Relativistic Fluctuation Theorems

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Introduction. — The physical basis of the direction of time is being discussed since Boltzmann’s H-Theorem in 1872. A priori, this thermodynamic arrow of time has to be distinguished from the possibility of a prime direction of time defined by the monotonous expansion of our Universe. Though we know by now that due to its dominating Dark Energy component our Universe is very likely to expand forever, a direct connection between these arrows of time seems unlikely, leaving the fascinating cosmological arrow of time in a speculative stage. Highly advanced is the situation of the arrow of time in thermodynamics. Since 1993 Fluctuation Theorems (FTs) have been derived for very general classes of systems, allowing statements about the entropy of systems far from thermal equilibrium. As a priori, these FTs are capable of proving the second law of thermodynamics for the entropy ∆S produced by an ensemble,

\[ \Delta S \geq 0 \ 	ext{at any time}, \tag{1} \]

they even allow statements about the probability of “violations” of (1) by a single member of the ensemble in finite regions of space and time. For the steady state of sufficiently chaotic dynamical systems, FTs of the form

\[ P(\Delta s = +a) = \frac{\exp \frac{a}{k_B}}{P(\Delta s = -a)} \tag{2} \]

valid for any a have been proven. Here P(Δs = a) is the probability to observe a production of entropy equal to a along a trajectory segment. In the context of dynamical systems, the dimensionless entropy Δs/k_B is defined by the contraction of phase space. The Steady State FT was derived in [10] for a stochastic systems in contact with a bath at temperature T. For this stochastic systems the entropy Δs = ΔQ/T is caused by an energy transfer ΔQ into the bath. In [11] a certain entropy production Δs is averaged exponentially to find Integral FTs of the form

\[ \langle e^{-\Delta s/k_B} \rangle = 1, \tag{3} \]

which remain valid even for non-stationary states in the presence of time-dependent external forces. Angle brackets denote averaging over trajectories by path integration. Equation (2) implies (3), and Eq. (3) implies (1) by virtue of the Jensen inequality using \( \Delta S = \langle \Delta s \rangle \).

Recently, the 1905 publications of Einstein on Brownian Motion [11] and Special Relativity [15] were unified [16, 17, 18]. Based on that, the results of this Letter are threefold. First, for the relativistic Brownian processes, we reconcile the FTs [16] and [18], which have become a paradigm of nonequilibrium physics, with Special Relativity. FTs for a similar process put forward in [17] can be derived readily by an analogous thread of reasoning.

In [17] and [18] it was pointed out that the relativistic time dilation leads to multiplicative coupling, necessitating a careful choice of the three discretization rules (according to Itô, Fisk-Stratonovich, as well as Hänggi-Klimontovich), since they lead to physically different processes. As second central result we show, that there is one relativistic Steady State FT [12] and one relativistic Integral FT [13] valid for all choices. Furthermore, we shall find the physically correct expression for the entropy production following from relativistic FTs when the Hänggi-Klimontovich discretization rule is applied.

The third result of this Letter is a general-relativistic Integral FT for the Cosmological Standard Model. This exposes clearly the role of cosmic expansion in entropy production. We shall identify the entropy production which is solely due to the Hubble expansion of space. Such entropy producing processes dominate when the expansion rate of the Universe exceeds the particle scattering rate, for instance in an early inflationary phase after the Big Bang.

Relativistic Brownian Motion. — To avoid technicalities we consider first the one-dimensional motion of a test particle in a heat bath. The generalization to higher spatial dimensions is straightforward. In the high temperature limit, the mean squared velocity of this particle with rest mass m may no longer obey the nonrelativistic law \( \sqrt{\frac{k_B T}{m}} \) since the finite speed of light defines an insurmountable upper bound. The authors of [16, 17, 18]...
have set forth the relativistic Brownian motion giving rise to relativistic velocity distributions in both, the language of stochastic differential equations (relativistic Langevin equations) and the language of probability densities (relativistic Fokker-Planck equations). Following [17, 18], the generalized deterministic force $F_\text{d}$ in the rest frame of the heat bath is

$$d p_\text{d} = F_\text{d} \, dt = -\nu p \, dt ,$$

so that the time scale of dissipation is $1/\nu$. Equation (4) is of the familiar form [20] with the nonrelativistic momentum $mv$ substituted by $p = p^\alpha = mv/\sqrt{1-v^2/c^2}$, which is the spatial component of the relativistic momentum vector $p^\alpha$. As is common, Greek indices refer to temporal ($\alpha = 0$) and spatial components. The signature of the Minkowski metric tensor is $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(-1,1)$. Moreover, Einstein’s summation convention is invoked throughout. Since the rest mass is not changed in collisions, $p^\alpha p_\alpha = -(mc)^2 = \text{const}$, the change in the momentum vector $dp^\alpha$ is always “orthogonal” to $p_\alpha$ in the sense of

$$p_\alpha dp^\alpha = 0 .$$

This means that the classical particle cannot leave its mass shell $p^\alpha p_\alpha = -(mc)^2$, which is nothing but its dispersion relation,

$$E = p^0 c = \sqrt{(mc^2)^2 + (pc)^2} .$$

The general solution of (5) is the projection

$$dp^\alpha = \left( \delta^\alpha_\beta + \frac{p^\alpha p_\beta}{(mc)^2} \right) \xi^\beta$$

of an arbitrary Lorentz vector $\xi^\beta$. It is readily seen that Eq. (4), valid in the rest frame of the bath, implies

$$dp_\text{d}^\alpha = -\nu v^\beta \left( \delta^\alpha_\beta + \frac{p^\alpha p_\beta}{(mc)^2} \right) \nu_\text{bath} \, d\tau ,$$

with the bath velocity vector $v^\beta_\text{bath}$ and the particle’s proper time $\tau$. Equation (7) is the generalized Lorentz-invariant deterministic part of the Brownian motion [22].

The description of relativistic Brownian motion is completed by Lorentz-invariant stochastic changes $dp^\alpha_\text{s}$ of the momentum caused by the impacts of the surrounding heat bath at temperature $T$. The derivation in [17, 18] rests on two ideas: The relativistic momentum is the proper quantity performing a Wiener process, since it is physically exchanged and additive, whereas the velocity is well-known to be not additive in Special Relativity. The second postulate demands that the distribution is a Gaussian in the instantaneous rest frame of the particle. This connects the relativistic Brownian motion to the nonrelativistic case. These principles determine the exchanged momenta $dp^\alpha_\text{s}$ do be distributed according to (cf. Eq. (35c) in [17])

$$P_{\text{coll}}(p^\alpha, dp^\alpha) = \frac{mc \delta (p^\beta dp_\beta^\alpha)}{2\sqrt{2\pi}D\tau} \exp \left( -\frac{dp^\alpha_\text{s} dp_\alpha}{4D\tau} \right) .$$

The Dirac distribution $\delta (p^\beta dp_\beta^\alpha)$ guarantees that the mass-shell condition [3] is also fulfilled by the stochastic impacts, since they are elastic. While the relativistic momentum $p$ is additive and unbounded, the velocity is restricted to the open interval $(-c,+c)$. This can be seen by the elegant relation $v/c^2 = p/E$ in the rest frame of the bath, which is equivalent to

$$dx = \frac{pc}{\sqrt{(mc)^2 + p^2}} \, dt .$$

The bath temperature $T$ is defined by the Einstein relation for the momentum diffusion constant $D$ (cf. Eq. (59) in [17]):

$$D = k_B T mc .$$

The Einstein relation is known to apply far from equilibrium [13]. This is not surprising since (8) relates the bath temperature to the strength of stochastic impacts $D$, and the particle damping rate $\nu$. These quantities define the coupling between the bath and a single particle, be it considered as part of an equilibrium ensemble or not.

**Relativistic Fluctuation Theorem.** — We have now the manifestly Lorentz-invariant Langevin equation

$$dp^\alpha = dp^\alpha_\text{d} + dp^\alpha_\text{s}$$

with the deterministic part given by (4) and the stochastic part described by (8) at hand. Specifying (11) to the rest frame of the bath, $v^\alpha_\text{bath} = (c,0)$, yields

$$dp = -\nu v \, dt + dp_\alpha .$$

The probability density of the exchanged momenta $dp_\text{s}$ is found by integrating out the $dp^\alpha_\text{s}$-component in (8), cf. [17]:

$$P_{\text{coll}}(p, dp_\alpha) = \frac{\exp \left( -\frac{dp_\alpha^2/4D}{\sqrt{1 + \frac{p^2}{(mc)^2}}} \right)}{2\sqrt{\pi}D\tau \sqrt{1 + \frac{p^2}{(mc)^2}}} .$$

The Eqs. (4), (12) and (13) establish the relativistic stochastic motion of the Brownian particle in phase space. The corresponding transition probability is easily determined by the discretization rule, for example in the case of Itô:

$$P_{\text{trans}} \left( \begin{array}{c} x \\ p \end{array} \rightarrow \begin{array}{c} x + dx \\ p + dp \end{array} \right) = \frac{\delta (dx - \frac{pc}{\sqrt{(mc)^2 + p^2}} \, dt)}{2\sqrt{\pi}D\tau \sqrt{1 + \frac{p^2}{(mc)^2}}} \times \exp \left( -\frac{(dp + p v dt)^2}{4D\tau \left( 1 + \frac{p^2}{(mc)^2} \right)} \right) .$$
It has been clarified in [13] that entropy fluctuations are caused by the particle entropy \( s_p = -k_B \ln P(x(t), p(t), t) \) with the particle’s nonequilibrium phase space density \( P(x, p, t) \), and the entropy \( s_m \) of the surrounding medium at temperature \( T \). From the equation of motion (cf. Eq. (7) in [13]) for \( s_p \),
\[
d s_p = ds_p|_{\lambda=0} + \lambda k_B \, d \ln E ,
\]
we have isolated the second term which depends on the discretization rule applied to [12]: Hänggi-Klimontovich, Fisk-Stratonovich, or Itô correspond to \( \lambda = 0, \frac{1}{2}, \) or 1 respectively.

The entropy production \( ds_m \) in the bath follows by contrasting the probability of a trajectory \( (x, p)^{\dagger}_t \) with its time-reverse \( (x, p)^{\dagger}(t) = (x(t_f - t), -p(t_f - t)) \) to extract the time-asymmetric part causing dissipation (cf. [13] and references therein):
\[
d s_m \equiv k_B \ln \frac{P^{\text{trans}}}{P^{\dagger \text{trans}}} = -\frac{dE}{T} - \lambda k_B \, d \ln E .
\]

From the Eqs. (15) and (16) we see that though the relativistic motion of the Brownian particle is physically inequivalent depending on \( \lambda \), the fluctuations of the total entropy \( s = s_p + s_m \) are independent of \( \lambda \). The Eqs. (15) and (16) can readily be integrated by the method developed in [13] over a finite time interval to the Steady State FT and for time-depend states to the Integral FT. Therewith we have proven relativistic FTs that are unaffected by the discretization dilemma.

We are now in the position to address the physical choice of \( \lambda \) by virtue of the FT. In the nonrelativistic regime, \( E \) is dominated by the mass so that the second term in (16) vanishes for all \( \lambda \). At arbitrary relativistic energies the Hänggi-Klimontovich rule, \( \lambda = 0 \), yields the correct expression for the entropy
\[
d s_m = -\frac{dE}{T} ,
\]
which is produced in the heat bath.

**Generalizations in the framework of Special Relativity.**

To generalize the FTs to \( n \) spatial dimensions, momentum and force in Eq. (4) are simply substituted by their spatial vectors and the Greek indices in the Lorentz-invariant Eqs. (7) and (8) take values up to \( n \). After integrating out the temporal component \( p^0 \), the distribution (16) is found to contain a quadratic form \( A \) instead of the square in the exponent (cf. Eq. (15) in [13]) with tensor components
\[
A_{ij} = \delta_{ij} - \frac{c^2}{E} p_i p_j .
\]
The FTs follow using the obvious fact that \( p \) is eigenvector of \( A \).

No complications are caused by allowing an inhomogeneous heat bath, where the dissipation rate \( \nu \) is a function of space and time. The dissipation rate \( \nu \) may also be an even function of the momentum, \( \nu(|p|) \). This of importance since we may not expect the embedding medium to behave as a Newtonian fluid at relativistic energies. A temperature varying in space and time does not pose a problem, nor does an additional time-dependent external force. In this case the produced heat \( dQ = T ds_m \) is the loss of particle energy in collisions, \(-dE\) reduced by the work extracted via the external force.

**Generalizations in the framework of General Relativity.**

The monotonic increase of entropy is a fundamental principle of physics and the Universe is known to expand, as was discovered by E. Hubble in 1929. The discussion whether there is a direct connection between these observations has never stopped [2, 3, 4, 5, 6]. Therefore we aspire a formulation of the FT consistent with General Relativity, but we restrict ourselves to the class of Friedmann-Lemaître models, which describe a spatially homogenous and isotropic, expanding or contracting Universe. The corresponding line element (as given by the Robertson-Walker metric) is \(-dt^2 + dr^2\). The important difference compared to Special Relativity is that the spatial part, \( dr^2 \), is scaled by a time dependent factor \( \sqrt{1 - \frac{\xi^2}{R^2}} \) describing the expansion or contraction of the Universe:
\[
dr^2 = R^2(t) h_{ij}(\xi) \, d\xi^i \, d\xi^j .
\]

The Latin indices describe spatial components numbered by 1 to \( n \). We do not have to deal with the details of the metric tensor \( h \) describing the spatial geometry. The result will be valid for arbitrary geometries. The expansion rate \( H(t) = \dot{R}(t)/R(t) \), named Hubble function, is one of the most important quantities in cosmology and its present value is a direct observable [21]. The typical frame for a cosmic heat bath is the frame of the cosmic microwave background.

In General Relativity, the correct equations of motion include the covariant differential \( Dp \) of the momentum.

\[\text{FIG. 1: A sketch of spacetime showing a spatial slice of the heat bath at fixed time and the world line of a Brownian particle in a (locally) expanding Universe.}\]
(Denoting by $p$ the 4-vector, the components of $Dp$ are $Dp^a = dp^a + \Gamma^a_{\mu\nu}p^\mu dx^\nu$.) Its spatial components replace the left hand side of (12), and can be split up into a spatially covariant part, $(n)Dp$, and a contribution due to the time-dependent scaling:

$$Dp = (n)Dp + H(t)p.$$  

(20)

Therefore the covariant Langevin equation, generalizing Eq. (12) to be valid in an expanding or contracting Universe of arbitrary spatial curvature, reads

$$(n)Dp = -[\nu(||p||, t) + H(t)] p \ dt + (n)Dp_s.$$  

(21)

The distribution of the stochastic impacts $(n)Dp_s$ is found after substituting the Euclidean metric $\delta_{ij}$ by $h_{ij}$ in (13). Applying the time-reversal map we find that Eq. (17) gains a second term:

$$ds_m = -\frac{dE}{T} - \left[ \frac{\|p c\|^2}{ET} \right] d\ln R$$

$$= d_{\text{static}} - H\left(p, \frac{d\ln R}{T}\right).$$  

(22)

The numerator $(p, d\ln R)$ in (22) is the canonical line integral (canonical one-form) in phase space. The Integral FT (17) extends to an expanding ($H > 0$) or contracting ($H < 0$) spacetime when this second term is taken into account. It has a clear geometric interpretation: The Hubble function is the external curvature of space,

$$DN = H \ dr,$$  

(23)

with $N$ being the time-like normal vector to the space of the heat bath as depicted in Fig. 1. This permits the second term in (22) to be written as

$$-\left(\frac{p, DN}{T}\right).$$

Since the particle energy $E = p^0 = -p_0 = -(p, N)$ is the zero component of the 4-vector $p$, the first term in (22) equals the differential

$$\frac{d(p, N)}{T} = \frac{(Dp, N) + (p, DN)}{T},$$

such that the sum of both terms is the projection

$$ds_m = \frac{(Dp, N)}{T}.$$  

(24)

As described by Eq. (28), the geodesic flow $N$ may be convergent or divergent, corresponding to a contracting or expanding Universe, but this does not imply a change of sign for the entropy production $ds_m$.

Conclusions. — Relativistic FTs have been established that remain valid in the regime of high temperatures or low masses, $mc^2 \ll k_B T$. By investigating the entropy production for particle and environment separately, we could determine the physically correct discretization rule, which had not been possible so far by relativistic invariance alone. These FTs were found to extend in the framework of General Relativity. Such a formulation reveals the influence of a time-depended gravitation field on the local entropy production. The dissipated heat $Ts_m$ is the exchanged 4-momentum $Dp$ projected on the local time direction of the heat bath.

On the theoretical road ahead, one may expect Integral FTs to hold for arbitrary time-dependent and inhomogeneous fields, such as gravitational waves, when the concise expression (24) is applied. For the process originally introduced in [10], the weaker inequality (1) has been proven recently [22] under general conditions. Experimentally, the relativistic FT is not only subject of high energy physics and cosmology. An ultra relativistic FT can be tested with a high-precision spectroscopy experiment by shining with a laser on an exited granulate of glass beads, so that the granulate serves as a heat bath and the photons are the relativistic “Brownian” particles.

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Equation (16) in [17] contains a vanishing term.