Some properties for the Steklov averages

J. Q. CHAGAS,¹ N. M. L. DIEHL,² AND P. L. GUIDOLIN³

¹Departamento de Matemática e Estatística
Universidade Estadual de Ponta Grossa
Ponta Grossa, PR 84030-900, Brazil

²Instituto Federal de Educação, Ciência e Tecnologia do Rio Grande do Sul
Canoas, RS 92412-240, Brazil

³Instituto Federal de Educação, Ciência e Tecnologia do Rio Grande do Sul
Viamão, RS, 94410-970, Brazil

Abstract

We derive and present a collection of properties about the Steklov averages, including some results about the derivation with respect to spatial variables, and with respect to time, and a form of the fundamental theorem of the calculus.

1 Introduction

In this work we’ll derive and present a collection of properties for the Steklov averages, which are an important regularization technique used currently in study of PDE’s theory, but let us start by some brief notes about this mathematical tool and its proponent.

The Steklov average (or Steklov mean function) was introduced by V. A. Steklov in 1907 (see [8]) for the study of the problem of expanding a given function into a series of eigenvalues defined by a 2nd-order ordinary differential operator; and its definition appears in §67 of [1] (along with some properties in §83). We reproduce this definition here:

Suppose the function $f(t)$, defined along the entire real axis, belongs to $L(a, b)$, for all finite values of $a, b$. Given any positive $h$, let us now construct the function

$$f_h(t) = \frac{1}{h} \int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f(u) \, du = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(t+v) \, dv.$$

Vladimir Andreevich Steklov (1864 - 1926) was an outstanding Russian mathematician who made many important contributions to mathematical physics (the Steklov average is only one of the mathematical notions associated with his name). Moreover, in 1921 Steklov founded the Physical-Mathematical Institute in Petrograd. Today, a famous institute of
mathematics in Moscow has Steklov’s name. More about life and work of V. A. Steklov can be seen in \[5\], \[6\] and \[7\].

In the present-day, Steklov averages are a very useful starting point for the derivation of a number of important solution properties for the PDE’s (see \[4\], \[9\] or \[10\]). In \[4\], the Steklov average is used to define a local weak solution that involves the time derivative \(u_t\) for quasilinear degenerate or singular parabolic equations, widely used to derive solution properties; in \[10\], it’s used in Chapter 2, which treats the non-newtonian filtration equations; and in \[9\] it’s used to define a weak solution for its central example, the parabolic p-Laplacian equation, because is proper to expose the idea of intrinsic scaling. Recent uses of the Steklov Average in treatment of PDE’s problems can be seen in \[2\] and \[3\].

However, only some properties of the Steklov averages are readily found in the literature. For this reason, in this work we’ve proposed to obtain and present a collection of important and useful operational properties for the Steklov averages.

Here is a brief description of what follows. In section 2, we present some convergence results for the Steklov average. In the section 3, we present the pointwise value form for the Steklov average. In section 4, we present some properties about the differentiability of the Steklov average, for spatial variables, and with respect to time \(t\). Finally, in the section 5, we present some properties about the integration of the Steklov average, including forms of the fundamental theorem of calculus and integration by parts.

## 2 About the convergence of Steklov averages

We’ll start this section presenting the definition of the Steklov average of a function. In the sequence, we’ll present results about the convergence of such averages.

The space of the measurable sets in \(\mathbb{R}^n\) it will be denoted by \(\mathcal{M}(\mathbb{R}^n)\).

**Definition 2.1.** Let \([a, b]\) a compact interval in \(\mathbb{R}\), \(E \in \mathcal{M}(\mathbb{R}^n)\) and \(1 \leq q \leq \infty\). Given \(v(\cdot, t) \in C^0([a, b], L^q(E))\), we define (for each \(0 < h < b - a\)) the **Steklov average** \(v_h(\cdot, t)\) of an function \(v\) by

\[
v_h(\cdot, t) = \frac{1}{h} \int_t^{t+h} v(\cdot, s) \, ds, \quad \text{for} \quad a \leq t \leq b - h.
\]  

(2.1)

The first result is presented in:

**Lemma 2.2.** Given \(v(\cdot, t) \in C^0([a, b], L^q(E))\), we have

\[
v_h(\cdot, t) \in C^0([a, b - h], L^q(E))
\]

\(\text{Moreover, } v_h(\cdot, t) \text{ is Lipschitz continuous in the interval } [a, b - h]\).
and, for each \(a \leq t < b\), we have

\[
\|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)} \to 0, \quad \text{when } h \to 0,
\]

uniformly in \(t \in [a, b - \varepsilon]\), for each \(0 < \varepsilon < h\).

**Proof.** Let’s prove this lemma in three assertions.

**Assertion (i) :** \(v_h(\cdot, t) \in L^q(E), \forall t \in [a, b - h]\).

Indeed, given \(a \leq t \leq b - h\), we have

\[
\|v_h(\cdot, t)\|_{L^q(E)} \leq \frac{1}{h} \left\| \int_t^{t+h} v(\cdot, s) \, ds \right\|_{L^q(E)} \leq \frac{1}{h} \int_t^{t+h} \|v(\cdot, s)\|_{L^q(E)} \, ds \\
\leq \frac{1}{h} \int_a^b \|v(\cdot, s)\|_{L^q(E)} \, ds \leq \infty,
\]

so that \(\|v_h(\cdot, t)\|_{L^q(E)} \leq M_h, \forall t \in [a, b - h]\), where \(M_h = \frac{1}{h} \int_a^b \|v(\cdot, s)\|_{L^q(E)} \, ds\).

**Assertion (ii) :** \(v_h(\cdot, t) \in C^0([a, b - h], L^q(E))\).

Let \(M := \max_{a \leq t \leq b} \|v(\cdot, t)\|_{L^q(E)}\). Then, given \(t_1 < t_2 \in [a, b - h]\), with \(t_2 - t_1 \leq h\), we have

\[
v_h(\cdot, t_2) - v_h(\cdot, t_1) = \frac{1}{h} \int_{t_2}^{t_2+h} v(\cdot, t) \, dt - \frac{1}{h} \int_{t_1}^{t_1+h} v(\cdot, t) \, dt \\
= \frac{1}{h} \int_{t_1+h}^{t_2+h} v(\cdot, t) \, dt - \frac{1}{h} \int_{t_1}^{t_2} v(\cdot, t) \, dt,
\]

and thus

\[
\|v_h(\cdot, t_2) - v_h(\cdot, t_1)\|_{L^q(E)} \leq \frac{1}{h} \int_{t_2}^{t_2+h} \|v(\cdot, t)\|_{L^q(E)} \, dt + \frac{1}{h} \int_{t_1}^{t_1+h} \|v(\cdot, t)\|_{L^q(E)} \, dt \\
\leq \frac{1}{h} \int_{t_1+h}^{t_2+h} M \, dt + \frac{1}{h} \int_{t_1}^{t_2} M \, dt = \frac{2}{h} M |t_2 - t_1|.
\]

**Assertion (iii) :** \(v_h(\cdot, t) \to v(\cdot, t)\) in \(L^q(E)\) as \(h \to 0\), for each \(t \in [a, b]\).

Indeed, for \(T \in [a, b]\), we define \(\varepsilon_T := b - T > 0\) and take \(0 < h < \varepsilon_T\). Given \(\varepsilon > 0\), let \(\delta > 0\) such that

\[
\|v(\cdot, t) - v(\cdot, s)\|_{L^q(E)} \leq \varepsilon, \quad \forall s, t \in [a, b] \text{ with } |s - t| \leq \delta \leq \varepsilon_T.
\]
Then, for each $0 < h \leq \delta$ and $\forall t \in [a, T]$, we obtain

$$
\|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)} \leq \left\| \frac{1}{h} \int_t^{t+h} \bar{v}(\cdot, s) \, ds - \frac{1}{h} \int_t^{t+h} v(\cdot, t) \, ds \right\|_{L^q(E)}
\leq \frac{1}{h} \int_t^{t+h} \|\bar{v}(\cdot, s) - v(\cdot, t)\|_{L^q(E)} \, ds
\leq \frac{1}{h} \int_t^{t+h} \varepsilon \, ds = \varepsilon,
$$

i.e., for all $t \in [a, T]$ we have that

$$
\|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)} \leq \varepsilon, \quad \forall \ 0 < h \leq \delta.
$$

(Lema 2.2) □

For the proof of the next lemma, it’s convenient define $v_h(\cdot, t)$ as follows:

\begin{definition}
Let $I \subset \mathbb{R}$ any interval, $E \in \mathcal{M}(\mathbb{R}^n)$, $1 \leq q, r \leq \infty$ and $h > 0$. Given $v(\cdot, t) \in L^r(I, L^q(E))$ we define $v_h(\cdot, t) \in C^0(I, L^q(E))$ by

$$
v_h(\cdot, t) = \frac{1}{h} \int_t^{t+h} \bar{v}(\cdot, s) \, ds, \quad \text{for each } a \leq t \leq b-h,
$$

where $\bar{v}(\cdot, t) \in L^r(\mathbb{R}, L^q(E))$ is defined by

$$
\bar{v}(\cdot, t) = \begin{cases} 
v(\cdot, t), & \text{se } t \in I, \\
0, & \text{se } t \in \mathbb{R} \setminus I. 
\end{cases}
$$

(2.3a)
(2.3b)
\end{definition}

\begin{lemma}
Given an interval $I \subset \mathbb{R}$, $E \in \mathcal{M}(\mathbb{R}^n)$, $1 \leq q, r \leq \infty$ and $h > 0$, let $v(\cdot, t) \in L^r(I, L^q(E))$. Then, $v_h(\cdot, t)$ as defined in (2.3a)-(2.3b) satisfies:

$$
v_h(\cdot, t) \in L^q(E), \quad \forall \ t \in I,
$$

(2.4a)

with

$$
\|v_h(\cdot, t)\|_{L^r(E)} \leq \frac{1}{h^{1/r}} \|v(\cdot, t)\|_{L^r(I, L^q(E))}, \quad \text{if } r \neq \infty,
\|v_h(\cdot, t)\|_{L^q(E)} \leq \|v(\cdot, t)\|_{L^q(I, L^q(E))}, \quad \text{if } r = \infty,
$$

$$
v_h(\cdot, t) \in C^0(I, L^q(E)) \cap L^\infty(I, L^q(E)),
$$

(2.4b)

$$
v_h(\cdot, t) : I \to L^q(E) \text{ is uniformly continuous on } I.
$$

(2.4b')

(\text{and, if } r = \infty, \text{ then } v_h(\cdot, t) : I \to L^q(E) \text{ is Lipschitz continuous on } I),
$$

(4)
\[ v_h(\cdot, t) \in L^r(I, L^q(E)), \quad (2.4c) \]

and

\[ \|v_h\|_{L^r(I, L^q(E))} \leq \|v\|_{L^r(I, L^q(E))}. \]

**Proof.**

**Proof of (2.4a):** If \( r = \infty \), for each \( t \in I \) we have

\[
\|v(\cdot, t)\|_{L^q(E)} \leq \frac{1}{h} \int_t^{t+h} \|\tilde{v}(\cdot, s)\|_{L^q(E)} ds \\
\leq \frac{1}{h} \int_t^{t+h} \|v(\cdot, s)\|_{L^q(E)} ds = \|v\|_{L^q(I, L^q(E))}.
\]

If \( 1 \leq r < \infty \), for each \( t \in I \) we have

\[
\|v(\cdot, t)\|_{L^q(E)} \leq \frac{1}{h} \int_t^{t+h} \|\tilde{v}(\cdot, s)\|_{L^q(E)} ds \leq \frac{1}{h^{1-1/r}} \left( \int_t^{t+h} \|\tilde{v}(\cdot, s)\|_{L^q(E)}^r ds \right)^{1/r} \\
= \frac{1}{h^{1/r}} \left( \int_t^{t+h} \|\tilde{v}(\cdot, s)\|_{L^q(E)}^r ds \right)^{1/r} \leq \frac{1}{h^{1/r}} \left( \int_t^{t+h} \|v(\cdot, s)\|_{L^q(E)}^r ds \right)^{1/r}
\]

i.e.,

\[
\|v(\cdot, t)\|_{L^q(E)} \leq \frac{1}{h^{1/r}} \|v\|_{L^r(I, L^q(E))} \quad \forall t \in I.
\]

In particular, for \( 1 \leq r \leq \infty \) we have \( v_h(\cdot, t) \in L^\infty(I, L^q(E)) \).

**Proof of (2.4b) and (2.4b):** If \( r = \infty \), for each \( t_1 < t_2 \in I \) with \( |t_2 - t_1| < h \), we have

\[
v_h(\cdot, t_2) - v_h(\cdot, t_1) = \frac{1}{h} \int_{t_1}^{t_2} \tilde{v}(\cdot, s) ds - \frac{1}{h} \int_{t_1}^{t_2} \tilde{v}(\cdot, s) ds \\
= \frac{1}{h} \int_{t_1}^{t_2} \tilde{v}(\cdot, s) ds - \frac{1}{h} \int_{t_1}^{t_2} \tilde{v}(\cdot, s) ds
\]

therefore

\[
\|v_h(\cdot, t_2) - v_h(\cdot, t_1)\|_{L^q(E)} \leq \frac{1}{h} \int_{t_1}^{t_2} \|\tilde{v}(\cdot, s)\|_{L^q(E)} ds + \frac{1}{h} \int_{t_1}^{t_2} \|\tilde{v}(\cdot, s)\|_{L^q(E)} ds \\
\leq \frac{1}{h} \int_{t_1}^{t_2} M ds + \frac{1}{h} \int_{t_1}^{t_2} M ds = \frac{2}{h} M|t_2 - t_1|,
\]

where \( M = \|v\|_{L^\infty(I, L^q(E))} \). This shows that

\[
\|v_h(\cdot, t) - v_h(\cdot, s)\|_{L^q(E)} \leq \frac{2}{h} M|t - s|, \quad \forall s, t \in I.
\]
If $1 \leq r < \infty$, for any $t_1 < t_2 \in I$ with $|t_2 - t_1| \leq h$, we have that
\[
v_h(\cdot, t_2) - v_h(\cdot, t_1) = \frac{1}{h} \int_{t_1}^{t_2 + h} \bar{v}(\cdot, s) \, ds - \frac{1}{h} \int_{t_1}^{t_1 + h} \bar{v}(\cdot, s) \, ds
\]
\[
= \frac{1}{h} \int_{t_1 + h}^{t_2 + h} \bar{v}(\cdot, s) \, ds - \frac{1}{h} \int_{t_1}^{t_2} \bar{v}(\cdot, s) \, ds
\]
as before, but now follows that
\[
\|v_h(\cdot, t_2) - v_h(\cdot, t_1)\|_{L^q(E)} \leq \frac{1}{h} \int_{t_1 + h}^{t_2 + h} \|\bar{v}(\cdot, s)\|_{L^q(E)} \, ds + \frac{1}{h} \int_{t_1}^{t_2} \|\bar{v}(\cdot, s)\|_{L^q(E)} \, ds
\]
\[
\leq \frac{1}{h^{1/r}} \left( \int_{t_1 + h}^{t_2 + h} \|\bar{v}(\cdot, s)\|^r_{L^q(E)} \, ds \right)^{1/r} + \frac{1}{h^{1/r}} \left( \int_{t_1}^{t_2} \|\bar{v}(\cdot, s)\|^r_{L^q(E)} \, ds \right)^{1/r}.
\]
As $\|\bar{v}(\cdot, s)\|_{L^q(E)} \in L^1(\mathbb{R})$, given $\varepsilon > 0$, there exists $\delta > 0$ such that
\[
\int_I \|\bar{v}(\cdot, s)\|^r_{L^q(E)} \, ds \leq \varepsilon^r
\]
whenever $J \in \mathcal{M}(\mathbb{R})$ and $|J| \leq \delta$.

Therefore, for any $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \delta$, we obtain
\[
\|v_h(\cdot, t_2) - v_h(\cdot, t_1)\|_{L^q(E)} \leq \frac{1}{h^{1/r}} (\varepsilon^r)^{1/r} + \frac{1}{h^{1/r}} (\varepsilon^r)^{1/r} = \frac{2}{h^{1/r}} \varepsilon.
\]
This shows that $v_h(\cdot, t) : I \to L^q(E)$ is uniformly continuous in $I$.

As $v_h(\cdot, t) : I \to L^q(E)$ is also limited in $I$, for each $1 \leq q, r \leq \infty$, we have that $v_h(\cdot, t) : I \to L^q(E)$ is limited and uniformly continuous in $I$; and for $1 \leq q \leq \infty, r = \infty$, we have that $v_h(\cdot, t) : I \to L^q(E)$ is limited and globally Lipschitz in $I$.

**Proof of (2.4c):** The case where $r = \infty$ already been shown in (2.4a). Consider then $1 \leq r < \infty$. From (2.3a) we obtain, $\forall \, t \in I$, that
\[
\|v_h(\cdot, t)\|_{L^q(E)} \leq \frac{1}{h} \int_t^{t+h} \|\bar{v}(\cdot, s)\|_{L^q(E)} \, ds \leq \frac{1}{h^{1/r}} \left( \int_t^{t+h} \|\bar{v}(\cdot, s)\|^r_{L^q(E)} \, ds \right)^{1/r}.
\]
Therefore
\[
\|v_h(\cdot, t)\|^r_{L^q(E)} \leq \frac{1}{h} \left( \int_t^{t+h} \|\bar{v}(\cdot, s)\|^r_{L^q(E)} \, ds \right), \quad \forall \, t \in I,
\]
and thus, if $I = [a, b], (a, b], [a, b)$, or $(a, b)$, for $-\infty < a < b < \infty$, follows that
\[
\int_I \|v_h(\cdot, t)\|^r_{L^q(E)} \, dt = \frac{1}{h} \int_a^b \left( \int_t^{t+h} \|\bar{v}(\cdot, s)\|^r_{L^q(E)} \, ds \right) \, dt
\]
\[
= \frac{1}{h} \int_a^b (V(t + h) - V(t)) \, dt
\]
\[
\begin{align*}
V(t) &:= \int_a^t \|v(\cdot, s)\|_{L^r(E)} \, ds, \quad \forall \, t \geq a; \text{ i.e., when } I \text{ is bounded, we have} \\
\int_I \|v_h(\cdot, t)\|_{L^r(E)} \, dt &\leq \int_I \|v(\cdot, t)\|_{L^r(E)} \, dt. \quad (2.4d)
\end{align*}
\]

We'll now extend (2.4d) for the cases \((-\infty, b), (-\infty, b]\); and \((a, \infty), [a, \infty)\); with \(a, b \in \mathbb{R}\).

In the cases \((-\infty, b)\) and \((-\infty, b]\), we obtain

\[
\int_I \|v_h(\cdot, t)\|_{L^r(E)} \, dt = \lim_{a \to -\infty} \int_a^b \|v_h(\cdot, t)\|_{L^r(E)} \, dt \leq \lim_{a \to -\infty} \int_a^b \|v(\cdot, t)\|_{L^r(E)} \, dt \\
= \int_{-\infty}^b \|v(\cdot, t)\|_{L^r(E)} \, dt = \int_I \|v(\cdot, t)\|_{L^r(E)} \, dt,
\]

and in the cases \((a, \infty)\) and \([a, \infty)\) we obtain

\[
\int_I \|v_h(\cdot, t)\|_{L^r(E)} \, dt = \lim_{b \to \infty} \int_a^b \|v_h(\cdot, t)\|_{L^r(E)} \, dt \leq \lim_{b \to \infty} \int_a^{b+h} \|v(\cdot, t)\|_{L^r(E)} \, dt \\
= \int_a^\infty \|v(\cdot, t)\|_{L^r(E)} \, dt = \int_I \|v(\cdot, t)\|_{L^r(E)} \, dt,
\]

where the inequality is obtained using \(V(t)\), again, and the fact that

\[
\frac{1}{h} \int_b^{b+h} V(t+h) \, dt \leq \frac{1}{h} \int_b^{b+h} V(b+h) \, dt = V(b+h) = \int_a^{b+h} \|v(\cdot, t)\|_{L^r(E)} \, dt.
\]

Finally, for the case \(I = \mathbb{R}\), we have

\[
\int_I \|v_h(\cdot, t)\|_{L^r(E)} \, dt = \lim_{b \to \infty} \int_a^b \|v_h(\cdot, t)\|_{L^r(E)} \, dt \leq \lim_{b \to \infty} \int_a^{b+h} \|v(\cdot, t)\|_{L^r(E)} \, dt \\
= \int_{-\infty}^\infty \|v(\cdot, t)\|_{L^r(E)} \, dt = \int_I \|v(\cdot, t)\|_{L^r(E)} \, dt.
\]

Thus, for each interval \(I \subset \mathbb{R}\), we have

\[
\int_I \|v_h(\cdot, t)\|_{L^r(E)} \, dt \leq \int_I \|v(\cdot, t)\|_{L^r(E)} \, dt.
\]

\((\text{Lemma } 2.4) \) □

The proof of the next lemma requires \(1 \leq r < \infty\).
Lemma 2.5. Given any interval $I \subset \mathbb{R}$, $E \in \mathcal{M}(\mathbb{R}^n)$, $1 \leq q \leq \infty$, $1 \leq r < \infty$ and $h > 0$, take $v(\cdot, t) \in L^r(I, L^q(E))$ and consider $v_h(\cdot, t) \in C^0(I, L^q(E)) \cap L^\infty(I, L^q(E))$ (see Lemma 2.4).

Then, when $h \to 0$ we have

$$v_h \to v \text{ in } L^r(I, L^q(E)), \tag{2.5a}$$

and

$$\int_I \|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)}^r dt \to 0. \tag{2.5a'}$$

Proof. Let $\bar{I}$ the interior of the set $I$.

Initially, let’s assume that $v(\cdot, t) \in C^0_c(\bar{I}, L^q(E))$, that is, $v(\cdot, t) \in C^0(I, L^q(E))$, with $v(\cdot, t) = 0 \ \forall \ t \in I \setminus [a, b]$, for some compact interval $[a, b] \subset \bar{I}$.

Then, we have that

$$\|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)} \to 0$$

uniformly in $t \in I$ as $h \to 0$, as has been proved in the item (2.2) of Lemma 2.2.

Indeed, let a compact interval $[a, b] \subset \bar{I}$ containing the support of $v(\cdot, t)$, and take a compact interval $[\alpha, \beta] \subset \bar{I}$ with $\alpha < a$ and $\beta > b$. For each $h > 0$ with

$$h \leq \min\{a - \alpha, \beta - b\} := \bar{h},$$

we have that $v_h(\cdot, t) = 0 = v(\cdot, t)$ if $t \in I$ satisfies $t \leq \alpha$ or $t \geq \beta$, and therefore,

$$\|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)} = 0 \ \forall \ 0 < h < \bar{h},$$

for each $t \in I$ with $t \leq \alpha$ or $t \geq \beta$.

On the other hand, for $t \in [\alpha, \beta]$, we may proceed as follows: given $\varepsilon > 0$, let $\delta > 0$ be small enough for that

$$\|v(\cdot, s) - v(\cdot, r)\| \leq \varepsilon \ \forall \ s, r \in [\alpha, \beta]$$

with $|s - r| \leq \delta$ and $\delta \leq \bar{h}$. Then, for every $t \in [\alpha, \beta]$ and $\forall \ 0 < h < \delta \leq \bar{h}$, we have that

$$\|v(\cdot, s) - v(\cdot, r)\|_{L^q(E)} \leq \frac{1}{h} \int_t^{t+h} \|v(\cdot, s) - v(\cdot, r)\|_{L^q(E)} ds \leq \frac{1}{h} \int_t^{t+h} \varepsilon \ ds = \varepsilon.$$

For $v(\cdot, t) \in C^0_c(\tilde{I}, L^q(E))$ this shows that

$$\int_I \|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)}^r = \int_0^\beta \|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)}^r \to 0 \text{ when } h \to 0, \tag{2.5b}$$

which results in (2.5a) e (2.5a'), in the case where $v(\cdot, t) \in C^0_c(\tilde{I}, L^q(E))$. 

8
In the general case \( v(\cdot, t) \in L^r(I, L^q(E)) \), we may proceed as follows: given \( \varepsilon > 0 \), we can take \( w(\cdot, t) \in C^0_c(\tilde{I}, L^q(E)) \) such that
\[
\int_I \| v(\cdot, t) - w(\cdot, t) \|^r_{L^q(E)} \leq \left( \frac{\varepsilon}{3} \right)^r,
\]
(this follows because \( C^\infty_c(\tilde{I}, L^q(E)) \) is dense in \( L^r(I, L^q(E)) \), \( \forall 1 \leq r < \infty \), i.e.,
\[
\| v - w \|_{L^r(I, L^q(E))} \leq \frac{\varepsilon}{3}.
\]
In particular, by (2.4e) of Lemma 2.4, we also have that \( \| v_h - w_h \|_{L^r(I, L^q(E))} \leq \frac{\varepsilon}{3} \).

Therefore, we have
\[
\| v_h - v \|_{L^r(I, L^q(E))} \leq \| v_h - w_h \|_{L^r(I, L^q(E))} + \| w_h - w \|_{L^r(I, L^q(E))} + \| w - v \|_{L^r(I, L^q(E))} \\
\leq \frac{\varepsilon}{3} + \| w_h - w \|_{L^r(I, L^q(E))} + \frac{\varepsilon}{3}.
\]

As \( w(\cdot, t) \in C^0_c(\tilde{I}, L^q(E)) \), from (2.5b), we know that \( \| w_h - w \|_{L^r(I, L^q(E))} \to 0 \) as \( h \to 0 \).

Therefore, taking \( h_0 > 0 \) small enough for occurs
\[
\| w_h - w \|_{L^r(I, L^q(E))} \leq \frac{\varepsilon}{3} \quad \forall 0 < h \leq h_0,
\]
we obtain that \( \| v_h - v \|_{L^r(I, L^q(E))} \leq \varepsilon, \quad \forall 0 < h \leq h_0. \)  

(Lemma 2.5) \( \square \)

Finally, we can show the next Lemma (since \( 1 \leq r < \infty \)), which has the harder proof to obtain among the presented results until here.

**Lemma 2.6.** Given \( I \subset \mathbb{R} \) (any interval), \( E \in \mathcal{M}(\mathbb{R}^n) \), \( 1 \leq q \leq \infty, 1 \leq r < \infty, h > 0 \), and \( v(\cdot, t) \in L^r(I, L^q(E)) \) arbitrary, consider \( v_h \in C^0(I, L^q(E)) \cap L^\infty(I, L^q(E)) \) defined as in (2.3a).

Then, there exists \( Z \subset I \) with zero measure such that, for each \( t \in I \setminus Z \), we have
\[
v(\cdot, t) \in L^q(E), \quad \text{and}
\]
\[
\| v_h(\cdot, t) - v(\cdot, t) \|_{L^q(E)} \to 0 \quad \text{as} \; h \to 0.
\]

**Proof.** Given \( v(\cdot, t) \in L^r(I, L^q(E)) \), with \( 1 \leq r < \infty \), we take a sequence of smooth approximations \( w_m(\cdot, t) \in C^0_c(\tilde{I}, L^q(E)) \), \( \forall m \in \mathbb{N} \), such that
\[
\| w_m - v \|_{L^r(I, L^q(E))} \to 0, \quad \text{as} \; m \to \infty,
\]
(2.6a)
and (passing to a subsequence, if necessary)

$$
\|w_m(\cdot, t) - v(\cdot, t)\|_{L^q(E)} \to 0, \quad m \to \infty, \tag{2.6a'}
$$

for each \( t \in I \setminus Z_0 \), with \( Z_0 \subseteq I \) of zero measure.

Observe that, because \( v(\cdot, t) \in L^r(I, L^q(E)) \), there exists \( Z_v \subseteq I \) with zero measure such that \( v(\cdot, t) \in L^q(E), \forall t \in I \setminus Z_v \). The null set \( Z_0 \) in (2.6a) satisfies, in particular, \( Z_v \subseteq Z_0 \).

For each \( m \geq 1 \), by (2.6a) we have that \( \|w_m(\cdot, t) - v(\cdot, t)\|_{L^q(E)} \in L^r(I) \), so that, by Hölder’s inequality, we have

$$
\|w_m(\cdot, t) - v(\cdot, t)\|_{L^q(E)} \in L^1_{loc}(I). \tag{2.6b}
$$

By Lebesgue’s differentiation theorem, there exists \( Z_m \subseteq I \), with \( |Z_m| = 0 \), and \( Z_v \subseteq Z_m \) such that

$$
\lim_{m \to 0} \frac{1}{h} \int_t^{t+h} \|w_m(\cdot, s) - v(\cdot, s)\|_{L^q(E)} ds = \|w_m(\cdot, t) - v(\cdot, t)\|_{L^q(E)}, \forall t \in I \setminus Z_m. \tag{2.6b}
$$

Let then \( Z := Z_0 \cup \left( \bigcup_{m=1}^{\infty} Z_m \right) \). Thus, we have \( Z_v \subseteq Z \subseteq I \), with \( Z \) having zero measure.

We claim that, for each \( t \in I \setminus Z \), we have

$$
\lim_{m \to 0} \|v_h(\cdot, t) - v(\cdot, t)\|_{L^q(E)} = 0. \tag{2.6c}
$$

Indeed, given \( \hat{t} \in I \setminus Z \) and \( \varepsilon > 0 \), we may proceed as follows: take \( m_0 \geq 1 \) big enough so that

$$
\|w_{m_0}(\cdot, \hat{t}) - v(\cdot, \hat{t})\|_{L^q(E)} \leq \frac{\varepsilon}{4}. \tag{2.6d}
$$

Let \( w(\cdot, t) \in C^0_c(\hat{t}, L^q(E)) \) be given by \( w(\cdot, t) = w_{m_0}(\cdot, t) \forall t \in I \). By (2.6b), since \( \hat{t} \in I \setminus Z_{m_0} \), we have

$$
\frac{1}{h} \int_{\hat{t}}^{\hat{t}+h} \|w(\cdot, s) - v(\cdot, s)\|_{L^q(E)} ds \to \|w(\cdot, \hat{t}) - v(\cdot, \hat{t})\|_{L^q(E)} \leq \frac{\varepsilon}{4}. \tag{2.6d}
$$

Hence, there exists \( h_\varepsilon > 0 \) (by (2.6d)) small enough that we have

$$
\frac{1}{h} \int_{\hat{t}}^{\hat{t}+h} \|w(\cdot, s) - v(\cdot, s)\|_{L^q(E)} ds \leq \frac{\varepsilon}{2}, \forall 0 < h \leq h_\varepsilon.
$$
Therefore, for any $0 < h \leq h_\varepsilon$, this gives
\[
\left\| w_h(\cdot, \hat{t}) - v_h(\cdot, \hat{t}) \right\|_{L^q(E)} = \frac{1}{h} \left\| \int_0^{\hat{t}} (w(\cdot, s) - v(\cdot, s)) \, ds \right\|_{L^q(E)} \\
\leq \frac{1}{h} \int_0^{\hat{t}} \left\| w(\cdot, s) - v(\cdot, s) \right\|_{L^q(E)} \, ds \leq \frac{\varepsilon}{2}
\]

Therefore, for all $h \in [0, h_\varepsilon]$, we have
\[
\left\| v_h(\cdot, \hat{t}) - v(\cdot, \hat{t}) \right\|_{L^q(E)} \leq \left\| v_h(\cdot, \hat{t}) - w_h(\cdot, \hat{t}) \right\|_{L^q(E)} + \left\| w_h(\cdot, \hat{t}) - w(\cdot, \hat{t}) \right\|_{L^q(E)} \\
\quad + \left\| w(\cdot, \hat{t}) - v(\cdot, \hat{t}) \right\|_{L^q(E)} 
\]
\[
\leq \frac{\varepsilon}{2} + \left\| w_h(\cdot, \hat{t}) - w(\cdot, \hat{t}) \right\|_{L^q(E)} + \frac{\varepsilon}{4}. \tag{2.6e}
\]

Because $w(\cdot, t) \in C^0_c(\hat{I}, L^q(E))$, we clearly have
\[
\lim_{h \to 0} \left\| w_h(\cdot, t) - w(\cdot, t) \right\|_{L^q(E)} = 0, \quad \text{for each } t \in I, \tag{2.6f}
\]
and hence, there exists $h_{\varepsilon\varepsilon} < 1$ such that
\[
\left\| w_h(\cdot, \hat{t}) - w(\cdot, \hat{t}) \right\|_{L^q(E)} \leq \frac{\varepsilon}{4}, \quad \forall \ 0 < h < h_{\varepsilon\varepsilon}. \tag{2.6f'}
\]

From (2.6e), (2.6f'), $\forall \ 0 < h \leq \min\{h_\varepsilon, h_{\varepsilon\varepsilon}\}$, we get
\[
\left\| v_h(\cdot, \hat{t}) - v(\cdot, \hat{t}) \right\|_{L^q(E)} \leq \frac{\varepsilon}{2} + \left\| w_h(\cdot, \hat{t}) - w(\cdot, \hat{t}) \right\|_{L^q(E)} + \frac{\varepsilon}{4} \\
\quad \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \quad \forall \ 0 < h \leq \min\{h_\varepsilon, h_{\varepsilon\varepsilon}\}.
\]

This shows (2.6c) for $t = \hat{t}$ (with $\hat{t} \in I \setminus Z$ arbitrary), as claimed. \hfill (Lemma 2.6)

### 3 Pointwise values of the Steklov averages

Regarding the **pointwise** values of $v_h(\cdot, t) \in C^0(I, L^q(E)) \cap L^\infty(I, L^q(E))$ (for a given $v(\cdot, t) \in L^r(I, L^q(E))$, where $1 \leq q, r \leq \infty$), we have, by Fubini’s theorem, that the following result holds:

**Lemma 3.1.** For each $J \subseteq I$, where $J$ is a bounded interval, there exists $Z \subseteq E$, with $|Z| = 0 \ (Z$ depending on $J$), such that
\[
\int_J |v(x, t)| \, dt < \infty, \quad \forall \ x \in E \setminus Z. \tag{3.1a}
\]
It follows that there exists $Z_\ast \subseteq E$, with $|Z_\ast| = 0$, such that

$$\int_{t}^{t+h} |\tilde{v}(x,t)| \, ds < \infty, \quad \forall \ x \in E \setminus Z_\ast, \ \forall \ t \in I, \ \forall \ h > 0. \quad (3.1b)$$

In particular, we have that the pointwise values of $v_h(\cdot, t)$ are given by

$$v_h(x,t) = \frac{1}{h} \int_{t}^{t+h} \tilde{v}(x,s) \, ds, \quad \forall \ x \in E \setminus Z_\ast, \ \forall \ t \in I. \quad (3.1c)$$

**Proof.** We start with the proof of (3.1a).

**Case I:** $1 \leq r < \infty$.

Let $E_N \to E$, with $|E_N| < \infty \ \forall \ N$ (if $|E| < \infty$, then simply take $E_N = E$ ahead). Then, if $1 \leq q < \infty$, we have

$$\int_{E_N} \left( \int_{J} |v(x,t)|^q \, dx \right)^{\frac{1}{q}} \, dt \leq |E_N|^{\frac{1}{q}} \left( \int_{J} \left( \int_{E_N} |v(x,t)|^q \, dx \right)^{\frac{1}{q}} \, dt \right)^{\frac{1}{q}} \, |J|^{1-\frac{1}{q}}$$

$$\leq |E_N|^{\frac{1}{q}} \left( \int_{J} \left( \int_{E_N} |v(x,t)|^q \, dx \right)^{\frac{1}{q}} \, dt \right)^{\frac{1}{q}} \, |J|^{1-\frac{1}{q}}$$

$$= |E_N|^{\frac{1}{q}} \cdot |J|^{1-\frac{1}{q}} \left( \int_{J} \|v(\cdot, t)\|_{L^q(E)} \, dt \right)^{\frac{1}{q}} < \infty,$$

i.e., (by Fubini and Hölder) we have

$$\int_{E_N} \left( \int_{J} |v(x,t)| \, dt \right)^{\frac{1}{q}} \, dx < \infty.$$  

This gives that there exists $Z_N \subseteq E_N$, with $|Z_N| = 0$, such that

$$\int_{J} |v(x,t)| \, dt < \infty, \quad \forall \ x \in E_N \setminus Z_N.$$  

In particular, setting $Z := \bigcup_{N=1}^{\infty} Z_N$, we have $Z \subseteq E$, $|Z| = 0$, and $\int_{J} |v(x,t)| \, dt < \infty$, for all $x \in E \setminus Z$. This shows (3.1a) for $1 \leq r < \infty$ and $1 \leq q < \infty$.

If $q = \infty$, we may proceed as follows: taking again $E_N \to E$, with $|E_N| < \infty, \ \forall \ N$, we have:
\[ \int_{E_N} \left[ \int_J |v(x,t)| dt \right] dx = \int_J \left[ \int_{E_N} |v(x,t)| dx \right] dt \]
\[ \leq \int_J \left( \int_{E_N} \|v(\cdot,t)\|_{L^\infty(E)} dx \right) dt \]
\[ = \int_J \|v(\cdot,t)\|_{L^\infty(E)} |E_N| dt \]
\[ \leq |E_N| \left( \int_J \|v(\cdot,t)\|_{L^\infty(E)}^r dt \right)^{\frac{1}{r}} |J|^{1 - \frac{1}{r}} \]
\[ \leq |E_N| |J|^{1 - \frac{1}{r}} \left( \int_J \|v(\cdot,t)\|_{L^\infty(E)}^r dt \right)^{\frac{1}{r}} < \infty, \]

i.e.,
\[ \int_{E_N} \left[ \int_J |v(x,t)| dt \right] dx < \infty. \]

This gives that there exists \( Z_N \subseteq E_N \), with \( |Z_N| = 0 \), such that \( \int_J |v(x,t)| dt < \infty \), \( \forall \ x \in E_N \setminus Z_N \). As before, setting \( Z := \bigcup_{N=1}^{\infty} Z_N \), it follows that \( Z \subseteq E \), \( |Z| = 0 \), and \( \int_J |v(x,t)| dt < \infty \), \( \forall \ x \in E \setminus Z \), which shows (3.1a) for \( 1 \leq r < \infty \) and \( q = \infty \).

Case II: \( r = \infty \).

As before, consider \( E_N \rightarrow E \), with \( |E_N| < \infty \) \( \forall \ N \) (if \( |E| < \infty \), then simply take \( E_N = E \)). Then, if \( 1 \leq q < \infty \), we have
\[ \int_{E_N} \left[ \int_J |v(x,t)| dt \right] dx = \int_J \left[ \int_{E_N} |v(x,t)| dx \right] dt \]
\[ \leq \int_J \left( \int_{E_N} |v(x,t)|^q dx \right)^{\frac{1}{q}} |E_N|^{1 - \frac{1}{q}} dt \]
\[ \leq |E_N|^{1 - \frac{1}{q}} \int_J \|v(\cdot,t)\|_{L^q(E)}^r dt \]
\[ \leq |E_N|^{1 - \frac{1}{q}} \int_J M dt = M |J| |E_N|^{1 - \frac{1}{q}} < \infty, \]

where \( M = \sup_{t \in I} \|v(\cdot,t)\|_{L^q(E)} \). Therefore,
\[ \int_J |v(x,t)| dt \in L^1(E_N) \]

(i.e., \( \int_{E_N} (\int_J |v(x,t)| dt) dx < \infty \)), and so there must exists \( Z_N \subseteq E_N \), with \( |Z_N| = 0 \), such that
\[ \int_J |v(x,t)| dt < \infty, \ \forall \ x \in E_N \setminus Z_N. \]
Setting \( Z := \bigcup_{N=1}^{\infty} Z_N \), then we have \( Z \subseteq E, |Z| = 0 \) and \( \int_J |v(x,t)| \, dt < \infty, \forall x \in E \setminus Z \). This shows (3.1a) for \( r = \infty \) and \( 1 \leq q < \infty \).

Now, consider the remaining case \( q = \infty \). Taking (again) \( E_N \rightarrow E \), with \( |E_N| < \infty \), \( \forall N \), or simply \( E_N = E \), if \( |E| < \infty \), setting \( M = \sup_{t \in I} \|v(\cdot, t)\|_{L^\infty(E)} \), we have that

\[
\int_{E_N} \left[ \int_J |v(x,t)| \, dx \right] dt = \int_J \left[ \int_{E_N} |v(x,t)| \, dx \right] dt \\
\leq \int_J \left( \int_{E_N} \|v(\cdot, t)\|_{L^\infty(E)} \, dx \right) dt \\
\leq |E_N| \int_J \|v(\cdot, t)\|_{L^\infty(E)} \, dt \\
\leq |E_N| \int_J M \, dt = M \cdot |E_N| \cdot |J| < \infty,
\]

i.e., we have

\[
\int_{E_N} \left[ \int_J |v(x,t)| \, dt \right] dx < \infty.
\]

Therefore, there exists \( Z_N \subseteq E_N \), with \( |Z_N| = 0 \), such that \( \int_J |v(x,t)| \, dt < \infty \), for all \( x \in E_N \setminus Z_N \). Setting \( Z := \bigcup_{N=1}^{\infty} Z_N \), we then have \( Z \subseteq E, |Z| = 0 \), and

\[
\int_J |v(x,t)| \, dt < \infty, \forall x \in E \setminus Z.
\]

This shows (3.1a) when \( r = \infty \) and \( q = \infty \); and completes the proof of (3.1a).

Proof of (3.1b):

Let \( v(\cdot, t) \in L^r(I, L^{q}(E)) \), where \( 1 \leq q,r \leq \infty \), and \( E \in \mathcal{M}(\mathbb{R}^n) \) for some interval \( I \subseteq \mathbb{R} \). Let \( \tilde{I} = \mathbb{R} \) and

\[
\tilde{v}(\cdot, t) = \begin{cases} 
 v(\cdot, t), & \text{se } t \in I; \\
 0, & \text{se } t \in \tilde{I} \setminus I,
\end{cases}
\]

we then have

\[
\tilde{v}(\cdot, t) \in L^r(\tilde{I}, L^{q}(E)).
\]

Taking \( \tilde{I}_l = [\tilde{a}_l, \tilde{b}_l] \) and making \( \tilde{I}_l \rightarrow \mathbb{R} \) as \( l \rightarrow \infty \) (i.e., as \( \tilde{a}_l \rightarrow -\infty \) and \( \tilde{b}_l \rightarrow +\infty \)), by (3.1a) we have that there exists \( Z_l \subseteq E \), with \( |Z_l| = 0 \), such that

\[
\int_{\tilde{I}_l} |\tilde{v}(x,t)| \, dt < \infty, \forall x \in E \setminus Z_l.
\]
Taking $Z_\ast := \bigcup_{l=1}^{\infty} Z_l$, we then have $Z_\ast \subseteq E$, with $|Z_\ast| = 0$, and

$$\int_{\tilde{I}_l} |\tilde{v}(x,t)| \, dt < \infty, \quad \forall \ x \in E \setminus Z_\ast, \quad \forall \ l. \quad (3.1d)$$

Now, given $\tilde{t} \in I$ and $h > 0$ arbitrary, taking $\tilde{\ell} \in \mathbb{N}$ large enough so that $[\tilde{t}, \tilde{t} + h] \subseteq \tilde{I}_{\tilde{\ell}}$, we then get for every $x \in E \setminus Z_\ast$, by (3.1d), that

$$\int_{\tilde{\ell}}^{t+h} |\tilde{v}(x,t)| \, dt \leq \int_{\tilde{I}_{\tilde{\ell}}} |\tilde{v}(x,t)| \, dt < \infty. \quad (3.1f)$$

This shows that

$$\int_{t}^{t+h} |\tilde{v}(x,t)| \, ds < \infty, \quad \forall \ x \in E \setminus Z_\ast, \quad \forall \ t \in I, \quad \forall \ h > 0$$

(with $Z_\ast \subseteq E$; $|Z_\ast| = 0$, and with $Z_\ast$ independent of $t \in I$ and of $h > 0$). This completes the proof of (3.1f).

As an immediate consequence, we obtain the validity of (3.1e). (Lemma 3.1) □

Observe, in the proof of (3.1e), that we also have proved the following:

If $v(\cdot, t) \in L^r(I, L^q(E))$, for some $1 \leq q, r \leq \infty$ (where $E \in \mathcal{M}(\mathbb{R}^n)$ and $I \subseteq \mathbb{R}$ is a interval), for each $J \subseteq I$ bounded and each $E_N \subseteq E$, with $|E_N| < \infty$, we have

$$\int_{E_N} \left( \int_J |v(x,t)| \, dt \right) \, dx < \infty. \quad (3.1e)$$

From (3.1e), it follows that there exists $Z_t \subseteq I$ and $Z_\ast \subseteq E$, with $|Z_t| = 0$ and $|Z_\ast| = 0$, such that

$$\int_J |v(x,t)| \, dt < \infty, \quad \forall \ x \in E \setminus Z_\ast, \quad \text{and} \quad \forall \ J \subseteq I, \quad \text{with} \ J \ \text{bounded}, \quad (3.1f)$$

and

$$\int_K |v(x,t)| \, dx < \infty, \quad \forall \ t \in I \setminus Z_t, \quad \forall \ K \subseteq E, \quad \text{with} \ |K| < \infty. \quad (3.1g)$$

4 About the differentiability of the Steklov averages

Let us now relate some properties concerning the differentiation of Steklov averages.
For some $1 \leq q_0, r_0 \leq \infty$, consider now

$$v(\cdot, t) \in L^{r_0}(I, L^{q_0}_{\text{loc}}(\Omega)), $$

where $I \subseteq \mathbb{R}$ is an interval and $\Omega \subseteq \mathbb{R}^n$ is an (arbitrary) open set.

In particular, for each $K \subseteq \Omega$ compact set, by Lemma 3.1 (see also (3.1e), (3.1f) and (3.1g)), we have

$$v(\cdot, t) \in L^{r_0}(I, L^{q_0}(K)).$$

It follows that there exists $Z_\ast \subseteq \Omega$, with $|Z_\ast| = 0$, such that

$$\int_{J} |v(x, t)| \, dt < \infty, \quad \forall \, x \in \Omega \setminus Z_\ast, $$

for every bounded interval $J \subseteq I$; and there exists $Z_t \subseteq I$, with $|Z_t| = 0$, such that

$$\int_{K} |v(x, t)| \, dx < \infty, \quad \forall \, t \in I \setminus Z_t, $$

for every compact set $K \subseteq \Omega$.

Setting $v_h(x, t)$ by

$$v_h(x, t) = \frac{1}{h} \int_{t}^{t+h} \tilde{v}(x, t) \, ds, \quad \forall \, x \in \Omega \setminus Z_\ast, \quad \forall \, t \in I$$

(where $h > 0$ is given, and $\tilde{v}(\cdot, t) = v(\cdot, t)$, if $t \in I$; or $\tilde{v}(\cdot, t) = 0$, if $t \in \mathbb{R} \setminus I$), we have

$$v_h(\cdot, t) \in C^0(I, L^{q_0}_{\text{loc}}(\Omega)) \cap L^\infty(I, L^{q_0}_{\text{loc}}(\Omega)).$$

Note that (4b) means that for each compact $K \subseteq \Omega$, one has

$$v_h(\cdot, t) \in C^0(I, L^{q_0}(K)) \cap L^\infty(I, L^{q_0}(K)).$$

Now, consider $v(\cdot, t) \in L^{r_0}(I, L^{q_0}_{\text{loc}}(\Omega))$, with $1 \leq q_0, r_0 \leq \infty$, such that

$$\nabla v(\cdot, t) \in L^{r_1}(I, L^{q_1}_{\text{loc}}(\Omega)) \quad \text{(with } 1 \leq q_1, r_1 \leq \infty),$$

where $\nabla v(\cdot, t)$ is meant in the \textit{distributional sense}: for each $t \in I \setminus Z_t$ (with $|Z_t| = 0$), by (4a) we have $v(\cdot, t) \in L^1_{\text{loc}}(\Omega)$. In particular, we can compute its distributional derivative $D_i v(\cdot, t)$, for $1 \leq i \leq n$, which are given by:

$$\left< D_i v(\cdot, t) \mid \phi \right> = -\left< v(\cdot, t) \mid \frac{\partial \phi}{\partial x_i} \right> = -\int_\Omega v(x, t) \frac{\partial \phi}{\partial x_i}(x) \, dx, \quad \forall \, \phi \in C^\infty_0(\Omega).$$
For $1 \leq i \leq n$, the assumption (4c) says that for almost all $t \in I$, $D_i v(\cdot, t)$ is given by some function $g_i(\cdot, t) \in L^q_{loc}(\Omega)$, and we have

$$
\begin{cases}
\int_I \|g_i(\cdot, t)\|_{L^q(N(K))}^r \, dt < \infty, \quad \text{se} \quad 1 \leq r_1 < \infty, \\
\sup \operatorname{ess}_{t \in I} \|g_i(\cdot, t)\|_{L^q(N(K))} < \infty, \quad \text{se} \quad r_1 = \infty,
\end{cases}
$$

for each given compact set $K \subseteq \Omega$. Follows then from (4c) (by Lemma 2.4 part (2.4b)) that we have

$$(D_i v)_h(\cdot, t) \in C^0(I, L^q_{loc}(\Omega)) \cap L^\infty(I, L^q_{loc}(\Omega)), \quad 1 \leq i \leq n.$$ 

Moreover, by Lemma 3.1, we have (enlarging the null sets $Z_t$ and $Z_s$, if necessary) that there exists $Z_t \subseteq I$, with $|Z_t| = 0$, such that

$$
\int_K |v(x, t)| \, dx < \infty, \quad \forall \ t \in I \setminus Z_t, \text{ for each compact set } K \subseteq \Omega,
$$

and

$$
\int_K |D_i v(x, t)| \, dx < \infty, \quad \forall \ t \in I \setminus Z_t, \quad \forall \ 1 \leq i \leq n, \quad \forall \ K \subseteq \Omega;
$$

and there exists $Z_s \subseteq \Omega$, with $|Z_s| = 0$, such that

$$
\int_I |v(x, t)| \, dt < \infty, \quad \forall \ x \in \Omega \setminus Z_s,
$$

and

$$
\int_I |D_i v(x, t)| \, dt < \infty, \quad \forall \ x \in \Omega \setminus Z_s, \quad (\forall \ 1 \leq i \leq n).
$$

In particular, for the pointwise values of $v_h(\cdot, t)$ and of $(D_i v)_h(\cdot, t)$ we have:

$$
v_h(x, t) = \frac{1}{h} \int_t^{t+h} \bar{v}(x, s) \, ds, \quad \forall \ x \in \Omega \setminus Z_s, \quad \forall \ t \in I;
$$

$$(D_i v)_h(x, t) = \frac{1}{h} \int_t^{t+h} \bar{D}_i v(x, s) \, ds, \quad \forall \ x \in \Omega \setminus Z_s, \quad \forall \ t \in I
$$

$$
= \frac{1}{h} \int_t^{t+h} \tilde{g}_i(x, s) \, ds, \quad \text{where} \quad \tilde{g}_i(x, s) = \begin{cases} g_i(\cdot, t), & t \in I, \\ 0, & t \in \mathbb{R} \setminus I. \end{cases}
$$

and, more importantly,

$$
(D_i v)_h(x, t) = \frac{1}{h} \int_t^{t+h} (D_i v)(x, s) \, ds, \quad \forall \ x \in \Omega \setminus Z_s, \quad \forall \ t \in I_h;
$$

In particular, for the pointwise values of $v_h(\cdot, t)$ and of $(D_i v)_h(\cdot, t)$ we have:
and, for each compact set $K \subseteq \Omega$, with:

$$v_h(\cdot, t) \in C^0(I, L^{q_0}(K)) \cap L^\infty(I, L^{q_0}(K)); \quad \text{and}$$

$$(D_i v)_h(\cdot, t) \in C^0(I, L^{q_1}(K)) \cap L^\infty(I, L^{q_1}(K)).$$

In the next result we’ll shows that the operators $D_i$ and $(\cdot)_h$ commute.

**Lemma 4.1.** Let $I \subseteq \mathbb{R}$ an interval and $\Omega \subseteq \mathbb{R}^n$ an open set. Let $v(\cdot, t) \in L^{r_0}(I, L^{q_0}_{\text{loc}}(\Omega))$ (for some $1 \leq q_0, r_0 \leq \infty$), such that $\nabla v(\cdot, t) \in L^{r_1}(I, L^{q_1}_{\text{loc}}(\Omega))$ (for some $1 \leq q_1, r_1 \leq \infty$).

Then, for each $1 \leq i \leq n$ we have

$$D_i(v_h(\cdot, t)) = (D_i v)_h(\cdot, t), \quad \forall \ t \in I_h, \text{ where } I_h = \{t \in I \mid t + h \in T\}. \quad (4.1)$$

Observe that: $v_h \in L^{q_0}_{\text{loc}}(\Omega)$, $\forall \ t \in I$; and $(D_i v)_h(\cdot, t) \in L^{q_1}_{\text{loc}}(\Omega)$, $\forall \ t \in I$. In particular, under the hypothesis of Lemma 4.1 we have

$$v_h(\cdot, t) \in C^0(I, \bar{L}^{q_0}(K)) \cap L^\infty(I, L^{q_0}(K)), \quad \text{and}$$

$$\nabla v_h(\cdot, t) \in C^0(I_h, \bar{L}^{q_1}(K)) \cap L^\infty(I_h, L^{q_1}(K))$$

where $I_h = \{t \in I \mid t + h \in T\}$, for each compact $K \subseteq \Omega$.

**Proof of Lemma 4.1.** Given $t \in I$, with $t + h \in T$, and $\phi \in C_0^\infty(\Omega)$ such that $\text{supp}(\phi) \subseteq K \subseteq \Omega$, where $K$ is an compact, we have $v_h(\cdot, t) \in L^{q_0}_{\text{loc}}(\Omega) \subseteq \mathcal{D}'(\Omega)$ and

$$\langle D_i(v_h(\cdot, t)) \mid \phi \rangle = -\langle v_h(\cdot, t) \mid D_i \phi \rangle = -\int_K v_h(x, t) \frac{\partial \phi}{\partial x_i}(x) \, dx$$

$$= -\frac{1}{h} \int_K \left( \int_t^{t+h} v(x, s) \frac{\partial \phi}{\partial x_i}(x) \, ds \right) \, dx = -\frac{1}{h} \int_t^{t+h} \left( \int_\Omega v(x, s) \frac{\partial \phi}{\partial x_i}(x) \, dx \right) \, ds$$

$$= -\frac{1}{h} \int_t^{t+h} \left( \int_\Omega v(x, s) \frac{\partial \phi}{\partial x_i} \, dx \right) \, ds = -\frac{1}{h} \int_t^{t+h} \left( \int_\Omega \nabla v(x, s) \cdot \frac{\partial \phi}{\partial x_i} \, dx \right) \, ds$$

$$= \frac{1}{h} \int_t^{t+h} \left( \int_\Omega g_i(x, s) \phi(x) \, dx \right) \, ds = \frac{1}{h} \int_t^{t+h} \left( \int_\Omega g_i(x, s) \phi(x) \, dx \right) \, ds$$

$$= \frac{1}{h} \int_K \phi(x) \left( \int_t^{t+h} g_i(x, s) \, ds \right) \, dx = \frac{1}{h} \int_K \phi(x) \left( \int_t^{t+h} (D_i v)(x, s) \, ds \right) \, dx$$

$$= \int_K (D_i v)_h(x, t) \cdot \phi(x) \, dx = \langle (D_i v)_h(\cdot, t) \mid \phi \rangle,$$

i.e.,

$$\langle D_i(v_h(\cdot, t)) \mid \phi \rangle = \langle (D_i v)_h(\cdot, t) \mid \phi \rangle, \quad \forall \ t \in I, \text{ with } t + h \in T,$$

where $\phi \in C_0^\infty(\Omega)$ is an arbitrary test function, and $\langle \cdot, \cdot \rangle$ denotes the natural pairing of
\( D'(\Omega) \) and \( D(\Omega) \). This shows that

\[
D_i(v_h(\cdot, t)) = (D_i v)_h(\cdot, t), \quad t \in I_h = \{ t \in I \mid t + h \in I \}.
\]

(Lemma 4.1) □

Another important operation is the differentiability of \( v_h(\cdot, t) \in L^q(E) \) with respect to \( t \) (in the Banach space \( L^q(E) \)):

**Lemma 4.2.** Given an interval \( I \subseteq \mathbb{R} \), \( E \in \mathcal{M}(\mathbb{R}^n) \), and \( 1 \leq q \leq \infty \); let \( v(\cdot, t) \in C^0(I, L^q(E)) \).

Let \( I_h = \{ t \in I \mid (t + h) \in I \} \) and \( \mathring{I}_h = \text{int}(I_h) = \{ t \in I \mid (t + h) \in I \} \), where \( h > 0 \) (small enough that \( \mathring{I}_h \neq \emptyset \)) is given.

Then, for every \( t \in \mathring{I}_h \), we have

\[
\left\| \frac{v_h(\cdot, t + \Delta t) - v_h(\cdot, t)}{\Delta t} - \frac{v(\cdot, t + h) - v(\cdot, t)}{h} \right\|_{L^q(E)} \rightarrow 0, \quad \text{as } \Delta t \rightarrow 0,
\]

uniformly on \( t \in [t_1, t_2] \subseteq \mathring{I}_h \).

In other words:

\[
v_h(\cdot, t) = \frac{1}{h} \int_t^{t+h} v(\cdot, s) \, ds \quad \text{(for } t \in I_h)\]

is (pointwise) strongly differentiable at \( t \in \mathring{I}_h \), with

\[
(v_h)_t(\cdot, t) = \frac{1}{h} \left( v(\cdot, t + h) - v(\cdot, t) \right) \quad (\in L^q(E)), \quad \forall \ t \in \mathring{I}_h.
\]

**Proof of Lemma [4.2]** Given \( \hat{t} \in [t_1, t_2] \), with \( [t_1, t_2] \subseteq \mathring{I}_h \) an compact interval, take \( \delta_0 > 0 \) small enough that

\[
[t_1 - \delta_0, t_2 + h + \delta_0] \subseteq \mathring{I}_h.
\]

Given \( \varepsilon > 0 \), (because \( v(\cdot, t) \) is uniformly continuous on \( [t_1 - \delta_0, t_2 + h + \delta_0] \)) we can take \( \delta = \delta(\varepsilon) \leq \delta_0 \) small enough that

\[
\left\| v(\cdot, t) - v(\cdot, s) \right\|_{L^q(E)} \leq \varepsilon, \quad \forall \ s, t \in [t_1 - \delta_0, t_2 + h + \delta_0], \text{ with } |s - t| \leq \delta.
\]
Then, for any \( \hat{t} \in [t_1, t_2] \) and any \( \Delta t \in \mathbb{R} \) with \( 0 < |\Delta t| < \delta \), we have

\[
\left\| \frac{1}{\Delta t} \left( v_h(\cdot, \hat{t} + \Delta t) - v_h(\cdot, \hat{t}) \right) - \frac{1}{h} \left( v(\cdot, \hat{t} + h) - v(\cdot, \hat{t}) \right) \right\|_{L^q(E)} = \\
= \frac{1}{h} \left\| \frac{1}{\Delta t} \int_{\hat{t} + \Delta t}^{\hat{t} + \Delta t + h} v(\cdot, s) \, ds - \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + h} v(\cdot, s) \, ds - v(\cdot, \hat{t} + h) + v(\cdot, \hat{t}) \right\|_{L^q(E)} \\
= \frac{1}{h} \left\| \left( \frac{1}{\Delta t} \int_{\hat{t} + \Delta t}^{\hat{t} + \Delta t + h} v(\cdot, s) \, ds - v(\cdot, \hat{t} + h) \right) - \left( \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} v(\cdot, s) \, ds - v(\cdot, \hat{t}) \right) \right\|_{L^q(E)} \\
\leq \frac{1}{h} \left\| \frac{1}{\Delta t} \int_{\hat{t} + \Delta t}^{\hat{t} + \Delta t + h} v(\cdot, s) \, ds - v(\cdot, \hat{t} + h) \right\|_{L^q(E)} + \frac{1}{h} \left\| \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} v(\cdot, s) \, ds - v(\cdot, \hat{t}) \right\|_{L^q(E)} \\
= \frac{1}{h} \left\| \frac{1}{\Delta t} \int_{\hat{t} + \Delta t}^{\hat{t} + \Delta t + h} (v(\cdot, s) - v(\cdot, \hat{t} + h)) \, ds \right\|_{L^q(E)} + \frac{1}{h} \left\| \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} (v(\cdot, s) - v(\cdot, \hat{t})) \, ds \right\|_{L^q(E)} \\
\leq \frac{1}{h} \left( \frac{1}{\Delta t} \int_{\hat{t} + \Delta t}^{\hat{t} + \Delta t + h} (v(\cdot, s) - v(\cdot, \hat{t} + h)) \, ds \right) + \frac{1}{h} \left( \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} (v(\cdot, s) - v(\cdot, \hat{t})) \, ds \right) \\
\leq \frac{1}{h} \left( \frac{1}{\Delta t} \int_{\hat{t} + \Delta t}^{\hat{t} + \Delta t + h} \varepsilon \, ds \right) + \frac{1}{h} \left( \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} \varepsilon \, ds \right) = \frac{1}{h} \varepsilon + \frac{1}{h} \varepsilon = \frac{2}{h} \varepsilon.
\]

i.e., for any \( \Delta t \in \mathbb{R} \) with \( 0 < |\Delta t| < \delta \) we have

\[
\left\| \frac{1}{\Delta t} \left( v_h(\cdot, \hat{t} + \Delta t) - v_h(\cdot, \hat{t}) \right) - \frac{1}{h} \left( v(\cdot, \hat{t} + h) - v(\cdot, \hat{t}) \right) \right\|_{L^q(E)} \leq \frac{2}{h} \varepsilon,
\]

for all \( \hat{t} \in [t_1, t_2] \), where \( \delta > 0 \) depends only on \( \varepsilon \) (and not on \( \hat{t} \in [t_1, t_2] \)). This shows (4.2a), so that the Lemma 4.2 is now proven. (Lemma 4.2) \( \square \)

Hence, we have that the mapping

\[
v_h : I_h \rightarrow L^q(E) \quad \left( \in C^0(I_h, L^q(E)) \right)
\]

is strongly differentiable at each \( t \in \hat{I}_h \) (when \( v(\cdot, t) \in C^0(I, L^q(E)) \), for \( 1 \leq q \leq \infty \), with:

\[
\left( v_h \right)_t (\cdot, t) \equiv \frac{\partial}{\partial t} v_h(\cdot, t) = \frac{v(\cdot, t + h) - v(\cdot, t)}{h}, \quad \forall \ t \in \hat{I}_h. \quad (4.2b)
\]

The next result shows that, in the case where we only have \( v(\cdot, t) \in L^r(I, L^q(E)) \), with \( 1 \leq r < \infty \), then (4.2b) still holds, but only almost everywhere on \( \hat{I}_h \).

**Lemma 4.3.** Given an interval \( I \subseteq \mathbb{R} \), \( E \in \mathcal{M}(\mathbb{R}^n) \), \( 1 \leq q \leq \infty \), \( 1 \leq r < \infty \) and \( h > 0 \) (such that \( \hat{I}_h \) is not empty), let

\[
v(\cdot, t) \in L^r(I, L^q(E))
\]
(and, as consequence, \(v_h(\cdot, t) \in C^0(\overline{T_h}, L^q(E))\)).

Let \(I_h = \{ t \in I \mid (t + h) \in I \}\) and \(\hat{I}_h = \text{int}(I_h)\) (as in Lemma 4.2).

Then, there exists \(Z_{**} \subseteq I\), with \(|Z_{**}| = 0\), such that

\[ v_h(\cdot, t) : \hat{I}_h \to L^q(E) \]

is (strongly) differentiable at every \(t \in \hat{I}_h \setminus Z_{**}\), with

\[ (v_h)_t(\cdot, t) = \frac{v(\cdot, t + h) - v(\cdot, t)}{h} \quad \forall \ t \in \hat{I}_h \setminus Z_{**}. \tag{4.3a} \]

Lemma 4.3 says that

\[ v(\cdot, t + h) \in L^q(E) \quad \text{and} \quad v(\cdot, t) \in L^q(E), \quad \forall \ t \in \hat{I}_h \setminus Z_{**}; \]

and that

\[ v_h(\cdot, t) = \frac{1}{h} \int_t^{t+h} v(\cdot, s) \, ds \quad (\in C^0(\overline{T_h}, L^q(E))) \]

is strongly differentiable at each \(t \in \hat{I}_h \setminus Z_{**}\), and (4.3a) holds, i.e.,

\[ \left\| \frac{v_h(\cdot, t + \Delta t) - v_h(\cdot, t)}{\Delta t} - \frac{v(\cdot, t + h) - v(\cdot, t)}{h} \right\|_{L^q(E)} \to 0, \]

as \(\Delta t \to 0\), for each \(t \in \hat{I}_h \setminus Z_{**}\).

**Proof of Lemma 4.3.** Given \(v(\cdot, t) \in L^r(I, L^q(E))\), let \(Z_{0,0} \subseteq I\) be such that \(|Z_{0,0}| = 0\) and

\[ v(\cdot, t) \in L^q(E), \quad \forall \ t \in I \setminus Z_{0,0}. \]

Let \(Z_{0,h} \equiv \{ t \in I_h \mid (t + h) \in Z_{0,0} \} \subseteq Z_{0,0} - h\). Then, we have \(Z_{0,h} \subseteq I_h\), with \(|Z_{0,h}| = 0\), and

\[ v(\cdot, t + h) \in L^q(E), \quad \forall \ t \in I_h \setminus Z_{0,h}. \]

(In particular, we have \(v(\cdot, t)\) and \(v(\cdot, t + h)\) in \(L^q(E), \forall \ t \in I_h \setminus (Z_{0,0} \cup Z_{0,h})\)).

Taking \(Z_0 := Z_{0,0} \cup Z_{0,h}\), we have \(Z_0 \subseteq I\), with \(|Z_0| = 0\), and

\[ v(\cdot, t), \ v(\cdot, t + h) \in L^q(E), \quad \forall \ t \in I_h \setminus Z_0. \]

Now, because \(v(\cdot, t) \in L^r(I, L^q(E))\) for some \(1 \leq r < \infty\), we can take a sequence of smooth approximations \(w_m(\cdot, t) \in C^0_c(\hat{I}, L^q(E))\) such that

\[ \left\| w_m - v \right\|_{L^r(I, L^q(E))} \to 0, \text{ (as } m \to \infty), \tag{4.3b} \]
and (passing to a subsequence, if necessary):

\[ \| w_m(\cdot, t) - v(\cdot, t) \|_{L^q(E)} \rightarrow 0, \quad (\text{as } m \rightarrow \infty), \forall t \in I \setminus Z_{*,0}, \quad (4.3c) \]

for some \( Z_{*,0} \subseteq I \), with \( |Z_{*,0}| = 0 \) and \( Z_{*,0} \supseteq Z_{0,0} \).

In particular, setting \( Z_{*,h} := \{ t \in I_h \mid t + h \in Z_{*,0} \} \subseteq Z_{*,0} - h \), we have \( Z_{*,h} \subseteq I_h \), with \( |Z_{*,h}| = 0 \), \( Z_{*,h} \supseteq Z_{0,h} \), and

\[ \| w_m(\cdot, t + h) - v(\cdot, t + h) \|_{L^q(E)} \rightarrow 0, \quad (\text{as } m \rightarrow \infty), \forall t \in I_h \setminus Z_{*,h}. \quad (4.3c') \]

In particular, letting \( Z_* \subseteq I \) be given by

\[ Z_* = Z_{*,0} \cup Z_{*,h}, \]

we have \( Z_* \subseteq I \), with \( |Z_*| = 0 \) and \( Z_* \supseteq Z_0 \); and

\[ \| w_m(\cdot, t) - v(\cdot, t) \|_{L^q(E)} \rightarrow 0, \quad \| w_m(\cdot, t + h) - v(\cdot, t + h) \|_{L^q(E)} \rightarrow 0, \]

as \( m \rightarrow \infty \), \( \forall t \in I_h \setminus Z_* \).

Finally, for each \( m = 1, 2, 3, \ldots \), by the standard Lebesgue’s differentiation theorem, we have that there exists some null set \( Z_{m,0} \subseteq I \), with \( |Z_{m,0}| = 0 \) and \( Z_{m,0} \supseteq Z_{0,0} \), such that

\[ \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \| v(\cdot, s) - w_m(\cdot, s) \|_{L^q(E)}^r \, ds = \| v(\cdot, t) - w_m(\cdot, t) \|_{L^q(E)}^r, \quad \forall t \in I \setminus Z_{m,0}. \quad (4.3d) \]

Letting

\[ Z_{m,h} := \{ t \in I_h \mid t + h \in Z_{m,0} \} \subseteq -h + Z_{m,0}, \]

we have \( Z_{m,h} \subseteq I_h \), with \( |Z_{m,h}| = 0 \) and \( Z_{m,h} \supseteq Z_{0,h} \); and (by (4.3d)):

\[ \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{t+h}^{t+h+\Delta t} \| v(\cdot, s) - w_m(\cdot, s) \|_{L^q(E)}^r \, ds = \| v(\cdot, t + h) - w_m(\cdot, t + h) \|_{L^q(E)}^r, \quad (4.3e) \]

\( \forall \ t \in I_h \setminus Z_{m,h} \). Then, if we taking \( Z_{ss} \subseteq I \) and the null set given by

\[ Z_{ss} := Z_{*,0} \cup Z_{*,h} \cup \left( \bigcup_{m=1}^{\infty} \left( Z_{m,0} \cup Z_{m,h} \right) \right), \]

given (any) \( t \in I_h \setminus Z_{ss} \), we will then have:

\[ \lim_{\Delta t \rightarrow 0} \frac{\| v_h(\cdot, t + \Delta t) - v_h(\cdot, t) \|_{L^q(E)} - \frac{v(\cdot, t + h) - v(\cdot, t)}{h}}{\Delta t} = 0, \quad \forall t \in I_h \setminus Z_{ss}. \quad (4.3f) \]
which will shows (4.3a), concluding the proof of the Lemma 4.3.

Claim: (4.3) is true.

Indeed, given \( \hat{t} \in \hat{I}_h \setminus Z_{**} \), we may proceed as follows: given \( \varepsilon > 0 \), let \( w \equiv w_m \in C_0^0(\hat{I}, L^q(E)) \) be some term of the sequence \( (w_m)_m \) given in (4.3b), (4.3c) and (4.3d), such that we have

\[
\| w - v \|_{L^q(I, L^q(E))} \leq \varepsilon,
\]

and

\[
\| w(\cdot, \hat{t}) - v(\cdot, \hat{t}) \|_{L^q(E)} \leq \varepsilon,
\]

and

\[
\| w(\cdot, \hat{t} + h) - v(\cdot, \hat{t} + h) \|_{L^q(E)} \leq \varepsilon.
\]

This gives

\[
\limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \left( v_{\hat{t} + \hat{t} + \Delta t} - v_{\hat{t}} \right) - \frac{1}{h} \left( v_{\hat{t} + h} - v_{\hat{t}} \right) \right\|_{L^q(E)} =
\]

\[
= \frac{1}{h} \limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \int_{\hat{t} + \Delta t}^{\hat{t} + \Delta t + h} v_\cdot s \ ds - \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + h} v_\cdot s \ ds - v_{\hat{t} + h} + v_{\hat{t}} \right\|_{L^q(E)}
\]

\[
= \frac{1}{h} \limsup_{\Delta t \to 0} \left\| \left( \frac{1}{\Delta t} \int_{\hat{t} + h}^{\hat{t} + h + \Delta t} v_\cdot s \ ds - v_{\hat{t} + h} \right) - \left( \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} v_\cdot s \ ds - v_{\hat{t}} \right) \right\|_{L^q(E)}
\]

\[
\leq \frac{1}{h} \limsup_{\Delta t \to 0} \left( \left\| \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t + h} v_\cdot s \ ds - \int_{\hat{t}}^{\hat{t} + \Delta t} w_\cdot s \ ds + \int_{\hat{t}}^{\hat{t} + \Delta t + h} w_\cdot s \ ds \right. \right.
\]

\[
\left. - w_{\hat{t} + h} + w_{\hat{t}} - w_{\hat{t} + h} \right\|_{L^q(E)}
\]

\[
+ \frac{1}{h} \limsup_{\Delta t \to 0} \left( \left\| \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} v_\cdot s \ ds - \int_{\hat{t}}^{\hat{t} + \Delta t} w_\cdot s \ ds + \int_{\hat{t}}^{\hat{t} + \Delta t} w_\cdot s \ ds \right. \right.
\]

\[
\left. - w_{\hat{t}} + w_{\hat{t}} - w_{\hat{t}} \right\|_{L^q(E)}
\]

\[
\leq \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \right) \left( \int_{\hat{t} + h}^{\hat{t} + h + \Delta t} \| v_\cdot s - w_\cdot s \|_{L^q(E)} ds \right)
\]

\[
+ \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \right) \left( \int_{\hat{t}}^{\hat{t} + \Delta t} \| v_\cdot s - w_\cdot s \|_{L^q(E)} ds \right)
\]

\[
+ \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \right) \left( \int_{\hat{t}}^{\hat{t} + \Delta t} \| w_\cdot s - w_\cdot s \|_{L^q(E)} ds \right)
\]

\[
+ \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \right) \left( \int_{\hat{t}}^{\hat{t} + \Delta t} \| w_\cdot s - w_\cdot s \|_{L^q(E)} ds \right)
\]

\[
+ \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \right) \left( \int_{\hat{t}}^{\hat{t} + \Delta t} \| w_\cdot s - w_\cdot s \|_{L^q(E)} ds \right)
\]

\[
+ \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \right) \left( \int_{\hat{t}}^{\hat{t} + \Delta t} \| w_\cdot s - w_\cdot s \|_{L^q(E)} ds \right)
\]
\[
\leq \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\tilde{t} + \Delta t}^{\tilde{t} + \Delta t + h} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds + 0 + \varepsilon \right) \\
+ \left( \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\tilde{t}}^{\tilde{t} + \Delta t} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds + 0 + \varepsilon \right).
\]

So that we have

\[
\limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \int_{\tilde{t}}^{\tilde{t} + \Delta t} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds \right\|_{L^q(E)} \leq \\
+ \left( \frac{1}{\Delta t} \int_{\tilde{t}}^{\tilde{t} + \Delta t} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds + \frac{2}{h} \varepsilon.\right.
\]

Now,

\[
\frac{1}{|\Delta t|} \left\| \int_{\tilde{t} + \Delta t}^{\tilde{t} + \Delta t + h} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds \right\| \leq \left( \frac{1}{|\Delta t|} \int_{\tilde{t} + \Delta t}^{\tilde{t} + \Delta t + h} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds \right) \frac{1}{\tau} |\Delta t|^{-\frac{1}{\tau}}
\]

and, similarly,

\[
\frac{1}{|\Delta t|} \left\| \int_{\tilde{t}}^{\tilde{t} + \Delta t} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds \right\| \leq \left( \frac{1}{|\Delta t|} \int_{\tilde{t}}^{\tilde{t} + \Delta t} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds \right) \frac{1}{\tau}. (4.3h')
\]

By (4.3d) and (4.3e), (since \( \tilde{t} \in \tilde{I}_h \setminus Z_s \)) we have

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\tilde{t} + \Delta t}^{\tilde{t} + \Delta t + h} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds = \|v(\cdot, \tilde{t} + h) - w(\cdot, \tilde{t} + h)\|_{L^p(E)} \leq \varepsilon^r
\]

and

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\tilde{t}}^{\tilde{t} + \Delta t} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds = \|v(\cdot, \tilde{t}) - w(\cdot, \tilde{t})\|_{L^p(E)} \leq \varepsilon^r,
\]

so that, by (4.3d) and (4.3h), we have

\[
\limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \int_{\tilde{t} + \Delta t}^{\tilde{t} + \Delta t + h} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds \right\| \leq \varepsilon
\]

and

\[
\limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \int_{\tilde{t}}^{\tilde{t} + \Delta t} \|v(\cdot, s) - w(\cdot, s)\|_{L^p(E)} \, ds \right\| \leq \varepsilon.
\]
Therefore, by (4.3g), we obtain
\[ \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \left( v_h(\cdot, \hat{t} + \Delta t) - v_h(\cdot, \hat{t}) \right) - \frac{1}{h} \left( v(\cdot, \hat{t} + h) - v(\cdot, \hat{t}) \right) \leq \left\| \frac{1}{\Delta t} \int_{\hat{t} + h}^{\hat{t} + \Delta t + h} v(\cdot, s) - w(\cdot, s) \right\|_{L^q(E)} ds \]
\[ + \frac{1}{h} \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t} + \Delta t} \left\| v(\cdot, s) - w(\cdot, s) \right\|_{L^q(E)} ds \]
\[ \leq \frac{1}{h} \varepsilon + \frac{1}{h} \varepsilon + 2 \frac{\varepsilon}{h} = \frac{4}{h} \varepsilon. \]

Because \( \varepsilon > 0 \) is arbitrary (and \( h > 0 \) is fixed), this shows (4.3f), and the proof of Lemma 4.3 is now complete. □

5 About the integration of the Steklov averages

From Lemma 4.3 if \( v(\cdot, t) \in L^1_{loc}(I, L^q(E)) \) (where \( 1 \leq r < \infty, 1 \leq q \leq \infty \)), we always obtain that
\[ v_h(\cdot, t) = \frac{1}{h} \int_{\hat{t}}^{\hat{t} + h} v(\cdot, s) \, ds \quad (\in C^0(\hat{I}_h, L^q(E))) \]
is (strongly) differentiable (as a map from \( \hat{I}_h \) to \( L^q(E) \)) at almost every point \( t \in \hat{I}_h \), with
\[ (v_h)_t(\cdot, t) = \frac{v(\cdot, t + h) - v(\cdot, t)}{h}, \quad \forall \ t \in \hat{I}_h \setminus Z, \]
for some \( Z \subseteq I \), with \( \vert Z \vert = 0 \), since
\[ \frac{v(\cdot, t + h) - v(\cdot, t)}{h} \in L^1_{loc}(\hat{I}_h, L^q(E)). \]

This gives that, for any compact interval \([t_1, t_2] \subseteq \hat{I}_h\), we have:
\[ \int_{t_1}^{t_2} (v_h)_t(\cdot, t) \, dt = \int_{t_1}^{t_2} \frac{v(\cdot, t + h) - v(\cdot, t)}{h} \, dt \]
\[ = \frac{1}{h} \int_{t_1}^{t_2} v(\cdot, t + h) \, dt - \frac{1}{h} \int_{t_1}^{t_2} v(\cdot, t) \, dt \]
\[ = \frac{1}{h} \int_{t_1 + h}^{t_2 + h} v(\cdot, s) \, ds - \frac{1}{h} \int_{t_1}^{t_1 + h} v(\cdot, s) \, ds \]
\[ = \frac{1}{h} \int_{t_1}^{t_2} v(\cdot, s) \, ds - \frac{1}{h} \int_{t_1}^{t_1 + h} v(\cdot, s) \, ds \]
\[ = v_h(\cdot, t_2) - v_h(\cdot, t_1), \quad \forall \ [t_1, t_2] \subseteq \hat{I}_h, \]
i.e.,
\[
\int_{t_1}^{t_2} (v_h)_t(\cdot,t) \, dt = v_h(\cdot,t_2) - v_h(\cdot,t_1), \quad \forall \, t_1 \leq t_2 \in \hat{I}_h.
\]

This is a special case of the followings more general results: Lemma 5.1 and Lemma 5.2 above.

**Lemma 5.1.** (*Fundamental Theorem of Calculus - version 1*)

Given an interval $I \subseteq \mathbb{R}$, $E \in \mathcal{M}((\mathbb{R}^n)$, $1 \leq q \leq \infty$, $t_0 \in \hat{I}$ and $F_0 \in L^q(E)$, let

\[ f(\cdot,t) \in C^0(\hat{I}, L^q(E)) \]

and

\[ F(\cdot,t) = F_0 + \int_{t_0}^{t} f(\cdot,s) \, ds, \quad \forall \, t \in \hat{I}. \]

Then

\[ F_t(\cdot,t) = f(\cdot,t), \quad \forall \, t \in \hat{I} \]

and

\[ \int_{t_1}^{t_2} f(\cdot,t) \, dt = F(\cdot,t_2) - F(\cdot,t_1), \quad \forall \, t_1 \leq t_2 \in \hat{I}. \]

**Proof.** [5.1a] follows by direct computation of

\[ F_t(\cdot,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [F(\cdot,t + \Delta t) - F(\cdot,t)] \]

and of the continuity of $f(\cdot,t)$, since, for each $t \in \hat{I}$, we have:

\[ F_t(\cdot,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [F(\cdot,t + \Delta t) - F(\cdot,t)] = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} f(\cdot,s) \, ds = f(\cdot,t). \]

On the other hand, [5.1a'] is trivial:

\[ \int_{t_1}^{t_2} f(\cdot,t) \, dt = \int_{t_0}^{t_2} f(\cdot,t) \, dt - \int_{t_0}^{t_1} f(\cdot,t) \, dt = F(\cdot,t_2) - F(\cdot,t_1). \]

(Lemma 5.1) □

The next extension of Lemma 5.1 when $f(\cdot,t) \in L^1_{\text{loc}}(\hat{I}, L^q(E))$ (for example, if $f(\cdot,t) \in L^r_{\text{loc}}(\hat{I}, L^q(E)$, for some $1 \leq r \leq \infty$), is also worth mentioning.

**Lemma 5.2.** (*Fundamental Theorem of Calculus - version 2*)

Let $I \subset \mathbb{R}$ an interval, $E \in \mathcal{M}((\mathbb{R}^n)$, $1 \leq q \leq \infty$, $t_0 \in \hat{I}$ and $F_0 \in L^q(E)$. Given

\[ f(\cdot,t) \in L^1_{\text{loc}}(\hat{I}, L^q(E)) \]

(5.2a)
and
\[ F(\cdot, t) := F_0 + \int_{t_0}^t f(\cdot, s) \, ds \quad (\forall \, t \in \hat{I}), \quad (5.2a') \]
then we have
\[ F_t(\cdot, t) = f(\cdot, t) \quad \text{a.e. } t \in \hat{I} \quad (5.2b) \]
and
\[ \int_{t_1}^{t_2} f(\cdot, t) \, dt = F(\cdot, t_2) - F(\cdot, t_1) \quad \forall \, t_1 \leq t_2 \in \hat{I}. \quad (5.2b') \]

Remark. If \( f(\cdot, t) \in L_{loc}^1(\hat{I}, L^q(E)) \) and \( G(\cdot, t) \in C^0(\hat{I}, L^q(E)) \) is such that
\[ G_t(\cdot, t) = f(\cdot, t), \quad \text{a.e. } t \in \hat{I}, \quad (5.2c) \]
it does not follow (in general) that
\[ \int_{t_1}^{t_2} f(\cdot, t) \, dt = G(\cdot, t_2) - G(\cdot, t_1), \quad \text{if } [t_1, t_2] \subseteq \hat{I}. \quad (5.2c') \]
In fact, recall the Cantor-Lebesgue function, which already shows that the result \( (5.2c) \) is not valid even for real-valued \( f \in L_{loc}^1(\hat{I}, \mathbb{R}) \).

The validity of \( (5.2c) \) requires that \( G(\cdot, t) \) be also absolutely continuous in \( \hat{I} \), i.e., that we have
\[ G(\cdot, t) := G_0 + \int_{t_0}^t g(\cdot, s) \, ds, \quad (\forall \, t \in \hat{I}), \]
for some \( G_0 \in L^q(E) \) and \( g(\cdot, t) \in L^1(\hat{I}, L^q(E)) \), and in this case, by \( (5.2b) \), we have that
\( g(\cdot, t) = f(\cdot, t) \) a.e. \( t \in \hat{I} \).

Proof of Lemma 5.2. Again, the proof of \( (5.2b) \) is a trivial consequence of \( (5.2a') \): we have
\[ \int_{t_1}^{t_2} f(\cdot, t) \, dt = \int_{t_0}^{t_2} f(\cdot, t) \, dt - \int_{t_0}^{t_1} f(\cdot, t) \, dt = F(\cdot, t_2) - F(\cdot, t_1). \]
If \( f(\cdot, t) \in C^0(\hat{I}, L^q(E)) \), then we have \( F_t(\cdot, t) = f(\cdot, t), \forall \, t \in \hat{I} \), (see Lemma 4.1).

In the general case where it is only assumed that \( f(\cdot, t) \in L_{loc}^1(\hat{I}, L^q(E)) \), we may proceed as follows (as in the proof of Lemma 4.3): first, let \( Z_0 \subseteq \hat{I} \) be a set with zero measure such that
\[ f(\cdot, t) \in L^q(E), \quad \forall \, t \in \hat{I} \setminus Z_0. \]
Taking a sequence of smooth approximations \( g_m(\cdot, t) \in C^0_c(\hat{I}, L^q(E)), m = 1, 2, 3, \ldots \), such that
\[ \int_{t_0}^{t} \|g_m(\cdot, t) - f(\cdot, t)\|_{L^q(E)} \, dt \longrightarrow 0 \quad (5.2d) \]
as $m \to \infty$, for each compact $[a, b] \subseteq \hat{I}$, and

$$\|g_m(\cdot, t) - f(\cdot, t)\|_{L^q(E)} \rightarrow 0$$  \hspace{1cm} (5.2e)

as $m \to \infty$, for each $t \in \hat{I} \setminus Z_\ast$ (where $Z_\ast \subseteq \hat{I}$ is some null set with $Z_\ast \supseteq Z_0$); and for each $m = 1, 2, 3, \ldots$, let $Z_m \subseteq \hat{I}$ be a null set such that

$$\frac{1}{\Delta t} \int_{t}^{t+\Delta t} \|f(\cdot, s) - g_m(\cdot, s)\|_{L^q(E)} ds \rightarrow \|f(\cdot, t) - g_m(\cdot, t)\|_{L^q(E)}$$

as $\Delta t \to 0$, for each $t \in \hat{I} \setminus Z_m$ (by the standard Lebesgue’s differentiation theorem on $L^1_{loc}(\hat{I}, \mathbb{R})$). Then, taking

$$Z_{**} := Z_0 \cup Z_\ast \cup \left( \bigcup_{m=1}^{\infty} Z_m \right),$$  \hspace{1cm} (5.2f)

we have $Z_{**} \subseteq \hat{I}$, $Z_{**} \supseteq Z_0$, $|Z_{**}| = 0$, and

$$\lim_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \left[ F(\cdot, t + \Delta t) - F(\cdot, t) \right] - f(\cdot, t) \right\|_{L^q(E)} = 0$$  \hspace{1cm} (5.2g)

for every $t \in \hat{I} \setminus Z_{**}$, thus showing (5.2b).

To finish the proof of Lemma 5.2, it remains to show (5.2g).

**Claim:** (5.2g) is true.

Indeed, given $\hat{t} \in \hat{I} \setminus Z_{**}$ (where $Z_{**} \subseteq \hat{I}$ is given in (5.2f)), we now show that (5.2g) holds at $t = \hat{t}$, i.e., we have

$$\limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \left[ F(\cdot, \hat{t} + \Delta t) - F(\cdot, \hat{t}) \right] - f(\cdot, \hat{t}) \right\|_{L^q(E)} = 0. $$  \hspace{1cm} (5.2h)

In fact, given $\varepsilon > 0$, let $g \equiv g_m$ be an approximant in the sequence $(g_m)_m$ such that, as $m \to \infty$, we have

$$\|f(\cdot, \hat{t}) - g(\cdot, \hat{t})\|_{L^q(E)} \leq \varepsilon.$$  \hspace{1cm} (5.2h')

(5.2h’) comes from the condition (5.2c) (the property (5.2d) will not be used here).

Writing (for $|\Delta t|$ small, namely, $0 < |\Delta t| \leq \hat{\delta}$, where $\hat{\delta} > 0$ is such that $[\hat{t} - \hat{\delta}, \hat{t} + \hat{\delta}] \subseteq \hat{I}$):

$$\frac{1}{\Delta t} \left[ F(\cdot, \hat{t} + \Delta t) - F(\cdot, \hat{t}) \right] - f(\cdot, \hat{t}) = \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t}+\Delta t} f(\cdot, s) ds - f(\cdot, \hat{t})$$

$$= \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t}+\Delta t} [f(\cdot, s) - g(\cdot, s)] ds + \frac{1}{\Delta t} \int_{\hat{t}}^{\hat{t}+\Delta t} g(\cdot, s) ds - g(\cdot, \hat{t}) + g(\cdot, \hat{t}) - f(\cdot, \hat{t}),$$
we get

\[
\left\| \frac{1}{\Delta t} \left[ F(\cdot, \hat{t} + \Delta t) - F(\cdot, \hat{t}) \right] - f(\cdot, \hat{t}) \right\|_{L^q(E)} \leq \\
\leq \frac{1}{\Delta t} \int_t^{t+\Delta t} \| f(\cdot, s) - g(\cdot, s) \|_{L^q(E)} ds + \frac{1}{\Delta t} \int_t^{t+\Delta t} g(\cdot, s) ds - g(\cdot, \hat{t}) \|_{L^q(E)} \\
+ \| g(\cdot, \hat{t}) - f(\cdot, \hat{t}) \|_{L^q(E)} \\
\leq \frac{1}{\Delta t} \int_t^{t+\Delta t} \| f(\cdot, s) - g(\cdot, s) \|_{L^q(E)} ds + \frac{1}{\Delta t} \int_t^{t+\Delta t} \| g(\cdot, s) - g(\cdot, \hat{t}) \|_{L^q(E)} ds + \varepsilon,
\]

for all \( \Delta t \in \mathbb{R} \) with \( 0 < |\Delta t| \leq \hat{t} \), where \( \hat{t} > 0 \) is such that \( [\hat{t} - \delta, \hat{t} + \delta] \subseteq \hat{I} \). This gives

\[
\limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \left[ F(\cdot, \hat{t} + \Delta t) - F(\cdot, \hat{t}) \right] - f(\cdot, \hat{t}) \right\|_{L^q(E)} \leq \\
\leq \limsup_{\Delta t \to 0} \left( \frac{1}{\Delta t} \int_t^{t+\Delta t} \| f(\cdot, s) - g(\cdot, s) \|_{L^q(E)} ds + \frac{1}{\Delta t} \int_t^{t+\Delta t} \| g(\cdot, s) - g(\cdot, \hat{t}) \|_{L^q(E)} ds \right) + \varepsilon \\
= \limsup_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \| f(\cdot, s) - g(\cdot, s) \|_{L^q(E)} ds + \varepsilon \\
= \| f(\cdot, \hat{t}) - g(\cdot, \hat{t}) \|_{L^q(E)} + \varepsilon \leq \varepsilon + \varepsilon
\]

i.e., we have

\[
\limsup_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \left[ F(\cdot, \hat{t} + \Delta t) - F(\cdot, \hat{t}) \right] - f(\cdot, \hat{t}) \right\|_{L^q(E)} \leq 2\varepsilon, \tag{5.2i}
\]

for any \( \varepsilon > 0 \).

Because \( \varepsilon > 0 \) in (5.2i) is arbitrary, this shows (5.2a), where \( t = \hat{t} \in \hat{I} \setminus Z_{ss} \) is also arbitrary. This completes the proof of Lemma 5.2.

\[\text{(Lemma 5.2)}\]

\[\square\]

Remark. Lemma 5.2 can be used to give a shorter proof to Lemma 4.3 since (picking \( t_0 \in \hat{I} \) arbitrary) we have

\[
v_h(\cdot, t) = \frac{1}{h} \int_t^{t+h} v(\cdot, s) ds = \frac{1}{h} \int_{t_0}^{t+h} v(\cdot, s) ds - \frac{1}{h} \int_{t_0}^t v(\cdot, s) ds,
\]

and, by Lemma 5.2 since \( v(\cdot, t) \in L^r(I, L^q(E)) \subseteq L^1_{loc}(\hat{I}, L^q(E)) \), we have

\[
\frac{\partial}{\partial t} \int_{t_0}^{t+h} v(\cdot, s) ds = v(\cdot, t + h), \quad \text{a.e. } t \in \hat{I}_h,
\]

(by a similar argument to that given in the proof of Lemma 5.2); and

\[
\frac{\partial}{\partial t} \int_{t_0}^t v(\cdot, s) ds = v(\cdot, t), \quad \text{a.e. } t \in \hat{I}.
\]
Lemma 5.3. (Integration by parts; special case: $F, G(\cdot, t) \in C^1(\dot{I}, L^q(E))$)

Let $I \subseteq \mathbb{R}$ (an interval), $E \in \mathcal{M}(\mathbb{R}^n)$, $1 \leq q < \infty$, $t_0, t_1 \in I$ and $F_0, G_1 \in L^q(E)$.

Given
\[ f(\cdot, t), \ g(\cdot, t) \in C^0(\dot{I}, L^q(E)), \]  
\[ F(\cdot, t) := F_0 + \int_{t_0}^t f(\cdot, s) \, ds, \quad \forall \ t \in \dot{I}, \]  
and
\[ G(\cdot, t) := G_1 + \int_{t_1}^t g(\cdot, s) \, ds, \quad \forall \ t \in \dot{I}, \]

then, for any compact interval $[a, b] \subseteq \dot{I}$, we have:
\[ \int_a^b f(\cdot, t)G(\cdot, t) \, dt = F(\cdot, b)G(\cdot, b) - F(\cdot, a)G(\cdot, a) - \int_a^b F(\cdot, t)g(\cdot, t) \, dt, \]  
i.e.,
\[ \int_a^b F_t(\cdot, t)G(\cdot, t) \, dt = F(\cdot, b)G(\cdot, b) - F(\cdot, a)G(\cdot, a) - \int_a^b F(\cdot, t) G_t(\cdot, t) \, dt. \]

Proof. Given $[a, b] \subset \dot{I}$, (using the Lebesgue’s dominated convergence theorem) we have
\[
\int_a^b f(\cdot, t)G(\cdot, t) \, dt = \int_a^b \left[ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} f(\cdot, s) \, ds \right] \cdot G(\cdot, t) \, dt
\]
\[ = \lim_{\Delta t \to 0} \int_a^b \frac{1}{\Delta t} \left[ \int_t^{t+\Delta t} f(\cdot, s) \, ds \right] \cdot G(\cdot, t) \, dt
\]
\[ = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_a^b [F(\cdot, t + \Delta t) - F(\cdot, t)]G(\cdot, t) \, dt
\]
\[ = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_a^b F(\cdot, t + \Delta t)G(\cdot, t) \, dt - \int_a^b F(\cdot, t)G(\cdot, t) \, dt \right]
\]
\[ = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_a^b F(\cdot, t)G(\cdot, t - \Delta t) \, dt + \int_b^{b+\Delta t} F(\cdot, t)G(\cdot, t - \Delta t) \, dt
\]
\[ - \int_a^{a+\Delta t} F(\cdot, t)G(\cdot, t - \Delta t) \, dt - \int_a^b F(\cdot, t)G(\cdot, t) \, dt \right]
\[
= \lim_{\Delta t \to 0} \left[ \int_a^b F(\cdot, t) \frac{G(\cdot, t) - G(\cdot, t - \Delta t)}{\Delta t} \, dt \right]
\]
\[ + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_b^{b+\Delta t} F(\cdot, t) G(\cdot, t - \Delta t) \, dt - \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_a^{a+\Delta t} F(\cdot, t) G(\cdot, t - \Delta t) \, dt
\]
\[ = - \lim_{\Delta t \to 0} \left[ \int_a^b F(\cdot, t) \frac{G(\cdot, t) - G(\cdot, t - \Delta t)}{\Delta t} \, dt \right] + F(\cdot, b)G(\cdot, b) - F(\cdot, a)G(\cdot, a)\]
This concludes the proof of Lemma 5.3.

**Lemma 5.4.** (Integration by parts; general case: \( F, G(\cdot, t) \) absolutely continuous in \( \tilde{I} \))

Let \( I \subseteq \mathbb{R} \) (an interval), \( E \in \mathcal{M}(\mathbb{R}^n) \), \( t_0, t_1 \in \tilde{I}, 1 \leq q \leq \infty \), and \( F_0, G_1 \in L^q(E) \).

Given

\[
f(\cdot, t), g(\cdot, t) \in L^1_{loc}(\tilde{I}, L^q(E)),
\]

\[
F(\cdot, t) := F_0 + \int_{t_0}^t f(\cdot, s) \, ds \quad (\forall \ t \in \tilde{I}),
\]

and

\[
G(\cdot, t) := G_1 + \int_{t_1}^t g(\cdot, s) \, ds \quad (\forall \ t \in \tilde{I}),
\]

then, for any compact \( [a, b] \subseteq \tilde{I} \), we have

\[
\int_a^b f(\cdot, t) G(\cdot, t) \, dt = F(\cdot, b) G(\cdot, b) - F(\cdot, a) G(\cdot, a) - \int_a^b F(\cdot, t) g(\cdot, t) \, dt,
\]

i.e.,

\[
\int_a^b F_t(\cdot, t) G(\cdot, t) \, dt = F(\cdot, b) G(\cdot, b) - F(\cdot, a) G(\cdot, a) - \int_a^b F(\cdot, t) G_t(\cdot, t) \, dt.
\]

**Proof.** Taking \( (f_m(\cdot, t))_m \) and \( (g_m(\cdot, t))_m \), with \( f_m(\cdot, t), g_m(\cdot, t) \in C^0_c(\tilde{I}, L^q(E)) \), such that

\[
\int_{\alpha}^{\beta} \| f_m(\cdot, t) - f(\cdot, t) \|_{L^q(E)} \, dt \to 0, \quad \int_{\alpha}^{\beta} \| g_m(\cdot, t) - g(\cdot, t) \|_{L^q(E)} \, dt \to 0, \quad \text{as} \ m \to \infty,
\]

for each compact interval \( [\alpha, \beta] \in \tilde{I} \) we set \( F_m(\cdot, t), G_m(\cdot, t) \in C(\tilde{I}, L^q(E)) \) given by

\[
F_m(\cdot, t) := F_0 + \int_{t_0}^t f_m(\cdot, s) \, ds, \quad G_m(\cdot, t) := G_1 + \int_{t_1}^t g_m(\cdot, s) \, ds \quad (\forall \ t \in \tilde{I}),
\]

so that we have \( F_m(\cdot, t) \to F(\cdot, t), G_m(\cdot, t) \to G(\cdot, t) \) in \( L^q(E) \), uniformly in \( t \in [\alpha, \beta] \) (for any compact \( [\alpha, \beta] \subseteq \tilde{I} \), as \( m \to \infty \).
By Lemma 5.3 (given any compact \([a, b] \subseteq \hat{I}\), \(\forall m\) we have
\[
\int_a^b f_m(\cdot, t)G_m(\cdot, t)\, dt = F_m(\cdot, b)G_m(\cdot, b)
- F_m(\cdot, a)G_m(\cdot, a) - \int_a^b F_m(\cdot, t)g_m(\cdot, t)\, dt.
\]
(5.4d’)

Letting \(m \to \infty\) in (5.4b’), we obtain (5.4a).

(Lemma 5.4) □

Acknowledgements. This work was partially supported by CNPq (Ministry of Science and Technology, Brazil), Grant # 154037/2011-7 and by CAPES (Ministry of Science and Technology, Brazil), Grant #1212003/2013. The authors also express their gratitude to Paulo R. Zingano (UFRGS, Brazil), for some helpful suggestions and discussions.

References

[1] N. I. Akhiezer, Theory of approximation, Dover, New York, 1992. [First published: Lektsii Po Teorii Approksimatsii (in Russian), OGIZ, Moskow-Leningrad, 1947. First English Translation: Frederick Ungar Publishing Co., New York, 1956.]

[2] J. Q. Chagas, P. L. Guidolin and J. P. Zingano, Some basic properties of bounded solutions of parabolic equations with \(p\)-Laplacian type diffusion Universidade Federal do Rio Grande do Sul. Porto Alegre. Brazil. (2017) (available at http://arXiv.org).

[3] J. Q. Chagas, P. L. Guidolin and P. R. Zingano, Global solvability results for parabolic equations with \(p\)-Laplacian type diffusion (submitted).

[4] E. DiBenedetto, Degenerate Parabolic Equations, Springer, New York, 1993.

[5] N. Kuznetsov et all., The Legacy of Vladimir Andreevich Steklov, Notices of the AMS, vol. 61, n. 1, (2014), 9-22.

[6] N. Kuznetsov, The Legacy of Vladimir Andreevich Steklov in Mathematical Physics: Work and School. Russian Academy of Sciences, St. Petersburg (2014). (Available at http://www.mathsoc.spb.ru/pantheon/steklov/Steklov__150.pdf)

[7] Y. Sinai (ed.), Vladimir Andreevich Steklov. In: Russian Mathematicians in the 20th Century, World Scientific, New Jersey, 2003. 37-59.

[8] W. Stekloff, Sur les expressions asymptotiques de certaines fonctions, définies par les équations différentielles linéaires du second ordre, et leurs applications au problème du développement d’une fonction arbitraire en séries procédant suivant les-dites fonctions. Communications de la Société mathématique de Kharkow, 2-ée série, vol 10 (1907), 97-199.
[9] J. M. Urbano, The Method of Intrinsic Scaling, Lecture Notes in Mathematics, vol. 1930, Springer, New York, 2008.

[10] Z. Wu, J. Zhao, J. Yin and H. Li, Nonlinear Diffusion Equations, World Scientific, Hong Kong, 2001.

Jocemar de Quadros Chagas
Departamento de Matemática e Estatística
Universidade Estadual de Ponta Grossa
Ponta Grossa, PR 84030-900, Brazil
E-mail: jocemarchagas@uepg.br

Nicolau Matiel Lunardi Diehl
Instituto Federal de Educação, Ciência e Tecnologia
Canoas, RS 92412-240, Brazil
E-mail: nicolau.diehl@canoas.ifrs.edu.br

Patrícia Lisandra Guidolin
Instituto Federal de Educação, Ciência e Tecnologia
Viamão, RS 94410-970, Brazil
E-mail: patricia.guidolin@viamao.ifrs.edu.br