Thermodynamical path integral and emergent symmetry

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We investigate a thermally isolated quantum many-body system with an external control represented by a step protocol of a parameter. The propagator at each step of the parameter change is described by thermodynamic quantities under some assumptions. For the time evolution of such systems, we formulate a path integral over the trajectories in the thermodynamic state space. In particular, for quasi-static operations, we derive an effective action of the thermodynamic entropy and its canonically conjugate variable. Then, the symmetry for the uniform translation of the conjugate variable emerges in the path integral. This leads to the entropy as a Noether invariant in quantum mechanics.

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I. INTRODUCTION

Thermodynamics and quantum mechanics are fundamental theories in physics. The universal behavior of macroscopic objects is described by thermodynamics, while the microscopic dynamics of any system is governed ultimately by quantum mechanics. Statistical mechanics connects them in equilibrium states; however, the relation between their dynamics is not established despite studies in many contexts, such as thermodynamic processes in quantum systems [1–6] and relaxation of pure quantum states to the thermal equilibrium [7–15]. Recently, state-of-the-art experiments for these studies are realized by using ultracold atoms [16, 17, 19, 20], nuclear magnetic resonance [21], trapped ions [22], and electronic circuits [23]. Given these backgrounds, we propose a theory connecting thermodynamical behavior to quantum mechanics.

Our strategy is to construct a thermodynamical path integral. In thermodynamics, an equilibrium state of a system is represented by a point in the thermodynamic state space. In quantum mechanics, on the other hand, the time evolution of a system is formulated in terms of a sum over all possible paths in a configuration space, weighted by the exponent of the action. In this paper, we combine these two concepts for a thermally isolated quantum many-body system under a time-dependent external control. We formulate the unitary evolution of quantum states by an integral over paths in the thermodynamic state space.

The path integral is constructed as follows. First, we introduce the projection operator to an energy shell that constitutes the micro-canonical ensemble of any energy $E$. In terms of the projection operators we express a decomposition of the identity operator. Next, we consider a series of step operations with the external control. At each step, the change of energy is much smaller than the energy itself but larger than the fluctuation. Then, we insert the decomposition formula at each step, evaluate a one-step propagator, and take a continuum limit.

We show that the propagator is expressed in terms of thermodynamic quantities. This is the key of our derivation. In order to evaluate the propagator, we introduce two assumptions. The first assumption is that the time interval between two successive step operations is so long that the phase of each energy eigenstate in the evolved state is interpreted as a uniform random variable. Now, for the time-evolved state, the state after the projection to an energy shell has the amplitudes of each energy eigenstate in the shell. The second assumption is that the amplitudes are equally weighted at each step. Such a class of non-equilibrium processes can be described by the thermodynamical path integral, which connects the concepts of thermodynamics and quantum mechanics in dynamical problems.

For quasi-static operations, we derive an effective action that has a symplectic structure for $(S, \hbar \theta)$, where $S$ is the thermodynamic entropy and $\theta$ is an auxiliary variable introduced in the path integral. The equations of motion are $dS/dt = 0$ and $d\theta/dt = 1/(\hbar \beta)$, where $\beta$ is the inverse temperature depending on time. In such slow operations, the symmetry for $\theta \to \theta + \eta$ emerges in the path integral, leading to entropy conservation in quantum mechanics, where $\eta$ is an infinitely small parameter. This provides a complementary view to the quantum adiabatic theorem [30, 31] because the operations are assumed to be slow yet so fast that transitions between different energy levels occur.

This emergent symmetry is related to the following topics. First, $\theta$ corresponds to a thermal time, which was introduced as a parameter of the flow determined by a statistical state [32, 33]. Through the relation $dt = \hbar \beta d\theta$,
the symmetry of the effective action for \( \theta \rightarrow \theta + \eta \) is connected to that for \( t \rightarrow t + \eta \hbar \beta \), which leads to entropy in classical systems \([38]\). Second, a similar symmetry has been phenomenologically studied for perfect fluids \([35\), 38\] and for effective field theories \([39\), 41\]. Finally, the entropy of stationary black holes is derived as the Noether charge for \( v \rightarrow v + \eta \hbar \beta \mathcal{H} \), where \( v \) is the Killing parameter and \( 1/\beta \mathcal{H} \) is the Hawking temperature \([42]\). Thus, our theory provides a unified perspective for studying the thermal time, perfect fluids, and black holes in terms of quantum mechanics.

This paper is organized as follows. In Sec. II we start with our setup. We describe isolated quantum many-body systems and employ an energy eigenstate as the initial state. In Sec. III we introduce a decomposition of the identity operator using projection operators onto energy shells, which plays a key role in our theory. By using this and the two assumptions, we formulate the thermodynamical path integral. In Sec. IV we construct quasi-static operations and derive the effective action in the thermodynamical phase space \( (S, h\theta) \). Then, we show the symmetry for \( \theta \rightarrow \theta + \eta \) and the entropy conservation, and discuss characterizations of the variable \( \theta \). In the final section, we provide concluding remarks.

II. SETUP

Although the theory developed in this paper is applicable to a wide class of quantum many-body systems, we specifically consider a Hamiltonian \( \hat{H}(h) \) consisting of \( N \) spins with spin-1/2 under a uniform magnetic field \( h > 0 \) so that the argument is explicit. We also assume that the system does not have any conserved quantities for any value of \( h \). It is straightforward to extend our result to the case with the existence of a small number of conserved quantities such as momentum and particle number \([43]\). The eigenvalues and eigenstates satisfy

\[
\hat{H}(h) \ket{n, h} = E(n, h) \ket{n, h},
\]

where \( n = 1, 2, \ldots, 2^N \). By incorporating the magnetic moment into \( h \), we assume the dimension of \( h \) to be energy. Then, \( h \) represents the characteristic energy scale per unit spin. We study the macroscopic behavior of the system by taking the large \( N \) limit.

We choose an energy shell \( I_E \equiv [E - \Delta/2, E + \Delta/2] \), where \( \Delta \) is much smaller than \( hN \) but should be large so that \( I_E \) contains \( e^{O(N)} \) energy levels. Here we choose \( \Delta = O(1) \) as an example. The number of eigenvalues in the shell is given by \( \sum_n \chi_{I_E}(E(n, h)) \), where \( \chi_{I_E}(x) = 1 \) for \( x \in I_E \) and zero otherwise. The density of states, \( D(E, h) \), is defined as

\[
D(E, h) = \sum_n \frac{1}{\Delta} \chi_{I_E}(E(n, h)).
\]

We assume the asymptotic form for large \( N \)

\[
D(E, h) = e^{N s(E/N, h) + o(N)}
\]

with a function \( s(u, h) \) whose functional form is independent of \( N \). This assumption is necessary for the consistency of statistical mechanics with thermodynamics. In fact, \([33]\) is satisfied for a wide class of systems with local interactions. For thermodynamic states \((E, h)\), the entropy \( S(E, h) \) is then defined as

\[
S(E, h) = N_\theta(E/N, h).
\]

The inverse temperature \( \beta(E, h) \) is defined by the thermodynamic relation

\[
\beta \equiv \left( \frac{\partial S}{\partial E} \right)_h.
\]

The Boltzmann constant is set to unity.

This means that the change of energy caused by the parameter change, which is \( O(N \Delta h) \), is much smaller than the energy itself, but it is larger than the fluctuation for large \( N \). In this sense, each quench is called small macroscopic. It should be noted here that the standard perturbation technique cannot be employed for this protocol.

In this section, we express the time-evolved state \( \ket{\Psi(t)} \) by

\[
\Psi(t) = \hat{H}(h(t)) \ket{\Psi(0)}.
\]

We study cases where the system is in a thermal equilibrium state at \( t = 0 \). We express the state by a single pure state, as per previous studies \([2, 29, 32, 43, 44]\). Unitary time evolution starting from such a thermal pure state is determined by \([27]\), which is in accordance with isolated quantum systems \([16, 17]\) and may provide an idealization of quantum dynamics in nature. In particular, we set

\[
\ket{\Psi(0)} = \ket{n_0, h_0}.
\]

Here, it should be noted that any single energy eigenstate may exhibit a thermal equilibrium state, according to the eigenstate thermalization hypothesis \([8]\).

III. THERMODYNAMICAL PATH INTEGRAL

In this section, we express the time-evolved state \( \ket{\Psi(t)} \) as a path integral over trajectories in the thermodynamic state space \((E, h)\). First, we use a formula of decomposition of identity operator \( \hat{1} \) and construct a path-integral-like form of \( \ket{\Psi(t)} \). Then, we introduce two assumptions, which enable us to evaluate the propagator, and we express it in terms of thermodynamic quantities. Finally, we reach the path-integral expression.
A. Projected states

We first express the identity operator $\hat{1}$ as an integration over energy $E$. We define a projection operator to $I_E$

$$\hat{P}_{E,h} = \sum_n \chi_{I_E}(E(n,h))|n,h\rangle \langle n,h|,$$  \hspace{1cm} (9) 

and we note that

$$1 = \frac{1}{\Delta} \int_{D(h)} dE \chi_{I_E}(E(n,h))$$  \hspace{1cm} (10) 

holds for each $n$. Here the interval of integration is defined as

$$D(h) = [E_{\text{min}}(h) - \Delta/2, E_{\text{max}}(h) + \Delta/2],$$  \hspace{1cm} (11) 

where $E_{\text{max}}(h)$ and $E_{\text{min}}(h)$ are the maximum and minimum energy eigenvalues for a given $h$, respectively. Using (10), we can express the complete relation $\hat{1} = \sum_n |n,h\rangle \langle n,h|$ as

$$\hat{1} = \frac{1}{\Delta} \int_{D(h)} \hat{P}_{E,h} dE$$  \hspace{1cm} (12) 

for each $h$.

Let us start with the evolution of (1) for $t_f = 2\Delta t$:

$$|\Psi(2\Delta t)\rangle = e^{-\frac{i}{\hbar}H_1 \Delta t} e^{-\frac{i}{\hbar}E_0 \Delta t} |n_0,h_0\rangle$$

$$= e^{-\frac{i}{\hbar}H_1 \Delta t} e^{-\frac{i}{\hbar}E(n_0,h_0) \Delta t} |n_0,h_0\rangle$$

$$= e^{-\frac{i}{\hbar}(E_0 + H_1) \Delta t} \int_{D(h_1)} dE_1 \frac{\Delta}{\hbar} \hat{P}_{E_1,h_1} |n_0,h_0\rangle,$$  \hspace{1cm} (13)

where we have set $H_j = \hat{H}(h_j)$ and $E_0 = E(n_0,h_0)$ and used (12) for $h_1$. Here, we define a projected state

$$|\mathcal{P}_{E_1,h_1}\rangle_{\Delta t} \equiv \frac{\hat{P}_{E_1,h_1}}{\sqrt{B_1(E_1,h_1)}} |n_0,h_0\rangle$$  \hspace{1cm} (14) 

with the notation $E_1 \equiv (E_0,E_1), h_1 \equiv (h_0,h_1)$ and the normalization factor

$$B_1(E_1,h_1) \equiv \langle n_0,h_0| \hat{P}_{E_1,h_1} |n_0,h_0\rangle.$$  \hspace{1cm} (15) 

Then, (13) is expressed as

$$|\Psi(2\Delta t)\rangle = e^{-\frac{i}{\hbar}(E_0 + H_1) \Delta t} \int_{D(h_1)} dE_1 \frac{\Delta}{\hbar} \sqrt{B_1(E_1,h_1)} |\mathcal{P}_{E_1,h_1}\rangle_{\Delta t}.$$  \hspace{1cm} (16) 

Now, we suppose that $|\mathcal{P}_{E_1,h_1}\rangle_{\Delta t}$ evolves by an energy-shifted Hamiltonian $\hat{H}_1 - E_1$ during $[\Delta t, 2\Delta t]$ to

$$|\mathcal{P}_{E_1,h_1}\rangle_{2\Delta t} \equiv e^{-\frac{i}{\hbar}(H_1 - E_1) \Delta t} |\mathcal{P}_{E_1,h_1}\rangle_{\Delta t}.$$  \hspace{1cm} (17) 

Thus, we reach

$$|\Psi(2\Delta t)\rangle = \int_{D(h_1)} dE_1 \frac{\Delta}{\hbar} \sqrt{B_1(E_1,h_1)} |\mathcal{P}_{E_1,h_1}\rangle_{2\Delta t} e^{-\frac{i}{\hbar}(E_1 + E_0) \Delta t}.$$  \hspace{1cm} (18) 

By repeating this procedure, we can construct the general form for any $M$. To do it, we use the notation

$$E_j \equiv (E_0, \cdots, E_j), \quad h_j \equiv (h_0, \cdots, h_j),$$  \hspace{1cm} (19) 

and define the projected state $|\mathcal{P}_{E_j,h_j}\rangle_{j\Delta t}$ in the following iterative manner:

$$|\mathcal{P}_{E_0,h_0}\rangle_0 \equiv |n_0,h_0\rangle,$$  \hspace{1cm} (20) 

$$|\mathcal{P}_{E_j,h_j}\rangle_{j\Delta t} \equiv \int dE_j \frac{1}{\sqrt{B_j(E_j,h_j)}} |\hat{P}_{E_j,h_j}\rangle |\mathcal{P}_{E_{j-1},h_{j-1}}\rangle_{(j-1)\Delta t},$$  \hspace{1cm} (21) 

with

$$B_j(E_j,h_j) \equiv j\Delta t \langle \mathcal{P}_{E_{j-1},h_{j-1}} | \hat{P}_{E_j,h_j} | \mathcal{P}_{E_{j-1},h_{j-1}} \rangle_{j\Delta t};$$  \hspace{1cm} (22) 

and

$$|\mathcal{P}_{E_j,h_j}\rangle_{(j+1)\Delta t} \equiv e^{-\frac{i}{\hbar}(H_j - E_j) \Delta t} |\mathcal{P}_{E_j,h_j}\rangle_{j\Delta t}.$$  \hspace{1cm} (23) 

Then, we can reexpress (18) as

$$|\Psi(2\Delta t)\rangle = \int_{D(h_2)} dE_2 \frac{\Delta}{\hbar} \sqrt{B_2(E_2,h_2)} |\mathcal{P}_{E_1,h_1}\rangle_{2\Delta t} e^{-\frac{i}{\hbar}(E_1 + E_0) \Delta t}$$

$$= \int_{D(h_2)} dE_2 \frac{\Delta}{\hbar} \int_{D(h_1)} dE_1 \frac{\Delta}{\hbar} e^{-\frac{i}{\hbar}(E_1 + E_0) \Delta t}$$

$$\sqrt{B_1(E_1,h_1)} \sqrt{B_2(E_2,h_2)} |\mathcal{P}_{E_2,h_2}\rangle_{2\Delta t};$$  \hspace{1cm} (24) 

where in the first line we insert (12) for $h_2$ and in the second line use (21). This is the path-integral-like representation of $|\Psi(2\Delta t)\rangle$. Now, by repeating the above procedure, we can obtain the formula for $t_f = M\Delta t$:

$$|\Psi(t_f)\rangle = \prod_{j=1}^{M} \int_{D(h_j)} dE_j \frac{\Delta}{\hbar} \sqrt{B_j(E_j,h_j)}$$

$$e^{-\frac{i}{\hbar} \sum_{j=0}^{M-1} E_j \Delta t} |\mathcal{P}_{E_M,h_M}\rangle_{M\Delta t}.$$  \hspace{1cm} (25) 

Here, $\sqrt{B_j(E_j,h_j)}$ in (25) can be expressed, from (21), (22) and (23), as

$$\sqrt{B_j(E_j,h_j)} = j\Delta t \langle \mathcal{P}_{E_j,h_j} | \mathcal{P}_{E_{j-1},h_{j-1}} \rangle_{j\Delta t};$$

$$= j\Delta t \langle \mathcal{P}_{E_j,h_j} | e^{-\frac{i}{\hbar}(H_{j-1} - E_{j-1}) \Delta t} | \mathcal{P}_{E_{j-1},h_{j-1}} \rangle_{(j-1)\Delta t}. $$  \hspace{1cm} (26)
This is the propagator from \(|P_{E_{j-1},h_{j-1}}(j-1)\Delta t}\) to \(|P_{E_j,h_j}(j)\Delta t|\), and [23] looks like the standard form of a path integral over \((E_1, E_2, \ldots, E_M)\). Note that, by the above construction, this propagator depends on the trajectory of thermodynamic states \((E_0, h_0), \ldots, (E_{j-2}, h_{j-2})\) in addition to the thermodynamic states \((E_{j-1}, h_{j-1}), (E_j, h_j)\).

**B. Evaluation of \(B_j\)**

We here show that the path dependence of \(\sqrt{B_j(E_j, h_j)}\) becomes negligible under two assumptions; we express \(\sqrt{B_j(E_j, h_j)}\) only in terms of \(E_{j-1}, E_j, h_{j-1}, h_j\). To be specific, we represent \(B_{j+1}\) in terms of \(q_{nm}\) defined by

\[
q_{nm} = \chi_{t_{E_{j+1}}}(E(n, h_{j+1}))\chi_{t_{E_j}}(E(m, h_j)) \langle n, h_{j+1}|m, h_j \rangle.
\]

Although \(q_{nm}\) depends on \((E_j, E_{j+1}, h_j, h_{j+1})\), we do not write this dependence explicitly. We then expand \(|P_{E_j,h_j}(j)\Delta t|\) as

\[
|P_{E_j,h_j}(j)\Delta t| = \sum_n \chi_{t_{E_j}}(E(n, h_j))d_{nj}(E_j, h_j) |n, h_j\rangle.
\]

By substituting this into [23], we have

\[
|P_{E_j,h_j}(j+1)\Delta t| = \sum_n \chi_{t_{E_j}}(E(n, h_j))e^{-\frac{i}{\hbar}(E(n, h_j) - E_j)\Delta t} d_{nj}(E_j, h_j) |n, h_j\rangle.
\]

Combining this expression with [22] leads to

\[
B_{j+1}(E_{j+1}, h_{j+1}) = \sum_n \chi_{t_{E_{j+1}}}(E(n, h_{j+1})) \left| \sum_m \chi_{t_{E_j}}(E(m, h_j))e^{-\frac{i}{\hbar}(E(m, h_j) - E_j)\Delta t} \langle n, h_{j+1}|m, h_j \rangle d_{mj}(E_j, h_j) \right|^2
\]

\[
= \sum_n \left| \sum_m e^{i\xi_{mj}} q_{nm} d_{mj}(E_j, h_j) \right|^2
\]

\[
= \sum_n |C_{nj}|^2,
\]

where we have defined

\[
\xi_{mj} = \frac{1}{\hbar}(E(m, h_j) - E_j)\Delta t,
\]

and

\[
C_{nj} = \sum_m e^{i\xi_{mj}} q_{nm} d_{mj}(E_j, h_j).
\]

We first study the properties of \(\xi_{nj}\), which is order of \((\Delta t)\Delta /\hbar\). In [29] for a fixed \(j\), a series of \(E(n, h_j)(e^{I_{E_j}})\) in terms of \(n\) is irregular as if it would follow some probability distribution. Therefore, when we choose a large \(\Delta t\) satisfying

\[
\Delta t \gg \frac{\hbar}{\Delta},
\]

we can assume that \(\xi_{mj}\) are independent uniform random variables on \([0, 2\pi]\). We then take the expectation value of \(|C_{nj}|^2\) with respect to the random variables \(\xi_{mj} (m = 1, \ldots, 2^N)\) as follows [18].

\[
|C_{nj}|^2 = \sum_m e^{i\xi_{mj}} q_{nm} d_{mj}^* q_{nm} d_{mj}^*.
\]

We also can show that

\[
\log |C_{nj}| = \log |C_{nj}| + o(N).
\]

The precise statement is stated as [31], and the proof is given in Appendix A. Therefore, [30] becomes

\[
B_{j+1}(E_{j+1}, h_{j+1}) = \sum_n |C_{nj}|^2 e^{o(N)}
\]

\[
= \sum_n |q_{nm}|^2 |d_{mj}(E_j, h_j)|^2 e^{o(N)}.
\]

Next, we consider the form of \(|d_{mj}(E_j, h_j)|\). From [24], [25] and [28], we can obtain

\[
|d_{mj}(E_j, h_j)|^2 = \left| \langle m, h_j | P_{E_{j-1},h_{j-1}}(j)\Delta t | \right|^2
\]

\[
|j\Delta t| P_{E_j,h_j} | P_{E_{j-1},h_{j-1}}(j)\Delta t |^2.
\]

This is the probability of transition from \(|P_{E_{j-1},h_{j-1}}(j)\Delta t|\) to \(|m, h_j\rangle\) having \(E(m, h_j) \in I_{E_j}\) when the quench \(h_{j-1} \rightarrow h_j\) is performed at \(t = j\Delta t\). The width of an energy shell is \(\Delta = O(1)\), while the quench is macroscopic in the sense that \(N|h_{j-1} - h_{j-1}| \gg O(\sqrt{N})\) because of [29]. Therefore, the transition can occur to any \(|m, h_j\rangle\) in \(I_{E_j}\). In addition, the energy shell contains \(e^{O(N)}\) energy eigenstates which do not have any particular structure. Motivated by this observation, we assume

\[
\log |d_{mj}(E_j, h_j)|^2 = - \log D(E_j, h_j) + o(N),
\]

which means that the overlap between \(|P_{E_j,h_j}\rangle\Delta t\) and each eigenstate in the shell is equally weighted up to the sub-exponential factor in \(N\). Thus, [36] becomes

\[
B_{j+1}(E_{j+1}, h_{j+1}) = \sum_n |q_{nm}|^2 D(E_j, h_j)^{o(N)},
\]

which depends only on \(E_j, E_{j+1}, h_j, h_{j+1}\).
C. Expression of $\sum_{m,n} |q_{mn}|^2$

In this subsection, we evaluate $\sum_{m,n} |q_{mn}|^2$. Here we set $E = E_j$, $E' = E_{j+1}$, $h = h_j$, and $h' = h_{j+1}$. The key idea is that we express $\sum_{m,n} |q_{mn}|^2$ in terms of thermodynamic quantities by introducing a probability density

$$P(E', h'|E, h) \equiv \frac{\sum_{m,n} |m, h'| n, h|^2 \chi_{I_E}(E, h') \chi_{I_E}(E, h)}{\Delta \sum_n \chi_{I_E}(E, h)}.$$  \hspace{1cm} (40)

$P(E', h'|E, h)\Delta$ is the probability of finding the energy in $I_E'$ when we instantaneously change the field from $h$ to $h'$ under the condition that the energy eigenstates satisfying $E(n, h) \in I_E$ are prepared with equal probability. From $22$, and $40$ we have

$$\sum_{m,n} |q_{mn}|^2 = P(E', h'|E, h)D(E, h)\Delta^2,$$  \hspace{1cm} (41)

from which we can evaluate $\sum_{m,n} |q_{mn}|^2$ if $P(E', h'|E, h)$ is determined.

Let us fix $P(E', h'|E, h)$. By employing the definition $40$ and recalling $6$, we can find a reasonable form of $P(E', h'|E, h)$ in terms of

$$\Delta S \equiv S(E', h') - S(E, h).$$  \hspace{1cm} (42)

We first show in quantum statistical mechanics that, for a given $E$ and $77$, the most probable transition $E \rightarrow E'$, which maximizes $\log P(E', h'|E, h)$, satisfies

$$\Delta S_* = \frac{1}{2} Na(E_{Ms}, h_M)(\Delta h)^2,$$  \hspace{1cm} (43)

where $E_{Ms} \equiv (E'_s + E)/2$, $h_M \equiv (h' + h)/2$, and $Na\beta^{-1}$ turns out to be the adiabatic susceptibility $17$. See Appendix $B$ for the derivation. Then, by using $40$ and expanding $\log P(E', h'|E, h)$ up to the second order of $\Delta S$, we can express $P(E', h'|E, h)$ as

$$P(E', h'|E, h) = e^{-\frac{1}{2N a(\Delta h)^2}(\Delta S - \frac{1}{2} Na(\Delta h)^2)^2 + o(N)},$$  \hspace{1cm} (44)

which is derived in Appendix $C$

D. Final expression

By combining $39$, $41$ and $44$, we get the final expression of the propagator

$$B_{j+1} = \Delta^2 e^{-\frac{1}{2N a_j(\Delta h_j^2)}(\Delta S_{j+1} - \frac{1}{2} Na_{j+1}(\Delta h_{j+1})^2)^2 + o(N)}.$$  \hspace{1cm} (45)

By substituting this into $23$, we obtain the final expression of the path integral in the thermodynamic state space $(E, h)$:

$$|\Psi(t_f)\rangle = \left[ \prod_{j=1}^{M} \int D(h_j) \right] |\mathcal{P}_{E_M, h_M}\rangle_{t_f}$$

$$\int M e^{-\frac{i}{\hbar} E_{j-1} \Delta t - \frac{1}{4N a(\Delta h)^2}(\Delta S_j - \frac{1}{2} Na(\Delta h_j)^2)^2 + o(N)}.$$  \hspace{1cm} (46)

We call this a thermodynamical path integral. This formula can be applied to a class of non-equilibrium processes which are consistent with the two assumptions.

Apparently, $|\mathcal{P}_{E_M, h_M}\rangle_{t_f}$ depends on path $(E_1, \cdots, E_{M-1})$. More precisely, $48$ indicates that $\log \langle |n, h_M|\mathcal{P}_{E_M, h_M}\rangle_{t_f}^2$ is independent of paths when $o(N)$ contribution is ignored, while $\psi \equiv \text{Arg} \langle |n, h_M|\mathcal{P}_{E_M, h_M}\rangle_{t_f}$ may be path-dependent. With the choice of $23$, the phase shift of $o(N)$ occurs at each time step. We thus assume that the phase $\psi$ is expressed as a function of $(E_1/N, \cdots, E_{M-1}/N)$. Then, in the large $N$ limit, the dominant contribution of the path integral $46$ may be estimated from the saddle point of

$$\sum_{j=1}^{M} \left[ -\frac{i}{\hbar} E_{j-1} \Delta t - \frac{1}{4N a(\Delta h_j)^2}(\Delta S_j - \frac{1}{2} Na(\Delta h_j)^2)^2 \right].$$  \hspace{1cm} (47)

In the following, we analyze only these terms as the dominant contribution $O(N)$.

IV. EMERGENT SYMMETRY IN QUASI-STATIC OPERATIONS

It would be difficult to take a continuum limit of the discretized expression of the path integral $40$. In this section, by introducing a variable $\theta$ and considering slow protocols referred to as quasi-static operations, we construct a continuum expression. Then, we derive an effective action in a thermodynamical phase space and find that a symmetry emerges in the path integral, which leads to the entropy conservation. Finally, we discuss characterizations of the variable $\theta$.

A. Continuous limit in quasi-static operations

First, we introduce a dimensionless variable $\theta$ through

$$e^{-\frac{1}{4N a(\Delta h)^2}(\Delta S - \frac{1}{2} Na(\Delta h)^2)^2} = \int d\theta e^{-\frac{Na(\Delta h)^2}{2} - i\theta(\Delta S - \frac{1}{2} Na(\Delta h)^2)^2 + o(N)}.$$  \hspace{1cm} (48)

Then, by substituting this into $46$, we obtain

$$|\Psi(t_f)\rangle = \int DE \int D\theta \left[ \mathcal{P}_{E_M, h_M}\right]_{t_f} e^{i\mathcal{J} + \frac{1}{2} \mathcal{L}_{\text{eff}}} \hspace{1cm} (49)$$
with
\[ J = \sum_{j=1}^{M} \left[ -N a_j (\Delta h_j)^2 \theta_j^2 + \alpha(N) \right], \tag{50} \]
\[ \mathcal{I}_{\text{eff}} = \sum_{j=1}^{M} \left[ -E_{j-1} \Delta t - h \theta_j \left( \Delta S_j - \frac{1}{2} N a_j (\Delta h_j)^2 \right) \right], \tag{51} \]
where \( \int \mathcal{D}E \int \mathcal{D}\theta = \prod_{j=1}^{M} \int dE_j \int d\theta_j \), \( \mathcal{J} \) determines the amplitude and \( \mathcal{I}_{\text{eff}} \) is the effective action for \( (E(t), \theta(t)) \).

Next, we define quasi-static operations. We consider \( 1 \ll M \ll \sqrt{N} \) with \( M \Delta t = t_f \) fixed. For simplicity, we assume that \( h_j \) increases monotonically, i.e., \( \Delta h_j / h_j = O(1/M) \). Then, from (50), \( 1/\sqrt{N} \ll \Delta h_j / h_j \ll 1 \) holds. For this \( h(t) \), we attempt to construct the quasi-static operation \( h^*(t) \) such that \( h^*(t) = h(t) \) is satisfied for \( 0 \leq t \leq t_f = \epsilon \), where \( \epsilon \) is a small dimensionless parameter that characterizes the slowness of the operation. We define the discrete protocol as \( h^*_j = \epsilon \left( 1 - \epsilon + [\epsilon j] \right) h_{[\epsilon j]} + (\epsilon - [\epsilon j]) h_{[\epsilon j]+1} \). \tag{52} 

Here, \( [x] \) represents the largest integer less than or equal to \( x \in \mathbb{R} \). Indeed, this \( h^*_j(t) \) satisfies \( h^*_{j+1} - h^*_j = \epsilon [\epsilon j+1 - h_{[\epsilon j]}] \), which implies that \( h^*_j \) changes slower than \( h_j \) by the factor \( \epsilon \), that is, \( h^*_j(t) = h(t) \) in the continuous limit. Note that, in order to use the formula (53), the condition \( 1/\sqrt{N} \ll \Delta h^*_j / h^*_j \ll 1 \) needs to be satisfied; this leads to \( 1 \ll M^* \ll \sqrt{N} \). Because of this condition and \( 1 \ll M \ll \sqrt{N} \), \( \epsilon \) should be small but finite so that \( M / \sqrt{N} \ll \epsilon \ll 1 \).

Let us take the continuous limit of the path integral (50) for such monotonically increasing protocol \( (h^*_j)_{j=1}^{M} \). Because \( \Delta h^*_j / h^*_j = O(1/M^*) \), \( J = \sum_{j=1}^{M} N a_j \Delta h_j^2 \theta_j^2 \) is estimated as \( O(N/M^*) = O(\epsilon N / M) \). It becomes smaller as \( \epsilon \) is decreased with \( N \) and \( M \) fixed. Therefore, \( J \) can be neglected for the quasi-static operations. Similarly, the third term of (51) is negligible. Thus, the path integral (51) becomes
\[ \Psi(t_f') = \int \mathcal{D}E \int \mathcal{D}\theta \left| \mathcal{P}_{E_M^*, h_M^*} \right| \left| t_f' \right| e^{i \varphi \mathcal{I}_{\text{eff}}} \tag{53} \]
with the effective action of \( (E(t), \theta(t)) \):
\[ \mathcal{I}_{\text{eff}} = \int_{0}^{t_f} dt \left[ -E(t) - \theta(t) \frac{dS(E(t), h^*(t))}{dt} \right]. \tag{54} \]

B. Emergent symmetry and entropy conservation

If we transform the integral variable as
\[ \theta(t) \rightarrow \theta(t) + \eta, \tag{55} \]
where \( \eta \) is a small parameter, (53) becomes
\[ \left| \Psi(t_f') \right| = \int \mathcal{D}E \int \mathcal{D}\theta \left| \mathcal{P}_{E_M^*, h_M^*} \right| \left| t_f' \right| e^{i \varphi \mathcal{I}_{\text{eff}}} \tag{56} \]
This means that the symmetry for (55) emerges in the path integral (53) for quasi-static operations.

Let us find the conservation law which is connected to this symmetry by the quantum-mechanical Noether theorem. We first introduce the entropy operator by
\[ \hat{S}(h) \equiv \log D(\hat{H}(h), h) = \sum_{n} \log D(E(n, h), h) |n, h \rangle \langle n, h|. \tag{57} \]
Then, we calculate
\[ \hat{S}(h^*_M) \left| \mathcal{P}_{E_M^*, h_M^*} \right| t_f' \]
\[ = \sum_{n} \log D(E(n, h^*_M), h^*_M) \times \chi_{I_{E_M^*}}(E(n, h^*_M)) d_{n,M} \left( E_M^*, h^*_M \right) |n, h^*_M, \rangle \]
\[ - \log D(E_M^*, h^*_M) \sum_{n} \chi_{I_{E_M^*}}(E(n, h^*_M)) \]
\[ \times d_{n,M} \left( E_M^*, h^*_M \right) |n, h^*_M, \rangle + o(N), \]
\[ = S(E_M^*, h^*_M) \left| \mathcal{P}_{E_M^*, h_M^*} \right| t_f' + o(N), \tag{58} \]
where we have used (28) and employed \( D(E(n, h), h) = D(E, h) + \partial D / \partial E \big| E \big| E(n, h) - E \big| = D(E, h)(1 + \beta(E, h) O(\Delta)) \) because \( E(n, h) \in I_E \) and \( \beta \equiv \partial \log D / \partial E \). Differentiating (55) with respect to \( \eta \) and setting \( \eta = 0 \), we obtain
\[ 0 = \int \mathcal{D}E \int \mathcal{D}\theta \left| \mathcal{P}_{E_M^*, h_M^*} \right| \left| t_f' \right| \left( S_M^* - S_0 \right) e^{i \varphi \mathcal{I}_{\text{eff}}} \tag{59} \]
\[ = \int \mathcal{D}E \int \mathcal{D}\theta e^{i \varphi \mathcal{I}_{\text{eff}}} \hat{S}(h^*_M) \left| \mathcal{P}_{E_M^*, h_M^*} \right| \left| t_f' \right| - S_0 |\Psi(t_f')\rangle + o(N), \]
where we have used the fact that \( S_0 \) is independent of the integration, and we have employed (53). By multiplying this by \( \langle \Psi(t_f') \rangle \) and noting \( \langle \Psi(0) \rangle = |n_0, h_0 \rangle \), we have
\[ \langle \Psi(t_f') \rangle \hat{S}(h^*_M) \langle \Psi(t_f') \rangle = \langle \Psi(0) \rangle \hat{S}(h^*_M) \langle \Psi(0) \rangle + o(N). \tag{60} \]
Thus, the expectation value of the entropy operator is conserved for the quasi-static operations.

C. Characterization of \( \theta \)

We study a mathematical structure of the effective action (54), which gives characterizations of the variable \( \theta \). First, \( E \) has one-to-one correspondence with \( S \) through the thermodynamic relation \( S = S(E, h) \) for a given \( h \).
We can choose $S(t)$ as an independent variable instead of $E(t)$. In this representation, (54) is expressed as

$$I_{\text{eff}} = \int_0^{t_f} dt \left[ -E(S(t), h'(t)) - \hbar \theta(t) \frac{dS(t)}{dt} \right],$$

where $h'(t)$ is a given time-dependent parameter. This can be seen as a canonical-form action with Hamiltonian $E(S(t), h'(t))$ and canonical variables $(S(t), \hbar \theta(t))$. Indeed, the symplectic structure $d(h \theta) \wedge dS$ can be obtained by taking the exterior derivative of the surface term in a general variation of (61):

$$\delta I_{\text{eff}} = \int_0^{t_f} dt \left[ -\beta^{-1} \frac{d\theta}{dt} \delta S - \frac{dS}{dt} \delta (h \theta) - \hbar \theta \delta S \right].$$

We can see that $\delta I_{\text{eff}} = 0$ is equivalent to the equations

$$\frac{d\theta}{dt} = \frac{1}{\hbar \beta \theta(S(t), h'(t))},$$
$$\frac{dS(t)}{dt} = 0,$$

with the boundary conditions $h \theta = 0$ or $\delta S = 0$ at $t = 0$ and $t = t_f$. Note here that the energy (or $S$) at $t = 0$ is fixed for the system we are considering. However, we cannot impose the energy at $t = t_f$, because there is generically no solutions for this condition. We thus impose $\theta = 0$ at $t = t_f$, which is possible because of the symmetry property (55). Since the initial energy is fixed, (61) may be called the microcanonical effective action in the thermodynamical phase space $(S, h \theta)$. In the view of (61), $h \theta$ is the canonically-conjugate variable to the entropy $S$. Previously, such a variable was referred to as thermacy [48], and effective actions for perfect fluids were constructed without microscopic derivation [57, 58]. Indeed, our action (61) takes the similar form as the previous ones for the spatially homogeneous cases; however, in these studies, the Planck constant does not appear, and $(d h \theta/d t) S$ is included instead of $h \delta (d S/d t)$ since $\theta$ is fixed at the both boundaries $t = 0$ and $t = t_f$. This effective action describes a different physical situation from ours [61]. Note also that (61) is derived from quantum mechanics.

Next, we discuss the concept of thermal time $\tau$, a dimensionless quantity that parameterizes the flow generated by $-\log \hat{\rho}$ with a statistical state $\hat{\rho}$ [32, 33]. In particular, $\tau$ is determined by

$$\frac{d \hat{A}}{d \tau} = [\hat{A}, -\log \hat{\rho}] / i$$

for Heisenberg operators $\hat{A}$ satisfying $\frac{d \hat{A}}{d t} = \frac{1}{i\hbar}[\hat{A}, \hat{H}]$. When $\hat{\rho} = e^{-\beta \hat{H}} / Z$,

$$\frac{d \hat{A}}{d \tau} = \hbar \beta \frac{d \hat{A}}{d t}$$

holds. Comparing this with (63) implies that $\theta$ corresponds to the thermal time.

Finally, expressing (65) as $dt = \hbar \beta d \theta$, we find that the symmetry for $\theta \to \theta + \eta$, (55), is equivalent to that for

$$t \to t + \eta \hbar \beta,$$

which appears in a different analysis of classical systems [30].

V. CONCLUDING REMARKS

Before ending this paper, we present a few remarks. First, as a working hypothesis for obtaining the thermodynamical path integral [16], we made the two assumptions in Sec. [11, 19]. The validity of these needs to be checked by applying them to various specific models and studying their properties more. We postpone this task.

Second, in order to evaluate physical quantities, we have to perform the integration of $E_j$ and $\theta_j$ in [19]. Here, considering that each term of $J$ and $I_{\text{eff}}$ is $O(N)$, one may employ a saddle point method with the analytic continuation of $J + i I_{\text{eff}} / \hbar$ for complex variables $E_j$ and $\theta_j$. One can then estimate the integral [49] for specific models and directly confirm the symmetry. Furthermore, it is an important future problem to study how entropy is not conserved for fast protocols through the saddle point estimation of $J + i I_{\text{eff}} / \hbar$.

Third, we remark on the quantum adiabatic theorem: the amplitude in each energy level remains constant (and thus $S$ is kept constant) if the operation speed is sufficiently slow [30, 31]. For excited states of nonintegrable many-body systems, such a speed becomes extraordinarily slow, which is $e^{-O(N)}$. The reason is as follows. Since the number of states in an energy shell is $e^{O(N)}$, the distances between neighboring energy levels are $e^{-O(N)}$. The operation time of the quantum adiabatic theorem in many-body system is much slower than the lifetime of the universe. Therefore, the quantum adiabatic theorem is insufficient to prove the second law of thermodynamics. In our theory, by contrast, the operation speed is so fast that transitions between different energy levels occur. Nevertheless, the entropy is conserved in [60] under such operations. It is a natural question how to unify the two theories.

Finally, we hope that experiments will be conducted to verify our theory. In particular, if one observes an entropic effect of the effective action, the measurement result is quite interesting. One of the most promising ways is an interference experiment. In the path-integral formulation, the action in the path integral represents the phase along the corresponding path. We thus expect that paths with different entropies (with the same energy) exhibit the interference pattern, which is predictable by our theory. In the future, we will propose a design of experiments for this observation.
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Appendix A: Proof of (35)

We prove (35) for any \((n, j)\). First, the precise statement of (35) is expressed as a probability:

\[
\lim_{N \to \infty} \text{Prob}(|\log |C|^2 - \log \langle |C|^2 \rangle|/N \geq \epsilon) = 0 \tag{A1}
\]

for any \(\epsilon > 0\), where we express \(C_{nj}\) as \(C\). To prove this, for \(X \equiv \log |C|^2\), we first show that

\[
\bar{X} = \log \langle |C|^2 \rangle + o(N), \tag{A2}
\]

\[
\bar{X}^2 - X^2 = o(N^2), \tag{A3}
\]

and then use Chebyshev’s inequality.

The strategy to show (A2) and (A3) is to use

\[
\bar{X} = \frac{\partial \langle |C|^{2K} \rangle}{\partial K} \bigg|_{K=0}.
\]

This is obtained by recalling \(d^l(a^x)/dx^l = a^x(\log a)^l\), setting \(a = |C|^2\) and \(x = K\), and taking the expectation value. Let us estimate \(\langle |C|^{2K} \rangle\). For \(K = 2\), by direct calculation, we confirm

\[
\langle |C|^2 \rangle = 2 \langle |C|^2 \rangle^2. \tag{A5}
\]

For general \(K\), we have

\[
\langle |C|^{2K} \rangle = K! \langle |C|^2 \rangle^K. \tag{A6}
\]

Thus, noting \(\log |C|^2 = O(N)\), we have

\[
\bar{X} = \frac{\partial \langle |C|^{2K} \rangle}{\partial K} \bigg|_{K=0} = \log \langle |C|^2 \rangle + o(N), \tag{A7}
\]

and

\[
\bar{X}^2 = \frac{\partial^2 \langle |C|^{2K} \rangle}{\partial K^2} \bigg|_{K=0} = \left(\log \langle |C|^2 \rangle\right)^2 + o(N^2), \tag{A8}
\]

for large \(N\). From these results, we obtain (A2) and (A3).

We thus conclude (A1).

Appendix B: Derivation of (43)

We start with the entropy defined by \(S(E, h) \equiv \log D(E, h)\) and consider the most probable value of the entropy change. \(\Delta S_\ast\), for a small parameter change \(h \rightarrow h + \Delta h\) with preparing an equilibrium state initially. We then show in the framework of quantum statistical mechanics that \(\Delta S_\ast\) is given by (43):

\[
\Delta S_\ast = \frac{1}{2} N a(\Delta h)^2, \tag{B1}
\]

where \(a\) is a non-negative intensive quantity. In particular, when the Hamiltonian is a linear function of \(h\), \(a\) is expressed in terms of the adiabatic susceptibility. This corresponds to (C8).

We begin with the setup. For any operator \(\hat{A}\), we define the expectation value with respect to the microcanonical ensemble by

\[
\langle \hat{A} \rangle^{mc}_{E,h} \equiv \frac{\sum_n \chi I_{E}(E(n, h), n)}{\sum_n \chi I_{E}(E(n, h))} \langle n, h \mid \hat{A} \mid n, h \rangle. \tag{B2}
\]

Let us define

\[
\hat{M}(h) \equiv -\frac{\partial \hat{H}(h)}{\partial h}, \tag{B3}
\]

and set

\[
M(E, h) = \langle \hat{M}(h) \rangle^{mc}_{E,h}. \tag{B4}
\]

This means that we consider \(\langle \hat{M}(h) \rangle^{mc}_{E,h}\) as the thermodynamic value of magnetization \(M(E, h)\). Indeed, we can show in quantum statistical mechanics that

\[
\frac{\partial S}{\partial h} \bigg|_{E} = \beta M, \tag{B5}
\]

which, together with the definition of \(\beta\), leads to

\[
dS = \beta dE + \beta M dh. \tag{B6}
\]

The proof of (B5) is given in the argument below (B10).

The most probable value of the energy change \(\Delta E_\ast\) for the small parameter change \(h \rightarrow h + \Delta h\) with preparing an equilibrium state initially is given by the expectation value of \(\hat{H}(h + \Delta h) - \hat{H}(h)\) with respect to the initial equilibrium state:

\[
\Delta E_\ast = \langle \hat{H}(h + \Delta h) \rangle^{mc}_{E,h} - \langle \hat{H}(h) \rangle^{mc}_{E,h} = -M(\Delta h) + \frac{1}{2} \frac{\partial^2 \hat{H}}{\partial h^2} \bigg|_{E,h}^{mc} (\Delta h)^2 + O((\Delta h)^3). \tag{B7}
\]

For this \(\Delta E_\ast\), we consider the entropy change \(\Delta S_\ast \equiv S(E + \Delta E_\ast, h + \Delta h) - S(E, h)\) and expand it in \(\Delta E_\ast\) and
$\Delta h$. Then, using (B17), we have

$$\Delta S_\ast \equiv \frac{\partial S}{\partial h} \Delta E_\ast + \left( \frac{\partial S}{\partial E} \right)_h \Delta h$$

$$+ \frac{1}{2} \left[ \frac{\partial^2 S}{\partial E^2} \right]_h (\Delta E_\ast)^2 + \left( \frac{\partial^2 S}{\partial h^2} \right)_E (\Delta h)^2$$

$$+ \frac{\partial^2 S}{\partial E \partial h} (\Delta E_\ast)(\Delta h) + O((\Delta h)^3)$$

$$= \frac{1}{2} \{ \Delta h \}^2 + O((\Delta h)^3), \quad (B8)$$

where

$$\{ \Delta h \} = M^2 \left( \frac{\partial^2 S}{\partial E^2} \right)_h - 2M \frac{\partial^2 S}{\partial E \partial h} + \left( \frac{\partial^2 S}{\partial h^2} \right)_E$$

+ $\beta \left( \frac{\partial^4 H}{\partial h^4} \right)$.

From now, we express $\beta$ in terms of experimentally measurable quantities. We start with the identity

$$\beta \frac{\partial M}{\partial h}_E = \left( \frac{\partial \beta M}{\partial h} \right)_E,$$

$$\left( \frac{\partial^2 S}{\partial h^2} \right)_E - M \frac{\partial^2 S}{\partial h \partial E} \quad (B9)$$

Here, we notice

$$\beta \frac{\partial M}{\partial h} = \left( \frac{\partial \beta M}{\partial h} \right)_E,$$

$$\left( \frac{\partial^2 S}{\partial h^2} \right)_E - M \frac{\partial^2 S}{\partial h \partial E} \quad (B10)$$

and

$$\beta \frac{\partial M}{\partial E}_h = \left( \frac{\partial \beta M}{\partial E} \right)_h,$$

$$\frac{\partial^2 S}{\partial E \partial h} - M \frac{\partial^2 S}{\partial E^2} \quad (B11)$$

where we have used (B6). We substitute (B10) and (B11) into (B10), compare the result with (B9), and then find

$$\{ \Delta h \} = M^2 \left( \frac{\partial^2 S}{\partial E^2} \right)_h - 2M \frac{\partial^2 S}{\partial E \partial h} + \left( \frac{\partial^2 S}{\partial h^2} \right)_E$$

$$+ \beta \left( \frac{\partial^4 H}{\partial h^4} \right) \quad (B12)$$

Note that $a \geq 0$ holds because

$$\{ \Delta h \} \geq \beta \sum_n \chi_{IE}(E(n, h)) | \langle n, h | (\hat{M}(h) - M) | n, h \rangle |^2$$

$$\geq 0, \quad (B13)$$

which is shown in Sec. (B2).

Finally, we consider the case where the Hamiltonian is a linear function of $h$ (as studied in many examples in statistical mechanics). Then, since the second term in the right-hand side of (B13) vanishes, $Na\beta^{-1}$ is the adiabatic susceptibility:

$$Na\beta^{-1} = \left( \frac{\partial M}{\partial h} \right)_S = - \left( \frac{\partial^2 E}{\partial h^2} \right)_S, \quad (B14)$$

where we have used (B6) at the last equality. Following a standard assumption for statistical mechanical models, we assume that Hamiltonians we study lead to the concavity of $E(S, h)$ in $h$, and then we conclude again that $a \geq 0$.

1. Proof of (B5)

Let $\Omega(E, h)$ be the number of eigenstates whose eigenvalues are less than $E$ for Hamiltonian $\hat{H}(h)$. That is,

$$\Omega(E, h) = \sum_n \chi(E > E(n, h)), \quad (B15)$$

where $\chi(X) = 1$ if $X$ holds and $\chi(X) = 0$ otherwise. From this definition, we obtain

$$\frac{\Omega(E, h + \Delta h) - \Omega(E, h)}{\Omega(E, h)}$$

$$= \frac{1}{\Omega} \left[ \sum_n \chi(E(n, h + \Delta h) < E < E(n, h)) - \sum_n \chi(E(n, h) < E < E(n, h + \Delta h)) \right]. \quad (B16)$$

For a small $\Delta h$, the right-hand side can be evaluated as

$$\frac{\partial E(n, h)}{\partial h} \left|_{E(n, h) \in IE} \right. \left( \Delta h \right) = o(N) \quad (B17)$$

where is independent of $n$ satisfying $E(n, h) \in IE$. Since the typical value of $\partial E(n, h)/\partial h$ in the energy shell $IE$ may be replaced by the expectation value with respect to the microcanonical ensemble, we have

$$\frac{\partial E(n, h)}{\partial h} \left|_{E(n, h) \in IE} \right. = \frac{\sum_n \chi_{IE}(E(n, h))}{\sum_n \chi_{IE}(E(n, h))} \chi_{IE}(E(n, h)) + o(N). \quad (B18)$$

By combining this with the identity

$$\langle n, h | \frac{\partial \hat{H}(h)}{\partial h} | n, h \rangle = \frac{\partial E(n, h)}{\partial h}, \quad (B19)$$

we obtain

$$\frac{\partial E(n, h)}{\partial h} \left|_{E(n, h) \in IE} \right. = -M(E, h) + o(N). \quad (B20)$$
Thus, \( (B18) \) becomes
\[
\frac{D(E,h)}{\Omega(E,h)} M(E,h)(\Delta h) + o(N). \tag{B22}
\]
By recalling \( S(E,h) \equiv \log D(E,h) = \log \Omega(E,h) + o(N) \) and \( \beta(E,h) = D(E,h)/\Omega(E,h) \), we can re-express \( (B17) \) as
\[
\frac{\partial S(E,h)}{\partial h} = \beta(E,h)M(E,h) \tag{B23}
\]
which is \( (B5) \).

2. proof of \( (B14) \)

We fix \( (E,h) \). For a given small \( \Delta h \), we choose \( \Delta E \) such that \( S(E,h) = S(E + \Delta E, h + \Delta h) \). This means
\[
\Delta E + M \Delta h = O((\Delta h)^2). \tag{B24}
\]
For this \( \Delta E \), we can have
\[
\left( \frac{\partial M}{\partial h} \right)_S \Delta h = \left\langle \tilde{M}(h + \Delta h) \right\rangle_{E + \Delta E, h + \Delta h}^{\text{mc}} - \left\langle \tilde{M}(h) \right\rangle_{E,h}^{\text{mc}} + O((\Delta h)^2). \tag{B25}
\]
From this and \( (B13) \), we have
\[
N a \beta^{-1} \Delta h = \left\langle \tilde{M}(h + \Delta h) \right\rangle_{E + \Delta E, h + \Delta h}^{\text{mc}} - \left\langle \tilde{M}(h) \right\rangle_{E,h}^{\text{mc}} - \left\langle \frac{\partial \tilde{M}}{\partial h} \right\rangle_{E,h}^{\text{mc}} \Delta h + O((\Delta h)^2),
\]
\[
= \left\langle \tilde{M}(h) \right\rangle_{E + \Delta E, h + \Delta h}^{\text{mc}} - \left\langle \tilde{M}(h) \right\rangle_{E,h}^{\text{mc}} + O((\Delta h)^2). \tag{B26}
\]
Now, we recall \( (B2) \) and re-express it as
\[
\left\langle \tilde{A} \right\rangle_{E,h}^{\text{mc}} = \frac{\sum_n \chi(E(n,h) < E) \left\langle n,h \right| \tilde{A} \left| n,h \right\rangle}{\Omega(E,h)} + o(N) \tag{B27}
\]
for any extensive variable \( \tilde{A} \). We start with
\[
\left\langle \tilde{M}(h) \right\rangle_{E + \Delta E, h + \Delta h}^{\text{mc}} = \sum_n \frac{\chi(E(n,h + \Delta h) < E + \Delta E)}{\Omega(E + \Delta E, h + \Delta h)} \langle n,h + \Delta h | \tilde{M}(h) | n,h + \Delta h \rangle + o(N). \tag{B28}
\]
We then have
\[
\left\langle \tilde{M}(h) \right\rangle_{E + \Delta E, h + \Delta h}^{\text{mc}} - \left\langle \tilde{M}(h) \right\rangle_{E,h}^{\text{mc}} = (\Delta M)_1 + (\Delta M)_2 + (\Delta M)_3 + o(N), \tag{B29}
\]
where
\[
\Omega(E,h)(\Delta M)_1 = \sum_n \left[ \chi(E(n,h) < E + \Delta E - \left( \frac{\partial E(n,h)}{\partial h} \right) \Delta h \right. \right.
\]
\[
- \chi(E(n,h) < E) \langle n,h | \tilde{M}(h) | n,h \rangle, \tag{B30}
\]
\[
\Omega(E,h)(\Delta M)_2 = \sum_n \chi(E(n,h) < E) \left[ \langle n,h + \Delta h | \tilde{M}(h) | n,h + \Delta h \rangle 
\]
\[
- \langle n,h | \tilde{M}(h) | n,h \rangle \right], \tag{B31}
\]
\[
\Omega(E,h)(\Delta M)_3 = - \sum_n \chi(E(n,h) < E)
\]
\[
\times \langle n,h | \tilde{M}(h) | n,h \rangle \left[ \frac{\partial \Omega}{\partial E} \Delta E + \frac{\partial \Omega}{\partial h} \Delta h \right]. \tag{B32}
\]
We first see
\[
(\Delta M)_3 = - \beta M \Delta E - \beta M^2 \Delta h, \tag{B33}
\]
and find that \( (\Delta M)_3 = 0 \) for \( \Delta E \) satisfying \( (B24) \). We then calculate \( (\Delta M)_1 \) as
\[
(\Delta M)_1 = \frac{1}{\Delta} \sum_n \frac{\chi_{E,n}(E(n,h))}{\Omega(E,h)} \times \left[ \Delta E - \frac{\partial E(n,h)}{\partial h} \Delta h \right] \langle n,h | \tilde{M}(h) | n,h \rangle
\]
\[
= \frac{D(E,h)}{\Omega(E,h)} \left[ M \Delta E + \sum_n \chi_{E,n}(E(n,h))(\langle n,h | \tilde{M}(h) | n,h \rangle)^2 \Delta h \right]
\]
\[
= \beta \Delta h \left[ - M^2 
\]
\[
+ \frac{\sum_n \chi_{E,n}(E(n,h))(\langle n,h | \tilde{M}(h) | n,h \rangle)^2}{D(E,h) \Delta} \right]. \tag{B34}
\]
where we have used \( (B24) \). Next, in order to evaluate \( (\Delta M)_2 \), we consider
\[
\langle n,h + \Delta h | \tilde{M}(h) | n,h + \Delta h \rangle - \langle n,h | \tilde{M}(h) | n,h \rangle
\]
\[
= \langle n,h | \tilde{M}(h) \frac{d}{dh} \rangle | n,h \rangle \Delta h + \langle \text{c.c.} \rangle
\]
\[
+ O((\Delta h)^2). \tag{B35}
\]
Noting that
\[
\langle m,h \rangle \left| \frac{d}{dh} \right| n,h \rangle = \begin{cases} 
\frac{1}{E(m,h) - E(n,h)} & \text{for } m \neq n, \\
0 & \text{for } m = n,
\end{cases} \tag{B36}
\]
where \( \theta(n, h) \) is a real number, we have
\[
\langle n, h | \hat{M}(h) \frac{d}{dh} | n, h \rangle = \sum_{m, m \neq n} | \langle n, h | \hat{M} | m, h \rangle |^2 \frac{E(m, h) - E(n, h)}{E(m, h) - E(n, h)} - i \frac{\partial E(n, h)}{\partial h} \theta(n, h).
\]
We thus obtain
\[
\Omega(E, h)(\Delta M)_2 = 2 \sum_{nm:n \neq m} \chi(E(n, h) < E) \frac{\langle m, h | \hat{M} \rangle | n, h \rangle^2}{E(m, h) - E(n, h)} \Delta h,
\]
where the contribution \( \sum_m \chi(E(m, h) < E) \) vanishes from the symmetry for the exchange of \( n \) and \( m \). Thus, from (B34) and (B38), we express (B29) as
\[
\Omega(E, h)(\Delta M)_2 = 2 \sum_{nm:n \neq m} \chi(E(n, h) < E) \chi(E(m, h) > E) \Delta h, \tag{B37}
\]
where (C11) has been introduced so that (C2) is respected.

From the symmetry property
\[
P(E', h'|E, h)D(E, h) = P(E, h'E, h')D(E', h'), \tag{C4}
\]
we can determine
\[
\phi_A(E', h'|E, h) = \frac{\Delta S}{2}. \tag{C5}
\]

Next we consider \( \phi_S(E', h'|E, h) \). From (B6) and the physical interpretation of \( \phi_A \), we find that the probability of large \( |E' - E| \) is small. Noting that for a given \( h \), \( E \) has one-to-one correspondence with \( S \) through the thermodynamic relation \( S = S(E, h) \) and seeing (C5), we expand \( \phi_S(E', h'|E, h) \) with respect to \( \Delta S \), instead of \( \Delta E \equiv E' - E \). Therefore, we ignore contribution of \( \Delta S^4 \) and higher order terms and write
\[
\phi_S(E', h'|E, h) = N f_0(\Delta h; E_M, h_M) + \frac{1}{N} f_2(\Delta h; E_M, h_M)(\Delta S)^2 + o(N^{-6}).
\]

Appendix C: Derivation of (44)

We first decompose \( \log P(E', h'|E, h) \) into
\[
\log P(E', h'|E, h) = \phi_S(E', h'|E, h) + \phi_A(E', h'|E, h), \tag{C1}
\]
with
\[
\phi_S(E', h'|E, h) = \phi_S(E, h'E, h'), \tag{C2}
\phi_A(E', h'|E, h) = -\phi_A(E, h'E, h'). \tag{C3}
\]
By recalling (B26), we arrive at
\[
\frac{\partial \log P(E', h'|E, h)}{\partial E'}|_{E'=E'} = 0. \tag{C7}
\]
Through (C3) and (C6), we obtain
\[
\frac{\partial^2 f_2}{\partial E_M^2} = 0, \tag{C8}
\]
where \( \beta' = \beta(E', h') \) and \( \star \) represents the evaluation at \( E' = E_*' \). Here, suppose that \( f_0 = O((\Delta h)^{\alpha} \) and \( f_2 = O((\Delta h)^{\alpha_2} \) for small \( \Delta h/h \). Then, the first, second, third, and fourth term of (C8) have the \( \Delta h \) dependence as \((\Delta h)^{\alpha}, (\Delta h)^{\alpha_2}, (\Delta h)^{\alpha_0}, \) and \((\Delta h)^{4+\alpha_2} \), respectively. By assuming \( \alpha_0 \geq 2 \) (otherwise (C6) would become singular when \( \Delta h \rightarrow 0 \)), we obtain \( \alpha_0 = 2 \) and \( \alpha_2 = -2 \). This leads to each bracket in (C8) vanishes, respectively:
\[
f_{2\star} = -\frac{N}{4(\Delta S)_+} = -\frac{1}{2\alpha_{\star}(\Delta h)^2}, \tag{C9}
\frac{\partial f_0}{\partial E_M}|_\star = \frac{(\Delta S)_+^2}{N^2} \frac{\partial f_0}{\partial E_M}|_\star = -\frac{1}{8} \frac{\partial f_0}{\partial E_M}|_\star (\Delta h)^2, \tag{C10}
\]
where (B13) has been used. We thus set
\[
f_{2\Delta h; E_M, h_M} = -\frac{1}{2\alpha(E_M, h_M)(\Delta h)^2}, \tag{C11}
f_{0\Delta h; E_M, h_M} = -\frac{1}{8} \alpha(E_M, h_M)(\Delta h)^2. \tag{C12}
\]
From these and (C1), (C5) and (C6), we obtain (44).
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[45] When the Hamiltonian has some conserved quantities, the Hilbert space is separated by the eigenvalues of them. By looking at a sector of the conserved quantities, our theory becomes applicable to this case. However, when the number of the conserved quantities is $O(N)$, which is the case of the integrable model, the whole story changes. Integrable models are outside of the scope of this paper.

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