ASYMPTOTIC LOWER BOUND FOR THE RADIUS OF SPATIAL ANALYTICITY TO SOLUTIONS OF KDV EQUATION

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Abstract. It is shown that the uniform radius of spatial analyticity $\sigma(t)$ of solutions at time $t$ to the KdV equation cannot decay faster than $|t|^{-4/3}$ as $|t| \to \infty$ given initial data that is analytic with fixed radius $\sigma_0$. This improves a recent result of Selberg and Da Silva, where they proved a decay rate of $|t|^{-4/3 + \epsilon}$ for arbitrarily small positive $\epsilon$. The main ingredients in the proof are almost conservation law for the solution to the KdV equation in space of analytic functions and space-time dyadic bilinear $L^2$ estimates associated with the KdV equation.

1. INTRODUCTION

Consider the Cauchy problem for KdV equation

\[
\begin{align*}
\begin{cases}
    u_t + uu_x + u_{xxx} = 0, \\
    u(0, x) = f(x),
\end{cases}
\end{align*}
\]

where the unknown is

\[u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.
\]

This equation was derived by Korteweg and de Vries [25] as a model for long wave propagating in a channel. The well-posedness theory of (1.1) has been extensively studied, for instance, Kenig, Ponce and Vega [23] proved local well-posedness in $H^s$ for $s > -3/4$. Later, this was extended to a global result by Colliander, Keel, Staffilani, Takaoka and Tao [3]. Moreover, Christ, Colliander and Tao [2] proved that the solution map of (1.1) fails to be uniformly continuous in $H^s$ for $s < -3/4$ which was first proved by Kenig, Ponce, and Vega [24] for the complex-valued problem. More recently, Guo [11] established a global well-posedness result in $H^{-3/4}$ which is sharp in the sense of [2].

In this work, we are interested in the persistence of spatial analyticity for the solutions of (1.1), given initial data in a class of analytic functions. This is motivated naturally by observing that many special solutions of (1.1) such as for instance solitary and cnoidal waves are analytic in a strip about the real axis. For real-analytic initial data $f$ with uniform radius of analyticity $\sigma_0 > 0$, so there is a holomorphic extension to a complex strip

\[S_{\sigma_0} = \{x + iy : |y| < \sigma_0\},
\]

it was established in [10] that for small $t$ the solution $u$ of (1.1) is analytic in $S_{\sigma(t)}$ with $\sigma(t) = \sigma_0$, i.e., the radius of analyticity remains constant for short times. For large times on the other hand it was shown in [1] that $\sigma(t)$ can decay no faster than $|t|^{-12}$.
as $t \to \infty$. This is improved greatly more recently by Selberg and Da Silva [30] to a decay rate of $|t|^{-\left(\frac{4}{3} + \epsilon\right)}$, where $0 < \epsilon \ll 1$ is sufficiently small. In the present paper we are able to remove the $\epsilon$ exponent, and thus improving the decay rate further to $|t|^{-\frac{4}{3}}$. The exponent $-\frac{4}{3}$ turn out to be related to the Sobolev regularity exponent to $H^{-\frac{3}{4}}$ (specifically, one is the reciprocal of the other) at which Guo [11] obtained a sharp well-posedness result. The main ingredients in our proof are almost conservation law for the solution to the KdV equation in spaces of analytic functions and space-time dyadic bilinear estimates associated with the KdV equation. For similar studies for the Dirac-Klein-Gordon system, generalized KdV and cubic NLS see [32, 16, 31, 34]. For studies on related issues for nonlinear partial differential equations see for instance [5, 6, 7, 9, 14, 17, 8, 15, 18, 28, 19, 29, 26].

A class of analytic function spaces suitable to study analyticity of solution is the analytic Gevrey class. These spaces are denoted $G^{\sigma,s} = G^{\sigma,s}(\mathbb{R})$ with a norm given by

$$\|f\|_{G^{\sigma,s}} = \|e^{\sigma|D_x|} \langle D_x \rangle^s f\|_{L^2_x},$$

where $D_x = -i\partial_x$ with Fourier symbol $\xi$ and $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. We write $G^\sigma := G^{\sigma,0}$.

For $\sigma = 0$ the Gevrey-space coincides with the Sobolev space $H^s$.

One of the key properties of the Gevrey space is that every function in $G^{\sigma,s}$ with $\sigma > 0$ has an analytic extension to the strip $S_\sigma$. This property is contained in the following Theorem which is proved in [20, p. 209] for $s = 0$; the argument applies also for $s \in \mathbb{R}$ with some obvious modifications.

**Paley-Wiener Theorem.** Let $\sigma > 0$ and $s \in \mathbb{R}$. Then the following are equivalent:

(i) $f \in G^{\sigma,s}$.

(ii) $f$ is the restriction to the real line of a function $F$ which is holomorphic in the strip $S_\sigma = \{x + iy : x, y \in \mathbb{R}, |y| < \sigma\}$ and satisfies

$$\sup_{|y| < \sigma} \|F(x + iy)\|_{L^2_x} < \infty.$$

Observe that the Gevrey spaces satisfy the following embedding property:

$$G^{\sigma,s} \subset G^{\sigma',s'} \quad \text{for all } 0 \leq \sigma' < \sigma \text{ and } s, s' \in \mathbb{R}. \quad (1.2)$$

As a consequence of this property and the existing well-posedness theory in $H^s$ we conclude that the Cauchy problem (1.1) has a unique, smooth solution for all time, given initial data $f \in G^{\sigma_0}$ for all $\sigma_0 > 0$. Our main result gives an algebraic lower bound on the radius of analyticity $\sigma(t)$ of the solution as the time $t$ tends to infinity.

**Theorem 1.** Assume $f \in G^{\sigma_0}$ for some $\sigma_0 > 0$. Let $u$ be the global $C^\infty$-solution of (1.1). Then $u$ satisfies

$$u(t) \in G^{\sigma(t)} \quad \text{for all } t \in \mathbb{R}$$

with the radius of analyticity $\sigma(t)$ satisfying an asymptotic lower bound

$$\sigma(t) \geq c|t|^{-\frac{4}{3}} \quad \text{as } |t| \to \infty,$$

where $c > 0$ is a constant depending on $\|f\|_{G^{\sigma_0}}$ and $\sigma_0$. 

By time reversal symmetry of (1.1) we may from now on restrict ourselves to positive times \( t \geq 0 \). The first step in the proof of Theorem 1 is to show that in a short time interval \( 0 \leq t \leq t_0 \), where \( t_0 > 0 \) depends on the norm of the initial data, the radius of analyticity remains strictly positive. This is proved using a standard contraction argument involving energy type estimates, and a bilinear estimate in Bourgain-Gevrey type space; the proofs are given in section 4. The next step is to improve the control on the growth of the solution in the time interval \([0, t_0]\), measured in the data norm \( G^{0\alpha} \). To achieve this we show that, although the conservation of \( G^{0\alpha} \)-norm of solution does not hold exactly, it does hold in an approximate sense (see Section 5.1). This approximate conservation law will allow us to iterate the local result and obtain the asymptotic lower bound on \( \sigma \) in Theorem 1 (see Section 5.2).

2. Preliminaries, Functions spaces and linear estimates

2.1. Preliminaries. First we fix notation. In equations, estimates and summations capitalized variables such as \( N \) and \( L \) are presumed to be dyadic with \( N, L > 0 \), i.e., these variables range over numbers of the form \( 2^k \) for \( k \in \mathbb{Z} \). In estimates we use \( A \lesssim B \) as shorthand for \( A \leq CB \) and \( A \ll B \) for \( A \leq C^{-1}B \), where \( C \gg 1 \) is a positive constant which is independent of dyadic numbers such as \( N \) and \( L \); \( A \sim B \) means \( B \lesssim A \lesssim B \); \( \mathbb{1}_i \) denotes the indicator function which is 1 if the condition in the bracket is satisfied and 0 otherwise; we write \( a \pm := a \pm \epsilon \) for sufficiently small \( 0 < \epsilon \ll 1 \). Finally, we use the notation

\[ \| \cdot \| = \| \cdot \|_{L^2_t(R^{1+1})}. \]

Consider an even function \( \chi \in C_0^\infty((-2, 2)) \) such that \( \chi(s) = 1 \) if \( |s| \leq 1 \). Define

\[ \beta_N(s) = \begin{cases} 
0, & \text{if } N < 1, \\
\chi(s), & \text{if } N = 1, \\
\chi\left(\frac{s}{N}\right) - \chi\left(\frac{s}{2N}\right), & \text{if } N > 1.
\end{cases} \]

Thus

\[ \text{supp } \beta_1 \subset \{ s \in \mathbb{R} : |s| \leq 2 \}, \quad \text{supp } \beta_N \subset \left\{ s \in \mathbb{R} : \frac{N}{2} \leq |s| \leq 2N \right\} \text{ for } N > 1. \]

Note that

\[ \sum_{N \geq 1} \beta_N(s) = 1 \quad \text{for } s \neq 0. \tag{2.1} \]

The Fourier transform in space and space-time are given by

\[ \mathcal{F}_x(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx, \]

\[ \mathcal{F}_{t,x}(u)(\tau, \xi) = \tilde{u}(\tau, \xi) = \int_{\mathbb{R}^{1+1}} e^{-i(\tau x + \tau^3)} u(t, x) \, dt \, dx. \]

Now define

\[ P_N u = \mathcal{F}_x^{-1} [\beta_N(\xi) \hat{u}] \quad \text{for } N \geq 1, \]

\[ Q_L u = \mathcal{F}_{t,x}^{-1} [\beta_L (\tau - \xi^3) \hat{u}] \quad \text{for } L \geq 1. \]

Here \( N \) and \( L \) measure the magnitude of the spatial frequency and modulation, respectively. We use the notation

\[ u_N := P_N u, \quad u_{N,L} := P_N Q_L u. \]
In view of (2.1) one can write
\[ u = \sum_{N \geq 1} u_N. \]

In addition to \( P_N \) and \( Q_N \) we also need the homogeneous projections \( P_N \) and \( Q_L \) defined by
\[
P_N u = \mathcal{F}^{-1}_x \left[ \| \frac{N}{2} \leq |\xi| \leq 2N \| \hat{u} \right] \text{ for } N > 0,
\]
\[
Q_L u = \mathcal{F}^{-1}_x \left[ \| |\xi| \leq L \| \hat{u} \right] \text{ for } L > 0.
\]

Note that
\[
P_N u = \hat{P}_N P_N u, \quad Q_L u = Q_L Q_L u \quad \text{ for } N, L > 1. \tag{2.2}
\]

Remark 1. We shall make a frequent use of the following dyadic summation estimate: For \( N \in 2^\mathbb{Z}, 1 \leq a < \beta \) and \( a \in \mathbb{R} \) we have
\[
\sum_{a \in N \leq \beta} N^a \sim \begin{cases} \beta^a & \text{if } a > 0, \\
\log(b/a) & \text{if } a = 0, \\
a^a & \text{if } a < 0. \end{cases} \tag{2.3}
\]

2.2. Function spaces. For \( 1 \leq q, r \leq \infty \) the mixed space-time Lebesgue space \( L_r^q L_s^b(\mathbb{R}^{1+1}) \) is defined with the norm
\[
\| u \|_{L_r^q L_s^b} = \left\| \| u(t, \cdot) \|_{L_r^q} \|_{L_s^b} \right\| = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |u(t,x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right)^{\frac{1}{q}}
\]
with an obvious modification when \( q = \infty \) or \( r = \infty \), and when the space is restricted to bounded intervals. Similarly
\[
\| u \|_{L_r^q L_s^b(I)} = \| u(\cdot, x) \|_{L_r^q L_s^b}.
\]

For an interval \( I \) we write \( L_r^q L_s^b(I) \) to denote \( L_r^q L_s^b(I \times \mathbb{R}) \), i.e., the time variable is restricted to \( I \).

The Bourgain space associated with the KdV equation, denoted \( X_r^s, b \), is defined as the completion of the Schwartz class \( \mathcal{S}(\mathbb{R}^{1+1}) \) with respect to the norm
\[
\| u \|_{X_r^s, b} = \left( \sum_{N,N_L \geq 1} N^{2s} L^{2b} \| u_{N,L} \|^2 \right)^{\frac{1}{2}}.
\]

The restriction to a time slab \( I \times \mathbb{R} \) of \( X_r^s, b \), denoted \( X_r^s, b^I \), is a Banach space when equipped with the norm
\[
\| u \|_{X_r^s, b^I} = \inf \{ \| v \|_{X_r^s, b} : \ v = u \text{ on } I \times \mathbb{R} \}.
\]

By a standard contraction argument in the \( X_r^s, b^I \)-space local well-posedness of (1.1) for \( H^s \) data reduces to the bilinear estimate
\[
\| \partial_x (uv) \|_{X_r^{s-1, b}} \leq \| u \|_{X_r^{s, b}} \| v \|_{X_r^{s, b}} \tag{2.4}
\]
for some \( b > 1/2 \).

In [23] Kenig, Ponce and Vega proved that (2.4) holds for \( s > -3/4 \), but fails for \( s < -3/4 \). Later, it was also shown by Nakanishi, Takaoka and Tsutsumi [27] that (2.4) also fails to hold at the borderline \( s = -3/4 \). A usual approach to resolve problems such as this
is to modify the Bourgain space by setting $b = \frac{1}{2}$ and replacing the $l^2$-summation in the modulation parameter, $L$, by $l^1$-summation. This space which we denote by $X^s$ is defined with respect to the norm

$$\|u\|_{X^s} = \left( \sum_{N \geq 1} N^{2s} \|u_N\|_X^2 \right)^{\frac{1}{2}},$$

where

$$\|v\|_X = \sum_{L \geq 1} L^\frac{1}{2} \|Q_L v\|.$$

Note that if $u_N \in X$ then

$$\|\gamma(M(t-t_0))u_N\|_X \lesssim \|u_N\|_X \quad (2.5)$$

for all $M, N \geq 1$, $t_0 \in \mathbb{R}$ and $\gamma \in \mathcal{S}(\mathbb{R})$. Indeed, by definition

$$\|\gamma(M(t-t_0))u_N\|_X = \sum_{L \geq 1} L^\frac{1}{2} \|\gamma(M(t-t_0))Q_L u_N\| \leq \|\gamma(M(\cdot-t_0))\|_{L^\infty} \sum_{L \geq 1} L^\frac{1}{2} \|Q_L u_N\| \lesssim \|u_N\|_X.$$

The restriction to a time slab $I \times \mathbb{R}$ of $X^s$, denoted $X^s_I$, is defined similarly as above. Now using $X^s_I$ as a contraction space local well-posedness in $H^{-3/4}$ will follow if one proves the bilinear estimate

$$\|\mathcal{B}(u, v)\|_{X^{-\frac{3}{4}}} \lesssim \|u\|_{X^{-\frac{3}{4}}} \|v\|_{X^{-\frac{3}{4}}} \quad (2.6)$$

where

$$\mathcal{B}(u, v)(t) = \chi(t/4) \int_0^t S(t-t') \partial_x \left( (\chi u \cdot \chi v)(t') \right) dt'.$$

is the time localized Duhamel term associated to the KdV equation.

However, as pointed out in [11] in trying to establish the bilinear estimate (2.6) a particular case of high:high-low frequency interaction introduces a logarithmic derivative loss, and thus (2.6) is an open problem. To resolve this problem a version of $X^s$ that is modified with respect to low frequency modes (corresponding to $N = 1$) is introduced. The new space, denoted $\bar{X}^s$, is defined with respect to the norm

$$\|u\|_{\bar{X}^s} = \left( \|u_1\|_{L^2_t L^\infty_x}^2 + \sum_{N \geq 1} N^{2s} \|u_N\|_X^2 \right)^{\frac{1}{2}},$$

where the additional $L^2_t L^\infty_x$-norm for the low frequency helps to avoid the logarithmic divergence in the bilinear estimate (2.6). The restriction to a time slab $I \times \mathbb{R}$ of $\bar{X}^s$, denoted $\bar{X}^s_I$, is defined similarly as before. By using this space Guo [11] proved the bilinear estimate

$$\|\mathcal{B}(u, v)\|_{\bar{X}^{-\frac{3}{4}}} \lesssim \|u\|_{\bar{X}^{-\frac{3}{4}}} \|v\|_{\bar{X}^{-\frac{3}{4}}} \quad (2.7)$$

thereby establishing an endpoint local well-posedness result for (1.1).

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1In [11] the spaces $X^s$ and $\bar{X}^s$ are denoted as $F^s$ and $\bar{F}^s$, respectively.
2.3. **Linear estimates.** Let \( S(t)f = e^{-it\partial_x^3}f \) be the solution to the Airy equation (free solution to the KdV equation). The following Lemma contains frequency localized Strichartz estimates, maximal function estimates and smoothing effect estimates for solution to the Airy equation (see e.g. \([12, 22, 21]\)). By the transfer principle (see e.g. \([11, \text{Lemma } 3.2]\)) these estimates can be extended to hold for any function in \( X \).

**Lemma 1.** Let \( I \) be a interval with \(|I| \leq 1\), \( N \geq 1 \) and \( M > 1 \) be dyadic numbers. Let the pair \((q, r)\) satisfies 
\[
2 \leq q, r \leq \infty, \quad \frac{3}{q} + \frac{1}{r} = \frac{1}{2}.
\]

(a) For all \( f \in \mathcal{S}(\mathbb{R}) \) we have the following:
\[
\begin{align*}
\| S(t)f_N \|_{L^q_t L^r_x} &\lesssim \| f_N \|_{L^2_x}, \quad (2.8) \\
\| S(t)f_N \|_{L^q_t L^\infty_x} &\lesssim N^{\frac{1}{q}} \| f_N \|_{L^1_x}, \quad (2.9) \\
\| S(t)f_N \|_{L^4_t L^4_x} &\lesssim N^{\frac{1}{4}} \| f_N \|_{L^1_x}, \quad (2.10) \\
\| S(t)f_M \|_{L^\infty_t L^2_x} &\lesssim M^{-1} \| f_M \|_{L^2_x}. \quad (2.11)
\end{align*}
\]

(b) For all \( u_N \in X \), we have
\[
\begin{align*}
\| u_N \|_{L^q_t L^r_x} &\lesssim \| u_N \|_{X}, \quad (2.12) \\
\| u_N \|_{L^q_t L^\infty_x} &\lesssim N^{\frac{1}{q}} \| u_N \|_{X}, \quad (2.13) \\
\| u_N \|_{L^4_t L^4_x} &\lesssim N^{\frac{1}{4}} \| u_N \|_{X}, \quad (2.14) \\
\| u_M \|_{L^\infty_t L^2_x} &\lesssim M^{-1} \| u_M \|_{X}. \quad (2.15)
\end{align*}
\]

We also have the following embedding estimates.

**Lemma 2.**

(i) Let \( 1 \leq N \lesssim 1 \). For all \( s \in \mathbb{R} \) and \( u \in \tilde{X}^s \) we have
\[
\| u_N \|_{L^q_t L^r_x} \lesssim \| u \|_{\tilde{X}^s}.
\]

(ii) For all \( u_1 \in X \) we have
\[
\| u_1 \|_{L^q_t L^\infty_x} \lesssim \| u_1 \|_{X}.
\]

(iii) For all \( s \in \mathbb{R} \) we have \( \tilde{X}^s \subset C(\mathbb{R}, H^s) \) and
\[
\sup_{t \in \mathbb{R}} \| u(t) \|_{H^s} \lesssim \| u \|_{\tilde{X}^s}.
\]

(iv) For all \( s_1 \leq s_2 \) we have \( \tilde{X}^{s_1} \subset \tilde{X}^{s_2} \).

**Proof.** First we prove (i). For \( 1 \leq N \lesssim 1 \) we have by \((2.12)\) with \((q, r) = (\infty, 2)\)
\[
\| u_N \|_{L^\infty_t L^2_x} \lesssim N^s \| u_N \|_{X} \quad \text{for all } \ s.
\]
Combining this with the definition of \( \tilde{X}^s \) and the simple estimate
\[
\| u_1 \|_{L^\infty_t L^2_x} \lesssim \| u_1 \|_{L^\infty_t L^\infty_x}
\]
we obtain (i).
The inequality (ii) follows from (2.13) with $N = 1$ whereas (iii) follows from the definition of $\tilde{X}^s$ and (2.12), i.e.,

$$
\|u\|^2_{L^2_t H^s} \sim \|u_t\|^2_{L^2_t L^2_x} + \sum_{N \geq 1} N^{2s} \|u_N\|^2_{L^2_t L^2_x} \lesssim \|u_t\|^2_{L^2_t L^2_x} + \sum_{N \geq 1} N^{2s} \|u_N\|^2_{X^s} = \|u\|^2_{X^s},
$$

Finally, (iv) is simple to prove.

Define the operator $\Lambda$ by

$$
\Lambda u = \mathcal{F}_{L^2_x}^{-1} \left( i(t - \xi^3) \hat{u} \right).
$$

We remark that since $|i + \tau - \xi^3| = |\tau - \xi^3|$ the operator $\Lambda^{-1}$ is not singular.

**Lemma 3** (Energy type estimates).

(a) Assume $f \in H^s$ for $s \in \mathbb{R}$. Then there exists a constant $C > 0$ such that

$$
\|\chi(t)S(t)f\|_{X^s} \leq C \|f\|_{H^s},
$$

(2.16)

(b) Assume $N \geq 1$ and $\Lambda^{-1} F_N \in X$. Then there exists a constant $C > 0$ such that

$$
\left\| \chi(t) \int_0^t S(t - t') F_N(t') \, dt' \right\|_{X} \leq C \|\Lambda^{-1} F_N\|_{X}.
$$

(2.17)

**Proof:** Part (a) follows from the definition of $\tilde{X}^s$ and Lemma 1, (2.9). Indeed, using (2.9) we obtain

$$
\|\chi(t)S(t)f_1\|_{L^2_t L^\infty} \lesssim \|f_1\|_{L^2_x}.
$$

(2.18)

On the other hand, we have

$$
\mathcal{F}_{L^2_x} \left[ Q_L \left( \chi(t)S(t)f_N \right) \right] (t, \xi) = \beta_L \left( t - \xi^3 \right) \hat{\chi}(t - \xi^3) \hat{f}_N(\xi),
$$

where we used the fact that $\hat{S(t)} f(\xi) = e^{i\xi^3 t} \hat{f}(\xi)$. Then by Plancherel

$$
\|Q_L \chi(t)S(t)f_N\| = \|\beta_L \left( t - \xi^3 \right) \hat{\chi}(t - \xi^3) \hat{f}_N(\xi)\|
$$

(2.19)

$$
= \|P_L \chi\|_{L^2_x} \|f_N\|_{L^2_x}.
$$

This in turn implies

$$
\|\chi(t)S(t)f_N\|_X \leq \sum_{L \geq 1} L^{\frac{s}{2}} \|P_L \chi\|_{L^2_x} \|f_N\|_{L^2_x} \lesssim \|\chi\|_{H^1_t} \|f_N\|_{L^2_x} \lesssim \|f_N\|_{L^2_x}.
$$

(2.20)

where to obtain the second inequality we used Cauchy-Schwarz, i.e.,

$$
\sum_{L \geq 1} L^{\frac{s}{2}} \|P_L \chi\|_{L^2_x} \lesssim \left( \sum_{L \geq 1} L^{-1} \right)^{\frac{1}{2}} \left( \sum_{L \geq 1} L^2 \|P_L \chi\|_{L^2_x}^2 \right)^{\frac{1}{2}} \lesssim \|\chi\|_{H^1_t}.
$$

(2.21)

Now using the estimates (2.18) and (2.20) in the definition of $\tilde{X}^s$ we obtain (a).

Variants of part (b) has appeared in the literature, see for instance [13]. For completeness we give the proof here by adapting the proof of ([4, Section 13.1]). To this end we let

$$
u_N(t) = \int_0^t S(t - t') F_N(t') \, dt'.
$$
Taking Fourier transform in space, 

\[
\hat{u}_N(t, \xi) = \int_0^t e^{i(t-t')\xi^3} \hat{F}_N(t', \xi) \, dt' = \int e^{i(t\lambda - \xi^3)} \hat{F}_N(\lambda, \xi) \, d\lambda \\
= \left( \int_{|\lambda-\xi^3| \leq 1} + \int_{|\lambda-\xi^3| > 1} \right) e^{i(t\lambda - \xi^3)} \hat{F}_N(\lambda, \xi) \, d\lambda \\
:= \hat{v}_N(t, \xi) + \hat{\nu}_N(t, \xi).
\]

**Estimate for \( v_N \).** Expanding we write 

\[
\hat{v}_N(t, \xi) = e^{it\xi^3} \sum_{k=1}^{\infty} \int_{|\lambda-\xi^3| \leq 1} \left[ i t(\lambda - \xi^3) \right]^k \hat{F}_N(\lambda, \xi) \, d\lambda
\]

and hence 

\[
v_N(t) = \sum_{k=1}^{\infty} \frac{i^k}{k!} S(t) g_k. \tag{2.22}
\]

where 

\[
\hat{g}_k(\xi) = \int_{|\lambda-\xi^3| \leq 1} \left[ i(\lambda - \xi^3) \right]^{k-1} \hat{F}_N(\lambda, \xi) \, d\lambda.
\]

Set \( \psi_k(t) = t^k \chi(t) \). In view of (2.22) and (2.21) we have 

\[
\left\| \chi(t) v_N \right\|_X \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\| L^+ P_k \psi_k \right\|_L^2 \left\| g_k \right\|_{L^2_x} \\
\leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\| \psi_k \right\|_{H^1_x} \left\| g_k \right\|_{L^2_x}.
\]

But \( \left\| \psi_k \right\|_{H^1_x} \lesssim 2^k + k2^{k-1} \), and by Cauchy-Schwarz

\[
|\hat{g}_k(\xi)|^2 = \left( \int_{|\lambda-\xi^3| \leq 1} |\lambda - \xi^3|^{2(k-1)} |i + \lambda - \xi^3|^2 \, d\lambda \right) \\
\cdot \left( \int_{|\lambda-\xi^3| \leq 1} |i + \lambda - \xi^3|^{-2} |\hat{F}_N(\lambda, \xi)|^2 \, d\lambda \right) \\
\lesssim \int_{|\lambda-\xi^3| \leq 1} |i + \lambda - \xi^3|^{-2} |\hat{F}_N(\lambda, \xi)|^2 \, d\lambda.
\]

By Plancrerel we have \( \left\| g_k \right\|_{L^2_x} \lesssim \left\| \Lambda^{-1} F_N \right\|_X \), and hence 

\[
\left\| \chi(t) v_N \right\|_X \lesssim \left\| \Lambda^{-1} F_N \right\| \left( \sum_{k=1}^{\infty} \frac{2^k + k2^{k-1}}{k!} \right) \\
\lesssim \left\| \Lambda^{-1} F_N \right\|_X \lesssim \left\| \Lambda^{-1} F_N \right\|_X,
\]

where in the last inequality we used 

\[
\left\| \Lambda^{-1} F_N \right\|^2 \sim \sum_{L=2} \left\| \Lambda^{-1} Q_L F_N \right\|^2 \lesssim \left\| \Lambda^{-1} F_N \right\|_X^2.
\]
Estimate for \( w_N \). Taking Fourier transform in time
\[
\hat{w}_N(t, \xi) = \int_{|\lambda - \xi^3| > 1} \frac{\delta(t - \lambda) - \delta(t - \xi^3)}{i(\lambda - \xi^3)} \hat{F}_N(\lambda, \xi) \, d\lambda
\]
\[
= \frac{1}{i(\tau - \xi^3)} \int_{|\lambda - \xi^3| > 1} \hat{F}_N(\tau, \xi) \, d\lambda
\]
\[
:= \hat{y}_N(t, \xi) - \hat{z}_N(t, \xi).
\]

Obviously, (see (2.5)) we can estimate \( y_N \) as
\[
\| \chi(t) y_N \|_X \lesssim \| y_N \|_X = \sum_{L \geq 1} L^1 \| Q_L y_N \|
\]
\[
\lesssim \sum_{L \geq 1} L^2 \| Q_L (\Lambda^{-1} F_N) \| = \| \Lambda^{-1} F_N \|_X.
\]

On the other hand, write \( \hat{z}_N(t, \xi) = \delta(t - \xi^3) \hat{h}_N(\xi) \), where
\[
\hat{h}_N(\xi) = \int_{|\lambda - \xi^3| > 1} \frac{\hat{F}_N(\lambda, \xi)}{i(\lambda - \xi^3)} \, d\lambda.
\]

By Plancherel
\[
\| Q_L (\chi(t) z_N) \| = \| \beta_L (t - \xi^3) \hat{h}_N(\xi) \| = \| P_L X \|_{L^2} \| h_N \|_{L^2}
\]

But by dyadic decomposition and Cauchy-Schwarz
\[
| \hat{h}_N(\xi) | \lesssim \int_{|\lambda - \xi^3| > 1} | (\Lambda^{-1} F_N)(\lambda, \xi) | \, d\lambda \lesssim \sum_{L \geq 1} \int_{|\lambda - \xi^3| > L} | (\Lambda^{-1} F_N)(\lambda, \xi) | \, d\lambda
\]
\[
\leq \sum_{L \geq 1} L^2 \left( \int_{|\tau - \xi^3| > L} | (\Lambda^{-1} F_N)(t, \xi) |^2 \, dt \right)
\]

and hence
\[
\| h_N \|_{L^2} \lesssim \sum_{L \geq 1} L^1 \| Q_L (\Lambda^{-1} F_N) \| = \| \Lambda^{-1} F_N \|_X.
\]

Therefore,
\[
\| \chi(t) z_N \|_X = \sum_{L \geq 1} L^1 \| Q_L (\chi(t) z_N) \|
\]
\[
= \sum_{L \geq 1} L^1 \| \beta_L (t - \xi^3) \hat{h}_N(\xi) \|
\]
\[
= \sum_{L \geq 1} L^1 \| P_L X \|_{L^2} \| h_N \|_{L^2}
\]
\[
\lesssim \| \chi \|_{H^1} \| h_N \|_{L^2} \lesssim \| \Lambda^{-1} F_N \|_X.
\]

3. Bilinear estimates

For dyadic numbers \( N_j > 0 \) (\( j = 1, 2, 3 \)) we denote by \( N_{\min}, N_{\med} \) and \( N_{\max} \) the minimum, median and maximum of \( (N_1, N_2, N_3) \). We use similar notation for \( L_j > 0 \) (\( j = 1, 2, 3 \)).
Following the methods in [33] the bilinear estimate in $X^{s,b}$-space that is needed to obtain local well-posedness of (1.1) reduces to establishing dyadic bilinear estimates of the form
\[
\left\| \hat{P}_{N_j} \hat{Q}_L \left( \left( \hat{P}_{N_j} \hat{Q}_L u_j \right) \left( \hat{P}_{N_j} \hat{Q}_L u_2 \right) \right) \right\| \leq C(N, L) \prod_{j=1}^2 \| \hat{P}_{N_j} \hat{Q}_L u_j \| \tag{3.1}
\]
for some $^2$ optimal constant $C(N, L)$ that is a function of $N_j, L_j > 0 \ (j = 1, 2, 3)$.

By checking the support properties in Fourier space of the bilinear term on the left hand side of (3.1) one can see that this term vanishes unless the following conditions are satisfied (see (29) and (30) in [33]):
\[
N_{\text{max}} \sim N_{\text{med}}, \tag{3.2}
\]
\[
L_{\text{max}} \sim \max(N_{\text{min}}N_{\text{max}}^2, L_{\text{med}}). \tag{3.3}
\]

We may thus assume (3.2) and (3.3) throughout the paper.

**Proposition 1** ([33], Proposition 6.1). Let $N_j, L_j > 0 \ (j = 1, 2, 3)$ be dyadic numbers. Then (3.1) holds with $C(N, L)$ as follows:

(a) If $N_{\text{max}} \sim N_{\text{min}}$ and $L_{\text{max}} \sim N_{\text{min}}N_{\text{max}}^2$, then
\[
C(N, L) \sim N_{\text{max}}^{-1} L_{\text{min}}^{-\frac{1}{2}} N_{\text{min}}^{\frac{1}{2}}. \tag{3.4}
\]

(b) If $N_2 \sim N_3 \gg N_1$ and $N_{\text{min}}N_{\text{max}}^2 \sim L_1 \gtrsim L_2, L_3$, then
\[
C(N, L) \sim N_{\text{max}}^{-1} L_{\text{min}}^{-\frac{1}{2}} \min \left( N_{\text{min}}N_{\text{max}}^2, N_{\text{max}} \right)^{\frac{1}{2}} N_{\text{min}} \tag{3.5}
\]

Similar estimates hold for any permutations of $(1, 2, 3)$.

(c) In all other cases, we have
\[
C(N, L) \sim N_{\text{max}}^{-1} L_{\text{min}}^{-\frac{1}{2}} \min \left( N_{\text{min}}N_{\text{max}}^2, L_{\text{med}} \right)^{\frac{1}{2}}. \tag{3.6}
\]

**Remark 2.** In view of (2.2) the bilinear estimate (3.1) still holds if we replace the projection
\[
\hat{P}_{N_j} \hat{Q}_L \mathbb{I}_{[N_j, L_j > 0]} \quad \text{by} \quad \hat{P}_{N_j} \hat{Q}_L \mathbb{I}_{[N_j, L_j > 1]}
\]
with $C(N, L)$ as in Proposition 1(a)–(c). Following the proof of [33, Proposition 6.1] we also see that (3.1) holds if we replace $\hat{Q}_L \mathbb{I}_{[L_j > 0]}$ by $\hat{Q}_L \mathbb{I}_{[L_j > 1]}$.

In view of Remark 2 we have the following:

**Corollary 1.** Let $N_j > 1$ and $L_j \geq 1 \ (j = 1, 2, 3)$ be dyadic numbers. The estimate
\[
\left\| P_{N_j} Q_L \left( \left( P_{N_j} Q_L u \right) \left( P_{N_j} Q_L u_2 \right) \right) \right\| \leq C(N, L) \prod_{j=1}^2 \| P_{N_j} Q_L u_j \| \tag{3.7}
\]
holds with $C(N, L)$ given as in Proposition 1(a)–(c).

This Corollary is used to prove the following Lemma.

---

$^2$ In [33] the optimal constant is denoted by $\| m \|_{\mathcal{B}(\mathbb{R}^+)}$, where
\[
m = m(\tau, \xi) = \prod_{j=1}^3 \prod_{\{\xi_j - N_j \leq |\tau - \xi_j| \leq L_j\}} 3 \left( \frac{1}{|\tau - \xi_j| - L_j} \right).
\]
Lemma 4 (See [11]). For dyadic numbers \( N_j \geq 1 \) \( j = 1, 2, 3 \) we have the following:

(i) The bilinear estimate

\[
\| \Lambda^{-1} P_{N_3} \partial_x (u_{N_1} v_{N_2}) \|_X \leq C(N) \| u_{N_1} \|_X \| v_{N_2} \|_X,
\]  

holds with \( C(N) \) as follows:

\[
C(N) \sim \begin{cases} 
N_1^{-1} N_2^{-\frac{1}{2}+}, & \text{if } N_3 \sim N_2 \gg N_1 > 1, \\
N_2^{-\frac{3}{2}}, & \text{if } N_1 \sim N_2 \gg N_3 > 1, \\
N_1^{-\frac{3}{2}}, & \text{if } N_1 \sim N_2 \gg N_3 = 1, \\
\max \left( N_1^{-\frac{1}{2}}, N_1^{-2+} N_3^{\frac{1}{2}} \right), & \text{if } N_1 \sim N_2 \gg N_3 > 1.
\end{cases}
\]  

(ii) If \( N_3 \gg N_2 \gg N_1 = 1 \) then

\[
\| \Lambda^{-1} P_{N_3} \partial_x (u_{N_1} v_{N_2}) \|_X \lesssim \| u_{N_1} \|_{L^3_t L^2_x} \| v_{N_2} \|_X.
\]  

(iii) Let \( I \) be a bounded interval. If \( 1 \leq N_1, N_2, N_3 \lesssim 1 \), then

\[
\| \int_I (t) \Lambda^{-1} P_{N_3} \partial_x (u_{N_1} v_{N_2}) \|_X \lesssim \| u_{N_1} \|_{L^3_t L^2_x} \| v_{N_2} \|_{L^3_t L^2_x}.
\]

For completeness, and since the notation and setup of this paper is slightly different from [11] we include the proof of Lemma 4 in Appendix A.

In the case of high-high/low frequency interaction, i.e., when \( N_1 \sim N_2 \gg N_3 = 1 \) the factor \( C(N) \sim N_1^{-3/2+} \) (see third line in (3.6)) in the dyadic bilinear estimate (3.5) is not good enough to obtain (2.7). Fortunately, Guo improved this estimate to \( C(N) \sim N_1^{-3/2} \) which is given as follows.

Lemma 5 ([11]: \( L^2_t L^\infty_x \)-estimate). Assume \( N_1 \sim N_2 \gg 1 \).

(i) Let \( u_{N_1}(t) = S(t) f_{N_1} \) and \( u_{N_2}(t) = S(t) g_{N_2} \) be two free solutions of the Airy equation, where \( f_{N_1}, g_{N_2} \in L^2 \). Then

\[
\| \chi(t) \int_0^t S(t-t') P_1 \partial_x \left( (u_{N_1} v_{N_2})(t') \right) \|_{L^3_t L^2_x} \lesssim N_1^{-\frac{3}{2}} \| f_{N_1} \|_{L^3_x} \| g_{N_2} \|_{L^\infty_x}.
\]  

(ii) For all \( u_{N_1}, v_{N_2} \in X \), we have by the transfer principle (see e.g. [11, Lemma 3.2])

\[
\| \chi(t) \int_0^t S(t-t') P_1 \partial_x \left( (u_{N_1} v_{N_2})(t') \right) \|_{L^3_t L^2_x} \lesssim N_1^{-\frac{3}{2}} \| u_{N_1} \|_X \| v_{N_2} \|_X.
\]

Lemma 4 and Lemma 5 together are key to obtain the bilinear estimate (2.7) which is used to prove the end-point well-posedness result for (1.1). We include the proof of the following Lemma (the proof will be reused later).

Lemma 6 (See [11]). Define the bilinear operator

\[
\mathcal{B}(u, v)(t) = \chi(t/4) \int_0^t S(t-t') \partial_x \left( (\chi u \cdot \chi v)(t') \right) \, dt'.
\]

Assume \( s \in [-3/4, 0] \). Then for all \( u, v \in \tilde{X}^s \) we have

\[
\| \mathcal{B}(u, v) \|_{\tilde{X}^s} \lesssim \left( \| u \|_{\tilde{X}^s} \| v \|_{\tilde{X}^s} + \| u \|_{\tilde{X}^s} \| v \|_{\tilde{X}^s} \right),
\]  

(3.11)
Proof. By definition
\[ \| B(u, v) \|_{X^s}^2 = J_1 + J_2, \]
where
\[ J_1 = \| P_1 B(u, v) \|_{L_x^2 L_t^\infty}^2, \]
\[ J_2 = \sum_{N_j > 1} N_j^2 \| P_{N_j} B(u, v) \|_{X^s}^2. \]

**Estimate for** \( J_1 \). It suffices to show
\[ J_1 \lesssim \| u \|_{\dot{X}^{\frac{3}{4}}}^2 \| v \|_{\dot{X}^{\frac{3}{4}}}^2. \]  
(3.12)

Decomposing \( u \) and \( v \) we have
\[ J_1 \lesssim \left( \sum_{N_j > 1} \| P_1 B(u_{N_1}, v_{N_2}) \|_{L_x^2 L_t^{\infty}}^2 \right)^2. \]

By symmetry we may assume \( N_1 \leq N_2 \). If \( N_2 \lesssim 1 \) we use Lemma 2(ii), Lemma 3(b), Lemma 4(iii) and Lemma 2(i) to obtain
\[ J_1 \lesssim \left( \sum_{1 \leq N_1 \leq N_2 \lesssim 1} \| P_1 \partial_x (\chi u_{N_1} \cdot \chi v_{N_2}) \|_X \right)^2 \]
\[ \lesssim \left( \sum_{1 \leq N_1 \leq N_2 \lesssim 1} \| u_{N_1} \|_{L_x^2 L_t^{\infty}} \| v_{N_2} \|_{L_x^2 L_t^{\infty}} \right)^2 \]
\[ \lesssim \| u \|_{\dot{X}^{\frac{3}{4}}}^2 \| v \|_{\dot{X}^{\frac{3}{4}}}^2. \]

If \( N_1 \sim N_2 \gg 1 \), then by Lemma 5(ii), and Cauchy Schwarz in \( N_1 \sim N_2 \) we have
\[ J_1 \lesssim \left( \sum_{N_1 \sim N_2 \gg 1} \| u_{N_1} \|_X \| v_{N_2} \|_X \right)^2 \]
\[ \lesssim \| u \|_{\dot{X}^{\frac{3}{4}}}^2 \| v \|_{\dot{X}^{\frac{3}{4}}}^2. \]

**Remark 3.** In the case \( N_1 \sim N_2 \gg N_3 = 1 \) if we use Lemma 4(i) with \( C(N) \) as in the third line of (3.6) instead of Lemma 5(ii) we would obtain
\[ J_1 \lesssim \left( \sum_{N_1 \sim N_2 \gg 1} \| u_{N_1} \|_X \| v_{N_2} \|_X \right)^2 \]
\[ \lesssim \| u \|_{\dot{X}^{\frac{3}{4}}}^2 \| v \|_{\dot{X}^{\frac{3}{4}}}^2. \]

Thus, using Lemma 4 in the case of high:high-low frequency interaction case introduces a logarithmic loss in the estimate for (2.7).
Estimate for $\mathcal{I}_2$. We want to show
\[
\mathcal{I}_2 \lesssim \left( \|u\|_{\mathcal{X}^1_\beta} \|v\|_{\mathcal{X}^{-\frac{1}{2}}_\beta} + \|u\|_{\mathcal{X}^{-\frac{1}{2}}_\beta} \|v\|_{\mathcal{X}^1_\beta} \right). \tag{3.13}
\]
Decomposing $u$ and $v$, and using Lemma 3(b) we obtain
\[
\mathcal{I}_2 = \sum_{N_j > 1} \left( \sum_{N_1, N_2 \geq 1} N_j^2 \|P_{N_j} \mathcal{R}(u_{N_1}, v_{N_2})\|_X \right)^2
\]
\[
= \sum_{N_j > 1} \left( \sum_{N_1, N_2 \geq 1} N_j^2 \|\Lambda^{-1} P_{N_j} \partial_x (\chi u_{N_1} \cdot \chi v_{N_2})\|_X \right)^2
\]
\[
\quad : = \mathcal{I}_3 + \mathcal{I}_4,
\]
where
\[
\mathcal{I}_3 = \sum_{N_j > 1} \left( \sum_{N_1, N_2 \geq 1} (\cdot) \right)^2, \quad \mathcal{I}_4 = \sum_{N_j > 1} \left( \sum_{N_1, N_2 \geq 1} (\cdot) \right)^2.
\]
Here $(\cdot)$ represents the argument in the inner summation.

By symmetry we may only estimate $\mathcal{I}_3$. Thus, it suffices to prove
\[
\mathcal{I}_3 \lesssim \|u\|_{\mathcal{X}^{-\frac{1}{2}}_\beta}^2 \|v\|_{\mathcal{X}^1_\beta}^2.
\]
In view of (3.2) this reduces further to
\[
\mathcal{I}_{3k} \lesssim \|u\|_{\mathcal{X}^{-\frac{1}{2}}_\beta}^2 \|v\|_{\mathcal{X}^1_\beta}^2 \quad (k = 1, \cdots, 5), \tag{3.14}
\]
where
\[
\mathcal{I}_{31} = \sum_{N_j > 1} \left( \sum_{1 \leq N_1 \leq N_2 \leq 1} (\cdot) \right)^2, \quad \mathcal{I}_{32} = \sum_{N_j > 1} \left( \sum_{1 = N_1 < N_2 - N_3} (\cdot) \right)^2,
\]
\[
\mathcal{I}_{33} = \sum_{N_j > 1} \left( \sum_{1 < N_1 \leq N_2 - N_3} (\cdot) \right)^2, \quad \mathcal{I}_{34} = \sum_{N_j > 1} \left( \sum_{N_1 - N_2 < N_3} (\cdot) \right)^2, \quad \mathcal{I}_{35} = \sum_{N_j > 1} \left( \sum_{N_1 - N_2 = N_3} (\cdot) \right)^2.
\]

We establish (3.14) as follows.

(i). $\mathcal{I}_{31}$: By Lemma 4(iii) and Lemma 2(i) we have
\[
\mathcal{I}_{31} \lesssim \sum_{N_j > 1} \left( \sum_{1 \leq N_1 \leq N_2 \leq 1} N_j^2 \|u_{N_1}\|_{L^\infty_t L^2_x} \|v_{N_2}\|_{L^\infty_t L^2_x} \right)^2
\]
\[
\lesssim \|u\|_{\mathcal{X}^{-\frac{1}{2}}_\beta}^2 \|v\|_{\mathcal{X}^1_\beta}^2.
\]

(ii). $\mathcal{I}_{32}$: By Lemma 4(ii) and (2.3) we have
\[
\mathcal{I}_{32} \lesssim \sum_{N_j > 1} \left( \sum_{1 = N_1 < N_2 - N_3} N_j^2 \|u_{N_1}\|_{L^2_t L^\infty_x} \|v_{N_2}\|_X \right)^2
\]
\[
\lesssim \|u\|_{\mathcal{X}^{-\frac{1}{2}}_\beta}^2 \sum_{N_j > 1} N_j^{2s} \left( \sum_{N_2 - N_3} \|v_{N_2}\|_X \right)^2
\]
\[
\lesssim \|u\|_{\mathcal{X}^{-\frac{1}{2}}_\beta}^2 \|v\|_{\mathcal{X}^1_\beta}^2.
\]
(iii). \( \mathcal{F}_{33} \): By Lemma 4(i) with \( C(N) \) as in the first line of (3.6) and (2.3) we have

\[
\mathcal{F}_{33} \lesssim \sum_{N_3 \gg 1} \left( \sum_{N_1 < N_2 < N_3} N_3^3 N_1^{-1} N_2^{1/2} \| u_{N_1} \|_X \| v_{N_2} \|_X \right)^2
\]

\[
\lesssim \| u \|_X^2 \| v \|_X^2 \sum_{N_3 > 1} \left( \sum_{N_2 < N_3} N_2^3 N_3^{1/2} \| v_{N_2} \|_X \right)^2
\]

where to obtain the second inequality we used Cauchy-Schwarz in \( N_1 \).

(iv). \( \mathcal{F}_{34} \): By Lemma 4(i) with \( C(N) \) as in the second line of (3.6) and (2.3) we have

\[
\mathcal{F}_{34} \lesssim \sum_{N_3 \gg 1} \left( \sum_{N_2 < N_3} N_3^3 N_1^{-1/2} \| u_{N_1} \|_X \| v_{N_2} \|_X \right)^2
\]

\[
\lesssim \| u \|_X^2 \| v \|_X^2 \sum_{N_3 > 1} \left( \sum_{N_2 < N_3} N_3^3 \| v_{N_2} \|_X \right)^2
\]

where to obtain the second inequality we used Cauchy-Schwarz in \( N_1 \sim N_2 \).

(v). \( \mathcal{F}_{35} \): By Lemma 4(i) with \( C(N) \) as in the fourth line of (3.6) and (2.3) we have

\[
\mathcal{F}_{35} \lesssim \sum_{N_3 > 1} \left( \sum_{N_2 < N_3} N_3^3 \max \left( N_1^{-1/2}, N_1^{-2+1/4} N_3^{1/4} \right) \| u_{N_1} \|_X \| v_{N_2} \|_X \right)^2
\]

\[
\lesssim \| u \|_X^2 \| v \|_X^2 \sum_{N_3 > 1} N_3^{2s} N_3^{-2s-1/4}
\]

\[
\lesssim \| u \|_X^2 \| v \|_X^2 \| v \|_X^2
\]

where to obtain the second inequality we used the fact that

\[
\max \left( N_1^{-1/2}, N_1^{-2+1/4} N_3^{1/4} \right) \lesssim N_3^{-s-1/4} N_1^{-2s} N_2^s
\]

and Cauchy-Schwarz in \( N_1 \sim N_2 \).

\[
\square
\]

4. ESTIMATES IN GEVREY TYPE SPACES AND WELL-POSEDNESS OF (1.1)

The Bourgain-Gevrey type space, denoted \( \check{X}^{\sigma,s} \), is defined with respect to the norm

\[
\| u \|_{\check{X}^{\sigma,s}} = \| e^{\sigma |D_x|} u \|_{\check{X}^s}.
\]

When \( \sigma = 0 \) the spaces \( \check{X}^{\sigma,s} \) coincides with \( \check{X}^s \). The restrictions of \( \check{X}^{\sigma,s} \) to a time slab \( I \times \mathbb{R} \) is defined in a similar way as before.
4.1. Linear estimates in Gevrey space. By substitution \( u \to e^{\alpha |D_x|} u \) and \( f \to e^{\alpha |D_x|} f \) in Lemma 2(iii), (iv) and Lemma 3(a), respectively, we easily get the following.

**Lemma 7.** Let \( \sigma \geq 0 \), \( s \in \mathbb{R} \). Then

(i) we have \( \tilde{X}^{\sigma,s} \subset C(\mathbb{R}, G^{\sigma,s}) \) and

\[
\sup_{t \in \mathbb{R}} \| u(t) \|_{G^{\sigma,s}} \leq C \| u \|_{\tilde{X}^{\sigma,s}}.
\]

for some absolute constant \( C > 0 \).

(ii) for all \( s_1 \leq s_2 \) we have \( \tilde{X}^{\sigma,s_2} \subset \tilde{X}^{\sigma,s_1} \).

(iii) for all \( f \in G^{\sigma,s} \) there exists a constant \( C > 0 \) such that

\[
\| \tilde{X}^{\sigma,s} \|_{G^{\sigma,s}} \leq C \| f \|_{G^{\sigma,s}}.
\]

4.2. Bilinear estimates in Gevrey space. From Lemma 6 and a simple triangle inequality we obtain the following.

**Corollary 2.** Let \( \mathcal{B}(u,v) \) be the bilinear form in Lemma 6. Then for all \( u,v \in \tilde{X}^{\sigma,s} \), where \( \sigma \geq 0 \) and \( s \in [-3/4,0] \), we have

\[
\| \mathcal{B}(u,v) \|_{\tilde{X}^{\sigma,s}} \lesssim \left( \| u \|_{\tilde{X}^{\sigma,s}} \| v \|_{\tilde{X}^{\sigma,-\frac{3}{4}}} + \| u \|_{\tilde{X}^{\sigma,-\frac{3}{4}}} \| v \|_{\tilde{X}^{\sigma,s}} \right) \quad (4.1)
\]

**Proof.** By definition of the \( \tilde{X}^{\sigma,s} \)-norm we have

\[
\| \mathcal{B}(u,v) \|_{\tilde{X}^{\sigma,s}}^2 = \| e^{\alpha |D_x|} \mathcal{B}(u,v) \|_{\tilde{X}^{\sigma,s}}^2 \lesssim L_1 + L_2,
\]

where

\[
L_1 = e^{\alpha} \| P_1 \mathcal{B}(u,v) \|_{L^2_x L^\infty_t}^2,
\]

\[
L_2 = \sum_{N_1 > 1} N_1^2 e^{2\alpha N_1} \| P_{N_1} \mathcal{B}(u,v) \|_{L^2_x}^2.
\]

But using the estimate for \( J_1 \) in (3.12) we obtain

\[
L_1 \lesssim e^{2\alpha} \| u \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2 \| v \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2 \lesssim \| u \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2 \| v \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2.
\]

Decomposing \( u \) and \( v \) we have

\[
L_2 \lesssim \sum_{N_1 > 1} \left( \sum_{N_1, N_2 > 1} N_1^2 e^{\alpha N_1} \| P_{N_1} \mathcal{B}(u_{N_1}, v_{N_2}) \|_{L^2_x} \right)^2.
\]

Let \( U = e^{\alpha |D_x|} u \) and \( V = e^{\alpha |D_x|} v \). Since \( N_3 \lesssim N_1 + N_2 \), by the triangle inequality, it follows that

\[
e^{\alpha N_1} \lesssim e^{\alpha N_1} e^{\alpha N_2}
\]

which can be combined with the estimate for \( J_2 \) in (3.13) above to obtain

\[
L_2 \lesssim \sum_{N_1, N_2 > 1} \left( \sum_{N_1, N_2 > 1} N_1^2 \| P_{N_1} \mathcal{B}(U_{N_1}, V_{N_2}) \|_{L^2_x} \right)^2
\]

\[
\lesssim \| U \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2 \| V \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2 + \| U \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2 \| V \|_{\tilde{X}^{\sigma,-\frac{3}{4}}}^2
\]

\[
= \| u \|_{\tilde{X}^{\sigma,s}}^2 \| v \|_{\tilde{X}^{\sigma,s}}^2 + \| u \|_{\tilde{X}^{\sigma,s}}^2 \| v \|_{\tilde{X}^{\sigma,s}}^2.
\]

\[\square\]
4.3. **Local well-posedness in Gevrey class.** Define the map

\[ \Phi(u)(t) = \chi(t/4)W(t)f + \mathcal{B}(u, u)(t). \]

Let \( s \in [-3/4, 0] \) and \( J = [-1, 1] \). By Lemma 7(ii), (iii) and Corollary 2 we have

\[ \|\Phi(u)\|_{\mathcal{X}_{J}^{s}} \leq C\left( \|f\|_{G^{\sigma,s}} + \|u\|_{\bar{X}_{J}^{s}}^{2}\right). \]

A similar estimate can be derived for the difference \( \Phi(u) - \Phi(v) \), where \( v \) is also a solution. Then by a standard fixed point argument (1.1) admits a unique solution

\[ u \in \bar{X}_{J}^{s} \subset C\left(J; G^{\sigma,s}(\mathbb{R})\right) \]

provided the data norm \( \|f\|_{G^{\sigma,s}} \) is sufficiently small. Moreover, the data to solution map \( f \mapsto u \) is Lipschitz continuous from \( \{f \in G^{\sigma,s} : \|f\|_{G^{\sigma,s}} \leq \epsilon\} \) to \( C(I; G^{\sigma,s}(\mathbb{R})) \). Moreover, the solution \( u \) satisfies the bound \( \|u\|_{\mathcal{X}_{J}^{s}} \lesssim \|f\|_{G^{\sigma,s}} \).

Finally, a local solution for (1.1) with arbitrarily large \( \|f\|_{G^{\sigma,s}} \) can be constructed using the scaling symmetry of KdV. Indeed, observe that if \( u \) solves (1.1) so does

\[ u_{\lambda}(t, x) = \lambda^{2}u(\lambda^{3}t, \lambda x) \]  

with initial data \( f_{\lambda}(x) = \lambda^{2}f(\lambda x) \) for some \( \lambda > 0 \). Now given \( f \) with arbitrarily large \( \|f\|_{G^{\sigma,s}} \) one can choose \( \lambda \) to be arbitrarily small \((0 < \lambda \ll 1)\) that

\[ \|f_{\lambda}\|_{G^{\sigma,s}} = \lambda^{2}\left( \int_{\mathbb{R}} e^{\lambda \alpha(x)}|\hat{f}(\xi)|^{2}d\xi\right)^{1/2} \lesssim \lambda^{2+\epsilon} \|f\|_{G^{\sigma,s}} \ll 1. \]  

By the above argument on local existence theory there exists a solution \( u_{\lambda} \in C\left(J; G^{\sigma,s}(\mathbb{R})\right) \) to (1.1) with initial data \( u_{\lambda}(0) = f_{\lambda} \). By the scaling (4.2) \( u \) solves (1.1) on \( I \times \mathbb{R} \), where \( I = \lambda^{3}J = [-\lambda^{3}, \lambda^{3}] \). In view of (4.3) the time of existence is given by

\[ t_{0} := \lambda^{3} = c\left( \|f\|_{G^{\sigma,s}}\right)^{-\frac{1}{3\alpha+1}}, \]

for some \( 0 < c \ll 1 \).

In conclusion, we have the following local well-posedness result in Gevrey class.

**Theorem 2** (Local well-posedness). Let \( \sigma > 0 \) and \( s \in [-3/4, 0] \). Then for any \( f \in G^{\sigma,s} \) there exists a time

\[ t_{0} = C_{0}(\|f\|_{G^{\sigma,s}}) > 0 \]

and a unique solution \( u \) of (1.1) on the time interval \( I = [-t_{0}, t_{0}] \) such that

\[ u \in C(I; G^{\sigma,s}). \]

Moreover, the solution depends continuously on the data \( f \), and satisfies the bound

\[ \|u\|_{\mathcal{X}_{J}^{s}} \leq C\|f\|_{G^{\sigma,s}}, \]  

where \( C \) depends only on \( s \). In particular, Lemma 7(i) and (4.4) gives the bound

\[ \sup_{t \in I} \|u(t)\|_{G^{\sigma,s}} \leq C(t)\|f\|_{G^{\sigma,s}}. \]

**Remark.** Theorem 2 shows that if the initial data \( f \) is analytic on the strip \( S_{\sigma} \) so is the solution \( u(t) \) on the same strip as long as \( t \in I \). Note also that in view of the embedding (1.2) we can allow all \( s \in \mathbb{R} \) in Theorem 2 but then the solution will be analytic only on a slightly smaller strip \( S_{\sigma-} \).
5. Almost conservation law and lower bound for $\sigma$

5.1. Almost conservation law. For a given $u(0) = f \in G^\sigma$ we have by the above local existence theory a solution $u(t) \in G^\sigma$ for $0 \leq t \leq t_0$, where (setting $s = 0$ in Theorem 2)

$$t_0 = C_0(\|f\|_{C^\sigma}) > 0$$

(5.1)

The solution $u$ satisfies the bound

$$\sup_{t \in [0, t_0]} \|u(t)\|_{C^\sigma} \leq C\|u(0)\|_{C^\sigma},$$

(5.2)

where the constant $C$ in (5.2) comes from (4.5) and is independent of $t_0$ and $\sigma$. The question is then whether we can improve on estimate (5.2). In what follows we will use equation (1.1) and the local existence theory mentioned above to obtain the approximate conservation law

$$\sup_{t \in [0, t_0]} \|u(t)\|_{C^\sigma}^2 = \|u(0)\|_{C^\sigma}^2 + \mathcal{E}_a(0),$$

(5.3)

where $\mathcal{E}_a(0)$ satisfies the bound $\mathcal{E}_a(0) \leq C\sigma^\frac{3}{2}\|u(0)\|_{C^\sigma}^\frac{3}{2}$. The quantity $\mathcal{E}_a(0)$ can be considered an error term since in the limit as $\sigma \to 0$, we have $\mathcal{E}_a(0) \to 0$, and hence recovering the well-known conservation of $L^2$-norm of solution:

$$\|u(t)\|_{L^2_x}^2 = \|u(0)\|_{L^2_x}^2 \quad \text{for all} \quad t \in [0, t_0].$$

Theorem 3. Let $I = [0, t_0]$, where $t_0$ is as in (5.1). Then there exists $C > 0$ such that for any $\sigma > 0$ and any solution $u \in X^{0,\sigma}_I$ to the Cauchy problem (1.1) on the time interval $I$, we have the estimate

$$\sup_{t \in [0, t_0]} \|u(t)\|_{C^\sigma}^2 \leq \|u(0)\|_{C^\sigma}^2 + C\sigma^\frac{3}{2}\|u\|_{X^{0,\sigma}_I}^3.$$ 

Moreover, we have

$$\sup_{t \in [0, t_0]} \|u(t)\|_{C^\sigma}^2 \leq \|u(0)\|_{C^\sigma}^2 + C\sigma^\frac{3}{2}\|u(0)\|_{C^\sigma}^3,$$

(5.4)

Proof. The estimate (5.4) follows from (5.3) and (4.4) with $s = 0$. Thus, it remains to prove (5.3). To this end set

$$w = e^{\sigma|D_x|^3}u,$$

where $u$ is the solution to (1.1). Since $u$ is real-valued so is $w$. Now set

$$f(w) = \frac{1}{2} \partial_x \left[ w \cdot w - e^{\sigma|D_x|^3} \left( e^{-\sigma|D_x|^3} w \cdot e^{-\sigma|D_x|^3} w \right) \right].$$

(5.5)

Then we use (1.1) to obtain

$$w_t + w_{xxx} + ww_x = f(w).$$

(5.6)

Multiplying (5.6) by $w$ and integrating in space we obtain

$$\frac{1}{2} \frac{d}{dt} \int_R w^2 dx + \int_R \partial_x \left(ww_{xx} - \frac{1}{2} w^2 + \frac{1}{3} w^3 \right) dx = \int_R w f(w) dx.$$

We may assume $w, w_x$ and $w_{xx}$ decays to zero as $|x| \to \infty$. This in turn implies

$$\frac{d}{dt} \int_R w^2 dx = 2 \int_R w f(w) dx.$$
Now integrating in time over the interval $I = [0, t_0]$, where $t_0 \leq 1$, we obtain
\[
\int_R w^2(t_0, x) \, dx = \int_R w^2(0, x) \, dx + 2 \int_{\mathbb{R}^{d+1}} \mathbb{1}_I(t) w f(w) \, dt \, dx.
\]
We conclude that
\[
\|u(t_0)\|_{C^{\sigma}}^2 \leq \|u(0)\|_{C^{\sigma}}^2 + \mathcal{R},
\]
where
\[
\mathcal{R} = 2 \left\{ \int_{\mathbb{R}^{d+1}} \mathbb{1}_I(t) w f(w) \, dt \, dx \right\}.
\]
Now combining (5.7) with the estimate for $\mathcal{R}$ in (5.8) below and using
\[
\|w\|_{\mathcal{X}_f^\sigma} = \|u\|_{\mathcal{X}_f^{\sigma,0}}
\]
we obtain (5.3).

The proof for the following Lemma is given in the next section.

**Lemma 8.** For all $\sigma \geq 0$ and $w \in \mathcal{X}_f^0$ we have
\[
\mathcal{R} \leq C \sigma^{\frac{3}{4}} \|w\|_{\mathcal{X}_f^\sigma}^\frac{3}{2},
\]
where $\sigma \in (0, \sigma_0]$ is a parameter to be chosen later.

### 5.2. Lower bound for $\sigma$

Let $f = u(0) \in G^{\sigma_0}$ for some $\sigma_0 > 0$. To construct a solution on $[0, T]$ for arbitrarily large $T$ we can apply the approximate conservation law (5.4) so as to repeat the local result on successive short time intervals to reach $T$, by adjusting the strip width parameter $\sigma$ according to the size of $T$. By employing this strategy one can show that the solution $u$ to (1.1) satisfies
\[
u(t) \in G^{\sigma(t)} \quad \text{for all} \quad t \in [0, T],
\]
with
\[
\sigma(t) \geq c T^{-\frac{1}{4}},
\]
where $c > 0$ is a constant depending on $\|f\|_{G^{\sigma_0}}$ and $\sigma_0$.

For completeness we include the proof (5.9)–(5.10) here which is similar to that of [30]. Define
\[
\Gamma_\sigma(t) = \|u(t)\|_{C^{\sigma}},
\]
where $\sigma \in (0, \sigma_0]$ is a parameter to be chosen later. By the local existence theory (see Theorem 2) there is a solution $u$ to (1.1) satisfying
\[
u(t) \in G^{\sigma_0} \quad \text{for all} \quad t \in [0, t_0]
\]
where
\[
t_0 = C_0 (\Gamma_{\sigma_0}(0)).
\]
Now fix $T$ arbitrarily large. We shall apply the above local result and (5.4) repeatedly, with a uniform time step $t_0$, and prove
\[
\sup_{t \in [0, T]} \Gamma_\sigma(t) \leq 2 \Gamma_{\sigma_0}(0)
\]
for $\sigma$ satisfying (5.10). Hence we have $\Gamma_\sigma(t) < \infty$ for all $t \in [0, T]$, which in turn implies $u(t) \in G^{\sigma(t)}$, and this completes the proof of (5.9)–(5.10).
It remains to prove (5.11). Choose $n \in \mathbb{N}$ so that $T \in [n t_0, (n + 1) t_0)$. Using induction we can show for any $k \in \{1, \ldots, n + 1\}$ that

$$\sup_{t \in [0, k t_0]} \Gamma^2_\sigma(t) \leq \Gamma^2_\sigma(0) + 2k^3 C \sigma^\frac{3}{2} \Gamma^3_\sigma(0),$$  \hfill (5.12)

$$\sup_{t \in [0, k t_0]} \Gamma^2_\sigma(t) \leq 2\Gamma^2_\sigma(0),$$  \hfill (5.13)

provided $\sigma$ satisfies

$$\frac{2T}{t_0} \frac{2^\frac{3}{2}}{3^2} C \sigma^\frac{3}{2} \Gamma^3_\sigma(0) \leq 1. \hfill (5.14)$$

Indeed, for $k = 1$, we have from (5.4) that

$$\sup_{t \in [0, t_0]} \Gamma^2_\sigma(t) \leq \Gamma^2_\sigma(0) + C \sigma^\frac{3}{2} \Gamma^3_\sigma(0) \leq \Gamma^2_\sigma(0) + C \sigma^\frac{3}{2} \Gamma^3_\sigma(0),$$

where we used $\Gamma_\sigma(0) \leq \Gamma_\sigma(0)$. This in turn implies (5.13) provided

$$C \sigma^\frac{3}{2} \Gamma^3_\sigma(0) \leq 1$$

which holds by (5.14) since $T > t_0$.

Now assume (5.12) and (5.13) hold for some $k \in \{1, \ldots, n\}$. Then applying (5.4), (5.13) and (5.12), respectively, we obtain

$$\sup_{t \in [k t_0, (k + 1) t_0]} \Gamma^2_\sigma(t) \leq \Gamma^2_\sigma((k - 1) t_0) + C \sigma^\frac{3}{2} \Gamma^3_\sigma((k - 1) t_0) \leq \Gamma^2_\sigma(0) + 2k^3 C \sigma^\frac{3}{2} \Gamma^3_\sigma(0) \leq \Gamma^2_\sigma(0) + (k + 1)2^\frac{3}{2} C \sigma^\frac{3}{2} \Gamma^3_\sigma(0).$$

Combining this with the induction hypothesis (5.12) (for $k$) we obtain

$$\sup_{t \in [0, (k + 1) t_0]} \Gamma^2_\sigma(t) \leq \Gamma^2_\sigma(0) + (k + 1)2^\frac{3}{2} C \sigma^\frac{3}{2} \Gamma^3_\sigma(0),$$

which proves (5.12) for $k + 1$. This also implies (5.13) for $k + 1$ provided

$$(k + 1)2^\frac{3}{2} C \sigma^\frac{3}{2} \Gamma^3_\sigma(0) \leq 1.$$

But the latter follows from (5.14) since

$$k + 1 \leq n + 1 \leq \frac{T}{t_0} + 1 \leq \frac{2T}{t_0}. $$

Finally, the condition (5.14) is satisfied for $\sigma$ such that

$$\frac{2T}{t_0} \frac{2^\frac{3}{2}}{3^2} C \sigma^\frac{3}{2} \Gamma^3_\sigma(0) = 1.$$

This implies

$$\sigma = c_0 T^{-\frac{1}{4}}, \quad \text{where} \quad c_0 = \left[ \frac{C_0 \Gamma_\sigma(0)}{2^\frac{3}{2} C \sigma^\frac{3}{2} \Gamma^3_\sigma(0)} \right]^{\frac{1}{4}}.$$

Thus, by choosing $c$ such that $c \leq c_0$ we obtain (5.10).
6. Proof of Lemma 8

We recall that
\[ R = 2 \left| \int_{\mathbb{R}^d} 1(t) \cdot w f(w) \, dt \right|, \]
where
\[ f(w) = \frac{1}{2} \partial_x \left[ w \cdot w - e^{\sigma|D_x|} \left( e^{-\sigma|D_x|} w \cdot e^{-\sigma|D_x|} w \right) \right]. \]

By Plancherel and (2.1) we have
\[ R = 2 \left| \int_{\mathbb{R}^d} 1(t) \cdot \hat{w}(\xi) f(w)(-\xi) \, dt d\xi \right| = 2 \left| \sum_{N_1 \geq 1} \int_{\mathbb{R}^d} 1(t) f(w) \cdot P_{N_1} w \, dt \right|, \]
where
\[ R_1 = \left\| \int_{\mathbb{R}^d} 1(t) f(w) \cdot P_{N_1} w \right\|_{L^1_t}, \]
\[ R_2 = \sum_{N_2 \geq 1} \left\| \int_{\mathbb{R}^d} 1(t) f(w) \cdot P_{N_2} w \right\|_{L^1_t}. \]

Moreover, since
\[ w = \sum_{N \geq 1} w_N \]
on one can also write
\[ 2P_{N_2} f(w) = \sum_{N_1, N_2 \geq 1} P_{N_1} \partial_x \left[ w_{N_1} \cdot w_{N_2} - e^{\sigma|D_x|} e^{-\sigma|D_x|} w_{N_1} \cdot w_{N_2} \right]. \]

Now taking the Fourier Transform of \( P_{N_2} f(w) \) we get
\[ 2|\mathcal{F}_x \left[ P_{N_2} f(w) \right] (\xi)| \leq \beta_{N_2} (\xi) \sum_{N_1, N_2 \geq 1} \int_{\mathbb{R}^2} |\xi| \left| 1 - e^{-\sigma(\xi_1 + \xi_2)} \prod_{j=1}^2 \beta_{N_j} (\xi_j) \right| d\mu(\xi), \]
where \( d\mu \) is the surface measure
\[ d\mu(\xi) = \delta(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2. \]

Note that for \( r \geq 0 \) and \( 0 \leq \theta \leq 1 \) we have the simple inequality
\[ 1 - e^{-r} \leq r^\theta. \]
Setting \(r = \sigma(|\xi_1| + |\xi_2| - |\xi_1 + \xi_2|) \geq 0\) and \(\theta = \frac{3}{4}\) we obtain
\[
1 - e^{-\sigma(|\xi_1| + |\xi_2| - |\xi_1 + \xi_2|)} 
\leq \sigma^\frac{3}{4} (|\xi_1| + |\xi_2| - |\xi_1 + \xi_2|)^\frac{3}{4} 
\leq \sigma^\frac{3}{4} \left( \frac{2(|\xi_1||\xi_2| - \xi_1 \xi_2)}{|\xi_1| + |\xi_2| + |\xi_1 + \xi_2|} \right)^\frac{3}{4} 
\leq \sigma^\frac{3}{4} (2 \min(|\xi_1|, |\xi_2|))^\frac{3}{4} 
\sim \sigma^\frac{3}{4} \min(N_1, N_2)^\frac{3}{4}.
\]
Thus, we have
\[
\left| \mathcal{F}_x \left[ P_{N_3} f(v) \right] (\xi) \right| 
\lesssim \sigma^\frac{3}{4} \sum_{N_1, N_2 \geq 1} \min(N_1, N_2)^\frac{3}{4} |\xi| \beta_{N_1}(\xi) \int_{\mathbb{R}^d} \prod_{j=1}^2 \beta_{N_j}(\xi_j) |\hat{w}(\xi_j)| d\sigma(\xi) 
= \sigma^\frac{3}{4} \sum_{N_1, N_2 \geq 1} \min(N_1, N_2)^\frac{3}{4} \left| \mathcal{F}_x \left[ P_{N_3} \partial_x \left( w_{N_1} w_{N_2} \right) \right] (\xi) \right|,
\]
and hence by Plancherel
\[
\| P_{N_3} f(w) \|_{L^2_x} \lesssim_{N_1, N_2 \geq 1} \sigma^\frac{3}{4} \min(N_1, N_2)^\frac{3}{4} \| P_{N_3} \partial_x \left( w_{N_1} w_{N_2} \right) \|_{L^2_x}. \tag{6.1}
\]
Now we give the estimate for \(\mathcal{R}_1\) and \(\mathcal{R}_2\).

6.1. **Estimate for \(\mathcal{R}_1\)**. Recall that \(I = [0, t_0]\), where \(t_0 = 1\). By Hölder inequality and Lemma 2(i) we have
\[
\mathcal{R}_1 \lesssim \left\| I(t)(P_1 w \cdot P_1 f(w)) \right\|_{L^1_t L^\infty_x} 
\lesssim t_0^\frac{1}{2} \| P_1 w \|_{L^\infty_t L^2_x} \left\| I(t) P_1 f(w) \right\| 
\lesssim \| w \|_{\tilde{X}_0} \left\| I(t) P_1 f(w) \right\|.
\]
Now we claim that
\[
\mathcal{R}_{11} := \left\| I(t) P_1 f(w) \right\| \lesssim \sigma^\frac{3}{4} \| w \|_{\tilde{X}_0}^2. \tag{6.2}
\]
This in turn implies the desired estimate for \(\mathcal{R}_1\), i.e.,
\[
\mathcal{R}_1 \lesssim \sigma^\frac{3}{4} \| w \|_{\tilde{X}_0}^3.
\]
Next we prove claim (6.2). By (6.1) we have
\[
\mathcal{R}_{11} \lesssim \sigma^\frac{3}{4} \sum_{N_1, N_2 \geq 1} \min(N_1, N_2)^\frac{3}{4} \left\| I(t) P_1 \partial_x \left( w_{N_1} \cdot w_{N_2} \right) \right\|.
\]
By symmetry we may assume \(N_1 \leq N_2\).
6.1.1. Case: $1 \leq N_1 \leq N_2 \lesssim 1$. By Sobolev, Hölder inequality and Lemma 2(ii)
\[
\mathcal{R}_{11} \lesssim \tau_0^3 \sigma^3 \sum_{i \leq N_1 \leq N_2 \leq 1} N_1^3 \| P_1 (w_{N_1} \cdot w_{N_2}) \|_{L_t^\infty L_x^1} \\
\lesssim \tau_0^3 \sigma^3 \sum_{i \leq N_1 \leq N_2 \leq 1} \| w_{N_1} \|_{L_t^\infty L_x^4} \| w_{N_2} \|_{L_t^\infty L_x^3} \\
\lesssim \sigma^3 \| w \|_{X_0}^2.
\]

6.1.2. Case: $N_1 \sim N_2 \gg 1$. Decomposing in modulation and in the output frequency we get
\[
\mathcal{R}_{11} \lesssim \sigma^3 \sum_{N_1 \sim N_2 \gg 1} \sum_{k \geq 1} N_1^3 \| P_1 Q_{L_k} \partial_x (P_{N_1} Q_{L_1} w \cdot P_{N_2} Q_{L_2} w) \| \\
\lesssim \sigma^3 \sum_{N_1 \sim N_2 \gg 1, 0 \leq M \leq 1} \sum_{k \geq 1} N_1^3 M \cdot \mathcal{K}_M(N_1, N_2)
\]
where
\[
\mathcal{K}_M(N_1, N_2) = \sum_{L_1, L_2, L_3 \geq 1} \| \hat{P}_M Q_{L_3} (P_{N_1} Q_{L_1} w \cdot P_{N_2} Q_{L_2} w) \|.
\]
Next we show that
\[
\mathcal{K}_M(N_1, N_2) \lesssim N_1^{-1} \| w_{N_1} \|_{X} \| w_{N_2} \|_{X} \tag{6.3}
\]
This in turn implies
\[
\mathcal{R}_{11} \lesssim \sigma^3 \sum_{N_1 \sim N_2 \gg 1} N_1^{-\frac{3}{2}} \| w_{N_1} \|_{X} \| w_{N_2} \|_{X} \\
\lesssim \sigma^3 \| w \|_{X_0}^2,
\]
where we used Cauchy-Schwarz in $N_1 \sim N_2$.

Now we prove (6.3). By Proposition 1(c) (see also Remark 2 and Corollary 1) we obtain
\[
\mathcal{K}_M(N_1, N_2) \lesssim N_1^{-1} \sum_{L_1, L_2, L_3 \geq 1} (L_{\min} L_{\med})^\frac{1}{2} \prod_{j=1}^2 \| P_{N_j} Q_{L_j} w \|.
\]
By symmetry we may assume $L_1 \leq L_2$. If $L_2 \geq L_3$, then by (2.3)
\[
\mathcal{K}_M(N_1, N_2) \lesssim N_1^{-1} \sum_{L_1, L_2 \geq 1, L_3 \leq L_2} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \prod_{j=1}^2 \| P_{N_j} Q_{L_j} w \| \\
\lesssim N_1^{-1} \sum_{j=1}^2 L_j^{\frac{3}{2}} \| P_{N_j} Q_{L_j} w \| \\
= N_1^{-1} \| w_{N_1} \|_{X} \| w_{N_2} \|_{X}.
\]
If $L_2 \leq L_3$ then $L_3 \sim N_1^3 M$, and hence by (2.3) (with $\alpha - \beta$) we obtain
\[
\mathcal{K}_M(N_1, N_2) \lesssim N_1^{-1} \sum_{j=1}^2 L_j^{\frac{3}{2}} \| P_{N_j} Q_{L_j} w_j \| \left( \sum_{L_3 \sim N_1^3 M} 1 \right) \\
= N_1^{-1} \| w_{N_1} \|_{X} \| w_{N_2} \|_{X}.
\]
6.2. Estimate for $R_2$. Decomposing in modulation and using Cauchy-Schwarz we obtain

$$R_2 \lesssim \sum_{N_3 > 1} \sum_{L_3 \geq 1} \left\| \Pi \left( P_{N_3} Q_{L_3} w \cdot P_{N_3} Q_{L_3} f(w) \right) \right\|_{L^2_x}$$

$$\lesssim \sum_{N_3 > 1} \left( \sup_{L_3 \geq 1} \left\| \Pi P_{N_3} Q_{L_3} w \right\| \right) \left( \sum_{L_3 \geq 1} L_3^{-1} \left\| \Pi P_{N_3} Q_{L_3} f(w) \right\| \right)$$

$$\lesssim \sum_{N_3 > 1} \left\| w_{N_3} \right\|_X \left\| \Lambda^{-1} P_{N_3} f(w) \right\|_X$$

$$\lesssim \left( \sum_{N_3 > 1} \left\| w_{N_3} \right\|_X^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{N_3 > 1} \left\| \Lambda^{-1} P_{N_3} f(w) \right\|_X^2 \right)^{\frac{1}{2}}$$

By (6.1) we have

$$\left\| \Lambda^{-1} P_{N_3} f(w) \right\|_X \lesssim \sigma_{N_1, N_2} \min(N_1, N_2) \left\| \Lambda^{-1} P_{N_3} \partial_x (w_{N_1} \cdot w_{N_2}) \right\|_X.$$

Then

$$R_2 \lesssim \sigma \left\| w \right\|_{X^0} \left( \sum_{N_3 > 1} \left( \sum_{N_1, N_2 \geq 1} \min(N_1, N_2) \left\| \Lambda^{-1} P_{N_3} \partial_x (w_{N_1} w_{N_2}) \right\|_X \right) \right)^{\frac{1}{2}}$$

$$\lesssim \sigma \left\| w \right\|_{X^0} \left( R_3 + R_4 \right),$$

where

$$R_3 = \sum_{N_3 > 1} \left( \sum_{1 \leq N_1 \leq N_2} N_1^{\frac{1}{2}} \left\| \Lambda^{-1} P_{N_3} \partial_x (w_{N_1} w_{N_2}) \right\|_X \right)^2,$$

$$R_4 = \sum_{N_3 > 1} \left( \sum_{N_1 \leq N_2 \geq 1} N_2^{\frac{3}{2}} \left\| \Lambda^{-1} P_{N_3} \partial_x (w_{N_1} w_{N_2}) \right\|_X \right)^2.$$

By symmetry we may only estimate $R_3$. Thus, it suffices to prove

$$R_3 \lesssim \left\| w \right\|_{X^0}^4.$$

In view of (3.2) this reduces further to

$$R_{3k} \lesssim \left\| w \right\|_{X^0}^4 \quad (k = 1, \ldots, 5),$$

where

$$R_{31} = \sum_{N_3 > 1} \left( \sum_{1 \leq N_1 \leq N_2 \leq 1} \left( \cdots \right) \right)^2, \quad R_{32} = \sum_{N_3 > 1} \left( \sum_{1 \leq N_1 < N_2 < N_3} \left( \cdots \right) \right)^2,$$

$$R_{33} = \sum_{N_3 > 1} \left( \sum_{1 < N_1 < N_2 < N_3} \left( \cdots \right) \right)^2, \quad R_{34} = \sum_{N_3 > 1} \left( \sum_{N_1 = N_2 < N_3} \left( \cdots \right) \right)^2, \quad R_{35} = \sum_{N_3 > 1} \left( \sum_{N_1 = N_2 = N_3} \left( \cdots \right) \right)^2.$$
(i). \( R_{31} \): By Lemma 4(iii) and Lemma 2(i) we have
\[
R_{31} \lesssim \sum_{N_1-1} \left( \sum_{1 \leq N_2 \leq N_1-1} \| w_{N_1} \|_{L_t \ell_x} \| w_{N_2} \|_{L_t \ell_x} \right)^2
\lesssim \| w \|_{X_0}^4.
\]

(ii). \( R_{32} \): By Lemma 4(ii) and (2.3) we have
\[
R_{32} \lesssim \sum_{N_3 \gg 1} \left( \sum_{1 \leq N_1 \ll N_3} \frac{N_1^{-\frac{1}{4}} N_2^{-\frac{1}{2}+}}{w_{N_1}} \| w_{N_2} \|_X \right)^2
\lesssim \| w \|_{X_0}^2 \sum_{N_3 \gg 1} \left( \sum_{N_2 \gg N_3} \frac{N_2^{-\frac{1}{2}+}}{w_{N_2}} \right)^2
\lesssim \| w \|_{X_0}^4.
\]

(iii). \( R_{33} \): By Lemma 4(i) with \( C(N) \) as in the first line of (3.6) and (2.3) we have
\[
R_{33} \lesssim \sum_{N_1 \gg 1} \left( \sum_{1 \leq N_1 \ll N_3} \frac{N_1^{-\frac{1}{4}}}{w_{N_1}} \| w_{N_2} \|_X \right)^2
\lesssim \| w \|_{X_0}^2 \sum_{N_3 \gg 1} \left( \sum_{N_2 \gg N_3} \frac{N_2^{-\frac{1}{2}+}}{w_{N_2}} \right)^2
\lesssim \| w \|_{X_0}^4
\]
where to obtain the second inequality we used Cauchy-Schwarz in \( N_1 \).

(iv). \( R_{34} \): By Lemma 4(i) with \( C(N) \) as in the second line of (3.6) and (2.3) we have
\[
R_{34} \lesssim \sum_{N_3 \gg 1} \left( \sum_{N_1 \sim N_3} \frac{N_1^{-\frac{1}{4}}}{w_{N_1}} \| w_{N_2} \|_X \right)^2
\lesssim \| w \|_{X_0}^2 \sum_{N_3 \gg 1} \left( \sum_{N_2 \sim N_3} \frac{N_2^{-\frac{1}{2}+}}{w_{N_2}} \right)^2
\lesssim \| w \|_{X_0}^4
\]
where to obtain the second inequality we used Cauchy-Schwarz in \( N_1 \sim N_2 \).

(v). \( R_{35} \): By Lemma 4(i) with \( C(N) \) as in the fourth line of (3.6) and (2.3) we have
\[
R_{35} \lesssim \sum_{N_3 \gg 1} \left( \sum_{N_1 \ll N_2 \gg N_3} \frac{N_1^{-\frac{1}{4}}}{w_{N_1}} \max \left( N_1^{-\frac{1}{2}}, N_1^{-\frac{1}{2}+}, N_3^{\frac{1}{2}} \right) \| w_{N_2} \|_X \right)^2
\lesssim \| w \|_{X_0}^2 \sum_{N_3 \gg 1} \frac{N_3^{-\frac{1}{2}}}{w_{N_2}}
\lesssim \| w \|_{X_0}^4,
\]
where to obtain the second inequality we used Cauchy-Schwarz in \( N_1 \sim N_2 \).
Appendix A. Proof of Lemma 4

First we prove (3.7) and (3.8). By definition of $X$, Hölder inequality, (2.15) we have
\[
\left\| \lambda^{-1} P_{N_3} \partial_x (u_{N_1} v_{N_2}) \right\|_{X} \leq N_2 \sum_{l \geq 1} L_{l}^{-\frac{3}{2}} \left\| P_{N_1} Q_L (u_{N_1} v_{N_2}) \right\|
\]
\[
\leq N_2 \left\| u_{N_1} v_{N_2} \right\|
\]
\[
\leq N_2 \left\| u_{N_1} \right\|_{L_{1}^{2} L_{x}^{\infty}} \left\| v_{N_2} \right\|_{L_{1}^{\infty} L_{x}^{2}}
\]
\[
\leq \left\| u_{N_1} \right\|_{L_{1}^{2} L_{x}^{\infty}} \left\| v_{N_2} \right\|_{X},
\]
where we also used the fact that $P_N$ and $Q_L$ are bounded in $L^2$. Thus, (3.7) is proved. Similarly, by definition of $X$, Hölder and Bernstein’s inequality we obtain
\[
\left\| \|I\| \lambda^{-1} P_{N_3} \partial_x (u_{N_1} v_{N_2}) \right\|_{X} \leq N_2 \sum_{l \geq 1} L_{l}^{-\frac{3}{2}} \left\| P_{N_1} Q_L (\|I\| u_{N_1} v_{N_2}) \right\|
\]
\[
\leq \left\| \|I\| P_{N_3} (u_{N_1} v_{N_2}) \right\|
\]
\[
\leq \left\| (u_{N_1} v_{N_2}) \right\|_{L_{1}^{\infty} L_{1}^{2}}
\]
\[
\leq \left\| u_{N_1} \right\|_{L_{1}^{2} L_{x}^{\infty}} \left\| v_{N_2} \right\|_{L_{1}^{\infty} L_{x}^{2}}
\]

which is (3.8).

To prove (3.5)–(3.6) we repeatedly use Corollary 1, Proposition 1, the constraints in (3.2) and (3.3). To this end we set
\[
\mathcal{J}(N) = \left\| \lambda^{-1} P_{N_3} \partial_x (u_{N_1} v_{N_2}) \right\|_{X}
\]
and denote
\[
u_{N_1, L_1} = P_{N_1} Q_{L_1} u, \quad \nu_{N_2, L_2} = P_{N_2} Q_{L_2} v.
\]
We now prove (3.5)–(3.6) by estimating $\mathcal{J}(N)$ case by case.

A.1. Case $N_3 \sim N_2 \gg N_1 > 1$. By definition of $X$, decomposition in modulation, Corollary 1 with $C(N, L)$ as in Proposition 1 we have
\[
\mathcal{J}(N) \lesssim N_2 \sum_{l \geq 1} L_{l}^{-\frac{3}{2}} \left\| P_{N_1} Q_{L_2} (u_{N_1} v_{N_2}) \right\|
\]
\[
\lesssim N_2 \sum_{l_1, L_2, L_3 \geq 1} L_{L_3}^{-\frac{3}{2}} \left\| P_{N_1} Q_{L_2} (u_{N_1, L_1} \cdot v_{N_2, L_2}) \right\|
\]
\[
\lesssim N_2 \sum_{l_1, L_2, L_3 \geq 1} L_{L_3}^{-\frac{3}{2}} C(N, L) \left\| u_{N_1, L_1} \right\|_{L_{X}} \left\| v_{N_2, L_2} \right\|_{L_{X}}
\]
By assumption, (3.3), we have
\[
L_{\max} \gtrsim N_1 N_2^2.
\]
If $L_{\max} \gtrsim N_2^6$, then we choose $C(N, L)$ as in Proposition 1(c) to obtain
\[
\mathcal{J}(N) \lesssim N_1^{\frac{1}{2}} N_2 \sum_{l_{\max} \gtrsim N_2^6} L_{L_{\min} (L_1 L_2)^{-\frac{1}{2}}} \left( L_{L_1}^{-\frac{3}{2}} \left\| u_{N_1, L_1} \right\|_{L_{X}} \right) \left( L_{L_2}^{-\frac{3}{2}} \left\| v_{N_2, L_2} \right\|_{X} \right)
\]
\[
\lesssim N_2^{\frac{3}{2}} \left\| u_{N_1} \right\|_{X} \left\| v_{N_2} \right\|_{X}.
\]
Next assume $L_{\text{max}} \ll N_2^3$. Choosing $C(N, L)$ as in Proposition 1(b), i.e.,

\[ C(N, L) \lesssim (N_1 N_2)^{-\frac{1}{2}} (L_{\text{med}})^{\frac{1}{2}}, \]

we obtain

\[ \mathcal{J}(N) \lesssim N_1^{-\frac{1}{2}} N_2^{\frac{1}{2}} \sum_{L_{\text{min}} \ll N_2^3, N_1 N_2 \lesssim L_{\text{max}} \ll N_2^6} C(L) \left( L_1^{\frac{1}{2}} \| u_{N_1, L_1} \| \right) \left( L_2^{\frac{1}{2}} \| v_{N_2, L_2} \| \right), \]

where

\[ C(L) = L_3^{-\frac{1}{2}} (L_{\text{med}})^{\frac{1}{2}} (L_1 L_2)^{-\frac{1}{2}} = L_{\text{max}}^{\frac{1}{2}}. \]

Now if $L_{\text{max}} \sim L_3$ we have

\[ \mathcal{J}(N) \lesssim N_1^{-1} N_2^{\frac{1}{2}} \| u_{N_1} \|_X \| v_{N_2} \|_X. \]

If $L_{\text{max}} \sim L_1$ or $L_2$, then

\[ \mathcal{J}(N) \lesssim N_1^{-1} N_2^{\frac{1}{2}} \| u_{N_1} \|_X \| v_{N_2} \|_X. \]

A.2. **Case** $N_3 \sim N_2 \sim N_1 \gg 1$. Proceeding as above, for $C(N, L)$ is as in Proposition 1, we obtain.

\[ \mathcal{J}(N) \lesssim N_1 \sum_{L_1, L_2, L_3 \geq 1} L_3^{-\frac{1}{2}} C(N, L) \| u_{N_1, L_1} \| \| v_{N_2, L_2} \|, \]

A.2.1. **Sub-case**: $L_{\text{max}} \sim N_1^3$. Choosing $C(N, L)$ as in Proposition 1(a), we get

\[ \mathcal{J}(N) \lesssim N_1^3 \sum_{L_{\text{med}} \sim N_1^3} C(L) \left( L_1^{\frac{1}{2}} \| u_{N_1, L_1} \| \right) \left( L_2^{\frac{1}{2}} \| v_{N_2, L_2} \| \right), \]

where

\[ C(L) = L_3^{-\frac{1}{2}} L_{\text{med}}^{-\frac{1}{2}} (L_1 L_2)^{-\frac{1}{2}} = L_{\text{med}}^{-\frac{1}{2}} L_{\text{max}}^{-\frac{1}{2}}. \]

By symmetry, we may assume $L_1 \geq L_2$. It suffices to consider the case $L_2 \geq L_3$ (the other cases are easier to deal with). Then

\[ \mathcal{J}(N) \lesssim N_1^3 \sum_{L_1 \sim N_1^3} L_1^{\frac{1}{2}} L_2^{-\frac{1}{2}} \left( L_1^{\frac{1}{2}} \| u_{N_1, L_1} \| \right) \left( L_2^{\frac{1}{2}} \| v_{N_2, L_2} \| \right) \]

\[ \lesssim N_1^{-\frac{1}{2}} \| u_{N_1} \|_X \sum_{L_3 \geq L_1 \geq L_2} L_2^{-\frac{1}{2}} L_2^{\frac{1}{2}} \| v_{N_2, L_2} \| \]

\[ \lesssim N_1^{-\frac{1}{2}} \| u_{N_1} \|_X \| v_{N_2} \|_X. \]

A.2.2. **Sub-case**: $L_{\text{med}} \sim L_{\text{med}} \gg N_1^3$. Choosing $C(N, L)$ as in Proposition 1(c), we have

\[ \mathcal{J}(N) \lesssim N_1^3 \sum_{L_{\text{med}} \sim L_{\text{med}} \gg N_1^3} L_{\text{med}}^{\frac{1}{2}} (L_1 L_2)^{-\frac{1}{2}} \left( L_1^{\frac{1}{2}} \| u_{N_1, L_1} \| \right) \left( L_2^{\frac{1}{2}} \| v_{N_2, L_2} \| \right). \]

By symmetry we may assume $L_1 \leq L_2 \leq L_3$ which in turn implies $L_2 \sim L_3 \gg N_1^3$. Then

\[ \mathcal{J}(N) \lesssim N_1^3 \sum_{L_2 \sim L_3 \gg N_1^3} L_2^{-\frac{1}{2}} \left( L_1^{\frac{1}{2}} \| u_{N_1, L_1} \| \right) \left( L_2^{\frac{1}{2}} \| v_{N_2, L_2} \| \right) \]

\[ \lesssim N_1^{-\frac{1}{2}} \| u_{N_1} \|_X \| v_{N_2} \|_X. \]
A.3. Case \( N_1 \sim N_2 \gg N_3 = 1 \). By definition of \( X \), decomposing in modulation and in the output frequency, and using Proposition 1 we obtain

\[
\mathcal{J}(N) \lesssim \sum_{L_3 \geq 1} L_3^{-\frac{1}{2}} \left\| \hat{P}_1 Q L_3 \partial_x (u_{N_1} v_{N_2}) \right\|
\]

\[
\lesssim \sum_{0 < M \leq 1} \sum_{L_1, L_2 \geq 1} L_3^{-\frac{1}{2}} M \left\| \hat{P}_M Q L_3 (u_{N_1, L_1} \cdot v_{N_2, L_2}) \right\|
\]

\[
\lesssim \sum_{0 < M \leq 1} \sum_{L_1, L_2 \geq 1} L_3^{-\frac{1}{2}} M \cdot C(N, L) \left\| u_{N_1, L_1} \right\| \left\| v_{N_2, L_2} \right\|
\]

We may assume \( M \geq N_1^{-2} \), since otherwise the desired estimate follows easily.

A.3.1. Sub-case: \( L_{\max} \sim N_1^2 M \). We choose \( C(N, L) \) as in Proposition 1(b), i.e.,

\[
C(N, L) \lesssim (N_1 M)^{-\frac{3}{2}} (L_{\min} L_{\med})^{\frac{1}{2}}.
\]

We may assume \( L_{\max} = L_3 \), since the other cases are easier. Then

\[
\mathcal{J}(N) \lesssim \sum_{N_1^2 < N \leq 1} \sum_{L_1, L_2 \geq 1} N_1^{-\frac{3}{2}} \left( L_1^\frac{1}{2} \left\| u_{N_1, L_1} \right\| \right) \left( L_2^\frac{1}{2} \left\| v_{N_2, L_2} \right\| \right)
\]

\[
\lesssim N_1^{-\frac{3}{2} +} \left\| u_{N_1} \right\| \left\| v_{N_2} \right\|.
\]

A.3.2. Sub-case: \( L_{\max} \sim L_{\med} \gg N_1^2 M \). We choose \( C(N, L) \) as in Proposition 1(c) we obtain

\[
\mathcal{J}(N) \lesssim \sum_{N_1^2 < N \leq 1} \sum_{L_1, L_2 \geq 1} M L_1^\frac{1}{2} (L_2 L_3)^{-\frac{1}{2}} \left( L_1^\frac{1}{2} \left\| u_{N_1, L_1} \right\| \right) \left( L_2^\frac{1}{2} \left\| v_{N_2, L_2} \right\| \right)
\]

\[
\lesssim N_1^{-\frac{3}{2} +} \left\| u_{N_1} \right\| \left\| v_{N_2} \right\|.
\]

A.4. Case \( N_1 \sim N_2 \gg N_3 > 1 \). Proceeding as in Subsection A.1 we obtain

\[
\mathcal{J}(N) \lesssim N_3 \sum_{L_{\max} \geq N_1^2 N_3} L_3^{-\frac{1}{2}} C(N, L) \left\| u_{N_1, L_1} \right\| \left\| v_{N_2, L_2} \right\|
\]

If \( L_{\max} \gtrsim N_1^6 \), then choosing \( C(N, L) \) as in Proposition 1(c), we obtain

\[
\mathcal{J}(N) \lesssim N_3^\frac{3}{2} \sum_{L_{\max} \geq N_1^6} L_3^\frac{1}{2} L_1^\frac{1}{2} \left( L_2 L_3 \right)^{-\frac{1}{2}} \left( L_1^\frac{1}{2} \left\| u_{N_1, L_1} \right\| \right) \left( L_2^\frac{1}{2} \left\| v_{N_2, L_2} \right\| \right)
\]

\[
\lesssim N_1^{-3} N_3^\frac{1}{2} \left\| u_{N_1} \right\| \left\| v_{N_2} \right\|.
\]

Next assume \( L_{\max} \ll N_1^6 \). In this case we choose \( C(N, L) \) as in Proposition 1(b), i.e.,

\[
C(N, L) \lesssim (N_1 N_3)^{-\frac{3}{2}} (L_{\min} L_{\med})^{\frac{1}{2}},
\]

to obtain

\[
\mathcal{J}(N) \lesssim N_1^{-\frac{3}{2}} N_3^\frac{1}{2} \sum_{N_1^2 N_3 \leq L_{\max} \ll N_1^6} C(L) \left( L_1^\frac{1}{2} \left\| u_{N_1, L_1} \right\| \right) \left( L_2^\frac{1}{2} \left\| v_{N_2, L_2} \right\| \right)
\]

where

\[
C_L = L_3^{-\frac{1}{2}} (L_{\min} L_{\med})^{\frac{1}{2}} (L_1 L_2)^{-\frac{1}{2}} = L_{\max}^{-\frac{1}{2}}.
\]
Now if \( L_{\text{max}} \sim L_3 \) we have
\[
\mathcal{J}(N) \lesssim N_{\frac{7}{4}}^{-\frac{1}{2}} \left\| u_{N_1} \right\|_X \left\| v_{N_2} \right\|_X.
\]
If \( L_{\text{max}} \sim L_1 \) or \( L_2 \), then
\[
\mathcal{J}(N) \lesssim N_{\frac{1}{4}}^{-\frac{3}{2}} \left\| u_{N_1} \right\|_X \left\| v_{N_2} \right\|_X.
\]
By symmetry, we may assume \( L_1 \geq L_2 \). It suffices to consider the case \( L_2 \geq L_3 \) (the other cases are easier to deal with). Then
\[
\mathcal{J}(N) \lesssim N_{\frac{1}{4}}^{-\frac{3}{2}} N_{\frac{3}{2}}^3 \sum_{N_1^2 N_2 \lesssim N_1^4} L_{\frac{1}{2}} \left( L_{\frac{1}{2}} \left\| u_{N_1} \right\|_1 \right) \left( L_{\frac{1}{2}} \left\| v_{N_2} \right\|_2 \right)
\]
\[
\lesssim N_{\frac{7}{4}}^{-\frac{1}{2}} \left\| u_{N_1} \right\|_X \sum_{L_2, L_3; L_2 \geq L_3} L_{\frac{1}{2}} \left( L_{\frac{1}{2}} \left\| v_{N_2} \right\|_2 \right)
\]
\[
\lesssim N_{\frac{1}{4}}^{-\frac{3}{2}} \left\| u_{N_1} \right\|_X \left\| v_{N_2} \right\|_X.
\]

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