Mean value property associated with the Dunkl Laplacian

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Abstract

Let $\Delta_k$ be the Dunkl Laplacian on $\mathbb{R}^d$. The main goal of this paper is to characterize $\Delta_k$-harmonic functions by means of a mean value property.

Keywords: Dunkl Laplacian, Mean value property, $\Delta_k$-harmonic functions.

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1 Introduction

Let $R$ be a root system of $\mathbb{R}^d$, $d \geq 1$, $k : R \rightarrow \mathbb{R}_+$ be a multiplicity function and $W$ be the group generated by the reflections $\sigma_\alpha$, $\alpha \in R$. The Dunkl Laplacian is defined in [1] for every function $f \in C^2(\mathbb{R}^d)$ by

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left( \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{|\alpha|^2}{2} \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right),$$

where $\Delta$ and $\nabla$ denote respectively the usual Laplacian and gradient on $\mathbb{R}^d$ and $R_+$ is a positive subsystem of $R$. Clearly, if $k$ is the identically vanishing function, then $\Delta_k$ is reduced to $\Delta$.

It is well known that a locally bounded function $f$ on an open subset $D$ of $\mathbb{R}^d$, is $\Delta$-harmonic (i.e., $f \in C^2(D)$ and $\Delta f = 0$ on $D$) if and only if

$$f(x) = \frac{1}{\sigma_{x,r}(S(x,r))} \int_{S(x,r)} f(y) d\sigma_{x,r}(y),$$

for every $x \in D$ and every $r > 0$ such that the closed ball $\overline{B}(x,r)$ of center $x$ and radius $r$ is contained in $D$. Here $\sigma_{x,r}$ is the surface area measure on the sphere $S(x,r)$ with center $x$ and radius $r$. 
H. Mejjaolli and K. Trimèche showed in [4] that every infinitely differentiable function $f$ on $\mathbb{R}^d$ is $\Delta_k$-harmonic on $\mathbb{R}^d$ if and only if for all $x \in \mathbb{R}^d$ and $r > 0$,

$$f(x) = \frac{1}{d_k} \int_{S(0,1)} \tau_x f(ry) \left( \prod_{\alpha \in \mathbb{R}^+} |\langle y, \alpha \rangle|^{2k(\alpha)} \right) d\sigma_{0,1}(y),$$

(1)

where $d_k$ is a normalized constant and $\tau_x$ is the Dunkl translation. The main goal of this paper is to investigate a mean value property which characterizes the $\Delta_k$-harmonicity of locally bounded functions on an open subset of $\mathbb{R}^d$.

Let $D \subset \mathbb{R}^d$ be an open set which is $W$-invariant. We shall say that a function $f : D \to \mathbb{R}$ satisfies the mean value property on $D$ if for every $x \in D$ and $r > 0$ such that $B(x, r) \subset D$,

$$f(x) = \int_{\mathbb{R}^d} f(y) d\sigma^k_{x,r}(y),$$

where $\sigma^k_{x,r}$ (see [1]) is the unique probability measure on $\mathbb{R}^d$ such that the right hand side of (1) coincides with

$$\int_{\mathbb{R}^d} f(y) d\sigma^k_{x,r}(y).$$

We shall prove that every locally bounded function $f$ on $D$ is $\Delta_k$-harmonic if and only if it satisfies the mean value property on $D$. To that end, we prove first the equivalence for infinitely differentiable functions on $D$. Next, we show that for a locally bounded function $f$ on $D$, if $f$ satisfies the mean value property then $f$ is infinitely differentiable on $D$. Thus, $f$ is $\Delta_k$-harmonic provided it satisfies the mean value property on $D$. To prove the converse, we need only show that if $f$ is $\Delta_k$-harmonic then it is infinitely differentiable on $D$. This will be proved once we have shown that the operator $\Delta_k$ is hypoelliptic. Thus, by means of convergence property of $\Delta_k$-harmonic functions, we prove that the operator $\Delta_k$ is hypoelliptic on $D$.

Note that the condition that $D$ is $W$-invariant is nearly optimal. In fact, in the case where $d = 1$, for every open set $D \subset \mathbb{R}$ which is not $W$-invariant, we can always construct a $\Delta_k$-harmonic function function $f$ on $D$ which does not satisfy the mean value property on $D$.

2 Preliminaries and some lemmas

Let $S(\mathbb{R}^d)$ be the Schwartz space and $C_0(\mathbb{R}^d)$ be the set of all continuous functions on $\mathbb{R}^d$ vanishing at infinity. For every open set $U \subset \mathbb{R}^d$, $C(U)$ and $C_c(U)$ will denote respectively the set of all continuous functions on $U$ and the set of all continuous functions with compact support on $U$. The
set of all bounded functions in $C(U)$ will be denoted by $C_b(U)$. For every $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $H_\alpha$ be the hyperplane of $\mathbb{R}^d$ orthogonal to $\alpha$ and let $\sigma_\alpha$ be the reflection in $H_\alpha$, i.e.,

$$\sigma_\alpha(x) := x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha,$$

where $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$ and $|x| := \sqrt{\langle x, x \rangle}$. A finite subset $R$ of $\mathbb{R}^d \setminus \{0\}$ is called a root system if $R \cap R\alpha = \{\pm \alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. For a given root system $R$, we denote by $W$ the finite group generated by all reflections $\sigma_\alpha$, $\alpha \in R$. A function $k : R \to \mathbb{R}^+$ is called a multiplicity function if it satisfies $k(w\alpha) = k(\alpha)$, for every $w \in W$ and every $\alpha \in R$.

Throughout this paper we fix a root system $R$, a multiplicity function $k$ and a $W$-invariant open subset $D$ of $\mathbb{R}^d$, that is, $w(D) \subset D$ for all $w \in W$. Let $w_k$ be the weight function on $\mathbb{R}^d$ defined by,

$$w_k(x) := \prod_{\alpha \in R_+} |\langle x, \alpha \rangle|^{2k(\alpha)},$$

where $R_+ := \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$ for some $\beta \in \mathbb{R}^d \setminus \cup_{\alpha \in R} H_\alpha$. Note that $w_k$ is homogeneous of degree $2\gamma$, with $\gamma := \sum_{\alpha \in R_+} k(\alpha)$. From now on, we assume that

$$\lambda := \gamma + \frac{d}{2} - 1 > 0.$$

The Dunkl Laplacian associated with the root system $R$ and the multiplicity function $k$ is the operator

$$\Delta_k := \sum_{i=1}^d T_i^2,$$

where for every $1 \leq i \leq d$ and $f \in C^1(D)$,

$$T_i f(x) := \partial_i f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad x \in D.$$
The Dunkl kernel associated with $R$ and $k$ is defined on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$E_k(x, y) := \int_{\mathbb{R}^d} e^{\langle y, \xi \rangle} d\mu_x^k(\xi).$$

It is well known that $E_k$ is positive, symmetric and admits a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$ satisfying $E_k(\xi z, \omega) = E_k(z, \xi \omega)$ for every $z, \omega \in \mathbb{C}^d$ and every $\xi \in \mathbb{C}$. The corresponding Dunkl transform is then given for every bounded measure $\mu$ on $\mathbb{R}^d$ by

$$\mathcal{F}_D(\mu)(x) := c_k \int_{\mathbb{R}^d} E_k(-i\xi, x) d\mu(\xi), \quad x \in \mathbb{R}^d,$$

where

$$c_k := \left( \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2}} w_k(y) dy \right)^{-1}.$$

If $\mu = f \, w_k dx$ where $f \in S(\mathbb{R}^d)$ and $dx$ is the Lebesgue measure on $\mathbb{R}^d$, then we shall write $\mathcal{F}_D(f)$ instead of $\mathcal{F}_D(\mu)$. Note that $\mathcal{F}_D$ is injective on the space of all bounded Borel measures $\mathcal{M}_b(\mathbb{R}^d)$ on $\mathbb{R}^d$ (see [8]) and is a topological isomorphism from $S(\mathbb{R}^d)$ into itself (see [3]). For each $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x$ is defined for every $f \in S(\mathbb{R}^d)$ by

$$\tau_x f = \mathcal{F}_D^{-1}(E_k(ix, \cdot) \mathcal{F}_D f),$$

where $\mathcal{F}_D^{-1}$ denotes the inverse of $\mathcal{F}_D$ on $S(\mathbb{R}^d)$. In [10], this translation was extended to $C^\infty(\mathbb{R}^d)$ by

$$\tau_x f(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_k^{-1} f(z + \eta) d\mu_x^k(z) d\mu_y^k(\eta),$$

where $V_k^{-1}$ is the inverse of $V_k$ on $C^\infty(\mathbb{R}^d)$. It was shown that, for every $f \in C^\infty(\mathbb{R}^d)$, the function $u : (x, y) \mapsto \tau_x f(y)$ is symmetric, infinitely differentiable on $\mathbb{R}^d \times \mathbb{R}^d$ and for every $x, y \in \mathbb{R}^d$,

$$(T_1)_x u(x, y) = (T_1)_y u(x, y). \quad (2)$$

Moreover, $\tau_x f(0) = f(0)$, $T_1 \tau_x f = \tau_x T_1 f$ and $\tau_x E_k(z, \cdot)(y) = E_k(x, z) E_k(y, z)$ for every $z \in \mathbb{C}^d$. Further, if the support of $f$ (noted $\text{supp} \ f$) is in $B(0, r)$ for some $r > 0$, then $\text{supp} \ \tau_x f \subset B(0, r + |x|)$.

According to [7], for each $x \in \mathbb{R}^d$ and $r > 0$, there exists a unique probability measure $\sigma_{x,r}^k$ on $\mathbb{R}^d$ which is supported by $\bigcup_{w \in W} \tau^k(wx, r) \setminus B(0, ||x| - r|)$ such that for every $f \in C^\infty(\mathbb{R}^d)$,

$$\frac{1}{d_k} \int_{S(0,1)} \tau_x f(r y) w_k(y) d\sigma_{0,1}(y) = \int_{\mathbb{R}^d} f(y) d\sigma_{x,r}^k(y). \quad (3)$$

where

$$d_k := \int_{S(0,1)} w_k(y) d\sigma_{0,1}(y) = \frac{1}{c_k 2^\lambda \Gamma(\lambda + 1)}.$$
Lemma 2.1. Let $\varphi \in S(\mathbb{R}^d)$ be radial. Then for every Borel set $A \subset \mathbb{R}^d$ and every $x \in \mathbb{R}^d$,

$$\int_A \tau_{-x} \varphi(y) w_k(y) dy = d_k \int_0^\infty \varphi(t)t^{2\lambda+1} \left( \int_A d\sigma_{x,t}^k(y) \right) dt.$$ \hspace{1cm} (4)

Proof. Let $x \in \mathbb{R}^d$ and denote by $\mu(A)$ and $\nu(A)$ the left hand side and the right hand side respectively of (4). Clearly, both $\mu$ and $\nu$ are bounded measures on $\mathbb{R}^d$. For every $y \in \mathbb{R}^d$,

$$\mathcal{F}_D(\mu)(y) = \mathcal{F}_D(\tau_{-x} \varphi)(y) = \mathcal{F}_D \varphi(y) E_k(-iy, x) = c_k E_k(-iy, x) \int_{\mathbb{R}^d} \varphi(z) E_k(-iy, z) w_k(z) dz = c_k \int_{\mathbb{R}^d} \tau_x E_k(-iy, \cdot)(z) \varphi(z) w_k(z) dz$$

Using spherical coordinates and (3), we deduce that,

$$\mathcal{F}_D(\mu)(y) = c_k d_k \int_0^\infty t^{2\lambda+1} \varphi(t) \int_{\mathbb{R}^d} E_k(\xi, -iy) d\sigma_{x,t}^k(\xi) dt = \mathcal{F}_D(\nu)(y).$$

Finally, we use the injectivity of $\mathcal{F}_D$ on $\mathcal{M}_b(\mathbb{R}^d)$ to conclude. \hfill \square

Let $\varphi \in S(\mathbb{R}^d)$ be a radial function with support in $\overline{B}(0, r), r > 0$. We claim that for every $x \in \mathbb{R}^d$,

$$\text{supp} \tau_x \varphi \subset \bigcup_{w \in W} \overline{B}(wx, r).$$ \hspace{1cm} (5)

Indeed, let $A$ be a Borel subset of $\mathbb{R}^d \setminus \bigcup_{w \in W} \overline{B}(wx, r)$. Then by (4),

$$\int_A \tau_{-x} \varphi(y) w_k(y) dy = d_k \int_0^r \varphi(t)t^{2\lambda+1} \left( \int_A d\sigma_{x,t}^k(y) \right) dt.$$ \hspace{1cm} (4)

Since for every $0 < t < r$, $\text{supp} \sigma_{x,t}^k \subset \bigcup_{w \in W} \overline{B}(wx, r)$ we deduce that,

$$\int_A \tau_{-x} \varphi(y) w_k(y) dy = 0.$$

This proves the claim.

In the sequel we shall write

$$M_{x,r}(f) = \int_{\mathbb{R}^d} f(y) d\sigma_{x,r}^k(y),$$

whenever the integral makes sense. A Borel function $f : D \to \mathbb{R}$ is said to satisfy the mean value property on $D$ if $M_{x,r}(f) = f(x)$ for every $x \in \mathbb{R}^d$ and $r > 0$ such that $\overline{B}(x, r) \subset D$. 

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Lemma 2.2. Let $f$ be a locally bounded function on $D$. If $f$ satisfies the mean value property on $D$, then $f \in C^\infty(D)$.

Proof. Without loss of generality we suppose that $f$ is bounded on $D$. Let $\phi$ be the function defined for every $t \in \mathbb{R}$ by $\phi(t) := cc^{-1}t^2\chi_{(0,\infty)}(t)$, where $\chi_{(0,\infty)}$ is the indicator function of $]0,\infty[$ and the constant $c$ is chosen so that

$$cd_k \int_0^1 \phi(1-t^2)t^{2\lambda+1}dt = 1.$$  

For every $n \geq 1$ we define the function $\phi_n$ by,

$$\phi_n(x) = n^{2\lambda+2} \phi(1-n^2|x|^2), \quad x \in \mathbb{R}^d. \quad (6)$$

Obviously $\phi_n$ is infinitely differentiable on $\mathbb{R}^d$ with support in $\overline{B}(0, \frac{1}{n})$.

Thus, by (5), for every $x \in \mathbb{R}^d$,

$$\text{supp } \tau_x \phi_n \subset \cup_{w \in W} \overline{B}(wx, \frac{1}{n}).$$

Let $D_n := \{x \in D : \overline{B}(x, \frac{1}{n}) \subset D\}$ and let

$$f_n(x) := \int_D f(y)\tau_{-x} \phi_n(y)w_k(y)dy, \quad x \in \mathbb{R}^d.$$  

Then $f_n \in C^\infty(D_n)$. On the other hand, it follows from (6) that for every $x \in D_n$,

$$f_n(x) = d_k \int_0^{\frac{1}{n}} \phi_n(t)t^{2\lambda+1}M_{x,t}(f)dt = f(x).$$

Hence $f \in C^\infty(D_n)$ and consequently $f \in C^\infty(D)$ as desired. \qed

3 Main result

$D$ will always denotes a $W$-invariant open subset of $\mathbb{R}^d$. Our main result is the following:

Theorem 3.1. Let $f$ be a locally bounded function on $D$. The following statements are equivalent:

(a) $f \in C^2(D)$ and $\Delta_k f = 0$ on $D$.
(b) $M_{x,r}(f) = f(x)$ for every $x \in D$ and $r > 0$ such that $\overline{B}(x, r) \subset D$.

The following proposition shows the equivalence between (a) and (b) whenever $f$ is infinitely differentiable on $D$. First, let us recall the Green formula associated with the Dunkl Laplacian (see [3]): For every $f \in C^2(\overline{B}(0, t))$, $t > 0$,

$$\int_{\overline{B}(0,t)} \Delta_k f(y)w_k(y)dy = \int_{S(0,t)} \frac{\partial}{\partial n} f(y)w_k(y)d\sigma_{0,t}(y), \quad (7)$$

where $\frac{\partial}{\partial n}$ is the partial derivation operator in the direction of the exterior unit normal.
Proposition 3.2. Assume that $f$ is infinitely differentiable on $D$. Then $f$ is $\Delta_k$-harmonic on $D$ if and only if $f$ satisfies the mean value property on $D$.

Proof. Let $x \in D$ and $r > 0$ such that $\overline{B}(x, r) \subset D$. We claim that $t \mapsto M_{x,t}(f)$ is derivable on $]0, r]$ and for every $t \in ]0, r]$, 

$$\frac{d}{dt} M_{x,t}(f) = \frac{1}{t^{2\lambda+1}} \int_0^t s^{2\lambda+1} M_{x,s}(\Delta_k f) ds. \quad (8)$$

Indeed, since for every $s \in ]0, r]$, the support of $\sigma_{x,s}^k$ is contained in $\bigcup w \in W B(w, x, r)$, it suffices to prove (8) replacing $f$ by a function $h \in C_{\infty}(\mathbb{R}^d)$ such that $h = f$ on $\bigcup w \in W B(w, x, r)$.

It is easily seen from (3) that for every $t \in ]0, r]$, 

$$\frac{d}{dt} M_{x,t}(h) = \frac{1}{d_k} \int_{S(0,1)} \langle \nabla (\tau_x h)(ty), y \rangle w_k(y) d\sigma_{0,1}(y)$$

$$= \frac{1}{d_k t^{2\lambda+1}} \int_{S(0,t)} \langle \nabla (\tau_x h)(u), \frac{u}{t} \rangle w_k(u) d\sigma_{0,1}(u)$$

$$= \frac{1}{d_k t^{2\lambda+1}} \int_{S(0,t)} \frac{\partial}{\partial n} (\tau_x h)(u) w_k(u) d\sigma_{0,1}(u).$$

Therefore, by the Green formula (7) and the fact that $\Delta_k \tau_x = \tau_x \Delta_k$, 

$$\frac{d}{dt} M_{x,t}(h) = \frac{1}{d_k t^{2\lambda+1}} \int_{B(0,t)} \tau_x (\Delta_k h)(u) w_k(u) du.$$

Hence, using spherical coordinates we deduce that, 

$$\frac{d}{dt} M_{x,t}(h) = \frac{1}{t^{2\lambda+1}} \int_0^t s^{2\lambda+1} M_{x,s}(\Delta_k h) ds.$$

Thus the claim is proved. Now assume that $\Delta_k f = 0$ on $D$. Then for all $t \in ]0, r]$, $\frac{d}{dt} M_{x,t}(f) = 0$, by (8). This yields that $M_{x,t}(f) = \lim_{s \to 0} M_{x,s}(f)$. On the other hand, it is known from [7] that the map $(x, s) \mapsto \sigma_{x,s}^k$ is continuous with respect to the weak topology on $\mathcal{M}_k(\mathbb{R}^d)$. Thus, 

$$\lim_{s \to 0} M_{x,s}(f) = f(x). \quad (9)$$

Whence $M_{x,t}(f) = f(x)$ which yields the necessity. Conversely, assume that $f$ satisfies the mean value property on $D$. Then, using (8) we deduce that $M_{x,t}(\Delta_k f) = 0$ for all $t \in ]0, r]$. Letting $t$ tend to 0 we obtain that $\Delta_k f(x) = 0$.

We then conclude, in virtue of Lemma 2.2, that every locally bounded function $f$ on $D$ which satisfies the mean value property on $D$ is necessarily $\Delta_k$-harmonic on $D$. The converse statement will be proved in the remainder of this section.
Lemma 3.3. Let \((h_n)_{n \geq 1} \subset C^\infty(D)\) be a locally uniformly bounded sequence of \(\Delta_k\)-harmonic functions on \(D\) with pointwise limit \(h\). Then \(h \in C^\infty(D)\) and \(\Delta_k h = 0\) on \(D\).

Proof. Let \(x \in D\) and let \(r > 0\) such that \(\overline{B}(x, r) \subset D\). Since for every \(n \geq 1\) the function \(h_n\) is \(\Delta_k\)-harmonic on \(D\), it follows from Proposition 3.2 that,

\[ h_n(x) = \int_{\mathbb{R}^d} h_n(y) d\sigma^k_{x, r}(y). \]

Applying the dominated convergence theorem, we get \(h(x) = M_{x, r}(h)\). Whence \(h\) satisfies the mean value property on \(D\) which finishes the proof, by Lemma 2.2 and Proposition 3.2.

Let \(g_k\) be the fundamental solution of the Dunkl Laplacian. That is, for every \(\varphi \in C^\infty_c(\mathbb{R}^d)\),

\[ \int_{\mathbb{R}^d} g_k(y) \Delta_k \varphi(y) w_k(y) dy = -\varphi(0). \]  

(10)

It is well known from [4] that,

\[ g_k(y) = c_k \Gamma(\lambda) 2^{\lambda - 1} |y|^{-2\lambda}. \]  

(11)

Theorem 3.4. Let \(h \in C(D)\) and \(f \in C^\infty(D)\). Assume that for every \(\varphi \in C^\infty_c(D)\),

\[ \int_D h(x) \Delta_k \varphi(x) w_k(x) dx = \int_D f(x) \varphi(x) w_k(x) dx. \]

Then \(h \in C^\infty(D)\).

Proof. It suffices to prove that \(h \in C^\infty(U)\), for every \(W\)-invariant open set \(U\) such that \(\overline{U} \subset D\).

Step 1. Assume first that \(f = 0\) on \(D\). Choose \(n_0 \geq 1\) such that for every \(x \in U, \overline{B}(x, \frac{1}{n_0}) \subset D\). For every \(n \geq n_0\), let \(\phi_n\) be as in (6). Then, the function \(h_n\) defined on \(U\) by

\[ h_n(x) := \int_D h(y) \tau_{-x} \phi_n(y) w_k(y) dy, \]

is infinitely differentiable on \(U\) and by (2) for every \(x \in U\),

\[ \Delta_k h_n = \int_D h(y) \Delta_k (\tau_{-x} \phi_n)(y) w_k(y) dy. \]

On the other hand, it follows from (11) that for every \(x \in U\),

\[ h_n(x) = d_k \int_0^\frac{1}{n} \phi(t) t^{2\lambda + 1} M_{x,t}(h) dt \]

\[ = c d_k \int_0^1 \phi(1 - u^2) u^{2\lambda + 1} M_{x,u}(h) du. \]
This yields that \((h_n)_{n \geq n_0}\) is uniformly bounded on \(U\) and converges pointwise to \(h\) on \(U\). Hence, in view of Lemma 3.3, \(h \in C^\infty(U)\) and \(\Delta h = 0\) on \(U\).

**Step 2.** We now turn to the general case where \(f\) is not trivial. Let \(v \in C^\infty_c(\mathbb{R}^d)\) such that \(v = f\) on \(U\) and define \(\psi\) on \(\mathbb{R}^d\) by

\[
\psi(x) := \int_{\mathbb{R}^d} g_k(y) \tau_x v(y) w_k(y) dy,
\]

where \(g_k\) is given by (11). Using spherical coordinates, it easily seen that the function \(g_k w_k\) is locally Lebesgue integrable on \(\mathbb{R}^d\). Thus, \(\psi \in C^\infty_c(\mathbb{R}^d)\). Furthermore, it follows from (2) and (10) that \(\Delta h = -f\) on \(U\). Then, for every \(\varphi \in C^\infty_c(U)\),

\[
\int_{\mathbb{R}^d} (h(x) + \psi(x)) \Delta_k \varphi(x) w_k(x) dx = \int_{\mathbb{R}^d} (f(x) + \Delta_k \psi(x)) \varphi(x) w_k(x) dx = 0.
\]

Whence, the first step yields that \(h + \psi\) is infinitely differentiable on \(U\) which finishes the proof.

We note that the previous theorem was already proved by H. Mejjaolli and K. Trimèche \[5\] using Sobolev spaces associated with the Dunkl operators.

Proof of Theorem 3.1 Statement (a) follows from (b) by means of Lemma 2.2 and Proposition 3.2. Assume now that (a) holds. Then, by \[7\], for every \(\varphi \in C^\infty_c(D)\),

\[
\int_{D} f(x) \Delta_k \varphi(x) w_k(x) dx = \int_{D} \Delta_k f(x) \varphi(x) w_k(x) dx = 0.
\]

Use now Theorem 3.4 and Proposition 3.2 to finish the proof. \(\square\)

In the following we shall give a counterexample proving that Theorem 3.1 does not hold true if the open set \(D\) is not \(W\)-invariant. To that end, let \(d = 1\) and consider the root system \(R = \{ \pm \sqrt{2} \}\). Then, the corresponding reflection group is given by \(W = \{ \pm id_{\mathbb{R}} \}\). Therfore, an open set \(U \subset \mathbb{R}\) is \(W\)-invariant if and only if it is symmetric.

**Proposition 3.5.** For every non symmetric open set \(U \subset \mathbb{R}\) there exists a function \(h : \mathbb{R} \to \mathbb{R}\) which is \(\Delta_k\)-harmonic on \(U\) but does not satisfy the mean value property on \(U\).

**Proof.** To abbreviate the notation we write \(I_{x,r} := |x-r, x+r|\). Let \(x \in U\) and \(r > 0\) such that \(\overline{I_{x,r}} \subset U\) and \(\overline{I_{-x,r}} \cap \overline{U} = \emptyset\). Choose \(f \in C^\infty_c(\mathbb{R}^d)\) such that \(f = -1\) on \(\overline{I_{-x,r}}\) and \(f = 0\) on \(U\). Since \(g_k w_k\) is locally Lebesgue integrable, we deduce that the function \(h\) defined on \(\mathbb{R}\) by

\[
h(z) = \int_{\mathbb{R}} g_k(y) \tau_z f(y) w_k(y) dy,
\]
is infinitely differentiable on $\mathbb{R}$. Moreover, by (10), for every $z \in \mathbb{R}$,
$$\Delta_k h(z) = -\tau_z f(0) = -f(z).$$
Hence, $\Delta_k h = 0$ on $U$. On the other hand, for every $t \in [0,r[$,
$$M_{x,t}(\Delta_k h) = -\int_{\mathbb{R}} f(y) d\sigma_{x,t}^k(y) = \sigma_{x,t}^k(T_{-x,t}).$$
Moreover, it follows from [7, Remarks 4.2] that
$$\text{supp } \sigma_{x,t}^k = T_{x,t} \cup T_{-x,t}.$$ 
Thus $\sigma_{x,t}^k(T_{-x,t}) > 0$ and consequently $M_{x,t}(\Delta_k h) > 0$. Whence, by (8) the function $t \mapsto \frac{1}{M_{x,t}} M_{x,t}(h)$ is positive on $[0,r[$. Hence, $M_{x,t}(h) \neq h(x)$ for every $t \in [0,r[$, which means that $h$ does not satisfy the mean value property on $U$. □

References

[1] C.F. Dunkl, Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc. 311 (1989) 167-183.
[2] C.F. Dunkl, Integrals kernels with reflection group invariance, Canad. J. Math, 43 (1991) 1213-1227.
[3] M.F. de Jeu, The Dunkl transform. Invent. Math, 113 (1993) 147-162.
[4] H. Mejjaoli, K. Trimèche, On a mean value property associated with the dunkl Laplacian operator and applications. Integral transform Spec. Funct., 12 (2001) 279-302.
[5] H. Mejjaoli, K. Trimèche, Hypoellipticity and hypoanalyticity of the Dunkl Laplacian operator. Integral transforms Spec. Funct., 15 (2004) 523-548.
[6] M. Rösler, Positivity of Dunkl’s intertwining operator. Duke Math. J. 98 (1999) 445-463.
[7] M. Rösler, A positive radial product formula for Dunkl kernel. Trans. Amer. Math. Soc. 355 (2003) 2413-2438.
[8] M. Rösler, M. Voit, Markov processes related with Dunkl operators. Advances in Applied Mathematics. 21 (1998) 575-643.
[9] K. Trimèche, The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual. Integral Transforms Spec. Funct., 12 (2001) 349-374.
[10] K. Trimèche, Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators. Integral Transforms Spec. Funct., 13 (2002) 17-38.