PROOF OF A CONJECTURE OF MÉSZÁROS AND MORALES ON THE VOLUME OF A FLOW POLYTOPE

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Abstract. We prove a conjecture of Mészáros and Morales on the volume of a flow polytope. Independently from our work, Zeilberger sketched a proof of their conjecture. In fact, our proof is the same as Zeilberger’s proof. The purpose of this note is to give a more detailed proof of the conjecture.

1. Introduction

We prove a conjecture of Mészáros and Morales [2] on the volume of a flow polytope, which is a type D analog of the Chan-Robbins-Yuen polytope [1]. Independently from our work, Zeilberger also proved the conjecture and sketched his proof in [3]. In fact, our proof is the same as Zeilberger’s proof. The purpose of this note is to give a more detailed proof of the conjecture.

For a function \( f(z) \) with a Laurent series expansion at \( z = 0 \), we denote by \( \text{CT}_z f(z) \) the constant term of the Laurent expansion of \( f(z) \) at 0. In other words, if \( f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \), then \( \text{CT}_z f(z) = a_0 \).

The conjecture of Mészáros and Morales can be stated as the following constant term identity.

**Conjecture 1.** [2, Conjecture 7.6] For an integer \( n \geq 2 \), we have

\[
\text{CT}_{x_n} \text{CT}_{x_{n-1}} \cdots \text{CT}_{x_1} \prod_{j=1}^{n} x_j^{-1} (1 - x_j)^{-2} \prod_{1 \leq j < k \leq n} (x_k - x_j)^{-1} (1 - x_k - x_j)^{-1} = 2^n \prod_{k=1}^{n} \text{Cat}(k),
\]

where \( \text{Cat}(k) = \frac{1}{k+1} \binom{2k}{k} \).

Conjecture 1 is a type \( D_n \) analog of the constant term identity

\[
\text{CT}_{x_n} \text{CT}_{x_{n-1}} \cdots \text{CT}_{x_1} \prod_{j=1}^{n} (1 - x_j)^{-2} \prod_{1 \leq j < k \leq n} (x_k - x_j)^{-1} = \prod_{k=1}^{n} \text{Cat}(k),
\]

which was conjectured by Chan, Robbins, and Yuen [1] and proved by Zeilberger [4].

In Section 2 we recall Zeilberger’s reformulation of Morris’ constant term identity. In Section 3 we prove Conjecture 1.

2. Zeilberger’s reformulation of Morris’ constant term identity

Zeilberger [4] rewrote Morris’ identity [3] as follows: For nonnegative integers \( a \) and \( b \) and a positive half integer \( c \), we have

\[
\text{CT}_{x_n} \text{CT}_{x_{n-1}} \cdots \text{CT}_{x_1} \prod_{i=1}^{n} (1-x_i)^{-a} x_i^{-b} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-2c} = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(a + b + (n - 1 + j)c)\Gamma(c)}{\Gamma(a + jc)\Gamma(c + jc)\Gamma(b + jc + 1)}.
\]

Zeilberger [4] showed that, when \( a = 2, b = 0, c = 1/2 \), the right hand side of (2) is

\[
\frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma\left(\frac{n+3+j}{2}\right)\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{4+j}{2}\right)\Gamma\left(\frac{4+j}{2}\right)\Gamma\left(\frac{4+j}{2}\right)} = \prod_{k=1}^{n} \text{Cat}(k),
\]

which together with (2) implies (1).

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By Cauchy’s integral formula, if \( f(z) \) has a Laurent series expansion at 0, we have

\[
CT_z f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z} \, dz,
\]

where \( C \) is the circle \( \{ z : |z| = \epsilon \} \) oriented counterclockwise for a real number \( \epsilon > 0 \) such that \( f(z) \) is holomorphic inside \( C \) except 0. Thus (2) can be rewritten as

\[
\frac{1}{(2\pi i)^n} \oint_{C_n} \cdots \oint_{C_1} \prod_{j=1}^{n} (1 - x_j)^{-a} x_j^{-b-1} \prod_{1 \leq j < k \leq n} (x_k - x_j)^{-2c} \, dx_1 \cdots dx_n
= \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(a + b + (n - 1 + j)c) \Gamma(c)}{\Gamma(a + jc) \Gamma(c + jc) \Gamma(b + jc + 1)}
\]

where \( C_j \) is the circle \( \{ z : |z| = j\epsilon \} \) oriented counterclockwise for a number \( 0 < \epsilon < \frac{1}{n} \).

3. Proof of the conjecture of Mészáros and Morales

We will prove the following theorem.

**Theorem 1.** For a nonnegative integer \( a \) and a positive half integer \( c \), we have

\[
CT_{x_n} CT_{x_{n-1}} \cdots CT_{x_1} \prod_{j=1}^{n} x_j^{-a+1} (1 - x_j)^{-a} \prod_{1 \leq j < k \leq n} (x_k - x_j)^{-2c} (1 - x_j - x_k)^{-2c}

= 2^{an + 4c(n^2)} \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(n + 2 + j) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + j) \Gamma(\frac{1}{2} + \frac{1}{2})}
\]

We can get Conjecture \( \textup{I} \) as the special case \( a = 2, c = 1/2 \) of Theorem \( \textup{I} \) as follows. When \( a = 2, c = 1/2 \), the right hand side of the formula in Theorem \( \textup{I} \) is

\[
\frac{2^{n^2 + n} \prod_{j=0}^{n-1} \frac{\Gamma(n + 2 + j) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + j) \Gamma(\frac{1}{2} + \frac{1}{2})}}{n!}
\]

Since

\[
\prod_{j=0}^{n-1} \frac{\Gamma(n + 2 + j) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + j) \Gamma(\frac{1}{2} + \frac{1}{2})} = \frac{\Gamma(2n + 2) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \frac{1}{2})} = 2^n,
\]

we obtain Conjecture \( \textup{I} \) by (3).

For the rest of this section we prove Theorem \( \textup{I} \). From now on we assume that \( \epsilon > 0 \) is a very small number.

By (4) we have

\[
CT_{x_n} CT_{x_{n-1}} \cdots CT_{x_1} \prod_{j=1}^{n} x_j^{-a+1} (1 - x_j)^{-a} \prod_{1 \leq j < k \leq n} (x_k - x_j)^{-2c} (1 - x_j - x_k)^{-2c}

= \frac{1}{(2\pi i)^n} \oint_{C_n} \cdots \oint_{C_1} \prod_{j=1}^{n} x_j^{-a} (1 - x_j)^{-a} \prod_{1 \leq j < k \leq n} (x_k - x_j)^{-2c} (1 - x_k - x_j)^{-2c} \, dx_1 \cdots dx_n,
\]

where \( C_j \) is the circle \( \{ z : |z| = j\epsilon \} \) oriented counterclockwise.
Using the change of variables $x_j = \frac{1-z_j}{2}$ or $z_j = 1-2x_j$, the above is equal to

$$\frac{1}{(2\pi i)^n} \oint_{C^n_1} \ldots \oint_{C^n_1} \prod_{j=1}^{n} \frac{(1-z_j)}{2}(-a) \frac{(1+z_j)}{2}(-a) \times \prod_{1 \leq j < k \leq n} \left(\frac{-z_k+z_j}{2}\right)^{-2a} \left(\frac{z_k+z_j}{2}\right)^{-2a} (-2)^{-n} dz_1 \cdots dz_n$$

where $C_j$ is the circle $\{z:|z-1|=2je\}$ oriented counterclockwise.

Using the change of variables $z_j^2 = y_j$ or $z_j = y_j^{1/2}$, the above is equal to

$$\frac{(-1)^n 2^{2n+4c^2/2}}{(2\pi i)^n} \oint_{C^n_1} \ldots \oint_{C^n_1} \prod_{j=1}^{n} (1-y_j)^{-a} y_j^{-1/2} \prod_{1 \leq j < k \leq n} (y_j - y_k)^{-2c} dy_1 \cdots dy_n,$$

where $C''_j$ is the circle $\{z:|z-1|=4je\}$ oriented counterclockwise. This is because if $C_j$ is parametrized by $1+2j\epsilon e^{i\theta}$ for $0 \leq \theta \leq 2\pi$, then the image of $C_j'$ under the map $z \mapsto z^2$ can be parametrized by $1 + 4j\epsilon e^{i\theta} + 4j^2 \epsilon^2 e^{2i\theta}$ for $0 \leq \theta \leq 2\pi$. Since $\epsilon$ is very small we can deform this image to the circle $C''_j$ without changing the contour integral.

Using the change of variables $t_j = 1 - y_j$, the above is equal to

$$\frac{2^{2n+4c^2/2}}{(2\pi i)^n} \oint_{C''_1} \ldots \oint_{C''_1} \prod_{j=1}^{n} t_j^{-a} (1-t_j)^{-1/2} \prod_{1 \leq j < k \leq n} (t_k - t_j)^{-2c} dt_1 \cdots dt_n,$$

where $C''_j$ is the circle $\{z:|z|=4j\epsilon\}$ oriented counterclockwise.

Using [5], we finish the proof of Theorem 1.

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