Exact meromorphic solutions of Schwarzian differential equations

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Abstract: This paper studies exact meromorphic solutions of the autonomous Schwarzian differential equations. All transcendental meromorphic solutions of five canonical types (among six) of the autonomous Schwarzian differential equations are constructed explicitly. In particular, the solutions of four types are shown to be elliptic functions. Also, all transcendental meromorphic solutions that are locally injective or possess a Picard exceptional value are characterized for the remaining canonical type.

1 Introduction and Lemmas

The Schwarzian derivative of a meromorphic function $f$ is defined as

$$S(f, z) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$ 

It is well-known that $S(f, z) \equiv 0$ if and only if $f$ is a Möbius transformation. This property reveals that the Schwarzian derivative $S(f, z)$ measures how much $f$ differs from being a Möbius transformation. Another basic property of the Schwarzian derivative is that it is invariant under the Möbius group in the sense that $S(f, z) = S(\gamma \circ f, z)$, where $\gamma$ can be any Möbius transformation. The converse is also true, namely, if $S(g, z) = S(f, z)$, where $f, g$ are meromorphic functions, then there exits a Möbius transformation $\gamma$ such that $g = \gamma \circ f$.

The Schwarzian derivative plays an essential role in various branches of complex analysis including univalent functions and conformal mappings. It has also been shown that the Schwarzian derivative has close connections with second-order linear differential equations and Lax pairs of certain integrable partial differential equations. In particular, it appears in the differential equation

$$S(f, z)^p = R(z, f) = \frac{P(z, f)}{Q(z, f)},$$

where $p$ is a positive integer, and $R(z, f)$ is an irreducible rational function in $f$ with meromorphic coefficients. The equation is known as the Schwarzian differential equation. Ishizaki obtained some Malmquist-type theorems of this equation and results concerning the deficiencies of its meromorphic solutions. The growth of meromorphic solutions of the equation with polynomial coefficients has been studied by Liao and Ye. A more complicated Schwarzian type differential equation was considered by Hotzel and Jank. If we restrict ourselves to the autonomous Schwarzian differential equation

$$S(f, z)^p = R(f) = \frac{P(f)}{Q(f)},$$

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where $P, Q$ are co-prime polynomials with constant coefficients, Ishizaki [7] obtained a Malmquist-Yosida-type result in which he gave a complete classification of the equation (2) possessing transcendental meromorphic solutions.

**Theorem A.** Suppose that the autonomous Schwarzian differential equation (2) admits a transcendental meromorphic solution. Then for some Möbius transformation $u = (af + b)/(cf + d), ad - bc \neq 0$, (2) reduces into one of the following types

$$S(u, z) = c\frac{(u - \sigma_1)(u - \sigma_2)(u - \sigma_3)(u - \sigma_4)}{(u - \tau_1)(u - \tau_2)(u - \tau_3)(u - \tau_4)}$$  \hspace{1cm} (3)

$$S(u, z)^3 = c\frac{(u - \sigma_1)^3(u - \sigma_2)^3}{(u - \tau_1)^3(u - \tau_2)^3(u - \tau_3)^3}$$ \hspace{1cm} (4)

$$S(u, z)^3 = c\frac{(u - \sigma_1)^3(u - \sigma_2)^3}{(u - \tau_1)^3(u - \tau_2)^3(u - \tau_3)^3}$$ \hspace{1cm} (5)

$$S(u, z)^2 = c\frac{(u - \sigma_1)^2(u - \sigma_2)^2}{(u - \tau_1)^2(u - \tau_2)^2(u - \tau_3)^2}$$ \hspace{1cm} (6)

$$S(u, z) = c\frac{u - \sigma_1}{u - \tau_1}$$ \hspace{1cm} (7)

$$S(u, z) = c$$ \hspace{1cm} (8)

where $c \in \mathbb{C}, \tau_j$ are distinct constants, and $\sigma_j$ are constants, not necessarily distinct, $j = 1, \ldots, 4$.

**Remark 1.** We remark that the conclusion of Theorem A does not hold for rational solutions of equation (3). For instance, the function

$$f(z) = -\frac{3}{2(z + a)^2},$$

where $a$ is an arbitrary constant, satisfies the equation $S(u, z) = u$ but it cannot be transformed into any type of (3)-(8) via Möbius transformations. It is also noted that $f$ can be viewed as a fixed point of the Schwarzian operator and we refer the readers to the reference [12] for the details on fixed points and $N$-cycles of the Schwarzian operator.

The above theorem intimates that to study the autonomous Schwarzian differential equation (2), it suffices to consider the equations (3)-(8). We will show that all transcendental meromorphic solutions of the equations (3)-(8) are elliptic functions and can be explicitly constructed. It is also shown that all transcendental meromorphic solutions of the equation (2) can be characterized by imposing some conditions on them. The precise statements of these results are as follows.

**Theorem 1.** If the Schwarzian differential equation (2) admits a transcendental meromorphic solution $f$ with a Picard exceptional value $\xi \in \hat{\mathbb{C}}$, then by some Möbius transformation $f = \gamma_1(u)$, (2) reduces into either

$$S(u, z) = c\frac{(u - \sqrt{2i})(u + \sqrt{2i})}{(u - 1)(u + 1)},$$

and the transcendental meromorphic solutions of (2) are $f(z) = \gamma_1(\sin(\alpha z + \beta))$, where $\alpha = \sqrt{2c}$ and $\beta$ is a constant; or

$$S(u, z) = c,$$

and all solutions of (2) are $f(z) = \gamma_2(e^{\alpha z})$, where $\alpha = \sqrt{-2c}$ and $\gamma_2$ is any Möbius transformation.

**Remark 2.** Theorem A shows that any transcendental meromorphic solution of equations (3)-(6) must have infinitely many poles.
The result below follows immediately from Theorem 1.

**Corollary 1.** If the Schwarzian differential equation (2) admits a transcendental entire solution \( f \), then by some Möbius transformation \( f = L_1(u) \), (2) reduces into either

\[
S(u, z) = c \frac{(u - \sqrt{2}i)(u + \sqrt{2}i)}{(u - 1)(u + 1)},
\]

and the entire solutions of (2) are \( f(z) = L_1(\sin(\alpha z + \beta)) \), where \( \alpha = \sqrt{2}c \) and \( \beta \) is a constant; or

\[
S(u, z) = c,
\]

and all entire solutions of (2) are \( f(z) = L_2(e^{\pm \alpha z}) \), where \( \alpha = \sqrt{-2}c \) and \( L_2 \) is any linear transformation.

**Theorem 2.** If the Schwarzian differential equation (2) admits a locally injective transcendental meromorphic solution, then by some Möbius transformation \( f = \gamma(u) \), (2) reduces into

\[
S(u, z) = c,
\]

and all solutions of (2) are \( f(z) = \gamma(e^{\alpha z}) \), where \( \alpha = \sqrt{-2}c \) and \( \gamma \) is any Möbius transformation.

Rewrite the equation (3) as

\[
\begin{align*}
\text{if and only if the following parameter relations for } & e_1 = \frac{4}{j=1} \tau_j, \\
e_2 = \sum_{1 \leq k \leq 4} \tau_j \tau_k, & e_3 = \sum_{1 \leq j < k < l \leq 4} \tau_j \tau_k \tau_l, \\
e_4 = \prod_{j=1}^4 \tau_j. &
\end{align*}
\]

Then we can construct all transcendental meromorphic solutions to the equation (3).

**Theorem 3.** All transcendental meromorphic solutions of the equation (3) are elliptic functions of the form

\[
u(z) = a - \frac{b}{\wp(z - z_0; g_2, g_3) - d},
\]

where \( \wp(z; g_2, g_3) \) is the Weierstrass elliptic function, \( z_0 \in \mathbb{C} \) is arbitrary, \( a = \tau_i \) and \( b, d, g_2, g_3 \) are constants that depend on \( c, \sigma_i \) and \( \tau_i, i = 1, 2, 3, 4 \). Further, with \( e_i (i = 1, 2, 3, 4) \) defined in (10) and

\[
q_i = \prod_{1 \leq j \leq 4, j \neq i} (\tau_i - \tau_j), \quad i = 1, 2, 3, 4,
\]

the equation (3) admits solutions of the form (11) if and only if the following parameter relations hold

\[
\begin{align*}
r_0 &= \frac{b}{2q_i} (3e_3^2 - 8e_2 e_4), \\
r_1 &= \frac{2b}{q_i} (6e_1 e_4 - e_2 e_3), \\
r_2 &= \frac{b}{q_i} (2e_2^2 - 3e_1 e_3 - 24e_4), \\
r_3 &= \frac{2b}{q_i} (6e_3 - e_1 e_2), \\
r_4 &= \frac{b}{2q_i} (3e_1^2 - 8e_2),
\end{align*}
\]

\[
\begin{align*}
d &= \frac{b}{6q_i} \left[ \sum_{1 \leq j < k \leq 4, j \neq i} (\tau_j - \tau_k)^2 - \sum_{1 \leq j < k \leq 4, j \neq i} 2(\tau_i - \tau_j)^2 \right],
\end{align*}
\]

\[
\begin{align*}
g_2 &= \frac{4b^2}{3q_i^2} (e_2^2 - 3e_1 e_3 + 12e_4), \\
g_3 &= \frac{4b^3}{27q_i^3} (2e_2^3 - 9e_1 e_2 e_3 - 72e_2 e_4 + 27e_3^2 + 27e_1^2 e_4),
\end{align*}
\]

\[
\Delta = g_2^2 - 27g_3^3 = \frac{16b^6}{q_i^6} \prod_{1 \leq j < k \leq 4} (\tau_j - \tau_k)^2 \neq 0.
\]

(16)
Remark 3. Theorem 3 indicates that the equation (9) has transcendental meromorphic solutions only if the parameters $c, \tau_1, i = 1, 2, 3, 4$ satisfy the conditions (12) and (13). In addition, the solution (11) has just one free parameter, which implies that the general solution of equation (11) should have more complicated singularities other than poles.

In view of the invariance of Schwarzian derivatives under the Möbius group, we may compose the solution $u$ of equations (11)-(13) with a Möbius transformation such that $\tau_1, \tau_2$ and $\tau_3$ can be any distinct desired numbers, and this allows us to derive all transcendental meromorphic solutions to the equation equations (11)-(13) explicitly.

Theorem 4. Let $\tau_1 = 4, \tau_2 = -3, \tau_3 = 0$, then all transcendental meromorphic solutions to the equation (11) are elliptic functions. Moreover, these solutions exist if and only if

$$\{\sigma_1, \sigma_2\} = \left\{ \sqrt[3]{5i}, -\sqrt[3]{5i} \right\},$$

and in this case, all the transcendental meromorphic solutions to the equation (11) are given by

$$u(z) = -\frac{3c}{c - 74088 \varphi(z - z_0; g_2, g_3)^3},$$

where $\varphi(z; g_2, g_3)$ is the Weierstrass elliptic function with $g_2 = 0, g_3 = c/10584$, and $z_0 \in \mathbb{C}$ is arbitrary.

Theorem 5. Let $\{\tau_1, \tau_2, \tau_3\} = \{0, 1, -1\}$, then all transcendental meromorphic solutions to the equation (11) are elliptic functions. Moreover, these solutions exist if and only if

$$\{\sigma_1, \sigma_2\} = \left\{ \frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}} \right\},$$

and in this case, all the transcendental meromorphic solutions to the equation (11) are given by

$$u(z) = \frac{9 [9 \varphi(z - z_0; g_2, g_3) + L^2 \varphi'(z - z_0; g_2, g_3)]}{2L \left[ 81 \varphi(z - z_0; g_2, g_3)^2 - 9L^2 \varphi(z - z_0; g_2, g_3) + L^4 \right]},$$

where $L^6 = -27c/64$, $\varphi(z; g_2, g_3)$ is the Weierstrass elliptic function with $g_2 = 0, g_3 = c/432$, and $z_0 \in \mathbb{C}$ is arbitrary.

Theorem 6. Let $\tau_1 = 0, \tau_2 = 1, \tau_3 = -1$, then all transcendental meromorphic solutions to the equation (11) are elliptic functions. Moreover, these solutions exist if and only if

$$\{\sigma_1, \sigma_2\} = \left\{ \frac{i}{2}, -\frac{i}{2} \right\},$$

and in this case, all the transcendental meromorphic solutions to the equation (11) are given by

$$u(z) = -\frac{1}{2L} \left( 8 \varphi(z - z_0; g_2, g_3) + L^2 \right)^2 \varphi'(z - z_0; g_2, g_3) \left( 64 \varphi(z - z_0; g_2, g_3)^2 + L^4 \right),$$

where $c = 9L^4/4$, $\varphi(z; g_2, g_3)$ is the Weierstrass elliptic function with $g_2 = -c/36, g_3 = 0$, and $z_0 \in \mathbb{C}$ is arbitrary.

Remark 4. It follows from Theorems 4-6 that all transcendental meromorphic solutions of the canonical Schwarzian differential equations (11)-(13) have been derived, except the solutions of equation (11) that have no Picard exceptional values. Although we are not able to prove that any transcendental meromorphic solution of equation (11) must have Picard exceptional value(s), we conjecture this is true.
2 Preliminaries

The important tools in our proof include Wiman-Valiron theorem and Wiman-Valiron theory. Let \( f \) be a transcendental entire function, and write

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n.
\]

As usual, for \( r > 0 \), we denote the maximum term by \( \mu(r, f) \), the central index by \( \nu(r, f) \), and the maximum modulus by \( M(r, f) \), i.e.,

\[
\mu(r, f) = \max_{|z|=r} |a_n z^n|, \quad \nu(r, f) = \sup\{|n||a_n| r^n = \mu(r, f)\}, \quad M(r, f) = \max_{|z|=r} |f(z)|.
\]

**Lemma 1** (Wiman-Valiron Theorem \[1\]). There exists a set \( F \subset [1, +\infty) \) satisfying

\[
\int_{F} \frac{dt}{t} < \infty
\]

with the following property: if \((z_k)\) is a sequence in \( \mathbb{C} \) with \( |f(z_k)| = M(|z_k|, f), |z_k| \not\in F \) and \( z_k \to \infty \), and if \( \nu_k = \nu(|z_k|, f) \), then

\[
\frac{f\left(z_k + \frac{z_k}{\nu_k} z\right)}{f(z_k)} \to e^z
\]

as \( k \to \infty \).

**Lemma 2** (\[8\]). Let \( f \) be a transcendental entire function, \( 0 < \delta < \frac{1}{4} \) and \( |z| = r \) such that

\[
|f(z)| > M(r, f) \nu(r, f)^{-\frac{1}{4} + \delta}
\]  

holds. Then there exists a set \( F \subset (0, +\infty) \) of finite logarithmic measure, i.e., \( \int_{F} dt/t < +\infty \) such that

\[
f^{(m)}(z) = \left(\frac{\nu(r, f)}{z}\right)^m (1 + o(1)) f(z)
\]  

holds for all \( m \geq 0 \) and all \( r \not\in F \).

The Schwarzian derivative has a fundamental relation with second-order linear ordinary differential equations.

**Lemma 3.** \[8, p. 110\] Let \( A(z) \) be analytic in a simply connected domain \( \Omega \). Then, for any two linearly independent solutions \( f_1, f_2 \) of

\[
f''(z) + A(z) f(z) = 0,
\]  

their quotient \( g = f_1/f_2 \) is locally injective and satisfies the differential equation

\[
S(g, z) = 2A(z).
\]

Conversely, let \( g \) be a locally injective meromorphic function in \( \Omega \) and define \( A(z) \) by \[18\]. Then \( A(z) \) is analytic in \( \Omega \) and the differential equation \[17\] admits two linearly independent solutions \( f_1, f_2 \) such that \( g = f_1/f_2 \).

**Remark 5.** The lemma above has crucial applications in differential equations. In particular, it has been used by Bergweiler and Eremenko \[3\] to solve the Bank-Laine conjecture, which concerns the zero distribution of solutions of equation \[17\] where \( A \) is an entire function of finite order.
Now we introduce some terminologies in Nevanlinna theory [8]. Let $f$ be a meromorphic function on $\mathbb{C}$ and $n(r, f)$ denote the number of poles of $f$ in the disk $\mathbb{D}(r) = \{ z \in \mathbb{C} | |z| < r \}$, counting multiplicity. The Nevanlinna characteristic function of $f$ is defined as
\[ T(r, f) = m(r, f) + N(r, f), \]
where
\[ m(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}, \]
\[ N(r, f) = n(0, f) \log r + \int_0^r [n(t, f) - n(0, f)] \frac{dt}{t}, \]
with $\log^+ x = \max\{0, \log x\}$. We note that $m(r, f)$ and $N(r, f)$ are called the proximity function and integrated counting function, respectively. Next, we define the order of $f$ by
\[ \rho(f) = \lim_{r \to \infty} \frac{\log T(r, f)}{\log r}. \]

The following result of Liao and Ye [10] says that the order of meromorphic solutions of equation (2) is bounded from above by 2.

**Lemma 4.** Let $f$ be a meromorphic solution of the autonomous Schwarzian differential equation (2), then $\rho(f) \leq 2$.

3 Proof of main results

We first recall the definition of totally ramified values: we call a point $a \in \overline{\mathbb{C}}$ a totally ramified value of a meromorphic function $f$ if all $a$-points of $f$ are multiple. According to a classical result of Nevanlinna, a non-constant function meromorphic in the plane can have at most four totally ramified values while a non-constant entire function can have at most two finite totally ramified values. We also need the following results.

**Lemma 5 ([8]).** Let $f(z)$ be a nonconstant meromorphic function. Then
\[ m \left( r, \frac{f'}{f} \right) = O(\log r), \]
if $f$ is of finite order, and
\[ m \left( r, \frac{f'}{f} \right) = O(\log(rT(r, f))), \]
possibly outside a set $E$ of $r$ with finite linear measure, if $f(z)$ is of infinite order.

**Lemma 6 ([14]).** If the differential equation
\[ w^2 + R(z)(w')^2 = Q(z), \] (19)
where $R, Q$ are nonzero rational functions, admits a transcendental meromorphic solution $f$, then $Q \equiv A$ is a constant, the multiplicity of zeros of $R(z)$ is no greater than 2 and $f(z) = \sqrt{A} \cos \alpha(z)$, where $\alpha(z)$ is a primitive of $1/\sqrt{R(z)}$ such that $\sqrt{A} \cos \alpha(z)$ is a transcendental meromorphic function.
3.1 Proof of Theorem 1

Let $f$ be a transcendental meromorphic solution with a Picard exceptional value of the equation (2). It follows from Theorem A that by some Möbius transformation

$$u = \frac{af + b}{cf + d}, \quad ad - bc \neq 0,$$

$u$ satisfies one of the equations (3)-(8).

If $u$ satisfies the equation (3), then $u$ has four totally ramified values $\tau_1, \tau_2, \tau_3, \tau_4$. This is impossible since $u$ has a Picard exceptional value. If $u$ satisfies the equation (4), then $u$ has three totally ramified values $\tau_1, \tau_2, \tau_3$. Thus, the Picard exceptional value of $u$ must be one of them. Without loss of generality, we may assume $\tau_3$ is a Picard exceptional value of $u$. Let

$$v = \frac{1}{u - \tau_3},$$

then $v$ has at most finitely many poles and satisfies the following differential equation

$$S(v, z) = \frac{e}{v - \sigma_1'(v - \sigma_2')^3},$$

Assume $\zeta_1, \cdots, \zeta_n$ are the poles (counting multiplicities) of $v$, then $v(z) = g(z)/P(z)$, where $g(z)$ is a transcendental entire function and $P(z) = (z - \zeta_1)\cdots(z - \zeta_n)$. We choose $z_k \to \infty$ such that $|z_k| \notin F$ and $|g(z_k)| = M(|z_k|, g)$. Let

$$h_k(z) = \frac{v(z_k + \rho_k z)}{v(z_k)},$$

where $\rho_k = \frac{z_k}{\nu_k}, \nu_k = \nu(|z_k|, g)$, then by Lemma 1 we have

$$\lim_{k \to \infty} h_k(z) = \lim_{k \to \infty} \frac{v(z_k + \rho_k z)}{v(z_k)} = \lim_{k \to \infty} \frac{g(z_k + \rho_k z)}{g(z_k)} \frac{P(z_k)}{P(z_k + \rho_k z)} = e^z.$$

Thus

$$\lim_{k \to \infty} \frac{\rho_k v'(z_k + \rho_k z)}{v(z_k)} = \lim_{k \to \infty} h'_k(z) = e^z,$$

and

$$\lim_{k \to \infty} \frac{\rho_k^2 v''(z_k + \rho_k z)}{v(z_k)} = \lim_{k \to \infty} h''_k(z) = e^z.$$

It follows from (20) that

$$\frac{1}{v(z_k)} \left( \frac{1}{\rho_k} \right)^2 \left( \frac{h''_k(z)}{h'_k(z)} - \frac{3}{2} \left( \frac{h'_k(z)}{h'_k(z)} \right)^2 \right) = c' \frac{(h_k(z) - \sigma_1'/v(z_k))^3(h_k(z) - \sigma_2'/v(z_k))^3}{(h_k(z) - \tau_1'/v(z_k))^3(h_k(z) - \tau_2'/v(z_k))^2}. \quad (21)$$

Noting the selection of $z_k$, we have

$$\lim_{k \to \infty} \frac{\nu_k}{v(z_k)} = 0$$

for any positive number $M$. Thus, the left side of the equation (21) tends to zero while the right side of equation (21) tends to $c'e^z$ as $k \to \infty$, which is a contradiction. Thus $u$ cannot satisfy (4). With similar arguments, we can prove that $u$ satisfies neither (5) nor (6).

If $u$ satisfies the equation (7), then $u$ has two totally ramified values $\tau_1, \tau_2$. Then we distinguish two cases.

Case 1: one of $\tau_1$ and $\tau_2$ is the Picard exceptional value of $u$, by the same arguments as above, we get a contradiction.

Case 2: both of $\tau_1$ and $\tau_2$ are not the Picard exceptional value of $u$. Without loss of generality,
we may assume the Picard exceptional value of \( u \) is infinity. Otherwise, we may consider a composition of a Möbius transformation and the function \( u \). Thus we can express \( u \) as

\[
u(r, g) \sim Ar \quad \text{and} \quad \rho(g) = 1.
\]

Hence \( \rho(u) = 1 \).

By computing the Laurent expansions on both sides of (22), we may obtain

- \( u'(z) = 0 \) if and only if \( u(z) = \tau_1 \) or \( u(z) = \tau_2 \).
- all the zeros of \( u' \) are simple.

Without loss of generality, we may assume \( \tau_1 = 1, \tau_2 = -1 \). Thus,

\[
\frac{(u')^2}{u^2 - 1}
\]

is a meromorphic function having only finitely many poles and no zeros. Noting \( \rho(u) = 1 \), we have

\[
Q^2(z) \frac{(u')^2}{u^2 - 1} = e^h(z),
\]

where \( Q(z) \) is a nonzero polynomial with simple zeros and \( h(z) \) is an entire function. Then, by Lemma [9] we have

\[
T(r, e^h) = m(r, e^h)
\]

\[
\leq 2m(r, Q) + m(r, \frac{u'}{u - 1}) + m(r, \frac{u'}{u + 1})
\]

\[
= O(\log r).
\]

This implies \( e^h \) is a polynomial and hence \( h \) is a constant. Without loss of generality, we may assume \( h = 1 \), then \( u \) satisfies the differential equation

\[
u^2 - Q(z)^2(u')^2 = 1. \quad (22)
\]
If \( \deg Q \geq 1 \), then by the equation (22) and Lemma 2 we have
\[
\nu(r, u) \sim Ar^{1 - \frac{\deg Q}{4}},
\]
where \( A \) is a positive number, but this contradicts with \( \rho(u) = 1 \). Hence \( Q(z) \) is a constant. It is easy to see that the solutions of (22) are of the form
\[
u = \sin(\alpha z + \beta),
\]
where \( \alpha, \beta \) are constants with \( Q^2 \alpha^2 = -1 \). Substituting \( u = \sin(\alpha z + \beta) \) into (7) and noting \( \tau_1 = 1, \tau_2 = -1 \), we obtain that
\[
\alpha = \sqrt{2}c, \quad \sigma_1 = \sqrt{2}i, \quad \sigma_2 = -\sqrt{2}i.
\]
Thus we get the conclusion.

Finally, if \( u \) satisfies equation (8), then \( R(f) \) must be a constant, say \( A \), and hence \( c^p = A \). It is easy to check that \( \lambda (z) = e^{\alpha z} \) is a solution of the equation (8), where \( \alpha = (\sqrt{2}c) \). Then it follows from the invariance property of the Schwarzian derivative under Möbius transformations that all the solutions of (8) are given by \( u = \gamma(e^{\alpha z}) \), where \( \gamma \) is a Möbius transformation and \( \alpha = (\sqrt{2}c) \). Hence, in this case, all the solutions of the equation (2) are \( f(z) = \gamma(e^{\alpha z}) \), where \( \gamma \) is a Möbius transformation and \( \alpha = (\sqrt{2}A^{1/2}) \).

### 3.2 Proof of Theorem 2

Suppose \( f \) is a locally injective transcendental meromorphic solution of the equation (2). According to Theorem A, all exists a Möbius transformation \( \gamma_1 \) such that \( u = \gamma_1(f) \) is also a locally injective transcendental meromorphic function and satisfies one of the equations (3)-(8). Then it follows from Lemma 3 that \( \lambda(u, z) \) is entire. This implies \( u \) cannot satisfy any of the equations (3)-(8). Otherwise, \( u \) has at least one Picard exceptional value. By Theorem 1 it indicates that \( u = \gamma_2(\sin(\alpha z + \beta)) \), where \( \alpha, \beta \) are constants and \( \gamma_2 \) is a Möbius transformation. Nevertheless, this contradicts with the fact that \( u \) is locally injective. As a consequence, \( u \) can only satisfy equation (8) and then the conclusion follows immediately from Theorem 1.

### 3.3 Proof of Theorem 3

Suppose \( u \) is a transcendental meromorphic solution to the equation (9), then Theorem 1 shows that \( u \) must have infinitely many poles. By comparing the Laurent expansions on both sides of the equation (9), we deduce that all the poles of \( u \) are simple and all the poles (if they exist) of \( \lambda(u, z) \) come from the zeros of \( u' \). Since all the poles of \( \lambda(u, z) \) are double, it follows that all zeros of \( u' \) should be simple, and at any zero of \( u' \), \( u(z) \) assumes one of the \( \tau_i, i = 1, 2, 3, 4 \). This means any zero of \( u - \tau_i \) must be double. Therefore,
\[
G(z) = \frac{u'^2}{(u - \tau_1)(u - \tau_2)(u - \tau_3)(u - \tau_4)}
\]
is a nonvanishing entire function, and there exists an entire function \( g(z) \) such that \( G = e^g \). According to Theorem 1 \( u \) has finite order of growth. Then we have
\[
T(r, e^g) = m(r, e^g) \leq m\left(r, \frac{u'}{(u - \tau_1)(u - \tau_2)}\right) + m\left(r, \frac{u'}{(u - \tau_3)(u - \tau_4)}\right) \leq \sum_{i=1}^{4} m\left(r, \frac{u'}{u - \tau_i}\right) + O(1) = O(\log r),
\]
where the last equality follows from Lemma 5. This implies $e^g$ is a polynomial and hence $g = C$ is a constant. As a consequence, $u$ satisfies the differential equation

$$u'^2 = K(u - \tau_1)(u - \tau_2)(u - \tau_3)(u - \tau_4), \quad K = e^C$$

whose general solution is given by

$$u(z) = K^{-1/2}\left(A - \frac{\wp'(w; g_2, g_3)}{\wp(z - z_0; g_2, g_3) - \wp(w; g_2, g_3)}\right)$$

$$= a - \frac{\wp(z - z_0; g_2, g_3) - d}{\wp(z - z_0; g_2, g_3)}$$

(24)

where $\wp(z; g_2, g_3)$ is the Weierstrass elliptic function, $z_0 \in \mathbb{C}$ is arbitrary and $a, b, d, g_2, g_3$ are constants that depend on $K$ and $\tau_i, i = 1, 2, 3, 4$. Finally, by substituting (24) into (3) and applying the differential equation satisfied by $\wp(z; g_2, g_3)$

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

where $\Delta = g_2^3 - 27g_3^2 \neq 0$, it can be computed that $a$ should be equal to one of the $\tau_i, i = 1, 2, 3, 4$, and other parameters should satisfy the relations (12) - (15).

### 3.4 Proof of Theorem 4

Let $u$ be a transcendental meromorphic solution to the equation (1), then Theorem 4 implies that $u$ must have infinitely many poles. With similar arguments as in Theorem 3 we find that

- all the poles of $u$ are simple;
- $u'(z) = 0$ if and only if $u(z) = \tau_i$ for some $i \in \{1, 2, 3\}$;
- if $u(z) = \tau_1$, then $z$ is a simple zero of $u'$;
- if $u(z) = \tau_2$, then $z$ is a double zero of $u'$;
- if $u(z) = \tau_3$, then $z$ is a zero of $u'$ of order 5.

It follows that

$$G(z) = \frac{u^6}{(u - \tau_1)^3(u - \tau_2)^4(u - \tau_3)^5}$$

is a nonvanishing entire function, and hence, there exists an entire function $g(z)$ such that $G = e^g$. According to Theorem 4 $u$ has finite order of growth. Then we have

$$T(r, e^g) = m(r, e^g)$$

$$\leq m\left(r, \frac{u^6}{(u - \tau_1)^3(u - \tau_2)^3(u - \tau_3)^3}\right) + m\left(r, \frac{u'}{u - \tau_2}\right) + m\left(r, \frac{u'^2}{(u - \tau_3)^2}\right)$$

$$\leq 3m\left(r, \frac{u'}{u - \tau_1}\right) + 4m\left(r, \frac{u'}{u - \tau_2}\right) + 5m\left(r, \frac{u'}{u - \tau_3}\right) + O(1)$$

$$= O(\log r).$$

This indicates that $g = C$ is a constant and hence $u$ satisfies the differential equation

$$u'^6 = K(u - \tau_1)^3(u - \tau_2)^4(u - \tau_3)^5, \quad K = e^C.$$

(26)

Since the elliptic curve parametrized by $u$ and $u'$ has genus one, the general solution of the above equation should be elliptic functions. Let

$$u(z) = \frac{1}{v(z)} + \tau_3,$$

(27)
then the equation \( (26) \) reduces to
\[
v^6 = K[(\tau_1 - \tau_3)v - 1]^3[(\tau_2 - \tau_3)v - 1]^4. \tag{28}\]
By using the singularity methods (see [4, 11] and the references therein), we find that the general solution to \( (28) \) reads
\[
v(z) = h - \frac{23328 [6\wp(z - z_0; g_2, g_3)^3 + \wp'(z - z_0; g_2, g_3)^2]}{5K(\tau_1 - \tau_3)^3(\tau_2 - \tau_3)^4}, \tag{29}\]
where \( \wp(z; g_2, g_3) \) is the Weierstrass elliptic function with \( g_2 = 0, \) \( z_0 \in \mathbb{C} \) is arbitrary and \( h, g_3 \) are constants depending on \( K \) and \( \tau_i, i = 1, 2, 3. \) Finally, with \( \tau_1 = 4, \tau_2 = -3, \tau_3 = 0, \) substituting \( (27) \) and \( (29) \) into \( (4) \) yields the solution of equation \( (4) \)
\[
u(z) = \frac{-3c}{c - 74088\wp(z - z_0; 0, g_3)^3},
\]
where \( g_3 = c/10584 \) and \( z_0 \in \mathbb{C} \) is arbitrary, provided that
\[
\{\sigma_1, \sigma_2\} = \left\{ \sqrt{5}i, -\sqrt{5}i \right\}.
\]
This completes the proof.

**3.5 Proof of Theorem 5**

Suppose \( u \) is a transcendental meromorphic solution to the equation \( (5) \), then Theorem 11 implies that \( u \) must have infinitely many poles. Using similar arguments as in Theorem 3, we can show that
- all the poles of \( u \) are simple;
- \( u'(z) = 0 \) if and only if \( u(z) = \tau_i \) for some \( i \in \{1, 2, 3\}; \)
- all the zeros of \( u' \) are double.

It follows that
\[
G(z) = \frac{u^3}{(u - \tau_1)^2(u - \tau_2)^2(u - \tau_3)^2} \tag{30}
\]
is a nonvanishing entire function, and hence, there exists an entire function \( g(z) \) such that \( G = e^g. \) Since the order of \( u \) is finite, we have
\[
T(r, e^g) = m(r, e^g)
\leq m\left(r, \frac{u'}{u - \tau_1}\right) + m\left(r, \frac{u'}{u - \tau_2}\right) + m\left(r, \frac{u'}{u - \tau_3}\right)
\leq 2\left[ \sum_{i=1}^{3} m\left(r, \frac{u'}{u - \tau_i}\right) \right] + O(1)
= O(\log r).
\]
This implies that \( g = C \) is a constant and hence \( u \) satisfies the differential equation
\[
u^3 = K(u - \tau_1)^2(u - \tau_2)^2(u - \tau_3)^2, \quad K = e^C. \tag{31}\]
Since the elliptic curve parametrized by \( u \) and \( u' \) has genus one, the general solution of the above equation should be elliptic functions. By using the singularity methods, we find that the general solution to \( (31) \) can be expressed as
\[
u(z) = \frac{1}{L}\left( \frac{(1 + i\sqrt{3})}{4}(\wp(z - z_0; g_2, g_3) - A_1) + \frac{(1 - i\sqrt{3})}{4}(\wp'(z - z_0; g_2, g_3) - A_2) \right)
+ \frac{1}{3}(\tau_1 + \tau_2 + \tau_3)
\tag{32}
\]
where \( L^3 = K \), \( \wp(z; g_2, g_3) \) is the Weierstrass elliptic function, \( z_0 \in \mathbb{C} \) is arbitrary and \( A_1, A_2, B_1, B_2, g_2, g_3 \) are constants depending on \( K \) and \( \tau_i, i = 1, 2, 3 \). Finally, with \( \{\tau_1, \tau_2, \tau_3\} = \{0, 1, -1\} \), substituting (32) into (5) yields that

\[
\{\sigma_1, \sigma_2\} = \left\{ \frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}} \right\}, \quad A_1 = A_2 = g_2 = 0, \quad g_3 = \frac{c}{432}, \quad B_1 = \frac{1}{18} (1 - i\sqrt{3}) L^2, \quad B_2 = \frac{1}{18} (1 + i\sqrt{3}) L^2, \quad L^6 = -\frac{27}{64} c.
\]

In this case, the equation (5) reduces to

\[
S(u, z)^3 = c \frac{(u^2 + 1/3)^3}{u^2(u^2 - 1)^2},
\]

and the solution (32) becomes

\[
u(z) = \frac{9 [9\wp(z; g_2, g_3) + L^2] \wp' (z; g_2, g_3)}{2L [81\wp(z; g_2, g_3)^2 - 9L^2 \wp(z; g_2, g_3) + L^4]}
\]

where \( L^6 = -27c/64, g_2 = 0, g_3 = c/432 \).

### 3.6 Proof of Theorem 6

Let \( u \) be a transcendental meromorphic solution to the equation (6), then Theorem 1 implies that \( u \) must have infinitely many poles. With similar arguments as in Theorem 2 we find that

- all the poles of \( u \) are simple;
- \( u'(z) = 0 \) if and only if \( u(z) = \tau_i \) for some \( i \in \{1, 2, 3\} \);
- if \( u(z) = \tau_1 \), then \( z \) is a simple zero of \( u' \);
- if \( u(z) = \tau_j, j = 2, 3 \), then \( z \) is a triple zero of \( u' \);

It follows that

\[
G(z) = \frac{u'^4}{(u - \tau_1)^2(u - \tau_2)^3(u - \tau_3)^3}
\]

is a nonvanishing entire function, and hence, there exists an entire function \( g(z) \) such that \( G = e^g \). As the order of \( u \) is finite, we have

\[
T(r, e^g) = m(r, e^g) \leq m \left( r, \frac{u'^2}{(u - \tau_1)^2(u - \tau_2)^2(u - \tau_3)^2} \right) + m \left( r, \frac{u'}{u - \tau_2} \right) + m \left( r, \frac{u'}{u - \tau_3} \right)
\]

\[
\leq 2m \left( r, \frac{u'}{u - \tau_1} \right) + 3m \left( r, \frac{u'}{u - \tau_2} \right) + 3m \left( r, \frac{u'}{u - \tau_3} \right) + O(1)
\]

\[
= O(\log r).
\]

This indicates that \( g = C \) is a constant and hence \( u \) satisfies the differential equation

\[
u'^4 = K(u - \tau_1)^2(u - \tau_2)^3(u - \tau_3)^3, \quad K = e^C.
\]

Since the elliptic curve parametrized by \( u \) and \( u' \) has genus one, the general solution of the above equation should be elliptic functions. Then the singularity methods indicate that the general solution to (34) can be expressed as

\[
u(z) = \frac{1}{2L} \wp'(z - z_0, g_2, g_3) - \frac{A_1}{\wp(z - z_0, g_2, g_3) - B_1} +
\]

\[
\frac{i}{2L} \left( \wp'(z - z_0, g_2, g_3) - \frac{A_2}{\wp(z - z_0, g_2, g_3) - B_2} - \frac{A_3}{\wp(z - z_0, g_2, g_3) - B_3} \right)
\]

(35)
where $L^4 = K$, $\wp(z; g_2, g_3)$ is the Weierstrass elliptic function, $z_0 \in \mathbb{C}$ is arbitrary and $A_j, B_j, g_2, g_3$ are constants depending on $K$ and $\tau_j, j = 1, 2, 3$. Finally, with $\tau_1 = 0, \tau_2 = 1, \tau_3 = -1$, substituting \(35\) into \(3\) yields that

\[
\left\{ \sigma_1, \sigma_2 \right\} = \left\{ \frac{i}{2}, -i \right\}, \quad g_2 = -\frac{c}{36}, \quad B_2 = -B_3 = \frac{L^2}{8}i,
\]

\[
c = \frac{9}{4}L^4, \quad A_1 = A_2 = A_3 = B_1 = g_3 = h = 0.
\]

In this case, the equation \(5\) reduces to

\[
S(u, z)^2 = c\frac{(u^2 + 1/4)^2}{u^2(u^2 - 1)},
\]

and the solution \(35\) becomes

\[
u(z) = -\frac{1}{2L} \frac{(8\wp(z - z_0; g_2, g_3) + L^2) \wp'(z - z_0; g_2, g_3)}{(64\wp(z - z_0; g_2, g_3)^2 + L^4)},
\]

where $g_2 = -c/36, g_3 = 0$ and $c = 9L^4/4$. This completes the proof.

**Remark 6.** Since elliptic functions are of order 2, Theorems 3-6 indicate that the estimate on the growth of meromorphic solutions of the equation \(2\) given in Lemma 4 is sharp.

### 3.7 Examples

We present some examples to illustrate all the possible configurations of the transcendental meromorphic solutions given in Theorem 4.

**Example 1.** The Schwarzian differential equation

\[
S(u, z) = \frac{3(25u^4 + 20u^3 + 14u^2 + 4u + 1)}{2u(u - 1)(u + 1)(3u + 1)}
\]

has the solution

\[
u(z) = \frac{1}{\wp(z - z_0; g_2, g_3) - 1},
\]

where $z_0 \in \mathbb{C}$ is arbitrary, $g_2 = 16$ and $g_3 = 0$.

**Example 2.** The Schwarzian differential equation

\[
S(u, z) = \frac{3(25u^4 + 20u^3 + 14u^2 + 4u + 1)}{u(u - 1)(u + 1)(3u + 1)}
\]

admits the solution

\[
u(z) = 1 - \frac{16}{\wp(z - z_0; g_2, g_3) + 12},
\]

where $z_0 \in \mathbb{C}$ is arbitrary, $g_2 = 64$ and $g_3 = 0$.

**Example 3.** The Schwarzian differential equation

\[
S(u, z) = -\frac{3(25u^4 + 20u^3 + 14u^2 + 4u + 1)}{u(u - 1)(u + 1)(3u + 1)}
\]

has the solution

\[
u(z) = -1 - \frac{8}{\wp(z - z_0; g_2, g_3) - 8},
\]

where $z_0 \in \mathbb{C}$ is arbitrary, $g_2 = 64$ and $g_3 = 0$. 

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Example 4. The Schwarzian differential equation
\[ S(u, z) = \frac{3 (225u^4 + 180u^3 + 126u^2 + 36u + 9)}{u(u - 1)(u + 1)(3u + 1)}, \]

admits the solution
\[ u(z) = -\frac{1}{3} - \frac{16}{\wp(z - z_0; g_2, g_3)} - 12, \]

where \( z_0 \in \mathbb{C} \) is arbitrary, \( g_2 = 5184 \) and \( g_3 = 0 \).

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