THE FIRST HOCHSCHILD COHOMOLOGY AS A LIE ALGEBRA

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Abstract. In this paper we study sufficient conditions for the solvability of the first Hochschild cohomology of a finite dimensional algebra as a Lie algebra in terms of its Ext-quiver in arbitrary characteristic. In particular, we show that if the quiver has no parallel arrows and no loops then the first Hochschild cohomology is solvable. For quivers containing loops, we determine easily verifiable sufficient conditions for the solvability of the first Hochschild cohomology. We apply these criteria to show the solvability of the first Hochschild cohomology space for large families of algebras, namely, several families of self-injective tame algebras including all tame blocks of finite groups and some wild algebras including most quantum complete intersections.

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1. Introduction. Let $K$ be an algebraically closed field and let $A$ be a finite dimensional $K$-algebra. The problem of describing the Hochschild cohomology of $A$ as a Gerstenhaber algebra and in particular the first Hochschild cohomology space as a Lie algebra, and how this structure is related to $A$, has been studied in several recent articles, see for example [1, 4, 8, 18, 23]. The Gerstenhaber bracket in Hochschild cohomology has been defined more than fifty years ago. Recently, new methods to explicitly compute it have been introduced rendering the problem more tractable. An interesting question arising is which Lie algebras can actually

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The results in this article contribute to an answer to this question. Namely, we give sufficient criteria for the solvability as a Lie algebra of the first degree Hochschild cohomology of $A$, denoted by $\text{HH}^1(A)$, in terms of its Ext-quiver and its relations.

The Hochschild cohomology $\text{HH}^\bullet(A) = \bigoplus_{n \geq 0} \text{HH}^n(A)$ of $A$ has a very rich structure. It is an associative, graded-commutative algebra with the cup product. It also has a graded Lie bracket of degree $-1$ and both structures are related by the graded Poisson identity. In particular, $\text{HH}^\bullet(A)$ is a Gerstenhaber algebra, $\text{HH}^1(A)$ is a Lie algebra with bracket induced by the usual commutator of derivations and $\text{HH}^\bullet(A)$ is a Lie module for this Lie algebra. All these structures are invariant under derived equivalence [16].

In this paper we study the solvability of the Lie algebra given by the first Hochschild cohomology of a finite dimensional algebra. We develop sufficient conditions on the Ext-quiver with relations so that this Lie algebra is solvable and we give several applications of our methods. In particular, we show that for wild algebras the first Hochschild cohomology is sometimes solvable and sometimes semi-simple. We show that it is solvable for most quantum complete intersections for arbitrary parameter and semi-simple for a family of algebras related to Beilinson algebras.

The first condition for solvability is based on the Ext-quiver having no parallel arrows and no loops.

**Theorem 1.1. (see Theorem 3.11)** Let $K$ be an algebraically closed field and let $A$ be a finite dimensional $K$-algebra such that $\dim_K(\text{Ext}^1_A(S,S)) = 0$ and $\dim_K(\text{Ext}^1_A(S,T)) \leq 1$, for all non-isomorphic simple $A$-modules $S$ and $T$. Then $\text{HH}^1(A)$ is a strongly solvable Lie algebra.

In particular, $\text{HH}^1(A)$ is a solvable Lie algebra.

The solvability of the first Hochschild cohomology has recently been extensively studied [4, 8, 17] and Theorem 1.1 also appears in [17] with a different proof. Our next main result concerns algebras whose Ext-quiver may have loops.

**Theorem 1.2. (see Theorem 3.13)** Let $K$ be an algebraically closed field and let $A$ be a finite dimensional $K$-algebra such that $\dim_K(\text{Ext}^1_A(S,T)) \leq 1$ for all simple $A$-modules $S$ and $T$. Assume that all derivations preserve the Jacobson radical. Then $\text{HH}^1(A)$ is a solvable Lie algebra.

We then give some further criteria for the solvability of $\text{HH}^1$ for algebras whose Ext-quiver has at most one loop at each vertex if there is more than one vertex and at most two loops in the local case.

Furthermore, we show some criteria for the solvability of the first Hochschild cohomology of graded algebras and we apply these to show that the first Hochschild cohomology of most quantum complete intersections is strongly solvable.

In [21] an extensive if not comprehensive classification of self-injective tame algebras up to derived equivalence has been given. We consider all families of symmetric algebras in this classification and we show that for almost all these algebras the first Hochschild cohomology is solvable in arbitrary characteristic. We
also show that almost all tame blocks of finite groups have solvable first Hochschild cohomology.

**Theorem 1.3.** (see Theorem 4.4) Let $K$ be an algebraically closed field of arbitrary characteristic. Let $A$ be a symmetric finite dimensional $K$-algebra of tame representation type appearing in the classification in [21] which is not derived equivalent to $K[X]/(X^r)$ when $\text{char}(K) \mid r$, the trivial extension of the Kronecker algebra if $\text{char}(K) \neq 2$ or $K[X,Y]/(X^2,Y^2)$ if the characteristic is 2. Then $\text{HH}^1(A)$ is a solvable Lie algebra.

In particular, let $A$ be a symmetric tame algebra of dihedral, semi-dihedral or quaternion type not derived equivalent to $K[X,Y]/(X^2,Y^2)$ if the characteristic is 2. Then $\text{HH}^1(A)$ is a solvable Lie algebra.

We note that there are two exceptions for the solvability of $\text{HH}^1(A)$ in the first part of the theorem and one in the second part. Namely, if $\text{char}(K) \mid r$ and $r \geq 3$, the Lie algebra $\text{HH}^1(K[X]/(X^r))$ is perfect. In particular, if $r = p = \text{char}(K)$ then it is simple and it is the so-called Jacobson-Witt algebra. Similarly, $\text{HH}^1(K[X,Y]/(X^2,Y^2))$ in characteristic 2 is also a Jacobson–Witt algebra and not solvable. The second exception in the first part is if $A$ is derived equivalent to the trivial extension of the Kronecker algebra and $\text{char}(K) \neq 2$, in which case the obtained Lie algebra is isomorphic to $\mathfrak{sl}_2(K)$, see [4].

We also show that the first Hochschild cohomology of a Brauer graph algebra with any multiplicity function is solvable in arbitrary characteristic, that is with the exception of the trivial extension of the Kronecker quiver in characteristic different from 2 and some other cases in characteristic 2.

The article is structured as follows. Section 2 contains background material on Lie algebras both in characteristic zero and in positive characteristic. It also contains a brief introduction of the Lie algebra structure of the first Hochschild cohomology of a finite dimensional algebra. Section 3 contains the main results of the paper, that is criteria for the solvability of the first Hochschild cohomology of a finite dimensional algebra in terms of its Ext-quiver and the relations on the quiver. In Section 4, we apply the results of Section 3 to several families of algebras such as symmetric algebras of tame representation type including Brauer graph algebras and blocks of groups algebras of finite groups as well as quantum complete intersections. We end the paper with an example of an infinite family of algebras for which the first Hochschild cohomology is semi-simple.

**Conventions.** Let $K$ be an algebraically closed field. For a finite dimensional $K$-algebra $A$, the Ext-quiver of $A$ is the quiver whose vertices are in bijection with the isomorphism classes of simple $A$-modules and where the number of arrows from a vertex corresponding to a simple module $S$ to a simple module $T$ is given by $\dim_K(\text{Ext}^1_A(S,T))$. Furthermore, if $A = KQ/I$ is a finite dimensional $K$-algebra, with quiver $Q = (Q_1,Q_0)$ with $Q_0$ the vertices of $Q$ and $Q_1$ the arrows in $Q$, then unless otherwise stated, we assume that the ideal $I$ is admissible. For an arrow $a$ in $Q$, we write $s(a)$ for the source of $a$ and $t(a)$ for the target of $a$. We assume that all modules will be right modules and we write $A^e = A \otimes_K A^{op}$. We denote by $J(A)$ the Jacobson radical of $A$. 
2. Background material. In this section we collect some background material on modular Lie algebras and the Lie bracket on the first Hochschild cohomology of a finite dimensional algebra (in any characteristic). For this let $L$ be a Lie algebra and recall that the derived Lie algebra $L^{(1)}$ is the Lie algebra defined by the linear span of all commutators $[x, y]$ for $x, y \in L$. We denote by $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ the $i$-th term of the derived series of $L$, where $i \geq 1$ and $L^{(0)} = L$.

2.1. Modular Lie algebras. We begin by collecting some well-known facts about modular Lie algebras.

**Definition 2.1.** A Lie algebra $L$ is strongly solvable if its derived subalgebra $L^{(1)}$ is nilpotent.

**Remark 2.2.** Note that any strongly solvable Lie algebra is solvable but the converse only holds in characteristic zero. Furthermore, just as in the case of solvable Lie algebras, we have the following.

**Lemma 2.3.** Every subalgebra and every factor of a strongly solvable Lie algebra is strongly solvable.

2.2. The Lie bracket on the first Hochschild cohomology. For a finite dimensional $K$-algebra $A = KQ/I$, we briefly recall the construction of the Lie bracket on $HH^1(A)$ as defined in [5]. We have $A = E \oplus J(A)$ where $E = KQ_0$ is a maximal semi-simple subalgebra of $A$ and $J(A)$ is the Jacobson radical of $A$. The set of paths is a basis of the path algebra $KQ$. When we consider the quotient $KQ/I$ this is not a basis anymore, but we can extract from the set of paths a basis consisting of some of these paths. In order to do so we need to fix a reduction system. For this, let $p, q$ be paths in $Q$. We say that $p$ divides $q$ if there exist paths $a, b$ in $Q$ such that $q = apb$. Two paths $p$ and $q$ in $Q$ are parallel if they have same source and target. Let $n \in \mathbb{N}$, then $Q_n$ denotes the set of paths of length $n$ in $Q$ and $Q_{\geq n}$ the set of paths of length at least $n$. A set $R \subseteq Q_{\geq 0} \times KQ$ is called a reduction system if for all $(s, f) \in R$, $s$ is parallel to $f$, $s \neq f$ and for any $(s', f') \in R$ we have that $s'$ does not divide $s$ if $s' \neq s$.

Let $\leq$ be a well-order on the set $Q_0 \cup Q_1$ such that $e < \alpha$, for all $e \in Q_0$ and $\alpha \in Q_1$. Let $\omega : Q_1 \to \mathbb{N}$ be a function which we extend to $Q_{\geq 0}$ by imposing that $\omega(e) = 0$, for all $e \in Q_0$ and $\omega(c_0 \ldots c_1) = \sum_{i=1}^{n} \omega(c_i)$, for $c_i \in Q_1$. Let $c, d \in Q_{\geq 0}$. We write that $c \leq_{\omega} d$ if one of the following conditions hold:

1. $c, d \in Q_0$ and $c \leq d$,
2. $\omega(c) < \omega(d)$,
3. $\omega(c) = \omega(d)$, $c = c_n \ldots c_1$, $d = d_m \ldots d_1 \in Q_{\geq 1}$ and there exists $j \leq \min(length(c), length(d))$ such that $c_i = d_i$ for all $1 \leq i \leq j - 1$ and $c_j < d_j$.

**Definition 2.4.** ([5, Definition 2.8]) Consider as before a well-order $\leq$ on $Q_0 \cup Q_1$ and $\leq_{\omega}$ on $Q_{\geq 0}$. Let $p \in KQ$ such that $p = \sum_{i=1}^{n} \lambda_i c_i$ with $\lambda_i \in K^*$, $c_i \in Q_{\geq 0}$
and $c_i < \omega c_1$ for all $i \neq 1$. We define $\text{tip}(p) := c_1$. More generally, if $X \subseteq KQ$, we define $\text{tip}(X) := \{ \text{tip}(x) \mid x \in X \setminus \{0\} \}$. Let

$$S := \text{Mintip}(I) = \{ p \in \text{tip}(I) \mid p' \notin \text{tip}(I) \text{ for all proper divisors } p' \text{ of } p \}.$$  

From Lemma 2.10 in [5] it follows that $I$ is the two sided ideal generated by \{ $s - f_s$ $\mid$ $s \in S$ $f_s = \sum p \lambda_p$ where the sum is over all $p \in Q_{\geq 2}$ parallel to $s$ and such that $p \leq \omega s$ and $\lambda_p \in K^*$. From the reduction system $R$ we obtain a minimal generating set $\mathcal{R}$ of $I$ by setting $\mathcal{R} = \{ s - f_s \mid (s, f_s) \in R \}$. We denote by $B$ the basis of $A$ whose elements are the images of paths under the canonical map $KQ \to KQ/I$ such that they cannot be reduced using our reduction system. We will freely refer to elements of $B$ as paths. Let $Q_i$ be the set of paths consisting of $i$ arrows and let $Q_{\geq i} = \bigcup_{j \geq i} Q_j$. We set $B_i = B \cap Q_i$.

If $X$ and $Y$ are sets of paths in $Q$, define

$$X||Y = \{(p, q) \in X \times Y \mid p$ and $q$ are parallel\}.

The vector spaces $K(X||Y)$ and $\text{Hom}_{E^*}(KX, KY)$ are isomorphic and we freely denote by $\alpha||\beta$ an element in $K(X||Y)$ as well as the morphism in $\text{Hom}_{E^*}(KX, KY)$ sending $\alpha$ to $\beta$ and any other basis element to zero.

Next we recall a construction of the Lie bracket on $HH^1(A)$ which can be deduced from the results in [5].

Since we are interested in $HH^1(A)$ and not in the higher Hochschild cohomology spaces, once the basis $B$ is fixed, we can work for cohomological purposes- with a set of relations $\mathcal{R}$ generating the ideal $I$, which does not need to be minimal.

The following is the start of a projective $A$-bimodule resolution of $A$:

$$\cdots \to A \otimes_E K \mathcal{R} \otimes_E A \xrightarrow{d_1} A \otimes_E KQ_1 \otimes_E A \xrightarrow{d_0} A \otimes_E A \to 0,$$

with differentials $d_0(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v$ and $d_1(1 \otimes r \otimes 1) = \sum_i \lambda_i \sum_{j=1}^{n_i} v_{ji} \cdots v_{j_i-1} \otimes v_{ji} \otimes v_{j_{i+1}} \cdots v_{j_{n_i}}$ where $r = \sum_i \lambda_i a_i \in \mathcal{R}$, with $a_i = v_{j_1} \cdots v_{j_{n_i}}$ and $v_{jk} \in Q_1$.

If we apply the functor $\text{Hom}_{A \otimes A^{op}}(-, A)$ to this chain complex and use the standard natural isomorphism $\text{Hom}_{A \otimes A^{op}}(A \otimes_E - \otimes_E A, A) \cong \text{Hom}_{E \otimes E^{op}}(-, A)$, we obtain the following cochain complex with the induced differentials:

$$0 \to \text{Hom}_{E \otimes E^{op}}(E, A) \xrightarrow{\delta^0} \text{Hom}_{E \otimes E^{op}}(KQ_1, A) \xrightarrow{\delta^1} \text{Hom}_{E \otimes E^{op}}(K\mathcal{R}, A) \to \cdots,$$

which can also be written as:

$$0 \to K(Q_0||\mathcal{B}) \xrightarrow{\delta^0} K(Q_1||\mathcal{B}) \xrightarrow{\delta^1} K(\mathcal{R}||\mathcal{B}) \to \cdots$$

where $\delta^0(\alpha||p) = \sum_{\alpha \in Q_1} \alpha||(\alpha p - p\alpha)$ and $\delta^1(\alpha||p) = \sum_{r \in \mathcal{R}, \alpha \mid r} r(||r^{\alpha,p})$ where $\alpha|\mathcal{R}$ means that $\alpha$ divides at least one of the paths occurring in $r$. In this last expression, if $r = \sum \lambda_q q$ then $r(||r^{\alpha,p}) = \sum \lambda_q ||q^{r^{\alpha,p}}$ where $q^{r^{\alpha,p}}$ is equal to the sum of all the nonzero paths obtained by subsequently replacing each appearance of $\alpha$ in $q$.
by \( p \). If \( q \) does not contain the arrow \( \alpha \) or if when replacing \( \alpha \) in \( q \) by \( p \) we obtain a zero element of \( A \) then we set \( q^{(\alpha,p)} = 0 \).

The Gerstenhaber structure defined on the cochains of the above complex and inducing the Gerstenhaber bracket in Hochschild cohomology is computed for example via comparison morphisms between a resolution whose first terms are described above and the \( E \)-reduced bar resolution introduced in [6]. Using the comparison maps between resolutions as in [1] – and taking into account that even if in that article the authors only deal with Toupie Algebras, the formulas for the comparison morphisms in low degrees do not change – we have that the Lie bracket on \( \text{Ker}(\delta_1) \subset K(Q_1||B) \) is induced by linearly extending the following

\[
[\alpha||h, \beta||b] = \beta||b^{(\alpha,h)} - \alpha||h^{(\beta,b)},
\]

with \( \alpha||h, \beta||b \in K(Q_1||B) \). This bracket induces a Lie algebra structure on \( \text{HH}^1(A) \).

### 2.3. Lie algebras of graded algebras.

Let now \( A = KQ/I \) with \( I \) an admissible ideal generated by homogeneous relations, so that the length of paths gives \( A \) the structure of a graded algebra with arrows in degree one, and the elements of the basis \( B \) are homogeneous. This induces a grading on the Lie algebra \( \text{Ker}(\delta^1) = K(Q_1||B) \cap \text{Ker}\delta^1 \) such that the elements in \( K(Q_1||B_i) \cap \text{Ker}\delta^1 \) are of degree \( i - 1 \) for all \( i \in \mathbb{N} \). Moreover, \( \text{Im}\delta^0 \) is a graded ideal of \( \text{Ker}\delta^1 \). We thus obtain an induced grading on \( \text{HH}^1(A) \). Set

\[
\mathcal{L}_0 := K(Q_1||Q_0) \cap \text{Ker}\delta^1
\]

\[
\mathcal{L}_1 := K(Q_1||Q_1) \cap \text{Ker}\delta^1 / \mathcal{I}_1
\]

\[
\mathcal{L}_i := K(Q_1||B_i) \cap \text{Ker}\delta^1 / \mathcal{I}_i
\]

where

\[
\mathcal{I}_1 = \{ \sum_{a \in Q_1} a||a - \sum_{a \in eQ_1} a||a \mid e \in Q_0 \}
\]

and for all \( i > 1 \)

\[
\mathcal{I}_i = \{ \sum_{a \in eQ_1, \gamma a \in B} a||\gamma a - \sum_{a \in eQ_1, a\gamma \in B} a||a\gamma \mid e, \gamma \in Q_0||Q_i \}
\]

With the above notation \( \mathcal{L} = \bigoplus_{i \geq 0} \mathcal{L}_i \) and \( [\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j-1} \) for all \( i, j \geq 0 \), where the bracket is induced by the brackets in Equation (1). Set \( \mathcal{N} := \bigoplus_{i \geq 2} \mathcal{L}_i \).

Then \( \mathcal{N} \) is a nilpotent Lie algebra. Moreover, \( \text{HH}^1(A) = \mathcal{L} \).

### 3. Criteria for solvability of \( \text{HH}^1 \) for finite dimensional algebras.

In this section we prove several criteria that if satisfied, imply the solvability of the Lie algebra given by the first Hochschild cohomology of a finite dimensional \( K \)-algebra.
3.1. Graded algebras without self-extensions of simples.

**Theorem 3.1.** Let $K$ be an algebraically closed field and let $A = KQ/I$ be a finite dimensional graded $K$-algebra. Suppose that $\text{Ext}^1_A(S,S) = \{0\}$ for every simple $A$-module $S$ and that $\dim_K(\text{Ext}^1_A(S,T)) \leq 1$ for all nonisomorphic simple $A$-modules $S$ and $T$. Then $\text{HH}^1(A)$ is a strongly solvable Lie algebra. In particular, $\text{HH}^1(A)$ is a solvable Lie algebra.

**Proof.** If $\text{Ext}^1_A(S,S) = \{0\}$ for every simple $A$-module $S$, then $Q$ has no loops. Therefore $K(Q_1||B_0) = \{0\}$. Since $\dim_K(\text{Ext}^1_A(S,T)) \leq 1$ for every simple $A$-modules $S$ and $T$, there are no parallel arrows in $Q$. Equation (1) then immediately gives that $L_1$ is an abelian Lie algebra. Consequently the derived subalgebra $L^{(1)}$ of $L$ is contained in $N$. Furthermore, $[L_i,L_j] \subset L_{i+j-1}$. As a consequence, $[L^{(1)},L^{(1)}] \subset \oplus_{i \geq 3} L_i$. By iterating this process, since $A$ is finite dimensional, it follows that $L^{(1)}$ is nilpotent and thus $L$ is strongly solvable; as a consequence $\text{HH}^1(A)$ is strongly solvable and hence solvable. $\Box$

3.2. Graded algebras with self-extensions of simples. Let us now consider the case of a graded algebra $A$ whose Ext-quiver has loops, that is there is a simple $A$-module $S$ such that $\text{Ext}^1_A(S,S) \neq \{0\}$.

For $i \in \mathbb{N}$, define the set

$$
\Sigma_i(A) := \{ \alpha||\beta \in Q_1||B_i \mid \alpha||\beta = \alpha_j||\beta_j \text{ for some } x = \sum \lambda_j \alpha_j||\beta_j \in \text{Ker}(\delta^1) \text{ with } \lambda_j \in K^* \}.
$$

If it is clear from the context, we will just write $\Sigma_i$ without specifying the algebra.

One can think of $\Sigma_i(A)$ as follows: if $\alpha||\beta \in Q_1||B_i$ belongs to this set, this means that there is a $K$-linear derivation of $A$ sending $\alpha$ to a linear combination of parallel paths such that $\beta$ appears in this linear combination.

Let $S$ and $T$ be simple $A$-modules and let $n = \dim_K(\text{Ext}^1_A(S,T))$. We will label the arrows, having source the corresponding vertex of the simple module $S$ and having target the corresponding vertex of the simple module $T$, by $\alpha_i$ for $1 \leq i \leq n$.

**Lemma 3.2.** Let $A = KQ/I$ be a finite dimensional graded $K$-algebra. Suppose further that for all simple $A$-modules $S$ and $T$ we have that $\alpha_j||\alpha_i \notin \Sigma_1$ for $i < j$ and that $\Sigma_0$ is empty. Then $L_1$ is a solvable Lie algebra.

**Proof.** Let $D_1 := K(Q_1||Q_1) \cap \text{Ker}(\delta^1)$. The key idea of the proof is to show that $D_1$ embeds into a direct sum of solvable Lie algebras. Each of these solvable Lie algebras is isomorphic to the Lie algebra of lower triangular matrices.

Let $\alpha||\beta$ and $\gamma||\epsilon$ be two summands of two different elements of $D_1$ such that $\alpha$ is not parallel to $\gamma$. Then by Equation 1 we have $[\alpha||\beta, \gamma||\epsilon] = 0$. Consequently, in order to compute the Lie bracket between two elements of $D_1$, it is enough to
calculate the Lie bracket between their corresponding summands $\alpha \| \beta$ and $\gamma \| \epsilon$ such that $\alpha$ is parallel to $\gamma$.

Let $m$ be the number of simple $A$-modules and let $h$ and $l$ be two positive integers such that $1 \leq h, l \leq m$. Let us denote $n_{h,l} = \dim_K(\text{Ext}_A^1(S_h, S_l))$ where $\mathfrak{S}_h$ and $\mathfrak{S}_l$ are simple $A$-modules. Note that $n_{h,l}$ is the number of arrows having source the corresponding vertex of the simple module $\mathfrak{S}_h$ and having target the corresponding vertex of the simple module $\mathfrak{S}_l$. We denote by $\mathfrak{b}(n_{h,l}, K)$ the Lie algebra generated by $\{\alpha_i \| \alpha_j\}_{i,j}$ where $1 \leq i, j \leq n_{h,l}$ and $i \leq j$. Note that $\mathfrak{b}(n_{h,l}, K)$ is isomorphic to the Lie algebra of lower triangular $n_{h,l} \times n_{h,l}$ matrices over a field $K$. The isomorphism sends $\alpha_i \| \alpha_j$ to $E_{ji}$, where $E_{ji}$ is the matrix that has a 1 in the $(j, i)$ position as its only nonzero entry. Note that $\alpha_i \| \alpha_j$ are elements of $K(Q_1 \| Q_1)$ and they do not need to belong to $\Sigma_1$.

Then, by the previous considerations, we have an embedding

$$\phi: \mathcal{D}_1 \hookrightarrow \mathcal{S} := \bigoplus_{h,l=1}^m \mathfrak{b}(n_{h,l}, K).$$

The map $\phi$ sends an element of $\mathcal{D}_1$, which is given by a linear combination of elements of the form $\alpha_i \| \alpha_j$ for $i \leq j$, to the same linear combination with summands in Lie algebras $\mathfrak{b}(n_{h,l}, K)$. Consequently, $\phi$ is clearly an injective Lie algebra homomorphism. The Lie algebra $\mathcal{S}$ is a solvable since it is the direct sum of solvable Lie algebras. Subalgebras of solvable Lie algebras are solvable, hence $\mathcal{D}_1$ is solvable. Since quotients of solvable Lie algebras are solvable, then $\mathcal{L}_1 := \mathcal{D}_1/\mathcal{I}_1$ is solvable.

We now state our first result on algebras with self-extensions of simples.

**Theorem 3.3.** Let $A = KQ/I$ be a finite dimensional graded $K$-algebra. Suppose further that for all simple $A$-modules $S$ and $T$ we have that $\alpha_j \| \alpha_i \notin \Sigma_1$ for $i < j$ and that $\Sigma_0$ is empty. Then $\text{HH}^1(A)$ is a solvable Lie algebra.

**Proof.** By the previous lemma we have that $\mathcal{L}_1$ is solvable, hence there exists a positive integer $n$ such that $\mathcal{L}_1^{(n)} = 0$. Since $\mathcal{L}_1^{(n)} = 0$ and since we also have that $\Sigma_0 = \emptyset$, we obtain that every element in the derived Lie algebra $\mathcal{L}^{(n+1)}$ of $\mathcal{L}^{(n)}$ has as summands elements in $\Sigma_i$, for $i \geq 2$ and thus $\mathcal{L}^{(n+1)} \subset \mathcal{N}$. Note that $\mathcal{N}$ is nilpotent, in particular, it is solvable. Subalgebras of solvable Lie algebras are solvable, hence $\mathcal{L}^{(n+1)}$ is solvable. In other words, there exist a positive integer $m$ such that $(\mathcal{L}^{(n+1)})^{(m)} = 0$. Note $(\mathcal{L}^{(n+1)})^{(m)} = \mathcal{L}^{(m+n+1)} = 0$. Consequently $\mathcal{L}$ is solvable. The statement follows since $\mathcal{L} = \text{HH}^1(A)$. \hfill $\square$

**Corollary 3.4.** Let $A = KQ/I$ be a finite dimensional graded $K$-algebra such that $\dim_K\text{Ext}_A^1(S, S) \leq 1$, for all simple $A$-modules $S$. Suppose further that for all simple $A$-modules $S$ and $T$ we have that $\alpha_j \| \alpha_i \notin \Sigma_1$ for $i < j$ and that $\alpha \| \alpha^2$ is not in $\Sigma_2$ for any loop $\alpha$ in $\mathcal{B}$. Then $\text{HH}^1(A)$ is a solvable Lie algebra.
Proof. Let \( \mathcal{L} = \text{HH}^1(A) \). If \( \mathcal{L}_0 = 0 \), the statement follows from Theorem 3.3. So assume now that \( \mathcal{L}_0 \neq 0 \). Our goal is to prove that \( \text{Ker}(\delta^1) \) is solvable, since \( \text{HH}^1(A) \) is a quotient of \( \text{Ker}(\delta^1) \) and quotients of solvable Lie algebras are solvable.

Note that \( \alpha||\alpha \) is not a summand of an element in \( (\text{Ker}(\delta^1))^{(1)} \) for any loop \( \alpha \), since the only way to obtain it is as \( [\alpha||e, \alpha||\alpha^2] \) but by hypothesis \( \alpha||\alpha^2 \) is not in \( \Sigma_2 \). Then \( \alpha||e \) is not a summand of an element in \( (\text{Ker}(\delta^1))^{(2)} \) since the only way to obtain \( \alpha||e \) is as \( [\alpha||e, \alpha||\alpha] \) and \( \alpha||\alpha \) is not a summand of an element in \( (\text{Ker}(\delta^1))^{(1)} \).

Now we focus our attention on the Lie algebra \( (\text{Ker}(\delta^1))^{(2)} \). Since \( \text{Ker}(\delta^1) \) is a graded Lie algebra, then also \( (\text{Ker}(\delta^1))^{(2)} \) is graded. Consequently, if we denote \( \mathcal{G} := (\text{Ker}(\delta^1))^{(2)} \), then we can write \( \mathcal{G} \) as \( \bigoplus_{l \geq 0} \mathcal{G}_l \) where \( l \) is a non-negative integer. Since for every loop \( \alpha \) we have that \( \alpha||e \) is not a summand of an element in \( (\text{Ker}(\delta^1))^{(2)} \), then \( \mathcal{G}_0 = 0 \). The next step is to prove that \( \mathcal{G}_1 = K(Q_1|Q_1) \cap (\text{Ker}(\delta^1))^{(2)} \) is a solvable Lie algebra. Since \( (\text{Ker}(\delta^1))^{(2)} \) is a Lie subalgebra of \( \text{Ker}(\delta^1) \), we have that \( \alpha_j||\alpha_j \) is not a summand of an element in \( \mathcal{G}_1 \) for \( i < j \).

Thus, using the same argument of Lemma 3.2, there is an injective Lie algebra homomorphism \( \phi : \mathcal{G}_1 \hookrightarrow \mathcal{S} := \bigoplus_{h,l=1}^m b(n_h,t,K) \). Hence the Lie algebra \( \mathcal{G}_1 \) is solvable.

Our aim now is to prove that \( \mathcal{G} = (\text{Ker}(\delta^1))^{(2)} \) is solvable. Since \( \mathcal{G}_1 \) is solvable, there exists a positive integer \( n \) such that \( \mathcal{G}_1^{(n)} = 0 \). In particular, we have that \( \mathcal{G}_1^{(n+1)} = 0 \). In addition, since \( \mathcal{G}_0 = 0 \), then also \( \mathcal{G}_0^{(n+1)} = 0 \). Consequently, \( \mathcal{G}^{(n+1)} \) is solvable since \( \mathcal{G}^{(n+1)} \) is a graded Lie algebra having \( \mathcal{G}_0^{(n+1)} = 0 \) and \( \mathcal{G}_1^{(n+1)} = 0 \). Consequently, there exists \( m \) such that \( (\mathcal{G}^{(n+1)})^{(m)} = 0 \). Since \( 0 = (\mathcal{G}^{(n+1)})^{(m)} = \mathcal{G}^{(n+1)+m} \), we have that \( \mathcal{G} = (\text{Ker}(\delta^1))^{(2)} \) is solvable.

Consequently there exist a positive integer \( k \) such that \( 0 = (\text{Ker}(\delta^1))^{(2)^{(k)}} = (\text{Ker}(\delta^1))^{(2+k)} \). Hence \( \text{Ker}(\delta^1) \) is solvable. The statement follows.

The following is a consequence of the proof of Corollary 3.4 in characteristic 2.

**Corollary 3.5.** Let \( A = KQ/I \) be a finite dimensional graded \( K \)-algebra over an algebraically closed field of characteristic 2 such that \( \dim K \text{Ext}^1_A(S,S) \leq 1 \), for all simple \( A \)-modules \( S \). Suppose further that for all simple \( A \)-modules \( S \) and \( T \) we have that \( \alpha_j||\alpha_i \notin \Sigma_1 \) for \( i < j \). Then \( \text{HH}^1(A) \) is a solvable Lie algebra.

**Proof.** Since the characteristic of \( K \) is 2, we have for any loop \( \alpha \) at vertex \( e \) that \( [\alpha||e, \alpha||\alpha^2] = 0 \). Therefore we can apply the same arguments as in Corollary 3.4 and the result follows.

3.3. Graded local algebras. We finish this section by considering the case when \( A \) is a graded local algebra.

**Proposition 3.6.** Let \( A = KQ/I \) be a local finite dimensional graded \( K \)-algebra such that \( \Sigma_0 = \emptyset \). Suppose \( Q \) has loops \( \alpha_1, \alpha_2, \ldots, \alpha_n \). If none of the \( \alpha_i||\alpha_j \), are in \( \Sigma_1 \) for \( i \neq j \) and \( 1 \leq i, j \leq n \), then \( \text{HH}^1(A) \) is solvable.
Proof. Let \( \mathcal{L} \) be as above. Since \( \Sigma_0 = \emptyset \), then \( \mathcal{L}_0 = 0 \). Because of the hypotheses on \( \Sigma_1 \), we have that \( \mathcal{L}_1 \) is abelian and consequently \( \mathcal{L}_1 \) is solvable. Furthermore, \( \mathcal{L} = \mathcal{L}_1 \oplus \mathcal{N} \) and since \( [\mathcal{L}_1, \mathcal{N}] \subset \mathcal{N} \) and \( \mathcal{N} \) is nilpotent, the statement follows. \( \square \)

The next result is for the more general case that \( \Sigma_0 \) is not empty.

**Theorem 3.7.** Let \( A = KQ/I \) be a local finite dimensional graded \( K \)-algebra with \( Q_1 = \{\alpha_1, \alpha_2\} \) such that \( \alpha_1||\alpha_2 \) and \( \alpha_2||\alpha_1 \) are not in \( \Sigma_1 \).

1. If \( \text{char}(K) = 2 \) then \( \text{HH}^1(A) \) is solvable.
2. If \( \text{char}(K) \neq 2 \) and \( \alpha_i||\alpha_i^2 \notin \Sigma_2 \) for \( i \in \{1, 2\} \) then \( \text{HH}^1(A) \) is solvable.

**Proof.** Let \( \mathcal{L} \) be as above. If \( \Sigma_0 \) is empty then the statement follows from Proposition 3.6. So assume that \( \Sigma_0 \) is not empty.

First we consider the case when both \( \alpha_1||e \) and \( \alpha_2||e \) are in \( \Sigma_0 \). Then the Lie algebra \( \mathcal{L} \) does not contain \( \alpha_i||\alpha_i\alpha_j \) for \( i,j \in \{1, 2\}, i \neq j \), because otherwise the derived subalgebra \( \mathcal{L}^{(1)} \) contains \( [\alpha_i||e, \alpha_i||\alpha_i\alpha_j] = \alpha_i||\alpha_i\alpha_j \) and \( [\alpha_j||e, \alpha_i||\alpha_i\alpha_j] = \alpha_j||\alpha_i \) which are not in \( \mathcal{L} \) since they are not in \( \Sigma_1 \) by hypothesis. Using the same argument as in Corollary 3.4, \( \mathcal{L}^{(1)} \) does not contain \( \alpha_i||\alpha_i \) for \( i = \{1, 2\} \) since the only way to obtain them is through the bracket \( [\alpha_i||\alpha_i^2, \alpha_i||e] = -2\alpha_i||\alpha_i \) for \( i \in \{1, 2\} \) which in characteristic 2 is equal to zero and if the characteristic is different from 2, by hypothesis, the elements \( \alpha_i||\alpha_i^2 \) are not in \( \mathcal{L} \) since they are not in \( \Sigma_2 \). The Lie algebra \( \mathcal{L}^{(2)} \) does not contain \( \alpha_i||e \) for \( i = \{1, 2\} \) since the only way to obtain them is through the bracket \( [\alpha_i||\alpha_i, \alpha_i||e] \) for \( i \in \{1, 2\} \) but \( \alpha_i||\alpha_i \) are not in \( \mathcal{L}^{(1)} \). Therefore by Corollary 3.4 the Lie algebra \( \mathcal{L}^{(2)} \) is solvable and consequently \( \mathcal{L} \) is solvable.

In case that only one of \( \alpha_1||e \) and \( \alpha_2||e \) are in \( \Sigma_0 \), the argument is similar. \( \square \)

### 3.4. The general case of not necessarily graded algebras.

In this subsection we will see how to deal with not necessarily graded finite dimensional algebras.

We start by recalling some definitions and basic facts.

**Definition 3.8.** Given a \( K \)-algebra \( A \), define

\[
\text{Der}_{rad}(A) := \{ \delta \in \text{Der}_K(A) | \delta(J(A)) \subseteq J(A) \},
\]

which is a Lie subalgebra of \( \text{Der}_K(A) \).

**Remark 3.9.** The inner derivations of \( A \) form a Lie ideal of \( \text{Der}_{rad}(A) \) so, there is an inclusion:

\[
\text{HH}^1_{rad}(A) := \text{Der}_{rad}(A)/\text{Inn}(A) \hookrightarrow \text{Der}_K(A)/\text{Inn}(A) = \text{HH}^1(A).
\]

As a consequence, \( \text{HH}^1_{rad}(A) \) is a Lie subalgebra of \( \text{HH}^1(A) \).
The first Hochschild cohomology as a Lie algebra

The problem then is: how far is $HH^1_{rad}(A)$ from $HH^1(A)$? In other words, how far is the inclusion in Remark 3.9 from being an isomorphism? In fact, in characteristic zero, this is always an isomorphism, as proved by Hochschild in [11, Theorem 4.2]. In Hochschild’s terminology, this means that $J(A)$ is a characteristic ideal of $A$.

This is no longer true in characteristic $p > 0$, as the following example shows: consider $A = K[x]/(x^p)$ and the $K$-linear derivation $\delta_0 : A \to A$ such that $\delta_0(x) = 1$. It is well defined, and clearly it does not preserve the radical. In this example any derivation is a scalar multiple of $\delta_0$ plus a derivation preserving the radical, so the quotient $HH^1(A)/HH^1_{rad}(A)$ is 1-dimensional.

In [8] the authors provide a sufficient condition for $HH^1_{rad}(A)$ to be equal to $HH^1(A)$. In particular, it follows from [8, Proposition 2.7] that the difference between $HH^1_{rad}(A)$ and $HH^1(A)$ comes from loops in $Q$.

It is worth noting that in order to show solvability of $HH^1(A)$ for most of the tame algebras considered in the next section, it is enough to verify that $HH^1(A) = HH^1_{rad}(A)$.

Going back to the problem of solvability, we first recall a result from [8] which we outline below.

**Proposition 3.10.** (see Proposition 2.4 and Corollary 2.5 in [8]) Let $A$ be a finite dimensional $K$-algebra, and $3 \leq n \in \mathbb{N}$. There is a morphism of Lie algebras

$$\text{Der}_{rad}(A/J(A)^n) \to \text{Der}_{rad}(A/J(A)^{n-1})$$

with abelian kernel inducing a morphism of Lie algebras

$$HH^1_{rad}(A/J(A)^n) \to HH^1_{rad}(A/J(A)^{n-1})$$

also with abelian kernel. As a consequence, if $HH^1_{rad}(A/J(A)^{n-1})$ is solvable, then $HH^1_{rad}(A/J(A)^n)$ is also solvable. In particular, if $HH^1_{rad}(A/J(A)^2)$ is solvable, then so is $HH^1_{rad}(A)$.

The main lines of the proof consist of verifying that any derivation $\delta : A/J(A)^n \to A/J(A)^n$ induces a well defined derivation $\hat{\delta} : A/J(A)^{n-1} \to A/J(A)^{n-1}$ such that $\hat{\delta}(\overline{a}) = \Pi(\delta((a)))$, where $\overline{a}$ is the class in $A/J(A)^{n-1}$ of an element $a \in A$, $\overline{a}$ denotes its class in $A/J(A)^n$ and $\Pi : A/J(A)^n \to A/J(A)^{n-1}$ denotes the canonical projection. It is straightforward to verify that it is a morphism of Lie algebras and the fact that any derivation which preserves the radical also preserves its powers leads to the kernel being abelian.

We now prove analogues in the general case of the results which we have already proved for the graded case.

We begin by showing that Theorem 3.1 still holds for ungraded algebras.

**Theorem 3.11.** Let $K$ be an algebraically closed field and let $A$ be a finite dimensional $K$-algebra. Suppose that $\text{Ext}^1_A(S,S) = \{0\}$ for every simple $A$-module $S$ and that $\text{dim}_K(\text{Ext}^1_A(S,T)) \leq 1$ for all nonisomorphic simple $A$-modules $S$ and $T$. Then $HH^1(A)$ is a strongly solvable Lie algebra. In particular, $HH^1(A)$ is a solvable Lie algebra.
Proof. Given $A$, since the Ext-quiver has no loops, we know that $\text{HH}^1(A)$ and $\text{HH}^1(\text{rad})(A)$ are equal, and if $\text{HH}^1(\text{rad})(A/J(A)^2)$ is solvable, then $\text{HH}^1(\text{rad})(A)$ is solvable too. Notice that

$$\text{Ext}^1_{A/J(A)^2}(S, S) = \text{Ext}^1_A(S, S) = \{0\} \quad \text{and} \quad \dim_K \text{Ext}^1_{A/J(A)^2}(S, T) = \dim_K \text{Ext}^1_A(S, T) \leq 1$$

for all simple $A$-modules $S, T$. So, $A$ satisfies the hypotheses if and only if $A/J(A)^2$, which is is a graded algebra, does for Theorem 3.1. As a consequence, $\text{HH}^1(A/J(A)^2)$ is solvable and so is $\text{HH}^1(A)$. \(\Box\)

Next we describe $\Sigma_0(A/J(A)^i)$, recall that:

$$\Sigma_0(A/J(A)^i) = \{\alpha||e| \alpha||e \text{ is a summand of an element in Ker}(\delta^1)\}$$

and that $\delta^1(\alpha||e) = \sum_{r \in \mathcal{R}, \alpha| r} r||r(\alpha, e)$. If we take $i$ to be equal to the Loewy length of $A$, we obtain also the definition of $\Sigma_0(A)$. Similarly we define $\Sigma_1(A)$. Since we are supposing that $A$ is finite dimensional and $\alpha$ is a loop, at least one relation containing $\alpha$ must exist, so that the indexing set of the sum is non empty. For $i = 2$,

$$\delta^1(\alpha||e) = \sum_{\gamma \in Q_1, \gamma \neq \alpha, t(\gamma) = e} r_{\alpha\gamma}||\gamma + \sum_{\mu \in Q_1, \mu \neq \alpha, s(\mu) = e} r_{\mu\alpha}||\mu + \alpha^2||2\alpha,$$

where $r_{\alpha\gamma}$ and $r_{\mu\alpha}$ are the relations corresponding to the fact that the respective compositions of these arrows are zero.

If the characteristic is not 2, then the last term is non zero and implies that $\alpha||e$ will never appear as a summand of an element in Ker$(\delta^1)$. In characteristic 2, the same conclusion can be obtained by looking either at the first sum or at the second one, since at least one of them is indexed over a non empty set, unless the algebra $A$ is $K[x]/(x^n)$, for some $n \geq 2$.

As a consequence, the only case where $\Sigma_0(A)$ is empty and $\Sigma_0(A/J(A)^2)$ is non empty is when $A = K[x]/(x^n)$, for some $n > 2$, char$(K) = 2$ and $n$ is odd.

In the following we show the ungraded analogues of Corollary 3.4, Corollary 3.5 and Theorem 3.3 in a more restricted form. We note that each time that we will use them in the sequel, we will first verify either by hand or that $\text{HH}^1(\text{rad})(A) = \text{HH}^1(A)$.

Note that $\Sigma_0(A) = \emptyset$ means that no $\alpha||e$ occurs as a non-trivial summand of a derivation. The following lemma characterizes the condition $\Sigma_0(A) = \emptyset$ in terms of outer derivations.

**Lemma 3.12.** Note that $\Sigma_0(A) = \emptyset$ if and only if $\text{HH}^1(A) = \text{HH}^1(\text{rad})(A)$, or equivalently, if and only if each derivation preserves the radical.

Proof. If no $\alpha||e$ occurs as a non-trivial summand of a derivation, then all derivations preserve the radical hence $\text{HH}^1(A) = \text{HH}^1(\text{rad})(A)$. Conversely, if there exists $\alpha||e$ occurring as a non-trivial summand of a derivation $f$, then $f$ is a derivation which does not preserve the radical. Since all inner derivations preserve the radical, then $f$ is an outer derivation. Consequently, $\text{HH}^1(A) \neq \text{HH}^1(\text{rad})(A)$. \(\Box\)
Theorem 3.13. Let \( A = KQ/I \) be a finite dimensional \( K \)-algebra such that \( \dim_K \Ext_A^1(S, T) \leq 1 \) for simple \( A \)-modules \( S \) and \( T \). Suppose that \( \Sigma_0(A) \) is empty. Then \( HH^1(A) \) is a solvable Lie algebra.

Proof. As before, we consider the graded algebra \( A/J(A)^2 \). We note that since there are no parallel arrows in \( Q \), the hypotheses of Theorem 3.3 are satisfied by \( A/J(A)^2 \), hence \( HH^1(A/J(A)^2) \) is solvable and so the same holds for \( HH^1_{\text{rad}}(A) \). Since \( \Sigma_0(A) \) is empty, all the derivations preserve the Jacobson radical, consequently \( HH^1(A) = HH^1_{\text{rad}}(A) \). The remaining case is when \( A = K[x]/(x^n) \) for some \( n > 2 \), \( \text{char}(K) = 2 \) and \( n \) is odd. In this case \( \Sigma_0(A) \) is empty but \( \Sigma_0(A/J(A)^2) \) is not, therefore we cannot apply the same method as before. However, \( HH^1(A) \) has as a \( K \)-basis the derivations \( f_i \) such that \( f_i(x) = x^i \) for \( 1 \leq i \leq n - 1 \). It is easy to show that \( HH^1(A) \) is solvable. \( \square \)

Proposition 3.14. Let \( A = KQ/I \) be a finite dimensional \( K \)-algebra such that \( \dim_K \Ext_A^1(S, T) \leq 1 \) for all simple \( A \)-modules \( S \) and \( T \). Then \( HH^1_{\text{rad}}(A) \) is a solvable Lie algebra.

Proof. Again, we consider the graded algebra \( A/J(A)^2 \) and we note that since there are no parallel arrows in \( Q \), it satisfies the hypotheses of Corollary 3.4. Note that \( \alpha||\alpha^2 \) is not in \( \Sigma_2(A/J(A)^2) \) for any loop \( \alpha \) since \( A/J(A)^2 \) is a radical square zero algebra and consequently \( \alpha^2 = 0 \). The result follows. \( \square \)

Finally we state the analogue of Corollary 3.5. After reducing to \( A/J(A)^2 \), its proof is analogous to the proof of Corollary 3.5.

Corollary 3.15. Let \( A = KQ/I \) be a finite dimensional algebra over a field \( K \) of characteristic 2 such that \( \dim_K \Ext_A^1(S, T) \leq 1 \) for all simple \( A \)-modules \( S \) and \( T \). Then \( HH^1_{\text{rad}}(A) \) is a solvable Lie algebra.

4. Applications. In this section we prove the solvability of \( HH^1(A) \) for the symmetric tame algebras \( A \) classified in [21], as an application of the results in Section 3. As before, throughout this section \( K \) is an algebraically closed field.

4.1. Dihedral, semi-dihedral and quaternion algebras. We first focus on symmetric tame algebras of dihedral, semi-dihedral and quaternion type, studied and classified up to Morita equivalence and up to scalars in [9]. This list of algebras contains all tame blocks of group algebras of finite groups. The classification in [9], has been extended up to derived equivalences in [13] and more recently most of the algebras of dihedral, semi-dihedral and quaternion have been distinguished up to stable equivalence of Morita type [25]. For algebras of dihedral type this classification is complete [24].

Theorem 4.1. Let \( K \) be an algebraically closed field of arbitrary characteristic. Let \( A \) be a symmetric tame algebra of dihedral, semi-dihedral or quaternion type not derived equivalent to \( K[X, Y]/(X^2, Y^2) \) if the characteristic of \( K \) is not 2. Then \( HH^1(A) \) is a solvable Lie algebra.
Proof. We organise the proof by the number of simple modules, and within each case we consider the dihedral, semi-dihedral and quaternion cases up to derived equivalence.

In the local algebra case all algebras have the same quiver $Q$ which has one vertex $e$ and two loops $X$ and $Y$.

Since the derived equivalence classification for the local algebras coincides with the Morita classification, it follows from [9] that for dihedral type we have that up to derived equivalence $A$ is one of the following:

1. $D(1A)_{1} = KQ/(X^2, Y^2, XY - YX)$,
2. $KQ/(XY, XY, X^m - Y^n)$: for $m \geq n \geq 2, m + n > 4$,
3. $D(1A)_{k} = KQ/(X^2, Y^2, (XY)^k - (YX)^k)$ in $k \geq 2$;
and when char$(K) = 2$, there are two more cases:
4. $KQ/(X^2, XY - YX, YX - Y^2)$,
5. $D(1A)_{d} = KQ/(X^2 - (XY)^k, Y^2 - d \cdot (XY)^k, (XY)^k - (YX)^k, (XY)^k X, (YX)^k Y)$, for $k \geq 2, d \in \{0, 1\}$.

In case (1), it is well-known, see for example [15], that $HH^1(A)$ is a Jacobson–Witt algebra which is a simple Lie algebra of Cartan type, so, in particular, it is not solvable. For char$(K) \neq 2$, $HH^1(K[X, Y]/(X^2, Y^2))$ has the following $K$-basis: $\{X|X, Y||Y\}$. Hence, $HH^1(K[X, Y]/(X^2, Y^2))$ is a 2-dimensional abelian Lie algebra consequently it is solvable.

For case (2), we note that the algebra $A/J(A)^3$ is graded for all $m \geq n \geq 2$ and $m + n > 4$. There are two different cases, (a) is for $m, n \geq 3$ and (b) is for $m \geq 3$ and $n = 2$. In both cases one easily checks that $\Sigma_0(A)$ is empty In case (a), it is also easy to verify that $X||Y$ and $Y||X$ are not in $\Sigma_1(A)$. Thus by Proposition 3.6 we have that $HH^1(A/J(A)^3)$ is solvable. Using Lemma 3.12 and Proposition 3.10 we obtain the result in this case.

Now suppose for case (b) that $n = 2$ and $m \geq 3$. Then an easy calculation shows that for $A/J(A)^3$, we have $\text{Ker}\delta^1 = \{X||X, X||Y, X||X^2, Y||Y, Y||X^2\}$. One easily verifies by repeatedly calculating the brackets of the elements in $\text{Ker}\delta^1$, that after a small number of iterations, they are all eventually zero. Therefore $HH^1(A/J(A)^3)$ and hence $HH^1(A)$ which coincides with $HH^1_{rad}(A)$ are solvable.

In case (3), the algebra $A$ is graded. A direct calculation shows that $X||Y$ and $Y||X$ are not $\Sigma_1$ and that if char$(K) \neq 2$, then $X||e$ and $Y||e$ are not in $\Sigma_0$. The result then follows from Theorem 3.7 (1) if char$(K) = 2$ and from Proposition 3.6 if char$(K) \neq 2$.

In case (4) since $A$ is graded, it is enough to verify that $X||e$ and $Y||e$ are not in $\Sigma_0$ and that the Lie algebra $L_1$ has $K$-basis $\{X||X + Y||Y, X||X + X||Y, Y||X\}$. Moreover, $L_1^{(1)}$ is abelian and thus $HH^1(A)$ is solvable.

For case (5) we note that the algebra $A/J(A)^3$ is isomorphic to $KQ/(X^2, Y^2, XYX, YXY)$, which is graded. Since $X||e$ and $Y||e$ are not in $\Sigma_0$, and $X||Y$ and $Y||X$ are not in $\Sigma_1$, the result follows by Proposition 3.6 and Lemma 3.12.
Next we consider local algebras of semi-dihedral type. In this situation, there are two possibilities, one of which occurs only in characteristic 2.

1. $SD(1A)^1_2 = KQ/((XY)^k - (YX)^k, (XY)^k X, Y^2, (XY)^{k-1} - (YX)^k)$, for $k \geq 2$. We consider $A/J(A)^3$ which is again given by $KQ/(X^2, Y^2, XYX, YXY)$. It is easy to verify that $X|Y$ and $Y|X$ are not in $\Sigma_1(A/J(A)^3)$. In addition, if $\text{char}(K) \neq 2$, $X|e$ and $Y|e$ are not in $\Sigma_0$. Hence the solvability of $HH^1(A)$ follows from Proposition 3.6 and Lemma 3.12.

2. $SD(1A)^2_2(c, d) = KQ/((XY)^k - (YX)^k, (XY)^k X, Y^2 - d(XY)^k, X^2 - (YX)^{k-1} Y + e(XY)^k)$, char($K$) = 2, $k \geq 2$, $(c, d) \neq (0, 0)$. In this case, we also focus on the algebra $A/J(A)^3$, which is again given by $KQ/(X^2, Y^2, XYX, YXY)$ and we proceed as before.

Finally, we consider local algebras of quaternion type. Again, there are two possibilities with one of them occurring only in characteristic 2.

1. $Q(1A)^1_1 = KQ/((XY)^k - (YX)^k, (XY)^k X, Y^2 - (XY)^{k-1} X, X^2 - (YX)^{k-1} Y)$, for $k \geq 2$ and

2. the algebras $Q(1A)^2_2(c, d)$, which occur only in characteristic 2 and are given by

$$KQ/(X^2 - (YX)^{k-1} Y - c(XY)^k, (XY)^k Y, Y^2 - (XY)^{k-1} X - d(XY)^k),$$

$$(XY)^k - (YX)^k, (XY)^k X, (YX)^k Y),$$

for $k \geq 2$ and $(c, d) \neq (0, 0)$.

In both cases $A/J(A)^3 = KQ/(X^2, Y^2, XYX, YXY)$ and the same arguments apply.

Using Holm’s classification [12], there are four families of symmetric tame algebras with two simple modules up to derived equivalence. They all have the quiver of type $2B$:

$$\begin{array}{ccc}
\alpha & \beta & \eta \\
\downarrow & & \circlearrowright \\
e_1 & \gamma & e_0
\end{array}$$

Aside from two exceptions, in characteristic 2 the solvability of $HH^1(A)$ will follow from Corollary 3.5, Corollary 3.15 and Lemma 3.12. For $\text{char}(K) \neq 2$, we will use Theorem 3.13 together with Lemma 3.12. Hence, aside from two exceptions, we will prove that $\eta||e_1$ and $\alpha||e_0$ are not in $\Sigma_0$. In the two cases where $\Sigma_0$ is non-empty, we will use a simple argument to show that $HH^1(A)$ is solvable.

1. $D(2B)^{k,s}(c) = KQ/(\beta \eta, \eta \gamma, \beta \alpha, \alpha^2 - c(\alpha \beta \gamma)^k, (\alpha \beta \gamma)^k - (\beta \gamma \alpha)^k, \alpha^s - (\gamma \alpha \beta)^k)$ with $k \geq s \geq 1$ and $c \in \{0, 1\}$. Note that $\eta||e_1$ is not in $\Sigma_0$ since $\delta^1(\eta||e_1)$ contains as non-zero summand $\beta \eta||\beta$. In addition, if $\text{char}(K) \neq 2$, then $\alpha||e_0$ is not in $\Sigma_0$ since $\delta^1(\alpha||e_0)$ contains as non-zero summand $2r||\alpha$ where $r = \alpha^2 - c(\alpha \beta \gamma)^k$. Hence, $\eta||e_1$ and $\alpha||e_0$ are not in $\Sigma_0$ and consequently $HH^1(A)$ is solvable. If $\text{char}(K) = 2$ and if $c = 1$, then $\delta^1(\alpha||e_0)$ contains as non-zero summand $r||\beta \gamma (\alpha \beta \gamma)^{k-1}$ and consequently $HH^1(A)$ is solvable.
If char(\(K\)) = 2 and if \(c = 0\), then \(\alpha || e_0 \in \Sigma_0\). It is easy to see that also \(\alpha || \alpha \in \Sigma_1\). Note that in this case \(\alpha^2 = 0\), hence \(\alpha || \alpha\) is not a summand of an element in the derived subalgebra of \(HH^1(A)\) since the only possible way to obtain \(\alpha || \alpha\) is through the bracket \([\alpha || e_0, \alpha || \alpha^2]\). Therefore \(\alpha || e_0\) is not a summand of an element in \(HH^1(A)^{(2)}\) since the only possible way to obtain \(\alpha || e_0\) is through the bracket \([\alpha || \alpha, \alpha || e_0]\). Consequently, every derivation in \(HH^4(A)^{(2)}\) preserves the radical. Hence \(HH^1(A)^{(2)}\) is a subalgebra of \(HH^1_{rad}(A)\). By Proposition 3.14 the Lie algebra \(HH^1_{rad}(A)\) is solvable and consequently \(HH^1(A)^{(2)}\) is solvable. Hence \(HH^1(A)\) is solvable.

\[(2)\] \(SD(2B)^{k,t}_1(c) = KQ/\langle \gamma \beta, \eta \gamma, \beta \eta, \alpha^2 - (\gamma \alpha \beta)^{k-1}\beta \gamma - c(\alpha \beta \gamma)^k, \eta^t - (\gamma \alpha \beta)^k, (\beta \gamma)^k - (\beta \gamma \alpha)^k \rangle\) with \(k \geq 1, t \geq 2\) and \(c \in \{0, 1\}\). Formally, there is the theorem \(D(2B)^{k,s}_1(c)\), we deduce that \(\eta || e_1\) is not in \(\Sigma_0\), and if char(\(K\)) \(\neq 2\) neither is \(\alpha || e_0\). Consequently \(HH^1(A)\) is solvable. Assume char(\(K\)) = 2. Then it is easy to check that \(\alpha || e_0 \in \Sigma_0\). Assume that also \(\alpha || \alpha\) and \(\alpha || \alpha^2\) are summands of elements in Ker(\(\delta^1\)). Note that \([\alpha || e_0, \alpha || \alpha^2] = -2\alpha || \alpha = 0\). Hence \(\alpha || \alpha\) is not a summand of an element of \(HH^1(A)^{(1)}\). In fact, the only possible way to obtain \(\alpha || \alpha\) is through the bracket \([\alpha || e_0, \alpha || \alpha^2]\). Consequently \(\alpha || e_0\) is not a summand of an element in \(HH^1(A)^{(2)}\), in other words, every derivation in \(HH^1(A)^{(2)}\) preserves the radical. Hence \(HH^1(A)^{(2)}\) is a subalgebra of \(HH^1_{rad}(A)\). By Proposition 3.14 the Lie algebra \(HH^1_{rad}(A)\) is solvable and consequently \(HH^1(A)^{(2)}\) is solvable. Hence \(HH^1(A)\) is solvable. Note that if \(\alpha || \alpha\) or \(\alpha || \alpha^2\) are not summands of elements in Ker(\(\delta^1\)) the proof above simplifies and it follows that \(HH^1(A)\) is solvable.

\[(3)\] \(SD(2B)^{k,t}_2(c) = KQ/\langle \beta \eta - (\alpha \beta \gamma)^{k-1} \alpha \beta, \eta \gamma - (\gamma \alpha \beta)^{k-1} \gamma \alpha, \gamma \beta - \eta^{t-1}, \alpha^2 - c(\alpha \beta \gamma)^k, (\beta \eta)^k, \eta^2 \gamma \rangle\) with \(k \geq 1, t \geq 2, k + t \geq 4\) and \(c \in \{0, 1\}\). The elements \(\eta || e_0\) and \(\alpha || e_0\) are not in \(\Sigma_0\).

\[(4)\] \(Q(2B)^{k,s}_1(a, c)\): the algebras are of the form

\[KQ/\langle \gamma \beta - \eta^{s-1}, \beta \eta - (\alpha \beta \gamma)^{k-1} \alpha \beta, \eta \gamma - (\gamma \alpha \beta)^{k-1} \gamma \alpha, \alpha^2 - a(\beta \gamma \alpha)^{k-1} \beta \gamma - c(\beta \gamma \alpha)^k, \alpha^2 \beta, \beta \gamma \alpha \rangle\]

with \(k \geq 1, s \geq 3\) and \(a \neq 0\). Note \(\eta || e_1\) is not in \(\Sigma_0\) since \(\delta^1(\eta || e_1)\) contains as non-zero summand \(r_2 || \gamma\) where \(r_2 = \eta \gamma - (\gamma \alpha \beta)^{k-1} \gamma \alpha\). Finally \(\alpha || e_0\) is not in \(\Sigma_0\) since \(\delta^1(\alpha || e_0)\) contains as non-zero summand \(r_3 || (\beta \gamma \alpha)^{k-1} \beta \gamma\) where \(r_3 = \beta \eta - (\alpha \beta \gamma)^{k-1} \alpha \beta\). Consequently \(HH^1(A)\) is solvable.

For tame symmetric algebras with 3 simple modules, there are three classes of algebras denoted by \(3K\), \(3A\) and \(D(3R)^{k,s,t,u}\) with \(s \geq t \geq u \geq k \geq 1, t \geq 2\). The quiver of the algebras in \(3K\) is of the form

\[\begin{array}{ccc}
& \gamma & \\
\lambda & \alpha & \delta \\
\eta & & \beta \\
& \kappa &
\end{array}\]

\[e_1 \leftrightarrow e_2 \leftrightarrow e_3\]
while for algebras of type 3A the quiver is:

```
+-----+ +-----+ +-----+
|     | |     | |     |
| ∂β | | δγ | | ∸η |
|     | |     | |     |
+-----+ +-----+ +-----+
```

Thus the solvability of \( HH^1 \) for algebras of type 3K and of type 3A follows from Theorem 3.11.

The algebras of type \( D(3R)^{k,s,t,u} \) with \( s \geq t \geq u \geq k \geq 1, t \geq 2 \) have the following quiver:

```
+-----+ +-----+ +-----+
|     | |     | |     |
| ∂α | | βρ | | δξ |
|     | |     | |     |
+-----+ +-----+ +-----+
```

and can be presented as follows: \( KQ/(\alpha\beta, \beta\rho, \rho\delta, \delta\xi, \xi\lambda, \lambda\alpha, \alpha^s - (\beta\delta\lambda)^k, \rho^t - (\delta\gamma\beta)^k, \xi^u - (\lambda\beta\delta)^k) \).

The result follows from Theorem 3.13, after verifying that \( \Sigma_0(A) = \emptyset \) due to the monomial quadratic relations \( \alpha\beta, \beta\rho \) and \( \delta\xi \). More precisely, \( \alpha||e_1 \) is not in \( \Sigma_0 \) since \( \delta^1(\alpha||e_1) \) contains as non-zero summand \( \alpha\beta||\beta \). Similarly, \( \delta^1(\rho||e_2) \) and \( \delta^1(\xi||e_3) \) contain as non-zero summands \( \beta\rho||\beta \) and \( \delta\xi||\delta \), respectively. Therefore \( HH^1(A) \) is solvable.

4.2. Brauer graph algebras. In this subsection we prove the solvability of \( HH^1(A) \) for \( A \) a Brauer graph algebra not derived equivalent to the trivial extension of the Kronecker algebra. In the latter case, \( HH^1(A) \) is isomorphic to \( gl_2(K) \) which is not solvable except in characteristic 2.

The solvability of the first Hochschild cohomology of Brauer graph algebras with multiplicity function identically equal to one and not derived equivalent to the trivial extension of the Kronecker algebra has been shown in [4]. We now prove that the first Hochschild cohomology of any Brauer graph algebra with any multiplicity function (apart from the trivial extension of the Kronecker algebra) is solvable in any characteristic, except possibly in characteristic 2. We call a relation of the form \( p - q \), for paths \( p, q \) in \( Q \), a binomial relation.

**Theorem 4.2.** Let \( A = KQ/I \) be a Brauer graph algebra. Then \( HH^1(A) \) is a solvable Lie algebra except if \( A \) is derived equivalent to the trivial extension of the Kronecker algebra in characteristic different from 2 or if the characteristic of \( K \) is 2 and there is a loop \( \alpha \) such that \( \alpha^2 = 0 \).
As we will see in the proof, for a Brauer graph algebra, we have in almost all cases that $\text{HH}^1_{\text{rad}}(A) = \text{HH}^1(A)$ and only if the characteristic of $K$ is 2, we might not have equality. However, we think that $\text{HH}^1(A)$ should be solvable for all cases also in characteristic 2 as is shown for some of these cases in Section 4.3 and for Brauer graph algebras with multiplicity function identically equal to one in [4].

Proof of Theorem 4.2. Let $A = KQ/I$ be a Brauer graph algebra not derived equivalent to the trivial extension of the Kronecker algebra. Brauer graph algebras with one simple module fall into the local dihedral algebras of type (1)-(3) in the proof of Theorem 4.1 and the result then follows in this case. So we now assume that $A$ has at least two non-isomorphic simple $A$-modules. Then $B := A/J(A)^3$ is a graded algebra with quiver $Q$ which might have loops and parallel arrows. Since $A$ is special biserial non-local, there is at most one loop at any given vertex. Denote by $\mathcal{R}_{\text{mon}}$ the set of monomial relations of $B$.

Suppose that $\alpha$ and $\beta$ are two parallel arrows in $Q_1$. Then there exists no monomial relation $r \in \mathcal{R}_{\text{mon}}$ such that both $\alpha$ and $\beta$ are in $r$. But there exists at least one monomial relation $r_0 \in \mathcal{R}_{\text{mon}}$ containing at least one of them, say, for example, $r_0$ contains $\alpha$. Then $r_0||r_0^{(\alpha,\beta)} \neq 0$. Suppose that $\gamma$ is another arrow in $r_0$. Then $\alpha$ is not parallel to $\gamma$ and $r_0||r_0^{(\alpha,\beta)} \neq r_0||r_0^{(\gamma,\delta)}$, for any $\delta \in Q_1$ with $\delta$ parallel to $\gamma$. Thus $\sum_{\alpha, r \in \mathcal{R}_{\text{mon}}} r || r^{(\alpha,\beta)} \neq 0$ and $\alpha || \beta \notin \Sigma_1$.

Hence, by Corollary 3.5 if the characteristic of $K$ is 2, we have that $\text{HH}^1(B)$ is solvable and thus $\text{HH}^1_{\text{rad}}(A)$ is solvable.

If the characteristic of $K$ is not 2, we show that the hypotheses of Theorem 3.3 hold for $B$. By the above it is enough to show that $\Sigma_0$ is empty. Let $\alpha \in Q_1$ be a loop at vertex $e$. Then $\alpha$ must appear in at least one relation $r$ generating $I$. This relation is either of the form $r = \alpha^2$ or of the form $r = \alpha^m - C^n$ for some integers $n, m \geq 1$ and a cycle $C = c_1 \ldots c_k$ in $Q$.

If $r = \alpha^2$ then $r$ is also a relation for $B$ and $\delta^1(\alpha || e)$ contains as non-zero summand $2r || \alpha$.

If $r = \alpha^m - C^n$ then $\alpha c_1$ and $c_k \alpha$ are monomial relations for $B$ and $\delta^1(\alpha || e)$ contains as non-zero summand $\alpha c_1 || c_1$.

Thus the hypotheses of Theorem 3.3 are verified and $\text{HH}^1(B)$ is solvable. It then follows from Proposition 3.10 that $\text{HH}^1_{\text{rad}}(A)$ is solvable.

Now suppose that every loop $\alpha$ in $A$ is in a binomial relation of the form $\alpha^m - C^n$ for some nonzero cycle $C = c_1 \ldots c_k$ and $m \geq 2$ and $n \geq 1$. Then both $\alpha c_1$ and $c_k \alpha$ as well as $\alpha^{m+1}$ are in $I$. Consequently, either the relation $\alpha c_1$ or the relation $c_k \alpha$ yields that there is no derivation sending the loop $\alpha$ to its source, and using Lemma 3.12 we conclude that $\text{HH}^1_{\text{rad}}(A) = \text{HH}^1(A)$.

Remark 4.3. (1) The proof of Theorem 4.2, shows that for all characteristics of $K$, $\text{HH}^1_{\text{rad}}(A)$ is solvable for a Brauer graph algebra $A$, unless $A$ is isomorphic to the trivial extension of the Kronecker algebra.

(2) In characteristic 2, if $A$ is a Brauer graph algebra which has a relation $\alpha^2 = 0$ for a loop $\alpha$, one can easily construct an example to show that $\text{HH}^1_{\text{rad}}(A) \neq \text{HH}^1(A)$. Consider, for instance, the algebra $KQ/I$ where $Q$ is the quiver.
and \( I = \langle \alpha^2, \gamma \beta, \beta \gamma \alpha - \alpha \beta \gamma \rangle \). The derivation that sends \( \alpha \) to its vertex and any other arrow to zero is well defined and does not preserve the radical. However as claimed above, one easily verifies by hand that \( \text{HH}^1(A) \) is solvable since its second derived Lie algebra is zero.

4.3. Symmetric tame algebras. In this subsection \( A \) is a symmetric tame algebra which is in the classification of Skowroński’s survey paper in [21]. Our aim is to prove the following.

**Theorem 4.4.** Let \( A \) be a symmetric tame algebra that appears in the classification in [21] not derived equivalent to \( K[X]/(X^r) \) when \( \text{char}(K) \mid r \) and not derived equivalent to the trivial extension of the Kronecker algebra if \( \text{char}(K) \neq 2 \). Then \( \text{HH}^1(A) \) is a solvable Lie algebra.

These algebras include the algebras of dihedral, semi-dihedral and quaternion type as well as Brauer graph algebras. We have seen in the previous sections that for these algebras the first Hochschild cohomology as a Lie algebra is solvable, except for a small number of special cases.

We start by recalling the following result from [21] (using the notation in that paper), which provides the derived equivalence classes of algebras of non-simple connected symmetric algebras of finite type.

**Theorem 4.5.** ([21]) The algebras \( N_{e,m}^n \), \( m \geq 2, e \geq 1 \), \( D(m), m \geq 2 \), \( T(K \Delta(A_n)) \) \( n \geq 1 \), \( T(K \Delta(D_n)) \) \( n \geq 4 \), \( T(K \Delta(E_n)) \), \( 6 \leq n \leq 8 \) and \( D'(m), m \geq 2 \) and \( \text{char}(K) = 2 \), form a complete family of representatives of the derived classes of the non-simple connected symmetric algebras of finite type.

**Proposition 4.6.** Let \( A \) be as in Theorem 4.5, excluding Nakayama algebras \( N_{e,m}^n \) where \( e = 1 \) and \( \text{char}(K) \) divides \( m + 1 \). The Lie algebra \( \text{HH}^1(A) \) is solvable.

**Proof.** Let \( N_{e,m}^n \), \( m \geq 2, e \geq 1 \) be a Nakayama algebra with \( e \) vertices and such that all the compositions of \( em+1 \) consecutive arrows generate the admissible ideal. If \( e = 1 \), then \( N_{e,m}^n \cong K[x]/(x^{m+1}) \). If \( p \) divides \( m + 1 \), then \( \text{HH}^1(K[x]/(x^{m+1})) \) is a perfect Lie algebra, therefore not solvable. If \( p \) does not divide \( m + 1 \), then \( x \mid e \) is not in \( \Sigma_0 \) and since the algebra is graded by Theorem 3.3 we have that \( \text{HH}^1(N_{e,m}^n) \) is a solvable Lie algebra. If \( e > 1 \) the statement follows from Theorem 3.11.

The trivial extension algebra \( T(K \Delta(A_n)) \) for \( n \geq 1 \) is the Nakayama algebra \( N_n^1 \) therefore the solvability of \( \text{HH}^1(T(K \Delta(A_n))) \) follows from the previous paragraph. For the other self-injective algebras of Dynkin type, we proceed as follows.

The algebras \( D(m) \) and \( D'(m) \) are defined in [21, Section 3.14] and a direct calculation shows that \( \text{HH}^1(A) \) is solvable for both \( D(m) \) and \( D'(m) \).

From Theorem 3.11 we directly obtain the solvability of \( \text{HH}^1(T(K \Delta(D_n))) \) and of \( \text{HH}^1(T(K \Delta(E_n))) \).
The description of symmetric algebras of Euclidean type given in [21] is as follows.

**Theorem 4.7.** ([21, Theorem 4.16]) Let $A$ be a symmetric algebra of Euclidean type. The following are equivalent

- $A$ is symmetric and has nonsingular Cartan matrix.
- $A$ is derived equivalent to an algebra of the form $A(p, q, \Lambda(n))$ or $\Gamma(n)$.

**Proposition 4.8.** The first Hochschild cohomology space of any symmetric algebra $A$ of Euclidean type with nonsingular Cartan matrix is a solvable Lie algebra.

**Proof.** Such an algebra is a Brauer graph algebra, therefore the statement follows from Theorem 4.2 except for $\Lambda(n)$ in characteristic 2. So suppose that $A = \Lambda(n)$ and that the characteristic of $K$ is 2. Then $A/J(A)^3$ is graded and it follows from Corollary 3.5 that $\text{HH}^1_{\text{rad}}(A)$ is solvable.

We verify by hand that $\text{HH}^1_{\text{rad}}(A) = \text{HH}^1(A)$. $\square$

In order to describe what happens when $A$ is a symmetric algebra of Euclidean type with singular Cartan matrix we need a preliminary result.

**Proposition 4.9.** ([21]) Let $A$ be a symmetric algebra of Euclidean type with singular Cartan matrix. There exists an Euclidean canonical algebra $C$ such that $A$ is isomorphic to the trivial extension $T(C)$.

**Proposition 4.10.** Let $C = C(p, q, r)$ be a canonical algebra with parameters $p, q$ and $r$ in $\mathbb{N}_{\geq 2}$. Then $\text{HH}^1(T(C))$ is solvable.

**Proof.** Let $A = T(C)$. The quiver of $A$ is given by the quiver of $C$, with two additional parallel arrows $\alpha$ and $\beta$, starting at the sink vertex of $A$ and ending at the source vertex of $A$. We begin by checking that $\text{HH}^1(A/J(A)^3)$ is solvable. For this note that $A/J(A)^3$ is graded and that since the quiver has no loops, $\Sigma_0$ is empty. Furthermore, we check by hand that neither $\alpha||\beta$ nor $\beta||\alpha$ are in $\Sigma_1$. Therefore by Theorem 3.3, $\text{HH}^1(A/J(A)^3)$ is solvable. By Proposition 3.10, we then have that $\text{HH}^1_{\text{rad}}(A)$ is solvable and since $A$ has no loops we have that $\text{HH}^1_{\text{rad}}(A) = \text{HH}^1(A)$. $\square$

**Corollary 4.11.** The first Hochschild cohomology $\text{HH}^1(A)$ for any symmetric algebra $A$ of Euclidean type with singular Cartan matrix is a solvable Lie algebra.

**Proof.** It is shown in [21] that $A$ is isomorphic to the trivial extension of an Euclidean canonical algebra of the form $C(2, 3, 3), C(2, 3, 4), C(2, 3, 5)$ or $C(2, 2, r)$ with $r \geq 2$. $\square$

Next we consider weakly symmetric algebras of tubular type. We first recall the following two results.
Theorem 4.12. ([21]) Let $A$ be a standard weakly symmetric algebra of tubular
type with non-singular Cartan matrix. Then $A$ is derived equivalent to an algebra
of the form $A_1(\lambda), A_2(\lambda)$ with $\lambda \in K \{0, 1\}, A_3, A_4, A_5$ or $A_{12}$.

Theorem 4.13. ([21]) Let $A$ be a non-standard non-domestic weakly symmetric
algebra of polynomial growth with non-singular Cartan matrix. Then $A$ is derived
equivalent to an algebra of the form $\Lambda_1, \Lambda_3(\lambda)$ where $\lambda \in K \{0, 1\}, \Lambda_4$ or $\Lambda_9$.

Proposition 4.14. Let $A$ be an algebra either as in Theorem 4.12 or as in Theo-
rem 4.13. The Lie algebra $HH^1(A)$ is solvable.

Proof. For the algebras $A_1(\lambda), A_3, A_4, A_{12}, \Lambda_4$ and $\Lambda_9$ the solvability of the first
Hochschild cohomology space follows from Theorem 3.11. For $A_5, \Lambda_1$, and $\Lambda_3(\lambda)$
we directly verify that $\Sigma_0$ is empty and the result follows from Theorem 3.13. For
$A_2(\lambda)$ we check that if the characteristic of $K$ is not 2, $\Sigma_0$ is empty and the result
follows from Theorem 3.3 and if characteristic is 2 then the result follows from
Corollary 3.5. 

Theorem 4.15. ([21, Theorem 4.14]) Let $A$ be a non-standard domestic self-injective algebra of infinite representation type. Then $A$ is derived equivalent to an algebra of the form $\Omega(n)$, for $n \geq 1$.

Proposition 4.16. Let $A$ be an algebra as in Theorem 4.15. Then $HH^1(A)$ is
solvable.

Proof. The algebras $\Omega(n)$ have the same quiver as a Brauer graph algebra with
one loop and one cycle of length at least 2. An argument similar to that for Brauer
graph algebras in the proof of Theorem 4.2 and in characteristic 2 similar to the
proof of Proposition 4.8 give the result.

The case where $A$ has singular Cartan matrix is covered by the following results.

Let us recall the following theorem.

Theorem 4.17. ([21]) Let $A$ be a self-injective algebra. The following statements
are equivalent:

- $A$ is symmetric of tubular type and has singular Cartan matrix.
- $A$ is derived equivalent to the trivial extension of a canonical tubular algebra.

Proposition 4.18. Let $A$ be derived equivalent to the trivial extension of a canonical
tubular algebra. Then $HH^1(A)$ is solvable.

Proof. If $A$ is symmetric of tubular type with singular Cartan matrix then $A$ is
derived equivalent to the trivial extension of a canonical algebra of type $C(2, 4, 4), C(3, 3, 3), C(2, 3, 6)$ or $C(2, 2, 2, 2, \lambda)$ for $\lambda \in K \{0, 1\}$. For the algebras of the form $C(p, q, r)$, the result directly follows from Proposition 4.10. For $C = C(2, 2, 2, 2, \lambda), \lambda \in K \{0, 1\}$,
we note that $T(C)$ is graded and its quiver has no loops, and it has two parallel arrows $\alpha$ and $\beta$ from the sink of the quiver of $C$ to its source. We check that neither $\alpha\|\beta$ nor $\beta\|\alpha$ are in $\Sigma_1$. Therefore the result follows from Theorem 3.3. \hfill \Box

The structure of arbitrary standard self-injective algebras of polynomial growth is described by the following theorem.

**Theorem 4.19.** ([21]) Let $A$ be a nonsimple basic connected self-injective algebra. The algebra $A$ is standard of polynomial growth if and only if $A$ is isomorphic to a self-injective algebra of Dynkin type, Euclidean type or tubular type.

**4.4. Quantum complete intersections.** The Hochschild cohomology of quantum complete intersections has been extensively studied, see for example [3, 2, 10, 19].

Recall that $KQ/I$ is a quantum complete intersection of rank $r$, if $KQ/I = K\langle X_1, \ldots, X_r \rangle / \langle X_jX_i - q_{ij}X_iX_j, X_i^{n_i}, \text{ for } 1 \leq i < j \leq r, \rangle$ where $n_i \in \mathbb{N}_{\geq 2}$ for all $i$ and $q_{ij} \neq 0$ for all $i, j$.

In this section we apply Proposition 3.6 to calculate the Lie algebra structure of the first Hochschild cohomology of those quantum complete intersections of rank $r$ with parameters different from 0 and 1, that is, quantum complete intersections with non commuting variables. In particular, it follows from our results that to a certain degree the solvability of $\text{HH}^1$ is independent of the rank and the parameters.

**Proposition 4.20.** Let $A$ be a quantum complete intersection of rank $r$ with $q_{ij} \neq 1$ for all $i < j$. Then $\text{HH}^1(A)$ is a strongly solvable Lie algebra and so, in particular, it is solvable.

**Proof.** We denote the relations as follows: $\rho_{ij}$ is the relation $X_jX_i - q_{ij}X_iX_j$ and $\rho_{ii}$ is the relation $X_i^{n_i}$, for $1 \leq i < j \leq r$. The first step in order to apply Proposition 3.6 is to show that $\mathcal{L}_0 = 0$. One of the terms appearing in the image of the differential of $X_i\|e_0$ is $(1 - q_{ii+1})\rho_{ii+1}\|X_{i+1}$. As a consequence, $X_i\|e_0 \not\in \Sigma_0$ since the only other element whose image under the differential contains $\rho_{ii+1}$ is $X_{i+1}\|e_0$, but in this case we get $(1 - q_{ii+1})\rho_{ii+1}\|X_{i+1}$. Since $X_i \neq X_{i+1}$ the previous statement follows. Now, we just have to check that $X_i\|X_j$ is not in $\Sigma_1$ for $i \neq j$. For this, notice that one of the nonzero terms in $\delta^1(X_i\|X_j)$ is $\rho_{ij}\|(1 - q_{ij})X_j^2$ or $\rho_{ji}\|(1 - q_{ji})X_j^2$, depending on whether $i < j$ or $i > j$. Again, $\rho_{ij}$ - respectively $\rho_{ji}$ appears again only in $\delta^1(X_j\|X_i)$, but, as before, neither occurrence cancels the other one out. It follows from the proof of Proposition 3.6 that $\text{HH}^1(A)$ is a strongly solvable Lie algebra. \hfill \Box

We also consider quantum complete intersections where we allow some of the quantum parameters to be 1. For convenience, we set $q_{ii} = 1$ for every $i \in \{1, \ldots, r\}$ and for any $i, j \in \{1, \ldots, r\}$ we set $q_{ij} = q_{ji}^{-1}$ for $1 \leq j < i \leq r$. 
Proposition 4.21. Let $A$ be a quantum complete intersection of rank $r$ over a field of characteristic $p$ such that $n_i = p^{m_i}$ for $m_i \in \mathbb{N}_{\geq 1}$. If there exists $i \in \{1, \ldots, r\}$ such that for every $j \in \{1, \ldots, r\}$ we have $q_{ij} = 1$, then $\text{HH}^1(A)$ is not solvable.

Proof. Assume there exists $i \in \{1, \ldots, r\}$ such that for every $j \in \{1, \ldots, r\}$ we have $q_{ij} = 1$. Then we show that $\text{HH}^1(A)$ contains a perfect Lie algebra and consequently $\text{HH}^1(A)$ is not solvable. We first prove that $X_i||e$ belongs to $\text{Ker}(\delta^1)$. In fact, the image of $X_i||e$ under $\delta^1$ is $\sum_{j=1}^{i-1} \rho_{ji}||(X_j - q_{ji}X_j) + \rho_{ii}||n_iX_i^{n_i-1} + \sum_{j=i+1}^{c} \rho_{ij}||(X_j - q_{ij}X_j)$. The first and the last sums are zero since $q_{ij} = 1$ for every $j \in \{1, \ldots, r\}$. The second expression is zero because $\text{char}(K)$ divides $n_i$. Therefore $X_i||e \in \text{Ker}(\delta^1)$. Since for a fixed $i$ all the $q_{ij} = 1$, it is easy to show that for every $h \in \{1, \ldots, p^{m_i} - 1\}$ we have that $X_i||X_i^h$ is an element of $\text{Ker}(\delta^1)$. If we denote by $X_i^0 := e$, then we have that the Lie subalgebra $S$ of $\text{HH}^1(A)$ with $K$-basis $X_i||X_i^h$ for $h \in \{0, \ldots, p^{m_i} - 1\}$ is perfect and therefore $\text{HH}^1(A)$ is not solvable.

4.5. Examples of algebras with nonsolvable first Hochschild cohomology.

In this section we give two examples of families of algebras for which the Lie algebra given by the first Hochschild cohomology is not solvable but rather semi-simple. They are a generalisation of the example of the Kronecker algebra given in [4], see also [8]. In characteristic zero, the Lie structure of the first cohomology space of both families can be deduced from [20].

These families consist of monomial algebras with radical square zero over a field of arbitrary characteristic. For the first one, let $n, m \in \mathbb{N}_{\geq 1}$. We denote by $Q_{n,m}$ the quiver with $n$ vertices $1, \ldots, n$ and a set of $m$ parallel arrows denoted by $\alpha_1, \ldots, \alpha_m$ from $i$ to $i+1$ for each pair of consecutive vertices $(i, i+1)$. For $n = 2$, the quiver $Q_{2,m}$ is the $m$-Kronecker quiver. Set $A_{n,m} = KQ_{n,m}/J(KQ_{n,m})^2$.

Proposition 4.22. There is an isomorphism of Lie algebras

$$\text{HH}^1(A_{n,m}) \cong \prod_{i=1}^{n} \mathfrak{sl}_m(K),$$

for $n, m \geq 2$. In particular, $\text{HH}^1(A_{n,m})$ is a solvable Lie algebra if and only if $\text{char}(K) = 2$ and $m = 2$.

Proof. A short calculation shows that, for $j \neq l$ where $j, l \in \{1, \ldots, m\}$, $h \in \{1, \ldots, m - 1\}$ and $i \in \{1, \ldots, n\}$, the set $\{\alpha_i,h||\alpha_i,h - \alpha_{i,h+1}||\alpha_{i,h+1}, \alpha_{i,j}||\alpha_{i,l}\}$ is a basis of $\text{HH}^1(A_{n,m})$. Since the Lie bracket of elements corresponding to parallel arrows starting at different vertices is zero, the Lie algebra structure of $\text{HH}^1(A_{n,m})$ decomposes as a product of Lie algebras. The next step is to show that for a fixed $i$ each of these Lie algebras is isomorphic to $\mathfrak{sl}_m(K)$. We consider the basis for $\mathfrak{sl}_m(K)$ given by the set of elementary matrices $\{e_{st}\}_{s,t}$ for $s \neq t$, together with $h_s = e_{ss} - e_{s+1,s+1}$. For $i$ fixed, we denote $\alpha_{i,j}$ by $\alpha_j$. Then if we write $e_{st} := \alpha_s||\alpha_t$ and $h_s := \alpha_s||\alpha_s - \alpha_s + 1||\alpha_{s+1}$ it is easy to show that the above statement follows. Therefore $\text{HH}^1(A)$ is isomorphic to $\prod_{i=1}^{n} \mathfrak{sl}_m(K)$. Since $\mathfrak{sl}_m(K)$ is solvable only
for \( m = 2 \) in \( \text{char}(K) = 2 \), the Lie algebra obtained is solvable if and only if \( \text{char}(K) = 2 \) and \( m = 2 \).

An analogous proof to the above shows the following generalisation of Proposition 4.22. Let \( n, m \) be such that \( m = (m_1, \ldots, m_k) \) and \( n, m_1, \ldots, m_k \in \mathbb{N}_{\geq 1} \). We denote by \( Q_{n,m} \) the quiver with \( n \) vertices \( 1, \ldots, n \) and a set of \( m_i \) parallel arrows denoted by \( \alpha_{i,1}, \ldots, \alpha_{i,m_i} \) from \( i \) to \( i + 1 \) for each pair of consecutive vertices \((i, i+1)\). For \( n = 2 \) and \( m = m \), the quiver \( Q_{2,m} \) is the \( m \)-Kronecker quiver. Set \( A_{n,m} = KQ_{n,m}/J(KQ_{n,m})^2 \).

**Corollary 4.23.** For \( A_{n,m} \) as above, we have

\[
\text{HH}^1(A_{n,m}) \cong \prod_{i=1}^{n} \text{sl}_{m_i}(K),
\]

for \( n, m_1, \ldots, m_k \geq 2 \).

It can be shown in a similar way that \( \text{HH}^1(KQ_{n,m}) \cong \prod_{i=1}^{n} \text{sl}_{m_i}(K) \), for \( n, m_1, \ldots, m_k \geq 2 \).

**Corollary 4.24.** Let \( A = KQ/I \) be a finite dimensional algebra such that \( Q \) contains \( Q_{n,m} \) as a subquiver and the other arrows of \( Q \) form a simple directed graph. Suppose that all degree 2 paths involving at least one arrow from \( Q_{n,m} \) are relations. Then

\[
\text{HH}^1(A)/\text{rad(\text{HH}^1(A))} \cong \prod_{i=1}^{n} \text{sl}_{m_i}(K).
\]

**Proof.** By assumption the ideal \( I \) contains all of the relations of degree 2 involving the arrows \( Q_{n,m} \). Hence \( \text{HH}^1(kQ_{n,m}) \cong \prod_{i=1}^{n} \text{sl}_{m_i}(K) \). Note that if \( \alpha \) is an arrow not in \( Q_{n,m} \) such that \( \alpha \) is parallel to a path \( p \) which contains at least one arrow in \( Q_{n,m} \), then \( p \) is zero in \( kQ/I \). This is because all degree 2 paths involving at least one arrow from \( Q_{n,m} \) are relations. Note that \( \alpha \) cannot be a loop since the arrows which are not in \( Q_{n,m} \) form a simply directed graph. By construction, \( \text{HH}^1(A) = \prod_{i=1}^{n} \text{sl}_{m_i}(K) \oplus S \), where \( S \) is a solvable Lie algebra. The statement follows.

It follows from Corollary 4.24, see also [8] in characteristic zero, that the first Hochschild cohomology of a special biserial algebra \( A \) (beyond the Kronecker algebra) is not necessarily solvable. This is the case, for example, when the quiver contains \( Q_{n,2} \) for some \( n \) as subquiver (with all relations of length two for all arrows in \( Q_{n,2} \)) with the relations as in Corollary 4.24. In that case \( \text{sl}_2(K) \) is a Lie subalgebra of \( \text{HH}^1(A) \).

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