Quantifying the Loss of Information from Binning List-Mode Data

Eric Clarkson

February 14, 2019

Abstract

List-mode data is increasingly being used in SPECT and PET imaging, among other imaging modalities. However, there are still many imaging designs that effectively bin list-mode data before image reconstruction or other estimation tasks are performed. Intuitively, the binning operation should result in a loss of information. In this work we show that this is true for Fisher information and provide a computational method for quantifying the information loss. In the end we find that the information loss depends on three factors. The first factor is related to the smoothness of the mean data function for the list-mode data. The second factor is the actual object being imaged. Finally, the third factor is the binning scheme in relation to the other two factors.

1 Introduction

Many imaging systems detect individual particles as they interact with the imaging hardware. These particles are usually photons, but there are also other choices such as neutrons, beta particles and alpha particles. A list-mode imaging system produces an attribute vector for each particle detected. The attribute vector may include spatial position, energy, time or other attributes that can be assigned to the particle [1-13]. When the particles are photons, list-mode systems are also called photon processing systems to indicate that the attributes are estimated from raw detector outputs via some data processing algorithm [14-17]. In this work we are only concerned with the fact that the imaging system produces an attribute vector for each particle, regardless of how these attributes are arrived at.

We may envision the more common type of imaging system, a binned system, as the result of resolving the space of all attribute vectors into a collection of non-overlapping bins. The system then counts how many attribute vectors fall into each bin and produces an integer vector whose dimension is the number of bins. Intuitively, this would seem to result in a loss of information. If we formulate the task of the imaging system as the estimate of a certain number of
parameters related to the object being imaged, then we may consider quantifying this loss of information, if indeed there is a loss of information.

If the parameter vector of interest has a known prior distribution, then the Shannon information between the parameter vector and the data may be used as a measure of information for the task at hand. In this case the data processing inequality implies that the Shannon information is not increased by the binning operation, but it does not quantify the loss of information due to binning. In this work we will use the Fisher Information Matrix (FIM) to quantify the information loss due to binning. The FIM does not require a prior distribution on the parameter vector of interest. We will show that the FIM always decreases when list-mode data is binned and provide an expression to calculate the information loss. We will find that the information loss depends on three factors, the smoothness of the mean data function for the list-mode data, the actual object being imaged, and the the binning scheme in relation to the other two factors.

2 List mode Fisher information

We will confine our calculations to photon imaging systems where we know that Poisson statistics are applicable. In a list mode imaging system the data is a list of $q$-dimensional attribute vectors $a_n$, one for each photon detected. These photon attributes contained in each of these vectors may include a two dimensional position on a the face of a detector, the depth of interaction in a scintillation detector, the energy of the photon, the direction the photon is travelling when detected for a plenoptic array, and polarization parameters. The collection of all possible attribute vectors is attribute space, $A$. We may arrange the data list into a matrix $A = [a_1, \ldots, a_N]$ and, for a fixed exposure time, the conditional probability distribution function (PDF) for the list is given by

$$p_r(A|\theta) = \frac{\hat{N}(\theta)}{N!} \exp \left[ -\hat{N}(\theta) \right] \prod_{n=1}^{N} p_r(a_n|\theta),$$

(1)

where $\theta$ is a $p$-dimensional parameter vector describing the object being imaged and $p_r(a|\theta)$ is the attribute space conditional PDF determined by $\theta$. The specific form for $p_r(a|\theta)$ depends on the imaging system. The FIM with respect to $\theta$ for list mode data is defined by

$$F_{LM}(\theta) = \left\langle [\nabla_\theta \ln p_r(A|\theta)] [\nabla_\theta \ln p_r(A|\theta)]^\dagger \right\rangle_{A|\theta}. $$

(2)

Using the specific form for $p_r(A|\theta)$ the list mode FIM can be written as

$$F_{LM}(\theta) = \hat{N}(\theta) \left\{ \left\langle [\nabla_\theta \ln p_r(a|\theta)] [\nabla_\theta \ln p_r(a|\theta)]^\dagger \right\rangle_{a|\theta} + [\nabla_\theta \ln \hat{N}(\theta)] [\nabla_\theta \ln \hat{N}(\theta)]^\dagger \right\}. $$

(3)

This is a $p \times p$ matrix which figures prominently in the task of estimating $\theta$ from the data list $A$ via the Cramer-Rao bound [18]. As we will discuss further
below, the FIM is also related to the performance of an ideal observer using the data list $A$ for the task of detecting a change in the parameter vector from $\theta$ to $\theta + \Delta \theta$ [19,20].

3 Binned Fisher information

List mode data can also be described as a Poisson Point Process [21] on attribute space via the generalized function $g(a)$ given by

$$g(a) = \sum_{n=1}^{N} \delta(a - a_n).$$

If we introduce binning functions $b_m(a)$ for $m = 1, \ldots, M$, then we get the components

$$g_m = \sum_{n=1}^{N} b_m(a_n) = \int g(a) b_m(a) d^q a$$

of a binned $M$-dimensional data vector $g$. We will assume that the functions $b_m(a)$ are binary with non-overlapping supports, so that $b_m(a) b_{m'}(a) = \delta_{mm'} b_m(a)$, and that they cover all of attribute space, i.e. for all $a \in A$ we have $b_1(a) + \ldots + b_M(a) = 1$. The PDF for the binned data vector is multivariate Poisson:

$$p_{r|g}(\theta) = \prod_{m=1}^{M} \frac{[\bar{g}_m(\theta)]^{g_m}}{g_m!} \exp \left[ -\bar{g}_m(\theta) \right]$$

with $\bar{g}_m(\theta) = \langle g_m \rangle_{A|\theta}$. The binned FIM is defined by

$$F_B(\theta) = \left\langle \left[ \nabla_{\theta} \ln p_{r|g}(\theta) \right] \left[ \nabla_{\theta} \ln p_{r|g}(\theta) \right]^\dagger \right\rangle$$

This matrix is relevant to the task of estimating $\theta$ from the data vector $g$ via the corresponding Cramer-Rao bound. As above, this FIM is also related to the performance of an ideal observer using the data vector $g$ for the task of detecting a change in the parameter vector from $\theta$ to $\theta + \Delta \theta$. Intuitively we expect better performance on the estimation task or the detection task with the list mode data than with the binned data, since there is an obvious loss of information about each photon in the transition from list mode to binned data. In the following we will show that this is true and derive an equation that quantifies this loss of information using the FIM matrices for the two data types.

4 Relation between the two FIMs

The attribute-space PDF can be written in terms of the conditional mean of the Poisson Point Process $\bar{g}(a|\theta) = \langle g(a) \rangle_{A|\theta}$ via the equation $\bar{N}(\theta) p_{r|g}(a|\theta) = \bar{g}(a|\theta)$. 


\( \bar{g}(a|\theta) \), where

\[
\bar{N}(\theta) = \int_A \bar{g}(a|\theta) \, d^3a = \sum_{m=1}^{M} \bar{g}_m(\theta). \tag{8}
\]

Now we may write the list mode FIM as

\[
F_{LM}(\theta) = \bar{N}(\theta) \left\langle \left[ \nabla_{\theta} \ln \bar{g}(a|\theta) \right] [\nabla_{\theta} \ln \bar{g}(a|\theta)]^\dagger \right\rangle_{a|\theta}, \tag{9}
\]

and this is the same as the integral expression

\[
F_{LM}(\theta) = \int_A \left[ \nabla_{\theta} \ln \bar{g}(a|\theta) \right] [\nabla_{\theta} \ln \bar{g}(a|\theta)]^\dagger \bar{g}(a|\theta) \, d^3a. \tag{10}
\]

Thus the list mode FIM is determined entirely by the conditional mean function \( \bar{g}(a|\theta) \).

Meanwhile, we have the relation between conditional means

\[
\bar{g}_m(\theta) = \int_A b_m(a) \bar{g}(a|\theta) \, d^3a \tag{11}
\]

and we can define a finite conditional probability distribution \( Pr(m|\theta) \) on \( \{1, \ldots, M\} \) via \( \bar{N}(\theta) Pr(m|\theta) = \bar{g}_m(\theta) \). Now the binned FIM is given by an expectation with respect to this finite probability distribution

\[
F_B(\theta) = \bar{N}(\theta) \left\langle \left[ \nabla_{\theta} \ln \bar{g}_m(\theta) \right] [\nabla_{\theta} \ln \bar{g}_m(\theta)]^\dagger \right\rangle_{m|\theta}. \tag{12}
\]

Notice the similarity with the corresponding expectation expression for the list mode FIM. The only difference is that a PDF for the attribute vector \( a \) has been replace by a finite probability distribution for the bin index \( m \). The binned FIM can also be written as

\[
F_B(\theta) = \sum_{m=1}^{M} \left[ \nabla_{\theta} \ln \bar{g}_m(\theta) \right] [\nabla_{\theta} \ln \bar{g}_m(\theta)]^\dagger \bar{g}_m(\theta) \tag{13}
\]

Thus, from one viewpoint, we get the binned FIM by using the bin functions \( b_m(a) \) to numerically perform the integration in the list mode FIM. It is not obvious at this point that this numerical procedure will always produce a lower value for the FIM.

The ideal observer detectability \( d(\theta_0, \theta_1) \) for the task of detecting a change in the parameter vector from \( \theta_0 \) to \( \theta_1 \) is defined by

\[
AUC(\theta_0, \theta_1) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{1}{2} d(\theta_0, \theta_1) \right), \tag{14}
\]

where \( AUC(\theta_0, \theta_1) \) is the area under the ROC curve for the ideal observer. It has been shown that, to lowest order, \( d^2(\theta, \theta + \Delta \theta) = \Delta \theta^\dagger F(\theta) \Delta \theta + \ldots \), where \( F(\theta) \) is the FIM for the conditional PDF of the data. Thus the scalar

\[
\Delta \theta^\dagger F_{LM}(\theta) \Delta \theta = \int_A \left[ \Delta \theta^\dagger \nabla_{\theta} \bar{g}(a|\theta) \right] \left[ \bar{g}(a|\theta) \right]^{-1} d^3a \tag{15}
\]
gives the square of the approximate ideal-observer detectability for this task when we use list mode data. Similarly, the scalar

$$
\Delta \theta^\dagger F_B (\theta) \Delta \theta = \sum_{m=1}^{M} \left[ \int \bar{\Delta} \nabla_\theta \bar{g} (a|\theta) d^q a \right]^2 [\bar{g}_m (\theta)]^{-1}
$$

(16)
gives the square of the approximate ideal-observer detectability for a small change in the parameter vector from \( \theta \) to \( \theta + \Delta \theta \) if we are using binned data. We will show that

$$
\Delta \theta^\dagger F_{LM} (\theta) \Delta \theta \geq \Delta \theta^\dagger F_B (\theta) \Delta \theta
$$

for all \( \theta \) and \( \Delta \theta \). By definition, this then implies that \( F_{LM} (\theta) \geq F_B (\theta) \) as matrices for all \( \theta \).

To simplify the calculations we will define \( \gamma (a) = \Delta \theta^\dagger \nabla_\theta \bar{g} (a|\theta) \) and suppress the fact that this function also depends on \( \theta \) and \( \Delta \theta \), since these vectors are fixed for the purposes of this computation. Then we have

$$
\Delta \theta^\dagger F_{LM} (\theta) \Delta \theta = \int \hat{A} [\gamma (a)]^2 [\bar{g} (a|\theta)]^{-1} d^q a
$$

(17)

This expression suggests that, for fixed \( \theta \), we define a weighted Hilbert space inner product for functions on attribute space via

$$
(\gamma, \gamma')_\theta = \int \gamma^* (a) \gamma' (a) [\bar{g} (a|\theta)]^{-1} d^q a = (\gamma, D^{-1} \theta \gamma')
$$

(18)

where \( D^{-1} \theta \gamma' (a) = \gamma' (a) [\bar{g} (a|\theta)]^{-1} \). The list-mode approximate detectability is then given by the corresponding weighted Hilbert-space norm as \( \Delta \theta^\dagger F_{LM} (\theta) \Delta \theta = ||\gamma||^2_\theta \).

For the binned data we have the summation

$$
\Delta \theta^\dagger F_B (\theta) \Delta \theta = \sum_{m=1}^{M} \left[ \int \bar{\Delta} \nabla_\theta \bar{g}_m (a|\theta) d^q a \right]^2 [\bar{g}_m (\theta)]^{-1}.
$$

(19)

We define the binning operator \( \mathcal{B} \) by

$$
(\mathcal{B} \gamma)_m = \int \hat{A} \gamma (a) d^q a
$$

(20)

and the ordinary Hilbert space adjoint of this operator by

$$
\mathcal{B}^\dagger g = \sum_{m=1}^{M} g_m \bar{b}_m (a).
$$

(21)

Then we have a simpler looking expression

$$
\Delta \theta^\dagger F_B (\theta) \Delta \theta = \sum_{m=1}^{M} (\mathcal{B} \gamma)_m^2 [\bar{g}_m (\theta)]^{-1}.
$$

(22)
This expression suggests introducing a weighted inner product in the $M$-dimensional data space by

\[
(g, g')_\theta = \sum_{m=1}^{M} g_m g'_m [\gamma_m(\theta)]^{-1} = (g, D^{-1}_\theta g')
\]

(23)

where $D^{-1}_\theta$ is a diagonal $M \times M$ matrix with the numbers $[\gamma_m(\theta)]^{-1}$ along the diagonal. With this notation the binned approximate detectability is given by the weighted norm $\Delta \theta^1 F_B(\theta) \Delta \theta = \|B\gamma\|^2_\theta$. Thus both $\Delta \theta^1 F_{LM}(\theta) \Delta \theta$ and $\Delta \theta^1 F_B(\theta) \Delta \theta$ are now expressed as weighted Hilbert space norms of the function $\gamma$ and the vector $B\gamma$, respectively.

We can now think of the binning operator as a map between two weighted Hilbert spaces: $B : L^2_\theta(\mathbb{A}) \rightarrow \mathbb{R}^M_\theta$. As a first step we want to find the pseudoinverse of this operator. We begin by finding the adjoint of this operator. Note that this is not the "ordinary adjoint" described above. The relevant calculation for this adjoint is given by

\[
(g, B\gamma')_\theta = (\gamma, D^{-1}_B B \gamma') = (D_B^{-1} D^{-1}_\theta g, \gamma') = (D_B^{-1} D^{-1}_\theta g, D^{-1}_\theta \gamma') = (D_B^{-1} D^{-1}_\theta g, \gamma')_\theta
\]

(24)

Thus $D_B^{-1} D^{-1}_\theta$ is the adjoint operator we are looking for. The pseudoinverse of $B$, as an operator between the weighted Hilbert spaces, is then given by

\[
B^+ = D_B^{-1} - D^{-1}_\theta (BD_B^{-1} D_B^{-1})^{-1} = D_B^{-1} (BD_B^{-1})^{-1}
\]

(25)

If we look at this expression in detail we first note that

\[
D_B^{-1} D_B g(a) = D_\theta \left\{ \sum_{m=1}^{M} g_m b_{m'}(a) \right\} = \tilde{g}(a|\theta) \sum_{m'=1}^{M} g_m b_{m'}(a)
\]

(26)

Now implementing the binning operator, and using the properties of the binning functions, gives us, in component form,

\[
(BD_B g)_m = g_m \int_{\mathbb{A}} \tilde{g}(a|\theta) b_m(a) d\theta = g_m \tilde{g}_m(\theta).
\]

(27)

Therefore we find that $BD_B g = D_\theta$. Now we have a simplified version of the needed pseudoinverse: $B^+ = D_B^{-1} D_\theta^{-1}$.

We may decompose the function $\gamma$ into two components $\gamma = \gamma_1 + \gamma_0$, where $\gamma_0$ is a null function with respect to the binning operator, i.e. $B\gamma_0 = 0$, and we have the orthogonality condition $\langle \gamma_1, \gamma_0 \rangle_\theta = 0$. The component $\gamma_1$ is given by $\gamma_1 = B^+ B \gamma$. Therefore we have $\gamma_1 = D_B^{-1} D_\theta^{-1} B \gamma$. Writing this equation out in detail we have

\[
\gamma_1(a) = B^+ B \gamma(a) = \tilde{g}(a|\theta) \sum_{m=1}^{M} \left[ \frac{g_m(\theta)}{\tilde{g}_m(\theta)} \right] b_m(a).
\]

(28)

The null component of $\gamma$ is then defined by $\gamma_0(a) = \gamma(a) - \gamma_1(a)$, and due to the orthogonality condition we have $\|\gamma\|^2_\theta = \|\gamma_1\|^2_\theta + \|\gamma_0\|^2_\theta$. 


Now we examine the square magnitude, in the weighted Hilbert space, of the $\gamma_1$ component of $\gamma$:

$$\|\gamma_1\|^2 = \int_{A} [\gamma_1(a)]^2 \left[ \tilde{g}(a|\theta) \right]^{-1} d^3a. \quad (29)$$

Substituting our expression for $\gamma_1(a)$ and then using the properties of the binning functions, we find that

$$\|\gamma_1\|^2 = \sum_{m=1}^{M} \left[ \frac{(B\gamma)_m}{\bar{g}_m(\theta)} \right]^2 \int_{A} \tilde{g}(a|\theta) b_m(a) d^3a. \quad (30)$$

After performing the integration we find that $\|\gamma_1\|^2 = \Delta \theta^1 F_B(\theta) \Delta \theta$.

Now we can find the difference between the list-mode and binned approximate detectabilities

$$\Delta \theta^1 F_{LM}(\theta) \Delta \theta - \Delta \theta^1 F_B(\theta) \Delta \theta = \|\gamma_0\|^2. \quad (31)$$

Using the definition of $\gamma(a)$ we have the final result

$$\Delta \theta^1 F_{LM}(\theta) \Delta \theta - \Delta \theta^1 F_B(\theta) \Delta \theta = \int_{A} \left\{ \left[ \Delta \theta^1 \nabla_{\theta} \tilde{g}(a|\theta) \right]_0 \right\}^2 \left[ \tilde{g}(a|\theta) \right]^{-1} d^3a. \quad (32)$$

Since $\Delta \theta$ is arbitrary, this equation gives us a matrix inequality between FIMs $F_{LM}(\theta) \geq F_B(\theta)$ with equality only if $\left[ \Delta \theta^1 \nabla_{\theta} \tilde{g}(a|\theta) \right]_0 = 0$. The equality condition can also be written as

$$\gamma(a) = \tilde{g}(a|\theta) \sum_{m=1}^{M} \left[ \frac{(B\gamma)_m}{\bar{g}_m(\theta)} \right] b_m(a) \quad (33)$$

where $\gamma(a) = \Delta \theta^1 \nabla_{\theta} \tilde{g}(a|\theta)$ and

$$(B\gamma)_m = \int_{A} b_m(a) \Delta \theta^1 \nabla_{\theta} \tilde{g}(a|\theta) d^3a. \quad (34)$$

The probability is zero that this condition will be satisfied in any real imaging situation, which means that binning always results in a loss of Fisher information.

Note that the condition for no loss of Fisher information due to binning can be written as

$$\gamma_0(a) = \tilde{g}(a|\theta) \sum_{m=1}^{M} \left[ \frac{\gamma(a)}{\tilde{g}(a|\theta)} - \frac{(B\gamma)_m}{\bar{g}_m(\theta)} \right] b_m(a) = 0. \quad (35)$$

This then gives us

$$\Delta \theta^1 F_{LM}(\theta) \Delta \theta - \Delta \theta^1 F_B(\theta) \Delta \theta = \sum_{m=1}^{M} \int_{A} \left[ \frac{\gamma(a)}{\tilde{g}(a|\theta)} - \frac{(B\gamma)_m}{\bar{g}_m(\theta)} \right]^2 \tilde{g}(a|\theta) b_m(a) d^3a. \quad (36)$$
Thus each bin contributes an amount to the loss of detectability according to three factors. The first factor is the deviation of the quantity in curly brackets from zero within that bin. The second factor is the value of the mean data function \( \bar{g}(a|\theta) \) within the bin. The third factor is the size of the bin itself. Therefore the efficiency of any particular choice of bins in preserving Fisher information depends on the actual parameter value \( \theta \) as well as the bin sizes.

Having derived this relationship it is actually straightforward to prove that it is valid without any discussion of weighted Hilbert spaces. However, the path we followed to get here demonstrates that the loss of Fisher information due to binning is caused by the null space of the binning operator \( B : L^2_{\theta}(A) \rightarrow R_M^d \), when viewed as an operator between weighted Hilbert spaces.

## 5 FIMs for object Reconstruction

In this section the parameter vector \( \theta \) is replaced with a function \( f(r) \) of spatial coordinates. This complication is mitigated by a linear relation between the object function and mean data function via a linear operator:

\[
\bar{g}(a|f) = Lf(a) = \int_S L(a, r) f(r) d^q r,
\]

where \( S \) is a support region for object functions in a \( q \)-dimensional space. The gradient operator \( \nabla_\theta \) is replaced by a functional derivative or Frechet derivative. The FIM matrices are now a Fisher information operators \( F_{LM} \) and \( F_B \). The simplicity of the connection between \( f(r) \) and \( \bar{g}(a|f) \) makes the functional derivative easy to compute.

The end result for the detectability calculation with list-mode data is then given by

\[
(\Delta f, F_{LM}(f) \Delta f) = \int_A [L \Delta f(a)]^2 [L f(a)]^{-1} d^q a.
\]

(38)

The weighted inner product for functions on attribute space is now defined by

\[
(g, g')_f = (g, D_f^{-1} g') = \int_A g^*(a) g'(a) [L f(a)]^{-1} d^q a
\]

(39)

With the resulting weighted Hilbert space norm we then have \((\Delta f, F_{LM}(f) \Delta f) = \|L \Delta f\|^2_f\).

The imaging operator for the binned imaging system is given by the concatenation of the list-mode system operator with the binning operator \( \mathcal{H} = \mathcal{B} \mathcal{L} \). The detectability calculation for the binned system then gives us

\[
(\Delta f, F_B(f) \Delta f) = \sum_{m=1}^{M} [(\mathcal{H} \Delta f)]^2_m [\mathcal{H} f]_m^{-1}.
\]

(40)

As before we introduce a weighted inner product in data space via

\[
(g, g')_f = (g, D_f^{-1} g') = \sum_{m=1}^{M} g^*_m g'_m [\mathcal{H} f]_m^{-1},
\]

(41)
and we then have \((\triangle f, \mathcal{F}_B (f) \triangle f) = \|\mathcal{H} \triangle f\|^2_f\).

The relevant operators are now the list mode system operator \(\mathcal{L} : L^2 (\mathbb{S}) \rightarrow L^2 (\mathbb{S})\), the binning operator \(\mathcal{B} : L^2 (\mathbb{S}) \rightarrow \mathbb{R}^M\), and their concatenation into the binned system operator \(\mathcal{H} : L^2 (\mathbb{S}) \rightarrow \mathbb{R}^M\). We have the decomposition in \(L^2 (\mathbb{S})\) of the function \(\mathcal{L} \triangle f\) as \(\mathcal{L} \triangle f = (\mathcal{L} \triangle f)_1 + (\mathcal{L} \triangle f)_0\), where \(\mathcal{B} (\mathcal{L} \triangle f)_0 = 0\) and \(((\mathcal{L} \triangle f)_1, (\mathcal{L} \triangle f)_0)_f = 0\).

As before we find the adjoint of the binning operator, as an operator between weight Hilbert spaces, via

\[
(g, B g')_f = (g, D_f^{-1} B g') = (B^\dagger D_f^{-1} g, g') = (D_f B^\dagger D_f^{-1} g, g') = (D_f B^\dagger f, g')_f.
\]

We then have the pseudoinverse of the binning operator \(B^+ = D_f B^\dagger D_f^{-1} (B D_f B^\dagger)^{-1}\), which simplifies to \(B^+ = D_f B^\dagger (B D_f B^\dagger)^{-1}\). Computing the operator in parentheses in this last expression leads to

\[
BD_f B^\dagger g = BD_f \sum_{m=1}^M g_m b_m (a) = BL f (a) \sum_{m=1}^M g_m b_m (a).
\]

Examining this equation componentwise then gives us

\[
(BD_f B^\dagger g)_m = g_m \int_B \mathcal{L} f (a) \ b_m (a) \ d^3 a = g_m (\mathcal{H} f)_m.
\]

Therefore we have \(BD_f B^\dagger = D_f\) and the needed pseudoinverse is given by \(B^+ = D_f B^\dagger D_f^{-1}\).

Now we have for the first term in the orthogonal decomposition \((\mathcal{L} \triangle f)_1 = B^+ B \mathcal{L} \triangle f\). If we write this equation out explicitly it becomes

\[
(\mathcal{L} \triangle f)_1 (a) = \mathcal{L} f (a) \sum_{m=1}^M \left[ \frac{(\mathcal{H} \triangle f)_m}{(\mathcal{H} f)_m} \right] b_m (a).
\]

Then the null component of \(\mathcal{L} \triangle f\) with respect to the binning operator in the weighted Hilbert space is \( (\mathcal{L} \triangle f)_0 (a) = \mathcal{L} f (a) - (\mathcal{L} \triangle f)_1 (a)\). Using the orthogonality of the decomposition we have \(\|\mathcal{L} \triangle f\|^2_f = \|(\mathcal{L} \triangle f)_1\|^2_f + \|(\mathcal{L} \triangle f)_0\|^2_f\).

The first term in the sum on the right is

\[
\|(\mathcal{L} \triangle f)_1\|^2_f = \int_B \{[\mathcal{L} f (a)]_1 \}^2 [\mathcal{L} f (a)]^{-1} d^3 a
\]

Using the properties of the bin functions we then have

\[
\|(\mathcal{L} \triangle f)_1\|^2_f = \sum_{m=1}^M \left[ \frac{(\mathcal{H} \triangle f)_m}{(\mathcal{H} f)_m} \right] \int_B \mathcal{L} f (a) b_m (a) d^3 a
\]

Thus we have \(\|(\mathcal{L} \triangle f)_1\|^2_f = (\triangle f, \mathcal{F}_B (f) \triangle f)\).
Now we see that the null component \((\mathcal{L}\triangle f)_0\) determines the loss of Fisher information: 
\[
(\triangle f, \mathcal{F}_{LM}(f) \triangle f) - (\triangle f, \mathcal{F}_B(f) \triangle f) = \|\mathcal{L}\triangle f\|_f^2.
\]
Alternatively we can write
\[
(\triangle f, \mathcal{F}_{LM}(f) \triangle f) - (\triangle f, \mathcal{F}_B(f) \triangle f) = \int_{\mathcal{A}} \left\{ \left[ \mathcal{L}\triangle f(a) \right]_0 \right\}^2 [\mathcal{L}f(a)]^{-1} d^q a. \quad (48)
\]
The two approximate detectabilities are equal only if
\[
\mathcal{L}\triangle f(a) = \mathcal{L}f(a) \sum_{m=1}^{M} \left[ \frac{[\mathcal{H}\triangle f]_m}{[\mathcal{H}f]_m} \right] b_m(a) \quad (49)
\]
This condition implies that for almost all perturbation functions \(\triangle f(a)\) the list-mode approximate detectability will be greater than the binned approximate detectability.

Note that the condition for no loss of information due to binning can also be written as
\[
(\mathcal{L}\triangle f)_0(a) = \mathcal{L}f(a) \sum_{m=1}^{M} \left[ \frac{\mathcal{L}\triangle f(a)}{\mathcal{L}f(a)} - \frac{[\mathcal{H}\triangle f]_m}{[\mathcal{H}f]_m} \right] b_m(a) = 0. \quad (50)
\]
This then gives us
\[
(\triangle f, \mathcal{F}_{LM}(f) \triangle f) - (\triangle f, \mathcal{F}_B(f) \triangle f) = \sum_{m=1}^{M} \int_{\mathcal{A}} \left[ \frac{\mathcal{L}\triangle f(a)}{\mathcal{L}f(a)} - \frac{[\mathcal{H}\triangle f]_m}{[\mathcal{H}f]_m} \right]^2 \mathcal{L}f(a) b_m(a) d^q a. \quad (51)
\]
Thus, as in the case described above for a finite dimensional parameter, each bin contributes to the loss of the detectability of a change in the object function according to three factors. The first factor is again the deviation of the quantity in curly brackets from zero within that bin. The second factor is the value of the function \(\mathcal{L}f(a)\) within the bin. The third factor is the size of the bin itself. The efficiency of any particular choice of bins in preserving Fisher information depends on the actual object function \(f\) as well as bin size.

Finally, note that, as in the previous section, this last equality can be proved directly. Again, the path followed in this derivation shows that the loss of Fisher information about the object function due to binning comes from the null space of \(\mathcal{B} : \mathcal{L}_f^2(\mathcal{A}) \rightarrow \mathbb{R}^M_f\) as an operator between weighted Hilbert spaces.

6 Example

For this example, consider the attribute space to be a symmetric interval on the real line: \(\mathcal{A} = [-L/2, L/2]\). The object functions will be square integrable functions of a real variable and the list-mode system operator is convolution with a point spread function (PSF): \(\mathcal{L}f(x) = p * f(x)\). We assume that the point spread function is band limited to the band \([-B/2, B/2]\).
Now let $M$ and $\triangle x$ be such that $L = M \triangle x$ and define the regularly spaced points in $\mathcal{A}$ via
\[ x_m = -\frac{L}{2} + \left( m - \frac{1}{2} \right) \triangle x \tag{52} \]
and the bin functions as
\[ b_m(x) = \text{rect}\left( \frac{x - x_m}{\triangle x} \right). \tag{53} \]

We now have the binning operator described by
\[ (B g)_m = \int g(x) \text{rect}\left( \frac{x - x_m}{\triangle x} \right) dx = \int_{x_m - \frac{\triangle x}{2}}^{x_m + \frac{\triangle x}{2}} g(x) dx. \tag{54} \]

The condition for no loss in the approximate detectability by binning is given by
\[ p \ast \triangle f(x) = p \ast f(x) \sum_{m=1}^{M} \left[ \frac{(H \triangle f)_m}{(H f)_m} \right] \text{rect}\left( \frac{x - x_m}{\triangle x} \right) \tag{55} \]

This condition is impossible to satisfy since the function on the left is band-limited and the function on the right, in general, is not. Thus, even with Nyquist sampling, when $B \triangle x = 1$, there is a loss in the detectability of a small change in the object function when we bin the list-mode data. The actual loss of Fisher information for a small change in the object function is given by
\[ (\triangle f, \mathcal{F}_{LM}(f) \triangle f) - (\triangle f, \mathcal{F}_B(f) \triangle f) = \sum_{m=1}^{M} \int_{x_m - \frac{\triangle x}{2}}^{x_m + \frac{\triangle x}{2}} \left\{ \frac{p \ast \triangle f(x)}{p \ast f(x)} - \frac{1}{2} \left[ \frac{(H \triangle f)_m}{(H f)_m} \right] \right\}^2 p \ast f(x) dx. \tag{56} \]

In general, loss of Fisher information is mitigated if $B$ is decreased since this will mean that $p \ast \triangle f(x)$ and $p \ast f(x)$ are smoother functions, and hence there will be a decrease the quantities in the curly brackets.

There is at least one circumstance in this example where there is no loss of Fisher information from binning the list-mode data. If $\triangle f(x) = \alpha f(x)$ for some constant $\alpha$, then $(\triangle f, \mathcal{F}_{LM}(f) \triangle f) - (\triangle f, \mathcal{F}_B(f) \triangle f) = 0$. This is true even if $M = 1$ and $\triangle x = L$. In other words, to detect a simple change in amplitude of the object function we might as well use one bin covering all of $\mathcal{A}$. There may also be other special situations where binning does not create a loss of Fisher information, but for generic functions $f(x)$ and $\triangle f(x)$ there will always be a loss.

7 Conclusion

We have shown that there is almost always a loss of Fisher information for any estimation task when list-mode data is binned. This loss of information is due to the null space of the binning operator when it is viewed as an operator.
between certain parameter dependent weighted Hilbert spaces. The magnitude of the loss can be quantified by finding the null component, with respect to the binning operator, of a directional derivative of the conditional PDF as an element of one of the weighted Hilbert spaces. We found that the information loss depends on the smoothness of the mean data function for the list-mode data, the actual object being imaged, and the the binning scheme in relation to the other two factors. We have shown that these conclusions apply even when the estimation problem is an object reconstruction problem, where the finite dimensional parameter vector is replaced with a function in an infinite dimensional Hilbert space.

As a final note the difference $\Delta^\dagger \mathbf{F}_{LM}(\theta) \Delta^\theta - \Delta^\dagger \mathbf{F}_B(\theta) \Delta^\theta$ can be written as

$$
\sum_{m=1}^{M} \int_{\mathcal{H}} \Delta^\theta \left\{ \frac{[\nabla \theta \bar{g}(\mathbf{a}|\theta)] [\nabla \theta \bar{g}(\mathbf{a}|\theta)]^\dagger}{\bar{g}(\mathbf{a}|\theta)} - \frac{[\nabla \theta \bar{g}_m(\theta)] [\nabla \theta \bar{g}_m(\theta)]^\dagger}{\bar{g}_m(\theta)} \right\} \bar{g}(\mathbf{a}|\theta) b_m(\mathbf{a}) d^q a.
$$

(57)

Therefore we have an expression for the difference $\mathbf{F}_{LM}(\theta) - \mathbf{F}_B(\theta)$ of FIMs:

$$
\sum_{m=1}^{M} \int_{\mathcal{H}} \left\{ \frac{[\nabla \theta \bar{g}(\mathbf{a}|\theta)] [\nabla \theta \bar{g}(\mathbf{a}|\theta)]^\dagger}{\bar{g}(\mathbf{a}|\theta)} - \frac{[\nabla \theta \bar{g}_m(\theta)] [\nabla \theta \bar{g}_m(\theta)]^\dagger}{\bar{g}_m(\theta)} \right\} \bar{g}(\mathbf{a}|\theta) b_m(\mathbf{a}) d^q a.
$$

(58)

Now if we have a nominal value for $\theta$, but there is some uncertainty in this value, then this is equivalent to making $\Delta^\theta$ a random vector with zero mean. If the covariance matrix for this vector is $\mathbf{K}_\theta$ then the average value for $\Delta^\dagger \mathbf{F}_{LM}(\theta) \Delta^\theta - \Delta^\dagger \mathbf{F}_B(\theta) \Delta^\theta$ is $\text{tr} \{ \mathbf{K}_\theta [\mathbf{F}_{LM}(\theta) - \mathbf{F}_B(\theta)] \}$. This may be a useful quantification of the average loss of Fisher information due to binning in this situation.

When $\mathbf{K}_\theta = \sigma^2 \mathbf{I}$ we end up with

$$
\text{tr} \{ \mathbf{K}_\theta [\mathbf{F}_{LM}(\theta) - \mathbf{F}_B(\theta)] \} = \sigma^2 \sum_{m=1}^{M} \int_{\mathcal{H}} \left[ \frac{[\nabla \theta \bar{g}(\mathbf{a}|\theta)]^2}{\bar{g}(\mathbf{a}|\theta)} - \frac{[\nabla \theta \bar{g}_m(\theta)]^2}{\bar{g}_m(\theta)} \right] \bar{g}(\mathbf{a}|\theta) b_m(\mathbf{a}) d^q a.
$$

This is a relatively compact expression that can be easily evaluated in many cases.

References

[1] L. Caucci and H. H. Barrett, “Objective assessment of image quality. V. Photon counting detectors and list-mode data,” JOSA A 29, 1003-1016 (2012).

[2] H. H. Barrett, T. White and L. C. Parra, “List-mode likelihood,” JOSA A 14, 2914-2923 (1997).

[3] L. Parra and H. H. Barrett, “List-mode likelihood: EM algorithm and image quality estimation demonstrated on 2-D PET,” IEEE Trans. Med. Imag. 17, 228–235 (1998).
[4] P. C. Johns, J. Dubeau, D. G. Gobbi, M. Li, and M. S. Dixit, “Photon-counting detectors for digital radiography and X-ray computed tomography,” in “Opto-Canada: SPIE Regional Meeting on Optoelectronics, Photonics, and Imaging,” (Proc. SPIE TD01) 367–369 (2002).

[5] P. M. Shikhaliiev, T. Xu, and S. Molloi, “Photon counting computed tomography: Concept and initial results,” Med. Phys. 32, 427–436 (2005).

[6] A. J. Reader, S. Ally, F. Bakatselos, R. Manavaki, R. J. Walledge, A. P. Jeavons, P. J. Julyan, S. Zhao, D. L. Hastings, and J. Zweit, “One-pass list-mode EM algorithm for high-resolution 3-D PET image reconstruction into large arrays,” IEEE Trans. Nucl. Sci. 49, 693–699 (2002).

[7] P. Khurd, I.-T. Hsiao, A. Rangarajan, and G. Gindi, “A globally convergent regularized ordered-subset EM algorithm for list-mode reconstruction,” IEEE Trans. Nucl. Sci. 51, 719–725 (2004).

[8] A. J. Reader, K. Erlandsson, R. J. Ott, and M. A. Flower, “Attenuation and scatter correction of list-mode data driven iterative and analytic image reconstruction algorithms for rotating 3D PET systems,” IEEE Trans. Nucl. Sci. 46, 2218–2226 (1999).

[9] D. L. Snyder and D. G. Politte, “Image reconstruction from list-mode data in an emission tomography system having time-of-flight measurements,” IEEE Trans. Nucl. Sci. 30, 1843–1849 (1983).

[10] C. Byrne, “Likelihood maximization for list-mode emission tomographic image reconstruction,” IEEE Trans. Med. Imag. 20, 1084–1092 (2001).

[11] R. H. Huesman, G. J. Klein, W. W. Moses, J. Qi, B. W. Reutter, and P. R. G. Virador, “List-mode maximum-likelihood reconstruction applied to positron emission mammography (PEM) with irregular sampling,” IEEE Trans. Med. Imag. 19, 532–537 (2000).

[12] R. Levkovitz, D. Falikman, M. Zibulevsky, A. Ben-Tal, and A. Nemirovski, “The design and implementation of COSEN, an iterative algorithm for fully 3-D listmode data,” IEEE Trans. Med. Imag. 20, 633–642 (2001).

[13] Joint reconstruction of activity and attenuation map using LM SPECT emission data, A. K.Jha, E. Clarkson, M. A. Kupinski, H. H. Barrett, Proceedings SPIE 8668, (2013).

[14] Caucci L, Jha AK, Furenlid LR, Clarkson EW, Kupinski MA, Barrett HH, “Image Science with Photon-Processing Detectors,” IEEE Nuclear Science Symposium conference record (2013).

[15] Luca Caucci and Yijun Ding and Harrison Barrett, “Computational Methods for Photon-Counting and Photon-Processing Detectors,” Chapter 5 in Photon Counting, InTech, (Rijeka, Britun, Nikolay, eds.) (2018).
[16] W. C. J. Hunter, H. H. Barrett, and L. R. Furenlid, “Calibration method for ML estimation of 3D interaction position in a thick gamma-ray detector,” IEEE Trans. Nucl. Sci. 56, 189–196 (2009).

[17] J. Y. Hesterman, L. Caucci, M. A. Kupinski, H. H. Barrett, and L. R. Furenlid, “Maximum-likelihood estimation with a contracting-grid search algorithm,” IEEE Trans. Nucl. Sci. 57, 1077–1084 (2010).

[18] J. Shao, *Mathematical Statistics*, Springer, New York (1999).

[19] E. Clarkson and F. Shen, “Fisher information and surrogate figures of merit for the task-based assessment of image quality,” JOSA A 27, 2313-2326 (2010).

F. Shen and E. Clarkson, “Using Fisher information to approximate ideal observer performance on detection tasks for lumpy-background images,” JOSA A 23, 2406-2414 (2006).

[20] H. H. Barrett and K. J. Myers, *Foundations of Image Science*, John Wiley & Sons, Hoboken, NJ (2004).