The pre-breakdown solutions of electrohydrodynamic equations for liquid insulators and comparison with experiment

M S Apfelbaum, R A Syrovatka and V I Vladimirov
Joint Institute for High Temperatures of the Russian Academy of Sciences, Izhorskaya 13
Bldg 2, Moscow 125412, Russia
E-mail: msa@ihed.ras.ru

Abstract. The system of electrohydrodynamic equations at a pre-breakdown liquid insulator is given. The influence of electric field on the partial dissociation molecules rate is taken into account. The stationary and nonstationary solutions for electric potential and hydrodynamic velocity distributions are obtained. Comparison of the experimental and theoretical results showed good agreement. The stationary velocity distribution and current-voltage characteristic are close to quadratic, as predicted by the theory.

1. Introduction

The purposes of the present work are analysis of the conduction processes in the liquid insulators under high voltage applied and the calculation of the electrohydrodynamic (EHD) flows by our solutions of the pre-breakdown macroscopic equations. The liquid insulator under high voltage reveals well-known features of its behavior, that is concluded in deviation of the current–field dependence from Ohm’s law [1, 2]. We suppose as well as in our previous works [3, 4] that the liquid dielectric is analogous to the weak electrolyte. In addition, the partial dissociation of its molecules has place. Besides, we consider that impurities can influence on the low-voltage conductivity of the medium in hand. This conductivity can be determined from the linear part of the current–voltage dependence, since Ohm’s law is valid under low voltage. The concentrations of ions $n_{\pm}$ and impurities $n_p$ are supposed to satisfy the following condition:

$$\begin{cases} n_{\pm} \ll n_a, \\ n_p \ll n_a, \end{cases}$$

where $n_a$ is the concentration of neutral particles (molecules). The rate of their dissociation, according to [1, 2] is

$$W_D = W_D(n_a, n_p, T, |E|) = W_D(n_a, n_p, T, 0)f(E).$$

Here $T$ is the absolute temperature; $E$ is the intensity of the electric field. The rate of the ions recombination is

$$\begin{cases} W_r = K_r n_{\pm} n_-, \\ K_r = \frac{(b_+ + b_- |e|)}{\varepsilon \varepsilon_0}. \end{cases}$$
Recombination rate $K_i$ was obtained by Langevin. According to Onsager [1] $K_i$ is not dependent of $|E|$; $b_\pm$ are mobilities of ions, $|e|$ is their charge, $\varepsilon$ is dielectric permittivity, $\varepsilon_0$ is the electric constant. We supposed that the influence of the field on $b_\pm$ and $\varepsilon$ is weak. In particular this consideration for $\varepsilon$ follows from Debye formulae: $(\varepsilon - 1) \propto \rho$. For our media $\rho \approx n_a m_a$, where $m_a$ is the mass of neutral particles. And the field does not influence on $n_a$ according to (1). We also use Einstein formulae for diffusion:

$$D_\pm = \frac{k_B T b_\pm}{|e|},$$

(4)

where $k_B$ is Boltzmann constant, $T$ is supposed to be constant and equal to 293 K, because the Joule heat is small. The energy spent in chemical reaction is also considered negligible for a weak electrolytes and substances analogous to them. For the function $f(E)$ in equation (2) there are two expressions. One of them was obtained by Plumley [2] for liquid insulators. Frenkel also independently derived it for solid insulator with possible covalent bonds. It looks like

$$f(E) = \exp(\beta |E|^2); \quad \beta = \frac{|e|^2}{\sqrt{\pi \varepsilon \varepsilon_0 k_B T}}.$$ 

(5)

If expression (5) is valid, then $W_D(n_a, n_p, T, 0)$ in equation (2) is reduced to well-known Arrhenius exponent: $W_D(n_a, n_p, T, 0) = A_0 \exp[-U_a/(k_B T)]$, where $U_a$ is the energy of destruction of chemical bonds, $A_0$ is the weakly dependent on temperature coefficient.

The second expression (5) was obtained by Onsager [1]. If the intensity of the field tends to infinity, Onsager formula from [1] is approximated by (5). Onsager as well as Plumley studied the media with only ion chemical bonds.

2. The model of EHD processes in the liquid insulator

We consider that the influence of hydrodynamic on process of conduction is negligible in our case. It is possible if the intensity of the field is not so strong to cause emission or injection of charges from electrodes. We also introduce the variables $q = (n_+ - n_-)|e|; \sigma = (n_- b_+ + n_+ b_-)|e|$ and potential $\phi$: $E = -\nabla \phi$. Then, we can write the system of macroscopic time-dependent equations for creation of ions. If we take $b_+ = b_- = b$ (this supposition is more correct for liquids than for gases) with (1)–(5) we will obtain

$$\frac{\partial q}{\partial t} + (V, \nabla q) - \frac{k_B T b}{|e|} \Delta q + (E, \nabla \sigma) = -\frac{q \sigma}{\varepsilon \varepsilon_0}, \quad \Delta \phi = -\frac{q}{\varepsilon \varepsilon_0},$$

$$\frac{\partial \sigma}{\partial t} + (V, \nabla \sigma) + b^2 (E, \nabla q) - \frac{k_B T b}{|e|} \Delta \sigma - \frac{\sigma^2}{\varepsilon \varepsilon_0} \exp(\beta |E|^{1/2}) + \frac{\sigma^2}{\varepsilon \varepsilon_0} = 0.$$ 

(6)

Here, $V$ is the hydrodynamic velocity of neutral component and it is approximately equal to the velocity of the whole mixture, if (1) is valid. To derive (6), we used (1)–(5) together with continuity equation for noncompressible liquid:

$$\nabla V = 0.$$ 

(7)

As indicated in [4], this model can also be generalized to non-isothermal processes. There was regarded the influence of charged particles, including impurities, on coefficient $A_0$ in Arrhenius exponent by means of initial conductivity $\sigma_0$. The general initial conditions for the system equations (6) and (7), independent of the electrodes geometry, are

$$q(t = 0) = V(t = 0) = 0; \quad \sigma(t = 0) = \sigma_0.$$ 

(8)

We also introduce a parameter $a = q_1/(n_1 |e|)$, where $q_1$ is the characteristic density of space charge, $n_1$ is the characteristic concentration of neutrals, which decay into ions. According to [4], outside the boundary layers, $2 b n_1 |e| = \sigma_0 \exp[(\beta/2)|E|^{1/2}]$. And, according to the Poisson
equation from system (6), \( q_1 = \varepsilon_0 U/|e|L^2 \), where \( L \) is the characteristic length of a considered task. Thus, outside the boundary layers, \( a = 2\varepsilon_0\beta /|\varepsilon_0| \sqrt{2L^2U/|\sigma_0\exp((\beta/2)|E|^{1/2})|} \), where \( U \) is the applied voltage. And for usual liquid insulator \( a \ll 1 \), i.e. the medium is neutral in spite of creation of space charge under strong fields (outside the boundary layers). The ratio of the second term in the 3rd equation of the system (6) to the fourth term is of the order of \( a^2 \) and we can neglect it. The ratio of the 3rd diffusion term of this equation to the fourth is of the order of \( (r_d/L)^2 \), \( r_d \) is the Debye radius. So, if the medium is quasineutral, then, as we know from plasma and electrolyte theory, this ratio is much smaller than unit. We can see also that hydrodynamic processes do not influence on electrical conductivity if \( \text{Re}_q = V_1/(bE_1) < 1 \). Under such conditions we can obtain from (6) Frenkel law for conductivity of the liquid insulators.

For the case of sphere under high voltage, there is existed only stationary solution outside boundary layers, consisting to the Frenkel conductivity law:

\[
\phi(r) = \left( \frac{I}{4\pi\sigma_0} \right)^{1/2} \left[ \frac{8}{\beta} - \frac{\varepsilon_0}{\sigma_0} \left( |E|^2 + \frac{8}{\beta} \right) \right] \text{sign} \phi(r_0), \quad |E| \exp \left( \frac{\beta}{2} |E| \right) = \frac{I}{4\pi\sigma_0 r^2}, \quad |\phi(r_0)| = U.
\]

Here \( I \) is the constant of integration that is equal to electric current; \( r \) is the distance from the center of the sphere.

3. Some applications of the electrodynamic model for the theoretical description of the hydrodynamic flows of the liquids caused the nonuniform high electric field strength

For the application results obtained above to the theoretical description of the flows we use the Felici conception [5]. According to [5] the influence of the volume dissociation on these flows is dominating under high fields of needle- or blade-electrodes.

We will show later that the velocities of the flows independent of the electrode radius \( r_0 \), if \( r_0 \) tends to zero. Therefore, the obtained results can be applied for the calculating of the flows under the high voltage point electrode field. The last phenomenon was observed in liquid dielectrics by Faraday and earlier in gases by Franklin. In previous publications of the author of this work the stationary flows were calculated by means of conductivity gradient. But in experiments (see [5]) the formation of steady flows has place later than that of the steady electric fields. The characteristic time of this formation \( t_1 \) is

\[
t_1 = \frac{L^2}{\nu} \gg \tau.
\]

Here \( \nu \) is the kinematics viscosity, \( \tau \) is the time of the electrodynamic processes relaxation. That is why in the present work we described not only stationary conduction processes, but non-stationary hydrodynamic flows under constant voltage \( U \) too.

Therefore, for our hydrodynamic model it is possible to describe the flows with high velocities. In the case of non-compressible medium, we have to add only the law of momentum conservation without heat transfer equation. We will solve axisymmetric problems, and will use cylindrical coordinate system \( R, z, \theta \). Write the momentum conservation law in the vortex form and introduce new variable \( \psi, \varphi \):

\[
x = \nabla \times V, \quad V_r = \frac{1}{R} \frac{\partial \varphi}{\partial r}, \quad V_R = -\frac{1}{R} \frac{\partial \varphi}{\partial z}, \quad r = \sqrt{R^2 + z^2}.
\]

Then momentum equation for three-dimensional case is

\[
\rho \frac{dx}{dt} - \rho(x, \nabla)V - \rho \nu \Delta V = [\nabla q, E].
\]
In the axisymmetric case, $V$ can be expressed according to equations (11). If the Coulomb force density sharply decreases with the increasing of the distance from the electrode, one can solve of (6), (7) and (12) without this term for the flows, caused by integral force:

$$F_E = \int G q EdG,$$

where $G$ is the volume of semispace, containing the liquid dielectrics. According to EHD model described above, we can calculate $F_E$ through (13). After this we can use the class of the solutions for hydrodynamic flows caused by a point sources of the force (not necessarily of electric origin). There is much publication on this topic. These works develop the idea, offered in [6]. So, with calculation $E, q$ in accordance with (9), we can analytically obtain $F_E$ from (13):

$$F_E = \varepsilon \varepsilon_0 \frac{J}{2\sigma_0} \left\{ 2 - \exp \left( \frac{\beta}{2} |E_0|^\frac{1}{2} \right) \left[ -\frac{\beta}{4} |E_0| + \beta |E_0|^\frac{1}{2} + 2 \right] \right\},$$

where $|E_0| \exp \left( -\frac{\beta}{2} |E_0|^\frac{1}{2} \right) = \frac{J \text{sign} \phi_W}{4\pi \sigma_0 r_0^2}.$

The expression for $F_E(U)$ s easily obtained by applying formulae (9) to (14).

Now we will describe the flows, which are caused by the Coulomb force $F_E$ in the form (14). In a hydrodynamics manual [6], there are some well-known solutions of Navie–Stocks stationary equations (7) and (12) as hydrodynamics boundary layer equations. For Shlichting stationary jet in cylindrical geometry [7],

$$V_z = \frac{\nu}{z} \left( 1 + \frac{\xi^2}{4} \right)^2, \quad V_R = \frac{\nu \gamma}{z} \left( 1 + \frac{\xi^2}{4} \right)^2, \quad \xi = \frac{R}{z},$$

$$\gamma = \frac{(3\pi \rho)^{\frac{1}{2}}}{4\nu z^2}, \quad p = 2\pi \rho \int_0^\infty V_z R dR.$$

Here, $p$ is the integral momentum of the jet, $R$ and $z$ are cylindrical coordinates for the case of an axisymmetric jet.

There is generalization of solution (15) in [6] as $\psi = \nu r f(\theta)$. And function $f(\theta)$ satisfies to an ordinary differential equation, obtained from (7) and (12). Squire proposed in [6] to explain the jet motion as a result of point force action of the delta function type. The proof of an expression

$$F_E = p$$

was given in our previous works. If $\beta |E_0|^{1/2} \gg 1$ it follows from (14) and (9) that

$$F_E = \frac{\varepsilon \varepsilon_0 U^2}{8}.$$

One can obtain an expression for the distribution of the axial velocity as a function of the applied voltage, substituting formulas (16) and (17) in (15):

$$V_z = \frac{3\pi \varepsilon \varepsilon_0 U^2}{8 \varepsilon_0 \nu}.$$

In conclusion of this part we describe the model of unsteady streams. Calculations [8] testified that the structure of streams developed in time is different from stationary type (15). The simplest explanation can be obtained with the aid of (17). In that case one can consider that the Coulomb density force $F_E$ is not dependent of time. The initial velocities one can take in [9]. For $\psi$ there obtains that the field of velocities at $t \to 0$ is the same as dipole field:

$$\psi = \frac{F_E t \sin^2 \theta}{4\pi \rho r}.$$
Therefore, we can calculate the pre-breakdown flow velocities when \( t \sim \tau \) and when \( t \ll L^2/\nu \). And it would be interesting to derive the calculation technique for velocities at time moment \( t \gg \tau \) and \( t \sim L^2/\nu \). And we also obtained breakdown condition of liquid insulators at high non-uniform electric field strength by applying well-known hydrodynamic explosion Sedov–Taylor theory. Then the EHD liquid breakdown analogue of Sedov point explosion maximum radius formulae near point high voltage electrode may be obtained:

\[
r_{\text{max}} = \left| e \right| U^3 \sqrt{\frac{21}{100\pi k_B^2 \rho_0 T^2}}.
\]

(20)

4. Experiment

Investigation of EHD flows in liquid dielectrics was carried out for transformer oil in the point-plane geometry. A negative electric potential was applied to a needle that was located 4.5 mm from a grounded metal plate 10 \times 10 \text{ cm}^2 in size. The radius of the tip of the needle was 70 \text{ µm}. The electrodes were placed in a container of polymethylmethacrylate. The oil level was equal to 1.5 cm.

Measurements of the voltage were made using a resistive divider with a transmission factor of 1 : 1000 and an Aktakom ABM-4085 multimeter. The current was determined from the voltage drop on the 200 kOhm resistor. To visualize EHD flows (figure 1), hollow plastic spheres with a diameter of 50–100 \text{ µm} were added to the oil. The motion of the particles was recorded using a HiSpec 1 video camera with a Nikon 60 mm f/2.8D AF Micro-Nikkor lens. The camera allowed us to make a video with a maximum resolution (1280 \times 1024) at up to 506 frames per second. The particles were illuminated by a laser with a wavelength of 536 nm.

The pre-breakdown current-voltage characteristic is shown in figure 2(a). Obtained results of experiments are compared with the one calculated from the formula obtained from (9) by the transition to a point electrode:

\[
I = \frac{e^3 \sigma_0 U^2}{16 \varepsilon \varepsilon_0 (k_B T)^2}.
\]

(21)

The ion charge \( e \) is considered equal to unity.

Figure 2(b) shows experimental and theoretical [calculated by the formula (18)] dependencies of the axial velocity on the voltage for \( z = 2.25 \text{ mm} \). Density of the oil was equal to 800 \text{ kg/m}^3,
kinematic viscosity was about $10^{-3}$ m$^2$/s. The obtained experimental dependence for the velocity is close to quadratic.

5. Discussion
According to our calculations and experiments the equations of Arrhenius type may be applied for the description of high voltage electrohydrodynamics. In addition, in this work we taken into account the influence of these EHD flows on the transformer cooling. The Nusselt number of heat transformer surface cooling by transformer oil according to dimensional analysis [7, 10] depend on convection transformer oil cooling flow Grasgoff number $Gr$, Prandtl number of this liquid $Pr$, and EHD criterion $a$:

$$Nu = f(Gr, Pr, a),$$

where $a = q_1/(n_1|e|)$. For considerable cooling $Gr$ and $Pr$ are constant and $a \ll 1$, because quasineutral regime is in transformer oil for this cooling. So, one can obtain from (22) linear dependence $Nu(U)$ with coefficient $k$: $Nu = C + kU$, where $C$ is constant, depended on $Gr$ and $Pr$, and $U$ is the effective transformer high voltage.

6. Conclusion
So, we have obtained the analytical expression for pre-breakdown current-voltage characteristic for liquid dielectrics in the point-plane geometry. The stationary and nonstationary velocity distributions of the EHD flows arising in these insulators at the pre-breakdown mode were also obtained. The experimental investigation of the current-voltage characteristic and EHD flows in the point-plane geometry was performed using transformer oil as a liquid dielectric. Comparison of the experimental and theoretical results showed good agreement. The stationary velocity distribution and current-voltage characteristic are close to quadratic, as predicted by the theory.

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