PRODUCTION OF MAGNETIC ENERGY BY MACROSCOPIC TURBULENCE IN GRB AFTERGLOWS

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ABSTRACT

Afterglows of gamma-ray bursts are believed to require magnetic fields much stronger than that of the compressed preshock medium. As an alternative to microscopic plasma instabilities, we propose amplification of the field by macroscopic turbulence excited by the interaction of the shock with a clumpy preshock medium, for example, a stellar wind. Using a recently developed formalism for localized perturbations to an ultrarelativistic shock, we derive constraints on the length scale, amplitude, and volume filling factor of density clumps required to produce a given magnetic energy fraction within the expansion time of the shock, assuming that the energy in the field achieves equipartition with the turbulence. Stronger and smaller scale inhomogeneities are required for larger shock Lorentz factors. Hence, it is likely that the magnetic energy fraction evolves as the shock slows. This could be detected by monitoring the synchrotron cooling frequency if the radial density profile ahead of the shock, smoothed over clumps, is known.

Subject headings: gamma rays: bursts — hydrodynamics — magnetic fields — shock waves — turbulence

1. INTRODUCTION

Since their discovery, gamma-ray burst (GRB) afterglows have been attributed to synchrotron radiation from the forward shock wave (Meszaros & Rees 1997), although it has been recently argued (Uhm & Beloborodov 2007; Genet et al. 2007) that observations might support a model in which the forward shock is invisible and the afterglow is emitted by a long-lived reverse shock in the burst ejecta. Assuming anyway a forward-shock origin for the afterglow emission, it is difficult to account for the magnetic energy density behind the forward shock by simple compression of the preshock field. Interstellar magnetic energies are typically comparable to thermal pressures and are therefore a fraction \( \epsilon_B \equiv \rho_B / \rho_0 = 10^{-9} \) to \( 10^{-7} \) of the total internal energy density when rest mass is included. It is possible that the preshock medium is the stellar wind of the burst progenitor; while the magnetic energy fraction in winds is less well known, it is unlikely to be much larger than this. Simply compressing the medium would produce the same ratio \( \rho_B / \rho_0 \) behind the shock. Instead, phenomenological models of afterglow light curves typically require \( \epsilon_B = 10^{-3} \) to \( 10^{-1} \) (Pania
tescu & Kumar 2002; Yost et al. 2003; Panaitescu 2005). It follows that the magnetic energy per baryon must be increased by \( \sim 10^{4} - 10^{8} \).

In fact, GRB afterglows present the most compelling case among astrophysical collisionless shocks for prompt creation of magnetic energy. The synchrotron emission from supernova remnants is generally consistent with compression of the interstellar field, although some modest additional amplification may be required in particular cases (Völk et al. 2005). We focus here on GRB afterglows rather than GRB prompt emission, because the latter is generally believed to be produced within the ejecta (but see Dermer & Mitman 1999), which may be magnetically dominated from the start (Coburn & Boggis 2003; Zhang et al. 2003).

The leading hypothesis for field amplification in GRB afterglows is the relativistic Weibel instability, which extracts free energy from the anisotropy of the particles’ velocity distribution function, producing filamentary currents aligned with the shock normal; these currents are responsible for the creation of transverse magnetic fields (Medvedev & Loeb 1999). This process is able to violate MHD flux-freezing because it occurs on a microscopic scale—the relativistic electron or ion skin depth—where the inertia of individual charged particles is significant. While the Weibel instability provides a plausible mechanism to isotropize the particle velocities, it is unclear whether the small-scale fields that it produces can survive mutual annihilation long enough to explain the observed synchrotron afterglow emission. Several groups (Silva et al. 2003; Frederiksen et al. 2004; Spitkovsky 2005) have attempted to simulate the long-term nonlinear outcome of the instability, but a consensus on this question has not been achieved yet (Waxman 2006 and references therein); recent results (Spitkovsky 2007) seem to indicate that the small-scale field produced by the Weibel instability decays rapidly over a few tens of ion skin depths and does not persist over distances from the shock transition where the emission originates. One might have thought that if this instability were the source of postshock fields, then \( \epsilon_B \) should have a universal value for highly relativistic, highly collisionless shocks. Yet, while \( \epsilon_B \) is modeled by a constant for individual GRB afterglows, it seems to vary from one afterglow to another (Panaitescu & Kumar 2002; Yost et al. 2003; Panaitescu 2005).

In this paper we explore a traditional MHD explanation for magnetic field growth: turbulence. It is well known in nonrelativistic fluid dynamics that oblique shocks produce or alter the vorticity of a fluid (Ishizuka et al. 1964). In this paper we show that the same is true for an ultrarelativistic shock passing over density inhomogeneities in the preshock circumburst medium. The formalism described in a previous paper (Goodman & MacFadyen 2007, hereafter Paper I) has let us define the vorticity created in an ultrarelativistic fluid in which the energy-momentum tensor can be approximated by that of an ideal fluid with pressure equal to one-third of the proper energy density \( P = \rho / 3 \). In the same work, we have introduced a remarkably simple but accurate general approximation for the local modulation of the shock Lorentz factor \( \Gamma \) by preshock density inhomogeneities; within this approximation, it is not necessary to follow the details of the flow far downstream in order to predict the evolution of the shock, provided that \( \Gamma \gg 1 \), that the preshock pressure is negligible, and that the postshock pressure satisfies \( \rho < \rho / 3 \). This approximation, which is modeled on nonrelativistic results described by Whitham (1974), reproduces exactly the self-similar evolution of \( \Gamma \) for a shock advancing into a cold preshock medium with a power-law density profile in planar, cylindrical, or spherical symmetry (Sari 2006).
More importantly for the present purpose, it allows us to estimate the postshock vorticity resulting from a prescribed preshock density that varies along as well as perpendicular to the shock normal. Given the vorticity, we divide the postshock velocities, which are marginally nonrelativistic in the average postshock rest frame, into vortical and nonvortical parts. We presume that the energy density in vortical motions is a measure of the magnetic energy density that will eventually result after the eddies wind up the field to the point where its backreaction on the turbulence becomes important. These methods are described in § 2.

In § 3 we briefly review the present state of knowledge concerning the density inhomogeneities that may exist ahead of the shock. Both the amplitude and the length scale of the inhomogeneities are important. The former controls the amount of vortical energy—and then of magnetic energy—that is produced, while the latter determines the eddy turnover time of the turbulence, which—when multiplied by the number of eddies necessary to amplify the field up to the observed $\epsilon_B$—must be less than the shock deceleration time, so that the field can be significantly amplified before adiabatic expansion reduces the particle energies available to be radiated. The uncertainties are large because one does not know whether the preshock medium is more like a stellar wind or like some component of the Galactic interstellar medium (ISM) and because the length scales of interest are too small ($\leq 10^{14}$ cm) to be directly resolved even in the ISM. Inhomogeneities on somewhat larger scales have been invoked to explain undulations in afterglow light curves (Wang & Loeb 2000; Lazzati et al. 2002; Schaefer et al. 2003; Nakar et al. 2003).

In § 4 we use the formalism of § 2 to characterize the density contrasts and length scales that preshock clumps should have in order to amplify the magnetic field up to the observed value, in the light of the circumburst picture outlined in § 3. We find that, for smaller shock Lorentz factors, the constraints on clump sizes and overdensities become less stringent; as a consequence, the magnetic energy fraction produced by preshock clumps via macroscopic turbulence is expected to evolve as the shock slows down. In § 5 we comment on the plausibility of our proposed mechanism to explain the magnetic field amplification in GRB afterglows, and we discuss how the results obtained in § 4 could be tested by inferring the time dependence of $\epsilon_B$ from the time evolution of the observed synchrotron cooling frequency as the shock ages.

2. GEOMETRICAL SHOCK DYNAMICS

The evolution of a shock advancing into an inhomogeneous medium depends, in principle, on the details of the downstream flow behind the shock and of the “piston” that drives it. Geometrical shock dynamics (GSD) is an approximation for this evolution in which only the conditions at the shock appear explicitly. Originally formulated by Whitham (1974) for nonrelativistic fluids, GSD has been extended by Paper I to strong ($\Gamma \gg 1$) ultrarelativistic shocks advancing into an ideal fluid whose pressure is negligible ahead of the shock, but one-third of its proper energy density behind the shock ($P = \rho/3$).

The fundamental approximation of GSD is to evaluate the forward-going Riemann characteristics in the postshock flow as if that flow were (1) isentropic and (2) homogeneous far downstream with properties determined by the mean shock speed and the mean preshock density. Actually, preshock density inhomogeneities lead to postshock entropy variations, so assumption 1 is wrong in principle, but it turns out to be a useful fiction. In this respect, an ultrarelativistic flow has the advantage that since pressure depends only on energy density and not on any other thermodynamic variable (such as the proper number density of baryons, $N$), the actual entropy is irrelevant to the Riemann characteristics, which therefore enjoy exact Riemann invariants. So assumption 1 is well justified except insofar as it may be compromised by secondary shocks created by the inhomogeneities themselves behind the main shock. Assumption 2 is reasonable when preshock density inhomogeneities are small in length scale, so that they may be expected to average out far downstream.

With assumption 1, the Riemann invariant on the forward characteristics has the same value just behind the shock as it does far downstream and, therefore, with assumption 2, the same value that it would have in the mean flow. Together with the jump conditions across the shock, this provides a relation between the local Lorentz factor of the shock, $\Gamma$, and the proper preshock energy density $\rho_0 \approx m N_0 c^2$, in which $N_0$ is the proper number density of nucleons ahead of the shock and $m$ is the rest mass per nucleon. In the essentially one-dimensional case that $\rho_0$ varies along the shock normal but not perpendicular to it, the ultrarelativistic GSD relation for the response of the shock Lorentz factor $\Gamma$ to localized and transitory variations in the preshock density $\rho_0$ becomes (Paper I)

$$\Gamma = \tilde{\Gamma} \left( \frac{\rho_0}{\bar{\rho}_0} \right)^{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \approx 0.232,$$

and the corresponding change in the postshock pressure is

$$P = \tilde{P} \left( \frac{\rho_0}{\bar{\rho}_0} \right)^{1-2\frac{k}{3}} \approx \tilde{P} \left( \frac{\rho_0}{\bar{\rho}_0} \right)^{0.536},$$

where the overbars indicate mean values. As in nonrelativistic GSD, these relations can be extended to multidimensional flows in which $\rho_0$ varies laterally as well as longitudinally (with respect to the shock normal), causing convergence or divergence of the shock normals. Equations (1) and (2) are then modified by factors involving the ratio of the local shock area to its mean value (Paper I). It is shown, however, that these corrections are of higher order in $\Gamma^{-1}$ unless the density contrasts are $\sim O(\Gamma)$. For the conditions contemplated in this paper, equations (1) and (2) are adequate even in two or three dimensions.

As in the original nonrelativistic theory, rigorous error estimates for ultrarelativistic GSD are difficult. Informally, the following conditions are probably necessary for the approximation to be useful. First, the preshock medium should be cold, meaning that preshock pressure satisfies $P_0 \ll \rho_0$ and that internal velocities are $\ll c$; this is very likely true of the external forward shocks of GRBs. Second, the length scales of the preshock inhomogeneities should be small compared to the shock radius, so that the shock responds to local perturbations before conditions far downstream have time to react. Third, since $\Gamma^{-1}$ is used as a small parameter, the preshock density fluctuations should not be so large as to cause the shock to become subrelativistic, i.e., one requires $\rho_0/\bar{\rho}_0 \ll \Gamma^{-1/2}$. Finally, transitions in density should not be so abrupt as to cause strong reverse shocks, which would alter the forward shock dynamics. Paper I describes tests of ultrarelativistic GSD by comparison with exact self-similar solutions (some of which it reproduces exactly) and with numerical simulations. The latter indicate, for example, that equations (1) and (2) are in error by only a few percent for a $\Gamma = 10^2$ shock encountering overdensities as large as $\rho_0/\bar{\rho}_0 \approx 30$.

2.1. Relativistic Vorticity

This subsection is independent of GSD. We review the meaning of enthalpy current and vorticity in ideal relativistic fluids, especially those with the ultrarelativistic equation of state $P = \rho/3$. 
When the shock passes over a local density excess—considered, for simplicity, in isolation from other inhomogeneities—the resulting postshock velocities are of two kinds. First, since the shocked clump is overpressured compared to its postshock surroundings (eq. [2]), it will expand and drive an outgoing pressure wave. If the density contrast of the clump is small, then the wave is essentially a linear disturbance from the start and travels at the sound speed, $c/\sqrt{3}$, in the rest frame of the mean postshock flow; waves launched by large overdensities will be somewhat faster and may steepen into secondary shocks, but whatever its strength, the pressure wave rapidly departs from its source. Overlapping pressure waves launched by many distant clumps may contribute significant local velocity perturbations, but because of their oscillatory nature, intuition suggests that these velocities will not secularly amplify the magnetic field (except insofar as secondary shocks may contribute to vorticity; see § 4). It would be interesting to test this expectation in numerical simulations.

Unless the density excess is constant along the shock front, the postshock velocity field will also contain a vortical component, whose strength is estimated below for an initially spherical over-density with a Gaussian radial profile. As shown in Paper I, the equation of state $P = \rho/3$ allows some freedom in how one defines relativistic vorticity, but to be useful in constraining the evolution of the flow, the vorticity should be associated with a conservation law such as Kelvin’s circulation theorem,

$$\frac{d}{dt} \int C \, H_{\mu} \, dx^\mu = 0,$$

where $C$ is a closed contour comoving with the fluid four-velocity $U^\mu$, and $H^\mu = h U^\mu$ for an appropriate thermodynamic function $h$ (see below). Equation (3) is equivalent to

$$\frac{\partial \omega}{\partial t} - \nabla \times (\mathbf{v} \times \omega) = 0, \quad \omega \equiv c \nabla \times \mathbf{H},$$

where $v^i \equiv c U^i / U^0$ is the three-velocity of the fluid, and $\mathbf{H}$ is the spatial part of $H^\mu$. Although formally identical to the nonrelativistic vorticity equation and written with three-vectors, equation (4) is actually relativistically covariant.

Because vorticity and circulation travel with the local fluid velocity and are nonoscillatory in the local fluid frame, they have the potential to twist up and secularly amplify any magnetic field frozen into the flow. A fundamental assumption of this paper is that the magnetic energy will eventually reach equipartition with the kinetic energy invested in the vortical part of the flow. This assumption would also be well worth testing numerically.

For a general equation of state $P = \rho/3$, where $N$ is the proper number density of conserved particles (e.g., baryons), it is conventional to take the quantity $h$ in the relation $H^\mu = h U^\mu$ to be the enthalpy per particle, $h \equiv (\rho + P)/N$. One assumes an ideal fluid, so that the energy-momentum tensor is

$$T^{\mu\nu} \equiv (\rho + P) U^\mu U^\nu + P g^{\mu\nu},$$

where $g^{\mu\nu} \rightarrow \text{diag}(-1, 1, 1, 1)$ in Minkowski coordinates, and the equations of motion are

$$T^{\mu\nu}_{\ ;\nu} = 0, \quad (NU^\mu)_{\ ;\mu} = 0.$$

The first law of thermodynamics $dh = T \, dS + N^{-1} \, dP$ implies $h_{\ ,\mu} = P_{\ ,\mu}/N$ if the fluid is isentropic; the first equation of motion in equation (6) can then be recast as

$$U^\nu H^\mu_{\ ;\nu} = -h_{\ ,\mu},$$

the “curl” of which implies equations (3) and (4) (e.g., Eshraghi 2003; Paper I). These relations do not hold across shocks, of course, since there is an entropy jump. But if the preshock medium is inhomogeneous, then even if the flow behind the shock is smooth, it will not be isentropic in general; that is, $P/N^{4/3}$ (for an ultrarelativistic equation of state) will not be uniform. Therefore, the conventional choice of $h$ does not lead to a conserved circulation under the circumstances contemplated in this paper. Fortunately, as pointed out in Paper I, if one defines $H^\mu$ using $h \propto P^{1/4}$ instead of the true enthalpy, then equation (7) always holds in smooth parts of the flow. This is a consequence of the equation of state $P = \rho/3$, which is “barytropic” if not isentropic. It is convenient to choose the constant of proportionality so that $H^\mu$ reduces to the fluid four-velocity when pressure is uniform. Therefore, we replace the conventional enthalpy current with

$$H^\mu \equiv (P/P) \, P^{1/4} \, U^\mu.$$

With this choice, circulation is conserved (eqs. [3] and [4]) everywhere except across shocks.

### 2.2. Vorticity Production by Shocks

The goal of this subsection is to use the one-dimensional GSD approximation to derive equations (15)–(16), which relate the postshock vorticity to the preshock fractional overdensity $\delta = (\rho/\rho_0) - 1$. Figure 1 illustrates four stages in the interaction of an ultrarelativistic shock with a density clump.

We begin by recalling some basic consequences of the shock jump conditions that will be needed below. Let $O$ denote the discontinuity in a fluid property $Q$ across the shock front. In the instantaneous local rest frame of the shock, where the outward unit normal to the shock front is $\mathbf{n}$, the jump conditions are $\{T^{\mu\nu} \eta_n \} = 0$. Using equation (5) for $T^{\mu\nu}$ (ideal fluid) and assuming that $P_0 \ll \rho_0$ ahead of the shock and $P = \rho/3$ behind it, one finds that the post-shock three-velocity of the fluid is $v \cdot n = -c/3$ in the shock frame (hence subsonic, since the sound speed is $c/\sqrt{3}$). The postshock energy density is $\rho = 2I^2/\rho_0$, where $\Gamma \gg 1$ is the local Lorentz
factor of the shock in the rest frame of the preshock fluid. Similarly, conservation of particles implies \(|NU^i n_i| = 0\), whence \(N = 2\sqrt{2}\Gamma N_0\). The quantities \(\rho, P\), and \(N\) always denote proper values, meaning that they are defined in the local fluid rest frame and are therefore Lorentz invariants by fiat, with subscript “0” denoting a preshock value rather than a spacetime index.

To facilitate Lorentz boosts between the preshock and postshock or shock frames, it is often convenient to use the rapidity parameter \(\tanh^{-1}(v/c)\), where \(v\) is the three-velocity in the preshock frame. The relativistic addition of colleen three-velocities is equivalent to addition of the corresponding rapidity parameters. Thus, for example, using \(\Phi = \cosh^{-1}\Gamma \approx \ln(2\Gamma)\) for the rapidity parameter of the shock and \(\phi\) for the rapidity of the postshock fluid, it follows from the above that \(\tanh(\phi - \Phi) = -1/3\), whence \(\phi = \Phi - \ln\sqrt{2}\). The Lorentz factor of the postshock fluid relative to the preshock frame is \(\cosh\phi = \cosh(\Phi - \ln\sqrt{2}) = \Gamma/\sqrt{2} + O(\Gamma^{-1})\).

Preshock clumps will typically have comparable longitudinal and lateral dimensions (meaning along and perpendicular to the mean direction of shock propagation) in their own rest frame. In the shock and postshock frames, the clumps will be longitudinally contracted by factors \(\sim \Gamma^{-1} \ll 1\). During the transit of the shock over a clump and even during the subsequent expansion of the shocked clump as it comes to pressure equilibrium with the surrounding postshock fluid, there will not be enough time for signals (sound waves) to communicate laterally from one end of the clump to the other. Therefore, the interaction of the shock with the clump can be calculated in a one-dimensional approximation, in which the area of the shock is constant and the preshock mass density \((\rho_0 c^2)\) is stratified on planes parallel to the shock front.

Nevertheless, lateral density gradients do produce postshock vorticity, \(\omega\), even in our one-dimensional approximation. First of all, the longitudinal component \(H_L\) of the nonconventional enthalpy current defined in equation (8) varies with lateral position behind the shock. Second, since the shock itself is delayed differently at different lateral positions, the shock normal develops a lateral component relative to the other. Therefore, the interaction of the shock with the clump is determined implicitly by \(\n = [\n_0]^{-1}\n_0\tau\n_\tau\) (the subscript reminds that derivatives are taken ahead of the shock). To leading order in \(\Gamma^{-1}\), the lateral components of the normal are therefore

\[
\n_\perp \approx \lambda^{-2}\n_\perp I = \frac{\lambda^{-2}}{c^2} \int_0^{\infty} \big(1 + \delta'^2\big)^{2-1} (\n_\perp \delta') dz'_0, \tag{11}
\]

where we have assumed that the typical size of a density clump is small compared to the characteristic length scale for variations in \(\Gamma\). The shock surface at time \(t\) is determined implicitly by \(\tau(r) = t\), so that the shock normal is \(\n = [\n_0]^{-1}\n_0\tau\n_\tau\) (the subscript reminds that derivatives are taken ahead of the shock).

Of course the lateral components of the enthalpy current take the same values in the mean postshock and preshock frames, since the Lorentz factor is the same in the preshock and postshock reference frames. The lateral part of the postshock fluid four-velocity is \(U_\perp \approx n_\perp U_\parallel = n_\perp \Gamma /\sqrt{2}\). With use of equations (8) and (9), the postshock lateral enthalpy current becomes

\[
H_\perp \approx 2^{-1/2} \lambda^{-1}(1 + \delta)'(1 - \delta')^{3/4} \n_\perp I. \tag{12}
\]

The lateral component of the enthalpy current takes the same values in the mean postshock and preshock frames, since these frames differ by a longitudinal boost. However, in order to compute the vorticity in the mean postshock fluid frame, we should remember that the longitudinal derivative in this reference system \(\n_\parallel\) is related to the corresponding derivative \(\n_\parallel\) in the mean postshock frame by \(\n_\parallel = \cosh\phi\n_\parallel\) because of Lorentz contraction. Thus, the contribution of \(H_\perp\) to the postshock vorticity is

\[
\n_\parallel \times H_\perp \approx \frac{1}{2} \lambda(1 + \delta)'^{2-3/4} (\n_\parallel \times \n_\parallel)
+ \frac{\lambda(1 - \delta)}{8} (1 + \delta)^{-3(1 + 2\delta)/4} \frac{\partial\n_\parallel}{\partial\n_\parallel} (\n_\parallel \times \n_\parallel); \tag{13}
\]

where \(\n_\parallel = e_\parallel\) is the mean shock normal and clearly the differential operator \(\n_\parallel \times \n_\parallel\) is the same in the preshock and postshock mean rest frames.

A comparable contribution to the postshock vorticity comes from \(H_L\). The longitudinal component of the postshock four-velocity is \(U_\parallel = \sinh(\phi - \Phi)e_\parallel\) when measured in the mean postshock frame. Since \(\phi - \Phi = \Phi - \Phi = -\lambda\ln(1 + \delta)\),

\[
U_\parallel = -\frac{1}{2} \left[(1 + \delta)^2 - (1 + \delta)^{-2}\right] e_\parallel \tag{14}
\]

in the mean postshock frame. This tends to be quite subluminal; for unit overdensity (\(\delta = 1\)), for example, \(U_\parallel \approx -0.1615e_\parallel\), and the corresponding fluid Lorentz factor is \(\approx 1.013\). The longitudinal postshock enthalpy current is \(H_L = \left(P/\rho\right)^{1/4} U_\parallel = (1 + \delta)^{(1 - 2\delta)/4} U_\parallel\).

Taking the lateral derivative of this and combining with equation (13) and the definition of \(I\) in equation (10), we can write the total postshock vorticity as

\[
\omega \approx \epsilon(\n_\parallel \times \n_\parallel)(f + \epsilon) \frac{(1 - \delta)}{8} (1 + \delta)^{-3(1 + 2\delta)/4}
\times \left[\frac{\partial\n_\parallel}{\partial\n_\parallel}\right] \int_0^\infty (1 + \delta)^{2-1}(\n_\parallel \times \n_\parallel) dz'_0, \tag{15}
\]

where the coordinate system is in the preshock rest frame and the derivatives are taken ahead of the shock. We have introduced the function

\[
f(\delta) \equiv (1 + \delta)^{-(1 - 2\delta)/4} \left[\frac{6\delta + 1}{8}(1 + \delta)^2 + \frac{6\delta - 1}{8}(1 + \delta)^{-2}\right]. \tag{16}
\]
For \( |\delta| < 1, f(\delta) \approx 3\lambda/2 \approx 0.348 \). It is interesting that in the total postshock vorticity there are no surviving factors of \( \Gamma \).

2.3. Vortical Energy

We have not been able to formulate a rigorous relativistically covariant way of dividing the energy of the postshock flow into vortical and nonvortical parts. In a nonrelativistic flow, however, this would be straightforward. One would divide the three-velocity field into potential and solenoidal parts,

\[
e = \nabla \psi + \nabla \times A, \tag{17}
\]

and then define the vortical energy by

\[
E_{\text{vort}} = \frac{1}{2} \int \rho_m |\nabla \times A|^2 d^3x, \tag{18}
\]

where \( \rho_m \) is the nonrelativistic mass density. To make the decomposition from equation (17) unique, some mild additional restrictions are necessary: for example, that the region of interest is simply connected and that \( \psi \) and \( A \) have some specified behavior on the boundary or at infinity. Then one can impose \( \nabla \cdot A = 0 \) and solve for \( A \) from

\[
\nabla^2 A = -\omega, \tag{19}
\]

with \( \omega = \nabla \times r \), using an appropriate Green’s function to invert \( \nabla^2 \). The nonvortical part follows similarly from \( \nabla^2 \psi = -\nabla \cdot r \).

It is not clear how to proceed in the relativistic case, because the coordinate energy density \( T^{00} \) is not simply quadratic in \( r \), but is a general function of state, \( \rho \), \( \Theta \), and \( T \). For the present purpose, however, we will assume that \( T^{00} \) is a linear function of \( \rho \) and \( \Theta \), and that \( \rho \) is a constant function of \( \Theta \).

As soon as the clump is at the same pressure as its surroundings, the vorticity of the fluid in turbulent motion and the fluid in turbulent motion is approximately equal [neglecting terms of order \( \rho/\Theta \)], with \( \rho/\Theta \) being approximately equal to \( \rho/\Theta_0 \), the average postshock value. Then the vortical energy of a single clump becomes

\[
E_{\text{vort}} = \frac{4}{3} \overline{\rho} \int \frac{|\nabla \times A|^2}{c^2} d^3x
\]

where \( J_1(kR) \) is the Bessel function of order 1 and \( \sigma(k) \) is the Hankel transform of the projected vorticity \( \sigma(R) \).

\[
\sigma(k) = \int_0^{+\infty} \overline{\rho} J_1(kR)R dR. \tag{22}
\]

As soon as the clamp is at the same pressure as its surroundings, if the vortical motions are subrelativistic, we can use the decomposition from equation (17) with \( v_{\text{vort}} = \nabla \times A \). Moreover, pressure equilibrium between the clamp and the average postshock medium implies that the clamp proper energy density \( \rho \) is approximately equal [neglecting terms of order \( O\left(\rho/v^2\right) \)] to the mean postshock value \( \overline{\rho} \). Then the vortical energy of a single clump becomes

\[
E_{\text{vort}} = \frac{4}{3} \overline{\rho} \int \frac{|\nabla \times A|^2}{c^2} d^3x
\]

In accordance with the discussion following equation (19), we have replaced the factor 1/2 in equation (18) with 4/3. In addition, we have used integration by parts to replace the integral over all space with an integral over the source of vorticity only. This is particularly convenient when the source is represented by a vortex.
We assume that the vortical energies of different clumps can simply be added. This is justified if the clumps are well separated compared to their larger (i.e., lateral) dimensions. Then if the number density of clumps in the preshock frame is \( N_{c,0} \) and all the clumps have the same axisymmetric density profile, the vortical energy density in the average postshock frame is \( \rho_{\text{vort}} = \alpha N_{c,0} E_{\text{vort}} \), and the vortical energy fraction becomes

\[
\epsilon_{\text{vort}} = \frac{\rho_{\text{vort}}}{\rho} = \frac{4\pi}{3} \alpha N_{c,0} \int_0^{+\infty} \frac{\tilde{\sigma}^2(k)}{c^2} dk.
\]

(25)

Recall from equation (20) that the projected vorticity is proportional to \( \alpha^{-1} = (2\sqrt{2}/\Gamma)^{-1} \). Since this factor is squared in computing the vortical energy, it follows from equation (25) that \( \epsilon_{\text{vort}} \propto \Gamma^{-1} \) for a fixed preshock density field. This scaling is perhaps the most important conclusion of our analysis up to this point.

3. THE CIRCUMBURST MEDIUM

Observations support the idea that long-duration GRBs are associated with the deaths of massive Wolf-Rayet (WR) stars, presumably arising from their core collapse (Woosley & Bloom 2006 and references therein). Then the circumburst environment is determined by the star’s mass-loss history. At the onset of the WR phase, the WR stellar wind is expected to expand with a typical velocity \( v_{\text{WR}} \approx 2000 \text{ km s}^{-1} \) inside the preexisting slower wind emitted during the red supergiant (RSG) phase, whose characteristic speed is \( v_{\text{RSG}} \approx 20 \text{ km s}^{-1} \). The winds from massive RSGs are characterized by a mass-loss rate \( \dot{M}_{\text{RSG}} \) between \( 10^{-6} \) and \( 10^{-4} M_{\odot} \text{ yr}^{-1} \) (Chevalier et al. 2006), while the mass-loss rates \( \dot{M}_{\text{WR}} \) of WR stars are between \( 10^{-5} \) and \( 10^{-4} M_{\odot} \text{ yr}^{-1} \) (Crowther 2007). Several solar masses are shed by the star during these evolutionary phases. The mass equivalent to the energy of a GRB, on the other hand, is only \( \approx 0.06 \dot{M}_{\text{iso},53} M_{\odot} \). Therefore, the GRB forward shock is expected to become nonrelativistic long before it escapes the wind to encounter the ISM. There are at least four regions of the wind that are relevant to the relativistic phase of the afterglow (Ramirez-Ruiz et al. 2005): from the inside out, these are an expanding WR wind \( (\rho_0/c^2 = \dot{M}_{\text{WR}}/4\pi v_{\text{WR}} r^2) \), where \( r \) is the distance from the star), the shocked WR wind \( (\rho_0 \approx \text{const}) \), the shocked RSG wind \( (\rho_0 \approx \text{const}) \), and a freely expanding RSG wind \( (\rho_0/c^2 = \dot{M}_{\text{RSG}}/4\pi v_{\text{RSG}} r^2) \). Beyond these lie another shocked part of the RSG wind, the shocked ISM, and finally the unshocked ISM.

Density inhomogeneities in such a stratified structure could be produced by several processes. First of all, the acceleration of the RSG wind, the shocked ISM, and finally the unshocked ISM. Therefore, in the estimates of the required turnover time made below, we simply assume that the magnetic energy fraction \( \epsilon_B \) in the preshock medium is comparable to that in the ISM, \( \approx 10^{-8} \), so that the number of eddy rotations necessary to explain the value of \( \epsilon_B \) inferred from afterglow observations is of order \( N_{\text{edd}} \approx 10 \). The microscopic magnetic field created by plasma instabilities at the shock might serve as a seed field for turbulent amplification if it cascades to larger scales before dissipation; fewer (if any) eddy turnovers would then be required, since the Weibel magnetic energy fraction at the shock peaks at a few tens of percent (Spitkovsky 2007). Under these circumstances, however, the Weibel instability might produce the required fields without the benefit of macroscopic amplification.
4. RESULTS

The formalism described in § 2 can be used to predict the eddy turnover time and vortical energy fraction of the turbulent motions produced by an ultrarelativistic shock sweeping up a clumpy medium; if the preshock average (i.e., smoothed over clumps) density profile is known, a comparison with afterglow models will then let us constrain the length scale, overdensity, and volume filling factor of the circumburst inhomogeneities. Unfortunately, observations have not yet clearly determined the density profile ahead of the GRB forward shock (Chevalier & Li 2000; Panaitescu & Kumar 2001, 2002; Chevalier et al. 2004), so that we consider both free winds (energy density $\rho_0 \propto r^{-2}$) and shocked uniform winds ($\rho_0 \approx \text{const}$) as possible circumburst media, keeping in mind that, as described by Ramirez-Ruiz et al. (2005) and outlined in § 3, the actual surrounding environment is much more complex.

Assuming that the blast wave is adiabatic and effectively spherical and that $E_{\text{iso}} = 10^{53}E_{\text{iso},53}$ erg is its isotropic equivalent energy, as derived from the gamma-ray output, we can compute the deceleration radius of the GRB forward shock when about half of the initial energy has been transferred to the shocked matter; for an initial Lorentz factor $\Gamma_0 = 10^2$, the typical swept-up mass where this happens is

$$M_{\text{dec}} = \frac{E_{\text{iso}}}{\Gamma_0^2 c^2} \approx 5.6 \times 10^{-6}E_{\text{iso},53}\Gamma_0^{-2}M_\odot.$$  \hspace{1cm} (26)

For the two circumburst density profiles mentioned above, the deceleration radius of the shock is then

$$r_{\text{dec}} = \begin{cases} r_{\text{dec},\text{free}} & \\
\frac{3M_{\text{dec},\text{free}}}{4\pi n_0 r_{\text{iso}}^2} & \text{free winds} \\
\frac{3M_{\text{dec}}}{4\pi n_0 r_{\text{iso}}^2} & \text{shocked uniform winds}
\end{cases},$$

where $\rho_0 \propto r^{-2}$, and

$$\Gamma_0 \approx 1.2 \times 10^{17} \left(\frac{E_{\text{iso},53}}{n_{\text{ISM,0}}\Gamma_0^{-2}}\right) \text{cm}, \quad \rho_0 \approx \text{const},$$

with $\rho_0$ the energy density of the interclump homogeneous medium. In order to use the formalism introduced in § 2—where the density contrast $\delta$ was defined with respect to the mean preshock energy density $\bar{\rho}_0$ (averaged over clumps), whereas here the interclump density $\rho_{\text{ext}}$ has been used—we should set

$$\delta(r, z_0) = \frac{\rho_{\text{ext}}}{\bar{\rho}_0} \left[1 + \delta_{\text{max}} e^{-(R^2 + z_0^2)/L^2}\right] - 1;$$

provided that $\delta_{\text{max}} N_{c,0} L^3 \lesssim 1$. In this section we always assume $\rho_{\text{ext}}/\bar{\rho}_0 \approx 1$, so that the formulae in § 2 can be used with a preshock overdensity profile

$$\delta(r, z_0) = \delta_{\text{max}} e^{-(R^2 + z_0^2)/L^2}.$$  \hspace{1cm} (32)

For small density contrasts, we can perform analytical calculations taking into account just the leading-order (first-order) term in $\delta$ within equation (15); the corresponding vorticity is referred to as the “leading-order vorticity” $\omega_{\text{lo}}$, and the Hankel transform of its projected vorticity is used in equation (25) to analytically compute the leading (second) order in $\delta_{\text{max}}$ of the vortical energy fraction ($e_{\text{vort,lo}}$). This approximation is certainly well justified for $\delta_{\text{max}} \ll 1$, but Figure 2 shows that the full numerical computation for $e_{\text{vort,lo}}$ is still in reasonable agreement with our analytical approximation $e_{\text{vort,lo}}$ even for $\delta_{\text{max}} \sim 1$. For larger central overdensities,
a numerical calculation is required, and we could fit the numerical results with a fitting function

\[ \epsilon_{\text{vert}} = \epsilon_{\text{vert}, \text{lo}} \frac{1}{1 + c_1 (\delta_{\text{max}})^{c_2}} \]  

(c_1 and c_2 are fitting parameters), so that for small overdensities (\(\delta_{\text{max}} \ll 1\)) we recover the result of the analytical computation.

4.2. Eddy Turnover Time

Vorticity embedded in clumps by the passage of the GRB forward shock can be responsible for the magnetic fraction inferred from afterglow models only if vortical motions are fast and energetic enough to amplify the field up to the observed value before the shock deceleration time, when adiabatic expansion would significantly reduce the particle energies available for the afterglow emission. In the leading-order approximation described above, an estimate of the eddy turnover time for the density contrast in equation (32) is

\[ \tau_{\text{eddy}} = \frac{1}{[\omega_{\text{lo}}(L, 0)]} \approx 6.5 \times 10^2 L_{13}^2 \delta_{\text{max}} \]  

where \(L = 10^{13} L_{13} \text{ cm}\); the reference value chosen for the central overdensity \(\delta_{\text{max}}\) is in agreement with observations (see §3) and reasonably satisfies the requirements for the leading-order approximation (see Fig. 2). If \(N_{\text{eddy}} = 10 N_{\text{eddy}, 1} \) is the number of eddy rotations necessary to explain the observed \(\epsilon_B\), the requirement \(N_{\text{eddy}} \tau_{\text{eddy}} \leq t_{\text{dec}}\) for a windlike or ISM-like circumburst medium gives, respectively,

\[ L_{13}^2 \delta_{\text{max}} \leq \begin{cases} 0.3 E_{\text{iso}, 53} V_{\text{WR}, 8.3} M_{\text{c}, 7}^{-1} \Gamma_{-5}^{-3} N_{\text{eddy}, 1}^{-1}, & \tilde{\rho}_0 \propto r^{-2}, \\ 8.7 (E_{\text{iso}, 53} / n_{\text{ISM}, 0})^{1/3} \Gamma_{-5}^{-2} N_{\text{eddy}, 1}^{-1}, & \tilde{\rho}_0 \approx \text{const}. \end{cases} \]  

Let us emphasize that, for smaller initial shock Lorentz factors \(\Gamma_0\), a larger size \(L\) and weaker overdensity \(\delta_{\text{max}}\) are enough to satisfy the constraint \(N_{\text{eddy}} \tau_{\text{eddy}} \leq t_{\text{dec}}\).

4.3. Vortical Energy Fraction

The kinetic energy density invested in the vortical part of the postshock flow should be a significant fraction of its proper energy density, since—as stated in §2—we assume that the amplified magnetic fraction, which is required to match the observational inference \(\epsilon_B \approx 10^{-3}\), will eventually be comparable to the vortical fraction, \(\epsilon_{\text{vert}} \sim \epsilon_{\text{vert}, \text{lo}}\). However, \(\epsilon_B\) might be smaller than \(\epsilon_{\text{vert}}\) if the backreaction of magnetic field on turbulent motions is important well before equipartition between the magnetic and vortical energy densities; on the other hand, \(\epsilon_{\text{vert}}\) may also be a lower limit for \(\epsilon_B\) if secondary shocks created by the overlapping sound waves of many different overpressured clumps significantly contribute to magnetic field amplification.

The results of a numerical calculation for the vortical fraction of clumps with overdensity profile in equation (32) are shown in Figure 3 (solid line), but for small central overdensities, the leading-order analytical computation described above gives a reliable estimate of the magnetic energy fraction produced by turbulence in GRB afterglows,

\[ \epsilon_B \sim \epsilon_{\text{vert}, \text{lo}} \approx 1.8 \times 10^{-3} \tilde{G}_{-5}^{-1} \left(\frac{\delta_{\text{max}}}{2}\right)^2 N_{c, 0} L_3^{3/2} / 0.25, \]  

where \(\tilde{G} = 10^2 \tilde{G}_{-5}\) and the reference values for the central overdensity \(\delta_{\text{max}}\) and the clump volume filling factor \(N_{c, 0} L_3\) have been chosen in order to match the observational constraint \(\epsilon_B \approx 10^{-3}\). So, clumps with moderate density contrasts (\(\delta_{\text{max}} \approx 2\), in agreement with the observational evidences reviewed in §3) can justify the lower limit of the magnetic energy fraction inferred from afterglow models. Higher density contrasts (see Fig. 3) would be necessary to achieve larger magnetic fractions; however, it is worth recalling that our model is applicable only under the assumptions that the clump overdensity is not larger than \(\Gamma^{1/2}\) (see §2) and that \(\delta_{\text{max}} N_{c, 0} L_3 \leq 1\), so that the overdensity profile can actually be described by equation (32).

Equation (36) suggests that smaller overdensities would be enough to satisfy the observational constraints on \(\epsilon_B\) as the forward shock slows down, since \(\epsilon_B \propto \tilde{G}^{-1}\); the magnetic fraction is then expected to change during the afterglow stage, whereas it is usually supposed constant, especially in models that invoke collisionless plasma instabilities to create the field. Therefore, the time dependence of \(\epsilon_B\) might be used as a test of our model. In §5 we discuss the prospects for constraining the evolution of \(\epsilon_B\) from afterglow observations.

5. SUMMARY AND DISCUSSION

We have proposed that the postshock magnetic fields of GRB afterglows may arise from macroscopic MHD turbulence rather than microscopic plasma instabilities. The source of turbulence is vorticity produced when the shock encounters density inhomogeneities in the preshock medium. We presume that the magnetic energy fraction (\(\epsilon_B\)) that results is comparable to the energy fraction of the turbulence. The ultrarelativistic geometrical shock dynamics formalism of Paper I allows an easy, although approximate, calculation of the vorticity produced by a given density inhomogeneity in the limit that the shock Lorentz factor \(\Gamma \gg 1\).

In this picture, the observational inference that \(\epsilon_B \approx 10^{-3}\) constrains both the amplitude and the length scale of inhomogeneities.
Equations (33) and (36) roughly relate the total energy fraction in vortical motions to the volume filling factor and density contrast of the clumps; this energy fraction must be comparable to the inferred postshock $\epsilon_B$. Filling factors and density contrasts of order unity are required when the shock is still highly relativistic. Equation (35), on the other hand, express the constraint on the length scale and density contrast of individual clumps (independently of their volume filling factor) so that the eddies can wind the magnetic field up to the observed value in less than the expansion time of the shock. This second constraint favors small length scales, so that the clumps responsible for field amplification would probably be too small, at least individually, to modulate the afterglow light curve unless the density contrasts of those clumps are very large. D"{e}rmer & Mitman (1999) have indeed proposed very dense clumps ($\bar{n}_\text{max} \approx 10^5$) in their model for prompt emission by the forward shock. While their scenario might also produce strong turbulence and field amplification, the density contrasts maybe beyond the range of validity of our approximations.

There is a question whether a fluid treatment of the postshock flow is justified at all, since the plasma is collisionless. The same question arises in supernova remnants, to which the standard answer is that magnetization of the plasma ensures a short effective mean free path. The present case is more extreme because the particles are relativistic, the length scales on which fluidlike behavior is required are smaller, especially in the present work, and the preshock field is energetically negligible. The relativistic Larmor radius of the postshock ions based on the compressed ambient field is $r_{L,i} \approx \Gamma m_p c^2/eB \approx 10^{12} (3 \mu G/B_0) \text{ cm}$, where $B_0$ is the preshock field strength. This is smaller than the maximum tolerable clump size for field amplification at the beginning of the afterglow phase, although only barely so ($\simeq 4.2$). Furthermore, whether or not the Weibel instability can produce persistent magnetic fields, it should enforce fluidlike behavior by isotropizing particle distribution functions whenever counterstreaming plasmas overlap.

A basic conclusion of this work is that vortical turbulence becomes easier to produce with decreasing shock Lorentz factor. Both the energy and timescale constraints become easier to satisfy as $\Gamma$ decreases (however, our ultrarelativistic approximations break down as $\Gamma \rightarrow 1$). Therefore, if the postshock magnetic energy density is produced by macroscopic turbulence, it is likely that $\epsilon_B$ will evolve as the shock ages, complicating the task of drawing physical inferences about the GRB environment from the observational data. The abundance of early X-ray light curves provided by the Swift satellite has already led to models that are more complicated than the rather simple theoretical description of the sparser BeppoSAX results (Galama et al. 1998). The light curves are not single power laws in time, but show breaks and sometimes “flares,” suggesting a need for extended energy input from the central source (Zhang et al. 2006 and references therein). But, to date, most models have assumed constant $\epsilon_B$ and $\epsilon_e$ (the postshock energy fraction in relativistic $e^\pm$) within individual afterglows, although these parameters are often allowed to vary from one afterglow to another.

The effect of an evolving $\epsilon_B$ depends on the relationship between the observed frequency and certain critical frequencies in the assumed spectrum. In our model, the coherence length of the field should be comparable to that of the eddies responsible for amplifying it (provided that the coherence length of the seed field is even larger), which is plausibly larger than the relativistic Larmor radius as noted above, so that standard synchrotron emission should dominate. On the other hand, for the small-scale field generated by Weibel-like plasma instabilities, the electron gyroradii are often larger than the magnetic field structures, so that jitter radiation (Medvedev 2000) might be the actual emission mechanism. A detailed analysis of the observed spectra at frequencies below the spectral peak, where jitter and synchrotron spectra are predicted to differ most strongly, might therefore distinguish between these two models for the amplification of magnetic field.

In order to compare our prediction of an evolving $\epsilon_B$ with the observed afterglow spectra, particularly important is the synchrotron cooling frequency, the Doppler-shifted synchrotron frequency of a postshock electron or positron that radiates much of its energy on a timescale comparable to the age of the shock. For a preshock medium with mass density profile $\rho_m(r) \equiv c^{-3} \rho_m = K r^{-\gamma}$ averaged over clumps and for an adiabatic relativistic shock with constant isotropically equivalent energy $E_{\text{iso}}$ (notwithstanding the above-cited inferences from Swift data), the cooling evolves as (up to dimensionless constants of order unity)

$$\nu_{\text{cool}} \approx \frac{e m_e}{(1 + z_{\text{GRB}})^{2/5} r_s^{5/2}} \frac{e m_e}{K (1 + z_{\text{GRB}})^{2/5} r_s^{5/2}} \times \left[ \frac{E_{\text{iso}}}{c} (1 + z_{\text{GRB}})^{-1} \Gamma \right]^{(3/5 - 4)/4} \approx \frac{e m_e}{K (1 + z_{\text{GRB}})^{2/5} r_s^{5/2}} \times \left[ \frac{E_{\text{iso}}}{c} (1 + z_{\text{GRB}})^{-1} \Gamma \right]^{(3/5 - 4)/4},$$

where $r_s$ is the shock radius, $\Gamma$ is the astronomical observer’s time, and $z_{\text{GRB}}$ is the GRB cosmological redshift. Evidently, the shock energy and Lorentz factor scale out of the cooling frequency when the latter is expressed in terms of the shock radius. If one could be confident that the early afterglow evolves in a freely expanding wind ($\omega = 2$), which seems a priori likely in collapsar models, then the evolution of $\epsilon_B$ could be inferred by measuring that of $\nu_{\text{cool}}$.

Present evidence suggests that the cooling frequency lies below the X-ray regime in the early afterglow phase. This conclusion rests on the usual assumption that the synchrotron-emitting electrons are injected with a power-law distribution of energies, $N(\gamma) \gamma^{-\alpha} \propto \gamma^{-\alpha} d\gamma$ for $\gamma > \gamma_{\text{min}} > 1$, in which $p > 2$ so that the total energy is dominated by electron energies near $\gamma_{\text{min}} m_e c^2$, whose characteristic observed frequency is $\nu_{\text{min}}$. The observed specific flux is often described as a power law in time and frequency, $F_\nu \propto \nu^{-\beta}$, despite various breaks and the aforementioned flares. If synchrotron emission dominates and $\nu_{\text{cool}} > \nu_{\text{min}}$ (slow-cooling regime), the spectral index is $\beta = (p - 1)/2$ if $\nu_{\text{min}} < \nu < \nu_{\text{cool}}$ and $\beta = p/2$ if $\nu > \nu_{\text{cool}}$. It is believed that the acceleration index $p$ is not much larger than 2, perhaps $p \approx 2.2 - 2.3$, which is then consistent with the observed X-ray indices $\beta = 1 - 1.5$ observed by Swift (Zhang et al. 2006 and references therein) only if $\nu_{\text{cool}} < 1$ keV. Evaluating equation (37) at the deceleration radius appropriate for a Wolf-Rayet wind (see eq. [27]), we find that the afterglow phase begins with a cooling frequency that is plausibly consistent with this constraint:

$$h \nu_{\text{cool, dec}} \approx 0.2 (1 + z_{\text{GRB}})^{-1} (10^4 \epsilon_B)^{-3/2} \times 5^{5/2} \Gamma_{\text{WR}}^{-5/2} M_{\text{WR}}^{5/2} \frac{E_{\text{iso}}}{5.3} \Gamma_{\text{iso}}^{-2} 0.2 \text{ keV}.$$  

(38)

Unless the cooling frequency passes through the observed band, one cannot learn much about the evolution of $\epsilon_B$ from observations at $\nu > \nu_{\text{cool}} > \nu_{\text{min}}$, because (up to numerical factors of order unity)

$$\nu F_\nu \approx \frac{\epsilon_e E_{\text{iso}}}{4 \pi d_L^2 (1 + z_{\text{GRB}})^{-1} \Gamma^{(\gamma_{\text{min}})^{2-p}}} \left( \frac{\gamma}{\gamma_{\text{min}}} \right)^{2-p}$$

in this regime, where $d_L$ is the luminosity distance; $\epsilon_B$ enters the above expression only via the correspondence $\nu \approx (1 + z_{\text{GRB}})^{-1} (\Gamma^{(\gamma_{\text{min}})^2}) (\epsilon_B \rho_m)^{1/2} e/m_e$ between observed frequency $\nu$ and
electron Lorentz factor $\gamma$ and, hence, is raised to the small exponent $(p - 2)/4 \leq 0.1$. On the other hand, equation (39) indicates that the flux above the cooling frequency provides an excellent measure of the energy in the electron population (Freedman & Waxman 2001).

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REFERENCES

Castor, J. I., Abbott, D. C., & Klein, R. I. 1975, ApJ, 195, 157
Chevalier, R. A., Fransson, C., & Nymark, T. K. 2006, ApJ, 641, 1029
Chevalier, R. A., & Li, Z.-Y. 2000, ApJ, 536, 195
Chevalier, R. A., Li, Z.-Y., & Fransson, C. 2004, ApJ, 606, 369
Coburn, W., & Boggs, S. E. 2003, Nature, 423, 415
Crowther, P. A. 2007, ARA&A, 45, 177
Dermer, C. D., & Mitman, K. E. 1999, ApJ, 513, L5
Dessart, L., & Owocki, S. P. 2005, A&A, 437, 657
Eshraghi, H. 2003, Phys. Plasmas, 10, 3577
Field, G. B. 1965, ApJ, 142, 431
Frederiksen, J. T., Hededal, C. B., Haugbølle, T., & Nordlund, Å. 2004, ApJ, 608, L13
Freedman, D. L., & Waxman, E. 2001, ApJ, 547, 922
Galama, T. J., Wijers, R. A. M. J., Bremer, M., Groot, P. J., Strom, R. G., Kouvetoilou, C., & van Paradijs, J. 1998, ApJ, 500, L97
Garcia-Segura, G., & Franco, J. 1996, ApJ, 469, 171
Genet, F., Daigne, F., & Mochkovitch, R. 2007, MNRAS, 381, 732
Goodman, J., & MacFadyen, A. 2007, J. Fluid. Mech., submitted (arXiv: 0706.1818) (Paper I)
Ishizuka, T., Hashimoto, Y., & Ono, Y. 1964, Prog. Theor. Phys., 32, 207
Lazzati, D., Rossi, E., Covino, S., Ghisellini, G., & Malesani, D. 2002, A&A, 396, L5
Medvedev, M. V. 2000, ApJ, 540, 704
Medvedev, M. V., & Loeb, A. 1999, ApJ, 526, 697
Meszaros, P., & Rees, M. J. 1997, ApJ, 476, 232
Moffat, A. F. J., Drissen, L., Lamontagne, R., & Robert, C. 1988, ApJ, 334, 1038
Nakar, E., Piran, T., & Granot, J. 2003, NewA, 8, 495
Panaitescu, A. 2005, MNRAS, 363, 1409
Panaitescu, A., & Kumar, P. 2001, ApJ, 554, 667
Panaitescu, A., & Kumar, P. 2002, ApJ, 571, 779
Ramirez-Ruiz, E., Garcia-Segura, G., Salmonson, J. D., & Pérez-Rendón, B. 2005, ApJ, 631, 435
Sari, R. 2006, Phys. Fluids, 18, 027106
Schaefer, B. E., et al. 2003, ApJ, 588, 387
Schechtkiniv, A. A., Cowley, S. C., Hammett, G. W., Maron, J. L., & McWilliams, J. C. 2002, New J. Phys., 4, 84
Schulz, N. S., Canizares, C., Huenemoerder, D., & Tibbets, K. 2003, ApJ, 595, 365
Silva, L. O., Fonseca, R. A., Tonge, J. W., Dawson, J. M., Mori, W. B., & Medvedev, M. V. 2003, ApJ, 596, L121
Spitkovsky, A. 2005, in AIP Conf. Proc. 801, Astrophysical Sources of High Energy Particles and Radiation, ed. T. Bulik, B. Rudak, & G. Madejski (New York: AIP), 345
———. 2007, ApJL, submitted (arXiv: 0706.3126)
ud-Doula, A., & Owocki, S. P. 2002, ApJ, 576, 413
Uhm, Z. L., & Beloborodov, A. M. 2007, ApJ, 665, L93
Vishniac, E. T. 1983, ApJ, 274, 152
Völk, H. J., Berezhko, E. G., & Ksenofontov, L. T. 2005, A&A, 433, 229
Wang, X., & Loeb, A. 2000, ApJ, 535, 788
Waxman, E. 2006, Plasma Phys. Controlled Fusion, 48, B137
Whitham, G. B. 1974, Linear and Nonlinear Waves (New York: Wiley)
Wolf, B., Stahl, O., & Fullerton, A. W., ed. 1999, Variable and Non-spherical Stellar Winds in Luminous Hot Stars (Berlin: Springer)
Woosley, S. E., & Bloom, J. S. 2006, ARA&A, 44, 457
Yost, S. A., Harrison, F. A., Sari, R., & Frail, D. A. 2003, ApJ, 597, 459
Zhang, B., Fang, Y. Z., Dyks, J., Kobayashi, S., Mészáros, P., Burrows, D. N., Nousek, J. A., & Gehrels, N. 2006, ApJ, 642, 354
Zhang, B., Kobayashi, S., & Mészáros, P. 2003, ApJ, 595, 950

Note added in proof.—A referee of Paper I calls our attention to M. H. Johnson & C. F. McKee (Phys. Rev. D, 3, 858 [1971]), who derived a relation equivalent to equation (1) for decreasing preshock densities, so that there is no reverse shock; Paper I shows that the relation is a good approximation also for quite sharp density increases.