\(γ\)– Operation & Decomposition of Some Forms of Fuzzy Soft Mappings on Fuzzy Soft Ideal Topological Spaces

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Abstract. In this paper, we extend the notion of \(γ\)–operation by using the fuzzy soft ideal notions. Furthermore, we introduce some forms of fuzzy soft continuity in fuzzy soft topological spaces via fuzzy soft ideals. The decomposition of these forms is investigated.

1. Introduction

It is well known that the conception of fuzzy sets, firstly defined by Zadeh [39] in 1965. Chronologically in 1968 and 1976, Chang [8] and Lowen [25] redound the concept of fuzzy topological spaces to literature substantively by using this conception. In 1999, Molodtsov [29] introduced the concept of soft set theory which is a completely new approach for modeling uncertainty. In his paper, Molodtsov established the fundamental results of this new theory and successfully applied the soft set theory into several directions, such as smoothness of functions, operations research, Riemann integration, game theory, theory of probability and so on. Maji et al. [27] defined and studied several basic notions of soft set theory in 2003. Shabir and Naz [33] introduced the concept of soft topological space and studied neighborhoods and separation axioms.

Maji et al. [28] initiated the study involving both fuzzy sets and soft sets. In his paper, the notion of fuzzy soft sets was introduced as fuzzy generalizations of soft sets and some basic properties of fuzzy soft sets are discussed in detail. In 2011, Tanay et al. [34] gave the topological structure of fuzzy soft sets. Kandil et al. introduced the concept of fuzzy soft connected sets[19–21], fuzzy soft hyperconnected spaces [22].

On the other hand; some properties of the concept of ideal or topological ideal obtained by delineate of Vaidyanathaswamy [38] in 1945 and Kuratowski [24] in 1966. In 1960 Vaidyanathaswamy [37], and Jankovic’ [12] in 1990 studied intensity on this subject. In 2015, Kandil et al.[16] initiated the notion of soft ideal topological spaces for the first time and studied its properties. Recently, in 2016, Kandil et al. [17, 18] introduced the concepts of fuzzy soft ideal and fuzzy soft local function. These concepts are discussed with a view to find new fuzzy soft topologies from the original one, called fuzzy soft topological spaces via fuzzy soft ideal \((X, τ, E, I)\).

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Our aim of this paper is to introduce and study some types of subsets of fuzzy soft topological spaces due to [10] to fuzzy soft topological spaces via fuzzy soft ideals. Also, we extend the idea of decomposition of continuity due to [9, 11, 23, 31, 40] to fuzzy soft topological spaces via fuzzy soft ideals.

2. Preliminaries

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by P(X). In this section, we present the basic definitions and results of fuzzy soft set theory which will be needed in the sequel.

Definition 2.1. [8] A fuzzy set A of a non-empty set X is characterized by a membership function \( \mu_A : X \rightarrow [0; 1] = I \) whose value \( \mu_A(x) \) represents the “degree of membership” of x in A for x \( \in X \). Let \( I_X \) denotes the family of all fuzzy sets on X.

Definition 2.2. [29] Let A be a non-empty subset of E. A pair (F, A) denoted by \( F_A \) is called a soft set over X, where F is a mapping given by F : A \( \rightarrow P(X) \). In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular e \( \in A \), F(e) may be considered the set of e-approximate elements of the soft set (F, A) and if e \( \notin A \), then F(e) = \( \emptyset \). I.e. F = \{F(e) : e \in A \subseteq E, F : A \rightarrow P(X)\}.

Proposition 2.1. [7] Every fuzzy set may be considered a soft set.

Definition 2.3. [28] Let \( \tilde{A} \subseteq E \). A pair (F, A), denoted by \( F_A \), is called fuzzy soft set over X, where F is a mapping given by F : A \( \rightarrow I_X \) defined by \( f_A(e) = \mu_A(e) \); where \( \mu_A(e) = 0 \) if e \( \notin A \), and \( \mu_A(e) = 1 \) if e \( \in A \), where \( \mu_A(x) = 0 \forall x \in X \). The family of all these fuzzy soft sets over X denoted by FSS(X)_E. Note that, a fuzzy soft set is a hybridization of fuzzy sets and soft sets, in which soft set is defined over fuzzy set. The family of all fuzzy soft sets over X with a fixed set of parameter E is denoted by FSS(X)_E.

Definition 2.4. [28] The complement of a fuzzy soft set \( (f, A) \), denoted by \( (f, A)^{\complement} \), is defined by \( (f, A)^{\complement} = (f^{\complement}, A) \), where \( f^{\complement} : A \rightarrow I_X \) is a mapping given by \( f^{\complement}_e(x) = 1 - f_e(x) \forall e \in A \), where \( 1_X(x) = 1 \forall x \in X \). Clearly, \( ((f, A)^{\complement})^{\complement} = f, A \).

Definition 2.5. [34] A fuzzy soft set \( f_E \) over X is said to be a null- fuzzy soft set, denoted by \( 0_E \), if for all e \( \in E \), \( f_e(e) = 0 \).

Definition 2.6. [34] A fuzzy soft set \( f_E \) over X is said to be an absolute fuzzy soft set, denoted by \( 1_E \), if \( f_e(e) = 1 \forall e \in E \). Clearly we have \( (0_E)^{\complement} = 1_E \) and \( (1_E)^{\complement} = 0_E \).

Definition 2.7. [34] Let \( f_A, g_B \in \text{FSS}(X)_E \). Then \( f_A \) is fuzzy soft subset of \( g_B \), denoted by \( f_A \subseteq g_B \), if A \( \subseteq B \) and \( \mu_A(e) \leq \mu_B(e) \forall x \in X \) and \( \forall e \in E \). Also, \( g_B \) is called fuzzy soft superset of \( f_A \) denoted by \( g_B \supseteq f_A \).

Definition 2.8. [36] Two fuzzy soft sets \( f_A \) and \( g_B \) on X are said to be equal if \( f_A \subseteq g_B \) and \( g_B \subseteq f_A \).

Definition 2.9. [32] The union of two fuzzy soft sets \( f_A \) and \( g_B \) over the common universe X, denoted by \( f_A \cup g_B \), is also a fuzzy soft set \( h_C \), where \( C = A \cup B \) and for all, e \( \in C \), \( h_C(e) = \mu_C(e) = \mu_A(e)^{\vee} \vee \mu_B(e)^{\vee} \forall e \in C \).

Definition 2.10. [34] The intersection of two fuzzy soft sets \( f_A \) and \( g_B \) over the common universe X, denoted by \( f_A \cap g_B \), is also a fuzzy soft set \( h_C \), where \( C = A \cap B \) and for all, e \( \in C \), \( h_C(e) = \mu_C(e) = \mu_A(e)^{\wedge} \wedge \mu_B(e)^{\wedge} \forall e \in C \).

Definition 2.11. [35] The difference of two fuzzy soft sets \( f_A \) and \( g_B \) over the common universe X, denoted by \( f_A - g_B \), is also a fuzzy soft set \( h_C \), where \( C = A \cap B \neq \emptyset \) and \( \forall e \in C \), \( x \in X \), \( h_C(e) = \mu_C(x) = \min[\mu_A(x), 1 - \mu_B(x)] \). Clearly, we have \( f_A - g_B = f_A \cap \bar{g}_B \).

Definition 2.12. [30, 34, 36] Let \( \tau \) be a collection of fuzzy soft sets over a Universe X with a fixed set of parameters \( E \), then \( \tau \) is called fuzzy soft topology on X if:

1. \( 0_E, 1_E \in \tau \), where \( 0_E(e) = 0 \) and \( 1_E(e) = 1 \forall e \in E \).
2. The union of any members of \( \tau \) belongs to \( \tau \).
3. The intersection of any two members of \( \tau \) belongs to \( \tau \).

The triplet \( (X, \tau, E) \) is called fuzzy soft topological space over X. Also, each member of \( \tau \) is called fuzzy soft open set in \( (X, \tau, E) \).

Definition 2.13. [34?] Let \( (X, \tau, E) \) be a fuzzy soft topological space. A fuzzy soft set \( f_A \) over X is said to
be fuzzy closed soft set in X, denoted by \( f_A \in \tau^c \), if its relative complement \( (f_A)^c \) is fuzzy open soft set.

**Definition 2.14.** [30, 32, 34] Let \( (X, \tau, E) \) be a fuzzy soft topological space and \( f_A \in \mathit{FSS}(X)_E \). The fuzzy soft closure of \( f_A \), denoted by \( Fcl(f_A) \), is the intersection of all fuzzy closed soft super sets of \( f_A \) i.e. \( Fcl(f_A) = \cap \{ h_C : h_C \text{ is fuzzy closed soft set and } f_A \subseteq h_C \} \). Clearly, \( Fcl(f_A) \) is the smallest fuzzy soft closed set over \( X \) which contains \( f_A \) and \( Fcl(f_A) \) is fuzzy closed soft set.

**Definition 2.15.**[30, 32, 34] The fuzzy soft interior of \( g_B \), denoted by \( \mathit{Fint}(g_B) \), is the fuzzy soft union of all fuzzy open soft subsets of \( g_B \) i.e. \( \mathit{Fint}(g_B) = \cup \{ h_C : h_C \text{ is fuzzy open soft set and } h_C \subseteq g_B \} \). Clearly, \( \mathit{Fint}(g_B) \) is the largest fuzzy soft open set contained in \( g_B \) and \( \mathit{Fint}(g_B) \) is fuzzy open soft set.

**Definition 2.16.** [26] The fuzzy soft set \( f_A \in \mathit{FSS}(X)_E \) is called fuzzy soft point if there exist \( x \in X \) and \( e \in E \) such that

\[
f_A(a) = \begin{cases} x_a & : a = e \\ 0 & : a \in E \setminus \{a\} \end{cases}^{a < 1}
\]

This fuzzy soft point is denoted by \( x^e_a \) or \( f_e \). The set of all fuzzy soft points in \( X \) will be denoted by \( \mathit{FSP}(X)_E \).

**Definition 2.17.** [26] The fuzzy soft point \( x^e_a \) is said to belong to the fuzzy soft set \( f_A \), denoted by \( x^e_a \in f_A \), if for the element \( e \in A, a \leq \mu^e_f_k(x) \).

**Definition 2.18.** [17, 18] A non-empty collection \( I \) of fuzzy soft sets over a universe \( X \) with a fixed set of parameters \( E \) is said to a fuzzy soft ideal on \( X \) if it satisfies the following conditions:

(i) If \( f_A \in I, g_B \subseteq f_A \Rightarrow g_B \in I \) (heredity)

(ii) If \( f_A, g_B \in I \Rightarrow f_A \cup g_B \in I \) (finite additivity)

We denote \((X, \tau, E, I)\) as a fuzzy soft ideal topological space on \( X \).

**Definition 2.19.** [17] Let \((X, \tau, E, I)\) be a fuzzy soft ideal topological space on \( X \). Then \( f_A^+I, \tau \) or \( f_A^+ = \cup \{ x^e_a \in \mathit{FSS}(X)_E : f_A \cap g_B \subseteq I \forall g_B \in \tau(x^e_a) \} \) is called the fuzzy soft local function of \( f_A \) with respect to \( I \) and \( \tau \) where \( \tau(x^e_a) \) is the set of all fuzzy soft open sets contains \( x^e_a \).

**Definition 2.20.**[17, 18] In a fuzzy soft ideal topological space \((X, \tau, E, I)\), the collection \( \tau^+I \) means an extension of fuzzy soft topological space finer than \( \tau \) via fuzzy soft ideal \( I \) which is constructed by considering the class \( \beta = \{ f_A \subseteq h_C : f_A \in \tau, h_C \in I \} \) as a base. Each member of \( \tau^+I \) is called fuzzy soft \( \star \)-open set and its relative complement is called fuzzy soft \( \star \)-closed set. Clearly, \( \tau \subseteq \tau^+I \).

**Theorem 2.1.**[17, 18] Let \((X, \tau, E, I)\) be a fuzzy soft ideal topological space on \( X \). Then the operator \( Fcl^+ : \mathit{FSS}(X)_E \rightarrow \mathit{FSS}(X)_E \) defined by: \( Fcl^+(f_A) = f_A \cup f_A^+ \) is a fuzzy soft closure operator satisfies Kuratowski’s axioms.

**Definition 2.21.**[17, 18] Let \((X, \tau, E, I)\) be a fuzzy soft ideal topological space on \( X \). The fuzzy soft \( \star \)-closure of a fuzzy soft set \( f_A \), denoted by \( Fcl^+(f_A) \), in a fuzzy soft ideal topological space \((X, \tau, E, I)\) defined as \( Fcl^+(f_A) = f_A \cup f_A^+ \). The fuzzy soft \( \star \)-interior of a fuzzy soft set \( f_A \), denoted by \( Fint^+(f_A) \), in a fuzzy soft ideal topological space \((X, \tau, E, I)\) is the largest fuzzy soft open set of \( f_A \) in \( \tau^+I \).

**Definition 2.22.**[6, 10] Let \( \mathit{FSS}(X)_E \) and \( \mathit{FSS}(Y)_K \) be families of fuzzy soft sets over \( X \) and \( Y \), respectively. Let \( u : X \rightarrow Y \) and \( p : E \rightarrow K \) be mappings. Then the map \( f_{pu} \) is called fuzzy soft mapping from \( \mathit{FSS}(X)_E \) to \( \mathit{FSS}(Y)_K \), denoted by \( f_{pu} : \mathit{FSS}(X)_E \rightarrow \mathit{FSS}(Y)_K \), such that:

(1) If \( g_B \in \mathit{FSS}(X)_E \) then the image of \( g_B \) under the fuzzy soft mapping \( f_{pu} \), denoted by \( f_{pu}(g_B) \), is a fuzzy soft set over \( Y \) defined by:

\[
f_{pu}(g_B)(y) = \begin{cases} \bigvee \{ (g_B(e))(x) : x \in u^{-1}(y) \} & \text{if } y \in p(E), y \in Y, \\ 0 & \text{otherwise} \end{cases}
\]

(2) If \( h_C \in \mathit{FSS}(Y)_K \), then the inverse image of \( h_C \) under the fuzzy soft mapping \( f_{pu} \), denoted by \( f_{pu}^{-1}(h_C) \), is a fuzzy soft set over \( X \) defined by:
Definition 2.23. [15] The fuzzy soft mapping \( f_{pu} \) is called surjective (respectively, injective, bijective) if \( p \) and \( u \) are surjective (respectively, injective, bijective), also \( f_{pu} \) is said to be constant if \( p \) and \( u \) are constant. 

Definition 2.24. [30] Let \((X, \tau, E)\) and \((Y, \sigma, K)\) be two fuzzy soft topological spaces and \( f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K\) be a fuzzy soft mapping. \( f_{pu} \) is called fuzzy soft continuous if \( f_{pu}^{-1}(h_C) \in \tau \) for every \( h_C \in \sigma \).

Definition 2.25. Let \((X, \tau, E)\) be a fuzzy soft topological space and \( f_A \in FSS(X)_E \).

1. \( f_A \) is called fuzzy soft semi-open set if \( f_A \subseteq Fcl(Fint(f_A)) \) [13, 14].
2. \( f_A \) is called fuzzy soft pre-open set if \( f_A \subseteq Fint(Fcl(f_A)) \) [1, 5].
3. \( f_A \) is called fuzzy soft \( \beta \)-open set if \( f_A \subseteq Fcl(Fint(Fcl(f_A))) \) [2, 19].
4. \( f_A \) is called fuzzy soft \( \alpha \)-open set if \( f_A \subseteq Fint(Fcl(Fint(f_A))) \) [3, 4].

The collection of all fuzzy soft semi-open (respectively, fuzzy soft pre-open, fuzzy soft \( \beta \)-open, fuzzy soft \( \alpha \)-open) sets will be denoted by \( FSSO(X)_E \) (respectively, \( FSP_O(X)_E, F\beta O(X)_E, F\alpha O(X)_E \)).

3. Subsets of fuzzy soft topological space via fuzzy soft ideal

In this section, we extend some spatial subsets of a fuzzy soft topological space \((X, \tau, E)\) via fuzzy soft ideal.

Definition 3.1. Let \((X, \tau, E, \overline{I})\) be a fuzzy soft ideal topological space and \( f_A \in FSS(X)_E \). Then, \( f_A \) is called fuzzy soft \( \overline{I} \)-open if \( f_A \subseteq Fint(Fcl(f_A)) \). A fuzzy soft set \( f_A \) is said to be fuzzy soft \( \overline{I} \)-closed if its complement is a fuzzy soft \( \overline{I} \)-open. We will denoted the set of all fuzzy soft \( \overline{I} \)-open sets (fuzzy soft \( \overline{I} \)-closed) by \( FSIO(X)_E \) (\( FSIC(X)_E \)).

Definition 3.2. Let \((X, \tau, E, \overline{I})\) be a fuzzy soft ideal topological space and \((X, \tau', E, \overline{I})\) be its fuzzy soft \( \ast \)-topological space. A mapping \( \gamma : FSS(X)_E \rightarrow FSS(X)_E \) is said to be an \( \gamma \)-operation on \( FSO(X)_E \) if \( f_A \subseteq \gamma(f_A) \) for every \( f_A \in FSO(X)_E \).

The collection of all fuzzy soft \( \gamma \)-open sets is denoted by \( FSO(\gamma)_E = \{ f_A : f_A \subseteq \gamma(f_A), f_A \in FSS(X)_E \} \). The complement of fuzzy soft \( \gamma \)-open set is called fuzzy soft \( \gamma \)-closed. i.e., \( FSC(\gamma)_E = \{ f_A : f_A \text{ is fuzzy soft } \gamma \text{-open} \} \).

Definition 3.3. Let \((X, \tau, E, \overline{I})\) be a fuzzy soft ideal topological space and \( f_A \in FSS(X)_E \). Different cases of \( \gamma \)-operations on \( FSS(X)_E \) are as follows:

1. If \( \gamma = Fint(Fcl') \), then \( \gamma \) is called fuzzy soft pre-\( \overline{I} \)-open operator. We will denoted the set of all fuzzy soft pre-\( \overline{I} \)-open sets by \( FSPIO(X)_E \) and the set of all fuzzy soft pre-\( \overline{I} \)-closed sets by \( FSPIC(X)_E \).
2. If \( \gamma = Fint(Fcl(Fint)) \), then \( \gamma \) is called fuzzy soft \( \alpha \)-\( \overline{I} \)-open operator. We will denoted the set of all fuzzy soft \( \alpha \)-\( \overline{I} \)-open sets by \( FSaIO(X)_E \) and the set of all fuzzy soft \( \alpha \)-\( \overline{I} \)-closed sets by \( FSaIC(X)_E \).
3. If \( \gamma = Fcl(Fint) \), then \( \gamma \) is called fuzzy soft semi-\( \overline{I} \)-open operator. We will denoted the set of all fuzzy soft semi-\( \overline{I} \)-open sets by \( FSIIIO(X)_E \) and the set of all fuzzy soft semi-\( \overline{I} \)-closed sets by \( FSIIIC(X)_E \).
4. If \( \gamma = Fcl(Fcl(Fcl')) \), then \( \gamma \) is called fuzzy soft \( \beta \)-\( \overline{I} \)-open operator. We will denoted the set of all fuzzy soft \( \beta \)-\( \overline{I} \)-open sets by \( FS\beta IO(X)_E \) and the set of all fuzzy soft \( \beta \)-\( \overline{I} \)-closed sets by \( FS\beta IC(X)_E \).

Theorem 3.1. Let \((X, \tau, E, \overline{I})\) be a fuzzy soft ideal topological space and \( \gamma : FSS(X)_E \rightarrow FSS(X)_E \) be an \( \gamma \)-operation on \( FSO(X)_E \). If \( \gamma = \{ Fint(Fcl'), Fint(Fcl(Fint)), Fcl(Fint(Fcl')) \} \), then:

1. \( \overline{I}_E \) and \( 0_E \) are fuzzy soft \( \gamma \)-open sets.
2. arbitrary union of fuzzy soft \( \gamma \)-open sets is a fuzzy soft \( \gamma \)-open.
3. arbitrary intersection of fuzzy soft \( \gamma \)-closed sets is a fuzzy soft \( \gamma \)-closed.

Proof.
1. Immediate.
We give the proof for the case of fuzzy soft pre-$\mathcal{T}$-open operator, i.e., $\gamma = \text{Fint}(\text{Fcl}')$. Let $\{f_{\lambda}; j \in J\} \subseteq \text{FSPIO}(X)_{E}$. Then, $\forall j \in J$, $(f_{\lambda})_{j} \subseteq \text{Fint}(\text{Fcl}')_{j}(f_{\lambda})$. It follows that $\bigcup_{j \in J} f_{\lambda} \subseteq \bigcup_{j \in J} \text{Fint}(\text{Fcl}')_{j}(f_{\lambda}) \subseteq \text{Fint}(\bigcup_{j \in J} f_{\lambda}) = \text{Fint}(\text{Fcl}'(\bigcup_{j \in J} f_{\lambda})).$ Hence, $(\bigcup_{j \in J} f_{\lambda}) \in \text{FSPIO}(X)_{E}. \text{ The other cases are similar.}$

(3) It follows directly by (2).

**Remark 3.1.** A finite intersection of fuzzy soft $\gamma$-open sets where $\gamma = \{\text{Fint}(\text{Fcl}'), \text{Fcl}'(\text{Fint}), \text{Fcl}(\text{Fint}(\text{Fcl}'))\}$ need not to be a fuzzy soft $\gamma$-open as shown by the following example:

**Example 3.1.**

(1) Let $X = [a, b], E = [e_{1}, e_{2}], \bar{I} = \{0_{E}\}$ and $\tau = \{1_{E}, 0_{E}, \{e_{1}, [0,0.6, 0.2]\}, \{e_{1}, [0,0.1, 0.2]\}\}$. Then, $f_{E} = \{\{e_{1}, [0,0.2, 0.5]\}\}$ and $g_{E} = \{\{e_{1}, [0,0.5, 0.2]\}\}$ are fuzzy soft pre-$\mathcal{T}$-open sets but $f_{E} \cap g_{E} \notin \text{FSPIO}(X)_{E}$. (2) Let $X = [a, b], E = [e_{1}, e_{2}], \bar{I} = \{0_{E}\}$ and $\tau = \{1_{E}, 0_{E}, \{e_{1}, [0,0.2]\}, \{e_{1}, [0,0.1, 0.2]\}\}$. Then, $f_{E} = \{\{e_{1}, [0,0.5, 0.2]\}\}$ and $g_{E} = \{\{e_{1}, [0,0.1, 0.2]\}\}$ are fuzzy soft semi-$\mathcal{T}$-open sets but $f_{E} \cap g_{E} \notin \text{FSPIO}(X)_{E}$. (3) Let $X = [a, b], E = [e_{1}, e_{2}], \bar{I} = \text{FSS}(X)_{E}$ and $\tau = \{1_{E}, 0_{E}, \{e_{1}, [0,0.2]\}, \{e_{1}, [0,0.1, 0.2]\}\}$. Then, $f_{E} = \{\{e_{1}, [0,0.2, 0.1]\}\}$ and $g_{E} = \{\{e_{1}, [0,0.2]\}\}$ are fuzzy soft $\beta$-$\mathcal{T}$-open sets but $f_{E} \cap g_{E} \notin \text{FSPIO}(X)_{E}$.

**Remark 3.2.** Note that the family of all fuzzy soft $\gamma$-open sets on a fuzzy soft ideal topological space $(X, \tau, E, I)$ forms a fuzzy soft supra topology, which is a collection of fuzzy soft sets contains $1_{E}, 0_{E}$ and closed under arbitrary union.

**Proposition 3.1.** Let $(X, \tau, E, I)$ be a fuzzy soft ideal topological space and $f_{A} \subseteq \text{FSS}(X)_{E}$. Then, we have:

(1) If $\bar{I} = \{0_{E}\}$, then $f_{A}$ is fuzzy soft pre-$\mathcal{T}$- (respectively, semi-$\mathcal{T}$-, $\alpha$-$\mathcal{T}$-, $\beta$-$\mathcal{T}$-) open $\iff f_{A}$ is fuzzy soft pre- (respectively, semi-, $\alpha$-, $\beta$-) open.

(2) If $\bar{I} = \text{FSS}(X)_{E}$, then $f_{A}$ is fuzzy soft pre-$\mathcal{T}$- (respectively, semi-$\mathcal{T}$-, $\alpha$-$\mathcal{T}$-, $\beta$-$\mathcal{T}$-) open $\iff f_{A}$ is fuzzy soft $\tau$-open.

**Proof.** As a sample we will prove the case of fuzzy soft $\alpha$-$\mathcal{T}$-open operator. i.e., $\gamma = \text{Fint}(\text{Fcl}')$.

(1) Let $\bar{I} = \{0_{E}\}$. Then, $f_{A} = \text{Fcl}(f_{A})$ and hence $\text{Fcl}'(f_{A}) = \text{Fcl}(f_{A})$ for every $f_{A} \subseteq \text{FSS}(X)_{E}$. Therefore, $\text{Fint}(\text{Fcl}'(f_{A})) = \text{Fint}(\text{Fcl}(f_{A}))$ and hence $\text{FSS}(X)_{E} = \tau_{A}$.

(2) Let $\bar{I} = \text{FSS}(X)_{E}$. Then, $f_{A} = \text{FSS}(X)_{E}$ and hence $\text{Fcl}'(f_{A}) = f_{A}$ for every $f_{A} \subseteq \text{FSS}(X)_{E}$. Therefore, $\text{Fint}(\text{Fcl}(f_{A})) = \text{Fint}(\text{Fcl}'(f_{A})) = \text{Fint}(f_{A})$ and hence $\tau = \text{FSS}(X)_{E}$.

The other cases are similar.

**Definition 3.3.** Let $(X, \tau, E, I)$ be a fuzzy soft ideal topological space and $f_{E} \subseteq \text{FSS}(X)_{E}$. Then:

(1) $x_{\gamma}^{E}$ is called $\gamma$-fuzzy soft interior point of $f_{E}$ if $\exists g_{E} \subseteq \text{FOS}(\gamma)_{E}$ such that $x_{\gamma}^{E} \in g_{E} \subseteq f_{E}$. The set of all $\gamma$-fuzzy soft interior points of $f_{E}$ is called the $\gamma$-fuzzy soft interior of $f_{E}$ and is denoted by $\gamma \text{Fint}(f_{E})$. Consequently, $\gamma \text{Fint}(f_{E}) = \bigcup \{g_{E}; g_{E} \subseteq \text{FOS}(\gamma)_{E}, g_{E} \subseteq f_{E}\}$. (2) $x_{\gamma}^{E}$ is called $\gamma$-fuzzy soft cluster point of $f_{E}$ if $f_{E} \cap g_{E} \neq \emptyset$ for every $g_{E} \subseteq \text{FOS}(\gamma)_{E}$. The set of all $\gamma$-fuzzy soft cluster points of $f_{E}$ is called the $\gamma$-fuzzy soft closure of $f_{E}$ and is denoted by $\gamma \text{Fcl}(f_{E})$. Consequently, $\gamma \text{Fcl}(f_{E}) = \bigcap \{g_{E}; g_{E} \subseteq \text{FSC}(\gamma)_{E}, f_{E} \subseteq g_{E}\}$.

**Theorem 3.2.** Let $(X, \tau, E, I)$ be a fuzzy soft ideal topological space, $\gamma : \text{FSS}(X)_{E} \rightarrow \text{FSS}(X)_{E}$ be one of the operations of Definition 3.3 and $f_{E}, g_{E} \subseteq \text{FSS}(X)_{E}$. Then, the following properties are satisfied for the $\gamma$-fuzzy soft interior operators denoted by $\gamma \text{Fint}(f_{E})$:

(1) $\gamma \text{Fint}(0_{E}) = 0_{E}$ and $\gamma \text{Fint}(1_{E}) = 1_{E}$.

(2) $\gamma \text{Fint}(f_{E}) \subseteq f_{E}$.

(3) $\gamma \text{Fint}(f_{E})$ is the largest fuzzy soft $\gamma$-open set contained in $f_{E}$.

(4) If $f_{E} \subseteq g_{E}$, then $\gamma \text{Fint}(f_{E}) \subseteq \gamma \text{Fint}(g_{E})$.

(5) $\gamma \text{Fint}(f_{E}) \cup \gamma \text{Fint}(g_{E}) \subseteq \gamma \text{Fint}(f_{E} \cup g_{E})$.

(6) $\gamma \text{Fint}(\gamma \text{Fint}(f_{E})) = \gamma \text{Fint}(f_{E})$.

(7) $\gamma \text{Fint}(f_{E} \cap g_{E}) \subseteq \gamma \text{Fint}(f_{E}) \cap \gamma \text{Fint}(g_{E})$.

**Proof.** Immediate.

**Theorem 3.3.** Let $(X, \tau, E, I)$ be a fuzzy soft ideal topological space, $\gamma : \text{FSS}(X)_{E} \rightarrow \text{FSS}(X)_{E}$ be one of the operations of Definition 3.3 and $f_{E}, g_{E} \subseteq \text{FSS}(X)_{E}$. Then, the following properties are satisfied for the
γ-fuzzy soft closure operators denoted by γFScI(f_e):
1. γFScI(0_E) = 0_E and γFScI(1_E) = 1_E.
2. f_e ⊆ γFScI(f_e).
3. γFScI(f_e) is the smallest fuzzy soft γ-closed set contains f_e.
4. If f_e ⊆ g_E, then γFScI(f_e) ⊆ γFScI(g_E).
5. γFScI(f_e) ∪ γFScI(g_E) ⊆ γFScI(f_e ∪ g_E).
6. γFScI(γFScI(f_e)) = γFScI(f_e).
7. γFScI(f_e ∩ g_E) ⊆ γFScI(f_e) ∩ γFScI(g_E).

Proof. Immediate.

4. Relations between subsets of fuzzy soft topological space via fuzzy soft ideal

Theorem 4.1. Let (X, τ, E, I) be a fuzzy soft ideal topological space. The following statements are hold:
1. Every fuzzy soft open set is fuzzy soft semi-I-open.
2. Every fuzzy soft semi-I-open set is fuzzy soft semi-open.
3. Every fuzzy soft pre-I-open set is fuzzy soft pre-open.

Proof.
1. Let A be a fuzzy soft open set. Then, Fint(A) = A. Therefore, A ⊆ Fcl*(Fint(A)) = Fcl*(A). Hence A is a fuzzy soft semi-I-open set.
2. Let A be a fuzzy soft semi-I-open set. Then, A ⊆ Fcl*(Fint(A)) = Fint(A) ∪ [Fint(A)]* ⊆ Fint(A) ∪ Fcl(Fint(A)) = Fcl(Fint(A)). Therefore, A is a fuzzy soft semi-open set.

The following example shows that the converse of Theorem 4.1 is not true in general.

Example 4.1. Let X = {a, b, c}, E = {e_1, e_2, e_3}, f_E = {(e_1, {a_0, b_0}, (e_2, {a_0, b_0, c_0})}, (e_3, {a_0, b_0, c_0})}, and g_E = {(e_1, {a_0, b_0, c_0})}. We put τ = {1_E, 0_E, f_E}. If we take I = {0_E}, then g_E is fuzzy soft semi-I-open set but g_E is fuzzy soft open, since g_E* = Fcl(g_E). If we take I = FSS(X)_E, then g_E is fuzzy soft semi-open set but is fuzzy soft semi-I-open, since g_E* = 0_E.

Theorem 4.2. In fuzzy soft ideal topological space (X, τ, E, I), the following statements hold:
1. Every fuzzy soft open set is a fuzzy soft pre-I-open set.
2. Every fuzzy soft pre-I-open set is a fuzzy soft pre-open set.
3. Every fuzzy soft I-open set is a fuzzy soft pre-I-open set.

Proof.
1. Let (X, τ, E, I) be a fuzzy soft ideal topological space and A ∈ τ. Then Fint(A) = A. Since A ⊆ Fcl*(Fint(A)), then Fint(A) ⊆ Fcl*(Fint(A)). Hence, A is a fuzzy soft pre-I-open set.
2. Let f_E ∈ FSPIO(X)_E. Then, f_E ⊆ Fint(Fcl*(f_E)). Since Fcl*(f_E) ⊆ Fcl(f_E), then f_E ⊆ Fint(Fcl(f_E)) and so f_E is a fuzzy soft pre-open set.
3. Let f_E ∈ FSIO(X)_E. Then, f_E ⊆ Fint(f_E). Since f_E ⊆ Fcl*(f_E), then f_E ⊆ Fint(Fcl*(f_E)) and so f_E ∈ FSPIO(X)_E.

The following example shows that the converse of Theorem 4.2 is not true in general.

Example 4.2. Let X = {a, b, c}, E = {e_1, e_2, ..., e_7} and τ = {1_E, 0_E, f_E = {(e_1, {a_0, b_0, c_0})}} be a fuzzy soft topology defined on X.
1. If E = {0_E} and g_E = {(e_1, {a_0, b_0, c_0})}, then g_E ∈ FSPIO(X)_E. But g_E ∉ τ.
2. If I = FSS(X)_E, then g_E = {(e_1, {a_0, b_0, c_0})} is a fuzzy soft pre-open set, but g_E ∉ FSPIO(X)_E.
3. If I = FSS(X)_E, then f_E ∈ FSPIO(X)_E. But f_E ∉ FSIO(X)_E.

Theorem 4.3. In a fuzzy soft ideal topological space (X, τ, E, I), the following statements hold:
1. Every fuzzy soft open set is fuzzy soft a-I-open.
2. Every fuzzy soft a-I-open set is fuzzy soft a-open.
3. Every fuzzy soft a-I-open set is fuzzy soft semi-I-open.
4. Every fuzzy soft semi-I-open set is fuzzy soft β-I-open.
(5) Every fuzzy soft $a$-$I$-open set is fuzzy soft pre-$I$-open.

(6) Every fuzzy soft pre-$I$-open set is fuzzy soft $\beta$-$I$-open.

**Proof.**

(1) Let $f_A \in \tau$. Then $\text{Fint}(f_A) = f_A$ and so $\text{Fcl}'(\text{Fint}(f_A)) = \text{Fcl}'(f_A) \supseteq f_A$. Therefore, $f_A = \text{Fint}(f_A) \subseteq \text{Fcl}(\text{Fcl}'(\text{Fint}(f_A)))$. Hence, $f_A \in \text{FSIO}_E$.

(2) Let $f_A \in \text{FSIO}_E$. Then, we have $f_A \subseteq \text{Fint}(\text{Fcl}'(\text{Fint}(f_A)))$. Since $\tau \subseteq \tau'$, then $\text{Fcl}'(\text{Fint}(f_A)) \subseteq \text{Fcl}(\text{Fint}(f_A))$. Therefore, $f_A \subseteq \text{Fcl}(\text{Fcl}'(\text{Fint}(f_A))) \subseteq \text{Fcl}(\text{Fint}(f_A)))$. This shows that $f_A$ is a fuzzy soft $a$-$I$-open set.

(3) Since $f_A \in \text{FSIO}_E$, then $f_A \subseteq \text{Fcl}(\text{Fcl}'(\text{Fint}(f_A))) \subseteq \text{Fcl}(\text{Fint}(f_A)))$. Hence, $f_A \in \text{FSSIO}_E$.

(4) Let $f_A \in \text{FSSIO}_E$. Then, $f_A \subseteq \text{Fcl}(\text{Fint}(f_A))$. Since $\tau \subseteq \tau'$, then $\text{Fcl}'(\text{Fint}(f_A)) \subseteq \text{Fcl}(\text{Fint}(f_A))$. Therefore, $f_A \subseteq \text{Fcl}(\text{Fint}(f_A)) \subseteq \text{Fcl}(\text{Fint}(f_A))$. Hence, $f_A \in \text{FSIO}_E$.

(5) Let $f_A \in \text{FSIO}_E$. Then, $f_A \subseteq \text{Fcl}(\text{Fcl}'(\text{Fint}(f_A))) \subseteq \text{Fcl}(\text{Fint}(f_A)))$. Hence, $f_A \in \text{FSIO}_E$.

(6) Let $f_A \in \text{FSIO}_E$. Then, $f_A \subseteq \text{Fint}(\text{Fcl}'(\text{Fint}(f_A))) \subseteq \text{Fcl}(\text{Fcl}(\text{Fint}(f_A)))$. Hence, $f_A \in \text{FSIO}_E$.

The following example shows that the converse of Theorem 4.3 is not true in general.

**Example 4.3.**

(1) Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{1_E, \bar{0}_E, \{(e_1, \{a_{u_01}, b_{0,1}, c_{u_01}\})\}, \{(e_1, \{a_{u_02}, b_{0,1}, c_{u_02}\})\}}$. The fuzzy soft set $f_{e_1} = \{(e_1, \{a_{u_01}, b_{0,1}, c_{u_01}\})\}$ is not fuzzy soft pre-$I$-open set but $f_{e_1}$ is fuzzy soft $a$-$I$-open set.

(2) The fuzzy soft semi-$I$-open set may not be fuzzy soft $a$-$I$-open. Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{\bar{1}_E, \bar{0}_E, \{(e_1, \{a_{u_01}, b_{0,1}, c_{u_01}\})\}}$. The fuzzy soft set $f_{\bar{1}_E} = \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}$ is fuzzy soft semi-$I$-open set but it is not fuzzy soft $a$-$I$-open.

(3) The fuzzy soft pre-$I$-open set may not be fuzzy soft $a$-$I$-open. Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{\bar{1}_E, \bar{0}_E, \{(e_1, \{a_{u_01}, b_{0,1}, c_{u_01}\})\}}$. The fuzzy soft set $f_{\bar{1}_E} = \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}$ is fuzzy soft pre-$I$-open set but it is not fuzzy soft $a$-$I$-open.

(4) Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{\bar{1}_E, \bar{0}_E, \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}}$. The fuzzy soft set $f_{\bar{1}_E} = \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}$ is not fuzzy soft pre-$I$-open set but $f_{\bar{1}_E}$ is fuzzy soft $a$-$I$-open set.

(5) Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{\bar{1}_E, \bar{0}_E, \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}}$. The fuzzy soft set $f_{\bar{1}_E} = \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}$ is not fuzzy soft pre-$I$-open set but $f_{\bar{1}_E}$ is fuzzy soft $a$-$I$-open set.

(6) Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{\bar{1}_E, \bar{0}_E, \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}}$. The fuzzy soft set $f_{\bar{1}_E} = \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}$ is not fuzzy soft pre-$I$-open set.

(7) Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{\bar{1}_E, \bar{0}_E, \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}}$. The fuzzy soft set $f_{\bar{1}_E} = \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}$ is a fuzzy soft semi-$I$-open set but $f_{e_1}$ is not fuzzy soft $a$-$I$-open.

(8) Let $X = \{a, b, c\}, E = \{e_1, e_2, \bar{e}_1 \in \{0, 1\}\}$ and $\tau = \{\bar{1}_E, \bar{0}_E, \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}}$. The fuzzy soft set $f_{\bar{1}_E} = \{(e_1, \{a_{u_03}, b_{0,6}, c_{u_01}\})\}$ is not fuzzy soft $a$-$I$-open.

**Lemma 4.1.** Let $(X, \tau, E, I)$ be a fuzzy soft ideal topological space and $f_A \in \text{FSSIO}_E$. Then, $f_A$ is a fuzzy soft $a$-$I$-open set if and only if $f_A$ is both fuzzy soft semi-$I$-open and fuzzy soft pre-$I$-open.

**Proof.**

**Necessity.** This is obvious by Theorem 4.3 (3, 5).

**Sufficiency.** Let $f_A$ be fuzzy soft semi-$I$-open and fuzzy soft pre-$I$-open set. Then, $f_A \subseteq \text{Fint}(\text{Fcl}'(f_A)) \subseteq \text{Fint}(\text{Fcl}'(f_A)))$. This shows that $f_A$ is a fuzzy soft $a$-$I$-open set.

**Theorem 4.4.** In any fuzzy soft ideal topological space $(X, \tau, E, I)$, the following properties are hold:

(1) Every fuzzy soft open set is a fuzzy soft $\beta$-$I$-open.

(2) Every fuzzy soft $\beta$-$I$-open set is a fuzzy soft $\beta$-$I$-open.

(3) Every fuzzy soft pre-$I$-open set is a fuzzy soft $\beta$-$I$-open.

(4) Every fuzzy soft semi-$I$-open set is a fuzzy soft $\beta$-$I$-open.

(5) Every fuzzy soft $I$-open set is a fuzzy soft $\beta$-$I$-open.

**Proof.**

(1) Let $f_A \in \tau$. Then, we have $f_A = \text{Fint}(f_A) \subseteq \text{Fcl}(\text{Fint}(f_A)) \subseteq \text{Fcl}(\text{Fint}(f_A \cup f_A')) = \text{Fcl}(\text{Fint}(f_A)))$. This
shows that $f_A \in \text{FS} \beta \text{IO}(X)_E$.

(2) Let $f_A \in \text{FS} \beta \text{IO}(X)_E$. Then, $f_A \subseteq \text{Fcl}(\text{Fint}(\text{Fcl}^*(f_A))) = \text{Fcl}(\text{Fint} (f_A \sqcup f_A^*)) \subseteq \text{Fcl}(\text{Fint}(f_A \sqcup \text{Fcl}(f_A))) = \text{Fcl}(\text{Fint}(\text{Fcl}(f_A)))$. This shows that $f_A$ is a fuzzy soft $\beta$-open.

(3) Let $f_A \in \text{FS} \beta \text{IO}(X)_E$. Then, $f_A \subseteq \text{Fcl}(\text{Fint}(\text{Fcl}^*(f_A))) \subseteq \text{Fcl}(\text{Fint}(\text{Fcl}^*(f_A)))$. This shows that $f_A \in \text{FS} \beta \text{IO}(X)_E$.

(4) Let $f_A \in \text{FS} \beta \text{IO}(X)_E$. Then, $f_A \subseteq \text{Fcl}(\text{Fint}(f_A)) \subseteq \text{Fcl}(\text{Fint}(f_A \sqcup f_A^*)) = \text{Fcl}(\text{Fint}(\text{Fcl}(f_A)))$. This shows that $f_A \in \text{FS} \beta \text{IO}(X)_E$.

(5) Let $f_A \in \text{FS} \beta \text{IO}(X)_E$. Then, $f_A \subseteq \text{Fcl}(f_A) \subseteq \text{Fcl}(\text{Fint}(f_A^*)) \subseteq \text{Fcl}(\text{Fint}(f_A \sqcup f_A^*)) = \text{Fcl}(\text{Fint}(\text{Fcl}(f_A)))$. This shows that $f_A \in \text{FS} \beta \text{IO}(X)_E$.

The following example shows that the converses of Theorem 4.4 is not true in general.

**Example 4.4.** Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $T = \{\emptyset\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_0.5, b_0.3, c_0.3\})\}$ is a fuzzy soft $\beta$-open set in $(X, \tau, T)$ but $f_E \notin \tau$.

(2) Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $s_E = \{(e_1, \{a_0.5, b_0.7, c_0.3\})\}$ and $u_E = \{(e_1, \{a_0.8, b_0.2, c_0.8\})\}$ are fuzzy soft sets in $X$. We put $\tau = \{\emptyset, e_1, e_2, u_E, s_E \land u_E, s_E \land u_E\}$ and we find that:

(i) If we take $I = \text{SS}(X)_E$, then $f_E = \{(e_2, \{a_0.2, b_0.3, c_0.4\})\}$ is a fuzzy soft $\beta$-open set but $f_E \notin \text{FS} \beta \text{IO}(X)_E$.

(ii) If we take $I = \{\emptyset\}$, then $f_E = \{(e_1, \{a_0.5, b_0.3, c_0.3\})\}$ is a fuzzy soft $\beta$-open set in $X$ but $f_E \notin \text{FS} \beta \text{IO}(X)_E$.

(iii) If we take $I = \{\emptyset\}$, then $f_E = \{(e_1, \{a_0.1, b_0.3, c_0.8\})\}$ is a fuzzy soft $\beta$-open set in $X$ but $f_E \notin \text{FS} \beta \text{IO}(X)_E$.

**Proposition 4.1.** Figure 1 shows the relation between different types of fuzzy soft open subsets of fuzzy soft topological space via fuzzy soft ideal.

5. Decompositions of some types of mappings on fuzzy soft topological spaces via fuzzy soft ideals

In this section, we introduce some types of continuity of fuzzy soft topological space with fuzzy soft ideal and study the relations between them.

**Definition 5.1.** Let $(X, \tau, E, T)$ be a fuzzy soft ideal topological space and $(Y, s, K)$ be a fuzzy soft topological space. Let $u : X \rightarrow Y$ and $p : E \rightarrow K$ be mappings. Let

$$f_{pu} : (X, \tau, E, T) \rightarrow (Y, s, K)$$

be a function. Then,

(1) $f_{pu}$ is called fuzzy soft $\tilde{T}$-continuous function if $f_{pu}^{-1}(g_B) \in \text{FSIO}(X)_E$ for every $g_B \in \sigma$.

(2) $f_{pu}$ is called fuzzy soft pre-$\tilde{T}$-continuous function if $f_{pu}^{-1}(g_B) \in \text{FPIO}(X)_E$ for every $g_B \in \sigma$.

(3) $f_{pu}$ is called fuzzy soft semi-$\tilde{T}$-continuous function if $f_{pu}^{-1}(g_B) \in \text{FSSIO}(X)_E$ for every $g_B \in \sigma$.

(4) $f_{pu}$ is called fuzzy soft $\alpha$-$\tilde{T}$-continuous function if $f_{pu}^{-1}(g_B) \in \text{FSaIO}(X)_E$ for every $g_B \in \sigma$.

(5) $f_{pu}$ is called fuzzy soft $\beta$-$\tilde{T}$-continuous function if $f_{pu}^{-1}(g_B) \in \text{FS} \beta \text{IO}(X)_E$ for every $g_B \in \sigma$.

**Theorem 5.1.** Let $(X, \tau, E, T)$ be a fuzzy soft ideal topological space and $(Y, s, K)$ be a fuzzy soft topological space. Let $u : X \rightarrow Y$ and $p : E \rightarrow K$ be mappings. Let

$$f_{pu} : (X, \tau, E, T) \rightarrow (Y, s, K)$$

be a function. Then every fuzzy soft $\tilde{T}$-continuous function is a fuzzy soft pre-$\tilde{T}$-continuous function.

**Proof.** This is an immediate consequence of Theorem 4.2 (3).
Theorem 5.2. Let \((X, \tau, E, \tilde{I})\) be a fuzzy soft ideal topological space and \((Y, \sigma, K)\) be a fuzzy soft topological space. Let \(u : X \to Y\) and \(p : E \to K\) be mappings. Let
\[
f_{pu} : (X, \tau, E, \tilde{I}) \to (Y, \sigma, K)
\]
be a function. Then every fuzzy soft pre-\(\tilde{I}\)-continuous function is fuzzy soft pre-continuous.

**Proof.** This is an immediate consequence of Theorem 4.2 (2).

Theorem 5.3. Let \((X, \tau, E, \tilde{I})\) be a fuzzy soft ideal topological space and \((Y, \sigma, K)\) be a fuzzy soft topological space. Let \(u : X \to Y\) and \(p : E \to K\) be mappings. A fuzzy soft function \(f_{pu} : (X, \tau, E, \tilde{I}) \to (Y, \sigma, K)\) is a fuzzy soft \(\alpha\tilde{I}\)-continuous function \iff it is fuzzy soft semi-\(\tilde{I}\)-continuous and fuzzy soft pre-\(\tilde{I}\)-continuous.

**Proof.** This is an immediate consequence of Lemma 4.1.

Theorem 5.4. Let \((X, \tau, E, \tilde{I})\) be a fuzzy soft ideal topological space and \((Y, \sigma, K)\) be a fuzzy soft topological space. Let \(u : X \to Y\) and \(p : E \to K\) be mappings. Let
\[
f_{pu} : (X, \tau, E, \tilde{I}) \to (Y, \sigma, K)
\]
be a function. Then:
(1) every fuzzy soft \(\alpha\tilde{I}\)-continuous function is a fuzzy soft semi-\(\tilde{I}\)-continuous.
(2) every fuzzy soft \(\alpha\tilde{I}\)-continuous function is a fuzzy soft pre-\(\tilde{I}\)-continuous.

**Proof.** This is an immediate consequence of Theorem 4.3 (3, 5).

Theorem 5.5. Let \((X, \tau, E, \tilde{I})\) be a fuzzy soft ideal topological space and \((Y, \sigma, K)\) be a fuzzy soft topological space. Let \(u : X \to Y\) and \(p : E \to K\) be mappings. Let
\[
f_{pu} : (X, \tau, E, \tilde{I}) \to (Y, \sigma, K)
\]
be a function. Then:
(1) every fuzzy soft pre-\(\tilde{I}\)-continuous function is a fuzzy soft \(\beta\tilde{I}\)-continuous.
(2) every fuzzy soft semi-\(\tilde{I}\)-continuous function is a fuzzy soft \(\beta\tilde{I}\)-continuous.

**Proof.** This is an immediate consequence of Theorem 4.3 (4, 6).

Proposition 5.1. Let \((X, \tau, E, \tilde{I})\) be a fuzzy soft ideal topological space and \((Y, \sigma, K)\) be a fuzzy soft topological space. Let \(u : X \to Y\) and \(p : E \to K\) be mappings. Let
\[
f_{pu} : (X, \tau, E, \tilde{I}) \to (Y, \sigma, K)
\]
be a fuzzy soft function. Then, the following properties hold:
(1) If \(\tilde{I} = [0_\tilde{I}]\), then \(f_{pu}\) is a fuzzy soft semi-\(\tilde{I}\)-continuous (respectively, fuzzy soft pre-\(\tilde{I}\)-continuous, fuzzy soft \(\beta\tilde{I}\)-continuous, fuzzy soft \(\alpha\tilde{I}\)-continuous) function \iff \(f_{pu}\) is a fuzzy soft semi-continuous (respectively, fuzzy soft pre-continuous, fuzzy soft \(\beta\)-continuous, fuzzy soft \(\alpha\)-continuous).

(2) If \(\tilde{I} = FSS(X)_E\), then \(f_{pu}\) is a fuzzy soft semi-\(\tilde{I}\)-continuous (respectively, fuzzy soft pre-\(\tilde{I}\)-continuous, fuzzy soft \(\tilde{I}\)-continuous, fuzzy soft \(\alpha\tilde{I}\)-continuous) function \iff \(f_{pu}\) is a fuzzy soft continuous.

**Proof.** This proof is obviously by using Proposition 3.1.

Proposition 5.2. The following diagram shows the decompositions of fuzzy soft mappings on fuzzy soft topological spaces via fuzzy soft ideals.

\[
\begin{array}{c}
\text{Fuzzy soft }\alpha\tilde{I}-\text{cts} \\
\downarrow \\
\text{Fuzzy soft semi-}\tilde{I}-\text{cts} \\
\downarrow \\
\text{Fuzzy soft }\beta\tilde{I}-\text{cts} \\
\end{array}
\quad 
\begin{array}{c}
\text{Fuzzy soft pre-}\tilde{I}-\text{cts} \\
\downarrow \\
\text{Fuzzy soft pre-cts} \\
\end{array}
\quad 
\begin{array}{c}
\text{Fuzzy soft }\beta\text{-cts} \\
\end{array}
\]

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References

[1] A. M. Abd El-Latif, Characterizations of Fuzzy Soft Pre Separation Axioms, Journal of New Theory 7 (2015) 47-63.
[2] A. M. Abd El-Latif, Fuzzy soft separation axioms based on fuzzy $\beta$-open soft sets, Annals of Fuzzy Mathematics and Informatics 11 (2) (2016) 221-237.
[3] A. M. Abd El-Latif, Fuzzy Soft $\alpha$-Connectedness in Fuzzy Soft Topological Spaces, Math. Sci. Lett. 5 (1) (2016) 85-91.
[4] A. M. Abd El-Latif and Rodyna A. Hosny, On Soft Separation Axioms via Fuzzy $\alpha$-Open Soft Sets, Inf. Sci. Lett. 5 (1) (2016) 1-9.
[5] A. M. Abd El-Latif and Rodyna A. Hosny, Fuzzy soft pre-connected properties in fuzzy soft topological spaces, South Asian Journal of Mathematics 5 (3) (2015) 202-213.
[6] B. Ahmad and A. Kharal, Mappings of fuzzy soft classes, Adv. Fuzzy Syst., (2009), Art. ID 407980, 6 pp.
[7] H. Aktas and N. Çaugman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726-2735.
[8] C. L. Chang, Fuzzy topological spaces, J. Math. Appl. 24 (1968) 182-193.
[9] S.A. El-Sheikh, Decompositions of Some Types of Soft Sets and Soft Continuity via Soft Ideals, Gen. Math. Notes 24 (2) (2014) 103-124.
[10] P. K. Gain, P. Mukherjee, and R. P. Chakraborty, On Some Decompositions of Fuzzy Soft Continuity, Journal of New Theory 4 (2015) 39-52.
[11] E. Hatir and T. Noiri, On Decompositions of Continuity Via Idealization, Acta Math. Hungar 96 (4) (2002) 341-349.
[12] D. Jankovic and T.R. Hamlett, New topologies from old via ideals, Am Math Monthly 97 (1990) 295-310.
[13] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Fuzzy Soft Semi Connected Properties in Fuzzy Soft Topological Spaces, Mathematical Sciences Letters 2 (2015) 171-179.
[14] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Fuzzy semi open soft sets related properties in fuzzy soft topological spaces, Journal of mathematics and computer science 13 (2014) 94-114.
[15] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Some fuzzy soft topological properties based on fuzzy semi open soft sets, South Asian Journal of Mathematics 4 (4) (2014) 154-169.
[16] A. Kandil, O.A. El Tantawy, S.A. El-Sheikh, and A.M. Abd El-latif, Soft Ideal Theory Soft Local Function and Generated Soft Topological Spaces, Appl. Math. Inf. Sci. 8 (4) (2014) 1595-1603.
[17] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Ideal Theory: Fuzzy Soft Local Function and Generated Fuzzy Soft Topological Spaces, The Journal of Fuzzy Mathematics 25 (2) (2017) 327-342.
[18] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Ideal topological spaces, South Asian Journal of Mathematics 6(4) (2016) 186-198.
[19] A. Kandil, O.A. El Tantawy, S.A. El-Sheikh, A. M. Abd El-latif and S. El-Sayed, Fuzzy soft connectedness based on fuzzy $\beta$-open soft sets, Journal of Mathematics and Computer Science 2 (1) (2015) 37-46.
[20] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Connected Sets in Fuzzy Soft Topological Spaces I, Journal of Advances in Mathematics 12 (8) (2016) 6473-6488.
[21] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Connected Sets in Fuzzy Soft Topological Spaces II, Journal of Egyptian Mathematical Society 25 (2017) 171-177.
[22] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Hyperconnected spaces, Annals of fuzzy mathematics and Informatics 13 (6) (2017) 689-702.
[23] A. Kandil, O.A. El Tantawy, S.A. El-Sheikh, and A.M. Abd El-latif, $\gamma$-operation and decomposition of some forms of soft continuity of soft topological spaces via soft ideals, Annals of Fuzzy Mathematics and Informatics 7 (2) (2014) 181-196.
[24] K. Kuratowski, Topology, vol. I. New York, Academic Press (1966), transl.
[25] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J Math Anal Appl. 56 (1976) 621-33.
[26] J. Mahanta and P.K. Das, Results on fuzzy soft topological spaces, arXiv:1203.0634v1 (2012).
[27] P. K. Maji, R. Biswas and A.R. Roy, Soft Set Theory, Computers Math. Appl. 45 (2003) 555-562.
[28] P.K. Maji, R. Biewas, and A.R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (3) (2001), 589-602.
[29] D. Molodtsov, Soft set theory-First results, Computers Math. Appl., 37(4-5) (1999), 19-31.
[30] B. Pazar Varol and H. Aygün, Fuzzy soft topology, Facettepe Journal of Mathematics and Statistics 5 (1) (2013) 87-96.
[31] V. Renuka Devi and D. Sivaraj, A Decomposition of Continuity via Ideals, Acta Math. Hungar 118 (1-2) (2008) 53-59.
[32] S. Roy and T.K. Samanta, An introduction to open and closed sets on fuzzy soft topological spaces, Annals of Fuzzy Mathematics and Informatics 6 (2) (2013) 425-431.
[33] M. Shabir and M. Naz, On Soft Topological Spaces, Computers and Mathematics with Applications 61 (2011) 1786-1799.
[34] B. Tanay and M.B. Kandemir, Topological structures of fuzzy soft sets, Computers and Mathematics with Applications 61 (2011) 412-418.
[35] Tridiv Jyotineog and Dusmanta Kumar Sut, Some new operations of fuzzy soft sets, J. Math. Comput. Sci. 2 (5) (2012) 1186-1199.
[36] Tugbahan Simsekler and Saziye Yuksel, Fuzzy Soft Topological spaces, Ann. Fuzzy Math. Inform 5 (1) (2013) 87-96.
[37] R. Vaidyanathaswamy, Set topology. New York: Chelsa (1960).
[38] R. Vaidyanathaswamy, The localization theory in set topology, Proc Indian Acad Sci 20 (1945) 51-61.
[39] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.
[40] A.M. Zahran, S.E. Abbas, S.A. Abd El-baki and Y.M. Saber, Decomposition of fuzzy continuity and fuzzy ideal continuity via fuzzy idealization, Chaos, Solitons and Fractals 42 (2009) 3064-3077.