Spin-orbit splittings in heavy-light mesons and Dirac equation

Riazuddin
National Centre for Physics,
Quaid-i-Azam University Campus,
Islamabad 45320, Pakistan

Sidra Shafiq
Centre for Advance Mathematics and Physics,
National University of Science and Technology,
Islamabad, Pakistan
(Dated: April 29, 2013)

The spin-orbit splitting in heavy-light mesons is seen to be suppressed experimentally. It is shown that it can be understood qualitatively in the frame work of Dirac theory. An alternative derivation of a relativistic dynamical symmetry for the Dirac Hamiltonian, which suppresses spin orbit splitting, is also given. However it is shown that such a symmetry is not needed since the spin-orbit splitting in Dirac theory with Coulomb like potential (as is the case for the one gluon exchange potential in pQCD) is small anyway.

I. INTRODUCTION

In the mass spectroscopy for heavy-light mesons as $qQ$ or $Qq$ bound states, one has both hyperfine splitting and spin-orbit splitting, which need to be understood in quantum chromodynamics (QCD). Writing a pseudoscalar(vector) heavy meson as $P_q(P_q^*)$, $P = D$ or $B$, $q = u$, $d$ or $s$, the hyperfine splittings in MeV experimentally are [1]

$$M_{D_d^*} - M_{D_d} = 140.64 \pm 0.10 \simeq M_{D_s^*} - M_{D_s} = 143.8 \pm 0.04$$
$$M_{B_d^*} - M_{B_d} = 45.78 \pm 0.35 \simeq M_{B_s^*} - M_{B_s} = 46.5 \pm 1.2 \quad (1)$$

On the other hand the spin orbit splittings seem to be suppressed (see below).

We can write the effective Hamiltonian for a bound hadron containing one heavy quark(antiquark) $Q(Q)$ as

$$H = H_q + H_Q \quad (2)$$

where $H_Q$ takes care of the residual momentum of the heavy quark(antiquark) and $H_q$ represent the motion of the light antiquark(quark) in a fixed potential provided by the heavy quark(antiquark). In heavy quark effective theory[2] (HQET), $H_Q$ contains a term $\bar{Q} \cdot \vec{B} / 2 m_Q$ which gives rise to color magnetic moment interaction of type $\vec{\mu}_q \cdot \vec{\mu}_Q$ which induces the conventional form of the Fermi-Breit potential

$$\frac{8\pi}{3} \alpha V(\mu) \frac{\sigma_Q \cdot \sigma_q}{4 m_Q m_q} \delta^3(r) \quad (3)$$

*Electronic address: riazuddin@ncp.edu.pk
where $\alpha_V(\mu) = \frac{1}{2} \alpha_G(\mu)$, $\alpha_G$ is the pQCD running coupling constant which depends on the energy scale $\mu$, in our case mass of the heavy quark. Here $m_Q$ and $m_q$ are effective constituent quark masses. The hyperfine splitting can be understood [3] in terms of the above term in $H_Q$, which shows that such splittings decrease with the increasing mass of $Q$ both because of $m_Q$ in the denominator and decrease of $\alpha_s(\mu)$ due to asymptotic freedom property of pQCD.

Since in HQET the spin of the heavy quark is decoupled it is natural to combine $\overrightarrow{j} = \overrightarrow{l} + \overrightarrow{S}_q$, the angular momentum of light degrees of freedom, with $\overrightarrow{S}_Q$ to give $\overrightarrow{J} = \overrightarrow{j} + \overrightarrow{S}_Q$ for the bound $Q\bar{q}$ system. Thus one can have the following multiplets

$$l = 0 \quad [P^+(1^-), P(0^-)]_{j=\frac{1}{2}}$$
$$l = 1 \quad [P^0(2^+), P_r(1^+)]_{j=\frac{3}{2}}$$
$$[P^r(1^+), P^r(0^+)]_{j=\frac{1}{2}}$$

(4)

where $P$ is $D$ or $B$ and $J^P$ gives the total angular momentum and parity quantum numbers. The splitting between $j = \frac{1}{2}$ and $j = \frac{3}{2}$ for $l = 1$ is due to the spin-orbit coupling $\overrightarrow{l}.\overrightarrow{S}_q$ while the hyperfine splitting between 2 members of each multiplet arise from the Fermi-Breit term as mentioned earlier. The spin-orbit splitting (in MeV)[1] for $D$ mesons, between $D_s^*(2^+)$ : 2462.8 $\pm$ 1.0 and $D(1^+)$ : 2422.3 $\pm$ 0.6MeV is 40, for the $B$ mesons, between $B^*_2(2^+)$ : 5743.9 $\pm$ 5.0 and $B(1^+)$ : 5723.4 $\pm$ 2.0 is 21, for $B^*_2(2^+)$ : 5839.7 $\pm$ 0.6 and $B_s(1^+)$ : 5829.4 $\pm$ 0.7 is 11. Thus these splittings are suppressed. A measure of this suppression is the parameter [4]

$$r = \frac{p_{3/2} - p_{1/2}}{(4p_{3/2} + 2p_{1/2})/6 - s_{1/2}}$$

(5)

which for the experimental data shown above is of order 0.07 both for $D$ and $B$ mesons.

II. A DYNAMICAL SPIN AND ORBITAL ANGULAR MOMENTUM SYMMETRY FOR THE DIRAC HAMILTONIAN

As it is well known, the solution of the Dirac equation with Coulomb potential for the hydrogen atom does give spin-orbit splitting between $2p_{\frac{3}{2}}$ and $2p_{\frac{1}{2}}$ energy levels in agreement with experiment. Since the one gluon exchange potential (OGE) in pQCD is Coulomb like, one would expect [5] such a splitting in the spectrum of hadrons, in particular heavy mesons considered here. Thus we take $H_q$ in the Eq. (2) as the Dirac Hamiltonian [setting $\hbar = c = 1$]

$$H = M_Q + \overrightarrow{\gamma} \cdot \overrightarrow{p} + \beta(m + V_q) + V_o$$

(6)

where $\overrightarrow{p} = -i\overrightarrow{\nabla}$ is the 3-momentum operator, $\overrightarrow{\gamma}$ and $\beta$ are the Dirac matrices, $m$ is the mass of the light quark and $M_Q$ that of the heavy quark $Q$. The above Hamiltonian as remarked earlier describes the motion of light quark(antiquark) in a fixed potential produced by the heavy antiquark(quark) as in the hydrogen atom where electron moves in Coulomb potential provided by the nucleus(proton). We have assumed that vector and scalar potentials are present, the latter for the reason to be stated shortly.

It has been observed [4,6] that if vector and scalar potentials satisfy the relation

$$V_o(\overrightarrow{r}) = V_s(\overrightarrow{r}) + U$$

(7)
where $U$ is independent of the position of the light quark relative to the heavy quark, then the Dirac Hamiltonian is invariant under a spin symmetry (called relativistic spin symmetry)

$$[H, S_i] = 0. \quad (8)$$

and if potentials are spherically symmetric, then there is an additional symmetry

$$[H, L_i] = 0 \quad (9)$$

The generators of these symmetries are given by

$$S_i = \begin{pmatrix} s_i & 0 \\ 0 & \tilde{s}_i \end{pmatrix}, \quad L_i = \begin{pmatrix} l_i & 0 \\ 0 & \tilde{l}_i \end{pmatrix}$$

here $s_i = \frac{\sigma_i}{2}$ are usual spin generators, $\sigma_i$ the Pauli matrices, $l_i = (r \times p)_i$, while $\tilde{s}_i = U^p s_i U^p$ and $\tilde{l}_i = U^p l_i U^p$, with $U^p = \frac{e^p}{2}$ as the helicity operator. Thus even though the system may be highly relativistic, the Dirac eigenstates can be labeled with orbital angular momentum as well as spin, and the states with the same orbital angular momentum are degenerate, e.g the states $n_r p_{\frac{3}{2}}$ and $n_r p_{\frac{1}{2}}$ are degenerate where $n_r$ is the radial quantum number[4].

Thus a symmetry has been identified in the heavy-light quark system which produces spin-orbit degeneracies independent of the details of the potential. It may be pointed out that for the hydrogen atom, such a symmetry would be contrary to experiment since the splitting between energy levels $2p_{3/2}$ and $2p_{1/2}$ as predicted by Dirac equation is in agreement with experiment which has been regarded as a great success of Dirac equation. The levels $s_{1/2}$ and $p_{1/2}$ are still degenerate (the Lamb shift) which requires quantum radiative corrections.

### III. DIRAC EQUATION AND SPIN SYMMETRY

By writing the Dirac equation in two component Pauli form, the symmetry discussed in the previous section can be derived in a much more transparent way which would also help us to solve exactly the Dirac equation if both the vector and scalar potentials are Coulomb like. Consider the Dirac equation in covariant form in the presence of a gauge field $V_\mu$

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \quad (10)$$

where $D_\mu = \partial_\mu + iV_\mu$ and $\gamma^0 = \beta$, $\gamma^i = \beta\alpha^i$. For our case taking the zeroth component of $V_\mu$ and a scalar potential $V_s$, the above equation become

$$[i\gamma^0(\partial_0 + iV_v(r)) + i\gamma^i\partial_i - m - V_s(r)]\Psi = 0. \quad (11)$$

Multiplying on the left by

$$[i\gamma^0(\partial_0 + iV_v(r)) + i\gamma^0\partial_0 + m + V_v(r)]$$

we obtain, since the potentials are independent of time,

$$[-\partial_0^2 + 2iV_v(r)\partial_0 + V_v^2(r) - \gamma^i\gamma^i(\partial_i\partial_i) + i\gamma^0\gamma^0[\partial_0, V_v] - i\gamma^i[\partial_i, V_v] - (m + V_v(r))^2] \Psi = 0.$$
For stationary states, \( \frac{\partial}{\partial t} \rightarrow iE \), and using the fact that \( \partial_j \partial_i \) is symmetric in \( j \) and \( i \), we obtain

\[
[\nabla^2 + V^2_v - V^2_s - 2EV_v - 2mV_s + i\gamma^0\gamma^j[\partial_i, V_v] - i\gamma^i[\partial_i, V_s] + (E^2 - m^2)]\Psi = 0,
\]

(13)

where

\[
[\partial_i, V_v] = \frac{\partial V_v}{\partial x^i} = \frac{\partial V_v}{\partial r}(\hat{r})^i,
\]

(14)

\[
[\partial_j, V_s] = \frac{\partial V_s}{\partial x^j} = \frac{\partial V_s}{\partial r}(\hat{r})^j.
\]

and the second equality holds for spherically symmetric potentials.

It is convenient to use the chiral representation of \( \gamma \)-matrices,

\[
\gamma^0 = \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
\alpha^i = \beta\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.
\]

(15)

Then we can write the above equation in two component matrix form

\[
\left( \hat{O} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sigma.\hat{r} \begin{pmatrix} -\frac{dV_v}{dr} & \frac{dV_s}{dr} \\ -\frac{dV_s}{dr} & \frac{dV_v}{dr} \end{pmatrix} \right) \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0,
\]

(16)

where

\[
\hat{O} = \nabla^2 + V^2_v - V^2_s - 2EV_v - 2mV_s + (E^2 - m^2).
\]

(17)

The diagonalization of the matrix multiplying \( i\sigma.\hat{r} \), since \( (\sigma.\hat{r})^2 = 1 \), gives the eigenvalues \( \eta \)

\[
(dV_v/dr - \eta)(-dV_v/dr - \eta) - (-dV_s/dr)^2 = 0,
\]

or

\[
\eta = \pm\sqrt{(dV_v/dr)^2 - (dV_s/dr)^2}.
\]

(18)

The matrix which diagonalizes this matrix does not affect the first term in the equation (16) as it is multiplied by a unit matrix which commutes with every matrix. Denoting the corresponding eigenfunctions by \( \Psi_\pm \) which are linear combinations of \( \Psi_L \) and \( \Psi_R \), we have

\[
[\hat{O} \pm i\eta\sigma.\hat{r}]\Psi_\pm = 0.
\]

(19)

We note that the eigenvalues \( \eta \) in Eq. (18) vanish for

\[
\frac{dV_v}{dr} = \pm \frac{dV_s}{dr},
\]

(20)

or

\[
V_v(r) = \pm V_s(r) + \text{constant}.
\]

(21)
Then
\[ \hat{O} \psi_{\pm} = 0, \] (22)
where \( \hat{O} \) is independent of spin. If plus sign is selected, the system is said to have relativistic spin-orbital angular momentum symmetry; for negative sign it is known as the pseudo spin symmetry\[6\] which has been observed in nuclei[7]. As a result there is no spin-orbit coupling and the results obtained in the previous section are derived in a different and more transparent way. Selecting the positive sign in Eqs. (21) and (22) we can solve the equation
\[ [\hat{O} \pm i \eta \sigma \cdot \vec{r}] \psi_{\pm} = 0, \] (23)
exactly for the energy eigenvalues as in hydrogen atom for
\[ V_v = -\frac{\alpha_v}{r} + U_v, \]
\[ V_s = -\frac{\alpha_s}{r} + U_s. \] (24)
Here \( V_v(r) \) is the vector OGE potential with \( \alpha_v = \frac{4}{3} \alpha_G \) in which we are interested and \( V_s(r) \) is the confining scalar potential which might arise from multi gluon effect. But its origin to be Coulomb like is not clear (see, however ref [4] and references there in).

For the above case we have
\[ \eta = \frac{(\alpha_v^2 - \alpha_s^2)^{1/2}}{r^2}. \] (25)
Using the spherical polar coordinates the operator \( \hat{O} \) becomes
\[ \hat{O} = \frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{L^2}{r^2} + \frac{\alpha_v^2 - \alpha_s^2}{r^2} + 2(E' \alpha_v + m' \alpha_s)\frac{1}{r} + (E'^2 - m'^2), \] (26)
where
\[ E' = E - U_v, \]
\[ m' = m + U_s. \] (27)
Accordingly the equation (19) becomes
\[ [\hat{O} + i \frac{(\alpha_v^2 - \alpha_s^2)^{1/2}}{r^2} \sigma \cdot \vec{r}] \psi_{\pm} = 0. \] (28)
The energy eigenstates can be now read of from those for the hydrogen atom [8] by making the substitutions
\[ \alpha \rightarrow \frac{\alpha_v E'}{m'} + \alpha_s, \] (29)
\[ E \rightarrow \frac{E'^2 - m'^2}{2m'}, \]
and are given by
\[ \frac{E'^2 - m'^2}{2m'} = -m' \frac{(\alpha_v E' + \alpha_s)^2}{2(n - \delta_j)^2}. \] (30)
Solving this quadratic equation for \( E' \) and re-substituting the values of primed quantities given in Eq. \( \text{(27)} \)

\[
E_{nj} = U_v + \frac{(m + U_s)}{(n - \delta_j)^2 + \alpha_v^2}[-\alpha_v \alpha_s \pm (n - \delta_j)(n - \delta_j)^2 + \alpha_v^2 - \alpha_s^2]^1/2, \tag{31}
\]

where \( \delta_j \) is

\[
\delta_j = (j \pm \frac{1}{2}) - [(j \pm \frac{1}{2})^2 - (\alpha_v^2 - \alpha_s^2)]^{1/2}. \tag{32}
\]

This is an exact result and \( n \) is the principle quantum number. When \( V_v = V_s \), so that \( \alpha_v = \alpha_s \) and \( U_v = U_s \), \( \delta_j \) becomes zero and

\[
E = U + (m + U)(1 - \frac{2\alpha_v^2}{\alpha_v^2 + n^2}).
\]

As the expression for energy is independent of \( j \), there is no spin-orbit-splitting for this case.

**IV. SPIN-ORBIT SPLITTING AND CONCLUSIONS**

We now discuss spin-orbit splitting based on equations \( \text{(31)} \) and \( \text{(32)} \). For our case \( l = 1, n = 2 \) and we have \( 1^+ \) and \( 2^+ \) mesons corresponding to \( j = \frac{1}{2} \) and \( j = \frac{3}{2} \). Since experimentally \( 2^+ \) mesons are heavier than \( 1^+ \) mesons, i.e. \( E_{2,\frac{3}{2}} > E_{2,\frac{1}{2}} \), one may conclude that \( \delta_{\frac{3}{2}} > \delta_{\frac{1}{2}} \). Then Eq. \( \text{(32)} \) implies that \( (\alpha_v^2 - \alpha_s^2) > 0 \) and further since \( \delta_{\frac{2}{2}} \) is real \( (\alpha_v^2 - \alpha_s^2) < 1 \). For \( V_v(r) \) we take OGE potential

\[
V_v(r) = -\frac{\alpha_v}{r} + U_v, \tag{33}
\]

where \( \alpha_v \) is treated as an effective coupling constant. We put \( \alpha_s = 0 \) and represent the scalar potential by a constant \( U_s \), the whole purpose of which is to renormalize the light quark mass in hadron to \( m + U_s \) [c.f Eq. \( \text{(31)} \)] where \( U_s \) is to be fixed by the data on spin-orbit splitting. How good is this approach? To see this, for the charmed sector, we compare the potential given in Eq. \( \text{(33)} \) with the Cornell potential\([9]\]

\[
V(r) = -\frac{K}{r} + \frac{r}{a^2} + C \tag{34}
\]
with $K = 0.48$, $a = 2.34\text{GeV}^{-1}$ and $C = -0.25\text{GeV}$. This comparison is shown in Fig 1 with the boundary condition ($f$ stands for fermi), $V(0.5f) = 0$ and the matching point $r = 0.14f$, which gives $\alpha_v = 0.8$, $U_v = 0.32\text{GeV}$. With these two conditions, which the Cornell potential also satisfies, it is matched with the lattice QCD potential[10] in ref. [11], showing that the Cornell potential gives the simplest extrapolation of the lattice QCD potential. The potential given in Eq. (33) with $\alpha_v = 0.8$ and $U_v = 0.32\text{GeV}$ almost give the same extrapolation as is clear from Fig.1.

We now proceed with the numerical results. First we note that the mass of $Qq$ or $qQ$ meson is given by ($n = 2$)

$$M_{2j} = M_Q + E_{2j}, \quad (35)$$

Then the mean mass of $j = \frac{1}{2}$, $j = \frac{3}{2}$ mesons is

$$\bar{M} = M_Q + \frac{1}{2}(E_{2,\frac{3}{2}} + E_{2,\frac{1}{2}}) = M_Q + U_v + \frac{1}{2}(m + U_s)(F_{2,\frac{3}{2}} + F_{2,\frac{1}{2}}) \quad (36)$$

while the mass splitting is given by

$$\Delta M = (E_{2,\frac{3}{2}} - E_{2,\frac{1}{2}}) = (m + U_s)(F_{2,\frac{3}{2}} - F_{2,\frac{1}{2}}) \quad (37)$$

where [with $\alpha_s = 0$]

$$F_{2j} = \frac{1}{(2 - \delta_j)^2 + \alpha^2}[(2 - \delta_j)[(2 - \delta_j)^2 + \alpha^2]^{\frac{1}{2}} \quad (38)$$

To carry out the numerical work we have to fix light quark $u$ or $s$ mass $m$, $M_Q$ and $U_s$. It is known from the mass spectra of $l = 0$ mesons that

$$m = m_u = 330\text{MeV}$$
$$m_s = 550\text{MeV}$$
$$M_c = 1480\text{MeV}$$
$$M_b = 4800\text{MeV}$$
$$\alpha_v = \frac{4}{3}\alpha_G, \quad (39)$$

where $\alpha_v$ and $U_v$ have already been fixed for the charmed sector. As the QCD coupling $\alpha_v$ is energy dependent, decreasing with increasing energy, one would expect

$$\alpha_v(M_{B_s}) < \alpha_v(M_B) < \alpha_v(M_D),$$

we will take this into consideration.

It is clear from Eqs. (36), (37) that once we have calculated $F_{2j}, (m + U_s)$ can be fixed from Eq. (36). Our numerical results are summarized in Tables I and II.

To conclude we see that the spin-orbit splittings, obtained from the Dirac equation with OGE potential and scalar confining potential to be a constant so that its role is to enhance the mass of light quark in a hadron, are small and qualitatively explain the data within a factor of about 2 (for charm) and 1.5 (for bottom). The values used for $\alpha_v$ for the OGE potential are some what larger from those obtained from the asymptotic freedom of QCD. However, in potential models $\alpha_v$ is usually treated as a phenomenological parameter. From Table 2, we see that $U_s$ for the scalar confining potential is almost independent of flavor. Another important conclusion one can draw is that as in hydrogen atom, the meson spectroscopy does not show relativistic spin and orbital angular momentum symmetry, since the splittings are small anyway.
TABLE I: Experimental data; all masses in MeV

| Mesons     | Mass $M$ | Mean Mass $\bar{M}$ | Mass Splitting $\Delta M$ |
|------------|----------|----------------------|---------------------------|
| $D_1^*(1^+)$ | 2422     | 2442                 | 40                        |
| $D_2^*(2^+)$ | 2462     |                      |                           |
| $B_1^*(1^+)$ | 5723     | 5733                 | 21                        |
| $B_2^*(2^+)$ | 5744     |                      |                           |
| $B_{s1}^*(1^+)$ | 5829     | 5834                 | 11                        |
| $B_{s2}^*(2^+)$ | 5840     |                      |                           |

TABLE II: Predicted mass splittings in MeV

| Flavor | $\alpha_v$ | $U_v$ | $F_\frac{1}{2} - F_\frac{3}{2}$ | $(F_\frac{1}{2} + F_\frac{3}{2})/2$ | $m + U_s$ | $\Delta M$(our) | $\Delta M$(expt) |
|--------|-------------|-------|-------------------------------|---------------------------------|-----------|-----------------|-----------------|
| $D$    | 0.8         | 320   | 0.029                         | 0.901                           | 712       | 21              | 40              |
| $B$    | 0.75        | 300   | 0.019                         | 0.917                           | 690       | 13              | 21              |
| $B_s$  | 0.65        | 260   | 0.009                         | 0.938                           | 814       | 7.3             | 11              |

Acknowledgments

One of the author (SS) would like to acknowledge the support of Higher Education Commission (HEC), Pakistan, under the indigenous scholarship.

[1] Particle Data Group, K. Nakamura et al., Journal of Physics, G37, 0750212 (2010).
[2] N. Isgur and M. B. Wise, Phys. Lett. B232, 113(1989); Phys. Rev. Lett. 66, 1130(1991); Ming-Lu, M. B. Wise and N. Isgur, Phys. Rev. D45, 553 (1994); For a review, see Falkyuddin and Riazuddin, A Modern Introduction to Particle Physics, 3rd. Edition, World Scientific (2011).
[3] Some of the earlier references are: A. DeRujula, H. Georgi and S. L. Glashow, Phys. Rev. D12, 147 (1976); G. Godfrey, and N. Isgur, Phys. Rev. D 32, 189 (1985); E. J. Eichten and C. Quigg, Phys. Rev. D 49, 5845 (1994); W. Buchmuller and S. H. H. Tye, Phys. Rev. D 24 (1981); M. DiPierro and E. J. Eichten, Phys. Rev. D 64, 114004 (2001); D. Ebert, R. N. Faustov, and V. O. Galkin, Mod. Phys. Lett. A 18, 601, 1597 (2003); S. Godfrey and J. L. Rosner, Phys. Rev. D 64, 097501 (2001); B. A. Knie et al., Phys. Rev. Lett. 92, 242001 (2001); S. Recksiegel and Y. Zumin, Phys. Lett. B 578, 369 (2004); N. Brambilla and A. Vairo, Acta Phys. Polon. B 38, 3429 (2007); A. M. Badalian and B. G. L. Bakker, Phys. Rev. D 67, 071901 (2003). For a recent reference, see A. M. Badalian, B. L. Bakker and I. V. Danikur, arXiv: 1006.4880 (2010). For a review see the last ref. in [2].
[4] P. R. Page, T. Goldman and J. W. Ginocchio, Phys. Rev. Lett., 86, 204(2001).
[5] N. Isgur, Phys. Rev. D62, 054026(2000), ibid,014025.
[6] J. S. Bell and H. Ruegg, Nucl. Phys. B 98, 151-153 (1975); A. L. Blokhin, C. Bahri, and J. P. Draayer, Phys. Rev. Lett. 74, 4149-4152 (1995); J. N. Ginocchio and A. Leviatan, Phys. Lett. B 425, 1-5 (1998); J. N. Ginocchio, Phys. Rep. 315, 213-240 (1999).
[7] A. Arima, M. Harrey, and K. Shimizu, Pseudo $LS$ coupling and pseudo $SU_3$ coupling schemes, Phys. Lett. B 30, 517-522.
(1969); K. Hecht and A. Adlen, Generalized seniority for favored $J \neq 0$ pairs in mixed configurations, Nucl. Physic. A 137, 129-143 (1969); B. D. Serot and J. D. Walecka, in The Relativistic Nuclear Many-Body Problem in Advances in Nuclear Physics, edited by J. W. Negele and E. Vogt (Plenum, New York, 1986), Vol. 16; B. A. Nikolaus, T. Hoch, and D. G. Madland, Phys. Rev. C 46, 1757-1781 (1992).

[8] C. Itzkson and Jean Bernard Zuber, Quantum Field Theory, McGraw-Hill, International Editions (1980), Chapter 2.

[9] E. Eichten et al. Phys. Rev. D 17, 3090 (1978), 21, 203 (1980).

[10] G. S. Bali et al, Phys. rev. D 62, 054503 (2000).

[11] A. Laschka, N. Kaishi and W. Weise, arXiv: hep-ph/1102.0945 (see in particular Fig. 6).