COHOMOLOGICALLY FULL RINGS

HAILONG DAO, ALESSANDRO DE STEFANI, AND LINQUAN MA

Abstract. Inspired by a question raised by Eisenbud-Mustaţă-Stillman regarding the injectivity of maps from Ext modules to local cohomology modules, we introduce a class of rings which we call cohomologically full rings. In positive characteristic, this notion coincides with that of F-full rings studied by Pham and the third author, while in characteristic 0, they include Du Bois singularities. We prove many basic properties of cohomologically full rings, including their behavior under flat base change. We show that ideals defining these rings satisfy many desirable properties, in particular they have small cohomological and projective dimension. Furthermore, we obtain Kodaira-type vanishing and strong bounds on the regularity of cohomologically full graded algebras.

1. Introduction

Throughout this paper, all rings we consider are commutative, Noetherian, and contain an identity element. For an ideal $I$ of a ring $S$, the local cohomology modules $H^i_I(S)$ can be described as $H^i_I(S) = \lim_{\to} \text{Ext}^i_S(S/I_e, S)$ for all $i \geq 0$, where $\{I_e\}$ is a decreasing sequence of ideals cofinal with the ordinary powers $\{I^e\}$, and the maps in the directed system are induced by the natural surjections. Clearly, it would be hugely beneficial if the direct limit above is actually a union, for then many questions about local cohomology can be reduced to understanding finitely generated modules. Motivated by such idea, when $S$ is a polynomial ring over a field, Eisenbud-Mustaţă-Stillman [EMS00, Question 6.2] asked for what ideals $I$ the natural map $\text{Ext}^n_S(S/I, S) \to H^n_I(S)$ is an injection. Since then, there have been many partial results towards answering this question, for example see [Mus00, SW07, MSS17].

We observe that, quite generally for any regular local ring $(S, m)$ of dimension $n$, the natural map $\text{Ext}^{n-i}_S(S/I, S) \to H^{n-i}_I(S)$ is an injection provided $H^i_m(S/J) \to H^i_m(S/I)$ is a surjection for all ideals $J$ that satisfy $J \subseteq I \subseteq \sqrt{J}$, by local duality (see Proposition 2.1 for more details). Motivated by this, we introduce the following intrinsic definition, which will be the main object of study of this article.

Definition 1.1. Let $(R, m, k)$ be a local ring, and let $R_{\text{red}} = R/\sqrt{0}$. We say that $R$ is $i$-cohomologically full if, for every surjection $(T, n) \to R$ with $T_{\text{red}} = R_{\text{red}}$, the natural map $H^i_n(T) \to H^i_m(R)$ is surjective. We say that $R$ is cohomologically full if it is $i$-cohomologically full for all $i$.

The definition can be adapted easily to the (standard) graded case, see Definition 2.3. In this paper we investigate the ubiquity and properties of cohomologically full rings. Our main findings can be summarized below.

The first author is partially supported by NSA Grant H98230-16-1-0012. The third named author was supported in part by NSF Grant #1836867/1600198 and NSF CAREER Grant DMS #1252860/1501102.
(1) Under very mild assumptions, ideals whose quotients are cohomologically full are precisely the ones that satisfy Eisenbud-Mustaţă-Stillman’s question mentioned above. In positive characteristics, they also answer completely the original motivational question: $S/I$ is cohomologically full if and only if the local cohomology module $H^i_I(S)$ can be written as a directed union of the Ext modules. See 2.1 and 3.6.

(2) The class of cohomologically full rings strictly include many classes of well-known singularities: Cohen-Macaulay local rings, F-pure rings in positive characteristics, Du Bois singularities in characteristics 0. In fact, we show that when $R$ has characteristic $p > 0$, cohomologically full rings are exactly F-full rings introduced in [MQ16]. We extend tight bounds on cohomological dimension and projective dimension of ideals defining these singularities to cohomologically full rings. See 2.2, 2.4, 2.5 and 2.7.

(3) We study carefully how cohomological fullness behaves under some basic operations: gluing, base change, reduction to positive characteristics. These properties allow us to classify this class of rings in small dimensions and give a wealth of examples. See 2.8, 2.10, 2.11 and Section 3.

(4) We establish a strong bound on regularity of a homogeneous ideal $I$ in a polynomial ring $S$ such as $S/I$ is cohomologically full. Roughly speaking, the bound is just the number of generators of $I$ times the maximal degree of the generators. Intriguingly, our proof uses reduction to characteristic $p$. See Subsection 4.1.

(5) Cohomologically full standard graded $k$-algebras that are Cohen-Macaulay on the punctured spectrum satisfy Kodaira-type vanishing theorem and certain Lyubeznik numbers can be read off as the 0-pieces of local cohomology modules. Such statements partially generalize previous results on the singularities mentioned above. See Subsections 4.2 and 4.3.

Beyond these results, there is more evidence that cohomologically full rings satisfy many desirable properties. For instance, in a recent preprint [CV18, Proposition 2.2.2.4], Conca and Vabaro have extended some results on Du Bois singularities by Kollár-Kovács to cohomologically full rings and use them to settle a conjecture by Herzog on ideals with square-free initial ideals.

This paper is organized as follows. In Section 2 we prove some basic properties of cohomologically full rings and in Section 3 we study the behavior of cohomologically full rings under various base change, properties (1)–(3) mentioned above will be proved in these sections. Finally in Section 4, we prove the aforementioned (4) and (5), as well as some other applications. Examples will be given throughout.

Acknowledgements: The authors would like to thank Pham Hung Quy and Matteo Varbaro for some useful discussions and comments. They would also like to thank Luis Núñez-Betancourt and Ilya Smirnov for sharing Proposition 4.12 with them.

2. Basic properties

In this section we prove some basic properties of cohomologically full rings. We begin with the following result which gives alternative characterizations of cohomologically full rings. For example we will see that, under very mild assumptions, cohomologically full rings are precisely those for which Eisenbud-Mustaţă-Stillman’s question has a positive answer. This will be our main tool for studying properties of this class of rings.
We recall that a regular local ring \((S, \mathfrak{m})\) is unramified if either \(S\) has equal characteristic, or \(S\) has mixed characteristic \((0, p)\) and \(p \notin \mathfrak{m}^2\). Equivalently, the \(\mathfrak{m}\)-adic completion of \(S\) is either a power series ring over a field, or a power series ring over a complete and unramified discrete valuation ring of mixed characteristic.

**Proposition 2.1.** Let \((R, \mathfrak{m}, k)\) be a local ring. Consider the following conditions:

1. \(R\) is \(i\)-cohomologically full.
2. For every surjection \(A \to R \cong A/I\) from a regular local ring \(A\), and every ideal \(J \subseteq I\) with \(\sqrt{J} = \sqrt{I}\), the natural map \(H^i_{\mathfrak{m}}(A/J) \to H^i_{\mathfrak{m}}(R)\) is surjective.
3. For every surjection \(A \to R \cong A/I\) from a regular local ring \(A\), and every sequence of ideals \(\{I_e\}\) of \(A\), cofinal with the ordinary powers \(\{I^e\}\), the natural map \(H^i_{\mathfrak{m}}(A/I_e) \to H^i_{\mathfrak{m}}(R)\) is surjective for all \(e\).
4. For every surjection \(A \to R \cong A/I\) from a regular local ring \(A\), the natural map \(\text{Ext}^{n-i}_R(A, A) \to H^{n-i}_R(S)\) is injective, where \(n = \dim(A)\).

Then we have \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)\).

Moreover, if \(R\) is a homomorphic image of an unramified regular local ring \(S\), say \(R = S/I\), then \((1) \Rightarrow (4)\) are all equivalent to:

5. The natural map \(\text{Ext}^{n-i}_S(R, S) \to H^{n-i}_R(S)\) is injective, where \(n = \dim(S)\).
6. If \(\{I_e\}\) is any sequence of ideals in \(S\) cofinal with the ordinary powers \(\{I^e\}\), the natural map \(H^i_{\mathfrak{m}}(S/I_e) \to H^i_{\mathfrak{m}}(R)\) is surjective for all \(e\).

**Proof.** \((1) \Rightarrow (2) \Rightarrow (3)\) are clear. \((4)\) follows from \((3)\) after applying Matlis duality to the surjections \(H^i_{\mathfrak{m}}(A/I_e) \to H^i_{\mathfrak{m}}(R)\), and taking the direct limit over \(e\). The implication \((4) \Rightarrow (5)\) is also clear. If \(\{I_e\}\) is any sequence of ideals in \(S\) cofinal with \(\{I^e\}\), then \(\text{Ext}^{n-i}_S(R, S) \to H^{n-i}_R(S)\) is injective if and only if \(\text{Ext}^{n-i}_S(R, S) \to \text{Ext}^{n-i}_S(S/I_e, S)\) is injective for all \(e\), since \(H^{n-i}_R(S)\) is the direct limit over \(e\) of the modules \(\text{Ext}^{n-i}_S(S/I_e, S)\). By Matlis duality, this is equivalent to \(H^i_{\mathfrak{m}}(S/I_e) \to H^i_{\mathfrak{m}}(R)\) being surjective for all \(e\), proving that \((5)\) and \((6)\) are equivalent.

Now, assuming that \(R = S/I\) is a homomorphic image of an unramified regular local ring \((S, \mathfrak{m}, k)\), we will show \((5) \Rightarrow (1)\), and this will complete the proof. Let \(T \to R\) be a surjective ring homomorphism with \(T_{\text{red}} = R_{\text{red}}\). We want to show that the induced map \(H^i_{\mathfrak{m}}(T) \to H^i_{\mathfrak{m}}(R)\) is surjective. Since passing to \(\mathfrak{m}\)-adic completions does not affect the modules \(H^i_{\mathfrak{m}}(T)\) and \(H^i_{\mathfrak{m}}(R)\), and it does not affect whether \(\text{Ext}^{n-i}_S(R, S) \to H^{n-i}_R(S)\) is injective or not, we may assume without loss of generality that \(R, S, T\) are all complete local rings. We pick a coefficient ring for \(T\): in equal characteristic \(V \cong k\) is a field; in mixed characteristic, \((V, pV)\) is a complete and unramified discrete valuation ring with \(V/pV \cong k\).

We have

\[
\begin{array}{ccc}
S & \downarrow & \\
V & \longrightarrow & T \\
& \longrightarrow & R.
\end{array}
\]

By [Gro64, (19.8.6.(i))], there exists a map \(V \to S\) making the diagram commute. Therefore \(V\) can be viewed as a coefficient ring for \(S\) since \(R, S, T\) have the same residue field. Since \(S\) is complete and unramified, and \(V\) is a coefficient ring for \(S\), by Cohen’s structure theorem we have \(S \cong V[\underline{a}]\), where \(\underline{a}\) denotes a set of \(n\) or \(n - 1\) variables over \(V\), depending on
whether $S$ has equal characteristic or mixed characteristic. We have a commutative diagram

$$
\begin{array}{ccc}
V & \cong & V[y] \\
\downarrow & & \downarrow \\
T & \longrightarrow & R
\end{array}
$$

Let $J$ be the kernel of $T \rightarrow R$. Since $S$ is formally smooth over $V$ and $J$ is nilpotent in $T$ (since $T_{\text{red}} = R_{\text{red}}$), there is a map $S \rightarrow T$ making the above diagram commutes. If we further let $y_1, \ldots, y_m$ be elements in $J$ that form a basis for $\frac{J+n^2}{n^2}$, we have a commutative diagram:

$$
\begin{array}{ccc}
S_0 = S[y_1, \ldots, y_m] & \longrightarrow & S \\
\downarrow & & \downarrow \\
T & \longrightarrow & R,
\end{array}
$$

where the map on the first line is the natural map sending $y_i$ to 0. We write $T = S_0/J_0$ and $R = S_0/I_0$ with $J_0 \subseteq I_0$ and $\sqrt{J_0} = \sqrt{I_0}$. Since $\dim S_0 = m + n$, by local duality, in order to prove that $H^i_y(T) \rightarrow H^i_y(R)$ is surjective, it is enough to show the map $\text{Ext}_{S_0}^{m+n-i}(S_0/I_0, S_0) \rightarrow \text{Ext}_{S_0}^{m+n-i}(S_0/J_0, S_0)$ is injective. This follows if we can show that the natural map $\text{Ext}_{S_0}^{m+n-i}(S_0/I_0, S_0) \rightarrow H_{J_0}^{m+n-i}(S_0)$ is injective. At this point we note that $I_0 = I + (y_1, \ldots, y_m)$, where $I$ is actually an ideal of $S$. Therefore, by induction on $m$, it is enough to prove that condition (5) implies that the map $\text{Ext}_{S[y]}^{n-i}(S[y]/(I + y), S[y]) \rightarrow H_{I+y}^{n-i}(S[y])$ is injective. We look at the exact sequence:

$$
0 \longrightarrow \text{Ext}_{S}^{n-i}(S/I, S) \longrightarrow H^{n-i}(S) \longrightarrow C \longrightarrow 0,
$$

where the first map is injective by assumption. Applying the functors $- \otimes_S S[y]$ first and $\Gamma_y(-)$ after that, we get an exact sequence:

$$
H^{0}_y(C \otimes_S S[y]) \rightarrow H^{1}_y(\text{Ext}_{S[y]}^{n-i}(S[y]/I, S[y])) \rightarrow H^{1}_y(H^{n-i}_y(S[y])) \rightarrow H^{1}_y(C \otimes_S S[y]).
$$

Since $y$ is a nonzerodivisor on $C \otimes_S S[y]$, we see that $H^{0}_y(C \otimes_S S[y]) = 0$. Moreover, since $y$ is a nonzerodivisor on $\text{Ext}_{S[y]}^{n-i}(S[y]/I, S[y])$ it is easy to see that

$$
\text{Ext}_{S[y]}^{n-i+1}(S[y]/(I + y), S[y]) \cong \frac{\text{Ext}_{S[y]}^{n-i}(S[y]/I, S[y])}{y \text{Ext}_{S[y]}^{n-i}(S[y]/I, S[y])} \rightarrow H^{1}_y(\text{Ext}_{S[y]}^{n-i}(S[y]/I, S[y])).
$$

Finally, a spectral sequence for local cohomology gives

$$
H^{1}_y(H^{n-i}_y(S[y])) \cong H^{n+1-i}_y(S[y]),
$$

and combining all the above we finally obtain injectivity of the map

$$
\text{Ext}_{S[y]}^{n+1-i}(S[y]/(I + y), S[y]) \rightarrow H^{n+1-i}_y(S[y]).
$$

This finishes the proof.  

Recall that a local ring $(R, m, k)$ of characteristic $p > 0$ is called $F$-full if the natural map $\mathcal{F}_R(H^i_m(R)) \rightarrow H^i_m(R)$ is surjective for all integers $i$, where $\mathcal{F}_R$ denote the Peskine-Szpiro base change functor of the Frobenius.
Corollary 2.2. If \( \text{char}(R) = p > 0 \) and \( R \) is a homomorphic image of an unramified regular local ring, then \( R \) is cohomologically full if and only if \( R \) is F-full.

Proof. We write \( R = S/I \) for an unramified regular local ring \( S \). After going modulo \( p \), if necessary, we can assume \( \text{char} \ (S) = p > 0 \). Setting \( I_e = I^{[p^e]} \) and using the equivalence \((1) \iff (6)\) in Proposition 2.1, we obtain that \( R \) is cohomologically full if and only if \( H^i_m(S/I^{[p^e]}) \to H^i_m(S/I) \) is surjective for all \( i \) and all \( e \). But the image of this map is the same as the image of \( \mathcal{F}_R(H^i_m(R)) \to H^i_m(R) \) by [Lyu06, Lemma 2.2]. Therefore \( R \) is cohomologically full if and only if \( R \) is F-full. \( \square \)

So far our Definition 1.1 and Proposition 2.1 are restricted to local rings. But they can be easily adapted to the graded set up.

Definition 2.3. Let \((R, m, k)\) be a standard graded \( k \)-algebra (i.e., \( R \) can be generated by finitely many elements of degree one over \( k \) and \( m \) denote the irrelevant maximal ideal). We say \( R \) is cohomologically full if \( R_m \) is cohomologically full.

If we write \( R = S/I \) where \( S = k[x_1, \ldots, x_n] \) is a standard graded polynomial ring, then by Definition 2.3 and Proposition 2.1, \( R \) is cohomologically full if and only if for any sequence of homogeneous ideals \( \{I_e\} \) in \( S \) cofinal with the ordinary powers \( \{I^e\} \), the natural map \( H^i_m(S/I_e) \to H^i_m(R) \) is surjective for all \( i \) and \( e \). Moreover, by graded local duality, this holds if and only if the natural map \( \text{Ext}^n_S(R, S) \to H^{n-i}_S(S) \) is injective for all \( i \), where \( n = \dim(S) \). Therefore the analog of Proposition 2.1 holds in this setup too.

Remark 2.4. We collect some immediate consequences of Proposition 2.1 and Corollary 2.2.

1. The proof of Proposition 2.1 \((5) \implies (1)\) actually shows that, when \( R \) is a homomorphic image of an unramified regular local ring \( S \), \( R \) is \( i \)-cohomologically full if and only if \( \hat{R} \) is \( i \)-cohomologically full.
2. Since F-pure local rings are F-full by [MQ16, Remark 2.4] (see also [Ma14, Theorem 1.1]), by Corollary 2.2 F-pure local rings are cohomologically full.
3. If \( (R, m, k) \) is a reduced local ring essentially of finite type over \( \mathbb{C} \). If \( R \) has Du Bois singularities, then \( R \) is cohomologically full by [MSS17, Lemma 3.3].
4. If \( (R, m, k) \) is a local ring of dimension \( d \), then \( R \) is always \( d \)-cohomologically full. In particular, Cohen-Macaulay rings are always cohomologically full.
5. Stanley-Reisner rings (i.e., quotient of polynomial or power series rings over a field by square-free monomials) are cohomologically full. This follows from (2) and (3) above, and also directly from [Lyu84, Theorem 1 (i)] and [Mus00, Theorem 1.1].

Recall that, given a proper ideal \( I \) inside a ring \( S \), the cohomological dimension of \( I \) in \( S \) is defined to be \( \text{cd}(I, S) = \sup\{j \in \mathbb{N} \mid H^j_I(S) \neq 0\} \). We always have inequalities \( \text{ht}(I) \leq \text{cd}(I, S) \leq \mu(I) \), where \( \mu(-) \) denotes the minimal number of generators of a module.

Proposition 2.5. Let \((S, m, k)\) be a regular local ring of dimension \( n \), and \( I \subseteq S \) be an ideal. Suppose \( S/I \) is cohomologically full. Then \( \text{depth}(S/I) \geq n - \text{cd}(I, S) \). Furthermore, equality holds if any of the following additional conditions is satisfied:

1. There exists a sequence of ideals \( \{I_e\} \), cofinal with \( \{I^e\} \), such that \( \text{depth}(S/I_e) = \text{depth}(S/I) \) for all \( e \). In particular, the equality holds when \( \text{char}(S) = p > 0 \).
2. \( \text{depth}(S/I) \leq 2 \).
3. \( S \) is essentially of finite type over a field of characteristic 0 and \( \text{depth}(S/I) \leq 3 \).
(4) $S$ is essentially of finite type over a field of characteristic 0, $\text{depth}(S/I) = 4$, and the local Picard group of the completion $\hat{S}/I$ is torsion.

Proof. By Proposition 2.1 (1) $\Rightarrow$ (4) we conclude that $\text{Ext}^j_S(S/I, S) \hookrightarrow H^j_I(S)$ is an injection for all $j$. It follows by local duality that

$$\text{depth}(S/I) = n - \max\{j \mid \text{Ext}^j_S(S/I, S) \neq 0\} \geq n - \text{cd}(I, S).$$

The reverse inequality is well-known to hold when $\text{depth}(S/I) \leq 2$ (see for example [Var13, Proposition 3.1]). When $\text{depth}(S/I) = 3$ or when $\text{depth}(S/I) = 4$ and the local Picard group of the completion $\hat{S}/I$ is torsion, this follows from [DT16, Corollary 2.8 and Theorem 2.9]. If there is a sequence of ideals $\{I_e\}$ that is cofinal with the ordinary powers $\{I^e\}$ such that $\text{depth}(S/I_e) = \text{depth}(S/I)$, then we obtain that $\text{Ext}^j_S(S/I, S) = 0$ if and only if $H^j_I(S) = \lim_{\rightarrow} \text{Ext}^j_S(S/I_e, S) = 0$ and when $\text{char}(S) = p > 0$ we can take $I_e = I^{[p^e]}$. \qed

Corollary 2.6. Let $(S, m, k)$ be a regular local ring of dimension $n$, and $I \subseteq S$ be an ideal. If $S/I$ has positive dimension, and is cohomologically full, then $\text{depth}(S/I) > 0$. In particular, if $S/I$ is one dimensional, then it is cohomologically full if and only if it is Cohen-Macaulay.

Proof. The lower bound for the depth is immediate from Proposition 2.5 and the Hartshorne-Lichtenbaum Vanishing Theorem [Har68]. The second claim follows from Remark 2.4 (4). \qed

Proposition 2.5 also recovers upper bounds to the projective dimension of a cohomologically full ring that are analogous to those obtained for F-pure or F-injective rings in characteristic $p > 0$ and for Du Bois rings in characteristic 0. For more details, see for example [SW07, DHS13, DSNB18, MSS17].

Corollary 2.7. Let $(S, m, k)$ be a regular local ring, and $I \subseteq S$ be an ideal. If $S/I$ is cohomologically full, then $\text{pd}_S(S/I) \leq \mu(I)$.

Proof. Let $n = \dim(S)$. It is sufficient to observe that $\text{pd}_S(S/I) = n - \text{depth}(S/I) \leq \text{cd}(I, S) \leq \mu(I)$, where the only nontrivial inequality follows from Proposition 2.5. \qed

We next present some general “gluing results” for cohomologically full rings.

Proposition 2.8. Let $(S, m)$ be an unramified regular local ring, and $J, K$ be two ideals of $S$. Let $R = S/I$, where $I = J \cap K$. Consider the following integers:

$$l = \max\{\text{pd}_S(S/J), \text{pd}_S(S/K), \text{pd}_S(S/I)\}, \quad l' = \max\{\text{pd}_S(S/J), \text{pd}_S(S/K)\}, \quad h = \text{ht}(J + K).$$

1. Suppose that $l < h$. Then $R$ is cohomologically full if and only if both $S/J$ and $S/K$ are cohomologically full.
2. Suppose that $l' < h$ and $S/J + K$ is cohomologically full. Then $R$ is cohomologically full if and only if $S/J$, $S/K$ are cohomologically full.
3. Suppose that $l' < h$ and there exists a sequence of ideals $\{I_e\}$, cofinal with the powers $\{I^e\}$, such that $\text{depth}(S/I) = \text{depth}(S/I_e)$ for all $e$. Then $R$ is cohomologically full if and only if $S/J$, $S/K$ and $S/J + K$ are cohomologically full.
Proof. Without loss of generality, we can assume that $S$ is complete. For all integers $j$, we have the following commutative diagram:

$$
\begin{array}{cccc}
\Ext^j_S(S/J+K, S) & \longrightarrow & \Ext^j_S(S/K, S) & \longrightarrow & \Ext^j_S(S/J, S) & \longrightarrow & \Ext^{j+1}_S(S/J+K, S) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^j_{J+K}(S) & \longrightarrow & H^j_{J}(S) & \oplus & H^j_{K}(S) & \longrightarrow & H^j_{J+K}(S)
\end{array}
$$

We first prove (1). To this end, we can assume that $j \leq l$ in the diagram above. But then $\Ext^j_S(S/J+K, S) = H^j_{J+K}(S) = 0$, as $j < h$. We also have $\Ext^{j+1}_S(S/J+K, S) \hookrightarrow H^{j+1}_{J+K}(S)$, since $j + 1 \leq h$. Chasing the above diagram immediately shows that the injectivity of the two middle vertical maps are equivalent.

The case of (2) is similar for $j \leq l'$. For $j > l'$, then $\Ext^j_S(S/J, S) \oplus \Ext^j_S(S/K, S) = 0$, so again we have that the vertical map $\Ext^j_S(S/I, S) \rightarrow H^j_I(S)$ must be injective since the one on the right is.

Finally, we prove (3). Given part (2), the only claim left to show is that, if $R$ is cohomologically full, then so is $S/J + K$. Assume that $R$ is cohomologically full. Since $S/I$ and $S/K$ are also cohomologically full, it follows from Proposition 2.5 that $\text{cd}(J, S) = \text{pd}_S(S/J)$ and $\text{cd}(K, S) = \text{pd}_S(S/K)$, so that $\max\{\text{cd}(J, S), \text{cd}(K, S)\} \leq l'$. For each $j \geq l' + 1$, the tail of the diagram above breaks into squares:

$$
\begin{array}{cc}
\Ext^j_S(S/I, S) & \longrightarrow & \Ext^{j+1}_S(S/J + K, S) \\
\downarrow & & \downarrow \\
H^j_I(S) & \longrightarrow & H^{j+1}_{J+K}(S)
\end{array}
$$

where the horizontal maps are isomorphisms. Since $S/I$ is assumed to be cohomologically full, it follows that $S/J + K$ must be cohomologically full as well. \qed

Remark 2.9. In Proposition 2.8, the assumptions regarding $l$ and $l'$ in relation to $h$ cannot be relaxed, even when $J + K$ is $m$-primary. For example, let $(S, m)$ be a regular local ring, $J$ be a prime ideal of dimension 1, and $K = m^r$, with $J \not\subseteq K$. Then $\text{depth}(S/J \cap K) = 0$, so $S/J \cap K$ cannot be cohomologically full by Corollary 2.6, even if $S/J$ and $S/K$ are. Note that Proposition 2.8 does not apply, since $l = l' = h$ in this case.

Proposition 2.10. Let $(S, m)$ be an unramified regular local ring, and $I$ be an ideal of $S$. Assume that $R = S/I$ has positive depth, and write $I = I_1 \cap I_2 \cap \cdots \cap I_r$, where each $I_i$ corresponds to a connected component of $\text{Spec } R \setminus \{m\}$. Then $R$ is cohomologically full if and only if each $S/I_i$ is.

Proof. If $\dim R = 1$, being cohomologically equivalent to being Cohen-Macaulay by Remark 2.4 (4) and Corollary 2.6. Since $R$ is assumed to have positive depth, the statement is then clear since each $S/I_i$ is also Cohen-Macaulay. We can therefore assume that $\dim R \geq 2$, and we induct on $r$, the number of connected components of $\text{Spec } R \setminus \{m\}$. If $r = 1$, the statement is a tautology. Suppose $r > 1$, and let $J = I_1$ and $K = I_2 \cap \cdots \cap I_r$, so that $I = J \cap K$. Observe that $S/J + K$ is cohomologically full, since $J + K$ is $m$-primary. Proposition 2.8 part (2) gives that $R$ is cohomologically full if and only if $S/J$ and $S/K$ are. The
claim now follows by induction, since the punctured spectrum of \( S/K \) has \( r-1 \) connected components. \( \square \)

**Corollary 2.11.** Let \((S, \mathfrak{m}, k)\) be a complete unramified regular local ring with separably closed residue field \( k \), and let \( I \) be an ideal of \( S \). Assume that \( R = S/I \) has dimension 2. Let 
\[
I = I_1 \cap I_2 \cap \cdots \cap I_r, \quad \text{where each } I_i \text{ corresponds to a connected components of } \text{Spec } R \setminus \{ \mathfrak{m} \}.
\]
Then \( R \) is cohomologically full if and only if \( S/I_i \) is Cohen-Macaulay for each \( i \).

**Proof.** By Proposition 2.10, we only need to show that if \( \text{Spec } R \setminus \{ \mathfrak{m} \} \) is connected, then \( R \) is Cohen-Macaulay. But the condition that \( \text{Spec}(R) \setminus \{ \mathfrak{m} \} \) is connected implies that \( \text{cd}(S, I) \leq n - 2 \) [Ogu73, HS77], and it follows that depth \( R \geq 2 \) by Proposition 2.5. \( \square \)

Let \( J, K \) be two ideals in an unramified regular local ring \( S \) such that \( J \cap K = JK \) and such that both \( S/J \) and \( S/K \) are cohomologically full. One can ask if the rings \( S/J \cap K \) and \( S/(J+K) \) are cohomologically full as well. This is true when \( S/J, S/K \) are Cohen-Macaulay. Because in this case we have \( S/J+K \) is also Cohen-Macaulay by the depth formula [Aus61, Theorem 1.2], and then we can use Proposition 2.8 for \( S/J \cap K \). However, the answer is no in general as the following example shows.

**Example 2.12.** Let \( S = k[[x, y, z]] \) and \( J = (xy, xz) \). Then \( S/J \) is cohomologically full by Remark 2.4 (5). Let \( r \in S \) be such that its image in \( S/J \) is a nonzerodivisor. Then \( S/(r) \) is cohomologically full (since it is Cohen-Macaulay) and \( (r) \cap J = rJ \). However, \( S/J + (r) \) cannot be cohomologically full for any choice of such \( r \), since depth\( (S/J + (r)) = 0 \) (see Corollary 2.6).

### 3. Cohomologically Full Rings under Base Change

In this section we study how cohomologically full rings behaves under various instances of base change.

**3.1. Deformation.** We begin by proving the following deformation result for cohomologically fullness. This recovers the statement for F-full rings [MQ16, Theorem 4.2 (2)], and generalizes it to rings of any characteristic. We first recall that a nonzerodivisor \( x \) in a local ring \((R, \mathfrak{m})\) is called a **surjective element** if the natural map on the local cohomology module 
\[
H^0_m(R/\langle x^n \rangle) \rightarrow H^0_m(R/\langle x \rangle)
\]
defined by \( R/\langle x^n \rangle \rightarrow R/\langle x \rangle \) is surjective for all \( n > 0 \) and \( i \geq 0 \). It is clear that if \( R/\langle x \rangle \) is cohomologically full, then \( x \) is a surjective element.

**Theorem 3.1.** Let \((R, \mathfrak{m}, k)\) be local ring that is a homomorphic image of an unramified regular local ring. Let \( x \in \mathfrak{m} \) be a nonzerodivisor on \( R \). If \( R/\langle x \rangle \) is cohomologically full, then \( R \) is cohomologically full.

**Proof.** By Remark 2.4 (1) we may assume \( R \) is complete. If \( R/\langle x \rangle \) is cohomologically full, then \( H^i_m(R/\langle x^k \rangle) \rightarrow H^i_m(R/\langle x \rangle) \) for all \( k > 0 \). In particular, \( x \) is a surjective element. It follows from [MQ16, Proposition 3.3] that the long exact sequence of local cohomology modules induced by 
\[
0 \rightarrow R \xrightarrow{x} R \rightarrow R/\langle x \rangle \rightarrow 0
\]
splits into short exact sequences. That is, for every \( i \), we have short exact sequences 
\[
0 \rightarrow H^{i-1}_m(R/\langle x \rangle) \rightarrow H^i_m(R) \xrightarrow{x} H^i_m(R) \rightarrow 0.
\]
Let $S \to R$ be a surjection where $S$ is a complete and unramified regular local ring. Taking the Matlis dual of the above sequence and applying local duality, we have the following commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Ext}^j_S(R, S) \\
\downarrow & & \downarrow \alpha \\
\cdots & \longrightarrow & H^i_j(S) \\
\downarrow & & \downarrow \beta \\
0 & \longrightarrow & H^i_{j+1}(S) \\
\end{array}
$$

where $j = \dim(S) - i$ and $\beta$ is injective by Proposition 2.1, because $R/(x)$ is cohomologically full. Now suppose there is $0 \neq \eta \in \text{Ext}^j_S(R, S)$ such that $\alpha(\eta) = 0$. We can write $\eta = x^n\eta'$ for some $\eta' \notin x\text{Ext}^j_S(R, S)$. Since $\alpha(\eta) = 0$ by assumption, and $\alpha(\eta)$ and $\alpha(\eta')$ only differ by $x^n$ inside $H^i_j(S)_x$, we must have $\alpha(\eta') = 0$, as well. By commutativity of the above diagram, we have $\beta(\varphi(\eta')) = 0$ and, since $\beta$ is injective, we have $\varphi(\eta') = 0$. However, this means that $\eta' \in x\text{Ext}^j_S(R, S)$, which contradicts the choice of $\eta'$. Therefore $\alpha$ is injective, and so is $\text{Ext}^j_S(R, S) \to H^i_j(S)$, since $\alpha$ factors through this map. Thus, $R$ is cohomologically full by Proposition 2.1.

**Definition 3.2.** Let $(R, \mathfrak{m}, k)$ be a local ring, and $M$ be a finitely generated $R$-module. The finiteness dimension of $M$ with respect to $\mathfrak{m}$ is defined as

$$f_{\mathfrak{m}}(M) = \inf\{t \in \mathbb{Z} \mid H^0_{\mathfrak{m}}(M) \text{ is not finitely generated}\}.$$ 

Recall that $f_{\mathfrak{m}}(M) < \infty$ if and only if $\dim(M) > 0$, since $H^0_{\mathfrak{m}}(M)$ is infinitely generated in this case. The following result generalizes [MQ16, Remark 5.4].

**Proposition 3.3.** Let $(R, \mathfrak{m}, k)$ be a local ring, and $x \in \mathfrak{m}$ be a nonzerodivisor. If $R/(x)$ is cohomologically full, then $\text{depth}(R) = f_{\mathfrak{m}}(R)$.

**Proof.** Let $t = \text{depth}(R)$. Clearly $\text{depth}(R) \leq f_{\mathfrak{m}}(R)$. Since $R/(x)$ is cohomologically full, $x$ is a surjective element. In particular, for all integers $n \geq 1$ we have an exact diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & H^t_{\mathfrak{m}}(R/(x^n)) \\
\downarrow & & \downarrow \cdot x^n \\
0 & \longrightarrow & H^t_{\mathfrak{m}}(R) \\
\downarrow & & \downarrow \cdot x^{n-1} \\
0 & \longrightarrow & H^t_{\mathfrak{m}}(R/(x)) \\
\end{array}
$$

where the leftmost vertical map is the one induced by the natural projection $R/(x^n) \to R/(x)$. If, by way of contradiction, we assume that $H^t_{\mathfrak{m}}(R)$ has finite length, then the middle map $\cdot x^{n-1} : H^t_{\mathfrak{m}}(R) \to H^t_{\mathfrak{m}}(R)$ is the zero map for $n > 0$. The Snake Lemma now implies that $H^t_{\mathfrak{m}}(R/(x^n)) \to H^t_{\mathfrak{m}}(R/(x))$ cannot be surjective for $n > 0$, contradicting the fact that $x$ is a surjective element.

### 3.2. Flat base change

In this subsection we consider flat base change. Our first result deals with flat extension of the ambient regular local ring.

**Lemma 3.4.** Let $(S, \mathfrak{m}) \to (T, \mathfrak{n})$ be a flat map of unramified regular local rings and $I \subseteq S$ be an ideal. If $S/I$ is cohomologically full, then so is $T/IT$. If $(S, \mathfrak{m}) \to (T, \mathfrak{n})$ is faithfully flat and $T/IT$ is cohomologically full, then so is $S/I$. 

In particular, if $R$ is a homomorphic image of an unramified regular local ring, then $R$ is cohomological full implies $R_Q$ is cohomologically full for every prime $Q \subseteq R$.

**Proof.** By Remark 2.4 (1) we may assume $S$, $T$ are both complete. For every positive integer $j$, consider the following commutative diagram:

$$
\begin{array}{ccc}
\Ext^j_S(S/I, S) & \xrightarrow{\varphi} & H^j_I(S) \\
\downarrow & & \downarrow \\
\Ext^j_S(S/I, S) \otimes T & \xrightarrow{\varphi \otimes \text{id}_T} & H^j_I(S) \otimes T \cong H^j_{IT}(T)
\end{array}
$$

Since $T$ is flat over $S$, it is clear that $\varphi$ is injective implies $\varphi \otimes \text{id}_T$ is injective. In addition, if $T$ is faithfully flat over $S$, then $\varphi \otimes \text{id}_T$ is injective also implies $\varphi$ is injective. The result now follows from (5) $\Rightarrow$ (1) of Proposition 2.1.

Finally, the last conclusion of the lemma follows because we can write $R = S/I$ for an unramified regular local ring $(S, m)$ and a localization of $S$ is obvious flat over $S$ and is still unramified: this is vacuous in equal characteristic, and in mixed characteristic, this follows from the fact that for every prime ideal $P \subseteq S$, we have $P^{(2)} \subseteq m^2$. \hfill \square

**Corollary 3.5.** Let $(S, m, k)$ be a regular local ring of characteristic $p > 0$, and $I \subseteq S$ be an ideal. Then $S/I$ is cohomologically full if and only if $S/I[p^\infty]$ is cohomologically full for all $e$.

**Proof.** This follows immediately from applying Lemma 3.4 to the $e$-th iteration of the Frobenius map of $S$, which is a faithfully flat map [Kum69]. \hfill \square

As a consequence, we observe that defining ideals of cohomologically full rings in positive characteristic give precisely the answer to [EMS00, Question 6.1].

**Corollary 3.6.** Let $(S, m, k)$ be a regular local ring of characteristic $p > 0$, and $I \subseteq S$ be an ideal. Then the following are equivalent:

1. $S/I$ is cohomologically full.
2. There is a decreasing sequence of ideals $\{I_e\}$ cofinal with the ordinary powers $\{I^e\}$ such that $H^j_I(S) = \bigcup_e \Ext^j_S(S/I_e, S)$.
3. $H^j_I(S) = \bigcup_e \Ext^j_S(S/I[p^e], S)$.

**Proof.** It is tautological that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (3) follows from Corollary 3.5 and Proposition 2.1. \hfill \square

We next obtain a stronger version of Corollary 2.6. This will help us construct examples and counterexamples later in this article. Recall that a ring $R$ satisfies Serre’s condition $(S_k)$ if $\text{depth}(R_P) \geq \min\{k, \text{ht}(P)\}$ for all $P \in \text{Spec}(R)$.

**Lemma 3.7.** Let $(S, m, k)$ be an unramified regular local ring of dimension $n$ and $I \subseteq S$ be an ideal. If $S/I$ is a cohomologically full ring, then $S/I$ satisfies Serre’s condition $(S_1)$. In particular, it has no embedded associated primes.

**Proof.** If $\dim(S/I) = 0$ there is nothing to show, so let us assume that $S/I$ has positive dimension. By Corollary 2.6, we have $\text{depth}(S/I) > 0$. The conclusion follows since (under our assumptions) the condition of being cohomologically full localizes by Lemma 3.4. \hfill \square

We next deal with more general faithfully flat base change. Using the result on deformation, we can prove the following:
Proposition 3.8. Let \((A, \mathfrak{m}) \to (B, \mathfrak{n})\) be a flat local extension between local rings that are homomorphic images of unramified regular local rings. Suppose \(B/\mathfrak{m}B\) is Cohen-Macaulay. If \(A\) is cohomologically full, then so is \(B\).

Proof. By Remark 2.4 (1) we may assume that \(A\) and \(B\) are both complete. Let \(x_1, \ldots, x_t\) be a maximal regular sequence in \(B/\mathfrak{m}B\), then \(x_1, \ldots, x_t\) is a regular sequence on \(B\) and \(A \to B' = B/(x_1, \ldots, x_t)\) is still faithfully flat with \(B'/\mathfrak{m}B'\) Artinian. If we can show \(B'\) is cohomologically full, then \(B\) will also be cohomologically full by Theorem 3.1. Hence we may assume \(\dim A = \dim B\).

We can form the following commutative diagram:

\[
\begin{array}{ccc}
S & \longrightarrow & T \\
\downarrow & & \downarrow \\
A = S/I & \longrightarrow & B = T/J
\end{array}
\]

where \(S, T\) are complete and unramified regular local rings that surject onto \(A, B\) respectively. Since \(I = J \cap S, I^n \subseteq J^n \cap S \subseteq I\). Hence \(J^n \cap S\) are thickenings of \(I\). Thus for every \(i, n \geq 0\), we have the induced commutative diagram on local cohomology (abuse of notation, \(\mathfrak{m}, \mathfrak{n}\) denote the maximal ideals of \(S, T\) respectively):

\[
\begin{array}{ccc}
H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{m}}(S/I) & \longrightarrow & H^i_{\mathfrak{n}}(B) = H^i_{\mathfrak{n}}(T/J) \\
\downarrow & & \downarrow \\
H^i_{\mathfrak{m}}(S/J^n \cap S) & \longrightarrow & H^i_{\mathfrak{n}}(T/J^n)
\end{array}
\]

The left vertical map is surjective because \(A\) is cohomologically full and the map \(\alpha\) is the base change \(- \otimes_A B\) (because \(\dim A = \dim B\)) hence it is surjective up to \(T\)-span. Therefore the \(T\)-span of the image of \(H^i_{\mathfrak{m}}(S/J^n \cap S)\) inside \(H^i_{\mathfrak{n}}(T/J)\) is equal to \(H^i_{\mathfrak{n}}(T/J)\). This implies \(H^i_{\mathfrak{n}}(T/J^n) \to H^i_{\mathfrak{n}}(T/J)\) is surjective for every \(i, n\) because its image is already a \(T\)-module. Therefore \(B = T/J\) is cohomologically full by Proposition 2.1 (6) \(\Rightarrow\) (1).

It is also quite natural to ask whether under a flat local extension \(A \to B\), \(B\) being cohomologically full implies \(A\) being cohomologically full? This question has a positive answer in characteristic \(p > 0\).

Proposition 3.9. Let \((A, \mathfrak{m}) \to (B, \mathfrak{n})\) be flat local extension of local rings of characteristic \(p > 0\) that are homomorphic images of unramified regular local rings. If \(B\) is cohomologically full, then so is \(A\).

Proof. First of all we can pick a minimal prime \(Q\) of \(\mathfrak{m}B\), the map \((A, \mathfrak{m}) \to (B_Q, Q\mathfrak{m}B)\) is still faithfully flat and \(B_Q\) is cohomologically full by Lemma 3.4. Thus without loss of generality we may assume \(\dim A = \dim B\). Since we are in characteristic \(p > 0\), cohomologically full is the same as \(F\)-full by Corollary 2.2. Thus it is enough to prove that, in this case, \(A\) is \(F\)-full if and only if \(B\) is \(F\)-full. We consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F}_A(H^i_{\mathfrak{m}}(A)) & \longrightarrow & B \otimes \mathcal{F}_A(H^i_{\mathfrak{m}}(A)) \xrightarrow{\cong} \mathcal{F}_B(H^i_{\mathfrak{n}}(B)) \\
\downarrow & & \downarrow \\
H^i_{\mathfrak{m}}(A) & \longrightarrow & B \otimes H^i_{\mathfrak{m}}(A) \xrightarrow{\cong} H^i_{\mathfrak{n}}(B).
\end{array}
\]
Since $B$ is faithfully flat over $A$, the left vertical map is surjective if and only if the right vertical map is surjective. This finishes the proof. □

At the moment, we do not know whether Proposition 3.9 holds true in general. We propose it here as a question.

Question 3.10. Let $(A, m) \to (B, n)$ be flat local extension of local rings that are homomorphic images of unramified regular local rings. If $B$ is cohomologically full, is $A$ cohomologically full?

Remark 3.11. To answer Question 3.10 in equal characteristic 0, we can reduce to the case that $A$ is cohomologically full on the punctured spectrum (by Lemma 3.4) and $B$ is a finite free $A$-module (via a reduction argument similar to [Ma17, Proof of Lemma 5.1]).

3.3. Thickening. In this subsection we study cohomologically full rings under thickenings. In general, if a thickening of $S/I$ is cohomologically full, then $S/I$ needs not be cohomologically full. We give an example.

Example 3.12. Let $S = k[[x, y, z]]$ and let $I = (x^4, x^3y, x^2y^2z, xy^3, y^4)$. Since $x^2y^2 \in I : (x, y, z)$, we have that depth $S/I = 0$. However, dim $S/I = 1$, so $S/I$ is not cohomologically full by Corollary 2.6. On the other hand, one can check that $I^2 = (x, y)^8S$, and thus $S/I^2$ is a one dimensional Cohen-Macaulay local ring. It follows that $S/I^2$ is cohomologically full.

Another natural question arising from our definition is the following: if $R$ is cohomologically full, then is $R$ red cohomologically full? This question also has negative answer.

Example 3.13. Let $k$ be a separably closed field of characteristic $p > 0$, and let $R = k[[s^4, s^3t, st^4, t^4]]$. It was shown by Hartshorne [Har79] that $R$ is a set-theoretic complete intersection, hence there is a thickening $T$ of $R$ that is Cohen-Macaulay, hence cohomologically full. However, $R = T_{\text{red}}$ is not cohomologically full: $R$ is a two-dimensional complete local domain with separably closed residue field that is not Cohen-Macaulay, so it cannot be cohomologically full by Corollary 2.11.

One could also ask whether the opposite implication holds, that is, whether $R_{\text{red}}$ being cohomologically full implies $R$ is cohomologically full. This implication, however, is even less likely than its converse, because it is very easy to find examples of rings $R$ that are not cohomologically full, but $R_{\text{red}}$ is even Cohen-Macaulay. One could then hope that, assuming $R = S/I$ and $R_{\text{red}} = S/\sqrt{T}$ have the same depth, $R$ is cohomologically full if $R_{\text{red}} = S/\sqrt{T}$ is cohomologically full. However, we next give an example which shows that this is not true, even if we consider a monomial thickening of a square-free monomial ideal.

Example 3.14. Let $S = k[[x, y, z, w]]$ and let $I = (x, y) \cap (z, w) \cap (x^2, z^2, w)$. It follows from Lemma 3.7 that $R = S/I$ is not cohomologically full, because it is not $(S_1)$. Moreover we have depth$(R) = \text{depth}(R_{\text{red}}) = 1$. However, we have $R_{\text{red}} = k[[x, y, z, w]]/(x, y) \cap (z, w)$ is cohomologically full by Remark 2.4 (5).

Our main result in this subsection shows a good control of thickenings by a nonzerodivisor:

Theorem 3.15. Let $(R, m, k)$ be a local ring that is a homomorphic image of an unramified regular local ring $S$ and let $x$ be a nonzerodivisor on $R$. Consider the following conditions:

1. $R/(x)$ is cohomologically full
(2) $R/(x^n)$ is cohomologically full for some $n \geq 1$
(3) $R/(x^n)$ is cohomologically full for all $n \geq 1$

Then we have (3) ⇒ (2) ⇒ (1). Write $R = S/I$, we have (1) ⇒ (3) if there exists a system of ideals $\{I_e\}$, cofinal with the ordinary powers $\{I^e\}$, such that $x$ is a surjective element for $S/I_e$ for all $n$. In particular, this holds if $R$ has characteristic $p > 0$.

**Proof.** First of all (3) ⇒ (2) is obvious. Next we show (2) ⇒ (1), we will actually prove the following:

**Claim.** If $R/(x^n)$ is cohomologically full, then $R/(x^m)$ is cohomologically full for all $m \leq n$.

**Proof of Claim.** Since $R/(x^n)$ is cohomologically full, we know that $x^n$ is a surjective element. It follows from [MQ16, Proposition 3.3] that $x$ is a surjective element and $H^i_m(R/(x^n)) \to H^i_m(R/(x^m))$ is surjective for all $m \leq n$. But $R/(x^n)$ is cohomologically full implies $H^i_m(T) \to H^i_m(R/(x^n))$ is surjective for all thickenings $T$ of $R/(x^n)$. Therefore $H^i_m(T) \to H^i_m(R/(x^m))$ is surjective for all thickenings $T$ that factors through $R/(x^n)$, this implies $R/(x^m)$ is also cohomologically full (for example, use Proposition 2.1 (1) ⇔ (6)).

Finally we prove (1) ⇒ (3) if there exists a system of ideals $\{I_e\}$, cofinal with the regular powers $\{I^e\}$, such that $x$ is a surjective element for $S/I_e$, for all $e$. By the above Claim, it is sufficient to prove that $R/(x)$ is cohomologically full implies $R/(x^2)$ is cohomologically full: then iterate this we know that $R/(x^{2^n})$ is cohomologically full for all $n$ and the Claim will establish (3). Note that the system $\{I_n + (x^n)\}$ is a thickening of $I + (x^2)$ that is cofinal with the regular powers $\{(I + (x^2))^n\}$, therefore it is sufficient to prove that $H^i_m(S/I_n + (x^n)) \to H^i_m(S/I + (x^2))$ is surjective for all $n$. This map factors as the composition $H^i_m(S/I_n + (x^n)) \to H^i_m(S/I_n + (x^2)) \to H^i_m(S/I + (x^2))$. The first map is surjective because, by assumption, $x$ is a surjective element for $S/I_n$. Thus, it is enough to show that $H^i_m(S/I_n + (x^2)) \to H^i_m(S/I + (x^2))$ is surjective. Because $x$ is a surjective element for $S/I$ and $S/I_n$, the following diagram has exact rows:

$$
\begin{array}{cccccc}
0 & \to & H^i_m(S/I_n + (x)) & \to & H^i_m(S/I_n + (x^2)) & \to & H^i_m(S/I_n + (x)) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^i_m(S/I + (x)) & \to & H^i_m(S/I + (x^2)) & \to & H^i_m(S/I + (x)) & \to & 0 \\
\end{array}
$$

The first and third columns are surjections because $S/I + (x)$ is cohomologically full by assumption. Therefore the map in the middle is surjective as well, as desired. The final claim follows from the fact that, in characteristic $p > 0$, the sequence of ideals $\{I^{[p^e]}\}$ satisfies the desired condition, since if $S/I + (x)$ is cohomologically full, then so is $S/I^{[p^e]} + (x^{p^e})$ for all $e$, by Corollary 3.5. The Claim then implies that $S/I^{[p^e]} + (x)$ is cohomologically full as well, and therefore $x$ is a surjective element for $S/I^{[p^e]}$. □

In general, $R$ is cohomologically full and $x$ is a surjective element does not imply $R/(x)$ is cohomologically full. We give an example.

**Example 3.16.** Let $S = k[[x, y, z]]$ and $R = S/J$, where $J = (xy, xz)$. Then $R$ is cohomologically full by Remark 2.4 (5). One can check that $\text{Ass Ext}_S^2(S/J, S) = \text{Ass} H^2_J(S) = \{(y, z)\}$. Pick a nonzerodivisor $r \notin (y, z)$, so that multiplication by $r$ on $\text{Ext}_S^2(S/J, S)$ is injective. Applying Matlis duality, we deduce that multiplication by $r$ is surjective on $H^1_m(R)$, and
thus \( r \) is a surjective element of \( R \). However, because \( \dim R/(r) = 1 \) and \( \depth R/(r) = 0 \), the ring \( R/(r) \) is not cohomologically full by Corollary 2.6.

3.4. **Reduction to characteristic** \( p > 0 \). For more details about the process of reduction to characteristic \( p \) we refer to [HH06, Sections 2.1 and 2.3] and [MSS17, Setup 5.1]. Here we use the same notation as in [MSS17, Setup 5.1].

Suppose \((R, \mathfrak{m})\) is a local ring essentially of finite type over \( \mathbb{C} \) (or any other field of characteristic zero). Then \( R \) is the homomorphic image of \( T_\mathfrak{m} \), where \( T = \mathbb{C}[x_1, \ldots, x_t] \) and \( P \) is a prime ideal of \( T \), so that \( R \cong (T/J)_P \) for some ideal \( J \subseteq T \). We pick a finitely generated regular \( \mathbb{Z} \)-algebra \( A \subseteq \mathbb{C} \), in a way that all the coefficients of the generators of \( P \) and of \( J \) belong to \( A \). Form \( T_A = A[x_1, \ldots, x_t] \), and let \( P_A = P \cap A \), and \( J_A = J \cap A \). Observe that \( P_A \otimes_A \mathbb{C} \cong P \), and \( J_A \otimes_A \mathbb{C} \cong J \). By generic flatness, we may replace \( A \) by the localization \( A_\alpha \), for some non-zero \( \alpha \in A \), and assume that \( R_A \cong (T_A/J_A)_P \) is flat over \( A \). If \( \mathfrak{n} \) is a maximal ideal of \( A \), and we let \( \kappa = A/\mathfrak{n} \), then we can consider \( R_\kappa = R_A \otimes_A \kappa \).

This is now a local ring, essentially of finite type over a field of characteristic \( p > 0 \). In what follows, by abusing notation, we will say that \( \kappa = A/\mathfrak{n} \) belongs to a set \( S \subseteq \operatorname{Max Spec}(A) \) to mean that \( \mathfrak{n} \) belongs to \( S \). We consider the following problem.

**Question 3.17.** Let \((R, \mathfrak{m})\) be a local ring essentially of finite type over \( \mathbb{C} \). With the notation introduced above, is it true that \( R \) is cohomologically full if and only if \( R_\kappa \) is cohomologically full for all \( \kappa \) in a dense subset \( S \) of \( \operatorname{Max Spec}(A) \)?

**Remark 3.18.** One could potentially hope that \( R \) cohomologically full implies that \( R_\kappa \) is cohomologically full for all \( \kappa \) in a dense open subset of \( \operatorname{Max Spec}(A) \). However, this is not the case. In fact, the ring \( R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)\#\mathbb{C}[s, t] \) is Du Bois and hence cohomologically full by [MSS17, Lemma 3.3]. However, setting \( A = \mathbb{Z} \), one can check that \( R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \) is cohomologically full if and only if \( \mathbb{Z}/(p)[x, y, z]/(x^3 + y^3 + z^3) \) is F-pure by [MQ16, Example 2.5]. It is well-known that this happens if and only if \( p \equiv 1 \mod 3 \).

We can answer Question 3.17 in one direction:

**Proposition 3.19.** Let \((R, \mathfrak{m})\) be a local ring essentially of finite type over \( \mathbb{C} \), and let \( A \), \( T_A \), \( P_A \) and \( J_A \) be as above. If \( R_\kappa \) is cohomologically full for all \( \kappa \) in a dense subset \( S \) of \( \operatorname{Max Spec}(A) \), then \( R \) is cohomologically full.

**Proof.** Fix an integer \( n \geq 1 \), and consider the natural map

\[
\operatorname{Ext}^i_{T_A}(R_A, T_A) \cong \operatorname{Ext}^i_{T_A}(T_A/J_A, T_A) \rightarrow \operatorname{Ext}^i_{T_A}(T_A/J_A^n, T_A).
\]

Denote by \( K \) its kernel. After inverting a non-zero element \( a \in A \), by generic freeness we can assume that \( K_a \) is free over \( A_a \). Furthermore, inverting more elements if needed, we may assume that for all \( \kappa = A/\mathfrak{n} \) with \( a \notin \mathfrak{n} \) we have

\[
\operatorname{Ext}^i_{T_A}(T_A/J_A, T_A)_a \otimes_{A_a} \kappa \cong \operatorname{Ext}^i_{T_\kappa}(T_\kappa/J_\kappa, T_\kappa),
\]

and also

\[
\operatorname{Ext}^i_{T_A}(T_A/J_A^n, T_A)_a \otimes_{A_a} \kappa \cong \operatorname{Ext}^i_{T_\kappa}(T_\kappa/J_\kappa^n, T_\kappa).
\]

See [HH06, Theorem 2.3.5] and the preceding discussion for more details on this. It follows that \( K_a \otimes_{A_a} \kappa \) is the kernel of the map \( \operatorname{Ext}^i_{T_\kappa}(T_\kappa/J_\kappa, T_\kappa) \rightarrow \operatorname{Ext}^i_{T_\kappa}(T_\kappa/J_\kappa^n, T_\kappa) \). Since \( S \subseteq \operatorname{Max Spec}(A) \) is dense, we can find a maximal ideal \( \mathfrak{n}' \in S \) such that \( a \notin \mathfrak{n}' \). Let \( \kappa' = A/\mathfrak{n}' \), so that \( R_{\kappa'} \) is cohomologically full by assumption. Then we have that \( K_a \otimes_{A_a} \kappa' = 0 \), since the
map $\text{Ext}^i_{T_p}(T\kappa/J\kappa, T\kappa) \to \text{Ext}^i_{T_p}(T\kappa/J\kappa, T\kappa)$ is an injection by assumption. However, this forces $K\kappa$ to be zero by freeness. In particular, $K$ is a torsion $A$-module, and thus $K \otimes_A \mathbb{C} = 0$. Since $\text{Ext}^i_{T_A}(R_A, T_A) \otimes_A \mathbb{C} \cong \text{Ext}^i_T(T/J, T)$ and $\text{Ext}^i_{T_A}(T_A/J^n_A, T_A) \otimes_A \mathbb{C} \cong \text{Ext}^i_T(T/J^n, T)$, we get that $\text{Ext}^i_T(T/J, T) \to \text{Ext}^i_T(T/J^n, T)$ is injective. This is still true after localizing at $P$, so that

$$\text{Ext}^i_{T_p}(R, T_P) \cong \text{Ext}^i_{T_p}((T/J)_P, T_P) \to \text{Ext}^i_{T_p}((T/J^n)_P, T_P)$$

is injective. The statement now follows by taking the direct limit over $n$, and using Proposition 2.1 (5) $\iff$ (1).

**Remark 3.20.** The other direction of Proposition 3.19 seems to be related to the weak ordinarity conjecture [MS11], and we expect it is difficult.

Proposition 3.19 can be used to prove that certain rings are cohomologically full in characteristic zero. In fact, sometimes it is easier to check cohomological fullness after reducing to characteristic $p > 0$ for infinitely many $p$, as in the following example.

**Example 3.21.** Let $R = \mathbb{C}[x, y, z]/(x^4 + y^4 + z^4)\# \mathbb{C}[s, t]$. Then $R_m$ is cohomologically full (where $m$ denotes the unique homogeneous maximal ideal of $R$). In fact, we let $A = \mathbb{Z}$, and we consider the rings

$$R_p = \mathbb{Z}/(p)[x, y, z]/(x^4 + y^4 + z^4)\# \mathbb{Z}/(p)[s, t],$$

for prime numbers $p \in \mathbb{Z}$. We will show that $(R_p)_m$ is $F$-full for all $p \equiv 1 \mod 4$. Since $\text{depth}(R_p) = 2 \leq \dim(R_p) = 3$, it is enough to show that for $p \equiv 1 \mod 4$ the following facts hold true:

1. The Frobenius map is surjective on $H^2_{m_p}(R_p)_0$.
2. $H^2_{m_p}(R_p)$ is generated by $H^2_{m_p}(R_p)_0$ as an $R_p$-module.

To verify (1) and (2), we compute the second local cohomology module of $R_p$ more explicitly, using the Künneth formula for local cohomology [GW78, Theorem 4.1.5]. Fix a prime $p$, let $B = \mathbb{Z}/(p)[x, y, z]/(x^4 + y^4 + z^4)$, with maximal ideal $m_B = (x, y, z)$, and let $C = \mathbb{Z}/(p)[s, t]$, with maximal ideal $m_C = (s, t)$. Since $H^i_{m_B}(B) = H^i_{m_C}(C) = 0$ for all $i \neq 2$, we obtain that

$$H^2_{m_p}(R_p) \cong (H^2_{m_B}(B) \# C) \oplus (B \# H^2_{m_C}(C)).$$

Furthermore, because $H^2_{m_C}(C)_j = 0$ for all $j > -2$, and $H^2_{m_B}(B)_j = 0$ for all $j > 1$, we have

$$H^2_{m_p}(R_p) = H^2_{m_p}(R_p)_0 \oplus H^2_{m_p}(R_p)_1 \cong \left[H^2_{m_B}(B)_0 \neq \mathbb{Z}/(p)\right] \oplus \left[H^2_{m_B}(B)_1 \neq \mathbb{Z}/(p)[s, t]_1\right].$$

We can explicitly compute a $\mathbb{Z}/(p)$-basis for $H^2_{m_B}(B)$:

$$H^2_{m_B}(B)_0 = \mathbb{Z}/(p)\left\langle \frac{z^2}{xy}, \frac{x^3}{x^2y}, \frac{z^3}{xy^2} \right\rangle \quad \text{and} \quad H^2_{m_B}(B)_1 = \mathbb{Z}/(p)\left\langle \frac{z^3}{xy} \right\rangle.$$

Therefore, as a $\mathbb{Z}/(p)$-vector space, we can write $H^2_{m}(R_p)$ as follows:

$$H^2_{m}(R_p) \cong \mathbb{Z}/(p)\left\langle \frac{z^2}{xy} \# 1, \frac{x^3}{x^2y} \# 1, \frac{z^3}{xy^2} \# 1 \right\rangle \oplus \mathbb{Z}/(p)\left\langle \frac{z^3}{xy} \# s, \frac{z^3}{xy} \# t \right\rangle.$$

We now check that (1) and (2) hold by direct computation:
(1) For $p = 4m + 1$ we have
\[
\left( \frac{z^2}{xy} \# 1 \right)^p = \frac{z^2 \cdot (z^4)^{2m}}{x^{4m+1}y^{4m+1}} \# 1 = \frac{z^2 \cdot (x^4 + y^4)^{2m}}{x^{4m+1}y^{4m+1}} \# 1 = \left( \frac{2m}{m} \right) \frac{z^2}{xy} \# 1,
\]
where the last equality follows from the fact that all the terms other than $(\frac{2m}{m}) x^{4m} y^{4m}$ that come from expanding $(x^4 + y^4)^{2m}$ contain a power of either $x$ or $y$ that exceeds $4m + 1$. Furthermore, since $2m < p$, the binomial coefficient $(\frac{2m}{m})$ is a unit. Similarly, one can check that:
\[
\left( \frac{z^3}{x^2y} \# 1 \right)^p = \left( \frac{3m}{m} \right) \frac{z^3}{x^2y} \# 1 \quad \text{and} \quad \left( \frac{z^3}{xy^2} \# 1 \right)^p = \left( \frac{3m}{m} \right) \frac{z^3}{xy^2} \# 1.
\]
Because $(\frac{3m}{m})$ is again a unit, we conclude that the Frobenius map is surjective on $H^2_{m^p}(R_p)_0$.

(2) Given that $H^2_{m^p}(R_p) = H^2_{m^p}(R_p)_0 \oplus H^2_{m^p}(R_p)_1$, it is enough to observe that
\[
\frac{z^3}{xy} \# s = (z \# s) \cdot \left( \frac{z^2}{xy} \# 1 \right) \quad \text{and} \quad \frac{z^3}{xy} \# t = (z \# t) \cdot \left( \frac{z^2}{xy} \# 1 \right).
\]
This shows that $H^2_{m^p}(R_p)_1 \subseteq R_p \cdot H^2_{m^p}(R_p)_0$, and concludes the proof.

4. Applications: Regularity, Kodaira-type vanishing, and Lyubeznik numbers

In this section we give some applications and study further nice properties of cohomologically full rings. In what follows, $(R, \mathfrak{m}, k)$ denotes either a local ring, or a standard graded algebra over a field $k$.

4.1. Bounds on the regularity of cohomologically full rings. Given a standard graded $k$-algebra $(R, \mathfrak{m}, k)$, a non-zero finitely generated $\mathbb{Z}$-graded $R$-module $M$, and an integer $i$, the module $H^i_{\mathfrak{m}}(R)$ is $\mathbb{Z}$-graded, and $H^i_{\mathfrak{m}}(M)_{\geq 0} = 0$. The $i$-th $a$-invariant of $M$ is defined as
\[
a_i(M) = \sup \{ s \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_s \neq 0 \},
\]
where $a_i(M) = -\infty$ if $H^i_{\mathfrak{m}}(M) = 0$. The Castelnuovo-Mumford regularity of $M$ is
\[
\text{reg}(M) = \max \{ a_i(M) + i \mid i \in \mathbb{Z} \}.
\]

It follows immediately from the definition that, if $(R, \mathfrak{m}, k)$ is a cohomologically full standard graded ring, then $a_i(T) \geq a_i(R)$ for all $i$ and all graded surjections $(T, \mathfrak{n}) \twoheadrightarrow (R, \mathfrak{m})$, with $T_{\text{red}} = R_{\text{red}}$. In particular, if $R = S/I$ for some standard graded polynomial ring $S$ over $k$, and $J \subseteq I$ is another ideal with $\sqrt{J} = \sqrt{I}$, then $a_i(S/J) \geq a_i(S/I)$ for all $i$.

We now focus on the positive characteristic case, and we recall the definition of F-thresholds. Given two ideals $\mathfrak{a} \subseteq \sqrt{J}$ in a ring $R$ of prime characteristic $p > 0$, and given an integer $e > 0$, we define
\[
\nu^J_{\mathfrak{a}}(p^e) := \max \{ t \in \mathbb{N} \mid \mathfrak{a}^t \nsubseteq J[p^e] \}.
\]
The F-threshold of $\mathfrak{a}$ with respect to $J$ is defined as
\[
c^J(\mathfrak{a}) = \lim_{e \to \infty} \frac{\nu^J_{\mathfrak{a}}(p^e)}{p^e}.
\]
F-thresholds were introduced in [MTW05] for regular rings, and then generalized and studied in the singular setup in [HMTW08]. The limit above was recently shown to exist in great generality [DSNBP17].

We now present an upper bound for the a-invariants of an ideal J inside a polynomial ring S over a field of prime characteristic. Comparing this result with [DSNBP17, Theorem 5.8], we observe that here we obtain a similar type of upper bound, but for all the a-invariants. Furthermore, we only need that the quotient S/I is F-full (rather than F-pure).

**Theorem 4.1.** Let (S, m, k) be a standard graded polynomial ring over k with char(k) = p > 0. Let a and J be graded ideals of S such that a ⊆ √J, and assume that S/J is cohomologically full. Let d(a) be the smallest maximal degree of a generator of some reduction of a. Then

\[ \max\{a_i(S/J) \mid i \in \mathbb{Z}\} \leq c^J(a)d(a). \]

In particular, \( \max\{a_i(S/J) \mid i \in \mathbb{Z}\} \leq \mu(J)\deg(J), \) where \( \deg(J) \) is the maximal degree of a minimal generator of J.

**Proof.** The final claim follows from setting \( a = J \) in the first inequality, and from the fact that \( \nu^J_d(p^e) \leq \mu(J)(p^e - 1) \) for all \( e \) by the pigeonhole principle, so that \( c^J(J) \leq \mu(J). \) Moreover, the inequality \( d(J) \leq \deg(J) \) is clear from the definition. To prove that \( a_i(S/J) \leq c^J(a)d(a) \) for all \( i \) I observe that, for all integers \( e > 0 \), we have graded containments \( a^{\nu^d(\nu^e)} \subseteq J^{\nu^e} \). Furthermore, since \( S/J \) is cohomologically full, so is \( S/J^{p^e} \) by Corollary 3.5. By our previous observations, we conclude that \( a_i(S/a^{\nu^d(\nu^e)}) \geq a_i(S/J^{p^e}) = p^ea_i(S/J) \), where the last equality follows from the fact that the Frobenius map is flat on \( S \). There exists a constant \( f \) such that \( \text{reg}(S/a^{\nu^d(\nu^e)}) = d(a)(\nu^d(\nu^e) + 1) + f \) for all \( e \gg 0 \) (for example, see [EU12, Theorem 0.1]). Therefore, for all \( e \gg 0 \) and all \( i \in \mathbb{Z} \), we get

\[ d(a)(\nu^d(\nu^e) + 1) + f - i = \text{reg}(S/a^{\nu^d(\nu^e)}) - i \geq a_i(S/a^{\nu^d(\nu^e)}) \geq p^ea_i(S/J). \]

Dividing by \( p^e \) and taking the limit as \( e \to \infty \) gives the desired inequality. \( \square \)

In order to obtain similar results for algebras over a field of characteristic zero, we need to be able to keep track of the invariants that appear in Theorem 4.1 as we perform the reduction to characteristic \( p > 0 \) process. We focus on the weaker inequality.

Let \( R \) be an algebra of finite type over \( \mathbb{C} \), and assume it is standard graded. Then \( R \) is the homomorphic image of a polynomial ring \( S = \mathbb{C}[x_1, \ldots, x_t] \), with \( \deg(x_i) = 1 \) for all \( i \). More specifically, we have that \( R \cong S/J \) as graded algebras, for some homogeneous ideal \( J \subseteq S \). We can choose a finitely generated regular \( \mathbb{Z} \)-algebra \( A \subseteq \mathbb{C} \), so that the coefficients of the generators of \( J \) lie inside \( A \). Consider the graded ring \( A[x_1, \ldots, x_t] \), with \( \deg(x_i) = 1 \) for all \( a \in A \), and \( \deg(x_i) = 1 \) for all \( i \). Let \( J_A = J \cap A \), and let \( R_A = S_A/J_A \). Observe that \( J_A \otimes_A \mathbb{C} \cong J \). By generic flatness, we may replace \( A \) by the localization \( A_a \), for some non-zero \( a \in A \), and assume that \( R_A \) is flat over \( A \). If \( n \) is a maximal ideal of \( A \), with \( \kappa = A/n \), we let \( S_\kappa = S_A \otimes_A \kappa \), \( J_\kappa = J_A \otimes_A \kappa \), and \( R_\kappa = R_A \otimes_A \kappa \). Given a subset \( S \subseteq \text{Max Spec}(A) \), by abuse of notation we will say that a property holds for \( \kappa \in S \), where \( \kappa = A/n \), to mean that it holds for \( n \in S \).

**Lemma 4.2.** Let \( S = \mathbb{C}[x_1, \ldots, x_t] \) be a standard graded polynomial ring, and let \( J \subseteq S \) be a homogeneous ideal. Let \( R = S/J \), and let \( A, S_A, J_A \) and \( R_A \) be as above. There exists a
dense open subset $S \subseteq \text{MaxSpec}(A)$ such that

$$
\mu(J) = \mu(J_\kappa), \quad \deg(J) = \deg(J_\kappa) \quad \text{and} \quad a_i(R) = a_i(R_\kappa)
$$

for all $i \in \mathbb{Z}$ and all $\kappa \in S$.

**Proof.** Recall that $\mu(J) = \dim_{\mathbb{C}}(\text{Tor}^1_{\mathbb{C}}(R, \mathbb{C}))$, and $\deg(J) = \max\{j \in \mathbb{Z} \mid \text{Tor}^1_{\mathbb{C}}(R, \mathbb{C})_j \neq 0\}$. By inverting an element of $A$ we may assume that $\text{Tor}^1_{\mathbb{C}}(R_A, S_A/(x_1, \ldots, x_t)S_A)$ is free over $A$. In particular, we have that

$$
\text{rank}_A(\text{Tor}^1_{\mathbb{C}}(R_A, S_A/(x_1, \ldots, x_t)S_A)) = \dim_{\mathbb{C}}(\text{Tor}^1_{\mathbb{C}}(R, S/(x_1, \ldots, x_t)S)) = \mu(J).
$$

Furthermore, by [HH06, Theorem 2.3.5 (e)], we have that

$$
\text{Tor}^1_{\mathbb{C}}(R_A, S_A/(x_1, \ldots, x_t)S_A) \otimes_A \kappa \cong \text{Tor}^1_{\mathbb{C}}(R_\kappa, S_\kappa/(x_1, \ldots, x_t)S_\kappa) \cong \text{Tor}^1_{\mathbb{C}}(R_\kappa, \kappa)
$$

for all $\kappa$ in a dense open subset of $\text{MaxSpec}(A)$. Counting vector space dimensions, it follows that $\mu(J_\kappa) = \mu(J)$ for all such $\kappa$. As for the maximal degree of a minimal generator of $J$, we consider the $A$-submodule $N := \text{Tor}^1_{\mathbb{C}}(R_A, S_A/(x_1, \ldots, x_t)S_A)_{\deg(J)}$ of $M := \text{Tor}^1_{\mathbb{C}}(R_A, S_A/(x_1, \ldots, x_t)S_A)$. Observe that $N \otimes_A \kappa \cong \text{Tor}^1_{\mathbb{C}}(R, \mathbb{C})_{\deg(J)} \neq 0$. By inverting an element in $A$, if needed, we may assume that the inclusion $N \subseteq M$ of $A$-modules splits. After tensoring with $\kappa = A/\mathfrak{n}$, we still have an inclusion $N/\mathfrak{n}N \subseteq M/\mathfrak{n}M \cong \text{Tor}^1_{\mathbb{C}}(R_\kappa, \kappa)$. Since $N$ is a finitely generated $A$-module, it follows by Nakayama’s Lemma that $N/\mathfrak{n}N \neq 0$. This shows that $\text{Tor}^1_{\mathbb{C}}(R_\kappa, \kappa)_{\deg(J)} \neq 0$, so that $\deg(J_\kappa) \geq \deg(J)$ for all $\kappa$ in a dense open subset of $\text{MaxSpec}(A)$. The other inequality can be obtained with analogous considerations. The argument for $a$-invariants is similar to that for the maximal degree of a minimal generator. Note that, by graded local duality on $S$, we have that

$$
a_i(R) = \max\{j \in \mathbb{Z} \mid H^j_{(x_1, \ldots, x_t)}(R)_{(x_1, \ldots, x_t)} \neq 0\} = \min\{j \in \mathbb{Z} \mid \text{Ext}^{t-i}_{S}(R, S(-t))_j \neq 0\}.
$$

By inverting an element of $A$, if needed, we may assume that the inclusion of $A$-modules $M' := \text{Ext}^{t-i}_{S}(R, S(-t))_{a_i(R)} \subseteq \text{Ext}^{t-i}_{S}(R, S(-t)) = M'$ splits. Observe that $M' \otimes_A \kappa \cong \text{Ext}^{t-i}_{S}(R, \mathbb{C})_{a_i(R)} \neq 0$. After tensoring the inclusion $N' \subseteq M'$ with $A/\mathfrak{n}$, we obtain an inclusion $N'/\mathfrak{n}N' \subseteq M'/\mathfrak{n}M' \cong \text{Ext}^{t-i}_{S}(R_\kappa, S_\kappa(-t))$, which shows that $\text{Ext}^{t-i}_{S}(R_\kappa, S_\kappa(-t))_{a_i(R)} \neq 0$. By local duality on $S_\kappa$ we then have

$$
a_i(R_\kappa) = \min\{j \in \mathbb{Z} \mid \text{Ext}^{t-i}_{S}(R_\kappa, S_\kappa(-t))_j \neq 0\} \geq a_i(R).
$$

As before, the other inequality is proved in a similar fashion. \qed

With the notation introduced above, we say that an algebra $R$ of finite type over $\mathbb{C}$ is of dense $F$-full type if $R_\kappa$ is cohomologically full (equivalently, $F$-full) for all $\kappa$ in a dense subset of $\text{MaxSpec}(A)$.

**Corollary 4.3.** Let $(S, \mathfrak{m}, \mathbb{C})$ be a standard graded polynomial ring. Let $J$ be an ideal of $S$ such that $S/J$ is of dense $F$-full type. Then $\max\{a_i(S/J) \mid i \in \mathbb{Z}\} \leq \mu(J) \deg(J)$.

**4.2. Kodaira Vanishing type results.** This subsection is devoted to the study of the following question.

**Question 4.4.** Let $S = k[x_1, \ldots, x_n]$ be an $n$-dimensional standard graded polynomial ring over a field $k$, and $I \subseteq S$ be a homogeneous ideal. When is $H^2_{x_j}(\text{Ext}^1_{S}(S/I, S(-n)))_{>0} = 0$ for all $i, j$?
The vanishing of $H^j_m(\text{Ext}^i_S(S/I, S(-n)))_{>0}$ when $j > 1$ is the local cohomology version of the Kodaira vanishing condition for $X = \text{Proj} S/I$. Question 4.4 has a positive answer when $R$ is F-pure in characteristic $p > 0$ or when $X$ is Du Bois in characteristic 0 (for example, see [DMV18]). We were not aware of any counter-examples to Question 4.4 when $R$ is F-injective in characteristic $p > 0$, or when $R$ is cohomologically full. Below we give some partial answers.

We recall that a local ring $(R, \mathfrak{m}, k)$ of characteristic $p > 0$ is called F-injective if, for all integers $i$, the map $F : H^i_m(R) \to H^i_m(R)$ induced by the Frobenius endomorphism on $R$ is injective.

**Proposition 4.5.** Let $S = k[x_1, \ldots, x_n]$ be an $n$-dimensional standard graded polynomial ring over a field $k$ of characteristic $p > 0$, and $I \subseteq S$ be a homogeneous ideal. Suppose $R = S/I$ is F-injective. Then $H^d_m(\text{Ext}^i_S(S/I, S(-n)))_{>0} = 0$ for $d_i = \dim(\text{Ext}^i_S(S/I, S(-n)))$. In particular, we have $H^i_m(\text{Ext}^i_S(S/I, S(-n)))_{>0} = 0$.

**Proof.** Given an $R$-module $M$, we denote by $F^e_M$ the $R$-module with action obtained by restriction of scalars via the $e$-th iterate of the Frobenius map. Since $R$ is F-injective, for all $0 \leq i \leq d$ we have graded injections $F^e_M \to F^e_{\text{Ext}^i_S(S/I, S(-n))}$ of $R$-modules. In turn, these induce graded surjections $F^e_{\text{Ext}^i_S(S/I, S(-n))} \twoheadrightarrow \text{Ext}^i_S(S/I, S(-n))$, by graded local duality. Thus, if $d_i = \dim(\text{Ext}^i_S(S/I, S(-n))) = \dim(F^e_{\text{Ext}^i_S(S/I, S(-n))})$, we get surjections $F^e_{H^d_m(\text{Ext}^i_S(S/I, S(-n)))} \twoheadrightarrow H^d_m(\text{Ext}^i_S(S/I, S(-n)))$ on the top local cohomology modules. Viewing $F^e_{H^d_m(\text{Ext}^i_S(S/I, S(-n)))}$ as a $p^{-e}$-graded module, the above surjection implies that $H^d_m(\text{Ext}^i_S(S/I, S(-n))) \neq 0$, then $H^d_m(\text{Ext}^i_S(S/I, S(-n)))_{>0} = 0$. This clearly implies $H^i_m(\text{Ext}^i_S(S/I, S(-n)))_{>0} = 0$, because $H^i_m(\text{Ext}^i_S(S/I, S(-n)))_{>0} = 0$. The last claim follows from the fact that $\dim(\text{Ext}^i_S(S/I, S(-n))) \leq i$. □

Recall that a local ring $(R, \mathfrak{m}, k)$ is said to have finite local cohomology, if $H^i_m(R)$ has finite length for $i \neq \dim(R)$. When $R$ is equidimensional, $R$ has finite local cohomology if and only if $R_P$ is Cohen-Macaulay for all $P \neq \mathfrak{m}$. We next show that Question 4.4 has a positive answer when $S/I$ is cohomologically full and has finite local cohomology. Our proof uses the theory of Eulerian graded $\mathcal{D}$-modules [MZ14].

**Theorem 4.6.** Let $S = k[x_1, \ldots, x_n]$ be an $n$-dimensional standard graded polynomial ring over a field $k$, and $I \subseteq S$ be a homogeneous ideal. Suppose $R = S/I$ is cohomologically full. Then $H^d_m(\text{Ext}^i_S(S/I, S(-n)))_{>0} = 0$ for every $i$. In particular, if $R$ has finite length cohomology, then $H^i_m(S)_{>0} = 0$ for all $j < d$.

**Proof.** By the graded version of Proposition 2.1, we obtain that the map $\text{Ext}^i_S(S/I, S) \to H^i_j(S)$ is a degree preserving inclusion. Furthermore, since $H^i_m(S/I)$ has finite length, so does $\text{Ext}^{n-i}_S(S/I, S)$. Applying the functor $H^i_m(-)$ to the above inclusion we see that $H^d_m(\text{Ext}^i_S(S/I, S)) = \text{Ext}^i_S(S/I, S) \to H^0_m(H^i_j(S))$ is still an inclusion. Since $H^0_m(H^i_j(S))$ is an Artinian Eulerian graded $\mathcal{D}$-module, it must be isomorphic to a finite direct sum of copies of $\ast E(n)$, where $\ast E$ is the graded injective hull of $k$ of $S$ [MZ14, Theorem 1.2]. In particular, the module $H^0_m(\text{Ext}^i_S(S/I, S))$ only lives in degrees $\leq -n$ and thus $H^i_m(\text{Ext}^i_S(S/I, S(-n)))_{>0} = 0$.

Finally, when $R$ has finite length cohomology, by graded local duality, $H^d_m(R)_{>0}$ is graded dual of $\text{Ext}^i_S(S/I, S(-n))_{>0} \cong H^0_m(\text{Ext}^i_S(S/I, S(-n)))_{>0}$, since $H^i_m(R)$ has finite length. □
4.3. Relations with (quasi-)Buchsbaum rings and Lyubeznik numbers.

**Definition 4.7.** Let \((R, m, k)\) be a \(d\)-dimensional local ring that is a homomorphic image of a regular local ring \(S\). We say that \(R\) is quasi-Buchsbaum if, for all \(i \neq d\), the module \(H^i_m(R)\) is a \(k\)-vector space. We say that \(R\) is Buchsbaum if the natural map \(\text{Ext}_S^i(k, R) \to H^i_m(R)\) is surjective for all \(i \neq d\).

The one given here is not the most common definition of Buchsbaum ring, but it is equivalent by [SV78, Theorem 1]. Note that Buchsbaum rings are quasi-Buchsbaum, since a surjection \(\text{Ext}_S^i(k, R) \to H^i_m(R)\) implies that \(H^i_m(R)\) is killed by \(m\).

**Proposition 4.8.** Let \((R, m, k)\) be a local ring of characteristic \(p > 0\) with finite local cohomology. If \(R\) is \(F\)-injective and \(k\) is perfect, then \(R\) is cohomologically full and Buchsbaum. If \(R\) is quasi-Buchsbaum and cohomologically full, then \(R\) is \(F\)-injective.

**Proof.** If \(R\) is \(F\)-injective and has finite local cohomology, then \(R\) is Buchsbaum by [Ma15, Corollary 1.3]. In particular, the local cohomology module \(H^i_m(R)\) is a vector space for all \(i \neq d\). Let \(v_1, \ldots, v_t\) form a basis of \(H^i_m(R)\). By assumption, the Frobenius map \(F : H^i_m(R) \to H^i_m(R)\) is injective, hence \(F(v_1), \ldots, F(v_t)\) are \(k^p\)-linearly independent. Since \(k\) is perfect, and \(H^i_m(R)\) is a \(k\)-vector space, the \(R\)-span of \(F(v_1), \ldots, F(v_t)\) must be \(H^i_m(R)\). This proves \(R\) is \(F\)-full and hence cohomologically full by Corollary 2.2.

For the second statement, assume that \(R\) is quasi-Buchsbaum and cohomologically full. Let \(v_1, \ldots, v_t\) be a \(k\)-basis of \(H^i_m(R)\), and assume \(\sum_i r_i F(v_i) = 0\) for some \(r_i \in R\), not all belonging to \(m\). It follows that \(\sum_i r_i F(v_i) = 0\), which shows that \(F(v_1), \ldots, F(v_t)\) are linearly dependent over \(k\). In particular, the \(R\)-span of \(F(v_1), \ldots, F(v_t)\), which is the \(k\)-span of \(F(v_1), \ldots, F(v_t)\), cannot be the whole \(H^i_m(R)\). This shows \(R\) is not \(F\)-full and hence not cohomologically full by Corollary 2.2, which is a contradiction. \(\square\)

In [MSS17, Example 3.5], it is shown that the assumption that \(k\) is perfect in the above cannot be removed. On the other hand, when \((k) \equiv 2\) modulo 3, \(R = k[x, y, z]/(x^3 + y^3 + z^3)\#k[s, t]\) is an example of a ring that is Buchsbaum [Miy89, Theorem A], but not \(F\)-injective. In particular, \(R\) is not cohomologically full either.

**Lemma 4.9.** Let \((R, m, k)\) be a standard graded cohomologically full \(k\)-algebra with finite local cohomology. Then \(H^i_m(R)_0\) generates \(H^i_m(R)\) as an \(R\)-module. Moreover, if \(R\) has characteristic \(p > 0\), then \(H^i_m(R)\) is generated by \(F^e(H^i_m(R)_0)\) for all \(e \in \mathbb{N}\) as an \(R\)-module.

**Proof.** Write \(R = S/I\) for some \(n\)-dimensional polynomial ring \(S\). By graded local duality, \(H^i_m(R)\) is generated in degree zero if and only if the socle of \(H^i_m(R)\) is concentrated in degree \(-n\). Since \(R\) is cohomologically full and \(\text{Ext}^{n-i}_S(S/I, S)\) has finite length, we have an injection \(\text{Ext}^{n-i}_S(S/I, S) \to H^i_m(H^{n-i}_m(S))\). However, the latter is an Eulerian \(D\)-module, hence isomorphic to a direct sum of \(*E(n)\), where \(*E(n)\) denotes the graded injective hull of \(k\) in \(S\) [MZ14, Theorem 1.2]. In particular, the socle of \(\text{Ext}^{n-i}_S(S/I, S)\) is contained in the socle of \(\bigoplus *E(n)\), which lies in degree \(-n\).

If \(R\) has characteristic \(p > 0\), since \(R\) is cohomologically full, we have that \(H^i_m(R)_0 = (R[F^e(H^i_m(R))]_0 = R_0[F^e(H^i_m(R)_0)]\) for all \(e \in \mathbb{N}\). By the first part of the Lemma, we finally obtain \(H^i_m(R) = R[H^i_m(R)_0] = R[F^e(H^i_m(R)_0)]\). \(\square\)

The degree zero part of local cohomology modules of cohomologically full rings with finite local cohomology are particularly relevant. We now explore some relations between
these dimensions and some Lyubeznik numbers. First, we recall the definition of Lyubeznik numbers.

**Definition 4.10.** Let \((S, m, k)\) be an unramified regular ring that is either local or standard graded. Let \(I\) be an ideal of \(S\), homogeneous in the latter case, and set \(R = S/I\). Assume that \(\text{char}(S) = \text{char}(R)\). The Lyubeznik number of \(R\) with respect to \(i, j\) is defined as

\[
\lambda_{i,j}(R) := \dim_k \text{Ext}^i_S(k, H^{n-j}_I(S)).
\]

In these assumptions, the Lyubeznik number \(\lambda_{i,j}(R)\) is an invariant of \(R\), meaning that it does not depend on the presentation of \(R\) as a quotient of a regular local or graded ring [Lyu93, NBW13]. The following result generalizes [DSGNB17, Theorem 5.3].

**Proposition 4.11.** Let \((R, m, k)\) be a standard graded cohomologically full \(k\)-algebra of characteristic \(p > 0\). If \(R\) has finite local cohomology, then for all \(j < d\)

\[
\lambda_{0,j}(R) = \dim_k([H^j_m(R)]_0).
\]

**Proof.** Let \(\ell\) be a field extension of \(k\), and set \(\overline{R} = R \otimes_k \ell\). Since \(R \to \overline{R}\) is flat with closed fiber a field, we have \(\lambda_{0,j}(R) = \lambda(\overline{R})\). Moreover, there is a graded isomorphism \(H^i_m(R) \otimes_R \overline{R} \cong H^i_m(\overline{R}) = H^i_m(\overline{R})\). Finally, if \(R\) is cohomologically full, then so is \(\overline{R}\) by Proposition 3.8. Therefore, we may assume that \(k\) is perfect. Write \(R = S/I\), where \(S\) is an \(n\) dimensional polynomial ring over \(k\). It is proved in [Zha11] that \(\lambda_{0,j}(R)\) is the \(k\)-vector space dimension of the \(F\)-stable part of \(\text{Ext}^n_S(\text{Ext}^{n-j}_S(S/I, S), S)_0\):

\[
\lambda_{0,j}(R) = \dim_k \left[ \bigcap_f \text{Ext}^e_S(\text{Ext}^{n-j}_S(S/I, S), S)_0 \right].
\]

Let \((-)^\vee\) denote the graded Matlis dual of a module. By graded local duality, we have graded isomorphisms

\[
\text{Ext}^n_S(\text{Ext}^{n-j}_S(S/I, S), S) \cong \left[ H^n_m(\text{Ext}^{n-j}_S(S/I, S)) \right]^{\vee} = \left[ \text{Ext}^{n-j}_S(S/I, S) \right]^{\vee} \cong H^j_m(S/I),
\]

where we used that \(\lambda(\text{Ext}^{n-j}_S(S/I, S)) < \infty\). By [Zha11, Proposition 5.7 and Lemma 7.3], the Lyubeznik number \(\lambda_{0,j}(R)\) is then the dimension of the \(F\)-stable part of \(H^j_m(R)_0\), where the Frobenius action is the one induced by the natural Frobenius action on \(R\). By Lemma 4.9, the module \(H^j_m(R)\) is the \(R\)-span of \(F(H^j_m(R)_0)\), showing that the entire vector space \(H^j_m(R)_0\) is \(F\)-stable. \(\square\)

We end this subsection showing that, under some assumptions, if \(R/(x)\) is cohomologically full then \(R\) and \(R/(x)\) share essentially the same Lyubeznik table. This argument was shown to us by Luis Núñez-Betancourt and Ilya Smirnov.

**Proposition 4.12.** Let \((S, m, k)\) be an unramified regular local ring that is either local or standard graded. Let \(I\) be an ideal of \(S\), homogeneous in the latter case, and set \(R = S/I\). Assume that \(\text{char}(S) = \text{char}(R)\). Let \(x \in m\) be an element whose image in \(R\) is a nonzerodivisor, and such that \(R/(x)\) is cohomologically full. In addition, assume that there exists a sequence of ideals \(\{I_e\}\), cofinal with \(\{I^e\}\), such that \(x\) is a surjective element for \(S/I_e\) for all \(e\) (this assumption is unnecessary when \(S\) has characteristic \(p > 0\)). Then

\[
\lambda_{i,j}(R) = \lambda_{i-1,j-1}(R/(x))
\]
Proof. Let \( n = \dim(S) \). Since \( x^e \) is a surjective element for \( S/I_e \) for all \( e \), for all \( j \in \mathbb{N} \) we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}_S^{n-j}(R, S) & \longrightarrow & \text{Ext}_S^{n-j}(R, S) & \longrightarrow & \text{Ext}_S^{n-(j-1)}(R/(x), S) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ext}_S^{n-j}(S/I_2, S) & \longrightarrow & \text{Ext}_S^{n-j}(S/I_2, S) & \longrightarrow & \text{Ext}_S^{n-(j-1)}(S/(I_2, x), S) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ext}_S^{n-j}(S/I_e, S) & \longrightarrow & \text{Ext}_S^{n-j}(S/I_e, S) & \longrightarrow & \text{Ext}_S^{n-(j-1)}(S/(I_e, x), S) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\end{array}
\]

with exact rows. Taking direct limits over \( e \), this gives rise to a short exact sequence

\[
0 \longrightarrow H_I^{n-j}(S) \longrightarrow H_I^{n-j}(S)_x \longrightarrow H_I^{n-(j-1)}(S) \longrightarrow 0.
\]

Applying \( \text{Hom}_S(k, -) \), and noting that \( \text{Ext}_S^i(k, H_I^{n-j}(S)_x) = 0 \) for all \( i \), we obtain

\[
\text{Ext}_S^i(k, H_I^{n-j}(S)) \cong \text{Ext}_S^{i-1}(k, H_{(I,x)}^{n-(j-1)}(S)).
\]

for all \( i \in \mathbb{N} \). It follows that \( \lambda_{i,j}(R) = \lambda_{i-1,j-1}(R/(x)) \) for all \( i, j \).

When \( S \) has characteristic \( p > 0 \), one can take \( I_e = I^{[p^e]} \), as shown in the proof of Theorem 3.15. \qed

### 4.4. Thickenings of cohomologically full rings

In this final subsection we point out that for a cohomologically full ring, many properties descend from its thickening. Let \( R = S/I \) where \( S \) is a regular (local or graded) ring. Let \( \mathcal{X}_i \) be subcategories of \( \text{mod}(R) \) closed under taking submodules. Let \( \mathcal{P} \) be a property defined by the following condition:

\[ R \text{ has property } \mathcal{P} \text{ if and only if } \text{Ext}_S^i(R, S) \in \mathcal{X}_i \text{ for all } i. \]

For example, \( \mathcal{P} \) could be: \( R \) has finite local cohomology, \( R \) is quasi-Buchsbaum, \( a_i(R) \leq 0 \) for (some or all) \( i \).

**Proposition 4.13.** Let \( R, \mathcal{P} \) and \( \mathcal{X}_i \) be as above. Let \( T = S/J \) be a thickening of \( R \). If \( T \) has property \( \mathcal{P} \) and \( R \) is cohomologically full, then \( R \) has property \( \mathcal{P} \).

**Proof.** Since \( R \) is cohomologically full, \( H_m^i(T) \) surjects to \( H_m^i(R) \) and by local duality \( \text{Ext}_S^i(R, S) \) is a submodule of \( \text{Ext}_S^i(T, S) \). Because \( \mathcal{X}_i \) is closed under taking submodules and \( T \) has property \( \mathcal{P} \), it follows that \( \text{Ext}_S^i(R, S) \in \mathcal{X}_i \), and hence \( R \) has property \( \mathcal{P} \) as well. \( \Box \)

We also have the analog for the Buchsbaum property.

**Proposition 4.14.** Let \((R, m, k)\) be a homomorphic image of a regular local ring \( S \). Suppose \( R \) is cohomologically full and some thickening \( T = S/J \) of \( R \) is Buchsbaum, then \( R \) is Buchsbaum.
Proof. For all $i < d$ we have a commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^i_S(k, T) & \longrightarrow & H^i_m(T) \\
\downarrow & & \downarrow \\
\text{Ext}^i_S(k, R) & \longrightarrow & H^i_m(R)
\end{array}
\]
where the top horizontal map is onto because $T$ is Buchsbaum, and the right vertical map is onto because $R$ is cohomologically full. It follows that the bottom horizontal map is surjective. Since this holds for all $i < d$, $R$ is Buchsbaum. \qed

References

[Aus61] M. Auslander. Modules over unramified regular local rings. *Illinois J. Math.*, 5:631–647, 1961.

[CV18] Aldo Conca and Matteo Varbaro. Square-free Groebner degenerations. *https://arxiv.org/abs/1805.11923*, 2018.

[DHS13] Hailong Dao, Craig Huneke, and Jay Schweig. Bounds on the regularity and projective dimension of ideals associated to graphs. *J. Algebraic Combin.*, 38(1):37–55, 2013.

[DMV18] Hailong Dao, Linquan Ma, and Matteo Varbaro. Serre’s condition and non-negativity of h-vectors. *in preparation*, 2018.

[DSGB17] Alessandro De Stefani, Eloísa Grifo, and Luis Núñez-Betancourt. Local cohomology and Lyubeznik numbers of F-pure rings. *https://arxiv.org/abs/1710.00462*, 2017.

[DSNB18] Alessandro De Stefani and Luis Núñez-Betancourt. F-thresholds of graded rings. *Nagoya Math. J.*, 229:141–168, 2018.

[DSNP17] Alessandro De Stefani, Luis Núñez-Betancourt, and Felipe Pérez. On the existence of F-thresholds and related limits. To appear in *Trans. Amer. Math. Soc.*, 2017.

[DT16] Hailong Dao and Shunsuke Takagi. On the relationship between depth and cohomological dimension. *Compos. Math.*, 152(4):876–888, 2016.

[EMS00] David Eisenbud, Mircea Mustață, and Mike Stillman. Cohomology on toric varieties and local cohomology with monomial supports. *J. Symbolic Comput.*, 29(4-5):583–600, 2000.

[EU12] David Eisenbud and Bernd Ulrich. Notes on regularity stabilization. *Proc. Amer. Math. Soc.*, 140(4):1221–1232, 2012.

[Gro64] A. Grothendieck. éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, (20):259, 1964.

[GW78] Shiro Goto and Keiichi Watanabe. On graded rings. I. *J. Math. Soc. Japan*, 30(2):179–213, 1978.

[Har68] Robin Hartshorne. Cohomological dimension of algebraic varieties. *Ann. of Math. (2)*, 88:403–450, 1968.

[Har79] Robin Hartshorne. Complete intersections in characteristic $p > 0$. *Amer. J. Math.*, 101(2):380–383, 1979.

[HH06] Melvin Hochster and Craig Huneke. Tight closure in equal characteristic zero. A preprint of a manuscript, 2006.

[HMTW08] Craig Huneke, Mircea Mustață, Shunsuke Takagi, and Kei-ichi Watanabe. F-thresholds, tight closure, integral closure, and multiplicity bounds. *Michigan Math. J.*, 57:463–483, 2008. Special volume in honor of Melvin Hochster.

[HS77] Robin Hartshorne and Robert Speiser. Local cohomological dimension in characteristic $p$. *Ann. of Math. (2)*, 105(1):45–79, 1977.

[Kun69] Ernst Kunz. Characterizations of regular local rings for characteristic $p$. *Amer. J. Math.*, 91:772–784, 1969.
Gennady Lyubeznik. On the local cohomology modules $H^i_a(R)$ for ideals $a$ generated by monomials in an $R$-sequence. In Complete intersections (Acireale, 1983), volume 1092 of Lecture Notes in Math., pages 214–220. Springer, Berlin, 1984.

Gennady Lyubeznik. Finiteness properties of local cohomology modules (an application of $d$-modules to commutative algebra). Inventiones mathematicae, 113(1):41–55, 1993.

Gennady Lyubeznik. On the vanishing of local cohomology in characteristic $p > 0$. Compos. Math., 142(1):207–221, 2006.

Linquan Ma. Finiteness properties of local cohomology for $F$-pure local rings. Int. Math. Res. Not. IMRN, (20):5489–5509, 2014.

Linquan Ma. $F$-injectivity and Buchsbaum singularities. Math. Ann., 362(1-2):25–42, 2015.

Linquan Ma. Lech's conjecture in dimension three. Adv. Math., 322:940–970, 2017.

Chikashi Miyazaki. Graded Buchsbaum algebras and Segre products. Tokyo J. Math., 12(1):1–20, 1989.

Linquan Ma and Pham Hung Quy. Frobenius actions on local cohomology modules and deformation. To appear in Nagoya Math. J., 2016.

Mircea Mustaţă and Vasudevan Srinivas. Ordinary varieties and the comparison between multiplier ideals and test ideals. Nagoya Math. J., 204:125–157, 2011.

Linquan Ma, Karl Schwede, and Kazuma Shimomoto. Local cohomology of Du Bois singularities and applications to families. Compos. Math., 153(10):2147–2170, 2017.

Mircea Mustaţă, Shunsuke Takagi, and Kei-ichi Watanabe. F-thresholds and Bernstein-Sato polynomials. In European Congress of Mathematics, pages 341–364. Eur. Math. Soc., Zürich, 2005.

Mircea Mustaţă. Local cohomology at monomial ideals. J. Symbolic Comput., 29(4-5):709–720, 2000. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).

Linquan Ma and Wenliang Zhang. Eulerian graded $\mathcal{D}$-modules. Math. Res. Lett., 21(1):149–167, 2014.

Luis Núñez-Betancourt and Emily E. Witt. Lyubeznik numbers in mixed characteristic. Math. Res. Lett., 20(6):1125–1143, 2013.

Arthur Ogus. Local cohomological dimension of algebraic varieties. Ann. of Math. (2), 98:327–365, 1973.

Jürgen Stückrad and Wolfgang Vogel. Toward a theory of Buchsbaum singularities. Amer. J. Math., 100(4):727–746, 1978.

Anurag K. Singh and Uli Walther. Local cohomology and pure morphisms. Illinois J. Math., 51(1):287–298, 2007.

Matteo Varbaro. Cohomological and projective dimensions. Compos. Math., 149(7):1203–1210, 2013.

Wenliang Zhang. Lyubeznik numbers of projective schemes. Adv. Math., 228(1):575–616, 2011.