Distributed Interval Optimization with Stochastic Zeroth-order Oracle

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Abstract

In this paper, we investigate a distributed interval optimization problem which is modeled with optimizing a sum of convex interval-valued objective functions subject to global convex constraints, corresponding to agents over a time-varying network. We first reformulate the distributed interval optimization problem as a distributed constrained optimization problem by scalarization. Then, we design a stochastic zeroth-order algorithm to solve the reformulated distributed problem, optimal solutions of which are also proved to be Pareto optimal solutions of the distributed interval optimization problem. Moreover, we construct the explicit convergence and the convergence rate in expectation of the given algorithm. Finally, a numerical example is given to illustrate the effectiveness of the proposed algorithm.

Index Terms

distributed interval optimization, Pareto optimal solution, zeroth-order algorithm, convergence rate, time-varying network.

I. INTRODUCTION

Recently, distributed optimization and control in a network environment, where agents only have the local information and exchange information with their neighbours, have attracted much attention, which maybe more effective in many large-scale problems than centralized designs. In fact, distributed first-order algorithms (which require subgradient information of local objective functions) and second-order algorithms (which require Hessian matrices information of local objective functions) for various (constrained) optimization problems have been widely studied for sensor networks, smart grids, and computation, etc [1]–[8]. Also, when the computation of first-order and second-order information of local objective functions is expensive, (distributed) zeroth-order or subgradient-free algorithms are designed (referring to [9]–[13] and references...
therein). Note that the connectivity is a key issue in the distributed design. Although fixed topologies are still required for distributed optimization designs in some situations, time-varying jointly connected networks have been considered in many algorithms such as [1], [3], [14]–[16].

In practice, local objective functions and constraints may not be accurately or explicitly described. For example, various uncertainties appear in power systems and related for operational security [17]. Interval optimization is an approach for dealing with these uncertainties. To solve optimization problems with uncertainties, the interval optimization problem (IOP), first proposed by [18] and further studied in [19], [20], has been widely studied in many different areas such as economics [21] and power systems [17]. In the interval optimization problem setup, objective functions are interval-valued, which are described by intervals rather than real numbers. The well-defined partial orderings and convexity of interval-valued maps [21]–[23] provide existence guarantees of solutions of maximization and minimization of interval optimization problems.

Up to now, the literature (referring to [24]–[27]) has provided various programming methods, including Wolfe’s method and Lamke’s algorithm, to deal with centralized interval optimization problems.

With this background, it is nature for us to consider how to effectively construct distributed algorithms for interval optimization problems over (time-varying) multi-agent networks. However, the partial order resulting from intervals makes the method based on gradients of objective functions become hard, especially when we only have local information in a distributed design, and in some cases, the subgradient of interval-valued objective functions may not be available. In fact, very few works were even done for centralized interval optimization without subgradients in the algorithm design. Up to now, although there are some works on distributed optimization problems without subgradient information of local objective functions, there is no zeroth-order design on distributed interval optimization without using subgradients of objective functions.

The motivation of this paper is to propose a distributed stochastic zeroth-order algorithm for interval optimization problems. Due to difficulties in distributed interval optimization (including that the gradient/subgradient information of interval-valued functions is, sometimes, computationally costly and even impracticable for some cases [28]), we actively employ a stochastic idea to solve distributed interval-valued problems. In fact, stochastic methods provide a way for subgradient-free designs to overcome the difficulty of obtaining subgradient information of local interval-valued functions. Also, stochastic ideas are employed to guarantee the almost sure convergence and stability of algorithms. Here we propose a distributed stochastic zeroth-order
algorithm for a class of interval optimization problems. The contributions of this paper are
summarized as follows:

(a) Following the rapid development of data science and engineering systems, we extend the
centralized interval optimization problem \cite{19}, \cite{20} to a distributed one. In fact, we refor-
mulate the distributed interval optimization problem as a distributed constrained non-smooth
optimization problem. In this reformulation, optimal solutions of the reformulated problem
are equivalent to Pareto optimal solutions of the distributed interval optimization problem.
Distributed randomization methods can be conveniently implemented for the reformulation,
while the well-known versions such as Wolfe’s and Lamke’s methods cannot be easily
extended to distributed versions due to the difficulty of step-size selections \cite{29}.

(b) We design a new distributed stochastic zeroth-order algorithm for the reformulated non-
smooth problem, since the subgradient of the interval optimization problem is hard to be
obtained. The algorithm adopts random differences to approximate subgradients of local
reformulated objective functions, which is also different from many existing distributed
stochastic zeroth-order or subgradient-free algorithms (c.f., \cite{13}, \cite{30}–\cite{32}) though it is
consistent with those algorithms when the local objective function is smooth.

(c) With the proposed algorithm, we prove the achievement of the global minimization with
probability one, and further provide its convergence rate in expectation. Moreover, the
convergence results of the proposed algorithm match the best rate of distributed zeroth-
order algorithms \cite{13}, \cite{30}–\cite{32} with diminishing step-sizes.

The rest of the paper is organized as follows. Preliminaries related to the analysis of the
distributed interval optimization problem are given in Section III. Then the distributed interval
optimization problem is formulated and the corresponding distributed algorithm is introduced in
Section III while the proposed algorithm is analyzed in Section IV. Following that, a numerical
example is given in Section V. Finally, some concluding remarks are addressed in Section VI.

II. MATHEMATICAL PRELIMINARIES

In this section, we introduce mathematical preliminaries about convex analysis \cite{3}, \cite{33}, \cite{34},
probability theory \cite{35}, \cite{36} and interval optimization, respectively.
A. Non-smooth analysis

Let \( \mathcal{R}^p \) be the \( p \)-dimensional Euclidean space. Denote \( \mathcal{R}^p_+ \) as its non-negative orthants. \( \| \cdot \| \) denotes the Euclidean norm. Denote the set of all non-empty compact intervals of \( \mathcal{R} \) by \( \mathcal{C}(\mathcal{R}) \).

**Definition 1.** \([33]\) Let \( f : \mathcal{R}^p \to \mathcal{R} \) be a non-smooth convex function. Vector-valued function \( \nabla f(x) \in \partial f(x) \subset \mathcal{R}^p \) is called the subgradient of \( f(x) \) if for any \( x, y \in \text{dom}(f) \), the following inequality holds:

\[
f(y) - f(x) - \langle \nabla f(x), y - x \rangle \geq 0.
\]

The next result is useful in the analysis of non-smooth functions.

**Lemma 1.** \([34]\) (Lebourg's Mean Value Theorem) Let \( x, y \in \mathcal{X} \). Suppose \( f(x) : \mathcal{R}^m \to \mathcal{R} \) is Lipschitz on an open set containing line segment \([x, y]\). Then there exists a point \( u \in (x, y) \) such that

\[
f(x) - f(y) \in \langle \partial f(u), x - y \rangle.
\]

Then we summarize some inequalities on Euclidean norm \([3], [34]\) to be used in this paper.

**Lemma 2.** \([4]\) Let \( x_1, x_2, \ldots, x_n \) be vectors in \( \mathcal{R}^p \). Then

\[
\sum_{i=1}^{n} \left\| x_i - \frac{1}{n} \sum_{i=1}^{n} x_j \right\|^2 \leq \sum_{i=1}^{n} \left\| x_i - x \right\|^2, \quad \forall x \in \mathcal{R}^p.
\]

Denote the projection of \( x \) onto set \( \mathcal{X} \) by \( P_X(x) \), i.e., \( P_X(x) = \arg \min_{y \in \mathcal{X}} \| x - y \| \), where \( \mathcal{X} \) is a closed bounded convex set in \( \mathcal{R}^p \). The following lemma introduces some results on projection operators:

**Lemma 3.** \([3], [33]\) Let \( \mathcal{X} \) be a closed convex set in \( \mathcal{R}^p \). Then for any \( x \in \mathcal{R}^p \), it holds that

1. \( \langle x - P_X(x), y - P_X(x) \rangle \leq 0 \), for all \( y \in \mathcal{X} \)
2. \( \| P_X(x) - P_X(y) \| \leq \| x - y \| \), for all \( x, y \in \mathcal{R}^m \).
3. \( \langle x - y, P_X(y) - P_X(x) \rangle \leq -\| P_X(x) - P_X(y) \|^2 \), for all \( y \in \mathcal{R}^m \).
4. \( \| x - P_X(x) \|^2 + \| y - P_X(x) \|^2 \leq \| x - y \|^2 \), for any \( y \in \mathcal{X} \).

B. Probability theory

Denote \((\Omega, \mathcal{F}, \mathbb{P})\) as the probability space, where \( \Omega \) is the whole event space, \( \mathcal{F} \) is the \( \sigma \)-algebra on \( \Omega \), and \( \mathbb{P} \) is the probability measure on \((\Omega, \mathcal{F})\).
Definition 2. \cite{35}

(a) \(x_1, x_2, \ldots, x_k \ldots\) is a sequence of random variables (r. v.) in \((\Omega, \mathcal{F}, \mathbb{P})\). If \(P(x_k \to x) = 1\), then \(x_k\) converges \(x\) almost surely (a. s.).

(b) \(x_1, x_2, \ldots, x_k \ldots\) is a sequence of random variables (r. v.) in \((\Omega, \mathcal{F}, \mathbb{P})\). If \(E\|x_k - x\|^p \to 0\), then \(x_k\) converges to \(x\) in \(L^p\).

In \((\Omega, \mathcal{F}, \mathbb{P})\), denote \(\{F(k)\}_{k \geq 1}\) as a sequence of increasing sub-\(\sigma\)-algebras on \(\mathcal{F}\). \(\{h(k)\}_{k \geq 1}, \{v(k)\}_{k \geq 1}\) and \(\{w(k)\}_{k \geq 1}\) are variable sequences in \(\mathcal{R}\) such that for each \(k\), \(h(k), v(k)\) and \(w(k)\) are \(F(k)\)-measurable. The following lemma is for the convergence of super-martingales:

Lemma 4. \cite{36} Suppose that \(\{v(k)\}_{k \geq 1}\) and \(\{w(k)\}_{k \geq 1}\) are nonnegative and \(\sum_{k=1}^{\infty} w(k) < \infty\), and \(\{h(k)\}_{k \geq 1}\) is bounded from below uniformly. If

\[E[h(k+1)|F(k)] \leq (1 + \eta(k))h(k) - v(k) + w(k), \quad \forall k \geq 1\]

holds almost surely, where \(\eta(k) \geq 0\) are constants with \(\sum_{k=1}^{\infty} \eta(k) < \infty\), then \(\{h(k)\}_{k \geq 1}\) converges almost surely with \(\sum_{k=1}^{\infty} v(k) < \infty\).

C. Interval Optimization

Denote \(A = [a_L, a_R]\) and \(B = [b_L, b_R]\) as two non-empty compact intervals in \(\mathcal{P}(\mathcal{R})\). Then we introduce quasi-orderings on \(\mathcal{C}(\mathcal{R})\) and some properties of interval-valued maps.

Definition 3. \cite{21}, \cite{22} For any \(A, B \in \mathcal{P}(\mathcal{R})\), denote

(a) \(A \leq_L B\) if \(a_L \leq b_L\);

(b) \(A \leq_U B\) if \(a_R \leq b_R\);

(c) \(A \leq B\) if \(A \leq_L B\) and \(A \leq_U B\).

Definition 4. \cite{21}, \cite{22} For any \(A, B \in \mathcal{P}(\mathcal{R})\), denote

(a) \(A <_L B\) if \(a_L < b_L\);

(b) \(A <_U B\) if \(a_R < b_R\);

(c) \(A < B\) if \(A <_L B\) and \(A <_U B\);

(d) \(A \leq B\) if \(A <_L B\) and \(A \leq_U B\), or \(A \leq_L B\) and \(A <_U B\).

Let \(G : \mathcal{R}^p \Rightarrow \mathcal{R}\) be an interval-valued map with respect to \(x\). Then we introduce Lipschitz continuity and convexity of the map \(G\).
Definition 5. [37] Let $G : \mathcal{R}^p \Rightarrow \mathcal{R}$ be an interval-valued map. $G$ is locally Lipschitz at $x$ if there exist $K > 0$ and a neighborhood $W$ of $x$ such that

$$G(x_1) \subseteq G(x_2) + K\|x_1 - x_2\|, \quad \forall x_1, x_2 \in W.$$

In fact, $G$ is locally Lipschitz at $x$ if there exist a neighborhood $W$ of $x$ and a constant $K \geq 0$, such that

$$G(x_1) \subseteq B(G(x_2), K\|x_1 - x_2\|).$$

Denote $B(A, \varrho) = \{y|d(y, A) \leq \varrho\}$, as the ball of radius $\varrho$ around subset $A$, where $y$ is chosen from a metric space.

Definition 6. [23] Let $G : \mathcal{R}^p \Rightarrow \mathcal{R}^q$ be an interval-valued map. $G$ is convex (lower-convex or upper convex) on $\Omega$ if, $\forall x_1, x_2 \in \Omega$, $\forall \alpha \in [0, 1]$,

$$G(\alpha x_1 + (1 - \alpha) x_2) \leq (\leq L \text{ or } \leq U) \alpha G(x_1) + (1 - \alpha) G(x_2).$$

Remark 1. Suppose that $G$ is compact-valued and convex with $G(\cdot) = [L(\cdot), R(\cdot)]$. Then, by Definitions [3] and [4] $L(\cdot), R(\cdot) : \mathcal{R}^p \rightarrow \mathcal{R}$ are convex functions with respect to $x \in \mathcal{R}^p$. Namely, for any $x_1, x_2 \in \mathcal{R}^p$ and $t \in [0, 1]$, following inequalities hold:

$$L(tx_1 + (1 - t)x_2) \leq tL(x_1) + (1 - t)L(x_2),$$

$$R(tx_1 + (1 - t)x_2) \leq tR(x_1) + (1 - t)R(x_2).$$

Then let us consider interval optimization problems. Let $G : \mathcal{R}^p \Rightarrow \mathcal{R}$ be an interval-valued map. Now the interval optimization problem is given as follows:

$$(IOP) \quad \min_x G(x) \quad s. t. \quad x \in \Omega$$

where $G(x) = [L(x), R(x)]$ is a non-empty compact interval in $\mathcal{R}$. For illustration, we introduce an example of an interval valued function ( [28]).

Example 1. Consider a function $G : \mathcal{R} \Rightarrow \mathcal{R}$. Without loss of generality, consider $c$ as an order set, which is influenced by orders maintained on the presence of components of $G(x)$. If $G(x_1, x_2) = \{c_1 x_1^2 + c_2 x_1 e^{c_3 x_2} : c_i \in C_i, i = 1, 2, 3\}$, where $C_i = [c_{iL}, c_{iR}], i = 1, 2, 3$, are intervals. Suppose $c = [c_1, c_2, c_3]^\top$, $t = [t_1, t_2, t_3]^\top$, and $C(t) = [c_1(t_1), c_2(t_2), c_3(t_3)]^\top$, 


where $c_i(t_i) = (1 - t_i)c^i_L + t_i c^i_R$ and $t_i \in [0, 1]$ for $i = 1, 2, 3$. For the given interval vector $C^3_v = \prod_{i=1}^{3} C_i$, $G(x_1, x_2) = [L(x), R(x)]$ is an interval, where $L(x) = \min_{t \in [0, 1]} G_{c(t)}(x_1, x_2)$, $R(x) = \max_{t \in [0, 1]} G_{c(t)}(x_1, x_2)$, and $G_{c(t)}(x_1, x_2) = c_1(t_1)x_1^2 + c_2(t_2)x_1 e^{c_3(t_3)x_2}$.

Recalling definitions of $L(x)$ and $R(x)$ of the example, we see that we cannot get explicit expressions of $L(x)$ and $R(x)$, and this IOP can be solved through set-valued optimization rather than vector valued optimization.

Based on quasi-orderings of compact intervals in $C(R)$ given in Definitions 3 and 4, we define a Pareto optimal solution to IOP.

**Definition 7.** [38]

(a) A point $x^* \in \Omega$ is said to be a solution to IOP if $G(x^*) \preceq G(x)$ for all $x \in \Omega$.

(b) A point $x^* \in \Omega$ is said to be a Pareto optimal solution to IOP if $G(\bar{x}) \preceq G(x^*)$ for some $\bar{x} \in \Omega$ implies $G(x^*) \preceq G(\bar{x})$.

Clearly, there is no solution to the interval optimization problem given in Fig. 1. However, $[x_1, x_2]$ are Pareto optimal solutions to this given problem.

(a) For $y < x_1$, we have $R(y) \geq R(x_1)$ and $L(y) \geq L(x_1)$, which means that $G(y) \geq G(x_1)$.

(b) For $y > x_2$, we have $R(y) \geq R(x_2)$ and $L(y) \geq L(x_2)$, which means that $G(y) \geq G(x_2)$.

(c) For $x_1 \leq y \leq x_2$, we have $R(y) \leq R(x_1)$, $L(y) \geq L(x_1)$, $R(y) \geq R(x_2)$ and $L(y) \leq L(x_2)$ according to Definition 7. Therefore, $[x_1, x_2]$ are Pareto optimal solutions to this given problem.

Associated with IOP, we consider the following scalarization of interval optimization problem:

$$SIOP: \quad \min_x \quad \lambda L(x) + (1 - \lambda)R(x)$$

s. t. $x \in \Omega$

where $\lambda \in [0, 1]$.

The following lemma is given in [38]. We give its proof here just for self-containment.

**Lemma 5.** Suppose that $G$ is compact-valued and convex with respect to $x$:

(a) If there exists a real number $\lambda \in (0, 1)$ such that $x^* \in \Omega$ is an optimal solution to SIOP, then $x^* \in \Omega$ is a Pareto optimal solution to IOP.

(b) If $x^* \in \Omega$ is a Pareto optimal solution to IOP, then there exists a real number $\lambda \in [0, 1]$ such that $x^* \in \Omega$ is an optimal solution to SIOP.
Proof. (a) Given a real number $\lambda \in (0, 1)$, let $x^* \in \Omega$ be an optimal solution to SIOP. Suppose that there is $\bar{x} \in \Omega$ such that $G(\bar{x}) \leq G(x^*)$, which implies $L(\bar{x}) \leq L(x^*)$ and $R(\bar{x}) \leq R(x^*)$. Therefore,

$$\lambda L(\bar{x}) + (1 - \lambda) R(\bar{x}) \leq \lambda L(x^*) + (1 - \lambda) R(x^*),$$

which contradicts that $x^*$ is an optimal solution to SIOP.

(b) Let $x^* \in \Omega$ be a Pareto optimal solution to IOP. Since $G$ is compact-valued and convex with respect to $x$, $L(x)$ and $U(x)$ are convex functions according to Remark 1. Following Definition 7 there exists a non-zero vector $\lambda = [a, b]^\top$ with $a \geq 0$ and $b \geq 0$, such that

$$\lambda^\top \begin{bmatrix} L(x^*) \\ R(x^*) \end{bmatrix} \leq \lambda^\top \begin{bmatrix} L(x) \\ R(x) \end{bmatrix},$$

holds for all $x \in \Omega$. Define $\bar{\lambda} = \left[ \frac{a}{a+b}, \frac{b}{a+b} \right]$ then

$$\bar{\lambda}^\top \begin{bmatrix} L(x^*) \\ R(x^*) \end{bmatrix} \leq \bar{\lambda}^\top \begin{bmatrix} L(x) \\ R(x) \end{bmatrix},$$

which implies the conclusion.
III. FORMULATION AND ALGORITHM

Consider the following distributed interval optimization problem over an \( n \)-agent network:

\[
\text{(DIOP)} \quad \min_x \ G(x) = \sum_{i=1}^{n} G_i(x_i)
\]

s. t. \( x_i = x_j, \ x_i \in X \) \hspace{1cm} (1)

where \( x = [x_1^T, x_2^T, \ldots, x_n^T]^T \in \mathbb{R}^{np}, \ x_i \in \mathbb{R}^p, \) and \( G_i : \mathbb{R}^p \rightarrow \mathbb{R} \) is a compact and convex interval-valued function. In this setting, the state of an agent \( i \) is the estimate of solution to \( \text{DIOP} \). Each agent \( i \) knows the local function \( G_i \) and global constraint \( X \).

We make the following assumption on local functions and constraints for \( \text{DIOP} \):

**Assumption 1.**

(a) \( G_i(x) \) is a convex, compact, Lipschitz continuous interval-valued function.

(b) \( X \) is a non-empty, compact, convex constraint set in \( \mathbb{R}^p \).

(c) The subgradient of \( G_i(x) \) is almost everywhere locally Lipschitz continuous.

Assumption 1(a) is consistent with assumptions in the centralized case [28], while Assumption 1(b) is a quite common assumption for the boundedness of distributed and centralized stochastic algorithms based on diminishing step-sizes [4], [39].

Consider \( \text{DIOP} \) over a time-varying multi-agent network, described by a directed graph \( G(k) = (\mathcal{N}, \mathcal{E}(k), W(k)) \), where \( \mathcal{N} = \{1, 2, \ldots, n\} \) is the agent set, the edge set \( \mathcal{E}(k) \subseteq \mathcal{N} \times \mathcal{N} \) represents information communication at time \( k \) and \( W(k) = [w_{ij}(k)]_{ij} \) represents the adjacency matrix at time \( k \). Each agent interacts with its neighbors in \( G(k) = (\mathcal{N}, \mathcal{E}(k), W(k)) \) at time \( k \). The following assumption is about communication topology

**Assumption 2.** The graph \( G(k) = (\mathcal{N}, \mathcal{E}(k), W(k)) \) satisfies:

(a) There exists a constant \( \eta \) with \( 0 < \eta < 1 \) such that, \( \forall k \geq 0 \) and \( \forall i, j, \ w_{ii}(k) \geq \eta; \ w_{ij}(k) \geq \eta \) if \( (j, i) \in \mathcal{E}(k) \).

(b) \( W(k) \) is doubly stochastic, i. e. \( \sum_{i=1}^{m} w_{ij}(k) = 1 \) and \( \sum_{j=1}^{m} w_{ij}(k) = 1 \).

(c) There is an integer \( \kappa \geq 1 \) such that \( \forall k \geq 0 \) and \( \forall (j, i) \in \mathcal{N} \times \mathcal{N}, \)

\[ (j, i) \in \mathcal{E}(k) \cup \mathcal{E}(k+1) \cup \cdots \cup \mathcal{E}(k+\kappa-1). \]

Assumption 2 reveals that agent \( i \) can collect information from all its neighbors “periodically”. It is also a widely used connectivity condition for distributed time-varying network designs (see [1], [3]).
Define the function $f : \mathbb{R}^{np} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $f_i : \mathbb{R}^p \times [0, 1] \rightarrow \mathbb{R}$ as

$$f(x, \lambda) \triangleq \sum_{i=1}^{n} f_i(x_i, \lambda_i)$$

$$f_i(x_i, \lambda_i) \triangleq \lambda_i L_i(x) + (1 - \lambda_i) R_i(x)$$

where $i = 1, 2, \ldots, n$, $x = [x_1^\top, x_2^\top, \ldots, x_n^\top]^\top \in \mathbb{R}^{nq}$ and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^\top \in \mathbb{R}^n$. Note that both $L(x)$ and $R(x)$ are separable, that is,

$$L(x) = \sum_{i=1}^{n} L_i(x_i), \quad R(x) = \sum_{i=1}^{n} R_i(x_i).$$

Let $\lambda = \lambda_0 1_n$ with $\lambda_0 \in (0, 1)$. We can write the distributed interval optimization problem as:

$$\min_x f(x, \lambda) = \sum_{i=1}^{n} f_i(x_i, \lambda_i)$$

s. t. $x_i = x_j, \quad x_i \in X$

$$\lambda_i = \lambda_j$$

where agent $i$ knows the information of $f_i, x_i, \lambda_i \in (0, 1)$ and its neighborhood information. Obviously, problem (5) degenerates to a conventional distributed constrained optimization problem [4] when each agent $i$ choose a common parameter $\lambda_i = 0$ or $\lambda_i = 1$. Some conclusions about the local objective function $f_i$ of (5) are listed in the following lemma.

**Lemma 6.** [28], [38] Suppose Assumption 1 holds. Then, for $i = 1, \ldots, n$,

(a) $f_i(x, \lambda)$ is convex with respect to $x$, that is, for any $x_1, x_2$,

$$f_i(\alpha x_1 + (1 - \alpha) x_2, \lambda) \leq \alpha f_i(x_1, \lambda) + (1 - \alpha) f_i(x_2, \lambda),$$

where $\alpha \in [0, 1]$.

(b) $f_i(x, \lambda)$ is convex with respect to $\lambda$.

(c) $f_i(x, \lambda)$ is Lipschitz continuous with respect to $x$, that is, for any $x_1, x_2$ and $\lambda$,

$$\| f_i(x_1, \lambda) - f_i(x_2, \lambda) \| \leq L \| x_1 - x_2 \|.$$

(d) $f_i(x, \lambda)$ is Lipschitz continuous with respect to $\lambda$, that is, for any $\lambda_1, \lambda_2$ and $x$,

$$\| f_i(x, \lambda_1) - f_i(x, \lambda_2) \| \leq K \| \lambda_1 - \lambda_2 \|.$$
The following lemma still holds for DIOP, whose proof is analogous to the proof of Lemma 5 and omitted here.

**Lemma 7.** If \((x^*, \lambda^*) \in \mathcal{R}^{np} \times \mathcal{R}^n\) is an optimal solution to problem (5), then \(x^*\) is a Pareto solution to problem (1).

Since the differentiability of \(f(x, \lambda)\) with respect to \(x\) may not hold, we propose a distributed zeroth-order interval-valued algorithm for problem (5).

**Algorithm 1 Distributed stochastic zeroth-order algorithm**

**Input:** Total numbers of iteration \(T\), step-size \(\iota(k)\).

**Initialize:** \(\xi_i \in X\) for all \(i = 1, 2, \ldots n\).

1: for \(k = 0, \ldots T\) do
2: Average of local observations \(x_i(k)\):
   \[
   \xi_i(k) = \sum_{j=1}^{n} w_{ij}(k) x_j(k).
   \]
3: Calculation of local measurement \(d_i(k)\)
   \[
   d_i(k) = \frac{[y_i^+(k) - y_i^-(k)] \triangle_i^-(k)}{2c(k)},
   \]
4: Descent Step:
   \[
   \hat{\xi}_i(k) = \xi_i(k) - \iota(k)d_i(k).
   \]
   Projection Step:
   \[
   x_i(k+1) = P_X(\hat{\xi}_i(k)).
   \]
5: Average of local observations \(\lambda_i(k)\):
   \[
   \lambda_i(k+1) = \sum_{j=1}^{n} w_{ij}(k) \lambda_i(k).
   \]
6: end for

where \(d_i(k)\) is used as an estimate for \(\partial f_{\xi_i(k)}(\xi_i(k), \lambda_i(k))\).

In (7), \(\triangle_i(k) = [\triangle_i^1(k), \triangle_i^2(k), \ldots, \triangle_i^p(k)]^\top\), \(\triangle_i^-(k) = \left[\frac{1}{\triangle_i^1(k)}, \frac{1}{\triangle_i^2(k)}, \ldots, \frac{1}{\triangle_i^p(k)}\right]^\top\), where \(\{\triangle_i^q(k)\}_{k \geq 0}, q = 1, 2, \ldots, p, k = 1, 2, \ldots\) is a sequence of mutually independent and identically
distributed random variables with zero mean. The measurements \( y_i^+(k) \) and \( y_i^-(k) \) are given by

\[
y_i^+(k) = f_i(\xi_i(k) + c(k) \triangle_i(k), \lambda_i(k)),
y_i^-(k) = f_i(\xi_i(k) - c(k) \triangle_i(k), \lambda_i(k)).
\]

Define \( F(k) = \sigma\{x_i(k), x_i(k-1), \ldots, x_i(0), i = 1, 2, \ldots, n; \lambda_i(k), \lambda_i(k-1), \ldots, \lambda_i(0), i = 1, 2, \ldots, n; \triangle_i(k-1), \triangle_i(k-2), \ldots, \triangle_i(0), i = 1, 2, \ldots, n\} \), where \( F(k) \) is the \( \sigma \)-algebra created by the whole history of Algorithm 1 up to moment \( k \) (referring to [4]).

In Algorithm 1 the following condition holds in the paper:

**Condition 1.** (a) let \( \{\triangle_q^q(k)\}_{k \geq 0} \) be a sequence of independent and identically distributed (i.i.d.) random variables for any fixed \((i, q)\), and for all \( k \geq 0 \) and \((i, q)\),

\[
|\triangle_q^q(k)| < M_1, \quad \left|\frac{1}{\triangle_q^q(k)}\right| < M_2, \quad E\left[\frac{1}{\triangle_q^q(k)}\right] = 0;
\]

(b) \( \{\triangle_q^q(k)\}_{k \geq 0} \) and \( \{\triangle_r^r(k)\}_{k \geq 0} \) are mutually independent of each other for \( i \neq j \) or \( q \neq r \); and

(c) take \( \iota(k) = \frac{1}{k^{1-\epsilon}} \) and \( c(k) = \frac{1}{k^\delta} \) with \( 0 \leq \epsilon < \frac{1}{4} \) and \( \frac{1}{2} - \delta > \epsilon \) in randomized difference (7).

**Remark 2.** (a) The step-size \( \iota(k) \) satisfies the following stochastic approximation step-size condition in [9], [39]:

\[
\iota(k) > 0, \quad \sum_{k=1}^{\infty} \iota(k) = \infty, \quad \sum_{k=1}^{\infty} \iota^2(k) < \infty.
\]

(b) \( c(k) \) used in randomized difference (7) satisfies

\[
c(k) > 0, \quad c(k) \to 0.
\]

(c) The chosen unit parameter \( \frac{\iota(k)}{c(k)} \) satisfies:

\[
\frac{\iota(k)}{c(k)} > 0, \quad \sum_{k=1}^{\infty} \frac{\iota^2(k)}{c^2(k)} > 0, \quad \sum_{k=1}^{\infty} \iota(k)c(k) < \infty.
\]

**IV. Main Results**

In this section, we first show that the estimate \((x_i(k), \lambda_i(k))\) converges to an optimal point \((x^*, \lambda^*)\) almost surely by Algorithm 1 and then discuss the convergence rate of Algorithm 1.
A. Convergence

Denote the transition matrix of \( W(k) \) as \( \Psi(k, s) = W(k)W(k-1) \cdots W(s), k \geq s \), where \( [\Psi(k, s)]_{ij} \) is the \( ij \)-th element of \( \Psi(k, s) \). The following result was given in Proposition 1 of [1].

**Lemma 8.** Under Assumptions [2] \( \left| [\Psi(k, s)]_{ij} \right| - \frac{1}{n} \leq \mu \beta^{k-s}, \forall k > s, \) where \( \mu = \frac{2(1 + \eta^{-K_0})}{1 - \eta^{-K_0}}, \) with \( K_0 = (n-1)\kappa \) and \( \beta = (1 - \eta^{-K_0})^{1/K_0} < 1 \).

Here is a theorem regarding convergence analysis of the proposed algorithm.

**Theorem 1.** With Assumptions [1,2]

(a) all sequences \( \{\lambda_i(k)\}, i \in N \) generated by Algorithm [7] reach the same point \( \lambda^* \) (which depends on initial parameters \( \lambda_i(0)'s \)).

(b) all sequences \( \{x_i(k)\}, i \in N \) generated by Algorithm [7] converge to the same optimal point \( x^* \) almost surely.

Before we give the proof of Theorem [1] let us introduce the following three lemmas. The first lemma gives an upper bound for the Euclidean norm of \( d_i(k) \) in expectation; the second lemma analyzes the consensus in \( L_1 \) norm of estimates \( x_i(k) \) in Algorithm [1] the third lemma analyzes the lower bound of the cross term of \( d_i(k) \) and \( (\xi_i(k) - x^*) \) in expectation and in conditional expectation with respect to \( F(k) \), where \( x^* \) is the optimal solution of [5] for fixed common point \( \lambda^* \). The proofs of these lemmas are given in Appendix.

**Lemma 9.** With Assumption [7] following statements hold:

(a) \( \| \partial f_i(x_i, \lambda) \| \leq L \) and \( \| \partial f_{i,\lambda}(x, \lambda) \| \leq K \).

(b) the first order moment and second moment of \( d_i(k) \) are bounded by

\[
\mathbb{E} \| d_i(k) \| \leq nM_1M_2L, \quad \mathbb{E} \| d_i(k) \|^2 \leq (nM_1M_2L)^2.
\]

\( L \) and \( K \) are Lipshitz constants with respect to \( x \) and \( \lambda \) in Lemma [2]

**Lemma 10.** With Assumptions [1,2] the consensus of estimate \( x_i(k) \) in \( L_1 \) is achieved by Algorithm [7] that is, for \( i, j = 1, 2, \ldots, n \),

\[
\lim_{k \to \infty} \mathbb{E} \| x_i(k) - x_j(k) \| = 0.
\]

**Lemma 11.** With Assumption [7] the cross term of \( d_i(k) \) and \( \xi_i(k) - \xi^* \) is lower bounded
(a) in conditional expectation with respect to $F(k)$ as follows:

$$
\mathbb{E} \left[ \langle d_i(k), x_i(k) - \xi^* \rangle \bigg| F(k) \right] \\
\geq f_i(\bar{x}(k), \bar{\lambda}(k)) - f_i(x^*, \lambda^*) - L \| \xi_i(k) - \bar{x}(k) \| - Bc(k) \\
- K \| \lambda_i(k) - \bar{\lambda}(k) \| - K \| \lambda_i(k) - \lambda^* \| - c(k) L \| \Delta_i(k) \|,
$$

(b) in expectation as follows:

$$
\mathbb{E} \left[ \langle d_i(k), \xi_i(k) - x^* \rangle \right] \\
\geq \mathbb{E} \left[ f_i(\bar{x}(k), \lambda^*) - f_i(x^*, \lambda^*) \right] - L \mathbb{E} \| \xi_i(k) - \bar{x}(k) \| \\
- 2K \mathbb{E} \| \lambda_i(k) - \lambda^* \| - c(k) L \mathbb{E} \| \Delta_i(k) \| - Bc(k),
$$

where $L$ is the Lipschitz constant with respect to $x$, $K$ is the Lipschitz constant with respect to $\lambda$ given in Lemma 6 and $B$ is a positive constant.

Then it is time to give the proof of Theorem 1.

**Proof.** (a) We claim that, for $i, j = 1, 2, \ldots, n$,

$$
\lim_{k \to \infty} \| \lambda_i(k) - \lambda_j(k) \| = 0 \quad \text{a. s.}
$$

Recalling the transition matrix $\Psi(k, s)$ and $\lambda_i(k+1)$ in (10), we have

$$
\lambda_i(k+1) = \sum_{j=1}^{n} [\Psi(k, 0)]_{ij} \lambda_j(0).
$$

Define $\bar{\lambda}(k+1) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i(k+1)$. According to Assumption 1 and by an analogous induction,

$$
\bar{\lambda}(k+1) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i(0).
$$

Therefore, for $i \in \mathcal{N},$

$$
\| \lambda_i(k+1) - \bar{\lambda}(k+1) \| \leq \sum_{j=1}^{n} \left| [\Psi(k, 0)]_{ij} \right| - \frac{1}{n} \| \lambda_j(0) \|.
$$

Plugging in the estimate of $\Psi(k, s)$ in Lemma 8 leads to

$$
\| \lambda_i(k+1) - \bar{\lambda}(k+1) \| \leq n \delta \beta^k \max_{1 \leq i \leq n} \| \lambda_i(0) \|.
$$

Therefore,

$$
\lim_{k \to \infty} \| \lambda_i(k) - \bar{\lambda}(k) \| = 0, \quad \forall i \in \mathcal{N}.
$$
(b) We claim that, for \( i, j = 1, 2, \ldots, n \),
\[
\lim_{k \to \infty} \|x_i(k) - x_j(k)\| = 0 \quad a. s.
\]
From Lemma 10, \( \lim_{k \to \infty} E \|x_i(k+1) - \bar{x}(k+1)\| = 0 \) holds. Still
\[
0 \leq E \left[ \liminf_{k \to \infty} \|x_i(k+1) - \bar{x}(k+1)\| \right]
\leq \liminf_{k \to \infty} E \|x_i(k+1) - \bar{x}(k+1)\| = 0,
\]
which implies \( E \left[ \liminf_{k \to \infty} \|x_i(k+1) - \bar{x}(k+1)\| \right] = 0 \). Therefore, \( \liminf_{k \to \infty} \|x_i(k+1) - \bar{x}(k+1)\| = 0 \) holds almost surely. Since \( \sum_{i=1}^{n} \|x_i(k+1) - \bar{x}(k+1)\|^2 \leq \sum_{i=1}^{n} \|x_i(k+1) - \bar{x}(k)\|^2 \) by Lemma 2 and \( \|x_i(k+1) - \bar{x}(k)\|^2 \leq \|\hat{\zeta}_i(k) - \bar{x}(k)\|^2 \) by Lemma 3, we have
\[
\sum_{i=1}^{n} \|x_i(k+1) - \bar{x}(k+1)\|^2
\leq \sum_{i=1}^{n} \|\hat{\zeta}_i(k) - \bar{x}(k)\|^2
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(k) \|x_j(k) - \bar{x}(k)\|^2 + \iota^2(k) \sum_{i=1}^{n} \|d_i(k)\|^2
+ 2\iota(k) \sum_{i=1}^{n} \|d_i(k)\| \sum_{j=1}^{n} w_{ij}(k) \|x_j(k) - \bar{x}(k)\|.
\]
According to Assumption 2(b),
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(k) \|x_j(k) - \bar{x}(k)\|^2 = \sum_{i=1}^{n} \|x_i(k) - \bar{x}(k)\|^2.
\]
Taking the conditional expectation of both sides of (17) yields
\[
\sum_{i=1}^{n} E \left[ \|x_i(k+1) - \bar{x}(k+1)\|^2 \mid F(k) \right]
\leq \sum_{i=1}^{n} \|x_j(k) - \bar{x}(k)\|^2 + \sum_{i=1}^{n} \iota^2(k) E \|d_i(k)\|^2
+ \sum_{i=1}^{n} 2\iota(k) E \|d_i(k)\| E \|x_i(k) - \bar{x}(k)\|.
\]
According to Remark 2 and Lemma 9(b), \( \sum_{k=1}^{\infty} \sum_{i=1}^{n} \iota^2(k) E \|d_i(k)\|^2 < \infty \). By Theorem 6.2 of [4], \( \sum_{k=1}^{\infty} \iota(k) \|x_i(k) - \bar{x}(k)\| < \infty \) with probability 1. From Lemma 9(a), \( \sum_{k=1}^{\infty} \sum_{i=1}^{n} 2\iota(k) E \|d_i(k)\| E \|x_i(k) - \bar{x}(k)\| < \infty \). Therefore, \( \lim_{k \to \infty} \|x_i(k) - \bar{x}(k)\| = 0 \) almost surely by Lemma 4.
(c) Clearly, $\|x_i(k+1) - x^*\|^2 \leq \|\xi_i(k) - x^*\|^2$ according to Lemma \[3\]. Then

$$
\|x_i(k+1) - x^*\|^2 \leq \|\xi_i(k) - x^*\|^2 + i^2(k)\|d_i(k)\|^2 - 2\epsilon(k)\langle d_i(k), \xi_i(k) - x^*\rangle.
$$

(20)

Taking conditional expectation on both sides of (20) gives

$$
\mathbb{E}\left[\|x_i(k+1) - x^*\|^2 \mid F(k)\right] 
\leq \mathbb{E}\left[\|\xi_i(k) - x^*\|^2 \mid F(k)\right] + i^2(k)\mathbb{E}\left[\|d_i(k)\|^2 \mid F(k)\right] - 2\epsilon(k)\mathbb{E}\left[\langle d_i(k), \xi_i(k) - x^*\rangle \mid F(k)\right]
$$

(21)

for all $k = 0, 1, 2, \ldots$. By the double stochasticity of matrix $W(k)$ in Assumption \[2\](b),

$$
\sum_{i=1}^{n} \mathbb{E}\left[\|\xi_i(k) - x^*\|^2 \mid F(k)\right] \leq \sum_{i=1}^{n} \|x_i(k) - x^*\|^2,
$$

$$
\sum_{i=1}^{n} \mathbb{E}\left[\|\xi_i(k) - \bar{x}(k)\| \mid F(k)\right] \leq \sum_{i=1}^{n} \|x_i(k) - \bar{x}(k)\|.
$$

(22)

Then, with probability 1, for $i \in \mathcal{N}$, it holds

$$
\sum_{i=1}^{n} \mathbb{E}\left[\|x_i(k+1) - x^*\|^2 \mid F(k)\right] 
\leq \sum_{i=1}^{n} \left[\|x_i(k) - x^*\|^2 + [O_i(k)]_1 + [O_i(k)]_2 + [O_i(k)]_3 + [O_i(k)]_4 + [O_i(k)]_5 + [O_i(k)]_6 - J_i(k)\right],
$$

(23)

where

$$
\begin{align*}
[O_i(k)]_1 &= i^2(k)\mathbb{E}\left[\|d_i(k)\|^2 \mid F(k)\right], \\
[O_i(k)]_2 &= 2\epsilon(k)L\mathbb{E}\|x_i(k) - \bar{x}(k)\|, \\
[O_i(k)]_3 &= 4\epsilon(k)c(k)L\mathbb{E}\|\Delta(k)_i\|, \\
[O_i(k)]_4 &= 2\epsilon(k)L\mathbb{E}\|\lambda_i(k) - \bar{\lambda}(k)\|, \\
[O_i(k)]_5 &= 2\epsilon(k)L\mathbb{E}\|\lambda_i(k) - \lambda^*\|, \\
[O_i(k)]_6 &= 2B\epsilon(k)c(k), \\
J_i(k) &= 2\epsilon(k)[f_i(\bar{x}(k), \bar{\lambda}(k)) - f_i(x^*, \lambda^*)].
\end{align*}
$$

Recalling Remark \[2\] and Lemma \[9\], \(\sum_{k=1}^{\infty} [O_i(k)]_1 < \infty\). By the proof in part (a), \(\sum_{k=1}^{\infty} [O_i(k)]_2 < \infty\), \(\sum_{k=1}^{\infty} [O_i(k)]_3 < \infty\). From Theorem \[1\], \(\sum_{k=1}^{\infty} [O_i(k)]_4 < \infty\) and \(\sum_{k=1}^{\infty} [O_i(k)]_5 < \infty\).
With Remark 2, \( \sum_{k=1}^{\infty} \left[ O_i(k) \right]_6 < \infty \). Therefore, \( \sum_{k=1}^{\infty} \sum_{i=1}^{n} \left[ \left[ O_i(k) \right]_1 + \left[ O_i(k) \right]_2 + \left[ O_i(k) \right]_3 + \left[ O_i(k) \right]_4 + \left[ O_i(k) \right]_5 + \left[ O_i(k) \right]_6 \right] < \infty \). From Lemma 8, \( \sum_{i=1}^{n} \| x_i(k) - x^* \|^2 \) converges almost surely with \( \sum_{k=1}^{\infty} \sum_{i=1}^{n} j_i(k) < \infty \). Since \( \sum_{i=1}^{\infty} \iota(k) = \infty \),

\[
\lim \inf_{k \to \infty} f_i(\bar{x}(k), \bar{\lambda}(k)) = f_i(x^*, \lambda^*)
\]

holds almost surely. Therefore, the sequence \( \lim_{k \to \infty} \sum_{i=1}^{n} \| \xi_i(k) - \xi^* \|^2 = 0 \) with probability 1. The proof is completed.

\[\square\]

**Remark 3.** Most of existing zeroth-order (distributed) algorithms \([13], [30]–[32]\) are based on the assumption that (local) objective functions are smooth. However, in our article, we assume that local interval-valued objective functions are nonsmooth. In this case, a direct application of the subgradient and the step-size selection for most of the existing distributed first-order algorithms \([1]–[6]\) are not applicable. In the proof of Theorem 1, we select a different parameter \( \iota(k) \) \( c(k) \), which guarantees the application of supermartingale convergence theorem in \([36]\). Also, we make use of Lebourg's mean value theorem \([34]\) to estimate local subgradient information.

**B. Convergence rate**

We further analyze the convergence rate of Algorithm 1. Denote \( (x^*, \lambda^*) \) as the optimal solution of problem (5), where \( \lambda^* \) is given in Theorem 1 and \( x^* \in \arg \min_{x_i=x_i \in X} f(x, \lambda^*) \). Here is the main result.

**Theorem 2.** With Assumptions 1,2 for Algorithm 1 we have

\[
R(T) - \rho \sim O\left( \frac{1}{T^\rho} \right).
\]

**Proof.** By taking expectation to both sides of (20), we obtain

\[
\mathbb{E}\| x_i(k+1) - x^* \|^2 \leq \mathbb{E}\| \xi_i(k) - x^* \|^2 + \iota^2(k)\mathbb{E}\| d_i(k) \|^2 - 2\iota(k)\mathbb{E}[\langle d_i(k), \xi_i(k) - x^* \rangle].
\]

(24)
By the double stochasticity of matrix $W(k)$ given in Assumption 2(b), we obtain

$$
\sum_{i=1}^{n} \mathbb{E}\|\xi_i(k) - x^*\|^2 = \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{j=1}^{n} w_{ij}(k)x_j(k) - x^* \right\|^2
\leq \sum_{i=1}^{n} \mathbb{E}\|x_i(k) - x^*\|^2,
$$

(25)

$$
\sum_{i=1}^{n} \mathbb{E}\|\xi_i(k) - \bar{x}(k)\| = \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{j=1}^{n} w_{ij}(k)x_j(k) - \bar{x}(k) \right\|
\leq \sum_{i=1}^{n} \mathbb{E}\|x_i(k) - \bar{x}(k)\|.
$$

(26)

By (25), (26) and Lemma 11, we have

$$
\sum_{k=1}^{T} \sum_{i=1}^{n} \mathbb{E}\|x_i(k+1) - x^*\|^2
\leq \sum_{k=1}^{T} \sum_{i=1}^{n} \mathbb{E}\|x_i(k) - x^*\|^2 + 4K \sum_{k=1}^{T} \sum_{i=1}^{n} \ell(s)\mathbb{E}\|\lambda_i(k) - \lambda^*\|
+ 2L \sum_{k=1}^{T} \sum_{i=1}^{n} \ell(k)\mathbb{E}\|x_i(k) - \bar{x}(k)\| + 2nB \sum_{k=1}^{T} \ell(k)c(k)
+ 4L \sum_{k=1}^{T} \sum_{i=1}^{n} \ell(k)c(k)\mathbb{E}\|\Delta_i(k)\| + \sum_{k=1}^{T} \sum_{i=1}^{n} \ell^2(k)\mathbb{E}\|d_i(k)\|^2
- 2 \sum_{k=1}^{T} \sum_{i=1}^{n} \ell(k)\mathbb{E}\left[ f_i(\bar{x}(k), \lambda^*) - f_i(x^*, \lambda^*) \right].
$$

(27)

Therefore, by taking summation of both sides of (24) for $k = 1, 2, \ldots T$ and $i = 1, 2, \ldots n$, we get

$$
\sum_{k=1}^{T} \sum_{i=1}^{n} \mathbb{E}\left[ f_i(x_i(k), \lambda_i(k)) - f_i(x^*, \lambda^*) \right]
\leq \sum_{k=1}^{T} \frac{1}{\ell(k)} \left[ \sum_{i=1}^{n} \mathbb{E}\|x_i(k+1) - x^*\|^2 - \sum_{i=1}^{n} \mathbb{E}\|x_i(k) - x^*\|^2 \right]
+ 3K \sum_{k=1}^{T} \sum_{i=1}^{n} \mathbb{E}\|\lambda_i(k) - \lambda^*\| + 2L \sum_{k=1}^{T} \sum_{i=1}^{n} \mathbb{E}\|x_i(k) - \bar{x}(k)\|
+ nB \sum_{k=1}^{T} c(k) + 2L \sum_{k=1}^{T} \sum_{i=1}^{n} c(k)\mathbb{E}\|\Delta_i(k)\|
+ \frac{1}{2} \sum_{k=1}^{T} \sum_{i=1}^{n} \ell(k)\mathbb{E}\|d_i(k)\|^2
$$

(28)
Note that \( \iota(k) = \frac{1}{k^{1-\epsilon}} \) and \( c(k) = \frac{1}{k^2} \), \( 0 \leq \epsilon < \frac{1}{4} \), and \( \frac{1}{2} - \epsilon > \delta > \epsilon \). Since \( X \) is bounded in \( \mathbb{R}^m \), for \( x \in X \), there exists a constant \( M_x \) such that \( \|x\| \leq M_x \). For the first term on the right hand side of (28), we have

\[
\sum_{k=1}^{T} \frac{1}{\iota(k)} \left[ \sum_{i=1}^{n} E \|x_i(k+1) - x^*\|^2 - \sum_{i=1}^{n} E \|x_i(k) - x^*\|^2 \right] \leq M_1 \frac{1}{\iota_T}.
\]

(29)

By Lemmas 8 and 9, for the second term and third term on the right hand side of (28), we have

\[
2L \sum_{k=1}^{T} \sum_{i=1}^{n} E \|x_i(s) - \bar{x}(s)\| \leq M_{21} \sum_{k=1}^{T} \iota(s) \leq M_{21} T^\epsilon,
\]

(30)

\[
3K \sum_{k=1}^{T} \sum_{i=1}^{n} E \|\lambda_i(s) - \lambda^*\| \leq M_{22} \sum_{k=1}^{T} \iota(s) \leq M_{22} T^\epsilon.
\]

(31)

Clearly, for the fourth term and fifth term on the right hand side of (28), we have

\[
nB \sum_{k=1}^{T} c(s) \leq M_{31} T^{1-\delta},
\]

(32)

and

\[
2L \sum_{k=1}^{T} \sum_{i=1}^{n} c(s) E \|\triangle_i(s)\| \leq M_{32} T^{1-\delta}.
\]

(33)

For the last term on the right hand side of (28), we have

\[
\frac{1}{2} \sum_{k=1}^{T} \sum_{i=1}^{n} \iota(s) E \|d_i(s)\|^2 \leq M_{23} T^\epsilon.
\]

(34)

Thus, the conclusion follows with \( M_{31} + M_{32} = M_3 \) and \( M_{21} + M_{22} + M_{23} = M_2 \). Therefore,

\[
\sum_{k=1}^{T} \sum_{i=1}^{n} E \left[ f_i(x_i(k), \lambda_i(k)) - f_i(x^*, \lambda^*) \right] \leq M_1 T^{1-\epsilon} + M_2 T^\epsilon + M_3 T^{1-\delta},
\]

(35)

where \( M_1, M_2 \) and \( M_3 \) are constants. Dividing both sides of (35) by \( T \) gives

\[
\frac{1}{T} \sum_{k=1}^{T} \sum_{i=1}^{n} E \left[ f_i(x_i(k), \lambda_i(k)) - f_i(x^*, \lambda^*) \right] \sim O \left( \max \left\{ \frac{1}{T^\epsilon}, \frac{1}{T^\delta}, \frac{1}{T^{1-\epsilon}} \right\} \right).
\]

(36)

The proof is completed. \( \square \)
Remark 4. The convergence rate in Theorem 2 is also corresponding to the regret bound, defined as
\[ R(T) = \frac{1}{T} \sum_{k=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ f_i(x_i(k), \lambda_i(k)) - f_i(x^*, \lambda^*) \right], \]
(as given in online optimization [40]) for the following interval optimization problem:
\[
\min_{x_i=x_j \in X} F(x, \lambda(k)) = \sum_{i=1}^{n} f_i(x_i, \lambda_i(k)), \quad T > 0. \tag{37}
\]
Also, the established convergence rate \(O\left(\frac{1}{T^v}\right)\) is the best convergence rate for distributed zeroth-order convex optimization. Note that this convergence rate is slower than that of distributed first-order methods [3], [4] for the limitation of parameter choices and the prior function knowledge.

V. SIMULATION

In this section, we demonstrate simulations of the distributed stochastic zeroth-order algorithm for the following distributed interval-valued quadratic problem:
\[
\min_{x_i=x_j \in X} G(x) = \sum_{i=1}^{5} [\nu_{1i}, \nu_{2i}] \|x - \rho_i\|^2 \quad s.t. \quad x \in X,
\]
where \(\nu_{1i}, \nu_{2i} \in \mathcal{R}\) and \(\rho_i \in \mathcal{R}^p\). This problem is motivated from centralized quadratic interval-valued learning [28] and distributed optimization [41].

Take \(X = \{x \| x \| \leq 100\}\), \(x \in \mathcal{R}\) with \([\nu_{1i}, \nu_{2i}] = [0.5, 2]\). Take \(\rho_1 = 3\), \(\rho_2 = 2\), \(\rho_3 = 1\), \(\rho_4 = 0\), \(\rho_5 = -1\). Then we consider parameters in the proposed algorithm by setting the step-size \(\iota(k) = \frac{1}{k^{7/8}}\) and \(c(k) = \frac{1}{k^{1/4}}\) used in randomized differences, along with \(\lambda_1(0) = 0.1\), \(\lambda_2(0) = 0.3\), \(\lambda_3(0) = 0.5\), \(\lambda_4(0) = 0.7\), \(\lambda_5(0) = 0.9\) and \(x_i(0)\)’s = 0.

Then we investigate the convergence performance of the distributed stochastic zeroth-order algorithm. Simulation results are based on a 5-agent time-varying network, whose communication topology between agents can be described by Fig. 2. Also, Figs. 3 and 4 show the convergence performance of the proposed algorithm. We can get a Pareto solution as \((0.500, 0.996)\) for 500 iterations.
VI. CONCLUSION

This paper investigated the distributed interval optimization problem, subject to local convex constraints. The objective functions are compact, interval-valued functions and the network for the distributed design is time-varying. Based on randomization technique, a distributed zeroth-order methodology was developed to find a Pareto optimal solution of distributed interval optimization problem. Moreover, we proved the convergence to a Pareto optimal solution with probability one over time-varying network, and finally gave a numerical example to illustrate the effectiveness of the proposed algorithm.

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(a). Suppose that there is a vector $x$ such that we can choose a subgradient $\nabla f_i(x, \lambda) \in \partial f_i(x, \lambda)$ with $\|\nabla f_i(x, \lambda)\| > L$. Suppose $y = x + \nabla f_i(x, \lambda)$. Recalling Definition 1 gives

$$f_i(y, \lambda) - f_i(x, \lambda) \geq \langle \nabla f_i(x, \lambda), y - x \rangle \geq \|\nabla f_i(x, \lambda)\|^2 > L\|\nabla f_i(x, \lambda)\|$$

which contradicts the Lipschitz continuity of $f_i(x, \lambda)$ with respect to $x$. By an analogous proof, $\|\partial f_i(x, \lambda, \lambda)\| \leq K$.

(b). For $d_i(k)$ in (7),

$$d_i(k) = \frac{[y_i^+(k) - y_i^-(k)] \Delta_i^-(k)}{2c(k)}$$

where $\|y_i^+(k) - y_i^-(k)\| = \|f_i(\xi_i(k) + c(k) \Delta_i(k), \lambda_i(k)) - f_i(\xi_i(k) - c(k) \Delta_i(k), \lambda_i(k))\| \leq 2Lc(k)\|\Delta_i(k)\|$ by Lemma 6. Due to Condition 1(a), we have

$$\mathbb{E}\left\| \frac{[y_i^+(k) - y_i^-(k)] \Delta_i^-(k)}{2c(k)} \right\| \leq nM_1M_2L,$$

and

$$\mathbb{E}\left\| \frac{[y_i^+(k) - y_i^-(k)] \Delta_i^-(k)}{2c(k)} \right\|^2 \leq (nM_1M_2L)^2.$$
Rewrite (10) compactly in terms of $\Psi(k, s)$ and the definition of $p_i(k + 1)$ as follows:

$$x_i(k + 1) = \sum_{j=1}^{n} \left[ \Psi(k, 0) \right]_{ij} x_j(0) + p_i(k + 1)$$

$$+ \sum_{s=1}^{k} \sum_{j=1}^{n} \left[ \Psi(k, s) \right]_{ij} p_j(s), \quad (42)$$

for $k \geq s$. Define $\bar{x}(k + 1) = \frac{1}{n} \sum_{i=1}^{n} x_i(k + 1)$. Moreover, with Assumption $\Pi(b)$, the following can be obtained similarly:

$$\bar{x}(k + 1) = \frac{1}{n} \sum_{i=1}^{n} x_i(0) + \frac{1}{n} \sum_{s=1}^{k + 1} \sum_{j=1}^{n} p_j(s) \quad (43)$$

Therefore, $\forall i \in \mathcal{N}$,

$$\|x_i(k + 1) - \bar{x}(k + 1)\| \leq \sum_{j=1}^{n} \left| \left[ \Psi(k, 0) \right]_{ij} - \frac{1}{n} \right| \|x_j(0)\|$$

$$+ \|p_i(k + 1)\| + \frac{1}{n} \sum_{j=1}^{n} \|p_j(k + 1)\|$$

$$+ \sum_{s=1}^{k} \sum_{j=1}^{n} \left| \left[ \Psi(k, s) \right]_{ij} - \frac{1}{n} \right| \|p_j(s)\|. \quad (44)$$

Taking the expectation of (44) yields

$$\mathbb{E}\left\|x_i(k + 1) - \bar{x}(k + 1)\right\|$$

$$\leq \sum_{j=1}^{n} \left| \left[ \Psi(k, 0) \right]_{ij} - \frac{1}{n} \right| \|x_j(0)\| + \mathbb{E}\|p_i(k + 1)\|$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\|p_j(k + 1)\| + \sum_{s=1}^{k} \sum_{j=1}^{n} \left| \left[ \Psi(k, s) \right]_{ij} - \frac{1}{n} \right| \mathbb{E}\|p_j(s)\|. \quad (45)$$

Plugging in the estimate of $\Psi(k, s)$ in Lemma 8 and the estimate of $p_i(k + 1)$ in (41), we have

$$\mathbb{E}\left\|x_i(k + 1) - \bar{x}(k + 1)\right\|$$

$$\leq n\delta^k \max_{1 \leq i \leq n} \left\| x_i(0) \right\| + \mathbb{E}\|d_i(k)\|$$

$$+ \frac{\omega(k)}{n} \sum_{i=1}^{n} \mathbb{E}\|d_i(k)\|$$

$$+ \delta \sum_{s=1}^{k} \beta^{k-s} \sum_{i=1}^{n} \mathbb{E}\|d_i(s - 1)\|. \quad (46)$$
From Lemma 9, 
\[ \mathbb{E}\|d_i(k)\| \leq L. \] Therefore,
\[ \mathbb{E}\|x_i(k + 1) - \bar{x}(k + 1)\| \leq n\delta \beta^k \max_{1 \leq i \leq n} \|x_i(0)\| + 2\delta(k)nM_1M_2L + \delta n^2M_1M_2 \sum_{s=1}^{k} \frac{\nu(s - 1)\beta^{k-s}L}{c(s - 1)} \beta^{k-s}L. \] (47)

Since \( \sum_{k=1}^{\infty} \nu(k)^2 < \infty \) with Remark 2(a) and \( \sum_{k=1}^{\infty} \frac{\nu(k)}{c(k)} < \infty \) with Remark 2(c), we obtain \( \lim_{k \to \infty} \nu(k) = 0 \) and \( \lim_{k \to \infty} \frac{\nu(k)}{c(k)} = 0 \). According to Lemma 3.1 in [4],
\[ \lim_{k \to \infty} \sum_{s=1}^{k} \nu(s - 1)\beta^{k-s} = 0, \lim_{k \to \infty} \sum_{s=1}^{k} \frac{\nu(s - 1)\beta^{k-s}}{c(s - 1)} = 0. \]
Thus, the conclusion follows.

**APPENDIX C**

**PROOF OF LEMMA 11**

(a). Define
\[ [C_i(k)]_1 = \xi_i(k) + c(k) \Delta_i(k), \]
\[ [C_i(k)]_2 = \xi_i(k) - c(k) \Delta_i(k). \]

According to Lemma 11
\[ f_i([C_i(k)]_1, \lambda_i(k)) - f_i([C_i(k)]_2, \lambda_i(k)) \in (\partial f_i \xi_i(k) + \theta_i c(k) \Delta_i(k), \lambda_i(k)), \]
\[ 2c(k) \Delta_i(k) \rangle, \] (48)
where \( \theta_i \in [-1, 1] \) is a constant. Therefore, there exists \( \varsigma_i \in \partial f_i \xi_i(k) + \theta_i c(k) \Delta_i(k), \lambda_i(k) \) such that
\[ f_i([C_i(k)]_1, \lambda_i(k)) - f_i([C_i(k)]_2, \lambda_i(k)) = \langle \varsigma_i, 2c(k) \Delta_i(k) \rangle. \]

By taking the conditional expectation of \( \langle d_i(k), \xi_i(k) - x^* \rangle \) with respect to \( F(k) \), we obtain
\[ \mathbb{E}[\langle d_i(k), \xi_i(k) - x^* \rangle | F(k)] = D_i(k), \] (49)
with \( D_i(k) = \mathbb{E}\left[ (\varsigma_i)^\top \triangle_i(k) [\triangle_i(k)]^{-\top} (\xi_i(k) - x^*)|F(k) \right] \), which can be further formulated as:

\[
D_i(k) = \mathbb{E}\left[ (\varsigma_i)^\top \left( \triangle_i(k) [\triangle_i(k)]^{-\top} - I \right) (\xi_i(k) - x^*)|F(k) \right] + \mathbb{E}\left[ \langle \varsigma_i, \xi_i(k) - x^* \rangle |F(k) \right].
\] (50)

By Definition 1 and Lemma 6, we obtain

\[
\mathbb{E}\langle \varsigma_i, \xi_i(k) - x^* \rangle |F(k) = \mathbb{E}\langle \varsigma_i, \xi_i(k) + \theta_i c(k) \triangle_i(k) - \theta_i c(k) \triangle_i(k) - x^* \rangle |F(k)
\]

\[
\geq \mathbb{E}\left[ f_i(\xi_i(k) + \theta_i c(k) \triangle_i(k), \lambda_i(k)) - f_i(x^*, \lambda_i(k)) \right] |F(k)
\]

\[
- |c(k)| \mathbb{E}\left\| \theta_i \triangle_i(k) \right\|
\]

\[
\geq \mathbb{E}\left[ f_i(\xi_i(k) + \theta_i c(k) \triangle_i(k), \lambda_i(k)) - f_i(x^*, \lambda_i(k)) \right] |F(k)
\]

\[
+ f_i(\bar{x}(k), \lambda_i(k)) - f_i(x^*, \lambda_i(k)) - |c(k)| \mathbb{E}\left\| \theta_i \triangle_i(k) \right\|
\]

\[
\geq f_i(\bar{x}(k), \bar{\lambda}(k)) - f_i(x^*, \lambda^*) + f_i(x^*, \lambda_i(k)) - f_i(x^*, \lambda_i(k)) + f_i(x^*, \lambda_i(k))
\]

\[
- L \|\xi_i(k) - \bar{x}(k)\| - 2|c(k)| \mathbb{E}\left\| \theta_i \triangle_i(k) \right\|
\]

\[
\geq f_i(\bar{x}(k), \bar{\lambda}(k)) - f_i(x^*, \lambda^*) - L \|\xi_i(k) - \bar{x}(k)\|
\]

\[
- K \|\lambda_i(k) - \bar{\lambda}(k)\| - K \|\lambda_i(k) - \lambda^*\| - 2|c(k)| \mathbb{E}\left\| \theta_i \triangle_i(k) \right\|
\] (51)

and

\[
\left| \mathbb{E}\left[ (\varsigma_i)^\top \left( \triangle_i(k) [\triangle_i(k)]^{-\top} - I \right) (\xi_i(k) - x^*) |F(k) \right] \right|
\]

\[
\leq Bc(k).
\] (52)

for a positive constant \( B \). Combining (51), (46) with (49) gives

\[
\mathbb{E}\left[ \langle d_i(k), x_i(k) - \xi^* \rangle |F(k) \right]
\]

\[
\geq f_i(\bar{x}(k), \bar{\lambda}(k)) - f_i(x^*, \lambda^*) - L \|\xi_i(k) - \bar{x}(k)\| - Bc(k)
\]

\[
- K \|\lambda_i(k) - \bar{\lambda}(k)\| - K \|\lambda_i(k) - \lambda^*\| - c(k) L \|\triangle_i(k)\|.
\] (53)
(b). Similar to the proof of part (a), we get
\[
E \left[ \langle d_i(k), x_i(k) - \xi^* \rangle \right] \geq f_i(\bar{x}(k), \lambda^*) - f_i(x^*, \lambda^*) - L \| \xi_i(k) - \bar{x}(k) \| - Bc(k) \\
-2K \| \lambda_i(k) - \lambda^* \| - c(k)L \| \Delta_i(k) \|.
\]

(54)

The proof of the second part of Lemma 11 is completed by taking the expectation to both sides of (54).

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Fig. 3: $\lambda_i(k)$ for agent $i$
Fig. 4: $x_i(k)$ for agent $i$