On the properties of multidimensional electrostatic oscillations of an electron plasma

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We consider the classical Cauchy problem for a system of equations describing 3D arbitrary electrostatic oscillations of the cold plasma and introduce an iteration procedure that allows estimating the blow-up time from below. This procedure is constructive provided one succeeds in obtaining a two-sided estimate of an additional quantity depending on the solution. We show that this is possible in the case of one and two dimensions, as well as for solutions with zero vorticity. For the particular case of two-dimensional initial data with radial symmetry, refined sufficient conditions for destruction and preservation of smoothness in the first period of oscillations are obtained. Moreover, we give the example of estimating the blow-up time for the data for which results of numerics exist and discuss the roughness of our estimate.

KEYWORDS
blowup, cold plasma, electrostatic oscillations, quasilinear hyperbolic system

MSC CLASSIFICATION
35Q60, 35L60, 35L67, 34M10

1 | INTRODUCTION

The equations of hydrodynamics of “cold” or electron plasma in the nonrelativistic approximation in dimensionless quantities take the form (see, e.g., Alexandrov et al\textsuperscript{1} and Ginzburg\textsuperscript{8})

\[
\frac{\partial n}{\partial t} + \text{div} \ (nV) = 0, \quad \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -E - [V \times B],
\]
\[
\frac{\partial E}{\partial t} = nV + \text{rot} \ B, \quad \frac{\partial B}{\partial t} = -\text{rot} \ E, \quad \text{div} \ B = 0,
\]

(1)

\[
(2)
\]

\(n\) and \(V = (V_1, V_2, V_3)\) are the density and velocity of electrons and \(E = (E_1, E_2, E_3)\) and \(B = (B_1, B_2, B_3)\) are vectors of electric and magnetic fields. All components of solution depends on \(t \in \mathbb{R}_+\) and \(x \in \mathbb{R}^3\).

At present, much attention is paid to the study of cold plasma in connection with the possibility of accelerating electrons in the wake wave of a powerful laser pulse\textsuperscript{7}; nevertheless, there are very few theoretical results in this area.

It is commonly known that the plasma oscillations described by (1) and (2) tend to break. Mathematically, the breaking process means a blowup of the solution and the appearance of a delta-shape singularity of the electron density.\textsuperscript{6} Among main interests is a study of possibility of existence of a smooth solution as long as possible.

Let us assume that the oscillations are electrostatic, that is, rot \(E\) and the magnetic field \(B\) do not change with time. For simplicity, we set \(B = 0\).
From the first equations of (1) and (2) under the assumption that the solution is sufficiently smooth and that the steady-state density is equal to 1, it follows
\[ n = 1 - \text{div } E, \quad (3) \]
thus, \( n \) can be removed from the system. Thus, we get
\[ \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -E, \quad \frac{\partial E}{\partial t} + \text{div } E = V, \quad (4) \]

- together with
\[ \text{curl } E = 0, \quad \text{curl } ((1 - \text{div } E)V) = 0. \quad (5) \]

Consider the initial data
\[ (V, E)|_{t=0} = (V_0, E_0)(x) \in C^2(\mathbb{R}^3), \quad x \in \mathbb{R}^3, \quad (6) \]
with properties (5).

**Definition 1.** We will say that the solution to the problem (4) and (6) does not blow up (or the oscillations do not break) on \([0, T], \quad T > 0\), if the density \( n \), found as (3) remains bounded for all \( t \in T \).

The main problem is that the hyperbolic system (4) and (6) has locally in time a smooth solution; however, it does not necessarily satisfies (5). In the general case, system (4) and (5) is overdetermined.

Nevertheless, as it will be shown below, there exist important classes of solutions such that (5) automatically hold. It is affine solutions and radially symmetric solutions (see Section 3.2). Therefore, below, we assume that we deal only with solutions with property (5).

We denote \( D = \text{div } V, \quad \Xi = (\xi_1, \xi_2, \xi_3) = \text{curl } V, \quad \lambda = \text{div } E, \quad J_1 = \text{det}(\lambda \partial_{x_i} V_j), \quad i, j = 1, 2, \quad J_2 = \text{det}(\lambda \partial_{x_i} V_j), \quad i, j = 2, 3, \quad J_3 = \text{det}(\lambda \partial_{x_i} V_j), \quad i, j = 1, 3, \quad J = J_1 + J_2 + J_3. \)

System (4) and (5) implies
\[ \frac{\partial D}{\partial t} + (V \cdot \nabla D) = -D^2 + 2J - \lambda, \quad \frac{\partial \lambda}{\partial t} + (V \cdot \nabla \lambda) = D(1 - \lambda). \]

Thus, along the characteristics \( \frac{dx}{dt} = V, \quad i = 1, 2 \), starting from the point \( x_0 \), we obtain the Cauchy problem for the nonlinear system of three ODEs (nonclosed due to \( J \)):
\[ \dot{D} = -D^2 + 2J - \lambda, \quad \dot{\lambda} = D(1 - \lambda), \]
\[ (D, \lambda(t, x_0)|_{t=0} = (D_0, \lambda_0(x_0)). \quad (7) \]

**Definition 2.** We will say that the breaking of oscillations does not occur in the first period for the initial data (6), if the projection of each phase trajectory starting at \((\lambda_0, D_0) = (\text{div } E_0(x_0), \text{div } V_0(x_0)), \quad x_0 \in \mathbb{R}^3 \) on the plane \((\lambda, D)\), intersects the semiaxis \( D = 0, \lambda < 0 \) at least at one point (the starting point of the trajectory does not count).

In other words, all trajectories make at least one revolution around the coordinate origin on the plane \((\lambda, D)\).

In Section 2, we show that if a two-sided estimate
\[ J_-(\lambda, D^2) \leq J \leq J_+(\lambda, D^2) \]
is known, it is possible to construct an iterative procedure that allows one to find the guaranteed number of oscillations before the blowup (Section 2.3) and to estimate from below the lifetime of a solution with bounded density (Section 2.4). In Section 3, we consider particular cases of the problem (4) and (6) and show that in some cases the solution remains smooth for all \( t > 0 \). Section 4 is dedicated to the cases, where the estimate of \( J \) can be obtained. For the case of plain oscillations (Section 4.1) and the case of irrotational oscillations in any dimensions (Section 4.2), we prove a sufficient condition for the boundedness of the density in the first period of oscillations. In Section 5, we construct two-sided estimates of the projection of the integral trajectory to the plane \((\lambda, D)\) for the case of radially symmetric oscillations in any dimensions. In Section 6, we consider the initial data in the form of a standard laser pulse corresponding to the 2D (plain) axisymmetric
solution and prove refined estimates allowing us to obtain, in particular, sufficient conditions for the preservation of smoothness by the solution to the Cauchy problem on several periods of oscillations and compare it with the results of direct numerics. In Section 7, we summarize the results obtained and discuss further hypotheses and open problems.

2 | METHOD FOR OBTAINING A SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF DENSITY

We denote for convenience smoothness by the solution to the Cauchy problem on several periods of oscillations and compare it with the results of ROZANOVA it is enough for us to solve the equations

\[ \frac{dD}{ds} = \frac{D^3 + 2s + 2J + 1}{Ds}. \]

Further, replacing \( Z = D^2 \geq 0 \), we obtain

\[ \frac{dZ}{ds} = \frac{2(Z + s + 1)}{s} + \frac{4J}{s} \equiv Q(s, Z, J). \]  

Let us assume that the following inequality is known:

\[ 2J_-(Z, s) \leq 2J \leq 2J_+(Z, s). \]

Then, taking into account the sign of \( s \), we obtain the estimate:

\[ Q_1(s, Z) \leq Q(s, Z, J) \leq Q_2(s, Z). \]  

where

\[ Q_1(s, Z) = \frac{2(Z + s + 2J_+(s, Z) + 1)}{s}, \]  

\[ Q_2(s, Z) = \frac{2(Z + s + 2J_-(s, Z) + 1)}{s}. \]

Now, we can apply Chaplygin’s theorem on differential inequalities, according to which the solution \( Z(s) \) of the Cauchy problem for (8) with initial conditions \( Z(s_0) = Z_0 \) for \( s > s_0 \) satisfies the inequality

\[ Z_1(s) \leq Z(s, J) \leq Z_2(s), \]  

and for \( s < s_0 \), the inverse inequality

\[ Z_2(s) \leq Z(s, J) \leq Z_1(s), \]  

where \( Z_k(s) \) are the solutions to problems \( \frac{dZ}{ds} = Q_k(s, Z), \) \( Z(s_0) = Z_0, k = 1, 2. \)

Since \( Z \geq 0 \), then the phase trajectory outgoing from the point \( (s_0, Z_0) \) is bounded, if the solution \( Z_1(s) \rightarrow -\infty \) as \( s \rightarrow -\infty \). If \( Z_2(s) \rightarrow +\infty \) as \( s \rightarrow -\infty \), without becoming zero, then the phase trajectory outgoing from this point is unbounded.

Thus, to obtain sufficient conditions for the global in time boundedness of the density and sufficient blow-up conditions, it is enough for us to solve the equations \( \frac{dZ}{ds} = Q_1(s, Z) \) and \( \frac{dZ}{ds} = Q_2(s, Z) \).

First of all, note that on the plane \( (s, D) \) the motion is clockwise (for positive \( D \), the value of \( s \) increases). Assume that the equation \( Z_2(s) = 0 \) has two roots on the semiaxis \( s < 0 \), we denote them \( S_- \) and \( S_+ \), \( S_- \leq S_+ \), or one root \( S_+ \).

In the latter case, \( Z_1(s) \rightarrow +\infty \) for \( s \rightarrow -\infty \).

If the equation \( Z_2(s) = 0 \) has two roots on the semiaxis \( s < 0 \), we denote them \( s_- \) and \( s_+ \), \( s_- \leq s_+ \).

The trajectory \( D(s, t) \) lies between the curves \( \sqrt{Z_2(s)} \) and \( \sqrt{Z_2(s)} \). Denote by \( s_k^t \) the point at which \( D(s, t) \) intersects the axis \( D = 0 \) for the \( k \)th time. If \( D > 0 \), then \( \sqrt{Z_1(s)} \leq D(s, t) \leq \sqrt{Z_2(s)}, s \) increases. If \( D < 0 \), then \( -\sqrt{Z_1(s)} \leq D(s, t) \leq -\sqrt{Z_2(s)}, s \) decreases.

Let us compose the curve \( L \) as follows: It consists of the part of trajectory \( \sqrt{Z_2(s)} \) if \( D > 0 \) and switches to the part of trajectory \( -\sqrt{Z_1(s)} \) if \( D < 0 \) at the axis \( D = 0 \). If \( -\sqrt{Z_1(s)} = 0 \) has two roots at \( s < 0 \), then \( L \) switches again to the axis \( D = 0 \) to the trajectory \( \sqrt{Z_2(s)} \), and so on. Denote by \( L^t_k \) the point at which \( L \) intersects the axis \( D = 0 \) for the \( k \)th time.
for $D_0 < 0$ and in the $k-1$th time for $D_0 \geq 0$. Thus, the trajectory $D(s, t)$ makes at least one revolution if the number of points $l^k$ is more than $2(L^2, L^3, \ldots)$ for $D_0 \geq 0$ and more then $1$ ($L^1, \ldots$) for $D_0 < 0$.

In its turn, we compose the curve $l$ as follows: It consists of the part of trajectory $\sqrt{Z}(s)$ if $D > 0$, switches to the part of trajectory $-\sqrt{Z}(s)$ if $D < 0$ at the axis $D = 0$ and switches once again the the axis $D = 0$ to the trajectory $\sqrt{Z}(s)$, and so on. Denote by $l^k$ the point at which $l$ intersects the axis $D = 0$ for the $k$th time for $D_0 < 0$ and in the $k-1$th time for $D_0 \geq 0$.

We note that $s_k$ for all $k \in \mathbb{N}$ lies between $l^k$ and $L^k$.

### 2.1 General estimates of $J$

**Lemma 2.1.**

$$\bar{\mathbf{E}}^2 - \tilde{D}^2 \leq 2J \leq \mathbf{E}^2,$$

where $\bar{\mathbf{E}}$ and $\tilde{D}$ are defined as $\mathbf{E}$ and $D$ with the change of all derivatives $\partial_iV_j$ to $\partial_iV_j$.

**Proof.** The estimate (15) is a corollary of the elementary inequality $-(a-b)^2 \leq 2ab \leq (a+b)^2$, for every $a, b \in \mathbb{R}$. ∎

Estimate (15) does not yet provide an opportunity to find explicit bounds, but we will use it for particular cases.

### 2.2 Behavior in the upper hyperplane $D > 0$

Now, we are going to prove that if $D(t)$ becomes unbounded, it happens when $D < 0$.

**Lemma 2.2.** The projection of the trajectory $D = D(s, t)$ to the plane $(s, D)$ is bounded in the upper hyperplane $D > 0$.

**Proof.** As follows from the second equation of (7), the motion along the projection $D = D(s, t)$ onto the plane $(s, D)$ is clockwise for $s < 0$. Therefore, in the upper hyperplane $D > 0$, there are two possibilities for a trajectory starting from any point $(s_0, D_0 > 0)$:

1. to cross the axis $D = 0$ and fall into the lower hyperplane, which means that the projection of the phase
   trajectory $D(s, t)$ was bounded for $D > 0$,
   or
2. to tend to $+\infty$ as $s \to -0$ and $t \to t_*$ $\leq \infty$. (This would mean that a vacuum point is formed along the
   characteristic.)

Assume that the latter possibility realizes, that is, $D(t) \to +\infty$ and $s(t) \to -0$ as $t \to t_* \leq \infty$. We see from (7) that $\tilde{D} = -D^2 + 2J - s - 1$, therefore $J \to +\infty$, if $t \to t_*$ and $s(t) \to -0$. Therefore, $J > K > 0$ for sufficiently small $s < 0$. Then, (12) implies

$$\frac{dZ}{ds} \leq Q_2(s, Z) = \frac{2(Z + s + 2K + 1)}{s}.$$

$$Z_2(s) = D^2 \leq -2K - 1 - 2s + C s^2 \text{ and } Z_2(0) \leq -2K - 1 < 0.$$ This contradiction shows that our assumption was wrong and $D > 0$ is bounded. ∎

### 2.3 Counting the number of periods for which the boundedness of density is guaranteed

Based on estimates (13) and (14), we can find the number of periods for which the breaking is guaranteed not to occur.

Let, for definiteness, $D_0$ be negative. Assume that the initial data are such that the solution does not blow up and on the curve $L$, there exist points $(s = L_1, D = 0)$ and $(s = L_2, D = 0)$, $L_1 < L_2$. However, we can use $(L_2, 0)$ as initial data and check whether the equation $Z_2(s) = 0$ has the second root $(L_3)$ at $s < 0$ besides $L_2$. This would mean that there $L$ (and $D(t, s)$) makes at least two revolutions and at the axis $D = 0$, there exist two points of intersection $s = L_3$ and

$s = L_4$, $L_3 < L_4$, where $L_3$ and $L_4$ are the roots of equation $Z_2(s) = 0$. Then, we can use $(L_4, 0)$ as initial data again and so on. We can continue this procedure until at some step $n$ for the point $(L_{2n}, 0)$, taken as initial data, we do not obtain a sufficient condition for boundedness of $D$. This means that we can guarantee $n$ revolutions of the trajectory $D(t, s)$. The iterative process of counting revolutions is easy to implement numerically, if we know $Z_1$ and $Z_2$ (see the example in Section 6.3). Nevertheless, the estimate of numbers of revolutions from below may be rough.
2.4 Estimates of the guaranteed time of existence of the bounded density

Let \( \text{div } \mathbf{V}_0 < 0 \), and for the characteristic starting from the point \( x_0 \), we can guarantee \( n \) revolutions of the trajectory \( D(t,s) \) around the origin. Then, the guaranteed lifetime of a solution with bounded density, denoted as \( T(x_0) \), can be estimated as

\[
T_l(x_0) < T(x_0) < T_L(x_0),
\]

where \( T_l \) and \( T_L \) is the time of passage of \( n \) turns along a compound spiral lines \( l \) and \( L \), respectively. Thus,

\[
T_l = -\int_{s_0}^l \frac{ds}{s\sqrt{Z_2(0;s)}} - \int_{l}^3 \frac{ds}{\sqrt{Z_2(1;s)}} - \int_{L}^2 \frac{ds}{\sqrt{Z_2(2;s)}} - \cdots - \int_{l_{m+1}} \frac{ds}{\sqrt{Z_2(n;s)}},
\]

\[
T_L = -\int_{s_0}^{L_1} \frac{ds}{s\sqrt{Z_1(0;s)}} - \int_{L_1}^{L_2} \frac{ds}{\sqrt{Z_2(1;s)}} - \int_{L_2}^{L_3} \frac{ds}{\sqrt{Z_2(2;s)}} - \cdots - \int_{L_{m+1}}^{s_0} \frac{ds}{\sqrt{Z_2(n;s)}},
\]

where we denoted \( Z_i(k;s), i = 1, 2, k = 0, 1, \ldots, n \), the functions \( Z_1(s) \) and \( Z_2(s) \), based on initial data at points \( (L_{2k}, 0) \) for \( Z_1(k,s) \) and \( (L_{2k-1}, 0) \) for \( Z_2(k,s) \).

For \( \text{div } \mathbf{V}_0 \geq 0 \), the formulas are similar, except for the sum \( T_l \) begins from \(- \int_{s_0}^L \frac{ds}{s\sqrt{Z_2(0;s)}} \) and the sum \( T_L \) begins from \(- \int_{s_0}^{L_1} \frac{ds}{\sqrt{Z_1(0;s)}} \).

The time \( T_* \) of existence of the smooth solution with a bounded density can be estimated as

\[
T_* > \inf_{x_0 \in \mathbb{R}} T_l(x_0),
\]

however, \( \sup_{x_0 \in \mathbb{R}} T_L(x_0) \) can be less than \( T_* \).

Remark 2.1. The technique of counting the number of oscillations before the blowup in the case of multidimensional electrostatic nonrelativistic oscillations is similar to the case of 1D relativistic oscillations.

3 PARTICULAR CASES

3.1 1D oscillations, a criterion of a singularity formation

In this case, \( (\mathbf{V}, \mathbf{E}) = (\mathbf{V}(x_1), \mathbf{E}(x_1)) \), \( V_2 = E_2 = 0 \), \( \mathbf{E} = 0 \), condition (5) evidently holds. It is easy to see that here \( J = 0 \) and system (7) turns into

\[
\dot{\lambda} = D(1 - \lambda), \quad \dot{D} = -D^2 - \lambda.
\]

Such a system has been considered in Rozanova and Chizhonkov,\(^9\) where the following criterion for the preservation of the global in time smoothness was obtained: At each point \( x_0 \in \mathbb{R} \) (here \( x = x_1 \)), the condition

\[
\Delta = (V_{10}')^2 + 2(E_{10})' - 1 < 0 \quad (16)
\]

holds.

3.2 Globally smooth affine solutions

Definition 3. A solution \( \mathbf{V}, \mathbf{E} \) is called an affine solution if it has the form

\[
\mathbf{V} = \mathbf{Q}(t) \mathbf{r}, \quad \mathbf{E} = \mathbf{R}(t) \mathbf{r},
\]

(17)

with \((3 \times 3)\) matrices \( \mathbf{Q} = (q_{ij}) \) and \( \mathbf{R} = (r_{ij}) \).
Condition (5) dictates $q_{ij} = q_{ji}$, $r_{ij} = r_{ji}$, so we get the matrix equation

$$\dot{Q} + Q^2 + R = 0, \quad R = (1 - \text{tr}R)Q$$

or quadratically nonlinear system of 12 ODEs.

For the plain oscillations $q_{3j} = q_{j3} = r_{3j} = r_{j3} = 0$, $i, j = 1, 2$, so the number of equations is 6.

**Definition 4.** A solution $V, E$ is called radially symmetric if

$$V = F(t, r)\mathbf{r}, \quad E = G(t, r)\mathbf{r}, \quad \mathbf{r} = (x_1, x_2, x_3), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

It is easy to check that for the affine solutions and radially symmetric solutions curl $V = 0$ and condition (5) holds.

Let us consider solutions that are both affine and radially symmetric, that is, $F = \alpha(t)$, $G = \beta(t)$. Here, $D = 3\alpha$, $J = 3\alpha^2$, $\lambda = 3\beta$, and the system (7) takes a closed form

$$\dot{\alpha} = -\alpha^2 - \beta, \quad \dot{\beta} = \alpha(1 - 3\beta),$$

the first integral is

$$\alpha^2 = 2\beta - 1 + K|1 - 3\beta|, \quad K = \frac{1 - 2\beta(0) + \alpha^2(0)}{|1 - 3\beta(0)|}.$$

Recall that the positivity of density requires $1 - 3\beta > 0$, see (3). The prevailing term of the right-hand side as $\beta \to -\infty$ is $2\beta$, it tends to $-\infty$ for any $K$. This means that $\alpha$ and $\beta$ are bounded for any initial data, the density is bounded, and the smooth in time solution to the Cauchy problem exists for all radially symmetric initial data.

In the case of plane oscillations, we have

$$V = F(t, r)\mathbf{r}, \quad E = G(t, r)\mathbf{r}, \quad \mathbf{r} = (x_1, x_2, 0), \quad r = \sqrt{x_1^2 + x_2^2}.$$

For the corresponding affine solution $D = 2\alpha$, $J = \alpha^2$, $\lambda = 2\beta$, therefore system (7) takes a closed form

$$\dot{\alpha} = -\alpha^2 - \beta, \quad \dot{\beta} = \alpha(1 - 2\beta),$$

the first integral is

$$2\alpha^2 = -(1 - 2\beta)\ln |1 - 2\beta| + K(1 - 2\beta) + 1, \quad K = \frac{1 - 2\alpha^2(0)}{1 - 2\beta(0)} - \ln |1 - 2\beta(0)|.$$

From the positivity of density, we have $1 - 2\beta > 0$; therefore, the prevailing term of the right-hand side as $\beta \to -\infty$ is the logarithmic one, it tends to $-\infty$ for any $K$. This means that $\alpha$ and $\beta$ are bounded for any initial data, the density is bounded, and the smooth in time solution to the Cauchy problem exists for all radially symmetric initial data (see also Chizhonkov and Rozanova\(^{11}\)).

These examples show, in particular, that there are three-dimensional initial data that do not lead a finite-time blowup. However, as was proved in Rozanova\(^{11}\), radially symmetric solutions of the cold plasma equation, which are not affine, in the general case blow up even if they are arbitrarily small perturbations of the zero stationary state.

### 4 | THE PLAIN AND IRROTATIONAL OSCILLATIONS

In this section, we specify two cases, where it is possible to obtain an estimate giving a possibility to study the behavior of the trajectory in the lower hyperplane.
4.1 Plain oscillations

For the case of plain oscillations $\mathbf{V} = \mathbf{V}(x_1, x_2)$, $\mathbf{E} = \mathbf{E}(x_1, x_2)$, $V_3 = E_3 = 0$, $\xi_1 = \xi_2 = 0$, and we have

$$\frac{\partial \xi_3}{\partial t} + (\mathbf{V} \cdot \nabla) \xi_3 = -D \xi_3.$$

Along characteristics, this equation can be written as $\dot{\xi}_3 = -D \xi_3$; therefore, (7) implies

$$\xi_3 = C_3 (\lambda - 1), \quad C_3 = \text{const}. \quad (18)$$

Thus, from (15), we have

$$2J \leq D^2 + C_3^2 s^2,$$

and (11) takes the form

$$Q_1(s, Z) = \frac{2(2Z + s + C_3^2 s^2 + 1)}{s}. \quad (19)$$

Therefore,

$$Z_1(s) = A_4 s^4 - A_1 s + A_0,$$

where

$$A_4 = \frac{Z_0 + \frac{2}{3}s_0 + \frac{1}{2}}{s_0^2}, \quad A_2 = -\frac{|\xi_30|^2}{s_0^2}, \quad A_1 = -\frac{2}{3}, \quad A_0 = -\frac{1}{2};$$

the coefficients are substituted with the value of the constant $C_3$, found from (18).

4.2 Irrotational oscillations

The case of irrotational oscillations is analogous. Since

$$\frac{\partial \xi_i}{\partial t} + (\mathbf{V} \cdot \nabla) \xi_i = -D \xi_i - (\mathbf{E} \cdot \nabla)V_i, \quad i = 1, 2, 3,$$

then the uniqueness of the solution of the Cauchy problem implies that if $\mathbf{E}_0 = 0$, then $\mathbf{E} = 0$ identically. Therefore, (9) gives

$$2J \leq D^2,$$

and (11) is

$$Q_1(s, Z) = \frac{2(2Z + s + 1)}{s}. \quad (19)$$

Thus,

$$Z_1(s) = A_4 s^4 + A_1 s + A_0,$$

where

$$A_4 = \frac{Z_0 + \frac{2}{3}s_0 + \frac{1}{2}}{s_0^2}, \quad A_1 = -\frac{2}{3}, \quad A_0 = -\frac{1}{2};$$

For both cases, the following lemma holds.
**Lemma 4.1.** Let at a point \( x_0 \in \mathbb{R}^3 \) the initial conditions (6) are such that \( V_0 = (V_1(x_1, x_2), V_2(x_1, x_2), 0), \ E_0 = (E_1(x_1, x_2), E_2(x_1, x_2), 0) \), or \( \text{curl} V_0 = 0 \) and

\[
\text{div} V_0 < 0, \ \Delta_\text{c} = (\text{div} V_0)^2 + |\text{curl} V_0|^2 + \frac{2}{3} \text{div} E_0 - \frac{1}{6} < 0, \tag{20}
\]

or

\[
\text{div} V_0 = 0, \ \text{div} E_0 > 0, \ \Delta_\text{c} = |\text{curl} V_0|^2 + \frac{2}{3} \text{div} E_0 - \frac{1}{6} < 0. \tag{21}
\]

Then, there exists a moment \( t_0 \) such that the trajectory \((s, D)\) turns out in the upper hyperplane \( D < 0 \) (a breaking of oscillations does not occur in the first period).

**Proof.** The sense of Lemma 2.1 is such that the trajectory \( D(s, t) \) does not go to infinity during the first oscillation in the lower hyperplane \( D < 0 \). As follows from Section 2, the trajectory in the lower hyperplane \( D < 0 \) is bounded from below by the curve \( -\sqrt{Z_3(s)} \). In its turn, \( Z_3(s) \) is bounded if and only if \( A_4 < 0 \), where \( A_4 \) is defined in (19). The condition \( A_4 < 0 \) is exactly (20). Further, since the motion along the phase trajectory is clockwise, then (21) signifies that the trajectory will occur in the lower hyperplane with the condition (20) at moment \( t > 0 \). \( \Box \)

Thus, as a corollary of Lemmas 2.2 and 4.1, we get the following theorem.

**Theorem 4.1.** Let at each point \( x_0 \in \mathbb{R}^3 \) the initial conditions (6) are such that \( V_0 = (V_1(x_1, x_2), V_2(x_1, x_2), 0), \ E_0 = (E_1(x_1, x_2), E_2(x_1, x_2), 0) \), or \( \text{rot} V_0 = 0 \) and one of conditions (20) and (21) holds. Then, the density \( n \), found as (3) from the solution of the problem (4) and (6), is bounded during the first period of oscillations.

**Proof.** According to condition (20) or (21), every trajectory \( D = D(s, t) \) starts from a point do not laying in the upper hyperplane and turns out in the upper hyperplane (Lemma 4.1) and then again turns out in the lower hyperplane (possibly, at the point, do not satisfying to 20 or 21) (Lemma 2.2). However, the trajectories starting from each point \( x_0 \in \mathbb{R}^3 \) make as least one full rotation around the origin \( \lambda = D = 0 \); therefore, \( D \) is bounded during the first period of oscillations, and (3) implies that \( n \) is also bounded. \( \Box \)

### 4.3 Affine solutions, an example

For the affine solutions (17), Theorem 4.1 gives the following sufficient conditions for maintaining smoothness in the first period of oscillations:

\[
\text{tr} Q(0) < 0, \ \Delta_\text{c} = (\text{tr} Q(0))^2 + \frac{2}{3} \text{tr} R(0) - \frac{1}{6} < 0,
\]

or

\[
\text{tr} Q(0) = 0, \ \text{tr} R(0) > 0, \ \Delta_\text{c} = 4 \text{tr} R(0) - 1 < 0. \tag{22}
\]

For the axisymmetric 3D case, (22) implies \( \beta < \frac{1}{12} \), whereas as is shown in Section 3.2, the solution is globally smooth for \( \beta < \frac{1}{3} \).

For the axisymmetric 2D and 1D cases, (22) gives \( \beta < \frac{1}{8} \) and \( \beta < \frac{1}{4} \), respectively. However, the solution is globally smooth for \( \beta < \frac{1}{2} \) for both these cases.

### 5 Radially Symmetric Oscillations in Any Dimensions

For the radially symmetric case, we can obtain a two-sided estimate (9) and find the time for which we guarantee the existence of the smooth solution to the problem (4) and (6). This result can be considered as an addition to the results of the recent paper, where we proved that any solution to the Cauchy problem for \( d = 2 \) and \( d = 3 \) generally blows up, even for arbitrary small perturbation to the zero steady state.
The calculations are similar in the space of any dimension \( d \geq 1 \), we assume

\[
V = F(t, r) \mathbf{r}, \quad E = G(t, r) \mathbf{r}, \quad \mathbf{r} = (x_1, x_2, \ldots, x_d), \quad r = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.
\]

However, we are interested mostly in physical dimension \( d = 3 \). For the plain oscillations \((d = 2)\),

\[
V = F(t, r) \mathbf{r}, \quad E = G(t, r) \mathbf{r}, \quad \mathbf{r} = (x_1, x_2, 0), \quad r = \sqrt{x_1^2 + x_2^2}.
\]

It is easy to check that \( \text{curl} \ V = 0 \) and condition (5) holds.

Further,

\[
D = dF + Fr, \quad \lambda = dG + Gr, \quad J = (d - 1)FF_r + \frac{(d - 1)d}{2} F^2,
\]

therefore,

\[
2J = 2(d - 1)DF - (d - 1)dF^2.
\]

Further, if we substitute (23) to (4), we get

\[
\dot{G} = F - dFG, \quad \dot{F} = -F^2 - G,
\]

where \( \dot{f} = \frac{df}{dt} + Fr. \)

On the phase plane \((G, F)\), system (24) implies one equation

\[
\frac{1}{2} \frac{dF^2}{dG} = \frac{F^2 + G}{1 - dF},
\]

which is linear with respect to \( Y = F^2 \) and can be explicitly integrated. Indeed, we have for \( d = 2 \)

\[
2Y = (2G - 1) \ln |1 - 2G| + C_2(2G - 1) - 1,
\]

\[
C_2 = \frac{1 + 2F^2(0, r_0)}{2G(0, r_0)} - 1 - \ln |1 - 2G(0, r_0)|,
\]

for \( d = 1 \) and \( d \geq 3 \)

\[
Y = \frac{2G - 1}{d - 2} + C_d|1 - dG|^\frac{1}{d}, \quad C_d = \frac{1 - 2G(0, r_0) + (d - 2)F^2(0, r_0)}{(d - 2)|1 - dG(0, r_0)| ^\frac{1}{d}}.
\]

If \( G(0, r_0) < \frac{1}{d}, r_0 \in \mathbb{R} \), then the phase trajectories of (24) are in the half-plane \( G < \frac{1}{d} \). In this half-plane the leading term in the right-hand side of (25) is \((2G - 1) \ln |1 - 2G|\), and the leading term in the right-hand side of (26) is \( \frac{2G - 1}{d - 2} \). Therefore, \( F \) and \( G \) are bounded for any \( t > 0 \).

Along every characteristic curve starting from \( r_0 \in \mathbb{R}^+ \), the functions \( F(t) \) and \( G(t) \), solution of (24), are periodic with the period

\[
T = 2 \int_{G_-}^{G_+} \frac{d\eta}{|1 - d\eta|F(\eta)}.
\]

\( F \) is given as (25) or (26), \( G_- < 0 \) and \( G_+ > 0 \) are the lesser and greater roots of the equation \( F(G) = 0 \). Moreover, \( \int_0^T F(r) \, dr = 0 \).

We refer to Rozanova\(^{11}\) for a detailed analysis of the phase portrait and the period of oscillations.

Thus, we can obtain the evaluating functions \( Q_1 \) and \( Q_2 \) in (10).

Indeed, the following upper estimate holds:

\[
2J \leq 2(d - 1)FD \leq - \left( \sigma_1 D - \frac{(d - 1)F}{\sigma_1} \right)^2 + \sigma_1^2 D^2 + \frac{(d - 1)^2}{\sigma_1^2} F^2 \leq \sigma_1^2 D^2 + \frac{(d - 1)^2}{\sigma_1^2} F^2, \quad \sigma_1 > 0.
\]
The lower estimate is analogous:

\[
2J \geq 2(d - 1)FD - d(d - 1)F_2^2 \\
\geq \left( \sigma_2^2D + \frac{(d - 1)F_2^2}{\sigma_2^2} \right)^2 - \sigma_2^2D^2 - \left( \frac{(d - 1)^2F_2^2}{\sigma_2^2} - d(d - 1)F_2^2 \right) \geq
\]

(28)

The parameters \( \sigma_i, i = 1, 2, \) in both estimates can be different and are chosen at our convenience. Thus, taking into account (27) and (28), we get estimating functions

\[
\tilde{Q}_1(s, Z) = \frac{2 \left( (1 + \sigma_1^2)Z + s + \frac{(d - 1)^2F_2^2}{\sigma_1^2} + 1 \right)}{s},
\]

\[
\tilde{Q}_2(s, Z) = \frac{2 \left( (1 - \sigma_2^2)Z + s - \frac{(d - 1)^2F_2^2}{\sigma_2^2} + 1 \right)}{s}.
\]

The equations \( \frac{dZ}{ds} = \tilde{Q}_1(s, Z) \) and \( \frac{dZ}{ds} = \tilde{Q}_2(s, Z) \) can be solved, their solutions are

\[
\tilde{Z}_1(s) = -\frac{2s}{1 + 2\sigma_1^2} - \frac{(d - 1)^2F_2^2 + \sigma_1^2}{\sigma_2^2(1 + \sigma_1^2)} + C_1 s^{2(1+\sigma_1^2)},
\]

(29)

\[
C_1 = s_0^{-2(1+\sigma_1^2)} \left( Z(0) + \frac{2\sigma_1^2(1 + \sigma_1^2)s_0 + (1 + 2\sigma_1^2)((d - 1)^2F_2^2 + \sigma_1^2)}{\sigma_2^2(1 + 2\sigma_1^2)(1 + \sigma_1^2)} \right),
\]

(30)

and

\[
\tilde{Z}_2(s) = -\frac{2s}{1 - 2\sigma_2^2} + \frac{K - 1}{1 - \sigma_2^2} + C_2 s^{2(1-\sigma_2^2)},
\]

(31)

\[
C_2 = s_0^{-2(1-\sigma_2^2)} \left( Z(0) + \frac{2(s_0 - 1)}{2\sigma_2^2 - 1} + \frac{K}{1 - \sigma_2^2} \right).
\]

(32)

Thus, we get

\[
\tilde{Q}_1(s, Z) \leq Q(s, Z, J) \leq \tilde{Q}_2(s, Z).
\]

The new estimate \( \tilde{Q}_1(s, Z) \leq Q(s, Z, J) \) is not necessarily better that the previous estimate \( Q_1(s, Z) \leq Q(s, Z, J) \); see (19). Nevertheless, for the initial data (6) sufficiently small in the uniform norm, the new estimate improves the lower bound for the existence time of a smooth solution.

6 | A MODEL EXAMPLE IN 2D

We restrict ourselves to a special kind of initial data, the most interesting from the point of view of physics. Motivated by the form of a standard laser pulse,\(^\text{12}\) we choose as the initial data (6)

\[
|E_0(\rho)| = \left( \frac{d_0^2}{\rho} \right)^2 \rho \exp \left( -\frac{\rho^2}{\rho_0^2} \right), \quad V_0(\rho) = 0, \quad \rho = \sqrt{x_1^2 + x_2^2},
\]
\( a_0 \) and \( \rho_* \) are parameters. For the sake of simplicity, we change the space variable to \( r = \rho / \rho_* \) and reduce the data to

\[
|E_0(r)| = K e^{-r^2} r, \quad V_0(r) = 0, \quad r = \sqrt{x_1^2 + x_2^2}, \quad K = \frac{a_*^2}{\rho_*} > 0. \tag{33}
\]

Figure 1 presents the curves \( C_1 = 0 \) and \( C_2 = 0 \) on the plane \((\sigma, \lambda) (\sigma = \sigma_1 \text{ or } \sigma_2)\) for \( D_0 = 0 \) for different values of \( F_+ \). Below (above) \( C_i \), lie such values of \( \lambda_0 = s_0 + 1 \) that \( C_i < 0 \) (\( C_i \geq 0 \)) and \( Z_i \) is bounded (unbounded), \( i = 1, 2 \). This conclusion follows from the analysis of leading terms in (29) and (31).

Since the trajectory on the phase plane can go to infinity only at \( \sigma < 0 \), we study the estimate function under this condition.

### 6.1 \( D < 0 \), above estimate \( \tilde{Z}_1 \)

Thus, we have to study the curve \( C_1 = 0 \), given as (30) for \( Z(0) = 0 \). We find \( s_0 \) from this equation and denote

\[
S_1 = -\frac{1}{2} \frac{(1 + 2\sigma_1^2)(F_+^2 + \sigma_1^2)}{\sigma_1^2(1 + \sigma_1^2)}.
\]

Condition (21) implies that the solution keeps smoothness at the first rotation provided \( \lambda_0 < \frac{1}{4} \left(s_0 < -\frac{3}{4}\right) \) at the most “dangerous” point \( r = 0 \) (\( G(r) \) has the maximum here).

We have to keep in mind that \( F_+ \) cannot be chosen arbitrary. Indeed, according to (23), at the point \( r = 0 \), we chose \( \lambda_0 \) together with \( G(0) = \frac{1}{2} \lambda_0 \).

Then, we can use (25), where the constant \( C_2 \) is found from \( Y(0) = 0, G(0) = \frac{1}{2} \lambda_0 \). Then,

\[
\tilde{F}_+ = F_+(\lambda_0) = \frac{1}{2} \sqrt{4e^{-C_1} - 2}, \quad C = \frac{1}{\lambda_0 - 1} - \ln \left(\frac{1 - \lambda_0}{2}\right), \tag{34}
\]

where \( F_+(\lambda_0) = \sqrt{Y(G_m)}, G_m = -e^{-C_1} \) is the maximum point of \( Y(G) \). Graph of \( F_+(\lambda_0) \) is presented in Figure 2, left.

**FIGURE 1** Graphs of \( C_1(\lambda, \sigma) = 0 \) (1) and \( C_2(\lambda, \sigma) = 0 \) (2), \( \lambda = s + 1 \), for \( F_+ = 0.05, F_+ < \frac{\sigma_1}{2\sigma_1 + 1} \), left. Graph of \( C_2(\lambda, \sigma) = 0 \) (2), for \( F_+ = 2.5, F_+ > \frac{\sigma_1}{2\sigma_1 + 1} \), right [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 2** Dependency \( F_+(\lambda) \) (see 34), left. Dependency \( \tilde{\sigma}_1(\sigma) \) (1) and \( \tilde{\sigma}_2(\sigma) \) (2); the maximum \((\tilde{\sigma}_1, \Lambda_1)\) and the minimum \((\tilde{\sigma}_2, \Lambda_2)\), right [Colour figure can be viewed at wileyonlinelibrary.com]
Let us introduce the function \( \lambda_1(F_+(\lambda_0), \sigma_1) = S_1(F_+(\lambda_0), \sigma_1) + 1 \lambda_0 \in (0, 1) \). Taking into account (34), we can write an explicit expression for this function

\[
\lambda_1(F_+(\lambda_0), \sigma_1) = \frac{1 + 4\sigma_1^2 - (2\sigma_1^2 + 1)(1 - \lambda_0)e^{\frac{\lambda_0}{\sigma_1}}}{4\sigma_1^2(\sigma_1^2 + 1)}.
\]

We look for \( \lambda_0 \leq \lambda_1(\lambda_0) \). If such \( \lambda_0 > \frac{1}{4} \), we obtain a sufficient condition for the smoothness of the solution on the first oscillation. Thus, we have to find the fixed point \( \lambda^*_1 \) of the mapping \( \lambda_1(F_+(\lambda_0)) \). This value can be expressed in terms of the Lambert \( W \) function:

\[
\lambda^*_1(\sigma_1) = \frac{(4\sigma_1^4 - 1)L_1(\sigma_1) - 4\sigma_1^2 - 1}{(4\sigma_1^4 - 1)L_1(\sigma_1) - 4\sigma_1^4(1 + \sigma_1^2)},
\]

\[
L_1 = \text{LambertW} \left( k, \frac{1}{2\sigma_1 - 1} e^{\frac{4\sigma_1^2 + 1}{4\sigma_1^2 - 1}} \right),
\]

where \( k = -1 \) for \( \sigma_1 < \frac{1}{\sqrt{2}} \) and \( k = 0 \) for \( \sigma_1 > \frac{1}{\sqrt{2}} \).

Let us maximize \( \lambda^*_1(\sigma_1) \) (this can be done numerically). The maximum point is \( \sigma_1^* = 0.5032 \ldots \), the maximum value \( \Lambda_1 \equiv \lambda^*_1(\sigma_1^*) = 0.3058 \ldots \). We can see that the sufficient condition (21) is improved, because now we can guarantee at least one revolution of every trajectory on the phase plane for the data (33) with \( \lambda_0 \in [0, \lambda^*_1] \), that is, \( K < \frac{\lambda^*_1}{2} = 0.1529 \ldots \) instead of \( K < 0.125 \).

### 6.2 | \( D < 0 \), below estimate \( \bar{Z}_2 \)

The analysis is analogous to the previous subsection. We find \( s_0 \) from the equation \( C_2 = 0 \), given as (32) for \( Z(0) = 0 \), and denote

\[
S_2 = \frac{1}{2}\left(\frac{2\sigma_2^2 - 1}{\sigma_2^2}F_2(2\sigma_2^2 + 1) - \sigma_2^4\right).
\]

Since the denominator of \( S_2 \) vanishes at \( \sigma_2 = 1 \), we restrict ourselves by the interval \( \sigma_2 \in (0, 1) \).

For \( F_+ < F_+ = \frac{\sigma_2^2}{2\sigma_1} \), the function \( S_2 \) has a minimum of \( S_2 \) on the interval \( \sigma_2 \in (0, 1) \); for \( F_+ > F_+ \), the function \( S_2 \) decays with \( \sigma_2 \) (see Figure 1).

We consider the function \( \lambda_2(F_+(\lambda_0), \sigma_2) = S_2(F_+(\lambda_0), \sigma_2) + 1 \lambda_0 \in (0, 1) \), its explicit expression is

\[
\lambda_2(F_+(\lambda_0), \sigma_2) = \frac{(2\sigma_2^2 - 1)}{2\sigma_2^2} \left( \frac{1}{\frac{1}{2}(1 - \lambda_0)e^{\frac{\lambda_0}{\sigma_2}} - 1} \right) \left( 2\sigma_2^2 + 1 - \sigma_2^4 \right)
\]

Now, we need to find \( \lambda_0 \geq \lambda_2(\lambda_0) \). If such \( \lambda_0 < 1 \), we obtain a sufficient condition for the blowup on the first oscillation. The fixed point \( \lambda^*_2 \) of the mapping \( \lambda_2(F_+(\lambda_0)) \) exists only for \( \sigma_2 > \frac{1}{\sqrt{2}} \), it can be expressed in terms of the Lambert \( W \) function:

\[
\lambda^*_2(\sigma_2) = \frac{(2\sigma_2^2 - 1)}{2\sigma_2^2} \left( \Sigma_1(\sigma_2)L_1(\sigma_2) + \Sigma_2(\sigma_2) \right)
\]

\[
L_1 = \text{LambertW} \left(-1, \frac{1}{\Sigma_1} e^{\frac{\sigma_2}{\Sigma_2}}\right).
\]

We minimize \( \lambda^*_2(\sigma_2) \) with respect to \( \sigma_2 \). The minimum point is \( \sigma_2^* = 0.9423 \ldots \), the minimum value \( \Lambda_2 \equiv \lambda^*_2(\sigma_2^*) = 0.5754 \ldots \).
Figure 2, right, presents the functions $\lambda_1^*(\sigma_1), \lambda_2^*(\sigma_2)$ and their extrema.

We summarize our results.

**Proposition 6.1.** Let us consider the solution to the Cauchy problem (4) and (33).

- If $K < \frac{1}{2} \Lambda_1 = 0.1529 \ldots$, then the solution keeps $C^1$-smoothness during at least the first oscillation;
- If $K > \frac{3}{2} \Lambda_2 = 0.2877 \ldots$, then the solution blows up within the first oscillation.

### 6.3 Example of the estimate of the number of oscillations before the blowup

Let us choose the data (33) with $K = 0.1$. We use the technique described in Sections 2.3 and 2.4 by means of functions $\tilde{Z}_1$ and $\tilde{Z}_2$.

Let us list its steps.

1. Given $K = \frac{1}{2} \lambda_0, K < \frac{1}{2} \Lambda_1$, find $F_+(\lambda_0)$ by formula (34).
2. For this value of $F_+$, we consider $\tilde{Z}_1 = \tilde{Z}_1(\lambda, \tilde{\sigma}_1)$ (see 29). The graph of $-\sqrt{\tilde{Z}_1}$ is the first part of the curve $L$.
3. We find $\lambda_0^1 < 0$, the second root of equation $\tilde{Z}_1(\lambda, \tilde{\sigma}_1) = 0$, and use it as the new initial data. Thus, we find new $F_+$ and construct the function $\tilde{Z}_2(\lambda, \tilde{\sigma}_2)$ (see 31). The graph of $\sqrt{\tilde{Z}_2}$ is the next link of the curve $L$.
4. We find $\lambda_0^2 > 0$, the second root of equation $\tilde{Z}_2(\lambda, \tilde{\sigma}_2) = 0$, and use it as the new initial data, and repeat Step 2. Thus, we get the third link of the curve $L$, and so on.
5. To find the guaranteed number of revolutions ($n$), we construct the curve $L$ as described in Section 2.3. The iteration procedure stops if on the next step $C_1 \geq 0$.
6. We construct the curve $l$ as described in Section 2.3 for this $n$. The constant $T_l$, computed by the formula from Section 2.4, is the estimate of the time of the existence of the smooth solution from below.

Figure 3, left, presents the curve $L$, which bounds the projection of the phase trajectory on the plane $(\lambda, D)$ from above. Figure 3, right, presents the curve $l$, which bounds the projection of the phase trajectory on the plane $(\lambda, D)$ from below. The method guarantees three oscillations before the blowup. The guaranteed time of smoothness

$$T_* > \inf_{r_0 \in \mathbb{R}} T_l(r_0) = T_l(0) = 18.8685 \ldots$$

The estimate of this guaranteed time from above is 19.1298 \ldots

**Remark 6.1.** A series of computations with the data (33) was performed in Chizhonkov² and Gorbunov et al.,³ so we have an opportunity to compare our results with the results of sophisticated numerics. In Gorbunov et al.,³ computations are made for $a_* = 0.365, \rho_* = 0.6$, that is, $K \approx 0.222$, the breaking time is about $\theta_* = 35$ (dimensionless units). Thus, the solution keeps smoothness within the first oscillation. However, Proposition 6.1 does not guarantee the preservation of smoothness during the first oscillation. This means that the sufficient condition from Proposition 6.1 is still too rough.
Remark 6.2. The method for refining estimates outlined in this section can be easily adapted to the 3D radially symmetric case.

7 | DISCUSSION

In this paper, we constructed a method that allows one to obtain a sufficient condition that guarantees the boundedness of the density component on a given time interval for the solution to the Cauchy problem of the system of PDE describing general 3D electrostatic oscillations. For particular cases (plain and irrotational oscillations), we find an explicit sufficient condition for preventing a blowup in the first period of oscillations. Next, we compare the sufficient condition with the known criteria for the formation of singularities, if such a possibility exists. Further, we consider the case of 2D axisymmetric oscillations with special initial data, where there is a possibility of comparison with the results of a numerical study of the blow-up process. For this case, we obtain a sufficient condition for the preservation of smoothness, which is less rough than in the general case, as well as a sufficient condition for a blowup.

As shown in Section 3, in the 1D case, as well as for the affine and radially symmetric cases in many dimensions, they exist globally in t smooth solutions of the Cauchy problem. It is very interesting to check whether there are globally smooth solutions in the multidimensional case, without these restrictive assumptions. The numerics suggest that any other solution necessarily blows up. Recently, a finite-time blowup for general radially symmetric initial data, even arbitrarily small, has been proved analytically. The only exception is the initial data in the form of so-called simple waves, where the solution either tends to the affine one or blows up. The present paper can be regarded as an addition to this result, since it makes it possible to estimate the lifetime of a smooth solution. Now, it is not known where affine solutions without radial symmetry necessarily blow up.

Another important open question is the description of the complete class of solutions corresponding to electrostatic oscillations. The solutions considered in this article belong to this class, and the hypothesis is that in the multidimensional case, electrostatic oscillations must necessarily be radially symmetric or affine. However, this fact has not yet been proven.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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