AN $L_p$-THEORY FOR DIFFUSION EQUATIONS RELATED TO
STOCHASTIC PROCESSES WITH NON-STATIONARY
INDEPENDENT INCREMENT

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Abstract. Let $X = (X_t)_{t \geq 0}$ be a stochastic process which has an (not necessarily stationary) independent increment on a probability space $(\Omega, \mathcal{F})$. In this paper, we study the following Cauchy problem related to the stochastic process $X$:

$$\frac{\partial u}{\partial t}(t, x) = A(t)u(t, x) + f(t, x), \quad u(0, \cdot) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \quad (0.1)$$

where $f \in L_p((0, T); L_p(\mathbb{R}^d)) = L_p((0, T); L_p)$ and

$$A(t)u(t, x) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[u(t, x + X_{t+h} - X_t) - u(t, x)].$$

We provide a sufficient condition on $X$ (see Assumptions 2.1 and 2.2) to guarantee the unique solvability of equation (0.1) in $L_p([0, T]; H_p^{\phi})$, where $H_p^{\phi}$ is a $\phi$-potential space on $\mathbb{R}^d$ (see Definition 2.9). Furthermore we show that for this solution,

$$\|u\|_{L_p([0, T]; H_p^{\phi})} \leq N\|f\|_{L_p((0, T); L_p)},$$

where $N$ is independent of $u$ and $f$.

1. Introduction

Roughly speaking, the second-order diffusion equations describe the motion of diffusion particles moving according to a law of stochastic process driven by a Brownian motion. Such equations are not suitable for natural phenomena with jumps, and accordingly there has been growing interest in equations with non-local operators related to pure jump processes owing to their applications in various models in physics, economics, engineering and many others involving long-range interactions.

If the non-local operators are close to fractional Laplacian operator, then there are considerable regularity results. See e.g. [1], [2], [3], [4] and [12] for the Harnack inequality and Hölder estimates. Regarding $L_p$-regularity theory, H. Dong and D. Kim [4] obtained a sharp $L_p$-estimate for the nonlocal elliptic equation

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d,$$

(1.1)

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where
\[ Lu(x) = \int_{\mathbb{R}^d} \left( u(x + y) - u(x) - y \cdot \nabla u(x) \chi^{(\alpha)}(y) \right) \frac{a(y)}{|y|^{d+\alpha}} dy, \]
\[ \alpha \in (0, 2), \chi^{(\alpha)} \text{ is a certain indicator function, and } a(y) \text{ is a measurable function} \]
with positive lower and upper bounds, that is, there exists \( \delta > 0 \)
\[ \frac{\delta}{|y|^{d+\alpha}} \leq \frac{a(y)}{|y|^{d+\alpha}} \leq \frac{\delta^{-1}}{|y|^{d+\alpha}} \quad \forall y \in \mathbb{R}^d. \tag{1.2} \]

Observe that, since \( \alpha \in (0, 2) \), \( a(y)|y|^{-d-\alpha}dy \) is a Lévy measure, i.e.
\[ \int_{\mathbb{R}^d} \left( 1 \wedge |y|^2 \right) \frac{a(y)}{|y|^{d+\alpha}} dy < \infty, \]
and \( C_{\alpha,d} \) is the Lévy measure of the rotationally invariant \( \alpha \)-stable process. In [14], X. Zhang introduced a generalization of (1.2). More precisely, he handled the Cauchy problem in \( L^p \)-space with the Lévy measure \( \nu(dy) \) (instead of \( a(y)|y|^{-d-\alpha}dy \)) with the condition \( \nu_0^\alpha(dy) \leq \nu(dy) \leq \nu_2^\alpha(dy) \), where \( \nu_i^\alpha, i = 1, 2 \), are the Lévy measures of two \( \alpha \)-stable processes taking the form
\[ \nu_i^\alpha(B) := \int_{S^{d-1}} \left( \int_0^\infty 1_B(r\theta) dr \right) \Sigma_i(d\theta), \]
\( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \) and \( \Sigma_i \) is a measure on \( S^{d-1} \). We also refer to a recent result [9], where \( L_p \)-theory is presented for the elliptic and parabolic equations
\[ Lu - \lambda u = g, \quad \partial_t u = Lu - \lambda u + f, \]
on \( \mathbb{R}^d \) and \( [0, T] \times \mathbb{R}^d \) respectively. Here
\[ L\phi(x) = L^\tau \phi(x) = \int_{\mathbb{R}^d} \left( \phi(x + y) - \phi(x) - \chi^{(\tau)}(y) \cdot \nabla \phi(x) \right) \pi(dy), \]
and \( \pi \) is supposed to satisfy a certain scaling property, which is called assumption \( D(\kappa, \ell) \) in [9].

In this article we prove the unique solvability of diffusion equation [2, 10] with the generator of stochastic processes beyond Lévy processes. In particular, we focus on diffusion equations with generators of stochastic processes with non-stationary independent increments. For instance, our stochastic processes \( X_t \) can be of type
\[ X_t = \int_0^t a(s) dY_s, \]
where \( Y_t \) is a subordinate Brownian motion, and \( X_t \) can also be an additive process. See Section 2.2 for more concrete examples. We adopt \( \phi \)-potential space (see [4]) for the space of solutions. This is because our operators are far away from \( \alpha \)-stable process and the classical Bessel potential space does not fit as a solution space.

We emphasize that even if the stochastic process \( X_t \) is a Lévy process, our result cannot be covered by above results. For instance, an example related to Subordinate Brownian motions is given in [9] Example 2.1 and Remark 2. In this example, there are conditions on weak scaling constants \( \delta_1 \) and \( \delta_2 \) such as \( 2\delta_1 > 1 \) and \( 2\delta_1 > \delta_2 \). However, we do not need this relation in our results (see Example 2.5).

Next we give a few remarks on our methods. Due to the non-local property of our operators, classical perturbation arguments are not available. Nonetheless, fortunately, our operators are still pseudo-differential operators. If the symbols of pseudo-differential operators are smooth enough then one can use classical tools
from Harmonic and Fourier analysis. However, if the moments of the given process are not finite, the symbol of the generator of the process loses the smoothness property. We overcome this difficulty using a probabilistic technique together with analytic tools. Technically our approach does not rely on the well-developed one-parameter semi-group theory since increments of our stochastic processes are not stationary.

The article is organized as follows. In section 2, we present our main result (Theorem 2.13), \( L_p \)-theory of PDEs with generators of non-stationary independent increment processes. In Section 3, we introduce a version of singular integral theory which fits our equations. In Section 4, we prove a maximal \( L_p \)-regularity theory for a class of pseudo differential operators. The result of this section is used to prove our main result when the symbol of the operator is smooth. Section 5 contains the proof of our main theorem, and finally in Appendix we prove a version of the Fefferman-Stein theorem.

We finish the introduction by introducing notations we will use in the article. \( \mathbb{N} \) and \( \mathbb{Z} \) denote the natural number system and the integer number system, respectively. Denote \( \mathbb{Z}_+ := \{ k \in \mathbb{Z}; k \geq 0 \} \). As usual \( \mathbb{R}^d \) stands for the Euclidean space of points \( x = (x^1, \ldots, x^d) \). For \( j = 1, \ldots, n \), multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \alpha_j \in \mathbb{Z}_+ \), and functions \( u(x) \) we set

\[
   u_{x^j} = \frac{\partial u}{\partial x^j} = D_j u, \quad D^\alpha u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u, \quad \nabla u = (u_{x^1}, u_{x^2}, \ldots, u_{x^d}).
\]

We also use the notations \( D_x^m \) (and \( D_x^\alpha \), respectively) for a partial derivative of order \( m \) (of multi-index \( \alpha \), respectively) with respect to \( x \). \( C(\mathbb{R}^d) \) denotes the space of bounded continuous functions on \( \mathbb{R}^d \). For \( n \in \mathbb{N} \), we write \( u \in C^n(\mathbb{R}^d) \) if \( u \) is \( n \)-times continuously differentiable in \( \mathbb{R}^d \), and \( \sup_{x \in \mathbb{R}^d, |\alpha| \leq n} |D^\alpha u| < \infty \). Simply we put \( C^n := C^n(\mathbb{R}^d) \). For \( p \in [1, \infty) \), a normed space \( F \) with norm \( \| \cdot \|_F \) and a measure space \((X, \mathcal{M}, \mu)\), \( L^p(X, \mathcal{M}, \mu; F) \) denotes the space of all \( F \)-valued \( \mathcal{M}^\mu \)-measurable functions \( u \) so that

\[
   \| u \|_{L^p(X, \mathcal{M}, \mu; F)} := \left( \int_X \| u(x) \|_F^p \mu(dx) \right)^{1/p} < \infty,
\]

where \( \mathcal{M}^\mu \) denotes the completion of \( \mathcal{M} \) with respect to the measure \( \mu \).

For \( p = \infty \), we write \( u \in L^\infty(X, \mathcal{M}, \mu; F) \) iff

\[
   \| u \|_{L^\infty(X, \mathcal{M}, \mu; F)} := \inf \{ \nu \geq 0 : \mu(\{ x : \| u(x) \|_F > \nu \}) = 0 \} < \infty.
\]

If there is no confusion for the given measure and \( \sigma \)-algebra, we usually omit the measure and the \( \sigma \)-algebra. In particular, for a domain \( U \subset \mathbb{R}^d \) we denote \( L_p(U) = L_p(U, \mathcal{L}, \ell; \mathbb{R}) \), where \( \mathcal{L} \) is the Lebesgue measurable sets, and \( \ell \) is the Lebesgue measure in \( \mathbb{R}^d \). We use the notation \( N \) to denote a generic constant which may change from line to line. While, throughout this paper the constants \( N_j \), \( j = 0, 1, \ldots \), will be fixed. We use \( N = N(a, b, \cdots) \) to indicate a positive constant that depends on the parameters \( a, b, \cdots \).

We use \( := \) or \( \equiv \) to denote a definition. \( |a| \) is the biggest integer which is less than or equal to \( a \). By \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) we denote the \( d \)-dimensional Fourier transform and the inverse Fourier transform, respectively. That is, \( \mathcal{F}[f](\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \) and \( \mathcal{F}^{-1}[f](\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi \) where \( i \) is the imaginary number, i.e. \( i^2 = -1 \). We use \( 1_B \) to denote the indicator of a set \( B \). For a Lebesgue measurable set \( A \), we use \( |A| \) to denote its Lebesgue measure. For a
complex number $z$, $\Re[z]$ is the real part of $z$ and $\bar{z}$ is the complex conjugate of $z$.
For a function space $\mathcal{H}(U)$ on an open set $U$ in $\mathbb{R}^d$, we let $\mathcal{H}_c(U) := \{ f \in \mathcal{H}(U) : f$ has compact support$\}$, $\mathcal{H}_0(U) := \{ f \in \mathcal{H}(U) : f$ vanishes at infinity$\}$.

2. PDEs with generators of independent increment process

2.1. Assumptions. Let $T < \infty$. Every stochastic process considered in this article is $\mathbb{R}^d$-valued. Recall that a measure $\mu$ on $\mathbb{R}^d$ is a Lévy measure if

$$\mu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (1 + |x|^2) \mu(dx) < \infty.$$  

Assumption 2.1. (i) The stochastic process $X$ has a pure jump component, that is there exist two independent stochastic process $X^1$ and $X^2$ such that for all $t \geq s \geq 0$, $X_t - X_s$ and $X^1_t - X^1_s + X^2_t - X^2_s$ have same distributions and

$$\mathbb{E}e^{i\xi \cdot (X^1_t - X^1_s)} = \exp \left( \int_s^t \int_{\mathbb{R}^d} (e^{i\xi \cdot x} - 1 - i\xi \cdot x1_{|x| < 1}) \mu_r(dx)dr \right),$$

where $\mu_r$ is a Lévy measure for each $r \in [0, \infty)$.

(ii) The paths of $X^2$ are locally bounded (a.s.). i.e.,

$$\mathbb{P} \left( \sup_{t \in [a,b]} |X^2_t| < \infty \right) = 1 \text{ for all } 0 < a < b < \infty.$$  

Denote

$$\Psi_{X^1}(t, \xi) := \int_{\mathbb{R}^d} (e^{i\xi \cdot x} - 1 - i\xi \cdot x1_{|x| < 1}) \mu_t(dx)$$

and

$$\Phi_{X^1}(s, t, \xi) := \int_s^t \Psi_{X^1}(r, \xi) dr.$$  

Assumption 2.2. Let $d_0 := \left\lfloor \frac{d}{2} \right\rfloor + 1$. (i) There exists a complex-valued function $\Psi_X(t, \xi)$ on $[0, \infty) \times \mathbb{R}^d$ so that for all $t \geq s > 0$ and $\xi \in \mathbb{R}^d$

$$\mathbb{E}e^{i\xi \cdot (X_t - X_s)} = \exp \left( \int_s^t \Psi_X(r, \xi) dr \right) =: \exp \left( \Phi_X(s, t, \xi) \right),$$

and furthermore $\Psi_X$ and $\Phi_X$ satisfy the followings:

- For each $\xi$, $\Psi_X(t, \xi)$ is locally integrable with respect to $t$ on $[0, T)$, i.e. $\Psi_X(\cdot, \xi) \in L_1([0, t))$ for all $t \in [0, T)$.
- $\xi \rightarrow \exp (\Phi_X(s, t, \xi))$ and $\xi \rightarrow \Psi_X(t, \xi) \cdot \exp (\Phi_X(s, t, \xi))$ are locally bounded and have at most polynomial growth at infinity with respect to $\xi$ uniformly for $0 < s < t < T$, i.e. there exists a $N > 0$ so that

$$\sup_{0 < s < t < T} (|\exp (\Phi_X(s, t, \xi))| + |\Psi_X(t, \xi) \cdot \exp (\Phi_X(s, t, \xi))|) \leq N (1 + |\xi|)^N. \quad (2.1)$$

(ii) There exists a nondecreasing function $\phi(\lambda) : (0, \infty) \mapsto (0, \infty)$ and positive constants $\delta_k$ and $N_j$ ($k = 1, 2, 3$ and $j = 1, 2, 3, 4$) such that

- for all $\xi \neq 0$

$$\Re[-\Psi_{X^1}(t, \xi)] \geq \delta_1 \phi(|\xi|^2), \quad (2.2)$$

- for all $\xi \neq 0$ and multi-index $|\alpha| \leq d_0$,  

$$|D^\alpha \Psi_{X^1}(t, \xi)| \leq N_1 \phi(|\xi|^2)|\xi|^{-|\alpha|}, \quad (2.3)$$
• \( \delta_3 \geq \delta_2 > 0 \) and for any \( \lambda_2 \geq \lambda_1 > 0 \),

\[
N_2 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_2} \leq \frac{\phi(\lambda_2)}{\phi(\lambda_1)} \leq N_3 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_3},
\]

(2.4)

• for all \( \lambda \in (0, \infty) \) and natural number \( n \leq d_0 \),

\[
|D^n\phi(\lambda)| \leq N_4 \lambda^{-n} \phi(\lambda).
\]

(2.5)

**Remark 2.3.** If \( X^2 = 0 \), then Assumption (ii) implies (i).

### 2.2. Examples.

To introduce examples satisfying above assumptions, we recall some definitions and facts on subordinate Brownian motion. A function \( \phi : (0, \infty) \to (0, \infty) \) is called a Bernstein function with \( \phi(0+) = 0 \) if \( \phi \) has a representation that

\[
\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt),
\]

(2.6)

where \( b \geq 0 \) and \( \mu \) is a measure on \((0, \infty)\) satisfying \( \int_0^\infty (1 + t)\mu(dt) < \infty \). Then it is well-known that (e.g. [11, Chapter 3] and [6, Lemma 3.2])

\[
\frac{\phi(\lambda_2)}{\phi(\lambda_1)} \leq \frac{\lambda_2}{\lambda_1}, \quad 0 < \lambda_1 < \lambda_2.
\]

(2.7)

and for any nonnegative integer \( n \),

\[
\lambda^n |D^n\phi(\lambda)| \leq N(n)\phi(\lambda), \quad \forall \lambda > 0.
\]

(2.8)

Thus \( \phi \) satisfies (2.7).

Let \( S = (S_t)_{t \geq 0} \) be a subordinator (i.e. an increasing Lévy process satisfying \( S_0 = 0 \)), then there is a Bernstein function \( \phi \) with \( \phi(0+) = 0 \) such that \( \mathbb{E}e^{-\lambda S_t} = e^{-\lambda \phi(\lambda)} \). Let \( W = (W_t)_{t \geq 0} \) be a Brownian motion in \( \mathbb{R}^d \), i.e. \( \mathbb{E}[e^{i\xi \cdot W_t}] = e^{-t|\xi|^2} \), \( \xi \in \mathbb{R}^d, t > 0 \), which is independent of \( S_t \). Then \( Y_t = W_{S_t} \), called the subordinate Brownian motion (SBM), is a rotationally invariant Lévy process in \( \mathbb{R}^d \) with characteristic exponent \( \phi(|\xi|^2) \), and by Lévy-Khintchine theorem,

\[
\mathbb{E}[e^{i\xi Y_t}] = e^{-t\phi(|\xi|^2)} = \exp \left( t \int_{\mathbb{R}^d} (e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{|x| < 1}) J(x) dx \right),
\]

(2.9)

where \( J(x) = j(|x|) \) and

\[
j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt).
\]

**Example 2.4 (Integral with respect to SBM).** Assume that the Bernstein function \( \phi \) satisfies the following weak-scaling conditions:

• There exist constants \( 0 < \delta_1 \leq \delta_2 < 1 \) and \( a_1 > 0 \) such that

\[
a_1 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_1} \leq \frac{\phi(\lambda_2)}{\phi(\lambda_1)} \leq a_2 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_2}, \quad 0 < \lambda_1 \leq \lambda_2 < \infty
\]

(2.10)

Note that \( \phi \) satisfies (2.7) and (2.8).

Let \( \sigma : (0, \infty) \to \mathbb{R} \) be a bounded measurable function such that for all \( t \in (0, \infty) \), \( |\sigma(t)| \in (\delta, \delta^{-1}) \) for some \( \delta \in (0, 1) \). Recall \( Y_t = B_{S_t} \) and define

\[
X_t^1 := \int_0^t \sigma(s) dY_s,
\]
Then
\[ \mathbb{E} e^{i\xi (X_t - X_s)} = \exp \left[ \int_s^t \left( \int_{\mathbb{R}^d} \left( e^{i\xi \cdot x} - 1 - i \xi \cdot x 1_{|x| < 1} \right) \mu_r(dx) \right) dr \right], \]
where \( \mu_r(B) = \int_{\mathbb{R}^d} 1_B(\sigma(r)x)J(x)dx \). Thus denoting
\[ \Psi_{X^1}(t, \xi) = \int_{\mathbb{R}^d} \left( e^{i\xi \cdot x} - 1 - i \xi \cdot x 1_{|x| < 1} \right) \mu_t(dx), \]
we see that \( X^1_t \) has a pure jump independent increment. Moreover, Due to the properties of the density function of Brownian motion and (2.6),
\[ \int \ldots \]

Example 2.5 (Additive process). Let \( X^1_t = B_{S_t} \) be a Subordinate Brownian motion. Assume the the Laplace exponent of \( S_t \) satisfies the following condition:

- (H): There exist constants \( 0 < \delta_1 \leq 1 \) and \( a_1 > 0 \) such that
\[ a_1 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_1} \leq \frac{\phi(\lambda_2)}{\phi(\lambda_1)} \quad 0 < \lambda_1 \leq \lambda_2 < \infty. \]

Then combining (2.7) and (H), we have
\[ a_1 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_1} \leq \frac{\phi(\lambda_2)}{\phi(\lambda_1)} \leq \left( \frac{\lambda_2}{\lambda_1} \right) \quad 0 < \lambda_1 \leq \lambda_2 < \infty. \quad (2.11) \]

Thus \( \phi \) satisfies (2.4).

Let \( a(t) : (0, \infty) \rightarrow (0, \infty) \) be a function which is bounded from both above and below. Define \( \Psi_{X^1}(t, \xi) = -a(t)\phi(|\xi|^2) \). Then obviously due to (2.8), \( \Psi_{X^1} \) satisfies (2.2) and (2.3). Moreover, there exists an additive process \( X^1_t \) (see [10] Theorem 9.8(ii) and Theorem 11.5) such that for all \( t > 0 \) and \( \xi \in \mathbb{R}^d \)
\[ \mathbb{E} e^{i\xi X^1_t} = \exp \left[ \int_0^t \Psi_{X^1}(s, \xi) ds \right], \]
which is because \( \phi(|\xi|^2) \) has the representation \( (\ref{2.9}) \).

The following well-known examples of subordinators satisfy \( (\ref{2.11}) \) but do not satisfy \( (\ref{2.10}) \):

1. Relativistic stable subordinator: \( \phi(\lambda) = (\lambda + m^{1/\alpha})^\alpha - m \), \( 0 < \alpha < 1 \) and \( m > 0 \), with \( \delta = 1 - \alpha \).

2. \( \phi(\lambda) = \frac{\lambda}{\log(1 + \lambda^{2/\beta})} \), where \( \beta \in (0, 2) \).

In the following example, we show that locally homogeneous additive process satisfies our assumption on \( X^2 \).

**Example 2.6.** Let \( X^2_t \) be an additive process. Then by \([10, \text{Theorem } 9.8]\), there exists a triple \((a(t), A(t), \mu_t)\) so that

\[
\mathbb{E}e^{i\xi \cdot X^2_t} = \exp \left( ia(t) \cdot \xi - \frac{1}{2} (A(t) \xi, \xi) + \int_{\mathbb{R}^d} \left( e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{|x|<1} \right) \mu_t(dx) \right),
\]

(2.12)

where \( t \in [0, \infty) \), \( \xi \in \mathbb{R}^d \), \( a(t) \in \mathbb{R}^d \), \( A(t) \) is a nonnegative symmetric matrix, and \( \mu_t \) is a Lévy measure for each \( t \in [0, \infty) \), and \( A(t) \), \( \mu_t \) are nondecreasing. If \( a_t \), \( A_t \), \( \mu_t \) are absolutely continuous with respect to \( dt \), say \( a_t = \int_0^t a^*(s)ds \), \( A(t) = \int_0^t A^*(s)ds \), \( \mu_t = \int_0^t \mu^*(s)ds \), then from \( (\ref{2.10}) \) we have

\[
\mathbb{E}e^{i\xi \cdot X^2_t} = \exp \left( \int_0^t \left( a^*(s) \cdot \xi - \frac{1}{2} (A^*(s) \xi, \xi) + \int_{\mathbb{R}^d} \left( e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{|x|<1} \right) \mu^*_s(dx) \right) ds \right)
\]

\[= \exp \left( \int_0^t \Psi_{X^2}(s, \xi)ds \right).\]

Due to the absolute continuity assumption, it is obvious that \( \Psi_{X^2} \) is locally integrable with respect to \( t \) for each \( \xi \in \mathbb{R}^d \). Moreover, for any \( 0 < s < t < T \),

\[
\left| \exp \left( \int_s^t \Psi_{X^2}(r, \xi)dr \right) \right| = \left| \mathbb{E}e^{i\xi \cdot X^2_t} \right| \leq 1
\]

and

\[
\leq \left| \Psi_{X^2}(t, \xi) \right|
\]

\[
\leq \left( \sup_{t \leq T} |A(t)||\xi|^2 + N(1 + |\xi|^2) \int_0^T \mathbb{E} \int_{\mathbb{R}^d} (1 + |x|^2)\mu^*(t)(dx)dt \right)
\]

\[
\leq N(T) \left( 1 + |\xi|^2 \right),
\]

which implies both \( \exp \left( \int_s^t \Psi_{X^2}(r, \xi)dr \right) \) and \( \Psi_{X^2}(t, \xi) \) \( \exp \left( \int_s^t \Psi_{X^2}(r, \xi)dr \right) \) are locally bounded and have a polynomial growth at infinity with respect to \( \xi \) uniformly for \( 0 < s < t < T \). Moreover obviously paths of \( X^2 \) are locally bounded (a.s.) since \( X^2 \) is a càdlàg process.

Let \( X^1_t \) be the process handled in Example \( 2.3 \) or Example \( 2.4 \). By considering product of probability spaces, we may assume \( X^1_t \) and \( X^2_t \) are independent without loss of generality. Set \( X_t = X^1_t + X^2_t \). Then \( X_t \) satisfies Assumption \( 2.3 \).
At first glance, conditions [2.2] and [2.3] seem to be quite strong since for each $t$ $\Psi_{X^1}(t, \xi)$ has to be smooth in $\mathbb{R}^d \setminus \{0\}$ with respect to $\xi$. However, these conditions are imposed only on the symbol $\Psi_{X^1}$. We would like to emphasize that our symbol $\Psi_X(t, \xi)$, which is the sum of $\Psi_{X^1}(t, \xi)$ and $\Psi_{X^2}(t, \xi) := \Psi_X(t, \xi) - \Psi_{X^1}(t, \xi)$, does not have to be smooth. We give a concrete simple example below.

**Example 2.7.** Let $\alpha \in (0, 2)$ and $a(t, x)$ be a positive measurable function on $(0, \infty) \times \mathbb{R}^d$ and assume that $a(t, x)$ is bounded from both above and below, i.e. there exists a constant $c \in (0, 1)$ such that

$$c \leq a(t, x) \leq c^{-1} \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^d.$$  

Using $a(t, x)$ and $c$, we define the following Lévy measures for each $t > 0$;

$$\mu_t(dx) := a(t, x)|x|^{-d-\alpha}dx, \quad \mu_1^1(dx) := \frac{c}{2}|x|^{-d-\alpha}dx$$

and

$$\mu_1^2(dx) := \left(a(t, x) - \frac{c}{2}\right)|x|^{-d-\alpha}dx.$$  

Define

$$\Psi_X(t, \xi) := \int_{\mathbb{R}^d} \left(e^{i\xi x} - 1 - i\xi \cdot x1_{|x|<1}\right) \mu_t(dx),$$

and ($k = 1, 2$)

$$\Psi_{X^k}(t, \xi) := \int_{\mathbb{R}^d} \left(e^{i\xi x} - 1 - i\xi \cdot x1_{|x|<1}\right) \mu^k_t(dx).$$

Observe that by the change of variables

$$\Psi_X(t, \xi) = \int_{\mathbb{R}^d} \left(e^{i\xi x} - 1 - i\xi \cdot x1_{|x|<1}\right) \mu_t(dx)$$

$$= -|\xi|^{\alpha} \int_{\mathbb{R}^d} \left(1 - \cos(x^1)\right) a \left(t, \frac{O_\xi x}{|\xi|}\right) |x|^{-d-\alpha}dx, \quad (2.13)$$

where $O_\xi$ is an orthonormal matrix such that $O^T_\xi \xi = |\xi|e_1$ and $e_1 = (1, 0, \ldots, 0)$. Since there is no regularity condition on the coefficient $a(t, x)$, $\Psi_X(t, \xi)$ is not smooth with respect to $\xi$ in general. On the other hand,

$$\Psi_{X^1}(t, \xi) := \int_{\mathbb{R}^d} \left(e^{i\xi x} - 1 - i\xi \cdot x1_{|x|<1}\right) \mu^1_t(dx)$$

$$= -N(d, \alpha, c)|\xi|^{\alpha}$$

and obviously $\Psi_{X^1}(t, \xi)$ satisfies [2.2] and [2.3]. Since for each $t > 0$, $\mu$, $\mu^1$, $\mu^2$ are Lévy measures, there exist additive processes $X_1$, $X^1_1$ and $X^2_2$ such that

$$\mathbb{E}e^{i\xi X_1} = \exp \left(\int_0^t \Psi_X(s, \xi)ds\right),$$

and

$$\mathbb{E}e^{i\xi X^1_1} = \exp \left(\int_0^t \Psi_{X^1}(s, \xi)ds\right), \quad \mathbb{E}e^{i\xi X^2_2} = \exp \left(\int_0^t \Psi_{X^2}(s, \xi)ds\right)$$

due to [10] Theorem 9.8(ii) and Theorem 11.5 again. We may assume that $X^1$ and $X^2$ are independent. One can easily check that $\Psi_{X^1}$ satisfies the assumptions in Example 2.4. Therefore our assumptions hold for the symbol $\Psi_X(t, \xi)$ which is defined in (2.13) and not smooth.
Remark 2.8. We acknowledge that there are still some singular symbols which our result cannot cover and were already considered previously by other authors. Here, we give an interesting simple example not covered in this paper.

Let $\alpha \in (0, 1)$ and $a(t)$ be a function on $[0, \infty)$ such that

$$0 < c \leq a(t) \leq c^{-1} \quad \forall t \in [0, \infty).$$

For each $t > 0$, define

$$\mu_t(dx) := a(t) \left( |x_1|^{-1-\alpha}dx_1 \cdot \epsilon_0(dx_2, \ldots, dx_d) + \cdots + |x_d|^{-1-\alpha}dx_d \cdot \epsilon_0(dx_1, \ldots, dx_{d-1}) \right).$$

Here $\epsilon_0$ is the Dirac measure centered at zero in $\mathbb{R}^{d-1}$. Then

$$\Psi_{X(t)}(t, \xi) := \int_{\mathbb{R}^d} \left( e^{i\xi \cdot x} - 1 - i \xi \cdot x I_{|x|<1} \right) \mu_t(dx)$$

$$= \mathcal{N}(d, \alpha)a(t) (|\xi|^\alpha + \cdots + |\xi_d|^\alpha).$$

This symbol does not satisfy (2.3). This example was handled by X. Zhang (see [14, Remark 2.7]) if $a(t)$ is independent of $t$. But the case when $a(t)$ depends on $t$ is not covered in literature as far as we know.

2.3. $L_p$-theory for diffusion equations in $\phi$-potential spaces. In this subsection we present our main result, the unique solvability of equation (2.16) in $\phi$-potential space.

**Definition 2.9** ($\phi$-potential space). For $\zeta \in C_0^\infty(\mathbb{R}^d)$, define

$$\phi(\Delta)\zeta(x) := -\phi(-\Delta)\zeta(x) := \mathcal{F}^{-1} \left[ \phi(|\xi|^2)\mathcal{F}(\zeta)(\xi) \right](x).$$

By $H^\phi_p$, we denote the space of functions $u \in L_p$ so that there exists a sequence of $u_n \in C_c^\infty(\mathbb{R}^d)$ such that

$$\|u_n - u\|_{L_p} \to 0$$

and

$$\|\phi(\Delta)u_n - \phi(\Delta)u_m\|_{L_p} \to 0$$

as $n, m \to \infty$. We call this sequence $u_n$ a defining sequence of $u$. For $u \in H^\phi_p$, we define

$$\phi(\Delta)u := \lim_{n \to \infty} \phi(\Delta)u_n,$$

where $u_n$ is a defining sequence of $u$ and the limit is understood in $L_p$-sense.

**Lemma 2.10.** (i) $\phi(\Delta)u$ is well defined for any $u \in H^\phi_p$, that is, it is independent of the choice of defining sequences.

(ii) $H^\phi_p$ is a Banach space equipped with the norm

$$\|u\|_{H^\phi_p} := \|u\|_{L_p} + \|\phi(\Delta)u\|_{L_p}.$$

(iii) Suppose that Assumption 2.2 holds. Then

$$\mathcal{A}(t)\zeta(x) := \lim_{h \to 0} \frac{B}{h} \left[ \zeta(x + X_{t+h} - X_t) - \zeta(x) \right]$$

is well defined for any $\zeta \in C_0^\infty(\mathbb{R}^d)$. Moreover there exists an adjoint operator $\mathcal{A}^*(t)$ so that

$$\int_{\mathbb{R}^d} \eta(x)\mathcal{A}(t)\zeta(x)dx = \int_{\mathbb{R}^d} \zeta(x)\mathcal{A}^*(t)\eta(x)dx$$

(2.15)

for all $\zeta, \eta \in C_0^\infty(\mathbb{R}^d)$. 
Proof. First we prove (i). Let \( u_n \) and \( v_n \) be defining sequences of \( u \in H_p^0 \), respectively. Then by the Plancherel theorem and Definition 2.11,\[
\int_{\mathbb{R}^d} \phi(\Delta) u(x) \zeta(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} \phi(\Delta) u_n(x) \zeta(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} u_n(x) \phi(\Delta) \zeta(x) \, dx
\]
and
\[
\int_{\mathbb{R}^d} \phi(\Delta) v(x) \zeta(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} \phi(\Delta) v_n(x) \zeta(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} \phi(\Delta) v_n(x) \zeta(x) \, dx
\]
for all \( \zeta \in C_0^\infty(\mathbb{R}^d) \). Thus \( \phi(\Delta) \) is well-defined.

(ii) is obvious due to the property of \( L_p \)-spaces.

Finally, we prove (iii). Recall \( \Phi_X(t, t, \xi) = 0 \). Then due to Assumption 2.2(i),\[
\mathcal{A}(t) \zeta(x) = \lim_{h \downarrow 0} \frac{\mathbb{E} [\zeta(x + X_{t+h} - X_t) - \zeta(x)]}{h}
\]
\[
= \lim_{h \downarrow 0} \mathbb{E} \left[ \mathcal{F}^{-1} \left[ \exp(\Phi_X(t, t + h, \xi) - \exp(\Phi_X(t, t, \xi)) \mathcal{F}[\zeta](\xi) \right] (x) \right]
\]
\[
= \mathcal{F}^{-1} \left[ \mathcal{F}[\zeta](\xi) \right] (x).
\]
Thus \( \mathcal{A}(t) \) is well-defined on \( C_0^\infty(\mathbb{R}^d) \) as a pseudo-differential operator since
\[
\Psi_X(t, \cdot, \mathcal{F}[\zeta](\cdot)) \in L_p(\mathbb{R}^d) \quad \forall p \in [1, \infty].
\]

Next, define
\[
\mathcal{A}^*(t) \zeta(x) = \mathcal{F}^{-1} \left[ \Psi_X(t, \xi) \mathcal{F}[\zeta](\xi) \right] (x).
\]
Then by the Plancherel theorem, (2.16) holds. The lemma is proved. \( \square \)

Definition 2.11 (Definition of solutions). For a given \( f \in L_p([0, T]; L_p) \), we say that \( u \in L_p([0, T]; L_p) \) is a solution to
\[
\frac{\partial u}{\partial t}(t, x) = \mathcal{A}(t) u(t, x) + f(t, x), \quad u(0, \cdot) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d,
\]
if there exists a sequence \( u_n \in C^\infty_c((0, T) \times \mathbb{R}^d) \) such that
\[
\frac{\partial u_n}{\partial t} - \mathcal{A}(t) u_n \to f \quad \text{in} \quad L_p([0, T]; L_p)
\]
and
\[
u_n \to u \quad \text{in} \quad L_p([0, T]; H_p^0)
\]
as \( n \to \infty \).

Remark 2.12. If \( u \) is a solution in the sense of Definition 2.11 then it becomes a solution in the usual weak-sense. Indeed, by the Plancherel theorem
\[
- \int_{(0,T) \times \mathbb{R}^d} u(t, x) \zeta_t(t, x) \, dt dx - \int_{(0,T) \times \mathbb{R}^d} u(t, x) \mathcal{A}^*(t) \zeta(t, x) \, dt dx
\]
\[
= \lim_{n \to \infty} \int_{(0,T) \times \mathbb{R}^d} \frac{\partial u_n}{\partial t}(t, x) \zeta(t, x) \, dt dx - \lim_{n \to \infty} \int_{(0,T) \times \mathbb{R}^d} \mathcal{A}(t) u_n(t, x) \zeta(t, x) \, dt dx
\]
\[
= \int_{(0,T) \times \mathbb{R}^d} f(t, x) \zeta(t, x) \, dt dx \quad \forall \zeta \in C_0^\infty((0, T) \times \mathbb{R}^d).
\]
Here is the main result of this section.
Theorem 2.13. Suppose that Assumptions 2.1 and 2.2 hold. Then for any \( f \in L_p([0,T];L_p) \), there exists a unique solution \( u \in L_p([0,T];H^p_T) \) to equation (2.10). Furthermore, for this solution \( u \), we have

\[
\|u\|_{L_p([0,T];H^p_T)} \leq C_1(d,p,\delta_k,N_j,T) \|f\|_{L_p([0,T];L_p)}
\]  

(2.17)

and

\[
\|\phi(\Delta)u\|_{L_p([0,T];L_p)} \leq C_2(d,p,\delta_k,N_j) \|f\|_{L_p([0,T];L_p)},
\]

(2.18)

where \( \delta_k \) and \( N_j \) (\( k = 1, 2, 3 \) and \( j = 1, 2, 3, 4 \)) are constants in Assumption 2.2.

The proof of this theorem will be given in Section 5.

3. \( L_p \)-boundedness of singular integral operators

In this section we introduce a version of Fefferman-Stein theorem and Hardy-Littlewood maximal theorem. We also prove an \( L_p \)-boundedness of singular integral operators related to certain pseudo-differential operators.

Let \( U = \mathbb{R}^d \) or \( U = \mathbb{R}^d_+ \). For a function \( \varphi : (0,\infty) \to (0,\infty) \), by \( Q_\varphi \) we denote the collection of all cubes

\[
Q_\varphi(t,x) = (t,t+\varphi(c)) \times B_c(x),
\]

where \( (t,x) \in [0,\infty) \times U \), \( c > 0 \), and \( B_c(x) = \{ y \in U : |x-y| < c \} \).

Definition 3.1. For locally integrable functions \( f \), denote

\[
f_{Q_\varphi(t_0,x_0)}(x) := \int_{Q_\varphi(t_0,x_0)} f(s,y)dsdy := \frac{1}{|Q_\varphi(t_0,x_0)|} \int_{Q_\varphi(t_0,x_0)} f(s,y)dsdy.
\]

The \( \varphi \)-type sharp function \( f^\varphi(t,x) \) and \( M_\varphi f(t,x) \) are defined as

\[
f^\varphi(t,x) := \sup_{Q_\varphi(t_0,x_0)} |f(s,y) - f_{Q_\varphi(t_0,x_0)}|dsdy
\]

and

\[
M_\varphi g(t,x) := \sup_{Q_\varphi(t_0,x_0)} |g(s,y)|dsdy,
\]

where \( (t,x) \in (0,\infty) \times U \), and sup is taken over all \( Q_\varphi(t_0,x_0) \in Q_\varphi \) containing \( (t,x) \).

Assumption 3.2. (i) \( \varphi : (0,\infty) \to (0,\infty) \) is a nondecreasing function so that

\[
\lim_{r \downarrow 0} \varphi(r) = 0, \quad \lim_{r \uparrow \infty} \varphi(r) = \infty,
\]

(3.1)

and

\[
\hat{c} := \sup_{r > 0} \frac{\varphi(2r)}{\varphi(r)} < \infty.
\]

(3.2)

(ii) There exists a constant \( \lambda_0 > 1 \) satisfying

\[
\varphi(\lambda_0 r) \geq 2\varphi(r) \quad \forall r > 0.
\]

(3.3)

Assumption 3.2(i) is sufficient to prove a \( \varphi \)-type Fefferman-Stein theorem, and condition (3.3) is additionally needed for \( \varphi \)-type Hardy-Littlewood maximal theorem.
Theorem 3.3 \((\varphi\text{-type Fefferman-Stein Theorem})\). Let \(p \in (1, \infty)\) and suppose Assumption 3.1 and 3.2 hold. Then there exists a constant \(N\) such that
\[
\|f\|_{L_p((0,\infty) \times U)} \leq N(d,p,\hat{c})\|f\|_{L_p((0,\infty) \times U)} \quad \forall f \in L_p((0,\infty) \times U).
\]

The proof of Theorem 3.3 will be given in Section 4.

Theorem 3.4 \((\varphi\text{-type Hardy-Littlewood Theorem})\). Let \(p \in (1, \infty)\) and suppose Assumption 3.5 holds. Then for some constant \(N = N(p,\varphi) > 0\),
\[
\|\mathcal{M}_\varphi g\|_{L_p((0,\infty) \times U)} \leq N\|g\|_{L_p((0,\infty) \times U)} \quad \forall g \in L_p((0,\infty) \times U). \quad (3.4)
\]

Proof. One can easily check that by \([13]\)

- \(\exists \tilde{N}_1 > 0\) s.t. \(Q_c^e(t,x) \cap Q_c^e(s,y) \neq \emptyset\) implies \(Q_c^e(s,y) \subset Q_c^e(t,x)\);
- \(\exists \tilde{N}_2 > 0\) s.t. \(|Q_c^e(t,x)| \leq \tilde{N}_2|Q_c^e(t,x)|\) for all \((t,x) \in (0,\infty) \times U\) and \(c > 0\);
- for each open set \(\mathcal{O}\) and \(c > 0\), the function \((t,x) \to |Q_c^e(t,x) \cap \mathcal{O}|\) is continuous.

Hence the theorem follows from the classical Hardy-Littlewood maximal theorem (see \([13]\) Theorem 1.1). Actually, \([13]\) Theorem 1.1 is proved on \(\mathbb{R}^{d+1}\). The key idea of the proof of \([13]\) Theorem 1.1 is Vitali’s covering lemma, which holds for arbitrary measurable subset of \(\mathbb{R}^{d+1}\), and by following the proof, one can easily check that the Hardy-Littlewood Theorem holds also on \((0,\infty) \times U\). \(\square\)

Let \(K(s,t,y,x)\) be a measurable function defined on \((0,\infty) \times (0,\infty) \times U \times U\) so that \(K(s,t,y,x) = 0\) if \(s \geq t\). For a locally integrable function \(f\) on \((0,\infty) \times U\), denote
\[
\mathcal{T}f(t,x) = \int_0^\infty \int_0^\infty K(s,t,y,x)f(s,y)dsdy
\]
\[
= \int_0^t \int_0^\infty K(s,t,y,x)f(s,y)dsdy
\]
\[
\leq \lim_{\varepsilon \downarrow 0} \int_0^{t-\varepsilon} \int_0^\infty K(s,t,y,x)f(s,y)dsdy
\]
\[
= \lim_{\varepsilon \downarrow 0} \mathcal{T}_\varepsilon f(t,x),
\]
where the sense of limit is specified in the following assumption.

Assumption 3.5. For any \(f \in L_2((0,\infty) \times U)\), \(\mathcal{T}f \to \mathcal{T}f\) in \(L_2((0,\infty) \times U)\) as \(\varepsilon \to 0\). Moreover, the operator \(f \mapsto \mathcal{T}f\) is bounded on \(L_2((0,\infty) \times U)\), i.e. there exists a constant \(N_5\) so that
\[
\|\mathcal{T}f\|_{L_2((0,\infty) \times U)} \leq N_5\|f\|_{L_2((0,\infty) \times U)} \quad \forall f \in L_2((0,\infty) \times U).
\]

The function \(\varphi\) in the next assumption is the one in Assumption 3.2.

Assumption 3.6 \((\varphi\text{-type Hörmander’s condition})\). (i) There exists a function \(\tilde{\varphi} : (0,\infty) \to (0,\infty)\) and constants \(c_1, c_2 > 0\) such that
\[
0 \leq c_1 \varphi(\tilde{\varphi}(r)), \quad \tilde{\varphi}(\varphi(r)) \leq c_2 r, \quad \forall r > 0. \quad (3.5)
\]
(ii) There exist constants \(c_0, N_0 > 0\) so that for all \((t,x), (s,y) \in (0,\infty) \times U\),
\[
\int_{|K(t-r,z,x) - K(r,s,z,y)| \geq c_0|\tilde{\varphi}(t-s) + |x-y|}} |K(t-z,x) - K(r,s,z,y)| \, drdz \leq N_0. \quad (3.6)
\]
Proof. Decompose \( L \). Suppose that Assumptions 3.2, 3.5 and 3.6 hold. Then for any \( N \), and Assumption 3.5, the closure of the complement of \( \phi \) is independent of \( \phi(T) \). Therefore one can still take \( \phi(T) = \phi^{-1}(\phi(t)) \). In general, even if \( \phi \) is neither strictly increasing nor continuous, due to (3.2) and (3.3) one can find a constant \( \delta > 0 \) so that
\[
\delta t \leq \phi^{-1}(\phi(t)) \leq \delta t \quad \forall t.
\]
Therefore one can still take \( \phi(t) = \phi^{-1}(\phi(t)) \).

Theorem 3.8. Let \( p > 2 \) and suppose that Assumptions 3.2, 3.5 and 3.6 hold. Then for any \( f \in L_2((0, \infty) \times U) \cap L_\infty((0, \infty) \times U) \),
\[
(i) \quad \| (Tf)^{\frac{p}{2}} \|_{L_\infty((0, \infty) \times U)} \leq N \| f \|_{L_\infty((0, \infty) \times U)},
\]
\[
(ii) \quad \| T f \|_{L_p((0, \infty) \times U)} \leq N \| f \|_{L_p((0, \infty) \times U)},
\]
where the constant \( N \) is independent of \( f \).

The proof of Theorem 3.8 is based on the following result.

Lemma 3.9. Suppose that Assumptions 3.2, 3.5 and 3.6 hold. Then for any \( f \in L_2((0, \infty) \times U) \cap L_\infty((0, \infty) \times U) \) and \( Q_\epsilon(t_0, x_0) \in Q_\phi \),
\[
\int_{Q_\epsilon(t_0, x_0)} \int_{Q_\epsilon(t_0, x_0)} |Tf(t, x) - T f(s, y)| \, dt \, ds \, dy \leq N \int_{Q_\epsilon(t_0, x_0)} \, dt \, ds \, dy \cdot \sup_{(0, \infty) \times U} |f|,
\]
where \( N \) depends only on \( d, \epsilon, N_\delta \), and \( N_0 \).

Proof. Decompose \( f \) into \( f = f_1 \cdot 1_{Q_\epsilon(t_0, x_0)} + (f - f_1 \cdot 1_{Q_\epsilon(t_0, x_0)}) =: f_1 + f_2 \), where \( \delta \) will be specified latter. Then obviously, \( f_1 \) has a support in \( Q_\epsilon(t_0, x_0) \) and \( f_2 \) has a support in the closure of the complement of \( Q_\epsilon(t_0, x_0) \). First we estimate \( T f_1 \). By Hölder’s inequality and Assumption 3.6,
\[
\int_{Q_\epsilon(t_0, x_0)} \int_{Q_\epsilon(t_0, x_0)} |Tf_1(t, x) - T f_1(s, y)| \, dt \, ds \, dy
\]
\[
\leq 2 |Q_\epsilon(t_0, x_0)| \int_{Q_\epsilon(t_0, x_0)} |T f_1(t, x)| \, dt \, dx
\]
\[
\leq 2 |Q_\epsilon(t_0, x_0)|^{\frac{1}{2}} \left( \int_{Q_\epsilon(t_0, x_0)} |T f_1(t, x)|^2 \, dt \, dx \right)^{\frac{1}{2}}
\]
\[
\leq 2 N_\delta |Q_\epsilon(t_0, x_0)|^{\frac{1}{2}} \left( \int_{Q_\epsilon(t_0, x_0)} |f_1(t, x)|^2 \, dt \, dx \right)^{\frac{1}{2}}
\]
\[
\leq N(d, N_\delta, \epsilon, c) |Q_\epsilon(t_0, x_0)|^{\frac{1}{2}} \sup_{(0, \infty) \times U} |f|,
\]
where in the last inequality we use the fact that there exists a \( n \in \mathbb{N} \) depending only on \( \delta \) so that \( 2^{n-1} \leq \delta \leq 2^n \) and thus \( \varphi(\delta c) \leq \varphi(2^n c) \leq (\delta c)^n \varphi(c) \).
Next we estimate $T f_2$. Recall

$$T f_2(t, x) - T f_2(s, y) = \int_0^\infty \int_U (K(r, t, z, x) - K(r, s, z, y))(f_2(r, z)) dr dz$$

and $f_2(r, z) = 0$ if $(r, z) \in Q_{c_0}^\infty(t_0, x_0)$. Note that if

$$(t, x), (s, y) \in Q_c^\infty(t_0, x_0) = (t_0, t_0 + \varphi(c)) \times B_c(x_0)$$

and

$$\tilde{\varphi}(|t-r|) + |x-y| < c_0(\tilde{\varphi}(|t-s|) + |x-y|),$$

then by (3.10) and (3.2),

$$|t_0 - r| \leq |t_0 - t| + |t - r| \leq \varphi(c) + c_1 \varphi(\tilde{\varphi}(|t-r|))$$

$$\leq \varphi(c) + c_1 \varphi(c_2 \tilde{\varphi}(r-s)) + c_0 |x-y|$$

$$\leq \varphi(c) + c_1 \varphi(c_2 \tilde{\varphi}(r-s) + 2c_0)$$

$$\leq \varphi(c) + c_1 \varphi(c_2 c_0 + 2c_0)$$

$$\leq \tilde{N}(c_0, c_1, c_2, \tilde{\varphi}(c))$$

and

$$|x_0 - z| \leq |x_0 - x| + |x - z|$$

$$\leq c + c_0 \tilde{\varphi}(|t-s|) + c_0 |x-y|$$

$$\leq c + c_0 \tilde{\varphi}(|c_2 c_0 + 2c_0|)$$

$$\leq \tilde{N}(c_0, c_1, c_2)$$

Thus taking $\delta > \bar{N} + \tilde{N}$, we have $(r, z) \in Q_{2c}^\infty(t_0, x_0)$ and

$$(K(r, t, z, x) - K(r, s, z, y))(f_2(r, z)) = 0.$$

Therefore by (3.10),

$$\left| \int_0^\infty \int_U (K(r, t, z, x) - K(r, s, z, y)) f_2(r, z) dr dz \right|$$

$$\leq \int \tilde{\varphi}(|t-r|) + |x-z| \geq c_0 \tilde{\varphi}(|t-s|) + |x-y|$$

$$\leq N_6 \sup_{(0, \infty) \times U} |f_2| \leq N_6 \sup_{(0, \infty) \times U} |f|,$$

which certainly implies

$$\int_{Q_c^\infty(t_0, x_0)} \int_{Q_c^\infty(t_0, x_0)} |T f_2(t, x) - T f_2(s, y)| \ dt ds dx dy$$

$$\leq N_6 |Q_c^\infty(t_0, x_0)|^2 \sup_{(0, \infty) \times U} |f|.$$

Combining (3.10) and (3.11), we have (3.9). The lemma is proved. \hfill \Box

**Proof of Theorem 3.8**

By Lemma 3.9

$$\|T f\|_{L_c^\infty(0, \infty) \times U} \leq N \|f\|_{L_c^\infty(0, \infty) \times U}.$$
Thus it is enough to prove (3.8).

Obviously, \((Tf)^2 \leq 2M_p(Tf)\). Thus by Assumption 3.3 and Theorem 3.3

\[
\|(Tf)^2\|_{L_p((0,\infty) \times U)} \leq N\|f\|_{L_p((0,\infty) \times U)}.
\]

Note that the map \(f \rightarrow (Tf)^2\) is subadditive since \(T\) is a linear operator. Hence by Marcinkiewicz’s interpolation theorem, for any \(p \in (2, \infty)\) there exists a constant \(N\) such that for all \(f \in L_p((0, \infty) \times U) \cap L_\infty((0, \infty) \times U),\)

\[
\|(Tf)^2\|_{L_p((0,\infty) \times U)} \leq N\|f\|_{L_p((0,\infty) \times U)}.
\]

Therefore by Theorem 3.3 (3.8) is proved.

\[\square\]

4. PDE WITH PSEUDO-DIFFERENTIAL OPERATORS

In this section we study PDEs with pseudo-differential operators. The result of this section is a generalization of Theorem 2.13 if \(X^2 = 0\).

Let \(\Psi\) be a complex-valued function defined for \(t > 0\) and \(\xi \in \mathbb{R}^d\). Consider the equation

\[
u_t = \Psi(t, iD)\nu + f, \quad \nu(0, x) = 0, \tag{4.1}
\]

where

\[
\Psi(t, iD)\nu(t, x) := \mathcal{F}^{-1} [\Psi(t, \xi)\mathcal{F} [\nu(t, \cdot)] (\xi)] (x).
\]

Then formally the solution \(\nu\) to equation (4.1) is given by

\[
\nu(t, x) = \mathcal{F}^{-1} \left[ \int_0^t \exp \left( \int_s^t \Psi(r, \xi)dr \right) \mathcal{F} [f(s, \cdot)] (\xi)ds \right] (x) \tag{4.2}
\]

Recall that \(d_0 = \left\lfloor \frac{d}{2} \right\rfloor + 1\).

**Assumption 4.1.**

(i) There exists a nondecreasing function \(\psi : (0, \infty) \mapsto (0, \infty)\) and positive constants \(\delta_5 \geq \delta_4, N_7, \) and \(N_8\) so that for any \(\lambda_2 \geq \lambda_1 > 0\)

\[
N_7 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_4} \leq \psi(\lambda_2) \psi(\lambda_1) \leq N_8 \left( \frac{\lambda_2}{\lambda_1} \right)^{\delta_5}. \tag{4.3}
\]

(ii) There exist positive constants \(\delta_6\) and \(N_9\) so that

\[
|D^2\psi(t, \xi)| \leq N_9|\psi(|\xi|^2)||\xi|^{-|\alpha|} \tag{4.5}
\]

for all \((t, \xi) \in (0, \infty) \times \mathbb{R}^d\) and multi-index \(|\alpha| \leq d_0\).

(iii) \(\psi(\lambda)\) is \(d_0\)-times continuously differentiable and there exists a constant \(N_{10}\) so that for all \(\lambda \in (0, \infty)\) and a natural number \(n \leq d_0\),

\[
|D^n\psi(\lambda)| \leq N_{10}\lambda^{-n}\psi(\lambda). \tag{4.6}
\]

Denote

\[
\psi^{-1}(t) := \inf \{s \geq 0 : \psi(s) \geq t\}
\]

Then \(\psi^{-1}\) is a nondecreasing function from \((0, \infty)\) into \((0, \infty)\) and there exist positive constants \(N_1,\) and \(N_2\) so that for any \(\lambda_2 \geq \lambda_1 > 0\),

\[
N_1 \left( \frac{\lambda_2}{\lambda_1} \right)^{1/\delta_5} \leq \psi^{-1}(\lambda_2) \psi^{-1}(\lambda_1) \leq N_2 \left( \frac{\lambda_2}{\lambda_1} \right)^{1/\delta_4}. \tag{4.7}
\]
where $N_1$ and $\tilde{N}_1$ depend only on $\delta_4$, $\delta_5$, $N_7$, and $N_8$. Furthermore, $\psi(\psi^{-1}(t)) \sim t$ and $\psi^{-1}(\psi(t)) \sim t$, that is for all $t > 0$

$$N^{-1}t \leq \psi^{-1}(\psi(t)) \leq t, \quad N^{-1}t \leq \psi(\psi^{-1}(t)) \leq Nt, \quad (4.8)$$

where $N$ depends only on $\delta_4$, $\delta_5$, $N_7$, and $N_8$.

Here is the main result of this section.

**Theorem 4.2.** Let $p \in (1, \infty)$ and suppose Assumption 4.1 holds. Then for any $f \in L_2((0, \infty) \times \mathbb{R}^d) \cap L_\infty((0, \infty) \times \mathbb{R}^d)$ and $u$ defined as in (4.2), we have

$$\|\psi(\Delta)u\|_{L_p((0, \infty) \times \mathbb{R}^d)} \leq N\|f\|_{L_p((0, \infty) \times \mathbb{R}^d)}, \quad (4.9)$$

where $N$ depends only on $d$, $p$, $\delta_k$ and $N_j$ ($k = 4, 5, 6$ and $j = 7, 8, 9, 10$), and

$$\psi(\Delta)u(t, x) := -\psi(-\Delta)u(t, x) := F^{-1}\left[\psi(\xi^2)F[u(t, \cdot)](\xi)\right](x).$$

We only show this theorem for $p \in [2, \infty)$ due to the duality argument. To prove this theorem, we apply Theorem 3.8. Define

$$p(s, t, x) = F^{-1}\left[\exp\left(\int_s^t \Psi(r, \xi)dr\right)\right](x),$$

and set

$$K(s, t, y, x) := 1_{0 < s < t}(\psi(\Delta))p(s, t, x - y)$$

$$:= -1_{0 < s < t}(\psi(-\Delta))p(s, t, x - y)$$

$$:= 1_{0 < s < t}F^{-1}\left[\psi(\xi)\exp\left(\int_s^t \Psi(r, \xi)dr\right)\right](x - y).$$

Note that due to Assumption 4.1 (iii), for each $t > s$, $\psi(\xi^2)\exp\left(\int_s^t \Psi(r, \xi)dr\right)$ is integrable with respect to $\xi$ and thus for any $g \in L_2(\mathbb{R}^d)$,

$$F^{-1}\left[\psi(\xi)\exp\left(\int_s^t \Psi(r, \xi)dr\right)Fg(\xi)\right](x)$$

$$= F^{-1}\left[\psi(\xi^2)\exp\left(\int_s^t \Psi(r, \xi)dr\right)\right]\cdot g(\cdot)(x)$$

$$:= \int_{\mathbb{R}^d} F^{-1}\left[\psi(\xi^2)\exp\left(\int_s^t \Psi(r, \xi)dr\right)\right][(x - y)g(y)dy$$

$$= \int_{\mathbb{R}^d} K(s, t, y, x)g(y)dy.$$

Therefore (at least formally)

$$\psi(\Delta)u(t, x) = \int_0^t \psi(\Delta)p(s, t, x) * f(s, x)ds = \int_0^t \int_{\mathbb{R}^d} K(s, t, y, x)f(s, y)dyds$$

$$= \lim_{\varepsilon \rightarrow 0} \int_0^{t - \varepsilon} \int_{\mathbb{R}^d} K(s, t, y, x)f(s, y)dyds$$

$$= \lim_{\varepsilon \rightarrow 0} T_\varepsilon f(t, x) =: T f(t, x),$$

where the limit is in $L_2((0, \infty) \times \mathbb{R}^d)$ (see Lemma 4.5).

Set

$$\varphi(c) := \psi^{-1}(c^{-1})^{-1/2}, \quad \tilde{\varphi}(c) := \psi(c^{-2})^{-1}.$$
Then due to (4.3) and (4.7), \( \varphi \) and \( \tilde{\varphi} \) satisfy
\[
\varphi(r) \downarrow 0 \ 	ext{as} \ r \downarrow 0, \quad \varphi(r) \uparrow \infty \ 	ext{as} \ r \uparrow \infty,
\]
\[
\tilde{c} := \sup_{r > 0} \frac{\varphi(2r)}{\varphi(r)} < \infty.
\]
\[
r \leq c_1 \varphi(\tilde{\varphi}(r)), \quad \tilde{\varphi}(\varphi(r)) \leq c_2 r \quad \forall r \in (0, \infty).
\]
Thus under this setting, Assumptions 3.2 and (3.5) hold. Therefore in order to prove (4.9), it suffices to show that Assumption 3.5 and (3.6) hold. For this, we need some preliminaries. Denote
\[
a_t := (\psi^{-1}(t^{-1}))^{1/2},
\]
\[
\delta_1 = \delta_1(\xi) = \begin{cases} 2\delta_4 & \text{if } |\xi| \geq 1 \\ 2\delta_5 & \text{if } |\xi| < 1, \end{cases}
\]
and
\[
\delta_2 = \delta_2(\xi) = \begin{cases} 2\delta_5 & \text{if } |\xi| \geq 1 \\ 2\delta_4 & \text{if } |\xi| < 1. \end{cases}
\]

**Lemma 4.3.** For any \( t \in (0, \infty) \) and \( \xi \in \mathbb{R}^d \),
\[
N^{-1}|\xi|^{\delta_1} \leq t\psi(|a_t\xi|^2) \leq N|\xi|^{\delta_2}
\]
where \( N \) depends only on \( \delta_4, \delta_5, \delta_6, N_7, \) and \( N_8 \).

**Proof.** Due to (4.3), there exists a \( N \) so that
\[
N^{-1}|\xi|^{\delta_1} \leq \frac{\psi(|a_t\xi|^2)}{\psi(a_t^2)} \leq N|\xi|^{\delta_2} \quad \forall (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{4.11}
\]
Combining (4.8) and (4.11), we have
\[
\delta_1^2N^{-1}|\xi|^{\delta_1} \leq t^{-1}\delta_1^{-1}t\psi(|a_t\xi|^2) \psi(a_t^{-2}) \leq \psi(a_t^{-1}) \psi(|a_t\xi|^2) \leq N|\xi|^{\delta_2}.
\]

**Corollary 4.4.** For any \( t \in (0, \infty) \) and \( \xi \in \mathbb{R}^d \),
\[
t \mathbb{R} \Psi(t, a_t\xi)] \leq -N|\xi|^{\delta_1},
\]
where \( N \) depends only on \( \delta_4, \delta_5, \delta_6, N_7, \) and \( N_8 \).

**Proof.** By (4.4) and Lemma 4.3,
\[
t \mathbb{R} [\Psi(t, a_t\xi)] \leq -\delta_6 t\psi(|a_t\xi|^2) \leq -N|\xi|^{\delta_1}.
\]

First, we prove that Assumption 3.5 holds.

**Lemma 4.5.** There exists a constant \( N(d, \delta_6) \) such that
\[
\|Tf\|_{L^2((0, \infty) \times \mathbb{R}^d)} \leq N\|f\|_{L^2((0, \infty) \times \mathbb{R}^d)} \quad \forall f \in L^2((0, \infty) \times \mathbb{R}^d). \tag{4.12}
\]
Proof. By Fubini’s theorem, Plancherel’s theorem, and Minkowski’s inequality,
\[
\|Tf\|_{L^2((0,\infty) \times \mathbb{R}^d)}^2 \\
\leq \int_{\mathbb{R}^d} \int_0^\infty \left( \int_0^t \psi(|\xi|^2) e^{\int_0^s \Psi(r,\xi) dr} \mathcal{F}(f)(s,\xi) ds \right)^2 dt d\xi \\
\leq \int_{\mathbb{R}^d} \int_0^\infty \left( \int_0^t \psi(|\xi|^2) e^{-\delta_s \psi(|\xi|^2)} |\mathcal{F}(f)(t-s,\xi)| ds \right)^2 dt d\xi \\
\leq \int_{\mathbb{R}^d} \int_0^\infty \left( \int_0^\infty |\mathcal{F}(f)(t-s,\xi)|^2 dt \right)^{1/2} \psi(|\xi|^2) e^{-\delta_s \psi(|\xi|^2)} ds \right)^2 d\xi \\
\leq N \int_0^\infty \int_{\mathbb{R}^d} |f(t,x)|^2 dt dx.
\]

The lemma is proved. \(\square\)

Next we show that \(K\) satisfies the condition. Denote
\[
q_1(s,t,x) = (t-s) \mathcal{F}^{-1} \left[ \psi(|a_{t-s} \xi|^2) \exp \left( \int_s^t \Psi(r, a_{t-s} \xi) dr \right) \right](x),
\]
\[
q_{2,\ell}(s,t,x) = (t-s) \mathcal{F}^{-1} \left[ \xi^\ell \psi(|a_{t-s} \xi|^2) \exp \left( \int_s^t \Psi(r, a_{t-s} \xi) dr \right) \right](x), \quad \ell = 1, \ldots, d,
\]
and
\[
q_3(s,t,x) \\
= (t-s)^2 \mathcal{F}^{-1} \left[ \Psi(t,a_{t-s} \xi) \psi(|a_{t-s} \xi|^2) \exp \left( \int_s^t \Psi(r, a_{t-s} \xi) dr \right) \right](x).
\]

By the change of variables,
\[
(t-s)(a_{t-s})^{-d} \psi(\Delta) p(s,t,\cdot)((a_{t-s})^{-1} x) = q_1(s,t,x), \tag{4.13}
\]
\[
(t-s)(a_{t-s})^{-d-1} \psi(\Delta) p_{x^\ell}(s,t,\cdot)((a_{t-s})^{-1} x) = q_{2,\ell}(s,t,x), \tag{4.14}
\]
and
\[
(t-s)^2(a_{t-s})^{-d} \frac{\partial}{\partial t} \psi(\Delta) p(s,t,\cdot)((a_{t-s})^{-1} x) = q_3(s,t,x). \tag{4.15}
\]

Note that by (4.6), (4.4), (4.5), Lemma 4.3 and Corollary 4.4, there exists a positive constant \(N\) such that for all \(\xi \neq 0\),
\[
\begin{align*}
|D_\xi^\alpha \left( \mathcal{F}(q_1(s,t,\cdot)(\xi)) \right) | \\
= (t-s) |D_\xi^\alpha \left( \psi(|a_{t-s} \xi|^2) \exp \left( \int_s^t \Psi(r, a_{t-s} \xi) dr \right) \right) | \\
\leq N \left| \xi^{\delta_s - |\alpha|} \exp \left( -N^{-1} |\xi|^\delta_1 \right) \right|, \tag{4.16}
\end{align*}
\]
Lemma 4.6. There exists a constant $N = N(d, \delta_k, N_j)$ ($k = 4, 5, 6$ and $j = 7, 8, 9, 10$) so that for any multi-index $\alpha$ with $|\alpha| \leq d_0$, $0 < s < t$, and $\ell = 1, \ldots, d$,

$$\int_{\mathbb{R}^d} |D_\xi^\alpha (\mathcal{F}[q_1(s, t, \cdot)](\xi))| \, d\xi + \int_{\mathbb{R}^d} |D_\xi^\alpha (\mathcal{F}[q_2, \ell(s, t, \cdot)](\xi))| \, d\xi + \int_{\mathbb{R}^d} |D_\xi^\alpha (\mathcal{F}[q_3(s, t, \cdot)](\xi))| \, d\xi \leq N.$$

Proof. The first term

$$\int_{\mathbb{R}^d} |D_\xi^\alpha (\mathcal{F}[q_1(s, t, \cdot)](\xi))| \, d\xi$$

is easily controlled by (4.10). Indeed, since

$$\left| |\xi|^{\delta_2-|\alpha|} \exp \left( -N^{-1} |\xi|^{\delta_1} \right) \right| \leq N \left| |\xi|^{\delta_2-d_0} \exp \left( -(2N)^{-1} |\xi|^{\delta_1} \right) \right|$$

and the latter function is integrable with respect to $\xi$, we have

$$\int_{\mathbb{R}^d} |D_\xi^\alpha (\mathcal{F}[q_1(s, t, \cdot)](\xi))| \, d\xi \leq N.$$

The other two terms are similarly controlled by the inequalities

$$|D_\xi^\alpha (\mathcal{F}[q_2, \ell(s, t, \cdot)](\xi))| \leq N \left| |\xi|^{\ell+\delta_2-d_0} \exp \left( -(2N)^{-1} |\xi|^{\delta_1} \right) \right|$$

and

$$|D_\xi^\alpha (\mathcal{F}[q_3(s, t, \cdot)](\xi))| \leq N \left| |\xi|^{2\delta_2-d_0} \exp \left( -(2N)^{-1} |\xi|^{\delta_1} \right) \right|.$$
Lemma 4.9. Let
\[ 2\delta_2 - 2|\alpha| - 2\varepsilon > 4\delta_4 - 2(d_0 - 1) - 2\varepsilon > -d. \]
The lemma is proved. \(\square\)

Lemma 4.8. There exists a constant \(N = N(d, \delta_k, N_j) > 0\) \((k = 4, 5, 6\) and \(j = 7, 8, 9, 10)\) so that for all \(c > 0\), multi-index \(|\alpha| \leq d_0\), \(0 < s < t\), and \(\ell = 1, \ldots, d\),
\[
\int_{|\xi| \geq c} |D^\alpha_\xi (\mathcal{F}[q_1(s, t, \cdot)](\xi))|^2 d\xi + \int_{|\xi| \geq c} |D^\alpha_\xi (\mathcal{F}[q_2, \xi(s, t, \cdot)](\xi))|^2 d\xi + \int_{|\xi| \geq c} |D^\alpha_\xi (\mathcal{F}[q_3(s, t, \cdot)](\xi))|^2 d\xi \leq N(1 + c^{4d_2 - 2d_0 + d}).
\]

Proof. Due to similarity, we only estimate the first term above.

By (4.16),
\[
|D^\alpha_\xi (\mathcal{F}[q_1(s, t, \cdot)](\xi))|^2 \leq N|\xi|^{2d_2 - 2|\alpha|} \exp\left(-N^{-1}|\xi|^{\delta_1}\right) \leq N|\xi|^{4\delta_4 - 2d_0} \exp\left(-(2N)^{-1}|\xi|^{\delta_1}\right).
\]
Therefore
\[
\int_{|\xi| \geq c} |D^\alpha_\xi (\mathcal{F}[q_1(s, t, \cdot)](\xi))|^2 d\xi \leq N(1 + c^{4\delta_4 - 2d_0 + d}).
\]
The lemma is proved. \(\square\)

Lemma 4.9. Let \(0 < \delta < (\delta_4 \wedge \frac{1}{2})\). Then there exists a constant \(N = N(d, \delta, \delta_k, N_j)\) \((k = 4, 5, 6\) and \(j = 7, 8, 9, 10)\) so that for any \(0 < s < t\) and \(\ell = 1, \ldots, d\)
\[
\int_{\mathbb{R}^d} \left|\mathbf{x}^{\frac{d}{2} + \delta} q_1(s, t, x)\right|^2 dx \leq N,
\]
(4.18)
\[
\int_{\mathbb{R}^d} \left|\mathbf{x}^{\frac{d}{2} + \delta} q_{2, \xi}(s, t, x)\right|^2 dx \leq N,
\]
(4.19)
and
\[
\int_{\mathbb{R}^d} \left|\mathbf{x}^{\frac{d}{2} + \delta} q_3(s, t, x)\right|^2 dx \leq N.
\]
(4.20)

Proof. As before, we only prove (4.18) since the proofs of (4.19) and (4.20) are similar.

Note that it suffices to show that for each \(\ell = 1, \ldots, d\),
\[
\int_{\mathbb{R}^d} \left|(ix^\ell)^{d_0 - 1} q_1(s, t, x)\right|^2 dx \leq N,
\]
(4.21)
where \(i\) is the imaginary number, i.e. \(i^2 = -1\). By a property of the Fourier inverse transform,
\[
(ix^\ell)^{d_0 - 1} \mathcal{F}^{-1} [f(\xi)](x) = (-1)^{d_0 - 1} \mathcal{F}^{-1} \left[D^{d_0 - 1}_{\xi^\ell} f(\xi)\right](x).
\]
Hence the left hand side of (4.21) is equal to
\[
\int_{\mathbb{R}^d} \left| (ix) \frac{d}{dx} + \frac{\delta}{(d_0 - 1)} \mathcal{F}^{-1} \left( D_{\xi}^{d_0-1} \mathcal{F} [q_1(s, t, \cdot)](\xi) \right)(x) \right|^2 \, dx \\
\leq \int_{\mathbb{R}^d} \left| x \frac{d}{dx} + \frac{\delta}{(d_0 - 1)} \mathcal{F}^{-1} \left( D_{\xi}^{d_0-1} \mathcal{F} [q_1(s, t, \cdot)](\xi) \right)(x) \right|^2 \, dx. \quad (4.22)
\]

Set
\[
\varepsilon := \varepsilon(\delta) = \frac{d}{2} + \delta - (d_0 - 1).
\]

Then by the Plancherel theorem, the right hand side of (4.22) equals
\[
N(d) \int_{\mathbb{R}^d} \left| (-\Delta)^{\varepsilon/2} \left( D_{\xi}^{d_0-1} q_1(s, t, \xi) \right) \right|^2 \, d\xi. \quad (4.23)
\]

Obviously, \( \varepsilon \in (0, 1 \wedge (2\delta_1 + \frac{d}{2} - (d_0 - 1))) \). Using the integral representation of the Fractional Laplacian operator \((-\Delta)^{\varepsilon/2}\) we get
\[
(-\Delta)^{\varepsilon/2} (D_{\xi}^{d_0-1} q_1(s, t, \xi)) = N \int_{\mathbb{R}^d} \frac{D_{\xi}^{d_0-1} q_1(s, t, \xi + \eta) - D_{\xi}^{d_0-1} q_1(s, t, \xi)}{|\eta|^{d+\varepsilon}} \, d\eta.
\]

We divide \((-\Delta)^{\varepsilon/2} (D_{\xi}^{d_0-1} q_1(s, t, \xi))\) into two terms:
\[
N \int_{|\eta| \geq 1} \frac{D_{\xi}^{d_0-1} q_1(s, t, \xi + \eta) - D_{\xi}^{d_0-1} q_1(s, t, \xi)}{|\eta|^{d+\varepsilon}} \, d\eta \\
+ N \int_{|\eta| < 1} \frac{D_{\xi}^{d_0-1} q_1(s, t, \xi + \eta) - D_{\xi}^{d_0-1} q_1(s, t, \xi)}{|\eta|^{d+\varepsilon}} \, d\eta =: \mathcal{I}_1(s, t, \xi) + \mathcal{I}_2(s, t, \xi).
\]

By Minkowski’s inequality and Lemma 4.7.
\[
\left[ \int_{\mathbb{R}^d} |\mathcal{I}_1(s, t, \xi)|^2 \, d\xi \right]^{1/2} \leq 2 \left\| D_{\xi}^{d_0-1} q_1(s, t, \cdot) \right\|_{L^2(\mathbb{R}^d)} \int_{|\eta| \geq 1} \frac{1}{|\eta|^{d+\varepsilon}} \, d\eta \leq N < \infty.
\]

We split \( \mathcal{I}_2 \) into \( \mathcal{I}_{2,1}, \mathcal{I}_{2,2}, \) and \( \mathcal{I}_{2,3} \), where
\[
\mathcal{I}_{2,1}(s, t, \xi) := \int_{|\eta| < 1} \int_{|\eta| < |\xi|} \frac{D_{\xi}^{d_0-1} q_1(s, t, \xi + \eta) - D_{\xi}^{d_0-1} q_1(s, t, \xi)}{|\eta|^{d+\varepsilon}} \, d\eta,
\]
\[
\mathcal{I}_{2,2}(s, t, \xi) := \int_{|\eta| < 1} \int_{|\eta| \geq |\xi|} \frac{D_{\xi}^{d_0-1} q_1(s, t, \xi + \eta)}{|\eta|^{d+\varepsilon}} \, d\eta,
\]
and
\[
\mathcal{I}_{2,3}(s, t, \xi) := -\int_{|\eta| < 1} \int_{|\eta| \geq |\xi|} \frac{D_{\xi}^{d_0-1} q_1(s, t, \xi)}{|\eta|^{d+\varepsilon}} \, d\eta.
\]

By the fundamental theorem of calculus and the Fubini theorem,
\[
|\mathcal{I}_{2,1}(s, t, \xi)| \leq \int_{0}^{1} \int_{|\eta| < 1} \int_{|\eta| < |\xi|} \left| \nabla D_{\xi}^{d_0-1} q_1(s, t, \xi + \theta\eta) \right| \, d\eta d\theta.
\]
Hence by Minkowski’s inequality and Lemma 4.8,
\[
\|\mathcal{I}_{2,1}(s, t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq \int_{|\eta| < 1} \left( \int_{|\eta| < |\xi|} \left| \nabla D_{\xi t}^{d_0 - 1} q_1(s, t, \xi) \right|^2 d\xi \right)^{1/2} \frac{1}{|\eta|^{d+\varepsilon-1}} d\eta
\]
\[
\leq N \int_{|\eta| < 1} \frac{1 + |\eta|^{2\delta_4-d_0+\frac{d}{2}}}{|\eta|^{d+\varepsilon-1}} d\eta \leq N
\]
since \(2\delta_4 - d_0 + \frac{d}{2} - d - \varepsilon + 1 > -d\).

Note that if \(|\xi| \geq 2\), then \(\mathcal{I}_{2,2}(s, t, \xi) = \mathcal{I}_{2,3}(s, t, \xi) = 0\) and thus we may assume \(|\xi| \leq 2\). Recalling the range of \(\varepsilon\), we have
\[
\varepsilon + \delta_4 < 2\delta_4 + \frac{d}{2} - (d_0 - 1).
\]

Hence by Hölder’s inequality and Lemma 4.7,
\[
|\mathcal{I}_{2,2}(s, t, \xi)|
\leq \left[ \int_{|\eta| < 1} \frac{|\xi + \eta|^{2\varepsilon+2\delta_4}}{|\eta|^{2d+2\varepsilon}} d\eta \right]^{1/2} \left[ \int_{\mathbb{R}^d} \left| \xi + \eta \right|^{-\varepsilon-\delta_4} D_{\xi t}^{d_0-1} q_1(s, t, \xi + \eta) \right]^{1/2}
\]
\[
\leq N \left[ \int_{|\eta| < 1} \frac{|\eta|^{-2d+2\delta_4}}{|\eta|^{2d+2\varepsilon}} d\eta \right]^{1/2} \left[ \int_{\mathbb{R}^d} \left| \eta \right|^{-\varepsilon-\delta_4} D_{\xi t}^{d_0-1} q_1(s, t, \eta) \right]^{1/2}
\]
\[
\leq N \left( 1 + |\xi|^{-\frac{d}{2}+\delta_4} \right).
\]
Therefore we have
\[
\|\mathcal{I}_{2,2}(s, t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq N \int_{|\xi| < 2} (1 + |\xi|^{-d+2\delta_4}) \, d\xi \leq N.
\]
Finally by Lemma 4.7, again,
\[
\|\mathcal{I}_{2,3}(s, t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq N \int_{|\xi| < 2} (1 + |\xi|^{-d+2\delta_4}) \, d\xi \leq N.
\]
Due to (4.22) and (4.23), combining all estimates for \(\mathcal{I}_1, \mathcal{I}_{2,1}, \mathcal{I}_{2,2}, \mathcal{I}_{2,3}\), we have (4.24). The lemma is proved.

Let \(q(s, t, x)\) be anyone of \(q_1, q_2, \ell\) and \(q_3\). Then by (4.17), Lemma 4.9 and Hölder’s inequality,
\[
\sup_{s < t} \|q(s, t, \cdot)\|_{L^1} \leq \sup_{s < t} \int_{|x| \leq 1} |q(s, t, x)| \, dx + N \sup_{s < t} \left( \int_{|x| < 1} |x|^{d/2+\delta} q(s, t, x)^2 \, dx \right)^{1/2} \leq \infty. \quad (4.24)
\]
Recall \(a_t := (\psi^{-1}(t^{-1}))^{1/2}\) and denote
\[
\psi(\Delta) p(r, t, z) := \psi(\Delta) [p(r, t, \cdot)](z) = \mathcal{F}^{-1} \left[ \psi(|\xi|^2) \mathcal{F} [p(r, t, \cdot)](\xi) \right](z).
\]

**Lemma 4.10.** Let \(0 < \delta < (\delta_4 \wedge \frac{d}{2})\). Then there exists a constant \(N = N(d, \delta, \delta_k, N_j)\) \((k = 4, 5, 6\) and \(j = 7, 8, 9, 10\) such that for all \(t > s > a > 0\), \(c > 0\),
\[
\int_s^t \int_{|z| \geq c} |\psi(\Delta) p(r, t, z)(z)| \, dz \, dr \leq N (a_{t-s} c)^{-\delta}, \quad (4.25)
\]
\[
\int_0^a \int_{\mathbb{R}^d} |\psi(\Delta)p(r, t, z + h) - \psi(\Delta)p(r, t, z)| \, dz \, dr \leq N|h|a_{t-a}, \quad (4.26)
\]
and
\[
\int_0^a \int_{\mathbb{R}^d} |\psi(\Delta)p(r, t, z) - \psi(\Delta)p(r, s, z)| \, dz \, dr \leq N(t-s)(s-a)^{-1}. \quad (4.27)
\]

Proof. (i) By (4.13), (4.18), and Hölder’s inequality,
\[
\int_{|z| \geq c} |\psi(\Delta)p(r, t, z)| \, dz
\]
\[
= (t-r)^{-1} \int_{|z| \geq a_{t-r}c} |q_1(r, t, z)| \, dz
\]
\[
\leq (t-r)^{-1} \left( \int_{|z| \geq a_{t-r}c} |z|^{-d-2\delta} \, dz \right)^{1/2} \left( \int_{|z| \geq a_{t-r}c} \left| \frac{\delta}{2} q_1(r, t, z) \right|^2 \, dz \right)^{1/2}
\]
\[
\leq N(t-r)^{-1} (a_{t-r}c)^{-\delta}.
\]

Therefore by (4.8) and changing the variable \( r \rightarrow t-(t-s)r \),
\[
\int_s^t \int_{|z| \geq c} |\psi(\Delta)p(r, t, z)| \, dz \, dr \leq N \int_s^t (t-r)^{-1} (a_{t-r}c)^{-\delta} \, dr
\]
\[
\leq N \int_0^1 r^{-1} (a_{(t-s)c})^{-\delta} \, dr.
\]

Thus by (4.7),
\[
\int_0^1 r^{-1} (a_{(t-s)c})^{-\delta} \, dr \leq N (a_{t-s}c)^{-\delta} \int_0^1 r^{-\delta/(2\delta_s)} \, dr \leq N (a_{t-s}c)^{-\delta}.
\]

(ii) Recall
\[
\psi(\Delta)p_{\varepsilon}(r, t, x) = (t-r)^{-1}(a_{t-r})^{d+1}q_2(\varepsilon, r, t, a_{t-r}x).
\]

Using the fundamental theorem of calculus, Fubini’s theorem, and (4.24),
\[
\int_0^a \int_{\mathbb{R}^d} |\psi(\Delta)p(t, r, z + h) - \psi(\Delta)p(t, r, z)| \, dz \, dr
\]
\[
\leq |h| \int_0^a \int_{\mathbb{R}^d} \int_0^1 |\nabla \psi(\Delta)p(r, t, z + \theta h)| \, d\theta \, dz \, dr
\]
\[
\leq |h| \int_0^a (t-r)^{-1} a_{t-r} \sum_{\ell=1}^d \int_{\mathbb{R}^d} |q_{2,\ell}(r, t, z)| \, dz \, dr \leq N|h| \int_0^a (t-r)^{-1} a_{t-r} \, dr.
\]

Moreover, by changing the variable \( r \rightarrow (t-a)r \) and (4.7),
\[
\int_0^a (t-r)^{-1} a_{t-r} \, dr \leq \int_1^\infty r^{-1} a_{(t-a)} \, dr = a_{t-a} \int_1^\infty r^{-1} \frac{a(t-a)}{a_{t-a}} \, dr
\]
\[
\leq Na_{t-a}.
\]

Hence (4.26) is proved.
(iii) By the fundamental theorem of calculus and (4.15),

$$|\psi(\Delta)p(r, t, z) - \psi(\Delta)p(r, s, z)|$$

$$\leq \int_0^1 |t - s|((\theta t + (1 - \theta)s - r)^d$$

$$\times |q_3(r, \theta t + (1 - \theta)s, a(\theta t + (1 - \theta)s - r))|d\theta.$$

Therefore, by (4.24),

$$\int_0^a \int_{\mathbb{R}^d} |\psi(\Delta)p(r, t, z) - \psi(\Delta)p(r, s, z)| \, dz \, dr \leq \int_0^a |t - s| (\theta t + (1 - \theta)s - r)^2dr$$

$$\leq |t - s|(s - a)^{-1}.$$ 

The lemma is proved. □

Recall

$$\varphi(c) = \psi^{-1}(c^{-1})^{-1/2} = 1/a_c$$

and observe that by (4.7), there exists a $\tilde{c}_0 \geq 1$ so that

$$\varphi(t + s) \leq \tilde{c}_0 (\varphi(t) + \varphi(s)) \quad \forall s, t \geq 0. \quad (4.28)$$

Denote

$$A(t, s, r, y, x) := \{z \in \mathbb{R}^d : \varphi(|t - r|) + |x - z| \geq 4\tilde{c}_0(\varphi(|t - s|) + |x - y|)\}.$$ 

Corollary 4.11. For all $(t, x), (s, y) \in (0, \infty) \times \mathbb{R}^d$, 

$$\int_0^\infty \int_{A(t, s, r, y, x)} |1_{0 < r < t}\psi(\Delta)p(r, t, x - z) - 1_{0 < r < s}\psi(\Delta)p(r, s, y - z)| \, dz \, dr \leq N, \quad (4.29)$$

where $N = N(d, \delta_k, N_j)$ ($k = 4, 5, 6$ and $j = 7, 8, 9, 10$).

Proof. Choose a $0 < \delta < (\delta_4 \wedge \frac{1}{2})$. Without loss of generality, we assume $t \geq s$. Denote

$$\mathcal{I}(t, s, r, y, x) = \int_{A(t, s, r, y, x)} |1_{0 < r < t}\psi(\Delta)p(r, t, x - z) - 1_{0 < r < s}\psi(\Delta)p(r, s, y - z)| \, dz.$$ 

If $r \geq t$, then $\mathcal{I}(t, s, r, y, x) = 0$. Thus

$$\int_0^\infty \mathcal{I}(t, s, r, y, x)dr = \int_{2s-t}^t \mathcal{I}(t, s, r, y, x)dr + \int_0^{2s-t} \mathcal{I}(t, s, r, y, x)dr$$

$$=: \mathcal{I}_1(t, s, r, y, x) + \mathcal{I}_2(t, s, r, y, x).$$

First we estimate $\mathcal{I}_1(t, s, r, y, x)$. Note that due to (4.28),

$$A(t, s, r, y, x) \subset \{z \in \mathbb{R}^d : |x - z| \geq \varphi(|t - s|)\} \quad (4.30)$$

...
if $2s - t < r < t$. By (4.30) and (4.25),

$$
\mathcal{I}_1(t, s, y, x) \leq \int_{2s-t}^{t} \int_{\varphi((t-r)|+|x-z|\geq 4\bar{c}_0(\varphi(t-s)+|x-y|)} |\psi(\Delta) p(r, t, x - z)| \, dz \, dr \\
+ \int_{2s-t}^{t} \int_{\varphi((t-r)|+|x-z|\geq 4\bar{c}_0(\varphi(t-s)+|x-y|)} |\psi(\Delta) p(r, s, x - z)| \, dz \, dr \\
\leq 2 \int_{2s-t}^{t} \int_{|z|\geq \varphi(t-s)} |\psi(\Delta) p(r, t, z)| \, dz \, dr
$$

We split $\mathcal{I}_2$. Observe

$$
\mathcal{I}_2 \leq \mathcal{I}_{2, 1} + \mathcal{I}_{2, 2}
$$

\[ := \int_{0}^{2s-t} \int_{A(t, s, r, y, x)} |1_{0<r<t} \psi(\Delta) p(r, t, x - z) - 1_{0<r<t} \psi(\Delta) p(r, t, y - z)| \, dz \, dr \]

$$
+ \int_{0}^{2s-t} \int_{A(t, s, r, y, x)} |1_{0<r<t} \psi(\Delta) p(r, t, y - z) - 1_{0<r<s} \psi(\Delta) p(r, s, y - z)| \, dz \, dr.
$$

If $|x - y| \leq \varphi((t-s))$ then by (4.26),

$$
\mathcal{I}_{2, 1} \leq N |x - y| a_{2(t-s)} \leq N.
$$

On the other hand, if $|x - y| > \varphi((t-s))$, then

$$
t - s \leq \frac{1}{\psi(|x - y|^{-2})}. \quad (4.31)
$$

Moreover by (4.28), if $s - (\psi(|x - y|^{-2}))^{-1} < r < t$ and (4.31) holds, then

$$
A(t, s, r, y, x) \subset \{|x - z| \geq \varphi((t-s)) + |x - y|\}.
$$

Therefore

$$
\mathcal{I}_{2, 1} \leq 2\mathcal{I}_{2, 1, 1} + \mathcal{I}_{2, 1, 2},
$$

where

$$
\mathcal{I}_{2, 1, 1} := \int_{s-(\psi(|x - y|^{-2}))^{-1}}^{t} \int_{|z|\geq \varphi((t-s)) + |x - y|} |\psi(\Delta) p(r, t, z)| \, dz \, dr,
$$

and

$$
\mathcal{I}_{2, 1, 2} := \int_{0}^{s-(\psi(|x - y|^{-2}))^{-1}} \int_{\mathbb{R}^d} 1_{0<r<t} |\psi(\Delta) p(r, t, x - z) - \psi(\Delta) p(r, t, y - z)| \, dz \, dr.
$$

Recalling

$$
a_t := (\psi^{-1}(t^{-1}))^{1/2} = 1/\varphi(t),
$$

we have by (4.25) again

$$
\mathcal{I}_{2, 1, 1} \leq N \left( a_{t-s+(\psi(|x - y|^{-2}))^{-1}} (\varphi((t-s)) + |x - y|) \right)^{-\delta} \leq N
$$

and by (4.26)

$$
\mathcal{I}_{2, 1, 2} \leq |x - y| a_{t-s+(\psi(|x - y|^{-2}))^{-1}} \leq N.
$$
It only remains to estimate \( I_{2,2} \), which is an easy consequence of (4.27). Indeed,
\[
I_{2,2} \leq N(t-s)(t-s)^{-1} \leq N
\]
since \( 2s - t < s \). The corollary is proved.

5. Proof of Theorem 2.13

In this section, \( X \) is a stochastic process satisfying Assumptions 2.1 and 2.2. First we introduce the representation of solutions and related estimates.

**Lemma 5.1.** Let \( f \) be a smooth function on \((0, T) \times \mathbb{R}^d\) such that for any multi-index \( \alpha \) and \( \beta \),
\[
\sup_{t \in (0, T)} \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha f(t, x)| < \infty \quad (5.1)
\]
and suppose that Assumptions 2.1 and 2.2 hold. Define
\[
u(t, x) := \int_0^t \mathbb{E} [f(s, x + X_t - X_s)] \, ds.
\]
Then
\[
u_t(t, x) = \mathcal{A}(t)\nu(t, x) + f(t, x), \quad \nu(0, x) = 0. \quad (5.2)
\]
for almost every \((t, x) \in (0, T) \times \mathbb{R}^d\). Moreover,
\[
\|\nu\|_{L^p([0, T]; L^p)} \leq N\|f\|_{L^p([0, T]; L^p)} \quad (5.3)
\]
and
\[
\|\phi(\Delta)\nu\|_{L^p([0, T]; L^p)} \leq \bar{N}\|f\|_{L^p([0, T]; L^p)}, \quad (5.4)
\]
where \( N = N(d, p, \delta_k, N_j, T) \) and \( \bar{N} = \bar{N}(d, p, \delta_k, N_j) \) \((k = 1, 2, 3 \text{ and } j = 1, 2, 3, 4)\).

**Proof.** Observe that by Fubini’s theorem and Assumption 2.2.1,
\[
u(t, x) = \int_0^t \mathbb{E} \left[ \mathcal{F}^{-1} \left[ \mathcal{F} \left[ f(s, \cdot + X_t - X_s) \right] (\xi) \right] (x) \right] \, ds
\]
\[
= \int_0^t \mathcal{F}^{-1} \left[ \mathbb{E} \left[ e^{i(\xi (X_t - X_s))} \right] \mathcal{F} \left[ f(s, \cdot) \right] (\xi) \right] (x) \, ds
\]
\[
= \int_0^t \mathcal{F}^{-1} \left[ \exp(\Phi_X(s, t, \xi)) \mathcal{F} \left[ f(s, \cdot) \right] (\xi) \right] (x) \, ds
\]
\[
= \mathcal{F}^{-1} \left[ \int_0^t \exp(\Phi_X(s, t, \xi)) \mathcal{F} \left[ f(s, \cdot) \right] (\xi) \, ds \right] (x). \quad (5.5)
\]
Recalling $\Phi_X(t, t, \xi) = 0$, by Assumption 2.2(i) again we have

$$\mathcal{A}(t)u(t, x) = \lim_{h \downarrow 0} \frac{E[u(t, x + X_{t+h} - X_t) - u(t, x)]}{h}$$

$$= \lim_{h \downarrow 0} \frac{\mathcal{F}^{-1} \left[ \left( e^{i\xi (X_{t+h} - X_t)} - 1 \right) \mathcal{F}[u(t, \cdot)](\xi) \right]}{h}(x)$$

$$= \lim_{h \downarrow 0} \frac{\mathcal{F}^{-1} \left[ \exp(\Phi_X(t, t+h, \xi)) - \exp(\Phi_X(t, t, \xi)) \mathcal{F}[u(t, \cdot)](\xi) \right]}{h}(x)$$

$$= \mathcal{F}^{-1} \left[ \Psi_X(t, \xi) \mathcal{F}[u(t, \cdot)](\xi) \right](x)$$

and

$$\frac{\partial}{\partial t} \int_0^t \exp(\Phi_X(s, t, \xi)) \mathcal{F}[f(s, \cdot)](\xi) ds$$

$$= \mathcal{F}[f(t, \cdot)](\xi) + \int_0^t \frac{\partial}{\partial t} \exp(\Phi_X(s, t, \xi)) \mathcal{F}[f(s, \cdot)](\xi) ds$$

$$= \mathcal{F}[f(t, \cdot)](\xi) + \int_0^t \Psi_X(t, \xi) \exp(\Phi_X(s, t, \xi)) \mathcal{F}[f(s, \cdot)](\xi) ds. \quad (5.7)$$

Since the last term above is integrable with respect to $\xi$ uniformly $t \in (0, T)$ for any $T \in (0, \infty)$, we get (5.2) by taking the inverse Fourier transform to both sides of (5.7).

Next we show (5.3) and (5.4). Due to the definition of $u$ and Minkowski's inequality,

$$\|u\|_{L_p(0, T); L_p)} = \left\| \int_0^t E[f(s, x + X_t - X_s)] ds \right\|_{L_p((0, T); L_p)}$$

$$\leq N(T) \|f\|_{L_p((0, T); L_p)}.$$

Thus it suffices to show (5.3). We now prove this estimate in the following two steps.

**Step 1:** Assume $X = X^1$. Note that if one takes $\psi = \phi$ and $\Psi = \Psi_X$, then Assumption 2.2(ii) is exactly same as Assumptions 4.1. Therefore due to (5.3) and Theorem 4.2

$$\phi(\Delta)u(t, x) = \mathcal{F}^{-1} \left[ \phi(|\xi|^2) \int_0^t \exp(\Phi_X(s, t, \xi)) \mathcal{F}[f(s, \cdot)](\xi) ds \right](x)$$

and

$$\|\phi(\Delta)u\|_{L_p((0, T); L_p)} \leq N \|f\|_{L_p((0, T); L_p)}.$$

**Step 2 (General case):** Recall that two processes $X^1$ and $X^2$ are independent. Thus by Assumption 2.2 and Fubini’s theorem,
\[ \phi(\Delta)u(t, x) = \mathcal{F}^{-1} \left[ \phi(|\xi|^2) \int_0^t \mathbb{E} [\mathcal{F} [f(s, \cdot + X_t - X_s)](\xi)] \, ds \right] (x) \]
\[ = \mathbb{E}' \left[ \mathcal{F}^{-1} \left( \phi(|\xi|^2) \times \int_0^t \mathbb{E} [\mathcal{F} [f(s, \cdot + X_t^1(\omega) - X_s^1(\omega) + X_t^2(\omega') - X_s^2(\omega')]](\xi)] \, ds \right) (x) \right] \]
\[ = \mathbb{E}' \left[ \mathcal{F}^{-1} \left( \phi(|\xi|^2) \times \int_0^t \mathbb{E} [\mathcal{F} [f(s, \cdot + X_t^1(\omega) - X_s^1(\omega) - X_t^2(\omega') - X_s^2(\omega')]](\xi)] \, ds \right) (x + X_t^2(\omega')) \right], \]

where \( \mathbb{E} \) and \( \mathbb{E}' \) are the expectations with respect to the variables \( \omega \) and \( \omega' \), respectively. Since the paths of \( X^2 \) are locally bounded (a.s.), one can easily check that \( f_{X^2}(s, x) := f(s, x - X_s^2) \) satisfies (5.1) (a.s.). For each fixed \( \omega' \in \Omega \), denote
\[ u_{X^2(\omega')}(t, x) := \int_0^t \mathbb{E} [f_{X^2(\omega')}(s, x + X_t^1 - X_s^1)] \, ds. \]

Then by Minkowski’s inequality, the change of variable \( x \to x - X_t^2(\omega') \) and the result of Step 1,
\[ \|\phi(\Delta)u\|_{L_p([0,T];L_p)} \leq \mathbb{E}'\|\phi(\Delta)u_{X^2(\omega')}\|_{L_p([0,T];L_p)} \leq NE'\|f_{X^2(\omega')}\|_{L_p([0,T];L_p)} = N\|f\|_{L_p([0,T];L_p)}. \]

The lemma is proved. \( \square \)

**Lemma 5.2.** Let \( u \in C^\infty_c((0, T) \times \mathbb{R}^d) \) and suppose that Assumption 2.2 holds. Then
\[ u(t, x) = \int_0^t \mathbb{E} [f(s, x + X_t - X_s)] \, ds \quad \forall (t, x) \in (0, T) \times \mathbb{R}^d, \quad (5.8) \]
where
\[ f(t, x) = u_t(t, x) - \mathcal{A}(t)u(t, x). \]

**Proof.** Recalling (5.6) and taking the Fourier transform, we have
\[ \mathcal{F}[f(t, \cdot)](\xi) = \frac{\partial}{\partial t} \mathcal{F}[u(t, \cdot)](\xi) - \Psi_X(t, \xi) \mathcal{F}[u(t, \cdot)](\xi). \]

For each \( \xi \), solving the above ODE with respect to \( t \), we have
\[ \mathcal{F}[u(t, \cdot)](\xi) = \int_0^t \exp(\Phi_X(s, t, \xi)) \mathcal{F}[f(s, \cdot)](\xi) \, ds. \]

Thus following (5.3) in the reverse order, we obtain (5.8) since the both sides of (5.8) are continuous on \((0, T) \times \mathbb{R}^d\). The lemma is proved. \( \square \)

**Proof of Theorem 2.13**

**Step 1 (Existence)**
Choose a sequence \( f_n \in C_c((0, T) \times \mathbb{R}^d) \) so that
\[
\|f_n - f\|_{L_p([0,T];L_p)} \to 0
\]
as \( n \to \infty \). Define
\[
u_n(t, x) := \int_0^t \mathbb{E}[f_n(s, x + X_t - X_s)] \, ds.
\]
Then by (5.3) and (5.4),
\[
\|\nu_n - \nu_m\|_{L_p([0,T];H^p)} \leq N \|f_n - f_m\|_{L_p([0,T];L_p)}
\]
and
\[
\|\phi(\Delta)(\nu_n - \nu_m)\|_{L_p([0,T];L_p)} \leq \bar{N} \|f_n - f_m\|_{L_p([0,T];L_p)}.
\]
Since \( L_p([0,T];H^p) \) is a Banach space, \( \nu_n \) converges to \( u \in L_p([0,T];H^p) \) and \( u \) becomes a solution to equation (2.16) according to Definition 2.11 and obviously \( u \) satisfies (2.17) and (2.18).

**Step 2** (Uniqueness)

Let \( u \) and \( v \) be solutions to equation (2.16). Then by Definition 2.11 one can find sequences \( u_n \in C_c((0, T) \times \mathbb{R}^d) \) and \( v_n \in C_c((0, T) \times \mathbb{R}^d) \) so that
\[
\frac{\partial u_n}{\partial t} - A(t)u_n \to f \quad \text{in} \quad L_p([0,T];L_p),
\]
\[
u_n \to u \quad \text{in} \quad L_p([0,T];H^p)
\]
and
\[
\frac{\partial v_n}{\partial t} - A(t)v_n \to f \quad \text{in} \quad L_p([0,T];L_p),
\]
\[
v_n \to v \quad \text{in} \quad L_p([0,T];H^p)
\]
as \( n \to \infty \). Denote
\[
f_n = \frac{\partial u_n}{\partial t} - A(t)u_n
\]
and
\[
g_n = \frac{\partial v_n}{\partial t} - A(t)v_n
\]
Then by Lemma 5.2
\[
u_n(t, x) = \int_0^t \mathbb{E}[f_n(s, x + X_t - X_s)] \, ds
\]
and
\[
v_n(t, x) = \int_0^t \mathbb{E}[g_n(s, x + X_t - X_s)] \, ds.
\]
Since both \( f_n \) and \( g_n \) converge to \( f \) in \( L_p([0,T];L_p) \), we have \( u = v \). The theorem is proved.
Throughout this section, let \((O, \mathcal{F}, \mu)\) be a complete measure space such that \(\mu(O) = \infty\).

By \(\mathcal{F}_0\) we denote the subset of \(\mathcal{F}\) consisting of all sets \(A\) such that \(\mu(A) < \infty\). \(L(O, \mathcal{F}, \mu)\) indicates the space of all locally integrable functions \(f\) on \((O, \mathcal{F}, \mu)\), i.e.,

\[
f \in L(O, \mathcal{F}, \mu) \iff f1_A \in L_1(O, \mathcal{F}, \mu) \quad \forall A \in \mathcal{F}_0.
\]

If the given measure space is clear, we simply use notation \(L\). We borrow terminologies from [7, Chapter 3].

**Definition 6.1.** We say that a collection \(\mathcal{P} \subset \mathcal{F}_0\) is a partition if and only if elements of \(\mathcal{P}\) are countable, pairwise disjoint, and

\[
\bigcup_{P \in \mathcal{P}} P = O.
\]

**Remark 6.2.** Due to the definition of the partition, the measure space \((O, \mathcal{F}, \mu)\) is \(\sigma\)-finite if there is a partition \(\mathcal{P}\) on \((O, \mathcal{F}, \mu)\).

**Definition 6.3.** Let \((\mathcal{P}_n, n \in \mathbb{Z})\) be a sequence of partitions. We say that \((\mathcal{P}_n, n \in \mathbb{Z})\) is a filtration of partitions on \((O, \mathcal{F}, \mu)\) if and only if

(i) \(\inf_{P \in \mathcal{P}_n} \mu(P) \to \infty\), as \(n \to -\infty\)

and

\[
\lim_{n \to \infty} \frac{1}{\mu(\mathcal{P}_n(x))} \int_{\mathcal{P}_n(x)} |f(y)| \mu(dy) = f(x) \quad (a.e.) \quad \forall f \in L_1,
\]

where \(\mathcal{P}_n(x)\) denote the element of \(\mathcal{P}_n\) containing \(x\); (ii) For each \(n \in \mathbb{Z}\) and \(P \in \mathcal{P}_n\), there is a (unique) \(P' \in \mathcal{P}_{n-1}\) such that \(P \subset P'\) and

\[
\mu(P') \leq N_0 \mu(P),
\]

where \(N_0\) is a constant independent of \(n, P,\) and \(P'\).

We introduce a general Fefferman-Stein sharp function related to the filtration of partition \((\mathcal{P}_n, n \in \mathbb{Z})\). For a locally integrable function \(f\) on \((O, \mathcal{F}, \mu)\), we define its sharp function \(f^#\) as

\[
f^#(x) := \sup_{n \in \mathbb{Z}} \int_{\mathcal{P}_n(x)} |f(y) - f_{[n]}(x)| \mu(dy) := \sup_{n \in \mathbb{Z}} \frac{1}{\mu(\mathcal{P}_n(x))} \int_{\mathcal{P}_n(x)} |f(y) - f_{[n]}(x)| \mu(dy),
\]

where

\[
f_{[n]}(x) := \frac{1}{\mu(\mathcal{P}_n(x))} \int_{\mathcal{P}_n(x)} f(y) \mu(dy).
\]

At last, we introduce a version of Fefferman-Stein theorem on a measure space \((O, \mathcal{F}, \mu)\) with a filtration.

**Theorem 6.4.** For any \(f \in L_p(O, \mathcal{F}, \mu)\),

\[
\|f\|_{L_p(O, \mathcal{F}, \mu)} \leq N \|f^#\|_{L_p(O, \mathcal{F}, \mu)},
\]

where \(p \in (1, \infty)\), \(q = p/(p-1)\), and \(N = (2q)^p N_0^{p-1}\).
We put $\varphi$ to the function Theorem 6.5. Suppose that Assumption 3.2(i) holds. Then there exists a sequence

$$\{\varphi(2^{-n})\sigma_n, (i+1)\varphi(2^{-n})\sigma_n\} \times B_{2^{-n}}(i_1, \ldots, i_d)$$

and define $P_i \in P$, $P \subset P_i$ become a filtration of partitions on $(0, \infty) \times B$.

Proof. See [7, Lemma 3.2.4] and [7, Theorem 3.2.10].

For $n, i_1, \ldots, i_d \in \mathbb{Z}$, denote

$$B_{2^{-n}}(i_1, \ldots, i_d) = (i_12^{-n}, (i_1 + 1)2^{-n}] \times \cdots \times (i_d2^{-n}, (i_d + 1)2^{-n}].$$

Recall $U = \mathbb{R}^d$ or $U = \mathbb{R}_+^d$. Finally we construct a filtration on $(0, \infty) \times U$ related to the function $\varphi$ in Assumption 3.2(i).

**Theorem 6.5.** Suppose that Assumption 3.2(i) holds. Then there exists a sequence $(\sigma_n, n \in \mathbb{Z})$ such that $\sigma_n \in [1, 2]$,

$$P_n := \{(i\varphi(2^{-n})\sigma_n, (i+1)\varphi(2^{-n})\sigma_n\} \times B_{2^{-n}}(i_1, \ldots, i_d), i \in \mathbb{Z}, i_1, \ldots, i_d \in \mathbb{Z}\}$$

and

$$P_n^+ := \{(i\varphi(2^{-n})\sigma_n, (i+1)\varphi(2^{-n})\sigma_n\} \times B_{2^{-n}}(i_1, \ldots, i_d), i, i_1 \in \mathbb{Z}, i_2, \ldots, i_d \in \mathbb{Z}\}$$

become a filtration of partitions on $(0, \infty) \times \mathbb{R}^d$ and $(0, \infty) \times \mathbb{R}_+^d$ respectively.

**Proof.** Because of similarity, we only construct the filtration $P_n$. We construct this filtration in inductive ways. Recall that $\varphi(r)$ is a nonnegative nondecreasing function from $(0, \infty)$ into $(0, \infty)$ so that

$$\varphi(r) \downarrow 0 \text{ as } r \downarrow 0, \quad \varphi(r) \uparrow \infty \text{ as } r \uparrow \infty,$$

and

$$\sup_{r>0} \frac{\varphi(2r)}{\varphi(r)} < \infty. \quad (6.2)$$

First, we set

$$P_0 := \{(i\varphi(1), (i+1)\varphi(1)) \times (i_1, i_1 + 1] \times \cdots \times (i_d, i_d + 1], \quad i \in \mathbb{Z}, i_1, \ldots, i_d \in \mathbb{Z}\}$$

and construct $P_n$ for $n = 1, 2, \ldots$ inductively. Suppose that $P_k$ is given for some $k \in \mathbb{Z}$ and

$$P_k = \{(i\varphi(2^{-k})\sigma_k, (i+1)\varphi(2^{-k})\sigma_k\} \times B_{2^{-k}}(i_1, \ldots, i_d), \quad i \in \mathbb{Z}, i_1, \ldots, i_d \in \mathbb{Z}\},$$

where $\sigma_k \in [1, 2]$ and

$$B_{2^{-k}}(i_1, \ldots, i_d) = (i_12^{-k}, (i_1 + 1)2^{-k}] \times \cdots \times (i_d2^{-k}, (i_d + 1)2^{-k}].$$

If $k = 0$, then obviously $\sigma_k = 1$. Since $\varphi$ is nondecreasing and $\varphi > 0$, there exists a $\mathbb{Z}_+$ so that

$$\frac{\varphi(2^{-k})\sigma_k}{\varphi(2^{-k+1})} \in [2^{d_k+1}, 2^{d_{k+1}+1}).$$

We put

$$\sigma_{k+1} = \frac{\varphi(2^{-k})\sigma_k}{\varphi(2^{-k+1})2^{d_{k+1}}}$$

and define $P_{k+1}$ as the collection of sets

$$\{(i\varphi(2^{-k+1})\sigma_{k+1}, (i+1)\varphi(2^{-k+1})\sigma_{k+1}\} \times B_{2^{-k+1}}(i_1, \ldots, i_d), \quad \forall i \in \mathbb{Z}, i_1, \ldots, i_d \in \mathbb{Z}. \quad (6.4)$$

Then obviously $\sigma_{k+1} \in [1, 2)$ and for any $P \in P_{k+1}$ there exists a unique $P' \in P_k$ so that

$$P \subset P'$$
and
\[
\frac{|P|}{|P|} = \frac{\varphi(2^{-k})\sigma_k}{\varphi(2^{-(k+1)})\sigma_{k+1}} 2^d = 2^{d+2^k} \leq 2^{d+1} \sup_{r>0} \frac{\varphi(r)}{\varphi(r)} < \infty.
\]
In order to confirm (6.4), observe that if \( \ell_{k+1} = 0 \) then for any \( i \in \mathbb{Z} \)
\[
(i \varphi(2^{-k})\sigma_k, (i + 1) \varphi(2^{-k})\sigma_k) = (i \varphi(2^{-(k+1)})\sigma_{k+1}, (i + 1) \varphi(2^{-(k+1)})\sigma_{k+1}],
\]
and on the other hand if \( \ell_{k+1} > 0 \) then
\[
(i \varphi(2^{-k})\sigma_k, (i + 1) \varphi(2^{-k})\sigma_k)
= \bigcup_{l=0}^{2^{\ell_{k+1}-1}} (i_l \varphi(2^{-(k+1)})\sigma_{k+1}, (i_l + 1) \varphi(2^{-(k+1)})\sigma_{k+1}],
\]

where \( i_l = i 2^{\ell_{k+1}} + l \).

Next we construct \( \mathcal{P}_n \) for \( n = -1, -2, \ldots \). Similarly, suppose that \( \mathcal{P}_k \) is given for some \( k \in \{0, -1, -2, \ldots \} \) and
\[
\mathcal{P}_k = \{ (i \varphi(2^{-k})\sigma_k, (i + 1) \varphi(2^{-k})\sigma_k) \times \mathcal{B}_{2^{-k}}(i_1, \ldots, i_d), \quad i \in \mathbb{Z}_+, i_1, \ldots, i_d \in \mathbb{Z} \},
\]
where \( \sigma_k \in [1, 2] \) and
\[
\mathcal{B}_{2^{-k}}(i_1, \ldots, i_d) = (i_1 2^{-k}, (i_1 + 1)2^{-k}] \times \cdots \times (i_d 2^{-k}, (i_d + 1)2^{-k}].
\]
Since \( \varphi \) is nondecreasing, \( \varphi > 0 \), and \( \sigma_k \in [1, 2] \), there exists a \( \ell_{k-1} \in \mathbb{N} \cup \{0\} \) so that
\[
\varphi(2^{-k})\sigma_k \left( 2^{-\ell_{k-1}-1} \right) \in [2^{-\ell_{k-1}-1}, 2^{-\ell_{k-1}+1}).
\]
We put
\[
\sigma_{k-1} = \frac{2^{\ell_{k-1}} \varphi(2^{-k})\sigma_k}{\varphi(2^{-(k-1)})}
\]
and define \( \mathcal{P}_{k-1} \) as the collection of sets
\[
(i \varphi(2^{-(k-1)})\sigma_{k-1}, (i + 1) \varphi(2^{-(k-1)})\sigma_{k-1}) \times \mathcal{B}_{2^{-(k-1)}}(i_1, \ldots, i_d),
\]
for all \( i \in \mathbb{Z}_+, i_1, \ldots, i_d \in \mathbb{Z} \). Then obviously
\[
\sigma_{k-1} \in [1, 2)
\]
and for any \( \mathcal{P} \in \mathcal{P}_k \) there exists a unique \( \mathcal{P}' \in \mathcal{P}_{k-1} \) so that
\[
\mathcal{P} \subset \mathcal{P}'
\]
and
\[
\frac{|\mathcal{P}'|}{|\mathcal{P}|} = \frac{\varphi(2^{-(k-1)})\sigma_{k-1} 2^d}{\varphi(2^{-k})\sigma_k} 2^{d+\ell_{k-1}} = 2^{d+1} \sup_{r>0} \frac{\varphi(r)}{\varphi(r)} < \infty.
\]

(6.5) is due to the followings: For any \( i \in \mathbb{Z}_+, i_1, \ldots, i_d \in \mathbb{Z} \), if \( \ell_{k-1} = 0 \) then
\[
(i \varphi(2^{-(k-1)})\sigma_{k-1}, (i + 1) \varphi(2^{-(k-1)})\sigma_{k-1}) = (i \varphi(2^{-k})\sigma_k, (i + 1) \varphi(2^{-k})\sigma_k],
\]
and on the other hand, unless \( \ell_{k-1} = 0 \) then
\[
(i \varphi(2^{-(k-1)})\sigma_{k-1}, (i + 1) \varphi(2^{-(k-1)})\sigma_{k-1})
= \bigcup_{l=0}^{2^{\ell_{k-1}-1}} (i_l \varphi(2^{-k})\sigma_k, (i_l + 1) \varphi(2^{-k})\sigma_k],
\]
where \( i_l = i 2^{\ell_{k-1}} + l \).
Theorem is proved. □

**Proof of Theorem 3.3**

This is an easy consequence of Theorem 6.4 with the filtration
\[ \mathcal{P}_k = \left\{ (i \varphi(2^{-k}) \sigma_k, (i + 1) \varphi(2^{-k}) \sigma_k) \times B_{2^{-k}}(i_1, \ldots, i_d) : i \in \mathbb{Z}_+, i_1, \ldots, i_d \in \mathbb{Z} \right\} \]
\[ =: \{ Q_{\varphi,k}(i, i_1, \ldots, i_d) : i \in \mathbb{Z}_+, i_1, \ldots, i_d \in \mathbb{Z} \}. \]

We only remark that for any \( Q_{\varphi,k}(i, i_1, \ldots, i_d) \in \mathcal{P} \), one can find a \( Q_{\varphi,c}(t_0, x_0) \in \mathcal{Q}_{\varphi} \) so that
\[ Q_{\varphi,k}(i, i_1, \ldots, i_d) \subset Q_{\varphi,c}(t_0, x_0) \]
and
\[ |Q_{\varphi,c}(t_0, x_0)| \leq N(d, \varphi)|Q_{\varphi,k}(i, i_1, \ldots, i_d)|. \]

The theorem is proved. □

7. ACKNOWLEDGEMENT

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**References**

[1] R.F. Bass, *Harnack inequalities for non-local operators of variable order*, Trans. Amer. Math. Soc. 357(2):837–850, 2005

[2] R.F. Bass, *Hölder continuity of harmonic functions with respect to operators of variable order*, Comm. Partial Differential Equations, 30:1249–1259, 2005

[3] H. Dong and D. Kim, *Schauder estimates for a class of non-local elliptic equations*, Discrete Contin. Dyn. Syst, 33(6):2319–2347, 2013

[4] H. Dong and D. Kim. On \( L^p \)-estimates for a class of non-local elliptic equations. *Journal of Functional Analysis*, 262(3):1166–1199, 2012.

[5] W. Farkas, N. Jacob, and R. L. Schilling. *Function spaces related to continuous negative definite functions: \( \psi \)-Bessel potential spaces*. Polska Akademia Nauk, Instytut Matematyczny, 2001.

[6] I. Kim, K.-H. Kim, and P. Kim. Parabolic Littlewood-Paley inequality for \( \phi (- \Delta) \)-type operators and applications to stochastic integro-differential equations. *Advances in Mathematics*, 249:161–203, 2013.

[7] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, volume 96. American Mathematical Society Providence, RI, 2008.

[8] P. Kim, R. Song, and Z. Vondráček, *Global uniform boundary Harnack principle with explicit decay rate and its application*, Stochastic processes and their applications, 124(1):235–267, 2014.

[9] R. Mikulevicius and C. Phonsom. On \( L^p \)-theory for parabolic and elliptic integro-differential equations with scalable operators in the whole space. *Stochastics and Partial Differential Equations: Analysis and Computation*, 1-48, 2016.

[10] K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge university press, 1999.

[11] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions: theory and applications*, volume 37. Walter de Gruyter, 2012.

[12] L. Silvestre, *Hölder estimates for solutions of integro-differential equations like the fractional Laplace*, Indiana Univ. Math. J, 55(3):1155–1174, 2006

[13] E. M. Stein and T. S. Murphy. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 3. Princeton University Press, 1993.

[14] X. Zhang. *Lp-maximal regularity of nonlocal parabolic equations and applications*. In *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 30:573–614, 2013.
