LIFTING SMOOTH CURVES OVER INVARIANTS FOR REPRESENTATIONS OF COMPACT LIE GROUPS, II

ANDREAS KRIEGL, MARK LOSIK, PETER W. MICHR, AND ARMIN RAINE R

Abstract. Any sufficiently often differentiable curve in the orbit space of a compact Lie group representation can be lifted to a once differentiable curve into the representation space.

1. Introduction

In [2] the following problem was investigated. Consider an orthogonal representation of a compact Lie group $G$ on a real finite dimensional Euclidean vector space $V$. Let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on $V$. Then the mapping $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$ induces a homeomorphism between the orbit space $V/G$ and the semialgebraic set $\sigma(V)$. Suppose a smooth curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space is given (smooth as curve in $\mathbb{R}^n$), does there exist a smooth lift to $V$, i.e., a smooth curve $\tilde{c} : \mathbb{R} \to V$ with $c = \sigma \circ \tilde{c}$?

It was shown in [2] that a real analytic curve in $V/G$ admits a local real analytic lift to $V$, and that a smooth curve in $V/G$ admits a global smooth lift, if certain genericity conditions are satisfied. In both cases the lifts may be chosen orthogonal to each orbit they meet and then they are unique up to a transformation in $G$, whenever the representation of $G$ on $V$ is polar, i.e., admits sections.

In this paper we treat the same problem under weaker differentiability conditions for $c : \mathbb{R} \to V/G$ and without the mentioned genericity conditions. In section 3 we show that a continuous curve in the orbit space $V/G$ allows a global continuous lift to $V$. As a consequence we can prove in section 4 that a sufficiently often differentiable curve in $V/G$ can be lifted to a once differentiable curve in $V$. What we mean by sufficiently often differentiable will be specified there.

In the special case that the symmetric group $S_n$ is acting on $\mathbb{R}^n$, in other words (see [2]), if smooth parameterizations of the roots of smooth curves of polynomials with all roots real are looked for, the following results were proved in [5]: Any differentiable lift of a $C^{2n}$-curve (of polynomials) $c : \mathbb{R} \to \mathbb{R}^n/S_n$ is actually $C^1$, and there always exists a twice differentiable but in general not better lift of $c$, if it is of class $C^{3n}$. Note that here the differentiability assumptions on $c$ are not the weakest possible which is shown by the case $n = 2$, elaborated in [1] 2.1. The proof there is based on the fact that the roots of a $C^n$-curve of polynomials $c : \mathbb{R} \to \mathbb{R}^n/S_n$ may be chosen differentiable with locally bounded derivative; this is due to Bronshtein [6] and Wakabayashi [12]. Therefore, our long-term objective is to prove the existence of a twice differentiable lift also in the general setting. The key is the generalization of Bronshtein’s and Wakabayashi’s result which seems to be difficult.
The polynomial results have applications in the theory of partial differential equations and perturbation theory, see [3].

2. Preliminaries

2.1. The setting. Let \( G \) be a compact Lie group and let \( \rho : G \to O(V) \) be an orthogonal representation in a real finite dimensional Euclidean vector space \( V \) with inner product \( \langle \cdot \mid \cdot \rangle \). By a classical theorem of Hilbert and Nagata, the algebra \( \mathbb{R}[V]^G \) of invariant polynomials on \( V \) is finitely generated. So let \( \sigma_1, \ldots, \sigma_n \) be a system of homogeneous generators of \( \mathbb{R}[V]^G \) of positive degrees \( d_1, \ldots, d_n \). We may assume that \( \sigma_1 : v \mapsto \langle v|v \rangle \) is the inner product. Consider the \textit{orbit map} \( \sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n \). Note that, if \( (y_1, \ldots, y_n) = \sigma(v) \) for \( v \in V \), then \( (t^{d_1}y_1, \ldots, t^{d_n}y_n) = \sigma(tv) \) for \( t \in \mathbb{R} \), and that \( \sigma^{-1}(0) = \{0\} \). The image \( \sigma(V) \) is a semialgebraic set in the categorical quotient \( V/G := \{ y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I \} \) where \( I \) is the ideal of relations between \( \sigma_1, \ldots, \sigma_n \). Since \( G \) is compact, \( \sigma \) is proper and separates orbits of \( G \), it thus induces a homeomorphism between \( V/G \) and \( \sigma(V) \).

2.2. The slice theorem. For a point \( v \in V \) we denote by \( G_v \) its isotropy group and by \( N_v = T_v(G.v)^\perp \) the normal subspace of the orbit \( G.v \) at \( v \). It is well known that there exists a \( G \)-invariant open neighborhood \( U \) of \( v \) which is real analytically \( G \)-isomorphic to the crossed product (or associated bundle) \( G \times_{G_v} S_v = (G \times S_v)/G_v \), where \( S_v \) is a ball in \( N_v \) with center at the origin. The quotient \( U/G \) is homeomorphic to \( S_v/G_v \). It follows that the problem of local lifting curves in \( V/G \) passing through \( \sigma(v) \) reduces to the same problem for curves in \( N_v/G_v \) passing through \( 0 \). For more details see [2], [8] and [10], theorem 1.1.

A point \( v \in V \) (and its orbit \( G.v \) in \( V/G \)) is called regular if the isotropy representation \( G_v \to O(N_v) \) is trivial. Hence a neighborhood of this point is analytically \( G \)-isomorphic to \( G/G_v \times S_v \cong G.v \times S_v \). The set \( V_{\text{reg}} \) of regular points is open and dense in \( V \), and the projection \( V_{\text{reg}} \to V_{\text{reg}}/G \) is a locally trivial fiber bundle. A non regular orbit or point is called singular.

2.3. Removing fixed points. Let \( V^G \) be the space of \( G \)-invariant vectors in \( V \), and let \( V' \) be its orthogonal complement in \( V \). Then we have \( V = V^G \oplus V' \), \( \mathbb{R}[V]^G = \mathbb{R}[V^G] \otimes \mathbb{R}[V']^G \) and \( V/G = V^G \times V'/G \).

\textbf{Lemma.} Any lift \( \bar{c} \) of a curve \( c = (c_0, c_1) \) of class \( C^k \) (\( k = 0, 1, \ldots, \infty, \omega \)) in \( V^G \times V'/G \) has the form \( \bar{c} = (\bar{c}_0, \bar{c}_1) \), where \( \bar{c}_1 \) is a lift of \( c_1 \) to \( V' \) of class \( C^\omega \) (\( k = 0, 1, \ldots, \infty, \omega \)). The lift \( \bar{c} \) is orthogonal if and only if the lift \( \bar{c}_1 \) is orthogonal. \( \square \)

2.4. Multiplicity. For a continuous function \( f \) defined near 0 in \( \mathbb{R} \), let the \textit{multiplicity} or \textit{order of flatness} \( m(f) \) at 0 be the supremum of all integers \( p \) such that \( f(t) = t^pg(t) \) near 0 for a continuous function \( g \). If \( f \) is \( C^n \) and \( m(f) < n \), then \( f(t) = t^{m(f)} g(t) \), where now \( g \) is \( C^{n-m(f)} \) and \( g(0) \neq 0 \). Similarly, one can define multiplicity of a function at any \( t \in \mathbb{R} \).

\textbf{Lemma.} Let \( c = (c_1, \ldots, c_n) \) be a curve in \( \sigma(V) \subseteq \mathbb{R}^n \), where \( c_i \) is \( C^{d_i} \), for \( 1 \leq i \leq n \), and \( c(0) = 0 \). Then the following two conditions are equivalent:

1. \( c_1(t) = t^2c_{1,1}(t) \) near 0 for a continuous function \( c_{1,1} \);
2. \( c_i(t) = t^{d_i}c_{i,i}(t) \) near 0 for a continuous function \( c_{i,i} \), for all \( 1 \leq i \leq n \).

\textbf{Proof.} The proof of the nontrivial implication (1) \( \Rightarrow \) (2) is the same as in the smooth case with \( r = 1 \), see [2] 3.3. for details. \( \square \)
3. Lifting continuous curves over invariants

Proposition 3.1. Let $c = (c_1, \ldots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be continuous. Then there exists a global continuous lift $\bar{c} : \mathbb{R} \to V$ of $c$.

This result is due to Montgomery and Yang [7] see also [3]. We present a short proof adapted to our setting:

Proof. We will make induction on the size of $G$. More precisely, for two compact Lie groups $G'$ and $G$ we denote $G' < G$, if

- $\dim G' < \dim G$ or
- if $\dim G' = \dim G$, then $G'$ has less connected components than $G$ has.

In the simplest case, when $G = \{e\}$ is trivial, we find $\sigma(V) = V/G = V$, whence we can put $\bar{c} := c$.

Let us assume that for any $G' < G$ and any continuous $c : \mathbb{R} \to V/G'$ there exists a global continuous lift $\bar{c} : \mathbb{R} \to V$ of $c$, and there is a nontrivial fixed point $\bar{c}(t)$ of $\bar{c}$ for all $t \in \mathbb{R}$.

We shall prove that then the same is true for $G$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be continuous. By lemma 2.4 we may remove the nontrivial fixed points of the $G$-action on $V$ and suppose that $V/G = \{0\}$. The set $c^{-1}(0)$ is closed in $\mathbb{R}$ and, consequently, $c^{-1}(\sigma(V) \setminus \{0\}) = \mathbb{R} \setminus c^{-1}(0)$ is open in $\mathbb{R}$. Thus, we can write $c^{-1}(\sigma(V) \setminus \{0\}) = \bigcup_{i \in I} (a_i, b_i)$, a disjoint union, where $a_i, b_i \in \mathbb{R} \cup \{\pm \infty\}$ with $a_i < b_i$ such that each $(a_i, b_i)$ is maximal with respect to not containing zeros of $c$, and $I$ is an at most countable set of indices. In particular, we have $c(a_i) = c(b_i) = 0$ for all $a_i, b_i \in \mathbb{R}$ appearing in the above presentation.

We assert that on each $(a_i, b_i)$ there exists a continuous lift $\bar{c} : (a_i, b_i) \to V \setminus \{0\}$ of the restriction $c|_{(a_i, b_i)} : (a_i, b_i) \to \sigma(V) \setminus \{0\}$. In fact, since $V/G = \{0\}$, for all $v \in V \setminus \{0\}$ the isotropy groups $G_v$, acting orthogonally on $N_v$, satisfy $G_v < G$. Therefore, by induction hypothesis and by 2.2 we find local continuous lifts of $c|_{(a_i, b_i)}$ near any $t \in (a_i, b_i)$ and through all $v \in \sigma^{-1}(c(t))$. Suppose $\bar{c}_1 : (a_i, b_i) \to V \setminus \{0\}$ is a local continuous lift of $c|_{(a_i, b_i)}$ with maximal domain $(a, b)$, where, say, $b < b_i$. Then there exists a local continuous lift $\bar{c}_2$ of $c|_{(a_i, b_i)}$ near $b$, and there is a $t_0 < b$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near $t_0$. Since $\bar{c}_1(t_0)$ and $\bar{c}_2(t_0)$ lie in the same orbit, there must exist a $g \in G$ such that $\bar{c}_1(t_0) = g \cdot \bar{c}_2(t_0)$. But then,

$$\bar{c}_{12}(t) := \begin{cases} \bar{c}_1(t) & \text{for } t \leq t_0 \\ g \cdot \bar{c}_2(t) & \text{for } t \geq t_0 \end{cases}$$

is a local continuous lift of $c|_{(a_i, b_i)}$ defined on a larger interval than $\bar{c}_1$. Thus we have shown that each local continuous lift of $c|_{(a_i, b_i)}$ defined on an open interval $(a, b) \subseteq (a_i, b_i)$ can be extended to a larger interval whenever $(a, b) \subseteq (a_i, b_i)$. This proves the assertion.

We put $\bar{c}|_{t_0-1} := 0$, since, by $\sigma^{-1}(0) = \{0\}$, this is the only choice. Then $\bar{c}$ is also continuous at points $t_0 \in c^{-1}(0)$ since $\langle \bar{c}(t) | \bar{c}(t) \rangle = \langle \sigma_1(\bar{c}(t)) \rangle = c_1(t)$ converges to 0 as $t \to t_0$.

4. Lifting differentially

Throughout the whole section we let $d \geq 2$ be the maximum of all degrees of systems of minimal generators of invariant polynomials of all slice representations of $\rho$. Of these there are only finitely many isomorphism types.

Lemma 4.1. A curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class $C^d$ admits an orthogonal $C^d$-lift $\bar{c}$ in a neighborhood of a regular point $c(t_0) \in V_{\text{reg}}/G$. It is unique up to a transformation from $G$. 

Proof. The proof works analogously as in the smooth case, see [2] 3.1. □

**Theorem 4.2.** Let \( c = (c_1, \ldots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n \) be a curve of class \( C^d \). Then for any \( t_0 \in \mathbb{R} \) there exists a local lift \( \tilde{c} \) of \( c \) near \( t_0 \) which is differentiable at \( t_0 \).

Proof. We follow partially the algorithm given in [2] 3.4. Without loss of generality we may assume that \( t_0 = 0 \). We show the existence of local lifts of \( c \) which are differentiable at \( 0 \) through any \( v \in \sigma^{-1}(c(0)) \). By lemma [2] we can assume \( V^G = \{0\} \).

If \( c(0) \neq 0 \) corresponds to a regular orbit, then unique orthogonal \( C^d \) lifts defined near \( 0 \) exist through all \( v \in \sigma^{-1}(c(0)) \), by lemma [2].

If \( c(0) = 0 \), then \( c_1 \) must vanish of at least second order at \( 0 \), since \( c_1(t) \geq 0 \) for all \( t \in \mathbb{R} \). That means \( c_1(t) = t^2c_{1,1}(t) \) near \( 0 \) for a continuous function \( c_{1,1} \) since \( c_1 \) is \( C^2 \). By the multiplicity lemma [2] we find that \( c_i(t) = t^{d_i}c_{i,i}(t) \) near \( 0 \) for \( 1 \leq i \leq n \), where \( c_{1,1}, c_{2,2}, \ldots, c_{n,n} \) are continuous functions. We consider the following curve in \( \sigma(V) \) which is continuous since \( \sigma(V) \) is closed in \( \mathbb{R}^n \), see [2]:

\[
c_i(t) : = (c_{1,1}(t), c_{2,2}(t), \ldots, c_{n,n}(t)) = (t^{-2}c_1(t), t^{-d_2}c_2(t), \ldots, t^{-d_n}c_n(t)).
\]

By proposition 3.1 there exists a continuous lift \( \tilde{c}_i(t) \) of \( c_i(t) \). Thus, \( \tilde{c}(t) := t \cdot \tilde{c}_i(t) \) is a local lift of \( c \) near \( 0 \) which is differentiable at \( 0 \):

\[
\sigma(\tilde{c}(t)) = \sigma(t \cdot \tilde{c}_i(t)) = (t^2c_{1,1}(t), \ldots, t^{d_n}c_{n,n}(t)) = c(t),
\]

and

\[
\lim_{t \to 0} \frac{t \cdot \tilde{c}_i(t)}{t} = \lim_{t \to 0} \tilde{c}_i(t) = \tilde{c}_i(0).
\]

Note that \( \sigma^{-1}(0) = \{0\} \), therefore we are done in this case.

If \( c(0) \neq 0 \) corresponds to a singular orbit, let \( v \) be in \( \sigma^{-1}(c(0)) \) and consider the isometry representation \( G_v \to O(N_v) \). By [2] the lifting problem reduces to the same problem for curves in \( N_v/G_v \) now passing through \( 0 \). □

**Lemma 4.3.** Consider a continuous curve \( c : (a,b) \to X \) in a compact metric space \( X \). Then the set \( A \) of all accumulation points of \( c(t) \) as \( t \searrow a \) is connected.

Proof. On the contrary suppose that \( A = A_1 \cup A_2 \), where \( A_1 \) and \( A_2 \) are disjoint open and closed subsets of \( A \). Since \( A \) is closed in \( X \), also \( A_1 \) and \( A_2 \) are closed in \( X \). There exist disjoint open subsets \( A'_1, A'_2 \subseteq X \) with \( A_1 \subseteq A'_1 \) and \( A_2 \subseteq A'_2 \). Consider \( F := X \setminus (A'_1 \cup A'_2) \) which is closed in \( X \) and hence compact. Since \( c \) is continuous, \( c^{-1}(A'_1) \) and \( c^{-1}(A'_2) \) are disjoint and open in \( \mathbb{R} \), there exists a sequence \( t_m \to a \) and \( c(t_m) \in F \) for all \( m \). By compactness of \( F \), this sequence has a cluster point \( y \) in \( F \). Hence \( y \) is in \( A \) by definition, which contradicts \( F \cap A = \emptyset \). □

**Theorem 4.4.** Let \( c = (c_1, \ldots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n \) be a curve of class \( C^d \). Then there exists a global differentiable lift \( \tilde{c} : \mathbb{R} \to V \) of \( c \).

Proof. The proof, as the one of proposition 3.1, will be carried out by induction on the size of \( G \).

If \( G = \{e\} \) is trivial, then \( \tilde{c} := c \) is a global differentiable lift.

So let us assume that for any \( G' < G \) and any \( c : \mathbb{R} \to V/G' \) satisfying the differentiability conditions of the theorem there exists a global differentiable lift \( \tilde{c} : \mathbb{R} \to V \) of \( c \), where \( G' \to O(V) \) is an orthogonal representation on an arbitrary real finite dimensional Euclidean vector space \( V \).

We shall prove that the same is true for \( G \). Let \( c = (c_1, \ldots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n \) be of class \( C^d \). We may assume that \( V^G = \{0\} \), by lemma [2]. As in the
proof of proposition \( \mathbb{4} \) we can write \( e^{-1}(\sigma(V) \setminus \{0\}) = \bigcup_{i}(a_i, b_i) \), a disjoint union, where \( a_i, b_i \in \mathbb{R} \cup \{ \pm \infty \} \) with \( a_i < b_i \). In particular, we have \( c(a_i) = c(b_i) = 0 \) for all \( a_i, b_i \in \mathbb{R} \) appearing in the above presentation.

Claim: On each \((a_i, b_i)\) there exists a differentiable lift \( \tilde{c} : (a_i, b_i) \rightarrow V \setminus \{0\} \) of the restriction \( c_{\{a_i, b_i\}} : (a_i, b_i) \rightarrow \sigma(V) \setminus \{0\} \). The lack of nontrivial fixed points guarantees that for all \( v \in V \setminus \{0\} \) the isotropy groups \( G_v \) acting on \( N_v \) satisfy \( G_v < G \). Therefore, by induction hypothesis and by \( \mathbb{2} \) we find local differentiable lifts of \( c_{\{a_i, b_i\}} \) near any \( t \in (a_i, b_i) \) and through all \( v \in \sigma^{-1}(c(t)) \). Suppose that \( \tilde{c}_1 : (a_i, b_i) \rightarrow V \setminus \{0\} \) is a local differentiable lift of \( c_{\{a_i, b_i\}} \) with maximal domain \((a, b)\), where, say, \( b < b_i \). Then there exists a local differentiable lift \( \tilde{c}_2 \) of \( c_{\{a_i, b_i\}} \) near \( b \), and there exists a \( t_0 < b \) such that both \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are defined near \( t_0 \). We may assume without loss that \( \tilde{c}_1(t_0) = \tilde{c}_2(t_0) = v_0 \), by applying a transformation \( g \in G \) to \( c_2 \), say. We want to show that we can arrange the lift \( \tilde{c}_2 \) in such a way that its derivative at \( t_0 \) matches with the derivative of \( \tilde{c}_1 \) at \( t_0 \). We decompose \( \tilde{c}_2(t_0) = \tilde{c}_2^\top(t_0) + \tilde{c}_2^\perp(t_0) \) into the parts tangent to the orbit \( G.v_0 \) and normal to it.

First we deal with the normal parts \( \tilde{c}_2^\perp(t_0) \subseteq V \). We consider the projection \( p : G.S_{v_0} \cong G \times_{G_{v_0}} S_{v_0} \rightarrow G/G_{v_0} \cong G.v_0 \) of the fiber bundle associated to the principal bundle \( \pi : G \rightarrow G/G_{v_0} \). Then, for \( t \) close to \( t_0 \), \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are differentiable curves in \( G.S_{v_0} \), whence \( p \circ \tilde{c}_i \) (\( i = 1, 2 \)) are differentiable curves in \( G/G_{v_0} \) which admit differentiable lifts \( g_i \) into \( G \) with \( g_i(t_0) = e \) (via the horizontal lift of a principal connection, say). Consequently, \( t \mapsto g_i(t)^{-1}.\tilde{c}_i(t) = : \tilde{c}_i(t) \) are differentiable lifts of \( c_{\{a_i, b_i\}} \) near \( t_0 \) which lie in \( S_{v_0} \), whence \( \tilde{c}_2(t_0) = \frac{d}{dt}_{t=t_0} (g(t)^{-1}.\tilde{c}_1(t)) = -g_1'(t_0).v_0 + \tilde{c}_2^\top(t_0) \in N_{v_0} \). So, \( \tilde{c}_2(t_0)^\top = (g_1'(t_0).v_0)^\top = g_1'(t_0).v_0 \), and so for the normal part we get \( \tilde{c}_2^\perp(t_0) = \tilde{c}_2^\perp(t_0) \).

Since \( \tilde{c}_i \) lie in \( S_{v_0} \) we can change to the isotropy representation \( G.v_0 \rightarrow O(N_{v_0}) \) (using the same letters \( \sigma_i \) for the generators of \( \mathbb{R}[N_{v_0}]^{G_{v_0}} \). We can suppose that \( v_0 = 0 \), i.e., \( c(t_0) = 0 \).

Recall the continuous curve in \( \sigma(V) \) defined in the proof of theorem \( \mathbb{4} \) which depends on the point \( t_0 \):

\[
c(1,t_0)(t) := (t - t_0)^{-2}c_1(t), (t - t_0)^{-d_2}c_2(t), \ldots, (t - t_0)^{-d_n}c_n(t).
\]

We find that for \( i = 1, 2 \):

\[
\sigma(\tilde{c}_i(t_0)) = \sigma \left( \lim_{t \to t_0} \frac{\tilde{c}_i(t) - \tilde{c}_i(t_0)}{t - t_0} \right) = \lim_{t \to t_0} \sigma \left( \frac{\tilde{c}_i(t)}{t - t_0} \right) = c(1,t_0)(t_0).
\]

So \( \tilde{c}_1(t_0) \) and \( \tilde{c}_2(t_0) \) are lying in the same orbit. This shows also that

\( \blacklozenge \) for any two lifts of \( c \) near \( t_0 \in c^{-1}(0) \) which are one-sided differentiable at \( t_0 \) the derivatives at \( t_0 \) lie in the same \( G \)-orbit.

Thus, there must exist a \( g_0 \in G_{v_0} \) such that \( \tilde{c}_1^\perp(t_0) = c^\perp(t_0) = g_0.\tilde{c}_2^\perp(t_0) \).

Now we deal with the tangential parts. We search for a differentiable curve \( t \mapsto g(t) \) in \( G \) with \( g(t_0) = g_0 \) and

\[
\tilde{c}_1^\top(t_0) = (\frac{d}{dt}_{t=t_0}(g(t).\tilde{c}_2(t)))^\top = g'(t_0).v_0 + g_0.\tilde{c}_2^\top(t_0) ^\top.
\]

But this linear equation can be solved for \( g'(t_0) \), and, hence, the required curve \( t \mapsto g(t) \) exists. Note that the normal parts still fit since

\[
(\frac{d}{dt}_{t=t_0}(g(t).\tilde{c}_2(t)))^\perp = (g'(t_0).v_0 + g_0.\tilde{c}_2^\perp(t_0))^\perp = 0 + g_0.\tilde{c}_2^\perp(t_0) = \tilde{c}_1^\perp(t_0).
\]

The two lifts \( \tilde{c}_1 \) for \( t \leq t_0 \) and \( g.\tilde{c}_2 \) for \( t \geq t_0 \) fit together differentiably at \( t_0 \). This proves the claim.

Now let \( \tilde{c} : (a_i, b_i) \rightarrow V \setminus \{0\} \) be the differentiable lift of \( c_{\{a_i, b_i\}} \) constructed above. For \( a_i \neq -\infty \), we put \( \tilde{c}(a_i) := 0 \), the only choice. Consider the expression
\(\gamma(t) := \frac{\tilde{c}(t)}{t-a_i}\), which is a differentiable curve in \(V \setminus \{0\}\) for \(t \in (a_i, b_i)\). We want to show that \(\lim_{t \to a_i} \gamma(t)\) exists. For \(t\) sufficiently close to \(a_i\) we have

\[\sigma(\gamma(t)) = \sigma \left( \frac{\tilde{c}(t)}{t-a_i} \right) = c_{(1,a_i)}(t) \to c_{(1,a_i)}(a_i) \quad \text{as} \quad t \to a_i,\]

where now \(c_{(1,a_i)}(t) := ((t - a_i)^{-2}c_1(t), (t - a_i)^{-d_2}c_2(t), \ldots, (t - a_i)^{-d_n}c_n(t))\). Let \(\tilde{c}_{(1,a_i)}\) be a corresponding continuous lift of \(c_{(1,a_i)}\) which exists by proposition 3.5. This shows that the set \(A\) of all accumulation points of \(\gamma(t)\) at \(a_i\) lies in the orbit \(G \cdot \tilde{c}_{(1,a_i)}(a_i)\) through \(\tilde{c}_{(1,a_i)}(a_i)\). By lemma 4.3 \(A\) is connected. In particular, the limit \(\lim_{t \to a_i} \gamma(t)\) must exist, if \(G\) is a finite group. In general let us consider the projection \(p : G \cdot S_{v_1} \cong G \times_{G_{v_1}} S_{v_1} \to G/G_{v_1} \cong G \cdot v_1\) of a fiber bundle associated to the principal bundle \(\pi : G \to G/G_{v_1}\), where we choose some \(v_1 \in A\). For \(t\) close to \(a_i\) the curve \(t \mapsto \gamma(t)\) is differentiable in \(G \cdot S_{v_1}\), whence \(t \mapsto p(\gamma(t))\) defines a differentiable curve in \(G/G_{v_1}\) which admits a differentiable lift \(t \mapsto \sigma(t)\) into \(G\). Now, \(t \mapsto \gamma(t)^{-1} \cdot \gamma(t)\) is a differentiable curve in \(S_{v_1}\) whose accumulation points for \(t \notin a_i\) have to lie in \(G \cdot v_1 \cap S_{v_1} = \{v_1\}\), since \(\sigma(\gamma(t)^{-1} \cdot \gamma(t)) = \sigma(\gamma(t))\). That means that \(t \mapsto \gamma(t)^{-1} \cdot \tilde{c}(t)\) defines a differentiable lift of \(\gamma((a_i,b_i))\), for \(t > a_i\) close to \(a_i\), whose one-sided derivative at \(a_i\) exists:

\[\lim_{t \to a_i} \frac{\gamma(t)^{-1} \cdot \tilde{c}(t)}{t - a_i} = \lim_{t \to a_i} \gamma(t)^{-1} \cdot \gamma(t) = v_1.\]

Let \(t \mapsto g(t)\) be extended smoothly to \((a_i,b_i)\) so that near \(b_i\) it is constant and replace \(t \mapsto \tilde{c}(t)\) by \(t \mapsto g(t)^{-1} \cdot \tilde{c}(t)\). Thus

\[\tilde{c}'(a_i) := \lim_{t \to a_i} \frac{\tilde{c}(t)}{t - a_i} = v_1.\]

The same reasoning is true for \(b_i \neq +\infty\). Thus we have extended \(\tilde{c}\) differentiably to the closure of \((a_i,b_i)\).

Let us now construct a global differentiable lift of \(c\) defined on the whole of \(\mathbb{R}\). For isolated points \(t_0 \in c^{-1}(0)\) the two differentiable lifts on the neighboring intervals can be made to match differentiably, by applying a fixed \(g \in G\) to one of them by \(\bullet\). Let \(E\) be the set of accumulation points of \(c^{-1}(0)\). For connected components of \(\mathbb{R} \setminus E\) we can proceed inductively to obtain differentiable lifts on them.

We extend the lift by \(0\) on the set \(E\) of accumulation points of \(c^{-1}(0)\). Note that every lift \(\tilde{c}\) of \(c\) has to vanish on \(E\) and is continuous there since \(\langle \tilde{c}(t) | \tilde{c}(t) \rangle = \sigma_1(\tilde{c}(t)) = c_1(t)\). We also claim that any lift \(\tilde{c}\) of \(c\) is differentiable at any point \(t' \in E\) with derivative \(0\). Namely, the difference quotient \(t \mapsto \frac{\tilde{c}(t)}{t-t'}\) is a lift of the curve \(c_{(1,t')}\) which vanishes at \(t'\) by the following argument: Consider the local lift \(\tilde{c}\) of \(c\) near \(t'\) which is differentiable at \(t'\), provided by theorem 4.2. Let \((t_m)_{m \in \mathbb{N}} \subseteq c^{-1}(0)\) be a sequence with \(t' \neq t_m \to t'\), consisting exclusively of zeros of \(c\). Such a sequence always exists since \(t' \in E\). Then we have

\[\tilde{c}'(t') = \lim_{t \to t'} \frac{\tilde{c}(t) - \tilde{c}(t')}{t - t'} = \lim_{m \to \infty} \frac{\tilde{c}(t_m)}{t_m - t'} = 0.\]

Thus \(c_{(1,t')} = \lim_{t \to t'} \sigma(\frac{\tilde{c}(t)}{t-t'}) = \sigma(\tilde{c}'(t')) = 0.\]

\(\square\)

**Remark 4.5.** Note that the differentiability conditions of the curve \(c\) in the current section are best possible: In the case when the symmetric group \(S_n\) is acting in \(\mathbb{R}^n\) by permuting the coordinates, and \(\sigma_1, \ldots, \sigma_n\) are the elementary symmetric polynomials with degrees \(1, \ldots, n\), there need not exist a differentiable lift if the differentiability assumptions made on \(c\) are weakened, see [I] 2.3. first example.
References

[1] D. Alekseevsky, A. Kriegl, M. Losik, P.W. Michor, Choosing roots of polynomials smoothly, Israel J. Math., 105 (1998), 203-233. arXiv:math.CA/9801026

[2] D. Alekseevsky, A. Kriegl, M. Losik, P.W. Michor, Lifting smooth curves over invariants for representations of compact Lie groups, Transformation Groups 5 (2000), 103-110. arXiv:math.AG/0312030

[3] G.E. Bredon, Introduction to compact transformation groups, Academic Press, New York, 1972.

[4] M.D. Bronshtein, Smoothness of polynomials depending on parameters, Sib. Mat. Zh. 20 (1979), 493-501 (Russian); English transl. in Siberian Math. J 20 (1980), 347-352.

[5] A. Kriegl, M. Losik, P.W. Michor, Choosing roots of polynomials smoothly, II, Israel J. Math. 139 (2004), 183-188. arXiv:math.CA/0208228

[6] A. Kriegl, P.W. Michor, Differentiable perturbation of unbounded operators, Math. Ann. 327 (2003), 191-201. arXiv:math.FA/0204060

[7] D. Montgomery, C.T. Yang, The existence of a slice, Ann. of Math., 65 (1957), 108-116.

[8] D. Luna, Sur certaines opérations différentiables des groupes de Lie, Amer. J. Math. 97 (1975), 172-181.

[9] C. Procesi, G. Schwarz, Inequalities defining orbit spaces, Invent. Math. 81 (1985), 539-554.

[10] G.W. Schwarz, Lifting smooth homotopies of orbit spaces, Publ. Math. IHES 51 (1980), 37-136.

[11] G.W. Schwarz, Smooth functions invariant under the action of a compact Lie group, Topology 14 (1975), 63-68.

[12] S. Wakabayashi, Remarks on hyperbolic polynomials, Tsukuba J. Math. 10 (1986), 17-28.

A. KRIEGL: INSTITUT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, A-1090 WIEN, AUSTRIA
E-mail address: Andreas.Kriegl@univie.ac.at

M. LOSIK: SARATOV STATE UNIVERSITY, UL. ASTRAKHANSKAYA, 83, 410026 SARATOV, RUSSIA
E-mail address: losikMV@info.sgu.ru

P.W. MICHIR: INSTITUT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, A-1090 WIEN, AUSTRIA; AND: ERWIN SCHÖDINGER INSTITUT FÜR MATHEMATISCHE PHYSIK, BOLTZMANNGASSE 9, A-1090 WIEN, AUSTRIA
E-mail address: Peter.Michor@esi.ac.at

A. RAINER: INSTITUT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, A-1090 WIEN, AUSTRIA
E-mail address: armin_rainer@gmx.net