MEAN DIMENSION OF $\mathbb{Z}^k$-ACTIONS

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Abstract. Mean dimension is a topological invariant for dynamical systems that is meaningful for systems with infinite dimension and infinite entropy. Given a $\mathbb{Z}^k$-action on a compact metric space $X$, we study the following three problems closely related to mean dimension.

(1) When is $X$ isomorphic to the inverse limit of finite entropy systems?

(2) Suppose the topological entropy $h_{\text{top}}(X)$ is infinite. How much topological entropy can be detected if one considers $X$ only up to a given level of accuracy? How fast does this amount of entropy grow as the level of resolution becomes finer and finer?

(3) When can we embed $X$ into the $\mathbb{Z}^k$-shift on the infinite dimensional cube $([0,1]^D)^{\mathbb{Z}^k}$?

These were investigated for $\mathbb{Z}$-actions in [Lindenstrauss, Mean dimension, small entropy factors and an embedding theorem, Inst. Hautes Études Sci. Publ. Math. 89 (1999) 227-262], but the generalization to $\mathbb{Z}^k$ remained an open problem. When $X$ has the marker property, in particular when $X$ has a completely aperiodic minimal factor, we completely solve (1) and a natural interpretation of (2), and give a reasonably satisfactory answer to (3).

A key ingredient is a new method to continuously partition every orbit into good pieces.

1. Introduction

1.1. Main results. Mean dimension is a topological invariant of dynamical systems introduced by Gromov [Gro99]. Just like topological entropy measures the number of bits per second to describe a point in a system, mean dimension measures the number of parameters per second. A basic example is the shift action on the Hilbert cube $[0,1]^\mathbb{Z}$,
whose mean dimension is 1. This system has infinite dimension and infinite topological entropy; mean dimension, however, provides a useful numerical invariant for such large dynamical systems. Soon after the introduction of mean dimension, some basic properties of mean dimension, in particular its relation to topological entropy and to embedding questions were studied by Benjamin Weiss and the second named author ([LW00, Lin99]).

Both [LW00, Lin99] focus on the case of $\mathbb{Z}$-actions. However, there is a substantial difference between these two papers in this respect: The paper [LW00] studied the basic theory of mean dimension, and the restriction to $\mathbb{Z}$-action in this paper was purely because of expositional reasons. As explained in that paper, all its main results can be generalized to the actions of discrete amenable groups without any essential change given a result due to Ornstein and Weiss on subadditive functions on amenable groups (cf. Lemma 2.3 below) that is given in [LW00, Appendix] (this lemma is implicit in Ornstein–Weiss [OW87, Ch. I, Sec. 2 and 3] and written explicitly for similar purposes also in Gromov [Gro99, p. 336]). Recently Hanfeng Li [Li13] successfully generalized these results even to the much larger class of sofic groups.

The paper [Lin99] studied more delicate questions and in that paper specific properties of $\mathbb{Z}$ were used. How to generalize these results to $\mathbb{Z}^k$-actions was one of the questions left open in [LW00, Lin99], and it is precisely this question which we are finally able to address in this paper.

Our motivation to develop the generalization is two-fold. The first is a purely theory motivated: There is a tradition in ergodic theory and dynamical systems to consider actions of more general group. A notable example in this vein is the paper [OW87], where much of the body of knowledge of ergodic theory of $\mathbb{Z}$-actions is generalized to the context of actions of amenable groups.

Moreover, some of the most natural and interesting examples of systems with nontrivial mean dimension arise in the context of $\mathbb{Z}^k$ actions (or $\mathbb{R}^k$-actions, which for our point of view are almost equivalent). Indeed, the concept of mean dimension was introduced by Gromov [Gro99] in order to study dynamical systems in geometric analysis from the viewpoint of mean dimension. In most of the systems considered in [Gro99], the acting groups are more complicated than $\mathbb{Z}$. For example, [Gro99, Chapter 4] studied a dynamical system consisting of complex subvarieties in $\mathbb{C}^n$. In this case $\mathbb{C}^n$ and its lattice $\mathbb{Z}^{2n}$ are the acting groups, the action being by translation. Readers can find many more examples in [Gro99, Chapters 3 and 4].
Fix a positive integer \( k \). Let \( X \) be a compact metric space with a continuous action \( T : \mathbb{Z}^k \times X \to X \). We call \((X, \mathbb{Z}^k, T)\) a dynamical system. We often abbreviate \((X, \mathbb{Z}^k, T)\) to \( X \) and denote \( T(n, x) \) by \( T^n x \) for \( n \in \mathbb{Z}^k \) and \( x \in X \). The mean dimension of the system \( X \) is denoted by \( \text{mdim}(X) \). Its definition is given in Section 2. The paper studies three problems closely related to \( \text{mdim}(X) \).

**The first problem:** *When can we approximate a dynamical system \( X \) arbitrarily well by finite topological entropy systems?* More precisely, when is \( X \) isomorphic to the inverse limit of \( \mathbb{Z}^k \)-actions

\[
\cdots \to X_n \to X_{n-1} \to \cdots \to X_2 \to X_1
\]

such that every \( X_n \) has a finite topological entropy?

The inverse limit of finite entropy systems is always zero mean dimensional ([Lin99, Proposition 6.11]). So \( \text{mdim}(X) = 0 \) is a necessary condition. We conjecture the following

**Conjecture 1.1.** Let \( \Gamma \) be a discrete amenable group (in particular, consider \( \Gamma = \mathbb{Z}^k \)). A \( \Gamma \)-system \((X, \Gamma, T)\) is an inverse limit of zero entropy systems iff it has zero mean dimension.

Our first main result is a partial result towards this conjecture for \( \Gamma = \mathbb{Z}^k \). The following definition is a variant of [Dow06, Def. 2]

**Definition 1.2.** Let \((X, \Gamma, T)\) be a dynamical system. It is said to have the **marker property** if for any finite subset \( F \subset \Gamma \) there exists an open set \( U \subset X \) such that

\[
X = \bigcup_{n \in \Gamma} T^n U
\]

and that \( U \cap T^n U = \emptyset \) for all non-identity \( n \in F \).

In particular it follows from the definition that if \((X, \Gamma, T)\) has the marker property the \( \Gamma \)-action is free, i.e. \( T^n x \neq x \) for all \( x \in X \) and \( n \in \Gamma \). To conform with the standard terminology for \( \mathbb{Z} \)-actions, we shall say a dynamical system \((X, \Gamma, T)\) is aperiodic if the underlying \( \Gamma \)-action is free. Clearly, the marker property is preserved by extensions: if \((X, \Gamma, T)\) is a \( \Gamma \)-system and \((Y, \Gamma, S)\) a factor with the marker property then so does \( X \). A large class of systems with this property is the class of **aperiodic minimal systems**: a system \((X, \Gamma, T)\) is said to be **minimal** if every \( \Gamma \)-orbit is dense; it is an **aperiodic minimal system** if the system is in addition aperiodic.

It is not clear whether any aperiodic \( \mathbb{Z}^k \)-system has the marker property or not. For further discussion of this question see [Gut12, Gut13]
Theorem 1.3 (Cf. [Lin99, Prop. 6.14]). Suppose a $\mathbb{Z}^k$-system $X$ has the marker property. Then $\text{mdim}(X) = 0$ if and only if $X$ is isomorphic to the inverse limit of finite topological entropy systems.

The result cited, [Lin99, Prop. 6.14], gives the theorem for actions of $\mathbb{Z}$; even though it is assumed there that $X$ has an aperiodic minimal factor, what is actually used in the proof is that $X$ has the marker property. Note that if a minimal system is not aperiodic then its topological entropy is zero, hence Theorem 1.3 completely answers the first problem for minimal $\mathbb{Z}^k$-systems.

The second problem: Suppose the topological entropy of $X$ is infinite. How much topological entropy can be detected if one considers $X$ only up to a given level of accuracy? How fast does this amount of entropy grow as the level of resolution becomes finer and finer?

Suppose $X$ is endowed with a distance $d$. For $\varepsilon > 0$ we define $S(X, \varepsilon, d)$ as the amount of entropy that can be detected in $X$ by considering it only up to accuracy $\varepsilon$ with respect to the metric $d$ — see Section 3 for the precise definition.

The topological entropy $h_{\text{top}}(X)$ is the limit of $S(X, \varepsilon, d)$ as $\varepsilon$ goes to zero; the limit, unlike the $S(X, \varepsilon, d)$, does not depend on the choice of $d$. When $h_{\text{top}}(X) = \infty$, we are interested in its growth. This motivates the introduction of the metric mean dimension $\text{mdim}_M(X, d)$:

$$\text{mdim}_M(X, d) = \liminf_{\varepsilon \to 0} \frac{S(X, \varepsilon, d)}{|\log \varepsilon|}.$$  

Weiss and the second named author ([LW00, Theorem 4.2]) proved $\text{mdim}_M(X, d) \geq \text{mdim}(X)$ for any distance $d$. Our second main result is the following.

Theorem 1.4 (Cf. [Lin99, Thm. 4.3]). Suppose $(X, \mathbb{Z}^k, T)$ is a $\mathbb{Z}^k$-system with the marker property. Then there exists a distance $d$ on $X$, compatible with the topology, satisfying

$$\text{mdim}_M(X, d) = \text{mdim}(X).$$

Therefore, at least for extensions of completely aperiodic minimal systems, the growth of $S(X, \varepsilon, d)$ can be controlled by the mean dimension $\text{mdim}(X)$.

Again we conjecture that (L1) holds for any $\mathbb{Z}^k$-system $X$, without having to assume the marker property, and indeed not just for $\mathbb{Z}^k$-systems but for any countable amenable group.
The third problem: Let $D$ be a positive integer, and consider the $\mathbb{Z}^k$-shift on $([0, 1]^D)^{\mathbb{Z}^k}$:

$$\mathbb{Z}^k \times ([0, 1]^D)^{\mathbb{Z}^k} \rightarrow ([0, 1]^D)^{\mathbb{Z}^k}, \quad (a, (x_n)_{n \in \mathbb{Z}^k}) \rightarrow (x_{n+a})_{n \in \mathbb{Z}^k}.$$ 

Given a dynamical system $X$, when can we equivariantly embed $X$ into $([0, 1]^D)^{\mathbb{Z}^k}$? Beboutov showed that any action of $\mathbb{R}$ on a compact metric space whose fixed points can be embedded in an interval can be embedded in the space of continuous function on $\mathbb{R}$, with the natural action of $\mathbb{R}$ (cf. [Kak68]). In his Ph.D. thesis, Jaworski [Jaw74] proved that if a system $(X, \mathbb{Z}, T)$ is finite dimensional and has no periodic points then there exists an embedding from $X$ into the $\mathbb{Z}$-shift $[0, 1]^\mathbb{Z}$. It was left unclear whether the periodic points are the only obstruction for such an embedding. Weiss and the second named author [LW00] observed that mean dimension is another obstruction. The mean dimension of $\mathbb{Z}^k$-shift $([0, 1]^D)^{\mathbb{Z}^k}$ is $D$. Hence if $X$ is embedded into it, its mean dimension $\text{mdim}(X)$ is not greater than $D$. The paper [LW00, Proposition 3.5] constructed an aperiodic minimal system for $\mathbb{Z}$ whose mean dimension is strictly greater than 1; this construction was generalized to arbitrary amenable groups in [Kri09]. These constructions give examples of aperiodic minimal systems which cannot be embedded into $[0, 1]^{\mathbb{Z}^k}$.

Even if the $\mathbb{Z}^k$-action is free, mean dimension is not the only obstruction for embedding. In [LW00] a $\mathbb{Z}$-minimal system is given with $\text{mdim}(X) = D/2$ that cannot be embedded in $([0, 1]^D)^{\mathbb{Z}}$ for any $D \in \mathbb{N}$, and it is straightforward to adapt this example to $\mathbb{Z}^d$ (and even to general discrete amenable groups).

Our third main theorem is a partial converse to this result.

**Theorem 1.5** (Cf. [Lin99, Thm. 5.1]). Suppose $(X, \mathbb{Z}^k, T)$ is a $\mathbb{Z}^k$-system with the marker property. If it satisfies

$$\text{mdim}(X) < \frac{D}{2^{k+1}},$$

then there exists an embedding from $X$ into $([0, 1]^{2D})^{\mathbb{Z}^k}$.

In [Lin99, Thm. 5.1] it was shown that a $\mathbb{Z}$-systems with the marker property with $\text{mdim}(X) < D/36$ can be embedded in $([0, 1]^D)^{\mathbb{Z}}$. For technical reasons, it is useful for us to have as a target the $\mathbb{Z}^k$-shift on $([0, 1]^{2D})^{\mathbb{Z}^k}$ rather than $([0, 1]^D)^{\mathbb{Z}^k}$.

Note also that in our condition $\text{mdim}(X) < D/2^{k+1}$, the constant involved is likely far from optimal. Presumably for an aperiodic $\mathbb{Z}^k$-system if $\text{mdim}(X) < \frac{D}{2}$ then $X$ can be embedded in $([0, 1]^D)^{\mathbb{Z}^k}$; for
Z-actions, it is conjectured in [LT14] that if a system \((X, \mathbb{Z}, T)\) satisfies
\[
\text{mdim}(X) < \frac{D}{2}, \quad \frac{1}{n} \dim \left( \{ x | T^n x = x \} \right) < \frac{D}{2} \quad (\forall n \geq 1)
\]
then there is an embedding from \(X\) into \(([0, 1]^D)^\mathbb{Z}\).

We can obtain an embedding result with the optimal constant of \(1/2\) if we make a stronger assumption about \(X\) than in Theorem 1.5 above. Any closed invariant subspace of \(\{1, 2, \ldots, l\}^\mathbb{Z}\) on which \(\mathbb{Z}\) acts freely (whether minimal or not) can be seen to satisfy the marker property. We shall call such a system an \textbf{aperiodic symbolic system}.

Following [GT14] (who treated the case of \(Z\)-actions) we show that if \(X\) is assumed to have an aperiodic symbolic factor one can get an optimal embedding constant:

\textbf{Theorem 1.6 (Cf. [GT14, Corollary 1.8])}. Suppose \(X\) has an aperiodic symbolic factor. If \(\text{mdim}(X) < D/2\) then there exists an embedding from \(X\) into \(([0, 1]^D)^\mathbb{Z}\).

We note that the proof of Theorem 1.6 is substantially less complicated than that of Theorem 1.5.

1.2. \textbf{Main ideas}. To explain the main ideas, we briefly review the proof of Jaworski’s theorem [Jaw74]: If a system \((X, \mathbb{Z}, T)\) is finite dimensional and has no periodic points, then it can be embedded into \([0, 1]^\mathbb{Z}\). We follow the presentation of Auslander [Aus88, Chapter 13].

For a continuous map \(f : X \to [0, 1]\) we define \(I_f : X \to [0, 1]^\mathbb{Z}\) by \(I_f(x) := (f(T^n x))_{n \in \mathbb{Z}}\). This is always \(\mathbb{Z}\)-equivariant. By the Baire category theorem, it is enough to prove that for any distinct \(x_1, x_2 \in X\) we can find closed neighborhoods \(A_i\) of \(x_i\) such that
\[
\{ f \in C(X, [0, 1]) | I_f(A_1) \cap I_f(A_2) = \emptyset \}
\]
is open dense in the space \(C(X, [0, 1])\) of continuous functions \(f : X \to [0, 1]\).

Fix a natural number \(N > 2 \dim X\). Since \(X\) has no periodic points, we can find \(j_1 < j_2 < \cdots < j_N\) such that \(2N\) points
\[
T^{j_1} x_1, \ldots, T^{j_N} x_1, T^{j_1} x_2, \ldots, T^{j_N} x_2
\]
are all distinct. (If \(x_2 = T^m x_1\) for some \(m\) then we take \(0, m+1, 2(m+1), \ldots, (N-1)(m+1)\). Otherwise we take \(0, 1, 2, \ldots, N-1\.) There exist open neighborhoods \(U_i\) of \(x_i\) such that \(2N\) sets \(T^{j_n} U_i (1 \leq n \leq N, i = 1, 2)\) are pairwisely disjoint. We take closed neighborhoods \(A_i \subset U_i\) of \(x_i\). Let \(\varphi : X \to [0, 1]\) be a cut-off function such that \(\varphi = 1\) on the union of the \(2N\) sets \(T^{j_n} A_i\) and its support is contained in the union of \(T^{j_n} U_i\).
Obviously the set (1.2) is open. Take arbitrary \( f \in C(X, [0, 1]) \) and \( \delta > 0 \). The condition \( N > 2 \dim X \) implies that generic continuous maps from \( X \) to \([0, 1]^N\) are topological embeddings. (See [HW41, Thm. V2]; it can be also deduced from Lemmas 2.1 and 3.3 below.) Hence we can find an embedding \( F : X \to [0, 1]^N \) such that the \( \ell^\infty \)-distance between \( F(x) \) and \( (f(T^{jn}x))_{1 \leq n \leq N} \) is less than \( \delta \) for all \( x \in X \).

We define a perturbation \( g \) of \( f \) as follows: If \( x \in T^{jn}U_i \) for some \( n \) and \( i \) then

\[
g(x) = (1 - \varphi(x))f(x) + \varphi(x)F(T^{-jn}x)_n.
\]

Otherwise set \( g(x) = f(x) \). This satisfies \( |g(x) - f(x)| < \delta \) and for \( x \in A_1 \cup A_2 \)

\[
(g(T^{j_1}x), \ldots, g(T^{j_N}x)) = F(x).
\]

Then \( g \) satisfies \( I_g(A_1) \cap I_g(A_2) = \emptyset \) because \( F \) is an embedding. This shows the density of (1.2) and finishes the proof.

The above proof has three important steps:

1. Find good pieces \( T^{j_1}x_1, \ldots, T^{j_N}x_1 \) and \( T^{j_1}x_2, \ldots, T^{j_N}x_2 \) of the orbits of \( x_1 \) and \( x_2 \).
2. Find an embedding \( F \) approximating \( I_f|_{\{j_1, \ldots, j_N\}} \) by using \( N > 2 \dim X \).
3. Define a perturbation \( g \) of \( f \) by “painting \( F \) on the good pieces of orbits”.

The proofs of our main theorems develop similar three steps for infinite dimensional systems. We do not need a substantial change in step (3). In step (2) we replace “embedding” with “\( \varepsilon \)-embedding”, which is an approximative version of embedding (see Section 2). The condition \( N > 2 \dim X \) is replaced with a condition on mean dimension. So we crucially need mean dimension theory in step (2). But machineries for this step were already developed for \( \mathbb{Z} \)-actions in [Lin99]; the \( \mathbb{Z}^k \)-case does not require a new idea.

The central issue is step (1). *We need to continuously partition every orbit into good pieces where steps (2) and (3) work well.* For dealing with this problem the paper [Lin99] introduced a new topological analogue of the Rokhlin tower lemma in ergodic theory. But this does not work for \( \mathbb{Z}^k \). The failure of the tower lemma technique is the main barrier to the generalization to \( \mathbb{Z}^k \).

The first idea to overcome this difficulty is the use of Voronoi tiling. Gutman [Gut11] is the first paper using Voronoi tiling in mean dimension theory. We develop this technique further. Suppose \( \mathbb{Z}^k \) continuously acts on a compact metric space \( X \). Take a small open set \( U \subset X \).
For each point $x \in X$ we consider the set
\[ C(x) = \{ n \in \mathbb{Z}^k | T^n x \in U \}, \]
and let
\[ \mathbb{R}^k = \bigcup_{n \in C(x)} V(x, n), \quad V(x, n) = \{ u \in \mathbb{R}^k | \forall m \in C(x) : |u-n| \leq |u-m| \}, \]
be the Voronoi tiling associated with $C(x)$. We try to use an appropriate piece $V(x, n)$ (or the lattice points $V(x, n) \cap \mathbb{Z}^k$) as a substitute for $\{j_1, \ldots, j_N\}$ in the proof of Jaworski’s theorem. This idea perfectly works if $X$ has an aperiodic symbolic factor. We prove Theorem 1.6 by this method. There is some analogy here to [Lig03, Lig04] where Voronoi tilings were used to obtain a $\mathbb{Z}^k$-analogue of the Krieger embedding theorem [Kri82] for symbolic subshifts.

In general the tiles $V(x, n)$ do not depend continuously on $x \in X$, and the above idea cannot be directly applied to Theorems 1.3, 1.4 or 1.5. So we introduce the second idea: **Adding one dimension.** This is the most important new idea in this paper. We take a cut-off function $\phi : U \to [0,1]$ supported in the above open set $U$, and consider the set
\[ \{(n, 1/\phi(T^n x)) | n \in \mathbb{Z}^k : \phi(T^n x) \neq 0\}. \]
Note that this is a subset of $\mathbb{R}^{k+1}$. So we go up one dimension higher. Let $\mathbb{R}^{k+1} = \bigcup_{n \in \mathbb{Z}^k} V(x, n)$ be the associated Voronoi decomposition. We take a large number $H$ and set $W(x, n) = V(x, n) \cap (\mathbb{R}^k \times \{-H\})$. Then we get the decomposition
\[ \mathbb{R}^k \times \{-H\} = \bigcup_{n \in \mathbb{Z}^k} W(x, n). \]
This does depend continuously on $x \in X$. Thus we can use $W(x, n)$ as a substitute for $\{j_1, \ldots, j_N\}$ in Jaworski’s theorem. This establishes step (1).

1.3. **Open problems.** The following three questions are the main open problems arising from the paper.

- **Can one remove the marker property assumption in Theorems 1.3 and 1.4?** We conjecture that $\text{mdim}(X) = 0$ is equivalent to the condition that $X$ is isomorphic to the inverse limit of finite entropy systems without any additional condition. We also conjecture that for any system $X$ there always exists a distance $d$ satisfying $\text{mdim}(X) = \text{mdim}_M(X, d)$. These conjectures are open even for $\mathbb{Z}$-actions.
• What is the optimal condition to ensure the existence of the embedding into the $\mathbb{Z}^k$-shift $([0,1]^D)\mathbb{Z}^k$? Probably it is enough to assume $\text{mdim}(X) < D/2$ and some conditions on the periodic points.

• How can one extend the theory to actions of non-commutative groups? Our new technique in this paper uses the Euclidean geometry in an essential way, and hence cannot be generalized to other groups directly.

1.4. Organization of the paper. In Section 2 we recall some basic definitions related to mean dimension. In Section 3 we prove the embedding theorem for extensions of aperiodic symbolic systems (Theorem 1.6) by using Voronoi tiling. Although this motivates more difficult arguments in later sections, it is logically independent of other theorems. In Section 4 we explain the technique of adding one dimension. In Section 5 we review the idea of small boundary property (a dynamical analogue of totally disconnectedness) and solve the problem of approximating a zero mean dimensional system by finite entropy ones (Theorem 1.3). In Section 6 we prove the existence of a distance $d$ satisfying $\text{mdim}_M(X, d) = \text{mdim}(X)$ (Theorem 1.4). In Section 7 we prove the embedding theorem for extensions of aperiodic minimal systems (Theorem 1.5). Sections 5, 6 and 7 are almost independent of each other; Section 7 is substantially more complicated than other sections.

2. Review of mean dimension

Here we recall some basic facts on mean dimension. For the details, see Gromov [Gro99] and Lindenstrauss–Weiss [LW00]. Throughout the paper we assume that $k$ is a fixed positive integer. For a positive integer $N$ we set

$$[N] = \{0, 1, 2, \ldots, N - 1\}^k \subset \mathbb{Z}^k.$$

All simplicial complexes are implicitly assumed to be finite, i.e. consist of only finitely many simplices.

Let $(X, d)$ be a compact metric space. Let $Y$ be a topological space, and $f : X \to Y$ a continuous map. For a positive number $\varepsilon$ the map $f$ is called an $\varepsilon$-embedding if $\text{diam}f^{-1}(y) < \varepsilon$ for all $y \in Y$. We define $\text{Widim}_\varepsilon(X, d)$ as the minimum integer $n \geq 0$ such that there exist an $n$-dimensional simplicial complex $P$ and an $\varepsilon$-embedding $f : X \to P$.

Let $P$ be a simplicial complex, and $V$ a Banach space. A map $f : P \to V$ is said to be linear if it has the following form on every face
\( \Delta \subset P: \)
\[
f\left( \sum_{i=0}^{n} \lambda_i v_i \right) = \sum_{i=0}^{n} \lambda_i f(v_i)
\]
where \( v_i \) are the vertices of \( \Delta \) and \( \lambda_i \) are nonnegative numbers satisfying \( \sum_{i=0}^{n} \lambda_i = 1. \)

**Lemma 2.1.** Let \( C \subset V \) be a convex subset, \((X, d)\) a compact metric space, and \( f: X \to C \) a continuous map. Suppose \( \varepsilon \) and \( \delta \) are positive numbers satisfying
\[
d(x, y) < \varepsilon \implies \|f(x) - f(y)\| < \delta.
\]

Let \( \pi: X \to P \) be an \( \varepsilon \)-embedding from \((X, d)\) to a simplicial complex \( P \). Then, after replacing \( P \) by a sufficiently finer subdivision, there exists a linear map \( g: P \to C \) satisfying
\[
\|f(x) - g(\pi(x))\| < \delta, \quad \forall x \in X.
\]

**Proof.** By subdividing \( P \), we can assume \( \text{diam} \pi^{-1}(O(v)) < \varepsilon \) for all vertices \( v \) of \( P \). Here \( O(v) \) is the open star of \( v \), namely the union of the relative interiors of faces containing \( v \). For each vertex \( v \in P \) we define \( g(v) \) as follows: If \( \pi^{-1}(O(v)) \neq \emptyset \) then we choose a point \( x_v \in \pi^{-1}(O(v)) \) and set \( g(v) = f(x_v) \). If \( \pi^{-1}(O(v)) = \emptyset \), then we choose \( g(v) \in C \) arbitrarily. We define a linear map \( g: P \to C \) by extending it linearly on every face.

Take \( x \in X \), and let \( v_0, \ldots, v_n \) be the vertices of the face of \( P \) containing \( \pi(x) \) in its relative interior. Suppose \( \pi(x) = \sum_i \lambda_i v_i \) with \( 0 < \lambda_i < 1 \) and \( \sum \lambda_i = 1. \) Since \( \pi(x) \in O(v_i) \) for each \( i \), we get \( d(x, x_{v_i}) < \varepsilon \). Hence \( \|f(x) - f(x_{v_i})\| < \delta. \) So
\[
\|f(x) - g(\pi(x))\| = \left\| \sum \lambda_i(f(x) - f(x_{v_i})) \right\| < \delta.
\]

Suppose \((X, d)\) is a compact metric space with a continuous action \( T: \mathbb{Z}^k \times X \to X \). For a subset \( \Omega \subset \mathbb{Z}^k \) we define a new distance \( d_\Omega \) on \( X \) by
\[
d_\Omega(x, y) = \sup_{n \in \Omega} d(T^n x, T^n y).
\]

We define the **mean dimension** \( \text{mdim}(X, \mathbb{Z}^k, T) \) (often abbreviated to \( \text{mdim}(X) \)) by
\[
\text{mdim}(X, \mathbb{Z}^k, T) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{1}{N^k} \text{Widim}_\varepsilon(X, d_{[N]}) \right).
\]
The limit with respect to \( N \) in the above equation exists because of the following sub-additivity and invariance:

\[
\text{Widim}_\varepsilon(X, d_{\Omega_1 \cup \Omega_2}) \leq \text{Widim}_\varepsilon(X, d_{\Omega_1}) + \text{Widim}_\varepsilon(X, d_{\Omega_2}), \quad (\Omega_1, \Omega_2 \subset \mathbb{Z}^k),
\]

\[
\text{Widim}_\varepsilon(X, d_{a + \Omega}) = \text{Widim}_\varepsilon(X, d_{\Omega}), \quad (a \in \mathbb{Z}^k, \Omega \subset \mathbb{Z}^k),
\]

where \( a + \Omega \) denotes the set \( \{a + n | n \in \Omega\} \). From this property and the standard division argument

\[
\lim_{N \to \infty} \frac{1}{N^k} \text{Widim}_\varepsilon(X, d_{[N]}) = \inf_{N \geq 1} \frac{1}{N^k} \text{Widim}_\varepsilon(X, d_{[N]}).
\]

Note that the value of \( \text{mdim}(X) \) is independent of the choice of a distance \( d \) compatible with the topology, hence mean dimension is a topological invariant.

**Example 2.2.** Let \( X = ([0,1]^D)^{\mathbb{Z}^k} \) with the standard shift action of \( \mathbb{Z}^k \). Then its mean dimension is \( D \).

In Section 3 we use more general Følner sequences. For a subset \( \Omega \subset \mathbb{Z}^k \) and \( R > 0 \) we define \( \partial^R_{\mathbb{Z}^k} \Omega \) as the set of \( n \in \mathbb{Z}^k \) such that there exist \( m \in \Omega \) and \( m' \in \mathbb{Z}^k \setminus \Omega \) satisfying \( |n - m| \leq R \) and \( |n - m'| \leq R \). We set \( \text{int}^R_{\mathbb{Z}^k} \Omega = \Omega \setminus \partial^R_{\mathbb{Z}^k} \Omega \). Here the symbol \( \mathbb{Z}^k \) indicates that these are defined for subsets of \( \mathbb{Z}^k \). In later sections we will consider \( \partial \) and \( \text{int} \) for subsets of \( \mathbb{R}^k \). A sequence \( \{\Omega_n\}_{n \geq 1} \) of finite subsets of \( \mathbb{Z}^k \) is called a Følner sequence if for every \( R > 0 \)

\[
\lim_{n \to \infty} \frac{|\partial^R_{\mathbb{Z}^k} \Omega_n|}{|\Omega_n|} = 0.
\]

For example the sequence \( \{|n|\}_{n \geq 1} \) is Følner. We use the following lemma (Gromov [Gro99, p. 336] and Lindenstrauss–Weiss [LW00, Appendix]). This lemma is originally due to Ornstein–Weiss [OW87, Chapter 1, Sections 2 and 3] and holds for general amenable groups.

**Lemma 2.3.** Let \( h : \{\text{finite subsets of } \mathbb{Z}^k\} \to \mathbb{R} \) be a nonnegative function satisfying the following three conditions.

- If \( \Omega_1 \subset \Omega_2 \), then \( h(\Omega_1) \leq h(\Omega_2) \).
- \( h(\Omega_1 \cup \Omega_2) \leq h(\Omega_1) + h(\Omega_2) \).
- For any \( a \in \mathbb{Z}^k \) and any finite subset \( \Omega \subset \mathbb{Z}^k \), we have \( h(a + \Omega) = h(\Omega) \).

Then for any Følner sequence \( \{\Omega_n\}_{n \geq 1} \) in \( \mathbb{Z}^k \), the limit of the sequence

\[
\frac{h(\Omega_n)}{|\Omega_n|} \quad (n \geq 1)
\]

exists and is independent of the choice of a Følner sequence.
For example the function \( h(\Omega) = \text{Widim}_\varepsilon(X, d_\Omega) \) (defined for a dynamical system \((X, \mathbb{Z}^k, T)\)) satisfies the above three conditions. So for any Følner sequence \( \{\Omega_n\}_{n \geq 1} \) in \( \mathbb{Z}^k \) we have

\[
\text{mdim}(X) = \lim_{\varepsilon \to 0} \left( \lim_{n \to \infty} \frac{\text{Widim}_\varepsilon(X, d_{\Omega_n})}{|\Omega_n|} \right).
\]

3. An embedding theorem: proof of Theorem 1.6

In this section we prove Theorem 1.6. Fix a positive integer \( D \). Let \((Z, \mathbb{Z}^k, S)\) be an aperiodic zero dimensional system. Here \( \dim Z = 0 \) means that clopen (closed and open) subsets form an open basis. Symmetric systems are zero dimensional. Let \( \pi : X \to Z \) be an extension. Namely \((X, \mathbb{Z}^k, T)\) is a dynamical system with a \( \mathbb{Z}^k \)-equivariant continuous surjection \( \pi \). Let \( C(X, [0, 1]^D) \) be the space of continuous maps \( f : X \to [0, 1]^D \) with the uniform norm topology. For \( f : X \to [0, 1]^D \) we define the map \( I_f : X \to ([0, 1]^D)^{\mathbb{Z}^k} \) by \( I_f(x) = (f(T^n x))_{n \in \mathbb{Z}^k} \).

**Theorem 3.1.** If \( \text{mdim}(X) < D/2 \) then for a dense \( G_\delta \) subset of \( f \in C(X, [0, 1]^D) \) the map

\[
(I_f, \pi) : X \to ([0, 1]^D)^{\mathbb{Z}^k} \times Z, \quad x \mapsto (I_f(x), \pi(x))
\]

is an embedding.

Theorem 1.6 in the introduction follows from this theorem.

**Proof of Theorem 1.6.** Suppose \( Z \) is an aperiodic subsystem of \( \{1, 2, \ldots, l\}^{\mathbb{Z}^k} \) and \( X \) is its extension with \( \text{mdim}(X) < D/2 \). We can topologically embed the space \([0, 1]^D \times \{1, 2, \ldots, l\} \) into \([0, 1]^D \). Hence we can dynamically embed the system \(([0, 1]^D)^{\mathbb{Z}^k} \times \{1, 2, \ldots, l\}^{\mathbb{Z}^k} \) into \(([0, 1]^D)^{\mathbb{Z}^k} \). So we can also embed \(([0, 1]^D)^{\mathbb{Z}^k} \times Z \) into \(([0, 1]^D)^{\mathbb{Z}^k} \). Then Theorem 3.1 implies that we can embed \( X \) into the system \(([0, 1]^D)^{\mathbb{Z}^k} \). \( \square \)

In the rest of this section we always assume that \( Z \) is an aperiodic zero dimensional system and \( \pi : X \to Z \) is an extension with \( \text{mdim}(X) < D/2 \). We set \( K = [0, 1]^D \) for simplicity of the notation. Let \( d \) be a distance on \( X \). For each \( \delta > 0 \) the set

\[
\left\{ f \in C(X, K) \left| (I_f, \pi) : X \to K^{\mathbb{Z}^k} \times Z \text{ is a } \delta\text{-embedding} \right. \right\}
\]

is open in \( C(X, K) \). Consider the \( G_\delta \) subset

\[
\bigcap_{n \geq 1} \left\{ f \in C(X, K) \left| (I_f, \pi) : X \to K^{\mathbb{Z}^k} \times Z \text{ is a } \frac{1}{n}\text{-embedding} \right. \right\}.
\]
This is equal to the set of $f \in C(X, K)$ such that $(I_f, \pi)$ is an embedding. Therefore Theorem 3.1 follows from the next proposition.

**Proposition 3.2.** Let $f : X \to K$ be a continuous map. For any $\delta > 0$ there exists a continuous map $g : X \to K$ satisfying

1. $|f(x) - g(x)| < \delta$ for all $x \in X$,
2. $(I_g, \pi) : X \to K^{Z^k} \times Z$ is a $\delta$-embedding with respect to the distance $d$.

We recall the following well-known fact about simplicial complexes:

**Lemma 3.3.** Let $n$ be a positive integer, and $P$ a simplicial complex with $\dim P < n/2$. Then almost every linear map $\varphi : P \to [0, 1]^n$ is a topological embedding.

**Proof.** Let $v_1, \ldots, v_s$ be the vertices of $P$. The space of linear maps $f : P \to [0, 1]^n$ is identified with

$$([0, 1]^n)^{\{v_1, \ldots, v_s\}}$$

which is endowed with the standard Lebesgue measure. The above “almost every” is defined with respect to this measure. For almost every choice of vectors $u_1, \ldots, u_s \in [0, 1]^n$, all $(n + 1)$-tuples $u_{i_1}, \ldots, u_{i_{n+1}}$ ($i_1 < \cdots < i_{n+1}$) are affinely independent. Then for such a choice of $u_1, \ldots, u_s$ the linear map $\varphi : P \to [0, 1]^n$ defined by $\varphi(v_i) = u_i$ becomes a topological embedding because $2 \dim P + 2 \leq n + 1$. □

The following lemma established in particular that an aperiodic zero-dimensional system has the marker property (cf. Definition 1.2). Note however that the set $U$ in the lemma below is not just an open set as in Definition 1.2 but clopen, which greatly simplifies the subsequent constructions.

**Lemma 3.4.** For any $L > 0$ there exists a clopen subset $U \subset Z$ such that

$$Z = \bigcup_{|n| < L} S^n U$$

and $U \cap S^n U = \emptyset$ for all non-zero $n \in \mathbb{Z}^k$ with $|n| < L$.

**Proof.** This lemma is close to the argument of Lightwood [Lig03, Section 4]. From $\dim Z = 0$, there exists a clopen covering $\{V_1, \ldots, V_a\}$ of $Z$ such that

$$V_i \cap S^n V_i = \emptyset \quad (\forall 1 \leq i \leq a, 0 < |n| < L).$$
We inductively define clopen sets $U_1, \ldots, U_a$ by $U_1 = V_1$ and

$$U_{i+1} = U_i \cup \left( V_{i+1} \setminus \bigcup_{|n|<L} S^n U_i \right).$$

Then $U_i \cap S^n U_i = \emptyset$ for $0 < |n| < L$ and

$$V_1 \cup \cdots \cup V_i \subset \bigcup_{|n|<L} S^n U_i.$$

Thus $U = U_a$ satisfies the required properties. \hfill \Box

Let $L$ be a positive integer, and $U \subset Z$ a clopen subset given by Lemma 3.4. For each point $x \in Z$ we define $C(x) \subset \mathbb{Z}^k$ as the set of $n \in \mathbb{Z}^k$ satisfying $S^n x \in U$. From (3.1), this is syndetic (coarsely dense) in $\mathbb{Z}^k$. From $U \cap S^n U = \emptyset$ $(0 < |n| < L)$ we have $|n-m| \geq L$ for any distinct $n$ and $m$ in $C(x)$. We have

(3.2) \quad \quad C(S^n x) = -n + C(x).

The set $C(x)$ depends continuously on $x$; for any $x \in Z$ and any finite set $\Omega \subset \mathbb{Z}^k$ if $y \in Z$ is sufficiently close to $x$ then $C(y) \cap \Omega = C(x) \cap \Omega$.

We consider the Voronoi decomposition associated with $C(x)$. We define a bounded convex polytope $V(x, n) \subset \mathbb{R}^k$ for each $n \in C(x)$ by

$$V(x, n) = \{ u \in \mathbb{R}^k | \forall m \in C(x) : |u-n| \leq |u-m| \}.$$ 

This contains the closed ball $B_L(n)$ of radius $L/2$ centered at $n$. These form a tiling:

$$\mathbb{R}^k = \bigcup_{n \in C(x)} V(x, n).$$

The tiles $V(x, n)$ are locally constant; for any $x \in Z$ and $n \in C(x)$ if $y \in Z$ is sufficiently close to $x$ then $n \in C(y)$ and $V(y, n) = V(x, n)$. We set $V_Z(x, n) = \mathbb{Z}^k \cap V(x, n)$.

**Lemma 3.5.** For any $R > 0$ we can choose $L$ sufficiently large so that all $V(x, n)$ satisfy

$$\frac{|\partial_R^2 V_Z(x, n)|}{|V_Z(x, n)|} < \frac{1}{R}.$$ 

**Proof.** For simplicity of the notation, we write $V = V(x, n)$. For a positive number $c$ we define $cV$ by

$$cV = \{ n + c(u-n) \in \mathbb{R}^k | u \in V \}.$$ 

When $c < 1$, $cV$ is contained in $V$. When $c > 1$, $cV$ contains $V$.

**Claim 3.6.** If $c < 1$ then all points $p$ in $cV$ satisfy $B_{(1-c)L/2}(p) \subset V$. If $c > 1$ then all points $p$ outside of $cV$ satisfy $B_{(c-1)L/2}(p) \cap V = \emptyset$. 


Proof. We consider the case $c < 1$. The case $c > 1$ is similar. Pick $p \in cV$ and let $q \in \partial V$ be any point in the boundary $\partial V$. Let $F$ be a facet ($(k - 1)$-dimensional face) of $V$ containing $q$. Let $cF = \{n + c(u - n) | u \in F\}$ be the corresponding facet of $cV$, and $H$ the hyperplane containing $cF$. Since $B_{1/2}(n) \subset V$, the distance between $H$ and $F$ is at least $(1 - c)L/2$. The line segment between $p$ and $q$ must intersects with $H$ because $p$ and $q$ are located on different sides of $H$. Hence $|p - q| \geq d(H, F) \geq (1 - c)L/2$. This implies $B_{(1-c)L/2}(p) \subset V$. □

Now that the claim has been established we can finish the proof of Lemma 3.5. Let $c = 1 - 2\sqrt{k}/L$ and $p \in cV$. Take $m \in \mathbb{Z}^k$ satisfying $p \in m + [0, 1)^k$. By $(1 - c)L/2 = \sqrt{k}$ and Claim 3.6

$$m \in B_{\sqrt{k}}(p) \subset V.$$ 

So $m \in V^\mathbb{Z} = V \cap \mathbb{Z}^k$. This means that

$$\bigcup_{m \in V^\mathbb{Z}} (m + [0, 1)^k) \supset cV.$$

Then $|V^\mathbb{Z}| \geq \text{vol}(cV) = c^k \text{vol}(V) = (1 - 2\sqrt{k}/L)^k \text{vol}(V)$ where $\text{vol}(V)$ is the $k$-dimensional Lebesgue measure of $V$.

Set $c_1 = 1 - 2(R + \sqrt{k})/L$ and $c_2 = 1 + 2(R + \sqrt{k})/L$. Note $(1 - c_1)L/2 = (c_2 - 1)L/2 = R + \sqrt{k}$. Take $m \in c_2 V \setminus c_1 V$. Then for every point $p$ in $m + [0, 1)^k$ the ball $B_{R + \sqrt{k}}(p)$ has non-empty intersections both with $V$ and $\mathbb{R}^k \setminus V$. This implies that $m + [0, 1)^k$ is contained in $c_2 V \setminus c_1 V$ by Claim 3.6. Hence

$$\bigcup_{m \in \partial c_1^\mathbb{Z} V} (m + [0, 1)^k) \subset c_2 V \setminus c_1 V.$$

Therefore $|\partial c_1^\mathbb{Z} V| \leq \text{vol}(c_2 V \setminus c_1 V) = (c_2^k - c_1^k) \text{vol}(V)$. As a conclusion,

$$\frac{|\partial c_1^\mathbb{Z} V|}{|V^\mathbb{Z}|} \leq \frac{(1 + 2(R + \sqrt{k})/L)^k - (1 - 2(R + \sqrt{k})/L)^k}{(1 - 2\sqrt{k}/L)^k}.$$ 

If $L$ is sufficiently large then the right-hand-side is smaller than $1/R$.

□

Proof of Proposition 3.2. Let $\delta > 0$ and $f \in C(X, K)$. We choose $0 < \varepsilon < \delta$ such that

$$d(x, y) < \varepsilon \implies |f(x) - f(y)| < \delta.$$ (3.3)
Using $\text{mdim}(X, T) < D/2$ and the definition of mean dimension in [2, 3], we can find $R > 0$ such that if a finite subset $\Omega \subset \mathbb{Z}^k$ satisfies
\[
\frac{|\partial_R^x \Omega|}{|\Omega|} < \frac{1}{R}
\]
then we have
\[
(3.4) \quad \frac{\text{Widim}_x(X, d_\Omega)}{|\text{int}_1^z \Omega|} < \frac{D}{2}.
\]
Here recall $\text{int}_1^z \Omega = \Omega \setminus \partial_1^z \Omega$.

We define an equivalence relation between finite subsets of $\mathbb{Z}^k$ by
\[
\Omega_1 \sim \Omega_2 \iff \exists a \in \mathbb{Z}^k : \Omega_2 = a + \Omega_1.
\]
We set
\[
\mathcal{A} := \left\{ \Omega \subset \mathbb{Z}^k : \text{finite set} \mid \frac{|\partial_R^x \Omega|}{|\Omega|} < \frac{1}{R} \right\} / \sim.
\]
Choose $\Omega_1, \Omega_2, \ldots \subset \mathbb{Z}^k$ so that
\[
\mathcal{A} = \{ [\Omega_1], [\Omega_2], \ldots \}, \quad \Omega_i \not\sim \Omega_j (i \neq j).
\]
For each $\Omega_i$ we consider
\[
I_f|_{\text{int}_1^z \Omega_i} : X \rightarrow K^{\text{int}_1^z \Omega_i}, \quad x \mapsto (f(T^n x))_{n \in \text{int}_1^z \Omega_i}.
\]
This satisfies (by (3.3))
\[
d_{\Omega_i}(x, y) < \varepsilon \implies \left\| I_f(x)|_{\text{int}_1^z \Omega_i} - I_f(y)|_{\text{int}_1^z \Omega_i} \right\|_\infty \overset{\text{def}}{=} \max_{n \in \text{int}_1^z \Omega_i} |f(T^n x) - f(T^n y)| < \delta.
\]
Choose an $\varepsilon$-embedding $p_i : (X, d_{\Omega_i}) \rightarrow P_i$ such that $P_i$ is a simplicial complex of dimension $\text{Widim}_x(X, d_{\Omega_i}) < (D/2)|\text{int}_1^z \Omega_i|$ (by (3.4)). Then by Lemmas 2.1 and 3.3 we can find a linear embedding $g_i : P_i \rightarrow K^{\text{int}_1^z \Omega_i}$ satisfying
\[
\left\| g_i(p_i(x)) - I_f(x)|_{\text{int}_1^z \Omega_i} \right\|_\infty < \delta.
\]
Set $F_i = g_i \circ p_i : X \rightarrow K^{\text{int}_1^z \Omega_i}$. This is an $\varepsilon$-embedding with respect to $d_{\Omega_i}$.

By Lemma 3.3 we can choose $L > 0$ so that all Voronoi tiles $V(x, n)$ ($x \in Z$) satisfy $|\partial_R^x V(x, n)|/|V(x, n)| < 1/R$. We want to define a perturbation $g : X \rightarrow K$ of $f$. Roughly speaking we define it by painting the functions $F_i$ inside the Voronoi tiles. Take $x \in X$. We look at the decomposition $\mathbb{Z}^k = \bigcup_{n \in C(\pi(x))} V^z(\pi(x), n)$ associated with $\pi(x) \in Z$, and we ask which tile contains the origin 0. There are two cases:
(1) For any $n \in C(\pi(x))$ the origin is not contained in $\text{int}^Z V^Z(\pi(x), n)$.

(2) There uniquely exists $n \in C(\pi(x))$ satisfying $0 \in \text{int}^Z V^Z(\pi(x), n)$.

In Case (1) we set $g(x) = f(x)$. In Case (2) we proceed as follows. The Voronoi tile $V(\pi(x), n)$ satisfies $|\partial^Z_{\mathbb{R}} V^Z(\pi(x), n)|/|V^Z(\pi(x), n)| < 1/R$. Hence there uniquely exist $i \geq 1$ and $a \in \mathbb{Z}^k$ satisfying $V^Z(\pi(x), n) = a + \Omega_i$. Then we set

$$g(x) = F_i(T^a x)_a.$$

Here $-a \in \text{int}^Z \Omega_i$ because $0 \in \text{int}^Z V^Z(\pi(x), n)$. The Voronoi tile $V(\pi(x), n)$ is locally constant with respect to $x$. Hence the map $g$ becomes continuous. (The above $i$ and $a$ are also locally constant.) It satisfies $|g(x) - f(x)| < \delta$ because

$$|F_i(T^a x)_a - f(x)| \leq \left\| F_i(T^a x) - I_f(T^a x)|_{\text{int}^Z \Omega_i} \right\| < \delta.$$

**Claim 3.7.** Let $x \in X$ and $n \in C(\pi(x))$. Take $i \geq 1$ and $a \in \mathbb{Z}^k$ satisfying $V^Z(\pi(x), n) = a + \Omega_i$. Then $I_g(x)|_{\text{int}^Z V^Z(\pi(x), n)} = F_i(T^a x)$.

Here we naturally consider $I_g(x)|_{\text{int}^Z V^Z(\pi(x), n)}$ as an element of $K_{\text{int}^Z \Omega_i}$ through $\text{int}^Z V^Z(\pi(x), n) = a + \text{int}^Z \Omega_i$.

**Proof.** Take $m \in \text{int}^Z V^Z(\pi(x), n) = a + \text{int}^Z \Omega_i$. By (3.2)

$$V^Z(\pi(T^m x), n - m) = -m + V^Z(\pi(x), n) = -m + a + \Omega_i.$$

Hence the origin is contained in $\text{int}^Z V^Z(\pi(T^m x), n - m)$, and

$$g(T^m x) = F_i(T^{-m+a}(T^m x))_{m-a} = F_i(T^a x)_{m-a}.$$

When $m$ runs over $\text{int}^Z V^Z(\pi(x), n)$, $m - a$ runs over $\text{int}^Z \Omega_i$. Thus we get $I_g(x)|_{\text{int}^Z V^Z(\pi(x), n)} = F_i(T^a x)$. \qed

We want to prove that the map $(I_g, \pi) : X \to K^{Z^k} \times Z$ is a $\delta$-embedding with respect to $d$. Suppose $(I_g(x), \pi(x)) = (I_g(y), \pi(y))$ for some $x, y \in X$. Set $z = \pi(x) = \pi(y)$, and take $n \in C(z)$ satisfying $0 \in V^Z(z, n)$. We can find $i \geq 1$ and $a \in \mathbb{Z}^k$ satisfying $V^Z(\pi(z, n) = a + \Omega_i$. (Note that this implies $-a \in \Omega_i$.) By Claim 3.7

$$I_g(x)|_{\text{int}^Z V^Z(z, n)} = F_i(T^a x) = I_g(y)|_{\text{int}^Z V^Z(z, n)} = F_i(T^a y).$$

Since $F_i$ is an $\varepsilon$-embedding with respect to $d_{\Omega_i}$, we get $d_{\Omega_i}(T^a x, T^a y) < \varepsilon$. Then $-a \in \Omega_i$ implies $d(x, y) < \varepsilon < \delta$. \qed
4. Adding one dimension

The key idea in Section 3 is the use of the Voronoi tiling. The purpose of this section is to present a modified version of Voronoi tiling technique. This has a wider applicability and will be used in the rest of the paper.

Suppose a system \((X, \mathbb{Z}^k, T)\) has the marker property (cf. Definition 1.2). Let \(M\) be a positive integer. From the marker property there exist an integer \(L \geq M\) and a continuous function \(\phi : X \to [0, 1]\) so that

- If \(\phi(x) > 0\) at some \(x \in X\), then \(\phi(T^n x) = 0\) for all non-zero \(n \in \mathbb{Z}^k\) with \(|n| < M\).
- For any \(x \in X\) there exists \(n \in \mathbb{Z}^k\) satisfying \(|n| < L\) and \(\phi(T^n x) = 1\).

Proof. Let \(U \subset X\) be an open set satisfying the definition of the marker property. Since \(X\) is compact, there exists a compact set \(K \subset U\) and an integer \(L \geq M\) satisfying

\[
X = \bigcup_{|n| < L} T^{-n} K.
\]

Take a continuous function \(\phi : X \to [0, 1]\) satisfying \(\phi = 1\) on \(K\) and \(\text{supp } \phi \subset U\). \(\square\)

Now we introduce the key technique: adding one dimension. Take a point \(x \in X\), and we consider the Voronoi tiling in \(\mathbb{R}^{k+1}\) associated with the set

\[
\{(n, 1/\phi(T^n x)) | n \in \mathbb{Z}^k : \phi(T^n x) \neq 0\}.
\]

We define the Voronoi cell \(V(x, n) \subset \mathbb{R}^{k+1}\) with the Voronoi center \((n, 1/\phi(T^n x))\) by

\[
V(x, n) = \left\{ u \in \mathbb{R}^{k+1} \middle| \forall m \in \mathbb{Z}^k : \right. \left. |u - (n, 1/\phi(T^n x))| \leq |u - (m, 1/\phi(T^m x))| \right\}.
\]

Here \(| \cdot |\) is the standard Euclidean norm. For \(n \in \mathbb{Z}^k\) with \(\phi(T^n x) = 0\) we set \(V(x, n) = \emptyset\). The space \(\mathbb{R}^{k+1}\) is decomposed into these convex subsets:

\[
\mathbb{R}^{k+1} = \bigcup_{n \in \mathbb{Z}^k} V(x, n).
\]

We choose a real number \(H \geq (L + \sqrt{k})^2\) and set

\[
W(x, n) = \pi_{\mathbb{R}^k} \left( V(x, n) \cap (\mathbb{R}^k \times \{-H\}) \right),
\]

with \(\pi_{\mathbb{R}^k}\) denoting the projection to the first \(k\) coordinates.
Then \( \mathbb{R}^k \) is decomposed into these \( W(x,n) \):

\[
\mathbb{R}^k = \bigcup_{n \in \mathbb{Z}^k} W(x,n).
\]

This construction is naturally \( \mathbb{Z}^k \)-equivariant:

\[
W(T^m x, n - m) = -m + W(x,n).
\]

For each \( n \in \mathbb{Z}^k \) the polytope \( W(x,n) \) depends continuously on \( x \) in the following sense: Suppose \( W(x,n) \) has a non-empty interior. For any \( \varepsilon > 0 \) if \( y \in X \) is sufficiently close to \( x \) then the Hausdorff distance between \( W(x,n) \) and \( W(y,n) \) are smaller than \( \varepsilon \). We gather basic properties of the Voronoi tiling in the next lemma.

**Lemma 4.1.** Let \( n \in \mathbb{Z}^k \) and \( a \in \mathbb{R}^k \).

1. If \( \phi(T^n x) > 0 \), then

\[
B_{M/2}(n,1/\phi(T^n x)) \subset V(x,n).
\]

Here \( B_{M/2}(\cdot) \) denotes the ball of radius \( M/2 \) with respect to the Euclidean norm.

2. If \( W(x,n) \) is non-empty, then \( 1 \leq 1/\phi(T^n x) \leq 2 \).

3. If \( (a,-H) \in V(x,n) \), i.e. \( a \in W(x,n) \), then \( |a - n| < L + \sqrt{k} \).

4. Let \( s > 1 \) and \( r > 0 \). We can choose \( M \) sufficiently large depending on \( s,r \) so that if \( (a,-sH) \in V(x,n) \) then

\[
B_r(a/s + (1 - 1/s)n) \subset W(x,n).
\]

Note that the choice of \( L, H \) depends on \( M \).

Taking \( s \approx 1 \) and \( r \) large, property (4) of Lemma 4.1 implies that if \( V(x,n) \) intersects \( \mathbb{R}^k \times \{-sH\} \) then \( W(x,n) \) contains a large ball.

**Proof.** (1) Since \( \phi(T^n x) > 0 \), we have \( \phi(T^{n+i} x) = 0 \) for \( 0 < |i| < M \). Hence for any \( m \neq n \) with \( \phi(T^m x) > 0 \) we have \( |n - m| \geq M \). For \( u \in B_{M/2}(n,1/\phi(T^n x)) \)

\[
|u - (m,1/\phi(T^m x))| \geq |(n,1/\phi(T^n x)) - (m,1/\phi(T^m x))| - |u - (n,1/\phi(T^n x))| \geq M/2.
\]

(2) and (3): Suppose \( (a,-H) \in V(x,n) \) and set \( t = 1/\phi(T^n x) \geq 1 \). Let \( b \in \mathbb{Z}^k \) be the nearest integer point to \( a \). There exists \( |i| < L \) satisfying \( \phi(T^{b+i} x) = 1 \). Set \( m = b + i \). We have \( |a - m| < L + \sqrt{k} \) and

\[
|(a,-H) - (n,t)| \leq |(a,-H) - (m,1)| < \sqrt{(L + \sqrt{k})^2 + (H + 1)^2}.
\]
We get $|a-n| < L+\sqrt{k}$ (by $t \geq 1$) and $(H+t)^2 < (L+\sqrt{k})^2 + (H+1)^2$. Recall $H \geq (L+\sqrt{k})^2$. Then

$$t \leq 1 + \frac{(L+\sqrt{k})^2 + 1}{2H} \leq 2.$$ 

This proves (2) and (3).

(4) The same argument as above shows $|a-n| < L + \sqrt{k}$. Set $t = 1/\phi(T^n x)$. For any $u \in \mathbb{R}^k$ with $|u-n| \leq M/2$ the point $(u,t)$ is contained in $V(x, n)$ by (1). Consider the line $\ell$ between $(a, -sH)$ and $(u, t)$. By the convexity of $V(x, n)$, the intersection between $\ell$ and $\mathbb{R}^k \times \{-H\}$ is contained in $W(x, n)$. It follows that

$$B_{\frac{(s-1)HM}{2(sH+t)}} \left( a + \frac{(s-1)H}{sH+t}(n-a) \right) \subset W(x, n).$$

The radius of the ball in the left hand side of (4.1) goes to infinity as $M \to \infty$ because

$$\frac{(s-1)HM}{2(sH+t)} = \frac{(s-1)M}{2(s+t/H)}$$

(recall $H$ depends on $M$, but as $M \to \infty$ so do both $L$ and $H$).

Our choice of parameters ensures that the center of the ball in (4.1) is close to $a/s + (1 - 1/s)n$. Indeed,

$$\left| \frac{a}{s} + \left( 1 - \frac{1}{s} \right) n - \left( a + \frac{(s-1)H}{sH+t}(n-a) \right) \right| =$$

$$= \left| \left( 1 - \frac{1}{s} - \frac{(s-1)H}{sH+t} \right) (n-a) \right|$$

$$\leq \left| 1 - \frac{1}{s} - \frac{s-1}{s+t/H} \right| (L+\sqrt{k})$$

$$= \left| \left( 1 - \frac{1}{s} \right) \frac{t}{H} \cdot \frac{1}{s+t/H} \right| (L+\sqrt{k})$$

$$\leq \frac{4(L+\sqrt{k})}{H} \leq \frac{4}{L+\sqrt{k}}.$$ 

Here we have used $|n-a| < L+\sqrt{k}$, $1 \leq t \leq 2$ and $H \geq (L+\sqrt{k})^2$. Since $L \geq M$, this goes to zero as $M \to \infty$. Therefore if $M$ is sufficiently large then $B_r(a/s + (1 - 1/s)n)$ is contained in $W(x, n)$. \hfill \Box

Let $A \subset \mathbb{R}^k$. We denote by $\partial A$ and $\text{int} A = A \setminus \partial A$ the standard boundary and interior of $A$. For $E > 0$ we define $\partial_E A$ as the set of
$u \in \mathbb{R}^k$ satisfying $B_E(u) \cap A \neq \emptyset$ and $B_E(u) \cap (\mathbb{R}^k \setminus A) \neq \emptyset$. Set $\text{int}_E A = A \setminus \partial E A$. For $x \in X$, we define a set $\partial(x, E)$ in $\mathbb{R}^k$ by

$$\partial(x, E) = \bigcup_{n \in \mathbb{Z}^k} \partial E W(x, n).$$

The next lemma is crucial in the proofs of Theorems 1.3 and 1.4.

**Lemma 4.2.** For any $\varepsilon > 0$ and $E > 0$ if we choose $M$ sufficiently large then

$$\limsup_{R \to \infty} \frac{1}{\text{vol}(B_R)} \sup_{x \in X} \text{vol}(\partial(x, E) \cap B_R) < \varepsilon.$$

Here $B_R = \{u \in \mathbb{R}^k \mid |u| \leq R\}$, and $\text{vol}(\cdot)$ is the $k$-dimensional Lebesgue measure.

**Proof.** Choose $s > 1$ with $1 - s^{-k} < \varepsilon$. By Lemma 4.1 (4), we can choose $M$ so that

$$(a, -sH) \in V(x, n) \implies B_E \left(\frac{a}{s} + \left(1 - \frac{1}{s}\right)n\right) \subset W(x, n).$$

For $n \in \mathbb{Z}^k$ and $x \in X$ we set

$$W'(x, n) = \pi_{\mathbb{R}^k} \left(\left(\mathbb{R}^k \times \{-sH\}\right) \cap V(x, n)\right).$$

For each $x \in X$ these $W'(x, n) \ (n \in \mathbb{Z}^k)$ form a tiling of $\mathbb{R}^k$. For any $a \in W'(x, n)$, the point $a/s + (1 - 1/s)n$ is contained in $\text{int}_E W(x, n)$. Hence

$$\text{vol}(\text{int}_E W(x, n)) \geq \text{vol}\left(\left\{a/s + \left(1 - \frac{1}{s}\right)n \mid a \in W'(x, n)\right\}\right) = s^{-k}\text{vol}(W'(x, n)).$$

Let $R > 0$. As in the proof of Lemma 4.1, all points $a \in W'(x, n)$ satisfy $|a-n| < L + \sqrt{k}$. Hence if $W'(x, n)$ has a non-empty intersection with the ball $B_{R-2L-2\sqrt{k}}$ then $|n| < R - L - \sqrt{k}$ and $W(x, n) \subset B_R$. Therefore when $n \in \mathbb{Z}^k$ runs over $\{n \mid W(x, n) \subset B_R\}$, the sets $W'(x, n)$ cover the ball $B_{R-2L-2\sqrt{k}}$. Thus

$$\sum_{n \in \mathbb{Z}^k : W(x, n) \subset B_R} \text{vol}(\text{int}_E W(x, n)) \geq s^{-k} \sum_{n \in \mathbb{Z}^k : W(x, n) \subset B_R} \text{vol}(W'(x, n)) \geq s^{-k}\text{vol}(B_{R-2L-2\sqrt{k}}).$$
Hence
\[ \text{vol}(\partial(x, E) \cap B_R) \leq \text{vol}(B_R) - \text{vol}\left( \bigcup_{n: W(x, n) \subset B_R} \text{vol}(\text{int}_E W(x, n)) \right) \leq \text{vol}(B_R) - s^{-k} \text{vol}(B_{R-2L-2\sqrt{k}}). \]

Thus
\[ \frac{1}{\text{vol}(B_R)} \sup_{x \in X} \text{vol}(\partial(x, E) \cap B_R) \leq 1 - s^{-k} \left( \frac{R - 2L - 2\sqrt{k}}{R} \right)^k, \]
\[ \limsup_{R \to \infty} \frac{1}{\text{vol}(B_R)} \sup_{x \in X} \text{vol}(\partial(x, E) \cap B_R) \leq 1 - s^{-k} < \varepsilon. \]

A main trick in the above proof was going down to \( \mathbb{R}^k \times \{-sH\} \) from \( \mathbb{R}^k \times \{-H\} \). This idea will be used again in Section 7.

5. Small boundary property: proof of Theorem 1.3

We prove Theorem 1.3 in this section. We need to introduce the notion small boundary property defined in [LW00, Lin99] and used implicitly by Shub–Weiss in [SW91] (a related, weaker condition was used in [Lin95] that allows for some periodic points). Let \((X, \mathbb{Z}^k, T)\) be a dynamical system. For a subset \(A \subset X\) its orbit capacity \(\text{ocap}(A)\) is defined by
\[ \text{ocap}(A) = \lim_{N \to \infty} \frac{1}{N^k} \sup_{x \in X} \sum_{n \in [N]} 1_A(T^n x). \]

This limit exists because of the sub-additivity:
\[ \sum_{x \in X} 1_A(T^n x) \leq \sup_{x \in X} \sum_{n \in \Omega_1} 1_A(T^n x) + \sup_{x \in X} \sum_{n \in \Omega_2} 1_A(T^n x) \quad (\Omega_1, \Omega_2 \subset \mathbb{Z}^k). \]

In particular
\[ (5.1) \quad \text{ocap}(A) = \inf_{N \geq 1} \left( \frac{1}{N^k} \sup_{x \in X} \sum_{n \in [N]} 1_A(T^n x) \right). \]

From Lemma 2.3 we also have
\[ (5.2) \quad \text{ocap}(A) = \lim_{R \to \infty} \left( \frac{1}{|B_R \cap \mathbb{Z}^k|} \sup_{x \in X} \sum_{n \in B_R \cap \mathbb{Z}^k} 1_A(T^n x) \right). \]

The system \(X\) is said to have the small boundary property if for any point \(x \in X\) and any open set \(U \ni x\) there is an open neighborhood \(V \subset U\) of \(x\) satisfying \(\text{ocap}(\partial V) = 0\). The importance of this notion is
clarified by the following (see Lindenstrauss [Lin95, Section 4]; many ideas of this theorem were already presented by Shub–Weiss [SW91]).

**Theorem 5.1.** If $X$ has the small boundary property, then for any $\varepsilon > 0$ and any two distinct points $x, y \in X$ there exists a factor $\pi : X \to Y$ such that $\pi(x) \neq \pi(y)$ and $h_{\text{top}}(Y) < \varepsilon$.

Therefore, if $X$ has the small boundary property, any two points can be distinguished by an arbitrarily small entropy factor. The paper [Lin95] discussed only $\mathbb{Z}$-actions, but its argument can be easily generalized to $\mathbb{Z}^k$-actions. (For a sketch of the proof, see also [Lin99, p. 257].)

**Corollary 5.2.** If $X$ has the small boundary property then it is isomorphic to the inverse limit of finite entropy systems.

**Proof.** Let $\Delta$ be the diagonal of $X \times X$. By Theorem 5.1 there exist a countable open covering $\{U_n \times V_n\}_{n \geq 1}$ of $X \times X \setminus \Delta$ and factors $\pi_n : X \to Y_n$ such that $\pi_n(U_n) \cap \pi_n(V_n) = \emptyset$ and $h_{\text{top}}(Y_n) < \infty$. Define $X_n = (\pi_1 \times \pi_2 \times \cdots \times \pi_n)(X)$ in $Y_1 \times Y_2 \times \cdots \times Y_n$. These $X_n$ naturally form an inverse system and its limit is isomorphic to $X$. Every $X_n$ has finite topological entropy.

We denote by $C(X)$ the Banach space of continuous functions $f : X \to \mathbb{R}$ with the uniform norm $\|\cdot\|$. The following is the main result of this section.

**Theorem 5.3.** Suppose $X$ has the marker property and $\operatorname{mdim}(X) = 0$. Then the set of continuous functions $f : X \to \mathbb{R}$ satisfying

\[ \operatorname{ocap} \{ x \in X \mid f(x) = 0 \} = 0 \]

is a dense $G_\delta$ subset of $C(X)$.

The following corollary contains Theorem 5.3.

**Corollary 5.4.** Suppose $X$ has the marker property and $\operatorname{mdim}(X) = 0$. Then $X$ has the small boundary property. In particular it is isomorphic to the inverse limit of finite entropy systems.

**Proof.** Take $x \in X$ and its open neighborhood $U$. There is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f = -1$ over $X \setminus U$. By Theorem 5.3 we can find a continuous function $g : X \to \mathbb{R}$ such that $\|f - g\| < 1$ and $\{g = 0\}$ has zero orbit capacity. Set $V = \{g > 0\}$. We have $x \in V \subset U$ and $\operatorname{ocap}(\partial V) = 0$.

The rest of this section consists of the proof of Theorem 5.3. Let $(X, \mathbb{Z}^k, T)$ be a dynamical system. From the equation (5.1), for any
closed set \( A \subseteq X \) and any \( \varepsilon > 0 \) there is an open set \( U \supset A \) satisfying \( \operatorname{ocap}(U) < \operatorname{ocap}(A) + \varepsilon \). This implies the next lemma.

**Lemma 5.5.** For any \( c > 0 \) the set

\[ \{ f \in C(X) | \operatorname{ocap}(\{ f = 0 \}) < c \} \]

is open in \( C(X) \).

**Proof.** Suppose a continuous function \( f : X \to \mathbb{R} \) satisfies \( \operatorname{ocap}(\{ f = 0 \}) < c \). Then there is an open set \( U \supset \{ f = 0 \} \) satisfying \( \operatorname{ocap}(U) < c \). Let \( \delta > 0 \) be the infimum of \( |f(x)| \) over \( x \in X \setminus U \). Suppose a continuous function \( g : X \to \mathbb{R} \) satisfies \( \| f - g \| < \delta \). Then \( \{ g = 0 \} \subset U \). Hence \( \operatorname{ocap}(\{ g = 0 \}) < c \). \( \square \)

It follows from Lemma 5.5 that the set

\[ \{ f \in C(X) | \operatorname{ocap}(\{ f = 0 \}) = 0 \} = \bigcap_{n \geq 1} \{ f \in C(X) | \operatorname{ocap}(\{ f = 0 \}) < 1/n \} \]

is a \( G_\delta \) subset of \( C(X) \). Therefore Theorem 5.3 follows from the next proposition.

**Proposition 5.6.** Suppose \( X \) has the marker property and \( \operatorname{mdim}(X) = 0 \). Let \( \delta \) be a positive number, and \( f : X \to \mathbb{R} \) a continuous function. Then there exists a continuous function \( g : X \to \mathbb{R} \) satisfying \( \| f - g \| < \delta \) and \( \operatorname{ocap}(\{ g = 0 \}) < \delta \).

We need one preparation:

**Lemma 5.7.** Let \( n \) be a positive integer, and \( P \) a simplicial complex. Then almost every linear map \( f : P \to \mathbb{R}^n \) satisfies

\[ |\{ 1 \leq i \leq n | f(x)_i = 0 \}| \leq \dim P \quad (\forall x \in P). \]

**Proof.** Let \( v_1, \ldots, v_s \) be the vertices of \( P \). For almost every choice of vectors \( u_1, \ldots, u_s \in \mathbb{R}^n \) the following holds: For every \( A \subset \{1, 2, \ldots, n\} \) and \( 1 \leq i_1 < i_2 < \cdots < i_{|A|} \leq s \) the convex hull of \( u_{i_1}|_A, \ldots, u_{i_{|A|}}|_A \) does not contain the origin in \( \mathbb{R}^A \). Suppose the vectors \( u_1, \ldots, u_s \) satisfy this condition, and define a linear map \( f : P \to \mathbb{R}^n \) by \( f(v_i) = u_i \). Take any \( x \in P \) and let \( A = \{ 1 \leq i \leq n | f(x)_i = 0 \} \). Then \( f(x)|_A = 0 \). The assumption on \( u_1, \ldots, u_s \) implies \( |A| \leq \dim P \). \( \square \)

**Proof of Proposition 5.6.** Let \( d \) be a distance on \( X \). Take \( 0 < \varepsilon < \delta/2 \) such that

\[ d(x, y) < \varepsilon \implies |f(x) - f(y)| < \delta. \]
Since \( \text{mdim}(X) = 0 \), we can find \( N > 0 \) satisfying

\[
\text{Wdim}_\varepsilon(X, d_{[N]}) < \varepsilon. 
\]

We set \( I_f(x) = (f(T^n x))_{n \in \mathbb{Z}^k} \) for \( x \in X \). If \( x, y \in X \) satisfy \( d_{[N]}(x, y) < \varepsilon \) then we have

\[
\|I_f(x)|_{[N]} - I_f(x)|_{[N]}\|_\infty \overset{\text{def}}{=} \max_{n \in [N]} |f(T^n x) - f(T^n y)| < \delta.
\]

As in Section 3 by Lemmas 2.1 and 5.7 we can construct a continuous map \( F : X \rightarrow \mathbb{R}^{[N]} \) satisfying

- \( \|F(x) - I_f(x)|_{[N]}\|_\infty < \delta \) for all \( x \in X \),
- \( |\{n \in [N] : F(x)_n = 0\}| \leq \text{Wdim}_\varepsilon(X, d_{[N]}) < \varepsilon N^k \) for all \( x \in X \).

Set \( E = \sqrt{k}(N + 1) \). We choose a sufficiently large positive integer \( M \) and apply the Voronoi tiling construction of Section 4 to \( X \). By Lemma 4.2 we can assume

\[
\limsup_{R \rightarrow \infty} \frac{1}{\text{vol}(B_R)} \sup_{x \in X} \text{vol}(\partial(x, E) \cap B_R) < \varepsilon.
\]

Here recall that we have the tiling \( \mathbb{R}^k = \bigcup_{n \in \mathbb{Z}^k} W(x, n) \) and \( \partial(x, E) = \bigcup_{n \in \mathbb{Z}^k} \partial E W(x, n) \). By Lemma 4.1 (3), \( \text{diam}(W(x, n)) < 2L + 2\sqrt{k} \).

We define a continuous function \( g : X \rightarrow \mathbb{R} \) by painting \( F \) in the tiles \( W(x, n) \). Let \( \alpha : [0, \infty) \rightarrow [0, 1] \) be a continuous function such that \( \alpha(0) = 0 \) and \( \alpha(t) = 1 \) for \( t \geq 1 \). Take \( x \in X \). If \( 0 \in \partial W(x, n) \) for some \( n \in \mathbb{Z}^k \) then we set \( g(x) = f(x) \). Otherwise there uniquely exists \( n \in \mathbb{Z}^k \) satisfying \( 0 \in \text{int} W(x, n) \). Choosing \( a \in \mathbb{Z}^k \) satisfying \( a \equiv n \) (mod \( N \)) and \( 0 \in a + [N] \), we set

\[
g(x) = (1 - \alpha(\text{dist}(0, \partial W(x, n))))f(x) + \\
+ \alpha(\text{dist}(0, \partial W(x, n)))F(T^n x)_{-a},
\]

with \( \text{dist} \) denoting the Euclidean distance. Since \( W(x, n) \) depends continuously on \( x \), the function \( g(x) \) is continuous. It satisfies \( \|f - g\| < \delta \) because \( |F(T^n x)_{-a} - f(x)| \leq \|F(T^n x) - I_f(T^n x)|_{[N]}\|_\infty < \delta \). We will show \( \text{ocap}\{g = 0\} < \delta \).

**Claim 5.8.** Let \( a, n \in \mathbb{Z}^k \) such that \( a \equiv n \) (mod \( N \)) and \( a + [N] \) is contained in \( \text{int} W(x, n) \). Then

\[
I_g(x)_{a+[N]} = F(T^n x).
\]

Here we naturally consider \( I_g(x)_{a+[N]} = (g(T^{n+n} x))_{n \in [N]} \) as a vector in \( \mathbb{R}^{[N]} \).
Proof. Take $m \in a + [N] \subset \text{int}_1 W(x, n)$. Then $W(T^m x, n - m) = -m + W(x, n)$ and hence $0 \in \text{int}_1 W(T^m x, n - m)$. Since $0 \in (a - m) + [N]$ and $a - m \equiv n - m \pmod{N}$, 

$$g(T^m x) = F(T^a x)^{a - m} = F(T^a x_{a + m}.$$  

Thus we get $I_g(x)|_{a + [N]} = F(T^a x)$.

Claim 5.9. For all $x \in X$ and $n \in \mathbb{Z}^k$

$$|\{m \in \mathbb{Z}^k \cap \text{int}_{\sqrt{k}} W(x, n) | I_g(x)m = 0 \}| \leq \varepsilon \text{vol}(W(x, n)).$$

Proof. The set $\mathbb{Z}^k \cap \text{int}_{\sqrt{k}} W(x, n)$ is contained in the disjoint union of $a + [N]$ where $a \in \mathbb{Z}^k$ satisfies $a \equiv n \pmod{N}$ and $a + [N] \subset \text{int}_{\sqrt{k}} W(x, n)$. Take such $a + [N]$. Then $I_g(x)|_{a + [N]} = F(T^a x)$ by Claim 5.8. The second condition of $F$ implies 

$$|\{m \in a + [N] | I_g(x)m = 0 \}| < \varepsilon N^k = \varepsilon \text{vol}(a + [0, N]^k).$$

Summing up this estimate, we get the result. 

Continuing with the proof of Proposition 5.6, let $R > 0$ and $x \in X$. The number of $m \in \mathbb{Z}^k$ satisfying $|m| \leq R$ and $I_g(x)m = 0$ is bounded by 

$$|\mathbb{Z}^k \cap B_R \cap \partial(x, N\sqrt{k})| +$$

$$+ \sum_{n:W(x,n)\cap B_R \neq \emptyset} |\{m \in \mathbb{Z}^k \cap \text{int}_{\sqrt{k}} W(x, n) | I_g(x)m = 0 \}|$$

$$\leq |\mathbb{Z}^k \cap B_R \cap \partial(x, N\sqrt{k})| + \varepsilon \sum_{n:W(x,n)\cap B_R \neq \emptyset} \text{vol}(W(x, n))$$

(by Claim 5.9) 

$$\leq |\mathbb{Z}^k \cap B_R \cap \partial(x, N\sqrt{k})| + \varepsilon \text{vol}(B_{R+2L+2\sqrt{k}})$$

(by diam($W(x, n)) < 2L + 2\sqrt{k}$).

Recall $E = \sqrt{k}(N + 1)$. If $m \in \partial(x, N\sqrt{k})$, then $m + [0, 1]^k \subset \partial(x, E)$. Hence 

$$|\mathbb{Z}^k \cap B_R \cap \partial(x, N\sqrt{k})| \leq \text{vol}(\partial(x, E) \cap B_{R+\sqrt{k}}).$$

Therefore 

$$\frac{|\{m \in \mathbb{Z}^k \cap B_R | I_g(x)m = 0 \}|}{|\mathbb{Z}^k \cap B_R|} \leq \frac{\text{vol}(\partial(x, E) \cap B_{R+\sqrt{k}})}{|\mathbb{Z}^k \cap B_R|} + \varepsilon \frac{\text{vol}(B_{R+2L+2\sqrt{k}})}{|\mathbb{Z}^k \cap B_R|}.$$
Using the formula (5.2) and the condition (5.3) we get
\[
\operatorname{ocap}(\{g = 0\}) = \lim_{R \to \infty} \sup_{x \in X} \left| \left\{ m \in \mathbb{Z}^k \cap B_R \mid I_g(x)_m = 0 \right\} \right| / |\mathbb{Z}^k \cap B_R| < \varepsilon + \varepsilon = 2\varepsilon < \delta.
\]
Here we have used \(|\mathbb{Z}^k \cap B_R| \sim \operatorname{vol}(B_R)| as \(R \to \infty\). □

6. Metric mean dimension: proof of Theorem 1.4

We prove Theorem 1.4 in this section. Let \((X, d)\) be a compact metric space, and \(\alpha\) an open covering of \(X\). We denote by \(\operatorname{mesh}(\alpha, d)\) the supremum of \(\operatorname{diam}U\) over \(U \in \alpha\). For \(\varepsilon > 0\) we define \(A(X, \varepsilon, d)\) as the minimum cardinality of open coverings \(\alpha\) of \(X\) satisfying \(\operatorname{mesh}(\alpha, d) < \varepsilon\). Let \(T : \mathbb{Z}^k \times X \to X\) be a continuous action. We have a natural sub-additivity: 
\[
\log A(X, \varepsilon, d_{\Omega_1 \cup \Omega_2}) \leq \log A(X, \varepsilon, d_{\Omega_1}) + \log A(X, \varepsilon, d_{\Omega_2}).
\]
We define
\[
(6.1) \quad S(X, \varepsilon, d) = \lim_{N \to \infty} \frac{1}{N^k} \log A(X, \varepsilon, d_{[N]}) = \inf_{N \geq 1} \frac{1}{N^k} \log A(X, \varepsilon, d_{[N]}).
\]
By Lemma 2.3 we also have
\[
(6.2) \quad S(X, \varepsilon, d) = \lim_{R \to \infty} \frac{1}{|B_R \cap \mathbb{Z}^k|} \log A(X, \varepsilon, d_{B_R \cap \mathbb{Z}^k}).
\]
We define the **metric mean dimension** \(\operatorname{mdim}_M(X, d)\) by
\[
\operatorname{mdim}_M(X, d) = \liminf_{\varepsilon \to 0} \frac{S(X, \varepsilon, d)}{\log \varepsilon}.
\]
This is a dynamical analogue of box-counting dimension in fractal geometry. Metric mean dimension is always an upper bound on mean dimension ([LW00, Theorem 4.2]):

**Theorem 6.1.**
\[
\operatorname{mdim}(X) \leq \operatorname{mdim}_M(X, d).
\]

The main question here is whether the equality holds for *generic* distances \(d\) or not. We apply to this question a machinery developed in Section 5. Let \(V\) be a Banach space (possibly infinite dimensional) with a norm \(\|\cdot\|_V\), and \(K \subset V\) a compact convex subset. Let \((K^{\mathbb{Z}^k}, \mathbb{Z}^k, \sigma)\) be the \(\mathbb{Z}^k\)-shift on \(K^{\mathbb{Z}^k}\) where
\[
\sigma^a(x) = y, \quad y_m = x_{m+n}.
\]
We define a distance \(D\) on \(K^{\mathbb{Z}^k}\) by
\[
D(x, y) = \sum_{n \in \mathbb{Z}^k} 2^{-|n|} \|x_n - y_n\|_V.
\]
We study maps from $X$ to $K^{\mathbb{Z}^k}$. Let $C(X, K)$ be the space of continuous maps from $X$ to $K$ equipped with the distance $\|f - g\|_V = \sup_{x \in X} \|f(x) - g(x)\|_V$. For a continuous map $f : X \to K$ we define $I_f : X \to K^{\mathbb{Z}^k}$ by $I_f(x) = (f(T^n x))_{n \in \mathbb{Z}^k}$. The image $I_f(X)$ is a subsystem of $K^{\mathbb{Z}^k}$. The next theorem is the main result of this section.

**Theorem 6.2.** Suppose $X$ has the marker property. Then for a dense $G_\delta$ subset of $f \in C(X, K)$

$$\operatorname{mdim}_M(I_f(X), D) \leq \operatorname{mdim}(X).$$

Assuming this theorem, we can easily prove Theorem 1.4.

**Proof of Theorem 1.4.** Let $V = \ell^2(\mathbb{N})$ be the standard $\ell^2$-space with $K = \{(u_n)_{n=1}^{\infty} \in \ell^2| 0 \leq u_n \leq 1/n\}$. Now $K$ is a compact convex subset homeomorphic to $[0, 1]^\mathbb{N}$. Generic continuous maps from compact metric spaces to $[0, 1]^\mathbb{N}$ are topological embeddings. This is well-known in classical dimension theory (Hurewicz–Wallman [HW41, Theorem V4]); it can be also deduced from Lemmas 2.1 and 3.3. So the map $I_f : X \to K^{\mathbb{Z}^k}$ becomes an embedding for generic $f \in C(X, K)$ because $f$ itself is an embedding. Then by Theorem 6.2 we can find a continuous map $f : X \to K$ such that $I_f$ is an embedding and $\operatorname{mdim}_M(I_f(X), D) \leq \operatorname{mdim}(X)$. Let $d$ be the pull-back of the distance $D$ by $I_f$. We get $\operatorname{mdim}_M(X, d) \leq \operatorname{mdim}(X)$. The inequality $\operatorname{mdim}_M(X, d) \geq \operatorname{mdim}(X)$ is always true. Thus $\operatorname{mdim}_M(X, d) = \operatorname{mdim}(X)$. \qed

Let $(X, \mathbb{Z}^k, T)$ be a dynamical system.

**Lemma 6.3.** For $\delta > 0$ and $c > 0$ the set of $f \in C(X, K)$ satisfying

$$\exists 0 < \varepsilon < \delta : \frac{S(I_f(X), \varepsilon, D)}{|\log \varepsilon|} < c$$

is open.

**Proof.** Suppose $f : X \to K$ and $0 < \varepsilon < \delta$ satisfy $S(I_f(X), \varepsilon, D) < c|\log \varepsilon|$. Then there exist $N > 0$ and an open covering $\alpha$ of $I_f(X)$ such that $\operatorname{mesh}(\alpha, D|_N) < \varepsilon$ and $|\alpha| < \exp(cN^k|\log \varepsilon|)$. For each $U \in \alpha$ we can find an open set $\tilde{U} \supset U$ of $K^{\mathbb{Z}^k}$ with $\operatorname{diam}(\tilde{U}, D|_N) < \varepsilon$. Set $\tilde{\alpha} = \{\tilde{U}\}_{U \in \alpha}$. If $g : X \to K$ is sufficiently close to $f$ then $I_g(X) \subset \bigcup \tilde{\alpha}$. Hence$$A(I_g(X), \varepsilon, D|_N) \leq |\tilde{\alpha}| < \exp(cN^k|\log \varepsilon|).$$

By (3.1), we get $S(I_g(X), \varepsilon, D) < c|\log \varepsilon|$. \qed
By this lemma the set
\[
\{ f \in C(X, K) \mid \text{mdim}_M(I_f(X), D) \leq \text{mdim}(X) \}
\]
\[
= \bigcap_{n \geq 1} \left\{ f \in C(X, K) \bigg| \exists 0 < \varepsilon < \frac{1}{n} : \frac{S(I_f(X), \varepsilon, D)}{\log \varepsilon} < \text{mdim}(X) + \frac{1}{n} \right\}
\]
is a $G_\delta$ subset of $C(X, K)$. Therefore Theorem 6.2 follows from the next proposition.

**Proposition 6.4.** Suppose $X$ has the marker property. Let $\delta$ be a positive number, and $f : X \to K$ a continuous map. There exist $0 < \varepsilon < \delta$ and a continuous map $g : X \to K$ satisfying $|f - g| < \delta$ and
\[
S(I_g(X), \varepsilon, D) < (\text{mdim}(X) + \delta) |\log \varepsilon|.
\]

We need some preparations. Let $Y \subset K^{\mathbb{Z}^k}$ be a closed invariant subset. For $R > 0$ we denote by $Y|_{B_R}$ the image of $Y$ under the natural projection $K^{\mathbb{Z}^k} \to K^{B_R \cap \mathbb{Z}^k}$. Let $\| \cdot \|_\infty$ be the $\ell^\infty$-distance on $K^{B_R \cap \mathbb{Z}^k}$ defined by $\|x - y\|_\infty = \max_{n \in B_R \cap \mathbb{Z}^k} \|x_n - y_n\|_V$.

**Lemma 6.5.** There exists a universal constant $\kappa > 0$ such that for every $\varepsilon > 0$
\[
A(Y, \varepsilon, D) \leq \limsup_{R \to \infty} \frac{1}{\text{vol}(B_R)} \log A(Y|_{B_R}, \kappa \varepsilon, \| \cdot \|_\infty).
\]

**Proof.** Set $c = \sum_{n \in \mathbb{Z}^k} 2^{-|n|}$, and take $L = L(\varepsilon) > 0$ satisfying
\[
\sum_{n \in \mathbb{Z}^k, |n| > L} 2^{-|n|} \text{diam} K < \varepsilon/2.
\]
Let $\pi : K^{\mathbb{Z}^k} \to K^{B_R + L \cap \mathbb{Z}^k}$ be the projection. For any $n \in B_R \cap \mathbb{Z}^k$ and $x, y \in K^{\mathbb{Z}^k}$
\[
D(\sigma^n(x), \sigma^n(y)) = \sum_{m \in \mathbb{Z}^k} 2^{-|m|} \|x_{n+m} - y_{n+m}\|_V
\]
\[
\leq \sum_{|m| \leq L} 2^{-|m|} \|x_{n+m} - y_{n+m}\|_V + \sum_{|m| > L} 2^{-|m|} \text{diam} K
\]
\[
< c \|\pi(x) - \pi(y)\|_\infty + \varepsilon/2.
\]
So $D_{B_R \cap \mathbb{Z}^k}(x, y) < c \|\pi(x) - \pi(y)\|_\infty + \varepsilon/2$. We set $\kappa = 1/(2c)$. Then for every $\varepsilon > 0$
\[
A(Y, \varepsilon, D_{B_R \cap \mathbb{Z}^k}) \leq A(Y|_{B_R + L}, \kappa \varepsilon, \| \cdot \|_\infty).
\]
Using (6.2) and $|B_R \cap \mathbb{Z}^k| \sim \text{vol}(B_R)$ as $R \to \infty$,

$$S(Y, \varepsilon, D) = \lim_{R \to \infty} \frac{1}{|B_R \cap \mathbb{Z}^k|} \log A(Y, \varepsilon, D_{B_R \cap \mathbb{Z}^k}) \leq \limsup_{R \to \infty} \frac{1}{\text{vol}(B_R)} \log A(Y|_{B_R}, \kappa \varepsilon, \|\cdot\|_{\infty}).$$

□

**Lemma 6.6.** Let $W$ be a Banach space, $P$ a simplicial complex, and $f : P \to W$ a linear map. Then for any $0 < \varepsilon \leq 1$

$$A(f(P), \varepsilon, \|\cdot\|) \leq \text{const} \cdot \varepsilon^{-\dim P},$$

where const is a positive constant depending only on diam$f(P)$, dim$P$ and the number of the simplices of $P$.

**Proof.** We can assume without loss of generality that diam$f(P) \leq 1$ and $P$ is an $n$-dimensional simplex. Then we can find $a, u_1, \ldots, u_n \in W$ with $\|u_i\| \leq 1$ such that

$$f(P) = \{a + x_1 u_1 + \cdots + x_n u_n | x_1, \ldots, x_n \geq 0, x_1 + \cdots + x_n \leq 1\}.$$ 

Consider the following $(1 + \lfloor 2n/\varepsilon \rfloor)^n$ points in $W$:

$$a + x_1 u_1 + \cdots + x_n u_n \left( x_i = 0, \frac{\varepsilon}{2n}, \frac{2\varepsilon}{2n}, \ldots, \left\lfloor \frac{2n}{\varepsilon} \right\rfloor \frac{\varepsilon}{2n} \right).$$

Now $f(P)$ is covered by the open $\varepsilon/2$-balls around these points. Hence

$$A(f(K), \varepsilon, \|\cdot\|) \leq \left(1 + \left\lfloor \frac{2n}{\varepsilon} \right\rfloor\right)^n \leq \left(\frac{2n + 1}{\varepsilon}\right)^n.$$ 

□

**Proof of Proposition 6.4.** We can assume mdim$(X) < \infty$. Let $f \in C(X, K)$. By Lemma 1.3 it is enough to prove that for any $\delta > 0$ there exist $0 < \varepsilon < \delta$ and a continuous map $g : X \to K$ satisfying $\|f - g\| < \delta$ and

$$\limsup_{R \to \infty} \frac{1}{\text{vol}(B_R)} \log A(I_g(X)|_{B_R}, \varepsilon, \|\cdot\|_{\infty}) < \left|\log \varepsilon\right|(\text{mdim}(X) + \delta).$$

Fix $0 < \tau < \delta/3$ satisfying

$$d(x, y) < \tau \implies \|f(x) - f(y)\|_V < \delta.$$ 

We choose $N > 0$ satisfying

$$\frac{1}{N^k} \text{Widim}_r(X, d_{[N]}) < \text{mdim}(X, T) + \tau.$$ 

By Lemma 2.1 there exists a continuous map $F : X \to K^{[N]}$ such that

- $\|F(x) - I_f(x)\|_{[N]} < \delta$ for all $x \in X$.
- The set $F(X)$ is contained in the image of a linear map from a Widim$(X, d_{[N]})$-dimensional simplicial complex to $K^{[N]}$. 

From the second condition and Lemma 6.6, we can find $0 < \varepsilon < \delta$ satisfying
\begin{equation}
A(F(X), \varepsilon, \| \cdot \|_{\infty}) < \varepsilon^{-Nk(\text{mdim}(X) + \tau)}.
\end{equation}

We also assume (for the later convenience) $\log 2 < \tau | \log \varepsilon |$ and $| \log \varepsilon | > 1$.

Set $E = \sqrt{r}(N + 1)$. As in Section 5 we choose a sufficiently large integer $M$ and apply the Voronoi tiling construction in Section 4 to $X$. From Lemma 4.2 we can assume
\begin{equation}
\limsup_{R \to \infty} \frac{1}{\text{vol}(B_R)} \sup_{x \in X} \text{vol}(\partial(x, E) \cap B_R) < \frac{\tau}{\log A(K, \varepsilon, \| \cdot \|_V)}.
\end{equation}

Here $\mathbb{R}^k = \bigcup_{n \in \mathbb{Z}^k} W(x, n)$ and $\partial(x, E) = \bigcup_{n \in \mathbb{Z}^k} \partial_E W(x, n)$. We define $g : X \to K$ in the same way as in Section 6. Let $\alpha : [0, \infty) \to [0, 1]$ be a continuous function such that $\alpha(0) = 0$ and $\alpha(t) = 1$ for $t \geq 1$. Take $x \in X$. If $0 \in \partial W(x, n)$ for some $n \in \mathbb{Z}^k$ then we set $g(x) = f(x)$. Otherwise there uniquely exists $n \in \mathbb{Z}^k$ satisfying $0 \in \text{int} W(x, n)$. Choosing $a \in \mathbb{Z}^k$ satisfying $a \equiv n \ (\text{mod } N)$ and $0 \in a + [N]$, we set
\begin{equation*}
g(x) = \{1 - \alpha(\text{dist}(0, \partial W(x, n)))\} f(x) + \alpha(\text{dist}(0, \partial W(x, n))) F(T^a x)_{-a}.
\end{equation*}

This satisfies $\| f(x) - g(x) \|_V < \delta$. We estimate $A(I_g(x)|_{B_R}, \varepsilon, \| \cdot \|_{\infty})$ for $R \gg 1$.

For $R > 0$ we define $\mathcal{C}_R$ as the set of subsets $C \subset B_R \cap \mathbb{Z}^k$ satisfying
\begin{itemize}
  \item $a + [N] \subset B_R$ for all $a \in C$,
  \item $(a + [N]) \cap (b + [N]) = \emptyset$ for any two distinct $a, b \in C$,
  \item $a + [N] \ (a \in C)$ cover most of $B_R \cap \mathbb{Z}^k$ in the following sense:
\end{itemize}
\begin{equation*}
|B_R \cap \mathbb{Z}^k \setminus \bigcup_{a \in C} (a + [N])| < \frac{\tau \text{vol}(B_R)}{\log A(K, \varepsilon, \| \cdot \|_V)}.
\end{equation*}

**Claim 6.7.** If we choose $R$ sufficiently large, then for any $x \in X$ there exists $C \in \mathcal{C}_R$ such that $I_g(x)|_{a + [N]} \in F(X)$ for all $a \in C$.

**Proof.** Let $R > 0$ and $x \in X$. For each $n \in \mathbb{Z}^k$ with $W(x, n) \subset B_R$ we define $C_n \subset \mathbb{Z}^k$ as the set of $a \in \mathbb{Z}^k$ satisfying $a \equiv n \ (\text{mod } N)$ and $a + [N] \subset \text{int} W(x, n)$. We define $C$ as the union of $C_n$ over $n \in \mathbb{Z}^k$ with $W(x, n) \subset B_R$. For every $a \in C$ we can prove $I_g(x)|_{a + [N]} = F(T^a x) \in F(X)$ as in Claim 6.8.

The set $C$ obviously satisfies the first and second conditions in the definition of $\mathcal{C}_R$. The problem is to confirm the third condition. For each $n \in \mathbb{Z}^k$ with $W(x, n) \subset B_R$ the set $\mathbb{Z}^k \cap \text{int}_{N\sqrt{k}} W(x, n)$ is covered
by $a + [N]$ ($a \in C_n$). Hence
\[
\bigcup_{n : W(x, n) \subset B_R} (\mathbb{Z}^k \cap \text{int}_N \sqrt{k} W(x, n)) \subset \bigcup_{a \in C} (a + [N]).
\]

By diam$W(x, n) < 2L + 2\sqrt{k}$ (Lemma 4.1 (3)), the union of $W(x, n)$ with $W(x, n) \subset B_R$ contains the ball $B_{R-2L-2\sqrt{k}}$. Therefore $B_R \cap \mathbb{Z}^k \setminus \bigcup_{a \in C} (a + [N])$ is contained in
\[
\{ \mathbb{Z}^k \cap (B_R \setminus B_{R-2L-2\sqrt{k}}) \} \cup \{ \mathbb{Z}^k \cap B_{R-2L-2\sqrt{k}} \cap \partial(x, N\sqrt{k}) \}.
\]
The first term is small compared to $\text{vol}(B_R)$:
\[
|\mathbb{Z}^k \cap (B_R \setminus B_{R-2L-2\sqrt{k}})| = O(R^{-1}) = o(\text{vol}(B_R)).
\]
The second term is bounded by
\[
|\mathbb{Z}^k \cap B_{R-2L-2\sqrt{k}} \cap \partial(x, N\sqrt{k})| \leq \text{vol}(B_{R-2L-2\sqrt{k}} \cap \partial(x, E)) \leq \text{vol}(B_R \cap \partial(x, E))
\]
(Recall $E = \sqrt{k}(N + 1)$). The last term can be estimated by (6.4).

Thus if $R$ is sufficiently large (uniformly in $x \in X$) then
\[
|B_R \cap \mathbb{Z}^k \setminus \bigcup_{a \in C} (a + [N])| < \frac{\tau \text{vol}(B_R)}{\log A(K, \varepsilon, \|\cdot\|_V)}.
\]

For each $C \in \mathcal{C}_R$ we set
\[
Q_C = \left\{ u \in K^{B_R \cap \mathbb{Z}^k} : \forall a \in C : u|_{a + [N]} \in F(X) \right\}.
\]
Every $C \in \mathcal{C}_R$ satisfies $|C| \leq |B_R \cap \mathbb{Z}^k|/N^k$. Hence
\[
\log A(Q_C, \varepsilon, \|\cdot\|_\infty) \leq |C| \cdot \log A(F(X), \varepsilon, \|\cdot\|_\infty) + |B_R \cap \mathbb{Z}^k| \setminus \bigcup_{a \in C} (a + [N]) \cdot \log A(K, \varepsilon, \|\cdot\|_V)
\]
\[
< \frac{|B_R \cap \mathbb{Z}^k|}{N^k} \log A(F(X), \varepsilon, \|\cdot\|_\infty) + \tau \text{vol}(B_R)
\]
\[
< |B_R \cap \mathbb{Z}^k| \cdot (\text{mdim}(X) + \tau)|\log \varepsilon| + \tau \text{vol}(B_R),
\]
where we have used (6.3) to pass to the last line of the above inequality. From Claim 6.7, the set $I_g(X)|_{B_R}$ is contained in the union of $Q_C$ over $C \in \mathcal{C}_R$ if $R$ is sufficiently large. Hence for $R \gg 1$
\[
\log A(I_g(X)|_{B_R}, \varepsilon, \|\cdot\|_\infty) < \log |\mathcal{C}_R| +
\]
\[
+ |B_R \cap \mathbb{Z}^k| \cdot (\text{mdim}(X) + \tau)|\log \varepsilon| + \tau \text{vol}(B_R).
\]
Obviously $|C_R| \leq 2^{|B_R\cap \mathbb{Z}^k|}$. Recall that we assumed $\log 2 < \tau |\log \varepsilon|$. Hence

$$\log A(I_g(X)|_{B_R}, \varepsilon, \|\cdot\|_\infty) < |B_R\cap \mathbb{Z}^k| \cdot (\text{mdim}(X) + 2\tau) |\log \varepsilon| + \tau \text{vol}(B_R).$$

Since $|B_R\cap \mathbb{Z}^k| \sim \text{vol}(B_R)$ as $R \to \infty$,

$$\limsup_{R \to \infty} \frac{1}{\text{vol}(B_R)} \log A(I_g(X)|_{B_R}, \varepsilon, \|\cdot\|_\infty) \leq (\text{mdim}(X) + 2\tau) |\log \varepsilon| + \tau$$

(recall $|\log \varepsilon| > 1$)

$$< (\text{mdim}(X) + 3\tau) |\log \varepsilon|$$

(recall $3\tau < \delta$).

□

7. A more difficult embedding theorem: Proof of Theorem 1.5

Theorem 1.5 follows from

**Theorem 7.1.** Let $D$ be a positive integer, and $(X, \mathbb{Z}^k, T)$ a dynamical system having the marker property. If $\text{mdim}(X) < D/2^{k+1}$, then for a dense $G_\delta$ subset of $f \in C(X, [0, 1]^{2D})$ the map

$$I_f : X \to ([0, 1]^{2D})^{\mathbb{Z}^k}, \quad x \mapsto (f(T^n x))_{n \in \mathbb{Z}^k},$$

is an embedding.

Throughout this section, we set $K = [0, 1]^D$. We always assume that $(X, \mathbb{Z}^k, T)$ has the marker property and $\text{mdim}(X) < D/2^{k+1}$. Let $d$ be a distance on $X$. As in Section 3 the $G_\delta$ subset

$$\bigcap_{n \geq 1} \left\{ f \in C(X, K^2) \mid I_f : X \to (K^2)^{\mathbb{Z}^k} \text{ is a } (1/n)\text{-embedding with respect to } d \right\}$$

is equal to the set of $f \in C(X, K^2)$ such that $I_f$ is an embedding. Therefore Theorem 7.1 follows from the next proposition.

**Proposition 7.2.** Let $f = (f_1, f_2) : X \to K^2$ be a continuous map. For any positive number $\delta$ there exists a continuous map $g = (g_1, g_2) : X \to K^2$ satisfying

1. $|f_i(x) - g_i(x)| < \delta$ for both $i = 1, 2$ and all $x \in X$,
2. the map $I_g : X \to (K^2)^{\mathbb{Z}^k}$ is a $\delta$-embedding with respect to the distance $d$.

The proof of this proposition occupies the rest of the section.
7.1. Linear maps on simplicial complexes. The purpose of this subsection is to prove Lemma 7.3 below. The argument here is elementary but notationally complicated.

**Lemma 7.3.** Let $V$ be a finite dimensional real vector space, $A$ a finite set, and $n$ a natural number. Let $M$ be an $(n, n \dim V + 1)$ matrix with entries in $A$ such that no value appears twice in a row or in a column. Then for almost every choice of $(u_a)_{a \in A} \in V^A$ the columns of the matrix

\[(u_{M_{ij}})_{1 \leq i \leq n, 1 \leq j \leq n \dim V + 1}\]

are affinely independent.

**Proof.**

**Case 1.** Suppose $V = \mathbb{R}$. We prove the statement by induction on $n$. The statement is trivial for $n = 1$. We consider the case $n \geq 2$. Take $(t_a)_{a \in A} \in \mathbb{R}^A$. Then the columns of the matrix $(t_{M_{ij}})_{1 \leq i \leq n, 1 \leq j \leq n + 1}$ are affinely independent if and only if the following $(n + 1, n + 1)$ matrix is regular:

\[
\begin{pmatrix}
  t_{M11} & \cdots & t_{M1n+1} \\
  \cdots & \cdots & \cdots \\
  t_{Mn1} & \cdots & t_{Mnn+1} \\
  1 & \cdots & 1
\end{pmatrix}
\]

Set $\alpha = M_{11}$, and suppose that $\alpha$ appears $r$ times in $M$. Then the determinant of the above matrix is a polynomial of $t_{\alpha}$ of degree $r$, and the coefficient of the $t_{\alpha}^r$ term is equal to (up to $\pm$) the determinant of the matrix of the same type (7.1) of smaller size. So by the induction the matrix (7.1) is regular for almost every choice of $(t_a)_{a \in A} \in \mathbb{R}^A$.

**Case 2.** Set $m = \dim V$, and take a basis $e_1, \ldots, e_m$ of $V$. Set $B = A \times \{1, 2, \ldots, m\}$ and define an $(nm, nm + 1)$ matrix $N$ valued in $B$ by

\[N_{ij} = (M_{qj}, r), \quad (i = (q - 1)m + r, 1 \leq q \leq n, 1 \leq r \leq m)\]

for $1 \leq i \leq nm$ and $1 \leq j \leq nm + 1$. We can apply Case 1 to $N$; for almost every choice of $(t_b)_{b \in B} \in \mathbb{R}^B$ the columns of the matrix $(t_{N_{ij}})$ are affinely independent. Define $(u_a)_{a \in A}$ in $V^A$ by

\[u_a = \sum_{r=1}^{m} t_{(a,r)} e_r.\]

Then the columns of $(u_{M_{ij}})_{1 \leq i \leq n, 1 \leq j \leq nm + 1}$ are affinely independent for almost every choice of $(t_b)$. This proves the lemma.

**Lemma 7.4.** Let $s$ and $n \leq N$ be positive integers. For almost every choice of vectors $u_1, \ldots, u_s \in V^N$, if we choose at most $(n^k \dim V + 1)$
distinct vectors in $V^{[n]}$ from
\[ u_a|_{b+[n]} \quad (1 \leq a \leq s, \ b \in [N - n + 1]) \]
then they are affinely independent. Here $u_a|_{b+[n]}$ is the restriction of $u_a$ to $b+[n] \subset [N]$. We consider it as a vector in $V^{[n]}$.

Proof. Set $m = \dim V$ and $A = \{1, 2, \ldots, s\} \times [N]$. We can assume $s \geq n^k m + 1$. Take distinct $(a_j, b_j) \in \{1, 2, \ldots, s\} \times [N - n + 1]$ for $1 \leq j \leq n^k m + 1$. We define an $(n^k, n^k m + 1)$ matrix $M$ valued in $A$ by $M_{ij} = (a_j, b_j + i)$ for $i \in [n]$ and $1 \leq j \leq n^k m + 1$. Now $M$ satisfies the condition of Lemma 7.4. Hence for almost every choice of $(u_c)_{c \in A} \in V^A$ the columns of the matrix $(u_{(a_j, b_j+i)})_{i \in [n], 1 \leq j \leq n^k m + 1}$ are affinely independent. We define vectors in $V^{[N]}$ by
\[ u_a = (u_{(a,b)})_{b \in [N]} \quad (1 \leq a \leq s). \]

Then $u_a|_{b_j+[n]}$ $(1 \leq j \leq n^k m + 1)$ are affinely independent. $\square$

The next lemma strengthens Lemma 3.3

**Lemma 7.5.** Let $n \leq N$ be positive integers, and $P$ a simplicial complex satisfying
\[ \dim P < \frac{n^k \dim V}{2}. \]

Then almost every linear map $f : P \to V^{[N]}$ satisfies the following. If $x, y \in P$ and $a, b \in [N - n + 1]$ satisfy
\[ f(x)|_{a+[n]} = f(y)|_{b+[n]}, \]
them $x = y$ and $a = b$.

Proof. Let $v_1, \ldots, v_s$ be the vertices of $P$. Take vectors $u_1, \ldots, u_s \in V^{[N]}$ and define a linear map $f : P \to V^{[N]}$ by $f(v_i) = u_i$. Let $x = \sum_{i=1}^s \alpha_i v_i$ and $y = \sum_{i=1}^s \beta_i v_i$ with $\sum_i \alpha_i = \sum_i \beta_i = 1$. Here $\alpha_i$ and $\beta_i$ are zero except for at most $\dim P + 1$ ones respectively. The equation $f(x)|_{a+[n]} = f(y)|_{b+[n]}$ implies
\[ \sum_{i=1}^s \alpha_i u_i|_{a+[n]} = \sum_{i=1}^s \beta_i u_i|_{b+[n]}. \]
The number of non-zero $\alpha_i$ and $\beta_i$ are at most $2(\dim P+1) \leq n^k \dim V + 1$. Now Lemma 7.4 implies the conclusion. $\square$
7.2. Idea of the proof. Throughout the rest of this section, we fix a positive number $\delta$ and a continuous map $f = (f_1, f_2) : X \to K^2$ (recall $K = [0, 1]^D$). We fix $0 < \varepsilon < \delta$ so that for both $i = 1, 2$
\begin{equation}
    d(x, y) < \varepsilon \implies |f_i(x) - f_i(y)| < \delta.
\end{equation}
Since $\text{mdim}(X) < D/2^{k+1}$, we can find $\eta > 0$ satisfying
\begin{equation}
    \text{mdim}(X) < D \left( \frac{1}{2^{k+1}} - \eta \right).
\end{equation}
We choose a positive integer $N$ sufficiently large so that
\begin{equation}
    n \geq N \implies \text{Widim}_\varepsilon(X, d_{[n]}) < D \left( \frac{1}{2^{k+1}} - \eta \right)(n-1)^k.
\end{equation}
We also assume that $N$ is even; this is just for simplicity of the exposition.

We now use the tiling construction of Section 4. Let $M$ be a positive integer. Since $X$ is assumed to have the marker property, we can find an integer $L \geq M$ and a continuous function $\phi : X \to [0, 1]$ so that
\begin{itemize}
    \item If $\phi(x) > 0$ at some $x \in X$, then $\phi(T^nx) = 0$ for all non-zero $n \in \mathbb{Z}^k$ with $|n| < M$.
    \item For any $x \in X$ there exists $n \in \mathbb{Z}^k$ satisfying $|n| < L$ and $\phi(T^nx) = 1$.
\end{itemize}
For each $x \in X$ let $\mathbb{R}^{k+1} = \bigcup_{n \in \mathbb{Z}^k} V(x, n)$ be the Voronoi decomposition associated with the set $\{(n, 1/\phi(T^nx))| n \in \mathbb{Z}^k\}$. We choose a real number $H \geq (L + \sqrt{k})^2$ and set $W(x, n) = \pi_{\mathbb{R}^k}(V(x, n) \cap (\mathbb{R}^k \times \{-H\}))$. These form a tiling of $\mathbb{R}^k$.

We choose $s > 1$ satisfying
\begin{equation}
    \frac{1}{2^{k+1}} - \eta < \frac{1}{2(1 + s^k)}.
\end{equation}
Choosing the above $M$ sufficiently large, we can assume (by Lemma 4.1 (4)) that for any $x \in X$, $n \in \mathbb{Z}^k$ and $a \in \mathbb{R}^k$
\begin{equation}
    (a, -sH) \in V(x, n) \implies B_{3\sqrt{\pi}N}(a/s + (1 - 1/s)n) \subset W(x, n).
\end{equation}
Now we have completed the setup for the proof of Proposition 7.2. Before going further, we explain its strategy here. The map $f = (f_1, f_2) : X \to K^2$ has two components $f_1$ and $f_2$. We first construct a perturbation $g_1$ of $f_1$ and next a perturbation $g_2$ of $f_2$. These functions $g_1$ and $g_2$ play different roles. Take $x \in X$. We try to encode the tiling $\mathbb{R}^k = \bigcup_{n \in \mathbb{Z}^k} W(x, n)$ by the value of $I_{g_1}(x) = (g_1(T^nx))_{n \in \mathbb{Z}^k}$. If all the (non-empty) tiles $W(x, n)$ are sufficiently large, then this encoding can be done almost without error. Such a good situation could be taken for granted in Section 3, but in the current context this is not so — some tiles will be small. We cannot encode such small tiles by $I_{g_1}(x)$, and
this is the main difficulty we need to overcome in this proof. Lemma [4.2] implies that such bad tiles are asymptotically negligible. But this is not enough here.

Our argument goes as follows. We give up on encoding all the information of the tiling $\bigcup_{n \in \mathbb{Z}^k} W(x, n)$. Instead we construct a \textbf{“pseudo-tiling”} of $\mathbb{R}^k$ from the value of $I_{g_1}(x)$:

\[
\begin{array}{ccc}
\text{tiling} & (W(x, n))_{n \in \mathbb{Z}^k} & \mapsto \text{encode} \\
I_{g_1}(x) & \in K^{\mathbb{Z}^k} & \mapsto \text{decode} \\
\text{pseudo-tiling} & W = (W_n)_{n \in \mathbb{Z}^k}. & \\
\end{array}
\]

The pseudo-tiling $W$ consists of $\ell^\infty$-functions $W_n \in \ell^\infty(\mathbb{Z}^k)$. When $W(x, n)$ is sufficiently large, the function $W_n$ is approximately equal to the characteristic function of $W(x, n) \cap \mathbb{Z}^k$. The definition of $W$ will be given later.

Next we construct a perturbation $g_2(x)$ of $f_2(x)$ by using the pseudo-tiling associated with $x$. Then the map $g = (g_1, g_2) : X \to K^2$ has been constructed. Take two points $x$ and $y$ in $X$, and suppose $(I_{g_1}(x), I_{g_2}(x)) = (I_{g_1}(y), I_{g_2}(y))$. The first equation $I_{g_1}(x) = I_{g_1}(y)$ implies that the pseudo-tilings associated with $x$ and $y$ are equal. So $I_{g_2}(x)$ and $I_{g_2}(y)$ are constructed from the same pseudo-tiling. Using this additional information, we can conclude $d(x, y) < \varepsilon$ from the equation $I_{g_2}(x) = I_{g_2}(y)$. This last step is analogous to the situation of Proposition [3.2]: in that case we also had two equalities $I_g(x) = I_g(y)$ and $\pi(x) = \pi(y)$. The equation $\pi(x) = \pi(y)$ implied that the Voronoi tilings associated with $x$ and $y$ are equal, and then we deduced $d(x, y) < \varepsilon$ from $I_g(x) = I_g(y)$.

\section*{7.3. Construction of $g_1$ and encoding the tiling.} In this subsection we construct a perturbation $g_1$ of $f_1$. We choose an $\varepsilon$-embedding $\pi : (X, d_{[N]}) \to P$ where $P$ is a simplicial complex of dimension $\text{Widim}_e(X, d_{[N]})$. Replacing $P$ with a sufficiently fine subdivision, we can apply Lemma [2.1] to $P$ and find a linear map $\tilde{f}_1 : P \to K^{[N]}$ satisfying

\[
\left\| \tilde{f}_1(\pi(x)) - I_{f_1}(x) \right\|_{[N]} \overset{\text{def}}{=} \max_{n \in [N]} |\tilde{f}_1(\pi(x)) - f_1(T^n x)| < \delta \quad (\forall x \in X).
\]

Set $P' = P \times \{n \in \mathbb{Z}^k \mid |n| < L + \sqrt{k}N\}$, and consider the map

\[
P' \to K^{[N]}, \quad (x, n) \mapsto \tilde{f}_1(x).
\]
Note \(\dim P' = \dim P = \text{Widim}_c(X, d_{\lceil N \rceil}) < DN^k/2^{k+1}\). We perturb the map (7.6) using Lemma 7.5 (applied with \(V = \mathbb{R}^D\) and \(n = N/2\)), and construct a linear map \(F : P' \to K^\lceil N \rceil\) satisfying the following two conditions:\footnote{Note that the map \(F\) has two variables \(x \in P\) and \(n \in \mathbb{Z}^k\); the second variable \(n\) of \(F\) will be used for encoding the positions of the Voronoi centers.}

(7.7) For all \(x \in X\) and \(n \in \mathbb{Z}^k\) with \(|n| < L + \sqrt{kN}\)
\[
\|F(\pi(x), n) - I_{f_1}(x)\|_\infty < \delta.
\]

(7.8) If \(x, y \in P'\) and \(a, b \in [N/2 + 1]\) satisfy
\[
F(x)|_{a+[N/2]} = F(y)|_{b+[N/2]},
\]
then \(x = y\) and \(a = b\).

Take \(x \in X\). We define \(g_1(x) \in K\) as follows. We take a cut-off function \(\alpha : [0, \infty) \to [0, 1]\) satisfying \(\alpha(0) = 0\) and \(\alpha(t) = 1\) for \(t \geq 1\).

If \(0 \in \partial W(x, n)\) for some \(n \in \mathbb{Z}^k\), then we set \(g_1(x) = f_1(x)\). Otherwise there uniquely exists \(n \in \mathbb{Z}^k\) satisfying \(0 \in \text{int} W(x, n)\). Taking \(a \in \mathbb{Z}^k\) with \(a \equiv n \pmod N\) and \(0 \in a + [N]\), we set
\[
g_1(x) = \{1 - \alpha(\text{dist}(0, \partial W(x, n)))\} f_1(x)
+ \alpha(\text{dist}(0, \partial W(x, n))) F(\pi(T^a x), n - a)_{-a}.
\]

Note that \(0 \in \text{Int} W(x, n)\) implies \(|n| < L + \sqrt{k}\) by Lemma 4.1 (3). Hence \(|n - a| < L + \sqrt{kN}\) and the term \(F(\pi(T^a x), n - a)_{-a}\) is well-defined. The map \(g_1 : X \to K\) is continuous since \(W(x, n)\) depends continuously on \(x\). From Condition (7.7), the function \(g_1\) satisfies \(|g_1(x) - f_1(x)| < \delta\) because
\[
|F(\pi(T^a x), n - a)_{-a} - f_1(x)| = |F(\pi(T^a x), n - a)_{-a} - I_{f_1}(T^a x)_{-a}| < \delta.
\]

We will need the following formula of \(I_{g_1}(x)\) later. The proof is the same as Claim 5.8 in Section 5.

Claim 7.6. Let \(a, n \in \mathbb{Z}^k\) such that \(a \equiv n \pmod N\) and \(a + [N] \subset \text{int} W(x, n)\). Then
\[
I_{g_1}(x)|_{a+[N]} = F(\pi(T^a x), n - a).
\]

\footnote{Recall that \(N\) is even.}
7.4. Decoding and the construction of pseudo-tiling. For \( n \in \mathbb{Z}^k \) with \(|n| < L + \sqrt{k}N\) we define \( Q_n \subset K^{[N]} \) as the set of \( F(x,n) \) \((x \in P)\). These \( Q_n \) can be thought as the decoder of the encoding \( I_{g_1} \). From Condition \((7.8)\) for \( m, n \in \mathbb{Z}^k \) with \(|m|, |n| < L + \sqrt{k}N\) and \( a, b \in [N/2 + 1] \) we have

\[
Q_m|_{a+[N/2]} \cap Q_n|_{b+[N/2]} = \emptyset \quad \text{if} \quad (m, a) \neq (n, b).
\]

We choose \( \tau > 0 \) so that

\[
\tau < \min_{(m,a)\neq(n,b)} \mathrm{dist}(Q_m|_{a+[N/2]}, Q_n|_{b+[N/2]})
\]

where the minimum is taken over all pairs of distinct \((m, a)\) and \((n, b)\) in \( \mathbb{Z}^k \times [N/2 + 1] \) with \(|m|, |n| < L + \sqrt{k}N\), and \( \mathrm{dist}(\cdot, \cdot) \) is the Euclidean distance.

Let \( 1_{[N]} \in \ell^\infty(\mathbb{Z}^k) \) be the characteristic function of \([N] = \{0, 1, \ldots, N-1\}^k\). We choose a cut-off function \( \beta : [0, \infty) \to [0, 1] \) such that \( \beta(0) = 1 \) and \( \beta(t) = 0 \) for \( t \geq \tau \). Take a point \( \omega \in K^{\mathbb{Z}^k} \). For \( n \in \mathbb{Z}^k \) we define an \( \ell^\infty \)-function \( \mathcal{W}_n : \mathbb{Z}^k \to [0, 1] \) by

\[
\mathcal{W}_n(t) = \min \left( 1, \sum_{|a-n|<L+\sqrt{k}N} \beta(\mathrm{dist}(\omega|_{a+[N]}, Q_{n-a}))1_{[N]}(t-a) \right).
\]

The function \( \mathcal{W}_n \) is supported in \( B_{L+\sqrt{k}N}(n + [N]) \). We set \( \mathcal{W}(\omega) = (\mathcal{W}_n)_{n \in \mathbb{Z}^k} \). This is the pseudo-tiling associated with \( \omega \). We sometimes denote \( \mathcal{W}_n \) by \( \mathcal{W}_n^\omega \). Now the map

\[
K^{\mathbb{Z}^k} \ni \omega \mapsto \mathcal{W}(\omega) \in (\ell^\infty(\mathbb{Z}^k))^{\mathbb{Z}^k}
\]

is continuous. It is also equivariant in the following sense: Let \( m \in \mathbb{Z}^k \), and take \( \omega' \in K^{\mathbb{Z}^k} \) with \( \omega'_n = \omega_{n+m} \). Then \( \mathcal{W}^\omega_n(t) = \mathcal{W}^\omega_{n+m}(t + m) \).

The meaning of the terminology “pseudo-tiling” is clarified by the next lemma.

**Lemma 7.7.** Let \( x \in X \) and set \((\mathcal{W}_n)_{n \in \mathbb{Z}^k} = \mathcal{W}(I_{g_1}(x)) \).

1. Let \( n \in \mathbb{Z}^k \). If \( m \in \mathbb{Z}^k \) satisfies \( B_{2\sqrt{k}N}(m) \subset W(x,n) \) then

\[
\mathcal{W}_n(m) = 1 \quad \text{and} \quad \mathcal{W}_n(m) = 0 \quad (\forall n' \neq n).
\]

2. There exists \( n \in \mathbb{Z}^k \) such that \( \mathcal{W}_n(m) = 1 \) and \( \mathcal{W}_n(m) = 0 \) for every \( m \in B_{\sqrt{k}N}((1 - 1/s)n) \cap \mathbb{Z}^k \), \( n' \neq n \). Here \( s > 1 \) is the positive constant introduced in Subsection \(7.2\).}

Roughly speaking, property (1) of the lemma says that the function \( \mathcal{W}_n \) looks like the characteristic function of \( W(x,n) \cap \mathbb{Z}^k \) for “nice” tiles \( W(x,n) \).
Proof. (1) Take \( a \in \mathbb{Z}^k \) satisfying \( a \equiv n \pmod{N} \) and \( m \in a + [N] \). The 1-neighborhood of \( a + [N] \) is contained in \( W(x, n) \). Hence Claim \[7.6\] implies
\[
I_{g_1}(x)|_{a+\{N\}} = F(\pi(T^n x), n-a) \in Q_{n-a},
\]
which implies that \( \mathcal{W}_n(m) = 1 \).

Next we prove \( \mathcal{W}_{n'}(m) = 0 \) for \( n' \neq n \) by showing
\[
\text{dist}(I_{g_1}(x)|_{b+[N]}, Q_{n'-b}) \geq \tau
\]
for all \( b \in \mathbb{Z}^k \) satisfying \( m \in b + [N] \) and \( |n'-b| < L + \sqrt{k}N \). We choose \( c \in \mathbb{Z}^k \) satisfying \( c \equiv n \pmod{N} \) and \( b \in c + [N] \). We define \( b', c' \in \mathbb{Z}^k \) by
\[
b'_i = b_i, \quad c'_i = c_i \quad \text{if } b_i - c_i < N/2,
b'_i = c'_i = c_i + N \quad \text{if } b_i - c_i \geq N/2.
\]
Now \( b' - c' \in [N/2], b' - b \in [N/2 + 1], c' \equiv n \pmod{N} \) and the 1-neighborhood of \( c' + [N] \) is contained in \( B_{2\sqrt{k}N}(m) \subset W(x, n) \). So
\[
(I_{g_1}(x)|_{c'+[N]})|_{b' - b + [N/2]} = (I_{g_1}(x)|_{c'+[N]})|_{b' + [N/2]}
\]
\[
= (I_{g_1}(x)|_{c'+[N]})|_{b' + [N/2]}
\]
\[
= F(\pi(T^{c' x}), n-c')|_{b' - b + [N/2]}.
\]
This is contained in \( Q_{n-c'}|_{b' - b + [N/2]} \). Suppose
\[
\text{dist}(I_{g_1}(x)|_{b+[N]}, Q_{n'-b}) < \tau.
\]
Then we have dist \((Q_{n-c'}|_{b' - b + [N/2]}, Q_{n'-b}|_{b' - b + [N/2]} \) \( < \tau \). By condition \([7.3]\) on \( \tau \) it follows that \( n-c' = n' - b \) and \( b'-c' = b' - b \), hence \( c' = b \) and \( n' = n \).

(2) Take \( n \in \mathbb{Z}^k \) with \( (0, -sH) \in V(x, n) \). From condition \([7.5]\),
\[
B_{3\sqrt{k}N}(1 - 1/s)n) \subset W(x, n).
\]
Then for \( m \in B_{\sqrt{k}N}(1 - 1/s)n) \cap \mathbb{Z}^k \) we have
\[
B_{2\sqrt{k}N}(m) \subset B_{3\sqrt{k}N}(1 - 1/s)n) \subset W(x, n).
\]
By (1) above we get \( \mathcal{W}_n(m) = 1 \) and \( \mathcal{W}_{n'}(m) = 0 \) for \( n' \neq n \). \( \square \)

7.5. Construction of \( g_2 \) and the proof of Proposition \[7.2\]. In this subsection we construct a perturbation \( g_2 \) of \( f_2 \) and prove Proposition \[7.2\]. For \( u = (u_1, \ldots, u_k) \in \mathbb{R}^k \) we set \([u] = ([u_1], \ldots, [u_k]) \in \mathbb{Z}^k \) (\([u_i] \) is the smallest integer not smaller than \( u_i \)). Set \( N' = [sN] \). For each \( n \in \mathbb{Z}^k \) we consider the distance \( d_{[N]}((1-s)n) \) on \( X \). Although this looks complicated, its geometric meaning is clear: Consider the projection from \( \mathbb{R}^k \times \{-H\} \) to \( \mathbb{R}^k \times \{-sH\} \) with respect to the
center \((n, 0)\). Then \([(1-s)n + [N']) \times \{-sH\}\) is approximately equal to the image of \([N] \times \{-H\}\) under this projection. So this is a version of our trick of going down to \(\mathbb{R}^k \times \{-sH\}\) from \(\mathbb{R}^k \times \{-H\}\).

Using (7.3) and (7.4), we have

\[
W \dim^e(X, d_{\lfloor(N-(1-s)n)\rfloor+\lfloor[N']\rfloor}) \leq W \dim^e(X, d_{\lfloor N \rfloor}) + W \dim^e(X, d_{\lfloor N' \rfloor})
\]

\[
< D \left( \frac{1}{2^{k+1}} - \eta \right) N^k + D \left( \frac{1}{2^{k+1}} - \eta \right) (N' - 1)^k
\]

\[
\leq D \left( \frac{1}{2^{k+1}} - \eta \right) (N^k + s^k N^k) < \frac{DN^k}{2},
\]

so there exists an \(\varepsilon\)-embedding

\[
\pi_n : (X, d_{\lfloor N \rfloor+\lfloor (1-s)n \rfloor+\lfloor N' \rfloor}) \to R_n
\]

where \(R_n\) is a simplicial complex of dimension \(< DN^k/2\). Let \(R\) be the disjoint union of \(R_n\) over \(|n| < L + 3\sqrt{kN}\). By Lemmas 2.1 and 3.3 we can find a linear embedding \(G : R \to K^{\lfloor N \rfloor}\) satisfying

\[
\|G(\pi_n(x)) - I_{f_2}(x)|_{\lfloor N \rfloor}\|_\infty < \delta \quad (x \in X, |n| < L + 3\sqrt{kN}).
\]

We define a continuous map \(g_2 : X \to K\) as follows. For a real number \(t\) we set \(\{t\} = \max(0, \min(1, t)) \in [0, 1]\), and for \(u = (u_1, \ldots, u_D) \in \mathbb{R}^D\) we set \(\langle u \rangle = (\langle u_1 \rangle, \ldots, \langle u_D \rangle) \in [0, 1]^D = K\). For each \(n \in \mathbb{Z}^k\) we take \(a_n \in \mathbb{Z}^k\) satisfying \(a_n \equiv n \pmod{N}\) and \(0 \in a_n + [N]\). Let \(x \in X\). Let \((W_n)_{n \in \mathbb{Z}^k} = W(I_{g_1}(x))\) be the pseudo-tiling associated with \(I_{g_1}(x) \in K^{\mathbb{Z}^k}\). We define \(A(x)\) as the set of \(n \in \mathbb{Z}^k\) with \(W_n(0) > 0\). Since \(W_n\) is supported in \(B_{L+\sqrt{kN}}(n + [N])\), every \(n \in A(x)\) satisfies \(|n| < L + 2\sqrt{kN}\). We set

\[
g_2(x) = \left( f_2(x) + \frac{\sum_{n \in A(x)} W_n(0) (G(\pi_n-a_n)(T^n x) - a_n - f_2(x))}{\max (1, \sum_{n \in A(x)} W_n(0))} \right),
\]

where the term \(G(\pi_n-a_n)(T^n x)\) is well-defined because

\[
|n - a_n| < L + 2\sqrt{kN} + \sqrt{kN} = L + 3\sqrt{kN}.
\]

This satisfies \(|g_2(x) - f_2(x)| < \delta\) because

\[
|G(\pi_n-a_n)(T^n x) - a_n - f_2(x)| = |G(\pi_n-a_n)(T^n x) - a_n - I_{f_2}(T^n x) - a_n|
\]

\[
\leq \|G(\pi_n-a_n)(T^n x) - I_{f_2}(T^n x)|_{\lfloor N \rfloor}\|_\infty < \delta.
\]

**Claim 7.8.** Let \(x \in X\) and \(a, n \in \mathbb{Z}^k\) with \(a \equiv n \pmod{N}\). Suppose the pseudo-tiling \((W_n)_{n \in \mathbb{Z}^k} = W(I_{g_1}(x))\) satisfies \(W_n = 1\) and \(W_n' = 0\) \((\forall n' \neq n)\) over \(a + [N]\). Then

\[
I_{g_2}(x)_{a+[N]} = G(\pi_n-a)(T^n x).
\]
Proof. Take $b \in a + [N]$. We have $a - b \equiv n - b \pmod{N}$ and $0 \in (a - b) + [N]$. Hence $a_{n-b} = a - b$. Let $W(I_{g_1}(T^b x)) = (W_m')_{m \in \mathbb{Z}^k}$. Then $W'_m(t) = W_{m+b}(t+b)$ and

$$W'_m(0) = \begin{cases} 1 & (m = n-b) \\ 0 & (m \neq n-b). \end{cases}$$

Therefore $A(T^b x) = \{n - b\}$ and

$$g_2(T^b x) = f_2(T^b x) + G(\pi_{n-b-a_{n-b}}(T^{a_{n-b}} T^b x)) - f_2(T^b x) = G(\pi_{n-a}(T^a x)) - a_{b}.$$ 

Hence $I_{g_2}(x)|_{a+[N]} = G(\pi_{n-a}(T^a x))$. 

The proof of Proposition 7.2 is completed by the next lemma.

Lemma 7.9. The map $I_{(g_1, g_2)}: X \to (K^2)^{\mathbb{Z}^k}$ is a $\delta$-embedding with respect to the distance $d$.

Proof. Suppose $I_{g_1}(x), I_{g_2}(y)) = (I_{g_1}(y), I_{g_2}(y))$ for some $x, y \in X$. Set

$$(W_n)_{n \in \mathbb{Z}^k} = W(I_{g_1}(x)) = W(I_{g_1}(y)).$$

From part (2) of Lemma 7.7, there exists $n \in \mathbb{Z}^k$ such that $W_n = 1$ and $W'_{n'} = 0 (n' \neq n)$ over $B_{\sqrt{\pi_N}}((1-1/s)n) \cap \mathbb{Z}^k$. Take $a \in \mathbb{Z}^k$ satisfying $a \equiv n \pmod{N}$ and $(1-1/s)n \in a+[0,N)^k$. Then $a+[N] \subset B_{\sqrt{\pi_N}}((1-1/s)n)$, and hence $W_n = 1$ and $W'_{n'} = 0 (n' \neq n)$ over $a+[N]$. By Claim 7.8

$$I_{g_2}(x)|_{a+[N]} = G(\pi_{n-a}(T^a x)) = I_{g_2}(y)|_{a+[N]} = G(\pi_{n-a}(T^a y)).$$

Since $G$ is an embedding, we get $\pi_{n-a}(T^a x) = \pi_{n-a}(T^a y)$. The map $\pi_{n-a}$ is an $\varepsilon$-embedding with respect to $d_{[N]|_{(1-s)(a-a)+[N']}}$. Hence

$$d_{[(1-s)(a-a)+[N']}(T^a x, T^a y) = d_{[(1-s)(a-a)+a+[N']} (x, y) < \varepsilon.$$ 

Setting $(1-1/s)n = a + t$ with $t \in [0,N)^k$,

$$[(1-s)(n-a)] + a = [(1-s)n + (s-1)a + a] = [(1-s)n + sa] = [-st] \quad \text{(by } (s-1)n = sa + st)$$

$$= [-st] \in -[N'] \quad \text{(by } N' = [sN]).$$

Therefore the origin is contained in $[(1-s)(n-a)] + a + [N']$. Thus $d(x,y) < \varepsilon < \delta$. 

Remark 7.10. By a little more careful argument, we can prove the following slightly stronger (but slightly more cumbersome to state)
variant of Theorem 7.1: Suppose $D_1$ and $D_2$ are positive numbers. Let $(X, Z^k, T)$ be a dynamical system having the marker property and

$$\text{mdim}(X) < \min\left(\frac{D_1}{2^{k+1}}, \frac{D_2}{4}\right).$$

Then for a dense $G_\delta$ subset of $f \in C(X, [0,1]^{D_1+D_2})$ the map $I_f : X \to ([0,1]^{D_1+D_2})^{Z^k}$ is an embedding.

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