GLOBAL UNIFORM IN N ESTIMATES FOR SOLUTIONS OF A SYSTEM OF HARTREE–FOCK–BOGOLIUBOV TYPE IN THE GROSS–PITAEVSKII REGIME.

XIAOQI HUANG

Abstract. We extend the recent work of Chong et al., 2022 [7] to the critical case. More precisely, we prove global in time, uniform in $N$ estimates for the solutions $\phi$, $\Lambda$ and $\Gamma$ of a coupled system of Hartree–Fock–Bogoliubov type with interaction potential $\frac{1}{N}V_N(x-y) = N^2v(N(x-y))$. We assume that the potential $v$ is small which satisfies some technical conditions, and the initial conditions have finite energy. The main ingredient is a sharp estimate for the linear Schrödinger equation with potential in 6+1 dimension, which may be of interest in its own right.

1. Introduction.

Consider the $N$–body linear Schrödinger equation which governs the time-evolution of $N$ boson systems

\begin{equation}
\left( \frac{1}{i} \frac{\partial}{\partial t} - \sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N} V_N(x_i-x_j) \right) \psi_N(t, \cdot) = 0,
\end{equation}

where $x_i \in \mathbb{R}^3$, $N$ is large and $V_N(x) = N^3v(Nx)$ for some Schwarz class potential $v$. The conditions on the potential will be discussed below. A physically appealing case concerns initial data forming a tensor product of the same one–particle state, in spirit of the Bose-Einstein condensation. We refer to [20] for extensive background on Bose-Einstein condensation.

The goal is to find a rigorous, simple approximation to $\psi_N$ which is consistent with

\begin{equation}
\psi_{\text{approx}}(t, x_1, \ldots, x_N) \sim \phi(t, x_1)\phi(t, x_2)\ldots\phi(t, x_N)
\end{equation}

in an appropriate sense, where $\phi$ is often called the mean–field limit.

In the stationary case, a survey of results concerning the ground state properties of the dilute bosonic gases can be found in [20]. In the time dependent case, in the work of Erdös, Schlein and Yau [10, 11, 12], by using the BBGKY hierarchies and the density matrix $\gamma_{N,t}$ formalism, the convergence of the exact dynamics to the mean–field limit is asserted in the trace norm as $N \to \infty$, provided that the the mean–field limit $\phi(t, x)$ satisfies the Gross–Pitaevskii equation

\begin{equation}
\frac{1}{i} \frac{\partial}{\partial t} \phi - \Delta \phi + 8\pi a_0 |\phi|^2 \phi = 0.
\end{equation}

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Here $a_0$ is the scattering length of the potential $v$, physically, the scattering length measures the effective range of the potential $V$, see e.g., \cite{12} for a precise definition of scattering length. Also in the recent work of Pickl \cite{21}, the Gross–Pitaevskii equation is derived using a different method. We refer to reader to \cite{3, 2, 23, 9, 4} for more backgrounds on the problem of approximating the many-body Schrödinger dynamics and related results in the Gross–Pitaevskii regime.

The Hartree-Fock-Bogoliubov type equations are derived in Fock space, which has been used in order to get estimates for the rate of convergence of the approximation to the exact solution. In the mean-field regime, we also refer to the earlier work of Rodnianski and Schlein \cite{23}, which gives the convergence rate for marginal densities $\gamma_{N,t}$, and the work of Lewin, Nam and Schlein \cite{19}, which gives efficient direct estimates in $L^2(\mathbb{R}^N)$ using the Hartree state approach. In the Gross–Pitaevskii regime, we refer to the work of Brennecke and Schlein \cite{6} for explicit rate of condensation.

The ideas behind the use of HFB system is to introduce pair excitations (given by a Bogoliubov transformation $e^B$) as a second order correction to the mean field approximation. The exact evolution of (1.1) is approximated by a construction involving two function: the condensate $\phi(t, x)$ and a pair excitation function $k(t, x, y)$, and it is

$$\psi_{\text{approx}} := e^{-\sqrt{N}A(\phi(t))}e^B(k(t))\Omega$$

where

$$A(\phi) = \int dx \{ \bar{\phi}(x)a_x - \phi(x)a_x^* \}$$

with $a_x^*$ and $a_x$ being the creation and annihilation operator in Fock space, and $e^{-\sqrt{N}A(\phi)}$ is a unitary operator on Fock space, the Weyl operator. And

$$B(k) = \frac{1}{2} \int dxdy \{ \bar{k}(t, x, y)a_xa_y - k(t, x, y)a_x^*a_y^* \}.$$  

The unitary operator $e^{B(k)}$ is the representation of an real symplectic matrix. Also, $\Omega$ is the vacuum. See for instance \cite{14} for background on this construction.

In order for $\psi_{\text{approx}}$ to be an approximation to the exact evolution, $\phi$ and $k$ must satisfy certain PDEs. They were introduced in \cite{14} and independently in a different context in \cite{1}, see also \cite{16}, \cite{15} for more details on the argument motivating the use of Hartree–Fock–Bogoliubov type equations. We also mention that in Benedikter, de Oliveira and Schlein \cite{2}, a similar approach is considered, where the authors impose the Gross–Pitaevskii equation equation for $\phi$ and define $k$ by an explicit formula, and give a rate of convergence result in terms of the marginal density.

In the remaining of this paper, we shall focus on the analysis of the systems of PDEs. In particular, our main result, which we used in the analysis of nonlinear system, is the estimate for linear Schrödinger equation stated in the next section. The linear estimate in section 2 generalizes the recent work of Chong et al., 2022 \cite{7} to the critical case.

The functions described by these equations are: the condensate $\phi(t, x)$ and the density matrices

$$\Gamma(t, x_1, x_2) = \frac{1}{N} \left( \text{sh}(k) \circ \text{sh}(k) \right)(t, x_1, x_2) + \bar{\phi}(t, x_1)\phi(t, x_2)$$
\begin{equation}
\Lambda(t, x_1, x_2) = \frac{1}{2N} sh(2k)(t, x_1, x_2) + \phi(t, x_1)\phi(t, x_2)
\end{equation}

where

\begin{equation}
sh(k) = k + \frac{1}{3!} k \circ k + \ldots \\
ch(k) = \delta(x - y) + \frac{1}{2!} k \circ k + \ldots
\end{equation}

Here \((u \circ v)(x, y) = \int u(x, z)v(z, y)dz\), the pair excitation function \(k\) is an auxiliary function, which does not explicitly appear in the system.

There are several equivalent ways of expressing the equations, in this section we shall use a compact, matrix formulation as in [7]. We separate the condensate part from the pair interaction part: define \(\Gamma_c = \phi \otimes \phi\), \(\Lambda_c = \phi \otimes \phi\), \(\Gamma_p = \frac{1}{N} sh(k) \circ sh(k)\) and \(\Lambda_p = \frac{1}{2N} sh(2k)\). Also denote \(\rho(t, x) = \Gamma(t, x, x)\)

To write the Hartree-Fock-Bogoliubov equations in matrix notation, denote

\[V_N(x - y) = N^3 v(N(x - y))\]

for some Schwarz class potential which will be discussed below. Define

\[\Omega = \begin{pmatrix} -\Gamma & -\Lambda \\ \Lambda & \Gamma \end{pmatrix} = \Psi + \Phi\]

where

\[\Psi = \begin{pmatrix} -\Gamma_p & -\Lambda_p \\ \Lambda_p & \Gamma_p \end{pmatrix}\]
\[\Phi = \begin{pmatrix} -\Gamma_c & -\Lambda_c \\ \Lambda_c & \Gamma_c \end{pmatrix}\]

Finally, let

\[S_3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\]

where \(I\) is the identity operator.

The evolution equations for \(\Psi\) and \(\Phi\) are

\begin{equation}
\frac{1}{i} \partial_t \Phi - [\Delta_x \delta(x - y)S_3, \Phi] = -[(V_N \ast \rho(t, x))\delta(x - y)S_3, \Phi] - [V_N \Psi^*, \Phi]
\end{equation}

\begin{equation}
\frac{1}{i} \partial_t \Psi - [\Delta_x \delta(x - y)S_3, \Psi] = -[(V_N \ast \rho(t, x))\delta(x - y)S_3, \Psi] - \frac{1}{2N}[S_3, V_N \Psi] - [V_N \Omega^*, \Psi] - \frac{1}{2N}[S_3, V_N \Phi]
\end{equation}
In addition, the condensate \( \phi \) satisfies

\[
\left\{ \frac{1}{i} \partial_t - \Delta_{x_1} \right\} \phi(x_1) = - \int dy \{ V_N(x_1 - y) \Gamma(y, y) \} \phi(x_1) - \int dy \{ V_N(x_1 - y) \Gamma_p(y, x_1) \} \phi(y) + \int dy \{ V_N(x_1 - y) \Lambda_p(x_1, y) \} \bar{\phi}(y)
\]

(1.12)

Here \( A^\ast(x, y) = \bar{A}(y, x), [A, B] = A \circ B - B \circ A \) and \( V_N \) acts as pointwise multiplication by \( V_N(x - y) \). See (5.1)-(5.4) for a scalar form of the above equations.

The solutions \( \phi, \Lambda_p, \Lambda_c, \Gamma_p \) and \( \Gamma_c \) all depend on \( N \). This has been suppressed to simplify the notation. However, we will always keep track of dependence on \( N \) in our estimates.

Next we review the conserved quantities of these equations, see [14] for more details. The first conserved quantities is the total number of particles (normalized by division by \( N \)):

\[
\text{tr}\{\Gamma(t)\} = \| \phi(t, \cdot) \|_{L^2(dx)}^2 + \frac{1}{N} \| \text{sh}(k(t, \cdot, \cdot)) \|_{L^2(dx, dy)}^2 = 1
\]

(1.13)

From here we see that

\[
\| \Lambda(t, \cdot, \cdot) \|_{L^2(dx, dy)} \leq C
\]

(1.14)

The second conserved quantity is the energy per particle

\[
E(t) = \text{tr}\{\nabla_{x_1} \cdot \nabla_{x_2} \Gamma(t)\}
\]

\[
= \frac{1}{2} \int dx_1 dx_2 \left\{ V_N(x_1 - x_2) |\Lambda(t, x_1, x_2)|^2 \right\}
\]

\[
+ \frac{1}{2} \int dx_1 dx_2 \left\{ V_N(x_1 - x_2) |\Gamma(t, x_1, x_2)|^2 \right\}
\]

\[
+ \frac{1}{2} \int dx_1 dx_2 \left\{ V_N(x_1 - x_2) \Gamma(t, x_1, x_1) \Gamma(t, x_2, x_2) \right\}
\]

\[
- \int dx_1 dx_2 \left\{ V_N(x_1 - x_2) |\phi(t, x_1)|^2 |\phi(t, x_2)|^2 \right\}
\]

(1.15)

The above holds for any Schwarz potential \( v \). In addition, in order to use the estimates of [8], we assume \( v \) is spherically symmetric and

\[
v \geq 0, \frac{\partial v}{\partial r}(r) \leq 0.
\]

(1.16)

For the initial conditions, we assume that

\[
\text{tr}\{\Gamma(0)\} \leq C_0
\]

\[
E(0) \leq C_0
\]

(1.17)
The size of the initial condition $C_0$ is important, as we shall see in the end of section 5, our assumption on the size of the potential $v$ depends on the size of initial data. Thus, we keep track of the constant $C_0$ from now on.

Note that the kinetic energy is

$$\text{tr}\{\nabla_{x_1} \cdot \nabla_{x_2} \Gamma(t)\} = \int dx \left\{ |\nabla_x \phi(t,x)|^2 \right\} + \frac{1}{2N} \int dx_1 dx_2 \left\{ |\nabla_{x_1} s(k)(t,x_1,x_2)|^2 + |\nabla_{x_2} s(k)(t,x_1,x_2)|^2 \right\}$$

(1.18)

If we assume $E \leq C_0$, then we have an $H^1$ estimate for $\Lambda$, uniformly in time (and $N$):

$$\int dx_1 dx_2 \left\{ |\nabla_{x_1} \Lambda(t,x_1,x_2)|^2 + |\nabla_{x_2} \Lambda(t,x_1,x_2)|^2 \right\} \leq CC_0$$

(1.19)

Also, $\Gamma$ satisfies the $H^2$ type estimate

$$|||\nabla_x|| \nabla_{xy} \nabla \Gamma(t)||_{L^2(dx dy)} \leq CC_0$$

(1.20)

See [15], [14], as well as [1] for these conserved quantities.

By Plancherel theorem, we see that (1.19) implies that for all time $t$

$$\|\langle \nabla_x \rangle^{\frac{3}{2}} \langle \nabla_y \rangle^\frac{1}{2} \Lambda_p(t)\|_{L^2(dx dy)} \leq CC_0.$$ 

(1.21)

as well as

$$\|\langle \nabla_x \rangle^{\frac{3}{2}} \langle \nabla_y \rangle^\frac{1}{2} \Lambda_c(t)\|_{L^2(dx dy)} \leq CC_0.$$ 

(1.22)

Here, and also later on, $\langle \nabla_x \rangle^\frac{3}{2}$ means $\left(1 - \Delta_x\right)^{\frac{3}{4}}$, which is a Fourier multiplier with symbol $(1 + |\xi|)^{\frac{3}{4}}$, and similarly for $\langle \nabla_y \rangle^\frac{3}{2}$.

Similarly, by (1.18) and Plancherel, we also have

$$\|\langle \nabla_x \rangle^{\frac{3}{2}} \langle \nabla_y \rangle^\frac{1}{2} \Gamma_p(t)\|_{L^2(dx dy)} \leq CC_0.$$ 

(1.23)

as well as

$$\|\langle \nabla_x \rangle^{\frac{3}{2}} \langle \nabla_y \rangle^\frac{1}{2} \Gamma_c(t)\|_{L^2(dx dy)} \leq CC_0.$$ 

(1.24)

In order to state the main result for this paper in the simplest possible form, we define the following partial Strichartz norms:

$$\|\Lambda\|_{S_{x,y}} = \sup_{(p,q) \text{ admissible}} \|\Lambda\|_{L^p(dt)L^q(dx)L^2(dy)} + \|\Lambda\|_{L^p(dt)L^q(dy)L^2(dx)}$$

(1.25)

Recall $(p,q)$ are admissible in $3 + 1$ dimensions if $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$, $2 \leq p \leq \infty$.

The main result of this paper is
Theorem 1.1. Let $\Lambda = \Lambda_p + \Lambda_c$, $\Gamma = \Gamma_p + \Gamma_c$ be solutions of (1.10), (1.11), while the potential satisfies (1.16) and (2.12) below, and the initial conditions satisfy (1.17). Then we have

$$\|(\nabla_x)^{\frac{1}{2}}(\nabla_y)^{\frac{1}{2}}\Lambda\|_{S_{x,y}} + \|(\nabla_x)^{\frac{1}{2}}(\nabla_y)^{\frac{1}{2}}\Gamma\|_{S_{x,y}} \leq C$$

for some constant $C$ independent of $N$. The above estimate still hold if we replace $\Lambda$ by $\Lambda_c$ or $\Lambda_p$, or replace $\Gamma$ by $\Gamma_c$ or $\Gamma_p$.

We also have a theorem for $\text{sh}(2k)$ (without dividing by $N$).

Theorem 1.2. Let $\Lambda, \Gamma, \phi$ be solutions of (1.10), (1.11), while the potential satisfies (1.16) and (2.12) below, and the initial conditions satisfy (1.17). Assume also that

$$\|\text{sh}(2k)(0, \cdot, \cdot)\|_{L^2} + \|\text{sh}(k) \circ \text{sh}(k)(0, \cdot, \cdot)\|_{L^2} \leq C$$

Then we have

$$\|\text{sh}(2k)\|_{S_{x,y}} + \|\text{sh}(k) \circ \text{sh}(k)\|_{S_{x,y}} \leq C$$

Remark 1.3. Here (1.27) improves the results in [7], Theorem 1.3 in two ways. First, the potential $N^2 v(N(x - y))$ we considered here represents a stronger interaction between particles, compared with the $N^{3\beta - 1}v(N^\beta(x - y))$, $\beta < 1$ type potentials considered in [7]. Second, the $\|\text{sh}(2k)\|_{S_{x,y}}$ norm stays bounded uniformly in $N$, compared with the log $N$ growth in [7]. Although our argument is written for the case where the potential is $N^2 v(N(x - y))$, it also works for the $N^{3\beta - 1}v(N^\beta(x - y))$ case. For example, the uniform in $N$ estimates in (1.27) still hold for the case where the potential is equal to $N^{3\beta - 1}v(N^\beta(x - y))$.

The above estimates also imply some estimates for $\text{sh}(k)$. In particular,

$$\|\text{sh}(k)\|_{L^p(dx)L^2(dy)} \leq C\|\text{sh}(2k)\|_{L^p(dx)L^2(dy)}$$

This is because $\text{sh}(k) = \frac{1}{2}\text{sh}(2k) \circ \text{ch}(k)^{-1}$ and $\text{ch}(k)^{-1}$ has bounded operator norm.

Finally, we also have estimates for $\phi$. Define the standard Strichartz spaces

$$\|\phi\|_S = \sup_{(p,q) \text{ admissible}} \|\phi\|_{L^p(dt)L^q(dx)}.$$ 

Corollary 1.4. Under the assumptions of Theorem 1.1, and the additional assumption $\|(\nabla)^{\frac{1}{2}}\phi(0, \cdot)\|_{L^2} \leq C$, we have

$$\|(\nabla)^{\frac{1}{2}}\phi\|_{S} \leq C.$$ 

We shall mention the difficulties surrounding equation (1.10) and (1.11). Denote

$$S = \frac{1}{i} \frac{\partial}{\partial t} - \Delta_x - \Delta_y$$

$$S_{\pm} = \frac{1}{i} \frac{\partial}{\partial t} - \Delta_x + \Delta_y$$
Schematically, treating $V_N$ as $\delta$ and ignoring the constants, the equations become

\begin{align}
S\Lambda_c &= \Gamma(t, x, x)\Lambda_c(t, x, y) + \Lambda_p(t, x, x)\Gamma_c(t, x, y) \\
S\Lambda_p &= \frac{V_N}{N}\Lambda_p = \Gamma(t, x, x)\Lambda_p(t, x, y) + \Lambda_p(t, x, x)\Gamma_p(t, x, y) - \frac{V_N}{N}\Lambda_c \\
S\Lambda_p &= \Gamma(t, x, x)\Lambda_p(t, x, y) + \Lambda_c(t, x, x)\Gamma_p(t, x, y) \\
S\Lambda_c &= \Gamma(t, x, x)\Gamma_c(t, x, y) + \Lambda_c(t, x, x)\Lambda_c(t, x, y)
\end{align}

Recall that $V_N(x) = N^3u(Nx)$, and $\frac{V_N}{N}$ in the $\Lambda_p$ equation satisfies the critical scaling in the sense that $\frac{V_N}{N} \in L^{3/2}$ uniformly in $N \geq 1$. In the case of one body problem, there is a vast amount of literature in the study of Schrödinger operators $-\Delta + V$ with critically singular potentials from different aspects see e.g., [24, 22, 18, 5, 17]. And $V \in L^{n/2}$ for dimension $n \geq 3$ is almost the minimal condition to ensure the Schrödinger operators $-\Delta + V$ is bounded from below and self-adjoint.

For the two-body problem in our case, borrowing ideas in [7], our method for treating the nonlinear term requires the Strichartz-type estimates for $\langle \nabla_x \rangle^\alpha \langle \nabla_y \rangle^\alpha \Lambda_p$ or $c$ and $\langle \nabla_x \rangle^\alpha \langle \nabla_y \rangle^\alpha \Gamma_p$ or $c$ with $\alpha \geq \frac{1}{2}$. However, the $\langle \nabla_x \rangle^\alpha \langle \nabla_y \rangle^\alpha$ derivative is the threshold for the linear equation in the sense that if we apply $\langle \nabla_x \rangle^\alpha \langle \nabla_y \rangle^\alpha$ with $\alpha > \frac{1}{2}$ to the terms $\frac{V_N}{N}\Lambda_p$ or $\frac{V_N}{N}\Lambda_c$ in the $\Lambda_p$ equation, we get a singularity which is essentially $N^{2\alpha-1}V_N\Lambda$. And since $\|N^{2\alpha-1}V_N\|_{L^p} \to \infty$ as $N \to \infty$ for any $1 \leq p \leq \infty$, such terms can not be treated using standard techniques.

The structure of the rest of the paper is the following. In section 2, we list the notations used in this paper and state our main estimate, Theorem 2.1 for the linear Schrödinger equation in $6 + 1$ dimension. In sections 3 and 4, we prove Theorem 2.1. In section 5, we prove Theorem 1.1 using the linear estimate Theorem 2.1. In section 6 and section 7, we use prove Theorem 1.2 and Corollary 1.4. In the last section, we use Theorem 2.1 to prove a “collapsing estimate” for the linear equation, which may be of interest on its own right.

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2. List of notations and statement of the main linear estimates.

Let us define the partial Strichartz norms

\begin{align}
\|\Lambda\|_{S_{x,y}} &= \sup_{(p,q) \text{ admissible}} \|\Lambda\|_{L^p(dt)L^q(dx)L^2(dy)} \\
&\quad + \sup_{(p,q) \text{ admissible}} \|\Lambda\|_{L^p(dt)L^q(dy)L^2(dx)}
\end{align}

where the pair $(p, q)$ is admissible in $3 + 1$ dimension if $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$, $2 \leq p \leq \infty$. 

Define the full Strichartz norm
\begin{equation}
\|\Lambda\|_S = \sup_{(p,q) \text{ admissible}} \|\Lambda\|_{L^p(dt)L^q(dx)L^2(dy)} + \sup_{(p,q) \text{ admissible}} \|\Lambda\|_{L^p(dt)L^q(dy)L^2(dx)} + \sup_{(p,q) \text{ admissible}} \|\Lambda\|_{L^p(dt)L^q(d(x-y))L^2(d(x+y))}.
\end{equation}

And define the restricted dual Strichartz norm, excluding the end-points \( p' = 2, p' = 1 \):

\begin{equation}
\|G\|_{S'} = \inf_{p_1 \geq p \geq p_0} \{ \|G\|_{L^{p_1}(dt)L^{p_2}(dx)L^2(dy)} \mid \|G\|_{L^{p_1}(dt)L^{p_2}(dy)L^2(dx)} \}.
\end{equation}

Let us also recall the standard Littlewood-Paley decomposition. Let \( \phi(x) \) such that \( \hat{\phi} \in C_0^\infty \) and \( \hat{\phi}(\xi) = 1 \) in \( |\xi| < 1 \), \( \hat{\phi}(\xi) = 0 \) in \( |\xi| > 2 \). Define the \( \phi_k \) for \( k \geq 0 \) by \( \hat{\phi}_k(\xi) = \hat{\phi}(\frac{\xi}{2^k}) \) and denote
\begin{equation}
P_{|\xi| < 2^k}f = f \ast \phi_k
\end{equation}
so that the inverse Fourier transform of \( \hat{\phi}(\frac{\xi}{2^k}) \hat{f} \) is \( P_{|\xi| < 2^k}f \).

Next let \( \psi_0 = \phi \) and define \( \psi_k \) for \( k \geq 1 \) by \( \hat{\psi}_k(\xi) = \hat{\phi}(\frac{\xi}{2^k}) - \hat{\phi}(\frac{\xi}{2^{k+1}}) \). We also denote
\begin{equation}
P_{|\xi| > 2^k}f = f \ast \psi_k
\end{equation}
For later use, we shall also abuse our notation a bit and define, for an arbitrary positive constant \( M \),
\begin{equation}
P_{|\xi| < M}f = f \ast \phi(Mx),
\end{equation}
for any fixed constant \( M \). And define,
\begin{equation}
P_{|\xi| \geq M}f = f - P_{|\xi| < M}f.
\end{equation}
So for any two fixed constants \( 0 < M < N \),
\[ P_{N \leq |\xi| < M}f = P_{|\xi| < M}f - P_{|\xi| < N}f. \]

Now we can define the following two “collapsing norms”. Let
\begin{equation}
\|\Lambda\|_{\text{collapsing}} = \|\Lambda\|_{L^\infty(d(x-y))L^2(dt)L^2(dx+y))'} \end{equation}
And define
\begin{equation}
\|\Lambda\|_{\text{low collapsing}} = \|P_{|\xi-\eta| < 20N}\Lambda\|_{\text{collapsing}} + \|P_{|\xi| < 20N}\Lambda\|_{\text{collapsing}} + \|P_{|\eta| < 20N}\Lambda\|_{\text{collapsing}}.
\end{equation}
If \( A \lesssim B \), there is a constant \( C \) such that \( A \leq CB \), and we use \( A \sim B \) to denote the case when \( A \lesssim B \) and \( B \lesssim A \).

Define \( \langle \nabla_x \rangle^{\frac{3}{2}}f = (1-\Delta_x)^{\frac{3}{4}}f \) such that the Fourier transform of \( \langle \nabla_x \rangle^{\frac{3}{2}}f \) is \( (1+|\xi|^2)^{\frac{3}{4}}\hat{f} \), and similarly the Fourier transform of \( \langle \nabla_y \rangle^{\frac{3}{2}}f \) is \( (1+|\eta|^2)^{\frac{3}{4}}\hat{f} \) for any \( f \in L^2(\mathbb{R}^6) \).
Let $x, y \in \mathbb{R}^3$, define
\[
S = \frac{1}{i} \frac{\partial}{\partial t} - \Delta_x - \Delta_y
\]
(2.10)
\[
S_\pm = \frac{1}{i} \frac{\partial}{\partial t} - \Delta_x \pm \Delta_y
\]

Consider the equation
\[
S\Lambda(t, x, y) = N^2 v(N(x - y))\Lambda(t, x, y) + G(t, x, y)
\]
(2.11)
\[
\Lambda(0, \cdot) = \Lambda_0
\]
where we shall assume that
\[
v \text{ is Schwarz sup } \parallel v \parallel_p < \varepsilon, \text{ and supp } \hat{v} \subset B_1(0).
\]
(2.12)
Here $\hat{v}$ denotes the Fourier transform and $B_1(0)$ denotes the unit ball in $\mathbb{R}^3$, and $\varepsilon$ is a fixed small constant to be specified later which depends on $C$.

Remark 2.2. The main difficulty in proving the theorem is the presence of the term $N^2 v(N(x - y))\Lambda(t, x, y)$, where $N^2 v(N(x - y)) \in L^{3/2}$ satisfies the critical scaling. The term $N^2 v(N(x - y))\Lambda(t, x, y)$ is a technical term which arises from the term $N^2 v(N(x - y))\Lambda(t, x, y)$, since in our application, we split $\Lambda = \Lambda_p + \Lambda_c$ and take $\Lambda = \Lambda_p$ and $H = \Lambda_c$ in (2.11). And the presence of $H(t, x, y)$ does not lead to any essential difficulty in the proof of the above theorem.

We also remark that in the case $H = 0$, by using the above theorem plus an abstract argument, one can replace the $\parallel \cdot \parallel_{\text{low collapsing}}$ norm on the left side of (2.13) by the $\parallel \cdot \parallel_{\text{collapsing}}$ norm and the same result still holds. See the discussion in section 8 for more details.

The simplest form of theorem is

Theorem 2.1. Let $\Lambda$ satisfy (2.11), and assume $v$ satisfy (2.12), we have
\[
\left\| \langle \nabla_x \rangle^{1/2} \langle \nabla_y \rangle^{1/2} \Lambda \right\|_{L^2(x,y)} + \left\| \langle \nabla_x \rangle^{1/2} \Lambda \right\|_{\text{low collapsing}} + \left\| \partial_t \langle \nabla_y \rangle^{1/2} \Lambda \right\|_{\text{low collapsing}}
\]
(2.13)
\[
\lesssim \left\| \langle \nabla_x \rangle^{1/2} \langle \nabla_y \rangle^{1/2} G \right\|_{L^6(x,y) L^2(d(x,y))} + \varepsilon \left\| \langle \nabla_x \rangle^{1/2} \langle \nabla_y \rangle^{1/2} H \right\|_{L^2(x,y) L^2(d(x,y))} + \varepsilon \left\| \partial_t \langle \nabla_y \rangle^{1/2} H \right\|_{\text{collapsing}} + \varepsilon \left\| \langle \nabla_x \rangle^{1/2} H \right\|_{\text{collapsing}}
\]
Remark 2.2. The main difficulty in proving the theorem is the presence of the term $N^2 v(N(x - y))\Lambda(t, x, y)$, where $N^2 v(N(x - y)) \in L^{3/2}$ satisfies the critical scaling. The term $N^2 v(N(x - y))\Lambda(t, x, y)$ is a technical term which arises from the term $N^2 v(N(x - y))\Lambda(t, x, y)$, since in our application, we split $\Lambda = \Lambda_p + \Lambda_c$ and take $\Lambda = \Lambda_p$ and $H = \Lambda_c$ in (2.11). And the presence of $H(t, x, y)$ does not lead to any essential difficulty in the proof of the above theorem.

We also remark that in the case $H = 0$, by using the above theorem plus an abstract argument, one can replace the $\parallel \cdot \parallel_{\text{low collapsing}}$ norm on the left side of (2.13) by the $\parallel \cdot \parallel_{\text{collapsing}}$ norm and the same result still holds. See the discussion in section 8 for more details.

All the implicit constants in $\lesssim$ are independent of $N$ and $\varepsilon$, and the choice of the small constant $\varepsilon$ in (2.12) will depend on the implicit constants and $C_0$ in (1.17).

3. Preliminary estimates for solutions to the linear Schrödinger equation.

We will use the following Strichartz estimate. In 6+1 dimensions,

Theorem 3.1 (Theorem 2.4, 2.5 of [3]). Let $Su = f + g$, $u(0, \cdot) = u_0$. Then
\[
\left\| u \right\|_S \lesssim \left\| f \right\|_{L^2(dt) L^{6}(x,y) L^2(d(x,y))} + \left\| g \right\|_{S'} + \left\| u_0 \right\|_{L^2}.
\]
Now we shall present the main theorem of this section.

**Theorem 3.2.** Let $Su = f$ $u(0, \cdot) = 0$ Then

$$\|u\|_{S_{x,y}} \lesssim \| \langle \nabla_{x+y} \rangle^{\frac{\alpha}{2}} f \|_{L^1(d(x-y))L^2(dt)\hat{L}^2(d(x+y))} + \| \| \partial_t \|_{L^1(d(x-y))L^2(dt)\hat{L}^2(d(x+y))}. $$

**Remark 3.3.** A frequency localized version of the above Theorem appears in [4], Proposition 4.7, which is also a motivation of the above theorem. The $S_{x,y}$ norm is crucial here, we do not expect the above estimate to be true if we replace it by the full Strichartz norm $\mathcal{S}$ as in (2.2).

**Proof.** The main ideas is to divide the frequency support of $u$ into several regions, and use Strichartz estimate for the regions where $\tau \sim |\xi|^2 + |\eta|^2$, and use Sobolev for the remaining regions. As we shall see later, the proof of Theorem 3.4-3.5 below uses essentially the same idea.

To begin with, we shall use the decomposition $u = \sum_{k=0}^{\infty} P|\xi-\eta|\sim 2^k u$, where for the case $k = 0$, we are abusing notations a bit by letting $P|\xi-\eta|\sim 1 u$ to denote the operator $P|\xi-\eta|\sim 1 u$. We have the square function estimate (see e.g., Lemma 3.5 in [7]).

$$\|u\|_{S_{x,y}} \sim \| \left( \sum_{k=0}^{\infty} |P|\xi-\eta|\sim 2^k u|\right)^2 \|_{S_{x,y}} \lesssim \left( \sum_{k=0}^{\infty} \|P|\xi-\eta|\sim 2^k u\|_{S_{x,y}} \right)^{\frac{1}{2}}. $$

(3.1)

We shall focus on the dyadic pieces where $k \geq 1$, since by the Strichartz estimate and the Sobolev estimate, one can easily show that

$$\|P|\xi-\eta|\sim 1 u\|_{S_{x,y}} \lesssim \| \langle \nabla_{x+y} \rangle^{\frac{\alpha}{2}} f \|_{L^1(d(x-y))L^2(dt)\hat{L}^2(d(x+y))}. $$

Now let $u_k = P|\xi-\eta|\sim 2^k u$, $f_k = P|\xi-\eta|\sim 2^k f$, and decompose $u_k = u_k^1 + u_k^2 + u_k^3$, where

$$ Su_k^1 = P_{10|^\tau|\sim 2^k} f_k, \text{ with initial conditions } 0 $$

$$ Su_k^2 = \mathcal{F} \left( P_{10|^\tau|\sim 2^k} f_k \right) \frac{1}{\tau + |\xi|^2 + |\eta|^2}, \text{ this no longer has initial conditions } 0 $$

$$ Su_k^3 = 0, \text{ a correction so that } u_k^2 + u_k^3 \text{ has initial condition } 0. $$

(3.2)

For $u_k^1$, by the Strichartz estimate

$$ \|u_k^1\|_{S} \lesssim \|f_k\|_{L^2(dt)\hat{L}^2(d(x-y))L^2(d(x+y))} \lesssim \|f_k\|_{L^\infty(dt)\hat{L}^2(d(x-y))L^2(d(x+y))} \lesssim \frac{\alpha}{2} \|f_k\|_{L^1(d(x-y))L^2(dt)\hat{L}^2(d(x+y))}, $$

where in the last line we used the fact that $f_k$ is frequency supported in $|\xi - \eta| \sim 2^k$ and Bernstein’s inequality, which is a (elementary) generalization of the classical Bernstein’s inequality to $L^2$ valued functions.
Now we make another dyadic decomposition, write
\[
(3.3) \quad f_k = \sum_{\ell \geq 0} P_{|\tau| \sim 2^{k+\ell}} f_k = \sum_{\ell} f_{k,\ell}.
\]

Note that for each fixed \(\ell, k\), we have
\[
\|f_{k,\ell}\|_{L^1(dx)} L^2(dt) L^2(dx) \sim 2^{-\frac{\ell}{2}} \|\partial_t \frac{1}{2} f_{k,\ell}\|_{L^1(dx-y)} L^2(dt) L^2(dx+y))
\]
\[
\lesssim 2^{-\frac{\ell}{2}} \|\partial_t \frac{1}{2} P_{|\tau| \sim 2^{k+\ell}} f\|_{L^1(dx-y)} L^2(dt) L^2(dx+y))
\]
where in the first line we used Bernstein’s inequality, and in the second line we used the fact that
\[
(3.4) \quad \|P_{|\xi-\eta| \sim 2^k} f\|_{L^1(dx-y)} L^2(dt) L^2(dx+y)) \lesssim \|f\|_{L^1(dx-y)} L^2(dt) L^2(dx+y)},
\]
which can be proved, for example, using the generalized Young’s inequality on the space of \(L^2\) valued functions.

Thus, by Minkowski’s inequality
\[
\left( \sum_{k=0}^{\infty} \|u_k^x \|_{L^2(x+y)}^3 \right)^{\frac{1}{3}} \lesssim \left( \sum_{k=0}^{\infty} \|f_{k,\ell}\|_{L^1(dx-y)} L^2(dt) L^2(dx+y)) \right)^{\frac{1}{3}}
\]
\[
\lesssim \sum_{\ell} \left( \sum_{k=0}^{\infty} \|f_{k,\ell}\|^2_{L^1(dx-y)} L^2(dt) L^2(dx+y)) \right)^{\frac{1}{2}}
\]
\[
\lesssim \sum_{\ell} 2^{-\frac{\ell}{4}} \left( \sum_{k} \|\partial_t \frac{1}{2} P_{|\tau| \sim 2^{k+\ell}} f\|_{L^1(dx-y)} L^2(dt) L^2(dx+y)) \right)^{\frac{1}{2}}
\]
\[
\lesssim \sum_{\ell} 2^{-\frac{\ell}{4}} \left( \left( \sum_{k} \|\partial_t \frac{1}{2} P_{|\tau| \sim 2^{k+\ell}} f\|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \|\partial_t \frac{1}{2} f\|_{L^1(dx-y)} L^2(dt) L^2(dx+y)) \right)^{\frac{1}{2}}
\]
\[
\lesssim \|\partial_t \frac{1}{2} f\|_{L^1(dx-y)} L^2(dt) L^2(dx+y)) \right)^{\frac{1}{2}}
\]
where in the last line we used the square function estimate in \(t\) variable.

For \(u_k^x\), the denominator is comparable with \(|\xi-\eta|^2 + |\xi+\eta|^2 \geq 2^{2k} \geq 100\tau\). Thus, by Sobolev’s estimates at an angle, which is Lemma 3.2 in [27], we have
\[
\|u_k^x\|_{L^2(dt) L^6(dx) L^2(dy)} + \|u_k^y\|_{L^2(dt) L^6(dy) L^2(dx)}
\]
\[
\lesssim \|\nabla_{x+y} u_k^x\|_{L^2(dt) L^2(dx) L^2(dy)}
\]
\[
\lesssim \|\left( \nabla_{x+y} + \nabla_{x+y} \right)^{-2} \frac{1}{\nabla_{x+y}} f_k\|_{L^2(dt) L^2(dx) L^2(dy)}
\]
This is also the place where we require the norm on the left side to be \(S_{x,y}\), since we do not have Sobolev-type estimates like
\[
\|u_k^x\|_{L^2(x-y)} L^2(dx+y)) \lesssim \|\nabla_{x+y} u_k^x\|_{L^2(dt) L^2(dx) L^2(dy)}
\]
Let us first assume \(|\xi+\eta| \leq |\xi-\eta|\), and make the decomposition
\[
(3.6) \quad f_k = \sum_{j=0}^{k} P_{|\xi+\eta| \sim 2^{k-j}} f_k = \sum_{j} f_{k,j}
\]
Then for each fixed \(k, j\), we have
where in the last line we used the square function estimate in (3.7)

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where in the third line we used Bernstein’s inequality, and in the last line we used the fact that

\[ \| f_k \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))} \lesssim \| f \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))}. \]

Thus, by Minkowski’s inequality

\[
\left( \sum_{k=0}^{\infty} \| u_k^2 \|_{L^2(dt)L^6(dx)L^2(dy)}^2 + \| u_k^2 \|_{L^2(dt)L^6(dy)L^2(dx)}^2 \right)^{\frac{1}{2}} \\
\lesssim \left( \sum_{k=0}^{\infty} \left( \sum_{j} \left( \| (\nabla_{x-y})^{-2}(\nabla_{x+y}) f_{k,j} \|_{L^2(dt)L^2(dx)L^2(dy)} \right) \right)^{\frac{1}{2}} \\
\lesssim \left( \sum_{j} \left( \sum_{k} \left( \| (\nabla_{x-y})^{-2}(\nabla_{x+y}) f_{k,j} \|_{L^2(dt)L^2(dx)L^2(dy)} \right) \right)^{\frac{1}{2}} \\
\lesssim \sum_{j} \left( \sum_{k} \| (\nabla_{x+y}) \frac{1}{2} P_{|\xi+\eta|\sim 2^{k-j}f} \|_{L^2(dt)L^2(dx)L^2(dy)} \right)^{\frac{1}{2}} \\
\lesssim \sum_{j} \left( \sum_{k} \| (\nabla_{x+y}) \frac{1}{2} P_{|\xi+\eta|\sim 2^{k-j}f} \|_{L^2(dt)L^2(dx)L^2(dy)} \right)^{\frac{1}{2}} \\
\lesssim \sum_{j} \left( \sum_{k} \| (\nabla_{x+y}) \frac{1}{2} P_{|\xi+\eta|\sim 2^{k-j}f} \|_{L^2(dt)L^2(dx)L^2(dy)} \right)^{\frac{1}{2}} \\
\lesssim \| (\nabla_{x+y}) \frac{1}{2} f \|_{L^2(dt)L^2(dx)L^2(dy)} \\
\] 

where in the last line we used the square function estimate in \( x+y \) variable.

The case \( |\xi+\eta| \geq |\xi-\eta| \) is similar.

For the other endpoint \( p=\infty \), define

\[
\| u_k^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim \| f_k \|_{L^\infty(dt)L^2(dx)L^2(dy)} \\
\lesssim \| f_k \|_{L^\infty(dt)L^2(dx)L^2(dy)} \]

and similarly

\[
f_k = \sum_{0 \leq \ell \leq k/2} P_{|\tau|\sim 2^{2k-\ell}} f_k = \sum_{\ell} f_{k,\ell}. \]

Note that for each fixed \( \ell, k \), we have

\[
\| u_{k,\ell}^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim \int \left| \frac{\mathcal{F}(f_{k,\ell})}{\tau + |\xi|^2 + |\eta|^2} \right|^2 d\tau \| L^2(dx)L^2(dy) \\
\lesssim 2^{k/2-\ell/4} \| (\nabla_{x-y})^{-2}(\partial_x) \frac{1}{2} f_{k,\ell} \|_{L^2(dt)L^2(dx)L^2(dy)} \\
\lesssim 2^{-\ell/4} \| (\nabla_{x-y})^{-\frac{1}{2}} \partial_x \frac{1}{2} f_{k,\ell} \|_{L^2(dt)L^2(dx)L^2(dy)} \\
\lesssim 2^{-\ell/4} \| \partial_x \frac{1}{2} P_{|\tau|\sim 2^{2k-\ell}} f \|_{L^2(dt)L^2(dx)L^2(dy)} \\
\lesssim 2^{-\ell/4} \| \partial_x \frac{1}{2} P_{|\tau|\sim 2^{2k-\ell}} f \|_{L^2(dt)L^2(dx)L^2(dy)},
\]
where in the third line we used Bernstein’s inequality, and in the last line we used the fact that
\[ \| f_k \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))} \lesssim \| f \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))}. \]

By Minkowski’s inequality,
\[
\left( \sum_{k=0}^{\infty} \| u_k^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)}^2 \right)^{\frac{1}{2}} = \left( \sum_{k=0}^{\infty} \left( \sum_{\ell} \| u_{k,\ell}^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \lesssim \left( \sum_{\ell=0}^{\infty} \left( \sum_{k:k \geq 2\ell} \| u_{k,\ell}^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell/4} \left( \sum_{k:k \geq 2\ell} \| \partial_t |x|^3 P_{|\tau| \sim 2^{k-\ell}} f \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))}^2 \right)^{\frac{1}{2}} \lesssim \| \partial_t |x|^3 f \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))}
\]
where in the last line we used the square function estimate in $t$ variable.

To deal with $u_k^3$, note that since $u_k^2$ is solution to free Schrödinger,
\[
\| u_k^3 \|_{S_{x,y}} \lesssim \| u_k^2(0, \cdot, \cdot) \|_{L^2(dx)L^2(dy)} = \| u_k^2(0, \cdot, \cdot) \|_{L^2(dx)L^2(dy)} \leq \| u_k^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)}.
\]

Thus it can be treated as in the previous case.

\[ \square \]

**Theorem 3.4.** Let $Su = f$ $u(0, \cdot) = 0$. Then
\[
\| \langle \nabla_{x-y} \rangle^{\frac{1}{2}} u \|_{S_{x,y}} \lesssim \| \langle \nabla_{x+y} \rangle^{\frac{1}{2}} f \|_{L^2(dt)L^{\frac{2}{3}}(d(x-y))L^2(d(x+y))} + \| \partial_t |x|^3 f \|_{L^2(dt)L^{\frac{2}{3}}(d(x-y))L^2(d(x+y))}.
\]

**Proof.** We shall use the decomposition $u = \sum_{k=0}^{\infty} P_{|\xi-\eta| \sim 2^k} u$, and the square function estimate
\[
\| u \|_{S_{x,y}} \sim \left( \sum_{k=0}^{\infty} \| P_{|\xi-\eta| \sim 2^k} u \|_{S_{x,y}}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{k=0}^{\infty} \| P_{|\xi-\eta| < 2^k} u \|_{S_{x,y}}^2 \right)^{\frac{1}{2}}.
\]

Since the right side of (3.12) only involves $L^{\frac{2}{3}}$-norm in the $x - y$ direction, we can add up the dyadic pieces in the right side using the square function estimate, thus it suffices to prove the Theorem for a single dyadic piece where $|\xi - \eta| \sim 2^k$ (which would simplify our argument compare with the previous theorem). Here again for the case $k = 0$, we are abusing notations a bit by letting $P_{|\xi-\eta| < 1} u$ to denote the operator $P_{|\xi-\eta| < 1} u$, and the case $k = 0$ is easy to handle by just using Strichartz.
Now let \( u_k = P_{|\xi - \eta| \sim 2^k} u \) and \( f_k = P_{|\xi - \eta| \sim 2^k} f \). We shall use the same decomposition as in (3.2), write \( u_k = u_k^1 + u_k^2 + u_k^3 \).

For \( u_k^1 \), by Strichartz and Bernstein’s inequality
\[
\| (\nabla_{x-y})^{\frac{1}{2}} u_k^1 \|_{S} \lesssim \| (\nabla_{x-y})^{\frac{1}{2}} f_k \|_{L^2(dt) L^{\frac{11}{5}}(d(x-y)) L^2(d(x+y))} \lesssim \| \partial_t \|_{S} f_k \|_{L^2(dt) L^{\frac{11}{5}}(d(x-y)) L^2(d(x+y))}.
\]
(3.13)

For \( u_k^2 \), the denominator is comparable with \( |\xi - \eta|^2 + |\xi + \eta|^2 \geq 2^{2k} \geq 10^2 \). Thus, by Sobolev estimates at an angle, we have
\[
\| (\nabla_{x-y})^{\frac{1}{2}} u_k^2 \|_{L^2(dt) L^6(dx)L^2(dy)} + \| (\nabla_{x-y})^{\frac{1}{2}} u_k^2 \|_{L^2(dt) L^6(dy)L^2(dx)} \lesssim \| (\nabla_{x-y})^{\frac{1}{2}} (\nabla_{x+y})^{\frac{1}{2}} u_k^2 \|_{L^2(dt) L^2(dx)L^2(dy)} \lesssim \| (\nabla_{x+y})^{\frac{1}{2}} f_k \|_{L^2(dt) L^2(dx)L^2(dy)} \lesssim \| (\nabla_{x+y})^{\frac{1}{2}} \|_{L^2(dt) L^{\frac{11}{5}}(d(x-y)) L^2(d(x+y))} \|
\]
(3.14)

For the other endpoint \( p = \infty \), since we are in the region \( 10^2 \frac{1}{r} \leq 2^k \), we have
\[
\| (\nabla_{x-y})^{\frac{1}{2}} u_k^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)} \leq 2^{k/2} \| (\nabla_{x-y})^{\frac{1}{2}} f_k \|_{L^2(dt)L^2(dx)L^2(dy)} \lesssim 2^k \| \partial_t \|_{S} (\nabla_{x-y})^{-2} f_k \|_{L^2(dt)L^2(d(x-y))L^2(d(x+y))} \lesssim \| \partial_t \|_{S} f_k \|_{L^2(dt)L^{\frac{11}{5}}(d(x-y)) L^2(d(x+y))} \|
\]
(3.15)

To deal with \( u_k^3 \), if we repeat the argument in (3.11), we have
\[
\| u_k^3 \|_{S_{x,y}} \lesssim \| u_k^2 \|_{L^\infty(dt)L^2(dx)L^2(dy)}.
\]
Thus it can be treated as in the previous case.

\[
\text{Theorem 3.5. Let } Su = f, \ u(0,\cdot) = u_0
\]

We have
\[
\| \partial_t \| \| u \|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))} \lesssim \min \{ \| \partial_t \| f \|_{L^2(dt) L^{\frac{11}{5}}(d(x-y)) L^2(d(x+y))} \},
\]
(3.16)
\[
\| (\nabla_{x-y})^{\frac{1}{2}} + (\nabla_{x+y})^{\frac{1}{2}} f \|_{L^2(dt) L^{\frac{11}{5}}(d(x-y)) L^2(d(x+y))} \| \| (\nabla_{x-y})^{\frac{1}{2}} + (\nabla_{x+y})^{\frac{1}{2}} f \|_{S_{x,y}'} \| u_0 \|_{L^2}.
\]
Proof. For simplicity, we shall only treat the case where $|\xi + \eta| < |\xi - \eta|$, the case $|\xi + \eta| \geq |\xi - \eta|$ is similar and to some extent simpler. We shall use the decomposition $u = \sum_{k=0}^{\infty} P_{|\xi-\eta| \sim 2^k} u$, by using the square function estimate, it suffices to prove the Theorem for a single dyadic piece where $|\xi - \eta| \sim 2^k$. The argument below also works for $k < 0$.

Now let $u_k = P_{|\xi-\eta| \sim 2^k} u$ and $f_k = P_{|\xi-\eta| \sim 2^k} f$. We decompose $u_k = u_k^1 + u_k^2 + u_k^3 + u_k^4$, where

$$\mathcal{S} u_k^1 = P_{|\tau| \sim 2^k} f_k, \quad \text{with initial conditions} \ 0$$

$$\mathcal{F} u_k^2 = \mathcal{F} \left( P_{|\tau| \leq 2^k} f_k + P_{|\tau| \geq 10^2} f_k \right) \frac{\tau + |\xi|^2 + |\eta|^2}{\mathcal{S} u_k^3}, \quad \text{this no longer has initial conditions} \ 0$$

$$\mathcal{S} u_k^3 = 0, \quad \text{a correction so that} \ u_k^2 + u_k^3 \ \text{has initial condition} \ 0$$

$$\mathcal{S} u_k^4 = 0, \quad u_k^4(0, \cdot) = P_{|\xi-\eta| \sim 2^k} u_0.$$

It is easy to handle $u_k^4$, since in the case where $|\xi + \eta| < |\xi - \eta|$,}

$$||\partial_t \frac{1}{2} u_k^4 ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \sim 2^{k/2} ||e^{it\Delta} P_{|\xi-\eta| \sim 2^k} u_0 ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))}$$

$$\lesssim ||\langle \nabla_x-y \rangle \frac{1}{2} P_{|\xi-\eta| \sim 2^k} u_0 ||_{L^2}.$$

For $u_k^1$, since for $f_k$, we have $|\xi - \eta|^2 + |\xi + \eta| \sim 2^{2k}$ and $\tau \sim 2^k$, it is straightforward to check that the dual variable $\tau$ to $t$ for $u_k^1$ is also supported where $|\tau|^{1/2} \sim 2^k$, by Strichartz estimates}

$$||\partial_t \frac{1}{2} u_k^1 ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim 2^{k/2} ||u_k^1 ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))}$$

$$\lesssim 2^{k/2} ||P_{|\tau| \sim 2^k} f_k ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))}$$

$$\lesssim ||\partial_t \frac{1}{2} f_k ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))}.$$

The same argument also gives}

$$||\partial_t \frac{1}{2} u_k^2 ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim ||\langle \nabla_x-y \rangle \frac{1}{2} f_k ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))},$$

as well as

$$||\partial_t \frac{1}{2} u_k^3 ||_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim ||\langle \nabla_x-y \rangle \frac{1}{2} f_k ||_{S_1^0}.$$
It remains to show that, if $10|\tau|^{1/2} \leq 2^k$,

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{S^k}.$$  \hspace{1cm} (3.19)

By interpolation, it suffices to show that

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{L^2(dt)\dot{F}^k(dz)L^2(dy)}$$  \hspace{1cm} (3.20)

and

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{L^2(dt)\dot{F}^k(dy)L^2(dx)}$$  \hspace{1cm} (3.21)

and

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{L^1(dt)L^2(dx)L^2(dy)}$$  \hspace{1cm} (3.22)

which would be stronger than (3.19) since it includes two endpoint cases.

The estimates (3.20) and (3.21) follow directly by Sobolev’s estimates at an angle. To prove (3.22), first by Sobolev in the $x-y$ direction, we have

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim 2^{-k/2}\|f_k\|_{L^2(dt)L^2(d(x-y))L^2(d(x+y))},$$  \hspace{1cm} (3.23)

and now by Bernstein’s inequality in the $t$ direction and $x-y$ direction.

$$2^{-k/2}\|f_k\|_{L^2(dt)L^2(dx)L^2(dy)} \lesssim 2^{k/2}\|f_k\|_{L^1(dt)L^2(dx)L^2(dy)} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{L^1(dt)L^2(dx)L^2(dy)}.$$  \hspace{1cm} (3.24)

If $|\tau|^{1/2} \geq 10 \cdot 2^k$, the denominator is comparable with $\tau$. Thus, by Sobolev in the $x-y$ direction, we have

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{L^2(dt)L^2(dx)L^2(dy)},$$  \hspace{1cm} (3.25)

The same argument also gives,

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{1/2} f_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))},$$

when $|\tau|^{1/2} \geq 10 \cdot 2^k$. It remains to show that (3.19) holds if $|\tau|^{1/2} \geq 10 \cdot 2^k$, which would be a consequence of (3.20)–(3.22). And as before, the estimates (3.21) and (3.22) in this case follow by Sobolev’s estimates at an angle.

To prove (3.22) when $|\tau|^{1/2} \geq 10 \cdot 2^k$, first by Sobolev in the $x-y$ direction, we have

$$\|\partial_t \dot{u}_k\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \lesssim \|\langle \nabla x-y\rangle^{-1/2} f_k\|_{L^2(dt)L^2(d(x-y))L^2(d(x+y))} = \|\langle \nabla x-y\rangle^{-1/2} f_k\|_{L^2(dt)L^2(dx)L^2(dy)}.$$
Now by Minkowski’s inequality for $dt$ integral,

$$
\|\partial_t^{-\frac{1}{2}} \langle \nabla_{x-y} \rangle f_k\|_{L^2(dt) L^2(dx) L^2(dy)}
= \|\tau^{-\frac{1}{2}} \int e^{-i\tau \langle \nabla_{x-y} \rangle f_k(t, \cdot) dt}\|_{L^2(d\tau) L^2(dx) L^2(dy)}
\lesssim 2^{-k/2} \| \langle \nabla_{x-y} \rangle f_k\|_{L^1(dt) L^2(dx) L^2(dy)}
\lesssim \| \langle \nabla_{x-y} \rangle \frac{1}{2} f_k\|_{L^1(dt) L^2(dx) L^2(dy)}.
$$

(3.26)

To deal with $u^2_k$, note that since $u^3_k$ is solution to free Schrödinger equation, and since we are assuming $|\xi - \eta|^2 + |\xi + \eta|^2 \sim 2^k$

$$
\|\partial_t\frac{1}{\tau} u^2_k\|_{L^2(dt) L^2(dx) L^2(dy)} \lesssim 2^{k/2} \| u^2_k(0, \cdot, \cdot)\|_{L^2(dx) L^2(dy)}
= 2^{k/2} \| u^2_k(0, \cdot, \cdot)\|_{L^2(dx) L^2(dy)}
\leq 2^{k/2} \| u^2_k\|_{L^\infty(dt) L^2(dx) L^2(dy)}.
$$

(3.27)

Thus, it suffices to control $\| u^2_k\|_{L^\infty(dt) L^2(dx) L^2(dy)}$.

First, if $|\tau|^{\frac{1}{2}} \geq 10 \cdot 2^k$

$$
\| u^2_k\|_{L^\infty(dt) L^2(dx) L^2(dy)} \lesssim \int \frac{P\left(\frac{P}{|\tau|^{\frac{1}{2}} \geq 10 \cdot 2^k} f_k\right)}{|\tau|^2 + |\xi|^2 + |\eta|^2} \| d\tau\|_{L^2(dx) L^2(dy)}
\lesssim 2^{-\frac{3k}{2}} \|\partial_t\frac{1}{\tau} f_k\|_{L^2(dt) L^2(dx) L^2(dy)}
\lesssim 2^{-k/2} \|\partial_t\frac{1}{\tau} \langle \nabla_{x-y} \rangle^{-1} f_k\|_{L^2(dt) L^2(dx) L^2(dy)}
\lesssim 2^{-k/2} \|\partial_t\frac{1}{\tau} f_k\|_{L^2(dt) L^2(dx) L^2(dy)}.
$$

(3.28)

Similarly,

$$
\| u^2_k\|_{L^\infty(dt) L^2(dx) L^2(dy)} \lesssim \int \frac{P\left(\frac{P}{|\tau|^{\frac{1}{2}} \geq 10 \cdot 2^k} f_k\right)}{|\tau|^2 + |\xi|^2 + |\eta|^2} \| d\tau\|_{L^2(dx) L^2(dy)}
\lesssim 2^{-k/2} \| f_k\|_{L^2(dt) L^2(dx) L^2(dy)}
\lesssim 2^{-k/2} \|\langle \nabla_{x-y} \rangle \frac{1}{2} f_k\|_{L^2(dt) L^2(dx) L^2(dy)}.
$$

(3.29)

It remains to show that

$$
\| u^2_k\|_{L^\infty(dt) L^2(dx) L^2(dy)} \lesssim 2^{-k/2} \|\langle \nabla_{x-y} \rangle \frac{1}{2} f_k\|_{S'_t}.
$$

(3.30)

In this case, we won’t prove (3.30) by interpolation since we do not know if one can show that

$$
\| u^2_k\|_{L^\infty(dt) L^2(dx) L^2(dy)} \lesssim 2^{-k/2} \|\langle \nabla_{x-y} \rangle \frac{1}{2} f_k\|_{L^1(dt) L^2(dx) L^2(dy)},
$$

(3.31)

which is also the reason why we have the restricted norm $S'_t$ in the statement of the Theorem.

Instead, we shall prove (3.30) by showing that

$$
\| u^2_k\|_{L^\infty(dt) L^2(dx) L^2(dy)} \lesssim 2^{-k/2} \|\langle \nabla_{x-y} \rangle \frac{1}{2} f_k\|_{L^p'(dt) L^q'(dx) L^2(dy)},
$$

(3.32)

as well as

\[\|u_k\|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim 2^{-k/2}\|\langle \nabla \rangle \frac{t}{2} f_k\|_{L^p(dt)L^q(dy)L^2(dx)},\]

for all admissible pairs \((p, q)\), with \( \frac{2}{p} = \frac{3}{2} - \frac{3}{q}, \ 2 \leq p < \infty \).

To prove (3.32) when \( |\tau|^\frac{1}{4} \geq 10 \cdot 2^k \), for admissible pair \((p, q)\) with \( p < \infty \),

\[
\|u_k\|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim \int \frac{\mathcal{F}\left( P_{|\tau|^\frac{1}{4} \geq 10 \cdot 2^k} f_k \right)}{\tau + |\xi|^2 + |\eta|^2} |d\tau| L^2(d\xi)L^2(d\eta)
\]

\[
\lesssim 2^{-\frac{k}{p}} \|f_k\|_{L^p(dt)L^2(dx)L^2(dy)}
\]

(3.34)

\[
\lesssim 2^{-\frac{k}{p}} \|f_k\|_{L^p(dt)L^2(dx)L^2(dy)}
\]

where we used Hölder’s inequality in the second line, the Hausdorff-Young inequality in the third line, and Bernstein’s inequality at an angle in the fourth and last line. The proof of (3.33) is similar.

If \( 10|\tau|^\frac{1}{4} \leq 2^k \),

\[
\|u_k\|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim \int \frac{\mathcal{F}\left( P_{10|\tau|^\frac{1}{4} \leq 2^k} f_k \right)}{\tau + |\xi|^2 + |\eta|^2} |d\tau| L^2(d\xi)L^2(d\eta)
\]

(3.35)

\[
\lesssim 2^{-\frac{k}{p}} \|\partial_t \frac{t}{2} f_k\|_{L^p(dt)L^2(dx)L^2(dy)}
\]

\[
\lesssim 2^{-k/2}\|\partial_t \frac{t}{2} \langle \nabla \rangle (\nabla x) - 1 f_k\| \|L^2(dt)L^2(d(x-y))L^2(d(x+y))\|
\]

\[
\lesssim 2^{-k/2}\|\partial_t \frac{t}{2} f_k\| \|L^2(dt)L^2(d(x-y))L^2(d(x+y))\|
\]

The same argument also gives,

\[
\|u_k\|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim 2^{-k/2}\|\langle \nabla \rangle \frac{t}{2} f_k\| \|L^2(dt)L^2(d(x-y))L^2(d(x+y))\|
\]

It remains to show that

(3.36) \[
\|u_k\|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim 2^{-k/2}\|\langle \nabla \rangle \frac{t}{2} f_k\| \|S_k\|
\]

for the case \( 10|\tau|^\frac{1}{4} \leq 2^k \), which would be a consequence of (3.21), (3.22).

To prove (3.21),

\[
\|u_k\|_{L^\infty(dt)L^2(dx)L^2(dy)} \lesssim \int \frac{\mathcal{F}\left( P_{|\tau|^\frac{1}{4} \geq 10 \cdot 2^k} f_k \right)}{\tau + |\xi|^2 + |\eta|^2} |d\tau| L^2(d\xi)L^2(d\eta)
\]

\[
\lesssim 2^{-k}\|f_k\| \|L^2(dt)L^2(dx)L^2(dy)\|
\]

\[
\lesssim 2^{-k/2}\|\langle \nabla \rangle \frac{t}{2} f_k\| \|L^2(dt)L^2(d(x-y))L^2(dx)L^2(dy)\|
\]

The proof of (3.21) is similar.
Lemma 3.7. If
\[ 2^{-k} \|f_k\|_{L^2(dt)L^2(dx)L^2(dy)} \lesssim \|f_k\|_{L^1(dt)L^2(dx)L^2(dy)} \lesssim 2^{-k/2} \|\langle\nabla x-y\rangle^{1/2} f_k\|_{L^1(dt)L^2(dx)L^2(dy)}. \]

Now we shall present several lemmas that involve the collapsing norm.

**Lemma 3.6.** If \( Su = g, u(0, \cdot) = u_0 \). Then
\[ \|u\|_{\text{collapsing}} \lesssim \min\{ \|\langle\nabla x\rangle^{1/2} g\|_{S^*_r} + \|\langle\nabla y\rangle^{1/2} u_0\|_{L^2}, \|\langle\nabla x\rangle^{1/2} u\|_{S_r} + \|\langle\nabla y\rangle^{1/2} u_0\|_{L^2} \}. \]

We record that the above implies

**Lemma 3.7.** If \( Su = g, u(0, \cdot) = u_0 \). Then
\[ \|\langle\nabla x\rangle^{1/2} u\|_{\text{collapsing}} + \|\langle\nabla y\rangle^{1/2} u\|_{\text{collapsing}} \lesssim \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} u\|_{S^*_r} + \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} u_0\|_{L^2}. \]

We will also need

**Lemma 3.8.** If \( Su = g, u(0, \cdot) = u_0 \). Then
\[ \|\langle\nabla x+y\rangle^{1/2} u\|_{\text{collapsing}} \lesssim \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} g\|_{S^*_r} + \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} u_0\|_{L^2}. \]

The proof of Lemma 3.6 and Lemma 3.8 are similar, for simplicity, we shall only present the proof of Lemma 3.8 here. The proof essentially follows from ideas in Lemma 5.1, 5.3 in [15].

**Proof.** We shall first prove the homogeneous estimate, let \( Su = 0 \), with \( u(0, \cdot) = u_0 \). Our goal is to show
\[ \sup_{x-y} \|\langle\nabla x+y\rangle^{1/2} u\|_{L^2(dt)L^2(dx+y)} \lesssim \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} u_0\|_{L^2}. \]
This is stronger than desired, since \( \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} u_0\|_{L^2} \lesssim \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} u_0\|_{L^2} \), and also by Lemma 3.6 we have
\[ \sup_{x-y} \|u\|_{L^2(dt)L^2(dx+y)} \lesssim \|\langle\nabla x\rangle^{1/2} u_0\|_{L^2} \lesssim \|\langle\nabla x\rangle^{1/2}\langle\nabla y\rangle^{1/2} u_0\|_{L^2}. \]

To prove (3.39), let \( \Lambda \) denote the space-time Fourier. For fixed \( x-y \), doing Cauchy-Schwarz with measures,
\[ \|\langle\nabla x+y\rangle^{1/2} \Lambda(t, x+y, \tilde{x} + \tilde{y})(\tau, \xi + \eta)^2 \]
\[ \lesssim \int \delta(\tau - |\xi|^2 - |\eta|^2) |\xi + \eta| d(\xi - \eta) \]
\[ \lesssim \int \delta(\tau - |\xi|^2 - |\eta|^2) |\nabla^2 \Lambda_0(\xi, \eta)|^2 d(\xi - \eta). \]
In order to prove the estimate, we must show
\[
\sup_{\tau, \xi} \int \delta(\tau - |\xi|^2 - |\eta|^2) \frac{\xi + \eta}{|\xi||\eta|} d(\xi - \eta) \lesssim 1.
\]
Without loss of generality, consider the region $|\xi| \leq |\eta|$. If $|\xi| \sim |\eta|$, $\frac{\xi + \eta}{|\xi||\eta|} \lesssim \frac{1}{|\xi - \eta|}$ and the integral can be evaluated in polar coordinates. If $|\xi| \ll |\eta|$ then $|\xi + \eta| \sim |\xi - \eta|$

Writing $\frac{|\xi + \eta|}{|\xi||\eta|} \lesssim \frac{1}{|\xi - \eta|} \lesssim \frac{1}{|\xi - \eta|\sqrt{1 - \cos(\theta)}}$ where $\theta$ is the angle between $\xi - \eta$ and $\xi + \eta$, we estimate

\[
(3.41) \quad \sup_{\tau} \int_0^\pi \int \delta(\tau - \rho^2) \frac{1}{\rho \sqrt{1 - \cos(\theta)}} \rho^2 d\rho d\theta \lesssim 1.
\]

The inhomogeneous estimate (3.38) just follows from the homogeneous estimate (3.39) and the Christ–Kiselev lemma. More precisely, let $T_1 = e^{it(\Delta_x + \Delta_v)}$, so $T_1 : L^p(\mathbb{R}^6) \to L^p(dt)L^q(dx)L^2(dy)$ and

\[
T_1^* : L^p(dt)W^{\alpha,q'}(dx)H^{\alpha}(dy) \to H^\alpha(dx)H^{\alpha}(dy).
\]

Fix $x - y$ and let $T_2 : H^\alpha(dx)H^{\alpha}(dy) \to L^2(dt)H^\alpha(dx+dy)$ be the operator $f \to (e^{it(\Delta_x + \Delta_v)} f)(t, x - y, x + y)$. Then the inhomogeneous estimate follows by applying the Christ–Kiselev lemma to $T_2T_1^*$.

**Lemma 3.9.** If $Su = g, u(0, \cdot) = u_0$. Then

\[
||| \partial_t^{\frac{1}{2}} u |||_{\text{collapsing}} \lesssim ||(\nabla_x)^{\frac{1}{2}} g||_{L_x^\infty} + ||(\nabla_x)^{\frac{1}{2}} (\nabla_y)^{\frac{1}{2}} u_0||_{L^2}.
\]

**Proof.** For the homogeneous estimate, it follows from the same argument as above. However, we can not apply Christ–Kiselev lemma here to get inhomogeneous estimate since $||\partial_t||^{\frac{1}{2}}$ does not commute with $1_{[0,t]}$ when we write out the solution using Duhamel’s formula. Let $Su = g$, with $u(0, \cdot) = 0$, it suffices to prove

\[
||| \partial_t^{\frac{1}{2}} u |||_{\text{collapsing}} \lesssim ||(\nabla_x)^{\frac{1}{2}} (\nabla_y)^{\frac{1}{2}} g||_{L_x^\infty}.
\]

To prove this, we shall decompose the Fourier support $\tau$ and $|\xi + \eta|$ of $u$ into finitely many regions.

Case 1: $\tau^{\frac{1}{2}} \leq 10(1 + |\xi + \eta|)$.

In this case, we have

\[
||| \partial_t^{\frac{1}{2}} u |||_{\text{collapsing}} \lesssim ||(\nabla_x + y)^{\frac{1}{2}} u||_{\text{collapsing}},
\]

thus the desired estimates follows from Lemma 3.8.

Case 2: $|\tau|^{\frac{1}{2}} > 2(|\xi| + |\eta|)$.

Write $u = u^1 + u^2$, where

\[
\mathcal{F} u^1 = \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2}, \text{ this no longer has initial condition 0}
\]
\[
Su^2 = 0, \text{ a correction so that } u^1 + u^2 \text{ has initial condition 0}.
\]
In this case, it suffices to control $u_1$ since $u_2$ is only supported where $|\tau| = |\xi|^2 + |\eta|^2$. The goodness about $u_1$ is that it has the same Fourier support with $f$. The strategy is based on

$$
\|\partial_t^{\frac{1}{2}} u_1\|_{L^\infty(d(x-y))L^2(d(x+y)dt)} = \|\tau^{\frac{1}{2}} \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2} \|_{L^\infty(d(x-y))L^2(\tau)d(\xi + \eta)} \lesssim \int |\tau|^{\frac{1}{2}} \left| \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2} \right| d(\xi - \eta) \|L^2(\tau d(\xi + \eta)).
$$

It suffices to show

$$
(3.45) \int |\tau|^{\frac{1}{2}} \left| \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2} \right| d(\xi - \eta) \|L^2(\tau d(\xi + \eta)) \lesssim \|\nabla_x \tau^{\frac{1}{2}} \nabla_y \tau^{\frac{1}{2}} f\|_{L^p(dt)L^{\sigma}(dx)L^2(dy)},
$$

as well as

$$
(3.46) \int |\tau|^{\frac{1}{2}} \left| \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2} \right| d(\xi - \eta) \|L^2(\tau d(\xi + \eta)) \lesssim \|\nabla_x \tau^{\frac{1}{2}} \nabla_y \tau^{\frac{1}{2}} f\|_{L^p(dt)L^{\sigma}(dy)L^2(dx)}.
$$

for all admissible pairs $(p, q)$, where $\frac{2}{p} = \frac{3}{2} - \frac{3}{q}$, $2 \leq p < \infty$.

For $2 \leq p < \infty$, by Cauchy-Schwarz, we have

$$
\text{LHS}(3.45) \lesssim \int_{|\xi - \eta| < |\tau|^{\frac{1}{2}}} |\mathcal{F} f| d(\xi - \eta) \|L^2(\tau d(\xi + \eta)) \lesssim A \|\partial_t^{\frac{1}{2}} - \frac{1}{2} |\nabla_y| \tau^{\frac{1}{2}} f\|_{L^2(dt)d(x-y)d(x+y)}
$$

where

$$
A = \sup_{\tau, \xi + \eta} \frac{|\tau|^{\frac{1}{2}} \frac{1}{2} |\nabla_y|^{\frac{1}{2}}}{|\tau|} \left( \int_{|\xi - \eta| < |\tau|^{\frac{1}{2}}} \left| \frac{|\xi|^{\frac{1}{2}} - 1}{|\eta|} \right| d(\xi - \eta) \right) \frac{1}{2}.
$$

Changing variables, this is something like

$$
A = \sup_{\tau, |u| < |\tau|^{\frac{1}{2}}} \frac{|\tau|^{\frac{1}{2}} \frac{1}{2} |\nabla_y|^{\frac{1}{2}}}{|\tau|} \left( \int_{|v| < |\tau|^{\frac{1}{2}}} \left| \frac{u + v}{|u - v|} \right|^{\frac{1}{2}} dv \right) \frac{1}{2}.
$$

After a change of variables this is reduced to $\tau = 1$, and $A$ is bounded.

Since by Sobolev,

$$
(3.49) \|\partial_t^{\frac{1}{2}} - \frac{1}{2} |\nabla_y|^{\frac{1}{2}} |\nabla_x|^{\frac{1}{2}} |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\frac{1}{2}} |\nabla_x|^{\frac{1}{2}} |\nabla_y|^{\frac{1}{2}} f\|_{L^2(dt)d(x-y)d(x+y)} \leq \|\nabla_x \tau^{\frac{1}{2}} \nabla_y \tau^{\frac{1}{2}} f\|_{L^p(dt)L^{\sigma}(dx)L^2(dy)}.
$$

Thus the proof of (3.45) is complete, and the proof of (3.46) is similar.

Case 3: $|\xi| + |\eta| > 2|\tau|^{\frac{1}{2}}$. 

In this case, due to Case 1, we can assume additionally that $|\tau|^{\frac{1}{2}} > 10(1 + |\xi + \eta|)$. Thus $|\xi - \eta| > |\xi + \eta|$, so also $|\xi - \eta| > |\tau|^{\frac{1}{2}}$. As before, it suffices to show (3.45) and (3.46).

$$\text{LHS (3.45)} \lesssim \left\| \int_{|\xi-\eta| > |\xi+\eta|+|\tau|^{\frac{1}{2}}} \frac{|\tau|^\frac{1}{2} |\mathcal{F}f|}{|\xi-\eta|^2} d(\xi - \eta) \right\|_{L^2(d\tau)}$$

$$\lesssim A \left\| \partial_t |\frac{1}{2} - \frac{2}{p} |\nabla_y |\frac{1}{2} |\nabla_x |^{\frac{1}{2} - \frac{2}{p}} f \right\|_{L^2(dt)dx dy}$$

In this case,

$$\text{A^2 = \sup_{\xi + \eta, \tau} \int_{|\xi-\eta| > |\xi+\eta|+|\tau|^{\frac{1}{2}}} |\tau|^{\frac{1}{2} - \frac{2}{p}} |\xi|^{\frac{1}{2} - \frac{1}{p}} d(\xi - \eta)}$$

Again we scale to $|\tau|^{\frac{1}{2}} + |\xi + \eta| = 1$ and have to estimate

$$\int_{|v| > 1} \frac{1}{|v|^4} \frac{|u+v|^{\frac{1}{p}-1}}{|u-v|} dv$$

This is bounded uniformly in $|u| < 1$. As before, the rest of the proof follow from Sobolev’s inequality, and the proof of (3.46) is similar.

Case 4: $2^{\frac{1}{p}}(|\xi| + |\eta|) < |\tau|^{\frac{1}{2}} < 2(|\xi| + |\eta|)$. In this case, due to Case 1, we can assume additionally that $|\tau|^{\frac{1}{2}} > 10(1 + |\xi + \eta|)$ so in this case $|\tau|^{\frac{1}{2}} \sim |\xi| \sim |\eta| \sim |\xi - \eta|$.

We shall use the decomposition $u = \sum_{k=0}^{\infty} P_{|\tau| \sim 2^k} u$, and the square function estimate

$$\left\| \partial_t \frac{1}{4} u \right\|_{L^\infty(d(x-y))L^2(d(x+y)dt)} \lesssim \left\| \left( \sum_{k=0}^{\infty} |P_{|\tau| \sim 2^k} \partial_t \frac{1}{4} u|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty(d(x-y))L^2(d(x+y)dt)} \lesssim \left( \sum_{k=0}^{\infty} 2^{2k} \left\| P_{|\tau| \sim 2^k} u \right\|_{L^\infty(d(x-y))L^2(d(x+y)dt)}^2 \right)^{\frac{1}{2}}.$$

For each fixed dyadic piece $P_{|\tau| \sim 2^k} u$, in the current case, we have

$$P_{|\tau| \sim 2^k} u = P_{|\tau| \sim 2^k} P_{|\xi| \sim 2^{k/2}} P_{|\eta| \sim 2^{k/2}} u,$$

which implies

$$\left\| P_{|\tau| \sim 2^k} u \right\| \lesssim \left\| \left( \nabla_x \frac{1}{2} P_{|\xi| \sim 2^{k/2}} P_{|\eta| \sim 2^{k/2}} u \right) L^2(dt)L^6(dx-y)L^2(dx+y) \right\|$$

$$\lesssim 2^{-k/2} \left\| \left( \nabla_x \frac{1}{2} \nabla_y \frac{1}{2} P_{|\eta| \sim 2^{k/2}} u \right) L^2(dt)L^6(dx-y)L^2(dx+y) \right\|$$

where we used Bernstein’s inequality in rotated coordinates twice, see e.g., Lemma 3.1 in [7] for more details.
Thus,
\[
\|\partial_t \frac{1}{4} u\|_{L^\infty(d(x-y))L^2(d(x+y))dt} \\
\lesssim \left( \sum_{k=0}^{\infty} \| (\nabla_x + \nabla_y)^{\frac{1}{2}} P_{|\eta| \sim 2^k/2} u\|_{L^2(d(x-y))L^2(d(x+y))}^2 \right)^{\frac{1}{2}} \\
\lesssim \left( \sum_{k=0}^{\infty} \| (\nabla_x + \nabla_y)^{\frac{1}{2}} P_{|\eta| \sim 2^{k/2}} f\|_{S'}^2 \right)^{\frac{1}{2}} \leq \| (\nabla_x + \nabla_y)^{\frac{1}{2}} f\|_{S'}
\]
where we used Strichartz (Theorem 3.1) in the second line, and square function estimates in $y$ the the last line.

\[\square\]

4. Proof of Theorem 2.1

In this section, we shall see how we can apply the theorems in the previous section to prove Theorem 2.1. Throughout this section, besides (2.12), we shall use extensively the fact that
\[\text{supp} \hat{v} \subset B_1(0)\]
This assumption implies that, the multiplication operator $T u = v_N u$ can at most enlarge the Fourier support of $u$ by a set of size $N$, which will greatly simplify our proof, especially the proof of Theorem 4.1 below. If we assume $v$ satisfies (2.12) without this condition, one can follow similar steps in this section to get the same conclusion. In that case, the multiplication operator $T u = v_N u$ can enlarge the Fourier support of $u$ by a set of arbitrary large size, but with a rapidly decay constant if the new Fourier support deviates from the Fourier support of $u$ by a large distance. For the sake of simplicity, we do not present the full details here for this general case.

To begin with, we shall first prove the following theorem involving collapsing norms at low frequency

**Theorem 4.1.** Let $\Lambda$ satisfy (2.11), we have
\[
\| (\nabla_x + \nabla_y)^{\frac{1}{2}} \Lambda\|_{\text{low collapsing}} + \|\partial_t \frac{1}{4} \Lambda\|_{\text{low collapsing}} \\
\lesssim \| (\nabla_x + \nabla_y)^{\frac{1}{2}} G\|_{S'}^2 + \varepsilon \| (\nabla_x + \nabla_y)^{\frac{1}{2}} H\|_{\text{collapsing}} + \varepsilon \|\partial_t \frac{1}{4} H\|_{\text{collapsing}} \\
+ \| (\nabla_x + \nabla_y)^{\frac{1}{2}} A_0\|_{L^2}
\]
where the norms $\| \cdot \|_{\text{collapsing}}$ and $\| \cdot \|_{\text{low collapsing}}$ are defined as in (2.8) and (2.9), and $\varepsilon$ is defined as in (2.12).

**Remark 4.2.** Due to the criticalness of the potential $N^2 v(N(x-y))$ in (2.11), it is still open to us if one can prove the above theorem without the frequency assumption on $\Lambda$, i.e., to replace on the left side the $\| \cdot \|_{\text{low collapsing}}$ by the full collapsing norm. However, in the case where $\Lambda$ satisfy (2.11) with $H = 0$, we do know how to control the full collapsing norm
\[
\| (\nabla_x + \nabla_y)^{\frac{1}{2}} \Lambda\|_{\text{collapsing}} + \|\partial_t \frac{1}{4} \Lambda\|_{\text{collapsing}}
\]
by using a different argument, the details are given in Section 8. Also in [7], a stronger version of the theorem is proved for the case when one replace $N^2v(N(x-y))$ by $N^{3\beta-1}v(N^{\beta}(x-y))$ for some $\beta < 1$.

**Proof.** We shall focus on $\|\partial_t^{\frac{1}{2}}\Lambda\|_{\text{low collapsing}}$, the proof for the term $\|\langle \nabla_{x+y} \rangle^{\frac{1}{2}} \Lambda\|_{\text{low collapsing}}$ is similar. And we will first show that

\begin{equation}
\|P_{|\xi-j|<20N} |\partial_t|^{\frac{1}{2}}\Lambda\|_{\text{collapsing}} \lesssim \|\langle \nabla_{x+y} \rangle^{\frac{1}{2}} G\|_{S^1} + \epsilon \|\partial_t|^{\frac{1}{2}} H\|_{\text{collapsing}} + \|\langle \nabla_{x+y} \rangle^{\frac{1}{2}} \Lambda_0\|_{L^2}.
\end{equation}

Let $\rho \in C^\infty_0(\mathbb{R}^3)$ be a smooth partition of unity, which satisfies

$$\sum_{j \in \mathbb{Z}^3} \rho(\xi - j) \equiv 1, \forall \xi \in \mathbb{R}^3.$$ 

We also assume that $0 \leq \rho \leq 1$, $\rho \equiv 1$ if $|\xi| \leq \frac{1}{2}$, and supp $\rho \subset B_1(0)$, the unit ball in $\mathbb{R}^3$ centered at origin. Let $\psi_j(x-y)$ be the inverse Fourier transform of $\rho(\frac{x-y}{40N} - j)$. For each fixed $j \in \mathbb{Z}^3$, $\psi_j$ is Fourier supported in a ball of radius $40N$ centered at $40N \cdot j$. Denote

$$P_j f = \psi_j * f$$

so that the Fourier transform of $P_j f = \rho(\frac{x-y}{40N} - j) \hat{f}$. In particular, let $P_0 f$ denote projection onto ball of radius $40N$ centered at origin, so that the Fourier transform of $P_0 f = \rho(\frac{x-y}{40N}) \hat{f}$.

Define

\begin{equation}
\|\partial_t|^{\frac{1}{2}}\Lambda\|_{\mathcal{N}} = \sum_{k=0}^{N} 2^{-k} \left( \sum_{|j| \leq |k| < k+1} \|P_j |\partial_t|^{\frac{1}{2}}\Lambda\|_{\text{collapsing}} \right).
\end{equation}

It is clear that

$$\|P_{|\xi-j|<20N} |\partial_t|^{\frac{1}{2}}\Lambda\|_{\text{collapsing}} \lesssim \|P_0 |\partial_t|^{\frac{1}{2}}\Lambda\|_{\text{collapsing}} \lesssim \|\partial_t|^{\frac{1}{2}}\Lambda\|_{\mathcal{N}}.$$ 

Thus it suffices to show that

\begin{equation}
\|\partial_t|^{\frac{1}{2}}\Lambda\|_{L^\infty} \lesssim \|\langle \nabla_{x+y} \rangle^{\frac{1}{2}} G\|_{S^1} + \epsilon \|\partial_t|^{\frac{1}{2}} H\|_{\text{collapsing}} + \|\langle \nabla_{x+y} \rangle^{\frac{1}{2}} \Lambda_0\|_{L^2}.
\end{equation}

We shall first deal with the last term in the norm, where $k = N$. Note that for each fixed $j \in \mathbb{Z}^3$, the Fourier transform in $x - y$ direction of $P_j |\partial_t|^{\frac{1}{2}}\Lambda$ is supported in a ball of radius $40N$, by Bernstein’s inequality,

\begin{equation}
\|P_j |\partial_t|^{\frac{1}{2}}\Lambda\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))} \lesssim N^\frac{1}{2} \|P_j |\partial_t|^{\frac{1}{2}}\Lambda\|_{L^6(d(x-y))L^2(dt)L^2(d(x+y))} \lesssim N^\frac{1}{2} \|\partial_t|^{\frac{1}{2}}\Lambda\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))}.
\end{equation}

Recall that

$$S\Lambda(t,x,y) = N^2v(N(x-y))\Lambda(t,x,y) + G(t,x,y) + N^2v(N(x-y))H(t,x,y)$$

$$\Lambda(0,\cdot) = \Lambda_0.$$
By Theorem 3.5 we have
\[
\|\partial_t \frac{1}{\varepsilon} \Lambda\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \\
\leq C\|N^2 v(N(x-y))\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} + C\|N^2 v(N(x-y))\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \\
+ C\|\partial_t \frac{1}{\varepsilon} H\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} + C\|\langle \nabla x-y \rangle^\frac{1}{2} + \langle \nabla x+y \rangle^\frac{1}{2} \rangle \|_{S'_c} \\
+ C\|\langle \nabla x-y \rangle^\frac{1}{2} + \langle \nabla x+y \rangle^\frac{1}{2} \rangle \Lambda_0\|_{L^2},
\]
(4.6)
where we used Hölder in the second inequality. By choosing \(\varepsilon\) small enough such that \(C\varepsilon < 1/2\), we have
\[
\|\partial_t \frac{1}{\varepsilon} \Lambda\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \\
\leq C\varepsilon N^{-1/2}\|\partial_t \frac{1}{\varepsilon} H\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))} + C\|\langle \nabla x-y \rangle^\frac{1}{2} + \langle \nabla x+y \rangle^\frac{1}{2} \rangle \|_{S'_c} \\
+ C\|\langle \nabla x-y \rangle^\frac{1}{2} + \langle \nabla x+y \rangle^\frac{1}{2} \rangle \Lambda_0\|_{L^2},
\]
(4.7)
If we combine (4.5) and (4.7), we get
\[
\|P_j \partial_t \frac{1}{\varepsilon} \Lambda\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))} \\
\leq C\varepsilon\|\partial_t \frac{1}{\varepsilon} H\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))} + C N^\frac{1}{2}\|\langle \nabla x \rangle^\frac{1}{2} \langle \nabla y \rangle^\frac{1}{2} \rangle \|_{S'_c} + C N^\frac{1}{2}\|\langle \nabla x \rangle^\frac{1}{2} \langle \nabla y \rangle^\frac{1}{2} \rangle \|_{L^2}.
\]
(4.8)
Since the number of \(j \in \mathbb{Z}^3\) such that \(N \leq |j| < N+1\) is bounded by \(CN^2\), we have
\[
\sum_{N \leq |j| < N+1} 2^{-N}\|P_j \partial_t \frac{1}{\varepsilon} \Lambda\|_{collapsing} \\
\leq C 2^{-N} N^2 \varepsilon\|\partial_t \frac{1}{\varepsilon} H\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))} + C 2^{-N} N^\frac{1}{2}\|\langle \nabla x \rangle^\frac{1}{2} \langle \nabla y \rangle^\frac{1}{2} \rangle \|_{S'_c} \\
+ C 2^{-N} N^\frac{1}{2}\|\langle \nabla x \rangle^\frac{1}{2} \langle \nabla y \rangle^\frac{1}{2} \rangle \|_{L^2}.
\]
(4.9)
Now we shall control the terms $0 \leq k \leq N - 1$ using a bootstrap argument. Write $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$, where

$$S\Lambda_1 = G, \quad \Lambda_1(0, \cdot) = \Lambda_0$$

$$S\Lambda_2 = N^2v(N(x - y))H(t, x, y), \quad \text{with initial conditions 0}$$

$$S\Lambda_3 = N^2v(N(x - y))\Lambda(t, x, y), \quad \text{with initial conditions 0.}$$

By Lemma 3.9, we have

$$\sum_{k=0}^{N-1} 2^{-k} \left( \sum_{k \leq |j| < k+1} \|P_j |\partial_t|^{\frac{1}{2}} \Lambda_1\|_{\text{collapsing}} \right)$$

$$\lesssim \sum_{k=0}^{N-1} 2^{-k} \left( \sum_{k \leq |j| < k+1} \|\partial_t |\partial_t|^{\frac{1}{2}} \Lambda_1\|_{\text{collapsing}} \right)$$

$$\lesssim \sum_{k=0}^{N-1} 2^{-k} \left( \sum_{k \leq |j| < k+1} \left( \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \eta_y \rangle^{\frac{1}{2}} G \|_{S_{\varepsilon}'} + \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \eta_y \rangle^{\frac{1}{2}} \Lambda_0\|_{L^2}\right) \right)$$

$$\lesssim \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \eta_y \rangle^{\frac{1}{2}} G \|_{S_{\varepsilon}'} + \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \eta_y \rangle^{\frac{1}{2}} \Lambda_0\|_{L^2}.$$

And if we repeat the argument in (4.5)-(4.8), for each fixed $j$, we have

$$\|P_j |\partial_t|^{\frac{1}{2}} \Lambda_2\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))}$$

$$\leq C_{\varepsilon} \|\partial_t |\partial_t|^{\frac{1}{2}} H\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))}$$

Thus,

$$\sum_{k=0}^{N-1} 2^{-k} \left( \sum_{k \leq |j| < k+1} \|P_j |\partial_t|^{\frac{1}{2}} \Lambda_2\|_{\text{collapsing}} \right)$$

$$\lesssim \sum_{k=0}^{N-1} \varepsilon 2^{-k} \left( \sum_{k \leq |j| < k+1} \|\partial_t |\partial_t|^{\frac{1}{2}} H\|_{\text{collapsing}} \right)$$

$$\lesssim \varepsilon \|\partial_t |\partial_t|^{\frac{1}{2}} H\|_{\text{collapsing}}.$$

It remains to control the terms involving $\Lambda_3$, note that since $\hat{v}$ is supported in the unit ball centered at origin, we have

$$S P_j \Lambda_3 = P_j N^2v(N(x - y)) \Lambda(t, x, y)$$

$$= P_j N^2v(N(x - y)) \sum_{j': |j' - j| \leq 1} P_{j'} \Lambda(t, x, y).$$

Thus we have the following analog of (4.11)

$$\|P_j |\partial_t|^{\frac{1}{2}} \Lambda_3\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))}$$

$$\leq C \sum_{j': |j' - j| \leq 1} \varepsilon \|P_{j'} |\partial_t|^{\frac{1}{2}} \Lambda\|_{L^\infty(d(x-y))L^2(dt)L^2(d(x+y))}.$$
Thus,

\[
\sum_{k=0}^{N-1} \sum_{k \leq |j| < k+1} 2^{-k} \left( \left\| P_j |\partial_t|^{\frac{3}{2}} \Lambda_3 \right\|_{collapsing} \right)
\]

(4.14)

\[
\lesssim \sum_{k=0}^{N-1} \epsilon 2^{-k} \left( \sum_{k \leq |j| < k+1} \sum_{|j'-j| \leq 1} \left\| P_{j'} |\partial_t|^{\frac{3}{2}} \Lambda \right\|_{collapsing} \right).
\]

By choosing \(\epsilon\) small enough, the right hand side of (4.14) is bounded by \(\frac{1}{2} \| \partial_t |^{\frac{3}{2}} \Lambda \|_N\), thus the proof of (4.4) is complete.

To prove (4.2) when \(P_{|\xi-\eta| < 20N}\) is replace by \(P_{|\xi| < 20N}\) or \(P_{|\eta| < 20N}\), we use a complete similar argument, the only necessary change is to replace the use of Bernstein’s inequality in (4.5) by Bernstein’s inequality in rotated coordinates.

\[
\square
\]

Proof of Theorem 2.1

To prove Theorem 2.1, it remains to control \(\| \Lambda \|_{S_{x,y}}\), where \(\cdot \|_{S_{x,y}}\) is defined as in (2.1).

Case 1. Let us first assume \(|\xi + \eta| \geq \frac{N}{100}\), the norms we are going to control are different for the case \(|\xi + \eta| < \frac{N}{100}\), but in both cases the norms contains \(\| \Lambda \|_{S_{x,y}}\). In this case, the norm with respect to which the potential is a perturbation should be

\[
\| \Lambda \|_N = \| (\nabla_x)^{\frac{3}{2}} (\nabla_y)^{\frac{3}{2}} \Lambda \|_S
\]

(4.15)

\[
+ \| P_{|\xi| \geq 10N} P_{|\eta| < 10N} (\nabla_x)^{\frac{3}{2}} \Lambda \|_{collapsing}
\]

(4.16)

\[
+ \| P_{|\xi| < 10N} P_{|\eta| \geq 10N} (\nabla_y)^{\frac{3}{2}} \Lambda \|_{collapsing}
\]

(4.17)

where \(\| \cdot \|_S\) is defined in (2.2) and \(\| \cdot \|_{collapsing}\) is defined as in (2.3). Here we are abusing notations a bit by using \(\Lambda\) to denote \(P_{|\xi + \eta| \geq \frac{N}{100}} \Lambda\). The projection on \(|\xi + \eta| \geq \frac{N}{100}\) is necessary, as we shall see later in the proof, we do not know how to control the collapsing norms in (4.10) and (4.17) without this assumption.

Let’s first control the low frequency, let \(P_{<N} = P_{|\xi| < 10N} P_{|\eta| < 10N}\) and \(P_{>N} = I - P_{<N}\). Note that

\[
S(\nabla_x)^{\frac{3}{2}} (\nabla_y)^{\frac{3}{2}} P_{<N} \Lambda
\]

\[
= P_{<N} (\nabla_x)^{\frac{3}{2}} (\nabla_y)^{\frac{3}{2}} N^2 v(N(x-y)) P_{|\xi-\eta| < 20N} \Lambda
\]

\[
+ (\nabla_x)^{\frac{3}{2}} (\nabla_y)^{\frac{3}{2}} P_{<N} G(t,x,y)
\]

(4.18)

\[
+ P_{<N} (\nabla_x)^{\frac{3}{2}} (\nabla_y)^{\frac{3}{2}} N^2 v(N(x-y)) H(t,x,y)
\]

\[
\sim P_{<N} N^3 v(N(x-y)) P_{|\xi-\eta| < 20N} \Lambda
\]

\[
+ (\nabla_x)^{\frac{3}{2}} (\nabla_y)^{\frac{3}{2}} P_{<N} G(t,x,y)
\]

\[
+ P_{<N} N^3 v(N(x-y)) H(t,x,y)
\]

Here we are abusing the notation a bit by writing

\[
P_{<N} (\nabla_x)^{\frac{3}{2}} (\nabla_y)^{\frac{3}{2}} N^2 v(N(x-y)) P_{|\xi-\eta| < 20N} \sim P_{<N} N^3 v(N(x-y)) P_{|\xi-\eta| < 20N},
\]
since, as a result of Bernstein’s inequality, for all \(1 \leq p, q \leq \infty\), we have

\[
\|P_{<N}\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} N^2 v(N(x-y))P_{[\xi-\eta]<20N} A\|_{L^p(x-y)L^q(x+y)} \\
\lesssim \|P_{<N}N^3 v(N(x-y))P_{[\xi-\eta]<20N} A\|_{L^p(x-y)L^q(x+y)},
\]

which is harmless for our purposes. We shall use the same notation \(\sim\) repeatedly in the later arguments. Also, strictly speaking, \(P_{<N}\) instead of \(\|\) will not make a essential difference in our argument. But it will not make a essential difference in our argument.

By Strichartz estimate (Theorem 3.1), we have

\[
\|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{<N} A\|_{S} \\
\lesssim \|P_{[\xi-\eta]<20N}N^3 v(N(x-y))A\|_{L^2(dt) L^2(d(x-y))L^2(d(x+y))} \\
+ \|P_{[\xi-\eta]<20N}N^3 v(N(x-y))H\|_{L^2(dt) L^2(d(x-y))L^2(d(x+y))} \\
+ \|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} \|P_{<N} G\|_{s_1'} + \|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} \Lambda_0 \|_{L^2} \\
\lesssim \|v\|_{L^\frac{p}{q}} \left(\|P_{[\xi-\eta]<20N} \langle \nabla_x + y \rangle^\frac{\epsilon}{2} H\|_{collapsing} + \|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} \Lambda_0 \|_{collapsing} \right) \\
+ \|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{<N} G\|_{s_1'} + \|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} \Lambda_0 \|_{L^2},
\]

where in the second inequality we used Bernstein’s inequality and the fact that \(|\xi + \eta| \geq \frac{N}{20}\).

By Theorem 4.1 we have

\[
\|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{<N} A\|_{S} \lesssim \epsilon \|\langle \nabla_x + y \rangle^\frac{\epsilon}{2} H\|_{collapsing} \\
+ \|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{<N} G\|_{s_1'} + \|\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} \Lambda_0 \|_{L^2}.
\]

At high frequency, write

\[
P_{>NA} = P_{[\xi] \geq 10N} P_{[\eta] \geq 10N} A + P_{[\xi] \geq 10N} P_{[\eta] < 10N} A \\
+ P_{[\xi] < 10N} P_{[\eta] \geq 10N} A
\]

(4.22)

\[= I + II + III.\]

To handle the first term \(I\), note that

\[
S\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{[\xi] \geq 10N} P_{[\eta] \geq 10N} A \\
\sim N^2 v(N(x-y))\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{[\xi] \geq 9N} P_{[\eta] \geq 9N} A \\
+ \langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{[\xi] \geq 10N} P_{[\eta] \geq 10N} G(t, x, y) \\
+ N^2 v(N(x-y))\langle \nabla_x \rangle^\frac{\epsilon}{2} \langle \nabla_y \rangle^\frac{\epsilon}{2} P_{[\xi] \geq 9N} P_{[\eta] \geq 9N} H.
\]

(4.23)

The lower bounds on \(|\xi|\) and \(|\eta|\) changed slightly due to convolution with \(\hat{v}_N\), since \(\hat{v}_N\) is compact supported in a set of size \(N\).
Using Strichartz (Theorem 3.1) and Hölder’s inequality, it is not hard to see that
\[
\|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{|\eta| \geq 10N} \|_S \\
\lesssim \varepsilon \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \Lambda\|_S \\
+ \varepsilon \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} H\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \\
+ \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} G\|_{S'_c} + \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_0\|_{L^2},
\]
where the $\varepsilon$ comes from $\|v\|_{L^3/2}$ when applying Hölder’s inequality.

To handle the second term $II$, \[ \mathcal{S}(\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{|\eta| < 10N} \Lambda \]
\[
\sim N^2 v(N(x-y)) \langle \nabla_x \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{|\eta| < 10N} \Lambda \\
+ N^2 v(N(x-y)) \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{10N \leq |\eta| < 11N} \Lambda \\
+ N^3 v(N(x-y)) P_{0N \leq |\xi| < 10N} P_{|\eta| < 10N} \Lambda \\
+ N^2 v(N(x-y)) \langle \nabla_x \rangle^{\frac{1}{2}} P_{|\xi| \geq 9N} P_{|\eta| < 11N} H \\
+ \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{|\eta| < 10N} G(t, x, y).
\]
Again the bounds on $|\xi|$ and $|\eta|$ changed slightly due to convolution with $\hat{v}_N$. By using Strichartz and Hölder’s inequality,
\[
\|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{|\eta| < 10N} \Lambda\|_S \\
\lesssim \varepsilon \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \Lambda\|_S \\
+ \varepsilon \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} H\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \\
+ \|N^3 v(N(x-y)) P_{0N \leq |\xi| < 10N} P_{|\eta| < 10N} \Lambda\|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \\
+ \varepsilon \|\langle \nabla_x \rangle^{\frac{1}{2}} H\|_{L^6(d(x-y))L^2(dt)L^2(d(x+y))} \\
+ \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} G\|_{S'_c} + \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_0\|_{L^2}.
\]
Again, since we are assuming $|\xi + \eta| \geq \frac{N}{10}$, by Bernstein’s inequality and Hölder’s inequality, the third term on the right side can be controlled by
\[
\|N^3 v(N(x-y)) P_{0N \leq |\xi| \leq 10N} P_{|\eta| < 10N} \Lambda\|_{L^2(dt)L^{6/5}(d(x-y))L^2(d(x+y))} \\
\lesssim \|P_{|\xi - \eta| < 20N} \langle \nabla_x + y \rangle^{\frac{1}{2}} \Lambda\|_{collapsing}.
\]
The third term $III$ can be handled in a similar way as the second term.

To finish to discussion for the case $|\xi + \eta| \geq \frac{N}{10}$, we are reduced to estimate
\[
\|P_{|\xi| \geq 10N} P_{|\eta| < 10N} \langle \nabla_x \rangle^{\frac{1}{2}} \Lambda\|_{collapsing} \\
+ \|P_{|\xi| < 10N} P_{|\eta| \geq 10N} \langle \nabla_y \rangle^{\frac{1}{2}} \Lambda\|_{collapsing}.
\]
We shall focus on the first term, since the other term involving $\langle \nabla_y \rangle^{\frac{1}{2}}$ can be dealt with similarly.
Write $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$, where

$$S\Lambda_1 = G, \quad \Lambda_3(0, \cdot) = \Lambda_0$$

(4.28) $S\Lambda_2 = N^2 v(N(x-y)) H(t, x, y)$, with initial conditions $0$

$S\Lambda_3 = N^2 v(N(x-y)) \Lambda(t, x, y)$, with initial conditions $0$.

By Lemma 3.7 we have

$$\|P_{|\xi| \geq 10N} P_{|\eta| < 10N} (\nabla_x)^{\frac{1}{2}} \Lambda_1\|_{\text{collapsing}},$$

(4.29) $\lesssim \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} G\|_{L^6} + \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_0\|_{L^2}$.

To handle $\Lambda_2$, since $|\eta| < 10N$, by Bernstein’s inequality at an angle, we have,

$$\|P_{|\xi| \geq 10N} P_{|\eta| < 10N} (\nabla_x)^{\frac{1}{2}} \Lambda_2\|_{\text{collapsing}}$$

(4.30) $\lesssim N^{1/2} \|P_{|\xi| \geq 10N} P_{|\eta| < 10N} (\nabla_x)^{\frac{1}{2}} \Lambda_2\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))}$.

Note that

$$S(\nabla_x)^{\frac{1}{2}} N^{1/2} P_{|\xi| \geq 10N} P_{|\eta| < 10N} \Lambda_2$$

(4.31) $\sim N^{5/2} v(N(x-y)) \langle \nabla_x \rangle^{\frac{1}{2}} P_{|\xi| \geq 9N} P_{|\eta| < 11N} H$.

By Strichartz and Hölder’s inequality

$$N^{1/2} \|P_{|\xi| \geq 10N} P_{|\eta| < 10N} (\nabla_x)^{\frac{1}{2}} \Lambda_2\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))}$$

(4.32) $\lesssim \|N^{\frac{1}{2}} v(N(x-y)) P_{|\xi| \geq 9N} P_{|\eta| < 11N} H\|_{L^2(dt) L^\infty(d(x-y)) L^2(d(x+y))}$

$\lesssim \varepsilon \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} H\|_{L^\infty(d(x-y)) L^2(dt) L^2(d(x+y))}$.

To handle $\Lambda_3$, since $|\eta| < 10N$, by Bernstein’s inequality at an angle, we still have,

$$\|P_{|\xi| \geq 10N} P_{|\eta| < 10N} (\nabla_x)^{\frac{1}{2}} \Lambda_3\|_{\text{collapsing}}$$

(4.33) $\lesssim N^{1/2} \|P_{|\xi| \geq 10N} P_{|\eta| < 10N} (\nabla_x)^{\frac{1}{2}} \Lambda_3\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))}$.

Note that

$$S(\nabla_x)^{\frac{1}{2}} N^{1/2} P_{|\xi| \geq 10N} P_{|\eta| < 10N} \Lambda_3$$

(4.34) $\sim N^{\frac{1}{2}} v(N(x-y)) \langle \nabla_x \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{|\eta| < 10N} \Lambda$

$+ N^{\frac{1}{2}} v(N(x-y)) \langle \nabla_x \rangle^{\frac{1}{2}} P_{|\xi| \geq 10N} P_{10N \leq |\eta| < 11N} \Lambda$

$+ N^{3} v(N(x-y)) P_{9N \leq |\xi| < 10N} P_{|\eta| < 10N} \Lambda$. 
By Strichartz and Hölder’s inequality
\begin{align}
N^\frac{1}{2}\|\langle\nabla_x\rangle^{\frac{1}{2}}P_{\xi \geq 10N}P_{\eta|<10N}\Lambda\| & L^2(dt)L^6(d(x-y))L^2(d(x+y)) \\
\lesssim & \|\langle\nabla_x\rangle^{\frac{1}{2}}P_{\xi \geq 10N}P_{\eta|<10N}\Lambda\| L^\infty(d(x-y))L^2(dt)L^2(d(x+y)) \\
& + \varepsilon N^{\frac{1}{2}}\|\langle\nabla_x\rangle^{\frac{1}{2}}\Lambda\| L^2(dt)L^6(d(x-y))L^2(d(x+y)) \\
& + \|N^3v(N(x-y))P_{\eta|\leq 10N}\Lambda\| L^2(dt)L^2(d(x+y)) \\
& + \varepsilon\|\langle\nabla_x\rangle^{\frac{1}{2}}(\langle\nabla_y\rangle^{\frac{1}{2}}\Lambda\| L^2(dt)L^6(d(x-y))L^2(d(x+y)) \\
& + \|P_{\xi-\eta}|<20N\langle\nabla_x\rangle^{\frac{1}{2}}\Lambda\|_{collapsing},
\end{align}
(4.35)
where for the second term on the right side we used Bernstein’s inequality and the fact that $|\eta| \geq 10N$, and for the third term in the right side, we used Bernstein’s inequality and the fact that $|\xi + \eta| \geq \frac{x}{10N}$.

**Case 2.** $|\xi + \eta| < \frac{x}{10N}, |\xi - \eta| < 10N$.

In this case, we are only able to control $\|\Lambda\|_{S_{x,y}}$ instead of $\|\Lambda\|_{S}$, recall that $\| \cdot \|_{S_{x,y}}$ is defined as in (2.1). Note that
\begin{align}
S\langle\nabla_x\rangle^{\frac{1}{2}}(\langle\nabla_y\rangle^{\frac{1}{2}}P_{\xi + \eta}|<\frac{x}{10N}P_{\xi - \eta}|<10N\Lambda \\
\sim P_{\xi + \eta}|<\frac{x}{10N}P_{\xi - \eta}|<10N\Lambda N^3v(N(x-y))P_{\xi - \eta}|<20N\Lambda \\
+ \langle\nabla_x\rangle^{\frac{1}{2}}(\langle\nabla_y\rangle^{\frac{1}{2}}P_{\xi + \eta}|<\frac{x}{10N}P_{\xi - \eta}|<10N\Lambda G(t,x,y) \\
+ P_{\xi + \eta}|<\frac{x}{10N}P_{\xi - \eta}|<10N\Lambda N^3v(N(x-y))H(t,x,y)
\end{align}
(4.36)
By using Theorem 3.2 for the first and third term, and the Strichartz estimate (Theorem 3.1) for the second term on the right side of (4.36), we have
\begin{align}
\|\langle\nabla_x\rangle^{\frac{1}{2}}(\langle\nabla_y\rangle^{\frac{1}{2}}P_{\xi + \eta}|<\frac{x}{10N}P_{\xi - \eta}|<10N\Lambda\|_{S_{x,y}} \\
\lesssim \|P_{\xi - \eta}|<20N\Lambda N^3v(N(x-y))\langle\nabla_x\rangle^{\frac{1}{2}}\Lambda\| L^1(d(x-y))L^2(dt)L^2(d(x+y)) \\
+ \|P_{\xi - \eta}|<20N\Lambda N^3v(N(x-y))\langle\partial_y\rangle^{\frac{1}{2}}\Lambda\| L^1(d(x-y))L^2(dt)L^2(d(x+y)) \\
+ \|P_{\xi - \eta}|<20N\Lambda N^3v(N(x-y))\langle\nabla_x\rangle^{\frac{1}{2}}H\| L^1(d(x-y))L^2(dt)L^2(d(x+y)) \\
+ \|\langle\nabla_x\rangle^{\frac{1}{2}}(\langle\nabla_y\rangle^{\frac{1}{2}}\Lambda\|_{S_{x,y}} + \|\langle\nabla_x\rangle^{\frac{1}{2}}(\langle\nabla_y\rangle^{\frac{1}{2}}\Lambda\|_{collapsing} \\
\lesssim \|\Lambda\|_{S_{x,y}} + \|\partial_y\|_{collapsing} + \|\partial_y\|_{collapsing} + \|\langle\nabla_x\rangle^{\frac{1}{2}}\Lambda\|_{collapsing} \\
+ \|\langle\nabla_y\rangle^{\frac{1}{2}}\Lambda\|_{collapsing} + \|\langle\nabla_y\rangle^{\frac{1}{2}}\Lambda\|_{collapsing} \\
+ \|\langle\nabla_y\rangle^{\frac{1}{2}}\Lambda\|_{collapsing} + \|\langle\nabla_y\rangle^{\frac{1}{2}}\Lambda\|_{collapsing}.
\end{align}
(4.37)
By Theorem 3.1 we have
\begin{align}
\|\langle\nabla_x\rangle^{\frac{1}{2}}(\langle\nabla_y\rangle^{\frac{1}{2}}P_{\xi + \eta}|<\frac{x}{10N}P_{\xi - \eta}|<10N\Lambda\| \lesssim \varepsilon\|\langle\nabla_x\rangle^{\frac{1}{2}}H\|_{collapsing} \\
+ \varepsilon\|\langle\partial_t\rangle^{\frac{1}{2}}H\|_{collapsing} + \|\langle\nabla_x\rangle^{\frac{1}{2}}\Lambda\|_{collapsing} + \|\langle\nabla_x\rangle^{\frac{1}{2}}\Lambda\|_{collapsing}.
\end{align}
(4.38)
**Case 3.** $|\xi + \eta| < \frac{N}{10}$, $|\xi - \eta| \geq 10N$.

In this case, $|\xi| \sim |\eta| \sim |\xi - \eta|$, define $P_{>N} = P_{|\xi + \eta| < \frac{N}{10}}P_{|\xi - \eta| \geq 10N}$ The norm with respect to which the potential is a perturbation should be

\begin{align}
        \|A\|_{N} &= \|\frac{1}{2} (\nabla_x \frac{1}{2} (\nabla_y) \frac{1}{2} P_{>N} A)\|_{S_{x,y}} \\
+ &\|\partial_t \frac{1}{2} (\nabla_x \frac{1}{2} P_{>N} A)\|_{L^2(dt) L^6(d(x-y))L^2(d(x+y))} \\
+ &\|\nabla \frac{1}{2} (\nabla_x \frac{1}{2} P_{>N} A)\|_{L^2(dt) L^6(d(x-y))L^2(d(x+y))},
\end{align}

where $P_{>N} = P_{|\xi + \eta| < \frac{N}{10}}P_{|\xi - \eta| \geq 9N}$.

In this case,

\begin{align}
S(\nabla_x) \frac{1}{2} (\nabla_y) \frac{1}{2} P_{>N} A \\
\sim N^2 v(N(x - y))(\nabla_x \frac{1}{2} P_{>N} A) \\
+ \langle \nabla_x \frac{1}{2} (\nabla_y) \frac{1}{2} P_{>N} G(t,x,y) + N^2 v(N(x - y))(\nabla_x \frac{1}{2} P_{>N} H).
\end{align}

Using Strichartz for the second and third terms on the right side, Theorem 3.4 for the first term on the RHS, and Hölder’s inequality, we get

\begin{align}
      \|\langle \nabla_x \frac{1}{2} (\nabla_y) \frac{1}{2} P_{>N} A\|_{S_{x,y}} \\
\leq \varepsilon \|\partial_t \frac{1}{2} (\nabla_x \frac{1}{2} P_{>N} A)\|_{L^2(dt) L^6(d(x-y))L^2(d(x+y))} \\
+ \varepsilon \|\nabla \frac{1}{2} (\nabla_x \frac{1}{2} P_{>N} A)\|_{L^2(dt) L^6(d(x-y))L^2(d(x+y))} \\
+ \|\nabla \frac{1}{2} (\nabla_y) \frac{1}{2} P_{>N} G\|_{S_{x,y}} \\
+ \|\nabla \frac{1}{2} (\nabla_y) \frac{1}{2} H\|_{L^2(dt) L^6(d(x-y))L^2(d(x+y))}.
\end{align}

Thus, it suffice to control

\begin{align}
      \|\partial_t \frac{1}{2} (\nabla_x \frac{1}{2} P_{>N} A)\|_{L^2(dt) L^6(d(x-y))L^2(d(x+y))} \\
+ \|\nabla \frac{1}{2} (\nabla_x \frac{1}{2} P_{>N} A)\|_{L^2(dt) L^6(d(x-y))L^2(d(x+y))} \\
= I + II.
\end{align}

For simplicity, we shall only give the details for the first term $I$, the second term is easier and can be handled in a similar way.
Note that

\[
S(\nabla_{x-y})^{\frac{1}{2}} P_{|\xi-\eta| \geq 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \Lambda \\
\sim N^2 v(N(x-y)) (\nabla_{x-y})^{\frac{1}{2}} P_{|\xi-\eta| \geq 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \Lambda \\
+ N^{\frac{5}{2}} v(N(x-y)) P_{8N \leq |\xi-\eta| < 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \Lambda \\
+ \langle \nabla_{x-y} \rangle^{\frac{1}{2}} P_{|\xi-\eta| \geq 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \Lambda \\
+ N^2 v(N(x-y)) (\nabla_{x-y})^{\frac{1}{2}} P_{|\xi-\eta| \geq 8N} P_{|\xi+\eta| < \frac{\Lambda}{10}} H,
\]

(4.45)

where the bounds on $|\xi-\eta|$ changed slightly due to convolution with $v_N$.

By Theorem 3.6 and Hölder’s inequality, we have

\[
\| \partial_1 (\nabla_{x-y})^{\frac{1}{2}} P_{|\xi-\eta| \geq 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \|_{L^2(dt)L^6(dx-y)L^2(dx+y)} \\
\lesssim \varepsilon \| \partial_1 (\nabla_{x-y})^{\frac{1}{2}} P_{|\xi-\eta| \geq 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \|_{L^2(dt)L^6(dx-y)L^2(dx+y)} \\
+ \| N^{\frac{5}{2}} v(N(x-y)) P_{8N \leq |\xi-\eta| < 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \|_{L^2(dt)L^6(dx-y)L^2(dx+y)} \\
+ \| \langle \nabla_{x-y} \rangle^{\frac{1}{2}} P_{|\xi-\eta| \geq 9N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \|_{L^2(dx-y)L^6(dx+y)L^2(dx+y)} \\
+ \| \langle \nabla_{x-y} \rangle^{\frac{1}{2}} P_{|\xi-\eta| \geq 8N} P_{|\xi+\eta| < \frac{\Lambda}{10}} \|_{L^2(dx-y)L^6(dx+y)L^2(dx+y)}
\]

(4.46)

where the first two terms on the right side of the first inequality corresponds to the first term on the right side of (3.10), and the last three terms on the right side of the first inequality corresponds to the remaining terms on the right side of (3.10).

5. Estimates for the nonlinear equation.

Recall the notation

\[
S_{\pm} = \frac{1}{i} \frac{\partial}{\partial t} - \Delta_x + \Delta_y
\]

From now on, $V_N(x) = N^3 v(Nx)$.

Define $\Gamma = \Gamma_c + \Gamma_p$, $\Lambda = \Lambda_c + \Lambda_p$, where $\Gamma_c = \delta \otimes \phi$, $\Lambda_c = \phi \otimes \phi$, $\Gamma_p = \frac{1}{2N} \sin(k) \circ \sin(k)$, and $\Lambda_p = \frac{1}{2N} \sin(2k)$. Let $\rho(t, x) = \Gamma(t, x, x)$. 
The four relevant equations are

\begin{align}
(5.1) \quad S\Lambda_p + \{V_N * \rho, \Lambda_p\} + & \frac{V_N}{N} \Lambda_p \\
+ \left((V_N \Gamma_p) \circ \Lambda_p + (V_N \Lambda_p) \circ \Gamma_p\right)_{\text{symm}} \\
+ \left((V_N \Gamma_c) \circ \Lambda_p + (V_N \Lambda_c) \circ \Gamma_p\right)_{\text{symm}} = & \frac{V_N}{N} \Lambda_c \\
(5.2) \quad S\pm \Gamma_p + [V_N * \rho, \Gamma_p] + & \left((V_N \Gamma_p) \circ \Lambda_p + (V_N \Lambda_p) \circ \Gamma_p\right)_{\text{skew}} \\
+ \left((V_N \Gamma_c) \circ \Gamma_p + (V_N \Lambda_c) \circ \Lambda_p\right)_{\text{skew}} = 0 \\
(5.3) \quad S\Lambda_c + \{V_N * \rho, \Lambda_c\} + & \left((V_N \Gamma_p) \circ \Lambda_c + (V_N \Lambda_p) \circ \Gamma_c\right)_{\text{symm}} = 0 \\
(5.4) \quad S\pm \Gamma_c + [V_N * \rho, \Gamma_c] + & \left((V_N \Gamma_p) \circ \Gamma_c + (V_N \Lambda_p) \circ \Lambda_c\right)_{\text{skew}} = 0.
\end{align}

Here \((A(x, y))_{\text{symm}} = A(x, y) + A(y, x)\) and \((A(x, y))_{\text{skew}} = A(x, y) - A(y, x)\).

The norm used for \(\Lambda_p\) is called \(N_1(\Lambda)\) and is

\[
\|\Lambda\|_{N_1(\Lambda)} = \|(\nabla_x)^{\frac{1}{2}}(\nabla_y)^{\frac{1}{2}} \Lambda\|_{S_{x,y}} + \|(\nabla_{x+y})^{\frac{1}{2}} \Lambda\|_{\text{low collapsing}} + \|(\partial_1)^{\frac{1}{2}} \Lambda\|_{\text{low collapsing}}
\]

\[
+ \|\|\langle \nabla \rangle^{\frac{1}{2}} P_{[-\eta/2, \eta/2]} \Lambda\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))}
\]

\[
+ \|\|\langle \nabla \rangle^{\frac{1}{2}} P_{[\xi-\eta/2, \xi+\eta/2]} \Lambda\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))}
\]

\[
+ \|\|\partial_1 \langle \nabla \rangle^{\frac{1}{2}} P_{[-\eta/2, \eta/2]} \Lambda\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))},
\]

where the norm \(\| \cdot \|_{\text{low collapsing}}\) is defined as in (2.9). The last three norms in (5.5) does not appear in the statement of Theorem 2.1 but as one can see from the proof of Theorem 2.1 in the previous section, they satisfy the same bounds as the first three norms on the right side of (5.5).

The norm used for \(\Lambda_c\) is called \(N_2(\Lambda)\) and is

\[
\|\Lambda\|_{N_2(\Lambda)} = \|(\nabla_x)^{\frac{1}{2}}(\nabla_y)^{\frac{1}{2}} \Lambda\|_{S_{x,y}} + \|(\nabla_{x+y})^{\frac{1}{2}} \Lambda\|_{\text{collapsing}} + \|\partial_1 \langle \nabla \rangle^{\frac{1}{2}} \Lambda\|_{\text{collapsing}}
\]

\[
+ \|\langle \nabla \rangle^{\frac{1}{2}} P_{[-\eta/2, \eta/2]} \Lambda\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))}
\]

\[
+ \|\|\partial_1 \langle \nabla \rangle^{\frac{1}{2}} P_{[-\eta/2, \eta/2]} \Lambda\|_{L^2(dt) L^6(d(x-y)) L^2(d(x+y))},
\]

We will use the following a priori estimates for \(\Gamma(t, x, \cdot)\) (proved in Lemma 6.2 in [7]).

**Lemma 5.1.** Let the potential \(v\) satisfies (1.16), and the initial conditions satisfy (1.17), we have for all \(0 \leq \alpha \leq 1\),

\[
\|\langle \nabla \rangle^{\alpha} \Gamma\|_{L^{\alpha}(dt) L^{\infty}(d(x-y)) L^{\frac{4}{\alpha}}(d(x+y))} \lesssim 1.
\]

The above estimates also hold for \(\Gamma_p, \Gamma_c\) and \(\Lambda_c\) separately.

We need to use a continuity argument, we have to localize our estimates to intervals \([0, T]\), where the right end of the interval must be a variable \(T\). Define \(\Lambda_{c,T}, \Lambda_{p,T}, \Gamma_{p,T}, \Gamma_{c,T}\)
to be solutions to the standard equations with the RHS multiplied by $\chi_{[0,T]}$:

$$
(5.8) \quad S\Lambda_p,T + \frac{V_N}{N} \Lambda_p,T
= \chi_{[0,T]} \left( - \{V_N \ast \rho, \Lambda_p\} - \left( (V_N \bar{\Gamma}_p) \circ \Lambda_p + (V_N \Lambda_p) \circ \Gamma_p \right)_{symm}
- \left( (V_N \bar{\Gamma}_c) \circ \Lambda_p + (V_N \Lambda_c) \circ \Gamma_p \right)_{symm} - \frac{V_N}{N} \Lambda_c \right)
$$

$$
(5.9) \quad S_\pm \Gamma_p,T = \chi_{[0,T]} \left( - \{V_N \ast \rho, \Gamma_p\} - \left( (V_N \bar{\Gamma}_p) \circ \Gamma_p + (V_N \bar{\Lambda}_p) \circ \Lambda_p \right)_{skew}
- \left( (V_N \bar{\Gamma}_c) \circ \Gamma_p + (V_N \bar{\Lambda}_c) \circ \Lambda_p \right)_{skew} \right)
$$

$$
(5.10) \quad S\Lambda_c,T = \chi_{[0,T]} \left( - \{V_N \ast \rho, \Lambda_c\} - \left( (V_N \bar{\Gamma}_p) \circ \Lambda_c + (V_N \Lambda_p) \circ \Gamma_c \right)_{symm} \right)
$$

$$
(5.11) \quad S_\pm \Gamma_c,T = \chi_{[0,T]} \left( - \{V_N \ast \rho, \Gamma_c\} - \left( (V_N \bar{\Gamma}_p) \circ \Gamma_c + (V_N \bar{\Lambda}_p) \circ \Lambda_c \right)_{skew} \right)
$$

with $\Lambda_c,T(0,\cdot) = \Lambda_c(0,\cdot)$, and similarly for the other three functions. Also, we $\Lambda_{c,T} = \Lambda_c$ in $[0,T]$ (but not outside this interval), and similarly for the other three functions.

**Theorem 5.2.** Let $[0,T]$ be as above, there exist a universal constant $C$ such that

$$
(5.12) \quad \|\Lambda_p,T\|_{N_1(\Lambda)} \leq C \left( \|\nabla_x\frac{d}{2}\nabla_y\frac{d}{2}\Lambda_p(0,\cdot)\|_{L^2} + \epsilon \|\Lambda_p,T\|_{N_2(\Lambda)} \right)
+ C\epsilon \|\Gamma_p,T\|_{S_{\epsilon,v}} + C\epsilon \|\Lambda_p,T\|_{N_1(\Lambda)} \|\Gamma_{p,T}\|_{S_{\epsilon,v}} + C\epsilon \|\Lambda_{c,T}\|_{N_2(\Lambda)}.
$$

The proof is based on Theorem 2.1, there exists a constant $C$ such that

$$
\|\Lambda_p,T\|_{N_1(\Lambda)}
\leq C \left( \|\nabla_x\frac{d}{2}\nabla_y\frac{d}{2}\chi_{[0,T]} \left( \{V_N \ast \rho, \Lambda_p\} + \left( (V_N \bar{\Gamma}_p) \circ \Lambda_p + (V_N \Lambda_p) \circ \Gamma_p \right)_{symm}
+ \left( (V_N \bar{\Gamma}_c) \circ \Lambda_p + (V_N \Lambda_c) \circ \Gamma_p \right)_{symm} \right) \|S_{\epsilon_x}\n+ \epsilon \left( \|\nabla_x\frac{d}{2}\nabla_y\frac{d}{2}\chi_{[0,T]} \left( \|\nabla_x\frac{d}{2}\nabla_y\frac{d}{2}\Lambda_p\|_{L^2(\delta t)L^6(x-y) L^2(\delta (x+y))}
+ \epsilon \left( \|\nabla_x\frac{d}{2}\nabla_y\frac{d}{2}\chi_{[0,T]} \Lambda_c\|_{collapsing} + \epsilon \left( \|\partial_t\chi_{[0,T]} \Lambda_c\|_{collapsing} + \epsilon \|\nabla_x\frac{d}{2}\nabla_y\frac{d}{2}\chi_{[0,T]} \Lambda_c\|_{collapsing} \right.
\right. + \epsilon \left( \|\partial_t\chi_{[0,T]} \Lambda_c\|_{collapsing} \right.
\left. + \epsilon \|\nabla_x\frac{d}{2}\nabla_y\frac{d}{2}\chi_{[0,T]} \Lambda_c\|_{collapsing} \right) \right. \right) \right) \right).
$$

For all terms other than $\|\partial_t\chi_{[0,T]} \Lambda_c\|_{collapsing}$, the subscript $T$ can be trivially added to $\Lambda, \Gamma$ on the RHS. And we also have

$$
(5.13) \quad \|\partial_t\chi_{[0,T]} \Lambda_c\|_{L^\infty(\delta (x-y)) L^2(\delta t) L^2(\delta (x+y))}
= \|\partial_t\chi_{[0,T]} \Lambda_c,T\|_{L^\infty(\delta (x-y)) L^2(\delta t) L^2(\delta (x+y))}
\lesssim \|\partial_t\chi_{[0,T]} \Lambda_{c,T}\|_{L^\infty(\delta (x-y)) L^2(\delta t) L^2(\delta (x+y))}
\lesssim \Lambda_{c,T}\|_{N_2(\Lambda)}.
$$

where in the third line we used the fact that

$$
\|\partial_t\chi_{[0,T]} F\|_{L^\infty(\delta (x-y)) L^2(\delta t) L^2(\delta (x+y))} \lesssim \|\partial_t\chi_{[0,T]} F\|_{L^\infty(\delta (x-y)) L^2(\delta t) L^2(\delta (x+y))},
$$
for any interval $[0,T]$. As remarked in [7], this can be shown by using the equivalent definition

\[
\|\partial_t \sqrt{t} u\|_{L^2} = \int \int \frac{|u(t) - u(s)|^2}{|t - s|^{1 + \frac{1}{4}}} \, dt \, ds,
\]

and the generalized Hardy's inequality from [25].

In the lemma that follows, we estimate the norm of the nonlinear terms in suitable dual Strichartz norms, using the bound (5.1) whenever possible.

Lemma 5.3. Let $[0,T]$ be as above, there exist a universal constant $C$ such that

\[
\|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \left\{ V_N * \rho_L, T \right\} \|_{L^\infty([0,T]) L^\frac{8}{3} (dx) L^2(dy)} \\
\leq C\|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_{p,T} \|_{S_{x,y}},
\]

where the extra $\varepsilon$ factor in (5.3) can be remedied by the smallness of $\|V_N\|_{L^1}$.

Proof. In this case, we essentially view $V_N$ as a $\delta$ distribution, by Minkowski integral inequality, and it suffices to show that

\[
\sup_z \|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \left( \Gamma(t, x, x + z) \Lambda_{p,T}(t, x + z, y) \right) \|_{L^\infty([0,T]) L^\frac{8}{3} (dx) L^2(dy)} \\
\leq C\|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_{p,T} \|_{S_{x,y}},
\]

Using the fractional Leibniz rule from Theorem 5.1 in [7], we have the following estimate, uniformly in $z$:

\[
\|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \left( \Gamma(t, x, x + z) \Lambda_{p,T}(t, x + z, y) \right) \|_{L^\infty([0,T]) L^\frac{8}{3} (dx) L^2(dy)} \\
\leq C\|\langle \nabla_x \rangle^{\frac{1}{2}}\Gamma(t, x, x + z) \|_{L^{\infty}(dt) L^{\frac{8}{3}}(dx)} \|\langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_{p,T} \|_{L^2(dt) L^{\infty}(dx) L^2(dy)} \\
+ C\|\Gamma(t, x, x + z) \|_{L^{\infty}(dt) L^{\frac{8}{3}}(dx)} \|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_{p,T} \|_{L^2(dt) L^2(dt) L^2(dx) L^2(dy)} \\
\leq C\|\langle \nabla_x \rangle^{\alpha} \Gamma(t, x, x + z) \|_{L^{\infty}(dt) L^{\frac{8}{3}}(dx)} \|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_{p,T} \|_{L^2(dt) L^2(dx)} \\
\leq C\|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \Lambda_{p,T} \|_{S_{x,y}^*.}
\]

Here $\alpha$ is can be any number in $(\frac{1}{4}, 1]$, $\frac{3}{4}+$ is a number that is bigger than but can be arbitrary close to $\frac{1}{4}$, similarly $\infty$ is any finite number but can be arbitrary large. We use $\infty$ since we do not have the sharp Sobolev estimate from $L^0$ to $L^\infty$. In the last inequality we used Lemma 5.1. \qed

Since $\Lambda_c$ satisfies the same a priori estimates as $\Gamma$, by the exact same argument we get

Lemma 5.4. Let $[0,T]$ be as above, there exist a universal constant $C$ such that

\[
\|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \left( (V_N \Lambda_c) \circ \Gamma_{p,T} \right) \|_{L^\infty([0,T]) L^\frac{8}{3} (dx) L^2(dy)} \\
\leq C\|\langle \nabla_x \rangle^{\frac{1}{2}}\langle \nabla_y \rangle^{\frac{1}{2}} \Gamma_{p,T} \|_{S_{x,y}^*}.
\]

The result depends on the $a$ priori bounds for $\Lambda_c$, but is true with $\Gamma_{p,T}$ replaced with any other function.
Thus, by Minkowski integral inequality it suffices to show that
\begin{equation}
\left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( (V_N \Lambda_{p,T}) \circ \Gamma_{p,T} \right) \right\rangle_{L^2((0,T))L^2(dy)} 
\leq C \varepsilon \| \Lambda_{p,T} \|_{\mathcal{N}_i(\Lambda)} \| \left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \Gamma_{p,T} \right\rangle_{S_{x,y}}.
\end{equation}

The result is still true if we replace \( \Gamma_{p,T} \) with any other function.

**Proof.** In this case, we shall not treat \( V_N \) as a \( \delta \) distribution. Recall that
\begin{equation}
\left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( (V_N \Lambda_{p,T}) \circ \Gamma_{p,T} \right) \right\rangle_{L^2((0,T))L^2(dy)} 
= \left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \right\rangle_{L^2((0,T))L^2(dy)} N^3 v \left( (N(x-z)) \Lambda_{p,T}(x,z) \Gamma_{p,T}(z,y) \right) dz 
= \int N^3 v \left( (N(z)) \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( \Lambda_{p,T}(x,x+z) \Gamma_{p,T}(x+z,y) \right) \right) dz.
\end{equation}

For fixed \( z \), the following holds, uniformly in \( z \)
\begin{align}
\| \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( \Lambda_{p,T}(t, x, x+z) \Gamma_{p,T}(t, x+z, y) \right) \|_{L^2((0,T))L^2(dy)} 
&\leq C \| \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( \Lambda_{p,T}(t, x, x+z) \Gamma_{p,T}(t, x, y) \right) \|_{L^2(dx)\Gamma_{p,T}(t, x,y)} 
&\leq C \| \Lambda_{p,T}(t, x, x+z) \|_{L^2(dx)\Gamma_{p,T}(t, x,y)} 
&\leq C \| \Lambda_{p,T}(t, x, x+z) \|_{L^2(dx)\Gamma_{p,T}(t, x,y)} 
&\leq C \| \Lambda_{p,T}(t, x, x+z) \|_{L^2(dx)\Gamma_{p,T}(t, x,y)} 
&\leq C \| \Lambda_{p,T}(t, x, x+z) \|_{L^2(dx)\Gamma_{p,T}(t, x,y)} 
&\leq C \| \Lambda_{p,T}(t, x, x+z) \|_{\mathcal{N}_i(\Lambda)}.
\end{align}

Thus, by Minkowski integral inequality it suffices to show that
\begin{equation}
\int \left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( \Lambda_{p,T}(t, x, x+z) \Gamma_{p,T}(t, x+z, y) \right) \right\rangle_{L^2((0,T))L^2(dx)} dz \leq C \| \Lambda_{p,T} \|_{\mathcal{N}_i(\Lambda)},
\end{equation}
which is equivalent to
\begin{equation}
\int \left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( \Lambda_{p,T}(t, x, x+z) \Gamma_{p,T}(t, x, y) \right) \right\rangle_{L^2(dx)\Gamma_{p,T}(t, x,y)} dx 
\leq C \| \Lambda_{p,T} \|_{\mathcal{N}_i(\Lambda)}.
\end{equation}

To see this, note that if for \( \Lambda_{p,T} \), we have \( |\xi - \eta| < 20N, |\xi| < 20N \) or \( |\eta| < 20N \), then the left side of (5.23) is easily controlled by
\begin{equation}
\int \left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( \Lambda_{p,T}(t, x, x+z) \Gamma_{p,T}(t, x, y) \right) \right\rangle_{L^2(dx)\Gamma_{p,T}(t, x,y)} dx 
\leq C \| \Lambda_{p,T} \|_{\text{low collapsing}}.
\end{equation}

Thus if we denote \( P_{>N} = P_{|\xi| \geq 20N} P_{|\xi| \geq 20N} P_{|\xi| \geq 20N} \), we are further reduced to showing that
\begin{equation}
\int \left\langle \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} \left( \Lambda_{p,T}(t, x, x+z) \Gamma_{p,T}(t, x, y) \right) \right\rangle_{L^2(dx)\Gamma_{p,T}(t, x,y)} dx 
\leq C \| \Lambda_{p,T} \|_{\mathcal{N}_i(\Lambda)}.
\end{equation}
As in the proof of main theorem for the linear equation, we shall divide our discussion into two cases.

**Case 1:** \( |\xi + \eta| < \frac{N}{10} \).
In this case, by Hölder’s inequality,
\[
\int |N^3v(N(x - y))| |(\nabla_{x+y})^{\frac{1}{2}} P_{\geq N} \Lambda_{p,T}(t, x, y)| L^2 dt L^2 \, d(x - y) \\
\leq C N^{\frac{1}{2}} \| (\nabla_{x+y})^{\frac{1}{2}} P_{\geq N} \Lambda_{p,T}(t, x, y) \|_{L^6(d(x-y))} \| L^2 \|_{L^2(d(x-y))} \\
\leq C N^{\frac{1}{2}} \| (\nabla_{x+y})^{\frac{1}{2}} P_{\geq N} \Lambda_{p,T}(t, x, y) \|_{L^2(d(x-y))} \| L^2 \|_{L^2(d(x-y))} \\
\leq C \varepsilon \| (\nabla_{x+y})^{\frac{1}{2}} \Lambda_{p,T}(t, x, y) \|_{L^2(d(x-y))} \| L^2 \|_{L^2(d(x-y))} \\
\leq C \varepsilon \| \Lambda_{p,T} \|_{N^1(\Lambda)},
\]
where in the third inequality we used Bernstein’s inequality along with the fact that $|\xi - \eta| \geq 20N$.

Case 2: $|\xi + \eta| \geq \frac{N}{20}.$

In this case, by Hölder’s inequality,
\[
\int |N^3v(N(x - y))| |(\nabla_{x+y})^{\frac{1}{2}} P_{\geq N} \Lambda_{p,T}(t, x, y)| L^2 dt L^2 \, d(x - y) \\
\leq C N^{\frac{1}{2}} \| (\nabla_{x+y})^{\frac{1}{2}} P_{\geq N} \Lambda_{p,T}(t, x, y) \|_{L^6(d(x-y))} \| L^2 \|_{L^2(d(x-y))} \\
\leq C N^{\frac{1}{2}} \| (\nabla_{x+y})^{\frac{1}{2}} P_{\geq N} \Lambda_{p,T}(t, x, y) \|_{L^2(d(x-y))} \| L^2 \|_{L^2(d(x-y))} \\
+ C N^{\frac{1}{2}} \| (\nabla_{x+y})^{\frac{1}{2}} P_{\geq N} \Lambda_{p,T}(t, x, y) \|_{L^2(d(x-y))} \| L^2 \|_{L^2(d(x-y))} \\
\leq C \varepsilon \| (\nabla_{x+y})^{\frac{1}{2}} \Lambda_{p,T}(t, x, y) \|_{L^2(d(x-y))} \| L^2 \|_{L^2(d(x-y))} \\
\leq C \varepsilon \| \Lambda_{p,T} \|_{N^1(\Lambda)}
\]
where in the third inequality we used Bernstein’s inequality in rotated coordinates along with the fact that $|\xi| \geq 20N$ and $|\eta| \geq 20N$.

\[\square\]

We continue with estimates for $\| \Lambda_{c,T} \|_{N^2(\Lambda)}$. This is an easy version of the previous theorem. Using Lemma 3.2-3.7, Lemma 3.9 and Strichartz estimates (Theorem 3.1), and then applying Lemma 5.3-5.5 in this section to handle the nonlinear terms, we get

**Theorem 5.6.** Let $[0, T]$ be as above, there exist a universal constant $C$ such that
\[
\| \Lambda_{c,T} \|_{N^2(\Lambda)} \leq C \| (\nabla_{x})^{\frac{1}{2}} (\nabla_{y})^{\frac{1}{2}} \Lambda_{c}(0, \cdot) \|_{L^2} + C \varepsilon \| \Lambda_{c,T} \|_{N^2(\Lambda)} + C \varepsilon \| \Lambda_{p,T} \|_{N^1(\Lambda)} \| \Gamma_{c,T} \|_{S_{x,y}}.
\]

Using Strichartz estimates for $S_{\pm}$ and Lemma 5.3-5.5, we get

**Theorem 5.7.** Let $[0, T]$ be as above, there exist a universal constant $C$ such that
\[
\| \Gamma_{c,T} \|_{S_{x,y}} \leq C \| (\nabla_{x})^{\frac{1}{2}} (\nabla_{y})^{\frac{1}{2}} \Gamma_{c}(0, \cdot) \|_{L^2} + C \varepsilon \| \Gamma_{c,T} \|_{S_{x,y}} + C \varepsilon \| \Lambda_{p,T} \|_{N^1(\Lambda)} \| \Lambda_{c,T} \|_{N^2(\Lambda)}
\]
\[
\| \Gamma_{p,T} \|_{S_{x,y}} \leq C \| (\nabla_{x})^{\frac{1}{2}} (\nabla_{y})^{\frac{1}{2}} \Gamma_{p}(0, \cdot) \|_{L^2} + C \varepsilon \| \Gamma_{p,T} \|_{S_{x,y}} + C \varepsilon \| \Lambda_{p,T} \|_{N^1(\Lambda)} + C \varepsilon \| \Lambda_{p,T} \|_{N^1(\Lambda)} \| \Lambda_{p,T} \|_{N^1(\Lambda)}.
\]
For later use, let us denote
\begin{align}
X(T) &= \|\Lambda_{c,T}\|_{N_2(\Lambda)} + \|\Gamma_{c,T}\|_{S_{x,y}} \\
Y(T) &= \|\Lambda_{p,T}\|_{N_1(\Lambda)} + \|\Gamma_{p,T}\|_{S_{x,y}}.
\end{align}
(5.29)
We want to show that \(X(T), Y(T)\) depends continuously on \(T\). To see this, for any fixed \(T \geq 0\), we have
\[
\left| \|\Lambda_{c,T}\|_{N_2(\Lambda)} - \|\Lambda_{c,T+\delta}\|_{N_2(\Lambda)} \right| \lesssim \|\Lambda_{c,T} - \Lambda_{c,T+\delta}\|_{N_2(\Lambda)}.
\]
(5.30)
And note that \(\Lambda_{c,T+\delta} - \Lambda_{c,T}\) satisfy
\[
S(\Lambda_{c,T+\delta} - \Lambda_{c,T}) = \chi_{[T,T+\delta]} \left( - \{V_N * \rho, \Lambda_c\} - \left( (V_N \tilde{\Gamma}_p) \circ \Lambda_c + (V_N \Lambda_p) \circ \Gamma_c \right)_{symm} \right),
\]
with 0 initial condition. It is not hard to see that, by crude energy estimates
\[
\|\Lambda_{c,T+\delta} - \Lambda_{c,T}\|_{N_2(\Lambda)} \leq \delta C(T,N)
\]
for some constant \(C(T,N)\) that depend on \(T, N\), which implies the continuity of \(\|\Lambda_{c,T}\|_{N_2(\Lambda)}\). The continuity of other norms in \(X(T)\) and \(Y(T)\) can be proved similarly.

We can now state and prove the main theorem of this section

**Theorem 5.8.** Assume \(\Lambda, \Gamma\) and \(\phi\) are smooth solutions to the HFB system, with finite energy per particle, uniformly in \(N\) (see \([1.17]\)), which implies \([1.21]-[1.24]\), we have
\[
\|\Lambda_c\|_{N_2(\Lambda)} + \|\Gamma_c\|_{S_{x,y}} \leq C
\]
(5.31)
and
\[
\|\Lambda_p\|_{N_1(\Lambda)} + \|\Gamma_p\|_{S_{x,y}} \leq C.
\]
Proof. By Theorem \([5.2, 5.6, 5.7]\) and the size of initial conditions \([1.21]-[1.24]\), we have
\[
X(T) \leq CC_0 + C\varepsilon X(T) Y(T)
\]
(5.32)
and
\[
Y(T) \leq CC_0 + C\varepsilon Y(T)^2 + C\varepsilon X(T).
\]
At this stage, we will need to assume \(\varepsilon\) is small, where the smallness may depend on \(C_0\). Without loss of generality, let’s assume \(CC_0 = 1\) and \(C\varepsilon \leq \frac{1}{10}\), then we have the following simplified version of (5.32)
\[
X(T) \leq 1 + \frac{1}{10} X(T) Y(T)
\]
(5.33)
and
\[
Y(T) \leq 1 + \frac{1}{10} Y(T)^2 + \frac{1}{10} X(T).
\]
If \(Y(T) \leq 4\), by the first line, we have \(X(T) \leq 2\), and if we plug this into the second line, we get \(Y(T) \leq 3\). By continuity, since \(Y(0) \leq 1\), we always have \(Y(T) \leq 3\), and thus \(X(T) \leq 2\), for all \(T \geq 0\), which concludes our proof. \(\Box\)

6. **Estimates for** \(sh(2k)\), \(p_2 = sh(k) \circ sh(k)\) and \(sh(k)\).

The equations for \(sh(2k)\) are \(N\Lambda_p\) and \(p_2 = N\Gamma_p\) are
\[
S_{\pm} [\Lambda_p] + \{V_N * \rho, sh(2k)\} + \left( (V_N \tilde{\Gamma}_p) \circ sh(2k) + (V_N \Lambda_p) \circ \Gamma_c \right)_{symm} = -\frac{V_N}{2}\Lambda
\]
\[
S_{\pm} p_2 + [V_N * \rho, p_2] + \left( (V_N \Gamma) \circ p_2 + (V_N \Lambda) \circ sh(2k) \right)_{skew} = 0.
\]
(5.34)
To handle the inhomogeneous term \(-\frac{V_N}{2}\Lambda\), we shall need the following lemma.
Lemma 6.1. Let $Su = -\frac{1}{p} \Lambda_p$ with $u(0, \cdot) = 0$, we have
\[ \|u\|_{S_{x,y}} \lesssim \|\Lambda_p\|_{N_1}, \]
where $\|\cdot\|_{N_1}$ is defined as in (5.5).

Note that the above result also hold if one replace $\Lambda_p$ by $\Lambda_c$ and replace $\|\Lambda_p\|_{N_1}$ by $\|\Lambda_c\|_{N_2}$, which is a direct consequence of Theorem 3.2 and Hölder’s inequality.

Proof. First note that if for $\Lambda$, we have $\|\xi - \eta\| < 20N$, $|\xi| < 20N$ or $|\eta| < 20N$, then by Theorem 3.2 and Hölder’s inequality, we have
\[ \|u\|_{S_{x,y}} \leq C \|\Lambda_p\|_{N_1} \]
Thus if we denote $P_{ > N} = P_{|\xi - \eta| \geq 20N} P_{|\xi| \geq 20N} P_{|\eta| \geq 20N}$, if suffices to prove Lemma 6.1 with $\Lambda$ replaced by $P_{ > N} \Lambda$. As in the proof of main theorem for the linear equation, we shall divide our discussion into two cases.

Case 1: $|\xi + \eta| < \frac{N}{10}$.

In this case, by Theorem 3.2 and Hölder’s inequality, we have
\[ \|u\|_{S_{x,y}} \leq C \|V_N \langle \nabla_{x+y} \rangle^\frac{1}{2} P_{ > N} P_{|\xi + \eta| < \frac{N}{10}} \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))} \]
\[ + C \|V_N \partial_t \|_{ > N} P_{|\xi + \eta| < \frac{N}{10}} \Lambda_p \|_{L^1(d(x-y))L^2(dt)L^2(d(x+y))} \]
\[ \leq C \|\Lambda_p\|_{N_1} \]
(6.1)

where in the fourth inequality we used Bernstein’s inequality along with the fact that $|\xi - \eta| \geq 20N$.

Case 2: $|\xi + \eta| \geq \frac{N}{10}$.

In this case, by Strichartz (Theorem 5.1) and Hölder’s inequality,
\[ \|u\|_{S_{x,y}} \leq C \|V_N P_{|\xi + \eta| \geq \frac{N}{10}} \|_{L^2(dt)L^\frac{4}{3}(d(x-y))L^2(d(x+y))} \]
(6.2)
\[ \leq C \|\Lambda_p\|_{N_1} \]
where in the third inequality we used Bernstein’s inequality in rotated coordinates along with the fact that $|\xi| \geq 20N$ and $|\eta| \geq 20N$. $$
Now we shall estimate the other nonlinear terms in dual Strichartz norms
\[
\| (V_N \ast \rho(t, x))\text{sh}(2k)(t, x, y) \|_{L^{\frac{2}{3}}(dt)L^{\frac{8}{3}}(dx)L^2(dy)}
+ \| (V_N \Gamma) \circ \text{sh}(2k) \|_{L^{\frac{2}{3}}(dt)L^{\frac{8}{3}}(dx)L^2(dy)}
\leq C\varepsilon \sup_z \| \Gamma(t, x + z, x) \|_{L^6(dt)L^{\frac{8}{3}}(dx)} \| \text{sh}(2k) \|_{L^2(dt)L^6(dx)L^2(dy)}
\leq C\varepsilon \| \text{sh}(2k) \|_{L^2(dt)L^6(dx)L^2(dy)}.
\]
(6.3)

where we used Sobolev and Lemma 5.1 in the last inequality. The above estimate still hold if we replace \( \Gamma \) by \( \Lambda_c \).

Similarly
\[
\| (V_N \ast \rho(t, x))p_2(t, x, y) \|_{L^{\frac{2}{3}}(dt)L^{\frac{8}{3}}(dx)L^2(dy)}
+ \| (V_N \Gamma) \circ p_2 \|_{L^{\frac{2}{3}}(dt)L^{\frac{8}{3}}(dx)L^2(dy)}
\leq C\varepsilon \| p_2 \|_{L^2(dt)L^6(dx)L^2(dy)}.
\]
(6.4)

And the above estimate still hold if we replace \( \Gamma \) by \( \Lambda_c \). Also, since for fixed \( z \), the following holds, uniformly in \( z \)
\[
\| \Lambda(t, x, x + z)p_2(t, x, z, y) \|_{L^{\frac{2}{3}}(dt)L^{\frac{8}{3}}(dx)L^2(dy)}
\leq \| \Lambda(t, x, x + z) \|_{L^2(dt)L^3(dx)} \| p_2(t, x, y) \|_{L^4(dt)L^3(dx)L^2(dy)}
\leq C\varepsilon \| \text{sh}(2k) \|_{L^2(dt)L^6(dx)L^2(dy)}
\]
(6.5)

If we repeat the argument in the proof of Lemma 5.2 and using 5.2.4, we have
\[
\| (V_N \Lambda_p) \circ p_2 \|_{L^{\frac{2}{3}}(dt)L^{\frac{8}{3}}(dx)L^2(dy)}
\leq C \int |N^3v(Nz)| \langle \nabla_x \rangle^\frac{1}{2} \Lambda_p(t, x, x + z) \| L^2(dt)L^3(dx) \| p_2 \| L^4(dt)L^3(dx)L^2(dy) dy dz
\leq C\varepsilon \| \text{sh}(2k) \|_{L^2(dt)L^6(dx)L^2(dy)}.
\]

Similarly, we also have
\[
\| (V_N \bar{\Lambda}_p) \circ \text{sh}(2k) \|_{L^{\frac{2}{3}}(dt)L^{\frac{8}{3}}(dx)L^2(dy)}
\leq C \int |N^3v(Nz)| \langle \nabla_x \rangle^\frac{1}{2} \bar{\Lambda}_p(t, x, x + z) \| L^2(dt)L^3(dx) \| \text{sh}(2k) \| L^4(dt)L^3(dx)L^2(dy) dy dz
\leq C\varepsilon \| \text{sh}(2k) \|_{L^2(dt)L^6(dx)L^2(dy)}.
\]

If we choose \( \varepsilon \) small enough such that \( C \leq \frac{1}{10} \), combing the above estimates, and using the fact that \( \| \Lambda_p \|_{N_1(\Lambda)} + \| \Lambda_c \|_{N_2(\Lambda)} \lesssim 1 \) from the main theorem of the last section, we get, by Strichartz
\[
\| \text{sh}(2k) \|_{S_{x,y}} + \| p_2 \|_{S_{x,y}}
\leq C \left( \| \text{sh}(2k) \|_{L^2} + \| p_2 \|_{L^2} \right) + C.
\]
(6.6)

7. Estimates for the condensate \( \phi \).
The non-linear equation for $\phi$ can be regarded as a linear equation on a background given by $\Gamma$ and $\Lambda$, for which we already have estimates:

$$\left\{ \frac{1}{\ell} \partial_t - \Delta_{x_1} \right\} \phi(x_1) \quad = -\int dy \{ V_N(x_1 - y) \Gamma(y, y) \} \phi(x_1) \quad -\int dy \{ V_N(x_1 - y) \Gamma_p(y, x_1) \} \phi(y) \quad +\int dy \{ V_N(x_1 - y) \Lambda_p(x_1, y) \} \bar{\phi}(y).$$

Define the standard Strichartz spaces

$$\| \phi \|_S = \sup_{(p,q) \text{ admissible}} \| \phi \|_{L^p(dt) L^q(dx)}.$$

**Proof.** We shall estimate the right hand side of the equation for $\phi$ in dual Strichartz norms, if we repeat the proof of Lemma 5.3-5.5, it is not hard to show

$$\| \langle \nabla \rangle^{\frac{1}{2}} \int dy \{ V_N(x_1 - y) \Gamma(y, y) \} \phi(x_1) \|_{L^6(dt) L^6(dx)} \leq C \varepsilon \| \langle \nabla \rangle^{\frac{1}{2}} \phi \|_{L^2(dt) L^6(dx)}$$

(7.2) $$\| \langle \nabla \rangle^{\frac{1}{2}} \int dy \{ V_N(x_1 - y) \Gamma_p(y, x_1) \} \phi(y) \|_{L^6(dt) L^6(dx)} \leq C \varepsilon \| \langle \nabla \rangle^{\frac{1}{2}} \phi \|_{L^2(dt) L^6(dx)}$$

$$\| \langle \nabla \rangle^{\frac{1}{2}} \int dy \{ V_N(x_1 - y) \Lambda_p(x_1, y) \} \bar{\phi}(y) \|_{L^6(dt) L^6(dx)} \leq C \varepsilon \| \langle \nabla \rangle^{\frac{1}{2}} \phi \|_{L^4(dt) L^4(dx)}.$$

Thus,

$$\| \langle \nabla \rangle^{\frac{1}{2}} \phi \|_S \leq C \| \langle \nabla \rangle^{\frac{1}{2}} \phi(0, \cdot) \|_{L^2} + 3C \varepsilon \| \langle \nabla \rangle^{\frac{1}{2}} \phi \|_S.$$

This gives us desired result by taking $\varepsilon C \leq \frac{1}{11}$.

8. **Remarks on the collapsing norm of $\Lambda$.**

In this section, we shall see how we can use Theorem 2.1 to prove the following

**Theorem 8.1.** Let $\Lambda$ satisfy

$$\text{SA}(t, x, y) + N^2 v(N(x - y)) \Lambda(t, x, y) = G(t, x, y), \quad \Lambda(0, \cdot) = \Lambda_0$$

where we assume $v$ satisfy (2.12), we have

$$\| \langle \nabla \rangle^{\frac{1}{2}} A \|_{\text{collapsing}} + \| \partial_t \| A \|_{\text{collapsing}}$$

$$\lesssim \| \langle \nabla \rangle^{\frac{1}{2}} (\nabla_y)^{\frac{1}{2}} G \|_{S^r} + \| \langle \nabla \rangle^{\frac{1}{2}} (\nabla_y)^{\frac{1}{2}} \Lambda_0 \|_{L^2}.$$

Recall that we already have desired estimates if we replace $\| \cdot \|_{\text{collapsing}}$ by $\| \cdot \|_{\text{low collapsing}}$, by using the results in Theorem 2.1

**Proof.** We shall first treat the homogeneous equation, let

$$\text{SA}(t, x, y) + N^2 v(N(x - y)) \Lambda(t, x, y) = 0$$

$$\Lambda(0, \cdot) = \Lambda_0.$$

Let $H = -\Delta_x - \Delta_y + N^2 v(N(x - y))$, $H_0 = -\Delta_x - \Delta_y$. If we let $W$ denote the wave operator in Yajima’s paper [24], acting in $x - y$ direction, we have $e^{itH} = W e^{itH_0} W^*$. 

where * denotes the dual operator. We also have, for the potential v satisfying \((2.12)\), \(W\) is a bounded operator from \(L^p \to L^p\) for any \(1 \leq p \leq \infty\), with bound independent of \(N\) (see [11] Proposition 5.1). Moreover, by calculating the integral kernel of \(W\) explicitly, the \(L^p \to L^p\) boundedness of \(W\) extends to the space of \(L^2\) valued function, ). Thus,

\[
\sup_{x-y} \| (\nabla x+y) \frac{\Lambda}{\Lambda} e^{-itH} \Lambda_0 \|_{L^2(dtd(x+y))} + \sup_{x-y} \| | \frac{\partial}{\partial t} | 1/4 e^{-itH} \Lambda_0 \|_{L^2(dtd(x+y))} \\
= \sup_{x-y} \| W (\nabla x+y) \frac{\Lambda}{\Lambda} e^{itH_0} W^* \Lambda_0 \|_{L^2(dtd(x+y))} + \sup_{x-y} \| W | \frac{\partial}{\partial t} | 1/4 e^{itH_0} W^* \Lambda_0 \|_{L^2(dtd(x+y))} \\
\lesssim \sup_{x-y} \| (\nabla x+y) \frac{\Lambda}{\Lambda} e^{itH_0} W^* \Lambda_0 \|_{L^2(dtd(x+y))} + \sup_{x-y} \| | \frac{\partial}{\partial t} | 1/4 e^{itH_0} W^* \Lambda_0 \|_{L^2(dtd(x+y))} \\
\lesssim \| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} W^* \Lambda_0 \|_{L^2(dx dy)},
\]

where we used Lemma 3.8 in the last inequality.

Recall \(W^* = \lim_{t \to -\infty} e^{itH_0} e^{-itH}\), where the limit exists in the strong operator topology, see e.g., Proposition 5.1 in [11] and also [13] for a proof of the existence of strong limit when \(v\) satisfy \((2.12)\). Thus, for each fixed \(g \in C_0^\infty (\mathbb{R}^6)\) with \(\| g \|_2 = 1\)

\[
\langle (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} W^* \Lambda_0, g \rangle \\
= (W^* \Lambda_0, (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} g) \\
= \lim_{t \to -\infty} (e^{itH_0} e^{-itH} \Lambda_0, (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} g) \\
= \lim_{t \to -\infty} (e^{-itH} \Lambda_0, (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} e^{-itH_0} g) \\
= \lim_{t \to -\infty} \| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} e^{-itH} \Lambda_0, e^{-itH_0} g \|_{L^2(dx dy)} \\
\leq \| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} e^{-itH} \Lambda_0 \|_{L^2(dx dy)} \\
\leq \| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} e^{-itH} \Lambda_0 \|_{S_{x,y}} \\
\leq \| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} \Lambda_0 \|_{L^2},
\]

where the second equality is a consequence of the existence of strong limit, and in the last inequality we used the special case of Theorem 2.4 with \(G = H = 0\). By taking supremum among all choices of \(g\), we have

\[
\| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} W^* \Lambda_0 \|_{L^2(dx dy)} \lesssim \| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} \Lambda_0 \|_{L^2}.
\]

Now we shall consider the inhomogeneous equation, let

\[
\Lambda(t, x, y) + N^2 v (N(x - y)) \Lambda(t, x, y) = G(t, x, y), \quad \Lambda(0, \cdot) = 0.
\]

If we let \(H\) be defined as above, we have by Duhamel’s formula

\[
\Lambda = \int_0^t e^{-i(t-s)H} G(s, \cdot) ds.
\]

We shall first show that

\[
\sup_{x-y} \| (\nabla x+y) \frac{\Lambda}{\Lambda} \int_0^t e^{-i(t-s)H} G(s, \cdot) ds \|_{L^2(dx dy)} \\
\lesssim \| (\nabla x) \frac{\Lambda}{\Lambda} (\nabla y) \frac{\Lambda}{\Lambda} G \|_{S_{x,y}}.
\]
To prove (8.6), it suffices to show that, for any fixed $x - y$ and fixed $T > 0$,

$$
\|\chi_{[0,T]}(t)(\nabla_x + y)^{\frac{1}{2}} \int_0^t e^{-i(t-s)H}G(s,\cdot)ds\|_{L^2(dtd(x+y))}
\lesssim \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} G\|_{S_T'},
$$

which, by the Christ–Kiselev lemma, is a consequence of

$$
(8.7)
\|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} G\|_{S_T'}.
$$

If we apply the homogeneous estimates proved above, the left side of (8.7) is bounded by

$$
(8.8)
\|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} e^{iTH} \int_0^T e^{-i(T-s)H}G(s,\cdot)ds\|_{L^2(dx'dy')}
\lesssim \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} G\|_{S_T'},
$$

where in the third line we used the special case of Theorem 2.1 with $G = H = 0$ (estimates for the homogeneous equation), and in the last line we used the special case of Theorem 2.1 with $\Lambda_0 = H = 0$ (inhomogeneous equation with zero initial data).

To prove inhomogeneous estimate for $|\partial_t|^{\frac{1}{2}}$ derivative, as before we cannot use the Christ–Kiselev lemma, we shall follow the ideas in the prove of Lemma 3.9. Write $\Lambda = \Lambda_1 + \Lambda_2$, where

$$
S \Lambda_1 = G(t, x, y), \text{ with initial conditions } 0
S \Lambda_2 = -N^2 v(N(x-y))\Lambda(t, x, y), \text{ with initial conditions } 0.
$$

For both $\Lambda_1$ and $\Lambda_2$, it suffices to consider the region where

$$
(8.9)
\tau^{\frac{1}{2}} \geq 10(1 + |\xi + \eta|), |\xi| \geq 20N, |\eta| \geq 20N \text{ and } |\xi - \eta| \geq 20N,
$$

since otherwise we have $|\partial_t|^{\frac{1}{2}} \Lambda\|_{collapsing} \lesssim \|\langle \nabla_x + y\rangle^{\frac{1}{2}} \Lambda\|_{collapsing}$, or we already know how to control collapsing norm when $|\xi| < 20N, |\eta| < 20N$, or $|\xi - \eta| < 20N$.

The estimate for $\Lambda_1$ just follows from Lemma 3.9 directly, and for $\Lambda_2$, we claim that it suffices to show

**Proposition 8.2.** Let $Su = f$ with $u(0,\cdot) = 0$, then if the Fourier support $\tau, \xi, \eta$ of $u$ satisfy (8.9), we have

$$
(8.10)
\|\partial_t |^{\frac{1}{2}} u\|_{collapsing} \lesssim \min \left\{ \|\langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} f\|_{L^2(dtdx'dy')} \right\},
$$

$$
\|\partial_t |^{\frac{1}{2}} \langle \nabla_x + y\rangle^{\frac{1}{2}} f\|_{L^2(dtdx'dy')} + N^{-\frac{1}{2}} \|\partial_t |^{\frac{1}{2}} f\|_{L^2(dx'dy')} \right\}.
$$

**Remark 8.3.** Note that the $\tau$ support of $u$ may be different from $f$, so $\tau$ support of $f$ may not satisfy (8.9), but the $\xi, \eta$ support of $f$ does satisfy (8.9).
Thus, by Proposition 8.2

\[
S \sum_{|\xi| \geq 20N} |P_{|\eta| \geq 20N} A_2 \sim -N^2 v(N(x-y)) P_{|\xi| \geq 19N} P_{|\eta| \geq 19N} \Lambda(t, x, y).
\]

Thus, by Proposition 8.2

\[
|||\partial_t \frac{1}{2} A_2|||_{\text{collapsing}} \\
\leq \ll \langle \nabla_x \rangle \frac{1}{2} \langle \nabla_y \rangle \frac{1}{2} N^2 v(N(x-y)) P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x, y) || L^2(dt) L^2(d(x-y)) L^2(d(x+y)) \\
+ |||\partial_t \frac{1}{2} N^2 v(N(x-y)) P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x, y) || L^2(dt) L^2(d(x-y)) L^2(d(x+y)) \\
\leq \ll |||\partial_t \frac{1}{2} \langle \nabla_x \rangle \frac{1}{2} P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x, y) || L^2(dt) L^2(d(x-y)) L^2(d(x+y)) \\
+ N^2 |||\partial_t \frac{1}{2} P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x) || L^2(dt) L^2(d(x-y)) L^2(d(x+y)) \\
\leq |||\partial_t \frac{1}{2} \langle \nabla_x \rangle \frac{1}{2} P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x) || L^2(dt) L^2(d(x-y)) L^2(d(x+y))
\]

where in the second inequality we used the fact that when $|\xi| \geq 19N$, $\langle \nabla_x \rangle \frac{1}{2}$ essentially only fall on $\Lambda$, as well as Hölder's inequality. Recall that when $|\xi + \eta| > \frac{N}{10}$, the last line appears as part of the norm for $\Lambda$ in the proof of Theorem 2.1 (see (4.15)), thus it is bounded by $|||\langle \nabla_x \rangle \frac{1}{2} \langle \nabla_y \rangle \frac{1}{2} G|||_{S'}$ for $\Lambda$ satisfying (8.5).

Case 2: $|\xi + \eta| < \frac{N}{10}$.

In this case,

\[
S \sum_{|\xi| < \frac{N}{10}} |P_{|\eta| \geq 20N} A_2 \sim -N^2 v(N(x-y)) P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x, y).
\]

Thus, by Proposition 8.2

\[
|||\partial_t \frac{1}{2} A_2|||_{\text{collapsing}} \\
\leq \ll \langle \nabla_x \rangle \frac{1}{2} \langle \nabla_y \rangle \frac{1}{2} N^2 v(N(x-y)) P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x, y) || L^2(dt) L^2(d(x-y)) L^2(d(x+y)) \\
+ |||\partial_t \frac{1}{2} N^2 v(N(x-y)) P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x, y) || L^2(dt) L^2(d(x-y)) L^2(d(x+y)) \\
\leq \ll |||\partial_t \frac{1}{2} \langle \nabla_x \rangle \frac{1}{2} P_{|\xi| < \frac{N}{10}} P_{|\eta| \geq 19N} \Lambda(t, x, y) || L^2(dt) L^2(d(x-y)) L^2(d(x+y))
\]

where in the second inequality we used the fact that when $|\xi - \eta| \geq 19N$, $\langle \nabla_x \rangle \frac{1}{2}$ essentially only fall on $\Lambda$, as well as Hölder’s inequality. And in the last inequality we used Bernstein’s inequality and the fact that $|\xi - \eta| \geq 20N$.

Recall that when $|\xi + \eta| < \frac{N}{10}$, the last line appears as part of the norm for $\Lambda$ in the proof of Theorem 2.1 (see (4.40)), thus it is bounded by $|||\langle \nabla_x \rangle \frac{1}{2} \langle \nabla_y \rangle \frac{1}{2} G|||_{S'}$ for $\Lambda$ satisfying (8.5).

Thus it remains to prove Proposition 8.2 to see this, we shall follow the ideas in the proof of Lemma 3.9.

Proof of Proposition 8.2

Case 1: If $|\tau| \frac{1}{2} > 2(|\xi| + |\eta|)$. 

Write \( u = u^1 + u^2 \), where
\[
\mathcal{F} u^1 = \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2}, \quad \text{this no longer has initial conditions 0}
\]
\( S u^2 = 0, \) a correction so that \( u^1 + u^2 \) has initial condition 0.

In this case, it suffices to control \( u_1 \) since \( u_2 \) is only supported where \( |\tau| = |\xi|^2 + |\eta|^2 \).

The goodness about \( u_1 \) is that it has the same Fourier support with \( f \). The strategy is based on
\[
\langle |\partial_t|^\frac{1}{2} u_1 \rangle_{L^2(d(x-y))L^2(d(x+y))dt} = \| \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2} \|_{L^\infty(d(x-y))L^2(\mathbb{R}^d)}
\]
(8.11)
\[
\lesssim \int |\tau|^\frac{1}{2} \left| \frac{\mathcal{F} f}{\tau + |\xi|^2 + |\eta|^2} \right| d(\xi - \eta)_{L^2(\mathbb{R}^d)}.
\]
(8.12)

By Cauchy-Schwarz, we have
\[
\text{RHS}(8.12) \lesssim \left| \frac{|\tau|^\frac{1}{2}}{|\tau|} \right| \int_{|\xi - \eta| < |\tau|^\frac{1}{2}} |\mathcal{F} f|d(\xi - \eta)_{L^2(\mathbb{R}^d)}
\]
\[
\lesssim A \| \nabla_y |\nabla_x|^{-\frac{1}{2}} f \|_{L^2(d(x-y))d(x+y)}
\]
where
\[
A = \sup_{\tau, \xi + \eta \in \mathbb{R}} \left| \frac{|\tau|^\frac{1}{2}}{|\tau|} \left( \int_{|\xi| < |\tau|^\frac{1}{2}} |\xi|_{\mathbb{R}^d}d(\xi - \eta) \right) \right|^\frac{1}{2}.
\]
(8.13)

Changing variables, this is something like
\[
A = \sup_{\tau, |u| < |\tau|^\frac{1}{2}} \left| \frac{|\tau|^\frac{1}{2}}{|\tau|} \left( \int_{|v| < |\tau|^\frac{1}{2}} |u + v|_{\mathbb{R}^d}dv \right) \right|^\frac{1}{2}.
\]
(8.14)

After a change of variables this is reduced to \( \tau = 1 \), and \( A \) is bounded. By Sobolev’s estimate at an angle, we have
\[
\| \nabla_y |\nabla_x|^{-\frac{1}{2}} f \|_{L^2(d(x-y))d(x+y)} \lesssim \| \nabla_x |\nabla_y|^{-\frac{1}{2}} f \|_{L^2(d(x-y))L^2(d(x+y))L^2(d(x+y))}
\]
(8.15)

Similarly, by Cauchy-Schwarz, we have
\[
\text{RHS}(8.12) \lesssim \left| \frac{|\tau|^\frac{1}{2}}{|\tau|} \right| \int_{|\xi - \eta| < |\tau|^\frac{1}{2}} |\mathcal{F} f|d(\xi - \eta)_{L^2(\mathbb{R}^d)}
\]
\[
\lesssim A \| \partial_t |\nabla_x|^{-\frac{1}{2}} f \|_{L^2(d(x-y))d(x+y)}
\]
where
\[
A = \sup_{\tau, \xi + \eta \in \mathbb{R}} \frac{1}{|\tau|} \left( \int_{|\xi| < |\tau|^\frac{1}{2}} |\xi - \eta| d(\xi - \eta) \right)^\frac{1}{2},
\]
(8.16)

which is bounded. By Sobolev’s inequality, we have
\[
\| \partial_t |\nabla_x|^{-\frac{1}{2}} f \|_{L^2(d(x-y))d(x+y)} \lesssim \| \partial_t |\nabla_x|^{-\frac{1}{2}} f \|_{L^2(d(x-y))L^2(d(x+y))L^2(d(x+y))}.
\]
Case 2: $|\xi| + |\eta| > 2|\tau|^{\frac{1}{2}}$

In this case, since we are assuming $|\tau|^{\frac{1}{2}} > 10(1 + |\xi + \eta|)$, we have $|\xi - \eta| > |\xi + \eta|$, and also $|\xi - \eta| > |\tau|^{\frac{1}{2}}$. As before, it suffices to bound the right hand side of (8.12). By Cauchy-Schwarz

$$\text{RHS} (8.12) \lesssim \| \int_{2|\xi - \eta| > |\xi + \eta| + |\tau|^{\frac{1}{2}}} \frac{|\tau|^{\frac{1}{2}} |F f|}{|\xi - \eta|^2} d(\xi - \eta) \|_{L^2(\tau^2 d(\xi + \eta))}$$

(8.17)

$$\lesssim A \| \langle \nabla_y, \nabla_x \rangle^{\frac{1}{2}} |\tau|^{\frac{1}{2}} |F f| \|_{L^2(\tau^2 d(x - y) d(x + y))}$$

$$\lesssim A \| \langle \nabla_x, \nabla_y \rangle^{\frac{1}{2}} |\tau|^{\frac{1}{2}} |F f| \|_{L^2(\tau^2 d(x - y) d(x + y))}$$

where we used Sobolev’s estimate at an angle in the last inequality. In this case,

$$A^2 = \sup_{\xi + \eta, \tau} \int_{2|\xi - \eta| > |\xi + \eta| + |\tau|^{\frac{1}{2}}} \frac{|\tau|^{\frac{1}{2}} |\xi|}{|\xi - \eta|^2} d(\xi - \eta).$$

Again we scale to $|\tau|^{\frac{1}{2}} + |\xi + \eta| = 1$ and have to estimate

$$\int_{|v| > 1} \frac{1}{|v|^2} \frac{|u + v|}{|u - v|} dv.$$

This is bounded uniformly in $|u| < 1$.

Similarly, since we are assuming $|\xi - \eta| \geq 20N$, by Cauchy-Schwarz

$$\text{RHS} (8.12) \lesssim \| \int_{|\xi - \eta| > 20N} \frac{|\tau|^{\frac{1}{2}} |F f|}{|\xi - \eta|^2} d(\xi - \eta) \|_{L^2(\tau^2 d(\xi + \eta))}$$

$$\lesssim N^{-\frac{1}{2}} \| \langle \partial_k, \nabla \rangle^{\frac{1}{2}} |F f| \|_{L^2(\tau^2 d(x - y) d(x + y))}.$$

Case 4 : $\frac{1}{2}(|\xi| + |\eta|) < |\tau|^{\frac{1}{2}} < 2(|\xi| + |\eta|)$, In this case, since we are assuming that $|\tau|^{\frac{1}{2}} > 10(1 + |\xi + \eta|)$, this implies $|\tau|^{\frac{1}{2}} \sim |\xi| \sim |\eta| \sim |\xi - \eta|$.

We shall use the decomposition $u = \sum_{k=0}^{\infty} P_{|\tau|^{\sim} 2^k} u$, and the square function estimate

$$\| \langle \partial_k, \nabla \rangle^{\frac{1}{2}} u \|_{L^\infty(\tau^2 (d(x - y)) L^2(\tau^2 (d(x + y)) d\tau)}$$

(8.19)

$$\sim \left( \sum_{k=0}^{\infty} \| P_{|\tau|^{\sim} 2^k} \partial_k \nabla u \|_{L^\infty(\tau^2 (d(x - y)) L^2(\tau^2 (d(x + y)) d\tau)} \right)^{\frac{1}{2}}$$

$$\lesssim \left( \sum_{k=0}^{\infty} 2^{2k} \| P_{|\tau|^{\sim} 2^k} u \|_{L^2(\tau^2 (d(x - y)) L^2(\tau^2 (d(x + y)) d\tau)}}^2 \right)^{\frac{1}{2}}.$$

For each fixed dyadic piece $P_{|\tau|^{\sim} 2^k} u$, in the current case, we have

$$P_{|\tau|^{\sim} 2^k} u = P_{|\tau|^{\sim} 2^k} P_{|\xi|^{\sim} 2^{k/2}} P_{|\eta|^{\sim} 2^{k/2}} u,$$

which implies

$$\| P_{|\tau|^{\sim} 2^k} u \|_{\text{collapsing}} \lesssim \| \langle \nabla_x \rangle^{\frac{1}{2}} P_{|\xi|^{\sim} 2^{k/2}} P_{|\eta|^{\sim} 2^{k/2}} P_{|\xi|^{\sim} 2^{k/2}} u \|_{L^2(\tau^2) L^6(\tau^2 (d(x - y)) L^2(\tau^2 (d(x + y)) d\tau)}$$

$$\lesssim 2^{-k/2} \| \langle \nabla_x \rangle^{\frac{1}{2}} \langle \nabla_y \rangle^{\frac{1}{2}} P_{|\xi|^{\sim} 2^{k/2}} P_{|\eta|^{\sim} 2^{k/2}} P_{|\xi|^{\sim} 2^{k/2}} u \|_{L^2(\tau^2) L^6(\tau^2 (d(x - y)) L^2(\tau^2 (d(x + y)) d\tau)}.$$
where we used Bernstein’s inequality in rotated coordinates twice, see e.g., Lemma 3.1 in [7]. Thus,

\[
\| \partial_t \hat{\tau} u \|_{L^\infty(d(x-y))L^2(d(x+y)dt)} \\
\lesssim \left( \sum_{k=0}^{\infty} \| (\nabla_x)^{\frac{\beta}{2}} (\nabla_y)^{\frac{\beta}{2}} P_{|\xi-\eta|>2^k} u \|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))} \right)^{\frac{1}{2}}
\]

(8.20)

\[
\lesssim \left( \sum_{k=0}^{\infty} \| (\nabla_x)^{\frac{\beta}{2}} (\nabla_y)^{\frac{\beta}{2}} P_{|\xi-\eta|>2^k} f \|_{L^2(dt)L^{6}(d(x-y))L^2(d(x+y))} \right)^{\frac{1}{2}}
\]

\[
\lesssim \| (\nabla_x)^{\frac{\beta}{2}} (\nabla_y)^{\frac{\beta}{2}} f \|_{L^2(dt)L^{6}(d(x-y))L^2(d(x+y))}
\]

where we used Strichartz (Theorem 3.1) in the second line, and square function estimates in \( y \) the the last line.

Similarly, for each fixed dyadic piece \( P_{|\tau|<2^k} \partial_t \hat{\tau} u \), in this case since \( |\xi-\eta| \sim 2^{\frac{k}{4}} \), by Bernstein’s inequality, we have

\[
\| P_{|\tau|<2^k} \partial_t \hat{\tau} u \|_{\text{collapsing}} \\
\lesssim \| \partial_t \hat{\tau} (\nabla_x \cdot y)^{\frac{\beta}{2}} P_{|\xi-\eta|<2^{\frac{k}{4}}} u \|_{L^2(dt)L^6(d(x-y))L^2(d(x+y))}
\]

(8.21)

\[
\lesssim \| \partial_t \hat{\tau} (\nabla_x \cdot y)^{\frac{\beta}{2}} P_{|\xi-\eta|<2^{\frac{k}{4}}} f \|_{L^2(dt)L^{6}(d(x-y))L^2(d(x+y))}
\]

where in the last inequality we used Theorem 3.5. Thus,

\[
\| \partial_t \hat{\tau} u \|_{L^\infty(d(x-y))L^2(d(x+y)dt)} \\
\lesssim \left( \sum_{k=0}^{\infty} \| \partial_t \hat{\tau} (\nabla_x \cdot y)^{\frac{\beta}{2}} P_{|\xi-\eta|<2^{\frac{k}{4}}} f \|_{L^2(dt)L^{6}(d(x-y))L^2(d(x+y))} \right)^{\frac{1}{2}}
\]

(8.22)

\[
\lesssim \| \partial_t \hat{\tau} (\nabla_x \cdot y)^{\frac{\beta}{2}} f \|_{L^2(dt)L^{6}(d(x-y))L^2(d(x+y))}
\]

where we used square function estimates in \( x - y \) the the last line. \( \square \)

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(X.H.) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK. MD 20742

Email address: xhuang49@umd.edu