The new gnn layer is provably more expressive than gnn based on global graph Laplacians, and to improve model robustness, regularization by local graph Laplacians is introduced. The representation stability against input graph data perturbation is theoretically proved, making use of the graph filter locality and the local graph regularization. Experiments on spherical mesh data, real-world facial expression recognition/skeleton-based action recognition data, and data with simulated graph noise show the empirical advantage of the proposed model.

1 Introduction

Graph neural network (gnn) has drawn intensive research interest recently and many gnn models have been developed. The tasks fall into two categories, the graph-level classification and the node-level classification\textsuperscript{51}. The latter, typically in the semi-supervised learning setting to propagate node labels, finds wide application, e.g., social network data, of which popular methods include GCN\textsuperscript{26} and many developments. For graph-level classification, there are again different application scenarios. One problem is to classify $A \mapsto y$, where $A$ stands for varying graph topologies, namely to distinguish graph structures, and the input data may also include node and/or edge features. Another class of applications consider learning representation of graph signals $x : V \rightarrow \mathbb{R}$ on a fixed or mostly fixed underlying graph $G = (V, E)$, that is, to classify $\{A, x\} \mapsto y$. Examples include spherical mesh data\textsuperscript{10,11,15,21}, data on manifolds in computer graphics\textsuperscript{3,16,35,37}, and landmark data on human face and body\textsuperscript{20,23,25,30,38,44,49,50,53}, the last one being a primary motivating application of our work. The problem relates to convolutional neural network (CNN) on non-Euclidean domain\textsuperscript{4}, and a challenge lies in that mesh can be irregular and coarse, e.g., the body landmarks in action recognition. Existing works have applied gnn models to such tasks, yet graph convolutional network (gcn) that is based on non-learnable graph Laplacian may fall short of model expressiveness, and other methods like GAT\textsuperscript{48} are mostly heuristic and lacking theoretical analysis. A detailed review is provided in Sec. 1.1.

In this paper, we approach gnn models from the angle of low-rank filter decomposition, see Fig. 1 (left). The proposed model, called $L3$-gnn, has trainable local graph filters with much reduced model complexity from traditional locally connected gnn\textsuperscript{5,9}. The proposed gnn layer provides a unified framework for gnn constructions, including spectral gnn like Chebnet\textsuperscript{13} and CNN/geometrical CNN\textsuperscript{16,35} (with

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low-rank filter) as special cases. Because our model allows neighborhood specialized local graph filters, regularization may be needed to prevent over-fitting. We introduce a regularization scheme based on local graph Laplacians, motivated by the eigen property of the latter.

In terms of model expressiveness, adaptive local filters are provably more expressive than graph filters based on adjacency or global Laplacian matrices, see Fig. 1 (right). We prove the strictly greater expressiveness of L3-gnn than Chebnet on an explicit example of 1D ring graph, and the theoretical prediction is validated on binary classification experiments of 1D simulated data. Using perturbation analysis, we also prove a Lipschitz-type representation stability of the new gnn layer. The locality of the construction guarantees a CNN-like Lipschitz bound, which is further provably improved by the local graph Laplacian regularization. Experimentally, the proposed L3-gnn model shows improved performance on classification of data on down-sampled spherical meshes, facial expression recognition and skeleton-based action recognition compared to other gnn benchmarks.

An important issue is the handling of changing underlying graph topology and other graph noise. Our basic setting assumes that the graph $A$ is the same across data samples $x_i$, while in practice, the graph may change such as missing edge or permutation of nodes. In the motivating applications of our work, the underlying graph is relatively fixed and only has occasional change. E.g., in facial expression and skeleton-based action recognition, the graph consists of tens of landmark points on human face or body, and the topology is intrinsic. The occasional graph data noise includes occlusions which give inaccurately detected landmarks or missing landmark points. In experiments, we examine the proposed gnn model on real-world landmark datasets which contain such noise, as well as MNIST dataset with simulated graph noise. Our model shows robustness to noise in graph data, especially with the proposed graph regularization.

In summary, the contributions of the work are the following: (1) We introduce a new gnn layer type which is low-rank decomposed into local basis filters linearly combined by coefficients. The gnn layer type is plug-and-play, and we incorporate it into space-temporal deep models in applications. (2) Regularization by local graph Laplacians is introduced to improve the robustness of the gnn model against graph noise, which is both validated in experiments and supported by theory. (3) Theoretical proof of the strictly greater expressiveness and the Lipschitz-type input-perturbation stability of the new gnn model. (4) Applications to object recognition of spherical data and facial expression/skeleton-based action recognitions using landmarks. Model robustness against graph data noise is validated on both real-world and simulated datasets.

1.1 Related works

Graph convolutional network. A systematic review can be found at [51]. Spectral graph convolution can be expressed as $U_k g(\Lambda)U^T x$, where $x$ is input signal on graph, $UAU^T$ is the eigen decomposition of the graph Laplacian, and $g_k : \mathbb{R} \to \mathbb{R}$ gives the graph spectral filter. Using general $k_\theta$ involves costly computation of the full eigen decomposition [5]. Chebnet [13] let $k_\theta$ be a Chebyshev polynomial of relatively
low degree, avoiding the eigen computation and achieving local filtering. \cite{27} applied Cayley polynomials to capture narrow frequency bands, and \cite{29} accelerated the spectral computation by Lanczos algorithm. GCN \cite{26}, the mostly-used gnn, is a variant of Chebnet using degree-1 polynomial. Spatially formulated graph convolution is performed by summing up neighbor nodes’ transformed features in NN4G \cite{45}, by borrowing thoughts from graph diffusion process in DCNN \cite{1}, and from message passing in MPNN \cite{17}. The above gnn models use fixed graph convolution filters based on adjacency matrix to extract node features, which hampers model expressiveness. Expressive power of gcn has been theoretically studied in \cite{24,33,34,39,52}, which mostly focus on distinguishing graph structures rather than feature representation of input signals on graph. To introduce data-adaptive graph convolution, AGCN \cite{28} used feature-based graph topology but the graph filter is not learnable. Learnable filter has been achieved by attention mechanism in GAT \cite{48} and variants \cite{31,54}, however, the learned local filters take non-negative values, which may not extract certain rich features. Our method learns local filters with possibly negative values and then largely and provably enhances model expressiveness.

CNN and geometrical cnn. A convolution layer can be viewed as a linear operator where the output signal at each spatial location has local receptive field in the input signal, and the local filters are shared across locations via translation. Widely used are 1D grid for time series data like audio signal and 2D grid for images, where the translation is defined on the Euclidean domain. CNN has been extended to non-Euclidean domains in several settings. Mesh on sphere is an important example, and CNN’s which perform sphere convolution have been studied in S2CNN \cite{10}, SphereNet \cite{11}, SphericalCNN \cite{15}, and UGSCNN \cite{21}, which all rely on the spherical geometrical information. By projecting 3D point clouds to 2D sphere, these methods have also been used for 3D object recognition, while other deep methods for 3D clouds include 3D convolutional \cite{41} and non-convolutional architectures \cite{40,42}. CNN’s on manifolds construct weight-sharing across local atlas making use of a mesh, e.g., by patch operator in \cite{35}, anisotropic convolution in ACNN \cite{3}, mixture model parametrization in MoNet \cite{37}, spline functions in SplineCNN \cite{16}, and manifold parallel transport in \cite{46}. These models of geometric CNN’s use information of regular or irregular meshes on a non-Euclidean domain which usually need sufficiently fine resolution.

Applications with face/body landmark data. Many applications in computer vision, such as facial expression recognition (FER) and skeleton-based action recognition, need to extract high-level feature from landmarked data which are sampled at irregular grid points on human face or at body joints. While CNN methods \cite{14,19,36} prevail in FER task, landmark methods have the potential advantage in lighter model size as well as more robustness to pose variation. Earlier methods based on facial landmarks used hand-crafted features \cite{20,38} rather than deep networks. Skeleton-based methods in action recognition have been developed intensively recently \cite{44}, including non-deep methods \cite{49,50} and deep methods \cite{23,25,30,53}. Facial and skeleton landmarks only give a coarse and irregular grid, and then mesh-based geometrical CNN’s are hardly applicable, while previous gcn models on such tasks may lack sufficient expressive power.

2 Method

2.1 Gnn with decomposed local filters

Consider an undirected graph $G = (V,E), |V| = n$. A gnn layer maps from input node features $X(u',c')$ to output $Y(u,c)$, where $u,u' \in V, c,c' \in [C] (c \in [C])$ is the input (output) channel index, the notation $[m]$ means $\{1, \cdots, m\}$. The gnn layer mapping can be generally written as

$$Y(u,c) = \sigma\left( \sum_{u' \in V, c' \in [C]} M(u',u,c',c)X(u',c') + \text{bias}(c) \right), \quad u \in V, c \in [C],$$  (1)
which includes both the spatial and spectral constructions as different ways of specifying $M$, as will be shown in Section 2.3. The proposed basis gun layer is defined as

$$M(u', u; c', c) = \sum_{k=1}^{K} a_k(c', c) B_k(u', u), \quad a_k(c', c) \in \mathbb{R},$$

(2)

where $B_k(u', u)$ is non-zero only when $u' \in N_u^{(d_k)}$, the $d_k$-th order neighborhood of $u$, and $K$ is the rank of the decomposition. In other words, $B_k$’s are $K$ basis of local filters around each $u$, and the order $d_k$ can differ with $1 \leq k \leq K$. Both $a_k$ and $B_k$ are trainable, so the number of parameters are

$$K \cdot CC' + \sum_{k=1}^{K} \sum_{u \in V} |N_u^{(d_k)}| \sim K \cdot CC' + Knp, \quad n = |V|,$$

(3)

where $p$ stands for the average local patch size. In practice, we use $K$ up to 5, and $d_k$ up to 3.

The construction (2) specifies the proposed gun model in one layer, and in practice it can be used in various larger gun architectures like other popular gun layer types, e.g., Chebnet and GCN. Pooling of graphs can be added between gun layers, and the choice of $K$ and neighborhood orders $(d_1, \ldots, d_K)$ can be adjusted accordingly across layers. The model may be extended in several ways to be discussed in the last section.

2.2 Regularization by local graph Laplacian

The proposed L3-gun layer enlarges the model capacity by allowing $K$ basis local filters at each location, and without proper regularization the model may be sensitive to noise in input graph signals like missing values. A natural way to regularize the basis is by the graph geometry, and by construction, only the local graph patch is concerned. We introduce the following regularization penalty of the basis filters $B_k$’s as

$$\mathcal{R}(\{B_k\}) = \sum_{k=1}^{K} \sum_{u \in V} (b_u^{(k)})^T L_u^{(k)} b_u^{(k)}, \quad b_u^{(k)}(v) := B_k(v, u), b_u^{(k)} : N_u^{(d_k)} \rightarrow \mathbb{R},$$

(4)

where $L_u^{(k)}$ equaling $(D - A)$ restricted to the subgraph on $N_u^{(d_k)}$, is the Dirichlet local graph Laplacian on $N_u^{(d_k)}$, shown in Fig. 2. The training objective is set as

$$\mathcal{L}({\{a_k, B_k\}}) + \lambda \mathcal{R}(\{B_k\}), \quad \lambda \geq 0,$$

(5)

where $\mathcal{L}$ is the classification loss. The theory of the eigen-modes of $L_u^{(k)}$ may further suggest to impose local orthogonality constraints on $B_k$’s to prevent linear dependence across $k$ to develop in training. As the classification loss encourages the diversity of the filters $B_k$’s, in practice the $K$-rankness remains a tight low-rank constraint, and thus we only include the quadratic loss in (4).

2.3 A unified framework for gun constructions

Gun constructions basically fall into two categories, the spatial and spectral constructions [51]. The proposed L3-gun layer belongs to spatial construction, and here we show that our construction (2) is a unified framework for various convolutional gun, both spatial and spectral. Details and proofs are given in Appendix A. Starting from (1):

- Locally connected gun [5, 9]: The only requirement is that for each pair of $c$ and $c'$, $M(u', u; c', c)$ is nonzero only when $u'$ is locally connected in $u$. The number of parameters in each layer $\sim np \cdot CC'$, $p$ standing for average patch size of local neighborhoods.
Figure 2: Local graph Laplacian \( L_u := D - A \) on a neighborhood around node \( u \), where \( N_u^{(d)} \) denotes the neighborhood of order \( d \) (including \( d \)-neighbors of \( u \)). Regularization by \( L_u \) encourages less variation of the gnn filters on local patch. The first Dirichlet eigenvector does not change sign on \( N_u \) and is envelope-like.

- **Chebnet** [13]: \( M_{\text{per}} (c', c) \) equals a degree-(\( L \)-1) polynomial of the graph adjacency/Laplacian matrix, where the polynomial coefficients are trainable. The number of parameters \( \sim L \cdot CC' \). Gcn [26] can be viewed as Chebnet with polynomial degree-1 and tied coefficients, reducing the model complexity to \( CC' \).

### Proposition 1.

*Chebnet (GCN) is a special case of L3-gnn \((2)\) when \( K \geq L \) \((K \geq 2)\).*

- **CNN**: When nodes lie on a geometrical domain that allows translation \((u' - u)\), in \((2)\) setting \( B_k(u', u) = b_k(u' - u) \) for some \( b_k(\cdot) \) enforces spatial convolutional. The convolutional kernel is decomposed as \( \sum_k a_k(c', c) b_k(\cdot) \) [43]. Extension of CNN to manifolds [16, 35] allows \((2)\) to contain such geometrical CNN's as well. The model complexity is reduced to \( \sim K \cdot CC' + KP \).

### Proposition 2.

*Mesh-based geometrical CNN's defined by linear patch operators, including standard CNN on \( \mathbb{R}^d \), and with low-rank decomposed filters are special cases of L3-gnn \((2)\).*

Compare to the traditional locally connected gnn, the model complexity in L3-gnn is reduced from \( np \cdot CC' \) to be additive \((np + CC')\) times \( K \). When the number of channels \( C, C' \) are large, e.g. in deep layers they \( \sim 10^2 \), and the graph size is not large, the addition of \( np \) to \( CC' \) is marginal. In case that \( np \ll CC' \), e.g., in landmark data applications, the complexity is dominated by \( KCC' \) which is comparable Chebnet if \( K \approx L \). The computational cost is also comparable, as shown in experiments in Sec. 4. Furthermore, the following proposition shows that in the strong regularization limit of \( \lambda \to \infty \) in \((5)\), L3-gnn reduces to be Chebnet-like.

### Proposition 3.

*Suppose the subgraphs on \( N_u^{(d_k)} \) are all connected, given \( \alpha_{u,k} > 0 \) for all \( u, k \), the minimum of \((4)\) with constraint \( \|b_{u,k}\|_2 \geq \alpha_{u,k} \) is achieved when \( b_{u,k}^{(k)} \) equals the first Dirichlet eigenvector on \( N_u^{(d_k)} \), which does not change sign on \( N_u^{(d_k)} \).*

The constraint \( \|b_{u,k}^{(k)}\|_2 \geq \alpha_{u,k} \) is included because otherwise the minimizer will be \( B_k \) all zero. The first Dirichlet eigenvector is envelope-like (Fig. 2), and basis \( B_k(\cdot, u) \) will be an averaging operator on the local patch, similar to \( A_{d_k}^{(k)} \). Thus the regularization parameter \( \lambda \) can be viewed as trading-off between the more expressiveness in the learnable \( B_k \), which maybe non-averaging, and the more stability of the averaging local filters, similar to Chebnet and GCN. We further analyze the expressiveness and robustness of the proposed gnn model in next section.

## 3 Analysis

### 3.1 Representation expressiveness

#### 3.1.1 An example on ring graph

As proved in Proposition \(1\) the L3-gnn model family contains that of Chebnet as long as \( K \geq \) the degree of Chebshev polynomials. Here we give an explicit example of graph local filter that cannot be expressed
by Chebnet for any polynomial degree, but can be expressed by L3-gnn with small $K$.

Consider the ring graph with $n$ nodes, each node has 2 neighbors, $n=8$ as shown in Fig. 1 (right). We index the nodes as $u=0,\ldots,n-1$ and allows addition/subtraction of $u-v$ (mod $n$). The graph has the property that for any node $u$, let $\pi_u$ be a permutation of the $n$ nodes such that $\pi_u(u+v) = \pi_u(u-v)$, i.e., mirror flip the ring around the axis through $u$, then the graph topology is preserved. In terms of matrix, it means that for $\tilde{L} = A$, or $A_{L3-gnn} = D^{-1/2}AD^{-1/2}$ ($D$ is constant on diagonal), or the graph Laplacian as well as any linear combination with identity matrix, all satisfies $\pi_u\tilde{L}\pi_u^T = \tilde{L}$, where we denote $\pi_u$ as the $n$-by-$n$ permutation matrix. Because $\pi_u^T\pi_u = I_n$, $\pi_u\tilde{L}^k\pi_u^T = \tilde{L}^k$ for any degree $k \geq 1$, that is, the graph filter in Chebnet preserves the symmetry that $\tilde{L}^k(u+v) = \tilde{L}^k(u-v)$ for any shift $v$ and any location $u$. This means that any desired filter which breaks the mirror symmetry cannot be expressed by Chebnet for any degree, e.g., the “difference” filter $B(u,u') = 1$ when $u' = u$ and $-1$ when $u' = u+1$, (Fig. 1 right). In contrast, setting this $B$ as the basis in (2) expresses the filter in L3-gnn with $K = 1$.

### 3.1.2 Experimental evidence and on general graphs

The argument on ring graph extends to other graphs. Generally, a Chebnet-like gnn applies linear combinations of local averaging filters in different orders, and then certain graph filters may be difficult to express by this family. Because polynomials uniformly approximate integrable functions on an interval, the difficulty remains for other gnn as long as the graph filter is based on graph adjacency/Laplacian matrix.

To verify the theory, we conduct experiments on ring and chain graph with a simulated dataset of up/down wind, as shown in Fig. 3. On a ring graph, an upwind signal after mirror flipping around any node becomes a downwind signal. As a result of this symmetry, a Chebnet layer cannot enhance feature discriminativeness to differentiate the two classes, while a “difference” filter like $B$ in Chebnet on a ring graph will not work. As a result of this symmetry, a Chebnet layer cannot enhance feature discriminativeness to differentiate the two classes, while a “difference” filter like $B$ in Chebnet on a ring graph will not work. Using a two-layer gnn, the results in Fig. 3(table) support the theoretical predictions. In the last row, we further impose shared basis across nodes which reduces L3-gnn to a 1D convolutional layer, and the learned basis shows a “difference” shape that works for both L3-gnn and Chebnet. Results are similar using a 1-layer gnn (Tab. A.1). Experimental details are provided in Appendix B.

### 3.2 Representation stability

We consider the change in the gnn layer output $Y$ defined in (1)(2) when the input $X$ changes. For simplicity, let $C = C' = 1$, and the argument extends. For any graph signal $x: V \rightarrow \mathbb{R}$ and $V' \subset V$, define $\|x\|_{2,V'} := (\sum_{u \in V'} x(u)^2)^{1/2}$ and $\langle x, y \rangle_{V'} = \sum_{u \in V'} x(u)y(u)$. The following perturbation bound holds for the L3-gnn layer with/without regularization. All proofs in Appendix A.

| Model     | order | #params  | ring graph Acc | chain graph Acc |
|-----------|-------|----------|----------------|-----------------|
| Chebnet   | L=5   | 6.5k     | 51.73 ± 0.24   | 51.05 ± 0.33    |
|           | L=9   | 19.0k    | 51.75 ± 0.40   | 50.76 ± 0.40    |
|           | L=30  | 62.7k    | 51.32 ± 0.38   | 51.01 ± 0.41    |
| L3-gnn    | 0/1/2 | 2.7k     | 99.82 ± 0.05   | 99.99 ± 0.09    |
|           | 1*    | 7.4k     | 99.93 ± 0.03   | 99.85 ± 0.04    |
|           |       | 2.3k     | 99.96 ± 0.01   | 99.94 ± 0.01    |

Figure 3: Up/down-wind classification. (Table) Test accuracy by Chebnet up to $L=30$ and L3-gnn $K=1$ and 3. Last row order 1 with star: L3-gnn with shared basis $B(\cdot,u)$ across all locations $u$. (Plots) Left: example data from two classes. Right: learned shared basis on the graph neighborhood of 3, corresponding to the last row in the table.

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Theorem 1. Suppose that $X = \{X(u)\}_{u \in V}$ is perturbed to be $\hat{X} = X + \Delta X$, the activation function $\sigma: \mathbb{R} \to \mathbb{R}$ is non-expansive, and $\sup_{u \in V} \sum_{k=1}^{K} |N_u^{(d_k)}| \leq Kp$, then the change in the output $\{Y(u)\}_{u \in V}$ in 2-norm is bounded by

$$\|\Delta Y\|_{2,V} \leq \beta^{(1)} \cdot \|a\|_2 \sqrt{Kp} \|\Delta X\|_{2,V}, \quad \beta^{(1)} := \sup_{k,u} \|B_k(\cdot, u)\|_{2,N_u^{(d_k)}}.$$ 

Note that $p$ indicates the averaged size of the $d_k$-order local neighborhoods. The proposition implies that when $K$ is $O(1)$, and the local basis $B_k$'s have $O(1)$ 2-norms on all local parches uniformly bounded by $\beta^{(1)}$, then the Lipschitz constant of the gnn layer mapping is $O(1)$, i.e., the product of $\|a\|_2$, $\beta^{(1)}$ and $\sqrt{Kp}$, which does not scale with $n$. This resembles the generalizes the 2-norm of a convolutional operator which only involves the norm of the convolutional kernel, which is possible due to the local receptive fields in the spatial construction of L3-gnn.

The local graph regularization introduced in Section 2.2 improves the stability of $Y$ w.r.t. $\Delta X$ by suppressing the response of the gnn filter towards local high-frequency perturbations in $\Delta X$. Specifically, the local Dirichlet graph Laplacian $L_u^{(k)}$ on the subgraph on $N_u^{(d_k)}$ is positive definite whenever the subgraph is connected and not isolated from the whole graph. We then define the weighted 2-norm on local patch $\|x\|_{L_u^{(k)}} := \langle x, L_u^{(k)} x \rangle_{N_u^{(d_k)}}$, and similarly $\|x\|_{(L_u^{(k)})^{-1}}$.

Theorem 2. Notation and setting as in Theorem 1 if furtherly, all the subgraphs on $N_u^{(d_k)}$ are connected within itself and to the rest of the graph, and there is $\rho \geq 0$ s.t.

$$\|\Delta X\|_{(L_u^{(k)})^{-1}} \leq \rho \|\Delta X\|_{2,N_u^{(d_k)}}, \quad \forall u, k,$$

then

$$\|\Delta Y\|_{2,V} \leq \rho \beta^{(2)} \cdot \|a\|_2 \sqrt{Kp} \|\Delta X\|_{2,V}, \quad \beta^{(2)} := \sup_{k,u} \|B_k(\cdot, u)\|_{L_u^{(k)}},$$

Comparing to Theorem 1, the bound improves if $\rho \beta^{(2)} < \beta^{(1)}$. The regularization penalty $R = \sum_{u,k} \|B_k(\cdot, u)\|_{L_u^{(k)}}^2$ leads to smaller $\beta^{(2)}$. On each $N_u^{(d_k)}$, the Dirichlet eigenvalues increases $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_{p_u,k}$, $p_u,k := |N_u^{(d_k)}|$, and thus the weight by $\lambda_i^{-1}$ in $\|x\|_{(L_u^{(k)})^{-1}}$ decreases the contribution from high-frequency modes, i.e. high-indexed eigenvectors of $L_u^{(k)}$. As a result, $\rho$ will be small if $\Delta X$ contains a significant high-frequency component on the local patch, e.g., i.i.d Gaussian noise on each node, or missing values. Note that in the weighted 2-norm of $\Delta X$ by $(L_u^{(k)})^{-1}$, only the relative amount of high-frequency component v.s. low-frequency component in $\Delta X$ matters, because any constant normalization of $L_u^{(k)}$ cancels in the product of $\rho$ and $\beta^{(2)}$. We experimentally show that the local graph regularization improves the stability and performance of the L3-gnn model, especially when graph data contain noise and corruptions.

4 Experiment

We test the proposed L3-gnn model on several datasets.

4.1 Object recognition of data on spherical mesh

We first examine the proposed L3-gnn method on the task of classifying data on a spherical mesh, including sphere MNIST and sphere ModelNet-40 (projecting 3D object data to a 2D sphere) \cite{21}. Both experiments follow the setting in literature. Though regular mesh on sphere is not the primary application scenario that

\footnote{Code available at: https://github.com/ZichenMiao/GNN-with-Low-rank-Learnable-Local-Filters}
motivates our gnn model, we include the experiments to compare with benchmarks and test the efficiency of L3-gnn model when the graph is a regular mesh.

Following [21], we implement different mesh resolution on a sphere, indicated by “mesh level” (Fig. 4), where number of nodes in different levels can vary from 2562 (level 4) to 12 (level 0). All the networks consist of three convolutional layers, and more details of mesh, dataset and network architecture are in Appendix C.1. Using the original mesh level (4;3;2) as in [21] which has finest resolution, L3-gnn gives among the best accuracies for sphere MNIST. On Modelnet-40 which follows similar setup, L3-gnn outperforms Chebnet and GCN and achieves a testing accuracy of 90.24, comparable to UGSCNN [21] which uses spherical mesh information (Tab. A.2). When the mesh becomes coarser, as shown in Fig. 4 (Table), L3-gnn improves over GCN and Chebnet ($L = 4$) and is comparable with UGSCNN under nearly all mesh settings. We observe that in some settings Chebnet can benefit from larger $L$, but the overall accuracy is still inferior to L3-gnn. The most right two columns give two cases of coarse meshes where L3-gnn shows the most significant advantage.

4.2 Facial expression recognition (FER)

We test on two FER datasets, Extended CohnKanade (CK+) [32] and FER13 [18]. 15 landmarks are selected from the standard 68 facial landmarks defined in AAM [12], and edges are connected according to prior information of human face, e.g., nearby landmarks on the eye are connected, see Fig. 5. Image pixel values on a patch around each landmark is used as the input node feature. Details about dataset and model setup are in Appendix C.2. Unlike spherical mesh, facial and body landmarks (next section) are coarse irregular grids where no clear pre-defined mesh operation is applicable. We benchmark L3-gnn with other gnn approaches including GAT [48], results in Fig. 5 (Table). The local graph regularization

Figure 5: (Table) Results on CK+ and FER13. The mean validation time on CK+: Chebnet ($L = 4$) 12.56ms, L3-gnn (order 1,1,2,3) 13.02ms. (Plots) Top: example image from CK+, showing facial landmarks and graph. Middle: occlusion case in FER13. Bottom: example frame from NTU, showing skeleton landmarks and graph.
strategy is applied on FER13, due to the severe outlier data of landmark detection caused by occlusion. On CK+, L3-gnn leads all non-CNN models by a large margin, and the best model (1,1,2,3) uses comparable number of parameters with the best Chebnet $L=4$. On FER13, gasis-gnn has lower performance than Chebnet but outperforms after adding regularization. The running times of best Chebnet and L3-gnn models are comparable.

### 4.3 Action recognition

We test on two datasets of skeleton-based action recognition, NTU-RGB+D [47] and Kinetics-Motion [22]. The irregular mesh is the 18/25-point body landmarks, with graph edges defined by body joints, shown in Fig. 5 and also see Fig. A2. We adopt ST-GCN [53], a ten-layer spatio-temporal model as the base architecture, and substitute the GCN layer with the new L3-gnn layer, call it ST-L3-gnn model. On Kinetics-Motion, we adopt the regularization mechanism to overcome the severe data missing caused by camera out-of-view. More experimental details in Appendix C.3. We benchmark performance with ST-GCN [53], ST-GCN (our implementation without using geometric information) and ST-Chebnet (replacing Chebnet but outperforms after adding regularization. The running times of best Chebnet and L3-gnn models are comparable.

#### Table 1: Results on NTU-RGB+D and Kinetics-Motion

| Model      | Bases order | #params (w/o FC) | x-view Acc | x-sub Acc | #params (w/o FC) | Acc   |
|------------|-------------|------------------|------------|-----------|------------------|-------|
| ST-GCN [53]| 1           | 2.6M             | 82.59      | 74.33     | 1.4M             | 72.85 |
| ST-ChebNet | $L=3$       | 3.1M             | 86.40      | 78.24     | 1.8M             | 77.91 |
| ST-L3-gnn  | 1;1;2       | 3.1M             | 90.78      | 83.64     | 1.8M             | 75.20 |
|            | +reg0.01    |                  | 88.38      | 81.54     |                  | 78.49 |
|            | 1;1;2/3     | 3.3M             | 91.52      | 82.46     | 2.1M             | 75.07 |

#### Table 2: Results on MNSIT with grid size $7 \times 7$ with different levels of Gaussian noise and Permutation noise

| Model   | bases order | #params (w/o FC) | Acc(original) | Acc (gaussian) (psnr 24.9) | Acc (gaussian) (psnr 19.1) | Acc (gaussian) (psnr 15.7) | Acc (permutation) |
|---------|-------------|------------------|---------------|-----------------------------|----------------------------|-----------------------------|-------------------|
| GCN     | 1           | 2.4k             | 90.02 ± 0.24  | 89.27 ± 0.09                | 85.70 ± 0.15               | 81.32 ± 0.18               | 83.09 ± 0.18      |
| ChebNet | L=3         | 6.5k             | 92.85 ± 0.09  | 91.13 ± 0.15                | 87.64 ± 0.23               | 82.70 ± 0.33               | 86.94 ± 0.06      |
|         | L=4         | 8.6k             | 93.12 ± 0.1   | 91.61 ± 0.06                | 87.78 ± 0.12               | 82.77 ± 0.11               | 87.01 ± 0.13      |
|         | L=5         | 10.7k            | 93.2 ± 0.07   | 91.92 ± 0.11                | 88.22 ± 0.10               | 83.04 ± 0.12               | 87.27 ± 0.23      |
|         | L=6         | 12.7k            | 93.42 ± 0.09  | 91.70 ± 0.06                | 87.94 ± 0.10               | 83.50 ± 0.10               | 87.42 ± 0.24      |
|         | L=7         | 14.8k            | 93.45 ± 0.06  | 91.80 ± 0.10                | 87.84 ± 0.15               | 83.75 ± 0.14               | 87.53 ± 0.19      |
| L3-gnn  | 1;1;2       | 8.1k             | 93.45 ± 0.10  | 92.10 ± 0.08                | 88.20 ± 0.13               | 83.00 ± 0.34               | 87.08 ± 0.19      |
|         | +reg0.5     | 8.4k             | 93.85 ± 0.13  | 92.31 ± 0.07                | **89.23 ± 0.10**           | 84.59 ± 0.23               | 88.08 ± 0.18      |
|         | 1;1;2/3     | 12.2k            | 93.67 ± 0.15  | 92.25 ± 0.15                | 88.28 ± 0.16               | 82.80 ± 0.37               | 87.66 ± 0.12      |
|         | +reg0.5     |                | 93.85 ± 0.15  | **92.56 ± 0.12**            | 89.15 ± 0.24               | **84.61 ± 0.25**           | **88.21 ± 0.15**   |

#### 4.4 Robustness to graph noise

To examine the robustness to graph noise, we experiment on down-sampled MNIST data on 2D regular grid with 4 nearest neighbors, using 10,000 training samples. With no noise, 28×28 data (Tab. A5, 14×14 data (Tab. A4, and 7×7 data (Tab. 2 “original” column), the performance of L3-gnn is comparable to Chebnet and better than GCN. We consider three types of noise, Gaussian noise added to the pixel value, missing nodes or equivalently missing value in image input, and permutation of the node indices, details in Appendix C.4. The results of adding different levels of gaussian noise and permutation noise are shown in Tab. 2 while results of adding missing value noise is provided in Appendix C.4. The results show that our regularization scheme improves the robustness to all three types of graph noise, supporting the theory
Specifically, L3-gnn without regularization may underperform than Chebnet, but catches up after adding the regularization, which is consistent with Proposition 3.

5 Conclusion and Discussion

The paper proposes a new gnn layer type using learnable local filters decomposed over a small number of basis. Strengths: Provable enhancement of model expressiveness with significantly reduced model complexity from locally connected gnn. Improved stability and robustness via local graph regularization, supported by theory. Plug-and-play layer type, suitable for gnn graph signal classification problems on relatively un-changing small underlying graphs, like face/body landmark data in FER and action recognition applications.

Limitations and extensions: (1) Scalability to larger graph. When $|V| = n$ is large, the complexity increase in the $npK$ term would be significant. The issue in practice can be remedied by mixing use of layer types, e.g., only adopting L3-gnn layers in upper levels of mesh which are of reduced size. (2) Changing underlying graph $A_i$ across sample $i$. The current solution is assuming fixed $A$ plus regularization, while for more severe change of $A_i$, other solutions include node registration or other pre-processing techniques, possibly by another neural network. (3) Incorporation of edge features. Edge features can be transformed into extra channels of node features by an additional layer in the bottom, and the low-rank graph operation can be similarly employed there.

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References

[1] James Atwood and Don Towsley. Diffusion-convolutional neural networks. In Advances in neural information processing systems, pages 1993–2001, 2016.

[2] John R Baumgardner and Paul O Frederickson. Icosahedral discretization of the two-sphere. SIAM Journal on Numerical Analysis, 22(6):1107–1115, 1985.

[3] Davide Boscaini, Jonathan Masci, Emanuele Rodol`a, and Michael Bronstein. Learning shape correspondence with anisotropic convolutional neural networks. In Advances in neural information processing systems, pages 3189–3197, 2016.

[4] Michael M Bronstein, Joan Bruna, Yann LeCun, Arthur Szlam, and Pierre Vandergheynst. Geometric deep learning: going beyond euclidean data. IEEE Signal Processing Magazine, 34(4):18–42, 2017.

[5] Joan Bruna, Wojciech Zaremba, Arthur Szlam, and Yann LeCun. Spectral networks and locally connected networks on graphs. arXiv preprint arXiv:1312.6203, 2013.

[6] Adrian Bulat and Georgios Tzimiropoulos. How far are we from solving the 2d & 3d face alignment problem? (and a dataset of 230,000 3d facial landmarks). In International Conference on Computer Vision, 2017.

[7] Zhe Cao, Tomas Simon, Shih-En Wei, and Yaser Sheikh. Realtime multi-person 2d pose estimation using part affinity fields. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 7291–7299, 2017.
Fan RK Chung and Fan Chung Graham. *Spectral graph theory*. Number 92. American Mathematical Soc., 1997.

Adam Coates and Andrew Y Ng. Selecting receptive fields in deep networks. In *Advances in neural information processing systems*, pages 2528–2536, 2011.

Taco S Cohen, Mario Geiger, Jonas Köhler, and Max Welling. Spherical cnns. *arXiv preprint arXiv:1801.10130*, 2018.

Benjamin Coors, Alexandru Paul Condurache, and Andreas Geiger. Spherenet: Learning spherical representations for detection and classification in omnidirectional images. In *Proceedings of the European Conference on Computer Vision (ECCV)*, pages 518–533, 2018.

Timothy F. Cootes, Gareth J. Edwards, and Christopher J. Taylor. Active appearance models. *IEEE Transactions on pattern analysis and machine intelligence*, 23(6):681–685, 2001.

Michaël Defferrard, Xavier Bresson, and Pierre Vandergheynst. Convolutional neural networks on graphs with fast localized spectral filtering. In *Advances in neural information processing systems*, pages 3844–3852, 2016.

Hui Ding, Shaohua Kevin Zhou, and Rama Chellappa. Facenet2expnet: Regularizing a deep face recognition net for expression recognition. In *2017 12th IEEE International Conference on Automatic Face & Gesture Recognition (FG 2017)*, pages 118–126. IEEE, 2017.

Carlos Esteves, Christine Allen-Blanchette, Ameesh Makadia, and Kostas Daniilidis. Learning so (3) equivariant representations with spherical cnns. In *Proceedings of the European Conference on Computer Vision (ECCV)*, pages 52–68, 2018.

Matthias Fey, Jan Eric Lenssen, Frank Weichert, and Heinrich Müller. Splinecnn: Fast geometric deep learning with continuous b-spline kernels. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 869–877, 2018.

Justin Gilmer, Samuel S Schoenholz, Patrick F Riley, Oriol Vinyals, and George E Dahl. Neural message passing for quantum chemistry. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 1263–1272. JMLR. org, 2017.

Ian J Goodfellow, Dumitru Erhan, Pierre Luc Carrier, Aaron Courville, Mehdi Mirza, Ben Hamner, Will Cukierski, Yichuan Tang, David Thaler, Dong-Hyun Lee, et al. Challenges in representation learning: A report on three machine learning contests. In *International Conference on Neural Information Processing*, pages 117–124. Springer, 2013.

Yanan Guo, Dapeng Tao, Jun Yu, Hao Xiong, Yaotang Li, and Dacheng Tao. Deep neural networks with relativity learning for facial expression recognition. In *2016 IEEE International Conference on Multimedia & Expo Workshops (ICMEW)*, pages 1–6. IEEE, 2016.

Mira Jeong and Byoung Chul Ko. Drivers facial expression recognition in real-time for safe driving. *Sensors*, 18(12):4270, 2018.

Chiyu Jiang, Jingwei Huang, Kartthik Kashinath, Philip Marcus, Matthias Niessner, et al. Spherical cnns on unstructured grids. *arXiv preprint arXiv:1901.02039*, 2019.

Will Kay, Joao Carreira, Karen Simonyan, Brian Zhang, Chloe Hillier, Sudheendra Vijayanarasimhan, Fabio Viola, Tim Green, Trevor Back, Paul Natsev, et al. The kinetics human action video dataset. *arXiv preprint arXiv:1705.06950*, 2017.
[23] Qiuhong Ke, Mohammed Bennamoun, Senjian An, Ferdous Sohel, and Farid Boussaid. A new representation of skeleton sequences for 3d action recognition. In Proceedings of the IEEE conference on computer vision and pattern recognition, pages 3288–3297, 2017.

[24] Nicolas Keriven and Gabriel Peyré. Universal invariant and equivariant graph neural networks. In Advances in Neural Information Processing Systems, pages 7090–7099, 2019.

[25] Tae Soo Kim and Austin Reiter. Interpretable 3d human action analysis with temporal convolutional networks. In 2017 IEEE conference on computer vision and pattern recognition workshops (CVPRW), pages 1623–1631. IEEE, 2017.

[26] Thomas N Kipf and Max Welling. Semi-supervised classification with graph convolutional networks. arXiv preprint arXiv:1609.02907, 2016.

[27] Ron Levie, Federico Monti, Xavier Bresson, and Michael M Bronstein. Cayleynets: Graph convolutional neural networks with complex rational spectral filters. IEEE Transactions on Signal Processing, 67(1):97–109, 2018.

[28] Ruoyu Li, Sheng Wang, Feiyun Zhu, and Junzhou Huang. Adaptive graph convolutional neural networks. In Thirty-second AAAI conference on artificial intelligence, 2018.

[29] Renjie Liao, Zhizhen Zhao, Raquel Urtasun, and Richard Zemel. Lanczosnet: Multi-scale deep graph convolutional networks. ICLR, 2019.

[30] Jun Liu, Amir Shahroudy, Dong Xu, and Gang Wang. Spatio-temporal lstm with trust gates for 3d human action recognition. In European conference on computer vision, pages 816–833. Springer, 2016.

[31] Ziqi Liu, Chaochao Chen, Longfei Li, Jun Zhou, Xiaolong Li, Le Song, and Yuan Qi. Geniepath: Graph neural networks with adaptive receptive paths. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 33, pages 4424–4431, 2019.

[32] Patrick Lucey, Jeffrey F Cohn, Takeo Kanade, Jason Saragih, Zara Ambadar, and Iain Matthews. The extended cohn-kanade dataset (ck+): A complete dataset for action unit and emotion-specified expression. In 2010 ieee computer society conference on computer vision and pattern recognition-workshops, pages 94–101. IEEE, 2010.

[33] Haggai Maron, Heli Ben-Hamu, Hadar Serviansky, and Yaron Lipman. Provably powerful graph networks. In Advances in Neural Information Processing Systems, pages 2153–2164, 2019.

[34] Haggai Maron, Heli Ben-Hamu, Nadav Shamir, and Yaron Lipman. Invariant and equivariant graph networks. 2019.

[35] Jonathan Masci, Davide Boscaini, Michael Bronstein, and Pierre Vandergheynst. Geodesic convolutional neural networks on riemannian manifolds. In Proceedings of the IEEE international conference on computer vision workshops, pages 37–45, 2015.

[36] Zibo Meng, Ping Liu, Jie Cai, Shizhong Han, and Yan Tong. Identity-aware convolutional neural network for facial expression recognition. In 2017 12th IEEE International Conference on Automatic Face & Gesture Recognition (FG 2017), pages 558–565. IEEE, 2017.

[37] Federico Monti, Davide Boscaini, Jonathan Masci, Emanuele Rodola, Jan Svoboda, and Michael M Bronstein. Geometric deep learning on graphs and manifolds using mixture model cnns. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 5115–5124, 2017.

[38] E Morales-Vargas, CA Reyes-García, and Hayde Peregrina-Barreto. On the use of action units and fuzzy explanatory models for facial expression recognition. PloS one, 14(10), 2019.
[39] Christopher Morris, Martin Ritzert, Matthias Fey, William L Hamilton, Jan Eric Lenssen, Gaurav Rattan, and Martin Grohe. Weisfeiler and leman go neural: Higher-order graph neural networks. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 33, pages 4602–4609, 2019.

[40] Charles R Qi, Hao Su, Kaichun Mo, and Leonidas J Guibas. Pointnet: Deep learning on point sets for 3d classification and segmentation. In Proceedings of the IEEE conference on computer vision and pattern recognition, pages 652–660, 2017.

[41] Charles R Qi, Hao Su, Matthias Nießner, Angela Dai, Mengyuan Yan, and Leonidas J Guibas. Volumetric and multi-view cnns for object classification on 3d data. In Proceedings of the IEEE conference on computer vision and pattern recognition, pages 5648–5656, 2016.

[42] Charles Ruizhongtai Qi, Li Yi, Hao Su, and Leonidas J Guibas. Pointnet++: Deep hierarchical feature learning on point sets in a metric space. In Advances in neural information processing systems, pages 5099–5108, 2017.

[43] Q Qiu, X Cheng, R Calderbank, and G Sapiro. Defnet: Deep neural network with decomposed convolutional filters. In International Conference Machine Learning, 2018.

[44] Bin Ren, Mengyuan Liu, Runwei Ding, and Hong Liu. A survey on 3d skeleton-based action recognition using learning method. arXiv preprint arXiv:2002.05907, 2020.

[45] Franco Scarselli, Marco Gori, Ah Chung Tsoi, Markus Hagenbuchner, and Gabriele Monfardini. The graph neural network model. IEEE Transactions on Neural Networks, 20(1):61–80, 2008.

[46] Stefan C Schonsheck, Bin Dong, and Rongjie Lai. Parallel transport convolution: A new tool for convolutional neural networks on manifolds. arXiv preprint arXiv:1805.07857, 2018.

[47] Amir Shahroudy, Jun Liu, Tian-Tsong Ng, and Gang Wang. Ntu rgb+ d: A large scale dataset for 3d human activity analysis. In Proceedings of the IEEE conference on computer vision and pattern recognition, pages 1010–1019, 2016.

[48] Petar Veličković, Guillem Cucurull, Arantxa Casanova, Adriana Romero, Pietro Lio, and Yoshua Bengio. Graph attention networks. arXiv preprint arXiv:1710.10903, 2017.

[49] Raviteja Venulapalli, Felipe Arrate, and Rama Chellappa. Human action recognition by representing 3d skeletons as points in a lie group. In Proceedings of the IEEE conference on computer vision and pattern recognition, pages 588–595, 2014.

[50] Jiang Wang, Zicheng Liu, Ying Wu, and Junsong Yuan. Mining actionlet ensemble for action recognition with depth cameras. In 2012 IEEE Conference on Computer Vision and Pattern Recognition, pages 1290–1297. IEEE, 2012.

[51] Zonghan Wu, Shirui Pan, Fengwen Chen, Guodong Long, Chengqi Zhang, and S Yu Philip. A comprehensive survey on graph neural networks. IEEE Transactions on Neural Networks and Learning Systems, 2020.

[52] Keyulu Xu, Weihua Hu, Jure Leskovec, and Stefanie Jegelka. How powerful are graph neural networks? ICLR, 2019.

[53] Sijie Yan, Yuanjun Xiong, and Dahua Lin. Spatial temporal graph convolutional networks for skeleton-based action recognition. In Thirty-second AAAI conference on artificial intelligence, 2018.

[54] Jiani Zhang, Xingjian Shi, Junyuan Xie, Hao Ma, Irwin King, and Dit-Yan Yeung. Gaan: Gated attention networks for learning on large and spatiotemporal graphs. arXiv preprint arXiv:1803.07294, 2018.
A Proofs

A.1 Details and proofs in Section 2.3

A.1.1 Locally connected gnn

Specifically, the construction in [5, 9] assumes that $u$ and $u'$ belongs to the graph of different scales, $u'$ is on the fine graph, and $u$ is on a coarse-grained layer produced by clustering of indices of the graph of the input layer. If one generalize the construction to allow over-lapping of the receptive fields, and assume no pooling or coarse-graining of the graph, then the non-zero parameters are of the number

$$\sum_{u \in V} |N_u| \cdot CC' = np \cdot CC',$$

where $n = |V|$, $p$ is the average patch size $|N_u|$, and $C$ and $C'$ are the number of input and output feature channels.

A.1.2 Chebnet and GCN

In view of (1), Chebnet [13] makes use of the graph adjacency matrix to construct $M$. Specifically, $A_{sym} := D^{-1/2}AD^{-1/2}$ is the symmetrized graph adjacency matrix (possibly including self-edge, then $A$ equals original $A$ plus $I$), and $L_{sym} := I - A_{sym}$ has spectral decomposition $L_{sym} = \Psi \Lambda \Psi^T$. Let $\tilde{L} = \alpha_1 I + \alpha_2 L_{sym}$ be the rescaled and re-centered graph Laplacian such that the eigenvalues are between $[-1, 1]$, $\alpha_1, \alpha_2$ fixed constants. Then, written in $n$-by-$n$ matrix form,

$$M_{c',c} = \sum_{l=0}^{L-1} \theta_l(c', c)T_l(\tilde{L}), \quad \theta_l(c', c) \in \mathbb{R}, \quad (6)$$

where $T_l(\cdot)$ is Chebyshev polynomial of degree $l$. As $A_{sym}$ and then $\tilde{L}$ are given by the graph, only $\theta_l$’s are trainable, thus the number of parameters are

$$L \cdot CC'.$$

GCN [26] is a special case of Chebnet. Take $L = 2$ in (6), and tie the choice of $\theta_0$ and $\theta_1$,

$$M_{c',c} = \theta(c', c)(\alpha_1 I + \alpha_2 A_{sym}) =: \theta(c', c)\tilde{A}, \quad \alpha_1, \alpha_2 \text{ fixed constants,}$$

where $\theta(c', c)$ is trainable. This factorized form leads to the linear part of the layer-wise mapping as $Y = \tilde{A}X\Theta$ written in matrix form, where $\tilde{A}$ is $n$-by-$n$ matrix defined as above, $X$ ($Y$) is $n$-by-$C'$-($C$) array, $\Theta$ is $C'$-by-$C$ matrix. The model complexity is $CC'$ which are the parameters in $\Theta$.

Proof of Proposition 1. Since GCN is a special case of Chebnet, it suffices to prove that (6) can be expressed in the form of L3-gnn (2) for some $K$. By definition of $\tilde{L}$, mathematically equivalently,

$$M_{c',c} = \sum_{l=0}^{L-1} \theta_l(c', c)T_l(\alpha_1 I + \alpha_2 L) = \sum_{l=0}^{L-1} \theta_l(c', c)T_l(\alpha_1 I + \alpha_2 (I - A_{sym})) = \sum_{l=0}^{L-1} \beta_l(c', c)A_{sym}^l, \quad (7)$$

where the coefficients $\beta_l$’s are determined by $\theta_l$’s, per $(c', c)$. Since $A_{sym}^l$ propagates to the $l$-th order neighborhood of any node, setting $B_k(u', u) = A_{sym}^{k-1}(u', u)$, $B_k(u', u)$ is non-zero when $u' \in N_u^{(k-1)}$, $1 \leq k \leq K := L$, and then setting $a_k(c', c) = \beta_{k-1}(c', c)$ gives (6) in the form of (2). \(\square\)
A.1.3 Standard and geometrical CNN’s

Standard cnn on $\mathbb{R}^d$, e.g. $d=1$ for audio signal and $d=2$ for image data, applies a discretized convolution to the input data in each convolutional layer, which can be written as (omitting bias which is added per $c$, and the non-linear activation)

$$y(u, c) = \sum_{c' \in [C']} \sum_{u' \in U} w_{c',c}(u' - u)x(u', c'),$$

(8)

where $U$ is a grid on $\mathbb{R}^d$. We write in the way of “anti-convolution”, which has “$u' - u$” rather than “$u - u'$”, but the definition is equivalent. For audio and image data, $U$ is usually a regular mesh with evenly sampled grid points, and proper boundary conditions are applied when computing $y(u, c)$ at a boundary grid point $u$. As the convolutional filters $w_{c',c}$ are compactly supported, the summation of $u'$ is on a neighborhood of $u$.

More generally, CNN’s on non-Euclidean domains are constructed when spatial points are sampled on an irregular mesh in $\mathbb{R}^d$, e.g., a 2D surface in $\mathbb{R}^3$. The generalization of (8) is by defining the “patch operator” [35] which pushes a template filter $w$ on a regular mesh on $\mathbb{R}^d$, $d$ being the intrinsic dimensionality of the sampling domain, to the irregular mesh in the ambient space that have coordinates on local charts. Specifically, for a mesh of 2D surface in 3D, $d=2$, and $w$ is a template convolutional filter on $\mathbb{R}^2$. For any local cluster of 3D mesh points $N_u$ around a point $u$, the patch operator $P_u$ provides $(P_u w)(u')$ for $u' \in N_u$, by certain interpolation scheme on the local chart. The operator $P_u$ is linear in $w$, and possibly trainable. As a result, in mesh-based geometrical cnn,

$$y(u, c) = \sum_{c' \in [C']} \sum_{u' \in U} (P_u w_{c',c})(u')x(u', c'),$$

(9)

and one can see that in Euclidean space taking $(P_u w)(u') = w(u' - u)$ reduces (9) to the standard cnn as in (8).

In both (8) and (9), spatial low-rank decomposition of the filters $w_{c',c}$ can be imposed [43]. This introduces a set of bases $\{b_k\}_k$ over space that linearly span the filters $w_{c',c}$. For standard cnn in $\mathbb{R}^d$, $b_k$ are basis filters on $\mathbb{R}^d$, and for geometrical cnn, they are defined on the reference domain in $\mathbb{R}^d$ same as $w_{c',c}$, where $d$ is the intrinsic dimension. Suppose $w_{c',c} = \sum_{k=1}^{K} \beta_{k,(c',c)} b_k$ for coefficients $\beta_{k,(c',c)}$, by linearity, (9) becomes

$$y(u, c) = \sum_{c' \in [C']} \sum_{u' k=1}^{K} \beta_{k,(c',c)} (P_u b_k)(u')x(u', c'),$$

(10)

and similarly for (8). The trainable parameters in (10) are $\beta_{k,(c',c)}$ and the basis filters $b_k$’s, the former has $KCC'$ parameters, and the latter has $\sum_k p_k$, where $p_k$ is the size of the support of $b_k$ in $\mathbb{R}^d$. Suppose the average size is $p$, then the number of parameters is $Kp$. This gives the total number of parameters as $KCC' + Kp$.

Proof of Proposition [3]. Since standard cnn is a special case of geometrical cnn [9] we only consider the latter. Assuming low-rank filter decomposition, the convolutional mapping is (10). Comparing to the cnn layer mapping defined in [14], one sees that

$$M(u', u; c', c) = \sum_{k=1}^{K} \beta_{k,(c',c)} (P_u b_k)(u'),$$

which equals [2] if setting $B_k(u', u) = (P_u b_k)(u')$ and $a_k(c', c) = \beta_{k,(c',c)}$. \qed


A.1.4 Strong regularization limit

Proof of Proposition [3]. The constrained minimization of $\mathcal{R}$ defined in [4] separates for each $u, k$, and the minimization of $b_u^{(k)}$ is given by
\[
\min_{w: N_u^{(d_k)} \to \mathbb{R}} w^T L_u^{(k)} w, \quad \text{s.t. } \|w\|_2 \geq \alpha_{u,k} > 0.
\]
(11)

For each $u, k$, the local Dirichlet graph Laplacian $L_u^{(k)}$ has eigen-decomposition $L_u^{(k)} = \Psi_u^{(k)} \Lambda_u^{(k)} (\Psi_u^{(k)})^T$, where $(\Psi_u^{(k)})^T \Psi_u^{(k)} = I$, and the diagonal entries of $\Lambda_u^{(k)}$ are eigenvalues of $L_u^{(k)}$, which are all $\geq 0$ and sorted in increasing order. By the variational property of eigenvalues, the minimizer of $w$ in (11) is achieved when $w = \Psi_u^{(k)} (\cdot, 1)$, i.e., the eigenvector associated with the smallest eigenvalue of $L_u^{(k)}$. By that the local subgraph is connected, this smallest eigenvalue has single multiplicity, and the eigenvector is the Perron-Frobenius vector which does not change sign. The claim holds for arbitrary $\alpha_{u,k} > 0$ since eigenvector is defined up to a constant multiplication.

A.2 Proofs in Section 3.2

Proof of Theorem 7. By definition,
\[
Y(u) = \sigma(\sum_{k=1}^{K} a_k \langle B_k(\cdot, u), X(\cdot) \rangle_{N_u^{(d_k)}} + \text{bias}),
\]
then since $\sigma$ is non-expansive, $\forall u \in V$,
\[
|\Delta Y(u)| \leq \left| \sum_{k=1}^{K} a_k \langle B_k(\cdot, u), \Delta X(\cdot) \rangle_{N_u^{(d_k)}} \right| \leq \|a\|_2 \left( \sum_{k=1}^{K} |\langle B_k(\cdot, u), \Delta X(\cdot) \rangle_{N_u^{(d_k)}}|^2 \right)^{1/2}.
\]
(12)

By that
\[
|\langle B_k(\cdot, u), \Delta X(\cdot) \rangle_{N_u^{(d_k)}}| \leq \|B_k(\cdot, u)\|_{2,N_u^{(d_k)}} \cdot \|\Delta X(\cdot)\|_{2,N_u^{(d_k)}},
\]
(13)
we have that
\[
\sum_{u \in V} |\Delta Y(u)|^2 \leq \|a\|^2_2 \sum_{u \in V} \sum_{k=1}^{K} |\langle B_k(\cdot, u), \Delta X(\cdot) \rangle_{N_u^{(d_k)}}|^2
\]
\[
\leq \|a\|^2_2 \sum_{u \in V} \sum_{k=1}^{K} \|B_k(\cdot, u)\|_{2,N_u^{(d_k)}} \cdot \|\Delta X(\cdot)\|_{2,N_u^{(d_k)}}^2
\]
\[
\leq (\|a\|_2 \beta^{(1)})^2 \sum_{u,k} \|\Delta X(\cdot)\|_{2,N_u^{(d_k)}}^2,
\]
(14)
and observe that
\[
\sum_{u,k} \|\Delta X(\cdot)\|_{2,N_u^{(d_k)}}^2 = \sum_{k=1}^{K} \sum_{u \in V} \sum_{v \in N_u^{(d_k)}} |\Delta X(v)|^2 = \sum_{k=1}^{K} \sum_{u \in V} 1_{\{v \in N_u^{(d_k)}\}} |\Delta X(v)|^2
\]
\[
= \sum_{k=1}^{K} \sum_{u \in V} 1_{\{u \in N_v^{(d_k)}\}} |\Delta X(v)|^2 = \sum_{k=1}^{K} \sum_{v \in V} |N_v^{(d_k)}| \cdot |\Delta X(v)|^2 \leq Kp \sum_{v \in V} |\Delta X(v)|^2,
\]
where we used the assumption on $kp$ to obtain the last $\leq$. Then (14) continues as
\[
\leq (\|a\|_2 \beta^{(1)})^2 Kp \|\Delta X\|_{2,V}^2,
\]
which proves that $\|\Delta Y\|_{2,V} \leq (\|a\|_2 \beta^{(1)}) \sqrt{Kp} \|\Delta X\|_{2,V}$ as claimed. \qed
Proof of Theorem 2: Same as in the proof of Theorem 1, we have (12). The eigen-decomposition $L_u^{(k)} = \Psi_u^{(k)} \Lambda_u^{(k)} (\Psi_u^{(k)})^T$ has that $(\Psi_u^{(k)})^T \Psi_u^{(k)} = I$, and, under the connectivity condition of the subgraph, the diagonal entries of $\Lambda_u^{(k)}$ all $> 0$. Thus

$$\langle u, v \rangle_{N_u^{(d_k)}} = \langle (\Lambda_u^{(k)})^{1/2} \Psi_u^{(k)} u, (\Lambda_u^{(k)})^{-1/2} \Psi_u^{(k)} v \rangle_{N_u^{(d_k)}}$$

which gives the Cauchy-Schwarz with weighted 2-norm as

$$|\langle B_k(\cdot, u), \Delta X(\cdot) \rangle_{N_u^{(d_k)}}| \leq \|B_k(\cdot, u)\|_{L_u^{(k)}} \cdot \|\Delta X(\cdot)\|_{(L_u^{(k)})^{-1}}.$$ (15)

Then similarly as in (14), using the definition of $\beta^{(2)}$ and the the condition with $\rho$, we obtain that

$$\sum_{u \in V} |\Delta Y(u)|^2 \leq (\|a\|_2 \beta^{(2)})^2 \sum_{u,k} \rho^2 \|\Delta X(\cdot)\|^2_{2,N_u^{(d_k)}},$$ (16)

and the rest of the proof is the same, which gives that

$$\sum_{u \in V} |\Delta Y(u)|^2 \leq (\|a\|_2 \beta^{(2)})^2 \rho^2 K^p \|\Delta X\|^2_{2,V},$$

which proves the claim. 

\[\square\]

B Up/down-wind Classification Experiment

B.1 Dataset Setup

We generate the Up/Down wind dataset on both ring graph and chain graph with 64 nodes. Every node is assigned to a probability drawn from $(0,1)$ uniform distribution. Node with probability less than $\text{threshold} = 0.1$ will be assigned with a gaussian distribution with $\text{std} = 1.5$. Each gaussian distribution added is masked half side. Distribution masked left half is the ‘Down Wind’ class, distribution masked right half is the ‘Up Wind’ class, as shown in left plot in Figure 3. We then sum up all half distributions from different locations in each sample. We generate 5000 training samples and 5000 testing samples.

B.2 Model architecture and training details

Network architectures.
- 2-gcn-layer model:
  GraphConv(1,32)-ReLU-MaxPool1d(2)-GraphConv(32,64)-ReLU-AvgPool(32)-FC(2),
- 1-gcn-layer model:
  GraphConv(1,32)-ReLU-AvgPool(64)-FC(2),
where GraphConv can be Chebnet or L3-gnn.

Training details.
We choose the Adam Optimizer, batch size of 100, set initial learning rate of $1 \times 10^{-3}$, make it decay by 0.1 at 80 epoch and train for 100 epoches.

B.3 Additional results

We report additional results using 1-gcn layer architecture in Table A.1. Our L3-GNN again shows stronger classification performance than Chebnet.
Table A.1: results of 1-gcn layer models

| Gnn model | order | #params | ring graph Acc | chain graph Acc |
|-----------|-------|---------|----------------|-----------------|
| Chebnet   | L=3   | 0.2k    | 50.80 ± 0.24   | 50.06 ± 0.21    |
| L=5       | 0.3k  | 51.14 ± 0.21 | 51.07 ± 0.35  |
| L=9       | 0.4k  | 51.68 ± 0.38 | 50.96 ± 0.29  |
| L=30      | 1.1k  | 51.37 ± 0.14 | 50.70 ± 0.16  |
| L3-GNN    | 1     | 0.3k    | 99.96 ± 0.08   | 99.07 ± 0.12    |
| 0;1;2     | 0.8k  | 99.96 ± 0.01 | 99.92 ± 0.01  |

C Experimental Details

C.1 Classification of sphere mesh data

Spherical mesh We conduct this experiment on icosahedral spherical mesh [2]. Like Cohen et al. [10], we project digit image onto surface of unit sphere, and follow Jiang et al. [21] by moving projected digit to equator, avoiding coordinate singularity at poles.

Here, we details the subdivision scheme of the icosahedral spherical mesh we used. Start with an unit icosahedron, this sphere discretization progressively subdivide each face into four equal triangles, which makes this discretization uniform and accurate. Plus, this scheme provides a natural downsampling strategy for networks, as it denotes the path for aggregating information from higher-level neighbor nodes to lower-level center node. We adopt the following naming convention for different mesh resolution: start with level-0($L_0$) mesh (i.e., unit icosahedron), each level above is associated with a subdivision. For level-$i(L_i)$, properties of spherical mesh are:

$$N_e = 30 \cdot 4 \times i, \ N_f = 20 \cdot 4 \times i, \ N_v = N_e - N_f + 2$$

in which $N_f$, $N_e$, $N_v$ denote number of edges, faces, and vertices.

To give a direct illustration of how many nodes each level of mesh has, we list them below,

- $L_0$ 12 nodes
- $L_1$ 42 nodes
- $L_2$ 162 nodes
- $L_3$ 642 nodes
- $L_4$ 2562 nodes
- $L_5$ 10242 nodes

Network architectures We use a three-stage GNN model for this sphereMNIST, with each stage conduct convolution on spherical mesh of a specific level. Detailed architecture (suppose mesh levels used are $L_i, L_j, L_k$):

```
Conv(1,16)$_{L_i}$-BN-ReLU-DownSamp-ResBlock(16,16,64)$_{L_j}$-DownSamp-ResBlock(64,64,256)$_{L_k}$-AvgPool-FC(10),
```

We use the 4-stage model architecture for SphereModelNet-40, where 4 mesh levels are: $L_5$, $L_4$, $L_3$, $L_2$. Detailed architecture are:

```
Conv(6,32)$_{L_5}$-BN-ReLU-DownSamp-ResBlock(32,32,128)$_{L_4}$-DownSamp
-ResBlock(128,128,512)$_{L_3}$-DownSamp-ResBlock(512,512,2048)$_{L_4}$-DownSamp-AvgPool-FC(40),
```

where the GraphConv($\text{feat\_in}, \text{feat\_out}$) in above model architectures can be either Mesh Convolution layer or Graph Convolution layer, and “ResBlock” is a bottleneck module with two $1 \times 1$ convolution layers and one GraphConv layer.
Training Details. For SphereMNIST experiments, we use batch size of 64, Adam optimizer, initial learning rate of 0.01 which decays by 0.5 every 10 epoches. We totally train model for 100 epoches. For SphereModelNet-40 experiment, we batch size of 16, Adam optimizer, initial learning rate of 0.005 which decay by 0.7 every 25 epoches. We totally train 300 epoches.

Results on fine mesh Table A.2 show the results of SphereMNIST and Sphere-ModelNet40 on fine meshes on the sphere. Specifically, the mesh used for SphereMNIST here is of levels $L_4, L_3, L_2$, and the SphereModelNet-40 mesh of levels $L_5, L_4, L_3, L_2$, same as in [21].

Table A.2: Results on SphereMNIST and SphereModelNet-40 following setup in [21]

| Model       | SphereMNIST Acc | SphereModelNet-40 Acc |
|-------------|-----------------|-----------------------|
| S2CNN [10]  | 96.0            | 85.0                  |
| UGSCNN [22] | 99.2            | 90.50                 |
| GCN         | 95.8            | 87.97                 |
| ChebNet($L=4$) | 99.3             | 88.85                 |
| ChebNet($L=5$) | -                 | 88.90                 |
| ChebNet($L=6$) | -                 | 88.70                 |
| ChebNet($L=7$) | -                 | 88.78                 |
| L3-gnn (1123) | 99.10              | 90.24                 |
| L3-gnn (112)  | 98.90            | 89.67                 |

C.2 Facial Expression Recognition

Dataset setup
- CK+: The CK+ dataset [32] is the mostly used laboratory-controlled FER dataset (downloaded from: [http://www.jeffcohn.net/resources/]). It contains 327 video sequences from 118 subjects with seven basic expression labels (anger, contempt, disgust, fear, happiness, sadness, and surprise). Every sequence shows a shift from neutral face to the peak expression. We extract the last three frames from each sequence in the CK+ dataset, form a dataset with 981 samples. Every facial image is aligned and resized to (120, 120) with face alignment model [6], and then we use this model again to get facial landmarks. As we describe in Section 4.2, we select 15 from 68 facial landmarks and build graph on them. The input feature for each node is an image patch centered at the landmark with size (20, 20), concatenated with the landmark’s coordinates, so the total input feature dimension is 402.
- FER13: FER13 dataset [18] is a large-scaled, unconstrained database collected automatically by Goole Image API (downloaded from: [https://www.kaggle.com/c/challenges-in-representation-learning-facial-expression-recognition-challenge/data]). It contains 28,709 training images, 3589 validation images and 3589 test images of size (48, 48) with seven common expression labels as CK+. We align facial images, get facial landmarks, and select nodes & build graph the same way as we do in CK+. Input features are local image patch centered at each landmark with size (8, 8) and landmark’s coordinates, so the total input feature dimension is 66.

Network architectures.
- CK+: GraphConv(402,64)-BN-ReLU-GraphConv(64,128)-BN-ReLU-FC(7),
- FER13: GraphConv(66,64)-BN-ReLU-GraphConv(64,128)-BN-ReLU-GraphConv(128,256)-BN-ReLU-FC(7), where GraphConvfeat_in, feat_out) here can be any type of graph convolution layer, including our L3-gnn.

Training details.
- CK+:
We use 10-fold cross validation as [14]. Batch size is set as 16, learning rate is 0.001 which decay by 0.1 if validation loss remains same for last 15 epoches. We choose Adam optimizer and train 100 epoches for each fold validation.

- FER13: We report results on test set. Batch size is set as 32, learning rate is 0.0001 which decay 0.1 if validation loss remains same for last 20 epoches. We choose Adam optimizer and train models for 150 epoches.

**Runtime analysis details.** In section 4.2, we report the running time of our L3-gnn(order 1,1,2,3), 13.02ms, with best ChebNet, 12.56ms, on CK+ dataset, which are comparable. Here, we provide more details about this. The time we use to compare is the time of model finishing inference on validation set with batch size of 16. For each model, we record all validation time usages in all folds and report the average of them. The Runtime analysis is performed on a single NVIDIA TITAN V GPU.

### C.3 Skeleton-based Action Recognition

#### Dataset setup

- **NTU-RGB+D:** NTU-RGB+D [47] is a large skeleton-based action recognition dataset with three-dimensional coordinates given to every body joint (downloaded from: http://rose1.ntu.edu.sg/datasets/requesterAdd.asp?DS=3). It comprises 60 action classes and total 56,000 action clips. Every clip is captured by three fixed Kineticsv2 sensors in lab environment performed by one of 40 different subjects. Three sensors are set at same height but in different horizontal views, $-45^\circ, 0^\circ, 45^\circ$. There are 25 joints tracked, as shown in Figure A.1. Two experiment setting are proposed by [47], cross-view (X-view) and cross-subject (X-sub). X-view consists of 37,920 clips for training and 18960 for testing, where training clips are from sensor on $0^\circ, 45^\circ$, testing clips from sensor on $-45^\circ$. X-sub has 40,320 clips for training and 16,560 clips for testing, where training clips are from 20 subjects, testing clips are from the other 20 subjects. We test our model on both settings.

- **Kinetics:** Kinetics [22] is a large and most commonly-used action recognition dataset with nearly 300,000 clips for 400 classes (downloaded from: https://deepmind.com/research/open-source/kinetics). We follow Yan et al. [53] to get 18-point body joints from each frame using OpenPose [7] toolkit. Input features for each joint to the Network is $(x, y, p)$, in which $x, y$ are 2D coordinates of the joint, and $p$ is the confidence for localizing the joint. To eliminate the effect of skeleton-based model’s inability to recognize objects in clips, we mainly focus on action classes that requires only body movements. Thus, we conduct our experiments

![Figure A.1: Illustration of 25-point body joints and graph.](image)
on Kinetics-Motion, proposed by [53]. This is a small dataset that contains 30 action classes strongly related to body motion. Note that there are severe data missing problem in landmark coordinates in Kinetics data, so we also use our regularization scheme in this experiment.

**Network Architectures**

- **NTU-RGB+D:**
  We follow the architecture in [53]:
  
  \[
  \text{STGraphConv}(3,64,9,s1)-\text{STGraphConv}(64,64,9,s1)-\text{STGraphConv}(64,64,9,s1)-\text{STGraphConv}(64,64,9,s1)-\text{STGraphConv}(64,128,9,s2)-\text{STGraphConv}(128,128,9,s1)-\text{STGraphConv}(128,128,9,s1)-\text{STGraphConv}(128,256,9,s2)-\text{STGraphConv}(256,256,9,s1)-\text{STGraphConv}(256,256,9,s1)-\text{STAvgPool-}f\text{c}(60). 
  \]

- **Kinetics:**
  We also design a computation-efficient architecture for Kinetics-Motion with larger temporal down-sampling rate, which results in less forward time:
  
  \[
  \text{STGraphConv}(3,32,9,s2)-\text{STGraphConv}(32,64,9,s2)-\text{STGraphConv}(64,64,9,s1)-\text{STGraphConv}(64,64,9,s1)-\text{STGraphConv}(64,128,9,s2)-\text{STGraphConv}(128,128,5,s1)-\text{STGraphConv}(128,128,5,s1)-\text{STGraphConv}(128,256,5,s2)-\text{STGraphConv}(256,256,3,s1)-\text{STGraphConv}(256,256,3,s1)-\text{STAvgPool-}f\text{c}(60), 
  \]

  where the structure of STGraphConv(\text{feat}_\text{in}, \text{feat}_\text{out}, \text{temporal kernel size}, \text{temporal stride}) is:
  
  \[
  \text{GraphConv} (\text{feat}_\text{in}, \text{feat}_\text{out})-\text{BN-ReLU-1DTemporalConv}(\text{feat}_\text{out}, \text{feat}_\text{out}, \text{temporal kernel size}, \text{temporal stride})-\text{BN-ReLU}. 
  \]

**Training Details**

- **NTU-RGB+D:**
  We use batch size of 32, initial learning rate of 0.001 which decay by 0.1 at (30, 80) epoch, and total train 120 epoches. SGD optimizer is selected. We padding every sample temporally with 0 to 300 frames.

- **Kinetics:**
  We use batch size of 32, initial learning rate of 0.01 which decay by 0.1 at (40, 80) epoch, and total train 100 epoches. SGD optimizer is selected. We padding every sample temporally with 0 to 300 frames, and during training, we perform data augmentation by randomly choosing 150 contiguous frames.

**C.4 Details of experiment on MNIST**

**C.4.1 Simulated graph noise on $7 \times 7$ MNIST**

Here we describe three types of noise in our experiments:

- **Gaussian noise.** Given a $7 \times 7$ image from MNIST, we sample 49 values from $\mathcal{N}(0, std)$. the std controls the strength of noise added. We conduct experiments under $std = 0.1, 0.2, 0.3$ as shown in Table 2. The amount of noise is also measured by PNSR which is standard for image data.

- **Missing value noise.** Given a image, we randomly sample 49 values from $U(0, 1)$, and select nodes with probabilities less than a threshold. This threshold is called noise level, which controls the percentage of nodes affected. Then, we remove the pixel value at those selected nodes. Experiments with noise level = 0.1, 0.2, 0.3 are conducted.

- **Graph node permutation noise.** For each sample, we randomly select a permutation center node which has exact 4 neighbors. Then, we rotate its neighbors clockwise by 90 degree, e.g., top neighbor becomes right neighbor, and then we update the indices of permuted nodes.

**C.4.2 Network architecture and training details**

We use the same architecture for different experiment settings:

\[
\text{GraphConv}(1,32)-\text{BN-ReLU-GraphConv}(32,64)-\text{BN-ReLU-FC}(10),
\]

where GraphConv can be different types of graph convolution layers. We set batch size to 100, use Adam optimizer, and set initial learning rate to 1e-3. Learning rate will drop by 10 if the least validation loss remains the same for the last 15 epoches. We set total training epoches as 200.
Here, we show experiments results on $28 \times 28$, $14 \times 14$ grid, as well as $7 \times 7$ grid with missing values. Table A.3 shows results on $28 \times 28$ image grid. Our model have better performance than other methods.

Table A.3: Results on MNSIT with grid size $28 \times 28$.

| Model  | bases order | #params(w/o FC) | Acc      |
|--------|-------------|----------------|----------|
| GCN    | 1           | 2.4k           | 93.30 ± 0.12 |
|        | L=3         | 6.5k           | 93.93 ± 0.18 |
|        | L=4         | 8.6k           | 94.97 ± 0.06 |
|        | L=5         | 10.7k          | 95.87 ± 0.09 |
|        | L=6         | 12.8k          | 96.64 ± 0.12 |
|        | L=7         | 14.8k          | 96.98 ± 0.19 |
|        | L=9         | 19.0k          | 97.43 ± 0.14 |
|        | L=15        | 31.5k          | 97.91 ± 0.08 |
|        | L=20        | 41.9k          | 97.90 ± 0.04 |
| ChebNet| 1:1:2       | 41.6k          | 96.78 ± 0.08 |
|        | 1:1:2:3     | 79.2k          | 97.32 ± 0.10 |
| L3-gnn | 1:1:2       | 13.8k          | 97.17 ± 0.09 |
|        | 1:1:2:3     | 14.8k          | 97.51 ± 0.07 |

Table A.4 shows results on $14 \times 14$ image grid, where our L3-gnn have comparable results with the best ChebNet [13] method.

Table A.4: Results on MNSIT with grid size $14 \times 14$.

| Model  | bases order | #params(w/o FC) | Acc      |
|--------|-------------|----------------|----------|
| GCN    | 1           | 2.4k           | 93.70 ± 0.09 |
|        | L=3         | 6.5k           | 96.06 ± 0.16 |
|        | L=4         | 8.6k           | 96.85 ± 0.11 |
|        | L=5         | 10.7k          | 97.24 ± 0.28 |
|        | L=6         | 12.8k          | 97.58 ± 0.10 |
|        | L=7         | 14.9k          | 97.74 ± 0.07 |
| ChebNet| 1:1:2       | 13.8k          | 97.17 ± 0.09 |
|        | 1:1:2:3     | 14.8k          | 97.43 ± 0.07 |
|        | 1:1:2:3     | 25.1k          | 97.51 ± 0.07 |

We shows our results on $7 \times 7$ image grid with missing values in Table A.5. With regularization, L3-gnn achieves the best performance in every experiment with different noise level.

Table A.5: Results on MNSIT with grid size $7 \times 7$ with different levels of missing value.

| Model  | bases order | reg | #params(w/o FC) | Acc(Original) | Acc(psnr 18.70) | Acc(psnr 15.33) | Acc(psnr 13.15) |
|--------|-------------|-----|----------------|---------------|-----------------|-----------------|-----------------|
| GCN    | 1           | -   | 2.4k           | 90.62 ± 0.24  | 83.44 ± 0.15    | 77.23 ± 0.13    | 71.67 ± 0.96    |
|        | L=3         | -   | 6.5k           | 92.83 ± 0.09  | 87.09 ± 0.18    | 82.11 ± 0.18    | 76.15 ± 0.26    |
|        | L=5         | -   | 8.6k           | 93.12 ± 0.1   | 87.09 ± 0.16    | 82.22 ± 0.28    | 75.95 ± 0.22    |
|        | L=6         | -   | 10.7k          | 93.2 ± 0.07   | 87.01 ± 0.14    | 82.04 ± 0.14    | 76.21 ± 0.38    |
|        | L=7         | -   | 12.7k          | 93.42 ± 0.09  | 87.20 ± 0.3     | 81.19 ± 0.29    | 75.24 ± 0.32    |
|        | 1:1:2       | 0.5 | 8.4k           | 93.56 ± 0.08  | 86.64 ± 0.16    | 81.14 ± 0.30    | 75.07 ± 0.08    |
|        | 1:1:2:3     | 0.5 | 12.2k          | 93.67 ± 0.15  | 86.51 ± 0.38    | 80.68 ± 0.11    | 74.24 ± 0.36    |
|        | 1:1:2:3     | 0.5 | 12.2k          | 93.85 ± 0.15  | 87.22 ± 0.08    | 82.64 ± 0.31    | 76.08 ± 0.38    |