Quantum Action Principle in Relativistic Mechanics

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A quantum version of the action principle is considered in the case of a free relativistic particle. The classical limit of the quantum action is obtained.

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I. INTRODUCTION

In the works [1, 2] a new form of non-relativistic quantum mechanics in terms of a quantum action principle was proposed. The quantum action principle was formulated for a new object - a wave functional \( \Psi [x(t)] \) which, unlike a wave function \( \psi(x,t) \), describes dynamics of a particle as a movement along a trajectory \( x(t) \). The wave functional has the meaning of a probability density in the space of trajectories. It is this description of dynamics that is most appropriate for relativistic quantum mechanics. In the relativistic mechanics a trajectory of a particle must be replaced by an invariant geometrical object - a world line, \( x^{\mu} (\tau), \mu = 0, 1, 2, 3 \) in the Minkowsky space. Here \( \tau \) is an arbitrary parameter along the world line. As a result, the wave functional \( \Psi [x^{\mu} (\tau)] \) becomes relativistic invariant preserving its probabilistic interpretation.

A special feature of relativistic mechanics is the presence of an additional symmetry - an independence of the action on the parametrization of a world line of a particle (see, for example, [3]). In ordinary quantum mechanics based on a wave function, it is necessary for its probabilistic interpretation to fix a time parameter by use of an additional gauge condition. In our approach the re-parametrization invariance must be unbroken. Gauge parameters which ensure invariance of the action must be added to a set of variational parameters of the quantum action principle. Therefore, the advantage of the new approach is the possibility of probabilistic interpretation of the quantum theory without a loss of its covariance. In the present paper the quantum action principle is considered in the case of a free relativistic particle. The classical limit of the quantum action is obtained.

II. QUANTUM ACTION PRINCIPLE

We begin with the action of a particle in the geometrical form (the velocity of light is equal to 1):

\[
I = -m \int ds,
\]

where \( m \) is a mass of a particle, and

\[
ds^2 = dx^2 \equiv dx^\mu dx_\mu
\]

is the interval in the Minkowsky space. Here the Greek indices are lowered and raised by means of metric tensor with the signature \((+,-,-,-)\). Introducing an arbitrary parametrization of a world line, \( x^{\mu} = x^{\mu} (\tau), \tau \in [0,1] \), and defining a 4-vector of the canonical momentum,

\[
p_\mu \equiv -m \frac{\dot{x}^\mu}{\sqrt{-x}}
\]

where the dot denotes the derivative with respect to the parameter \( \tau \), one can write action (1) in the canonical form:

\[
I = \frac{1}{2} \int_0^1 (p_\mu \dot{x}^\mu - \chi H) d\tau.
\]

Here \( \chi \) is a new variable which ensures the re-parametrization invariance of the action (1) and

\[
H \equiv p^2 - m^2.
\]

At this stage an invariant parameter along a world line can be introduced:

\[
c(\tau) = \int_0^\tau \chi d\tau.
\]

Then action (1) takes a form:

\[
I = \int_0^C (p_\mu \dot{x}^\mu - H) dc,
\]

where now the dot denotes the derivative with respect to the parameter \( c \), and \( C \equiv c(1) \).

Let us quantize action (1) following [1]. In the space of functionals \( \Psi [x^{\mu} (c)] \) we define a functional-differential operator:

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where \( \tilde{h} \) is a constant with the dimensionality \( D_j \cdot s^2 \). For an action operator \( \mathcal{I} \) which is obtained by substitution of \( \mathbf{v} \) into \( \mathbf{u} \), we consider the following secular equation:

\[
\lambda = \frac{\tilde{h}}{i} \int \frac{d\lambda}{\delta x^\mu} \left[ x^\mu \frac{\delta \sigma}{\delta x^\nu} - \left( \frac{\delta \lambda}{\delta x^\nu} \right)^2 \right] + \tilde{h}^2 \left( \frac{\delta r}{\delta x^\nu} \right)^2 + \frac{\delta^2 r}{(\delta x^\nu)^2} \right] \, dc + m^2 \mathcal{C}, \tag{11}
\]

and, in addition, a condition of its reality,

\[
\int_0^C \left[ x^\mu \frac{\delta r}{\delta x^\nu} - 2 \frac{\delta \sigma}{\delta x^\nu} \frac{\delta r}{\delta x^\nu} - \frac{\delta^2 \sigma}{(\delta x^\nu)^2} \right] \, dc = 0. \tag{12}
\]

Representation (11) is not final because eigenvalues must be independent on a world line \( x^\mu (c) \) except for boundary points \( b^\mu \equiv x^\mu (C) \) and \( a^\mu \equiv x^\mu (0) \) which are supposed to be fixed in the action principle. This demand imposes a set of differential equations on coefficients of series which are represented by the functionals \( \sigma [x^\mu (c)] \) and \( r [x^\mu (c)] \). A solution of this set of equations depends on initial values of these coefficients at the moment \( c = 0 \). Therefore, we obtain an eigenvalue \( \lambda \) as a function of initial data. It is this function that must be stationary in the quantum action principle. The variable \( C \) also must be added to the set of variational parameters. In the next section, a quasi-classical approach for the quantum action principle will be considered, and the classical limit of the quantum action of a free relativistic particle will be obtained.

### III. CLASSICAL LIMIT OF QUANTUM ACTION

In the classical limit, the eigenvalue \( \lambda \) of the quantum action, Eq. (11), is completely defined by the functional \( \sigma [x^\mu (c)] \). In the case of a free relativistic particle considered here, one can take into account only integral functionals in the following form:

\[
\sigma [x^\mu (c)] = \int_0^C \left[ \sigma_1 (c) x^\mu (c) + \frac{1}{2} \sigma_2 (c) (x)^2 + \ldots \right] \, dc. \tag{13}
\]

In the classical limit, one can consider functionals which are not higher than quadratic in \( x^\mu (c) \). Substituting (13) into Eq. (11), after integration by parts one obtains the final form of the eigenvalue \( \lambda \),

\[
\lambda = \left( \sigma_1 x^\mu + \frac{1}{2} \sigma_2 (x)^2 \right) \bigg|_0^C - \int_0^C \sigma_2 \, dc + m^2 \mathcal{C}, \tag{14}
\]

and the condition of its independence on a world line \( x^\mu (c) \) in terms of two differential equations,

\[
\dot{\sigma}_1 + 2 \sigma_2 \sigma_1 = 0, \tag{15}
\]

\[
\dot{\sigma}_2 + 2 \sigma_2 = 0. \tag{16}
\]

A general solution of equations (15) and (16) is

\[
\sigma_1 (c) = \frac{\sigma_1 (0)}{1 + 2 \sigma_2 (0) C}, \tag{17}
\]

\[
\sigma_2 (c) = \frac{\sigma_2 (0)}{1 + 2 \sigma_2 (0) C}, \tag{18}
\]

where \( \sigma_1 (0) \) and \( \sigma_2 (0) \) are initial values of the coefficients \( \sigma_1 \) and \( \sigma_2 \). Substitution of this solution into Eq. (14) gives the eigenvalue \( \lambda \) as a function of the initial data, \( \sigma_1 (0), \sigma_2 (0) \), and the invariant time parameter \( C \):

\[
\lambda = \sigma_1 \left( \frac{b^\mu}{1 + 2 \sigma_2 (0) C} - a^\mu \right) + \frac{\sigma_2 (0)}{2} \left( \frac{(b^\mu)^2}{1 + 2 \sigma_2 (0) C} \right.

\]

\[
- \left( a^\mu \right)^2 - \left( \frac{\sigma_1 (0)}{1 + 2 \sigma_2 (0) C} \right)^2 + m^2 \mathcal{C}. \tag{19}
\]

It is this function that must be stationary with respect to the initial data, \( \sigma_1 (0), \sigma_2 (0), \) and the invariant time parameter \( C \) in the quantum action principle. The extremum condition with respect to the initial data gives
\[ \sigma_{\mu}^{(0)} = \frac{1}{2C} \left[ b_\mu - a_\mu \left( 1 + 2\sigma_2^{(0)} C \right) \right]. \quad (20) \]

Therefore, one of the initial data parameters, \( \sigma_2^{(0)} \) in this case, is not fixed in the classical limit of the quantum action principle, and the eigenvalue \( \lambda \) is degenerate. The extremum condition with respect to \( C \) gives

\[ C = \pm \frac{1}{2m} \sqrt{(b-a)^2}. \quad (21) \]

Substituting (20), (21) into Eq. (19), one obtains the quantum action eigenvalue in the classical limit:

\[ \lambda = \pm m \sqrt{(b-a)^2}. \quad (22) \]

This result coincides with classical action (1) calculated on the classical trajectory of a free particle. The wave functional corresponding to eigenvalue (22) has a phase which in the classical limit is proportional to:

\[ \sigma [x^\mu] = \frac{1}{4} \int_0^Q \left( x^\mu (q) - \bar{x}^\mu \right)^2 dq, \quad (23) \]

where

\[ \bar{x}^\mu = -\frac{b^\mu - e^Q a^\mu}{e^Q - 1}, \quad (24) \]

\[ Q \equiv \ln \left( 1 + 2\sigma_2^{(0)} C \right). \quad (25) \]

IV. CONCLUSIONS

We conclude that in the classical limit the quantum action principle returns us to the original action of a relativistic particle calculated on a classical trajectory. Quantum corrections to this action will give us essential predictions of the new theory and define a new "Plank" constant \( \bar{\hbar} \). The parameter \( \sigma_1^{(0)} \), which is indefinite in the classical limit, plays the role of a degree of excitation of a quantum particle.

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[2] Natalya Gorobey, and Alexander Lukyanenko, arXiv: 0810.2255 (October 2008).
[3] Michael B. Green, John H. Schwarz, and Edward Witten, *Superstring Theory* (Cambridge Univ. Press, N.Y., 1987).