Revisiting Le Cam’s Equation: Exact Minimax Rates over Convex Density Classes

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Abstract

We study the classical problem of deriving minimax rates for density estimation over convex density classes. Building on the pioneering work of Birgé [1983], Birgé [1986], Le Cam [1973], Wong and Shen [1995], Yang and Barron [1999], we determine the exact (up to constants) minimax rate over any convex density class. This work thus extends these known results by demonstrating that the local metric entropy of the density class always captures the minimax optimal rates under such settings. Our bounds provide a unifying perspective across both parametric and nonparametric convex density classes, under weaker assumptions on the richness of the density class than previously considered. Our proposed ‘multistage sieve’ MLE applies to any such convex density class. We further demonstrate that this estimator is also adaptive to the true underlying density of interest. We apply our risk bounds to rederive known minimax rates including bounded total variation, and Lipschitz density classes. We further illustrate the utility of the result by deriving upper bounds for less studied classes, e.g., convex mixture of densities.

1 Introduction

It is well known that (global) metric entropy often times determines the minimax rates for density estimation. Specifically, the following equation sometimes informally referred to as the ‘Le Cam equation’, is used to heuristically determine the minimax rate of convergence

$$\log M^{\text{glo}}_F(\varepsilon) \asymp n\varepsilon^2,$$

where $n$ is the sample size, $\log M^{\text{glo}}_F(\varepsilon)$ is the global metric entropy of the density set $F$ at a Hellinger distance $\varepsilon$ (see Definition 5), and $\varepsilon^2$ determines the order of the minimax rate. In this paper we complement these known results, by establishing that local metric entropy always determines the minimax rate for convex density classes, where the densities are assumed to be (uniformly) bounded from above and below.

In detail, under the setting of density estimation just described, we suggest a small revision to the Le Cam equation: namely, change the global entropy to local entropy, and the Hellinger metric to the $L_2$-metric. Furthermore, the same result holds when the convex density class contains densities only (uniformly) bounded from above, and a single density which is bounded from below. Unlike previous known results, our result unites minimax density estimation under both parametric and nonparametric convex density classes. A further contribution is that our proposed ‘multistage sieve’ maximum likelihood estimator (MLE) achieves these bounds regardless of the density class (as long as it is convex). In addition, we demonstrate that this multistage sieve estimator is adaptive to
the true density of interest. To the best of our knowledge we are not aware of any other estimators with such a property, under such general settings.

We will now formally describe the setting we consider. To that end, we first define a general class of bounded densities, i.e., \( \mathcal{F}_B^{[\alpha, \beta]} \). Later, we will assume that the true density of interest belongs to a known convex subset of this general ambient density class.

**Definition 1 (Ambient density class \( \mathcal{F}_B^{[\alpha, \beta]} \)).** Given constants \( 0 < \alpha < \beta < \infty \), for some fixed dimension \( p \in \mathbb{N} \), and a common known (Borel measurable) compact support set \( B \subseteq \mathbb{R}^p \) (with positive measure), we then define the class of density functions, \( \mathcal{F}_B^{[\alpha, \beta]} \), as follows:

\[
\mathcal{F}_B^{[\alpha, \beta]} := \left\{ f : B \to [\alpha, \beta] \mid \int_B f \, d\mu = 1, f \text{ measurable} \right\},
\]

where \( \mu \) is the dominating finite measure on \( B \). We always take \( \mu \) to be a (normalized) probability measure on \( B \).

Furthermore, we can endow \( \mathcal{F}_B^{[\alpha, \beta]} \) with the \( L_2 \)-metric. That is, for any two densities \( f, g \in \mathcal{F}_B^{[\alpha, \beta]} \), we denote the \( L_2 \)-metric between them to be

\[
\| f - g \|_2 := \left( \int_B (f - g)^2 \, d\mu \right)^{\frac{1}{2}}.
\]

**Remark 1.** Qualitatively, we have that \( \mathcal{F}_B^{[\alpha, \beta]} \) is the class of all densities that are uniformly \( \alpha \)-lower bounded and \( \beta \)-upper bounded, on a common compact support \( B \subseteq \mathbb{R}^p \). Furthermore, Definition 1 implies that \( \mathcal{F}_B^{[\alpha, \beta]} \) forms a convex set, and that the metric space \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \) is complete, bounded, but may not be totally bounded\(^1\).

In this paper we will focus on the scenario where it is known that the true density \( f \in \mathcal{F} \subset \mathcal{F}_B^{[\alpha, \beta]} \), where \( \mathcal{F} \) represents our knowledge on the true density, before observing any data. With these mathematical preliminaries, we formalize our core density estimation problem of interest as follows.

**Core problem:** Suppose that we observe \( n \) observations \( X := (X_1, \ldots, X_n)^\top \) i.i.d. \( \sim f \), for some (fixed but unknown) \( f \in \mathcal{F} \). Here \( \mathcal{F} \subset \mathcal{F}_B^{[\alpha, \beta]} \) is a convex set, which is known to the observer. Can we propose a universal estimator for \( f \), and derive the exact (up to constants) squared \( L_2 \)-minimax rate of estimation, in expectation?

For convenience, we can illustrate the generating process for a univariate example of our density estimation problem of interest in Figure 1. It will serve as a useful conceptual guide to later help visualize our proposed estimator over such general convex class of densities \( \mathcal{F} \).

Now, without further ado, we will informally state our main result as a direct affirmative answer to our core question of interest. Namely, there does exist a likelihood-based estimator (one can think of it as a multistage sieve MLE), i.e., \( \nu^*(X) \), which achieves the following rate of estimation error

\[
\sup_{f \in \mathcal{F}} \mathbb{E}\| \nu^*(X) - f \|_2^2 \lesssim \varepsilon^2 \wedge \text{diam}_2(\mathcal{F})^2.
\]

\(^1\) These fundamental (and additional) analytic properties of \( \mathcal{F}_B^{[\alpha, \beta]} \) are formally justified in Appendix A.1.
Figure 1: Illustrative generating process for a univariate density $f \in \mathcal{F} \subseteq \mathcal{F}_{B}^{[\alpha,\beta]}$. 

Here $\varepsilon^* := \sup \{ \varepsilon : n\varepsilon^2 \leq \log M_{\mathcal{F}}^{loc}(\varepsilon, c) \}$, with $\log M_{\mathcal{F}}^{loc}(\varepsilon, c)$ being the $L_2$-local metric entropy of $\mathcal{F}$ (see Definition 5). The quantity $\text{diam}_2(\mathcal{F})$, refers to the $L_2$-diameter of $\mathcal{F}$, which is finite by the boundedness of $\mathcal{F}_{B}^{[\alpha,\beta]}$ in our setting. In addition, the rate above is minimax optimal, as there is a matching (up to constants) lower bound.

Remark 2. We will later see that we can largely relax the $\alpha$-lower boundedness condition on $\mathcal{F}$. That is, the results we are about to derive can be readily generalized to convex subsets $\mathcal{F} \subseteq \mathcal{F}_{B}^{[0,\beta]}$. This is so long as the class $\mathcal{F}$ contains a single density which is bounded away from 0.

Next, we turn our attention to reviewing some relevant literature.

1.1 Relevant Literature

Classical work

As noted, density estimation is a classical statistical estimation problem with a rich history. Lively accounts of the key references, particularly for nonparametric density estimation as relevant to our setting, are already covered in Yang and Barron [1999, Section 1] and Bilodeau et al. [2021, Section 6.1]. We similarly begin with a brief panoramic overview of these references in regard to minimax risk bounds for density estimation, before comparing and contrasting the results from the most relevant references to our work.

In terms of minimax lower bounds on density estimation, Boyd and Steele [1978] prove a fundamental $n^{-1}$ rate in the mean integrated $p^{th}$ power error (with $p \geq 1$), for any arbitrary density estimator. Such generalized lower bounds on density estimation were also further studied in Devroye [1983]. In the case of density estimation over classes with more assumed structure (e.g., smoothness, or regularity assumptions) minimax lower bounds have been developed based on hypothesis testing approaches coupled with information-theoretic techniques. We now provide brief highlights of such key works in this direction.

In Bretagnolle and Huber [1979], the authors derive sharp lower bounds for Sobolev smooth densities in $\mathbb{R}^d$ ($d \in \mathbb{N}$), with risk measured with respect to a power of the $L_q$-metric ($q \geq 1$). In Birgé [1986], sharp risk bounds for more general classes of such smooth families were provided using
metric entropy based methods, with an emphasis on the Hellinger loss. The work of Efroïmovich and Pinsker [1982] provided precise (asymptotic) analysis for an ellipsoidal class of densities in the $L_2$-metric. Across a wide-ranging series of related and collaborative efforts Has’minskiĭ [1978], Ibragimov and Has’minskiĭ [1977, 1978], Ibragimov and Khas’minskij [1980] used Fano’s inequality type arguments to establish lower bounds over a variety of density estimation settings. These range from deriving lower bounds on nonparametric density estimation in the uniform metric, to minimax risk bounds for the Gaussian white noise model, for example. The authors also develop metric entropy based techniques in Has’minski˘ı and Ibragimov [1990] to derive minimax lower bounds for a wide variety of density classes defined on $\mathbb{R}^d$ ($d \in \mathbb{N}$), in $L_q$-loss ($q \geq 1$). Numerous applications of optimal lower bounds using both Assouad’s and Fano’s lemma arguments for densities on a compact support, are demonstrated in [Yu, 1997, Section 29.3]. Later Yang and Barron [1999] demonstrated that global metric entropy bounds capture minimax risk for sufficiently rich density classes over a common compact support. Classical reference texts on minimax lower bound techniques with an emphasis on nonparametric density estimation include Devroye [1987], Devroye and Györfi [1985], Le Cam [1986]. More modern such references include Tsybakov [2009] and Wainwright [2019, Chapter 15]. The latter in particular, also incorporates metric entropy based lower bound techniques.

In addition, there is a large body of work in deriving upper bounds for specific density estimators using metric entropy methods. This includes Barron and Cover [1991], Yatracos [1985], who employ the minimum distance principle to derive density estimators and their metric entropy-based upper bounds in the Hellinger and $L_1$-metric, respectively. In a similar spirit to Birgé [1983], Birgé [1986], van de Geer [1993] is also concerned with density estimation using Hellinger loss. However, its focus is to use techniques from empirical process theory in order to specifically establish the Hellinger consistency of the nonparametric MLE, over convex density classes. Upper bounds for density estimation based on the ‘sieve’ MLE technique is studied in Wong and Shen [1995]. Recall that a ‘sieve’ estimator effectively estimates the parameter of interest via an optimization procedure (e.g., maximum likelihood) over a constrained subset of the parameter space [Grenander, 1981, Chapter 8]. In Birgé and Massart [1993] the authors study ‘minimum contrast estimators’ (MCEs), which include the MLE, least squares estimators (LSEs) etc., and apply them to density estimation. This is further developed in Birgé and Massart [1998] where they analyze convergence of MCEs using sieve-based approaches.

**Comparison to our work**

By stating our main result early in the introduction, we now turn to contrasting it with the most relevant results in the literature. These include both the aforementioned classical references, and more recent work on convex density estimation, which have most directly inspired our efforts in this work.

First we would like to comment on the closely related landmark papers [Birgé, 1983, Birgé, 1986, Le Cam, 1973]. These works consider very abstract settings and show upper bounds based on Hellinger ball testing. Although widely believed that they do, whether these results lead to bounds that are minimax optimal is unclear. Moreover, their estimator is quite involved and non-constructive. In contrast, in this paper we offer a simple to state, constructive multistage sieve MLE type of estimator, which is provably minimax optimal over any convex density class $F$. A crucial difference is that we metrize the space $F$ with the $L_2$-metric as we mentioned above. Even though in our instance the two distances are equivalent, in contrast to the Hellinger distance, the $\varepsilon$-local metric entropy of the convex density class in the $L_2$-metric can be shown to be monotonic in $\varepsilon$. 


This key observation enables us to match the upper and lower bounds exactly.

Next, we will compare our work with the celebrated paper of Yang and Barron [1999], who inspect a very similar problem. Yang and Barron [1999] demonstrate a lower and upper bound which need not match in general but do match under certain sufficient conditions. Notably their bounds involve only quantities depending on the global entropies of the set $\mathcal{F}$ (which is also assumed to be convex for some results of Yang and Barron [1999]). This is convenient as often times global metric entropy is easier to work with compared to local metric entropy, however under Yang and Barron [1999]'s sufficient condition it can be seen that the two notions are equivalent. Hence, our work can be thought of as removing the sufficient condition requirement from Yang and Barron [1999] and also unifying parametric and nonparametric density estimation problems (over convex classes) for which one typically needs to use different tools to obtain the accurate rates. Finally we would like to mention Wong and Shen [1995]. In that paper the authors propose a sieve MLE estimator and demonstrate that it is nearly minimax optimal under certain conditions. Our estimator is not the same as the one considered by Wong and Shen [1995], and we can provably match the minimax rate over whatever be the convex set $\mathcal{F}$. A notable difference is that Wong and Shen [1995] work with the Hellinger metric and KL divergence, which although equivalent to $L^2$-metric in our problem, are actually less practical in terms of matching the bounds exactly as we explained above. We will now turn our attention to reviewing some further relevant literature.

Recent work

Our estimator and proof techniques thereof, are inspired by the recent work of Neykov [2022] on the Gaussian sequence model. We would like to stress on the fact that the sequence model is a very distinct problem from density estimation. In particular, our underlying metric space of interest is $(\mathcal{F}, \| \cdot \|_2)$, as compared to $\left( \mathbb{R}^n, \| \cdot \|_2 \right)$ for the sequence model. Both of these metric spaces differ vastly from each other in their underlying geometric structure. Furthermore, unlike our setting, the sequence model contains additional Gaussian information on the underlying generating process, which can be directly exploited for estimation purposes. As such, given that Neykov [2022] provides a guiding template for our analysis, some resulting structural similarities to their work are to be expected. However, all corresponding proofs, and estimators thereof, have to be non-trivially adapted to our nonparametric density estimation setting. A notable example of such required modifications, is that our estimator presented in this paper does not use proximity in Euclidean norm, but is a likelihood-based estimator.

We additionally note that density estimation in both abstract and more concrete settings, continues to be an active area of research. It is not feasible to detail such a large and growing body of references. However, we provide a selective overview of some interesting recent directions in density estimation, to simply indicate the diversity of the research efforts thereof. For example, Baldi et al. [2009], Cleanthous et al. [2020] study convergence properties of density estimators using wavelet-based methods. The papers Birgé [2014], Efroymovich [2008], Goldenshluger and Lepski [2014], Rigollet [2006], Rigollet and Tsybakov [2007], Samarov and Tsybakov [2007] study adaptive minimax density estimation on $\mathbb{R}^d (d \geq 1)$ under $L^p$-loss ($p \geq 1$). Here, ‘adaptive’ refers to the fact that the density class is defined by an unknown tuning hyperparameter, which must be explicitly accounted for during the estimation process. Recently Wang and Marzouk [2022] used techniques from optimal transport to study the convergence properties of various nonparametric density estimators. Interestingly, Bilodeau et al. [2021] applied empirical (metric) entropy methods to establish

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Note that $\| \cdot \|_2$ here is the Euclidean metric on $\mathbb{R}^n$. 

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minimax optimal rates in the adjacent setting of conditional density estimation. Although these works do not directly study our core problem of interest, we note that they represent new and important perspectives on classical minimax density estimation, and related problems.

1.2 Notation

We outline some commonly used notation here. We use \( a \lor b \) and \( a \land b \) for the max and min of two numbers \( \{a, b\} \), respectively. Throughout the paper \( \| \cdot \|_2 \) denotes the \( L_2 \)-metric in \( \mathcal{F} \). Constants may change values from line to line. For an integer \( m \in \mathbb{N} \), we use the shorthand \( [m] := \{1, \ldots, m\} \).

We use \( B_2(\theta, r) \) to denote a closed \( L_2 \)-ball centered at the point \( \theta \) with positive radius \( r \). We use \( \lesssim \) and \( \gtrsim \) to mean \( \leq \) and \( \geq \) up to absolute (positive) constant factors, and for two sequences \( a_n \) and \( b_n \) we write \( a_n \asymp b_n \) if both \( a_n \lesssim b_n \) and \( a_n \gtrsim b_n \) hold. Throughout the paper we use \( \log \) to denote the natural logarithm, or we specify the base explicitly otherwise. Our use of \( \{\alpha, \beta\} \) is only used to refer to the constants in Definition 1, of \( \mathcal{F}_B^{[\alpha, \beta]} \) (and thus \( \mathcal{F} \)). We will introduce additional section-specific notation as needed.

1.3 Organization

The rest of this paper is organized as follows. In Section 2 we prove risk bounds for our underlying setting. We first establish the key topological equivalence between the \( L_2 \)-metric and the Kullback-Leibler divergence in \( \mathcal{F}_B^{[\alpha, \beta]} \). We then proceed to derive minimax lower bounds for our setting in Section 2.1, introducing additional relevant mathematical background as needed, e.g., local metric entropy. In Section 2.2 we define our likelihood-based estimator, and provide intuition behind its construction. We then derive its (matching) minimax risk upper bound. We then demonstrate in Section 2.3 that our estimator is adaptive to the true density \( f \). In Section 3, we apply our results to specific examples of commonly used convex density classes. We then conclude in Section 4 by summarizing our results, and discuss some future research directions.

2 Minimax Lower and Upper Bounds

Before establishing our main results, we establish a key technical lemma which drives much of the geometric arguments in our analysis to follow. Note that for any two densities \( f, g \in \mathcal{F}_B^{[\alpha, \beta]} \), the KL-divergence between them is defined to be

\[
d_{\text{KL}}(f \| g) := \int_B f \log \left( \frac{f}{g} \right) \, d\mu =: \mathbb{E}_f \log \left( \frac{f(X)}{g(X)} \right),
\]

where \( X \sim f \) in (4).

Remark 3. We observe that \( d_{\text{KL}}(f \| g) \) is well-defined in (4) for our setting, since \( \inf_{x \in B} g(x) \geq \alpha > 0 \), by Definition 1. We further emphasize that KL-divergence is not valid metric in general, since it is not symmetric in its arguments.

The crucial fact in the risk bounds we will soon derive, is the ‘topological equivalence’ of the \( L_2 \)-metric and KL-divergence, on the density class \( \mathcal{F}_B^{[\alpha, \beta]} \). Since it is hard to find a concrete reference for this folklore fact, we formalize this equivalence for our setting in Lemma 2.
Lemma 2 (KL-L₂ equivalence on \( F_B^{[\alpha,\beta]} \)). For each pair of densities \( f, g \in F_B^{[\alpha,\beta]} \), the following relationship holds:
\[
c(\alpha, \beta)\|f - g\|_2^2 \leq d_{\text{KL}}(f \| g) \leq (1/\alpha)\|f - g\|_2^2,
\]
where we denote \( c(\alpha, \beta) := \frac{h(\beta/\alpha)}{\beta} > 0 \). Here \( h : (0, \infty) \to \mathbb{R} \) is defined to be
\[
h(\gamma) := \begin{cases} \frac{\gamma-1-\log \gamma}{(\gamma-1)^2} & \text{if } \gamma \in (0, \infty) \setminus \{1\} \\ \frac{1}{2} = \lim_{x \to 1} \frac{x-1-\log x}{(x-1)^2} & \text{if } \gamma = 1, \end{cases}
\]
and is positive over its entire support. It is also easily seen that on \( F_B^{[\alpha,\beta]} \), \( d_{\text{KL}} \) (and hence the \( L_2 \)-metric) is also equivalent to the Hellinger metric. Furthermore, these properties are also inherited by \( \mathcal{F} \subset F_B^{[\alpha,\beta]} \), which is our density class of interest.

Remark 4. We note that both the upper and lower bounds in (5) are stated without proof and without tracking constants in Klemelä [2009, Lemma 11.6]. We formally prove this claim in Appendix A. Importantly, the validity of (5) relies on the assumption of the boundedness of the densities, which holds in our setting.

2.1 Minimax Lower Bound

We will first establish a lower bound. For completeness, we need to introduce some additional relevant background and notation. We start by stating Fano’s inequality for our convex density class, \( \mathcal{F} \) [see Tsybakov, 2009, Lemma 2.10].

Lemma 3 (Fano’s inequality for \( \mathcal{F} \)). Let \( \{f^1, \ldots, f^m\} \subset \mathcal{F} \) be a collection of \( \varepsilon \)-separated densities (i.e., \( \|f^i - f^j\|_2 > \varepsilon \) for \( i \neq j \)), in the \( L_2 \)-metric. Suppose \( J \) is uniformly distributed over the index set \([m]\), and \( (X_i | J = j) \overset{i.i.d.}{=} f^j \) for each \( i \in [n] \). Then
\[
\inf_{\hat{f}} \sup_{f} \mathbb{E}\|\hat{f}(X) - f\|_2^2 \geq \frac{\varepsilon^2}{4} \left( 1 - \frac{nI(X_1; J) + \log 2}{\log m} \right).
\]

In the above \( I(X_1; J) := \frac{1}{m} \sum_{j=1}^{m} d_{\text{KL}}(f^j \| \hat{f}) \), where \( \hat{f} = \frac{1}{m} \sum_{j=1}^{m} f^j \) is the mutual information between \( X_1 \) and the randomly sampled index \( J \). Further, the infimum is taken over all measurable functions of the data. Next, we define the important notion of a packing set for \( \mathcal{F} \) [see Section 5.2 Wainwright, 2019, e.g., for more details].

Definition 4 (Packing sets and packing numbers of \( \mathcal{F} \) in the \( L_2 \)-metric). Given any \( \varepsilon > 0 \), an \( \varepsilon \)-packing set of \( \mathcal{F} \) in the \( L_2 \)-metric, is a set \( \{f^1, \ldots, f^m\} \subset \mathcal{F} \) of \( \varepsilon \)-separated densities (i.e., \( \|f^i - f^j\|_2 > \varepsilon \) for \( i \neq j \)) in the \( L_2 \)-metric. The corresponding \( \varepsilon \)-packing number, denoted by \( M(\varepsilon, \mathcal{F}) \), is the cardinality of the largest (maximal) \( \varepsilon \)-packing of \( \mathcal{F} \). We refer to \( \log M \mathcal{F}^{\text{blo}}(\varepsilon) := \log M(\varepsilon, \mathcal{F}) \) as the global metric entropy of \( \mathcal{F} \).

Remark 5. Note that we are not assuming here that \( \mathcal{F} \) is totally bounded, hence some (or perhaps all) of the packing numbers may be infinite; this however does not cause a problem in what follows. Henceforth, all packing sets (or packing numbers) of \( \mathcal{F} \), will be assumed to be with reference to the \( L_2 \)-metric, unless stated otherwise. We will use the standard fact that a \( \varepsilon \)-maximal packing of \( \mathcal{F} \), is also a \( \varepsilon \)-covering set of \( \mathcal{F} \).
We will now define the notion of *local metric entropy*, which will play a key role in the development of our risk bounds.

**Definition 5** (Local metric entropy of $\mathcal{F}$). Let $c > 0$ be fixed, and $\theta \in \mathcal{F}$ be an arbitrary point. Consider the set $^3 \mathcal{F} \cap B_2(\theta, \varepsilon)$. Let $M(\varepsilon/c, \mathcal{F} \cap B_2(\theta, \varepsilon))$ denote the $\varepsilon/c$-packing number of $\mathcal{F} \cap B_2(\theta, \varepsilon)$, in the $L_2$-metric. Let

$$M_{\mathcal{F}}^{loc}(\varepsilon, c) := \sup_{\theta \in \mathcal{F}} M(\varepsilon/c, \mathcal{F} \cap B_2(\theta, \varepsilon)) =: \sup_{\theta \in \mathcal{F}} M_{\mathcal{F} \cap B_2(\theta, \varepsilon)}^{\text{glob}}(\varepsilon/c).$$

We refer to $\log M_{\mathcal{F}}^{loc}(\varepsilon, c)$ as the *local metric entropy* of $\mathcal{F}$.

We show the following minimax lower bound for our convex density estimation setting over $\mathcal{F}$. It is a direct consequence of Fano’s inequality per Lemma 3.

**Lemma 6** (Minimax lower bound). Let $c > 0$ be fixed, and independent of the data samples $X$. Then the minimax rate satisfies

$$\inf_{\hat{p}} \sup_{f \in \mathcal{F}} \mathbb{E}_f \|\hat{p}(X) - f\|_2^2 \geq \frac{\varepsilon^2}{8c^2},$$

if $\varepsilon$ satisfies $\log M_{\mathcal{F}}^{loc}(\varepsilon, c) > 2n\varepsilon^2/\alpha + 2 \log 2$.

### 2.2 Upper Bound

We now turn our attention to the upper bound. We note that our universal estimator over $\mathcal{F}$, will be a likelihood-based estimator for $f$. As such for any two densities $g, g' \in \mathcal{F}$, we will routinely work with the *log-likelihood difference* for the $n$ observed samples $X := (X_1, \ldots, X_n) \stackrel{i.i.d.}{\sim} f \in \mathcal{F}$. We will denote this by

$$\psi(g, g', X) := \log \left( \prod_{i=1}^n \frac{g(X_i)}{g'(X_i)} \right) = \sum_{i=1}^n \log \left( \frac{g(X_i)}{g'(X_i)} \right) = \sum_{i=1}^n \log g(X_i) - \sum_{i=1}^n \log g'(X_i). \quad (7)$$

**Remark 6.** We note that the log-likelihood difference $\psi(g, g', X)$ in (7), is well-defined. This follows since for each $i \in [n]$, the individual random variables $\log g(X_i)/g'(X_i)$ are well-defined (as $\alpha > 0$), and bounded. That is, $-\infty < \log \alpha/\beta \leq \log g(X_i)/g'(X_i) \leq \log \beta/\alpha < \infty$, for each $i \in [n]$.

We will use the log-likelihood difference to help us decide which of the two densities is “more” correct, given the observed data samples $X$. Given this, we will first need a concentration result on the density log-likelihood difference. We do this by establishing the following lemma.

**Lemma 7** (Log-likelihood difference concentration in $F$). Let $\delta > 0$ be arbitrary, and let $X := (X_1, \ldots, X_n) \stackrel{i.i.d.}{\sim} f \in \mathcal{F}$, be the $n$ observed samples. Suppose we are trying to distinguish between two densities $g, g' \in \mathcal{F}$. Let $\psi(g, g', X)$ denote their log-likelihood difference per (7). We then have

$$\sup_{g, g': \|g-g'\|_2 \leq \delta} \mathbb{P}(\psi(g, g', X) > 0) \leq \exp \left( -nL(\alpha, \beta, C) \delta^2 \right) \quad (8)$$

$^3$Observe that this set may also fail to be totally bounded, since while the ball $B_2(\theta, \varepsilon)$ is a bounded set, it is not totally bounded.
where
\[
C > 1 + \sqrt{1/(\alpha c(\alpha, \beta))}
\]
\[
L(\alpha, \beta, C) := \left(\frac{\sqrt{c(\alpha, \beta)}(C - 1) - \sqrt{1/\alpha}}{2 \{2K(\alpha, \beta) + \frac{3}{4} \log \beta/\alpha\}}\right)^2,
\]
with \(K(\alpha, \beta) := \beta/(\alpha^2 c(\alpha, \beta))\), and \(c(\alpha, \beta)\) is as defined in Lemma 2. In the above \(\mathbb{P}\) is taken with respect to the true density function \(f\), i.e., \(\mathbb{P} = \mathbb{P}_f\).

From Lemma 7, we derive a key concentration result concerning a packing set in \(\mathcal{F}\), as summarized in Lemma 8. The relevance of such a result will become clearer later, when we introduce our sieve-based MLE for \(f\). Our sieve estimator will be constructed using packing sets of \(\mathcal{F}\), thus Lemma 8 will be an important tool to enable us to handle the concentration properties of our estimator.

**Lemma 8 (Maximum likelihood concentration in \(\mathcal{F}\)).** Let \(\delta > 0\) be arbitrary, and let \(X := (X_1, \ldots, X_n)^\top \overset{\text{i.i.d.}}{\sim} f \in \mathcal{F}\), be the \(n\) observed samples. Suppose further that we have a maximal \(\delta\)-packing set of \(\mathcal{F}' \subset \mathcal{F}\), i.e., \(\{g_1, \ldots, g_m\} \subset \mathcal{F}'\) such that \(\|g_i - g_j\|_2 > \delta\) for all \(i \neq j\), and it is known that \(f \in \mathcal{F}'\). Now let \(j^* \in [m]\), denote the index of a density whose likelihood is the largest. We then have
\[
\mathbb{P}(\|g_{j^*} - f\|_2 > (C + 1)\delta) \leq m \exp\left(-nL(\alpha, \beta, C)\delta^2\right),
\]
where \(C\) is assumed to satisfy (9), and \(L(\alpha, \beta, C)\) is defined as per (10).

Next we establish that the map \(\varepsilon \mapsto \log M_{\mathcal{F}}^{\text{loc}}(\varepsilon, c)\) is non-increasing. This lemma is made possible by the fact that the set \(\mathcal{F}\) is convex by assumption, and that we are using the \(L_2\)-metric. This monotonicity property of the \(\varepsilon\)-local metric entropy in the \(L_2\)-metric is a critical technical ingredient used in the proofs establishing our upper bound.

**Lemma 9 (Monotonicity of local metric entropy).** The map \(\varepsilon \mapsto \log M_{\mathcal{F}}^{\text{loc}}(\varepsilon, c)\) is non-increasing.

We now turn our attention to describing our proposed likelihood-based estimator, i.e., \(\nu^*(X)\), of \(f \in \mathcal{F}\). In the discussion that follows we let \(d := \text{diam}_2(\mathcal{F})\), which is finite by the boundedness of \(\mathcal{F}\). The estimator is directly inspired by a recent construction used in Neykov [2022], who applied it to the Gaussian sequence model. However, there the underlying space used is \((\mathbb{R}^n, \|\cdot\|_2)\), whereas in our case it is \((\mathcal{F}, \|\cdot\|_2)\), which has a vastly different underlying geometric structure. Importantly since we are performing density estimation, our proposed estimator uses a fundamentally different log-likelihood-based selection criteria, compared to the projection-based sequence model estimator in Neykov [2022]. Similar to Neykov [2022], although our estimator can also be described constructively, it is not intended to be practically computable. Our estimator will be shortly described as a four-stage constructive procedure (i.e., **Step 1–Step 4**). Since the qualitative description of the construction may appear to be quite involved, we provide a corresponding visual representation in Figure 2 as a helpful guide for the reader. We emphasize however, that Figure 2 is not drawn to any precise scale.
**Step 1** Initialize inputs.
Let $X := (X_1, \ldots, X_n)^T$ denote our $n$ observed i.i.d. data samples. Fix some sufficiently large $c > 0$, and then define $C$ such that $c := 2(C + 1)$. Importantly, the constant $c$ should be set without looking at the data samples, i.e., independently of $X$.

**Step 2** Construct a maximal packing set tree of depth $\mathcal{J}$ before seeing the data.
Construct a tree of packing sets of depth $\mathcal{J} \in \mathbb{N}$, which is independent of the data samples $X$. Here, $\mathcal{J}$ is as defined in Theorem 11. The explicit construction of such a packing set tree proceeds as follows. First, fix any arbitrary point $\Upsilon_1 \in \mathcal{F}$, which is the root node, i.e., the first level of the packing set tree. In the case where $\mathcal{J} = 1$, the tree construction stops at this single root node.

Assuming the (more interesting) case where $\mathcal{J} > 1$, we then let $d := \text{diam}(\mathcal{F})$, and construct a maximal $\frac{d}{2(C+1)}$-packing set of $B_2(\Upsilon_1, d) \cap \mathcal{F} = \mathcal{F}$. Denote this packing set by $P_{\Upsilon_1} := \{m_1, m_2, m_3, \ldots, m_{|P_{\Upsilon_1}|}\}$. The set $P_{\Upsilon_1}$ forms the children (densities) of our root node, that is the second level of the tree. Now, for each density in $P_{\Upsilon_1}$, we again construct a maximal packing set as follows. For example, taking the density $m_3 \in P_{\Upsilon_1}$, we construct a maximal $\frac{d}{4(C+1)}$-packing set of $B_2(m_3, d/2) \cap \mathcal{F}$, which we denote as $P_{m_3} := \{m_{3,1}, m_{3,2}, m_{3,3}, \ldots, m_{3,|P_{m_3}|}\}$. Here, the (finite) packing set $P_{m_3}$ again forms the children of the node density $m_3$. Iterating this process over each density in $P_{\Upsilon_1}$, forms the complete second level of the tree.

Now we can further iterate this process over each density in the second level of the tree to construct the third level of the tree. For example, taking the density $m_{3,3}$, we construct a maximal $\frac{d}{8(C+1)}$-packing set of $B_2(m_{3,3}, d/4) \cap \mathcal{F}$, which we denote as $P_{m_{3,3}} := \{m_{3,3,1}, m_{3,3,2}, m_{3,3,3}, \ldots, m_{3,3,|P_{m_{3,3}}|}\}$, which forms the children of node $m_{3,3}$. This process is iterated so that for the $k$th-level of the tree, we construct $\frac{d}{2^k(C+1)}$-packing sets, with closed balls $B_2(\cdot, d/2^{k-1}) \cap \mathcal{F}$. In particular the packing set tree is extended for each depth level $k \in \{2, 3, \ldots, \mathcal{J} - 1\}$. This process results in a maximal packing set tree of depth $\mathcal{J}$, as claimed.

**Step 3** Build a finite sequence of densities by traversing our packing set tree.
Now, after observing our data sample $X$, we construct a finite sequence of densities, i.e., $\Upsilon := (\Upsilon_k)_{k=1}^\mathcal{J}$, using our packing set tree construction in **Step 2**. First, we initialize the first term of our sequence to $\Upsilon_1$, i.e., the root node already chosen in **Step 2**. If $\mathcal{J} = 1$, then the sequence $\Upsilon := (\Upsilon_1)$. Otherwise, if $\mathcal{J} > 1$, we traverse down one level of our packing set tree, and assign $\Upsilon_2$ to be the density from $P_{\Upsilon_1}$ which maximizes the log-likelihood given the data. That is, set $\Upsilon_2 := \arg\max_{\nu \in P_{\Upsilon_1}} \sum_{i=1}^n \log \nu(X_i)$. Since $P_{\Upsilon_1}$ is a finite set, this will be exhausted for each such iteration in finitely many steps.

Moreover, we note that when assigning $\Upsilon_2$, there may be ties in children densities who all simultaneously maximize the log-likelihood. To break ties, by convention, we always select the left-most child from our packing set tree. Once the $\Upsilon_2$ is assigned from our packing set tree, once again assign $\Upsilon_3$ from its children by again maximizing the log-likelihood. Keep iterating.
in this manner for each index \( k \in \{2, 3, \ldots, J\} \), and construct the finite, i.e., terminating sequence \( \Upsilon \).

**Step 4 Output estimator as the \( J^{th}\)-term of the sequence.**

Finally, we note that the finite sequence \( \Upsilon := (\Upsilon_k)_{k=1}^J \) satisfies\(^d\) \( \|\Upsilon_J - \Upsilon_J'\|_2 \leq \frac{d}{2^{J'}} \), for each pair of positive integers \( J' < J \). Our multistage sieve MLE, i.e., \( \nu^*(X) \), can be taken as the final term of this sequence. That is \( \nu^*(X) := \Upsilon_J \). The estimator \( \nu^*(X) \) is readily understood by comparing\(^e\) Figure 2 with the qualitative description in **Step 1-Step 4**.

---

\(^a\)By convention, the children forming the packing set densities are arbitrarily indexed in an increasing alphanumeric manner, from left child node to right child node.

\(^b\)This selection rule thus effectively assigns the child density maximizing log-likelihood with the smallest such alphanumeric index.

\(^c\)Note that \( k \) here refers to index of the \( k^{th}\)-term our sequence \( \Upsilon \).

\(^d\)We will formally justify this in the appendix in Lemma 28.

\(^e\)We note that in Figure 2 if \( J = 1 \), the estimator would just output \( \Upsilon_1 \). In the case where \( J > 1 \), the maximal packing sets for each level of the tree are illustrated on the left, and the corresponding constructed tree level is shown on the right. In this instance the finite sequence of \( J \) densities is given by \( \Upsilon = (\Upsilon_1, m_3, m_3, m_3, 2, \ldots, m_3, 3, 2, \ldots, 5) \). The estimator then takes the \( J^{th}\)-term of \( \Upsilon \), i.e., \( \nu^*(X) = m_{3,2}, \ldots, 5_{(J-1)-term} \).

**Remark 7.** We reiterate that Figure 2 is not drawn to any precise scale. In reality the \( L_2\)-balls should be much “wider” than the set \( \mathcal{F} \) (and \( \mathcal{F}_B^{[\alpha,\beta]} \)). This is because they do not impose that their elements are proper densities, unlike the elements of the set \( \mathcal{F} \) (and \( \mathcal{F}_B^{[\alpha,\beta]} \)) which are non-negative and integrate to 1. It is intended to be useful conceptual guide to understanding the construction of our multistage sieve MLE.

We observe that our proposed estimator \( \nu^*(X) \) can be thought of as an “multistage sieve MLE” in the spirit of Wong and Shen [1995]. Broadly speaking a ‘sieve’ MLE effectively takes the MLE over a strategically constrained subset of the parameter space, i.e., \( \mathcal{F} \) in our setting [see Chapter 8 Grenander, 1981, e.g., for more details]. Specifically, as we traverse the down the finite-depth maximal packing set tree, each group of children densities along with the MLE selection rule can be thought of as a “sieve”. We note that the sieve MLE proposed in Wong and Shen [1995] is a construction which is also not practically computable for general density classes \( \mathcal{F} \).

**Remark 8 (An online finite packing set tree construction).** We note that the finite-depth maximal packing set tree described in **Step 2**, can be replaced with a conceptually simpler online finite-depth maximal packing set tree construction. This proceeds as follows. Once again, as per **Step 2**, we can initialize \( \Upsilon_1 \in \mathcal{F} \) to be the root node independently of the data. We then construct the second level of our packing set tree, i.e., \( P_{\Upsilon_1} := \{m_1, m_2, m_3, \ldots, m_{|P_{\Upsilon_1}|}\} \), as the previously described maximal packing set. This first level is constructed without looking at the data samples \( X \). This time however, we can traverse down the first level of the tree and set \( \Upsilon_2 := \arg \max_{\nu \in P_{\Upsilon_1}} \sum_{i=1}^{n} \log \nu(X_i) \), i.e., by using the data samples \( X \). Given \( \Upsilon_2 \) selected in this data driven manner, we can construct the second level of the tree as the children of \( \Upsilon_2 \), i.e., the maximal packing set \( P_{\Upsilon_2} \) without using the data samples. We can then set \( \Upsilon_3 := \arg \max_{\nu \in P_{\Upsilon_2}} \sum_{i=1}^{n} \log \nu(X_i) \), once again using the data. We can thus repeat this recursive process for \( J \) iterations, whereby the maximal packing set of children
of each parent node are constructed without seeing the data. The specific child node is selected after seeing the data, and then the estimator can traverse to one of these children. This does not require the all possible children of all possible parent nodes of the maximal packing set tree to be constructed up front as described in Step 2. Instead, we only construct the children as required in a simple sequential manner.

We next show that our multistage sieve MLE is a measurable function of the data with respect to the Borel $\sigma$-field on $F$ in $L_2$-metric topology. This is important, because all upper bound risk rates in expectation for $\nu^*(X)$ that follow, are with respect to the $L_2$-metric topology on $F$.

**Proposition 10** (Measurability of $\nu^*(X)$). The multistage sieve MLE, i.e., $\nu^*(X)$, is a measurable function of the data with respect to the Borel $\sigma$-field on $F$ in the $L_2$-metric topology.

With the measurability of $\nu^*(X)$ established, the main theorem establishing the performance of $\nu^*(X)$ is Theorem 11 below.

**Theorem 11** (Upper bound rate for the multistage sieve MLE $\nu^*(X)$). Let, $\nu^*(X) = \Upsilon_\mathcal{T}$ be the output of the multistage sieve MLE which is run for $J \in \mathbb{N}$ steps. Here $J$ is defined as the maximal integer $J \in \mathbb{N}$, such that

$$n\varepsilon_J^2 > 2 \log M^\text{loc}_F \left( \varepsilon_J, \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)d}} \right) \lor \log 2,$$

or $J = 1$ if no such $J$ exists. Then

$$\mathbb{E}\|\nu^*(X) - f\|_2^2 \leq \bar{C}\varepsilon^2,$$

for some universal constant $\bar{C}$, and where $\varepsilon^* := \varepsilon_\mathcal{T}$. We remind the reader that $c := 2(C + 1)$ is the constant from the definition of local metric entropy, which is assumed to be sufficiently large. Here $C$ is assumed to satisfy (9), and $L(\alpha, \beta, C)$ is defined as per (10).

We will now formally illustrate that the above estimator achieves the minimax rate. The precise expression of the rate is quantified in the following result.

**Theorem 12** (Minimax rate). Define $\varepsilon^* := \sup\{\varepsilon : n\varepsilon^2 \leq \log M^\text{loc}_F (\varepsilon, c)\}$, where $c$ in the definition of local metric entropy is a sufficiently large absolute constant. Then the minimax rate is given by $\varepsilon^{*2} \land d^2$ up to absolute constant factors.

**Remark 9** (Extending results to loss functions in $\text{KL}$-divergence and the Hellinger metric). Recall that by Lemma 2 we have the “topological equivalence” of the $\text{KL}$-divergence and squared Hellinger metric with the squared $L_2$-metric on $F$. This means that we can readily extend our minimax risk bounds in Theorem 12 to loss functions measured via $\text{KL}$-divergence and the squared Hellinger metric. The important consideration is that (11) is still solved (in both cases) using the local metric entropy of $F$ using the squared $L_2$-metric. Note that for the $\text{KL}$-divergence to be well-defined, we require that all densities are strictly positively lower bounded over the common compact support.

---

4Observe that by the definition of $\varepsilon_\mathcal{T}$ and (11) we have that all packing sets used in the construction of the estimator must be finite, even though we are not assuming that the set $F$ is totally bounded.
We now argue that the minimax rate for a class $\mathcal{F} \subset \mathcal{F}_B^{[0,\beta]}$ which is convex and not necessarily lower bounded by $\alpha > 0$ is given by the same equation, as long as there exists a single density in $f_\alpha \in \mathcal{F}$ which is $\alpha$-lower bounded. The argument used to establish this claim essentially the same as used in Yang and Barron [1999, Lemma 1], which we formalize for our setting in Proposition 13. For completeness, we provide all details for our setting in the Appendix.

**Proposition 13** (Extending results to $\mathcal{F}_B^{[0,\beta]}$). Let $\mathcal{F} \subset \mathcal{F}_B^{[0,\beta]}$ be a convex class of densities, with at least one $f_\alpha \in \mathcal{F}$ that is $\alpha$-lower bounded, with $\alpha > 0$. Then the minimax rate in the squared $L_2$-metric is $\varepsilon^* \leq d^2$, where $\varepsilon^* := \sup\{\varepsilon : n\varepsilon^2 \leq \log M_{\mathcal{F}_x}^\text{loc}(\varepsilon, c)\}$.

### 2.3 Adaptivity

In this section we illustrate that the estimator, $\nu^*(X)$, as defined in Section 2.2 is adaptive to the true density $f$. Before that, similar to Neykov [2022] we re-define the notion of adaptive $L_2$-local metric entropy for any density $\theta \in \mathcal{F}$.

**Definition 14** (Adaptive Local Entropy). Let $\theta \in \mathcal{F}$ be a density. Let $M(\theta, \varepsilon, c)$ denote the maximal cardinality of a packing set of the set $B_2(\theta, \varepsilon) \cap \mathcal{F}$ at an $L_2$ distance $\varepsilon/c$. Let

$$M_{\mathcal{F}}^\text{adloc}(\theta, \varepsilon, c) := M(\varepsilon/c, \mathcal{F} \cap B_2(\theta, \varepsilon)) =: M_{\mathcal{F} \cap B_2(\theta, \varepsilon)}^{\text{glo}}(\varepsilon/c).$$

We refer to $\log M_{\mathcal{F}}^\text{adloc}(\theta, \varepsilon, c)$ as the adaptive $L_2$-local metric entropy of $\mathcal{F}$ at $\theta$. We note that in contrast to Definition 5, here we do not take the supremum over all $\theta \in \mathcal{F}$, i.e., $M_{\mathcal{F}}^\text{adloc}(\theta, \varepsilon, c)$ here depends on input $\theta$ value.

**Theorem 15** (Adaptive upper bound rate for the multistage sieve MLE $\nu^*(X)$). Let, $\nu^*(X) = \bar{\Upsilon}_J$ be the output of the multistage sieve MLE which is run for $J$ iterations where $J$ is defined as the maximal solution to

$$n\varepsilon_J^2 > 2 \inf_{f \in \mathcal{F}} M_{\mathcal{F}}^\text{adloc}(f, 2\varepsilon_J \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)}}, 2c) \vee \log 2,$$

where $\varepsilon_J := \frac{\sqrt{L(\alpha, \beta, c/2 - 1)} d}{2J - 2\varepsilon_J} c$ and $J = 1$ if no such $J$ exists. Let $J^*$ be defined as the maximal integer $J \in \mathbb{N}$, such that $\varepsilon_J := \frac{\sqrt{L(\alpha, \beta, c/2 - 1)} d}{2J^2 c}$ such that

$$n\varepsilon_J^2 > 2 M_{\mathcal{F}}^\text{adloc}(f, 2\varepsilon_J \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)}}, 2c) \vee \log 2,$$

and $J^* = 1$ if no such $J$ exists. Then

$$\mathbb{E}\|\nu^*(X) - f\|_2^2 \leq \hat{C} \varepsilon^*^2,$$

for some universal constant $\hat{C}$, and where $\varepsilon^* := \varepsilon_J^*$. We remind the reader that $c := 2(C + 1)$ is the constant from the definition of local metric entropy, which is assumed to be sufficiently large. Here $C$ is assumed to satisfy (9), and $L(\alpha, \beta, C)$ is defined as per (10).

---

Note that running the estimator with $\bar{\Upsilon}$ many steps, may result into having non-finite packing sets — that is not an issue however.

Observe that by the definition of $\bar{\Upsilon}$ and (11) we have that some packing sets used in the construction of the adaptive estimator may not be finite, but will be at most countable. This follows from the $L_2$-separability of $\mathcal{F}$ and is formalized in Lemma 25 in the appendix. We note that the measurability of the adaptive estimator still holds as per Proposition 10 in this case.
3 Examples

We will now apply our work to derive risk bounds for density estimation (under the squared $L_2$-metric) for various examples of convex density classes $F$. To that end, per Proposition 13 our risk bounds only require us to establish that the stated class $F$ is indeed convex, and importantly that there exists at least one density $f_\alpha \in F$ that is positively bounded away from 0 over the entire support $B$. In order to establish the latter fact we can usually take $f_\alpha \sim \text{Unif}[B]$, and check that it lies in our density class $F$, and by suitably expanding our ambient space $^7F_B^{[a, b]}$. We will also use the following key fact relating $L_2$-local and $L_2$-global metric entropies.

$$\log M_F^{\text{glo}}(\varepsilon/c) - \log M_F^{\text{glo}}(\varepsilon) \leq \log M_F^{\text{loc}}(\varepsilon, c) \leq \log M_F^{\text{glo}}(\varepsilon/c) \quad (13)$$

Here, (13) follows directly from Yang and Barron [1999, Lemma 2], where it is only proved for the case $c = 2$. However, their proof directly extends to the more general case for each $c > 0$, which is required for our setting. For the various examples of $F$ that follow below, we will formally show$^8$ that a stronger sufficient condition on global entropy is satisfied, namely

$$\log M_F^{\text{glo}}(\varepsilon/c) - \log M_F^{\text{glo}}(\varepsilon) \propto \log M_F^{\text{glo}}(\varepsilon/c), \quad (14)$$

provided we take $c$ to be sufficiently large enough, which is within our control to do, per our packing set tree construction. In short, (14) will enable us to bound the local metric entropy via (13). To illustrate this, we initially consider two examples from Yang and Barron [see 1999, Section 6]. We begin with the class $F := \text{Lip}_{\gamma,q}(\Psi)$, i.e., the $(\gamma, q, \Psi)$-Lipschitz density class defined as per (15). As noted in Yang and Barron [1999, Section 6.4], with fixed constants $\max\{1/q - 1/2, 0\} < \gamma \leq 1$, and $1 \leq q \leq \infty$, the $\varepsilon$-global metric entropy of $\text{Lip}_{\gamma,q}(\Psi)$ is of the order $\varepsilon^{-1/\gamma}$ per Birman and Solomjak [1980].

Example 16 (Lipschitz density class $F$). Let $1 < \Psi < \beta < \infty$, $\max\{1/q - 1/2, 0\} < \gamma \leq 1$, and $1 \leq q \leq \infty$ be fixed constants, and $B := [0, 1]$. Now, let $F := \text{Lip}_{\gamma,q}(\Psi)$ denote the space of $(\gamma, q, \Psi)$-Lipschitz densities with total variation at most $\beta$. That is,

$$\text{Lip}_{\gamma,q}(\Psi) := \left\{ f : B \rightarrow [0, \Psi] \middle| \|f(x + h) - f(x)\|_q \leq \Psi h^\gamma, \|f\|_q \leq \Psi, \int_B f \, d\mu = 1, f \text{ measurable} \right\}, \quad (15)$$

and $\|f\|_q := \left(\int_B |f(x)|^q \, d\mu\right)^{1/q}$. Note that in (15) we have that $x \in B$, and only consider $h > 0$, and further $f(x + h) = f(1)$, for $x + h > 1$, so that the predicate of $\text{Lip}_{\gamma,q}(\Psi)$ is well-defined. Then $\text{Lip}_{\gamma,q}(\Psi)$ is a convex density class, there exists a density $f_\alpha \in \text{Lip}_{\gamma,q}(\Psi)$ that is strictly positively bounded away from 0, and the minimax rate (in the squared $L_2$-metric) for estimating $f \in \text{Lip}_{\gamma,q}(\Psi)$ is of the order $n^{-2\gamma/(2\gamma + 1)}$.

Another well studied density estimation problem is the case where $F := \text{BV}_\zeta$ is total bounded variation at most $\zeta$, defined as per (16). Importantly we note that the $\varepsilon$-global $L_2$-metric entropy of this well studied function class is of the order $\varepsilon^{-1}$ [see Section 6.4 Yang and Barron, 1999, e.g., for more details].

---

$^7$We reiterate that our use of $\{a, b\}$ in this section (and throughout the paper) is only used to refer to the constants in Definition 1, of $F_B^{[a, b]}$ and thus $F$.

$^8$See Appendix C for these details.
Example 17 (Bounded total variation density class \( \mathcal{F} \)). Let \( 1 < \zeta < \beta < \infty \) be a fixed constant, and \( B := [0, 1] \). Now, let \( \mathcal{F} := \text{BV}_\zeta \) denote the space of univariate densities with total variation at most \( \beta \). That is,

\[
\text{BV}_\zeta := \left\{ f : B \to [0, \zeta] \left| \|f\|_\infty \leq \zeta, V(f) \leq \zeta, \int_B f \, d\mu = 1, f \text{ measurable} \right. \right\},
\]

(16)

where we define the total variation of \( f \), i.e., \( V(f) \) as

\[
V(f) := \sup_{\{x_1, \ldots, x_m \mid 0 \leq x_1 < \cdots < x_m \leq 1, m \in \mathbb{N}\}} \sum_{i=1}^{m-1} |f(x_{i+1}) - f(x_i)|,
\]

(17)

and \( \|f\|_\infty := \sup_{x \in B} |f(x)| \). Then the minimax rate (in the squared \( L_2 \)-metric) for estimating \( f \in \text{BV}_\zeta \) is of the order \( n^{-2/3} \).

Another interesting example illustrating the use case of our bounds is that where \( \mathcal{F} := \text{Quad}_\gamma \), forms the density class of \( \gamma \)-quadratic functionals defined as per (18). Importantly we note that the \( \varepsilon \)-global \( L_2 \)-metric entropy of this well studied function class is of the order \( \varepsilon^{-1/4} \) [see Example 15.8 and Example 15.22 Wainwright, 2019, e.g., for more details].

Example 18 (Quadratic functional density class \( \mathcal{F} \)). Let \( 0 < \alpha < 1 < \beta < \infty \), and \( \gamma > 1 \) be fixed constants, with \( B := [0, 1] \). Now, let \( \mathcal{F} := \text{Quad}_\gamma \) denote the space of univariate quadratic functional densities. That is,

\[
\text{Quad}_\gamma := \left\{ f : B \to [\alpha, \beta] \left| \|f''\|_\infty \leq \gamma, \int_B f \, d\mu = 1, f \text{ measurable} \right. \right\}.
\]

(18)

Then \( \text{Quad}_\gamma \) is a convex density class, there exists a density \( f_\alpha \in \text{Quad}_\gamma \) that is strictly positively bounded away from 0, and the minimax rate (in the squared \( L_2 \)-metric) for estimating \( f \in \text{Quad}_\gamma \) is of the order \( n^{-4/5} \).

We now turn our attention to an interesting example, which demonstrates that our results can yield useful bounds in cases where \( L_2 \)-global metric entropy of \( \mathcal{F} \) may be unknown (or difficult to compute), but the \( L_2 \)-local metric entropy can be controlled.

Example 19 (Convex mixture density class \( \mathcal{F} \)). Let \( \mathcal{F} := \text{Conv}_k \) where

\[
\text{Conv}_k := \left\{ \sum_{i=1}^k \alpha_i f_i \left| \sum_{i=1}^k \alpha_i = 1, \alpha_i \geq 0, f_i \in \mathcal{F}_B^{[\alpha, \beta]} \right. \right\},
\]

(19)

for some fixed \( k \in \mathbb{N} \) and \( f_i \in \mathcal{F}_B^{[\alpha, \beta]} \) for each \( i \in [k] \). Further, let \( \mathbf{G} = (G_{ij})_{i,j \in [k]} \) denote the Gram matrix with \( G_{ij} := \int_B f_i f_j \mu(dx) \), which we assume is positive definite, i.e., \( \mathbf{G} > 0 \). Then the minimax rate for estimating \( f \in \text{Conv}_k \) is bounded from above by \( \sqrt{\frac{k}{n}} \) up to absolute constant factors.
4 Discussion

In this paper we derived exact minimax rates for density estimation over convex density classes. Our work builds on seminal research of Birgé [1983], Le Cam [1973], Wong and Shen [1995], Yang and Barron [1999]. More directly, we non-trivially adapted the techniques of Neykov [2022], who used it for deriving exact rates for the Gaussian sequence model. Our results demonstrate that the $L_2$-local metric entropy always determines that minimax rate under squared $L_2$-loss in this setting. We thus provide a unifying perspective across parametric and nonparametric convex density classes, and under weaker assumptions than those used by Yang and Barron [1999].

An important open question that we would like to think further about is whether there exists a computationally tractable estimator which is also minimax optimal in our setting. We can also consider applying our techniques to the nonparametric regression setting (with Gaussian noise) where $f$ is a uniformly bounded regression function of interest. We leave these exciting directions for future work. Finally, we hope that this research stimulates further activity in approximating $L_2$-local metric entropy for various convex density classes.

5 Acknowledgments

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Figure 2: Maximal packing set tree construction in Step 2. The color of nodes corresponds to the color of the points. The nodes with a white background correspond to the points we select at each iteration of the multistage sieve estimation. The leaf node with the yellow background represents the final multistage sieve estimator.
A Preliminary

We begin with some basic mathematical preliminaries for our work.

A.1 Properties of $\mathcal{F}_B^{[\alpha,\beta]}$

Here we provide some basic analytic properties of our core density class $\mathcal{F}_B^{[\alpha,\beta]}$, as per Definition 1. Many of these facts will be used, sometimes implicitly, in our proofs. We hope that by documenting them rigorously, they provide the reader with a much richer understanding of the geometry of this broader density class. This may also be a useful reference for researchers in working in similar density estimation settings. We provide suitable references where the properties follow from standard real analysis theory.

**Lemma 20** (Convexity of $\mathcal{F}_B^{[\alpha,\beta]}$). The density class $\mathcal{F}_B^{[\alpha,\beta]}$, forms a convex set, in the $L_2$-metric.

**Proof of Lemma 20.** In order to show the convexity of $\mathcal{F}_B^{[\alpha,\beta]}$, Let $f, g \in \mathcal{F}_B^{[\alpha,\beta]}$, and let $\kappa \in [0, 1]$ be arbitrary. Then for each $x \in B$, we observe that

$$
(\kappa f + (1 - \kappa) g)(x) := \kappa f(x) + (1 - \kappa) g(x) \geq \kappa \alpha + (1 - \kappa) \alpha \geq \alpha
$$

(20)

$$
(\kappa f + (1 - \kappa) g)(x) := \kappa f(x) + (1 - \kappa) g(x) \leq \kappa \beta + (1 - \kappa) \beta \leq \beta
$$

(21)

From (20) and (21), it follows that $\kappa f + (1 - \kappa) g : B \rightarrow [\alpha, \beta]$. Moreover, since $\int_B f \, d\mu = \int_B g \, d\mu = 1$, we have

$$
\int_B (\kappa f + (1 - \kappa) g) \, d\mu = \kappa \int_B f \, d\mu + (1 - \kappa) \int_B g \, d\mu = 1.
$$

(22)

Since $f, g$ are measurable functions, then so is their convex combination, i.e., $\kappa f + (1 - \kappa) g$. Combining the above we have shown that $\kappa f + (1 - \kappa) g \in \mathcal{F}_B^{[\alpha,\beta]}$, which proves the convexity of $\mathcal{F}_B^{[\alpha,\beta]}$, as required.

**Lemma 21** (Boundedness of $\mathcal{F}_B^{[\alpha,\beta]}$). The density class $\mathcal{F}_B^{[\alpha,\beta]}$, is bounded, in the $L_2$-metric.

**Proof of Lemma 21.** We now show that $\mathcal{F}_B^{[\alpha,\beta]}$ is bounded in the $L_2$-metric. To see this observe that for any $f, g \in \mathcal{F}_B^{[\alpha,\beta]}$:

$$
\|f - g\|_2^2 := \int_B (f - g)^2 \, d\mu \leq \int_B |f - g| \, 2\beta \, d\mu \leq 2\beta \left( \int_B |f| \, d\mu + \int_B |g| \, d\mu \right) = 4\beta.
$$

(23)

It follows that $\text{diam}_2 \left( \mathcal{F}_B^{[\alpha,\beta]} \right) := \sup \left\{ \|f - g\|_2 \mid f, g \in \mathcal{F}_B^{[\alpha,\beta]} \right\} \leq 2\sqrt{\beta} < \infty$, as required.

**Lemma 22** ($\mathcal{F}_B^{[\alpha,\beta]}$ lies in $L^2(B)$). The density class $\mathcal{F}_B^{[\alpha,\beta]}$, satisfies $\mathcal{F}_B^{[\alpha,\beta]} \subset L^2(B)$, where

$$
L^2(B) := \left\{ f : B \rightarrow \mathbb{R} \mid \int_B f^2 \, d\mu < \infty, f \text{ measurable} \right\}.
$$

(24)

As such $(\mathcal{F}_B^{[\alpha,\beta]}, \| \cdot \|_2)$ is an induced metric subspace of $(L^2(B), \| \cdot \|_2)$. 

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Proof of Lemma 22. Let \( f \in \mathcal{F}_B^{[\alpha, \beta]} \) be arbitrary. We then observe that

\[
\int_B f^2 \, d\mu \leq \int_B f \beta \, d\mu = \beta < \infty,
\]

since \( f \leq \beta \) by definition of \( \mathcal{F}_B^{[\alpha, \beta]} \). Given that \( f : B \to [\alpha, \beta] \subset \mathbb{R} \), we have that \( f \in L^2(B) \), i.e., \( \mathcal{F}_B^{[\alpha, \beta]} \subset L^2(B) \) as required.

\[\square\]

Lemma 23 (Completeness and separability of \( L^2(B) \)). The metric space \( (L^2(B), \| \cdot \|_2) \), with \( L^2(B) \) defined as per Equation (24), is complete and separable.

Proof of Lemma 23. We note that completeness of \( (L^2(B), \| \cdot \|_2) \) follows directly from Brezis [2011, Theorem 4.8], and separability follows from Brezis [2011, Theorem 4.13].

\[\square\]

Lemma 24 (Completeness and separability of \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \)). The metric space \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \) is complete and separable.

Proof of Lemma 21. Firstly we note that \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \) is an induced metric subspace of \( (L^2(B), \| \cdot \|_2) \) per Lemma 22. Now separability of \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \) follows, since it is inherited from \( (L^2(B), \| \cdot \|_2) \) by applying Shirali and Vasudeva [2006, Proposition 2.3.16]. We now show the completeness of \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \). Take an arbitrary Cauchy sequence \( (f_k)_{k=1}^\infty \) in \( \mathcal{F}_B^{[\alpha, \beta]} \). Since \( (L^2(B), \| \cdot \|_2) \) is complete per Lemma 23, it follows that the \( L_2 \) limit of \( (f_k)_{k=1}^\infty \) exists in \( L^2(B) \). Let \( f \) be that limit, i.e., \( \lim_{k \to \infty} f_k = f \in L^2(B) \). We will show that \( f \in \mathcal{F}_B^{[\alpha, \beta]} \). First let us show that it is a density, i.e., it integrates to 1. By Cauchy-Schwartz \( \int_B |f_k(x) - f(x)| \mu(\, dx) \leq \sqrt{\int_B |f_k(x) - f(x)|^2 \mu(\, dx)} \to 0 \) so that \( \int f(x) \mu(\, dx) = 1 \). Next consider the function \( f' = (f \wedge \alpha) \vee \beta \). Since for any \( x \in [\alpha, \beta] \) and any \( y \) we have \( |x - y| \geq |x - (y \wedge \alpha) \vee \beta| \) then that implies (since \( f_k \in \mathcal{F}_B^{[\alpha, \beta]} \)) that \( \int_B |f_k(x) - f'(x)|^2 \mu(\, dx) \leq \int_B |f_k(x) - f(x)|^2 \mu(\, dx) \to 0 \) and so \( f' \) must also be a limit of \( f_k \). Since the limits are unique (up to considering equivalence classes modulo sets of measure 0 with respect to \( \mu \)) then \( f = f' \) and hence it belongs to \( \mathcal{F}_B^{[\alpha, \beta]} \).

\[\square\]

Lemma 25 (Separability of \( (\mathcal{F}, \| \cdot \|_2) \)). The metric space \( (\mathcal{F}, \| \cdot \|_2) \) is separable. Furthermore, if \( A \subset \mathcal{F} \), then \( (A, \| \cdot \|_2) \) is separable.

Proof of Lemma 25. We first observe by Lemma 24 that the metric space \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \) is separable. It then follows that since \( \mathcal{F} \subset \mathcal{F}_B^{[\alpha, \beta]} \), that \( (\mathcal{F}, \| \cdot \|_2) \) is a restriction of \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \), and thus a separable metric space by Shirali and Vasudeva [2006, Proposition 2.3.16]. By a similar argument, it follows that if \( A \subset \mathcal{F} \), then \( (A, \| \cdot \|_2) \) is separable.

\[\square\]

The following lemma shows that in general \( \mathcal{F}_B^{[\alpha, \beta]} \) is not totally bounded. We consider a restricted case of \( B := [0, 1] \), to construct a suitable counterexample.

Lemma 26 (Non total boundedness of \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \)). Suppose \( \beta \geq 2 - \alpha \). Then the metric space \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \) is not totally bounded, and hence not compact.
Proof of Lemma 26. We note that per Shirali and Vasudeva [2006, Theorem 5.1.12] a metric space is totally bounded if and only if every sequence contains a Cauchy subsequence. We will use this characterization to construct a counterexample to demonstrate that \((F_{[\alpha,\beta]}^{[\alpha,\beta]}, \| \cdot \|_2)\) is not totally bounded. In particular, we will define a sequence in \((F_{[\alpha,\beta]}^{[\alpha,\beta]}, \| \cdot \|_2)\) which can’t contain any Cauchy subsequence.

Specifically we consider the sequence of functions \((1, \{x \mapsto \sin(2\pi jx)\}_{j \in \mathbb{N}})\). These functions are orthonormal in \(L^2([0,1])\). Construct the sequence of functions \(f_j(x) = 1 + (1 - \alpha)\sin(2\pi jx)\) for \(j \in \mathbb{N}\). By the orthogonality of 1 and \(\sin(2\pi jx)\) we have that \(\int_0^1 f_j(x)dx = 1\). Furthermore, \(\alpha \leq f_j(x) \leq 2 - \alpha\) for all \(x \in [0,1]\), hence since \(\beta \geq 2 - \alpha\) we have \(f_j(x) \in F_{[\alpha,\beta]}^{[\alpha,\beta]}\). Take any two \(j \neq k \in \mathbb{N}\), and consider

\[
\|f_j - f_k\|_2^2 = (1 - \alpha)^2\|\sin(2\pi jx) - \sin(2\pi kx)\|_2^2 = 2(1 - \alpha)^2 > 0.
\]

This shows that there cannot be a Cauchy subsequence and hence the set is not totally bounded. \(\Box\)

A.2 Elementary inequalities

We will state and prove Lemma 27, which will provide the key fact to will assist us in the proof of the lower bound in Lemma 2.

Lemma 27 (Elementary log inequality). For each \(\gamma > 0\), and for any \(x \in (0,\gamma]\), the following relationship holds:

\[
\log x \leq (x - 1) - h(\gamma)(x - 1)^2. \tag{26}
\]

Here \(h: (0,\infty) \to \mathbb{R}\) is defined as in (6), and is positive over its entire support.

Proof of Lemma 27. We first argue that \(h(x) > 0\) for \(x \in (0,\infty)\). This is by the elementary inequality \(\log(x + 1) \leq x\) for all \(x \geq -1\). Next, it suffices to show that the map \(x \mapsto h(x)\) is decreasing for \(x > 0\) where \(h\) is defined in (6). This is because (26) holds for \(x = 1\), and if \(x \neq 1\) it is equivalent to

\[
h(\gamma) \leq \frac{(x - 1) - \log x}{(x - 1)^2},
\]

for \(x \leq \gamma\). It is simple to verify that

\[
h'(x) = \frac{-x^2 + 2x\log x + 1}{(x - 1)^3x}.
\]

We will show that the above function is negative on \((0,\infty)\) which will complete the proof. First we will evaluate it at \(x = 1\). By a triple application of L’Hôpital’s rule it is simple to verify that \(\frac{d}{dx} h(x)|_{x=1} = -\frac{1}{3} < 0\). Thus, it remains to show that for \(x \neq 1\),

\[(-x^2 + 2x\log x + 1)(x - 1) < 0.\]

Now let \(f(x) := \frac{x^2 - 1}{2x} - \log x\). We want to show that \(f(x) > 0\), for each \(x > 1\) and \(f(x) < 0\) for \(x < 1\). First observe that \(f(1) = 0\). Moreover, we have that

\[f'(x) = \frac{(x - 1)^2}{2x^2} > 0. \tag{27}\]

That is, \(f(x)\) is strictly increasing, which implies that \(f(x) > 0\) for each \(x \in (1,\infty)\), and \(f(x) < 0\) for each \(x < 1\) as required. \(\Box\)
B Proofs of Section 2

B.1 Proof of Lemma 2

Lemma 2 ([KL-\(L_2\) equivalence on \(\mathcal{F}_B^{[\alpha,\beta]}\)). For each pair of densities \(f, g \in \mathcal{F}_B^{[\alpha,\beta]}\), the following relationship holds:

\[ c(\alpha, \beta)\|f - g\|_2^2 \leq d_{\text{KL}}(f\|g) \leq (1/\alpha)\|f - g\|_2^2, \tag{5} \]

where we denote \(c(\alpha, \beta) := \frac{h(\beta/\alpha)}{\beta} > 0\). Here \(h : (0, \infty) \to \mathbb{R}\) is defined to be

\[ h(\gamma) := \begin{cases} \frac{\gamma - 1 - \log \gamma}{(\gamma - 1)^2} & \text{if } \gamma \in (0, \infty) \setminus \{1\} \\ \frac{1}{2} = \lim_{x \to 1} \frac{x - 1 - \log x}{(x - 1)^2} & \text{if } \gamma = 1, \end{cases} \tag{6} \]

and is positive over its entire support. It is also easily seen that on \(\mathcal{F}_B^{[\alpha,\beta]}\), \(d_{\text{KL}}\) (and hence the \(L_2\)-metric) is also equivalent to the Hellinger metric. Furthermore, these properties are also inherited by \(\mathcal{F} \subset \mathcal{F}_B^{[\alpha,\beta]}\), which is our density class of interest.

Proof of Lemma 2. We will prove the upper and lower bound in turn.

(Upper bound in (5)): We seek to show that \(d_{\text{KL}}(f\|g) \leq \frac{1}{\alpha}\|f - g\|_2^2\). First, for any two densities \(f, g \in \mathcal{F}\), we define the \(\chi^2\)-divergence between \(f\) and \(g\), as follows:

\[ \chi^2(f\|g) := \int_B \frac{(f - g)^2}{g} d\mu. \tag{28} \]

Per Remark 3, we note that \(\chi^2(f\|g)\) in (28) is similarly well-defined. We then have:

\[
\begin{align*}
    d_{\text{KL}}(f\|g) &\leq \chi^2(f\|g) \\
    &\overset{(\text{per Gibbs and Su [2002, Theorem 5]})}{=} \int_B \frac{(f - g)^2}{g} d\mu \\
    &\overset{(\text{using (28))}}{=} \frac{1}{\alpha} \int_B (f - g)^2 d\mu \\
    &\overset{(\text{since } \inf_{x \in B} g(x) \geq \alpha > 0)}{=} \frac{1}{\alpha} \|f - g\|_2^2, \\
    &\overset{(\text{by definition})}{=} \frac{1}{\alpha} \|f - g\|_2^2.
\end{align*}
\]

As required.

(Lower bound in (5)): We seek to show that \(d_{\text{KL}}(f\|g) \geq c(\alpha, \beta)\|f - g\|_2^2\). We proceed as
follows: First observe that for any \( f, g \in \mathcal{F} \), we have that \( 0 < \frac{f}{g} \leq \frac{\beta}{\alpha} < \infty \)

\[
d_{\text{KL}}(f||g) := \int_B f \log \left( \frac{f}{g} \right) d\mu \tag{per (4)}
\]

\[
= \int_B -f \log \left( \frac{g}{f} \right) d\mu \quad \text{(since } \inf_{x \in B} f(x) \geq \alpha > 0)\n\]

\[
\geq \int_B -f \left( \frac{g}{f} - 1 - h(\beta/\alpha) \left( \frac{g}{f} - 1 \right)^2 \right) d\mu \quad \text{(using Lemma 27, with } C = \frac{\beta}{\alpha} \text{ and } x = \frac{g}{f})\n\]

\[
= \int_B (f - g) d\mu + h(\beta/\alpha) \int_B \frac{(g - f)^2}{f} d\mu \quad \text{(since } \int_B (f - g) d\mu = 0, \text{ and } 0 < \sup_{x \in B} f(x) \leq \beta)\n\]

\[
\geq \frac{h(\beta/\alpha)}{\beta} \int_B (g - f)^2 d\mu \quad \text{(by definition)}\n\]

\[
= c(\alpha, \beta) \| f - g \|_2^2, \tag{29}\]

where we define \( c(\alpha, \beta) := \frac{h(\beta/\alpha)}{\beta} > 0 \). This proves the lower bound in (5), as required.

We now show the following equivalence between the Hellinger, i.e., \( d_H \)-metric, and the \( L_2 \) metric in \( \mathcal{F}[\alpha, \beta] \).

\[
(1/4\beta) \| f - g \|_2^2 \leq d_H(f||g)^2 \leq (1/\alpha) \| f - g \|_2^2. \tag{30}\]

To prove the upper bound in (30), we note that

\[
d_H(f||g)^2 \leq d_{\text{KL}}(f||g) \quad \text{(from Gibbs and Su [2002])}\n\]

\[
\leq (1/\alpha) \| f - g \|_2^2, \quad \text{(per (5))}\n\]

as required. In order to prove the lower bound we observe that

\[
\| f - g \|_2^2 = \int_B (f - g)^2 d\mu
\]

\[
= \int_B (\sqrt{f} + \sqrt{g})^2 (\sqrt{f} - \sqrt{g})^2 d\mu
\]

\[
\leq 4\beta \int_B (\sqrt{f} - \sqrt{g})^2 d\mu \quad \text{(since } f, g \leq \beta)\n\]

\[
=: 4\beta d_H(f||g)^2, \quad \text{(by definition)}\n\]

which implies the required lower bound in (30). We have thus established the required upper and lower bounds in both (5) and (30).

Finally, we note that \( (\mathcal{F}, \| \cdot \|_2) \) is metric space, since it is the restriction of the metric space \( (\mathcal{F}_B^{[\alpha, \beta]}, \| \cdot \|_2) \). And so the bounds (5) and (30), are also inherited by \( \mathcal{F} \subset \mathcal{F}_B^{[\alpha, \beta]} \).
B.2 Proof of Lemma 6

Lemma 6 (Minimax lower bound). Let $c > 0$ be fixed, and independent of the data samples $X$. Then the minimax rate satisfies

$$\inf \sup \mathbb{E}_f \| \hat{\nu}(X) - f \|^2_2 \geq \frac{\varepsilon^2}{8c^2},$$

if $\varepsilon$ satisfies $\log M^\text{loc}_F (\varepsilon, c) > 2n\varepsilon^2/\alpha + 2 \log 2$.

Proof of Lemma 6. Let $c > 0$ be fixed, and $\theta \in \mathcal{F}$ be an arbitrary point. Consider maximal packing the set $\{f^1, \ldots, f^m\} \subset \mathcal{F} \cap B_2(\theta, \varepsilon)$ at a $L_2$-“distance” at least $\varepsilon/c$. Here $B_2(\theta, \varepsilon)$ denotes a closed $L_2$-ball around the point $\theta$, with radius $\varepsilon$. Suppose it has $m$ elements. Then we know that

$$I(X; J) \leq \frac{1}{m} \sum_{j=1}^m d_{\mathsf{KL}}(f_j^j \mid \mid \theta) \leq \max_{j \in [m]} d_{\mathsf{KL}}(f_j^j \mid \mid \theta) \leq \max_{j \in [m]} (1/\alpha) \| f_j^j - \theta \|^2_2 \leq \varepsilon^2/\alpha.$$

Here the first two inequalities follow by applying (5), and using the fact that $\{f^1, \ldots, f^m\} \subset \mathcal{F} \cap B_2(\theta, \varepsilon)$, respectively. Hence, if the packing number satisfies $\log m \geq 2n\varepsilon^2/\alpha + 2 \log 2$ we will have a lower bound proportional to $\varepsilon^2$ (it will be $\varepsilon^2/(8c^2)$). By taking the supremum over $\theta$, we conclude that if $\log M^\text{loc}_F (\varepsilon, c) > 2n\varepsilon^2/\alpha + 2 \log 2$ we have a lower bound proportional to $\varepsilon^2$. \hfill \square

B.3 Proof of Lemma 7

Lemma 7 (Log-likelihood difference concentration in $\mathcal{F}$). Let $\delta > 0$ be arbitrary, and let $X := (X_1, \ldots, X_n)^\top \overset{\text{i.i.d.}}{\sim} f \in \mathcal{F}$, be the $n$ observed samples. Suppose we are trying to distinguish between two densities $g, g' \in \mathcal{F}$. Let $\psi(g, g', X)$ denote their log-likelihood difference per (7). We then have

$$\sup_{g, g': \| g - g' \|_2 \geq C \delta, \| g' - f \|_2 \leq \delta} \mathbb{P}(\psi(g, g', X) > 0) \leq \exp \left( -nL(\alpha, \beta, C)\delta^2 \right) \tag{8}$$

where

$$C > 1 + \sqrt{1/(\alpha c(\alpha, \beta))} \tag{9}$$

$$L(\alpha, \beta, C) := \frac{\left(\sqrt{c(\alpha, \beta)}(C - 1) - \sqrt{1/\alpha} \right)^2}{2 \left\{ 2K(\alpha, \beta) + \frac{3}{4} \log \beta/\alpha \right\}}, \tag{10}$$

with $K(\alpha, \beta) := \beta/(\alpha^2 c(\alpha, \beta))$, and $c(\alpha, \beta)$ is as defined in Lemma 2. In the above $\mathbb{P}$ is taken with respect to the true density function $f$, i.e., $\mathbb{P} = \mathbb{P}_f$.

Proof of Lemma 7. We first observe per Remark 6 that the log-likelihood, $\psi(g, g', X)$, is well-defined. Next the mean of these variables, for each $i \in [n]$, is

$$\mathbb{E}_f \left[ \log \frac{g(X_i)}{g'(X_i)} \right] = \mathbb{E}_f \left[ \log \left( \frac{f(X_i)}{g'(X_i)} \right) / \frac{f(X_i)}{g(X_i)} \right] \quad \text{(which is well-defined by Remark 6.)}$$

$$= \mathbb{E}_f \left[ \log \frac{f(X_i)}{g'(X_i)} \right] - \mathbb{E}_f \left[ \log \frac{f(X_i)}{g(X_i)} \right]$$

$$= d_{\mathsf{KL}}(f \mid \mid g') - d_{\mathsf{KL}}(f \mid \mid g). \tag{31}$$
Where the last line follows by definition using (4). We then have

\[
\mathbb{P}(\psi(g, g', X) > 0) = \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log \frac{g(X_i)}{g'(X_i)} > 0 \right) 
\]

(using (7))

\[
= \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log \frac{g(X_i)}{g'(X_i)} - \mathbb{E}_f \log \frac{g(X_1)}{g'(X_1)} > \mathbb{E}_f \log \frac{g'(X_1)}{g(X_1)} \right) 
\]

(using (31))

\[
\leq \exp\left( - \frac{n^2 \epsilon^2}{2 \left\{ \sum_{i=1}^{n} \mathbb{E} [Y_i^2] + \frac{1}{2} n \kappa t \right\}} \right) 
\]

\[
= \exp\left( - \frac{n^2 \epsilon^2}{2 \left\{ n \mathbb{E} [Y_1^2] + \frac{1}{3} n \kappa t \right\}} \right) 
\]

(since \( Y_i \) are i.i.d.)

\[
= \exp\left( - \frac{n^2 \epsilon^2}{2 \left\{ \mathbb{E} [Y_1^2] + \frac{1}{3} \kappa t \right\}} \right) 
\]

(32)

where \( \kappa := 2 \log \beta/\alpha, t := d_{KL}(f||g) - d_{KL}(f||g'), \) and \( Y_i := \log \frac{g(X_i)}{g'(X_i)} - \mathbb{E}_f \log \frac{g(X_1)}{g'(X_1)}. \) This follows by the boundedness of \( \log \frac{g(X_i)}{g'(X_i)}, \) and then by applying Bernstein’s inequality, provided that \( t > 0. \) In order to check this final positivity condition, we first note that there exists a \( C > 0 \) such that \( \|g - g'\|_2 \geq C\delta, \) and \( \|g' - f\|_2 \leq \delta \) both hold. We then have

\[
\|f - g\|_2 \geq (C - 1)\delta. 
\]

(33)

To see this we observe that by assumption, and the triangle inequality respectively that \( C\delta \leq \|g - g'\| \leq \|g - f\| + \|f - g'\|. \) Then using \( \|f - g'\| \leq \delta \) by assumption and re-arranging, we obtain (33) as required. As a result we obtain the following two inequalities

\[
\sqrt{d_{KL}(f||g)} \geq \sqrt{c(\alpha, \beta)} \|f - g\|_2 \geq \sqrt{c(\alpha, \beta)} (C - 1)\delta \]

(34)

\[
\sqrt{d_{KL}(f||g')} \leq \sqrt{1/\alpha} \|f - g'\|_2 \leq \sqrt{1/\alpha} \delta, 
\]

(35)

where \( C > 0 \) is defined to be a constant satisfying \( c(\alpha, \beta)(C - 1)^2 > 1/\alpha, \) i.e.,

\[
C > 1 + \sqrt{1/(\alpha c(\alpha, \beta))}. 
\]

(36)

Under the condition specified by (36), and by squaring and subtracting (35) from (34), we obtain

\[
t := d_{KL}(f||g) - d_{KL}(f||g') \geq (c(\alpha, \beta)(C - 1)^2 - 1/\alpha)\delta^2 > 0 
\]

(37)
Now we show that $E_f(Y^2_1) \lesssim d_{KL}(f\|g) + d_{KL}(f\|g')$. To see this

$$E_f(Y^2_1) \leq E_f\left[\left(\log \frac{g(X_1)}{g'(X_1)}\right)^2\right]$$

$$= E_f\left[\log \left(\frac{f(X_1)}{g'(X_1)}\right)^2\right]$$

(which is well-defined by Remark 6.)

$$= E_f\left[\left(\log \frac{f(X_1)}{g(X_1)} - \log f(X_1)\right)^2\right]$$

$$\leq 2E_f\left[\left(\log \frac{f(X_1)}{g(X_1)}\right)^2\right] + 2E_f\left[\left(\log \frac{f(X_1)}{g'(X_1)}\right)^2\right]$$

$$\overset{=: A}{=} A \overset{=: B}{=} (\text{using } (a - b)^2 \leq 2(a^2 + b^2), \text{ for } a, b \geq 0.)$$

We now bound the $A$ term above, with $B$ handled similarly. We observe that:

$$A := E_f\left[\left(\log \frac{f(X_1)}{g(X_1)}\right)^2\right]$$

(by definition)

$$= \int f \left(\frac{\log f}{g}\right)^2 d\mu$$

$$= \int_{f \leq g} f \left(\frac{\log g}{f}\right)^2 d\mu + \int_{g < f} f \left(\frac{\log \frac{f}{g}}{g}\right)^2 d\mu.$$

(38)

Now using $\log x \leq x - 1$, for each $x \in \mathbb{R}_{>0}$, we have that

$$\left(\frac{\log g}{f}\right)^2 \leq \left(\frac{g - f}{f}\right)^2 \quad \text{and} \quad \left(\frac{\log \frac{f}{g}}{g}\right)^2 \leq \left(\frac{f - g}{g}\right)^2,$$

(39)

which hold for $f \leq g$ (i.e., $\frac{g}{f} \geq 1$), and $g < f$ (i.e., $\frac{f}{g} > 1$), respectively. Now we have:

$$A \leq \int_{f \leq g} \frac{(g - f)^2}{f} d\mu + \int_{g < f} \frac{(f - g)^2 f}{g^2} d\mu$$

(using (38) and (39).)

$$\leq (1/\alpha) \int_{f \leq g} (g - f)^2 d\mu + (\beta/\alpha^2) \int_{g < f} (f - g)^2 d\mu$$

(since $0 < \alpha < f, g \leq \beta$)

$$\leq (\beta/\alpha^2) \|f - g\|_2^2$$

(since $\beta/\alpha^2 \geq 1/\alpha$.)

$$\leq K(\alpha, \beta) d_{KL}(f\|g),$$

(40)

where $K(\alpha, \beta) := \beta/(\alpha^2 c(\alpha, \beta))$, where $c(\alpha, \beta)$ is as defined in Lemma 2. By a similar argument, we also have that

$$B \leq K(\alpha, \beta) d_{KL}(f\|g').$$

(41)

Let $z := d_{KL}(f\|g) + d_{KL}(f\|g')$. Then using (40) and (41), we obtain

$$E_f(Y^2_1) \leq 2K(\alpha, \beta)[d_{KL}(f\|g) + d_{KL}(f\|g')] =: 2zK(\alpha, \beta)$$

(42)

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Now we use the basic inequality $a + b \leq \left(\sqrt{a^2 + b^2}\right)^2 \leq 2(a + b)$, to obtain

$$z \leq \left(\sqrt{d_{KL}(f||g)} + \sqrt{d_{KL}(f||g')}\right)^2 \leq 2z. \quad (43)$$

Now, $t^2 := (d_{KL}(f||g) - d_{KL}(f||g'))^2 = \left(\sqrt{d_{KL}(f||g)} - \sqrt{d_{KL}(f||g')}\right)^2 \left(\sqrt{d_{KL}(f||g)} + \sqrt{d_{KL}(f||g')}\right)^2$
we have:

$$\left(\sqrt{d_{KL}(f||g)} - \sqrt{d_{KL}(f||g')}\right)^2 z \leq t^2 \leq 2 \left(\sqrt{d_{KL}(f||g)} - \sqrt{d_{KL}(f||g')}\right)^2 z. \quad (44)$$

We then conclude using (32),(37), that

$$\mathbb{P}(\psi(g,g',X) > 0) \leq \exp\left(-\frac{nt^2}{2\left\{\mathbb{E}[Y^2_1] + \frac{1}{3}\kappa t\right\}}\right) \quad \text{(per (32))}$$

$$\leq \exp\left(-\frac{n\left(\sqrt{d_{KL}(f||g)} - \sqrt{d_{KL}(f||g')}\right)^2 z}{2\left\{2zK(\alpha, \beta) + \frac{1}{3}\kappa z\right\}}\right) \quad \text{(since } t \leq z \text{ and (42))}$$

$$= \exp\left(-\frac{n\left(\sqrt{d_{KL}(f||g)} - \sqrt{d_{KL}(f||g')}\right)^2}{2\left\{2K(\alpha, \beta) + \frac{1}{3}\kappa\right\}}\right)$$

$$\leq \exp\left(-\frac{n\left(\sqrt{c(\alpha, \beta)(C-1) - \sqrt{1/\alpha}}\right)^2 \delta^2}{2\left\{2K(\alpha, \beta) + \frac{1}{3}\kappa\right\}}\right) \quad \text{(by subtracting (35) from (34))}$$

$$= \exp\left(-nL(\alpha, \beta, C)\delta^2\right),$$

whenever condition (36) holds, and $L(\alpha, \beta, C) := \frac{\left(\sqrt{c(\alpha, \beta)(C-1) - \sqrt{1/\alpha}}\right)^2}{2\left\{2K(\alpha, \beta) + \frac{1}{3}\kappa\right\}}$. Now, taking the supremum over all $g, g': ||g - g'||_2 \geq C\delta, ||g' - f||_2 \leq \delta$, the required result follows. \qed

### B.4 Proof of Lemma 8

Recall that Lemma 8 is concerning a packing set. Suppose we have a maximal packing set of $\mathcal{F}' \subset \mathcal{F}$, i.e., $\{g_1, \ldots, g_m\} \subset \mathcal{F}' \subset \mathcal{F}$ such that $||g_i - g_j||_2 > \delta$ for all $i \neq j$, and it is known that $f \in \mathcal{F}'$. We then obtain a key concentration result as per Lemma 8.

**Lemma 8** (Maximum likelihood concentration in $\mathcal{F}$). Let $\delta > 0$ be arbitrary, and let $X := (X_1, \ldots, X_n)^\top \iid f \in \mathcal{F}$, be the $n$ observed samples. Suppose further that we have a maximal $\delta$-packing set of $\mathcal{F}' \subset \mathcal{F}$, i.e., $\{g_1, \ldots, g_m\} \subset \mathcal{F}'$ such that $||g_i - g_j||_2 > \delta$ for all $i \neq j$, and it is known that $f \in \mathcal{F}'$. Now let $j^* \in [m]$, denote the index of a density whose likelihood is the largest. We then have

$$\mathbb{P}(||g_{j^*} - f||_2 > (C + 1)\delta) \leq m \exp\left(-nL(\alpha, \beta, C)\delta^2\right),$$

where $C$ is assumed to satisfy (9), and $L(\alpha, \beta, C)$ is defined as per (10).
Proof of Lemma 8. We first define the intermediate thresholding random variables

\[ T_k := \begin{cases} \max_{j \in [m]} \|g_j - g_k\|_2 & \text{, s.t. } \sum_{i=1}^n \log g_j(X_i) \geq \sum_{i=1}^n \log g_k(X_i), \|g_j - g_k\|_2 > C\delta \\ 0 & \text{, otherwise,} \end{cases} \]

for each \( k \in [m] \). Without loss of generality suppose that \( \|g_k - f\|_2 \leq \delta \). Next

\[ \mathbb{P}(\|g_j^* - f\|_2 > (C + 1)\delta) \leq \mathbb{P}(j^* \in \{ j : \|g_j - g_k\|_2 > C\delta \}) \leq \mathbb{P}(T_k > 0). \]

On the other hand

\[ \mathbb{P}(T_k > 0) = \mathbb{P}\left( \exists j \in [m] : \sum_{i=1}^n \log g_j(X_i) \geq \sum_{i=1}^n \log g_k(X_i), \|g_j - g_k\|_2 > C\delta \right) \]

\[ = \mathbb{P}\left( \bigcup_{j=1}^m \left\{ \sum_{i=1}^n \log g_j(X_i) \geq \sum_{i=1}^n \log g_k(X_i), \|g_j - g_k\|_2 > C\delta \right\} \right) \]

\[ \leq m \exp\left( -nL(\alpha, \beta, C)\delta^2 \right), \quad \text{(using union bound and Lemma 7.)} \]

where \( C \) is assumed to satisfy (9), and \( L(\alpha, \beta, C) \) is defined as per (10).

\( \square \)

B.5 Proof of Lemma 9

Lemma 9 (Monotonicity of local metric entropy). The map \( \varepsilon \mapsto \log M_F^{\text{loc}}(\varepsilon, c) \) is non-increasing.

Proof of Lemma 9. It suffices to show that if \( g_1, \ldots, g_m \in \mathcal{F} \cap B_2(\theta, \varepsilon) \) is a maximal packing set at a distance \( \varepsilon/c \), then we can pack \( B_2(\theta, \varepsilon') \cap \mathcal{F} \) at a distance \( \varepsilon'/c \) with at least \( m \) points where \( \varepsilon' < \varepsilon \). Consider the points \( \theta(1 - \varepsilon'/\varepsilon) + \varepsilon'/\varepsilon g_j \). These points clearly are densities since \( \theta, g_j \in \mathcal{F} \). We will show that these points are an \( \varepsilon'/c \) packing of \( B_2(\theta, \varepsilon') \cap \mathcal{F} \). First let us convince ourselves that the points belong to the set. We have

\[ \|\theta(1 - \varepsilon'/\varepsilon) + \varepsilon'/\varepsilon g_j - \theta\|_2 = \varepsilon'/\varepsilon \|g_j - \theta\|_2 \leq \varepsilon', \]

and using the fact that \( \mathcal{F} \) is convex (by assumption) grants the conclusion. Next

\[ \|\theta(1 - \varepsilon'/\varepsilon) + \varepsilon'/\varepsilon g_j - \theta(1 - \varepsilon'/\varepsilon) - \varepsilon'/\varepsilon g_k\|_2 = \varepsilon'/\varepsilon \|g_j - g_k\|_2 > \varepsilon'/c, \]

which completes the proof.

\( \square \)

B.6 Proof of Proposition 10

Proposition 10 (Measurability of \( \nu^*(X) \)). The multistage sieve MLE, i.e., \( \nu^*(X) \), is a measurable function of the data with respect to the Borel \( \sigma \)-field on \( \mathcal{F} \) in the \( L_2 \)-metric topology.

Proof of Proposition 10. Recall our multistage sieve estimator \( \nu^*(X) := \Upsilon_T \), where \( X := (X_1, \ldots, X_n)^\top \) is a fixed data sample. Here \( \Upsilon_T \) denotes the last term of the finite sequence \( \Upsilon := (\Upsilon_k)_{k=1}^T \) as described in Section 2.2.
In order to show the measurability of \( \nu^*(\mathbf{X}) \) we need to formalize our setting. We note that our estimator \( \nu^*: B^n \to \mathcal{F} \), is more precisely a map from the measurable space \((B^n, \sigma(B^n))\) to the measurable space \((\mathcal{F}, \sigma(\mathcal{F}))\). Here \( \sigma(B^n) \) and \( \sigma(\mathcal{F}) \) denote the Borel \( \sigma \)-field with respect to the Euclidean and \( L_2 \)-metric topologies on \( B^n \) and \( \mathcal{F} \), respectively.

Our proof strategy will be to proceed by induction on \( k \in [J] \) over the sequence \( \Upsilon \). We will show that each \( k \)-th indexed map in \( \Upsilon \), i.e., \( \Upsilon_k \), is Borel measurable, which in turn will imply the measureability of the \( \nu^*(\mathbf{X}) \). Following our (maximal) packing set construction as described in Section 2.2 and Figure 2, we need to consider the case where the traversal down the tree is not necessarily unique at each level, i.e., there may be collisions (ties) in the packing set children nodes, where the likelihood is equal. We do always ensure a unique path down the maximal packing set tree, by selecting the smallest alphanumerically indexed children node at each level. However, our measurability proof must account for this selection rule explicitly.

In order to proceed by induction, we consider the base case for \( k = 1 \), i.e., \( \Upsilon_1 \in \mathcal{F} \). Importantly, we note that \( \Upsilon_1 \) is chosen arbitrarily from \( \mathcal{F} \) independently of the data samples, \( \mathbf{X} \). Let \( A \in \sigma(\mathcal{F}) \) be any Borel set. Since all samples \( \mathbf{X} \in B^n \) are mapped to \( \Upsilon_1 \) in our setting, then \( \Upsilon_1^{-1}[A] = B^n \) if \( \Upsilon_1 \in A \in \sigma(\mathcal{F}) \), or \( \Upsilon_1^{-1}[A] = \emptyset \), otherwise. In either case we have \( \emptyset, B^n \in \sigma(B^n) \), which shows that \( \Upsilon_1 \) is Borel measurable. Now consider the event \( \{ \Upsilon_2 = m_s \} := \{(X_1, \ldots, X_n)^\top \in B^n \mid \Upsilon_2(X_1, \ldots, X_n) = m_s \} \subset B^n \), for some index \( s \in \mathbb{N} \). Then we have

\[
\{ \Upsilon_2 = m_s \} := \left\{ (X_1, \ldots, X_n)^\top \in B^n \mid \Upsilon_2(X_1, \ldots, X_n) = m_s \right\} = \bigcap_{g \in P_{\Upsilon_1}} \left\{ (X_1, \ldots, X_n)^\top \in B^n \left| \sum_{i=1}^n \log(m_s(X_i)) \geq \sum_{i=1}^n \log(g(X_i)) \right. \right\} \bigcap_{s=1}^{s-1} \left\{ (X_1, \ldots, X_n)^\top \in B^n \left| \sum_{i=1}^n \log(m_s(X_i)) > \sum_{i=1}^n \log(m_j(X_i)) \right. \right\}.
\]

In (46), we observe that \( \{ \Upsilon_2 = m_s \} \subset B^n \) is represented as the intersection of 2 separate (finite) set intersections. Note that the second intersection set explicitly accounts for our alphanumerical index selection rule in the children densities of \( P_{\Upsilon_1} \). Consider the first finite intersection term. Here, each \( g \in P_{\Upsilon_1} \subset \mathcal{F} \) are Borel measurable by (1). We note that the log and the addition (i.e., \( \log + \sum \)) functions are both continuous and measurable, and therefore, so is their composition. Thus the resulting finite sum, \( \sum_{i=1}^n \log f(X_i) \), is a measurable function, for any density \( f \in \mathcal{F} \) (which is always measurable). As such the \( \Upsilon_2 \) is measurable since all these inequalities give rise to measurable sets and when one intersects them (they are finitely many) one obtains another measurable set. Once again, let \( A \in \sigma(\mathcal{F}) \) be any Borel set. Then such an \( A \) contains either no such densities \( m_s \), or at most finitely many (since the number of children of our maximal packing set tree is always finite). If no such \( m_s \in A \), then \( \Upsilon_2^{-1}[A] = \emptyset \in \sigma(B^n) \). Thus \( \Upsilon_2 \) is indeed Borel measurable in this case. In the case where there exist finitely many such \( m_s \in A \), it follows that

\[
\Upsilon_2^{-1}[A] = \bigcup_{\{s \mid m_s \in A\}} \{ \Upsilon_2 = m_s \} := \bigcup_{\{s \mid m_s \in A\}} \left\{ (X_1, \ldots, X_n)^\top \in B^n \mid \Upsilon_2(X_1, \ldots, X_n) = m_s \right\}.
\]

In (47) we note that \( \Upsilon_2^{-1}[A] \) represents a finite union of Borel measurable sets as per (46), which is again Borel measurable. That is, we have shown that \( \Upsilon_2^{-1}[A] \in \sigma(B^n) \), which indeed implies the Borel measurability of \( \Upsilon_2 \), as required.
Similarly, consider the event \( \{ \gamma_3 = m_{s,t} \} \subset B^n \), for some \( t \in \mathbb{N} \) and \( s \in \mathbb{N} \) taken as per (45). Here, the indexed density \( m_{s,t} \) signifies that \( \gamma_3 \) is derived from the children of the packing set of \( \gamma_2 = m_s \), as denoted by \( P_{m_s} \) in our work. Once again we can write this \( \gamma_3 \) as

\[
\{ \gamma_3 = m_{s,t} \} := \left\{ (X_1, \ldots, X_n)^\top \in B^n \mid \gamma_3(X_1, \ldots, X_n) = m_{s,t} \right\}
\]

\[
= \bigcap_{g \in P_{m_s}} \left\{ (X_1, \ldots, X_n)^\top \in B^n \left| \sum_{i=1}^{n} \log(m_{s,t}(X_i)) \geq \sum_{i=1}^{n} \log(g(X_i)) \right. \right\} \bigcap_{t=1}^{n-1} \left\{ (X_1, \ldots, X_n)^\top \in B^n \left| \sum_{i=1}^{n} \log(m_{s,t}(X_i)) > \sum_{i=1}^{n} \log(m_{s,j}(X_i)) \right. \right\} \bigcap \{ \gamma_2 = m_s \}.
\] (48)

By a similar argument to the measurability of \( \gamma_2 \) it follows that \( \gamma_3 \) is also measurable. As such, given the recursive construction of the finite sequence \( \gamma := (\gamma_k)_{k=1}^{J} \) via our maximal packing set tree traversal, this pattern inductively repeats for each \( k \in \{4, \ldots, J\} \). Since \( \nu^*(X) := \gamma_J \), this implies the measurability of \( \nu^*(X) \), as required.

**Remark 10.** We note that the arguments in the proof above hold, even if the cardinality of the set of children densities at any iteration were at most countable (not just finite). That is, (46) would still return a measurable set even if \( |P_{\gamma_1}| = \infty \), since Borel measurability is preserved over countable intersections and unions. The packing sets in our construction are necessarily at most countable, since all of the subsets of \( \mathcal{F} \) we consider are separable in the \( L_2 \)-metric (i.e., contain a countably dense subset). This follows from Lemma 25.

### B.7 Proof of Theorem 11

We begin with a useful result, which will enable us to construct upper bounds for estimator \( \nu^*(X) \).

**Lemma 28.** The finite sequence \( \gamma := (\gamma_k)_{k=1}^{J} \), as defined in the construction of our estimator \( \nu^*(X) \), satisfies

\[
\|\gamma_J - \gamma_{J'}\|_2 \leq \frac{d}{2^{J'-2}},
\] (49)

for each pair of positive integers \( J' < J \).

**Proof of Lemma 28.** Let \( \gamma_{J'}, \gamma_J \in \gamma \), for any positive integers \( J > J' \geq 1 \). We then have

\[
\|\gamma_J - \gamma_{J'}\|_2 \leq \sum_{i=J'}^{J-1} \|\gamma_{i+1} - \gamma_i\|_2 \leq \sum_{i=1}^{J-1} \frac{d}{2^{i-1}} \leq \frac{d}{2^{J'-2}}.
\] (50)

As required.

**Lemma 29** (Telescoping sum of conditional probabilities). Let \( n \geq 2 \) be a fixed integer, and \( \{A_1, A_2, \ldots, A_n\} \) denote events on a common probability space, with \( \mathbb{P}(A_j^c) > 0 \) for each \( j \geq 1 \). We then have

\[
\mathbb{P}(A_n) \leq \sum_{j=n}^{2} \mathbb{P}(A_j \mid A_{j-1}^c) + \mathbb{P}(A_1).
\] (51)
Proof of Lemma 29. We will prove this by induction on \( n \geq 2 \). We check the induction base case for \( n = 2 \). We first observe that

\[
A_2 \subseteq A_1 \cup A_2 = (A_2 \cap A_1^c) \cup A_1,
\]

where the latter set is a *disjoint* union. It then follows that

\[
\mathbb{P}(A_2) \leq \mathbb{P}(A_2 \cap A_1^c) + \mathbb{P}(A_1) \quad \text{(by monotonicity of } \mathbb{P} \text{ applied to (52))}
\]

\[
\leq \frac{\mathbb{P}(A_2 \cap A_1^c)}{\mathbb{P}(A_1^c)} + \mathbb{P}(A_1) \quad \text{(since } \mathbb{P}(A_1^c) \in (0, 1], \text{ by assumption)}
\]

\[
=: \mathbb{P}(A_2 \mid A_1^c) + \mathbb{P}(A_1),
\]

which proves the base case for \( n = 2 \). Now, by induction assume the result is true for each integer \( n = k > 2 \). We then have for \( n = k + 1 \) that:

\[
\mathbb{P}(A_{k+1}) \leq \mathbb{P}(A_{k+1} \mid A_k^c) + \mathbb{P}(A_k) \quad \text{(using induction base case)}
\]

\[
\leq \mathbb{P}(A_{k+1} \mid A_k^c) + \sum_{j=k}^{2} \mathbb{P}(A_j \mid A_{j-1}^c) + \mathbb{P}(A_1) \quad \text{(using induction hypothesis)}
\]

\[
= \sum_{j=k+1}^{2} \mathbb{P}(A_j \mid A_{j-1}^c) + \mathbb{P}(A_1),
\]

as required. So the result is true for \( n = k + 1 \), and thus by induction holds for each integer \( n \geq 2 \). \( \square \)

Theorem 11 (Upper bound rate for the multistage sieve MLE \( \nu^*(X) \)). Let, \( \nu^*(X) = Y_J \) be the output of the multistage sieve MLE which is run for \( J \in \mathbb{N} \) steps. Here \( J \) is defined as the maximal integer \( J \in \mathbb{N} \), such that \( \varepsilon_J := \sqrt{L(\alpha, \beta, c/2 - 1)} \) satisfies

\[
\frac{n}{\varepsilon_J^2} > 2 \log M_{\mathcal{F}}(\varepsilon_J c) \left( \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)}}, \varepsilon_J \right) \lor \log 2,
\]

or \( J = 1 \) if no such \( J \) exists. Then

\[
\mathbb{E}||\nu^*(X) - f||_2^2 \leq \bar{C}\varepsilon^2,
\]

for some universal constant \( \bar{C} \), and where \( \varepsilon^* := \varepsilon_J \). We remind the reader that \( c := 2(C + 1) \) is the constant from the definition of local metric entropy, which is assumed to be sufficiently large. Here \( C \) is assumed to satisfy (9), and \( L(\alpha, \beta, C) \) is defined as per (10).

Proof of Theorem 11. Combining the results of Lemma 8 (with \( c := 2(C + 1) \) where \( c \) is the constant from the definition of local packing entropy) and Lemma 9 we conclude that for each \( j \in \{2, \ldots, J\} \)

\[\text{Observe that by the definition of } \varepsilon_J \text{ and (11) we have that all packing sets used in the construction of the estimator must be finite, even though we are not assuming that the set } \mathcal{F} \text{ is totally bounded.}\]
we have

\[
P\left(\|f - \Upsilon_j\|_2 > \frac{d}{2^{j-1}} \bigg| \|f - \Upsilon_{j-1}\|_2 \leq \frac{d}{2^{j-2}} , \Upsilon_{j-1}\right) \\
\leq |P_{\Upsilon_{j-1}}| \exp\left(\frac{-nL(\alpha, \beta, C)d^2}{2^{2(j-1)}(C + 1)^2}\right) \\
\leq M_{f,loc}^{\text{loc}}\left(\frac{d}{2^{j-2}}, c\right) \exp\left(\frac{-nL(\alpha, \beta, C)d^2}{2^{2(j-1)}(C + 1)^2}\right) 
\]  

(54)

(55)

where \(P_{\Upsilon_j}\) are the maximal packing sets described in the construction of \(\nu^*(X)\). Crucially, we observe that the RHS of (55) does not depend on the conditioned random variables, i.e., \(\Upsilon_{j-1}\), for each \(j \in \{2, \ldots, J\}\) hence we can drop \(\Upsilon_{j-1}\) from the conditioning. Now let denote \(A_j := \{\|f - \Upsilon_j\|_2 > \frac{d}{2^{j-2}}\}\), for each integer \(j \geq 1\). Then we can proceed by working with the unconditional events \(A_j\) in (55).

Moreover, we then have that \(A_{j-1}^c := \{\|f - \Upsilon_{j-1}\|_2 \leq \frac{d}{2^{j-2}}\}\) for each integer \(j \geq 2\). In particular \(P(A_{j-1}^c) = \{\|f - \Upsilon_1\|_2 \leq d\} = 1\), since \(f, \Upsilon_1 \in F\), so indeed \(\|f - \Upsilon_1\|_2 \leq \text{diam}_2(F) = d\) almost surely. By aligning our notation directly with Lemma 29, we can apply the telescoping bound to \(P(A_j)\) as follows

\[
P(A_j) := P\left(\|f - \Upsilon_j\|_2 > \frac{d}{2^{J-1}}\right) \\
\leq M_{f,loc}^{\text{loc}}\left(\frac{d}{2^{J-2}}, c\right) \sum_{j=1}^{J-1} \exp\left(\frac{-nL(\alpha, \beta, C)d^2}{2^{2(j-1)}(C + 1)^2}\right) \\
\leq M_{f,loc}^{\text{loc}}\left(\frac{d}{2^{J-2}}, c\right) a(1 + a^{4-1} + a^{16-1} + \ldots)1(J > 1) \\
\leq M_{f,loc}^{\text{loc}}\left(\frac{d}{2^{J-2}}, c\right) a(1 + a + a^2 + \ldots)1(J > 1) \\
\leq M_{f,loc}^{\text{loc}}\left(\frac{d}{2^{J-2}}, c\right) \frac{a}{1 - a} 1(J > 1), 
\]  

(56)

(57)

where for brevity in (56) we denote

\[
a := \exp\left(\frac{-nL(\alpha, \beta, C)d^2}{2^{2(J-1)}(C + 1)^2}\right). 
\]

Since \(C\) is assumed to satisfy (9), and \(L(\alpha, \beta, C)\) is defined as per (10), it follows that \(a < 1\). Note here that the above bound (57) holds, provided that \(P(A_j^c) > 0\) for \(j < J\) as required by Lemma 29. Suppose that the RHS of (57) is strictly smaller than 1. In that case for all \(j\), \(P(A_j^c) > 0\) since bound (57) holds inductively for all \(P(A_j)\) for \(j \leq J\). On the other hand, if the RHS of (57) is \(\geq 1\) then (57) trivially holds. In both cases we conclude that (57) holds.

If one sets \(\varepsilon_j := \frac{\sqrt{L(\alpha, \beta, C)}d}{2^{2(j-1)}(C + 1)}\), we have that if

\[
n\varepsilon_j^2 > 2 \log M_{f,loc}^{\text{loc}}\left(\varepsilon_j \frac{2(C + 1)}{\sqrt{L(\alpha, \beta, C)}}, c\right) = 2 \log M_{f,loc}^{\text{loc}}\left(\frac{d}{2^{J-2}}, c\right),
\]

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and $a := \exp(-n\varepsilon_J^2) < 1/2 \iff n\varepsilon_J^2 > \log 2$, the above probability in (57) will be bounded from above by $2\exp(-n\varepsilon_J^2/2)$. This condition is implied when
\[
 n\varepsilon_J^2 > 2\log M_{\mathcal{F}}^{\text{loc}} \left( \varepsilon_J \frac{2(C+1)}{\sqrt{L(\alpha, \beta, C)}}, c \right) \vee \log 2.
\] (58)
We now have
\[
\|\nu_J^* - f\|_2 \leq \|Y_J - Y_J\|_2 + \|Y_J - f\|_2 \leq 3\varepsilon_J \frac{C+1}{\sqrt{L(\alpha, \beta, C)}},
\] (59)
with probability at least $1 - 2\exp(-n\varepsilon_J^2/2)$ which holds for all $J$ satisfying (58) (including $J$). Here we want to clarify that the last inequality in (59) follows from the fact that $\|Y_J - Y_J\|_2 \leq d/2^{J-2}$, as seen when we verified that $Y$ forms a Cauchy sequence in Lemma 28 (and since $J \geq J$). Let $J^*$ be selected as the maximum integer $J$ such that (58) holds, or otherwise if such $J$ does not exist $J^* = 1$, i.e. $J^* \equiv J$. Let $\eta = \frac{3C+1}{\sqrt{L(\alpha, \beta, C)}}$, $C = 2$ and $C' = 1/2$. We have established that the following bound holds
\[
\mathbb{P}(\|f - \nu_J^*\|_2 > \varepsilon_j) \leq C \exp(-C'n\varepsilon_J^2)1(J > 1) \leq C \exp(-C'n\varepsilon_J^2)1(J^* > 1),
\] for all $1 \leq J \leq J^*$, where this bound also holds in the case when $J^* = 1$ by exception. Observe that we can extend this bound to all $J \in \mathbb{Z}$ and $J \leq J^*$, since for $J < 1$ we have $\eta \varepsilon_J \geq 6d$ and so
\[
\mathbb{P}(\|f - \nu_J^*\|_2 > \varepsilon_j) \leq 0 \leq C \exp(-C'n\varepsilon_J^2)1(J^* > 1).
\] We conclude that
\[
\mathbb{P}(\|f - \nu_J^*\|_2 > \varepsilon_j) \leq 0 \leq C \exp(-C'n\varepsilon_J^2)1(J^* > 1),
\] for any $J \leq J^*$. Now for any $\varepsilon_{J-1} > x \geq \varepsilon_J$ for $J \leq J^*$ we have that
\[
\mathbb{P}(\|f - \nu_J^*\|_2 > 2\eta x) \leq \mathbb{P}(\|f - \nu_J^*\|_2 > \varepsilon_{J-1})
\leq C \exp(-C'n\varepsilon_J^2)1(J^* > 1)
\leq C \exp(-C'n\varepsilon_J^2)1(J^* > 1),
\]
where the last inequality follows due to the fact that the map $x \mapsto C \exp(-C'n\varepsilon_J^2)$ is monotonically decreasing for positive reals. We will now integrate the tail bound:
\[
\mathbb{P}(\|f - \nu_J^*\|_2 > 2\eta x) \leq C \exp(-C'n\varepsilon_J^2)1(J^* > 1),
\] (60)
which holds true for $x \geq \varepsilon^* := \varepsilon_{J^*}$, where $\varepsilon_J = \frac{\sqrt{L(\alpha, \beta, C)}d}{2^{J-1}(C+1)}$, always (since even if $J^* = 1$ by exception, this bound is still valid). We then have
\[
\mathbb{E}\|f - \nu_J^*\|_2^2 = \int_0^{\infty} 2x\mathbb{P}(\|f - \nu_J^*\|_2 > x) \, dx
\leq C'' \varepsilon^2 + \int_{2\eta \varepsilon^*}^{\infty} 2x C \exp(-C''nx^2)1(J^* > 1) \, dx
= C'' \varepsilon^2 + C'' n^{-1} \exp(-C'''n\varepsilon^2 x^2)1(J^* > 1).
\]
Now $n\varepsilon^2$ is bigger than a constant (i.e., $\log 2$) otherwise $J^* = 1$. Hence, the above is smaller than $C \varepsilon^2$ for some absolute constant $C$. 

\[ \square \]

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B.8 Proof of Theorem 12

**Theorem 12** (Minimax rate). Define \( \varepsilon^* := \sup \{ \varepsilon : n\varepsilon^2 \leq \log M_F^{\text{loc}}(\varepsilon, c) \} \), where \( c \) in the definition of local metric entropy is a sufficiently large absolute constant. Then the minimax rate is given by \( \varepsilon^{*2} \land d^2 \) up to absolute constant factors.

**Proof of Theorem 12.** First suppose that \( \varepsilon^* \) satisfies \( n\varepsilon^{*2} > 4 \log 2 \). Then for \( \delta^* := \varepsilon^*/\sqrt{4(1/\alpha \lor 1)} \) we have \( \log M_F^{\text{loc}}(\delta^*, c) \geq \log M_F^{\text{loc}}(\varepsilon^*, c) \geq n\varepsilon^{*2}/2 + n\varepsilon^{*2}/2 > 2n\delta^{*2}/\alpha + 2 \log 2 \) and so this implies the sufficient condition for the lower bound per Lemma 6. Let \( \eta := \frac{\sqrt{\varepsilon}}{\sqrt{L(\alpha, \beta, c/2 - 1)}} \land 1 \). For a constant \( C \) such that \( C\eta > 1 \), we have

\[
C^2 n\varepsilon^{*2} \geq 1/\eta^2 \log M_F^{\text{loc}}(C\eta \varepsilon^*, c) \geq \log M_F^{\text{loc}}(C\eta \varepsilon^*, c) \geq \log M_F^{\text{loc}}(C\eta \varepsilon^*, c) \geq \log M_F^{\text{loc}}(C\eta \varepsilon^*, c)
\]

Setting \( \delta := C\sqrt{\varepsilon}^{*} \) we obtain that

\[
n\delta^2 \geq 2 \log M_F^{\text{loc}}(\delta^{*} \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)}}, c).
\]

In addition since \( C > 1 \), \( \delta \) satisfies (11) (taking into account that \( n\varepsilon^{*2} > 4 \log 2 \), which implies \( n\delta^2 \geq 4 \log 2C^2 > \log 2 \)). We note that the map \( 0 < x \mapsto nx^2 - \log M_F^{\text{loc}}(x - \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)}}, c) \) is non-decreasing by Lemma 9. Now, with \( \varepsilon_J^* \) defined as per Theorem 11, this implies that \( \delta \geq \varepsilon_J^*/2 \). This shows that the rate in this case is of the order \( \varepsilon^{*2} \).

Next, suppose that \( \varepsilon^* \) defined by \( \sup \{ \varepsilon : n\varepsilon^2 \leq \log M_F^{\text{loc}}(\varepsilon, c) \} \) satisfies \( n\varepsilon^{*2} \leq 4 \log 2 \). For \( 2\varepsilon^* \), we have \( 16 \log 2 \geq 4\varepsilon^{*2}n \geq \log M_F^{\text{loc}}(2\varepsilon^*, c) \). If \( c \) in the definition of local packing is large enough, we could put points in the diameter of the ball with radius \( 2\varepsilon^* \) such that the packing set has more than \( \exp(16 \log 2) \) points. But that implies that the set \( \mathcal{F} \) is entirely inside a ball of radius \( \sqrt{16 \log 2}n^{-1/2} \) (as \( \varepsilon^{*2} \leq (4 \log 2)n^{-1} \)). To see the latter, one can take the midpoint of the line segment connecting the endpoints of a diameter of \( \mathcal{F} \) and position a ball of radius \( 2\varepsilon^* \) there. In such a case, for the lower bound, we could pick \( \varepsilon \) to be proportional to the diameter of the set (with a small proportionality constant). That will ensure that \( \varepsilon\sqrt{n} \) is upper bounded by some constant (as \( 2\sqrt{16 \log 2}n^{-1/2} \) is bigger than the diameter), and at the same time \( \log M_F^{\text{loc}}(\varepsilon, c) \) can be made bigger than a constant (provided that \( c \) in the definition of a local packing is large enough) – by taking \( \theta \) (where \( \theta \) is the center of the localized set \( B_2(\theta, \varepsilon) \cap \mathcal{F} \)) to be the midpoint of a diameter of the set \( \mathcal{F} \) and then placing equispaced points on the diameter. Hence, the diameter of the set is a lower bound (up to constant factors) in this case, which is of course always an upper bound too (up to constant factors). So we conclude that either for \( \varepsilon^* \) defined by \( \sup \{ \varepsilon : n\varepsilon^2 \leq \log M_F^{\text{loc}}(\varepsilon, c) \} \) satisfies \( n\varepsilon^2 \geq 4 \log 2 \) or the lower and upper bounds are of order the diameter of the set. In summary the rate is given by \( \varepsilon^{*2} \land d^2 \). This is true since in the second case, \( 4\varepsilon^* \) is bigger than the diameter of the set. \( \square \)

B.9 Proof of Proposition 13

**Proposition 13** (Extending results to \( \mathcal{F}_B^{[0, \beta]} \)). Let \( \mathcal{F} \subset \mathcal{F}_B^{[0, \beta]} \) be a convex class of densities, with at least one \( f_\alpha \in \mathcal{F} \) that is \( \alpha \)-lower bounded, with \( \alpha > 0 \). Then the minimax rate in the squared \( L_2 \)-metric is \( \varepsilon^{*2} \land d^2 \), where \( \varepsilon^* := \sup \{ \varepsilon : n\varepsilon^2 \leq \log M_F^{\text{loc}}(\varepsilon, c) \} \).
Proof of Proposition 13. We argue this as follows. Let $f_\alpha \in \mathcal{F}$, which is lower bounded by some $\alpha > 0$. Now consider the following set of $f_\alpha$-mixture densities, i.e., $\mathcal{F}' = \{(1/2)f_\alpha + (1/2)f : f \in \mathcal{F}\} \subset \mathcal{F}$. By construction, all densities in $\mathcal{F}'$ are thus lower bounded by $\alpha/2$, i.e., $\mathcal{F}' \subset \mathcal{F}_{B}^{[\alpha/2, \beta]}$. Moreover, $\mathcal{F}'$ forms a convex density class. Hence, the minimax rate would be given by $\varepsilon^2 \wedge \text{diam}_2(\mathcal{F}')^2$ where $\varepsilon = \sup\{\varepsilon : n\varepsilon^2 \leq \log M_{\mathcal{F}}^{\text{loc}}(\varepsilon, c)\}$. We can artificially create variables from the class $\mathcal{F}'$ by randomizing $X_i$ as follows

$$Z_i = \begin{cases} T_i \overset{\text{i.i.d.}}{\sim} f_\alpha & \text{with probability 1/2,} \\ X_i & \text{with probability 1/2.} \end{cases}$$

Then let $\hat{f}$ be our estimator of $(1/2)f_\alpha + (1/2)f$. We know:

$$\mathbb{E}_{Z} \|\hat{f} - ((1/2)f_\alpha + (1/2)f)\|^2 \lesssim \varepsilon^2 \wedge \text{diam}(\mathcal{F}')^2,$$

so that

$$\mathbb{E}_{X,T,V} \|\hat{f} - f_\alpha - f\|^2 \lesssim 4\varepsilon^2 \wedge \text{diam}(\mathcal{F}')^2,$$

where $T = (T_1, \ldots, T_n)$ and $V = (V_1, \ldots, V_n)$ are the values of the coin flips in the definition of $Z_i$. Hence, $\mathbb{E}_{T,V} 2\hat{f} - f_\alpha$ achieves the same rate for $f$ since by Jensen’s inequality

$$\mathbb{E}_{Y} \|\mathbb{E}_{T,V}(2\hat{f} - f_\alpha) - f\|^2 \leq \mathbb{E}_{Y,T,V} \|2\hat{f} - f_\alpha - f\|^2 \lesssim 4\varepsilon^2 \wedge \text{diam}(\mathcal{F}')^2.$$

Moreover, note that since $\hat{f} \in \mathcal{F}'$ for each of value of $T,V$ we have $\mathbb{E}_{T,V} 2\hat{f} - f_\alpha \in \mathcal{F}$. Thus, the upper bound is the same for the two sets. On the other hand since $\mathcal{F}' \subset \mathcal{F}$ the lower bound is also of the same rate. Finally, observe that $\log M_{\mathcal{F}}^{\text{loc}}(\varepsilon, c) = \log M_{\mathcal{F}}^{\text{loc}}(2\varepsilon, c)$ so that the order of $\varepsilon^* = \sup\{\varepsilon : n\varepsilon^2 \leq \log M_{\mathcal{F}}^{\text{loc}}(\varepsilon, c)\}$ is the same as that of the equation $\varepsilon^* = \sup\{\varepsilon : n\varepsilon^2 \leq \log M_{\mathcal{F}}^{\text{loc}}(\varepsilon, c)\}$. In addition, it is also clear that $2\text{diam}_2(\mathcal{F}') = \text{diam}_2(\mathcal{F})$.

\[ \Box \]

B.10 Proof of Theorem 15

We first prove the following simple lemma.

Lemma 30. Suppose $\nu, \mu \in \mathcal{F}$ are two densities such that $\|\nu - \nu\|_2 \leq \delta$. If $\delta \leq \varepsilon$ then have $M(\nu, \varepsilon, c) \leq M(\mu, 2\varepsilon, 2c)$.

Proof of Lemma 30. It suffices to show that $B(\nu, \varepsilon) \subseteq B(\mu, 2\varepsilon)$. For any $x \in B(\nu, \varepsilon)$ we have $\|x - \nu\|_2 \leq \varepsilon$, and hence by the triangle inequality we obtain

$$\|x - \mu\|_2 \leq \|x - \nu\|_2 + \|\nu - \mu\|_2 \leq \varepsilon + \delta \leq 2\varepsilon,$$

which completes the proof. \[ \Box \]

Theorem 15 (Adaptive upper bound rate for the multistage sieve MLE $\nu^*(X)$). Let, $\nu^*(X) = \nu^J$ be the output of the multistage sieve MLE which is run for $J$ iterations where $J$ is defined as the maximal solution to

$$n\varepsilon^2 \geq \inf_{f \in F} M_{\mathcal{F}}^{\text{dloc}} \left( f, 2\varepsilon J, \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)}}, 2c \right) \vee \log 2,$$
where \( \varepsilon_J := \frac{\sqrt{L(\alpha, \beta, c/2 - 1)} d}{2(\beta - 2)c} \) and \( J = 1 \) if no such \( J \) exists. Let \( J^* \) be defined as the maximal integer \( J \in \mathbb{N} \), such that \( \varepsilon_J := \frac{\sqrt{L(\alpha, \beta, c/2 - 1)} d}{2(\beta - 2)c} \) such that\(^{11}\),

\[
n \varepsilon_J^2 > 2M_{\mathcal{F}}^{\text{dloc}} \left( f, 2\varepsilon_J \frac{c}{\sqrt{L(\alpha, \beta, c/2 - 1)}} , 2c \right) \vee \log 2, \tag{12}\]

and \( J^* = 1 \) if no such \( J \) exists. Then

\[
\mathbb{E}\|\nu^*(X) - f\|_2^2 \leq C\varepsilon^*^2,
\]

for some universal constant \( C \), and where \( \varepsilon^* := \varepsilon_{J^*} \). We remind the reader that \( c := 2(C + 1) \) is the constant from the definition of local metric entropy, which is assumed to be sufficiently large. Here \( C \) is assumed to satisfy (9), and \( L(\alpha, \beta, C) \) is defined as per (10).

**Proof of Theorem 15.** Combining the results of Lemma 8 (with \( c := 2(C + 1) \) where \( c \) is the constant from the definition of local packing entropy) and Lemma 9 we conclude that for each \( j \in \{2, \ldots, J\} \) we have

\[
\mathbb{P}\left( \|f - \Upsilon_j\|_2 > \frac{d}{2^{j-1}} \left\| f - \Upsilon_{j-1}\|_2 \leq \frac{d}{2^{j-2}} \right\|_{\Upsilon_{j-1}} \right)
\leq |P_{\Upsilon_{j-1}}| \exp \left( -\frac{nL(\alpha, \beta, C)d^2}{2^{2(j-1)}(C + 1)^2} \right) \tag{61}
\leq M(f, \frac{d}{2^{j-3}}, 2c) \exp \left( -\frac{nL(\alpha, \beta, C)d^2}{2^{2(j-1)}(C + 1)^2} \right) \tag{62}
\leq M(f, \frac{d}{2^{j-3}}, 2c) \exp \left( -\frac{nL(\alpha, \beta, C)d^2}{2^{2(j-1)}(C + 1)^2} \right) \tag{63}
\]

where \( P_{\Upsilon_j} \) are the maximal packing sets described in the construction of \( \nu^*(X) \). Furthermore, inequality (62) follows from Lemma 30. Crucially, we observe that the RHS of (63) does not depend on the conditioned random variables, i.e., \( \Upsilon_{j-1} \), for each \( j \in \{2, \ldots, J\} \) hence we can drop \( \Upsilon_{j-1} \) from the conditioning. Now let denote \( A_j := \{\|f - \Upsilon_j\|_2 > \frac{d}{2^{j-1}}\} \) for each integer \( j \geq 1 \). Then we can proceed by working with the unconditional events \( A_j \) in (55).

Moreover, we then have that \( A^c_{j-1} := \{\|f - \Upsilon_{j-1}\|_2 \leq \frac{d}{2^{j-2}}\} \) for each integer \( j \geq 2 \). In particular \( \mathbb{P}(A_1^c) \ldots \{\|f - \Upsilon_1\|_2 \leq d\} = 1 \), since \( f, \Upsilon_1 \in \mathcal{F} \), so indeed \( \|f - \Upsilon_1\|_2 \leq \text{diam}_2(\mathcal{F}) =: d \) almost surely. By aligning our notation directly with Lemma 29, we can apply the telescoping bound to

\(^{11}\)Note that running the estimator with \( J \) many steps, may result into having non-finite packing sets — that is not an issue however.

\(^{12}\)Observe that by the definition of \( \varepsilon_J \) and (11) we have that some packing sets used in the construction of the adaptive estimator may not be finite, but will be at most countable. This follows from the \( L_2 \)-separability of \( \mathcal{F} \) and is formalized in Lemma 25 in the appendix. We note that the measurability of the adaptive estimator still holds as per Proposition 10 in this case.
\[ \mathbb{P}(A_j) \text{ as follows} \]

\[
\mathbb{P}(A_j) := \mathbb{P}\left( \|f - \Upsilon_J\|_2 > \frac{d}{2^{j-1}} \right) \quad \text{(by definition)}
\]

\[
\leq M_{\mathcal{F}}^{\text{adloc}} \left( f, \frac{d}{2^{j-\frac{3}{2}}}, 2c \right) \sum_{j=1}^{J-1} \exp \left( -\frac{nL(\alpha, \beta, C)d^2}{2^{2j}(C+1)^2} \right) \quad \text{(per (55))}
\]

\[
\leq M_{\mathcal{F}}^{\text{adloc}} \left( f, \frac{d}{2^{j-\frac{3}{2}}}, 2c \right) a(1 + a^{4-1} + a^{16-1} + \ldots) \mathbb{1}(J > 1)
\]

\[
\leq M_{\mathcal{F}}^{\text{adloc}} \left( f, \frac{d}{2^{j-\frac{3}{2}}}, 2c \right) a(1 + a + a^2 + \ldots) \mathbb{1}(J > 1)
\]

\[
\leq M_{\mathcal{F}}^{\text{adloc}} \left( f, \frac{d}{2^{j-\frac{3}{2}}}, 2c \right) \frac{a}{1-a} \mathbb{1}(J > 1),
\]

where for brevity in (64) we denote

\[
a := \exp \left( -\frac{nL(\alpha, \beta, C)d^2}{2^{2j-1}(C+1)^2} \right).
\]

Since \( C \) is assumed to satisfy (9), and \( L(\alpha, \beta, C) \) is defined as per (10), it follows that \( a < 1 \). Note here that the above bound (65) holds, provided that \( \mathbb{P}(A^*_j) > 0 \) for \( j < J \) as required by Lemma 29. Suppose that the RHS of (65) is strictly smaller than 1. In that case for all \( j \), \( \mathbb{P}(A^*_j) > 0 \) since bound (65) holds inductively for all \( \mathbb{P}(A_j) \) for \( j \leq J \). On the other hand, if the RHS of (65) is \( \geq 1 \) then (65) trivially holds. In both cases we conclude that (65) holds.

If one sets \( \varepsilon_j := \sqrt{\frac{nL(\alpha, \beta, C)}{2^{J-1}(C+1)}} \), we have that if

\[
n \varepsilon_j^2 > 2M_{\mathcal{F}}^{\text{adloc}} \left( f, \varepsilon_j, \frac{4(C+1)}{\sqrt{L(\alpha, \beta, C)}} \right)
\]

and \( a := \exp(-n \varepsilon_j^2) < 1/2 \iff n \varepsilon_j^2 > \log 2 \), the above probability in (65) will be bounded from above by \( 2 \exp(-n \varepsilon_j^2/2) \). This condition is implied when

\[
n \varepsilon_j^2 > 2M_{\mathcal{F}}^{\text{adloc}} \left( f, \varepsilon_j, \frac{4(C+1)}{\sqrt{L(\alpha, \beta, C)}} \right) \vee \log 2.
\]

We now have

\[
\|\nu^*_j - f\|_2 \leq \|\Upsilon_J - \Upsilon_J\|_2 + \|\Upsilon_J - f\|_2 \leq 3\varepsilon_j \frac{C+1}{\sqrt{L(\alpha, \beta, C)}},
\]

with probability at least \( 1 - 2 \exp(-n \varepsilon_j^2/2) \) which holds for all \( J \) satisfying (66). Here we want to clarify that the last inequality in (67) follows from the fact that \( \|\Upsilon_J - \Upsilon_J\|_2 \leq d/2^{J-2} \), as per Lemma 28 (and since \( J \geq J \)). Let \( J^* \) be selected as the maximum integer \( J \) such that (66) holds, or otherwise if such \( J \) does not exist \( J^* = 1 \). Let \( \eta = 3\sqrt{\frac{C+1}{L(\alpha, \beta, C)}} \), \( C = 2 \) and \( C' = 1/2 \). Observe that by the definition of \( J^* \) it follows that all packing sets encountered prior \( J^* \), will have been finite packing sets. We have established that the following bound holds

\[
\mathbb{P}(\|f - \nu^*_J\|_2 > \eta \varepsilon_j) \leq C \exp(-C' n \varepsilon_j^2) \mathbb{1}(J > 1) \leq C \exp(-C' n \varepsilon_j^2) \mathbb{1}(J^* > 1),
\]

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for all $1 \leq J \leq J^*$, where this bound also holds in the case when $J^* = 1$ by exception. Observe that we can extend this bound to all $J \in \mathbb{Z}$ and $J \leq J^*$, since for $J < 1$ we have $\eta \varepsilon J \geq 6d$ and so
\[
\mathbb{P}(\| f - \nu_J^* \|_2 > \eta \varepsilon J) \leq 0 \leq C \exp(-C' n \varepsilon_J^2) 1(J^* > 1).
\]
We conclude that
\[
\mathbb{P}(\| f - \nu_J^* \|_2 > \eta \varepsilon J) \leq 0 \leq C \exp(-C' n \varepsilon_J^2) 1(J^* > 1),
\]
for any $J \leq J^*$. Now for any $\varepsilon_{J-1} > x \geq \varepsilon_J$ for $J \leq J^*$ we have that
\[
\mathbb{P}(\| f - \nu_J^* \|_2 > 2\eta x) \leq \mathbb{P}(\| f - \nu_J^* \|_2 > \eta \varepsilon_{J-1}) \\
\leq C \exp(-C' n \varepsilon_{J-1}^2) 1(J^* > 1) \\
\leq C \exp(-C' n x^2) 1(J^* > 1),
\]
where the last inequality follows due to the fact that the map $x \mapsto C \exp(-C' n x^2)$ is monotonically decreasing for positive reals. We will now integrate the tail bound:
\[
\mathbb{P}(\| f - \nu_J^* \|_2 > 2\eta x) \leq C \exp(-C' n x^2) 1(J^* > 1),
\]
(68)
which holds true for $x \geq \varepsilon^*$, where $\varepsilon_J = \sqrt{\frac{L(\alpha, \beta; C)}{2(\alpha-1)(\alpha+1)}}$, always (since even if $J^* = 1$ by exception, this bound is still valid). We then have
\[
\mathbb{E}\| f - \nu_J^* \|_2 = \int_0^{\infty} 2x \mathbb{P}(\| f - \nu_J^* \|_2 > x) \, dx \\
\leq C'' \varepsilon^2 + \int_{2\eta \varepsilon^*}^{\infty} 2x C \exp(-C'' n x^2) 1(J^* > 1) \, dx \\
= C'' \varepsilon^2 + C''' n^{-1} \exp(-C'''' n \varepsilon^2) 1(J^* > 1).
\]
Now $n \varepsilon^2$ is bigger than a constant (i.e., $\log 2$) otherwise $J^* = 1$. Hence, the above is smaller than $\bar{C} \varepsilon^2$ for some absolute constant $\bar{C}$.

Remark 11 (Early stopping in adaptive estimation). Suppose that in our adaptive estimation, that we traverse the maximal packing set tree construction and encounter a density $\Upsilon_i$, such that the cardinality of the set of its children densities is countably infinite, i.e., $|P \Upsilon_i| = \infty$. Then we can simply return $\nu^*(X) = \Upsilon_i$, in such a case. The reason for this is that the index $i$ will be necessarily at least equal to $J^*$ as defined in (12), which is what is required for (67) to hold.
C Proofs of Section 3

C.1 Formal justification for Example 16

Before proving Examples 16 to 18, we first prove a useful lemma. This lemma will provide a sufficient condition to ensure that $L_2$-local and $L_2$-global metric entropies are of the same order for various forms of the density class $\mathcal{F}$, as specified in our chosen examples.

**Lemma 31** (Asymptotic order global metric entropy). Let $\mathcal{F} \subset \mathcal{F}^{[\alpha, \beta]}_B$, such that for any fixed $\eta > 0$, we have $0 < \varepsilon \mapsto \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \asymp \varepsilon^{-1/\eta}$. Then there exists a $c > 0$, such that the following holds

$$\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) - \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \asymp \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c)$$

(69)

**Proof of Lemma 31.** We firstly note that (69) has the following equivalence

$$\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) - \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \asymp \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c)$$

$$\iff \exists 0 < k_1 < k_2 \text{ s.t. } k_1 \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) \leq \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) - \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \leq k_2 \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c)$$

(70)

In general, for Equation (70) we observe that since $\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) > 0$, it follows that $\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) - \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \leq \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c)$. So taking $k_2 = 1$ will always suffice to ensure (70) holds. It remains to check that we can also find a $k_1 \in (0, 1)$ such that (70) also holds. In our case, since $\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \asymp \varepsilon^{-1/\eta}$ by assumption, we have that $\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) \geq C_1(\varepsilon/c)^{-1/\eta}$ and $\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \leq C_2\varepsilon^{-1/\eta}$ for some universal constants $C_1, C_2 > 0$. It then follows

$$\log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c) - \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon) \geq 1 - \frac{C_2}{C_1} \varepsilon^{-\frac{1}{\eta}} \geq \log M_{\mathcal{F}}^{\text{glo}}(\varepsilon/c)$$

(71)

That is, there exists such a $k_1 \in (0, 1)$, if we choose $c \geq \left(\frac{C_2}{C_1}\right)^\eta$, for each $\eta > 0$. So indeed (69) holds, for the specified class $\mathcal{F}$, as required.

**Example 16** (Lipschitz density class $\mathcal{F}$). Let $1 < \Psi < \beta < \infty$, max $\{1/q - 1/2, 0\} < \gamma \leq 1$, and $1 \leq q \leq \infty$ be fixed constants, and $B := [0, 1]$. Now, let $\mathcal{F} := \text{Lip}_{\gamma,q}(\Psi)$ denote the space of $(\gamma, q, \Psi)$-Lipschitz densities with total variation at most $\beta$. That is,

$$\text{Lip}_{\gamma,q}(\Psi) := \left\{ f : B \to [0, \Psi] \mid \|f(x + h) - f(x)\|_q \leq \Psi h^\gamma, \|f\|_q \leq \Psi, \int_B f \, d\mu = 1, f \text{ measurable} \right\},$$

(15)

and $\|f\|_q := (\int_B |f(x)|^q \, d\mu)^{1/q}$. Note that in (15) we have that $x \in B$, and only consider $h > 0$, and further $f(x + h) = f(1)$, for $x + h > 1$, so that the predicate of $\text{Lip}_{\gamma,q}(\Psi)$ is well-defined. Then $\text{Lip}_{\gamma,q}(\Psi)$ is a convex density class, there exists a density $f_0 \in \text{Lip}_{\gamma,q}(\Psi)$ that is strictly positively bounded away from 0, and the minimax rate (in the squared $L_2$-metric) for estimating $f \in \text{Lip}_{\gamma,q}(\Psi)$ is of the order $n^{-\frac{2\gamma}{2\gamma + 1}}$. 

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Proof of Example 16. In order to establish the minimax rate for $\text{Lip}_{\gamma,q}(\Psi)$, we need to show that $\text{Lip}_{\gamma,q}(\Psi)$ is a convex density class, and that there exists a density $f_\alpha \in \text{Lip}_{\gamma,q}(\Psi)$ that is strictly positively bounded away from 0. We can then apply Proposition 13. We first verify that $\text{Lip}_{\gamma,q}(\Psi)$ here is a convex density class. To that end, let $f, g \in \text{Lip}_{\gamma,q}(\Psi)$, and let $\kappa \in [0, 1]$, be arbitrary. Then for each $x \in B := [0, 1]$, we observe that

$$(\kappa f + (1 - \kappa)g)(x) := \kappa f(x) + (1 - \kappa)g(x) \geq \kappa(0) + (1 - \kappa)(0) = 0$$

(72)

$$(\kappa f + (1 - \kappa)g)(x) := \kappa f(x) + (1 - \kappa)g(x) \leq \kappa \Psi + (1 - \kappa)\Psi = \Psi$$

(73)

From (72) and (73), it follows that $\kappa f + (1 - \kappa)g : B \to [0, \Psi]$. Moreover, since $\int_B f \, d\mu = \int_B g \, d\mu = 1$, we have

$$\int_B (\kappa f + (1 - \kappa)g) \, d\mu = \kappa \int_B f \, d\mu + (1 - \kappa) \int_B g \, d\mu = 1.$$  

(75)

Since $f, g \in \text{Lip}_{\gamma,q}(\Psi)$, we have both $\|f\|_q, \|g\|_q \leq \Psi$. Then by the triangle inequality it follows

$$\|\kappa f + (1 - \kappa)g\|_q \leq \|\kappa f\|_q + \|(1 - \kappa)g\|_q \leq \kappa \Psi + (1 - \kappa)\Psi = \Psi.$$  

(76)

Since $f, g$ are measurable functions, then so is their convex combination, i.e., $\kappa f + (1 - \kappa)g$. Now we observe

$$
\|\kappa f + (1 - \kappa)g\|_q (x + h) - (\kappa f + (1 - \kappa)g)(x)
= \|\kappa (f(x + h) - f(x)) + (1 - \kappa)(g(x + h) - g(x))\|_q \\
\leq \|\kappa (f(x + h) - f(x))\|_q + \|(1 - \kappa)(g(x + h) - g(x))\|_q \\
\leq \kappa h^\gamma + (1 - \kappa)h^\gamma \\
= h^\gamma,
$$

as required. Combining (74), (75), (76), and (77) we have shown that $\kappa f + (1 - \kappa)g \in \text{Lip}_{\gamma,q}(\Psi)$. This proves the convexity of $\text{Lip}_{\gamma,q}(\Psi)$, as required.

Now let $f_\alpha \sim \text{Unif}[B]$, i.e. $f_\alpha(x) := \mathbb{I}_{[0,1]}(x)$. Therefore, $\|f_\alpha(x + h) - f(x)\|_q = 0 \leq \Psi h^\gamma$, for each $x \in B$, and $h > 0$ such that $f_\alpha(x + h)$ is defined. Moreover, $\|f_\alpha(x)\|_q = 1 < \Psi$, by assumption, for each $1 \leq q \leq \infty$. Now we have that $\int_B f_\alpha \, d\mu = \int_B \mathbb{I}_{[0,1]}(x) \, d\mu(x) = 1$, and $f$ is measurable since it is a simple function. So indeed we have found $f_\alpha \in \text{Lip}_{\gamma,q}(\Psi)$, such that it is $\alpha$-lower bounded (with $\alpha = 1$).

We now proceed to check that $L_2$-global metric entropy is of the same order as the $L_2$-local metric entropy for $\text{Lip}_{\gamma,q}(\Psi)$. That is, we want to check that (69) holds. Here $\mathcal{F} = \text{Lip}_{\gamma,q}(\Psi)$, with $0 < \varepsilon \mapsto \log M^\delta_{\mathcal{F}}(\varepsilon) \gtrsim \varepsilon^{-\gamma}$. Thus, we can apply Lemma 31 with $\eta := \gamma \in (0, 1]$ to conclude that indeed (69) holds, as required.

Since we have checked all the sufficient conditions in order to apply Proposition 13 for $\text{Lip}_{\gamma,q}(\Psi)$, we can obtain the minimax rate of density estimation by solving

$$n\varepsilon^2 \gtrsim \varepsilon^{-\frac{1}{\gamma}} \iff \varepsilon \asymp n^{-\frac{\gamma}{2\gamma+1}} \iff \varepsilon^2 \asymp n^{-\frac{2\gamma}{2\gamma+1}}.$$  

(78)

So the minimax rate is (up to constants) the order of $n^{-\frac{2\gamma}{2\gamma+1}}$ as required.
C.2 Formal justification for Example 17

Example 17 (Bounded total variation density class \(\mathcal{F}\)). Let \(1 < \zeta < \beta < \infty\) be a fixed constant, and \(B := [0,1]\). Now, let \(\mathcal{F} := \text{BV}_\zeta\) denote the space of univariate densities with total variation at most \(\beta\). That is,

\[
\text{BV}_\zeta := \left\{ f: B \to [0,\zeta] \mid \|f\|_\infty \leq \zeta, V(f) \leq \zeta, \int_B f \, d\mu = 1, f \text{ measurable} \right\},
\]

where we define the total variation of \(f\), i.e., \(V(f)\) as

\[
V(f) := \sup_{\{x_1, \ldots, x_m \mid 0 \leq x_1 < \cdots < x_m \leq 1, m \in \mathbb{N}\}} \sum_{i=1}^{m-1} |f(x_{i+1}) - f(x_i)|,
\]

and \(\|f\|_\infty := \sup_{x \in B} |f(x)|\). Then the minimax rate (in the squared \(L_2\)-metric) for estimating \(f \in \text{BV}_\zeta\) is of the order \(n^{-2/3}\).

Proof of Example 17. In order to establish the minimax rate for \(\text{BV}_\zeta\), we need to show that \(\text{BV}_\zeta\) is a convex density class, and that there exists a density \(f_\alpha \in \text{BV}_\zeta\) that is strictly positively bounded away from 0. We can then apply Proposition 13. We first verify that \(\text{BV}_\zeta\) here is a convex density class. To that end, let \(f, g \in \text{BV}_\zeta\), and let \(\kappa \in [0,1]\), be arbitrary. Then for each \(x \in B := [0,1]\), it follows by an identical argument to (72) and (73) that

\[
\kappa f + (1 - \kappa)g: B \to [0,\zeta].
\]

Moreover, since \(\int_B f \, d\mu = \int_B g \, d\mu = 1\), we have

\[
\int_B (\kappa f + (1 - \kappa)g) \, d\mu = \kappa \int_B f \, d\mu + (1 - \kappa) \int_B g \, d\mu = 1.
\]

Since \(f, g \in \text{BV}_\zeta\), we have both \(\|f\|_\infty, \|g\|_\infty \leq \zeta\). Then by the triangle inequality it follows

\[
\|\kappa f + (1 - \kappa)g\|_\infty \leq \kappa \|f\|_\infty + \|(1 - \kappa)g\|_\infty \leq \kappa \zeta + (1 - \kappa)\zeta = \zeta.
\]

Since \(f, g\) are measurable functions, then so is their convex combination, i.e., \(\kappa f + (1 - \kappa)g\). Finally, fix any \(m \in \mathbb{N}\), and let \(a \leq x_1 < \cdots < x_m \leq b\) be any fixed partition of \(B\). Now we observe

\[
\sum_{i=1}^{m-1} |(\kappa f + (1 - \kappa)g)(x_{i+1}) - (\kappa f + (1 - \kappa)g)(x_i)|
\]

\[
= \sum_{i=1}^{m-1} |\kappa(f(x_{i+1}) - f(x_i)) + (1 - \kappa)(g(x_{i+1}) - g(x_i))|
\]

\[
\leq \kappa \sum_{i=1}^{m-1} |f(x_{i+1}) - f(x_i)| + (1 - \kappa) \sum_{i=1}^{m-1} |g(x_{i+1}) - g(x_i)|
\]

(by the triangle inequality.)

\[
\leq \kappa V(f) + (1 - \kappa) V(g)
\]

\[
\leq \kappa (\zeta) + (1 - \kappa)(\zeta)
\]

(since \(V(f), V(g) \leq \zeta\), by definition of \(\text{BV}_\zeta\).)

\[
= \zeta.
\]
Taking the supremum over all $m \in \mathbb{N}$ and all partitions of length $m$ of $B$ of the LHS sum we obtain:

$$V(\kappa f + (1 - \kappa)g) \leq \zeta,$$

(82)
as required. Combining (79),(80),(81), and (82) we have shown that $\kappa f + (1 - \kappa)g \in BV_\zeta$. This proves the convexity of $BV_\zeta$, as required.

Similar to the proof of Example 16, we let $f_\alpha \sim \text{Unif}[B]$, i.e. $f_\alpha(x) := \mathbb{I}_{[0,1]}(x)$. Therefore, $\|f\|_\infty = 1 \leq \zeta$ by assumption. Also, $V(f) = 0 < \zeta$, by assumption. Now we have that $\int_B f_\alpha \, d\mu = \int_B \mathbb{I}_{[0,1]}(x) \, d\mu(x) = 1$, and $f$ is measurable since it is a simple function. So indeed we have found $f_\alpha \in BV_\zeta$, such that it is $\alpha$-lower bounded (with $\alpha = 1$).

We now proceed to check that $L_2$-global metric entropy is of the same order as the $L_2$-local metric entropy for $BV_\zeta$. That is, we want to check that (69) holds. Here $F = BV_\zeta$, with $0 < \varepsilon \mapsto \log M^{\text{glo}}_F(\varepsilon) \asymp \varepsilon^{-1}$. Thus, we can apply Lemma 31 with $\eta := 1$ to conclude that indeed (69) holds, as required.

Since we have checked all the sufficient conditions in order to apply Proposition 13 for $BV_\zeta$, we can obtain the minimax rate of density estimation by solving

$$n \varepsilon^2 \asymp \varepsilon^{-1} \iff \varepsilon \asymp n^{-\frac{1}{2}} \iff \varepsilon^2 \asymp n^{-\frac{2}{3}}.$$ 

(83)

So the minimax rate is (up to constants) the order of $n^{-\frac{2}{3}}$ as required.

C.3 Formal justification for Example 18

Example 18 (Quadratic functional density class $\mathcal{F}$). Let $0 < \alpha < 1 < \beta < \infty$, and $\gamma > 1$ be fixed constants, with $B := [0,1]$. Now, let $\mathcal{F} := \text{Quad}_\gamma$ denote the space of univariate quadratic functional densities. That is,

$$\text{Quad}_\gamma := \left\{ f : B \to [\alpha, \beta] \left| \|f''\|_\infty \leq \gamma, \int_B f \, d\mu = 1, f \text{ measurable} \right. \right\}.$$ 

(18)

Then $\text{Quad}_\gamma$ is a convex density class, there exists a density $f_\alpha \in \text{Quad}_\gamma$ that is strictly positively bounded away from 0, and the minimax rate (in the squared $L_2$-metric) for estimating $f \in \text{Quad}_\gamma$ is of the order $n^{-4/5}$.

Proof of Example 18. In order to establish the minimax rate for $\text{Quad}_\gamma$, we need to show that $\text{Quad}_\gamma$ is a convex density class. We can then apply Proposition 13. We first verify that $\text{Quad}_\gamma$ here is a convex density class. To that end, let $f, g \in \text{Quad}_\gamma$, and let $\kappa \in [0,1]$, be arbitrary. Then for each $x \in B := [0,1]$, it follows by an identical argument to (72) and (73) that

$$\kappa f + (1 - \kappa)g : B \to [0,\beta].$$

(84)

Moreover, since $\int_B f \, d\mu = \int_B g \, d\mu = 1$, we have

$$\int_B (\kappa f + (1 - \kappa)g) \, d\mu = \kappa \int_B f \, d\mu + (1 - \kappa) \int_B g \, d\mu = 1.$$

(85)
Since $f, g$ are measurable functions, then so is their convex combination, i.e., $\kappa f + (1 - \kappa)g$. Now we observe

$$\| (\kappa f + (1 - \kappa)g)'' \|_\infty = \| \kappa f'' + (1 - \kappa)g'' \|_\infty$$

(by linearity of 2nd derivative.)

$$\leq \| \kappa f'' \|_\infty + \| (1 - \kappa)g'' \|_\infty$$

(by the triangle inequality.)

$$= \kappa \| f'' \|_\infty + (1 - \kappa)\| g'' \|_\infty$$

$$\leq \kappa \gamma + (1 - \kappa)\gamma$$

(since $f, g \in \text{Quad}_\gamma$)

$$= \gamma,$$

(86)
as required. Combining (84),(85), and (86) we have shown that $\kappa f + (1 - \kappa)g \in \text{Quad}_\gamma$. This proves the convexity of $\text{Quad}_\gamma$, as required.

Similar to the proof of Example 16, we let $f_\alpha \sim \text{Unif}[B]$, i.e. $f_\alpha(x) := \|_{\text{[0,1]}}(x)$. Since $\| f'' \|_\infty = 0 \leq \gamma$. Here, for the boundary points of $B := [0,1]$, we are careful to take all derivatives of $f_\alpha(x)$ at $x = 0$ from the right, and all derivatives from the left at $x = 1$. Now we have that $\int_B f_\alpha \, d\mu = \int_B \|_{\text{[0,1]}}(x) \, d\mu(x) = 1$, and $f$ is measurable since it is a simple function. So indeed we have found $f_\alpha \in \text{Quad}_\gamma$, such that it is $\alpha$-lower bounded (with $\alpha = 1$).

We now proceed to check that $L_2$-global metric entropy is of the same order as the $L_2$-local metric entropy for $\text{Quad}_\gamma$. That is, we want to check that (69) holds. Here $\mathcal{F} = \text{Quad}_\gamma$, with $0 < \varepsilon \mapsto \log M_\mathcal{F}^\text{gl} (\varepsilon) \asymp \varepsilon^{-1/4}$. Thus, we can apply Lemma 31 with $\eta := 4$ to conclude that indeed (69) holds, as required.

Since we have checked all the sufficient conditions in order to apply Proposition 13 for $\text{Quad}_\gamma$, we can obtain the minimax rate of density estimation by solving

$$n\varepsilon^2 \asymp \varepsilon \asymp n^{-\frac{4}{7}} \iff \varepsilon \asymp n^{-\frac{4}{7}}.$$  

(87)

So the minimax rate is (up to constants) the order of $n^{-\frac{4}{7}}$ as required.

\section{C.4 Formal justification for Example 19}

\textbf{Example 19} (Convex mixture density class $\mathcal{F}$). Let $\mathcal{F} := \text{Conv}_k$ where

$$\text{Conv}_k := \left\{ \sum_{i=1}^k \alpha_i f_i \middle\vert \sum_{i=1}^k \alpha_i = 1, \alpha_i \geq 0, f_i \in \mathcal{F}_B^{[\alpha,\beta]} \right\},$$

(19)

for some fixed $k \in \mathbb{N}$ and $f_i \in \mathcal{F}_B^{[\alpha,\beta]}$ for each $i \in [k]$. Further, let $\mathbf{G} = (G_{ij})_{i,j \in [k]}$ denote the Gram matrix with $G_{ij} := \int_B f_i f_j \mu(dx)$, which we assume is positive definite, i.e., $\mathbf{G} \succ 0$. Then the minimax rate for estimating $f \in \text{Conv}_k$ is bounded from above by $\sqrt{\frac{k}{n}}$ up to absolute constant factors.

\textit{Proof of Example 19.} Let $\mathbf{G} = (G_{ij})_{i,j \in [k]}$ denote the Gram matrix $G_{ij} := \int_B f_i f_j \mu(dx)$. Then it is simple to see that for some point $\theta \in \mathcal{F}$ which can be represented as the convex combination $\theta = \sum_{i \in [k]} \alpha_i f_i$, the packing set should consist of functions $g_i = \sum_{j \in [k]} \beta_{ij} f_j$ satisfying both

$$\alpha - \beta_i, \beta_i \leq \varepsilon^2,$$

$$(\beta_i - \beta_j, \beta_i - \beta_j) \geq \varepsilon^2 / c^2,$$

for $i \neq j$,
where $\beta_i$ are vectors from the $k$-dimensional unit simplex, i.e., $\sum_{j \in [k]} \beta_{ij} = 1$, $\beta_{ij} \geq 0$. Now suppose that $G \succ 0$. Then upon substituting $\alpha' = \sqrt{G} \alpha$, $\beta'_i = \sqrt{G} \beta_i$ and dropping the simplex requirements on the $\beta$ we obtain the set

\[
\|\alpha' - \beta'_i\| \leq \varepsilon \\
\|\beta'_i - \beta'_j\| > \varepsilon / c,
\]

which is like packing the unit sphere at a distance $1/c$. Hence, the log cardinality of such a packing is always $\lesssim k$ [Wainwright, 2019, see Chapter 5]. If $k$ is not allowed to scale with $n$, we conclude therefore that the minimax rate is upper bounded by $n^{-1/2}$ which is the parametric rate as we would expect. If $k$ is allowed to scale with $n$ the rate is smaller than $\sqrt{\frac{\varepsilon}{n}}$. \qed
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