Existence and Regularity of Solutions for Unbounded Elliptic Equations with Singular Nonlinearities

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1. Introduction

Consider the Dirichlet problem for some nonlinear elliptic equations:

\[-\text{div}([1 + |u|^q] \nabla u) = (f/|u|^c) \quad \text{in} \Omega, \]
\[-u = 0 \quad \text{on} \partial \Omega,\]

for almost every $x \in \Omega$, where $\alpha$ and $\beta$ are positive constant, and $0 \leq f \in L^m(\Omega)$, with $m \geq 1$.

A possible motivation for studying the existence of these types of problems arises from the calculation of variations and stochastic control. For example, if we consider the functional

\[ J(v) = \frac{1}{2} \int_\Omega [a(x) + |v|^{1-\theta}] |\nabla v|^2 - \int_\Omega f(x)v, \]

the Euler–Lagrange equation associated to the functional $J$ is

\[-\text{div}([a(x) + |v|^{1-\theta}] \nabla v) + \frac{1 - \theta}{2} \frac{|\nabla v|^2}{|v|^\theta} \text{sign}(v) = f. \]

Several papers deal with existence of solutions to the singular elliptic problems with lower order terms having a quadratic growth with respect to the gradient (for example, [1–9]), namely, with the model problem

\[-\text{div}(M(x,u) \nabla u) + \frac{|\nabla u|^2}{|u|^\theta} \text{sign}(u) = f(x), \quad x \in \Omega, \]
\[-u(x) = 0, \quad x \in \partial \Omega,\]

where $\theta$ is a positive constant and $M: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function. More precisely, existence of positive solutions for (7) was shown in [1–3], for $M(x,t) = 1$ and $0 < \theta < 1$, and the uniqueness of positive solution, for $M(x,t) = 1$ and $0 < \theta < 1$, in [4]. On the contrary, the existence of positive solutions of (7) is shown in [6] for $0 < \theta < 1$, provided $M$ is a bounded uniformly elliptic matrix and $0 \leq f \in L^m(\Omega)$ ($m > (2N/N + 2)$). Later, in [9], it is
proved the existence of solution for (7) with $0 < \theta < 1$, where
\( M(x,t) = 1 \) and the data \( f \in L^m(\Omega) \) with \( m > (N/2) \), and
does not satisfy any sign assumption. Recently, a problem
introduced by L. Boccardo (see [7, 10]) has given a strong
impulse to the study of quasilinear problems having the
unbounded divergence operator. In particular, in [7], the
authors have proved the existence of positive solutions to
problem (7) under the assumption that $0 < \theta < 1,$
\( M(x,t) = 1 + |t|^q \), and \( 0 \leq f \in L^m(\Omega) \). We refer also that, in
[5], the author has shown the same result as in [7], in the case
$0 < \theta < 1$ and without any sign restriction over \( f \).

Let us now consider the Dirichlet boundary value problem (7) in the simple case:

\[
\begin{cases}
-2\Delta u + \frac{|\nabla u|^2}{u} = f(x), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega.
\end{cases}
\] (8)

If we define \( v = 2(u/\sqrt{|u|}) \), then the function \( v \) is solution of

\[
\begin{cases}
-\Delta v = \frac{f(x)}{|v|}, & x \in \Omega, \\
v(x) = 0, & x \in \partial \Omega,
\end{cases}
\] (9)

which is singular on the right-hand side. Let us remark that,
in the case of nonnegative \( f \), in [11], the authors considered
the elliptic semilinear problems whose model is

\[
\begin{cases}
-\Delta u = \frac{f}{u}, & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega,
\end{cases}
\] (10)

where \( \gamma > 0 \). More precisely, they have shown that the term
\( (f/|u|^\gamma) \) has a regularizing effect on the solutions \( u \). In [12],
the author has shown the existence of solutions to the
following elliptic problem with degenerate coercivity:

\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{(1 + |u|)^p} \right) = \frac{f}{|u|}, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\] (11)

where \( p, \gamma > 0 \).

The purpose of this paper is to study the same kind of
lower order term as in problems (7) and (9) (indeed,
\( (f/|u|^\gamma) \)) in the case of an elliptic operator with unbounded
coefficients. The main difficulties posed by this problem were
that the principal part of the differential operator
\( \text{div}((a(x) + |u|^p)\nabla u) \) is not well defined on the whole
\( H^1_0(\Omega) \); the solutions did not belong, in general, to \( H^1_0(\Omega) \)
and the lower order term has a singularity at \( u = 0 \). Despite
these difficulties, we prove that, in our case too, the lower
order term \( (f/|u|^\gamma) \) has a regularizing effect.

Our main existence results are as follows.

**Theorem 1.** Assume that (2) and (3) hold true. If
\( 0 \leq f \in L^m(\Omega) \) with \( m > (N/2) \), then there is a positive solution
\( u \in L^\infty(\Omega) \) of (1), in the sense of distributions, that is,

\[
\int_{\Omega} [a(x) + u^q] \nabla u \nabla \varphi = \int_{\Omega} \frac{f \varphi}{u},
\] (12)

for any test function \( \varphi \in C_0^\infty(\Omega) \). Moreover, we have the following
summability results for \( u \):

(1) Let \( 0 < q < 1 \):

(i) If \( 0 < \gamma \leq 1 - q \), then \( u \in H^1_0(\Omega) \).

(ii) If \( \gamma > 1 - q \), then \( u \in H^1_{1,\text{loc}}(\Omega) \).

(2) Let \( q = 1 \):

(i) If \( 0 < \gamma \leq 1 \), then \( u \in H^1_0(\Omega) \).

(ii) If \( \gamma > 1 \), then \( u \in H^1_{1,\text{loc}}(\Omega) \).

(3) Let \( q > 1 \), then \( u \in H^1_{1,\text{loc}}(\Omega) \).

When \( f \in L^m(\Omega) \), \( 1 < m < (N/2) \), we will prove the
following regularizing effects.

**Theorem 2.** We suppose that \( 0 \leq f \in L^m(\Omega) \), \( 1 < m < (N/2) \)
and that (2) and (3) are satisfied. If \( 0 < q < 1 \), then, there exists
a solution \( u \) of (1) in the sense (19), such that

(1) If \( \gamma < 1 - q \) and \( (2^{*}/2^{*} + q - 1 + \gamma) \leq m < (N/2) \),
then \( u \in H^1_0(\Omega) \cap L^{m^{*}}(\Omega) \), where

\[
m^{*} = \left( \frac{N}{N - 2m} \right).
\] (13)

(2) If \( \gamma = 1 - q \), then \( u \in H^1_0(\Omega) \).

(3) If \( \gamma > 1 - q \), then \( u \in L^{(1 + q)(1 + q)\gamma}(\Omega) \cap H^1_{1,\text{loc}}(\Omega) \).

Notation: throughout this paper, we fix an integer \( N \geq 3 \).
For any \( p > 1 \), \( p' = (p/p - 1) \) will be the Hölder conjugate
exponent of \( p \), and if \( 1 \leq p < N \), we will denote by \( p^* \)
\((Np/N - p)\) the Sobolev conjugate exponent of \( p \). As usual,
let us denote by \( \mathcal{S} \) the Sobolev constant, i.e.,

\[
\mathcal{S} = \inf_{u \in H^1_0(\Omega) \cap L^{2^{*}}(\Omega)} \frac{||\nabla u||_2^2}{||u||_2^2}.
\] (14)

We denote by \( \mathcal{D} \) the Poincaré constant given by

\[
\mathcal{D} = \inf_{u \in H^1_0(\Omega) \cap L^{2^{*}}(\Omega)} \frac{||\nabla u||_2^2}{||u||_2^2}.
\] (15)

For all \( k > 0 \), we recall the definition of a truncated
function \( T_k(s) \) defined by

\[
T_k(s) = \max\{\min\{k, s\} - k\}.
\] (16)

We also consider

\[
G_k(s) = s - T_k(s).
\] (17)

As usual, we consider the positive and negative part of a
measurable function \( u(x) \)
\[ u(x) = u^+(x) - u^-(x), \quad \text{where } u^+(x) = u(x) \chi_{\{u > 0\}} \text{ and } u^-(x) = -u(x) \chi_{\{u < 0\}}. \] (18)

2. The Approximated Problem

To prove our existence results, we will use the following approximating problems:

\[ -\text{div}\left( [a(x) + |u_n|^q] \nabla u_n \right) = \frac{f_n}{(|u_n| + (1/n))^\gamma}, \quad x \in \Omega, \]

where \( n \in \mathbb{N}^* \), and

\[ f_n(x) = \frac{f(x)}{1 + (1/n) |f(x)|}. \] (20)

As in [11], we prove existence of positive solution of the approximated problem.

**Lemma 1.** Let \( g \) be positive function belonging to \( L^\infty(\Omega) \). Suppose that (2) and (3) are satisfied. Then, there exists a positive solution \( u_n \in H_0^1(\Omega) \cap L^\infty(\Omega) \) of the problem

\[ -\text{div}\left( [a(x) + |u_n|^q] \nabla u_n \right) = \frac{g}{(|u_n| + (1/n))^\gamma}, \quad x \in \Omega, u_n \in H_0^1(\Omega). \] (21)

**Proof.** To prove it, we define the following operator \( S_n : L^2(\Omega) \to L^2(\Omega) \) which associates to every \( v \in L^2(\Omega) \) the solution \( u_n \in H_0^1(\Omega) \) to

\[ \begin{aligned}
\begin{cases}
-\text{div}\left( [a(x) + |u_n|^q] \nabla u_n \right) = \frac{g}{(|u_n| + (1/n))^\gamma}, & \text{in } \Omega, \\
\quad w_n = 0, & \text{on } \partial\Omega.
\end{cases}
\end{aligned} \] (22)

From the results of [13], the operator \( S_n \) is well defined and \( w_n \) is bounded by the results of [14]. We take \( w_n \) as a test function in (19), and we use Hölder’s inequality and (3) to deduce that

\[ \alpha \int_{\Omega} |\nabla w_n|^2 \leq \int_{\Omega} \left[ a(x) + |T_n(u_n)|^q \right] |\nabla w_n|^2 \leq \int_{\Omega} \frac{g|w_n|}{(|v| + (1/n))^\gamma} \leq n\|g\|_{L^\infty(\Omega)} \int_{\Omega} |w_n| \leq n\|g\|_{L^\infty(\Omega)} \sqrt{\Omega} \|\nabla w_n\|_{L^2(\Omega)}^\gamma \] (23)

Thanks to Poincaré’s inequality, we deduce

\[ \alpha \int_{\Omega} |w_n|^2 \leq \frac{n}{\gamma} \|n\|_{L^\infty(\Omega)} \sqrt{\Omega} \|w_n\|_{L^2(\Omega)}^\gamma \] (24)

Hence, there exists an invariant ball for \( S_n \). On the contrary, from the \( H_0^1(\Omega) \to L^2(\Omega) \) embedding, it is easily seen that \( S_n \) is continuous and compact. The Schauder theorem shows that \( S_n \) has a fixed point or equivalently, and there exists a solution \( u_n \in H_0^1(\Omega) \) to problems

\[ \begin{aligned}
\begin{cases}
-\text{div}\left( [a(x) + |T_n(u_n)|^q] \nabla u_n \right) = \frac{g}{(|u_n| + (1/n))^\gamma}, & \text{in } \Omega, \\
\quad u_n = 0, & \text{on } \partial\Omega.
\end{cases}
\end{aligned} \] (25)

Moreover, by the maximum principle, it is clear that the sequence \( u_n \) is nonnegative since \( g \) is nonnegative, and we choose \( G_k(u_n) \) as test function in (25) and use (3) to obtain

\[ \alpha \int_{A_k} |G_k(u_n)|^2 \leq \frac{1}{K} \int_{A_k} gG_k(u_n), \] (26)

where \( A_k = \{ x \in \Omega: |u_n| > k \} \). By the method of Stampacchia (see [14]), the sequence \( u_n \) is bounded in \( L^\infty(\Omega) \). Supposing that \( u_n \) is bounded by \( d_n \) in \( L^\infty(\Omega) \), we have that \( u_n = u_{n+1} \in L^\infty(\Omega) \cap H_0^1(\Omega) \) is a solution of (13).

By Lemma 1, it follows the existence of a solution \( u_n \in L^\infty(\Omega) \cap H_0^1(\Omega) \) of (19).

Now, we are going to prove that the sequence \( u_n \) is not 0 in \( \Omega \). For this, we are going to prove that it is uniformly away from zero in every compact set in \( \Omega \). We will follow a similar technique to that one in [12].

**Lemma 2.** Assume that (2) and (3) hold true. If \( 0 \preceq g \in L^1(\Omega) \) and \( u_n \) is the solution of problem (19), then for every \( n \in \mathbb{N}^* \), \( u_n \leq u_{n+1} \) a.e. in \( \Omega \). Furthermore, if \( \omega \subset \Omega \), then, for every \( n \in \mathbb{N}^* \), there exists \( c_\omega > 0 \) such that \( u_n \geq c_\omega > 0 \) a.e. in \( \omega \).

**Proof.** Let us consider \( T_k \left[ (u_n - u_{n+1})^+ \right] \) as a test function in problems (19). Then,
Observing that \( f_n \leq f_{n+1} \), we have

\[
\int_{\Omega} \frac{f_n}{(u_n + (1/n))} T_k [(u_n - u_{n+1})^+] \leq \int_{\Omega} \frac{f_{n+1}}{(u_{n+1} + (1/(n+1)))} T_k [(u_n - u_{n+1})^+] \\
= \int_{\Omega} [a(x) + u_{n+1}^a] \nabla u_{n+1} \nabla T_k [(u_n - u_{n+1})^+] \\
\leq \int_{\Omega} [a(x) + u_n^a] \nabla u_{n+1} \nabla T_k [(u_n - u_{n+1})^+].
\]

Therefore, by (3), we deduce that

\[
al \int_{\Omega} \| \nabla T_k [(u_n - u_{n+1})^+] \|^2 \leq \int_{\Omega} [a(x) + u_n^a] \| \nabla T_k [(u_n - u_{n+1})^+] \|^2 \leq 0.
\]

Consequently, we obtain \( \int_{\Omega} | \nabla T_k [(u_n - u_{n+1})^+] |^2 = 0 \), so by Poincaré’s inequality, we have \( T_k [(u_n - u_{n+1})^+] = 0 \) for every \( k > 0 \). Thus, \( u_n \leq u_{n+1} \) a.e. in \( \Omega \).

We remark that \( u_1 \) is bounded; indeed, \( |u_1| \leq c \), for some positive constant \( c \). Then, it follows that

\[
-\text{div} \left( \left[ a(x) + |u_1|^q \right] \nabla u_1 \right) \geq f_1 \left( \frac{1}{c+1} \right)^q, \quad x \in \Omega.
\]

Thanks to (3), we have \( a \leq a(x) + |u_1|^q \leq \beta + c \). Thus, we infer that \( u_1 \) is a supersolution of a linear Dirichlet problem with a strictly positive and bounded, measurable coefficient. The strong maximum principle implies that \( u_1 > 0 \). In addition, Harnack’s inequality gives the stronger conclusion: for every \( \omega \subset \subset \Omega \), there exists \( c_\omega \) such that \( u_1 \geq c_\omega \) a.e. in \( \omega \). Finally, using that the sequence \( u_n \) is increasing, one deduces that \( u_n \geq c_\omega \) a.e. in \( \omega \) for every \( n \in \mathbb{N}^* \). □

2.1. Existence of Bounded Solutions. In this section, we will prove existence of bounded weak solutions for (1).

**Lemma 3.** Let \( 0 \leq f \in L^m(\Omega) \) with \( m > (N/2) \). Suppose that (2) and (3) hold true. Let \( \{u_n\} \) be a sequence solutions of (19) with \( f_n = f \) for every \( n \in \mathbb{N}^* \). Then, the norm of the sequence \( \{u_n\} \) is bounded by a constant which depends on \( q, m, N, \alpha, \gamma, \text{meas}(\Omega) \) and on the norm of \( f \) in \( L^m(\Omega) \).

\[
(1-q)\min(\alpha, 1) \int_{\Omega} |\nabla u_n|^2 \leq C \int_{\Omega} f |u_n|^{1-q-\gamma} \quad (32)
\]

and thus (since \( q \leq 1 \),

\[
(1-q)\min(\alpha, 1) \int_{\Omega} |\nabla u_n|^2 \leq C \int_{\Omega} f |u_n|^{1-q-\gamma} \leq C \|u_n\|_{L^\infty(\Omega)}^{1-q-\gamma} \int_{\Omega} f \leq C.
\]

from which the sequence \( u_n \) is bounded in \( H^1_0(\Omega) \). □
Lemma 5. Let \( 0 < f \in L^m(\Omega) \) with \( m > (N/2) \), and we suppose that (2) and (3) are satisfied. If \( q \leq 1 \) and \( \gamma > 1 - q \) and \( u_n \) is a solution to problem (19), then \( u_n \) is uniformly bounded in \( H^1_{loc}(\Omega) \).

Proof. Let \( \varphi \in C^1_c(\Omega) \) and \( \omega = \text{Supp} \varphi \) be the support of \( \varphi \); then, from Lemma 2, there exists \( c_\omega > 0 \) such that \( u_n \geq c_\omega \) for a.e. \( x \in \omega \).

Choosing \([u_n(1)-1]^{1/2} \varphi^2 \) as test function in (19) and using (3), we obtain

\[
\alpha (1-q) \int_{\Omega} |\nabla u_n|^2 \varphi^2 + 2 \int_{\Omega} [a(x) + u_n^q] [(u_n + 1)^{1-q} - 1] \nabla u_n \nabla \varphi \\
\leq \int_{\Omega} \frac{f_n}{(u_n + (1/m))^{\gamma}} [(u_n + 1)^{1-q} - 1] \varphi^2 \leq \frac{\|\varphi\|_{L^q(\Omega)}}{c_\omega} \int_\Omega f,
\]

which then implies

\[
\alpha (1-q) \int_{\Omega} |\nabla u_n|^2 \varphi^2 \\
\leq \frac{\|\varphi\|_{L^q(\Omega)}}{c_\omega} \int_{\Omega} f - 2 \int_{\Omega} [a(x) + u_n^q] [(u_n + 1)^{1-q} - 1] \nabla u_n \nabla \varphi.
\]

Using (3), we have

\[
a(x) + t^q \leq c_0 (1 + t)^q,
\]

for every \( q > 0 \) and \( t \geq 0 \) (and for a suitable \( c_0 \) independent on \( n \)). We then have

\[
2 \int_{\Omega} [a(x) + u_n^q] [(u_n + 1)^{1-q} - 1] \nabla u_n \nabla \varphi \\
\leq \varepsilon \int_{\Omega} |\nabla u_n|^2 \varphi^2 + C(\varepsilon) \int_{\Omega} [a(x) + u_n^q]^2 [(u_n + 1)^{1-q} - 1] \varphi^2.
\]

Applying (38) to (35) and letting \( \varepsilon = (\alpha (1-q)/2) \), we obtain

\[
\int_{\Omega} |\nabla u_n|^2 \varphi^2 \leq C + C \int_{\Omega} u_n^q |\nabla \varphi|^2 \leq C + C \|\varphi\|_{L^q(\Omega)}^2 \int_{\Omega} |\nabla \varphi|^2 \leq C,
\]

and this gives that \( u_n \) is bounded in \( H^1_{loc}(\Omega) \). \( \square \)

Lemma 6. Let \( q = 1 \). Suppose that (2) and (3) hold. If \( 0 < f \in L^m(\Omega) \) with \( m > (N/2) \), then the sequence \( \{u_n\} \) defined by (19) satisfies the following summability:

1. If \( 0 < \gamma \leq 1 \), then \( u_n \) is uniformly bounded in \( H^1(\Omega) \).
2. If \( \gamma > 1 \), then \( u_n \) is uniformly bounded in \( H^1_{loc}(\Omega) \).

Proof. (1) Let us take \( \log (1 + u_n) \) as test function in (19) and use (3) to obtain that

\[
\min(1, \alpha) \int_{\Omega} |\nabla u_n|^2 \leq \int_{\Omega} f \log (1 + u_n) \leq \int_{\Omega} f u_n^{1-\gamma} \\
\leq \|u_n\|_{L^\infty(\Omega)} \int_{\Omega} f \leq C.
\]
(2) Let $\varphi \in C^1_c(\Omega)$ and choose $\log(1 + u_n)\varphi^2$, as a test function in problem (19). From assumption (19), one has

$$\min(1, \alpha) \int_\Omega |\nabla u_n|^2 \varphi^2 + 2 \int_\Omega [a(x) + u_n^q] \log(1 + u_n)\nabla u_n \nabla \varphi$$

$$\leq \int_\Omega f \log(1 + u_n) \varphi^2 \leq \int_\Omega f \frac{\varphi^2}{(1 + u_n^q)^{1/2}} \leq \frac{\|\varphi\|_{L^2(\Omega)}^2}{c_{\omega}} \int_\Omega f, \quad (41)$$

where $\omega = \text{Supp}\varphi$. By Young's inequalities, it is easy to prove

$$2 \int_\Omega [a(x) + u_n^q] \log(1 + u_n)\nabla u_n \nabla \varphi \leq \epsilon \int_\Omega |\nabla u_n|^2 \varphi^2 + C(\epsilon). \quad (42)$$

Hence, equality (41) implies that

$$\min(1, \alpha) \int_\Omega |\nabla u_n|^2 \varphi^2 \leq \frac{1}{(1 + u_n^q)^{1/2}} \int_\Omega f + \epsilon \int_\Omega |\nabla u_n|^2 \varphi^2 + C(\epsilon). \quad (43)$$

Letting $\epsilon = (\min(1, \alpha)/2)$, we get that $u_n$ is bounded in $H^1_{\text{loc}}(\Omega)$. \qed

**Lemma 7.** Let $q > 1$. Assume that (2) and (3) hold true. If $0 \leq f \in L^{2q}(\Omega)$ with $m > (N/2)$, then the solution $u_n$ of (19) is uniformly bounded in $H^1_{\text{loc}}(\Omega)$.

**Proof.** Let $\varphi$ be a function in $C^1_c(\Omega)$ and $\omega = \text{Supp}\varphi$. Take $[1 - (u_n + 1)^{1-q}]\varphi^2$ as test function in (19) and use (3) to obtain

$$\min(1, \alpha) \int_\Omega |\nabla u_n|^2 \varphi^2 \leq (q - 1)\min(1, \alpha) \int_\Omega \frac{1}{1 + u_n^q} |\nabla u_n|^2 \varphi^2$$

$$\leq \frac{f}{(1 + u_n)^{1/2}} \int_\Omega \varphi^2 - 2 \int_\Omega [a(x) + u_n^q] [1 - (u_n + 1)^{-q}]\nabla u_n \nabla \varphi. \quad (44)$$

Using Young's inequality with $\epsilon$, we have by (3) and Lemma 3 that

$$2 \int_\Omega [a(x) + u_n^q] [1 - (u_n + 1)^{-q}]\nabla u_n \nabla \varphi$$

$$\leq \epsilon \int_\Omega |\nabla u_n|^2 \varphi^2 + C(\epsilon) \int_\Omega |\varphi|^2. \quad (45)$$

Taking the above estimate in (44) and letting $\epsilon = (\min(1, \alpha)/2)$, we obtain

$$\min(1, \alpha) \int_\Omega |\nabla u_n|^2 \varphi^2 \leq \frac{\|\varphi\|_{L^2(\Omega)}^2}{c_{\omega}} \int_\Omega f + C, \quad (46)$$

and thus, Lemma 7 is proved. \qed

**Proof.** of Theorem 1.

We start by proving point (1.i), the rest of the proof of the theorem can be proven similarly. According to Lemmas 3 and 4, there exists a subsequence $u_n$ and a function $u \in H^1_0(\Omega) \cap L^{\infty}(\Omega)$ such that $u_n$ weakly converges to $u$ in $H^1_0(\Omega)$. Now, we can pass to the limit in the equation satisfied by the approximated solutions $u_n$:

$$\int_\Omega [a(x) + u_n^q] \nabla u_n \nabla \varphi = \int_\Omega \frac{f_n \varphi}{(u_n + (1/n))^\gamma}, \quad \forall \varphi \in C^1_c(\Omega), \quad (47)$$

where $f_n(x) = (f(x)/1 + (1/n)) f(x)$.

For the term of the left-hand side, it is sufficient to observe that $\nabla u_n$ converge to $\nabla u$ weakly in $L^2(\Omega)$ and $[a(x) + u_n^q]$ a.e. (and weakly $\ast$ in $L^{\infty}(\Omega)$) converges towards $[a(x) + u^q]$. On the contrary, for the limit of the right-hand side of (47), let $\omega = \text{Supp}\varphi$, and one can use Lebesgue's dominated convergence theorem, since

$$\frac{f \varphi}{(u + (1/n))^\gamma} \leq \frac{|\varphi|}{c_{\omega}}. \quad (48)$$

Finally, passing to the limit as $n$ goes to infinity in equation (47), we conclude that

$$\int_\Omega [a(x) + u^q] \nabla u \nabla \varphi = \int_\Omega \frac{f \varphi}{u^\gamma}, \quad \forall \varphi \in C^1_c(\Omega). \quad (49)$$

**2.2. Further Existence Result.** In this section, we suppose (2) and (3) and we assume that

$$0 < q < 1 \quad (50)$$

holds true.

**Lemma 8.** We suppose that (2), (3), and (50) hold true. Let $\gamma < 1 - q$ and $\theta_n f \in L^{2q}(\Omega)$, with

$$\frac{2^*}{2^* + q - 1 + \gamma} \leq m < \frac{N}{2} \quad (51)$$

Then, the solutions $u_n$ to problem (19) are uniformly bounded in $H^1_0(\Omega) \cap L^{m^{\gamma}(1+q/\gamma)}(\Omega)$.

**Proof.** Let us take $(1 + u_n)^{1-q} - 1$ as a test function in (19) and use assumption (3) to obtain
\[(1 - q)\min(1, a) \int_\Omega |\nabla u_n|^2 \leq (1 - q) \int_\Omega \frac{a(x) + u_n^q}{(1 + u_n)^q} |\nabla u_n|^2 \leq C \int_\Omega f |u_n|^{1-q} \gamma. \]

\[(52)\]

We can use Hölder’s inequality on the right-hand side with exponent \(p = (2^*/2^* + q - 1 + \gamma) = (2N/N(\gamma + 1 + q) + 2(1 - q - \gamma)) > 1\), and Sobolev inequality on the left-hand side to deduce

\[
\min(1, a) (1 - q) \left( \int_\Omega u_n^{2^*/2^*} \right)^{2/2^*} \leq C \left( \int_\Omega u_n^{p^*(1-q-\gamma)} \right)^{1/p^*}.
\]

\[(53)\]

We note that \(2^* = p^*(1 - q - \gamma)\); moreover, \((2/2^*) \geq (1/p^*)\) (thanks to the fact that \(\gamma < 1 - q\)). This last estimate imply that \(u_n\) is uniformly bounded in \(L^{2^*} (\Omega)\) and in \(H^1_0 (\Omega)\).

We are going to prove now that the sequence \(u_n\) is bounded in \(L^{m^*} ((1 + q)^{-1}) (\Omega)\). Let \(\lambda = (N + m - 1 + ym(2N - 2)/N - 2m)\) and \((1 + u_n)^{1-q} - 1\) as a test function for problem (19), we can deduce

\[
\lambda \min(1, a) \int_\Omega \left| \frac{|\nabla u_n|^2}{(1 + u_n)^{-1-q}} \right| \leq C + \int_\Omega f |u_n|^{m^*(1-q-\gamma)} \leq C + \int_\Omega f |u_n|^{m^*(1-q-\gamma)} \leq C.
\]

\[(54)\]

Now, we rewrite

\[
\frac{4\lambda \min(1, a)}{(1 + q + \lambda)^2} \int_\Omega |\nabla \left[ (1 + u_n)^{1+q}\lambda/2 \right] - 1|^{2/2^*} \leq \lambda \min(1, a) \int_\Omega |\nabla u_n|^2 (1 + u_n)^{-1-q} \gamma.
\]

\[(55)\]

and use the Sobolev inequality and the Hölder inequality in (54) to obtain

\[
\left( \int_\Omega |\nabla \left[ (1 + u_n)^{1+q}\lambda/2 \right] - 1|^{2/2^*} \right)^{2/2^*} \leq \left( \int_\Omega |u_n + 1|^{m^*(1-q-\gamma)} \right)^{1/m^*} \gamma.
\]

\[(56)\]

We note that the choice of \(\lambda\) is equivalent to require \((2/2^*) (1 + q + \lambda) = m^*(1 - q)\); furthermore, \((2/2^*) \geq (1/m^*)\) and \((2/2^*) (1 + q + \lambda) = m^*(1 + q + \gamma)\). Thus, the sequence \(\{u_n\}\) is uniformly bounded in \(L^{m^*} ((1+q)^{-1}) (\Omega)\).

\section*{Lemma 9.}

Under the hypotheses \(0 \leq f \in L^1 (\Omega)\), (2), (3), and (50), if \(\gamma > 1 - q\), then the solutions \(u_n\) are uniformly bounded in \(H^1_0 (\Omega)\).

\section*{Proof.}

We choose \((1 + u_n)^{1-q} - 1\) as test function in (19) to obtain, by hypothesis (3), that

\[
(1 - q) \min(1, a) \int_\Omega |\nabla u_n|^2 \leq (1 - q) \int_\Omega \frac{a(x) + u_n^q}{(1 + u_n)^q} |\nabla u_n|^2 \leq C \int_\Omega f.
\]

\[(57)\]

We are going to prove now that the sequence \(u_n\) is bounded in \(L^{m^*} (\Omega)\). Let \(\varphi \in C^1_0 (\Omega)\) and choose \((1 + u_n)^{1-q} - 1\) as a test function in problems (19). From assumption (19), one has

\[
\min(1, a) \int_\Omega |\nabla u_n|^2 \varphi^2 + 2 \int_\Omega \left[ a(x) + u_n^q \right] |\nabla u_n|^2 \leq C \int_\Omega \|\varphi\|_{L^2(\Omega)}^2 + \frac{f_n}{C^{q-1}} \int_\Omega \varphi.
\]

\[(60)\]
where \( \omega = \text{Supp} \varphi \). We can use Young’s inequality with \( \varepsilon \) and both (37) and (59) to obtain
\[
2\int_{\Omega} \left[ (a(x) + u_n^2) \left( (u_n + 1)^{1-q} - 1 \right) \nabla u_n \cdot \nabla \varphi \right] + C(\varepsilon) \int_{\Omega} [a(x) + u_n^q] \left( (u_n + 1)^{1-q} - 1 \right) |\nabla \varphi|^2
\leq \varepsilon \int_{\Omega} |\nabla u_n|^2 \varphi^2 + C(\varepsilon) \int_{\Omega} u_n^2 |\nabla \varphi|^2
\leq \varepsilon \int_{\Omega} |\nabla u_n|^2 \varphi^2 + C(\varepsilon).
\]
\]
Hence, equality (60) implies that
\[
\min(1, \alpha) \int_{\Omega} |\nabla u_n|^2 \varphi^2 \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{\varepsilon \omega} \int_{\Omega} f + \int_{\Omega} |\nabla u_n|^2 \varphi^2 + C(\varepsilon).
\]
(62)

Letting \( \varepsilon = (\min(1, \alpha)/2) \), we get that \( u_n \) is bounded in \( H^1_{\text{loc}}(\Omega) \).

**Lemma 11.** Under the assumptions of Theorem 2, let \( u_n \) be a solution to problem (19). Then, the sequence \( u_n \) is uniformly bounded in \( L^\infty_{\text{loc}}(\Omega) \), for every \( \sigma < (N/N - 1) \).

**Proof.** We will prove our proof in two steps:

Step 1: we want to prove that, for every \( \lambda > 1 \), \( (1 + u_n)^{\sigma - 1} |\nabla u_n|^2 \in L^r_{\text{loc}}(\Omega) \). Indeed, let \( \lambda > 1 \), \( \varphi \in C^\infty(\Omega) \) and \( \omega = \text{Supp} \varphi \). Thanks to (3), we have from (19) with test function \( 1 - (1/(1 + u_n)^{1-1}) |\varphi|^2 \)

Thus, by the above estimate and since \( u_n \) is uniformly bounded in \( H^1_{\text{loc}}(\Omega) \), this proves Step 1.

Step 2: here, we show that \( u_n^\sigma |\nabla u_n| \) is uniformly bounded in \( L^r_{\text{loc}}(\Omega) \) for every \( r < (N/Nt - n1) \). For this, let \( \sigma < 2 \), \( 0 < \varphi \in C^\infty(\Omega) \), and \( \omega = \text{Supp} \varphi \). We use Hölder inequality with exponent \( 2/\sigma \) and by step 1, and we obtain

\[
\int_{\Omega} u_n^\sigma |\nabla u_n|^2 \varphi^\sigma
\leq C(\omega) \left( \int_{\Omega} (1 + u_n)^{\sigma(1+q)/2} |\nabla \varphi|^2 \right)^{\sigma/2}
\leq C(\omega) \left( \int_{\Omega} (1 + u_n)^{\sigma(1+q)/2} \varphi^\sigma \right)^{2-\sigma/2}.
\]
(65)
Using the Sobolev inequality, we obtain

$$\left( \int_\Omega \left( u_n^{(\sigma^2+\sigma)} \right)^{\sigma^2} \right)^{\frac{\sigma^2}{\sigma}} \leq C(\omega) \left( \int_\Omega \left( 1 + u_n \right)^{(\frac{\sigma^2}{\sigma}+1)} \right)^{\frac{\sigma^2}{\sigma}+1} + C(\omega).$$

(66)

Noticing that \((\sigma^2+\sigma) \geq 2 - \sigma^2/2\) and choosing \(\sigma\) such that

\((q+1)^{\sigma^2} = (\sigma + q)/(2 - \sigma)\)

yields

\(\sigma = N (2 + q - \lambda)/N (q + 1) - (\lambda + q)).\)

Using Young’s inequality with \(\varepsilon\), we obtain

$$\left( \int_\Omega \left( u_n^{(\sigma^2+\sigma)} \right)^{\sigma^2} \right)^{\frac{\sigma^2}{\sigma}} \leq \varepsilon \left( \int_\Omega \left( 1 + u_n \right)^{(\sigma^2+\sigma)} \right)^{\frac{\sigma^2}{\sigma}} + C(\omega, \varepsilon).$$

(67)

It is easy to check that the hypotheses \(\lambda > 1\) imply \(\sigma < (N/N - 1) < 2\).

**Proof.** of Theorem 2.

The proof of the theorem is similar to the proof of the previous theorem with just a small change for the convergence of the term on the left side of equation (47). Indeed, using Lemma 11, we have that

\([a(x) + \varepsilon_\omega^2]u_n \rightharpoonup [a(x) + \varepsilon]u\]

is weak in \((L^\infty_{\text{loc}}(\Omega))\) for every \(\varepsilon < (N/N - 1).\)

Hence, for every \(\phi \in C_0^1(\Omega)\), we can pass to the limit with respect to \(n\) in the integral on the left-hand side of (47). □

**Remark 1.** Assume that (2) and (3) are satisfied. We can choose \(u_n\), as test function in (19), using (3), and we obtain that

$$\frac{4\gamma}{(\gamma + q + 1)^2} \int_\Omega |\nabla (u_n^{(\gamma+q+1)})|^2 = \gamma \int_\Omega |\nabla u_n|^2 u_n^{(\gamma+q-1)},$$

\(\leq \gamma \int_\Omega [a(x) + u_n^2]|\nabla u_n|^2 u_n^{(\gamma+q-2)} \leq \int_\Omega f.$$

(68)

We deduce from (68) that the sequence \(u_n^{(\gamma+q+1)}\) is bounded in \(H^1_0(\Omega)\). Therefore, \(u^{(\gamma+q+1)}\) belongs to \(H^1_0(\Omega)\).

**Data Availability**

No data were used to support the study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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