Abstract. In the present paper we thoroughly investigate theoretical properties of the SI-method, which was firstly introduced in [2] and proved to be remarkably stable when applied to a certain class of stiff boundary value problems. In particular, we provide sufficient conditions for the method to be applicable to the given two point boundary value problem for the second order differential equation as well as the corresponding error estimates. The implementation details of the method are addressed.

1. Introduction

The aim of the present paper is to provide a thorough theoretical justification of the SI-method introduced in [2] specifically for the case of boundary value problems. In what follows, we give a slightly different view on the SI-method as compared to that from [2] and provide sufficient conditions that guarantee the method's applicability (existence of the method's approximation) to the given two point boundary value problem for the second order ordinary differential equation (ODE). Using results from [4], [5], [6] we prove error estimates for the SI-method applied to BVP, which have been stated as propositions in [2] (without proof).

Continuing the research begun in [2], we focus on the boundary value problem

\begin{align*}
  u''(x) &= \mathcal{N}(u(x), x), \\
  u(a) &= 0, \quad u(b) = u_b, \quad a < b, \quad 0 < u_b \in \mathbb{R},
\end{align*}

where

\begin{equation}
  \mathcal{N}(u, x) \equiv \mathcal{N}(u, x)u, \quad \mathcal{N}(u, x) \in C^1(\mathbb{R} \times [a, b]), \quad \mathcal{N}''_u(u, x) \geq 0, \quad \forall x \in [a, b], \quad \forall u \in \mathbb{R}.
\end{equation}

Condition (3) guarantees existence and uniqueness of the solution to BVP (1), (2) (see, [1, p. 331, Theorem 7.26]).

**Lemma 1.** Let

\begin{equation}
  \mathcal{N}(u, x) \geq 0, \quad \forall u \in [0, +\infty), \quad x \in [a, b]
\end{equation}

and condition (3) holds true. Then the solution \( u(x) \) to BVP (1), (2) is monotonically increasing on \([a, b]\).

**Proof.** First of all, let us point out that

\begin{equation}
  u'(a) \neq 0.
\end{equation}

Otherwise, according to the Pickard-Lindelof Theorem (see, for example, [1, p.350]), whose conditions are fulfilled, \( u(x) \) must totally coincide with 0 on \([a, b]\), which contradicts the condition \( u(b) = u_b > 0 \).

Second, let us prove that \( u(x) > 0, \ \forall x \in (a, b] \). Assume that the latter is not true and there exists at least one point \( x_1 \in (a, b) \) such that \( u(x_1) \leq 0 \). This immediately implies the existence of a point \( b_1 \in [x_1, b) \), such that

\begin{equation}
  u(a) = u(b_1) = 0, \quad x_1 \in (a, b_1].
\end{equation}

Obviously, function \( u(x) \equiv 0 \) also satisfies equation (1), which, in conjunction with condition (3), allows us to apply the result of Theorem 21 from [3, p. 48] (the maximum principle) and prove that neither \( u(x) \) nor...
Lemma 1 allows us to re-state the BVP (1), (2) in an equivalent form as a BVP with respect to inverse function \( x(u) \):

\[
x''(u) = -N(u, x(u)) \left( x'(u) \right)^2, \\
x(0) = a, \; x(u_b) = b,
\]

2. SI-method for boundary value problems and its properties

In order to define the SI-approximation of the problem (1), (2) let us pick some \( c \in (a, b) \) and divide the intervals \([a, c]\) and \([0, u_b]\) into subintervals

\[
[x_{i-1}, x_i], \; i = 1, \ldots, N_1, \\
x_0 = a, \; x_{N_1} = c, \; x_{i-1} < x_i, \; \forall i \in 1, N_1
\]

and

\[
[u_{i-1}, u_i], \; i \in 1, N_2, \\
u_0 = 0, \; u_{N_2} = u_b, \; u_{i-1} < u_i, \; \forall i \in 1, N_2
\]

respectively.

We consider a pair of functions \( \tilde{u}(x) \) and \( \tilde{x}(u) \) satisfying the following conditions:

1. Function \( \tilde{u}(x) \) is a solution to the equation

\[
\tilde{u}''(x) = \alpha(\mathbb{P}_x(\tilde{u}'(x)), \mathbb{P}_x(\tilde{u}(x)), x)\tilde{u}(x), \; x \in [a, c], \; \tilde{u}(x) \in C^1([a, c]),
\]

where

\[
\mathbb{P}_x(f(x)) = f(x_i), \; x \in [x_i, x_{i+1}], \; \forall i \in 0, N_1 - 1,
\]

2. Function \( \tilde{x}(u) \) is a solution to the equation

\[
\tilde{x}''(u) = \beta(\mathbb{P}_u(\tilde{x}'(u)), \mathbb{P}_u(\tilde{x}(u)), u) \left( \tilde{x}'(u) \right)^3, \; u \in [\tilde{u}(c), b], \; \tilde{x}(u) \in C^1([\tilde{u}(c), b]),
\]

where

\[
\mathbb{P}_u(f(u)) = f(\tilde{u}_i), \; u \in [\tilde{u}_i, \tilde{u}_{i+1}],
\]

\[
\beta(u) = \beta_i(x'(\tilde{u}_i), x(\tilde{u}_i), u), \; u \in [\tilde{u}_i, \tilde{u}_{i+1}],
\]

\[
\beta_i(x', x, u) = - \left( N_{x_i}(\tilde{u}_i, x) + N'_{x}(\tilde{u}_i, x)x' \right) (u - \tilde{u}_i) - N(\tilde{u}_i, x),
\]

\[
\mathbb{P}_u(\tilde{u}(x_i)) = \tilde{u}_i, \; \forall i \in 1, N_2
\]

\[
\mathbb{P}_u(\tilde{x}(u_{i+1})) = \tilde{x}_{i+1}, \; \forall i \in 1, N_2
\]

\[
\mathbb{P}_u(\tilde{x}(u_{N_2})) = \tilde{x}(u_b)
\]

\[
\mathbb{P}_u(\tilde{u}(x_0)) = \tilde{u}_0 = 0
\]

\[
\mathbb{P}_u(\tilde{u}(x_{N_1})) = \tilde{u}(c)
\]

\[
\mathbb{P}_u(\tilde{x}(u_1)) = \tilde{x}(a)
\]

\[
\mathbb{P}_u(\tilde{x}(u_{N_2})) = \tilde{x}(b)
\]

The latter means that

\[
\alpha(\mathbb{P}_x(\tilde{u}'(x)), \mathbb{P}_x(\tilde{u}(x)), x)\tilde{u}(x) = 0
\]

\[
\beta(\mathbb{P}_u(\tilde{x}'(u)), \mathbb{P}_u(\tilde{x}(u)), u) \left( \tilde{x}'(u) \right)^3 = 0
\]

\[
\alpha(\mathbb{P}_x(\tilde{u}'(x)), \mathbb{P}_x(\tilde{u}(x)), x)\tilde{u}(x) \neq 0, \; \forall \alpha \in [a, b], \; \tilde{u}(x) \in C^1([a, b]),
\]

\[
\beta(\mathbb{P}_u(\tilde{x}'(u)), \mathbb{P}_u(\tilde{x}(u)), u) \left( \tilde{x}'(u) \right)^3 \neq 0, \; \forall \beta \in [\tilde{u}(c), b], \; \tilde{x}(u) \in C^1([\tilde{u}(c), b])
\]
\[ \bar{u}_i = \begin{cases} u_i, & \bar{u}_i > \bar{u}(c), \quad \forall i \in 0, N_2 - 1 \end{cases} \]

(3) functions \( \bar{u}(x) \) and \( \bar{x}(u) \) satisfy the boundary conditions

\[ \bar{u}(0) = 0, \quad \bar{x}(u_b) = b. \]

and the "matching" conditions

\[ \bar{x}(\bar{u}(c)) = c, \quad \bar{x}'(\bar{u}(c)) = \frac{1}{\bar{u}'(c)}. \]

Lemma 2. Let a pair of functions, \( \bar{u}(x) \) and \( \bar{x}(u) \), satisfy the conditions (1) \( \bar{u}(x) \) and \( \bar{x}(u) \) satisfy the boundary conditions (17) \( \bar{u}(0) = 0, \quad \bar{x}(u_b) = b. \)

Proof. Indeed, from (14), (15) it follows that

\[ \bar{x}'(u) = \frac{\bar{u}'(c)}{1 - 2(\bar{u}'(c))^2} \left( \sum_{i|\bar{u}_i < u} \bar{u}_i \right) \]

The fact that \( \bar{u}(x) \) is continuously differentiable on \([a, c]\) and condition (15) yield us

\[ \bar{u}'(c) \neq 0, \]

which, in the light of formula (19), automatically guarantees that \( \bar{x}(u) \) is monotone on \([\bar{u}(c), u_b]\). This completes the proof. \( \square \)

Definition 1. Let functions \( \bar{u}(x), \bar{x}(u) \) satisfy conditions (1), (2) Then function

\[ u(x) = \begin{cases} \bar{u}(x), & x \in [a, c], \\ \bar{x}^{-1}(x), & x \in [c, b] \end{cases} \]

is called an SI-approximation of the solution to BVP (1), (2).

Theorem 1. Let conditions (3), (4) and

\[ N_x'(u, x) \geq 0, \quad \forall u \in [0, u_b], \quad \forall x \in [a, b] \]

hold true. Then the SI-approximation \( u(x) \) (20) exists for arbitrary

\[ c \in (a, b). \]

In order to prove Theorem 1 we first need to prove a few auxiliary statements below.

Lemma 3. Let \( \bar{u}_\nu(x) \in C^1([a, c]) \) denote the solution to equation (11), (12) subjected to initial conditions

\[ \bar{u}_\nu(0) = 0, \quad \bar{u}'_\nu(0) = \nu > 0 \]

and let conditions (3), (4) and (21) hold true. Then \( \forall \nu, \bar{\nu} \in (0, +\infty), \quad \forall x \in [a, c] \)

\[ \bar{u}_\nu(x) > \bar{u}_\nu(x), \]

\[ \bar{u}'_\nu(x) > \bar{u}'_\nu(x), \]

provided that

\[ \nu > \bar{\nu}. \]

1As one can notice, \( \bar{u}_i = \bar{u}_j, \quad \forall i, j \in 0, N_2 - 1 \mid u_i, u_j \leq \bar{u}(c) \)
Proof. Let $\tilde{u}_{\nu, i}(x) \in C^1([a, c])$ denote the solution of equation (11), (12) subjected to initial conditions
\begin{equation}
\tilde{u}_{\nu, i}(x_i) = \mu, \quad \tilde{u}'_{\nu, i}(x_i) = \nu, \quad \mu \geq 0, \quad \nu > 0, \quad \forall i \in 0, N_1 - 1
\end{equation}
so that
\begin{equation}
\tilde{u}_\nu(x) \equiv \tilde{u}_{\nu, 0, 0}(x).
\end{equation}
Let us fix some arbitrary
\begin{equation}
j \in 0, N_1 - 1
\end{equation}
and assume that
\begin{equation}
\tilde{u}'_{\nu, j, j}(x_j), \tilde{u}'_{\nu, j, j}(x_j) > 0, \quad \tilde{u}'_{\nu, j, j}(x_j) > \tilde{u}'_{\nu, j, j}(x_j),
\end{equation}
\begin{equation}
\tilde{u}_{\nu, j, j}(x_j), \tilde{u}_{\nu, j, j}(x_j) \geq 0, \quad \tilde{u}_{\nu, j, j}(x_j) \geq \tilde{u}_{\nu, j, j}(x_j).
\end{equation}
Under the conditions of Lemma and assumptions (27), (28) we are going to prove that
\begin{equation}
\tilde{u}'_{\nu, j, j}(x_j) > \tilde{u}'_{\nu, j, j}(x_j), \quad \forall x \in (x_j, x_{j+1}],
\end{equation}
\begin{equation}
\tilde{u}'_{\nu, j, j}(x_j) > \tilde{u}'_{\nu, j, j}(x_j), \quad \forall x \in (x_j, x_{j+1}].
\end{equation}
By definition, functions $\tilde{u}_{\nu, j, j}(x)$, $\tilde{u}_{\nu, j, j}(x)$ satisfy equations
\begin{equation}
\tilde{u}''_{\nu, j, j}(x) - \alpha_j(\nu_j, \mu_j, x)\tilde{u}_{\nu, j, j}(x) = 0, \quad \forall x \in [x_j, x_{j+1}],
\end{equation}
\begin{equation}
\tilde{u}''_{\nu, \bar{\nu}, \bar{j}, \bar{j}}(x) - \alpha_j(\bar{\nu}_j, \bar{\mu}_j, x)\tilde{u}_{\nu, \bar{\nu}, \bar{j}, \bar{j}}(x) = 0, \quad \forall x \in [x_j, x_{j+1}]
\end{equation}
respectively. It is easy to verify, that under conditions (3), (11), (21) the inequality
\begin{equation}
\alpha_j(\nu_j, \mu_j, x) \geq \alpha_j(\bar{\nu}_j, \bar{\mu}_j, x) \geq 0, \quad \forall x \in (x_j, x_{j+1}]
\end{equation}
holds true. The latter means that the maximum principle (see, for example, [3, Theorem 3, p. 6]) is applicable to functions $\tilde{u}_{\nu, j, j}(x)$, $\tilde{u}_{\nu, j, j}(x)$, saying that neither of the two can attain a nonnegative maximum on $[x_j, x_{j+1}]$, which, in turn, yields us the inequalities
\begin{equation}
\tilde{u}_{\nu, j, j}(x), \tilde{u}_{\nu, \bar{\nu}, \bar{j}, \bar{j}}(x) > 0, \quad \forall x \in (x_j, x_{j+1}], \quad n = 1, 2.
\end{equation}
Subtracting (32) from (31) and using inequalities (33), (34), we get the estimate
\begin{equation}
w'(x) - \alpha_j(\nu_j, \mu_j, x)w(x) \geq 0, \quad w(x) = \tilde{u}_{\nu, j, j}(x) - \tilde{u}_{\nu, j, j}(x), \quad \forall x \in [x_j, x_{j+1}].
\end{equation}
From (27) and (28) it follows that
\begin{equation}
w(x) \geq 0, \quad w'(x) > 0,
\end{equation}
which, in conjunction with the maximum principle (which is applicable to $w(x)$ as well, see, for example, [3, Theorem 3, 4, p. 6–7]), yields us a fact that
\begin{equation}
w'(x) > 0, \quad \forall x \in [x_j, x_{j+1}].
\end{equation}
The latter automatically implies inequalities (29) (30).

By now we proved that if conditions (27), (28) hold true for some $j \in 0, N_1 - 1$ then they are also fulfilled for $j + 1$ with
\begin{equation}
\nu_{j+1} = \tilde{u}'_{\nu, j, j}(x_{j+1}), \quad \mu_{j+1} = \tilde{u}_{\nu, j, j}(x_{j+1})
\end{equation}
\begin{equation}
\bar{\nu}_{j+1} = \tilde{u}'_{\nu, j, j}(x_{j+1}), \quad \bar{\mu}_{j+1} = \tilde{u}_{\nu, j, j}(x_{j+1}).
\end{equation}
Under the conditions of the Lemma, inequality (25) implies conditions (27), (28) for $j = 0$ with
\begin{equation}
\nu_0 = \nu, \quad \mu_0 = 0, \quad \bar{\nu}_0 = \bar{\nu}, \quad \bar{\mu}_0 = 0
\end{equation}
and the rest obviously follows from what was proved above and the principle of mathematical induction. This completes the proof.

Lemma 4. Let the conditions of Lemma [3] hold true. Then the functions $\tilde{u}_\nu(x)$ and $\tilde{u}'_\nu(x)$, as functions of parameter $\nu$, are continuous on $[0, +\infty), \forall x \in [a, c]$. 

□
Proof. The statement of the Lemma almost immediately follows from the corresponding theorem about continuity of solutions of IVPs with respect to initial conditions and parameters (see, for example, [11, Theorem 8.40, p 372]).

**Lemma 5.** Let the conditions of Lemma 3 hold true. Then there exists a unique value \(\nu^* > 0\) such that
\[
\tilde{u}_{\nu^*}(c) = u_b.
\]

**Proof.** From conditions (3), (4) and (21) and the maximum principle it follows that
\[
\tilde{u}_{\nu}(x) > \nu(x - a), \forall x \in [a, c], \forall \nu > 0.
\]
The latter yields us the inequality
\[
\tilde{u}_{\nu}(c) > u_b
\]
provided that
\[
\nu \geq \frac{u_b}{c - a},
\]
which, in conjunction with the obvious equality
\[
\tilde{u}_0(c) = 0,
\]
Lemma 4 and the Bolzano’s theorem, provides us the existence of \(\nu^*\) mentioned in the Lemma. The uniqueness follows from the monotonicity properties of \(\tilde{u}_{\nu}(x)\) as a function of \(\nu\) (Lemma 3). □

**Lemma 6.** Let the conditions of Lemma 3 hold true and let \(\tilde{x}_{\nu}(u) \in C^1([\tilde{u}_{\nu}(c), u_b])\) denote the solution to equation (14), (15) subjected to initial conditions

\[
\tilde{x}_{\nu}(\tilde{u}_{\nu}(c)) = c, \quad \tilde{x}''_{\nu}(\tilde{u}_{\nu}(c)) = \frac{1}{\tilde{u}_{\nu}'(c)}, \ \nu \in [0, \nu^*]
\]

where \(\nu^*\) was introduced in Lemma 3. Then \(\phi(\nu) = \tilde{x}_{\nu}(u_b)\) is a continuous function of \(\nu \in (0, \nu^*)\) and

\[
\lim_{\nu \uparrow \nu^*} \phi(\nu) = c,
\]

\[
\lim_{\nu \downarrow 0} \phi(\nu) = +\infty.
\]

**Proof.** We start by proving that the function \(\phi(\nu) = \tilde{x}_{\nu}(u_b)\) is continuous on \((0, \nu^*)\).

It is easy to see that on each interval \([\tilde{u}_i, \tilde{u}_{i+1}], i \in 0, N_2 - 1\) function \(\tilde{x}_{\nu}(u)\) can be expressed in a recursive way

\[
\tilde{x}_{\nu}(u) = \tilde{x}_{\nu,i}(u) = \int_{\tilde{u}_i}^{u} \frac{\tilde{x}_{\nu,i-1}(\tilde{u}_i)d\eta}{\sqrt{1 - 2(\tilde{x}_{\nu,i-1}(\tilde{u}_i))^2 \int_{\tilde{u}_i}^{\eta} \beta_{\nu,i}(\tilde{x}_{\nu,i-1}(\tilde{u}_i), \tilde{x}_{\nu,i-1}(\tilde{u}_i), \xi)d\xi}} + \tilde{x}_{\nu,i-1}(\tilde{u}_i),
\]

where

\[
\tilde{u}_0 = \tilde{u}_{\nu}(c), \quad \tilde{x}_{\nu,-1}(\tilde{u}_0) = c, \quad \tilde{x}_{\nu,-1}(\tilde{u}_0) = \frac{1}{\tilde{u}_{\nu}'(c)}.
\]

According to the definition of \(\tilde{u}_i\) given in (16), some intervals \([\tilde{u}_i, \tilde{u}_{i+1}]\) has zero measure, containing a single point \(\tilde{u}_{\nu}(c)\). This, however, does not affect correctness of the reasoning below.

From (40) it follows that

\[
\begin{align*}
\tilde{x}_{\nu,i}(\tilde{u}_{i+1}) &= \phi_{\nu,i}(\tilde{x}_{\nu,i-1}(\tilde{u}_i), \tilde{x}_{\nu,i-1}(\tilde{u}_i)), \\
\tilde{x}_{\nu,i}'(\tilde{u}_{i+1}) &= \psi_{\nu,i}(\tilde{x}_{\nu,i-1}(\tilde{u}_i), \tilde{x}_{\nu,i-1}(\tilde{u}_i), \tilde{u}_i), \\
\tilde{u}_i &= \tilde{u}_{\nu}(\tilde{u}_{i-1}),
\end{align*}
\]
where

\begin{equation}
\phi_i(x', x, u) = \frac{x'}{\sqrt{1 - 2 (x')^2}} \int_u^{\bar{u}_{i+1}} \beta_i(x', x, \xi) d\xi + x, \ u \leq \bar{u}_{i+1},
\end{equation}

\begin{equation}
\psi_i(x', x, u) = \frac{x'}{\sqrt{1 - 2 (x')^2}} \int_u^{\bar{u}_{i+1}} \beta_i(x', x, \xi) d\xi, \ u \leq \bar{u}_{i+1},
\end{equation}

\begin{equation}
\bar{u}_i(u) = \begin{cases} u_i, & u < u_i, \\ u, & u \geq u_i, \end{cases} \quad i \in \{0, N_2 - 1\}.
\end{equation}

It is easy to see that the functions (43), (44) and (45) are all the continuous functions of their arguments, which, in conjunction with the recursive formulas (42) and initial conditions (41), implies that \( \tilde{x}_\nu(u_b) \) is continuously dependent on \( \bar{u}_\nu(c) \), \( \bar{u}_\nu(c) \). On the other hand, according to Lemma 4 the latter two quantities are also continuous functions of the parameter \( \nu \), which completes the first part of the proof.

To prove equality (38) we can, without loss of generality, to assume that \( \nu < \nu^* \) is so close to \( \nu^* \) that \( \bar{u}_{N_2-1} \leq \bar{u}_\nu(c) < u_1 \) and prove that

\begin{equation}
\lim_{\nu \uparrow \nu^*} \tilde{x}_\nu(u_1) = +\infty.
\end{equation}

The latter automatically implies equality (39), since \( \tilde{x}_\nu(u_b) > \tilde{x}_\nu(u_1), \ \forall \nu \in (0, \nu^*) \).

Under the assumption (46) we have that

\begin{equation}
\lim_{\nu \downarrow 0} \tilde{x}_\nu(u_1) = c + \lim_{\nu \downarrow 0} \int_{\bar{u}_\nu(c)}^{u_b} \left[ p_\nu + q_\nu (\eta - \bar{u}_\nu(c)) + r_\nu (\eta - \bar{u}_\nu(c))^2 \right]^{-\frac{1}{2}} d\eta,
\end{equation}

where

\begin{align*}
p_\nu &= \left( \bar{u}'_\nu(c) \right)^2, \\
q_\nu &= 2\mathcal{N}(\bar{u}_\nu(c), c), \\
r_\nu &= \mathcal{N}'(\bar{u}_\nu(c), c) + \frac{\mathcal{N}'(\bar{u}_\nu(c), c)}{\bar{u}'(c)}.
\end{align*}
From Lemma 4 it is easy to see that to evaluate \( \lim_{\nu \downarrow 0} r_{\nu} \) one should deal with an indeterminate form
\[
\lim_{\nu \downarrow 0} \frac{\tilde{u}_\nu(c)}{\tilde{u}'_\nu(c)} = 0,
\]
and the limit itself might not even exist. However, what is known for sure is that
\[
0 \leq \lim_{\nu \downarrow 0} \frac{\tilde{u}_\nu(c)}{\tilde{u}'_\nu(c)} \leq \lim_{\nu \downarrow 0} \frac{c \tilde{u}''_\nu(c)}{\tilde{u}'_\nu(c)} = c,
\]
which, in the other words, mean that \( r_\nu \) is nonnegative and bounded from above when \( \nu \) tends to 0 from the right, i.e,
\[
0 \leq \lim_{\nu \downarrow 0} r_{\nu} \leq \lim_{\nu \downarrow 0} r_{\nu} < +\infty.
\]
Now let us consider an arbitrary sequence
\[
\{ \nu_i \mid 0 < \nu_i < \nu^*, \lim_{i \to +\infty} \nu_i = 0 \}
\]
and prove that
\[
\lim_{i \to +\infty} \tilde{x}_{\nu_i}(u_1) = +\infty
\]
which will imply the equality (47). To that end we need to deal somehow with the fact that the limit \( r_{\nu_i} \) as \( i \) tends to +\( \infty \) might not exist. Given that, however, if we manage to prove that for every sub-sequence \( \{\nu_{i(j)}\} \), such that the limit
\[
\lim_{j \to +\infty} r_{\nu_{i(j)}}
\]
does exist (and is finite, according to (48)), the equality
\[
\lim_{j \to +\infty} \tilde{x}_{\nu_{i(j)}}(\tilde{u}_1) = +\infty
\]
holds true, then this would automatically guarantee the fulfillment of equality (49).

It is enough to consider the two mutually exclusive cases:
\[
\lim_{j \to +\infty} r_{\nu_{i(j)}} \neq 0,
\]
and
\[
\lim_{j \to +\infty} r_{\nu_{i(j)}} = 0.
\]
For a subsequence \( \{\nu_{i(j)}\} \), such that (50) holds true, we have that
\[
\lim_{j \to +\infty} \tilde{x}_{\nu_{i(j)}}(\tilde{u}_1) = c + \lim_{j \to +\infty} \frac{1}{\sqrt{r_{\nu_{i(j)}}}} \ln \left( 2 r_{\nu_{i(j)}} \eta + q_{\nu_{i(j)}} + 2 \sqrt{r_{\nu_{i(j)}} \left( r_{\nu_{i(j)}} \eta^2 + q_{\nu_{i(j)}} \eta + p_{\nu_{i(j)}} \right)} \right)_{\eta=0} = +\infty,
\]
since
\[
\lim_{j \to +\infty} p_{\nu_{i(j)}} = \lim_{j \to +\infty} q_{\nu_{i(j)}} = 0
\]
If subsequence \( \{\nu_{i(j)}\} \) implies equality (51) then we get the estimate
\[
\lim_{j \to +\infty} \tilde{x}_{\nu_{i(j)}}(\tilde{u}_1) \geq \lim_{j \to +\infty} \int_{\tilde{u}_{\nu_{i(j)}}(c)}^{u} \left[ p_{\nu_{i(j)}} + q_{\nu_{i(j)}} \tilde{u}_1 + r_{\nu_{i(j)}} \tilde{u}_1^2 \right]^{-\frac{1}{2}} d\eta = \lim_{j \to +\infty} \left[ p_{\nu_{i(j)}} + q_{\nu_{i(j)}} \tilde{u}_1 + r_{\nu_{i(j)}} \tilde{u}_1^2 \right]^{-\frac{1}{2}} \left( \tilde{u}_1 - \tilde{u}_{\nu_{i(j)}}(c) \right) = +\infty,
\]
since (51) and (52).
This completes the proof of the Lemma. □

Proof of Theorem 4. The statement of Theorem 4 immediately follows from Lemma 6 and the Bolzano’s intermediate value theorem being applied to function

\[ f(\nu) = \tilde{x}_\nu(u_b) - b, \ \nu \in [0, \nu^*], \]

where \( \nu^* \) is defined in Lemma 5. □

3. Error analysis of the SI-method applied to BVP

To appear soon.

4. Implementation details of the SI-method

The general idea of the SI method was introduced in [2] as well as one of its possible implementations. A key role in the definition of an SI-method’s implementation is played by, so called, step functions. In the current paper we define the step functions in a slightly different way to how they were defined in [2].

Throughout the rest of the paper we will refer to \( U(x) = U(A, B, C, D, x) \) as the forward step function and define it to be the linear solution to the Cauchy problem

\[ U''(s) = (As + B)U(s), \ U(0) = D, \ U'(0) = C. \tag{53} \]

Whereas the inverse step function \( V(s) = V(\bar{A}, \bar{B}, \bar{C}, \bar{D}, s) \) is defined as the solution to the nonlinear Cauchy problem

\[ V''(s) = (\bar{A}s + \bar{B})(V'(s))^3, \ V(0) = \bar{D}, \ V'(0) = \bar{C}. \tag{54} \]

References

[1] Walter G. Kelley and Allan C. Peterson. The theory of differential equations. Universitext. Springer, New York, second edition, 2010. Classical and qualitative.

[2] Volodymyr L. Makarov and Denys V. Dragunov. An efficient approach for solving stiff nonlinear boundary value problems. Journal of Computational and Applied Mathematics, 345:452 – 470, 2019.

[3] Murray H. Protter and Hans F. Weinberger. Maximum principles in differential equations. Springer-Verlag, New York, 1984. Corrected reprint of the 1967 original.

[4] Giovanni Vidossich. On the continuous dependence of solutions of boundary value problems for ordinary differential equations. J. Differential Equations, 82(1):1–14, 1989.

[5] Giovanni Vidossich. A general existence theorem for boundary value problems for ordinary differential equations. Nonlinear Anal., 15(10):897–914, 1990.

[6] Giovanni Vidossich. Differentiability of solutions of boundary value problems with respect to data. J. Differential Equations, 172(1):29–41, 2001.