The massive $CP^{N-1}$ model for frustrated spin systems

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Abstract

We study the classical $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ Non Linear Sigma model which is the continuous low energy effective field theory for $N$ component frustrated spin systems. The $\beta$ functions for the two coupling constants of this model are calculated around two dimensions at two loop order in a low temperature expansion. Our study is completed by a large $N$ analysis of the model. The $\beta$ functions for the coupling constants and the mass gap are calculated in all dimensions between 2 and 4 at order $1/N$. As a main result we show that the standard procedure at the basis of the $1/N$ expansion leads to results that partially contradict those of the weak coupling analysis. We finally present the procedure that reconciles the weak coupling and large $N$ analysis, giving a consistent picture of the expected scaling of frustrated magnets.

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1 INTRODUCTION

The Renormalization Group (RG) approach to Critical Phenomena has provided one of the greatest success of Theoretical Physics in the past twenty years\(^1\). The most important concept that emerged is that of effective field theory: the recognition that, whatever complicated the microscopic physics of a system is, its low energy, long distance properties can be well described by a much simpler and universal theory. The concept of low energy effective theory is particularly transparent in Statistical Mechanics, through the celebrated idea of block spin procedure which allows to generate a long distance effective action by integrating out the short distance degrees of freedom of a microscopic spin system defined at the scale of some lattice size \(a = \Lambda^{-1}\). In practice, one has to have recourse to approximate methods among which the quantum field theoretical approach has been one of the most prolific and successful one\(^2, 3\). In the latter, the block spin procedure is bypassed by a continuous limit in which is made the choice of the fluctuating fields together with operators satisfying the criterion of perturbative renormalizability. The resulting renormalizable action is then seen as the low energy effective theory, i.e. at scale \(p \ll \Lambda\), of the microscopic theory since non renormalizable - or irrelevant - terms correspond to those that are discarded by taking the long distance limit \(\Lambda \to \infty\)\(^4\). Critical and, more generally, long distance quantities are then obtained from perturbative expansions of this low energy theory. As well known, the consistency of the perturbative approach relies on the existence of a fixed point, around which the strength of the interactions and, in particular, that of non renormalizable interactions can be carefully controlled. But the main task is then to ensure that the results obtained so far persist beyond perturbation theory. This generally consists in comparing results obtained from different perturbative approaches. A celebrated example for which such an approach has been particularly successful is the Heisenberg ferromagnet\(^1, 2, 5, 6, 7\).

However, many issues remain open in the context of more involved systems among which are frustrated spin systems (for a review see \(^8\)). Indeed, the effect of competing interactions on the classical Heisenberg antiferromagnet has been the subject of a great interest, especially focused on the question of the universality class for its phase transition.
in three dimensions. Previous studies based on the Non Linear Sigma (NLσ) model approach by Azaria et al. in the classical \[9, 10\] as well as in the quantum \[11, 12\] case have revealed the existence of a non trivial UV fixed point with \(O(4)\) symmetry which governs the phase transition above two dimensions. But no stable fixed point has been found in the \(\epsilon = 4 - d\) expansion of the Landau-Ginzburg-Wilson effective action so that a first order phase transition is expected around four dimensions \[13, 14\]. This situation is even more confused when one takes into account of experimental results that do not show any evidence of universality \[15, 16, 17\] and those obtained with Monte Carlo simulations that are in favour of a continuous phase transition in \(d = 3\) with exponents that differ from those of the \(O(4)\) universality class \[18, 19\]. Various scenarios have been proposed to account for these results but none of them is fully satisfactory. In any case, at the center of the debate is the existence of the \(O(4)\) fixed point found in the NLσ model analysis. More precisely, the very question is to what extent can we believe on the results obtained with help of the NLσ model beyond perturbation theory.

The validity of the low temperature expansion at the basis of the NLσ model approach has been intensively questioned. The possibility that massive or irrelevant operators as well as non perturbative configurations drastically affect the RG properties of frustrated and non frustrated systems is not excluded \[20, 21, 22\]. In this work we would like to deal with the specific situation of frustrated spin systems. At the heart of the problem is the existence, in their RG phase diagram, of antagonistic fixed points that result from the competition between different kinds of order. The main consequence of the presence of several fixed points is that the status of some operator may change: being relevant with respect to a given fixed point, it may become irrelevant with respect to another one. In the low temperature expansion made around two dimensions, such an operator remains marginal so that one can describe this cross-over by means of a renormalizable theory interpolating between the different fixed points of the theory. However, when changing the perturbative scheme the power counting changes and there is no guarantee that irrelevant operators do not destabilize the \(O(4)\) fixed point or, at least, reduce its basin of attraction so that the \(O(4)\) scaling behavior becomes hardly observable. With this respect, one may question the existence of a universal theory that interpolates between the different fixed
points of the phase diagram. In order to investigate the reliability of the perturbative approach in such a multi-fixed point situation we shall, in the following, study extensively the NL$\sigma$ model relevant for frustrated systems by means of low temperature and large $N$ expansions.

In the second section of this paper, we present the effective action that describes the low energy properties of frustrated spin systems which is the $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ NL$\sigma$ model. This model differs from that used by Azaria et al. in their investigation of the low temperature properties of frustrated spin systems by the fact that the fundamental fields, in terms of which the theory is written, span a representation of $SU(N)$ rather than of $O(N)$. The reason we shall focus on this model will be explained in this section.

In the third section, we proceed to an extensive analysis of the RG properties of the $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ NL$\sigma$ model in a double expansion in powers of the temperature $T$ and $\epsilon = d - 2$. We give the $\beta$ functions for the two coupling constants entering in this model, calculated at two loop order. The RG flow is solved and analyzed in $d = 2$ and $d = 2 + \epsilon$ dimensions. We show that the $\beta$ functions and the correlation length $\xi$ interpolate between two fixed points, one with the $CP^{N-1}$ symmetry, associated with high energy degrees of freedom, the other one with the $O(2N)$ symmetry, associated with the long distance behavior, which drives the finite temperature phase transition of frustrated spin systems in $d > 2$. As a main result, we show that a new scale $\Xi$, distinct from the correlation length, is generated in the RG flow. This scale, that characterizes the cross-over between low and high energy scaling behaviors, appears as the main feature of the low temperature RG analysis of frustrated models.

It is the purpose of the fourth section of this work to address the question of the validity of the weak coupling analysis beyond the formal expansion in powers of $T$ and $\epsilon$. This is why we study the $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ NL$\sigma$ model in an expansion in powers of $1/N$. There appears here an important point, related to the way the cross-over length scale $\Xi$ takes place in this approach. Indeed, in the weak coupling analysis, this length scale is identified in the RG flow analysis but is absent of the perturbative calculation which is completely free of any scale dependence. In contrast, in the $1/N$ expansion the situation is different and this length scale defines itself the way to perform
the perturbative expansion. As a result we show that, according to the value of $\Xi$, one has to perform different expansions using different effective field theories in the vicinity of the two - $CP^{N-1}$ and $O(2N)$ - fixed points. Then, the $\beta$ functions for the coupling constants and the mass gap are calculated in all dimensions between 2 and 4 at order $1/N$. A careful comparison between the low temperature and large $N$ expansions shows that there is an agreement of the RG quantities in the very neighborhood of the fixed points. In the last section of this paper we finally discuss the implications of our findings on the physics of frustrated spin systems.

2 THE MASSIVE CP$^{N-1}$ MODEL

The most striking feature with frustrated systems is that long wavelength fluctuations are different, in nature, from the short distance ones. This is typically the case with the Heisenberg antiferromagnetic model on the triangular lattice. Starting with Heisenberg spins on the lattice, the strong correlations due to geometrical frustration induce a local order in which the spins adopt a $120^\circ$ structure on each elementary cell. Whereas the fluctuating field at the scale of the lattice size $a$ is a vector, it turns out to be, because of frustration, a rotation matrix $R$ of $SO(3)$ at larger scales. It has been shown by Dombre and Read$^{23}$ that, as the consequence of the matrix character of the order parameter, the relevant action for frustrated Heisenberg models is a NL$\sigma$ model defined on the homogeneous space $O(3) \otimes O(2)/O(2)$. This model has been studied in the past with help of a low temperature expansion. As a main result of this study is the existence of a stable fixed point in the neighborhood of two dimensions with increased symmetry $O(3) \otimes O(3)/O(3) \sim O(4)/O(3)$$^{9,10}$. Our aim being to investigate the reliability of the low temperature expansion, we need a generalization of this model that allows a $1/N$ expansion. One of them is obtained when the action derived by Dombre and Read is written in terms of $N$ component real basic fields. This leads to the $O(N) \otimes O(2)/O(N-2) \otimes O(2)$ NL$\sigma$ model which has been studied by Azaria et al.$^{9,10}$. Another option is to parametrize the long distance fluctuations with help of $N$ component complex fields of $SU(N)$. Doing so we obtain the $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ NL$\sigma$
model which we shall extensively study in this paper. The reason to favour this model is threefold: first, although we shall be interested in the classical case in the following, it appears that the $SU(N) \otimes U(1)/SU(N - 1) \otimes U(1)$ NL$\sigma$ model is more appropriate to investigate the quantum case. Indeed, there are reasons to believe that the low energy excitations in the disordered phase of quantum frustrated spin systems are spin one half deconfined excitations $z$ called spinons\cite{24}. On the other hand, this model has a more natural large $N$ expansion than the $O(N) \otimes O(2)/O(N - 2) \otimes O(2)$ version. Indeed, as it will be shown below, it exhibits the phenomenon of enlarge symmetry at the fixed point for any value of $N$ whereas this property is lost for the $O(N) \otimes O(2)/O(N - 2) \otimes O(2)$ NL$\sigma$ model which has no special symmetry properties for $N > 3$. Note finally that the $SU(2) \otimes U(1)/U(1)$ and $O(3) \otimes O(2)/O(2)$ NL$\sigma$ models, which are relevant for the case of Heisenberg spins, differ by their topological properties. Indeed, $SO(3)$ having a non trivial first homotopy group, there exist point and line defects in two and three dimensions respectively which are not taken into account in the $SU(2)$ formulation. It is not excluded that these topological defects drastically affect the physics of frustrated spin systems\cite{25}.

We shall make however the hypothesis that it is not the case and concentrate ourselves on the multi-fixed point situation problem, leaving for future work the understanding of the role of topology.

The partition function of the $SU(N) \otimes U(1)/SU(N - 1) \otimes U(1)$ NL$\sigma$ model writes\cite{26,27,28,29}:

$$Z = \int Dz^\dagger Dz \delta(z^\dagger z - 1) \exp [-S]$$

with:

$$S = \int d^d x \left[ \frac{\Lambda^{d-2}}{2f_1} \partial_\mu z^\dagger \partial_\mu z + \frac{\Lambda^{d-2}}{2f_2} \left( z^\dagger \partial_\mu z \right)^2 \right].$$

Here $z$ is a $N$ component complex vector submitted to the constraint $z^\dagger z = 1$, $\Lambda$ being the inverse of the lattice spacing $a$ of the underlying spin system. The action (2) is invariant under the action of left $SU(N)$ and right $U(1)$ global transformations:

$$z \rightarrow U \, z \, e^{i\theta}, \quad U \in SU(N) \quad \text{and} \quad \theta \in \mathbb{R}.$$
The ground state of (2) is:

\[
  z_0 = \begin{pmatrix}
    0 \\
    \cdot \\
    \cdot \\
    1
  \end{pmatrix}
\]  \tag{4}

and is invariant under the action of a left $SU(N-1)$ subgroup and a diagonal $U(1)$ group:

\[
  z_0 \rightarrow e^{D_{N-1}\theta} z_0 e^{i \sqrt{\frac{2(N-1)}{N}} \theta},
\]  \tag{5}

where $D_{N-1}$ is given by:

\[
  D_{N-1} = i \sqrt{\frac{2}{N(N-1)}} \begin{pmatrix}
    1 \\
    \cdot \\
    \cdot \\
    1 - N
  \end{pmatrix}
\]  \tag{6}

Action (2) thus describes the symmetry breaking pattern $SU(N) \otimes U(1) \rightarrow SU(N-1) \otimes U(1)$. It is a NL$\sigma$ model defined on the homogeneous coset space $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ that accounts for the physics of the resulting $2N-1$ interacting Goldstone modes.

Let us describe the symmetries associated with action (2) according to the values taken by the coupling constants $f_1$ and $f_2$. When $4f_1 = f_2$, action (2) becomes gauge invariant with respect to the right $U(1)$ local group and one recovers the $CP^{N-1}$ model relevant for non frustrated magnets. In all other cases the gauge invariance is broken as a consequence of frustration on the microscopic spin system. In the limit $f_2 \to \infty$, the model becomes $O(2N)$ invariant as can be seen when parametrizing the $z$ field in terms of a $2N$ component real vector field $\phi$. The model thus interpolates smoothly, as one varies $f_2$, between the two symmetric $CP^{N-1}$ and $O(2N)/O(2N-1)$ NL$\sigma$ models.

The partition function (1) can be rewritten into another form which enlightens the symmetry properties and the role played by frustration in the model. As in the $CP^{N-1}$
model, one can decouple the current-current interaction:

$$\omega^2 = (z^\dagger \partial_\mu z)^2$$

(7)

with help of a Lagrange gauge field $A_\mu$. After some standard manipulations, we obtain so a form that brings out more clearly gauge invariance properties:

$$Z = \int Dz^\dagger Dz \, DA_\mu \, \delta\left(z^\dagger z - \frac{1}{2f_1}\right) \exp[-S]$$

(8)

with:

$$S = \int d^d x \left[ D_\mu z^\dagger D_\mu z + \Lambda^{d-2} \frac{M^2}{2} A_\mu^2 \right],$$

(9)

where $D_\mu = \partial_\mu + i A_\mu$ and

$$M^2 = -\frac{1}{f_1} \left(1 - \frac{f_2}{4f_1}\right).$$

In the absence of the mass term $M^2$, i.e. for $4f_1 = f_2$, action (8) is gauge invariant with respect to the right $U(1)$ local group and describes the pure $CP^{N-1}$ model. When $M^2 \neq 0$, i.e. $4f_1 \neq f_2$, action (9) corresponds to a “massive $CP^{N-1}$” model. We thus see that, in this parametrization, the effect of frustration is to give a mass to the gauge field $A_\mu$. As an immediate consequence we can already anticipate that a new length scale has been generated by frustration. The next sections will be largely devoted to discuss in great details the physical and technical implications of the existence of this new length scale.

3 WEAK COUPLING ANALYSIS

3.1 Introduction

The renormalizability of NL$\sigma$ models defined on coset space $G/H$ around two dimensions has been studied by Friedan [32]. He has shown that the RG properties of a NL$\sigma$ model only depend on the geometry of the coset $G/H$ viewed as a metric manifold. In particular, the $\beta$ functions for the coupling constants entering in the model are given, at two loop order, in terms of the Ricci and Riemann tensors on the manifold $G/H$ so that renormalization depends only on the local properties of $G/H$ and is insensitive to its global structure. It is
worth stressing that this weak coupling expansion is, in fact, a loop expansion ordered in powers of the temperature $T$. In symmetric models such as $O(N)$ or $CP^{N-1}$ models, there is only one coupling which identifies with the temperature. However, in non-symmetric models such as the massive $CP^{N-1}$ model, there are at least two coupling constants. In this case, the resulting expansion is ordered in powers of an effective temperature but is not perturbative with respect to the ratio of the coupling constants. This is the reason why the RG functions may interpolate between fixed points that lie at finite distance.

With this respect, the low temperature approach of the NL$\sigma$ model provides the natural framework to study frustrated systems whose phase diagram exhibits several fixed points. However, one has to keep in mind that the low temperature expansion misses terms of order $e^{-1/T}$ that include the global nature of the manifold - or of the massive modes of the theory - that may be important for the physics. We shall return to this point in the next section when dealing with the $1/N$ expansion.

Starting from the partition function (1) we can take advantage of the constraint of unit modulus to integrate out one degree of freedom of $z$. To this end, let us parametrize $z$ as:

$$z = \left( \frac{\pi}{e^{i\pi} \sqrt{1 - \pi^\dagger \pi}} \right),$$

where $\pi$ is a $N - 1$ component complex vector field which, together with the scalar field $\pi$, represent the $2N - 1$ Goldstone modes. With this parametrization, the partition function can be rewritten in a more suggestive form:

$$Z = \int_{|\pi| \leq 1} D\pi \exp \left[ -\frac{1}{2} \int d^d x \ g_{ij}(\pi) \partial\pi^i \partial\pi^j \right]$$

which exhibits the geometrical nature of the theory. Here, $\pi^i$ is a short notation for the $2N - 1$ Goldstone modes. The action $g_{ij}(\pi) \partial\pi^i \partial\pi^j$ represents the line element on $G/H$ equipped with the metric $g_{ij}(\pi)$. Thus, in the absence of the constraint $|\pi| \leq 1$, action (12) would describe a free theory on the manifold $G/H$. In the low temperature expansion this constraint is irrelevant at any finite order of perturbation theory since it gives contributions of order $e^{-1/T}$. Friedan has shown that the $\beta$ functions which give the evolution of the metric with the scale $\lambda$:

$$\beta_{ij} = \lambda \frac{\partial g_{ij}}{\partial \lambda}$$

(13)
are given at two loop order by \[ \beta_{ij}(g) = \epsilon g_{ij} - \frac{1}{2\pi} R_{ij} - \frac{1}{8\pi^2} R_{i}^{pq} R_{jpqr} \] (14)

where \( R_{ij} \) and \( R_{jpqr} \) are respectively the Ricci and Riemann tensors of the manifold \( G/H \) equipped with the metric \( g_{ij} \). As we shall now see, these intrinsic properties can be formulated in the language of group theory which enlightens the symmetry properties and provides a powerful framework for practical computations.

3.2 Vielbein basis formulation

The main advantage of the geometrical interpretation of the RG properties of NL\( \sigma \) models is that the Goldstone modes \( \pi^i \) (respectively the tangent vectors \( \partial_{\mu} \pi^i \)) can be viewed just as a particular coordinate frame on the manifold \( G/H \) (respectively on the tangent space of \( G/H \)). For practical calculations, it is extremely convenient to work, in the tangent space, in the vielbein basis \( \omega^I_\mu \) related to the \( \partial_{\mu} \pi^i \) via \( \omega^I_\mu = \omega^I_i \partial_{\mu} \pi^i \) such that the vielbein basis metric \( \eta^{IJ} = g_{ij}(\pi) \omega^i_I \omega^j_J \) is free of any coordinate dependence. From the RG point of view, the main advantage of the vielbein formulation is that the geometrical quantities such as the Riemann tensor only depend on the Lie algebras \( \text{Lie}(G) \) and \( \text{Lie}(H) \), i.e. on the structure constants defined by the following commutation rules:

\[
\begin{align*}
[T_a, T_I] &= f_{ai}^J T_J , \\
[T_a, T_b] &= f_{ab}^c T_c , \\
[T_I, T_J] &= f_{IJ}^K T_K + f_{IJ}^a T_a ,
\end{align*}
\] (15)

where \( T_a \in \text{Lie}(H) \) and \( T_I \in \text{Lie}(G) \)-\text{Lie}(H) are normalized according to \( \text{Tr} T_i T_j = -2\delta_{ij} \).

We shall now see how this works in the massive \( CP^{N-1} \) model.

Let us parametrize the coset \( SU(N) \otimes U(1)/SU(N-1) \otimes U(1) \) by choosing one unique element, i.e. fixing the gauge:

\[
L(\pi^i) = \begin{pmatrix}
\sqrt{1 - \pi^i \pi^i} & \pi^i \\
-e^{-i\pi} \pi^i & e^{i\pi} \sqrt{1 - \pi^i \pi^i}
\end{pmatrix}.
\] (16)

This matrix is the same as for the coset space \( SU(N)/SU(N-1) \). This is the consequence of the fact that the coset space \( G \otimes X/H \otimes X \) - where \( X \) is the maximal subgroup of \( G \)
that commutes with $H$ - and $G/H$ are topologically equivalent. In the following, we shall work directly in the coset $SU(N)/SU(N - 1)$ keeping in mind that we look for an action which has $SU(N) \otimes U(1)$ as symmetry (isometry) group.

The quantity $L^{-1} \partial L$ belongs to $\text{Lie}(G)$ of $G = SU(N)$ and we have:

$$L^{-1} \partial L = (L^{-1} \partial L)_{G-H} + (L^{-1} \partial L)_H$$

(17)

where $(L^{-1} \partial L)_H$ is in $\text{Lie}(H)$ of $H = SU(N - 1)$. One can now express the action (2) in terms of the currents (17) as:

$$S = -\frac{1}{2} \int d^d x \, \text{Tr} [(L^{-1} \partial L)_{G-H}]^2$$

(18)

which is the general expression of NL$\sigma$ models. It depends only on the currents in $\text{Lie}(G)$-$\text{Lie}(H)$, the tangent space of $G/H$. This reflects the gauge invariance under the $H$ group, i.e. the arbitrary in the choice of the ground state. Since $L^{-1} \partial L$ belongs to $\text{Lie}(G)$, eq.(17) can be rewritten as:

$$L^{-1} \partial \mu L = \omega^I_\mu T_I + \Omega^a_\mu T_a ,$$

(19)

where $\omega^I_\mu$ and $\Omega^a_\mu$ are respectively the vielbein and the connection in the tangent space of $G/H$. They are given by:

$$\left\{ \begin{array}{l}
\omega^I_\mu = -\frac{1}{2} \text{Tr} [L^{-1} \partial \mu L \, T_I] \\
\Omega^a_\mu = -\frac{1}{2} \text{Tr} [L^{-1} \partial \mu L \, T_a].
\end{array} \right.$$ 

(20)

Among all these currents, the $U(1)$ one plays a crucial role. It is defined in terms of the $z$ field as:

$$\omega^X_\mu = -\frac{1}{2} \text{Tr} [L^{-1} \partial \mu L \, D_{N-1}] = \frac{i}{4} \sqrt{\frac{2N}{N-1}} \left( z^\dagger \frac{\leftrightarrow}{\leftrightarrow} \partial_\mu z \right)$$

(21)

while the currents $\omega^A_\mu$ in $\text{Lie}(SU(N))$-$\text{Lie}(SU(N - 1))$-$\text{Lie}(U(1))$ verify:

$$\sum_A (\omega^A_\mu)^2 = \left[ \partial_\mu z^\dagger \partial_\mu z + \frac{1}{4} \left( z^\dagger \frac{\leftrightarrow}{\leftrightarrow} \partial_\mu z \right)^2 \right].$$

(22)

We can now write action (11) in terms of these currents:

$$S = \frac{1}{2} \int d^d x \left[ \eta_1 \sum_A (\omega^A_\mu)^2 + \eta_2 (\omega^X_\mu)^2 \right]$$

(23)
which defines the vielbein basis metric $\eta_{IJ}$. Indeed we have:

$$S = \frac{1}{2} \int d^d x \, \eta_{IJ} \omega^I_\mu \omega^J_\mu ,$$

(24)

where

$$\eta_{IJ} = -\frac{1}{2} \left[ \eta_1 \text{Tr}(T^I T^J) + (\eta_2 - \eta_1) \delta_{I\alpha} \delta_{J\beta} \text{Tr}(T^\alpha T^\beta) \right] ,$$

(25)

with:

$$\left\{ \begin{array}{l} \eta_1 = \frac{1}{f_1} \\
\eta_2 = 2 \frac{N - 1}{N f_1} \left( 1 - \frac{4 f_1}{f_2} \right) . \end{array} \right.$$  

(26)

We recall that in our case $T_I \in \text{Lie}(SU(N))$-$\text{Lie}(SU(N-1))$, $T_a \in \text{Lie}(U(1))$ and $T_a \in \text{Lie}(SU(N-1))$ and that the corresponding algebra is given in (13). As already pointed out, whereas non frustrated models are defined with one coupling constant, the frustrated case is characterized by two independent coupling constants $\eta_1$ and $\eta_2$.

The crucial advantage of the vielbein basis formulation is that the Riemann tensor only depends on the structure constants $f^k_{ij}$ of $\text{Lie}(G)$ so that the $\beta$ functions:

$$\beta_{IJ} = \lambda \frac{\partial \eta_{IJ}}{\partial \lambda} ,$$

(27)

writes at two loop order:

$$\beta_{IJ}(\eta) = \epsilon \eta_{IJ} - \frac{1}{2\pi} R_{IJ} - \frac{1}{8\pi^2} R^{PQR}_I R_{JPQR}$$

(28)

where the Riemann tensor writes (10):

$$R_{IJKL} = f_{IJ}^a f_{aKL} + \frac{1}{2} f_{IJ}^M (f_{MKL} + f_{LMK} - f_{KLM})$$

$$+ \frac{1}{4} (f_{IKM} + f_{MIK} - f_{KMI}) (f_{JL}^M + f_{LJ}^M - f_{MJ}^M)$$

$$- \frac{1}{4} (f_{JKM} + f_{MKJ} - f_{KJM}) (f_{IL}^M + f_{LI}^M - f_{MJ}^M).$$

(29)

The indices $a$ and $\{ I, J \ldots \}$ refer to $H$ and $G - H$ respectively. The $G - H$ indices are raised and lowered by means of $\eta^{IJ}$ and $\eta_{IJ}$ and repeated indices are summed over.
3.3 Results for the SU(N) ⊗ U(1)/SU(N − 1) ⊗ U(1) model

We are now in a position to study the RG properties of the massive $CP^{N-1}$ model. The computation of the Riemann tensor (eq.(29)) leads to the following $\beta$ functions:

$$\begin{align*}
\lambda \frac{\partial \eta_1}{\partial \lambda} &= \epsilon \eta_1 + \beta_{\eta_1} \\
\lambda \frac{\partial \bar{\eta}_2}{\partial \lambda} &= \epsilon \bar{\eta}_2 + \beta_{\bar{\eta}_2}
\end{align*}$$

(30)

with:

$$\begin{align*}
\beta_{\eta_1} &= -\frac{1}{\pi} \left( N - \frac{\bar{\eta}_2}{\eta_1} \right) - \frac{1}{\eta_1} \frac{1}{2\pi^2} \left( 4N - 6 N \frac{\bar{\eta}_2}{\eta_1} + (3N - 1) \left( \frac{\bar{\eta}_2}{\eta_1} \right)^2 \right) \\
\beta_{\bar{\eta}_2} &= -\frac{N - 1}{\pi} \left( \frac{\bar{\eta}_2}{\eta_1} \right)^2 - \frac{1}{\eta_1} \frac{1}{2\pi^2} \left( \frac{\bar{\eta}_2}{\eta_1} \right)^3
\end{align*}$$

(31)

where $\bar{\eta}_2 = \frac{N}{2(N - 1)} \eta_2$.

In eq.(30), the one and two loop order contributions to the $\beta$ functions are ordered in powers of $\frac{1}{\eta_1} = T$, the effective temperature of the model. The long distance, low energy, physics is determined by the behavior of the RG flow in the $\lambda \to \infty$ limit while critical behavior is associated with the existence of UV stable fixed points of (30). To exhibit the low $T$ properties, it is convenient to make the following change of variables:

$$\begin{align*}
T &= \frac{1}{\eta_1} = f_1 \\
x &= 1 - \frac{\bar{\eta}_2}{\eta_1} = \frac{4f_1}{f_2}
\end{align*}$$

(32)

In terms of these new coupling constants, eq.(30) reads:

$$\begin{align*}
\lambda \frac{\partial T}{\partial \lambda} &= -\epsilon T + \beta_T \\
\lambda \frac{\partial x}{\partial \lambda} &= \beta_x
\end{align*}$$

(33)

where:

$$\begin{align*}
\beta_T &= \frac{T^2}{\pi} \left( N - 1 + x \right) + \frac{T^3}{2\pi^2} \left( N - 1 + 2x + (3N - 1) x^2 \right) \\
\beta_x &= -N \frac{T}{\pi} x \left( 1 - x \right) - N \frac{T^2}{\pi^2} x \left( 1 - x^2 \right)
\end{align*}$$

(34)
As usually, $T$ is the relevant coupling constant that orders perturbation theory. The variable $x$ measures the anisotropy of the model: as $x$ varies one interpolates between the pure $CP^{N-1}$ model, when $x = 1$, and the pure $O(2N)$ model, when $x = 0$. In the following, we shall study separately the $d = 2$ and $d > 2$ cases.

### 3.3.1 Two dimensional case

When $\epsilon = 0$, eq.(33) exhibits a line of UV fixed points at $T = 0$ for any values of $x$. However they are not all true fixed points in the sense of limit of RG trajectories. To get the complete RG properties one has to solve the flow resulting from eq.(33). In the following, we restrict ourselves to one loop accuracy since inclusion of two loops terms does not modify qualitatively our conclusions.

The RG trajectories are parametrized by the RG invariant $K_N$, given at one loop by:

$$K_N = \frac{x^{1-1/N}}{1-x} T,$$

in terms of which the solutions of eq.(33) are given by the implicit equation:

$$\begin{cases}
K_N \ln \lambda &= A_N (x(1)) - A_N (x(\lambda)) \\
T(\lambda) &= K_N \frac{1-x(\lambda)}{x(\lambda)^{1-1/N}}
\end{cases}$$

with initial conditions $x(1)$ and $T(1)$ satisfying eq.(33). The function $A_N(x)$ is given in terms of the hypergeometric function $2F_1$:

$$A_N(x) = \frac{\pi}{N-1} x^{1-1/N} 2F_1 \left(2, 1 - \frac{1}{N}; 2 - \frac{1}{N}; x \right).$$

We show in Fig.1 the corresponding flow diagram.

In the UV regime, i.e. when $\lambda \to 0$, all the flow lines with $K_N \neq 0$ reach the $CP^{N-1}$ fixed point at $x = 1$ and $T = 0$ which thus governs the short distance behavior of models with $x \neq 0$. On the invariant line $K_N = 0$, corresponding to the $O(2N)$ model, the flow is governed in the UV by the $O(2N)$ fixed point at $x = 0$ and $T = 0$. In the infrared limit, when $\lambda$ increases, $x(\lambda)$ decreases to its asymptotic value at $x = 0$. This means that all models with $x(1) \neq 1$ are asymptotically equivalent at long distance to the $O(2N)$
model. We thus conclude that, while a priori we have a line of fixed points at $T = 0$, only two of them determine the RG properties. They are the $CP^{N-1}$ and $O(2N)$ fixed points that govern the short and long distance behaviors respectively. The presence of these two competing fixed points is a characteristic feature of frustration where fluctuations of different nature compete. We are thus led to expect that apart from the correlation length $\xi$, there should exist a new scale $\Xi$, characterizing the cross-over between short and long wavelength associated with the $CP^{N-1}$ and $O(2N)$ fixed points respectively. As a consequence, the scaling behavior of correlation functions will depend, in a non trivial way, on the relative magnitude of $\xi$, $\Xi$ and $\Lambda^{-1}$. To see this, we have to go further in the flow analysis.

First of all, in order to perturbation theory remains valid one must have $T(\lambda) \ll 1$. Given initial conditions on the flow $(x(1), T(1))$ at scale $\Lambda^{-1}$, this implies that $\lambda$ has to be much smaller than $\lambda_0$ defined by:

$$A_N(x(1)) = K_N \ln \lambda_0.$$  

This defines the correlation length $\xi = \lambda_0 \Lambda^{-1}$ of the model:

$$\xi = \Lambda^{-1} \exp \left( \frac{A_N(x(1))(1-x(1))}{x(1)^{1-1/N}T(1)} \right),$$  

which follows the standard RG equation:

$$\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta_x \frac{\partial}{\partial x} + \beta_T \frac{\partial}{\partial T} \right) \xi = 0.$$  

The general solution of eq.(40) writes in fact $\Xi = \Phi(K_N)\xi$ where $\Phi(K_N)$ is an arbitrary function of the flow invariant $K_N$. This means that in a model with several coupling constants there may be others physical scales necessary to characterize the complete scaling behavior. We shall indeed see that if $\lambda_0$ defines the critical regime through the relation $1 \ll \lambda \ll \lambda_0$ it does not specify which fixed point governs the scaling behavior of the correlation functions. Let us now precise this point by defining three regimes according to the initial conditions on the flow:

- Given initial conditions very close to the $CP^{N-1}$ fixed point with $x(1) \sim 1$, we define the region governed by the $CP^{N-1}$ fixed point to be such that $x(\lambda)$ is still close to 1. Since
in this regime $A_N(x) \sim \frac{\pi}{N(1-x)}$, one finds that $\lambda$ has to be much smaller than

$$\lambda_1 = \lambda_0 \exp \left( -\frac{\pi}{NK_N} \right).$$  \hspace{1cm} (41)

We so obtain a new RG invariant scale $\Xi = \lambda_1 \Lambda^{-1}$, solution of eq.(40), that defines the region governed by the $CP^{N-1}$ fixed point: $1 \ll \lambda \ll \lambda_1$ or, in terms of the momentum $p$, $\Xi^{-1} \ll p \ll \Lambda$. Notice that in this regime one has $1 - x(\lambda) \sim T(\lambda)/K_N$.

• Independently of the initial conditions $x(1)$, the $O(2N)$ regime is obtained for values of $\lambda \ll \lambda_0$ such that $x(\lambda) \sim 0$. One sees that it is sufficient to have $\lambda \gg \lambda_1$ since $A_N(x) \sim \frac{\pi}{N-1} x^{1-1/N}$ when $x \to 0$. The latter condition not only implies $K_N \ll 1$ but also defines the region governed by the $O(2N)$ fixed point: $\lambda_1 \ll \lambda \ll \lambda_0$ or $\xi^{-1} \ll p \ll \Xi^{-1}$.

• Finally, when $K_N \ll 1$ and $x(1) \sim 1$, one has $\xi \gg \Xi$ so that the $O(2N)$ and $CP^{N-1}$ regimes can be observed when $\xi^{-1} \ll p \ll \Xi^{-1}$ and $\Xi^{-1} \ll p \ll \Lambda$ respectively.

Let us resume our findings. We first found that the perturbation theory remains valid if one stays in a regime $\lambda \ll \Lambda \xi$ so that $T(\lambda) \ll 1$. In symmetric NL$\sigma$ models, such as when $x = 1$ or $x = 0$, the correlation functions display scaling behavior with logarithmic corrections that are governed by the unique non trivial fixed point of the theory. In non symmetric NL$\sigma$ models, i.e. $x(1) \neq 0, 1$, even when $\lambda \ll \Lambda \xi$, there remains a non trivial scale in the theory. In both regimes $\xi^{-1} \ll p \ll \Xi^{-1}$ and $\Xi^{-1} \ll p \ll \Lambda$, i.e. when one lies near the $O(2N)$ or $CP^{N-1}$ fixed points, the field theoretical approach to the RG is expected to give the correct asymptotic behavior of the correlations functions.

The previous discussion can be formulated in terms of the gauge field formulation of the theory. This is most easily done in the limit $N \to \infty$, where $\Xi$ takes the nice form: \hspace{1cm} (42)

$$\Xi = \xi e^{-\pi M^2 / N}$$

where $M^2$ is the gauge field mass (see eq.(10)). We see that $\xi$ being fixed, the length $\Xi$ is associated with the energy scale where the effects of the $U(1)$ current-current interaction become strong. We shall return to this point in more details when studying the $1/N$ expansion. Let us just finally observe that the status of the $U(1)$ current-current interaction of eq.(22) changes between the $CP^{N-1}$ and $O(2N)$ fixed points. While it is marginally
relevant near the former, it becomes marginally irrelevant near the latter fixed point and
is therefore a “dangerous irrelevant” variable.

All the previous discussion is not qualitatively modified by inclusion of two loop order
terms in the \( \beta \) functions. For completeness, we now give the expression of the correlation
length \( \xi \) at two loop order:

\[
\xi = \xi_{\text{1-loop}} \ G(x) \left( \frac{1}{\psi_0 \ T^{2(N-1)}} + \psi_1 \ T \frac{\frac{3N - 4}{2N^2}}{3N - 4} \right) \tag{43}
\]

where \( \xi_{\text{1-loop}} \) is the one loop correlation length given by eq.(39), \( \Psi_0 \) and \( \Psi_1 \) being constants
and \( G(x) \) being a pure function of \( x \) given by:

\[
G(x) = \exp \int dx \left( \frac{2F_1 \left( 2, 1 - \frac{1}{N}; 2 - \frac{1}{N}; x \right)}{2Nx} \right) \frac{(1-x)(3x+1)}{2N(N-1)x(1-x)} - \frac{N-1 + (2N-3)x}{2N(N-1)x(1-x)} \tag{44}
\]

obtained by solving eq.(40).

In the very neighborhood of \( O(2N) \) fixed point, eq.(43) becomes to leading order in \( x \):

\[
\xi = \xi_{O(2N)} \left( 1 + \frac{x}{4N^2} \right) \exp \left( \frac{-x\pi}{T(2N-1)(N-1)} \right), \tag{45}
\]

while in the neighborhood of the \( CP^{N-1} \) fixed point, where \( 1 - x = \frac{T}{K_N} + O(T^2) \), one
finds to leading order in \( 1 - x \) and \( T \):

\[
\xi = \xi_{CP^{N-1}} \exp \left( -\frac{(1-x)\pi}{N^2T} \ln \frac{T}{2\pi} \right) \tag{46}
\]

where \( \xi_{O(2N)} \) and \( \xi_{CP^{N-1}} \) are the two loop correlation lengths of the \( O(2N) \) and \( CP^{N-1} \)
models:\[33\]

\[
\left\{ \begin{array}{l}
\xi_{O(2N)} = C_{1\xi} \Lambda^{-1} \ T^{2(N-1)} \exp \left( \frac{\pi}{(N-1)T} \right) \\
\xi_{CP^{N-1}} = C_{2\xi} \Lambda^{-1} \ T \frac{2}{N} \exp \left( \frac{\pi}{NT} \right) 
\end{array} \right. \tag{47}
\]

where \( C_{1\xi} \) and \( C_{2\xi} \) are constants that depend on the regularization scheme.
3.3.2 Above two dimensions

In dimension \( d > 2 \), there is a phase transition, as one varies \( T \), from an ordered phase, with the breaking of the \( SU(N) \otimes U(1) \) symmetry, to a disordered phase. Indeed, eq.\((33)\) admits, apart from the trivial \( T = 0 \) fixed point, two non trivial fixed points with \( (T_{CPN-1}^* = \frac{\pi \epsilon}{N} - \frac{2 \pi \epsilon^2}{N^2}, x^* = 1) \) and \( (T_{O(2N)}^* = \frac{\pi \epsilon}{N - 1} - \frac{\pi \epsilon^2}{2(N - 1)^2}, x^* = 0) \) which are respectively \( CP^{N-1} \) and \( O(2N) \) symmetric. In the infrared regime, the latter has one direction of instability and thus governs the long distance behavior while the former has two unstable directions and governs the short distance behavior. On the critical line connecting the two non trivial fixed points, the flow drives the system towards the \( O(2N) \) fixed point which is thus responsible of the phase transition of frustrated models above two dimensions (see Fig.2). The symmetry is increased at the stable IR fixed point for all values of \( N \). In particular when \( N = 2 \), we find the \( O(4)/O(3) \) universality class for frustrated Heisenberg spin systems in agreement with previous studies of the \( O(N) \otimes O(2)/O(N - 2) \otimes O(2) \) NL\( \sigma \) model with \( N = 3^{10} \).

As in two dimensions, there exists a specific scale \( \Xi_\epsilon = \lambda_1 \Lambda^{-1} \) associated with the cross-over between the \( O(2N) \) and \( CP^{N-1} \) behavior. The scale \( \Xi_\epsilon \) can be again obtained by a careful analysis of the RG flow. As previously, we shall restrict ourselves to one loop accuracy.

The one loop RG invariant flow takes now the form:

\[
K_N = \frac{x^{1-1/N}}{1-x} (T - T_c(x)),
\]

with:

\[
T_c(x) = \frac{\pi \epsilon}{N - 1} (1 - x) \frac{1}{2} F_1 \left( \frac{2, 1 - \frac{1}{N}, 2 - \frac{1}{N}; x \right).
\]

The critical line \( T_c(x) \) is obtained when \( K_N = 0 \), and \( K_N > 0 \) label the lines in the disordered phase. With help of these quantities, the solutions of eq.\((33)\) are given by:

\[
\begin{align*}
\lambda^{-\epsilon} &= \frac{\epsilon A_N(x(\lambda)) + K_N}{\epsilon A_N(x(1)) + K_N} \\
T(\lambda) &= \frac{1 - x(\lambda)}{x(\lambda)^{1-1/N}} (K_N + \epsilon A_N(x(\lambda)))
\end{align*}
\]
where $A_N$ is given by eq.\(\text{(37)}\). In the symmetric phase, when $K_N > 0$, the analysis of the RG flow follows straightforwardly from that of $d = 2$. We obtain for the correlation length and the cross-over scale:

\[
\begin{align*}
\xi &= \Lambda^{-1} \left(1 + \epsilon \frac{A_N(x(1))}{K_N}\right)^{1/\epsilon} \\
\Xi &= \xi \left(1 + \frac{\epsilon \pi}{NK_N}\right)^{-1/\epsilon}
\end{align*}
\] (51)

which define, exactly as in $d = 2$, the regimes of energy governed by either the $CP^{N-1}$ or the $O(2N)$ fixed point. On the critical line, i.e. when $K_N \to 0$, $\xi$ diverges but $\Xi$ has a finite limit: $\Xi_c = \Lambda^{-1} \left(\frac{N A_N(x(1))}{\pi}\right)^{1/\epsilon}$. This means that even on the critical line, the $O(2N)$ critical behavior of frustrated systems can only be seen asymptotically in the regime $p \ll \Xi_c^{-1}$. This fact may be of importance when studying frustrated spin systems on the lattice where one has to work with lattice sizes $L \gg \Xi_c$ to be able to observe $O(2N)$ universal scaling. In the limit $N \to \infty$, $\Xi_c$ can be expressed in terms of $M^2$ of eq.\(\text{(10)}\) as:

\[
\Xi_c = \Lambda^{-1} \left(\frac{N}{\pi \epsilon M^2}\right)^{1/\epsilon}.
\] (52)

Let us finally give the expressions of the critical exponents:

- At the $O(2N)$ fixed point, $x$ is an irrelevant variable and scales to zero as:

\[
x(\lambda) \sim \left(\frac{\lambda}{\lambda_1}\right)^{-\phi_{O(2N)}}, \quad \lambda \gg \lambda_1.
\] (53)

This defines the cross-over exponent $\phi_{O(2N)}$ which is obtained, at two loop, from eq.\(\text{(34)}\):

\[
\phi_{O(2N)} = \frac{N}{N - 1} \epsilon + \frac{N}{2 (N - 1)^2} \epsilon^2 + O\left(\epsilon^3\right).
\] (54)

The exponent $\nu_{O(2N)}$ itself is, of course, given by that of the $O(2N)$ model:

\[
\nu_{O(2N)}^{-1} = \epsilon + \frac{\epsilon^2}{2N - 2} + O\left(\epsilon^3\right).
\] (55)

- Near the $CP^{N-1}$ fixed point, $T$ and $x$ are relevant. While the scaling exponent is given in the $T$ direction by:

\[
\nu_{CP^{N-1}}^{-1} = \epsilon + \frac{2}{N} \epsilon^2 + O\left(\epsilon^3\right),
\] (56)
we find that $x(\lambda)$ scales trivially:

$$x(\lambda) \sim 1 - \left(\frac{\lambda}{\lambda_1}\right)^\epsilon, \quad \lambda \ll \lambda_1$$

so that $\phi_{CP^{N-1}} = \epsilon$. This property holds at all orders of perturbation theory. This is easily seen on eq.(31) where one can observe that the $\beta$ function for $\bar{\eta}_2$ orders with increasing powers of $(\bar{\eta}_2/\eta_1)^p$, $p \geq 2$, so that all its partial derivatives with respect to $\bar{\eta}_2$ taken at $\bar{\eta}_2 = 0$ vanishes. Hence the scaling of $\bar{\eta}_2$ follows simply from dimensional analysis. This means that $\bar{\eta}_2$ does not renormalize at sufficiently high energy and that the $U(1)$ current has no anomalous dimension at the $CP^{N-1}$ fixed point, a result which is completely consistent with the $U(1)$ gauge invariance.

3.4 Concluding remarks

In this section, we have studied the massive $CP^{N-1}$ model by means of a low temperature expansion around two dimensions. We have expressed the RG properties of this model in terms of local, geometrical, quantities associated with the manifold $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ viewed as a metric space. We have found that the $\beta$ functions and the mass gap, calculated up to two loop order, interpolate smoothly between those of $CP^{N-1}$ and $O(2N)$ models as one varies $x$. In addition, a careful analysis of the RG flow have revealed the existence of a new length scale $\Xi$, characterizing the cross-over from short distance behavior, governed by the $CP^{N-1}$ fixed point, to the long distance one governed by the $O(2N)$ fixed point. The appearance of this length results from the competition between fluctuations of different nature and is reminiscent of the frustration of the lattice spin system. An important point is that, contrarily to the correlation length $\xi$, the length $\Xi$ is not related to a singular behavior of the RG flow so that perturbation theory is a priori well defined in all the regime $\xi^{-1} \ll p \ll \Lambda$ independently of the relative value of $p$ and $\Xi$, or equivalently, for any value of $x$ between 0 and 1. Indeed, the low temperature expansion which is at the basis of the NL$\sigma$ model analysis is a loop expansion ordered in powers of $T$ and $\epsilon$ and is insensitive to the value of $x$.

It is however worth to stress that all our results rely precisely on the perturbative renormalizability of the model in this double expansion in $T$ and $\epsilon$. Under this hypothesis,
we expect that, at sufficiently low energy, i.e. when $p \ll \Lambda$, the physics of the spin system is well described in terms of only two coupling constants $x$ and $T$ and is therefore universal. In this regime, the high energy degrees of freedom have decoupled from the low energy ones and all the remaining irrelevant operators give subdominant contributions to the scaling of physical quantities. The question that naturally arises is whether or not this universality property holds beyond perturbation theory and more precisely, to what extent can we control the effects of irrelevant operators beyond the double expansion in $\epsilon$ and $T$. Indeed, one possible source of failure of perturbation theory in NL$\sigma$ models is the neglected $O(e^{-1/T})$ terms that take into account of the global nature of the manifold. The best that one can do is to check the consistency of the low $T$ expansion against other perturbatives approaches such as the $1/N$ expansion. It is the purpose of the next section to investigate the properties of the massive $CP^{N-1}$ model through an expansion in the number of components of the fields and to study some aspects of the relationship between the large $N$ and weak coupling analysis of this model.

4 THE LARGE N APPROACH.

The $1/N$ expansion is a powerful tool on several accounts. On the one hand, it reveals by itself non perturbative informations about the model: mass generation, existence of a phase transition, etc. On the other hand, it provides a test of consistency of perturbative approaches, like weak coupling analysis. However, up to now, because of their strong analogies with gauge field theories in four dimensions and their simplicity, only symmetric models like $CP^{N-1}$ or $O(2N)$ NL$\sigma$ models - with sometimes their supersymmetric counterpart - and Gross-Neveu model in two dimensions have been subject to a detailed analysis. In particular, it is just for the two last ones that the precise relationship between the large $N$ and weak coupling analysis has really been understood beyond leading order. Such a relationship is still lacking in the case of more involved models like the massive $CP^{N-1}$ model.

As already emphasized, the main feature of this model is that the current-current operator $(z^{\dagger} \partial_{\mu} z)^2$ which is relevant at the $CP^{N-1}$ fixed point becomes irrelevant at the
$O(2N)$ one. In the double expansion in $T$ and $\epsilon$ which is at the basis of the low temperature approach of the NL$\sigma$ model around two dimensions, such a term stays marginal and is consequently renormalizable. This is why the low $T$ expansion allows to interpolate between the two fixed points of the theory. However this situation is particular to the weak coupling expansion and one does not expect that this will be the case in other perturbative scheme. Indeed, we shall see that the standard $1/N$ expansion is unable to describe the whole phase diagram of the massive $CP^{N-1}$ model. The reason for this is that the cross-over scale that emerged from the RG flow analysis of the weak coupling expansion enters in the $1/N$ expansion in such a way that it provides a boundary to the domain of validity of the standard large $N$ perturbative approach. We shall show however that by carefully taking into account of the presence of this new length scale, it is possible to capture the correct renormalization of the parameters of the massive $CP^{N-1}$ model in the large $N$ analysis at least around each of the two fixed points of the model.

Let us first explicit the $N$ dependence of the coupling constants $f_1$ and $f_2$. The action (2) writes now:

$$S = \int d^d x \left[ \Lambda d^{-2} \frac{N}{2 f_1} \partial_\mu z^\dagger \partial_\mu z + \Lambda d^{-2} \frac{N}{2 f_2} \left( z^\dagger \stackrel{\leftrightarrow}{\partial_\mu} z \right)^2 \right],$$

where the $z$ is a $N$ component complex field still subject to the constraint $z^\dagger z = 1$. The starting point of the large $N$ expansion is to take into account of the latter constraint with help of a Lagrange multiplier field $\lambda$ with a resulting effective action:

$$S = \int d^d x \left[ \partial_\mu z^\dagger \partial_\mu z - i \lambda \left( z^\dagger z - \Lambda d^{-2} \frac{N}{2 t} \right) + \frac{x t}{2 N} \Lambda^{d-2} \left( z^\dagger \stackrel{\leftrightarrow}{\partial_\mu} z \right)^2 \right]$$

with $t = f_1$ and $x = 4 f_1/f_2$. In the absence of the current-current interaction, quartic in $z$, one could have integrated exactly over the $z$ field to obtain an effective, non local, action for the $\lambda$ field from which results the well known $1/N$ expansion of the $O(2N)$ model. The difficulty with the massive $CP^{N-1}$ model comes precisely from the $U(1)$ current-current interaction. The standard prescription for dealing with such a term is to introduce a Lagrange field $A_\mu$ as in the pure $CP^{N-1}$ model. In this case, the resulting action is given by:

$$S = \int d^d x \left[ z^\dagger \left( -D^2_\mu - i \lambda \right) z + \Lambda d^{-2} \frac{N M^2}{2} A_\mu^2 + i \Lambda d^{-2} \frac{N}{2 t} \lambda \right].$$
where \( M^2 = (1 - x)/xt \) and \( D_\mu = \partial_\mu + iA_\mu \) is the covariant derivative introduced in section(2). Now the integration over the \( z \) field can be exactly done and the resulting partition function writes in terms of \( \lambda \) and \( A_\mu \) fields as:

\[
Z = \int D\lambda DA_\mu e^{-NS_0}
\]

with:

\[
S_0 = \text{Tr} \ln(-D^2_\mu - i\lambda) + \Lambda^{d-2} \frac{i}{2t} \int d^dx \lambda + \Lambda^{d-2} \frac{M^2}{2} \int d^dx A^2_\mu .
\]

The set of eqs.(61,62) have been obtained first by Campostrini and Rossi\(^27\) who have studied the resulting \( 1/N \) expansion in dimension \( d = 2 \). They have computed the mass gap and the \( \beta \) functions for the coupling constants \( t \) and \( x \) at leading order in \( 1/N \). As a result, they argued that the mass term \( M^2 \) does not renormalize, a result which contradicts our two loop expressions of eq.(34). The origin of this contradiction between the weak coupling and large \( N \) analysis lies in the fact that the decoupling of the current-current interaction is not an innocent procedure. In particular, the use of the field theoretical approach on the resulting field theory (62) to obtain the correct evolution of the two coupling constants makes sense only in a regime of energy much greater than any other mass scale of the theory, i.e. in the vicinity of the \( CP^N-1 \) fixed point. This fact was missed by Campostrini and Rossi. As a consequence, their results for the mass gap or the \( \beta \) functions are not correct in the large mass \( M^2 \) limit. In this case indeed, one has to return to the form (59) of the action and consider the current-current term \( \Lambda^{2-d} \frac{xt}{2N} \left( \partial^\dagger_\mu z \right)^2 \) as an irrelevant operator in the neighborhood of the \( O(2N) \) fixed point.

In the following, we shall study action (58) in the large \( N \) expansion in \( d = 2 \) and \( d > 2 \). Some of our results in \( d = 2 \) have already been obtained by Campostrini and Rossi\(^27\). To be self contained, we shall present them together with new ones.

### 4.1 The \( N \to \infty \) limit

As readily seen on eq.(51), the theory is exactly soluble in the limit \( N \to \infty \) since the functional integral is dominated by the saddle point:
\[
\begin{aligned}
\frac{\delta S_0}{\delta \lambda} &= 0 \\
\frac{\delta S_0}{\delta A_\mu} &= 0.
\end{aligned}
\] (63)

Searching for solutions with \(A_\mu\) constant one finds \(A_\mu = 0\) so that the saddle point is the same as for \(O(2N)\) and \(CP^{N-1}\). Eq. (63) then leads to:

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} = \frac{\Lambda^{d-2}}{2t}
\] (64)

with \(\lambda = im^2\).

In dimension \(d \geq 2\), the integral (64) is UV divergent, and we shall use a cut-off regularization procedure. One has to consider separately the \(d = 2\) and \(d > 2\) cases.

- In \(d = 2\), eq.(64) gives the mass gap equation:

\[
\frac{\pi}{t} = \ln \frac{\Lambda}{m}
\] (65)

from which follows the \(\beta\) function for \(t\) to leading order (\(N = \infty\)):

\[
\beta_t = \frac{t^2}{\pi}
\] (66)

which expresses the property of asymptotic freedom in two dimensions [5].

- In \(d > 2\), eq.(64) writes:

\[
\frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d-2} = \frac{\Lambda^{d-2}}{2} \left(1 - \frac{1}{t_c t}\right)
\] (67)

with \(t_c = \frac{d - 2}{2K_d}\) and \(K_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\).

One can then deduce the \(\beta\) function for \(t\):

\[
\beta_t = -(d - 2) \frac{t}{t_c} t \left(1 - \frac{t}{t_c}\right).
\] (68)

The \(\beta\) function for \(x\) follows trivially from the fact that \(M^2\) does not renormalize at this level since \(A_\mu\) does not participate at the saddle point. One thus have:

\[
\beta_{M^2} = (d - 2) M^2.
\] (69)
The set of eqs.(66-69) gives a complete description of the RG properties of the model at $N = \infty$ and is completely consistent with those obtained in the weak coupling analysis of the previous section:

- In dimension $d = 2$, the expressions of the mass gap and the $\beta$ function for $t$ agree with those obtained from the weak coupling expanded at leading order in $N$. Moreover, the non renormalization of the mass term $M^2$ at this order is consistent with the fact that $M^2$ identifies with the weak coupling one loop flow invariant at $N = \infty$ since $M^2 = \frac{1 - x}{xt} = K^{-1}_\infty$ (see eq.(35)).

- In dimension $d > 2$, there exist two non trivial fixed points with $t = t_c$ and $M^2 = 0$ or $M^2 = \infty$. The fixed point with $M^2 = \infty$ governs the phase transition for all models with $M^2 \neq 0$ since $M^2$ is a relevant variable. The critical exponent $\nu^{-1} = d - 2$ is trivially the same as for the $O(2N)$ and $CP^{N-1}$ models while the cross-over exponent $\phi = d - 2$ is just given by dimensional analysis. Both results agree with those of eqs.(54-57) at leading order in $\epsilon$ and $N$.

As seen, there is a complete agreement between weak coupling and large $N$ approach to leading order in $1/N$ and $\epsilon$. However, the $N = \infty$ limit is special since it does not involve the $U(1)$ current-current interaction which is crucial in the $CP^{N-1}$ model. This interaction will express itself as soon as $1/N$ corrections to the saddle point will be taken into account and will reveal the non trivial structure of the RG properties of the model.

4.2 The 1/N corrections: general discussion

Let us now investigate the effects of fluctuations around the saddle point. The Feynman rules for the $1/N$ expansion of action (60) follow from those of the pure $CP^{N-1}$ model (see for instance Ref.[31]). They are summerized in Fig.3.

The inverse propagator of the $\lambda$ field is given as in the $CP^{N-1}$ model by:

$$\Gamma_{\lambda}(k) = \int \frac{d^{d}q}{(2\pi)^{d}} \frac{1}{(q^{2} + m^{2})(q^{2} + k^{2} + m^{2})}.$$  \hfill (70)

The only difference with the $CP^{N-1}$ model comes from the inverse gauge field propagator $A_\mu$ which acquires a longitudinal part:

$$\Gamma_{\mu\nu}(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Gamma_{\perp}(k) + \Lambda^{d-2} M^{2} \frac{k_\mu k_\nu}{k^2}.$$  \hfill (71)
with:
\[
\Gamma_{\perp}(k) = \Lambda^{d-2} M^2 - \frac{4}{d-1} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} m^{d-2} + \frac{k^2 + 4m^2}{d-1} \Gamma_{\lambda}(k) .
\] (72)

With help of the rules of Fig.3, the $1/N$ corrections can be derived in a standard way for all Green’s functions of the theory with the restriction that no internal spinon loop is allowed since they are already taken into account in the definition of the $\lambda$ propagator. Before turning to the practical computations of the $1/N$ corrections to the mass gap $m$ and to the $\beta$ functions for $t$ and $M^2$ or $x$, it is worth at this stage to precise from eq.(72) what is the expected domain of validity of the decoupling procedure. To see this, let us consider the short distance behavior of the inverse gauge field propagator involved in the computation of the correlation functions. In the large $k$ limit, one has:
\[
\begin{align*}
\Gamma_{\perp}(k) &\sim M^2 - \frac{1}{\pi} + \frac{1}{2\pi} \ln \frac{k^2}{m^2} & \text{in } d=2 \\
\Gamma_{\perp}(k) &\sim \Lambda^{d-2} M^2 - \frac{4}{d-2} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} m^{d-2} + A_d k^{d-2} & \text{in } d>2
\end{align*}
\] (73)

with $A_d = \frac{K_d}{2(d-1)S_d}$ and $S_d = \frac{\Gamma(d-1) \sin [\pi(d-2)/2]}{2\pi (\Gamma(d/2))^2}$.

It appears from expression (73) that the scaling behavior of the coupling constants $(x,t)$ can be extracted from the UV behavior of the field theory (62) when:
\[
\begin{align*}
M^2 &\ll \frac{1}{2\pi} \ln \frac{\Lambda^2}{m^2} & \text{in } d=2 \\
M^2 &\ll A_d & \text{in } d>2
\end{align*}
\] (74)

This defines the energy regime (I) where the decoupling procedure is meaningful:
\[
\begin{align*}
p &\gg m \exp (\pi M^2) & \text{in } d=2 \\
p &\gg \Lambda \left(\frac{M^2}{A_d}\right)^{1/d-2} & \text{in } d>2
\end{align*}
\] (75)

Expanding the second equation to leading order in $\epsilon$, one sees that the bounds defined in eq.(75) identify with those obtained in the weak coupling analysis - eqs.(12,22) - that define the regime governed by the $CP^{N-1}$ fixed point. Using the saddle point equations
(see eqs. (65,67)), one can translate eq.(74) in terms of \( t \) and \( M^2 \) as:

\[
\begin{cases}
    tM^2 \ll 1 & \text{in } d = 2 \\
    tM^2 \ll \frac{d-2}{4(d-1)S_d} & \text{in } d > 2
\end{cases}
\]

(76)

There exists a second regime (II) where it is possible to extract the renormalization of \( t \) and \( x \). This is the regime governed by the \( O(2N) \) fixed point defined by:

\[
\begin{cases}
    p \ll m \exp(\pi M^2) & \text{in } d = 2 \\
    p \ll \Lambda \left( \frac{M^2}{A_d} \right)^{1/d-2} & \text{in } d > 2
\end{cases}
\]

(77)

which, in terms of \( t \) and \( M^2 \), writes:

\[
\begin{cases}
    tM^2 \gg 1 & \text{in } d = 2 \\
    tM^2 \gg \frac{d-2}{4(d-1)S_d} & \text{in } d > 2
\end{cases}
\]

(78)

In this regime the \( A_\mu \) field does not propagate anymore and the decoupling (60) is of no use to determine the renormalization of \( t \) and \( x \). One has to return to the original form of the action (59). Near the \( O(2N) \) fixed point, the current-current \( U(1) \) interaction is irrelevant and can be considered as a composite operator in the \( O(2N) \) theory so that its effect on the physics can be computed perturbatively in a double expansion in \( x \) and \( 1/N \) [28, 29].

In the following, we shall consider perturbation theory in the regimes (I) and (II) separately. We shall compute the \( \beta \) functions for the coupling constants \( x \) and \( t \) and the mass gap \( m \) at leading order in \( 1/N \) in \( d = 2 \) and \( d > 2 \). Comparison with the results obtained from the weak coupling analysis of the preceding section of this work will also be made.

### 4.3 The CP\(^{N-1}\) regime

This is the regime (I) defined by eq.(74). As already quoted above, perturbation theory, as given by the Feynman rules of Fig.3, is well defined. We shall assume in the following
that the theory is renormalizable so that only renormalization of spinon and gauge field masses and fields are needed. We shall thus concentrate on the two point functions of the spinon and gauge field at order $1/N$:

$$
\begin{align*}
G^{(2)}_{1/N}(p) &= \langle z^\dagger(0)z(p) \rangle \\
G^{(2)}_{\mu,\nu1/N}(p) &= \langle A_\mu(0)A_\nu(p) \rangle.
\end{align*}
$$

(79)

The renormalization procedure is standard. As usual, we shall consider the leading order of the Taylor expansion near $p = 0$ of $G^{(2)}_{1/N}(p)$ and $G^{(2)}_{\mu,\nu1/N}(p)$ which suffer from divergent corrections which will be eliminated by appropriate redefinitions of the masses and fields.

4.3.1 The two point spinon function

The diagrams that contribute to the $1/N$ correction of $G^{(2)}_{1/N}(p)$ are given in Fig.4. The self energy $\Sigma(p)$ is defined by:

$$
G^{(2)}_{1/N}(p) = \frac{1}{p^2 + m^2} - \frac{1}{N(p^2 + m^2)} \Sigma(p) \frac{1}{p^2 + m^2}
$$

(80)

or in terms of the 1-P.I. function:

$$
\Gamma^{(2)}_{1/N}(p) = p^2 + m^2 + \frac{1}{N} \Sigma(p).
$$

(81)

After lengthly but straightforward calculations we have obtained for $\Sigma(p)$:

$$
\Sigma(p) = Ap^2 + Bm^2 + O(p^4)
$$

(82)

with:

$$
\begin{align*}
A &= \frac{4 - d}{d} \int \frac{d^dk}{(2\pi)^d} \frac{\Gamma^{-1}_\lambda(k)}{(k^2 + m^2)^2} - \frac{4(d - 1)}{d} \int \frac{d^dk}{(2\pi)^d} \frac{\Gamma^{-1}_\perp(k)}{k^2 + m^2} + \frac{\Lambda^{2-d}}{M^2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + m^2} \\
B &= 3 \int \frac{d^dk}{(2\pi)^d} \frac{\Gamma^{-1}_\lambda(k)}{(k^2 + m^2)^2} + \left( \frac{\Lambda^{2-d}}{M^2} + (d - 2)(d + 1) \frac{\Gamma^{-1}_\perp(0)}{m^2} \right) \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + m^2} + (d - 1) \left( 4 - (d - 1)\Lambda^{d-2}M^2 \frac{\Gamma^{-1}_\perp(0)}{m^2} \right) \int \frac{d^dk}{(2\pi)^d} \frac{\Gamma^{-1}_\perp(k)}{k^2 + m^2}.
\end{align*}
$$

(83)
Defining:

$$\Gamma_R^{(2)}(p) = Z \Gamma_{1/N}^{(2)}(p)$$

with the following prescriptions:

$$\begin{cases} 
\Gamma_R^{(2)}(p^2 = 0) = m_R^2 \\
\frac{\partial \Gamma_R^{(2)}(p)}{\partial p^2} \bigg|_{p=0} = 1
\end{cases}$$

we obtain:

$$\begin{cases} 
Z = 1 - \frac{A}{N} \\
m_R^2 = m^2 \left( 1 + \frac{B - A}{N} \right)
\end{cases}$$

We shall now consider the cases $d = 2$ and $d > 2$ separately.

- **Two dimensional case.**

  In $d = 2$, we find for $Z$ and $m_R^2$:

  $$\begin{cases} 
  Z = 1 - \frac{1}{N} \left[ \frac{1}{2} \ln \frac{\Lambda^2}{m^2} - \ln(2\pi M^2 - 2 + \ln \frac{\Lambda^2}{m^2}) + \frac{1}{4\pi M^2} \ln \frac{\Lambda^2}{m^2} \right] \\
  m_R^2 = m^2 \left[ 1 + \frac{1}{N} \left( \ln \frac{\Lambda^2}{m^2} + (3 - 2\pi M^2) \ln(2\pi M^2 - 2 + \ln \frac{\Lambda^2}{m^2}) \right) \right].
\end{cases}$$

  These expressions agree with those of Ref. [27]. Using the saddle point equation (65), one can express the mass gap $m_R^2$ as a function of $t$ and $M^2$:

  $$m_R^2 = m^2 \left[ 1 + \frac{1}{N} \left( \ln \frac{2\pi}{t} + (3 - 2\pi M^2) \ln(2\pi M^2 - 2 + \frac{2\pi}{t}) \right) \right].$$

  (88)

  Using the $N = \infty$ result (106) and the fact that $\beta_{M^2} = O(1/N)$ since $M^2$ does not renormalize at the saddle point one obtains the $\beta$ function for $t$:

  $$\beta_t = \frac{t^2}{\pi} \left[ 1 + \frac{t}{2\pi N} \left( 1 + \frac{3 - 2\pi M^2}{1 + t(M^2 - 1/\pi)} \right) \right].$$

  (89)

  This result has been first obtained by Campostrini and Rossi [27].

- **Above two dimensions.**

  Let us begin by giving the explicit expressions of the field and mass renormalizations in $d = 3$:
\[ Z = 1 - \frac{1}{N} \left[ \frac{4}{3\pi^2} \ln \frac{\Lambda}{m} + \frac{1}{2tM^2} - \frac{64}{3\pi^2} \ln \left( \frac{\Lambda}{16} + M^2 \Lambda - \frac{m}{2\pi} \right) \right] \]

\[ m_R^2 = m^2 \left[ 1 + \frac{32}{3\pi^2 N} \ln \frac{\Lambda}{m} + \frac{16}{\pi N} \frac{\Lambda}{m} + \frac{256}{\pi^2 N} \left( \frac{1}{3} - \frac{\pi \Lambda M^2}{m} \right) \ln \left( \frac{\Lambda}{16} + M^2 \Lambda - \frac{m}{2\pi} \right) \right] \]

which leads to the \( \beta \) function for \( t \):

\[ \beta_t = - \left[ t \left( 1 + \frac{4}{N} \frac{M^2}{1/16 + M^2 + 1/t - 1/\pi^2} \right) \right] \left( 1 + \frac{16}{3N\pi^2} + \frac{8}{3N\pi^2} \frac{1}{1/16 + M^2 + 1/t - 1/\pi^2} \right) \]  

In \( d > 2 \), we can obtain the \( \beta \) function for \( t \) from eqs. (B4) and (B3):

\[ \beta_t = - \left[ t - \frac{t^2}{t_c} \left( 1 + \frac{d^2 - d - 2}{N} - \frac{(d - 1)^2}{N} \frac{M^2}{A_d + M^2 + 1/t - 1/t_c} \right) \right] \left( d - 2 \frac{4(d - 1)(d - 2)S_d}{Nd} + \frac{2(d^2 - 1)(d - 2)K_d}{Nd} \frac{1}{A_d + M^2 + 1/t - 1/t_c} \right) \]

4.3.2 Gauge field renormalization

We shall now investigate the role of fluctuations on the gauge field \( A_\mu \) at order \( 1/N \) in two and three dimensions. This will provide the renormalization of the gauge field mass \( M^2 \) from which will follow the renormalization of the complete set of coupling constants of action (62). From the renormalization point of view, the situation is formally equivalent to what happens in Quantum Electrodynamics. In particular, Ward identities stated in the treatment of gauge field with a non-vanishing mass for the photon ensure the absence of a counter-term for the longitudinal part of the gauge field propagator in such a way that the renormalization constant for the gauge field \( A_\mu \), \( Z_{A_\mu} \) and for its mass, \( Z_{M^2} \) verify \( Z_{A_\mu} Z_{M^2} = 1 \). It follows from this that the inverse gauge field propagator at order \( 1/N \) can be written:

\[ \Gamma_{\mu\nu1/N}(p) = \left( \delta_{\mu\nu} - \frac{P_{\mu}P_{\nu}}{p^2} \right) \Gamma_{\perp1/N}(p) + M^2 \frac{P_{\mu}P_{\nu}}{p^2} \]  

(93)
where $\Gamma_{\perp 1/N}(p)$ is the transversal part of the inverse gauge field propagator at order $1/N$.

We define the renormalized inverse gauge field propagator by:

$$
\Gamma_{\mu\nu R}(p) = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Gamma_{\perp R}(p) + M^2_R \frac{p_\mu p_\nu}{p^2}
$$

(94)

with:

$$
\left\{ \begin{array}{l}
\Gamma_{\perp R}(p) = Z_{A_\mu}^{-1} \Gamma_{\perp 1/N}(p) \\
M^2_R = Z_{A_\mu}^{-1} M^2
\end{array} \right.
$$

(95)

where $Z_{A_\mu}$ is determined by the prescriptions:

$$
\Gamma_{\perp R}(p = 0) = M^2_R
$$

(96)

and:

$$
\left\{ \begin{array}{l}
\left. \frac{\partial \Gamma_{\perp R}}{\partial p^2} \right|_{p=0} = \frac{1}{12\pi m^2_R} \quad \text{in } d = 2 \\
\left. \frac{\partial \Gamma_{\perp R}}{\partial p^2} \right|_{p=0} = \frac{1}{24\pi m_R} \quad \text{in } d = 3.
\end{array} \right.
$$

(97)

The diagrams contributing to the renormalization of the gauge field $A_\mu$ at this order can be found in Refs. [27, 44, 45] and they are reproduced in Fig.5. The renormalization of $\Gamma_{\perp}(p)$ is provided by a Taylor expansion in power of the external moment $p$ of the diagrams. We then need the expressions of the transversal part of the bare inverse gauge field propagator at small $p$:

$$
\begin{align*}
\Gamma_{\perp}(p) & \sim \frac{p^2}{12\pi m^2} + M^2 + O(p^4) \quad \text{in } d = 2 \\
\Gamma_{\perp}(p) & \sim \frac{p^2}{24\pi m} + M^2 + O(p^4) \quad \text{in } d = 3
\end{align*}
$$

(98)

which, taking into account of the spinon mass renormalization at order $1/N$ given by eq.(86), leads to:

$$
\begin{align*}
\Gamma_{\perp}(p) & \sim \frac{p^2}{12\pi m^2} + \frac{p^2}{12\pi m^2} \frac{B-A}{N} + M^2 + O(p^4) \quad \text{in } d = 2 \\
\Gamma_{\perp}(p) & \sim \frac{p^2}{24\pi m_R} + \frac{p^2}{48\pi m_R} \frac{B-A}{N} + M^2 + O(p^4) \quad \text{in } d = 3.
\end{align*}
$$

(99)
The calculation of the counter-terms for the inverse gauge field propagator at order $1/N$ is very cumbersome and details are given in the Appendix.

We have obtained the following expressions:

\[
\begin{align*}
\Gamma_{\perp 1/N}(p) &= \Gamma_{\perp}(p) - \frac{p^2}{12\pi m_R^2} \frac{B-A}{N} \quad \text{in } d = 2 \\
\Gamma_{\perp 1/N}(p) &= \Gamma_{\perp}(p) - \frac{p^2}{48\pi m_R} \frac{B-A}{N} \quad \text{in } d = 3
\end{align*}
\]

which, together with eq.(99), show that the counter-terms associated with the transversal part of the inverse gauge field propagator are just cancelled by the renormalization of the spinon mass in such a way that the gauge field renormalization is trivial:

\[
Z_{A\mu} = 1.
\]

We thus find that the gauge field mass $M^2$ does not renormalize at order $1/N$ in $d = 2$ and $d = 3$. This result probably holds for any dimension between 2 and 4. However, it is worth stressing that this result is true only in the small $tM^2$ limit where the above calculation makes sense. The non renormalization of $M^2$ at order $1/N$ and the relation between $M^2$ and $x - M^2 = (1 - x)/xt$ - induce a non trivial renormalization for the variable $x$ given by:

\[
\beta_x = -\frac{M^2}{(1 + tM^2)^2} \beta_t.
\]

### 4.3.3 Comparison between large $N$ and weak coupling analysis

At first sight, the $\beta$ functions for $t$ and $x$ seem to interpolate between the $CP^{N-1}$ and $O(2N)$ models: they behave smoothly as functions of $tM^2$. The $\beta$ function (89) for $t$ allows to recover that of the $CP^{N-1}$ and $O(2N)$ models as $M^2$ goes to zero or infinity respectively. However, as discussed above, the whole calculation and, in particular, that involving gauge field loops is valid in a narrow range of energy defined by $tM^2 \ll 1$ which also implies $1 - x \ll 1$. We shall now see that it is only in that case that the results of the $1/N$ approach agree with those of the weak coupling analysis. In contrast, we shall show that the naive extrapolation of the results for large $tM^2$ disagrees with the weak coupling ones.
Let us first show that there is, in the $CP^{N-1}$ regime, a good agreement with the weak coupling results of the preceding section. We consider now separately the cases $d = 2$ and $d > 2$.

Let us consider the $\beta$ functions (34) obtained in the weak coupling analysis in two dimensions. These equations can be expanded in powers of $1/N$ by doing the substitution $T \rightarrow t/N$. The resulting expressions can then be expanded in powers of $1 - x$. Keeping only terms of first order in this parameter one obtains the following expressions:

$$
\beta_t = \frac{t^2}{\pi} \left( 1 - \frac{1}{N} + \frac{x}{N} \right) + \frac{2t^3}{\pi^2N} \\
\beta_x = -\frac{t}{\pi} \left( 1 - x \right) - \frac{2t^2}{\pi^2N} \left( 1 - x \right).
$$

(103)

Now one can formally expand the $\beta$ functions (89) and (102) obtained in the $1/N$ approach in powers of $t$ up to order $t^3$. Then $M^2$ is replaced by its expression as a function of $x$ and $t$. Keeping again just terms of order $1 - x$, we recover the set of equations (103).

This matching between the large $N$ and weak coupling expansions also works for the mass gap. The expression (88) of the mass gap obtained in the large $N$ analysis can be expanded in powers of $t$ and re-exponentiated in powers of $1/N$. This leads to:

$$
\xi = C \Lambda^{-1} \frac{2}{tN} \exp \left( \frac{\pi}{t} \right) \exp \left( -\frac{(1-x)}{Nt} \ln \frac{t}{2\pi} \right)
$$

(104)

which coincides at leading order in $t$ with the expression (10) of $\xi$ obtained in the weak coupling analysis in the $CP^{N-1}$ regime.

In dimension greater than two, one can compare the exponents $\nu_{CP^{N-1}}$ and $\phi_{CP^{N-1}}$ that govern the leading scaling behavior of $x$ and $t$ at the $CP^{N-1}$ fixed point. The exponent $\nu_{CP^{N-1}}$ obtained in the $1/N$ expansion from eq.(92):

$$
\nu_{CP^{N-1}} = \frac{1}{d-2} \left( 1 - \frac{4d(d-1)S_d}{N} \right)
$$

(105)

coincides with that obtained in the weak coupling analysis (eq.(56)) at order $1/N$ and $\epsilon^2$. The cross-over exponent $\phi_{CP^{N-1}}$ can be obtained in the large $N$ limit from eq.(102) - $\phi_{CP^{N-1}} = d - 2$ - in agreement with the weak coupling result. We can therefore conclude that, in the $CP^{N-1}$ regime, the large $N$ and weak coupling expansions coincide.
Such an agreement does not exist in the $O(2N)$ regime. In two dimensions, let us consider the $\beta$ functions obtained in the weak coupling analysis and re-expanded in powers of $1/N$:

$$\begin{align*}
\beta_t &= t^2 \frac{2}{\pi} \left(1 - \frac{1}{N} + \frac{x}{N}\right) + \frac{t^3}{2\pi^2 N} \\
\beta_x &= -\frac{x t}{\pi} \left(1 + \frac{t}{\pi N}\right).
\end{align*}$$

(106)

These expressions have to be compared to the $\beta$ functions obtained in the large $N$ analysis expanded in powers of $1/tM^2$ since the condition $tM^2 \gg 1$ defines the $O(2N)$ regime:

$$\begin{align*}
\beta_t &= \frac{t^2}{\pi} \left(1 - \frac{1}{N} + \frac{x}{N}\right) + \frac{t^3}{2\pi^2 N} (1 + x) \\
\beta_x &= -\frac{x t}{\pi} \left(1 - \frac{1}{N}\right) - \frac{x t^2}{2\pi^2 N}.
\end{align*}$$

(107)

As readily seen, the two set of equations (106) and (107) differ for the $\beta$ function for $x$ even at first order in $t$. This disagreement persists, of course, for the mass gap by a term which is linear in $x/N$.

In dimension greater than two, let us compare the critical exponents obtained from the $\beta$ functions of eqs.(92, 102) with those obtained in the weak coupling expansion.

The critical exponent $\nu_{O(2N)}$ as obtained from eq.(106):

$$\nu_{O(2N)} = \frac{1}{d - 2} \left(1 - \frac{4(d - 1)S_d}{Nd}\right)$$

(108)

coincides with that obtained in the weak coupling expansion (eq.(53)) at order $1/N$ and $\epsilon^2$. However, since the gauge field mass does not renormalize at order $1/N$, the crossover exponent is given by dimensional analysis: $\phi_{O(2N)} = d - 2 + O(1/N^2)$, and thus disagrees with the weak coupling expression of eq.(54). There is, in the $O(2N)$ regime, a disagreement between the $1/N$ expansion performed with help of the decoupling procedure and the weak coupling expansion. The reason is that the decoupling procedure does not take properly into account the dangerously irrelevant character of the current-current operator. The agreement we have obtained for the critical exponent $\nu_{O(2N)}$ stems from the fact that the exponent $\nu$ is determined by the asymptotic scaling in the relevant direction $t$ at $x = 0$. 

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Naturally, this disagreement was expected since as previously discussed, the decoupling procedure is valid provided \( tM^2 \ll 1 \). We shall now see that, provided one performs a suitable \( 1/N \) expansion in the vicinity of the \( O(2N) \) fixed point - i.e. when \( tM^2 \gg 1 \), the weak coupling and the large \( N \) analysis are consistent.

### 4.4 The \( O(2N) \) regime

In this regime, the gauge field \( A_\mu \) no longer propagates and perturbation theory defined by the Feynman rules of Fig.3 is meaningless. In particular, the decoupling of the current-current term \( \left( z^\dagger \partial_\mu z \right)^2 \) is of no use for exploring this low energy regime and one has to come back to the original form of action \((59)\). Simple power counting around the \( O(2N) \) theory indicates that \( \left( z^\dagger \partial_\mu z \right)^2 \) has scaling dimension \( 2 - d \) and is therefore an irrelevant (resp. marginal irrelevant) operator in dimension \( d > 2 \) (resp. \( d = 2 \)). Its effects on the low energy physics can however be computed in a double expansion in \( x \) and \( 1/N \) around the \( O(2N) \) theory as soon as \( x \ll 1 \) and \( N \gg 1 \)\[28, 29\]. In the following, we shall compute the corrections to the RG functions at first order in \( x \) and \( 1/N \). The Feynman rules for the double expansion follow from those of the \( 1/N \) expansion of the \( O(2N) \) sigma model. Propagators of the spinon \( z \) and \( \lambda \) fields as well as the \( \lambda \bar{z}z \) vertex are unchanged.

The only modification is the presence of a new four point vertex \( \Gamma_\mu \) which has been first introduced by Chubukov et al.\[28, 29\] in their study of the quantum version of the model in \( d = 2 + 1 \) (see Fig.6):

\[
\Gamma_\mu (k_1, k_2, k_3, k_4) = \Lambda^{2-d} \frac{xt}{2N} (k_1 + k_3)_\mu (k_2 + k_4)_\mu z_\alpha^\dagger (k_1) z_\beta^\dagger (k_2) z_\alpha (k_3) z_\beta (k_4). \tag{109}
\]

This operator being irrelevant we expect that higher order derivatives irrelevant operators will be generated by renormalization. However as we are only interested in the leading correction in \( x \) we can omit these possible additional terms.

As in the preceding subsection, we shall consider the renormalization of the spinon two point function and the four point vertex function \( \Gamma_\mu^{(4)} \) at first order in \( x \) and \( 1/N \) from which the RG functions will be computed and finally compared with those previously obtained in the weak coupling analysis.

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4.4.1 The two point spinon function

The diagrams contributing to the two point function $G_{1/2}(p)$ at first order in $x$ and $1/N$ are depicted in Fig.7. The coefficients $A$ and $B$ entering in the expression of the self energy $\Sigma(p)$ of eq.(82) are now given by:

$$
\begin{cases}
A = \frac{4-d}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{m^2} - x t \Lambda^{2-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \\
B = 3 \int \frac{d^d k}{(2\pi)^d} \frac{1}{m^2} - \left( (3-d) \frac{1}{m^2} + \Lambda^{2-d} x t \right) \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \\
&+ x t \Lambda^{2-d} \Gamma^{-1}_\lambda(0) \left( \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \right)^2.
\end{cases}
$$

(110)

In terms of this quantities, the field and mass renormalizations are given by eq.(86).

- In two dimensions, we find:

$$
\begin{align*}
Z &= 1 - \frac{1}{N} \left[ \frac{1}{2} \ln \ln \frac{\Lambda^2}{m^2} - \frac{x t}{4\pi} \ln \frac{\Lambda^2}{m^2} \right] \\
m_R^2 &= m^2 \left[ 1 + \frac{1}{N} \ln \ln \frac{\Lambda^2}{m^2} - \frac{1}{N} \ln \frac{\Lambda^2}{m^2} + \frac{1}{N} \frac{x t}{4\pi} \left( \ln \frac{\Lambda^2}{m^2} \right)^2 \right].
\end{align*}
$$

(111)

With use of the saddle point equation (65), the mass gap $m_R^2$ writes in terms of $x$ and $t$ as:

$$
m_R^2 = m^2 \left[ 1 + \frac{1}{N} \left( \ln \frac{2\pi}{t} - \frac{2\pi}{t} + \frac{x\pi}{t} \right) \right],
$$

(112)

from which follows the $\beta$ function for $t$:

$$
\beta_t = \frac{t^2}{\pi} \left( 1 - \frac{1}{N} + \frac{x}{N} \right) + \frac{t^3}{2\pi^2 N}.
$$

(113)

- In dimension greater than two, the divergent part of $A$ and $B$ are easily computed and, omitting the details, we find for $\beta_t$:

$$
\beta_t = t^2 \left[ (d-2) \left( \frac{1}{t_c} - \frac{1}{t} \right) \left( 1 + \frac{4(d-1)}{Nt_c} S_d \right) + \frac{(d-2)(d-3)}{Nt_c} + \frac{x(d-2)}{Nt_c} \right].
$$

(114)
4.4.2 The four point vertex function

The expression of $\beta_x$ can be obtained by considering the renormalization of the vertex function at first order in $x$ and $1/N$. At this order, the four point vertex function is given using the diagrams of Fig.8 by:

$$\Gamma^{(4)}_{\mu_1/N} (k_1, k_2, k_3, k_4) = -\frac{\Lambda^{2-d} x t}{2N} (k_1 + k_2)_\mu (k_3 + k_4)_\mu \left[ 1 - \frac{6}{N} \frac{d-2}{d} \int {d^d k \over (2\pi)^d} \frac{\Gamma^{-1}_\lambda (k)}{(k^2 + m^2)^2} \right].$$  \hfill (115)

The renormalized vertex function is defined through:

$$\Gamma^{(4)}_{\mu R} (k_1, k_2, k_3, k_4) = Z^2 \Gamma^{(4)}_{\mu_1/N} (k_1, k_2, k_3, k_4)$$ \hfill (116)

where $Z$ is the spinon field renormalization. At this order we just need the field renormalization $Z$ of the $O(2N)$ model. Defining $Z_{xt}$ by:

$$(xt)_R = Z_{xt} \ x t$$  \hfill (117)

we deduce from eq.(115):

$$Z_{xt} = 1 - \frac{4 (d-1)}{Nd} \int {d^d k \over (2\pi)^d} \frac{\Gamma^{-1}_\lambda (k)}{(k^2 + m^2)^2}$$ \hfill (118)

and explicitly:

$$\begin{align*}
Z_{xt} &= 1 - \frac{1}{N} \ln \ln \frac{\Lambda^2}{m^2} & \text{in } d = 2 \\
Z_{xt} &= 1 - \frac{8 (d-1)}{Nd} S_d \ln \frac{\Lambda}{m} & \text{in } d > 2 .
\end{align*}$$ \hfill (119)

Using these results, together with the expressions of the $\beta$ functions for $t$ given by eqs.(113,114), we deduce the $\beta$ function for the variable $x$:

$$\begin{align*}
\beta_x &= -\frac{xt}{\pi} \left( 1 + \frac{t}{2\pi N} \right) & \text{in } d = 2 \\
\beta_x &= - \left[ d - 2 + \frac{8 (d-1)}{Nd} S_d + \frac{\beta^{O(2N)}_t}{t} \right] x & \text{in } d > 2 ,
\end{align*}$$  \hfill (120)

where in the last equation, $\beta^{O(2N)}_t$ is the $\beta$ function for $t$ for the $O(2N)$ model.
4.4.3 Comparison between weak coupling and large $N$ analysis

In dimension two, one sees that the $\beta$ functions of eqs. (113,120) agree, at first order in $x$ and $1/N$, with the weak coupling expressions of eq. (106). There is still a disagreement for the two loop term of the $\beta$ function for $x$ but this corresponds to subdominant corrections to the scaling behavior at the $O(2N)$ fixed point. Anyway, this term does not contribute to the mass gap at this order. Indeed, one can check that the expression of the correlation length obtained from eq. (112):

$$\xi = C' \Lambda^{-\frac{1}{2}} (N-1) \exp \left(\frac{\pi N}{t(N-1)}\right) \exp \left(\frac{-x\pi}{2Nt}\right)$$  \hspace{1cm} (121)

identifies with the correlation length (eq. (45)) obtained in the weak coupling expansion.

In dimension greater than two, the cross-over exponent $\phi_{O(2N)}$ is obtained from eq. (120):

$$\phi_{O(2N)} = d - 2 + \frac{8 (d-1)}{Nd} S_d.$$ \hspace{1cm} (122)

For example, in $d = 3$, we find $\phi_{O(2N)} = 1 + 32/3\pi^2 N$, in agreement with the result of Chubukov et al\cite{28,29}. As seen, $\phi_{O(2N)}$ is not given by dimensional analysis but displays a non trivial correction at order $1/N$ contrarily to what was found in the preceding subsection. This is the result of the important fact that, in the $O(2N)$ regime, the gauge field mass $M^2$ does not renormalize contrarily to what asserted by Campostrini and Rossi\cite{27}.

Finally, it is easy to show that eq. (122) identifies, at order $1/N$ and $\epsilon^2$ with the weak coupling result (eq. (54)):

$$\phi_{O(2N)} = \epsilon + \frac{\epsilon}{N} + \frac{\epsilon^2}{2N}.$$ \hspace{1cm} (123)

Altogether these results show that the double expansion in $1/N$ and $x$ agrees with the weak coupling analysis provided one is sufficiently closed to the $O(2N)$ fixed point.

5 CONCLUSION

In this work we have studied the $SU(N) \otimes U(1)/SU(N-1) \otimes U(1)$ NL$\sigma$ model which is relevant for the low energy physics of frustrated spin systems. We have shown that this model interpolates between two different fixed points which are $O(2N)$ and $CP^{N-1}$.
symmetric. Our main goal, in this paper, was to inquire the reliability of perturbation theory in such a multi-fixed point situation about which little is known at present. To this end, we have performed a rather complete analysis of the RG properties of the model using the weak coupling and $1/N$ expansions at next to leading order.

The main characteristic of frustrated systems is the presence of an additional length scale $\Xi$ that determines the scaling regimes governed by the different fixed points of the theory. While the presence of this scale does not alter the weak coupling expansion, it has strong consequences for the $1/N$ expansion. Indeed, the standard approach of the $1/N$ expansion fails to explore all the phase diagram: the decoupling procedure used to tackle with the $U(1)$ current-current interaction is meaningless as one removes from the basin of attraction of the $CP^{N-1}$ fixed point. We have shown that another strategy has to be used to explore the physics for large values of the gauge field mass $M^2$: around the $O(2N)$ fixed point a double expansion in $1/N$ and $x$ is necessary to obtain the correct renormalization of the parameters of the massive $CP^{N-1}$ model. Doing this, the results obtained from weak coupling and $1/N$ expansions agree in the very neighborhood of the two fixed points of the theory, giving us confidence on perturbation theory in the vicinity of these fixed points. However this procedure also addresses the question of universality for frustrated systems and for models where several antagonistic fixed points coexist. Indeed, the weak coupling expansion of NL$\sigma$ models, as we know from the work of Friedan, is renormalizable. This implies that a priori only two coupling constants $x$ and $T$ are sufficient to determine the low energy behavior when $\zeta^{-1} \ll p \ll \Lambda$, i.e., in the whole range of variation of $x$, $0 \leq x \leq 1$. In this sense we expect the behavior of frustrated systems to be universal. This scheme appears to be more questionable in view of the large $N$ expansion since one may suspect that in the double expansion in $1/N$ and $x$, higher orders corrections in $x$ involve more and more irrelevant operators and coupling constants. If it is the case the existence of a universal field theory describing the whole phase diagram of frustrated systems should be ruled out and only a local idea of universality, bounded to the vicinity of the fixed points, would be meaningful.

A pratical consequence of the previous discussion is that it could be very difficult to observe the $O(2N)$ scaling behavior when dealing with numerical simulations with systems
of size $L \ll \Xi$. Indeed, for sizes of order $L \sim \Xi$ one can expect to observe rather than the $O(2N)$ scaling some effective exponents that result from the competition of $O(2N)$ and $CP^{N-1}$ fluctuations. This might be one reason why, up to now, the $O(4)$ critical behavior has not been observed in the Monte Carlo simulations of frustrated Heisenberg spin systems. A better understanding of finite size scaling in the model is in progress.

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Appendix : Gauge field renormalization.

We give here the asymptotic UV behavior of the diagrams contributing to the renormalization of the transversal part of the inverse gauge field propagator $\Gamma_{\perp}(p)$ in 2 and 3 dimensions. Since there is no renormalization of the longitudinal part of the gauge field propagator we can forget, in the diagrams of Fig.5 involving internal loop of the gauge field, the longitudinal part $S_{i\parallel}$ and keep only their transversal part $S_{i\perp}$. Moreover to determine the renormalization of the propagator, we can restrict to search for the counter-term proportional to $p^2$ since it defines completely the renormalization of the propagator itself.

- In $d = 2$, we have obtained:

\[
S_1 = p^2 \left( \frac{-1}{12\pi m_R^2} \ln \ln \frac{\Lambda^2}{m_R^2} \right)
\]

\[
S_2 = p^2 \left( \frac{1}{24\pi m_R^2} \ln \ln \frac{\Lambda^2}{m_R^2} + \frac{1}{48\pi m_R^2} \text{li} \frac{\Lambda^2}{m_R^2} \right)
\]

\[
S_3 = p^2 \left( \frac{-1}{24\pi m_R^2} \ln \ln \frac{\Lambda^2}{m_R^2} - \frac{1}{48\pi m_R^2} \text{li} \frac{\Lambda^2}{m_R^2} + \frac{1}{24\pi m_R^2} \ln \ln \frac{\Lambda^2}{m_R^2} \right)
\]

$S_{4\perp} = 0$

\[
S_{5\perp} = p^2 \left( \frac{-1}{24\pi m_R^2} \ln \left( 2\pi m^2 - 2 + \ln \frac{\Lambda^2}{m_R^2} \right) \right)
\]

\[
S_{6\perp} = p^2 \left( \frac{1}{24\pi m_R^2} \ln \frac{\Lambda^2}{m_R^2} - \frac{M^2}{12m_R^2} \ln \left( 2\pi M^2 - 2 + \ln \frac{\Lambda^2}{m_R^2} \right) + \frac{1}{12\pi m_R^2} \ln \left( 2\pi M^2 - 2 + \ln \frac{\Lambda^2}{m_R^2} \right) \right)
\]

\[
S_{7\perp} = p^2 \left( \frac{1}{24\pi m_R^2} \ln \left( 2\pi M^2 - 2 + \ln \frac{\Lambda^2}{m_R^2} \right) \right).
\]

(124)
The global contribution to $\Gamma_\perp(p)$ at order $1/N$ thus writes:

$$S_1 + 2S_2 + 2S_3 + S_{4\perp} + 2S_{5\perp} + 2S_{6\perp} + 4S_{7\perp} =$$

$$p^2 \left( \frac{1}{12\pi m_R^2} \ln \frac{\Lambda^2}{m_R^2} + \frac{1}{12\pi m_R^2} \left( 3 - 2\pi M^2 \right) \ln \left( 2\pi M^2 - 2 + \ln \frac{\Lambda^2}{m_R^2} \right) \right)$$

and therefore we have:

$$\Gamma_{1/1/N}(p) = \Gamma_\perp(p) - \frac{p^2}{12\pi m_R^2} \frac{1}{N} \left( \ln \ln \frac{\Lambda^2}{m_R^2} + (3 - 2\pi M^2) \ln \left( 2\pi M^2 - 2 + \ln \frac{\Lambda^2}{m_R^2} \right) \right) .$$

\[126\]

• In $d = 3$, the cut-off dependence of the diagrams writes:

$$S_1 = p^2 \left( -\frac{1}{18\pi^3 m_R^2} \ln \frac{\Lambda^2}{m_R^2} \right)$$

$$S_2 = p^2 \left( \frac{\Lambda}{6\pi^4 m_R^2} + \frac{\Lambda^2}{48\pi^3 m_R^2} + \frac{1}{3\pi^5 m_R} \ln \frac{\Lambda^2}{m_R^2} \right)$$

$$S_3 = p^2 \left( -\frac{\Lambda}{6\pi^4 m_R^2} - \frac{\Lambda^2}{48\pi^3 m_R^2} - \frac{1}{3\pi^5 m_R} \ln \frac{\Lambda^2}{m_R^2} + \frac{1}{12\pi^3 m_R} \ln \frac{\Lambda^2}{m_R^2} \right)$$

$$S_{4\perp} = 0$$

$$S_{5\perp} = p^2 \left( -\frac{2}{3\pi^3 m_R} \ln \left( \frac{\Lambda}{16} + M^2 \Lambda - \frac{m}{2\pi} \right) \right)$$

$$S_{6\perp} = p^2 \left( \frac{\Lambda}{6\pi^2 m_R^2} + \frac{2}{3\pi^3 m_R^2} (m_R - 4\pi AM^2) \ln \left( \frac{\Lambda}{16} + M^2 \Lambda - \frac{m}{2\pi} \right) \right)$$

$$S_{7\perp} = p^2 \left( \frac{4}{9\pi^3 m_R} \ln \left( \frac{\Lambda}{16} + M^2 \Lambda - \frac{m}{2\pi} \right) \right)$$

and the complete contribution to $\Gamma_\perp(p)$ at order $1/N$ is:

$$S_1 + 2S_2 + 2S_3 + S_{4\perp} + 2S_{5\perp} + 2S_{6\perp} + 4S_{7\perp} =$$

$$p^2 \left( \frac{\Lambda}{3m_R^2 \pi^2} + \frac{2}{9m_R \pi^3} \ln \frac{\Lambda^2}{m_R^2} + \frac{1}{9\pi^3 m_R^2} (16m_R - 48\pi AM^2) \ln \left( \frac{\Lambda}{16} + M^2 \Lambda - \frac{m}{2\pi} \right) \right)$$

\[128\]
so that:

$$\Gamma_{\perp 1/N}(p) = \Gamma_{\perp}(p) - \frac{p^2}{48\pi m_R} \frac{1}{N} \left( \frac{32}{3\pi^2} \ln \frac{\Lambda}{m_R} + \frac{16\Lambda}{\pi m_R} + \frac{256}{\pi^2} \left( \frac{1}{3} - \frac{\pi M^2}{m_R} \right) \ln \left( \frac{\Lambda}{16} + M^2 \Lambda - \frac{m}{2\pi} \right) \right).$$

(129)
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Figure Captions

Fig.1: The infrared RG flow in $d = 2$ in the $(x, T)$ plane.

Fig.2: The infrared RG flow in $d = 2 + \epsilon$ in the $(x, T)$ plane.

Fig.3: Feynman rules for the $1/N$ expansion in the $CP^{N-1}$ regime. The solid line denotes spinon propagator while dashed and wavy lines represent respectively the $\lambda$ and $A_\mu$ propagators.

Fig.4: Diagrams contributing to the two point function at order $1/N$ in the $CP^{N-1}$ regime.

Fig.5: Diagrams contributing to the gauge field propagator at order $1/N$ in the $CP^{N-1}$ regime.

Fig.6: The four point vertex occurring in the double expansion in $x$ and $1/N$ in the $O(2N)$ regime.

Fig.7: Diagrams contributing to the two point function at first order in $1/N$ and $x$ in the $O(2N)$ regime.

Fig.8: Diagrams contributing to the four point vertex function at first order in $1/N$ and $x$ in the $O(2N)$ regime.
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
$\Lambda^{2-d} \frac{x^t (k_1 + k_3)_{\mu} (k_2 + k_4)_{\mu}}{2N}$

Fig. 6.
Fig. 7.
