Noncommutative 2–Dimensional Models of Gravity

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Abstract

A review is given of some 2-dimensional metrics for which noncommutative versions have been found. They serve partially to illustrate a noncommutative extension of the moving-frame formalism.

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1 Introduction

There is a very simple argument due to Pauli that the quantum effects of a gravitational field will in general lead to an uncertainty in the measurement of space coordinates. It is based on the observation that two ‘points’ on a quantized curved manifold can never be considered as having a purely space-like separation. If indeed they had so in the limit for infinite values of the Planck mass, then at finite values they would acquire for ‘short time intervals’ a time-like separation because of the fluctuations of the light cone. Since the ‘points’ are in fact a set of four coordinates, that is scalar fields, they would not then commute as operators. This effect could be considered important at least at distances of the order of Planck length, and perhaps greater. This is one motivation to study noncommutative geometry. A second motivation, which is the one we consider ours, is the fact that it is possible, since the work of Connes [1] and others to study noncommutative differential geometry, and there is no reason to assume that even classically coordinates commute at all length scales. One can consider for example coordinates as order parameters as in solid-state physics and suppose that singularities in the gravitational field become analogs of core regions; one must go beyond the classical approximation to describe them.

A straightforward and conservative way to render coordinates noncommutative is to represent them by operators. Thus the space-time manifold is replaced with an algebra \( \mathcal{A} \) (noncommutative ‘space’), generated by a set of noncommutative ‘coordinates’ \( x^i \). These coordinates are not completely arbitrary; they are restricted by relations such as commutation rules. We would like to think of \( x^i \) as linear operators on some vector space, and therefore we shall assume that the multiplication in the algebra is associative. The essential element which allows us to interpret a noncommutative algebra as a space-time is the possibility to introduce a differential structure on the former.

As in the commutative case, many non equivalent differential structures on a given algebra exist. This means that there are several ‘geometries’ which one can associate to the underlying noncommutative space. A possible interpretation is that the gravitational field is intimately related to the choice of differential calculus. Once the differential calculus is chosen, some more or less obvious assumptions as hermiticity and bilinearity fix the linear connection almost uniquely. This is to be contrasted with the commutative case, where for each differential structure linear connection can be chosen almost arbitrarily. The choice therefore of a differential is one of the more important steps in the ‘quantization’. We stress that we do not think that all noncommutative geometries are suitable as noncommutative models of space-time, any more so than in the commutative limit. We believe, for example, that for every sufficiently small open region around a classical point there is a sufficiently strong noncommutative extension such that to this region there is associated at most one degree of freedom. We refer to such spaces as ‘fuzzy’. A notable counter example would be the irrational noncommutative torus. In fact, in this case the dimension of the algebra is infinite whereas the limiting space is compact.

We use a noncommutative version of Cartan’s frame formalism [2]; the subject has been also studied from other points of view [3, 4, 5]. In order to develop some intuition in the complete absence of experimental evidence, one is obliged to consider examples. Several of these have been found. In the next four sections we shall introduce as illustrations the quantum plane, 2-dimensional anti-de Sitter space and the 2-dimensional Rindler space. Their simplicity will allow us to bypass the general formalism and will permit a more intuitive presentation. A series of models in all dimensions has been
found [6], as well as some models in dimensions two [2, 7] and four [8, 9]. We shall argue that the moving frame formalism is in this respect a natural way to implement gravity. It enables one to introduce a sort of correspondence principle as a guide of how to construct the frame from its commutative limit. The examples of Sections 2 and 3 have been considered elsewhere; we believe the Rindler example, in all given variants to be new. The parameter $k$ with the dimensions of length squared is introduced in Section 4 to facilitate the discussion of the commutative limit. A parameter $\mu$, related to the acceleration is also introduced in this section.

2 Differential calculus: Quantum plane

We introduce here the quantum plane to illustrate the basic features of noncommutative geometry; more details can be found for example in the books mentioned above. A noncommutative ‘space’ is an associative $*$-algebra $A$ generated by a set of hermitean ‘coordinates’ $x^i$ which in some limit tend to the (real) coordinates $\tilde{x}^i$ of a manifold which we identify as the classical limit of the geometry. That is, in the classical or weak-field limit we impose the condition

$$x^i \to Z^{-1/2} \tilde{x}^i$$

for some $Z$, perhaps singular. Elements of $A$ will be denoted by $x^i$, $f$, $g$, $p_i$, and so forth. In general the coordinates satisfy a set of commutation relations. We shall consider the algebra as a formal algebra and not attempt to represent it as an algebra of operators.

The simplest relation which can be used to define the algebra is

$$[x^i, x^j] = iJ^{ij}, \quad (2.1)$$

where $J^{ij}$ are real numbers defining a canonical or symplectic structure. Often in the literature the notation $\theta^{ij}$ is used instead of $J^{ij}$. Another associative algebra is defined using the commutation relations

$$[x^i, x^j] = iC^{ijk} x^k. \quad (2.2)$$

For simplicity we shall assume that the center of the algebra, the set of elements which commute with all generators, consists of complex multiples of the identity. The third important special case is a quantum space, defined by a homogeneous quadratic relation:

$$[x^i, x^j] = iC^{ijkl} x^k x^l. \quad (2.3)$$

A combination of these three commutation relations will be satisfied by a set of generators $p_i$ of $A$ which we shall refer to as momenta; the commutation relations obeyed by the ‘coordinates’ in general are not even necessarily polynomial.

In ordinary geometry a vector field can be defined as a derivation of the algebra of smooth functions. This definition can be used also when the algebra is noncommutative. A derivation, we recall, is a linear map $f \mapsto Xf$ which satisfies the Leibniz rule, $X(fg) = (Xf)g + fXg$. Derivations will be denoted by $X$, $Y$, $e_i$, and so forth and the set of all derivations by $\text{Der}(A)$. A simple example is the algebra of $2 \times 2$ complex matrices $M_2$ with (redundant) generators the Pauli matrices. The algebra is
of dimension four, the center is of dimension one and \( \text{Der}(M_2) \) is of dimension three with basis consisting of three derivations \( e_i = \text{ad} \sigma_i \):

\[
e_i f = [\sigma_i, f].
\]

We notice that the Leibniz rule is here the Jacobi identity. We see also that the left multiplication \( \sigma_i e_j \) of the derivation \( e_j \) by the generator \( \sigma_i \) no longer satisfies the Jacobi identity: it is not a derivation. The vector space \( \text{Der}(M_2) \) is not a left \( M_2 \) module.

This property is generic. If \( X \) is a derivation of an algebra \( \mathcal{A} \) and \( h \) an element of \( \mathcal{A} \), then \( hX \) is not necessarily a derivation:

\[
hX(fg) = h(Xf)g + hfXg \neq h(Xf)g + fhXg
\]

if \( hf \neq fh \). Notice that the derivations of \( M_2 \) are inner: they were defined as a commutator with an element of the algebra. It is a simple theorem that all derivations of a complete matrix algebra are inner. On the other hand, the derivations on an algebra of functions are not inner; they are known as outer.

Although derivations do not form a left module, one can introduce associated elements known as differential forms which form a bimodule; they can be multiplied from the left and from the right. We shall therefore express as much as possible physical quantities using the latter. We define here a 1-form \( \omega \) as a linear map \( \omega : \text{Der}(\mathcal{A}) \rightarrow \mathcal{A} \). The set of 1-forms \( \Omega^1(\mathcal{A}) \) has a bimodule structure, that is, if \( \omega \) is a 1-form, \( f\omega \) and \( \omega f \) are also 1-forms. The elements of \( \Omega^1(\mathcal{A}) \) will be typically denoted by \( \omega, \theta, \xi, \eta \).

So, the essential step is the definition of a differential \( d \); it is a linear map from functions to 1-forms, \( d : \mathcal{A} \rightarrow \Omega^1(\mathcal{A}) \) which obeys the Leibniz rule. In general \( fdg \neq dgf \) but we shall introduce later special forms \( \theta^i \) which commute with the algebra. The exterior product \( \xi\eta \) of two 1-forms \( \xi \) and \( \eta \) is a 2-form. There is no reason to assume the exterior product antisymmetric. We mention also that one can deduce the structure of the algebra of all forms from that of the module of 1-forms. The map \( d \) can be extended to all forms if one require that \( d^2 = 0 \). We should stress that in general one can associate many differential calculi to a given algebra.

To illustrate these notions, we construct a differential for the canonical structure (2.1): \( [x^i, x^j] = iJ^{ij} \). From the Leibniz rule it follows that the differential of the unit element must vanish. Therefore the differential must satisfy the constraint

\[
0 = d[x^i, x^j] = dx^i x^j + x^i dx^j - dx^j x^i - x^j dx^i.
\]

A possible but not unique solution to this equation is

\[
dx^i x^j - x^j dx^i = [dx^i, x^j] = 0.
\]

Furthermore, the relations of the algebra imply that

\[
0 = d[dx^i, x^j] = d^2 x^j x^i - dx^i dx^j - dx^j dx^i + x^j d^2 x^i,
\]

that is, the differentials anticommute as if they were defined on a manifold,

\[
dx^i dx^j = -dx^j dx^i.
\]

For the Lie algebra (2.2), however, we see that we could not have imposed the condition \( [dx^i, x^j] = 0 \) as it is inconsistent with the relation \( [x^i, x^j] = iC^{ij}_{\quad k} x^k \). It would imply \( C^{ij}_{\quad k} dx^k = 0 \).
The first example we discuss in detail is the quantum plane. It has two generators $x$ and $y$ related by
\[ xy = qyx, \tag{2.9} \]
where $q$ is a constant which we shall assume not to be a root of unity. For example, two derivations $e_1, e_2$ can be defined by the formulae
\[
\begin{align*}
  e_1 x &= x, & e_2 x &= 0, \\
  e_1 y &= 0, & e_2 y &= y. 
\end{align*} \tag{2.10}
\]
These would necessarily be outer derivations. There are other possibilities. Let $e_i$ be defined by
\[
\begin{align*}
  e_1 x &= \frac{q^2 - 1}{q^2 + 1} x^{-1} y^2, & e_1 y &= \frac{q^4}{(q^2 + 1)} x^{-2} y^3, \\
  e_2 x &= 0, & e_2 y &= -\frac{q^2}{(q^2 + 1)} x^{-2} y. 
\end{align*} \tag{2.11}
\]
These derivations are, as we shall see, inner.

A differential must satisfy the constraint
\[
d(xy - qyx) = dxy + xdy - qdyx - qydx = (dxy - qdyx) + (xdy - qydx) = 0. \tag{2.12}
\]
This we can satisfy, for example, by setting
\[
\begin{align*}
  dxy - qydx &= 0, & qdyx - xdy &= 0, \tag{2.13}
\end{align*}
\]
which defines the commutation rules of $dx$ and $y$, and $dy$ and $x$. In order to complete the definition one must add the rules for $dx$ and $x$ and $dy$ and $y$. For example,
\[
\begin{align*}
  dxy &= qdyx, & xdy &= qdyx, \\
  xdx &= qdx, & qdy &= dyy. \tag{2.14}
\end{align*}
\]
Applying $d$ once more, one thus obtains the exterior product
\[
(dx)^2 = 0, \quad (dy)^2 = 0, \quad dxdy = -qdydx. \tag{2.15}
\]

For reasons [10] which do not concern us here (deformed symmetries), one prefers another differential calculus constructed by setting instead of (2.13)
\[
\begin{align*}
  dxy - qydx &= (1 - q^2)dxyqdyx - xdy = (1 - q^2)dxy. 
\end{align*} \tag{2.16}
\]
The full set of relations for the 1-differential forms would be in this case
\[
\begin{align*}
  qdyx &= ydx, & xdy &= qdyx + (q^2 - 1)dxy, \\
  xdx &= q^2 dx, & ydy &= q^2 dyy. \tag{2.17}
\end{align*}
\]
In this case the exterior product is given by
\[
(dx)^2 = 0, \quad (dy)^2 = 0, \quad qdxdy = -dydx. \tag{2.18}
\]
We shall see that it is based on the inner derivations (2.11) defined above. The relation between the unusual structure of these derivations and the deformed symmetries is not completely understood.
3 Frame: 2-d de Sitter

As we have seen, there is a variety of possibilities to define a differential. One problem is how to determine or at least restrict it by imposing some physical requirements. We shall use here a modification of the moving frame formalism and show that so defined differential calculi over an algebra admit essentially a unique metric and linear connection. We shall fix therefore the differential calculus by requiring that the metric have the desired classical limit. The idea is to define an analogue of a parallelizable manifold, which has therefore a globally defined frame. The frame is defined either as a set of vector fields $e_i$ or as a set of 1-forms $\theta^i$ dual to them. The metric components with respect to the frame are then constant.

We choose a set of $n$ derivations $e_i$ which we assume to be inner generated by ‘momenta’ $p_i$:

$$e_i f = [p_i, f].$$

(3.1)

We suppose that the momenta generate also the whole algebra $A$. Since the center is trivial, this means that an element which commutes with all momenta must be a complex number. An alternative way is to use the 1-forms $\theta^i$ dual to $e_i$ such that relation

$$\theta^i(e_j) = \delta^i_j$$

(3.2)

holds. To define the left hand side of this equation we define first the differential, exactly as in the classical case, by the condition

$$df(e_i) = e_i f \theta^i.$$  

(3.3)

The left and right multiplication by elements of the algebra $A$ are defined by

$$fg = fe_i g \theta^i, \quad dg = e_i gf \theta^i.$$  

(3.4)

Since every 1-form can be written as sum of such terms the definition is complete. In particular, since

$$f \theta^i(e_j) = f \delta^i_j = (\theta^i f)(e_j),$$  

(3.5)

we conclude that the frame necessarily commutes with all the elements of the algebra $A$; this is a characteristic feature.

In the case of the algebra $M_2$ considered above, the module of 1-forms is generated by three elements $d\sigma_i$ defined as the maps

$$d\sigma_i(e_j) = e_j \sigma_i = [\sigma_j, \sigma_i].$$

The maps $\sigma_k d\sigma_i$ and $d\sigma_i \sigma_k$ are defined respectively as

$$\sigma_k d\sigma_i(e_j) = \sigma_k [\sigma_j, \sigma_i], \quad d\sigma_i \sigma_k(e_j) = [\sigma_j, \sigma_i] \sigma_k.$$  

Obviously, $\sigma_k d\sigma_i \neq d\sigma_i \sigma_k$. The 1-form $\theta$ defined as

$$\theta = -p_i \theta^i$$

(3.6)

can be considered as an analog of the Dirac operator in ordinary geometry. It implements the action of the exterior derivative on elements of the algebra. That is

$$df = -[\theta, f] = [p_i \theta^i, f] = [p_i, f] \theta^i.$$  

(3.7)
The differential is real if \((df)^* = df^*\). This is assured if the derivations \(e_i\) are real: \(e_i f^* = (e_i f)^*\), which is the case if the momenta \(p_i\) are antihermitean. From the definitions one has \(\theta^* = \theta^\dagger\), \(\theta^* = -\theta\). Furthermore, \((f\xi)^* = \xi^* f^*\), \((\xi f)^* = f^* \xi^*\), and \((\xi \eta)^* = -\eta^* \xi^*\). Note that whereas the product of two hermitean elements is hermitean only if they commute, the product of two hermitean 1-forms is hermitean only if they anticommute.

Consider once more the quantum plane introduced in the previous section. The momenta \(p_i\) can be defined as

\[ p_1 = \frac{q}{q-1} y, \quad p_2 = \frac{q}{q-1} x. \]  

(3.8)

From these expressions one easily finds the relations

\[ e_1 x = -xy, \quad e_2 x = 0, \]
\[ e_1 y = 0, \quad e_2 y = xy. \]

Using the definition \( df = e_i (f) \theta^i \), one obtains for \( \theta^1 \) and \( \theta^2 \),

\[ \theta^1 = -y^{-1} x^{-1} dx, \quad \theta^2 = y^{-1} x^{-1} dy. \]

(3.9)

From (3.5) the module structure (2.14) can be reconstructed. The momenta \( p_i \) satisfy the quadratic relation

\[ p_2 p_1 = qp_1 p_2. \]

(3.10)

The second differential calculus ([10]) on the quantum plane also has a frame. The corresponding momentum generators are

\[ p_1 = \frac{1}{q^{3} - 1} x^{-2} y^2, \quad p_2 = \frac{1}{q^{4} - 1} x^{-2}. \]

(3.11)

They satisfy

\[ p_1 p_2 = q^4 p_2 p_1. \]

(3.12)

Note that the momenta are singular in the limit \( q \to 1 \). In quantum mechanics the relation between the differential and the momentum is given by

\[ \frac{\partial f}{\partial x} = \frac{i}{\hbar} [p_{Q.M.}, f], \]

whereas it is given here by

\[ \frac{\partial f}{\partial x} = [p, f]. \]

That is, \( p = \frac{i}{\hbar} p_{Q.M.} \). The singularity of the classical limit \( \hbar \to 0 \) has been included in the definition of the momentum.

The implementation of the differential structure as we have given is just as arbitrary as before since it amounts to a choice of the momenta. In some cases, the construction of the frame is not difficult. In the example (2.1) one can choose the differential such that \([x^i, dx^j] = 0\): a frame is \( \theta^i = dx^i \) since \( dx^i \) commute with all elements of the algebra. The most general form is \( \theta^i = \Lambda_j^i dx^j \) with \( \Lambda_j^i \) real numbers. The duality relations give the momenta

\[ \delta_j^i = \theta^i (e_j) = dx^i (e_j) = e_j (x^i) = [p_j, x^i], \]

(3.13)
that is,

\[ p_j = -iJ_{ji}^{-1}x^i. \]  
(3.14)

In fact, in order to discuss noncommutative limit (2.1) should be rewritten as

\[ [x^i, x^j] = i\kappa J^{ij}. \]  
(3.15)

where the parameter \( \kappa \) describes the fundamental area scale on which noncommutativity becomes important. It is presumably of order of the Planck area \( G\hbar \); the commutative limit is then defined by \( \kappa \to 0 \). The momenta read

\[ p_j = -\frac{i}{\kappa}J_{ji}^{-1}x^i, \]  
(3.16)

and they are singular in the limit \( \kappa \to 0 \). Since the frame is given by \( \theta^i = dx^i \), this space can be naturally thought of as the noncommutative generalization of flat space. The momenta are linear in the coordinates and hence

\[ [p_i, p_j] = -\frac{1}{\kappa^2}J_{im}^{-1}J_{jn}^{-1}[x^m, x^n] = \frac{i}{\kappa}J_{ij}^{-1}. \]  
(3.17)

In general only by explicit construction can one show that the frame exists. In the case of the Lie algebra (2.2), for example, one sees that the 1-forms \( dx^i \) do not define a frame because they do not commute with the algebra. In the example of \( M_2 \) with Pauli matrices as momenta, the frame which is the solution to the equation (3.2) is seen to be

\[ \theta^i = \frac{1}{2}\sigma_j\sigma^i d\sigma^j. \]

This construction can be repeated [11] for the algebra \( M_n \) of \( n \times n \) complex matrices.

As the main example of this section we consider the algebra generated by two hermitean elements \( u \) and \( v \) related by

\[ [u, v] = -2\hbar v. \]  
(3.18)

This is the Jordanian deformation [12] of \( \mathbb{R}^2 \). To find the frame, we rewrite this as follows

\[ (u + h)v = v(u - h). \]  
(3.19)

The differential must satisfy

\[ dvu + (u + h)dv = dv(u - h) + vdu. \]  
(3.20)

One can impose separately the conditions

\[ dvu = vdu, \]

\[ (u + h)dv = dv(u - h). \]  
(3.21)

The first of these relations suggests that \( du \) can be taken as a frame element, in fact \( f(v)du \) as well as \( du \). We set \( \theta^1 = f(v)du \). Rewriting (3.21) as

\[ (u + h)vv^{-1}dv = vv^{-1}dvv^{-1}v(u - h), \]  
(3.22)

we see that we can take \( \theta^2 = -v^{-1}dv \). Of course, we assume that \( \theta^i \) commute with \( u \) and \( v \). The duality relations (3.2) determine \( f(v) \). They read

\[ f(v)[p_1, u] = 1, \quad f(v)[p_2, u] = 0, \]

\[ v^{-1}[p_1, v] = 0, \quad v^{-1}[p_2, v] = -1, \]
and obviously reduce to (3.21) for
\[ p_2 = \frac{1}{2\hbar} u, \quad p_1 = \frac{1}{2\hbar} v, \quad f(v) = v^{-1}. \] (3.23)
The frame therefore is given by
\[ \theta^1 = v^{-1} du, \quad \theta^2 = -v^{-1} dv. \] (3.24)
In this case, too, the momenta are proportional to the coordinates so their commutation relation is:
\[ [p_1, p_2] = p_1. \] (3.25)
A short calculation shows that the frame elements anticommute. One can show in fact that the corresponding metric is that of the Lobachevski plane or of (anti) de Sitter space depending on the choice of signature.

4 Momentum algebra: Rindler frame

In the last example of the previous section we saw that typically there is little freedom in finding the solution to duality and consistency equations. This is due to the relations among the momentum generators which we shall now derive. In the basis of 1-forms \( \theta^i \) the exterior product can be written explicitly
\[ \theta^i \theta^j = P^{ij}_{kl} \theta^k \otimes \theta^l. \] (4.1)
The \( P^{ij}_{kl} \) are complex numbers which satisfy the projector condition
\[ P^{ij}_{kl} = P^{ij}_{mn} P^{mn}_{kl}. \]
The basis 1-forms anticommute for \( P^{ij}_{kl} = \frac{1}{2} (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \). The exterior derivative of \( \theta^i \) is a 2-form, so it can be written as
\[ d\theta^i = -\frac{1}{2} C^i_{jk} \theta^j \theta^k. \] (4.2)
The \( C^i_{jk} \) are called the structure elements. They can be chosen to satisfy \( C^i_{jk} = C^i_{lm} P^{lm}_{jk} \).

Impose now the condition \( d^2 = 0 \). It gives
\[ 0 = d(df) = d(-[\theta, f]) = -[d\theta + \theta^2, f], \] (4.3)
so it implies that \( d\theta + \theta^2 \) commutes with all elements of the algebra. Since \( d\theta + \theta^2 \) is a 2-form, in the frame basis it can be written as
\[ d\theta + \theta^2 = -\frac{1}{2} K_{ij} \theta^i \theta^j, \] (4.4)
where the elements \( K_{ij} \) are complex numbers. One can impose \( K_{ij} P^{ij}_{kl} = K_{kl} \). A straightforward calculation shows that
\[ d\theta = -dp_i \theta^i - p_i d\theta^i = [p_j, p_i] \theta^j \theta^i + \frac{1}{2} p_i C^i_{jk} \theta^j \theta^k, \]
\[ \theta^2 = p_i p_j \theta^i \theta^j, \] (4.5)
and hence (4.4) reduces to
\[
(p_kp_j + \frac{1}{2} C^i_{jk}p_i + \frac{1}{2} K_{jk})\theta^i\theta^k = 0
\] (4.6)
which can be written as
\[
2p_m p_l p^{lm}_{jk} + (p_i C^i_{jk} + K_{jk}) = 0.
\] (4.7)
The relation \(d(f\theta^i - \theta^i f) = 0\) written in terms of the momenta gives further restrictions. It reads
\[
[p_j\delta^i_k + p_k\delta^i_j + \frac{1}{2} C^i_{jk}, f]\theta^j\theta^k = 0
\] (4.8)
which means
\[
(C^i_{jk} + 2p_j\delta^i_k + 2p_k\delta^i_j - F^i_{jk})\theta^j\theta^k = 0,
\] (4.9)
where \(F^i_{jk}\) are complex numbers. Thus the structure elements are linear in the momenta:
\[
C^i_{jk} = F^i_{jk} - 2p_k P^{(kl)}_{ij}.
\] (4.10)
The \(()\) denotes symmetrization on the indices. Combining (4.7) and (4.10) we obtain the relation
\[
p_k p_l P^{kl}_{ij} - p_k F^k_{ij} - K_{ij} = 0.
\] (4.11)
The coefficients in (4.11) are complex numbers. The momentum generators \(p_i\) satisfy a quadratic relation.

In order to understand the restrictions which (4.11) brings we shall construct noncommutative generalizations of \(2-d\) Rindler space, that is one-half of \(2-d\) Minkowski space in Rindler coordinates. We use this example also to clarify that different moving frames produce different calculi. The commutative Rindler frame is given by \(\theta^0 = \mu x dt\) and \(\theta^1 = dx\); the commutative Minkowski frame is \(\theta^0 = dt\) and \(\theta^1 = dx\). The classical coordinate transformation
\[
x' = x \cosh \mu t, \quad t' = x \sinh \mu t
\]
induces the transformation \(\theta^i = \Lambda^i_{\ j}\theta^j\) with
\[
\Lambda = \begin{pmatrix}
\cosh \mu t & -\sinh \mu t \\
-\sinh \mu t & \cosh \mu t
\end{pmatrix}
\]
from the latter to the former. We shall construct a noncommutative version of both of these frames and show that their commutative limits differ. In the Minkowski case the limit is the original frame whereas in the Rindler case the limit is quite different.

We shall first now show that the Rindler frame does not fit into the general formalism as outlined in the beginning of the section; there are no dual momenta. If the momenta \(p_i\) did exist then they would necessarily satisfy the relations
\[
[p_0, \mu] = (\mu x)^{-1}, \quad [p_0, x] = 0,
\]
\[
[p_1, \mu] = 0, \quad [p_1, x] = 1.
\] (4.12)
But one easy sees that these are not consistent with the Leibniz rules. In fact if we set \([p_0, p_1] = L_{01}\) and \([t, x] = ikJ^{01}\), we find from the Jacobi identities that
\[
[L_{01}, t] = [p_0, [p_1, t]] - [p_1, [p_0, t]] = \mu^{-1} x^{-2},
\] (4.13)
\[
[L_{01}, x] = [p_0, [p_1, x]] - [p_1, [p_0, x]] = 0.
\] (4.14)
The $L_{01}$ commutes with $x$ and therefore belongs to the algebra generated by $x$. From the commutation with $t$ we see that

$$i\kappa \mu L_{01} J^{01} = -x^{-2}. \quad (4.13)$$

Similarly we find that

$$i\kappa [p_0, J^{01}] = [[p_0, t], x] - [[p_0, x], t] = 0, \quad (4.15)$$

$$i\kappa [p_{01}, J^{01}] = [[p_{01}, t], x] - [[p_{01}, x], t] = 0 \quad (4.16)$$

from which we conclude that $J^{01}$ is constant. We deduce therefore that

$$L_{01} = \frac{1}{i\kappa \mu J^{01}} x^{-1}. \quad (4.17)$$

But the duality relations (4.12) require

$$p_0 = \frac{1}{i\kappa \mu J^{01}} \log(\mu x), \quad p_1 = \frac{1}{i\kappa J^{01}} t \quad (4.17)$$

and thus one easily sees that

$$L_{01} = \frac{1}{i\kappa \mu J^{01}} x^{-1} = \frac{1}{i\kappa \mu J^{01}} e^{-i\kappa \mu J^{01} p_0} \quad (4.17)$$

which is not a quadratic expression in $p_0$ and $p_1$.

Although the momenta $p_i$ dual to the frame which we have used do not satisfy quadratic relation it is easy to introduce another set $\tilde{p}_i$ which do. To alleviate the formula we choose units such that $\kappa \mu = 1$. We have then

$$[p_0, p_1] = \frac{1}{i J^{01}} x^{-1} = \frac{1}{i J^{01}} e^{-i J^{01} p_0}.$$  

We define the new momenta by the equations

$$i \tilde{p}_0 = e^{i J^{01} p_0}, \quad \tilde{p}_1 = p_1.$$  

They obey the commutation relation

$$[\tilde{p}_0, \tilde{p}_1] = i J^{01} \tilde{p}_0 [p_0, p_1] = \tilde{p}_0 e^{-i J^{01} p_0} = -\frac{1}{i \kappa}.$$  

From (4.17) one see also that $\tilde{p}_i$ are related to the coordinate generators by the transformations

$$x = i \kappa \tilde{p}_0, \quad t = i \kappa J^{01} \tilde{p}_1$$

and therefore

$$[t, x] = i \kappa J^{01},$$

as well as

$$[\tilde{p}_0, t] = 1, \quad [\tilde{p}_1, x] = 1.$$  

The dual frame is the Minkowski coordinate frame; note that transforming from $(p_0, p_1)$ to $(\tilde{p}_0, \tilde{p}_1)$ we obtained the flat differential calculus.

One can show that conjugate momenta exist for a family of simple modifications of the original Rindler frame but which have not a satisfactory commutative limit.
Consider a modification obtained by replacing the coordinate \( x \) in the definition of the frame by an arbitrary function \( f(x) \). That is

\[
\theta^0 = f(x) \, dt, \quad \theta^1 = dx.
\]  

(4.18)

The frame formalism determines completely the structure of the algebra of forms. With the above Ansatz the differential calculus reads

\[
dx x = x \, dx, \quad dx t = t \, dx,
\]

\[
dt x = x \, dt, \quad dt t = (t + ikJ^{01}(\frac{d}{dx} \log f)) \, dt.
\]  

(4.19)

Also

\[
(dx)^2 = 0, \quad dx dt = -dt dx, \quad (dt)^2 = -\frac{1}{2} ikJ^{01}(\frac{d^2}{dx^2} \log f) \, dx \, dt,
\]  

(4.20)

as (4.19) imply \( dJ^{01} = 0 \). We shall see below that these relations are always valid in spite of the fact that the actual value of \( f \) is not that of the Rindler frame.

The commutative metric given by \( f(x) \) is curved with the scalar curvature

\[
R = -2f^{-1} f''.
\]

We have

\[
[p_0, t] = f^{-1}, \quad [p_0, x] = 0,
\]

\[
[p_1, t] = 0, \quad [p_1, x] = 1.
\]  

(4.21)

If we denote as before \( [p_0, p_1] = k^{-1}L_{01} \) and \( [t, x] = ikJ^{01} \), the Jacobi identities imply

\[
[p_0, J^{01}] = 0, \quad [p_1, J^{01}] = 0
\]

\[
[t, L_{01}] = -f' f^{-2}, \quad [x, L_{01}] = 0.
\]  

(4.22)

From (4.22) one can conclude that \( J^{01} \) is constant and that \( L_{01} \) is a function of \( x \) alone. It follows that for some constants \( c_i \)

\[
p_0 = \frac{1}{ikJ^{01}} \int f^{-1} + c_0, \quad p_1 = \frac{1}{ikJ^{01}} t + c_1
\]  

(4.23)

and thus each of the pairs \((t, x)\) and \((p_0, p_1)\) generates the algebra.

The assumption that the momenta exist restricts almost uniquely the function \( f \). Following (4.11), \( L_{01} \) is at most a quadratic function of the momenta. Assume that it is of the form

\[
[p_0, p_1] = ik^{-1} + \mu^2 kp_0^2.
\]  

(4.24)

Then (4.24) combined with (4.23), that is

\[
[p_0, p_1] = -\frac{dp_0}{dx},
\]

determines \( p_0 = p_0(x) \). One finds the relations

\[
i\mu kp_0 = \tanh(\mu x), \quad f(x) = \frac{1}{J_{01}} \cosh^2(\mu x).
\]  

(4.25)

In the limit \( \mu \to 0 \) these become

\[
i\mu p_0 \to x, \quad f \to \frac{1}{J_{01}}(1 - \mu^2 x^2).
\]
Although these values are not those of the classical limit, the relation (4.23) is satisfied. The frames of similar types appear [13] in two dimensional dilaton gravity theories. See also the review article by Grumiller et al. [14]. Analogous analysis can be performed assuming that the relation (4.11) is linear in the momenta; one can show that the resulting algebra is the one of anti-de Sitter space discussed at the end of the previous section.

Let us compare two differential calculi given by different frames. Suppose given a map

$$x'^i = F^i(x^j)$$

between two sets of generators $(t, x)$ and $(t', x')$ of a given algebra. It induces a map

$$\theta'^i = \Lambda^j_i \theta^j$$

of the frame $\theta^i = dx^i$ onto the frame $\theta'^i = dx'^i$. Each of the frames defines a differential calculus: each of the commutators defines a Poisson structure. To distinguish a frame from an arbitrary basis of a module of 1-forms we call the latter $\eta^i$. We set then

$$\eta^i = \Lambda^j_i \theta^j$$

when we wish to consider the left-hand side as elements of the differential calculus defined by $\theta^i$. This means in particular that

$$[f, \eta^i] = [f, \Lambda^j_i] \theta^j. \quad (4.26)$$

This illustrates that in a natural way one can construct a basis of a module of 1-forms which is not a frame.

The transformation from flat to Rindler coordinates depends on $t$, and as $x$ and $t$ do not commute the analysis can be carried out to first order in $\kappa$. For simplicity we put $\mu = 1$. We shall choose $(t, x)$ as Rindler coordinates and $(t', x')$ as flat coordinates. The relation between the two is

$$x' = x \cosh t, \quad t' = x \sinh t.$$

The Rindler frame is, replacing $x$ by $f(x)$, $\theta^0 = f dt$, $\theta^1 = dx$. The flat frame is $\theta'^i = d' x'^i$; the flat frame in Rindler generators

$$\eta^0 = f d't, \quad \eta^1 = f' d' x.$$

The differential $d$ is that of the Rindler calculus; the differential $d'$ is that of the flat calculus. The Rindler differential calculus is defined by the module relations

$$[x^j, \theta^i] = 0.$$

The flat differential calculus can be defined by the module relations

$$[x^j, \theta'^i] = 0.$$

One can now explore the geometry of the noncommutative space using the Rindler coordinates $(t, x)$ and the flat differential calculus $d'$ defined by the $\theta'^i$ and the corresponding momenta $p'_i$. The calculations can be performed only to first order. We have

$$[p'^0_0, t] = x^{-1} \cosh t, \quad [p'^0_0, x] = - \sinh t$$

$$[p'^1_0, t] = -x^{-1} \sinh t, \quad [p'^1_0, x] = \cosh t \quad (4.27)$$
One can verify easily that the Jacobi identities for \( k^{-1}L_{01}' = [p_0', p_1'] \)
\[
[t, L_{01}'] = 0, \quad [x, L_{01}'] = 0
\]
as it should be. On the other hand, for the commutator \( ikJ^{01} = [t, x] \) we get
\[
[p_0', J^{01}] = x^{-1} \sinh t J^{01}, \quad [p_1', J^{01}] = -x^{-1} \cosh t J^{01}.
\]
The solution to these equations to the first order is \( J^{01} = cx \) for some constant \( c \).

The relations in the module of 1-forms in the basis \( \eta^i \) can be also obtained. From the general formula (4.26) we see
\[
[t, \eta^0] = 0, \quad [t, \eta^1] = 0,
\]
\[
[x, \eta^0] = [x, \cosh t](\cosh \eta^0 + \sinh \eta^1) - [x, \sinh t](\sinh \eta^0 + \cosh \eta^1),
\]
\[
[x, \eta^1] = -[x, \sinh t](\cosh \eta^0 + \sinh \eta^1) + [x, \sinh t](\sinh \eta^0 + \cosh \eta^1).
\]
To within the lowest-order corrections of the classical limit we find that
\[
[x, \cosh t] = ikcx^{-1} \sinh t, \quad [x, \sinh t] = ikcx^{-1} \cosh t.
\]
and therefore that
\[
[t, \eta^0] = 0, \quad [t, \eta^1] = 0,
\]
\[
[x, \eta^0] = -ikcx^{-1} \eta^1, \quad [x, \eta^1] = -ikcx^{-1} \eta^0.
\]
In terms of the differentials
\[
t \, dt' = dt \, t - icx^{-2} d't \quad x \, dt = dt \, x + icx^{-2} d'x
\]
\[
t \, dx' = dx \, t \quad x \, dx = dx \, x + icdt.
\]
Also,
\[
d'x \, dx = 0, \quad d't \, dx' = -d'x \, dt', \quad d't \, dt' = -\frac{1}{2} icd'(x^{-2}) \, d't.
\]

5 Calculi without derivations: Fuzzy spheres

Let us finally observe that we can define a differential calculus using the frame \( \theta^0 = f(x)dx, \theta^1 = dx \) for any function \( f(x) \) including \( f(x) = \mu x \), if only we abandon the existence of the momenta. All one need to impose is that the frame commutes with all functions in the algebra, the Leibniz rules and \( d^2 = 0 \). We obtain the relations which define the algebra of forms:
\[
\theta^0 \theta^1 = -\theta^1 \theta^0, \quad (\theta^1)^2 = 0
\]
as well as
\[
(\theta^0)^2 = -\frac{1}{2} i kJ^{01} f(\frac{d^2}{dx^2} \log f) \theta^0 \theta^1
\]
with constant \( J^{01} \); the algebra is the same as (4.19). In the case \( f(x) = \mu x \) these relations read
\[
(dt)^2 = \frac{1}{2} i eJ^{01}(\mu x)^{-2} dt \, dx, \quad dx \, dt = -dt \, dx, \quad (dx)^2 = 0.
\]

In order to understand the calculus-without-derivations better we mention that a similar calculus on the fuzzy sphere. The parameter \( \mu \) is the inverse radius of the sphere.
and we set $\epsilon = \mu^2 k$. Consider an algebra with two generators $\chi$, $\phi$ and introduce a frame

$$\theta^1 = d\chi, \quad \theta^2 = \sin \chi d\phi.$$  

Assume as usual that the frame elements commute with the generators $\chi$, $\phi$. This gives immediately

$$[\chi, d\phi] = 0, \quad [\chi, d\chi] = 0, \quad [\phi, d\chi] = 0 \quad (5.2)$$

which implies

$$(d\chi)^2 = (\theta^2)^2 = 0, \quad \theta^1 \theta^2 = -\theta^2 \theta^1$$

as well as

$$dJ^{12} = 0.$$  

As usual we denote $[\chi, \phi] = i \epsilon J^{12}$; we shall set

$$J^{12} = 1.$$  

The rest of the relations yield the differential identities

$$[\phi, d\phi] = i \epsilon \cot \chi d\phi. \quad (5.3)$$

We can take the differential of this identity to obtain the relation

$$(d\phi)^2 = -\frac{1}{2} i \epsilon \sin^{-2} \chi d\chi d\phi, \quad (\theta^2)^2 = -\frac{1}{2} i \epsilon \sin^{-1} \chi \theta^1 \theta^2.$$  

These differentials do not seem to be associated with momenta but one can look for a ‘Dirac operator’. We introduce

$$Q = -p_i \theta^i$$

and we look for $p_i$ such that

$$d\phi = -[Q, \phi] = [p_1, \phi] \theta^1 + [p_2, \phi] \theta^2; \quad (5.4)$$

$$d\chi = -[Q, \chi] = [p_1, \chi] \theta^1 + [p_2, \chi] \theta^2. \quad (5.5)$$

From the relations

$$[p_1, \phi] = 0, \quad [p_1, \chi] = 1,$$

$$[p_2, \phi] = \sin^{-1} \chi, \quad [p_2, \chi] = 0$$

we can conclude that

$$i \epsilon p_1 = -\phi, \quad i \epsilon p_2 = G,$$  

$$\frac{dG}{d\chi} = \frac{1}{\sin \chi}.$$  

The function $G(\chi)$ is therefore given by

$$G(\chi) = \frac{1}{2} \log \frac{1 - \cos \chi}{1 + \cos \chi}.$$  

Finally the Dirac operator $Q$ reduces to

$$Q = \frac{1}{i \epsilon} \phi d\chi - \frac{1}{i \epsilon} G(\chi) \sin \chi d\phi.$$  

We have then

$$dQ = -\frac{1}{i \epsilon} (2 \sin^{-1} \chi + G \cot \chi) \theta^1 \theta^2.$$  

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\[ Q^2 = -\frac{1}{2} \frac{1}{i\epsilon} G^2 \sin^{-1} \chi \theta^1 \theta^2. \]

For any \( f(\chi, \phi) \) one has
\[ d^2 f = [dQ + Q^2, f] \]

The differential \( d \) is well defined then only on \( f \) such that
\[ [dQ + Q^2, f] = 0. \]

It would be well-defined on all \( f \) only if for some \( c \in \mathbb{R} \),
\[ dQ + Q^2 = c, \]

and we see that this condition does not hold, that is, it is not possible to introduce the momenta in a consistent way.

A solution to this problem [15, 2] is to consider only 2-forms modulo the image of \( d^2 \). This would result in a consistent differential calculus but with only 2-forms depending on \( \chi \), with perhaps special functions of \( \phi \). One should not of course be too attached to the condition \( d^2 = 0 \). A non-vanishing value for this operator could be interpreted as some sort of ‘micro-curvature’. In a subsequent article [16] the authors will examine the relation between this ‘micro-curvature’ and ordinary curvature using the WKB approximation.

This differential calculus can be compared with the usual one on the fuzzy sphere by introducing variables \( x^\lambda \) by the usual equations
\[ x^\lambda = (\sin \chi \cos \phi, \sin \chi \sin \phi, \cos \chi). \]

We have then
\[ dx^\lambda = (\cos \chi \cos \phi, \cos \chi \sin \phi, -\sin \chi) \theta^1 - (\sin \phi, \cos \phi, 0) \theta^2 \]

The new differential calculus would not be suitable to describe a round sphere since the \( SO_3 \) symmetry is not respected.

### 6 Differential geometry

We have presented several noncommutative ‘blurings’ of classical geometries, all of which are of dimension two. We have concentrated our attention on the new aspects on the noncommutative theory, especially the multiple choices of differential calculus and the relation of the geometry to the symplectic structures. We have not, in fact, introduced the metric, the connection or the curvature on the noncommutative space. This can be done by taking the commutative limit and using the definition of a metric in terms of the frame. It can also be done [2] before the limit is taken. To complete the analysis of the Rindler example, we mention briefly linear connections and curvature without defining them in the full rigor; for details we refer to [8]. Note that when the momenta exist the metric is given; otherwise there is a certain ambiguity which must be determined by field equations. For the Rindler frame the torsion-free metric-compatible connection and curvature are classically given by
\[ \omega^0_1 = F \theta^0 = f' dt, \quad \Omega^0_1 = -f^{-1} f'' \theta^0 \theta^1 = -f'' dt dx, \quad f' = \frac{df}{dx}. \]
The geometry is flat only if $f$ is linear in $x$. We have here set
\[ F = \mu^{-1} \frac{d}{dx} \log f \]
to simplify the presentation.

The fuzzy calculus differs from the commutative limit only in the relation (5.1) which we rewrite as
\[ (\theta^0)^2 = i\epsilon \kappa \theta^0 \theta^1, \quad \kappa = -\frac{1}{2} F^{01} \mu^{-1} f F'. \]
We designate by $i\epsilon \zeta^{ij}_j$ the noncommutative correction to the connection. It must satisfy the equations
\[
\begin{align*}
\omega^0_0 \theta^0 + \zeta^0_1 \theta^1 &= 0, \quad (6.1) \\
\omega^1_0 \theta^0 + \zeta^1_0 \theta^0 + \zeta^1_1 \theta^1 &= 0. \quad (6.2)
\end{align*}
\]
To determine the solution of (6.1–6.2) we must impose that the metric be symmetric and real. For this we need the array $P^{ij}_{kl}$ which we write as
\[ P^{ij}_{kl} = \frac{1}{2} \delta^{ij}_{kl} + \epsilon Q^{ij}_{kl} \]
with the only non-vanishing components
\[ -Q^{01}_{00} = Q^{10}_{00} = Q^{00}_{01} = -Q^{00}_{10} = \kappa. \]
It follows [8] that the metric components are of the form
\[ g^{ij} = \eta^{ij} + i\epsilon \kappa \epsilon^{ij}. \]
The condition that the connection be metric-compatible is to first order given by the formula
\[ \zeta^{(ij)} + \omega^{(ijk)} \epsilon^{kj} - 2 \omega^{(i} \nu_{jk} \theta^m = 0. \]
That is,
\[
\begin{align*}
\zeta^{(00)} - 2 \kappa \omega^0_1 - 2 \omega^0_0 \nu_{01} \theta^0 &= 0, \quad (6.3) \\
\zeta^{(01)} - \omega^0_0 \nu_{01} \theta^0 - \omega^1_0 \nu_{00} \theta^1 &= 0, \quad (6.4) \\
\zeta^{(11)} + 2 \kappa \omega^1_0 - 2 \omega^1_0 \nu_{01} \theta^0 &= 0. \quad (6.5)
\end{align*}
\]
This simplifies to
\[
\begin{align*}
\zeta^0_0 &= -2 \kappa \omega^0_1, \quad (6.6) \\
\zeta^0_1 - \zeta^1_0 &= \kappa F(\theta^0 + \theta^1), \quad (6.7) \\
\zeta^1_1 &= -2 \kappa \omega^1_0. \quad (6.8)
\end{align*}
\]
The form of these expressions would seem to confirm the choice made [8] previously of a twisted definition of symmetry. From (6.1) and (6.2) we find that for some $a$ and $b$
\[ \zeta^0_1 = a \theta^1, \quad \zeta^1_0 = b \theta^0 - \kappa F \theta^1 \]
from which we can conclude that $a = 0$, $b = -\kappa F$ and therefore the connection is to first order given by
\[
\begin{align*}
\omega^0_0 &= -2 \epsilon \kappa F \theta^0, \\
\omega^1_0 &= F \theta^0 - i \epsilon \kappa F(\theta^0 + \theta^1), \\
\omega^0_1 &= F \theta^0, \\
\omega^1_1 &= -2 \epsilon \kappa F \theta^0.
\end{align*}
\]
This can more simply be written in terms of the natural basis $dx^i$ as

$$
\begin{align*}
\omega^0_0 &= -2i\epsilon f'dt, & \omega^0_1 &= f'dt, \\
\omega^1_0 &= f'dt - i\epsilon f'(dt + f^{-1}dx), & \omega^1_1 &= -2i\epsilon f'dt.
\end{align*}
$$

The curvature, also to first order in $\epsilon = \mu^2k$ reads

$$
\begin{align*}
\Omega^0_0 &= 2i\epsilon f^{-1}(\kappa f')\theta^0\theta^1 \\
\Omega^0_1 &= -f^{-1}f''\theta^0\theta^1 \\
\Omega^1_0 &= -f^{-1}f''\theta^0\theta^1 + i\epsilon f^{-1}(\kappa f')\theta^0\theta^1 \\
\Omega^1_1 &= 2i\epsilon f^{-1}(\kappa f')\theta^0\theta^1 + i\epsilon kF^2\theta^0\theta^1
\end{align*}
$$

This geometry could furnish convenient model to study noncommutative effects, for example in the colliding-$D$-brane description of the Big-Bang proposed by Turok & Steinhardt [17]. The 2-space describing the time evolution of the separation of the branes has been shown to be conveniently described using Rindler coordinates. One can blur this geometry by using the description given in Section 4. The flat curvature would have to be replaced by the expression above.

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