Barrier Pairs for Safety Control of Uncertain Output Feedback Systems

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Abstract—The barrier function method for safety control typically assumes the availability of full state information. Unfortunately, in many scenarios involving uncertain dynamical systems, full state information is often unavailable. In this paper, we aim to solve the safety control problem for an uncertain single-input single-output system with partial state information. First, we develop a synthesis method that simultaneously creates a barrier function and a dynamic output feedback safety controller. This safety controller guarantees that the unit sub-level set of the barrier function is an invariant set under the uncertain dynamics and disturbances of the system. Then, we build an identifier-based estimator that provides a state estimate affine to the uncertain model parameters of the system. To detect the potential risks of the system, a fault detector uses the state estimate to find an upper bound for the barrier function. The fault detector triggers the safety controller when the system’s original action leads to a potential safety issue and resumes the original action when the potential safety issue is resolved by the safety controller.

I. INTRODUCTION

Barrier functions [1], [2] are commonly used for safety control and verification of dynamical systems. However, the standard theory assumes that the full state information is available to compute the barrier function values. Unfortunately, full state information can only be estimated in many safety control scenarios involving disturbances and uncertainties, such as wearable robots [3] coupled with time-varying human dynamics and self-driving vehicles [4] in an uncertain environments.

There are multiple methods [5], [6], [7] for synthesizing a full state feedback controller that enforces a valid barrier function. Thus, safety control for a system without full state measurements naturally begins with finding a state observer and then builds a safety controller using the state estimate from the state observer. However, how to estimate the barrier function values using the state estimate from the state observer remains a question. Although some observer-based output feedback controllers can enforce safety constraints without knowing the barrier function values [8], [9], we cannot blindly use these controllers at all times. For example, a human operator may give a human assistant robot [3] an input that potentially violates some safety constraints of the human-robot coupled system. If we replace the original robot controller entirely with a safety controller, this robot will maintain safety but not assist the human operator. For this reason, we must estimate the barrier function value in real-time so that the safety controller only corrects the original system when necessary.

In this paper, we aim to solve the safety control problem for an uncertain single-input single-output (SISO) system with partial state information. The main contributions of this paper are summarized as follows.

1) In Sec. III, we develop a control synthesis method that simultaneously creates a barrier function and a dynamic output feedback safety controller. Using the controller parameter transformation scheme in [10], our dynamic output feedback safety controller guarantees that the unit sub-level set of our barrier function is an invariant set with bounded model uncertainty and disturbance. The proposed control synthesis method in this paper builds significantly upon the method in [7], which only focuses on full state feedback safety control.

2) In Sec. IV, we propose a robust fault detector that consists of an identifier-based estimator [11]. Similar to the method in [12], this identifier-based estimator provides us with a robust state estimate, which helps us find the upper bound for our vector norm barrier function using partial state information. However, unlike the estimator in [12], our fault detector in this paper does not require the system to be originally stable or have a stable static output feedback controller.

3) In Sec. V, we showcase our fault detector and safety controller, which work together to protect an uncertain SISO system from potential risks. In particular, we demonstrate how our fault detector helps us correctly trigger our safety controller when the system is about to violate the safety constraints and resume the original mission of the system when it is safe to do so.

Notation. We define $S_p \triangleq \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix}$ and $S_k \triangleq \begin{bmatrix} 0_{n \times n} & I_{n \times n} \end{bmatrix}$ as two selection matrices, which extract the plant state $x_p$ and the controller state $x_k$ from the closed-loop state vector $x_{CL}$. Supposing $P \succ 0$, \begin{equation} \|x\|_P \triangleq \sqrt{x^TPx} \end{equation} (1) is a vector norm function based on $P$. In our LMIs, we define $\mathcal{H}(\star) \triangleq \star + \star^T$.

II. PRELIMINARIES

In this section, we introduce models of an uncertain dynamical system $\Sigma_p$ and a full-order dynamic output feedback...
controller \( \Sigma_k \). The uncertain dynamical system is described as a polytopic linear differential inclusion (PLDI) [13]. Then we give an overview of barrier pairs, which will be used for safety verification and control of the uncertain dynamical system. Based on the concept of barrier pairs, we present our formal problem statement.

### A. State Model

In this paper, we consider a SISO system. We will explore how we can extend our study to a multiple-input multiple-output (MIMO) system in the future.

First, assume that the transfer function of our system \( \Sigma_p \) from input \( u \) to output \( y \) is expressed as

\[
G_p(s) = \frac{y(s)}{u(s)} = \frac{\beta_1 s^{n-1} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n},
\]  

where \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( \beta_1, \beta_2, \ldots, \beta_n \) are the uncertain parameters of our plant. Let us define

\[
A_0 \triangleq \begin{bmatrix} 0^T & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad c_0 \triangleq \begin{bmatrix} 0_1 \times (n-1) \end{bmatrix}.
\]

In Sec. [\textbullet{}\textbullet{}\textbullet{}], we will conduct a robust fault detection of \( \Sigma_p \) using an identifier-based estimator [11]. In order to build this identifier-based estimator, we need to use the following state space realization of \( \Sigma_p \):

\[
\dot{x}_p = A_0 x_p + [b_y \quad b_u] \begin{bmatrix} y \\ u \end{bmatrix},
\]

\[
y = c_0 x_p + w
\]

where \( b_y \triangleq [-\alpha_n \cdots -\alpha_2 -\alpha_1]^T \) and \( b_u \triangleq [\beta_n \cdots \beta_2 \beta_1]^T \) contain all the uncertain plant parameters and \( w \) is a disturbance signal. Let us suppose that

\[
b_y = \bar{b}_y + \sum_{i=1}^{n_p} \delta_i \cdot \theta_i^y \cdot \tilde{b}_i,
\]

\[
b_u = \bar{b}_u + \sum_{i=1}^{n_p} \delta_i \cdot \theta_i^u \cdot \tilde{b}_i
\]

where \( \bar{b}_y \triangleq [-\bar{\alpha_n} \cdots -\bar{\alpha_2} -\bar{\alpha_1}]^T \) and \( \bar{b}_u \triangleq [\bar{\beta_n} \cdots \bar{\beta_2} \bar{\beta_1}]^T \) are the nominal plant parameters, \( n_p \) is the number of dimensions of the parameter uncertainties, \( \bar{b}_i \in \mathbb{R}^{n \times 1}, \theta_i^y \in \mathbb{R}^{1 \times 1}, \theta_i^u \in \mathbb{R}^{1 \times 1} \) describe the direction of the parameter uncertainties in each dimension, \( \delta_i \in \mathbb{R} \) is a scalar variable such that \( |\delta_i| \leq 1 \) for all \( i = 1, 2, \ldots, n_p \). Based on [\textbullet{}\textbullet{}], \( [b_y \quad b_u] \) is in a polytopic region in \( \mathbb{R}^{n \times 2} \) and (5) becomes a PLDI.

Next, let the state-space model of a full-order dynamic output feedback controller \( \Sigma_k \) be

\[
\dot{x}_k = A_k x_k + b_k y
\]

\[
u = c_k x_k
\]

where \( A_k \in \mathbb{R}^{n \times n}, b_k \in \mathbb{R}^{n \times 1}, \) and \( c_k \in \mathbb{R}^{1 \times n} \) are the controller parameters to be determined.

Let us define

\[
A_{CL} \triangleq \begin{bmatrix} \bar{A}_p & \bar{b}_yc_0 \\ \bar{b}_kc_0 & \bar{A}_k \end{bmatrix}, \quad B_{wp} \triangleq \begin{bmatrix} \bar{b}_y \\ \bar{b}_k \end{bmatrix}, \quad \Theta_y \triangleq \begin{bmatrix} \theta_y^1 \\ \theta_y^2 \\ \vdots \\ \theta_y^{n_p} \end{bmatrix},
\]

\[
C_q \triangleq \begin{bmatrix} \Theta_y c_0 \\ \Theta_u c_0 \end{bmatrix}, \quad D_{wp} \triangleq \begin{bmatrix} \Theta_u \\ 0 \end{bmatrix},
\]

our formal problem statement.

### B. Barrier Pairs

The concept of barrier pairs [7] describes the relationship between a barrier function and a feedback controller in a safety control problem. In this paper, we extend the definition of barrier pairs to output feedback systems.

**Definition 1.** A barrier pair is a pair \((B, \Sigma_k)\) consisting of a barrier function \( B : x_{CL} \rightarrow \mathbb{R} \) and a controller \( \Sigma_k \) satisfying the following conditions:

(a) \( \varepsilon \leq B(x_{CL}) \leq 1, u = \Sigma_k(y) \implies \dot{B}(x_{CL}) < 0 \),

(b) \( B(x_{CL}) \leq 1 \implies x_p \in \mathcal{X}_s, \Sigma_k(y) \in \mathcal{U} \).

In particular, \( x_p \in \mathcal{X}_s \) and \( u \in \mathcal{U} \) are the state and input constraints. Intuitively, (a) and (b) mean the invariance and constraint satisfaction properties of a barrier pair.

### C. Problem Statement

In this paper, we consider an uncertain dynamical system \( \Sigma_p \) described in [\textbullet{}\textbullet{}\textbullet{}\textbullet{}] with assumption \( \|w\| \leq \bar{w} \) and safety constraints \( x_p \in \mathcal{X}_s \) and \( u \in \mathcal{U} \).

**Problem.** Assuming that \( y \) and \( u \) are the only available measurements of \( \Sigma_p \) and \( x_p = 0 \) at \( t = 0 \), find a barrier pair \((B, \Sigma_k)\) and a switching system \( \Sigma_p \) where \((B, \Sigma_k)\) satisfies conditions (a) and (b) in Definition 1 for \( \Sigma_p \) and where \( \Sigma_p \) switches \( \Sigma_p \) from its original input \( u = \hat{u} \) to \( u = \Sigma_k(y) \) whenever \( B \) is close to 1 and switches back to \( u = \hat{u} \).

Fig. 1 shows a block diagram of \( \Sigma_p \) that is similar to the switching systems proposed in [2], [7]. This type of switching system seeks to maintain the safety of \( \Sigma_p \) through a minimum intervention in its original input \( u = \hat{u} \). However, unlike the previous works, the full state of \( \Sigma_p \) is not available in our problem. Therefore, our switching system cannot switch between \( u = \hat{u} \) and \( u = \Sigma_k(y) \) based on the exact value of our barrier function \( B \).
To address this limitation, we will solve our problem in two steps. In Sec. [III] we show a synthesis method that creates a barrier pair for $\Sigma_p$. In Sec. [IV] we propose a robust fault detector that finds an upper bound for $B$ using only the measurements of $y$ and $u$. Based on this upper bound for $B$, we will construct $\Sigma_s$ in Sec. [IV-C].

III. BARRIER PAIR SYNTHESIS

In this section, we focus on the barrier pair synthesis for our system $\Sigma_p$ described in (4) and (5). First we formulate conditions (a) and (b) in Definition [I] as linear matrix inequality (LMI) constraints. Then we introduce our LMI optimization problems for barrier pair synthesis.

A. LMIs for Invariance Property

In this paper, we define our barrier function as a vector norm function

$$B(x_{CL}) \triangleq \|x_{CL}\|_p.$$  \hspace{1cm} (9)

Let us partition $P$ and $Q \triangleq P^{-1}$ as

$$P = \begin{bmatrix} X & V \\ V^T & * \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Y & W \\ W^T & * \end{bmatrix},$$

where $X, Y, V, W \in \mathbb{R}^{n \times n}$. In addition, we define

$$\Pi_1 \triangleq \begin{bmatrix} I & X \\ 0 & V^T \end{bmatrix} \quad \Pi_2 \triangleq \begin{bmatrix} Y & I \\ W^T & 0 \end{bmatrix},$$

where $\Pi_1 = P\Pi_2$ and $\Pi_2 = Q\Pi_1$. In Proposition [I] we will use $\Pi_1$ and $\Pi_2$ to perform a controller parameter transformation [10] and derive the LMI constraints for condition (a) in Definition [I].

**Proposition 1.** Supposing that $|w| \leq \bar{w}$, there exist a barrier function $B(x_{CL}) = \|x_{CL}\|_p$ and a controller $\Sigma_k$ in the form of (6) such that $(B(x_{CL}), \Sigma_k)$ satisfies condition (a) in Definition [I] if there exist $X \succ 0$, $Y \succ 0$, $E \in \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{n \times 1}$, $G \in \mathbb{R}^{1 \times n}$, and $\mu_w, \mu_1, \mu_2, \cdots, \mu_{n_p} \geq 0$ such that

$$\begin{bmatrix} Y & I & X \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succ 0.$$  \hspace{1cm} (12)

and

$$\begin{bmatrix} H_A & * & * \\ * & -M_{w} & \ast \ast \ast \\ M_pHC & M_pD_{wp} & -M_p \end{bmatrix} \prec 0,$$  \hspace{1cm} (13)

where $M_p \triangleq \text{diag}(\mu_1, \mu_2, \cdots, \mu_{n_p})$, $M_{wp} \triangleq \text{diag}(\mu_w, \mu_{n_p})$,

$$H_A \triangleq \text{He} \left\{ \begin{bmatrix} \hat{A}_pY + \hat{b}_uG & \hat{A}_pX + F_{c0}Y \end{bmatrix} + \mu_w \bar{w}^2 \begin{bmatrix} Y \\ X \\ 0 \end{bmatrix} \right\},$$  \hspace{1cm} (14)

and

$$E \triangleq VA_kW^T + F_{c0}Y + X\hat{b}_uG + X\hat{A}_pY,$$

$$F \triangleq Vb_k, \quad G \triangleq c_hW^T.$$  \hspace{1cm} (15)

**Proof.** Notice that (13) is obtained by performing a congruence transformation with $\text{diag}(\Pi_1, I, I)$ on

$$\begin{bmatrix} \Phi_Q + \mu_w \bar{w}^2Q & \ast & \ast \\ B_{wp}^T & -M_{wp} & \ast \ast \ast \\ M_pC_qQ & M_pD_{wp} & -M_p \end{bmatrix} \prec 0,$$  \hspace{1cm} (16)

where $\Phi_Q \triangleq A_{CL}Q + QA_{CL}$. Thus, we focus on the following two steps to complete the proof. First, we will show that condition (a) in Definition [I] holds for $(B(x_{CL}), \Sigma_k)$ if (16) holds. Then, we will show that (16) holds if (12) and (13) hold.

In the first step, we consider $B^2(x_{CL}) = x_{CL}^TP_{CL}x_{CL}$ as a quadratic Lyapunov function candidate for our closed-loop system with $u = \Sigma_k(y)$. Then, $(B(x_{CL}), \Sigma_k)$ satisfies condition (a) in Definition [I] for $\Sigma_p$ in (4) and (5) under the assumption $|w| \leq \bar{w}$ if and only if $\frac{d}{dt}B^2(x_{CL}) < 0$, or equivalently

$$\begin{bmatrix} x_{CL}^T \\ w \\ p \end{bmatrix} \begin{bmatrix} A_{CL}^TP + P A_{CL} & \ast & \ast \\ B_{CL}^TP & 0 & \ast \ast \ast \\ B_{CL}^TP & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{CL}^T \\ w \\ p \end{bmatrix} < 0,$$  \hspace{1cm} (17)

for all $x_{CL}, w$, and $p$ that

$$x_{CL}^TP_{CL}x_{CL} \geq \varepsilon^2, \quad w^2 \leq \bar{w}^2, \quad \text{and} \quad p_i^2 \leq q_i^2, \quad \forall i = 1, \cdots, n_p.$$  \hspace{1cm} (18)

Using the S-procedure, (17) holds under the conditions in (18) if there exist $\mu_{CL}, \mu_w, \mu_1, \cdots, \mu_{n_p} \geq 0$ such that

$$\begin{bmatrix} x_{CL}^T \\ w \\ p \end{bmatrix} H_p \begin{bmatrix} x_{CL}^T \\ w \\ p \end{bmatrix} + \mu_w \bar{w}^2 - \mu_{CL} < 0,$$  \hspace{1cm} (19)

where

$$H_p \triangleq \begin{bmatrix} \Phi_p + \mu_w \bar{w}^2P + C_q^TM_pC_q^T & B_{wp}^TP & -M_{wp} + D_{wp}^TM_pD_{wp} \end{bmatrix}$$

and $\Phi_p \triangleq A_{CL}^TP + P A_{CL}$. Then, (19) holds if $H_p \prec 0$ and $\mu_{CL} = \mu_w \bar{w}^2$. Through a congruence transformation with $\text{diag}(Q, I)$ on $H_p$ and the Schur complement, $H_{CL} \prec 0$ for $\mu_{CL} = \mu_w \bar{w}^2$ is equivalent to (16).

Now, let us establish the second step. Since (13) is obtained by performing a congruence transformation with $\text{diag}(\Pi_1, I, I)$ on (16), (16) holds if there exists non-singular $\text{diag}(\Pi_1, I, I)$ such that (12) holds. Obviously, $\text{diag}(\Pi_1, I, I)$ is non-singular if and only if $V$ is non-singular. Since $FQ = I$, we have $VW^T = I - XY$. Then, there exists non-singular $V$ and $W$ if $I - XY$ is non-singular. $I - XY$ is non-singular if $Y > 0$ and $X - Y^{-1} > 0$. Using the Schur complement, $Y > 0$ and $X - Y^{-1} > 0$ if (12) holds. Consequently, we obtain that there exists non-singular $\text{diag}(\Pi_1, I, I)$ if (12) holds.

Therefore, condition (a) in Definition [I] holds for $(B(x_{CL}), \Sigma_k)$ if there exist $X \succ 0$, $Y \succ 0$, $E \in \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{n \times 1}$, $G \in \mathbb{R}^{1 \times n}$, and $\mu_w, \mu_1, \mu_2, \cdots, \mu_{n_p} \geq 0$ such that (12) and (13) hold.

If we define the scalar variables $\mu_w, \mu_1, \mu_2, \cdots, \mu_{n_p}$ a
priori, \( \text{Proposition 1} \) becomes an LMI in \( (X, Y, E, F, G) \). Even though \( \text{Proposition 1} \) is not an LMI with respect to these variables jointly, we can consider methods such as D-K iteration \( \text{Proposition 14} \) for searching the values of these scalar variables.

B. LMIs for State and Input Limits

Now, let us focus on condition (b) in Definition 1 for barrier function \( B(x_{CL}) = \|x_{CL}\| \) and safety controller \( \Sigma_k \) in the form of \( \text{Proposition 6} \). In this paper, we let define \( \mathcal{X}_k \) and \( \mathcal{U} \) as

\[
\mathcal{X}_k \triangleq \{ x_k : \| f_i^T x_k \| \leq 1, \quad i = 1, 2, \ldots, n_f \},
\]

\[
\mathcal{U} \triangleq \{ u : \| u \| \leq \bar{u} \}.
\]

Since \( f_i^T x_k = f_i^T S_p x_{CL} \) and \( u = c_k S_k x_{CL} \), condition (b) in Definition 1 holds for \( (B(x_{CL}), \Sigma_k) \) if

\[
f_i^T S_p S_k^T f_i \leq 1, \quad \text{for } i = 1, 2, \ldots, n_f,
\]

\[
c_k S_k^T c_k \leq \bar{u}^2.
\]

Since \( \Sigma_k \) is undetermined (i.e., \( A_k, b_k \) and \( c_k \) are also decision variables), \( \text{Proposition 24} \) is not an LMI. We address this issue as follows.

Since \( Q = P^{-1} \), we can use the Schur complement to obtain that \( \text{Proposition 24} \) holds if

\[
\begin{bmatrix}
    P & c_k S_k \\
    c_k^T & \bar{u}^2
\end{bmatrix} \succeq 0.
\]

By performing a congruence transformation with \( \text{Proposition 25} \), we obtain that \( \text{Proposition 24} \) holds if

\[
\begin{bmatrix}
    Y & * \\
    I & X & * \\
    G^T & 0 & \bar{u}^2
\end{bmatrix} \succeq 0,
\]

where \( \text{Proposition 25} \) is an LMI in our new variable set \( (X, Y, E, F, G) \) introduced in Proposition 1. In addition, we can use the Schur complement to obtain that \( \text{Proposition 26} \) also implies condition (12) in Proposition 1.

C. Barrier Pair Construction

Through a convex optimization

\[
\begin{align*}
\text{maximize} & \quad \log(\det(Y)) \\
\text{subject to} & \quad X > 0, \quad Y > 0,
\end{align*}
\]

we obtain a solution of \( (X, Y, E, F, G) \) that maximize the volume of the \( x_p \) space projection \( \{ x_p : x_p = S_p x_{CL}, \ B(x_{CL}) \leq 1 \} \) of the unit sub-level set of \( B(x_{CL}) \).

There are multiple methods to construct a controller \( \Sigma_k \) based on a solution of \( (X, Y, E, F, G) \). For example, Ref. [15, Lemma 7.9] defines \( V V^T = X - Y^{-1} \) and \( W = -YV \). After obtaining \( V \) and \( W \), we can construct our controller parameters \( A_k, b_k, \) and \( c_k \) according to [15].

IV. ROBUST FAULT DETECTION

Although barrier pair \( (B, \Sigma_k) \) is obtained in the previous section, the calculation of \( B \) relies on knowing the full state \( x_{CL} \) of the closed-loop system. Since \( x_{CL} \) is not available, the true value of \( B \) is unknown. In this section, we will introduce a robust fault detector that provides an upper bound for \( B \) using only the measurements of \( y \) and \( u \).

A. Identifier-Based Estimator

In [11], the concept of an identifier-based estimator was developed for the purpose of model identification and adaptive control. However, as a byproduct, it also provides us with a robust state estimate \( \hat{x}_p \) to the model uncertainty of \( \Sigma_p \). The identifier-based estimator for our system \( \Sigma_p \) is a pair of sensitivity function filters

\[
\begin{align*}
\dot{z}_y &= A^T_z \hat{z}_y + c^T_0 y \\
\dot{z}_u &= A^T_z \hat{z}_u + c^T_0 u
\end{align*}
\]

where \( A_z \triangleq A_0 - b_z c_0 \) and \( b_z \in \mathbb{R}^{n \times 1} \) is defined by the user such that \( A_z \) is a Hurwitz matrix. Let us define

\[
\begin{align*}
E_y &\triangleq c_y S_0^T, & E_u &\triangleq c_u S_0^{-1}
\end{align*}
\]

where \( c_0 \) is the controllability matrix of \( (A^T_z, c^T_0) \), \( c_y \) is the controllability matrix of \( (A^T_z, z_y) \), and \( c_u \) is the controllability matrix of \( (A^T_z, z_u) \).

Lemma 1. \( z_y \) and \( z_u \) are the states of the identifier-based estimator in (28). \( E_y \) and \( E_u \) are defined as (29). If we define a state estimate \( \hat{x}_p \) for \( \Sigma_p \) in (4) as

\[
\hat{x}_p = E^T_y (b_y + b_z) + E^T_u b_u,
\]

the state estimation error \( e \triangleq x_p - \hat{x}_p \) follows

\[
e = A_z e - b_z w.
\]

Proof. This is similar to the proofs of [11, Lemma 2] and [12, Lemma 1]. Subtracting (31) from (4), we obtain that

\[
\dot{\hat{x}}_p = A_z \hat{x}_p + (b_y + b_z) y + b_u u.
\]

Since \( C_0 \) is the controllability matrix of \( (A^T_z, c^T_0) \), we can derive from (28) that

\[
\begin{align*}
\dot{E}_y &= A^T_z E_y + y \cdot I, \quad E_y = A^T_z E_u + u \cdot I.
\end{align*}
\]

Notice that \( A_z \) is in a canonical form, which leads to \( E_y A_z = A_z E^T_y \) and \( E^T_u A_z = A_z E^T_u \). By taking the transpose of (33), we obtain \( \dot{E}^T_y = A_z E^T_y + y \cdot I \) and \( E^T_u = A_z E^T_u + u \cdot I \). Therefore, (32) holds if \( \dot{\hat{x}}_p = E^T_y (b_y + b_z) + E^T_u b_u \).

Notice that in (30), \( \hat{x}_p \) is affine in \( b_y \) and \( b_u \), which are defined in (5). Substituting (5) into (30), we have

\[
\begin{align*}
\hat{x}_p &= \bar{x}_p + \sum_i \delta_i \cdot \hat{x}_i^p \\
\hat{x}_p &\triangleq E^T_y (b_y + b_z) + E^T_u b_u, \\
\hat{x}_i^p &\triangleq \theta^y_i \cdot E^T_y \hat{b}_i + \theta^u_i \cdot E^T_u \hat{b}_i
\end{align*}
\]

where \( \delta \triangleq [\delta_1, \delta_2, \ldots, \delta_{n_p}] \) and \( |\delta_i| \leq 1 \) for all \( i = 1, 2, \ldots, n_p \). Let us define a set

\[
\Delta \triangleq \{ [\delta_1, \delta_2, \ldots, \delta_{n_p}] : |\delta_i| = 1, \quad i = 1, 2, \ldots, n_p \},
\]

which consists of all the \( 2^{n_p} \) extreme values of \( \delta \). Then, we have

\[
\hat{x}_p(\delta) \in \text{Co} \{ \bar{x}_p + \sum_i \delta_i \cdot \hat{x}_i^p, \forall \delta \in \Delta \}.
\]
Although we do not know the exact value of \( \hat{x}_p \), due to the uncertainty in \( b_p \) and \( b_u \), (36) shows that the possible value of \( \hat{x}_p(\delta) \) is in a polytopic region in \( \mathbb{R}^n \).

B. Barrier Function Estimation

Next, we will explain how (31) and (36) help us find a computable upper bound for our barrier function \( B \).

**Proposition 2.** Let us define
\[
B(\bar{\delta}) \triangleq \max_{\delta \in \Delta} B(\hat{x}_CL(\bar{\delta})) + r_e,
\]
where \( \hat{x}_CL(\bar{\delta}) \triangleq [\hat{x}_p(\bar{\delta})^T, x_k^T]^T \) and \( r_e > 0 \). Supposing that \( |w| \leq \bar{w}, B(xCL) \leq B \) if there exists \( \mu_e \geq 0 \) such that
\[
\begin{bmatrix}
A^T_x + X A_z + \mu_e \cdot X & * \\
-b_z^T X & -\mu_e \cdot \frac{r_e^2}{\bar{w}^2}
\end{bmatrix} \preceq 0.
\]
(38)

**Proof.** Through the triangle inequality of \( \| \cdot \|_P \), we have
\[
B(xCL) = \|xCL\|_P \leq \|\hat{x}_CL(\bar{\delta})\|_P + \|S^T_P e\|_P,
\]
(39)
where \( e \triangleq x_p - \hat{x}_p \). According to (36),
\[
\|\hat{x}_CL(\bar{\delta})\|_P \leq \max_{\delta \in \Delta} \|\hat{x}_CL(\bar{\delta})\|_P.
\]
(40)
Based on the definition of \( P \) in (10),
\[
\|S^T_P e\|_P \leq \|e\|_X.
\]
Thus, \( B(xCL) \leq B \) for all \( w \) that \( |w| \leq \bar{w} \) if \( \max_{\delta \in \Delta} \|\hat{x}_CL(\bar{\delta})\|_P + \|e\|_X \leq r_e \), or equivalently \( \|e\|_X \leq r_e \), for all \( w \) that \( |w| \leq \bar{w} \).

At \( t = 0 \), \( e \triangleq x_p - \hat{x}_p = 0 \) for \( x_p = z_y = z_u = 0 \). Since \( e = 0 \) at \( t = 0 \) and \( \bar{e} = A^T_x \bar{e} + b_z w \), \( \|e\|_X \leq r_e \) for all \( w \) that \( |w| \leq \bar{w} \) if and only if \( \frac{\|e\|_X^2}{\bar{e}^2} < 0 \), or equivalently
\[
\begin{bmatrix}
e^T & \bar{e} \\
\bar{w}^T & b_z^T X \bar{e} + -\mu_e \cdot \frac{r_e^2}{\bar{w}^2}
\end{bmatrix} < 0,
\]
(42)
for all \( e \) and \( w \) that
\[
e^T X \bar{e} \geq r_e^2 \quad \text{and} \quad \bar{w}^2 \leq \bar{w}^2.
\]
(43)
Using the S-procedure, we obtain that (42) holds under the conditions in (33) if there exists \( \mu_e \geq 0 \) such that (38) holds. Therefore, \( B(xCL) \leq B \) for all \( w \) that \( |w| \leq \bar{w} \) if there exists \( \mu_e \geq 0 \) such that (38) holds.

Notice that \( X \) is already obtained from optimization (27). So, if we fix the value of \( \mu_e \), (38) becomes an LMI in \( b_z \). Through convex optimization
\[
\begin{align*}
\text{minimize} & \quad r_e^2 \\
\text{subject to} & \quad \text{(38)}
\end{align*}
\]
\( b_z \) in (28) can be defined for minimizing \( r_e \). Since \( \hat{x}_CL(\bar{\delta}) \) for all \( \delta \in \Delta \) can be obtained from our identifier-based estimator, \( B \) is available to us.

C. Switching Logic of \( \Sigma_s \)

Based on the value of \( B \) (Fig. 2), we can now define the switching logic of \( \Sigma_s \). Let us define two thresholds \( \varepsilon \) and \( \bar{\varepsilon} \) (with \( \varepsilon < \bar{\varepsilon} < \bar{\varepsilon} \leq 1 \)). \( \Sigma_s \) switches from the original input

\[
u = \hat{u} \quad \text{if} \quad B \geq \bar{\varepsilon}
\]
and switches back to \( u = \hat{u} \) if \( B < \varepsilon \).

\[
\begin{array}{c}
\text{start} \\
\downarrow \quad \text{u = u} \quad \text{if} \quad B \geq \bar{\varepsilon} \\
\downarrow \quad \text{u = u} \quad \text{if} \quad B < \varepsilon
\end{array}
\]

**Fig. 2.** \( \Sigma_s \) switches from the original input \( u = \hat{u} \) to \( u = \Sigma_k(y) \) if \( B \geq \bar{\varepsilon} \) and switches back to \( u = \hat{u} \) if \( B < \varepsilon \).

In this section, we provide an example to illustrate the robust fault detection and safety control of our proposed method.

A. System Model

Here we consider an uncertain mass-spring system \( \Sigma_p \) (Fig. 3) with a transfer function
\[
G_p(s) = \frac{y(s)}{u(s)} = \frac{k}{m \cdot s^2 + k}
\]
where the system has a unit mass \( m = 1 \) and a spring stiffness \( k \), the output \( y = k(x - x_0) \) measures the spring force, and the input \( u \) represents an adjustable force exerting to the mass. The spring stiffness is uncertain and defined as \( k = k + \delta \cdot k \), where \( k = 10 \), \( \bar{k} = 1 \), and \( |\delta| \leq 1 \). The state model of \( \Sigma_p \) can be expressed as
\[
\begin{align*}
\dot{x}_p &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_p + \begin{bmatrix} -k \bar{k} \\ 0 \end{bmatrix} y + \begin{bmatrix} k m \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_p + w
\end{align*}
\]
(46)
where \( x_p = [\dot{x} \ x]^T \) and \( w \) is an unknown exogenous input. If we consider this system as a simplified 1-DOF human-robot interaction model in a wearable robot control problem [3], then \( m \) is the robot inertia, \( k \) is an uncertain human joint stiffness, and \( x_0 = \frac{k}{m} \) is a desired joint position where the human operator tends to move to.

The safe regions \( \mathcal{X} \) and \( \mathcal{U} \) are defined as
\[
\begin{align*}
\mathcal{X}_s &\triangleq \{ |x| \leq 2, |x| \leq 2 \}, \\
\mathcal{U} &\triangleq \{ |u| \leq 10 \}.
\end{align*}
\]
(47)
(48)
Assuming that \( |w| \leq \bar{w} = 0.05 \), we aim to achieve a residual set \( \{xCL : B(xCL) \leq \bar{\varepsilon} \} \) of our barrier function with \( \varepsilon = 0.5 \).
The robustness of our switching system is difficult to implement the exact worst case of the disturbance input and bound in the test. We implement \( \| \bar{\Sigma} \| \) provides us with a state estimate \( \bar{x}_p \) to calculate a barrier function upper bound \( \bar{B} \) with \( r_e = 0.05 \).

In our safety control test, we want to demonstrate the robustness of our switching system \( \Sigma_{k} \) with respect to bounded model uncertainty and disturbance in the test. Unfortunately, it is difficult to implement the exact worst case of the disturbance signal \( w(t) \) that maximizes the peak value of the estimation error residue \( \| e(t) \|_X \). Instead, we implement \( w(t) \) as a sinusoidal signal in this example. Let us define a transfer function

\[
G_e(s) \triangleq \frac{\| e(s) \|_X}{w(s)}
\]

(49)

from the disturbance \( w(s) \) to the estimation error residue \( \| e(s) \|_X \). In the frequency domain (Fig. 4b), \( \| G_e(s) \|_\infty = 1.09 \) at \( f_e = 0.09 \) Hz. Therefore, we define the disturbance as a sinusoidal signal \( w(t) = \bar{w} \cdot \sin(2\pi f_c t) \), which gives us the maximum sinusoidal response of \( G_e(s) \).

To test the fault detection, we implement the original input \( \bar{u} \) as a reference tracking controller, which lets \( x \) follow a trapezoidal reference (Fig. 4c). The reference is designed to violate the safety limits of \( x \) and \( \dot{x} \) on purpose in such a way that \( \bar{u} \) can cause potential risks. Fig. 4b-d show that the safety controller \( \Sigma_{k} \) is correctly triggered when the system is about to violate the constraints of \( x \) and \( \dot{x} \). Fig. 4d shows that the true value of \( B \) is strictly lower than \( \bar{B} \) at all times.

VI. DISCUSSION

In our problem statement, we assume that the initial state of \( x_p \) is 0. Similar to Ref. [12], our fault detector can also be extended to cases where the initial states of \( x_p \) are not 0. The Hurwitz matrix \( A_z \) in our identifier-based estimator in [28] guarantees that the state estimation error due to an unknown initial state of \( x_p \) converges to 0 exponentially.

Fig. 4d in our example shows a slight over-conservatism of the barrier function upper bound \( B \). This is partly because the worst case of the disturbance input \( w(t) \) is difficult to find. In our example, we implement the disturbance input \( w(t) \) as a sinusoidal signal, which only leads to the worst case of the sinusoidal response of \( G_e(s) \) in [28]. Note that as long as \( w \leq \bar{w} \), constraint [38] guarantees \( \| e(t) \|_X \leq r_e \) no matter what type of signal we implement \( w(t) \) as.

In this paper, we considered the safety control problem for an uncertain SISO system with partial state information. By knowing the limits of the uncertain model parameters and disturbance a priori, our fault detector and safety controller work together to protect the uncertain SISO system from potential risks. In the future, we will extend our safety control method to MIMO systems.

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