Gowers norms for singular measures

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Abstract

Gowers introduced the notion of uniformity norm $\|f\|_{U^k(G)}$ of a bounded function $f : G \to \mathbb{R}$ on an abelian group $G$ in order to provide a Fourier-theoretic proof of Szemeredi’s Theorem, that is, that a subset of the integers of positive upper density contains arbitrarily long arithmetic progressions. Since then, Gowers norms have found a number of other uses, both within and outside of Additive Combinatorics. The $U^k$ norm is defined in terms of an operator $\triangle^k : L^\infty(G) \to L^\infty(G^{k+1})$. In this paper, we introduce an analogue of the object $\triangle^k f$ when $f$ is a singular measure on the torus $\mathbb{T}^d$, and similarly an object $\|\mu\|_{U^k}$. We provide criteria for $\triangle^k \mu$ to exist, which turns out to be equivalent to finiteness of $\|\mu\|_{U^k}$, and show that when $\mu$ is absolutely continuous with density $f$, then the objects which we have introduced are reduced to the standard $\triangle^k f$ and $\|f\|_{U^k(T)}$. We further introduce a higher-order inner product between measures of finite $U^k$ norm and prove a Gowers-Cauchy-Schwarz inequality for this inner product.

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1 Introduction

In 2001, Gowers developed a new proof of Szemeredi’s Theorem that every dense enough subset of the integers contains arbitrarily long arithmetic progressions $a, a + b, \ldots, a + (k - 1)b$, see [Gowers(2001)]. His method revolved around the introduction of uniformity norms $\| \cdot \|_{U^k(\mathbb{Z}_N)}$, which measure the extent to which a bounded function on $\mathbb{Z}_N$ is $(k + 1)$st degree polynomially “structured”. Since then, uniformity norms have found applications in diverse topics, notably progressions in primes [Green and Tao(2008)], probabilistically checkable proofs [Samorodnitsky and Trevisan(2009)], multi-linear oscillatory integrals [Christ et al.(2005) Christ, Li, Tao, and Thiele], the bi-linear Hilbert transform along curves [Fan and Li(2009)], boundedness of paraproducts [Kovač(2012)], and others.

Gower’s original definition of the $U^k(\mathbb{Z}_N)$ norms proceeded as follows: for a bounded function $f : \mathbb{Z}_N \rightarrow \mathbb{R}$ and a $u = (u_1, \ldots, u_{k+1}) \in \mathbb{Z}_N^{k+1}$, inductively define

$$\Delta^1_{u_1} f(x) = f(x)f(x - u_1)$$
$$\Delta^k_{u} f(x) = \Delta^1_{u_{k+1}} \Delta^k_{u'} f(x)$$

where $u' = (u_1, \ldots, u_k)$.

Then the $k$-th order uniformity norm of $f$ is given by
\[ \|f\|_{U^k(\mathbb{Z}_n)} = \left( \sum_{x \in \mathbb{Z}_N, u \in \mathbb{Z}_N^k} \Delta_k^u f(x) \right)^{\frac{1}{2k}} \]

In this paper, we extend the domains of definition of \( \Delta^k \) and \( \| \cdot \|_{U^k} \) to the class of positive finite singular Radon measures on \( \mathbb{T}^d \). For a measure \( \mu \) on \( \mathbb{T}^d \) we construct a measure \( d\Delta^k \mu(x; u) \) on \( \mathbb{T}^{d(k+1)} \) and provide a definition for \( \|\mu\|_{U^k} \) which (we show but cannot at first assume) reduces to \( \left( \Delta^k \mu(\mathbb{T}^{d(k+1)}) \right)^{\frac{1}{2k}} \).

Let us say that \( \mu \in U^{k+1} \) if the finite measure \( \Delta^k \mu \) exists on \( \mathbb{T}^{d(k+1)} \) and \( \|\mu\|_{U^{k+1}} < \infty \). Then our main result can be summarized as the assertion that \( \Delta^{k+1} \mu \) exists if \( |\mu| \in U^{k+1} \) (Theorem 4.1).

Our motivation for this work stems from potential applications in Geometric Measure Theory, and particularly from the paper [Laba and Pramanik(2009)] in which Laba and Pramanik demonstrate that a measure supported in \([0,1]\) with Fourier dimension close enough to 1 contains in its support three-term arithmetic progressions \(a, a+b, a+2b\). Here, the Fourier dimension of a measure \( \mu \) is defined as

\[ \dim_F \mu := \sup\{\beta \in [0,1] : \exists C \text{ such that } |\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\beta} \} \]

In following papers, we use the machinery of uniformity norms to demonstrate that for a given \( k \in \mathbb{N} \), measures satisfying an appropriate generalization of the above Fourier dimension assumption contain \( k \)-term arithmetic progressions.

### 1.1 Outline

In Section 2 we introduce, given a measure \( \mu \) on \( \mathbb{T}^d \) and a \( k \in \mathbb{N} \), the objects \( \Delta^k \mu \) and \( \|\mu\|_{U^k} \). These definitions involve a choice of a mollifier \( \Phi_n \) and the evaluation of a limit, and it is not immediately clear whether these limits exist. We do not, in this paper, show that \( \| \cdot \|_{U^k} \) defines a norm.

Section 3 collects certain computations involving the Fourier transform of \( \Delta^k \mu \) together, and shows that regardless of whether \( \Delta^{k+1} \mu \) were to exist, \( \|\mu\|_{U^{k+1}} \) is well-defined (see Proposition 3.1).

In Section 4 we prove the main result, Theorem 4.1, which guarantees that \( \Delta^{k+1} \mu \) exists provided that the (necessary) condition \( \|\mu\|_{U^{k+1}} < \infty \) is met. This theorem also guarantees that the definition of \( \Delta^k \mu \) is independent of the choice of mollifier \( \Phi_n \).

The definition of uniformity norms extends to any compact abelian group \( G \). In Section 5, we verify that \( \Delta^k \mu \) is a true extension of Gowers’ \( \Delta^k f \) in the sense that if \( f \) is a positive function with \( \|f\|_{U^k(\mathbb{T}^d)} < \infty \), then when \( d\mu(x) := f(x) \, dx \), we have

\[ d\Delta^k \mu(x; u) = \Delta^k_u f(x) \, dx \, du \]

(this is Theorem 5.1) and that \( \|\mu\|_{U^k} = \|f\|_{U^k(\mathbb{T}^d)} \) (see Lemma 5.2).
In Section 6 we define a higher-order inner product between measures in $U_k$ and extend the results of Sections 3 and 4 to this setting.

Finally, in the Appendix we collect together some calculations not strictly related to uniformity norms but which we use in our work.

## 2 Definitions and Conventions

By the term measure, we will always mean a finite Radon measure on $\mathbb{T}^m$ for some $m \in \mathbb{N}$. Fix the parameter $d \in \mathbb{N}$; throughout this paper, we work primarily over $\mathbb{T}^d$ or cartesian products of $\mathbb{T}^d$.

Recall that an approximate identity $(\Psi_n)_{n \in \mathbb{N}}$ on $\mathbb{T}^m$ is a sequence of positive functions $\Psi_n \in L^1$ with $\|\Psi_n\|_1 = 1$ and $\int_{[-\epsilon, \epsilon]^c} \Psi_n \to 0$ for every $\epsilon > 0$.

For each non-negative integer $k$, fix an approximate identity $\Phi = \Phi^{(k)} = (\Phi^{(k)}_n)_{n \in \mathbb{N}}$ on $\mathbb{T}^d \times \mathbb{T}^k$. Several objects in this paper will initially be defined in terms of $\Phi^{(k)}$ before being shown to be independent of this choice.

For any vector or tuple $u = (u_1, \ldots, u_{k+1})$, we let $u' := (u_1, \ldots, u_k)$.

We make the following definitions.

**Definition 2.1.** Let $\mu$ be a measure on $\mathbb{T}^d$. Define

$$\triangle^0 \mu := \mu$$

If for some non-negative integer $k$, the measure $\triangle^k \mu$ exists on $\mathbb{T}^{d(k+1)}$, and if the mapping $A : C(\mathbb{T}^{d(k+2)}) \to \mathbb{R}$ given by

$$A : f \mapsto \lim_{n \to \infty} \int f(x; u_1, \ldots, u_{k+1}) \Phi_n \ast \triangle^k \mu(x - u_{k+1}; u_1, \ldots, u_k) \, du_{k+1}$$

$$= \lim_{n \to \infty} \int f(x; u) \phi_n \ast \triangle^k \mu(x - u_{k+1}; u') \, du_{k+1}$$

is positive, linear, and bounded on $C(\mathbb{T}^{d(k+2)})$, then we denote by $\triangle^{k+1} \mu$ the measure on $\mathbb{T}^{d(k+2)}$ which corresponds to $A$ by the Riesz Representation Theorem.

For instance,

$$\int_{\mathbb{T}^d \times \mathbb{T}^d} f(x; u_1) \, d\triangle^1 \mu(x; u_1) := \lim_{n \to \infty} \int_{\mathbb{T}^d \times \mathbb{T}^d} f(x; u_1) \Phi_n^{(0)} \ast \mu(x - u_1) \, d\mu(x) \, du_1$$

$$= \lim_{n \to \infty} \int_{\mathbb{T} \times \mathbb{T}} f(x; x - u_1) \Phi_n^{(0)} \ast \mu(u_1) \, du_1 \, d\mu(x)$$

$$= \int_{\mathbb{T}^d \times \mathbb{T}^d} f(x; x - u_1) \, d\mu(x) \, du_1$$

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In Lemma 3.3, we will see that this definition is independent of the choice of approximate identity \((\Phi_n)\) on \(\mathbb{T}^{d(k+1)}\).

Our main result, Theorem 4.3, gives criteria for the measure \(\Delta^{k+1}\mu\) to exist. It relies on the following definition, which in the event that \(\Delta^{k+1}\mu\) exists, is the mass of the measure \(\Delta^{k+1}\mu\).

**Definition 2.2.** Let \(\mu\) be a measure on \(\mathbb{T}\), \(k\) be a non-negative integer, and suppose that the measure \(\Delta^{k}\mu\) exists on \(\mathbb{T}^{d(k+1)}\). Then we make the definition

\[
\|\mu\|_{U^{k+1}}^{2k+1} := \lim_{n \to \infty} \int \Phi_n * \Delta^k \mu(x - u_{k+1}; u_1, \ldots, u_k) \, d\Delta^k \mu(x; u_1, \ldots, u_k) \, du_{k+1}
\]

\[
= \lim_{n \to \infty} \int \Phi_n * \Delta^k \mu(x - u_{k+1}; u') \, d\Delta^k \mu(x; u') \, du_{k+1}
\]

provided that this limit exists.

In Section 7 we show that \(\| \cdot \|_{U^{k+1}}\) defines a norm. Proposition 3.1 will tell us that the limit defining \(\|\mu\|_{U^{k+1}}\) always exists in \([0, \infty]\), and that the value obtained does not depend on which approximate identity \((\Phi_n)\) we fixed at the beginning (see the remark following the statement of Proposition 3.1).

We also place here a comment regarding interpretation of the above. We write

\[(x; u) = (x; u_1, \ldots, u_{k+1}) \in \mathbb{T}^d \times \mathbb{T}^{d(k+1)}\]

for the variables against which \(\Delta^{k+1}\mu\) integrates. It is probably most natural to think of \(x\) as the zero-th index variable, that is as though \(x = u_0\). In the construction of the measure

\[d\Delta^{k+1}\mu(x; u) = \lim_{n \to \infty} \Phi_n * \Delta^k \mu(x - u_{k+1}; u') \, d\Delta^k \mu(x; u') \, du_{k+1}\]

the components of the variable \(u\) are the parameters by which the “base point” \(x\) has been shifted - on the right side of the above, shifts of the basepoint \(x\) by \(u_1, u_2, \ldots\) and \(u_k\) have already been evaluated in forming the \(d\Delta^k \mu(x; u')\), and now we combine this with a copy of itself which has had its basepoint shifted by \(u_{k+1}\), to form \(d\Delta^{k+1}\mu(x; u)\). This can be more easily seen if we consider an analogy.

For a function \(f\) on \(\mathbb{T}^d\), define \(\Delta^0 f(x) = f(x)\), and inductively

\[\Delta^{k+1} f(x; u_1, \ldots, u_{k+1}) = \Delta^k f(x; u_1, \ldots, u_k) \Delta^k f(x - u_{k+1}; u_1, \ldots, u_k)\]

(the similarity to the measure \(d\Delta^{k+1} \mu(x; u) = \lim_{n \to \infty} \Phi_n * \Delta^k (x - u_{k+1}; u) \, d\Delta^k (x; u) \, du_{k+1}\) defined above should be clear). If we iteratively expand this out, we obtain the formula

\[\Delta^{k+1} f(x; u) = \prod_{\iota \in \{0,1\}^{k+1}} f(x - \iota \cdot u)\]
and we see that \( \Delta^{k+1} f(x; u) \) is obtained from \( f \) by evaluating \( f \) at \( x \) shifted by each of the possible combinations of the components of \( u \). For instance, we have

\[
\Delta^2 f(x; u_1, u_2) = f(x) f(x - u_1) f(x - u_2) f(x - u_1 - u_2)
\]

We can think of the tuple \((x, x - u_1, x - u_2, x - u_1 - u_2) \in T^4\) as one of the vertices of a cube with a base point at \((x, x, x, x)\), and \(\Delta^2 f(x; u_1, u_2)\) as the product of the values of \( f \) at these vertices, and analogous statements hold for higher \( k \). The measure \( d\Delta^{k+1} \mu(x; u) \) is the analogue of \( \Delta^{k+1} f(x; u) \), now measuring the “size” of the measure \( d\mu \) on the vertices of the \(2^{k+1}\)-dimensional cube defined by a vertex at \((x, \ldots, x)\) and \((x, x - u_1, \ldots, x - \sum_{i=1}^{k+1} u_i)\), or alternatively, a measure of how much weight \( \mu \) assigns to the configuration of points \( x, x - u_1, x - u_2, \ldots, x - u_1 - u_2 - \cdots - u_{k+1} \) (1)

### 3 Fourier Identities

**Proposition 3.1.** Let \( \mu \) be a measure and suppose that the measure \( \Delta^k \mu \) exists for some \( k \geq 0 \). For any approximate identity \((\Psi_n)\) on \( T^{d(k+1)} \), consider the following expressions

\[
(A) \quad \lim_{n \to \infty} \int \Psi_n * \Delta^k \mu(x - u_{k+1}; u') \, d\Delta^k \mu(x; u') \, du_{k+1}
\]

\[
(B) \quad \| \mu \|_{U^{k+1}}^{2^{k+1}}
\]

Both \((A)\) and \((B)\) exist with a value in \([0, \infty]\) that is equal to

\[
(D) \quad \sum_{c \in Z^d} |\hat{\Delta^k \mu}(0; c)|^2
\]

Remark: That \((A)\) equals \((B)\) for any choice of approximate identity \((\Psi_n)\) means that any approximate identity may be used in the definition of \( \| \mu \|_{U^{k+1}} \) (which is \((A)\) for \( \Psi \equiv \Phi \)) without affecting the value of \( \| \mu \|_{U^{k+1}} \).

**Proof.** The claim that \((A)\) equals \((D)\) follows from item 1 of Lemma 10.2 with the choice of \( \nu = \Delta^k \mu \). Since \((B)\) equals \((A)\) for the choice of \( \Psi_n \equiv \Phi_n \), this shows that \((B)\) equals \((D)\), completing the proof.

**Lemma 3.2.** Suppose that \( \mu \) is a measure on \( T^d \) and that \( k \in \mathbb{N} \) is such that \( \Delta^k \mu \) exists and that \( \| \mu \|_{U^{k+1}} < \infty \). Then for all \( \xi \in \mathbb{Z}^d \) we have the inequality

\[
\sum_{c \in Z^d} |\hat{\Delta^k \mu}(\xi; c)|^2 \leq \sum_{c \in Z^d} |\hat{\Delta^k \mu}(0; c)|^2
\]

\[
= \| \mu \|_{U^{k+1}}^{2^{k+1}}
\]
Proof. The equality in line (3) follows from the equality of items (D) and (B) of Proposition 3.1, so we need to show the inequality (2).

To do so, we will rewrite the left-hand side of the inequality on the spatial-side.

Let \((\Psi_n)\) be an approximate identity on \(T^{dk}\), with the further property that \(\hat{\Psi}_n \geq 0\). For \(a \in \mathbb{Z}^d\), write

\[
f_{n,a}(u) = \int e^{-2\pi i a \cdot x} \Psi_n \ast \triangle^k \mu(x; u) \, dx
d\]

so that

\[
\hat{f}_{n,a}(c) = \int f_{n,a}(u) e^{-2\pi i c \cdot u} \, du
\]

\[
= \int \int e^{-2\pi i a \cdot x} \Psi_n \ast \triangle^k \mu(x; u) dx e^{-2\pi i c \cdot u} \, du = \int \int e^{-2\pi i (a \cdot x + c \cdot u)} \Psi_n \ast \triangle^k \mu(x; u) dx \, du
\]

\[
= \hat{\Psi}_n(a; c) \triangle^k \mu(a; c)
\]

Parseval’s Identity tells us that

\[
\int |f_{n,a}(u)|^2 = \sum_c |f_{n,a}(c)|^2 = \sum_c |\hat{\Psi}_n(a; c) \triangle^k \mu(a; c)|^2
\]

(7)

Note also that by the triangle inequality and positivity of \(\Psi_n\),

\[
\int |f_{n,a}(u)|^2 \leq \int |f|_{n,0}(u)|^2
\]

(8)

Plugging (7) into (8) gives

\[
\sum_{c \in \mathbb{Z}^{dk}} |\hat{\Psi}_n(\xi; c) \triangle^k \mu(\xi; c)|^2 \leq \sum_{c \in \mathbb{Z}^{dk}} |\hat{\Psi}_n(0; c) \triangle^k |\mu|(0; c)|^2
\]

(9)

In the limit as \(n \to \infty\), the left-hand side of (9) converges to the left-hand side of (2), since finiteness of the left-hand side of (2) would permit the invocation of Dominated Convergence to obtain this equality, we can instead assume that the left-hand side of (2) is infinite. In this case its partial sums must be arbitrarily large, and as \(n\) grows, these partial sums provide a lower-bound on the left-hand side of (9) since we \(\hat{\Psi}_n \geq 0\), so the left-hand side of (9) is also infinite.

Next we show that the right-hand side of (9) converges to the right-hand side of (2). By assumption \(\| |\mu| \|_{U^{k+1}} < \infty\), which by Proposition 3.1 (items (D) and (B)) means that

\[
\sum_{c \in \mathbb{Z}^{dk}} |\triangle^k |\mu|(0; c)|^2 = \| |\mu| \|_{U^{k+1}}^{2k+1} < \infty
\]

(10)
Since $\widehat{\Psi}_n \leq 1$, we may then apply Dominated Convergence to evaluate the limit as $n \to \infty$ of (9) and conclude that it also equals (10).

So we may take limits on both sides of (9) to obtain (2).

**Proposition 3.3.** Suppose that $\mu$ is a measure on $\mathbb{T}^d$ such that for some $k \geq 0$, $\Delta^k \mu$ exists and $\|\mu\|_{U^{k+1}} < \infty$. Then for any $\xi \in \mathbb{Z}^d$ and $\eta \in \mathbb{Z}^{d(k+1)}$, the series

$$\sum_{c \in \mathbb{Z}^{dk}} \widehat{\Delta^k \mu}(-\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(\eta_{k+1} + \xi; c)$$

is absolutely summable and for any approximate identity $(\Psi_n)$ on $\mathbb{T}^{d(k+1)}$,

$$\lim_{n \to \infty} \int e^{-2\pi i (\xi, \eta') \cdot (x, u')} \Psi_n \ast \Delta^k \mu(x - u_{k+1}, u') \, d\Delta^k \mu(x; u') \, du_{k+1}$$

is equal to

$$\sum_{c \in \mathbb{Z}^{dk}} \widehat{\Delta^k \mu}(\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(\xi - \eta_{k+1}; c)$$

In particular, if the measure $\Delta^{k+1} \mu$ exists, then with the choice of $\Psi_n \equiv \Phi_n$ in the above,

$$\widehat{\Delta^{k+1} \mu}(\xi; \eta) = \sum_{c \in \mathbb{Z}^{dk}} \widehat{\Delta^k \mu}(\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(\xi - \eta_{k+1}; c)$$

**Proof.** We first address the claim of absolute summability. Applying Cauchy-Schwarz, we have

$$\sum_{c \in \mathbb{Z}^{dk}} |\widehat{\Delta^k \mu}(-\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(\eta_{k+1} + \xi; c)|$$

$$\leq \left( \sum_{c \in \mathbb{Z}^{dk}} |\widehat{\Delta^k \mu}(-\eta_{k+1}; \eta' - c)|^2 \right)^{\frac{1}{2}} \left( \sum_{c \in \mathbb{Z}^{dk}} |\widehat{\Delta^k \mu}(\eta_{k+1} + \xi; c)|^2 \right)^{\frac{1}{2}}$$

An application of Lemma 3.2 then bounds both sums in (15) by $\|\mu\|_{U^{k+1}}^{2^{k+1}}$ which is by assumption finite, showing the absolute summability of (11).

Now we evaluate (12).

We apply Lemma 10.1 of the Appendix, letting

$$g(x - u_{k+1}; u') := e^{-2\pi i (\eta_{k+1}, \eta') \cdot (x - u_{k+1}, u')} \Psi_n \ast \Delta^k \mu(x - u_{k+1}, u')$$

and

$$d\nu(x; u') = e^{-2\pi i (\xi + \eta_{k+1}) \cdot x} \, d\Delta^k \mu(x; u')$$

Since then

$$\hat{g}(c_0; c) = \hat{\Psi}_n(c_0 - \eta_{k+1}; c + \eta') \widehat{\Delta^k \mu}_n(c_0 - \eta_{k+1}; c + \eta')$$

$$\hat{\nu}(c_0; c) = \widehat{\Delta^k \mu}(c_0 + \xi + \eta_{k+1}; c)$$

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Lemma 10.1 gives
\[
\int e^{-2\pi i (\xi, \eta) \cdot (x, u)} \Psi_n \ast \Delta^k \mu(x - u_{k+1}, u') \, d\Delta^k \mu(x; u') \, du_{k+1}
= \sum_{c \in Z^{dk}} \hat{\Psi}_n(-\eta_{k+1}; c + \eta') \widehat{\Delta^k \mu}(-\eta_{k+1}; c + \eta') \widehat{\Delta^k \mu}(\eta_{k+1} + \xi; -c)
\]
or rewriting \(c \mapsto -c\), that
\[
\int e^{-2\pi i (\xi, \eta) \cdot (x, u)} \Psi_n \ast \Delta^k \mu(x - u_{k+1}, u') \, d\Delta^k \mu(x; u') \, du_{k+1}
= \sum_{c \in Z^{dk}} \hat{\Psi}_n(-\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(-\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(\eta_{k+1} + \xi; c)
\] (16)

Since we have already shown that \(\sum_{c \in Z^{dk}} |\widehat{\Delta^k \mu}(-\eta_{k+1}; \eta' - c)\widehat{\Delta^k \mu}(\eta_{k+1} + \xi; c)| < \infty\) and since \(|\hat{\Psi}_n| \leq 1\), we may apply Dominated Convergence and the fact that
\[
\lim_{n \to \infty} \hat{\Psi}_n(-\eta_{k+1}; \eta' - c) = 1
\]
for each \(c \in Z^{dk}\) to obtain from (16) that
\[
\lim_{n \to \infty} \int e^{-2\pi i (\xi, \eta) \cdot (x, u)} \Psi_n \ast \Delta^k \mu(x - u_{k+1}, u') \, d\Delta^k \mu(x; u') \, du_{k+1}
= \lim_{n \to \infty} \sum_{c \in Z^{dk}} \hat{\Psi}_n(-\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(-\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(\eta_{k+1} + \xi; c)
= \sum_{c \in Z^{dk}} \widehat{\Delta^k \mu}(-\eta_{k+1}; \eta' - c) \widehat{\Delta^k \mu}(\eta_{k+1} + \xi; c)
\]
which is the equality between (62) and (13).

Finally, the last assertion of the Proposition, (14), then follows from setting \(\Psi_n \equiv \Phi_n\) in the above and comparing to the definition of the measure \(\Delta^{k+1} \mu\).

\[
\] 4 Existence Results

In this section we will prove the following

**Theorem 4.1.** Given a measure \(\mu\) on \(\mathbb{T}^d\) and an integer \(k \geq 0\), suppose that the measure \(\Delta^k \mu\) exists. Then existence of \(\Delta^{k+1} \mu\) is equivalent to the finiteness of \(\|\mu\|_{TV^{k+1}}\), and both the finiteness of \(\|\mu\|_{TV^{k+1}}\) and the identity of the measure \(\Delta^{k+1} \mu\) itself are independent of the choice of approximate identity \((\Phi_n)\) in terms of which they are defined.

We first prove the following lemma.

**Lemma 4.2.** Let \(\mu\) be a measure on \(\mathbb{T}^d\) and suppose that for some \(k \geq 0\), the measure \(\Delta^k \mu\) exists and is finite. Suppose further that for some approximate identity \(\Psi_n\) on \(\mathbb{T}^{d(k+1)}\), the measure \(\nu\) on \(\mathbb{T}^d \times \mathbb{T}^{d(k+1)}\) defined by the mapping
\[
f \mapsto \lim_{n \to \infty} \int_{\mathbb{T}^d \times \mathbb{T}^{d(k+1)}} f(x; u) \Psi_n \ast \Delta^k \mu(x - u_{k+1}; u') \, d\Delta^k \mu(x; u') \, du_{k+1}
\]
exists and is finite.

Then the measure $\triangle^k|\mu|$ exists and

$$|\triangle^k|\mu| = \triangle^k|\mu|$$

Further,

$$\|\mu\|_U^{k+1} = \|\nu\|$$

where $\|\nu\|$ denotes the total variation norm of the measure $\nu$.

**Proof.** Let $j \geq 0 \in \mathbb{Z}$. For the duration of this proof only, for $(\Psi^{(j)})$ an approximate identity on $T^{d(j+1)}$ make the definition

$$\int f \triangle^j_{\Psi^{(j)}} \mu := \lim_{n \to \infty} \int_{T^d \times T^d(j+1)} f(x; u)\Psi^{(j)}_n * \triangle^k \mu(x - u_{k+1}; u') \, d\triangle^k \mu(x; u') \, du_{k+1}$$

Suppose that the finite measure $\triangle^j_{\Psi^{(j)}} \mu$ exists.

**Step 1.** For a measure $\rho$ on $T^d \times T^d(j)$, let $P$ denote the projection onto $T^d(j)$, so that

$$\int_{T^j} f(u) \, dP \rho(u) := \int_{T^d \times T^d(j)} f(u) \, d\rho(x; u)$$

Then given a measure $\mu$ we have

$$\triangle^j_{\Psi^{(j)}} \mu(T^d(j+1)) < \infty \text{ if and only if } \|P \triangle^j_{\Psi^{(j-1)}} \mu\|_2 < \infty$$

**Proof of Step 1.** By Proposition 3.1 we have that $\triangle^j_{\Psi^{(j)}} \mu(T^d(j+1)) = \sum_{\eta \in \mathbb{Z}^d} |\hat{\triangle}^j_{\Psi^{(j-1)}} \mu(0; \eta)|^2$. Since $\hat{\triangle}^j_{\Psi^{(j-1)}} \mu(0; \eta) = P \triangle^j_{\Psi^{(j-1)}} \mu(\eta)$, the claim follows.

**Step 2.** Assume inductively that $|\triangle^j_{\Psi^{(j-1)}} \mu| = \triangle^j_{\Psi^{(j-1)}} |\mu|$. Let $\sigma_n \to \sigma = \text{sgn}(\triangle^j_{\Psi^{(j-1)}} \mu)$ be a sequence of trigonometric polynomials. Then

$$P(\sigma_n \triangle^j_{\Psi^{(j-1)}} \mu) \in L^2$$

with $\|P(\sigma_n \triangle^j_{\Psi^{(j-1)}} \mu)\|_2 \leq \|\triangle^j_{\Psi^{(j-1)}} \mu\|$

**Proof of Step 2.** Suppose first that $\sigma_m(x; u') = e^{-2\pi i (\kappa_0 \cdot x + \kappa \cdot u')} = e^{\kappa_0(x)} e^{\kappa(u')}$ for some $(\kappa_0; \kappa) \in T^d \times T^d(j)$.

By Lemma 10.2 of the Appendix, we have

$$\sum_{\eta \in \mathbb{Z}^d} |\sigma_m \triangle^j_{\Psi^{(j-1)}} \mu(0; \eta)|^2$$

(19)
\[
\Phi_n e_{-(\kappa_0, \kappa)}(0; \eta) | \sigma_m \Delta^j_{\psi(j-1)} \mu(0; \eta)|^2
\]

Using that \(\sigma_m = e_{(\kappa_0, \kappa)}\), we then write

\[
(19) = \lim_{n \to \infty} \sum_{\eta \in \mathbb{Z}^d} \Phi_n(-\kappa_0; \eta - \kappa) \Delta^j_{\psi(j-1)} \mu(-\kappa_0; \eta - \kappa) \sigma_m \Delta^j_{\psi(j-1)} \mu(0; -\eta)
\]

\[
= \lim_{n \to \infty} \sum_{\eta \in \mathbb{Z}^d} \Phi_n * \Delta^j_{\psi(j-1)} \mu(-\kappa_0; \eta - \kappa) \sigma_m \Delta^j_{\psi(j-1)} \mu(0; -\eta)
\]

\[
= \lim_{n \to \infty} \int e_{(\kappa_0, \kappa)}(x - u_{j+1}; u') \Phi_n * \Delta^j_{\psi(j-1)} \mu(x - u_{j+1}; u') \sigma_m(x; u') d(\Delta^j_{\psi(j-1)} \mu)(x; u') du_{j+1}
\]

where the last line above follows from Plancheral.

Using the definition of the measure \(\Delta^{j+1}_{\psi(j)} \mu\) and that \(e_{(\kappa_0, \kappa)} = \sigma_m\), this means that we in fact have

\[
(19) = \int e_{(\kappa_0, \kappa)}(x - u_{j+1}; u') \sigma_m(x; u') d\Delta^{j+1}_{\psi(j)} \mu(x; u)
\]

\[
\leq ||\sigma_m||^2_\infty ||\Delta^{j+1}_{\psi(j)} \mu||
\]

The same estimate then holds when \(\sigma_m\) is instead any trigonometric polynomial.

\[\square\]

Step 3.

\[P(\sigma_n \Delta^j_{\psi(j-1)} \mu) \overset{L^2}{\to} P(\sigma \Delta^j_{\psi(j-1)} \mu)\]

**Proof of Step 3.** Since \(\sigma_m \Delta^j_{\psi(j-1)} \mu \to \sigma \Delta^j_{\psi(j-1)} \mu\) weak*, we have weak* convergence of \(P(\sigma_m \Delta^j_{\psi(j-1)} \mu)\) to \(P(\sigma \Delta^j_{\psi(j-1)} \mu)\). Since we have by Step 2 that \(||P(\sigma_m \Delta^j_{\psi(j-1)} \mu)||_2 \leq ||\Delta^{j+1}_{\psi(j)} \mu||\), we have (by weak compactness of the unit ball) that any subsequence of \(P(\sigma_m \Delta^j_{\psi(j-1)} \mu)\) has a subsubsequence which converges weakly in \(L^2\), and so necessarily the whole sequence converges to \(P(\sigma \Delta^j_{\psi(j-1)} \mu) \in L^2\) with

\[
||P(\sigma \Delta^j_{\psi(j-1)} \mu)||_2 \leq ||\Delta^{j+1}_{\psi(j)} \mu||
\]

\[\square\]

Step 4. Therefore \(\Delta^{j+1}_{\psi(j)} |\mu| (\mathbb{T}^{d(j+1)}) < \infty\).

**Proof of Step 4.** By Step 3, we have \(||\Delta^{j+1}_{\psi(j)} |\mu||_2 < \infty\), which means by Step 1 that \(\Delta^{j+1}_{\psi(j)} |\mu| (\mathbb{T}^{d(j+1)}) < \infty\).
We will now be done once we close the induction introduced in Step 2 by showing that 
\[ |\Delta_{\Psi^{(j)}}^{j+1}| \mu | = |\Delta_{\Psi^{(j)}}^{j}| \mu |. \] Continuing

Step 5. Write \( d\Delta_{\Psi^{(j-1)}}^{j}(x; \mathbf{u'}) = d\mu_{\mathbf{w'}}(x) \ d\mu_{\mathbf{w}}(x) \ dP(\Delta_{\Psi^{(j-1)}}^{j}(\mathbf{u}')(\mathbf{u}') \ \text{if} \ P\Delta_{\Psi^{(j-1)}}^{j}| \mu | \in L^2 \) then for any trigonometric polynomials \( f_1, f_2 \in C(\mathbb{T}^{d(j+1)}) \)
\[ \int f_1(x; \mathbf{u'}) f_2(x - u_{j+1}; \mathbf{u'}) d\Delta_{\Psi^{(j-1)}}^{j+1}(x; \mathbf{u}) = \int \int f_1(x; \mathbf{u'}) d\mu_{\mathbf{w}}(x) \int f_2(y; \mathbf{u'}) d\mu_{\mathbf{w'}}(y) \ d\mathbf{u'} \]

Proof of Step 5. Since \( P(\Delta_{\Psi^{(j-1)}}^{j}| \mu | \in L^2 \) by Step 1, we have that \( d\Delta_{\Psi^{(j-1)}}^{j}(x; \mathbf{u'}) = d\mu_{\mathbf{w'}}(x) \ d\mathbf{u'} \) for some measures \( \mu_{\mathbf{w'}} \) defined for almost every \( \mathbf{u}' \).

We can calculate that
\[ \int \int f_1(x; \mathbf{u'}) d\mu_{\mathbf{w'}} \int f_2(y; \mathbf{u'}) d\mu_{\mathbf{w'}}(y) \ d\mathbf{u'} \]
\[ = \sum_{\eta} \overline{f_1}(\mu_{\mathbf{w'}}(0; \eta)) \overline{f_2}(\mu_{\mathbf{w'}}(0; -\eta)) \]
which is the same as
\[ \sum_{\eta} (\overline{f_1}(\Delta_{\Psi^{(j-1)}}^{j})(0; \eta)) (\overline{f_2}(\Delta_{\Psi^{(j-1)}}^{j})(0; -\eta)) \]

For \( f_1, f_2 \) exponentials, this is easily seen to equal \( \int f_1(x - u_{j+1}; \mathbf{u'}) f_2(x; \mathbf{u'}) d\Delta_{\Psi^{(j-1)}}^{j+1}(x; \mathbf{u}) \), and we have this equality also in the case that the \( f_i \) are trigonometric polynomials. \( \square \)

Step 6. For any trigonometric polynomials \( f_1, f_2 \in C(\mathbb{T}^{d(j+1)}) \)
\[ \int (f_1 \sigma)(x; \mathbf{u'}) (f_2 \sigma)(x - u_{j+1}; \mathbf{u'}) d\Delta_{\Psi^{(j)}}^{j+1}(x; \mathbf{u}) \]
\[ = \int f_1(x; \mathbf{u'}) f_2(x - u_{j+1}; \mathbf{u'}) d\Delta_{\Psi^{(j)}}^{j+1}| \mu |(x; \mathbf{u}) \]

Proof of Step 6. Since \( \Delta_{\Psi^{(j)}}^{j+1}| \mu |(\mathbb{T}^{d(j+1)}) < \infty \) by Step 4, we have by Step 5 that
\[ \int f_1(x - u_{j+1}; \mathbf{u'}) f_2(x; \mathbf{u'}) d\Delta_{\Psi^{(j)}}^{j+1}| \mu |(x; \mathbf{u}) = \int \int f_1(x; \mathbf{u'}) d|\mu|_{\mathbf{w'}}(x) \int f_2(y; \mathbf{u'}) d|\mu|_{\mathbf{w'}}(y) \ d\mathbf{u'} \]
(20)

The inductive hypothesis that \( |\Delta_{\Psi^{(j-1)}}^{j}| \mu | = \Delta_{\Psi^{(j-1)}}^{j}| \mu | \) introduced in Step 2 means that \( d|\mu|_{\mathbf{w'}}(x) = \sigma(x; \mathbf{u'}) d\mu_{\mathbf{w'}}(x) \), so
\[ (20) = \int \int f_1(x; \mathbf{u'}) \sigma(x; \mathbf{u'}) d\mu_{\mathbf{w'}}(x) \int f_2(y; \mathbf{u'}) \sigma(y; \mathbf{u'}) d\mu_{\mathbf{w'}}(y) \ d\mathbf{u'} \]

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Since \(|f\sigma_m| \leq c|\) for some constant \(c\), \(f\sigma_m \to f\), and since \(\int (\int c d|\mu|_u)^2 du' = c^2 \int 1 d\triangle^{j+1}|\mu| = c^2 \triangle^{j+1}_{\Psi(\omega)}|\mu| (\mathbb{T}^{(j+1)})^{2j+1} < \infty\) by \((20)\) and Step 4, we may apply Dominated Convergence to conclude that

\[
(20) = \lim_{m \to \infty} \int \int f_1(x; u') \sigma_m(x; u') d\mu_u(x) \int f_2(y; u') \sigma_m(y; u') d\mu_{u'}(y) \, du'
\]

Since \(f\sigma_m\) is a trigonometric polynomial, Step 3 then tells us that

\[
(20) = \lim_{m \to \infty} \int f_1(x; u') \sigma_m(x; u') f_2(x - u_{j+1}; u') \sigma_m(x; u') d\triangle^{j+1}_{\Psi(\omega)} \mu(x; u)
\]

\[
= \int f_1(x; u') \sigma(x; u') f_2(x - u_{j+1}; u') \sigma(x; u') d\triangle^{j+1}_{\Psi(\omega)} \mu(x; u)
\]

where the last line follows from Dominated Convergence since the integrand is bounded and \(\triangle^{j+1}_{\Psi(\omega)} \mu\) is a finite measure.

So in fact,

\[
\int f(x; u) \sigma(x; u') \sigma(x - u_{j+1}; u') \, d\triangle^{j+1}_{\Psi(\omega)} \mu(x; u) = \int f(x; u) \, d\triangle^{j+1}_{\Psi(\omega)} |\mu| (x; u)
\]

(21)

for every trigonometric polynomial \(f\) since any such can be written in the form \(f(x; u) = f_1(x; u') f_2(x - u_{j+1}; u')\).

Step 7. Thus

\[
|\triangle^{j+1}_{\Psi(\omega)} \mu| = \triangle^{j+1}_{\Psi(\omega)} |\mu|
\]

Proof of Step 7. Since the trigonometric polynomials are dense in the space of continuous functions, we see that in fact (21) holds for all continuous functions \(f\), which (since \(\triangle^{j+1}_{\Psi(\omega)} |\mu|\) is a positive measure) means that the sign of the measure \(\triangle^{j+1}_{\Psi(\omega)} \mu\) is \(\sigma(x; u') \sigma(x - u_{j+1}; u')\); that \(|\triangle^{j+1}_{\Psi(\omega)} \mu| = \triangle^{j+1}|\mu|\) then follows from Step 6.

To summarize Steps 2 through 7, we have shown that if we assume that \(|\triangle^j_{\Psi(\omega)} |\mu|\) for some \(j\) and that \(\triangle^{j+1}_{\Psi(\omega)} \mu\) exists, then \(\triangle^j_{\Psi(\omega)} |\mu|\) exists and \(|\triangle^{j+1}_{\Psi(\omega)} |\mu|\) = \(\triangle^{j+1}_{\Psi(\omega)} |\mu|\). Thus by induction, since the base case for \(j = 0\) is trivial, we have if \(\triangle^j_{\Psi(\omega)} |\mu|\) exists for \(j = 0, \ldots, k + 1\), then so does \(\triangle^j_{\Psi(\omega)} |\mu|\) and \(\triangle^{j+1}_{\Psi(\omega)} |\mu|\) = \(|\triangle^{j+1}_{\Psi(\omega)} |\mu|\) for \(j = 0, \ldots, k + 1\). Further, the conclusions of each of Steps 1-7 hold true.

Since \(\triangle^j_{\Psi(\omega)} |\mu|\) for \(\Psi^{(j)} = \Phi^{(j-1)}\), and by assumption \(\triangle^j |\mu|\) must exist for each \(j \leq k\), and also by assumption \(\triangle^{k+1}_{\Psi(\omega)} |\mu| = \triangle^{k+1}_{\Psi(\omega)} |\mu|\) exists, we obtain that \(\triangle^j |\mu| = |\triangle^j |\mu|\) and for all \(j \leq k + 1\). This gives us (17).

To obtain (18), note that by Proposition 3.1, we have that \(||\mu||_{U^{k+1}}\) exists, and is given by the expression

\[
\lim_{n \to \infty} \int \Psi_n \star \triangle^{k+1}_{\Psi(\omega)} |\mu| (x - u_{k+1}; u') \, du_{k+1}
\]
which is $\Delta_{\Psi_k}^{k+1} |\mu| (T_d^{(k+2)})$.

By Step 7, this must be the same as $\|\Delta_{\Psi_k}^{k+1} \mu\| = \|\nu\|$, which is (18). In particular, the theorem is proved.

**Corollary 4.3.** Let $\mu$ be a measure on $T^d$ and suppose that for some $k \geq 0$, the measure $\Delta^k \mu$ exists and is finite. Then exactly one of the following holds

(A) The measure $\Delta^{k+1} \mu$ exists with a finite variation norm $\|\Delta^{k+1} \mu\|$ equal to $\|\mu\|^{2^{k+1}}_{U^{k+1}}$

(B) $\|\mu\|_{U^{k+1}} = \infty$

Further, if (A) holds, then in fact for any approximate identity $(\Psi_n)$ on $T^{d(k+1)}$, the mapping

$$f \mapsto \lim_{n \to \infty} \int f(x, u) \Psi_n * \Delta^k \mu(x - u_{k+1}; u') d\Delta^k \mu(x - u_{k+1}; u') du_{k+1} \quad (22)$$

corresponds to a finite measure via the Riesz Representation Theorem.

**Proof.** If for some approximate identity $(\Psi_n)$ (22) defines a (finite) measure $\nu$, then by Lemma 4.2, we have that $\|\mu\|_{U^{k+1}} \leq \|\nu\|$. Since the reverse inequality follows from applying the triangle inequality to (22) and Proposition 3.1, we obtain that $\|\Delta^{k+1} \mu\| = \|\mu\|^{2^{k+1}}_{U^{k+1}}$. In particular, if $\Delta^{k+1} \mu$ exists as a finite measure, then $\|\mu\|_{U^{k+1}}$ is finite and equal to $\|\Delta^{k+1} \mu\|$.

So we are tasked with showing that if $\|\mu\|_{U^{k+1}} < \infty$, then for any $\Psi_n$, the measure given by (22) exists (note that $\Delta^{k+1} \mu$ is such a measure for the choice $\Psi_n \equiv \Phi_n$).

By the Riesz Representation Theorem, it is enough to check that

$$f \mapsto \lim_{n \to \infty} \int f(x, u) \Psi_n * \Delta^k \mu(x - u_{k+1}; u') d\Delta^k \mu(x; u') du_{k+1} \quad (23)$$

defines a bounded linear functional on $C(T^{(d(k+2)})$. Where defined, this is clearly linear. And if the limit exists for some $f$, then we have the bound

$$(23) \leq \|f\|_\infty \lim_{n \to \infty} \int \Psi_n * \Delta^k |\mu| (x - u_{k+1}; u') d\Delta^k |\mu| (x; u') du_{k+1} = \|f\|_\infty \|\mu\|^{2^{k+1}}_{U^{k+1}} \quad (24)$$

by the equivalence of (A) and (B) of Proposition 3.1.

So it suffices to show that the limit exists, and indeed that it exists for a dense subset of $C(T^{d(k+2)})$, such as the class of exponential functions. But given that $\|\mu\|_{U^{k+1}} < \infty$, that the limit in (23) exists for $f$ an exponential is exactly what (62) in Proposition 3.3 guarantees.

**Corollary 4.4.** Let $\mu$ be a measure on $T^d$ and suppose that for some $k \geq 0$, the measure $\Delta^k \mu$ exists and is finite. Suppose further that for some approximate identity $(\Psi_n)$ on $T^{d(k+1)}$, the mapping

$$\nu : f \mapsto \lim_{n \to \infty} \int f(x, u) \Psi_n * \Delta^k \mu(x - u_{k+1}; u') d\Delta^k \mu(x - u_{k+1}; u') du_{k+1} \quad (25)$$

defines a (finite) measure $\nu$ on $T^{d(k+1)}$. Then the measure $\Delta^{k+1} \mu$ exists and $\nu = \Delta^{k+1} \mu$. 

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Proof. We first show that if the (finite) measure $\nu$ exists, then $\triangle^{k+1} \mu$ must exist as well.

If (25) defines a finite measure, then in particular

$$\lim_{n \to \infty} \int 1 \Psi_n \ast \triangle^k \mu(x - u_{k+1}; u') d\triangle^k \mu(x - u_{k+1}; u') du_{k+1}$$

must be finite. By the equivalence of (A) and (D) in Proposition 3.1, this means that $\|\mu\|_{U^{k+1}}$ is finite. According to Corollary 4.3, we must then have that the finite measure $\triangle^{k+1} \mu$ exists.

The hypotheses of Proposition 3.3 are satisfied, and so according to (62) and (13) of that proposition,

$$\lim_{n \to \infty} \int e^{-2\pi(\xi, \eta) \cdot (x, u)} \Psi_n \ast \triangle^k \mu(x - u_{k+1}; u') d\triangle^k \mu(x; u') du_{k+1}$$

$$= \sum_{c \in \mathbb{Z}^{dk}} \hat{\triangle^k \mu}(\eta_{k+1}; \eta' - c) \hat{\triangle^k \mu}(\xi - \eta_{k+1}; c)$$

(26)

which means that the Fourier transform of $\nu$ is (26).

Since Proposition 3.3 also guarantees that (26) is the Fourier transform of $\triangle^{k+1} \mu$, we conclude that $\nu = \triangle^{k+1} \mu$.

Proof of Theorem 4.1. Suppose that the measure $\triangle^{k+1} \mu$ exists. Lemma 4.2 tells us that $\|\mu\|_{U^{k+1}} = \|\triangle^{k+1} \mu\| < \infty$. Conversely, if $\|\mu\|_{U^{k+1}} \neq \infty$, then (A) of Corollary 4.3 must hold, so that $\triangle^{k+1} \mu$ exists.

So existence of $\triangle^{k+1} \mu$ is equivalent to finiteness of $\|\mu\|_{U^{k+1}}$.

Of course, as the remark following the statement of Proposition 3.1 reminds us, the value of $\|\mu\|_{U^{k+1}}$ is independent of $(\Phi_n)$. And according to Corollary 4.4, if we replaced $(\Phi_n)$ by any other approximate identity in the construction of $\triangle^{k+1} \mu$, we would obtain the same measure $\triangle^{k+1} \mu$.

5 $\triangle^k \mu$ and $\triangle^k f$

Recall that given any compact abelian group $G$ with Haar measure $dx$, the $k$-th order uniformity norm $U^k$ of a function $f$ on $G$ is given by

$$\|f\|_{U^k(G)}^{2k} := \int_{G \times G^k} \Delta^k f(x; u) dx du$$

(27)

where $\Delta^0 f(x) := f(x)$ and inductively, $\Delta^{k+1} f(x; u) := \Delta^k f(x; u) \Delta^k f(x - u_{k+1}; u')$.

In this section, we show that if $f$ is a function on $\mathbb{T}^d$ with finite $U^{k+1}(\mathbb{T}^d)$ norm, then the uniformity norm $\|f\|_{U^{k+1}(\mathbb{T}^d)}$ coincides with $\|f dx\|_{U^{k+1}}$ as defined in previous sections, and that the measure $\triangle^{k+1}(f dx)$ has a density given by $\triangle^{k+1} f(x; u)$. In more detail, our main result is the following.
**Theorem 5.1.** Suppose that the positive function $f : \mathbb{T}^d \to \mathbb{R}$ has a finite $U^{k+1}$ norm for some $k$. Then the measure $d\mu = f dx$ satisfies

$$\|\mu\|_{U^{k+1}} = \|f\|_{U^{k+1}(\mathbb{T}^d)}$$

and the finite measure $\Delta^{k+1} \mu$ exists, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^{d(k+2)}$, and has a density given by $\Delta^{k+1} f$.

**Proof.** We induct on $k$. That $\Delta^0(f dx) = (\Delta^0 f) dx$ is a tautology. Fix $k \geq 0$ and let us state the inductive assumption

(I) $d\Delta^k(f dx)(x; \mathbf{u}') = \Delta^k f(x; \mathbf{u}') dx d\mathbf{u}'$

Since we are assuming that for this $k$, (I) is true, the conclusion of Lemma 5.2 below is valid. Part (a) of Lemma 5.2 tells us that $\|\Delta^0 f\|_{U^{k+1}} = \|f\|_{U^{k+1}(\mathbb{T}^d)}$, which is by assumption finite. Corollary 4.3 tells us that since $\|\Delta^0 f\|_{U^{k+1}}$ is finite, the measure $\Delta^{k+1}(f dx)$ exists. By Proposition 3.3, the Fourier transform of $\Delta^{k+1}(f dx)$ is

$$\Delta^{k+1}(f dx)(\xi; \mathbf{c}) = \sum_{c \in \mathbb{Z}^d} \Delta^k(f dx)(-\eta_k+1; \mathbf{c}) \Delta^k(f dx)(\eta_k+1; \xi; \mathbf{c}) \tag{28}$$

and by Lemma 5.2, the Fourier transform of the function $\Delta^{k+1} f$ is

$$\Delta^{k+1}(f dx)(\xi; \mathbf{c}) = \sum_{c \in \mathbb{Z}^d} \Delta^k f(-\eta_k+1; \mathbf{c}) \Delta^k f(\eta_k+1; \xi; \mathbf{c}) \tag{29}$$

Since by our inductive hypothesis, $d(\Delta^k(f dx))(x; \mathbf{u}) = \Delta^k f(x; \mathbf{u}) dx d\mathbf{u}$ and hence $\Delta^k f = \Delta^k(f dx)$, the right-hand sides of (28) and (29) coincide, which shows that the measure $d\Delta^{k+1}(f dx)(x; \mathbf{u})$ must equal the measure $\Delta^{k+1} f(x; \mathbf{u}) dx d\mathbf{u}$, completing the induction.  

\[\square\]

**Lemma 5.2.** Let $k \geq 0$, $f : \mathbb{T}^d \to \mathbb{R}$ be a positive function with finite $U^{k+1}(\mathbb{T}^d)$ norm, and assume that the assumption (I) holds. Then for any $\xi \in \mathbb{Z}^d$ and $\mathbf{c} \in \mathbb{Z}^{d(k+1)}$, we have

(a) $\|f dx\|_{U^{k+1}} = \|f\|_{U^{k+1}(\mathbb{T}^d)}$

(b) $\Delta^{k+1} f(\xi; \mathbf{c}) = \sum_{c \in \mathbb{Z}^d} \Delta^k f(-\eta_k+1; \mathbf{c}) \Delta^k f(\eta_k+1; \xi; \mathbf{c}) \tag{29}$

**Proof.** Let $(\Psi_n)$ be an approximate identity with $\Psi_n \in L^1$. We claim that both (a) and then (b) would follow if we knew that

$$\Delta^{k+1} f(\xi; \mathbf{c}) = \lim_{n \to \infty} \sum_{c \in \mathbb{Z}^d} \Psi_n(0; -\mathbf{c}) \Delta^k f(-\eta_k+1; \mathbf{c}) \Delta^k f(\eta_k+1; \xi; \mathbf{c}) \tag{30}$$

To see this, consider first (a). We have

$$\|f\|_{U^{k+1}(\mathbb{T}^d)}^2 \equiv \int \Delta^{k+1} f(x; \mathbf{u}) dx d\mathbf{u}$$
Applying (30) with $\xi, \eta = 0$ to the last line above, we would have

$$\|f\|^{2k+1}_{U^k+1(\mathbb{T}^d)} = \lim_{n \to \infty} \sum_{c \in \mathbb{Z}^{dk}} \hat{\Psi}_n(0; c)|\hat{\Delta}^k f(0; c)|^2$$

(31)

We then use assumption [(I)] to write $\hat{\Delta}^k f = \hat{\Delta}^k (f dx)$ in (31) and conclude that

$$\|f\|^{2k+1}_{U^k+1(\mathbb{T}^d)} = \lim_{n \to \infty} \sum_{c \in \mathbb{Z}^{dk}} \hat{\Psi}_n(0; c)|\hat{\Delta}^k (f dx)(0; c)|^2$$

(32)

$$= \lim_{n \to \infty} \int \Psi_n * \Delta^k (f dx)(x - u_{k+1}; \mathbf{u}') d\Delta^k (f dx)(x; \mathbf{u}') du_k$$

(33)

Setting $d\mu = f dx$, we have by the equivalence of [(A)] and $\|\mu\|^{2k+1}_{U^k+1}$ in Proposition 3.1 that (33) is $\|f\|^{2k+1}_{U^k+1}$, which shows that

and we have proved [(a)].

Now, [(b)] clearly follows from (30) if we could evaluate the limit in (30).

In order to compute the limit, we need to know that the sum

$$\sum_{c \in \mathbb{Z}^{dk}} |\hat{\Delta}^k f(-\eta_{k+1}; \mathbf{u}') - \hat{\Delta}^k f(\eta_{k+1} + \xi; c)|$$

(34)

is finite.

Since $\|f dx\|^{2k+1}_{U^k+1} = \|f\|^{2k+1}_{U^k+1(\mathbb{T}^d)}$ is finite by hypothesis and we have that $\hat{\Delta}^k f = \hat{\Delta}^k (f dx)$, we can apply Proposition 3.3 for the measure $d\mu = f dx$ to conclude that (34) is finite.

Now that we know that (34) is finite, we can use Dominated Convergence and the fact that $|\hat{\Psi}_n| \leq 1$, $\hat{\Psi}_n \to 1$, to evaluate the limit in (30) and conclude that

$$\sum_{c \in \mathbb{Z}^{dk}} \hat{\Delta}^k f(-\eta_{k+1}; \mathbf{u}') \hat{\Delta}^k f(\eta_{k+1} + \xi; c)$$

(35)

and this is [(b)].

So we are left with showing that (30) holds.

For $(\xi, \eta) \in \mathbb{Z}^d \times \mathbb{Z}^{d(k+1)}$, write

$$g(x - u_{k+1}; \mathbf{u}') := \Delta^k f(x - u_{k+1}; \mathbf{u}') e^{-2\pi i (\eta' \cdot \mathbf{u}' + (-\eta_{k+1})(x - u_{k+1}))}$$

$$d\nu(x; \mathbf{u}') = \Delta^k f(x; \mathbf{u}') e^{-2\pi i (\eta_{k+1} + \xi) x}$$

Note that

$$\int g(x - u_{k+1}; \mathbf{u}') d\nu(x; \mathbf{u}') = \int \Delta^{k+1} f(x; \mathbf{u}) e^{-2\pi i (\eta \cdot \mathbf{u} + \xi u_{k+1})} dx d\mathbf{u}$$
We would like to use Lemma 10.1 of the Appendix, but first we must mollify \( g \) to ensure an \( L^1 \) Fourier transform so that the hypotheses of Lemma 10.1 are satisfied. By Lemma 10.4 of the Appendix

\[
\int g(x - u_{k+1}; u') \, d\nu(x; u')
= \lim_{n \to \infty} \int \Psi_n * g(x - u_{k+1}; u') \, d\nu(x; u') \, du_{k+1}
\]  

(36)

so

\[
\hat{\triangle}^{k+1} f(\xi; \eta) = \int \triangle^{k+1} f(x; u) e^{-2\pi i \eta \cdot u_{k+1}} \, dx \, du
= \int g(x - u_{k+1}; u') \, d\nu(x; u') \, du_{k+1}
= \lim_{n \to \infty} \int \Psi_n * g(x - u_{k+1}; u') \, d\nu(x; u') \, du_{k+1}
\]  

(38)

Since we have for any \( c_0 \in \mathbb{Z}^d, c \in \mathbb{Z}^{dk} \)

\[
\hat{\Psi}_n * g(c_0; c) = \hat{\Psi}_n(c_0; c) \hat{\triangle}^{k} f(c_0 - \eta_{k+1}; c + \eta') \quad \text{and} \quad \hat{\nu}(c_0; c) = \hat{\triangle}^{k} f(\eta_{k+1} + \xi + c_0; c)
\]

Lemma 10.1 now tells us that

\[
(38) = \lim_{n \to \infty} \sum_{c \in \mathbb{Z}^{dk}} \hat{\Psi}_n(0; c) \hat{\triangle}^{k} f(-\eta_{k+1}; c + \eta') \hat{\triangle}^{k} f(\eta_{k+1} + \xi; -c)
\]  

(39)

and this is the same as (30).

\[ \square \]

6 The Gowers-Cauchy-Schwarz Inequality

In this section we introduce a higher-order inner product which gives rise to \( \| \cdot \|_{U^k} \) and prove a higher-order Cauchy-Schwarz inequality. This will allow us to prove in Section 7 that \( \| \cdot \|_{U^k} \) obeys the triangle inequality and so is indeed a norm.

For now on, let us define the space \( U^k = U^k(\mathbb{T}^d) \) by

\[
U^k := \{ \mu \text{ a measure on } \mathbb{T}^d : \triangle^k \mu \text{ exists} \}
\]

We will make use of the following notation in the sequel.

For \( u \in \mathbb{T}^{d(k+1)} \) and \( l \leq k + 1 \), define

\[
u_{\leq l} = (u_1, \ldots, u_l)
\]
and for \( \iota \in \{0,1\}^{k+1} \), we define

\[
\iota \leq l = (\iota_1, \ldots, \iota_l) \\
\iota > l = (\iota_{l+1}, \ldots, \iota_{k+1}) \\
\iota > k+1 = \emptyset
\]

And we concatenate \( \kappa \in \{0,1\}^i \) and \( \sigma \in \{0,1\}^j \) as

\[
\kappa \sigma = (\sigma_1, \ldots, \sigma_j, \kappa_1, \ldots, \kappa_i) \in \{0,1\}^{i+j}
\]

Further, if \( \mu_i \) in \( U^k \) for each \( \iota \in \{0,1\}^{k+1} \), \( j \in [0, \ldots, k] \) and \( \kappa \in \{0,1\}^j \), we set

\[
\mu := \mu_\emptyset := \{\mu_i\}_{i \in \{0,1\}^{k+1}} \\
\mu_\kappa := \{\mu_{\kappa\sigma}\}_{\sigma \in \{0,1\}^{k+1-j}}
\]

**Definition 6.1.** For \( k \in \mathbb{N} \), let \( \mu_i, \iota \in \{0,1\}^{k+1} \) be \( 2^{k+1} \) measures in \( U^{k+1}(\mathbb{T}^d) \).

If the measures \( \Delta^k(\mu_i) \equiv \Delta^k \mu_i \) exist on \( \mathbb{T}^{d(k+1)} \) for \( i = 0,1 \), we define their inner product as

\[
< \mu > := \lim_{n \to \infty} \int \Phi_n * \Delta^k(\mu_1)(x - u_{k+1}; u') \, d\Delta^k(\mu_0)(x; u') \, du_{k+1}
\]

provided this limit exists, and we define the measure

\[
\Delta^{k+1}(\mu) \equiv \Delta^{k+1}(\{\mu_i\}_{i \in \{0,1\}^{k+1}})
\]

on \( \mathbb{T}^{d(k+2)} \) as the weak* limit

\[
w^* \lim_{n \to \infty} \Phi_n * \Delta^k(\mu_1)(x - u_{k+1}; u') \, d\Delta^k(\mu_0)(x; u') \, du_{k+1}
\]

provided that this limit exists.

Of course, \( \Delta^0(\mu) \) is already defined. When each \( \mu_i \) in the above is equal to the same measure \( \mu \), \( \Delta^{k+1}(\mu) \) reduces to \( \Delta^{k+1}\mu \). Uncoincidentally, the proof that \( \Delta^{k+1}(\mu) \) always exists for any \( \mu \in (U^{k+1})^{2^{k+1}} \) follows the same steps as the proof that \( \Delta^{k+1}\mu \) exists. We postpone an outline of this proof until Section 8, as it requires the replacement of the assumption that \( \|\mu\|_{U^{k+1}} < \infty \) used throughout Section 4, with the assumption that \( |< \mu >| < \infty \) when each \( \mu_i \in U^{k+1} \) - a consequence of the following proposition, which is our main result in the present section.
Proposition 6.2. Given $2^{k+1}$ measures $\mu_i \in U^{k+1}$, $i \in \{0,1\}^{k+1}$, suppose that $\Delta^k(\mu^{(i)})$ exists for $i = 0, 1$. Then

$$\limsup_{n \to \infty} \left| \int 1 \Phi_n * \Delta^k(\mu_1)(x - u_{k+1}; u') d\Delta^k(\mu_0)(x; u') d\mu_{k+1} \right| \leq \prod_{i \in \{0,1\}^{k+1}} \|\mu_i\|_{U^{k+1}}$$

An immediate Corollary is

Proposition 6.3. Suppose for $k \in \mathbb{N}$ that the $2^k$ measures $\mu_i \in U^{k+1}$ and that

$$\Delta^{k+1}(\mu)(I^{k+2})$$

exists. Then

$$| < \mu > | \leq \prod_{i \in \{0,1\}^{k+1}} \|\mu_i\|_{U^{k+1}}$$

Before proving Proposition 6.2, we must introduce more notation.

Let $\kappa = \iota_{>1}$, so that

$$\Delta^{j+1} \mu_i = w^* \lim_{n \to \infty} \Phi_{n_1} * \Delta(\mu_{k+1})(x - u_{j+1}; u_{<j}) d\Delta(\mu_0)(x; u_{<j}) d\mu_{j+1}$$

$$= w^* \lim_{n \to \infty} w^* \lim_{n_0 \to \infty} \Phi_{n_1} * \Delta(\mu_{k+1})(x - u_{j+1}; u_{<j}) \Phi_{n_2} * \Delta(\mu_{k+1})(x; u_{<j}) d\mu_{j+1} d\mu_{<j}$$

Then it can be seen that starting from a collection of $\mu_i$, $i \in \{0,1\}^k$, there are a total of $\prod_{j=1}^k 2 = 2^k$ explicit or implicit limits in the construction of $\Delta^k(\mu)$. This count can be verified if we index those $n$ which appear in each of these limits by writing

$$\Delta^{j+1} \mu_i = w^* \lim_{n \to \infty} \Phi_{n_1} * \Delta(\mu_0)(x - u_{j+1}; u_{<j}) \Phi_{n_2} * \Delta(\mu_{1+i})(x; u_{<j}) d\mu_{j+1} d\mu_{<j}$$

and note that there are $| \{n_{i,j} \}_{0 \leq j \leq k, i \in \{0,1\}^k} | = 2^k$ distinct indices.

We set

$$\Phi_{i,j} := \Phi_{n_{i,j+1}}$$

$$\vec{n}' = \{n_{i,j} \}_{0 \leq j \leq k, i \in \{0,1\}^k}$$

$$\vec{n} = \{n_{i,j} \}_{0 \leq j \leq k+1, i \in \{0,1\}^{k+1}}$$

and stipulate that $\lim_{\vec{n}' \to \infty}$ refers to each of the $2^k$ limits $\lim_{n_{i,j} \to \infty}$, $j \leq k$, taken in lexicographic order (and similarly for $\lim_{\vec{n} \to \infty}$). The reader should convince herself that if we inductively define

$$A_{0}^{(1)}[\mu_{i>1}](x; u_1) = \prod_{i \in \{0,1\}} \Phi_{i} * \mu_{i \to 1}(x - \iota_1 u_1)$$

$$A_{0}^{(j)}[\mu_{i>1}](x; u_{<j}) = \prod_{i \in \{0,1\}} \Phi_{i \to j-1} * A_{0}^{(j-1)}[\mu_{i \to j-1}](x - \iota_j u_j; u_{<j-1})$$

then we have
Lemma 6.4. Suppose that $\mu_\iota, \iota \in \{0, 1\}^{k+1}$ are measures in $U^k$ and that $\Delta^{k-1}(\mu_{\iota_{>k-1}})$ exists for $\iota_k = 0, 1$. Then for $\iota_{k+1} = 0, 1$

$$w^* \lim_{n' \to \infty} A^{(k)}_0[\mu_{\iota_{k+1}}](x; u) = w^* \lim_{n_1 \to \infty} w^* \lim_{n_0 \to \infty} \prod_{\iota_k = 0}^1 \Phi_{\iota_{>k-1}} \ast \Delta^{k-1}(\mu_{\iota_{>k-1}})(x - \iota_k u_k; u')$$

One should think of the significance of this lemma as a means of expressing $\Delta^k(\mu_{\iota_{k+1}})$ directly in terms of the $\mu_\iota$.

One may check that if for each $j \leq k + 1$ and $\iota \leq j \in \{0, 1\}^j$, we define the variable $T^\iota_{\leq j}$ by

$$T^\iota_1 \equiv 0$$
$$T^\iota_{\leq j} = T^\iota_{\leq j-1} + \iota_0^{(\iota_{>j-1})} + \iota_{\leq j} \cdot t^{(\iota_{>j-1})}$$

set

$$t^j = \{t^{(\geq n)}\}_{1 \leq n \leq j, \iota \in \{0, 1\}^{k+1}}$$

and define

$$\Phi^{(0)} \equiv 1$$
$$\Phi^{(j)}(t^j) = \prod_{\iota \leq j \in \{0, 1\}^j} \Phi_{\iota_{>j}}(t^{(\iota_{>j})}) \Phi^{(j-1)}(t^{j-1})$$

then we have

Lemma 6.5. Suppose that $\mu_\iota, \iota \in \{0, 1\}^{k+1}$ are measures on $\mathbb{T}^d$. Then

$$A^{(k)}_0[\mu_{\iota_{k+1}}](x; u) = A^{(k)}_0(x; u^{(\iota_{k+1})}) = \int \Phi^{(k)}(t^k) \prod_{\iota \in \{0, 1\}^k} \Phi_{\iota} \ast \mu_\iota(x - \iota \cdot u - T^\iota) \, dt$$

Further, if we select an approximate identity $(\phi_m)_{m \in \mathbb{N}}$ on $\mathbb{T}^d$, then

$$A^{(k)}_0[\mu_{\iota_{k+1}}](x; u) = w^* \lim_{n \to \infty} \int \Phi^{(k)}(t^k) \prod_{\iota \in \{0, 1\}^k} \Phi_{\iota} \ast \phi_m \ast \mu_\iota(x - \iota \cdot u - T^\iota) \, dt$$

Proof of Proposition 6.2. By Lemma 6.4, for each $\iota_{k+1} \in \{0, 1\}$

$$w^* \lim_{n \to \infty} A^{(k)}_0[\mu_{\iota_{k+1}}](x; u) = w^* \lim_{n_1 \to \infty} w^* \lim_{n_0 \to \infty} \prod_{\iota_k = 0}^1 \Phi_{\iota_{>k-1}} \ast \Delta^{k-1}(\mu_{\iota_{>k-1}})(x - \iota_k u_k; u')$$
\[= \Delta^k(\mu_{k+1})\]

Choosing an approximate identity \( (\phi_n) \) on \( T^d \), then since \( (\phi_n^{k+1}) \) is still an approximate identity, we have by the second part of Lemma 6.5 that

\[
\Delta^k(\mu_{k+1}) = w^* \lim_{n' \to \infty} \lim_{m \to \infty} \int \Phi^{(k)}(t^k) \prod_{i \leq k \in \{0, 1\}} \Phi_{i} * \phi_{m_i}^{k+1} * \mu_i(x - i \cdot u - T^i) \, dt
\]

Since these are all weak* limits and \( \Phi_n * \Delta^k(\mu_1) \) is continuous, we have

\[
\limsup_{n \to \infty} \left| \int 1 \Phi_n * \Delta^k(\mu_1)(x - u_{k+1}; u') \, d\Delta^k(\mu_0)(x; u') \, du_{k+1} \right|
\]

\[= \limsup_{n_1 \to \infty} \lim_{n_0 \to \infty} \lim_{n' \to \infty} \lim_{m \to \infty} \int \Phi^{(k+1)}(t^{k+1}) \prod_{i \in \{0, 1\}^{k+1}} \Phi_{i} * \phi_{m_i}^{k+1} * \mu_i(x - i \cdot u - T^i) \, dt
\]

Applying Fubini’s Theorem, we have that this is the same as

\[
\limsup_{n \to \infty} \lim_{m \to \infty} \int \Phi^{(k+1)}(t^{k+1}) \Phi_{i}(t^{(i)}) \prod_{i \in \{0, 1\}^{k+1}} \phi_{m_i}^{k+1} * \mu_i(x - i \cdot u - T^i - t^{(i)}) \, dx \, du \, dt
\]

Set \( dv_i(x) = d\mu_i(x - T^i - t^{(i)}) \).

Now we apply Corollary \( 9.11 \) from Section 9, obtaining that this is bounded by

\[
\limsup_{n \to \infty} \lim_{m \to \infty} \int \Phi^{(k+1)}(t^{k+1}) \Phi_{i}(t^{(i)}) \prod_{i \in \{0, 1\}^{k+1}} \left[ \phi_{m_i}^{[k+1]} * \Delta^{k+1} \nu_i(T^{d(k+2)}) \right]^{1/2^{k+1}} \, dt \quad (47)
\]

Of course,

\[
\prod_{i \in \{0, 1\}^{k+1}} \left[ \phi_{m_i}^{[k+1]} * \Delta^{k+1} \nu_i(T^{d(k+2)}) \right]^{1/2^{k+1}} = \prod_{i \in \{0, 1\}^{k+1}} \left[ \phi_{m_i}^{[k+1]} * \Delta^{k+1} \mu_i(T^{d(k+2)}) \right]^{1/2^{k+1}}
\]

since \( \nu_i \) is a shift of \( \mu_i \). And integrating over all the \( t \)'s in (47) leaves us with

\[
\lim_{m \to \infty} \prod_{i \in \{0, 1\}^{k+1}} \left[ \phi_{m_i}^{[k+1]} * \Delta^{k+1} \mu_i(T^{d(k+2)}) \right]^{1/2^{k+1}} = \prod_{i \in \{0, 1\}^{k+1}} \| \mu \|_{U^{k+1}}
\]

since the \( F_{i < j} \)'s all integrate out to 1, completing the proof. \( \Box \)
7 The $U^k$ Norm

We are now in a position to show $\| \cdot \|_{U^k}$ is indeed a norm.

We follow the usual approach in such matters by using a Gowers-Cauchy-Schwarz type inequality in order to obtain the triangle inequality (the other requirements for a norm being obvious).

Monotonicity of the $U^k$ norms, or that $\| \mu \|_{U^k} \leq \| \mu \|_{U^{k+1}}$, follows from the Gowers-Cauchy-Schwarz inequality by setting $\mu_i = 1$ for each $i$, and this shows the non-negativity of the $U^k$ norms for $k > 1$.

Of course, we did not need monotonicity to show that $\| \mu \|_{U^k} > 0$ for nonzero $\mu$ and $k > 1$, since we have the identity

$$\| \mu \|_{U^k}^2 = \sum_{c \in \mathbb{Z}} |\hat{\mu}(0; c)|^2 \geq |\hat{\mu}(0; 0)|^2 = \| \mu \|_{U^{k-1}}^2$$

from Proposition 3.1. But we will use Gowers-Cauchy-Schwarz type inequality to show that the $U^k$ norm satisfies the triangle inequality.

**Proposition 7.1.** $\| \cdot \|_{U^k}$ defines a norm on the space of measures $\mu$ on $T^d$ for which $\| \mu \|_{U^k} < \infty$.

**Proof.** Homogeneity of $\| \cdot \|_{U^k}$ is immediate, and that $\| \mu \|_{U^k} = 0$ only if $\mu = 0$ follows from Proposition 3.1. So we are left only to check the triangle inequality.

To do this, let $\mu_1$ and $\mu_2$ be two measures in $U^k$. Then by Lemma 6.4,

$$\| \mu_1 + \mu_2 \|_{U^k}^2 = \lim_{n' \to \infty} \int A_0^{(k)}[\mu_1 + \mu_2](x; u) \, dx \, du$$

and by Lemma 6.5, this is the limit as $\lim_{n' \to \infty}$ of

$$\int \int \Phi^{(k)}(t^k) \prod_{i \in \{0,1\}^k} \Phi_i \ast (\mu_1 + \mu_2)(x - i \cdot u - T^v) \, dt \, dx \, du$$

Since convolution is additive, we may write

$$\int \int \Phi^{(k)}(t^k) \prod_{i \in \{0,1\}^k} \Phi_i \ast (\mu_1 + \mu_2)(x - i \cdot u - T^v) \, dt \, dx \, du$$

$$= \int \int \Phi^{(k)}(t^k) \prod_{i \in \{0,1\}^k} [\Phi_i \ast (\mu_1)(x - i \cdot u - T^v) + \Phi_i \ast (\mu_2)(x - i \cdot u - T^v)]$$

and expanding out the product over $i$, this is the same as

$$\int \int \Phi^{(k)}(t^k) \sum_{\mu \in \{\mu_1, \mu_2\}^k} \prod_{i \in \{0,1\}^k} [\Phi_i \ast (\mu_i)(x - i \cdot u - T^v)] \, dt \, dx \, du$$

By Dominated Convergence, we have

$$\lim_{m' \to \infty} \int \int \Phi^{(k)}(t^k) \sum_{\mu \in \{\mu_1, \mu_2\}^k} \prod_{i \in \{0,1\}^k} [\Phi_i \ast (\phi^k_{m_i} \ast \mu_i)(x - i \cdot u - T^v)] \, dt \, dx \, du$$
\[
\lim_{\vec{m} \to \infty} \int \int \Phi^{(k)}(t^k) \sum_{\mu \in \{\mu_1, \mu_2\}^k} \prod_{i \in \{0,1\}^k} \int \Phi_i(t^{(i)}) (\phi_{m_i} \ast \mu_i)(x - t \cdot u - T_k - t^{(i)}) dt^{(i)} dx du
\]

Letting \(d\nu_i(x) = d\mu_i(x - T_k - t^{(i)})\) for \(i = 1, 2\), and applying Fubini’s Theorem, this is

\[
\begin{equation}
(52) = \lim_{\vec{m} \to \infty} \int \Phi^{(k)}(t^k) \left[ \prod_{i \in \{0,1\}^k} \Phi_i(t^{(i)}) \right] \sum_{\nu \in \{\nu_1, \nu_2\}^k} \prod_{i \in \{0,1\}^k} (\phi_{m_i}^{[k]} \ast \nu_i)(x - t \cdot u) dx du dt
\end{equation}
\]

We use Corollary 9.1 from Section 9 to obtain the bound

\[
(53) \leq \lim_{\vec{m} \to \infty} \int \Phi^{(k)}(t^k) \left[ \prod_{i \in \{0,1\}^k} \Phi_i(t^{(i)}) \right] \sum_{\nu \in \{\nu_1, \nu_2\}^k} \prod_{i \in \{0,1\}^k} \left[ \int \phi_{m_i}^{[k]} \ast \Delta^k \nu_i(x; u) dx du \right]^{\frac{1}{2}} dt
\]

Since

\[
\prod_{i \in \{0,1\}^k} \left[ \int \phi_{m_i}^{[k]} \ast \Delta^k \nu_i(x; u) dx du \right]^{\frac{1}{2}} = \prod_{i \in \{0,1\}^k} \left[ \int \phi_{m_i}^{[k]} \ast \Delta^k \nu_i(x; u) dx du \right]^{\frac{1}{2}}
\]

\[
\int \Phi^{(k)}(t^k) \left[ \prod_{i \in \{0,1\}^k} \Phi_i(t^{(i)}) \right] dt = 1
\]

we have

\[
(54) = \lim_{\vec{m} \to \infty} \sum_{\mu \in \{\mu_1, \mu_2\}^k} \prod_{i \in \{0,1\}^k} \left[ \int \phi_{m_i}^{[k]} \ast \Delta^k \mu_i(x; u) dx du \right]^{\frac{1}{2}}
\]

or

\[
(55) = \sum_{\mu \in \{\mu_1, \mu_2\}^k} \prod_{i \in \{0,1\}^k} \|\mu_i\|_{U^k}
\]

This sum is the same as

\[
(56) = [\|\mu_1\|_{U^k} + \|\mu_2\|_{U^k}]^{2k}
\]

Plugging this back into (48), we have shown that

\[
\|\mu_1 + \mu_2\|_{U^k} \leq [\|\mu_1\|_{U^k} + \|\mu_2\|_{U^k}]
\]

which is the triangle inequality.

Thus \(\|\cdot\|_{U^k}\) is a norm.

\(\square\)
8 Existence of the Inner Product

Now that we have Corollary 9.1, it is possible to extend the results of Section 3 and Section 4 on $\Delta^{k+1} \mu$ and $\|\mu\|_{U^{k+1}}$ to $\Delta^{k+1}(\mu)$ and $<\mu_0, \mu_1>$. The hypotheses used to obtain the results of Sections 3 and 4 were that

1. $\Delta^k|\mu|, \Delta^k \mu$ exist
2. $\lim_{n \to \infty} \int 1 \Phi_n * \Delta^k|\mu|(x-u_{k+1}; u') d\Delta^k|\mu|(x; u') du_{k+1} < \infty$

Given $\mu_i \in U^{k+1}$ for $i \in \{0, 1\}^{k+1}$, set $|\mu| = \{|\mu_i|\}_{i \in \{0, 1\}^{k+1}}$ and $|\mu_i| = \{|\mu_{i'}|\}_{i' \in \{0, 1\}^k}$ for $i = 0, 1$.

Then replacing the assumptions [1] and [2] by the assumptions

1. $\Delta^k(|\mu|_i), \Delta^k(\mu_i), i = 0, 1$ exist
2. $\lim \sup_{n \to \infty} \int 1 \Phi_n * \Delta^k(\mu_0)(x-u_{k+1}; u') d\Delta^k(\mu_0)(x; u') du_{k+1} < \infty$

(note that [2] follows from the statement that $\mu_i \in U^{k+1}$ for $i \in \{0, 1\}^{k+1}$ by Corollary 6.2) we can establish in turn the analogues of each of the propositions leading up to the proof that $\Delta^{k+1} \mu$ exists when $\|\mu\|_{U^{k+1}} < \infty$.

In this section, we state what these analogues are, and give some words about their proofs. Since these proofs are, mutatis mutandi, identical to the proofs in these previous sections, we do not provide all the details.

We introduce one more piece of notation before proceeding. Given measures $\mu_i, i \in \{0, 1\}^{k+1}$, we set

$$<\mu_0, \mu_1> := <\mu>$$

8.1 Fourier Identities for the Inner Product

Proposition 8.1. Let $\mu_i, i \in \{0, 1\}^{k+1}$ be measures and suppose that the measures $\Delta^k \mu_i, \Delta^k |\mu|_i$, $i = 0, 1$ exist for some $k \geq 0$. For any approximate identity $(\Psi_n)$ on $T^{d(k+1)}$, let $i \in \{0, 1\}$ and consider the following three expressions

(A) $\lim_{n \to \infty} \int \Psi_n * \Delta^k \mu_i(x-u_{k+1}; u') d\Delta^k \mu_i(x; u') du_{k+1}$

(B) $<\mu_i, \mu_i>$

Both of the expressions (A) and (B) exists with a value in $[0, \infty]$ that is equal to

(D) $\sum_{c \in \mathbb{Z}^k} |\Delta^k \mu_i(0; c)|^2$

Remark: That (A) equals (B) for any choice of approximate identity $(\Psi_n)$ means that any approximate identity may be used in the definition of $<\mu_i, \mu_i>$ (which is (A) for $\Psi = \Phi$), and so by extension via polarization, any approximate identity may be used in the definition of $<\mu_0, \mu_1> = <\mu>$. 
Lemma 8.2. Suppose that \( \mu_i, i \in \{0, 1\}^{k+1} \) are in \( U^{k+1} \), for some \( k \in \mathbb{N} \) such that \( \Delta^k \mu_i, \Delta^k |\mu_i|, i = 0, 1 \) exist. Then for all \( \xi \in \mathbb{Z} \) we have the inequality
\[
\sum_{c \in \mathbb{Z}^k} |\Delta^k \mu_i(\xi; c)|^2 \leq \sum_{c \in \mathbb{Z}^k} |\Delta^k \mu_i(0; c)|^2
\]
(59)
\[\Rightarrow |< \mu_i, \mu_i >| \]
(60)

Theorem 8.3. Given a measures \( \mu_i \in U^{k+1}, i \in \{0, 1\}^{k+1} \) for some integer \( k \geq 0 \), suppose that the measures \( \Delta^k \mu_i, \Delta^k |\mu_i| \) exist for \( i = 0, 1 \). Then the measure \( \Delta^{k+1} \mu \) exists, and is independent of the choice of approximate identity \( (\Phi_n) \) in terms of which it is defined.

Proposition 8.4. Suppose that \( \mu_i \in U^{k+1}, i \in \{0, 1\}^{k+1} \) for some non-negative integer \( k \). Then for any \( \xi \in \mathbb{Z} \) and \( \eta \in \mathbb{Z}^{k+1} \), the series
\[
\sum_{c \in \mathbb{Z}^k} \Delta^k \mu_0(-\eta_{k+1}; \eta' - c) \Delta^k \mu_1(\eta_{k+1} + \xi; c)
\]
(61)
is absolutely summable and for any approximate identity \( (\Psi_n) \) on \( \mathbb{T}^{d(k+1)} \),
\[
\lim_{n \to \infty} \int e^{-2\pi i (\xi, \eta)} \Psi_n(\Delta^k \mu_1(x - u_{k+1}, u')) d\Delta^k \mu_0(x; u') du_{k+1}
\]
(62)
\[\Rightarrow \exists \sum_{c \in \mathbb{Z}^k} \Delta^k \mu_0(\eta_{k+1}; \eta' - c) \Delta^k \mu_1(\xi - \eta_{k+1}; c)
\]
(63)
In particular, if the measure \( \Delta^{k+1} (\mu) \) exists, then with the choice of \( \Psi_n \equiv \Phi_n \) in the above,
\[
\Delta^{k+1} \mu(\xi; \eta) = \sum_{c \in \mathbb{Z}^k} \Delta^k \mu_0(\eta_{k+1}; \eta' - c) \Delta^k \mu_1(\xi - \eta_{k+1}; c)
\]
(64)

8.2 Existence Results for the Inner Product

Lemma 8.5. Let \( \mu \) be measures in \( U^{k+1} \) for some \( k \geq 0 \). Suppose further that for some approximate identity \( \Psi_n \) on \( \mathbb{T}^{d(k+1)} \), the measure \( \nu \) on \( \mathbb{T}^d \times \mathbb{T}^{d(k+1)} \) defined by the mapping
\[
f \mapsto \lim_{n \to \infty} \int_{\mathbb{T}^d \times \mathbb{T}^{d(k+1)}} f(x; u) \Psi_n(x - u_{k+1}, u') d\Delta^k \mu_0(x; u') du_{k+1}
\]
exists and is finite.
Then the measures \( \Delta^k |\mu_i| \) exist for \( i = 0, 1 \) and
\[
|\Delta^k \mu_i| = \Delta^k |\mu_i|
\]
(65)

Further,
\[
< |\mu_i| > = \|\nu\|
\]
(66)
where \( \|\nu\| \) denotes the total variation norm of the measure \( \nu \).
Corollary 8.6. Suppose that \( \mu_i \in U^{k+1}, i \in \{0, 1\}^{k+1} \) for some non-negative integer \( k \) and that \( \Delta^k \mu_i, \Delta^k \mu_{i'} \) exist and are finite for \( i = 0, 1 \). Then

(A) The measure \( \Delta^{k+1} \mu \) exists with a finite variation norm \( \| \Delta^{k+1} \mu \| \) equal to \( < |\mu|> \).

In fact for any approximate identity \( (\Psi_n) \) on \( \mathbb{T}^{d(k+1)} \), the mapping

\[
\nu : f \mapsto \lim_{n \to \infty} \int f(x, u) \Psi_n \ast \Delta^k \mu_1(x - u_{k+1}; u') d\Delta^k \mu_0(x - u_{k+1}; u') du_{k+1}
\]

(67)
corresponds to a finite measure via the Riesz Representation Theorem.

Corollary 8.7. Suppose that \( \mu_i \in U^{k+1}, i \in \{0, 1\}^{k+1} \) for some non-negative integer \( k \) and that \( \Delta^k \mu_i, \Delta^k \mu_{i'} \) exist and are finite for \( i = 0, 1 \). Suppose further that for some approximate identity \( (\Psi_n) \) on \( \mathbb{T}^{d(k+1)} \), the mapping

\[
\nu : f \mapsto \lim_{n \to \infty} \int f(x, u) \Psi_n \ast \Delta^k \mu_1(x - u_{k+1}; u') d\Delta^k \mu_0(x - u_{k+1}; u') du_{k+1}
\]

(68)
defines a (finite) measure \( \nu \) on \( \mathbb{T}^{d(k+1)} \). Then the measure \( \Delta^{k+1} \mu \) exists and \( \nu = \Delta^{k+1} \mu \).

9 Mollification and Van der Corput’s Trick

In this section, we introduce for every integer \( n \geq 0 \) an approximate identity \( (\phi_m^{[n]})_{m \in \mathbb{N}} \) on \( \mathbb{T}^{d(n+1)} \) built from a given approximate identity \( (\phi_m) \) on \( \mathbb{T}^d \), and prove the following key lemma used in Sections 6, 7, and 8.

Before we proceed further, we set up some notation.

Throughout this section, let \( (\phi_m) \) be an approximate identity. Fix \( m \) and write \( \phi \) for \( \phi_m \). The symbol \( \phi^n \) refers to \( \phi \ast \cdots \ast \phi \) where \( n \) copies of \( \phi \) are convolved, and \( \phi \ast g(x; u) \) for some multivariate function \( g : \mathbb{T}^d \times \mathbb{T}^n \to \mathbb{T} \) will always refer to the (partial) convolution of \( \phi \) and \( g \) with respect to the \( x \) variable.

Corollary 9.1. For each \( i \in \{0, 1\}^k \), let \( \mu_i \in U^k \). Then

\[
\int \prod_{i \in \{0, 1\}^k} \phi^k \ast \mu_i(x - i \cdot u) dx du
\]

(69)

\[
\leq \prod_{i' \in \{0, 1\}^{k-1}} \left[ \int \phi^{[k-1]} \ast \Delta^k \mu_{1'}(x - u_{k'}'; u') \phi^{[k-1]} \ast \Delta^k \mu_{0'}(x; u') dx du \right]^\frac{1}{k+1}
\]

(70)

\[
\leq \prod_{i \in \{0, 1\}^k} \left[ \phi^{[k]} \ast \Delta^k \mu_i(\mathbb{T}^{d(k+1)}) \right]^\frac{1}{k+1} \left( \prod_{i \in \{0, 1\}^k} \| \mu_i \|_{U^k} \right)
\]

(71)

Crucial in the derivation of this result is the following observation, a corollary to the work in Sections 3 and 4.
Corollary 9.2. Let $\Psi_n : \mathbb{T}^{d(k+1)} \to \mathbb{T}^d$ be an approximate identity and $\mu \in U^{k+1}$. Then for any $f \in C(\mathbb{T}^{d(k+2)})$,

$$\int f \, d\mu^{[k+1]}(x; u) = \lim_{n \to \infty} \int f(x, u_1, \ldots, u_{k+1}) \Psi_n * \triangle^k \mu(x - u_{k+1}; u') \Psi_n * \triangle^k \mu(x - u_{k+1}; u') \, dx \, du$$

(72)

Proof. Both the Fourier transform of $\triangle^{n+1} \mu$ and the measure defined by the right side of (72) coincide, since taking Fourier transforms, we have by Proposition 3.3) that the first is

$$\sum_{c \in \mathbb{Z}^k} \hat{\triangle}^k \mu(-\eta_{k+1}; \eta-c) \hat{\triangle}^k \mu(\eta_{k+1} + \xi; c)$$

(73)

while the second is

$$\lim_{n \to \infty} \sum_{c \in \mathbb{Z}^k} \hat{\Psi}_n(-\eta_{k+1}, \eta-c) \hat{\Psi}_n(\eta_{k+1} + \xi, c) \hat{\triangle}^k \mu(-\eta_{k+1}; \eta-c) \hat{\triangle}^k \mu(\eta_{k+1} + \xi; c)$$

(74)

by an application of Lemma 10.1 with $f(x - u_{k+1}; u) := \Psi_n * \triangle^k \mu(x - u_{k+1}; u')$ and $d\nu(x, u') = \Psi_n * \triangle^k \mu(x; u') \, dx \, du'$.

Then exactly as in the proof of (13) of Proposition 3.3), Dominated Convergence allows us to pass the limit to the inside of the sum since the terms of (73) are absolutely summable by Proposition 3.3 and their absolute values dominate the terms in (74), and so we have shown that the measures defined by both sides of (72) have Fourier transforms equal to (73).

We begin in this section by introducing $n + 1$-dimensional convolution operators $\phi^{[n]}$ arising from certain “intertwined” convolutions of $2n + 1$ copies of the convolution operator $\phi$. These objects are useful because they allow us to deal with a mollified $\triangle^n \mu$ as opposed to $\triangle^n(\phi \ast \mu)$, which is important since convolution does not commute with the operator $\triangle$.

Given $t = (t_0; t_1, \ldots, t_k)$, let $\tilde{t} = (t_1, \ldots, t_k) \in \mathbb{T}^{dk}$.

Let $\phi^{[0]} = \phi$.

We define

$$\phi^{[n]}(t) = \int \phi(t_0 + \sum_{j=1}^n c_j) \phi(-c_1) \phi(-t_1 - c_1) \cdots \phi(-c_n) \phi(-t_n - c_n) \, dc$$

Notice that

$$\phi \ast \phi^{[n]}(t_0, \ldots, t_n) = \int \phi^{[n+1]}(t_0, \ldots, t_{n+1}) \, dt_{n+1}$$

(75)
This ability to expand from $\phi \ast \phi^{[n]}$ to $\phi^{[n+1]}$, and, as we will see in Lemma 9.5, from $\ast^n \phi$ to $\phi^{[n]}$, underlies $\phi^{[n]}$’s utility.

The following is essentially the statement that the Fourier transform of a convolution is the product of the Fourier transforms, and the proof is precisely the same, though we pay in additional notational weight for the fact that we are not strictly dealing with a convolution when writing $\phi^{[n]}$ in terms of $\phi$.

**Lemma 9.3.** Let $(\xi; \eta) \in \mathbb{T}^d \times \mathbb{T}^{dn}$. Then

$$\hat{\phi}^{[n]}(\xi; \eta) = \hat{\phi}(\xi) \cdot \hat{\phi}(-\eta_1) \cdots \hat{\phi}(-\eta_n) \cdot \hat{\phi}(\xi - \eta_1) \cdots \hat{\phi}(\xi - \eta_n) \quad (76)$$

**Proof.** Expanding $\hat{\phi}^{[n]}$ according to its definition and integrating first in $t_0$ we have

$$\hat{\phi}^{[n]}(\xi; \eta) = \int \phi^{[n]}(t) e^{-2\pi i t \cdot (\xi; \eta)} dt$$

$$= \int \phi(t_0 + \sum_{j=1}^{n} c_j) \prod_{i=1}^{n} \phi(-c_i) \phi(-t_i - c_i) \exp(\xi t_0 + \sum_{j=1}^{n} \eta_j t_j) dt_0 d\vec{\eta} dc$$

$$= \hat{\phi}(\xi) \int \exp\left(\sum_{j=1}^{n} c_j(-\xi)\right) \prod_{i=1}^{n} \phi(-c_i) \phi(-t_i - c_i) \exp(-2\pi i \left(\sum_{j=1}^{n} \eta_j t_j\right)) d\vec{\eta} dc \quad (77)$$

then integrating in the remaining $t_i$

$$= \hat{\phi}(\xi) \hat{\phi}(-\eta_1) \cdots \hat{\phi}(-\eta_n) \int \exp\left(\sum_{j=1}^{n} c_j(-\xi)\right) \prod_{i=1}^{n} \phi(-c_i) \exp(\eta_1 c_1) \cdots \exp(\eta_n c_n) dc \quad (78)$$

Collecting terms, this is

$$= \hat{\phi}(\xi) \hat{\phi}(-\eta_1) \cdots \hat{\phi}(-\eta_n) \int \exp\left(\sum_{j=1}^{n} c_j(\eta_j - \xi)\right) \phi(c_1) \cdots \phi(-c_n) dc$$

and finally integrating on $c$ yields the conclusion. $\square$

We will need to define a component of $\phi^{[n]}$,

$$\tilde{\phi}^{[n]}(\vec{t}) := \tilde{\phi}^{[n]}(\vec{t}, c) := \prod_{j=1}^{n} \phi(-c_j) \phi(-t_1 - c_1) \cdots \phi(-t_n - c_n)$$

so that

$$\phi^{[n]}(t) = \int \phi(t_0 + \sum_{j=1}^{n} c_j) \tilde{\phi}^{[n]}(\vec{t}) dc \quad (79)$$

and also (letting now $t = (t_0, \ldots, t_{n+1})$)

$$\phi^{[n+1]}(t) = \int \phi(-t_{n+1} - c_{n+1}) \phi(-c_{n+1}) \phi(t_0 + \sum_{j=1}^{n+1} c_j) \tilde{\phi}^{[n]}(\vec{t}) dc \quad (80)$$
\[ \phi^{[n+1]}(t) = \int \phi(-t_{n+1} - c_{n+1})\phi(-c_{n+1})\phi^{[n]}(t_0 + c_{n+1}; \bar{t}) \, dc_{n+1} \]

As a consequence, we have the following.

**Lemma 9.4.** For any \( n \in \mathbb{N} \) and function bounded function \( f : \mathbb{T}^d \times \mathbb{T}^{dn} \to \mathbb{T}^d \) set

\[ \Delta f : \mathbb{T}^d \times \mathbb{T}^{d(n+1)}(x; v) \mapsto f(x; v')f(x - v_{n+1}; v') \]

Then

\[ \int \phi \ast f(x - t_0; v' - \bar{t})\phi \ast f(x - v_{n+1} - t_0; v' - \bar{t})\phi^{[n]}(t_0; \bar{t}) \, dt' = \phi^{[n+1]} \ast \Delta f(x; v) \quad (81) \]

**Proof.** We expand the convolutions on the left side of (81), obtaining

\[ \int \phi(-c_{n+1})f(x + c_{n+1} - t_0; v' - \bar{t}) \cdot \phi(-t_{n+1})f(x - v_{n+1} + t_{n+1} - t_0; v' - \bar{t})\phi^{[n]}(t_0; \bar{t}) \, dc_{n+1} \, dt \]

Sending \( t_{n+1} \mapsto t_{n+1} + c_{n+1} \) and \( t_0 \mapsto t_0 + c_{n+1} \), this becomes

\[ (82) = \int \phi(-c_{n+1})\phi(-t_{n+1} - c_{n+1})\phi^{[n]}(t_0; \bar{t}) \cdot f(x - t_0; v' - \bar{t})f(x - v_{n+1} + t_{n+1} - t_0; v' - \bar{t}) \, dc_{n+1} \, dt \]

Then applying Fubini’s theorem, we have

\[ (83) = \int \left[ \int \phi(-c_{n+1})\phi(-t_{n+1} - c_{n+1})\phi^{[n]}(t_0; \bar{t}) \right] \, dc_{n+1} \]

\[ \cdot f(x - t_0; v' - \bar{t})f(x - v_{n+1} + t_{n+1} - t_0; v' - \bar{t}) \, dt \]

By (80), this is

\[ (84) = \int \phi^{[n+1]}(t) f(x - t_0; v' - \bar{t})f(x - t_0 - (v_{n+1} - t_{n+1}); v' - \bar{t}) \, dt \]

which is what we sought to show. \( \square \)

We use the above lemma to show the following, from which our main result in this section is immediate.

**Lemma 9.5.** For \( j \leq J \in \mathbb{N} \) and \( i \in \{0, 1\}^J \), let \( \mu_i \) be measures in \( U^{J+1} \). Then for any \( \iota \leq j \in \{0, 1\}^j \), we have

\[ \int \prod_{i>j \in \{0, 1\}^k\setminus j} \phi^{[k-j]} \ast \phi^{[j]} \ast \Delta^j \mu_i(x - \iota_{>j} \cdot u_{\leq j}; u_{\leq j}) \, dx \, du \]

\[ \leq \prod_{i_{>j+1}=0} \left[ \int \prod_{i_{>j+1} \in \{0, 1\}^{k-j-1}} \phi^{[k-j-1]} \ast \phi^{[j+1]} \ast \Delta^{j+1} \mu_i(x - \iota_{>j+1} \cdot u_{\leq j+1}; u_{\leq j+1}) \, dx \, du \right] ^{\frac{1}{2}} \]

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Proof. Use Fubini’s Theorem to rewrite the left side of (86) as

\[
\int \prod_{\ell_j \in \{0,1\}^{k-j}} \phi^{k-j-1}_\ell \star \phi^j_\ell \star (\phi \star \Delta_j \mu_x)(x - \ell_j \cdot u_{\leq j}; u_{\leq j}) \ dx \ du
\]  

(87)

Letting \((\Psi_n)\) denote an approximate identity, by Dominated Convergence this is the same as

\[
(87) = \int \prod_{\ell_j \in \{0,1\}^{k-j}} \phi^{k-j-1}_\ell \star \phi^j_\ell \star (\phi \star \Psi_n \star \Delta_j \mu_x)(x - \ell_j \cdot u_{\leq j}; u_{\leq j}) \ dx \ du
\]  

(88)

Next, we expand the convolution out by writing

\[
(88) = \int \prod_{\ell_j \in \{0,1\}^{k-j}} \phi^{k-j-1}_\ell \star \phi^j_\ell (t^{(i)})(\phi \star \Psi_n \star \Delta_j \mu_x)(x - t^{(i)}_0 - \ell_j \cdot u_{\leq j}; u_{\leq j} - t^{(i)}_0) \ dt^{(i)} \ dx \ du
\]  

(89)

By applying Fubini’s Theorem, changing variables so that \(x\) is labeled as \(x_0\) and \(u_{j+1} \mapsto -x_1 + x_0\), and then applying Fubini’s Theorem to move the integrals over \(x_0\) and \(x_1\), this becomes

\[
(89) = \lim_{n \to \infty} \int \prod_{\ell_j = 0}^{1} [p(t^0) p(t^1)]^\frac{1}{2} g_{\ell_{j+1}}(u_{\not= j+1}, t^{j+1}) \ du_{\not= j+1} \ dt
\]  

(90)

where \(t^i = (t^{(i)}_{\ell_j \leq j})_{\ell_j \in \{0,1\}}^i\), \(dt = dt^0 \ dt^1\),

\[p(t^{j+1}) := \prod_{\ell_j \in \{0,1\}^{k-j}} \phi^{k-j-1}_\ell \star \phi^j_\ell (t^{(i)})\]

\[g_{\ell_{j+1}}(u_{\not= j+1}, t^{j+1}) := \int \prod_{\ell_j \in \{0,1\}^{k-j-1}} \phi \star \Psi_n \star \Delta_j \mu_x(x_{\ell_{j+1}} - t^{(i)}_0 - \ell_{j+1} \cdot u_{\not= j+1}; u_{\leq j} - t^{(i)}_0) \ dx_{\ell_{j+1}}\]

Applying Cauchy-Schwarz, we have

\[
(90) \leq \lim_{n \to \infty} \prod_{\ell_j = 0}^{1} \left[ \int p(t^0) p(t^1) g_{\ell_{j+1}}^2(u_{\not= j+1}, t^{j+1}) \ du_{\not= j+1} \ dt \right]^\frac{1}{2}
\]  

(91)

\[
= \lim_{n \to \infty} \prod_{\ell_j = 0}^{1} \left[ \int p(t^{j+1}) g_{\ell_{j+1}}^2(u_{\not= j+1}, t^{j+1}) \ du_{\not= j+1} \ dt^{j+1} \right]^\frac{1}{2}
\]  

(92)
by Fubini’s Theorem since \( \int p = 1 \).

Now we may write, upon a change of variables, that

\[
g_{i+1}^2 (u_{\neq i+1}, t^{i+1}) = \int \prod_{\iota > j_1 \in \{0,1\}^{k-j-1}} \phi * \Psi_n * \Delta_j \mu_i (x - t_0 - t_{j+1} : u_{j+1}; u_{\leq j} - \bar{t}(\iota)) \, dx 
\cdot \int \prod_{\iota > j_1 \in \{0,1\}^{k-j-1}} \phi * \Psi_n * \Delta_j \mu_i (x - u_{j+1} - t_0 - t_{j+1} : u_{j+1}; u_{\leq j} - \bar{t}(\iota)) \, du_{j+1}
\]

and using Fubini’s Theorem to integrate through each term by \( t^{(i)} \), and invoking Lemma 9.4, we have

\[
\int p(t^{i+1}) g_{i+1}^2 (u_{\neq i+1}, t^{i+1}) \, dt^{i+1} = \int \prod_{\iota > j_1 \in \{0,1\}^{k-j-1}} \phi^{*k-j-1} \ast \phi^{[j+1]} \ast \Delta \left( \Psi_n \ast \Delta_j \mu_i \right) (x - t_{j+1} : u_{j+1}; u_{\leq j}) \, dx \, du_{j+1}
\]

Thus

\[
(91) = \lim_{n \to \infty} \prod_{j+1=0}^{1} \left[ \int \prod_{\iota > j_1 \in \{0,1\}^{k-j-1}} \phi^{*k-j-1} \ast \phi^{[j+1]} \ast \Delta \left( \Psi_n \ast \Delta_j \mu_i \right) (x - t_{j+1} : u_{j+1}; u_{\leq j}) \, dx \, du_{j+1} \, du_{\neq j} \right]^{1/2}
\]

We may now apply Dominated Convergence in order to invoke Lemma 9.2, and conclude that

\[
(93) = \lim_{n \to \infty} \prod_{j+1=0}^{1} \left[ \int \prod_{\iota > j_1 \in \{0,1\}^{k-j-1}} \phi^{*k-j-1} \ast \phi^{[j+1]} \ast \Delta \left( \Psi_n \ast \Delta_j \mu_i \right) (x - t_{j+1} : u_{j+1}; u_{\leq j}) \, dx \, du_{j+1} \, du_{\neq j} \right]^{1/2}
\]

And this is precisely what we sought to show.

\[
\square
\]

Proof of Corollary 9.1. The proof follows directly from an induction using Lemma 9.5.

\[
\square
\]

10 Appendix

**Lemma 10.1.** Let \( \nu_i, i = 0,1 \) be finite signed measures on \( \mathbb{T}^{d(k+1)} \). If \( (\Psi_n) \) is an approximate identity on \( \mathbb{T}^{d(k+1)} \), then

\[
\int \Psi_n \ast \nu_1 (x - u_{k+1}; u') \, d\nu_0 (x; u') \, du_{k+1} = \sum_{\eta \in \mathbb{Z}^d} \widehat{\Psi_n} (0; \eta) \widehat{\nu_1} (0; \eta) \widehat{\nu_0} (0; -\eta)
\]
Proof. Consider Parseval’s theorem in the form that for a function \( g \) with \( \hat{g} \in L^1 \) and a measure \( \mu \),

\[
\int g \, d\mu = \sum \hat{g}\mu
\]

We apply it to the integral \( \int g(x - u_{k+1}; u') \, d\nu(x; u') \) to get that

\[
\int g(x - u_{k+1}; u') \, d\nu(x; u') = \sum_{(\xi; \eta) \in \mathbb{Z}^{d(k+1)}} \hat{g}(\xi; \eta) e^{2\pi i \xi u_{k+1}} \hat{\nu}(-\xi; -\eta)
\]

Since \( \hat{g} \in L^1 \) and \( |\hat{\nu}|(0; 0) = |\nu|_d(\mathbb{T}^{d(k+1)}) < \infty \), this sum converges uniformly, so if we integrate with respect to \( u_{k+1} \), we can take the integral inside the sum. This means that

\[
\int g(x - u_{k+1}; u') \, d\nu(x; u') \, du_{k+1} = \sum_{\eta \in \mathbb{Z}^d} \hat{g}(0; \eta) \hat{\nu}(0; -\eta)
\]

since \( \xi \in \mathbb{Z}^d \) so that \( e^{2\pi i \xi u_{k+1}} \, du_{k+1} = 0 \) unless \( \xi = 0 \).

\[\square\]

Lemma 10.2. Let \( \nu \) be a measure on \( \mathbb{T} \times \mathbb{T}^j \) for some \( j \in \mathbb{N} \). Let \( (\Psi_n) \) be an approximate identity on \( \mathbb{T}^d \times \mathbb{T}^j \). Then

1. \( \int \Psi_n * \nu(x; u') \, d\nu(y; u') \, dx \to \sum_{\eta \in \mathbb{Z}^j} |\hat{\nu}(0; \eta)|^2 \)
   and for any \((\kappa_0, \kappa) \in \mathbb{Z}^d \times \mathbb{Z}^j\),

2. \( \int (\Psi_n e_{(\kappa_0, \kappa)}) * \nu(x; u') \, d\nu(y; u') \, dx \to \sum_{\eta \in \mathbb{Z}^j} |\hat{\nu}(0; \eta)|^2 \) where \( e_{(\kappa_0, \kappa)}(x; u') := e^{-2\pi i (x\kappa_0 + u' \cdot \kappa)} \).

Proof. It suffices to prove the second statement, since the first is a special case of the second. For a measure \( \rho \) on \( \mathbb{T} \times \mathbb{T}^j \), let \( P \) denote the projection onto \( \mathbb{T}^j \), so that

\[
\int_{\mathbb{T}} f(u) \, dP\rho(u) := \int_{\mathbb{T} \times \mathbb{T}} f(u) \, d\rho(x; u)
\]

We have

\[
\int (\Psi_n e_{(\kappa_0, \kappa)}) * \nu(x; u') \, d\nu(y; u') \, dx
\]

\[= \int \int \Psi_n(x - z; u' - v) e_{(\kappa_0, \kappa)}(x - z; u' - v) \, d\nu(z; v) \, d\nu(y; u') \, dx\]
\[
\begin{align*}
&= \int e_{(\kappa_0,\kappa)}(x; u') \Psi_n(x - z; u' - v)e_{-(\kappa_0,\kappa)}(z; v) d\nu(z; v) d\nu(y; u') dx \\
&= \int e_{(\kappa_0,\kappa)}(x; u') \Psi_n \ast (e_{-(\kappa_0,\kappa)}\nu)(x; u') d\nu(y; u') dx
\end{align*}
\]

This last integral is

\[
= \int P(e_{(\kappa_0,\kappa)} \Psi_n \ast (e_{-(\kappa_0,\kappa)}\nu))(u') dP(\nu)(u')
\]

Let \( P(\nu)_s \) denote the singular part of \( P(\nu) \). Since \( P(e_{(\kappa_0,\kappa)} \Psi_n \ast (e_{-(\kappa_0,\kappa)}\nu))(u') \) diverges to infinity \( P(\nu)_s \)-a.e, we surmise that if the singular part \( P(\nu)_s \) of \( P(\nu) \) is non-trivial, then (95) diverges to \( \infty \). Since such a \( P(\nu) \) cannot be in \( L^2 \), we have that in this case

\[
\sum_{\eta \in \mathbb{Z}^j} |\hat{P}(\nu)(\eta)|^2 = \sum_{\eta \in \mathbb{Z}^j} |\hat{\nu}(0; \eta)|^2 = \infty = (95)
\]

as we sought to show.

In the case that \( P(\nu) \) is absolutely continuous (say with density equal to the function \( g \)), we may set the integrand \( P(e_{\kappa_0} \Psi_n \ast (e_{-(\kappa_0,\kappa)}\nu))(u') g(u') \). Then we know that \( f_n \to |g|^2 \) at Lebesgue-almost every point.

If \( g \in L^2 \), then expressing things on the Fourier side and applying Dominated Convergence, we arrive at the desired equality.

If \( g \notin L^2 \), then applying Egorov’s Theorem we see that \( \int f_n \to \infty \), which is again equal to

\[
\infty = \sum_{\eta \in \mathbb{Z}^j} |\hat{g}(\eta)|^2 = \sum_{\eta \in \mathbb{Z}^j} |\hat{P}(\nu)(\eta)|^2 = \sum_{\eta \in \mathbb{Z}^j} |\hat{\nu}(0; \eta)|^2
\]

which completes the proof.

\[\square\]

**Lemma 10.3.** Suppose that \( f \) is a positive function on \( \mathbb{T}^d \) with finite \( U^{k+1} \)-norm, which is to say that

\[
\int \Delta^k f(x - u_{k+1}; u') \Delta^k f(x; u') dx du < \infty
\]

(96)

Define \( F_\zeta \) for \( \zeta = (\eta_{k+1}, \eta') \in \mathbb{Z}^d \times \mathbb{Z}^{dk} \) by

\[
F_\zeta(u') := \int \Delta^k f(x; u') e^{-2\pi i \zeta \cdot (x, u')} dx
\]

set

\[
F := F_0
\]

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and for $\mathbf{t} \in \mathbb{T}^{dk}$ set

$$P_n(\mathbf{t}) = \int \Phi_n(t_0; \mathbf{t}) \, dt_0$$

$$e_\zeta(t_0; \mathbf{t}) := e^{-2\pi i \zeta \cdot (t_0, \mathbf{t})}$$

Then given $\zeta \in \mathbb{Z}^{d(k+1)}$,

(i) $F \in L^2(\mathbb{T}^{dk})$

(ii) $F_\zeta \in L^2(\mathbb{T}^{dk})$

(iii) $\int \Phi_n * (\Delta^k f \, e_\zeta)(u_{k+1}; \mathbf{u}') \, du_{k+1} = P_n * F_\zeta(\mathbf{u}')$

(iv) $P_n * F_\zeta(\mathbf{u}') \to F_\zeta(\mathbf{u}') \text{ in } L^2(\mathbb{T}^{dk})$

**Proof.** Applying Fubini’s Theorem and changing variables $u_{k+1} \mapsto -u_{k+1} + x$, (96) is equivalent to the assumption

$$\int \left[ \int \Delta^k f(x; \mathbf{u}') \, dx \right] \, du'$$

$$= \int \left[ \int \Delta^k f(x; \mathbf{u}') \, dx \right]^2 \, du' = \int F(u')^2 < \infty$$

which is (i).

Since $|F_\zeta| \leq F$, $F_\zeta \in L^2(\mathbb{T}^{dk})$, which is (ii).

To obtain (iii) expand the convolution

$$\int \Phi_n * (\Delta^k f \, e_\zeta)(u_{k+1}; \mathbf{u}') \, du_{k+1}$$

$$= \int \int \Phi_n(t_0; \mathbf{t}) \Delta^k f(u_{k+1} - t_0; \mathbf{u}' - \mathbf{t}) e_\zeta(u_{k+1} - t_0, \mathbf{u}' - \mathbf{t}) \, dt_0 \, dt \, du_{k+1}$$

and use Fubini’s Theorem to send $u_{k+1} \mapsto u_{k+1} + t_0$ so that

$$(99) = \int \left[ \int \Phi_n(t_0; \mathbf{t}) \, dt_0 \right] \left[ \int \Delta^k f(u_{k+1}; \mathbf{u}' - \mathbf{t}) e_\zeta(u_{k+1}, \mathbf{u}' - \mathbf{t}) \, du_{k+1} \right] \, dt$$

$$= \int P_n(\mathbf{t}) F_\zeta(\mathbf{u}' - \mathbf{t}) \, dt$$

$$= P_n * F_\zeta(\mathbf{u}')$$

which is (iii).

Notice that for $\mathbf{c} \in \mathbb{Z}^{dk}$,

$$\int P_n(\mathbf{t}) e^{-2\pi i \mathbf{c} \cdot \mathbf{t}} \, dt = \int \left[ \int \Phi_n(t_0; \mathbf{t}) \, dt_0 \right] e^{-2\pi i \mathbf{c} \cdot \mathbf{t}} \, dt$$

$$= \hat{\Phi}_n(0; \mathbf{c})$$
so (100) belongs to $L^2(\mathbb{T}^{dk})$ since its Fourier transform is $\hat{\Phi}_n(0; \cdot \hat{F}_\zeta(\cdot))$, and this is majorized by $\hat{\Phi}_n(0; 0)|\hat{F}| = |\hat{F}|$.

From the Fourier transform $\hat{\Phi}_n(0; \cdot \hat{F}_\zeta(\cdot))$ of (100) we also see that for fixed $\xi$, in the limit as $n \to \infty$

$$P_n * F_\zeta \to F_\zeta \quad \text{in } L^2(\mathbb{T}^{dk})$$

since by Plancherel,

$$\int |P_n * F_\zeta - F_\zeta|^2\,du' = \sum_{c \in \mathbb{Z}^{dk}} |\hat{\Phi}_n(0; c)\hat{F}_\zeta(c) - \hat{F}_\zeta(c)|^2$$

$$= \sum_{c \in \mathbb{Z}^{dk}} |1 - \hat{\Phi}_n(0; c)|^2|\hat{F}_\zeta(c)|^2$$

and the fact that $|1 - \hat{\Phi}_n(0; c)| \leq 2$, $1 - \hat{\Phi}_n(0; c) \to 0$, and $|\hat{F}_\zeta|^2 \in L^1$ means that we may apply Dominated Convergence, obtaining (iv).

\[\square\]

**Lemma 10.4.** Suppose that $f$ is a positive function on $\mathbb{T}^d$ with finite $U^{k+1}$-norm, which is to say that

$$\int \nabla^k f(x - u_{k+1}; u') \nabla^k f(x; u')\,dx\,du < \infty \quad (101)$$

For $x, y \in \mathbb{T}^d$, $u' \in \mathbb{T}^{dk}$, and $(\xi; \eta) \in \mathbb{Z}^d \times \mathbb{Z}^{dk}$, let

$$g(y; u') := \nabla^k f(y; u')e^{-2\pi i [\eta \cdot u' + (-\eta_{k+1})y]}$$

$$d\nu(x; u') = \nabla^k f(x; u')e^{-2\pi i (\eta_{k+1} + \xi)x}$$

Then

$$\int \nabla^k f(x - u_{k+1}; u') \nabla^k f(x; u')e^{-2\pi i [\eta \cdot u' + \xi]}\,dx\,du$$

$$= \lim_{n \to \infty} \int \Phi_n * g(x - u_{k+1}; u')\,d\nu(x; u')\,du_{k+1} \quad (102)$$

**Proof.** To obtain the convergence that we want, we reduce this to a question about convergence in $L^2(\mathbb{T}^{dk})$.

Recall the notation

$$F_\zeta(u') = \int \nabla^k f(x; u')e^{-2\pi i \zeta(x, u')}\,dx, \quad \zeta \in \mathbb{Z}^d \times \mathbb{Z}^{dk}$$

$$P_n(t) = \int \Phi_n(t_0; t)\,dt_0$$

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By Lemma 10.3, \( F_\xi \in L^2(\mathbb{T}^{dk}) \).

By the hypothesis (101) we may apply Fubini’s Theorem to (102) to send \( u_{k+1} \mapsto -u_{k+1} + x \), obtaining

\[
(102) = \int \left[ \int \Delta^k f(u_{k+1}; u') e^{-2\pi i \left[-\eta_{k+1} u_{k+1} + \eta' u'\right]} du_{k+1} \right] \left[ \Delta^k f(x; u') e^{-2\pi i \left[(\eta_{k+1} + \xi)x + 0 \cdot u'\right]} dx \right] du'
\]

\[
= \int F(-\eta_{k+1} \eta')(u') F(\eta_{k+1} + \xi, 0)(u') du'
\]

Similarly,

\[
(103) = \lim_{n \to \infty} \int \left[ \int \Phi_n \ast (\Delta^k f e^{(-\eta_{k+1} \eta')})(u_{k+1}; u') du_{k+1} \right] F(\eta_{k+1} + \xi, 0)(u') du'
\]

By (iii) of Lemma 10.3, (105) can be restated as

\[
(103) = \lim_{n \to \infty} \int P_n \ast F(-\eta_{k+1} \eta')(u') F(\eta_{k+1} + \xi, 0)(u') du'
\]

According to (iv) from Lemma 10.3, \( P_n \ast F(-\eta_{k+1} \eta') \) converges to \( F(-\eta_{k+1} \eta') \) in \( L^2(\mathbb{T}^{dk}) \); since strong convergence implies weak convergence in a Hilbert space, writing \( \langle a, b \rangle \) for the \( L^2(\mathbb{T}^{dk}) \) inner product \( \int a(u) \overline{b(u)} du \) between \( a \) and \( b \), we have

\[
(103) = \lim_{n \to \infty} \langle P_n \ast F(-\eta_{k+1} \eta'), F(\eta_{k+1} + \xi, 0) \rangle
\]

\[
= \langle F(-\eta_{k+1} \eta'), F(\eta_{k+1} + \xi, 0) \rangle
\]

So we obtain that

\[
(103) = \int F(-\eta_{k+1} \eta')(u') F(\eta_{k+1} + \xi, 0)(u') d\mu(u')
\]

which according to (104) is the same as (102), and we have shown that (103)=(102), concluding the proof.

\[\square\]

References

[Christ et al.(2005)] Michael Christ, Xiaochun Li, Terence Tao, and Christoph Thiele. On multilinear oscillatory integrals, nonsingular and singular. Duke Math. J., 130(2):321–351, 2005. ISSN 0012-7094.

[Fan and Li(2009)] Dashan Fan and Xiaochun Li. A bilinear oscillatory integral along parabolas. Positivity, 13(2):339–366, 2009. ISSN 1385-1292. doi: 10.1007/s11117-008-2270-3. URL http://dx.doi.org/10.1007/s11117-008-2270-3.

[Gowers(2001)] W. T. Gowers. A new proof of Szemerédi’s theorem. Geom. Funct. Anal., 11(3):465–588, 2001. ISSN 1016-443X. doi: 10.1007/s00039-001-0332-9. URL http://dx.doi.org/10.1007/s00039-001-0332-9.

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Ben Green and Terence Tao. The primes contain arbitrarily long arithmetic progressions. *Ann. of Math. (2)*, 167(2):481–547, 2008. ISSN 0003-486X. doi: 10.4007/annals.2008.167.481. URL http://dx.doi.org/10.4007/annals.2008.167.481

Vjekoslav Kovač. Boundedness of the twisted paraproduct. *Rev. Mat. Iberoam.*, 28(4):1143–1164, 2012. ISSN 0213-2230. doi: 10.4171/RMI/707. URL http://dx.doi.org.proxy.lib.ohio-state.edu/10.4171/RMI/707

Izabella Laba and Malabika Pramanik. Arithmetic progressions in sets of fractional dimension. *Geom. Funct. Anal.*, 19(2):429–456, 2009. ISSN 1016-443X. doi: 10.1007/s00039-009-0003-9. URL http://dx.doi.org/10.1007/s00039-009-0003-9

Alex Samorodnitsky and Luca Trevisan. Gowers uniformity, influence of variables, and PCPs. *SIAM J. Comput.*, 39(1):323–360, 2009. ISSN 0097-5397. doi: 10.1137/070681612. URL http://dx.doi.org.proxy.lib.ohio-state.edu/10.1137/070681612

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