LEFT DETERMINED MODEL STRUCTURES FOR
LOCALLY PRESENTABLE CATEGORIES

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ABSTRACT. We extend a result of Cisinski on the construction of
cofibrantly generated model structures from (Grothendieck) top-
oses to locally presentable categories and from monomorphism to
more general cofibrations. As in the original case, under additional
conditions, the resulting model structures are "left determined" in
the sense of Rosický and Tholen.

1. INTRODUCTION

Given a Quillen model structure on a category, any two of the three
classes of maps involved (cofibrations, fibrations and weak equiva-
lences) determine the remaining one and hence the whole model struc-
ture. Going one step further, one can ask for model structures where
already one of the classes determine the other two.

Rosický and Tholen [18] introduced the notion of a left determined
model category, where the class $W$ of weak equivalences is determined
by the class $C$ of cofibrations as the smallest class of maps satisfying
some closure conditions. For such a model category, $W$ is then the
smallest possible class of weak equivalences for which $C$ and $W$ yield a
model structure.

Independently, Cisinski [4] considered classes of maps (under the
name localizer) that satisfy (almost) the same closure conditions for
the case where the underlying category is a (Grothendieck) topos and
$C$ is the class of monomorphisms. Moreover, he gave an explicit con-
struction of model structures for this case, and showed that under
suitable conditions the resulting class of weak equivalences is a small-
est localizer (w.r.t. monomorphisms). This model structure is then left
determined.

Our aim is to extend this construction and the corresponding results
to a more general context, where the class of cofibrations may not be the
monomorphisms and where the underlying category is not necessarily
a topos. The necessary assumptions for such a generalization to work
fall into three sorts:

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• general conditions on the underlying category. We assume, that the underlying category is locally presentable. Since every Grothendieck topos is locally presentable, this will include the original examples.

• conditions on the class of cofibrations in spe. We assume, that these are already part of a cofibrantly generated weak factorization system and that every object is cofibrant.

• conditions on the cylinder used for the construction. These will be discussed later.

The remaining sections of this paper are as follows: Section 2 contains the needed definitions and facts about accessible categories, weak factorization systems and model structures (mostly without proofs). In Section 3 we show that, given a cofibrantly generated weak factorization system together with a cylinder satisfying suitable assumptions, Cisinski’s construction produces a cofibrantly generated model structure. In Section 4 we compare the weak equivalences produced by that construction with smallest localizers and identify conditions under which these coincide. Finally, Section 5 contains some well known examples in order to illustrate the construction. For the case of module categories we also describe the used cylinders in terms of pure submodules.

Notation is almost standard; but we write composition in reading order and denote identity morphisms by the name of their objects.

2. Accessible categories and model structures

We first turn to accessible and locally presentable categories. The main source for this material is the book of Adámek and Rosický [2].

Definition 2.1. Let $\lambda$ be a regular cardinal.

(a) an object $X$ in a category $\mathcal{K}$ is $\lambda$-presentable if the functor $\mathcal{K}(X, -) : \mathcal{K} \to \text{Set}$ preserves $\lambda$-directed colimits.

(b) A category $\mathcal{K}$ is $\lambda$-accessible if it satisfies the following two conditions:

1. $\mathcal{K}$ has $\lambda$-directed colimits.
2. there is a set $\mathcal{A}$ of $\lambda$-presentable objects of $\mathcal{K}$ such that every object of $\mathcal{K}$ is a $\lambda$-directed colimit of objects from $\mathcal{A}$.

It is accessible if it is $\lambda$-accessible for some regular cardinal $\lambda$.

(c) A category $\mathcal{K}$ is $\lambda$-locally presentable if it is $\lambda$-accessible and cocomplete. It then follows that it is also complete, see e.g. [2, Corollary 1.28]. It is locally presentable if it is $\lambda$-presentable for some regular cardinal $\lambda$.

(d) A functor $F : \mathcal{K} \to \mathcal{L}$ is $\lambda$-accessible if both $\mathcal{K}$ and $\mathcal{L}$ are $\lambda$-accessible and $F$ preserves $\lambda$-directed colimits. It is accessible if it is $\lambda$-accessible for some regular cardinal $\lambda$.

(e) A full subcategory $\mathcal{K}$ of $\mathcal{L}$ is accessibly embedded if it is closed under $\lambda$-directed colimits for some regular cardinal $\lambda$. 
Notation 2.2. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be any functor. We write $F\mathcal{A}$ for the full image of $\mathcal{A}$ under $F$, i.e. the full subcategory of $\mathcal{B}$ determined by all objects $FX$ ($X \in \mathcal{A}$). If $\mathcal{K}$ is a full subcategory of $\mathcal{B}$, we write $F^{-1}\mathcal{K}$ for its full preimage under $F$, i.e. the full subcategory of $\mathcal{A}$ determined by all those objects $X \in \mathcal{A}$ with $FX \in \mathcal{K}$.

Lemma 2.3. Let $F: \mathcal{A} \rightarrow \mathcal{C}$ be an accessible functor and let $\mathcal{K}$ be a full subcategory of $\mathcal{C}$.

(a) If $\mathcal{K}$ is accessible and accessibly embedded in $\mathcal{C}$ then $F^{-1}\mathcal{K}$ is also accessible and accessibly embedded in $\mathcal{A}$.

(b) If $\mathcal{K}$ is the full image of an accessible functor and also isomorphism-closed in $\mathcal{C}$ then the same holds for $F^{-1}\mathcal{K}$.

Proof. Part (a) is \cite[Remark 2.50]{2}. For part (b), let $G: \mathcal{B} \rightarrow \mathcal{C}$ be an accessible functor with $\mathcal{K} = G\mathcal{B}$.

1. The comma category $(F \downarrow G)$ is accessible and the projection $(F \downarrow G) \rightarrow \mathcal{A}$ is accessible by \cite[Theorem 2.43]{2}.

2. $F$ and $G$ induce an accessible functor $H: (F \downarrow G) \rightarrow \mathcal{C}^2$ via $H(A, B, u: FA \rightarrow GB) = u$. Since the full subcategory of $\mathcal{C}^2$ given by isomorphisms is accessible and accessibly embedded in $\mathcal{C}$, the same holds for its preimage under $H$, by part (a). This preimage is the full subcategory $\text{Iso}(F,G)$ of $(F \downarrow G)$ whose objects are those $(A, B, u: FA \rightarrow GB)$ for which $u$ is an isomorphism.

3. $F^{-1}(G\mathcal{B})$ is the full image of the composite

\[ \text{Iso}(F,G) \hookrightarrow (F \downarrow G) \rightarrow \mathcal{A}. \]

We now turn to model structures. We follow Adámek, Herrlich, Rosický, Tholen \cite{1} in introducing these via the notion of a weak factorization system. Other sources include the article of Beke \cite{3} and the books of Hirschhorn \cite{4} and Hovey \cite{6}. Most definitions do not need the underlying category to be complete and cocomplete as is usually assumed when working with model structures. For now we tacitly assume that the relevant limits and colimits exist for the various statements to make sense.

Notation 2.4. For two maps $f$ and $g$ in a category $\mathcal{K}$ we write $f \square g$ if for every solid square

\[
\begin{array}{ccc}
& & 1 \\
& 1 & \downarrow 1 \\
1 & \downarrow 1 & 1 \\
\end{array}
\]

the (dotted) diagonal exists. For a class $\mathcal{H}$ of maps we set $\mathcal{H}^\square = \{g \in \mathcal{K} \mid \forall h \in \mathcal{H}: h \square g\}$ and $\square \mathcal{H} = \{f \in \mathcal{K} \mid \forall h \in \mathcal{H}: f \square h\}$.

Remark 2.5. (1) Any class of the form $\square \mathcal{H}$ is stable under pushouts, retracts in $\mathcal{K}^2$ and transfinite compositions of smooth chains, where a smooth chain is a colimit preserving functor $D: \alpha \rightarrow \mathcal{K}$.
from some ordinal and its transfinite composition is the induced map from $D_0$ to $\text{colim}_{\beta<\alpha} D_\beta$. The dual results hold for classes of the form $\mathcal{H}\square$. We write $\text{cell}(\mathcal{H})$ for the class of those maps that are transfinite compositions of pushouts of maps from $\mathcal{H}$. Hence the above observation in particular gives $\text{cell}(\mathcal{H}) \subseteq \square(\mathcal{H}\square)$.

(2) Suppose $f = xy$. If $f \square y$, then by redrawing
\[
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \quad \downarrow \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\quad \text{as}
\quad
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \quad \downarrow \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

one obtains $f$ as a retract of $x$. Dually, if $x \square f$ then $f$ is a retract of $y$.

(3) The relation $\square$ gives a Galois-connection on classes of maps, i.e. one always has $\mathcal{L} \subseteq \square \mathcal{R} \iff \mathcal{L}\square \supseteq \mathcal{R}$.

Definition 2.6. A weak factorization system in a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of maps such that the following two conditions are satisfied:

1. $\mathcal{L} = \square \mathcal{R}$ and $\mathcal{L}\square = \mathcal{R}$.
2. Every map $f$ has a factorization as $f = \ell r$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$.

The weak factorization system $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated if $\mathcal{L} = \square(I\square)$ for some subset $I \subseteq \mathcal{L}$. It is functorial if there is a functor $F: \mathcal{K}^2 \to \mathcal{K}$ together with natural maps $\lambda: \text{dom} \to F$ and $\rho: F \to \text{cod}$ such that $\lambda_f \in \mathcal{L}$, $\rho_f \in \mathcal{R}$ and $f = \lambda_f \rho_f$ for all $f \in \mathcal{K}^2$.

Definition 2.7. A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{K}$ consists of three classes $\mathcal{C}$ (cofibrations), $\mathcal{F}$ (fibrations) and $\mathcal{W}$ (weak equivalences) such that the following conditions are satisfied:

1. $\mathcal{W}$ is closed under retracts in $\mathcal{K}^2$ and has the 2-3 property: if in $f = gh$ two of the maps lie in $\mathcal{W}$ then so does the third.
2. Both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems. The classes $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{W} \cap \mathcal{F}$ are called trivial cofibrations and trivial fibrations respectively. The model structure is cofibrantly generated or functorial if the two weak factorization systems in (2) are. An object $X$ is called cofibrant if the map $(0 \to X)$ from the initial object is a cofibration and fibrant if the map $(X \to 1)$ to the terminal object is a fibration. For a functorial model structure, one obtains the cofibrant replacement functor and the fibrant replacement functor by restricting the two functorial factorizations to $(0\downarrow \mathcal{K})$ and $(\mathcal{K} \downarrow 1)$ respectively.

Remark 2.8. Any weak factorization system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{K}$ gives a model structure with $\mathcal{C} = \mathcal{L}$, $\mathcal{F} = \mathcal{R}$ and $\mathcal{W} = \mathcal{K}$ for which Definition 2.6
and Definition 2.7 produce the same notions of "cofibrantly generated" and "functorial". Any notion about model structures in general (like e.g. "(co)fibrant objects" or "(co)fibrant replacement functor" from above) can be applied to weak factorization systems by considering this special model structure.

**Definition 2.9.** Let \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) be a model structure in a category \(\mathcal{K}\).

(a) For an object \(X\), a **cylinder object** \(C_X\) for \(X\) is given by a \((\mathcal{C}, \mathcal{W})\)-factorization of the codiagonal \((X|X): X + X \to X\) as \(X + X \xrightarrow{\gamma_X} C_X \xrightarrow{\sigma_X} X\). The cylinder object \(C_X\) is **final** if \(\sigma_X \in \mathcal{C}^\square\). Given two cylinder objects \(C_X\) and \(C'_X\), we call \(C'_X\) **finer than** \(C_X\) if there is a \(\varphi_X: C_X \to C'_X\) making the following diagram commutative:

\[
\begin{array}{ccc}
X + X & \xrightarrow{\gamma_X} & C_X \\
\downarrow{\gamma_X} & & \downarrow{\varphi_X} \\
C_X & \xrightarrow{\sigma_X} & C'_X
\end{array}
\]

(b) A **(functorial) cylinder** \((C, \gamma, \sigma)\) is a functor \(C: \mathcal{K} \to \mathcal{K}\) together with natural maps \(\gamma\) and \(\sigma\) whose \(X\)-components make \(C_X\) into a cylinder object as in (a). Together with the (natural) coproduct inclusions one then obtains natural maps with \(X\)-components as in the diagram below:

\[
\begin{array}{ccc}
X & \xrightarrow{\delta_X} & X + X \\
\downarrow{\gamma_X} & & \downarrow{\gamma_X} \\
C_X & \xleftarrow{\sigma_X} & C'_X
\end{array}
\]

The cylinder is **final** if all \(\sigma_X\) are in \(\mathcal{C}^\square\). A cylinder \((C', \gamma', \sigma')\) is **finer than** \((C, \gamma, \sigma)\) iff \(C_X\) is finer than \(C'_X\) for all \(X\).

(c) Given a cylinder \((C, \gamma, \sigma)\), two maps \(f, g: X \to Y\) are **homotopic** if the induced map \((f|g): X + X \to Y\) from the coproduct factors through \(\gamma_X: X + X \to CX\). This will be written as \(f \sim g\) or sometimes as \(f \sim g \pmod{\mathcal{C}}\).

(d) The symmetric transitive closure of \(\sim\) is written as \(\approx\). Since \(\sim\) is reflexive and compatible with composition, \(\approx\) is a congruence relation. The quotient category will be denoted by \(\mathcal{K}/\approx\). A map \(f: X \to Y\) is a **homotopy equivalence**, if its image in \(\mathcal{K}/\approx\) is an isomorphism, or equivalently, if there exists a \(g: Y \to X\) with \(fg \approx X\) and \(gf \approx Y\).
For a weak factorization system \((\mathcal{L}, \mathcal{R})\), cylinder objects, functorial cylinders and homotopy are defined as those for the trivial model structure \((\mathcal{L}, \mathcal{K}, \mathcal{R})\).

**Observation 2.10.** Let \((\mathcal{L}, \mathcal{R})\) be a weak factorization system (similar observations apply to model structures).

(a) Suppose that in part (a) of Definition 2.9 the object \(X\) is cofibrant. Then the coproduct injections \(\iota_0^X\) and \(\iota_1^X\) are in \(\mathcal{L}\), being pushouts of the map \((0 \to X)\). Consequently, not only \(\gamma_X\), but also \(\gamma_0^X\) and \(\gamma_1^X\) are in \(\mathcal{L}\).

(b) The \((\mathcal{L}, \mathcal{R})\)-factorizations of codiagonals provide enough final cylinder objects and every cylinder object \(CX\) can be refined to a final one by a \((\mathcal{L}, \mathcal{R})\)-factorization of \(\sigma_X: CX \to X\). Also every final cylinder object is a finest one: if a \(CX\) is final and \(C'X\) is any other cylinder object, then \(\gamma'_X \Box \sigma_X\) will give a diagonal in

\[
\begin{array}{ccc}
X + X & \xymatrix{\ar[rr]^\gamma X & & CX} & \\
\gamma'_X & \ar[u] & \\
\downarrow & & \downarrow \sigma_X \\
C'X & \xymatrix{\ar[rr]_{\sigma'_X} & & X}
\end{array}
\]

so that \(CX\) is finer than \(C'X\).

(c) Suppose that \((\mathcal{L}, \mathcal{R})\) is functorial. Then one always has enough final cylinders and every cylinder \((C, \gamma, \sigma)\) can be refined to a final cylinder by a functorial factorization of \(\sigma\).

(d) If \((C', \gamma', \sigma')\) is finer than \((C, \gamma, \sigma)\) then the implication

\[
f \sim g \pmod{C'} \implies f \sim g \pmod{C}
\]

holds for any two maps \(f, g: X \to Y\). In particular any two final cylinders determine the same homotopy relation.

**Remark 2.11.** When functorial factorizations are not available, one can still define homotopy as in Part (c) of Definition 2.9 for a nonfunctorial choice of cylinder objects \(CX\) without any naturality condition on the maps \(\gamma_X\) or \(\sigma_X\).

One can also relax the definition by not fixing a choice for a cylinder object: two maps \(f, g: X \to Y\) are homotopic if \((f|g)\) factors through some \(\gamma_X: X + X \to CX\) of a cylinder object. This is known as "left homotopy" in the literature on model categories (see e.g. [5, Definition 7.3.2] or [6, Definition 1.2.4]). But the resulting homotopy relation is not necessarily compatible with precomposition.

An alternative approach is to use a fixed choice of final cylinder objects. The existence of certain diagonals then works as a substitute for the missing naturality. The homotopy relation with respect to such a choice will always be symmetric and compatible with composition. Moreover (by an argument as in Observation 2.10) it does not depend
on the choice of cylinder objects. This approach was introduced by Kurz and Rosický [9].

Since we will only meet situations where functorial factorizations are available, we will not need this added generality.

We now turn to weak factorization systems in locally presentable categories. The following theorem should indicate, why these categories are a convenient setting.

**Theorem 2.12.** Let $\mathcal{K}$ be a locally presentable category and $I$ a set of maps in $\mathcal{K}$.

(a) Every map $f$ can be factored as $f = xy$ with $x \in \text{cell}(I)$ and $y \in I^\square$. Moreover this factorization can be made functorial. In particular $(\square(I^\square), I^\square)$ is a functorial factorization system.

(b) In the situation of (a), the factorization functor $\mathcal{K}^2 \to \mathcal{K}$ is accessible.

(c) The full subcategory of $\mathcal{K}^2$ given by the homotopy equivalences with respect to a final cylinder is the full image of an accessible functor.

**Proof.** Part (a) is shown e.g. in [3, Proposition 1.3]. Part (b) is due to J.H. Smith; for a published proof see e.g. Rosický [16, Proposition 3.1]. The statements therein are phrased for model structures but apply to weak factorization systems via Remark 2.8. Part (c) is [17, Proposition 3.8]. □

The last ingredient will be a theorem of Smith which describes conditions under which two classes $C$ and $W$ of maps in a locally presentable category are part of a cofibrantly generated model structure.

**Definition 2.13.** A functor $F: \mathcal{A} \to \mathcal{B}$ satisfies

(a) the **solution set condition at an object** $B$ of $\mathcal{B}$ if there is a set of maps $\{f_i: B \to FA_i \mid i \in I\}$ such that every map $f: B \to FA$ factors as $f = f_i(Fu)$ for some $f_i$ and $u: A_i \to A$.

(b) the **solution set condition at a class of objects**, if it satisfies the solution set condition at every element of that class.

(c) the **solution set condition**, if it satisfies the solution set condition at all objects of $\mathcal{B}$.

A full subcategory $\mathcal{K}$ of $\mathcal{B}$ satisfies the conditions above if its inclusion functor does.

**Lemma 2.14.** Every accessible functor $F: \mathcal{K} \to \mathcal{L}$ (and hence its full image) satisfies the solution set condition.

**Proof.** [2] Corollary 2.45] □

**Theorem 2.15** (Smith’s Theorem). Let $\mathcal{K}$ be a locally presentable category, $I$ a set of maps and $W$ a class of maps in $\mathcal{K}$. Suppose that the following conditions are satisfied:
(1) $\mathcal{W}$ has the 2-3 property and is closed under retracts in $\mathcal{K}^2$.
(2) $I \subseteq \mathcal{W}$
(3) $\Box(I \subseteq) \cap \mathcal{W}$ is closed under pushouts and transfinite composition.
(4) $\mathcal{W}$ satisfies the solution set condition at $I$.

Then setting $\mathcal{C} := \Box(I \subseteq)$ and $\mathcal{F} := (\mathcal{C} \cap \mathcal{W}) \Box$ gives a cofibrantly generated model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{K}$.

Proof. [3, Theorem 1.7] □

Remark 2.16. Conditions (1)–(3) in the above Theorem are necessary for any cofibrantly generated model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ with $I$ being the set of generating cofibrations. Rosický [17, Theorem 4.3] has recently shown that condition (4) is also necessary.

3. Cisinski’s construction

We now present the construction of a cofibrantly generated model structure from a suitable cofibrantly generated weak factorization system and cylinder. As in the original case, we need additional conditions on the cylinder used. Our conditions in Definition 3.8 are different from those of Cisinski [4, Définition 2.3]. Nevertheless, they are equivalent in the case of $(\text{Mono}, \text{Mono} \Box)$ in a Grothendieck topos.

Before turning to the actual construction, we first look at one particular ingredient in a more general setting.

Definition 3.1. Let $\mathcal{A}$ be a category with pushouts. Given a natural map $\alpha: \mathcal{X} \to \mathcal{A}$ and a map $f: X \to Y$ let $f \star \alpha$ be the connecting map in the diagram below:

\[
\begin{array}{c}
FX \xrightarrow{\alpha_X} F'X \\
FY \xrightarrow{\alpha_Y} FY + F'X \xrightarrow{F'f} F'Y
\end{array}
\]

Dually, let $\mathcal{X}$ be a category with pullbacks. Given a natural map $\beta: \mathcal{A} \to \mathcal{X}$ and a map $g: A \to B$ let $\beta \star g$ be the connecting map in the diagram:

\[
\begin{array}{c}
F'Y \xrightarrow{\beta_B} G' \xrightarrow{\beta_A} \mathcal{A}
\end{array}
\]
For a class $I$ of maps, we write $I \star \alpha$ for \{ $f \star \alpha$ | $f \in I$ \} and $\beta \star I$ for \{ $\beta \star f$ | $f \in I$ \}.

For the next Lemma, recall the notion of a conjugate pair of natural maps between two adjunctions from e.g. Mac Lane [12, IV-7]: given two adjunctions $F : \mathcal{X} \rightleftarrows \mathcal{A} : G$ and $F' : \mathcal{X} \rightleftarrows \mathcal{A} : G'$, two natural maps $\alpha : F \rightarrow F'$ and $\beta : G' \rightarrow G$ are conjugate if the diagram

$$
\begin{array}{ccc}
\mathcal{A}(F'A, \mathcal{X}) & \xrightarrow{\alpha} & \mathcal{X}(X', GA) \\
\mathcal{A}(FX, \mathcal{X}) & \xrightarrow{\sim} & \mathcal{X}(X, GA)
\end{array}
$$

commutes for all $X \in \mathcal{X}$ and $A \in \mathcal{A}$.

**Lemma 3.2.** Suppose $\alpha : F \rightarrow F'$ and $\beta : G' \rightarrow G$ are two conjugate natural maps. Then for all $f : X \rightarrow Y$ and $g : A \rightarrow B$ one has

$$(f \star \alpha) \Box g \iff f \Box (\beta \star g)$$

**Proof.** We will show the direction "$\Rightarrow$". The opposite direction then follows by duality. So assume $(f \star \alpha) \Box g$ and consider any diagram

$$
\begin{array}{c}
X \xrightarrow{u} G'A \\
Y \xrightarrow{v} P \xrightarrow{q} GA \\
G'B \xrightarrow{\beta_B} GB
\end{array}
$$

where $P$ is the pullback of $\beta_B$ and $Gg$. We need a diagonal for the left upper square. Switching via the adjunctions (indicated by $\hat{\text{ }}$) in both
directions) gives the solid arrows of the diagram

\[
\begin{array}{c}
FX \xrightarrow{\alpha_X} F'X \\
Ff \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
FY \xrightarrow{i} Q \xrightarrow{r} A \\
\quad \alpha_Y \quad \downarrow \quad \downarrow \quad \downarrow \\
F'Y \xrightarrow{\beta \alpha} \xrightarrow{d} B
\end{array}
\]

where \( Q \) is the pushout of \( Ff \) and \( \alpha_X \). Now \( r: Q \to A \) is induced by \( \hat{v}q \) and \( \hat{u} \). Testing against \( i \) and \( j \) yields the commutativity of the right lower square (i.e. \( rg = (f \alpha) \hat{v}p \)), which therefore has a diagonal \( d: F'Y \to A \). Switching back via the adjunction gives

\[
\begin{array}{c}
X \xrightarrow{u} GA \\
\quad \hat{d} \quad \downarrow \quad \downarrow \quad \downarrow \\
Y \xrightarrow{v} P \xrightarrow{q} GA \\
\quad \beta g \quad \downarrow \quad \downarrow \quad \downarrow \\
\quad \quad \quad G'B \xrightarrow{\beta_B} GB
\end{array}
\]

where the equality \( \hat{d}(\beta \ast g) = v \) can be verified by testing against \( p \) and \( q \). Hence \( \hat{d}: Y \to G'A \) is the desired diagonal. \( \square \)

**Corollary 3.3.** In the situation of the previous Lemma, let \( I \) be a class of maps in \( X \) and \( J \) be a class of maps in \( A \). Then

\[
I \ast \alpha \subseteq J \implies (\Box(I^\Box)) \ast \alpha \subseteq \Box(J^\Box)
\]

**Proof.**

\[
I \ast \alpha \subseteq J \implies I \ast \alpha \subseteq \Box(J^\Box) \implies I^\Box \supseteq \beta \ast (J^\Box) \implies (\Box(I^\Box))^\Box \supseteq \beta \ast (J^\Box) \implies (\Box(I^\Box))^\Box \ast \alpha \subseteq \Box(J^\Box) \square
\]

**Remark 3.4.** Corollary 3.3 applies to any natural map between left adjoints (assuming that the necessary pushouts and pullbacks exist) because any such map determines a conjugate map between the respective right adjoints.

**Definition 3.5.** Let \((L, R)\) be a cofibrantly generated weak factorization system in a locally presentable category \( \mathcal{K} \). For a functorial cylinder \((C, \gamma, \sigma)\), a generating set \( I \) and a subset \( S \subseteq \Box(I^\Box) \) define
\( \Lambda(C, S, I) \) via the following construction:

\[
\begin{align*}
\Lambda^0(C, S, I) & := S \cup (I \ast \gamma^0) \cup (I \ast \gamma^1) \\
\Lambda^{n+1}(C, S, I) & := \Lambda^n(C, S, I) \ast \gamma \\
\Lambda(C, S, I) & := \bigcup_{n \geq 0} \Lambda^n(C, S, I)
\end{align*}
\]  

(3.1)(3.2)(3.3)

Lemma 3.6. Suppose a cylinder functor \( C \) for \( (\mathcal{L}, \mathcal{R}) \) is a left adjoint. Then for any two generating subsets \( I, J \subseteq \mathcal{L} \) one has

\[
\square(\Lambda(C, S, I)) = \square(\Lambda(C, S, J))
\]

Proof. We will drop \( C \) and \( S \) from the notation for \( \Lambda \) and show \( \Lambda^n(I) \subseteq \square(\Lambda(J)) \) for all \( n \geq 0 \).

1. We have \( J \ast \gamma^k \subseteq \Lambda(J) \) (for \( k = 0, 1 \)). Corollary 3.3 then gives \( \mathcal{L} \ast \gamma^k \subseteq \square(\Lambda(J)) \). So in particular \( \Lambda^0(I) \subseteq \square(\Lambda(J)) \).

2. Assume \( \Lambda^n(I) \subseteq \square(\Lambda(J)) \).
   Corollary 3.3 then gives \( \Lambda^{n+1}(I) = \Lambda^n(I) \ast \gamma \subseteq \square(\Lambda(J)) \). \( \square \)

Remark 3.7. In general one cannot expect \( \Lambda(C, S, I) \subseteq \mathcal{L} \) without any further assumptions. However, if \( C \) is a left adjoint, Lemma 3.6 shows, that this property does not depend on the choice of the generating subset. This motivates the following definition.

Definition 3.8. Let \( (\mathcal{L}, \mathcal{R}) \) be weak factorization system in a category \( \mathcal{K} \). A functorial cylinder \( (C, \gamma, \sigma) \) is cartesian if

(a) The cylinder functor \( C: \mathcal{K} \rightarrow \mathcal{K} \) is a left adjoint

(b) \( \mathcal{L} \ast \gamma \subseteq \mathcal{L} \) and \( \mathcal{L} \ast \gamma^k \subseteq \mathcal{L} \) (\( k = 0, 1 \))

Remark 3.9. Condition (a) allows using Lemma 3.2 and Corollary 3.3. In particular, if \( (\mathcal{L}, \mathcal{R}) \) is cofibrantly generated by some subset \( I \subseteq \mathcal{L} \), Condition (b) already holds whenever \( I \ast \gamma^0, I \ast \gamma^1 \) and \( I \ast \gamma \) lie in \( \mathcal{L} \). Also for any \( f \in \mathcal{L} \) we have \( Cf = f'(f \ast \gamma^0) \) where \( f' \) is a pushout of \( f \), so that \( Cf \) is again in \( \mathcal{L} \).

We now insert a comparison of Definition 3.8 with [4, Définition 2.3]. Let \( \mathcal{E} \) be a Grothendieck topos. We recall the following properties:

1. Colimits in \( \mathcal{E} \) are universal: given a colimit cocone \( x_i: X_i \rightarrow X \) and a map \( f: Y \rightarrow X \), the induced maps \( f^*(x_i): f^*(X_i) \rightarrow Y \) obtained from pulling back the \( x_i \) along \( f \) again form a colimit cocone. This is [7, Lemma 1.51].

2. \( \mathcal{E} \) is locally presentable. This follows from [2, Theorem 1.46] together with the fact that the sheaves with respect to a site form a small orthogonality class (in the sense of [2, Definition 1.35]) inside the respective presheaf topos.
(3) Whenever one has a diagram

\[
\begin{array}{c}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
A & B & X \\
\uparrow a & \downarrow b & \downarrow y \\
P & Q & \downarrow x \lor y \\
\rightarrow & \rightarrow & \\
\end{array}
\]

where \(x\) and \(y\) are monomorphisms, \(P\) is the pullback of \(x\) and \(y\), and \(Q\) is the pushout of \(a\) and \(b\), then the induced map \(x \lor y : Q \to X\) is also a monomorphism. This follows from [7, Proposition 1.55].

(4) Monomorphisms are closed under transfinite composition. This follows from repeated application of [2, Corollary 1.60].

From the last three items above, it follows by [3, Proposition 1.12] that (Mono, Mono\(\Box\)) is a cofibrantly generated weak factorization system. Now suppose \((C, \gamma, \sigma)\) is a cylinder for (Mono, Mono\(\Box\)) and consider the following conditions:

**DH1:** The functor \(C\) preserves monomorphisms and all colimits.

**DH2:** If \(f : X \to Y\) is a monomorphism then

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma^k} & CX \\
\downarrow f & \downarrow C_f & \downarrow C_f \\
Y & \xrightarrow{\gamma^k} & CY \\
\end{array}
\]

are pullback squares \((k = 0, 1)\).

**DH3:** If \(f : X \to Y\) is a monomorphism then

\[
\begin{array}{ccc}
X + X & \xrightarrow{\gamma^x} & CX \\
\downarrow f + f & \downarrow C_f & \downarrow C_f \\
Y + Y & \xrightarrow{\gamma^y} & CY \\
\end{array}
\]

is a pullback square.

Conditions DH1 and DH2 were introduced by Cisinski [4, Définition 2.3]. We first observe, that it is enough to restrict attention to DH1:

**Lemma 3.10.** Given a cylinder \((C, \gamma, \sigma)\) for (Mono, Mono\(\Box\)), one has the implications \(DH1 \implies DH2 \implies DH3\).
Proof. Assume that the cylinder satisfies DH1. For every \( f : X \to Y \), the outer rectangle in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma^k_X} & CX \\
\downarrow{f} & & \downarrow{Cf} \\
Y & \xrightarrow{\gamma^k_Y} & CY
\end{array}
\]

is always a pullback. If \( f \) is a monomorphism then so is \( Cf \) and hence the left square is also a pullback. So the cylinder satisfies DH2.

Assume that the cylinder satisfies DH2. Given a monomorphism \( f : X \to Y \), consider for \( k = 0, 1 \) the diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{p^k} & P \\
\downarrow{f} & & \downarrow{h} \\
Y & \xrightarrow{\iota^k_Y} & Y + Y \\
\end{array}
\]

where the right square is a pullback and \( p^k \) is induced by the maps \( f \iota^k_X \) and \( \gamma^k_X \). By DH2 the outer rectangle is also pullback and hence the left square is a pullback too. Because coproducts are universal, the maps \( p^0 \) and \( p^1 \) make \( P \) into a coproduct of \( X \) and \( X \). The canonical isomorphism \( u : X + X \to P \) with \( \iota^k_X u = p^k \) then satisfies \( uh = f + f \) and \( ug = \gamma_X \). So the cylinder satisfies DH3.

\[\square\]

Corollary 3.11. In a Grothendieck topos a cylinder for \((\text{Mono}, \text{Mono}^\Box)\) is cartesian iff it satisfies DH1 (and hence DH2 and DH3) above.

Proof. Let \((C, \gamma, \sigma)\) be a cylinder.

Suppose it is cartesian. Then the left adjoint \( C \) preserves all colimits and we already remarked before that \( f \in \text{Mono} \) implies \( Cf \in \text{Mono} \). Therefore condition DH1 is satisfied, as well as conditions DH2 and DH3.

Conversely, suppose that condition DH1 is satisfied. Now, any locally presentable category is cocomplete (by definition), co-wellpowered (by [2, Theorem 1.58]) and has a (small) generator (by [2, Theorem 1.20]). Therefore it satisfies the dual form of the conditions in Freyd’s Special Adjoint Functor Theorem, and the colimit preserving functor \( C \) is indeed a left adjoint.

To check that \( \text{Mono} \) is stable under the \((-) \star \gamma^k \) and \((-) \star \gamma \), match diagram (3.4) above with the diagrams (3.5) and (3.6). More precisely, for a monomorphism \( f : X \to Y \) let \( a = f \), \( b = \gamma^k_X \), \( x = \gamma^k_Y \), \( y = Cf \) in diagram (3.4). Then \( f \star \gamma^k \) coincides (up to isomorphism) with \( x \lor y \) and because condition DH2 is satisfied, \( x \lor y \) is a monomorphism. Similarly, conditions DH3 gives that \( f \star \gamma \) is a monomorphism.

\[\square\]

We now resume the description of the construction.
Definition 3.12. Let \((\mathcal{L}, \mathcal{R})\) be a weak factorization system, cofibrantly generated by a subset \(I \subseteq \mathcal{L}\). Let \((\mathcal{C}, \gamma, \sigma)\) be a functorial cylinder and \(S \subseteq \mathcal{L}\) be any subset. Define \(\mathcal{W}(\mathcal{C}, S, I)\) as the class of all those maps \(f : X \rightarrow Y\) such that for all objects \(T\) with \((T \rightarrow 1) \in \Lambda(\mathcal{C}, S, I)\) the induced map \(f^* : \mathcal{K}(Y, T)/\approx \rightarrow \mathcal{K}(X, T)/\approx\) is bijective.

Remark 3.13. Clearly \(\mathcal{W}(\mathcal{C}, S, I)\) contains all isomorphisms, has the 2-3 property and is closed under retracts in \(\mathcal{K}^2\). Furthermore, whenever \(fg\) and \(gf\) lie in \(\mathcal{W}(\mathcal{C}, S, I)\), then so do \(f\) and \(g\). All these properties follow from the corresponding properties of bijections. Also note, that for \(f \sim g\), one has \(f \in \mathcal{W}(\mathcal{C}, S, I) \iff g \in \mathcal{W}(\mathcal{C}, S, I)\) because the induced maps \(f^*, g^* : \mathcal{K}(Y, T)/\approx \rightarrow \mathcal{K}(X, T)/\approx\) coincide.

Besides being cofibrantly generated, the weak factorization system \((\text{Mono}, \text{Mono}^\square)\) in a Grothendieck topos has the property that each object is cofibrant, i.e. that each map \((0 \rightarrow X)\) is in \(\mathcal{L}\). For convenience, we combine these two properties into one definition:

Definition 3.14. A model structure (weak factorization system) is cofibrant if it is cofibrantly generated and every object is cofibrant.

Lemma 3.15. Let \((\mathcal{L}, \mathcal{R})\) be a cofibrant weak factorization system, let \((\mathcal{C}, \gamma, \sigma)\) be a cartesian cylinder and let \(\Lambda := \Lambda(\mathcal{C}, S, I)\) as in Definition 3.5. Then the natural maps \(\gamma^0\) and \(\gamma^1\) have their components in \(\square(\Lambda^\square)\).

Proof. Application of Corollary 3.3 to \(I \star \gamma^k \subseteq \Lambda\) gives \(\mathcal{L} \star \gamma^k \subseteq \square(\Lambda^\square)\). Because the left adjoint \(C\) must preserve the initial object, \(\gamma^k_X\) differs from \((0 \rightarrow X) \star \gamma^k\) only by composition with some isomorphism (due to the choice involved in Definition 3.1). Hence \(\gamma^k_X \in \square(\Lambda^\square)\). □

We are now ready to state the main result of the section.

Theorem 3.16. Let \(\mathcal{K}\) be a locally presentable category and \((\mathcal{L}, \mathcal{R})\) a cofibrant weak factorization system generated by a set \(I \subseteq \mathcal{L}\). Let \((\mathcal{C}, \gamma, \sigma)\) be a cartesian cylinder and \(S \subseteq \mathcal{L}\) an arbitrary subset. Then, setting

\[
\mathcal{C} := \mathcal{L} \quad \mathcal{W} := \mathcal{W}(\mathcal{C}, S, I) \quad \mathcal{F} := (\mathcal{C} \cap \mathcal{W})^\square
\]

(3.7)
gives a cofibrant model structure \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) on \(\mathcal{K}\). Moreover, \((\mathcal{C}, \gamma, \sigma)\) is also a cylinder for this model structure.

Remark 3.17. Theorem 3.16 does not remain valid if ”cofibrant” is weakened to ”cofibrantly generated” in its statement. Let \(\mathcal{G}\) be a (small) generator in \(\mathcal{K}\) and consider the set of codiagonal maps \(I := \{(G|G) : G + G \rightarrow G \mid G \in \mathcal{G}\}\).

(1) \(I^\square\) is the class Mono of monomorphisms and \(\square(I^\square)\) is the class \(\text{StrEpi}\) of strong epimorphisms.
(2) The \((\text{StrEpi}, \text{Mono})\)-factorization of every codiagonal \((X|X)\) as

\[
X + X \xrightarrow{(X|X)} X \xrightarrow{X} X
\]

gives a cylinder \((C, \gamma, \sigma)\) where \(C\) and \(\sigma\) are the identity and \(\gamma_X = (X|X)\). In particular, \(C\) is a left adjoint and the homotopy relation is equality.

(3) If \(f : X \to Y\) is a strong epimorphism, then \(f \star \gamma^0\), \(f \star \gamma^1\) and \(f \star \gamma\) are also strong epimorphisms. This is clear for \(\gamma^0\) and \(\gamma^1\) because they are identity transformations. In the case of \(\gamma\), it is enough to observe that \(f = g(f \star \gamma)\), where \(g\) is the pushout of \(f + f\) along \(\gamma_X\). (Alternatively one can check that \(\gamma^* \star (-)\) preserves monomorphisms and apply Lemma 3.2).

Altogether, \((\text{StrEpi}, \text{Mono})\) is cofibrantly generated and \((C, \gamma, \sigma)\) is cartesian. Going through the construction of \(\Lambda = \Lambda(\emptyset, I)\) in this case, one obtains that \(\Lambda^0\) consists only of isomorphisms and therefore all \(\Lambda^n\) consist only of isomorphisms. Consequently, every object \(X\) satisfies \((X \to 1) \in \Lambda^\square\) and \(W(\emptyset, I)\) is the class of isomorphisms. In particular \(\text{StrEpi}^\square\) is not included in \(W(\emptyset, I)\).

The rest of this section will consist of the proof of Theorem 3.16 via Smith’s Theorem 2.15. It turns out that almost all steps in the proof of \([4, \text{Théorème 2.13}]\) can be reused with only minor modifications to verify conditions (1)–(3) of Theorem 2.15. However, in verifying condition (4) we will depart from \([4]\) and use Part (c) of 2.12 (i.e. \([17, \text{Proposition 3.8}]\)). Condition (1) already already holds by Remark 3.13.

We now turn to condition (2).

By Lemma 3.6, \(\Lambda(C, S, I)^\square\) and hence \(W(C, S, I)\) do not depend on \(I\). While they do depend on \(C\) and \(S\) (it will turn out that \(S\) is contained in \(C \cap W\) and the components of \(\sigma\) lie in \(W\), the particular choices of \(C\) and \(S\) do not play any role in the proof. Therefore we will simply write \(\Lambda\) for \(\Lambda(C, S, I)\) and \(W\) for \(W(C, S, I)\). We call an object \(X\) fibrant if \((X \to 1) \in \Lambda^\square\). In Lemma 3.30 we will show that these objects coincide with the fibrant objects of the resulting model structure, so that the terminology is justified.

**Definition 3.18** \([4, \text{Définition 2.15}]\). A map \(f : X \to Y\) is a dual strong deformation retract if there exist maps \(g : Y \to X\) and \(h : CX \to X\) such that the following diagram commutes

\[
\begin{array}{ccc}
X + X & \xrightarrow{(X|f,g)} & X \\
\downarrow{\gamma_X} & & \downarrow{g} \\
CX & \xrightarrow{\sigma_X} & X \\
\end{array}
\]

\[
\begin{array}{ccc}
& & Y \\
\end{array}
\]

(3.8)

**Lemma 3.19.** Every element of \(C^\square\) is a dual strong deformation retract.
Proof. Let \( f : X \to Y \in C^\square \). Because every object is cofibrant, \( f \) is a retraction, so there is a \( g : Y \to X \) such that the right triangle in diagram (3.8) commutes. Because of \((X|fg)f = (f|f) = (X|X)f = \gamma_X \sigma_X f\) the left square of that diagram also commutes. Now \( \gamma_X \square f \) gives the desired diagonal \( h: CX \to X \). \( \square \)

Corollary 3.20. \( C^\square \subseteq \mathcal{W} \).

Proof. By the previous Lemma, it is enough to check that every dual strong deformation retract is in \( \mathcal{W} \). If \( f \) and \( g \) are as in Diagram (3.8), then \( X \sim fg \) and \( Y = gf \). Using Remark 3.13 one obtains that \( fg \) and \( gf \) are in \( \mathcal{W} \) and hence \( f \in \mathcal{W} \). \( \square \)

Remark 3.21. In fact, one has \( C^\square = (C \cap \mathcal{W})^\square \cap \mathcal{W} \). For the direction not covered by the Corollary, factor a given \( f \in (C \cap \mathcal{W})^\square \cap \mathcal{W} \) as \( f = \ell r \) with \( \ell \in C \) and \( r \in C^\square \). Then \( r \in \mathcal{W} \) and hence \( \ell \in C \cap \mathcal{W} \). Therefore \( \ell \square f \) and \( f \) is a retract of \( r \). So in the language of model structures, the "trivial fibrations are indeed those fibrations that are trivial".

Condition (2) holds by Corollary 3.20. Verifying condition (3) will occupy us until Corollary 3.31.

Lemma 3.22. Let \( X \) and \( T \) be objects with \( T \) fibrant. Then the homotopy relation \( \sim \) is an equivalence relation on \( \mathcal{K}(X,T) \).

Proof. The relation is clearly reflexive. For symmetry and transitivity let \( u,v,w \in \mathcal{K}(X,T) \) and suppose \( v \sim u \) and \( v \sim w \) via maps \( h,k : CX \to X \) with \( \gamma_X h = (v|u) \) and \( \gamma_X k = (v|w) \). This gives the solid arrows in the following diagram

\[
\begin{array}{ccc}
X \oplus X & \xrightarrow{\gamma_X} & CX \\
\gamma_X^0 + \gamma_X^0 & \downarrow p & \downarrow \gamma_X^0 \star \gamma \\
CX + CX & \xrightarrow{Q} & CCX \\
\end{array}
\]

where \( Q \) is the pushout of \( \gamma_X^0 + \gamma_X^0 \) and \( \gamma_X \) and where \( t \) is induced by the commuting outer rectangle. By Lemma 3.13 we have \( \gamma_X^0 \in \square(\Lambda^\square) \). Applying Corollary 3.15 to \( \Lambda \star \gamma \subseteq \Lambda \) gives \( \gamma_X^0 \star \gamma \in \square(\Lambda^\square) \). Hence \( (\gamma_X^0 \star \gamma) \square (T \to 1) \) and \( d : CCX \to T \) exists. Therefore the following diagram commutes

\[
\begin{array}{ccc}
X \oplus X & \xrightarrow{\gamma_X} & CX + CX \\
\gamma_X & \downarrow & \gamma_{CX} \\
CX & \xrightarrow{\gamma_X^0 \star \gamma} & CCK \\
\end{array}
\]

exhibiting a homotopy from \( u \) to \( w \). \( \square \)
**Remark 3.23.** With the previous Lemma, the condition for \( f : X \to Y \) to be in \( \mathcal{W} \) can be rephrased in terms of the homotopy relation instead of its transitive closure: for any given \( t : X \to T \) with \( T \) fibrant there is a \( u : Y \to T \) with \( t \sim fu \) and such a \( u \) is determined up to homotopy. In particular one obtains the following description for maps between fibrant objects:

**Corollary 3.24.** Suppose \( X \) and \( Y \) are fibrant. Then \( f : X \to Y \) is in \( \mathcal{W} \) if and only if there exist a \( g : Y \to X \) with \( X \sim fg \) and \( Y \sim gf \).

**Proof.** One direction is clear. If \( f : X \to Y \) is in \( \mathcal{W} \) then using the remark with \( t = X \): \( X \to X \) gives a \( g : Y \to X \) with \( X \sim fg \). Therefore \( f \sim fgf \) and using the remark with \( t = f : X \to Y \) yields \( gf \sim Y \).

**Lemma 3.25.** \( \square (\Lambda \square) \subseteq \mathcal{W} \)

**Proof.** Suppose \( f : X \to Y \) is in \( \square (\Lambda \square) \) and let \( t : X \to T \) be a map with \( T \) fibrant.

1. **Existence:** Because \( f \square (T \to 1) \), there exists a \( u : Y \to T \) with \( t = fu \), so in particular \( t \sim fu \).

2. **Uniqueness:** Assume \( u, v : Y \to T \) with \( t \sim fu \) and \( t \sim fv \).

   By Lemma 3.22, \( fu \sim fv \) and there is some \( h : CX \to X \) with \( \gamma_X h = (fu|fv) = (f + f)(u|v) \). Therefore one has the following diagram

\[
\begin{array}{ccc}
X + X & \xrightarrow{\gamma_X} & CX \\
\downarrow f + f & & \downarrow h \\
Y + Y & \xrightarrow{X+X} & CX \\
\downarrow (u|v) & & \downarrow r \\
& & T
\end{array}
\]

where \( r \) is the induced map from the pushout. By Corollary 3.3, \( f \star \gamma \in \square (\Lambda \square) \) and hence \( f \star \gamma \square (T \to 1) \), so that \( r \) factors through \( f \star \gamma \) via some \( d : CY \to T \). Therefore \( (u|v) = \gamma_Y d \) and \( u \sim v \).

**Corollary 3.26.** The natural maps \( \gamma^0 \) and \( \gamma^1 \) have their components in \( \mathcal{C} \cap \mathcal{W} \). The natural map \( \sigma \) has its components in \( \mathcal{W} \).

**Proof.** Let \( X \) be any object. Lemma 3.15 and Lemma 3.25 together give \( \gamma^X \in \square (\Lambda \square) \subseteq \mathcal{C} \cap \mathcal{W} \). The 2-3 property of \( \mathcal{W} \) then implies \( \sigma_X \in \mathcal{W} \).

The two implications obtained in Lemma 3.19 and in Corollary 3.20 can be strengthened to equivalences under some conditions.
Lemma 3.27. Suppose \( f \in \Lambda \Box \). Then
\[
f \in \mathcal{C} \Box \iff f \text{ is a dual strong deformation retract}
\]

Proof. The direction “\( \Rightarrow \)” is Lemma 3.19. For the direction “\( \Leftarrow \)”, assume \( f : X \to Y \) to be a strong dual deformation retract with maps \( g : Y \to X \) and \( h : CX \to X \) as in diagram (3.8), i.e. \( gf = X, (X|fg) = \gamma_X h \) and \( hf = \sigma_X f \). Any commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{u} & X \\
\downarrow{c} & & \downarrow{f} \\
L & \xrightarrow{v} & Y
\end{array}
\]

with \( c \in \mathcal{C} \) gives rise to the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma^1_K} & CX \\
\downarrow{u} & & \downarrow{h} \\
K & \xrightarrow{\gamma^0_K} & CK \\
\downarrow{c} & & \downarrow{p} \\
L & \xrightarrow{q} & P \\
\downarrow{\gamma^1_L} & & \downarrow{g} \\
CL & \xrightarrow{\sigma_L} & L & \xrightarrow{v} & Y \\
\end{array}
\]

where \( P \) is the pushout of \( c \) and \( \gamma^1_K \) and \( x : P \to X \) is induced by \( \gamma^1_K C(u)h = u \gamma^1_X h = uf g = cvg \). Testing against \( p \) and \( q \) gives the commutativity of the lower right square in

\[
\begin{array}{ccc}
K & \xrightarrow{u} & X & \xrightarrow{\gamma^0_X} & CX \\
\downarrow{c} & & \downarrow{C(u)} & & \downarrow{h} \\
CK & \xrightarrow{\gamma^1_K} & P & \xrightarrow{x} & X \\
\downarrow{p} & & \downarrow{\gamma^1} & & \downarrow{g} \\
L & \xrightarrow{\gamma^1_L} & CL & \xrightarrow{\sigma_L} & L & \xrightarrow{v} & Y
\end{array}
\]

and hence \((c \star \gamma^1) \Box f\) gives a diagonal \( d : CL \to X \). The outer diagram then shows that \( d' := \gamma^0_L d : L \to Y \) is the desired diagonal. \( \square \)

Lemma 3.28. Suppose \( f \in \Lambda \Box \) with fibrant codomain. Then
\[
f \in \mathcal{C} \Box \iff f \in \mathcal{W}
\]

Proof. The direction ”\( \Rightarrow \)” is Corollary 3.20. For the direction ”\( \Leftarrow \)”, assume \( f : X \to Y \in \mathcal{W} \) and \( Y \) fibrant. By Lemma 3.27, it is sufficient to show that \( f \) is a dual strong deformation retract. We will construct
$g: Y \to X$ and $h: CX \to X$, such that the equations in diagram (3.8) are satisfied.

Because $f$ and $(Y \to 1)$ are in $\Lambda\Box$, the same holds for $(X \to 1)$. By Corollary 3.24 there exists a $g: Y \to X$ with $X \sim fg$ and $Y \sim gf$. Let $k: CX \to X$ be the homotopy from $X$ to $fg$.

1. One may assume $Y = gf$. Consider the following diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow{\gamma_Y^1} & \nearrow{d} & \downarrow{f} \\
Y & \xrightarrow{\gamma_Y^0} & CY \\
\end{array}
$$

where the right square comes from $Y \sim gf$. The diagonal $d: CY \to X$ exists because $\gamma_Y^1 \in \Box(\Lambda\Box)$ by Lemma 3.15. Let $g' := \gamma_Y^0d$. Then $gf' = Y$ and $(g'f) = \gamma_Yd$. Hence $X \sim fg \sim fg'$ and by Lemma 3.22 we have $X \sim fg'$ via some homotopy $k'$. Now replace $g$ and $k$ by $g'$ and $k'$.

2. There are maps $x: CX + CX \xrightarrow{x} X$ and $d: CCX \to X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X + X & \xrightarrow{\gamma_X} & CX \\
\downarrow{\gamma_X + \gamma_X^1} & \nearrow{d} & \downarrow{f} \\
CCX & \xrightarrow{\sigma_{CCX}} & CX \\
\end{array}
$$

The equation

$$(\gamma_X^1 + \gamma_X)(k|fg) = (\gamma_X^1 k|\gamma_X kfg) = (fg|fgf) = (X|X)fg = \gamma_X \sigma_X fg$$

induces $x$ in the following diagram

$$
\begin{array}{ccc}
X + X & \xrightarrow{\gamma_X} & CX \\
\downarrow{\gamma_X + \gamma_X^1} & \nearrow{j} & \downarrow{f} \\
CX + CX & \xrightarrow{i} & Q \\
\downarrow{(k|fg)} & \nearrow{x} & \downarrow{g} \\
& X & \\
\end{array}
$$

where $Q$ is the pushout of $\gamma_X^1 + \gamma_X$ and $\gamma_X$ with coprojections $i: CX + CX \to Q$ and $j: CX \to Q$. The commutativity of the outer rectangle of diagram $(*)$ now follows from the following
two equations
\[ ix f = (k|kfg)f = (kf|kf) = (CX|CX)kf = \gamma_X \sigma_{CX} kf \]
\[ = i(\gamma_X \star \gamma_X) \sigma_{CX} kf \]
\[ jx f = \sigma_X fgf = \sigma_X \gamma_X^1 kf = C(\gamma_X^1) \sigma_{CX} kf \]
\[ = j(\gamma_X^1 \star \gamma_X) \sigma_{CX} kf \]

Finally the existence of the diagonal \( d \) in diagram (\(
\star
\)) follows from \( \gamma_X^1 \star \gamma_X \in \Box (\Lambda \Box) \).

(3) With \( x \) and \( d \) as in (2), let \( h := C(\gamma_X^0) d : CX \to X \). Then the following diagram commutes:

\[
\begin{array}{ccc}
X + X & \xrightarrow{(X|fg)} & X \\
\downarrow \gamma_X & & \downarrow f \\
CX & \xrightarrow{\sigma_X} & X \\
\end{array}
\]

The lower triangle is the equation
\[ C(\gamma_X^0) df = C(\gamma_X^0) \sigma_{CX} kf = \sigma_X \gamma_X^0 kf = \sigma_X f \]

The upper triangle is the equation
\[ \gamma_X C(\gamma_X^0) d = (\gamma_X^0 + \gamma_X^0) \gamma_X d \]
\[ = (\gamma_X^0 + \gamma_X^0) i(\gamma_X^1 \star \gamma_X) d \]
\[ = (\gamma_X^0 + \gamma_X^0) ix \]
\[ = (\gamma_X^0 + \gamma_X^0)(k|kf g) \]
\[ = (\gamma_X^0 k|\gamma_X^0 kf g) \]
\[ = (X|fg) \]

Altogether, \( h \) and \( g \) satisfy the equations in diagram (3.38). \( \square \)

**Corollary 3.29.** Let \( f : X \to Y \in \mathcal{C} \) with fibrant codomain. Then
\[ f \in \mathcal{W} \iff f \in \Box (\Lambda \Box) \]

**Proof.** The direction ” \( \Leftarrow \)” is Lemma 3.25. For the direction ” \( \Rightarrow \)” suppose \( f \in \mathcal{W} \). Factor \( f \) as \( ip \) with \( i \in \Box (\Lambda \Box) \) and \( p \in \Lambda \Box \). Then \( p \) satisfies the condition of the previous Lemma and hence
\[ f \in \mathcal{W} \iff p \in \mathcal{W} \iff p \in \mathcal{C} \Box \]
so that in particular \( f \Box p \). Therefore \( f \) is a retract of \( i \) and lies in \( \Box (\Lambda \Box) \). \( \square \)

**Lemma 3.30.** Let \( \mathcal{N} = \{ p \in \Lambda \Box \mid p \text{ has a fibrant codomain} \} \). Then
(a) \( \mathcal{C} \cap \mathcal{W} = \mathcal{C} \cap \Box \mathcal{N} \).
(b) \( \mathcal{N} \subseteq (\mathcal{C} \cap \mathcal{W}) \Box \)
(c) \((X \to 1) \in \Lambda \sqsubseteq (\mathcal{C} \cap \mathcal{W}) \sqsubseteq (\mathcal{C} \cap \mathcal{W}) \sqsupseteq (\mathcal{C} \cap \mathcal{W})\)

**Proof.** First observe that because of \((\Lambda \sqsubseteq) \subseteq (\mathcal{C} \cap \mathcal{W}) \sqsubseteq (\mathcal{C} \cap \mathcal{W}) \sqsupseteq (\mathcal{C} \cap \mathcal{W})\) we have \(\Lambda \sqsubseteq (\mathcal{C} \cap \mathcal{W}) \sqsubseteq (\mathcal{C} \cap \mathcal{W})\) and hence the implication “\(\Rightarrow\)” in (c) always holds. The implication “\(\Leftarrow\)” in (c) follows from (b). Moreover, (a) implies (b) via \(\mathcal{C} \cap \mathcal{N} \subseteq \mathcal{N}\). So it is enough to show (a). Let \(c: K \to L\) be any map in \(\mathcal{C}\).

Factor \((L \to 1)\) through some \(u: L \to L'\) with \(u \in \mathcal{N}\) and \(L'\) fibrant. Then in particular \(u \in \mathcal{C} \cap \mathcal{W}\) by Corollary 3.29. Therefore \(c \in \mathcal{W} \iff cu \in \mathcal{W} \iff cu \in \mathcal{N}(\Lambda)\) (\(\ast\)) where the second equivalence again results from Corollary 3.29.

(1) Suppose \(c \in \mathcal{W}\). Consider any \(p \in \mathcal{N}\) and maps \(x: K \to X\) and \(y: L \to Y\) as in the following diagram:

\[
\begin{array}{ccc}
K & \xrightarrow{c} & L \\
\downarrow x & & \downarrow u \\
X & \xrightarrow{p} & Y
\end{array}
\]

Then \(y': L' \to Y\) exists because \(u \sqsubseteq (Y \to 1)\) and \(d: L' \to X\) exists because of the above (\(\ast\)). The equations \(cud = x\) and \(udp = uyp = y\) then exhibit \(ud: L \to X\) as the desired diagonal.

(2) Suppose \(c \in \mathcal{N}\). Factor \(cu\) as \(cu = xp\) with \(x \in \mathcal{N}(\Lambda)\) and \(p \in \Lambda\). Because \(u\) has fibrant codomain, the same holds for \(p\) and hence \(p \in \mathcal{N}\). Because \(u \in \mathcal{N}(\Lambda) \subseteq \mathcal{N}\), also \(cu \in \mathcal{N}\). Therefore \(cu\) is a retract of \(p\) and hence \(cu \in \mathcal{N}(\Lambda) \subseteq \mathcal{W}\). Now by (\(\ast\)) above, \(c \in \mathcal{W}\). \(\square\)

**Corollary 3.31.** \(\mathcal{C} \cap \mathcal{W}\) is stable under pushouts, transfinite composition and retracts.

**Proof.** By part (a) of the previous Lemma, \(\mathcal{C} \cap \mathcal{W}\) can be expressed as the intersection of two classes, each of which is stable under these operations. \(\square\)

It now remains to verify condition (4). We want to express \(\mathcal{W}\) as the full preimage (under some accessible functor) of the class of homotopy equivalences with respect to some final cylinder. Observe that the cylinder used in the construction may not be final.

**Lemma 3.32.** There is a final refinement \((\mathcal{C}', \gamma', \sigma')\) of \((\mathcal{C}, \gamma, \sigma)\) such that for any two maps \(f, g: X \to Y\) with fibrant codomain we have \(f \sim g \mod (\mathcal{C}') \iff f \sim g \mod (\mathcal{C})\).

In particular, the two cylinders agree on the notion of homotopy equivalences between fibrant objects.
Proof. Let $\sigma = \lambda \rho$ be a functorial $(\mathcal{C}, \mathcal{C}^\square)$-factorization of $\sigma$ and for each object $X$ set $C'X = \text{cod}(\lambda_X)$, $\gamma'_X = \gamma_X \lambda_X$ and $\sigma'_X = \rho_X$. Then $(C', \gamma', \sigma')$ is a final refinement of $(\mathcal{C}, \gamma, \sigma)$ and the direction $\Rightarrow$ was already noted in part (d) of Observation 2.10.

Now assume $f \sim g \mod C$ for maps $f, g : X \to Y$ with $Y$ cofibrant. Let $h : CX \to Y$ be a homotopy from $f$ to $g$ and consider the square:

$$
\begin{array}{ccc}
CX & \xrightarrow{h} & Y \\
\downarrow{\lambda_X} & & \downarrow{\lambda_X} \\
C'X & \longrightarrow & 1
\end{array}
$$

Corollary 3.20 gives $\rho_X \in C^\square \subseteq \mathcal{W}$ and Corollary 3.26 gives $\lambda_X \rho_X = \sigma_X \in \mathcal{W}$. Therefore the 2-3 property of $\mathcal{W}$ forces $\lambda_X \in \mathcal{W}$ and hence $\lambda_X \in \mathcal{C} \cap \mathcal{W}$. By part (c) of Lemma 3.30 we have $(Y \to 1) \in (\mathcal{C} \cap \mathcal{W})^\square$. This gives the desired diagonal $d : C'X \to Y$ of the above square, establishing $f \sim g \mod C'$.

Corollary 3.33. The class $\mathcal{W}$ satisfies the solution set condition.

Proof. By Lemma 2.14, it is sufficient to exhibit $\mathcal{W}$ as the full image of some accessible functor. Let $L : \mathcal{K} \to \mathcal{K}$ be the fibrant replacement functor given by the weak factorization system $(\square(\Lambda^\square), \Lambda^\square)$, which is accessible by part (b) of Theorem 2.12. Via composition, $L$ induces a functor $L_* : \mathcal{K}^2 \to \mathcal{K}^2$, which is also accessible because colimits in $\mathcal{K}^2$ are calculated pointwise.

Let $f : X \to Y$ be any map.

(1) $f \in \mathcal{W} \iff Lf \in \mathcal{W}$

Consider the square:

$$
\begin{array}{ccc}
X & \xrightarrow{\ell_X} & LX \\
\downarrow{f} & & \downarrow{Lf} \\
Y & \xrightarrow{\ell_Y} & LY
\end{array}
$$

where $\ell_X, \ell_Y \in \square(\Lambda^\square)$ are given by the functorial factorization. By Lemma 3.25 $\ell_X$ and $\ell_Y$ lie in $\mathcal{W}$. Now the 2-3 property of $\mathcal{W}$ gives the above equivalence.

(2) $Lf \in \mathcal{W} \iff Lf$ is a homotopy equivalence $\mod C$

By construction, $Lf$ has fibrant domain and codomain. The equivalence now follows from Corollary 3.24.

Let $(C', \gamma', \sigma')$ be a final refinement of $(\mathcal{C}, \gamma, \sigma)$ as in the previous Lemma. Then point (2) still remains valid with $C'$ in place of $C$. Therefore $\mathcal{W}$ is the preimage, under the accessible functor $L_*$, of the class of homotopy equivalences determined by $C'$. By part (c) of Theorem 2.12 that class is the full image of an accessible functor. It is also isomorphism-closed. Hence the same holds for $\mathcal{W}$ by Lemma 2.3. □
Proof of Theorem 3.16. By Remark 3.13, Corollary 3.20, Corollary 3.31 and Lemma 3.33, the classes $\mathcal{C}$ and $\mathcal{W}$ satisfy the conditions of Smith’s Theorem 2.15.

4. Left determination

Let $\mathcal{K}$ be any complete and cocomplete category. Given a fixed class $\mathcal{C}$ of maps in $\mathcal{K}$, consider the following conditions on a class $\mathcal{W}$ of maps:

(i) $\mathcal{W}$ has the 2-3 property.
(ii) $\mathcal{W}$ is closed under retracts in $\mathcal{K}^2$.
(iii) $\mathcal{C} \subseteq \mathcal{W}$.
(iv) $\mathcal{C} \cap \mathcal{W}$ is closed under pushouts and transfinite composition.

Then each condition is stable under intersections, i.e., if it is satisfied by every $\mathcal{W}_i$ in some (possibly large) family $\mathcal{W}_i$ ($i \in I$), then it is also satisfied by their intersection. Also, whenever $\mathcal{C}$ and $\mathcal{W}$ are part of a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, then $\mathcal{W}$ satisfies all of the above conditions.

The following Definition was given by Cisinski [4, Définition 3.4] for the special case where $\mathcal{K}$ is a (Grothendieck) topos and $\mathcal{C}$ is the class of all monomorphisms.

Definition 4.1. Let $\mathcal{C}$ be a fixed class of maps in $\mathcal{K}$. The class $\mathcal{W}$ is a localizer for $\mathcal{C}$ if it satisfies conditions (i),(iii) and (iv) above. For any given class $S$ of maps, $\mathcal{W}(S)$ denotes the smallest localizer containing $S$. In particular $\mathcal{W}(\emptyset)$ is the smallest localizer.

The following Definition was given by Rosický and Tholen [18, Définition 2.1].

Definition 4.2. Given a class $\mathcal{C}$ of maps in $\mathcal{K}$, write $\mathcal{W}_C$ for the smallest class satisfying conditions (i)–(iv) above and containing a class of maps $S$. Then $\mathcal{W}(\emptyset) \subseteq \mathcal{W}_C$ and $\mathcal{W}_C \subseteq \mathcal{W}$ for any model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. As in Definition 4.1, one can also consider the smallest class $\mathcal{W}_{S,C}$ of maps satisfying conditions (i)–(iv) and containing a class of maps $S$. Then $\mathcal{W}(S) \subseteq \mathcal{W}_{S,C}$ and $\mathcal{W}_{S,C} \subseteq \mathcal{W}$ for any model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ satisfying $S \subseteq \mathcal{W}$. In particular, whenever $\mathcal{C}$ and $\mathcal{W}(S)$ give a model structure then $\mathcal{W}(S) = \mathcal{W}_{S,C}$.

We now return to the situation of the previous section, so we assume from now on that $\mathcal{K}$ is locally presentable. The following Lemma and Theorem are adapted from [4, Proposition 3.8] and [4, Théorème 3.9].

Lemma 4.4. Let $(\mathcal{C}, \mathcal{C}^\square)$ be a cofibrant weak factorization system in $\mathcal{K}$, generated by a subset $I \subseteq \mathcal{C}$. Let $(\mathcal{C}, \gamma, \sigma)$ be a cartesian cylinder and let $S \subseteq \mathcal{C}$ be a set of maps. Then $\mathcal{W}(\mathcal{C}, S, I) = \mathcal{W}(\Lambda(\mathcal{C}, S, I))$.

Proof. We will again write $\Lambda$ for $\Lambda(\mathcal{C}, S, I)$ and $\mathcal{W}$ for $\mathcal{W}(\mathcal{C}, S, I)$. The inclusion $\mathcal{W}(\Lambda) \subseteq \mathcal{W}$ holds because $\Lambda \subseteq \mathcal{W}$ by Lemma 3.25.
Now given any \( f : X \to Y \in \mathcal{W} \), use \((\text{cell}(\Lambda), \Lambda^\Box)\)-factorizations of \((X \to 1)\) and \((Y \to 1)\) to obtain a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\ell_X} & X' \\
\downarrow f & & \downarrow z \\
Y & \xrightarrow{\ell_Y} & Y'
\end{array}
\]

where \( \ell_X \) and \( \ell_Y \) are in \( \text{cell}(\Lambda) \), \( X' \) and \( Y' \) are fibrant, \( f' \) is induced by this factorization and \( f' = zy \) is in turn a factorization with \( z \in \text{cell}(\Lambda) \) and \( y \in \Lambda^\Box \). In particular \( \ell_X, \ell_Y \) and \( z \) are in \( \mathcal{W}(\Lambda) \). Then the 2-3 property gives

\[
f \in \mathcal{W} \implies y \in \mathcal{W} \iff y \in \mathcal{C}^{\Box} \implies y \in \mathcal{W}(\Lambda) \implies f \in \mathcal{W}(\Lambda)
\]

where the equivalence in the middle is given by Lemma 3.28. □

**Theorem 4.5.** Let \((\mathcal{C}, \mathcal{C}^\Box)\) be a cofibrant weak factorization system in \( \mathcal{K} \) and \( S \) be an arbitrary set of maps (not necessarily included in \( \mathcal{C} \)). Suppose that \((\mathcal{C}, \gamma, \sigma)\) is a cartesian cylinder such that all components of \( \sigma \) lie in \( \mathcal{W}(S) \). Then, setting \( \mathcal{W} := \mathcal{W}(S) \) and \( \mathcal{F} := (\mathcal{C} \cap \mathcal{W})^{\Box} \) gives a cofibrant model structure \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) on \( \mathcal{K} \). Also \( \mathcal{W}(S) = \mathcal{W}_{SC} \).

**Proof.** First observe, that one may assume \( S \subseteq \mathcal{C} \): factor each \( s \in S \) as \( s = c_s r_s \) with \( c_s \in \mathcal{C} \) and \( r_s \in \mathcal{C}^{\Box} \) and consider \( S' := \{c_s \mid s \in S\} \). Any given localizer contains \( S \) if and only if it contains \( S' \), because all the \( r_s \) lie in it. Therefore \( \mathcal{W}(S') = \mathcal{W}(S) \).

Now assume \( S \subseteq \mathcal{C} \). Let \( I \) be some generating subset of \( \mathcal{C} \). By the previous Lemma, it is enough to show \( \mathcal{W}(\Lambda(\mathcal{C}, S, I)) = \mathcal{W}(S) \). We will write \( \Lambda(S) \) for \( \Lambda(\mathcal{C}, S, I) \).

The inclusion \( S \subseteq \Lambda(S) \) already forces \( \mathcal{W}(S) \subseteq \mathcal{W}(\Lambda(S)) \) and therefore it remains to show \( \Lambda(S) \subseteq \mathcal{W}(S) \).

By assumption, the components of \( \sigma \) lie in \( \mathcal{W}(S) \). Consequently the components of \( \gamma^0 \) and \( \gamma^1 \) lie in \( \mathcal{C} \cap \mathcal{W}(S) \). We will now show \( \Lambda^n(S) \subseteq \mathcal{W}(S) \) for all \( n \geq 0 \).

(1) We already have \( S \subseteq \mathcal{W}(S) \). Let \( f : X \to Y \) be in \( \mathcal{C} \) and consider the following diagram used for the definition of \( f \star \gamma^0 \).
where $Q$ is the pushout of $f$ and $\gamma_X^0$. Because $\gamma_X^0 \in C \cap W(S)$ we have $q \in W(S)$. Together with $\gamma_Y^0 \in W(S)$ this gives $f \star \gamma^0 \in W(S)$. In the same way $f \star \gamma^1 \in W(S)$. Hence $I \star \gamma^0$ and $I \star \gamma^1$ are contained in $W(S)$.

(2) Assume $\Lambda^n(S) \subseteq W(S)$ and let $f: X \to Y$ be in $\Lambda^n(S)$. By assumption $f \in W(S)$ and hence $f$ lies in $C \cap W(S)$. Then the same holds for $f + X$ and $Y + f$ (being pushouts of $f$), as for their composition $f + f = (f + X)(Y + f)$. Moreover $f \in C \cap W(S)$ together with $\gamma_X^0, \gamma_Y^0 \in W(S)$ force $Cf \in W(S)$ by the 2-3 property. Altogether, in the following diagram used for the definition of $f \star \gamma$

\[
\begin{array}{ccc}
X + X & \xrightarrow{\gamma_X} & CX \\
\downarrow f + f & & \downarrow r \\
Y + Y & \xrightarrow{\gamma_Y} & Q \\
& & \downarrow Cf \\
& & \gamma \rightarrow CY
\end{array}
\]

both maps $r$ and $Cf$ lie in $W$, and hence $f \star \gamma \in W$. \hfill \Box

In view of Corollary 3.26 it is clear that the condition of $\sigma$ having its components in $W(S)$ cannot be omitted from the Theorem. This condition will always be satisfied (regardless of the $W(S)$ in question) whenever the cylinder is final, i.e. when $\sigma$ has its components in $C^\square$.

Corollary 4.6. Let $(C, C^\square)$ be a cofibrant weak factorization system in $K$ and suppose that there is a final cartesian cylinder for $(C, C^\square)$. Then $C$, $W(S)$ and $(C \cap W(S))^\square$ form a cofibrantly generated model structure. In particular for $S = \emptyset$, the construction of Theorem 3.16 gives a left determined model structure.

Remark 4.7. The above result also shows, that the construction of the model structure from $(C, C^\square)$ and $S$ does not depend on the choice of the cylinder used. For example, if the underlying category is distributive and if the class $C$ is stable under pullbacks along product projections, then any factorization of the codiagonal $(1|1): 2 = 1 + 1 \to 1$ as a composition of some $g: 2 \to V$ and $s: V \to 1$ with $g \in C$ and $s \in C^\square$ will provide a final cylinder with $C = (-) \times V$, $\gamma = (-) \times g$ and $\sigma = (-) \times s$. If $V$ is exponentiable then $C$ is a left adjoint.

Example 4.8. Let $\top: 1 \to \Omega$ be the subobject classifier of a Grothendieck topos $\mathcal{E}$ and let $\bot: 1 \to \Omega$ be the characteristic map of $0 \to 1$,
which means that $\bot$ is the uniquely determined map in the pullback:

\[
\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \Omega
\end{array}
\]

Then the induced map $\left(\bot|\top\right) : 1 + 1 \rightarrow \Omega$ is a monomorphism (this is just another instance of Diagram (3.4)). Since $\Omega$ is injective, this gives a $(\text{Mono}, \text{Mono}^\square)$-factorization of the codiagonal $(1|1) : 1 + 1 \rightarrow 1$. Therefore $\left(-\right) \times \Omega$ gives a final cylinder and the natural map $\gamma$ is given as $\left(-\right) \times (\bot, \top)$.

Because $\mathcal{E}$ is cartesian closed, $\left(-\right) \times \Omega$ is a left adjoint and it clearly preserves monomorphisms. By Corollary 3.11 the resulting cylinder is cartesian.

5. Examples

In this section, we will examine examples, where the underlying categories are locally presentable, but not toposes. However, except for the last one, they are still cartesian closed and cylinders can be obtained from suitable factorizations of the codiagonals $2 \rightarrow 1$ as indicated in Remark 4.7.

Moreover, the homotopy relation is already determined by $C(1)$ in the sense that two maps $f, g : X \rightarrow Y$ are homotopic if and only if their exponential adjoints $\gamma f, \gamma g : 1 \rightarrow Y^X$ are homotopic. This latter condition often has a direct description in terms of the structure of $Y^X$, so that it is sufficient to know when two elements $x, y : 1 \rightarrow X$ are homotopic.

The first example also provides an instance of the second line of generalization, in that the class of cofibrations is not the class of monomorphisms.

Example 5.1. Consider $\mathcal{K} = \textbf{Cat}$, the category of small categories and functors. It has a model structure, the so called ”folk model structure”, where the cofibrations are those functors that are injective on objects, and the weak equivalences are the usual categorical equivalences. This model structure has been known for some time (hence the name), the first published source seems to be Joyal and Tierney [8]. It has also been later reproved and described in detail by Rezk [15]. We will show that this model structure is left determined by rebuilding it from a generating set of cofibrations and a final cartesian cylinder.

Recall that for any set $S$ one has the discrete category on its elements (written also as $\mathbf{S}$) and the indiscrete category (i.e. the connected groupoid with trivial object groups) on its elements, which we will write as $\overline{\mathbf{S}}$. These two constructions give functors in the obvious way to provide left and right adjoints for the underlying object functor.
Ob: \textbf{Cat} \to \textbf{Set}. In particular we write \(2\) and \(\overline{2}\) for the discrete and the indiscrete category on two objects. Moreover, we write \(\overline{2}\) for the linearly ordered set \(\{0, 1\}\) and \(P\) for the ”parallel pair”, i.e. the pushout of the inclusion \(2 \hookrightarrow \overline{2}\) with itself.

Consider \(I = \{(0 \hookrightarrow 1), (2 \twoheadrightarrow 2), p: P \to \overline{2}\}\), where the last functor maps both nontrivial arrows of \(P\) to the nontrivial arrow of \(\overline{2}\).

1. We first check that \(I\) is a set of generating cofibrations. Clearly \(I \square\) consists of all those functors, which are full, faithful and surjective on objects. Moreover, for any map \(f\) one has \(f \in \square(I \square) \iff \text{Ob}(f)\) is a monomorphism. For the direction ”\(\Rightarrow\)”, observe that the functor \((2 \twoheadrightarrow 1)\) is in \(I \square\) and that \(f \square (2 \twoheadrightarrow 1)\) forces \(\text{Ob}(f) \square (2 \to 1)\) in \(\textbf{Set}\).

Conversely, consider a square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow i & & \downarrow p \\
B & \xrightarrow{g} & Y
\end{array}
\]

where \(p \in I \square\) and \(i\) is injective on objects. Define \(h: B \to X\) on objects by \(h(i(a)) = f(a)\) and \(h(b) \in p^{-1}(g(b))\) for \(b \notin i(A)\). This can be done because \(\text{Ob}(i)\) is injective and \(\text{Ob}(p)\) is surjective. For a morphism \(u: b \to b'\) in \(B\), define \(h(u): h(b) \to h(b')\) to be the unique element of \(X(h(b), h(b')) \cap p^{-1}(g(u))\). This works because \(p\) is full and faithful. Then \(h\) is the desired diagonal.

2. The cylinder functor \(C = (-) \times \overline{2}\) is obtained from the factorization \(2 \hookrightarrow \overline{2} \to 1\) and \(\gamma_X: X \times 2 \to X \times \overline{2}\) is the usual inclusion. Because \((\overline{2} \to 1)\) is in \(I \square\), the resulting cylinder is final. Two objects \(x, y: 1 \to X\) of a category \(X\) are homotopic iff they are isomorphic. Therefore two functors \(f, g: X \to Y\) are homotopic iff they are naturally isomorphic.

3. It remains to check condition (b) of Definition 3.8, i.e. stability of \(I\) under \((-) \ast \gamma\) and \((-) \ast \gamma^k\).

For the case of \(\gamma\), consider a diagram

\[
\begin{array}{ccc}
X + X & \xrightarrow{\gamma_X} & CX \\
\downarrow f + f & & \downarrow c_f \\
Y + Y & \xrightarrow{Q} & CY
\end{array}
\]

where \(Q\) is a pushout of \(f + f\) and \(\gamma_X\). The maps \(\text{Ob}(\gamma_X)\) and \(\text{Ob}(\gamma_Y)\) are bijective. Because the functor \(\text{Ob}\) preserves pushouts, the map \(\text{Ob}(q)\) is also bijective and hence \(\text{Ob}(f \ast \gamma)\) is bijective.
For the case of $\gamma^0$ and $\gamma^1$ one can calculate directly that the following two diagrams
\[
\begin{array}{ccc}
2 & \xrightarrow{\gamma^k_2} & 2 \times \overline{2} \\
\downarrow & & \downarrow \\
\gamma_2 & \xrightarrow{\gamma^k_2} & 2 \times \overline{2}
\end{array} \quad \begin{array}{ccc}
P & \xrightarrow{\gamma^k} & P \times \overline{2} \\
\downarrow & & \downarrow \\
P & \xrightarrow{p \times \overline{2}} & P \times \overline{2}
\end{array}
\]
are pushout squares and hence $(2 \twoheadrightarrow 2) \ast \gamma^k$ and $p \ast \gamma^k$ are isomorphisms. Moreover, $(0 \rightarrow 1) \ast \gamma^k = \gamma^k_1$.

(4) Now for the computation of $\Lambda(\emptyset, I)$. By (3) above, $\Lambda^0(\emptyset, I)$ consists of isomorphisms and the two inclusions $\gamma^0_1, \gamma^1_1 : 1 \rightarrow \overline{2}$. A direct computation gives that
\[
\begin{array}{ccc}
1 + 1 & \xrightarrow{\gamma_1} & 1 \times \overline{2} \\
\downarrow & & \downarrow \\
\gamma^k_1 + \gamma^k_1 & \xrightarrow{\gamma^k_1 \times \overline{2}} & \gamma^k_1 \times \overline{2}
\end{array}
\]
is a pushout square and hence $\gamma^k_1 \ast \gamma$ is an isomorphism. Therefore $\Lambda(\emptyset, I) = \Lambda^0(\emptyset, I) = \{\gamma^0_1, \gamma^1_1\}$ and every object of $\textbf{Cat}$ is fibrant.

(5) From Corollary 3.24 we obtain that $W = W(\emptyset, I)$ consists of the categorical equivalences, which completes the construction.

The following Lemma gives a method to build new examples from old ones by inducing Cisinski’s construction on certain reflective subcategories.

**Lemma 5.2.** Let $\mathcal{K}$ be a locally presentable category with a cofibrant weak factorization system generated by a set $I$, a cylinder $(C, \gamma, \sigma)$ and a reflection $R: \mathcal{K} \rightarrow \mathcal{A}$ onto a full subcategory $\mathcal{A}$ which is also locally presentable. Then the restriction of $R: \mathcal{K} \rightarrow \mathcal{A}$ provides a cylinder $(RC, R\gamma, R\sigma)$ for the cofibrant weak factorization system generated by $RI$ in $\mathcal{A}$. Moreover, the following holds:

(a) The two cylinders $(C, \gamma, \sigma)$ and $(RC, R\gamma, R\sigma)$ determine the same homotopy relation on $\mathcal{A}$.

(b) For any $S \subseteq \square(I^\square)$ one has $\Lambda(RC, RS, RI) = RA(C, S, I)$. Consequently $\Lambda(RC, RS, RI)$ and $\Lambda(C, S, I)$ determine the same class of fibrant objects in $\mathcal{A}$.

(c) Suppose that $(C, \gamma, \sigma)$ is cartesian and that the right adjoint of $C$ leaves $\mathcal{A}$ invariant. Then the cylinder $(RC, R\gamma, R\sigma)$ is also cartesian.

(d) Given $S \subseteq \square(I^\square)$, if in the situation of (c) every object of $\mathcal{A}$ is fibrant w.r.t. $\Lambda(C, S, I)$ then $\mathcal{W}(RC, RS, RI) = \mathcal{A} \cap \mathcal{W}(C, S, I)$. 

Proof. First observe that by Part (a) of Theorem 2.12, the set $RI$ indeed generates a weak factorization system in $\mathcal{A}$, which is cofibrant because $\mathcal{A}$ is full. We will repeatedly use the equivalence

$$Rf \square g \iff f \square g \quad \text{for all } f \in \mathcal{K}, g \in \mathcal{A} \quad (\ast)$$

which holds by adjointness between $R$ and the inclusion of $\mathcal{A}$. Given any object $A \in \mathcal{A}$, its coproduct with itself in $\mathcal{A}$ is $R(A + A)$ and also $RA \cong A$. Application of $R$ to diagram (2.1) in Definition 2.9 therefore shows that $RCA$ is indeed a cylinder object for $A$.

(a) Consider any two maps $f, g : A \to B$ in $\mathcal{A}$ and the induced map $(f \vert g) : A + A \to B$ from the coproduct in $\mathcal{K}$. Then $(f \vert g) : R(A + A) \to B$ is the induced map from the coproduct in $\mathcal{A}$. Now $f \sim g \pmod{C} \iff f \sim g \pmod{RC}$ follows with $(\ast)$.

(b) Because $R$ preserves pushouts, we have $Rf \ast R\gamma = R(f \ast \gamma)$ and $Rf \ast R\gamma^k = R(f \ast \gamma^k)$ (for $k = 0, 1$), which gives the equality $\Lambda(RC, RS, RI) = RA(C, S, I)$. By $(\ast)$ we have $\Lambda(C, S, I) \square (A \to 1) \iff \Lambda(C, S, I) \square (A \to 1)$ and hence $\Lambda(C, S, I)$ and $\Lambda(RC, RS, RI)$ determine the same class of fibrant objects.

(c) Let $G : \mathcal{K} \to \mathcal{K}$ be a right adjoint of $C$ with $GA \subseteq \mathcal{A}$. The isomorphisms (natural in $A, B \in \mathcal{A}$)

$$\mathcal{A}(RCA, B) \cong \mathcal{K}(CA, B) \cong \mathcal{K}(A, GB) \cong \mathcal{A}(A, GB)$$

exhibit the cylinder functor as a left adjoint. The second condition in Definition 3.8 holds because of (b).

(d) By Corollary 3.24 and part (a) above, both $\mathcal{W}(RC, RS, RI)$ and $\mathcal{A} \cap \mathcal{W}(C, S, I)$ coincide with the class of homotopy equivalences in $\mathcal{A}$.

In the situation of the above Lemma, one cannot expect in general that a final cylinder on $\mathcal{K}$ will induce a final cylinder on the subcategory $\mathcal{A}$. Therefore the induced model structure may fail to be left determined even if the original one was. Nevertheless, in the next three examples one can check directly that the induced cylinders are final and hence the induced model structures are left determined.

**Example 5.3.** Let $\mathcal{K} = \text{Cat}$ and $\mathcal{A} = \text{PrOrd}$, the category of preordered sets (i.e. sets with a reflexive and transitive relation) and monotone maps. $\text{PrOrd}$ has a model structure where the cofibrations are the monomorphisms and the weak equivalences are the categorical equivalences. We will obtain it from the previous one on $\text{Cat}$.

The reflection $R : \text{Cat} \to \text{PrOrd}$ is bijective on objects and identifies parallel arrows. We will keep the notation from Example 5.1. Discarding the isomorphism $Rp$ from $RI$, we obtain the generating set $I' = RI \setminus \{Rp\} = \{(0 \to 1), (2 \hookrightarrow 2)\}$. One has $\square(I') = \text{Mono}$, which
is obtained exactly as in Example 5.1, keeping in mind that functors between preorders are always faithful and that the monomorphisms in \( \text{PrOrd} \) are exactly the functors that are injective on objects. The right adjoint to \((-) \times \bar{2}\) is \((-)^{\bar{2}}\) which leaves \( \text{PrOrd} \) invariant. Every object is fibrant and therefore \( \mathcal{W}' = \mathcal{W}(\emptyset, I') \) consists of the categorical equivalences.

**Example 5.4.** Let \( \mathcal{K} = \text{PrOrd} \) and \( \mathcal{A} = \text{Ord} \), the category of ordered sets (i.e. sets with a reflexive, transitive and antisymmetric relation) and monotone maps. \( \text{Ord} \) has a model structure where the cofibrations are all maps and the weak equivalences are the isomorphisms. We will obtain it from the previous one on \( \text{PrOrd} \).

The reflection \( R : \text{PrOrd} \to \text{Ord} \) assigns to every preordered set \( X \) the quotient \( X/\sim \) obtained from identifying homotopic elements. The generating set \( I' = \{(0 \to 1), (2 \twoheadrightarrow 2)\} \) is already contained in \( \text{Ord} \) and hence \( I' = RI' \). Because a full surjective functor between ordered sets must be an isomorphism, the class \( I^{\square} \) consists of all isomorphisms and consequently \( \square(I^{\square}) = \text{Ord} \). For any ordered set \( P \) one has \( P^{\bar{2}} = P \). Therefore \( \text{Ord} \) is invariant under \((-)^{\bar{2}}\). Every object is fibrant and therefore \( \mathcal{W}' = \mathcal{W}(\emptyset, I') \) is the class of isomorphisms.

**Example 5.5.** Let \( \mathcal{K} = \text{PrOrd} \) and \( \mathcal{A} = \text{Set} \). Here we identify \( \text{Set} \) with the full subcategory of indiscrete preordered sets. It has a model structure where the cofibrations are the monomorphisms and the weak equivalences are the maps between nonempty sets together with the identity map of the empty set. This (almost trivial) model structure is also mentioned in [4, Exemple 3.7] and [18, Section 3]. It can be constructed with the cylinder in Example 4.8, with the set of generating cofibrations given by the proof in [3, Proposition 1.12]. Instead we will obtain it from the one on \( \text{PrOrd} \) in Example 5.3.

The reflection \( R : \text{PrOrd} \to \text{Set} \) assigns to every preordered set \( X \) the identity map \( \bar{2} \) on its elements. Let \( I' \) be as in Example 5.3. Discarding the identity map \( \bar{2} \) from \( RI' \), we obtain the generating set \( I'' = \{(0 \to 1)\} \) in \( \text{Set} \). Then \( I''^{\square} \) is the class of surjective maps and \( \square(I''^{\square}) = \text{Mono} \). For any indiscrete preorder \( X \), the preorder \( X^{\bar{2}} \) is again indiscrete. Therefore \( \text{Set} \) is invariant under \((-)^{\bar{2}}\). Every object is fibrant and therefore \( \mathcal{W}'' = \mathcal{W}(\emptyset, I'') \) consists of the identity map of the empty set and of all maps with nonempty domain.

In the previous examples, all objects were fibrant and consequently the homotopy relation already determined the weak equivalences via Corollary 3.24. Here is an example where this does not happen.

**Example 5.6.** Let \( \mathcal{K} = \text{rsRel} \), the category of plain undirected graphs (i.e. sets with a reflexive and symmetric relation together with maps preserving such relations). We will construct a left determined model structure on \( \text{rsRel} \) where the cofibrations are the monomorphisms and
the weak equivalences are those maps that induce bijections between path components. It can be seen as the one-dimensional version of the left determined model structure on simplicial complexes as described in [18, Remark 3.7].

We will write \( n \) for the discrete graph on \( n \) vertices, \( K_n \) for the indiscrete (i.e. complete) graph on \( n \) vertices and \( K_n^- \) for the graph obtained from \( K_n \) by deleting one edge.

Consider the set \( I = \{(0 \to 1), (2 \hookrightarrow K_2)\} \), where the second map is the usual inclusion.

(1) We first check that \( I \) is a set of generating cofibrations. The class \( I^\square \) consists of those maps \( f: (X, \alpha) \to (Y, \beta) \) that are surjective and full (i.e. satisfy \( f(x)\beta f(x') \implies x\alpha x' \)).

Moreover one has \( \square(I^\square) = \text{Mono} \). This follows by the same argument as in the case of categories (step (1) in Example 5.1) with \( K_2 \) in place of \( \mathbb{2} \).

(2) The cylinder functor \( C = (-) \times K_2 \) is obtained from the factorization \( 2 \hookrightarrow K_2 \to 1 \) and \( \gamma_X: X \times 2 \to X \times K_2 \) is the usual inclusion. Because \( (K_2 \to 1) \) is in \( I^\square \), the resulting cylinder is final. Two vertices \( x, y: 1 \to X \) of a graph are homotopic iff they are joined by an edge in \( X \). Therefore, for two maps \( f, g: (X, \alpha) \to (Y, \beta) \) one has \( f \sim g \iff \forall x, x' \in X : (x\alpha x' \implies f(x)\beta g(x')) \) because \( Y^X \) is \( \text{rsRel}(X, Y) \) equipped with the relation \( \beta^\alpha \) defined by the condition on the right side of the above equivalence.

In particular the homotopy relation is not transitive in general. The homotopy relation on \( \text{rsRel}(X, Y) \) is transitive whenever \( Y \) (i.e. its relation) is transitive. Moreover, if \( Y \) is discrete then homotopy coincides with equality.

(3) For a partial description of \( \Lambda = \Lambda(\emptyset, I) \) first observe, that the forgetful functor \( \text{rsRel} \to \text{Set} \) preserves pushouts. In particular, in a pushout diagram

\[
\begin{array}{ccc}
A \times 2 & \xrightarrow{\gamma_A} & A \times K_2 \\
\downarrow f \times 2 & & \downarrow \\
B \times 2 & \to & Q
\end{array}
\]

one can assume that the underlying set of \( Q \) is \( B \times 2 \), that the horizontal underlying maps are identity maps and that the two vertical underlying maps coincide. Now suppose that \( A \) is nonempty and \( B \) is indiscrete.

Then \( Q \) is path connected: given any \( b, b' \in B \) and \( i, j \in 2 \), take some \( a \in A \) with \( b \xrightarrow{f(a)} b' \). Then

(i) \( (b, i) \xrightarrow{(f(a), i)} (f(a), i) \) in \( B \times 2 \)

(ii) \( (a, i) \xrightarrow{(a, j)} (a, j) \) in \( A \times K_2 \)
(iii) \((f(a), j) \rightarrow (b', j)\) in \(B \times 2\) and passing to \(Q\) gives a path
\[
\begin{array}{ccc}
(b, i) & \rightarrow & (f(a), i) \\
& \rightarrow & (f(a), j) \\
& \rightarrow & (b', j)
\end{array}
\]
in \(Q\). Hence, if \(f : A \rightarrow B\) is an inclusion then \(f \ast \gamma\) is the inclusion of the (nonempty) path connected \(Q\) into the indiscrete \(B \times K_2\).

As in Example 5.1 we have \((0 \rightarrow 1) \ast \gamma^k = \gamma^k_1 : 1 \rightarrow K_2\). From the inclusion \(\gamma_1 : 2 \rightarrow K_2\) we obtain the following diagram

\[
\begin{array}{ccc}
2 & \rightarrow & 2 \times K_2 \\
\gamma_1 & & \downarrow \\
K_2 & \rightarrow & K_4^- \\
\gamma_1 \times K_2 & & \\
\gamma_1 \ast \gamma^0 & & \gamma_1 \ast \gamma^0 \\
\gamma_1 \times K_2 & & \downarrow \\
\gamma_1 \times K_2 & \rightarrow & K_2 \times K_2
\end{array}
\]

where (according to the notation introduced) \(K_4^-\) is the graph

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\bullet & \leftarrow & \bullet \\
\bullet & \rightarrow & \bullet
\end{array}
\]

and \(\gamma_1 \ast \gamma^0\) is the inclusion of \(K_4^-\) into \(K_4 = K_2 \times K_2\). Up to a permutation of vertices, the same inclusion is obtained as \(\gamma_1 \ast \gamma^1\).

Hence each map in \(\Lambda^0\) is the inclusion of a nonempty path connected subgraph of some suitable \(K_n\). Applying the above observation gives (via induction) that each \(\Lambda^n\) consists only of maps of this type. Except for the two inclusions \(\gamma_1^0\) and \(\gamma_1^1\), the included subgraph of \(K_n\) is wide, i.e. it has the maximal number of vertices.

Consequently, every transitive graph \(T\) is fibrant: given some inclusion \(P \hookrightarrow K_n\) with \(P\) path connected and \(|P| = n\), any map \(f : P \rightarrow T\) can be extended to \(h : K_n \rightarrow T\) by \(h(x) := f(x)\).

Conversely, assume that \(X\) is fibrant. Observe that \(K_3^- \hookrightarrow K_3\) is in \(\square(\Lambda^2)\) because it can be obtained from \(K_4^- \hookrightarrow K_4\) as a pushout

\[
\begin{array}{ccc}
K_4^- & \rightarrow & K_4^- \\
\gamma_3^- & \downarrow & \downarrow \\
K_4 & \rightarrow & K_3
\end{array}
\]

where \(p\) is the surjection that collapses the two vertices of degree 3. Therefore, every map \(f : K_3^- \rightarrow X\) can be extended to a map \(f' : K_3 \rightarrow X\), which is precisely the definition of transitivity.
In summary, the fibrant graphs are exactly the transitive graphs.

(4) For a graph \((X, \alpha)\), a path component is an equivalence class of the transitive closure \(\alpha^*\) of the relation \(\alpha\). We write \([x]\) for the equivalence class of any \(x \in X\) and \(\pi_0 X\) for the discrete graph on the set \(\{[x] \mid x \in X\}\). Setting \(\pi_0 f([x]) := [f(x)]\) for any \(f : X \to Y\) makes \(\pi_0\) into a functor and the canonical map \(r_X : X \to \pi_0 X\) with \(r(x) = [x]\) gives a reflection into the subcategory of discrete graphs. For two maps \(f, g : (X, \alpha) \to (Y, \beta)\) one has:

\[
\pi_0 f = \pi_0 g \iff \forall x, x' \in X : (x \alpha^* x' \implies f(x) \beta^* g(x'))
\]

Comparing this with the homotopy condition

\[
f \sim g \iff \forall x, x' \in X : (x \alpha x' \implies f(x) \beta g(x'))
\]

one obtains that always \(f \sim g\) \(\implies\) \(\pi_0 f = \pi_0 g\) and that the converse implication \(\pi_0 f = \pi_0 g\) \(\implies\) \(f \sim g\) holds whenever \(\beta\) is already transitive. In the general case of a map \(f : X \to Y\) one has:

\[
f \in \mathcal{W} \iff \pi_0 f\text{ is an isomorphism}
\]

For the direction \(\Rightarrow\) assume \(f \in \mathcal{W}\). Remark 3.23 with \(t = r_X\) and \(T = \pi_0 X\) gives a map \(u : X \to \pi_0 X\) such that in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow r_X & & \downarrow r_Y \\
\pi_0 X & \xrightarrow{\pi_0 f} & \pi_0 Y
\end{array}
\]

we have \(r_X \sim f u\). Then also \(f r_Y = r_X(\pi_0 f) \sim f u(\pi_0 f)\) and by Remark 3.23 with \(t = r_X(\pi_0 f)\) and \(T = \pi_0 Y\) this forces \(r_Y \sim u(\pi_0 f)\). But for discrete codomains, homotopy means equality and hence the above diagram strictly commutes. Applying the functor \(\pi_0\) to that diagram exhibits \(\pi_0 u\) as the two-sided inverse of \(\pi_0 f\).

For the direction \(\Leftarrow\) assume that \(\pi_0 f\) is an isomorphism and let \(t : X \to T\) be a map to a transitive graph \(T\). Uniqueness up to homotopy follows from the equivalence

\[
fh \sim fh' \iff (\pi_0 f)(\pi_0 h) = (\pi_0 f)(\pi_0 h')
\]

\[
\iff \pi_0 h = \pi_0 h' \iff h \sim h'
\]

for any \(h, h' : Y \to T\) because \(T\) is transitive.

For existence, let \(s : \pi_0 T \to T\) be a section of \(r_T\) with \(\pi_0 s = \pi_0 T\) (i.e. a choice of representatives of the path components) and define \(h : Y \to T\) as the composite \(h = r_Y(\pi_0 f)^{-1}(\pi_0 t)s\). Then \(\pi_0(fh) = \pi_0 t\) and hence \(fh \sim t\).
Example 5.7. Keep the notation of the previous example and consider the full reflective subcategory $\text{eqRel}$ of transitive graphs, i.e. sets equipped with an equivalence relation. It has a model structure where the cofibrations are the monomorphisms and the weak equivalences are those maps that induce bijections between equivalence classes. This model structure has been described in detail by Lárusson [11]. We will obtain it via Lemma 5.2 from the previous one on $\text{rsRel}$.

The reflection $R: \text{rsRel} \to \text{eqRel}$ assigns to every graph $(X, \alpha)$ its transitive closure $(X, \alpha^*)$. Because the graphs $0$, $1$, $2$ and $K_2$ are already transitive, one obtains $RI = I$ and also $\square(RI) = \text{Mono} \cap \text{eqRel}$ as in step (1) above. Moreover, if $X$ is transitive then so is $X^K_2$ and we already noted in step (3) that all transitive graphs are fibrant. From Lemma 5.2 we now obtain that $W' = W(\emptyset, I)$ consists of those maps $f$ where $\pi_0 f$ is an isomorphism, i.e. those maps that induce a bijection between equivalence classes. Finally observe, that $R$ preserves full surjections. Therefore the induced cylinder is again final and the induced model structure is left determined.

We now turn from “space-like” to “linear” examples. Let $R$ be a ring and let $\mathcal{K} = R\text{Mod}$, the category of left $R$-modules. We also write $\text{Mod}_R$ and $R\text{Mod}_R$ for the categories of right and two-sided $R$-modules respectively. We always have a cofibrant weak factorization system $(\text{Mono}, \text{Mono} \square)$ in $\mathcal{K}$, which is generated by the set $I$ of all inclusions $a \hookrightarrow R$ of left ideals. Also $\text{Mono} \square$ consists of all those epimorphisms with injective kernel (for details see [1, Example 1.8(i)]).

We will only be concerned with model structures constructed from the above weak factorization system, i.e. where Mono is the class of cofibrations. Hence it remains to find cartesian cylinders.

In order to find possible examples, we first characterize cartesian cylinders for the weak factorization system $(\text{Mono}, \text{Mono} \square)$ in $\mathcal{K}$. Recall that a map $f: U \to V$ of right modules is pure (or equivalently that $f(U)$ is a pure submodule of $V$) if for every (finitely generated) left module $M$, the map $f \otimes_R M: U \otimes_R M \to V \otimes_R M$ is a monomorphism. We use another characterization of pure submodules: $U \subseteq V$ is pure iff every finite system of equations

$$u_j = \sum_i x_i r_{ij} \quad (u_j \in U, \ r_{ij} \in R)$$

which has a solution with $x_i \in V$ also has a solution with $x_i \in U$. For a direct proof, which can easily be adapted to the non-commutative setting, see e.g. Matsumura [13, Theorem 7.13].

Proposition 5.8. Suppose $V$ is a two-sided $R$-module together with a map $v: R \to V$ in $R\text{Mod}_R$ and let $C_v: \mathcal{K} \to \mathcal{K}$ be the functor with $C_v(M) = (R + V) \otimes_R M = M + V \otimes_R M$. Let $\gamma^0_R: R \to R + V$ be the coproduct injection, $\sigma_R: R + V \to R$ be the product projection and
Then the following holds:

(a) \((C_v, \gamma, \sigma)\) is a cylinder if and only if \(v: R \to V\) is pure (in \(\text{Mod}_R\)).

Moreover, two maps \(f, g: M \to N\) are homotopic iff \(g - f: M \to N\) factors through \(v \otimes_R M: M \to V \otimes_R M\).

(b) Every left adjoint cylinder \((C, \gamma, \sigma)\) arises in this way from a suitable pure monomorphism \(v: R \to \ker(\sigma_R)\).

(c) In the situation of (a) we have \((C_v, \gamma, \sigma)\) is cartesian if and only if \(V\) is a flat right module.

(d) In the situation of (a) we have \((C_v, \gamma, \sigma)\) is final if and only if \(V \otimes_R M\) is injective for every \(M\).

**Proof.** We use familiar matrix notation for maps between (co)products and omit the object names for identities and zero maps. Then the maps introduced above can be written as \(\gamma_R^0 = (1 \ 0), \gamma_R^1 = (1 \ v), \gamma_R = (1 \ v)\) and \(\sigma_R = (1 \ 0)\). Abbreviating \(v \otimes_R M\) as \(v_M\) and \(V \otimes f\) as \(f_V\), we can also write \(\gamma_M = (1 \ v_M)\) and \(C(f) = (f \ v)\).

(a) Because of \((1 \ v_M) (v) = (1)\) the maps \(\gamma_M\) and \(\sigma_M\) clearly factor the codiagonal. Moreover, \(\gamma_M\) is a monomorphism iff \(v_M\) is a monomorphism, from which the equivalence follows.

Given two maps \(f, g: M \to N\), the map \((f \ g): M + M \to N\) can be extended along \(\gamma_M: M + M \to M + V \otimes_R M\) iff the equation
\[
\begin{pmatrix}
1 & 0 \\
1 & v_M
\end{pmatrix}
\begin{pmatrix}
h_1 \\
h_2
\end{pmatrix} = 
\begin{pmatrix}
f \\
g
\end{pmatrix}
\]
can be solved with some \(h_1: M \to N\) and \(h_2: V \otimes_R M \to N\). This is equivalent to the condition that \(g - f: M \to N\) extends along \(v_M: M \to V \otimes_R M\).

(b) Let \((C, \gamma, \sigma)\) be a cylinder such that \(C\) has a right adjoint \(G\).

Application of \(C\) to the right translations \(\rho_r: R \to R\) for each \(r \in R\) gives a right action of \(R\) on \(C(R)\) which makes \(C(R)\) into a two-sided module such that the isomorphisms
\[
K(C(R), M) \cong K(R, G(M)) \cong G(M)
\]
are isomorphisms of left modules and hence \(C \cong C(R) \otimes_R (-)\).

Moreover, the diagrams
\[
\begin{array}{ccc}
R & \xrightarrow{\gamma^k_R} & C(R) \\
\rho_r & \downarrow & \downarrow C(\rho_r) \\
R & \xrightarrow{\gamma^k_R} & C(R)
\end{array}
\]
\[
\begin{array}{ccc}
R & \xrightarrow{\sigma_R} & R \\
\rho_r & & \rho_r \\
R & \xrightarrow{\sigma_R} & R
\end{array}
\]
show that \(\sigma_R\) and the \(\gamma^k_R\) are maps of two-sided modules. Consequently, \(C(R) = \gamma^0_R(R) + \ker(\sigma_R)\) is a decomposition as two-sided modules. With respect to this decomposition, we obtain
\( \gamma_R^0 = (1, 0) \), and \( \sigma_R = (\frac{1}{0}) \). Moreover, \( \gamma_R^1 = (1, v) \) for some \( v: R \to \ker(\sigma_R) \). Application of naturality of \( \gamma \) and \( \sigma \) to an \( m: R \to M \) then gives \( \gamma_M = \gamma_R \otimes_R M \) and \( \sigma_M = \sigma_R \otimes_R M \).

(c) Let \( i: M \to N \) be a monomorphism.

The pushout of \( i \) and \( \gamma_M^0 \) is \( N + V \otimes_R M \) and \( i \star \gamma^0 \) is the map \( (\frac{1}{0} \, \nu_M) : N + V \otimes_R M \to N + V \otimes_R N \). Therefore \( i \star \gamma^0 \) is a monomorphism iff \( i \nu_V \) is a monomorphism. In particular, flatness of \( V \) is necessary for \((C, B, \sigma)\) to be cartesian.

Now suppose \( V \) is flat. As seen above, \( i \star \gamma^0 \) is a monomorphism. Because of \( (1 \, v_M) = (1 \, 0) (\frac{1}{0} \, \nu_M) \) the maps \( \gamma_M^0 \) and \( \gamma_M^1 \) differ only by an automorphism of their codomain. Moreover, for any \( f: M \to N \) one has \( v_M f v = v \otimes_R f = f v \). Therefore these automorphisms are part of a natural automorphism on the cylinder functor. Consequently \( i \star \gamma^1 \) is the pushout of \( i \star \gamma^0 \) along an isomorphism and hence \( i \star \gamma^1 \) is also a monomorphism.

For \( i \star \gamma \), it is enough to consider the special case where \( i \) is the inclusion \( a \hookrightarrow R \) of a left ideal. Let \( j: V \otimes_R a \to V \) be the map with \( j(w \otimes a) = wa \). The pushout \( Q \) of \( i \) and \( \gamma_M \) can be calculated as the cokernel in the exact row below

\[
\begin{array}{cccccc}
0 & \rightarrow & a + a & \xrightarrow{k} & R + a + V \otimes_R a & \xrightarrow{Q} & 0 \\
& & h & \downarrow & i \star \gamma & \downarrow & \\
& & R + V & & & \\
\end{array}
\]

where 
\[
k = \begin{pmatrix} -i & 0 & 1 & 0 \\ 0 & -i & 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 1 & i & 0 \\ 0 & v & 0 & j \end{pmatrix}^\top
\]

and \( i \star \gamma \) is induced by \( h \) because \( \text{im}(k) \subseteq \ker(h) \). To show that \( i \star \gamma \) is a monomorphism, it remains to verify \( \ker(h) \subseteq \text{im}(k) \).

Assume \((x, y, a, w) \in \ker(h)\) for some \( x, y \in R \), \( a \in a \) and \( w = \sum_n w_n \otimes b_n \in V \otimes_R a \). This corresponds to equations \( x + y + a = 0 \) and \(-vy = \sum_n w_n b_n \). Because \( v \) is a monomorphism, we have \( y = -\sum_n r_n b_n \in a \) and \( x \in a \). Therefore \((x, y, a, w) = (-x, -y)(\begin{pmatrix} -i & 0 \\ 0 & -i & 1 \end{pmatrix} \, v_a) \in \text{im}(k) \).

(d) Tensoring the split exact sequence

\[
0 \rightarrow V \xrightarrow{(0 \, 1)} R + V \xrightarrow{(1 \, 0)} R \rightarrow 0
\]

with \( M \), we obtain \( \ker(\sigma_M) = V \otimes_R M \) from which the equivalence follows.

Observe that in the situation of \( \text{b.8 \, d} \), two maps \( f, g: M \to N \) are homotopic iff \( g - f: M \to N \) factors through some injective module.
This relation is known as stable equivalence (see e.g. [9 Section 4] or [6 Definition 2.2.2]) and the homotopy equivalences will then also be called stable equivalences.

**Corollary 5.9.** Let \((C, \gamma, \sigma)\) be a final cartesian cylinder in \(R\text{-Mod}\) and suppose that the ring \(R\) is injective. Then each map in \(\Lambda = \Lambda(C, \emptyset, I)\) has injective domain and codomain. In particular, every object is fibrant and \(W = W(C, \emptyset, I)\) is the class of stable equivalences.

**Proof.** By part (b) of Proposition 5.8 one can assume \(C = C_v\) for some \(v: R \to V\). Moreover, \(C_v\) preserves injective objects by part (d).

We prove by induction that each map in \(\Lambda^n\) has injective domain and codomain.

For an inclusion \(i: a \to R\) of a left ideal, we already remarked in the proof of part (c) that \(i \star \gamma^0\) and \(i \star \gamma^1\) have isomorphic domains and also calculated \(i \star \gamma^0: R + V \otimes_R a \to R + V \otimes_R R\). Therefore every map in \(\Lambda^0\) has injective domain and codomain.

Now assume that the claim holds for \(\Lambda^n\) and let \(f: M \to N\) be a map in \(\Lambda^n\). Then the codomain of \(f \star \gamma\) is \(N + V \otimes_R N\), which is injective. Its domain \(Q\) is the cokernel of a split exact sequence

\[
0 \to M + M \to N + N + M + V \otimes_R M \to Q \to 0
\]

and is therefore also injective. \(\Box\)

**Example 5.10.** Let \(H\) be a finite dimensional Hopf algebra over a field \(k\), i.e. a (finite dimensional) \(k\)-algebra together with algebra maps \(\Delta: H \to H \otimes_k H\) (comultiplication) and \(\varepsilon: H \to k\) (counit), and an anti-algebra map \(S: H \to H\) (antipode) satisfying certain conditions (for details see e.g. Montgomery [14]). \(H\text{-Mod}\) has a model structure where the weak equivalences are the stable equivalences [6 Theorem 2.2.12 and Proposition 4.2.15] We will show that this model structure is left determined by verifying the conditions of Proposition 5.8 and Corollary 5.9.

1. Due to results of Larson and Sweedler [10] Theorem 2 (p79) and Proposition 2 (p83)] on finite dimensional Hopf algebras over a field, \(H\) satisfies the following conditions:
   (a) the antipode \(S: H \to H\) is invertible.
   (b) there exists a nonzero \(d \in H\) with \(hd = \varepsilon(h)d\) for all \(h \in H\).
   Giving \(k\) a left \(H\)-module structure via \(\varepsilon: H \to k\), such a \(d\) corresponds to a (nonzero) \(H\)-linear map \(d: k \to H\).
   (c) a left \(H\)-module is injective iff it is projective

2. Let \(M\) and \(N\) be two \(H\)-modules. Then \(M \otimes_k N\) has an \(H \otimes_k H\)-module structure with \((c \otimes c')(m \otimes n) = cm \otimes c'n\). Via the map \(\Delta: H \to H \otimes_k H\) this induces an \(H\)-module structure on \(M \otimes_k N\). Observe that with this definition \(k \otimes_k M \cong M \cong M \otimes_k k\) and for a two sided module \(V\) also \(M \otimes_k (V \otimes_H N) \cong (M \otimes_k V) \otimes_H N\) as \(H\)-modules.
Let $\text{Hom}(M, N)$ be the group of all $k$-linear maps from $M$ to $N$. Then $\text{Hom}(M, N)$ has a $H \otimes_k H^{op}$-module structure with $((c \otimes c')f)m = c(f(c'm))$. From this one obtains two different $H$-module structures on $\text{Hom}(M, N)$:

The first one is induced via $H \xrightarrow{\Delta} H \otimes_k H \xrightarrow{H \otimes S} H \otimes_k H^{op}$. We write $\text{Hom}^r(M, N)$ for this module structure. The second one is induced via $H \xrightarrow{\Delta} H \otimes_k H \xrightarrow{tw} H \otimes_k H \xrightarrow{H \otimes S^{-1}} H \otimes_k H^{op}$, where $tw$ is defined by $tw(c \otimes c') = c' \otimes c$. We write $\text{Hom}^l(M, N)$ for this module structure.

Then one can verify that this gives bifunctors on $H\text{Mod}$ and that for any given $M$, the $k$-linear evaluation maps $e_N: \text{Hom}^r(M, N) \otimes_k M \to N$ and $e'_N: M \otimes_k \text{Hom}^l(M, N) \to N$ defined by $e_N(f, m) = fm = e'_N(m, f)$ are indeed $H$-linear and provide counits of two adjunctions $(-) \otimes_k M \dashv \text{Hom}^r(M, -)$ and $M \otimes_k (-) \dashv \text{Hom}^l(M, -)$.

(3) We fix some $d: k \to H$ as in (1b) above. Set $V = H \otimes_k H$. Then $V$ is a two sided $H$-module. Define $v: H \to V$ by the composition $H \cong k \otimes_k H \xrightarrow{d \otimes H} H \otimes_k H$. Then this gives a map of two sided $H$-modules.

(4) Tensoring over the field $k$ with a fixed module preserves monomorphisms. In particular the above $v: H \to V$ is a monomorphism. Moreover the natural isomorphisms $v \otimes_H (-) \cong d \otimes_k (-)$ and $V \otimes_H (-) \cong H \otimes_k (-)$ yield that $v: H \to V$ is pure and $V$ is flat.

(5) For a fixed module $M$, both $\text{Hom}^l(M, -)$ and $\text{Hom}^r(M, -)$ preserve epimorphisms. Therefore their left adjoints $M \otimes_k (-)$ and $(-) \otimes_k M$ preserve projective $H$-modules. In particular, $V \otimes_H M \cong H \otimes_k M$ is projective and therefore injective.

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