Wakimoto realization of the elliptic algebra $U_{q,p}(\widehat{sl}_N)$

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Takeo KOJIMA

Department of Mathematics, College of Science and Technology, Nihon University, Surugadai, Chiyoda-ku, Tokyo 101-0062, JAPAN

Abstract

We construct a free field realization of the elliptic quantum algebra $U_{q,p}(\widehat{sl}_N)$ for arbitrary level $k \neq 0, -N$. We study Drinfeld current and the screening current associated with $U_{q,p}(\widehat{sl}_N)$ for arbitrary level $k$. In the limit $p \to 0$ this realization becomes $q$-Wakimoto realization for $U_q(\widehat{sl}_N)$.

1 Introduction

The elliptic quantum group has been proposed in papers [1, 2, 3, 4, 5]. There are two types of elliptic quantum groups, the vertex type $A_{q,p}(\widehat{sl}_N)$ and the face type $B_{q,\lambda}(g)$, where $g$ is a Kac-Moody algebra associated with a symmetrizable Cartan matrix. Not only the quantum group but also the elliptic quantum groups have the structure of quasi-triangular quasi-Hopf algebras introduced by V.Drinfeld [6]. H.Konno [7] introduced the elliptic quantum algebra $U_{q,p}(\widehat{sl}_2)$ as an algebra of the elliptic analogue of Drinfeld current in the context of the fusion SOS model [8]. M.Jimbo, H.Konno, S.Odake, J.Shiraishi [9] continued to study the elliptic quantum
algebra $U_{q,p}(\hat{sl}_2)$. They identified $U_{q,p}(\hat{sl}_2)$ with the tensor product of $B_{q,\lambda}(\hat{sl}_2)$ and a Heisenberg algebra $\mathcal{H}$. The elliptic quantum group $B_{q,\lambda}(\hat{sl}_2)$ is a quasi-Hopf algebra while the elliptic algebra $U_{q,p}(\hat{sl}_2)$ is not. The intertwining relation of the vertex operator of $B_{q,\lambda}(\hat{sl}_2)$ is based on the quasi-Hopf structure of $B_{q,\lambda}(\hat{sl}_2)$. By the above isomorphism $U_{q,p}(\hat{sl}_2) \simeq B_{q,\lambda}(\hat{sl}_2) \otimes \mathcal{H}$, we can understand "intertwining relation" of the vertex operator for the elliptic algebra $U_{q,p}(\hat{sl}_2)$.

Going along the isomorphism $U_{q,p}(\hat{g}) \simeq B_{q,\lambda}(\hat{g}) \otimes \mathcal{H}$, the elliptic analogue of Drinfeld current of $U_{q,p}(\hat{sl}_2)$ is extended to those of $U_{q,p}(\hat{g})$ for non-twisted affine Lie algebra $\hat{g}$ [9, 10]. In this paper we are interested in higher-rank generalization of level $k$ free field realization of the elliptic quantum algebra. For the elliptic algebra $U_{q,p}(\hat{sl}_2)$, there exist two kind of free field realizations for arbitrary level $k$, the one is parafermion realization [7, 9], the other is Wakimoto realization [17]. In this paper we are interested in the higher-rank generalization of Wakimoto realization of $U_{q,p}(\hat{sl}_2)$. We construct level $k$ free field realization of Drinfeld current associated with the elliptic algebra $U_{q,p}(\hat{sl}_N)$. This gives the higher-rank generalization of the author’s previous work on $U_{q,p}(\hat{sl}_3)$ [18]. It is supposed that this free field realization can be applied for construction of the level $k$ integrals of motion for the elliptic algebra $U_{q,p}(\hat{sl}_N)$. For this purpose, see references [20, 21, 22].

The organization of this paper is as follows. In section 2 we set the notation and introduce bosons. In section 3 we review the level $k$ free field realization of the quantum group $U_q(\hat{sl}_N)$ [15, 16]. In section 4 we give the free field realization of the dressing operator $U^i(z), U^*i(z)$, which cause the elliptic deformation of Drinfeld current. We study the screening current, too. In appendix A, we explain a systematic way of construction for a free field realization of the dressing operators $U^i(z)$ and $U^*i(z)$. In appendix B, we summarize the normal ordering of the basic operators.

After finishing this work, I noticed a paper on $U_{q,p}(\hat{sl}_N)$ by W.Chang and X.Ding [17] [math.QA:0812.1147], which seemed to be submitted to arXiv a day after my submitting $U_{q,p}(\hat{sl}_3)$ paper [18] [nlin.SI:0812.0890, proceedings of the 27-th International Colloquium, Armenia, August 2008].

## 2 Bosons

The purpose of this section is to set up the basic notation and to introduce the boson. In this paper we fix three parameters $q, k, r \in \mathbb{C}$. Let us set $r^* = r - k$. We assume $k \neq 0, -N$ and $\text{Re}(r) > 0, \text{Re}(r^*) > 0$. We assume $q$ is generic with $|q| < 1, q \neq 0$. Let us set a pair of
parameters \( p \) and \( p^* \) by

\[
p = q^{2r}, \quad p^* = q^{2r^*}.
\]

We use the standard symbol of \( q \)-integer \([n]\) by

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

Following [15, 16] we introduce free bosons \( a^i_n, (1 \leq i \leq N - 1; n \in \mathbb{Z}_{\neq 0}), b^{i,j}_{n}, (1 \leq i < j \leq N; n \in \mathbb{Z}_{\neq 0}), c^{i,j}_{n}, (1 \leq i < j \leq N; n \in \mathbb{Z}_{\neq 0}) \), and the zero-mode operators \( a^i, (1 \leq i \leq N - 1), b^{i,j}, (1 \leq i < j \leq N), c^{i,j}, (1 \leq i < j \leq N) \).

\[
[a^i_n, a^j_m] = \frac{[(k + N)n][A_{i,j}n]}{n}\delta_{n+m,0}, \quad [p^i_n, q^j_n] = (k + N)A_{i,j},
\]

\[
[b^{i,j}_n, b^{k,l}_m] = -\frac{[n]^2}{n}\delta_{i,k}\delta_{j,l}\delta_{n+m,0}, \quad [p^{i,j}_n, q^{k,l}_n] = -\delta_{i,k}\delta_{j,l},
\]

\[
[c^{i,j}_n, c^{k,l}_m] = \frac{[n]^2}{n}\delta_{i,k}\delta_{j,l}\delta_{n+m,0}, \quad [p^{i,j}_n, q^{k,l}_n] = \delta_{i,k}\delta_{j,l}.
\]

Here the matrix \((A_{i,j})_{1 \leq i,j \leq N - 1}\) represents the Cartan matrix of classical \( sl_N \). For parameters \( a_i \in \mathbb{R}, (1 \leq i \leq N - 1), b_{i,j} \in \mathbb{R}, (1 \leq i < j \leq N) \), \( c_{i,j} \in \mathbb{R}, (1 \leq i < j \leq N) \), we set the vacuum vector \(|a,b,c\rangle\) of the Fock space \( \mathcal{F}_{a,b,c} \) as following.

\[
a^i_n|a,b,c\rangle = b^{i,k}_n|a,b,c\rangle = c^{i,k}_n|a,b,c\rangle = 0, \quad (n > 0; 1 \leq i \leq N - 1; 1 \leq j < k \leq N),
\]

\[
p^i_n|a,b,c\rangle = a^i|a,b,c\rangle, \quad p^{i,k}_b|a,b,c\rangle = b^{j,k}_n|a,b,c\rangle, \quad p^{i,j}_c|a,b,c\rangle = c^{j,k}_n|a,b,c\rangle,
\]

\[(1 \leq i \leq N - 1; 1 \leq j < k \leq N).
\]

The Fock space \( \mathcal{F}_{a,b,c} \) is generated by bosons \( a^i_n, b^{i,k}_n, c^{i,k}_n \) for \( n \in \mathbb{N}_{\neq 0} \). The dual Fock space \( \mathcal{F}^*_{a,b,c} \) is defined as the same manner. In this paper we construct the elliptic analogue of Drinfeld current for \( U_{q,h}(sl_N) \) by these bosons \( a^i_n, b^{i,k}_n, c^{i,k}_n \) acting on the Fock space.

Let us set the elliptic theta function \( \Theta_p(z) \) by

\[
\Theta_p(z) = (z;p)\infty(p/z;p)\infty(p;p)\infty, \quad (z;p)\infty = \prod_{n=0}^{\infty}(1 - p^n z).
\]

It is convenient to work with the additive notation. We use the parametrization

\[
q = e^{-\pi \sqrt{-1}/r\tau}, \quad p = e^{-2\pi \sqrt{-1}/r\tau}, \quad p^* = e^{-2\pi \sqrt{-1}/r^*\tau}, \quad (r\tau = r^*\tau^*),
\]

\[
z = q^{2u}.
\]
Let us set Jacobi elliptic theta function \([u]_r\) by
\[
[u]_r = q^{\frac{u^2}{r}} \Theta_{q^2r}(z) / (q^{2r}; q^{2r})_\infty^3.
\]

The function \([u]_r\) has a zero at \(u = 0\), enjoys the quasi-periodicity property
\[
[u + r]_r = -[u]_r, \quad [u + r\tau]_r = -e^{-\pi\sqrt{-1}r} q^{\frac{r^2}{4\tau^2}} [u]_r.
\]

Let us set the \(q\)-difference \((\alpha \partial_z f)(z)\) by
\[
(\alpha \partial_z f)(z) = \frac{f(q^\alpha z) - f(q^{-\alpha} z)}{(q - q^{-1}) z}.
\]

Let us set the delta-function \(\delta(z)\) as formal power series.
\[
\delta(z) = \sum_{n \in \mathbb{Z}} z^n.
\]

### 3 Free field realization of \(\widehat{U}_q(sl_N)\)

The purpose of this section is to give a review on the free field realization of the quantum affine algebra \(\widehat{U}_q(sl_N)\) [16], which is a basis of those of the elliptic algebra \(\widehat{U}_{q,p}(sl_N)\).

#### 3.1 Drinfeld current

Let us set the bosonic operators \(a^i_{\pm}(z), a^i(z), (1 \leq i \leq N - 1), b^{ij}_{\pm}(z), b^{ij}(z), c^{ij}(z), (1 \leq i < j \leq N)\) by
\[
a^i_{\pm}(z) = \pm (q - q^{-1}) \sum_{n > 0} a^i_{\pm n} z^n \mp p^i_a \log q,
\]
\[
b^{ij}_{\pm}(z) = \pm (q - q^{-1}) \sum_{n > 0} b^{ij}_{\pm n} z^n \pm p^{ij}_b \log q,
\]
\[
a^i(z) = - \sum_{n \neq 0} \frac{a^n_i}{[(k + N)n]} q^{k+1} z^n + \frac{1}{k + N} (q^i_p + p^i_a \log z),
\]
\[
b^{ij}(z) = - \sum_{n \neq 0} \frac{b^{ij}_i}{[n]} z^n + q^{ij}_b + p^{ij}_b \log z,
\]
\[
c^{ij}(z) = - \sum_{n \neq 0} \frac{c^{ij}_i}{[n]} z^n + q^{ij}_c + p^{ij}_c \log z,
\]

Let us set the auxiliary operators \(\gamma^{i,j}(z), \beta^{i,j}_{1}(z), \beta^{i,j}_{2}(z), \beta^{i,j}_{3}(z), \beta^{i,j}_{4}(z), (1 \leq i < j \leq N)\) by
\[
\gamma^{i,j}(z) = - \sum_{n \neq 0} \frac{(b + c)^{ij}_n}{[n]} z^n + (q^{ij}_b + q^{ij}_c) + (p^{ij}_b + p^{ij}_c) \log(-z),
\]
\[
\beta^{i,j}_{1}(z) = b^{ij}_{\pm}(z) - (b^{ij} + c^{ij})(qz), \quad \beta^{i,j}_{2}(z) = b^{ij}_{\pm}(z) - (b^{ij} + c^{ij})(q^{-1} z),
\]
\[
\beta^{i,j}_{3}(z) = b^{ij}_{\pm}(z) + (b^{ij} + c^{ij})(q^{-1} z), \quad \beta^{i,j}_{4}(z) = b^{ij}_{\pm}(z) + (b^{ij} + c^{ij})(qz).
\]
We give a free field realization of Drinfeld current for $U_q(sl_N)$.

**Definition 3.1** Let us set the bosonic operators $E^{±,i}(z), (1 \leq i \leq N - 1)$ by

\[
E^{+,i}(z) = \frac{-1}{(q - q^{-1})z} \sum_{j=1}^{i} E^{+,i}_{j}(z),
\]

\[
E^{-,i}(z) = \frac{-1}{(q - q^{-1})z} \sum_{j=1}^{N-1} E^{-,i}_{j}(z),
\]

where we have set

\[
E^{+,i}_{j}(z) = e^{\gamma^+_{j}(q^{j-1}z)}(e^{\beta^{+,i+1}_{j}(q^{j-1}z)} - e^{\beta^{+,i}_{j}(q^{j-1}z)})e_{j}^{\sum_{l=1}^{j-1}(b^{+,i+1}_{l}(q^{l-1}z) - b^{+,i}_{l}(q^{l}z))},
\]

\[
E^{-,i}_{j}(z) = e^{\gamma^{-,i}_{j}(q^{j-1}z)}(e^{-\beta^{-,i}_{j}(q^{j-1}z)} - e^{-\beta^{-,i}_{j+1}(q^{j-1}z)})
\times e_{j}^{\sum_{l=i+1}^{j}(b^{-,i+1}_{l}(q^{l-1}z) - b^{-,i}_{l}(q^{l}z)) + \sum_{l=i+1}^{N-1}(b^{-,i+1}_{l}(q^{l-1}z) - b^{-,i+1}_{l}(q^{l}z))},
\]

for $1 \leq j \leq i - 1,

\[
E^{-,i}_{i}(z) = e^{\gamma^{-,i+1}_{i}(q^{i-1}z) + a^{i}_{i}(q^{i-1}z)}
\times e_{i}^{\sum_{l=i+1}^{N-1}(b^{-,i+1}_{l}(q^{l-1}z) - b^{-,i+1}_{l}(q^{l}z))},
\]

\[
E^{-,i}_{i-1}(z) = e^{\gamma^{-,i}_{i-1}(q^{i-1}z) - e^{\beta^{-,i+1}_{i}(q^{i-1}z)} - e^{\beta^{-,i}_{i}(q^{i-1}z)}
\times e_{i-1}^{\sum_{l=i+1}^{N-1}(b^{-,i+1}_{l}(q^{l-1}z) - b^{-,i+1}_{l}(q^{l}z))},
\]

for $i + 1 \leq j \leq N - 1$.

Let us set the bosonic operators $\psi^{±,i}_{i}(z), (1 \leq i \leq N - 1)$ by

\[
\psi^{±,i}_{i}(q^{±,i-1}z) = e^{\sum_{j=1}^{i}(b^{±,i+1}_{j}(q^{±,(k+j-1)}z) - b^{±,i}_{j}(q^{±,(k+j)}z)) + a^{i}_{i}(q^{±,k+N}z)}
\times e^{\sum_{l=i+1}^{N-1}(b^{±,i+1}_{l}(q^{±,(k+j)}z) - b^{±,i+1}_{l}(q^{±,(k+j+1)}z))}.
\]

Let us set

\[
h_{i} = \sum_{j=1}^{i} (p^{j,i}_{b} - p^{i,j}_{b}) + \sum_{j=i+1}^{N} (p^{i,j}_{b} - p^{i+1,j}_{b}).
\]

Here the symbol $\mathcal{O}$ represents the normal ordering of $\mathcal{O}$. For example we have

\[
b^{i,j}_{k} : b^{i,j}_{k} := \begin{cases} b^{i,j}_{k} & \text{if } k < 0 \\ \delta^{i,j}_{k}b^{i,j}_{k} & \text{if } k > 0 \end{cases}.
\]

**Theorem 3.1** The operators $E^{±,i}(z), \psi^{±,i}_{i}(z), h_{i}, (1 \leq i \leq N - 1)$ give a free field realization of $U_q(sl_N)$ for arbitrary level $k \neq 0, -N$. In other words, they satisfy the following commutation relations.

\[
[h_{i}, E^{±,i}(z)] = \pm A_{i,j} E^{±,i}(z),
\]

(3.17)
Definition 3.2
Let us introduce the bosonic operator

\[(1 - q^{\pm A_{i,j}} z_2) E^{\pm,i}(z_1) E^{\pm,j}(z_2) = (q^{\pm A_{i,j}} z_1 - z_2) E^{\pm,j}(z_2) E^{\pm,i}(z_1), \quad (3.18)\]

\[\{ E^{\pm,i}(z_1) E^{\pm,j}(z_2) E^{\pm,j}(z_3) - (q - q^{-1}) E^{\pm,i}(z_1) E^{\pm,j}(z_3) E^{\pm,i}(z_2) + E^{\pm,i}(z_3) E^{\pm,i}(z_1) E^{\pm,j}(z_2) \} + \{ z_1 \leftrightarrow z_2 \} = 0, \quad \text{for } A_{i,j} = -1, \quad (3.23)\]

\[\left[ E^{+,i}(z_1), E^{-,j}(z_2) \right] = \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} \left( \delta \left( q^{-k} z_1 \right) \psi_+^{i} \left( q^{-\frac{k}{2}} z_1 \right) - \delta \left( q^{\frac{k}{2}} z_1 \right) \psi_-^{i} \left( q^{-\frac{k}{2}} z_2 \right) \right). \quad (3.24)\]

When we take the limit \( q \to 1 \), we recover Wakimoto realization for \( \hat{s}l_N \) [12].

3.2 Screening current

Following [16], we define the screening current \( S^i(z) \), which commute with \( U_q(\hat{s}l_N) \).

Definition 3.2 \textit{Let us introduce the bosonic operator} \( S^i(z) \), \((1 \leq i \leq N - 1) \) ny

\[ S^i(z) = \frac{-1}{(q - q^{-1}) z} : e^{-a^i(z)} \tilde{S}^i(z) : , \quad (3.25)\]

\textit{where we have set}

\[ \tilde{S}^i(z) = \sum_{j=i+1}^{N} : e^{(i+1)j}(q^{N-j} z) (e^{-\beta^i_j(q^{N-j} z)} - e^{-\frac{\beta^i_j}{2}(q^{N-j} z)}) e^{\sum_{l=i+1}^{j} (b^i_{l+1} (q^{N-l+1} z) - b^i_l (q^{N-l} z))} : . \]

Proposition 3.2 \textit{The bosonic operator} \( S^i(z) \), \( E^{\pm,i}(z) \) \textit{satisfy the following commutation relations.}

\[ \left[ u_1 - u_2 - \frac{A_{i,j}}{2} \right]_{k+N} S^i(z_1) S^j(z_2) = \left[ u_1 - u_2 + \frac{A_{i,j}}{2} \right]_{k+N} S^j(z_2) S^i(z_1) \sim \text{reg.}, \quad (3.26)\]

\[ E^{+,i}(z_1) S^j(z_2) = S^j(z_1) E^{+,i}(z_2) \sim \text{reg.}, \quad (3.27)\]

\[ E^{-,i}(z_1) S^j(z_2) = S^j(z_2) E^{-,i}(z_1) \sim \text{reg.} + \delta_{i,j} \times k+N \partial_{z_2} \left( \frac{1}{z_1 - z_2} : e^{\sum_{n \neq 0} \frac{a^i_n}{(k+N) n!} (\frac{k+N}{2})_{|n|} z_2^{-n} - \frac{1}{k+N} (q^{n} + p^i_n \log z_2) :} \right). \quad (3.28)\]

\textit{The symbol} \( \sim \text{reg.} \) \textit{means equality modulo regular function.}
The equalities (3.17), (3.18), (3.19), (3.20), (3.21), (3.22), (3.23), (3.26), (3.27) hold in “∼ reg.” sense. The exceptional cases are (3.24) and (3.28), which do not exist inside regular function. Note that the elliptic theta function \([u]_{k+N}\) has already appeared in trigonometric symmetry \(U_q(\hat{sl}_N)\).

4 Free field realization of \(U_{q,p}(\hat{sl}_N)\)

The purpose of this section is to give a free field realization for the elliptic algebra \(U_{q,p}(\hat{sl}_N)\) for arbitrary level \(k \neq 0, -N\).

4.1 Drinfeld current

Following [17], let us introduce the auxiliary operators \(B_{\pm}^{i,j}(z), B_{\pm}^{i,j}(z), (1 \leq i < j \leq N)\) by

\[
B_{\pm}^{i,j}(z) = \exp \left( \pm \sum_{n>0} \frac{1}{n} b_{-n}^{i,j}(q^{r^* - 1}z)^n \right),
\]

(4.1)

\[
B_{\pm}^{i,j}(z) = \exp \left( \pm \sum_{n>0} \frac{1}{n} b_{n}^{i,j}(q^{-r^* + 1}z)^{-n} \right).
\]

(4.2)

Let us introduce the auxiliary operators \(A^{*i}(z), A^i(z), (1 \leq i \leq N - 1)\) by

\[
A^{*i}(z) = \exp \left( \sum_{n>0} \frac{1}{n} a_{-n}^{i}(q^{r^*}z)^n \right),
\]

(4.3)

\[
A^i(z) = \exp \left( -\sum_{n>0} \frac{1}{n} a_{n}^{i}(q^{-r^*}z)^{-n} \right).
\]

(4.4)

**Definition 4.1** We define the dressing operators \(U^{*i}(z), U^i(z), (1 \leq i \leq N - 1)\).

\[
U^{*i}(z) = \left( \prod_{j=1}^{i-1} B_{+}^{j,i+1}(q^{2-j}z)B_{-}^{j,i}(q^{1-j}z) \right)
\]

\[
\times \left( \prod_{j=1}^{i-1} B_{+}^{j,i+1}(q^{2-j}z)B_{+}^{j,i+1}(q^{-j}z) \prod_{j=i+2}^{N} B_{+}^{i,j}(q^{-j+1}z)B_{-}^{i,j+1}(q^{-j+2}z) \right) A^i(q^{\frac{k-N}{2}}z),
\]

(4.5)

\[
U^i(z) = \left( \prod_{j=1}^{i-1} B_{+}^{j,i+1}(q^{2-j}z)B_{-}^{j,i}(q^{1-j}z) \right)
\]

\[
\times \left( \prod_{j=1}^{i-1} B_{-}^{j,i+1}(q^{-2+j}z)B_{+}^{j,i+1}(q^{1-j}z) \right)
\]

\[
\times \left( \prod_{j=1}^{i-1} B_{-}^{j,i+1}(q^{-2+j}z)B_{-}^{j,i+1}(q^{-j}z) \prod_{j=i+2}^{N} B_{-}^{i,j}(q^{-j+1}z)B_{+}^{i,j+1}(q^{-j+2}z) \right) A^i(q^{\frac{k-N}{2}}z).
\]

(4.6)

Formulae (4.5) and (4.6) are main result of this paper. In appendix A we explain a systematic construction of the dressing operators \(U^{*i}(z), U^i(z)\).
Proposition 4.1  They satisfy Serre relation.

Definition 4.2  We define the elliptic deformation of Drinfeld current \( e_i(z) \), \( f_i(z) \), \( \Psi_i^\pm(z) \), \( 1 \leq i \leq N - 1 \), by

\[
e_i(z) = U^{*i}(z)E^{+i}(z), \tag{4.7}
\]

\[
f_i(z) = E^{-i}(z)U^i(z), \tag{4.8}
\]

\[
\Psi_i^+(z) = U^{*i}(q^{\frac{b}{2}}z)\psi_i^+(z)U^i(q^{-\frac{b}{2}}z), \tag{4.9}
\]

\[
\Psi_i^-(z) = U^{*i}(q^{-\frac{b}{2}}z)\psi_i^-(z)U^i(q^{\frac{b}{2}}z). \tag{4.10}
\]

Example  Upon specialization \( N = 3 \) we recover the dressing operator of \( U_{q,p}(sl_3) \) [18].

\[
\begin{align}
U^{*1}(z) & = B_{+}^{1,2}(qz)B_{+}^{1,2}(q^{-1}z)B_{+}^{13}(q^{-2}z)B_{+}^{23}(q^{-1}z)A^1(q^{\frac{k-3}{2}}z), \tag{4.11} \\
U^{*2}(z) & = B_{+}^{1,3}(qz)B_{-}^{1,2}(z)B_{+}^{2,3}(z)B_{+}^{23}(q^{-2}z)A^{2}(q^{\frac{k-3}{2}}z), \tag{4.12} \\
U^1(z) & = B_{-}^{1,2}(q^{-1}z)B_{+}^{1,3}(z)B_{+}^{2,3}(z)B_{-}^{2,3}(q^2z)A^1(q^{\frac{k-3}{2}}z), \tag{4.13} \\
U^2(z) & = B_{-}^{1,3}(q^{-1}z)B_{+}^{1,2}(z)B_{-}^{2,3}(z)B_{+}^{2,3}(q^2z)A^2(q^{\frac{k-3}{2}}z), \tag{4.14}
\end{align}
\]

The notation of this paper is slightly different from those of [18]. For example, \( B_{+}^{1,2}(z) = B_{+}^{1,2}(q^{r-1}z) \), \( B_{+}^{1,3}(z) = B_{+}^{1,3}(q^{r-1}z) \), \( B_{+}^{2,3}(z) = B_{+}^{2,3}(q^{r-1}z) \).

Proposition 4.1  The bosonic operators \( e_i(z) \), \( f_i(z) \), \( \Psi_i^\pm(z) \), \( 1 \leq i \leq N - 1 \) satisfy the following commutation relations.

\[
\begin{align}
\Theta_p(q^{-A_{i,j}}z_1/z_2)e_i(z_1)e_j(z_2) & = q^{-A_{i,j}}\Theta_p(q^{A_{i,j}}z_1/z_2)e_j(z_2)e_i(z_1), \tag{4.15} \\
\Theta_p(q^{A_{i,j}}z_1/z_2)f_i(z_1)f_j(z_2) & = q^{A_{i,j}}\Theta_p(q^{-A_{i,j}}z_1/z_2)f_j(z_2)f_i(z_1), \tag{4.16} \\
\Theta_p(q^{A_{i,j}}z_1/z_2)\Theta_p(q^{-A_{i,j}}z_1/z_2)\Psi_i^+(z_1)\Psi_j^+(z_2) & = \Theta_p(q^{-A_{i,j}}z_1/z_2)\Theta_p(q^{A_{i,j}}z_1/z_2)\Psi_i^+(z_1), \tag{4.17}
\end{align}
\]

\[
\Theta_p(q^{-A_{i,j}-k}z_1/z_2)\Theta_p(p^*q^{A_{i,j}+k}z_1/z_2)\Psi_i^+(z_2) \Psi_j^+(z_2) = \Theta_p(p^*q^{-A_{i,j}-k}z_1/z_2)\Theta_p(q^{A_{i,j}+k}z_1/z_2)\Psi_i^+(z_1), \tag{4.18}
\]

\[
\Theta_p(q^{-A_{i,j}+\frac{k}{2}}z_1/z_2)\Psi_i^+(z_1)e_j(z_2) = \Theta_p(q^{A_{i,j}+\frac{k}{2}}z_1/z_2)e_j(z_2)\Psi_i^+(z_1), \tag{4.19}
\]

\[
\Theta_p(q^{A_{i,j}+\frac{k}{2}}z_1/z_2)\Psi_i^+(z_1)f_j(z_2) = \Theta_p(q^{-A_{i,j}+\frac{k}{2}}z_1/z_2)f_j(z_2)\Psi_i^+(z_1), \tag{4.20}
\]

\[
[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2} \left( \delta \left(q^{-\frac{k}{2}z_1}{z_2}\right) \Psi_i^+(q^{-k/z_2}z_1) - \delta \left(q^{\frac{k}{2}z_1}{z_2}\right) \Psi_i^-(q^{-k/z_2}z_2) \right). \tag{4.21}
\]

They satisfy Serre relation.

\[
(p^*q^2z_2/z_1 : p^*)_{\infty}(p^*q^{-2}z_1/z_2; p^*)_{\infty}
\]
Theorem 4.2

\begin{align}
&(p q^{-2} z_1 : p) \infty (p q^{-2} z_1 / z_2; p) \\
&\times \{ (p q z_1 / z_2; p) \infty (p q z_2 / z_1; p) \infty f_i(z_1) f_j(z_2) \\
&- [2] (p q z_1 / z_2; p) \infty (p q z_2 / z_1; p) \infty f_i(z_1) f_j(z_2) \\
&+ (p q z_1 / z_2; p) \infty (p q z_2 / z_1; p) \infty f_j(z_1) f_i(z_2) \} \\
+(z_1 \leftrightarrow z_2) = 0, & \text{for } A_{i,j} = -1.
\end{align}

Following [7, 9], we introduce the Heisenberg algebra \( \mathcal{H} \) generated by the following \( P_i, Q_i \), \( 1 \leq i \leq N - 1 \).

\[ [P_i, Q_j] = \frac{A_{i,j}}{2}. \]

Definition 4.3

Let us define the bosonic operators \( E_i(z), F_i(z), H_i^\pm(z) \in U_q(\mathfrak{sl}_N) \otimes \mathcal{H}, 1 \leq i \leq N - 1 \) by

\begin{align}
E_i(z) &= e_1(z) e^{2Q_i} z^{-\frac{P_i-1}{r}}, \\
F_j(z) &= f_1(z) z^{\frac{\lambda_i + P_i-1}{r}}, \\
H_i^\pm(z) &= \Psi_i^\pm(z) e^{2Q_i} q^{\pm \lambda_i} (q^{\pm \frac{1}{2}} z)^{\frac{\lambda_i + P_i-1}{r} - \frac{P_i-1}{r}}.
\end{align}

Theorem 4.2

The bosonic operators \( E_i(z), F_i(z), H_i^\pm(z), 1 \leq i \leq N - 1 \) satisfy the following commutation relations.

\begin{align}
\left[ u_1 - u_2 - \frac{A_{i,j}}{2} \right] r^\ast E_i(z_1) E_j(z_2) &= \left[ u_1 - u_2 + \frac{A_{i,j}}{2} \right] r^\ast E_j(z_2) E_i(z_1), \\
\left[ u_1 - u_2 + \frac{A_{i,j}}{2} \right] r^\ast F_i(z_1) F_j(z_2) &= \left[ u_1 - u_2 - \frac{A_{i,j}}{2} \right] r^\ast F_j(z_2) F_i(z_1), \\
\left[ u_1 - u_2 + \frac{A_{i,j}}{2} \right] r^\ast \left[ u_1 - u_2 - \frac{A_{i,j}}{2} \right] r^\ast H_i^\pm(z_1) H_j^\pm(z_2) &= \left[ u_1 - u_2 - \frac{A_{i,j}}{2} \right] r^\ast \left[ u_1 - u_2 + \frac{A_{i,j}}{2} \right] r^\ast H_j^\pm(z_2) H_i^\pm(z_1), \\
\left[ u_1 - u_2 + \frac{A_{i,j}}{2} - \frac{k}{2} \right] r^\ast \left[ u_1 - u_2 - \frac{A_{i,j}}{2} + \frac{k}{2} \right] r^\ast H_i^\pm(z_1) H_j^\pm(z_2) &= \left[ u_1 - u_2 - \frac{A_{i,j}}{2} - \frac{k}{2} \right] r^\ast \left[ u_1 - u_2 + \frac{A_{i,j}}{2} + \frac{k}{2} \right] r^\ast H_j^\pm(z_2) H_i^\pm(z_1),
\end{align}
\[
\begin{align*}
\left[ u_1 - u_2 \pm \frac{k}{4} \right] r_i H_i^{\pm}(z_1) E_j(z_2) &= \left[ u_1 - u_2 \pm \frac{k}{4} \right] r_i E_j(z_2) H_i^{\pm}(z_1), \\
\left[ u_1 - u_2 \pm \frac{k}{4} + \frac{A_{ij}}{2} \right] r_i H_i^{\pm}(z_1) F_j(z_2) &= \left[ u_1 - u_2 \pm \frac{k}{4} - \frac{A_{ij}}{2} \right] r_i F_j(z_2) H_i^{\pm}(z_1),
\end{align*}
\]
\[ (4.32) \]

They satisfy Serre relation.
\[
\begin{align*}
&z_1 \frac{1}{z_2} (p^* q^2 z_2 / z_1 : p^*) \infty (p^* q^2 z_1 / z_2 : p^*) \infty \\
&\times \left\{ z_2 \frac{1}{z_1} (p q^2 z_2 / z_1 : p) \infty (pq^2 z_2 / z_1 : p) \infty (pq^2 z_2 / z_1 : p) \infty (pq^2 z_2 / z_1 : p) \infty \right\} E_i(z) E_i(z) E_i(z) E_i(z) \\
&\quad - \left[ 2 \{(pq^2 z_1 / z_2 : p) \infty (pq^2 z_1 / z_2 : p) \infty (pq^2 z_2 / z_1 : p) \infty \right\} E_i(z) E_i(z) E_i(z) E_i(z) \\
&\quad + \left( z_1 \leftrightarrow z_2 \right) = 0, \quad \text{for } A_{ij} = -1, \\
&z_1 \frac{1}{z_2} (pq^{-1} z_2 / z_1 : p) \infty (pq^{-1} z_2 / z_1 : p) \infty \\
&\times \left\{ z_2 \frac{1}{z_1} (pq z_1 / z_2 : p) \infty (pq z_1 / z_2 : p) \infty (pq z_1 / z_2 : p) \infty \right\} F_i(z) F_i(z) F_i(z) F_i(z) \\
&\quad - \left[ 2 \{(pq z_1 / z_2 : p) \infty (pq z_1 / z_2 : p) \infty (pq z_1 / z_2 : p) \infty \right\} F_i(z) F_i(z) F_i(z) F_i(z) \\
&\quad + \left( z_1 \leftrightarrow z_2 \right) = 0, \quad \text{for } A_{ij} = -1.
\end{align*}
\]
\[ (4.35) \]

Now we have constructed level \( k \) free field realization of Drinfeld current \( E_i(z), F_i(z), H_i^{\pm}(z) \) for the elliptic algebra \( U_{q,p}(sl_N) \) [9, 10].

### 4.2 Screening current

In this section we study the screening current for \( U_{q,p}(sl_N) \). In the paper [7] it was recognized that the screening current of \( U_{q,p}(sl_2) \) was exactly the same as those of \( U_{q}(sl_2) \). Hence we select the same definition of screening current of \( U_{q}(sl_N) \) [16] as the screening current of \( U_{q,p}(sl_N) \).

\[
S_i(z) = \frac{-1}{(q - q^{-1})z} e^{-\alpha_i(z)} \\
\times \sum_{j=1+1}^{N} e^{\frac{i}{2} \sum_{j=1+1}^{N} (\beta_{ij}^+(q^{N-j}z) - \beta_{ij}^+(q^{N-j}z))}.
\]

**Proposition 4.3** The bosonic operator \( S_i(z), E_i(z), F_i(z), (1 \leq i \leq N - 1) \) satisfy the following commutation relations.
\[
\left[ u_1 - u_2 - \frac{A_{ij}}{2} \right]_{k+N} S_i(z_1) S_j(z_2) = \left[ u_1 - u_2 + \frac{A_{ij}}{2} \right]_{k+N} S_j(z_2) S_i(z_1) \sim \text{reg.},
\]
\[ (4.37) \]
\[
\begin{align*}
\left[u_1 - u_2 - \frac{A_{i,j}}{2}\right] E_i(z_1) E_j(z_2) &= \left[u_1 - u_2 + \frac{A_{i,j}}{2}\right] E_j(z_2) E_i(z_1) \sim \text{reg.}, \quad (4.38) \\
\left[u_1 - u_2 + \frac{A_{i,j}}{2}\right] F_i(z_1) F_j(z_2) &= \left[u_1 - u_2 - \frac{A_{i,j}}{2}\right] F_j(z_2) F_i(z_1) \sim \text{reg.} \quad (4.39)
\end{align*}
\]

\[E_i(z_1) S_j(z_2) = S_j(z_2) E_i(z_1) \sim \text{reg.}, \quad (4.40)\]

\[F_i(z_1) S_j(z_2) = S_j(z_2) F_i(z_1) \sim \text{reg.} + \delta_{i,j} \times k+N \partial_{z_2} \left( \frac{1}{z_1 - z_2} : e^{\sum_{n \neq 0} \frac{\alpha_i^*}{\beta_{i,j} N} q^{b_N N|n|} z_2^{-n} - \frac{1}{k+N} (q_n + p_n \log z_2)} U^i(z_2) z_2^{b_i + p_i - 1} : \right). \quad (4.41)\]

The symbol \(\sim \text{reg.}\) means equality modulo regular function.

The equalities (4.28), (4.29), (4.30), (4.31), (4.32), (4.33), (4.35), (4.36), (4.37), (4.38) hold in "\(\sim \text{reg.}\)" sense. The exceptional cases are (4.34) and (4.41), which do not exist inside regular function. It seems to be possible to construct three kind of infinitely many commutative operators, which are based on the commutation relations (4.37), (4.38), (4.39). See references [20, 21, 22].

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**A Construction of dressing operators \(U^i(z), U^*i(z)\)**

In this appendix we explain a systematic way to find the dressing operators \(U^i(z)\) and \(U^*i(z)\) associated with the elliptic algebra \(U_{q,p}(\hat{sl}_N)\). Wakimoto realization is not symmetric with Cartan subalgebra. In other words, Wakimoto realization of Drinfeld current \(E^{+i}(z)\) is very different from those of \(E^{-i}(z)\). The realization of \(E^{+i}(z)\) is simpler than those of \(E^{-i}(z)\). Hence it is better to consider the dressing operator \(U^*i(z)\) associated with \(E^{+i}(z)\) at first. We construct the dressing operator \(U^*i(z)\) by products of the basic operator \(B_{\pm i,j}(z)\). The
The remaining non-commutative commutation relation is given by

\[ [\tilde{B}^{\star,i,j}_+(z)_+; E^{+,i}_+(z_1); E^{+,i}_+(z_2)] \neq 0, \quad [\tilde{B}^{\star,i,j}_+(z)_+; E^{+,k}_+(z_2); E^{+,i}_+(z_1)] = 0, \quad \text{for} \; (k, l) \neq (i, j). \quad (A.1) \]

For example, the explicit formulae of \( \tilde{B}^{\star,i,j}_+(z) \) for \( U_{q,p}(sl_4) \) are given as follows.

\[
\begin{align*}
\tilde{B}^{1,2}_+(z) &= B^{1,2}_+(z)B^{1,3}_+(q^{-1}z)B^{1,4}_+(q^{-2}z), \\
\tilde{B}^{1,3}_+(z) &= B^{1,3}_+(z)B^{1,4}_+(q^{-1}z)B^{2,3}_+(q^{-2}z)B^{2,4}_+(q^{-2}z), \\
\tilde{B}^{1,4}_+(z) &= B^{1,4}_+(z)B^{2,4}_+(q^{-1}z)B^{3,4}_+(q^{-2}z), \\
\tilde{B}^{2,4}_+(z) &= B^{2,4}_+(z)B^{3,4}_+(q^{-1}z), \\
\tilde{B}^{3,4}_+(z) &= B^{3,4}_+(z).
\end{align*}
\]

The remaining non-commutative commutation relation is given by

\[
E^{+,j-1}_+(z_1)B^{\star,i,j}_+(q^{j-1}z_1) = (p^* q^{-1}z_2/z_1; p^*)\infty \tilde{B}^{\star,i,j}_+(q^{j-1}z_1)E^{+,j-1}_+(z_1).
\]

For simplicity, we demonstrate this construction in \( U_{q,p}(sl_4) \) case. The commutation relation between \( e^{\tilde{B}^{i,j}_+(z)} \) and \( B^{\star,i,j}_+(z) \) is exactly the same as those between \( e^{\tilde{B}^{i,j}_+(z)} \) and \( B^{\star,i,j}_+(z) \).

Hence, in what follows, we can regard

\[
\begin{align*}
E^{+,1}_1(z) &\sim e^{\tilde{B}^{12}_+(z)}; \\
E^{+,2}_1(z) &\sim e^{\gamma^{12}(z) + \beta^{13}_1(z)}; \\
E^{+,3}_1(z) &\sim e^{\gamma^{23}(qz) + b^{13}_1(z) - b^{12}_1(qz)}; \\
E^{+,3}_2(z) &\sim e^{\gamma^{23}(qz) + \beta^{24}_2(z) + b^{13}_1(z) - b^{13}(qz)}; \\
E^{+,3}_3(z) &\sim e^{\gamma^{23}(qz) + b^{14}_3(qz) - b^{23}_1(qz)}.
\end{align*}
\]

There exists lexicographical ordering structure for index \((i, j)\) of \( b^{i,j}_m \) inside \( E^{+,i}_+(z) \). Hence we assume the formulae of \( \tilde{B}^{i,j}_+(z) \) as following.

\[
\tilde{B}^{\star,i,j}_+(z) = B^{\star,i,j}_+(z) \times \prod_{(k, l); (i, j) < (k, l)} B^{k, l}_+(q^{-m^{k,l}_+(z)}; q^{m^{k,l}_+(z)}; t^{k,l}_+). \quad (A.2)
\]

Here \( m^{\pm}_k, l \in \mathbb{Z} \) and \( t^{\pm}_k, l \in \mathbb{N} \). Here \((i, j) < (k, l)\) means the lexicographical ordering, i.e. \((1, 2) < (1, 3) < (1, 4) < (2, 3) < (2, 4) < (3, 4)\).

• Let’s determine \( \tilde{B}^{\star,12}_+(z) = B^{\star,12}_+(z) \times \cdots \). In order to satisfy the commutativity \([\tilde{B}^{\star,12}_+(z), E^{+,2}_1(z_2)] = \)
0, the auxiliary operator should be \( \mathcal{B}^{+12}_{+}(z) = \mathcal{B}^{+12}_{+}(z)\mathcal{B}^{+13}_{+}(q^{-1}z) \cdots \). Upon this assumption, the commutativity \([\mathcal{B}^{+12}_{+}(z_1), E^{+,2}_{+}(z_2)] = 0\) holds automatically. In order to satisfy the commutativity \([\mathcal{B}^{+12}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0\), the auxiliary operator should be \( \mathcal{B}^{+12}_{+}(z) = \mathcal{B}^{+12}_{+}(z)\mathcal{B}^{+13}_{+}(q^{-1}z)\mathcal{B}^{+14}_{+}(q^{-2}z) \cdots \). Upon this assumption, the commutation relation \([\mathcal{B}^{+12}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0 \) and \([\mathcal{B}^{+12}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0\) hold automatically. Hence we conclude \( \mathcal{B}^{+12}_{+}(z) = \mathcal{B}^{+12}_{+}(z)\mathcal{B}^{+13}_{+}(q^{-1}z)\mathcal{B}^{+14}_{+}(q^{-2}z) \). The auxiliary operator \( \mathcal{B}^{+i,i+1}_{+}(z) \) is determined as the same manner.

- Let’s determine \( \mathcal{B}^{+13}_{+}(z) = \mathcal{B}^{+13}_{+}(z) \times \cdots \). Because of the assumption (A.2), commutativity \([\mathcal{B}^{+13}_{+}(z_1), E_{+}^{+,1}(z_2)] = 0\) holds. In order to satisfy the commutativity \([\mathcal{B}^{+13}_{+}(z_1), E^{+,2}_{+}(z_2)] = 0\), the auxiliary operator should be \( \mathcal{B}^{+13}_{+}(z) = \mathcal{B}^{+13}_{+}(z)\mathcal{B}^{+23}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-2}z) \cdots \). In order to satisfy the commutativity \([\mathcal{B}^{+13}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0\), the dressing operator should be \( \mathcal{B}^{+13}_{+}(z) = \mathcal{B}^{+13}_{+}(z)\mathcal{B}^{+23}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-2}z) \cdots \). In order to satisfy the commutativity \([\mathcal{B}^{+13}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0\), the auxiliary operator should be \( \mathcal{B}^{+13}_{+}(z) = \mathcal{B}^{+13}_{+}(z)\mathcal{B}^{+23}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-2}z) \times \cdots \). Upon these assumption, the commutativity \([\mathcal{B}^{+13}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0\) holds, automatically. Hence we conclude \( \mathcal{B}^{+13}_{+}(z) = \mathcal{B}^{+13}_{+}(z)\mathcal{B}^{+23}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-2}z) \mathcal{B}^{+24}_{+}(q^{-3}z) \). The auxiliary operator \( \mathcal{B}^{+i,i+2}_{+}(z) \) is determined as the same manner.

- Let’s determine \( \mathcal{B}^{+14}_{+}(z) = \mathcal{B}^{+14}_{+}(z) \times \cdots \). Because of the assumption (A.2), the commutation relations \( [\mathcal{B}^{+14}_{+}(z_1), E^{+,1}(z_2)] = [\mathcal{B}^{+14}_{+}(z_1), E^{+,2}(z_2)] = 0 \) hold. In order to satisfy the commutativity \([\mathcal{B}^{+14}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0\), the auxiliary operator should be \( \mathcal{B}^{+14}_{+}(z) = \mathcal{B}^{+14}_{+}(z)\mathcal{B}^{+24}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-2}z) \cdots \). In order to satisfy the commutativity \([\mathcal{B}^{+14}_{+}(z_1), E^{+,3}_{+}(z_2)] = 0\), the dressing operator should be \( \mathcal{B}^{+14}_{+}(z) = \mathcal{B}^{+14}_{+}(z)\mathcal{B}^{+24}_{+}(q^{-1}z)\mathcal{B}^{+24}_{+}(q^{-2}z) \mathcal{B}^{+24}_{+}(q^{-3}z) \). The dressing operator \( \mathcal{B}^{+i,i+3}_{+}(z) \) is determined as the same manner.

We have determined the auxiliary operators \( \mathcal{B}^{ij}_{+}(z) \) for \( U_{q,p}(sl_N) \).

As you have seen the above, the lexicographical ordering structure inside \( E_{+}^{+,i}(z) \) plays an important role in construction of the auxiliary operator \( \mathcal{B}^{+ij}_{+}(z) \). As the same manner as the above, we have the explicit formulae of the auxiliary operator \( \mathcal{B}^{+ij}_{+}(z) \) for the elliptic algebra \( U_{q,p}(sl_N) \) as following.

\[
\mathcal{B}_{+}^{+ij}(z) = \prod_{s=i}^{j-1} \prod_{t=j}^{N} \mathcal{B}_{+}^{s,t}(q^{i+j-s-t}z) \prod_{s=i+1}^{j-1} \prod_{t=j}^{N} \mathcal{B}_{-}^{s,t}(q^{i+j+2-s-t}z), \quad (A.3)
\]

We have the commutation relation.

\[
\Theta_{p^*}(q^{-1}z_1/z_2)E_{+}^{+,i}(z_1)\mathcal{B}_{+}^{+i+1,j}(z_1)E_{+}^{+,i}(z_2)\mathcal{B}_{+}^{+i+1,j}(z_2) = q^{-1}\Theta_{p^*}(qz_1/z_2)E_{+}^{+,i}(z_2)\mathcal{B}_{+}^{+i+1,j}(z_2)E_{+}^{+,i}(z_1)\mathcal{B}_{+}^{+i+1,j}(z_1). \quad (A.4)
\]

Let us set the auxiliary operator \( \mathcal{B}_{+}^{-ij}(z) = \prod_{s=i}^{j-1} \prod_{t=j}^{N} \mathcal{B}_{+}^{s,t}(q^{i+j-s-t}z) \prod_{s=i+1}^{j-1} \prod_{t=j}^{N} \mathcal{B}_{+}^{s,t}(q^{i+j+2-s-t}z) \). Considering about the equation (A.4) and the structure of Cartan matrix of the classical \( sl_N \),
we set the dressing operator

\[
\tilde{U}^i(z) = \prod_{j=1}^{i-1} B_+^{j,i+1}(q^{j-2}z) \tilde{B}_+^{j,i+1}(q^{j-1}z) \prod_{j=1}^{i-1} \tilde{B}_+^{j,i+1}(q^{j-1}z) \prod_{j=1}^{i-1} \tilde{B}_+^{j,i+1}(q^{j-1}z).
\]

(A.5)

Let us set \( \tilde{e}_i(z) = \tilde{U}^i(z)E^+i(z) \), \( 1 \leq i \leq N - 1 \). We have the commutation relations

\[
\Theta_{p^r}(q^{-A_{i,j}z_1/z_2}) \tilde{e}_i(z_1) \tilde{e}_j(z_2) = q^{-A_{i,j}} \Theta_{p^r}(q^{A_{i,j}z_1/z_2}) \tilde{e}_j(z_2) \tilde{e}_i(z_1).
\]

Clearing up overlap, we have

\[
\tilde{U}^i(z) = \left( \prod_{j=1}^{i-1} B_+^{j,i+1}(q^{2-j}z) B_+^{j,i}(q^{-1+j}z) \right) \times B_+^{i,i+1}(q^{2-i}z) B_+^{i,i}(q^{-i}z) \left( \prod_{j=i+2}^{N} B_+^{i,j}(q^{-j+1}z) B_+^{i+1,j}(q^{-j+2}z) \right).
\]

Next we consider the dressing operator \( U^i(z) \). The structure of \( E^{-i}(z) \) is more complicated than those of \( E^{+i}(z) \). It is difficult to use lexicographical ordering structure for \( E^{-i}(z) \). Now let’s go back to the explicit formulae of the dressing operator for \( U_{q,p}(sl_3) \), (4.11), (4.12), (4.13), (4.14) [18]. There exists "duality" relation \( B_\pm^{i,j}(q^sz) \leftrightarrow B_\pm^{j,i}(q^{-s}z) \) between the dressing operators \( U^i(z) \) and \( U^j(z) \) for \( U_{q,p}(sl_3) \). Hence we set

\[
\tilde{U}^i(z) = \left( \prod_{j=1}^{i-1} B_-^{j,i+1}(q^{2-j}z) B_-^{j,i}(q^{-1+j}z) \right) \times B_-^{i,i+1}(q^{2-i}z) B_-^{i,i}(q^{-i}z) \left( \prod_{j=i+2}^{N} B_-^{i,j}(q^{-j+1}z) B_-^{i+1,j}(q^{-j+2}z) \right).
\]

Let us set \( e_i(z), f_i(z), \Psi^\pm_i(z), (1 \leq i \leq N - 1) \) by

\[
e_i(z) = U^i(z)E^{+i}(z), \quad f_i(z) = E^{-i}(z)U^i(z),
\]

\[
\Psi^+_i(z) = U^i(q^{-\frac{k}{2}}z)\psi^+_i(z)U^i(q^{-\frac{k}{2}}z), \quad \Psi^-_i(z) = U^i(q^{-\frac{k}{2}}z)\psi^-_i(z)U^i(q^{-\frac{k}{2}}z).
\]

where we have set

\[
U^i(z) = \tilde{U}^i(z)A^i(q^{s_i}z), \quad U^i(z) = \tilde{U}^i(z)A^i(q^{-s_i}z), \quad (s_i \in \mathbb{R}).
\]

By necessary condition on commutation relations we get the parameters \( s_i = \frac{k-2N}{2} \). Now we have gotten conjecturable formulae of the dressing operators \( U^i(z) \) and \( U^i(z) \). Using appendix B, we can show every commutation relations of \( e_i(z), f_i(z), \Psi^\pm_i(z) \), by direct calculation. It seems that the method explained above can be applied to the elliptic algebra \( U_{q,p}(\mathfrak{g}) \) for arbitrary \( \mathfrak{g} \).
Here we summarize the normal ordering of the basic operators:

\[ e^{\beta_1 i}(z_1) : B^{\nu i j}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* + 1} z_2 / z_1; q^{2r^*}} \]

\[ e^{\beta_2 i}(z_1) : B^{\nu i j}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* + 1} z_2 / z_1; q^{2r^*}} \]

\[ e^{\beta_3 i}(z_1) : B^{\nu i j}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* - 3} z_2 / z_1; q^{2r^*}} \]

\[ e^{\beta_4 i}(z_1) : B^{\nu i j}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* - 3} z_2 / z_1; q^{2r^*}} \]

\[ e^{\nu i j}(z_1) : B^{\nu i j}(z_2) = \cdot \frac{(q^{2r^* - 1} z_2 / z_1; q^{2r^*})^2}{q^{2r^* + 1} z_2 / z_1; q^{2r^*}} \]

\[ B^{\nu i j}(z_1) : e^{\beta_1 i}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* + 1} z_2 / z_1; q^{2r^*}} \]

\[ B^{\nu i j}(z_1) : e^{\beta_2 i}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* + 1} z_2 / z_1; q^{2r^*}} \]

\[ B^{\nu i j}(z_1) : e^{\beta_3 i}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* - 3} z_2 / z_1; q^{2r^*}} \]

\[ B^{\nu i j}(z_1) : e^{\beta_4 i}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* - 3} z_2 / z_1; q^{2r^*}} \]

\[ B^{\nu i j}(z_1) : e^{\nu i j}(z_2) = \cdot \frac{q^{2r^* - 1} z_2 / z_1; q^{2r^*}}{q^{2r^* - 1} z_2 / z_1; q^{2r^*}} \]

\[ A^{\nu i j}(z_1) : e^{\alpha i j}(z_2) = \cdot \frac{q^{2r^* + N + A_i j} z_2 / z_1; q^{2r^*}}{q^{2r^* - 2k - N - A_i j} z_2 / z_1; q^{2r^*}} \]
Here we have used the notation
\[
(z;p_1,p_2)_\infty = \prod_{n_1,n_2=0}^\infty (1 - p_1^{n_1} p_2^{n_2} z).
\]

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