ON TORI TRIANGULATIONS ASSOCIATED WITH TWO-DIMENSIONAL CONTINUED FRACTIONS OF CUBIC IRRATIONALITIES.

O. N. KARPENKOV

INTRODUCTION.

A series of properties for ordinary continued fractions possesses multidimensional analogues. H. Tsuchihashi [7] showed the connection between periodic multidimensional continued fractions and multidimensional cusp singularities. The relation between sails of multidimensional continued fractions and Hilbert bases is describe by J.-O. Moussafr in the work [6].

In his book [1] dealing with theory of continued fractions V. I. Arnold gives various images for the sails of two-dimensional continued fraction generalizes the golden ratio. In the article [5] E. I. Korkina investigated the sales for the simplest two-dimensional continued fractions of cubic irrationalities, whose fundamental region consists of two triangles, three edges and one vertex.

We consider the same model of the multidimensional continued fraction as was considered by the authors mentioned above. In the present work the examples of new tori triangulation of the sails for two-dimensional continued fractions of cubic irrationalities for some special families possessing the fundamental regions with more complicated structure are obtained.

In §1 the necessary definitions and notions are given. In §2 the properties of two-dimensional continued fractions constructed using Frobenius operators are investigated, the relation between the equivalence classes of tori triangulations and cubic extensions for the field of rational numbers is discussed. (The detailed analysis of the properties of cubic extensions for the rational numbers field and their classification is realized by B. N. Delone and D. K. Faddeev in the work [3].) In §3 the appearing examples of tori triangulations are discussed.

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1. Definitions.

Points of the space $\mathbb{R}^k$ ($k \geq 1$) whose coordinates are all integer numbers are called integer points.

Consider a set of $n+1$ hyperplanes passing through the origin in general position in the space $\mathbb{R}^{n+1}$. The complement to these hyperplanes consists of $2^{n+1}$ open orthants. Let us choose an arbitrary orthant.

The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the sail.

The union of all $2^{n+1}$ sails defined by these hyperplanes of the space $r^{n+1}$ is called $n$-dimensional continued fraction constructed according to the given $n+1$ hyperplanes in general position in $n+1$-dimensional space.

Two $n$-dimensional sails (continued fractions) are called equivalent if there exists a linear integer lattice preserving transformation of the $n+1$-dimensional space such that it maps one sail (continued fraction) to the other.

To construct the whole continued fraction up to the equivalence relation in one-dimensional case it is sufficiently to know some integer characteristics of one sail (that is to say the integer lengths of the edges and the integer angles between the consecutive edges of one sail).

**Hypothesis 1. (Arnold)** There exist the collection of integer characteristics of the sail that uniquely up to the equivalence relation determines the continued fraction.

Let $A \in GL(n+1, \mathbb{R})$ be an operator whose roots are all real and distinct. Let us take the $n$-dimensional spaces that spans all possible subsets of $n$ linearly independent eigenvectors of the operator $A$. As far as eigenvectors are linearly independent, the obtained $n+1$ hyperspaces will be $n+1$ hyperspaces in general position. The multidimensional continued fraction is constructed just with respect to these hyperspaces.

**Proposition 1.1.** Continued fractions constructed by operators $A$ and $B$ of the group $GL(n+1, \mathbb{R})$ with distinct real irrational eigenvalues are equivalent iff there exists such an integer operator $X$ with the determinant equals to one that the operator $\tilde{A}$ obtained from the operator $A$ by means of the conjugation by the operator $X$ commutes with $B$.

**Proof.** Let the continued fractions constructed by operators $A$ and $B$ of the group $GL(n+1, \mathbb{R})$ with distinct real irrational eigenvalues are equivalent, i. e. there exists linear integer lattice preserving transformation of the space that maps the continued fraction of the operator $A$ to the continued fraction of the operator $B$ (and the orthants of the
first continued fraction maps to the orthants of the second one). Under such transformation of the space the operator $A$ conjugates by some integer operator $X$ with the determinant equals to one. All eigenvalues of the obtained operator $\tilde{A}$ are distinct and real (since the characteristic polynomial of the operator is preserving). As far as the orthants of the first continued fraction maps to the orthants of the second one, the sets of the proper directions for the operators $\tilde{A}$ and $B$ coincides. Thus, given operators are diagonalizable together in some basis and hence they commutes.

Let us prove the converse. Suppose there exists such an integer operator $X$ with the determinant equals to one that the operator $\tilde{A}$ obtained from the operator $A$ by means of the conjugation by the operator $X$ commutes with $B$. Note that the eigenvalues of the operators $A$ and $\tilde{A}$ coincide. Therefore all eigenvalues of the operator $\tilde{A}$ (just as for the operator $B$) are real, distinct, and irrational. Let us consider such basis that the operator $\tilde{A}$ is diagonal in it. Simple verification shows that the operator $B$ is also diagonal in this basis. Consequently the operators $\tilde{A}$ and $B$ defines the same orthant decomposition of the $n+1$-dimensional space and the operators corresponding to this continued fractions coincide. It remains just to note that a conjugation by an integer operator with the determinant equals to one corresponds to the linear integer lattice preserving transformation of the $n+1$-dimensional space. □

Later on we will consider only continued fractions constructed by some invertible integer operator of the $n+1$-dimensional space such that its inverse is also integer. The set of such operators form the group denoted by $GL(n+1, \mathbb{Z})$. This group consist of the integer operators with the determinants equal to $\pm 1$.

The $n$-dimensional continued fraction constructed by the operator $A \in GL(n+1, \mathbb{Z})$ whose characteristic polynomial over the rational numbers field is irreducible and eigenvalues are real is called the $n$-dimensional continued fraction of $(n+1)$-algebraic irrationality. The cases of $n = 1, 2$ correspond to one(two)-dimensional continued fractions of quadratic (cubic) irrationalities.

Let the characteristic polynomial of the operator $A$ be irreducible over the rational numbers field and its roots be real and distinct. Under the action of the integer operators commuting with $A$ whose determinants are equal to one and preserving the given sail the sail maps to itself. These operators form an Abelian group. It follows from Dirichlet unity elements theorem (\cite{2}) that this group is isomorphic to $\mathbb{Z}^n$ and that its action is free. The factor of a sail under such group action is isomorphic to $n$-dimensional torus at that. (For the converse
The polyhedron decomposition of \( n \)-dimensional torus is defined in the natural way, the affine types of the polyhedrons are also defined (in the notion of the affine type we include the number and mutual arrangement of the integer points for the faces of the polyhedron). In the case of two-dimensional continued fractions for cubic irrationalities such decomposition is usually called torus triangulation.

By a fundamental region of the sail we call a union of some faces that contains exactly one face from each equivalence class.

2. Conjugacy classes of two-dimensional continued fractions for cubic irrationalities.

Two-dimensional continued fractions for cubic irrationalities constructed by the operators \( A \) and \(-A\) coincide. In that way the study of continued fractions for integer operators with the determinants equal to \( \pm 1 \) reduces to the study of continued fractions for integer operators with the determinants equal to one (i.e. operators of the group \( SL(3, \mathbb{Z}) \)).

An operator (matrix) with the determinant equals to one

\[
A_{m,n} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -m & -n \end{pmatrix},
\]

where \( m \) and \( n \) are arbitrary integer numbers is called a Frobenius operator (matrix). Note the following: if the characteristic polynomial \( \chi_{A_{m,n}}(x) \) is irreducible over the field \( \mathbb{Q} \) than the matrix for the left multiplication by the element \( x \) operator in the natural basis \( \{1, x, x^2\} \) in the field \( \mathbb{Q}[x]/(\chi_{A_{m,n}}(x)) \) coincides with the matrix \( A_{m,n} \).

Let an operator \( A \in SL(3, \mathbb{Z}) \) has distinct real irrational eigenvectors. Let \( e_1 \) be some integer nonzero vector, \( e_2 = A(e_1) \), \( e_3 = A^2(e_1) \). Then the matrix of the operator in the basis \( (e_1, e_2, ce_3) \) for some rational \( c \) will be Frobenius. However the transition matrix here could be non-integer and the corresponding continued fraction is not equivalent to initial one.

**Example 2.1.** The continued fraction constructed by the operator

\[
A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -7 & 0 & 29 \end{pmatrix},
\]

is not equivalent to the continued fraction constructed by any Frobenius operator with the determinant equals to one.
Thereupon the following question is of interest. *How often the continued fractions that don’t correspond to Frobenius operators can occur?*

In any case the family of Frobenius operators possesses some useful properties that allows us to construct whole families of nonequivalent two-dimensional periodic continued fractions at once, that extremely actual itself.

It is easy to obtain the following statements.

**Statement 2.1.** The set $\Omega$ of operators $A_{m,n}$ having all eigenvalues real and distinct is defined by the inequality $n^2m^2 - 4m^3 + 4n^3 - 18mn - 27 \leq 0$. For the eigenvalues of the operators of the set to be irrational it is necessary to subtract extra two perpendicular lines in the integer plain: $A_{a,-a}$ and $A_{a,a+2}$, $a \in \mathbb{Z}$.

**Statement 2.2.** The two-dimensional continued fractions for the cubic irrationalities constructed by the operators $A_{m,n}$ and $A_{-n,-m}$ are equivalent.

Further we will consider all statements modulo this symmetry.

**Remark.** Example 2.3 given below shows that among periodic continued fractions constructed by operators in the set $\Omega$ equivalent continued fractions can happen.

Let us note that there exist nonequivalent two-dimensional periodic continued fractions constructed by operators of the group $GL(n+1, \mathbb{R})$ whose characteristic polynomials define isomorphic extensions of the rational numbers field. In the following example the operators with equal characteristic polynomials but distinct continued fractions are shown.

**Example 2.2.** The operators $(A_{-1,2})^3$ and $A_{-4,11}$ have distinct two-dimensional continued fractions (although their characteristic polynomials coincide).

At the other hand similar periodic continued fractions can correspond to operators with distinct characteristic polynomials.

**Example 2.3.** The operators $A_{0,-a}^2$ and $A_{-2a,-a^2}$ are conjugated by the operator in the group $GL(3, \mathbb{Z})$ and hence the periodic continued fractions (including the torus triangulations) corresponding to the operators $A_{0,-a}$ and $A_{-2a,-a^2}$ are equivalent.

Let us note that distinct cubic extensions of the field $\mathbb{Q}$ possess nonequivalent triangulations.
3. Torus triangulations and fundamental regions for some series of operators $A_{m,n}$

Torus triangulations and fundamental domains for several infinite series of Frobenius operators are calculated here. In this paragraph it considers only the sails containing the point $(0, 0, 1)$ in its convex hull.

The ratio of the Euclidean volume for an integer $k$-dimensional polyhedra in $n$-dimensional space to the Euclidean volume for the minimal simplex in the same $k$-dimensional subspace is called its the integer $k$-dimensional volume (if $k = 1$ — the integer length of the segment, if $k = 2$ — the integer area of the polygon).

The ratio of the Euclidean distance from an integer hyperplane (containing an $n - 1$-dimensional integer sublattice) to the integer point to the minimal Euclidean distance from the hyperplane to some integer point in the complement of this hyperplane is called the corresponding integer distance.

By the integer angle between two integer rays (i.e. rays that contain more than one integer point) with the vertex at the same integer point we call the value $S(u, v)/(|u| \cdot |v|)$, where $u$ and $v$ are arbitrary integer vectors passing along the rays and $S(u, v)$ is the integer volume of the triangle with edges $u$ and $v$.

Remark. Our integer volume is an integer number (in standard parallelepiped measuring the value will be $k!$ times less). The integer $k$-dimensional volume of the simplex is equal to the index of the lattice subgroup generated by its edges having the common vertex.

Since the integer angles of any triangle with all integer vertices can be uniquely restored by the integer lengths of the triangle and its integer volume we will not write the integer angles of triangles below.

Hypothesis 2. The specified invariants distinguish all nonequivalent torus triangulations of two-dimensional continued fractions for the cubic irrationalities.

In the formulations of the propositions 3.1—3.5 we say only about homeomorphic type for the torus triangulations although the description of the fundamental regions allowing us to calculate any other invariant including affine types of the faces is given in the proofs. (As an example we calculate integer volumes and distances to faces in propositions 3.1 and 3.2.) The affine structure examples of all triangulation faces are shown on the figures.

Proposition 3.1. Let $m = b - a - 1$, $n = (a + 2)(b + 1)$ $(a, b \geq 0)$ then the torus triangulation corresponding to the operator $A_{m,n}$ is
homeomorphic to the following one:

![Diagram](attachment:image.png)

(in the figure $b = 6$).

Proof. The operators

$$X_{a,b} = A_{m,n}^{-2}, \quad Y_{a,b} = A_{m,n}^{-1}(A_{m,n}^{-1} - (b + 1)I)$$

commutes with the operator $A_{m,n}$ without transpose of the sails (note that the operator $A_{m,n}$ transpose the sails). Here $I$ is the identity element of the group $SL(3, \mathbb{Z})$.

Let us describe the closure for one of the fundamental regions obtaining by factoring of the sail by the operators $X_{a,b}$ and $Y_{a,b}$. Consider the points $A = (1, 0, a + 2)$, $B = (0, 0, 1)$, $C = (b - a - 1, 1, 0)$ and $D = ((b + 1)^2, b + 1, 1)$ of the sail containing the point $(0, 0, 1)$. Under the operator $X_{a,b}$ action the segment $AB$ maps to the segment $DC$ (the point $A$ maps to the point $D$ and $B$ to $C$). Under the operator $Y_{a,b}$ action the segment $AD$ maps to the segment $BC$ (the point $A$ maps to the point $B$ and $D$ to $C$). The integer points $((b + 1)i, i, 1)$, where $i \in \{1, \ldots, b\}$ belong to the interval $BD$.

As can be easily seen the integer lengths of the segments $AB$, $BC$, $CD$, $DA$ and $BD$ are equal to 1, 1, 1, 1 and $b + 1$ correspondingly; the integer areas of both triangles $ABD$ and $BCD$ are equal to $b + 1$. The integer distances from the origin to the plains containing the triangles $ABD$ and $BCD$ are equal to 1 and $a + 2$ correspondingly.

The operators $X_{a,b}$ and $Y_{a,b}$ mapping the sail to itself, since all their eigenvectors are positive (or in this case it is equivalent to say that values of their characteristic polynomials on negative semi-axis are always negative). Furthermore this operators are the generators of the group of integer operators mapping the sail to itself, since it turn out that torus triangulation obtaining by factoring the sail by this operators contains the unique vertex (zero-dimensional face), and hence the torus triangulation have no smaller subperiod.

Let us show that all vertices for the fundamental domain of the arbitrary periodic continued fraction can be chosen from the closed convex hull of the following points: the origin; $A$; $X(A)$; $Y(A)$ and $XY(A)$, where $A$ is the arbitrary zero-dimensional face of the sail, and
operators \(X\) and \(Y\) are generators of the group of integer operators mapping the sail to itself.

Consider a tetrahedral angle with the vertex at the origin and edges passing through the points \(A, X(A), Y(A),\) and \(XY(A)\). The union of all images for this angle under the transformations of the form \(X^mY^n\), where \(m\) and \(n\) are integer numbers, covers the whole interior of the orthant. Hence all vertices of the sail can be obtained by shifting by operators \(X^mY^n\) the vertices of the sail lying in our tetrahedral angle. The convex hull for the integer points of the form \(X^mY^n(A)\) is in the convex hull of all integer points for the given orthant at that. Therefore the boundary of the convex hull for all integer points of the orthant is in the complement to the interior points of the convex hull for the integer points of the form \(X^mY^n(A)\). The complement in its turn is in the unit of all images for the convex hull of the following points: the origin, \(A, X(A), Y(A),\) and \(XY(A)\), under the transformations of the form \(X^mY^n\), where \(m\) and \(n\) are integer numbers.

It is obviously that all points of the constructed polyhedron except the origin lie at the concerned open orthant at that.

**Proposition 3.2.** Let \(m = -a, n = 2a + 3\ (a \geq 0)\), then the torus triangulation corresponding to the operator \(A_{m,n}\) is homeomorphic to the following one:

![Diagram](image)

**Proof.** Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

\[
X_a = A_{m,n}^{-2}; \quad Y_a = (2I - A_{m,n}^{-1})^{-1}.
\]

As in the previous case let us make the closure of one of the fundamental regions of the sail (containing the point \((0,0,1)\)) that obtains by factoring by the operators \(X_a\) and \(Y_a\). Let \(A = (0,0,1), B = (2,1,1), C = (7,4,2)\) and \(D = (-a,1,0)\). Besides this points the vertex \(E = (3,2,1)\) is in the fundamental region. Under the operator \(X_a\) action the segment \(AB\) maps to the segment \(DC\) (the point \(A\) maps to the point \(D\) and \(B \rightarrow C\)). Under the operator \(Y_a\) action the segment \(AD\) maps to the segment \(BC\) (the point \(A\) maps to the point \(B\) and \(D \rightarrow C\)).

If \(a = 0\) then the integer length of the sides \(AB, BC, CD\) and \(DA\) are equal to 1, and the integer areas of the triangles \(ABD\) and \(BCD\)
are equal to 1 and 3 correspondingly. The integer distances from the origin to the plains containing the triangles $ABD$ and $BCD$ are equal to 2 and 1 correspondingly.

If $a > 0$ then all integer length of the sides and integer areas of all four triangles are equal to 1. The integer distances from the origin to the plains containing the triangles $ABD$, $BDE$, $BCE$ and $CED$ are equal to $a + 2$, $a + 1$, 1 and 1 correspondingly.

Here and below the proofs of the statements on the generators are similar to the proof of the corresponding statements in the proof of proposition 3.1.

□

**Proposition 3.3.** Let $m = 2a - 5$, $n = 7a - 5$ ($a \geq 2$), then the torus triangulation corresponding to the operator $A_{m,n}$ is homeomorphic to the following one:

\[
\begin{array}{c}
D \\
\downarrow \\
E \\
\downarrow \\
A
\end{array}
\quad
\begin{array}{c}
C \\
\downarrow \\
F \\
\downarrow \\
B
\end{array}
\]

(in the figure $a = 5$).

**Proof.** Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

\[
X_a = 2A_{m,n}^{-1} + 7I; \quad Y_a = A_{m,n}^2.
\]

Let us make the closure of the fundamental regions of the sail (containing the point $(0, 0, 1)$) that obtains by factoring by the operators $X_a$ and $Y_a$. Let $A = (-14, 4, -1)$, $B = (-1, 1 - a, 7a^2 - 10a + 4)$, $C = (1, 5 - 7a, 49a^2 - 72a + 30)$ and $D = (0, 0, 1)$. Under the operator $X_a$ action the segment $AB$ maps to the segment $DC$ (the point $A$ maps to the point $D$ and the $B$ — to $C$). Under the operator $Y_a$ action the segment $AD$ maps to the segment $BC$ (the point $A$ maps to the point $B$ and $D$ — to $C$). Besides this points the vertices $E = (-1, 0, 2a - 1)$ and $F = (0, -a, 7a^2 - 5a + 1)$ are in the fundamental region. The interval $BE$ contains $a - 2$ integer points, the interval $DF$ — $a - 1$, $AD$ and $CB$ — one point for each.

□

**Proposition 3.4.** Let $m = a - 1$, $n = 3 + 2a$ ($a \geq 0$), then the torus triangulation corresponding to the operator $A_{m,n}$ is homeomorphic to
the following one:

\[
\begin{array}{c}
G \\
\quad D \\
\quad \quad C \\
\quad \quad \quad E \\
\quad \quad \quad \quad A \\
\quad \quad \quad \quad \quad B
\end{array}
\]

\(\text{(in the figure } a = 4)\).

**Proof.** Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

\[
X_a = (2I + A_{m,n}^{-1})^{-2}; \quad Y_a = A_{m,n}^{-2}.
\]

Let us make the closure of one of the fundamental regions of the sail (containing the point \((0,0,1)\)) that obtains by factoring by the operators \(X_a\) and \(Y_a\). Let \(A = (1,-2a-3,4a^2+11a+10), B = (0,0,1), C = (-4a-11,2a+5,-a-2)\) and \(D = (-a-2,0,a^2+3a+3)\). Besides this points the vertices \(E = (-2,1,0), F = (-2a-3,a+1,1)\) and \(G = (0,-1-a,2a^2+5a+4)\) are in the fundamental region. The intervals \(BG\) and \(DF\) contains \(a\) integer points each. In the interior of the pentagon \(BEFDG\) \((a+1)^2\) integer points of the form: \((-j,-i+j,(2a+3)i-(a+2)j+1)\), where \(1 \leq i \leq a+1, 1 \leq j \leq 2i-1\) are contained. Under the operator \(X_a\) action the segment \(AB\) maps to the segment \(DC\) (the point \(A\) maps to the point \(D\) and \(B\) to \(C\)). Under the operator \(Y_a\) action the broken line \(AGD\) maps to the broken line \(BEC\) (the point \(A\) maps to the point \(B\), the point \(G\) maps to the point \(E\), and the point \(D\) — to the point \(C\)). \(\square\)

**Proposition 3.5.** Let \(m = -(a+2)(b+2)+3, n = (a+2)(b+3)-3\) \((a \geq 0, b \geq 0)\), then the torus triangulation corresponding to the operator \(A_{m,n}\) is homeomorphic to the following one:

\[
\begin{array}{c}
D \\
\quad C \\
\quad \quad D \\
\quad \quad \quad C \\
\quad \quad \quad \quad F \\
\quad \quad \quad \quad \quad E \quad E \\
\quad \quad \quad \quad \quad \quad A \quad B \quad A \quad B
\end{array}
\]

\(\text{(in the figure } b = 5)\).

**Proof.** Let us choose the following generators of the subgroup of integer operators mapping the sail to itself:

\[
X_{a,b} = ((b+3)I - (b+2)A_{m,n}^{-1})A_{m,n}^{-2}; \quad Y_{a,b} = A_{m,n}^{-2}.
\]
Let us make the closure of one of the fundamental regions of the sail (containing the point \((0,0,1)\)) that obtains by factoring by the operators \(X_{a,b}\) and \(Y_{a,b}\). Let \(A = (b^2 + 3b + 3, b^2 + 2b - a + 1, a^2b + 3a^2 + 4ab + b^2 + 6a + 5b + 4), B = (b^2 + 5b + 6, b^2 + 4b + 4), C = (-ab - 2a - 2b - 1, 1, 0)\) and \(D = (0, 0, 1)\). The interval \(BD\) contains \(b + 1\) integer points. Besides this points the vertices \(E = (b + 4, b + 3, b + 2), F = (b + 2, b + 1, a + b + 2)\) and \(G = (1, 1, 1)\) are in the fundamental region. Under the operator \(X_{a,b}\) action the segment \(AB\) maps to the segment \(DC\) (the point \(A\) maps to the point \(D\) and the point \(B\) — to the point \(C\)). Under the operator \(Y_{a,b}\) action the broken line \(AFD\) maps to the broken line \(BEC\) (the point \(A\) maps to the point \(B\), the point \(F\) maps to the point \(E\), and the point \(D\) — to the point \(C\)). □

Note that the generators of the subgroup of operators commuting with the operator \(A_{m,n}\) that do not transpose the sails can be expressed by the operators \(A_{m,n}\) and \(\alpha I + \beta A_{m,n}^{-1}\), where \(\alpha\) and \(\beta\) are nonzero integer numbers.

It turns out that in general case the following statement is true: the determinants of the matrices for the operators \(\alpha I + \beta A_{m,n}^{-1}\) and \(\alpha I + \beta A_{m+k\beta,n+\alpha k}^{-1}\) are equal. In particular, if the absolute value of the determinant of the matrix for the operator \(\alpha I + \beta A_{m,n}^{-1}\) is equal to one then the absolute value of the determinant of the matrix for the operator \(\alpha I + \beta A_{m+k\beta,n+\alpha k}^{-1}\) is also equal to one for an arbitrary integer \(k\).

Seemingly torus triangulation for the other sequences of operators \(A_{m_0+k\beta,n_0+\alpha s}\), where \(s \in \mathbb{N}\), (besides considered in the propositions 3.1—3.5) have much in common (for example, number of polygons and their types).

Note that the numbers \(\alpha\) and \(\beta\) for such sequences satisfy the following interesting property. Since

\[|\alpha I + \beta A_{m,n}^{-1}| = \alpha^3 + \alpha^2 \beta m - \alpha \beta^2 n + \beta^3,\]

the integer numbers \(m\) and \(n\) such that \(|\alpha^3 + \alpha^2 \beta m - \alpha \beta^2 n + \beta^3| = 1\) exist iff \(\alpha^3 - 1\) is divisible by \(\beta\) and \(\beta^3 - 1\) is divisible by \(\alpha\), or \(\alpha^3 + 1\) is divisible by \(\beta\) and \(\beta^3 + 1\) is divisible by \(\alpha\).

For instance such pairs \((\alpha, \beta)\) for \(10 \geq \alpha \geq \beta \geq -10\) (besides described in the propositions 3.1—3.5) are the following: \((3, 2), (7, -2), (9, -2), (9, 2), (7, -4), (9, 4), (9, 5), (9, 7)\).

In conclusion we show the table with squares filled with torus triangulation of the sails constructed in this work whose convex hulls contain the point with the coordinates \((0,0,1)\), see fig 1. The torus triangulation for the sail of the two-dimensional continued fraction for the cubic


irrationality, constructed by the operator $A_{m,n}$ is shown in the square sited at the intersection of the string with number $n$ and the column with number $m$. If one of the roots of characteristic polynomial for the operator is equal to 1 or -1 at that than we mark the square $(m, n)$ with the sign $*$ or # correspondingly. The squares that correspond to the operators which characteristic polynomial has two complex conjugate roots we paint over with light gray color.

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*E-mail address*, Oleg Karpenkov: karpenk@mccme.ru