How small can thermal machines be? Towards the smallest possible refrigerator

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We investigate the fundamental dimensional limits to thermodynamic machines. In particular we show that it is possible to construct self-contained refrigerators (i.e. not requiring external sources of work) consisting of only a small number of qubits and/or qutrits. We present three different models, one consisting of two qubits, one consisting of a qubit and a qutrit with nearest-neighbour interactions, and one consisting of a single qutrit. We then investigate fundamental limits to their performance; in particular we show that it is possible to cool towards absolute zero.

When over two centuries ago Sadi Carnot [1] set out to study the physics of steam engines – and in the process established thermodynamics, one of the cornerstones of physics – the key to his progress was to abstract from real machines to idealised, “model independent” heat machines. What he found is that although the properties of each machine depend on the details of its construction, the fundamental limit to their efficiency is independent on these details. But can physics indeed be completely left out? Here we return to physics and ask about other fundamental limits, specifically, how small can thermal machines be? Is there a fundamental limit to their size? And, when they are very small, are there additional constraints on their performance? Can small machines work as well as macroscopic machines, or is there a complementarity between size and performance? For example, can small machines be constructed that can cool arbitrarily close to absolute zero, or does size impose a fundamental limit?

In the present paper we approach these questions in the framework of quantum mechanics. Most importantly, quantum mechanics provides a natural and universal notion of “size”, namely the dimension of the Hilbert space that describes the system. It is this measure of size that we will use here, to characterise the thermal machines, not their extent in space or their mass. Nevertheless, it is very often the case that small Hilbert space dimensions are associated to microscopic systems, (atoms, spins, etc.) and hence with small space dimensions and small mass.

The particular thermodynamic machines we study here are “self-contained” refrigerators. By “self-contained” we mean that they do not use any outside source of work. That cooling can be achieved when work is performed on a microscopic system is well-known. Indeed, laser cooling [2, 3, 4] is the work-horse of the entire field of atomic and molecular optics. Such externally driven refrigerators have also been studied theoretically, where they have received much attention recently, e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. However, in this instance one can hardly call the microscopic system that is getting cooled a self-contained fridge – one also needs, for example, a macroscopic laser. In this paper we are interested in accounting for all the degrees of freedom that constitute our fridge; this is the reason for focussing on machines that do not require external work. On the other hand, of course, no thermal engine can work without a source of free energy. As in all thermal engines we provide this by allowing the fridge to be in contact with two heat reservoirs at different temperatures $T_c < T_h$.

Apart from the implications for fundamental physics, we envisage that this work may be relevant to a number of fields. One such field is biology. Here, for example, if cooling of the active site of a protein can be achieved then increased catalysis rates may be possible [18]. It is an intriguing question as to whether simple mechanisms, such as those we describe in this paper, are used by biological systems. A second field is the rapidly developing field of nanotechnology where the benefits from cooling at the atomic scale are clear.

In this work we will present three models. The first refrigerator consist of two qubits (two level systems). Building upon this idea we present a second fridge, consisting of a qubit and a qutrit (three level system) and where all interactions are nearest neighbour. As a last example we present a refrigerator consisting of a single qutrit; arguably the smallest refrigerator possible. Finally we prove that there are no fundamental limits to how close to absolute zero such small refrigerators can cool.

1. MODEL I: TWO QUBITS

The first model we present consists of three qubits, two constituting the refrigerator, and one being the object to be cooled. It is inspired by the method of algorithmic cooling [19], particularly by the few qubit version in [20].

A. Functioning principle

Consider first two qubits, immersed in a bath at temperature $T_c$ (the ‘cold’ bath). Qubit 1 is the object to be cooled, while qubit 2 will eventually play the role of the “spiral” that takes heat from qubit 1 and dissipates it into the environment.

The “free” Hamiltonian $H_0$ for the two qubits is

$$H_0 = E_1 \Pi^{(1)} + E_2 \Pi^{(2)}$$

where $\Pi^{(i)} = \langle 1_i \rangle_1 \langle 1_i \rangle$ is a projector for qubit $i$, and we denote by $|0\rangle$ and $|1\rangle$ the ground state and excited states respectively. The energies of the excited states of the two qubits are $E_1$ and $E_2$ respectively, while their ground state energies are zero.
We take \( E_2 > E_1 \), hence qubit 2 has a larger energy level separation than qubit 1.

At equilibrium each qubit is in a thermal state, \( \tau_i \), which has the form
\[
\tau_i = N_i e^{-E_i \Pi^{(i)}/kT_c}
\]  
where \( N_i = (1 + e^{-E_i/kT_c})^{-1} \) and \( k \) is Boltzmann’s constant.

Since the qubits do not interact the total thermal state is simply the direct product state
\[
\rho_{12} = \tau_1 \otimes \tau_2
\]
\[
= N_1 N_2 e^{-E_1 \Pi^{(1)}/kT_c} e^{-E_2 \Pi^{(2)}/kT_c}. 
\]

A convenient way to represent the thermal state (2) is in terms of the probabilities, \( r_i \) and \( 1 - r_i \), to find the qubit in the eigenstates \( |0\rangle \) and \( |1\rangle \) respectively,
\[
\tau_i = r_i |0\rangle \langle 0| + (1 - r_i) |1\rangle \langle 1|,
\]
where \( r_i = 1/N_i \).

We will denote by \( T_{Eq}^{int} \) the equilibrium temperature of each qubit when refrigeration is occurring. Cooling of qubit 1 means reaching a temperature \( T_{Eq}^{int} < T_{c} \), which corresponds to a larger probability \( r_1^{Eq} > r_1 \) of being in the ground state.

The idea of algorithmic cooling is to increase \( r_1 \) by transferring excitations to the second qubit. Specifically, two particular eigenstates of \( \mathcal{H}^{int} \) are \( |10\rangle \) and \( |01\rangle \). The former appears with probability \( (1 - r_1)r_2 \) while the latter with \( r_1(1 - r_2) \). Suppose now that we perform a unitary \textsc{swap} operation \( U \), which swaps these two states,
\[
|10\rangle \rightarrow |01\rangle
\]
\[
|01\rangle \rightarrow |10\rangle
\]

while leaving the others unchanged. Since we take \( E_2 > E_1 \), the probability \( (1 - r_1)r_2 \) is larger than \( (1 - r_2)r_1 \), that is we are more likely to be in the state \( |10\rangle \) (where qubit 1 is excited) than in \( |01\rangle \) (where it is in the ground state). Thus after the swap we increase the probability of qubit 1 being in the ground state (we cool it) and we increase the probability of qubit 2 being in the excited state (we heat it).

Since the qubits are in contact with a thermal bath, if we don’t do anything else, the qubits will eventually return to the environmental temperature \( T_c \). However, if we keep on repeating the unitary transformation \( U \), qubit 1 will reach an equilibrium temperature \( T_{1}^{Eq} \) lower than \( T_c \) and qubit 2 a temperature \( T_{2}^{Eq} \) larger than \( T_c \).

The procedure described above however requires external work to be performed on the qubits. Indeed, the energy \( E_2 \), of \( |01\rangle \), is larger than the energy \( E_1 \), of \( |10\rangle \). This is accomplished by some external system that induces the unitary transformation \( U \); for example, in an NMR experiment this could be done via a sequence of pulses of an external magnetic field.

The idea behind our fridge model is then to replace the external work by a different source of free energy. Specifically, free energy can be obtained whenever a system has access to two thermal baths at different temperatures. In our model this is accomplished by adding a third qubit, in contact with a thermal bath (the ‘hot’ bath) at a higher temperature \( T_h > T_c \). This qubit plays the role of the “engine”. Thus our fridge consists in qubits 2 and 3 (the spiral and engine respectively); qubit 1 is the object to be cooled.

To enable transitions between different states without an input of external energy, we take the engine qubit to have the energy level spacing \( E_3 \), such that \( E_3 = E_2 - E_1 \). With this condition we now have two degenerate energy eigenstates \( |010\rangle \) and \( |101\rangle \) and hence we can swap the two states without requiring work. The interchange
\[
|01\rangle \rightarrow |010\rangle
\]
\[
|010\rangle \rightarrow |101\rangle
\]
accomplishes on qubits 1 and 2 the transformations (5) and (6).

For transitions between \( |010\rangle \) and \( |101\rangle \) to occur we introduce an interaction Hamiltonian
\[
H_{int} = g (|010\rangle \langle 01 | + |101\rangle \langle 01|)
\]
where \( g \) determines the strength of the interaction.

Furthermore, for simplicity, we will assume that this interaction is taken to be weak compared to the energy level spacing, \( E_i \gg g \). In this regime the interaction will not significantly alter the energy eigenvalues or eigenvectors of the system (which remains governed by \( H_0 \)) and hence we can meaningfully talk about the temperature of the individual qubits, as each qubit will remain in the standard form of a thermal state (2), with \( E_1 \) and \( E_2 \) the same as they were in the absence of interaction.

To understand the functioning of the fridge first note that the interaction Hamiltonian can swap, without impediment, the states \( |010\rangle \) and \( |101\rangle \), since they are degenerate in energy in-so-far as the free Hamiltonian is concerned. However, if all qubits were kept at the same temperature, the system would be at equilibrium since the the probability of the flip (7) is equal to that of (8). In order to drive the transitions in one direction, such as to cool qubit 1, we place the third qubit in contact with a hotter bath. In this way the probability of the state \( |101\rangle \) becomes larger than that of \( |010\rangle \) and so we enhance the probability of the flip (7) and diminish that of (8). It is this biasing of the interaction which has the effect of taking heat from qubit 1 into qubit 2, creating a refrigerator.

In the next section we will describe our model in detail.
this scenario is to imagine that with some probability density \( p_i \) per unit time that each qubit may be thermalised back to its initial thermal state \( \rho_0, i = 2 \). (Note that for qubit 3 the bath temperature is \( T_h \) and not \( T_c \).) Mathematically we model this by the non-unitary evolution

\[
\rho \mapsto \tau_i \text{Tr}_i \rho;
\]

Here \( p_i \) quantifies how well insulated each particle is relative to the bath; when \( p_i \) is small the particle is hardly ever thermalised by the bath.

All together this leads to the master equation

\[
\frac{\partial \rho}{\partial t} = -i[H_0 + H_{\text{int}}, \rho] + \sum_{i=1}^{3} p_i (\tau_i \text{Tr}_i \rho - \rho),
\]

(see Appendix A). Note that (12) can easily be rewritten explicitly in Lindblad form.

C. Equilibrium Solution

We will be interested in solving for the equilibrium state, \( \rho_{\text{Eq}} \), which satisfies

\[
0 = -i[H_0 + H_{\text{int}}, \rho_{\text{Eq}}] + \sum_{i=1}^{3} p_i (\tau_i \text{Tr}_i \rho_{\text{Eq}} - \rho_{\text{Eq}}).
\]

The equilibrium solution can be easily found analytically, but it has a complicated dependence on all the parameters, so it is much more illuminating to present a numerical analysis.

Figure 1 shows the dependence of the temperature of qubit 1 (the system to be cooled) as a function of the temperature of the hot bath, \( T_h \). We see that when the temperature of the hot bath is larger than that of the cold bath, \( T_h > T_c \), i.e. in the normal regime of functioning, qubit 1 is cooled to a temperature \( T_{1,\text{Eq}} < T_c \), hence our refrigerator indeed cools.

We notice that when \( T_h = T_c \), that is, when we supply no free energy, the temperature of qubit 1 is unchanged. Finally we also notice that if the temperature of the “hot” bath, \( T_h \) is taken to be actually smaller than that of the “cold” bath, \( T_c \), than our refrigerator works effectively in reverse and it warms qubit 1 to a temperature higher than the hottest environmental temperature (which in this case is \( T_c \)).

D. Parameter dependence and Zeno effects

A natural question to ask is how the behaviour of the fridge changes as we vary the parameters \( p_i \) independently. Indeed, in a “standar” refrigerator we do not want all parts to interact with the environment equally: the inside of the fridge has to be well insulated to maintain a low temperature while the spiral at the back of the fridge has to interact strongly with the environment so that it dissipates heat quickly.

Concentrating first on qubit 2, (i.e. the spiral) we expect that as \( p_2 \) becomes larger the performance of the fridge should increase (\( T_{1,\text{Eq}} \) should decrease). Furthermore, qubit 3 plays the role of the “engine” of the refrigerator, which it achieves by pumping heat from the hot environment into the system. We expect that the best performance is therefore achieved when it interacts strongly with its environment, as this allows it to extract heat at the highest rate.

In Fig. 2 we plot separately the dependence of \( T_{1,\text{Eq}} \) on \( p_2 \) and \( p_3 \). For small values of \( p_2 \) and \( p_3 \) we indeed observe the expected behaviour, however as we increase them further the performance degrades. The reason behind this is that there are quantum effects which come into play in the regime of strong

\[\text{FIG. 1: Equilibrium temperature of qubit 1, } T_{1,\text{Eq}}, \text{ in contact with a bath at temperature } T_{c} = 1, \text{ versus bath temperature } T_{h} \text{ of qubit 3, with all insulation parameters taken to be equal, } p_{i} = p. \text{ The different curves correspond to different values of } g/p. \text{ We take the energy level spacings } E_{1} = E_{2} = 1, E_{3} = 2. \text{ Inset: Schematic diagram showing energy levels and interaction.} \]

\[\text{FIG. 2: Equilibrium temperature } T_{1,\text{Eq}} \text{ of qubit 1 as a function of the insulation parameter: (a) } p_{2} \text{ and (b) } p_{3}. \text{ When } p_{2} \text{ or } p_{3} \text{ vanish then we are unable to achieve cooling. For large values of } p_{2} \text{ or } p_{3} \text{ the performance of the fridge degrade – in this regime a Zeno effect inhibits the functioning of the refrigerator.} \]
coupling between the qubits and the environment.

Specifically, the effect that we are observing is an instance of a Zeno effect [21]. Thermalising each qubit is closely related to a measurement; it is as though the environment ‘measures’ the qubit as it puts it in a thermal state. Now, as we increase the rate of thermalisation we soon enter a regime whereby the interaction Hamiltonian $H_{int}$ will not have time to work between successive thermalisations and hence the whole refrigerator is no longer able to function.

E. Approaching absolute zero

We would now like to look more closely at the performance of our refrigerator. A first important question is whether or not there are fundamental limitations to the temperature to which we are able to cool down the first qubit. We show that there is no such limitation.

The minimal achievable temperature is limited by two effects: a heat current flowing into the fridge due to imperfect insulation, and the actual cooling ability (i.e., the ability to cool given perfect insulation). It is the second aspect which we are interested in.

We fix $E_1$, since this is a characteristic of the object to be cooled and not of the refrigerator and also fix $T_c$, as this is the environmental temperature. We increase $E_2$ and $T_h$ such that the ratio $E_2/T_h$ remains constant and much less than 1. This results in increasing the population of the ground state for qubit 2, while maintaining a high probability of finding qubit 3 in the excited state. Put together these two elements mean that the interaction (9) becomes ever more biased as we increase $E_2$. This leads to cooling as close as we want to absolute zero, as seen in Fig. 3.

2. MODEL II: ONE QUBIT, ONE QUTRIT

One drawback of the previous model is that the interaction Hamiltonian (9) is a three body interaction. Here we present a model with only two-body nearest neighbour interactions.

The model consists of three particles, where one is to be cooled and two to construct the fridge. Particle 1 and 3 are qubits and particle 2 is a qutrit (three level system). The energy levels of each particle are as depicted in Fig. 4 (inset) and are such that the energy eigenstates $|020\rangle$ and $|101\rangle$ are degenerate in energy. By introducing an interaction which can take the population of the latter into the former we can cool down qubit 1.

We do this by introducing two separate interactions between the particles via the Hamiltonians

\[
H_{int}^{(12)} = g \left( |02\rangle \langle 11| + |11\rangle \langle 02| \right) \otimes \mathbb{1}^{(3)} \tag{14}
\]

\[
H_{int}^{(23)} = h \mathbb{1}^{(1)} \otimes (|01\rangle \langle 10| + |10\rangle \langle 01|) \tag{15}
\]

Neither (14) or (15) induces transitions between the two desired states. However, (14) causes transitions between $|020\rangle$ and $|110\rangle$ and (15) between $|100\rangle$ and $|101\rangle$. Therefore, in second order we indeed induce the desired transition.

Finally we bias this interaction as in the previous model, by taking particles 1 and 2 to be in contact with a bath at temperature $T_c$ and qubit 3 at $T_h$.

The details of this model are given in Appendix B. Here we simply display the results obtained in Fig. 4. We observe that this refrigerator behaves qualitatively the same as in our previous model.

3. THE SMALLEST POSSIBLE FRIDGE

In the previous model particle 3 is required only so that we can heat the lower transition on qubit 2 (between $|0\rangle$ and $|1\rangle$) independently of the upper transition (between $|1\rangle$ and $|2\rangle$) as we have taken $p_1 = p_2 = p_3 = p$, $T_c = 1$ and $E = 1$. Inset: Schematic diagram showing energy levels and interaction.
It is possible to quantify the efficiency? If so is our construction
we shall mention a few. The first is to explore what can be
tions which we have not yet even begun to explore, of which
shown that it is possible to cool towards absolute zero.
Moreover we have called that by ‘small’ we mean precisely that we take quantum
self contained systems which act as refrigerators. We re-
that there is no fundamental difficulty in constructing small,
smaller refrigerator, by discarding the third qubit – the fridge
now contains only a single qutrit and the interaction [14] (See
Fig. 5). We believe this is the smallest possible system which
may be called a refrigerator.

4. CONCLUSIONS

We have presented three simple models which demonstrate
that there is no fundamental difficulty in constructing small,
self contained systems which act as refrigerators. We re-
call that by ‘small’ we mean precisely that we take quantum
systems consisting of very few states. Furthermore we have
shown that it is possible to cool towards absolute zero.
There are however many interesting questions and direc-
tions which we have not yet even begun to explore, of which
we shall mention a few. The first is to explore what can be
said about the efficiency of the refrigerators we presented. Is
it possible to quantify the efficiency? If so is our construction
the most efficient or are there other Hamiltonian’s which are
better for cooling?
Moreover, it is fundamental to ask whether or not there
exists a complementarity between small dimension and effi-
ciency – is it the case that you can only be large and efficient
or small and inefficient? In other words, can a small machine
reach the efficiency of an ideal Carnot engine? Our particu-
lar models do not reach this efficiency; indeed, both the spi-
ral and the engine qubits reach equilibrium temperatures that
are different by a finite (instead of infinitesimal) amount from
the temperatures of their environments, which leads to irre-
versible heat exchanges. However, could it be possible to find
a better model? This is not clear, since in a Carnot cycle the
system transitions through very many states. Here, however,
it is not the case that we can pass through many states in a
single cycle of refrigeration. The question is whether or not
this affects the achievable efficiency of the refrigerator.

Here we focused on refrigerators, but studying other ther-
mostatic machines would be an interesting direction to ex-
lore. Given a supply of free energy we have already shown
how to cool a system, but are there other interesting tasks that
we can also achieve given only an interaction with two ther-
mal baths? Could we somehow extract the ‘work’ and use it
to drive other molecular machines?

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APPENDIX A: DERIVING MASTER EQUATION

To find the master equation consider a small time interval
$\delta t$ around a time $t_0$. First to order in $\delta t$, the evolution of the
density matrix $\rho$ is given by

$$
\rho(t_0 + \delta t) = (1 - \delta t(p_1 + p_2 + p_3))\rho(t_0) \\
+ \delta t(p_1 T_1 \rho(t_0) + p_2 T_2 \rho(t_0)) \\
+ p_3 T_3 \rho(t_0)) - i\delta[H_0 + H_{int}, \rho(t_0)].
$$

(A1)

from which it follows that

$$
\frac{\partial \rho}{\partial t} = -i[H_0 + H_{int}, \rho] + \sum_{i=1}^{3} p_i(\tau_i T_i \rho - \rho),
$$

(A2)
Note that (12) can easily be rewritten explicitly in Lindblad form.

APPENDIX B: DETAILS FOR TWO-QUBIT-ONE-QUTRIT REFRIGERATOR

The free Hamiltonian of the 3 qubits is taken to be

\[ H_0 = E_1 \Pi_1^{(1)} + E_2 \Pi_1^{(2)} + (E_1 + E_2) \Pi_2^{(2)} + E_2 \Pi_1^{(3)} \]  

(B1)

where \( \Pi_i^{(j)} = |1\rangle_i \langle 1| \) and \( \Pi_2^{(i)} = |2\rangle_i \langle 2| \). The interaction Hamiltonian is now

\[ H_{int} = g \left( \langle 02 | 11 \rangle + |11 \rangle \langle 02 | \right) \otimes \mathbb{1}^{(3)} 
\]
\[ + h \mathbb{1}^{(1)} \otimes \left( |01 \rangle \langle 10 | + |10 \rangle \langle 01 | \right) \]  

(B2)

where \( g \) and \( h \) are the coupling constants for each interaction. We again assume that the interaction strength between the qubits is small in comparison to the energy level spacing, \( E_i \), that is \( g, h \ll E_i \). Each particle again interacts with a thermal bath. For qubits 1 and 3, the thermal states are given, as previously, by (2). For particle 2 the thermal state is now given by

\[ \tau_2 = N_2' \exp\left( -\left( E_2 \Pi_1^{(2)} + (E_1 + E_2) \Pi_2^{(2)} \right)/kT_c \right) \]  

(B3)

where \( N_2' = (1 + e^{-E_1/kT_c} + e^{-(E_1 + E_2)/kT_c})^{-1} \).

The master equation governing the evolution is

\[ \frac{\partial \rho}{\partial t} = -i[H_0 + H_{int}, \rho] + \sum_{i=1}^{3} p_i (\tau_i \text{Tr}, \rho - \rho) \]  

(B4)

for which we will be interested in solving for the equilibrium solution \( \rho_{Eq} \). As before, we will not present an analytic form for \( \rho_{Eq} \) but only study properties of it numerically. Figure 4 displays the dependence of the equilibrium temperature of qubit 1, \( T_{1,Eq} \), on \( T_h \), the temperature of the hot bath, for various values of the parameters \( g/p \) and \( h/p \).