Abstract. Gas-like $\alpha$-cluster states are investigated in $^{16}$O with the use of 4$\alpha$ OCM (Orthogonality Condition Model). The $0^+_6$ state in $^{16}$O is shown to have a character of a 4$\alpha$ condensate, an analogue to the Hoyle state in $^{12}$C. The decay properties of this state and the $\alpha$+Hoyle-like rotatinal states are also investigated.

1. Introduction
It is well established that $\alpha$ clustering plays a very important role in the structure of lighter nuclei [1, 2]. The importance of $\alpha$-cluster formation has also been discussed in infinite nuclear matter, where $\alpha$-particle type condensation is expected at low density [3, 4]. Although in nuclei there are only a few bosonic constituents, they may give rise to clear condensation characteristics, as is well known from nuclear pairing [5]. Concerning $\alpha$-particle condensation, the Hoyle state, i.e. the $0^+_2$ state in $^{12}$C, has clearly been established. Several papers of the past [6, 7, 8, 9] and also more recent ones [10, 11, 12, 13, 14, 15] have by now established beyond doubt that the Hoyle state has only about one third of saturation density and can be described to good approximation as a product state of three $\alpha$ particles, condensed into the lowest mean field $0S$-orbit [16, 17]. This shall be the definition of an $\alpha$-particle condensed state in finite nuclei, clearly reflecting the situation found in infinite matter in Ref. [3]. The emergence of this novel aspect of the Hoyle state naturally leads us to speculations about 4$\alpha$-particle condensation in $^{16}$O.

In the present contribution, the recent progress on the investigation of 4$\alpha$-particle condensate states in $^{16}$O is shown.

2. Four-alpha condensation in $^{16}$O
It is well documented [18] that in $^{16}$O the $0^+_3$ state at 6.06 MeV contains an $\alpha$ particle orbiting in a $0S$-wave around the ground state of $^{12}$C. However, there are many more possibilities. The $\alpha$ particle can be in higher nodal $S$-wave, or can orbit in a $0D$-wave around the $2^+_1$ state of $^{12}$C and couple to a $0^+$ state. It can also orbit in an odd parity wave around the $1^-$ and $3^-$ states in $^{12}$C etc. The $^{12}$C core can also be in the Hoyle state which then leads us to the 4$\alpha$ gas state...
in $^{16}$O, which is the state of our interest. Experimentally, there are six $0^+$ states in $^{16}$O up to around the 4$\alpha$-disintegration threshold at 14.4 MeV: the ground state $0^+_1$, $0^+_2$ at 6.06 MeV, $0^+_3$ at 12.05 MeV, $0^+_4$ at 13.6 MeV [19], $0^+_5$ at 14.01 MeV, and $0^+_6$ at 15.1 MeV.

We recently performed a quite complete OCM calculation for $^{16}$O, including many of the cluster configurations [21]. We were able to reproduce the full spectrum of the observed $0^+$ states. As shown in Fig. 1, we tentatively make a one-to-one correspondence of our states with the six lowest $0^+$ states of the experimental spectrum. In view of the complexity of the situation, the agreement can be considered as very satisfactory.

In Table 1 we show the calculated rms radii and monopole matrix elements to the ground state, together with the corresponding experimental values. The $M(E0)$ values for the $0^+_2$ and $0^+_5$ states are consistent with the corresponding experimental values. The consistency for the
$0^+_3$ state is within a factor of two. The structures of the $0^+_2$ and $0^+_3$ states are well established as having $\alpha+^{12}$C$(0^+_1)$ and $\alpha+^{12}$C$(2^+_1)$ cluster structures, respectively [18, 22, 23, 24]. These structures of the $0^+_2$ and $0^+_3$ states are confirmed in the present calculation. We also mention that the ground state is described as having a shell-model configuration consistent with the present framework, the calculated rms value agreeing with the observed one ($2.71 \text{ fm}$). On the contrary, the structures of the observed $0^+_4$, $0^+_5$ and $0^+_6$ states have, in the past, not clearly been understood. As shown in Fig. 1, the present calculation succeeds in reproducing the $0^+_4$, $0^+_5$ and $0^+_6$ states, together with the $0^+_1$, $0^+_2$ and $0^+_3$ states. This puts us in a favorable position to discuss the $4\alpha$-condensate state, expected to exist around the $4\alpha$ threshold. In Table 1, the largest rms value of about $5 \text{ fm}$ is found for the $0^+_6$ state. Compared with the relatively smaller rms radii of the $0^+_4$ and $0^+_5$ states, this large size indicates that the $0^+_6$ state is composed of a weakly interacting gas of $\alpha$ particles of the condensate type.

The analysis of the diagonalisation of the $\alpha$-particle density matrix $\rho(r,r')$ (as was done in [16, 17, 25, 26]) showed that the newly discovered $0^+$ state at $13.6 \text{ MeV}$ [19], as well as the well known $0^+$ state at $14.01 \text{ MeV}$, corresponding to our states at $12.6 \text{ MeV}$ and $14.1 \text{ MeV}$, respectively, have very little condensate occupancy of the $0S$-orbit (about 20%). On the other hand, the sixth $0^+$ state, whose calculated energy is $16.5 \text{ MeV}$, to be identified with the experimental state at $15.1 \text{ MeV}$, has 61% of the $\alpha$ particles being in the $0S$-orbit. The corresponding single-$\alpha$ $0S$ orbit is shown in Fig. 2. It has a strong spatially extended behaviour, and in the interior no node ($0S$), showing almost no effect of the Pauli principle exchanging nucleons belonging to different $\alpha$ particles. This indicates that the $\alpha$ particles are condensed into the very dilute $0S$ single-$\alpha$ orbit (see also Ref. [27]). Thus, the $0^+_6$ state clearly has a $4\alpha$-condensate character. We should note that the orbit is very similar to the single-$\alpha$ orbit of the Hoyle state [17]. In Fig. 2 we also show the single-$\alpha$ orbit for the ground state. It has maximum amplitude at around $3 \text{ fm}$ and oscillations in the interior with two-nodal (2S) behaviour, due to the Pauli principle and reflecting the shell-model configuration. The $0^+_6$ state also has a very large radius of $5.6 \text{ fm}$, though this value may be somewhat overestimated because of the too high energy of the $0^+_6$ state.

The condensate nature for the $0^+_6$ state can also be seen clearly by the following analysis. We
calculate the reduced-width amplitude, which is defined as follows:

\[
Y(r) = \sqrt{\frac{4!}{3!1!}} \left( \frac{\delta(r' - r)}{r^2} Y_L(\vec{r}') \Phi_L^{(12C)} \right) |\phi_0^+\rangle
\]  

(1)

Here, \( \Phi_L^{(12C)} \) is the wave function of \( ^{12C} \), given by the 3\( \alpha \) OCM calculation [17], and \( r \) is the relative distance between the centre-of-mass of \( ^{12C} \) and the \( \alpha \) particle. From this quantity we can see how large is the component in a certain \( \alpha+^{12C} \) channel which is contained in the \( 0^+_6 \) state. The amplitudes for the \( 0^+_6 \) state are shown in Fig. 3. It only has a large amplitude in the \( \alpha+^{12C}(0^+_6) \) channel, whereas the amplitudes in other channels are much suppressed. The amplitude in the Hoyle-state channel has no oscillations and its long tail stretches out to \( \sim 20 \) fm.

\[L=0^+_4 \quad \text{---} \]
\[L=2^+_4 \quad \text{---} \]
\[L=4^+_4 \quad \text{---} \]
\[L=1^+_4 \quad \text{---} \]
\[L=3^+_4 \quad \text{---} \]
\[L=0^+_4 \quad \text{---} \]

**Figure 3.** The reduced-width amplitudes \( rY(r) \) for the \( 0^+_6 \) state, with \( Y(r) \) defined by Eq. (1); \( L \) denotes the orbital angular momentum of the remaining \( \alpha \)-particle coupled to \( ^{12C} \).

3. Decay properties
The \( \alpha \)-particle condensate states occur near the \( \alpha \)-particle disintegration threshold, which rapidly grows in energy with mass, and thus the level density in which such a condensate state is embedded raises enormously. For example the \( \alpha \)-disintegration threshold in \( ^{12C} \) is at \( E_x = 7.27 \) MeV and in \( ^{16O} \) it is already at \( E_x = 14.4 \) MeV. Under ordinary circumstances this could mean that the \( \alpha \)-particle condensate state in \( ^{16O} \) has a very short life-time. However, on the one hand, it is an experimental fact that the supposed \( ^{16O} \) ‘Hoyle’-state at \( E_x = 15.1 \) MeV has a startlingly small width, 160 keV, for such a high excitation energy, and, on the other hand, it is easily understandable that such an exotic configuration as four \( \alpha \) particles moving almost independently within the common Coulomb barrier, has great difficulties to decay into states, lower in energy, which all have very different configurations.

We make a quantitative estimate of the decay width of the \( E_x = 15.1 \) MeV state. Based on the \( R \)-matrix theory [28], the decay width \( \Gamma_L \) can be given by the following formulae:

\[
\Gamma_L = 2P_L(a)\gamma_L^2(a), \quad P_L(a) = \frac{ka}{F_L^2(ka) + G_L^2(ka)},
\]

\[ka \quad F_L^2(ka) + G_L^2(ka) \]
\[ \gamma_L^2(a) = \theta_L^2(a) \gamma_W(a), \quad \gamma_W(a) = \frac{3\hbar^2}{2\mu a^2}, \quad (2) \]

where \( k, a \) and \( \mu \) are the wave number of the relative motion, the channel radius, and the reduced mass, respectively, and \( F_L, G_L, \) and \( P_L(a) \) are the regular and irregular Coulomb wave functions and the corresponding penetration factor, respectively. The reduced width of \( \theta_L^2(a) \) is related to Eq. (1) like \( \theta_L^2(a) = (a^3/3)\gamma_L^2(a) \). In Table 2, we show the partial \( \alpha \)-decay widths of the \( 0^+_6 \) state \( \Gamma_L \) decaying into the \( \alpha +^{12}C(0^+_1), \alpha +^{12}C(2^+_1) \) and \( \alpha +^{12}C(0^+_2) \) channels, the total \( \alpha \)-decay width, which is obtained as a sum of the partial widths, and the reduced widths \( \theta_L^2(a) \). The wave numbers \( k \) were calculated from the experimental decay energies. Thus the excitation energy of the \( 0^+_6 \) state was assumed to be 15.1 MeV, the one of the observed \( 0^+_6 \) state.

| \( \alpha \)-decay Widths | \( 0^+_6 \) Channel | \( 0^+_6 \) Channel | \( 0^+_6 \) Channel | Total |
|--------------------------|---------------------|---------------------|---------------------|-------|
| \( \Gamma_L \) (keV)    | 104                 | 32                  | \( 8 \times 10^{-7} \) | 136   |
| \( \theta_L^2(a) \)     | 0.024               | 0.016               | 0.6                 |       |

The very small total \( \alpha \)-decay width of 136 keV obtained is in good agreement with the corresponding experimental value of 160 keV, and indicates that this state is unusually long-lived. This fact can be explained in terms of the present analysis as follows: Since this state has a very exotic structure composed of gas-like four \( \alpha \) particles, the overlap between this state and the \( \alpha +^{12}C(0^+_1) \) or \( \alpha +^{12}C(2^+_1) \) wave functions with a certain channel radius becomes very small, as this is, indeed, indicated by the small \( \theta_L^2(a) \) values, 0.024 and 0.016, respectively, which implies small \( \gamma_L^2(a) \) values. These largely suppress the decay widths expressed by Eq. (2) in spite of the large values of the penetration factors caused by the large decay energies 7.9 MeV and 3.5 MeV into the \( \alpha +^{12}C(0^+_1) \) and \( \alpha +^{12}C(2^+_1) \) channels, respectively. On the other hand, the decay into \( \alpha +^{12}C(0^+_2) \) is also suppressed due to the very small penetration caused by the very small decay energy, 0.28 MeV, into this channel, even though the corresponding reduced width takes a large value, \( \theta_L^2(a) = 0.6 \), which is natural since the \( 0^+_6 \) state of \( ^{12}C \) has a gas-like three-\( \alpha \)-particle structure. It is very likely that the above mechanism holds generally for the \( \alpha \)-gas states in heavier \( \alpha \) systems, and therefore such states can also be expected to exist in heavier systems as relatively long-lived resonances.

4. Alpha + Hoyle Rotational States

We further investigated the non-zero-spin excited states in \( ^{16}O \), which are analogous to the Hoyle state, i.e., are members of the same family as the \( 0^+_6 \) state. We found that some states characteristically have large components of \( \alpha + \) Hoyle state, where the \( \alpha \)-particle orbits around the Hoyle state in \( S, P, D, F, \) and \( G \) waves for \( J^\pi = 0^+, 1^-, 2^+, 3^-, \) and \( 4^+ \), respectively. In Fig. 4, we show the spectra as a function of \( J(J + 1) \). One can suppose most naturally that these form rotational bands with \( \alpha + \) Hoyle-state structure, with \( K^\pi = 0^+ \) for the positive parity states and \( K^\pi = 0^- \) for the negative parity states. This was also very recently discussed in Ref. [29] and provides another possibility of interpreting the old [30] and rather recent [31] experimental data, which have been discussed for a long time in terms of a 4\( \alpha \)-linear-chain structure.

It is interesting to see that the \( 0^+_6 \) state at 16.5 MeV strays off the \( J(J + 1) \) rotational line and gets more strongly bound. This indicates that it has a rather different structure from the
Figure 4. Rotational band of the $\alpha$+Hoyle state. The dashed line starts from the average excitation energy, 17.9 MeV, of the four $0^+$ states over the $0^+_6$ state at 16.5 MeV.

other rotational members. One can say that when the $\alpha$ particle rotates around the Hoyle state in an $S$-wave, it seems to drop into the $S$-orbit which is already occupied by the three $\alpha$ particles in the Hoyle state, leading to the energy gain for the state. This might be interpreted as a transition from the local condensate to the complete condensate (see also Ref. [29]). On the other hand, we found four $0^+$ states above the $0^+_6$ state, which all have relatively large overlaps with an $\alpha$+Hoyle configuration. They can be the real bandhead of the $\alpha$+Hoyle rotational band, instead of the $0^+_6$ state. However, the average excitation energy, 17.9 MeV, is 0.5 MeV higher than the $J(J+1)$ rotational line. Due to the appearance of the $0^+_6$ state, the real bandhead seems to be pushed up energetically and fragmented, since the $\alpha$+Hoyle state component must already be rather exhausted in the $0^+_6$ state.

5. Conclusion
The theoretical investigation of the Hoyle state in $^{12}$C has established beyond doubt that it is a dilute gas-like state of three $\alpha$ particles, held together only by the Coulomb barrier, where the three bosons are condensed into the $0S$-orbital. There is no objective reason, why in $^{16}$O, $^{20}$Ne,$\cdots$ there should not exist similar ‘Hoyle-like’ states. In this contribution, we have given the results of a complete OCM calculation which reproduces the first six $0^+$ states of $^{16}$O. In that calculation the $0^+_6$ state at 16.5 MeV, to be identified with the experimental $0^+$ state at 15.1 MeV, shows characteristics typical for the Hoyle state. We argued that the $0^+_6$ state has a very unusual structure and thus couples only very weakly to states at lower energies, leading to the very small width of 160 keV. Going on with this reasoning, it is not at all excluded that $\alpha$-gas states in even heavier self-conjugate nuclei will have unusually long life-times, given their excitation energy high up in the continuum. We further discussed the rotational band of the $\alpha$+Hoyle state. The interplay between such rotational states and the $4\alpha$-condensate state is very interesting to investigate in much more detail. Further experimental information is also very much required to confirm this novel interpretation of these states.

References
[1] Wildermuth K and Tang Y C 1977 A Unified Theory of the Nucleus (Braunschweig: Vieweg)
[2] Ikeda K et al 1980 Prog. Theor. Phys. Suppl. 68 1
[3] Röpke G et al 1998 Phys. Rev. Lett. 80 3177
[4] Sogo T, Lazauskas R, Röpke G and Schuck P 2009 Phys. Rev. C 79 051301(R)
[5] Ring P and Schuck P 1980 The Nuclear Many-Body Problem (Berlin: Springer-Verlag)
[6] Horiuchi H 1974 Prog. Theor. Phys. 51 1266; 1975 Prog. Theor. Phys. 53 447
[7] Kamimura M 1981 Nucl. Phys. A 351 456
[8] Uegaki E et al 1977 Prog. Theor. Phys. 57 1262; 1978 Prog. Theor. Phys. 59 1031; 1979 Prog. Theor. Phys. 62 1621
[9] Descouvemont P et al 1987 Phys. Rev. C 36 54
[10] Tohsaki A et al 2001 Phys. Rev. Lett. 87 192501
[11] Funaki Y et al 2003 Phys. Rev. C 67 051306(R)
[12] Funaki Y et al 2005 Eur. Phys. J. A 24 321
[13] Funaki Y et al 2006 Eur. Phys. J. A 28 259
[14] Chernykh M et al 2007 Phys. Rev. Lett. 98 032501; 2010 Phys. Rev. Lett. 105 022501
[15] Funaki Y et al 2009 Phys. Rev. C 80 064326
[16] Matsumura H et al 2004 Nucl. Phys. A 739 238
[17] Yamada T et al 2005 Eur. Phys. J. A 26 185
[18] Horiuchi H and Ikeda K 1968 Prog. Theor. Phys. 40 277
[19] Wakasa T et al 2007 Phys. Lett. B 653 173
[20] Ajzenberg-Selove F 1986 Nucl. Phys. A 460 1
[21] Funaki Y et al 2008 Phys. Rev. Lett. 101 082502
[22] Suzuki Y 1976 Prog. Theor. Phys. 55, 1751; Prog. Theor. Phys. 1976 56 111
[23] Libert-Heinemann M et al 1980 Nucl. Phys. A 339 429
[24] Fukatsu K et al 1992 Prog. Theor. Phys. 87 151
[25] Suzuki Y et al 2002 Phys. Rev. C 65 064318
[26] Yamada T et al 2008 Phys. Rev. A 78 035603
[27] Funaki Y et al 2008 Phys. Rev. C 77 064312
[28] Lane A M and Thomas R G 1958 Rev. Mod. Phys. 30 257
[29] Ohkubo S and Hirabayashi Y 2010 Phys. Lett. B 684 127
[30] Chevallier P et al 1961 Phys. Rev. 180 827
[31] Freer M et al 1995 Phys. Rev. C 51 1682