Observability for parabolic equations from a measurable set in time

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EXACT CONTROL FOR THE HEAT EQUATION

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, $\omega \subset \Omega$, $T > 0$. Given $y_0 \in L^2(\Omega)$, there is $(y, f)$ satisfying

\[
\begin{align*}
\begin{cases}
\partial_t y - \Delta y = 1_{\omega} f & \text{in } \Omega \times (0, T), \\
y = 0 & \text{on } \partial \Omega \times (0, T), \\
y(\cdot, 0) = y_0 & \text{in } \Omega, \\
y(\cdot, T) = 0 & \text{in } \Omega, \\
\|f\|_{L^2(\Omega \times (0, T))} \leq c \|y_0\|_{L^2(\Omega)}.
\end{cases}
\end{align*}
\]

G. Lebeau, L. Robbiano, CPDE (1995).

A. Fursikov, O. Imanuvilov, Lecture Note Series (1996). Also OK for parabolic equations.
EQUIVALENTLY

\[
\begin{aligned}
\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T), \\
u = 0 & \text{on } \partial\Omega \times (0, T), \\
u (\cdot, 0) \in L^2(\Omega),
\end{cases}
\end{aligned}
\]

\[
\|u (\cdot, T)\|_{L^2(\Omega)}^2 \leq c \int_\omega \int_0^T |u (x, t)|^2 \, dx \, dt.
\]
Hölder type Observation for elliptic

\[ \| w \|_{H^1(\Omega \times (\gamma, T-\gamma))} \leq C \left( \| w \|_{L^2(\omega \times (0, T))} \right)^{1-\alpha} \left( \| w \|_{H^1(\Omega \times (0, T))} \right)^\alpha \]

for any \( w \in H^2(\Omega \times (0, T)) \), \( (\partial^2_t + \Delta) w = 0 \) and \( w|_{\partial \Omega} = 0 \).

\[ \Rightarrow \] Controllability in finite and infinite dim

\[ \Rightarrow \] Observability
Hölder type Observation for elliptic

\[ \Rightarrow \text{Sum of eigenfunctions estimate} \]

\[ \Rightarrow \text{Observability} \]

G. Lebeau, E. Zuazua, *ARMA (1998).*

L. Miller, *DCDS (2010).*
I. A similar way for parabolic equations

II. Replace \((0, T)\) by a measurable set \(E\) in time, \(|E| > 0\)

III. Time optimal bang-bang control
NEW OBSERVABILITY INEQUALITY

\[ a \in L^\infty (0, T; L^q (\Omega)), \; q \geq 2 \; \text{and} \; b \in L^\infty (\Omega \times (0, T))^n \]

\[
\begin{cases}
\partial_t u - \Delta u + au + b \cdot \nabla u = 0 & \text{in } \Omega \times (0, T), \\
u = 0 & \text{on } \partial\Omega \times (0, T), \\
u (\cdot, 0) \in L^2 (\Omega),
\end{cases}
\]

\( \Omega \) convex, \( \omega \subset \Omega, \; E \subset (0, T) \; \text{and} \; |E| > 0 \)

\[ \|u (\cdot, T)\|_{L^2(\Omega)} \leq c \int_\omega \int_E |u (x, t)| \; dx \; dt. \]
Hölder type Observation from one point in time

\[ \| u (\cdot, \ell) \|_{L^2(\Omega)} \leq C e^{\frac{c}{T}} \left( \| u (\cdot, \ell) \|_{L^2(\omega)} \right)^{1-\alpha} \left( \| u (\cdot, 0) \|_{L^2(\Omega)} \right)^{\alpha} \]

\[ \implies \]

Observability for a measurable set in time

\[ \| u (\cdot, T) \|_{L^2(\Omega)} \leq c \int_\omega \int_E |u(x,t)| \, dx dt \]
Proof

Step I. **Observation from one point in time**
   → here $\Omega$ convex or star-shaped domain
   → interpolation inequality, parameter $\varepsilon > 0$

Step II. **Usage of set of positive measure,** $E \subset (0, T)$ and $|E| > 0$
   → Construction of a sequence of points $\{\ell_j\}_{j \geq 1}$
   → parameter $z > 1$

Step III. **Choice of the parameters:** $\varepsilon$ and $z$
Observation from a point in time

\[
\|u (\cdot, \ell)\|_{L^2(\Omega)} \leq C e^{C \frac{\gamma}{\ell}} \left( \|u (\cdot, \ell)\|_{L^2(\omega)} \right)^{1-\alpha} \left( \|u (\cdot, 0)\|_{L^2(\Omega)} \right)^{\alpha}
\]

\[
\begin{align*}
\|u (\cdot, \ell)\|_{L^2(\Omega)} & \leq \frac{1}{\varepsilon^\gamma} C e^{C \frac{\gamma}{\ell}} \|u (\cdot, \ell)\|_{L^1(\omega)} + \varepsilon \|u (\cdot, 0)\|_{L^2(\Omega)} \\
\forall \varepsilon & > 0
\end{align*}
\]
Set of positive measure

Let $T > 0$ and $E \subset (0, T)$ be a set of positive measure. Let $L$ be a point of density for $E$. Then for any $z > 1$, there exists $\ell_1 \in (L, T)$ such that the decreasing sequence $\{\ell_j\}_{j \geq 1}$ given by

$$\ell_{j+1} = L + \frac{1}{z^j} (\ell_1 - L)$$

converges to $L$ and satisfies

$$\ell_j - \ell_{j+1} \leq 3 |E \cap (\ell_{j+1}, \ell_j)| .$$
End of proof 1/2

Take $0 < \ell_{j+2} < \ell_{j+1} \leq t \leq \ell_j < T$,

\[
\| u (\cdot, t) \|_{L^2(\Omega)} \leq \varepsilon \| u (\cdot, \ell_{j+2}) \|_{L^2(\Omega)} + \frac{1}{\varepsilon \gamma} C e^{C/(t-\ell_{j+2})} \| u (\cdot, t) \|_{L^1(\omega)}
\]

We know that $\| u (\cdot, \ell_{j}) \|_{L^2(\Omega)} \leq c \| u (\cdot, t) \|_{L^2(\Omega)}$.

We integrate over $E \cap (\ell_{j+1}, \ell_j)$

\[
| E \cap (\ell_{j+1}, \ell_j) | c \| u (\cdot, \ell_{j}) \|_{L^2(\Omega)} \\
\leq | E \cap (\ell_{j+1}, \ell_j) | \varepsilon \| u (\cdot, \ell_{j+2}) \|_{L^2(\Omega)} \\
+ \frac{C}{\varepsilon \gamma} e^{C/(\ell_{j+1}-\ell_{j+2})} \int_{E \cap (\ell_{j+1}, \ell_j)} \| u (\cdot, t) \|_{L^1(\omega)} \, dt
\]
End of proof 2/2

We use $\ell_j$

$$\varepsilon \gamma e^{-C \left[ \frac{1}{\ell_1-L} \frac{z^{j+2}}{z(z-1)} \right]} \|u(\cdot, \ell_j)\|_{L^2(\Omega)} - \varepsilon \gamma + 1 e^{-C \left[ \frac{1}{\ell_1-L} \frac{z^{j+2}}{z(z-1)} \right]} \|u(\cdot, \ell_{j+2})\|_{L^2(\Omega)} \leq c \int_{E \cap (\ell_{j+1}, \ell_j)} \|u(\cdot, t)\|_{L^1(\omega)} \, dt$$

Set $d = C \left[ \frac{1}{\ell_1-L} \frac{1}{z(z-1)} \right]$.

We choose $\varepsilon = e^{-dz^{j+2}}$ and $(\gamma + 1) z^2 = (\gamma + 2)$

$$e^{-d(\gamma + 2)z^j} \|u(\cdot, \ell_j)\|_{L^2(\Omega)} - e^{-d(\gamma + 2)z^{j+2}} \|u(\cdot, \ell_{j+2})\|_{L^2(\Omega)} \leq c \int_{E \cap (\ell_{j+1}, \ell_j)} \|u(\cdot, t)\|_{L^1(\omega)} \, dt$$

We take $j = 2j'$ and make the sum for $j' = 1$ until infinity.
Proof of Observation from a point in time 1/3

\[ G_\lambda (x, t) = e^{-\frac{|x - x_0|^2}{4 (\ell - t + \lambda)}} \frac{e}{(\ell - t - \lambda)^{n/2}} \]

\[ N_{\lambda, u} (t) = \frac{\int_{\Omega} |\nabla u (x, t)|^2 G_\lambda (x, t) \, dx}{\int_{\Omega} |u (x, t)|^2 G_\lambda (x, t) \, dx} \]
Proof of Observation from a point in time 2/3

$\Omega$ convex

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 G_\lambda \, dx + \int_\Omega |\nabla u|^2 G_\lambda \, dx = \int_\Omega u (\partial_t - \Delta) u G_\lambda \, dx$$

$$\frac{d}{dt} N_{\lambda,u}(t) \leq \frac{1}{\ell - t + \lambda} N_{\lambda,u}(t) + \frac{\int_\Omega |(\partial_t - \Delta) u|^2 G_\lambda \, dx}{\int_\Omega |u|^2 G_\lambda \, dx}$$
Proof of Observation from a point in time 3/3

Step I. Make appear \( B_r \subset \omega \)

\[
\left\lceil -\frac{16\lambda}{r^2} \left( \lambda N_{\lambda,u}(\ell) + \frac{n}{4} \right) + 1 \right\rceil \int_{\Omega} |u(\ell)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx \leq \int_{B_r} |u(\ell)|^2 \, dx
\]

Step II. Estimate \( \lambda N_{\lambda,u}(\ell) + \frac{n}{4} \)

\[
\lambda N_{\lambda,u}(\ell) + \frac{n}{4} \leq C \left( \frac{\lambda}{\ell} + 1 \right) e^{C\ell} \ln \left[ e^{C(1+\frac{1}{\ell})} \frac{\int_{\Omega} |u(0)|^2 \, dx}{\int_{\Omega} |u(\ell)|^2 \, dx} \right]
\]

Step III. Choice of \( \lambda \)

\[
\frac{1}{2} = \frac{16\lambda}{r^2} C \left( \frac{\lambda}{\ell} + 1 \right) e^{C\ell} \ln \left[ e^{C(1+\frac{1}{\ell})} \frac{\int_{\Omega} |u(0)|^2 \, dx}{\int_{\Omega} |u(\ell)|^2 \, dx} \right]
\]
Exact control and cost imply Bang-bang control

For any \( y_0 \in L^2(\Omega) \), there exists a control \( f \in L^\infty(\Omega \times E) \) such that the unique solution \( y \) of

\[
\begin{aligned}
\partial_t y - \Delta y + a(x, t) y &= 1_{\Omega \times E} f \\
y &= 0 \\
y(\cdot, 0) &= y_0
\end{aligned}
\]

in \( \Omega \times (0, T) \),
on \( \partial \Omega \times (0, T) \),
in \( \Omega \),

satisfies

\[
y(\cdot, T) = 0 \quad \text{and} \quad \| f \|_{L^\infty(\Omega \times E)} \leq c \| y_0 \|_{L^2(\Omega)}.
\]
Known result

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, $\omega \subset \Omega$. Given $y_0 \in L^2(\Omega)$ and $M > 0$, if $(y^*, 1_{(0,T^*)}f^*)$ is a couple $(y, f)$ satisfying

\[
\begin{aligned}
    \partial_t y - \Delta y &= 1_{\omega} f & \text{in } \Omega \times (0, +\infty), \\
    y &= 0 & \text{on } \partial \Omega \times (0, +\infty), \\
    y(\cdot, 0) &= y_0 & \text{in } \Omega, \\
    y(\cdot, T^*) &= 0 & \text{in } \Omega, \\
    \|f(\cdot, t)\|_{L^2(\Omega)} &\leq M & \text{a.e. } t \in (0, +\infty),
\end{aligned}
\]

and $T^* > 0$ is the smallest of the admissible time among all the couples $(y, f)$, then

\[\|f^*(\cdot, t)\|_{L^2(\Omega)} = M \quad \text{a.e. } t \in (0, T^*) .\]

Further, $(y^*, 1_{(0,T^*)}f^*)$ is unique.
NEW RESULT

Let \( \Omega \) be a convex or star-shaped domain in \( \mathbb{R}^n \), \( \omega \subset \Omega \), \( T > 0 \), \( a \in L^\infty (\Omega \times (0, T)) \). Given \( y_0 \in L^2 (\Omega) \) and \( M > 0 \), if \((y^*, 1_{|(\tau^*, T)} f^*) \) is a couple \((y, f)\) satisfying

\[
\begin{aligned}
\partial_t y - \Delta y + a (x, t) y &= 1 \mathbb{1}_\omega f & \text{in } \Omega \times (0, T) , \\
y &= 0 & \text{on } \partial \Omega \times (0, T) , \\
y (\cdot, 0) &= y_0 & \text{in } \Omega , \\
y (\cdot, T) &= 0 & \text{in } \Omega , \\
\|f (\cdot, t)\|_{L^2 (\Omega)} &\leq M & \text{a.e. } t \in (0, T) , \\
\end{aligned}
\]

and \( \tau^* \in [0, T) \) is the largest admissible time among all the couples \((y, f)\), then

\[
\|f^* (\cdot, t)\|_{L^2 (\Omega)} = M \quad \text{a.e. } t \in (\tau^*, T) .
\]

Further, \((y^*, 1_{|(\tau^*, T)} f^*)\) is unique.
Proof by contradiction

If \( \| f^* (\cdot, t) \|_{L^2(\Omega)} = M \) a.e. \( t \in (\tau^*, T) \) is false, then \( \exists \, \varepsilon > 0 \) and \( E \subset (\tau^*, T) \), \( |E| > 0 \) such that

\[
\| f^* (\cdot, t) \|_{L^2(\Omega)} \leq M - \varepsilon \quad \text{a.e.} \ t \in E.
\]

Aim : Contradiction with the optimality of \( \tau^* \)
i.e., construction of a couple \( (z, 1|_{(\tau^* + \delta, T)} v) \) solution of

\[
\begin{align*}
\partial_t z - \Delta z + az &= 1|_{\omega \times (\tau^* + \delta, T)} v \quad \text{in } \Omega \times (0, T), \\
z &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
z (\cdot, 0) &= y_0 \quad \text{in } \Omega, \\
z (\cdot, T) &= 0 \quad \text{in } \Omega, \\
\|v (\cdot, t)\|_{L^2(\Omega)} &\leq M \quad \text{a.e. } t \in (0, T),
\end{align*}
\]

for a small \( \delta > 0 \) with \( E \subset (\tau^* + \delta_o, T) \) for a sufficiently small \( \delta_o > 0 \).
Thank You!