Nonlinear structure of the wakefield generated by relativistic intense ion bunch

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Abstract. The resonant excitation of the nonlinear wakefield by a single proton bunch is investigated with the parameters characteristic of the AWAKE experiment. It is shown that obtained structure of the wakefield at a distance more than twenty periods behind the driver proton bunch can be suitable for the side injection and further acceleration of the witness electron bunch in the wakefield.

1. Introduction
The electron acceleration to high energies in plasma wakefields generated by a relativistic proton bunch (proton-driven plasma wakefield acceleration, PDPWA) is now under intensive investigation [1]. These studies are motivated by the possibility to use proton bunches of tens kilojoules energies [2] for a single stage acceleration of electron bunches caring nC charge to teraelectron-volt energies [3–5]. Such proton bunches are routinely produced by the CERN Super Proton Synchrotron (SPS, 450 GeV, $3 \times 10^{11}$ protons, $\approx 20$ kJ) or Large Hadron Collider (LHC, 6.5 TeV, $1.2 \times 10^{11}$ protons, $\approx 125$ kJ).

However, the short driver length required for efficient excitation of the plasma wave (which are much shorter than CERN proton bunches available today) presents a serious obstacle to a realization of the PDPWA concept [6, 7]. This problem can be solved using self-modulation instability of the proton driver bunch [5]. Nowadays, the experiment AWAKE is planed and preparing at CERN on the base of this approach, when the self-modulation instability transfers a long proton beam into a train of short bunches, each effectively generates wakefield [7].

In this paper, the resonant excitation of the nonlinear wakefield by a single short proton bunch is investigated with the plasma, proton bunch and wakefield parameters characteristic of the AWAKE experiment.

2. Basic equations
The hydrodynamic model of cold relativistic ideal plasma is used for description of the electron dynamics. Plasma ions are immobile (on the considered time scale, which does not exceed inverse ion plasma frequency) and provide a neutralizing background.
Hydrodynamic equations of electron motion are complemented by the Maxwell equations [8]:

\[
\frac{\partial n}{\partial t} + \text{div}(n\vec{v}) = 0, \quad (1)
\]

\[
\frac{\partial \vec{p}}{\partial t} = e\vec{E} - mc^2\nabla \gamma, \quad (2)
\]

\[
\frac{\partial \vec{E}}{\partial t} = -4\pi(n\vec{v} + \vec{j}_b) - \frac{e}{m} \text{rot rot} \vec{p}, \quad (3)
\]

\[
\gamma = \sqrt{1 + \frac{p^2}{m^2c^2}}, \quad (4)
\]

\[
\vec{v} = \frac{\vec{p}}{mc\gamma}, \quad (5)
\]

where \( e, m \) are electron charge and mass, \( c \) is the speed of light; \( n, \vec{p}, \vec{v} \) are concentration, specific impulse and speed of electrons, \( \gamma \) is the Lorentz factor of background electrons; \( \vec{E} \) is the electric field strength, \( \vec{j}_b = n_b q_b \vec{v}_b \hat{OZ} \) is the ion beam current driving the wake fields in plasma. In the axisymmetric case, all quantities depend only on \( z, r \) and \( t \).

The high energy ultra-relativistic driving ion bunch considered below evolves slowly in rarefied plasma. Neglecting scattering and stopping of ions, it is possible to assume that speed and shape of the ion bunch does not change during investigated process of the wake field generation. In this case, all quantities can be expressed through only two independent variables (that implies the self-similar solution with the phase velocity of wakefields \( v_{\text{ph}} \) equals to the velocity of driving ion bunch \( v_b \)):

\[
\xi = k_p(z - v_b t), \quad \rho = k_p r, \quad (6)
\]

where \( k_p = \omega_p/c, \omega_p = \sqrt{4\pi n_0 e^2/m_e} \) is the electron plasma frequency and \( n_0 \) is a background electron plasma density.

We will also use the new dimensionless functions:

\[
\vec{q} = \frac{\vec{p}}{mc}, \quad \vec{w} = \frac{\vec{v}}{c}, \quad \vec{\epsilon} = \frac{e\vec{E}}{mc\omega_p}, \quad \nu = \frac{n}{n_0}, \quad \nu_b = \frac{n_b}{n_0}, \quad \psi = q_\nu \beta^{-1}_b - \gamma, \quad \beta = \frac{\partial \psi}{\partial \xi}, \quad \varphi = \frac{\nu}{\gamma}, \quad (7)
\]

where \( \beta_b = v_b/c \).

Using transformations described in [9] we obtain the following equations:

\[
\frac{\partial \beta}{\partial \xi} + (1 - \beta_b^{-2})\frac{\partial^2 \gamma}{\partial \xi^2} + \frac{1}{\rho} \frac{\partial q_\rho}{\partial \rho} - \beta_b^{-1} \Delta \gamma + \beta_b \varphi \gamma - \beta_b \nu + \beta_b n_0 = 0, \quad (8)
\]

\[
q_\nu \varphi + \beta_b \frac{\partial \beta}{\partial \rho} = (\beta_b^{-2} - 1) \frac{\partial^2 q_\nu}{\partial \xi^2}, \quad (9)
\]

\[
\beta_b \varphi \psi - \beta_b^{-1} \Delta \psi + \beta_b \nu + (\beta_b^{-2} - 1) \frac{1}{\rho} \frac{\partial q_\rho}{\partial \rho} = 0, \quad (10)
\]

\[
2\beta_b^2 \gamma \psi + \beta_b^2 \varphi^2 + q_\nu^2 + 1 = \gamma^2 (1 - \beta_b^2), \quad (11)
\]

\[
\frac{\partial \psi}{\partial \xi} - \beta = 0. \quad (12)
\]

For the given driver with proton energy of 450 GeV, the terms proportional to \((1 - \beta_b^2) \approx 10^{-5}\) are small and can be neglected.
In this case equations can be simplified to:

\[
\frac{\partial \beta}{\partial \xi} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial q_r}{\partial \xi} - \Delta_{\perp} \gamma + \varphi \gamma - \nu + \nu_0 = 0, 
\]

(13)

\[
q_r \varphi + \frac{\partial \beta}{\partial \rho} = 0,
\]

(14)

\[
\varphi \psi - \Delta_{\perp} \psi + \nu = 0,
\]

(15)

\[
2 \gamma \psi + \psi^2 + q_r^2 + 1 = 0,
\]

(16)

\[
\frac{\partial \psi}{\partial \xi} - \beta = 0.
\]

(17)

Estimates of the neglected terms for the solution obtained below have shown that these terms are indeed of the order of \((1 - \beta_b^2)\) in comparison with the other terms of equations and so they can be safely omitted.

In this paper, the wakefield structure was modeled using equations (13)–(17) with the help of the program elaborated in [9], which was modified for the ion beam driver (instead of the laser pulse). The nonlinear wakefield generated by a short relativistically-intense laser pulse was investigated in [10, 11] using original algorithm described in [9].

3. Modeling results and discussions

In line with the AWAKE experiment, in simulations the plasma density were chosen \(n_0 = 7 \times 10^{14} \text{ cm}^{-3}\). This corresponds to \(k_p = 49.7 \text{ cm}^{-1}\). The shape of the driver ion bunch was Gaussian in both radial and longitudinal (time) directions:

\[
n_b = 0.7 n_0 \exp \left( -\frac{\rho^2 + \xi^2}{2(k_p \sigma)^2} \right),
\]

(18)

with \(k_p \sigma = 1\). Protons in bunch were ultra-relativistic with the energy \(E_p = 450 \text{ Gev}\). This corresponds to

\[
\gamma_b = 1/\sqrt{1 - v^2/c^2} = 480.
\]

Charge of the beam was 10 nC.

Simulations were made from \(\xi = 10\) till \(\xi = -330\) with steps \(\Delta \xi = 10^{-4}\), \(\Delta \rho = 2 \times 10^{-3}\) and accuracy of internal Newton iterations \(\delta = 10^{-5}\) [9]. To analyze the convergence of the solution, simulations with different steps (increased two times on \(\rho\) and five times on \(\xi\)) were made: (i) \(\Delta \xi = 10^{-4}\), \(\Delta \rho = 4 \times 10^{-3}\); (ii) \(\Delta \xi = 5 \times 10^{-4}\), \(\Delta \rho = 4 \times 10^{-3}\). The results converge everywhere, except the half of latest period (\(\xi\) from 330 till 325). The convergence means here that the difference of the solutions with different steps is smaller than \(\delta = 10^{-5}\).

It should be emphasized that a noise-free high accuracy numerical solution is necessary for correct description of the nonlinear oscillating structure of the wakefield, especially over a long enough distance behind the driver [12]. This aim was achieved using a low numerical hydrocode dispersion, and with the help of monitoring the accuracy of Newton iterations and changes of the time and radial steps, as long as the resulting solution did not change at any point within a given precision \(\delta\) (for results presented below \(\delta = 10^{-5}\)).

An important feature of the three-dimensional nonlinear wake wave is that its characteristics are changing with the distance from the driver (laser pulse or bunch of charged particles). Namely, due to the nonlinear dependence of the frequency of the plasma wave on its amplitude, the wakefield phase front is bending with distance from the driver [13]. As a result, at some distance from the trailing edge of the diver the wake wave breaks [14], transferring its energy to the plasma particles. Breaking of the wake wave leads to uncontrolled capture and acceleration of electrons, as well as to the generation of an intense shortwave electromagnetic radiation.
Figure 1. Normalized density of electrons $\nu = n/n_0$ on the plane $(\xi, \rho)$ at different distances behind the driver bunch of protons (18). The scale of the horizontal axis is specified in the wake wave periods, measured back from the driver position. Upper image shows the density immediately after trailing edge of the driver bunch, over first two periods, the second image shows the density around 27th period behind the driver bunch, and the third lower image shows the wave near the break point around 47th period of the wake.
Figure 1 shows the normalized electron density of the plasma wake wave $\nu = n/n_0$ on $(\xi, \rho)$-plane behind the driver bunch of protons (18). Upper image shows the density immediately after trailing edge of the driver bunch, over first two periods, $\xi$ from 12 till 0. The second image ($\xi$ from 172 to 160) shows the density around 27th period behind the driver bunch, and the third lower image ($\xi$ from 325 to 313) shows the wave near the break point around 47th period of the wake.

The wakefield structure is changed substantially with the distance from the driver in such a way that the curvature of the wave front is changed to opposite direction after a tens of periods (compare upper and middle images in figure 1). Approaching the wave breaking point (see lower image in figure 1), the non-axial maxima of plasma density appear. After this, the amplitude of density perturbations begins to grow until wake breaks.

The pilot modelling of trapping of test electrons has shown that obtained structure of the wakefield at a distance more than twenty periods behind the driver proton bunch can be suitable for the side injection of the witness electron bunch into the wakefield [12, 15]. Test electrons were injected with the energy 10 MeV under the angle of one degree to the driver propagation axis. None of the injected electrons were trapped in the first ten periods of the wakefield behind the driver. The relative number of trapped electrons (normalized to the injected number) was 15% in the twentieth wakefield period and about 40% in the fortieth period behind the driver.

A more detailed analysis of trapping and acceleration of externally injected electrons in the nonlinear wakefield generated by the proton driver will be the subject of further investigation.

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