Stabilization of second-order evolution equations with time delay

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Abstract We consider second-order evolution equations in an abstract setting with damping and time delay and give sufficient conditions ensuring exponential stability. Our abstract framework is then applied to the wave equation, the elasticity system and the Petrovsky system.

Keywords Second-order evolution equations · Delay feedbacks · Stabilization

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1 Introduction

Let $H$ be a real Hilbert space with norm and inner product denoted respectively by $\| \cdot \|_H$ and $(\cdot, \cdot)_H$, and let $A : D(A) \to H$ be a positive self-adjoint operator with a compact inverse in $H$. Denote by $V := D(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$. Moreover, for $i = 1, 2,$, let $U_i$ be real Hilbert spaces with norm and inner product denoted respectively by $\| \cdot \|_{U_i}$ and $(\cdot, \cdot)_{U_i}$ and let $B_i : U_i \to V'$ be linear operators. In this setting, we consider the problem

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\[ u_{tt}(t) + Au(t) + B_1B_1^*u(t) + B_2B_2^*u(t - \tau) = 0, \quad t > 0, \quad (1.1) \]
\[ u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1, \quad (1.2) \]
\[ B_2^*u_t(t) = f^0(t) \quad t \in (-\tau, 0), \quad (1.3) \]

where the constant \( \tau > 0 \) is the time delay. We assume that the delay feedback operator \( B_2 \) is bounded, that is, \( B_2 \in \mathcal{L}(U_2, H) \), while the standard one \( B_1 \in \mathcal{L}(U_1, V') \) may be unbounded.

Time delays are often present in applications and practical problems, and it is by now well known that even an arbitrarily small delay in the feedback may destabilize a system which is uniformly exponentially stable in absence of delay. For some examples in this sense, we refer to \([10, 11, 23, 29]\).

We are interested in giving stability results for the above problem under a suitable assumption on the “size” of the feedback operator \( B_2 \), when the feedback \( B_1 \) is a stabilizing one. More precisely, we will show that for a system which is exponentially stable in absence of time delay, i.e., for \( B_2 = 0 \), the exponential stability is preserved if \( \|B_2^*\| \) is sufficiently small.

On one hand, this paper extends and generalizes the result from \([27]\) that treated the particular case of the wave equation with two local and internal damping terms, one having a time delay, while here we consider an abstract setting and allow unbounded operators \( B_1 \). On the other hand, it completes the analysis of \([1, 23, 25]\). Indeed here, we do not assume

\[ \exists \alpha < 1 \text{ such that } \|B_2^*u\|_{U_2} \leq \alpha \|B_1^*u\|_{U_1}, \quad \forall u \in V; \quad (1.4) \]

as in \([25]\) (cfr. assumption (1.8) of \([23]\) for the wave equation and the assumption (1.14) of \([1]\)). This extension is then significant and not trivial. Indeed, without assuming (1.4) the energy functional associated with the system is no more decreasing in time. This makes the problem difficult to deal with, and we have to use a completely different approach with respect to \([1, 23, 25]\). Furthermore, by removing this assumption, we are able to consider a wide class of concrete models and problems not covered by \([1, 23, 25]\) (see Sects. 4–6 below).

More precisely assuming that an observability inequality holds for the system (1.1), (1.2) when \( B_2 = 0 \), through the definition of a suitable energy [see (3.2)] and the use of a perturbation argument as in \([27]\), we obtain sufficient conditions ensuring exponential stability. Our abstract framework is then applied to some concrete examples, namely the wave equation, the elasticity system and the Petrovsky system with damping terms that do not necessarily satisfy (1.4) and hence not treated in \([23, 25]\).

In \([1]\), the method introduced in \([2]\) is used to investigate the stabilization problem of the system (1.1), (1.2) with \( B_2 = \alpha B_1 \) and \( 0 < \alpha < 1 \) [see (1.4)], i.e., the exponential stability is obtained by combining an observability estimate for the system (1.1), (1.2) when \( B_2 = 0 \) with a boundedness property of a transfer function of the associated open-loop system. It would be interesting to see if this approach can be adapted to our setting.

The paper is organized as follows. In Sect. 2, a well-posedness result of the abstract system is proved. In Sect. 3, we obtain exponential stability results for the abstract
system under suitable conditions. Finally, in Sects. 4–6, we apply our abstract results to the wave equation with local and boundary dampings, the elasticity system and the Petrovsky system, respectively. Other examples (like wave or beam equations on networks) could be given, and we skip them for shortness.

2 Well-posedness

In this section, we will give well-posedness results for problem (1.1)–(1.3) using semigroup theory.

As in [23], we introduce the function

$$z(\rho, t) := B_2^* u_t(t - \tau \rho), \quad \rho \in (0, 1), \ t > 0. \quad (2.1)$$

Then, problem (1.1)–(1.3) can be rewritten as

$$u_{tt}(t) + A u(t) + B_1 B_1^* u(t) + B_2 z(1, t) = 0, \quad t > 0, \quad (2.2)$$
$$\tau z_t(\rho, t) + z_\rho(\rho, t) = 0, \quad \rho \in (0, 1), \ t > 0, \quad (2.3)$$
$$u(0) = u_0 \text{ and } u_t(0) = u_1, \quad (2.4)$$
$$z(\rho, 0) = f^0(-\tau \rho), \quad \rho \in (0, 1), \quad (2.5)$$
$$z(0, t) = B_2^* u_t(t), \quad t > 0. \quad (2.6)$$

If we denote $U := (u, u_t, z)^T$, then $U' = (u_t, u_{tt}, z_t)^T$ and $U$ satisfies

$$\begin{cases} U' = A U \\ U(0) = (u_0, u_1, f^0(-\tau .))^T, \end{cases} \quad (2.7)$$

where the operator $A$ is defined by

$$A \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ -A u - B_1 B_1^* v - B_2 z(1) \\ -\tau^{-1} z_\rho \end{pmatrix}, \quad (2.8)$$

with domain

$$\mathcal{D}(A) := \left\{ (u, v, z)^T \in V \times V \times H^1((0, 1); U_2) : Au + B_1 B_1^* v \in H \text{ and } z(0) = B_2^* v \right\}. \quad (2.9)$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := V \times H \times L^2((0, 1); U_2), \quad (2.10)$$
equipped with the inner product

\[
\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_H := \left\langle A^\frac{1}{2} u, A^\frac{1}{2} \tilde{u} \right\rangle_H + \langle v, \tilde{v} \rangle_H + \xi \int_0^1 \langle z(\rho), \tilde{z}(\rho) \rangle_{U_2} d\rho, \tag{2.11}
\]

where \( \xi \) is any fixed positive number.

The following well-posedness result holds.

**Proposition 2.1** For any initial datum \( U_0 \in H \) there exists a unique solution \( U \in C([0, +\infty), H) \) of problem (2.7). Moreover, if \( U_0 \in D(A) \), then

\[
U \in C([0, +\infty), D(A)) \cap C^1([0, +\infty), H).
\]

**Proof** We will show that the operator \( A \) defined by (2.8), (2.9) generates a strongly continuous semigroup in the Hilbert \( H \) defined in (2.10), (2.11).

Denoting by \( I \) the identity operator, we first show that there exists a positive constant \( c \) such that \( A - cI \) is dissipative (cfr. [3]). Let \( (u, v, z)^T \in D(A) \), then

\[
\left\langle A \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_H := \left\langle \begin{pmatrix} v \\ -Au - B_1 B_1^* v + B_2 z(1) \\ -\tau^{-1} z_\rho 
\end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_H 
\]

\[
= \left\langle A^\frac{1}{2} v, A^\frac{1}{2} u \right\rangle_H - \langle Au, v \rangle_{V^\prime} - \langle B_1 B_1^* v, v \rangle_{V^\prime} - \langle B_2 z(1), v \rangle_{V^\prime} + \xi \int_0^1 \langle z_\rho(\rho), z(\rho) \rangle_{U_2} d\rho.
\]

Since \( Au + B_1 B_1^* v + B_2 z(1) \in H \subset V^\prime \), by duality, we have

\[
\left\langle A \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_H = \left\langle A^\frac{1}{2} v, A^\frac{1}{2} u \right\rangle_H - \langle Au, v \rangle_{V^\prime}, V - \langle B_1 B_1^* v, v \rangle_{V^\prime, V} 
\]

\[
- \langle B_2 z(1), v \rangle_{V^\prime} - \xi \int_0^1 \langle z_\rho(\rho), z(\rho) \rangle_{U_2} d\rho 
\]

\[
= - \| B_1^* v \|_{U_1} - \langle z(1), B_2^* v \rangle_{U_2} - \xi \int_0^1 \langle z_\rho(\rho), z(\rho) \rangle_{U_2} d\rho.
\]
Integrating by parts and using the relation \( z(0) = B_2^*v \), we get
\[
\int_0^1 \langle z_{\rho}(\rho), z(\rho) \rangle_{U_2} d\rho = \frac{1}{2} \left( \|z(1)\|_{U_2}^2 - \|B_2^*v\|_{U_2}^2 \right),
\]
thus, using also Young’s inequality
\[
\langle A \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \rangle_{\mathcal{H}} \leq \left( \frac{\xi}{2} + \frac{1}{2\xi} \right) \|B_2^*v\|_{U_2}^2 \leq c \|v\|_{\mathcal{H}}^2,
\]
for a suitable constant \( c > 0 \). Hence, the operator \( A - cI \) is dissipative.

Now, we show that \( \lambda I - A \) is surjective for some \( \lambda > 0 \). Given \( (f, g, h)^T \in \mathcal{H} \), we seek \( (u, v, z)^T \in D(A) \) such that
\[
(\lambda I - A) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}.
\]
This is equivalent to
\[
\begin{align*}
\lambda u - v &= f; \\
\lambda v + Au + B_1 B_1^* v + B_2 z(1) &= g; \\
\lambda z + \tau^{-1} z_{\rho} &= h.
\end{align*}
\]
Analogously to [23], suppose that we have found \( u \) with the appropriate regularity. Then, by (2.12),
\[
v = \lambda u - f \in V. 
\]
Moreover, from (2.6), (2.14) and (2.15), \( z \) is given by
\[
z(\rho) = \lambda B_2^* u e^{-\lambda \tau \rho} - B_2^* f e^{-\lambda \tau \rho} + \tau e^{-\lambda \tau \rho} \int_0^\rho e^{\lambda \sigma \tau} h(\sigma) d\sigma, \quad \rho \in (0, 1).
\]
In particular,
\[
z(1) = \lambda B_2^* u e^{-\lambda \tau} + z_0,
\]
where
\[
z_0 = -B_2^* f e^{-\lambda \tau} + \tau e^{-\lambda \tau} \int_0^1 e^{\lambda \sigma \tau} h(\sigma) d\sigma,
\]
is a fixed element of \( U_2 \) depending only on \( f \) and \( h \).
It remains only to determine \( u \). From (2.12) and (2.13), \( u \) satisfies
\[
\lambda^2 u + Au + \lambda B_1 B_1^* u + B_2 z(1) = g + \lambda f + B_1 B_1^* f,
\]
and then, by (2.17),
\[
\lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda \tau} B_2 B_2^* u = g + \lambda f + B_1 B_1^* f - B_2 z^0.
\] (2.18)

We denote the right-hand side of (2.18) by \( w \), namely
\[
w := g + \lambda f + B_1 B_1^* f - B_2 z^0 \in H \subset V'.
\]

Then, from (2.18), we have
\[
\left\langle \lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda \tau} B_2 B_2^* u, \varphi \right\rangle_{V', V} = \langle w, \varphi \rangle_{V', V}.
\]

Since \( u \in V \subset H \), we can rewrite
\[
\left\langle \lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda \tau} B_2 B_2^* u, \varphi \right\rangle_{V', V} = \lambda^2 \langle u, \varphi \rangle_H + \langle A \frac{1}{\lambda} u, A \frac{1}{\lambda} \varphi \rangle_H + \lambda \langle B_1^* u, B_1^* \varphi \rangle_{U_1} + \lambda e^{-\lambda \tau} \langle B_2^* u, B_2^* \varphi \rangle_{U_2}.
\]

Therefore, we obtain
\[
\lambda^2 \langle u, \varphi \rangle_H + \langle A \frac{1}{\lambda} u, A \frac{1}{\lambda} \varphi \rangle_H + \lambda \langle B_1^* u, B_1^* \varphi \rangle_{U_1} + \lambda e^{-\lambda \tau} \langle B_2^* u, B_2^* \varphi \rangle_{U_2} = \langle w, \varphi \rangle_{V', V}. \tag{2.19}
\]

The left-hand side of (2.19) is a continuous and coercive bilinear form on \( V \). Then, Lax–Milgram’s lemma implies the existence of a unique solution \( u \in V \) of (2.19) that satisfies
\[
\lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda \tau} B_2 B_2^* u = w \text{ in } V'.
\]

This implies that \( Au \in H \), and by defining \( v \) by (2.15) and \( z \) by (2.16), we have found \( (u, v, z)^T \in D(A) \) satisfying (2.12)–(2.14). This implies that \( \lambda I - A \) is surjective for all \( \lambda > 0 \) and the same holds for the operator \( \lambda I - (A - c I) \).

Then, the Lumer–Phillips theorem implies that \( A - c I \) generates a strongly continuous semigroup of contraction in \( H \). Hence, the operator \( A \) generates a strongly continuous semigroup in \( H \). \( \square \)
3 Stability result

For a fixed constant $\xi$ satisfying

$$\xi > 1,$$  \hfill (3.1)

we define the energy functional for solutions to problem (1.1)–(1.3) as

$$E(t) := E(u, t) = \frac{1}{2} \left( \| A^{1/2} u(t) \|_H^2 + \| u_t(t) \|_H^2 \right) + \frac{\xi}{2} \int_{t-\tau}^t \| B^*_2 u_t(s) \|_{U_2}^2 \, ds. \hfill (3.2)$$

We can obtain a first estimate.

**Proposition 3.1** For any regular solution of problem (1.1)–(1.3), it holds

$$E'(t) \leq - \| B^*_1 u_t(t) \|_{U_1}^2 + \frac{1 + \xi}{2} \| B^*_2 u_t(t) \|_{U_2}^2 - \frac{\xi - 1}{2} \| B^*_2 u_t(t - \tau) \|_{U_2}^2. \hfill (3.3)$$

**Proof** Differentiating $E(t)$, we get

$$E'(t) = \left( A^{1/2} u(t), A^{1/2} u_t(t) \right)_H + \left( u_t(t), u_{tt}(t) \right)_H$$

$$+ \frac{\xi}{2} \| B^*_2 u_t(t) \|_{U_2}^2 - \frac{\xi}{2} \| B^*_2 u_t(t - \tau) \|_{U_2}^2.$$  

Hence, using the definition of $A$ and (1.1), we get successively

$$E'(t) = \langle Au(t), u_t(t) \rangle_{V', V} - \langle u_t(t), Au(t) \rangle_{V', V'} - \langle u_t(t), B_1 B^*_1 u_t(t) \rangle_{V', V'}$$

$$- \langle u_t(t), B_2 B^*_2 u_t(t - \tau) \rangle_{V', V'} + \frac{\xi}{2} \| B^*_2 u_t(t) \|_{U_2}^2 - \frac{\xi}{2} \| B^*_2 u_t(t - \tau) \|_{U_2}^2.$$  

Then,

$$E'(t) = - \| B^*_1 u_t(t) \|_{U_1}^2 - \left( B^*_2 u_t(t), B^*_2 u_t(t - \tau) \right)_{U_2}$$

$$+ \frac{\xi}{2} \| B^*_2 u_t(t) \|_{U_2}^2 - \frac{\xi}{2} \| B^*_2 u_t(t - \tau) \|_{U_2}^2,$$

and (3.3) follows from Cauchy–Schwarz’s inequality. \hfill \Box

Note that, from (3.3), the energy of solutions to problem (1.1)–(1.3) is not decreasing in general. Indeed, the second term in the right-hand side of (3.3), coming from the delay term in (1.1), is nonnegative. We now consider, as in [27], the next auxiliary problem, which is close to the first one but whose energy is decreasing.

$$\varphi_{tt}(t) + A \varphi(t) + B_1 B^*_1 \varphi_t(t) + B_2 B^*_2 \varphi(t - \tau) + \xi B_2 B^*_2 \varphi_t(t) = 0, \quad t > 0, \hfill (3.4)$$

$$\varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \hfill (3.5)$$

$$B^*_2 \varphi_t(t) = g^0(t), \quad t \in (-\tau, 0), \hfill (3.6)$$
where $\xi$ is the same constant as in (3.2).

The well-posedness of system (3.4)–(3.6) can be proved using standard semigroup theory as in Proposition 2.1. Analogously to above, we introduce the function

$$\eta(\rho, t) = B_2^* \varphi(t - \tau \rho), \quad \rho \in (0, 1), \ t > 0;$$

and we rewrite the problem in the abstract form

$$\begin{cases}
\Phi' = A^0 \Phi, \\
\Phi(0) = (\varphi_0, \varphi_1, g^0(-\tau \cdot))^T,
\end{cases} \tag{3.7}$$

where the operator $A^0$ is defined by

$$A^0 \begin{pmatrix} \varphi \\ \psi \\ \eta \end{pmatrix} := \begin{pmatrix} \psi \\ -A \varphi - B_1 \psi - B_2 \eta(1) - \xi B_2^* \psi \\ -\tau^{-1} \eta \rho \end{pmatrix},$$

with domain $\mathcal{D}(A^0) = \mathcal{D}(A)$ [see (2.9)] in the Hilbert space $\mathcal{H}$ defined by (2.10) and (2.11).

**Proposition 3.2** For any initial datum $\Phi_0 \in \mathcal{H}$, there exists a unique solution $\Phi \in C([0, +\infty), \mathcal{H})$ of problem (3.7). Moreover, if $\Phi_0 \in \mathcal{D}(A^0)$, then

$$\Phi \in C([0, +\infty), \mathcal{D}(A^0)) \cap C^1([0, +\infty), \mathcal{H}).$$

For solutions of problem (3.4)–(3.6), the energy $F(\cdot)$,

$$F(t) := F(\varphi, t) = \frac{1}{2} \left( \| A^1_2 \varphi(t) \|_H^2 + \| \varphi_t(t) \|_H^2 \right) + \frac{\xi}{2} \int_{t-\tau}^{t} \| B_2^* \varphi_t(s) \|_{U_2}^2 \, ds, \tag{3.8}$$

with $\xi$ satisfying (3.1), is decreasing in time.

More precisely, we have the following result.

**Proposition 3.3** For any regular solution of problem (3.4)–(3.6), we have

$$F'(t) \leq -\| B_1^* \varphi_t(t) \|_{U_1}^2 - \frac{\xi - 1}{2} \| B_2^* \varphi_t(t) \|_{U_2}^2 - \frac{\xi - 1}{2} \| B_2^* \varphi_t(t - \tau) \|_{U_2}^2. \tag{3.9}$$

Then, if $\xi$ satisfies (3.1), the energy $F(\cdot)$ is decreasing.
Proof In order to have (3.9), we differentiate (3.8). Hence, using the definition of $A$ and (3.4), we obtain

$$
F'(t) = \left\langle A^{1/2} \varphi, A^{1/2} \varphi_t \right\rangle_H + \langle \varphi_t, \varphi_{tt} \rangle_H + \frac{\xi}{2} \left\| B^*_2 \varphi_t(t) \right\|^2_{U_2} - \frac{\xi}{2} \left\| B^*_2 \varphi_t(t-\tau) \right\|^2_{U_2} 
$$

$$
= \langle A \varphi(t), \varphi_t(t) \rangle_{V',V} - \langle \varphi_t(t), A \varphi(t) \rangle_{V,V'} 
- \langle \varphi_t(t), B_1 B^*_1 \varphi_t(t) \rangle_{V',V'} - \xi \langle \varphi_t(t), B_2 B^*_2 \varphi_t(t) \rangle_{V,V'} 
- \langle \varphi_t(t), B_2 B^*_2 \varphi_t(t-\tau) \rangle_{V',V'} + \frac{\xi}{2} \left\| B^*_2 \varphi_t(t) \right\|^2_{U_2} - \frac{\xi}{2} \left\| B^*_2 \varphi_t(t-\tau) \right\|^2_{U_2}.
$$

Then,

$$
F'(t) = - \left\| B^*_1 \varphi_t(t) \right\|^2_{U_1} - \xi \left\| B^*_2 \varphi_t(t) \right\|^2_{U_2} - \langle B^*_2 \varphi_t(t), B^*_2 \varphi_t(t-\tau) \rangle_{U_2} 
+ \frac{\xi}{2} \left\| B^*_2 \varphi_t(t) \right\|^2_{U_2} - \frac{\xi}{2} \left\| B^*_2 \varphi_t(t-\tau) \right\|^2_{U_2},
$$

and therefore, (3.9) follows from Cauchy–Schwarz’s inequality. \hfill \Box

Consider now the following damped system associated with (1.1) and (1.2),

$$
w_{tt}(t) + A w(t) + B_1 B^*_1 w_t = 0, \quad t > 0, \tag{3.10}
$$

$$
w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1, \tag{3.11}
$$

with $(w_0, w_1) \in V \times H$. For our stability result, we need that this system is exponentially stable or equivalently that the next observability inequality holds (see Lemma 3.2 of [22]). Namely, we assume that there exists a time $\overline{T} > 0$ such that for every time $T > \overline{T}$, there is a constant $c$, depending on $\overline{T}$ but independent of the initial data, such that

$$
E_S(0) \leq c \int_0^T \left\| B^*_1 w(t) \right\|^2_{U_1} dt, \tag{3.12}
$$

for every weak solution of problem (3.10), (3.11) with initial data $(w_0, w_1) \in V \times H$. Here, $E_S(\cdot)$ denotes the standard energy for wave-type equations, that is

$$
E_S(t) = E_S(w, t) := \frac{1}{2} \left( \left\| A^{1/2} w(t) \right\|^2_H + \left\| w_t(t) \right\|^2_H \right).
$$

For shortness, let us denote by $C_2$ the norm of $B_2$

$$
\|B_2\| = \left\| B^*_2 \right\| = C_2. \tag{3.13}
$$

We can prove an exponential stability result for the perturbed problem (3.4)–(3.6).
**Theorem 3.4** Assume that (3.1) holds and that the observability estimate (3.12) holds for problem (3.10)–(3.11). Then, there are two positive constants \( K, \tilde{\mu} \) such that

\[
F(t) \leq Ke^{-\tilde{\mu}t} F(0), \quad t > 0,
\]

(3.14)

for any solution of problem (3.4)–(3.6). In particular,

\[
K = \frac{C_0 + 1}{C_0},
\]

(3.15)

\[
\tilde{\mu} = \frac{1}{2T} \ln \frac{C_0 + 1}{C_0},
\]

(3.16)

with \( T \) any fixed time satisfying \( T > \max \{T, \tau\} \), \( T \) being an observability time for (3.12), and

\[
C_0 = \max \left\{ 2c, \frac{32cTC_2^2 + \xi}{\xi - 1}, \frac{32cC_2^2T\xi^2}{\xi - 1} \right\},
\]

(3.17)

where \( C_2 \) is as in (3.13) and \( c := c(T) \) is the observability constant in (3.12).

**Proof** Following a classical argument (see [30]), we can decompose the solution \( \varphi \) of (3.4)–(3.6) as

\[
\varphi = w + \tilde{w}
\]

where \( w \) is the solution of system (3.10), (3.11) with \( w_0 = \varphi_0, w_1 = \varphi_1 \); while \( \tilde{w} \) solves

\[
\tilde{w}_{tt}(t) + A\tilde{w}(t) + B_1B_1^*\tilde{w}_t(t) = -\xi B_2B_2^*\varphi_t(t) - B_2B_2^*\varphi_t(t - \tau), \quad t > 0, \quad \tilde{w}(0) = 0 \quad \text{and} \quad \tilde{w}_t(0) = 0.
\]

(3.18)

(3.19)

By (3.8),

\[
F(0) = E_S(w, 0) + \frac{\xi}{2} \int_{-\tau}^{0} \left\| B_2^*\varphi_t(s) \right\|_{U_2}^2 ds
\]

\[
= E_S(w, 0) + \frac{\xi}{2} \int_{0}^{\tau} \left\| B_2^*\varphi_t(t - \tau) \right\|_{U_2}^2 dt.
\]
Therefore, from (3.12), if \( T > \max\{\bar{T}, \tau\} \) we obtain

\[
F(0) \leq c \int_0^T \left\| B_1^* w_t(t) \right\|_{U_1}^2 dt + \frac{\xi}{2} \int_0^T \left\| B_2^* \varphi_t(t - \tau) \right\|_{U_2}^2 dt
\]

\[
\leq 2c \int_0^T \left( \left\| B_1^* \varphi_t(t) \right\|_{U_1}^2 + \left\| B_1^* \tilde{w}_t(t) \right\|_{U_1}^2 \right) dt
\]

\[
+ \frac{\xi}{2} \int_0^T \left\| B_2^* \varphi_t(t - \tau) \right\|_{U_2}^2 dt,
\]

(3.20)

where \( c \) is the observability constant for the damped system (3.10), (3.11).

Now, observe that from (3.18),

\[
\frac{d}{dt} \frac{1}{2} \left( \left\| \tilde{w}_t(t) \right\|_H^2 + \left\| A^{\frac{1}{2}} \tilde{w}(t) \right\|_H^2 \right) + \left\| B_1^* \tilde{w}_t \right\|_{U_1}^2
\]

\[
= \langle \tilde{w}_t, \tilde{w}_{tt} + A \tilde{w} + B_1 B_1^* \tilde{w}_t \rangle
\]

\[
= \langle \tilde{w}_t, -\xi B_2 B_2^* \varphi_t(t) - B_2 B_2^* \varphi_t(t - \tau) \rangle_H.
\]

Integrating in time from 0 to \( t \), for \( t \in (0, 2T] \), and using (3.19), we have

\[
\frac{1}{2} \left( \left\| \tilde{w}_t(t) \right\|_H^2 + \left\| A^{\frac{1}{2}} \tilde{w}(t) \right\|_H^2 \right) + \int_0^t \left\| B_1^* \tilde{w}_t(s) \right\|_{U_1}^2 ds
\]

\[
= \int_0^t \langle \tilde{w}_t, -\xi B_2 B_2^* \varphi_t(s) - B_2 B_2^* \varphi_t(s - \tau) \rangle_H ds,
\]

and then

\[
\left\| \tilde{w}_t(t) \right\|_H^2 + 2 \int_0^t \left\| B_1^* \tilde{w}_t(s) \right\|_{U_1}^2 ds
\]

\[
\leq \frac{1}{8TC^2} \int_0^t \left\| B_2^* \tilde{w}_t(s) \right\|_{U_2}^2 ds + 8T C^2 \xi^2 \int_0^t \left\| B_2^* \varphi_t(s) \right\|_{U_2}^2 ds
\]

\[
+ \frac{1}{8TC^2} \int_0^t \left\| B_2^* \tilde{w}_t(s) \right\|_{U_2}^2 ds + 8T C^2 \xi^2 \int_0^t \left\| B_2^* \varphi_t(s - \tau) \right\|_{U_2}^2 ds,
\]

(3.21)

where \( C_2 \) was defined in (3.13).
This estimate directly implies that for all \( t \in [0, 2T] \), one has

\[
\| \tilde{w}_t(t) \|_H^2 + 2 \int_0^t \| B_1^* \tilde{w}_t(s) \|_{U_1}^2 \, ds \leq \frac{1}{4TC_2^2} \int_0^{2T} \| B_2^* \tilde{w}_t(s) \|_{U_2}^2 \, ds \\
+ 8TC_2^2 \xi^2 \int_0^{2T} \| B_2^* \varphi_t(s) \|_{U_2}^2 \, ds \\
+ 8TC_2^2 \int_0^{2T} \| B_2^* \varphi_t(s - \tau) \|_{U_2}^2 \, ds, \tag{3.22}
\]

and so, integrating in \([0, 2T]\),

\[
\frac{1}{2} \int_0^{2T} \| \tilde{w}_t(t) \|_H^2 \, dt + 2 \int_0^{2T} \int_0^t \| B_1^* \tilde{w}_t(s) \|_{U_1}^2 \, ds \, dt \\
\leq \frac{1}{2C_2^2} \int_0^{2T} \| B_2^* \tilde{w}_t(s) \|_{U_2}^2 \, ds + 16T^2C_2^2 \xi^2 \int_0^{2T} \| B_2^* \varphi_t(s) \|_{U_2}^2 \, ds \\
+ 16T^2C_2^2 \int_0^{2T} \| B_2^* \varphi_t(s - \tau) \|_{U_2}^2 \, ds. \tag{3.23}
\]

Therefore,

\[
\frac{1}{2} \int_0^{2T} \| \tilde{w}_t(t) \|_H^2 \, dt + 2 \int_0^{2T} \int_0^t \| B_1^* \tilde{w}_t(s) \|_{U_1}^2 \, ds \, dt \\
\leq 16T^2C_2^2 \xi^2 \int_0^{2T} \| B_2^* \varphi_t(s) \|_{U_2}^2 \, ds + 16T^2C_2^2 \int_0^{2T} \| B_2^* \varphi_t(s - \tau) \|_{U_2}^2 \, ds,
\]

from which follows

\[
\int_0^{2T} \int_0^t \| B_1^* \tilde{w}_t(s) \|_{U_1}^2 \, ds \, dt \leq 8T^2C_2^2 \xi^2 \int_0^{2T} \| B_2^* \varphi_t(s) \|_{U_2}^2 \, ds \\
+ 8T^2C_2^2 \int_0^{2T} \| B_2^* \varphi_t(s - \tau) \|_{U_2}^2 \, ds.
\]
Using the fact that
\[
\int_0^t \int_0^t ( \| B_1^* \tilde{w}_t(s) \|^2_{U_1} ) \ ds \ dt = \int_0^t \int_0^{2T} ( \| B_1^* \tilde{w}_t(s) \|^2_{U_1} (2T - s) ) \ ds \ dt
\]
\[
\geq \int_0^t \int_0^{2T} ( \| B_1^* \tilde{w}_t(s) \|^2_{U_1} ) \ ds \ dt
\]
\[
\geq T \int_0^T ( \| B_1^* \tilde{w}_t(s) \|^2_{U_1} ) \ ds,
\]
we deduce that
\[
\int_0^T ( \| B_1^* \tilde{w}_t(s) \|^2_{U_1} ) \ ds \leq \frac{2T}{8} \xi \int_0^T ( \| B_2^* \phi(t) \|^2_{U_1} ) \ ds + 8T C_1^2 \int_0^T ( \| B_2^* \phi(t) \|^2_{U_1} ) \ ds. \tag{3.24}
\]

Using (3.24) in (3.20), we obtain
\[
F(0) \leq 2c \int_0^T ( \| B_1^* \phi(t) \|^2_{U_1} ) \ ds + 16cT C_2^2 \xi \int_0^T ( \| B_2^* \phi(t - \tau) \|^2_{U_2} ) \ ds + 16cC_2^2 T \xi \int_0^T ( \| B_2^* \phi(t) \|^2_{U_2} ) \ ds, \tag{3.25}
\]
that we rewrite as
\[
F(0) \leq 2c \int_0^{2T} ( \| B_1^* \phi(t) \|^2_{U_1} ) \ ds + \frac{32cT C_2^2 + \xi}{\xi - 1} \left( \int_0^{2T} ( \| B_2^* \phi(t - \tau) \|^2_{U_2} ) \ ds \right)
\]
\[
+ \frac{32cC_2^2 T \xi}{\xi - 1} \left( \frac{\xi - 1}{2} \int_0^{2T} ( \| B_2^* \phi(t) \|^2_{U_2} ) \ ds \right) \leq -C_0 \int_0^{2T} F'(t) \ ds. \tag{3.26}
\]
with $C_0$ as in (3.17).

Therefore, from (3.26), using also that $F(\cdot)$ is decreasing, we obtain
\[
F(2T) \leq F(0) \leq C_0 (F(0) - F(2T)),
\]
Then,
\[ F(2T) \leq \frac{C_0}{C_0 + 1} F(0), \]
and this implies the exponential estimate (3.14) with \( K, \tilde{\mu} \) as in (3.15), (3.16), due to the semigroup property together with the fact that \( F \) is non increasing. \( \square \)

Now, let us recall the following classical result of Pazy (Theorem 1.1 in Ch. 3 of [26]).

**Theorem 3.5** Let \( X \) be a Banach space and let \( A \) be the infinitesimal generator of a \( C_0 \) semigroup \( T(t) \) on \( X \), satisfying \( \|T(t)\| \leq Me^{\omega t} \). If \( B \) is a bounded linear operator on \( X \), then \( A + B \) is the infinitesimal generator of a \( C_0 \) semigroup \( S(t) \) on \( X \), satisfying \( \|S(t)\| \leq Me^{(\omega + M\|B\|)t} \).

Using this perturbation result, we are ready to prove our main result, namely the asymptotic stability result for problem (1.1)–(1.3) when the norm of the delay feedback operator \( B_2^* \) is sufficiently small.

**Theorem 3.6** Assume that the observability estimate (3.12) holds for problem (3.10)–(3.11). For all \( \xi > 1 \) in the definition (3.2), there is \( \beta > 0 \) depending on \( T, \tau, \xi \) and on the operator \( B_1 \), such that if the delay feedback operator satisfies \( \|B_2^*\| < \beta \), then there exist positive constants \( K, \mu \) for which we have
\[ E(t) \leq Ke^{-\mu t} E(0), \quad t > 0, \quad (3.27) \]
for any solution of (1.1)–(1.3).

**Proof** We can see problem (1.1)–(1.3) as a perturbation of the auxiliary one. Therefore,
\[ A \begin{pmatrix} u \\ v \\ z \end{pmatrix} = (A^0 + B) \begin{pmatrix} u \\ v \\ z \end{pmatrix} \]
with
\[ B \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -\xi B_2^* v \\ 0 \end{pmatrix}. \]

From Theorems 3.4 and 3.5, we know that if
\[ -\tilde{\mu} + K \|B\| < 0, \quad (3.28) \]
where \( \tilde{\mu} \) and \( K \) are defined by (3.15)–(3.17), then (3.27) holds with \( \mu = \tilde{\mu} - K \|B\| \).
It remains to prove that (3.28) is satisfied for \( \| B_2 \| \) sufficiently small. We can rewrite (3.28) as

\[
\xi \| B_2 \|^2 < \frac{\tilde{\mu}}{K},
\]

that is

\[
\xi C_2^2 < \frac{1}{2T} \frac{C_0}{C_0 + 1} \ln \frac{C_0 + 1}{C_0}.
\]

The difficulty is that the constant \( C_0 \) [defined by (3.17)] appearing in the right-hand side of this estimate depends on \( \xi \) and \( C_2 \) as well. So let us consider the continuous function \( h : (0, +\infty) \to (0, +\infty) \),

\[
h(s) := \frac{s}{s + 1} \ln \frac{s + 1}{s}.
\]

Then, \( h \) tends to zero for \( s \to 0^+ \) and for \( s \to +\infty \). Moreover, \( h \) assumes the maximal value \( \frac{1}{e} \) at \( \frac{1}{e-1} \), is increasing before \( \frac{1}{e-1} \) and decreasing after.

Now, it is easy to check that \( \frac{\xi}{\xi - 1} > \frac{1}{e-1} \). Considering that \( \xi \) is fixed > 1 and \( T \) is fixed as well, we consider \( C_0 \) as a function of \( C_2 \geq 0 \) that we write \( C_0(C_2) \). But we remark that from its definition, \( C_0 \) is nondecreasing (in \( C_2 \)) and

\[
C_0(0) = \max \left\{ 2c, \frac{\xi}{\xi - 1} \right\} > \frac{1}{e - 1}.
\]

Hence, \( h(C_0(C_2)) \) is nonincreasing as a function of \( C_2 \) with \( h(C_0(0)) > 0 \) and since the left-hand side of (3.29) is increasing in \( C_2 \) and is zero at \( C_2 = 0 \), there exists a point \( \beta > 0 \) such that

\[
2\xi \beta^2 T = h(C_0(\beta)),
\]

and for which (3.29) holds for all \( C_2 \in [0, \beta) \).

Obviously, \( \beta \) depends on \( T \) (and then on \( T \) and \( \tau \)), on \( \xi \) and, through the observability constant \( c \) and the time \( T \), on the feedback operator \( B_1 \).

**Remark 3.7** If \( B_1 \) is bounded, namely if \( B_1 \in \mathcal{L}(U_1, H) \), then by Proposition 1 of [13], the system (3.10)–(3.11) is exponentially stable [or equivalently the observability estimate (3.12) holds for problem (3.10)–(3.11)] if and only if the observability estimate

\[
E_S(0) = \frac{1}{2} \left( \| A^\frac{1}{2} w_0 \|_H^2 + \| w_1 \|_H^2 \right) \leq c \int_0^T \| B_1^* \varphi(t) \|_{U_1}^2 \, dt,
\]  

(3.30)
holds for some $T > 0$ and $c > 0$, for every weak solution $\varphi$ of the conservative system

\begin{align}
\varphi_{tt}(t) + A \varphi(t) &= 0, \quad t > 0, \\
\varphi(0) &= w_0 \quad \text{and} \quad \varphi_t(0) = w_1,
\end{align}

(3.31)

(3.32)

with initial data $(w_0, w_1) \in V \times H$.

**Remark 3.8** The previous Theorem 3.6 states that for all $\xi > 1$, there exists $\beta^*(\xi) > 0$ (depending on $\xi$) such that exponential stability holds if the delay feedback satisfies $C_2 = \|B_2^*\| < \beta^*(\xi)$. But an optimal upper bound can be easily obtained. Indeed from the proof of Theorem 3.6, we see that for any fixed $\xi > 1$, $\beta^*(\xi)$ satisfies

$$
\beta^*(\xi)^2 = \frac{h(C_0(\beta^*(\xi)))}{2\xi T},
$$

and as $\max_{s>0} h(s) = 1/e$, we deduce that

$$
\beta^*(\xi)^2 \leq \frac{1}{2Te}.
$$

By setting

$$
\beta^* = \sup_{\xi > 1} \beta^*(\xi),
$$

we deduce that for all $C_2 < \beta^*$, there exists at least one $\xi > 1$ such that $C_2 < \beta^*(\xi)$ and hence such that (3.29) holds. In other words, for every $B_2$ with $\|B_2^*\| < \beta^*$, the exponential decay is valid.

### 4 The wave equation

#### 4.1 Internal dampings

Our first application concerns the wave equation with locally distributed internal dampings. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a Lipschitz boundary $\partial \Omega$. We suppose given $b_1, b_2$ in $L^\infty(\Omega)$ such that

$$
b_1(x), b_2(x) \geq 0 \quad \text{a.e.} \ x \in \Omega.
$$
Let us consider the initial boundary value problem

\[
\begin{align*}
\dot{u}_{tt}(x,t) - \Delta u(x,t) + b_1(x)u_t(x,t) + b_2(x)u_t(x,t - \tau) &= 0 \quad \text{in } \Omega \times (0, +\infty), \quad (4.1) \\
u(x,t) &= 0 \quad \text{on } \partial \Omega \times (0, +\infty), \quad (4.2) \\
\frac{\partial u}{\partial v}(x,t) &= -k(x)u_t(x,t), \quad x \in \Gamma_0, \ t > 0 \quad (4.3) \\
u(x,0) &= u_0(x) \ \text{and} \ \dot{u}(x,0) = u_1(x) \ \text{in } \Omega, \quad (4.4) \\
\sqrt{b_2}u_t(x,t) &= f_0(x,t) \ \text{in } \omega_2 \times (-\tau, 0), \quad (4.5)
\end{align*}
\]

with initial data \((u_0, u_1, f_0) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2((-\tau, 0); L^2(\omega_2))\), where \(\omega_i = \{x \in \Omega : b_i(x) > 0\}\) is the support of \(b_i, i = 1 \text{ or } 2\).

This problem enters into our previous framework, if we take \(H = L^2(\Omega)\) and the operator \(A\) defined by

\[A : \mathcal{D}(A) \to H : u \to -\Delta u,\]

where \(\mathcal{D}(A) = \{u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega)\}\). This operator \(A\) is a self-adjoint and positive operator with a compact inverse in \(H\) and is such that \(V = \mathcal{D}(A^{1/2}) = H^1_0(\Omega)\).

We then define \(U_1 = L^2(\omega_1), U_2 = L^2(\omega_2)\) and the operators \(B_i, i = 1, 2,\) as

\[B_i : U_i \to H : v \to \sqrt{b_i(x)}\tilde{v}, \quad (4.6)\]

where \(\tilde{v} \in L^2(\Omega)\) is the extension of \(v\) by zero outside \(\omega_i\). It is easy to verify that

\[B_i^* \varphi = \sqrt{b_i} \varphi_{|\omega_i} \quad \text{for } \varphi \in H.\]

As \(B_i B_i^* \varphi = b_i \varphi\), for any \(\varphi \in H\) and \(i = 1, 2\), we deduce that problem \((4.1)-(4.5)\) enters in the abstract framework \((1.1)-(1.3)\).

In this setting, the energy functional is

\[
E(t) = \frac{1}{2} \int_{\Omega} \left\{ \dot{u}_{tt}^2(x,t) + |\nabla u(x,t)|^2 \right\} \, dx + \frac{\xi}{2} \int_{t-\tau}^t \int_{\Omega} b_2(x)u_t^2(x,s) \, dx \, ds, \quad (4.7)
\]

which is the standard energy for wave equation

\[
E_S(t) = E_S(w, t) := \frac{1}{2} \int_{\Omega} \left( \dot{w}_{tt}^2 + |\nabla w|^2 \right) \, dx,
\]

plus an integral term due to the presence of a time delay.
Since \( B_1 \) is bounded, according to Remark 3.7, our main assumption concerns the existence of an observability estimate for the standard wave equation:

\[
\begin{align*}
\varphi_{tt}(x,t) - \Delta \varphi(x,t) &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
\varphi(x,t) &= 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
\varphi(x,0) &= \varphi_0(x) \text{ and } \varphi_t(x,0) = \varphi_1(x) \quad \text{in } \Omega,
\end{align*}
\]

with \((\varphi_0, \varphi_1) \in H^1_0(\Omega) \times L^2(\Omega)\).

We then assume that there exists a time \( T > 0 \) such that for every time \( T > \bar{T} \), there is a constant \( c \), depending on \( T \) but independent of the initial data, such that

\[
E_S(0) \leq c \int_0^T \int_\Omega b_1(x) \varphi_t^2(x,t) \, dx \, ds,
\]

for every weak solution of problem (4.8)–(4.10).

According to (3.13), we have

\[
\|B_2\| = \|b_2\|^{1/2}_\infty,
\]

where for \( v \in L^\infty(\Omega) \), we denote \( \|v\|_\infty = \sup_{x \in \Omega} |v(x)| \), the \( L^\infty \) norm of \( v \).

Therefore, according to Theorem 3.6, we have the next result:

**Theorem 4.1** Assume that the observability estimate (4.11) holds for the wave equation (4.8)–(4.10). For all \( \xi > 1 \) in the definition (4.7), there is \( \beta > 0 \) depending on \( \bar{T}, \tau, \xi \) and \( b_1 \) such that if \( \|b_2\|_\infty < \beta \), then there exist positive constants \( K, \mu \) for which we have

\[
E(t) \leq Ke^{-\mu t}E(0), \quad t > 0,
\]

for any solution of (4.1)–(4.5).

**Remark 4.2**

1. From Lemma VII.2.4 of [20] (see also [16,17,19,21,30]), the observability estimate (4.11) holds for the wave equation (4.8)–(4.10) if the boundary of \( \Omega \) is of class \( C^3 \), if \( T \) is bigger than the diameter of \( \Omega \) and if

\[
b_1(x) \geq b^0 > 0, \quad \text{a.e. } x \in \omega,
\]

when the open subset \( \omega \) of \( \Omega \) is a neighborhood of \( \bar{\Gamma}_0 \), where

\[
\Gamma_0 = \{ x \in \partial \Omega : (x - x_0) \cdot \nu(x) > 0 \},
\]

for some \( x_0 \in \mathbb{R}^n \) and \( \nu(x) \) is the outer unit normal vector at \( x \in \partial \Omega \).

2. From [4], the observability estimate (4.11) also holds for the wave equation (4.8)–(4.10) if the boundary of \( \Omega \) is of class \( C^\infty \) and if (4.13) holds when the open subset \( \omega \) of \( \Omega \) satisfies the geometric control property.

\( \square \)
Remark 4.3 According to point 1 of the previous remark, Theorem 4.1 allows to recover the results from Theorem 1.2 of [27] in a larger setting. □

4.2 Internal and boundary dampings

We assume here that the boundary ∂Ω of Ω is splitted up as ∂Ω = Γ₀ ∪ Γ₁, where Γ₀, Γ₁ are closed subsets of ∂Ω with Γ₀ ∩ Γ₁ = ∅. Moreover, we assume that Γ₀ and Γ₁ have a nonempty interior (on ∂Ω). We suppose given k ∈ L∞(Γ₀) and b ∈ L∞(Ω) such that b(x) ≥ 0 a.e. x ∈ Ω and

\[ k(x) ≥ k₀ > 0 \quad \text{a.e. } x ∈ Γ₀. \]

We here consider the problem

\[ u_{tt}(x, t − Δu(x, t) + b(x)u_t(x, t − τ) = 0, \quad x ∈ Ω, \ t > 0, \quad (4.15) \]

\[ u(x, t) = 0, \quad x ∈ Γ₁, \ t > 0 \quad (4.16) \]

\[ \frac{∂u}{∂ν}(x, t) = −k(x)u_t(x, t), \quad x ∈ Γ₀, \ t > 0 \quad (4.17) \]

\[ u(x, 0) = u₀(x), \quad u_t(x, 0) = u₁(x), \quad x ∈ Ω, \quad (4.18) \]

\[ \sqrt{b}u_t(x, t) = f₀(x, t), \quad x ∈ Ω, \ t ∈ (−τ, 0), \quad (4.19) \]

with initial data \((u₀, u₁, f₀) ∈ H₁^Γ₀(Ω) × L²(Ω) × L²((−τ, 0); L²(ω₂)), \) where

\[ H₁^Γ₀ := \{ u ∈ H¹(Ω) : u = 0 \text{ on } Γ₁ \}. \]

This problem enters into our previous framework, if we take \( H = L²(Ω) \) and the operator \( A \) defined by

\[ A : \mathcal{D}(A) → H : u → −Δu, \]

where

\[ \mathcal{D}(A) := \left\{ u ∈ H₁Γ₀(Ω) : Δu ∈ L²(Ω) \text{ and } \frac{∂u}{∂ν} = 0 \text{ on } Γ₀ \right\}. \]

We then define \( U₁ := L²(Γ₀), \quad U₂ := L²(ω₂) \) (ω₂ being the support of b) and the operators \( B₁, B₂ \) as

\[ B₂ ∈ \mathcal{L}(U₂; H), \quad B₂u = √b(x)u, \quad ∀ u ∈ L²(ω₂), \]

and

\[ B₁ ∈ \mathcal{L}(U₁; V'), \quad B₁u = \sqrt{k}A_{−1}Nu, \quad ∀ u ∈ L²(Γ₀), \]

\[ B₁^*w = \sqrt{k}uw|Γ₀, \quad ∀ w ∈ V := \mathcal{D}(A^{1/2}), \]
where $A_{-1}$ is the extension of $A$ to $H$, namely for all $h \in H$ and $\varphi \in \mathcal{D}(A)$, $A_{-1}h$ is the unique element in $(\mathcal{D}(A))'$ (the duality is in the sense of $H$), such that (see for instance [28])

$$\langle A_{-1}h; \varphi \rangle_{(\mathcal{D}(A))',\mathcal{D}(A)} = \int_{\Omega} hA\varphi \, dx.$$ 

Here and below, $N \in \mathcal{L}(L^2(\Gamma_0); L^2(\Omega))$ is defined as follows: for all $v \in L^2(\Gamma_0)$, $Nv$ is the unique solution (transposition solution) of

$$\Delta Nv = 0, \quad Nv|_{\Gamma_1} = 0, \quad \frac{\partial Nv}{\partial \nu}|_{\Gamma_0} = v.$$

With these definitions, we can show that problem (4.15)–(4.19) enters in the abstract framework (1.1)–(1.3).

Now, the energy functional is

$$E(t) = \frac{1}{2} \int_{\Omega} \left\{ u_t^2(x,t) + |\nabla u(x,t)|^2 \right\} \, dx + \frac{\xi}{2} \int_{t-\tau}^{t} \int_{\Omega} b(x)u_t^2(x,s)dxds. \quad (4.20)$$

As $B_1$ is not bounded, we need to consider the nondelayed system

$$w_{tt}(x,t) - \Delta w(x,t) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (4.21)$$

$$w(x,t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (4.22)$$

$$\frac{\partial w}{\partial \nu}(x,t) = -k(x)w_t(x,t), \quad x \in \Gamma_0, \quad t > 0, \quad (4.23)$$

$$w(x,0) = w_0(x) \text{ and } w_t(x,0) = w_1(x) \quad \text{in } \Omega, \quad (4.24)$$

with $(w_0, w_1) \in H^1_{0}(\Omega) \times L^2(\Omega)$.

Hence, our main assumption will be: There exists a time $T > 0$ such that for every time $T > \overline{T}$, there is a constant $c$, depending on $T$ but independent of the initial data, such that

$$E_S(0) \leq c \int_{0}^{T} \int_{\Gamma_0} k(x)w_t^2(x,s)dxds, \quad (4.25)$$

for every weak solution of problem (4.21)–(4.24).

Then, our previous results apply also to this model, and we can restate Theorem 3.6.

**Theorem 4.4** Assume that the observability estimate (4.25) holds for every weak solution of problem (4.21)–(4.24). For all $\xi > 1$ in the definition (4.20), there is $\beta > 0$
depending on $\overline{t}$, $\tau$, $\xi$ and $k$, such that if $\|b(x)\|_\infty < \beta$, then there exist positive constants $K, \mu$ for which we have
\[ E(t) \leq Ke^{-\mu t} E(0), \quad t > 0, \]
for any solution of (4.15)–(4.19).

**Remark 4.5**
1. From Theorem 1 and Remark 1 of [15] (see also [7–9,18]), the observability estimate (4.25) holds for the damped wave equation (4.21)–(4.24) if the boundary of $\Omega$ is of class $C^2$, if $T$ is large enough and if $\Gamma_0$ is given by (4.14) for some $x_0 \in \mathbb{R}^n$.
2. From [4], the observability estimate (4.25) also holds for the damped wave equation (4.21)–(4.24) if the boundary of $\Omega$ is of class $C^\infty$ and if the part $\Gamma_0$ satisfies the geometric control property.
3. If we suppress the assumption $\Gamma_0 \cap \Gamma_1 = \emptyset$, then Theorem 1 of [15] shows that the observability estimate (4.25) holds for the damped wave equation (4.21)–(4.24) under the same assumptions than in point 1 but with the choice $k(x) = (x - x_0) \cdot \nu(x)$ and if $n \leq 3$ (see also Proposition 6.4 of [12] in dimension 2). For this example, $k$ is no more uniformly positive on $\Gamma_0$, nevertheless it enters into our abstract framework.

**Remark 4.6**
This result, namely exponential decay of the energy for solutions to problem (4.15)–(4.19) for “small” internal delay feedback, has been first proved in [3], for $b$ and $k$ constant and $\Gamma_0$ given by (4.14), by constructing a suitable Lyapunov functional and using the multiplier method. We give here a simpler proof by using a more general method, allowing to weaken the assumptions on $b$, $k$ and $\Gamma_0$.

5 The elasticity system

5.1 Internal dampings

Let $\Omega \subset \mathbb{R}^n$ an open bounded domain with a Lipschitz boundary $\partial \Omega$. We consider the following elastodynamic system:

\[ u_{tt}(x, t) - \mu \Delta u(x, t) - (\lambda + \mu)\nabla \operatorname{div} u \]
\[ + b_1(x)u_t(x, t) + b_2(x)u_t(x, t - \tau) = 0 \quad \text{in } \Omega \times (0, +\infty), \]
\[ u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \]
\[ u(x, 0) = u_0(x) \quad \text{and } u_t(x, 0) = v_0(x) \quad \text{in } \Omega, \]
\[ \sqrt{b_2}u_t(x, t) = f^0(t) \quad \text{in } \omega_2 \times (-\tau, 0), \]

with initial data $(u_0, v_0, f^0) \in H^1_0(\Omega)^n \times L^2(\Omega)^n \times L^2((\tau, 0); L^2(\omega_2)^n)$ and $b_1, b_2$ satisfying the same assumptions as in Sect. 4.1. Note that in this case, the state variable $u$ is vector-valued and takes values in $\mathbb{R}^n$, while $\lambda$, $\mu$ are the Lamé coefficients that are positive real numbers.
As before this problem enters into our abstract setting, once we take $H = L^2(\Omega)^n$, and $A$ defined by

$$A : \mathcal{D}(A) \to H : u \to -\mu \Delta u(x, t) - (\lambda + \mu) \nabla \text{div } u,$$

where $\mathcal{D}(A) = \{ u \in H^1_0(\Omega)^n : \mu \Delta u + (\lambda + \mu) \nabla \text{div } u \in L^2(\Omega)^n \}$.

The operator $A$ is a self-adjoint and positive operator with a compact inverse in $H$ and is such that $V = \mathcal{D}(A^{1/2}) = H^1_0(\Omega)^n$ equipped with the inner product

$$(u, v)_V = \int_{\Omega} \left( \mu \sum_{i,j=1}^{n} \partial_i u_j \partial_i v_j + (\lambda + \mu) \text{div } u \text{ div } v \right) \text{d}x, \quad \forall u, v \in H^1_0(\Omega)^n.$$

We then define $U_i = L^2(\omega_i)^n$ and the operators $B_i$, $i = 1, 2$, as

$$B_i : U_i \to H : v \to \sqrt{b_i} \tilde{v},$$

where $\tilde{v}$ is the extension of $v$ by zero outside $\omega_i$. As before

$$B_i^*(\varphi) = \sqrt{b_i} \varphi_{|\omega_i} \text{ for } \varphi \in H,$$

and thus $B_i B_i^*(\varphi) = b_i \varphi$, for any $\varphi \in H$ and $i = 1, 2$. So, problem (5.1)–(5.4) enters in the abstract framework (1.1)–(1.3).

Therefore, in order to apply the abstract results of Sect. 3, we only need to check the observability estimate for the associated conservative system: There exists a time $T > 0$ and a constant $c > 0$ such that

$$\frac{1}{2} \left( (\varphi_0, \varphi_0)_V + \int_{\Omega} |\psi_0|^2 \text{d}x \right) \leq c \int_0^T \int_{\Omega} b_1(x)|\varphi_i|^2(x, s) \text{d}x \text{d}s, \quad (5.5)$$

for every weak solution $\varphi$ of

$$\varphi_{tt}(x, t) - \mu \Delta \varphi(x, t) - (\lambda + \mu) \nabla \text{div } \varphi(x, t) = 0 \text{ in } \Omega \times (0, +\infty),$$
$$\varphi(x, t) = 0 \text{ on } \partial \Omega \times (0, +\infty),$$
$$\varphi(x, 0) = \varphi_0(x) \text{ and } \varphi_t(x, 0) = \psi_0(x) \text{ in } \Omega,$$

with initial data $(\varphi_0, \psi_0) \in H^1_0(\Omega)^n \times L^2(\Omega)^n$.

If such an estimate holds, the stability result from Sect. 3 can be applied to the above system.

\textbf{Remark 5.1} Under the assumptions of point 1 of Remark 4.2, the observability estimate (5.5) is obtained in the proof of Theorem 3.1 of [6] (estimate (3.2) of [6]). \qed
5.2 Internal and boundary dampings

Under the assumptions of Sect. 4.2, we consider the following elastodynamic system

\[ u_{tt}(x, t) - \mu \Delta u(x, t) - (\lambda + \mu) \nabla \text{div} u + b(x)u_t(x, t - \tau) = 0 \quad \text{in} \ \Omega \times (0, +\infty), \]  
\[ u(x, t) = 0, \quad x \in \Gamma_1, \ t > 0, \]  
\[ \sigma(u(x, t)) \cdot v(x) = -k(x)u_t(x, t), \quad x \in \Gamma_0, \ t > 0, \]  
\[ u(x, 0) = u_0(x) \text{ and } u_t(x, 0) = v_0(x) \quad \text{in} \ \Omega, \]  
\[ \sqrt{b}u_t(x, t) = f^0(t) \quad \text{in} \ \omega_2 \times (-\tau, 0), \]  
with initial data \((u_0, v_0, f^0) \in H^1_0(\Omega)^n \times L^2(\Omega)^n \times L^2((-\tau, 0); L^2(\omega_2)^n)\) and

\[ \sigma(u) = \mu \left( \sum_{i=1}^{n} \partial_i (u_j)v_j \right) + (\lambda + \mu)(\text{div} u) v \quad \text{on} \ \Gamma_0. \]  

This problem enters into our abstract setting, once we take \(H = L^2(\Omega)^n, A\) defined in the previous subsection, and \(B_1\) and \(B_2\) defined as in Sect. 4.2.

As \(B_1\) is not bounded, we need to assume that there exists a time \(T > 0\) such that for every time \(T > \overline{T}\), there is a constant \(c\), depending on \(T\) but independent of the initial data, such that

\[ \frac{1}{2} \left( w_0, w_0 \right)_V + \int_{\Omega} |\psi_0|^2 \, dx \leq c \int_{0}^{T} \int_{\Gamma_0} k(x)|w_t|^2(x, s) \, dx \, ds, \]  
for every weak solution \(w\) of the nondelayed system

\[ w_{tt}(x, t) - \mu \Delta w(x, t) - (\lambda + \mu) \nabla \text{div} w = 0 \quad \text{in} \ \Omega \times (0, +\infty), \]  
\[ w(x, t) = 0, \quad x \in \Gamma_1, \ t > 0, \]  
\[ \sigma(w(x, t)) \cdot v(x) = -k(x)w_t(x, t), \quad x \in \Gamma_0, \ t > 0, \]  
\[ w(x, 0) = w_0(x) \text{ and } w_t(x, 0) = \psi_0(x) \quad \text{in} \ \Omega, \]  
for initial data \((w_0, \psi_0) \in H^1_0(\Omega)^n \times L^2(\Omega)^n\).

Again if such an estimate holds, the stability result from Sect. 3 can be applied to the system (5.7)–(5.10).

Remark 5.2  1. Under the assumptions of point 1 of Remark 4.5, the observability estimate (5.11) is proved in [5].

2. If we assume that the boundary of \(\Omega\) is smooth and that

\[ (x - x_0) \cdot v(x) \leq 0 \quad \text{on} \ \Gamma_1, \]  
then the observability estimate (5.11) is proved in Lemma 3.2 of [14]. □
6 The Petrovsky system

6.1 Hinged boundary conditions

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a boundary $\partial \Omega$ of class $C^4$ (as before this regularity could be weakened).

Let us consider the initial boundary value problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(x,t) + \Delta u(x,t) + b_1(x)u_t(x,t) + b_2(x)u_t(x,t - \tau) &= 0 & \text{in } \Omega \times (0, +\infty), \\
u(x, t) &= \Delta u(x, t) = 0 & \text{on } \partial \Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x) \text{ and } u_t(x, 0) = u_1(x) & \text{in } \Omega, \\
u_t(x, t) &= f^0(x, t) & \text{in } \omega_2 \times (-\tau, 0),
\end{align*}
\]

with initial data $(u_0, u_1, f^0) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega) \times L^2((-\tau, 0); L^2(\omega_2))$ and $b_1, b_2$ satisfying the same assumptions as in Sect. 4.1.

Now, we take $H = L^2(\Omega)$ and let $A$ be the operator

\[
A : D(A) \rightarrow H : u \rightarrow \Delta^2 u,
\]

where

\[
D(A) = \left\{ v \in H^1_0(\Omega) \cap H^4(\Omega) : \Delta u = 0 \text{ on } \partial \Omega \right\}.
\]

The operator $A$ is self-adjoint and positive, has a compact inverse in $H$ and satisfies $D(A^{1/2}) = H^2(\Omega) \cap H^1_0(\Omega)$. We then define $U_i = L^2(\omega_i)$ and the operators $B_i, i = 1, 2$, by (4.6). So, problem (6.1)–(6.4) enters in the abstract framework (1.1)–(1.3).

If an observability estimate of the associated conservative system holds, then the results of Sect. 3 apply also to the plate model.

Remark 6.1 Under the assumptions of point 1 of Remark 4.2 and the additional regularity of the boundary, it is well known that an observability estimate of the associated conservative system holds, see Proposition 7.5.7 (see also Example 11.2.4) of [28].

6.2 Clamped boundary conditions

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with a boundary $\partial \Omega$ of class $C^4$.
Here, we consider the initial boundary value problem

\[ u_{tt}(x, t) + \Delta^2 u(x, t) + b_1(x)u_t(x, t) + b_2(x)u_t(x, t - \tau) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{6.6} \]

\[ u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{6.7} \]

\[ u(x, 0) = u_0(x) \text{ and } u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \tag{6.8} \]

\[ u_t(x, t) = f^0(x, t) \quad \text{in } \Omega_2 \times (-\tau, 0), \tag{6.9} \]

with initial data \((u_0, u_1, f^0) \in H^2_0(\Omega) \times L^2(\Omega) \times L^2((-\tau, 0); L^2(\omega_2))\) where

\[ H^2_0(\Omega) := \left\{ \varphi \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \]

and \(b_1, b_2\) satisfy the same assumptions than in the previous subsection.

As usual, if an observability estimate holds for the associated conservative system, the results of Sect. 3 can be applied to this model.

**Remark 6.2** Under the assumptions of point 1 of Remark 4.2 and the additional assumptions of this subsection, the observability estimate for the associated conservative system has been recently proved by the authors (see [24], Theorem 6.1). \( \square \)

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