Scaling Exponents in the Incommensurate Phase of the Sine-Gordon and U(1) Thirring Models

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In this paper we study the critical exponents of the quantum sine-Gordon and U(1) Thirring models in the incommensurate phase. This phase appears when the chemical potential $h$ exceeds a critical value and is characterized by a finite density of solitons. The low-energy sector of this phase is critical and is described by the Gaussian model (Tomonaga-Luttinger liquid) with the compactification radius dependent on the soliton density and the sine-Gordon model coupling constant $\beta$.

For a fixed value of $\beta$, we find that the Luttinger parameter $K$ is equal to 1/2 at the incommensurate-incommensurate transition point and approaches the asymptotic value $\beta^2/8\pi$ away from it. We describe a possible phase diagram of the model consisting of an array of weakly coupled chains. The possible phases are Fermi liquid, Spin Density Wave, Spin-Peierls and Wigner crystal.

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In this paper we study the critical exponents of the quantum sine-Gordon (SG) and the U(1) Thirring models in the presence of soliton condensate. The Lagrangian of SG model is given by

$$L = \int dx \left\{ \frac{1}{2} \left[ (\partial_\tau \Phi)^2 - (\partial_x \Phi)^2 \right] - h\beta/\pi \partial_x \Phi + \frac{\mu^2}{\beta^2} (\cos \beta \Phi - 1) \right\},$$

where $\mu$ and $\beta$ are real parameters, mass-like and coupling constant respectively and $h$ is chemical potential.

SG model is closely related to the U(1) Thirring model (it describes its (iso)spin sector). The Lagrangian density for the latter model is given by

$$\mathcal{L} = i \Psi_\alpha \gamma^\mu \partial_\mu \Psi_\alpha - \frac{1}{4} \sum_{\alpha=1,2,3} g_\alpha j_\alpha^a j^{a\mu},$$

where $j_\alpha^a = (1/2) \bar{\Psi}_\alpha \gamma^\mu \tau_\alpha^a \Psi_\beta$ are the (iso)spin currents.

The relationship between the two models is established by bosonization (this procedure and the relationship between the operators is discussed at length in [1]). The coupling constants $g_\parallel$ and $g_\perp$ are related to $\beta$ and $\mu$.

The Hamiltonian of the U(1) Thirring model, which as a model of two species of fermions possesses central charge $c = 2$, can be written equivalently as a sum of two charge- and spin- Hamiltonians, each having central charge $c = 1$, i.e. $H_c = v_c/2 \int dx \left\{ :JJ : + :\bar{J}J : \right\}$ and

$$H_s = \frac{v_s}{2} \int dx \left\{ :JJ : + :\bar{J}J : + g_\parallel :J_x J_z : + g_\perp :J_y J_y : \right\}.$$

In the above equations, $J$, $\bar{J}$ and $J_\parallel$, $J_\perp$ are chiral charge- and spin-currents given as usual by $J = R_\parallel R_\alpha$ and $\bar{J} = R_\perp (\bar{\sigma}_{\alpha\beta}/2) R_\beta$ for the right currents and similarly for the left ones.

The U(1) Thirring model may be used as a continuous model of interacting lattice fermions. Here one can imagine two situations. First, one may use the (iso)spin part of the Thirring model as a model describing the charge sector of the Mott insulator. In the latter case the currents belong to the charge sector and the spin sector is gapless. It is natural to impose an additional requirement of the SU(2) symmetry in the spin sector, which fixes the corresponding compactification radius $\beta_s = \sqrt{8\pi}$. In the different interpretation the model describes the continuous limit of lattice fermions with a gapless charge sector and an anisotropic interaction in the spin sector. In this case one does not expect any restrictions on the compactification radii.

In both cases one has the following relationship between the order parameter fields and the bosonic fields:

$$\rho_{\text{st}}(x) \sim a_c \sin \left[ \frac{\beta_c \Phi_c(x)}{2} + 2k_F x \right] \cos \frac{\beta_c \Phi_c(x)}{2} + a_s \cos \left[ \frac{\beta_s \Phi_s(x) + 4k_F x}{2} \right],$$

$$n^\pm(x) \sim \exp \left[ \pm i \beta_s \Theta_s(x)/2 \right],$$

$$\Delta_0(x) \sim \sin \frac{\beta_c \Theta_c(x)}{2} \cos \frac{\beta_c \Phi_c(x)}{2},$$

$$\Delta(x) \sim \sin \frac{\beta_c \Theta_c(x)}{2} \cos \frac{\beta_c \Phi_c(x)}{2},$$

with $\beta_c \beta_s = 8\pi$ and $\beta_c \beta_s = 8\pi$.

To be definite, let us stick to the Mott insulator version. In this case the field $\Phi$ of SG model [1] is $\Phi_0$ and the $\Phi_s$ sector is gapless with $\beta_s = \sqrt{8\pi}$ and $\beta_c \equiv \beta$.
When $\beta^2 < 8\pi$ the cosine term in Eq.(1) is relevant and generates a spectral gap in the low-energy sector. However, when the chemical potential $h$ exceeds a certain threshold, the system becomes gapless again.

In this paper we discuss the case $\beta \to 0$. This case is seldom discussed in the literature because such limit cannot be achieved by short-range interactions (recall that for the repulsive Hubbard model $\beta$ varies between $\sqrt{8\pi}$ (weak repulsion) and $\sqrt{4\pi}$ (infinite repulsion). In [3] is found that the Luttinger liquid parameter of the charge sector $K_c$ takes values monotonically increasing in the interval $1/2 < K_c < \zeta^2(B)/2 < 1$ as $u = U/t$ decreases (the asymptotic dependence of the dressed charge $\zeta(B)$ on $u$ is also given in [3]).

One can achieve small $\beta$ considering long-range interactions (such as unscreened Coulomb interaction which can be realized, for instance, in quantum wires).

The limit of small $\beta$ is very curious from the mathematical point of view because solitons in SG model become in this case infinitely heavy. Thus in the absence of solitons (that is in the insulating phase) SG model becomes a model of a free massive bosonic field with mass $\mu$. This field can be thought about as an optical phonon. One may wonder what happens when one forces solitons to appear applying the chemical potential. As we shall demonstrate, the scaling dimensions in the regime of small $\beta$ are determined not by the soliton mass, but by the mass of the optical phonon $\mu$.

Let us describe our results. In the Mott insulator interpretation where the field $h$ truly corresponds to chemical potential we get the following. The gapless (incommensurate or doped) phase of SG model has scaling dimensions dependent on $\beta$ and soliton concentration $n_{sol}$ (which is related to deviation of the electron concentration from half-filling). Looking at scaling dimensions of various operators, one can identify different regions of the phase diagram.

The most singular operators are components of the staggered magnetization $n^z, n^\pm$, the $2k_F$ and the $4k_F$ components of the staggered charge density $\rho_{St}$. The first four operators all have the same scaling dimension

$$ d_n = \frac{1}{2} + \frac{K_c}{2} , $$

where $K_c(n_{sol}, \beta)$ varies from $1/2$ at $n_{sol} \to 0$ to $\beta^2/8\pi$ in the limit of large chemical potential. The $4k_F$ charge density component has scaling dimension

$$ d_{4k_F} = 2K_c . $$

Therefore $d_n, d_{4k_F} < 1$ in the entire area of interest and the corresponding susceptibilities remain singular throughout this area. However, starting from $K_c < 1/3$ (with $d_{4k_F} < d_n$) the $4k_F$-component is more singular. Its correlation function decays more slowly being also oscillatory with wavelength $\lambda = 2\pi/4k_F = a$, the intersite spacing. In this case in the presence of other chains the material will undergo a phase transition to a Wigner crystal ($4k_F$-Charge Density Wave). For $K_c > 1/3$ the material will undergo a phase transition either of Spin Density Wave (SDW) or Spin Peierls (SP) type. The importance of the single-electron interchain tunneling is determined by the magnitude of the scaling dimension of the single fermion creation (annihilation) operator:

$$ d_\psi = \frac{1}{8} \left( \sqrt{K_c} + 1/\sqrt{K_c} \right)^2 . \tag{11} $$

Since the operator describing single-electron interchain tunneling has scaling dimension $2d_\psi$, it becomes relevant at $K_c > \sqrt{2} - 1$.

From the exact solution [4,2] the excitation spectrum is given by

$$ E = \epsilon(\theta) , \quad P = 2\pi \int_0^\theta d\theta' \sigma(\theta') , \tag{12} $$

where $\epsilon(\theta)$ is obtained from the solution of the following Bethe-ansatz integral equation:

$$ \epsilon(\theta) + \int_{-B}^B d\theta' G(\theta - \theta') \epsilon(\theta') = M_s \cosh \theta - h . \tag{13} $$

$M_s$ is the kink’s mass and the Fourier image of the kernel is

$$ G(\omega) = \frac{\sinh \frac{\beta \omega (1-2\tau)}{2(1-\tau)}}{2 \cosh(\frac{\beta \omega}{2}) \sinh \frac{\beta \omega \tau}{2(1-\tau)}} , \quad \tau = \frac{\beta^2}{8\pi} . \tag{14} $$

The soliton’s mass is a function of the coupling constant $\beta$ and at small values of $\beta$ it grows like $1/\tau$, where $\tau$ here and in the following is the short notation for $\beta^2/8\pi$. A plot of $M_s$ versus $\tau$ can be found in [3].

The value of $B$ in Eq. (13) is determined by the condition $\epsilon(\pm B) = 0$, and depends on the coupling constant $\beta$ and the field $h$.

At the point $\tau = 1/2$ ($\beta^2 = 4\pi$) the kernel (4) vanishes and $\epsilon(\theta)$ is given simply by the right-hand side of (13). For other values of $\tau$, Eq. (13) can be solved numerically. A plot of $\epsilon(\theta)$, for a fixed value of the coupling constant $\beta$ and different values of the field $h$, is shown on Fig. 1.

The soliton condensate appears when the field $h$ exceeds the critical value, found from (13) to be $h_c = M_s$. The soliton contribution to the ground state energy of the sine-Gordon model is given by

$$ \mathcal{E}_0 = \frac{M_s}{2\pi} \int_{-B}^B d\theta \cosh \theta \epsilon(\theta) . \tag{15} $$

The average number of solitons $n$ and the susceptibility $\chi$ are given by

$$ n_{sol} = -\frac{\partial \mathcal{E}_0}{\partial h} , \quad \chi = \frac{\partial n_{sol}}{\partial h} . \tag{16} $$
Another way to determine $K$ is to use the identity relating $K$ to the susceptibility $\chi$ and the Fermi velocity $v_F$:

$$K = \frac{1}{2\pi} \chi v_F .$$

We can use the above formula as a check of the result obtained by (21).

First, consider the case $h \to +\infty$ where the limit $B$ goes to infinity. In this limit the integral equation Eq. (13), as $\theta \to B$, can be written in the following form

$$\epsilon(\theta) + \int_{-B}^{B} d\theta' G(\theta - \theta') \epsilon(\theta') = M_\epsilon \frac{\epsilon^2}{2} - h ,$$

and the derivative $\epsilon'(\theta) = d\epsilon(\theta)/d\theta$, taking into account the boundary conditions for $\epsilon(\theta)$, fulfills

$$\epsilon'(\theta) + \int_{-B}^{B} d\theta' G(\theta - \theta') \epsilon'(\theta') = M_\epsilon \frac{\epsilon^2}{2} .$$

As can be seen from Eq (13) and Eq. (25), $\epsilon'(\theta)$ and $2\pi \sigma(\theta)$ fulfill the same integral equation, as $\theta \to +\infty$, meaning that the Fermi velocity in the limit $h \to +\infty$ goes to 1, $v_F \to 1$.

According to (21) the ground state energy has the following $h$-asymptotic behavior:

$$E_0 = -\tau h^2/\pi, \quad n_{sol} = 2\tau h/\pi, \quad \chi = \beta^2/(2\pi)^2,$$

which, for the Luttinger liquid parameter, gives

$$K = \beta^2/8\pi = \tau .$$

On the other hand, we can derive the same result from Eq. (21). In the limit $B \to +\infty$ this equation can be written as a Wiener-Hopf equation:

$$\zeta(\theta) + \int_{-\infty}^{0} d\theta' G(\theta - \theta') \zeta(\theta') = 1 \quad (28)$$

(We have shifted $\theta$ by $B$, so that now we need to calculate $\zeta(0)$). The solution for

$$\zeta^{(-)}(\omega) = \frac{1}{G^{(+)}(0)G^{(+)}(\omega)(i\omega + 0)$$

is

$$\zeta^{(-)}(\omega) = \frac{1}{G^{(+)}(0)G^{(+)}(\omega)(i\omega + 0)} \quad (29)$$

where

$$1 + G(\omega) = G^{(+)}(\omega)G^{(-)}(\omega)$$

with $G^{(+)}(\omega)$ being analytic in the upper (lower) half-plane and $G(\infty) = 1$. The limit at $\theta = 0$ is given by

$$\Delta_{n,m} = \frac{1}{16} \left[ n\zeta(B) \pm \frac{2m}{\zeta(B)} \right]^2 .$$

Finally, we can also calculate the number of solitons using the density distribution function $\sigma(\theta)$:

$$n_{sol} = \int_{-B}^{B} d\theta \sigma(\theta) .$$

with $\sigma(\theta)$ satisfying the following integral equation:

$$\sigma(\theta) + \int_{-B}^{B} d\theta' G(\theta - \theta') \sigma(\theta') = M_\sigma \cosh \theta/2\pi . \quad (18)$$

At low-energies the excitation spectrum can be linearized at the Fermi points $\theta = \pm B$, with the Fermi velocity given by:

$$v_F = \frac{\partial \epsilon(\theta)}{\partial \theta} \bigg|_{\theta = B} \frac{1}{2\pi \sigma(B)} . \quad (19)$$

The boundary $B$ is the same as the one of Eq. (13).

The scaling exponents of the Luttinger liquid are expressed in terms of a single parameter $K$. This parameter can be calculated in two different ways.

First of all, it is related to the dressed charge $\zeta(B)$ which can be found from the solution of the following integral equation (3):

$$\zeta(\theta) + \int_{-B}^{B} d\theta' G(\theta - \theta') \zeta(\theta') = 1 \quad (20)$$

where $G(\theta)$ and $B$ are the same as in Eq. (13). The parameter $K$ is related to $\zeta(B)$ by

$$K = c\zeta(B)^2 \quad (21)$$

up to a constant factor which, as results from (27) and (31), is $c = 1/2$. The scaling dimensions are

$$\Delta_{n,m} = \frac{1}{16} \left[ n\zeta(B) \pm \frac{2m}{\zeta(B)} \right]^2 .$$

FIG. 1. Plot of $-\epsilon(\theta)$ on $\theta$ and $h$ dependence obtained from the numerical solution of Eq. (13), only inside the interval $(-B, B)$, is shown. The coupling constant $\tau$ is taken to have value 0.1.
The value of the coupling constant $\tau$ is kept fixed.

$$\zeta(0) = \lim_{\omega \to -\infty} i\omega \zeta^{(-)}(\omega) = \left[G^{(+)}(0)\right]^{-1}$$

$$= [1 + G(\omega = 0)]^{-1/2} = \sqrt{2\tau}. \quad (30)$$

Thus we get

$$K = \frac{1}{2} \zeta(0)^2 = \tau = \frac{\beta^2}{8\pi}, \quad (31)$$

which is consistent with Eq. (27).

Near the critical line the interval $(-B, B)$ closes to a point and from (24) the dressed charge becomes 1. Otherwise, for all values of $h$ we have solved the integral equation for the dressed charge, Eq. (24), numerically. A plot of $\zeta(\theta)$, only inside the interval $(-B, B)$, is shown on Fig. 2. The value of $\zeta(B)$ decreases (increases) fast near $B = 0$ for $\tau < 1/2$ ($\tau > 1/2$). For larger values of $B$, $\zeta(B)$ reaches quickly values close to its asymptotic value $\sqrt{2\tau}$.

In the fermionic sector corresponding to the sine-Gordon model the chiral components $R_\uparrow, L_\uparrow$ of the fermion operator $\psi_\uparrow$ are given by the expressions

$$R_\uparrow = \exp \left[-i\sqrt{\pi/2}(\zeta + \zeta^{-1})\phi_c - i\sqrt{\pi/2}(\zeta - \zeta^{-1})\bar{\phi}_c - i\sqrt{2\pi}\phi_s\right], \quad (32)$$

$$L_\uparrow = \exp \left[i\sqrt{\pi/2}(\zeta - \zeta^{-1})\phi_c + i\sqrt{\pi/2}(\zeta + \zeta^{-1})\bar{\phi}_c + i\sqrt{2\pi}\bar{\phi}_s\right], \quad (33)$$

where $\phi_c = (\Phi_c + \Theta_c)/2$, $\bar{\phi}_c = (\Phi_c - \Theta_c)/2$ and similarly for $\phi_s$. The correlation function of the right chiral component is given by

$$\left\langle R_\uparrow(x, \tau)R_\uparrow^+(0, 0)\right\rangle = \frac{1}{(v_c\tau - ix)^2} \left(\frac{a^2}{v_c^2\tau^2 + x^2}\right)^{\theta_c/2} \times$$

$$\times \frac{1}{(v_s\tau - ix)^{\frac{\theta}{2}}} \cdot \quad (34)$$

For $\theta > 1$ the single-fermion density of states (DOS)
exhibits pseudogap-type behavior. From Eq. (36) this happens only when \( \tau < \tau_c \), where
\[
\tau_c = (3 - 2\sqrt{2}) \approx 0.17157 .
\] (37)
For all values of \( h \) in the interval \( (h_c, +\infty) \) we find \( \theta \) numerically.

Plots of the dependence of the Luttinger parameter \( K \) and the exponent \( \theta \) on the field \( h \) are represented on Fig. 3 and Fig. 4.

\( \beta \approx 1, \beta^2 < 8\pi \) Behavior of \( K, \theta, n_{\text{sol}}, \chi \) near the critical line (CL). It is interesting to have the \( h \) dependence of the above quantities near the commensurate-incommensurate transition as \( h \to h_c^\pm \).

This can be achieved by solving the integral equation \[(33)\] by iterations. After two iterations the boundary \( B \) can be written as a series in powers of \( (h/h_c - 1) \)
\[
B = \sqrt{\frac{2}{3}} \left( \frac{h}{h_c} - 1 \right)^{1/2} \left[ 1 - G(0) \frac{2\sqrt{2}}{3} \left( \frac{h}{h_c} - 1 \right)^{1/2} - \left( \frac{1}{12} - \frac{20}{9} G(0)^2 \right) \left( \frac{h}{h_c} - 1 \right) + \cdots \right] ,
\] (38)
with coefficients which, up to the order shown, do not change in higher order iterations. In this way we get for the soliton contribution to the ground state energy:
\[
\varepsilon_0 = -\frac{2\sqrt{2} M_s^2}{3\pi} \left( \frac{h}{h_c} - 1 \right)^{3/2} \left[ 1 - 2\sqrt{2} G(0) \left( \frac{h}{h_c} - 1 \right)^{1/2} \right.
\]
\[\left. + \left( \frac{3}{20} + \frac{28G(0)}{3} \right) \left( \frac{h}{h_c} - 1 \right) \cdots \right] ,
\] (39)
and for the dressed charge:
\[
\varepsilon_{\text{dressed}} = -\frac{2\sqrt{2} M_s^2}{3\pi} \left( \frac{h}{h_c} - 1 \right)^{3/2} \left[ 1 - 2\sqrt{2} G(0) \left( \frac{h}{h_c} - 1 \right)^{1/2} \right.
\]
\[\left. + \left( \frac{3}{20} + \frac{28G(0)}{3} \right) \left( \frac{h}{h_c} - 1 \right) \cdots \right] ,
\] (40)
Formulae (39) and (40) show that (to first order) the number of solitons and the susceptibility have square root dependence and square root singularity as functions of \( h \), respectively.

Eq. (39) and Eq. (40) show, on the other hand, that the Luttinger parameter \( K \) decreases, from \( K = 1/2 \) at \( h = h_c \), as a square root function of \( (h/h_c - 1) \), whereas the exponent \( \theta \) increases linearly having value 0 at \( h = h_c \). Explicitly, \( K \) and \( \theta \) as functions of \( (h/h_c - 1) \), near the critical line, are given by:
\[
\left\{ \begin{array}{ll}
K = \frac{1}{2} - 2\sqrt{2} G(0) \left( \frac{h}{h_c} - 1 \right)^{1/2} + \cdots , \\
\theta = \frac{1}{8} + \frac{3\sqrt{2}}{2} G(0) \left( \frac{h}{h_c} - 1 \right)^{1/2} + \cdots .
\end{array} \right.
\] (41)
As functions of the number of solitons
\[
n_{\text{sol}} \approx \frac{\sqrt{2} M_s}{\pi} \left( \frac{h}{h_c} - 1 \right)^{1/2} + \cdots ,
\] (42)
they (to a first approximation) are given by:

\[
\begin{align*}
K &= \frac{1}{2} - 2\pi \frac{G(0)}{M_s} n_{\text{sol}} \quad (n_{\text{sol}} \to 0) ; \\
\theta &= \frac{1}{8} + 3\pi \frac{G(0)}{2M_s} n_{\text{sol}} \quad (n_{\text{sol}} \to 0) .
\end{align*}
\]  

A plot of the dependence of \( \theta \) on the field \( h \), near the critical line, is shown on Fig. 3. The inset, in Fig. 4, confirms the square root \( h \)-dependence of \( \theta \) in that region.

In Fig. 5 we have represented the dependence of the exponent \( \theta \) on the number of solitons, outside the region, near the critical line, is shown on Fig. 4. The inset, in Fig. 4, shows also in the plot. The parameter \( \tau \) is taken to have value 0.2 in both plots.

For smaller values of \( \tau \) the expansion (39) holds only for values of \( h \) outside this small interval \( h \) above \( h_c \). For values of \( h \) outside this small interval, a crossover to a different behavior takes place. In this case one has to take into account the presence of the breathers in the ground state (39).

(\( \beta \to 0 \) The limit of small \( \tau \), \( M_s \tau = \text{const.} \)) At \( \beta^2 < 4\pi \) in the sine-Gordon spectrum, in addition to solitons, breathers (kink-antikink bound states) will appear. The mass spectrum of the breathers is described by (43):

\[
M_n = 2M_s \sin \left( n \frac{\pi \beta^2}{8\pi - 2\beta^2} \right) ,
\]

with \( n \) – integer, taking the following values:

\[
n = 1, \ldots, \left[ \frac{8\pi}{\beta^2} - 1 \right] .
\]

When \( \beta \to 0 \) the bound-state energy vanishes and at small \( \beta^2 \) one can expand the sine in (45) and obtains

\[
M_n = n\pi M_s \tau .
\]

As \( \beta \to 0 \) the soliton mass grows to infinity while \( \beta^2 M_s \) converges to a finite value \( \left[ \frac{8\pi}{\beta^2} - 1 \right] \). The linearity in the mass spectrum means that the breathers instead of being in

\[
\begin{align*}
\pi^2 \tau / \ln(1/\tau) &\gg (h/h_c - 1) .
\end{align*}
\]

\[
E = -\frac{\mu^2}{4\pi \tau} \sinh^2 B \approx -\frac{\pi \mu}{2} \ln(h_c/(\pi(h_h_c))).
\]
In order to calculate the Luttinger parameter in this region, we would need to have in addition to $\chi$ the value of the Fermi velocity $v_F$. We achieve that through the following calculations.

The excitation spectrum in this region is given by

$$\epsilon(\theta) = -\mu \sqrt{\sinh(B - \theta) \sinh(B + \theta)} ,$$  

(55)

whereas the equation for the momentum is:

$$\int_{-B}^{B} \frac{du}{\sinh(u - v)} = \pi \mu \cosh v +$$

$$+ 2\pi n_{\text{sol}} \ln |\cosh [(v + B)/2] \cosh [(v - B)/2]| .$$

After the substitutions

$$x = \tanh v , \quad Y(x) = \cosh v p(v) ,$$

we get

$$P \int_{-b}^{b} \frac{dy}{x - y} = g(x) ,$$  

(57)

where $b = \tanh B$ and

$$g(x) = \frac{\pi \mu}{1 - x^2} - \frac{2\pi n_{\text{sol}}}{\sqrt{1 - x^2}} \ln \left[ \frac{\sqrt{1 - b^2} + \sqrt{1 - x^2}}{-\sqrt{1 - b^2} + \sqrt{1 - x^2}} \right] .$$

The general solution of such equation is

$$Y(x) = \frac{1}{\pi} \sqrt{b^2 - x^2} P \int_{-b}^{b} \frac{g(y) dy}{y^2 - x^2} .$$  

(58)

The solution at small $b \approx B \ll 1$ is

$$p_F - p = \frac{4n_{\text{sol}}}{b} \sqrt{b^2 - x^2} , \quad \epsilon(x) = -\mu \sqrt{b^2 - x^2} ,$$  

(59)

which gives the following dispersion law:

$$E = v_F(p - p_F) ,$$  

(60)

with

$$v_F = \frac{b \mu}{4n_{\text{sol}}} \approx \frac{1}{2} \left( \frac{\mu}{n_{\text{sol}}} \right)^{1/2} \exp \left( -\frac{\pi \mu}{4n_{\text{sol}}} \right) .$$  

(61)

The maximal value of $-\epsilon$ is $\mu \sinh B \approx \mu b$. This means that $E(p)$ deviates from the linear form at $p \sim p_F$ indicating that the ultraviolet cut-off in the momentum space is of order of $p_F \sim n_{\text{sol}}$.

According to the above results in the region (53) we have

$$K = \frac{1}{2} \pi \chi v_F \approx \frac{1}{2} \left( \frac{n_{\text{sol}}}{\mu} \right)^{3/2} \exp \left( \frac{\pi \mu}{4n_{\text{sol}}} \right) \approx \frac{1}{2} \left( \frac{n_{\text{sol}}}{\mu} \right)^{2} .$$  

(62)
When \( n_{\text{sol}} \) approaches the boundary of validity of this expression given by Eq. (54), which corresponds to very small \( n_{\text{sol}}/\mu \), \( K \) is still small: \( K \sim (n_{\text{sol}}/\pi \mu)^2/2 \). At even smaller \( n_{\text{sol}} \), it asymptotically approaches 1/2. In this region Eq. (25) can be solved by iterations.

Thus at \( \tau \ll 1 \) (the classical regime) the Luttinger parameter \( K \) decreases quickly from 1/2 towards its asymptotic value \( \tau \). This rapid change takes place for \( n_{\text{sol}} < \mu \), that is, it is determined not by the soliton mass which goes to infinity at \( \beta^2 \to 0 \), but by the mass of the breather which remains constant in this limit.

**DISCUSSION**

In this paper we have studied the scaling exponents of the isospin sector of the U(1) Thirring model in the incommensurate phase when its energy spectrum is gapless. This model may be used as an effective theory for a low-energy limit of fermionic theory with long range repulsion on a lattice close to half-filling. Exactly at half-filling the charge sector is gapped and is described by the sine-Gordon model with a relevant cosine term. This gap is suppressed by a finite chemical potential and the system goes into a gapless incommensurate phase. In this phase, which is the phase of concern in this paper, the correlation functions of the operators (88) and (84) decay as power laws of some critical exponents which are given in terms of a single parameter \( K \), the Luttinger liquid parameter. \( K \) is related to the dressed charge which satisfies an integral equation characteristic for exactly solvable models.

We are mainly interested in the region of small \( \beta \). This is an interesting limit which has not been much discussed in literature. At \( \beta \to 0 \) the solitons become infinitely heavy and in the insulating phase (absence of solitons) SG model becomes a model of free massive bosonic field with the mass \( \mu \). It turns out that even in the incommensurate phase when the chemical potential forces solitons to appear, \( K \) does not depend on the soliton mass but on the mass of the optical phonon \( \mu \) (see Eq. (83)).

As we have shown, the value of \( K \) depends on doping (that is the soliton density) with \( K = 1/2 \) in the limit of zero doping and reaching its asymptotic value \( \beta^2/8\pi \) at large doping. Therefore changing the number of solitons one can explore different phases of the model (see Fig. 3). Thus for small doping the single-particle hopping is relevant. In this case one might expect that the system of many coupled chains will become a Fermi liquid. When doping increases \( K \) decreases, the single particle tunneling processes are suppressed, but the spin susceptibility becomes singular. This is a region where interchain coupling will establish Spin Density Wave (SDW) ordering. For \( K < 1/3 \) the \( 4k_F \)-component of the charge density operator has the smallest dimension with the corresponding susceptibility being the most singular. This notifies that the Wigner crystal order parameter will be stabilized by a Coulomb interaction in a three dimensional array of coupled chains.

The number of solitons and the susceptibility are found analytically and numerically and display, at the transition point, square-root behavior and square-root singularity respectively. As \( \beta \to 0 \), the above dependence holds only in a decreasingly small interval \( h \) above \( h_c \). For larger \( h \) the dependence of \( n_{\text{sol}} \) and \( \chi \) on \( h \) is given in Eq. (61). Plots of the dependence of \( n_{\text{sol}} \) and \( \chi \) on \( h \) are given on Fig. 3 and Fig. 7.

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**APPENDIX**

Expansion near the critical line

The kernel \( G(\theta) \) of (13) can be expanded about \( \theta = 0 \) in a Taylor series with convergence radius \( r_0 = \pi \tau/(1 - \tau) \). For small \( h_c - h = M_h - h \), using the Taylor expansion of the inhomogeneity \( h_c \cosh(\theta) - h \), the integral equation can be solved for a given accuracy by a power series about \( \theta = 0 \) implying also a series in \( B \) about \( B = 0 \):

\[
G(\theta) \sim \sum_{i=0}^{N} \frac{G_{2i}}{(2i)!} \theta^{2i} , \quad G_{2i} = G^{(2i)}(0) ,
\]

\[
h_c \cosh(\theta) - h \sim h_c \sum_{i=0}^{N} \frac{1}{(2i)!} \theta^{2i} - h ,
\]

\[
\epsilon(\theta) \sim \sum_{i=0}^{N} \epsilon_{2i} \theta^{2i} , \quad \epsilon_{2i} \sim \sum_{j=0}^{2(N-i+1)} \epsilon_{2i,j} B^j .
\]

Beginning with \( \epsilon_{2i,j} = 0 \) the iteration for the coefficients converges after \( 2N + 1 \) cycles, when actually \( N = 4 \) was used and results for \( N = 2 \) are shown below. The solution for \( B \) from the condition \( \epsilon(0) = 0 \), where \( \epsilon(B) \) is now a sole power series in \( B \), is parameterized as a power series in \( \sqrt{h/h_c - 1} \), cf. (38).

\[
B \sim \sum_{j=1}^{2N+1} \alpha_j x^j , \quad x = \sqrt{h/h_c - 1} ,
\]

\[
\epsilon(B) \sim \sum_{i=0}^{2N+2} \epsilon_i^{(B)} B^i \sim \sum_{i=0}^{2N+4} \epsilon_i^{(h)} x^i ,
\]

where also \( h \) is replaced by \( h_c(1 + x^2) \) in \( \epsilon_i^{(h)} \) which yields \( \epsilon_1^{(h)} = 0 \) and \( \epsilon_2^{(h)} = h_c(\alpha_1^2 - 2)/2 \). From \( \epsilon_{i+1}^{(h)} = 0 \), \( i = 2, \ldots, 2N + 3 \), the coefficients \( \alpha_i \) can be read off recursively.
In the soliton contribution to the ground-state energy, the power expansion of \( \cosh \theta \) and its replacement (20) is used, where the integration leads to a power series in \( B \) and replacement (20) to a series in \( \sqrt{h/h_c - 1} \):

\[
E_0 \sim \frac{2\sqrt{2}M^2}{3\pi} x^3 \sum_{i=0}^{2N+2} E^{(i)}_0 x^i, \quad x = \sqrt{\frac{h}{h_c} - 1}.
\]

(68)

The integral equation (20) for \( \zeta \) is treated exactly as (21), leading to a series in \( \theta \) and \( B \). For \( \zeta(B) \) applying the previous replacement for \( B \) the series reads

\[
\zeta(B) \sim \sum_{i=0}^{2N+2} \zeta^{(i)}_0 x^i, \quad x = \sqrt{\frac{h}{h_c} - 1}.
\]

(69)

Finally, this series is replaced into (21), which then is inserted into (23), leading to the following series:

\[
K \sim \sum_{i=0}^{2N+2} K_i x^i, \quad \theta \sim \sum_{i=0}^{2N+2} \theta_i x^i.
\]

(70)

**Coefficients \( \alpha_i \) of \( B \) in (66)**

\[
\begin{align*}
\alpha_1 &= \sqrt{2}, \\
\alpha_2 &= -4G_0/3, \\
\alpha_3 &= \sqrt{2}(-1/12 + 20G_0^3/9), \\
\alpha_4 &= 2G_0/5 - 256G_0^3/27 - 8G_2/5, \\
\alpha_5 &= \sqrt{2}(3/160 - 49G_0^3/45 + 616G_0^3/27 + 256G_0G_2/45), \\
\alpha_6 &= 8(-2G_0/35 + 64G_0^3/27 - 3584G_0^3/81 + (29/35 - 128G_0^2/3)G_2/3 - 4G_4/21)/3, \\
\alpha_7 &= \sqrt{2}(-5/896 + 769G_0^2/1400 - 286G_0^2/15 + 7779G_0^2/243 + 32(-193/35 + 176G_0^2)G_0G_2/45 + 272G_0^2/75 + 736G_0G_4/315) \\
\end{align*}
\]

**Coefficients \( \hat{E}^{(i)}_0 \) of \( E_0 \) in (23)**

\[
\begin{align*}
\hat{E}^{(0)}_0 &= 1, \\
\hat{E}^{(1)}_0 &= -2\sqrt{2}G_0, \\
\hat{E}^{(2)}_0 &= 3/20 + 28G_0^2/3, \\
\hat{E}^{(3)}_0 &= -2\sqrt{2}(G_0/5 + 320G_0^3/27 + 8G_2/15), \\
\hat{E}^{(4)}_0 &= -3/224 + 9G_0^2/5 + 1144G_0^4/9 + 64G_0G_2/5, \\
\hat{E}^{(5)}_0 &= 2\sqrt{2}(G_0/28 - 16G_0^3/9 - 14336G_0^3/81 + 4(-1/35 - 64G_0^2/9)G_2 - 4G_4/45).
\end{align*}
\]

\[\hat{E}^{(6)}_0 = 1/384 - 2321G_0^2/4200 + 286G_0^3/27 + 1478048G_0^5/729 + 16(61/175 + 2288G_0^3/27)G_0G_2/3 + 1232G_0^2/225 + 352G_0G_4/105\]

**Coefficients \( \zeta^{(i)}_0 \) of \( \zeta(B) \) in (63)**

\[
\begin{align*}
\zeta^{(0)}_0 &= 1, \\
\zeta^{(1)}_0 &= -2\sqrt{2}G_0, \\
\zeta^{(2)}_0 &= 32G_0^2/3, \\
\zeta^{(3)}_0 &= \sqrt{2}(G_0/2 - 280G_0^3/3 - 8G_2)/3, \\
\zeta^{(4)}_0 &= 32(-G_0^2/5 + 16G_0^3/9 + 14G_0G_2/5)/3, \\
\zeta^{(5)}_0 &= \sqrt{2}(-3G_0/80 + 154G_0^3/15 - 16016G_0^3/27 + 2(1/3 - 336G_0^3/5)G_2 - 16G_4)/15, \\
\zeta^{(6)}_0 &= 16(13G_0^3/35 - 448G_0^3/9 + 57344G_0^3/27 + 8(-29G_0/35 + 704G_0^3/9)G_2 + 58G_2^2/5 + 272G_0G_4/35)/9.
\end{align*}
\]

**Coefficients \( K_i \) of \( K \) in (70)**

\[
\begin{align*}
K_0 &= 1/2, \\
K_1 &= -2\sqrt{2}G_0, \\
K_2 &= 44G_0^2/3, \\
K_3 &= (G_0 - 944G_0^3/3 - 16G_2)/3\sqrt{2}, \\
K_4 &= 2(-7G_0^2/5 + 5008G_0^4/27 + 304G_0G_2/15), \\
K_5 &= -3G_0/80 + 734G_0^3/45 - 35216G_0^5/27 + 2(1 - 5008G_0^3/15)G_2/3 - 16G_4/15, \\
K_6 &= 8(11G_0^3/35 - 8236G_0^4/135 + 277600G_0^6/81 + 4(-151/7 + 27016G_0^3/9)G_0G_2/15 + 52G_2^2/5 + 712G_0G_4/105)/3.
\end{align*}
\]

**Coefficients \( \theta_i \) of \( \theta \) in (70)**

\[
\begin{align*}
\theta_0 &= 1/8, \\
\theta_1 &= 3\sqrt{2}G_0/2, \\
\theta_2 &= 5G_0^2, \\
\theta_3 &= \sqrt{2}(-G_0 + 16(-7G_0^3 + G_2))/8, \\
\theta_4 &= (-51G_0 + 16(535G_0^3 + 69G_2))G_0/90, \\
\theta_5 &= \sqrt{2}(81G_0 + 32(355G_0^3 - 9(5G_2 - 8G_4) - 40(743G_0^3 + 130G_2G_0^2))/2880, \\
\theta_6 &= 2(39G_0^2 + 84(43G_0^2 - 227G_0^2 + 32(33985G_0^3 + 8001G_2)G_0^2 - 12(107G_2 - 138G_4)G_0)/945.
\end{align*}
\]
Coefficients $\theta_i$ of $\theta$ in (70), for fermions without spin

$\theta_0 = \theta_1 = 0$

$\theta_2 = 16G_0^2$

$\theta_3 = -160\sqrt{2}G_0^3/3$

$\theta_4 = 8(-G_0^2 + 140G_0^4 + 16G_0G_2)/3$

$\theta_5 = 8\sqrt{2}(91G_0^3 - 7360G_0^5 - 1264G_0^2G_2)/45$

$\theta_6 = 32(G_0^2/5 - 137G_0^4/3 + 23176G_0^6/9 + 4(-1 + 452G_0^2/3)G_0G_2 + 8G_2^2 + 24G_0G_4/5)/9$

$\theta_7 = 2\sqrt{2}(-613G_0^3/70 + 10048G_0^5/9 - 431744G_0^7/9 + 16(1103/35 - 8032G_0^2/3)G_0G_2/3 - 1408G_0G_2^2/3 - 5504G_0^2G_4/35)/3$

Our results can be generalized for the case of spinless fermions. Here instead of Eq. (23) and Eq. (35) one has to use

$$K = \pi \chi v_F, \quad \theta = \frac{1}{2} \left( \sqrt{K} - \frac{1}{\sqrt{K}} \right)^2.$$  (71)

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