STANDING AND TRAVELLING WAVES IN A PARABOLIC-HYPERBOLIC SYSTEM

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Abstract. We consider a nonlinear system of partial differential equations which describes the dynamics of two types of cell densities with contact inhibition. After a change of variables the system turns out to be parabolic-hyperbolic and admits travelling wave solutions which solve a 3D dynamical system. Compared to the scalar Fisher-KPP equation, the structure of the travelling wave solutions is surprisingly rich and to unravel part of it is the aim of the present paper. In particular, we consider a parameter regime where the minimal wave velocity of the travelling wave solutions is negative. We show that there exists a branch of travelling wave solutions for any nonnegative wave velocity, which is not connected to the travelling wave solution with minimal wave velocity. The travelling wave solutions with nonnegative wave velocity are strictly positive, while the solution with minimal one is segregated in the sense that the product $uv$ vanishes.

1. Introduction. A travelling wave solution is a special type of solution which propagates with a constant velocity and a fixed profile. Travelling wave phenomena are ubiquitous in many fields of science. The propagation of impulse in nerve fibers and the spread of fronts in combustion and chemical reactions are examples of such waves ([13, 14]). Besides, travelling waves often characterise the large...
time behaviour of solutions in PDEs. Therefore, travelling waves have attracted researchers’ attention.

It is well known that the *Fisher-KPP equation* ([10, 12])
\[ u_t = u_{xx} + u(1 - u) \quad x \in \mathbb{R}, \ t > 0 \]
has travelling wave solutions (TWs for short), \( u = U(z) = z = x - ct \), for any wave velocity \( c \geq 2 \); here \( U'' + cU' + U(1 - U) = 0 \) in \(-\infty < z < \infty\) and \( U(-\infty) = 1 \) and \( U(\infty) = 0 \). TWs with different wave velocities have different decay rates at \( \infty \): for larger velocities the decay of \( U(z) \) as \( z \to \infty \) is slower. In addition, it is known that, given an initial function \( u_0(x) \geq 0 (x \in \mathbb{R}) \) which satisfies \( u_0(-\infty) > 0 \) and \( u_0(\infty) = 0 \), the corresponding solution of the evolution problem converges to the TW with minimal velocity as \( t \to \infty \) provided that, for example, \( u_0(x) = 0 \) for sufficiently large \( x \) ([12, 7]). The *degenerate Fisher-KPP equation*
\[ u_t = (uu_x)_x + u(1 - u), \]
exhibits similar phenomena. In this case, the minimal wave velocity is \( 1/\sqrt{2} \), and the TW with minimal velocity vanishes for sufficiently large \( z \) ([6, 11]).

In the present paper, we consider nonnegative solutions of the system
\[
\begin{cases}
u_t = (u(v + v_x))x + u(1 - (u + \alpha v)) & x \in \mathbb{R}, \ t > 0, \\
u_t = (u(v + v_x))x + \gamma v(1 - \frac{u + \alpha v}{k}) & x \in \mathbb{R}, \ t > 0,
\end{cases}
\]
where \( \alpha, k \) and \( \gamma \) are positive constants. If \( v \equiv 0 \), \( u \) satisfies the degenerate Fisher-KPP equation. System (1) describes the growth and the motility of two types of cell densities \( u(x, t) \) and \( v(x, t) \) with contact inhibition ([9, 15, 16]). A feature of system (1) is the mobility term which is not a usual diffusion. This means that each cell density moves towards less populated regions by sensing the population pressure which is described by the gradient of total cell density. In a future paper we shall extensively discuss the biological interpretation of the mathematical results obtained in this and other papers. Here, we concentrate on the mathematical novelties of the results.

Setting \( w = u + v \), the parabolic-hyperbolic character of system (1) becomes clear:
\[
\begin{cases}
u_t = (ww_x)_x + u(1 - \alpha w + (\alpha - 1)u) + \gamma (w - u) \left(1 - \frac{\alpha w - (\alpha - 1)u}{k}\right), \\
u_t = (ww_x)_x + u(1 - \alpha w + (\alpha - 1)u).
\end{cases}
\]
Given nonnegative and bounded initial data \( u_0(x) \) and \( v_0(x) \), the evolution problem for (1) has a solution \( (u(x, t), v(x, t)) \), and if \( u_0 + v_0 = 0 \) a.e. in \( \mathbb{R} \), then \( u(x, t) - v(x, t) = 0 \) for all \( t > 0 \) and a.e. \( x \in \mathbb{R} \) (see [1, 2, 4]). In other words, if \( u \) and \( v \) are initially segregated, they remain segregated for later time. The segregation property reflects the hyperbolic character of the system. Recently, a similar result for the Neumann boundary value problem in 1D was obtained by Carrillo et al([8]).

In view of the segregation property, it is natural to ask whether or not there exist segregated TWs. In [5], it was shown that there exists a unique wave velocity \( \overline{c} \), depending on \( \gamma \) and the quotient \( k/\alpha \), for which system (1) has a unique segregated TW up to translation: for each positive \( \alpha \), \( k \) and \( \gamma \) there exist unique nonnegative functions \( U(z) \) and \( V(z) \) and \( \overline{c} \in \mathbb{R} \) such that \( u(x, t) = U(x - \overline{c}t) \) and \( v(x, t) = V(x - \overline{c}t) \) satisfy (1),
\[ V(-\infty) = \frac{k}{\alpha}, \quad U(\infty) = 1, \quad U + V \text{ is continuous and monotonic}, \]
and
\[
\begin{cases}
U(z) = 0, \ V(z) > 0 & \text{if } z < 0 \\
U(z) > 0, \ V(z) = 0 & \text{if } z > 0.
\end{cases}
\]
Here, we fix the interface at \( z = 0 \). In addition
\[-\sqrt{\frac{1}{2}} < \varepsilon < \sqrt{\frac{k}{2\alpha}}\] (here one can recognise the minimal wave velocity of the degenerate Fisher-KPP equation), and
\( \varepsilon > 0 \) if \( \alpha < k \), \( \varepsilon = 0 \) if \( \alpha = k \) and \( \varepsilon < 0 \) if \( \alpha > k \).

Figure 1 shows how the wave velocity of segregated TWs depends numerically on \( \alpha \):

\[ \begin{array}{c}
\alpha \\
\bar{c}
\end{array} \]

Figure 1. Dependency of the wave velocity of segregated traveling wave solutions on \( \alpha \). The horizontal and the vertical axes are respectively the wave velocity \( \bar{c} \) and the value \( \alpha \). The parameter values are \( \gamma = 1 \) and \( k = 2 \).

it is monotonically decreasing in \( \alpha \) and its sign (i.e. the moving direction) changes at \( \frac{k}{\alpha} = 1 \). We refer to [5] for further details. In Figure 2 we display typical profiles of segregated TWs.

It was shown in [3] that if
\[
\alpha = 1, \ k > 1 \text{ and } \gamma > 0 \quad (\Rightarrow \varepsilon > 0),
\]
for all \( c > \varepsilon \), there exists an overlapping TW: there exist continuous functions \( U(z) \) and \( V(z) \) such that \( u(x,t) = U(x-ct) \) and \( v(x,t) = V(x-ct) \) satisfy (1), \( V(-\infty) = \frac{k}{\alpha}, U(\infty) = 1, U(-\infty) = V(\infty) = 0, U + V \) is decreasing, and
\[
U(z) > 0 \text{ and } V(z) > 0 \quad \text{for all } z \in \mathbb{R}.
\]

Examples of profiles of overlapping TWs are shown in Figure 3. Numerical evidence suggests that the overlapping TWs are unique up to translation and there are no TWs if \( c < \varepsilon \). In numerical simulations, we use a shooting method to find TWs. For each simulation, only one trajectory connecting between two equilibria, which corresponds to a TW, can be numerically obtained. This suggests the uniqueness of TWs up to translation. On the other hand, if we can not capture any trajectory connecting between two equilibria numerically, this implies that there are no TWs. In this way, we obtain numerical evidence about the nonexistence and the uniqueness of TWs in numerical simulations. So, if \( k > \alpha = 1 \), the situation reminds that of the Fisher-KPP equation. The analogy goes even further: numerical evidence suggests that for a large class of (not necessarily segregated) initial data, the corresponding solutions of the evolution problem converge to a segregated TW, i.e. to a TW with
Figure 2. Typical profiles of segregated TWs. The parameter values are $\gamma = 1$ and $k = 2$, the same as the ones in Figure 1. The solid and the dashed curves respectively denote $U$ and $V$. (a) $\alpha = 4/3$ (b) $\alpha = 2$ (c) $\alpha = 4$

Figure 3. Profiles of overlapping TWs. The solid and the dashed curves respectively denote $U$ and $V$ and the gray curve denotes $U + V$. The parameter values are $\alpha = 1$, $k = 2$ and $\gamma = 1$. For this parameter setting, the velocity of segregated TW is $\bar{c} = 0.42$.

Figure 3. Profiles of overlapping TWs. The solid and the dashed curves respectively denote $U$ and $V$ and the gray curve denotes $U + V$. The parameter values are $\alpha = 1$, $k = 2$ and $\gamma = 1$. For this parameter setting, the velocity of segregated TW is $\bar{c} = 0.42$.

minimal wave velocity (more precisely, the large time behaviour of the solution of the evolution problem depends critically on the decay rate of the initial function $v_0(x)$ at $x \to \infty$; see [3]).

A first guess could be that the result for $k > \alpha = 1$ can be generalised to the case in which $k > 1$ and $0 < \alpha < k$, when the wave velocity $\bar{c}$ of the segregated TW is positive. However, as we discuss in a future paper, this is false for two reasons: (1) there is a parameter regime where the wave velocity of the segregated TW is not minimal and the branch of overlapping TWs has a turning point; (2) if $\alpha < 1$, for either sufficiently small or sufficiently large values of $\gamma$, the branch of overlapping TWs does not connect to the segregated TW, but to a partially overlapping TW (i.e. $UV$ vanishes on the halflines $(-\infty, 0)$ or $(0, \infty)$), which is connected to the
segregated TW through a branch of partially overlapping TWs. So, the structure of the TWs is much richer than that of the scalar Fisher-KPP equation.

In the present paper, we consider the case 
\[ \alpha > k > 1 \text{ and } \gamma > 0. \]
Then, the velocity of the segregated TWs is negative: \( \tau < 0 \). We shall prove that there exists a branch of overlapping TWs for all \( c \geq 0 \) (if \( c = 0 \) it is more appropriate to call it an overlapping standing wave). See Figures 2(c) and 4 for profiles of segregated and overlapping TWs in this parameter setting. Moreover as shown in Figure 5 overlapping TWs with non-monotone in \( U + V \) are also exhibited in accordance with the parameter values.

**Theorem 1.1.** Let \( \alpha > k > 1 \) and \( \gamma > 0 \). For each \( c \geq 0 \) the problem
\[
\begin{align*}
(U(U + V))' + cU' + U(1 - (U + \alpha V)) &= 0 \quad \text{in } \mathbb{R} \\
(V(U + V))' + cV' + \gamma V(1 - \frac{U + \alpha V}{k}) &= 0 \quad \text{in } \mathbb{R} \\
U(\infty) &= 1, \quad V(-\infty) = \frac{k}{\alpha}, \quad U(-\infty) = V(\infty) = 0
\end{align*}
\]
has a solution \( (U_c(z), V_c(z)) \), where \( U_c, V_c \in C^2(\mathbb{R}) \) and \( U_c(z) > 0 \) and \( V_c(z) > 0 \) for all \( z \in \mathbb{R} \).
In addition \((U_c(z), V_c(z))\) converges to a standing wave as \(c \to 0\) in the following sense: let \((U_c, V_c)\) satisfy
\[
 w_c(0) = U_c(0) + V_c(0) = \frac{1}{2}(1 + \frac{c}{k}).
\]
Then any sequence \(c_k \to 0\) has a subsequence (denoted again by \(c_k\)), such that:

- \(w_{c_k}\) converges uniformly in \(\mathbb{R}\) to a function \(w_0 \in C^2(\mathbb{R})\);
- \(w_{c_k}'\) converges uniformly in \(\mathbb{R}\) to \(w_0'\), and \(w_0' > 0\) in \(\mathbb{R}\);
- \(U_{c_k}\) converges in \(L^2_{\text{loc}}(\mathbb{R})\) to a function \(U_0 \in BV(\mathbb{R})\);
- \(V_{c_k}\) converges in \(L^2_{\text{loc}}(\mathbb{R})\) to \(V_0 = w_0 - U_0\);
- \((U_0, V_0)\) is a (weak) standing wave.

Condition (3) is a way to eliminate the translation invariance of the TWs. We conjecture that the weak standing wave is unique up to translation, but since we have not proved it, the convergence as \(c \to 0\) is stated up to subsequences.

As we shall see in Section 2 (see Remark 2.4), the standing wave has extremely fast decay properties (one could call it “almost segregated”). The fast decay is also visible in the constructed overlapping TWs: if \(c > 0\), \(V_c\) has exponential decay as \(z \to \infty\), but the exponent explodes as \(c \to 0\) and in the limit one recovers the fast decay of the standing wave (see Section 4). Again the structure is surprisingly different from that of the scalar Fisher-KPP equation: there is a continuum of wave velocity, \([0, \infty)\), but the velocity \(c = 0\) is not minimal, and the segregated TW with negative velocity is not connected to the branch of overlapping TWs defined by Theorem 1.1. Indeed, numerical evidence suggests that there are no other overlapping TWs for \(c < 0\) (this means that we could not numerically capture any trajectory connecting between two equilibria) and that the pair \((U_c, V_c)\) defined by Theorem 1.1 is unique up to translation. However, the segregated TW is not isolated: in a forthcoming paper, we shall show that it can be connected with a new type of partially segregated TWs which possess at least two interfaces and negative wave velocity larger than that of the segregated TW.

We are not yet able to address the stability of the TWs analytically. In a future paper, we present numerical evidence for the stability of the segregated TW. For example, it attracts those solutions of the evolution problem for which \(u, v, \gamma = \frac{1}{\bar{\alpha}}, \tilde{\alpha} = \frac{1}{\tilde{k}}, \tilde{t} = \gamma t, \tilde{x} = -\sqrt{\frac{\alpha}{k}}\) leaves system (1) invariant if we set
\[
 \tilde{u} = \tilde{\alpha} v, \quad \tilde{v} = \frac{\bar{\alpha}}{\bar{k}} u, \quad \tilde{\gamma} = \frac{\gamma}{\gamma}, \quad \tilde{\alpha} = \frac{1}{\gamma}, \quad \tilde{k} = \frac{k}{\gamma}, \quad \tilde{t} = \gamma t, \quad \tilde{x} = -x \sqrt{\frac{\alpha}{k}}.
\]

**Corollary 1.2.** Let \(0 < \alpha < k < 1\) and \(\gamma > 0\). Then problem (2) has a solution \((U_c(z), V_c(z))\) for each \(c \leq 0\), where \(U_c, V_c \in C^2(\mathbb{R})\) and \(U_c(z) > 0\) and \(V_c(z) > 0\) for all \(z \in \mathbb{R}\).

The proof of Theorem 1.1 is given in sections 2 (if \(c = 0\)), 3 (if \(c > 0\)) and 4 (the limit \(c \to 0\)). The proofs in section 3 are based on the phase plane analysis of a system of three ODEs, as might be expected from the parabolic-hyperbolic structure of system (1). This also explains the mathematical relevance of system (1) in the context of travelling waves: it is considerably easier to analyse than coupled systems of two parabolic equations, where one needs to analyse systems of four ODEs. Nevertheless, it leads to completely new phenomena, in particular its structure of TWs is much richer than that of the scalar Fisher-KPP equation.
2. **Standing waves.** In this section we look for overlapping standing waves and prove Theorem 1.1 in the case \( c = 0 \): given \( \alpha > k > 1 \) and \( \gamma > 0 \), we must find a pair of smooth solutions \( U \) and \( V \) of problem (2) if \( c = 0 \). Mathematically it is convenient to set \( w = U + V \) and to introduce the variables

\[
\begin{aligned}
&s = (ww')^2 \\
t &= V/w
\end{aligned}
\]

(observe that \( ww' \) can be interpreted as a flux and \( t \) as a mass fraction). We look for a solution for which \( w' > 0 \) in \( \mathbb{R} \) and use \( w \in [\frac{k}{\alpha}, 1] \) as an independent variable for the functions \( s \) and \( t \). Then \( s(w) \) and \( t(w) \) satisfy, for \( \frac{k}{\alpha} < w < 1 \), the system

\[
\begin{aligned}
s_w &= 2w^2 \left( w - 1 + (1 - \gamma)t + tw \left( \alpha + \frac{\gamma}{k} - 2 - (1 - \frac{\gamma}{k}) (\alpha - 1)t \right) \right) \\
t_w &= -\frac{w^2t(1-t)}{s} \left( \gamma - 1 + w (1 - \frac{\gamma}{k}) (1 + (\alpha - 1)t) \right)
\end{aligned}
\]

(4)

with boundary conditions

\[
s \left( \frac{k}{\alpha} \right) = s(1) = 0, \quad t \left( \frac{k}{\alpha} \right) = 1, \quad t(1) = 0.
\]

First we study the sign of \( \gamma \), setting \( \nu = \frac{w}{\alpha} \)

\[
B^{(\gamma)}(w, t) = w - 1 + (1 - \gamma)t + tw \left( \alpha + \frac{\gamma}{k} - 2 - (1 - \frac{\gamma}{k}) (\alpha - 1)t \right).
\]

Then there exists a smooth and strictly decreasing function \( t_\gamma : [\frac{k}{\alpha}, 1] \to [0, 1] \) such that for all \( w \in [\frac{k}{\alpha}, 1] \)

\[
B^{(\gamma)}(s(w, t)) \left\{ \begin{array}{ll}
> 0 & \text{if } t_\gamma(w) < t \leq 1 \\
< 0 & \text{if } 0 \leq t < t_\gamma(w).
\end{array} \right.
\]

In addition the map \( \gamma \mapsto t_\gamma(w) \) is strictly increasing for all \( w \in [\frac{k}{\alpha}, 1] \).

**Proof.** We have that for all \( (w, t) \in [\frac{k}{\alpha}, 1] \times [0, 1] \)

\[
B^{(\gamma)}(w, t) = 0 \iff w = \zeta_\gamma(t) = \frac{1 + (\gamma - 1)t}{1 + (\alpha - 1)t} \left( 1 - \frac{1}{(1 - \frac{\gamma}{k})t} \right).
\]

Since \( \frac{\partial \zeta_\gamma(t)}{\partial \gamma} = \frac{(k-1)(1-t)}{k(1+(\alpha-1)t)(1-\frac{1}{1-\frac{\gamma}{k}}t)} > 0 \) for \( 0 < t < 1 \), we may define \( \zeta_0(t) := \lim_{\gamma \to 0} \zeta_\gamma(t) \) and \( \zeta_\infty(t) := \lim_{\gamma \to \infty} \zeta_\gamma(t) \) for all \( \gamma > 0 \) for all \( t \in [0, 1] \). Hence

\[
\zeta_0(t) = \frac{1}{1 + (\alpha - 1)t} < \zeta_\gamma(t) < \frac{k}{1 + (\alpha - 1)t} = \zeta_\infty(t) \quad \text{for } 0 < t < 1.
\]

(6)

Observe that \( \zeta_0(t) \to 1 \) as \( t \to 0 \) and \( \zeta_\infty(t) \to \frac{k}{\alpha} \) as \( t \to 1 \), but, on the other hand, \( \zeta_\infty(t) \to k > \zeta_\gamma(0) = 1 \) as \( t \to 0 \) and \( \zeta_0(t) \to \frac{k}{\alpha} < \zeta_\gamma(1) = \frac{k}{\alpha} \) as \( t \to 1 \).

Setting \( \nu_\gamma(t) = 1 + t \left( \alpha + \frac{\gamma}{k} - 2 - (1 - \frac{\gamma}{k}) (\alpha - 1)t \right) \), we have that \( \zeta_\gamma = (1 + (\gamma - 1)t)/\nu_\gamma \), \( \zeta'_\gamma = (1 - 1/\nu_\gamma) \zeta_\gamma \) and

\[
\zeta''_\gamma = -\frac{2\nu'_\gamma}{\nu_\gamma^2} \zeta'_\gamma + 2 \frac{(k-\gamma)(\alpha-1)}{k \nu_\gamma} \zeta_\gamma.
\]

One easily checks that, for all \( t \in (0, 1) \), the coefficient of \( \zeta_\gamma \) in the latter equation is positive if \( k > \gamma \) and negative if \( \gamma < k \). Hence \( \zeta_\gamma(t) \) has no local maxima (minima)
in $(0, 1)$ if $k \geq \gamma$ ($k \leq \gamma$), so $\zeta_\gamma(t)$ can change monotonicity at most one time in $[0, 1]$. Since $\zeta_\gamma(0) = 1$ and $\zeta_\gamma(1) = \frac{k}{\alpha}$, this implies that
\[
\zeta_\gamma'(t) < 0 \quad \text{if } 0 < t < 1 \text{ and } \frac{k}{\alpha} < \zeta_\gamma < 1.
\]

Since
\[
\zeta_\gamma(t) = \frac{k}{\alpha} + (1 - t) \frac{1 - \frac{k}{\alpha} - (k + \frac{\gamma}{\alpha} - \frac{k}{\alpha} - \gamma)t}{1 + t (\alpha + \frac{\gamma}{\alpha} - 2 - (1 - \frac{\gamma}{\alpha}) (\alpha - 1)t)},
\]
we distinguish three cases (see Figure 6):

(i) if $\frac{\alpha(k-1)}{\alpha-1} \leq \gamma \leq \frac{\alpha(k-1)}{\alpha-k}$, then $\frac{k}{\alpha} \leq \zeta_\gamma(t) \leq 1$ for all $0 \leq t \leq 1$; in this case we define $t_\gamma$ as the inverse of $\zeta_\gamma$ in $[0, 1]$;

(ii) if $0 < \gamma < \frac{\alpha(k-1)}{\alpha-1}$, then
\[
\begin{cases} 
\frac{k}{\alpha} \leq \zeta_\gamma(t) \leq 1 & \text{if } 0 \leq t \leq \frac{\alpha-k}{(\alpha-1)(k-\gamma)}, \\
\zeta_\gamma(t) < \frac{k}{\alpha} & \text{if } \frac{\alpha-k}{(\alpha-1)(k-\gamma)} < t < 1,
\end{cases}
\]
and $t_\gamma$ is the inverse of $\zeta_\gamma$ restricted to the interval $\left[0, \frac{\alpha-k}{(\alpha-1)(k-\gamma)} \right]$;

(iii) if $\gamma > \frac{\alpha(k-1)}{\alpha-1}$, then
\[
\begin{cases} 
\frac{k}{\alpha} \leq \zeta_\gamma(t) \leq 1 & \text{if } \frac{\gamma(k-1)-(\alpha-1)k}{(\gamma-k)(\alpha-1)} \leq t \leq 1, \\
\zeta_\gamma(t) > 1 & \text{if } 0 < t < \frac{k(k-1)-(\alpha-1)k}{(\gamma-k)(\alpha-1)},
\end{cases}
\]
and $t_\gamma$ is the inverse of $\zeta_\gamma$ restricted to the interval $\left[\frac{\gamma(k-1)-(\alpha-1)k}{(\gamma-k)(\alpha-1)}, 1 \right]$. One easily checks that $t_\gamma$ has all properties claimed in Lemma 2.1.

**Lemma 2.2.** Let $\alpha > k > 1$, $\gamma > 0$ and
\[
B^{(t)}(w, t) = 1 - \gamma - w (1 + (\alpha - 1)t).
\]

(i) If $\frac{\alpha-k}{\alpha-1} \leq \gamma \leq \frac{\alpha(k-1)}{\alpha-k}$, then $B^{(t)} < 0$ in $\Omega := \left[\frac{k}{\alpha}, 1 \right] \times (0, 1)$.

(ii) If $\gamma > \frac{\alpha(k-1)}{\alpha-k}$ ($> k$), then
\[
B^{(t)} > 0 \quad \text{in } \Omega_1 = \left\{(w, t) : \frac{\gamma(k-1)-(\alpha-1)k}{(\gamma-k)(\alpha-1)} < t < 1, \frac{k(k-1)-(\alpha-1)k}{(\gamma-k)(1+(\alpha-1)t)} < w \leq 1 \right\}
\]
and
\[
B^{(t)} < 0 \quad \text{in } \Omega \setminus \Omega_1, \quad B^{(s)} \geq \delta > 0 \quad \text{in } \overline{\Omega_1} \quad (7)
\]
for some $\delta = \delta(\gamma, \alpha, k)$.

(iii) If $0 < \gamma < \frac{\alpha-k}{\alpha-1} < 1$, then
\[
B^{(t)} > 0 \quad \text{in } \Omega_2 = \left\{(w, t) : 0 < t < \frac{\alpha-k-(\alpha-1)\gamma}{(k-\gamma)(\alpha-1)}, \frac{k}{\alpha} \leq w < \frac{k(1-\gamma)}{(k-\gamma)(1+(\alpha-1)t)} \right\}
\]
and
\[
B^{(t)} < 0 \quad \text{in } \Omega \setminus \Omega_2, \quad B^{(s)} \leq -\delta < 0 \quad \text{in } \overline{\Omega_2} \quad (8)
\]
for some $\delta = \delta(\gamma, \alpha, k)$.

See also Figure 7 for parts (ii) and (iii) of Lemma 2.2.
Proof. Let $\zeta_\gamma$ be defined as in the proof of Lemma 2.1. By (6),

$$\zeta_\gamma(t) < \frac{k}{1 + (\alpha - 1)t} < \frac{k(\gamma - 1)}{(\gamma - k)(1 + (\alpha - 1)t)} \text{ if } \gamma > \frac{k(\alpha - 1)}{\alpha - k},$$

which implies (7), and

$$\zeta_\gamma(t) > \frac{1}{1 + (\alpha - 1)t} > \frac{k(1 - \gamma)}{(k - \gamma)(1 + (\alpha - 1)t)} \text{ if } \gamma < \frac{\alpha - k}{\alpha - 1},$$

which implies (8). The remainder of the proof is immediate.

To complete the proof of the existence of overlapping standing waves it is enough to prove the following result.

**Proposition 2.3.** Let $\alpha > k > 1$. Then system (4)-(5) has a solution for all $\gamma > 0$. 

Figure 6. Profiles of $t_\gamma$ in cases (i), (ii) and (iii) of Lemma 2.1.
Proof. Let \( \varepsilon > 0 \) and \( \tau \in [0, 1] \), and consider the problem

\[
\begin{align*}
    s_w &= 2w^2 (w - 1 + (1 - \gamma)t) + tw (\alpha + \frac{\gamma}{k} - 2 - (1 - \frac{\gamma}{k}) (\alpha - 1)t) \\
    t_w &= -\frac{w^2 t (1 - t)}{(\gamma - 1 + w (1 - \frac{\gamma}{k}) (1 + (\alpha - 1)t))} \\
    s \left( \frac{k}{\alpha} \right) &= \varepsilon, \ t \left( \frac{k}{\alpha} \right) = \tau.
\end{align*}
\]

The solution \((s_{\varepsilon, \tau}, t_{\varepsilon, \tau})\) is defined in a maximum interval of existence \([\frac{k}{\alpha}, w_{\varepsilon, \tau}] \) \(\subset [0, 1]\) and, since \(0 \leq t_{\varepsilon, \tau} \leq 1\) in \([\frac{k}{\alpha}, w_{\varepsilon, \tau}]\), it can be continued as long as \(s_{\varepsilon, \tau} > 0\). Hence we may assume that the interval is maximal: \(s_{\varepsilon, \tau} > 0\) in \([\frac{k}{\alpha}, w_{\varepsilon, \tau}]\) and, if \(w_{\varepsilon, \tau} < 1\), \(s_{\varepsilon, \tau}(w) \to 0\) as \(w \to w_{\varepsilon, \tau}^{-}\). (Here \(\eta \to \xi^{-}\) and \(\eta \to \xi^{+}\) respectively mean that \(\eta\) approaches \(\xi\) from below and that \(\eta\) approaches \(\xi\) from above.) Since \((s_{\varepsilon, \tau})_w\) is uniformly Lipschitz continuous, we may set \(s_{\varepsilon, \tau}(w_{\varepsilon, \tau}) := \lim_{w \to w_{\varepsilon, \tau}^{-}} s_{\varepsilon, \tau}(w)\).

Let \(E_{\varepsilon}^{+} = \{\tau \in [0, 1] : w_{\varepsilon, \tau} = 1, s_{\varepsilon, \tau}(1) > 0\}\), \(E_{\varepsilon}^{-} = \{\tau \in [0, 1] : w_{\varepsilon, \tau} < 1\}\). If \(\tau = 0\), \(t_{\varepsilon, 0}(w) = 0\) in \([\frac{k}{\alpha}, w_{\varepsilon, 0}]\), so \((s_{\varepsilon, 0})_w = -2w^2 (1 - w) < 0\) in \([\frac{k}{\alpha}, w_{\varepsilon, 0}]\) and a simple integration shows that

\[
0 \in E_{\varepsilon}^{-} \quad \text{if} \quad \varepsilon > 0 \quad \text{is sufficiently small}. \tag{10}
\]

Similarly, if \(\tau = 1\), then \(t_{\varepsilon, 1}(w) = 1\) and \((s_{\varepsilon, 1})_w = \frac{2\alpha \gamma}{k} w^2 (w - \frac{k}{\alpha}) > 0\) in \([\frac{k}{\alpha}, w_{\varepsilon, 1}]\), so

\[
1 \in E_{\varepsilon}^{+} \quad \text{for all} \quad \varepsilon > 0. \tag{11}
\]

Let \(\tau \in E_{\varepsilon}^{-}\). Then \(w_{\varepsilon, \tau} < 1\) and \(s_{\varepsilon, \tau}(w_{\varepsilon, \tau}) = 0\). By Lemma 2.2, \(t_{\varepsilon, \tau}\) is decreasing in a left neighbourhood of \(w_{\varepsilon, \tau}\). Since \(s_{\varepsilon, \tau}\) is uniformly Lipschitz continuous, \(1/s_{\varepsilon, \tau}\) is not integrable near \(w_{\varepsilon, \tau}\), and it follows from the equation for \(t_{\varepsilon, \tau}\) that \(t_{\varepsilon, \tau}(w) \to 0\) as \(w \to w_{\varepsilon, \tau}^{-}\). By Lemma 2.1, this implies that \((s_{\varepsilon, \tau})_w(w_{\varepsilon, \tau}) < 0\), so we have proved that

\[
\text{if} \ \tau \in E_{\varepsilon}^{-}, \ \text{then} \ (s_{\varepsilon, \tau})_w(w) \neq 0 \quad \text{as} \quad w \to w_{\varepsilon, \tau}^{-}. \tag{12}
\]

Since the solution \((s_{\varepsilon, \tau}, t_{\varepsilon, \tau})\) continuously depends on the parameter \(\tau\) by general theory of ODEs, \(w_{\varepsilon, \tau}\) also depends continuously on \(\tau\). It easily follows from (12) that the sets \(E_{\varepsilon}^{+}\) and \(E_{\varepsilon}^{-}\) are open in the relative topology of \([0, 1]\). Since \(E_{\varepsilon}^{-} \cap E_{\varepsilon}^{+} = \emptyset\),

![Figure 7](image-url)
it follows from (10) and (11) that, for $\varepsilon$ small enough, there exists $\tau_\varepsilon \in (0, 1)$ which does not belong to $E^+_\varepsilon \cup E^-_\varepsilon$.

For sufficiently small $\varepsilon$, we denote the solution $(s_{\varepsilon, \tau_\varepsilon}, t_{\varepsilon, \tau_\varepsilon})$ by $(s_\varepsilon, t_\varepsilon)$. Since $\tau_\varepsilon \notin E^-_\varepsilon$, we have $s_{\varepsilon, \tau_\varepsilon} = 1$, and, since $\tau_\varepsilon \notin E^+_\varepsilon$, $s_\varepsilon(1) = 0$.

We claim that there exists $w_\varepsilon \in \left[\frac{k}{\alpha}, 1\right)$ such that

$$t_\varepsilon \text{ is decreasing in } (w_\varepsilon, 1) \quad \text{and} \quad w_\varepsilon \to \frac{k}{\alpha} \text{ as } \varepsilon \to 0.$$  \hspace{1cm} (13)

If $\frac{\varepsilon - k}{\alpha - k} \leq \gamma \leq \frac{k(\alpha - 1)}{\alpha - k}$, this follows from Lemma 2.2(ii) (with $w_\varepsilon = \frac{k}{\alpha}$). If $\gamma > \frac{k(\alpha - 1)}{\alpha - k}$, we use Lemma 2.2(ii): $t_{\varepsilon w}(w) < 0$ for $w$ near $\frac{k}{\alpha}$; if the orbit $(w, t_\varepsilon(w))$ enters the set $\Omega_1$ for a certain value $w < 1$, it remains in $\Omega_1$ and, by (7), $s_\varepsilon(1) > 0$. Hence we have found a contradiction and (13) also holds if $\gamma > \frac{k(\alpha - 1)}{\alpha - k}$ (with $w_\varepsilon = \frac{k}{\alpha}$).

Finally, if $0 < \gamma < \frac{\varepsilon - k}{\alpha - k}$, we use Lemma 2.2(iii). Then $t_{\varepsilon w}(w) < 0$ for $w$ near 1 and we can follow the orbit $(w, t_\varepsilon(w))$ backwards, i.e. for decreasing values of $w$: if it enters $\Omega_2$ for a certain $w_\varepsilon > \frac{k}{\alpha}$, it remains in $\Omega_2$ for all $\frac{k}{\alpha} < w < w_\varepsilon$ and, by (8), $\varepsilon = s_\varepsilon(1) > \delta(w_\varepsilon - \frac{k}{\alpha})$. Hence $w_\varepsilon < \frac{k}{\alpha} + \frac{\varepsilon}{k} \to \frac{k}{\alpha}$ as $\varepsilon \to 0$, and we have proved (13).

By (13), the total variation of $t_\varepsilon$ in $(\frac{k}{\alpha}, 1]$ is uniformly bounded. By Helly’s selection theorem, and since $s_\varepsilon$ is uniformly Lipschitz continuous, there exist a sequence $\varepsilon_n \to 0$, a nonincreasing function $t \in L^\infty\left(\left[\frac{k}{\alpha}, 1\right]\right)$ and a Lipschitz continuous function $s$ such that, as $n \to \infty$,

$$s_{\varepsilon_n} \to s \text{ in } C\left(\left[\frac{k}{\alpha}, 1\right]\right), \quad t_{\varepsilon_n} \to t \text{ in } L^1\left(\left(\frac{k}{\alpha}, 1\right)\right) \text{ and a.e. in } \left(\frac{k}{\alpha}, 1\right),$$

and

$$(t_{\varepsilon_n})_w \to t_w \text{ weakly*} \quad \text{in the sense of bounded measures on } \left(\frac{k}{\alpha}, 1\right).$$

In particular $s\left(\frac{k}{\alpha}\right) = s(1) = 0$, and $(s, t)$ satisfies the differential equations in (9) in the open set where $s > 0$. Observe that the latter set is non-empty: if $s = 0$ in $\left(\frac{k}{\alpha}, 1\right)$, then $s_{\varepsilon w} = s_w = 0$ for a.e. $w \in \left(\frac{k}{\alpha}, 1\right)$, but by Lemma 2.2 this is impossible.

Let $(a, b) \subseteq \left(\frac{k}{\alpha}, 1\right)$ be an open interval in the set where $s > 0$ such that $s(a) = s(b) = 0$. Since $s$ is Lipschitz continuous in $(a, b)$, $1/s$ is not integrable near $a$ and $b$, and it follows from the equation for $t$ that

$$(w, t) = (a^+, t(a^+)) \quad \text{and} \quad (b^-, t(b^-)) \quad \text{both satisfy} \quad t(1 - t)B^t(w, t) = 0.$$  \hspace{1cm} (14)

In addition $s_w(a^+) \geq 0$ and $s_w(b^-) \leq 0$, i.e.

$$B^s(a^+, t(a^+)) \geq 0 \quad \text{and} \quad B^s(b^-, t(b^-)) \leq 0.$$  \hspace{1cm} (15)

If $a > \frac{k}{\alpha}$, it follows from (14), (15), Lemma 2.2 and the fact that $t(w)$ is nonincreasing, that $s_w > 0$ a.e. in some neighborhood $(a - \delta_1, a)$, and since $s(a) = 0$ this yields a contradiction. Hence $a = \frac{k}{\alpha}$ and, by (14) and Lemma 2.2,

$$t\left(\left(\frac{k}{\alpha}\right)^+\right) = 1.$$  

Reasoning similarly we prove that if $b < 1$, $s_w < 0$ a.e. in some neighborhood $(b, b + \delta_1)$, and since $s(b) = 0$ this yields a contradiction. Hence $b = 1$ and $t(1^-) = 0$.

We have proved that $s > 0 \text{ in } (0, 1), \quad t\left(\left(\frac{k}{\alpha}\right)^+\right) = 1 \quad \text{and} \quad t(1) = 0.$

One easily proves that $0 < t < 1 \quad \text{in } (0, 1)$, which completes the proof. \hfill $\square$
Remark 2.4. The decay of $V(z)$ and $U(z)$ is extremely fast: there exist positive constants $C_0$ and $C_1$ such that

$$\log(V(z)) \approx -C_0 e^{-z} \quad \text{as } z \to \infty$$

$$\log(U(z)) \approx -C_1 e^{-\sqrt{\frac{2}{\gamma} z}} \quad \text{as } z \to -\infty.$$  \hspace{1cm} (16)

We proceed formally. Approximately, as $z \to \infty$ and $w \to 1$,

$$t w \approx -\gamma \left(1 - \frac{1}{k}\right) \frac{1}{s(w)}.$$  \hspace{1cm} (17)

Since $s_w \to 0$, $(1 - w) (\log t)_w \to -\infty$. Hence $V \approx t \to 0$ faster than any power of $1 - w$ and

$$s_w \approx -2(1 - w) \Rightarrow s \approx (w - 1)^2.$$  \hspace{1cm} (18)

Then

$$t w \approx -\gamma \left(1 - \frac{1}{k}\right) \frac{1}{(w - 1)^2} \Rightarrow \log(t(w)) \approx -\gamma \left(1 - \frac{1}{k}\right) \frac{1}{1 - w}.$$  \hspace{1cm} (19)

Since $w'' \approx w - 1$ and $w(z) \approx 1 - C e^{-z}$ as $z \to \infty$, this implies (16).

Similarly, as $z \to -\infty$ and $w \to k\alpha$,

$$t w \approx -\frac{k^2(k - 1)}{\alpha^2 s}.$$  \hspace{1cm} (20)

Since $s_w \to 0$, $s = o \left(\frac{w - k\alpha}{\alpha}\right)$ and $(w - k\alpha) (\log(1 - t))_w \to +\infty$. Hence $U \approx 1 - t \to 0$ faster than any power of $w - k\alpha$ and

$$s_w \approx \frac{2k\gamma}{\alpha} (w - k\alpha) \Rightarrow s \approx \frac{k\gamma}{\alpha} (w - k\alpha)^2.$$  \hspace{1cm} (21)

Then

$$t w \approx -\frac{k(k - 1)}{\alpha\gamma (w - k\alpha)^2} \Rightarrow \log(t(w)) \approx -\frac{k(k - 1)}{\alpha\gamma (w - k\alpha)}.$$  \hspace{1cm} (22)

Since $w'' \approx 2\alpha \left(\frac{k\alpha - w}{\alpha}\right)$ and $w(z) \approx \frac{k}{\alpha} + C e^{\sqrt{\frac{2}{\gamma} z}}$ as $z \to -\infty$, (17) follows.

3. Travelling waves: $c > 0$. In this section we prove that, given $\alpha > k > 1$ and $\gamma > 0$, for any velocity $c > 0$ there exists a one-parameter family of overlapping travelling waves, i.e. strictly positive and smooth solutions $(U(z), V(z))$ of problem (2).

The main ingredients of the proof are the following:

(i) The introduction of new variables, chosen for mathematical convenience:

$$w = U + V, \quad R = w \left(\frac{w'}{c} + 1\right), \quad r = \frac{U}{w},$$

so $U = rw$ and $V = (1 - r)w$. Observe that $cR$ can be interpreted as a flux, $r$ as a mass fraction, and $R$ satisfies

$$(rR)' = -\frac{1}{c}U[1 - (U + \alpha V)] \quad \text{for } z \in \mathbb{R}.$$
For overlapping TWs the functions $R$, $r$ and $w$ are smooth and satisfy $0 < r < 1$, $w > 0$, $R > 0$ ($\Leftrightarrow w' > -c$) and

$$
\begin{cases}
R' = \frac{w}{c} \left( \gamma \frac{\alpha}{k} w - 1 \right) + \left( \gamma - 1 + \frac{2\alpha\gamma}{k} + \frac{\gamma}{k} \right) w r \\
r' = \frac{wr}{cR} \left( \gamma - 1 - \frac{\gamma}{k} - 1 + (\alpha - 1)(1 - r) \right) w \\
w' = \frac{c}{w}(R - w) \\
R(\infty) = r(\infty) = w(\infty) = 1 \\
R(-\infty) = w(-\infty) = \frac{k}{\alpha}, \ r(-\infty) = 0.
\end{cases}
$$

(iii) The linearisation of problem (I) at $z = \pm \infty$: there exists a 2-dimensional stable (unstable) manifold at $\infty$ ($-\infty$). Due to translation invariance, the corresponding 2-parameter families of local solutions are reduced to 1-parameter families.

A similar method was used in [3] in the case that $\alpha = 1$ (with a different choice of variables), but if $\alpha > k > 1$ the analysis is much more complex. For example, in [3] we used $w$ as an independent variable since $w(z)$ is monotonic if $\alpha = 1$. As we shall see below, if $\alpha > k > 1$ there are parameter values for which $w(z)$ is not monotonic.

This section is organised as follows. In subsections 3.1 and 3.2 we discuss the linearisation at $z = \pm \infty$, and in 3.3 we collect some auxiliary results. In the remaining subsections we use a shooting method to solve problem (I) in different parameter regimes. The distinction in different regimes is dictated by the linearisation problems, and often leads to different qualitative properties of the TWs.

3.1. Linearisation around $(1, 1, 1)$. We linearise the equations around $(R, r, w) = (1, 1, 1)$:

$$
\begin{cases}
R' = \frac{1}{2}(w - 1) + \frac{1}{c} \left( \frac{\gamma(k - 1)}{k} - \alpha + 1 \right)(r - 1) \\
r' = \frac{\gamma(1-k)}{kc}(r - 1) \\
w' = c(R - 1) - c(w - 1) \\
R(\infty) = r(\infty) = w(\infty) = 1.
\end{cases}
$$

The eigenvalues are

$$
\lambda_1 = -\frac{\gamma(k-1)}{kc} < 0, \ \lambda_2 = -\frac{\gamma}{2} - \frac{1}{2}\sqrt{c^2 + 4} < 0, \ \lambda_3 = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{c^2 + 4} > 0,
$$

and if $\lambda_1 \neq \lambda_2$ and $\gamma(k - 1) \neq k(\alpha - 1)$, the corresponding eigenvectors are

$$
\begin{align*}
v_1 &= \left( 1 + \frac{\lambda_1}{c}, \frac{\gamma(k-1)(\lambda_1^2 + c\lambda_1 - 1)}{c\lambda_1(\gamma(k-1) - k(\alpha - 1))}, 1 \right) = \left( 1 - \frac{\gamma(k-1)}{kc^2}, \frac{\gamma^2(k-1)^2 - k^2c^2 - \gamma k(k-1)c^2}{kc^2(\gamma(k-1) - k(\alpha - 1))}, 1 \right) \\
v_2 &= (1 + \frac{\lambda_2}{c}, 0, 1) = \left( \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{c}{\alpha}}, 0, 1 \right) \\
v_3 &= (1 + \frac{\lambda_3}{c}, 0, 1) = \left( \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{c}{\alpha}}, 0, 1 \right)
\end{align*}
$$

(if $\gamma(k - 1) = k(\alpha - 1)$, then $v_1 = (0, \gamma^2(k-1)^2 - k^2c^2 - \gamma k(k-1)c^2, 0)$). The first components of $v_1$ and $v_2$ are smaller than the third components,

$$(v_1)_1 < (v_1)_3 = 1 \quad \text{and} \quad (v_2)_1 < (v_2)_3 = 1,$$
and
\[(v_1)_1 > (v_2)_1 \iff \lambda_1 > \lambda_2.\]

Observe that
\[\lambda_1^2 + c\lambda_1 - 1 > 0 \iff \lambda_1 < \lambda_2 \iff \gamma > \frac{k\gamma^2}{2c(k-1)} \left(1 + \sqrt{1 + \frac{4}{c^2}}\right), \quad (18)\]
since
\[\lambda_1^2 + c\lambda_1 - 1 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) = -\lambda_1 - \lambda_2 \left(\frac{\gamma(k-1)}{kc} + \frac{c}{2} \left(\sqrt{1 + \frac{4}{c^2}} - 1\right)\right). \quad (19)\]

The sign of the second component of \(v_1\) is determined by
\[(v_1)_2 > 0 \iff (\lambda_1 - \lambda_2)(\gamma(k-1) - k(\alpha - 1)) < 0.\]

For each \(k > 1\) and \(\gamma > 0\), there exists a unique \(c_0 > 0\) such that \(\lambda_1 < \lambda_2\) if \(0 < c < c_0\), \(\lambda_1 > \lambda_2\) if \(c > c_0\) and \(\lambda_1 = \lambda_2\) if \(c = c_0\).

Since \(\lambda_1, \lambda_2 < 0\), there exists a local 2-dimensional stable manifold \(M_s\) for the nonlinear system around \((R, r, w) = (1, 1, 1)\), which, if \(\lambda_1 \neq \lambda_2\), is tangent to the plane orthogonal to
\[v_1 \times v_2 = \left(-\frac{\gamma(k-1)(\lambda_1^2 + c\lambda_1 - 1)}{c\lambda_1(\gamma(k-1) - k(\alpha - 1))}, \frac{-\lambda_1 - \lambda_2}{c}, \frac{\gamma(k-1)(\lambda_1^2 + c\lambda_1 - 1)(\lambda_2 - c)}{c\lambda_1(\gamma(k-1) - k(\alpha - 1))}\right).\]

By the translation invariance with respect to \(z\), the local stable manifold \(M_s\) consists of a one-parameter family of invariant \(C^1\)-curves.

\(M_s\) is transversal to the invariant set \(\{(R, r, w); R > 0, r = 1, w > 0\}\). Therefore there exists a (local) solution such that the corresponding curve \(C_0\) lies on \(M_s\) and is tangent to \(v_2\). This solution is determined by the equation
\[\frac{dR}{dw} = -\frac{w^2(w - 1)}{c^2(w - R)} \quad (20)\]

If \(\lambda_1 \neq \lambda_2\), it follows from the fast stable manifold theorem (see the Appendix) that each invariant curve in \(M_s\) is tangent to \(v_1\) or \(v_2\). More precisely, exactly two incoming orbits are tangential to the eigenvector which corresponds to the most negative eigenvalue. Distinguishing the cases \(k(\alpha - 1) - \gamma(k - 1) > 0(< 0)\) and \(\lambda_1 > \lambda_2(\lambda_2)\), this leads to 4 different pictures for the orbits near \((1, 1, 1)\). See Figures 8, 9, 10 and 11.

3.2. Linearisation around \((\frac{k}{\alpha}, 0, \frac{k}{\alpha})\). We linearise the equations around \((\frac{k}{\alpha}, 0, \frac{k}{\alpha})\):

\[
\begin{cases}
R' = \frac{k}{\alpha} (k - 1 - \gamma + \frac{2}{c}) r + \frac{2}{c} (w - \frac{k}{\alpha}) \\
r' = \frac{k - 1}{c} r \\
w' = \frac{\alpha c}{R} (R - \frac{k}{\alpha}) - \frac{\alpha c}{\alpha} (w - \frac{k}{\alpha}) \\
R(-\infty) = w(-\infty) = \frac{k}{\alpha}, \quad r(-\infty) = 0.
\end{cases}
\]

The eigenvalues are
\[\Lambda_1 = \frac{k - 1}{c} > 0, \quad \Lambda_2 = \frac{\alpha c}{2R} \left(1 + \frac{4k\gamma}{\alpha c^2} - 1\right) > 0, \quad \Lambda_3 = -\frac{\alpha c}{2R} \left(1 + \frac{4k\gamma}{\alpha c^2} + 1\right) < 0\]
and, if \( \alpha(k - 1) \neq \gamma(\alpha - 1) \) and \( \Lambda_1 \neq \Lambda_2 \), the corresponding eigenvectors are

\[
V_1 = \left(1 + \frac{k(k-1)}{\alpha c^2}, \frac{\alpha c^2 (k-1 - \gamma) + k(k-1)^2}{k c^2 (\alpha(k-1) - \gamma(\alpha-1))}, 1\right)
\]

\[
= \left(1 + \frac{k}{\alpha c} \Lambda_1, \frac{\Lambda_1^2 + \frac{\alpha c^2}{k} \Lambda_1 - \frac{\alpha}{k}}{\alpha(k-1) - \gamma(\alpha-1)}, 1\right)
\]

\[
V_2 = \left(\frac{1}{2} \left(1 + \sqrt{1 + \frac{4k\gamma}{\alpha c^2}}\right), 0, 1\right) = \left(1 + \frac{k}{\alpha c} \Lambda_2, 0, 1\right)
\]

\[
V_3 = \left(\frac{1}{2} \left(1 - \sqrt{1 + \frac{4k\gamma}{\alpha c^2}}\right), 0, 1\right) = \left(1 + \frac{k}{\alpha c} \Lambda_3, 0, 1\right)
\]

(observe that \( V_1 = (0, \alpha c^2 (k-1 - \gamma) + k(k-1)^2, 0) = (0, k(k-1)^2 - \gamma c^2, 0) \) if \( \alpha(k-1) = \gamma(\alpha-1) \)). The first components of \( V_1 \) and \( V_2 \) are greater than the third
components,

$$(V_1)_1 > (V_1)_3 = 1 \quad \text{and} \quad (V_2)_1 > (V_2)_3 = 1,$$

and

$$(V_1)_1 > (V_2)_1 \iff \Lambda_1 > \Lambda_2.$$  

Since

$$\Lambda_1^2 + \frac{ac}{k} \Lambda_1 - \frac{\gamma \alpha}{k} = (\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_3)$$

$$= (\Lambda_1 - \Lambda_2) \left( \frac{k-1}{c} + \frac{ac}{2k} + \frac{ac}{2k} \sqrt{1 + \frac{4\kappa^2}{ac^2}} \right),$$  

we have that

$$\Lambda_1^2 + \frac{ac}{k} \Lambda_1 - \frac{\gamma \alpha}{k} < 0 \iff 0 < \Lambda_1 < \Lambda_2 \iff \gamma > \frac{k(k-1)^2}{\alpha c^2} + k - 1,$$  

FIGURE 10. The case $\lambda_1 < \lambda_2$ and $0 < \gamma < k(\alpha - 1)/(k - 1)$:

$(\gamma, c) = (2, 1) \ (k = 4 \text{ and } \alpha = 10)$

FIGURE 11. The case $\lambda_1 > \lambda_2$ and $0 < \gamma < k(\alpha - 1)/(k - 1)$:

$(\gamma, c) = (3, 1) \ (k = 4 \text{ and } \alpha = 10)$
and the sign of the second component of \( V_1 \) is determined by
\[
(V_1)_2 > 0 \Leftrightarrow (\Lambda_1 - \Lambda_2)(\alpha(k - 1) - \gamma(\alpha - 1)) > 0.
\]

For each \( k > 1 \) and \( \gamma > 0 \), there exists a unique \( c_1 > 0 \) such that \( \Lambda_1 < \Lambda_2 \) if \( 0 < c < c_1 \), \( \Lambda_1 > \Lambda_2 \) if \( c > c_1 \), and \( \Lambda_1 = \Lambda_2 \) if \( c = c_1 \).

Since \( \Lambda_1, \Lambda_2 > 0 \), there exists a local 2-dimensional unstable manifold \( \mathcal{M}_u \) for the nonlinear problem around \( (R, r, w) = (\frac{k}{\alpha}, 0, \frac{k}{\alpha}) \), which, if \( \Lambda_1 \neq \Lambda_2 \) and \( \alpha(k - 1) \neq \gamma(\alpha - 1) \), is tangent to the plane orthogonal to
\[
V_1 \times V_2 = \left( \frac{\alpha(\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k})}{\alpha(k - 1) - \gamma(\alpha - 1)}, \frac{k}{\alpha c} (\Lambda_2 - \Lambda_1), -\frac{\alpha(\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k})}{\alpha(k - 1) - \gamma(\alpha - 1)} \right).
\]

By the translation invariance with respect to \( z \), the local unstable manifold \( \mathcal{M}_u \) consists of a one-parameter family of invariant \( C^1 \)-curves.

Since \( \mathcal{M}_u \) is transversal to the invariant set \( \{(R, r, w); R > 0, r = 0, w > 0\} \), there exist 2 (local) solutions which form a curve \( \mathcal{C}_1 \) which lies on \( \mathcal{M}_u \) and is tangent to \( V_2 \). This solution is determined by the equation
\[
\frac{dR}{dw} = \frac{\alpha \gamma w^2 (w - \frac{k}{\alpha})}{c^2 k (R - w)}.
\]

If \( \Lambda_1 \neq \Lambda_2 \), it follows from the fast stable manifold theorem (see the Appendix) that each invariant curve in \( \mathcal{M}_u \) is tangent to \( V_1 \) or \( V_2 \). More precisely, exactly two outcoming orbits are tangential to the eigenvector which corresponds to the largest eigenvalue. Distinguishing the cases \( \alpha(k - 1) - \gamma(\alpha - 1) > 0(<0) \) and \( \Lambda_1 > \Lambda_2(< \Lambda_2) \), this leads to 4 different pictures for the orbits near \( (\frac{k}{\alpha}, 0, \frac{k}{\alpha}) \). See Figures 12, 13, 14 and 15.

Finally we observe that \( \mathcal{M}_u \) is transversal to the plane \( \{R = w\} \). On the intersection line, with direction
\[
(1, 0, -1) \times (V_1 \times V_2) = \left( \frac{k}{\alpha c} (\Lambda_2 - \Lambda_1), \frac{k}{\alpha c} (\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k}) \Lambda_2, \frac{k}{\alpha c} (\Lambda_2 - \Lambda_1) \right),
\]
we have that \( w - \frac{k}{\alpha} = \frac{\alpha k(\Lambda_2 - \Lambda_1)(\alpha(k - 1) - \gamma(\alpha - 1))}{\alpha(\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k}) \Lambda_2} r + o(r) \) as \( r \to 0 \) and so, by (21),
\[
w - \frac{k}{\alpha} = \frac{\alpha(k - 1) - \gamma(\alpha - 1)}{\alpha(\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k})} r + o(r).
\]

By direct calculation, \( \frac{\gamma}{\alpha(\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k})} + \frac{k}{\alpha} = \frac{k(k - 1)}{\alpha c(\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k})} \)
and hence
\[
R' = \frac{\alpha}{\alpha c} (w - \frac{k}{\alpha}) + \frac{k}{\alpha c} (k - 1 - \gamma + \frac{2}{\alpha}) r + o(r)
\]
\[
= \frac{k(k - 1)}{\alpha c(\Lambda_1^2 + \frac{\alpha c}{\kappa} \Lambda_1 - \frac{\gamma}{k})} (\alpha(k - 1) - \gamma(\alpha - 1)) r + o(r) \quad \text{as } r \to 0.
\]

3.3. Preliminary remarks. We rewrite the equations for \( r \) and \( R \) of problem (1) as
\[
r' = \frac{w r(1 - r)}{c R} f(w, r) \quad \text{and} \quad R' = \frac{w}{k c} h(w, r),
\]
where
\[
f(w, r) := \gamma - 1 - \frac{k}{\alpha}(\gamma - k)(\alpha - (\alpha - 1)r)w
\]
and
\[
h(w, r) := \gamma(\alpha w - k) + (\gamma k - k + (\alpha k - 2\alpha \gamma + \gamma)w) r + (\alpha - 1)(\gamma - k)w^2
\]
\[
= (\alpha \gamma(1 - r)^2 + (\gamma + \alpha k)r(1 - r) + kr^2) w - k(\gamma(1 - r) + r)
\]
\[
= ((\gamma - k)r - \gamma)((\alpha - 1)r - \alpha)w - k(\gamma(1 - r) + r).
\]
Below we collect some properties of the functions $f$ and $h$:

$$f(w, r) = 0 \iff w = F(r) := \frac{k(\gamma - 1)}{(\gamma - k)(\alpha - (\alpha - 1)r)} = \frac{k(\gamma - 1)}{(\gamma - k)(1 + (\alpha - 1)(1 - r))} \text{ if } \gamma \neq k$$

$$(f(w, r) = k - 1 > 0 \text{ if } \gamma = k);$$

$$F'(r) = \frac{k(\gamma - 1)(\alpha - 1)}{(\gamma - k)(1 + (\alpha - 1)(1 - r))^2} \begin{cases} > 0 & \text{if } \gamma > k \text{ or } \gamma < 1 \\ < 0 & \text{if } 1 < \gamma < k; \end{cases}$$

$$F(0) = \frac{k(\gamma - 1)}{\alpha(\gamma - k)}, \quad F(1) = \frac{k(\gamma - 1)}{\gamma - k};$$

**Figure 12.** The case $\Lambda_1 < \Lambda_2$ and $\gamma > \alpha(k - 1)/(\alpha - 1)$: $(\gamma, c) = (10, 1)$

**Figure 13.** The case $\Lambda_1 > \Lambda_2$ and $\gamma > \alpha(k - 1)/(\alpha - 1)$: $(\gamma, c) = (25, 0.5)$ ($k = 8$ and $\alpha = 20$)
Figure 14. The case $\Lambda_1 < \Lambda_2$ and $0 < \gamma < \alpha(k-1)/(\alpha-1)$: $(\gamma, c) = (0.00066, 0.2475) \ (k = 1.0004 \text{ and } \alpha = 2.5020)$

Figure 15. The case $\Lambda_1 > \Lambda_2$ and $0 < \gamma < \alpha(k-1)/(\alpha-1)$: $(\gamma, c) = (3, 1) \ (k = 4 \text{ and } \alpha = 10)$

$$f(1,1) = \frac{\gamma(k-1)}{k} > 0, \quad f(\frac{k}{\alpha}, 0) = k - 1 > 0;$$

$$h_w(w,r) = \alpha \gamma (1-r)^2 + (\gamma + \alpha k)(1-r) + k r^2 > 0 \ \text{if } 0 \leq r \leq 1; \quad (26)$$

$$h \leq 0 \iff w \leq H(r) := \frac{k(\gamma - (\gamma - 1)r)}{((\gamma - k)r - \gamma)((\alpha - 1)r - \alpha)}; \quad (27)$$

$$h(1,1) = h\left(\frac{k}{\alpha}, 0\right) = 0, \quad \text{or} \quad H(0) = \frac{k}{\alpha}, \quad H(1) = 1;$$

$$H'(0) = \frac{k}{\gamma^2}((\alpha - 1)\gamma - \alpha(k-1)); \quad (28)$$

$$H'(1) = \frac{1}{k}(k(\alpha - 1) - \gamma(k-1)). \quad (29)$$
Since \( H(r) \) is defined if
\[
 r \neq \frac{\alpha}{k - 1} > 1 \text{ and } r \neq \frac{\gamma - k}{\gamma - k} \left\{ \begin{array}{ll}
 > 1 & \text{if } \gamma > k \\
 < 0 & \text{if } \gamma < k,
\end{array} \right.
\]
the function \( H \) is well-defined and smooth in the interval \([0, 1]\). Since for all \( C \in \mathbb{R} \) the equation \( H(r) = C \) leads to a quadratic equation in \( r \), \( H' \) changes sign at most once in \([0, 1]\):
\[
 H'(0)H'(1) > 0 \Rightarrow H' \text{ does not change sign in } [0, 1]
\]
and
\[
 H'(0)H'(1) < 0 \Rightarrow H' \text{ changes sign once in } [0, 1].
\]
Finally, for all \( 0 \leq r \leq 1 \)
\[
f(H(r), r) = \frac{\gamma(k - 1)}{\gamma(1 - r) + kr} > 0.
\]

3.4. **The case** \( \gamma > \frac{k(\alpha - 1)}{\alpha - 1} \).

3.4.1. **Case 1:** \( \gamma > \frac{k(\alpha - 1)}{\alpha - 1} \), \( \lambda_1 > \lambda_2 \), \( \Lambda_1 < \Lambda_2 \). Let
\[
 \gamma > \frac{k(\alpha - 1)}{\alpha - 1}, \quad \frac{k(k - 1)}{\alpha c^2} + k - 1 < \gamma < \frac{k^2}{2(\alpha - 1)} \left( 1 + \sqrt{1 + \frac{4}{\gamma^2}} \right). \tag{33}
\]
Then \( \gamma > \alpha > k \) and, by (18) and (22), \( \lambda_1 > \lambda_2 \) and \( \Lambda_1 < \Lambda_2 \). See also Figures 9 and 12.

We shall prove that
\[
(33) \Rightarrow \text{problem (I) has a solution for which } w \text{ is non monotonic.}
\]
More precisely we shall show that \( w \) has exactly one maximum with a value larger than 1 and does not possess local minima.

Let \( H \) be defined by (27). By (28), (29), (31) and (33), and since \( \gamma > \frac{\alpha(\alpha - 1)}{\alpha - 1} \), there exists \( \rho_0 \in (0, 1) \) such that
\[
 H'(r) \left\{ \begin{array}{ll}
 > 0 & \text{if } 0 \leq r < \rho_0 \\
 < 0 & \text{if } \rho_0 < r \leq 1.
\end{array} \right.
\]

Let \( \mathcal{M}_u \) be the local unstable manifold around \( \left( \frac{k}{\alpha}, 0, \frac{k}{\alpha} \right) \). We are interested in those orbits contained in \( \mathcal{M}_u \) for which \( r \geq 0 \). We use a parameter \( \mu \in [0, 1] \) to label them, where \( \mu = 0 \) (\( \mu = 1 \)) corresponds to the part of the curve \( \mathcal{C}_1 \) for which \( w < \frac{k}{\alpha} \left( w > \frac{k}{\alpha} \right) \). We recall that the solutions corresponding to \( \mu = 0 \) and \( \mu = 1 \) satisfy \( r = 0 \) and equation (23).

To characterise \( \mu \) precisely, we define \( \mathcal{U}_\delta \) as the intersection of \( \mathcal{M}_u \) and the sphere of radius \( \delta > 0 \) and center \( \left( \frac{k}{\alpha}, 0, \frac{k}{\alpha} \right) \). We fix \( \delta \) so small that \( \mathcal{U}_\delta \) is a closed and simple curve, which intersects each outgoing orbit contained in \( \mathcal{M}_u \) exactly once. We know already which orbits correspond to \( \mu = 0 \) and \( \mu = 1 \), and we denote their intersection points with \( \mathcal{U}_\delta \) by \( x_0 \) and, respectively, \( x_1 \). Let \( \mathcal{U}_\delta^+ \) be the part of \( \mathcal{U}_\delta \) where \( r \geq 0 \), so \( \mathcal{U}_\delta^+ \) is a curve connecting \( x_0 \) and \( x_1 \). We denote its length by \( L_\delta \). Given an outgoing orbit which intersects \( \mathcal{U}_\delta^+ \) at a point \( x \), we define its label \( \mu \) as \( 1/L_\delta \) times the length of the curve in \( \mathcal{U}_\delta^+ \) which connects \( x_0 \) and \( x \). By construction, \( \mu \) varies between 0 and 1. We represent each of these orbits by a solution \((R_\mu(z), r_\mu(z), w_\mu(z))\) which is well defined for \( z \leq 0 \) and crosses \( \mathcal{U}_\delta^+ \) at \( z = 0 \):
\[
 (R_\mu(0), r_\mu(0), w_\mu(0)) \in \mathcal{U}_\delta^+ \quad \text{for all } \mu \in [0, 1].
\]
Let \((-\infty, z_\mu)\) denote the maximal interval of existence of \((R_\mu, r_\mu, w_\mu)\) (so \(0 < z_\mu \leq \infty\) and \(R_\mu > 0\) and \(w_\mu > 0\) in \((-\infty, z_\mu)\)). Observe that \(0 < r_\mu < 1\) in \((-\infty, z_\mu)\) for all \(\mu \in (0, 1)\). In addition it easily follows from (23) that

\[
R_1(z) > w_1(z) > \frac{k}{\alpha} \quad \text{for } z \in (-\infty, z_1).
\]  

(35)

In Figures 9 and 12 we have indicated, for \(\mu \in (0, 1)\), the local and global behaviour of \((R_\mu, r_\mu, w_\mu)\) near \((k/\alpha, 0, k/\alpha)\) through projections on the \((R, w)\) and \((r, w)\) planes. The figure is justified by the fast stable manifold theorem and uses the properties of the eigenvectors (see section 3.2). Observe that we may assume that \(\delta\) is so small that there exists \(\mu_0 \in (0, 1)\) such that

\[
\begin{cases}
R_\mu(0) > w_\mu(0) & \text{if } \mu_0 < \mu \leq 1 \\
R_\mu(0) = w_\mu(0) & \text{if } \mu = \mu_0 \\
R_\mu(0) < w_\mu(0) & \text{if } 0 < \mu < \mu_0.
\end{cases}
\]  

(36)

We define

\[
\mathcal{A} = \{\mu \in (0, 1]; \text{ there exists } z_\mu^* < z_\mu \text{ such that } R_\mu(z_\mu^*) = w_\mu(z_\mu^*),\text{ and } R_\mu > w_\mu \text{ in } (-\infty, z_\mu^*)\}
\]

and

\[
\mathcal{A}_0 = \{\mu \in \mathcal{A}; R_\mu < w_\mu \text{ in a right neighbourhood of } z_\mu^*\}.
\]

Observe that

\[
w_\mu' > 0 \text{ in } (-\infty, z_\mu^*) \quad \text{if } \mu \in \mathcal{A},
\]  

(37)

and, by the definition of \(z_\mu^*\) and the equation for \(w_\mu\),

\[
R_\mu'(z_\mu^*) \leq w_\mu'(z_\mu^*) = 0 \quad \text{and hence } \quad w_\mu(z_\mu^*) \leq H(r_\mu(z_\mu^*)) \quad \text{if } \mu \in \mathcal{A}.
\]  

(38)

By (35), \(1 \notin \mathcal{A}\). By (36) and the “unstable manifold theorem”,

\[
(0, \mu_0] \subseteq \mathcal{A} \quad \text{and } \quad z_\mu^* > 0 \text{ if } \mu \in \mathcal{A} \setminus (0, \mu_0].
\]  

(39)

By (24), we may assume that \(\delta\) is so small that \(R_\mu'(z_\mu^*) < 0\) for \(\mu \in (0, \mu_0]\). By the following result, this implies that

\[
(0, \mu_0] \subseteq \mathcal{A}_0.
\]  

(40)

![Figure 16. Global portrait of the system (I) \((\gamma, c) = (10, 1)\) ](image_url)
Lemma 3.1. Let $\mu \in \mathcal{A}$. Then $\mu \in \mathcal{A}_0$ if and only if $R'_\mu(z^*_\mu) < 0$, and, if $\mu \notin \mathcal{A}_0$, 

$$w_\mu(z^*_\mu) = H(r_\mu(z^*_\mu)), \quad r_\mu(z^*_\mu) > \rho_0 \quad \text{and} \quad R''_\mu(z^*_\mu) > w'''(z^*_\mu) = 0.$$ \hfill (41)

Proof. Let $\mu \in \mathcal{A}$. If $R'_\mu(z^*_\mu) < 0$, then $\mu \in \mathcal{A}_0$. If $R'_\mu(z^*_\mu) \geq 0$, then, by (38), $R'_\mu(z^*_\mu) = w'(z^*_\mu) = 0$, $w_\mu(z^*_\mu) = H(r_\mu(z^*_\mu))$, and, by (32), $r'_\mu(z^*_\mu) > 0$. Differentiation of the equations yields that $w''_\mu = (\frac{\mu}{w_\mu}(R_\mu - w_\mu))'$ is not possible since it would imply that $\mu \notin \mathcal{A}_0$ (this is the case which corresponds to (41)); finally, if $R''_\mu(z^*_\mu) = 0$, we differentiate the equations again and find that $w'''(z^*_\mu) = 0$ and 

\[
R''_\mu = \frac{1}{\kappa(z^*_\mu)} h_{rr}(H(r_\mu), r_\mu)r'_\mu
\]

where we have used (26), the equality $h_{rr}(H(r), r) = -h_{w}(H(r), r)H'(r)$, and (34).

We distinguish three cases: if $R''_\mu(z^*_\mu) < 0$, then $R_\mu < w_\mu$ in a left neighbourhood of $z^*_\mu$, which is not possible since $\mu \in \mathcal{A}$; if $R''_\mu(z^*_\mu) > 0$, then $R_\mu > w_\mu$ in a right neighbourhood of $z^*_\mu$ and $\mu \notin \mathcal{A}_0$ (this is the case which corresponds to (41)); finally, if $R''_\mu(z^*_\mu) = 0$, we differentiate the equations again and find that $w''''(z^*_\mu) = 0$ and, since $\gamma > k$,

\[
R'''(z^*_\mu) = \frac{w_\mu(z^*_\mu)}{\kappa(z^*_\mu)} h_{rr}(w_\mu(z^*_\mu), r_\mu(z^*_\mu))(r'_\mu(z^*_\mu))^2
\]

which is not possible since it would imply that $R_\mu < w_\mu$ in a left neighbourhood of $z^*_\mu$.

\[
\mu \in \mathcal{A}_0 \subseteq \mathcal{A} \subseteq (0, 1).
\]

Since $1 \notin \mathcal{A}$, $\mathcal{A}_0 \subset \mathcal{A} \subset (0, 1)$. The following result follows at once from Lemma 3.1 and the continuous dependence of the solutions on $\mu$.

Corollary 3.2. $\mathcal{A}_0$ is an open subset of $(0, 1)$.

By (40) we may define

\[
\mu_1 := \sup\{\mu \in (0, 1) : (0, \mu) \subseteq \mathcal{A}_0 \} \in (\mu_0, 1].
\hfill (42)

Lemma 3.3. Let $\mu_1$ be defined by (42). Then $\mu_1 \in \mathcal{A} \setminus \mathcal{A}_0$. In particular (41) is satisfied if $\mu = \mu_1$ and

$$R_{\mu_1} > w_{\mu_1} \quad \text{in a right neighbourhood of } z^*_\mu_1.$$ \hfill (43)

Proof. If $\mu \in \mathcal{A}$, it follows from (38) and the equation for $R_\mu$ that $h(w_\mu, r_\mu) \leq 0$ at $z = z^*_\mu$. Hence, by (27) and (34),

$$R_\mu(z^*_\mu) = w_\mu(z^*_\mu) \leq H(\rho_0) \quad \text{if } \mu \in \mathcal{A}.$$ \hfill (44)

We claim that

$$\limsup_{\mu \to \mu_1^-} z^*_\mu < \infty.$$ \hfill (45)

Arguing by contradiction we assume that $z^*_\mu_k \to \infty$ as $\mu_k \to \mu_1^-$ for some sequence \{\mu_k\} $\subseteq \mathcal{A}_0$. By (37) and (44), $w_{\mu_k} \leq w_{\mu_k}(z^*_\mu_k) \leq H(\rho_0)$ in $(-\infty, z^*_\mu_k)$, and it easily follows from the equations in problem (I) that $(R_{\mu_1}, r_{\mu_1}, w_{\mu_1})$ is a solution of problem (I) in $(-\infty, \infty)$ which converges to $(1, 1, 1)$ as $z \to \infty$. Since $R_{\mu_k} > w_{\mu_k}$ and $w_{\mu_k} < 1$ in $(-\infty, z^*_\mu_k)$, such behaviour is not compatible with the linearisation around $1, 1, 1$ (see also Figure 9; here we have used that, by the fast stable manifold theorem, the solutions in $\mathcal{A}_\omega$, which we consider, are tangent to $v_1$ as $z \to \infty$) and we have found a contradiction.
Since $\mu_1 > \mu_0$, it follows from (39) that $\liminf_{\mu \to \mu_1^-} z_{\mu}^* \geq 0$. Hence, by (45) there exists a sequence $\mu_k \to \mu_1$ such that $z_{\mu_k}^* \to z$ as $k \to \infty$ for some $z \geq 0$. By the continuous dependence of the solutions on $\mu$, we have that $R_{\mu_1}(z) = w_{\mu_1}(z)$, whence $\mu_1 \in A$.

Since, by Corollary 3.2, $A_0$ is open, it follows from the definition of $\mu_1$ that $\mu_1 \notin A_0$. Inequality (43) follows from (41).

We set $B = \{\mu \in (0, \mu_1) \colon$ there exists $z_{\mu}^{**} \in (z_{\mu}, z_{\mu})$ such that $R_{\mu}(z_{\mu}^{**}) = w_{\mu}(z_{\mu}^{**})$ and $R_{\mu} < w_{\mu}$ in $(z_{\mu}, z_{\mu}^{**}) \subseteq A_0\}$.

Observe that for all $\mu \in B$

$$R_{\mu}'(z_{\mu}^{**}) \geq w_{\mu}'(z_{\mu}^{**}) = 0 \quad \text{and hence} \quad w_{\mu}(z_{\mu}^{**}) \geq H(r_{\mu}(z_{\mu}^{**})) \geq \frac{k}{\alpha}. \quad (46)$$

Since $w_{\mu}' < 0$ in $(z_{\mu}, z_{\mu}^{**})$, it follows from (25), (32) and (38) that $r_{\mu} > 0$ in $(z_{\mu}, z_{\mu}^{**})$.

Hence, by (38), (46) and the graph of the function $H$,

$$r_{\mu}(z_{\mu}^{**}) > \rho_0 \quad \text{if} \quad \mu \in B. \quad (47)$$

In particular, by (46),

$$w_{\mu}(z_{\mu}^{**}) \geq 1. \quad (48)$$

Lemma 3.4. $\mu \in B$ if $\mu_1 - \mu > 0$ is sufficiently small.

Proof. We argue by contradiction and suppose that there exists a sequence $\{\mu_k\} \subset (0, \mu_1) \setminus B$ such that $\mu_k \to \mu_1$ as $k \to \infty$. Then $\mu_k \in A_0 \setminus B$, $R_{\mu_k} > w_{\mu_k}$ in $(-\infty, z_{\mu_k}^*)$ and $R_{\mu_k} < w_{\mu_k}$ in $(z_{\mu_k}, z_{\mu_k})$. Letting $k \to \infty$, (43) and the continuous dependence of the solutions on $\mu$ imply that $z_{\mu_k}^* \to z_{\mu_1}^*$ as $k \to \infty$ and we obtain a contradiction from the sign of $R_{\mu_1} - w_{\mu_1}$ in a right neighbourhood of $z_{\mu_1}^*$.

It follows from the linearisation around $(\frac{k}{\alpha}, 0, \frac{k}{\alpha})$ that $\mu \notin B$ for sufficiently small $\mu > 0$, and in view of Lemma 3.4 we have that

$$\mu_2 := \inf B \in (0, \mu_1).$$

Lemma 3.5. $\mu_2 \notin B$.

Proof. Arguing by contradiction we suppose that $\mu_2 \in B$. If $\mu \in (0, \mu_2)$, then $\mu \in A_0$ and $R_{\mu} < w_{\mu}$ in $(z_{\mu}, z_{\mu})$. Since $\mu_2 \in B$, it follows from the definition of $\mu_2$ and the continuous dependence on $\mu$, that $R_{\mu_2} \leq w_{\mu_2}$ in $(z_{\mu_2}^{**}, z_{\mu_2})$ and $R_{\mu_2} < w_{\mu_2}$ in $(z_{\mu_2}, z_{\mu_2}^{**})$. Hence $R(z_{\mu_2}^{**}) = w(z_{\mu_2}^{**})$ and $R'(z_{\mu_2}^{**}) = w'(z_{\mu_2}^{**}) = 0$, i.e. $h(w(z_{\mu_2}^{**}, r(z_{\mu_2}^{**})) = 0$. By (47), $r_{\mu_2}(z_{\mu_2}^{**}) > \rho_0$ and we may proceed as in the proof of Lemma 3.1 (cf. (41)): $R_{\mu_2} \leq w_{\mu_2}$ in a neighbourhood of $z_{\mu_2}^*$ and we have found a contradiction.

Since $\mu_2 \in A_0$, Lemma 3.5 and its proof imply that

$$R_{\mu_2} > w_{\mu_2} \quad \text{in} \quad (-\infty, z_{\mu_2}^*) \quad \text{and} \quad R_{\mu_2} < w_{\mu_2} \quad \text{in} \quad (z_{\mu_2}, z_{\mu_2}).$$

The discussion of Case 1 is completed by the following result.

Lemma 3.6. $z_{\mu_2} = \infty$ and $(R_{\mu_2}, r_{\mu_2}, w_{\mu_2})$ is a solution of problem (I).
Proof. To show that $z_{\mu_2} = \infty$ we have to exclude that $R_{\mu_2}(z_{\mu_2}) = 0$. Arguing by contradiction we assume that $z_{\mu_2} < \infty$ and $R_{\mu_2}(z_{\mu_2}) = 0$. Then necessarily $R'_{\mu_2}(z_{\mu_2}) \leq 0$, whence, by (32), $w_{\mu_2}(z_{\mu_2}) \leq H(r_{\mu_2}(z_{\mu_2})) < F(r_{\mu_2}(z_{\mu_2}))$. Hence $r_{\mu_2} > 0$ in a left neighbourhood of $z_{\mu_2}$ and $r_{\mu_2}(z_{\mu_2}) = 1$. Since $H(1) = 1$, $w_{\mu_2}(z_{\mu_2}) \leq 1$. On the other hand, by (48), the definition of $\mu_2$ and the continuous dependence on $\mu$, $w_{\mu_2}(z_{\mu_2}) \geq 1$, so $w_{\mu_2}(z_{\mu_2}) = 1$. Let $\mu_{2k} \rightarrow \mu_2^-$ for some sequence $\{\mu_{2k}\} \subset B$. By continuous dependence on $\mu$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for sufficiently large $k$,

$$R_{\mu_{2k}}(z_{\mu_2} + \delta) < \varepsilon, \quad r_{\mu_{2k}}(z_{\mu_2} + \delta) > 1 - \varepsilon \quad \text{and} \quad w_{\mu_{2k}}(z_{\mu_2} + \delta) < 1 + \varepsilon.$$

Since $z_{\mu_2} \in (z_{\mu_2}^*, z_{\mu_2}^{*\ast})$ for $k$ large enough, $w_{\mu_{2k}} < 0$ in a right neighbourhood of $z_{\mu_2} + \delta$ which does not depend on $k$, so, in that neighbourhood, $r_{\mu_{2k}}$ is near to 1 and $R_{\mu_{2k}}$ and $w_{\mu_{2k}}$ are near to the solution curve of

$$
\begin{align*}
\frac{dR}{dw} &= -\frac{w^2(w-1)}{c^2(w-R)} \quad \text{if} \; w < 1 \\
R(1) &= 0.
\end{align*}
$$

But the latter curve satisfies $R < 0$ if $w \neq 1$, and we have found a contradiction. Hence $z_{\mu_2} = \infty$. This implies that $w_{\mu_2} \geq 1$ in $z_{\mu_2}^*, \infty$. Finally, it easily follows that $(R_{\mu_2}(z), r_{\mu_2}(z), w_{\mu_2}(z)) \rightarrow (1, 1, 1)$ as $z \rightarrow \infty$. \hfill \Box

Remark 3.7. In a forthcoming paper on homoclinic TWs we shall use the following observation, which follows easily from the above construction and the stable manifold theorem. If $\mu < \mu_2$ (so $\mu \in A_0 \setminus B$) and $\mu_2 - \mu$ is sufficiently small, then $w_{\mu}$ in decreasing in $(z_{\mu}^*, z_{\mu})$ and crosses the plane $w = 1$ at some $z \in (z_{\mu}^*, z_{\mu})$. In particular, the solution orbit will be close to that of the part of the one-dimensional unstable manifold of $(1, 1, 1)$ for which $R < w$, i.e. the orbit defined by $\frac{dR}{dw} = \frac{w^2(1-w)}{c^2(w-R)}$ for $w < 1$ and $R(1) = 1$ (the convergence to this orbit, as $\mu \rightarrow \mu_2^-$, is not uniform: the solutions remain for ever larger values of $z$ near the equilibrium point $(1, 1, 1)$ before following the unstable manifold).

3.4.2. Case 2: $\gamma > \frac{k(\alpha-1)}{k-1}, \lambda_1 < \lambda_2, \Lambda_1 < \Lambda_2$. Let

$$\gamma > \frac{k(\alpha-1)}{k-1}, \quad \gamma > \frac{k(k-1)^2}{\alpha c^2} + k - 1, \quad \gamma > \frac{k c^2}{2(k-1)} \left(1 + \sqrt{1 + \frac{4}{c^2}}\right). \quad (49)$$

We claim that

$$\text{(49) \Rightarrow problem (I) has a solution (R(z), r(z), w(z)).}$$

In addition $w$ is either increasing or non-monotonic. In the latter case $w$ has exactly one maximum with a value larger than 1 and it does not possess local minima. We do not know whether both cases really occur.

To prove our claim we follow the procedure in the previous section. The difference is that now $\lambda_1 < \lambda_2$ instead of $\lambda_1 > \lambda_2$ (see Figure 8). In section 3.4.1 we only used the latter inequality to prove (45). In this case we are not able to prove (45) and distinguish two cases: if (45) is satisfied, we follow the proof in section 3.4.1 and obtain a solution for which $w$ is non-monotonic; if (45) is not satisfied, it follows from what is written just below (45) that problem (I) has a solution for which $w$ is monotonic.
Remark 3.8. Remark 3.7 can be easily adapted to the present case: let \((R_\mu, r_\mu, w_\mu)\) be the constructed overlapping TW. If \(\mu < \bar{\mu}\) and \(\bar{\mu} - \mu\) is sufficiently small, then \((R_\mu, r_\mu, w_\mu)\) enters the trapezium
\[
\mathcal{T} = \{(R, w) : 0 < R < w, \frac{k}{\alpha} < w < 1\}
\]
after some value \(z_\mu < z_0\) near the equilibrium \((1, 1, 1)\). Indeed, if (45) is not satisfied, then \(w_\mu\) is monotonically \(\mu = \mu_1\), and, if \(\mu_1 - \mu\) is sufficiently small, \(\mu \in \mathcal{A}_0\) and the solution crosses the plane \(R = w\) close to \((1, 1, 1)\). Again it follows from the stable manifold theorem that the solution orbit will be close to that of the part of the one-dimensional unstable manifold of \((1, 1, 1)\) for which \(R < w\).

In the present version the remark also applies to the cases 3, 4 and 5 discussed below.

3.4.3. Case 3: \(\gamma > \frac{k(\alpha - 1)}{k - 1}, \Lambda_1 > \Lambda_2\). Until now we have assumed that \(\gamma > \frac{k(k-1)^2}{\alpha c^2} + k - 1(> 0)\). In this section we show that the existence and monotonicity results for the solution of problem (I) which we have proved in sections 3.4.1 and 3.4.2 remain valid if
\[
0 < \gamma < \frac{k(k-1)^2}{\alpha c^2} + k - 1.
\]
More precisely: if
\[
\gamma > \frac{k(\alpha - 1)}{k - 1}, \quad \gamma < \frac{k(k-1)^2}{\alpha c^2} + k - 1, \quad \gamma < \frac{kc^2}{2(k - 1)} \left(1 + \sqrt{1 + \frac{4}{c^2}}\right),
\]
problem (I) has a solution for which \(w\) is non-monotonic with exactly one maximum large than 1, and if
\[
\gamma > \frac{k(\alpha - 1)}{k - 1}, \quad \frac{kc^2}{2(k - 1)} \left(1 + \sqrt{1 + \frac{4}{c^2}}\right) < \gamma < \frac{k(k-1)^2}{\alpha c^2} + k - 1,
\]
problem (I) has a solution for which \(w\) is either increasing or non-monotonic with exactly one maximum large than 1.

Condition (50) implies that \(\Lambda_1 > \Lambda_2\) instead of \(\Lambda_1 < \Lambda_2\), a condition which concerns the local structure of the solutions near \((\frac{k}{\alpha}, 0, \frac{k}{\alpha})\), as \(z \to -\infty\). Below we indicate where we have previously used the sign of \(\Lambda_1 - \Lambda_2\), and how we can adapt the various proofs.

The following quantities can be defined as before: the local two-dimensional manifold \(\mathcal{M}_u\), the solutions \((R_\mu, r_\mu, w_\mu)\) for \(\mu \in [0, 1]\), their maximal interval of existence \((-\infty, z_\mu)\), the sets \(\mathcal{A}_0 \subseteq \mathcal{A} \subseteq (0, 1)\) and the points \(z_\mu^*\). In sections 3.4.1 and 3.4.2 we have used the condition that \(\Lambda_1 < \Lambda_2\) to conclude that \(\mu \in \mathcal{A}_0\) if \(\mu > 0\) is sufficiently small (see also (36)). This is no longer true if \(\Lambda_1 > \Lambda_2\) (see Figure 13).

So let \(\Lambda_1 > \Lambda_2\). Since \(\gamma > \frac{\alpha(k-1)}{\alpha^2}\), the linearisation near \((\frac{k}{\alpha}, 0, \frac{k}{\alpha})\) implies that there exists a minimal value \(\mu^* \in (0, 1)\) such that
\[
\text{for all } \mu \in (\mu^*, 1) \text{ there exists } \zeta_\mu \leq z_\mu \text{ such that } R_\mu(z) > w_\mu(z) > \frac{k}{\alpha} \text{ for } z \in (\zeta_\mu, \zeta_\mu).
\]
In particular it follows from the local analysis near \(z = -\infty\) that \(\mu \in \mathcal{A}_0\) if \(\mu - \mu^* > 0\) is sufficiently small. This implies that definition (42) of \(\mu_1\) must be changed in
\[
\mu_1 := \sup \{\mu \in (\mu^*, 1) : (\mu^*, \mu) \subset \mathcal{A}_0\}.
\]
For the rest the proofs are identical (with some obvious changes, such as \((\mu^*, \mu_1) \subset \mathcal{A}_0\) instead of \((0, \mu_1) \subset \mathcal{A}_0\)).
3.4.4. A remark on exceptional cases. There are exceptional wave velocities $c$ for which the strict inequalities in the previous sections are not satisfied (for example, it may happen that $\lambda_1 = \lambda_2$). Such velocities, say $c_0$, are isolated, and it is possible to approximate them by a sequence of velocities for which we already know that problem (I) has a solution. For a different parameter regime, in [3] we have indicated how to prove that the corresponding TWs converge to a solution of problem (I) with velocity $c_0$ (of course one should control the translation invariance in doing so). Therefore we omit the proofs in the present. Similarly, we shall not pay attention to exceptional velocities in the cases which will be treated below.

3.5. The case $\frac{\alpha(k-1)}{\alpha-1} < \gamma < \frac{k(\alpha-1)}{k-1}$. We observe that $\frac{\alpha(k-1)}{\alpha-1} < \frac{k(\alpha-1)}{k-1}$.

3.5.1. Case 4: $\frac{\alpha(k-1)}{\alpha-1} < \gamma < \frac{k(\alpha-1)}{k-1}$, $A_1 < A_2$. Let

$$\frac{\alpha(k-1)}{\alpha-1} < \gamma < \frac{k(\alpha-1)}{k-1}, \quad \gamma > \frac{k(k-1)^2}{\alpha^2} + k - 1. \quad (51)$$

In this section we prove that

$$(51) \Rightarrow \text{problem (I) has a solution for which } w \text{ is increasing.}$$

It follows from (28), (29), (30) and (51) that

$$A_1 < A_2, \quad H'(r) > 0 \quad \text{for all } r \in [0,1].$$

The local analysis near $\left(\frac{\alpha}{\alpha-1}, 0, \frac{k}{\alpha} \right)$ is identical to that in Case 1. So we can follow the procedure in section 3.4.1 and define the solutions $(R_{\mu}, r_{\mu}, w_{\mu})$ for $\mu \in [0,1]$, their maximal interval of existence $(-\infty, z_{\mu})$ and the set $A \subseteq (0,1)$ of parameters $\mu$ for which $R_{\mu} = w_{\mu}$ for the first time at some $z_{\mu}^*$ $(R_{\mu} > w_{\mu} \text{ in } (-\infty, z_{\mu}^*))$. As before $1 \notin A$. In addition we have that, for all $\mu \in A$,

$$w_{\mu}' > 0 \text{ in } (-\infty, z_{\mu}^*), \quad R_{\mu}'(z_{\mu}^*) \leq w_{\mu}'(z_{\mu}^*) = 0, \quad w_{\mu}(z_{\mu}^*) \leq H(r_{\mu}(z_{\mu}^*)) < 1.$$

Let

$$\mu_3 := \sup\{\mu \in (0,1) : (0, \mu) \subset A\}. \quad (52)$$

Then

$$R_{\mu_3} \geq w_{\mu_3} \text{ in } (-\infty, z_{\mu_3}). \quad (53)$$

Lemma 3.9. $\mu_3 \notin A$.

Proof. Suppose that $\mu_3 \in A$. Then $R_{\mu_3}(z_{\mu_3}^*) = w_{\mu_3}(z_{\mu_3}^*)$ and $R_{\mu_3}'(z_{\mu_3}^*) = w_{\mu_3}'(z_{\mu_3}^*) = 0$ (We already know that $R_{\mu_3}'(z_{\mu_3}^*) \leq w_{\mu_3}'(z_{\mu_3}^*) = 0$, and if $R_{\mu_3}'(z_{\mu_3}^*) < 0$, we get a contradiction with the definition of $\mu_3$ as a supremum). Arguing as in the proof of Lemma 3.1, we find that $w_{\mu_3}'(z_{\mu_3}^*) = 0$ and, since $h_{\mu}(H(r), r) = -h_{\mu}(H(r), r)H'(r) < 0$ for $r \in [0,1]$,

$$R_{\mu_3}' = \frac{w_{\mu_3}r_{\mu_3}(1 - r_{\mu_3})}{k c^2 R_{\mu_3}} h_{\mu}(H(r_{\mu_3}), r_{\mu_3)f(H(r_{\mu_3}), r_{\mu_3}) < 0 \text{ at } z = z_{\mu_3}^*.$$  

Then (53) is violated in a neighbourhood of $z_{\mu_3}^*$ and we have found a contradiction.

It remains to prove the following result.

Lemma 3.10. $z_{\mu_3} = \infty, (R_{\mu_3}, r_{\mu_3}, w_{\mu_3})$ is a solution of problem (I), and $w_{\mu_3}'(z) > 0$ for all $z$. 

Proof. By Lemma 3.9 and the definition of $\mathcal{A}$, $R_{\mu_3} > w_{\mu_3}$ in $(-\infty, z_{\mu_3})$. Then, by the equation for $w, w'_{\mu_3} > 0$ in $(-\infty, z_{\mu_3})$. We claim that
\[
\lim_{\mu \to \mu_3} z^*_\mu = \infty. \tag{54}
\]
Arguing by contradiction we suppose that $\lim_{\mu_k \to \mu_3} z^*_{\mu_k} < \infty$ for some sequence $\{\mu_k\} \subset (0, \mu_3)$. As described above, the solution $(R_{\mu}, r_{\mu}, w_{\mu})$ continuously depends on $\mu$, and the definition of $z^*_\mu$ gives $w_{\mu}(z^*_\mu) = R_{\mu}(z^*_\mu)$ and $R_{\mu} > w_{\mu}$ in $(-\infty, z^*_\mu)$. Therefore, the assumption that $\lim_{\mu_k \to \mu_3} z^*_{\mu_k} < \infty$ for some sequence $\{\mu_k\} \subset (0, \mu_3)$ leads to $\mu_3 \in \mathcal{A}$, and we have found a contradiction.

Since $w_{\mu} < 1$ in $(-\infty, z^*_\mu)$ for all $\mu \in (0, \mu_3)$, it follows easily from (54) and the equations in problem (I) that $(R_{\mu_3}, r_{\mu_3}, w_{\mu_3})$ is a solution of problem (I) in $(-\infty, \infty)$ which converges to $(1, 1, 1)$ as $z \to \infty$. \qed

3.5.2. **Case 5**. Let $\frac{\alpha(k-1)}{\alpha - 1} < \gamma < \frac{k(\alpha - 1)}{k - 1}, \Lambda_1 > \Lambda_2$. Let
\[
\frac{\alpha(k-1)}{\alpha - 1} < \gamma < \frac{k(\alpha - 1)}{k - 1}, \quad \gamma < \frac{k(k-1)^2}{\alpha\epsilon^2} + k - 1. \tag{55}
\]
Then the result in section 3.5.1 remains valid:

(55) \Rightarrow problem (I) has a solution for which $w$ is increasing.

The proof is similar to that in section 3.4.3: the linearisation around $(\frac{k}{\alpha}, 0, \frac{k}{\alpha})$ implies that there exists a minimal value $\mu^* \in (0, 1)$ such that for all $\mu \in (\mu^*, 1)$ there exists $\zeta^*_\mu \leq z_\mu$ such that $R_{\mu}(z) > w_{\mu}(z) > \frac{k}{\alpha}$ for $z \in (-\infty, \zeta^*_\mu)$, and we can follow the proof in section case 4 with the definition (52) of $\mu_3$ replaced by
\[
\mu_3 := \sup \{\mu \in (\mu^*, 1) : (\mu^*, \mu) \subset \mathcal{A} \}.
\]

3.6. **The case $0 < \gamma < \frac{\alpha(k-1)}{\alpha - 1}$**. The case $0 < \gamma < \frac{\alpha(k-1)}{\alpha - 1}$ is very similar to the case $\gamma > \frac{k(\alpha - 1)}{k - 1}$ considered in section 3.4. In the proofs it is enough to interchange the roles of the points $(\frac{k}{\alpha}, 0, \frac{k}{\alpha})$ and $(1, 1, 1)$ and therefore we omit them. Below we list the results in the various cases, and it should be clear from the corresponding phase space pictures how the proofs in section 3.4 should be adapted.

3.6.1. **Case 6**: if $\gamma < \frac{\alpha(k-1)}{\alpha - 1}, \Lambda_1 < \Lambda_2, \lambda_1 > \lambda_2$, or, equivalently,
\[
0 < \gamma < \frac{\alpha(k-1)}{\alpha - 1}, \quad \frac{k(k-1)^2}{\alpha\epsilon^2} + k - 1 < \gamma < \frac{k^2}{2(k-1)} \left(1 + \sqrt{1 + \frac{4}{\epsilon^2}}\right),
\]
we proceed as in section 3.4.1: problem (I) has a solution for which $w$ is non-monotonic (see Figures 11 and 14). More precisely, $w$ has exactly one minimum with a value smaller than $\frac{k}{\alpha}$ and does not possess local maxima.

3.6.2. **Case 7**: if $\gamma < \frac{\alpha(k-1)}{\alpha - 1}, \Lambda_1 > \Lambda_2, \lambda_1 > \lambda_2$, or, equivalently,
\[
\gamma < \frac{\alpha(k-1)}{\alpha - 1}, \quad \gamma < \frac{k(k-1)^2}{\alpha\epsilon^2} + k - 1, \quad \gamma < \frac{k^2}{2(k-1)} \left(1 + \sqrt{1 + \frac{4}{\epsilon^2}}\right),
\]
we proceed as in section 3.4.2: problem (I) has a solution $(R(z), r(z), w(z))$ (see Figures 11 and 15). In addition $w$ is either increasing or non-monotonic. In the latter case $w$ has exactly one minimum with a value smaller than $\frac{k}{\alpha}$ and it does not possess local maxima. We do not know whether both cases really occur.
3.6.3. Case 8: if $\gamma < \frac{\alpha(k-1)}{\alpha-1}$, $\lambda_1 < \lambda_2$, we proceed as in section 3.4.3: the existence and monotonicity results for the solution of problem (I) considered in cases 6 and 7 remain valid if

$$\gamma > \frac{k^2}{2(k-1)} \left(1 + \sqrt{1 + \frac{4}{\epsilon^2}}\right).$$

More precisely: if

$$\gamma < \frac{\alpha(k-1)}{\alpha-1}, \quad \gamma > \frac{k(k-1)^2}{\alpha \epsilon^2} + k - 1, \quad \gamma > \frac{k^2}{2(k-1)} \left(1 + \sqrt{1 + \frac{4}{\epsilon^2}}\right),$$

problem (I) has a solution for which $w$ is non-monotonic with exactly one minimum smaller than $\frac{k}{\alpha}$ (see Figures 10 and 14), and if

$$\gamma < \frac{\alpha(k-1)}{\alpha-1}, \quad \frac{k^2}{2(k-1)} \left(1 + \sqrt{1 + \frac{4}{\epsilon^2}}\right) < \gamma < \frac{k(k-1)^2}{\alpha \epsilon^2} + k - 1,$$

problem (I) has a solution for which $w$ is either increasing or non-monotonic with exactly one minimum smaller than $\frac{k}{\alpha}$ (see Figures 10 and 15).

4. **Convergence to standing waves as $c \to 0$.** We briefly discuss the behaviour of the constructed overlapping TWs $(U_c, V_c)$ for small values of $c$. Let $0 < c \leq 1$ and set $w_c = U_c + V_c$. To eliminate translation invariance we require that

$$w_c(0) = \frac{1}{2} \left(1 + \frac{k}{\alpha}\right).$$

It easily follows from the equation for $w$ which is obtained by adding the equations for $U$ and $V$ in (2) and the uniform Lipschitz continuity of $w_c$ that $|w''_c| < C$ for some constant $C$ which does not depend on $c$. In addition it follows from their construction that $U_c$ (and so $V_c = w_c - U_c$) are uniformly bounded in $BV(\mathbb{R})$. Therefore any sequence $c_k \to 0$ has a subsequence, which we denote again by $c_k$, such that:

- $w_{c_k}$ converges uniformly in $\mathbb{R}$ to a function $w_0 \in C^1(\mathbb{R})$;
- $w'_{c_k}$ converges uniformly in $\mathbb{R}$ to $w'_0$;
- $U_{c_k}$ converges in $L^2_{loc}(\mathbb{R})$ to a function $U_0 \in BV(\mathbb{R})$;
- $V_{c_k}$ converges in $L^2_{loc}(\mathbb{R})$ to $V_0 = w_0 - U_0$.

Let $\zeta \in C^2(\mathbb{R})$ have compact support. Since

$$\int_{\mathbb{R}} \left(-U_c w'_c \zeta' - cU_c \zeta' + U_c(1 - (U_c + \alpha V_c))\zeta\right) dz = 0$$

$$\int_{\mathbb{R}} \left(-V_c w'_c \zeta' - cV_c \zeta' + \gamma V_c (1 - \frac{U_c + \alpha V_c}{k}) \zeta\right) dz = 0,$$

we may pass to the limit $c_k \to 0$ and obtain that

$$\int_{\mathbb{R}} \left(-U_0 w'_0 \zeta' + U_0(1 - (U_0 + \alpha V_0))\zeta\right) dz = 0$$

$$\int_{\mathbb{R}} \left(-V_0 w'_0 \zeta' + \gamma V_0 (1 - \frac{U_0 + \alpha V_0}{k}) \zeta\right) dz = 0.$$

So $(U_0, V_0)$ is a (weak) standing wave.

It also follows from the construction in the previous section that, for fixed $\alpha$, $k$ and $\gamma$, $w'_c > 0$ in $\mathbb{R}$ for sufficiently small $c > 0$, so $w'_0 \geq 0$ in $\mathbb{R}$. By the strong maximum principle, $w'_0 > 0$ in $\mathbb{R}$, in particular $\frac{k}{\alpha} < w_0 < 1$ in $\mathbb{R}$. 

Finally we consider the decay rates of the constructed overlapping TWs for small values of $c > 0$. Setting $r_c = U_c/w_c$, it follows from the linearisation of the equation for $r_c$ around $(\frac{k}{\alpha}, 0, \frac{k}{\alpha})$ that

$$U_c \approx Ce^{-\frac{k-1}{c}z} \quad \text{as } z \to -\infty,$$

so, for fixed but small $c$, $U_c$ has exponential decay as $z \to -\infty$ but the exponent explodes as $c \to 0$. Similarly,

$$V_c \approx Ce^{-\frac{\gamma(k-1)}{kc}z} \quad \text{as } z \to \infty.$$

These decay rates are not suitable to pass to the limit $c \to 0$. Formally, a more precise result can be obtained as follows. For small values of $c > 0$ the decay rates as $z \to \pm\infty$ are determined by $\lambda_2$ and $\Lambda_2$. We focus here on the decay as $z \to -\infty$. Since $\Lambda_2 \to \sqrt{\gamma \alpha k}$ as $c \to 0^+$, it follows from a straightforward calculation that for sufficiently small $c$

$$w_c' \approx \sqrt{\frac{2\alpha}{k}(w_c - \frac{k}{\alpha})} \quad \text{as } z \to -\infty. \quad (57)$$

On the other hand $r_c$ satisfies, as a function of $w$, the equation

$$\frac{1}{r_c} \frac{dr_c}{dw} = \frac{w(1 - r_c)}{cR_c w_c'}(\gamma - 1 - (\frac{k}{\alpha} - 1)(1 + (\alpha - 1)(1 - r)))w$$

$$= \frac{1 - r_c}{(w_c' + c)w_c'}(\gamma - 1 - (\frac{k}{\alpha} - 1)(1 + (\alpha - 1)(1 - r)))w$$

$$\approx \frac{k - 1}{(\sqrt{\frac{2\alpha}{k}}(w - \frac{k}{\alpha}) + c)\sqrt{\frac{2\alpha}{k}}(w - \frac{k}{\alpha})} \approx \frac{k(k - 1)}{\gamma \alpha (w - \frac{k}{\alpha})^2}.$$

Integrating we obtain that, in the limit $c \to 0$, $U_c$ and $w_c$ satisfy the relation

$$-\log U_c \approx \frac{k(k - 1)}{\gamma \alpha (w_c - \frac{k}{\alpha})} \quad \text{as } z \to -\infty.$$

Together with $(57)$ this result is compatible with the decay rate $(17)$ for the standing wave.

A similar procedure leads to a limiting (as $c \to 0$) decay rate for $V_c(z)$ as $z \to \infty$ which is compatible with $(16)$.

5. **Concluding remarks.** When in a healthy tissue environment, abnormal tissue (which is slightly different from healthy tissue) appears as a result of mutation, it is biologically and medically relevant to know how the abnormal tissue expands. To address this question, mathematical models are proposed and analysed. In this paper, we consider a system which describes between normal and abnormal cell densities, say $u$ and $v$, respectively. Since TWs often describe the large time behaviour of solutions, it is important to consider TWs to answer the above question. Particularly, we focus on TWs of $(1)$ in the parameter regime where $\alpha > k > 1$ and $\gamma > 0$. It is known ([5]) that segregated TWs, which satisfy $UV = 0$ in $\mathbb{R}$, have negative wave velocities $c < 0$. In the present paper we discuss the existence of overlapping TWs satisfying $U, V > 0$ in $\mathbb{R}$ and $c \geq 0$. If $c = 0$, $(2)$ possesses standing waves with extremely fast decay tails. To our best knowledge, this type of TWs has not been reported so far, and it seems that this feature comes form a parabolic-hyperbolic nature of $(2)$. If $c > 0$, the overlapping TWs can be either monotonic or nonmonotonic in $U + V$ according to the parameter values. But then a subsequent question arises spontaneously: is there any TW for $c \in (\bar{c}, 0)$? Or is
the segregated TWs with negative wave velocity isolated? In numerical tests we could not find any overlapping TWs for $c \in (\tau, 0)$. However, in a forthcoming paper we will prove the existence of TWs for $c \in (\tau, 0)$ which are neither segregated nor overlapping: we call them partially overlapping TWs.

Though we are gradually unraveling TWs of (2), the global structure of TWs is not yet completely understood. This will be the subject of future work.

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Appendix A. The fast stable manifold theorem. In the paper we have used the following version of the fast stable manifold theorem. The theorem can be easily generalised to a more general setting. The essential ingredient of the proof is that the stable manifold is 2-dimensional, which makes it possible, as Joost Hulshof kindly explained us, to use polar coordinates. This reduces the proof to the standard stable manifold theorem.

Proposition (fast stable manifold theorem). Let $(x_0, y_0, z_0)$ be an equilibrium point of the system
\[
\begin{align*}
x' &= f(x, y, z) \\
y' &= g(x, y, z) \\
z' &= h(x, y, z),
\end{align*}
\]
where $f$, $g$ and $h$ are of class $C^2$ in a neighbourhood of $(x_0, y_0, z_0)$. Let $A$ be the matrix which is obtained after linearisation around $(x_0, y_0, z_0)$. Assume that $A$ has three different real eigenvalues,
\[
\mu < \lambda < 0 < \Lambda,
\]
with corresponding eigenvectors $v_\mu$, $v_\lambda$ and $v_\Lambda$, and let $S$ be the local two-dimensional stable manifold (of class $C^2$) near $(x_0, y_0, z_0)$. Then $S$ contains a one-dimensional manifold $S_{\text{fast}}$ (of class $C^2$) which is tangent to $v_\mu$, and all orbits contained in $S \setminus S_{\text{fast}}$ are tangent to $v_\lambda$.

Proof. Without loss of generality we may assume that $(x_0, y_0, z_0)$ is the origin and that $A$ is diagonal:
\[
\begin{align*}
x' &= \lambda x + F(x, y, z) \\
y' &= \mu y + G(x, y, z) \\
z' &= \Lambda z + H(x, y, z),
\end{align*}
\]
where $F$, $G$, $H$ are of class $C^2$ and satisfy
\[
|F(x, y, z)|, |G(x, y, z)|, |H(x, y, z)| \leq C_0(x^2 + y^2 + z^2)
\]
in a neighbourhood of $(0, 0, 0)$. In particular $v_\lambda = (1, 0, 0)$, $v_\mu = (0, 1, 0)$ and $v_\Lambda = (0, 0, 1)$, and the tangent plane of $S$ at the origin is horizontal: locally $S$ is described by the equation $z = Z(x, y)$, where $Z$ is of class $C^2$ and satisfies
\[
|Z(x, y)| \leq C_1(x^2 + y^2)
\]
in a neighbourhood of $(0, 0)$. 

From now on we restrict ourselves to orbits contained in $S$: as long as $x(0)^2 + y(0)^2$ is small enough, $(x(t), y(t), z(t))$ lies on $S$ (at least) locally in time, and $(x(t), y(t))$ satisfies

$$\begin{cases} x' = \lambda x + F(x, y, Z(x, y)) \\ y' = \mu y + G(x, y, Z(x, y)). \end{cases}$$

In polar coordinates, $(r, \varphi)$, the system becomes

$$\begin{cases} r' = (\lambda \cos^2 \varphi + \mu \sin^2 \varphi)r + \cos \varphi F(r \cos \varphi, r \sin \varphi, Z(r \cos \varphi, r \sin \varphi)) \\ \varphi' = \frac{1}{2}(\mu - \lambda) \sin(2\varphi) - \sin \varphi G(r \cos \varphi, r \sin \varphi, Z(r \cos \varphi, r \sin \varphi)) + \cos \varphi \frac{F(r \cos \varphi, r \sin \varphi, Z(r \cos \varphi, r \sin \varphi))}{r}. \end{cases}$$

There exists a constant $C_2$ depending on $C_0$ and $C_1$ such that for all $\varphi$ and sufficiently small values of $r$,

$$\left| \frac{F(r \cos \varphi, r \sin \varphi, Z(r \cos \varphi, r \sin \varphi))}{r} \right| \leq C_2 r$$

and

$$\left| \frac{G(r \cos \varphi, r \sin \varphi, Z(r \cos \varphi, r \sin \varphi))}{r} \right| \leq C_2 r.$$ 

It follows from the equation for $r$ in system (II) and the inequality $\mu < \lambda$ that

$$r' \leq \lambda r + 2C_2 r^2 < \frac{1}{2} \lambda r$$

if $r(0) < r_0$ with a small $r_0 > 0$; so $r \to 0$ monotonically (and uniformly with respect to $\varphi$) as $t \to \infty$. This fact also implies that

$$S_0 := \{(x, y, z) \in S \mid x^2 + y^2 < r_0^2\}$$

is positively invariant for $t \geq 0$.

Since $\mu \neq \lambda$, it follows easily from the equation for $\varphi$ that $\varphi(t)$ converges to a multiple of $\frac{1}{2} \pi$ as $t \to \infty$. By periodicity we may restrict ourselves to the cases that $\varphi(t)$ converges to $0, \frac{1}{2} \pi, \pi$ and $\frac{3}{2} \pi$.

Linearising system (II) around $(0, 0)$ and $(0, \pi)$, we obtain, respectively, the systems

$$\begin{cases} \bar{r}' = \lambda \bar{r} \\ \bar{\varphi}' = \frac{1}{2} G_{xx}(0, 0, 0) \bar{r} + (\mu - \lambda) \bar{\varphi} \end{cases}$$

and

$$\begin{cases} \bar{r}' = \lambda \bar{r} \\ \bar{\varphi}' = \frac{1}{2} G_{xx}(0, 0, 0) \bar{r} + (\mu - \lambda) \bar{\varphi} \end{cases}$$

so $(0, 0)$ and $(0, \pi)$ are stable nodes since $\mu < \lambda < 0$. Linearisation around $(0, \frac{1}{2} \pi)$ and $(0, \frac{3}{2} \pi)$ yields, respectively,

$$\begin{cases} \bar{r}' = \mu \bar{r} \\ \bar{\varphi}' = \frac{1}{2} F_{yy}(0, 0, 0) \bar{r} + (\lambda - \mu) \bar{\varphi} \end{cases}$$

and

$$\begin{cases} \bar{r}' = \mu \bar{r} \\ \bar{\varphi}' = \frac{1}{2} F_{yy}(0, 0, 0) \bar{r} + (\lambda - \mu) \bar{\varphi} \end{cases}$$

so $(0, \frac{1}{2} \pi)$ or $(0, \frac{3}{2} \pi)$ are saddle points. Since, for sufficiently small $r$, the nonlinearities in system (I) are of class $C^1$, these saddles have both a one-dimensional differential stable manifold which is tangent to the eigenvector

$$\begin{pmatrix} 1, \frac{\pm \frac{1}{2} F_{yy}(0, 0, 0)}{2\mu - \lambda} \end{pmatrix}$$

(this eigenvector is still expressed in polar coordinates).
Near $\varphi = \frac{1}{2}\pi$ the stable manifold is described by the equation $\varphi = \Phi(r)$, where

$$\Phi(r) = \frac{1}{2}\pi + ar + o(r) \quad \text{as} \quad r \to 0^+, \quad a := \frac{F_{yy}(0,0,0)}{2(2\mu - \lambda)}.$$ 

In the original variables this becomes

$$\begin{align*}
x &= r \cos \Phi(r) = r \sin(-ar + o((r)) = -ar^2 + o(r^2) \\
y &= r \sin \Phi(r) = r \cos(ar + o((r)) = r + o(r^2).
\end{align*}$$

So $r = y + o(y^2)$ as $y \to 0^+$ and

$$x = X(y) := -ay^2 + o(y^2) \quad \text{as} \quad y \to 0^+.$$  \hspace{1cm} (58)

Similarly, near $\varphi = \frac{3}{2}\pi$, $\varphi = \Phi_0(r)$, where $\Phi_0(r) = \frac{3}{2}\pi - ar + o(r)$ as $r \to 0^+$. In the original variables this becomes again

$$\begin{align*}
x &= r \cos \Phi_0(r) = r \sin(-ar + o((r)) = -ar^2 + o(r^2) \\
y &= r \sin \Phi_0(r) = -r \cos(ar + o((r)) = -r + o(r^2).
\end{align*}$$

So $r = -y + o(y^2)$ as $y \to 0^-$ and

$$x = X(y) := -ay^2 + o(y^2) \quad \text{as} \quad y \to 0^-.$$  \hspace{1cm} (59)

From (58) and (59) we obtain the one-dimensional differential manifold $S_{\text{fast}}$ of class $C^2$, and the proof is complete.

\[\square\]

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