HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR S–CONVEX FUNCTION IN THE SECOND SENSE VIA FRACTIONAL INTEGRALS

ERHAN SET*, IMDAT ISCAN▲, AND HASAN HÜSEYIN KARA■

Abstract. In this paper, we established Hermite-Hadamard-Fejer type inequalities for s–convex functions in the second sense via fractional integrals. The some results presented here would provide extensions of those given in earlier works.

1. INTRODUCTION

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality [10]:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

where \(f : I \subset \mathbb{R} \rightarrow \mathbb{R}\) is a convex function on the interval \(I\) of real numbers and \(a,b \in I\) with \(a < b\). A function \(f : [a,b] \subset \mathbb{R} \rightarrow \mathbb{R}\) is said to be convex if whenever \(x,y \in [a,b]\) and \(t \in [0,1]\) the following inequality holds

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).
\]

In [9], Fejér gave a generalization of the inequalities (1.1) as the following:

If \(f : [a,b] \rightarrow \mathbb{R}\) is a convex function, and \(g : [a,b] \rightarrow \mathbb{R}\) is nonnegative, integrable and symmetric about \(\frac{a+b}{2}\), then

\[
f\left(\frac{a + b}{2}\right) \int_a^b g(x) \, dx \leq \int_a^b f(x) g(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx.
\]

If \(g \equiv 1\), then we are talking about the Hermite–Hadamard inequalities. More about those inequalities can be found in a number of papers and monographies (for example, see [7]-[19]).

In [11], Hudzik and Maligrada considered among others the class of functions which are s-convex in the second sense.

Definition 1. A function \(f : [0, \infty) \rightarrow \mathbb{R}\) is said to be s-convex in the second sense if

\[
f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y).
\]

for all \(x, y \in [0, \infty)\), \(\lambda \in [0,1]\) and for some fixed \(s \in (0,1]\).

2000 Mathematics Subject Classification. 26D07, 26D15.
Key words and phrases. s-convex Function, Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Riemann-Liouville fractional integral.
It can be easily seen that \( s = 1 \), \( s \)-convexity reduces to ordinary convexity of functions defined on \([0, \infty)\).

In [7], Dragomir and Fitzpatrick proved Hadamard’s inequality which holds for \( s \)-convex functions in the second sense.

**Theorem 1.** Suppose that \( f : [0, \infty) \rightarrow [0, \infty) \) is an \( s \)-convex functions in the second sense, where \( s \in (0, 1) \), and let \( a, b \in [0, \infty), \, a < b \). If \( f \in L[a, b] \), then the following inequalities hold:

\[
2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s + 1}
\]

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 2.** Let \( f \in [a, b] \). The Riemann–Liouville integrals \( J_{a+}^\alpha f \) and \( J_{b-}^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by integrals hold:

\[
J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) \, dt, \quad x > a
\]

and

\[
J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) \, dt, \quad x < b
\]

respectively where \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} \, dt \). Here is \( J_{a+}^\alpha f(x) = J_{b-}^\alpha f(x) = f(x) \)

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral. The recent results and the properties concerning this operator can be found ([11, 13])

In [15], Sarıkaya et al. represented Hermite-Hadamard’s inequalities in fractional integral forms as follows.

**Theorem 2.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be positive function with \( 0 \leq a < b \) and \( f \in L[a, b] \). If \( f \) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold

\[
f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}
\]

with \( \alpha > 0 \).

In [12], İscan gave the following Hermite-Hadamard-Fejer integral inequalities via fractional integrals:

**Theorem 3.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be convex function with \( a < b \) and \( f \in L[a, b] \). If \( g : [a, b] \rightarrow \mathbb{R} \) is nonnegative, integrable and symmetric to \((a + b)/2\), then the following inequalities for fractional integrals hold

\[
f \left( \frac{a + b}{2} \right) \left[ J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a) \right] \leq \left[ J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a) \right] \leq \frac{f(a) + f(b)}{2} \left[ J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a) \right]
\]

with \( \alpha > 0 \).

Set et al. established some inequalities connected with the left-hand side of the inequality ([13]) used the following lemma.
Lemma 1. [19] Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$ and let $g : [a, b] \to \mathbb{R}$. If $f', g \in L[a, b]$, then the following identity for fractional integrals holds:

\begin{align}
(1.8) \quad f \left( \frac{a + b}{2} \right) \left[ J_{\frac{\alpha + \beta}{\alpha}}^\alpha g(a) + J_{\frac{\alpha + \beta}{\alpha}}^\alpha g(b) \right] \\
- \left[ J_{\frac{\alpha + \beta}{\alpha}}^\alpha f(g)(a) + J_{\frac{\alpha + \beta}{\alpha}}^\alpha f(g)(b) \right] \\
= \frac{1}{\Gamma(\alpha)} \int_a^b k(t) f'(t) \, dt
\end{align}

where

$$
k(t) = \begin{cases} 
\frac{t}{a} \int_a^t (s - a)^{\alpha - 1} g(s) \, ds, & t \in [a, \frac{a + b}{2}] \\
\frac{b - t}{b} \int_t^b (b - s)^{\alpha - 1} g(s) \, ds, & t \in [\frac{a + b}{2}, b].
\end{cases}
$$

Set al. proved the following three theorems.

Theorem 4. [19] Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ and $f' \in L[a, b]$ with $a < b$ and $g : [a, b] \to \mathbb{R}$ is continuous. If $|f'|$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:

\begin{align}
(1.9) \quad \left| f \left( \frac{a + b}{2} \right) \left[ J_{\frac{\alpha + \beta}{\alpha}}^\alpha g(a) + J_{\frac{\alpha + \beta}{\alpha}}^\alpha g(b) \right] \\
- \left[ J_{\frac{\alpha + \beta}{\alpha}}^\alpha f(g)(a) + J_{\frac{\alpha + \beta}{\alpha}}^\alpha f(g)(b) \right] \right| \\
\leq \frac{(b - a)^{\alpha + 1}}{2\alpha + 1} \frac{\|g\|_{[a, b], \infty}}{\Gamma(\alpha + 1)} \left( |f'(a)| + |f'(b)| \right)
\end{align}

with $\alpha > 0$.

Theorem 5. [19] Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ and $f' \in L[a, b]$ with $a < b$ and $g : [a, b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality for fractional integrals holds:

\begin{align}
(1.10) \quad \left| f \left( \frac{a + b}{2} \right) \left[ J_{\frac{\alpha + \beta}{\alpha}}^\alpha g(a) + J_{\frac{\alpha + \beta}{\alpha}}^\alpha g(b) \right] \\
- \left[ J_{\frac{\alpha + \beta}{\alpha}}^\alpha f(g)(a) + J_{\frac{\alpha + \beta}{\alpha}}^\alpha f(g)(b) \right] \right| \\
\leq \frac{(b - a)^{\alpha + 1}}{2\alpha + 1} \frac{\|g\|_{[a, b], \infty}}{\Gamma(\alpha + 1)} \left\{ \left( (\alpha + 1) |f'(a)|^q + (\alpha + 1) |f'(b)|^q \right)^{1/q} \\
+ \left( (\alpha + 1) |f'(a)|^q + (\alpha + 3) |f'(b)|^q \right)^{1/q} \right\}
\end{align}

with $\alpha > 0$.

Theorem 6. [19] Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ and $f' \in L[a, b]$ with $a < b$ and $g : [a, b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then
the following inequality for fractional integrals holds:

\[
\left| f\left(\frac{a+b}{2}\right) \left[ J_{\alpha}^{\alpha}g(a) + J_{\alpha}^{\alpha}g(b) \right] - \left[ J_{\alpha}^{\alpha}g(a) + J_{\alpha}^{\alpha}g(b) \right] \right| \\
\leq \frac{(b-a)^{\alpha+1} ||g||_\infty}{2^{\alpha+1/2} (\alpha+1)^{1/4} \Gamma(\alpha+1)} \\
\times \left\{ B_{1/2} (\alpha + 1, s + 1) + \frac{1}{2^{\alpha+s+1} (\alpha + s + 1)} \right\} ||f'(a)| + |f'(b)||.
\]

where \(1/p + 1/q = 1\).

We recall the following function:

The incomplete Beta function by

\[
B_x (\alpha, \beta) = \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt.
\]

In this paper, motivated by the recent results given in [11, 19], we establish
Hermite-Hadamard-Fejer type inequalities for \(s\)-convex functions in the second sense
via fractional integral. An interesting feature of our results is that they provide new
estimates on these types of inequalities for fractional integrals.

\section{Main Results}

Now, by using the Lemma [1] we prove our main theorems.

\textbf{Theorem 7.} Let \(f : I \subseteq [0, \infty) \to \mathbb{R}\) be a differentiable mapping on \(I^o\) and
\(f' \in L[a, b]\) with \(a < b\) and \(g : [a, b] \to \mathbb{R}\) is continuous. If \(|f'|\) is \(s\)-convex on \([a, b]\) for some fixed \(s \in (0, 1]\), then the following inequality for fractional integrals holds:

\[
\left| f\left(\frac{a+b}{2}\right) \left[ J_{\alpha}^{\alpha}g(a) + J_{\alpha}^{\alpha}g(b) \right] - \left[ J_{\alpha}^{\alpha}g(a) + J_{\alpha}^{\alpha}g(b) \right] \right| \\
\leq \frac{(b-a)^{\alpha+1} ||g||_{[a,b],\infty}}{\Gamma(\alpha+1)} \\
\times \left\{ B_{1/2} (\alpha + 1, s + 1) + \frac{1}{2^{\alpha+s+1} (\alpha + s + 1)} \right\} ||f'(a)| + |f'(b)||.
\]

\textbf{Proof.} Since \(|f'|\) is \(s\)-convex on \([a, b]\) for some fixed \(s \in (0, 1]\), we know that for
\(t \in [a, b]\)

\[
|f'(t)| = \left| f'\left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b\right) \right| \leq \left( \frac{b-t}{b-a}\right)^s |f'(a)| + \left( \frac{t-a}{b-a}\right)^s |f'(b)|
\]
From Lemma 1 we have

\[
\left| f\left(\frac{a+b}{2}\right) \left[ J_{\frac{a+b}{2}}^\alpha - g(a) + J_{\frac{a+b}{2}}^\alpha - g(b) \right] - \left[ J_{\frac{a+b}{2}}^\alpha - (f g)(a) + J_{\frac{a+b}{2}}^\alpha - (f g)(b) \right] \right| \\
\leq \frac{1}{\Gamma (\alpha )} \left\{ \int_a^b \left[ \int_a^{(s-a)^{\alpha -1}} g(s) \, ds \right] |f'(t)| \, dt \\
+ \int_a^b \left[ \int_t^{(b-s)^{\alpha -1}} g(s) \, ds \right] |f'(t)| \, dt \right\} \\
\leq \frac{\|g\|_{[a, \frac{a+b}{2}], \infty}}{(b-a) \Gamma (\alpha + 1)} \int_a^{\frac{a+b}{2}} \left( \int_a^{(s-a)^{\alpha -1}} ((b-t)^{\alpha} |f' (a)| + (t-a)^{\alpha} |f' (b)|) \, dt \\
+ \int_a^b \left[ \int_s^{(b-s)^{\alpha -1}} g(s) \, ds \right] |f'(s)| \, ds \right\} \\
+ \frac{\|g\|_{[\frac{a+b}{2}, b], \infty}}{(b-a) \Gamma (\alpha + 1)} \int_{\frac{a+b}{2}}^b \left[ \int_s^{(b-s)^{\alpha -1}} g(s) \, ds \right] |f'(s)| \, ds \right\} \\
= \frac{\|g\|_{[a, \frac{a+b}{2}], \infty}}{(b-a) \Gamma (\alpha + 1)} \int_a^{\frac{a+b}{2}} \left( (b-t)^{\alpha} |f' (a)| + (t-a)^{\alpha} |f' (b)| \right) \, dt \\
+ \frac{\|g\|_{[\frac{a+b}{2}, b], \infty}}{(b-a) \Gamma (\alpha + 1)} \int_{\frac{a+b}{2}}^b \left[ (b-t)^{\alpha} |f' (a)| + (t-a)^{\alpha} |f' (b)| \right) \, dt \\
= \frac{\|g\|_{[a, \frac{a+b}{2}], \infty}}{(b-a) \Gamma (\alpha + 1)} \left[ \int_a^{\frac{a+b}{2}} (t-a)^{\alpha} (b-t)^{\alpha} \, dt + \int_{\frac{a+b}{2}}^b (t-a)^{\alpha} (b-t)^{\alpha} \, dt \right] \\
+ \frac{\|g\|_{[\frac{a+b}{2}, b], \infty}}{(b-a) \Gamma (\alpha + 1)} \left[ \int_a^{\frac{a+b}{2}} (b-t)^{\alpha} (t-a)^{\alpha} \, dt + \int_{\frac{a+b}{2}}^b (b-t)^{\alpha} (t-a)^{\alpha} \, dt \right] \\
\]
Theorem 8. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^o \) and \( f' \in L[a, b] \) with \( a < b \) and let \( g : [a, b] \to \mathbb{R} \) is continuous. If \( |f'|^q \) is \( s \)-convex on \([a, b] \) for some fixed \( s \in (0, 1] \), \( q > 1 \), then the following inequality for fractional integrals holds:
Using Lemma 1, power mean inequality and convexity of \(|\cdot|\),

\[
\left| f \left( \frac{a + b}{2} \right) \left[ J^\alpha_{\frac{a+b}{2}} - g(a) + J^\alpha_{\frac{a+b}{2}} + g(b) \right] \\
- \left[ J^\alpha_{\frac{a+b}{2}} - (fg)(a) + J^\alpha_{\frac{a+b}{2}} + (fg)(b) \right] \right| \\
\leq \frac{(b - a)^{\alpha + 1} \|g\|_{[a,b],\infty}}{2^{\alpha+1+\frac{q}{2}} (\alpha + 1) (\alpha + 2)^{1/q} (\alpha + s + q)^{1/q} \Gamma (\alpha + 1)}
\]

\[
(2.2) \quad \times \left\{ ((\alpha + s + 1) (\alpha + 3) |f'(a)|^q + 2^{1-s} (\alpha + 1) (\alpha + 2) |f'(b)|^q \right\}^{1/q}
\]

\[
+ (2^{1-s} (\alpha + 1) (\alpha + 2) |f'(a)|^q + (\alpha + s + 1) (\alpha + 3) |f'(b)|^q \right\}^{1/q}
\]

Proof. Since \(|f'|\) is \(s-\)convex on \([a, b]\) for some fixed \(s \in (0, 1]\), we know that for \(t \in [a, b]\)

\[
|f'(t)|^q = \left| f' \left( \frac{b - t}{b - a} + \frac{t - a}{b - a} \right) \right|^q \leq \left( \frac{b - t}{b - a} \right)^s |f'(a)|^q + \left( \frac{t - a}{b - a} \right)^s |f'(b)|^q
\]

Using Lemma 1, power mean inequality and convexity of \(|f'|^q\), it follows that

\[
\left| f \left( \frac{a + b}{2} \right) \left[ J^\alpha_{\frac{a+b}{2}} - g(a) + J^\alpha_{\frac{a+b}{2}} + g(b) \right] \\
- \left[ J^\alpha_{\frac{a+b}{2}} - (fg)(a) + J^\alpha_{\frac{a+b}{2}} + (fg)(b) \right] \right| \\
\leq \frac{1}{\Gamma (\alpha)} \left( \int_a^b \left| s - a \right|^{\alpha - 1} g(s) \; ds \right)^{1-1/q}
\]

\[
\times \left( \int_a^b \int_a^t \left| (s - a) \right|^{\alpha - 1} g(s) \; ds \; dt \right)^{1/q}
\]

\[
+ \frac{1}{\Gamma (\alpha)} \left( \int_a^b \int_b^t \left| (b - s) \right|^{\alpha - 1} g(s) \; ds \; dt \right)^{1-1/q}
\]

\[
\times \left( \int_a^b \int_a^t \left| f'(t)|^q \; dt \right| \right)^{1/q}
\]

\[
\leq \frac{\|g\|_{[a,b],\infty}}{\Gamma (\alpha)} \left( \int_a^b \left| (s - a) \right|^{\alpha - 1} \; ds \right)^{1-1/q}
\]

\[
\times \left( \int_a^b \left| f'(t)|^q \; dt \right| \right)^{1/q}
\]
\[
\begin{align*}
&+ \frac{\|g\|_{[a, b], \infty}}{\Gamma (\alpha)} \left( \int_{a}^{b} \left| \int_{t}^{b} (b - s)^{\alpha - 1} \, ds \right| \, dt \right) \left( \int_{a}^{b} \left| f' (t) \right|^q \, dt \right)^{1/q} \\
\times &\left( \int_{a}^{b} \left| (b - s)^{\alpha - 1} \right| \, ds \right)^{1/q} \\
\leq &\frac{1}{\alpha \Gamma (\alpha)} \left( \frac{(b - a)^{\alpha+1}}{2^{\alpha+1} (\alpha + 1)} \right)^{1-1/q} \\
\times &\left\{ \frac{\|g\|_{[a, b], \infty}}{(b - a)^{s/q}} \left[ \int_{a}^{b} \left( (t - a)^{\alpha} (b - t)^s |f' (a)|^q + (t - a)^{\alpha + s} |f' (b)|^q \right) \, dt \right]^{1/q} \\
+ &\frac{\|g\|_{[a, b], \infty}}{(b - a)^{s/q}} \left[ \int_{a}^{b} \left( (b - t)^{\alpha + s} |f' (a)|^q + (b - t)^{\alpha} (t - a)^s |f' (b)|^q \right) \, dt \right]^{1/q} \right\} \\
\times &\left[ \left( \frac{(b - a)^{\alpha+1}}{2^{\alpha+1} (\alpha + 1)} \right)^{1-1/q} \left\{ \frac{\|g\|_{[a, b], \infty}}{(b - a)^{s/q}} \left[ \int_{a}^{b} \left( (b - t)^{\alpha+1} B_{1/2} (\alpha + 1, s + 1) |f' (a)|^q + \frac{(b - a)^{\alpha+s+1}}{2^{\alpha+s+1} (\alpha + s + 1)} |f' (b)|^q \right) \, dt \right]^{1/q} \right\} \\
+ &\left( \frac{(b - a)^{\alpha+1+\frac{1}{q}}}{2^{\alpha+1+\frac{1}{q}} (\alpha + 1)} \right)^{1-1/q} \left\{ \frac{\|g\|_{[a, b], \infty}}{(b - a)^{s/q}} \left[ \int_{a}^{b} \left( (b - t)^{\alpha+s+1} (\alpha + s + 1) B_{1/2} (\alpha + 1, s + 1) |f' (b)|^q \right) \, dt \right]^{1/q} \right\} \right\} \\
\times &\left\{ (\alpha + s + 1) (\alpha + 3) |f' (a)|^q + 2^{1-s} (\alpha + 1) (\alpha + 2) |f' (b)|^q \right\}^{1/q} \\
+ &\left( 2^{1-s} (\alpha + 1) (\alpha + 2) |f' (a)|^q + (\alpha + s + 1) (\alpha + 3) |f' (b)|^q \right)^{1/q}.
\end{align*}
\]
Remark 2. In Theorem 8, if we choose \( s = 1 \), then \( \|g\|_{[a,b],\infty} \) reduces inequality \( \text{[1.10]} \) of Theorem 5.

Theorem 9. Let \( f : I \subseteq [0,\infty) \to \mathbb{R} \) be a differentiable mapping on \( I^o \) and \( f' \in L^2[a,b] \) with \( a < b \) and let \( g : [a,b] \to \mathbb{R} \) is continuous. If \( |f'|^q \) is \( s \)-convex on \( [a,b] \) for some fixed \( s \in (0,1], q > 1 \), then the following inequality for fractional integrals holds:

\[
\begin{align*}
(2.3) \quad & \left| f \left( \frac{a+b}{2} \right) \left[ J^\alpha_{(a+b)} g(a) + J^\alpha_{(a+b)} g(b) \right] \\
& - \left[ J^\alpha_{(a+b)} (fg)(a) + J^\alpha_{(a+b)} (fg)(b) \right] \right| \\
& \leq \frac{(b-a)^{-1+2}}{2^{\alpha+1}+\frac{p}{q}} (\alpha q+1)(\alpha+2)^{1/p} (s+1)^{1/q} \Gamma(\alpha+1) \\
& \times \left( (|f'(a)|^q (2^{s+1} - 1) + |f'(b)|^q) \right)^{1/q} + \left( (|f'(a)|^q + |f'(b)|^q (2^{s+1} - 1)) \right)^{1/q}
\end{align*}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. Using Lemma [1] Hölder’s inequality and the \( s \)-convex of \( |f'|^q \) it follows that

\[
\begin{align*}
& \left| f \left( \frac{a+b}{2} \right) \left[ J^\alpha_{(a+b)} g(a) + J^\alpha_{(a+b)} g(b) \right] \\
& - \left[ J^\alpha_{(a+b)} (fg)(a) + J^\alpha_{(a+b)} (fg)(b) \right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^b \left( (s-a)^{-1} g(s) \right)^p \, ds \right)^{1/p} \left( \int_a^b \left( f'(t) \right)^q \, dt \right)^{1/q} \\
& \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^b \left( (s-a)^{-1} g(s) \right)^p \, ds \right)^{1/p} \left( \int_a^b \left( f'(t) \right)^q \, dt \right)^{1/q} \\
& \quad + \frac{1}{\Gamma(\alpha)} \left( \int_a^b \left( (s-a)^{-1} g(s) \right)^p \, ds \right)^{1/p} \left( \int_a^b \left( f'(t) \right)^q \, dt \right)^{1/q} \\
& = \frac{1}{\Gamma(\alpha)} \left( \int_a^b \left( (s-a)^{-1} g(s) \right)^p \, ds \right)^{1/p} \left( \int_a^b \left( f'(t) \right)^q \, dt \right)^{1/q} \\
& \quad \times \left( \int_a^b \left( f'(t) \right)^q \, dt \right)^{1/q} \left( \int_a^b \left( f'(t) \right)^q \, dt \right)^{1/q} \\
& \leq \frac{(b-a)^{-1+2}}{2^{\alpha+1}+\frac{p}{q}} (\alpha q+1)(\alpha+2)^{1/p} (s+1)^{1/q} \Gamma(\alpha+1) \\
& \times \left( (|f'(a)|^q (2^{s+1} - 1) + |f'(b)|^q) \right)^{1/q} + \left( (|f'(a)|^q + |f'(b)|^q (2^{s+1} - 1)) \right)^{1/q}.
\end{align*}
\]
Here we use

\[
\int_{a}^{b} \left( \int_{a}^{t} (s-a)^{\alpha-1} g(s) \, ds \right)^{p} dt = \frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1} (\alpha p + 1) \alpha^{p}} \int_{a}^{b} \left| f'(t) \right|^{q} dt \leq \frac{b-a}{2^{s+1} (s+1)} \left[ \left| f'(a) \right|^{q} (2^{s+1} - 1) + \left| f'(b) \right|^{q} \right] \]

\[
\int_{a}^{b} \left| f(t) \right|^{q} dt \leq \frac{b-a}{2^{s+1} (s+1)} \left[ \left| f'(a) \right|^{q} + \left| f'(b) \right|^{q} (2^{s+1} - 1) \right].
\]

\[\Box\]

**Remark 3.** In Theorem 9, if we choose \( s = 1 \), then \( \Box \) reduces inequality \((1.11)\) of Theorem 6.

**References**

[1] G. Anastassiou, M.R. Hooshmandasl, A. Ghasemi and F. Moftakharzadeh, Montgomery identities for fractional integrals and related fractional inequalities, J. Ineq. Pure and Appl. Math., 10(4) (2009), Art. 97.

[2] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.

[3] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Science, 9(4) (2010), 493-497.

[4] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1(1) (2010), 51-58.

[5] Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, Nonl. Sci. Lett. A, 1(2) (2010), 155-160.

[6] Z. Dahmani, L. Tabharit, S. Taf, New generalizations of Gruss inequality usin Riemann-Liouville fractional integrals, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.

[7] S.S. Dragomir, S. Fitzpatrick, The Hadamard’s inequality for s-convex functions in the second sense, Demonstratio Math. 32 (4) (1999) 687–696.

[8] S.S. Dragomir, C.E.M. Pearce, Selected topics on Hermite–Hadamard inequalities and applications, in: RGMIA Monographs, Victoria University, 2000. Online: [http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html](http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html).

[9] L. Fejér, Über die Fourierreihen, II, Math. Naturwiss. Anz. Ungar. Akad. Wiss., 24 (1906), 369–390. (In Hungarian).

[10] J. Hadamard, Étude sur les propriétés des fonctions entières et en particular d’une fonction considérée par Riemann, J. Math. Pures Appl., 58 (1893), 171-215.

[11] H. Hudzik, L. Maligranda, Some remarks on s-convex functions, Aequationes Math. 48 (1994) 100–111.

[12] İ. İşcan, Hermite-Hadamard-Fejér type inequalities for convex function via fractional integrals, 2014, [arXiv:1404.7722v1](http://arxiv.org/abs/1404.7722).

[13] İ. İşcan, Generalization of different type integral inequalities for s-convex functions via fractional integrals, Applicable Analysis: An Int. J., 93 (9) (2014), 1846–1862.

[14] J.E. Pečarić, F. Proschan, Y.L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, San Diego, 1992.

[15] M.Z. Sarıkaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57(9) (2013), 2403-2407.

[16] M.Z. Sarıkaya, On new Hermite Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57 (3) (2012), 377-386.

[17] M.Z. Sarıkaya and S. Erden, On The Hermite- Hadamard-Fejér Type Integral Inequality for Convex Function, Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 3, 85-89.
[18] M.Z. Sarikaya and S. Erden, On the Weighted Integral Inequalities for Convex Functions, RGMIA Research Report Collection, 17(2014), Article 70, 12 pp.

[19] E. Set, I. İşcan, M.E. Özdemir and M.Z. Sarıkaya, On New Inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals, submitted.

⋆DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, ORDU UNIVERSITY, 52200, ORDU, TURKEY
E-mail address: erhanset@yahoo.com

▲DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, GİRESUN UNIVERSITY, GİRESUN, TURKEY
E-mail address: imdati@yahoo.com

■DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, ORDU UNIVERSITY, 52200, ORDU, TURKEY
E-mail address: h.husein.kara61@gmail.com