High Order Regularity Obstacle
of Geodesics in Space of Kähler Potentials

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Abstract

In this paper we address following questions regarding regularity of geodesics in space of Kähler potentials. First, is the regularity of a geodesic stable under smooth boundary value perturbation? Second, can we expect that any sufficiently regular geodesic with smooth boundary value is actually smooth? We construct examples to show that answers to both questions are negative.

1 Introduction

Given \((V,\omega_0 > 0)\), a smooth Kähler manifold, we consider space of Kähler potentials

\[ \mathcal{H} = \{ \phi \in C^\infty(V) | \omega_0 + \sqrt{-1} \partial\bar{\partial}\phi > 0 \}. \]

At any point \(\phi \in \mathcal{H}\), tangent space \(T_\phi \mathcal{H}\) can be identified with \(C^\infty(V)\), following Mabuchi [15], we define following Riemannian metric in \(\mathcal{H}\), for \(\psi_1, \psi_2 \in T_\phi \mathcal{H}\),

\[ <\psi_1, \psi_2 > \phi = \int_V \psi_1 \psi_2 (\omega_0 + \sqrt{-1} \partial\bar{\partial}\phi)^n. \]

With this metric, the energy of a differentiable curve \(\varphi : [0,1] \to \mathcal{H}\) is

\[ \int_0^1 \int_V \left( \frac{d\varphi}{dt} \right)^2 (\omega_0 + \sqrt{-1} \partial\bar{\partial}\varphi(t, *))^n dt. \]

Then the geodesic equation for a smooth curve is

\[ \varphi_{tt} - g_{\varphi}^{\bar{\varphi}} \varphi_{ti} \bar{\varphi}_{\bar{i}} = 0. \] (1.1)

As discovered by Semmes[18], Donaldson[10], Mabuchi[15], (1.1) can be written as a homogenous complex Monge-Ampère equation. Denote

\[ S = \{ \tau = t + \sqrt{-1} \theta \in \mathbb{C} | 0 \leq t \leq 1 \}. \]

Then we can consider a curve \(\varphi \in C^1([0,1]; \mathcal{H})\) as a function defined on \(S \times V\), by letting

\[ \Phi(\tau, *) = \varphi(\text{Re } \tau, *). \]
Let \( \pi \) be the trivial projection from \( S \times V \to V \), and \( \Omega_0 = \pi^*(\omega_0) \). Then a smooth curve \( \varphi : [0, 1] \to \mathcal{H} \) satisfying (1.1) is equivalent to the corresponding \( \Phi \) satisfying

\[
(\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi)^n = 0.
\]

So the problem of finding a geodesic, in \( \mathcal{H} \), connecting 0 and \( \varphi \), can be related to solving following Dirichlet problem of homogenous complex Monge-Ampère equation on \( S \times V \).

**Problem 1.1.**  
*Dirichlet Problem of HCMA Equation on \( S \times V \)*  
Given \( \varphi \in \mathcal{H} \), find a \( \Phi \in C^1(S \times V) \), s.t.

\[
\begin{align*}
(\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi)^n &= 0, \quad \text{in } S \times V; \\
\Phi(\tau, \ast) &= 0, \quad \text{for } \text{Re } \tau = 0; \\
\Phi(\tau, \ast) &= \varphi, \quad \text{for } \text{Re } \tau = 1; \\
\partial_\theta \Phi &= 0, \quad \text{in } S \times V; \\
\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, \ast) &\geq 0, \quad \text{for } \tau \in S.
\end{align*}
\]

Due to work of Chen\[^3\], Chu-Tossati-Weinkove\[^6\], we know above problem always have a \( C^{1,1} \) solution. But in general a solution may not correspond to a curve in \( \mathcal{H} \). First \( \Phi(\tau, \ast) \) may not be \( C^\infty \). As shown by Liz Vivas-Lempert-Darvas\[^8\][\[^13\][\[^14\]], \( C^{1,1} \) is the optimal global regularity, for general \( \varphi \). Second it’s expected that \( \omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, \ast) \) may degenerate for some \( \tau \in S \), which makes \( \Phi(\tau, \ast) \notin \mathcal{H} \). But however we can consider solution to Problem 1.1 as a weak or generalized geodesic.

In this paper, if \( \Phi \), a solution to Problem 1.1, satisfies

\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, \ast) > 0, \quad \text{for } \tau \in S,
\]

we say \( \Phi \) is a non-degenerate geodesic. And if \( \Phi \in C^k(S \times V) \) we say that \( \Phi \) is a \( C^k \) geodesic. Similarly, if \( \Phi \in C^\infty(S \times V) \), we say \( \Phi \) is a smooth geodesic.

A problem analogous to Problem 1.1 is the following Dirichlet problem of HCMA equation on the product of disc and manifold.

**Problem 1.2.**  
*Dirichlet Problem of HCMA Equation on the Product of Disc and \( V \)*  
Let \( D \) be the unit disc in complex plane. Given \( (V, \omega_0) \), a smooth Kähler manifold, and \( F \in C^\infty(\partial D \times V) \), satisfying

\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} F(\tau, \ast) > 0, \quad \text{for } \tau \in \partial D,
\]

find \( \Phi \in C^2(D \times V) \), satisfying

\[
\begin{align*}
(\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi)^n &= 0, \quad \text{in } D \times V; \\
\Phi(\tau, \ast) &= 0, \quad \text{for } \tau \in \partial D; \\
\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, \ast) &= F, \quad \text{for } \tau \in D.
\end{align*}
\]

In \[^{10}\], by relating Dirichlet problem of HCMA equation to the existence and stability of a family of holomorphic discs with boundaries attached to a totally real submanifold, Donaldson proved

\[^1\] In this paper, we abbreviate homogenous complex Monge-Ampère equation as HCMA equation.
Theorem 1.1. The set of boundary values \( F \), for which Problem 1.2 has smooth solution, is an open set in \( C^\infty(\partial D \times V) \) w.r.t. some topology.

Another consequence of applying technique of [10] is

Theorem 1.2. Given \( F \in C^\infty(\partial D \times V) \), if \( \Phi \in C^3(D \times V) \) is a solution to Problem 1.2, then \( \Phi \in C^\infty(D \times V) \).

In [4], by partially generalizing technique of [10] to strip case, we proved

Theorem 1.3. (Chen-Feldman-Hu, Theorem 1.2 [4])
For any fixed \( k > 4 \), there exists a \( \delta > 0 \), s.t. if \( \phi \in C^\infty(V) \) satisfies \( |\phi|_B < \delta \), the geodesic connecting 0 and \( \varphi \) is \( C^4 \) and non-degenerate.

The essential technique of [4] is making use of foliation structure to derive a high order apriori estimate for near constant geodesic. Comparing Theorem 1.1 and Theorem 1.2 with our Theorem 1.3, it’s nature to ask, if it’s possible to fully generalize the holomorphic disc technique in [10] to strip case, and prove some analogues of Theorem 1.1 and Theorem 1.2 for geodesic problem? For example we ask:

**Question 1.1.** Given \((V, \omega_0)\) a Kähler manifold, with \( \omega_0 > 0 \), can we find a \( B = B(V, \omega_0) \) and \( \varepsilon = \varepsilon(V, \omega_0) \), s.t. if \( \varphi \in C^\infty(V) \) satisfies

\[
|\varphi|_B \leq \varepsilon,
\]

then there exists a non-degenerate smooth geodesic \( \Phi \) connecting 0 and \( \varphi \)?

**Question 1.2.** Given \((V, \omega_0)\) a Kähler manifold, with \( \omega_0 > 0 \), can we find a \( B = B(V, \omega_0) \) and \( \delta = \delta(V, \omega_0) \), s.t. if a \( C^B \) geodesic \( \Phi \) satisfies

\[
\Phi(0, *) \in C^\infty(V), \Phi(1, *) \in C^\infty(V),
\]

\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, *) \geq \delta \omega_0, \text{ for all } \tau \in S,
\]

then \( \Phi \) is actually smooth?

**Question 1.3.** Given \((V, \omega_0)\) a Kähler manifold, with \( \omega_0 > 0 \), and \( \Phi \in C^\infty([0,1] \times V) \), a non-degenerate smooth geodesic, is there a \( B = B(V, \omega_0, \Phi) \) and \( \varepsilon = \varepsilon(V, \omega_0, \Phi) \), s.t. if \( \varphi \in C^\infty(V) \) satisfies

\[
|\varphi|_B \leq \varepsilon,
\]

then \( \Phi(0, *) \) and \( \Phi(1, *) + \varphi \) can be connected by a non-degenerate smooth geodesic?

We will construct following examples to show that answers to Question 1.1, Question 1.2, Question 1.3 are all negative.

**Example 1.1.** On torus \( T = S^1 \times S^1 \), with flat background metric

\[
\omega_0 = \frac{1}{2} dx \wedge dy,
\]

there exists a sequence of analytic functions \( \varphi_k \), for \( k = 1, 2, \ldots \), s.t.

\[
|\varphi_k|_B \to 0, \text{ as } k \to \infty,
\]

for any fixed \( B \), but none of \( \varphi_k \) can be connected with 0 by a smooth non-degenerate geodesic.
Remark 1.1. For Example 1.1, if we denote the geodesic connecting 0 and \( \varphi_k \) by \( \Phi_k \), by Theorem 1.8 of [4], we know for \( k \) big enough, \( \Phi_k \)'s are non-degenerate and \( \Phi_k \) gets more and more regular as \( k \to \infty \). But even though, none of \( \Phi_k \) can be \( C^\infty \). This implies answer to Question 1.2 is negative.

Example 1.2. Let the background Kähler manifold be \( \mathcal{T} = S^1 \times S^1 \), with flat metric 
\[
\omega_0 = \frac{1}{2} dx \wedge dy,
\]
there exists a non-degenerate analytic geodesic \( \Phi \), an analytic function \( \varphi \) on \( V \) and a possibly large constant \( B \), s.t.
\[
\Phi(0,*) + \chi \varphi \text{ cannot be connected by a non-degenerate geodesic } \Psi \in C^B([0,1] \times V), \text{ for any } \chi \neq 0.
\]

2 Preparation

Our background manifold will be the torus
\[
\mathcal{T} = [-\pi, \pi] \times [-\pi, \pi] / \sim,
\]
where 
\[
(-\pi, y) \sim (\pi, y), \text{ for any } y \in [-\pi, \pi],
\]
and 
\[
(x, -\pi) \sim (x, \pi), \text{ for any } x \in [-\pi, \pi].
\]
Coordinate of \( \mathcal{T} \) will be denoted by \( (x, y) \). The background Kähler form \( \omega_0 \) is
\[
\frac{dx \wedge dy}{2}.
\]

In this and next section, we will consider a curve in \( \mathcal{H} \) as a function defined on \([0,1] \times \mathcal{T}\). The geodesic equation is
\[
(1 + \Delta_T \Phi) \Phi_{tt} = |\nabla_T \Phi|^2. \tag{2.1}
\]
Here, \( \Delta_T = \partial_x^2 + \partial_y^2 \) and \( \nabla_T \cdot = (\partial_x \cdot, \partial_y \cdot) \).

Given two potentials \( \varphi_0, \varphi_1 \), symmetric w.r.t. both \( x \) and \( y \)-axis, i.e. satisfying
\[
\varphi_0(x, y) = \varphi_0(-x, y), \quad \varphi_0(x, y) = \varphi_0(x, -y),
\]
\[
\varphi_1(x, y) = \varphi_1(-x, y), \quad \varphi_1(x, y) = \varphi_1(x, -y),
\]
we have, according to uniqueness theorem of boundary value problem of [9], if there is a geodesic \( \Phi \) connecting \( \varphi_0 \) and \( \varphi_1 \), \( \Phi \) should also be symmetric w.r.t. both \( x \), \( y \)-axis, i.e. satisfy
\[
\Phi(t, x, y) = \Phi(t, -x, y), \quad \Phi(t, x, y) = \Phi(t, x, -y),
\]
for any \( t \in [0,1] \).

And for initial value problem, according to uniqueness theorem of Appendix B, if the initial data of a geodesic \( \Phi \) is symmetric w.r.t. both \( x \), \( y \)-axis i.e. \( \Phi(0,*) \) and \( \frac{d}{dt} \Phi(0,*) \) are both symmetric w.r.t. \( x \), \( y \)-axis, we have \( \Phi \) is also symmetric w.r.t. \( x \), \( y \)-axis.
If $\Phi$ is symmetric w.r.t. both $x$ and $y$-axis, all odd order $x$ or $y$-derivatives of $\Phi$ vanish along $[0,1] \times \{0\}$, i.e.

$$D^k_x D^i_x D^j_y \Phi = 0, \text{ on } [0,1] \times \{0\}, \text{ if } 2 \nmid i, \text{ or } 2 \nmid j.$$ 

Based on above observation, in Subsection 2.1, we study the ODE system satisfied by second order derivatives of $\Phi$ along $[0,1] \times \{0\}$. In Subsection 2.2, assuming geodesic $\Phi$ is regular enough, we analysis the ODE system satisfied by higher order derivatives of $\Phi$ along $[0,1] \times \{0\}$, and show that either these derivatives are determined by boundary data or the boundary data should satisfy some condition.

### 2.1 ODE System of Second Order Derivatives

Suppose we have a non-degenerate geodesic $\Phi \in C^6([0,1] \times V)$, connecting $\varphi_0$ and $\varphi_1$, with both $\varphi_0$ and $\varphi_1$ symmetric w.r.t. both $x$ and $y$-axis. Since $\Phi$ is symmetric w.r.t. both $x,y$-axis, $\Phi_t$ should also be symmetric w.r.t. both $x,y$-coordinate. So

$$\Phi_{tt} = \frac{\left| \nabla_T \Phi_t \right|^2}{1 + \Delta_T \Phi} = 0, \text{ on } [0,1] \times \{0\}.$$ 

And in addition, we assume $\varphi_0(0) = \varphi_1(0) = 0$, so we would have

$$\Phi \equiv 0, \text{ on } [0,1] \times \{0\}.$$ 

**Notation 1.** From now on, “′” will be used to denote $\frac{d}{dt}$.

Let the $T$-direction 2nd order Taylor expansion of $\Phi$ along $[0,1] \times \{0\}$ be

$$\Phi = a_{11}x^2 + a_{22}y^2 + O(x^4 + y^4).$$

In above $a_{11}$ and $a_{22}$ are $C^2$ functions of $t$. By simply plugging above into equation (2.1), we get, along $[0,1] \times \{0\}$,

$$a_{11}'' = \frac{4(a_{11}')^2}{1 + 2a_{11} + 2a_{22}}, \quad (2.2)$$

$$a_{22}'' = \frac{4(a_{22}')^2}{1 + 2a_{11} + 2a_{22}}. \quad (2.3)$$

Denote

$$\sigma_2 = \frac{a_{11}'a_{22}'}{(1 + 2a_{11} + 2a_{22})^2}, \quad \sigma_1 = \frac{a_{11}' + a_{22}'}{1 + 2a_{11} + 2a_{22}}, \quad (2.4)$$

computation shows

$$\sigma_2' = 0. \quad (2.5)$$

$$\sigma_1' = \frac{2(a_{11}' - a_{22}')^2}{(1 + 2a_{11} + 2a_{22})^2} = 2\sigma_1^2 - 8\sigma_2 \geq 0. \quad (2.6)$$

Now, with the help of above equations, we can solve Dirichlet problem and initial value problem of $a_{11}, a_{22}$, along $[0,1] \times \{0\}$. 

5
Our examples will be constructed in the situation that $\sigma_2 < 0$. In the following, we show that some boundary data would lead to $\sigma_2 < 0$.

First, for initial value problem, since $\frac{d}{dt}\Phi$ is known on $\{t = 0\} \times T$, we can easily make $\sigma_2 < 0$, by letting $a'_{11}$ and $a'_{22}$ have opposite signs.

For Dirichlet problem, we assert that if

$$a_{11}(1) - a_{11}(0) > 0 > a_{22}(1) - a_{22}(0),$$

then $\sigma_2 < 0$. From (2.2) and (2.3) we easily know that, with above boundary condition, $a'_{11}$ and $a'_{22}$ cannot equal to 0 anywhere, so

$$a'_{11} > 0 > a'_{22}$$

which implies $\sigma_2 < 0$.

In the following, we denote $\sigma_2 \equiv -\epsilon^2$, for some constant $\epsilon > 0$, and

$$a'_{11} \left(1 + 2a_{11} + 2a_{22}\right) = \epsilon A, \quad a'_{22} \left(1 + 2a_{11} + 2a_{22}\right) = -\frac{\epsilon}{A}, \quad (2.7)$$

with $A$ a positive real valued function of $t$.

Plugging above into (2.6), we get

$$\epsilon(1 + \frac{1}{A^2})A' = (\epsilon A - \frac{\epsilon}{A})' = \sigma_1' = 2(\epsilon A - \frac{\epsilon}{A})^2 + 8\epsilon^2 = 2\epsilon(\epsilon A + \frac{1}{A})^2, \quad (2.8)$$

which can be reduced to

$$A' = 2\epsilon(A^2 + 1). \quad (2.9)$$

Then taking $\frac{d}{dt}$ derivative of (2.7) gives

$$a''_{11} \left(1 + 2a_{11} + 2a_{22}\right) = 4\epsilon^2 A^2, \quad a''_{22} \left(1 + 2a_{11} + 2a_{22}\right) = 4\epsilon^2 \frac{1}{A^2}. \quad (2.10)$$

In Appendix A, Theorem A.1 shows that, if $a_{11}, a_{22} \in C^2([0, 1])$ satisfy equation (2.2), (2.3), then they are uniquely determined by boundary value, i.e. determined by $a_{11}(0), a_{11}(1), a_{22}(0), a_{22}(1)$. This means that if there is a $C^6$ geodesic connecting $\varphi_0$ and $\varphi_1$ then, along $[0, 1] \times \{0\}$, all 2nd-order derivatives of $\Phi$ are determined by 2-jets of $\varphi_0$ and $\varphi_1$ at $0 \in T$.

### 2.2 ODE System of High Order Derivatives

Now for any $n \geq 2$, assume geodesic $\Phi$ is in $C^{2n+4}([0, 1] \times T)$, and we adopt all the other assumptions and notations of last subsection.

Along $[0, 1] \times \{0\}$, let the $2n-$th order Taylor expansion of $\Phi$ be

$$\Phi = a_{11}x^2 + a_{22}y^2 + P(4, 2n-2) + b_{2n}x^{2n} + b_{2n-2}x^{2n-2}y^2 + \ldots + b_0y^{2n} + O(x^{2n+2} + y^{2n+2}), \quad (2.11)$$

where $P(z_1, z_2)$ denotes a polynomial of $x, y$ with terms of degree between $z_1$ and $z_2$, and $b_k$’s are all $C^2$ functions of $t$. By simply plugging (2.11), (2.7) and (2.10) into (2.1), we found,

$$b''_{2n} + 8\epsilon^2 A^2 b_{2n-2} + 2n(2n-1)4\epsilon^2 A^2 b_{2n} = 8n\epsilon Ab'_{2n} + Q_{2n}, \quad (2.12)$$
and for $k \in \mathbb{Z}$, $1 \leq k < n$,
\[
b''_{2n-2k} + (2n-2k)(2n-2k-1)4\epsilon^2 A^2 b_{2n-2k} + (2k+2)(2k+1)4\epsilon^2 A^2 b_{2n-2k-2} + (2n-2k+1)4\epsilon^2 A^2 b_{2n-2k+2} + 2k(2k-1)\frac{4\epsilon^2}{A^2} b_{2n-2k} \\
= [8(n-k)\epsilon A - 8k\frac{\epsilon}{A}] b'_{2n-2k} + Q_{2n-2k},
\]
(2.13)
\[
b''_0 + 2n(2n-1)\frac{4\epsilon^2}{A^2} b_0 + 8\frac{\epsilon^2}{A^2} b_2 = -\frac{8n\epsilon}{A} b'_0 + Q_0.
\]
(2.14)

In above $Q_0, Q_2, Q_4, \ldots, Q_{2n}$ are some quantities determined by derivatives of $\Phi$ of order less than $2n$, along $[0,1] \times \{0\}$. Then we will simplify (2.12)(2.13)(2.14) to the ODE system of following quantities
\[
W_{2k} \equiv \frac{A^{2k}}{(1 + A^2)^n} b_{2n-2k}, \text{ for } k = 0, 1, \ldots, n.
\]

Computation shows:
\[
W_0'' = -8n\epsilon^2 W_0 - 8\epsilon^2 W_2 + G_0,
\]
(2.15)
and for $k \in \mathbb{Z}$, $1 \leq k \leq n-1$,
\[
W_{2k}'' = -(n-k+1)(2n-2k+1)8\epsilon^2 W_{2k-2} - [32\epsilon^2 k(n-k) + 8n\epsilon^2] W_{2k} - (k+1)(2k+1)8\epsilon^2 W_{2k+2} + G_{2k},
\]
(2.16)
\[
W_{2n}'' = -8\epsilon^2 W_{2n-2} - 8n\epsilon^2 W_{2n} + G_{2n},
\]
(2.17)
where $G_0, G_2, \ldots, G_{2n}$ are determined by $T$-directional derivatives of $\Phi$, along $[0,1] \times \{0\}$, of order less than $2n$, i.e. not depending on $b_0, \ldots, b_{2n}$.

For convenience of later discussion we write above equations in matrix form
\[
W_{2j}'' = \sum_{i=0}^{n} W_{2i} M_j + G_{2j},
\]
(2.18)

It’s important to know eigenvalues of $M = (M_j)$, because it determines if Dirichlet problem of equation (2.18) is solvable or not. Suppose that $v_i = (v^i)$ is an eigenvector of $M$, with eigenvalue $-\pi^2$, i.e.
\[
\sum_{j=0}^{n} M_j v^j = -\pi^2 v^j
\]
then
\[
Y = \sum_{j=0}^{n} W_{2j} v^j
\]
satisfies
\[
Y'' = \sum_{j=0}^{n} W_{2j}'' v^j = \sum_{j=0}^{n} \left( \sum_{i=0}^{n} W_{2i} M_j v^j + G_{2j} v^j \right) = -\pi^2 Y + \sum_{j=0}^{n} G_{2j} v^j.
\]
This would imply

$$Y(0) + Y(1) = - \int_0^1 [Y(t) \cos(\pi t)]' dt = \frac{1}{\pi} \int_0^1 \sum_{j=0}^n G_{2j} v^j \sin(\pi t) dt. \quad (2.19)$$

It shows that, in some situation, to make Dirichlet problem of \( \{W_{2k}\}_{k=0}^n \), with equation (2.18), solvable, boundary data have to satisfy some condition. On the other hand it’s easy to see, if none of \( M \)'s eigenvalue is an integer square multiple of \(-\pi^2\), then Dirichlet problem of \( \{W_{2k}\} \) with equation (2.18) is always solvable.

We find that matrix \( M \) is related to another problem, and with following approach, it’s easier to compute eigenvalues of \( M \).

Denote, for \( k \in \mathbb{Z}, \ 0 \leq k \leq n \),

$$V_k = (-1)^k \cos^{2n-2k} \theta \sin^{2k} \theta,$$

and denote

$$\mathcal{P}^{2n} = \text{Span}_\mathbb{R} < \cos^{2n} \theta, \cos^{2n-2} \theta \sin^2 \theta, \cos^{2n-4} \theta \sin^4 \theta, \ldots, \sin^{2n} \theta >.$$

It’s easy to see that

$$\mathcal{P}^{2n} = \{ p | \{x^2 + y^2 = 1\} | p \ being \ polynomial \ of \ x, y \ of \ degree \ \leq 2n \ and \ symmetric \ w.r.t. \ both \ x, y-axis \}.$$

We consider the action of operator \( D_{\theta \theta} \) on \( \mathcal{P}^{2n} \). According to basic analysis theory, operator \( D_{\theta \theta} \) has \( n + 1 \) different eigenvectors:

$$1, \cos(2\theta), \cos(4\theta), \ldots, \cos(2n\theta),$$

corresponding to following eigenvalues respectively

$$0, -4, -16, \ldots, -4n^2.$$

On the other hand we compute action of \( D_{\theta \theta} \) on \( \{V_k\}_{k=0}^n \), which is a base of \( \mathcal{P}^{2n} \). Computation shows

$$D_{\theta \theta} V_0 = -2n V_0 - 2n(2n - 1)V_1,$$

and for \( k = 1, \ldots, n - 1 \),

$$D_{\theta \theta} V_k = -2k(2k - 1)V_{k-1} - [(2n - 2k)4k + 2n]V_k - (2n - 2k)(2n - 2k - 1)V_{k+1},$$

$$D_{\theta \theta} V_n = -2n V_n - 2n(2n - 1)V_{n-1}.$$

If we write above equations as

$$D_{\theta \theta} V_j = \sum_{i=0}^n V_i N_{ij},$$

we would find that

$$M_i^j = 4\epsilon^2 N_{ij}.$$

So \( M \) has following eigenvalues

$$0, -1 \cdot 16\epsilon^2, -4 \cdot 16\epsilon^2, \ldots, -n^2 \cdot 16\epsilon^2.$$

Then, we reach following conclusion of this section
Theorem 2.1. For $n \in \mathbb{Z}$, $n \geq 2$, given $\Phi \in C^{2n+4}([0,1] \times \mathcal{T})$, a geodesic symmetric w.r.t. both $x$-axis and $y$-axis, and $\Phi(0,\ast) = \varphi_0$, $\Phi(1,\ast) = \varphi_1$, we have

$$\frac{\Phi_{xx}' \Phi_{yy}'}{4(1 + \Phi_{xx} + \Phi_{yy})^2}$$

(2.20)

is a constant along $[0,1] \times \{0\}$. And in some situations, we can make (2.20) negative, which we denote by $-\epsilon^2$. When (2.20) is negative we have

(1) if $\epsilon$ is small enough, s.t. $n^2 \cdot 16\epsilon^2 < \frac{\pi}{2}$, then, along $[0,1] \times \{0\}$ all $\mathcal{T}$-directional derivatives of $\Phi$ of order less or equal to $2n$ are determined by $2n$-jets of $\varphi_0, \varphi_1$ at $0 \in \mathcal{T}$;

(2) if $\epsilon$ or $n$ is specifically chosen, s.t. $n^2 \cdot 16\epsilon^2 = \frac{\pi}{2}$, then, along $[0,1] \times \{0\}$ all $\mathcal{T}$-directional derivatives of $\Phi$ of order less than $2n$ are determined by $(2n-2)$-jets of $\varphi_0, \varphi_1$ at $0 \in \mathcal{T}$ and there are two non-zero vectors, $u$ and $v$, so that

$$\sum_{i=0}^{n} v_i \cdot D_x^{2n-2i} \varphi_0(0) \cdot D_y^{2i} \varphi_0(0) + \sum_{i=0}^{n} u_i \cdot D_x^{2n-2i} \varphi_1(0) \cdot D_y^{2i} \varphi_1(0) = K(J_0^{2n-2} \varphi_0, J_0^{2n-2} \varphi_1),$$

(2.21)

where $u$, $v$ and $K$ are all determined by $(2n-2)$-jets of $\varphi_0, \varphi_1$ at $0 \in \mathcal{T}$.

3 Construction

In this section we construct examples, the main idea is try to make (2.21) be violated.

3.1 Example 1. Short Geodesic may not be Smooth

Given $n \in \mathbb{Z}$, $n > 2$, suppose we can connect $0$ and $\varphi_n = \frac{1}{2} \sin \left(\frac{\pi}{2n}\right) \cdot (\sin^2 x - \sin^2 y)$, by a $C^{2n+4}$ geodesic $\Phi$, then according to Theorem A.1 and analysis of section 2.1 we have that along $[0,1] \times \{0\}$

$$\frac{\Phi_{xx}' \Phi_{yy}'}{4(1 + \Phi_{xx} + \Phi_{yy})^2} = -\frac{\pi^2}{16n^2}.$$  

Then using Theorem 2.1 we can find $v$, with $v^\kappa \neq 0$, for some $0 \leq \kappa \leq n$, s.t.

$$\sum_{i=0}^{n} v_i (J_0^{2n-2} \varphi_n) \cdot D_x^{2n-2i} \varphi_n(0) \cdot D_y^{2i} \varphi_n(0) = K(J_0^{2n-2} \varphi_n).$$

(3.1)

In above, $v$ and $K$ only depends on $(2n-2)$-jet of $\varphi_n$ at $0 \in \mathcal{T}$. 

Then for any $\chi \neq 0$, if
\[ \psi_n = \frac{1}{2} \sin \left( \frac{\pi}{2n} \right) \cdot (\sin^2 x - \sin^2 y) + \chi \sin^{2n-2\kappa} x \sin^{2\kappa} y, \] (3.2)
can also be connected with 0 by a $C^{2n+4}$ non-degenerate geodesic, we have
\[ \sum_{i=0}^{n} v_i (J_0^{2n-2} \psi_n) \cdot D_x^{2n-2i} \psi_n(0) \cdot D_y^{2i} \psi_n(0) = K(J_0^{2n-2} \psi_n). \] (3.3)
Note that $\psi_n$ and $\varphi_n$ have same $(2n-2)$-jets at $0 \in T$, so the $v_i$ (and $K$) in (3.1) and the $v_i$ (and $K$) in (3.3) are the same. If we take difference of (3.1) and (3.3), we get
\[ v_i \cdot (2n-2\kappa)! (2\kappa)! = 0, \]
which is a contradiction. So we have for any $n \in \mathbb{Z}$, $n \geq 3$, either
\[ \varphi_n = \frac{1}{2} \sin \left( \frac{\pi}{2n} \right) \cdot (\sin^2 x - \sin^2 y), \]
or
\[ \tilde{\varphi}_n = \frac{1}{2} \sin \left( \frac{\pi}{2n} \right) \cdot (\sin^2 x - \sin^2 y) + e^{-10n} \sin^{2n-2\kappa} x \sin^{2\kappa} y, \]
cannot be connected with 0 by a non-degenerate $C^{2n+4}$ geodesic.

It’s easy to see, for any fixed $B > 0$, $|\varphi_n|_B + |\tilde{\varphi}_n|_B \to 0$ as $n \to \infty$. So we can pick a sequence from $\{\varphi_n\}_{n=3}^\infty \cup \{\tilde{\varphi}_n\}_{n=3}^\infty$ satisfying requirement of Example 1.1.

3.2 Example 2. Perturbation of an Analytic Geodesic

According to [1], for some $n \in \mathbb{Z}$, big enough, we can find an analytic geodesic $\bar{\Psi}$ defined on $[0, \frac{\pi}{4n}] \times T$ satisfying
\[ \bar{\Psi}(0,*) = 0; \]
\[ \frac{d}{dt} \bar{\Psi}(0,*) = \sin^2 x - \sin^2 y. \]
Then by scaling, we can get a geodesic $\Psi$ on $[0,1] \times V$, with
\[ \Psi(0,*) = 0; \]
\[ \frac{d}{dt} \Psi(0,*) = \frac{\pi}{4n} (\sin^2 x - \sin^2 y). \]
For this geodesic, along $[0,1] \times \{0\}$
\[ \frac{\Psi'_x \Psi'_y}{4(1 + \Psi_x + \Psi_y)^2} = -\frac{\pi^2}{16n^2}, \]
so we can repeat what we did in last subsection, and find $0 \leq \kappa \leq n$, s.t. $\Psi(0,*)$ cannot be connected with $\Psi(1,*) + \chi \sin^{2n-2\kappa} x \sin^{2\kappa} y$, by a $C^{2n+4}$ non-degenerate geodesic, for any $\chi \neq 0$. 10
A  Dirichlet Problem of ODE of 2nd Order Derivatives

In this appendix, we discuss the solvability of following problem

Problem A.1.  \textit{Given the boundary value of } $a_{11}$ \textit{and } $a_{22}$ \textit{at } $t = 0, 1$, \textit{satisfying}

$$a_{11}(1) + a_{22}(1) + \frac{1}{2} > 0; \quad (A.1)$$
$$a_{11}(0) + a_{22}(0) + \frac{1}{2} > 0, \quad (A.2)$$

\textit{can we find } $a_{11}, a_{22} \in C^2([0, 1]; \mathbb{R})$, \textit{satisfying}

$$a^{''}_{11} = \frac{4(a'_{11})^2}{1 + 2a_{11} + 2a_{22}}; \quad (A.3)$$
$$a^{''}_{22} = \frac{4(a'_{22})^2}{1 + 2a_{11} + 2a_{22}}, \quad (A.4)$$

\textit{and}

$$a_{11} + a_{22} + \frac{1}{2} > 0, \quad \text{on } [0, 1]. \quad (A.5)$$

We will prove following theorem

Theorem A.1. \textit{The necessary and sufficient condition for Problem A.1 to have solution is that boundary data satisfy}

$$a_{11}(0) + a_{22}(1) + \frac{1}{2} > 0; \quad (A.6)$$
$$a_{11}(1) + a_{22}(0) + \frac{1}{2} > 0. \quad (A.7)$$

\textit{When above conditions are satisfied, we can write down solution explicitly and the solution is unique.}

\textbf{Proof.} Given (A.2), we can transform conditions (A.1) (A.6) (A.7) to following

$$a_{11}(1) - a_{11}(0) > -(a_{11}(0) + a_{22}(0) + \frac{1}{2}); \quad (A.8)$$
$$a_{22}(1) - a_{22}(0) > -(a_{11}(0) + a_{22}(0) + \frac{1}{2}); \quad (A.9)$$
$$(a_{11}(1) - a_{11}(0)) + (a_{22}(1) - a_{22}(0)) > -(a_{11}(0) + a_{22}(0) + \frac{1}{2}). \quad (A.10)$$

Above conditions can be illustrated by Figure 1, i.e. $(a_{11}(1) - a_{11}(0), a_{22}(1) - a_{22}(0))$, as a point, stays in the interior of the shadowed area.

And we only need to consider the cases when $(a_{11}(1) - a_{11}(0), a_{22}(1) - a_{22}(0))$ stays

- in region I, $a_{11}(1) - a_{11}(0) > 0 > a_{22}(1) - a_{22}(0);$  
- in region II, $a_{11}(1) - a_{11}(0) > a_{22}(1) - a_{22}(0) > 0;$  
- on $\Gamma$, $a_{11}(1) - a_{11}(0) = a_{22}(1) - a_{22}(0) > 0;$
\[
\begin{align*}
\frac{a_{11}(1) - a_{11}(0)}{1 + 2a_{11} + 2a_{22}} &= \frac{a_{22}(1) - a_{22}(0)}{1 + 2a_{11} + 2a_{22}} = -(a_{11}(0) + a_{22}(0) + \frac{1}{2})
\end{align*}
\]

Figure 1: Boundary Condition

- on \(\Gamma_{+}\), \(a_{11}(1) - a_{11}(0) \geq 0 = a_{22}(1) - a_{22}(0)\).

For other cases, we can get solution by either changing role of \(a_{11}\) and \(a_{22}\) or by reversing the direction of time. In the following we discuss solvability separately

**Case 1.** \((a_{11}(1) - a_{11}(0), a_{22}(1) - a_{22}(0)) \in I\)

As discussed in section 2.1, we know in this situation,

\[
\frac{a'_{11}}{1 + 2a_{11} + 2a_{22}} = \epsilon A,
\]

\[
\frac{a'_{22}}{1 + 2a_{11} + 2a_{22}} = -\frac{\epsilon}{A},
\]

(A.11)

is a negative constant, so we can denote

\[
\frac{a'_{11}}{1 + 2a_{11} + 2a_{22}} = \epsilon A,
\]

\[
\frac{a'_{22}}{1 + 2a_{11} + 2a_{22}} = -\frac{\epsilon}{A},
\]

(A.11)

for some positive constant \(\epsilon\), and positive function \(A^2\).

As shown in Section 2.1, \(A\) should satisfy (2.9) so for some constant \(C_p\)

\[A = \tan(2\epsilon t + C_p).\]

(A.12)

To make \(A\) a positive function on \([0, 1]\), we should have

\[0 < 2\epsilon < \frac{\pi}{2}, \quad 0 < C_p < \frac{\pi}{2}, \quad 0 < 2\epsilon + C_p < \frac{\pi}{2},\]

(A.13)

Integrating following equation

\[
\frac{1}{2}[\log(1 + 2a_{11} + 2a_{22})]' = \frac{a'_{11} + a'_{22}}{1 + 2a_{11} + 2a_{22}} = \epsilon(A - \frac{1}{A}),
\]

gives

\[1 + 2a_{11} + 2a_{22} = C_v(A + \frac{1}{A}),\]

(A.14)

\[^2\text{Here we have assumed } a_{11}(1) - a_{11}(0) > 0.\]
for some constant $C_v$. Plugging (A.14) into (A.11) gives
\[ a'_{11} = C_v \epsilon(A^2 + 1), \quad a'_{22} = -C_v \epsilon(1 + \frac{1}{A^2}), \]
for some constant $C_v$. Integrating above expressions gives
\[ a_{11} = \frac{C_v A}{2} + C, \quad a_{22} = \frac{C_v}{2A} - (\frac{1}{2} + C). \]
Then plug (A.12) into above equation and using boundary condition, we found
\[ \tan^2 C_p = \frac{a_{11}(1) - a_{11}(0)}{a_{22}(0) - a_{22}(1)} \cdot \frac{a_{11}(0) + a_{22}(1) + 1/2}{a_{11}(1) + a_{22}(0) + 1/2}. \]
\[ \tan^2(2\epsilon + C_p) = \frac{a_{11}(1) - a_{11}(0)}{a_{22}(0) - a_{22}(1)} \cdot \frac{a_{11}(1) + a_{22}(0) + 1/2}{a_{11}(0) + a_{22}(1) + 1/2}. \]
Above equations and (A.13) would uniquely determine $C_p$, $\epsilon$. Then with other equations we can determine $C$ and $C_v$.

**Case 2.** $(a_{11}(1) - a_{11}(0), a_{22}(1) - a_{22}(0)) \in \Pi$

Using the similar argument of section 2.1, we know in this situation,
\[ \frac{a'_{11}}{1 + 2a_{11} + 2a_{22}} = \frac{a'_{22}}{1 + 2a_{11} + 2a_{22}} \]
is a positive constant, so we can denote
\[ \frac{a'_{11}}{1 + 2a_{11} + 2a_{22}} = \epsilon A, \quad \frac{a'_{22}}{1 + 2a_{11} + 2a_{22}} = \frac{\epsilon}{A}, \quad (A.15) \]
for some positive constant $\epsilon$, and positive function $A$.

Plugging (A.15) into (2.6), we found, when $A \neq \pm 1$, $A$ satisfies
\[ A' = 2\epsilon(A^2 - 1). \]
Equation of $A$ can be explicitly solved, and for positive $A$ there are only three possibilities: $0 < A < 1$, $1 < A$ or $A \equiv 1$. It turns out that only $A > 1$ fits with our boundary condition
\[ a_{11}(1) - a_{11}(0) > a_{22}(1) - a_{22}(0) > 0. \quad (A.16) \]
So $A$ takes following form
\[ A = \frac{1 + e^{4\epsilon(t + C_p)}}{1 - e^{4\epsilon(t + C_p)}}, \quad (A.17) \]
To make $A$ a positive function on $[0, 1]$, we should have
\[ 1 + C_p < 0. \quad (A.18) \]
\[ ^3\text{Here we have assumed } a_{11}(1) - a_{11}(0) > 0. \]
Integrating following equation
\[
\frac{1}{2} \log(1 + 2a_{11} + 2a_{22})' = \frac{a_{11}' + a_{22}'}{1 + 2a_{11} + 2a_{22}} = \epsilon(A + \frac{1}{A}),
\]
gives
\[1 + 2a_{11} + 2a_{22} = C_v(A + \frac{1}{A}), \quad (A.19)\]
for some constant \(C_v\). Plugging (A.19) into (A.15) gives
\[a_{11}' = C_v\epsilon(A^2 - 1), \quad a_{22}' = -C_v\epsilon(1 - \frac{1}{A}),\]
for some constant \(C_v\). Integrating above expressions gives
\[a_{11} = \frac{C_vA}{2} + C, \quad a_{22} = -\frac{C_v}{2A} - (\frac{1}{2} + C).\]

Then plug (A.17) into above equation and using boundary condition, we found
\[\tanh^2(2\epsilon ) = \frac{a_{22}(1) - a_{22}(0)}{a_{11}(1) - a_{11}(0)} \left( \frac{a_{11}(1) + a_{22}(0) + 1/2}{a_{11}(0) + a_{22}(1) + 1/2} \right), \quad (A.20)\]
\[\tanh^2(2\epsilon + 2\epsilon C_p) = \frac{a_{22}(1) - a_{22}(0)}{a_{11}(1) - a_{11}(0)} \left( \frac{a_{11}(0) + a_{22}(1) + 1/2}{a_{11}(1) + a_{22}(0) + 1/2} \right), \quad (A.21)\]

Our boundary conditions (A.16) would guarantee right hand side of (A.20) and (A.21) be smaller than 1. So above equations and (A.18) would uniquely determine \(C_p, \epsilon\). Then with other equations we can determine \(C\) and \(C_v\).

**Case 3.** \((a_{11}(1) - a_{11}(0), a_{22}(1) - a_{22}(0)) \in \Gamma\)

In this case, note that using (A.3) (A.4) we can derive
\[\left[ \frac{a_{11}' - a_{22}'}{(1 + 2a_{11} + 2a_{22})^2} \right]' = 0,
\]
so
\[a_{11}' - a_{22}' = C(1 + 2a_{11} + 2a_{22})^2.\]

Integrating above equation gives
\[a_{11}(1) - a_{22}(1) - [a_{11}(0) - a_{22}(0)] = \int_0^1 C(1 + 2a_{11} + 2a_{22})^2,
\]
for some constant \(C\). And note that with current boundary condition, we have left hand side of above equation equals 0, so \(C = 0\), and
\[a_{11}(t) - a_{22}(t) = a_{11}(0) - a_{22}(0).
\]

With above relation, (A.3) (A.4) can be reduced to a single ODE. Then we get solution to Problem A.1 in this case, is
\[a_{11} = \frac{Ce^{C_1t} - 1 + 2[a_{11}(0) - a_{22}(0)]}{4}, \quad a_{22} = a_{11} + a_{22}(0) - a_{11}(0),\]
where
\[ C = 2a_{22}(0) + 2a_{11}(0) + 1, \]
\[ C_1 = \log \left( \frac{2a_{11}(1) + 2a_{22}(1) + 1}{2a_{11}(0) + 2a_{22}(0) + 1} \right). \]

**Case 4.** \((a_{11}(1) - a_{11}(0), a_{22}(1) - a_{22}(0)) \in \Gamma_+\)

In this case,
\[ a_{22}(1) = a_{22}(0). \]

And according to (A.4), \(a_{22}\) is a convex function, so \(a'_{22}\) must equal to zero somewhere. But then (A.4) would imply \(a'_{22}\) is identically zero, so \(a_{22}\) is a constant.

So, in this situation, solution to Problem A.1 is
\[ a_{11} = \frac{1}{2(C_1 t + C_2)} - \frac{1}{2} - a_{22}(0), \quad a_{22} \equiv a_{22}(0), \]
where
\[ C_2 = \frac{1}{1 + 2a_{11}(0) + 2a_{22}(0)}, \]
\[ C_1 = \frac{2(a_{11}(0) - a_{11}(1))}{(1 + 2a_{11}(0) + 2a_{22}(0))(1 + 2a_{11}(1) + 2a_{22}(1))}. \]

And based on the construction we know in all four cases solutions are unique. The theorem is proved.

**B Uniqueness of Initial Value Problem**

In this appendix we prove following uniqueness theorem of initial value problem

**Lemma B.1.** Given \((V, \omega_0)\) a Kähler manifold, with \(\omega_0 > 0\). If two \(C^4\) non-degenerate geodesics \(\Phi_1, \Phi_2\) have same initial data
\[ \Phi_1(0, *) = \Phi_2(0, *); \]
\[ \frac{d}{dt}\Phi_1(0, *) = \frac{d}{dt}\Phi_2(0, *), \]
then
\[ \Phi_1 = \Phi_2. \]

**Proof.** According to analysis of [10], kernels of \(\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi_i\) form a foliation in \([0, 1] \times \mathbb{R} \times V\), for \(i = 1, 2\). We denote the foliation on \([0, 1] \times \mathbb{R} \times V\) corresponding to \(\Phi_i\) as \(\mathcal{F}_i\).

Now for a point \(p\) on \(V\), there is one and only one leaf in \(\mathcal{F}_i\) passing \((0, 0, p)\), for \(i = 1, 2\), which we denote as \(\mathcal{L}_i\). \(\mathcal{L}_i\) should be the graph of a holomorphic map from \([0, 1] \times \mathbb{R}\) to \(V\), because \(\Phi_i\) is non-degenerate. We denote this holomorphic map as \(l_i\).

According to [9], \(\mathcal{L}_i \cap \{\tau|\text{Re}\tau = 0\} \times V\) is the trajectory of a Hamiltonian flow, in the product of time space \(\mathbb{R}\) and phase space \(V\), with Hamiltonian \(\frac{d}{dt}\Phi_i(0, *)\) and symplectic form \(\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi_i(0, *)\). Since two geodesics have same initial data, the corresponding Hamiltonian flows have same Hamiltonians and symplectic forms, so these two trajectories coincide. This implies
\[ l_1 = l_2, \quad \text{on} \quad \{\tau|\text{Re}\tau = 0\}. \]
And since leaf is holomorphic, we get
\[ l_1 \equiv l_2, \text{ on } [0,1] \times \mathbb{R}, \]
so $\Phi_1$ and $\Phi_2$ correspond to same foliation structure. So $\frac{d}{dt}(\Phi_1 - \Phi_2)$ is harmonic on each leaf. Using equation and boundary condition, we know $\frac{d}{dt}(\Phi_1 - \Phi_2)$ has vanishing value and derivative on one side of each leaf.

Then we need to use following Proposition, which is Problem 2.2 of [12].

**Proposition B.1.** Given a harmonic function $u \in C^2(\Omega \cup T)$, where $T$ is an open and smooth portion of $\partial \Omega$, if
\[ u = u_n = 0, \text{ on } T, \]
then
\[ u \equiv 0. \]

So apply above Proposition to $\frac{d}{dt}(\Phi_1 - \Phi_2)$, we get $\Phi_1 \equiv \Phi_2$.  

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