Emergence of Kinematic Space from Quantum Modular Geometric Tensor

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Abstract

We generalize the Quantum Geometric Tensor by replacing a Hamiltonian with a modular Hamiltonian. The symmetric part of the Quantum Geometric Tensor provides a Fubini-Study metric, and its anti-symmetric sector gives a Berry curvature. Now the generalization or Quantum Modular Geometric Tensor gives a Kinematic Space and a modular Berry curvature. Here we demonstrate the emergence by focusing on a spherical entangling surface. We also use the result of the identity Virasoro block to relate the connected correlator of two Wilson lines to the two-point function of a modular Hamiltonian. This result realizes a novel holographic entanglement formula for two intervals of a general separation. This formula does not only hold for a classical gravity sector but also Quantum Gravity. The formula also provides a new Quantum Information interpretation to the connected correlators of Wilson lines as the mutual information. Our study provides an opportunity to explore Quantum Kinematic Space through Quantum Modular Geometric Tensor and hence go beyond symmetry.
1 Introduction

The *Emergent Spacetime* proposes that space and time are not fundamental but emergent (at a macroscopic scale). The successful approach is based on this idea, like \((d+1)\)-dimensional Anti-de Sitter/\(d\)-dimensional Conformal Field Theory (AdS\(_{d+1}/\text{CFT}_d\)) correspondence, which avoids the difficulty of non-renormalizability in Quantum Gravity. Therefore, one can consider Quantum Field Theory (QFT) on a flat spacetime for an alternative approach of Quantum Gravity. The correspondence transforms the problem into the building of a dictionary of bulk gravity theory and QFT. The classical gravity theory and CFT \cite{1} are both calculable. Therefore, it is easier than non-renormalizability. Recently, people were interested in using a picture of Quantum Information to interpret the holographic principle. It was motivated by the emergence of an AdS\(_{d+1}\) minimum surface from CFT\(_d\) entanglement entropy. Indeed, people did not understand how a bulk gravity emerges from physical degrees of freedom in QFT \cite{2,3}. The holographic formula implies that emergent spacetime comes from the degrees of Quantum Entanglement.

The quantum correction in the bulk gravity generates a *conformal anomaly*. The conformal symmetry is *not* enough in the AdS/CFT correspondence. Therefore, it is hard to understand why the correspondence still holds in a quantum regime. The gauge formulation of 3d Einstein gravity theory has an exact correspondence from a non-conformal field theory. A bulk Wilson line operator \cite{4} (or a quantum minimum-surface) is dual to entanglement entropy for a single interval \cite{5,6}. Therefore, at the quantum level, we still have a geometry that emerged from entanglement entropy. It is a non-trivial realization of the holographic entanglement formula beyond a classical gravity sector. One can also calculate a connected two-point function of modular Hamiltonian or mutual information of two intervals for detecting quantum bulk gravity theory \cite{7}. It is hard to know whether the emergence can happen in a quantum regime because it is hard to compute.

It is convenient to calculate \(H_{\text{mod}}\) for the two regions are separated by a spherical entangling surface:

\[
H_{\text{mod}} \equiv -\ln \rho_A, \quad \rho_A \equiv \text{Tr}_B \rho_{AB},
\]

where \(\rho_{AB}\) is a density matrix with a Hilbert space \(H_A \otimes H_B\), \(\rho_A\) is a reduced density matrix of a region \(A\), and \(\text{Tr}_B\) is a partial trace operation acting on a region \(B\). People
used the operator product expansion (OPE) to organize the CFT data and extract the OPE block $B^{jk}_l$ \[8\],

$$
\mathcal{O}_j(x_1)\mathcal{O}_k(x_2) \equiv |x_1 - x_2|^{-\Delta_j - \Delta_k} \sum_l C_{jkl} B^{jk}_l(x_1, x_2),
$$

where $\Delta_j$ and $\Delta_k$ are the conformal dimensions of $\mathcal{O}_j$ and $\mathcal{O}_k$, respectively, and $C_{jkl}$ is an OPE coefficient. The $B^{jk}_l$ is proportional to a modular Hamiltonian, and it satisfies a Laplacian equation on a 2$d$-dimensional Kinematic Space (KS) \[8, 9\]

$$
ds^2 \equiv g_{\mu\nu} dx^\mu dy^\nu = \alpha \frac{4}{(x - y)^2} \left( -\eta_{\mu\nu} + \frac{2(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x - y)^2} \right) dx^\mu dy^\nu,
$$

where

$$(x - y)^2 \equiv \eta^{\mu\nu}(x_\mu - y_\mu)(x_\nu - y_\nu); \quad \eta_{\mu\nu} \equiv \text{diag}(-1, 1, \ldots, 1).$$

The $\alpha$ is an arbitrary constant. We label the indices of boundary spacetime as $\mu = 0, 1, \cdots, d - 1$. For $d = 2$, the metric of KS is given by a second-order derivative of geodesic length (kinematic measure) \[10\]. In a higher-dimensional CFT, the KS metric remains the second derivative of a similar logarithmic function \[9\], whose connection with the minimum surface is unclear. Moreover, the definition of KS for a general entangling surface is still an open question.

When a spin-1/2 electron adiabatically follows a magnetic field, a gauge field provides a Berry curvature to affect the motion. Nowadays, people discovered that this phenomenon is just a holonomy. The difference of a quantum state is by a U(1) phase. One can geometrize usual Quantum Mechanics through an emergent gauge potential (defines a covariant derivative). Because a quantum distance between different states should be invariant for a different holonomy, the Hilbert space has a U(1) quotient. Requiring the U(1) gauge invariance on Fubini-Study metric and Berry curvature can formulate the combination, Quantum Geometric Tensor (QGT) \[11\]. The QGT gives a beautiful unification of a metric and curvature. Recently, one provided a modular extension to a Berry curvature \[12, 13\], similar to replacing a Hamiltonian by $H_{\text{mod}}$. One could also show the equivalence between the modular Berry geometry and Riemann geometry on KS \[14, 15\]. Therefore, we expect to obtain KS and modular Berry curvature (MBC) from Quantum Modular Geometric Tensor \[11\]

$$
g_{jk}^{(n)}(\vec{\lambda}) \equiv \langle \partial_j n(\vec{\lambda})|\partial_k n(\vec{\lambda})\rangle - \langle \partial_j n(\vec{\lambda})|n(\vec{\lambda})\rangle\langle n(\vec{\lambda})|\partial_k n(\vec{\lambda})\rangle,
$$

2
where $|n(\vec{\lambda})\rangle$ is an eigenstate of $H_{\text{mod}}$, and the derivative of the eigenstate is given by

$$|\partial_j n(\vec{\lambda})\rangle \equiv \frac{\partial}{\partial \lambda_j} |n(\vec{\lambda})\rangle.$$  

The central question that we would like to address in this letter is the following: *What is the relation between KS and QMGT?*

In this letter, we realize $\text{KS}_{2d}/\text{CFT}_d$ correspondence through QMGT. Our result implies that KS and MBC are given by the symmetric and anti-symmetric parts of QMGT, respectively. On a CFT$_2$ vacuum state, we obtain a novel entanglement formula for two intervals of a general separation. This formula provides a new interpretation of connected Wilson lines from mutual information.

## 2 QMGT and Emergent Geometry

We first calculate QMGT in CFTs for a modular Hamiltonian with a spherical entangling surface, which satisfies the following algebra:

$$
[H_{\text{mod}}, H_{\text{mod}}] = 0; \\
[H_{\text{mod}}, \partial_{\nu,x} H_{\text{mod}}] = -2\pi i \partial_{\nu,x} H_{\text{mod}}; \\
[H_{\text{mod}}, \partial_{\nu,y} H_{\text{mod}}] = 2\pi i \partial_{\nu,y} H_{\text{mod}},
$$

where

$$
\partial_{\nu,x} \equiv \frac{\partial}{\partial x^\nu}, \quad \partial_{\nu,y} \equiv \frac{\partial}{\partial y^\nu}.
$$

The $x$ and $y$ are the tips of a causal diamond as in Fig. [1][9]. A direct calculation shows that the anti-symmetric part of QMGT is MBC. We then use CFT$_2$ and CFT$_3$ to demonstrate our method for extracting the metric of KS [8][9] from the symmetric part of QMGT and present the results in general dimensions.

### 2.1 QMGT in CFTs

Because $H_{\text{mod}}$ is diagonal for the eigenstate $|n\rangle$, we obtain that:

$$
0 = \partial_j (\langle m|H_{\text{mod}}|n\rangle) = E_{\text{mod}}^{(n)} \langle \partial_j m|n\rangle + E_{\text{mod}}^{(m)} \langle m|\partial_j n\rangle + \langle m|\partial_j H_{\text{mod}}|n\rangle, \; m \neq n;
$$

$$
\langle m|\partial_j n\rangle = \frac{\langle m|\partial_j H_{\text{mod}}|n\rangle}{E_{\text{mod}}^{(n)} - E_{\text{mod}}^{(m)}},
$$
where $E^{(n)}_{\text{mod}}$ is an eigenvalue of $H_{\text{mod}}$ for $|n\rangle$. Therefore, the QMGT becomes

$$g^{(n)}_{jk} = \sum_{m \neq n} \frac{\langle n | \partial_j H_{\text{mod}} | m \rangle \langle m | \partial_k H_{\text{mod}} | n \rangle}{(E^{(n)}_{\text{mod}} - E^{(m)}_{\text{mod}})^2}.$$  \hspace{1cm} (10)

Now we show $E^{(n)}_{\text{mod}} = E^{(m)}_{\text{mod}} \pm 2\pi i$ when $\langle m | \partial_{\nu,x,y} H_{\text{mod}} | n \rangle \neq 0$ as in the following:

$$\langle m | [H_{\text{mod}}, \partial_{\nu,x,y} H_{\text{mod}}] | n \rangle = (E^{(m)}_{\text{mod}} - E^{(n)}_{\text{mod}}) \langle m | \partial_{\nu,x,y} H_{\text{mod}} | n \rangle = \mp 2\pi i \langle m | \partial_{\nu,x,y} H_{\text{mod}} | n \rangle. \hspace{1cm} (11)$$

We use Eq. (7) in the second equality. This result implies $\langle n | \partial_{\nu,x,y} H_{\text{mod}} | n \rangle = 0$. Hence the QMGT is the same as a two-point function of $H_{\text{mod}}$ for the spherical case

$$g^{(n)}_{jk} = \sum_{m \neq n} \frac{\langle n | \partial_j H_{\text{mod}} | m \rangle \langle m | \partial_k H_{\text{mod}} | n \rangle}{(2\pi i)^2} = \frac{\langle n | \partial_j H_{\text{mod}} \partial_k H_{\text{mod}} | n \rangle}{(2\pi i)^2}. \hspace{1cm} (12)$$

When $\partial_j = \partial_{\mu,x}$ ($\partial_{\mu,y}$) and $\partial_k = \partial_{\nu,x}$ ($\partial_{\nu,y}$), we cannot find $|m\rangle$ to satisfy $E^{(n)}_{\text{mod}} - E^{(m)}_{\text{mod}} = \pm 2\pi i$. Therefore, it implies $\langle m | \partial_{\nu,x,y} H_{\text{mod}} | n \rangle = 0$. Hence we only need to consider $\partial_j = \partial_{\mu,x}$ ($\partial_{\mu,y}$) and $\partial_k = \partial_{\nu,y}$ ($\partial_{\nu,x}$).
2.2 KS and MBC

The anti-symmetric part of QMGT for $\partial_{\mu,x}$ and $\partial_{\mu,y}$ provides the only non-trivial component

$$R_{\mu,x;\nu,y} = -\frac{1}{4\pi^2} [\partial_{\mu,x} H_{\text{mod}}, \partial_{\nu,y} H_{\text{mod}}],$$

(13)

which is consistent with MBC [12]. Because MBC is an operator, it is unnecessary to calculate the expectation value by a regularization.

Now we calculate the symmetric part of QMGT. In general dimensions, the 2-point function of stress tensor shows that

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle = \frac{C_T I_{\mu\nu,\rho\sigma}}{x^{2d}},$$

(14)

where

$$I_{00,00} = 1 - \frac{1}{d}.\quad (15)$$

The $C_T$ is related to the $c$-anomaly $c_d$ as

$$C_T = \frac{(d + 1)!}{(d - 1)! d^d c_d}.\quad (16)$$

In the holographic CFT, the $c_d$ equals to the $a$-anomaly $a_d$ [2].

For a spherical entangling surface and $d > 1$, the modular Hamiltonian is [3]

$$H_{\text{mod}} = 2\pi \int_A d^{d-1}x' \frac{R^2 - |\vec{x} - \vec{x}'|^2}{2R} T_{00}(\vec{x}'),\quad (17)$$

where $T_{00}$ is the $(00)$-component of a stress tensor, and $R$ is the radius of the $(d - 1)$-dimensional sphere centered at $x$. When $d = 1$, we can the OPE block to define the modular Hamiltonian, but the integration variable becomes time [14, 15]. When the radius $R$ approaches zero, the modular Hamiltonian becomes:

$$H_{\text{mod}} \rightarrow \frac{4\pi^{\frac{d-1}{2}+1}R^d}{(d^2 - 1)\Gamma(\frac{d-1}{2})} T_{00} \equiv A_H \left( - (x^\mu - y^\mu)(x_\mu - y_\mu) \right)^{\frac{d}{2}} T_{00} = 2^d R^d A_H T_{00},\quad (18)$$

where

$$A_H = \frac{4\pi^{\frac{d-1}{2}+1}}{2^d (d^2 - 1)\Gamma(\frac{d-1}{2})}.$$  

(19)
Hence the 2-point function of $H_{\text{mod}}$ is different from the conformal block with asymptotic behavior $z^d$ in $z \to 0$ (the definition is in Eq. (23)) by following factor:

$$C_1 \equiv A_H^2 C_T I_{00,00} = A_H^2 C_T \left(1 - \frac{1}{d}\right). \quad (20)$$

Now we show a regularization method for extracting KS metric in CFT$_2$ and CFT$_3$. The 2-point function of $H_{\text{mod}}$ is given by:

$$\langle H_{\text{mod}}(x_1, x_2) H_{\text{mod}}(x_3, x_4) \rangle = \frac{c}{72} z^2 F_1(2, 2; 4, z) + \frac{c}{72} \bar{z}^2 F_1(2, 2; 4, \bar{z})$$

$$= \frac{c}{72} \left(6(1-z)(1-z) - 2z\right) + \frac{c}{72} \left(6(1-\bar{z})(1-\bar{z}) - 2\bar{z}\right), \quad (21)$$

where $c$ is a central charge. Now we do an expansion as the following:

$$x_1 = x_4 + \delta x; \quad x_3 = x_2 + \delta y. \quad (22)$$

Therefore, we obtain that:

$$u \equiv \frac{x_{12} x_{24}^2}{x_{13} x_{24}^2} = z \bar{z}$$

$$= \left(x_{42}^2 + 2(x_{42} \cdot \delta x) + (\delta x)^2\right) \left(x_{24}^2 + 2x_{24} \cdot \delta y + (\delta y)^2\right)$$

$$\left(x_{42}^2 + 2x_{42} \cdot (\delta x - \delta y) + (\delta x - \delta y)^2\right) x_{24}^2$$

$$= 1 + \frac{2}{x_{42}^2} \left(\eta_{\mu\nu} - \frac{2(x_{42} \mu)(x_{42} \nu)}{x_{42}^2}\right) \delta x^\mu \delta y^\nu + \cdots; \quad (23)$$

$$v = \frac{x_{14} x_{23}^2}{x_{13} x_{24}^2} = (1 - z)(1 - \bar{z}) \to 0,$$

where $x_{jk} \equiv x_j - x_k$. Now we introduce $z$ as that $z \equiv 1 + \epsilon - \bar{\epsilon}$, where $1 \gg \epsilon/\bar{\epsilon} \gg \epsilon$. Up to the first-order term of $\epsilon$ and $\bar{\epsilon}$, the $\bar{z}$ is one. The $\bar{\epsilon}$ is an additional regularization parameter to avoid the divergence. Therefore, we choose $\epsilon$ as that

$$\epsilon = \frac{2}{x_{42}^2} \left(\eta_{\mu\nu} - \frac{2(x_{42} \mu)(x_{42} \nu)}{x_{42}^2}\right) \delta x^\mu \delta y^\nu + \cdots. \quad (24)$$

We can observe the following universal term from $\langle 0| H_{\text{mod}}(x + \delta x, y) H_{\text{mod}}(y + \delta y, x)|0 \rangle$,

$$\langle 0| \partial_{\mu,x} H_{\text{mod}} \partial_{\nu,y} H_{\text{mod}} |0\rangle \delta x^\mu \delta y^\nu \sim \frac{c}{6} \epsilon \ln(\bar{\epsilon}). \quad (25)$$
Now we do the identification as that: \( x_1 = x + \delta x; \ x_2 = y; \ x_3 = y + \delta y; \ x_4 = x. \) Therefore, the symmetric part of QMGT is given by:

\[
ds^2 = \frac{\langle 0|\partial_{\mu,x}H_{\text{mod}}\partial_{\nu,y}H_{\text{mod}}|0 \rangle}{(2\pi i)^2} \delta x^\mu \delta y^\nu - \frac{1}{4\pi^2} \epsilon \ln(\bar{\epsilon}) \]

\[
= \frac{1}{4\pi^2} a_2 \ln(\bar{\epsilon}) \frac{4}{x_{42}^2} \left( -\eta_{\mu\nu} + \frac{2(x_{42})_{\mu}(x_{42})_{\nu}}{x_{42}^2} \right) \delta x^\mu \delta y^\nu,
\]

where \( a_2 = c/12. \)

The vacuum state is also the eigenstate of the modular Hamiltonian for the spherical case. Therefore, we can directly use the vacuum state to calculate QMFT. Hence we obtain KS metric from the symmetric part of QMGT. For the CFT\(_1\) and other even-dimensional CFTs, we can use the same regularization method to obtain KS.

For CFT\(_3\), the 2-point function of modular Hamiltonian is \( (G_{\Delta, l} \text{ is a global conformal block of conformal dimension } \Delta \text{ and spin } l, \text{ whose explicit form is obtained from the recursion relation}):\)

\[
\langle 0|H_{\text{mod}}(x_1, x_2)H_{\text{mod}}(x_3, x_4)|0 \rangle = C_1 G_{3, 2} = \frac{\pi}{2} a_3 \frac{z(z + 4\sqrt{1 - \bar{z}} - 8) - 8\sqrt{1 - \bar{z}} + 8}{z\sqrt{1 - \bar{z}}},
\]

When we consider the same expansion of \( \epsilon \) and \( \bar{\epsilon} \), the 2-point function of modular Hamiltonian for the linear term of \( \epsilon \) becomes

\[
\langle 0|H_{\text{mod}}(x + \delta x, y)H_{\text{mod}}(y + \delta y, x)|0 \rangle \sim \frac{\pi}{4} a_3 \left( \frac{\epsilon}{2\bar{\epsilon}^{3/2}} - 12\sqrt{\bar{\epsilon}} \epsilon - \frac{7\epsilon}{2\sqrt{\bar{\epsilon}}} + 8\epsilon \right).
\]

Because we have \( \bar{z} = 1 \) up to the first-order term of \( \epsilon \) and \( \bar{\epsilon} \), it is equivalent to doing a direct expansion of \( \epsilon \) and \( \bar{\epsilon} \) from \( C_1 G_{3, 2}/2 \). Ignoring the divergent terms, the symmetric part of QMGT also gives the KS metric

\[
ds^2 \sim \frac{1}{4\pi^2} \frac{\pi}{2} a_3 \frac{4}{x_{42}^2} \left( -\eta_{\mu\nu} + \frac{2(x_{42})_{\mu}(x_{42})_{\nu}}{x_{42}^2} \right) \delta x^\mu \delta y^\nu.
\]

One can use the same regularization in odd-dimensional CFTs, except for \( d = 1 \). Hence we have the regularization method for all dimensions. Because we always take limit \( z \to 1 \) for extracting KS metric, we only need the recursion relation of the conformal block \( G_{\Delta, l} \) at \( z = \bar{z} \) [1],

\[
(l + d - 3)(2\Delta + 2 - d)G_{\Delta, l}(z) = (d - 2)(\Delta + l - 1)G_{\Delta, l-2}(z) + \frac{2 - z}{2z}(2l + d - 4)(\Delta - d + 2)G_{\Delta+1, l-1}(z) - \frac{\Delta(2l + d - 4)(\Delta + 2 - d)(\Delta + 3 - d)(\Delta - l - d + 4)^2}{16(\Delta + 1 - \frac{d}{2})(\Delta - \frac{d}{2} + 2)(l - \Delta + d - 5)(l - \Delta + d - 3)}G_{\Delta+2, l-2}(z). \]

\[7\]
The boundary condition $G_{\Delta,1}$ and $G_{\Delta,0}$ are known in terms of generalized hypergeometric functions. After solving for $G_{d,2}$, one can do the regularization to obtain KS metric from the symmetric part of QMGT. The spacetime interval for $d > 1$, in general, is given by

\[ds^2 \sim \frac{1}{4\pi^2}(-1)^{\frac{d-1}{2}}a_d \ln(c) \frac{4}{x_{42}^2} \left(-\eta_{\mu\nu} + \frac{2(x_{42})_{\mu}(x_{42})_{\nu}}{x_{42}^2}\right) \delta x^\mu \delta y^\nu\]  

(31)

for the even-dimensional CFT;

\[ds^2 \sim \frac{1}{4\pi^2}(-1)^{\frac{d+1}{2}}\pi a_d \frac{4}{x_{42}^2} \left(-\eta_{\mu\nu} + \frac{2(x_{42})_{\mu}(x_{42})_{\nu}}{x_{42}^2}\right) \delta x^\mu \delta y^\nu\]  

(32)

for the odd-dimensional CFT. When $d = 1$, the spacetime interval is that

\[ds^2 \sim \frac{1}{4\pi^2}a_1 \frac{4}{x_{42}^2} \left(-\eta_{\mu\nu} + \frac{2(x_{42})_{\mu}(x_{42})_{\nu}}{x_{42}^2}\right) \delta x^\mu \delta y^\nu,\]  

(33)

where $a_1 \equiv a_2/2$.

### 3 Wilson Line

Here we want to establish the relation between KS metric [9] and Wilson lines correlator in the gauge formulation of 3d Einstein gravity. One can expand the logarithm of the expectation value of Wilson lines correlator as in the following:

\[
\ln\langle0|W(x_1,x_2)W(x_3,x_4)|0\rangle \\
\equiv \ln\left(\langle0|W(x_1,x_2)W(x_3,x_4)|0\rangle + \langle0|W(x_1,x_2)W(x_3,x_4)|0\rangle_{\text{connected}}\right) \\
= \ln\left(\langle0|W(x_1,x_2)|0\rangle\right) + \ln\left(\langle0|W(x_3,x_4)|0\rangle\right) + \frac{\langle0|W(x_1,x_2)W(x_3,x_4)|0\rangle_{\text{connected}}}{\langle0|W(x_1,x_2)|0\rangle\langle0|W(x_3,x_4)|0\rangle} \\
+ \mathcal{O}\left(\frac{1}{c}\right),
\]

(34)

where $W(x_1,x_2)$ is a Wilson line with two ending points, $x_1$ and $x_2$, and $\langle W(x_1,x_2)W(x_3,x_4)\rangle_{\text{connected}}$ is the connected part of the Wilson lines correlator. In the AdS$_3$ Einstein gravity theory, one realized the following identity [5] [6]

\[S_{EE} = \lim_{n \to 1} \frac{1}{1-n} \ln\langle0|W_K|0\rangle,\]  

(35)

where $S_{EE}$ is entanglement entropy of CFT$_2$, and $W_K$ is an $n$-sheet Wilson line operator. It is easy to show that

\[\ln\langle0|W_K|0\rangle = \frac{c}{6} (n-1) \ln\langle0|W|0\rangle + \cdots,\]  

(36)

8
where \( \cdots \) does not contribute to entanglement entropy under the one-sheet limit \( n \to 1 \). Hence we obtain that

\[
S_{EE} = -\frac{c}{6} \ln \langle 0 | W | 0 \rangle. \tag{37}
\]

The Wilson line operator, in general, depends on the conformal dimension \( h \). Using the result of entanglement entropy can fix the coefficient of \( h \):

\[
-\frac{c}{6} \ln \langle 0 | W(x_1, x_2) | 0 \rangle = \frac{ch}{3} \ln (x_1 - x_2)^2 = \frac{c}{3} \ln |x_1 - x_2|. \tag{38}
\]

Therefore, we obtain \( h = 1/2 \). We can also include the quantum correction of Wilson lines by the identification \( c = \tilde{c} + 26 \) \([5, 6]\), where \( \tilde{c} = 3l/(2G_3) \), \( l \equiv 1/\sqrt{-\Lambda} \), and \( G_3 \) is 3d gravitational constant. The \( \Lambda \) is a cosmological constant. Hence the one-loop correction of bulk gravity theory does not modify our result.

Now we define a new quantity from the connected correlator of Wilson lines:

\[
I[A : B] = S_{EE}[A] + S_{EE}[B] + \frac{c}{6} \ln \langle 0 | W(x_1, x_2) W(x_3, x_4) | 0 \rangle
= \frac{c}{6} \langle 0 | W(x_1, x_2) W(x_3, x_4) | 0 \rangle_{\text{connected}} + O(c^0), \tag{39}
\]

in which the ending points of the region \( A \) are \( x_1 \) and \( x_2 \) and the region \( B \) are \( x_3 \) and \( x_4 \). From the relation between Wilson line and the identity Virasoro OPE block \([4]\), we obtain the connected correlator

\[
\frac{c}{6} \langle W(x_1, x_2) W(x_3, x_4) \rangle_{\text{connected}} = \frac{1}{12} \, 2 \, 2 \, 2 \, 2 \, 2 \, 4 \, 4 \, \frac{1}{12} \, 2 \, 2 \, 2 \, 2 \, 2 \, 4 \, \bar{z} + O(1/c). \tag{40}
\]

Hence we obtain the relations between \( I \) or the correlator of Wilson lines and the two-points of \( H_{\text{mod}} \)

\[
\frac{c}{6} I[(x_1, x_2) : (x_3, x_4)] = \langle H_{\text{mod}}(x_1, x_2) H_{\text{mod}}(x_3, x_4) \rangle + O(c^0). \tag{41}
\]

From the computation of QMG, the KS metric is given by:

\[
\begin{align*}
\frac{ds^2}{\propto} & \sim -\left\langle 0 \left| \frac{\partial H_{\text{mod}}}{\partial x} \frac{\partial H_{\text{mod}}}{\partial y} \delta x \delta y \right| 0 \right\rangle \\
& \sim \langle 0 | H_{\text{mod}}(x - \delta x, y) H_{\text{mod}}(x, y + \delta y) | 0 \rangle - \langle 0 | H_{\text{mod}}(x - \delta x, y) H_{\text{mod}}(x, y) | 0 \rangle \\
& \quad - \langle 0 | H_{\text{mod}}(x, y) H_{\text{mod}}(x, y + \delta y) | 0 \rangle + \langle 0 | H_{\text{mod}}(x, y) H_{\text{mod}}(x, y) | 0 \rangle. \tag{42}
\end{align*}
\]
We can rewrite the kinematic space in terms of correlators of Wilson lines on a given time slice:

\[
\begin{align*}
\frac{c}{6}(I(L_1 \cup L_2 : L_2 \cup L_3) - I(L_1 \cup L_2 : L_2) - I(L_2 : L_2 \cup L_3) + I(L_2 : L_2)) \\
= \frac{c^2}{36} \left( \ln \left( \langle 0 | W(x - \delta x, y) W(x, y + \delta y) | 0 \rangle \right) - \ln \left( \langle 0 | W(x - \delta x, y) W(x, y) | 0 \rangle \right) \\
- \ln \left( \langle 0 | W(x, y) W(x, y + \delta y) | 0 \rangle \right) + \ln \left( \langle 0 | W(x, y) W(x, y) | 0 \rangle \right) \right),
\end{align*}
\]

where

\[
L_1 \equiv (x - \delta x, x), \quad L_2 \equiv (x, y), \quad L_3 \equiv (y, y + \delta y).
\] (44)

Because the KS metric on a time slice is also given by the second-order derivative of a geodesic line [10]:

\[
\begin{align*}
ds^2 &= \frac{\partial^2 S_{EE}(x, y)}{\partial x \partial y} \delta x \delta y \\
&\sim S_{EE}(x - \delta x, y) + S_{EE}(x, y + \delta y) - S_{EE}(x, y) - S_{EE}(x - \delta x, y + \delta y) \\
&= S_{EE}(L_1 \cup L_2) + S_{EE}(L_2 \cup L_3) - S_{EE}(L_2) - S_{EE}(L_1 \cup L_2 \cup L_3).
\end{align*}
\] (45)

Combining the above results, we obtain the following equivalence:

\[
\begin{align*}
S_{EE}(L_1 \cup L_2) + S_{EE}(L_2 \cup L_3) - S_{EE}(L_2) - S_{EE}(L_1 \cup L_2 \cup L_3) \\
&\sim \frac{c^2}{36} \left( \ln \left( \langle 0 | W(x - \delta x, y) W(x, y + \delta y) | 0 \rangle \right) - \ln \left( \langle 0 | W(x - \delta x, y) W(x, y) | 0 \rangle \right) \\
- \ln \left( \langle 0 | W(x, y) W(x, y + \delta y) | 0 \rangle \right) + \ln \left( \langle 0 | W(x, y) W(x, y) | 0 \rangle \right) \right) \\
&= \frac{c}{6}(I(L_1 \cup L_2 : L_2 \cup L_3) - I(L_1 \cup L_2 : L_2) - I(L_2 : L_2 \cup L_3) + I(L_2 : L_2)).
\end{align*}
\] (46)

We can observe that the above equality still holds if one does the following replacement

\[
\frac{c}{6} I(A : B) \rightarrow I_M(A : B) \equiv S_{EE}(A) + S_{EE}(B) - S_{EE}(A \cup B),
\] (47)

where \( I_M \) is the mutual information of the regions \( A \) and \( B \). The result is non-trivial because we do not assume that two intervals are far from each other and also include the quantum correction of Wilson lines. Our derivation only guarantees the equivalence up to the second-order of \( \delta x \) and \( \delta y \). Note that the connected Wilson line correlators do not vanish for two disjoint intervals with a far separation. Hence our result at least holds up to the one-loop correction for \( c \) (beyond the minimum surface).
4 Outlook

We showed that QMGT is a combination of KS metric (symmetric part) \[8, 9\] and MBC (anti-symmetric part) for a spherical case in all CFTs. Therefore, we realized the geometrization of CFTs, similar to the story of QGT \[11\]. Now one can obtain Quantum Kinematic Space by calculating the two-points of $H_{\text{mod}}$ without relying on conformal symmetry.

In CFT$_1$ and CFT$_2$, we can compute the KS metric for an excited state using the connected 2-point correlator of the identity Virasoro OPE block. On a vacuum state, this correlator reduces to the 2-point function of modular Hamiltonian \[40\]. The correlator becomes the identity Virasoro block in an excited state. The connected part is the global block of stress tensor after a coordinate transformation. It is not difficult to show that the resulting KS metric agrees with the known result.

We formulated the quantum relation between the connected correlator of Wilson lines and QMGT in the gauge formulation of 3d Einstein gravity. This relation provides a new Quantum Information interpretation to the connected correlator of Wilson lines as the mutual information. However, the derivation is only valid up to a one-loop quantum correction for $c$ and second-order expansion for $\delta x$ and $\delta y$. Hence verifying the following generalized form in the 3d Einstein gravity should teach us more about Quantum Gravity from Quantum Information:

$$\frac{c^2}{36} \left( \ln(\langle W_{L_1 \cup L_2} W_{L_2 \cup L_3} \rangle) - \ln(\langle W_{L_1} W_{L_2} W_{L_3} \rangle) \right)$$

$$= I_M(L_1 \cup L_2 : L_2 \cup L_3) - I_M(L_1 \cup L_2 : L_2) - I_M(L_2 : L_2 \cup L_3) + I_M(L_2 : L_2)$$

$$= S_{EE}(L_1 \cup L_2) + S_{EE}(L_2 \cup L_3) - S_{EE}(L_2) - S_{EE}(L_1 \cup L_2 \cup L_3). \quad (48)$$

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