Quantifying coherence of Gaussian states

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Coherence arises from the superposition principle and plays a key role in quantum mechanics. Recently, Baumgratz et al. [T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. 113, 140401 (2014)] established a rigorous framework for quantifying the coherence of finite dimensional quantum states. In this work we provide a framework for quantifying the coherence of Gaussian states and explicitly give a coherence measure based on the relative entropy.

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I. INTRODUCTION

Coherence is a basic feature in quantum mechanics, it is a common necessary condition for both entanglement and other types of quantum correlations. Many works have been undertaken to theoretically formulate quantum coherence [1–7], but up to now there has been no well-accepted efficient method for quantifying coherence. Recently, Baumgratz et al. established a comprehensive framework of coherence quantification [8], by which coherence is considered to be a resource that can be characterized, quantified, and manipulated in a manner similar to quantum entanglement [9–12]. This seminal work has revealed the condition of coherence transformations for pure states, and then built a universal method for quantifying the coherence of mixed states via the convex roof scheme.

All above results for quantifying quantum coherence are implicitly assumed the finite dimensional setting, which is neither necessary nor desirable. In relevant physical situations such as quantum optics, it must require the quantum states in infinite dimensional systems, especially the Gaussian states [24–26]. In this work we investigate the quantification of coherence for Gaussian states.

This work is organized as follows. In section II, we discuss the necessary conditions any measure of coherence for Gaussian states should satisfy. In section III, we prove that the incoherent one-mode Gaussian states are just thermal states. In section IV, we determine the structure of one-mode incoherent Gaussian operations. In section V, we explicitly provide a coherence measure for one-mode Gaussian states based on the relative entropy. In section VI we consider the multi-mode case. Section VII is a brief summary.

II. HOW TO QUANTIFY THE COHERENCE OF GAUSSIAN STATES

A state \( \rho \) (finite or infinite dimensional) is said to be incoherent if it is diagonal when expressed in a fixed orthonormal basis. We denote the set of all incoherent states by \( \mathcal{I} \). A quantum map is called incoherent operation (ICPTP) if it is completely positive, trace-preserving, and maps any incoherent states into incoherent states. Ref. [8] presented the postulates that any proper measure of the coherence \( C(\rho) \) for finite-dimensional state \( \rho \) must satisfy as follows.

1. \( C(\rho) \geq 0 \) and \( C(\rho) = 0 \) iff \( \rho \in \mathcal{I} \).

2a) Monotonicity under all incoherent completely positive and trace-preserving (ICPTP) maps: \( C(\rho) \geq C(\text{ICPTP}(\rho)) \).

2b) Monotonicity for average coherence under subselection based on measurement outcomes: \( C(\rho) \geq \sum_n p_n C(\rho_n) \), where \( p_n = K_n \rho K_n^+ / p_n \) and \( p_n = tr(K_n \rho K_n^+) \) for all \( n \), \( \sum_n K_n^+ K_n = I \), \( K_n \mathcal{I} K_n^+ \subset \mathcal{I} \), with \( + \) the adjoint and \( I \) identity operator.

3) Nonincreasing under the mixing of quantum states: \( \sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n) \).

Note that (2b) and (3) together imply (2a).

For the case of Gaussian states, we adopt (1) and (2a) as necessary conditions that any coherence measure should satisfy, while give up (2b) and (3). Gaussian states do not form a convex set, then it seems hard to establish the counterparts of (2b) and (3).

III. INCOHERENT STATES OF ONE-MODE GAUSSIAN STATES

In this section, we find out the incoherent states of one-mode Gaussian states. We first note that, coherence is basis dependent, so wherever we talk about coherence we must be clear which basis is presupposed.

Theorem 1. For fixed orthonormal basis \( \{|n\rangle\}_{n=0}^\infty \), a one-mode Gaussian state is diagonal iff it is a thermal state.

Proof. A state \( \rho \) is called Gaussian if its characteristic
function $\chi(\rho, \lambda) = \text{tr}[\rho D(\lambda)]$ is of the form
\[
\chi(\rho, \lambda) = \exp\left\{ -\frac{1}{2}(x_\lambda, y_\lambda)\Omega \mathcal{V} \Omega^t \left( x_\lambda \atop y_\lambda \right) - i\Omega \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}^t \left( x_\lambda \atop y_\lambda \right) \right\},
\] (1)
where $D(\lambda) = e^{\lambda a^+ - \lambda^* a}$ is the displacement operator, $a, a^+$ are annihilation and creation operator, $\lambda_\alpha$ and $\lambda_\beta$ are the real and imaginary parts of $\lambda$, $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, d = (d_1, d_2)^t$ with $d_1$ and $d_2$ real numbers, $t$ denotes transpose, $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ is a real symmetric matrix and satisfies $V + i\Omega \geq 0$. A Gaussian state $\rho$ is fully described by the covariance matrix $V$ and the displacement vector $d$. $\det V \geq 1$ and $\det d \rho = 1$ if $\rho$ is pure. We write a Gaussian state as $\rho(V, d)$.

A state is called thermal if it has the form
\[
\rho_{th}(\pi) = \sum_{n=0}^{\infty} \frac{\pi^n}{n!} |n\rangle \langle n|,
\] (2)
where $\pi = \text{tr}[a^+ a \rho_{th}(\pi)] \geq 0$ is the mean number.

It is easy to check that the characteristic function of the thermal state $\rho_{th}(\pi)$ is Gaussian with covariance matrix $(2\pi + 1)I$ and zero displacement vector. Hence we only need to prove that the diagonal Gaussian states must be thermal states. To this aim, we calculate the elements $\rho_{mn} = \langle m | \rho | n \rangle$ from Eq.(1) and its inverse relation
\[
\rho = \int \frac{d^2 \lambda}{\pi} \chi(\rho, \lambda) D(-\lambda),
\] (3)
where $d^2 \lambda = dx_\lambda dy_\lambda$ and $J = \int_{-\infty}^{\infty}$. We thus have
\[
\langle m | \rho | n \rangle = \int \frac{d^2 \lambda}{\pi} \chi(\rho, \lambda) \langle m | D(-\lambda) | n \rangle, \tag{4}
\]
\[
\langle m | D(-\lambda) | n \rangle = \int \frac{d^2 \alpha d^2 \beta}{\pi^2} \langle m | \alpha | D(-\lambda) | \beta \rangle \langle \beta | n \rangle, \tag{5}
\]
\[
\langle \alpha | D(-\lambda) | \beta \rangle = \left( \frac{\alpha^{\alpha^*} - \beta^{\beta^*}}{\sqrt{\alpha^{\alpha^*} \beta^{\beta^*} \sqrt{\alpha^{\alpha^*}}} \right) \rangle | \beta \rangle = e^{-\frac{1}{2} \sqrt{\alpha^{\alpha^*} \beta^{\beta^*} \sqrt{\alpha^{\alpha^*}}} n \rangle, \tag{6}
\]
where $|\alpha\rangle = e^{-\frac{1}{2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} n \rangle}$ are coherent states. Using the formula
\[
D(\alpha)D(\beta) = e^{\alpha^* \beta - \beta^* \alpha} D(\alpha + \beta),
\] (7)
and after direct algebras, we get
\[
\langle m | \rho | n \rangle = \int \int \frac{d^2 \lambda d^2 \alpha d^2 \beta}{\pi^2} \frac{\alpha^n \beta^m}{\sqrt{n!m!}} \exp b, \tag{8}
\]
\[
b = -\alpha^* \lambda + \alpha^* \beta + \lambda^* \beta - |\alpha|^2 - |\beta|^2 - \frac{1}{2} |\lambda|^2 - \chi(\rho, \lambda).
\] (9)
Eq.(8) is somewhat similar to the results of Refs.[27–29], but in fact not the same thing. To calculate Eq.(8), we introduce the integration
\[
J = \int \int d^2 \alpha d^2 \beta \exp \{ b + ua + v\beta^* \}, \tag{10}
\]
where $u, v$ are real numbers. As a result,
\[
\rho_{mn} = \frac{1}{\pi^3} \left( \frac{\partial^m}{\partial u^m} \frac{\partial^n}{\partial \beta^*} J \right)_{u=v=0}. \tag{11}
\]
$J$ is a Gaussian integration, we can calculate as follows.
Write $b + ua + v\beta^*$ as
\[
b + ua + v\beta^* = -\frac{1}{2}(x_\alpha, y_\alpha, x_\beta, y_\beta, x_\lambda, y_\lambda) A
\cdot (x_\alpha, y_\alpha, x_\beta, y_\beta, x_\lambda, y_\lambda)^t + B(x_\alpha, y_\alpha, x_\beta, y_\beta, x_\lambda, y_\lambda)^t, \tag{12}
\]
where $A$ is a $6 \times 6$ symmetric complex matrix, $B$ is a $6 \times 1$ row complex vector,
\[
A = \begin{pmatrix} 2 & 0 & -1 & -i & 1 & i \\ 0 & 2 & i & -1 & -i & 1 \\ -1 & i & 2 & 0 & -1 & 1 \\ -i & 1 & 0 & 2 & 2 & -i & -1 \\ 1 & -i & -1 & -1 & 1 + V_{22} & -V_{12} \\ i & 1 & i & -1 & -V_{12} & 1 + V_{11} \end{pmatrix}, \tag{13}
\]
\[
det A = 16(\text{det} V + V_{11} + V_{22} + 1) > 0, \tag{14}
\]
\[
B = (u, iu, v, -iv, -id_2, id_1). \tag{15}
\]
Apply the Gaussian integration formula we get
\[
J = \frac{(2\pi)^3}{\sqrt{\det A}} \exp \{ \frac{1}{2} BA^{-1} B^t \}. \tag{16}
\]
Let
\[
\xi = \frac{1}{2} BA^{-1} B^t = \frac{1}{2} (u, v) B_2(u, v)^t + B_1(u, v)^t + B_0, \tag{17}
\]
where
\[
B_2 = \begin{pmatrix} V_{11} - V_{12} + 2 V_{12} & V_{11} V_{12} - V_{12}^2 \\ V_{11} V_{12} - V_{12}^2 & V_{11} V_{12} + V_{12}^2 \end{pmatrix}, \tag{18}
\]
\[
B_1 = 2(1 - i V_{12} + V_{22}) d_1 + i(1 + V_{11} + V_{12}) d_2, \tag{19}
\]
\[
(1 + i V_{12} + V_{22}) d_1 - i(1 + V_{11} - i V_{12}) d_2, \tag{20}
\]
\[
B_0 = -(1 + V_{22}) d_2 - 2 V_{12} d_1 + (1 + V_{11}) d_2. \tag{21}
\]
Introduce the symbols
\[
J_{k_1 k_2 \ldots k_m} = \frac{\partial^m J}{\partial k_1 \partial k_2 \ldots \partial k_m}, \tag{22}
\]
\[
\xi_{k_1 k_2 \ldots k_m} = \frac{\partial^m \xi}{\partial k_1 \partial k_2 \ldots \partial k_m}. \tag{23}
\]
where \( k_1, k_2, \ldots, k_m \in \{ u, v \}, m \in \{ 1, 2, 3, \ldots \} \), and let \( J(0), J_{k_1k_2 \ldots k_m}(0), \xi(0), \xi_{k_1k_2 \ldots k_m}(0) \) represent the corresponding values when \( u = v = 0 \). Direct calculations show that

\[
\xi_{k_1}(0) = \frac{1}{2}(B_1)_{k_1},
\]

(20)

\[
\xi_{k_1k_2}(0) = (B_2)_{k_1k_2},
\]

(21)

\[
\xi_{k_1k_2 \ldots k_m}(0) = 0 \text{ when } m \geq 3,
\]

(22)

\[
J_{k_1}(0) = J(0)\xi_{k_1}(0),
\]

(23)

\[
J_{k_1k_2}(0) = J(0)[\xi_{k_1k_2}(0) + \xi_{k_1}(0)\xi_{k_2}(0)],
\]

(24)

\[
J_{k_1k_2 \ldots k_m}(0) = J(0)[\xi_{k_1k_2 \ldots k_m}(0) + \xi_{k_1}(0)\xi_{k_2}(0)\xi_{k_3}(0) + \ldots + \xi_{k_1}(0)\xi_{k_2}(0)\xi_{k_3}(0)\xi_{k_4}(0) + \ldots + \xi_{k_2 \ldots k_m}(0)\xi_{k_1}(0)\xi_{k_2}(0)\xi_{k_3}(0)\xi_{k_4}(0) + \ldots],
\]

(25)

where \([\frac{m}{2}] = \frac{m(m-1)}{2}\) when \( m \) is even and \([\frac{m+1}{2}] = \frac{m+1}{2}\) when \( m \) is odd, \( \sigma(k_1k_2 \ldots k_m) \) is any permutation of \( k_1k_2 \ldots k_m \), \( \sum_{\sigma(k_1k_2 \ldots k_m)} \) sums all permutations of \( k_1k_2 \ldots k_m \).

From Eqs. (11,26) we can calculate any \( \rho_{mn} \) in principle.

If the Gaussian state \( \rho \) is diagonal thus \( \rho_{01} = \rho_{02} = 0 \),

\[
\rho_{01} = \frac{2^3}{\sqrt{\det A}}J(0)\frac{1}{2}(B_1)_{2} = 0 \Rightarrow (B_1)_{2} = 0.
\]

(27)

\[
\rho_{02} = \frac{2^3}{\sqrt{\det A}}J(0)(B_2)_{22} + \frac{1}{2}(B_1)_{2} \frac{1}{2}(B_1)_{2} = 0
\]

\[
\Rightarrow (B_2)_{22} = 0.
\]

(28)

Similarly \( \rho_{10} = \rho_{20} = 0 \) yield

\[
(B_1)_{11} = 0,
\]

(29)

\[
(B_2)_{11} = 0.
\]

(30)

Taking Eqs. (27-30) into \( B_2, B_1 \), we get

\[
V_{11} - V_{12} = V_{12} = d_1 = d_2 = 0,
\]

(31)

hence \( \rho \) is a thermal state. We then complete this proof.

**V. A COHERENCE MEASURE OF ONE-MODE GAUSSIAN STATES BASED ON RELATIVE ENTROPY**

For any one-mode Gaussian state \( \rho(V, d) \), we define a coherence measure as

\[
C(\rho) = \inf_{\delta} \{ S(\rho \mid \delta) : \delta \text{ is an incoherent state} \},
\]

(40)

where \( S(\rho \mid \delta) = tr(\rho \log_2 \rho) - tr(\rho \log_2 \delta) \) is the relative entropy, \( \inf \) runs over all incoherent Gaussian states. The entropy of \( \rho, S(\rho) = -tr(\rho \log_2 \rho) \) is [31]

\[
S(\rho) = g(\nu) = \frac{\nu + 1}{2} \log_2 \frac{\nu + 1}{2} - \frac{\nu - 1}{2} \log_2 \frac{\nu - 1}{2}.
\]

(41)

where \( \nu = \sqrt{\det V} \). We now calculate \( \sup_{\delta} tr(\rho \log_2 \delta) \). Suppose

\[
\delta(\pi) = \sum_{n=0}^{\infty} \frac{\pi^n}{(\pi + 1)^{n+1}} |n\rangle \langle n|,
\]

(42)
then
\[
tr[\rho \log_2 \delta] = tr[\rho_{\text{diag}} \log_2 \delta] = \sum_{n=0}^{\infty} \rho_{nn} \log_2 \left( \frac{\pi}{\pi + 1} \right)^{n+1}
\]
\[
= (\sum_{n=0}^{\infty} n \rho_{nn}) \log \pi - (\sum_{n=0}^{\infty} n \rho_{nn} + 1) \log(\pi + 1),
\]
where \(\rho_{\text{diag}} = \sum_{n=0}^{\infty} \rho_{nn}|n\rangle \langle n|\). It follows that
\[
\frac{\partial}{\partial \pi} tr[\rho \log_2 \delta] = \frac{1}{\ln 2} \frac{1}{\pi + 1} \left[ \sum_{n=0}^{\infty} n \rho_{nn} \pi - 1 \right].
\]
Let \(\frac{\partial}{\partial \pi} tr[\rho \log_2 \delta] = 0\), we get
\[
\pi = \sum_{n=0}^{\infty} \rho_{nn}n.
\]
\[\text{(44)}\]

The remaining is how to calculate \(\pi = \sum_{n=0}^{\infty} \rho_{nn}n\).
\[
\pi = \sum_{n=0}^{\infty} (n|\rho|n) = \sum_{n=0}^{\infty} (n|a^n+|a^n) = tr(\rho a^n+ a\rho a^n)
\]
\[
= \int d^2 \alpha \pi (\alpha|a^n+ a\alpha = \int d^2 \alpha |\alpha|a^n+ |a\alpha
\]
\[
= \int d^2 \alpha \alpha \int d^2 \lambda \pi (\rho(\lambda)(\alpha|D(-\lambda)a^n+|a\lambda),
\]
where we have used \(a^n+|a^n = n|n\), the coherent state \(|\alpha\rangle, a|\alpha\rangle = |\alpha\rangle,\) and Eq.(3).
\[\text{(46)}\]

It is easy to check that
\[
|a^n+\rangle = e^{-n|a|^2} \sum_{n=1}^{\infty} n!a^n |n\rangle,
\]
\[\text{(47)}\]

thus direct algebras show that
\[
|a^n+\rangle = e^{-n|a|^2} \sum_{n=1}^{\infty} \frac{2^2}{\sqrt{n!}} (\alpha + \lambda|n\rangle
\]
\[\text{(48)}\]

Using the result of Ref.[29] with some algebras we get
\[\pi = \frac{1}{4}(V_{11} + V_{22} + d^2_1 + d^2_2 - 2).
\]
\[\text{(54)}\]

In conclusion, we get
\[
C[\rho(V, d)] = \nu - \frac{1}{2} \log \nu - \frac{1}{2} \log \nu + \frac{1}{2}
\]
\[\text{(55)}\]
\[
= \sqrt{\det V} = \sqrt{V_{11}V_{22} - V_{12}^2},
\]
\[\text{(56)}\]
\[
\pi = \frac{1}{4}(V_{11} + V_{22} + d^2_1 + d^2_2 - 2).\]
\[\text{(57)}\]

We next prove \(C[\rho]\) is nondecreasing under any incoherent operation. For any incoherent operation \(O\), suppose \(C[\rho] = S[\rho]|\pi\) with the thermal state \(|\pi = \sum_{n=0}^{\infty} \rho_{nn}n\) as specified in Eq.(57), then we have
\[
C[O(\rho)] \leq S[O(\rho)||O(\pi)] \leq S[\rho]|\pi] = C[\rho].
\]
\[\text{(58)}\]

In Eq.(58) above, the first inequality comes from the definition of \(C[O(\rho)]\) and the fact that \(O(\rho)\) is a thermal state, the second inequality comes from the monotonicity of relative entropy under completely positive and trace preserving mapping [32].

From Eqs.(55-57), we see that the coherence measure \(C[\rho(V, d)]\) is strictly monotonically decreasing in \(\nu\) while strictly monotonically increasing in \(\pi\). For pure Gaussian states \(\nu = 1\) reaches the minimum of \(\nu\). In this sense, we say that the maximally coherent states are pure.

VI. MULTI-MODE GAUSSIAN STATES

We extend the results of one-mode Gaussian states into multi-mode Gaussian states. For the positive integer \(m \geq 2\), an \(m\)-mode Gaussian state \(\rho(V, d)\) is described by [26] its covariance matrix \(V, a 2m \times 2m\) real symmetric positive matrix, and its displacement vector \(d, 2m\) dimensional real vector. \(V\) satisfies \(V + i\Omega \geq 0\) with \(\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes^m\)

A. Incoherent states

For fixed orthonormal basis \((|m\rangle)_{m=0}^{\infty}\) with positive integer \(m \geq 2\), the diagonal states is of the form
\[
\rho_{A_1A_2...A_m} = \sum_{n_1,...,n_m} \rho_{n_1n_2...n_m}|n_1\rangle \otimes \cdots \otimes |n_m\rangle |n_m\rangle
\]
\[\text{(59)}\]

where \(A_i\) denotes the \(i\)th mode, and
\[
\rho_{n_1n_2...n_m} = \langle n_1|n_2|...|n_m|\rho_{A_1A_2...A_m}|n_m\rangle ... |n_2\rangle |n_1\rangle.
\]
\[\text{(60)}\]
It is easy to check that
\[ [\rho^{A_1A_2},\rho^{A_1}] = 0, \]
where \( \rho^{A_1} \) is the reduced state with respect to the \( A_1 \) mode, \( \rho^{A_1A_2} \) is the reduced state with respect to the \( A_1A_2 \) modes, and \([\cdot]\) denotes commutator. Recall that for two-mode Gaussian states [33]
\[ [\rho^{A_1A_2},\rho^{A_1}] = 0 \Leftrightarrow \rho^{A_1A_2} = \rho^{A_1} \otimes \rho^{A_2}. \]
Together with Theorem 1 above, we get that the incoherent two-mode Gaussian states are of the form
\[ \rho^{A_1A_2} = \rho_{th}^{A_1}(\pi_1) \otimes \rho_{th}^{A_2}(\pi_2), \]
where \( \rho_{th}^{A_1}(\pi_1) \) and \( \rho_{th}^{A_2}(\pi_2) \) are all thermal states with mean numbers \( \pi_1 \) and \( \pi_2 \).

For \( m \)-mode incoherent Gaussian state \( \rho^{A_1A_2...A_m} \), we consider its covariance matrix \( V \). Since \( \rho^{A_iA_j} \), for any \( 1 \leq i \leq j \) is of the form \( \rho^{A_iA_j} = \rho^{A_i} \otimes \rho^{A_j} \), hence the covariance matrix \( V \) must be of diagonal form, that is
\[ \rho^{A_1A_2...A_m} = \otimes_{i=1}^{m} \rho_{th}^{A_i}(\pi_i). \]

\section*{B. Incoherent operation}

An \( m \)-mode Gaussian operation is described by \( (T,N,d) \), it performs on the Gaussian state \( \rho(V,d) \) and get the Gaussian state with the covariance matrix and displacement vector as [30]
\[ d \rightarrow Td + \overrightarrow{7}, V \rightarrow TVT^t + N, \]
where \( \overrightarrow{7} \in R^{2m} \), \( N, T \) are \( 2m \times 2m \) real matrices satisfying
\[ N + i\Omega - iT\Omega T^t \geq 0. \]
Similar to the one-mode case, it is easy to determine the incoherent operation is of the form as follows. \( T \) consists of \( \{t_iO_i\}_{i=1}^{m} \) with \( t_i \) real number, \( O_i \) \( 2 \times 2 \) real matrix satisfying \( O_iO_j^t = I \), each \( (2i-1,2i) \) row has just \( t_iO_i \), each \( (2j-1,2j) \) column has just one of \( \{t_iO_i\}_{i=1}^{m} \), and other elements are all zero. \( N = diag\{w_1I, w_2I, ..., w_mI\} \) with \( w_i \geq 0 \) and \( I \) being \( 2 \times 2 \) identity. The condition Eq.(65) then reads
\[ w_i \geq |t_i^2 \det O_i - 1| \quad \text{for all } i. \]

\section*{C. A coherence measure}

Similar to the one-mode case, we can generalize Eqs.(55-57) into \( m \)-mode case, that is
\[ C[\rho(V,d)] = -S(\rho) \]
\[ + \sum_{i=1}^{m} (\nu_i - 1) \log_2 (\nu_i + 1) - \nu_i \log_2 \nu_i, \]
where \( S(\rho) \) is the entropy of \( \rho \) [31], \( \{\nu_i\}_{i=1}^{m} \) are symplectic eigenvalues of \( V \) [26], \( \pi_i \) is determined by the \( i \)-mode covariance matrix \( V^{(i)} \) and displacement vector \( d^{(i)} \). We can prove Eq.(68) fulfills (C2a) in the similar way as the one-mode case.

\section*{VII. SUMMARY}

In summary, along the line of quantifying coherence of finite-dimensional quantum states, we provided a measure for Gaussian states. To this aim, we proved that the incoherent Gaussian states are just thermal states. We defined the Gaussian incoherent operations as the Gaussian operations which maps incoherent states into incoherent states and found out the structure of Gaussian incoherent operations. The central result is that we provided a coherence measure for Gaussian states based on the relative entropy, it satisfies (C1) and (C2a).

There remain many questions for future investigations. Firstly, whether or not we can establish the counterparts of (C2b) and (C3) for Gaussian states, in present work we only adopt (C1) and (C2a) as necessary conditions for any coherence measure. Secondly, how about the behaviors of coherence in Gaussian dynamical systems, such as frozen coherence [15], sudden change etc.

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