ON SEPARABILITY OF SCHUR RINGS OVER ABELIAN $p$-GROUPS

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Abstract. An $S$-ring (Schur ring) is called separable with respect to a class of $S$-rings $\mathcal{K}$ if it is determined up to isomorphism in $\mathcal{K}$ only by the tensor of its structure constants. An abelian group is said to be separable if every $S$-ring over this group is separable with respect to the class of $S$-rings over abelian groups. Let $C_n$ be a cyclic group of order $n$ and $G$ be a noncyclic abelian $p$-group. From the previously obtained results it follows that if $G$ is separable then $G$ is isomorphic to $C_p \times C_p^k$ or $C_p \times C_p \times C_p^k$, where $p \in \{2, 3\}$ and $k \geq 1$. We prove that the groups $D = C_p \times C_p^k$ are separable whenever $p \in \{2, 3\}$. From this statement we deduce that a given Cayley graph over $D$ and a given Cayley graph over an arbitrary abelian group one can check whether these graphs are isomorphic in time $|D|^{O(1)}$.

Keywords: Cayley graphs, Cayley graph isomorphism problem, Cayley schemes, Schur rings, permutation groups.

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Let $G$ be a finite group. A subring of the group ring $\mathbb{Z}G$ is called an $S$-ring (a Schur ring) over $G$ if it is a free $\mathbb{Z}$-module spanned on a special partition of $G$ (exact definitions are given in Section 1). The elements of this partition are called the basic sets of the $S$-ring and the number of the basic sets is called the rank of the $S$-ring. The notion of the $S$-ring goes back to Schur and Wielandt. They used “the $S$-ring method” to study a permutation group having a regular subgroup [1, 2].

Let $A$ and $A'$ be $S$-rings over groups $G$ and $G'$ respectively. A (combinatorial) isomorphism from $A$ to $A'$ is defined to be a bijection $f : G \to G'$ such that for every basic set $X$ of $A$ the set $X' = f(X)$ is a basic set of $A'$ and $f$ is an isomorphism of the Cayley graphs Cay$(G, X)$ and Cay$(G', X')$. An algebraic isomorphism from $A$ to $A'$ is defined to be a ring isomorphism of them inducing the bijection between the basic sets of $A$ and the basic sets of $A'$. It can be checked that every combinatorial isomorphism induces an algebraic isomorphism. If every algebraic isomorphism from $A$ to another $S$-ring is induced by a combinatorial isomorphism then $A$ is said to be separable. More precisely, we say that $A$ is separable with respect to a class of $S$-rings if every algebraic isomorphism from $A$ to an $S$-ring from this class is induced by a combinatorial isomorphism. Every separable $S$-ring is determined up to isomorphism only by the tensor of its structure constants. For more details see [3, 4].

Denote the classes of $S$-rings over cyclic and abelian groups by $\mathcal{K}_C$ and $\mathcal{K}_A$ respectively. A cyclic group of order $n$ is denoted by $C_n$. It was proved in [3] that every

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S-ring over a cyclic $p$-group is separable with respect to $\mathcal{K}_C$. Infinite series of S-rings over cyclic groups that are nonseparable with respect to $\mathcal{K}_C$ were constructed in [4].

An abelian group $G$ is said to be separable if every S-ring over $G$ is separable with respect to $\mathcal{K}_A$. One can prove that a subgroup of a separable group is separable. [5, Section 3] implies that the group $H \times H$ is nonseparable for every group $H$ of order at least 4. Let $G$ be a noncyclic abelian $p$-group. It follows that if $G$ is separable then $G$ is isomorphic to $C_p \times C_{p^k}$ or $C_p \times C_p \times C_{p^k}$, where $p \in \{2, 3\}$ and $k \geq 1$. In the present paper we prove that the groups from the first family are separable. We will consider the question on a separability of the groups from the second family in a more general context in the next paper.

**Theorem 1.** The group $C_p \times C_{p^k}$ is separable for $p \in \{2, 3\}$ and $k \geq 1$.

The proof of Theorem 1 is based on the description of all S-rings over the group $D$ from this theorem that was obtained for $p = 2$ in [6] and for $p = 3$ in [7]. If $\mathcal{A}$ is an S-ring over $D$ of rank at least 3 then one of the following statements holds: (1) $\mathcal{A}$ is the tensor product or the generalized wreath product of two smaller S-rings; (2) $\mathcal{A}$ is cyclotomic (it means that $\mathcal{A}$ is determined by a suitable subgroup of Aut($D$)). The detailed description of S-rings over $D$ is given in Lemma 6.2 and Lemma 6.3. The separability of tensor products and generalized wreath products follows from the separability of operands. The most difficult task here is to check that cyclotomic S-rings are separable (see Section 7).

There is a relationship between the separability of S-rings and the isomorphism problem for Cayley graphs (see [8, Section 6.2]). In the case when all S-rings over a group of order $n$ are separable the isomorphism problem for Cayley graphs over this group can be solved in time $n^{O(1)}$ by using the Weisfeiler-Leman algorithm [9, 10]. By Theorem 1 this implies (see Section 8) the following statement.

**Theorem 2.** Suppose that the group $D \cong C_p \times C_{p^k}$, where $p \in \{2, 3\}$ and $k \geq 1$, is given explicitly. Then for every Cayley graph $\Gamma$ over $D$ and every Cayley graph $\Gamma'$ over an arbitrary explicitly given abelian group one can check in time $|D|^{O(1)}$ whether $\Gamma$ and $\Gamma'$ are isomorphic.

It should be mentioned that the isomorphism problem for Cayley graphs over cyclic groups was solved in [11] and [12] independently.

In Sections 1-3 we recall some definitions and facts concerned with S-rings and Cayley schemes. The most part of them is taken from [13].

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1. S-rings and Cayley schemes

Let $G$ be a finite group and $\mathbb{Z}G$ be the integer group ring. Denote the identity element of $G$ by $e$. If $X \subseteq G$ then the formal sum $\sum_{x \in X} x$ is considered as an element of $\mathbb{Z}G$ and denoted by $X$. The set $\{x^{-1} : x \in X\}$ is denoted by $X^{-1}$.

A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an S-ring over $G$ if there exists a partition $S = S(\mathcal{A})$ of $G$ such that:
1) \( \{e\} \in S \).
2) if \( X \in S \) then \( X^{-1} \in S \).
3) \( \mathcal{A} = \text{Span}_Z \{X : X \in S\} \).

The elements of \( S \) are called the basic sets of \( \mathcal{A} \) and the number \( |S| \) is called the rank of \( \mathcal{A} \). If \( X, Y, Z \in S \) then the number of distinct representations of \( z \in Z \) in the form \( z = x y \) with \( x \in X \) and \( y \in Y \) is denoted by \( c^Z_{X,Y} \). Note that if \( X \) and \( Y \) are basic sets of \( \mathcal{A} \) then \( \sum_{Z \in S(\mathcal{A})} c^Z_{X,Y} Z \). Therefore the numbers \( c^Z_{X,Y} \) are structure constants of \( \mathcal{A} \) with respect to the basis \( \{X : X \in S\} \).

Let \( \mathcal{R} \) be a partition of \( G \times G \). A pair \( \mathcal{C} = (G, \mathcal{R}) \) is called a Cayley scheme over \( G \) if the following properties hold:

1) \( \text{Diag}(G \times G) = \{(g, g) : g \in G\} \in \mathcal{R} \);
2) if \( R \in \mathcal{R} \) then \( R^* = \{(h, g) : (g, h) \in R\} \in \mathcal{R} \);
3) if \( R, S, T \in \mathcal{R} \) then the number \( c^S_{R,S} = |\{h \in G : (g, h) \in R, (h, f) \in S\}| \) does not depend on the choice of \( (g, f) \in T \);
4) \( \{(hg, fg) : (h, f) \in R\} = R \) for every \( R \in \mathcal{R} \) and every \( g \in G \).

The numbers \( c^S_{R,S} \) are called the intersection numbers of \( \mathcal{C} \), the elements of \( \mathcal{R} \) are called the basic relations of \( \mathcal{C} \), and the number \( |\mathcal{R}| \) is called the rank of \( \mathcal{C} \). If \( R \in \mathcal{R} \) and \( g \in G \) then the number \( n(R) = \{h : (g, h) \in R\} \) does not depend on the choice of \( g \) and it is called the valency of \( R \). Note that \( n(R) = c^S_{R,R^*}, \) where \( S = \text{Diag}(G \times G) \).

There is a one-to-one correspondence between \( S \)-rings and Cayley schemes over \( G \). Namely, if \( \mathcal{A} \) is an \( S \)-ring over \( G \) then the pair \( \mathcal{C}(\mathcal{A}) = (G, \mathcal{R}(\mathcal{A})) \), where \( \mathcal{R}(\mathcal{A}) = \{R(X) : X \in S(\mathcal{A})\} \) and \( R(X) = \{(g, xg) : g \in G, x \in X\} \), is a Cayley scheme over \( G \). Conversely, if \( \mathcal{C} = (G, \mathcal{R}) \) is a Cayley scheme over \( G \) then \( S(\mathcal{C}) = \{X(R) : R \in \mathcal{R}\} \), where \( X(R) = \{x \in G : (e, x) \in R\} \subseteq G \), is a partition of \( G \) that defines the \( S \)-ring \( \mathcal{A}(\mathcal{C}) \) over \( G \). If \( \mathcal{A} \) is an \( S \)-ring and \( \mathcal{C}(\mathcal{A}) \) is the corresponding Cayley scheme then

\[
c^Z_{R^*(X), R^*(Y)} = c^Z_{X,Y} \quad (1)
\]

for all \( X, Y, Z \in S(\mathcal{A}) \).

**§ 2. ISOMORPHISMS**

Let \( \mathcal{A} \) and \( \mathcal{A}' \) be \( S \)-rings over groups \( G \) and \( G' \) respectively and \( \mathcal{C} = (G, \mathcal{R}) \) and \( \mathcal{C}' = (G', \mathcal{R}') \) be Cayley schemes over \( G \) and \( G' \) respectively. Set \( S = S(\mathcal{A}) \) and \( S' = S(\mathcal{A}') \). A (combinatorial) isomorphism from \( \mathcal{C} \) to \( \mathcal{C}' \) is defined to be a bijection \( f : G \rightarrow G' \) such that \( \mathcal{R}' = \mathcal{R}' \), where \( \mathcal{R}' = \{R' : R \in \mathcal{R}\} \) and \( R' = \{(g^f, h^f) : (g, h) \in R\} \). A (combinatorial) isomorphism from \( \mathcal{A} \) to \( \mathcal{A}' \) is defined to be a bijection \( f : G \rightarrow G' \) which is an isomorphism of the corresponding Cayley schemes \( \mathcal{C}(\mathcal{A}) \) and \( \mathcal{C}(\mathcal{A}') \).

The group \( \text{Iso}(\mathcal{A}) \) of all isomorphisms from \( \mathcal{A} \) onto itself has a normal subgroup

\[
\text{Aut}(\mathcal{A}) = \{f \in \text{Iso}(\mathcal{A}) : R(X)^f = R(X) \text{ for every } X \in S(\mathcal{A})\}.
\]

This subgroup is called the automorphism group of \( \mathcal{A} \) and denoted by \( \text{Aut}(\mathcal{A}) \); the elements of \( \text{Aut}(\mathcal{A}) \) are called automorphisms of \( \mathcal{A} \).

An algebraic isomorphism from \( \mathcal{A} \) to \( \mathcal{A}' \) is defined to be a bijection \( \varphi : S \rightarrow S' \) such that \( c^Z_{X,Y} = c^{Z\varphi}_{X\varphi,Y\varphi} \) for all \( X, Y, Z \in S \). The mapping \( X \rightarrow X^\varphi \) is extended by linearity to the ring isomorphism of \( \mathcal{A} \) and \( \mathcal{A}' \). An algebraic isomorphism from \( \mathcal{C} \) to \( \mathcal{C}' \) is defined
to be a bijection $\varphi : \mathcal{R} \to \mathcal{R}'$ such that $c_{R,S}^{T} = c_{R',S'}^{T'}$ for all $R, S, T \in \mathcal{R}$. If $\varphi$ is an algebraic isomorphism of $A$ and $A'$ then the mapping $R(X) \mapsto R(X')$ is an algebraic isomorphism of the corresponding Cayley schemes $\mathcal{C}(A)$ and $\mathcal{C}(A')$ by (1).

Every isomorphism $f$ of $S$-rings (Cayley schemes) preserves the structure constants (intersection numbers) and hence $f$ induces the algebraic isomorphism $\varphi_f$.

We say that a Cayley scheme $\mathcal{C}$ is separable with respect to a class of Cayley schemes $\mathcal{K}$ if for every $\mathcal{C}' \in \mathcal{K}$ every algebraic isomorphism $\varphi : \mathcal{C} \to \mathcal{C}'$ is induced by an isomorphism. The next statement immediately follows from (1).

**Proposition 2.1.** If $A$ is an $S$-ring and $\mathcal{C}(A)$ is the corresponding Cayley scheme then $A$ is separable with respect to a class of $S$-rings $\mathcal{K}$ if and only if $\mathcal{C}(A)$ is separable with respect to the class of Cayley schemes corresponding to $S$-rings from $\mathcal{K}$.

For a fixed algebraic isomorphism $\varphi$ from $A$ to $A'$ (from $\mathcal{C}$ to $\mathcal{C}'$) the set of all isomorphisms $f$ such that $\varphi_f = \varphi$ is denoted by $\text{Iso}(A, A', \varphi)$ (iso($\mathcal{C}$, $\mathcal{C}'$, $\varphi$)) . If $H$ is a group then put $H_{\text{right}} = \{x \mapsto xh, \ x \in H : h \in H\}$. From the definitions it follows that

$$G_{\text{right}} \text{Iso}(A, A', \varphi)G_{\text{right}}' = \text{Iso}(A, A', \varphi).$$

(2)

Note that an $S$-ring $A$ is separable with respect to a class of $S$-rings $\mathcal{K}$ if and only if

$$\text{Iso}(A, A', \varphi) \neq \emptyset$$

for every $A' \in \mathcal{K}$ and every algebraic isomorphism $\varphi : A \to A'$. The group ring $\mathbb{Z}G$ and the $S$-ring of rank 2 over $G$ are separable with respect to the class of all $S$-rings. In the former case every basic set is singleton and hence every algebraic isomorphism is induced by an isomorphism in a natural way. In the latter case there exists the unique algebraic isomorphism from the $S$-ring of rank 2 over $G$ to the $S$-ring of rank 2 over a given group and this algebraic isomorphism is induced by every isomorphism.

Another type of isomorphism between $S$-rings (Cayley schemes) arises from group isomorphism. A Cayley isomorphism from $A$ to $A'$ (from $\mathcal{C}$ to $\mathcal{C}'$) is defined to be a group isomorphism $f : G \to G'$ such that $S' = S' (\mathcal{R}' = \mathcal{R}')$. If there exists a Cayley isomorphism from $A$ to $A'$ we write $A \cong_{\text{Cay}} A'$. Every Cayley isomorphism is a (combinatorial) isomorphism, however the converse statement is not true.

§ 3. $S$-rings: basic facts and constructions

Let $A$ be an $S$-ring over a group $G$. A set $X \subseteq G$ is called an $A$-set if $X \in A$. A subgroup $H \leq G$ is called an $A$-subgroup if $H$ is an $A$-set. Let $L \subseteq U \leq G$. A section $U/L$ is called an $A$-section if $U$ and $L$ are $A$-subgroups. If $S = U/L$ is an $A$-section then the module

$$A_S = \text{Span}_{\mathbb{Z}} \{X^\pi : X \in S(A), \ X \subseteq U\},$$

where $\pi: U \to U/L$ is the quotient homomorphism, is an $S$-ring over $S$.

If $X \subseteq G$ then the set

$$\text{rad}(X) = \{g \in G : Xg = gX = X\}$$
is a subgroup of \( G \) and it is called the **radical** of \( X \). If \( X \) is an \( \mathcal{A} \)-set then the groups \( \langle X \rangle \) and \( \text{rad}(X) \) are \( \mathcal{A} \)-subgroups of \( G \).

Let \( \mathcal{A} \) and \( \mathcal{A}' \) be \( S \)-rings over \( G \) and \( G' \) respectively and \( \varphi : \mathcal{A} \to \mathcal{A}' \) be an algebraic isomorphism. It is easy to see that \( \varphi \) is extended to a bijection between \( \mathcal{A} \)- and \( \mathcal{A}' \)-sets and hence between \( \mathcal{A} \)- and \( \mathcal{A}' \)-sections. The images of an \( \mathcal{A} \)-set \( X \) and an \( \mathcal{A} \)-section \( S \) under the action of \( \varphi \) are denoted by \( X^{\varphi} \) and \( S^{\varphi} \) respectively. If \( S \) is an \( \mathcal{A} \)-section then \( \varphi \) induces the algebraic isomorphism

\[
\varphi_S : \mathcal{A}_S \to \mathcal{A}'_S,
\]

where \( S' = S^{\varphi} \). Since \( c_{X,Y}^{(e)} = \delta_{Y,X^{-1}} |X| \) and \( |X| = c_{X,X^{-1}}^{(e)} \), where \( \delta_{X,X^{-1}} \) is the Kronecker delta, we conclude that \( (X^{-1})^{\varphi} = (X^{\varphi})^{-1} \) and \( |X| = |X^{\varphi}| \) for every \( \mathcal{A} \)-set \( X \). In particular, \( |G| = |G'| \). It can be checked that

\[
\langle X^{\varphi} \rangle = \langle X \rangle^{\varphi}, \quad \text{rad}(X^{\varphi}) = \text{rad}(X)^{\varphi}
\]

for every \( \mathcal{A} \)-set \( X \) (see [14, p.10]).

We say that an \( S \)-ring \( \mathcal{A} \) is **symmetric** if \( X = X^{-1} \) for every \( X \in \mathcal{S}(\mathcal{A}) \). Clearly that if \( \mathcal{A} \) is symmetric and \( \varphi : \mathcal{A} \to \mathcal{A}' \) is an algebraic isomorphism then \( \mathcal{A}' \) is also symmetric.

If \( X \subseteq G \) and \( m \in \mathbb{Z} \) then the set \( \{ x^m : x \in X \} \) is denoted by \( X^{(m)} \). Sets \( X,Y \subseteq G \) are called **rationally conjugate** if there exists \( m \in \mathbb{Z} \) coprime to \( |G| \) such that \( Y = X^{(m)} \). Further we formulate two statements on \( S \)-rings over abelian groups that were proved, in fact, by Schur in [1]. We give these statements in the form which can be found in [2, Section 23].

**Lemma 3.1.** Let \( \mathcal{A} \) be an \( S \)-ring over an abelian group \( G \). Then \( X^{(m)} \in \mathcal{S}(\mathcal{A}) \) for every \( X \in \mathcal{S}(\mathcal{A}) \) and every \( m \in \mathbb{Z} \) coprime to \( |G| \). Other words, the mapping \( \sigma_m : g \mapsto g^m \) is a Cayley isomorphism from \( \mathcal{A} \) onto itself.

**Lemma 3.2.** Let \( \mathcal{A} \) be an \( S \)-ring over an abelian group \( G \), \( p \) be a prime divisor of \( |G| \), and \( H = \{ g \in G : g^p = e \} \). Then for every \( \mathcal{A} \)-set \( X \) the set \( X^{[p]} = \{ x^p : x \in X, |X \cap Hx| \neq 0 \mod p \} \) is an \( \mathcal{A} \)-set.

Let \( K \leq \text{Aut}(G) \). Then the set \( \text{Orb}(K,G) \) of all orbits of \( K \) on \( G \) forms a partition of \( G \) that defines an \( S \)-ring \( \mathcal{A} \) over \( G \). In this case \( \mathcal{A} \) is called **cyclotomic** and denoted by \( \text{Cyc}(K,G) \).

Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be \( S \)-rings over \( G_1 \) and \( G_2 \) respectively. Then the set

\[
\mathcal{S} = \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2) = \{ X_1 \times X_2 : X_1 \in \mathcal{S}(\mathcal{A}_1), X_2 \in \mathcal{S}(\mathcal{A}_2) \}
\]

forms a partition of \( G = G_1 \times G_2 \) that defines an \( S \)-ring over \( G \). This \( S \)-ring is called the **tensor product** of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) and denoted by \( \mathcal{A}_1 \otimes \mathcal{A}_2 \).

Let \( e_1 \) and \( e_2 \) be the identity elements of \( G_1 \) and \( G_2 \) respectively. Then the set

\[
\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2, \quad \text{where}
\]

\[
\mathcal{S}_1 = \{ X_1 \times \{ e_2 \} : X_1 \in \mathcal{S}(\mathcal{A}_1) \}, \quad \mathcal{S}_2 = \{ G_1 \times \{ X_2 \} : X_2 \in \mathcal{S}(\mathcal{A}_2) \setminus \{ e_2 \} \}
\]
forms a partition of $G = G_1 \times G_2$ that defines an $S$-ring over $G$. This $S$-ring is called the \textit{wreath product} of $A_1$ and $A_2$ and denoted by $A_1 \wr A_2$.

**Lemma 3.3.** $S$-rings $A_1 \otimes A_2$ and $A_1 \downarrow A_2$ are separable if and only if $S$-rings $A_1$ and $A_2$ are separable.

\textit{Proof.} Follows from [15, Theorem 1.20]. \hfill \Box

**Lemma 3.4.** [3, Lemma 2.1] Let $A$ and $A'$ be $S$-rings over $G$ and $G'$ respectively. Let $B$ be the $S$-ring generated by $A$ and an element $\xi \in \mathbb{Z}G$ and $B'$ be the $S$-ring generated by $A'$ and an element $\xi' \in \mathbb{Z}G'$. Then given algebraic isomorphism $\varphi : A \rightarrow A'$ there is at most one algebraic isomorphism $\psi : B \rightarrow B'$ extending $\varphi$ and such that $\psi(\xi) = \xi'$.

Following [16] we say that an $S$-ring $A$ is quasi-thin if $|X| \leq 2$ for every $X \in S(A)$.

**Lemma 3.5.** Let $A$ be a quasi-thin $S$-ring over $G$. Suppose that there are no $A$-subgroups $H$ in $G$ such that $H \cong C_2 \times C_2$, $A_H = \mathbb{Z}H$, and $A_{G/H} = \mathbb{Z}(G/H)$. Then $A$ is separable with respect to the class of all $S$-rings.

\textit{Proof.} If $A$ is quasi-thin then the corresponding Cayley scheme $\mathcal{C}(A)$ is also quasi-thin, i.e. every basic relation of $\mathcal{C}(A)$ has valency at most 2. From [16, Theorem 1.1] it follows that every quasi-thin Cayley scheme which is not a Klein scheme (see [16, p.2]), is separable with respect to the class of all Cayley schemes. If $\mathcal{C}(A)$ is a Klein scheme then there exists an $A$-subgroup $H$ of $G$ such that $H \cong C_2 \times C_2$, $A_H = \mathbb{Z}H$, and $A_{G/H} = \mathbb{Z}(G/H)$. However, this contradicts to the assumption of the lemma. Therefore $\mathcal{C}(A)$ and $A$ are separable with respect to the class of all Cayley schemes and the class of all $S$-rings respectively. \hfill \Box

\section*{4. Generalized wreath product}

Let $A$ be an $S$-ring over $G$ and $U/L$ be an $A$-section. We say that $A$ is the \textit{generalized wreath product} or \textit{U/L-wreath product} if $L \unlhd G$ and $L \leq \text{rad}(X)$ for every $X \in S(A)$ outside $U$. The generalized wreath product is called \textit{nontrivial} or \textit{proper} if $L \neq 1$ and $U \neq G$. If $U = L$ then $A$ coincides with the wreath product of $A_L$ and $A_{G/L}$.

The main goal of this section is to prove a sufficient condition of separability for the generalized wreath product over an abelian group. Before this we formulate some additional statements required for the proof.

If $f : G \rightarrow G'$ is a bijection and $X \subseteq G$ then the restriction of $f$ on $X$ is denoted by $f^X$. Let $H \leq G$. Given $X,Y \in G/H$ put

$$G_{X \rightarrow Y} = \{f^X : f \in G_{\text{right}}, X^f = Y\}.$$

In the following three lemmas $A$ and $A'$ are $S$-rings over groups $G$ and $G'$ respectively and $\varphi : A \rightarrow A'$ is an algebraic isomorphism.

**Lemma 4.1.** [14, Lemma 3.4] If $f \in \text{Iso}(A,A',\varphi)$, $H$ is an $A$-subgroup of $G$, and $H' = H^\varphi$ then

$$h f^X h'^X \in \text{Iso}(A_H,A'_H,\varphi_H)$$

for all $X \in G/H$, $h \in G_{H \rightarrow X}$, and $h' \in G'_{X' \rightarrow H'}$, where $X' = X^f$. 

Lemma 4.2. \cite{14} Theorem 3.3, 1] Let \( G \) and \( G' \) be abelian and \( U/L \) be an \( A \)-section of \( G \). Suppose that \( A \) is the \( U/L \)-wreath product, \( U' = U^φ \), and \( L' = L^φ \). Then \( A' \) is the \( U'/L' \)-wreath product.

Lemma 4.3. \cite{14} Theorem 3.5] In the conditions of Lemma 4.2 the set \( \text{Iso}(A, A', ϕ) \) consists of all bijections \( f : G → G' \) possessing the following properties:

1) \((G/U)^f = G'/U'\), \((G/L)^f = G'/L'\),
2) \(f^{G/L} ∈ \text{Iso}(A_{G/L}, A'_{G'/L'}, ϕ_{G/L})\),
3) if \( X ∈ G/U \) and \( X' ∈ G'/U' \), then there exist \( g ∈ G_{U→X} \) and \( g' ∈ G'_{X'→U'} \) such that \( gf^Xg' ∈ \text{Iso}(A_U, A'_U, ϕ_U)\).

Lemma 4.4. Let \( A \) be the \( U/L \)-wreath product over an abelian group \( G \). Suppose that \( A_U \) and \( A_{G/L} \) are separable with respect to \( K_A \) and \( \text{Aut}(A_U)^{U/L} = \text{Aut}(A_{U/L}). \) Then \( A \) is separable with respect to \( K_A \).

Proof. Let \( A' \) be an \( S \)-ring over an abelian group \( G' \), \( ϕ : A → A' \) be an algebraic isomorphism, and \( U' = U^φ, L' = L^φ \). Then from Lemma 4.2 it follows that \( A' \) is the \( U'/L' \)-wreath product. Since \( A_U \) and \( A_{G/L} \) are separable with respect to \( K_A \), the algebraic isomorphisms

\[
ϕ_U : A_U → A'_U, \quad ϕ_{G/L} : A_{G/L} → A'_{G'/L'}
\]

are induced by some isomorphisms \( f_1 \) and \( f_2 \) respectively. Let \( X ∈ G/U \). We can consider \( X \) as a subset of \( G/L \) because \( X \) is a union of some cosets of \( L \). Put \( X' = X^f_2 \). Choose \( g ∈ G_{U→X} \) and \( g' ∈ G'_{X'→U'} \). The bijection \( g'^{U/L}f_2^{X/L}g^X/L' \) induces the algebraic isomorphism \( ϕ_{U/L} \) by Lemma 4.2 applied to \( A_{G/L} \)-subgroup \( U/L \) of \( G/L \). Put

\[
f_0 = g'^{U/L}f_2^{X/L}g^X/L'(f_1^{U/L})^{-1}.
\]

Since \( f_1^{U/L} \) also induces \( ϕ_{U/L}, \) we conclude that \( f_0 ∈ \text{Aut}(A_{U/L}) \). There exists \( h_X ∈ \text{Aut}(A_U) \) such that \( h_X^{U/L} = f_0 \) because \( \text{Aut}(A_{U/L}) = \text{Aut}(A_U)^{U/L}. \) Put

\[
f_X = g^{-1}h_Xf_1(g')^{-1}.
\]

Let \( f : G → G' \) be the bijection whose restriction on \( X \) coincides with \( f_X \) for every \( X ∈ G/U \). Let us check that \( f \) possesses Properties 1-3 from Lemma 4.3. It is clear that \((G/U)^f = G'/U'\) and \((G/L)^f = G'/L'\) and hence \( f \) possesses Property 1. By the definition of \( f \) we have \( gf^Xg' = g_{fX}g' = h_Xf_1 \). So from (2) it follows that for every \( X ∈ G/U \) the bijection \( gf^Xg' \) induces the algebraic isomorphism \( ϕ_U \). It proves that \( f \) possesses Property 3. The straightforward computations show that

\[
f^{X/L} = (g^{-1}h_Xf_1(g')^{-1})^{X/L} = (g^{U/L})^{-1}g^{U/L}f_2^{X/L}g^X/L'(f_1^{U/L})^{-1}f_1^{U/L}(g^X/L')^{-1} = f_2^{X/L}
\]

for every \( X ∈ G/U \). Therefore \( f^{G/L} = f_2 \) and \( f^{G/L} \) induces \( ϕ_{G/L} \). So \( f \) possesses Property 2. Thus \( f ∈ \text{Iso}(A, A', ϕ) \) by Lemma 4.3.

It is worth noting that in the proof of this lemma we followed, in general, the scheme of the proof of separability of coset \( S \)-rings over cyclic groups (\cite{14} p.33)).
§ 5. S-rings over cyclic groups

Lemma 5.1. Let $G$ be a cyclic group of order $n \neq 4$ and $A$ be an $S$-ring over $G$ such that $A = \mathbb{Z}G$ or $A = \text{Cyc}(K, G)$, where $K = \{ \varepsilon, \sigma \}$, $\sigma : x \to x^{-1}$. Suppose that $\varphi$ is an algebraic isomorphism from $A$ to an $S$-ring $A'$ over an abelian group $G'$. Then $G' \cong G$.

Proof. From the properties of an algebraic isomorphism it follows that $|G| = |G'| = n$. Suppose that $X \in \mathcal{S}(A)$ contains a generator of $G$. Then $G' = \langle X^\varphi \rangle$ by (3). If $|X| = 1$ then $|X^\varphi| = 1$ and hence $G'$ is cyclic. If $|X| = 2$ then $X = X^{-1}$. So $|X^\varphi| = 2$ and $(X^\varphi)^{-1} = X^\varphi$ by the properties of an algebraic isomorphism. Therefore either $X^\varphi = \{x, x^{-1}\}$ for some $x \in G'$ or $X^\varphi = \{x, y\}$ for some $x, y \in G'$ such that $|x| = |y| = 2$. In the former case $G'$ is cyclic and hence it is isomorphic to $G$; in the latter case $|G| = |G'| = 4$ that contradicts to the assumption of the lemma.

Let $A$ be an $S$-ring over a cyclic group $G$. Put $\text{rad}(A) = \text{rad}(X)$, where $X$ is a basic set of $A$ containing a generator of $G$. Note that $\text{rad}(A)$ does not depend on the choice of $X$. Indeed, if $Y \in \mathcal{S}(A)$, $\langle Y \rangle = G$, and $Y \neq X$ then $X$ and $Y$ are rationally conjugate by Theorem 3.1 and hence $\text{rad}(X) = \text{rad}(Y)$.

Lemma 5.2. Let $p$ be a prime and $A$ be an $S$-ring over a cyclic $p$-group $G$. Suppose that $\text{rad}(A) > e$. Then there exists an $A$-section $U/L$ such that $A$ is the proper $U/L$-wreath product and $\text{rad}(A_U) = e$.

Proof. Let $X$ be the union of all basic sets of $A$ with the trivial radical. Then $U = \langle X \rangle$ is a proper $A$-subgroup and $\text{rad}(A_U) = e$. There exists the least nontrivial $A$-subgroup $L$ of $G$ because $G$ is a cyclic $p$-group. All basic sets of $A$ outside $U$ have nontrivial radical. Since the radical of every basic set is an $A$-subgroup, we conclude that $L \leq \text{rad}(X)$ for every $X \in \mathcal{S}(A)$ outside $U$. Thus $A$ is the proper $U/L$-wreath product.

Let $p \in \{2, 3\}$ and $k \geq 1$. Set $A = \langle a \rangle$ and $a_1 = a^{p^{k-1}}$, where $|a| = p^k$. These notations are valid until the end of this section.

Lemma 5.3. Let $A$ be an $S$-ring over $A$. Suppose that $\text{rad}(A) = e$. Then one of the following statements holds:

1) $\text{rk}(A) = 2$;
2) $A = \mathbb{Z}A$;
3) $A = \text{Cyc}(K, A)$, where $K = \{ \varepsilon, \sigma \}$, $\sigma : x \to x^{-1}$;
4) $p = 2$ and $A = \text{Cyc}(K, A)$, where $K = \{ \varepsilon, \sigma \}$, $\sigma : x \to a_1 x^{-1}$.

In all cases $A$ is separable with respect to the class of all $S$-rings.

Proof. From [17] Theorem 4.1 and [17] Theorem 4.2 it follows that every $S$-ring with the trivial radical over a cyclic group is the tensor product of cyclotomic $S$-rings with the trivial radical and $S$-rings of rank 2. Since $A$ is a $p$-group, we conclude that either $\text{rk}(A) = 2$ or $A = \text{Cyc}(K, A)$ for some $K \leq \text{Aut}(A)$. In the former case it is obvious that $A$ is separable. In the latter case [18] Lemma 5.1] implies that one of Statements 2-4 holds. In particular, $A$ is quasi-thin. The group $A$ is cyclic and hence it does not contain $A$-subgroups $H$ such that $H \cong C_2 \times C_2$. Therefore $A$ is separable by Lemma 3.5.
Denote the symmetric group of a set $V$ by $\text{Sym}(V)$. If $\Gamma \leq \text{Sym}(V)$ then denote the set of all orbits of the componentwise action of $\Gamma$ on $V^2$ by $\text{Orb}(\Gamma, V^2)$. Permutation groups $\Gamma$, $\Gamma' \leq \text{Sym}(V)$ are called 2-equivalent if $\text{Orb}(\Gamma, V^2) = \text{Orb}(\Gamma', V^2)$. A permutation group $\Gamma \leq \text{Sym}(V)$ is called 2-isolated if it is the only group which is 2-equivalent to $\Gamma$.

**Lemma 5.4.** Let $\mathcal{A}$ be the proper $U/L$-wreath product over $A$ and $\text{rad}(\mathcal{A}_U) = e$. Then $\text{Aut}(\mathcal{A}_U)^{U/L} = \text{Aut}(\mathcal{A}_{U/L})$.

**Proof.** To prove the lemma it is sufficient to prove that $\text{Aut}(\mathcal{A}_{U/L})$ is 2-isolated. Indeed, the orbits of the componentwise action of the groups $\text{Aut}(\mathcal{A}_{U/L})$ and $\text{Aut}(\mathcal{A}_U)^{U/L}$ on $(U/L)^2$ coincide with the basic relations of the Cayley scheme corresponding to $\mathcal{A}_{U/L}$. This implies that $\text{Aut}(\mathcal{A}_{U/L})$ and $\text{Aut}(\mathcal{A}_U)^{U/L}$ are 2-equivalent. So if $\text{Aut}(\mathcal{A}_{U/L})$ is 2-isolated then $\text{Aut}(\mathcal{A}_U)^{U/L} = \text{Aut}(\mathcal{A}_{U/L})$.

Since $\text{rad}(\mathcal{A}_U) = e$, one of the statements of Lemma 5.3 holds for $\mathcal{A}_U$. If $\text{rk}(\mathcal{A}_U) = 2$ then $U = L$ and obviously $\text{Aut}(\mathcal{A}_{U/L})$ is 2-isolated. If one of Statements 2-4 of Lemma 5.3 holds for $\mathcal{A}_U$ then $\mathcal{A}_{U/L} = \mathbb{Z}(U/L)$ or every basic set of $\mathcal{A}_{U/L}$ is of the form $\{x, x^{-1}\}$, $x \in U/L$. Therefore the stabilizer of $L$ in $\text{Aut}(\mathcal{A}_{U/L})$ has a faithful regular orbit and hence $\text{Aut}(\mathcal{A}_{U/L})$ is 2-isolated by [6, Lemma 8.2].

Note that the above lemma does not hold for cyclic $p$-groups, where $p > 3$.

In [3] it was proved that every $S$-ring over a cyclic $p$-group, where $p$ is a prime, is separable with respect to $\mathcal{K}_C$. Further we prove that all $S$-rings over cyclic 2- and 3-groups are separable with respect to $\mathcal{K}_A$.

**Lemma 5.5.** Let $\mathcal{A}$ be an $S$-ring over $A$. Then $\mathcal{A}$ is separable with respect to $\mathcal{K}_A$.

**Proof.** We proceed by induction on $k$. If $k = 1$ then the statement of the lemma holds because $\text{rk}(\mathcal{A}) = 2$ or $\mathcal{A} = \mathbb{Z}A$. Now let $k \geq 2$. If $\text{rad}(\mathcal{A}) = e$ then $\mathcal{A}$ is separable by Lemma 5.3. Suppose that $\text{rad}(\mathcal{A}) > e$. Then from Lemma 5.2 it follows that $\mathcal{A}$ is the proper $U/L$-wreath product for some $\mathcal{A}$-section $U/L$ such that $\text{rad}(\mathcal{A}_U) = e$. By the induction hypothesis $S$-rings $\mathcal{A}_U$ and $\mathcal{A}_{A/L}$ are separable with respect to $\mathcal{K}_A$. Lemma 5.4 yields that $\text{Aut}(\mathcal{A}_U)^{U/L} = \text{Aut}(\mathcal{A}_{U/L})$. Thus all conditions of Lemma 4.4 hold for $\mathcal{A}$ and hence $\mathcal{A}$ is separable with respect to $\mathcal{K}_A$. $\square$

§ 6. S-RINGS OVER $C_p \times C_p^k$

Let $p \in \{2, 3\}$ and $k \geq 1$. Put $D = A \times B$, where $A = \langle a \rangle$, $|a| = p^k$, $B = \langle b \rangle$, $|b| = p$. Let $a_1 = a^{p^{k-1}}$ and $a_2 = a^{p^{k-2}}$. If $l \leq k$ then denote the subgroups $\{g \in A : |g| \leq p^l\}$ and $\{g \in D : |g| \leq p^l\}$ of $D$ by $A_l$ and $D_l$ respectively. In these notations $A = A_k$ and $D = D_k$.

In the next lemma we describe sections of $D$ such that $D$ is determined up to isomorphism by these sections.

**Lemma 6.1.** Let $q$ be a prime, $m \geq 3$, and $D'$ be an abelian group of order $q^{m+1}$. Suppose that the following conditions hold:

1) $D'$ contains at least two subgroups of order $q^{m-1}$ and one of them, say $A'$, is cyclic;
2) $A'$ contains a subgroup $A'_1$ of order $q$ such that $D'/A'_1$ is isomorphic to $C_q \times C_{q^{m-1}}$. 
Then $D'$ is isomorphic to $C_q \times C_{q^m}$ or $C_{q^2} \times C_{q^{m-1}}$. Moreover, if $m \geq 4$ or $D'$ contains a noncyclic subgroup $W'$ of order $q^2$ such that $|W' \cap A'| = q$ and $D'/W'$ is cyclic then $D' \cong C_q \times C_{q^m}$.

Proof. Since $D'$ is abelian, it is the direct product of cyclic groups. Moreover, $D'$ is isomorphic to one of the following groups

$$C_{q^{m+1}}, C_q \times C_{q^m}, C_{q^2} \times C_{q^{m-1}}, C_q \times C_q \times C_{q^{m-1}}$$

because $A'$ is the cyclic group of order $q^{m-1}$. Note that $D'$ is noncyclic because $D'$ contains at least two subgroups of order $q^{m-1}$. Suppose that $D' \cong C_q \times C_q \times C_{q^{m-1}}$. Then $D' = H' \times A'$, where $H' \cong C_q \times C_q$. If $A'_1 \leq A'$ has order $q$ then $D'/A'_1$ contains a subgroup isomorphic to $C_q \times C_q \times C_q$ because $m \geq 3$. We obtain a contradiction with $D'/A'_1 \cong C_q \times C_{q^{m-1}}$. Therefore $D' \cong C_q \times C_{q^m}$ or $D' \cong C_{q^2} \times C_{q^{m-1}}$.

If $X \subseteq G \times H$ then denote the projections of $X$ on $G$ and $H$ by $pr_G(X)$ and $pr_H(X)$ respectively. Prove the second part of the lemma. Suppose that $D' = H' \times A'$, where $H' \cong C_{q^2}$. If $m \geq 4$ then $D'/A'_1$ contains a subgroup isomorphic to $C_{q^2} \times C_{q^2}$ and we obtain a contradiction with $D'/A'_1 \cong C_q \times C_{q^{m-1}}$. If there is a noncyclic subgroup $W'$ of order $q^2$ in $D'$ such that $|W' \cap A'| = q$ and $D'/W'$ is cyclic then $|pr_{H'}(W')| = q$ because $|W' \cap A'| = q$. Since $W'$ is noncyclic, we have $|pr_{H'}(W')| = q$. So $W' = pr_{H'}(W') \times pr_{A'}(W') \cong C_q \times C_q$. This implies that $D'/W'$ is noncyclic, a contradiction with the assumption of the lemma. Thus $D' \cong C_q \times C_{q^m}$. \hfill $\square$

Let $A$ be an $S$-ring over $D$. A basic set $X \in S(A)$ is called highest if it contains an element of order $p^k$. By the radical of $A$ we mean the subgroup $\text{rad}(A)$ generated by the subgroups $\text{rad}(X)$, where $X$ runs over all highest basic sets of $A$. A subset of $D$ is called regular if it consists of elements of the same order.

The description of all $S$-rings over $D$ was obtained for $p = 2$ in [6] and for $p = 3$ in [7].

Lemma 6.2. If $A$ is an $S$-ring over $D$ and $k = 1$ then one of the following statements holds:

1) $\text{rk}(A) = 2$;
2) $A$ is the tensor product of two $S$-rings over cyclic groups of order $p$;
3) $A$ is the wreath product of two $S$-rings over cyclic groups of order $p$;
4) $p = 3$ and $A = \text{Cyc}(K, D)$, where $K = \{e, \delta\}$, $\delta : x \mapsto x^{-1}$;
5) $p = 3$ and $A \cong \text{Cay} \text{Cyc}(K, D)$, where $K = \{\sigma\}$ and $\sigma : (a_1, b) \mapsto (b, a_1^2)$.

Proof. Follows from the computer calculations that made by using the package COCO2P [19]. \hfill $\square$

Lemma 6.3. Let $A$ be an $S$-ring over $D$ and $k \geq 2$. Then one of the following statements holds:

1) $\text{rad}(A) = e$ and there exist $A$-subgroups $L, H \leq D$ such that $A = A_H \otimes A_L$, $\text{rk}(A_H) = 2$, and $|L| \leq p \leq |H|$;
2) $\text{rad}(A) > e$ and there exist an $A$-section $U/L$ such that $A$ is the proper $U/L$-wreath product. Moreover, $A_{U/L} = \mathbb{Z}(U/L)$, or $|U/L| \leq 4$, or $\text{rad}(A_U) = e$ and $|L| = p$;
3) \( \text{rad}(A) = e \) and \( A \cong \text{C} \text{yc}(K,D) \), where \( K \leq \text{Aut}(D) \) is one of the groups listed in Table 1 for \( p = 2 \) and in Table 2 for \( p = 3 \).

| group | generators | order | \( k \) |
|-------|------------|-------|-------|
| \( K_0 \) | \((a,b) \to (a,b)\) | 1 | \( k \geq 2 \) |
| \( K_1 \) | \((a,b) \to (a^k,b)\) | 2 | \( k \geq 3 \) |
| \( K_2 \) | \((a,b) \to (a_1,a^{-1},b)\) | 2 | \( k \geq 3 \) |
| \( K_3 \) | \((a,b) \to (a^{-1},b a_1)\) | 2 | \( k \geq 3 \) |
| \( K_4 \) | \((a,b) \to (a_1 a^{-1}, b a_1)\) | 2 | \( k \geq 3 \) |
| \( K_5 \) | \((a,b) \to (b a_2 a, b a_1), (a,b) \to (a_1^{-1},b)\) | 4 | \( k \geq 4 \) |
| \( K_6 \) | \((a,b) \to (b a_2 a, b a_1), (a,b) \to (a_1 a^{-1},b)\) | 4 | \( k \geq 4 \) |
| \( K_7 \) | \((a,b) \to (b a^{-1}_2, b)\) | 2 | \( k \geq 4 \) |
| \( K_8 \) | \((a,b) \to (b a_1 a^{-1}, b)\) | 2 | \( k \geq 4 \) |
| \( K_9 \) | \((a,b) \to (b a_2 a, b a_1)\) | 2 | \( k \geq 3 \) |
| \( K_{10} \) | \((a,b) \to (b a_2 a^{-1}, b a_1)\) | 2 | \( k \geq 4 \) |

Table 1.

| group | generators | order | \( k \) |
|-------|------------|-------|-------|
| \( K_0 \) | \((a,b) \to (a,b)\) | 1 | \( k \geq 2 \) |
| \( K_1 \) | \((a,b) \to (a,b^2)\) | 2 | \( k \geq 2 \) |
| \( K_2 \) | \((a,b) \to (a^{-1},b)\) | 2 | \( k \geq 2 \) |
| \( K_3 \) | \((a,b) \to (a^{-1},b), (a,b) \to (a,b^2)\) | 4 | \( k \geq 2 \) |
| \( K_4 \) | \((a,b) \to (a^{-1},b^2)\) | 2 | \( k \geq 2 \) |
| \( K_5 \) | \((a,b) \to (b a b a_1)\) | 6 | \( k \geq 3 \) |
| \( K_6 \) | \((a,b) \to (b a_2 b a_1)\) | 6 | \( k \geq 3 \) |
| \( K_7 \) | \((a,b) \to (b a_1 b a_1), (a,b) \to (a,b^2 a_1)\) | 6 | \( k \geq 3 \) |
| \( K_8 \) | \((a,b) \to (b a_1 b a_1), (a,b) \to (a^{-1},b a_1)\) | 6 | \( k \geq 3 \) |
| \( K_9 \) | \((a,b) \to (b a_2 b a_1), (a,b) \to (a^{-1},b^2)\) | 6 | \( k \geq 3 \) |

Table 2.

**Proof.** The lemma summarizes the statements of [6] Theorem 6.1, Theorem 7.1, Theorem 9.1] in case \( p = 2 \) and the statements of [7] Theorem 4.1, Theorem 5.1, Theorem 6.1] in case \( p = 3 \). \( \square \)

**Lemma 6.4.** Let \( A \) be an \( S \)-ring over \( D \) and Statement 2 of Lemma \( \ref{lem:2} \) holds for \( A \). Then \( \text{Aut}(A_U/U/L) = \text{Aut}(A_U/L) \).

**Proof.** To prove the lemma we show that \( \text{Aut}(A_U/U/L) = \text{Sym}(U/L) = \text{Aut}(A_U/U/L) \). In the former case \( \text{Aut}(A_U/U/L) = \text{Aut}(A_U/L) \) because \( \text{Aut}(A_U/L) \) and \( \text{Aut}(A_U/U/L) \) are 2-equivalent (see the proof of Lemma \( \ref{lem:2} \)).

If \( A_U/U/L = \mathbb{Z}(U/L) \) or \( |U/L| \leq 4 \) then it is obvious that \( \text{Aut}(A_U/L) \) is 2-isolated. Further we assume that \( \text{rad}(A_U) = e \) and \( |L| = p \). Suppose that \( U \) is cyclic. Then \( L = A_1 \) is the unique \( A \)-subgroup of order \( p \) and one of the statements of Lemma \( \ref{lem:2} \) holds for \( A_U \). If \( \text{rk}(A_U) = 2 \) then \( U = L \) and hence \( \text{Aut}(A_U/L) \) is 2-isolated. If one of Statements 2-4 of Lemma \( \ref{lem:2} \) holds for \( A_U \), then \( A_{U/L} = \mathbb{Z}(U/L) \) or every basic set of \( A_{U/L} \) is of the form \( \{ x, x^{-1} \} \), \( x \in U/L \). So the stabilizer of \( L \) in \( \text{Aut}(A_{U/L}) \) has a faithful regular orbit and \( \text{Aut}(A_{U/L}) \) is 2-isolated by [6] Lemma 8.2.

Suppose now that \( U \) is noncyclic. Then \( U \cong D_l \) for some \( l \leq k \). If \( A_U \) is regular then \( \text{Aut}(A_{U/L}) \) is 2-isolated by [6] Theorem 8.1] for \( p = 2 \) and by [7] Corollary 5.2] for \( p = 3 \).
If $\mathcal{A}_U$ is nonregular then Lemma 6.3 implies that $\mathcal{A}_U = \mathcal{A}_H \otimes \mathcal{A}_L$, where $\text{rk}(\mathcal{A}_H) = 2$. Note that $\text{rk}(\mathcal{A}_L) = 2$ or $\mathcal{A}_L = \mathbb{Z}L$ since $|L| = p$ and $p \in \{2, 3\}$. If $\text{rk}(\mathcal{A}_L) = 2$ then

$$\text{Sym}(U/L) \geq \text{Aut}(\mathcal{A}_{U/L}) \geq \text{Aut}(\mathcal{A}_U)^{U/L} = (\text{Sym}(H) \times \text{Sym}(L))^{U/L} = \text{Sym}(U/L);$$

if $\mathcal{A}_L = \mathbb{Z}L$ then

$$\text{Sym}(U/L) \geq \text{Aut}(\mathcal{A}_{U/L}) \geq \text{Aut}(\mathcal{A}_U)^{U/L} = (\text{Sym}(H) \times L_{\text{right}})^{U/L} = \text{Sym}(U/L).$$

Thus in both cases $\text{Aut}(\mathcal{A}_{U/L}) = \text{Sym}(U/L) = \text{Aut}(\mathcal{A}_U)^{U/L}$. \hfill $\square$

§ 7. Proof of Theorem 1

All notations from the previous section are valid throughout this section.

Let $\mathcal{A}$ be an arbitrary $S$-ring over $D$. Let us prove that $\mathcal{A}$ is separable with respect to $\mathcal{K}_\mathcal{A}$. From now on throughout this section we write for short “separable” instead “separable with respect to $\mathcal{K}_\mathcal{A}$”. Let $\mathcal{A}'$ be an $S$-ring over an abelian group $D'$ and $\varphi : \mathcal{A} \to \mathcal{A}'$ be an algebraic isomorphism. We proceed by induction on $k$. Let $k = 1$. Then one of the statements of Lemma 6.2 holds for $\mathcal{A}$. If $\text{rk}(\mathcal{A}) = 2$ then, obviously, $\mathcal{A}$ is separable. If $\mathcal{A}$ is the tensor product or the wreath product of two $S$-rings over cyclic groups of order $p$ then $\mathcal{A}$ is separable by Lemma 3.3. If Statement 4 of Lemma 6.2 holds for $\mathcal{A}$ then $\mathcal{A}$ satisfies the conditions of Lemma 3.5 and hence $\mathcal{A}$ is separable.

Now suppose that Statement 5 of Lemma 6.2 holds for $\mathcal{A}$. Then $|D'| = |D| = 9$ and $\text{rk}(\mathcal{A}') = \text{rk}(\mathcal{A}) = 3$. From (3) it follows that $\text{rad}(\mathcal{A}')$ is trivial since $\text{rad}(\mathcal{A}) = e$. If $D'$ is cyclic then Lemma 5.3 yields that $\text{rk}(\mathcal{A}') = 2$ or $\mathcal{A}'$ and $\mathcal{A}$ are quasi-thin that is not true. So $D'$ is noncyclic and hence $D' \cong D$. Further we assume that $D' = D$. From Lemma 6.2 it follows that $\mathcal{A}$ is the unique up to Cayley isomorphism $S$-ring of rank 3 over $D$ with basic sets of cardinalities $1, 4, 4$. Therefore $\mathcal{A}' \cong \text{Cay} \mathcal{A}$. Let $X = \{x, x^{-1}, y, y^{-1}\}$ be a nontrivial basic set of $\mathcal{A}$ and $X^e = \{x', (x')^{-1}, y', (y')^{-1}\}$. Then the Cayley isomorphism

$$\sigma : (x, y) \to (x', y') \in \text{Aut}(D)$$

induces $\varphi$.

Now let $k \geq 2$. Then one of Statements 1-3 of Lemma 6.3 holds for $\mathcal{A}$. Every $S$-ring over the group of order $p$, where $p \in \{2, 3\}$, is separable. So if $\mathcal{A} = \mathcal{A}_H \otimes \mathcal{A}_L$, where $\text{rk}(\mathcal{A}_H) = 2$ and $|L| \leq p$, then $\mathcal{A}$ is of rank 2 and hence separable or $\mathcal{A}$ is separable by Lemma 3.3.

Suppose that Statement 2 of Lemma 6.3 holds for $\mathcal{A}$. From Lemma 6.4 it follows that

$$\text{Aut}(\mathcal{A}_U)^{U/L} = \text{Aut}(\mathcal{A}_{U/L}).$$

So by Lemma 1.1 it is sufficient to prove a separability of $\mathcal{A}_U$ and $\mathcal{A}_{D/L}$. If $U = D_l$ for some $l < k$ then $\mathcal{A}_U$ is separable by the induction hypothesis. If $U$ is cyclic then $\mathcal{A}_U$ is separable by Lemma 5.5. Similarly, $\mathcal{A}_{D/L}$ is separable by the induction hypothesis whenever $D/L$ is noncyclic and by Lemma 5.5 whenever $D/L$ is cyclic.

Suppose that Statement 3 of Lemma 6.3 holds for $\mathcal{A}$. Then $\mathcal{A} \cong \text{Cay} \text{Cyc}(K, D)$, where $K \leq \text{Aut}(D)$ is one of the groups listed in Table 1 for $p = 2$ and in Table 2 for $p = 3$. If $K = K_0$ then $\mathcal{A} = \mathbb{Z}D$ is separable. If $p = 2$ and $K \in \{K_1, K_2, K_3, K_4, K_7, K_8, K_9, K_{10}\}$ or $p = 3$ and $K \in \{K_1, K_2, K_4, K_5\}$ then $\mathcal{A}$ is quasi-thin and it is easy to check directly that there are no $\mathcal{A}$-subgroups $H$ in $D$ such that $H \cong C_2 \times C_2$ and $\mathcal{A}_{D/H} = \mathbb{Z}(D/H)$.
So in these cases \( A \) is separable by Lemma 3.3. If \( p = 3 \) and \( K = K_3 \) then \( A \) is the tensor product of two quasi-thin \( S \)-rings over cyclic \( 3 \)-groups and hence \( A \) is separable by Lemma 3.3 and Lemma 3.3.

Consider the remaining cases. Let \( p = 2 \) and \( K \in \{ K_5, K_6 \} \) or \( p = 3 \) and \( K \in \{ K_6, K_7, K_8, K_9 \} \).

**Lemma 7.1.** \( D' \cong D \).

*Proof.* Let us check that \( D' \) satisfies the conditions of Lemma 6.1. At first consider the case \( p = 2 \).

1. The inequality \( k \geq 4 \) holds because \( K \in \{ K_5, K_6 \} \) (see Table 1).
2. Note that \( \{ bu, bu^{-1} \} \in S(A) \) for every \( u \in A_{k-1} \setminus A_{k-2} \) as \( K \in \{ K_5, K_6 \} \). Choose a basic set \( Y \subseteq b(A_{k-1} \setminus A_{k-2}) \). The group \( F = \langle Y \rangle \) is a cyclic \( A \)-subgroup of order \( 2^{k-1} \) and \( A_F = \text{Cyc}(M, F) \), where \( M = \{ \varepsilon, \sigma \} \) and \( \sigma : x \to x^{-1} \). Since \( k \geq 4 \), we conclude that \( |F| > 4 \). Clearly that \( \varphi \) induces the algebraic isomorphism

\[
\varphi_F : A_F \to A_{F' \varphi}.
\]

From Lemma 5.1 it follows that \( F' \varphi \) is a cyclic subgroup of \( D' \) of order \( 2^{k-1} \).

3. The group \( D_{k-2} \) is an \( A \)-subgroup of order \( 2^{k-1} \) distinct from \( F \). So \( D_2 \) is an \( A' \)-subgroup of order \( 2^{k-1} \) distinct from \( F' \).

4. The group \( A_2 \) is an \( A \)-subgroup of order 2. So \( A_{2} \) is an \( A' \)-subgroup of order 2. Let \( \pi : D \to D/A_1 \) be the quotient epimorphism and \( X \) be a highest basic set of \( A \). Then \( X = \{ x, x^{-1}, ba_2x, ba_2^{-1}x^{-1} \} \) whenever \( K = K_5 \) and \( X = \{ x, a_1x^{-1}, ba_2x, ba_2x^{-1} \} \) whenever \( K = K_6 \) for some generator \( x \) of \( A \). The set \( \pi(X) \) is a generating set of \( D/A_1 \) and the following properties hold

\[
|\pi(X)| = 4, \ \pi(X) = \pi(X)^{-1}, \ |\text{rad}(\pi(X))| = 2.
\]

Let

\[
\varphi_{D/A_1} : A_{D/A_1} \to A_{D'/A_1}.
\]

be the algebraic isomorphism induced by \( \varphi \). From (3) it follows that \( \pi(X)^{\varphi_{D/A_1}} \) is a generating set of \( D'/A_1 \) and (4) also holds for \( \pi(X)^{\varphi_{D/A_1}} \). Let \( \pi(X)^{\varphi_{D/A_1}} = \{ x', b x', y', b y' \} \), where \( e, b' \) is \( \text{rad}(\pi(X)^{\varphi_{D/A_1}}) \). If \( (x')^{-1} = b x' \) then \( (x')^2 = (y')^2 = b' \) and hence \( |D'/A_1| = 8 \). So \( |D| = |D'| = 16 \). We obtain a contradiction because \( k \geq 4 \) and \( |D| \geq 32 \). Therefore we may assume that \( y = (x')^{-1} \). This implies that \( D'/A_1 \) is generated by at most two elements one of which has order 2. Note that \( D'/A_1 \) is noncyclic because it contains at least two subgroups \( A_2/A_1 \) and \( D'/A_1 \) of order 2. We conclude that \( D'/A_1 \cong C_2 \times C_{2^{k-1}} \). Thus \( D' \cong D \cong C_2 \times C_{2^{k}} \) by Lemma 6.1.

Now let us check that the conditions of Lemma 6.1 hold for \( D' \) whenever \( p = 3 \).

1. Since \( K \in \{ K_6, K_7, K_8, K_9 \} \), the group \( A_{k-1} \) is a cyclic \( A \)-subgroup of order \( 3^{k-1} \). Moreover, \( A_{A_{k-1}} = \mathbb{Z}, \) whenever \( K \in \{ K_6, K_7 \} \) and \( A_{A_{k-1}} = \text{Cyc}(M, A_{k-1}) \), where \( M = \{ \varepsilon, \sigma \} \), \( \sigma : x \to x^{-1} \), whenever \( K \in \{ K_8, K_9 \} \). Clearly that \( \varphi \) induces the algebraic isomorphism

\[
\varphi_{A_{k-1}} : A_{A_{k-1}} \to A_{(A_{k-1})' \varphi}.
\]

Lemma 5.1 yields that \( A_{k-1}' \) is a cyclic \( A \)-subgroup of order \( 3^{k-1} \).

2. The group \( D_{k-2} \) is an \( A' \)-subgroup of order \( 3^{k-1} \) distinct from \( A_{k-1}' \).
3. Note that $A_{1}^{ϕ}$ is an $A'$-subgroup of order 3. Let $π : D → D/A_{1}$ be the quotient epimorphism and $X$ be a highest basic set of $A$. If $K ∈ \{K_{6}, K_{7}\}$ then $X = \{x, bx, b^{2}a_{1}x\}$ and if $K ∈ \{K_{8}, K_{9}\}$ then $X = \{x, x^{-1}, bx, b^{2}x^{-1}, b^{2}a_{1}^2x, ba_{1}x^{-1}\}$ for some generator $x$ of $A$. The set $π(X)$ is a generating set of $D/A_{1}$,

\[|π(X)| = 3, \ |rad(π(X))| = 3,\]

whenever $K ∈ \{K_{6}, K_{7}\}$, and

\[|π(X)| = 6, \ π(X) = π(X)^{-1}, \ |rad(π(X))| = 3,\]

whenever $K ∈ \{K_{8}, K_{9}\}$. Let

\[ϕ_{D/A_{1}} : A_{D/A_{1}} → A'_{D/A_{1}}^{ϕ}\]

be the algebraic isomorphism induced by $ϕ$. From the properties of an algebraic isomorphism it follows that $π(X)^{ϕ_{D/A_{1}}}$ is a generating set of $D'/A_{1}^{ϕ}$, (5) holds for $π(X)^{ϕ_{D/A_{1}}}$ if $K ∈ \{K_{6}, K_{7}\}$, and (6) holds for $π(X)^{ϕ_{D/A_{1}}}$ if $K ∈ \{K_{8}, K_{9}\}$. Since $k ≥ 3$, we conclude that $π(X)^{ϕ_{D/A_{1}}} = x'B'$ or $π(X)^{ϕ_{D/A_{1}}} = x'B' \cup (x')^{-1}B'$, where $B' = rad(π(X)^{ϕ_{D/A_{1}}})$. Therefore $D'/A_{1}^{ϕ}$ is generated by at most two elements one of which has order 3. Note that $D'/A_{1}^{ϕ}$ is noncyclic because it contains at least two subgroups $A_{2}^{ϕ}/A_{1}^{ϕ}$ and $D_{1}^{ϕ}/A_{1}^{ϕ}$ of order 3. This implies that $D'/A_{1}^{ϕ} ≅ C_{3} × C_{3k-1}$.

By the first part of Lemma 6.1 the group $D'$ is isomorphic to $C_{3} × C_{3k}$ or $C_{9} × C_{3k-1}$. If $k ≥ 4$ then $D' ≅ C_{3} × C_{3k}$ by the second part of Lemma 6.1. Now let $k = 3$. Put $D'_{1} = \{x ∈ D' : |x| = 3\}$. Clearly that $|D'_{1}| = 9$. Suppose that $D'_{1}$ is an $A'$-subgroup. Then $D'_{1} = D_{1}^{ϕ}$ or $D'_{1} = A_{2}^{ϕ}$ since only the groups $D_{1}$ and $A_{2}$ are $A$-subgroups of order 9. However, $A_{2}^{ϕ}$ is cyclic by Lemma 5.1. So $D'_{1} = D_{1}^{ϕ}$. The group $D/D_{1}$ is cyclic, $A_{D/D_{1}} = Z(D/D_{1})$ or $A_{D/D_{1}} = Cyc(M, D/D_{1})$, where $M = \{ε, σ\}$, $σ : x → x^{-1}$. Therefore the group $D'/D_{1}$ is also cyclic by Lemma 5.1. Note that $|D_{1} \cap A_{2}| = 3$ and hence $|D'_{1} \cap A_{2}^{ϕ}| = 3$. Thus $D' ≅ D ≅ C_{3} × C_{9}$ by the second part of Lemma 6.1.

Suppose that $D' ≅ C_{9} × C_{9}$. Then by the above discussion $D'_{1}$ is not an $A'$-subgroup. The group $D_{2}$ is an $A$-subgroup as $A$ is regular. So $D_{2}^{ϕ}$ is an $A'$-subgroup and $D' \setminus D_{2}^{ϕ}$ is an $A'$-set. Since $|D_{2}| = |D_{2}^{ϕ}| = 27$, the inclusion $D_{1}^{ϕ} \subset D_{2}^{ϕ}$ holds. Let $X ⊆ D \setminus D_{2}$ be a highest basic set of $A$. Then $|X| ∈ \{3, 6\}$, rad($X$) is trivial, $＜X⟩ = D$, and if $|X| = 6$ then $X = x^{-1}$. These properties also hold for $X^{ϕ}$.

Suppose that $|xD'_{1} \cap X^{ϕ}| = 3$, where $x' ∈ X^{ϕ}$. If $|X^{ϕ}| = 3$ then $Y' = X^{ϕ}(x'^{-1})$ is an $A'$-set and $Y' ⊆ D'_{1}$. Moreover, $Y' ⊄ A_{1}^{ϕ}$ since otherwise rad($X^{ϕ}$) = $A_{1}^{ϕ}$. So $D'_{1} = A_{1}^{ϕ}$ is an $A'$-subgroup, a contradiction. Let $|X^{ϕ}| = 6$. If $((x')^{2} ∪ (x')^{-2})D_{1}^{ϕ} \cap D_{2}^{ϕ} = ∅$ then $((x')^{2} ∪ (x')^{-2})D'_{1} \subset D_{2}^{ϕ}$ and hence $x' ∈ D_{2}^{ϕ}$. On the other hand, $x' ∈ D' \setminus D_{2}^{ϕ}$, a contradiction. Therefore $((x')^{2} ∪ (x')^{-2})D_{1} \cap D_{2}^{ϕ} = ∅$. This implies that $Y' = X^{ϕ} \cap D_{2}^{ϕ}$ is an $A'$-set and $Y' ⊆ D'_{1}$. Note that $Y' \not∈ A_{1}^{ϕ}$ because otherwise rad($X^{ϕ}$) = $A_{1}^{ϕ}$. We conclude that $D'_{1} = A_{1}^{ϕ}$, a contradiction. So $|xD'_{1} \cap X^{ϕ}| ≠ 3$ for every highest basic set $X ∈ S(A)$. Then $Y' = (X^{ϕ})^{[3]}$ is an $A'$-set by Lemma 3.2. Since $D' ≅ C_{9} × C_{9}$, we obtain that $Y' ⊆ D'_{1}$. If $Y' \not∈ A_{1}^{ϕ}$ then $D'_{1} = A_{1}^{ϕ}$, a contradiction. Therefore $(X^{ϕ})^{[3]} ⊆ A_{1}^{ϕ}$ for every highest basic set $X$. The union of all highest basic sets of $A$ has cardinality 54. So $|\{x ∈ D' : x^{3} ∈ A_{1}^{ϕ}\}| ≥ 54$. 

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Lemma 7.3. \( A \) is not isomorphic to \( C_9 \times C_9 \) and hence \( D' \cong D \cong C_3 \times C_9 \).

Further we may assume without loss of generality that \( D = D' \).

Lemma 7.2. \( \mathcal{A}' \cong_{\text{Cay}} \mathcal{A} \).

Proof. One of the statements of Lemma 6.3 holds for \( \mathcal{A}' \). Since \( \text{rad}(\mathcal{A}) \) is trivial, from (3) it follows that \( \text{rad}(\mathcal{A}') \) is also trivial. So Statement 2 of Lemma 6.3 does not hold for \( \mathcal{A}' \). Clearly that \( |\mathcal{S}(\mathcal{A}')| = |\mathcal{S}(\mathcal{A})| > 6 \). This implies that Statement 1 of Lemma 6.3 does not hold for \( \mathcal{A}' \). Thus Statement 3 of Lemma 6.3 holds for \( \mathcal{A}' \). It means that \( \mathcal{A}' \cong_{\text{Cay}} \text{Cyc}(K', D) \), where \( K' \leq \text{Aut}(D) \) is one of the groups listed in Table 1 for \( p = 2 \) and in Table 2 for \( p = 3 \).

At first consider the case \( p = 2 \). There are basic sets of \( \mathcal{A} \) of cardinality 4. So \( \mathcal{A}' \) also has basic sets of cardinality 4. This yields that \( \mathcal{A}' \) is not quasi-thin. Therefore \( K' \in \{K_5, K_6\} \). Not that \( \text{Cyc}(K_5, D) \) is symmetric whereas \( \text{Cyc}(K_6, D) \) is not symmetric. If \( \mathcal{A} \) is symmetric then by the properties of an algebraic isomorphism \( \mathcal{A}' \) is also symmetric and hence \( \mathcal{A}' \cong_{\text{Cay}} \text{Cyc}(K_5, D) \cong_{\text{Cay}} \mathcal{A} \). If \( \mathcal{A} \) is not symmetric then \( \mathcal{A}' \) is also not symmetric and \( \mathcal{A}' \cong_{\text{Cay}} \text{Cyc}(K_6, D) \cong_{\text{Cay}} \mathcal{A} \).

Now let \( p = 3 \). Given an \( S \)-ring \( \mathcal{B} \) put

\[
\mathcal{N}(\mathcal{B}) = \{|X| : X \in \mathcal{S}(\mathcal{B})\}.
\]

It is clear that \( \mathcal{N}(\mathcal{B}) \) is invariant under algebraic isomorphisms. Therefore the statement of the lemma follows from the next observation: \( \mathcal{B} = \text{Cyc}(K_i, D) \) is the unique up to Cayley isomorphism cyclotomic \( S \)-ring over \( D \) such that

1) \( \mathcal{N}(\mathcal{B}) = \{1, 3\} \) if \( i = 6 \);
2) \( \mathcal{N}(\mathcal{B}) = \{1, 3, 6\} \) if \( i = 7 \);
3) \( \mathcal{N}(\mathcal{B}) = \{1, 2, 3, 6\} \) if \( i = 8 \);
4) \( \mathcal{N}(\mathcal{B}) = \{1, 2, 6\} \) if \( i = 9 \).

Let \( X \in \mathcal{S}(\mathcal{A}) \) be a highest basic set. If \( p = 3 \) and \( K \in \{K_6, K_8\} \) then put \( Z = bA_1 \) and \( Y = X \cup Z \); otherwise put \( Y = X \).

Lemma 7.3. \( \mathcal{A} = \langle Y \rangle \) and \( \mathcal{A}' = \langle Y^e \rangle \).

Proof. Put \( \mathcal{A}_1 = \langle Y \rangle \). From Lemma 6.3 it follows that \( X \) contains a generator \( x \) of \( \mathcal{A} \) and

\[
X = \{x, x^{-1}, ba_2x, ba_2^{-1}x^{-1}\},
\]

if \( p = 2 \) and \( K = K_5 \);

\[
X = \{x, a_1x^{-1}, ba_2x, ba_2x^{-1}\},
\]

if \( p = 2 \) and \( K = K_6 \);

\[
X = \{x, bx, b^2a_1x\},
\]

if \( p = 3 \) and \( K \in \{K_6, K_7\} \);

\[
X = \{x, x^{-1}, bx, b^2x^{-1}, b^2a_1^2x, ba_1x^{-1}\},
\]

if \( p = 3 \) and \( K \in \{K_8, K_9\} \).

Statement 1 of Lemma 6.3 does not hold for \( \mathcal{A}_1 \) because otherwise every element of order \( p^k \) must lie in a basic set of cardinality at least \( p^k - 1 \), where \( k \geq 4 \) if \( p = 2 \) and
$k \geq 3$ if $p = 3$. It is easy to check that every subset of $Y$ consisting of elements of order $p^k$ has the trivial radical. So $A_1$ has a highest basic set with the trivial radical and Statement 2 of Lemma 6.3 does not hold for $A_1$. Thus $A_1$ is cyclotomic. From the classification of all cyclotomic $S$-rings over $D$ that is given in Lemma 6.3 it follows that $A_1 = A$. It should be mentioned that if $p = 3$ then $\text{Cyc}(K_6, D)$ and $\text{Cyc}(K_7, D)$ as well as $\text{Cyc}(K_8, D)$ and $\text{Cyc}(K_9, D)$ have the same highest basic sets. In the cases when $K \in \{K_7, K_9\}$ the $S$-ring $A$ is generated by the highest basic set $X$ and in the cases when $K \in \{K_6, K_8\}$ the $S$-ring $A$ is generated by $X \cup Z$.

From (3) it follows that $X^\varphi$ is a highest basic set of $A'$. If $p = 3$ then there are exactly two subgroups $A_2$ and $D_1$ of order 9 in $D$. These subgroups are $A$-subgroups. The group $A_2^\varphi$ is a cyclic $A'$-subgroup by Lemma 5.1 and hence $A_2^\varphi = A_2$. So $D_1 = D_1$. If $K \in \{K_6, K_8\}$ then $Z^\varphi \in \{Z, Z^{-1}\}$ because only $Z$ and $Z^{-1}$ are basic sets of cardinality 3 inside $D_1$. Thus if $Y = X \cup Z$ then $Y^\varphi = X^\varphi \cup Z$ or $Y^\varphi = X^\varphi \cup Z^{-1}$.

Since $A' \cong_{\text{Cay}} A$, by using the above arguments it can be proved that $A' = (Y^\varphi)$. □

**Lemma 7.4.** The algebraic isomorphism $\varphi$ is induced by a Cayley isomorphism.

**Proof.** Lemma 7.2 implies that there exists a Cayley isomorphism $f$ from $A$ to $A'$. The sets $X^\varphi$ and $X^f$ are highest basic sets of $A'$. If $p = 2$ then every highest basic set of $A'$ is of the form

$$X_0 \cup bX_1,$$

where $X_0 \subseteq A$, $X_1 \neq \emptyset$, $i \in \{0, 1\}$, because $K \in \{K_5, K_6\}$. Similarly, if $p = 3$ then every highest basic set of $A'$ is of the form

$$X_0 \cup bX_1 \cup b^2X_2,$$

where $X_0 \subseteq A$, $X_1 \neq \emptyset$, $i \in \{0, 1, 2\}$. So by Lemma 3.1 the sets $X^\varphi$ and $X'^f$ are rationally conjugate and there exists a Cayley isomorphism $g$ from $A'$ onto itself such that $X'^fg = X^\varphi$. The Cayley isomorphism $fg$ from $A$ to $A'$ induces the algebraic isomorphism $\varphi_{fg}$. If $p = 2$ or $p = 3$ and $K \in \{K_7, K_9\}$ then $A = (X)$ and $A' = (X^\varphi)$ by Lemma 7.3. In these cases from Lemma 3.4 it follows that $\varphi = \varphi_{fg}$.

Now let $p = 3$ and $K \in \{K_6, K_8\}$. From Lemma 7.3 it follows that $A = (Y)$ and $A' = (Y^\varphi)$. If $Z'^fg = Z^\varphi$ then $Y'^fg = Y^\varphi$ and hence Lemma 3.4 implies that $\varphi = \varphi_{fg}$. Suppose that $Z'^fg \neq Z^\varphi$. Without loss of generality we may assume that $Z'^fg = \{b, ba_1, ba_3\}$ and $Z^\varphi = \{b^2, b^2a_1, b^2a_3^2\}$. If $K = K_6$ then $X^\varphi = \{y, by, b^2a_1y\}$ or $X^\varphi = \{y, ba_1y, b^2y\};$ if $K = K_8$ then $X^\varphi = \{y, y^{-1}, by, ba_1y^{-1}, b^2a_1y, b^2y^{-1}\}$, where $y$ is a generator of $A$. Put

$$h : (y, b) \rightarrow (y, b^2a_1) \in \text{Aut}(D)$$

if $K = K_6$ and

$$h : (y, b) \rightarrow (y, b^2a_1^2) \in \text{Aut}(D)$$

if $K = K_8$. The straightforward check shows that $X'^fg = (X^\varphi)^h = X^\varphi$ and $Z'^fg = Z^\varphi$. Since $X^\varphi$ is a highest basic set of the cyclotomic $S$-ring $(A')^h$, we conclude that $(A')^h = A'$. Therefore $fg$ is a Cayley isomorphism from $A$ to $A'$ such that $Y'^fg = Y^\varphi$. From Lemma 3.4 it follows that $\varphi = \varphi_{fg}$, where $\varphi_{fg}$ is the algebraic isomorphism induced by $fg$. □
Thus if \( A = \text{Cyc}(K, D) \), where \( K \in \{ K_5, K_6 \} \) whenever \( p = 2 \) and \( K \in \{ K_6, K_7, K_8, K_9 \} \) whenever \( p = 3 \), every algebraic isomorphism of \( A \) is induced by a Cayley isomorphism. So \( A \) is separable and the proof of Theorem 1 is complete.

§ 8. Separability and the isomorphism problem for Cayley graphs

Let \( \Gamma = \text{Cay}(G, X) \) and \( \Gamma' = \text{Cay}(G', X') \) be Cayley graphs over groups \( G \) and \( G' \) respectively. Denote the set of all isomorphisms from \( \Gamma \) to \( \Gamma' \) by \( \text{Iso}(\Gamma, \Gamma') \). Fix classes of groups \( \mathcal{K} \) and \( \mathcal{K}' \). The isomorphism problem for Cayley graphs can be formulated as follows.

**ISO.** Given Cayley graphs \( \Gamma \) over \( G \in \mathcal{K} \) and \( \Gamma' \) over \( G' \in \mathcal{K}' \) determine whether \( \text{Iso}(\Gamma, \Gamma') \neq \emptyset \).

Further we consider the reduction of ISO to the following problem:

**ALISO.** Given Cayley schemes \( \mathcal{C} \) over \( G \in \mathcal{K} \) and \( \mathcal{C}' \) over \( G' \in \mathcal{K}' \) and an algebraic isomorphism \( \varphi : \mathcal{C} \to \mathcal{C}' \) determine whether \( \text{Iso}(\mathcal{C}, \mathcal{C}', \varphi) \neq \emptyset \).

**Proposition 8.1.** ISO is reduced to ALISO in time \( |G|^{O(1)} \).

**Proof.** Suppose that there is an algorithm \( Al_1 \) solving ALISO. We assume that \( |G| = |G'| = n \) since otherwise, obviously, \( \Gamma \) and \( \Gamma' \) are not isomorphic. Denote the sets of edges of \( \Gamma \) and \( \Gamma' \) by \( E \) and \( E' \) respectively. Let

\[
\mathcal{T} = (\text{Diag}(G \times G), E, G \times G \setminus (E \cup \text{Diag}(G \times G))
\]

and

\[
\mathcal{T}' = (\text{Diag}(G' \times G'), E', G' \times G' \setminus (E' \cup \text{Diag}(G' \times G'))
\]

be the corresponding to \( \Gamma \) and \( \Gamma' \) ordered partitions of \( G \times G \) and \( G' \times G' \). By using the Weisfeiler-Leman algorithm \([9, 10]\) we can construct in time \( n^{O(1)} \) starting from \( \mathcal{T} \) and \( \mathcal{T}' \) the ordered partitions \( \mathcal{R} = (P_1, P_2, \ldots, P_k) \) and \( \mathcal{R}' = (Q_1, Q_2, \ldots, Q_l) \) defining Cayley schemes \( \mathcal{C} \) and \( \mathcal{C}' \) over \( G \) and \( G' \) respectively. These schemes are the least schemes for which \( E \) and \( E' \) are unions of basic relations.

If \( f \in \text{Iso}(\Gamma, \Gamma') \) then by properties of the Weisfeiler-Leman algorithm \( k = l \), \( f \) is an isomorphism from \( \mathcal{C} \) to \( \mathcal{C}' \) such that \( P_i^f = Q_i \), \( i = 1, \ldots, k \), and hence the bijection \( \varphi : P_i \to Q_i \), \( i = 1, \ldots, k \), is an algebraic isomorphism. Conversely, if \( \varphi : P_i \to Q_i \) is an algebraic isomorphism and \( f \in \text{Iso}(\mathcal{C}, \mathcal{C}', \varphi) \) then \( E^f = E' \) and hence \( f \in \text{Iso}(\Gamma, \Gamma') \).

Therefore \( \text{Iso}(\mathcal{C}, \mathcal{C}', \varphi) = \text{Iso}(\Gamma, \Gamma') \).

One can check in time \( n^{O(1)} \) whether the mapping \( \varphi : P_i \to Q_i \), \( i = 1, \ldots, k \), is an algebraic isomorphism because \( \mathcal{C} \) has at most \( n^3 \) intersection numbers. If \( \varphi \) is not an algebraic isomorphism then \( \Gamma \) and \( \Gamma' \) are not isomorphic. If \( \varphi \) is an algebraic isomorphism then applying \( Al_1 \) it can be determined whether the set \( \text{Iso}(\mathcal{C}, \mathcal{C}', \varphi) = \text{Iso}(\Gamma, \Gamma') \) is not empty. \( \square \)

Now let \( \mathcal{K} \) be the class of groups isomorphic to \( D = C_p \times C_{p^k} \), where \( p \in \{ 2, 3 \} \) and \( k \geq 1 \), and \( \mathcal{K}' \) be the class of all abelian groups. If \( \mathcal{C} \) is a Cayley scheme over \( G \in \mathcal{K} \) that is separable with respect to the class of Cayley schemes over groups from \( \mathcal{K}' \) and \( \mathcal{C}' \in \mathcal{K}' \) then ALISO is trivial because for every algebraic isomorphism \( \varphi : \mathcal{C} \to \mathcal{C}' \) the set \( \text{Iso}(\mathcal{C}, \mathcal{C}', \varphi) \) is not empty. Therefore ISO can be solved in time \( |G|^{O(1)} \). Thus
Theorem 2 follows from Theorem 1, Proposition 2.1 and Proposition 8.1 applying to the classes $\mathcal{K}$ and $\mathcal{K}'$.

It should be mentioned that the material of this section is based on the concepts suggested in [9] and developed in [8].
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