Space of Infinitesimal Isometries and Bending of Shells

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Abstract  We discuss infinitesimal isometries of the middle surfaces and present some characteristic conditions for a function to be the normal component of an infinitesimal isometry. Our results show that those characteristic conditions depend on the Gaussian curvature of the middle surfaces: Normal components of infinitesimal isometries satisfy an elliptic problem, or a parabolic one, or a hyperbolic one according to the middle surface being elliptic, or parabolic, or hyperbolic, respectively. In those cases, a problem of determining an infinitesimal isometry is changed into that of 1-dimension. Then we apply those results to the energy functionals of bending of shells which has been obtained as two-dimensional problems by the limit theory of $\Gamma$-convergence from the three-dimensional nonlinear elasticity. Therefore the limit theory of $\Gamma$-convergence reduces to be a one-dimensional problem in the those cases.

Keywords  material nonlinearity, strain energy function, Riemannian geometry

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1 Introduction

Let $M \subset \mathbb{R}^3$ be a smooth surface and let $\Omega \subset M$ be a bounded, open set. A map $V : \Omega \to \mathbb{R}^3$ is said to be an infinitesimal isometry on $\Omega$ if

$$\langle \hat{D}_X V, X \rangle = 0 \quad \text{for} \quad X \in T_x \Omega, \ x \in \Omega,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric of $\mathbb{R}^3$ and $\hat{D}$ denotes the covariant differential of the Euclidean space $\mathbb{R}^3$. We denote by $\text{IS}^1(\Omega, \mathbb{R}^3)$ all $H^1$ infinitesimal isometries on $\Omega$.

The study of infinitesimal isometries has been a long history, see [1, 14, 47, 48, 52] and many others. Their purposes were to establish "infinitesimal rigidity" for some closed surfaces and their interests were not on the structure of infinitesimal isometries themselves. For a detail survey along this direction, we refer to [52].

Our interests in the space $\text{IS}(\Omega, \mathbb{R}^3)$ of infinitesimal isometries are motivated by the recent lower dimensional models for thin structures (such as membrane and shells) through $\Gamma$-convergence. This approach has lead to the derivation of a hierarchy of limiting theories and provides a rigorous justification of convergence of three-dimensional minimizers to minimizers of suitable lower dimensional limit energies.

Given a 2-dimensional surface $\Omega$, consider a shell $S^h$ of middle surface $\Omega$ and thickness $h$, and associate to its deformation $u$ the scaled per unit thickness three dimensional nonlinear elastic energy $E(u, S^h)$. The $\Gamma$-limit $I^\beta$ of the energies

$$h^{-\beta} E(u, S^h)$$

are identified as $h \to 0$ for a given scaling $\beta \geq 0$.

When $\Omega$ is a subset of $\mathbb{R}^2$ (i.e., a plate), such $\Gamma$-convergence was first established by [32] for $\beta = 0$, then by [22, 23] for $\beta \geq 2$ (also see [43] for the results for $\beta = 2$ under additional conditions). In case of $0 < \beta < 5/3$ the convergence was obtained by [18], see also [15]. Other significant results for plates concern the derivation of limit theories for incompressible materials [16, 17, 54], for heterogeneous materials [50], and through establishing convergence of equilibria, rather than strict minimizers [38, 39].

When $\Omega$ is an any surface, the first result by [33] relates to scaling $\beta = 0$ and models membrane shells. The limit energy $I^0$ depends only on the stretching and shearing
produced by the deformation of the middle surface $\Omega$. Then the limit energy $I^2$ in the case of $\beta = 2$ was given by [21]. This scaling corresponds to a flexural shell model, where the only admissible deformations are those preserving the metric of $\Omega$. Then the energy $I^2$ depends on the change of curvature by the deformation. The limit energies $I^\beta$ are obtained by [35] for scaling $\beta \geq 4$. Based on some quantitative rigidity estimate due to [22], [35] demonstrates that the first term in the expansion $u - R$, in terms of $h$, belongs to the space $\text{IS}(\Omega, \mathbb{R}^3)$ of infinitesimal isometries. That means that there is no first order change in the induced metric of the middle surface $\Omega$. The corresponding limit energy $I^\beta$ consists of the bending energy which is given by the first order change of the second fundamental form of $\Omega$ for $\beta > 4$. In the case of $\beta = 4$, [35] also shows that, if the middle surface $\Omega$ is approximately robust, the $\Gamma$-limit is still a bending term. Moreover, in the scaling regime of $2 < \beta < 4$ the limit $I^\beta$ is given by [34] which reduces to be the pure bending energy again.

As shown by [34, 35], the limit energy functionals $I^\beta$ of the $\Gamma$-convergence for all $\beta > 2$ are over the space $\text{IS}(\Omega, \mathbb{R}^3)$ of infinitesimal isometries. Then the space $\text{IS}(\Omega, \mathbb{R}^3)$ naturally plays a crucial role in the analysis of shells. The aim of the present paper is to understand the space $\text{IS}(\Omega, \mathbb{R}^3)$.

We now give heuristic overview of our results, whose precise formulations will be presented in the sections later. Let $N$ be the unit normal field of surface $M$ and let $\mathcal{X}(\Omega)$ be all vector fields on $\Omega$. For $V \in H^1(\Omega, \mathbb{R}^3)$, we decompose as

$$V = W + wN \quad \text{for} \quad W \in \mathcal{X}(\Omega), \quad w \in H^1(\Omega).$$

We look for conditions on functions $w$ such that there are vector fields $W \in \mathcal{X}(\Omega)$ to guarantee $V \in \text{IS}^1(\Omega, \mathbb{R}^3)$.

First, Section 2 is devoted to treating the structure of $V = W \in \text{IS}^1(\Omega, \mathbb{R}^3)$ corresponding to the zero normal component $w = 0$. Such an infinitesimal isometry is said to be a Killing field. Through there are rich results on Killing fields ([46]), we focus on the relations between a Killing field and the Gaussian curvature function. In particular, we show that the dimension of the Killing field space is 3 if $\Omega$ is of constant curvature and is not larger than 1 in the case of non-constant curvature (Corollaries 2.1, 2.2, and Theorem 2.3). Furthermore the explicit formulas of Killing fields are given in terms of the Gaussian curvature function (Theorem 2.2).

Let $H^1_{\text{is}}(\Omega)$ denote all functions $w \in H^1(\Omega)$ such that there are $W \in \mathcal{X}(\Omega)$, which are perpendicular to all Killing fields in $H^1(\Omega, \mathbb{R}^3)$, to ensure that $V = W + wN \in \text{IS}^1(\Omega, \mathbb{R}^3)$. Section 3 shows that $w \in H^1_{\text{is}}(\Omega)$ if and only if $w$ satisfies an equation (3.18) (Theorem 3.1).

The type of the equation (3.18) is subject to the Gaussian curvature function: It is elliptic, or parabolic, or hyperbolic according to ellipticity, or parabolicity, or hyperbolicity.
of the middle surface $\Omega$, respectively. The three cases are studied, respectively, in Sections 4, 5, and 6. Our results show that the problem to determine whether $w \in H^1_{\text{is}}(\Omega)$ is actually that of 1-dimension in the above three types, respectively.

As a consequence of those theories, we present a condition for the middle surface $\Omega$ which can guarantee that $H^1_{\text{is}}(\Omega) \cap C^\infty(\Omega)$ is dense in $H^1_{\text{is}}(\Omega)$ in the norm of $H^1(\Omega)$ (Theorems 4.3 and 6.3): There is a point $o \in \Omega$ such that $\Omega$ is star-shaped with respect to $o$ and

$$\Omega \subset \exp_o \Sigma(o),$$

where $\exp_o \Sigma(o)$ is the interior of the cut locus of $o$. Such an issue is actually not trivial. In general, even though $\Omega$ is elliptic, an element $V \in IS^1(\Omega, \mathbb{R}^3)$ may not be approximated by smooth infinitesimal isometries. An interesting example, discovered by [14] (also see [52]), is a closed smooth surface of non-negative curvature for which the infinitesimal rigidity holds true: All $C^\infty$ infinitesimal isometries are trivial. But there is a $C^2$ non-trivial infinitesimal isometry. Therefore $H^1_{\text{is}}(\Omega) \cap C^\infty(\Omega)$ is not dense in $H^1_{\text{is}}(\Omega)$ for this surface.

In Section 7 we apply the above theories to the limit energy $I^\beta$ of $\Gamma$-convergence for the scaling $\beta > 2$. Then the limit energy functional is changed into a one-dimensional formula over a function space with one variable (Theorem 7.1). In particular, we present the explicit formulas of the limit energy functionals for a spherical shell (Theorem 7.2) and a cylinder shell (Theorem 7.3), respectively, under the nonlinear isotropic materials.

Here we do not use the traditional methods, adopted in the classical linear thin shell theories. Their starting point is to assume that the middle surface is given by a coordinate path: $\Omega$ is the image in $\mathbb{R}^3$ of a smooth map defined on a connected domain of $\mathbb{R}^2$, rooted from classical differential geometry. The classical models use the traditional geometry and end up with highly complicated resultant equations. In these, the explicit presence of the Christoffel symbols, makes some necessary computations too complicated. We view the middle surface $\Omega$ as a 2-dimensional Riemannian manifold with the induced metric to make everything coordinates free as far as possible. When necessary, some special coordinates are chosen to simplify computations as in modelling and control for the classical thin shells, see [4, 5, 6, 7, 8, 9, 20, 31, 37, 57, 58, 59, 60] and many others.

2 Killing Fields in Dimension 2

We shall present explicit formulas of Killing fields in terms of the Gaussian curvature function (Proposition 2.1 and Theorem 2.2).

Let $M \subset \mathbb{R}^3$ be a smooth surface with the induced metric $g$ from the Euclidean metric of $\mathbb{R}^3$. Let $\Omega \subset M$ be an open set with smooth boundary $\Gamma$. Denote by $\mathcal{X}(\Omega)$ all vector fields on $\Omega$. Let $W \in \mathcal{X}(\Omega)$ be given. Let $\alpha(t)$ be the 1-parameter group generated by $W$,
\[ \dot{\alpha}(t,x) = W(\alpha(t,x)), \quad \alpha(0,x) = x \quad \text{for} \quad x \in \Omega. \quad (2.1) \]

\( W \) is said to be a Killing field on \( \Omega \) if and only if \( \alpha(t) \) are local isometries. It is easy to check that \( W \) is a Killing field on \( \Omega \) if and only if \( V = W \) is a \( C^\infty \) infinitesimal isometry on \( \Omega \), that is,

\[ DW(X,Y) + DW(Y,X) = 0 \quad \text{for} \quad X,Y \in M_x, \quad \text{for all} \quad x \in \Omega, \quad (2.2) \]

where \( D \) is the Levi-Civita connection of the induced metric \( g \) and \( DW \) is the covariant differential of vector field \( W \).

Let \( KF(\Omega,T) = \{ \text{all } C^\infty \text{ Killing fields on } \Omega \} \). Then ([46])

\[ \dim KF(M,T) \leq 3. \]

For the purpose of application to bending of shells here, we only need to consider \( H^1 \) Killing fields. To this end, we introduce some common notions in Riemannian geometry. Let \( T^k(\Omega) \) be all \( k \)-th order tensor fields on \( \Omega \) where \( k \) is a nonnegative integer. In particular, \( T^0(\Omega) \) is all functions on \( \Omega \) and \( T(\Omega) = \mathcal{X}(\Omega) \). For each \( x \in M \), the \( k \)-th order tensor space \( T^k_x \) on \( M_x \) is an inner product space defined as follows. Let \( e_1, e_2 \) be an orthonormal basis of \( M_x \). For any \( \alpha, \beta \in T^k_x, x \in M \), the inner product is given by

\[ \langle \alpha, \beta \rangle_{T^k_x} = \sum_{i_1=1, \ldots, i_k=1}^2 \alpha(e_{i_1}, \ldots, e_{i_k}) \beta(e_{i_1}, \ldots, e_{i_k}) \text{ at } x. \quad (2.4) \]

Note that the right hand side of (2.4) is free of choice of orthonormal bases. In particular, for \( k = 1 \) the definition (2.4) becomes

\[ g(\alpha, \beta) = \langle \alpha, \beta \rangle_{T^1_x} = \langle \alpha, \beta \rangle, \quad \forall \alpha, \beta \in M_x, \]

that is, the induced inner product of \( M_x \) of \( M \) from \( \mathbb{R}^3 \). Let \( L^2(\Omega,T^k) \) be the Soblev spaces of all \( k \)-th order tensor fields on \( \Omega \) with inner products

\[ (T_1, T_2)_{L^2(\Omega,T^k)} = \sum_{i=0}^k \int_{\Omega} \langle T_1, T_2 \rangle_{T^k_x} dg \quad \text{for} \quad T_1, T_2 \in L^2(\Omega,T^k). \]

Let

\[ H^1(\Omega,T) = \{ W \mid W \in \mathcal{X}(\Omega), \ W \in L^2(\Omega,T), \ DW \in L^2(\Omega,T^2) \} \]

with norm

\[ \| W \|_{H^1(\Omega,T)} = \left( \| w \|^2_{L^2(\Omega,T)} + \| DW \|^2_{L^2(\Omega,T^2)} \right)^{1/2}. \]
Denote by $H_{Kf}^1(\Omega, T)$ all $W \in H^1(\Omega, T)$ with the relations (2.2) being true for almost everywhere on $\Omega$ and the norm of $H^1(\Omega, T)$. Then

$$K_{f}(M, T) \subset K_{f}(\Omega, T) \subset H_{Kf}^1(\Omega, T).$$

(2.5)

We have

**Theorem 2.1** Let $\Omega \subset M$ be an open set. Then

$$H_{Kf}^1(\Omega, T) = K_{f}(\Omega, T)$$

and

$$\dim K_{f}(M, T) \leq \dim H_{Kf}^1(\Omega, T) \leq 3.$$  

(2.6)

**Proof** Let $W \in H_{Kf}^1(\Omega, T)$. By Lemma 4.4 in [58], we have

$$\Delta W = 2\kappa W \quad \text{for} \quad x \in \Omega,$$

(2.7)

where $\Delta$ is the Hodge-Laplace operator in the metric $g$ and $\kappa$ is the Gaussian curvature function of $M$. Then the ellipticity of the operator $\Delta$ implies that $W$ is $C^\infty$ on $\Omega$, that is, $W \in K_{f}(\Omega, T)$. Then the left hand side of the inequality (2.6) follows from (2.5). Moreover, the right hand side of the inequality (2.6) is given by the equations (2.14)-(2.16) later. $\square$

**Remark 2.1** In general, $\dim K_{f}(M, T) \neq \dim H_{Kf}^1(\Omega, T)$, see Corollary 2.2 and Example 2.1 later.

Let $o \in M$ be fixed and let $\exp_o : M_o \to M$ be the exponential map in the metric $g$. For any $v \in M_o$ with $|v| = 1$, then there is a unique $t_0(v) > 0$ (or $t_0(v) = \infty$) such that the normal geodesic $\gamma(t) = \exp_o tv$ is the shortest on the interval $[0, t_0]$. Let

$$C(o) = \{ t_0(v)v \mid v \in M_o, \ |v| = 1 \} , \quad \Sigma(o) = \{ tv \mid v \in M_o, \ |v| = 1, \ 0 \leq t < t_0(v) \}.$$  

The set $\exp_o C(o) \subset M$ is said to be the cut locus of $o$ and the set $\exp_o \Sigma(o) \subset M$ is called the interior of the cut locus of $o$. Then

$$M = \exp_o \Sigma(o) \cap \exp_o C(o).$$

Furthermore, $\exp_o : \Sigma(o) \to \exp_o \Sigma(o)$ is a diffeomorphism and $C(o)$ is a zero measure set on $M_o$. Then $\exp_o C(o)$ is a zero measure set on $M$ since it is the image of the zero measure set $C(o)$, that is, $\exp_o \Sigma(o)$ is $M$ minus a zero measure set.

We introduce the polar coordinate system at $o \in M$ as follows. Let $e_1, e_2$ be an orthonormal basis of $M_o$. Set

$$\sigma(\theta) = \cos \theta e_1 + \sin \theta e_2 \quad \text{for} \quad \theta \in [0, 2\pi).$$

(2.8)
Consider a family of two parameter curves on $M$ given by

$$F(t, \theta) = \exp_o t\sigma(\theta)$$

for $t\sigma(\theta) \in \Sigma(o)$. Then

$$\partial t = \frac{\partial}{\partial t} F(t, \theta) = \exp_o * \sigma(\theta), \quad \partial \theta = \frac{\partial}{\partial \theta} F(t, \theta) = t \exp_o \dot{\sigma}(\theta).$$

(2.9)

In particular,

$$g = dt^2 + f^2(t, \theta) d\theta^2$$

for $x = \exp_o t\sigma(\theta) \in \exp_o \Sigma(o)$,

where $f(t, \theta)$ is the solution to the problem

$$\begin{cases}
  f_{tt}(t, \theta) + \kappa(t, \theta)f(t, \theta) = 0, \\
  f(0, \theta) = 0, \quad f_t(0, \theta) = 1,
\end{cases}$$

(2.10)

where $\kappa$ is the Gaussian curvature function on $M$ and $\kappa(t, \theta) = \kappa(F(t, \theta))$.

Let

$$T = \partial t, \quad E = \frac{1}{f} \partial \theta$$

for $x \in \exp_o \Sigma(o) - \{o\}$. Then $T, E$ is a frame field on $\exp_o \Sigma(o) - \{o\}$. Let $D$ denote the Livi-Civita connection of the induced metric $g$ on $M$. We have

$$D_T T = 0, \quad D_T E = 0, \quad D_E T = \frac{f_t}{f} E, \quad D_E E = -\frac{f_t}{f} T$$

(2.12)

for $x \in \exp_o \Sigma(o) - \{o\}$.

Let

$$W = \varphi(t, \theta) T + \phi(t, \theta) E,$$

where $\varphi = \langle W, T \rangle$ and $\phi = \langle W, E \rangle$. Then

$$D_T W = \varphi_t T + \phi_t E, \quad D_E W = \frac{1}{f}(\varphi_\theta - f_t \phi) T + \frac{1}{f}(\phi_\theta + f_t \varphi) E.$$

(2.13)

Then the relation (2.2) is equivalent to

$$\begin{cases}
  \varphi_t(t, \theta) = 0, \\
  f \varphi_t - f_t \phi + \varphi_\theta = 0, \\
  \phi_\theta + f_t \varphi = 0.
\end{cases}$$

(2.14)

The first equation in (2.14) yields

$$\varphi(t, \theta) = \langle W, \sigma(\theta) \rangle (o).$$

(2.15)

Next, we calculate the first order derivative of the second equation in (2.14) with respect to $t$ and use the equations (2.15) and (2.10) to have

$$\begin{cases}
  \phi_{tt} + \kappa \phi = 0, \\
  \phi(0) = \langle W, \dot{\sigma}(\theta) \rangle, \quad \phi_t(0) = DW(e_2, e_1).
\end{cases}$$

(2.16)

It follows from (2.15) and (2.16) that
Proposition 2.1 Let \((M,g)\) be of constant curvature \(\kappa\). Let \(\Omega \subset M\) be an open set and let \(o \in \Omega\). For \(W \in \text{KF}(\Omega,T)\), we have

\[
W = \langle W(o), \sigma(\theta) \rangle T + (af + ft \langle W(o), \dot{\sigma}(\theta) \rangle) E
\]

(2.17)

for \(x\) in a neighborhood of \(o\), where \(f\) is given by (2.10) and \(a = DW(e_2,e_1)\).

Let \(M\) be a sphere with curvature \(\kappa > 0\). Let \(W \in \text{KF}(M,T)\). Then \(f = \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa}t\) for \(t \in (0, \frac{\pi}{\sqrt{\kappa}})\) and the formula (2.17) holds true for all \(x \in \exp_o \Sigma(o)\). It is easy to check that \(W\), given by (2.17), is \(C^\infty\) at \(\exp_o C(o)\), which is the antipodal point of \(o\). Then

\[
\dim \text{KF}(M,T) = 3.
\]

It follows Theorem 2.1 that

**Corollary 2.1** Let \(M \subset \mathbb{R}^3\) be a closed sphere of constant curvature \(\kappa > 0\) and let \(\Omega \subset M\) be an open set. Then

\[
\dim H^1_{\text{kf}}(\Omega,T) = 3.
\]

Let \(o \in \Omega\) be given. \(\Omega\) is said to be star-shaped with respect to \(o\) if for any \(x \in \Omega\) there is a shortest geodesic contained in \(\Omega\) connecting \(x\) and \(o\). Since for any \(W(o) \in M_o\) and a number \(a = DW(e_2,e_1)\) given, the problems (2.15) and (2.16) have solutions for all \(x \in \exp_o \Sigma(o)\), it follows that

**Corollary 2.2** Let \(M \subset \mathbb{R}^3\) be a surface with zero curvature. Let \(\Omega \subset M\) be star-shaped with respect to \(o \in \Omega\) and

\[
\Omega \subset \exp_o \Sigma(o).
\]

(2.18)

Then

\[
\dim H^1_{\text{kf}}(\Omega,T) = 3.
\]

(2.19)

**Remark 2.2** The condition (2.18) is necessary for the equation (2.19). This is because a vector field \(W\), which is given by (2.17), can not guarantee \(W\) is \(C^\infty\) on \(\Omega \cap \exp_o C(o)\), see Example 2.1 below.

**Example 2.1** Consider a cylinder

\[
M = \{ (x,z) \mid x = (x_1,x_2) \in \mathbb{R}^2, \ |x| = 1, \ z \in \mathbb{R} \}.
\]

Let \(b > 0\) and let

\[
\Omega = \{ (x,z) \mid |x| = 1, \ |z| < b \}.
\]
Let \( o = (1,0,0) \). Then
\[
\Omega \cap \exp_o C(o) = \{ (-1,0,z) \mid |z| < b \}, \quad f(t,\theta) = t,
\]
and the vector field
\[
W = tE
\]
is not well defined on \( \Omega \cap \exp_o C(o) \). In this case, it is easy to check that
\[
\dim H^1_{\text{Kf}}(\Omega,T) = 2.
\]
In particular,
\[
\dim \text{KF}(M,T) = 2.
\]

**Lemma 2.1** Let \( \kappa \) be the Gaussian curvature function of \( M \) and let \( W \) be a Killing field on \( \Omega \). Then
\[
\langle \nabla \kappa, W \rangle = 0 \quad \text{for} \quad x \in \Omega,
\]
\[
D^2 \kappa(\nabla \kappa, W) = 0 \quad \text{for} \quad x \in \Omega.
\]

**Proof** Let \( o \in \Omega \) be given. We have
\[
\kappa_{\theta t} = (f \langle \nabla \kappa, E \rangle)_t = f_t \langle \nabla \kappa, E \rangle + f D^2 \kappa(E, T),
\]
which gives
\[
\kappa_{\theta t}(0) = \langle \nabla \kappa, \dot{\sigma}(\theta) \rangle.
\]
Let \( \varphi \) and \( \phi \) be given by (2.15) and (2.16), respectively. Using the equations (2.10) and (2.16) and the third equation in (2.14), we obtain
\[
0 = \phi^{(3)}_\theta(0) + f^{(4)}(0) \varphi(0) = -\kappa_{\theta t}(0) \phi(0) - \kappa_t(0) \phi_{\theta t}(0) - \kappa(0) \phi_{\theta t}(0) - 2 \kappa_t(0) \varphi(0)
\]
\[
= - \langle \nabla \kappa, W \rangle(0),
\]
that is, the formula (2.20) is true at \( o \), where the following formula is used
\[
\phi_{\theta t}(0) = \langle W, \dot{\sigma}(\theta) \rangle = - \langle W, \sigma(\theta) \rangle,
\]
\[
\phi_{\theta t}(0) = [DW(\dot{\sigma}(\theta), \sigma(\theta))]_\theta = -DW(\sigma(\theta), \sigma(\theta)) + DW(\dot{\sigma}(\theta), \dot{\sigma}(\theta)) = 0.
\]
Since \( o \in \Omega \) can be any point, the formula (2.20) follows.

Finally using (2.20), we have
\[
0 = \nabla \kappa \langle \nabla \kappa, W \rangle = D^2 \kappa(\nabla \kappa, W) + \langle \nabla \kappa, D_{\nabla \kappa} W \rangle = D^2 \kappa(\nabla \kappa, W).
\]

Let \( (M,g) \) be orientable. Let \( X \) be a vector field on \( \Omega \). We define a vector field \( QX \) on \( \Omega \) by
\[
QX = \langle X, e_2 \rangle e_1 - \langle X, e_1 \rangle e_2 \quad \text{for} \quad x \in \Omega,
\]
\[\text{(2.22)}\]
where $e_1, e_2$ is an orthonormal basis of $M_x$ with an positive orientation. It is easy to check that the vector field $QX$ is well defined.

We have

**Theorem 2.2** Let $(M, g)$ be orientated and let $\kappa$ be the Gaussian curvature function. Let $\Omega \subset M$ be a connected open set and let

$$|\nabla \kappa| > 0 \text{ for } x \in \Omega.$$

Then

$$\dim H_1^{\text{kf}}(\Omega, T) = 1 \quad (2.23)$$

holds true if and only if the following formulas are true

$$D^2 \kappa(\nabla \kappa, Q \nabla \kappa) = 0 \quad \text{for } x \in \Omega, \quad (2.24)$$

$$\langle Q \nabla \kappa, \nabla \Delta \kappa \rangle = 0 \quad \text{for } x \in \Omega, \quad (2.25)$$

where $\Delta$ is the Laplacian of the metric $g$. Moreover, $W \in H_1^{\text{kf}}(\Omega, T)$ has a formula

$$W = ce^{h_0} Q \nabla \kappa, \quad (2.26)$$

where $c$ is a constant and $h_0$ is a solution to the problem

$$\nabla h = \frac{|\nabla \kappa|^2 \Delta \kappa - 2D^2 \kappa(\nabla \kappa, \nabla \kappa)}{|\nabla \kappa|^4} \nabla \kappa. \quad (2.27)$$

**Proof** By Lemma 2.1, we look for Killing fields in the form

$$W = e^h Q \nabla \kappa, \quad (2.28)$$

where $h$ is a function on $\Omega$.

Let $o \in \Omega$ be given. Let $e_1, e_2$ be an orthonormal basis of $M_o$ with an positive orientation. Let $\sigma(\theta)$ be given by (2.8). Then

$$T = T(t, \theta), \quad E = E(t, \theta)$$

forms an orthonormal basis of $M_{\mathcal{F}(t, \theta)}$ with the positive orientation for $\mathcal{F}(t, \theta) \in \Omega$. For convenience, we denote

$$E_1 = T, \quad E_2 = E,$$

and

$$p_i = \langle \nabla p, E_i \rangle, \quad p_{ij} = D^2 p(E_i, E_j), \quad p_{ijk} = D^3 p(E_i, E_j, E_k),$$

for $i, j, k = 1, 2$, where $p$ is a function on $\Omega$. Then

$$W = \kappa_2 E_1 - \kappa_{11} E_2.$$
Using the relations
\[ D_{E_1}E_1 = D_{E_1}E_2 = 0, \quad D_{E_2}E_1 = \frac{f_t}{f} E_2, \quad D_{E_2}E_2 = -\frac{f_t}{f} E_1, \]
we obtain
\[
\begin{align*}
\left\{ \begin{array}{l}
e^{-h}D_{E_1}W = (h_1\kappa_2 + \kappa_{12})E_1 - (h_1\kappa_1 + \kappa_{11})E_2, \\
ne^{-h}D_{E_2}W = (h_2\kappa_2 + \kappa_{22})E_1 - (h_2\kappa_1 + \kappa_{12})E_2.
\end{array} \right.
\end{align*}
\]
It follows from the above formulas that \( W \) is a Killing field if and only if
\[
\begin{align*}
\left\{ \begin{array}{l}
 h_1\kappa_2 + \kappa_{12} = 0, \\
 h_2\kappa_1 + \kappa_{12} = 0, \\
 -h_1\kappa_1 + h_2\kappa_2 + \kappa_{22} - \kappa_{11} = 0.
\end{array} \right. \tag{2.29}
\end{align*}
\]
Then the formula (2.28) defines a Killing field if and only if there is a solution \( h \) to the problem (2.29).

Now we solve the problem
\[
\left\{ \begin{array}{l}
 h_1\kappa_2 + h_2\kappa_1 + 2\kappa_{12} = 0, \\
 -h_1\kappa_1 + h_2\kappa_2 + \kappa_{22} - \kappa_{11} = 0,
\end{array} \right. \tag{2.30}
\]
to have
\[
\left( \begin{array}{l}
h_1 \\
h_2
\end{array} \right) = \frac{1}{|\nabla \kappa|^2} \left( \begin{array}{c}
2\kappa_2\kappa_{12} + \kappa_1(\kappa_{11} - \kappa_{22}) \\
2\kappa_1\kappa_{12} - \kappa_2(\kappa_{11} - \kappa_{22})
\end{array} \right). \tag{2.31}
\]
It is easy to check that a solution of (2.29) is a solution of (2.31) if and only if the formula (2.24) holds. Then the formula (2.28) defines a Killing field if and only if the formula (2.24) holds true and \( h \) satisfies the problem (2.31).

Next, let us show that there is a solution to the problem (2.31) if and only if the formula (2.25) holds true. To this end, we let
\[
X = h_1E_1 + h_2E_2,
\]
where \((h_1, h_2)^T\) is given by the formula (2.31). We review the vector field \( X \) as a 1-form. It follows from the Poincare lemma that the problem (2.31) has a solution if and only if
\[
dX = 0 \quad \text{for} \quad x \in \Omega,
\]
since \( \Omega \) is star-shaped with respect to 0, where \( d \) denotes the exterior differentiation.

We assume that the formula (2.24) holds true to compute \( dX \). By [56], we have
\[
dX = E_1 \wedge D_{E_1}X + E_2 \wedge D_{E_2}X = [E_1(h_2) - E_2(h_1) + h_2f_t/f]E_1 \wedge E_2.
\]
Using (2.29) and (2.31), we obtain
\[
E_2(h_1)\kappa_2^2 = \kappa_{12}\kappa_{22} - \kappa_2\kappa_{12} + [h_2|\nabla \kappa|^2 + \kappa_1\kappa_{12}]f_t/f, \tag{2.32}
\]

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\begin{equation} \label{eq:2.33}
E_2(h_1)(\kappa_2^2 - \kappa_1^2) = 2h_1\kappa_1\kappa_{12} + \kappa_{12}(\kappa_{11} + \kappa_{22}) + \kappa_1(\kappa_{112} - \kappa_{222}) + \kappa_2(\kappa_{11} - \kappa_{22}) f_1/f.
\end{equation}

It follows from (2.32), (2.33), and (2.31) that
\begin{equation} \label{eq:2.34}
E_2(h_1)|\nabla \kappa|^2 = -2h_1\kappa_1\kappa_{12} - \kappa_{12}(\kappa_{11} - \kappa_{22}) - \kappa_1(\kappa_{112} - \kappa_{222}) + h_2|\nabla \kappa|^2 f_1/f.
\end{equation}

Similarly, we have
\begin{equation} \label{eq:2.35}
E_1(h_2)|\nabla \kappa|^2 = -2h_2\kappa_2\kappa_{12} + \kappa_{12}(\kappa_{11} - \kappa_{22}) - 2(\kappa_{11} - \kappa_{22}) f_1/f.
\end{equation}

From (2.34), (2.35), and (2.30), we obtain
\begin{equation} \label{eq:2.36}
[E_1(h_2) - E_2(h_1) + h_2 f_1/f]|\nabla \kappa|^2
= 2\kappa_{12}[h_1\kappa_1 - h_2\kappa_2 + \kappa_{11} - \kappa_{22}] - \kappa_1(\kappa_{112} + \kappa_{222}) + \kappa_2(\kappa_{111} + \kappa_{221})
= \langle Q\nabla \kappa, \nabla \Delta \kappa \rangle,
\end{equation}

where the following formulas have been used
\[
\kappa_{121} = \kappa_{112} + \kappa_{212}, \quad \kappa_{122} = \kappa_{212} = \kappa_{221} + \kappa_{11}.
\]

\begin{equation} \label{eq:2.37}
E_2(\Delta \kappa) = \kappa_{112} + \kappa_{222} + 2(\kappa_{12} - \kappa_{12}) f_1/f = \kappa_{112} + \kappa_{222}, \quad E_1(\Delta \kappa) = \kappa_{111} + \kappa_{221}.
\end{equation}

To complete the proof, it remains to show that the formula (2.31) is the same as (2.27).

Let
\[ X = [2\kappa_2\kappa_{12} + \kappa_1(\kappa_{11} - \kappa_{22})] E_1 + [2\kappa_1\kappa_{12} - \kappa_2(\kappa_{11} - \kappa_{22})] E_2. \]

Since \( \langle X, Q\nabla \kappa \rangle = 2D^2\kappa(\nabla \kappa, Q\nabla \kappa) = 0 \), we have
\[ X = \frac{\langle X, \nabla \kappa \rangle}{|\nabla \kappa|^2} \nabla \kappa. \]

A simple computations shows that
\[ \langle X, \nabla \kappa \rangle = 2D^2\kappa(\nabla \kappa, \nabla \kappa) - |\nabla \kappa|^2\Delta \kappa, \]

which completes the proof. \( \square \)

It follows from Lemma 2.1 and Theorem 2.2 that

**Theorem 2.3** If the Gaussian curvature function \( \kappa \) is not constant on \( \Omega \), then
\[ \dim H^1_{\text{hf}}(\Omega, T) \leq 1. \]

Moreover, if there is a point \( o \in \Omega \) such that
\[ [\langle Q\nabla \kappa, \nabla \Delta \kappa \rangle] + [D^2\kappa(\nabla \kappa, Q\nabla \kappa)]^2 > 0 \quad \text{at} \quad o, \]

then \( H^1_{\text{hf}}(\Omega, T) = \{0\} \).
3 Infinitesimal Isometries

We shall give some characteristic conditions on a function \( w \) for which there exists a vector field \( W \) such that \( V = W + wN \) is to be an infinitesimal isometry (Theorem 3.1).

Let \( M \) be a surface with the induced metric \( g \) from \( \mathbb{R}^3 \). Let \( N \) be the unit normal field of \( M \). Let \( \Pi \) be the second fundamental form of \( M \). Let \( \Omega \subset M \) be an open set. For \( w \in H^1_{1s}(\Omega) \), there exists a unique vector field \( W \) on \( \Omega \), which is perpendicular to \( H^1_{1s}(\Omega,T) \) in \( H^1(\Omega,T) \), such that \( (W,w) \) is to be an infinitesimal isometry on \( \Omega \), that satisfies

\[
DW(X,X) + w\Pi(X,X) = 0 \quad \text{for} \quad X \in M_x, \quad x \in \Omega,
\]

where \( D \) is the Levi-Civita connection of the induced metric \( g \).

Let \( o \in \Omega \) be such that \( \Omega \) is star-shaped with respect to \( o \). Let the frame field \( T \) and \( E \) be given in (2.11). Let

\[
W = \varphi(t,\theta)T + \phi(t,\theta)E \quad \text{for} \quad x = F(t,\theta) \in \Omega \cap \exp_o \Sigma(o),
\]

where \( \varphi = \langle W, T \rangle \) and \( \phi = \langle W, E \rangle \). In the sequel all our computations are made on the region \( \Omega \cap \exp_o \Sigma(o) \).

Similar to (2.14), the relation (3.1) is equivalent to

\[
\begin{align*}
\varphi_t + w\Pi(T,T) &= 0, \\
f\phi_t - f_t\varphi + \varphi_\theta + 2fw\Pi(T,E) &= 0, \\
\phi_\theta + f_t\varphi + f\Pi(E,E) &= 0, \\
\varphi(0) &= \langle W, \sigma(\theta) \rangle, \quad \phi(0) = \langle W, \dot{\sigma}(\theta) \rangle.
\end{align*}
\]

Let \( \varphi \) solve the first equation in (3.2) with the initial data \( \varphi(0) = \langle W, \sigma(\theta) \rangle \). Then

\[
\varphi = \langle W_0, \sigma(\theta) \rangle - \int_0^t w\Pi_{11}ds.
\]

As in (2.16), a similar computation shows that \( \phi \) solves the second equation in (3.2) if and only if it satisfies

\[
\begin{align*}
\phi_{tt} + \kappa \phi &= P(w), \\
\phi(0) &= \langle W, \dot{\sigma}(\theta) \rangle,
\end{align*}
\]

where

\[
P(w) = -2w_1\Pi_{12} + w_2\Pi_{11} - w\Pi_{121} \quad \text{for} \quad x \in \Omega,
\]

\[
w_1 = \langle Dw, T \rangle, \quad w_2 = \langle DW, E \rangle, \quad \Pi_{12} = \Pi(T, E), \quad \Pi_{121} = D\Pi(T, E, T),
\]

etc. Furthermore, differentiating the third equation in (3.2) with respect to the variable \( t \) and using the first equation of (3.2) yield

\[
0 = \phi_{t\theta} + f_{tt}\varphi + f_t[w\Pi(E,E) - w\Pi(T,T)] + f[w\Pi(E,E)]_t \quad \text{for} \quad t > 0.
\]
Letting $t \to 0$ in (3.6), we obtain another initial data for the problem (3.4)
\[ \phi_t(0) = -w(o)\Pi(\sigma(\theta), \dot{\sigma}(\theta)). \] (3.7)

Let $k$ be an integer. Let $T^k(M)$ be all tensor fields of rank $k$ on $M$. Let
\[ R_{XY} : T^k(M) \to T^k(M) \]
be the curvature operator where $X, Y \in \mathcal{X}(M)$ are vector fields. For $K \in T^k(M)$, we have the following formulas, called the Ricci identities,
\[ D^2K(\cdots, X, Y) = D^2K(\cdots, Y, X) + (R_{XY}K)(\cdots). \] (3.8)
The above formulas are very useful when we have to exchange the order of the covariant differentials of a tensor field.

We seek some conditions on $w$ such that the problem (3.1) has a vector field solution $W$.

**Lemma 3.1** Let $M$ be orientable. Let $(W, w)$ be an infinitesimal isometry of $\Omega$. Then
\[ \langle D^2w, Q^*\Pi \rangle + w\kappa \text{tr } \Pi = \langle \nabla \kappa, W \rangle \quad \text{for} \quad x \in \Omega, \] (3.9)
where $Q$ is defined by (2.22), $\kappa$ is the Gaussian curvature function, and $\nabla, \text{tr}$ are the gradient, the trace of the induced metric of $M$, respectively.

**Proof** Let $o \in \Omega$ be any point. Then there is $\varepsilon > 0$ such that the geodesic ball centered at $o$ with the radius $\varepsilon$ is contained in $\Omega$. Therefore, the systems (3.2) and (3.4) make sense for $(t, \theta) \in [0, \varepsilon] \times [0, 2\pi)$.

From (3.4) and using the symmetry of $D\Pi$, we have
\[ \phi_{t\theta} + \kappa \phi + \kappa \dot{\phi}_\theta = -2(w_1\Pi_{12})_\theta + (w_2\Pi_{11})_\theta - (w\Pi_{121})_\theta \]
\[ = -2f(w_{12}\Pi_{12} + w_1\Pi_{122}) - 2f_t[w_2\Pi_{12} + w_1(\Pi_{22} - \Pi_{11})] \]
\[ + f(w_{22}\Pi_{11} + w_2\Pi_{112}) + f_t(2w_2\Pi_{12} - w_1\Pi_{11}) \]
\[ - f(w_2\Pi_{121} + w_1\Pi_{122}) - f_t w(2\Pi_{221} - \Pi_{111}) \]
\[ = f(w_{22}\Pi_{11} - 2w_1\Pi_{12} - 2w_1\Pi_{122} - w\Pi_{1212}) \]
\[ + f_t[w_1(\Pi_{11} - 2\Pi_{22}) + w(\Pi_{111} - 2\Pi_{221})], \] (3.10)
which yields
\[ \phi^{(3)}_\theta(0) = w_{22}\Pi_{11} - 2w_1\Pi_{12} - 2w_1\Pi_{122} - w\Pi_{1212} \]
\[ + [w_1(\Pi_{11} - 2\Pi_{22}) + w(\Pi_{111} - 2\Pi_{221})]'(0) \]
\[ - \kappa \phi(0) - \kappa' \phi(0) - \kappa \phi(0) \]
\[ = w_1(\Pi_{11} - 2\Pi_{22}) + w_{22}\Pi_{11} - 2w_1\Pi_{12} + 2w_1(\Pi_{111} - 3\Pi_{221}) \]
\[ + w[\Pi_{1111} - 3\Pi_{2211} + 2\kappa(\Pi_{22} - \Pi_{11})] \]
\[ - \langle \nabla \kappa, \dot{\sigma}(\theta) \rangle \langle W, \dot{\sigma}(\theta) \rangle + \langle \nabla \kappa, \sigma(\theta) \rangle \langle W, \sigma(\theta) \rangle, \] (3.11)
where the following formulas have been used

\[
\Pi_{1212} = \Pi_{2211} + R_{TE} D^2 \Pi(T, E) = \Pi_{2211} + \kappa(\Pi_{11} - \Pi_{22}) \quad \text{(by (3.8))},
\]

\[
\phi_0'(0) = w\kappa(\Pi_{11} - \Pi_{22}).
\]

On the other hand, using the equation (2.10) and the first equation in (3.2), we obtain

\[
(f_t \phi)(3)(0) = [f^{(4)} \phi + 3f^{(3)} \phi' + 3f'' \phi'' + f''\phi^{(3)}](0)
= -2\kappa' \phi(0) - 3\kappa(0)\phi'(0) + \phi^{(3)}(0)
= -2 \langle \nabla \kappa, \sigma(\theta) \rangle \langle W, \sigma(\theta) \rangle - w_{11} \Pi_{11} - 2w_{1} \Pi_{111}
+ w(3\kappa_{11} - \Pi_{1111}) \quad \text{at } o. 
\]

Moreover, we have

\[
(f_w \Pi_{22})^{(3)}(0) = f^{(3)}(0)w(o)\Pi_{22}(o) + 3(w\Pi_{22})''(0)
= 3w_{11} \Pi_{22} + 6w_{1} \Pi_{221} + w(3\Pi_{2211} - \kappa\Pi_{22}) \quad \text{at } o. \tag{3.12}
\]

Finally, using the third equation in (3.2), we obtain from (3.11)-(3.13)

\[
0 = (\phi_t + f_t \phi + f_w \Pi_{22})^{(3)}(0)
= w_{11} \Pi_{22} - 2w_{12} \Pi_{12} + w_{22} \Pi_{11} + w\kappa(\Pi_{11} + \Pi_{22})
- \langle \nabla \kappa, \sigma(\theta) \rangle \langle W, \sigma(\theta) \rangle - \langle \nabla \kappa, \sigma(\theta) \rangle \langle W, \sigma(\theta) \rangle
= \langle D^2 w, Q^* \Pi \rangle + w\kappa \text{tr } \Pi - \langle \nabla \kappa, W \rangle \quad \text{at } o. \tag{3.13}
\]

\[
\phi = \Phi_0(t) \langle W_o, \dot{\sigma}(\theta) \rangle - w(o)\Pi(\dot{\sigma}(\theta), \sigma(\theta))f + \int_0^t \Phi(t, s)P(w)(s) ds, \tag{3.16}
\]

where \(P(w)\) is given by (3.5). Then \(\phi\) solves the problem (3.4) and (3.7).

Let \(s \geq 0\) be given. Let \(\Phi_0(t)\) and \(\Phi(t, s)\) solve the problem

\[
\begin{cases}
\Phi_{0t}(t) + \kappa(t)\Phi_0(t) = 0 \quad \text{for } t \geq 0, \\
\Phi_0(0) = 1, \quad \Phi_{0t}(0) = 0, 
\end{cases} \tag{3.14}
\]

and

\[
\begin{cases}
\Phi_{tt}(t, s) + \kappa(t)\Phi(t, s) = 0 \quad \text{for } t \geq s, \\
\Phi(s, s) = 0, \quad \Phi_t(s, s) = 1, 
\end{cases} \tag{3.15}
\]

respectively. Note that

\[
\Phi(t, 0) = f.
\]

Let \(w\) be a function on \(\Omega\) and \(W_o \in M_o\). Let

We have
Theorem 3.1 Let $M$ be orientable and let $\Omega$ be star-shaped with respect to a point $o \in \Omega$. Then $w \in H^1_{is}(\Omega)$ if and only if it is in the form of

$$w = u(x) + \langle W_o, N \rangle(x) \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o),$$

(3.17)

where $W_o \in M_o$ is a constant vector and $u$ is a solution to the problem

$$A_o u + u(o) \Pi(\sigma(\theta), \dot{\sigma}(\theta)) \kappa_2 f = 0 \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o),$$

(3.18)

where

$$A_o u = \langle D^2 u, Q^* \Pi \rangle + u \kappa \text{tr} \Pi + \kappa_1 \int_0^t u \Pi_{11} ds - \kappa_2 \int_0^t \Phi(t, s) P(u)(s) ds.$$  

(3.19)

**Remark 3.1** Since $\Omega$ is star-shaped with respect to $o$, a simple computation shows that

$$|u(o)| \leq C(\|u\|_{H^1(\Omega)} + \|u\|_{L^2(\Gamma)}) \quad \text{for} \quad u \in H^1(\Omega).$$

(3.20)

Then the second term in the left hand side of (3.18) makes sense for $u \in H^1(\Omega)$.

**Remark 3.2** As a constant vector $M_o$ on $\Omega$, or a translation displacement of $\Omega$, $(\hat{W}_o, \langle W_o, N \rangle)$ is a trivial infinitesimal isometry where $W_o = \hat{W}_o + \langle W_o, N \rangle N$. Then a solution $u$ to the problem (3.18) is itself in $H^1_{is}(\Omega)$.

**Remark 3.3** The formula (3.17) depends on the choice of the point $o \in \Omega$. If the point $o$ can be chosen to be an umbilical point of $M$ ([19]), then $\kappa(o) \geq 0$ and

$$\Pi(o) = \sqrt{\kappa(o)} g,$$

which yields

$$\Pi(o)(\sigma(\theta), \dot{\sigma}(\theta)) = 0 \quad \text{for} \quad \theta \in (0, 2\pi].$$

In this case the equation (3.18) becomes

$$A_o u = 0 \quad \text{for} \quad x \in \Omega.$$  

(3.21)

Another case for which (3.21) holds true is that $o \in \Omega$ can be chosen such that

$$\kappa_2 = 0 \quad \text{for} \quad x \in \Omega.$$  

(3.22)

**Proof of Theorem 4.1** Necessity Let $w \in H^1_{is}(\Omega)$. Let a vector field $W \perp H^1_{kf}(\Omega, T)$ be such that $(W, w)$ is an infinitesimal isometry. Let

$$W(o) = \hat{W}(o) + \langle W(o), N \rangle N \quad \text{for} \quad x \in \Omega.$$  

Let

$$U = W - \hat{W}(o), \quad u = w - \langle W(o), N \rangle \quad \text{for} \quad x \in \Omega.$$
Then \((U, u)\) is an infinitesimal isometry field with \(U(o) = 0\). Using the formulas (3.3) and (3.16) in the formula (3.9) where \((W, w)\) is replaced by \((U, u)\), we have the formula (3.19) for \(u\).

**Sufficiency** Let \(u\) solve the problem (3.18). It will suffice to prove that there is a vector field \(U \in \mathcal{X}(\Omega)\) such that \((U, u)\) is an infinitesimal isometry. We define

\[
U = \varphi T + \phi E \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o),
\]

where

\[
\varphi = -\int_0^t u\Pi_{11} ds, \quad \phi = -u(o)\Pi(\sigma(\theta), \sigma(\theta))f + \int_0^t \Phi(t, s)P(u)ds.
\]  

(3.23)

Then the equation (3.18) means that

\[
\langle D^2 u, Q^* \Pi \rangle + u\kappa \text{tr} \Pi = \langle \nabla \kappa, U \rangle \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o).
\]

(3.24)

Clearly, \(\varphi\) and \(\phi\) satisfy the first equation and the second equation in (3.2). To complete the proof, it remains to show that \(\varphi\) and \(\phi\), given by (3.23), solve the third equation in (3.2). For this end, we let

\[
\eta = \phi_\theta + f_1\varphi + fu_{12} \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o).
\]

Using (2.10), (3.10), and (3.24), we compute

\[
\eta'' = \phi''_\theta + f(3)\varphi + 2f''\varphi' + f'\varphi'' + f(u_{122})'' + 2f'(u_{122})' + f''u_{122}
\]

\[
= \phi''_\theta - (f\kappa' + f'\kappa)\varphi - 2f\kappa\varphi' + f'\varphi'' + f(u_{122})'' + 2f'(u_{122})' - f\kappa u_{122}
\]

\[
= \phi''_\theta + f[(u_{122})'' - \kappa u_{122} - 2\kappa\varphi' - \kappa'\varphi] + f'[2(u_{122})' - \kappa\varphi + \varphi']
\]

\[
= -\kappa\theta\phi + \kappa\phi_\theta + f\kappa'\varphi + f'\varphi'
\]

\[
+ f[(u_{122})'' - \kappa u_{122} - 2\kappa\varphi'] + f'[2(u_{122})' + \varphi']
\]

\[
+ f[u_{2211} - 2u_{12} + 2u_{12} - u_{122} - u_{1122}]
\]

\[
+ f'[u_{11}(\Pi_{11} - 2\Pi_{22}) + u(\Pi_{11} - 2\Pi_{22})]
\]

\[
= -[f \langle \nabla \kappa, U \rangle + \kappa(\phi_\theta + f'\varphi + fu_{122}) + f(\langle D^2 u, Q^* \Pi \rangle + u\kappa \text{tr} \Pi)
\]

\[
+ f'(u_{11} + u_{11} + \varphi')
\]

\[
= f(\langle D^2 u, Q^* \Pi \rangle + u\kappa \text{tr} \Pi - \langle \nabla \kappa, U \rangle) - \kappa\eta + f'(u_{11} + \varphi')
\]

\[
= -\kappa\eta,
\]

(3.25)

where the following formula has been used

\[
R_{E_1, E_2} \Pi(E_1, E_2) = \kappa(\Pi_{11} - \Pi_{22})(\text{by (3.8)}).
\]

Moreover, we have the initial data

\[
\eta(0) = \phi_\theta(0) + \varphi(0) = 0, \quad \eta'(0) = \phi_\theta(0) + \varphi'(0) + u(o)u_{22}(o) = 0,
\]

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which imply by the equation (3.25) that \((U,u)\) is an infinitesimal isometry.

If surface \(M\) is given as a graph, an infinitesimal isometry function \(w \in H^1_{\text{in}}(\Omega)\) can be written as an explicit formula in the Cartesian orthogonal coordinate system. Let

\[
M = \{(x,h(x)) \mid x = (x_1,x_2) \in \mathbb{R}^2\},
\]

where \(h\) is a smooth function on \(\mathbb{R}^2\). Let

\[
V(p) = (u_1,u_2,u) \quad \text{for} \quad p \in M.
\]

We then have

**Theorem 3.2** ([1], [47], [48], [52]) Let \(\tilde{\Omega} \subset \mathbb{R}^2\) be a star-shaped with respect to a point \(\tilde{o} \in \tilde{\Omega}\). Then there are functions \(u_1, u_2\) such that \(V\) is an infinitesimal isometry on

\[
\Omega = \{ (x,h(x)) \mid x \in \tilde{\Omega} \}
\]

if and only if \(u\) solves the problem

\[
\tilde{\text{div}} A(x) \tilde{\nabla} u = 0 \quad \text{for} \quad x \in \tilde{\Omega},
\]

where \(\tilde{\text{div}}\) and \(\tilde{\nabla}\) are the divergence and gradient of \(\mathbb{R}^2\) in the Euclidean metric, respectively, and

\[
A(x) = \begin{pmatrix}
h_{x_2 x_2} & -h_{x_1 x_2} \\
-h_{x_1 x_2} & h_{x_1 x_1}
\end{pmatrix} \quad \text{for} \quad x \in \tilde{\Omega}.
\]

Based on Theorem 3.2, we shall work out a formula for \(w\) as follows.

It follows from (3.26), we shall work out a formula for \(w\) as follows.

\[
M_p = \{ (\alpha, \alpha(h)) \mid \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \} \quad \text{for} \quad p = (x,h(x)) \in M.
\]

Moreover,

\[
N = \eta(\tilde{\nabla} h, -1), \quad \eta = \frac{1}{\sqrt{1 + |\tilde{\nabla} h|^2}}
\]

\[
\Pi((\alpha, \alpha(h)), (\beta, \beta(h))) = \eta \tilde{D}^2 h(\alpha, \beta) \quad \text{for} \quad (\alpha, \alpha(h)), (\beta, \beta(h)) \in M_p,
\]

\[
\kappa(p) = \eta^4(h_{x_1 x_1} h_{x_2 x_2} - h_{x_1 x_2}^2) \quad \text{for} \quad p \in M,
\]

where \(\tilde{D}^2 h\) is the Hessian of \(h\) in the Euclidean metric of \(\mathbb{R}^2\).

Let \(o = (0,0,0)\). Consider the polar coordinates \((t, \theta)\) in the induced metric \(g\) of \(M\) initiating from \(o\). Consider a 1-parameter family of geodesics given by

\[
\mathcal{F}(t, \theta) = \exp_o t \sigma(\theta),
\]
where
\[
\sigma(\theta) = \frac{\cos \theta}{\sqrt{1 + h_{x_1}^2}}(1, 0, h_{x_1}) + \frac{\sin \theta}{\sqrt{(1 + h_{x_1}^2)(1 + |Dh|^2)}}(-h_{x_1}, h_{x_2}, 1 + h_{x_1}^2, h_{x_2}).
\] (3.31)

Let
\[
\mathcal{F}(t, \theta) = (r(t), h(r(t))),
\]
where \(r(t) = (r_1(t), r_2(t))\) is a curve in \(\mathbb{R}^2\). Then
\[
\left(\ddot{r}, D^2h(\dot{r}, \dot{r}) + \langle \dot{D}h, \dot{r} \rangle\right) = \dot{D}_{\hat{x}} \dot{\hat{r}} = D_{\hat{x}} \dot{\hat{r}} - \Pi(\dot{\hat{r}}, \dot{\hat{r}})N = -\Pi(\dot{r}, \dot{r})N,
\]
which yields
\[
\ddot{r}(t) + \eta^2 \dot{D}^2h(\dot{r}(t), \dot{r}(t)) \dot{D}h = 0 \quad \text{for} \quad t > 0,
\] (3.32)
with the initial data
\[
r(0) = 0, \quad \dot{r}(0) = \sigma(\theta).
\] (3.33)

Set
\[
X_i(t, s) = \hat{D}_{r(s)}[(r_i(s) - r_i(t))Q\nabla h(s)] \quad \text{for} \quad x = r(t) \in \mathbb{R}^2, \quad i = 1, 2,
\] (3.34)
where \(Q\nabla h = (h_{x_2}, -h_{x_1})\).

**Theorem 3.3** Let \(M\) be given by (3.26) and let \(w \in H^1_{\text{is}}(\Omega)\) be given. Then there is a solution \(u\) to the problem (3.28) such that
\[
w/\eta = \langle Z, \dot{D}h \rangle + h_{x_1}(x) \int_0^t [X_2(t, s)(u) - u_{x_1} \dot{r}(h)]ds
\]
\[
- h_{x_2}(x) \int_0^t [X_1(t, s)(u) + u_{x_2} \dot{r}(h)]ds - u(x) \quad \text{for} \quad x = r(t) \in \mathbb{R}^2,
\] (3.35)
where \(Z \in \mathbb{R}^2\) is a constant vector.

**Proof** Since \(w \in H^1_{\text{is}}(\Omega)\), there is a unique displacement field \(V = (u_1, u_2, u)\) in \(\mathbb{R}^3\) such that
\[
w = \langle V, N \rangle, \quad \langle \dot{D}_X V, X \rangle = 0 \quad \text{for} \quad X \in M_x, x \in \Omega.
\]
By Theorem 3.2, the third component \(u\) of \(V\) solves the problem (3.28). We shall obtain \(u_1\) and \(u_2\) in terms of \(u\).

By Lemma 3 of Chapter 12 in [52], there is a vector field \(Y = (\psi_1, \psi_2, \psi)\) such that
\[
\dot{D}_X V = X \times Y \quad \text{for} \quad X \in M_x, x \in \Omega,
\] (3.36)
where \(\times\) denotes the exterior product. Letting \(X = (1, 0, h_{x_1})\) and \(X = (0, 1, h_{x_2})\) in (3.36), respectively, we obtain
\[
Y = (-u_{x_2}, u_{x_1}, \psi),
\] (3.37)
\[ u_{1x_1} = -h_{x_1}u_{x_1}, \quad u_{1x_2} = \psi - h_{x_2}u_{x_1}, \quad u_{2x_1} = -\psi - h_{x_1}u_{x_2}, \quad u_{2x_2} = -h_{x_2}u_{x_2}. \quad (3.38) \]

It follows from (3.38) that

\[
\begin{cases}
  u_1 = z_1 + \int_0^t [\psi\dot{r}_2 - u_{x_1}\dot{r}(h)] ds, \\
  u_2 = z_2 - \int_0^t [\psi\dot{r}_1 + u_{x_2}\dot{r}(h)] ds,
\end{cases} \quad (3.39)
\]

where \( z_1 \) and \( z_2 \) are constants.

Next, we compute \( \psi \). Since the curvature operator of \( \mathbb{R}^3 \) in the Euclidean metric is zero,

\[-\tilde{D}_X \tilde{D}_Z V + \tilde{D}_Z \tilde{D}_X V + \tilde{D}_{[X,Z]} V = 0 \quad (3.40)\]

for vector fields \( X, Z \) on \( \Omega \).

Using the formula (3.36) in the identity (3.40), we obtain

\[ X \times \tilde{D}_Z Y = Z \times \tilde{D}_X Y. \]

In particular, taking \( X = (1, 0, h_{x_1}) \) and \( Z = (0, 1, h_{x_2}) \), respectively, yield

\[
\begin{cases}
  \psi_{x_1} = -h_{x_1}u_{x_1} + h_{x_2}u_{x_2}, \\
  \psi_{x_2} = -h_{x_1}u_{x_2} + h_{x_2}u_{x_1},
\end{cases}
\]

which give

\[ \psi = \langle \tilde{\nabla} u, Q \tilde{\nabla} h \rangle - \int_0^t \langle \tilde{\nabla} u, \tilde{D}_t Q \tilde{\nabla} h \rangle ds. \quad (3.41) \]

Inserting the formula (3.41) into the formula (3.39), we have

\[
\begin{cases}
  u_1 = z_1 + \int_0^t [X_2(t,s)(u) - u_{x_1}\dot{r}(h)] ds, \\
  u_2 = z_2 - \int_0^t [X_1(t,s)(u) + u_{x_2}\dot{r}(h)] ds.
\end{cases} \quad (3.42)
\]

Then the formula (3.35) follows from (3.42).

\[ \square \]

4 Elliptic Surfaces

Let \( M \) be a surface in \( \mathbb{R}^3 \). \( M \) is said to be elliptic if the fundamental form \( \Pi \) is positive for all \( x \in M \). Assume that \( M \) be elliptic throughout this section. Then the problem (3.18) will become an elliptic one (Theorem 4.1).

We introduce another metric on \( M \) by

\[ \hat{g} = \Pi \quad \text{for} \quad x \in M. \]

**Proposition 4.1** Let \( M \) be elliptic. Then for \( w \in C^2(M) \),

\[ \kappa \Delta_{\Pi} w + \frac{1}{2\kappa} Q^* \Pi (\nabla \kappa, \nabla w) = \langle D^2_w, Q^* \Pi \rangle \quad \text{for} \quad x \in M, \quad (4.1) \]

where \( \Delta_{\Pi} \) is the Laplacian of the metric \( \hat{g} = \Pi \) and \( Q : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega) \) is the operator, given by (2.22).
Proof Let $o \in M$ be fixed. Consider the polar coordinates in the induced metric $g$

\[
\partial t = T, \quad \partial \theta = f E.
\]

Note that the above $(\partial t, \partial \theta)$ is no longer the polar coordinates in the metric $\hat{g} = \Pi$.

In the coordinate system $(\partial t, \partial \theta)$, we have

\[
\hat{g} = \left(\begin{array}{c}
\Pi_{11} \ f \Pi_{12} \\
\Pi_{12} \ f^2 \Pi_{22}
\end{array}\right), \quad \det \hat{G} = \kappa f^2, \quad \hat{G}^{-1} = \frac{1}{\kappa f^2} \left(\begin{array}{c}
f^2 \Pi_{22} - f \Pi_{12} \\
-f \Pi_{12} \ \Pi_{11}
\end{array}\right).
\]

Moreover,

\[
w_{t\theta} = f w_{12} + f' w_2, \quad w_{\theta\theta} = f^2 w_{22} - f f' w_1 + f_\theta w_2.
\]

Using those formulas, we obtain

\[
\kappa \Delta_{\Pi} w = \frac{\kappa}{\sqrt{\kappa}} \left[\left(\sqrt{\kappa} f \frac{\Pi_{22}}{\kappa} w_t\right) - \left(\sqrt{\kappa} f \frac{\Pi_{12}}{\kappa} w_t\right) \theta - \left(\sqrt{\kappa} f \frac{\Pi_{12}}{\kappa} w_\theta\right) t + \left(\sqrt{\kappa} f \frac{\Pi_{11}}{\kappa} f w_\theta\right) \theta\right]
\]

\[
= \langle D^2 w, Q^* \Pi \rangle + \left\{ \frac{\sqrt{\kappa}}{f} \left[\left(\frac{\Pi_{22}}{\sqrt{\kappa}}\right) t - \left(\frac{\Pi_{12}}{\sqrt{\kappa}}\right) \theta\right] - \Pi_{11}\right\} w_1 - 2 \frac{f'}{f} \Pi_{12} w_2
\]

\[
+ \left\{ \frac{\sqrt{\kappa}}{f} \left[\left(\frac{\Pi_{11}}{\sqrt{\kappa}}\right) \theta - \left(\frac{\Pi_{12}}{\sqrt{\kappa}}\right) t\right] + \frac{f_\theta}{f^2} \Pi_{11}\right\} w_2
\]

\[
= \langle D^2 w, Q^* \Pi \rangle + \left\{ \frac{1}{2\kappa} (\kappa_2 \Pi_{12} - \kappa_1 \Pi_{22}) w_1 - 2 \frac{f'}{f} \Pi_{12} w_2 + \left[ 2 \frac{f'}{f} \Pi_{12} + \frac{1}{2\kappa} (\kappa_1 \Pi_{12} - \kappa_2 \Pi_{11})\right] w_2\right\}
\]

\[
= \langle D^2 w, Q^* \Pi \rangle - \frac{1}{2\kappa} \Pi (Q \nabla \kappa, Q \nabla w).
\]

\[
\square
\]

Let $\Omega \subset M$ with boundary $\Gamma$. Instead of the usual inner product of $L^2(\Omega)$, we use the following inner product on $L^2(\Omega)$

\[
(w, v)_{L^2(\Omega)} = \int_{\Omega} w v d\Pi \quad \text{for} \quad w, v \in L^2(\Omega).
\]

We denote by $L^2_{\Pi}(\Omega)$ the above space.

It is well known that the negative Laplacian operator $-\Delta_{\Pi}$ on $\Omega$ with the Dirichlet boundary condition is a positive selfadjoint operator on $L^2_{\Pi}(\Omega)$ and

\[
D(\Delta_{\Pi}) = H^2(\Omega) \cap H^1_0(\Omega).
\]

Moreover, we extend the domain $H^2(\Omega) \cap H^1_0(\Omega)$ of $\Delta_{\Pi}$ to $H^1_0(\Omega)$ such that

\[
\Delta_{\Pi} : H^1_0(\Omega) \to H^{-1}(\Omega)
\]

(4.2)

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is an isomorphism. We let
\[ Bw = B_\omega w + w(\sigma(\theta), \dot{\sigma}(\theta))f \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o), \] (4.3)
where
\[ B_\omega w = \frac{1}{2\kappa^2} Q^* Q(\nabla \kappa, \nabla w) + w \tr \Pi + \frac{\kappa_1}{\kappa} \int_0^t \Pi w_{11} ds - \frac{\kappa_2}{\kappa} \int_0^t \Phi(t, s) P(w)(s) ds \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o). \] (4.4)

**Remark 4.1** Let \( o \in \Omega \). Since \( \Omega \cap \exp_o C(o) \) is zero measurable and
\[ \Omega = [\Omega \cap \exp_o \Sigma(o)] \cup [\Omega \cap \exp_o C(o)], \]
\( Bw \) is defined by (4.3) on \( \Omega \) almost everywhere.

Consider the operator \( A_\omega \), defined by (3.19). It follows from (3.19) and (4.1) that
\[ A_\omega w + w(\sigma(\theta), \dot{\sigma}(\theta)) = \kappa(\Delta_\Pi w + Bw). \] (4.5)

Since for \( \bar{W}_0 \in M_o \), \( (\bar{W}_0, (W_0, N)) \) is a trivial, smooth infinitesimal isometry where \( W_0 = \tilde{W}_0 + (W_0, N) \), it follows from Theorem 3.1 and (4.5) that

**Theorem 4.1** Let \( \Omega \subset M \) be elliptic and star-shaped with respect to \( o \in \Omega \). Then
\[ H^1_{\text{loc}}(\Omega) = \{ w \mid w \in H^1(\Omega), \Delta_\Pi w + Bw = 0 \}. \] (4.6)

Next, we consider the structure of solutions to the equation \( \Delta_\Pi w + Bw = 0 \) in \( H^1(\Omega) \).
Since \( \Delta_\Pi w \in H^1_0(\Omega) \) for \( w \in H^{-1}(\Omega) \), we have the following estimates
\[ \| B_\omega \Delta_\Pi^{-1} w \|_{H^{-1}(\Omega)} \leq C \| \Delta_\Pi^{-1} w \|_{L^2_0(\Omega)} \leq C \| \Delta_\Pi^{-1} w \|_{H^1_0(\Omega)} \leq C \| w \|_{H^{-1}(\Omega)} \quad \text{for} \quad w \in L^2(\Omega), \]
which yield

**Lemma 4.1** The operator \( B_\omega \Delta_\Pi^{-1} : H^{-1}(\Omega) \to H^{-1}(\Omega) \) is a compact operator.

Consider the operator \( \Delta_\Pi + B \) with the domain \( D(\Delta_\Pi + B) = H^1_0(\Omega) \). Denote by \( B^* \) the adjoint operator of \( B \) with respect to the inner product of \( L^2_\Pi(\Omega) \). Then
\[ B^* = B_\omega^* + \left( \frac{\kappa_2}{\kappa} [\Pi(\sigma(\theta), \dot{\sigma}(\theta))f], \right)_L^2(\Omega) \delta(o), \] (4.7)
where \( \delta(o) \in H^{-1}(\Omega) \) is the Dirac function at \( o \) and
\[ D(\Delta_\Pi + B^*) = H^1_0(\Omega). \]
Let

\[ V_0(\Omega) = \{ \varphi \mid \varphi \in H^1_0(\Omega), \Delta_{\Pi} \varphi + B \varphi = 0 \} \],

(4.8)

\[ V_{0*}(\Omega) = \{ \varphi \mid \varphi \in H^1_0(\Omega), \Delta_{\Pi} \varphi + B^* \varphi = 0 \} \],

(4.9)

\[ V_{0*}(\Gamma) = \{ \varphi_\nu \mid \varphi \in V_{0*}(\Omega) \}. \]

(4.10)

It follows from Lemma 4.1 and the formula (4.7) that

\[ \Delta_{\Pi}^{-1} B^* = (B_o \Delta_{\Pi}^{-1})^* + \left( \frac{K^2}{\kappa^2} |\Pi(\sigma(\theta), \dot{\sigma}(\theta)) f| \right)_{L^2_\Pi(\Omega)} \Delta_{\Pi}^{-1} \delta(a) : L^2_\Pi(\Omega) \rightarrow L^2_\Pi(\Omega) \]

is a compact operator. Then \( V_0(\Omega) \) and \( V_0(\Gamma) \) are subspaces of finite dimension.

We discompose \( H^1_{\text{is}}(\Omega) \) as a direct sum in \( H^1(\Omega) \) as

\[ H^1_{\text{is}}(\Omega) = V_0(\Omega) \oplus V_{0^\perp}(\Omega). \]

(4.11)

We have

**Theorem 4.2** Let \( V_{0*}(\Omega) = \{ 0 \} \). Then \( w \in V_{0^\perp}(\Omega) \) if and only if there is a unique \( \psi \in H^{1/2}(\Gamma) \) such that

\[ w = w_0 - \Delta_{\Pi}^{-1}(I + B \Delta_{\Pi}^{-1})^{-1} B w_0, \]

(4.12)

where \( w_0 \in H^1(\Omega) \) is the unique solution to the problem

\[ \begin{cases} \Delta_{\Pi} w_0 = 0 & \text{for } x \in \Omega, \\ w_0 = \psi & \text{for } x \in \Gamma. \end{cases} \]

(4.13)

If \( V_{0*}(\Omega) \neq \{ 0 \} \), then \( w \in V_{0^\perp}(\Omega) \) if and only if there is a unique \( \psi \in H^{1/2}(\Gamma) \) satisfying

\[ (\psi, \varphi_\nu)_{L^2_\Pi(\Gamma)} = 0 \quad \text{for all } \varphi \in V_{0*}(\Omega), \]

(4.14)

such that (4.13) and (4.12) hold.

**Proof** By Theorem 3.1, what we are looking for is a solution \( w \in H^1_{\text{is}}(\Omega) \) to the problem

\[ \begin{cases} \Delta_{\Pi} w + B w = 0 & \text{for } x \in \Omega, \\ w = \psi & \text{for } x \in \Gamma. \end{cases} \]

(4.15)

Let \( w_0 \in H^1(\Omega) \) be the solution to the problem (4.13) and let \( v = w - w_0 \). Then the problem (4.15) is equivalent to solve

\[ \Delta_{\Pi} v + B v = -B w_0 \quad \text{for some } v \in H^1_0(\Omega). \]

(4.16)

Let \( u = \Delta_{\Pi} v \). Then the problem (4.16) is the same to the problem

\[ u + B \Delta_{\Pi}^{-1} u = -B w_0. \]

(4.17)
By the Fredholm theorem ([30]), the problem (4.17) is solvable if and only if
\[
(\mathcal{B} w_0, \varphi)_{L^2_\Pi(\Omega)} = 0
\]  
(4.18)
for all \( \varphi \in \mathcal{V} \) where
\[
\mathcal{V} = \{ \varphi \in L^2_\Pi(\Omega) \mid \varphi + (\mathcal{B} \Delta^{-1}_\Pi)^* \varphi = 0 \}.
\]  
(4.19)

It is easy to check that
\[
\mathcal{V} = \mathcal{V}_{0*}(\Omega) = \{ \varphi \in H^1_0(\Omega) \mid \Delta_\Pi \varphi + \mathcal{B}^* \varphi = 0 \}.
\]

Clearly, if \( \mathcal{V}_{0*}(\Omega) = \{ 0 \} \), the claim is true. We assume that \( \mathcal{V}_{0*}(\Omega) \neq \{ 0 \} \). Let \( \psi \in H^{1/2}(\Gamma) \) be such that (4.14) is true. Then it follows from (4.14) that
\[
(\mathcal{B} w_0, \varphi)_{L^2_\Pi(\Omega)} = (w_0, \mathcal{B}^* \varphi)_{L^2_\Pi(\Omega)} = -(w_0, \Delta_\Pi \varphi)_{L^2_\Pi(\Omega)} = -\langle \psi, \varphi \rangle_{L^2_\Pi(\Gamma)} = 0,
\]
for all \( \varphi \in \mathcal{V}_{0*}(\Omega) \). Then the problem (4.15) has a solution \( w \), given by (4.12). Moreover, a simply computation shows that if \( w \) is a solution to the problem (4.15), then the conditions (4.14) hold. Then the proof is complete. \( \square \)

If \( \Omega \) is of constant curvature, it follows from the formulas (4.3) and (4.4) that
\[
\mathcal{B} w = \frac{1}{2\kappa^2} Q^* \Pi(\nabla \kappa, \nabla w) + \mathrm{tr} \Pi.
\]

We then have \( \mathcal{B} w \in C^\infty(\Omega) \) whenever \( w \in C^\infty(\Omega) \). Let \( \Omega \) be star-shaped respect to \( o \) and be not of constant curvature. Consider a solution \( w_0 \) to the problem (4.13). If \( \psi \in C^\infty(\Gamma) \), then \( w_0 \in C^\infty(\Omega) \). Furthermore, by the formula (4.3), \( \mathcal{B} w_0 \) is \( C^\infty \) on \( \Omega \cap \exp_o \Sigma(o) \). Therefore from the formula (4.12), \( w \) is also \( C^\infty \) on \( \Omega \cap \exp_o \Sigma(o) \). Since \( C^\infty(\Gamma) \) is dense in \( H^{1/2}(\Gamma) \), the ellipticity of the operator \( \Delta_\Pi \) implies the following density result.

**Theorem 4.3** Let \( \Omega \subset M \) be elliptic which is star-shaped with respect to \( o \in \Omega \). Moreover, suppose that one of the following assumptions holds true: \( \Omega \) is of constant curvature, or
\[
\Omega \subset \exp_o \Sigma(o).
\]  
(4.20)
Then the strong \( H^1(\Omega) \) closure of
\[
H^1_{ib}(\Omega) \cap C^\infty(\Omega)
\]
agrees with \( H^1_{ib}(\Omega) \).
Remark 4.2 Let $\Omega$ be bounded and be not of constant curvature. If $\Omega \cap \exp_o C(o) \neq \emptyset$, we only have

$$Bu \in L^2(\Omega)$$

when $u \in C^\infty(\Omega)$ where the operator $B$ is defined by (4.3). A condition like (4.20) is necessary for the above density result. An interesting example is given by [14] (see [34] or [52]), where $\Omega$ is a closed smooth surface of non-negative curvature for which $C^\infty$ infinitesimal isometries consist only of trivial fields, whereas there exist non-trivial $C^2$ infinitesimal isometries. Therefore $H^1_{is}(\Omega) \cap C^\infty(\Omega)$ is not dense in $H^1_{is}(\Omega)$ for this surface.

By Theorem 4.4, if $V_0(\Omega) = \{0\}$, an infinitesimal isometry function $w \in H^1_{is}(\Omega)$ is completely given by its boundary trace $w \in H^{1/2}(\Gamma)$. However, in general $V_0(\Omega) \neq \{0\}$ even for a spherical cap, see Theorem 4.6 later. Next, we consider several cases for which the relations $V_0(\Omega) = \{0\}$ hold. This problem closely relates to the first eigenvalue of $-\Delta_{\Pi}$ with the Dirichlet boundary condition. Let $\lambda_1$ be the first positive eigenvalue of $-\Delta_{\Pi}$ on $L^2_{\Pi}(\Omega)$.

Let $\kappa_{\Pi}$ be the curvature function of $M$ in the metric $g_{\Pi} = \Pi$. Let $\rho_{\Pi} = \rho_{\Pi}(x, o)$ be the distance function from $x \in M$ to $o \in M$ in the metric $g_{\Pi} = \Pi$. For $a > 0$, let

$$\mu(a) = \sup_{\rho_{\Pi} \leq a} \kappa_{\Pi}.$$ 

Then $\mu(a)$ is an increasing function in $a \in [0, \infty)$. Let $a_0 > 0$ be given by

$$\mu(a_0) a_0^2 = \frac{\pi}{2}.$$ 

Lemma 4.2 Assume that there is $0 < a < a_0$ such that

$$\Omega \subset \{x \mid x \in M, \rho_{\Pi}(x) < a\}.$$ 

Then

$$\lambda_1 \geq \frac{1}{4} \mu(a) \ctg^2 \sqrt{\mu(a)} a \quad (4.21)$$

Proof The Laplace operator comparison theorem yields

$$\Delta_{\Pi} \rho_{\Pi} \geq \sqrt{\mu(a)} \ctg \sqrt{\mu(a)} a > 0 \quad \text{for} \quad x \in M, \rho_{\Pi} < a. \quad (4.22)$$

Let $O \subset \subset \Omega$ be an open set with a boundary $\partial O$. It follows from (4.22) that

$$\text{Vol}(\partial O) = \int_{\partial O} 1 d\Gamma \geq \int_{\partial O} \langle \nabla \rho, \nu \rangle_{\Pi} d\Gamma_{\Pi} = \int_O \Delta_{\Pi} \rho_{\Pi} dg_{\Pi} \geq \sqrt{\mu(a)} \ctg \sqrt{\mu(a)} a \text{Vol}(O).$$

Then the estimate (4.21) follows from the Cheeger theorem ([51]). \hfill \Box

We have
Theorem 4.4 There is $a > 0$ such that, if 
\[ \Omega \subset \{ x \mid x \in M, \rho_{\Pi}(x) < a \}, \]
then 
\[ \mathcal{V}_0(\Omega) = \{ 0 \}, \]
where $\mathcal{V}_0(\Omega)$ is given by (4.8).

Proof For $w \in H^2(\Omega) \cap H^1_0(\Omega)$, using the estimate (4.21), we have
\[ -\left( w, \Delta_{\Pi}w + B^*w \right)_{L^2(\Omega)} = -\left( \Delta_{\Pi}w, w \right)_{L^2(\Omega)} - \left( Bw, w \right)_{L^2(\Omega)} \]
\[ \geq \frac{1}{2} \| \nabla_{\Pi}w \|_{L^2(\Omega)}^2 - C \| w \|_{L^2(\Omega)}^2 \]
\[ \geq \frac{1}{8} \mu(a) \cot^2 \sqrt{\mu(a)a} - C \| w \|_{L^2(\Omega)}^2. \]
The proof is complete. \(\square\)

A Elliptic Surface of Revolution Let $h$ be a smooth function on $[0, b)$ with $h(0) = 0$ such that
\[ \frac{1}{s} h''(s) h'(s) > 0 \quad \text{for} \quad s \in [0, b). \quad (4.23) \]
Let 
\[ M = \{ (x, h(|x|)) \mid x = (x_1, x_2) \in \mathbb{R}^2, \ |x| < b \}. \quad (4.24) \]
We have

Theorem 4.5 Let $o = (0, h(0))$ and let $\Omega \subset M$ be a bounded open set which is star-shaped with respect to $o$. Then 
\[ \mathcal{V}_0(\Omega) = \{ 0 \}, \quad (4.25) \]
where $\mathcal{V}_0(\Omega)$ is given by (4.8).

Proof We shall do a careful computation by the formula (3.35). For this end, we make some preparations.

By the formula (3.31), we have 
\[ \sigma(\theta) = (\cos \theta, \sin \theta). \]
Let $X, Y$ be vector fields on $\mathbb{R}^2$. Then
\[ \tilde{D}^2 h(X, Y) = h''(|x|) \frac{\langle X, x \rangle \langle Y, x \rangle}{|x|^2} + \frac{h'(|x|)}{|x|} \langle X, Y \rangle. \quad (4.26) \]
We look for a solution to the problem (3.32)-(3.33) in a form of $r(t) = \alpha(t)\sigma(\theta)$ where $\alpha(t) > 0$ for $t > 0$. It is easy to check from (4.26) that such $\alpha(t)$ is a positive solution to the problem

$$\begin{cases} \alpha''(t) + \eta^2(\alpha(t)\sigma(\theta))h''(\alpha(t))h'(\alpha(t))\alpha'^2(t) = 0 \quad \text{for} \quad t > 0, \\ \alpha(0) = 0, \quad \alpha'(0) = 1. \end{cases}$$

(4.27)

Moreover, the solution $\alpha(t)$ to the problem (4.27) is actually the solution to the problem

$$\begin{cases} \alpha'(t) = \frac{1}{\sqrt{1 + h'^2(\alpha(t))}} \quad \text{for} \quad t > 0, \\ \alpha(0) = 0. \end{cases}$$

(4.28)

Furthermore, a simple computation shows that $\alpha(t)$ is also the solution to the problem (2.10). Then

$$\alpha(t) = f(t), \quad F(t, \theta) = \left(r(t), h(r(t))\right) = \left(f(t)\sigma(\theta), h(f(t))\right) \quad \text{for} \quad t \geq 0.$$ 

We obtain

$$T = \tilde{D}_{\theta t} F = f'(t)\left(\sigma(\theta), h'(f(t))\right), \quad E = \frac{1}{f} \tilde{D}_{\theta \theta} F = \left(\dot{\sigma}(\theta), 0\right).$$

(4.29)

We shall prove that the problem

$$\begin{cases} \Delta_{\Pi} w + B w = 0 \quad \text{for} \quad x \in \Omega, \\ w = 0 \quad \text{for} \quad x \in \Gamma, \end{cases}$$

(4.30)

has the unique zero solution. Let $w$ be a solution to the problem (4.30). By the proof of Theorem 3.1, $(W, w)$ is an infinitesimal isometry, where

$$W = \varphi T + \phi E,$$

$$\varphi = -\int_0^t w_{11} ds, \quad \phi = -w(o)\Pi(\dot{\sigma}(\theta), \sigma(\theta))f + \int_0^t \Phi(t, s) P(w) ds.$$ 

Then

$$W(o) = 0.$$ 

(4.31)

Let

$$W + w N = (u_1, u_2, u),$$

where $u$ is a solution to the problem (3.28) and $u_1, u_2$ are given by (3.42). Since $N(o) = (0, 0, 1)$, it follows from (4.31) that

$$u_1(o) = 0, \quad u_2(o) = 0, \quad u(o) = w(o).$$

By (3.42), we have

$$z_1 = z_2 = 0.$$
Moreover, using (3.35), we obtain
\[ w/\eta = h'(f(t)) \cos \theta \int_0^t X_2(t, s)(u)ds - h'(f(t)) \sin \theta \int_0^t X_1(t, s)(u)ds \]
\[ -[1 + h'^2(f(t))]u(x) + h'(f(t)) \int_0^t u(r(s))h''(f(s))f'(s)ds, \tag{4.32} \]
where \( X_i \) are given by (3.34). On the other, a simple computation shows that
\[ \cos \theta X_2(t, s)(u) = -\{h'(f(s)) + [f(s) - f(t)]h''(f(s)) \frac{f'(s)}{f(s)} \} u \theta \cos \theta \sin \theta \]
\[ = \sin \theta X_1(t, s)(u). \tag{4.33} \]
It follows from (4.32) and (4.33) that
\[ w\eta = \eta^2 h'(f) \int_0^t u(r(s))h''(f(s))f'(s)ds - u(x) \quad \text{for} \quad x = r(t) \in \Omega. \tag{4.34} \]

We now apply the maximum principle to the elliptic problem (3.28) to know that there is \( x_0 \in \Gamma \) such that
\[ u(x_0) = \sup_{x \in \Omega} u(x). \]
We may assume that \( u(x_0) \geq 0 \). Otherwise, we consider \(-u\). Let \( r(t_0) = x_0 \). Then the formula (4.34) yields
\[ u(x_0) = \eta^2 h'(f) \int_0^{t_0} u(r(s))h''(f(s))f'(s)ds \leq u(x_0) \frac{h'^2(f(t_0))}{1 + h'^2(f(t_0))}, \]
which gives \( u(x_0) = 0 \). Next, we consider \(-u\) and have \( \inf_{x \in \Omega} u = 0 \). Then \( u \equiv 0 \) on \( \Omega \). Finally, we obtain \( w \equiv 0 \) on \( \Omega \) by (4.34).

**A Spherical Cap** Let \( M \) be a sphere of constant curvature \( \kappa > 0 \) with the induced metric \( g \) from \( \mathbb{R}^3 \). Then the second fundamental form of \( M \) is given by
\[ \Pi = \sqrt{\kappa} g. \tag{4.35} \]
Then
\[ \sqrt{\kappa} \Delta_{\Pi} w = \Delta w, \quad \mathcal{B} w = 2\sqrt{\kappa}w, \]
where \( \Delta \) is the Laplacian of \( M \) in the induced metric \( g \) from \( \mathbb{R}^3 \).

Let \( o \in M \) be given. Let \( \rho(x) = \rho(x, o) \) be the distance from \( x \in M \) to \( o \) in the induced metric \( g \) of \( M \). Set
\[ \Omega(a) = \{ x \mid x \in M, \rho(x) < a \} \quad \text{for} \quad 0 < a \leq \frac{\pi}{\sqrt{\kappa}}. \]
Then for $0 < a < \frac{\pi}{\sqrt{\kappa}}, \Omega(a)$ is a spherical cap with a nonempty smooth boundary

$$
\Gamma(a) = \{ x | x \in M, \ \rho(x) = a \}.
$$

It follows Theorem 4.1 that $w \in H^1_{ib}(\Omega(a))$ if and only if $w$ satisfies the problem

$$
\Delta w + 2\kappa w = 0 \quad \text{for} \quad x \in \Omega(a).
$$

(4.36)

Moreover,

$$
V_0(\Omega(a)) = V_{0a}(\Omega(a)) = \{ \varphi | \Delta \varphi + 2\kappa \varphi = 0, \ \varphi|_{\Gamma(a)} = 0 \}.
$$

We have

**Theorem 4.6**

\[
\begin{cases}
V_0(\Omega(a)) = \{ 0 \} & \text{for} \quad 0 < a < \frac{\pi}{2\sqrt{\kappa}}, \\
V_0(\Omega(a)) \neq \{ 0 \} & \text{for} \quad \frac{\pi}{2\sqrt{\kappa}} \leq a \leq \frac{\pi}{\sqrt{\kappa}}.
\end{cases}
\]

(4.37)

**Proof** Let $\lambda_1(a)$ be the first positive eigenvalue of $-\Delta$ on $\Omega(a)$ with the Dirichlet boundary condition on $\Gamma(a)$. By [11], [42], the first positive eigenvalue of $-\Delta$ of the sphere $M$ without boundary is $2\kappa$. Since $C_0^\infty(\Omega(a_1)) \subset C_0^\infty(\Omega(a_2)) \subset C_0^\infty(M)$ for all $0 < a_1 \leq a_2 \leq \frac{\pi}{\sqrt{\kappa}}$, then $H_0^1(\Omega(a_1)) \subset H_0^1(\Omega(a_2)) \subset H^1(M)$ in the following sense: For $h \in H_0^1(\Omega(a))$, we define $h = 0$ for $x \in M/\Omega(a)$. Then

\[
2\kappa = \inf \left\{ \frac{\int_M |\nabla h|^2 dg}{\int_M h^2 dg} | h \in H^1(M) \right\}
\]

\[
\leq \inf \left\{ \frac{\int_{\Omega(a_2)} |\nabla h|^2 dg}{\int_{\Omega(a_2)} h^2 dg} | h \in H_0^1(\Omega(a_2)) \right\} = \lambda_1(a_2) \leq \lambda_1(a_1).
\]

(4.38)

Since

$$
\Delta \rho = \sqrt{\kappa} \cot \sqrt{\kappa} \rho \quad \text{for} \quad x \in M, \ x \neq o,
$$

it is easy to check that the following function

$$
\varphi(x) = \cos \sqrt{\kappa} \rho(x) \quad \text{for} \quad x \in M
$$

is an eigenfunction of $-\Delta$ of the sphere $M$ without boundary corresponding to the eigenvalue $2\kappa$. Clearly, $\varphi$ is also an eigenfunction of $-\Delta$ on $\Omega(\frac{\pi}{2\sqrt{\kappa}})$ with the Dirichlet boundary condition on $\Gamma(\frac{\pi}{2\sqrt{\kappa}})$ corresponding to an eigenvalue $2\kappa$, which implies, by (4.38), that

$$
\lambda_1(a) = 2\kappa \quad \text{for} \quad \frac{\pi}{2\sqrt{\kappa}} \leq a \leq \frac{\pi}{\sqrt{\kappa}},
$$

which means

$$
V_0(\Omega(a)) \neq \{ 0 \} \quad \text{for} \quad \frac{\pi}{2\sqrt{\kappa}} \leq a \leq \frac{\pi}{\sqrt{\kappa}}.
$$
Next, we assume that
\[ 0 < a < \frac{\pi}{2\sqrt{\kappa}}. \]
Let \(o = (0,0,0)\) and let the semi-sphere \(\Omega(\frac{\pi}{2\sqrt{\kappa}})\) be given by
\[
\Omega(\frac{\pi}{2\sqrt{\kappa}}) = \{ (x,h(|x|)) \mid x \in \mathbb{R}^2, \ |x| < \frac{1}{\sqrt{\kappa}} \},
\]
where
\[ h(s) = \frac{1}{\sqrt{\kappa}} - \sqrt{\frac{1}{\kappa} - s^2} \quad \text{for} \quad s \in [0, \frac{1}{\sqrt{\kappa}}). \]
Since
\[ h''(s)h'(s)s^{-1} = \frac{1}{1 - \kappa s^2} \quad \text{for} \quad s \in [0, \frac{1}{\sqrt{\kappa}}), \]
it follows from Theorem 4.5 that \(\mathcal{V}_0(\Omega(a)) = \{ 0 \}\) for \(0 < a < \frac{\pi}{2\sqrt{\kappa}}. \)

**Remark 4.3** The relations (4.37) mean that, for the first eigenvalue \(\lambda_1(a)\) of \(-\Delta\) on \(\Omega(a)\) with the Dirichlet boundary condition on \(\Gamma(a)\),
\[
\begin{aligned}
\lambda_1(a) &> 2\kappa \quad \text{for} \quad 0 < a < \frac{\pi}{2\sqrt{\kappa}}, \\
\lambda_1(a) &= 2\kappa \quad \text{for} \quad \frac{\pi}{2\sqrt{\kappa}} \leq a \leq \frac{\pi}{\sqrt{\kappa}}.
\end{aligned}
\]

**Remark 4.4** If \(a = \frac{\pi}{\sqrt{\kappa}}\), then \(\Omega(a) = M\). Since a sphere is rigid, any infinitesimal isometry of \(M\) is trivial, see [48].

5 **Parabolic Surfaces**

A surface \(M\) is said to be parabolic if
\[ \kappa = 0, \quad \Pi \neq 0 \quad \text{for all} \quad x \in M. \]
Let \(M\) be parabolic and orientable. Let \(\Omega \subset M\). It follows from Theorem 3.1 that \(w \in H^1_{\text{loc}}(\Omega)\) if and only if \(w \in H^1(\Omega)\) solves the problem
\[ \langle D^2w, Q^*\Pi \rangle = 0 \quad \text{for} \quad x \in \Omega. \]

We assume that there is a vector field \(E \in \mathcal{X}(M)\) such that
\[ \dot{D}_E N = 0, \quad |E| = 0 \quad \text{for} \quad x \in M. \quad (5.1) \]
Let \(p_0 \in M\) be given. We consider a parabolic coordinates \((t,s)\) on \(M\) as follows. Let curves \(r\) and \(\zeta : \mathbb{R} \rightarrow M\) be given by
\[
\begin{aligned}
\dot{r}(t) &= E(r(t)) \quad \text{for} \quad t \in \mathbb{R}, \\
r(0) &= p_0.
\end{aligned}
\]
and
\[
\begin{aligned}
\dot{\zeta}(s) &= QE(\zeta(s)) \quad \text{for} \quad s \in \mathbb{R}, \\
\zeta(0) &= p_0,
\end{aligned}
\]
respectively, where the operator \( Q : M_p \to M_p \) for \( p \in M \) is given by (2.22). Let two parameters families \( \alpha(t,s) \) and \( \beta(t,s) \) be given by
\[
\left\{
\begin{aligned}
\frac{\partial \alpha}{\partial t}(t,s) &= E(\alpha(t,s)) \quad \text{for} \quad t \in \mathbb{R}, \\
\alpha(0,s) &= \zeta(s),
\end{aligned}
\right.
\]
and
\[
\left\{
\begin{aligned}
\frac{\partial \beta}{\partial s}(t,s) &= QE(\beta(t,s)) \quad \text{for} \quad s \in \mathbb{R}, \\
\zeta(t,0) &= r(t),
\end{aligned}
\right.
\]
respectively. Then
\[
\alpha(t,s) = \beta(t,s) \quad \text{for} \quad (t,s) \in \mathbb{R}^2,
\]
\[
\partial t = \frac{\partial \alpha}{\partial t}(t,s) = E, \quad \partial s = \frac{\partial \beta}{\partial s}(t,s) = Q \frac{\partial \alpha}{\partial t}(t,s).
\]
We have

**Theorem 5.1** Let \( M \) be a parabolic surface and orientable. Let \((t,s)\) be the parabolic coordinates on \( M \). Then
\[
H^1_{ib}(M) = \set{ w_0(s) + w_1(s)t | w_1, w_0 \in H^1(\mathbb{R}), t \in \mathbb{R} }.
\]

**Proof** Consider the frame field \( E_1 = E, E_2 = QE \). By (5.1), we have
\[
D_{\partial t} \partial t = D_{\partial t} t + \Pi(\partial t, \partial t) N = \frac{\partial^2 \alpha}{\partial t^2}(t,s)
\]
\[
= \left\langle \frac{\partial^2 \alpha}{\partial t^2}(t,s), \frac{\partial \alpha}{\partial t}(t,s) \right\rangle \frac{\partial \alpha}{\partial t}(t,s) + \left\langle \frac{\partial^2 \alpha}{\partial t^2}(t,s), \frac{\partial \beta}{\partial s}(t,s) \right\rangle \frac{\partial \beta}{\partial s}(t,s)
\]
\[
+ \left\langle \frac{\partial^2 \alpha}{\partial t^2}(t,s), N \right\rangle N
\]
\[
= \frac{1}{2} \frac{\partial}{\partial t} \left| \frac{\partial \alpha}{\partial t}(t,s) \right|^2 \frac{\partial \alpha}{\partial t}(t,s) + \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial \alpha}{\partial t}(t,s), Q \frac{\partial \alpha}{\partial s}(t,s) \right) \frac{\partial \beta}{\partial s}(t,s)
\]
\[
+ \frac{\partial}{\partial t} \left( \frac{\partial \alpha}{\partial t}(t,s), N \right) N = 0.
\]

It follows from (5.1) and (5.3) that
\[
0 = \langle D^2 w, Q^* \Pi \rangle = D^2 w(E,E) \Pi(QE,QE) = \frac{\partial^2 w}{\partial t^2} \Pi(QE,QE).
\]
Since \( \Pi(QE,QE) \neq 0 \), we have the formula (5.2). \( \Box \)
**A Cylinder** Let $a > 0$ be given. Consider a cylinder

$$M = \{ (x, z) \mid x = (x_1, x_2) \in \mathbb{R}^2, \ |x| = a, \ z \in \mathbb{R} \}.$$  

Then

$$N = \frac{1}{a^2}(x, 0).$$

Let $E = (0, 0, 1) = \partial z$. Then

$$\hat{D}_E N = 0, \ |E| = 1.$$  

Consider the parabolic coordinates $(z, \theta)$, given by

$$(x, z) = (a \cos \theta, a \sin \theta, z).$$

Let $b > 0$ be given and let

$$\Omega = \{ (x, z) \mid |x| = a, \ |z| < b \}, \ T = \{ x \mid x \in \mathbb{R}^2, \ |x| = a \}. \quad (5.4)$$

Then, by Theorem 5.1,

$$H^1_{1s}(\Omega) = \{ w_0 + w_1 z \mid w_0, w_1 \in H^1(T), \ |z| < b \}. \quad (5.5)$$

**Remark 5.1**

(i) Clearly, $H^1_{1s}(\Omega) \cap C^\infty(\Omega)$ is dense in $H^1_{1s}(\Omega)$.

(ii) Let $\Omega$ be given in (5.4) with $a = 1$. Let $w_0 \in H^2(T)$ and $w_1 \in H^3(T)$ be given. Then an infinitesimal isometry corresponding to $w = -w_0'(\theta) + zw_1''(\theta) \in H^1_{1s}(\Omega)$ is given by

$$V = (w_0(\theta) - zw_1'(\theta)) \partial \theta + w_1(\theta) \partial z + wN$$

$$= \left( -w_0(\theta) \sin \theta - w_0'(\theta) \cos \theta, \ w_0(\theta) \cos \theta - w_0'(\theta) \sin \theta, \ w_1(\theta) \right)$$

$$+ z \left( w_1'(\theta) \sin \theta + w_1''(\theta) \cos \theta, \ -w_1'(\theta) \cos \theta + w_1''(\theta) \sin \theta, \ 0 \right)$$

$$= \left( R_{w_0, w_0'}, w_1 \right) + z \left( -R_{w_1', w_1''}, 0 \right)$$

where

$$R = \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}.$$  

**A Conical Surface** Let $a > 0$ and let

$$M = \{ (x, z) \mid |x| = a|z|, \ x = (x_1, x_2) \in \mathbb{R}^2, \ z \in \mathbb{R} \}.$$  

Then

$$N = \frac{1}{\sqrt{1 + a^2}(|x|, -1)}.$$
Consider the parabolic coordinates \((z, \theta)\), given by
\[
(x, z) = z(a \cos \theta, a \sin \theta, 1).
\]

Let \(b_1, b_2 > 0\) be given and let
\[
\Omega = \{ (x, z) \mid |x| = az, \ b_1 < z < b_2 \}, \quad T = \{ x \mid x \in \mathbb{R}^2, \ |x| = 1 \}.
\]

Since \(\hat{D}_{\partial z} N = 0\), we have from Theorem 5.1
\[
H^1_{\text{is}}(\Omega) = \{ w_0 + w_1 z \mid w_0, w_1 \in H^1(T), \ b_1 < z < b_2 \}.
\]

(5.6)

6 Hyperbolic Surfaces

A surface \(M\) is said to be a hyperbolic surface if
\[
\kappa < 0 \quad \text{for} \quad x \in M.
\]

Let \(M\) be a hyperbolic surface and orientable. We assume that surface \(M\) is given by a family of two parameter curves
\[
M = \{ \alpha(s, \varsigma) \in \mathbb{R}^3 \mid (s, \varsigma) \in \mathbb{R} \times T \},
\]
which satisfies

\[
\langle \partial s, \partial \varsigma \rangle = 0, \quad \Pi(\partial s, \partial s) < 0, \quad \Pi(\partial s, \partial \varsigma) = 0 \quad \text{for} \quad x \in M,
\]

(6.2)

where
\[
T = \{ x \in \mathbb{R}^2 \mid |x| = 1 \}.
\]

Let \(\varsigma = (\cos \vartheta, \sin \vartheta)\) for \(\vartheta \in [0, 2\pi]\). Then
\[
\partial \varsigma = \frac{\partial \alpha}{\partial \vartheta}(s, \varsigma).
\]

We consider the structure of the operator \(\langle D^2 w, Q^* \Pi \rangle\). Let
\[
E_1 = \frac{\partial s}{|\partial s|}, \quad E_2 = \frac{\partial \varsigma}{|\partial \varsigma|}.
\]

By (6.2), we have
\[
|\partial s|^2 |\partial \varsigma|^2 \langle D^2 w, Q^* \Pi \rangle = D^2 w(\partial s, \partial s)\Pi(\partial \varsigma, \partial \varsigma) + D^2 w(\partial \varsigma, \partial \varsigma)\Pi(\partial s, \partial s)
\]
\[
= \Pi(\partial \varsigma, \partial \varsigma)w_{ss} + \Pi(\partial \varsigma, \partial s)w_{\vartheta \vartheta}
\]
\[
+ \Pi(\partial \varsigma, \partial \varsigma)D_{\partial s} \partial \varsigma w + \Pi(\partial \varsigma, \partial s)D_{\partial \varsigma} \partial \varsigma w.
\]
Let \( \Omega \subset M \) be given by
\[
\Omega = \{ \alpha(s, \vartheta) \mid (t, \varsigma) \in (0, b) \times T \},
\]
where \( b > 0 \) is given. We fix \( o \in \Omega \) to be such that \( \Omega \) is star-shaped with respect to \( o \).

Let
\[
a(s, \vartheta) = -\frac{\Pi(\partial s, \partial s)}{\Pi(\partial \varsigma, \partial \varsigma)}.
\]
By Theorem 3.1, \( w \in H^1_{is}(\Omega) \) if and only if \( w \in H^1(\Omega) \) is a solution to the problem
\[
w_{ss} = (a(s, \vartheta)w_{\vartheta})_{\vartheta} + \tilde{B} w,
\]
where
\[
\tilde{B} w = -a_{\vartheta}(s, \vartheta)w_{\vartheta} - \Pi^{-1}(\partial \varsigma, \partial \varsigma)[\Pi(\partial \varsigma, \partial \varsigma)D_{\partial s} \partial sw + \Pi(\partial s, \partial s)D_{\partial \varsigma} \partial \varsigma w]
- |\partial s|^{-1} |\partial \varsigma|^{-1}[w_{\kappa} tr \Pi + \kappa_1 \int_0^s w_{\Pi 11} ds + \kappa_2 \int_0^t \Phi(t, s) P(w)(s) ds
+ w(o) \Pi(\sigma(\vartheta), \dot{\sigma}(\vartheta))\kappa_2 f],
\]
where the four factor of the third term in the right hand side of (6.5) is given by Theorem 3.1. Clearly, the linear operator
\[
\tilde{B} : H^1(\Omega) \to L^2(\Omega),
\]
is bounded.

We introduce a family of self-adjoint operators on \( L^2(T) \) by
\[
A(s)u = -(a(s, \vartheta)u_{\vartheta})_{\vartheta}, \quad D(A(s)) = H^2(T) \quad \text{for} \quad s \in [0, b].
\]
Consider a family of operators on \( H^1(T) \times L^2(T) \)
\[
A(s) = \begin{pmatrix} 0 & I \\ A(s) & 0 \end{pmatrix}, \quad D(\tilde{A}(s)) = H^2(T) \times H^1(T) \quad \text{for} \quad s \in [0, b].
\]
Let
\[
\dot{H}^m(T) = \{ u \in H^m(T) \mid (u, 1)_{L^2(T)} = 0 \} \quad \text{for} \quad m = 0, 1.
\]

Lemma 6.1 The operator family \( \{ A(s) \}_{0 \leq s \leq b} \) generates a unique evolution system \( \dot{U}(s, \lambda) \) for \( 0 \leq \lambda \leq s \leq b \) on \( H^1(T) \times \dot{L}^2(T) \). In particular, there exist constants \( C(b) > 0 \) and \( \omega(b) > 0 \) such that
\[
\| U(s, \lambda) \| \leq C(b)e^{\omega(b)(s-\lambda)} \quad \text{for} \quad 0 \leq \lambda \leq s \leq b.
\]
Proof We introduce the equivalent norms on \( H^m(T) \times H^{m-1}(T) \) by

\[
\|(u, v)\|_m = \left( \frac{1}{2\pi}(u, 1)^2 + \sup_{0 \leq s \leq b} \left( (A^m(s)u, u)_{L^2(T)} + (A^{m-1}(s)v, v)_{L^2(T)} \right) \right)^{1/2}
\]

for \((u, v) \in H^m(T) \times H^{m-1}(T), m = 1, 2, \) and \(3, \) respectively, where \( H^0(T) = L^2(T) \) and \( A^0(s) = I. \) Note that \( \left( H^m(T) \times H^{m-1}(T), \| \cdot \|_m \right) \) is not a Hilbert space in general.

Let \( 0 < \lambda_1(s) \leq \lambda_2(s) \leq \cdots \leq \lambda_k(s) \leq \cdots \)
be all positive eigenvalues of \( A(s) \) and let their corresponding eigenfunctions be \( \{ \varphi_k \} \)
such that \( \{(2\pi)^{-1/2}, \varphi_k\} \) forms an orthonormal basis of \( L^2(T). \) Then for each \( s \in [0, b] \)
the operator \( A(s) \) generates a \( C_0 \) group semigroup \( S_s(t) \) on \( H^1(T) \times L^2(T), \) given by

\[
S_s(t) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_0 \\ 0 \end{pmatrix} + \sum_k a_k e^{\sqrt{\lambda_k}it} \left( \frac{1}{\sqrt{\lambda_k}} \varphi_k \right) + \sum_k b_k e^{-\sqrt{\lambda_k}it} \left( \frac{1}{\sqrt{\lambda_k}} \varphi_k \right), \quad (6.8)
\]

where

\[
c_0 = \frac{1}{2\pi} (u, 1)_{L^2}, \quad \left\{ \begin{array}{l} a_k + b_k = \sqrt{\lambda_k}(u, \varphi_k)_{L^2}, \\ a_k - b_k = -i(v, \varphi_k)_{L^2}, \end{array} \right.
\]

and \((u, v) \in H^1(T) \times \dot{L}^2(T)\) is real.

Let \( S_s(t)(u, v) = \left( S_s(t), S_s(t)^2 \right). \) It follows from (6.8) that

\[
\left( A(s)S_s(t), S_s(t)^2 \right)_{L^2} + \|S_s(t)^2\|_{L^2}^2 = \sum_k (|a_k e^{\sqrt{\lambda_k}it} + b_k e^{-\sqrt{\lambda_k}it}|^2 + |a_k e^{\sqrt{\lambda_k}it} - b_k e^{-\sqrt{\lambda_k}it}|^2)
\]

\[
= 2 \sum_k (|a_k|^2 + |b_k|^2) = \frac{1}{2} \sum_k [\lambda_k (u, \varphi_k)_{L^2}^2 + (v, \varphi_k)_{L^2}^2] = \frac{1}{2}[(A(s)u, u)_{L^2} + (v, v)_{L^2}],
\]

which yield

\[
\|S_s(t)(u, v)\|_1 \leq \|(u, v)\|_1 \quad \text{for} \quad (u, v) \in H^1(T) \times \dot{L}^2(T), \quad s \in [0, b].
\]

Similarly, we have

\[
\|S_s(t)(u, v)\|_2 \leq \|(u, v)\|_2 \quad \text{for} \quad (u, v) \in H^2(T) \times \dot{H}^1(T), \quad s \in [0, b].
\]

Then the proof is complete by Theorem 3.1 of Chapter 5 in [44].}

**Theorem 6.1** For any \( w_0 \in H^1(T) \) and \( w_1 \in L^2(T) \) with \( (w_1, 1)_{L^2(T)} = 0, \) there is a unique \( w \in H^1_{is}(\Omega) \) such that

\[
w(0, \vartheta) = w_0(\vartheta), \quad w_s(0, \vartheta) = w_1(\vartheta). \quad (6.9)
\]
Proof The problem (6.4) and (6.9) is equivalent to the first order system
\[
\begin{aligned}
\begin{cases}
\frac{\partial}{\partial s} \begin{pmatrix} w \\ w_s \end{pmatrix} = 
\begin{pmatrix} 0 & I \\ A(s) & 0 \end{pmatrix} \begin{pmatrix} w \\ w_s \end{pmatrix} + 
\begin{pmatrix} 0 \\ \tilde{B}w \end{pmatrix} \quad &\text{for } (s, \varsigma) \in (0, b) \times T, \\
(w(0), w_s(0)) = (w_0, w_1) \quad &\text{for } \varsigma \in T.
\end{cases}
\end{aligned}
\]

(6.10)

Let \((w_0, w_1) \in H^1(T) \times L^2(T)\) be given. Let \(b \geq \eta > 0\) be given. Consider a Banach space, given by
\[
X_\eta = L^1 \left(0, \eta; H^1 \left((0, \eta) \times T \right) \right).
\]
Consider a linear operator \(F : X_\eta \to X_\eta\), given as follows. For \(u \in X_\eta\), \(Fu\) is defined as the first component of
\[
\begin{aligned}
U(s, 0)(w_0, w_1) + \int_0^s U(s, \lambda)(0, \tilde{B}u) d\lambda \quad &\text{for } 0 \leq s \leq \eta,
\end{aligned}
\]
where the operator \(\tilde{B}\) is given by (6.5) and \(U\) is the evolution system in Lemma 6.1. Then a solution \(w \in X_\eta\) to the problem (6.10) for \(0 \leq s \leq \eta\) if and only if \(w\) is a fixed point of the operator \(F\) in \(X_\eta\).

Let \(u_0\) be the first component of the first term in (6.11). For \(u \in X_\eta\), let \(Gu\) be the first component of the second term in (6.11). Then
\[
F u = u_0 + Gu \quad \text{for } u \in X_\eta.
\]
Moreover, we have, by (6.7) and (6.6),
\[
\begin{aligned}
\|(Gu)(s)\|_{H^1((0, \eta) \times T)} &= \| \int_0^s U(s, \lambda)(0, \tilde{B}u) d\lambda \|_{H^1(T) \times L^2(T)} \\
&\leq C(b) \int_0^\eta e^{\omega(b)(s-\lambda)} d\lambda \sup_{0 \leq s \leq \eta} \| \tilde{B}u \|_{L^2(T)} \\
&\leq C(b) \frac{e^{\omega(b)\eta} - 1}{\omega(b)} \| u \|_{X_\eta} \quad \text{for } 0 \leq s \leq \eta, \quad u \in X_\eta.
\end{aligned}
\]
Then for \(\eta > 0\) small, \(G\) is a strictly contractive map on \(X_\eta\) which implies that \(F\) has a unique fixed point \(w \in X_\eta\), given by
\[
w = \sum_{k=0}^{\infty} G^k u_0 \quad \text{for } 0 \leq s \leq \eta.
\]
Then \(w\) is a solution to the problem (6.10) for \(0 \leq s \leq \eta\). Moreover, the solution \(w\) can be extended to \(s \in [0, b]\) since the problem (6.10) is linear. \(\square\)

Remark 6.1 Let \(\Omega\) be given by (6.3) and let \(o \in \Omega\) be fixed. In general
\[
\Omega \cap \exp_o C(0) \neq \emptyset,
\]
where \(\exp_o C(0)\) is the cut locus of \(o\). Then the operator \(\tilde{B}\), given by (6.5), does not map \(C^\infty(\Omega)\) into \(C^\infty(\Omega)\). Then Theorem 6.1 does not imply density results.
Let $M$ be given by (6.1). To obtain density results, we need to choose $\Omega$ such that $\Omega \subset \exp_o \Sigma(o)$. For this end, we let

$$\Omega = \{ \alpha(s, \vartheta) \in M \mid s \in (0, b), \ \vartheta \in (0, \vartheta_0) \}, \quad (6.12)$$

where $b > 0$ and $0 < \vartheta_0 < 2\pi$. Let $o \in \Omega$ be fixed. Let

$$M_1 = M/\{ \alpha(s, \vartheta_1) \mid s \in \mathbb{R} \}, \ \vartheta_0 < \vartheta_1 < 2\pi.$$

Since $(M_1, g)$ is simply connected and curvature negative, we have

$$\exp_o \Sigma(o) = M_1.$$

In this sense

$$\bar{\Omega} \subset \exp_o \Sigma(o). \quad (6.13)$$

By similar arguments as for Theorem 6.1, we have

**Theorem 6.2** Let $\Omega$ be given by (6.12). For $h_1, h_2 \in H^1(0, b), w_0 \in H^1(0, \vartheta_0)$, and $w_1 \in L^2(0, \vartheta_0)$ given, there is a unique $w \in H^1_{ib}(\Omega)$ such that

$$w(s, 0) = h_1(s), \ w(s, \vartheta_0) = h_2(s), \ w(0, \vartheta) = w_0(\vartheta), \ w_s(0, \vartheta) = w_1(\vartheta).$$

By similar arguments as in Theorem 4.3, it follows from Theorem 6.2 and the relation (6.13) that

**Theorem 6.3** Let $\Omega$ be given by (6.12). Then the strong $H^1(\Omega)$ closure of

$$H^1_{ib}(\Omega) \cap C^{\infty}(\Omega)$$

agrees with $H^1_{ib}(\Omega)$.

We present two examples which satisfy the assumptions (6.2) to end this section.

**A Segment Surface of Revolution** Let

$$\tilde{M} = \{ \alpha(r, \vartheta) \mid r \geq 0, \ \vartheta \in (0, 2\pi] \},$$

where

$$\alpha(r, \vartheta) = (r \cos \vartheta, r \sin \vartheta, \log(1 + r^2)).$$

Then

$$\partial r = (\cos \vartheta, \sin \vartheta, \frac{2r}{1 + r^2}), \ \partial \vartheta = r(- \sin \vartheta, \cos \vartheta, 0),$$

$$N = \frac{2r}{\sqrt{1 + 6r^2 + r^4}}(\cos \vartheta, \sin \vartheta, -\frac{1 + r^2}{2r}).$$
We have
\[ \Pi(\partial r, \partial r) = \frac{2(1 - r^2)}{(1 + r^2)\sqrt{1 + 6r^2 + r^4}}, \]
\[ \Pi(\partial \vartheta, \partial \vartheta) = \frac{2r^2}{\sqrt{1 + 6r^2 + r^4}}, \quad \Pi(\partial r, \partial \vartheta) = 0, \]
\[ \kappa = \frac{1}{|\partial r|^2|\partial \vartheta|^2}\Pi(\partial r, \partial r)\Pi(\partial \vartheta, \partial \vartheta) = \frac{4(1 - r^4)}{(1 + 6r^2 + r^4)^2} \begin{cases} > 0 & \text{for } 0 < r < 1; \\ = 0 & \text{for } r = 1; \\ < 0 & \text{for } r > 1. \end{cases} \]

We let
\[ M = \{ \alpha(r, \vartheta) \mid r > 1, \ \vartheta \in [0, 2\pi) \}. \]

Then the assumptions (6.2) hold true.

**A Hyperboloid of one Sheet** Let
\[ M = \{ (x, z) \mid x = (x_1, x_2) \in \mathbb{R}^2, \ z^2 + 1 = x_1^2 + x_2^2 \}. \]

Consider a family of two parameter curves
\[ \alpha(r, \vartheta) = \left(r \cos \vartheta, r \sin \vartheta, \sqrt{r^2 - 1}\right) \text{ for } r > 1, \ \vartheta \in [0, 2\pi). \]

Then
\[ \partial r = (\cos \vartheta, \sin \vartheta, \frac{r}{\sqrt{r^2 - 1}}), \quad \partial \vartheta = r(-\sin \vartheta, \cos \vartheta, 0), \]
\[ N = \eta(\cos \vartheta, \sin \vartheta, -\frac{\sqrt{r^2 - 1}}{r}), \]
\[ \eta = \frac{r}{\sqrt{2r^2 - 1}}, \]
\[ \Pi(\partial r, \partial r) = -\frac{\eta}{r(r^2 - 1)}, \quad \Pi(\partial r, \partial \vartheta) = 0, \quad \Pi(\partial \vartheta, \partial \vartheta) = r\eta. \]

The assumptions (6.2) hold.

### 7 Bending of Shells

We shall apply the theories in Sections 3-6 to the limit energy functionals of the \( \Gamma \)-convergence to reduce bending of shells to a one-dimensional problem in the elliptic case, or parabolic case, or hyperbolic case, respectively.

Let \( M \) be a connected, oriented surface in \( \mathbb{R}^3 \) with the normal field \( N \). Suppose that \( g \) is the induced metric of the surface \( M \) from the standard metric of \( \mathbb{R}^3 \). A family \( \{ S^h \}_{h>0} \) of shells of small thickness \( h \) around \( \Omega \) is given through
\[ S^h = \{ p \mid p = x + zN(x), \ x \in \Omega, \ -h/2 < z < h/2 \}, \quad 0 < h < h_0. \]
The projection onto $\Omega$ along $N$ will be denoted by $\pi$. We will assume that $0 < h < h_0$, with $h_0$ sufficiently small to have $\pi$ well defined on each $S^h$.

To a deformation $u \in W^{1,2}(S^h, IR^3)$, we associate its elastic energy (scaled per unit thickness):

$$E^h(u) = \frac{1}{h} \int_{S^h} W(\hat{\nabla} u) dp,$$

where $\hat{\nabla}$ denotes the gradient of the Euclidean space $IR^3$. Here, the stored energy density $W : IR^{3 \times 3} \rightarrow [0, \infty]$ is assumed to be $C^2$ in a neighborhood of $SO(3)$, and to satisfy the following normalization, frame indifference and nondegeneracy conditions

$$\forall F \in IR^{3 \times 3}, \forall R \in SO(3), \quad W(R) = 0, \quad W(RF) = W(F), \quad W(F) \geq C \text{dist}^2(F, SO(3))$$

(with a uniform constant $C > 0$). In the study of the elastic properties of thin shells $S^h$, a crucial step is to describe the limiting behavior, as $h \rightarrow 0$, of minimizers $u^h$ to the total energy functional

$$J(u) = E^h(u) - \frac{1}{h} \int_{S^h} \left\langle f^h, u \right\rangle dp,$$

subject to applied forces $f^h$. It can be shown that if the forces $f^h$ scale like $h^\alpha$, then $E^h(u) \sim h^\beta$ where $\beta = \alpha$ if $0 \leq \alpha \leq 2$ and $\beta = 2\alpha - 2$ if $\alpha > 2$. The main part of the analysis consists, therefore, of characterizing the limiting behavior of the scaled energy functionals $\frac{E^h}{h^\beta}$, or more generally, that of $\frac{E^h}{e^h}$, where $e^h$ is a given sequence of positive numbers obeying a prescribed scaling law.

The first result in this framework is due to [33], who studied the scaling $\beta = 0$. This leads to a membrane shell model with energy depending only on stretching and shearing of the mid-surface. The case $\beta = 2$ has been analyzed in [21] and it corresponds to geometrically nonlinear bending theory, where the only admissible deformations are the isometries of the mid-surface, while the energy expresses the total change of curvature produced by the deformation.

In [35], the limiting model has been identified for the range of scalings $\beta \geq 4$, based on some estimates in [22]. In these cases, the admissible deformations $u$ are only those which are close to a rigid motion $R$ and whose first order term in the expansion of $u - R$ with respect to $h$ is given by $RV$, where $V \in IS^1(\Omega, IR^3)$ is an infinitesimal isometry on $\Omega$.

Let $V \in IS^1(\Omega, IR^3)$. Then there exists a matrix $A$ such that

$$A^T(x) = -A(x), \quad \hat{D}X V = A(x)X, \quad X \in M_x, \quad x \in \Omega. \quad (7.3)$$

For $\beta > 4$ the limiting energy is given only by a bending term, that is, the first order change in the second fundamental form of $\Omega$, produced by $V$,

$$I(V) = \frac{1}{24} \int_{\Omega} Q_2 \left( x, \Xi(V) \right) dg \quad \text{for} \quad V \in IS^1(\Omega, IR^3), \quad (7.4)$$
\[ \Xi(V) = (\hat{D}^*(AN) - A\Pi)_{\tan}, \quad (7.5) \]

and corresponds to the linear pure bending theory derived in [13] from linearized elasticity. In (7.5), \( \hat{D}^*(AN) \) is the transpose of \( \hat{D}(AN) \), given by

\[ \hat{D}^*(AN)(\tau, \eta) = \left\langle \hat{D}(AN), \eta \right\rangle \text{ for } \tau, \eta \in M_x, x \in \Omega. \]

In (7.4), the quadratic forms \( Q_2(x, \cdot) \) are defined as follows:

\[ Q_2(x, F_{\tan}) = \min_{a \in \mathbb{R}^3} Q_3(F + a \otimes N), \quad Q_3(F) = D^2W(I)(F, F). \]

The form \( Q_3 \) is defined for all \( F \in \mathbb{R}^{3 \times 3} \), while \( Q_2(x, \cdot) \) for a given \( x \in \Omega \) is defined on tangential minors \( F_{\tan} = (\left\langle F\tau, \eta \right\rangle)_{\tau, \eta \in M_x} \) of such matrices.

For \( \beta = 4 \) the \( \Gamma \)-limit, which turns out to be the generalization of the von Kármán functional [23] to shells, also contains a stretching term measuring the second order change in the metric of \( \Omega \),

\[ \tilde{I}(V, B_{\tan}) = \frac{1}{2} \int_{\Omega} Q_2 \left( x, B_{\tan} - \frac{1}{2}(A^2)_{\tan} \right) dg + \frac{1}{24} \int_{\Omega} Q_2 \left( x, \Xi(V) \right) dg \]

for \( V \in IS^1(\Omega, \mathbb{R}^3) \). This involves a symmetric matrix field \( B_{\tan} \) belonging to the finite strain space

\[ B = \left\{ L^2 - \lim_{h \to 0} \text{sym } \nabla w^h \mid w \in W^{1,2}(\Omega, \mathbb{R}^3) \right\}, \]

where

\[ \text{sym } \nabla w(\tau, \eta) = \frac{1}{2} \left( \left\langle \nabla w\tau, \eta \right\rangle + \left\langle \nabla w\eta, \tau \right\rangle \right) \text{ for } \tau, \eta \in M_x, x \in \Omega. \]

The space \( B \) emerges as well in the context of linear elasticity and ill-inhibited surfaces [24, 49].

It was further shown in [35] that for a certain class of surfaces, referred to as approximately robust surfaces, the limiting energy for \( \beta = 4 \) reduces to the purely linear bending functional (7.4). Elliptic surfaces happen to belong to this class [35].

Moreover, [34] has proved that the limit energy of the range of scalings \( 2 < \beta < 4 \) for elliptic surfaces is still given by (7.4).

Here we focus on the limit energy (7.4) and reduce it from over the space \( IS^1(\Omega, \mathbb{R}^3) \) to over the space \( H^1_{\text{ls}}(\Omega) \) to give mathematical formulas, as in [58].

Let \( T_0 \in T^2(M) \) be the third fundamental form of surface \( M \), given by

\[ T_0(\tau, \eta) = \left\langle \hat{D}\tau N, \hat{D}\eta N \right\rangle \text{ for } \tau, \eta \in M_x, x \in M. \]

We now describe the limiting energy formula (7.4) in the common denotation in Riemannian geometry. For simplicity, we restrict ourselves to the case when the stored-energy
function is isotropic (that is to say, \( W(F) = W(R_1 FR_2) \) for all \( F \in M^{3 \times 3} \) and all \( R_1, R_2 \in \text{SO}(3) \)). In this case, the second derivative of \( W \) at the identity is

\[ D^2W(I)(A,A) = 2\mu|E|^2 + \lambda(\text{tr} E)^2, \quad E = \frac{A+A^T}{2}, \]

for some constants \( \mu, \lambda \in \mathbb{R} \).

**Lemma 7.1** Let \( \mu > 0 \) and \( 2\mu + \lambda > 0 \). For \( G \in T^2(\Omega) \) symmetric,

\[ Q_2(x,G) = 2\mu|G|_{\gamma_x}^2 + \frac{\lambda\mu}{\mu + \lambda/2}\text{tr}^2 G \quad \text{for} \quad x \in \Omega, \quad (7.6) \]

where \( | \cdot |_{\gamma_x} \) is given by (2.4) and \( \text{tr} \) is the trace in the induced metric \( g \).

**Proof** Let \( x \in \Omega \) be given and let

\[ F = \{ F \mid F \in M^{3 \times 3}, \text{ symmetric} \}, \quad F_0 = \{ a \otimes N + N \otimes a \mid a \in \mathbb{R}^3 \}. \]

We introduce an inner product on \( F \) by

\[ \langle F_1,F_2 \rangle_* = 2\mu \langle F_1,F_2 \rangle + \lambda \text{tr} F_1 \text{tr} F_2 \quad \text{for} \quad F_1, F_2 \in F. \]

Then \((F, \langle \cdot, \cdot \rangle_*)\) is an inner product space.

Let \( e_1, e_2 \) be an orthonormal basis of \( M_x \). Then \( F_1, F_2, F_3 \) forms an orthonormal basis of \((F_0, \langle \cdot, \cdot \rangle_*)\), where

\[ F_i = \frac{e_i \otimes N + N \otimes e_i}{2\sqrt{\mu}}, \quad F_3 = \frac{1}{\sqrt{2\mu + \lambda}}N \otimes N. \]

Then for \( F \in M^{3 \times 3} \) symmetric with \( G = F_{\text{tan}} \), we have

\[ Q_2(x,G) = \min_{a \in \mathbb{R}^3} Q_3(F + a \otimes N) = |F| - \sum_{i=1}^3 \langle F, F_i \rangle_* F_i|^2 \]

\[ = 2\mu|F| - \sum_{i=1}^3 \langle F, F_i \rangle_* F_i|^2 + \lambda(\text{tr} F - \langle F, F_3 \rangle_* \text{tr} F_3)^2 \]

\[ = 2\mu\left( \sum_{i,j=1}^2 \langle F, e_i \otimes e_j \rangle^2 + \frac{\lambda}{2\mu + \lambda}\left( \sum_{i=1}^2 \langle F, e_i \otimes e_i \rangle \right)^2 \right) \]

\[ + \lambda\left( \frac{2\mu}{2\mu + \lambda}\right)^2 \left( \sum_{i=1}^2 \langle F, e_i \otimes e_i \rangle \right)^2 \]

\[ = 2\mu|G|_{\gamma_x}^2 + 2\mu\lambda \text{tr}^2 G. \]

\[ \square \]

Let \( k \) be a nonnegative integer and let \( T \in T^k(\Omega) \) be a \( k \)-th order tensor field on \( \Omega \). The internal product of \( X \) with \( T \) is a \( k - 1 \)-th order tensor field \( i(X)T \), defined by

\[ i(X)T(X_1, \cdots, X_{k-1}) = T(X, X_1, \cdots, X_{k-1}) \quad \text{for} \quad X_1, \cdots, X_{k-1} \in \mathcal{X}(M). \quad (7.7) \]
Lemma 7.2 Let $V \in IS^1(\Omega, \mathbb{R}^3)$ with $V = W + wN$. Then
\[
\Xi(V) = i(W)D\Pi + \Pi(DW, \cdot) + \Pi(\cdot, DW) + wT_0 - D^2w,
\]
where $\Xi(V)$ is given by (7.5), $D$ is the Levi-Civita connection of the induced metric $g$, and $\cdot$ denotes the position of variables.

Proof It follows from (7.3) that
\[
AX = DXW + w\hat{D}XN + [X(w) - \Pi(W, X)]N \quad \text{for} \quad X \in M_x, \, x \in \Omega.
\]
Then
\[
\langle AN, X \rangle = -\langle N, AX \rangle = \Pi(W, X) - W(w) \quad \text{for} \quad X \in M_x, \, x \in \Omega.
\]
Since $\langle AN, N \rangle = 0$, the identity (7.10) yields
\[
AN = i(W)\Pi - Dw.
\]

Let $x \in \Omega$ be given. We compute the identity (7.8) at the point $x$. Let $e_1, e_2$ be an orthonormal basis of $M_x$ such that
\[
\hat{D}e_iN = \lambda_iN, \quad \lambda_i = \Pi(e_i, e_i) \quad \text{for} \quad i = 1, 2.
\]
Let $E_1, E_2$ be a frame field normal at $x$ such that
\[
E_i = e_i \quad \text{at} \quad x \quad \text{for} \quad i = 1, 2.
\]
We have at $x$
\[
\Xi(V)(\tau_i, \tau_j) = \hat{D}^*(AN)(\tau_i, \tau_j) - A\Pi(\tau_i, \tau_j) = \langle D_{\tau_i}(AN), \tau_j \rangle - \langle A\Pi_{\tau_i}, \tau_j \rangle
\]
\[
= D\Pi(W, \tau_j, \tau_i) + \Pi(D_{\tau_i}W, \tau_j) - \tau_i\tau_j(w) + \lambda_i \langle \tau_i, A\tau_j \rangle
\]
\[
= D\Pi(W, \tau_j, \tau_i) + \Pi(D_{\tau_i}W, \tau_j) + \Pi(D_{\tau_j}W, \tau_i) + w\lambda_i\lambda_j - \tau_i\tau_j(w),
\]
which yields the identity (7.8). \qed

Remark 7.1 The identity (7.8) shows that the tensor field $\Xi(V)$, given by (7.4), is exactly the change of the linearized curvature tensor of the middle surface $\Omega$, introduced by [29], also see [3, 10] or [58], in the case of infinitesimal deformations.

Consider a deformation $\varphi : \Omega \rightarrow \mathbb{R}^3$. After the deformation, the middle surface becomes
\[
\bar{\Omega} = \{ \varphi(x) \mid x \in \Omega \}.
\]
Let $\bar{\Pi}$ be the second fundamental form of $\bar{\Omega}$. Then the change of curvature tensor of the middle surface is defined by
\[
G = \varphi^*\bar{\Pi} - \Pi,
\]

which is a 2-th tensor field on \( \Omega \).

Consider a small deformation
\[
\varphi(x) = x + V(x) \quad \text{for} \quad x \in \Omega
\]
with \( V = W + wN \in IS^1(\Omega, \mathbb{R}^3) \). Let \( N \) be the normal of \( \Omega \). After linearization ([58]), we have
\[
\tilde{N}(\varphi(x)) = i(W)\Pi - Dw + N \quad \text{for} \quad x \in \Omega.
\]

Let \( x \in \Omega \) be given and let \( E_1, E_2 \) be a frame field normal at \( x \) with the positive orientation. Then
\[
\begin{align*}
\text{Linearization} (\varphi^*\Pi - \Pi)(E_i, E_j) &= \text{Linearization} \left[ \left( \hat{D}_{\varphi, E_i} \tilde{N}, \varphi_* E_j \right) - \Pi(E_i, E_j) \right] \\
&= \left\langle \hat{D}_{E_i} [i(W)\Pi - Dw], E_j \right\rangle + \left\langle \hat{D}_i N, \varphi_* E_j \right\rangle - \Pi(E_i, E_j) \\
&= D\Pi(W, E_i, E_j) + \Pi(D_{E_i}W, E_j) + \Pi(D_{E_j}W, E_i) \\
&\quad + wT_0(E_i, E_j) - D^2 w(E_i, E_j) \quad \text{at} \quad x,
\end{align*}
\]
that is, by (7.8),
\[
\Xi(V) = \text{Linearization} (\varphi^*\Pi - \Pi).
\]

Let \( \Omega \subset M \) be elliptic and star-shaped with respect to \( o \in \Omega \). We further assume that for any \( \psi \in H^{1/2}(\Gamma) \) the problem (4.15) has a unique solution \( w = \lambda(\psi) \in H^1(\Omega) \). By Theorem 4.1, there is a unique \( W = \Lambda(\psi) \in H^1(\Omega, T) \) which is perpendicular to \( H^1_{\text{kl}}(\Omega, T) \) such that \( V = \Lambda(\psi) + \lambda(\psi)N \) is an infinitesimal isometry. Then for any \( V \in IS^1(\Omega, \mathbb{R}^3) \), we have a formula in the form of
\[
V = W + \Lambda(\psi) + \lambda(\psi)N \quad \text{for} \quad W \in H^1_{\text{kl}}(\Omega, T), \psi \in H^{1/2}(\Gamma).
\]
By Theorem 2.1, \( \dim H^1_{\text{kl}}(\Omega, T) \leq 3 \). Then the limit energy (7.4) of the \( \Gamma \)-convergence becomes a functional over a one-dimensional space
\[
I(V) = \hat{I}(\alpha, \psi) \quad \text{for} \quad (\alpha, \psi) \in \mathbb{R}^3 \times H^1(\Gamma).
\]
(7.10)

Similar situations happen when the meddle surface \( \Omega \) is parabolic or hyperbolic. It follows from Theorems 4.1, 5.1, and 6.1 that

**Theorem 7.1** Let the meddle surface \( \Omega \) be elliptic, or parabolic, or hyperbolic. Then the limit energy formula (7.4) of the \( \Gamma \)-convergence reduces to be a one-dimensional problem.

We shall write out explicit formulas of (7.10) for spherical shells and cylinder shells, respectively, before ending this section.
Bending of Spherical Shells Let $M$ be the sphere of curvature $\kappa > 0$ and let $g$ be the induced metric of $M$ from $\mathbb{R}^3$. Then the third fundamental form of $M$ is given by

$$T_0 = \kappa g.$$ 

Let $o \in M$ be fixed. Let $\rho(x) = \rho(x, o)$ be the distance function from $x \in M$ to $o$ in the induced metric $g$. For $0 < a \leq \frac{\pi}{\sqrt{\kappa}}$, let

$$\Omega(a) = \{ x \mid x \in M, \rho(x) < a \}, \quad \Gamma(a) = \{ x \mid x \in M, \rho(x) = a \}. \quad (7.11)$$

Let $V = W + wN$ be an infinitesimal isometry on $\Omega(a)$. By the formulas (7.8) and (4.35), we have

$$\Xi(V) = \sqrt{\kappa}(DW + D^*W) + \kappa wg - D^2w = -\kappa wg - D^2w. \quad (7.12)$$

In particular, for $V = W \in H^1_{1k}(\Omega, T)$ a Killing field,

$$\Xi(V) = 0.$$ 

By Theorem 4.1, $w \in H^1_{1k}(\Omega)$ if and only if $w$ solves the problem

$$\begin{cases} \Delta w + 2\kappa w = 0 & \text{for } x \in \Omega(a), \\ w = \psi & \text{for } x \in \Gamma(a). \end{cases} \quad (7.13)$$

Then it follows from (7.12) and (7.13) that

$$\text{tr} \Xi(V) = 0 \quad \text{for } x \in \Omega. \quad (7.14)$$

Furthermore, we have

**Lemma 7.3** Let $V = W + wN$ be an infinitesimal isometry with $w \in H^1_{1k}(\Omega)$. Then

$$|\Xi(V)|^2_{T^2} = \frac{1}{2} \Delta |Dw|^2 + \kappa \text{div} w \nabla w \quad \text{for } x \in \Omega. \quad (7.15)$$

**Proof** Recall that the Weitzenböck formula (Theorem 1.27 in [58]) reads

$$|D^2w|^2_{T^2} = \frac{1}{2} \Delta |Dw|^2 + \langle \Delta Dw, Dw \rangle - \text{Ric} \langle Dw, Dw \rangle \quad \text{for } x \in \Omega, \quad (7.16)$$

where $\Delta$ is the Hodge-Laplacian in the metric $g$ applying to vector fields and $\text{Ric} \langle \cdot, \cdot \rangle$ is the Ricci curvature tensor. Since $\text{Ric} = \kappa g$ and $\langle \Delta Dw, Dw \rangle = -\langle D(\Delta w), Dw \rangle$, we have, by (7.13) and (7.16),

$$|D^2w|^2_{T^2} = \frac{1}{2} \Delta |Dw|^2 + \kappa |Dw|^2 \quad \text{for } x \in \Omega. \quad (7.17)$$
From (7.12) and (7.17), we obtain
\[
|\Xi(V)|^2_{T^2} = |\kappa w g + D^2 w|^2_{T^2} = 2\kappa^2 w^2 + 2\kappa w \langle g, D^2 w \rangle_{T^2} + |D^2 w|^2_{T^2}
\]
\[
= \frac{1}{2} \Delta |Dw|^2 + \kappa (|Dw|^2 - 2\kappa w^2)
\]
\[
= \frac{1}{2} \Delta |Dw|^2 + \kappa \operatorname{div} w \nabla w \quad \text{for } x \in \Omega.
\]
\[\square\]

Let \( V_0^\perp(\Omega(a)) \) be given in (4.11). By Theorem 4.6, \( V_0^\perp(\Omega(a)) = H^1 h(\Omega(a)) \) for \( 0 < a < \frac{\pi}{2\sqrt{\kappa}} \). We define a linear operator \( \Theta : L^2(\Gamma(a)) \to L^2(\Gamma(a)) \) by
\[
\Theta \psi = w_\rho,
\]
where \( w \in V_0^\perp(\Omega) \) is the solution to the problem (7.13). Then \( D(\Theta) = H^{1/2}(\Gamma(a)) \) for \( 0 < a \leq \frac{\pi}{\sqrt{\kappa}} \).

**Theorem 7.2** Let \( \Omega(a) \) and \( \Gamma(a) \) be given in (7.11). Then the bending energy (7.4) of the \( \Gamma \)-convergence becomes the following one-dimensional problem
\[
\tilde{I}(\psi) = \frac{\mu}{12} \int_{\Gamma(a)} [2\psi_\tau(\Theta \psi) \tau - \kappa \psi \Theta \psi - \sqrt{\kappa} \cotg(\sqrt{\kappa})(|\Theta \psi|^2 + |\psi_\tau|^2)]d\Gamma \quad (7.18)
\]
for \( \psi \in H^{1/2}(\Gamma(a)) \), where \( \tau \) is the unit tangential vector field along \( \Gamma(a) \).

**Proof** Let \( \tau = \tau(\rho) \) be the unit tangential vector field along \( \Gamma(\rho) \) for \( 0 < \rho \leq a \). Then \( D\rho, \tau \) forms a frame field on \( \Omega(a) \). We have
\[
D_{D\rho} D\rho = 0, \quad D_{D\rho} \tau = 0, \quad (7.19)
\]
\[
D_\tau D\rho = \sqrt{\kappa} \rho \cotg(\sqrt{\kappa}\rho)\tau, \quad D_\tau \tau = -\sqrt{\kappa} \rho \cotg(\sqrt{\kappa}\rho)D\rho. \quad (7.20)
\]
Moreover, the equation in (7.13) gives
\[
D^2 w(D\rho, D\rho) = -2\kappa w - D^2 w(\tau, \tau)
\]
\[
= -w_{\tau\tau} - \sqrt{\kappa} \rho \cotg(\sqrt{\kappa}\rho)w_\rho - 2\kappa w \quad \text{for } x \in \Gamma(a). \quad (7.21)
\]
It follows from the formulas (7.15) and (7.19)-(7.21) that
\[
\int_{\Omega(a)} |\Xi(V)|^2_{T^2}d\Gamma = \int_{\Gamma(a)} [D^2 w(D\rho, D\rho) + \kappa w w_\rho]d\Gamma
\]
\[
= \int_{\Gamma(a)} [w_\rho D^2 w(D\rho, D\rho) + w_\tau (w_{\rho\tau} - \langle D\rho, D_\tau \rangle) + \kappa w w_\rho]d\Gamma
\]
\[
= \int_{\Gamma(a)} [2w_\tau w_{\rho\tau} - \sqrt{\kappa} \rho \cotg(\sqrt{\kappa}\rho)(w^2_\rho + w^2_\tau) - \kappa w w_\rho]d\Gamma. \quad (7.22)
\]
Finally, we use the formulas (7.22), (7.14) and (7.6) in the formula (7.4) to obtain (7.18).

\[ \Omega = \{ (\cos \theta, \sin \theta, z) \mid \theta \in [-\pi, \pi), |z| < a \}. \] (7.23)

Then
\[ \partial z = (0, 0, 1), \quad \partial \theta = (-\sin \theta, \cos \theta, 0). \]

Let \( w \in H^1_\text{is}(\Omega) \) be given. By Theorem 5.1,
\[ w = w_0 + w_1 z, \quad w_0, w_1 \in H^1(T). \]

Let \( W \in \mathcal{X}(\Omega) \) be such that \( V = W + wN \) is an infinitesimal isometry. A simple computation shows that
\[ W = \left[ \int_0^\theta (\theta - \eta)w_1(\eta)d\eta + c_1 \right] \partial z - \left[ \int_0^\theta [w_0(\eta) + w_1(\eta)z]d\eta + c_2 \right] \partial \theta, \]
where \( c_1, c_2 \) are constants and
\[ \Xi(V)(\partial z, \partial z) = 0, \quad \Xi(V)(\partial z, \partial \theta) = -\int_0^\theta w_1(\eta)d\eta - w_1 \theta, \]
\[ \Xi(V)(\partial \theta, \partial \theta) = -w - w_{\theta \theta}. \]

Using the above formulas, we obtain

**Theorem 7.3** Let \( \Omega \) be given by (7.23). Then the bending energy (7.4) of the Gamma-convergence becomes the following one-dimensional formula
\[ \tilde{I}(w_0, w_1) = \int_{-\pi}^\pi \left\{ \frac{\mu a}{3} \left[ \frac{\mu + \lambda}{2\mu + \lambda} (w_0 + w_{\theta \theta})^2 + (w_{1\theta} + \int_0^\theta w_1(\eta)d\eta)^2 \right] + \frac{\mu(\mu + \lambda)a^3}{3(2\mu + \lambda)}(w_1 + w_{1\theta \theta})^2 \right\} d\theta \quad \text{for} \quad (w_0, w_1) \in H^1(T) \times H^1(T). \]

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