Variable Light-Cone Theory of Gravity

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Abstract

We show how to reformulate Variable Speed of Light Theories (VSLT) in a covariant fashion as Variable Light-Cone Theories (VLCT) by introducing two vierbein bundles each associated with a distinct metric. The basic gravitational action relates to one bundle while matter propagates relative to the other in a conventional way. The variability of the speed of light is represented by the variability of the matter light-cone relative to the gravitational light-cone. The two bundles are related locally by an element $M$, of $SL(4, R)$ . The dynamics of the field $M$ is that of a $SL(4, R)$-sigma model gauged with respect to local (orthochronous) Lorentz transformations on each of the bundles. Only the “massless” version of the model with a single new coupling, $F$, that has the same dimensions as Newton’s constant $G_N$, is considered in this paper. When $F$ vanishes the theory reduces to standard General Relativity.

We verify that the modified Bianchi identities of the model are consistent with the standard conservation law for the matter energy-momentum tensor in its own background metric.

The implications of the model for some simple applications are examined. We discuss the Newtonian limit, the appropriate generalisations of the flat FRW universe and the spherically symmetric static solution. We conclude that the variability of the speed of light in the early universe is a possible homogenising mechanism. However an examination of the post-Newtonian approximation shows that on the basis of the results of VLBI measurements, the new coupling satisfies $F/G_N < 3.2 \times 10^{-4}$ . We point out that other “massive” versions of the model with different asymptotic properties may still permit consistency with General Relativity at large distances while predicting departures at short distances.
1 Introduction

Albrecht, Magueijo and Barrow [1]-[6] have proposed Variable Speed of Light Theories (VSLT) to provide a stable basis for the observed isotropy and flatness of the universe. In these models the speed of light is parametrised as a kind of equation of state tied to the radius of the universe. They demonstrate that with the appropriate dependence of the velocity of light it is possible to create a cosmic dynamics containing long time attractors for the evolution of the universe that provide an explanation for its current state. However the formulation of the theories is not obviously covariant and the implication of the theories for regimes other than the early universe is not immediately clear.

Moffat [7, 8] earlier presented related ideas and Clayton and Moffat [9] have proposed an ingenious bi-metric theory that overcomes some of the problems in formulating covariant VSLT. This is more fully dynamical than the bi-metric theory of Rosen [10, 11, 12].

The possibility of “superluminal propagation” induced by quantum effects has been discussed previously [13]-[20]. The results are intriguing but remain controversial.

In a spirit of exploration we propose in this paper a theory with variable speed of light that is both geometrical in structure and explicitly covariant in the same way as standard General Relativity. Furthermore the energy momentum tensor of matter is conserved in the conventional way. In this way it achieves some of the same aims as ref [9], which appeared while this work was in preparation. The proposal is to introduce into the space-time manifold, two vierbein bundles. One is associated with matter and the other with gravity. The matter vierbein can be strained relative to the gravitational vierbein giving a geometrical meaning to variability of the speed of light. The propagation of light is of course associated with the light-cone of the matter vierbein. Because of this unconventional approach to light and matter propagation we will refer to the Variable Light-Cone Theory (VLCT). It has a more general structure than just a varying speed of light. The theory in some aspects has the flavour of the bi-metric theories [3, 14, 6] but the emphasis is more on the relationship of the two vierbein bundles rather than on the two implied metrics.

The dynamics of the theory is specified by giving gravity its standard curvature based action, $I_G$, and matter its standard action, $I_M$, in the appropriate metric, and by introducing a linking action, $I_L$, that controls the relationship between the two vierbein bundles. The full action, $I$, is the sum of all three terms, $I = I_G + I_L + I_M$. (1)
2 General Structure

The theory is most clearly formulated in the vierbein formalism and makes fundamental use of the resulting Lorentz gauge invariance. We introduce a vierbein bundle appropriate to gravity, \( \{e_\mu^a\} \), with the associated metric

\[
 g_{\mu\nu} = e_\mu^a e_\nu^a ,
\]  

(2)

where raising and lowering of \( a \)-indices is carried out with the standard Lorentz metric \( \eta_{ab} = \{1, -1, -1, -1\} \). The inverse vierbein is \( \{e^a_\mu\} \) so that

\[
 e_\mu^a e_\nu^a = \delta_\nu^\mu , \quad e^{a\mu} e_{\mu b} = \delta^a_b ,
\]  

(3)

and

\[
 g^{\mu\nu} = e^{a\mu} e^a_\nu .
\]  

(4)

The vierbein associated with matter is \( \{\bar{e}_\mu^\bar{a}\} \) and the raising and lowering of \( \bar{a} \)-indices is by means of the Lorentz metric, \( \eta_{\bar{a}\bar{b}} = \{1, -1, -1, -1\} \). The associated metric is

\[
 b r g_{\mu\nu} = \bar{e}_\mu^\bar{a} \bar{e}_\nu^\bar{a} .
\]  

(5)

The two vierbein bundles are related by a local linear transformation

\[
 \bar{e}_\mu^\bar{a} = e_\mu^a M^a_\bar{a} ,
\]  

(6)

where we assume that the matrix \( M \) is an element of \( SL(4, R) \). This implies that the volume elements in the two bundles are the same. We denote the common value of the two determinants by \( J \).

\[
 J = \det\{e_\mu^a\} = \det\{\bar{e}_\mu^\bar{a}\} .
\]  

(7)

It would be interesting to know if the assumption that \( \det M = 1 \) could be relaxed. It seems to play a role in ensuring the conservation of the matter energy-momentum tensor.

We denote the inverse matrix by \( M^a_\bar{a} \) so that

\[
 M^a_\bar{a} M^\bar{a}_b = \delta^a_b , \quad M^a_\bar{a} M^\bar{a}_\bar{b} = \delta^a_\bar{b} .
\]  

(8)

We introduce vierbein connections for both bundles and associated coordinate connections. The relationship between them is achieved by requiring that the vierbeins are covariantly constant in the appropriate way.

\[
 D_\mu e_\nu^a = \partial_\mu e_\nu^a + \omega^b_\mu e_\nu^b - \Gamma^\lambda_\mu\nu e_\lambda^a = 0 ,
\]  

(9)

and

\[
 \bar{D}_\mu \bar{e}_\nu^\bar{a} = \partial_\mu \bar{e}_\nu^\bar{a} + \bar{\omega}^b_\mu \bar{e}_\nu^b - \bar{\Gamma}^\lambda_\mu\nu \bar{e}_\lambda^\bar{a} = 0 .
\]  

(10)
The requirement that $\eta_{ab}$ and $\eta_{\bar{a}\bar{b}}$ be covariantly constant implies that $\omega_{\mu ab} = -\omega_{\mu ba}$ and $\bar{\omega}_{\mu \bar{a}\bar{b}} = -\bar{\omega}_{\mu \bar{b}\bar{a}}$.

It is convenient to define a covariant derivative of $M$ that includes both the right and left vierbein connections,

$$D_\mu M^a_{\bar{a}} = \partial_\mu M^a_{\bar{a}} + \omega_{\mu a}^b M^b_{\bar{a}} - M^a_{\bar{b}} \bar{\omega}_{\mu \bar{b}}^\bar{a} ,$$  \hspace{1cm} (11)

However in differentiating a second time the appropriate spatial connection, $\Gamma^\lambda_{\mu\nu}$, must be used. The covariant derivative $\bar{D}_\mu$ can be extended in a similar way. Its effect on $M$ is the same as that of $D_\mu$ but a second differentiation must use the spatial connection $\bar{\Gamma}^\lambda_{\mu\nu}$.

Gravitational curvature tensors are defined so that

$$[D^L_\mu, D^L_\nu] V_a = R_{ab\mu\nu} V^b ,$$  \hspace{1cm} (12)

and

$$[D^R_\mu, D^R_\nu] V_{\bar{a}} = R_{\bar{a}b\mu\nu} V^\bar{b} ,$$  \hspace{1cm} (13)

where $D^L_\mu$ includes the left vierbein connection field, $\omega_{\mu ab}$ but not the spatial connection, $\Gamma^\lambda_{\mu\nu}$ . Similarly $D^R_\mu$ includes the only the right vierbein connection, $\bar{\omega}_{\mu \bar{a}\bar{b}}$ . We have then

$$R_{ab\mu\nu} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu a}^c \omega_{\nu cb} - \omega_{\nu a}^c \omega_{\mu cb} ,$$  \hspace{1cm} (14)

with a similar definition for $\bar{R}_{\bar{a}b\mu\nu}$ . It follows that

$$[D_\mu, D_\nu] M^a_{\bar{a}} = R^a_{\mu b\nu} M^b_{\bar{a}} - M^a_{\bar{b}} \bar{R}^\bar{b}_{\mu \bar{a}\nu} - 2 C^\lambda_{\mu\nu} \Gamma^\lambda_{\mu\nu} M^a_{\bar{a}} ,$$  \hspace{1cm} (15)

where

$$C^\lambda_{\mu\nu} = \frac{1}{2} \left( \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) .$$  \hspace{1cm} (16)

The quantity $C^\lambda_{\mu\nu}$ is the torsion tensor in the coordinate basis.

### 3 Gravitational Action

The gravitational action is the standard action

$$I_G = -\frac{1}{16\pi G} \int d^4x J R ,$$  \hspace{1cm} (17)

where

$$R = e^{a\mu} e^{b\nu} R_{ab\mu\nu} .$$  \hspace{1cm} (18)

The vierbeins and the connections are treated as independent variables. If we vary the former then

$$\delta I_G = \frac{1}{8\pi G} \int d^4x J \delta e_{ac} \left( e^{a\mu} R^c_{\mu} - \frac{1}{2} e^{a\sigma} R \right) .$$  \hspace{1cm} (19)
The variation of the vierbein connection yields
\[ \delta I_G = -\frac{1}{16\pi G} \int d^4x J e^{a\mu} e^{b\nu} \left(D^L_{\mu} \delta \omega_{\nu ab} - D^L_{\nu} \delta \omega_{\mu ab} \right). \]  
(20)

Switching to the full covariant derivative we get
\[ \delta I_G = -\frac{1}{16\pi G} \int d^4x J e^{a\mu} e^{b\nu} \left(D_{\mu} \delta \omega_{\nu ab} - D_{\nu} \delta \omega_{\mu ab} + 2C^\lambda_{\mu\nu \delta} \delta \omega_{\lambda ab} \right). \]  
(21)

Using the result
\[ JD_{\mu} V^\mu = \partial_{\mu}(JV^\mu) + 2JC^\lambda_{\mu\nu} V^\mu. \]  
(22)
we can integrate by parts and obtain finally
\[ \delta I_G = -\frac{1}{8\pi G} \int d^4x J e^{a\mu} e^{b\nu} \left(C^\lambda_{\mu\nu \delta} \delta \omega_{\lambda ab} - C^\lambda_{\lambda\nu \delta} \delta \omega_{\mu ab} + C^\sigma_{\mu\nu} \right) \delta \omega_{\sigma ab}. \]  
(23)

If there is no other interaction in the theory we can deduce from the vanishing of these variations that \( C^\lambda_{\mu\nu} = 0, \) and
\[ R^\sigma_{\mu} - \frac{1}{2} \delta^\sigma_{\mu} R = 0, \]
the standard equations for matterless gravity.

4 Linking Action

We arrive at the action for the matrix, \( M, \) that links the two vierbein bundles by treating it as the spin variable in a sigma model. This makes possible the gauging away of local (orthochronous) Lorentz transformations in each bundle separately. The point is that only the local distortion of one bundle relative to the other is of physical significance and not the particular choices of local frame in the two bundles.

Because it is gauge invariant under local (orthochronous) Lorentz transformations in either bundle, the action that achieves this is
\[ I_L = \frac{1}{16\pi F} \int d^4x J g^{\mu\nu} \text{Tr}(j_\mu j_\nu), \]  
(24)
where \( F \) is a new gravitational constant with the same dimensions as \( G. \) The matrix valued current \( j_\mu \) is given by
\[ j_\mu = (D_\mu M) M^{-1}, \]  
(25)
More explicitly
\[ j^{ab}_\mu = (D_\mu M^a_\mu) M^{bb}. \]  
(26)
It is also convenient to define an alternative version of the current, appropriate to the barred vierbein bundle, \( \bar{j}_{\mu}^{\bar{a}\bar{b}} \), as
\[
\bar{j}_{\mu}^{\bar{a}\bar{b}} = \left( M^{-1} D_{\mu} M \right)^{\bar{a}\bar{b}} = M_{\bar{a}}^a D_{\mu} M^{a\bar{b}} .
\]  
(27)

We treat the vierbein, which enters through \( g^{\mu\nu} \), the matrix, \( M \), and the connections \( \omega_{\mu a b} \) and \( \bar{\omega}_{\mu \bar{a} \bar{b}} \) as independent variables. The result for the linking action from the vierbein variation is,
\[
\delta I_L = -\frac{1}{8\pi F} \int d^4x J \delta e_{\sigma c} \left( e^{\sigma \mu} g^{\rho \mu} - \frac{1}{2} e^{\sigma \rho} g^{\mu\nu} \right) \text{Tr}(j_{\mu} j_{\nu}) ,
\]  
(28)

From the left vierbein connection we have
\[
\delta I_L = \frac{1}{8\pi F} \int d^4x J \delta \omega_{\mu a b} j^{\mu a b} ,
\]  
(29)

and from the right vierbein
\[
\delta I_L = -\frac{1}{8\pi F} \int d^4x J \delta \bar{\omega}_{\mu \bar{a} \bar{b}} \bar{j}^{\mu \bar{a} \bar{b}} .
\]  
(30)

On varying the matrix \( M \) we obtain
\[
\delta I_L = -\frac{1}{8\pi F} \int d^4x J \text{Tr} \left[ (\delta MM^{-1})(D_{\mu} j^{\mu} - 2C_{\lambda \mu}^{\lambda} j^{\mu}) \right] .
\]  
(31)

The quantity \( \delta MM^{-1} \) is an arbitrary element of the \( SL(4, R) \) Lie algebra and is sufficiently general to identify the other factor in the trace which is also an element of the algebra.

5 Matter Action

We assume that matter is propagated in the vierbein background \( \{ \bar{e}_{\mu\bar{a}} \} \). This seems a consistent approach since it implies that matter behaves in a conventional way in relation to the gravitational field it experiences. However the theory does change the relationship of this observed gravitational field to the distribution of matter density.

If we assume that the matter action does not depend explicitly on the right connection \( \bar{\omega}_{\mu \bar{a} \bar{b}} \), then the most general form of the variation of the matter action is
\[
\delta I_M = -\int d^4x J \left( \delta e_{\sigma c} T^{\sigma c} + \text{Tr}(\delta MM^{-1} U) \right) .
\]  
(32)

However because we assume that matter behaves conventionally in the effective gravitational field we may write
\[
\delta I_M = -\frac{1}{2} \int d^4x J \delta g_{\mu\nu} \bar{T}^{\mu\nu} ,
\]  
(33)
where $\bar{T}^{\mu\nu} = \bar{T}^{\nu\mu}$ is the symmetric energy momentum tensor for matter. Since

$$\bar{g}_{\mu\nu} = \bar{e}_{\mu\bar{a}}\bar{e}_{\nu\bar{a}}$$ \hspace{1cm} (34)

it follows that

$$\delta I_M = - \int d^4x J \delta \bar{e}_{\mu\bar{a}}\bar{e}_{\nu\bar{a}} \bar{T}^{\mu\nu} .$$ \hspace{1cm} (35)

However since

$$\delta \bar{e}_{\mu\bar{a}} = \delta e_{\mu a} M^a_{\bar{a}} + e_{\mu a} \delta M^a_{\bar{a}}$$ \hspace{1cm} (36)

it follows that

$$T^{\sigma c} = M_c^{\bar{a}} \bar{e}_{\nu\bar{a}} \bar{T}^{\sigma\nu} ,$$ \hspace{1cm} (37)

and hence

$$T^\sigma_\lambda = e_{\lambda\bar{c}} T^{\sigma c} = \bar{e}_{\lambda\bar{a}} \bar{e}_{\nu\bar{a}} \bar{T}^{\sigma\nu} = \bar{T}^{\sigma\nu} \bar{g}_{\lambda\nu} .$$ \hspace{1cm} (38)

If we adopt the convention that barred quantities, that is those appropriate to the gravitational background of the matter, have spatial indices raised and lowered with the barred metric we can define

$$\bar{T}^{\sigma\lambda} = \bar{T}^{\sigma\nu} \bar{g}_{\lambda\nu} .$$ \hspace{1cm} (39)

Hence we get the simple seeming result

$$T^\sigma_\lambda = T^\sigma_\lambda .$$ \hspace{1cm} (40)

However it is important to recall that

$$T^{\sigma\tau} = T^\sigma_\lambda g^{\lambda\tau} \neq \bar{T}^{\sigma\tau} .$$ \hspace{1cm} (41)

In fact $T^{\sigma\tau}$ is not necessecarily symmetric. This does not cause any difficultry.

We see also that

$$U^{ba} = \left[ e_{\mu a} M_b^{\bar{a}} \bar{e}_{\nu\bar{a}} \bar{T}^{\mu\nu} \right] ,$$ \hspace{1cm} (42)

where $[\cdots]$ indicates the projection of the contained quantity onto the Lie algebra. Using the result $M^b_{\bar{a}} = e^{b\lambda} \bar{e}_{\lambda\bar{a}}$ we can show that

$$U^{ba} = e^{b\lambda} e_{\mu a} \bar{T}^{\mu}_{\lambda} - \frac{1}{4} \eta^{ba} \bar{T} .$$ \hspace{1cm} (43)

where $\bar{T} = \bar{T}^{\mu}_{\mu}$. Note that $T = T^{\mu}_{\mu} = \bar{T}$.

6 Equations of Motion

We obtain the equations of motion by requiring that the variation of the total action is stationary. The result is

$$\frac{1}{8\pi G} \left( R^\sigma_\rho - \frac{1}{2} \delta^\sigma_\rho R \right) - \frac{1}{8\pi F} \left( \text{Tr}(j^\sigma j_\rho) - \frac{1}{2} \delta^\sigma_\rho \text{Tr}(j^\lambda j_\lambda) \right) - T^\sigma_\rho = 0 .$$ \hspace{1cm} (44)
\[
\frac{1}{8\pi F} \left( D_\mu j^\mu - 2C^\lambda_\mu j^\mu \right) + U = 0 . \tag{45}
\]

\[
\bar{j}^{[\bar{b}, \bar{a}]} = \frac{F}{G} e^{a\mu} e^{b\nu} \left( C^\lambda_{\mu\nu} \delta^a_\nu - C^\lambda_{\nu\mu} \delta^a_\mu + C^a_{\mu\nu} \right) . \tag{46}
\]

\[
\bar{j}^{[\bar{b}, \bar{a}]} = 0 . \tag{47}
\]

Eq(44) implies that
\[
\frac{1}{8\pi G} R^\rho_\rho - \frac{1}{8\pi F} \text{Tr}(j^\rho j_\rho) = \left( T^\rho_\rho - \frac{1}{2} \delta^\rho_\rho T \right) . \tag{48}
\]

\section{Bianchi Identity}

Just as in standard General Relativity it is necessary to check that the theory satisfies the integrability conditions associated with the Bianchi identity. In the presence of torsion these are changed to the following
\[
R_{\lambda\tau\nu;\rho} + R_{\lambda\tau\sigma;\mu} + R_{\lambda\tau\mu;\nu} = -2 \left( R_{\lambda\tau\rho\mu} C^\rho_\nu + R_{\lambda\tau\rho\mu} C^\rho_\sigma + R_{\lambda\tau\rho\sigma} C^\mu_\nu \right) , \tag{49}
\]
where \(;\mu\) indicates the covariant derivative \(D_\mu\). In the contracted version this is
\[
\left( R^\mu_\sigma - \frac{1}{2} \delta^\mu_\sigma R \right) ;\mu = R^{\mu\nu}_\rho C^\rho_\nu + 2R^\mu_\rho C^\rho_\sigma . \tag{50}
\]

If we set
\[
G^\sigma_\rho = R^\sigma_\rho - \frac{1}{2} \delta^\rho_\sigma R , \tag{51}
\]
and
\[
F^\sigma_\rho = \text{Tr} \left\{ j^\sigma j_\rho - \frac{1}{2} \delta^\sigma_\rho j^\lambda j_\lambda \right\} , \tag{52}
\]
then eq(44) can be expressed as
\[
\frac{1}{8\pi G} G^\sigma_\rho - \frac{1}{8\pi F} F^\sigma_\rho - T^\sigma_\rho = 0 . \tag{53}
\]

We now take the covariant derivative, \(D_\sigma\), of this equation in order to test the relationship of the Bianchi identity to the appropriate divergence law for the matter energy-momentum tensor. Eq(50) gives immediately \(G^{\lambda\sigma}_\sigma\). From eq(52) we obtain
\[
F^{\lambda\sigma}_\sigma = \text{Tr}(j^{\sigma;\sigma} j^\lambda) + g^{\sigma\lambda} \text{Tr}(j^\mu (j_{\sigma\mu} - j_{\mu\sigma})) . \tag{54}
\]

From eq(13) we have
\[
\text{Tr}(j^{\sigma;\sigma} j^\lambda) = 2C^\rho_\rho \text{Tr}(j^\sigma j^\lambda) - 8\pi F \text{Tr}(U j^\lambda) , \tag{55}
\]
and using eq(48) we find
\[
\text{Tr}(j^{\sigma;\sigma} j^\lambda) = \frac{F}{G} C^\rho_\rho R^\lambda_\rho - 16\pi FC^\rho_\rho (T^\lambda_\rho - \frac{1}{2} g^{\sigma\lambda} T) - 8\pi F \text{Tr}(U j^\lambda) . \tag{56}
\]
The cyclic property of traces allows us to write
\[ \text{Tr}(j^\mu (j_{\sigma\mu} - j_{\mu\sigma})) = \text{Tr}(j^\mu ([D_{\mu}, D_{\sigma}] M) M^{-1}) . \]  
(57)

If we introduce \( R_{\mu\sigma} \) and \( \bar{R}_{\mu\sigma} \) as Lie algebra matrices with the definitions
\[ (R_{\mu\sigma})_{ab} = R_{ab\mu\sigma} \quad \text{and} \quad (\bar{R}_{\mu\sigma})_{\bar{a}\bar{b}} = \bar{R}_{\bar{a}\bar{b}\mu\sigma} , \]  
then eq(55) gives
\[ \text{Tr}(j^\mu (j_{\sigma\mu} - j_{\mu\sigma})) = \text{Tr}(j^\mu R_{\mu\sigma} - \bar{j}^\mu \bar{R}_{\mu\sigma} - 2C^\rho_{\mu\sigma} j^\mu j^\rho) . \]  
(59)

Eq(57) implies that the second term on the right yields zero under the trace. From eq(46) and eq(48) we obtain the result
\[ \text{Tr}(j^\mu (j_{\sigma\mu} - j_{\mu\sigma})) = \frac{F}{G} (2C^\rho_{\mu\sigma} R^\tau_{\rho} + 2C^\rho_{\mu\sigma} R^\mu_{\rho} - C^\mu_{\tau\nu} R^\nu_{\mu\sigma}) + 16\pi F C^\rho_{\mu\sigma} (T^\mu_{\rho} - \frac{1}{2} \delta^\mu_{\rho} T) . \]  
(60)

Combining these results we find
\[ F^\sigma\lambda_{;\sigma} = -\frac{F}{G} g^{\sigma\lambda} (2C^\rho_{\mu\sigma} R^\mu_{\rho} - C^\mu_{\tau\nu} R^\nu_{\mu\sigma}) - 16\pi F (C^\rho_{\rho\sigma} T^\sigma\lambda - g^{\sigma\lambda} C^\rho_{\mu\sigma} T^\mu_{\rho}) - 8\pi F \text{Tr}(U j^\lambda) . \]  
(61)

The covariant divergence of eq(53) then yields
\[ T^\sigma\lambda_{;\sigma} = \text{Tr}(U j^\lambda) + 2(C^\rho_{\rho\sigma} T^\sigma\lambda - g^{\sigma\lambda} C^\rho_{\mu\sigma} T^\mu_{\rho}) , \]  
(62)

alternatively
\[ T^\sigma_{\lambda;\sigma} = \text{Tr}(U j^\lambda) + 2(C^\rho_{\rho\sigma} T^\sigma_{\lambda} - C^\rho_{\mu\lambda} T^\mu_{\rho}) . \]  
(63)

It is useful to note that
\[ j_{\lambda ab} = -e^\rho_a (\Gamma^\tau_{\lambda\rho} - \bar{\Gamma}^\tau_{\lambda\rho}) e^\tau_b . \]  
(64)

We have then
\[ \text{Tr}(U j^\lambda) = - (\Gamma^\tau_{\lambda\rho} - \bar{\Gamma}^\tau_{\lambda\rho}) (e^\tau_b U^b_{\rho a} e^\rho_a) . \]  
(65)

Using eq(43) we find
\[ \text{Tr}(U j^\lambda) = - (\Gamma^\tau_{\lambda\rho} - \bar{\Gamma}^\tau_{\lambda\rho}) \bar{T}^\rho_{\tau} , \]  
(66)

where we have used the result \( \text{Tr} j^\lambda = 0 \). In turn this implies \( \Gamma^\sigma_{\tau\sigma} = \bar{\Gamma}^\sigma_{\tau\sigma} \). We conclude that
\[ D^\sigma T^\sigma_{\lambda} = -(\Gamma^\tau_{\lambda\rho} - \bar{\Gamma}^\tau_{\lambda\rho}) \bar{T}^\rho_{\tau} + 2(C^\rho_{\rho\sigma} T^\sigma_{\lambda} - C^\rho_{\mu\lambda} T^\mu_{\rho}) . \]  
(67)

The relationship between the covariant derivatives \( D_\mu \) and \( \bar{D}_\mu \), implies
\[ D^\sigma T^\sigma_{\lambda} = \bar{D}^\sigma T^\sigma_{\lambda} + (\Gamma^\sigma_{\tau\sigma} - \bar{\Gamma}^\sigma_{\tau\sigma}) \bar{T}^\tau_{\lambda} - (\Gamma^\tau_{\sigma\lambda} - \bar{\Gamma}^\tau_{\sigma\lambda}) \bar{T}^\sigma_{\tau} . \]  
(68)
We obtain then
\[ D_\sigma T^\sigma_\lambda = (\tilde{\Gamma}^\sigma_\sigma - \Gamma^\sigma_\sigma)T^\tau_\lambda - (\tilde{\Gamma}^\tau_\lambda - \tilde{\Gamma}^\tau_{\lambda \sigma})T^\sigma_\tau . \] (69)

However we have \( \Gamma^\sigma_\tau = \tilde{\Gamma}^\sigma_\tau \), and find, replacing \( T^\sigma_\lambda \) by \( \tilde{T}^\sigma_\lambda \),
\[ D_\sigma \tilde{T}^\sigma_\lambda = 2\tilde{C}^\sigma_\sigma \tilde{T}^\tau_\lambda - 2\tilde{C}^\sigma_\tau \tilde{T}^\tau_\sigma . \] (70)

It is readily checked that this is equivalent to the standard conservation law
\[ \hat{D}_\sigma \tilde{T}^\sigma_\lambda = 0 , \] (71)
where \( \hat{D}_\sigma \) is the covariant derivative incorporating the metric connection for the metric \( \bar{g}_{\mu \nu} \). We conclude therefore that the extended equations we have proposed for VLC gravity are consistent with conservation of the energy-momentum tensor for matter in the appropriate (barred) metric.

8 Connection Structure

One of the features of the theory is the rôle played by torsion. Here we show that the equations of motion do permit an analysis of the vierbein connections.

From eq(47) we can obtain \( \bar{\omega}^\mu_\alpha \bar{a} \bar{b} \) in terms of the other variables. We can therefore eliminate \( \bar{\omega}^\mu_\alpha \bar{a} \bar{b} \) from the equations of motion. We have
\[ j^\mu_\alpha \bar{a} \bar{b} = M^\alpha_\bar{a} j^\mu_\bar{a} \bar{b} = M^\alpha_\bar{a} j^\mu_{\{\bar{a}, \bar{b}\}} M^\bar{b} \] . (72)

We find then
\[ j^\mu_\alpha \bar{a} \bar{b} = \frac{1}{2} M^\alpha_\bar{a} \left( M^\alpha_{\bar{c}} \partial_\mu M^\bar{c} \bar{b} + M^\alpha_{\bar{a}} e^\nu_{\mu \bar{c}} dM^\bar{c} \bar{b} + M^\alpha_{\bar{b}} e^\nu_{\mu \bar{a}} dM^\bar{c} \bar{b} + M^\alpha_{\bar{b}} e^\nu_{\mu \bar{a}} dM^\bar{c} \bar{b} \right) M^\bar{b} \] . (73)

This equation exhibits \( j^\mu_\alpha \bar{a} \bar{b} \) as a linear function of \( \omega^\mu_\alpha \bar{a} \bar{b} \). Eq(46) relates the torsion tensor \( C^\lambda_\mu \nu \) linearly to \( j^\mu_\alpha \bar{a} \bar{b} \). The resulting equation determines \( \omega^\mu_\alpha \bar{a} \bar{b} \) in terms of \( M^\alpha_\bar{a} \) and \( e^\mu_{\alpha a} \) and their derivatives.

From the covariant constancy of \( e^\mu_{\alpha a} \) we have
\[ \Gamma^\lambda_\mu \nu = e^\alpha\lambda \left( \partial_\mu e^\nu_{\alpha a} + \omega^\nu_{\mu a} e^\nu_{\alpha b} \right) . \] (74)

Hence
\[ C^\lambda_\mu \nu = \frac{1}{2} e^\alpha\lambda \left( \partial_\mu e^\nu_{\alpha a} + \omega^\nu_{\mu a} e^\nu_{\alpha b} - \partial_\nu e^\mu_{\alpha a} - \omega^\mu_{\nu a} e^\nu_{\alpha b} \right) . \] (75)

If we define
\[ C_{\lambda \mu \nu} = g_{\lambda \sigma} C^\sigma_\mu \nu , \] (76)
and
\[ \omega^\lambda_\mu \nu = e^\alpha_{\lambda a} e^\nu_{\beta b} \omega^\mu_{\alpha \beta} , \] (77)
then we find
\[ \omega_{\mu\lambda\nu} = C_{\lambda\mu\nu} - C_{\nu\mu\lambda} + C_{\mu\lambda\nu} + \hat{\omega}_{\mu\lambda\nu} \],
where \( \hat{\omega}_{\mu\lambda\nu} \) is the metric version of \( \omega_{\mu\lambda\nu} \) and is given by
\[ \hat{\omega}_{\mu\lambda\nu} = \frac{1}{2} \left( e_{\nu}^a \partial_{\mu} e_{\lambda a} - e_{\lambda}^a \partial_{\mu} e_{\nu a} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\nu\mu} \right). \]

Eq(78) can be expressed in the form
\[ C_{\sigma\mu\nu} + C_{\lambda\mu\nu}^\lambda g_{\nu\sigma} - C_{\lambda\nu\mu}^\lambda g_{\mu\sigma} = -X_{\sigma\mu\nu}, \]
where
\[ X_{\sigma\mu\nu} = G_{\sigma[\mu,\nu]}^{-1} \],
and
\[ j_{\sigma\mu\nu} = e_{\mu}^a e_{\nu}^b j_{\sigma ab}. \]
It follows that
\[ C_{\sigma\mu\nu} = -X_{\sigma\mu\nu} - \frac{1}{2} X_{\lambda\mu\nu}^\lambda g_{\nu\sigma} + \frac{1}{2} X_{\lambda\nu\mu}^\lambda g_{\mu\sigma}. \]
Finally we have solution
\[ \omega_{\mu\lambda\nu} = -X_{\lambda\mu\nu} + X_{\nu\mu\lambda} - X_{\mu\lambda\nu} - X_{\tau\lambda\nu}^\tau g_{\mu\nu} + X_{\tau\nu\mu}^\tau g_{\mu\lambda} + \hat{\omega}_{\mu\lambda\nu}. \]

9 Special Cases

The above solution for \( \omega_{\mu\lambda\nu} \) can be used rather straightforwardly in special cases.

We consider a situation in which the vierbeins and the linking matrix have diagonal forms, namely
\[ e_{\mu a} = A_a \eta_{\mu a} \quad \text{and} \quad M^{\alpha}_{\bar{a}} = \Lambda_\alpha \delta^{\alpha}_{\bar{a}}. \]
It follows that
\[ e^{\alpha \mu} = A_a^{-1} \eta^{\alpha \mu} \quad \text{and} \quad M^{\bar{a}}_{\alpha} = \Lambda_a^{-1} \delta^{\bar{a}}_{\alpha}, \]
and that
\[ \bar{e}_{\mu \bar{a}} = \Lambda_a A_a \eta_{\mu a} \quad \text{and} \quad \bar{e}^{\bar{a} \mu} = \Lambda_a^{-1} A_a^{-1} \eta^{\bar{a} \mu}. \]
We enforce an obvious correspondence between the values of the symbols \( \mu, a \) and \( \bar{a} \) to give meaning to the \( \eta \) and \( \delta \) symbols. In an appropriate sense we can write \( A_a = A_{\bar{a}} = A_{\mu} \) and \( \Lambda_\alpha = \Lambda_{\bar{a}} = \Lambda_{\mu} \) where this is convenient.

In this special case we can compute \( j_{\mu\alpha} \) from eq(73) to yield
\[ j_{\lambda\alpha\beta} = \frac{\partial\Lambda_a}{\Lambda_a} \eta_{\lambda\alpha\beta} + \frac{1}{2} \omega_{\lambda\alpha\beta} \left( 1 - \frac{\Lambda^2_a}{\Lambda^2_b} \right), \]
Hence
\[ j_{\lambda[a,b]} = \frac{1}{4} \omega_{\lambda ab} \left( 2 - \frac{\Lambda_a^2}{\Lambda_b^2} - \frac{\Lambda_b^2}{\Lambda_a^2} \right) , \]  
and
\[ X_{\lambda \mu \nu} = \frac{G}{4F} \delta^a_\mu \delta^b_\nu A_a A_b \omega_{\lambda ab} \left( 2 - \frac{\Lambda_a^2}{\Lambda_b^2} - \frac{\Lambda_b^2}{\Lambda_a^2} \right) . \]
We also have
\[ \hat{\omega}_{\mu ab} = \frac{\partial_\nu A_a}{A_a} \left( \frac{A_a}{A_b} \right) \delta^\nu_\mu \eta_{\mu a} - \frac{\partial_\nu A_b}{A_b} \left( \frac{A_b}{A_a} \right) \delta^\nu_\mu \eta_{\mu b} . \]

### 9.1 Flat Expanding Universe

An application of the above equations is to a flat expanding universe where we set
\[ A_0 = 1 , \quad A_i = A(t) , \quad \text{and} \quad \Lambda_0 = \Lambda(t) , \quad \Lambda_i = \Lambda_S(t) , \]  
and \( i \) runs over orthogonal spatial directions. The metric is therefore
\[ ds^2 = dt^2 - A^2(t)dr^2 . \]
The barred metric appropriate to matter is
\[ d\bar{s}^2 = \Lambda^2(t)dt^2 - \Lambda^2_S(t)A^2(t)dr^2 . \]
If we adopt the convention that a coordinate label \( \mu \) can take a time value which we denote by \( t \) or acquire a spatial character which we denote by the variables \( x, y \) or \( z \), then rotational invariance tells us that the only non-vanishing components of \( \omega_{\lambda \mu \nu} \) are \( \omega_{xty} = -\omega_{xyt} \) and that
\[ \omega_{xty} = \omega \delta_{xy} . \]
The same is true for the metric connection, and we find that
\[ \hat{\omega}_{xty} = A \dot{A} \delta_{xy} . \]
The same considerations of rotational invariance imply that only \( X_{xty} = -X_{xyt} \) are nonvanishing and from eq(90) we find
\[ X_{xty} = \frac{G}{4F} \omega_{xty} \left( 2 - \frac{\Lambda_a^2}{\Lambda_b^2} - \frac{\Lambda_b^2}{\Lambda_a^2} \right) . \]
We find also
\[ X^r_{rt} = \frac{1}{A^2} \delta_{x'y'} X_{x'ty'} . \]
From eq(84) we have then
\[
\omega \delta_{xy} = \frac{G}{4F} \left( -2\omega \delta_{xy} + \delta_{xy} A^2 \times \frac{1}{A^2} \delta'_{xy} \omega \delta'_{xy} \right) \left( 2 - \frac{\Lambda^2}{\Lambda^2} - \frac{\Lambda^2}{E^2} \right) + A \dot{A} \delta_{xy} .
\] (99)

Finally
\[
\omega = \frac{A \dot{A}}{E} ,
\] (100)
where
\[
E = 1 - \frac{G}{4F} \left( 2 - \frac{\Lambda^2}{\Lambda^2} - \frac{\Lambda^2}{E^2} \right) .
\] (101)

We will use this result later when we consider flat FRW space-time.

9.2 Static Rotationally Invariant Space-time

Another application of the formalism is to the equivalent of the Schwarzschild solution. We will use isotropic coordinates, so that
\[
A_0 = A(r) , \quad A_i = B(r) , \quad \text{and} \quad \Lambda_0 = \Lambda(r) , \quad \Lambda_i = \Lambda_S(r) .
\] (102)
The unbarred metric is then
\[
ds^2 = A^2(r) dt^2 - B^2(r) dr^2 ,
\] (103)
and the barred metric is
\[
ds^2 = \Lambda^2(r) A^2(r) dt^2 - \Lambda^2_S(r) B^2(r) dr^2 .
\] (104)

Rotational invariance implies that only the components \( \omega_{xty} = -\omega_{xyt} \), \( \omega_{ttx} = -\omega_{txt} \) and \( \omega_{xyz} \) may be non-zero. In the case of the metric connection we find
\[
\dot{\omega}_{xty} = 0 ,
\] (105)
\[
\dot{\omega}_{ttx} = AA' \hat{r}_x ,
\] (106)
where \( \hat{r}_x \) is the \( x \)-component of the radial unit vector. We have also
\[
\dot{\omega}_{xyz} = BB' (\hat{r}_y \delta_{xz} - \hat{r}_z \delta_{xy}) .
\] (107)
The components of \( X_{\lambda\mu\nu} \) that may be non-zero have the values,
\[
X_{xty} = \frac{G}{4F} \omega_{xty} \left( 2 - \frac{\Lambda^2}{\Lambda^2} - \frac{\Lambda^2}{E^2} \right) ,
\] (108)
\[
X_{ttx} = \frac{G}{4F} \omega_{ttx} \left( 2 - \frac{\Lambda^2}{\Lambda^2} - \frac{\Lambda^2}{E^2} \right) ,
\] (109)
and
\[
X_{xyz} = 0 .
\] (110)
Solving for \( \omega_{\lambda \mu \nu} \) we find a pattern similar to that for the metric connection,
\[
\omega_{xty} = 0 ,
\] (111)
\[
\omega_{tx} = \frac{AA'}{P} \delta_{x} ,
\] (112)
and
\[
\omega_{xyz} = Q (\delta_{x} \delta_{xz} - \delta_{z} \delta_{xy}) ,
\] (113)
where
\[
P = 1 + \frac{G}{4F} \left( 2 - \frac{\Lambda^2}{\Lambda^2} - \frac{\Lambda^2}{\Lambda^2} \right) ,
\] (114)
and
\[
Q = BB' + \frac{G}{4F} \left( 2 - \frac{\Lambda^2}{\Lambda^2} - \frac{\Lambda^2}{\Lambda^2} \right) \frac{B^2 A'}{AP} .
\] (115)
We will use these results in analysing the Schwarzschild-type metric in VLC theory.

10 Weak Field Limit

The weak field limit has the form
\[
e_{\mu a} = e_{\mu a}^{(0)} + h_{\mu a} ,
\] (116)
where
\[
e_{\mu a}^{(0)} e_{\nu b}^{(0)} = \eta_{\mu \nu} ,
\] (117)
Similarly we can set
\[
\bar{e}_{\mu \bar{a}} = \bar{e}_{\mu \bar{a}}^{(0)} + h_{\mu \bar{a}} ,
\] (118)
where
\[
\bar{e}_{\mu \bar{a}}^{(0)} \bar{e}_{\nu \bar{b}}^{(0)} = \eta_{\mu \nu} .
\] (119)
The connections \( \omega_{\mu ab} \) and \( \bar{\omega}_{\mu \bar{a} \bar{b}} \) are first order quantities. The matrix \( M \) has the form
\[
M_{a}^{\bar{a}} = M^{(0)}_{a}^{\bar{a}} + m_{a}^{\bar{a}} ,
\] (120)
where \( M^{(0)} \) is a Lorentz transformation. We have then
\[
M^{(0)}_{a}^{\bar{a}} = M^{(0)}_{a}^{\bar{a}} .
\] (121)
We can use \( M \) and its inverse and \( e \) and \( \bar{e} \) to convert superfixes and suffixes between the various bases. For example we have
\[
m_{a}^{b} = m_{a}^{\bar{a}} M^{(0)}_{\bar{a}}^{\bar{b}} .
\] (122)
The requirement that $\text{det} M = 1$ implies that
\[ m^a_a = m^\bar{a}_\bar{a} = m^{\mu}_\mu = 0 \] (123)
The relationship between $e$ and $\bar{e}$ implies that
\[ \bar{h}_{\mu\bar{a}} = h_{\mu\bar{a}} + m_{\mu\bar{a}} \] (124)
To lowest order
\[ R_{\alpha\beta\gamma\delta} = \frac{1}{2} \bar{e}^{\alpha\beta} \partial_{\gamma\delta} \bar{e} - \frac{1}{2} \partial_{\alpha\beta} \bar{e}^{\gamma\delta} \] (125)
Hence
\[ R^\sigma_\mu = e^{(0)\sigma\alpha} e^{(0)\lambda\mu} R_{\alpha\beta\gamma\delta} \] (126)
so that
\[ R^\sigma_\mu = \partial_\mu \omega_\sigma - \partial_\sigma \omega_\mu \] (127)
and
\[ R = 2 \partial_\mu \omega_\mu \] (128)
Again in the lowest order approximation
\[ j_{\mu\bar{a}} = \partial_\mu m_{\bar{a}b} + \omega_{\mu\bar{a}b} - \bar{\omega}_{\mu\bar{a}b} \] (129)
If we convert to the coordinate basis we have
\[ j_{\mu\lambda\tau} = \partial_\mu m_{\lambda\tau} + \omega_{\mu\lambda\tau} - \bar{\omega}_{\mu\lambda\tau} \] (130)
We can also evaluate $\bar{j}_\mu$. In this lowest approximation it coincides with $j_\mu$. From the equation of motion we have
\[ \bar{j}_{\mu[\lambda,\tau]} = j_{\mu[\lambda,\tau]} = 0 \] (131)
That is
\[ \partial_\sigma m_{[\lambda,\tau]} + \omega_{\sigma\lambda\tau} - \bar{\omega}_{\sigma\lambda\tau} = 0 \] (132)
This result tells us that the torsion in the unbarred, gravitational vierbein bundle is zero.
Eq(15) yields
\[ \eta^{\mu\sigma} \partial_\mu (\partial_\sigma m_{\lambda\tau} + \omega_{\sigma\lambda\tau} - \bar{\omega}_{\sigma\lambda\tau}) = -8\pi F U_{\lambda\tau} \] (133)
where
\[ U_{\lambda\tau} = \bar{T}_{\lambda\tau} - \frac{1}{4} \eta_{\lambda\tau} \bar{T} \] (134)
Making use of eq(132) we obtain the result
\[ \eta^{\mu\sigma} \partial_\mu \partial_\sigma m_{[\lambda,\tau]} = -8\pi F \left( \bar{T}_{\lambda\tau} - \frac{1}{4} \eta_{\lambda\tau} \bar{T} \right) \] (135)
The remaining equations are to lowest order

\[ R_{\sigma\lambda} - \frac{1}{2} \eta_{\sigma\lambda} R = 8\pi G \bar{T}_{\sigma\lambda} , \]  

(136)

and

\[ C^\lambda_{\mu\nu} = \frac{1}{2} \left( \partial_{\mu} h_{\nu}^\lambda - \partial_{\nu} h_{\mu}^\lambda + \omega^\lambda_{\mu \nu} - \omega^\lambda_{\nu \mu} \right) = 0 . \]  

(137)

In the present approximation \( m_{[\mu,\nu]} \) can be absorbed by gauge transformations of the form

\[ \omega_{\sigma\lambda\tau} \to \omega_{\sigma\lambda\tau} + \partial_{\sigma} \phi_{\lambda\tau} , \quad \text{and} \quad \bar{\omega}_{\sigma\lambda\tau} \to \bar{\omega}_{\sigma\lambda\tau} + \partial_{\sigma} \bar{\phi}_{\lambda\tau} . \]  

(138)

We can assume therefore that in this approximation \( m_{[\mu,\nu]} \) vanishes. Therefore \( m_{\mu\nu} \) may be assumed symmetric. It satisfies

\[ \partial^2 m_{\mu\nu} = -8\pi F(\bar{T}_{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} \bar{T}) . \]  

(139)

The gauge invariance referred to above means also the we are free to choose \( h_{\mu\nu} \) to be symmetric with the result that \( \bar{h}_{\mu\nu} \) is also symmetric.

Under these circumstances we can solve the vanishing torsion equation to give

\[ \omega_{\nu\lambda\mu} = \partial_{\mu} h_{\nu\lambda} - \partial_{\lambda} h_{\nu\mu} . \]  

(140)

Eq(14) now yields

\[ \partial_{\mu} \partial_{\nu} h^\nu_{\sigma} + \partial_{\sigma} \partial_{\mu} h^\mu_{\nu} - \partial^2 h_{\mu\sigma} - \partial_{\mu} \partial_{\sigma} h_{\nu\mu} - \eta_{\mu\sigma} (\partial_{\nu} \partial_{\tau} h^\nu_{\tau} - \partial^2 h^\nu_{\nu}) = 8\pi G \bar{T}_{\sigma\mu} . \]  

(141)

We now refine our coordinate system by choosing the harmonic gauge.

\[ g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0 . \]  

(142)

In the lowest approximation it yields

\[ \partial_{\mu} h^\mu_{\lambda} = \frac{1}{2} \partial_{\lambda} h^\mu_{\mu} . \]  

(143)

The equation of motion then becomes

\[ - \partial^2 \left( h_{\mu\sigma} - \frac{1}{2} \eta_{\mu\sigma} h^\tau_{\tau} \right) = 8\pi G \bar{T}_{\mu\sigma} , \]  

(144)

or

\[ - \partial^2 h_{\mu\sigma} = 8\pi G \left( \bar{T}_{\mu\sigma} - \frac{1}{2} \eta_{\mu\sigma} \bar{T} \right) . \]  

(145)
10.1 Static Case

In a static situation

\[ \partial^2 = -\nabla^2 , \quad \text{and} \quad \bar{T}_{00} = \bar{T} = \rho , \quad (146) \]

where \( \rho \) is the density of matter. We have then

\[ \nabla^2 h_{00} = 4\pi G \rho , \quad \text{and} \quad \nabla^2 m_{00} = 6\pi F \rho . \quad (147) \]

It follows that

\[ \nabla^2 \bar{h}_{00} = 4\pi G_N \rho , \quad (148) \]

where

\[ G_N = G + \frac{3}{2} F . \quad (149) \]

We interpret \( \bar{h}_{00} \) as the gravitational potential seen by matter. Therefore \( G_N \) can be identified with Newton’s constant. Both gravitational constants \( G \) and \( F \) enter into the structure of weak gravity but only in a particular linear combination.

10.2 Gravitational Waves

A time-dependent matter distribution can act as a source of gravitational radiation. Clearly from eq(145) we see that there are such waves of a conventional type. However eq(139) shows that we may also have unconventional gravitational radiation associated with the oscillations of the field \( m_{\mu\nu} \). It is interesting to count the degrees of freedom introduced in this way.

Dealing first with the conventional degrees of freedom represented by \( h_{\mu\nu} \) the standard argument is that the conservation of the energy momentum tensor implies that the gauge condition eq(143) is maintained. This is a necessary consistency check. Far from the source the wave, in the weak field limit, may be treated as a plane wave of the form

\[ h_{\mu\nu} = A_{\mu\nu}e^{ik\cdot x} , \quad (150) \]

and the gauge condition leads to the result

\[ k_\mu A^\mu_{\nu} - \frac{1}{2} k_\nu A^\tau_{\tau} = 0 . \quad (151) \]

These four conditions reduce the original ten degrees of freedom to six. Finally four of these degrees of freedom may be removed by appropriate coordinate transformations that preserve the gauge condition. This leaves the standard two polarizations for the gravitational wave.

When we come to the degrees of freedom represented by the \( m_{\mu\nu} \) we see that the conservation law for the matter energy-momentum tensor implies

\[ \partial^2 \partial_\nu m^{\mu}_{\nu} = 2\pi F \partial_\nu \bar{T} . \quad (152) \]
However we know that
\[ T = \frac{1}{8\pi G} \partial^2 h^\tau \tau \quad . \] (153)
Therefore
\[ \partial^2 \partial_\mu m^\mu_\nu = \frac{F}{4G} \partial^2 \partial_\nu h^\tau \tau . \] (154)
This implies, for waves that have their source in matter, that
\[ \partial_\mu m^\mu_\nu - \frac{F}{4G} \partial_\nu h^\tau \tau = 0 \quad . \] (155)

If we assume we are far from the source so that
\[ m_{\mu\nu} = B_{\mu\nu} e^{ik.x} \quad , \] (156)
then we find
\[ k_\mu B^\mu_\nu - \frac{F}{4G} k_\nu A^\tau \tau = 0 \quad . \] (157)
The symmetric matrix \( m_{\mu\nu} \) has ten degrees of freedom. However one is removed by the vanishing trace condition. Four others are removed by the conditions above, leaving five remaining degrees of freedom. This implies that there are seven degrees of freedom left in the combined \( \{ h_{\mu\nu}, m_{\mu\nu} \} \)-system. However the matter metric does not see one of these degrees of freedom. We can easily deduce the result that
\[ \partial_\mu \tilde{h}^\mu_\nu - \frac{1}{2} \left( 1 + \frac{F}{2G} \right) \partial_\nu \tilde{h}^\tau \tau = 0 \quad . \] (158)
These four equations reduce the original ten degrees of freedom in \( \tilde{h}_{\mu\nu} \) to six. Because the coefficient of the second term departs from 1/2 when \( F \neq 0 \), we cannot find a change of coordinates that removes further degrees of freedom. Matter interacting with the observed gravitational waves will see only six degrees of freedom. This is the conventional result for non-standard gravity theories.

11 FRW Universe

A strong motivation for the VLC theory is the possible modification that the theory may make to the evolution of the early universe. We examine this issue in this section.

The equation of motion, eq(143), yields the result
\[ \partial_\mu j^\mu + \Gamma^\mu_\lambda_\mu j^\lambda + \omega_\mu j^\mu - j^\mu \omega_\mu = -8\pi F U . \] (159)
In the present case the only non-vanishing component of \( \Gamma^\mu_\lambda_\mu \) is
\[ \Gamma^\mu_\tau_\mu = \frac{\dot{J}}{J} = 3 \frac{\dot{A}}{A} . \] (160)
In taking the \{00\} component of eq(159) we find in the present case

\[ \partial_t \left( \frac{\dot{A}}{A} \right) + 3 \frac{\dot{A} \dot{A}}{A^2} - \frac{3}{2} \left( \frac{\dot{A}}{AE} \right)^2 \left( \frac{\Lambda^2}{A^2} - \frac{\Lambda_S^2}{A^2} \right) = -8\pi F \left( T^t_t - \frac{1}{4} \bar{T} \right) = -6\pi F (\rho + p) . \]

(161)

The spatial components of the equation yield the same information. We find also

\[ R^t_t = -\frac{3}{A} \partial_t \left( \frac{\dot{A}}{E} \right) , \]

(162)

and

\[ \text{Tr} (j^i_j) = \frac{4}{3} \left( \frac{\dot{A}}{A} \right)^2 , \]

(163)

yielding the equation of motion

\[ -\frac{3}{A} \partial_t \left( \frac{\dot{A}}{E} \right) - \frac{G}{F} \frac{4}{3} \left( \frac{\dot{A}}{A} \right)^2 = 4\pi G (\rho + 3p) . \]

(164)

The spatial components are

\[ R^x_y = -\frac{1}{A} \partial_t \left( \frac{\dot{A}}{E} \right) \delta_{xy} - 2 \left( \frac{\dot{A}}{AE} \right)^2 \delta_{xy} , \]

(165)

and

\[ \text{Tr} (j^x_jy) = -\frac{1}{2} \left( 2 - \frac{\Lambda^2}{A^2} - \frac{\Lambda_S^2}{A^2} \right) \left( \frac{\dot{A}}{AE} \right)^2 \delta_{xy} , \]

(166)

leading to the equation of motion

\[ -\frac{1}{A} \partial_t \left( \frac{\dot{A}}{E} \right) - 2E \left( \frac{\dot{A}}{AE} \right)^2 = -4\pi G (\rho - p) . \]

(167)

These equations reduce to the standard equations for the flat FRW universe when \( F = 0 \). They may be manipulated to yield the first order equation

\[ \left( \frac{\dot{A}}{E} \right)^2 - \frac{2G A^2}{9F} \left( \frac{\dot{A}}{A} \right)^2 = \frac{8\pi G}{3} \frac{A^2}{E} \rho . \]

(168)

It is also possible to verify, consistently with the Bianchi identities, that they imply the appropriate energy conservation law

\[ \partial_t \left( \frac{A^3}{\Lambda} \rho \right) + p \partial_t \left( \frac{A^3}{\Lambda} \right) = 0 . \]

(169)

If we set

\[ c = \frac{\Lambda}{\Lambda_S} = \Lambda^{\frac{3}{2}} , \]

(170)
then c is the velocity, seen from the gravitational background, of a signal travelling with the velocity of light in the matter background. It is therefore the variable speed of light generated by the variable light-cone structure of the theory. We have
\[ \frac{\dot{c}}{c} = \frac{4 \dot{\Lambda}}{3 \Lambda}. \] (171)

The equations of motion can be rewritten in the form
\[ \partial_t \left( \frac{\dot{c}}{c} \right) + 3 \frac{\dot{A}}{A} \frac{\dot{c}}{c} - 2 \left( \frac{\dot{A}}{AE} \right)^2 \left( c^2 - \frac{1}{c^2} \right) = -8\pi F (\rho + p), \] (172)

and
\[ E \left( \frac{\dot{A}}{AE} \right)^2 - \frac{1}{8F} \frac{(\dot{c})^2}{c^2} = \frac{8\pi G}{3} \rho. \] (173)

### 11.1 Steady Expansion

An interesting consequence of these equations is that if the pressure satisfies
\[ p = \lambda \rho, \] (174)

where \( \lambda \) is a constraint then there exist solutions with constant values of \( c \neq 1 \).

We find
\[ \left( \frac{\dot{A}}{AE} \right)^2 \left( c^2 - \frac{1}{c^2} \right) = 4\pi F (1 + \lambda) \rho, \] (175)

and
\[ E \left( \frac{\dot{A}}{AE} \right)^2 = \frac{8\pi G}{3} \rho. \] (176)

We have then
\[ \frac{1}{E} \left( c^2 - \frac{1}{c^2} \right) = \frac{3}{2} (1 + \lambda) \frac{F}{G}. \] (177)

For relativistic matter \( \lambda = 1/3 \) and we obtain the quadratic equation
\[ \left( c^2 \right)^2 + 2 \left( 1 - \frac{2F}{G} \right) c^2 - 3 = 0, \] (178)

with the physical solution
\[ c^2 = c_0^2 = -\left( 1 - \frac{2F}{G} \right) + 2\sqrt{1 - \frac{F}{G} + \frac{F^2}{G^2}} \approx 1 + \frac{F}{G}. \] (179)

For pressureless matter \( \lambda = 0 \) and we obtain similarly
\[ c^2 = c_0^2 = -\frac{3}{5} \left( 1 - \frac{2F}{G} \right) + \frac{1}{2} \sqrt{\frac{36}{25} \left( 1 - \frac{2F}{G} \right)^2 + \frac{44}{5}} \approx 1 + \frac{3F}{4G}. \] (180)
We take the relativistic case as an illustration. From eq(176) we see that the expansion is controlled by
\[
\left(\frac{\dot{A}}{A}\right)^2 = \frac{8\pi (GE_0)}{3} \rho ,
\]
where \(E_0 = E(c_0^2)\). Because the speed of light is constant we have from eq(169)
\[
\rho \propto \frac{1}{A^4} .
\]
This is exactly like the standard scenario for expansion in the early universe with the change that \(G_N\) has been replaced by \(GE_0\). It follows that
\[
A \propto \sqrt{t} .
\]
It is clearly true also that in the matter coordinates
\[
\bar{A} \propto \sqrt{\bar{t}} ,
\]
where \(\bar{A} = c_0^{-1/4} A\) and \(\bar{t} = c_0^{3/4} t\). This steady expansion is essentially the same as the conventional case.

11.2 Non-steady Expansion

It very difficult to investigate the solutions of eqs(172) and (173) in general. However some feeling for their nature can be obtained by examining solutions near the above steady expansion. If we set
\[
c = c_0 e^\xi ,
\]
and assume that \(\xi\) is a small quantity and that we can neglect quantities quadratic in \(\xi\) and \(\dot{\xi}\) so that eq(173) becomes
\[
E \left(\frac{\dot{A}}{AE}\right)^2 = \frac{8\pi G}{3} \rho .
\]
If we examine the case of relativistic matter, then when we combine this result with eq(172) we get
\[
\ddot{\xi} + 3\frac{\dot{A}}{A} \dot{\xi} - 2E \left(\frac{\dot{A}}{AE}\right)^2 \left(\frac{1}{E} \left(c^2 - \frac{1}{c^2}\right) - 2 \frac{F}{G}\right) = 0 .
\]
Keeping only terms \(O(\xi)\) we obtain
\[
\ddot{\xi} + 3\frac{\dot{A}}{A} \dot{\xi} - 2 \left(\frac{\dot{A}}{AE_0}\right)^2 \left(c_0^2 + \frac{3}{c_0^2}\right) \xi = 0 ,
\]
where, in keeping with the approximation, $A$ may be assumed to be the steady expansion solution obtained previously. As a further simplification we will assume that $F$ is very small and set $F = 0$ in the above equation. We obtain then
\[ \ddot{\xi} + \frac{3}{2t} \dot{\xi} - \frac{2}{t^2} \xi = 0 \quad . \tag{189} \]
This homogeneous equation has two linearly independent solutions
\[ \xi = t^{\alpha} \quad , \quad \text{and} \quad \xi = \frac{1}{t^{\beta}} \quad . \tag{190} \]
where $\alpha = (\sqrt{33} - 1)/4 = 1.1861$ and $\beta = (\sqrt{33} + 1)/4 = 1.6861$. Using these solutions we can construct a range of scenarios for a (small) departure from the steady expansion. A solution of the form
\[ \xi = \frac{b}{t^{\beta}} \quad , \tag{191} \]
will yield
\[ c = c_0 e^{b/t^{\beta}} \quad , \tag{192} \]
which for $b > 0$ results in the speed of light increasing in the far past and tending to the steady expansion in the future. If $b < 0$ then the speed of light will increase to the steady value from a smaller value in the past. If we choose a solution
\[ \xi = a t^{\alpha} \quad \tag{193} \]
then when $a > 0$ the speed of light will increase from the steady value or when $b < 0$ it will decrease from this value. By choosing arbitrary combinations of the two solutions any mixture of these scenarios can be obtained. Of course the analysis being perturbative, the results cannot be trusted beyond the point where $\xi \approx 1$, so the ultimate behaviour at very small or large times remains unresolved.

12 Static Rotationally Invariant Case

We derive the equations for the static rotationally invariant system in isotropic coordinates. They can be used to study the equivalent of the Schwarzschild solution. However we will confine our attention here to the asymptotic properties of the solution as a way of investigating the post-Newtonian approximation.

The equation of motion, eq(15), in the absence of matter yields
\[ \partial_{\mu} j^{\mu} + \Gamma^{\mu}_{\lambda \mu} j^{\lambda} + \omega_{\lambda} j^{\mu} - j^{\mu} \omega_{\mu} = 0 \quad . \tag{194} \]
Making use of the fact that
\[ \Gamma^{\mu}_{\lambda \mu} = \frac{\partial_{\mu} J}{J} = \frac{\partial_{\mu} A}{A} + 3 \frac{\partial_{\mu} B}{B} \quad , \tag{195} \]
we we see that
\[ \Gamma_{\mu}^\nu = 0 , \quad \text{and} \quad \Gamma_{x\mu}^\nu = \left( \frac{A'}{A} + 3 \frac{B'}{B} \right) \hat{r}_x . \]  \tag{196}

From the \{00\} component of eq() we find
\[ \left( \frac{\Lambda'}{\Lambda} \right)' + \left( \frac{2}{r} + \frac{A'}{A} + 3 \frac{B'}{B} \right) \frac{\Lambda}{\Lambda} - \frac{1}{2} \left( \frac{A'}{AP} \right)^2 \left( \frac{\Lambda^2}{\Lambda^3} - \frac{\Lambda^2}{\Lambda^2} \right) = 0 . \]  \tag{197}

The other non-trivial components yield the same equation. Using the definition
\[ F_{\mu\nu} = \text{Tr}\{j_\mu j_\nu\} \] we find
\[ F_{tt} = \frac{A}{B} \left( \frac{A'}{P} \right)^2 \left( \frac{\Lambda^2}{\Lambda^3} - \frac{\Lambda^2}{\Lambda^2} \right) , \]  \tag{198}
and
\[ F_{xy} = \frac{4}{3} \left( \frac{\Lambda'}{\Lambda} \right)^2 \hat{r}_x \hat{r}_y . \]  \tag{199}

We also obtain for the relevant components of \( R_{\mu\nu} \) the results
\[ R_{tt} = \frac{A}{B} \left[ \left( \frac{A'}{BP} \right)' + \frac{2}{r} \left( \frac{A'}{BP} \right) - 2 \left( \frac{A'}{BP} \right) \left( \frac{Q}{B^2} \right) \right] , \]  \tag{200}
and
\[ R_{xy} = R_{xy}^{(1)} + R_{xy}^{(2)} + R_{xy}^{(3)} , \]  \tag{201}
where
\[ R_{xy}^{(1)} = - \left[ \frac{B}{A} \left( \frac{A'}{BP} \right)' + \left( \frac{Q}{B^2} \right)' \right] \hat{r}_x \hat{r}_y , \]  \tag{202}
\[ R_{xy}^{(2)} = - \left[ \left( \frac{Q}{B^2} \right)' + \frac{2}{r} \frac{Q}{B^2} \right] \delta_{xy} , \]  \tag{203}
and
\[ R_{xy}^{(3)} = - \left[ \left( \frac{A'}{AP} \right) \frac{Q}{B^2} - \frac{1}{r} \frac{B}{A} \left( \frac{A'}{BP} \right) - \frac{1}{r} \left( \frac{Q}{B^2} \right) - \left( \frac{Q}{B^2} \right)^2 \right] \left( \delta_{xy} - \hat{r}_x \hat{r}_y \right) . \]  \tag{204}

From the \{tt\} component of the equation of motion we obtain
\[ \left( \frac{A'}{BP} \right)' + \frac{2}{r} \left( \frac{A'}{BP} \right) - 2 \left( \frac{A'}{BP} \right) \left( \frac{Q}{B^2} \right) - \frac{G}{F} \left( \frac{A'}{P} \right)^2 \left( 2 - \frac{\Lambda^2}{\Lambda^3} - \frac{\Lambda^2}{\Lambda^2} \right) = 0 . \]  \tag{205}

If we contract the \{xy\} components of the equation of motion with \( \hat{r}_y \) we obtain
\[ \frac{B}{A} \left( \frac{A'}{BP} \right)' + 2 \left( \frac{Q}{B^2} \right)' + \frac{2}{r} \left( \frac{Q}{B^2} \right) + \frac{4G}{3F} \left( \frac{A'}{\Lambda} \right)^2 = 0 . \]  \tag{206}

On contracting with a unit vector orthogonal to \( \hat{r}_y \) we obtain
\[ \left( \frac{Q}{B^2} \right)' + \frac{3}{r} \left( \frac{Q}{B^2} \right) + \frac{1}{r} \frac{B}{A} \left( \frac{A'}{BP} \right) - \left( \frac{A'}{BP} \right) \frac{Q}{B^2} + \left( \frac{Q}{B^2} \right)^2 = 0 . \]  \tag{207}
12.1 Asymptotic behaviour

We assume that, at large \( r \), the leading asymptotic behaviour of the solution to these equations has the form

\[
\Lambda = 1 + \frac{\Lambda_1}{r} + \frac{\Lambda_2}{r^2} + O\left(\frac{1}{r^3}\right),
\]

(208)

\[
A = 1 + \frac{A_1}{r} + \frac{A_2}{r^2} + O\left(\frac{1}{r^3}\right),
\]

(209)

and

\[
B = 1 + \frac{B_1}{r} + \frac{B_2}{r^2} + O\left(\frac{1}{r^3}\right).
\]

(210)

On evaluating the two leading order contributions to each of the equations we find the results

\[
A_1 + B_1 = 0.
\]

(211)

This result holds in the conventional theory.

\[
\Lambda_2 = \frac{1}{2}\Lambda_1^2 + A_1\Lambda_1,
\]

(212)

\[
A_2 = \frac{1}{2}A_1^2,
\]

(213)

\[
B_2 = -\frac{1}{4}A_1^2 - \frac{1}{6}G\Lambda_1^2.
\]

(214)

If we compare this asymptotic behaviour with the weak field approximation then we see that

\[
A_1 = -GM, \quad \text{and} \quad \Lambda_1 = -\frac{3}{2}FM,
\]

(215)

where \( M \) is the mass of the object at the centre of the spherically symmetric system. The implication of these results for the barred (matter) metric is easily checked. We find

\[
A^2\Lambda^2 \simeq 1 - \frac{2G_N}{r} + \frac{(2G_N^2 + 3GF)M^2}{r^2},
\]

(216)

and

\[
B^2\Lambda_S^2 \simeq 1 + \frac{2(G_N - F)M}{r}.
\]

(217)

The standard post-Newtonian parametrisation is

\[
A^2\Lambda^2 \simeq 1 - \frac{2G_N}{r} + \frac{2\beta G_N^2 M^2}{r^2},
\]

(218)

and

\[
B^2\Lambda_S^2 \simeq 1 + \frac{2\gamma G_N M}{r},
\]

(219)
where for standard General Relativity we have $\beta = \gamma = 1$. In the theory we have outlined we find

$$\beta = 1 + \frac{3 GF}{2 G_N^2} \simeq 1 + \frac{3}{2} \frac{F}{G_N} + O(F^2), \quad (220)$$

and

$$\gamma = 1 - \frac{F}{G_N}. \quad (221)$$

The time-delay results yield $\gamma = 1.00 \pm .002$ [21, 22] which places a strong restriction on $f/G_N$. An even stronger restriction results from the very accurate VLBI data we have $\gamma = 1.000 \pm .00032$ [23, 24]. It follows immediately that $F < 2 \times 10^{-3} G_N$ with the consequence $\beta - 1 < 3 \times 10^{-3}$. The appropriate conclusion is that on the scale of $G$ or $G_N$, $F$ is very small. This however does not prevent it having an effect in the very early universe as indicated in the discussion of the FRW model. Nevertheless these experiments could be taken to rule out the model as we have presented it here. Variations of the model in which extra terms are introduced into the linking Lagrangian that give a mass to the new degrees of freedom will lead to them having a finite range. This would leave the long distance asymptotic behaviour of the model coincident with that of standard General Relativity at least in the weak field limit. It would still permit modifications of the standard theory at short distances. In particular it would allow the existence of very transparent modifications of the effective metric which could be interpreted as a novel form of matter. The potential of such a model to explain the presence of dark matter in the universe is intriguing and requires further study.

13 Stability

Because of the indefinite character of the local Lorentz metric it is not obvious that the degrees of freedom in the linking field $M$ all contribute positively to the energy, in other words that they correspond to a stable theory. While we cannot at present give a final answer to this question we find it illuminating to consider the issue in the case of two dimensions. Here the matrix $M$ has the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (222)$$

with $ad - bc = 1$. The quantity $f = \text{Tr} M M^T$ is invariant under separate left and right Lorentz transformations. We have

$$f = a^2 + d^2 - b^2 - c^2. \quad (223)$$

If we set

$$M = LSL', \quad (224)$$
where \( L \) and \( L' \) are independent Lorentz transformations then of course
\[
f = \text{Tr}MM^T = \text{Tr}SS^T. \tag{225}
\]
If we choose the special form for \( S \)
\[
S = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}, \tag{226}
\]
then
\[
f = \lambda^2 + \lambda^{-2} \geq 2. \tag{227}
\]
If we choose
\[
S = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}, \tag{228}
\]
then
\[
f = 2 \cos^2 \theta - 2 \sin^2 \theta = 2 \cos 2\theta. \tag{229}
\]
In this case
\[
-2 \leq f \leq 2. \tag{230}
\]
third possibility is
\[
S = \begin{pmatrix}
0 & -\lambda^{-1} \\
\lambda & 0
\end{pmatrix}. \tag{231}
\]
We then find
\[
f = -\left(\lambda^2 + \lambda^{-2}\right) \leq -2. \tag{232}
\]
The three ranges together cover all possibilities for the value of \( f \). We therefore can conclude that the matrix \( M \) will be gauge equivalent to the appropriate special form \( S \) provided that form is chosen according to the value of \( f \). Indeed it is possible to check explicitly that given \( f \) the real matrices \( L, L' \) and \( S \) can be computed provided \( S \) is chosen in the appropriate manner.

It is therefore clear that the object that results from equivalencing matrices \( M \in SL(2, R) \) by simultneous left and right Lorentz transformations, although made up of pieces of manifold, is not itself a manifold. With the sign conventions we have adopted for the linking action, oscillations for which the field strengths maintain \( f \geq 2 \) will yield positive energies. The remaining two cases yield negative and then positive energy oscillations.

Similar yet more complicated possibilities exist for the four dimensional case. The completely positive mode case corresponds to matrices \( M \) that are Lorentz gauge equivalent to the special form \( S \) where
\[
S = \begin{pmatrix}
\Lambda_0 & 0 & 0 & 0 \\
0 & \Lambda_1 & 0 & 0 \\
0 & 0 & \Lambda_2 & 0 \\
0 & 0 & 0 & \Lambda_3
\end{pmatrix}, \tag{233}
\]
where $\Lambda_0 \Lambda_1 \Lambda_2 \Lambda_3 = 1$. This was the assumption we made implicitly in the special cases we worked out above. It is not known whether solutions of the equations of motion remain in the purely in the positive sector. It is not known whether it is important for this to be the case. In the sense we have elucidated here, therefore, the stability of the theory remains undecided. However the examples discussed in detail show that possible solutions with stable characteristics exist.

14 Conclusions

We have shown that it is possible to formulate a purely geometrical theory with a variable speed of light in a covariant fashion. The theory gives a meaning to variations of the speed of light by identifying two vierbein bundles one associated with gravity and one associated with matter. Matter moves according a conventional Lagrangian in its own gravitational background. Its energy momentum tensor is conserved in the usual way. In this way it similar to other bi-metric theories [3, 10, 11, 12] Gravity is also formulated in a conventional way in terms of its own vierbein bundle. The extra dynamics is supplied by a linking action that treats the matrix relating the two bundles as a dynamical variable. The form of the action is based on a sigma-model construction and requires the introduction of a new coupling constant with the same dimensions as the Newtonian constant. The action is not unique and can be modified by the addition of terms that effectively give a mass to the new degrees of freedom at the expense of introducing new parameters.

In this paper we have examined the the minimal version of the theory without extra mass inducing terms. As a result we find the asymptotic properties of the theory are modified relative to the standard Einstein theory. The parameter $\gamma$ of the post-Newtonian approximation is not equal to unity. This is inconsistent with the time-delay and VLBI measurements [21, 22, 23, 24]. Since our theory reduces to the standard one when the new coupling constant, $F$, is set to zero we can of course achieve agreement with experiment by reducing this coupling to a sufficiently low value. A very low value for $F$ does not by itself preclude strong effects in the very early universe. However the most natural conclusion is that the a departure from standard GR along the lines suggested is unlikely. It is possible that with the introduction of mass terms in the linking action the resulting modification of the asymptotic properties will leave the theory with essentially standard properties at long distances and with modifications only at short distances. We have not studied this point in this paper.

We have also shown that a (flat) FRW universe can be found in which it is possible to see the speed of light behaving in the way envisaged by previous authors. This would indeed permit the uniformisation of cosmic temperature associated with the cosmic microwave background. However we have not been able to analyse the model sufficiently closely to establish precisely what may
happen in the very early, or indeed late, universe. The results must therefore be treated as suggestive rather than definitive. Nevertheless the compatibility of the variable speed of light idea with covariance and energy and momentum conservation for matter has been established within a completely geometrical theory.

Finally we must draw attention to the fact that the stability of the theory against the production of negative energy densities associated with the new degrees of freedom has not been satisfactorily established. It would be interesting to check that matter oscillations do not give rise to negative energy radiation.

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