Indefinite Linear Quadratic Mean Field Social Control Problems with Multiplicative Noise

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Abstract

This paper studies uniform stabilization and social optimality for linear quadratic (LQ) mean field control problems with multiplicative noise, where agents are coupled via dynamics and individual costs. The state and control weights in cost functionals are not limited to be positive semi-definite. This leads to an indefinite LQ mean field control problem, which may still be well-posed due to deep nature of multiplicative noise. We first obtain a set of forward-backward stochastic differential equations (FBSDEs) from variational analysis, and construct a feedback control by decoupling the FBSDEs. By using solutions to two Riccati equations, we design a set of decentralized control laws, which is further shown to be asymptotically social optimal. Some equivalent conditions are given for uniform stabilization of the systems with the help of linear matrix inequalities. A numerical example is given to illustrate the effectiveness of the proposed control laws.

Index Terms

Mean field game, stabilization control, variational analysis, forward-backward stochastic differential equation, generalized Riccati equation

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I. INTRODUCTION

A. Background and motivation

The topic of mean field games and control has drawn increasing attention in many disciplines including system control, applied mathematics and economics [6], [7], [12]. A mean field game involves a very large number of small interacting players. While the influence of each player is negligible, the impact of the overall population is significant. By combining mean field approximations and individual best response, the dimensionality difficulty can be overcome.

Mean field games and control have found wide applications, including smart grids [27], [9], finance, economics [13], [8], [37], [15], and social networks [4], [25], etc.

Depending on the state-cost setup of a mean field game, it can be classified into linear-quadratic (LQ) type and more general nonlinear type. By now, the LQ type has been commonly adopted in mean field studies because of its analytical tractability and close connection to practical applications. In this aspect, some relevant works include [18], [24], [39], [5], [28]. Huang et al. developed the Nash certainty equivalence (NCE) based on the fixed-point method and designed an $\epsilon$-Nash equilibrium for LQ games with discount costs [18]. The NCE approach was then applied to the (general) cases with stochastic ergodic costs [24] and with Markov jump parameters [39], respectively. The works [10], [5] employed the adjoint equation approach and the fixed-point theorem to obtain sufficient conditions for the existence of the equilibrium strategy over a finite horizon. For other aspects of mean field games, readers are referred to [20], [22], [10] for nonlinear mean field games, [44] for oblivious equilibrium in dynamic games, [17], [40], [41] for mean field games with major players, [16], [28] for robust mean field games.

Apart from noncooperative games, team optimization forms another research branch for studying cooperative behavior among multiple decision makers. In particular, social optima in mean field models with weak coupling have drawn more research interests. By social optimization, all players in a large population system (endowed with some weak-coupling structure in either cost or dynamics) will cooperate to minimize a common social cost— the sum of individual costs. Accordingly, we formulate a type of team decision problem [30]. Different from Nash games, all the agents in a team problem are cooperative and share the same cost functional, although they may have different information sets [14]. Huang et al. considered social optima in mean field LQ control, and provided an asymptotic team-optimal solution [19]. Wang and Zhang [42].
investigated a mean field social optimal problem where the Markov jump parameter appears as a common source of randomness. For further literature on social control, for instance, see \[21\] for social optima in mixed games, \[3\] for team-optimal control with finite population and partial information, and \[33\] for the dynamic collective choice by finding a social optimum.

Concerned with mean field games and control for stochastic systems, most existing literature focused on the case with additive noise (i.e., the intensity of noise is independent of the state). Sometimes, such kind of noise is not sufficient to depict practical situations. Alternatively, multiplicative noise is another realistic description for stochastic disturbance. Mean field control with multiplicative noise has attracted much attention due to its wide applications in engineering, economics, and etc \[11\], \[15\], \[38\], \[43\]. This paper investigates uniform stabilization and social optimality for mean field LQ control systems with multiplicative noises, where subsystems are coupled via both dynamics and individual costs. The intensities of multiplicative noises depends on both system states and control inputs. The state weight $Q$ and control weight $R$ in the cost functional are not limited to be positive semi-definite. In fact, an indefinite $Q$ or $R$ may naturally occur in a wide class of practical problems, including production adjustment \[37\], uncertain systems \[16\], and portfolio selection \[50\]. This problem leads to generalized Riccati equations, which is essentially different from the classical Riccati equation due to indefinite weights and multiplicative noise appearing in the problem.

B. Challenge and main contributions

Most previous results on mean field games and control were given by virtue of the fixed-point analysis \[18\], \[24\], \[19\], \[10\], \[5\], \[42\]. However, the fixed-point assumption may be not easy to tackle, particularly for high-dimensional systems. In this paper, we solve the problem by decoupling forward-backward stochastic differential equations (FBSDEs) instead of fixed-point analysis. In recent years, some substantial progress for the optimal LQ control has been made by solving the FBSDEs. See \[45\], \[47\], \[48\], \[29\], \[36\], \[31\] for details.

For the finite-horizon mean field LQ control problem, we first obtain a set of FBSDEs by examining the social cost variation, and give a centralized feedback control by decoupling the FBSDEs. Applying mean field approximations, we design decentralized control laws. By exploiting the uniform convexity property of the optimal control problem, we further show that the decentralized controls have asymptotic social optimality. For the infinite-horizon case, we
construct a set of decentralized control laws by using solutions of two Riccati equations, and further show decentralized controls are asymptotically social optimal. Some equivalent conditions are further given for uniform stabilization of all the subsystems with the help of linear matrix inequalities.

For the mean field control systems with multiplicative noise, it is more difficult to show the uniform stabilization of all the subsystems than the case with additive noise. Due to the appearance of multiplicative noise, the approximation error between population state average $\hat{x}^{(N)}$ and aggregate effect $\bar{x}$ relies on the states of all the agents while the mean square of the state $\hat{x}_i$ conversely depends on the approximation error. Thus, we need to analyze jointly the approximation error and states of all the agents. By tackling the corresponding integral inequalities, we obtain that all the subsystems are uniformly stabilizable and the mean field approximation is consistent. Moreover, since the weights $Q$ and $R$ in the cost functional are indefinite, the prior boundedness of the state is not implied directly by the finiteness of the cost, which brings about extra difficulty to show the social optimality of decentralized control. Here we first obtain the prior upper bounds of states and controls by exploiting the uniform convexity property of the problem, and further prove that decentralized strategies have asymptotic social optimality by perturbation analysis.

The main contributions of the paper are summarized as follows.

• For the finite-horizon problem, we first obtain necessary and sufficient existence conditions of centralized optimal control based on FBSDEs, and then design a feedback-type decentralized control by decoupling FBSDEs and applying mean field approximations.

• By exploiting the uniform convexity of the problem, the decentralized control laws are shown to have asymptotic social optimality.

• The necessary and sufficient conditions are given for uniform stabilization of the systems by virtue of the system’s observability and linear matrix inequalities.

• An explicit expression of the asymptotic average social cost is given in terms of the solutions of two Riccati equations.

C. Organization and notation

The organization of the paper is as follows. In Section II, the indefinite LQ mean field social control problem is formulated. In Section III, we first construct a set of decentralized
control laws for the finite-horizon case, and then show its asymptotic social optimality. In Section IV, we design asymptotically optimal control for the infinite-horizon case and further give some equivalent conditions of uniform stabilization. In Section V, we give the value of asymptotic average optimal social cost. In Section VI, a numerical example is provided to show the effectiveness of the proposed controls. Section VII concludes the paper.

The following notation will be used throughout this paper. Denote by $\| \cdot \|$ the Euclidean vector norm or matrix spectral norm, and $\otimes$ the Kronecker product. For a vector $z$ and a matrix $Q$, $\| z \|^2_Q = z^T Q z$; $Q > 0$ $(Q \geq 0)$ means that the matrix $Q$ is positive definite (positive semi-definite). $Q^\dagger$ is the Moore-Penrose pseudoinverse of the matrix $Q$, $\mathcal{R}(Q)$ denotes the range of a matrix (or an operator) $Q$, and $\ker(Q)$ is the kernel of $Q$. For two vectors $x, y$, $\langle x, y \rangle = x^T y$. $L^2([0, \infty), \mathbb{R}^k)$ is given by $\{ f : [0, \infty) \rightarrow \mathbb{R}^k | \int_0^\infty \| f(t) \|^2 dt < \infty \}$. $L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$ is the space of all $\mathcal{F}_t$-adapted $\mathbb{R}^k$-valued processes $x(\cdot)$ such that $\mathbb{E} \int_0^T \| x(t) \|^2 dt < \infty$. For convenience of presentation, we use $c, c_1, c_2, \ldots$ to denote generic positive constants, which may vary from place to place.

II. Problem Description

Consider a large population systems with $N$ agents. Agent $i$ evolves by the following stochastic differential equation:

$$dx_i(t) = [Ax_i(t) + Bu_i(t) + Gx^{(N)}(t) + f(t)]dt + [Cx_i(t) + Du_i(t) + \sigma(t)]dW_i(t), \ 1 \leq i \leq N,$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^r$ are the state and input of the $i$th agent. $A, B, G, C, D$ are constant matrices with appropriate dimensions. $x^{(N)}(t) = \frac{1}{N} \sum_{j=1}^N x_j(t)$, $f, \sigma \in L^2([0, \infty), \mathbb{R}^n)$. $\{W_i(t), 1 \leq i \leq N\}$ are a sequence of independent 1-dimensional Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$. The cost functional of agent $i$ is given by

$$J_i(u) = \mathbb{E} \int_0^\infty \left\{ \| x_i(t) - \Gamma x^{(N)}(t) - \eta(t) \|^2_Q + \| u_i(t) \|^2_R \right\} dt,$$

where $Q, R, \Gamma \in \mathbb{R}^{n \times n}$ are constant matrices, and $\eta \in L^2([0, \infty), \mathbb{R}^n)$. $Q$ and $R$ are symmetric (generally indefinite). Denote $u = \{u_1, \ldots, u_i, \ldots, u_N\}$. The decentralized control set is given

$^1$ $Q^\dagger$ is a unique matrix satisfying $QQ^\dagger Q = Q^\dagger, Q^\dagger QQ^\dagger = Q, (Q^\dagger Q)^T = Q^\dagger Q$, and $(QQ^\dagger)^T = QQ^\dagger$. 

March 31, 2022  DRAFT
Also, denote
\[ d \text{ynamics of all agents can be written in the more compact form:} \]
\[ \mathcal{U}_d = \left\{ (u_1, \cdots, u_N) \mid u_i(t) \text{ is adapted to } \sigma(x_i(s), 0 \leq s \leq t), \mathbb{E} \int_0^\infty \| x_i(t) \|^2 dt < \infty, \forall i \right\}. \]

For comparison, define the centralized control set as
\[ \mathcal{U}_c = \left\{ (u_1, \cdots, u_N) \mid u_i(t) \text{ is adapted to } \mathcal{F}_t, \mathbb{E} \int_0^\infty \| x_i(t) \|^2 dt < \infty, \forall i \right\}, \]
where \( \mathcal{F}_t \triangleq \sigma\{\bigcup_{i=1}^N \mathcal{F}^i_t\} \) and \( \mathcal{F}^i_t = \sigma(x_i(0), W_i(s), 0 \leq s \leq t), i = 1, \cdots, N. \)

In this paper, we mainly study the following problem.

(P0) Seek a set of decentralized control laws to optimize social cost for the system (1)-(2), i.e., \( \inf_{u \in \mathcal{U}_d} J_{soc}(u) \), where
\[ J_{soc}(u) = \sum_{i=1}^N J_i(u). \]

We first make the assumption on the initial values of agents’ states.

A1) \( x_i(0), i = 1, \ldots, N \) are mutually independent and have the same mathematical expectation. \( x_i(0) = x_{i0}, \mathbb{E} x_i(0) = \bar{x}_0, i = 1, \cdots, N \). There exists a constant \( c_0 \) (independent of \( N \)) such that \( \max_{1 \leq i \leq N} \mathbb{E} \| x_i(0) \|^2 < c_0 \). Furthermore, \( \{ x_i(0), i = 1, \ldots, N \} \) and \( \{ W_i, i = 1, \ldots, N \} \) are independent of each other.

Remark 2.1: Since the weights \( Q \) and \( R \) are indefinite, Problem (P0) is called an indefinite LQ mean field social control problem. Due to the indefiniteness of \( Q \) and \( R \), the convexity may be lost, and the problem may have no solutions. Thus, we need to discuss the convexity of Problem (P0), which is related to the generalized Riccati equation.

To facilitate the discussion for the convexity of Problem (P0), we write the problem in a high-dimensional form.

Let \( x = (x_1^T, \cdots, x_N^T)^T, u = (u_1^T, \cdots, u_N^T)^T, 1 = (1, \cdots, 1)^T, \sigma_i = (0, \cdots, 0, \sigma^T, 0, \cdots, 0)^T, \)
\( \tilde{A} = \text{diag}(A, \cdots, A) + \frac{1}{N}(11^T \otimes G), B = \text{diag}(B, \cdots, B), C_i = \text{diag}(0, \cdots, 0, C, 0, \cdots, 0), \)
\( D_i = \text{diag}(0, \cdots, 0, D, 0, \cdots, 0), \) and \( R = \text{diag}(R, \cdots, R). \) With the above notations, the dynamics of all agents can be written in the more compact form:
\[ dx(t) = [\tilde{A}x(t) + Bu(t) + 1 \otimes f(t)] dt + \sum_{i=1}^N [C_i x(t) + D_i u(t) + \sigma_i(t)] dW_i(t). \]

Also, denote
\[
\begin{align*}
Q_T & \triangleq \Gamma^T Q + Q \Gamma - \Gamma^T Q \Gamma \\
\bar{\eta} & \triangleq \bar{Q} \eta - \Gamma^T Q \eta \\
\end{align*}
\]
By rearranging the integrand of $J_{soc}$, we have

$$J_{soc} = \mathbb{E} \int_0^\infty \left( \|x(t)\|_Q^2 - 2(1 \otimes \eta(t))^T x(t) + N \|\eta(t)\|^2 + \|u(t)\|^2_R \right) dt,$$

where $Q = (Q_{ij})$ is given by

$$Q_{ii} = Q - Q_{\Gamma} / N, \quad Q_{ij} = -Q_{\Gamma} / N, \quad 1 \leq i \neq j \leq N.$$

**Remark 2.2:** Hereafter, we may exchange the usage of notation $u = (u_1, \ldots, u_N) \in \mathbb{R}^{r \times N}$ and $u = (u_1^T, \ldots, u_N^T)^T \in \mathbb{R}^{rN}$. Both notations represent the control laws among all agents, but only differ in their formations.

**III. MEAN FIELD LQ SOCIAL CONTROL OVER A FINITE HORIZON**

For the convenience of design, we first consider the following finite-horizon problem.

**Theorem 3.1:**

**Proof.** The proof is similar to [16], [26].

**Proposition 3.2:** The following statements are equivalent:

(i) Problem (P1) is uniformly convex in $u$;
(ii) For any \( u_i \in L^2_F(0, T; \mathbb{R}^r) \), \( i = 1, \ldots, N \), there exists a constant \( \gamma > 0 \) such that
\[
\sum_{i=1}^{N} \mathbb{E} \int_0^T \left\{ \| y_i - \Gamma y^{(N)} \|^2_Q + \| u_i \|^2_R \right\} dt + \sum_{i=1}^{N} \mathbb{E} \| y_i(T) - \Gamma_0 y^{(N)}(T) \|^2_H \geq \gamma \sum_{i=1}^{N} \mathbb{E} \int_0^T \| u_i \|^2 dt,
\]

(iii) The equation
\[
\dot{P} + \bar{A}^T P + PA + \sum_{i=1}^{N} C_i^T P C_i + Q - \left( B^T P + \sum_{i=1}^{N} D_i^T P C_i \right)^T \Upsilon \left( B^T P + \sum_{i=1}^{N} D_i^T P C_i \right) = 0, \quad (6)
\]
with \( P(T) = \bar{H} \) admits a solution such that \( \Upsilon = R + \sum_{i=1}^{N} D_i^T P D_i \geq 0 \) and \( \mathcal{R} \left( B^T P + \sum_{i=1}^{N} D_i^T P C_i \right) \subseteq \mathcal{R}(\Upsilon) \), where \( \bar{H} = (\bar{H}_{ij}) \) is given by
\[
\bar{H}_{ii} = H - H_{\Gamma_0}/N, \quad \bar{H}_{ij} = -H_{\Gamma_0}/N, \quad 1 \leq i \neq j \leq N.
\]

**Proof.** (i)\(\Leftrightarrow\)(ii) is implied from [16], [26]. (i)\(\Leftrightarrow\)(iii) is given by Theorem 4.5 of [34]. \(\square\)

By examining the variation of \( J_{Soc}^{F} \), we obtain some necessary and sufficient conditions for the existence of centralized optimal control of (P1).

**Theorem 3.1:** Assume A1) holds. Then we have the following results:

(i) Problem (P1) has a set of optimal control laws if and only if Problem (P1) is convex in \( u \) and the following equation system admits a set of solutions \((x_i, p_i, \beta_i^j, i, j = 1, \ldots, N)\):
\[
\begin{align*}
\dot{x}_i &= (Ax_i + Bu_i + Gx^{(N)} + f) dt + (Cx_i + Du_i + \sigma) dW_i, \\
\dot{p}_i &= - \left( A^T p_i + C^T \beta_i^j + G^T p^{(N)} \right) dt - \left( Q x_i - Q_{\Gamma} x^{(N)} - \tilde{\eta} \right) dt + \sum_{j=1}^{N} \beta_i^j dW_j, \\
x_i(0) &= x_{i0}, \quad p_i(T) = H x_i - H_{\Gamma_0} x^{(N)} - \tilde{\eta}_0, \quad 1 \leq i \leq N,
\end{align*}
\]
where \( p^{(N)} = \frac{1}{N} \sum_{i=1}^{N} p_i \), and the optimal control \( u_i, 1 \leq i \leq N \) satisfies the stationary condition
\[
Ru_i + B^T p_i + D^T \beta_i^j = 0.
\]

(ii) If Problem (P1) is uniformly convex, then (P1) admits a set of optimal control laws.

**Proof.** See Appendix A \(\square\)

To ensure the solvability of the problem (P1), we assume

A2) Problem (P1) is uniformly convex in \( u \).
We now use the idea inspired by [47], [48] to solve the FBSDE (7). Let \( p_i = P_N x_i + K_N x^{(N)} + s_N \). It follows from (7) that
\[
\begin{align*}
\frac{dx^{(N)}}{(N)} &= \left[ (A + G)x^{(N)} + Bu^{(N)} + f \right] dt + \frac{1}{N} \sum_{i=1}^{N} (Cx_i + Du_i + \sigma) dW_i, \\
\frac{dp^{(N)}}{(N)} &= - \left[ (A + G)^T p^{(N)} + \frac{1}{N} \sum_{i=1}^{N} C^T \beta_i^i + (I - \Gamma) Q (I - \Gamma)x^{(N)} - \bar{\eta} \right] dt \\
&\quad + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_j^i dW_j, \\
\end{align*}
\]
(9)

Then by (7), (9) and Itô’s formula,
\[
\begin{align*}
\frac{dp}{i} &= \hat{P}_N x_i + P_N \left[ (Ax_i + Bu_i + Gx^{(N)} + f) dt + (Cx_i + Du_i + \sigma) dW_i \right] + (\dot{s}_N + \hat{K}_N x^{(N)}) dt \\
&\quad + K_N \left\{ [(A + G)x^{(N)} + Bu^{(N)} + f] dt + \frac{1}{N} \sum_{i=1}^{N} (Cx_j + Du_j + \sigma) dW_j \right\} \\
&= - \left[ A^T (P_N x_i + K_N x^{(N)} + s_N) + G^T ((P_N + K_N)x^{(N)} + s_N) + C^T \beta_i^i \\
&\quad + Q x_i - Q \Gamma x^{(N)} - \bar{\eta} \right] dt + \sum_{j=1}^{N} \beta_j^i dW_j. \\
\end{align*}
\]
(10)

This implies that \( \beta_i^i = (P_N + \frac{1}{N} K_N)(Cx_i + Du_i + \sigma) \), and \( \beta_j^i = \frac{1}{N} K_N(Cx_j + Du_j + \sigma) \), \( j \neq i \).

From the stationary condition (8),
\[
Ru_i + B^T (P_N x_i + K_N x^{(N)} + s_N) + D^T (P_N + \frac{1}{N} K_N)(Cx_i + Du_i + \sigma) = 0. \\
\]
(11)

Let \( \Upsilon_N \overset{\Delta}{=} R + D^T (P_N + \frac{K_N}{N}) D \). If (11) admits a solution, then the optimal control can be given by
\[
\begin{align*}
u_i &= - \Upsilon_N^{-1} \left[ \left( B^T P_N + D^T (P_N + \frac{K_N}{N}) C \right) x_i + B^T K_N x^{(N)} + B^T s_N + D^T (P_N + \frac{K_N}{N}) \sigma \right]. \\
\end{align*}
\]
(12)
This together with (10) gives

\[
\begin{aligned}
\dot{P}_N + A^T P_N + P_N A + C^T (P_N + \frac{K_N}{N}) C + Q - \left( B^T P_N + D^T (P_N + \frac{K_N}{N}) C \right)^T \\
\times \Upsilon_N^\dagger \left(B^T P_N + D^T (P_N + \frac{K_N}{N}) C \right) = 0, \quad P_N(T) = H,
\end{aligned}
\]

(13)

\[
\begin{aligned}
\dot{K}_N + (A + G)^T K_N + K_N (A + G) - K_N B \Upsilon_N^\dagger B^T K_N \\
- \left(B^T P_N + D^T (P_N + \frac{K_N}{N}) C \right)^T \Upsilon_N^\dagger B^T K_N + G^T P_N + P_N G \\
- K_N B \Upsilon_N^\dagger \left(B^T P_N + D^T (P_N + \frac{K_N}{N}) C \right) - Q_T = 0, \quad K_N(T) = -H\Gamma_0,
\end{aligned}
\]

(14)

\[
\begin{aligned}
\dot{s}_N + \left[A + G - B \Upsilon_N^\dagger \left(B^T (P_N + K_N) + D^T (P_N + \frac{K_N}{N}) C \right) \right]^T s_N + (P_N + K_N) f - \bar{\eta} \\
+ \left[C - D \Upsilon_N^\dagger \left(B (P_N + K_N) + D^T (P_N + \frac{K_N}{N}) C \right) \right]^T (P_N + \frac{1}{N} K_N) \sigma = 0, \quad s_N(T) = -\bar{\eta}_0.
\end{aligned}
\]

(15)

From the above discussion combined with Theorem 3.1 we have the following result.

**Proposition 3.3:** Assume that A1)-A2) hold. If (13)-(15) admit solutions such that

\[
\mathcal{R} \left(B^T P_N + D^T (P_N + \frac{K_N}{N}) C \right) \cup \mathcal{R} (B^T K_N) \subseteq \mathcal{R} (\Upsilon_N),
\]

\[
B^T s_N + D^T (P_N + \frac{K_N}{N}) \sigma \in \mathcal{R} (\Upsilon_N), \quad \Upsilon_N \geq 0,
\]

then Problem (P1) has an optimal control given by (12).

Let \( P, K, s \) satisfy

\[
\begin{aligned}
\dot{P} + A^T P + PA + C^T PC + Q \\
- \left(B^T P + D^T PC \right)^T \Upsilon_N^\dagger \left(B^T P + D^T PC \right) = 0, \quad P(T) = H,
\end{aligned}
\]

(16)

\[
\begin{aligned}
\dot{K} + (A + G)^T K + K (A + G) + G^T P + PG - \left(B^T P + D^T PC \right)^T \Upsilon_N^\dagger B^T K \\
- KB \Upsilon_N^\dagger (B^T P + D^T PC) - KB \Upsilon_N^\dagger B^T K - Q_T = 0, \quad K(T) = -H\Gamma_0,
\end{aligned}
\]

(17)

\[
\begin{aligned}
\dot{s} + \left[A + G - B \Upsilon_N^\dagger \left(B^T (P + K) + D^T PC \right) \right]^T s + (P + K) f \\
+ \left[C - D \Upsilon_N^\dagger \left(B (P + K) + D^T PC \right) \right]^T P \sigma - \bar{\eta} = 0, \quad s(T) = -\bar{\eta}_0,
\end{aligned}
\]

(18)

where \( \Upsilon \triangleq R + D^T P D \). For further analysis, we assume

**A3)** (16)-(18) have solutions such that \( \Upsilon \geq 0, \) and

\[
\mathcal{R} \left(B^T P + D^T PC \right) \cup \mathcal{R} (B^T K) \subseteq \mathcal{R} (\Upsilon), \quad B^T s + D^T P \sigma \in \mathcal{R} (\Upsilon).
\]

(19)
Remark 3.1: If (16)-(18) have solutions such that $\Upsilon > 0$, then $\Upsilon^\dagger = \Upsilon^{-1}$ and $\mathcal{R}(\Upsilon) = \mathbb{R}^n$. Thus, assumption A3) holds necessarily. This corresponds to the case considered in [43].

As an approximation to $x^{(N)}$ in (9), we obtain
\[ \frac{d\bar{x}}{dt} = (A + G)\bar{x} - B\Upsilon^\dagger[B^T(P + K) + D^TPC]\bar{x} - B\Upsilon^\dagger(B^T s + D^TP\sigma) + f, \quad \bar{x}(0) = \bar{x}_0. \] (20)

Then, by Proposition 3.3, the decentralized control law for agent $i$ can be taken as
\[ \hat{u}_i(t) = -\Upsilon^\dagger(t)\left[(B^TP(t) + D^TP(t)C)x_i(t) + B^TK(t)\bar{x}(t) + B^Ts(t) + D^TP(t)\sigma(t)\right], \] (21)
where $P, K, s$ and $\bar{x}$ are determined by (16)-(20).

Remark 3.2: In previous works [19], [42], the mean field term $x^{(N)}$ in cost functions (dynamics) is first substituted by a deterministic function $\bar{x}$. By solving an optimal tracking problem subject to consistency requirements, a fixed-point equation of $\bar{x}$ is obtained. The decentralized control is constructed by handling the fixed-point equation. Here, we first obtain the centralized solution by the variational analysis, and then design decentralized control laws by tackling the FBSDEs combined with mean field approximations. Note that in this case $s$ and $\bar{x}$ are fully decoupled and no fixed-point equation is needed.

Remark 3.3: By the local Lipschitz continuous property of the quadratic function, (16)-(17) must admit a unique local solution in a small time duration $[T_0, T]$. The global existence of the solution for $t \in [0, T]$ can be referred to [1]. Particularly, if $Q \geq 0$ and $R > 0$, then (16)-(17) admits solutions such that $\Upsilon > 0$. Indeed, letting $\Pi = P + K$, $\Pi$ satisfies the following equation
\[ \dot{\Pi} + (A + G)^T\Pi + \Pi(A + G) - (B^T\Pi + D^TPC)^T\Upsilon^\dagger(B^T\Pi + D^TPC) \\
+ C^TPC + (I - \Gamma)^TQ(I - \Gamma) = 0, \quad \Pi(T) = 0. \] (22)

By [46], if $Q \geq 0$ and $R > 0$, then (16) and (22) admit solutions such that $\Upsilon > 0$, which implies (16)-(17) admit a solution, respectively. Besides, from [34], the solvability of (16)-(17) is equivalent to the uniform convexity of two optimal control problems.

After the decentralized control laws (21) is applied, we have the following closed-loop system
\[ d\hat{x}_i = [\hat{A}\hat{x}_i - B\Upsilon^\dagger(B^T(K\bar{x} + s) + D^TP\sigma) + G\bar{x}^{(N)} + f]dt \\
+ [\hat{C}\hat{x}_i - D\Upsilon^\dagger(B^T(K\bar{x} + s) + D^TP\sigma) + \sigma]dW_i, \] (23)
where $\bar{A} \triangleq A - BY^\dagger(B^T P + D^T PC)$, and $\bar{C} \triangleq C - DT^\dagger(B^T P + D^T PC)$.

**Theorem 3.2:** Let A1)-A3) hold. Then for Problem (P1), the set of decentralized control laws $\{\hat{u}_1, \cdots, \hat{u}_N\}$ given by (21) has asymptotic social optimality, i.e.,

$$\left| \frac{1}{N} J_{soc}^F(\hat{u}) - \frac{1}{N} \inf_{u \in L_2^F(0,T;\mathbb{R}^{nr})} J_{soc}^F(u) \right| = O\left( \frac{1}{\sqrt{N}} \right).$$

**Proof.** See Appendix B. 

**Remark 3.4:** The works [19], [38] considered the above mean field model with positive (semi-) definite $Q$ and $R$ by the fixed point approach. To achieve asymptotic optimality, an additional condition is needed, like well-posedness of a fixed point equation, which is not easy to verify.

Note that in the case $Q \geq 0$ and $R > 0$, by Proposition 3.2 and Remark 3.3, assumptions A1)-A3) hold necessarily. Hence, we get rid of the fixed point condition thoroughly.

IV. MEAN FIELD LQ SOCIAL CONTROL OVER AN INFINITE HORIZON

Based on the similar discussion and analysis in Section III, we may design the following decentralized control laws for Problem (P0):

$$\hat{u}_i(t) = - Y^\dagger \left[ (B^T P + D^T PC)x_i(t) + B^T (\Pi - P)\bar{x}(t) ight. $$

$$+ B^T s(t) + D^T P\sigma(t) \right], \ i = 1, \cdots, N,$$

where $Y = R + D^T PD$, $P$ and $\Pi$ are determined by

$$A^T P + PA + C^T PC - (B^T P + D^T PC)^T Y^\dagger B^T P + D^T PC + Q = 0,$$

$$\left(A + G\right)^T \Pi + (A + G) - (B^T \Pi + D^T PC)^T Y^\dagger (B^T \Pi + D^T PC) + C^T PC + Q - Q_f = 0,$$

and $s, \bar{x} \in L_2([0, \infty), \mathbb{R}^n)$ are determined by

$$\frac{ds}{dt} + [A + G - BY^\dagger (B^T \Pi + D^T PC)]^T s + \Pi f + [C - DT^\dagger (B\Pi + D^T PC)]^T P\sigma - \eta = 0,$$

$$\frac{d\bar{x}}{dt} = [A + G - B Y^\dagger (B^T \Pi + D^T PC)]\bar{x} - B Y^\dagger (B^T s + D^T P\sigma) + f, \ \bar{x}(0) = \bar{x}_0.$$  

Here the existence conditions of $P, \Pi, s$ and $\bar{x}$ are to be ensured later.

For further analysis, we first introduce some definitions. Consider the following system

$$dy(t) = (Ay(t) + Bu(t))dt + (Cy(t) + Du(t))dW(t),$$

$$z(t) = Fy(t),$$

March 31, 2022 DRAFT
where \( y(t) \in \mathbb{R}^n \), and \( W(t) \) is a 1-dimensional Brownian motion.

**Definition 4.1:** The system (29) with \( u = 0 \) (or simply \([A, C]\)) is said to be mean-square stable, if for any initial value \( y(0) \), \( \lim_{t \to \infty} \mathbb{E}[y^T(t)y(t)] = 0. \)

**Definition 4.2:** The system (29) (or simply \([A, B; C, D]\)) is said to be stabilizable (in the mean-square sense), if there exists a control law \( u(t) = Ky(t) \) such that for any initial \( y(0) \in \mathbb{R}^n \), the closed-loop system \( dy(t) = (A + BK)y(t)dt + (C + DK)y(t)dW(t) \) is mean-square stable. In this case \( u(t) \) is called a stabilizer. If \( C = D = 0 \), then the system, abbreviated as \((A, B)\), is stabilizable.

**Definition 4.3:** [49] The system (29)-(30) (or simply \([A, C; F]\)) is said to be exactly observable, if there exists a \( T_0 \geq 0 \) such that for any \( T > T_0, z(t) = 0, u(t) = 0, a.s., 0 \leq t \leq T \) implies \( y(0) = 0. \) If \( C = 0 \), then the system, abbreviated as \((A, F)\), is observable.

**Definition 4.4:** [49] The system (29)-(30) (or simply \([A, C; F]\)) is said to be exactly detectable, if there exists a \( T_0 \geq 0 \) such that for any \( T > T_0, z(t) = 0, u(t) = 0, a.s., 0 \leq t \leq T \) implies \( \lim_{t \to \infty} \mathbb{E}[y^T(t)y(t)] = 0. \)

Some basic assumptions are listed for reference:

**A4** The system \([A, B; C, D]\) is stabilizable, and the system \((A + G, B)\) is stabilizable.

**A5** \( S_1 = \{ \bar{P} = \bar{P}^T : \mathcal{H}(\bar{P}) \geq 0, \ker(R_P) \subseteq \ker(B) \cap \ker(D), [A, C, Q_{\bar{P}}^{1/2}] \text{ is exactly detectable} \} \neq \emptyset, S_2 = \{ \bar{\Pi} = \bar{\Pi}^T : \mathcal{M}(\bar{\Pi}) \geq 0, [A + G, Q_{\bar{\Pi}}^{1/2}] \text{ is detectable} \} \neq \emptyset, \) where

\[
\mathcal{H}(\bar{P}) = \begin{bmatrix} Q_P & \bar{P}B + C^T \bar{P}D \\ B^T \bar{P} + D^T \bar{P}C & R_P \end{bmatrix},
\]

\[
\mathcal{M}(\bar{\Pi}) = \begin{bmatrix} Q_{\bar{\Pi}} & \bar{\Pi}B + C^T \bar{\Pi}D \\ B^T \bar{\Pi} + D^T \bar{\Pi}C & R_P \end{bmatrix},
\]

with

\[
Q_P = A^T \bar{P} + \bar{P}A + C^T \bar{P}C + Q,
\]

\[
R_P = R + D^T \bar{P}D,
\]

\[
Q_{\bar{\Pi}} = (A + G)^T \bar{\Pi} + \bar{\Pi}(A + G) + C^T \bar{\Pi}C + Q - Q_{\Gamma}.
\]

**Lemma 4.1:** Under A4)-A5), the following holds:

(i) (25) admits a unique solution \( P \) such that \( \Upsilon \geq 0 \) and \([\bar{A}, \bar{C}]\) is mean-square stable, where

\[
\bar{A} = A - B^T \Upsilon(\bar{B}^T \bar{P} + D^T \bar{P}C), \quad \text{and} \quad \bar{C} = C - D^T \Upsilon(\bar{B}^T \bar{P} + D^T \bar{P}C);
\]
(ii) \((26)\) admits a unique solution \(\Pi\) such that \(A + G - BY^\top (B^T \Pi + D^T PC)\) is Hurwitz;
(iii) \((27)-(28)\) admits a set of unique solutions \(s, \bar{x} \in L_2([0, \infty), \mathbb{R}^n)\);
(iv) \(\mathcal{R}(B^T P + D^T PC) \cup \mathcal{R}(B^T (\Pi - P)) \subseteq \mathcal{R}(\Upsilon), B^T s + D^T P \sigma \in \mathcal{R}(\Upsilon)\).

Proof. Applying Theorem 2 in [23], we obtain that under A4)-A5), \((25)\) admits a unique solution \(P\) such that the system \(\ddot{A}, \ddot{C}\) is mean-square stable. Note that in \((26)\), \(P\) is known. Since \((A + G, B)\) is stabilizable, then from [23] Theorem 2, \((26)\) admits a unique solution \(\Pi\) such that \(A + G - BY^\top (B^T \Pi + D^T PC)\) is Hurwitz. From an argument in [40, Appendix A], we obtain \(s \in L_2([0, \infty), \mathbb{R}^n)\) if and only if

\[
s(0) = \int_0^\infty e^{[A + G - BY^\top (B^T \Pi + D^T PC)](t-\tau)} (\Pi f + \dot{C} P \sigma - \bar{\eta}) d\tau.
\]

Under this initial condition, we have

\[
s(t) = \int_t^\infty e^{-[A + G - BY^\top (B^T \Pi + D^T PC)](t-\tau)} (\Pi f + \dot{C} P \sigma - \bar{\eta}) d\tau.
\]

From the argument in [23] Theorem 1, one can show that \((B^T s + D^T P \sigma)^T (I - \Upsilon \Upsilon^\top) = 0\), which implies \(B^T s + D^T P \sigma \in \mathcal{R}(\Upsilon).\) Similarly, we have \(\mathcal{R}(B^T P + D^T PC) \cup \mathcal{R}(B^T K) \subseteq \mathcal{R}(\Upsilon)\). \(\square\)

We now introduce an additional assumption. Later, the assumption is shown to be necessary and sufficient for the uniform stabilization of all the subsystems.

**A6** \(\ddot{A} + G\) is Hurwitz, where \(\ddot{A} = A - BY^\top (B^T P + D^T PC)\).

It is shown that the decentralized control laws \((21)\) uniformly stabilize the systems \(\Pi\).

**Theorem 4.1:** Let A1), A4)-A6) hold. Then there exists an \(N_0\) such that for \(N \geq N_0\), the following hold:

\[
\max_{1 \leq i \leq N} \mathbb{E} \int_0^\infty (||\dot{x}_i(t)||^2 + ||\ddot{x}_i(t)||^2) dt < \infty \quad (31)
\]

\[
\mathbb{E} \int_0^\infty ||\ddot{x}^{(N)}(t) - \ddot{x}(t)||^2 dt = O\left(\frac{1}{N}\right) \quad (32)
\]

Proof. After the control \((24)\) is applied, we have

\[
d\ddot{x}_i = [\ddot{A}\ddot{x}_i + G\dot{x}^{(N)} + \bar{f}] dt + [\dot{C}\ddot{x}_i + \bar{\sigma}]dW_i, \quad (33)
\]

where \(\bar{f} \triangleq f - BY^\top (B^T (K\ddot{x} + s) + D^T P \sigma)\), and \(\bar{\sigma} \triangleq \sigma - D Y^\top (B^T (K\ddot{x} + s) + D^T P \sigma)\). Let \(\xi(t) = \dot{x}^{(N)}(t) - \ddot{x}(t)\). From \((33)\) and \((28)\),

\[
\xi(t) = e^{(A+G)t} \xi(0) + \frac{1}{N} \sum_{i=1}^N \int_0^t e^{(A+G)(t-\tau)} (\bar{C}\ddot{x}_i + \bar{\sigma}) dW_i. \quad (34)
\]
Thus, we have
\[
\mathbb{E} \int_0^T (\|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2) \, dt
\]
\[
\leq 2 \mathbb{E} \int_0^T \left\| e^{(\hat{A} + G)t} \left( \left\| \hat{x}^{(N)}(0) - \bar{x}(0) \right\|^2 + 2 \int_0^T \left\| e^{(\hat{A} + G)(t-\tau)} \left( \tilde{C} \hat{x}_i + \tilde{\sigma} \right) dW_i(\tau) \right\|^2 \right) \, dt
\]
\[
\leq 2 \int_0^T \left\| e^{(\hat{A} + G)t} \right\|^2 \mathbb{E} \| \hat{x}^{(N)}(0) - \bar{x}(0) \|^2 \, dt 
+ \frac{2}{N} \mathbb{E} \int_0^T \left\| e^{(\hat{A} + G)(t-\tau)} \right\|^2 \| \tilde{C} \hat{x}_i + \tilde{\sigma} \|^2 d\tau dt
\]
\[
\leq \frac{2}{N} \max_{1 \leq i \leq N} \mathbb{E} \int_0^T \| \hat{x}_i \|^2 \, dt + \frac{c_2}{N} \int_0^T \left\| e^{(\hat{A} + G)(t-\tau)} \right\|^2 d\tau dt
\]
\[
\leq \frac{C_1}{N} \max_{1 \leq i \leq N} \mathbb{E} \int_0^T \| \hat{x}_i \|^2 \, dt + \frac{c_1}{N}. \tag{35}
\]
Let \( P \) satisfy
\[
P \hat{A} + \hat{A}^T P + \hat{C}^T P \hat{C} = -2I.
\]
From Lemma \[4.11\] and \[32\], we have \( P > 0 \). By Itô’s formula and \( 33\),
\[
\mathbb{E} [\hat{x}_i^T(T) P \hat{x}_i(T) - \hat{x}_i^T(0) P \hat{x}_i(0)]
= \mathbb{E} \int_0^T \left[ \hat{x}_i^T P (\hat{A} \hat{x}_i + G \hat{x}^{(N)} + \hat{f}) + (\hat{A} \hat{x}_i + G \hat{x}^{(N)} + \hat{f})^T P \hat{x}_i \right] \, dt 
+ \mathbb{E} \int_0^T (\tilde{C} \hat{x}_i + \tilde{\sigma})^T P (\tilde{C} \hat{x}_i + \tilde{\sigma}) \, dt. \tag{36}
\]
From \( 36 \), we have
\[
\mathbb{E} [\hat{x}_i^T(T) P \hat{x}_i(T) - \hat{x}_i^T(0) P \hat{x}_i(0)]
= \mathbb{E} \int_0^T \left[ \hat{x}_i^T (P \hat{A} + \hat{A}^T P + \hat{C}^T P \hat{C}) \hat{x}_i + (\hat{x}^{(N)})^T (PG + GT P) \hat{x}^{(N)} \right. 
+ 2 (P \hat{f} + C^T P \hat{\sigma})^T \hat{x}_i + \sigma^T P \hat{\sigma} \right] \, dt
\leq \mathbb{E} \int_0^T \left[ \hat{x}_i^T (P \hat{A} + \hat{A}^T P + \hat{C}^T P \hat{C}) \hat{x}_i + \| \hat{x}_i \|^2 + (\hat{x}^{(N)})^T (PG + GT P) \hat{x}^{(N)} \right. 
+ \| P \hat{f} + C^T P \hat{\sigma} \|^2 + \sigma^T P \hat{\sigma} \right] \, dt
\leq - \mathbb{E} \int_0^T (\hat{x}_i^T \dot{\hat{x}}_i) \, dt + \alpha_T, \tag{37}
\]
where
\[
\alpha_T = \mathbb{E} \int_0^T \left[ (\hat{x}^{(N)})^T (PG + GT P) \hat{x}^{(N)} + \| P \hat{f} + C^T P \hat{\sigma} \|^2 + \sigma^T P \hat{\sigma} \right] \, dt.
\]
This with (35) gives
\[ \mathbb{E} \int_0^T \| \hat{x}_i \|^2 dt \leq \mathbb{E} [x_{i0}^T P x_{i0}] + \alpha_T \leq c_2 \mathbb{E} \int_0^T \| x^{(N)}_i \|^2 dt + c_2 \]
\[ \leq 2c_2 \mathbb{E} \int_0^T \left( \| \bar{x}(t) \|^2 + \| \xi(t) \|^2 \right) dt + c_2 \]
\[ \leq 2c_2 \left[ \mathbb{E} \int_0^T \| \bar{x}(t) \|^2 dt + \frac{c_1}{N} \max_{1 \leq i \leq N} \mathbb{E} \int_0^T \| \hat{x}_i \|^2 dt \right] + \frac{2c_1 c_2}{N} + c_2. \] (38)

Thus, there exists \( N_0 \) such that for any \( N > N_0 \),
\[ \max_{1 \leq i \leq N} \mathbb{E} \int_0^T \| \hat{x}_i \|^2 dt \leq 2c_2 \mathbb{E} \int_0^T \| \bar{x}(t) \|^2 dt + 2c_1 c_2 + c_2. \]
Note \( \bar{x} \in L_2([0, \infty), \mathbb{R}^n) \). We have
\[ \max_{1 \leq i \leq N} \mathbb{E} \int_0^\infty \| \hat{x}_i \|^2 dt \leq c. \]
This together with (35) gives (32). \( \square \)

We now give two equivalent conditions for uniform stabilization of all the subsystems.

**Theorem 4.2:** For (P0), let A5) hold. Assume that (25)-(26) have solutions. Then for (P0) the following statements are equivalent:

(i) there exists an \( N_0 \) such that for \( N \geq N_0 \) and any initial condition \((\hat{x}_1(0), \ldots, \hat{x}_N(0))\) satisfying A1),
\[ \sum_{i=1}^N \mathbb{E} \int_0^\infty \left( \| \hat{x}_i(t) \|^2 + \| \hat{u}_i(t) \|^2 \right) dt < \infty; \] (39)

(ii) (25)-(27) admit solutions such that \( R + D^T PD \geq 0, \mathcal{R}(B^T P + D^T PC) \cup \mathcal{R}(B^T (I - P)) \subseteq \mathcal{R}(Y), B^T s + D^T P \sigma \in \mathcal{R}(Y), \) and \( A + G \) is Hurwitz;

(iii) A4) and A6) hold.

*Proof.* See Appendix C. \( \square \)

For the case \( Q \geq 0, R > 0 \), when the assumption A5) is strengthened to A5)', we can give the following equivalent conditions for uniform stabilization of the systems.

A5') \( Q \geq 0, R > 0, [A, C, \sqrt{Q}] \) is exactly observable, and \( (A + G, \sqrt{Q}(I - \Gamma)) \) is observable.

**Theorem 4.3:** Let A5') hold. Assume that (25)-(26) have solutions. Then for (P0) the following statements are equivalent:

(i) For any initial condition \((\hat{x}_1(0), \ldots, \hat{x}_N(0))\) satisfying A1), the following holds,
\[ \sum_{i=1}^N \mathbb{E} \int_0^\infty \left( \| \hat{x}_i(t) \|^2 + \| \hat{u}_i(t) \|^2 \right) dt < \infty; \]
(ii) (25) and (26) admit unique solutions such that $P > 0$, $\Pi > 0$, and $\bar{A} + G$ is Hurwitz;
(iii) A4) and A6) hold.

Proof. See Appendix C.

Remark 4.1: In [31], some similar results were given for the stabilization of mean field systems. However, only the limiting problem was considered in their work and the mean field term in dynamics and costs is $\mathbb{E}x(t)$ instead of $x^{(N)}(t)$. Here we study large-population multiagent systems and the number of agents is large but not infinite. The errors of mean field approximations need to be further analyzed. In this case, an additional assumption A6) is needed to obtain uniform stabilization.

To compare the optimal social costs under decentralized and centralized strategies, we need the presumption that Problem (P0) admits a centralized solution. Thus, we set an assumption on the following generalized Riccati equation:

A7) The equation

$$\dot{P} = A^T P + PA + \sum_{i=1}^{N} C_i^T P C_i + Q - \left(B^T P + \sum_{i=1}^{N} D_i^T P C_i\right)^T \Upsilon \left(B^T P + \sum_{i=1}^{N} D_i^T P C_i\right) = 0$$

admits a solution such that $\Upsilon = R + \sum_{i=1}^{N} D_i^T P D_i \geq 0$, $\mathcal{R} \left(B^T P + \sum_{i=1}^{N} D_i^T P C_i\right) \subseteq \mathcal{R}(\Upsilon)$ and the following system is mean-square stable:

$$dx = \left[\dot{A} - B \Upsilon \left(B^T P + \sum_{i=1}^{N} D_i^T P C_i\right)\right] x dt + \sum_{i=1}^{N} \left[C_i - D_i \Upsilon \left(B^T P + \sum_{i=1}^{N} D_i^T P C_i\right)\right] dW_i.$$

We now are in a position to state the asymptotic social optimality of the decentralized control.

Theorem 4.4: Let A1), A4)-A7) hold. For Problem (P0), the set of decentralized control laws $\{\hat{u}_1, \cdots, \hat{u}_N\}$ given by (24) has asymptotic social optimality, i.e.,

$$\left| \frac{1}{N} J_{soc}(\hat{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}_c} J_{soc}(u) \right| = O\left(\frac{1}{\sqrt{N}}\right).$$

We first provide a preliminary lemma, which plays an important role in showing asymptotic optimality of decentralized control.

Lemma 4.2: For the system (29), assume $[A, B; C, D]$ is stabilizable. Then for any $u \in L_2([0, \infty), \mathbb{R}^n)$ and a stabilizer $Ky$, there exist constants $\alpha_i, c_i > 0, i = 1, 2$ such that

$$\mathbb{E} \int_{0}^{\infty} \|y(t)\|^2 dt \leq \alpha_1 \mathbb{E} \int_{0}^{\infty} \|u(t) - Ky(t)\|^2 dt + c_1,$$

$$\mathbb{E} \int_{0}^{\infty} \|u(t)\|^2 dt \leq \alpha_2 \mathbb{E} \int_{0}^{\infty} \|u(t) - Ky(t)\|^2 dt + c_2.$$
Proof. Define $u^* = u - Ky$, where $y$ satisfies (29). Then $u^* \in L_2([0, \infty), \mathbb{R}^n)$ and $y$ satisfies
\[ dy(t) = [(A + BK)y(t) + Bu^*(t)]dt + [(C + DK)y(t) + u^*(t)]dW(t), \quad y(0) = y_0. \]
Since $Ky$ is a stabilizer, then by [35], there exists a constant $\alpha_1$ such that $\mathbb{E}\int_0^\infty \|y(t)\|^2dt \leq \alpha_1 \mathbb{E}\int_0^\infty \|u^*(t)\|^2dt + c_1$. Hence,
\[
\mathbb{E}\int_0^\infty \|u(t)\|^2dt = \mathbb{E}\int_0^\infty \|u^*(t) + Ky(t)\|^2dt \\
\leq \alpha_2 \mathbb{E}\int_0^\infty \|u^*(t)\|^2dt + c_2 \\
= \alpha_2 \mathbb{E}\int_0^\infty \|u(t) - Ky(t)\|^2dt + c_2,
\]
where $\alpha_2 = 2\alpha_1\|K\|^2 + 2$, and $c_2 = 2c_1$. \qed

Proof of Theorem 4.4 We first prove that for $u \in \mathcal{U}_c$, $J_{soc}(u) < c_1$ implies that there exists a constant $c_2$ such that
\[
\mathbb{E}\int_0^\infty (\|x_i\|^2 + \|u_i\|^2)dt < c_2, \tag{40}
\]
for all $i = 1, \ldots, N$. From A7), the following equation admits a unique solution $s \in L_2([0, \infty), \mathbb{R}^{Nn})$,
\[
\dot{s} + [\dot{\mathbf{A}} - \sum_{i=1}^N \mathbf{B} \mathbf{Y}^\dagger (\mathbf{B}^T \mathbf{P} + \mathbf{D}_i^T \mathbf{P} \mathbf{C}_i)] \mathbf{T}s + \mathbf{P}(f \otimes 1) \\
+ \sum_{i=1}^N [\mathbf{C}_i - \mathbf{D}_i \mathbf{Y}^\dagger (\mathbf{B}^T \mathbf{P} + \mathbf{D}_i^T \mathbf{P} \mathbf{C}_i)] \mathbf{T}\mathbf{P}\mathbf{\sigma}_i - \bar{\eta} \otimes 1 = 0.
\]
By Itô’s formula, we have
\[
J_{soc}(u) = \limsup_{T \to \infty} \mathbb{E}[\mathbf{x}^T(0) \mathbf{P} \mathbf{x}(0) - \mathbf{x}^T(T) \mathbf{P} \mathbf{x}(T)] \\
+ \mathbb{E}\int_0^\infty \|\mathbf{u} + \mathbf{Y}^\dagger (\mathbf{B}^T \mathbf{P} + \sum_{i=1}^N \mathbf{D}_i^T \mathbf{P} \mathbf{C}_i) \mathbf{x} + \mathbf{B}^T \mathbf{s} + \sum_{i=1}^N \mathbf{D}_i^T \mathbf{P} \mathbf{\sigma}_i\|^2 dt \\
\geq \mathbb{E}\int_0^\infty \|\mathbf{u} + \mathbf{Y}^\dagger (\mathbf{B}^T \mathbf{P} + \sum_{i=1}^N \mathbf{D}_i^T \mathbf{P} \mathbf{C}_i) \mathbf{x}\|^2 dt - c.
\]
By Lemma 4.2 there exist constants $\alpha, c > 0$ such that
\[
\sum_{i=1}^N \mathbb{E}\int_0^\infty (\|x_i\|^2 + \|u_i\|^2)dt \\
\leq \alpha \mathbb{E}\int_0^\infty \|\mathbf{u} + \mathbf{Y}^\dagger (\mathbf{B}^T \mathbf{P} + \sum_{i=1}^N \mathbf{D}_i^T \mathbf{P} \mathbf{C}_i) \mathbf{x}\|^2 dt + c \leq \alpha J_{soc}(u) + c \leq c_2. \tag{41}
\]
By a similar argument to the proof of Theorem 3.2 combined with Theorem 4.1, the conclusion follows.

Remark 4.2: If A5) is replaced by A5′), then it can be shown that the decentralized control (24) still has asymptotic social optimality.

V. Asymptotically Social Optimal Cost

We now give an explicit expression of the asymptotic average social optimum in terms of the solutions of two Riccati equations.

Theorem 5.1: Assume i) A1), A4-A7) hold; ii) \( \{x_{i0}\} \) have the same variance. Then the asymptotic average social optimum is given by

\[
\lim_{N \to \infty} \frac{1}{N} J_{soc}(\hat{u}) = \mathbb{E} \left[ (x_{i0} - \bar{x}_0)^T P(x_{i0} - \bar{x}_0) + \bar{x}_0^T \Pi \bar{x}_0 + 2s^T(0)\bar{x}_0 \right] + m,
\]

where \( P \) and \( \Pi \) are given by (25)-(26), respectively, and

\[
m = \int_0^\infty \left[ \|\sigma(t)\|_P^2 - \|B^T s(t) + D^T P \sigma(t)\|_{\Upsilon_1}^2 + 2s^T(t)f(t) + \|\eta(t)\|_Q^2 \right] dt.
\]

To prove Theorem 5.1 we need two lemmas.

Consider the mean-field type system

\[
dz_i = (Az_i + Bu_i + GE[z_i] + f)dt + (Cz_i + Du_i + \sigma)dW_i, \quad z_i(0) = x_{i0},
\]

with the cost function

\[
J_i(u_i) = \mathbb{E} \int_0^\infty (\|z_i - \Gamma E[z_i] - \eta\|_\bar{Q}^2 + \|u_i\|_\bar{P}^2)dt.
\]

The admissible control set is given by

\[
U_i = \left\{ u_i \mid u_i(t) \text{ is adapted to } \sigma(z_i(s), 0 \leq s \leq t), \mathbb{E} \int_0^\infty \|z_i(t)\|^2dt < \infty, \forall i \right\}.
\]

Lemma 5.1: For the system (42)-(43), the optimal control is given by

\[
\hat{u}_i = -\Upsilon_1^{-1}[(B^T P + D^T PC)z_i + B^T (\Pi - P)E[z_i] + B^T s + D^T P \sigma],
\]

and the optimal cost is

\[
\inf_{u_i \in U_i} J_i(u_i) = \mathbb{E} \left[ (x_{i0} - \bar{x}_0)^T P(x_{i0} - \bar{x}_0) + \bar{x}_0^T \Pi \bar{x}_0 + 2s^T(0)\bar{x}_0 \right] + m.
\]
Proof. From (42),
\[ d\mathbb{E}[z_i] = [(A + G)\mathbb{E}[z_i] + B\mathbb{E}[u_i] + f]dt, \quad \mathbb{E}[z_i](0) = x_{i0}. \tag{45} \]

Applying Itô’s formula to \( \|z_i - \mathbb{E}[z_i]\|^2_P \), we have
\[
\mathbb{E}\left[\|z_i(T) - \mathbb{E}[z_i(T)]\|^2_P - \|x_{i0} - \bar{x}_0\|^2_P\right]
= \mathbb{E} \int_0^T \left\{ 2\langle z_i - \mathbb{E}[z_i], P[A(z_i - \mathbb{E}[z_i]) + B(u_i - \mathbb{E}[u_i])] + \|Cz_i + Du_i + \sigma\|^2_P \right\} dt
= \mathbb{E} \int_0^T \left\{ \langle (A^T P + PA + C^T PC)(z_i - \mathbb{E}[z_i]), z_i - \mathbb{E}[z_i] \rangle 
+ 2\langle (B^T P + D^T PC)(z_i - \mathbb{E}[z_i]), u_i - \mathbb{E}[u_i] \rangle 
+ \langle u_i - \mathbb{E}[u_i], D^T PD(u_i - \mathbb{E}[u_i]) \rangle + \langle \mathbb{E}[u_i], D^T PDE[u_i] \rangle + \langle \sigma, P\sigma \rangle 
+ \langle C^T P\mathbb{E}[z_i] + 2C^T P\sigma, \mathbb{E}[z_i] \rangle + 2\langle D^T PC\mathbb{E}[z_i] + D^T P\sigma, \mathbb{E}[u_i] \rangle \right\} dt. \tag{46} \]

From (24) and (45),
\[
\mathbb{E}[z_i(T)]^T \Pi \mathbb{E}[z_i(T)] - \bar{x}_0^T \Pi \bar{x}_0 = \mathbb{E} \int_0^T \left\{ \langle [(A + G)^T \Pi + \Pi(A + G)]\mathbb{E}[z_i], \mathbb{E}[z_i] \rangle 
+ 2\langle B^T \Pi \mathbb{E}[z_i], \mathbb{E}[u_i] \rangle + 2\langle \Pi f, \mathbb{E}[z_i] \rangle \right\} dt. \tag{47} \]

Also, applying Itô’s formula to \( \langle s, \mathbb{E}[z_i] \rangle \), we have
\[
\mathbb{E}[z_i(T)]^T s(T) - \bar{x}_0^T s(0) = \mathbb{E} \int_0^T \left\{ \langle - [A + G - BT^T (B^T \Pi + D^T PC)]^T s, \mathbb{E}[z_i] \rangle 
- \langle [C - DT^T (B^T \Pi + D^T PC)]^T P\sigma + \Pi f - \bar{\eta}, \mathbb{E}[z_i] \rangle \right\} dt
= \mathbb{E} \int_0^T \left\{ \langle \Pi B + C^T PD \rangle Y^T (B^T s + D^T P\sigma), \mathbb{E}[z_i] \rangle + \langle s, f \rangle 
- \langle C^T P\sigma + \Pi f - \bar{\eta}, \mathbb{E}[z_i] \rangle + \langle B^T s, \mathbb{E}[u_i] \rangle \right\} dt. \tag{48} \]
Denote $\Psi \triangleq B^T P + D^T PC$. By (46)-(48), we obtain
\[
J_i(u_i) = \mathbb{E} \int_0^\infty \left( \|z_i - \Gamma E[z_i] - \eta\|^2_Q + \|u_i\|^2_R \right) dt \\
= \mathbb{E} \int_0^\infty \left[ \|z_i - E[z_i]\|^2_Q + \|(I - \Gamma)E[z_i]\|^2_Q - 2\eta^T E[z_i] + \|\eta\|^2_Q \\
+ \|u_i - E[u_i]\|^2_R + \|E[u_i]\|^2_R \right] dt \\
= \mathbb{E} \left[ \|x_{i0} - \tilde{x}_0\|^2_P + \tilde{x}_0^T \Pi \tilde{x}_0 + 2s^T(0)\tilde{x}_0 \right] - \lim_{T \to \infty} \mathbb{E} \left\{ \|z_i(T) - E[z_i(T)]\|^2_P \\
+ \mathbb{E}[z_i(T)]^T \Pi E[z_i(T)] + 2z_i(T)^T s(T) \right\} + \mathbb{E} \int_0^\infty \left[ \langle \Psi^T \Upsilon^\dagger \Psi(z_i - E[z_i]), z_i - E[z_i] \rangle \\
+ 2\langle \Psi, z_i - E[z_i] \rangle + \langle \Upsilon(u_i - E[u_i]), u_i - E[u_i] \rangle \\
+ \langle (B^T \Pi + D^T PC)^T \Upsilon^\dagger (B^T \Pi + D^T PC)E[z_i], E[z_i] \rangle \\
+ 2\langle (B^T \Pi + D^T PC)E[z_i] + B^T s + D^T P\sigma, E[u_i] \rangle + \langle \Upsilon E[u_i], E[u_i] \rangle \\
+ \langle (B^T \Pi + D^T PC)^T \Upsilon^\dagger (B^T s + D^T P\sigma), E[z_i] \rangle + 2\langle s, f \rangle + \langle P\sigma, \sigma \rangle + \langle Q\eta, \eta \rangle \right] dt \\
\geq \mathbb{E} \left[ \|x_{i0} - \tilde{x}_0\|^2_P + \tilde{x}_0^T \Pi \tilde{x}_0 + 2s^T(0)\tilde{x}_0 \right] + m.
\]

\[\square\]

**Lemma 5.2:** Let A1), A4)-A7) hold. Then
\[
\mathbb{E} \int_0^\infty \|\hat{x}_i - \hat{z}_i\|^2 dt = O\left(\frac{1}{N}\right),
\]
where $\hat{z}_i$ is the closed-loop state of $z_i$ in (42).

**Proof.** After applying the control (44) into the dynamics (42), we have
\[
d\hat{z}_i = [A\hat{z}_i - B\Upsilon^\dagger ([B^T P + D^T PC] \hat{z}_i + B^T (\Pi - P)E[\hat{z}_i] + B^T s + D^T P\sigma) + G\Pi E[\hat{z}_i] + f)] dt \\
+ [C\hat{z}_i - D\Upsilon^\dagger ([B^T P + D^T PC] \hat{z}_i + B^T (\Pi - P)E[\hat{z}_i] + B^T s + D^T P\sigma) + \sigma] dW_i,
\]
which leads to
\[
dE[\hat{z}_i] = [(A + G - (B\Upsilon^\dagger B^T \Pi + D^T PC))E[\hat{z}_i] + f] dt, \ E[\hat{z}_i(0)] = \tilde{x}_0.
\]
By comparing this with (28), we can verify that $E[\hat{z}_i] = \bar{x}$. From (33),
\[
d(\hat{x}_i - \hat{z}_i) = \bar{A}(\hat{x}_i - \hat{z}_i) dt + G(\hat{x}^{(N)} - E[\hat{z}_i]) dt + \bar{C}(\hat{x}_i - \hat{z}_i) dW_i.
\]
This implies
\[ \hat{x}_i(t) - \hat{z}_i(t) = \int_0^t \Phi_i(t - \tau)G[\hat{x}^{(N)}(\tau) - E[\hat{z}_i(\tau)]]d\tau, \]
where \( \Phi_i \) satisfies
\[ d\Phi_i(t) = \tilde{A}\Phi_i(t)dt + \tilde{C}\Phi_i(t)dW_i, \quad \Phi_i(t) = I. \]

By Schwarz’s inequality and Theorem 4.1,
\[
E \int_0^\infty \|\hat{x}_i(t) - \hat{z}_i(t)\|^2 dt = E \int_0^\infty \left\| \int_0^t \Phi_i(t - \tau)G[\hat{x}^{(N)}(\tau) - E[\hat{z}_i(\tau)]]d\tau \right\|^2 dt \\
\leq E \int_0^\infty t \int_0^t \|\Phi_i(t - \tau)\|^2 \|G[\hat{x}^{(N)}(\tau) - E[\hat{z}_i(\tau)]]\|^2 d\tau dt \\
= E \int_0^\infty \|G(\hat{x}^{(N)}(\tau) - E[\hat{z}_i(\tau)])\|^2 \int_\tau^\infty t \|\Phi_i(t - \tau)\|^2 dt d\tau \\
\leq cE \int_0^\infty \|\hat{x}^{(N)}(\tau) - E[\hat{z}_i(\tau)]\|^2 d\tau = O\left(\frac{1}{N}\right). 
\]

**Proof of Theorem 5.1** Note that \( E[\hat{z}_i] = \bar{x} \). We have
\[
\frac{1}{N} J_{soc}(\hat{u}) = \frac{1}{N} \sum_{i=1}^N E \int_0^\infty \left[ \|\hat{x}_i - \Gamma\hat{x}^{(N)} + \eta\|^2_Q \\
+ \|Y^t[(B^T P + D^T PC)\hat{x}_i + B^T(\Pi - P)\bar{x} + B^T s + D^T P\sigma]\|^2_R \right] dt \\
= \frac{1}{N} \sum_{i=1}^N E \int_0^\infty \left[ \|\hat{z}_i - \Gamma E[\hat{z}_i] + \eta + \hat{x}_i - \hat{z}_i + \Gamma\hat{x}^{(N)} - \Gamma E[\hat{z}_i]\|^2_Q dt \\
+ \|Y^t[(B^T P + D^T PC)(\hat{z}_i + \hat{x}_i - \hat{z}_i) + B^T(\Pi - P)E[\hat{z}_i] + B^T s + D^T P\sigma]\|^2_R \right] dt.
\]

By Schwarz’s inequality, and Lemma 5.2, one can obtain
\[
\left| \frac{1}{N} J_{soc}(\hat{u}) - \frac{1}{N} J_{soc}(\hat{u}) \right| \\
\leq \frac{1}{N} \sum_{i=1}^N E \int_0^\infty \left[ \|\hat{x}_i - \hat{z}_i\|^2_Q + \|\Gamma(x^{(N)} - E[\hat{z}_i])\|^2_Q \right] dt \\
+ c_1 \frac{1}{N} \sum_{i=1}^N \left( E \int_0^\infty \|\hat{x}_i - \hat{z}_i\|^2_Q dt \right)^{1/2} + c_2 \frac{1}{N} \sum_{i=1}^N \left( E \int_0^\infty \|\Gamma(x^{(N)} - E[\hat{z}_i])\|^2_Q dt \right)^{1/2} \\
\leq O(1/\sqrt{N}).
\]

From this and Lemma 5.1, the theorem follows. \(\square\)
VI. NUMERICAL EXAMPLE

In this section, a numerical example is given to illustrate the effectiveness of the proposed decentralized control laws.

We consider a scalar system with 50 agents in Problem (P0). Take $A = 0.1, B = C = D = Q = 1, R = -0.2, G = -0.1, f = e^{-t}, \eta = \frac{1}{t+1}, \sigma = 0.1, \text{ and } \Gamma = -0.2$. The initial states of 50 agents are taken independently from a normal distribution $N(1, 0.1)$. The Riccati equations (25)-(26) admit solutions $P = 0.6808$ and $\Pi = 0.3290$, respectively. Then, under the control law (24), the state trajectories of agents are shown in Fig. 1. After the transient phase, the states of agents achieve an agreement. The trajectories of $\bar{x}$ and $\hat{x}^{(N)}$ in (P0) are shown in Fig. 2. It can be seen that $\bar{x}$ and $\hat{x}^{(N)}$ coincide well, which illustrates the consistency of mean field approximations.

![Fig. 1: Curves of 30 agents.](image-url)
The cost gap $\varepsilon$ between centralized and decentralized optimal controls is demonstrated in Fig. 3 where the agent number $N$ grows from 1 to 50.
VII. CONCLUDING REMARKS

In this paper, we have considered uniform stabilization and asymptotic optimality for indefinite mean field LQ social control systems with multiplicative noises. By decoupling FBSDEs, we design the decentralized control laws, which are further shown to be asymptotically optimal. Some equivalent conditions are further given for uniform stabilization of all the subsystems.

The interesting generalization is to consider mean field LQ control systems with model uncertainty by handling FBSDEs. Also, the variational analysis may be applied to leader-follower models to construct decentralized social control.

APPENDIX A

PROOF OF THEOREM 3.1

Proof of Theorem 3.1 (i) Suppose that $\tilde{u}_i$ satisfies $R\tilde{u}_i + B^T p_i + D^T \beta_i = 0$, where $\{p_i, \beta_i, i, j = 1, \ldots, N\}$ is a set of solutions to the equation system

$$dp_i = \alpha_i dt + \beta_i dW_i + \sum_{j \neq i} \beta_{ij} dW_j, p_i(T) = p_{iT}, \; i = 1, \ldots, N. \quad (A.1)$$

Here $\alpha_i, \beta_i, p_{iT}, i, j = 1, \ldots, N$ are to be determined. Denote by $\tilde{x}_i$ the state of agent $i$ under the control $\tilde{u}_i$. For any $u_i \in L_2^2(0, T; \mathbb{R}^r)$ and $\theta \in \mathbb{R}$ ($\theta \neq 0$), let $u_i^\theta = \tilde{u}_i + \theta u_i$. Denote by $x_i^\theta$ the solution of the following perturbed state equation:

$$dx_i^\theta = (Ax_i^\theta + B(\tilde{u}_i + \theta u_i) + \frac{G}{N} \sum_{i=1}^N x_i^\theta) dt + (Cx_i^\theta + Du_i^\theta + \sigma) dW_i,$$

$$x_i^\theta(0) = x_{i0}, \; i = 1, 2, \ldots, N.$$ 

Let $y_i = (x_i^\theta - \tilde{x}_i)/\theta$. It can be verified that $y_i$ satisfies (5). Then by Itô’s formula, for any $i = 1, \ldots, N$,

$$\mathbb{E}[\langle p_{iT}, y_i(T) \rangle] = \mathbb{E}[\langle p_i(T), y_i(T) \rangle - \langle p_i(0), y_i(0) \rangle]$$

$$= \mathbb{E} \int_0^T [\langle \alpha_i, y_i \rangle + \langle p_i, Ay_i + Gy_i(N) + Bu_i \rangle + \langle \beta_i, Cy_i + Du_i \rangle] dt,$$
which implies
\[
\sum_{i=1}^{N} \mathbb{E}\{p_i T, y_i(T)\} \\
= \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \left[ \alpha_i, y_i + \langle p_i, A y_i + G y^{(N)} + B u_i \rangle + \langle \beta_i, C y_i + D u_i \rangle \right] dt \\
= \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \left[ \alpha_i + A^T p_i + C^T \beta_i, y_i \right] + \langle B^T p_i + D^T \beta_i, u_i \rangle \right] dt + \mathbb{E} \left[ \sum_{i=1}^{N} \frac{G_N}{N} \sum_{i=1}^{N} y_i \right] dt \\
= \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \left[ \alpha_i + A^T p_i + G^T p^{(N)} + C^T \beta_i, y_i \right] + \langle B^T p_i + D^T \beta_i, u_i \rangle \right] dt. \tag{A.2}
\]

From (4), we have
\[
\hat{J}^{F}_{soc}(\bar{u} + \theta u) - \hat{J}^{F}_{soc}(\bar{u}) = 2\theta I_1 + \theta^2 I_2 \tag{A.3}
\]

where \( \bar{u} = (\bar{u}_1, \ldots, \bar{u}_N) \), and
\[
I_1 \triangleq \sum_{i=1}^{N} \mathbb{E} \left\{ \int_{0}^{T} \left[ \langle Q(\bar{x}_i - (\Gamma \hat{x}^{(N)} + \eta)), y_i - \Gamma y^{(N)} \rangle + \langle R \bar{u}_i, u_i \rangle \right] dt \\
+ \langle H(\bar{x}_i(T) - (\Gamma_0 \hat{x}^{(N)}(T) + \eta_0)), y_i(T) - \Gamma_0 y^{(N)}(T) \rangle \right\},
\]
\[
I_2 \triangleq \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \left[ \|y_i - \Gamma y^{(N)}\|_Q^2 + \|u_i\|_R^2 \right] dt + \sum_{i=1}^{N} \mathbb{E} \|y_i(T) - \Gamma_0 y^{(N)}(T)\|_H^2.
\]

Note that
\[
\sum_{i=1}^{N} \mathbb{E} \left\{ \int_{0}^{T} \left[ Q(\bar{x}_i - (\Gamma \hat{x}^{(N)} + \eta)), y_i \right] dt + \langle H(\bar{x}_i(T) - (\Gamma_0 \hat{x}^{(N)}(T) + \eta_0)), \Gamma_0 y^{(N)}(T) \rangle \right\}
\]
\[
= \sum_{j=1}^{N} \mathbb{E} \left\{ \int_{0}^{T} \left[ \frac{\Gamma^T Q}{N} \sum_{i=1}^{N} (\bar{x}_i - (\Gamma \hat{x}^{(N)} + \eta)), y_j \right] dt \\
+ \langle \frac{\Gamma_0^T H}{N} \sum_{i=1}^{N} (\bar{x}_i(T) - (\Gamma_0 \hat{x}^{(N)}(T) + \eta_0)), y_j(T) \rangle \right\}
\]
\[
= \sum_{j=1}^{N} \mathbb{E} \left\{ \int_{0}^{T} \left[ \Gamma^T Q((I - \Gamma) \hat{x}^{(N)} - \eta), y_j \right] dt + \langle \Gamma_0^T H((I - \Gamma_0) \hat{x}^{(N)}(T) - \eta_0), y_j(T) \rangle \right\}.
\]
From (A.2), one can obtain that

\[
I_1 = \sum_{i=1}^{N} \mathbb{E} \int_0^T \left[ \langle Q(\ddot{x}_i - (\Gamma \ddot{x}^{(N)}) + \eta), y_i - \Gamma y^{(N)} \rangle + \langle R\ddot{u}_i + B^T p_i + D^T \beta^i_t, u_i \rangle \right] dt
\]

\[
+ \sum_{i=1}^{N} \mathbb{E} \left[ \langle H(\ddot{x}_i(T) - (\Gamma_0 \ddot{x}^{(N)})(T) + \eta_0), y_i(T) - \Gamma_0 y^{(N)}(T) \rangle - \langle p_i(T), y_i(T) \rangle \right]
\]

\[
+ \sum_{i=1}^{N} \mathbb{E} \int_0^T \langle \alpha_i + A^T p_i + G^T p^{(N)} + C^T \beta^i_t, y_i \rangle dt
\]

\[
= \sum_{i=1}^{N} \mathbb{E} \int_0^T \left( R\ddot{u}_i + B^T p_i + D^T \beta^i_t, u_i \right) dt
\]

\[
+ \sum_{i=1}^{N} \mathbb{E} \left\{ \int_0^T \left( Q\ddot{x}_i - Q\Gamma \ddot{x}^{(N)} - \bar{\eta} + \alpha_i + A^T p_i + G^T p^{(N)} + C^T \beta^i_t, y_i \right) dt
\]

\[
+ \langle H\ddot{x}_i(T) - H\Gamma_0 \ddot{x}^{(N)}(T) - \bar{\eta}_0 - p_i(T), y_i(T) \rangle \right\}.
\]

(A.4)

From (A.3), \( \hat{u} = (\hat{u}_1, \cdots, \hat{u}_N) \) is a minimizer to Problem (P1) if and only if \( I_2 \geq 0 \) and \( I_1 = 0 \).

By Proposition 5.1, \( I_2 \geq 0 \) if and only if (P1) is convex. \( I_1 = 0 \) is equivalent to

\[
\begin{align*}
\alpha_i &= -[A^T p_i + C^T \beta^i_t - Q\ddot{x}_i - Q\Gamma \ddot{x}^{(N)} - \bar{\eta} + G^T p^{(N)}], \\
p_i(T) &= H\ddot{x}_i(T) - H\Gamma_0 \ddot{x}^{(N)}(T) - \bar{\eta}_0, \\
R\ddot{u}_i + B^T p_i + D^T \beta^i_t &= 0.
\end{align*}
\]

Thus, we have the following optimality system:

\[
\begin{align*}
\dot{x}_i &= (A\ddot{x}_i + B\ddot{u}_i + G\ddot{x}^{(N)} + f) dt + (C\ddot{x}_i + D\ddot{u}_i + \sigma)dW_i, \\
\dot{p}_i &= -[A^T \ddot{p}_i + G^T \ddot{p}^{(N)} + C^T \ddot{\beta}^i_t + Q\ddot{x}_i - Q\Gamma \ddot{x}^{(N)} - \bar{\eta}] dt + \sum_{j=1}^{N} \ddot{\beta}^j_t dW_j, \quad (A.5) \\
R\ddot{u}_i + B^T \ddot{p}_i + D^T \ddot{\beta}^i_t &= 0, \quad i = 1, \cdots, N. \\
\ddot{x}_i(0) &= x_{i0}, \quad \ddot{p}_i(T) = H\ddot{x}_i(T) - H\Gamma_0 \ddot{x}^{(N)}(T) - \bar{\eta}_0.
\end{align*}
\]

This implies that FBSDE (7) admits a solution \((\ddot{x}_i, \ddot{p}_i, \ddot{\beta}^i_t, i, j = 1, \cdots, N)\).

On the other hand, if the equation system (7) admits a solution \((\ddot{x}_i, \ddot{p}_i, \ddot{\beta}^i_t, i, j = 1, \cdots, N)\). Let \( \hat{u}_i \) satisfy \( R\ddot{u}_i + B^T \ddot{p}_i + D^T \ddot{\beta}^i_t = 0 \). If (P1) is convex, then by (A.3), \( \hat{u} \) is a minimizer to Problem (P1).

(ii) By Proposition 5.2 the fact that (P1) is uniformly convex implies (6) admits a solution. This with (46) further gives FBSDE (7) admits a solution. Thus, (ii) follows. \( \square \)
APPENDIX B
PROOF OF THEOREMS 3.2

To prove Theorem 3.2, we need two lemmas.

Lemma B.1: Let A1)-A3) hold. Under the control (21), we have
\[
\max_{0 \leq t \leq T} \mathbb{E}\|\hat{x}_i(t)\|^2 \leq c. \tag{B.1}
\]

Proof. Let \(\Phi_i(t)\) is the solution to the following stochastic differential equation:
\[
d\Phi_i(t) = \bar{A}\Phi_i(t)dt + \bar{C}\Phi_i(t)dW_i(t), \quad \Phi_i(0) = I. \tag{B.2}
\]
From (23), we have
\[
\hat{x}_i(t) = \Phi_i(t)x_{i0} + \Phi_i(t)\int_0^t \Phi_i^{-1}(\tau)(\bar{G}x^{(N)}(\tau) + g(\tau))d\tau + \Phi_i(t)\int_0^t \Phi_i^{-1}(\tau)\sigma(\tau)dW_i(\tau),
\]
where
\[
g \overset{\Delta}{=} (\bar{C}^T \bar{D} - B)\Upsilon^\dagger B^T(\bar{K}\bar{x} + s) + f - \bar{C}^T \sigma.
\]
It can be verified that \(\int_0^T \|g(t)\|^2 dt \leq c\). Note that \(\mathbb{E}\int_0^T tr[\Phi_i^T(t)\Phi_i(t)]dt < c\). We have
\[
\mathbb{E}\|\hat{x}_i(t)\|^2 \leq 3\mathbb{E}\|\Phi_i(t)x_{i0}\|^2 + 3\mathbb{E}\int_0^T tr[\Phi_i^T(t-\tau)\sigma^T \sigma \Phi_i(t-\tau)]d\tau
\]
\[
+ 3\mathbb{E}\int_0^T tr[\Phi_i^T(t-\tau)\Phi_i(t-\tau)]d\tau \mathbb{E}\int_0^T \|\bar{G}\hat{x}^{(N)}(\tau) + g(\tau)\|^2 d\tau
\]
\[
\leq c_0 + 6c_1 \left( c_2 \mathbb{E}\int_0^T \frac{1}{N} \sum_{i=1}^N \|\hat{x}_i(\tau)\|^2 d\tau + c_3 \right)
\]
\[
= 6c_1c_2 \max_{1 \leq i \leq N} \mathbb{E}\int_0^T \|\hat{x}_i(\tau)\|^2 d\tau + c.
\]
By Gronwall’s inequality, \(\max_{1 \leq i \leq N} \mathbb{E}\|\hat{x}_i(t)\|^2 \leq ce^{6c_1c_2t}\). This implies (B.1). □

Lemma B.2: Let A1)-A3) hold. Under the control (21), we have
\[
\max_{0 \leq t \leq T} \mathbb{E}\|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2 = O\left(\frac{1}{N}\right). \tag{B.3}
\]

Proof. It follows by (23) that
\[
d\hat{x}^{(N)} = \left[(\bar{A} + G)\hat{x}^{(N)} - \bar{B} \Upsilon^\dagger B^T(\Upsilon \bar{x} + s) + f\right]dt + \frac{1}{N} \sum_{i=1}^N [\bar{C}\hat{x}_i - D\Upsilon^\dagger B^T(\bar{K}\bar{x} + s) + \sigma]dW_i.
\]
From this and (20), we have
\[
d(\hat{x}^{(N)} - \bar{x}) = (\bar{A} + G)(\hat{x}^{(N)} - \bar{x})dt + \frac{1}{N} \sum_{i=1}^N [\bar{C}\hat{x}_i - D\Upsilon^\dagger B^T(\bar{K}\bar{x} + s) + \sigma]dW_i,
\]
which leads to
\[ \hat{x}^{(N)}(t) - \bar{x}(t) = e^{(\tilde{A}+G)^T}[\hat{x}^{(N)}(0) - \bar{x}(0)] \]
\[ + \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} e^{(A+G)(t-\tau)}[\tilde{C}\hat{x}_i - D \Upsilon^T B^T (K \bar{x} + s)]dW_i(\tau). \]

By A1), one can obtain
\[ \mathbb{E}\|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2 \]
\[ \leq 2 \|e^{(\tilde{A}+G)t}\|^2 \left\{ \mathbb{E}\|\hat{x}^{(N)}(0) - \bar{x}_0\|^2 + \frac{1}{N} \int_{0}^{t} \|e^{-(A+G)(t-\tau)}\|^2 (c_1 \mathbb{E}\|\hat{x}_i\|^2 + c_2) d\tau \right\} \]
\[ \leq \frac{2}{N} \|e^{(A+G)t}\|^2 \left\{ \max_{1 \leq i \leq N} \mathbb{E}\|\hat{x}_i\|^2 + c \int_{0}^{t} \|e^{-(A+G)(t-\tau)}\|^2 d\tau \right\}, \]
which completes the proof. \( \square \)

**Proof of Theorem 3.2** We first prove that for \( u \in \mathcal{U}_e, J_{soc}^F(u) < \infty \) implies that \( \mathbb{E} \int_{0}^{T} (\|x_i\|^2 + \|u_i\|^2)dt < \infty \), for all \( i = 1, \ldots, N \). By A2), we have
\[ \delta_0 \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \|u_i\|^2 dt - c \leq J_{soc}^F(u) < \infty, \]
which implies \( \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \|u_i\|^2 dt < c_1 \). By (I) and Schwarz’s inequality,
\[ \mathbb{E}\|x_i(t)\|^2 \leq c_1 \mathbb{E} \int_{0}^{t} \|x^{(N)}(\tau)\|^2 d\tau + c_2 \leq \frac{c_1}{N} \mathbb{E} \int_{0}^{t} \sum_{j=1}^{N} \|x_j(\tau)\|^2 d\tau + c_2 \]
which further gives that
\[ \sum_{j=1}^{N} \mathbb{E}\|x_j(t)\|^2 \leq c_1 \int_{0}^{t} \sum_{j=1}^{N} \mathbb{E}\|x_j(\tau)\|^2 d\tau + Nc_2. \]
By Gronwall’s inequality,
\[ \sum_{j=1}^{N} \mathbb{E}\|x_j(t)\|^2 \leq Nc_2 e^{c_1 t} \leq Nc_2 e^{c_1 T}. \]
Let \( \tilde{x}_i = x_i - \hat{x}_i, \bar{u}_i = u_i - \hat{u}_i \) and \( \tilde{x}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i \). Note that it follows by Lemma B.1 that
\[ \mathbb{E} \int_{0}^{T} (\|\tilde{x}_i\|^2 + \|\bar{u}_i\|^2)dt < \infty. \]
Then we have
\[ \mathbb{E} \int_{0}^{T} (\|\tilde{x}_i\|^2 + \|\bar{u}_i\|^2)dt < \infty. \] (B.5)
By (1) and (23),
\[ d\tilde{x}_i = (A\tilde{x}_i + G\tilde{x}^{(N)} + B\tilde{u}_i)dt + (C\tilde{x}_i + D\tilde{u}_i)dW_i, \quad \tilde{x}_i(0) = 0. \tag{B.6} \]

From (4), we have
\[
\tilde{J}_i^F(u) = \sum_{i=1}^{N} (\tilde{J}_i^F(\tilde{u}) + \tilde{J}_i^F(\tilde{u}) + 2\tilde{I}_i), \tag{B.7}
\]
where
\[
\tilde{J}_i^F(\tilde{u}) \triangleq \mathbb{E} \left[ \int_0^T \right] ||\tilde{x}_i - \Gamma\tilde{x}(N)||_{\mathcal{Q}}^2 + ||\tilde{u}_i||_{\mathcal{H}}^2 dt + \mathbb{E}[||\tilde{x}_i(T) - \Gamma_0(\tilde{x}^{(N)}(T))||_{\mathcal{H}}^2] \\
\tilde{I}_i = \mathbb{E} \left\{ \int_0^T \left[ (\tilde{x}_i - \Gamma\tilde{x}(N) - \eta)^T Q (\tilde{x}_i - \Gamma\tilde{x}(N)) + \tilde{u}_i^T R\tilde{u}_i \right] dt \\
+ \left[ \tilde{x}_i(T) - (\Gamma_0\tilde{x}(N)(T) + \eta_0) \right]^T H [\tilde{x}_i(T) - \Gamma_0\tilde{x}(N)(T)] \right\}.
\]

By A2, $\tilde{J}_i^F(\tilde{u}) \geq 0$. We now prove $\frac{1}{N} \sum_{i=1}^{N} \tilde{I}_i = O(\frac{1}{\sqrt{N}})$.
\[
\sum_{i=1}^{N} \tilde{I}_i = \sum_{i=1}^{N} \mathbb{E} \left[ \int_0^T \right] \tilde{x}_i^T (Q\tilde{x}_i - Q\Gamma\tilde{x} - \tilde{\eta}) + \tilde{u}_i^T R\tilde{u}_i \right] dt + \sum_{i=1}^{N} \mathbb{E} \left[ \int_0^T (\tilde{x}^{(N)} - \tilde{x})^T Q\Gamma\tilde{x} dt \\
+ \sum_{i=1}^{N} \mathbb{E} \left[ \tilde{x}_i^T (T)(H\tilde{x}_i(T) - H\Gamma_0\tilde{x}(T) - \tilde{\eta}_0) + (\tilde{x}^{(N)}(T) - \tilde{x}(T))^T H\Gamma_0\tilde{x}_i(T) \right].
\tag{B.8}
\]

Denote $\hat{p}_i(t) = P\hat{x}_i(t) + K\hat{x}(t) + s(t)$. Then by (13)-(15) and Itô’s formula,
\[
d\hat{p}_i = - \left[ A^T P + PA + CT^T PC + Q - (B^T P + D^T PC)^T \Gamma^T (B^T P + D^T PC) \right] \hat{x}_i dt \\
+ P \left[ A\hat{x}_i - B\Gamma^T (B^T (K\tilde{x} + s) + D^T P\sigma) + G\tilde{x}^{(N)} + f \right] dt \\
+ P [C\hat{x}_i - D\Gamma^T (B^T (K\tilde{x} + s) + D^T P\sigma) + \sigma] dW_i \\
- \left[ (A + G)^T K + K(A + G) - (B^T P + D^T PC)^T \Gamma^T B^T P - KBY^T B^T K + G^T P + PG \right. \\
- KBY^T (B^T P + D^T PC) - Q\Gamma] \tilde{x}_i dt + K \left\{ (A + G)\hat{x}_i - B\Gamma^T [B^T (P + K) + D^T PC] \tilde{x} \\
- B\Gamma^T (B^T s + D^T P\sigma) + f \right\} dt - \left\{ [A + G - B\Gamma^T (B(P + K) + D^T PC)]^T s \\
+ (P + K)f + [C - D\Gamma^T (B(P + K) + D^T PC)]^T P\sigma - \tilde{\eta} \right\}
= - (A^T \hat{p}_i + G^T \hat{p}_i^{(N)} + CT^T \hat{\beta}_i + Q\hat{x}_i - Q\Gamma\tilde{x} - \tilde{\eta}) dt \\
+ (G^T P + PG)(\hat{x}^{(N)} - \tilde{x}) dt + \hat{\beta}_i dW_i, \tag{B.9}
\]
where $\hat{\beta}_i^2 = P(C\hat{x}_i + D\hat{u}_i + \sigma)$. By (21), we have $R\hat{u}_i = -(B\hat{p}_i + D\hat{\beta}_i^2)$. Note that $\hat{p}_i(T) = H\hat{x}_i(T) - H_{\Gamma_0}\bar{x}(T) - \tilde{\eta}_0$. From (B.6) and (B.9),

$$\sum_{i=1}^N \mathbb{E}[\hat{x}_i^T(T)(H\hat{x}_i(T) - H_{\Gamma_0}\bar{x}(T) - \tilde{\eta}_0)] = \mathbb{E} \int_0^T \sum_{i=1}^N \left\{ -\hat{x}_i^T \left[ Q\hat{x}_i - Q_T\bar{x} - \bar{\eta} \right] - \hat{u}_i^T R\hat{u}_i \right\} dt$$

$$+ N\mathbb{E} \int_0^T (\hat{x}^{(N)} - \bar{x})^T (G^T P + PG)\hat{x}^{(N)} dt.$$  

This and (B.8) lead to

$$\frac{1}{N} \sum_{i=1}^N \hat{I}_i = \mathbb{E} \int_0^T (\hat{x}^{(N)} - \bar{x})^T (G + G^T P + PG)\hat{x}^{(N)} dt + \mathbb{E}[(\hat{x}^{(N)}(T) - \bar{x}(T))^T H_{\Gamma_0}\bar{x}^{(N)}(T)].$$

By Lemma B.2 and (B.5), we obtain

$$\left| \frac{1}{N} \sum_{i=1}^N \hat{I}_i \right|^2 \leq c\mathbb{E} \int_0^T ||\hat{x}^{(N)} - \bar{x}||^2 dt \cdot \mathbb{E} \int_0^T ||\hat{x}^{(N)}||^2 dt$$

$$\times \mathbb{E}[(||\hat{x}^{(N)}(T) - \bar{x}(T)||^2 \cdot \mathbb{E}||\hat{x}^{(N)}(T)||^2],$$

which implies $\frac{1}{N} \sum_{i=1}^N \hat{I}_i = O(1/\sqrt{N}).$ \hfill $\square$

APPENDIX C

PROOF OF THEOREMS 4.2 AND 4.3

Proof of Theorem 4.2 (iii) $\Rightarrow$ (i) was given in Theorem 4.1. We now prove (i) $\Rightarrow$ (iii). By (33),

$$\frac{d\mathbb{E}[\hat{x}_i]}{dt} = \bar{A}\mathbb{E}[\hat{x}_i] - B\bar{T}^T((\Pi - P)\bar{x} + s) + G\mathbb{E}[\hat{x}^{(N)}] + f, \ \mathbb{E}[\hat{x}_i(0)] = \bar{x}_0.$$

(C.1)

It follows from A1) that

$$\mathbb{E}[\hat{x}_i] = \mathbb{E}[\bar{x}_j] = \mathbb{E}[\hat{x}^{(N)}], \ j \neq i.$$  

By comparing (38) and (C.1), we obtain

$$\frac{d(\mathbb{E}[\hat{x}_i] - \bar{x})}{dt} = (\bar{A} + G)(\mathbb{E}[\hat{x}_i] - \bar{x}), \ \mathbb{E}[\hat{x}_i(0)] - \bar{x}(0) = 0,$$

which implies

$$\mathbb{E}[\hat{x}_i] = \bar{x} = \mathbb{E}[\hat{x}^{(N)}].$$  

(C.2)

Note that $||\bar{x}||^2 \leq \mathbb{E}||\hat{x}_i||^2$. It follows from (39) that

$$\int_0^\infty ||\bar{x}(t)||^2 dt < \infty.$$  

(C.3)
By (28), we have
\[ \bar{x}(t) = e^{[A+G-BY^\dagger(B^T\Pi+D^TPC)]t} \left[ \bar{x}_0 + \int_0^t e^{-(A+G-BY^\dagger B^T\Pi)\tau} h(\tau) d\tau \right], \]
where \( h = -B\Upsilon^\dagger(B^T s + D^T P\sigma) + f. \) By the arbitrariness of \( \bar{x}_0 \) with (C.3) and (C.5) we obtain that \( A+G-B\Upsilon^\dagger(B^T\Pi+D^TPC) \) is Hurwitz. That is, \( (A,G,B) \) is stabilizable. Note that \( \mathbb{E}[\hat{x}(N)]^2 \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\hat{x}_i^2]. \) Then from (39) we have
\[ \mathbb{E} \int_0^\infty \| \hat{x}(N)(t) \|^2 dt < \infty. \] (C.4)
This leads to \( \mathbb{E} \int_0^\infty \| k(t) \|^2 dt < \infty, \) where \( k = -B\Upsilon^\dagger[B^T((\Pi - P)\bar{x} + s) + D^TP\sigma] + G\bar{x}(N) + f. \)
By (33), we obtain
\[ \mathbb{E}\| \hat{x}_i(t) \|^2 = \mathbb{E}\left\| \Phi_i(t) \left( x_{i0} + \int_0^t \Phi_i^{-1}(\tau) k(\tau) d\tau \right) \right\|^2, \]
where \( \Phi_i \) satisfies (B.2). By (39) and the arbitrariness of \( x_{i0} \) we obtain that \( \mathbb{E} \int_0^\infty \| \Phi_i(t) \|^2 dt < \infty, \) i.e., \( [A,B;C,D] \) is stabilizable. From (C.3) and (C.4),
\[ \mathbb{E} \int_0^\infty \| \hat{x}(N)(t) - \bar{x}(t) \|^2 dt < \infty. \] (C.5)
On the other hand, it follows from (34) that
\[ \mathbb{E}\| \hat{x}(N)(t) - \bar{x}(t) \|^2 \]
\[ = \mathbb{E}\left\| e^{(\bar{A}+G)t}(\hat{x}(N)(0) - \bar{x}_0) \right\|^2 + \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \int_0^t \| e^{(\bar{A}+G)(t-\tau)}(C\hat{x}_i(\tau) + \bar{\sigma}(\tau)) \|^2 d\tau. \]
By (C.5) and the arbitrariness of \( x_{i0}, i = 1, \ldots, N, \) we obtain that \( \bar{A}+G \) is Hurwitz.

(iii)\( \Rightarrow \) (ii) was given in Lemma 4.1 (ii)\( \Rightarrow \) (iii) was implied from [23, Theorem 2].

**Proof of Theorem 4.3** (iii)\( \Rightarrow \) (i) has been proved in Theorem 4.1. Following (i)\( \Rightarrow \) (iii) of Theorem 4.2 together with [2], [49], we obtain (i)\( \Rightarrow \) (ii).

(ii)\( \Rightarrow \) (iii). Define \( V(t) = \mathbb{E}[y^T(t)Py(t)], \) where \( y \) satisfies (29). Denote \( V \) by \( V^* \) when \( u = u^*(t) = -\Upsilon^\dagger(B^T P + D^T P C)y(t). \) By (26) we have
\[ V^*(T) - V^*(0) = \mathbb{E}\left\{ y^T(t) \left[ -Q - (B^T P + D^T P C)^T \Upsilon^\dagger(B^T P + D^T P C) \\
+ (B^T P + D^T P C)^T \Upsilon^\dagger(D^T PD) \Upsilon^\dagger(B^T P + D^T P C) \right] y(t) \right\} \]
\[ = \mathbb{E}\left\{ y^T(t) \left[ -Q - (B^T P + D^T P C)^T \Upsilon^\dagger R \Upsilon^\dagger(B^T P + D^T P C) \right] y(t) \right\} \]
\[ \leq 0. \]
Note that $V^* \geq 0$. Then $\lim_{t \to \infty} V^*(t)$ exists, which implies
\[
\lim_{t_0 \to \infty} [V^*(t_0) - V^*(t_0 + T)] = 0. \tag{C.6}
\]
Rewrite $P(t)$ in [16] by $P_T(t)$. Then we have $P_{T+t_0}(t_0) = P_T(0)$. By (22),
\[
\mathbb{E} \int_{t_0}^{T+t_0} [y^T(t)Qy(t) + u^T(t)Ru(t)] dt = \mathbb{E}[y^T(t_0)P_{T+t_0}(t_0)y(t_0)] + \mathbb{E} \int_0^T \|u(t) + \Upsilon(B^TP_{T+t_0}(t_0) + D^TP_{T+t_0}(t_0)C)y(t)\|^2 dt \\
\geq \mathbb{E}\|y(t_0)\|^2_{P_{T+t_0}(t_0)} = \mathbb{E}\|y(t_0)\|^2_{P_T(0)}.
\]
This with (C.6) implies
\[
\lim_{t_0 \to \infty} \mathbb{E}\|y(t_0)\|^2_{P_T(0)} \leq \lim_{t_0 \to \infty} \mathbb{E} \int_{t_0}^{T+t_0} (\|y(t)\|^2_Q + \|u^*(t)\|^2_R) dt = \lim_{t_0 \to \infty} [V^*(t_0) - V^*(t_0 + T)] = 0.
\]
By A5', one can obtain that there exists $T > 0$ such that $P_T(0) > 0$ (See e.g. [48], [49]). Thus, we have $\lim_{t \to \infty} \mathbb{E}\|\bar{y}(t)\|^2 = 0$, which implies $[A, B; C, D]$ is stabilizable.

To show that $(A + G, B)$ is stabilizable, we consider to optimize
\[
\bar{J}(u) = \int_0^T [\bar{y}^T(s)(C^TPD + Q - Q_T)\bar{y}(s) + 2\bar{y}^T(s)C^TP\bar{u}(s) + \bar{u}^T(s)\Upsilon \bar{u}(s)] ds,
\]
where $\bar{y}$ evolve by
\[
d\bar{y}(t) = [(A + G)\bar{y}(t) + B\bar{u}(t)] dt, \quad \bar{y}(0) = \bar{y}_0. \tag{C.7}
\]
Let $\bar{u}^*(t) = \Upsilon B^T\Pi(t)\bar{y}(t)$, where $\Pi_T(t)$ satisfies
\[
\dot{\Pi} + (A + G)^T\Pi + \Pi(A + G) - (B^T\Pi + D^TPD)^T \Upsilon(B^T\Pi + D^TPD) + C^TPD + Q - Q_T = 0, \quad \Pi(T) = 0. \tag{C.8}
\]
By direct calculations,
\[
\bar{y}_0^T \Pi_T(0) \bar{y}_0 = \int_0^T [\bar{y}^T(s)(C^TPD + Q - Q_T)\bar{y}(s) + 2\bar{y}^T(s)C^TP\bar{u}^*(s) + (\bar{u}^*)^T(s)\Upsilon \bar{u}^*(s)] ds \\
= \int_0^T [\bar{y}^T(s)(\bar{u}^*)^T(s) \left[ \begin{array}{cc} Q - Q_T + C^TPD & C^TPD \\ D^TPD & \Upsilon \end{array} \right] \begin{bmatrix} \bar{y}(s) \\ \bar{u}^*(s) \end{bmatrix} ds. \tag{C.9}
\]
Note that
\[
\begin{bmatrix}
P & PD \\
D^TP & R + D^TPD
\end{bmatrix} = 
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix} 
\begin{bmatrix}
P & 0 \\
0 & R
\end{bmatrix} 
\begin{bmatrix}
I & D \\
0 & I
\end{bmatrix}.
\]

Thus, we have
\[
\begin{bmatrix}
P & PD \\
D^TP & R + D^TPD
\end{bmatrix} > 0.
\]

By Schur’s lemma \cite{32}, \( P - P D Y^\dagger D^T P \geq 0 \). This gives \( C^T (P - P D Y^\dagger D^T P) C \geq 0 \). Using Schur’s lemma again, we obtain
\[
\begin{bmatrix}
C^T P C & C^T P D \\
D^T P C & Y
\end{bmatrix} \geq 0.
\]

Assume \( \tilde{y}_0^T \Pi_T(0) \tilde{y}_0 = 0 \). Then from (C.9), we have \( \int_0^T \tilde{y}^T (s)(Q - Q\Gamma)\tilde{y}(s) dt = 0 \), which implies \( (I - \Gamma)^{1/2}Q\tilde{y}(s) = 0 \), \( 0 \leq s \leq T \). This together with A5’ gives \( \tilde{y}_0 = 0 \). Hence, we obtain \( \Pi_T(0) > 0 \). By a similar argument as the above proof, we can obtain the stabilizability of \( (A + G, B) \).

\[\Box\]

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