Trapped Bose-Einstein Condensates: Role of Dimensionality

Yuri S. Kivshar and Tristram J. Alexander
Optical Sciences Center, Research School of Physical Sciences and Engineering
Australian National University, Canberra ACT 0200, Australia

We analyse systematically, from the viewpoint of the nonlinear physics of solitary waves, the effect of the spatial dimension \(D = 1, 2, 3\) on the structure and stability of the Bose-Einstein condensates (BECs) trapped in an external anisotropic parabolic potential. While for the positive scattering length the stationary ground-state solutions of the Gross-Pitaevskii equation are shown to be always stable independently of the spatial dimension, for the negative scattering length the ground-state condensate is stable only in the 1D and 2D cases, whereas in the 3D case it becomes unstable. A direct link between nonlinear modes of BECs and (bright and dark) solitary waves of the nonlinear Schrödinger equation is demonstrated for all the dimensions.

I. INTRODUCTION

Experimental realizations of Bose-Einstein condensation (BEC) in ultracold and dilute atomic gases \[1,2\] generated extensive theoretical studies of the macroscopic dynamics of condensed atomic clouds (see, e.g., Ref. 3 and references therein). In experiments, the BEC atoms are trapped in a three-dimensional, generally anisotropic, external potential created by a magnetic trap, and their collective dynamics can be described by the well-known Gross-Pitaevskii (GP) equation,

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{r})\Psi + U_0 |\Psi|^2 \Psi, \tag{1}\]

where \(\Psi(\vec{r},t)\) is the macroscopic wave function of a condensate in the \(D\)-dimensional space, \(V(\vec{r})\) is a parabolic trapping potential, and the parameter \(U_0 = 4\pi\hbar^2a/m\) characterizes the two-particle interaction proportional to the s-wave scattering length \(a\). When \(a > 0\), the interaction between the particles in the condensate is repulsive, as in most current experiments \[1\], whereas for \(a < 0\) the interaction is attractive.

From the viewpoint of the nonlinear dynamics of solitary waves, the case of the negative scattering length is the most interesting one. Indeed, it is well known that the solutions of Eq. (1) without a parabolic potential display collapse for both \(D = 2\) and \(D = 3\), so that no stable spatially localised solutions exist in those dimensions at all. A trapping potential allows a stabilisation of otherwise unstable and collapsing solutions and, as has been already shown for the 3D case with the help of different approximate techniques \[4,5\], there exists a critical number \(N_{cr}\) of the total condensate particles \(N\), defined through the normalization of the condensate wavefunction as

\[
N = \int |\Psi|^2 d^3r, \tag{2}\]

such that the condensate is stable for \(N < N_{cr}\), and it becomes unstable for \(N \geq N_{cr}\), and then its local density \(|\Psi|^2\) grows to infinity. This critical value of the particles' number is readily observed in experiments \[2\].

In this paper we analyse systematically, in the framework of the GP equation (1), the structure and stability of a trapped condensate depending on its dimension \(D\). In spite of the fact that the first experimental results were obtained for the BEC in a 3D anisotropic trap, the cases of lower dimensions are also of great importance and, moreover, they have a clear physical motivation. First of all, Eq. (1) with \(D = 1\) appears as the fundamental model of BEC in highly anisotropic traps of the axial symmetry

\[
V(\vec{r}) = m\omega^2_2 (r_\perp^2 + \lambda z^2), \quad r_\perp = \sqrt{x^2 + y^2}, \tag{3}\]

provided \(\lambda \equiv \omega_2^2/\omega_1^2 \ll 1\), i.e. for cigar-shaped traps. In this case, the transverse structure of the condensate is controlled by a trapping potential, and it can be described by the eigenmodes of a two-dimensional isotropic harmonic oscillator. Then, after averaging over the cross-section, Eq. (1) appears as a 1D model with renormalised parameters, and it describes the longitudinal dynamics of the condensate (see Ref. 6 and Sec. 2 below).

Secondly, in the case \(D = 2\), the model (1) can be employed to describe the condensate dynamics in the so-called pancake traps \((\lambda \gg 1)\), and it can be related to recent experiments with quasi-two-dimensional condensates or coherent atomic systems \[6,7\].
The paper is organised as follows. In Sec. 2 we consider the 1D case and find numerically the ground-state solutions for the trapped condensate in both $a < 0$ and $a > 0$ cases. In the former case, the problem is found to be remarkably similar to that of the soliton propagation in optical fibers in the presence of the dispersion management [9]. The most interesting case of a 2D trap is considered in Sec. 3, where we discuss the necessary conditions for the instability of the radially symmetric condensate in a trap, and also analyse the invariants of the stationary solutions found numerically. We point out, for the first time to our knowledge, that 2D condensates can be stable, in spite of the possibility of collapse in the model described by the 2D GP equation. Similar results are briefly presented in Sec. 4 for the 3D BECs of radial symmetry, and they provide a rigorous background for the variational results obtained earlier by different methods [4]. In all the cases, we also find the condensate structure for the positive scattering length where the instability is absent in all the dimensions.

II. QUASI-ONE-DIMENSIONAL TRAP

A. Model

To derive the 1D GP equation from Eq. (1), we assume that the parabolic potential $V(\vec{r})$ describes a cigar-shaped trap. For the potential (3), this means that $\lambda \ll 1$, and the transverse structure of the condensate, being close to a Gaussian in shape, is mostly defined by the trapping potential. We are interested in the stationary solutions of the form $\Psi(\vec{r},t) = \Psi(\vec{r}) e^{-i\mu \bar{\hbar} t}$, where $\mu$ has the meaning of the chemical potential. Measuring the spatial variables in the units of the longitudinal harmonic oscillator length $a_{ho} = (\bar{\hbar} / m \omega \sqrt{\lambda})^{1/2}$, and the wavefunction, in units of $(\bar{\hbar} \omega / 2 U_0 \sqrt{\lambda})^{1/2}$, we obtain the following stationary dimensionless equation:

$$\nabla_D \Phi + \mu' \Phi - \left[ \frac{1}{\lambda} (\xi^2 + \eta^2) + \zeta^2 \right] \Phi + \sigma |\Phi|^2 \Phi = 0,$$

(4)

where $\mu' = 2 \mu \sqrt{\lambda} / \hbar \omega$, $(\xi, \eta, \zeta) = (x,y,z) / a_{ho}$, and the sign $\sigma = \text{sgn}(a) = \pm 1$ in front of the nonlinear term is defined by the sign of the s-wave scattering length of two-body interaction.

We assume that in Eq. (1) the nonlinear interaction is weak relative to the trapping potential force, i.e. $U_0 / m \omega \sim \lambda \ll 1$. Then, it follows from Eq. (3) that the transverse structure of the condensate is of order of $\lambda$, and the condensate has a cigar-like shape. Therefore, we can look for solutions of Eq. (4) in the form,

$$\Phi(r,\zeta) = \phi(r) \psi(\zeta),$$

(5)

where $r = \sqrt{\xi^2 + \eta^2}$, and $\phi$ is a solution of the auxiliary problem for the 2D quantum harmonic oscillator

$$\nabla_\perp \phi + \gamma \phi - \lambda^{-1} r^2 \phi = 0,$$

(6)

which we take in the form of the no-node ground state,

$$\phi_0(r) = C \exp \left( - \frac{r^2}{2 \sqrt{\lambda}} \right), \quad \gamma = \frac{2}{\sqrt{\lambda}},$$

(7)

$C$ being constant. To preserve all the information about the 3D condensate in an asymmetric trap describing its properties by the 1D GP equation for the longitudinal profile, we impose the normalisation,

$$\int_{-\infty}^{\infty} \phi(r) dx dy = 2\pi \int_0^{\infty} \phi(r) r dr = 1$$

(8)

that yields $C^2 = 1 / \pi \sqrt{\lambda}$.

After substituting the solution (5) with the function (7) into Eq. (4), dividing by $\phi$ and integrating over the transverse cross-section $(\xi, \eta)$ of the cigar-shaped condensate, we finally obtain the following 1D stationary GP equation (see also the similar results in Ref. 6)

$$\frac{d^2 \psi}{d\xi^2} + \beta \psi - \zeta^2 \psi + \sigma |\psi|^2 \psi = 0,$$

(9)

where $\beta = \mu' - \gamma$, and the normalisation (8) for $\phi(r)$ is used.
Importantly, the number of the condensate particles \( N \) is now defined as \( N = \frac{\hbar \omega}{2 U_0 \sqrt{\lambda}} \), where

\[
Q = \int_{-\infty}^{\infty} |\psi|^2 d\zeta
\]

is the integral of motion for the normalised nonstationary GP equation.

### B. Stationary Ground-State Solutions

Equation (4) includes all the terms of the same order, and it describes the longitudinal profile of the condensate state in an anisotropic trap. In the linear limit, i.e. when formally \( \sigma \to 0 \), Eq. (4) becomes the well-known equation for a harmonic quantum oscillator. Its localised solutions exist only for \( \beta = \beta_n = 1 + 2n \ (n = 0, 1, 2, \ldots) \), and they are defined through the Gauss-Hermite polynomials, \( \psi_n(\zeta) = c_n e^{-\zeta^2/2} H_n(\zeta) \), where

\[
H_n(\zeta) = (-1)^n e^{\zeta^2/2} \frac{d^n(e^{-\zeta^2/2})}{d\zeta^n},
\]

so that \( H_0 = 1, H_1 = 2\zeta, \) etc.

To describe the effect of weak nonlinearity, we can employ a perturbation theory based on the expansion of the general solution of Eq. (4) in the infinite set of eigenfunctions defined by Eq. (11). The similar approach has been used earlier in the theory of the dispersion-managed optical solitons (see, e.g., Ref. 10), and it allows us to describe the effect of weak interaction on the condensate structure and stability. These results will be presented elsewhere [11].

In the opposite limit, i.e. when the nonlinear term is much larger than the parabolic potential, localised solutions exist only for \( \beta < 0 \) and \( \sigma = +1 \), and they are described by the stationary nonlinear Schrödinger (NLS) equation. The one-soliton solution is

\[
\psi_s(\zeta) = \frac{\sqrt{-2\beta}}{\cosh(\zeta\sqrt{-\beta})},
\]

so that \( Q_s = 4\sqrt{-\beta} \) coincides with the soliton invariant.

In general, the ground-state solution of Eq. (4) can be found only numerically. Figures 1(a) and 1(b) present some examples of the numerically found solutions of Eq. (4) for several values of the dimensionless parameter \( \beta \), for both negative (a) and positive (b) scattering length. For \( \beta \to 1 \), i.e. in the limit of the harmonic oscillator mode, the solution is close to Gaussian for both the cases. When \( \beta \) deviates from 1, the solution profile is defined by the type of nonlinearity. For attraction (\( \sigma = +1 \)), the profile approaches the sech-type soliton (12), whereas for repulsion (\( \sigma = -1 \)) the solution flattens, cf. Fig. 1(a) and Fig. 1(b).

In Fig. 2 we show the dependence of the invariant \( Q \) on the parameter \( \beta \), for both the families of localised solutions, corresponding to two different signs of the scattering length. The dotted line shows the limit of the soliton solution of the NLS equation without a trapping potential. In the asymptotic, i.e. say for \( \beta < -2 \), the curve \( Q_s(\beta) \) coincides with the invariant \( Q \) for the BEC condensate in a trap. This means that for such localised condensates the effect of the trap is negligible and the condensate function becomes localised mostly due to an attractive interparticle interaction.

To discuss the stability of localised solutions, we refer to a number of the corresponding problems well analysed in nonlinear optics of guided modes and solitary waves [12]. In terms of the so-called Vakhitov-Kolokolov stability criterion, the condition \( dQ/d(-\beta) > 0 \) gives the solution stability for the case \( \sigma = +1 \). Therefore, all 1D solitons for the attraction case are stable. A detailed discussion of the soliton stability for this case can be found in Ref. 9.

Formally, the case \( \sigma = -1 \) corresponds to the opposite sign of the derivative \( dQ/d\beta \). However, this case of the so-called defocusing nonlinearity is also well-known in nonlinear optics of guided waves, and it has been analysed for different types of the trapping potential. In particular, it is known that the Vakhitov-Kolokolov criterion does not apply to this case, and all the family of spatially localised solutions are stable [13]. Extension of all those results to the case of a parabolic trap is a straightforward technical problem.

### C. Higher-Order Modes and Multi-Soliton States

It is well known that in the linear limit \( \sigma \to 0 \), Eq. (4) possesses a discrete set of localised modes corresponding to the different orders of the Gauss-Hermite polynomials. We demonstrate that all those modes can be readily obtained...
for the nonlinear problem as well, giving a continuation of the Gauss-Hermite modes to a nonlinear regime. Figure 3 shows several examples of those modes for both negative ($\sigma = +1$) and positive ($\sigma = -1$) scattering length, respectively. In the limit $\beta \to 1$, those modes are defined by the eigenfunctions of the linear harmonic oscillator. The effect of nonlinearity is different for the negative and positive scattering length. For the negative scattering length (attraction), the higher-order modes transform into a sequence of dark solitons $[14]$, so that the first-order mode corresponds to a single dark soliton, the second-order mode to a pair of dark solitons, etc. [see Figs. 3(c) and 3(d)]. Again, these stationary solutions satisfy the force balance condition - repulsion between dark solitons is exactly compensated by an attractive force of the trapping potential.

For the positive scattering length ($\sigma = -1$), the higher-order modes transform into a sequence of dark solitons $[4]$, so that the first-order mode corresponds to a single dark soliton, the second-order mode to a pair of dark solitons, etc. [see Figs. 3(c) and 3(d)]. Again, these stationary solutions satisfy the force balance condition - repulsion between dark solitons is exactly compensated by an attractive force of the trapping potential.

### III. QUASI-TWO-DIMENSIONAL TRAP

#### A. Model and Stationary Ground-State Solutions

The case of the two-dimensional (2D) GP equation can be associated with a quasi-condensate of atomic hydrogen $[4]$ or quasi-2D gas of laser cooled atoms $[8]$. Derivation of the GP equation from the first principles of scattering theory in the two-dimensional geometry is not trivial, and it has been recently shown $[13]$ that the correct form of the 2D GP equation $[4]$ should have the 2D interaction potential $U$ in the form $U_0 = -(2\hbar^2/m)\ln^{-1}(ka)$, where $0 < ka \leq 1$, $a$ is the scattering length, and the characteristic wavenumber $k$ can be approximated as $k \sim 1/a_0$. Here, we derive the 2D GP equation directly from its 3D form of Eq. (1), for the case of a pancake trap when $\lambda \gg 1$.

In the case of a pancake trap, in the parabolic potential $[4]$ we assume the parameter $\lambda$ is large, i.e. $\lambda \gg 1$. Then, the longitudinal profile of the condensate is controlled by the parabolic potential $\sim (m\omega^2/2)z^2$. Measuring the spatial variables in the units of the transverse harmonic oscillator length ($\hbar/\omega$) and the wavefunction in units of ($\hbar/2U_0$)$^{1/2}$, from Eq. (1) we obtain the following stationary equation,

$$\nabla_D \Phi + \mu' \Phi - \left[ (\xi^2 + \eta^2) + \lambda \zeta^2 \right] \Phi + \sigma |\Phi|^2 \Phi = 0,$$

where $\mu' = 2\mu/\hbar \omega$. For $\lambda \gg 1$, the longitudinal structure of the condensate is squeezed in the $z$-direction and its size in this direction is of the order of $\lambda^{-1}$. Therefore, we can look for stationary solutions in the form,

$$\Phi(r, \zeta) = \psi(r)\phi(\zeta),$$

where $\psi(r)$ depends on the radial coordinate $r = \sqrt{x^2 + y^2}$, and this time the function $\phi(\zeta)$ is a solution of the 1D quantum harmonic oscillator,

$$\frac{d^2 \phi}{d\zeta^2} + \gamma \phi - \lambda \zeta^2 \phi = 0,$$

with the Gaussian form,

$$\phi_0(\zeta) = C \exp \left( -\frac{\zeta^2 \sqrt{\lambda}}{2} \right), \quad \gamma = \sqrt{\lambda}. \tag{16}$$

Normalisation condition $\int_{-\infty}^{\infty} \phi_0^2(\zeta)d\zeta = 1$ yields $C = \pi^{1/4}\lambda^{1/8}$.

Substituting the solution (14) into Eq. (13) and averaging over $\zeta$, we obtain the following 2D equation for $\psi(r)$,

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \psi + \beta \psi - r^2 \psi + \sigma |\psi|^2 \psi = 0, \tag{17}$$

Therefore, we can look for stationary solutions in the form,
where \( \sigma = \pm 1 \) and \( \beta = \mu' - \gamma \). Equation (15) is the 2D GP equation for the stationary solutions without any angular dependence. In the linear limit, i.e. when we neglect the nonlinear term \( \sigma \to 0 \), the solution exists only at \( \beta = 2 \), and it can be written in the form \( \psi_0(r) = C_0 \exp(-r^2/2) \), where \( C_0 \) is defined by normalisation. In the opposite limit, the localised solution exists only for \( \sigma = +1 \), and it is described by the radially symmetric solitary wave of the 2D NLS equation.

We integrate Eq. (17) numerically and find a family of radially symmetric localised solutions \( \psi(r) \) for the condensate ground state, for both \( \sigma = +1 \) (attraction) and \( \sigma = -1 \) (repulsion). Some results are presented in Figs. 5(a,b), for different values of the parameter \( \beta \). The structure of the localised solutions is similar to those in the 1D case. However, in the limit of large negative \( \beta \), the solution transforms into the 2D NLS soliton, known to be a self-similar radially symmetric solution corresponding to a critical collapse. This is further illustrated in Fig. 6, where we present the dependence of the invariant \( Q \) on the parameter \( \beta \). The 2D NLS soliton exists only for \( \beta < 0 \) and \( \sigma = +1 \), and it corresponds to the fixed value of the invariant, \( Q_s \approx 11.7 \), shown as a dotted straight line in Fig. 6.

B. Stability and collapse

Stability of the 2D condensates in a parabolic trap is an important issue. In particular, as was shown by Bergé [14] and Tsurumi and Wadati [3], the presence of a trapping potential does not remove collapse from the 2D GP equation. This result is very much expected because it has been known for a long time that, in the case of the 2D NLS equation with a parabolic potential, there exists an exact analytical transformation that allows the potential to be removed from the 2D NLS equation [17]. This means that for some classes of the input conditions of the GP equation, the collapse should be observed even in the presence of a trapping potential. Nevertheless, as follows from our results summarised in Fig. 6, for the attractive case \( (\sigma = +1) \) the derivative \( dQ/d(-\beta) \) is positive and it does not change the sign. Moreover, according to the collapse conditions derived in Ref. 15, for the radius \( R \) of the time-dependent 2D condensate the following equation holds

\[
\frac{d^2 \langle R^2 \rangle}{dt^2} = 8H - 4\langle R^2 \rangle, \tag{18}
\]

where \( H \) is the system Hamiltonian

\[
H = 2\pi \int_0^\infty \left( \frac{|d\psi|}{dr}^2 + r^2|\psi|^2 - \frac{1}{2}(|\psi|^4) \right) r dr \tag{19}
\]

and \( R \) is defined as

\[
\langle R^2 \rangle = 2\pi \int_0^\infty r^3|\psi|^2 dr. \tag{20}
\]

Equation (18) means that the condition \( H < 0 \) is a sufficient condition for collapse, because the value \( \langle R^2 \rangle \) surely becomes negative for some value of \( t \) and, therefore, the wavefunction becomes singular and collapses as \( \langle R^2 \rangle \) tends to zero.

When \( H \) is zero or positive, the dynamics depend on the applied perturbation, so that for a large enough perturbation we can lower the stationary value of the Hamiltonian \( H \). In Fig. 7 we present the dependences \( H(\beta) \) and \( H(Q) \) for the family of stationary solutions found numerically for the attraction case \( \sigma = +1 \). It is clear that the whole family of stationary solutions corresponds to \( H > 0 \). Therefore, the ground-state is expected to be stable everywhere, but its dynamics should be sensitive to the perturbation amplitude for larger \( \beta \) since a perturbation can make the Hamiltonian negative. This statement should be further verified by numerical simulations of the 2D GP equation.

C. Higher-Order Modes and Vortices

Similar to the 1D case discussed above, the higher-order modes of a harmonic oscillator can be found in the 2D case too, thus providing an analytical continuation of the 2D Gauss-Hermite modes. The set of such solutions is broader because it includes a continuation of a superposition of different modes. For example, the mode \( \psi_{00}(r) \) is the fundamental ground-state solution, that continues the zero-order Gauss-Hermite mode \( \sim H_0(x)H_0(y) \), as discussed above. In general, the modes \( \psi_{nm}(x, y) \) have no radial symmetry and they include both dipole-like and vortex-like states.
The simplest higher-order radially symmetric mode describes the condensate with a single vortex, shown in Fig. 8. Figures 8(a) and 8(b) show the density profile of the condensate with a vortex state, for both negative ($\sigma = +1$) and positive ($\sigma = -1$) scattering length, respectively. Figure 8(c) presents the dependence of the invariant $Q$ vs. $\beta$ for the vortex state in comparison with the ground-state mode. Because the vortex state is a nonlinear mode that extends the corresponding mode of the linear system, it approaches the value $\beta = 4$.

IV. THREE-DIMENSIONAL TRAP OF RADIAL SYMMETRY

The case of a radially symmetric 3D trap is the most studied in the literature, so that the corresponding results are well known, and therefore we reproduce some of them just for the completeness of the general picture of the stationary states. Similar results for the stationary states can be found, e.g. in Refs. 3 and 17, whereas the BEC stability has been recently discussed by Huepe et al. [19].

Figures 9(a) and 9(b) show the density profiles of the 3D condensate of radial symmetry for different values of $\beta$, for both negative ($\sigma = +1$) and positive ($\sigma = -1$) scattering length. While in the case of repulsion ($\sigma = -1$), the condensate broadens for larger $\beta$ remaining always stable, for the attractive interparticle interaction ($\sigma = +1$), the condensate is stable only for $\beta < \beta_{cr}$, where $\beta_{cr} = 0.72$ corresponds to the maximum value $Q_{cr} \approx 14.45$ [see Fig. 10]. This result is consistent with the Vakhitov-Kolokolov stability criterion $dQ/d(\beta) > 0$. The critical value $\beta_{cr}$ corresponds to a critical value of the particles $N_{cr}$ and it has been already observed in experiment [2]. Without the trapping potential i.e. in the limit of the 3D NLS equation, all localised solutions for $\sigma = +1$ are unstable (shown as a dotted curve in Fig. 10).

V. CONCLUSIONS

We have presented a systematic study of the ground-state and higher-order spatially localised solutions of the GP equation for the Bose-Einstein condensates in a parabolic trap of different dimensions. While many results for the radially symmetric ground-state and vortex modes in 3D are available in the literature for the condensates with the positive scattering length (i.e., for repulsive interaction of the condensate atoms), little is known for the condensates with the negative scattering length, especially for the 2D traps. In particular, we have presented the results indicating that, in spite of the collapse condition in the 2D case, the family of 2D localised solutions of the GP equation can be stable. Additionally, we have clarified the meaning of higher-order localised modes, i.e. nonlinear Gauss-Hermite eigenmodes, and demonstrated their link to multi-soliton states. We believe the systematic study of the stationary states in all the dimensions carried out here from the viewpoint of the nonlinear physics of localised states and solitary waves allows us to close a gap in the literature and deepens our knowledge about the condensate properties and stability.

REFERENCES

[1] M.H. Anderson et al., Science 269, 198 (1995); K.B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995); M.O. Mewes et al., Phys. Rev. Lett. 77, 416 (1996).
[2] C.C. Bradley et al., Phys. Rev. Lett. 75, 1687 (1995); Phys. Rev. Lett. 78, 985 (1997).
[3] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, e-print cond-mat/9806038 to appear in Rev. Mod. Phys. (1999).
[4] See, e.g., G. Baym and C.J. Pethick, Phys. Rev. Lett. 76, 6 (1996); E.V. Shuryak, Phys. Rev. A 54, 3151 (1996).
[5] T. Tsurumi and M. Wadati, J. Phys. Soc. Jpn. 66, 3031 (1997).
[6] V.M. Perez-Garcia, H. Michinel, and H. Herrero, Phys. Rev. A. 57, 3837 (1998).
[7] H. Ganeev, M. Hartl, D. Schmelke, H. Schnitzler, T. Pfau, and J. Mlynek, Phys. Rev. Lett. 81, 5298 (1998).
[8] A.I. Safonov, S.A. Vasilyev, I.S. Yasnokov, I.I. Lukashevich, and S. Jaakkola, Phys. Rev. Lett. 81, 4545 (1998).
[9] See, e.g., S.K. Turitsyn, Phys. Rev. E 56, R3784 (1997).
[10] S.K. Turitsyn, Phys. Rev. E 58, R1256 (1998); S.K.Turitsyn and V.K. Mezentsev, JETP Lett. 67, 640 (1998).
FIG. 1. (a,b) Examples of the longitudinal condensate density $|\psi|^2$ described by the solution of Eq. (9) for the negative ($\sigma = +1$) and positive ($\sigma = -1$) scattering length, respectively. The values of $\beta$ are given next to the curves.

FIG. 2. Dependence of the invariant $Q$, defined by Eq. (10), on the parameter $\beta$ for the family of the ground-state localised solutions of Eq. (9). Dotted - result for the 1D NLS soliton (12) without a trapping potential.
FIG. 3. Examples of higher-order (first and second) nonlinear modes of BEC in a parabolic trap shown for the negative ($\sigma = +1$, upper row) and positive ($\sigma = -1$, lower row) scattering length.
FIG. 4. Invariant $Q$ vs. $\beta$ for the first- and second- order nonlinear modes, shown together with the corresponding dependence for the fundamental mode of Fig. 2. Dotted lines are given by the values $Q_s$, $2Q_s$, and $3Q_s$, respectively.

FIG. 5. (a,b) Examples of the transverse condensate density $|\psi|^2$ described by Eq. (17) for the attractive ($\sigma = +1$) and repulsive ($\sigma = -1$) interaction, respectively.

FIG. 6. Dependence of the invariant $Q$ on the parameter $\beta$ for the family of the ground-state solutions of Eq. (17). Dotted - the limit of the 2D NLS soliton without a trapping potential.
FIG. 7. Dependence of the Hamiltonian $H$ on (a) the parameter $\beta$, and (b) on the invariant $Q$ for the family of 2D radially symmetric solutions of Eq. (17) at $\sigma = +1$.

FIG. 8. (a,b) Vortex profiles and (c) the corresponding invariant $Q$ for the ground-state and vortex localised modes of the 2D GP equation.
FIG. 9. (a,b) Examples of the condensate density in a 3D trap of radial symmetry for (a) attractive ($\sigma = +1$) and (b) repulsive ($\sigma = -1$) interaction, respectively.

FIG. 10. Invariant $Q$ vs. the parameter $\beta$ for the family of the radially symmetric ground-state solutions of the 3D GP equation.