LETTERS FROM WILLIAM BURNSIDE TO ROBERT FRICKE: AUTOMORPHIC FUNCTIONS, AND THE EMERGENCE OF THE BURNSIDE PROBLEM

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Abstract. Two letters from William Burnside have recently been found in the Nachlass of Robert Fricke that contain instances of Burnside’s Problem prior to its first publication. We present these letters as a whole to the public for the first time. We draw a picture of these two mathematicians and describe their activities leading to their correspondence. We thus gain an insight into their respective motivations, reactions, and attitudes, which may sharpen the current understanding of professional and social interactions of the mathematical community at the turn of the 20th century.

1. The simple group of order 504 – a first meeting of minds

Until 1902, when the publication list of the then fifty-year-old William Burnside already encompassed 90 papers, only one of these had appeared in a non-British journal: a three-page article [5] entitled “Note on the simple group of order 504” in volume 52 of Mathematische Annalen in 1898. In this paper Burnside deduced a presentation of that group in terms of generators and relations which is based on the fact that it is isomorphic to the group of linear fractional transformations with coefficients in the finite field with 8 elements. The proof presented is very concise and terse, consisting only of a succession of algebraic identities, while calculations in the concrete group of transformations are omitted.

In the very same volume of Mathematische Annalen, only one issue later, there can be found a paper [18] by Robert Fricke entitled “Über eine einfache Gruppe von 504 Operationen” (On a simple group of 504 operations), which is on exactly the same subject. But though the subject is the same and the line of thought presented ultimately leads to the identical result, the flavour and style of exposition are completely different: Fricke’s paper consists of 19 pages, and is accompanied by six very detailed pictures, most of which are larger than half a printed page. Here the author concentrated on giving a geometric description of the group in question as a discrete group acting on the hyperbolic plane. Most of the necessary calculations are read off the illustrations.

1Burnside’s presentation is given by two generators $A, B$ and relations

$$A^7 = B^2 = (AB)^3 = (A^3BA^5BA^3B)^2 = E,$$

where $E$ denotes the unit element in the group.
Two mathematicians publishing in the same journal on the same subject—per se, this would not justify the need for any special attention. The different approaches observed may be seen as a testament to the fact that one of the authors has used “superior” methods to simplify and shorten his line of arguments. This opinion may be further supported by taking into consideration that William Burnside today is known by many, and not by only experts, as “the first to develop the theory of groups from a modern abstract point of view”, and a “giant of the subject” and thus has become both a proponent of mathematical modernity and a figure of nearly mythic proportions. Robert Fricke, on the other hand, who was as industrious as Burnside and was held in high esteem by his contemporaries, was for the most part considered no more than a faithful disciple of his academic teacher, Felix Klein. Thus Fricke’s achievements were not only mostly overshadowed by the reputation of Klein, but large parts of his work were regarded as old-fashioned even by the end of his life. Consequently these two articles might be seen as nothing more than a display of the difference between two mathematicians, style and content giving evidence for Burnside’s “modernity” and Fricke’s “old-fashionedness”.

Still, there is more to this chance meeting of two at first sight so dissimilar mathematicians: the way in which their two articles complement each other so well, without the authors having met before, indicates that Burnside and Fricke might have had more in common than is visible at a cursory glance. If only these articles are taken as evidence, this statement may seem to be a little far-fetched. Nevertheless, in the sequel we will report on another, up until now unknown, but significant occasion, when the paths of Burnside and Fricke crossed again: in 1900 and 1901 the two mathematicians exchanged several letters. Two of Burnside’s letters have recently been discovered by the authors in Fricke’s Nachlass.

The mere existence of this correspondence is in itself remarkable, since Burnside’s workstyle was characterized by a later biographer as having been conducted with no apparent “extensive direct contacts with other mathematicians interested in the subject [of group theory]”, to the point that Burnside worked “in isolation, possibly even more so than was normal for his times, with little opportunity (or, perhaps, inclination) to discuss his ideas with others”. Burnside’s letters prompt us to now see his isolationist stance (if it existed at all) in a slightly different light: it was not universal, but seems to have been only with respect to certain mathematicians, as e.g. Frobenius. Others, like Fricke, were able to draw strong reactions from Burnside.

Furthermore, and maybe even more noteworthy, these letters contain the first known instance of Burnside formulating what was later to be called the Burnside Problem, and show the state of his research in this regard several months before the publication of the actual article.

In our paper, besides presenting these letters as a whole to the public for the first time, we will try to explain why Burnside might have put his trust in Robert Fricke in such an uncharacteristic way. At the same time we will draw a picture of these two quite similar 19th century mathematicians on the brink of mathematical modernity, and demonstrate their common mathematical interests in the, at that

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2See [38] and [31], p. 465, respectively.
3See e.g. the opinion of K. O. Friedrichs (1901–1982), Fricke’s successor at the Technische Hochschule in Braunschweig, as described in [42], p. 75.
4See [33], p. 32.
5See [31], pp. 472ff, and [34], p. 31.
time still interconnected, fields of discrete groups and automorphic functions, as well as their attitudes and their interactions with the mathematical community.

2. Robert Fricke, Felix Klein, and Automorphic Functions

Since it was Fricke who seems to have initiated the contact with Burnside, we start with him by giving a short description of his life and work.

Born in Helmstedt, Germany, on 24 September 1861, as the second of four children of a civil servant, Robert Fricke grew up in Braunschweig, where he finished school in summer 1880. He then studied mathematics, physics and philosophy for a teacher’s degree at the universities in Göttingen, Zürich, Berlin, Straßburg, and finally Leipzig, where he moved in winter 1883. There he attended the lectures and seminars of Felix Klein (1849–1925), and was immediately attracted by Klein’s approach to mathematics. Klein on the other hand soon recognized Fricke’s talent, and took him under his wing. In 1885 Fricke received both a teacher’s degree and a Dr. phil. from the University of Leipzig. He returned to Braunschweig, where he first became a school teacher, but then was granted leave to work as a private teacher for the sons of the Prince Regent of the Duchy of Braunschweig. As a result, having more time to advance his mathematical studies, he kept in touch with Klein, who by 1886 had accepted a chair in Göttingen. In 1891 Fricke decided to give up school teaching completely in favour of an academic career. He habilitated at the university of Kiel in the same year, and there became a Privatdozent. In 1892 he moved to Göttingen to work nearer to Klein. In 1894 he was appointed professor of higher mathematics at the Technische Hochschule in Braunschweig, where he succeeded Richard Dedekind (1831–1916). He held this position until his death in 1930. The bond between Fricke and Klein, which was furthered by Fricke marrying a niece of Klein’s in 1894, remained strong during their lifetimes.

To understand the development leading to the correspondence between Fricke and Burnside we have to go back in time to the early 1880s, when Klein had just been appointed professor of geometry in Leipzig. At that time, in 1881 and 1882, Klein was in scientific competition with the French mathematician Henri Poincaré (1854–1912). The common subject of their research was the theory of what would later be called automorphic functions. Because of excessive overwork and asthma attacks Klein suffered a breakdown of health in 1882, from which he only slowly recovered. After that, he felt unable to continue his research as before and decided to concentrate on promoting the subject in lectures and textbooks, thus trying to attract young mathematicians into this field.

During that time Klein wrote his Vorlesungen über das Ikosaeder, which was published in 1884. This book was to be the first of a multivolume project on “regular solids, modular functions, and automorphic functions”. In the autumn of 1887 Klein invited Fricke to join these efforts. While Fricke’s prowess as a mathematician was already apparent in his thesis of 1885, this project would be the stepping stone for his academic career. Klein and Fricke went on to write a two-volume book on Elliptic Modular Functions published in 1890 and 1892.

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6 A more detailed biographical account can be found in [26].
7 More than 400 letters between them are kept in the archives in Braunschweig and Göttingen.
8 Klein coined the term automorphic function about 1890, see [16], p. 577.
9 See Klein’s own recollections in [15], p. 258 and [16], p. 585.
10 See [24], p. V of the Vorrede, and [16], p. 742.
the first of which Fricke used to fulfill the requirements for his *Habilitation* in Kiel. These were followed in 1897 by *Automorphic Functions I* [24].

At the start of their cooperation, Klein, as a well established expert, was the main author, while Fricke as a young Ph.D. and aspiring mathematician meticulously worked out the details. Over the years, as Fricke gained experience and his scientific reputation grew, the character of their collaboration changed. While Fricke more and more exerted influence on contents and style, thus in fact becoming the main, and in large part the only author and the driving force, Klein remained as the initiator and public figurehead of the project and acted as some kind of spiritual guide in the background, every once in a while slightly changing the direction or contents. This change of role was reflected by the change of order in which the names of these two authors appeared on the title page of *Automorphic Functions I*.

3. William Burnside, and automorphic functions

The life of William Burnside is fairly well documented in [14, 25], so we may limit ourselves to a short survey.

Burnside was born on 2 July 1852 in London where he also grew up and went to school. In October 1871 he started studying mathematics in Cambridge. He graduated from there in 1875, and became a fellow of Pembroke College. In 1885 Burnside was appointed professor of mathematics at the Royal Naval College at Greenwich, a position which he held until his retirement in 1919. He continued his research until his death in 1927.

Burnside’s preoccupation with automorphic functions began about 1891. In that and the following year he authored a sequence of three papers [1, 2, 3] on this subject. These marked a turning point in Burnside’s development as a mathematician. Starting out in the Cambridge tradition of applied mathematics, with a personal emphasis laid on kinematics, kinetics, and especially hydrodynamics, he had now devoted large parts of these articles, which were going to exceed all of his other research publications in length, to a more formal approach. Furthermore the papers stood at the beginning of his growing involvement in the theory of groups, mainly in discrete and finite groups [12].

Both to elaborate the connection between automorphic functions and group theory, and to assess Burnside’s achievements, at this point it is advisable to give a rough outline of the mathematics involved:

Let $\Gamma$ be a discrete group of linear fractional transformations $z \mapsto \frac{az + \beta}{\gamma z + \delta}$ with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha\delta - \beta\gamma = 1$, the topology being induced by the Euclidean metric.

If $d$ is an arbitrarily chosen integer, an *automorphic form* of dimension $d$ with respect to $\Gamma$ is a meromorphic function $\theta$ defined on a domain in $\mathbb{C}$ satisfying the functional equation

$$\theta \left( \frac{az + \beta}{\gamma z + \delta} \right) = (\gamma z + \delta)^d \cdot \theta(z)$$

for all elements $z \mapsto \frac{az + \beta}{\gamma z + \delta}$ of $\Gamma$, whenever $\theta$ is defined.

An automorphic form of dimension 0 is called an *automorphic function*. Thus, automorphic functions with respect to $\Gamma$ are invariant under the group action of $\Gamma$.

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11See [25], p. VI et seq. of the Vorrede.
12This estimation was already given in Burnside’s obituary [14], p. 76ff.
Given a rational function \( z \mapsto H(z) \) and a negative integer \( d \), the Poincaré series of dimension \( d \) (with respect to \( \Gamma \) and \( H \)) is defined as

\[
\sum_k H \left( \frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k} \right) \cdot (\gamma_k z + \delta_k)^d,
\]

where the summation is taken over all of the elements \( z \mapsto \frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k} \) in the group \( \Gamma \) (because \( \Gamma \) is discrete the summation is taken over at most countably many elements).

Whenever such a series is uniformly convergent in an open domain, it represents an automorphic form of dimension \( d \) in that domain. Therefore the question arises quite naturally of how the convergence of Poincaré series depends on the group \( \Gamma \) and the dimension \( d \). Poincaré himself [39, 40] had already shown that for dimensions \( d \leq -4 \) these series are uniformly convergent where defined.

In [1, 2, 3] Burnside significantly extended Poincaré’s results. Burnside started from a concrete physical problem in hydrodynamics: the uniform streaming motion in two dimensions around a set of circles. The physical description of uniform flow together with the boundary conditions given by the circles leads to a differential equation the solution of which might be represented by a Poincaré series. Burnside subsequently obtained results on the convergence of Poincaré series of dimension \( d = -2 \) which also extend to \( d = -1 \).

Although with these papers he entered a field of research which was new to him, Burnside took notice of other important publications in this direction besides Poincaré’s. During the course of his articles he incorporated Klein’s freshly created terminology of automorphic function and Primform. Furthermore he mentioned a paper [43] by Friedrich Schottky (1851–1935), which had advanced the theory of Poincaré series considerably. Nevertheless Burnside’s methods differ substantially from Schottky’s, and thus lead to parallel and in some instances improved results.

4. Fricke composing Automorphic Functions II

In the autumn of 1899, the year after his and Burnside’s article on the simple group of order 504, Fricke focussed on working out the structure and the details of Automorphic Functions II. This volume was planned to be the completion of his and Klein’s project. At the end of that year, Fricke had written the first one and a half chapters, approximately 100 pages, and had sketched an outline of the third, which was intended to cover the theory of Poincaré series [13]. For this chapter, Fricke collected the published convergence results known to him, starting from Poincaré’s fundamental papers [39, 40]. For dimension \( d = -2 \), he took into consideration Schottky’s already mentioned article [43] from 1887, and the Ph.D. thesis [41] of Ernst Ritter (1867–1895) from 1892, which had been written in Göttingen under the auspices of Klein. By extending the ideas which he found in these sources, mainly using Schottky’s method of proof, Fricke managed to fill several gaps by the summer of 1900. His progress and the development of the book can be traced almost minutely through a series of letters to Klein [14].

Although Burnside’s results from 1891 and 1892 were highly relevant to the subject, they are not mentioned in Fricke’s correspondence, and seem to not have been

\[\text{Sources:}\]

[13] Fricke to Klein, 9 Sept, 8 Nov, 27 Dec 1899, Klein Nachlass 9, SUB Göttingen.

[14] Fricke to Klein, 25 Apr, 12, 19, 25 May, 1, 8, 17 July, 11 Oct 1900, Klein Nachlass 9, SUB Göttingen.
part of the initial manuscript. Fricke did report to Klein on an article of Whittaker on automorphic functions\textsuperscript{15} in which Burnside’s achievements were explicitly mentioned, but he only referred to a preliminary abstract in \textsuperscript{15}. If he had considered the later article \textsuperscript{46} from 1899, he certainly would have come across Burnside’s papers.

While streamlining the exposition of his manuscript, Fricke had also excerpted two short articles \textsuperscript{19, 20}, which he published in \textit{Göttinger Nachrichten}. So exultant was he about his accomplishments that on 17 July 1900, after months of intensive work, he wrote to Klein\textsuperscript{16}: “Only now have I gained confidence that ‘Automorphic Functions II, Part I’ will actually succeed.”

Despite this show of optimism, Fricke and Klein both lived in constant trepidation that the general interests of the mathematical community had gradually but surely shifted away from automorphic functions towards other, more foundational issues.\textsuperscript{17} Therefore Fricke planned to present his new results on Poincaré series to the general mathematical public at the meeting of the \textit{Deutsche Mathematiker-Vereinigung}, which was to take place in Aachen from 16 to 23 September 1900. He regarded this talk as an opportunity both for promoting his work and for probing the possible reaction towards the forthcoming book on automorphic functions.

However, when he returned from Aachen, Fricke was in a state of turmoil. He wrote to Klein\textsuperscript{18}: “This year’s meeting did not turn out very propitious for me, as I did not find the right kind of support for the areas of interest with which I came. I came there ‘automorphically charged’, so to speak, but in Aachen, as well as now afterwards, I was left with the impression that I will commit an anachronism with the second volume on automorphic functions.”

Fricke further described how he reacted to this blow to his confidence and which actions he took: “It is nevertheless a case of ‘the sooner the better’ and I would have to wage a pretty fierce battle if I intended to definitively abandon this project. My policy therefore was to make myself as free as possible from other hindrances so as to develop automorphic functions more quickly. I have therefore asked Lampe\textsuperscript{19} to nominate another colleague as reviewer for number theory starting next year [...]. On the other hand, following my mood I have begun every possible correspondence concerning automorphic functions.”

Thus in defiance of both his own disenchantment and the seeming lack of interest of the mathematical community, Fricke concentrated more than ever on concluding the first part of \textit{Automorphic Functions II}, which at that time he only had to put the finishing touches to. After returning from a short journey to Paris, he sent letters to several experts on automorphic functions, among them David Hilbert\textsuperscript{20}.

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\textsuperscript{15}Fricke to Klein, 23 Dec 1898, Klein Nachlass 9, SUB Göttingen.
\textsuperscript{16}Fricke to Klein, 17 July 1900, Klein Nachlass 9, SUB Göttingen (our translation).
\textsuperscript{17}Only a month after the enthusiastic letter from Fricke, David Hilbert (1862–1943) delivered his famous talk \textit{Mathematische Probleme} before the Second International Congress of Mathematicians in Paris, which included a list of open problems that would subsequently be considered as the most important to “Modern Mathematics”.
\textsuperscript{18}Fricke to Klein, 11 Oct 1900, Klein Nachlass 9, SUB Göttingen (our translation).
\textsuperscript{19}Emil Lampe (1840–1918) was the editor of \textit{Jahrbuch über die Fortschritte der Mathematik}, the main reviewing journal in mathematics at that time, which Fricke voluntarily worked for as a reviewer of articles in number theory.
\textsuperscript{20}Fricke to Hilbert, 30 Sept 1900, Hilbert Nachlass 107, SUB Göttingen.
5. A LETTER OF RESPONSE AND ENQUIRY

It seems that until the Aachen meeting Fricke was completely unaware of Burnside’s work on automorphic functions. As a mathematician, Burnside was known to him at least since the episode of their nearly simultaneous working on the simple group of order 504 of two years before. In fact, Fricke had been chosen by Klein during the process of editing Burnside’s paper to evaluate its correctness, and thus had been spurred into activity, which had resulted in his own article in Mathematische Annalen shortly afterwards.24

After Aachen, however, Fricke at last was aware of Burnside’s results, as can be seen by the summary of his Aachen talk in [21], where he clearly states them. We may assume that he was informed about Burnside’s articles during the meeting by Heinrich Burkhardt (1861–1914), a close associate of Fricke and Klein. Burkhardt had reviewed Burnside’s papers for Jahrbuch über die Fortschritte der Mathematik in 1895, which was immediately followed by another review from him discussing Ritter’s thesis.25 Presumably this incident in Aachen might have caused Fricke to start his correspondence in order to avoid other such embarrassing situations due to ignorance of relevant literature.

Thus, when in short succession he sent off the series of letters mentioned above, he also started an exchange of ideas with Burnside. Unfortunately, of this particular correspondence, only two of Burnside’s letters seem to have survived. The first one of these starts as follows:

Bromley Road
Catford
Oct 28, 1900.

Dear Professor Fricke

Many thanks for your last very kind letter. I hope very much you will carry out your intention of making a trip to England; and if you do, I trust you will let me shew you some hospitality. It would give me great pleasure if you would come and stay here for a few days so that we might become personally acquainted. I can read German with comparative ease, but I cannot speak it at all & I fear now it is rather late to learn.

At the risk of wasting your time I must return once more to the subject of my last letter, to free myself from the reproach of having forgotten the main point of my argument. Since I wrote you I have looked up Poincaré’s article in Acta Math. Vol I, with which I was more familiar in 1892 than I am now; and I have reconstructed

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21Lindemann to Fricke, 22 Nov 1900, Fricke Nachlass, UA Braunschweig.
22Wellstein to Fricke, 29 Oct 1900, Fricke Nachlass, UA Braunschweig.
23Wirtinger to Fricke, 8, 15 Oct 1900, Fricke Nachlass, UA Braunschweig.
24Fricke to Klein, 1, 12, 13 Sept, 13 Oct 1898, Klein Nachlass 9, SUB Göttingen.
25See Jahrbuch über die Fortschritte der Mathematik, Volume 24, p. 391ff.
26Burnside to Fricke, 28 Oct 1900, Fricke Nachlass, UA Braunschweig.
It is safe to assume that the papers from *Göttinger Nachrichten* which Fricke sent to Burnside are those articles [19, 20] mentioned above which Fricke published while working out the details of *Automorphic Functions II*.

Returning Fricke’s kindness, Burnside sent to him his contribution [6] to volume 18 of *Transactions of the Cambridge Philosophical Society* which had been reserved for articles dedicated to George Gabriel Stokes (1819–1903) on the occasion of the fiftieth anniversary of his being Lucasian Professor of Mathematics at the University of Cambridge in October 1899.

The reconstructed argument that Burnside mentioned constitutes the contents of the second page of Burnside’s letter:

Putting

\[ f(z) = \sum N(z_i)(\gamma_i z + \delta_i)^{-2m-2} \]

\[ \sum D(z_i)(\gamma_i z + \delta_i)^{-2m} \]

take \( \sum D(z_i)(\gamma_i z + \delta_i)^{-2m} \) to be free from poles then p. 227, it has \( 2m(n-1) \) zeros.

Again taking \( \sum N(z_i)(\gamma_i z + \delta_i)^{-2m-2} \) free of poles it can (p. 282) be expressed linearly in terms of \((2m+1)(n-1)\) linearly independent functions: i.e. the numerator is a linear homogeneous function of \((2m+1)(n-1)\) arbitrary constants. If each zero of the denominator is a zero of the numerator there are \( 2m(n-1) \) linear relations among the constants, leaving over \( n-1 \) arbitraries. But from the physical considerations \( f(z) \) ought to contain \( n \) arbitraries, viz. the \( n \) circulation constants. Hence I inferred that the function postulated physically cannot be represented in the form

\[ \frac{\sum N(z_i)(\gamma_i z + \delta_i)^{-2m-2}}{\sum D(z_i)(\gamma_i z + \delta_i)^{-2m}}. \]

[On the margin of this page we find the remark:] references to Poincaré’s paper in Vol I of Acta Mathematica.

Let us try to put this proof in the context of Burnside’s original paper. Obviously, Fricke’s question pertained to certain discontinuous groups, the fundamental domains of which are bounded by \( n > 1 \) pairs of circles. In the last parts of his paper [2], p. 287 et seq., Burnside had asserted by counting dimensions that there exist everywhere finite automorphic forms of dimension \(-2\) which cannot be represented by Poincaré series. Fricke seems to have asked if such an automorphic form could be represented by a quotient of Poincaré series of appropriate dimensions, thus avoiding any difficult convergence considerations. Burnside replied that such a representation is not possible in general, giving as a reason an extended version of his dimension argument. Since Fricke at that time still collected any information...
concerning automorphic functions in connection with Poincaré series, he was clearly interested in Burnside’s argument.  

These pages of Burnside’s letter enable us to restore the basics of the correspondence so far: Fricke started by asking a mathematical question on the representability of automorphic functions by Poincaré series. Burnside replied by stating the general impossibility, admitting that he could not remember the details of his argument. Fricke answered with a “very kind” letter in which he included personal details, such as his plan to travel to England. Maybe he expressed his wish to get personally acquainted with John Perry (1850–1920), whose *Calculus for Engineers* he was concurrently translating into German. Perhaps he mentioned as well that he had been invited to Cambridge in June 1899 to join the celebration in honour of Stokes, but had not attended. He also included his latest scientific publications. Burnside felt himself obliged to look up the missing details of his proof, and supplemented his reply with one of his latest publications.

But Burnside’s letter does not end with his more comprehensive answer to Fricke’s question, as would usually be the case in a correspondence between two mathematicians who barely know each other. Instead, Burnside added a third page which reads as follows:

*I take the opportunity of asking you, whether the following question has ever presented itself to you; and if it has, whether you have come to any conclusion about it.*

*Can a group, generated by a finite number of operations, and such that the order of every one of its operations is finite and less than an assigned integer, consist of an infinite number of operations.*

*E.g. as a very particular case:–*

*If $S_1$ and $S_2$ are operations, and if $\Sigma$, representing in turn any and every combination or repetition of $S_1$ and $S_2$, such as $S_1^a S_2^b S_1^c \ldots S_2^e$, is such that $\Sigma^m = 1$, where $m$ is a given integer, is the group generated by $S_1$ and $S_2$ a group of finite order or not. Of course if $m$ is 2, the group is of order 4 and if $m$ is 3 the group is of order 27; but for values of $m$ greater than 3, the question seems to me to present serious difficulties however one looks at it.*

In a more specified form, the question raised above would later be called the *Burnside Problem*. Although it is presented here in statu nascendi and for the special case of two generators, the fundamental problem is already clearly stated.

6. A letter of thanks and report on progress

After Fricke and Klein had finished their manuscript in December 1900, the first part of *Automorphic Functions II* was published in May 1901. As was customary at that time, they sent off a number of copies of their book to colleagues who worked in that particular area of mathematics or to whom they wanted to express some measure of gratitude or friendship. Due to the high costs, the list of people thus

27Fricke himself had explicitly expressed an analogous assertion in one of his earlier major papers in 1892 (see [17], p. 453), where he referred to [11]. Remarkably, Fricke would later use the findings of that paper in his research on the simple group of order 504.
recognised had to be restricted in number to a select few. On that specific occasion, Burnside belonged to the recipients of such a copy. Usually, this honour was answered with short polite letters of thanks, the tone of these missives determined by the degree of personal acquaintance to the author. Again, Burnside’s reply goes beyond this:

The Croft,
Bromley Road,
Catford.
June 9, 1901.

Dear Prof. Fricke

Very many thanks to you and to Prof. Klein for your kindness in sending me a copy of the new instalment of your book on automorphic functions. I got the first volume when it appeared and have read the greater part of it; and I look forward with great pleasure to studying the new part.

I have recently returned to the question I wrote you about in the winter, viz. that of the discontinuous group defined by

\[ S^m = 1 \]

when \( S \) represents any and every combination of \( n \) independent generating operations \( A_1, A_2, \ldots, A_n \); and \( m, n \) are given integers.

I find that when \( m = 3 \) and \( n \) is given, the order can be determined by a kind of recurring formula. In particular if \( n = 3 \) the order is \( 3^{17} \).

For \( m = 4 \), \( n = 2 \) I find \( 2^{12} \) for the order. These results, if correct as I believe them to be, would seem to show that a graphical method, i.e. the consideration of the network of polygons, is practically out of the question from the extreme complexity of the figure with so great a number of polygons.

So far I am quite baffled by the case \( n = 2, m = p \), a prime greater than 3; but it is easy to shew that the order cannot be less than \( p^{p+2} \) and I think it is probably greater, if finite at all.

Believe me

Yours very sincerely

W. Burnside.

With the letters from Burnside at hand, we can try to reconstruct Burnside’s progress and the genesis of his famous paper [7] of 1902 where Burnside’s results were published in a coherent and detailed form. Below we let \( B(m, n) \) denote the universal group with \( m \) generators, all elements of which have an order dividing \( n \), which today is appreciatively called the Burnside group of exponent \( n \) (with \( m \) generators).

It seems that in October 1900 Burnside had not progressed very far in determining answers to his own question. In the first letter Burnside presented the two examples \( B(2, 2) \) and \( B(2, 3) \), the first of which is quite obvious while the other one was known to him for several years: \( B(2, 3) \) had already appeared as an illustrative example of the graphical method of analysing groups in his article [4] in 1893, and

\[^{28}\]Burnside to Fricke, 9 June 1901, Fricke Nachlass, UA Braunschweig.
was used again in the same context in his monograph on finite groups [12] in 1897. This method, which Burnside described quite thoroughly in his book, starts from an abstract finitely presented group and leads to a representation of it as a group of linear fractional transformations acting on the plane [29] Burnside’s 1893 paper [4], which was also his first dedicated solely to group theory, may be interpreted as containing the seeds from which the Burnside Problem originated.

When Burnside wrote to Fricke again in June 1901, he could report on some substantial advance, and he was even quite close to the final article. Having recapitulated the main problem in the special form that can also be found in the article (though by 1902 the role of the integer parameters \(n\) and \(m\) would be interchanged), Burnside continued to give an account of his results up to that time, leaving out any proofs: First of all Burnside told of a “recurring formula” for the order of \(B(m, 3)\). If we presume that the erroneous value 3\(^{17}\) was inadvertently written in the letter (and not simply due to an ink spot), we contend that he intended to give as a particular example \(|B(3, 3)| = 3^7\). While the recurrence formula which Burnside presented in his later article leads to the correct order of \(B(3, 3)\), it still only gives an upper bound for greater numbers of generators, as was shown by later authors in [32]. The next result Burnside stated in his letter, \(|B(2, 4)| = 2^{12}\), was right on the spot (although we now know that the proof in [7] only suffices to show that the order of this group divides 2\(^{12}\)). Finally Burnside wrote of a lower bound for the order of \(B(2, p)\), \(p\) being a prime greater than 3, always keeping in mind that the order might even be infinite. Using facts in elementary group theory, he was able to improve on this result by the time of the final version.

It seems that, while Burnside was on the right track in all cases, sometimes he was a little too confident and stepped into hidden pitfalls. While the methods and proofs presented in the final paper were correct, they did not suffice to show all parts of the stated claims. With hindsight, this does not diminish Burnside’s achievements. As Burnside told Fricke in his second letter, the orders of the groups appearing force the researchers to go beyond the scope of any visually oriented method. Thus here a purely abstract approach was truly necessary, which Burnside provided splendidly [31].

Both in the letters and in the article [7] Burnside treated special instances of the question which today is justifiably called the Burnside Problem: Is a group with a finite number \(m\) of generators necessarily finite if the orders of its elements all divide a given integer \(n\)?

However, the introductory phrase of Burnside’s first preserved letter addresses a related, but slightly different question: Is a finitely generated group finite if the orders of its elements are less than an integer \(n\)? This particular variant did not see

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\(^{29}\) Burnside attributed the origin of this method to the fundamental paper Gruppentheoretische Studien [13] of Walther (von) Dyck (1856–1934). Dyck had been a student and an assistant of Klein in Munich and in Leipzig, where he wrote [13] as his Habilitation thesis. The title of nobility “von” was bestowed on Dyck in 1901.

\(^{30}\) In [4] Burnside considered the infinite group given by generators \(P\) and \(Q\) which satisfy

\[P^3 = Q^3 = (PQ)^3 = E\]

and asked which further relations enforce the resulting group to be finite.

\(^{31}\) Even the proof readers of the final article seemed to be slightly overwhelmed by its content, since they missed Burnside’s accidental switch in the name of generators from \(A, B\) to \(P, Q\) in the paragraph before last.
print before the publication of the textbook of Harold Hilton (1876–1974) on finite groups \cite{27} in 1908. There it could be found in the appendix, where it was part of a list of twelve research problems, most of which Hilton expressly attributed to Burnside \cite{26}.

Moreover when in the second letter Burnside repeated the problem as a small reminder, he explicitly put it within the context of discontinuous groups, just as in the final article. For somebody belonging to a later generation of mathematicians this seems to be quite unnecessary, because the Burnside Problem can be seen as a problem in abstract group theory, totally divorced from any geometric interpretation.

Strangely enough, in the initial sentence of his 1902 article, Burnside kept his reference to discontinuous groups and instead dropped both the assumption of existence of a universal upper bound for the orders of each group element, and, more grievously, the prerequisite that the group in question be finitely generated. The last omission is substantial, because obviously any infinitely generated abstract group is infinite, even if all element orders are finite. Abandoning only the first condition leads to a generalisation of the original problem, which nowadays is called the General Burnside Problem: Is a finitely generated group necessarily finite if the orders of its elements are finite?

The vagueness of the first sentence of \cite{7} has given rise to a number of interpretations including a discussion if Burnside’s reference to discontinuous groups already implied those essential conditions \cite{33}. To shed some more light on this particular reference, we have to take a closer look on Burnside’s understanding of groups.

7. Burnside’s attitude towards groups – and towards Fricke

Nearly all of Burnside’s publications suggest that he thought of groups as being composed of “operations”. This observation has already been pointed out in case of his book by Neumann in \cite{34}. There it was contrasted with the definition of a group in the second volume of *Lehrbuch der Algebra* \cite{44} of Heinrich Weber (1842–1913), where the author more abstractly only speaks of “elements” as constituents of a group.

When at the turn of the century Burnside’s main interests shifted to abstract group theory, his approach to (countably) infinite groups always tended to be in connection with what at that time were called “discontinuous groups”. The contemporary understanding of this term differed from today’s: it was mainly used to discriminate this class of groups from “continuous groups”, which at that time more or less meant Lie groups. Burnside tried to give a precise definition of both concepts when he was given the opportunity of writing an expository article on the theory of groups in the Encyclopaedia Britannica \cite{8,11}.

According to our analysis his 1893 paper \cite{4} seems to be his only research paper where Burnside speaks of the elements of a group as “symbols” and not as “operations”. Its central problem is indeed formulated as an abstract word problem. However, the solution is deduced in large part via the graphical method, which is

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\[32\] The first to address this problem was B. H. Neumann in 1937 \cite{33}, who termed it the *bounded order problem*.

\[33\] See e.g. \cite{37}. In the case of complex matrix groups, Burnside himself has shown in 1905 \cite{9} that the bounded order problem has a positive solution without resorting to the further assumption of the group being finitely generated.
directly connected to discontinuous groups. In his 1901 letter to Fricke we find the explicit statement about the unsuitability of this approach, which again confirms our conclusion that [4] can be seen as a direct precursor of [7].

Although Burnside was well aware of the difference between abstract finitely generated groups and concrete discontinuous groups, most of the time he did not seem to feel the necessity to make a clear-cut distinction between these two classes. Both the evidence of the letters to Fricke and his 1893 paper show that Burnside’s interest in the Burnside Problem originally stemmed from concretely given discontinuous groups. Thus, while today his 1902 paper [7], where he first presented his set of problems to a general public, for the most part is read as a piece of pure abstract group theory, he himself explicitly put his “unsettled question” into the context of discontinuous groups. We assume that he did so, as he had done several months before in his letter, without giving a second thought to this wording.

These insights into Burnside’s view about groups enable us to explain why he had used the occasion of answering a mathematical question on automorphic functions to present his current research problems in the field of group theory, together with the results he had achieved, to a virtual stranger such as Robert Fricke. At first sight, his behaviour seems surprising, because, as we noted above, he was known to have worked more or less in seclusion, without many personal contacts with other non-British mathematicians, and his contact with Fricke happened rather by chance than by his own choice.

Although he did not know Fricke personally, Burnside was well acquainted with the subjects and results of Fricke’s work. Burnside was said to be “well informed about the published literature” [34] and he himself wrote in his letter that he could “read German with comparative ease”. We can thus safely assume that the remark in his letter that he had read the greater part of Automorphic Functions I was not just a set phrase, but a fact. Indeed, the circle of ideas encompassing automorphic functions, discrete groups and physical applications thereof was not only well known to him, but for some time in his life formed one of his central research interests. We agree with Forsyth’s assessment [35] that Burnside initially encountered infinite discrete groups in this context.

Moreover, Burnside’s opinion of how expository mathematical writing should be done was quite close to Klein’s and Fricke’s illustrative approach. On the occasion of a presidential address [10] to the London Mathematical Society in 1908, Burnside characterized the succinct way of exposition in research papers on abstract group theory as “driest formalism”, made up of “a series of conundrums”. If his own article [5] on the simple group of order 504 were assessed according to the criteria voiced in the address, only the last paragraph, where Burnside made the connection with a concrete example, would save this paper from being “a merely curious illustration of non-commutative multiplication”.

On the other hand, Burnside’s presidential address also describes the way in which such an article should be written: “The reader is led naturally from the concrete to the abstract, and acquires by actual instances the new ideas necessary to the theory.” Fricke’s article [18] on the simple group of order 504 exemplifies to the utmost Burnside’s demand for mooring abstract results in concrete examples, and appears as if written in full accordance with Burnside’s guidelines.

34 See [34], p. 32.
35 See [14], p. 77.
When Burnside himself wrote on the theory of groups for the 10th edition of *Encyclopaedia Britannica* in 1902, he quoted as authorities on both discontinuous and finite groups Felix Klein with his book on the icosahedron as well as several others of Klein’s school: Dyck with his *Gruppentheoretische Studien*, Josef Gierster (1854–1893), and, finally, Fricke with *Elliptic Modular Functions I* and *Automorphic Functions I*. Later on, in the next edition of *Encyclopaedia Britannica* in 1911, while severely cutting down the number of references, Burnside left the books of Klein and Fricke on his list, and added *Automorphic Functions II*.

In summary, Burnside’s personal development towards group theory paralleled that of a number of Klein’s former students. The mathematicians affiliated to this group were the natural choice as addressees for Burnside’s questions. When the occasion arose and Burnside came into direct contact with one of its more exposed members, namely Robert Fricke, who had shown a profound versatility in those areas close to his heart, he seized the opportunity and allowed a deeper insight into his own current research with high hopes of having found a like-minded spirit and highly competent partner.

As a matter of fact, Burnside’s inquiry was to produce a prompt reaction from Fricke, as is shown by the reverse sides of the pages of the letter from 1900. While they initially had been left blank by Burnside, Fricke had used them as some kind of scratch paper to sketch his immediate ideas on Burnside’s question. Thus on the reverse of the third page we find a pencil sketch of a regular hexagon consisting of 54 alternating black and white equilateral triangles, which forms the graphical representation of the group $B(2,3)$ and which can also be found as part of the only accompanying figure of Burnside’s 1893 paper [4]. We are safe to assume that Fricke did not need to look it up but was able to produce it by himself. Furthermore on the reverse of the second page there are several computations in Fricke’s handwriting. Obviously they were done to work out a hint on the solution of the first open case, the group of exponent 4 generated by two elements, by calculating all $2 \times 2$ matrices with entries in the integers modulo 8 generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Since this did not lead to any useful insights, Fricke eventually abandoned this line of research.

The remarkable similarity between Burnside’s and Fricke’s grasp of the concept of a group became visible again several years later. When after Weber’s death his *Lehrbuch der Algebra* went out of print, the publisher offered Fricke the opportunity to elaborate a new revision with the same title, based on the original. At the point where Fricke gave the definition of a group, he closely followed Weber, but added the following – from a mathematical point of view unnecessary – introduction which sounds only too reminiscent of Burnside: “Let there be given a system of like operations, or like analytic expressions, or mathematical entities described in any other way, which to refrain from any interpretation we call ‘elements’ [...]”

8. Aftermath

In *Automorphic Functions II*, Fricke explicitly acknowledges Burnside’s contribution to the convergence of Poincaré series of dimension $-2$, but the results are only mentioned in a reporting tone and seem to have been worked into the manuscript of the book in a very late phase. The proof presented there follows the lines

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36See [23], p. 267 (our translation and emphasis).
37See [25], pp. 157, 160, 166.
of Schottky. However, after finishing his manuscript, Fricke seems to have worked out, simplified, and generalized Burnside’s results to prove the convergence of certain Poincaré series of dimension \(-1\). How important Fricke deemed these results to be can be seen by the fact that he took the chance to include these achievements in the *Festschrift* [22] which was published on the occasion of Richard Dedekind’s 70th birthday in October 1901.

The publication of the further parts of *Automorphic Functions* II was considerably delayed: Fricke and Klein wanted to include some key theorems, proofs of which were independently given by Poincaré and Klein in the early 1880s. Unfortunately, by the turn of the century, these proofs were not considered sufficiently rigorous anymore.\(^{38}\) Only after 1907, when L. E. J. Brouwer (1881–1966) and Paul Koebe (1882–1945) had achieved new results on the uniformization of algebraic curves, based on the advances of modern point-set topology, could Fricke continue completing *Automorphic Functions* II. The final two parts were published in 1911 and 1912, respectively.

After their exchange of letters, despite their similar attitudes, Burnside and Fricke seem to have lost contact with each other. Although Fricke travelled to England in spring 1903 to meet John Perry, we have found no evidence that he took up Burnside’s invitation. On the other hand, Burnside became a member of the *Deutsche Mathematiker-Vereinigung* in 1904, and remained so for the rest of his life, even through the First World War, thus always keeping in touch with the mathematical community in Germany.

Both the Burnside Problem and the problem of convergence of Poincaré series were the source of inspiration for later mathematicians. In both cases there are still open questions awaiting their definitive solutions.

### 9. Acknowledgements

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