Squeezed Dirac and Topological Magnons in a Bosonic Honeycomb Optical Lattice

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Quantum information storage using charge-neutral quasiparticles are expected to play a crucial role in the future of quantum computers. In this regard, magnons or collective spin-wave excitations in solid-state materials are promising candidates in the future of quantum computing. Here, we study the quantum squeezing of Dirac and topological magnons in a bosonic honeycomb optical lattice with spin-orbit interaction by utilizing the mapping to quantum spin-1/2 XYZ Heisenberg model on the honeycomb lattice with discrete $Z_2$ symmetry and a Dzyaloshinskii-Moriya interaction. We show that the squeezed magnons can be controlled by the $Z_2$ anisotropy and demonstrate how the noise in the system is periodically modified in the ferromagnetic and antiferromagnetic phases of the model. Our results also apply to solid-state honeycomb (anti)ferromagnetic insulators.

I. INTRODUCTION

Quantum squeezing is the mechanism for reducing the noise of a given quantum observable at the expense of enhancing the noise of its conjugate observable \cite{14}. The spin squeezing \cite{5} in particular plays a vital role in the detection of quantum entanglement \cite{6-8} and also present itself as a promising candidate for quantum-information processing \cite{9}. In recent years, quantum squeezing has expanded tremendously to different systems such as photons \cite{10-13} and phonons \cite{14, 15}. Recently, squeezed magnons (collective spin-wave excitation) in solid-state materials have garnered much attention \cite{16-19} as reported in the cubic antiferromagnetic insulators \textsuperscript{1}XPS, \textsuperscript{2}Mn and \textsuperscript{3}Fe \cite{24-27}. In the antiferromagnetic phase, we find that the squeeze coherent oscillations of magnons depend on the nature of phases in 2D quantum magnetism \cite{28, 31}. In particular, the $p$-orbital bosons trapped in an optical lattice can be used as a model for quantum spin-1/2 XYZ Heisenberg model \cite{32} with a discrete $Z_2$ symmetry. For a particular choice of 2D optical lattice spin-orbit interaction (SOI) or fictitious gauge fluxes can be engineered with laser beams and provide topologically non-trivial band structures with integer Chern numbers \cite{33}. In the corresponding quantum spin model, this would correspond to a synthetic Dzyaloshinskii-Moriya (DM) SOI \cite{34, 35}, therefore the associated magnetic excitations would correspond to topological magnons.

In this paper, we study the squeezed coherent oscillations (periodic modulation of noise) of Dirac and topological magnons in a $p$-orbital bosonic atoms trapped in a honeycomb optical lattice. This system maps to a quantum spin-1/2 XYZ Heisenberg model with discrete $Z_2$ symmetry \cite{32}. We study the magnon squeezing of the corresponding quantum spin system in the ferromagnetic and antiferromagnetic phases with a DM SOI, which introduces topological features in the associated magnon dispersions. We show that the squeeze coherent oscillations of magnons in the ferromagnetic phase requires no dipolar interactions \cite{21, 23}. This is a consequence of the $Z_2$ symmetry of the Hamiltonian.

In the antiferromagnetic phase, we find that the squeeze coherent oscillations of magnons depend on the counter-precession of magnon intrinsic spins in the system. Furthermore, we map the system to a $Z_2$-invariant hardcore bosons on the honeycomb lattice and uncover the mean-field phase diagram with gapped Goldstone modes in each phase. Our results are applicable to solid-state materials such as honeycomb antiferromagnetic in-
sulators XPS \text{3} (X ≡ Mn and Fe) \text{24}\text{27}. We hope that these results will pave the way towards the utilization of Dirac and topological magnons in quantum information storage and spintronics.

The organization of this paper is as follows. In Sec. \text{II} we introduce the p-orbital bosonic atoms trapped in a 2D optical lattice and the mapping to XYZ quantum spin-1/2 Heisenberg model. We also show the symmetry transformations associated with the quantum spin system. In Sec. \text{IIA} and Sec. \text{IIB} we derive the squeeze Hamiltonian of the XYZ quantum spin-1/2 Heisenberg model on the honeycomb lattice with DM interaction and discuss the associated magnon band structures. Sec. \text{III} discusses the squeezing properties and the coherent oscillations of magnon in our model. In Sec. \text{IV} we present the concluding remarks. Appendix \text{A} analyzes the topological aspects of magnons in our model and Appendix \text{B} discusses the mapping to Z$_2$-invariant hardcore bosons; we also uncover the complete mean-field phase diagram.

\section{MODEL}

The p-orbital bosonic atoms of mass $m$ trapped in a 2D optical lattice can be described by a tight binding Hamiltonian \text{32}. At half-filling (zero magnetic field) it maps to a quantum spin-1/2 XYZ Heisenberg model

\begin{equation}
\hat{H}_{\text{XYZ}} = \sum_{\langle i,j \rangle} [J (1 + \gamma) \hat{S}^z_i \hat{S}^z_j + (1 - \gamma) \hat{S}^y_i \hat{S}^y_j] + J_2 \hat{S}^z_i \hat{S}^z_j,
\end{equation}

where the symbol $\langle i,j \rangle$ represents the sum over nearest neighbour (NN) sites and $J, J_2 > 0$ are exchange constants and $\gamma \neq 0$ is an anisotropy. The most important feature of the p-orbital Bose system is the manifestation of Z$_2$ symmetry. In the spin language, this corresponds to the transformations $\hat{S}^x_{ij} \rightarrow -\hat{S}^x_{ij}$, $\hat{S}^y_{i} \rightarrow -\hat{S}^y_{i}$ and $\hat{S}^z_{ij} \rightarrow -\hat{S}^z_{ij}$ for $\gamma \neq 0$. This is synonymous with the fact that no spin component commutes with the Hamiltonian. Note that the sign of $\gamma$ in Eq. (1) can be changed by the canonical transformation $\hat{S}^x_{ij} \rightarrow -\hat{S}^x_{ij}$, $\hat{S}^y_{ij} \rightarrow -\hat{S}^y_{ij}$, that is $\pi/2$-rotation about the z-axis. Therefore the ground state of Eq. (1) is independent of the sign of $\gamma$.

The mapping from the bosonic p-orbital atoms in a 2D optical lattice to quantum spin 1/2 XYZ system makes no assumptions regarding the geometry of the 2D lattice \text{32}. Here, we study this model on a honeycomb lattice. In the limit $J_2 < J (1 + \gamma)$ the spins would prefer to anti-align along the x-axis and the other terms in Eq. (1) act as quantum fluctuations. In this paper, we work in this limit and set $J_2 = J$ and $0 < \gamma < 1$, which preserves the Z$_2$ symmetry of the Hamiltonian. The quantization axis will be chosen along the x-direction. However, since a $\pi/2$ rotation about the y-axis transforms $\hat{S}^x_{ij} \rightarrow \hat{S}^z_{ij}$ and $\hat{S}^z_{ij} \rightarrow -\hat{S}^x_{ij}$, the quantization axis can equally be chosen along the z-axis after the transformation. The XYZ Heisenberg model can also be mapped to Z$_2$-invariant hardcore bosons (see Appendix \text{B}).

On the honeycomb lattice spin-orbit interaction (SOI) can be allowed. In the quantum spin language with x-axis as the quantization axis the linear order term in the perturbative expansion of the SOI corresponds to the DM interaction

\begin{equation}
\hat{H}_{\text{so}} = \Delta_{\text{so}} \sum_{\langle i,j \rangle} \nu_{ij} \hat{x}_i \times \hat{S}_j,
\end{equation}

where $\langle \langle i,j \rangle \rangle$ represents sum over next-nearest neighbour (NNN) sites and $\nu_{ij} = \pm$ depending on the hopping along the NNN sites. For magnetic insulators, the SOI term is present due to lack of inversion symmetry of the lattice according to the Moriya rules \text{35}. This occurs on the NNN sites for the honeycomb lattice \text{39}. In the bosonic language, the SOI maps to a fictitious gauge flux for the bosons, which is analogous to a bosonic version of the Haldane model \text{40}. The presence of gauge flux makes the system topologically nontrivial and can be controlled by laser beams \text{33}. Note that the SOI also preserves the Z$_2$ symmetry of the original Hamiltonian \text{1}. Hence, the total quantum spin Hamiltonian can be written as

\begin{equation}
\hat{H} = \hat{H}_{\text{XYZ}} + \hat{H}_{\text{so}}.
\end{equation}

\section{A. Antiferromagnetic phase at half-filling}

In this section, we commence with the antiferromagnetic phase at half-filling (zero magnetic field). In this model there is no geometric spin frustration and the quantum fluctuations about the mean-field ground state can be represented by the standard Holstein-Primakoff (HP) transformations \text{50}. \begin{align*}
\hat{S}^z_{iA} &= S - \hat{n}_{iA}, \quad \hat{S}^+_{iA} \approx \sqrt{2S} \hat{b}_{iA}, \quad \hat{S}^-_{iA} = (\hat{S}^+_{iA})^\dagger, \\
\hat{S}^z_{iB} &= -S + \hat{n}_{iB}, \quad \hat{S}^+_{iB} \approx \sqrt{2S} \hat{b}_{iB}, \quad \hat{S}^-_{iB} = (\hat{S}^+_{iB})^\dagger,
\end{align*}

where $\hat{n}_{i\alpha} = \hat{b}^\dagger_{i\alpha} \hat{b}_{i\alpha}$; $\alpha = A, B$ sublattices of the honeycomb lattice in Fig. (1) and $S^z = S^z \pm i S^y$ are the raising and lowering spin operators respectively. We substitute the HP transformations into Eq. (3) and drop the constant mean-field energy. After Fourier transform, the Hamiltonian in momentum space can be written as

\begin{equation}
\hat{H} = S \sum_{k,\alpha,\beta} \left[ \Omega_{\alpha,\beta} \hat{b}^\dagger_{k\alpha} \hat{b}_{k\beta} + \frac{\Delta_{\alpha,\beta}}{2} \left( \hat{b}^\dagger_{k\alpha} \hat{b}_{k-\beta} + \hat{b}_{-k\alpha} \hat{b}_{k\beta} \right) \right],
\end{equation}

where

\begin{equation}
\Omega_{\alpha,\beta} = \begin{pmatrix} \nu_{0} - m_{k} & \nu_{1} \lambda_{k} \\ \nu_{1} \lambda_{k} & \nu_{0} - m_{k} \end{pmatrix}_{\alpha,\beta},
\end{equation}

\begin{equation}
\Delta_{\alpha,\beta} = \begin{pmatrix} 0 & \nu_{2} \lambda_{k} \\ \nu_{2} \lambda_{k} & 0 \end{pmatrix}_{\alpha,\beta},
\end{equation}

The mapping from the bosonic p-orbital atoms in a 2D optical lattice to quantum spin 1/2 XYZ system makes no assumptions regarding the geometry of the 2D lattice \text{32}. Here, we study this model on a honeycomb lattice. In the limit $J_2 < J (1 + \gamma)$ the spins would prefer to anti-align along the x-axis and the other terms in Eq. (1) act as quantum fluctuations. In this paper, we work in this limit and set $J_2 = J$ and $0 < \gamma < 1$, which preserves the Z$_2$ symmetry of the Hamiltonian. The quantization axis will be chosen along the x-direction. However, since a $\pi/2$ rotation about the y-axis transforms $\hat{S}^x_{ij} \rightarrow \hat{S}^z_{ij}$ and $\hat{S}^z_{ij} \rightarrow -\hat{S}^x_{ij}$, the quantization axis can equally be chosen along the z-axis after the transformation.
with $\alpha$ would be to find the eigenvalues of Eqs. (5) and (6): In order to diagonalize the Hamiltonian (4), a first step site respectively. $\delta_1 = a(0, -\hat{y})$, $b_1 = -\sqrt{3a\hat{x}}$, $b_2 = a(\sqrt{3\hat{x}}, -3\hat{y})/2$. The sublattices $A$ and $B$ are labeled by different colors.

$$\lambda_k = \sum_{j=1}^{3} e^{ik_j \delta_j}, \quad m_k = 2\Delta_{so} \sum_{j=1}^{3} \sin k_j \cdot b_j. \quad (7)$$

The coefficients are given by

$$v_1 = \frac{J\gamma}{2}, \quad v_2 = J(1-\gamma/2), \quad v_0 = 3JS(1+\gamma). \quad (8)$$

In order to diagonalize the Hamiltonian [4], a first step would be to find the eigenvalues of Eqs. (5) and (6): $\Omega_{\alpha\beta} = \Omega_{k\alpha} \delta_{\alpha\beta}$, $P_{k\alpha} = P_{k\alpha} = \delta_{\alpha\beta}$, where

$$\Omega_{k\alpha} = v_0 + m_k + (-1)^\alpha |v_1 \lambda_k|, \quad (9)$$

$$\Delta_{k\alpha} = (-1)^\alpha |v_2 \lambda_k|, \quad (10)$$

with $\alpha = 1, 2$ for $A, B$ sublattices respectively. Now, Eq. (4) can be written as

$$\hat{H} = \frac{1}{2} \left( \hat{H}_0 + \hat{H}_S \right), \quad (11)$$

where

$$\hat{H}_0 = \sum_{k,\alpha} \left( \Omega_{k\alpha} \hat{b}_{k\alpha} \hat{b}_{k\alpha} + \Omega_{-k\alpha} \hat{b}_{-k\alpha} \hat{b}_{-k\alpha} \right), \quad (12)$$

$$\hat{H}_S = \sum_{k,\alpha} \Delta_{k\alpha} \left( \hat{b}_{+k\alpha} \hat{b}_{k\alpha} + \hat{b}_{-k\alpha} \hat{b}_{k\alpha} \right), \quad (13)$$

and $\Omega_{k\alpha} \neq \Omega_{-k\alpha}$ for $\Delta_{so} \neq 0$. The magnon pair are correlated with wave vectors of equal magnitude but opposite in direction $k$ and $-k$. They constitute a squeezed Hamiltonian with two-magnon modes. As shown in Sec. [1B] the off-diagonal term is nonzero even in the ferromagnetic phase due to the Z$_2$ symmetry of the Hamiltonian. Now, Eq. (11) can be brought to a diagonal form by the Bogoliubov transformation

$$\begin{pmatrix} \hat{b}_{k\alpha} \\ \hat{b}_{-k\alpha} \end{pmatrix} = P_{k\alpha} \begin{pmatrix} \hat{d}_{k\alpha} \\ \hat{d}^\dagger_{-k\alpha} \end{pmatrix} \quad (14)$$

where $\hat{d}_{k\alpha}$ are the creation (annihilation) operators of the quasiparticles. They obey the commutation relation $[\hat{d}_{k\alpha}, \hat{d}^\dagger_{k'\alpha'}] = \delta_{k,k'} \delta_{\alpha,\alpha'}$, if $|u_{k\alpha}|^2 - |v_{k\alpha}|^2 = 1_{N \times N}$. $P_{k\alpha}$ is the paraunitary operator given by

$$P_{k\alpha} = \begin{pmatrix} u_{k\alpha} & -v_{k\alpha}^* \\ -u_{k\alpha}^* & u_{k\alpha} \end{pmatrix}, \quad (15)$$

and satisfy the relation $P_{k\alpha}^\dagger \tau_j P_{k\alpha} = \tau_j$, where $\tau_3 = \text{diag}(1_{N \times N}, -1_{N \times N})$. The quantities $u_{k\alpha} = \text{diag}(u_{k1}, u_{k2})$, $v_{k\alpha} = \text{diag}(v_{k1}, v_{k2})$ can be expressed as

$$u_{k\alpha} = e^{i\phi_k} \cosh \theta_{k\alpha}, \quad v_{k\alpha} = \sinh \theta_{k\alpha}, \quad (16)$$

and

$$\phi_k = -\phi_{-k} = \tan^{-1} \left[ \frac{3\text{Im} \lambda_k}{3\text{Re} \lambda_k} \right]. \quad (17)$$

Here, $\text{Re}$ and $\text{Im}$ denote the real and imaginary parts.

$$\tanh 2\theta_{k\alpha} = \frac{\Delta_{k\alpha}}{\Omega_{k\alpha}}. \quad (18)$$

The band structures of magnon are discussed in Appendix A. In the antiferromagnetic phase at half filling, the magnon bands are doubly degenerate at the SU(2) rotationally symmetric point $\gamma = 0$ with a Dirac node protected by the linear Goldstone mode at $k = 0$ with energy $\omega_{k=0} = 0$. In the Z$_2$-invariant phase $\gamma \neq 0$ the linear Goldstone (Dirac) mode at $k = 0$ is gapped ($\omega_{k=0} \neq 0$) and the degeneracy of the magnon bands is also lifted by Z$_2$ symmetry of the Hamiltonian. In this case the Dirac nodes appear at the corners of the Brillouin zone $K_{\pm} = (\pm 4\pi/3, \sqrt{3})$ at finite energy regardless of the SOI. As we will show later, the squeezed coherent oscillations of magnons in the antiferromagnetic phase is independent of the SOI, but depend on the degenerate (at $\gamma = 0$) and non-degenerate (at $\gamma \neq 0$) up and down spins of the propagating magnons on the two sublattices.

### B. Ferromagnetic phase

The fully polarized (FP) ferromagnetic phase can be obtained from the antiferromagnetic phase in the presence of large magnetic field applied along the quantization axis. In this case the Hamiltonian can be modeled ferromagnetically at zero field by

$$\hat{H}_{XYZ} = -J \sum_{\langle i,j \rangle} \left( (1 + \gamma) \hat{S}_i^x \hat{S}_j^x + (1 - \gamma) \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z \right), \quad (19)$$

where $J > 0$ and $0 < \gamma < 1$. We apply the standard Holstein-Primakoff (HP) transformations [3D]: $\hat{S}_{i\alpha} = \hat{S}_i - \hat{n}_{i\alpha}, \hat{S}_{i\alpha}^+ \approx \sqrt{2} \hat{b}_{i\alpha}, \hat{S}_{i\alpha}^- = (\hat{S}_{i\alpha}^+)\dagger$. With the inclusion of $\hat{H}_{so}$ the resulting momentum space Hamiltonian is given by

![FIG. 1: Color online. Schematics of the honeycomb lattice with the gauge flux ($\phi = \pi/2$) treading the NNN bonds. Here, $\delta_1$ and $b_i$ are the vectors connecting the NN and NNN sites respectively. $\delta_3 = a(\sqrt{3}\hat{x}, \hat{y})/2$, $\delta_2 = a(-\sqrt{3}\hat{x}, \hat{y})/2$ and $\delta_1 = a(0, -\hat{y})$, $b_1 = -\sqrt{3a\hat{x}}$, $b_2 = a(\sqrt{3\hat{x}}, -3\hat{y})/2$. The sublattices $A$ and $B$ are labeled by different colors.](image_url)
where $J_z$ no longer an exact eigenstate of the Hamiltonian due to

Now, Eq. (20) can be written as Eq. (11) with

SU(2) rotationally invariant ferromagnets ($\gamma = 0$) or $\mathrm{U}(1)$-invariant ferromagnets the off-diagonal term $\Delta_{k\alpha}$ is zero. It can only be induced when the dipolar interaction is taken into account [22]. In the present model, however, $\Delta_{k\alpha}$ is nonzero provided $\gamma \neq 0$. The $\Delta_{k\alpha}$ term is the hallmark of magnon squeezing in magnetic systems. We have discussed the topological aspects of magnon in Appendix A. In the ferromagnetic phase the magnon bands form Dirac nodes at $\mathbf{K}_\pm = (\pm 4\pi/3\sqrt{3}, 0)$ when SOI is neglected with a quadratic Goldstone mode at $\mathbf{k} = 0$ for $\gamma = 0$ and a gapped Goldstone mode at $\mathbf{k} = 0$ for $\gamma \neq 0$. Unlike the antiferromagnetic phase, topological magnons are present in the ferromagnetic phase once SOI is introduced.

III. SQUEEZING OF MAGNON

A. Magnon squeezed states

Having derived the two-magnon modes squeezed Hamiltonian, we now turn to the squeezing properties of magnons in our model. The $\mathbb{Z}_2$ symmetry of the Hamiltonian ($i.e., \gamma \neq 0$) provides an interesting squeezing property in this system. We note that up to an irrelevant phase factor, the quasiparticle transformation in Eq. (14) can be written as $\hat{d}_{k\alpha} = S_{k,-k}(z_{k\alpha})\hat{b}_{k\alpha}S_{k,-k}^{-1}(z_{k\alpha})$ where $S_{k,-k}(z_{k\alpha}) = \exp\left[z_{k\alpha}^{\dagger}\hat{b}_{k\alpha}\hat{b}_{-k\alpha} - z_{k\alpha}^{\dagger}\hat{b}_{-k\alpha}\hat{b}_{k\alpha}\right]$, and $z_{k\alpha} = \theta_{k\alpha}e^{-i\phi_{k\alpha}}$. The unitary operator $S_{k,-k}(z_{k\alpha})$ with the property $S_{k,-k}^{\dagger}(z_{k\alpha}) = S_{k,-k}^{-1}(z_{k\alpha})$ is called a two-magnon mode squeeze operator similar to that of phonons [10] [11].

The distinguishing feature of the present model is that the $\mathbb{Z}_2$ symmetry of the Hamiltonian makes the quasiparticle transformations to be well-defined even for the $\mathbf{k} = 0$ mode. The two-magnon mode vacuum is given by $|0\rangle_{k\alpha} \otimes |0\rangle_{-k\alpha}$ with $b_{k\alpha}|0\rangle_{k\alpha} = b_{-k\alpha}|0\rangle_{-k\alpha} = 0$. Applying the squeezed operator to a vacuum gives a squeezed vacuum defined as $|\psi_{\alpha}\rangle_0 = S_{k,-k}(z_{k\alpha})|0\rangle_{k\alpha} \otimes |0\rangle_{-k\alpha}$, where $d_{k\alpha}|\psi_{\alpha}\rangle_0 = d_{-k\alpha}|\psi_{\alpha}\rangle_0 = 0$. Therefore the quasiparticle excitations $d_{\pm k\alpha}$ are generated by squeezing the $\hat{b}_{\pm k\alpha}$. Hence, they are called two-mode magnon squeezing operators. We can define a squeezed magnon entangled state as

$$|\Psi\rangle = c_1|\psi_A,\psi_B\rangle_0 + c_2|\psi_A + 1,\psi_B - 1\rangle_0,$$  (27)

where $|c_1|^2 + |c_2|^2 = 1$. An entangled magnon state of this form can be utilized in quantum memory [14] [28]. We also note that the total $x$-component of the spins carried by the magnons $\hat{S}^x = \sum_i(\hat{S}_{i,A}^x + \hat{S}_{i,B}^x) = \sum_i(-\hat{n}_{i,A} + \hat{n}_{i,B})$ is not a conserved quantity for $\gamma \neq 0$. In the HP spin-2 boson mapping it is easily shown that $\langle \psi_A|\hat{S}_{x}^z|\psi_A\rangle_0 = -1$ and $\langle \psi_B|\hat{S}_{x}^z|\psi_B\rangle_0 = 1$. Therefore the two-magnon modes in the $A$ and $B$ sublattices carry equal and opposite non-degenerate spins precessing along the $x$-quantization axis for $\gamma \neq 0$. The ideas of magnon qubit, magnon spintronics, and magnon quantum computing are based on the manipulation of these intrinsic magnon spins.

The squeezing of the magnetization components can be calculated in the squeezed vacuum states. For instance the variance (squared uncertainty) of the $y$ and $z$ components of the magnetization can be written as

$$\langle \Delta S_{\alpha\gamma,y}^2 \rangle_0 = \langle S_{\alpha\gamma,y}^2 \rangle_0 - \langle S_{\alpha\gamma,y} \rangle_0^2,$$  (28)

where the average of the magnetization along the $y$ and $z$ directions vanish, $\langle S_{\alpha\gamma,y,z} \rangle_0 = 0$. For the $\mathbf{k} = 0$ mode we find

$$\langle \Delta S_{\alpha\gamma,z}^2 \rangle_0 / N = \frac{1}{4} \exp[\pm 4\theta_{0\alpha}],$$  (29)

where $\theta_{0\alpha}$ is real as given in Eq. (18). $N$ is the total number of sites and $\pm$ sign applies to $y$ and $z$ components respectively. For ferromagnet (FM) and antiferromagnet
FIG. 2: Color online. The symmetric off-diagonal coherent oscillations in the antiferromagnetic phase as a function of time for several values of $\gamma$. The coherent oscillations are independent of SOI.

FIG. 3: Color online. The antisymmetric off-diagonal coherent oscillations in the antiferromagnetic phase as a function of time for several values of $\gamma$. The coherent oscillations are independent of SOI.

(AFM) in mode 1 on sublattice A we obtain
\[
\theta_{01}^{\text{FM}} = \frac{1}{2} \tanh^{-1} \left( -\frac{1}{3} \right),
\]
\[
\theta_{01}^{\text{AFM}} = \frac{1}{2} \tanh^{-1} \left( -\frac{2 + \gamma}{2 + \gamma} \right).
\]
In the antiferromagnetic case the squeezing is dependent on $\gamma$, but not in the ferromagnetic case. We see that the reduction of the quantum noise in $z$ component of the magnetization increases the noise in the $y$ component.

B. Coherent oscillations of squeezed magnons

In this section, we study the periodic modulation of noise in the system. We note that the Hamiltonian is similar to those generated through impulsive stimulated Raman scattering between magnons and light interactions, where a laser pulse is applied on the magnetic insulator [16, 17]. We imagine this scenario in a bosonic honeycomb optical lattice or honeycomb magnetic insulators [24–27]. After the pulse is applied the system will evolve in time to new excitations. Suppose a delta function laser pulse is applied [16, 17, 51], the integration of the Schrödinger equation at $t > 0$ gives
\[
|\psi_t\rangle = e^{iH_0t} \exp \left[ \sum_{\alpha} \xi_{\alpha}^{*} \hat{b}_{\alpha}^{\dagger} \hat{b}_{-\alpha} - \xi_{\alpha} \hat{b}_{\alpha}^{\dagger} \hat{b}_{-\alpha} \right] |\psi_0\rangle,
\]
where $\xi_{\alpha} = iI_0 \Delta_{\alpha}$, and $I_0$ is a constant that is proportional to the refractive index and the intensity of the laser pulse.
In the squeezing of magnons the system should contain both diagonal and off-diagonal contributions, but it...
is the off-diagonal terms, rather than the diagonal ones, which are responsible for the coherent oscillations in the system. Therefore, we calculate the off-diagonal correlation function of the magnonic operators in the evolved wave function at finite time. We define a symmetric ($S$) and antisymmetric ($A$) off-diagonal correlation functions generated by

$$C_S(\gamma, t) = \sum_{j,\alpha,\beta} \delta_{\alpha\beta} \langle \hat{S}_{j,\alpha}^+ \hat{S}_{j,\beta}^- + \hat{S}_{j,\beta}^- \hat{S}_{j,\alpha}^+ \rangle_t,$$

$$C_A(\gamma, t) = \sum_{j,\alpha,\beta} \nu_{\alpha\beta} \langle \hat{S}_{j,\alpha}^+ \hat{S}_{j,\beta}^- + \hat{S}_{j,\beta}^- \hat{S}_{j,\alpha}^+ \rangle_t,$$

where $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$ and 0 otherwise; $\nu_{\alpha\beta} = 1$ for $\alpha = \beta \in A$ sublattice and $\nu_{\alpha\beta} = -1$ for $\alpha = \beta \in B$ sublattice. The symmetric function can be regarded as a measure of the $Z_2$ symmetry of the Hamiltonian. Using the HP transformations and the Baker-Campbell-Hausdorff Lemma we obtain the following expressions

$$C_S(\gamma, t) = 2I_0 S \sum_{k,\alpha} \Delta_{\kappa\alpha} \sin(2\Omega_{\kappa\alpha} t),$$

$$C_A(\gamma, t) = 2I_0 S \sum_{k,\alpha} (-1)^\alpha \Delta_{\kappa\alpha} \sin(2\Omega_{\kappa\alpha} t).$$

In the antiferromagnetic phase, $\Omega_{\kappa\alpha} = (\Omega_{\kappa\alpha} + \Omega_{-\kappa\alpha})/2$ and it is independent of the SOI mass term $m_k$. The same expression holds for $C_S(A)(\gamma, t)$ in the ferromagnetic phase, but the functions $\Omega_{\kappa\alpha}$ and $\Delta_{\kappa\alpha}$ are different as given in Sec. [11B] and they also depend on the SOI mass term $m_k$.

In Figs. [2] and [3] we have shown the symmetric and antisymmetric off-diagonal coherent oscillations in the antiferromagnetic phase respectively. In the former, the coherent oscillations vanishes at the SU(2) rotationally symmetric point $\gamma = 0$, because the degeneracy of the magnon modes comes with equal and opposite degenerate intrinsic magnon spins and the counter-precession of the spins cancels each other at the SU(2) symmetric point. However, for $\gamma \neq 0$ the intrinsic magnon spins are no longer degenerate resulting in non-verlapping of the symmetric coherent magnon oscillations. In the latter, the two degenerate magnon modes with equal and opposite intrinsic magnon spins at $\gamma = 0$ add and the antisymmetric coherent magnon oscillations are nonzero. As noted above, the SOI does not have any effects on the coherent oscillations in the antiferromagnetic phase. We note that the SU(2) rotationally symmetric point $\gamma = 0$ is a good approximation to the honeycomb antiferromagnetic insulators XPS$_3$ (X = Mn and Fe) [24, 27].

The ferromagnetic phase behaves differently from the antiferromagnetic phase as one would expect. In this case the symmetric and antisymmetric off-diagonal coherent oscillations of magnons are shown in Figs. [4] and [5] respectively. In contrast to antiferromagnetic phase, they depend on the SOI as well as the $Z_2$ anisotropy $\gamma$. In this case, the vanishing of the off-diagonal coherent oscillations at the rotationally symmetric point $\gamma = 0$ is not related to the counter-precessions of the magnon intrinsic spins, but due to the fact that the off-diagonal magnon mode vanishes at $\gamma = 0$ (see Sec. [11B]).

IV. CONCLUSION

In this paper, we have shown that the utilization of Bose atoms in 2D honeycomb optical lattice or equivalently spin-orbit coupling magnetic insulators could play a prominent role in quantum information. We showed that the correspondence between Bose atoms in 2D honeycomb optical lattice and quantum magnetism leads to interesting features. For the $p$-orbital Bose atoms, the discrete $Z_2$ symmetry of the corresponding XYZ quantum spin-1/2 Hamiltonian on the honeycomb lattice leads to lifted magnon mode degeneracy in the antiferromagnetic phase and gapped Goldstone modes in all phases. In the degenerate modes at the rotationally symmetric point, we found that the coherent oscillations of magnons in the squeezed magnon states depend on the opposite precession of the two-magnon modes with equal and opposite spins. This degeneracy is lifted by a discrete $Z_2$ anisotropy of the Hamiltonian. For the ferromagnetic phase, the $Z_2$ symmetry of the Hamiltonian naturally allows an off-diagonal term necessary for magnon squeezing to exist in stark contrast to rotationally symmetric ferromagnets in which a dipolar interaction is required [21, 23]. In solid-state materials, spontaneous Raman scattering or femtosecond optical pulses measurements in honeycomb antiferromagnetic insulators XPS$_3$ (X = Mn and Fe) [24, 27] could provide evidence of magnon squeezing similar to those found in the cubic antiferromagnetic insulators XF$_2$ (X = Mn and Fe) [16, 17].

We also discussed the $Z_2$-invariant bosonic (magnetic) phases in the hardcore boson mapping (see Appendix B) where the model is devoid of quantum Monte Carlo (QMC) sign problem in all the parameter regimes on the honeycomb lattice. We note that the one-dimensional version of the XYZ quantum spin-1/2 Hamiltonian is a playground for exploring quantum entanglement of spin qubit states [52, 53].

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Appendix A: Topological aspects of magnons

First we study the magnon band structures. At the continuous rotationally symmetric antiferromagnetic point $\gamma = 0$ the magnon bands $\omega_{\kappa\alpha} = \Omega_{\kappa\alpha} - \Delta_{\kappa\alpha}$...
are doubly degenerate and they possess a gapless linear Goldstone mode near $k \to 0$ as expected for a rotationally invariant system as shown Figs. 6(i) and (ii). For $\gamma \neq 0$ the continuous rotational symmetry is broken down to $Z_2$ symmetry. Quite interestingly, the degeneracy of the magnon bands is lifted [54] as well as the Goldstone mode as shown Figs. 6(iii) and (iv) with a gap of $\Delta_{\text{gap}}^{\text{AFM}} = 3\sqrt{2}|\gamma|$. This is one of manifestations of the $Z_2$ symmetry of the Hamiltonian [1]. However, SOI is unable to open a topological gap at the Dirac points $K_x = (\pm 4\pi/3\sqrt{3}, 0)$, but induces asymmetry in the magnon bands. The absence of a topological gap is as a result of the magnetic flux configuration in the half-filled antiferromagnetic phase. In other words, the DM-induced fictitious magnetic flux is destructive due to opposite sign of the spins in the Néel state. However, this can be lifted by introducing a moderate external magnetic field perpendicular to the lattice plane.

In the ferromagnetic phase which can be achieved from the antiferromagnetic phase by applying a strong magnetic field, the magnon bosonic operators have off-diagonal terms away from the rotationally symmetric point ($\gamma = 0$) (see Sec. IIIB). Therefore, the quadratic Goldstone mode near $k \to 0$ in Figs. 7(i) and (ii) are gapped ($\Delta_{\text{gap}}^{\text{FM}} = 3|\gamma|\sqrt{2}$) by the $Z_2$ symmetry of the Hamiltonian as shown in Figs. 7(iii) and (iv). In this case, however, SOI opens a topological gap at the Dirac points $K_x = (\pm 4\pi/3\sqrt{3}, 0)$.

The topological aspects of magnons can be studied by defining the Berry curvatures and Chern numbers of the magnon dispersions. The Berry curvature can be defined as

$$ B_{ij;ak} = -2\Im[\tau_3(\partial_k, p^\dagger_{ko})\tau_3(\partial_k, p_{ko})]_{ii}, \quad (A1) $$

where $i, j = \{x, y\}$. We can alternatively write the Berry curvature as

$$ B_{ij;ak} = -\sum_{\alpha \neq \alpha'} 2\Im[(P_{ko}|v_i|P_{ko'}) (P_{ko'}|v_j|P_{ko})], $$

where $v_i = \partial|\tau_3H_k|/\partial k_i$ defines the velocity operators. The Chern number is given by the integration of the
Berry curvature over the momentum space Brillouin zone

\[ n_\alpha = \frac{1}{2\pi} \int_{BZ} dk_i dk_j B_{ij;\alpha k}. \]  

(A3)

Topologically, the top and bottom magnon bands in the ferromagnetic phase carry Chern numbers of \( n_{\pm} = \pm 1 \) respectively. Whereas the top and bottom magnon bands in the antiferromagnetic phase at half-filling are topologically trivial with vanishing Chern numbers, but with a Berry phase or winding number of \( W = \pm 1 \) for a closed loop encircling the Dirac nodes for \( \gamma \neq 0 \). These results are consistent with the zig-zag magnon edge modes in Fig. 9. The topologically trivial bands have only one chiral edge state connecting the Dirac magnon points in the bulk bands, whereas the topologically nontrivial bands have gapless magnon edge states at the time-reversal-invariant momentum \( k_x = \pm \pi/\sqrt{3} \) and \( k_y = 0 \).

Appendix B: Hardcore bosons

The quantum spin-1/2 XYZ Heisenberg model can be mapped to hardcore bosons. In the limit \( J_z = J < J(1 + \gamma) \), the transformation has the form \( \hat{\alpha}_i^\dagger \leftrightarrow \hat{S}_z^\dagger \), \( \hat{\alpha}_i \leftrightarrow \hat{S}_z \), and \( \hat{n}_i \leftrightarrow \hat{S}_z^2 + 1/2 \), where \( \hat{S}_z^\pm = \hat{S}_x^z \pm i\hat{S}_y^z \) and \( \hat{n}_i = \hat{\alpha}_i^\dagger \hat{\alpha}_i \). They obey the algebra \( [\hat{\alpha}_i, \hat{\alpha}_j^\dagger] = 0 \) for \( i \neq j \) and \( \{\hat{\alpha}_i, \hat{\alpha}_j\} = 1 \). Hence, the spin-1/2 XYZ Hamiltonian maps to the bosonic Hamiltonian

\[ \hat{H}_{XYZ} = t \sum_{\langle ij \rangle} (\hat{\alpha}_i^\dagger \hat{\alpha}_j + \text{h.c.}) + t' \sum_{\langle ij \rangle} (\hat{\alpha}_i^\dagger \hat{\alpha}_j^\dagger + \text{h.c.}) \]

+ \( V \sum_{\langle ij \rangle} \hat{n}_i \hat{n}_j - \mu \sum_i \hat{n}_i, \] (B1)

where the constant terms have been dropped. Here, \( t = J(1 - \gamma)/2 \), \( t' = J\gamma/4 \), \( V = J(1 + \gamma) \), \( \mu = H_z \). In the opposite limit \( J_z > J(1 + \gamma) \), we have that \( t = J/2 \), \( t' = J\gamma/2 \), and \( V = J_z \) with \( \mu = H_z \). Therefore, the \( p \)-orbital bosonic atoms in a 2D optical lattice can be also be studied in terms of hardcore bosons.

We note that unlike frustrated systems the model (B1) is devoid of the debilitating quantum Monte Carlo (QMC) sign problem in all the parameter regimes on the honeycomb lattice. The mean-field phase diagram is depicted in Fig. 9. In the \( Z_2 \)-invariant model, quantum fluctuations are suppressed due to gapped Goldstone modes. Thus, we expect that the mean-field phase diagram will capture the essential features of the quantum phase diagram. The only difference is that the classical phase boundaries will be slightly modified.

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