Numerical Methods for Three-Dimensional Analysis of Shock Instability in Supernova Cores

Wakana Iwakami¹, Naofumi Ohnishi¹,², Kei Kotake³,⁴, Shoichi Yamada⁵,⁶ and Keisuke Sawada¹

¹. Department of Aerospace Engineering, Tohoku University, 6-6-01 Aramaki-Aza-Aoba, Aoba-ku, Sendai 980-8579, Japan
². Center for Research Strategy and Support, Tohoku University, 6-6-01 Aramaki-Aza-Aoba, Aoba-ku, Sendai 980-8579, Japan
³. Division of Theoretical Astronomy, National Astronomical Observatory Japan, 2-21-1, Osawa, Mitaka, Tokyo 181-8588, Japan
⁴. Max-Planck-Institut für Astrophysik, Karl-Schwarzshild-Str. 1, D-85741, Garching, Germany
⁵. Science & Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan
⁶. Advanced Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan

E-mail: iwakami@rhd.mech.tohoku.ac.jp

Abstract. We studied the standing accretion shock instability (SASI) for a core-collapse supernova explosion. SASI induces a nonspherically symmetric motion of a standing spherical shock wave. In order to investigate the growth of SASI, we solved the three-dimensional compressible Euler equations using ZEUS-MP/2 code based on the finite-difference method with a staggered mesh of spherical polar geometry. Although the von Neumann and Richtmyer artificial viscosity is used in ZEUS-MP/2 code to capture shock waves, we propose utilizing a tensor artificial viscosity in order to overcome the numerical instability regarded as the carbuncle phenomenon. This numerical instability emerges around the grid polar axis and precludes mode analysis of SASI.

1. Introduction
A core-collapse supernova is a spectacular explosion of a massive star in the final stage of evolution. However, the explosion mechanism has not been fully understood. The study of the explosion mechanism of this astrophysical phenomenon is important because it is expected to contribute to nuclear physics and elementary particle physics as well as to the forthcoming neutrino astronomy and gravitational wave astronomy. The recent progress in the research of core-collapse supernovae focusing on the explosion mechanism has been reviewed by Kotake et al. [1]. Although this explosion needs a spherical shock wave generated by gravitational collapse in the central iron core to reach the surface of a star, the results of reliable one-dimensional numerical simulations have indicated that the spherical shock wave stalls or falls back to the center of a star. It is currently thought that multidimensional effects are the crucial factors of the shock revival. SASI drives nonspherical motions of the standing shock wave in the spherically accreting flow and is the likely phenomenon causing the multi-dimensional effects.
Foglizzo [2, 3] originally studied the instability of a standing shock wave in the context of linear analysis for accreting black holes. Blondin et al. applied it for the first time to supernova explosion mechanisms such as SASI [4]. Ohnishi et al. performed more elaborate linear analysis of the shock wave surface configuration with two-dimensional numerical simulations [5]. The present work is an extension of Ohnishi et al. to three-dimensional simulations.

The numerical simulation of SASI requires a scheme that can capture shock waves in an inviscid flow. There are many schemes introducing an artificial viscosity produced by means of an additional viscous term or estimating numerical fluxes with a Riemann solver. However, shock-capturing solvers may yield unreliable results under some conditions. For high-resolution simulations, numerical instability called the carbuncle phenomenon tends to occur where a relatively strong shock wave is parallel to one axis in two dimensions or a plane composed of two axes in three dimensions. The carbuncle phenomenon is considered to be caused by the odd-even decoupling instability found by Quirk [6] or a shock wave instability in purely inviscid flow suggested by Robinet [7]. In order to control the carbuncle phenomenon, a certain amount of dissipation is added to a shock-capturing scheme in an appropriate manner.

Our numerical code is a Eulerian code based on ZEUS-MP/2 code [8] modified according to our preceding 2D study [5]. ZEUS-MP/2 code is built with the finite-difference method using a staggered mesh. The von Neumann and Richtmyer artificial viscosity is employed to capture shock waves in the original code. We selected a spherical coordinate grid, which is suitable for mode analysis of the spherical shock instability. In this paper, we demonstrate that the carbuncle phenomenon appears only around the grid polar axis in our code and that the use of the tensor artificial viscosity [9] successfully suppresses this phenomenon and yields the correct mode analysis results.

2. Numerical Procedure

We performed three-dimensional numerical simulations of the accretion flow through a spherical shock wave surrounding a central object termed a proto-neutron star (PNS) in the iron core. The flow is gravitationally attracted by the PNS and irradiated by neutrinos emitted from the neutrino sphere near the PNS. The basic evolution equations are the three-dimensional compressive Euler equations to which the terms associated with the gravitational force and the neutrino heating and cooling are added.

We consider the spherical coordinates \( (r, \theta, \phi) \) with the origin at the center of the PNS. We use 300 radial mesh points to cover \( r_{\text{in}} \leq r \leq r_{\text{out}} \), where \( r_{\text{in}} \approx 50 \text{km} \) is the radius of the inner boundary located roughly at the neutrino sphere and \( r_{\text{out}} = 2000 \text{km} \) is the radius of the outer boundary at which the supersonic flow is falling inwards. The radial grid width \( \Delta r \) increases with increasing \( r \) to resolve the fluid motion inside the shock wave. We adopt 30 polar and 60 azimuthal mesh points to cover the whole spherical surface. The polar grid width \( \Delta \theta \) and azimuthal grid width \( \Delta \phi \) are constant.

In order to induce non-spherical instability, we add the non-spherically symmetric radial velocity perturbations \( \delta v_r(\theta, \phi) \propto -\frac{3}{8r^2} \sin \theta \cos \phi \) as the \( l = 1, m = 1 \) single-mode perturbation to the steady spherically symmetric flow [10], where \( l \) and \( m \) are the indexes of the spherical harmonics \( Y_l^m(\theta, \phi) \). The perturbation amplitude is set to less than 1% of the unperturbed velocity. The readers should refer to the papers by Ohnishi et al. [5] and Iwakami et al. [11] for the detailed numerical conditions.

3. Artificial Viscosity

The von Neumann and Richtmyer artificial viscosity in the original ZEUS-MP/2 is

\[
q = \begin{cases} 
\frac{\rho}{\Delta x} \left( \frac{\partial v}{\partial x} \right)^2 & \text{if } \left( \frac{\partial v}{\partial x} \right) < 0 \\
0 & \text{otherwise}
\end{cases}
\]
where \( q \) is the artificial viscosity, \( v \) is the velocity, \( \rho \) is the density, and \( l \) is a constant with dimensions of length. The von Neumann and Richtmyer artificial viscosity extended to multi-dimensions is obtained by solving Eq. (1) in each direction. However, this extension is ad hoc.

The tensor artificial viscosity proposed by Stone and Norman [12] is

\[
Q = \begin{cases} 
 l^2 \rho \nabla \cdot v |\nabla v - \frac{1}{3}(\nabla \cdot v)e | & \text{if } \nabla \cdot v < 0 \\
0 & \text{otherwise} 
\end{cases},
\]

where \( Q \) is the artificial viscosity tensor, \( v \) is the velocity tensor, \( e \) the unit tensor, and \( \nabla v = [v_{i,j} + v_{j,i}]/2 \) is the symmetrized velocity gradient tensor. The tensor artificial viscosity is produced by analogy to the stress tensor for molecular viscosity of a Newtonian fluid. This tensor has the property of \( \text{Tr}(Q) = 0 \), and the off-diagonal components are dropped in this study. This tensor artificial viscosity gives the proper definition for the extension to multi-dimensions and the curvilinear coordinates system. More details can be found in the papers by Stone and Norman [12] and Iwakami et al. [11].

4. Results and Discussion

First, we show that the carbuncle phenomenon occurs when the von Neumann and Richtmyer artificial viscosity is used. The density field at \( t = 35.0 \text{ ms} \) is displayed in the upper left panel of Fig. 1. The shock wave radius grows only around the grid polar axis. The \( \phi \) component of the velocity distribution at \( \theta \sim 0 \) and \( r = 1.4 \times 10^7 \text{ cm} \) where the shock wave is formed is shown in the upper right panel of Fig. 1. A saw-tooth shaped velocity distribution appears as time progresses. This means that the numerical error grows with the opposite sign to adjacent grid points. Such distribution can be observed when the carbuncle phenomenon occurs [6].

Next, we show that the carbuncle phenomenon does not occur when the tensor artificial viscosity is used. The two bottom panels in Fig. 1 display the results for the tensor artificial viscosity. The carbuncle phenomenon described above is suppressed. The discrepancy of the

![Figure 1](image-url)
velocity distributions may result in the difference of the phase and the growth rate of the amplitude due to the difference of artificial viscosity. It is not essential for this problem.

Finally, we investigate how the carbuncle phenomenon affects on the mode analysis of SASI. The mode amplitudes are obtained from the expansion coefficients $c_{lm}$ of the spherical harmonics $Y_{lm}^m(\theta, \phi)$ for the shock wave surface configuration. The left panel in Fig. 2 shows the time development of the mode amplitude for the von Neumann and Richtmyer artificial viscosity; the right panel shows that for the tensor artificial viscosity. A linear phase lasting $\sim 100$ ms in which the fundamental $l = 1$, $m = 1$ mode amplitude grows exponentially appears for the tensor artificial viscosity in the right panel, but does not appear for the von Neumann and Richtmyer artificial viscosity in the left panel. The behavior of the non-linear phase from $\sim 100$ ms is also different between the two cases.

Two reasons can be considered for the success of the tensor artificial viscosity. One is the tensor artificial viscosity is defined by analogy to the Newtonian fluid and has the property of $\text{Tr}(Q) = 0$ from Stoke’s hypothesis. The other is that the tensor artificial viscosity is defined correctly in a curvilinear coordinate system such as spherical coordinates.

![Figure 2. Left panel: Time history of the normalized mode amplitude $|c_{lm}^n/c_{00}^n|$ associated with the shock wave surface configuration for introducing the von Neumann and Richtmyer artificial viscosity. Right panel: Same as left panel but for introducing the tensor artificial viscosity.](image)

5. Conclusion
In order to study the standing shock instability for a core-collapse supernova explosion, we performed three-dimensional hydrodynamic simulations with a Eulerian code based on ZEUS-MP/2 using the von Neumann and Richtmyer artificial viscosity to capture shock waves. The carbuncle phenomenon emerges only around the grid polar axis and affects the mode analysis in investigations into the nature of SASI. We found that the tensor artificial viscosity effectively suppresses this phenomenon.

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