Equivalence between spectral properties of graphs with and without loops

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Abstract

In this paper we introduce a spectra preserving relation between graphs with loops and graphs without loops. This relation is achieved in two steps. First, by generalizing spectra results got on \((m, k)\)-stars to a wider class of graphs, the \((m, k, s)\)-stars with or without loops. Second, by defining a covering space of graphs with loops that allows to remove the presence of loops by increasing the graph dimension. The equivalence of the two class of graphs allows to study graph with loops as simple graph without loosing information.

Keywords: Graph loop, Multigraph, Graph reduction, Laplacian spectra, Laplacian matrices, Covering graph

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1. Introduction

In graph theory there are many areas of social networks and biological networks, where multigraph, or multivariate networks, arise more naturally than simple graphs [1, 2, 3]. Beyond the appearance in many natural contexts, these multigraph structures also emerge from several kinds of aggregation, scaling and blocking, [4, 5, 6].

A common procedures and extensively treated approach in network analysis considers \textit{simple graphs}. These are defined by binary relations, i.e. only single edges between two different vertices are allowed, and the possibility for self relations is excluded. A \textit{loop} is the most common mathematical object for representing self relations where a vertex is both the sender and receiver of an edge, [7, 8, 9].

In practice, on the one hand, simple graphs can be derived from multigraphs

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by collapsing multiple edges into a single one and removing the loops \([6, 10]\). These approaches discards inherent information in the original network. On the other hand, methods properly working for simple graphs does not guarantee to preserve the complete information of the original multigraph; for example, methods based on the Laplacian matrix properties are useless to provide more information since the Laplacian matrix is invariant with respect presence of loops.

As matter of principle, all the information of a multigraph are retained within the multiple and self relations of vertices. However, this multistructure has not been treated in literature as extensively as well as the simple graphs.

In this manuscript, we propose a suitable method to handle graphs with loops as simple graphs keeping the same eigenvalues and as much as possible the eigenvectors of their adjacency and transition matrices. One of the result of the paper is the possibility to associate a Laplacian matrix to a graph with loops that allows to study its topological properties.

Because eigenvalues and eigenvectors describe completly the matrix, by preserving the adjacency (or transition) matrix spectra (eigenvalues and eigenvectors) we maintain the informations and properties of the graphs as much as possible. In this way, graphs with loops can be dealt with tools extensively treated for simple graphs.

To define the corresponce between graphs and simple graphs we introduce an extension of the structure and the results discussed in the paper \([11]\); then we succeed to build a correspondence between two classes of subgraphs, namely the \((m, k, s)-\text{star with loops}\) and the \((m, k, s)-\text{star}\) (without loops).

The paper is organized as follows: after some preliminary remarks (section 2), in section 3 we generalize the class of \((m, k)-\text{star}\) in graphs to a wider class of graph, the \((m, k, s)-\text{star}\) with or without loop, in order to extend the results obtained in \([11]\). By giving conditions on the graph structure which implies the presence of multiple eigenvalues. Then we get a connection between eigenvector and eigenvalue of graph with loop and graph without loop. In particular, in a graph, each vertex with loop can be described as an \((1, k, -)-\text{star}\) with loop: we will give a useful tool to replace the looped vertex with an \((2, k, s)-\text{star}\) without loop by maintaining the same spectra.

Thanks to these results it is possible to describe a graph with loop by a graph without loop and to define the Laplacian matrix of the correspondence graph. Finally, in section 4 we draw some conclusions and discuss an outlooks on future developments.

### 2. Preliminar definitions

We consider an undirected weighted connected graph \(G := (\mathcal{V}, \mathcal{E}, w)\), where the \(n\) vertices \(\mathcal{V}\) are connected by the \(\mathcal{E}\) edges with \(w\) the weight function: \(w : \mathcal{E} \rightarrow \mathbb{R}^+\). Let \(A\) be the weighted adjacency matrix, which is symmetric since
the graph is undirected \((A \in Sym_n(\mathbb{R}^+))\),

\[
A_{ij} = \begin{cases} 
  w(i, j), & \text{if } i \text{ is connected to } j \ (i \sim j) \\
  0 & \text{otherwise}
\end{cases}
\]

where \(i, j \in \mathcal{V}\).

Because of the graph is not necessarily simple any diagonal element of \(A\) could be positive.

If the graph \(G\) is simple, i.e. it is an undirected graph in which both multiple edges and loops are disallowed, then we can also define the Laplacian matrix \(L \in Sym_n(\mathbb{R})\) and normalized Laplacian matrix \(\mathcal{L} \in Sym_n(\mathbb{R})\), by means of the strength diagonal matrix \(D\) that is

\[
D_{ij} = \begin{cases} 
  \sum_{k=1}^{n} w(i, k), & \text{if } i = j \\
  0 & \text{otherwise}
\end{cases}
\]

and the Laplacian and normalized Laplacian are respectively defined

\[
L := D - A, \quad \mathcal{L} := D^{-1/2}(D - A)D^{-1/2}.
\]

Whenever we refer to the \(k\)-th eigenvalue of a Laplacian matrix, we will refer to the \(k\)-th nonzero eigenvalue according to a increasing order.

Furthermore we observe that by defining the transition matrix \(T\) as \(T := D^{-1}A\) we can link the spectra of \(T\) and the spectra of \(\mathcal{L}\). This relation will be very useful later in order to link the Laplacian of graph with loop and Laplacian of graph without loop.

First of all we observe that \(T\) is similar to \(\tilde{A} := D^{-1/2}AD^{-1/2}\) by means of the invertible matrix \(D^{1/2}\) :

\[
D^{-1/2}\tilde{A}D^{1/2} = D^{-1/2}D^{-1/2}AD^{-1/2}D^{1/2} = D^{-1}A = T.
\]

So that \(\sigma(T) = \sigma(\tilde{A})\), and it is easy to prove that the following statements are equivalent

**S.1** \(v\) is an eigenvector of \(\tilde{A}\) with eigenvalue \(\lambda\)

**S.2** \(v^TD^{1/2}\) is a left eigenvector of \(T\) with the eigenvalue \(\lambda\)

**S.3** \(D^{-1/2}v\) is a right eigenvector of \(T\) with eigenvalue \(\lambda\)

Thus, linking the spectra of \(T\) and the spectra of \(\mathcal{L}\) is equivalent to link the spectra of \(\tilde{A}\) and the spectra of \(\mathcal{L}\).

Then one can easily prove that the following statements are equivalent

**S.1** \(v\) is an eigenvector of \(\tilde{A}\) with eigenvalue \(\lambda\)

**S.4** \(v\) is an eigenvector of \(\mathcal{L}\) with the eigenvalue \(1 - \lambda\).

For the classical results on Laplacian matrices theory, one may refer to [9, 12, 13, 14, 15].
3. \((m, k, s)\)-star with and without loop

In the present section, we extend the results of the Theorem (3.1) of the paper [11]: by defining a wider class of weighted \((m, k)\)-stars in graph we are able to generalize the results obtained on multiple eigenvalues on Laplacian matrices, transition and adjacency matrices. For simplicity we call this class the \textit{weighted} \((m, k, s)\)-\textit{stars}.

A second result concerns the problem of \textit{weighted} \((m, k, s)\)-\textit{stars} with loops and its multiple eigenvalues.

Then it is possible to introduce a correspondence between the two classes of graph: the third result of this section, that is the main result of this work, concerns the possibility of removing loops from a \textit{weighted} \((m, k, s)\)-\textit{stars with loops in graph} through a replacing of \textit{weighted} \((m, k, s)\)-\textit{stars in the graph}. In particular a corollary of this result will be useful in order to remove any loop from the graph by increasing the dimension of graph, where the growth of the dimension is the number of loops at most.

3.1. \((m, k, s)\)-star: eigenvalues multiplicity

Let us initially focus on graphs without loop. We recall that a \((m, k)\)-star is a graph \(G = (V, E, w)\) whose vertex set \(V\) is bi-partite in two subsets \((V_1, V_2)\) of cardinalities \(m\) and \(k\) respectively, such that the vertices in \(V_1\) have no connections among them, and each of these vertices is connected with all the vertices in \(V_2\): i.e

\[
\forall i \in V_1, \forall j \in V_2, \ (i, j) \in E \\
\forall i, j \in V_1, \ (i, j) \notin E.
\]

By \(S_{m,k}\) we denote a \((m,k)\)-star graph bi-partite in two sets of cardinality \(|V_1| = m\) and \(|V_2| = k\).

In Fig.3.1 are represented an example of \((m,k,1)\)-star graph and \((m,k,0)\)-star graph.

We define a \((m, k, s)\)-\textit{star of a graph} \(G = (V, E, w)\) as the \((m, k, s)\)-\textit{star of partitions} \(V_1, V_2 \subset V\) such that only the vertices in \(V_2\) can be connected with
the rest of the graph $V \setminus (V_1 \cup V_2)$: i.e.

$$\forall i \in V_1, \forall j \in V_2, \quad (i, j) \in E$$

if $s = 0$ then $\forall i \in V_1, \forall j \in V \setminus V_2, \quad (i, j) \notin E$,

if $s = 1$ then $\forall i \in V_1, \forall j \in V \setminus (V_1 \cup V_2), \quad (i, j) \notin E$

and $\forall i, j \in V_1, i \neq j, \quad (i, j) \in E$.

By defining the concepts of degree, weight and central weight of a $(m, k, s)$-star we simplify the statement of the theorems on eigenvalues multiplicity.

![Diagram](image-url)

**Figure 1:** In the left picture we show a $S_{3,4,0}$ graph and in the right picture we show a $S_{3,4,1}$ graph. The subsets of vertices $V_1$ and $V_2$ are respectively colored in yellow and blue. Grey-edges are the edges between vertices belonging to different sets, yellow-edges are the edges between vertices in $V_1$ and blue-edges are the edges between vertices in $V_2$.

**Definition 3.2** (Degree of a $(m, k, s)$-star: $\deg(S_{m,k,s})$). The degree of a $(m, k, s)$-star is $\deg(S_{m,k,s}) := m - 1$ and the degree of a set $S$ of $(m, k, s)$-stars, as $m$ and $k$ vary in $\mathbb{N}$, such that $|S| = l$, is defined as the sum over each $(m, k, s)$-star degree, i.e.

$$\deg(S) := \sum_{i=1}^{l} \deg(S_{m_i,k_i,s_i}).$$

**Definition 3.3** (Weight of a $(m, k, s)$-star: $w(S_{m,k,s})$). The weight of a $(m, k, s)$-star of vertices set $V_1 \cup V_2$ is defined when the following condition holds: let $\{i_1, ..., i_m\} = V_1$, and $w(i_1, j) = ... = w(i_m, j), \forall j \in V_2$ whereas all the vertices in $V_1$ are connected to each other by links with the same weight, $w(i_p, i_1) = ... = w(i_p, i_{p-1}) = w(i_p, i_{p+1}) = ... = w(i_p, i_m), \forall i_p \in V_1$, then we denote the weight of a $(m, k, s)$-star by $w(S_{m,k,s})$:

$$w(S_{m,k,s}) := \sum_{j \in V} w(i, j)$$

for any $i \in V_1$. 

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**Definition 3.4** (Central weight of a \((m,k,s)\)-star: \(w_c(S_{m,k,s})\)). The central weight of a \((m,k,s)\)-star of vertices set \(\mathcal{V}_1 \cup \mathcal{V}_2\) is defined when the following condition holds: let \(\{i_1, ..., i_m\} = \mathcal{V}_1\), and \(w(i_1,j) = ... = w(i_m,j), \forall j \in \mathcal{V}_2\) whereas all the vertices in \(\mathcal{V}_1\) are connected to each other by links with the same weight, \(w(i_p,i_1) = ... = w(i_p,i_p-1) = w(i_p,i_p+1) = ... = w(i_p,i_m), \forall i_p \in \mathcal{V}_1\), then we denote the central weight of a \((m,k,s)\)-star by \(w_c(S_{m,k,s})\):

\[
w_c(S_{m,k,s}) := w(i, i) + w(i, \tilde{i})\]

for any \(i, \tilde{i} \in \mathcal{V}_1, i \neq \tilde{i}\).

In the previous definition the weight of loop, \(w(i, i)\), is clearly set to zero and it plays no role in the present section, but it will be useful in the next sections.

We are ready to enunciate the first theorem, that is an extension to \((m,k,s)\)-stars of the Theorem in [11]. Given a graph \(\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)\) associated with the Laplacian matrix \(L\), and denoting \(\sigma(L)\) the set of the eigenvalues of \(L\) and \(m_L(\lambda)\) the algebraic multiplicity of the eigenvalue \(\lambda\) in \(L\), the following theorem holds.

**Theorem 3.1.** Let

- \(p\) be the number of all the \(S_{m,k,s}\) as \(m\) and \(k\) vary in \(\mathbb{N}\) and \(m + k \leq n\), of \(\mathcal{G}\);
- \(r\) be the number of \(S_{m,k,s}\) with different weight, \(w_1, ..., w_r\), i.e. \(w_i \neq w_j\) for each \(i \neq j\), where \(i, j \in \{1, ..., r\}\);

then for any \(i \in \{1, ..., r\}\),

\[\exists \lambda \in \sigma(L) \text{ such that } \lambda = w_i \text{ and } m_L(\lambda) \geq \deg(S_{w_i})\]

where \(S_{w_i} := \{S_{m,k,s} \in \mathcal{G} | w_c(S_{m,k,s}) + w(S_{m,k,s}) = w_i\}\).

In order to prove the previous theorem we have need of the following Lemma on the weighted adjacency matrix \(A\), where \(\sigma(A)\) is the spectrum of \(A\) and \(m_A(\lambda)\) the algebraic multiplicity of \(\lambda\).

**Lemma 3.2.** Let

- \(p\) be the number of all the \(S_{m,k,s}\) as \(m\) and \(k\) vary in \(\mathbb{N}\) and \(m + k \leq n\), of \(\mathcal{G}\);
- \(r\) be the number of \(S_{m,k,s}\) with different weight, \(w_1, ..., w_r\), i.e. \(w_i \neq w_j\) for each \(i \neq j\), where \(i, j \in \{1, ..., r\}\);

then for any \(i \in \{1, ..., r\}\),

\[\exists \lambda \in \sigma(A) \text{ such that } \lambda = -w_i \text{ and } m_A(\lambda) \geq \deg(S_{w_i})\]

where \(S_{w_i} := \{S_{m,k,s} \in \mathcal{G} | w_c(S_{m,k,s}) = w_i\}\).
Proof. Without loss of generality we consider only connected graphs; indeed if a graph is not connected the same result holds, since the \((m,k,s)\)-star degree of the graph is the sum of the star degrees of the connected components and the characteristic polynomial of \(L\) is the product of the characteristic polynomials of the connected components.

Let a \((m,k,s)\)-star of the graph \(G\). Under a suitable permutation of the rows and columns of weighted adjacency matrix \(A\), we can label the vertices in \(\mathcal{V}_1\) with the indices \(1,\ldots,m\), and the vertices in \(\mathcal{V}_2\) with the indeces \(m+1,\ldots,m+k\).

Let \(v_1(A),\ldots,v_m(A)\) be the rows corresponding to vertices in \(\mathcal{V}_1\), then the adjacency matrix has the following form

\[
A = \begin{pmatrix}
0 & w(1,2) & \cdots & w(1,m) & w(1,m+1) & \cdots & w(1,m+k) & 0 & \cdots & 0 \\
- & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
- & - & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
- & - & \cdots & w(m-1,m) & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
- & - & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

where the block \(A_{22}\) is any \((n-m) \times (n-m)\) symmetric matrix with zero diagonal and nonnegative elements.

Because \(w(1,2) = \ldots = w(1,m) = w(2,3) = \ldots = w(2,m) = \ldots = w(m-1,m) = w_c(S_{m,k,s})\) the matrix \(\hat{A} := A + w_c(S_{m,k,s})I_n\) has \(m\) rows (and \(m\) columns) \(v_1(\hat{A}),\ldots,v_m(\hat{A})\) linearly dependent such that \(v_1(\hat{A}) = \ldots = v_m(\hat{A})\), then \(v_1(\hat{A}),\ldots,v_{m-1}(\hat{A}) \in ker(\hat{A})\).

Hence

\[\exists \mu_1,\ldots,\mu_{m-1} \in \sigma(\hat{A}) \quad \text{such that} \quad \mu_1 = \ldots = \mu_{m-1} = 0.\]

Let \(\mu_i\) be one of these eigenvalues, then

\[0 = \det((A + w_c(S_{m,k,s})I_n) - \mu_i I_n) = \det(A - (-w_c(S_{m,k,s}) + \mu_i)I_n)\]

so that \(\lambda := -w_c(S_{m,k,s}) \in \sigma(A)\) with multiplicity greater or equal to \(\deg(S_{m,k,s})\).

Let \(p\) be the number of \(S_{m,k,s}\) in the graph \(G\) denoted by \(S_{m_1,k_1,s_1},\ldots,S_{m_p,k_p,s_p}\).

Denoting \(w_1^k,\ldots,w_r^k\) the different central weights of such \((m,k,s)\)-stars, and \(r \leq p\), we prove that for any \(i \in \{1,\ldots,r\}\),

\[\exists \lambda \in \sigma(A) \quad \text{such that} \quad \lambda = -w_i^k.\]
and the multiplicity of \( \lambda \geq \text{deg}(S_{w^i}) = \sum_{S_{m_j,k_j} \in S_{w^i}} \text{deg}(S_{m_j,k_j,s_j}) \),

where \( S_{w^i} := \{ S_{m,k,s} \in G|w_c(S_{m,k,s}) = w^i_c \} \).

Let \( R_i \), with \( i \in \{1, ..., r\} \), be the number of \((m, k, s)\)-stars in \( S_{w^i} \), and \( \sum_{i=1}^r R_r = p \), we assume that the first \( R_1 \) indexes, namely \( 1, ..., R_1 \), refer to the \((m, k, s)\)-stars in \( S_{w^1} \), whereas the indexes \( R_1 + 1, ..., R_1 + R_2 \) refer to the \((m, k, s)\)-stars in \( S_{w^2} \), and so on.

We focus on the \( R_i \) \((m, k, s)\)-stars in \( S_{w^i} \). The rows in \( \hat{A} := A + w^i_c I_n \) corresponding to the vertices \( x_j \) in \( V_1(S_{w^i}) \) with \( j \in \{ \sum_{q=1}^{i-1} R_q + 1, ..., \sum_{q=1}^i R_q \} \), are \( m_j \) vectors \( (v^{(j)}_{j_1}(\hat{A}), ..., v^{(j)}_{j_{m_j}}(\hat{A})) \), linearly dependent and such that \( v^{(j)}_{j_1}(\hat{A}) = ... = v^{(j)}_{j_{m_j}}(\hat{A}) \), whose indexes are

\[
j_1 = \sum_{t=1}^{j-1} m_t + 1, ..., j_{m_j} = \sum_{t=1}^{j-1} m_t + m_j
\]

when \( j > 1 \), or

\[
j_1 = 1, ..., j_{m_j} = m_j
\]

when \( j = 1 \).

Then we get

\[
v^{(j)}_{j_1}(\hat{A}), ..., v^{(j)}_{j_{m_j-1}}(\hat{A}) \in \text{ker}(\hat{A}), \quad \forall j \in \{ \sum_{q=1}^{j-1} R_q + 1, ..., \sum_{q=1}^j R_q \}
\]

and

\[\exists \mu_{j_1}, ..., \mu_{j_{m_j-1}} \in \sigma(\hat{A}) \quad \text{such that} \quad \mu_{j_1} = ... = \mu_{j_{m_j-1}} = 0.\]

This is true for each \( j \in \{ \sum_{q=1}^{j-1} R_q + 1, ..., \sum_{q=1}^j R_q \} \), so that

\[\exists \mu_1, ..., \mu_{\text{deg}(S_{w^i})} \in \sigma(\hat{A}) \quad \text{such that} \quad \mu_1 = ... = \mu_{\text{deg}(S_{w^i})} = 0.\]

Finally, let \( \mu_t \) be one of these eigenvalues, then

\[0 = \text{det}((A + w^i_c I_n) - \mu_t I_n) = \text{det}(A - (-w^i_c + \mu_t) I_n)\]

and \( \lambda := -w^i_c \in \sigma(A) \) with multiplicity greater or equal to \( \text{deg}(S_{w^i}) \).

The proof for the Laplacian version of the Lemma 3.2 is similar to that for the adjacency matrix, in fact using the same arguments as in the proof of 3.2 wa can say that the Theorem 3.1 is true.
Some corollaries on the signless and normalized Laplacian matrices can be obtained by using similar proofs. Let $B$ and $L$ be the signless and normalized Laplacian matrices of $G = (V, E, w)$ respectively and $\sigma(B)$, $\sigma(L)$ the spectrum of $B$ and $L$ with algebraic multiplicity $m_B(\lambda)$, $m_L(\lambda)$ for the eigenvalue $\lambda$ in $B$ and $L$ respectively.

**Corollary 1.** If

- $p$ is the number of all the $S_{m,k,s}$ as $m$ and $k$ vary in $\mathbb{N}$ and $m+k \leq n$, of $G$,
- $r$ is the number of $S_{m,k,s}$ with different weights, $w_1, \ldots, w_r$,

then for any $i \in \{1, \ldots, r\}$,

$$\exists \lambda \in \sigma(B) \text{ such that } \lambda = w_i \text{ and } m_B(\lambda) \geq \deg(S_{w_i})$$

where $S_{w_i} := \{S_{m,k,s} \in G | w(S_{m,k,s}) - w_c(S_{m,k,s}) = w_i \}$.

**Corollary 2.** If

- $p$ is the number of all the $S_{m,k,s}$ as $m$ and $k$ vary in $\mathbb{N}$ and $m+k \leq n$, of $G$,
- $r$ is the number of $S_{m,k,s}$ with different weights, $w_1, \ldots, w_r$,

then for any $i \in \{1, \ldots, r\}$,

$$\exists \lambda \in \sigma(L) \text{ such that } \lambda = 1 + w_i \text{ and } m_L(\lambda) \geq \sum_{i=1}^{r} \deg(S_{w_i})$$

where $S_{w_i} := \{S_{m,k,s} \in G | \frac{w_c(S_{m,k,s})}{w(S_{m,k,s})} = w_i \}$.

From relations S.1 S.4 in the above situation we also have that

$$\exists \lambda \in \sigma(T) \text{ such that } \lambda = w_i \text{ and } m_T(\lambda) \geq \sum_{i=1}^{r} \deg(S_{w_i}).$$

We observe that when $s = 0$, and thus $w_c = 0$, each of the above results can be reduced to the results obtained in [11].

### 3.2. $(m, k, s)$-star with loops: eigenvalues multiplicity

In this section we consider $(m, k, s)$-star with loops and we generalize the previous results.

First of all we state some useful definitions:
Figure 2: A $S_{3,3,1}$ in a graph, where the subsets $V_1$ (the red vertices) and $V_2$ (the blue vertices), are respectively with cardinality $m = 3$ and $k = 3$. The weights of the edges between vertices belonging to $V_1$ are colored in red, the weights of the edges between vertices belonging to two different sets are colored in purple. In the Laplacian matrix there is the eigenvalue $\lambda = 6$ with multiplicity 2.

**Definition 3.5** ($(m, k, s)$-star with loop: $\tilde{S}_{m,k,s}$). A $(m, k, s)$-star with loops is a $(m, k, s)$-star whose each vertex in the set $V_1$ has a loop.

We denote a $(m, k, s)$-star graph with partitions of cardinality $|V_1| = m$ and $|V_2| = k$ by $\tilde{S}_{m,k,s}$.

We define a $(m, k, s)$-star with loop of a graph $G = (V, E, w)$ as the $(m, k, s)$-star with loop of partitions $V_1, V_2 \subset V$ such that

\[
\forall i \in V_1, \forall j \in V_2 \cup \{i\}, \quad (i, j) \in E
\]

if $s = 0$ then $\forall i \in V_1, \forall j \in V \setminus (V_2 \cup \{i\}), \quad (i, j) \notin E$,

if $s = 1$ then $\forall i \in V_1, \forall j \in V \setminus (V_1 \cup V_2), \quad (i, j) \notin E$

and $\forall i, j \in V_1, \quad (i, j) \in E$

In other words, a $(m, k, s)$-star with loop of a graph $G = (V, E, w)$ is a $(m, k, s)$-star of a graph $G$ whose each vertex in the set $V_1$ has a loop.

By defining the degree, weight and central weight of a $(m, k, s)$-star with loop as in the previous section we simplify the stating of the theorems on eigenvalues multiplicity. For the $(m, k, s)$-stars with loops the Lemma 3.2 has to be modified as follows

**Lemma 3.3.** Let

- $p$ be the number of all the $\tilde{S}_{m,k,s}$ as $m$ and $k$ vary in $\mathbb{N}$ and $m + k \leq n$, of $G$;
• $r$ be the number of $\tilde{S}_{m,k,s}$ with different weight, $w_1, ..., w_r$, i.e. $w_i \neq w_j$ for each $i \neq j$, where $i, j \in \{1, ..., r\}$:

then for any $i \in \{1, ..., r\}$,

$$\exists \lambda \in \sigma(A) \text{ such that } \lambda = -w_i \text{ and } m_A(\lambda) \geq \deg(S_{w_i})$$

where $S_{w_i} := \{\tilde{S}_{m,k,s} \in G | w_c(\tilde{S}_{m,k,s}) = w_i\}$.

For graphs with loop we can’t apply the results on spectra of Laplacian matrices, but we can prove an analogous result for the transition matrix.

**Corollary 3.** If

• $p$ is the number of all the $\tilde{S}_{m,k,s}$ of a graph $G$ where as $m$ and $k$ vary in $\mathbb{N}$ and $m + k \leq n$,

• $r$ is the number of $\tilde{S}_{m,k,s}$ with different weights, $w_1, ..., w_r$,

then for any $i \in \{1, ..., r\}$,

$$\exists \lambda \in \sigma(T) \text{ such that } \lambda = -w_i \text{ and } m_T(\lambda) \geq \sum_{i=1}^{r} \deg(S_{w_i})$$

where $S_{w_i} := \{\tilde{S}_{m,k,s} \in G | w_c(\tilde{S}_{m,k,s}) = w_i\}$.

3.3. Correspondence between a graph with loops and a graph with $(m,k,s)$-stars

In this section we define a correspondence between $(m,k,s)$-stars with loops and the $(m,k,s)$-stars without loops by keeping the same spectra of adjacency and transition matrices. In particular, in a graph, each vertex with loop is equivalent to an $(1,k,-)$-star with loop and we will give a procedure to replace the looped vertex with an $(2,k,s)$-star without loop by maintaining the same spectra (see Fig.(3.3)).

The following definitions are useful:

**Definition 3.6 ((m,k,s)-star q-reduced: $S_{m,k,s}^{(-q)}$).** A $q$-reduced $(m,k,s)$-star is a $(m,k,s)$-star (with or without loops) of vertex sets $\{V_1, V_2\}$, such that the cardinality of $V_1$ is decreased to $m - q$, with $m > q$.

Hence the order and degree of the $S_{m,k,s}^{(-q)}$ are $m + k - q$ and $m - q - 1$ respectively. Furthermore, let $w$ be the weights between vertices in the original $(m,k,s)$-star
Figure 3: In the left \( S_{3,3,1} \), in a graph, in the middle its \( S_{3,3,1}^{(-2)} \) that is a \( S_{1,3,1} \) in a graph; and in the right its \( S_{3,3,1}^{(+1)} \) that is a \( S_{2,3,1} \) in a graph.

and \( i, j \) any vertex in \( V_1 \), \( i \neq j \), of the \((m,k,s)\)-star, we define the weights of vertices in \( S_{m,k,s}^{(-q)} \) as follows

\[
w^{(-q)}(i,j) = \begin{cases} 
\frac{q}{m-q} w(\hat{i}, \hat{j}) + w(\hat{i}, \hat{i}), & \text{if } i, j \in V_1, \ i = j \\
\frac{m}{m-q} w(\hat{i}, \hat{j}), & \text{if } i \in V_1, j \in (V_1 \cup V_2) \setminus \{i\}, \ \\
w(i, j), & \text{if } i, j \in V_2 \\
0, & \text{otherwise}
\end{cases}
\] (1)

**Definition 3.7** \((1, k, -)\)-star \( q \)-enlarged: \( S_{1,k,-}^{(+q)} \). A \( q \)-enlarged \((1,k,-)\)-star is a \((1,k,-)\)-star (with or without loop) of vertex sets \( \{V_1, V_2\} \), such that the cardinality of \( V_1 \) is increased to \( 1 + q \) and the loop is removed.

Hence the order and degree of the \( S_{1,k,-}^{(+q)} \) are \( 1 + k + q \) and \( q \) respectively.

Furthermore, let \( w \) be the weights between vertices in the original \((1,k,\-\)\)-star and \( i \) the vertex in \( V_1 \) of the \((1,k,\-\)\)-star, we define the weights of vertices in \( S_{1,k,-}^{(+q)} \) as follows

\[
w^{(+q)}(i,j) = \begin{cases} 
\frac{1}{q} w(\hat{i}, \hat{i}), & \text{if } i \in V_1, j \in V_1 \setminus \{i\} \\
\frac{1}{q} w(\hat{i}, \hat{j}), & \text{if } i \in V_1, j \in V_2 \\
w(i, j), & \text{if } i, j \in V_2 \\
0, & \text{if } i = j \in V_1
\end{cases}
\] (2)

And then we define the \( q \)-enlarged graph

**Definition 3.8** \((q \)-enlarged graph: \( G^{(+q)} \), A \( q \)-enlarged graph \( G^{(+q)} \) is obtained from a graph \( G \) with some \((1,k,\-\)\)-stars adding \( q \) of the vertices in the set \( V_1 \) of \( G \), removing the loops and defining the weights as in (2).

Similarly one can define the \( q \)-reduced graph. Now we state the second main result of this paper. The main idea at the base of our theorem is illustrated in Fig.(3.3): starting from a graph with \((m,k,s)\-\)star with loops (or even simply
with vertices with loops) we firstly obtain a reduced graph (middle plot), and secondly an enlarged graph without loops (right plot).

**Theorem 3.4** (Loops removing theorem (adjacency matrix)). Let

- \( \mathcal{G} \) be a graph, of \( n \) vertices, with a \( S_{m,k,s} \),
- \( \mathcal{H} := \mathcal{G}^{(m-1)} \) be the \((m-1)\)-reduced graph with a \( S_{m,k,s}^{(m-1)} \) instead of \( S_{m,k,s} \),
- \( \mathcal{I} := \mathcal{H}^{(q)} \) be the \( q \)-enlarged graph with a \( S_{1,k,1}^{(q)} \) instead of \( S_{1,k,-} \),
- \( A \) be the adjacency matrix of \( \mathcal{G} \),
- \( A^{(m-1)} \) be the adjacency matrix of \( \mathcal{H} \), defined as in [1]
- \( A^{(q)} \) be the adjacency matrix of \( \mathcal{I} \), defined as in [2]

then

1. \( \sigma(A^{(m-1)}) \subset \sigma(A) \),
2. \( \sigma(A^{(m-1)}) \subset \sigma(A^{(q)}) \),
3. There exists a matrix \( H \in \mathbb{R}^{n \times (n-(m-1))} \) such that \( A^{(m-1)} = H^T A H \) and \( H^T H = I \). Therefore, if \( v \) is an eigenvector of \( A^{(m-1)} \) for an eigenvalue \( \mu \), then \( Hv \) is an eigenvector of \( A \) for the same eigenvalue \( \mu \).
4. There exists a matrix \( K \in \mathbb{R}^{n-(m-1) \times (n-(m-1)+q)} \) such that \( A^{(m-1)} = K^T A^{(q)} K \) and \( K^T K = I \). Therefore, if \( v \) is an eigenvector of \( A^{(m-1)} \) for an eigenvalue \( \mu \), then \( Kv \) is an eigenvector of \( A^{(q)} \) for the same eigenvalue \( \mu \).

Before proving Theorem 3.4, we recall the well known result for eigenvalues of symmetric matrices, [16].

**Lemma 3.5** (Interlacing theorem). Let \( A \in \text{Sym}_{n_A}(\mathbb{R}) \) with eigenvalues \( \mu_1(A) \geq ... \geq \mu_{n_A}(A) \). For \( n_B < n_A \), let \( K \in \mathbb{R}^{n_A \times n_B} \) be a matrix with orthonormal columns, \( K^T K = I \), and consider the \( B = K^T A K \) matrix, with eigenvalues \( \mu_1(B) \geq ... \geq \mu_{n_B}(B) \). If

- the eigenvalues of \( B \) interlace those of \( A \), that is,
  \[ \mu_i(A) \geq \mu_i(B) \geq \mu_{n_A-n_B+i}(A), \ i = 1, ..., n_B, \]
- if the interlacing is tight, that is, for some \( 0 \leq k \leq n_B \),
  \[ \mu_i(A) = \mu_i(B), \ i = 1, ..., k \text{ and } \mu_i(B) = \mu_{n_A-n_B+i}(A), \ i = k+1, ..., n_B \]
then \( KB = AK \).
Proof. We will prove only points 2. and 4., because using the same arguments the statements 1. and 3. follow and the matrix $H$ exists. First we prove the existence of the $K$ matrix:

let $\tilde{n} := n - (m - 1)$ and $P = \{P_1, ..., P_{\tilde{n}}\}$ be a partition of the vertex set $\{1, ..., \tilde{n} + q\}$. The characteristic matrix $K$ is defined as the matrix where the $j$-th column is the characteristic vector of $P_j$ ($j = 1, ..., \tilde{n}$).

Let $A^{\{+q\}}$ be partitioned according to $P$

$$A^{\{+q\}} = \begin{pmatrix} A^{\{+q\}}_{1,1} & \cdots & A^{\{+q\}}_{1,\tilde{n}} \\ \vdots & & \vdots \\ A^{\{+q\}}_{\tilde{n},1} & \cdots & A^{\{+q\}}_{\tilde{n},\tilde{n}} \end{pmatrix},$$

where $A^{\{+q\}}_{i,j}$ denotes the block with rows in $P_i$ and columns in $P_j$. The matrix $A^{\{-(m-1)\}} = (a^{\{-(m-1)\}}_{i,j})$ whose entries $a^{\{-(m-1)\}}_{i,j}$ are the averages of the $A^{\{+q\}}_{i,j}$ rows is the matrix where the $j$-th column is the characteristic vector of $P_j$ with respect $\mathcal{P}$, i.e. $a^{\{-(m-1)\}}_{i,j}$ denote the average number of neighbours in $P_j$ of the vertices in $P_i$.

The partition is equitable if for each $i, j$, any vertex in $P_i$ has exactly $a^{\{-(m-1)\}}_{i,j}$ neighbours in $P_j$. In such a case, the eigenvalues of the quotient matrix $A^{\{-(m-1)\}}$ belong to the spectrum of $A^{\{+q\}}$ ($\sigma(A^{\{-(m-1)\}}) \subset \sigma(A^{\{+q\}})$) and the spectral radius of $A^{\{-(m-1)\}}$ equals the spectral radius of $A^{\{+q\}}$: for more details cfr. [17], chapter 2.

Then we have the relations

$$M^{1/2}A^{\{-(m-1)\}}M^{1/2} = \tilde{K}^TA^{\{+q\}}\tilde{K}, \quad \tilde{K}^T\tilde{K} = M.$$

Considering an $(1, k, -)$–star (with loop) in a graph with adjacency matrix $A^{\{-(m-1)\}}$, we weight it by a diagonal mass matrix $M$ of order $\tilde{n}$ whose diagonal entries are one except for the entry of the vertex in $V_1$,

$$M_{ii} = \begin{cases} \frac{1}{1+q}, & \text{if } i \in V_1 \\ 1, & \text{otherwise} \end{cases}, \quad (3)$$

and we get

$$A^{\{-(m-1)\}} = K^TA^{\{+q\}}K, \quad K^TK = I,$$

where $K := \tilde{K}M^{-1/2}$. In addition to the th. [3.1], the eigenvalues of $A^{\{-(m-1)\}}$ are a subset of the eigenvalues of $A^{\{+q\}}$,

$$\sigma(A^{\{-(m-1)\}}) \subset \sigma(A^{\{+q\}}).$$

Finally, if $v$ is an eigenvector of $A^{\{-(m-1)\}}$ with eigenvalue $\mu$, then $Kv$ is an eigenvector of $A^{\{+q\}}$ with the same eigenvalue $\mu$.

Indeed form the equation $A^{\{-(m-1)\}}v = \mu v$ an taking into account that the partition is equitable, we have $KA^{\{-(m-1)\}} = A^{\{+q\}}K$, and

$$A^{\{+q\}}(Kv) = (A^{\{+q\}}K)v = (KA^{\{-(m-1)\}})v = \mu(Kv).$$
We obtain a similar result for the transition matrix $T$, and more in general for each $D^{-1}A$ where $A$ is the adjacency matrix of the graph $\mathcal{G}$ with a $\tilde{S}_{m,k,s}$ and $D$ any real diagonal matrix such that $d_{ii} = d_{jj}$ for any $i, j \in V_1$.

**Theorem 3.6** (Loops removing theorem and transition matrix). Let

- $\mathcal{G}$ be a graph, of $n$ vertices, with a $\tilde{S}_{m,k,s}$,
- $\mathcal{H} := \mathcal{G}^{(-(m-1))}$ be the $(m-1)$-reduced graph with a $\tilde{S}_{m,k,s}^{(-(m-1))}$ instead of $\tilde{S}_{m,k,s}$,
- $\mathcal{I} := \mathcal{H}^{(+q)}$ be the $q$-enlarged graph with a $S_{1,k,-}^{(+q)}$ instead of $\tilde{S}_{1,k,-}$,
- $A$ and $D$ be, respectively, the adjacency matrix and the strength diagonal matrix of $\mathcal{G}$,
- $A^{(-(m-1))}$ and $D^{(-(m-1))}$ be, respectively, the adjacency matrix and the strength diagonal matrix of $\mathcal{H}$, defined as in [1]
- $A^{(+q)}$ and $D^{(+q)}$ be, respectively, the adjacency matrix and the strength diagonal matrix of $\mathcal{I}$, defined as in [2]

then

1. $\sigma(T^{(-(m-1))}) \subset \sigma(T)$, where $T^{(-(m-1))} := (D^{(-(m-1))})^{-1}A^{(-(m-1))}$ and $T := D^{-1}A$

2. $\sigma(T^{(-(m-1))}) \subset \sigma(T^{(+q)})$, where $T^{(+q)} := (D^{(+q)})^{-1}A^{(+q)}$

3. There exists a matrix $H \in \mathbb{R}^{n \times (n-(m-1))}$ such that $T^{(-(m-1))} = HTTH$ and $HTH = I$. Therefore, if $v$ is an eigenvector of $T^{(-(m-1))}$ for an eigenvalue $\mu$, then $Hv$ is an eigenvector of $T$ for the same eigenvalue $\mu$.

4. There exists a matrix $K \in \mathbb{R}^{n-(m-1) \times (n-(m-1)+q)}$ such that $T^{(-(m-1))} = KT^{(+q)}K$ and $KTK = I$. Therefore, if $v$ is an eigenvector of $T^{(-(m-1))}$ for an eigenvalue $\mu$, then $Kv$ is an eigenvector of $T^{(+q)}$ for the same eigenvalue $\mu$.

The proof for the transition matrix version of the Loops removing Theorem 3.4 is similar to that for the adjacency matrix, in fact using the same arguments as in the proof of 3.4 and the equivalences S.1–S.3, in order to work with symmetric matrices, we can say that 1. and 2. are true and that $H$ and $K$ matrices exist.
Corollary 4. Let $G$ be a graph, of $n$ vertices, with a $\tilde{S}_{m,k,s}$, if $v$ is a right eigenvector of $T$ with eigenvalue $\lambda \in \sigma(T) \setminus \{-w_c(\tilde{S}_{m,k,s})/w(\tilde{S}_{m,k,s})\}$ then $D^{1/2}v$ is an eigenvector of $L$ with eigenvalue $(1 - \lambda)$.

The proof directly follows from the Theorem 3.6.

4. Conclusions

The Laplacian matrix associated to undirected graph provides powerful tools to study the geometrical and dynamical properties of the graph [7, 9]. In particular in a previous work [11] we have correlated the presence $(m,k)$-stars in a graph to the eigenvalue multiplicity in the Laplacian matrix spectrum. In this work, we have extended the previous results for the $(m,k)$-stars to the $(m,k,s)$-stars with or without loops. Our approach allows to introduce relations for the spectral properties and for the eigenvectors of adjacency or transition matrices associated to graphs containing $(m,k,s)$-stars with loops and to graphs containing $(m,k,s)$-stars without loops. In this way it is possible to extend to multigraphs methods developed for simple graphs. The results discussed in the paper give the possibility to associate a Laplacian matrix to graphs with loops, to reduce the size of a graph (with or without loops) preserving the spectral properties and to describe a graph with loops as a simple graph without discarding relevant information inherent in the original graph. Despite of the fact that graphs with loops appear in many natural contexts and may be obtained by several kinds of aggregation, scaling and blocking, they have not been treated as extensively as simple graphs. Possible applications of our results could be to the organisational networks, where different kinds of ties may appear within the same branch creating loops [13] or in citation and co-authorship networks, in which self-citations are recurrent and the link weights between two authors in co-authorship networks can increase over time if they have further collaboration [19, 20], as well as in opinion networks, where individuals are subject to vanity [21, 22]. Finally our results could be relevant in neural networks over undirected graphs, where loops tend to freeze the dynamics that converges to fixed points [23].

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