Abstract. Consider a diagram of quasi-categories that admit and functors that preserve limits or colimits of a fixed shape. We show that any weighted limit whose weight is a projective cofibrant simplicial functor is again a quasi-category admitting these (co)limits and that they are preserved by the functors in the limit cone. In particular, the Bousfield-Kan homotopy limit of a diagram of quasi-categories admit any limits or colimits existing in and preserved by the functors in that diagram. In previous work, we demonstrated that the quasi-category of algebras for a homotopy coherent monad could be described as a weighted limit with projective cofibrant weight, so these results immediately provide us with important (co)completeness results for quasi-categories of algebras. These generalise most of the classical categorical results, except for a well known theorem which shows that limits lift to the category of algebras for any monad, regardless of whether its functor part preserves those limits. The second half of this paper establishes this more general result in the quasi-categorical setting: showing that the monadic forgetful functor of the quasi-category of algebras for a homotopy coherent monad creates all limits that exist in the base quasi-category, without further assumption on the monad. This proof relies upon a more delicate and explicit analysis of the particular weight used to define quasi-categories of algebras.

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1. Introduction

Let $\qCat_\infty$ denote the simplicial category of quasi-categories, a full subcategory of the cartesian closed category of simplicial sets. Because the quasi-categories are the fibrant...
objects in a model category enriched over the Joyal model structure (in this case over itself), the simplicial category \( \mathfrak{qCat}_\infty \) is closed under a certain class of weighted limits: those whose weights are projective cofibrant simplicial functors. Here, a subcategory of an enriched category is *closed* under a certain class of limits if it possesses those limits and if moreover they are preserved by the subcategory inclusion.

Our aim in this paper is to prove the following theorem:

**1.1. Theorem.** Let \( X \) be a simplicial set. The quasi-categorically enriched subcategory of \( \mathfrak{qCat}_\infty \) spanned by those quasi-categories admitting (co)limits of shape \( X \) and those functors preserving them is closed in \( \mathfrak{qCat}_\infty \) under all projective cofibrant weighted limits.

This paper is a continuation of [3] and [4], which contain all the necessary preliminaries; references therein will have the form I.x.x.x. or II.x.x.x. Projective cofibrant weights are discussed in §II.3, and a recognition theorem is proven. This class of weights plays a fundamental role in our development of the quasi-categorical monadicity theorem, which constructs the quasi-category of algebras for a homotopy coherent monad and characterises it up to equivalence.

The class of *projective cofibrant weighted limits*, that is, weighted limits with projective cofibrant weights, includes familiar Bousfield-Kan-style homotopy limits such as comma quasi-categories or mapping cocylinders (see example II.3.1.10). A weight is projective cofibrant if and only if it can be built as a retract of a countable composite of pushouts of coproducts of basic “cells” defined by tensoring representable functors with the boundary inclusions \( \partial \Delta^n \rightarrow \Delta^n \) of simplicial sets. To prove that projective cofibrant weighted limits of quasi-categories are again quasi-categories, it suffices to show that \( \mathfrak{qCat}_\infty \) is closed under the corresponding limit notions—namely, splittings of idempotents, countable composites of isofibrations, pullbacks of isofibrations, products, and cotensors with simplicial sets—and that moreover the cotensor of a quasi-category with a monomorphism is an isofibration. Here an *isofibration* is a functor between quasi-categories that is a fibration in the Joyal model structure; see I.2.2.4. Section 2 contains a general discussion of projective cofibrant weighted limits.

Our first key result, theorem [1.1], shows that if each quasi-category in a simplicial diagram has (co)limits of a fixed shape and each functor in that diagram preserves these, then the projective weighted limit again posses these (co)limits and they are preserved by the functors defining the limit cone. For example, the quasi-category of algebras \( A[t] \) for a homotopy coherent monad \( T \) on a quasi-category \( A \) is defined in §II.7 via a projective cofibrant weighted limit. Consequently, as an immediate corollary of our first key result we can show that the forgetful functor \( u^t : A[t] \rightarrow A \) creates any colimits that are preserved by the functor part \( t : A \rightarrow A \) of the monad—precisely as is the case in ordinary category theory. However, the immediate application of that result to the corresponding question of the creation of limits by \( u^t : A[t] \rightarrow A \) does not yield a true generalisation of the corresponding classical result, since that theorem does not require any assumption that the functor part of our monad should preserve limits. Our second key result, theorem [5.23], rectifies this deficiency. It makes a much more detailed and concrete analysis of the specific weight used in building quasi-categories of algebras to show that \( u^t : A[t] \rightarrow A \) does indeed create any
limits that exist in \( A \) without having to constrain our homotopy coherent monad by any continuity assumptions.

In 2-category theory the classes of flexible and PIE limits (cf. [2]) are of great importance. They are, for example, the 2-limits that are inherited by 2-categories of algebras and pseudo-morphisms for a 2-monad and are therefore the ones that exist in common 2-algebraic situations such as in 2-categories of (co)complete categories, monoidal categories, Grothendieck toposes, accessible categories, and so forth. They also enjoy the familiar homotopy theoretic property that pseudo-transformations of diagrams whose legs are equivalences induce equivalences of such limits. These classes are not only parsimonious but they also encompass most, if not all, 2-limit types of practical importance, such as pseudo, lax and oplax limits, comma categories, cotensors, gluing constructions, descent and lax descent constructions, and so forth. The class of limits weighted by projective cofibrant functors or projective cell complexes are direct quasi-categorical analogues of the classes of flexible and PIE limits respectively. They too contain analogues of all of the 2-limit types discussed above and much more besides. Consequently the completeness results we develop in this paper are ripe for wide application.

Our interest in the questions discussed here originally arose from our work on homotopy coherent monads presented in [4]. We had also been interested in some specific (co)completeness results for functor quasi-categories (hom-spaces), comma constructions, homotopy limits and the like, but had not sought to unify the results we had developed. However, an email sent to us by Tom Fiore encouraged us to look at the more general results we present here. Specifically, he asked whether it was possible to select certain families of limits in a quasi-category in a suitably functorial manner. Our approach to this problem was to observe that his result could be reduced to asking whether a particular limit of quasi-categories weighted by a some projective cofibrant weight possessed limits of that kind. The results presented here show that it does.

**Limits and colimits in a quasi-category.** In §I.5, we show that limits (resp. colimits) in a quasi-category are characterised by certain absolute right (resp. left) lifting diagrams in \( q\text{Cat}_2 \), André Joyal’s 2-category of quasi-categories; absolute right liftings are introduced in definition I.5.0.1. For ease of exposition, we restrict our discussion to limits from here on; the corresponding results for colimits are dual.

I.5.2.8. **Definition.** A quasi-category \( A \) has all limits of shape \( X \) if and only if there is an absolute lifting

\[
\begin{array}{ccc}
A^X & \xrightarrow{\text{id}_{A^X}} & A^X \\
\downarrow^{\text{id}_A} & & \downarrow^{c} \\
A & \xrightarrow{\lambda} & \lim \
\end{array}
\]
of the identity through the constant diagram functor \( c: A \to A^X \). A functor \( f: A \to B \) preserves all limits of shape \( X \) if the composite diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow^c & & \downarrow^c \\
A^X & \xrightarrow{id_{A^X}} & A^X
\end{array}
\]

\[
\begin{array}{ccc}
A^X & \xrightarrow{f^X} & B^X
\end{array}
\]

is again an absolute right lifting; see, for example, I.5.2.12.

Taking \( X = \emptyset \), definition I.5.2.8 specialises to say that a quasi-category \( A \) has a terminal object if and only if the constant functor \( A \to 1 \) has a right adjoint. The unit of this adjunction is represented by a 1-simplex in \( A^A \) from the identity functor to the constant functor at the terminal object whose component at the terminal vertex is degenerate; see lemma I.4.1.3.

**Outline of the proof.** The special case of terminal objects will be of particular interest: the general case for limits of shape \( X \) may be reduced to this special one. We shall prove theorem [1.1] in two steps, which occupy sections [3] and [4] respectively:

(I) Prove the special case \( X = \emptyset \) of quasi-categories which admit terminal objects.

(II) Reduce the general problem to that special case “fibre-wise.”

The key ingredient for this second step is:

I.6.1.3. **Theorem.** In \( \qCat_\infty \), a functor \( g: C \to A \) admits an absolute right lifting through the functor \( f: B \to A \) if and only if for all objects \( c \) of \( C \) the quasi-category \( f \downarrow gc \) has a terminal object.

In the special case of absolute lifting diagrams of the form given in definition I.5.2.8, the comma quasi-categories of theorem I.6.1.3 are quasi-categories of cones over a fixed diagram. A terminal object encodes a limiting cone; see observation I.5.2.7.

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2. **Weighted limits in \( \qCat_\infty \)**

The theory of weighted limits is reviewed in §II.3.1. For the reader’s convenience, and as a warm-up to our proof of theorem [1.1] recall:

II.3.2.4. **Proposition.** The full simplicial subcategory \( \qCat_\infty \) of quasi-categories is closed in \( \sSet \) under each of the following classes of limits:

1) arbitrary small products,
2) pullbacks of isofibrations,
3) countable composites of isofibrations,
4) splitting of idempotents, and
5) cotensors with any simplicial set $X$.

Moreover the cotensor with a monomorphism $X \hookrightarrow Y$ is an isofibration. Hence, $\mathcal{q}\text{Cat}_\infty$ is closed under limits weighted by projective cofibrant weights.

Proof. Let $A$ be a small simplicial category. A weight $W \in s\text{Set}^A$ is projective cofibrant just when it is expressible as a retract of a countable composite of pushouts of coproducts of maps $i \times A_a : X \times A_a \to Y \times A_a$ built by tensoring a covariant representable by an inclusion $X \hookrightarrow Y$ of simplicial sets. By cocontinuity of the weighted limit bifunctor in the weight, the weighted limit $\{W, D\}_A$ of a diagram $D : A \to \mathcal{q}\text{Cat}_\infty$ is a retract of a countable tower of pullbacks of products of maps
\[
\{i \times A_a, D\}_A \cong Da^i : Da^Y \to Da^X.
\]

Because the quasi-categories are the fibrant objects in a monoidal model structure with all objects cofibrant, $\mathcal{q}\text{Cat}_\infty$ is closed under simplicial cotensors, and in particular the domains and codomains of these maps are again quasi-categories. Moreover, monoidalness of the Joyal model structure implies that each of these maps is an isofibration (a fibration between fibrant objects).

As a category of fibrant objects, the quasi-categories are closed under products and splittings of idempotents as well as pullbacks and composites of isofibrations. In this way, we see that the weighted limit $\{W, D\}_A$ is again a quasi-category. \qed

2.1. Remark. In the proof of proposition II.3.2.4 just given, we have used a slightly different definition of projective cofibrant weight than appears in II.3.2.1. These are of course equivalent: by the algebraic small object argument, a cofibrant object in the projective model structure on $s\text{Set}^A$ is expressible as a retract of a countable composite of pushouts of coproducts of maps of the form $\partial \Delta^n \times A_a \to \Delta^n \times A_a$. Only countable composites are needed because the simplicial sets appearing here are $\omega$-small.

3. Quasi-categories with terminal objects

The basic theory of terminal objects is developed in §I.4.1. We begin by introducing some special notation for the quasi-categorically enriched category described in the statement of theorem 1.1.

3.1. Definition. For any simplicial set $X$, let $\mathcal{q}\text{Cat}_{X,\infty}$ denote the quasi-categorically enriched subcategory of $\mathcal{q}\text{Cat}_\infty$ with:

- **Objects** the quasi-categories $A$ which admit all limits of shape $X$,
- **$n$-arrows** all those $n$-arrows whose vertices are functors $f : A \to B$ which preserve all limits of shape $X$.

We shall write $\mathcal{q}\text{Cat}_X$ for the underlying category of $\mathcal{q}\text{Cat}_{X,\infty}$.

Our aim in this section is to prove that $\mathcal{q}\text{Cat}_{\emptyset,\infty}$, the subcategory of quasi-categories admitting terminal objects and terminal-object preserving functors, is closed in $\mathcal{q}\text{Cat}_\infty$ under limits with projective cofibrant weights.
3.2. Observation. A slightly banal but surprisingly useful fact is that, in order to show that a functor \( f : A \to B \) of quasi-categories preserves terminal objects, it is enough to show that there is some terminal object \( t \) in \( A \) such that \( ft \) is terminal in \( B \): all terminal objects in \( A \) are isomorphic, and any functor between quasi-categories preserves isomorphisms.

3.3. Observation. Our first order of work will be to prove that \( \qCat_\emptyset \) is closed in \( \qCat \) under each of the following classes of conical limits:

1) arbitrary small products,
2) pullbacks of isofibrations,
3) countable composites of isofibrations, and
4) splitting of idempotents.

The “isofibrations” referred to here are isofibrations that preserve terminal objects (i.e., isofibrations that lie in the subcategory \( \qCat_\emptyset \)). We shall prove these closure properties in a series of lemmas. Eventually, we will prove that the quasi-categorically enriched category \( \qCat_{\emptyset\infty} \) is closed in \( \qCat_{\infty} \) under:

5) cotensors with any simplicial set \( X \).

Moreover, we will see that the map induced by cotensoring with an inclusion \( X \hookrightarrow Y \) is an isofibration in \( \qCat_\emptyset \). As in the proof of proposition II.3.2.4, it will follow that \( \qCat_{\emptyset\infty} \) is closed in \( \qCat_{\infty} \) under limits weighted by projective cofibrant weights.

3.4. Remark (proof-schema). For each class of conical limits listed in observation 3.3, we shall prove the desired closure result using a similar argument. Specifically, given a diagram in \( \qCat_\emptyset \), we shall take its limit \( L \) in \( \qCat \), which comes equipped with projection functors \( \pi_i : L \to A_i \), and establish the following results:

(i) there is some object \( t \) of \( L \) which is pointwise terminal, in the sense that each \( \pi_i t \) is terminal in \( A_i \), and

(ii) every pointwise terminal object in \( L \) is a terminal object in \( L \).

Now (i) delivers a pointwise terminal \( t \) which is terminal in \( L \) by (ii). The pointwise terminality of \( t \) tells us that each \( \pi_i \) maps that particular terminal to a terminal in \( A_i \), so it follows that \( \pi_i \) preserves all terminal objects by observation 3.2. Together these arguments show that the vertex \( L \) and its limiting cone are in \( \qCat_\emptyset \).

Finally, if \( f : A \to L \) is a functor induced by another cone in \( \qCat_\emptyset \), then we know that the components \( f_i = \pi_i f : A_i \to A_i \) are all in \( \qCat_\emptyset \) and thus preserve terminal objects. It follows that a terminal object \( s \) of \( A \) maps to an object \( fs \) of \( L \) for which \( \pi_i fs = f_i s \) is terminal in \( A_i \) for each \( i \). In other words \( fs \) is pointwise terminal in \( L \) and hence is thus terminal by (ii). This demonstrates that \( f \) preserves terminal objects as required.

Universal properties of terminal objects. In proving the following results we will make repeated use of a few simple lemmas expressing the (relative) universal property of terminal objects and their behaviour with respect to isofibrations.

3.5. Recall (terminal objects). Propositions I.4.3.5 and I.4.1.6 tell us that if \( a \) is an object of a quasi-category \( A \) then the following are equivalent:
• $a$ is a terminal object of $A$ in the sense of Joyal [1]. In other words it enjoys the universal property that any sphere $\Delta^n \to A$, $n \geq 1$, which maps the final vertex $[n]$ to $a$

\[
\begin{array}{c}
\Delta^0 \\
\alpha
\end{array} \xrightarrow{[n]} \partial \Delta^n \xrightarrow{\alpha} A
\]

has a filler, this being the dashed arrow in the diagram above,

• for all quasi-categories $B$, the composite functor $B \xrightarrow{!} \Delta^0 \xrightarrow{a} A$ is terminal in the hom-category $\text{hom}_2(B, A) = h(A^B)$ of $\text{qCat}_2$,

• the functor $a: \Delta^0 \to A$ is right adjoint to the unique functor $!: A \to \Delta^0$.

II.5.3.5. Lemma. Suppose that $E$ and $B$ are quasi-categories which possess terminal objects and that $p: E \to B$ is an isofibration which preserves terminal objects. Assume also that $e$ is terminal in $E$. Then any lifting problem

\[
\begin{array}{c}
\Delta^0 \\
\alpha
\end{array} \xrightarrow{[n]} \partial \Delta^n \xrightarrow{u} E
\]

with $n > 0$ in which $u$ carries the vertex $[n]$ to $e$ has a solution.

3.6. Observation. While we will not use this fact here, we might observe in passing that the lifting property of the last lemma is equivalent to the statement that

\[
\begin{array}{c}
E \\
\alpha
\end{array} \xrightarrow{\Delta^0} \Delta^n \xrightarrow{e} \Delta^n \xrightarrow{v} B
\]

is an absolute right lifting diagram in $\text{qCat}_A$.

Lemma II.5.3.5 possesses the following immediate converse:

3.7. Lemma. Suppose that $p: E \to B$ is a isofibration of quasi-categories and that $e$ is an object of $E$ for which $pe$ is a terminal object in $B$. Assume also that the lifting property of lemma II.5.3.5 holds. Then $e$ is a terminal object of $E$.

Proof. Suppose that $u: \partial \Delta^n \to E$ is a sphere in $E$ which carries the vertex $[n]$ to the object $e$. Then the sphere $pu: \partial \Delta^n \to B$ carries $[n]$ to the terminal object $pe$ of $B$ and so it has a filler $v: \Delta^n \to B$. This provides us with a lifting problem of the form given in the statement of lemma II.5.3.5, which we may solve to provide a filler in $E$ for the original sphere $u$. This then establishes the universal property which shows that $e$ is terminal in $E$. \qed
The following lemma is little more than a trivial observation that will nonetheless receive repeated use in the sequel:

3.8. **Lemma.** Suppose that $E$ and $B$ are quasi-categories which possess terminal objects and that $p: E \rightarrow B$ is an isofibration which preserves terminal objects. If $s$ is a terminal object in $B$ then there exists a terminal object $t$ in $E$ with $pt = s$.

**Proof.** The hypotheses imply that $E$ admits a terminal object $t'$ and that $pt'$ is terminal in $B$. By uniqueness of terminal objects, there exists some isomorphism $s \cong pt'$ in $B$ which we can lift along the isofibration $p$ to give $t \cong t'$ which lies over $s \cong pt'$. It follows that $t$ is also terminal in $E$, because it is isomorphic to our original terminal object $t'$, and that $pt = s$ as required. \[\square\]

**Conical limits in $\mathcal{qCat}_\emptyset$.** We will now show that $\mathcal{qCat}_\emptyset$ is closed under the conical limits of observation 3.3.

3.9. **Lemma.** The subcategory $\mathcal{qCat}_\emptyset$ is closed in $\mathcal{qCat}$ under arbitrary small products.

**Proof.** Let $\{A_i\}_{i \in I}$ be an indexed family of quasi-categories in $\mathcal{qCat}_\emptyset$, and let $\prod_{i \in I} A_i$ denote their product in $\mathcal{qCat}$ with projection functors $\pi_i: \prod_{i \in I} A_i \rightarrow A_i$. It is a trivial matter to establish the properties discussed in the remark 3.4:

(i) Any choices of terminal object $t_i$ in $A_i$ for each $i \in I$ can be collected together to give a pointwise terminal $(t_i)_{i \in I}$ in $\prod_{i \in I} A_i$.

(ii) Suppose that $(t_i)_{i \in I}$ is pointwise terminal and that we have a lifting problem

\[
\begin{array}{c}
\partial \Delta^n \quad \xrightarrow{f} \quad \prod_{i \in I} A_i \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Proof. Given an isofibration $p: E \to B$ and a functor $f: A \to B$ in $q\text{Cat}_0$, we form their pullback

$$
\begin{array}{c}
E \times_B A \\
\downarrow \pi_0 \\
A
\end{array}
\quad
\begin{array}{c}
\downarrow p \\
B
\end{array}
\quad
\begin{array}{c}
\downarrow \pi_1 \\
\downarrow f
\end{array}
$$

in $q\text{Cat}$. We establish the conditions of remark [3.4] as follows:

(i) Given a terminal object $t \in A$, $ft$ is terminal in $B$, and we may apply lemma [3.8] to obtain a terminal object $s$ in $E$ with $ps = ft$. The pair $(s, t)$ is an object of $E \times_B A$ which is pointwise terminal by construction.

(ii) Suppose we are given a pointwise terminal $(s, t)$ in $E \times_B A$ and a lifting problem

$$\partial \Delta^n \xrightarrow{g} E \times_B A \xrightarrow{\pi_0} E \xrightarrow{\pi_1} A \xrightarrow{f} B$$

in which $g$ maps $[n]$ to $(s, t)$. To solve this problem consider the following diagram

$$\partial \Delta^n \xrightarrow{g} E \times_B A \xrightarrow{\pi_0} E \xrightarrow{\pi_1} A \xrightarrow{f} B$$

in which the dashed arrow $\bar{g}_1$ is a solution to the lifting problem $\pi_1 g: \partial \Delta^n \to A$; this solution exists because $\pi_1 g$ maps $[n]$ to $t$, which is terminal in $A$ by the assumption that $(s, t)$ is pointwise terminal.

The dashed solution $\bar{g}_0$ exists by application of lemma II.5.3.5, which applies since $p$ is a terminal object preserving isofibration by assumption and $\pi_0 g$ maps $[n]$ to the terminal object $s$ of $E$. The lower right triangle in this diagram tells us that $p\bar{g}_0 = f\bar{g}_1$, so the pair $(\bar{g}_0, \bar{g}_1)$ induces a map $\bar{g}: \Delta^n \to E \times_B A$ which provides a solution to the original lifting problem (3.12) as required. □

3.13. Lemma. The subcategory $q\text{Cat}_0$ is closed in $q\text{Cat}$ under countable composites of isofibrations.

Proof. Suppose that we are given a countable sequence

$$
\begin{array}{c}
P_i \\
\rightarrow A_i \\
P_{i+1} \\
\rightarrow A_{i+1} \\
\end{array}
\quad
\begin{array}{c}
P_2 \\
\rightarrow A_2 \\
P_1 \\
\rightarrow A_1 \\
P_0 \\
\rightarrow A_0 \\
\end{array}
$$

of quasi-categories and isofibrations in $q\text{Cat}_0$, and let $A$ be their limit in $q\text{Cat}$. We know that the projection functors $\pi_i: A \to A_i$ are isofibrations. Once again, we establish the conditions of remark [3.4]

(i) Our task is to build a family $(t_i)_{i \in \mathbb{N}}$ of terminal objects $t_i$ in $A_i$ which satisfies the compatibility property that $p_i t_{i+1} = t_i$ for all $i \in \mathbb{N}$. To do this we simply proceed inductively, picking any terminal object $t_0$ of $A_0$ to start things off and
then extending our family from level $i$ to level $i + 1$ by applying lemma 3.8 to the terminal object-preserving-isofibration $p_i: A_{i+1} \to A_i$ in order to find a $t_{i+1}$ which is terminal in $A_{i+1}$ and which has $p_i t_{i+1} = t_i$.

(ii) Suppose $t$ in $A$ is a pointwise terminal object projecting to the family $(t_i)_{i \in \mathbb{N}}$. To show that $t$ is terminal, we must solve any lifting problem

$$\partial \Delta^n \xrightarrow{f} A \xrightarrow{\pi_0} A_0$$

that maps the vertex $[n]$ to $t$. We construct a solution $\bar{f}$ by inductively defining its components. Start by using the fact that $t_0$ is terminal to find a lift

$$\partial \Delta^n \xrightarrow{f} A \xrightarrow{\pi_0} A_0$$

$$\Delta^n \xrightarrow{\bar{f}_0} A_0$$

Suppose now that we have defined maps $\bar{f}_i: \Delta^n \to A_i$ so that $p_i \bar{f}_{i+1} = \bar{f}_i$ for all $i < k$. By lemma II.5.3.5 and the hypothesis that $t_k$ is terminal, there is a solution to the lifting problem

$$\partial \Delta^n \xrightarrow{f} A \xrightarrow{\pi_k} A_k$$

$$\Delta^n \xrightarrow{\bar{f}_k} A_k$$

$$\Delta^n \xrightarrow{\bar{f}_{k-1}} A_{k-1}$$

$$\Delta^n \xrightarrow{\bar{f}} A$$

defining $\bar{f}_k$. The family $(\bar{f}_i)_{i \in \mathbb{N}}$ defines the desired solution $\bar{f}$ to (3.14). □

3.15. Lemma. The subcategory $q\text{Cat}_0$ is closed in $q\text{Cat}$ under splitting of idempotents.

Proof. Suppose that $e: A \to A$ is an idempotent in $q\text{Cat}_0$ and let $A^e$ denote the sub-quasi-category of $A$ stabilised by that idempotent—i.e., the equaliser of $e$ and $\text{id}_A$—whose single limit cone projection is the inclusion $A^e \hookrightarrow A$. We verify the conditions of remark 3.4 as follows:

(i) For any terminal object $t$ of $A$, the object $et$ of $A^e$ is terminal in $A$, since $e$ is assumed to preserve terminal objects. Thus $et$ is pointwise terminal in $A^e$.

(ii) Now suppose that $g: \partial \Delta^n \to A^e$ is a lifting problem which maps $[n]$ to an object $s$ which is pointwise terminal in $A^e$. This means that $s$ is terminal in $A$, which ensures the existence of the dashed solution in the following lifting problem:

$$\partial \Delta^n \xrightarrow{g} A^e$$

$$\Delta^n \xrightarrow{s} A$$
It is now easy to check that \( e\bar{g}: \Delta^n \to A^e \) is a solution to our original lifting problem defined by the sphere \( g: \partial\Delta^n \to A^e \).

We defer the proof that \( q\text{Cat}_{/\emptyset} \) is closed under cotensors by a simplicial set \( X \) to lemma 4.17, which proves a more general version of this statement. Modulo this step, which appears as corollary 4.20, we may now prove our desired result.

3.16. Theorem. \( q\text{Cat}_{/\emptyset,\infty} \) is closed in \( q\text{Cat}_{\infty} \) under limits weighted by projective cofibrant weights.

Proof. A projective cofibrant weighted limit may be expressed as a retract of the limit of a countable tower of pullbacks of products of maps defined by cotensoring an object in the diagram with a monomorphism of simplicial sets. Corollary 4.20 shows that these maps are terminal-object preserving isofibrations between quasi-categories admitting terminal objects. Lemmas 3.9, 3.11, 3.13, and 3.15 show that the rest of the diagram lies in \( q\text{Cat}_{/\emptyset} \) as well.

4. Absolute lifting diagrams via terminal objects

We now turn our attention to step (II), the reduction of the general case of theorem 1.1 to the special one of quasi-categories admitting and functors preserving terminal objects. First we must set up the precise result that we intend to prove.

Absolute right lifting diagrams and right exact transformations.

4.1. Remark. As in I.5.2.21, we let \( \downarrow \) denote the category \( c \to a \leftarrow b \) and write \( q\text{Cat}_{/\downarrow} \) for the simplicially enriched category of simplicial functors \( \downarrow \to q\text{Cat}_{\infty} \). Objects of \( q\text{Cat}_{/\downarrow} \) are pairs of maps

\[
\begin{array}{ccc}
B & \to & A \\
\downarrow^f & & \\
C & \to & A
\end{array}
\]

and 0-arrows are natural transformations

\[
\begin{array}{ccc}
B & \to & A & \to & B' \\
\downarrow^g & \Rightarrow & \downarrow^g & \Rightarrow & \downarrow^f \\
C & \to & A & \to & A'
\end{array}
\]

We say that a transformation of this kind is a pointwise isofibration if each of its components \( u, v, \) and \( w \) is an isofibration.

As with any enriched functor category, \( q\text{Cat}_{/\downarrow} \) inherits limits pointwise from \( q\text{Cat}_{\infty} \). In particular, it has limits weighted by projective cofibrant weights, which are then preserved by the simplicial projection functors \( P_a, P_b, P_c: q\text{Cat}_{/\downarrow} \to q\text{Cat}_{\infty} \) which evaluate at each of the objects \( a, b, \) and \( c \) of \( \downarrow \) respectively.
4.4. **Remark.** Our particular interest is in a subcategory of $\text{qCat}_\infty^{\text{uni}}$ whose objects are those diagrams \((4.2)\) in which \(g\) admits an absolute right lifting through \(f\)

\[
\begin{array}{c}
B \\
\downarrow \psi \lambda \\
C \rightarrow g \rightarrow A
\end{array}
\] (4.5)

in $\text{qCat}_\infty^{\text{uni}}$. Transformations of the kind depicted in \((4.3)\) between diagrams which satisfy this condition give rise to the following diagram

\[
\begin{array}{c}
B \\
\downarrow \psi \lambda \\
C \rightarrow g \rightarrow A \\
\downarrow \psi \lambda' \\
B' \rightarrow g' \rightarrow A'
\end{array}
\]

\[
\begin{array}{c}
B \\
\downarrow \psi \tau \\
C \rightarrow g \rightarrow A \\
\downarrow \psi \tau' \\
B' \rightarrow g' \rightarrow A'
\end{array}
\]

\[
\begin{array}{c}
B \\
\downarrow \psi \tau \\
C \rightarrow g \rightarrow A \\
\downarrow \psi \tau' \\
B' \rightarrow g' \rightarrow A'
\end{array}
\]

in which the triangles are absolute right liftings and the 2-cell \(\tau\) is induced by the universal property of the triangle on the right. We say that the transformation \((4.6)\) is right exact if and only if the induced 2-cell \(\tau\) is an isomorphism. This right exactness condition holds if and only if, in the diagram on the left, the whiskered 2-cell \(u\lambda\) displays \(v\ell\) as the absolute right lifting of \(g'w\) through \(f'\).

4.7. **Definition.** Let $\text{qCat}_\infty^{\text{uni} \downarrow}$ denote the simplicial subcategory of $\text{qCat}_\infty^{\text{uni}}$ with:

- **objects** those diagrams \((4.2)\) in which \(g\) admits an absolute right lifting through \(f\), and

- **\(n\)-arrows** those \(n\)-arrows of $\text{qCat}_\infty^{\text{uni} \downarrow}$ whose vertices are right exact transformations.

As usual, we let $\text{qCat}_\infty^{\text{uni} \downarrow}$ denote the underlying category of $\text{qCat}_\infty^{\text{uni} \downarrow}$.

4.8. **Observation.** To check that this definition makes sense, we need to show that the composite of a pair of right exact transformations is itself right exact. This follows from the following calculation

\[
\begin{array}{c}
B \\
\downarrow \psi \lambda \\
C \rightarrow g \rightarrow A \\
\downarrow \psi \lambda' \\
B' \\
\downarrow \psi \tau \\
C' \rightarrow g \rightarrow A' \\
\downarrow \psi \tau' \\
B'' \\
\downarrow \psi \lambda'' \\
C'' \rightarrow g \rightarrow A''
\end{array}
\]

\[
\begin{array}{c}
B \\
\downarrow \psi \lambda \\
C \rightarrow g \rightarrow A \\
\downarrow \psi \lambda' \\
B' \\
\downarrow \psi \tau \\
C' \rightarrow g \rightarrow A' \\
\downarrow \psi \tau' \\
B'' \\
\downarrow \psi \lambda'' \\
C'' \rightarrow g \rightarrow A''
\end{array}
\]

which shows that the 2-cell induced by the composite of a pair of transformations may be computed as the pasted composite, shown on the right, of the 2-cells induced by each
individual transformation. Now if those induced 2-cells are isomorphisms then so is their pasted composite, from which our desired composition result follows.

**Projective cofibrant weighted limits in** $\mathcal{qCat}_{\infty}$. Our aim is now to prove:

**4.9. Proposition.** The quasi-categorically enriched subcategory $\mathcal{qCat}_{\infty}$ is closed in $\mathcal{qCat}_{\infty}$ under limits weighted by projective cofibrant weights.

Before embarking on its proof, let us explain how it will provide us with a proof of theorem 1.1.

**Proof of theorem 1.1.** For any fixed simplicial set $X$, there exists a simplicially enriched functor $F_X : \mathcal{qCat}_{\infty} \to \mathcal{qCat}_{\infty}$ which carries a quasi-category $A$ to the diagram

$$
\begin{array}{ccc}
A & \to & A^X \\
\downarrow^c & & \downarrow^\text{id} \\
A^X & \to & A^X 
\end{array}
$$

and carries a functor $f : A \to B$ to the transformation:

$$
\begin{array}{ccc}
A & \to & B \\
\downarrow^c & & \downarrow^\text{id} \\
A^X & \to & B^X
\end{array}
$$

By definition I.5.2.8, the quasi-category $A$ possesses limits of shape $X$ if and only if (4.10) admits an absolute right lifting. Moreover, a functor $f : A \to B$ preserves limits of shape $X$ if and only if (4.11) is right exact. In other words, the simplicial category $\mathcal{qCat}_{\infty}$ of definition 3.1 is the inverse image of the simplicial subcategory $\mathcal{qCat}_{\infty}$ along the simplicial functor $F_X : \mathcal{qCat}_{\infty} \to \mathcal{qCat}_{\infty}$.

Now $\mathcal{qCat}_{\infty}$ inherits limits pointwise from $\mathcal{qCat}_{\infty}$ and the exponentiation functor $(-)^X$ is right adjoint, so we know that $F_X$ preserves those limits under which $\mathcal{qCat}_{\infty} \subset \mathcal{Sset}$ is closed. In particular, it preserves all limits weighted by projective cofibrant weights. It is a general and easily demonstrated fact that if $\mathcal{A}$ and $\mathcal{B}$ are enriched categories admitting a certain class of limits, $F : \mathcal{A} \to \mathcal{B}$ is an enriched functor preserving those limits, and $\mathcal{B}'$ is an enriched subcategory of $\mathcal{B}$ which is closed under those limits, then the enriched inverse image subcategory $\mathcal{A}' := F^{-1}(\mathcal{B}')$ is also closed in $\mathcal{A}$ under those limits.

In this case, proposition 4.9 tells us that the simplicial subcategory $\mathcal{qCat}_{\infty}$ is closed in $\mathcal{qCat}_{\infty}$ under weighted by projective cofibrant weights, and $F_X : \mathcal{qCat}_{\infty} \to \mathcal{qCat}_{\infty}$ preserves these limits. So it follows that the inverse image $\mathcal{qCat}_{\infty}$ of $\mathcal{qCat}_{\infty}$ along $F_X$ is closed in $\mathcal{qCat}_{\infty}$ under projective cofibrant weighted limits.

Now to prove proposition 4.9 we will characterise the objects and 0-arrows of $\mathcal{qCat}_{\infty}$ in terms of the possession and preservation of terminal objects in certain comma quasi-categories.
4.12. Observation. As described in example II.3.1.9, the comma quasi-category construction is one example of a limit weighted by a projective cofibrant weight \( W: \downarrow \to \text{sSet} \) so, in particular, it gives rise to a simplicial functor \( \downarrow = \{ W, - \}: \qCat_\infty \to \qCat_\infty \). Weighted limits commute with weighted limits so it follows that this comma construction preserves all limits that are constructed pointwise in \( \qCat_\infty \), including, in particular, those with projective cofibrant weights.

4.13. Observation (a pointwise characterisation of right exactness). Theorem I.6.1.3 tells us that the functor \( g \) in the diagram (4.2) admits an absolute right lifting through \( f \) if and only if the comma quasi-category \( f \downarrow gc \) has a terminal object for all objects \( c \) of \( C \). A companion result, corollary I.6.1.7, tells us that a triangle (4.5) is an absolute right lifting diagram if and only if each triangle obtained by restricting along \( c: \Delta^0 \to C \) defines an absolute right lifting diagram.

Suppose that (4.3) is a transformation of diagrams which admit absolute right liftings. Then we may pick any absolute right lifting (4.5) of its domain and apply corollary I.6.1.7 to show that for all objects \( c \) of \( C \) the 2-cell \( \lambda c: f \ell c \Rightarrow gc \) induces an object which is terminal in \( f \downarrow gc \). Applying that same result to the composite triangle on the left of (4.6) we find that (4.3) is right exact transformation if and only if for all objects \( c \) in \( C \) the 2-cell \( u \lambda c: f' \ell c = uf \ell c \Rightarrow ugc = g' wc \) induces an object which is terminal in \( f' \downarrow g' wc \).

Now observe that for each object \( c \) of \( C \) the transformation (4.3) restricts to define a transformation \( B \Delta^0 \to \downarrow \to A \) \( \downarrow \to B' \) \( A' \) which is mapped to a functor \( v \downarrow \text{id}: f \downarrow gc \to f' \downarrow g' wc \) under the comma construction. This functor maps any object of \( f \downarrow gc \) which is induced by \( \lambda c: f \ell c \Rightarrow gc \) to an object of \( f' \downarrow g' wc \) which is induced by \( u \lambda c: f' \ell c \Rightarrow g' wc \).

Combining and summarising these facts we get the following lemma.

4.14. Lemma. The diagram \( C \xrightarrow{g} A \xleftarrow{f} B \) is in the subcategory \( \qCat_\infty \), if and only if for all objects \( c \) of \( C \) the comma quasi-category \( f \downarrow gc \) has a terminal object. A transformation (4.3) between two such diagrams is in the subcategory \( \qCat_\infty \) if and only if for all objects \( c \) in \( C \) the induced functor \( v \downarrow \text{id}: f \downarrow gc \to f' \downarrow g' wc \) preserves terminal objects.

Proof. The first part is the content of theorem I.6.1.3. For the second part, fix an absolute right lifting as in (4.5) and an object \( c \) of \( C \). By corollary I.6.1.7, we know that an object induced by \( \lambda c: f \ell c \Rightarrow gc \) is terminal in \( f \downarrow gc \). The functor \( v \downarrow \text{id}: f \downarrow gc \to f' \downarrow g' wc \) carries this terminal object to an object induced by \( u \lambda c: f' \ell c \Rightarrow g' wc \) in \( f' \downarrow g' wc \). Hence, it preserves terminal objects if and only if any object induced by the 2-cell \( u \lambda c \) is terminal in \( f' \downarrow g' wc \). As discussed in observation 4.13, this latter condition holds for all objects \( c \) in \( C \) if and only if the transformation in (4.3) is right exact, as required. \( \square \)
The characterisation of the 0-arrows of $\mathbf{qCat}_{\infty}$ provided by lemma 4.14 enables the proof of the following result:

4.15. Lemma. The subcategory $\mathbf{qCat}_{\infty}$ is closed in $\mathbf{qCat}$ under arbitrary small products, pullbacks of pointwise isofibrations, countable composites of pointwise isofibrations, and splitting of idempotents.

Proof. Suppose that we are given a diagram $D : A \to \mathbf{qCat}_{\infty}$ of one of the kinds described in the statement. We shall adopt the notation

$$
\begin{array}{ccc}
B^i & \xrightarrow{f^i} & C^i \\
\downarrow & & \downarrow g^i \\
A^i & \rightarrow & \Delta^0
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
\pi^a_{i,j} & \xrightarrow{g^i} & \pi^b_{i,j} \\
\downarrow w^i & & \downarrow v^i \\
\Delta^0 & \rightarrow & \Delta^0
\end{array}
$$

for the diagram and transformation obtained by evaluating $D : A \to \mathbf{qCat}_{\infty}$ at an object $i$ and at an arrow $\phi : i \to j$ of $A$ respectively. Of course, the diagram $D$ possesses a pointwise limit in $\mathbf{qCat}_{\infty}$ and we shall adopt the notation

$$
\begin{array}{ccc}
limit_{i \in A} B^i & \xrightarrow{f} & \lim_i B^i \\
\downarrow & & \downarrow f^i \\
\lim_i A^i & \rightarrow & \lim_i A^i
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
limit_{i \in A} C^i & \xrightarrow{g} & \lim_i A^i \\
\downarrow & & \downarrow f^i \\
\lim_i C^i & \rightarrow & \lim_i C^i
\end{array}
$$

for the component of its limit cone at the object $i$ of $A$.

Given an object $c$ of $\lim_{i \in A} C^i$ we may apply the limit projection at the object $i$ in $A$ to obtain an object $c^i := \pi^{c,i} c$ of $C^i$. The limit projections commute with the arrows in the diagram $D$, so if $\phi : i \to j$ is an arrow in $A$ then we know that $w^i c^i = w^i \pi^{c,j} c = \pi^{c,j} c = c^j$. Consequently we have arrows as displayed on the left-hand side

$$
\begin{array}{ccc}
\Delta^0 & \xrightarrow{g^i c^i} & A^i \\
\downarrow & & \downarrow \pi^{i,j} \\
\Delta^0 & \rightarrow & \Delta^0
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
\Delta^0 & \xrightarrow{g^i} & \lim_i A^i \\
\downarrow & & \downarrow f^i \\
\Delta^0 & \rightarrow & \Delta^0
\end{array}
$$

that assemble to give a diagram $D_c : A \to \mathbf{qCat}_{\infty}$ and a cone over that diagram, as displayed on the right. As limits in $\mathbf{qCat}_{\infty}$ are constructed pointwise in $\mathbf{qCat}_{\infty}$, this cone is a limit cone. Because the limit of any diagram of terminal objects is a terminal object, the object in the lower-left-hand corner of the limit diagram is $\Delta^0$. 

(4.16)
Applying the comma construction \( \downarrow: q\text{Cat} \downarrow \rightarrow q\text{Cat} \), we obtain a diagram \( A \rightarrow q\text{Cat} \) which maps each object \( i \) of \( A \) to \( f^i \downarrow g^i c^i \) and maps an arrow \( \phi: i ightarrow j \) of \( A \) to the induced functor \( v^\phi \downarrow_{v^\phi}^j \): \( f^i \downarrow g^i c^i \rightarrow f^j \downarrow g^j c^j \). The comma construction, as a weighted limit, preserves limits so it maps the limit cone \([4.16]\) to a limit cone in \( q\text{Cat} \), which displays \( f \downarrow gc \) as a limit of the diagram \( A \rightarrow q\text{Cat} \).

Now \( D \) is a diagram which lands in \( q\text{Cat} \), mapping objects to diagrams which admit an absolute right lifting and arrows to right exact transformations. So lemma \([4.14]\) tells us that for each object \( i \) of \( A \) the quasi-category \( f^i \downarrow g^i c^i \) has a terminal object and that for each arrow \( \phi: i ightarrow j \) of \( A \) the functor \( v^\phi \downarrow_{v^\phi}^j \): \( f^i \downarrow g^i c^i \rightarrow f^j \downarrow g^j c^j \) preserves terminal objects. Furthermore, if an arrow \( \phi: i ightarrow j \) of \( A \) is mapped to a pointwise isofibration under \( D \) then it is also mapped to a pointwise isofibration under \( D_c \). Observation I.3.3.18 demonstrates that the comma quasi-category functor carries pointwise isofibrations to isofibrations, so it follows that the functor \( v^\phi \downarrow_{v^\phi}^j \): \( f^i \downarrow g^i c^i \rightarrow f^j \downarrow g^j c^j \) is an isofibration. Summarising these facts we find that the composite functor \( A \overset{D}{\rightarrow} q\text{Cat} \downarrow \rightarrow q\text{Cat} \) lands in the subcategory \( q\text{Cat}_g \) and parametrises one of the conical limits listed in observation \([3.3]\).

Now depending on the particular limit involved, one of lemmas \([3.9]\), \([3.11]\), \([3.13]\), or \([3.15]\) applies to show that the limit \( f \downarrow gc \) of the diagram \( \downarrow \circ D_c \) possesses a terminal object and that each of the limit cone projections \( \pi^{b,i} \downarrow_{\pi^{b,i}} \): \( f \downarrow gc \rightarrow f^i \downarrow g^i c^i \) preserves terminal objects. Consequently, lemma \([4.14]\) implies that the limit of the diagram \( D \) and each of its limit cone projections are in \( q\text{Cat}_g \).

Finally, if the transformation on the left of the following display

![Diagram](image)

is induced by the cone of right exact transformations on the right, we must show that it too is right exact. To that end, suppose that \( c' \) is an object of \( C' \) and recall from the earlier argument that \( f \downarrow gwc' \) is a limit of the diagram \( \downarrow \circ D_{wc'}: A \rightarrow q\text{Cat}_g \). Furthermore, it is clear that the functor \( v \downarrow_{u} \): \( f^i \downarrow g^i c^i \rightarrow f \downarrow gwc' \) is induced by applying the universal property of that limit to the cone of functors \( v^i \downarrow_{v^i} \): \( f^i \downarrow g^i c^i \rightarrow f^i \downarrow g^i w^i c' \). However, each of those latter functors preserves terminal objects by lemma \([4.14]\) because they are induced by right exact transformations. Hence, by appeal to an appropriate one of the closure lemmas cited above, this induced functor \( v \downarrow_{u} \): \( f^i \downarrow g^i c^i \rightarrow f \downarrow gwc' \) also preserves terminal objects. To conclude, we apply lemma \([4.14]\) to show that the induced transformation in the display above is indeed right exact, as required.

We complete the proof of proposition \([4.9]\) with the following lemma:
4.17. **Lemma.** The simplicial subcategory $\mathbf{qCat}_{\infty}^{\downarrow}$ is closed in $\mathbf{qCat}_{\infty}^{\downarrow}$ under cotensoring by an arbitrary simplicial set $X$. Moreover, the transformation induced by cotensoring with a monomorphism $X \hookrightarrow Y$ is a right exact pointwise isofibration.

**Proof.** Suppose that the diagram (4.2) is in $\mathbf{qCat}_{\infty}^{\downarrow}$ and has the absolute right lifting (4.5). Then its cotensor by $X$ in $\mathbf{qCat}_{\infty}^{\downarrow}$ is determined pointwise, viz

\[
\begin{array}{ccc}
B^X & \xrightarrow{f^X} & A^X \\
\downarrow & & \downarrow \\
C^X & \xrightarrow{g^X} & A^X
\end{array}
\]

and it is a matter of a routine argument using properties of the 2-adjunction $\mathbf{−} \times X \dashv (\mathbf{−})^X$ on the 2-category $\mathbf{qCat}_2$, as discussed in observation I.5.2.14, to show that the triangle

\[
\begin{array}{ccc}
B^X & \xrightarrow{f^X} & A^X \\
\downarrow & & \downarrow \\
C^X & \xrightarrow{g^X} & A^X
\end{array}
\]

is also an absolute right lifting. This shows that the cotensor (4.18) is an object of $\mathbf{qCat}_{\infty}^{\downarrow}$.

The projection transformations of this cotensor are indexed by the vertices $x$ of $X$, with the projection given by evaluation at that vertex (precomposition by $x: \Delta^0 \to X$). The 2-functoriality properties of cotensoring imply that the following pasting equality holds:

\[
\begin{array}{ccc}
& & B^X \\
& & \downarrow \\
& \phi_{\lambda^X} & \downarrow \\
\downarrow & & \downarrow \\
& \downarrow & & \downarrow \lambda^X \\
C^X & \xrightarrow{g^X} & A^X & \Rightarrow & B^X \\
\downarrow & & \downarrow \\
C^X & \xrightarrow{g^X} & A^X
\end{array}
\]

This demonstrates that the comparison 2-cell induced as in (4.6) is an identity and thus that the projection indexed by $x$ is right exact.

All that remains is to show that if we are given a transformation

\[
\begin{array}{ccc}
B' & \xrightarrow{f'} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{v} & A \\
\downarrow & & \downarrow v \\
C & \xrightarrow{g^X} & A^X \\
\downarrow & & \downarrow \\
C^X & \xrightarrow{g^X} & A^X
\end{array}
\]

whose composite with each of the projection transformations discussed above is right exact then that transformation itself is right exact. Write $\tau: v\ell \Rightarrow \ell^X w$ for the comparison 2-cell induced as in (4.6) from (4.19). Observation 4.8 tells us that the comparison 2-cell
associated with the composite of (4.19) with the projection indexed by a vertex \( x \) of \( X \) is given by the pasting:

\[
\begin{array}{ccc}
B & \xrightarrow{\ell} & B^X \\
\downarrow \psi \tau & & \downarrow \ell^X \\
C & \xrightarrow{w} & C^X
\end{array}
\]

Consequently the hypothesis that the transformation (4.19) composes with each projection to give a right exact transformation reduces to the equivalent condition that the whiskered 2-cell

\[
\begin{array}{ccc}
C & \xrightarrow{\psi \tau} & B^X \\
\downarrow \ell^X & & \downarrow \ell \\
C^X & \xrightarrow{w} & C
\end{array}
\]

is an isomorphism for all vertices \( x \) in \( X \). Taking transposes under the 2-adjunctions \(- \times X \dashv (\dashv (-)^X\) and \(- \times C \dashv (\dashv (-)^C\) on the 2-category of all simplicial sets (discussed in I.3.2.5), this 2-cell corresponds to the 2-cell

\[
\Delta^0 \xrightarrow{x} X \xrightarrow{\psi \hat{\tau}} BC
\]

Now these transpositions preserve the isomorphism property of 2-cells, so this latter 2-cell is an isomorphism for all vertices \( x \) of \( X \). It follows, by observation I.3.2.3, that \( \hat{\tau} \) is an isomorphism. Consequently, its transpose \( \tau \) is also an isomorphism, which completes our proof that the transformation (4.19) is right exact as required.

4.20. **Corollary.** The category \( q\text{Cat}_{\infty} \) is closed in \( q\text{Cat}_{\infty} \) under cotensoring by an arbitrary simplicial set \( X \). Moreover, the cotensor with an inclusion \( X \hookrightarrow Y \) induces a terminal-object preserving isofibration of quasi-categories.

**Proof.** Specialising lemma 4.17 to a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{1} & A \\
\downarrow & & \downarrow \\
1 & = & 1
\end{array}
\]
in \(q\text{Cat}_r\) constructs cotensors for a quasi-category \(A\) with a terminal object. The top right-hand component of the right exact pointwise isofibration induced by cotensoring with an inclusion \(X \hookrightarrow Y\) is a terminal object preserving isofibration of quasi-categories. □

We now complete the proof of proposition 4.9 and, in doing so, finish proving theorem 1.1.

**Proof of proposition 4.9.** Lemmas 4.15 and 4.17 demonstrate that \(q\text{Cat}_r\) is closed in \(q\text{Cat}_\infty\) under all of the limit types used in the proof of proposition II.3.2.4. Consequently, it follows, by the argument given in section 2, that \(q\text{Cat}_r\) is closed in \(q\text{Cat}_\infty\) under all limits weighted by projective cofibrant weights. □

5. Limits and colimits in the quasi-category of algebras

5.1. **Recall** (homotopy coherent monads and their algebras). Recall, from §II.6, that \(\text{Mnd}\) denotes the full simplicial subcategory of the free homotopy coherent adjunction \(\text{Adj}\) determined by its object \(+\). Its single hom-space \(\text{Mnd}(+,-)\) is the category \(\Delta_+\), which we think of as a simplicial set by identifying it with its nerve (as always). The endo-composition on this hom-space is the ordinal sum functor \(\oplus:\Delta_+ \times \Delta_+ \rightarrow \Delta_+\) and its identity is the object \([-1]\). Consequently, any simplicial functor \(X: \text{Mnd} \rightarrow s\text{Set}\) may be specified by giving a simplicial set \(X\), the image of the object \(+ \in \text{Mnd}\), equipped with a left action \(\cdot:\Delta_+ \times X \rightarrow X\) of the simplicial monoid \((\Delta_+,\oplus,[-1])\). Natural transformations of such are simply equivariant maps, that is simplicial maps which preserve the left action of \(\Delta_+\).

A *homotopy coherent monad* on a quasi-category \(A\) is a simplicial functor \(T: \text{Mnd} \rightarrow q\text{Cat}_\infty\) which maps the object \(+\) to \(A\). Equivalently, such structures correspond to left actions \(\cdot:\Delta_+ \times A \rightarrow A\) of the simplicial monoid \((\Delta_+,\oplus,[-1])\) on \(A\). The endomorphism \(t := [0] \cdot -: A \rightarrow A\) descends to a monad on the homotopy category of \(A\).

The *quasi-category of algebras* \(A[t]\) for a homotopy coherent monad is defined to be the limit of \(T: \text{Mnd} \rightarrow q\text{Cat}_\infty\) weighted by the simplicial functor \(W_-: \text{Mnd} \rightarrow s\text{Set}\) which maps \(+\) to \(\Delta_\infty\) and whose left action by \(\Delta_+\) is again the ordinal sum operation \(\oplus:\Delta_+ \times \Delta_\infty \rightarrow \Delta_\infty\). That this construction delivers us a quasi-category of algebras, rather than just a mere simplicial set of such, is a consequence of the fact that the weight \(W_-\) is actually a projective cell complex (see lemma II.6.1.9). This quasi-category of algebras comes equipped with a forgetful functor \(u^t: A[t] \rightarrow A\), which is the component of its defining limit cone at \([0] \in \Delta_\infty\) (see definition II.6.1.15).

5.2. **Observation.** The weighted limit construction \(\{W_-, -\}: q\text{Cat}_\text{Mnd} \rightarrow q\text{Cat}_\infty\) is functorial, so it maps a natural transformation \(f: A \rightarrow B\) of homotopy coherent monads to a functor \(f[t]: A[t] \rightarrow B[t]\) of quasi-categories of algebras. The family of underlying functors are natural with respect to this simplicial functor structure, so in particular the following square

\[
\begin{array}{ccc}
A[t] & \xrightarrow{f[t]} & B[t] \\
\downarrow{u^t} & & \downarrow{u^t} \\
A & \xrightarrow{f} & B
\end{array}
\]

(5.3)
commutes. The simplicial category of homotopy coherent monads \( \text{qCat}^\text{Mnd} \) possesses all limits weighted by projective cofibrant weights, because it is a functor category. These are computed as in \( \text{qCat}^{\infty} \), and the quasi-category of algebras simplicial functor \(-[t] := \{W_-, -\} : \text{qCat}_\text{Mnd}^{\infty} \to \text{qCat}_\infty \) preserves them, because limits commute.

5.4. Observation. The defining universal property of weighted limits tells us that simplicial maps \( h : X \to A[t] \) correspond to equivariant maps \( \bar{h} : X \times W_- \to A \), where \( X \times W_- \) denotes the tensor of \( W_- \) by the simplicial set \( X \). These corresponding maps are related by the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{c}
\text{id}_X \times [0] \\
X \times W_-
\end{array}
\end{array}
\begin{array}{c}
\text{id}_X \times [0] \\
\begin{array}{c}
X \times W_- \\
\downarrow \text{id}_X \times [0]
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A[t] \\
\downarrow u^t
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A \\
\downarrow \bar{h}
\end{array}
\end{array}
\]

(5.5)

This result also provides an explicit description of the action of the functor \( f[t] : A[t] \to B[t] \) constructed from a natural transformation \( f : A \to B \) of homotopy coherent monads. This follows by composing the square above with the commutative square in (5.3), which reveals that the composite \( f[t]h : X \to B[t] \) corresponds to the equivariant map \( f\bar{a} : \Delta^n \times W_- \to B \).

5.6. Observation. Every 0-arrow of \( \text{Mnd} \) is an iterated (possibly empty) composite of the 0-arrow \([0] \in \text{Mnd}(+, +)\), so it follows that all of the functors in the image of a homotopy coherent monad \( T : \text{Mnd} \to \text{qCat}_\infty \) preserve some limit or colimit in the quasi-category \( A \) if and only if the single functor \( t : A \to A \) of \( T \) preserves that limit or colimit. Incidentally, we call \( t : A \to A \) the functor part of the homotopy coherent monad \( T \).

Now theorem 1.1 admits the following immediate corollary:

5.7. Corollary. Let \( T : \text{Mnd} \to \text{qCat}_\infty \) be a homotopy coherent monad on a quasi-category \( A \) and let \( X \) be a simplicial set. Suppose also that \( A \) admits and the functor part \( t : A \to A \) of \( T \) preserves all \( (co)\)limits of shape \( X \). Then the forgetful functor \( u^t : A[t] \to A \) creates all \((co)\)limits of shape \( X \). \( \square \)

In the case of colimits, this is a direct analogue of the usual categorical result. However, in the case of limits we expect the monadic forgetful functor to create all limits that \( A \) admits, regardless of whether or not they are preserved by the functor part of the monad. Our aim in this section is to prove the analogous quasi-categorical result, viz:

5.8. Theorem. Let \( T : \text{Mnd} \to \text{qCat}_\infty \) define a homotopy coherent monad on a quasi-category \( A \). Then \( u^t : A[t] \to A \) creates any limits that \( A \) admits.

Our proof of this result, which takes up the remainder of this work, parallels that of theorem 1.1: we first dispense with the special case of terminal objects and then extend this result to the general case of limits of any shape.
Creation of terminal objects. For the remainder of this paper we fix a homotopy coherent monad $T : \text{Mnd} \to \text{qCat}_\infty$ on a quasi-category $A$.

5.9. Observation. We let $W_+ : \text{Mnd} \to \text{sSet}$ denote the unique representable weight, which maps $+$ to $\Delta_+$ and whose left action by $\Delta_+$ is the ordinal sum $\oplus : \Delta_+ \times \Delta_+ \to \Delta_+$. Applying Yoneda’s lemma, in the form given in example II.3.1.5, we find that the weighted limit $\{W_+, T\}$ is isomorphic to $A$. Indeed the proof of corollary II.6.2.3 tells us that if $W_+ \hookrightarrow W_-$ is the natural transformation which corresponds, under Yoneda’s lemma, to the vertex $[0]$ of $W_- (+) = \Delta_\infty$ then it induces a map of weighted limits $\{W_-, T\} \to \{W_+, T\}$ which is isomorphic to the functor $u^t : A[t] \to A$. It is also worth recalling, from remark II.6.2.4, that we may describe the inclusion $W_+ \hookrightarrow W_-$ more explicitly as the equivariant map $- \oplus [0] : \Delta_+ \hookrightarrow \Delta_\infty$.

Now the proof of corollary II.6.2.3 also reveals that this inclusion $W_+ \hookrightarrow W_-$ is a projective cell complex. In other words, it may be expressed as a transfinite (indeed countable) composite of pushouts of projective cells $\partial \Delta^m \times W_+ \hookrightarrow \Delta^m \times W_+$. In particular it follows, from proposition II.3.2.2, that the functor $u^t : A[t] \to A$ is an isofibration of quasi-categories.

Our aim in this section is to prove the following result:

5.10. Proposition. The functor $u^t : A[t] \to A$ creates terminal objects. Explicitly, we show that if $a$ is a terminal object in $A$ then any commutative diagram

$$
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{v} & A[t] \\
\downarrow & & \downarrow \\
\Delta^0 & \xrightarrow{w} & A
\end{array}
$$

(5.11)

of solid arrows admits the dashed diagonal filler shown.

5.12. Remark. The $n = 0$ case of the lifting property asserted by proposition [5.10] implies that terminal vertices lift along $u^t$. The $n \geq 1$ cases then allow us to apply lemma [3.7] to show that any such lifted vertex is terminal in $A[t]$.

Before proving this result, observe that it immediately entails the following corollary:

5.13. Corollary. Suppose that $f : A \to B$ is a natural transformation of monads, that the quasi-categories $A$ and $B$ possess terminal objects, and that the functor $f$ preserves terminal objects. Then the induced functor $f[t] : A[t] \to B[t]$ preserves the terminal objects that are guaranteed by proposition [5.10].

Proof. This is an easy argument involving the commutative square (5.3) and the fact that, by proposition [5.10], the underlying functors there create (and thus both preserve and reflect) terminal objects. □

5.14. Observation. Examining the proofs of corollary II.6.2.3 and proposition II.3.3.3 in greater detail, we obtain an explicit presentation of $W_-$ as a colimit of a countable sequence
of pushouts of projective cells. Each one of those cells corresponds to an atomic and non-degenerate arrow of the collage $\text{coll } W_-$ which is not in the image of the inclusion $\text{coll } W_+ \hookrightarrow \text{coll } W_-$. Lemma II.6.1.9 reminds us that $\text{coll } W_-$ may be identified with the subcategory of the homotopy coherent adjunction category $\mathbf{Adj}$ consisting of those arrows with codomain $+$, so we may describe the arrows identified in the last sentence using the graphical calculus description of $\mathbf{Adj}$. Specifically they are those non-degenerate strictly undulating squiggles with domain $-$ and codomain $+$ whose dimensions are greater than $0$ and which do not contain the symbol $+$ in their interiors.

The atomic arrows discussed in the last paragraph may be indexed as a countable sequence $\{a_i\}_{i \in \mathbb{N}}$ in such a way that each arrow is preceded by all of those of strictly smaller width (see definition II.4.1.2) and by those of the same width but smaller dimension. Then we can define $W_k$ to be the smallest sub-weight of $W_-$ which contains the atomic $0$-arrow $u \in \mathbf{Adj}(-, +)$ along with all of the arrows $a_i$ with $i < k$. It is then straightforward to see that $W_0$ is the image of $W_+ \hookrightarrow W_-$ and that $W_- = \bigcup_{k \in \mathbb{N}} W_k$. Furthermore, we selected the order of our sequence in a way that ensures that the simplicial boundary of the $m$-arrow $a_k$ is contained within $W_k$. Consequently, it is a straightforward matter to check that $W_{k+1}$ may be constructed from $W_k$ as a pushout

$$\begin{align*}
\partial \Delta^m \times W_+ & \hookrightarrow \Delta^m \times W_+ \\
W_k & \subseteq \Delta^m \times W_+ \\
\Delta^m \times W_+ & \rightarrow W_{k+1}
\end{align*}$$

in which the map $\Delta^m \times W_+ \rightarrow W_{k+1}$ on the right corresponds to the $m$-simplex $a_k \in W_k$ by Yoneda’s lemma.

5.16. Observation (The key fact). Observe that $u$ is the final vertex of each arrow $a_k$. This follows because $a_k$ has $+$ at its left-hand end, $-$ and its right-hand end, and does not contain $+$ in its interior, so when drawn as a squiggle it must have the following form

in which it crosses the lowest line exactly once, with that crossing situated at the far left of the squiggle. From this description of the shape of $a_k$ it is now easy to apply observation II.4.1.10 to show that its final vertex is $u$ as stated. This tells us that the inclusion $W_+ \hookrightarrow W_k$ factors through the attaching map of (5.15) via the inclusion $[m] : \Delta^0 \hookrightarrow \partial \Delta^m$. 


of the final vertex:

$$\partial \Delta^m \times W_+ \quad \text{(5.17)}$$

We should point out that this fact is absolutely key to our proof that no assumptions are required of $t: A \to A$ in order for the quasi-category of algebras $A[t]$ to inherit limits from $A$. To build our intuition for why this is the case we consider a simpler sub-problem, that of showing that a terminal object $a$ of $A$ admits a canonical $T$-algebra structure. We know that $u^*: A[t] \to A$ is the functor induced on weighted limits by the inclusion $W_+ \hookrightarrow W_-$, so this problem reduces to constructing the dashed lift in the following diagram

$$W_+ \xrightarrow{a} A \quad \text{of equivariant maps.}$$

Here the horizontal map is the one which corresponds to the terminal object $a$ of $A$ by Yoneda’s lemma, and which therefore maps the representing element $u$ to that object. We construct this extension inductively, filling a simplex boundary at each step to find a simplex in $A$ to which we might map the next member of our sequence $\{a_k\}_{k \in \mathbb{N}}$ of atomic and non-degenerate arrows; closing up at each step under the action of $\Delta_+$. It is the result of the last paragraph which ensures that we can find these fillers at each step: it tells us that $u$ is the terminal vertex of each $a_k$ and, thus, that its boundary maps to a sphere in $A$ whose terminal vertex is the terminal object $a$, this being the image of $u$ under the original map we are extending. Consequently, we can use the universal property of $a$ to find the filler we seek.

Notice that the same argument does not deliver us a side-condition-free result which supplies canonical algebra structures for initial objects. Such a result would require us to know that the initial vertex of each $a_k$ must map to an initial object of $A$ under the extension we are building. However, all we know is that these map to some object obtained by applying an iterated composite of the functor part $t: A \to A$ of our monad to that initial object. This is not, in general, good enough for our purposes here, unless we happen to know that $t$ preserves initial objects.

The results discussed in observations 5.14 and 5.16 provide all of the tools that we need to prove the lifting result described in the statement of proposition 5.10. Our argument directly generalises that given at the end of the last observation, by giving an inductive construction of a lift along the tower of isofibrations induced by applying the weighted limit functor $\{−, T\}$ to the relative projective cell decomposition of $W_+ \hookrightarrow W_-$. 

**Proof of proposition 5.10** The contravariant weighted limit functor $\{−, T\}$ carries colimits in $\mathcal{S}et^{\text{Mnd}}$ to limits in $\mathcal{S}et$. So it carries the sequence $W_0 \hookrightarrow W_1 \hookrightarrow \cdots \hookrightarrow W_k \hookrightarrow \cdots$ of observation 5.14 to a tower of functors

$$A[t] \cong \{W_-, T\} \to \cdots \to \{W_{k+1}, T\} \to \{W_k, T\} \to \cdots \to \{W_0, T\} \cong A. \quad \text{(5.18)}$$
whose countable composite is $u^t: A[t] \to A$. Furthermore, it carries each pushout (5.15) to a pullback
\[
\begin{align*}
\{W_{k+1}, T\} & \longrightarrow \{\Delta^m \times W_+, T\} \cong A^\Delta^m \\
\downarrow & \\
\{W_k, T\} & \longrightarrow \{\partial\Delta^m \times W_+, T\} \cong A^{\partial\Delta^m}
\end{align*}
\] (5.19)
which describes the $k^{th}$ step in the tower decomposition of $u^t$ as a pullback of an isofibration $A^\Delta^m \to A^{\partial\Delta^m}$. In particular, this shows that each functor in that tower is an isofibration and thus that its countable composite $u^t$ is also an isofibration, as asserted above.

Now suppose we are given a terminal object $a \in A$ and a lifting problem (5.11). We construct a diagonal filler by inductively lifting along the tower of fibrations (5.18). By the inductive hypothesis, we may assume that we have already constructed a lift $l_k: \Delta^n \to \{W_k, T\}$, which features in the following commutative diagram
\[
\begin{align*}
\Delta^0 & \longrightarrow \Delta^n \longrightarrow A \\
\downarrow^a & \downarrow^w & \downarrow^\text{ev}_{[m]} \\
\partial\Delta^n & \longrightarrow \{W_{k+1}, T\} & \longrightarrow A^\Delta^m \\
\downarrow^{l_{k+1}} & \downarrow & \downarrow \\
\Delta^n & \longrightarrow \{W_k, T\} & \longrightarrow A^{\partial\Delta^m}
\end{align*}
\] (5.20)
in which the right hand square is the pullback displayed in (5.19) and the lower right-hand commutative triangle is derived by applying $\{-, T\}$ to the commutative triangle (5.17).

The inductive step of our argument requires us to demonstrate the existence of the dashed lift $l_{k+1}: \Delta^n \to \{W_{k+1}, T\}$ filling the left hand square in this diagram. By the pullback property of the right hand square, it suffices to construct a lift $m: \Delta^n \to A^\Delta^m$ for the composite of these two squares. Transposing that latter lifting problem, we find in turn that we must solve a corresponding lifting problem:
\[
\begin{align*}
\partial\Delta^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m & \longrightarrow A \\
\downarrow & \\
\Delta^n \times \Delta^m & \longrightarrow
\end{align*}
\]
Observe that the commutativity of the triangles at the bottom of (5.20) imply that the horizontal map $f$ at the top of this transposed problem carries the final vertex $([n], [m])$ to the terminal object $a$ of $A$. However, all of the simplices of the prism $\Delta^n \times \Delta^m$ that are not contained in its boundary $\partial\Delta^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m$ have final vertex $([n], [m])$, so it follows that we can use the universal property of the terminal object $a$ to extend $f$ from
one skeleton of $\Delta^n \times \Delta^m$ to the next in order to construct the desired filler and thereby complete our proof.

□

**Creation of limits.** We now move on to the general case. Given a diagram $g: X \to A[t]$, we wish to show that if $u'tg: X \to A$ has a limit, then this lifts to a limit of $g$ in $A[t]$ that is preserved by $u't: A[t] \to A$. That is, our aim will be to construct an absolute right lifting

\[
\begin{array}{c}
A[t] \\
\downarrow c \\
\mathbb{1} \xrightarrow{g} A[t]^X \\
\end{array}
\]

of $g$ along the constant diagram map. It will be most economical to describe the construction of this absolute right lifting diagram in the following slightly more general context.

5.22. **Observation.** A homotopy coherent monad on $A$ induces a pointwise-defined homotopy coherent monad on $A^X$ in such a way that the constant diagram map $c: A \to A^X$ is a natural transformation of homotopy coherent monads. As weighted limits commute, the diagram (5.21) is isomorphic to the diagram:

\[
\begin{array}{c}
A[t] \\
\downarrow c[t] \\
\mathbb{1} \xrightarrow{g} A^X[t] \\
\end{array}
\]

Consequently, theorem [5.8] may be obtained as an immediate corollary to the following proposition, simply by specialising it to the natural transformation $c: A \to A^X$.

Our aim is prove the following general result.

5.23. **Theorem.** Suppose $f: B \to A$ is a natural transformation of homotopy coherent monads, and suppose that

\[
\begin{array}{c}
B \\
\downarrow f \\
A \\
\end{array}
\]

is an absolute right lifting diagram. Then

\[
\begin{array}{c}
B[t] \\
\downarrow f[t] \\
\mathbb{1} \xrightarrow{g} A[t] \\
\end{array}
\]
admits an absolute right lifting. Furthermore that lifting is preserved by underlying functors, in the sense that the transformation

\[
\begin{align*}
\begin{array}{ccc}
B[t] & \xrightarrow{u^t} & A[t] \\
\downarrow f[t] & & \downarrow f \\
1 & \xrightarrow{g} & B \\
\end{array}
\end{align*}
\]

is right exact (cf. remark \ref{4.4}).

5.27. Remark. Observation \ref{4.13} reminds us that the diagrams \eqref{5.24} and \eqref{5.25} admit absolute right liftings if and only if the comma quasi-categories \( f \downarrow u^t g \) and \( f[t] \downarrow g \) (respectively) admit terminal objects. Furthermore, lemma \ref{4.14} tells us that the isofibration \( f[t] \downarrow g \rightarrow f \downarrow u^t g \) induced by the pointwise isofibration \eqref{5.26} preserves these terminal objects if and only if \eqref{5.26} is right exact. These observations reduce theorem \ref{5.23} to the following equivalent result:

5.28. Proposition. Suppose that the comma quasi-category \( f \downarrow u^t g \) possesses a terminal object. Then the comma quasi-category \( f[t] \downarrow g \) also possesses a terminal object and it is preserved by the isofibration \( f[t] \downarrow g \rightarrow f \downarrow u^t g \) induced by \eqref{5.26}.

We now proceed to prove this latter proposition, but first we require a few preparatory observations:

5.29. Observation. The weight \( W_- \) may equally be thought of as giving us a monad on the category \( \Delta_\infty \). Here we have dropped the adjective “homotopy coherent” because a homotopy coherent monad on a category is indeed no-more-nor-less than a classical monad, a fact which follows immediately from the characterisation of \( (\Delta_+, \oplus, [-1]) \) as the free monoidal category containing a monoid. It is also easily checked that the quasi-category of algebras for a monad on a category is indeed the usual Eilenberg-Moore category of algebras for that monad. In the particular case of \( W_- \), we may show that the category of algebras \( \Delta_\infty[t] \) can be identified with the subcategory of \( \Delta_\infty \) of top and bottom preserving maps. However these last two results are not ones we shall need here.

All we will need is the simple fact that the identity map \( \text{id} : W_- = \Delta_\infty \rightarrow \Delta_\infty \) corresponds, via observation \ref{5.4}, to an object \([0]\) of \( \Delta_\infty[t] \) which is mapped to the object \([0]\) of \( \Delta_\infty \) by the underlying functor \( u^t : \Delta_\infty[t] \rightarrow \Delta_\infty \). Applying proposition \ref{5.10} we know that \( u^t \) creates terminal objects, and we also know that \([0]\) is terminal in \( \Delta_\infty \), so it follows that \([0]\) is terminal in \( \Delta_\infty[t] \). Furthermore, if \( a \) is an object of \( A[t] \) given as a monad transformation \( \bar{a} : \Delta_\infty \rightarrow A \) then the induced functor \( \bar{a}[t] : \Delta_\infty[t] \rightarrow A[t] \) of quasi-categories of algebras maps the object \([0]\) of \( \Delta_\infty[t] \) to the object \( a \) of \( A[t] \). This is an immediate consequence of the fact that \([0]\) is defined to be the object corresponding to the identity on \( \Delta_\infty \) and the explicit description of \( \bar{a}[t] \) given at the end of observation \ref{5.4}.
5.30. **Observation.** Returning now to the diagrams in the statement of theorem 5.23, we can apply observation 5.4 to the map \( g : 1 \to A[t] \) of (5.25) to obtain a corresponding equivariant map \( \bar{g} : W_\ast \to A \). Applying the quasi-category of algebras construction to the monads on \( A, B, \) and \( \Delta_\infty \) and to the monad transformations \( f \) and \( \bar{g} \) we obtain the following commutative diagram:

\[
\begin{array}{ccc}
1 & \xrightarrow{\Delta_\infty[t]} & A[t] \\
\downarrow & & \downarrow f[t] \\
1 & \xrightarrow{\bar{g}[t]} & A & \xleftarrow{B[t]} \downarrow u^t \\
\end{array}
\]

Here the equality \( \bar{g}[0] = u^t g \) indicated by the curved arrow at the bottom of this diagram is simply that expressed by the commutative diagram (5.5), and the equality \( (\bar{g}[t])[0] = g \) indicated by the curved arrow at its top is that discussed at the end of observation 5.29.

A routine pullback computation, starting from definition I.3.3.17, reveals that if we are given three functors \( f : B \to A, g : C \to A, \) and \( h : D \to C \) then we may express the comma quasi-category \( f \downarrow gh \) as the following pullback of the comma quasi-category \( f \downarrow g \):

\[
\begin{array}{ccc}
D & \xrightarrow{h} & C \\
\downarrow & & \downarrow \\
A & \xleftarrow{f} & B
\end{array}
\]

Here the horizontal arrow at the top of this square is obtained by applying the comma construction functor \( \downarrow : q\text{Cat}_\infty \to q\text{Cat}_\infty \) to the following transformation:

\[
\begin{array}{ccc}
D & \xrightarrow{gh} & A \\
\downarrow h & & \downarrow f \\
C & \xleftarrow{g} & B
\end{array}
\]

Applying this result to the rows of (5.31), we obtain the following diagram of commutative squares:

\[
\begin{array}{ccc}
1 & \xrightarrow{\Delta_\infty} & A[t] \\
\downarrow & & \downarrow f[t] \\
1 & \xrightarrow{\bar{g}[t]} & A & \xleftarrow{B[t]} \downarrow u^t \\
\end{array}
\]

Here the pullbacks labelled (B) and (C) are those obtained by applying the result of the last paragraph to the bottom and top rows of (5.31) respectively. The vertical isofibrations of the square labelled (A) are those induced by the obvious transformations whose legs are various of the vertical isofibrations in (5.31). Most crucially, the vertical isofibration on
the left of square (A) is precisely the induced functor that features in the statement of proposition 5.28.

This provides us with almost all of the machinery we need to establish proposition 5.28. The last piece of jigsaw is the following lemma:

5.33. **Lemma.** Under the assumption that (5.24) is an absolute right lifting diagram, the comma quasi-category \( f \downarrow \bar{g} \) possesses a terminal object which is preserved by the projection functor \( p_1: f \downarrow \bar{g} \to \Delta_\infty \).

We postpone the proof of this technical lemma in our brimming enthusiasm to get to our ultimate goal:

**Proof of proposition 5.28.** Our proof revolves around an analysis of the squares displayed in (5.32). First consider the square labelled (B). The domain and codomain of its lower horizontal functor \([0]: 1 \to \Delta_\infty \) both possess terminal objects and the inclusion preserves such, since \([0]\) is the terminal object of \( \Delta_\infty \). What is more lemma 5.33 tells us that the right hand vertical \( f \downarrow \bar{g} \to \Delta_\infty \) of (B) is also a functor whose domain and codomain possesses and which preserves terminal objects. So applying lemma 3.11, we find that the top left vertex \( f \downarrow \bar{u} \) of (B) also possesses a terminal object and that its projection functors preserve such.

The cancellation lemma for pullbacks tells us that the square labelled (A) is also a pullback. So our intention is again to apply lemma 3.11 to (A) to show that its top left hand vertex \( f \downarrow \bar{g} \) possesses a terminal object and that this is preserved by the left hand vertical isofibration \( f[t] \downarrow \bar{g} \to f \downarrow \bar{u} \bar{g} \), these being precisely the conclusions of proposition 5.28. We have just shown that the lower horizontal map of (A) is a functor whose domain and codomain possesses and which preserves terminal objects, so all that remains in order to apply lemma 3.11 and complete our proof is to demonstrate the same properties for its right hand vertical \( f[t] \downarrow \bar{g}[t] \to f \downarrow \bar{g} \).

Now observation 5.2 points out that \( \qCat^{\text{Mon}}_{\infty} \) possesses all limits weighted by projective cofibrant weights, that these are constructed as in \( \qCat_\infty \), and that the quasi-category of algebras construction preserves them. So applying that result to the comma construction for the monad transformations \( f \) and \( \bar{g} \) we find that the comma quasi-category \( f \downarrow \bar{g} \) inherits a monad structure and that there exists a canonical isomorphism between the category of algebras \( (f \downarrow \bar{g})[t] \) associated with that monad and the comma quasi-category \( f[t] \downarrow \bar{g}[t] \). What is more, that isomorphism makes the following triangle commute

\[
\begin{array}{ccc}
 f[t] \downarrow \bar{g}[t] & \cong & (f \downarrow \bar{g})[t] \\
 \downarrow & & \downarrow \\
 f \downarrow \bar{g} & \to & f \downarrow \bar{g}
\end{array}
\]

where the diagonal map on the left is the right hand vertical of square (A). Consequently, the required result is equivalent to showing that the quasi-category of algebras \( (f \downarrow \bar{g})[t] \) has a terminal object which is preserved by the underlying functor down to \( f \downarrow [g] \). These facts
follow immediately from proposition \ref{prop:5.10} and the fact that we know, from lemma \ref{lem:5.33}, that $f \downarrow g$ possesses a terminal object.

Finally, we complete our argument by establishing the technical lemma \ref{lem:5.33}, which is a special case of the following result:

5.34. **Lemma.** Suppose that $c$ is a terminal object in $C$ and that we have an absolute right lifting diagram

$$
\begin{array}{ccc}
 C & \xrightarrow{f} & A \\
 \downarrow^g & & \downarrow^p_1 \\
 1 \xrightarrow{\lambda} C & \xrightarrow{\ell} & B
\end{array}
$$

Then the comma quasi-category $f \downarrow g$ admits a terminal object which is preserved by the projection $p_1: f \downarrow g \to C$.

**Proof.** All of the components that we need are contained in the proof of theorem I.6.1.3. Firstly the existence of the absolute right lifting in the statement is equivalent to asking for the comma quasi-category $f \downarrow gc \simeq f/gc$ to have a terminal object, this being the object of $f \downarrow gc$ induced by $\lambda$. Given that result, the first part of the proof of theorem I.6.1.3 demonstrates that any lifting problem

$$
\begin{array}{ccc}
 \partial \Delta^n & \longrightarrow & f \downarrow g \\
 \downarrow & & \downarrow^{p_1} \\
 \Delta^n & \longrightarrow & C
\end{array}
$$

which maps the terminal vertex of $\partial \Delta^n$ to the object of $f \downarrow g$ induced by $\lambda$ has a solution as marked. However, that object is mapped to the terminal object $c$ of $C$ by the projection $p_1: f \downarrow g \to C$, so it follows that we can apply lemma \ref{lem:3.7} to complete our proof. \qed

The desired theorem \ref{thm:5.8} is an immediate corollary of proposition \ref{prop:5.10} and theorem \ref{thm:5.23}, or equivalently, proposition \ref{prop:5.28}.

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