A FAMILY OF SCHröDINGER OPERATORS WHOSE SPECTRUM IS AN INTERVAL

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Abstract. By approximation, I show that the spectrum of the Schrödinger operator with potential \( V(n) = f(n\rho \text{ (mod } 1)) \) for \( f \) continuous and \( \rho > 0 \), \( \rho \not\in \mathbb{N} \) is an interval.

1. Introduction

In this short note, I wish to describe a family of Schrödinger operators on \( l^2(\mathbb{N}) \) whose spectrum is an interval. To set the stage introduce for a bounded sequence \( V : \mathbb{N} \rightarrow \mathbb{R} \) and \( u \in l^2(\mathbb{N}) \) the Schrödinger operator \( H_V \) defined by

\[
(H_V u)(n) = u(n + 1) + u(n - 1) + V(n)u(n) \quad n \geq 2
\]

\[
(H_V u)(1) = u(2) + V(1)u(1).
\]

We will denote by \( \sigma(V) \) the spectrum of the operator \( H_V \).

It is well known that if \( V(n) \) is a sequence of independent identically distributed random variables with distribution \( \mu \) satisfying \( \text{supp}(\mu) = [a, b] \), we have that for almost every \( V \) the spectrum \( \sigma(V) \) is \([a-2, b+2] \). For the Almost–Mathieu Operator with potential

\[
V_{\lambda, \alpha, \theta}(n) = 2\lambda \cos(2\pi(n\alpha + \theta)),
\]

where \( \lambda > 0 \), \( \alpha \not\in \mathbb{Q} \), the set \( \sigma(V_{\lambda, \alpha, \theta}) \) contains no interval \([2] \).

Bourgain conjectured in \([4]\), that if one considers the potential

\[
V_{\lambda, \alpha, x, y}(n) = \lambda \cos \left( \frac{n(n - 1)}{2} \alpha + nx + y \right)
\]

with \( \lambda > 0 \), \( \alpha \not\in \mathbb{Q} \), the spectrum \( \sigma(V_{\lambda, \alpha, x, y}) \) is an interval.

Denote by \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) the circle. I will prove the following result

**Theorem 1.1.** For any continuous function \( f : \mathbb{T} \rightarrow \mathbb{R} \), any \( \alpha \neq 0, \theta \), and \( \rho > 0 \) not an integer, introduce the potential

\[
V(n) = f(an\rho + \theta).
\]

Then we have that

\[
\sigma(V) = [\min(f) - 2, \max(f) + 2].
\]
Potentials of the type (1.4), where already discussed in Bourgain [3], Griniasty–Fishman [6], Last–Simon [7], and Stolz [8]. In particular, the case $0 < \rho < 1$ is due to Stolz [8] under an additional regularity assumption on $f$. The proof of this theorem depends essentially on the following lemma on the distribution of $n^\rho$, which is a consequence of a result of Boshernitzan [5].

**Lemma 1.2.** Let $r \geq 0$ be an integer and $r < \rho < r + 1$. Given any $\alpha \neq 0$, $\theta$, $K \geq 1$, $\varepsilon > 0$, $a_0, \ldots, a_r$, there exists an integer $n \geq 1$ such that

\begin{equation}
(1.6) \sup_{|k| \leq K} \|\alpha(n + k)^\rho + \theta - \sum_{j=0}^r a_j k^j\| \leq \varepsilon,
\end{equation}

where $\|x\| = \text{dist}(x, \mathbb{Z})$ denotes the distance to the closest integer.

We will prove this lemma in the next section. $\|\|$ is not a norm, but it obeys the triangle inequality

\begin{equation}
(1.7) \|x + y\| \leq \|x\| + \|y\|
\end{equation}

for any $x, y \in \mathbb{R}$. In particular for any integer $N$, we have that $\|Nx\| \leq |N|\|x\|$.

**Proof of Theorem 1.1.** By Lemma 1.2 we can find for any $x \in [0, 1)$ a sequence $n_l$ such that

\[ \sup_{|k| \leq l} \|\alpha(n_l + k)^\rho + \theta - x\| \leq 1/l. \]

Hence, the sequence $V_l(n) = V(n - n_l)$ converges pointwise to $f(x)$. The claim now follows from a Weyl–sequence argument. \hfill $\square$

It is remarkable that combined with the Last–Simon semicontinuity of the absolutely continuous spectrum [7], one also obtains the following result

**Theorem 1.3.** For $r \geq 0$ an integer, $r < \rho < r + 1$, and $f$ a continuous function on $\mathbb{T}$, introduce the set $\mathcal{B}_r(f)$ as

\begin{equation}
(1.8) \quad \mathcal{B}_r(f) = \bigcap_{\alpha \neq 0, \ldots, a_r} \sigma_{ac}(f(\sum_{j=0}^r a_j n^j)).
\end{equation}

Then for $\alpha \neq 0$ and any $\theta$

\begin{equation}
(1.9) \quad \sigma_{ac}(f(\alpha n^\rho + \theta)) \subseteq \mathcal{B}_r(f).
\end{equation}

Here $\sigma_{ac}(V)$ denotes the absolutely continuous spectrum of $H_V$.

We note that for $r = 0$, we have that

\begin{equation}
(1.10) \quad \mathcal{B}_0(f) = [-2 + \max(f), 2 - \min(f)].
\end{equation}

Under additional regularity assumptions on $f$ and $r = 0$, Stolz has shown in [8] that we have equality in (1.9).

Furthermore note that

\begin{equation}
(1.11) \quad \mathcal{B}_{r+1}(f) \subseteq \mathcal{B}_r(f).
\end{equation}

In [7], Last and Simon have stated the following conjecture

\begin{equation}
(1.12) \quad \mathcal{B}_1(2\lambda \cos(2\pi \cdot)) = \emptyset
\end{equation}
for $\lambda > 0$. They phrased this in poetic terms as 'Does Hofstadter’s Butterfly have wings?'. The best positive result in this direction as far as I know, is by Bourgain [3] showing

$$|B_1(2\lambda \cos(2\pi x))| \to 0, \quad \lambda \to 0.$$  

It is an interesting question if Theorem 1.1 holds for $\rho \geq 2$ an integer. In the particular case of $f(x) = 2\lambda \cos(2\pi x)$, $\rho = 2$, this would follow from Bourgain’s conjecture. However, there is also the following negative evidence. Consider the skew-shift $T : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$T^n(x, y) = (x + n\alpha, n\alpha + ny),$$

where $\alpha \not\in \mathbb{Q}$. Then Avila, Bochi, and Damanik [1] have shown that for generic continuous $f : \mathbb{T}^2 \to \mathbb{R}$, the spectrum $\sigma(f(T^n(x, y)))$ contains no interval. So, it is not clear what to expect in this case.

As a final remark, let me comment on a slight extension. If one replaces $V$ by the following family of potentials

$$V(n) = f(\alpha n^\rho + \sum_{k=1}^{K} \alpha_k n^{\beta_k}),$$

where $\beta_k < \rho$ and $\alpha_k$ are any numbers, then Theorem 1.1 and 1.3 remain valid.

2. PROOF OF LEMMA 1.2

Let in the following $r$ be an integer such that $r < \rho < r+1$. By Taylor expansion, we have that

$$\alpha(n+k)^\rho = \sum_{j=0}^{r} x_j(n)k^j + \alpha \frac{\rho \cdots (\rho-r)}{(r+1)!} (n + \xi)^{\rho-r-1}k^{r+1}$$

for some $|\xi| \leq k$ and

$$x_j(n) = \alpha \frac{\rho \cdots (\rho-j+1)}{j!} n^{\rho-j}.$$  

We now first note the following lemma

**Lemma 2.1.** For any $K \geq 1$ and $\varepsilon > 0$, there exists an $N_0(K, \varepsilon)$ such that

$$|\alpha(n+k)^\rho - \sum_{j=0}^{r} x_j(n)k^j| \leq \varepsilon$$

for $|k| \leq K$ and $n \geq N_0(K, \varepsilon)$.

**Proof.** This follows from (2.1) and that $\rho - r - 1 < 0$.  

A sequence $x(n)$ is called uniformly distributed in $\mathbb{T}^{r+1}$ if for any $0 \leq a_j < b_j \leq 1$, $j = 0, \ldots, r$, we have that

$$\lim_{n \to \infty} \frac{1}{n} \#\{1 \leq k \leq n : x_j(k) \in (a_j, b_j), \quad j = 0, \ldots, r\} = \prod_{j=0}^{r}(b_j - a_j).$$
If \( x(n) \) is a sequence in \( \mathbb{R}^{r+1} \), we can view it as a sequence in \( \mathbb{T}^{r+1} \) by considering \( x(n) \mod 1 \), and call it uniformly distributed if \( x(n) \mod 1 \) is. We need the following consequence of Theorem 1.8 in [5].

**Theorem 2.2** (Boshernitzan). Let \( (f_1, \ldots, f_s) \) be functions \( \mathbb{R} \to \mathbb{R} \) of subpolynomial growth, that is, there is an integer \( N \) such that

\[
\lim_{x \to \infty} f_j(x)x^{-N} = 0, \quad 1 \leq j \leq s.
\]

Then the following two conditions are equivalent

(i) The sequence \( \{f_1(n), \ldots, f_s(n)\}_{n \geq 1} \) is uniformly distributed in \( \mathbb{T}^s \).

(ii) For any \( (m_1, \ldots, m_s) \in \mathbb{Z}^s \setminus \{0\} \), and for every polynomial \( p(x) \) with rational coefficients, we have that

\[
\lim_{x \to \infty} \frac{\sum_{j=1}^s m_j f_j(x) - p(x)}{\log(x)} = \pm \infty.
\]

We will use the following consequence of this theorem

**Lemma 2.3.** The sequence

\[
x(n) = (x_0(n) \ldots x_r(n))
\]

is uniformly distributed in \( \mathbb{T}^{r+1} \).

**Proof.** This follows from the fact that for any polynomial \( p(n) \) and integer vector \( (m_0, \ldots, m_r) \) we have that

\[
|\sum_{j=0}^r m_j x_j(n) - p(n)|
\]

grows at least like \( n^{a-r} \), which grows faster than \( \log(n) \).

Now we come to

**Proof of Lemma 1.2.** By Lemma 2.1 there exists an \( N_0 \) such that

\[
||\alpha(n+k)\rho - \sum_{j=0}^r x_j(n)\rho^j|| \leq \frac{\varepsilon}{2}
\]

for any \( n \geq N_0 \) and \( |k| \leq K \). By Lemma 2.3 we can now find \( n \geq N_0 \) such that

\[
||x_l(n) - \tilde{a}_l|| \leq \frac{\varepsilon}{2(r+1)K^r}, \quad l = 0, \ldots, r,
\]
where \( \tilde{a}_0 = a_0 - \theta \) and \( \tilde{a}_l = a_l \), \( l \geq 1 \). For \( |k| \leq K \), we now have that using (1.7)

\[
\| \alpha(n + k)^\rho + \theta - \sum_{j=0}^r a_j k^j \|
\]

\[
\leq \| \alpha(n + k)^\rho - \sum_{j=0}^r x_j(n) k^j \| + \| \sum_{j=0}^r x_j(n) k^j - \sum_{j=0}^r a_j k^j + \theta \|
\]

(2.8)

\[
\leq \frac{\varepsilon}{2} + \sum_{j=0}^r \| (x_j(n) - \tilde{a}_j) k^j \|
\]

(2.9)

\[
\leq \frac{\varepsilon}{2} + \sum_{j=0}^r |k^j|\|x_j(n) - \tilde{a}_j\|
\]

\[
\leq \frac{\varepsilon}{2} + \sum_{j=0}^r K^j \frac{\varepsilon}{2(r + 1) K^j} = \varepsilon,
\]

where we used the definition of \( \tilde{a}_l \) in (2.8), and that \( k \) is an integer in (2.9). This finishes the proof. \( \square \)

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