Stochastic Homogenization for Reaction-Diffusion Equations

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Joint Work with Andrej Zlatoš

June 18, 2018
Motivation: Forest Fires
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1. A PDE

Let \( u(t, x) \) denote the temperature at \( (t, x) \in (0, \infty) \times \mathbb{R}^d \).

\[
\begin{align*}
  & \quad \begin{cases} 
    u_t - \Delta u = f(x, u) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\
    u(0, x) \approx \chi_{\Theta_0} \quad \text{on} \quad \mathbb{R}^d,
  \end{cases} \\
  \text{for} \ \Theta_0 \subseteq \mathbb{R}^d \ \text{open and bounded.}
\end{align*}
\]

For each \( x \in \mathbb{R}^d \), \( f(x, \cdot) \) : Ignition KPP
1. A PDE to Model Combustion

Let $u(t, x)$ denote the temperature at $(t, x) \in (0, \infty) \times \mathbb{R}^d$.

\[
\begin{cases}
  u_t - \Delta u = f(x, u, \omega) & \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\
  u(0, x) = \chi_{\Theta_0} & \text{on} \quad \mathbb{R}^d,
\end{cases}
\]

for $\Theta_0 \subseteq \mathbb{R}^d$ open and bounded. For each $x \in \mathbb{R}^d$, $f(x, \cdot)$:

\[
\begin{align*}
\text{Ignition} & \quad \text{KPP} \\
\end{align*}
\]
2. Conveying a Random Environment

$(\Omega, \mathcal{F}, \mathbb{P})$
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\((\Omega, \mathcal{F}, \mathbb{P})\)

For each \(\omega \in \Omega\), \(f(x, u, \omega)\) satisfies

- \(f(x, u, \omega)\) is an ignition reaction OR KPP reaction,
- \(f_0(u) \leq f(x, u, \omega) \leq f_1(u)\), where \(f_0, f_1 : [0, 1] \rightarrow \mathbb{R}\) are some fixed deterministic, homogeneous reactions of the same type as \(f(x, u, \omega)\).
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Stationarity and Ergodicity (SE):

- \(f(\cdot, u, \cdot)\) is stationary, i.e. there exists a measure-preserving group of transformations \(\{\mathcal{J}_y\}_{y \in \mathbb{R}^d} : Ω \to Ω\) so that for all \(u \in \mathbb{R}\),

\[
f(x + y, u, ω) = f(x, u, \mathcal{J}_y ω).
\]
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\((\Omega, \mathcal{F}, \mathbb{P})\)

For each \(\omega \in \Omega\), \(f(x, u, \omega)\) satisfies

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Stationarity and Ergodicity (SE):

- \(f(\cdot, u, \cdot)\) is stationary, i.e. there exists a measure-preserving group of transformations \(\{\mathcal{I}_y\}_{y \in \mathbb{R}^d} : \Omega \rightarrow \Omega\) so that for all \(u \in \mathbb{R}\),
  \[f(x + y, u, \omega) = f(x, u, \mathcal{I}_y \omega).\]

- \((\Omega, \mathcal{F}, \mathbb{P})\) is ergodic with respect to \(\mathcal{I}_y\). In other words, if there exists an event \(E \in \mathcal{F}\) so that
  \[E = \mathcal{I}_y E \quad \text{for all} \quad y \in \mathbb{R}^d,\]
  then \(\mathbb{P}[E]\) is either 0 or 1.
3. Describing the Asymptotics

\[ u(\cdot, \cdot, \omega) \text{ solves} \]

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\[ u^\varepsilon(t, x, \omega) := u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega\right) \]
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What about the initial condition?

$$u^\varepsilon(0, x, \omega) \approx \chi_{\Theta_{0}}(x)$$
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This implies

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\begin{aligned}
    u_t - \Delta u &= f(x, u, \omega) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\
    u(0, x, \omega) &\approx \chi_{\frac{1}{\varepsilon} \Theta_0}(x) \quad \text{on} \quad \mathbb{R}^d.
\end{aligned}
\]

So initial fire is large compared to the size of the heterogeneities.
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So initial fire is large compared to the size of the heterogeneities

Q: What happens as \( \varepsilon \to 0? \)
Goal of Homogenization

Identify deterministic open sets \( \{ \Theta_t \}_{t>0} \) such that almost surely and locally uniformly away from the boundary \( \Gamma_t := \partial \Theta_t \),

\[
\lim_{\varepsilon \to 0} u^\varepsilon (t, x, \omega) = \begin{cases} 
1 & \text{if } x \in \Theta_t \\
0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Theta_t}.
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\( \{ \Theta_t \}_{t>0} \) represents the effective front propagation taking place on average in the random, heterogeneous environment.
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The normal velocities for the tangent planes.
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   Yes.

Q: What happens if the set \( \Gamma_0 \) has a singularity, or if \( \Gamma_t \) develops a singularity?
   Use Viscosity Solutions interpretation.
Equivalent Goal:

Identify a deterministic function $c^* : \mathbb{S}^{d-1} \to (0, \infty)$ such that almost surely and locally uniformly in space-time (away from certain boundaries),

$$
\lim_{\varepsilon \to 0} u^\varepsilon(t, x, \omega) = \overline{u}(t, x),
$$

where $\overline{u}$ is the unique viscosity solution of

$$
\begin{cases}
\overline{u}_t = c^* \left( - \frac{D\overline{u}}{|D\overline{u}|} \right) |D\overline{u}| & \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\
\overline{u}(0, x) = \chi_{\Theta_0}(x) & \text{on} \quad \mathbb{R}^d.
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c^*(e) = \text{the normal velocity in direction } e \in \mathbb{S}^{d-1} \text{ governing the front propagation}
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$c^*(e)$ is the normal velocity in direction $e \in \mathbb{S}^{d-1}$ governing the front propagation.

Barles, Soner, and Souganidis: $\bar{u}(t, x) = \chi_{\Theta_t}(x)$
Results

Theorem (Lions, Souganidis, ’05)

Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is (SE), \(f(\cdot, \cdot, \omega)\) is KPP. Then for \(\mathbb{P}\)-a.e. \(\omega\), homogenization holds.
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**Approach:**

- KPP Reaction-Diffusion Equations can be compared to solutions of

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- Hopf-Cole transformation: Converts this PDE into a viscous Hamilton-Jacobi equation with a convex Hamiltonian.
- Stochastic homogenization for viscous HJ equations with convex Hamiltonians is well-understood.
Theorem (L., Zlatoš, ’17)

Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is (SE), \(f(\cdot, \cdot, \omega)\) is ignition, \(d \leq 3\), and certain additional assumptions*. Then for \(\mathbb{P}\)-a.e. \(\omega\), homogenization holds.
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Theorem (Lions, Souganidis, ’05; L., Zlatoš, in prep)
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Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is \((SE)\), \(f(\cdot, \cdot, \omega)\) is KPP. Then for \(\mathbb{P}\text{-a.e. } \omega\), homogenization holds.
Why $d \leq 3$?

How can we expect to see a sharp interface ($\bar{u}(t,x) = \chi_{\Theta_t}(x)$) in the limit as $\varepsilon \to 0$?
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We need to control the width of the transition zone.
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1D:

$$\begin{cases} u_t - u_{xx} = f(x, u), \\ u(0, x, \omega) = \chi_{\Theta_0}. \end{cases}$$

For $\eta \in (0, \frac{1}{2})$, let

$$L_{u, \eta}(t) := dist_H \left( \{ x : u(t, x) \geq 1 - \eta \}, \{ x : u(t, x) \geq \eta \} \right)$$
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**1D:**

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$$L_{u, \eta}(t) := \text{dist}_H(\{x : u(t,x) \geq 1 - \eta\}, \{x : u(t,x) \geq \eta\})$$

$$x \to \frac{x}{\varepsilon} \approx xt \quad \Rightarrow \quad L_{u, \eta}(t) \sim o(t).$$
Theorem (Zlatoš, '14)

Let $u$ solve

$$
\begin{aligned}
&u_t - \Delta u = f(x, u) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\
u_t \geq 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d.
\end{aligned}
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If $d \leq 3$, then

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\limsup_{t \to \infty} L_{u, \eta}(t) < \infty.
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In fact, for $d \leq 3$, there exists $C > 0$ such that for $\mathbb{P}$-a.e. $\omega$,

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\limsup_{t \to \infty} L_{u, \eta, \omega}(t) < C.
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In fact, for $d \leq 3$, there exists $C > 0$ such that for $\mathbb{P}$-a.e. $\omega$,

\[
\limsup_{t \to \infty} L_{u, \eta, \omega}(t) < C.
\]

For $d > 3$, this is not in general true! There exist reactions $f(\cdot, \cdot, \omega)$ with $\omega \in \Omega$ such that

\[
L_{u, \eta, \omega}(t) \sim Ct
\]
Main Steps

1. Identify a deterministic candidate $c^* : \mathbb{S}^{d-1} \rightarrow (0, \infty)$.

2. Given $c^*$, show that $u^\varepsilon \rightarrow u$ in the appropriate sense.
Main Steps

1. Identify a deterministic candidate \( c^* : \mathbb{S}^{d-1} \rightarrow (0, \infty) \). How will we relate \( c^* \) to the reaction-diffusion PDE?
2. Given \( c^* \), show that \( u^\varepsilon \rightarrow u \) in the appropriate sense.
Main Steps

1. Identify a deterministic candidate $c^* : S^{d-1} \rightarrow (0, \infty)$. How will we relate $c^*$ to the reaction-diffusion PDE?

2. Given $c^*$, show that $u^\varepsilon \rightarrow u$ in the appropriate sense. This is a completely deterministic PDE argument relying upon the theory of viscosity solutions and generalized front propagation.
Definition: Front Speeds

Fix $e \in \mathbb{S}^{d-1}$, and let $u(\cdot, \cdot, \omega)$ solve

$$
\begin{cases}
    u_t - \Delta u = f(x, u, \omega) & \text{in } (0, \infty) \times \mathbb{R}^d, \\
    u(0, x, \omega) = \chi_{\{x \cdot e \leq 0\}}(x) & \text{on } \mathbb{R}^d.
\end{cases}
$$

The front speed $c^*(e) > 0$ is the deterministic constant such that for $\mathbb{P}$-a.e. $\omega$, for any $K \subseteq \mathbb{R}^d$ compact, for any $\delta > 0$,

$$
\lim_{t \to \infty} \inf_{K \subseteq \{x \cdot e \leq c^*(e) - \delta\}} u(t, xt, \omega) = 1
$$

and

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\lim_{t \to \infty} \sup_{K \subseteq \{x \cdot e \geq c^*(e) + \delta\}} u(t, xt, \omega) = 0.
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Roughly speaking, this says that for \(\mathbb{P}\text{-a.e. } \omega\),

$$u(t, x, \omega) \xrightarrow{t \to \infty} \chi_{\{x \cdot e < c^*(e)t\}}(x)$$
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\]

Observe: Initial data and front speeds are invariant with respect to hyperbolic scaling.
Key Difficulties

- **Heterogeneous Setting.** If the right hand side is $f(u)$, a traveling front with speed $c$ satisfies

\[ u(t, x) = U(x \cdot e - ct) \]

solves the PDE and

\[ \lim_{s \to -\infty} U(s) = 1 \quad \lim_{s \to \infty} U(s) = 0. \]

If $(U, c)$ is a traveling front pair, then $c$ satisfies our definition of front speeds.
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If $(U, c)$ is a traveling front pair, then $c$ satisfies our definition of front speeds.

There is an analogous type of solution (pulsating front) for right hand side $f(x, u)$ when $f(\cdot, u)$ is periodic.
Key Difficulties

- **Heterogeneous Setting.** If the right hand side is \( f(u) \), a traveling front with speed \( c \) satisfies

\[
u(t, x) = U(x \cdot e - ct)
\]

solves the PDE and

\[
\lim_{s \to -\infty} U(s) = 1 \quad \lim_{s \to \infty} U(s) = 0.
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No such solutions exist for general heterogeneous right hand side \( f(x, u) \).
Front-Like Initial Data and Higher Dimensions:

Front speeds in random media in one dimension: Nolen and Ryzhik, Zlatoš
Definition: Spreading Speeds

Fix $e \in S^{d-1}$, and let $u(\cdot, \cdot, \omega)$ solve

$$
\begin{cases}
  u_t - \Delta u = f(x, u, \omega) & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  u(0, x) = \theta_0 \chi_{B_R} & \text{on } \mathbb{R}^d,
\end{cases}
$$

for $R$ sufficiently large. Then we say $w(e)$ is the spreading speed in direction $e$ if for $\mathbb{P}$-a.e. $\omega$, for any $\delta > 0$,

$$
\lim_{t \to \infty} u(t, (w(e) - \delta)te, \omega) = 1,
$$

$$
\lim_{t \to \infty} u(t, (w(e) + \delta)te, \omega) = 0.
$$
First Passage Times for Reaction-Diffusion Equations

Define
\[ \tau(0, y, \omega) := \inf \left\{ t : u(t, x, \omega) \geq \theta_0 \chi_{B_R(y)} \right\}. \]

By the subadditive ergodic theorem, there exists a deterministic \( \bar{\tau}(e) \) such that for \( \mathbb{P}\text{-a.e. } \omega, \)
\[ \lim_{n \to \infty} \frac{\tau(0, ne, \omega)}{n} = \bar{\tau}(e). \]

Then
\[ w(e) := \frac{1}{\bar{\tau}(e)} \]
satisfies the definition of spreading speed.
All Directions at Once: The Wulff Shape

Proposition

Let $u(\cdot, \cdot, \omega)$ solve

$$
\begin{aligned}
&u_t - \Delta u = f(x, u, \omega) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\
&u(0, x) = \Theta_0 \chi_{B_R} \quad \text{on} \quad \mathbb{R}^d,
\end{aligned}
$$

for $R$ sufficiently large. Define

$$
S := \{se : 0 \leq s \leq w(e)\},
$$

a convex set. For $\mathbb{P}$-a.e. $\omega$, for every $\delta > 0$, for $t$ sufficiently large,

$$
(1 - \delta) tS \subseteq \left\{ x : u(t, x, \omega) = \frac{1}{2} \right\} \subseteq (1 + \delta) tS.
$$
All Directions at Once: The Wulff Shape

Proposition

Let \( u(\cdot, \cdot, \omega) \) solve

\[
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  u_t - \Delta u &= f(x, u, \omega) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, \\
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\]

Question: How do we move from a speed for compactly-supported initial data to a speed for half-space initial data?
Recovery of Front Speeds

In the periodic setting, Freidlin-Gärtner formula says:

\[ w(e) = \inf_{e' \in \mathbb{S}^{d-1}, \ e' \cdot e > 0} \frac{c^*(e')}{e' \cdot e} \]
Recovery of Front Speeds

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For us, we do not have front speeds, but we DO have spreading speeds!
Recovery of Front Speeds

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For us, we do not have front speeds, but we DO have spreading speeds! Let

\[ c^*(e) := \sup_{e' \in \mathbb{S}^{d-1}, e' \cdot e > 0} w(e') e' \cdot e \]

The additional assumptions* guarantee that the Wulff Shape \( S \) has no corners, so it has tangents in all directions. This is enough to show that \( c^*(e) \) defined in this way is the front speed.
Example where Homogenization Holds: Isotropic Environment

(I) The random environment is isotropic. This guarantees that $\mathbb{P}$ is invariant with respect to rotations in physical space.
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**Canonical Example: Poisson Point Process**

Let $\mathcal{P}(\omega) := \{x_n(\omega)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d$ denote a collection of points distributed by a Poisson point process with intensity 1. Then we have

$$f(x, u, \omega) \approx f_1(u) \chi_{B_1(\mathcal{P}(\omega))} + f_0(u)(1 - \chi_{B_1(\mathcal{P}(\omega))})$$
Common Theme: Convexity

Let

\[ \overline{H}(p) := c^* \left( \frac{p}{|p|} \right) |p|. \]

- For all solvable cases of stochastic homogenization for reaction-diffusion equations (solvable ignition and all KPP), \( \overline{H}(p) \) is convex.

- For the stochastic homogenization of Hamilton-Jacobi equations, there are counterexamples to homogenization when the random Hamiltonians are nonconvex (Ziliotto ['16], Feldman-Souganidis ['16]).

- For general ignition, will likely need to strengthen some assumptions to obtain general homogenization.
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- Can we impose stronger assumptions on the random environment (finite range of dependence) to eliminate some of the restrictions (dimension, no corners on Wulff shape, etc.)?
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- Can we extend to time-dependent reactions? More general coefficients?
- Can we quantify the convergence in these statements? In particular, can we quantify the fluctuations to the front-like interface?
Thank you very much for your attention!