Finite size scaling of the correlation length above the upper critical dimension

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We show numerically that correlation length at the critical point in the five-dimensional Ising model varies with system size $L$ as $L^{5/4}$, rather than proportional to $L$ as in standard finite size scaling (FSS) theory. Our results confirm a hypothesis that FSS expressions in dimension $d$ greater than the upper critical dimension of 4 should have $L$ replaced by $L^{d/4}$ for cubic samples with periodic boundary conditions. We also investigate numerically the logarithmic corrections to FSS in $d = 4$.

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I. INTRODUCTION

Finite size scaling\textsuperscript{1,2} (FSS) has been extremely useful in extrapolating numerical results on finite systems in the vicinity of a critical point to the thermodynamic limit, in order to get information on critical singularities. The basic hypothesis of FSS is that the linear size of the system $L$ enters in the ratio $L/\xi_\infty$ where $\xi_\infty$ is the correlation length of the infinite system (which we will call the “bulk” correlation length for convenience) and which diverges as the critical temperature $T_c$ is approached like

$$\xi_\infty \approx c_0 t^{-\nu},$$

where

$$t \equiv \frac{T - T_c}{T_c}$$

measures the deviation from criticality. Here $c_0$ is a non universal “metric factor”\textsuperscript{3,4} and we use the symbol $\approx$ to signify “asymptotically equal to”. Hence, if a quantity $X$ diverges in the bulk like $t^{-y_\nu}$, the FSS form for the behavior of $X$ is

$$\frac{X}{X_0} \approx L^{y_\nu} \tilde{X} \left( \frac{L}{\xi_\infty} \right) \approx L^{y_\nu} \tilde{X} \left( c_1 L^{1/\nu} t \right),$$

where $X_0$ and $c_1$ are non-universal scale factors, and $\pm$ refers to $t \gtrless 0$. The scaling functions $P^\pm$ and $\tilde{X}$ are universal.\textsuperscript{5,6} In the last expression in Eq. (3) we have taken the argument of the function $P^\pm$ in the first expression to the power $1/\nu$, in order that temperature appears linearly. This has the advantage that a single smooth function $\tilde{X}$, applies both above and below $T_c$, whereas two functions $P^\pm$ are needed in the first expression in Eq. (3).

It is often convenient to consider dimensionless quantities, because these are expected to have $y_\nu = 0$. Two commonly studied examples are (i) the “Binder ratio”\textsuperscript{7,8},

$$g \equiv \frac{1}{2} \left( 3 - \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2} \right) \approx W^\pm \left( \frac{L}{\xi_\infty} \right) \approx \tilde{g} \left( c_1 L^{1/\nu} t \right),$$

where $m$ is the order parameter, and (ii) the ratio of the correlation length of the finite system $\xi_L$ to the system size\textsuperscript{3,5,6}

$$\frac{\xi_L}{L} \approx U^\pm \left( \frac{L}{\xi_\infty} \right) \approx \tilde{\xi} \left( c_1 L^{1/\nu} t \right).$$

The definition of $\xi_L$ is not unique (though any reasonable definition will give the same scaling form). We shall give one definition, which is often used in numerical work, in the next section. Again, the scaling functions, $W^\pm, \tilde{g}, U^\pm$ and $\tilde{\xi}$ are universal.

Note from Eqs. (4) and (5) that, for dimensionless quantities like $g$ and $\xi_L/L$, data for different sizes intersect at the critical temperature. Hence dimensionless quantities are very convenient because they locate the critical temperature in a simple way, from the crossing point, without needing to know the values of other quantities such as exponents. Furthermore, since the scaling functions $\tilde{g}(x)$ and $\tilde{\xi}(x)$ are universal the values of $g$ and $\xi_L/L$ at the crossing point (i.e. at $T_c$) are also universal.

Finite size scaling, as represented here by Eqs. (3)–(5), is expected to be valid in the limit $L \to \infty, t \to 0$, with $L^{1/\nu} t$ arbitrary. Originally proposed on phenomenological grounds, a justification for FSS was later provided by Brézin\textsuperscript{7} using renormalization group (RG) arguments, at least for the case of systems without disorder (which is the only case we discuss here). However, Brézin\textsuperscript{7} also noted that FSS breaks down at the “upper critical dimension” $d_u = 4$. For $d > 4$ the critical exponents are given by mean field theory, e.g. $\nu = 1/2$, and the corresponding field theory is a free theory (i.e. the fluctuations are Gaussian) since the effective coupling constant vanishes at long length scales. This coupling constant is irrelevant in the RG sense, but singularities occur when it tends to zero, and so it cannot simply be set to its “fixed point” value of zero. It is the singularities which come from this dangerous irrelevant variable that lead to a breakdown of the FSS expressions in Eqs. (3)–(5) for $d > 4$.

Nonetheless, since the bulk behavior for $d > 4$ is trivial, one might imagine that, in this limit, the size dependence can also be expressed in fairly simple way and this turns out to be the case. As seems to be at least implicit in much of the earlier work\textsuperscript{7,8,9,10,11} we make the hypothesis that, for cubic samples with periodic boundary conditions:
for $d > 4$, FSS formulae can still be applied but with the system size $L$ replaced by a larger length $A L^{d/4}$

$$\ell = A_1 L^{d/4}$$

where $A_1$ is non-universal.

We shall see that physically $\ell$ is the correlation length at the critical point. With this replacement, Eqs. (3)–(5) become (remember $d > 4$ here)

$$\frac{X}{X_0} \approx \nu^p \left( \frac{\ell}{\xi_\infty} \right) \approx L^{d/4} \tilde{X} \left( c_2 L^{d/2} t \right),$$

$$g \approx W^c \left( \frac{\ell}{\xi_\infty} \right) \approx \tilde{g} \left( c_2 L^{d/2} t \right),$$

$$\frac{\xi_L}{L} \approx U^c \left( \frac{\ell}{\xi_\infty} \right), \text{ i.e. } \frac{\xi_L}{L^{d/4}} \approx A_1 \tilde{\xi} \left( c_2 L^{d/2} t \right)$$

with $c_2$ non-universal, where we have noted that $\nu = 1/2$ for $d > 4$. As before, the scaling functions are universal, so the value of $g$ at the crossing point at $T_c$ is universal. Furthermore, this universal value has been calculated. We see that, at criticality, $\xi_L$ is of order $L^{d/4}$ which is much greater than $L$ for large sizes, a result which, at first, seems surprising. The value of $\xi_L/L^{d/4}$ at criticality, however, is non universal because of the factor of $A_1$ in Eq. (6). This factor occurs because $\ell$ has dimensions of length, and so, for Eq. (6) to be dimensionally correct, $A_1$ must be proportional to $a^{(4-d)/4}$, where $a$ is a microscopic length scale, e.g. the lattice spacing. Quantities involving microscopic length scales are not universal and so $A_1$ is not universal.

There has been extensive discussion as to whether Eq. (6) applies to the five-dimensional Ising model in the limits $L \to \infty, t \to 0$. Apparently it does not, though there appear to be several corrections to FSS which conspire to give a “crossing” for small sizes at a value of $g$ which differs from the calculated universal value.

As noted above, a surprising feature of Eq. (9) is that the correlation length of the finite system at the critical point is greater than the system size. To our knowledge there does not appear to have been any direct verification of this prediction for $d > 4$ by numerical simulations. In this paper, we confirm the prediction in Eq. (9) by Monte Carlo simulations on the five-dimensional Ising model. We also carry out similar simulations for the four-dimensional Ising model, for which logarithmic corrections to standard FSS are expected.

In Sec. II describe the model and some aspects of the simulations. The results in five dimensions are presented in Sec. III and the results in four dimension are presented in Sec. IV. We summarize our results in Sec. V.
FIG. 2: Data for $\xi L/L^{d/4}$ in $d = 5$. Clearly the data intersect close to a common point, as expected for the modified FSS expression in Eq. (9). The vertical line is at $T = 8.7785$ which is our best estimate for $T_c$.

is the Fourier transform of the spin-spin correlation function, and $k_{\text{min}} = (2\pi/L)(1,0,0)$ is the smallest non-zero wave vector on the lattice. Above $T_c$ and for $L \to \infty$, Eq. (12) gives the usual second moment definition of the correlation length.

We perform Monte Carlo simulations using the Wolff cluster algorithm to reduce the effects of critical slowing down.

III. RESULTS IN 5 DIMENSIONS

Data for $\xi L/L$ is shown in Fig. 1 for sizes $4 \leq L \leq 16$. According to standard FSS, Eq. (6), the data would intersect at a common point which is clearly not the case. However, according to the modified FSS expression in Eq. (9), it is data for $\xi L/L^{5/4}$ which should intersect at a common point, and Fig. 2 shows that this works pretty well. Fig. 2 therefore provides convincing evidence that the correlation length at the critical point varies as $L^{5/4}$ in 5 dimensions, rather than being proportional to $L$ as would be expected in standard FSS.

In Fig. 3 we show a plot for $\xi L/L$ (i.e. without the logarithmic factor). Clearly the data does not show a common intersection. However, including the logarithmic factor, the plot in Fig. 6 shows a good intersection with only small corrections to FSS. The factor $L \log L$ can be replaced by $\ln(L/L_0)$ where $L_0$ is a microscopic scale, and presumably because of corrections to FSS, the data scales well with $T_c = 8.7785$. By considering different choices for $T_c$ we estimate that $T_c = 8.7785(5)$, consistent with the more accurate result $8.77844(2)$ in Ref. [16].

For completeness we also show results for the Binder ratio in Fig. 4. As found in other work [8,13,14,15,16], the data for small sizes intersect at a value of $g$ larger than the predicted universal value of $0.4058 \cdots$. The data for larger sizes have intersections at somewhat smaller values and presumably would reach the universal value for $L \to \infty$.

IV. RESULTS IN 4 DIMENSIONS

In four dimensions, Brézin argued that $\xi L \propto L^{(d-4)/4}$ at criticality, and so we expect that FSS expressions should be modified by the replacement

$$L \to \ell = A_2L(\ln L)^{1/4} \quad (d = 4).$$

In Fig. 5 we show a plot for $\xi L/L$ (i.e. without the logarithmic factor). Clearly the data does not show a common intersection. However, including the logarithmic factor, the plot in Fig. 8 shows a good intersection with only small corrections to FSS. The factor $\ln L$ can be replaced by $\ln(L/L_0)$ where $L_0$ is a microscopic scale, and
FIG. 4: Data for the Binder ratio in $d = 5$. The vertical dashed line corresponds to $T = 8.7785$ which is our best estimate of $T_c$ from the correlation length data, see Fig. 3. The horizontal dashed line corresponds to $g = 0.4058 \cdots$, the predicted universal value.

FIG. 5: Data for $\xi_L/L$ in $d = 4$. According to conventional FSS, Eq. (3), the data should have a common intersection. This is clearly not the case.

FIG. 6: Data for $\xi_L/(L \ln L)^{1/4}$ in $d = 4$. The data intersect at close to a common point.

with an appropriate choice of $L_0$ we get sharper intersections. However, $\ln(L/L_0) = (\ln L)(1 + \ln L_0/\ln L)$ and so including $L_0$ corresponds to an additive correction to FSS (which vanishes only logarithmically). It is difficult to separate this from other corrections to FSS, and so we don’t feel we can give a reliable estimate for $L_0$.

V. CONCLUSIONS

We have demonstrated that the FSS behavior of the correlation length (for a cubic sample with periodic boundary conditions) in five dimensions follows Eq. (9), which is the expected modification of FSS for the case $d > 4$. This provides confirmation that the standard FSS expressions, e.g. Eqs. (3)–(5), can be simply modified above $d = 4$ by the replacement $L \rightarrow \ell \propto L^{d/4}$, which gives Eqs. (7)–(9). This had been verified before for the Binder ratio, but not, to our knowledge, for the correlation length. It is interesting that the correlation length at the critical point is of order $\ell$ and hence much bigger than the system size $L$. This is possible because the long wavelength fluctuations are non-interacting near criticality for $d > 4$. We also demonstrated the expected logarithmic modification to FSS of the correlation length for $d$ precisely equal to 4.

It is also interesting to ask what are the corresponding results with $d > 4$ for other geometries and boundary conditions. For the “strip” geometry, where the sample is infinite in one direction and of size $L$ in the others, Brézin showed that the correlation length at the critical point varies as $L^{(d-1)/2}$ (which is reasonable since FSS is
done only with respect to the \(d - 1\) finite dimensions). It is then natural to expect that FSS will then work with \(L\) replaced throughout by \(L^{(d-1)/3}\).

For free boundary conditions, it seems obvious that even for \(d > 4\) the behavior of the system will be affected when \(\xi_L\) becomes of order \(L\), rather than only change when \(\xiL\) becomes of order the much larger length \(\ell\). Hence we expect that the standard FSS expressions, Eqs. \(3\)–\(5\) would apply with \(\nu\) may also enter but, since \(L\), such terms would presumably be corrections to the scaling terms which involve \(\xiL/L\). Since FSS for models with free boundary conditions in \(d > 4\) is poorly understood, it would be interesting to investigate such models in some detail.

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