Analytic matrix technique for boundary value problems in applied plasticity

L. Novozhilova
Department of Mathematics,
Western Connecticut State University, Danbury, CT, USA
S. Urazhdin
Department of Physics and Astronomy,
West Virginia University, Morgantown, WV, USA

Abstract

An efficient matrix formalism for finding power series solutions to boundary value problems typical for technological plasticity is developed. Hyperbolic system of two first order quasilinear PDEs that models two-dimensional plastic flow of von Mises material is converted to the telegraph equation by the hodograph transformation. Solutions to the boundary value problems are found in terms of hypergeometric functions. Convergence issue is also addressed. The method is illustrated by two test problems of metal forming.

1 Introduction

A model of incompressible, ideal, rigid-plastic material was developed about a hundred years ago. In the two-dimensional case the model is described by a hyperbolic system of quasilinear equations. In 1920s Prandl, Hencky, and Mises, among others, suggested a systematic way, named slip-line method (SLM), for finding stress fields and associated velocity fields for two-dimensional plain strain deformation problems for rigid-plastic body. The method is presented in many classic books on plasticity theory (cf. [1], [2], [4]) along with its applications to metal forming processes and problems of plastic failure.
(limit analysis). Solution of a 2D rigid-plastic flow problem by this method is reduced to construction of the field of characteristics (slip-lines). Assumed domain of the plastic flow is to be decomposed into a set of subdomains (patches) of a priori specified types, and appropriate boundary value problems (BVPs), consistent with physics of the process, are stated for each of the subdomains. The subproblems were usually solved numerically. In 1980s a matrix implementation of the SLM, called matrix method, was developed within engineering community [5]. The matrix method is based on the assumption that boundary data are defined by real analytic functions. The solution to each subproblem is obtained by applying appropriate matrix operators to the vector(s) of coefficients of the data. In [6] a new, simpler approach to implementation of the matrix method was developed, mathematically justified, and equipped with recurrent definitions of five matrix operators needed for solving main boundary value problems typical in technological applications (versus twenty operators in the original engineering version of the method). It was also shown that solutions to the subproblems can be written in terms of hypergeometric functions.

In this work explicit expressions for the five matrix operators of the slip-line analytic technique (SLAT) developed in [6] are presented. Convergence issues are addressed and model examples are provided.

SLAT can be used as a source of test problems for numerical methods aimed at problems with discontinuities. The fact that solutions to problems in plasticity may have singularities along characteristics is well known and experimentally confirmed. Presence of discontinuities is an important feature that cannot be easily detected by numerical methods. In general, capturing singularities is very challenging mathematical problem that is of great interest in many applications, including phase transitions and microstructure formation. SLAT also provides examples of analytically solved problems with free boundaries since only a structure of the plastic domain is "guessed" a priori and finding the boundary of the domain is part of the solution. Furthermore, although analytic solutions have been superseded with powerful numerical methods, in particular FEM, the slip-line analysis is a working tool in metal forming [3], granular flow modelling [6], and geomechanics [10]. The velocity field in the hodograph plane can also be found by this method [6]. Exact solutions for stress and velocity fields can be used to derive analytic expressions for the plastic power, which provides a framework for finding optimal in a certain sense geometric or material parameters of the process under consideration.
SLM is also a classic topic in standard courses on plasticity theory, and so is the method of characteristics in PDE theory. Therefore the authors believe that an elegant matrix formulation of the method presented in this paper has a good potential in engineering and mathematical education due to simplicity of the material model, lucidity of the basic ideas, and classical character of mathematical machinery.

The paper is organized as follows.

In Section 2 the governing equations for von Mises ideal rigid-plastic material under conditions of plain strain deformation are introduced and a classic reduction to quasilinear hyperbolic system of PDEs is given. Using hodograph transformation, the system is transformed into the telegraph equation, which means that the original system is C-integrable. This fact was known long ago [2], but here the transformation to the linear problem is done by appropriate change of variables and does not involve any considerations from mechanics. In Section 3 three main BVPs typical for applied plasticity are defined. Using Riemann function for the telegraph equation, the initial characteristic problem with analytic data can be solved in terms of Bessel functions [8]. We show that this solution can be expressed in terms of the hypergeometric functions depending on the product of independent variables, which is more practical for computations. Exact solutions for Cauchy and mixed problems are found by reducing the problems to equivalent initial characteristic problem. In Section 4 the method is applied to two test problems from the theory of plastic flow:

- Calculation of the slip-line field generated by two circular arcs, and
- Computation of the stress state near an elliptic hole loaded with constant normal pressure.

Solutions obtained for the test problems are in excellent agreement with those known from literature.

To simplify exposition, the following notations and conventions will be used hereafter.

- \( t_n = t^n / n! \), where \( t \) is a variable.
- \([t] = [1, t_1, t_2, t_3, \ldots]^T\) (vector-column).
- Any given function is real analytic (i.e., it can be represented as the sum of a power series).
Summation is taken over the range from 0 to $\infty$ unless specified otherwise.

Notation

$$\sum_{k}^{\infty} \frac{z^k}{k!(k+n)!} = F_1(n+1; z)$$

stands for the hypergeometric functions. The subindexes will be omitted for simplicity.

Given a power series $\sum a_k t_k$, notation $\mathbf{a}$ stands for the row of its coefficients.

## 2 Governing equations

Let a homogeneous isotropic body made of ideal rigid-plastic material be in a state of plane strain plastic deformation. Assume that the flow is parallel to $xy$-plane. Let $D$ denote the projection of the body onto this plane. It is assumed that the stress tensor components $\sigma_{ij}, \ i, j = 1, 2,$ and the strain rate tensor components $\epsilon_{ij}$ in the domain $D$ satisfy the following equations:

1. Equilibrium equations

$$\sigma_{ij, i} = 0, \ j = 1, 2. \quad (1)$$

Here the Einstein rule of summation over repeated indices is assumed.

2. von Mises yield criterion

$$(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 = 4k^2, \quad (2)$$

where $k$ is the shear yield stress of the material.

3. Constitutive relations

$$\frac{\sigma_{11} - \sigma_{22}}{\sigma_{12}} = \frac{\epsilon_{11} - \epsilon_{22}}{\epsilon_{12}}, \ j = 1, 2. \quad (3)$$

4. Incompressibility constraint

$$\epsilon_{11} + \epsilon_{22} = 0. \quad (4)$$
Two perpendicular lines, called slip-lines, are passing through any interior point of the plastic domain. Each of these lines is tangent to a maximum shear stress direction at this point. Lines in the first and the second shear directions \[7\] are called \(\alpha\)- and \(\beta\)-lines, respectively. Under the plane strain conditions, the mean stress \(\sigma\) is given by \(\sigma = (\sigma_{11} + \sigma_{22})/2\). The stress components at a point \((x, y)\) are determined by the equations

\[
\begin{align*}
\sigma_{11} &= \sigma - k \sin 2\phi, \\
\sigma_{22} &= \sigma + k \sin 2\phi, \\
\sigma_{12} &= k \cos 2\phi,
\end{align*}
\]

where \(\phi = \phi(x, y)\) is the angle from the \(x\)-axis to the \(\alpha\)-line passing through the point.

Substituting these equations into \((1)\) yields

\[
\begin{align*}
\sigma_x - 2k (\phi_x \cos 2\phi + \phi_y \sin 2\phi) &= 0, \\
\sigma_y - 2k (\phi_x \sin 2\phi - \phi_y \cos 2\phi) &= 0,
\end{align*}
\]

where lower indices indicate corresponding partial derivative. Quasilinear system \((3)\) is hyperbolic, and its characteristics coincide with slip-lines. It is shown below that the system can be converted into a linear PDE by a hodograph transform reversing the roles of the dependent, \((\sigma, \phi)\), and independent, \((x, y)\), variables.

Assuming that the Jacobian

\[
J = \sigma_x \phi_y - \sigma_y \phi_x
\]

does not vanish, one derives

\[
\begin{align*}
\phi_y &= J x_\sigma, \\
\phi_x &= -J y_\sigma, \\
\sigma_x &= J y_\phi, \\
\sigma_y &= -J x_\phi,
\end{align*}
\]

and the system \((7)\) transforms into

\[
\begin{align*}
x_\phi &= 2k (x_\sigma \cos 2\phi + y_\sigma \sin 2\phi), \\
y_\phi &= 2k (x_\sigma \sin 2\phi - y_\sigma \cos 2\phi).
\end{align*}
\]

Introduce point-dependent rectangular coordinate system with axes directed along characteristics,

\[
X = x \cos \phi + y \sin \phi, \quad Y = -x \sin \phi + y \cos \phi,
\]

and the characteristic coordinates \((\alpha, \beta)\),

\[
\begin{align*}
\alpha &= \phi/2 + (\sigma - \sigma_0)/(4k), \\
\beta &= \phi/2 - (\sigma - \sigma_0)/(4k),
\end{align*}
\]
where $\sigma_0$ is the value of the mean stress at the origin. Then, after algebraic simplifications, the governing system of equations takes the form

$$Y_\alpha + X = 0, \quad X_\beta - Y = 0. \quad (10)$$

Geometric meaning of the characteristic coordinates is clear from Fig.1. If the functions $X, Y$ are smooth enough, each of them satisfies the telegraphy equation

$$\frac{\partial f}{\partial \alpha \partial \beta} + f = 0. \quad (11)$$

The same equation holds for the radii $R, S$ of $\alpha$- and $\beta$-line curvatures, respectively, and for the components of the velocity field $\mathbf{v}$. The stress components at a point $(x(\alpha, \beta), y(\alpha, \beta))$ are defined by (5), with

$$\phi = \alpha + \beta, \quad \sigma = \sigma_0 + 2k(\alpha - \beta). \quad (12)$$

The main stresses are given by the equations $\sigma_1 = \sigma + k, \ \sigma_2 = \sigma - k$. 

Figure 1: Curvilinear characteristics rectangle with initial $\alpha$-line $OA$ and initial $\beta$-line $OB$. 
3 Main boundary value problems

3.1 The initial characteristic problem

It is well known that there exists a unique solution to the following initial characteristics problem [7]:

Find a function \( f(\alpha, \beta) \) satisfying (11) in the domain OACB (Fig.1) and the initial conditions

\[
f(\alpha, 0) = \sum c_n \alpha^n, \quad \alpha \in (0, \alpha_1), \quad f(0, \beta) = \sum d_n \beta^n, \quad \beta \in (0, \beta_1).
\]

The coefficients \( c_n, d_n \) are given real numbers and the compatibility condition \( c_0 = d_0 \) holds. A classic solution in terms of Bessel functions [8] can be rewritten in terms of hypergeometric functions as

\[
f(\alpha, \beta) = \sum (c_n \alpha^n + d_n \beta^n) \, F(n + 1; -\alpha \beta) - c_0 \, F(1; -\alpha \beta).
\] (13)

For a plasticity problem with curvatures of the initial characteristics given by the equations

\[
R(\alpha, 0) = \sum a_n \alpha^n, \quad S(0, \beta) = \sum b_n \beta^n, \quad (14)
\]

missing data \( R(0, \beta), \, S(\alpha, 0) \) are obtained from equations (10) (with \( X, Y \) replaced with \( R, S \))

\[
R(0, \beta) = \sum b_n \beta_{n+1} + a_0, \quad S(\alpha, 0) = \sum a_n \alpha_{n+1} + b_0. \quad (15)
\]

Then it follows from (13) that the slip-line field in the characteristic rectangle OACB is defined by the curvatures

\[
R(\alpha, \beta) = \sum (a_n \alpha_n + b_{n-1} \beta_n) \, F(n + 1; -\alpha \beta), \quad (16)
\]

\[
S(\alpha, \beta) = \sum (-a_n \alpha_n + b_n \beta_n) \, F(n + 1; -\alpha \beta), \quad (17)
\]

where \( a_{-1} = 0 = b_{-1} = 0 \).

The following theorem summarizes these results.

**Theorem 1** Let two arcs of intersecting slip-lines be given by the equations (14). Then the slip-line field in the characteristic rectangle generated by the two slip-lines is uniquely defined by series (16), (17).
Furthermore, if the coefficients $a_k, b_k$ are bounded, the series converge as exponential series. If for some $q > 0$

$$|a_n| \leq q^n n!, \quad |b_n| \leq q^n n!,$$

then for any $p$, $0 < p < 1/q$, the series converge as geometric series with ratio $r = pq$ provided $|\alpha| \leq p$, $|\beta| \leq p$.

The estimates of the rate of convergence follow from the inequality

$$|F(n + 1; z)| \leq \exp(|z|).$$

**Remark.** A particular slip-line field generated by two circular arcs is often used in technological plasticity. In this case, $R(\alpha, 0) = a_0$, $S(0, \beta) = b_0$ and solution takes the form

$$R = a_0 F(1; -\alpha^2 \beta) + b_0 F(2; -\alpha^2 \beta), \quad S = b_0 F(1; -\alpha^2) - a_0 \alpha F(2; -\alpha^2 \beta).$$

Solution (16), (17) to the initial characteristic problem can be written as double power series

$$R(\alpha, \beta) = \sum_{n,k} a_n \alpha^{n+1} (-\beta)^k + b_n \beta^{n+1} (-\alpha)^k, \quad (18)$$

$$S(\alpha, \beta) = \sum_{n,k} -a_n \alpha^{n+1} (-\beta)^k + b_n \beta^{n+1} (-\alpha)^k, \quad (19)$$

or in a matrix form as

$$R(\alpha, \beta) = (aA(\beta) + bB(\beta)) [\alpha], \quad S(\alpha, \beta) = (-aB(\alpha) + bA(\alpha)) [\beta]. \quad (20)$$

Here $a, b$ are the vectors of coefficients in the initial conditions (14) and the matrix-functions $A, B$ are defined by the formulas

$$A(t) = \begin{bmatrix} 1 & -t & t_2 & -t_3 & \ldots \\ 0 & 1 & -t & t_2 & \ldots \\ 0 & 0 & 1 & -t & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$B(t) = \begin{bmatrix} t_1 & -t_2 & t_3 & -t_4 & \ldots \\ t_2 & -t_3 & t_4 & -t_5 & \ldots \\ t_3 & -t_4 & t_5 & -t_6 & \ldots \\ t_4 & -t_5 & t_6 & -t_7 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$
If an initial slip-line shrinks to a point, the characteristic rectangle degenerates into a fan with slip-line field defined by (20) with appropriate row of coefficients set to zero.

3.2 The Cauchy problem

Consider a smooth non-characteristic curve $OC$. Let the radius of curvature of $OC$ be $r(\gamma)$, $\gamma$ being the angle between the curve and $x$–axis. Given two functions $\sigma(\gamma)$, $\phi(\gamma)$, the problem is to find functions $\sigma(x,y)$, $\phi(x,y)$ satisfying equations (6), (7) in the characteristic triangle $OAC$ (Fig. 2) and boundary conditions $\sigma = \sigma(\gamma)$, $\phi = \phi(\gamma)$ on $OC$. There exists a unique solution of this problem in the curvilinear triangle $OAC$ [7].

This BVP can be reduced to the equivalent initial characteristic problem on the fictitious characteristic rectangle $OACB$. Coefficients $a_k$, $b_k$ for the initial slip-lines are found as follows. From an infinitesimal characteristic triangle with hypotenuse lying on $OC$ (Fig. 2) one has the following elementary identities

$$ds_\alpha/ \cos(\eta) = ds_\beta/ \sin(\eta) = r(\gamma)d\gamma,$$

(21)

where $ds_\alpha$, $ds_\beta$ are the length differentials of the bounding $\alpha$– and $\beta$-lines, and $\eta = \gamma - \varphi$. Using (4), functions $\alpha = \alpha(\gamma)$ and $\beta = \beta(\gamma)$ can be determined.
at all points of $OC$. Substituting the expressions

$$ds_\alpha = R(\alpha(\gamma), \beta(\gamma))d\alpha, \quad ds_\beta = -S(\alpha(\gamma), \beta(\gamma))d\beta$$  \hspace{1cm} (22)

with $R, S$ defined by (16), (17) into (21), one obtains a system for the unknown coefficients $\alpha_k, \beta_k$. A particular case when both the normal stress $\sigma_n$ and the tangential stress $\tau_n$ are constant along $OC$ is detailed below. In this case

$$\alpha = \beta, \quad \eta = \pi/2 - 0.5 \cos^{-1}(\tau_n/k) = \text{const}, \quad \gamma = \eta + 2\alpha.$$  \hspace{1cm} (23)

Writing the radius of curvature of $OC$ as a function of $\alpha$,

$$r(2\alpha + \eta) = r[\alpha],$$

one obtains from (21)

$$R(\alpha, \alpha)/\cos \eta = -S(\alpha, \alpha)/\sin \eta = \pm 2r[\alpha].$$  \hspace{1cm} (24)

These identities (with $R, S$ determined by (16), (17)) imply after some mathematical manipulations

$$a = 2rC, \quad b = 2rD,$$  \hspace{1cm} (25)

where the upper triangular matrices $C, D$, derived from the recurrent relations in [6], read

$$C = \begin{bmatrix} c & s & c & s & c & s & \cdots \\ 0 & c & s & 2c & 2s & 3c & \cdots \\ 0 & 0 & 2c & 1 \cdot 2s & 2 \cdot 3c & 2 \cdot 3s & \cdots \\ 0 & 0 & 0 & 3!c & 1 \cdot 2 \cdot 3 \cdot s & 2 \cdot 3 \cdot 4 \cdot c & \cdots \\ 0 & 0 & 0 & 0 & 4!c & 1 \cdot 2 \cdot 3 \cdot 4 \cdot s & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

$$D = \begin{bmatrix} -s & c & -s & c & -s & c & \cdots \\ 0 & -s & c & -2s & 2c & -3s & \cdots \\ 0 & 0 & -2!s & 1 \cdot 2c & -2 \cdot 3s & 2 \cdot 3c & \cdots \\ 0 & 0 & 0 & -3!s & 1 \cdot 2 \cdot 3 \cdot c & -2 \cdot 3 \cdot 4 \cdot s & \cdots \\ 0 & 0 & 0 & 0 & -4!s & 1 \cdot 2 \cdot 3 \cdot 4 \cdot c & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$
For simplicity, notation $c$, $s$ have been used for $\cos \eta$ and $\sin \eta$, respectively. Formulas (25) determine initial data for the initial characteristic problem equivalent to the original Cauchy problem in the domain $OAC$.

The solution for Cauchy problem is tested below on a classic example with known slip-line field formed by logarithmic spirals.

**Example.** Consider a circular arc of radius one loaded with constant normal pressure and zero tangent stress component. This implies $\eta = \pi/4$ and $r = (1, 0, 0, ...)$. It follows from (25) that

$$a = \sqrt{2}(1, 1, 1, ...) \quad b = \sqrt{2}(1, -1, 1, -1, ...).$$

Therefore the radii of curvature for the initial $\alpha$- and $\beta$-lines are $R(\alpha, 0) = \sqrt{2}\exp(\alpha)$ and $S(0, \beta) = \sqrt{2}\exp(-\beta))$. These are the radii of curvature of logarithmic spirals.

Stress-free surface boundary ($\sigma_n = \tau_n = 0$ on $OC$) can be analyzed similarly. However, in this case the shape of the free boundary is not known a priori, and to start a process of solution based on the identities (24) one needs to know one of the initial slip-lines. Assuming for certainty that the initial slip-line $OA$ (Fig. 2) is known (for example, vector $a$ could be found from solution of BVP on the adjacent patch), one finds the row $b$ of coefficients for the initial $\beta$-line and free surface radius of curvature $r(\alpha)$ from the identities (24). It can be shown that

$$b = aF, \quad r(2\alpha + \eta) = R(\alpha, \alpha)/\sqrt{2}, \quad (26)$$

where the entries of the matrix $F$ are defined as

$$f_{ii} = -1, \quad f_{ij} = 0, \text{ for } j < i, \quad f_{ij}/2 = (-1)^{i+j+1} \text{ for } j > i, \quad (27)$$

and $R(\alpha, \beta)$ is given by (16). Finding the shape of a free surface may be of interest in applications.

### 3.3 Mixed problem

Consider a smooth non-characteristic curve $OC$. Let the radius of curvature of $OC$ be $r(\gamma)$, $\gamma$ being the angle between the curve and $x$–axis. Given the radius of curvature for the initial $\alpha$-line $OA$ and a function $\phi(\gamma)$ on $OC$, the problem is to find functions $\sigma(x, y), \phi(x, y)$ satisfying equations (6), (7) in the characteristic triangle $OAC$ and boundary condition $\phi = \phi(\gamma)$ on $OC$ (Fig.3). It is known that the problem has a unique solution [7].
Figure 3: Characteristic triangle for the mixed problem. One of the bounding slip-lines and the values of the function $\phi$ on the non-characteristic curve $OC$ are given.

Such formulation arises in applied plasticity if a tangential stress $\tau_n$ is given on a contact line $OC$. It is assumed here that $\tau_n = \mu k$, $\mu \in [0,1]$, (Prandtl friction law). Note that constancy of $\tau_n$ implies that the angle $\eta$ also remains constant on $OC$. Assuming that the curve $OC$ is defined by an equation $\beta = \beta(\alpha) = c[\alpha]$, one obtains elementary identities

$$R(\alpha, \beta(\alpha))/\cos \eta = -S(\alpha, \beta(\alpha))\beta/\sin \eta = r(\alpha + \beta(\alpha) + \eta)(1 + \beta').$$  \hspace{1cm} (28)

The unknown vector $c$ and vector $b$ for the initial $\beta$-line of the equivalent initial characteristic problem can be obtained from these identities. A particular case is analyzed below.

Let $OC$ be a straight line. If $x$-axis is directed along the $\alpha$-line passing through the origin, then $\phi = \alpha + \beta = 0$ on $OC$, or equivalently, $\beta = -\alpha$. The rightmost term in (28) takes the form $\infty \cdot 0$ and becomes useless. The first identity takes the form

$$R(\alpha, -\alpha)/\cos \eta = -S(\alpha, -\alpha)/\sin \eta \hspace{1cm} (29)$$

Using (18), one obtains after some mathematical manipulations the missing boundary data for the initial $\beta$-line of the equivalent initial characteristic problem. Specifically, if $\eta < \pi/2$, the row of the coefficients for the $\beta$-line radius of curvature is given by the equation $b = aT(\eta)$, where the elements
of the upper triangular matrix $T$ are defined by the equations

$$
t_{ij} = (-1)^i (\tan \eta)^{j-i-1} (\tan^2 \eta - 1) \text{ for } j > i,
$$

$$
t_{ii} = (-1)^i \tan \eta \text{ for } i = j
$$

$$
t_{ij} = 0 \text{ for } j < i.
$$

It can be shown that for perfectly rough boundary ($\eta = \pi/2$) $b_n = (-1)^n a_n$.

4 Test problems

In this section SLAT is applied to two test problems. The first example is classical and its exact solution is well known [9]. The second problem was treated before only numerically. Since the main mathematical operation used in SLAT is multiplication of matrices and vectors of small size, computer time needed for each problem is negligible. However, some preliminary analytic work is needed to present initial data in the required form.

4.1 Extrusion through a short wedge-shaped die

Consider an extrusion of a plastic material through a short wedge-shaped die of angle $\gamma$ (Fig. 4). For comparability of this example with solution given
in [3], the die is assumed to be frictionless (i.e., $\eta = \pi/4$), and the following values of parameters are used: $\gamma = 10^\circ$, $\alpha_1 = 30^\circ$, $\beta_1 = -20^\circ$, $OA = 2$ (dimensionless units). Thus the initial data are $\mathbf{a} = (\sqrt{2}, 0, 0, \ldots)$ and $\mathbf{b} = (-\sqrt{2}, 0, 0, \ldots)$. The initial characteristic problem with the data was solved in the characteristic rectangle (Fig. 4) using (14). Then the coordinates of the point $E$ relative to the origin $O$ were found using simple geometric considerations. The hydrostatic stress components $p_B = -\sigma(B)$, $p_D = -\sigma(D)$, and the extrusion pressure $P/H$ were obtained using the solution for the initial characteristic problem and Hencky’s first theorem in a standard static manner [5]. The value of $H$ in this example is found to be 2.28774.

|       | $x_E$ | $y_E$ | $P/H$ |
|-------|-------|-------|-------|
| [5]   | 0.9065 | -2.2877 | 0.4117 |
| SLAT  | 0.90648 | -2.28774 | 0.41164 |

For accuracy of $10^{-5}$, five-dimensional truncation of matrices and vectors was found to be sufficient. The results for some parameters obtained by the original matrix method [5] and SLAT are practically identical (Table 1). Hydrostatic and extrusion pressures are normalized by the shear yield stress of the material.

4.2 Stress state calculation near loaded elliptic hole

Consider planar plastic flow near elliptic hole loaded with constant normal pressure (Fig. 5). Firstly, Cauchy problem in the domain 1 was solved. Secondly, initial characteristic problem was solved in the domain 2 (note that the domain 2 is the upper half of a characteristic rectangle). Equation for the radius of curvature of the elliptic contour is obtained as follows. Let lengths be normalized by the semimajor axis of the ellipse. Then using parametric equations $x = \cos t$, $y = b \sin t$, $0 < b \leq 1$, of the ellipse and the standard formula for the curvature, one obtains the radius of curvature $\rho(t)$ in the form

$$\rho(t) = 2 \sqrt{2} ((1 + b^2) + (1 - b^2) \cos 2t)^{3/2}/b.$$  (30)
From elementary geometry it follows that
\[ \gamma = 2\alpha + \pi/4 = \tan^{-1}(b \cot t) - \pi/4. \]  
(31)

From this equation the variable \( t \) can be expressed in terms of \( \alpha \), and after some mathematical manipulations one obtains the radius of the curvature for the elliptic hole as a function of \( \alpha \):
\[ r(\alpha) = \rho(t(\alpha)) = 2\sqrt{2}b^2(1 + d^2)^{-3/2}(1 + q \cos 4\alpha)^{-3/2}, \]
(32)

where \( q = (1 - b^2)/(1 + b^2) \). For comparability with [11], value \( b = 0.4 \) have been chosen. For the domain 1, the row of the coefficients for the initial \( \alpha \)-line of the equivalent initial characteristic problem was found using the first equation in (25). For accuracy of \( 10^{-4} \), fifteen-dimensional truncation of vectors was found to be sufficient. In general, numerical experimentations show that the number of terms needed for a given accuracy grows significantly when the value of \( b \) decreases. Coefficients for the initial \( \beta \)-line, which is symmetric to the \( \alpha \)-line with respect to the horizontal axis, are given by the equation \( b_k = (-1)^{k+1}a_k \). The main stresses on the \( \xi \)-axis normalized by \( 2k \) are given by the equations \( \sigma_1 = \sigma + (2\alpha + 1) \), \( \sigma_2 = \sigma + 2\alpha \). The comparative results are shown in Fig. 6, where \( \Delta = \sigma_1 + p \).
Mathematically accurate and efficient analytic implementation of the slip-line method aimed at computing the stress fields for plane strain deformation of the rigid-plastic medium has been presented. It can be used as a source of reliable test problems for numerical methods and in engineering applications.

The velocity field in the hodograph plane can also be found by this method. Using solutions in the physical and hodograph planes, exact expressions for the plastic power can also be derived. This gives an alternative way for computing the technological pressure, an important parameter for engineering applications, and also provides a framework for solving relevant optimization problems. Detailed explicit formulas for computation of energetic characteristics of slip-line fields will be given elsewhere.

Although only Prandtl friction law was analyzed in the mixed problem presented in this work, Coulomb friction requires just technical modifications and can be treated in a similar way. This may be of interest in the theory of granular flow, where using this kind of friction law is customary. This modification of SLAT will be presented in a later paper.
Acknowledgements
This research was partially supported by the CSU/AAUP grant # 242414.

References

[1] 1. R. Hill, The Mathematical Theory of Plasticity, Oxford University Press, London, 1950.

[2] L.M. Kachanov, Foundations of the Theory of Plasticity, North-Holland, Amsterdam, 1971.

[3] A. Drescher, Analytical Methods in Bin-Load Analysis, Elsevier, Amsterdam, 1991.

[4] J. Lubliner, Plasticity Theory, Macmillan Publishing Company, New York, 1990.

[5] W. Johnson, R. Sowerby, and R. Venter, Plane Strain Slip-Line Theory and Bibliography, Pergamon Press, Oxford, 1982.

[6] L.S. Novozhilova and S.V. Urazhdin, Analytic calculation of the energy characteristics of slip-line fields in plane plastic deformation problems, J. Appl. Maths Mech., (Elsevier Science Ltd), 1997, 61, No. 2, 311–318.

[7] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 2, Wiley-Interscience, 1962.

[8] H. Geiringer, Fondements mathematiques de la theorie des corps plastiques isotropes, Gauthier - Villars, Paris, 1937.

[9] B.A. Druyanov, R.I. Nepershin, Problems of Technological Plasticity, Elsevier Science Ltd, London, 1994.

[10] R. O. Davis, A. P. S. Selvadurai, Plasticity and Geomechanics, Cambridge University Press, New York, 2002.

[11] V.V. Sokolovskii, The Theory of Plasticity, High School, Moscow, 1969 (in Russian).