Global Convergence Analysis of Deep Linear Networks with A One-neuron Layer

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Abstract

In this paper, we follow Eftekhari [12]'s work to give a non-local convergence analysis of deep linear networks. Specifically, we consider optimizing deep linear networks which have a layer with one neuron under quadratic loss. We describe the convergent point of trajectories with arbitrary starting point under gradient flow, including the paths which converge to one of the saddle points or the original point. We also show specific convergence rates of trajectories that converge to the global minimizer by stages. To achieve these results, this paper mainly extends the machinery in [12] to provably identify the rank-stable set and the global minimizer convergent set. We also give specific examples to show the necessity of our definitions. Crucially, as far as we know, our results appear to be the first to give a non-local global analysis of linear neural networks from arbitrary initialized points, rather than the lazy training regime which has dominated the literature of neural networks, and restricted benign initialization in [12]. We also note that extending our results to general linear networks without one hidden neuron assumption remains a challenging open problem.

1 Introduction

Deep neural networks have been successfully trained with simple gradient-based methods, which require optimizing highly non-convex functions. Many properties of the learning dynamic for deep neural networks are also present in the idealized and simplified case of deep linear networks. It is widely believed that deep linear networks could capture some important aspects of optimization in deep learning ([37]). Therefore, many works [16, 3, 2, 6, 38, 11, 18, 45, 5, 12] have tried to study this problem in recent years. However, previous understanding mainly adopts local analysis or lazy training [8], and there are few findings of the non-local analysis, even for linear networks.

Local analysis of deep linear networks with quadratic loss. Several works analyzed linear networks with the quadratic loss. Bartlett et al. [6] provided a linear convergence rate of gradient descent with identity initialization by assuming that the target matrix is either close to identity or positive definite. Bartlett et al. [6] also showed the necessity of the positive definite target under identity initialization (see Bartlett et al. [6, Theorem 4]). Arora et al. [2] also proved linear convergence of deep linear networks, by assuming that the initialization has a positive deficiency margin and is nearly balanced. Later, a few works followed a similar idea with the neural tangent kernel (NTK) [19] or lazy training [8] to establish convergence analysis. Du and Hu [11] demonstrated that, if the width of hidden layers is all larger than the depth, gradient descent with Gaussian random initialization could converge to a global minimum at a linear rate. Hu et al. [15] improved the lower bound of width to be independent of depth, by utilizing orthogonal weight initialization, but requiring each layer to have the same width. Moreover, Wu et al. [13], Zou et al. [45] obtained a linear rate

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convergence for linear ResNet [17] with zero-(asymmetric) initialization, i.e., deep linear network with identity initialization. Specifically, Wu et al. [43] adopted zero-asymmetric initialization requiring a zero-initialized output layer and identity initialization for other layers. Such unbalanced weight matrices lead to a small variation of output layer weight compared to the other layers, which is similar to local analysis. And Zou et al. [45] applied identity initialization (for deep linear networks), but still required a lower bound for the width. All the above works are not non-local analyses.

Non-local analysis of deep linear networks with quadratic loss. The non-local analysis requires a more comprehensive analysis. As far as we know, current works mainly focused on gradient flow, i.e., gradient descent with an infinitesimal small learning rate. From the manifold viewpoint, Bah et al. [5] showed that the gradient flow always converges to a critical point of the underlying functional. Moreover, they established that, for almost all initialization, the flow converges to a global minimum on the manifold of rank-\(k\) matrices, where \(k\) can be smaller than the largest possible rank of the induced weight. Hence, their work only ensured the convergence to minimizers in a constrained subset, which is not necessarily the global minimizer. Additionally, they also provided a concrete example to display the existence of such rank unstable trajectories (see Bah et al. [5, Remark 42]). Following Bah et al. [5], Eftekhari [12] provided a non-local convergence analysis for deep linear nets with quadratic loss. By assuming that one layer has only one neuron (including scalar output case) and the initialization is balanced, Eftekhari [12] elaborated that gradient flow converges to global minimizers starting from a restricted set. Moreover, Eftekhari [12] also confirmed that gradient flow could efficiently solve the problem by showing concrete linear convergence rates in the restricted set he defined.

In this work, we are interested in the efficient solve the problem by showing concrete linear convergence rates in the restricted set he defined. In this paper, we analyze gradient flow for deep linear nets in such a scheme.

1.1 Our Contributions

In this paper, we analyze gradient flow for deep linear networks with quadratic loss following the setting of [12]. The main contributions of this paper are summarized as follows:

- **Convergent result.** We first analyze the convergent behavior of trajectories. Compared to Eftekhari [12], we define a more general rank-stable set of initialization to give almost surely convergence guarantee to the global minimizer (Theorem 3.3). Moreover, we also describe a more general global minimizer convergent set to guarantee convergence to the global minimizer (Theorem 3.6).

  Furthermore, inherited from the above results, we introduce the indicator of arbitrary beginning point to decide the convergent point of the trajectory (Theorem 3.8). Our analysis covers the trajectories that converge to saddle points. We emphasize that our analysis is beyond the lazy training scheme, and does not require the constrained initialization region mentioned in [12].

- **Convergence rate.** We also establish explicit convergence rates of the trajectories converging to the global minimizer. Our convergence rates build on the fact that the dynamic of the singular value \(s(t)\) can be divided into two stages: \(s(t)\) decreases in the first stage and increases in the second stage. In the case where our convergent results declare that \(W(t) \to u_1 v_1^T\), we show that in the worse case, there are three stages of convergence:\footnote{We denote \(s_1 u_1 v_1^T\) as the SVD of the best rank-one approximation of target, \(W(t) = s(t) u(t) v(t)^T\) as the SVD of the induced weight matrix at time \(t\), \(a_1(t) = u_1^T u(t), b_1(t) = v_1^T v(t)\), and \(N\) is the number of layers, see Section 4 for details.}

  - **Stage 1:** 
    \[
    1 - a_1(t) b_1(t) = \mathcal{O} \left( \left\lfloor (N - 2) t \right\rfloor^{-\frac{s_1}{s_2}} \right), \quad s(t) = \Omega \left( \left\lfloor (N - 2) t \right\rfloor^{-\frac{N}{s_2}} \right) \quad \text{until} \quad a_1(t) b_1(t) \geq 0;
    \]

  - **Stage 2:** 
    \[
    1 - a_1(t) b_1(t) = \mathcal{O} \left( \left\lfloor (N - 2) t \right\rfloor^{-\frac{s_1}{s_2}} \right), \quad s(t) = \Omega \left( \left\lfloor (N - 2) t \right\rfloor^{-\frac{N}{s_2}} \right) \quad \text{until} \quad s(t) \geq 0;
    \]

  - **Stage 3:** 
    \[
    1 - a_1(t) b_1(t) = \mathcal{O} \left( e^{-c_5 t} \right), \quad s_1 - s(t) = \mathcal{O} \left( e^{-\min(c_5, c_6) t} \right),
    \]

  where \(c_1, c_2\) are positive constants related to previous stages, target, initialization, and \(c_5, c_6\) are analogous positive constants additionally related to depth \(N\). We conclude that the rates begin from
polynomial to linear convergence, and are heavily dependent on the initial magnitude of $a_1(0) + b_1(0)$. And our analysis is more comprehensive since Eftekhari [12] only gave linear convergence rates for the last stage.

- We conduct numerical experiments to verify our findings. Though gradient descent seldom converges to strict saddle points [25], we discover that our analysis of gradient flow reveals the long stuck period of trajectory under gradient descent in practice and the transition of the convergence rates for trajectories.

1.2 Additional Related work

**Exponentially-tailed loss.** There is much literature [15, 33, 21, 20, 32] focusing on classification tasks under exponentially-tailed loss, such as logit loss or cross-entropy loss. Though this paper mainly focuses on the quadratic loss, we also list some findings related to linear networks. Specifically, Gunasekar et al. [15], Nacson et al. [33] proved the convergence to a max-margin solution assuming the loss converged to global optima. Lyu and Li [32], Ji and Telgarsky [20, 21] also demonstrated the convergence to the max-margin solution under weaker assumptions that the initialization has zero classification error. These analyses focused on the final phase of training, which is still not a global analysis. Lin et al. [30] showed a global analysis for directional convergence of deep linear networks. Their results also covered arbitrary initialization, but they required the spherically symmetric data assumption.

**Global landscape analysis.** Except for the non-local trajectory analysis, there is another line of works on non-local landscape analysis (see, e.g., [23, 31, 24, 35, 44, 34, 26, 10, 28, 27, 29, 41, 36, 1] and the surveys [40, 39]) which analyze the properties of stationary points, local minima, strict saddle points, etc. These works draw a whole picture of the benign landscape of deep linear networks, which provides a potential guarantee of the trajectory analysis, and motivates our work.

1.3 Organization

The remainder of the paper is organized as follows. We present some preliminaries in Section 2, including the notation, assumptions, and some preparation of our narration. In Section 3 we analyze the convergent points of trajectories progressively. We show the rank-stable initialized set in Subsection 3.1, the global minimizer convergent set of initialization in Subsection 3.2, and the convergent behavior of arbitrary initialization in Subsection 3.3. We list some examples to support our results in Subsection 3.4. In Section 4, we give explicit convergence rates for the trajectories converging the global minimizer. In Section 5, we perform numerical experiments to support our theoretical results. Finally, we conclude our work in Section 6.

2 Preliminaries

In this paper, we consider the optimization of deep linear network under squared loss:

$$\min_{\mathbf{W}_1, \ldots, \mathbf{W}_N} L^N(\mathbf{W}_1, \ldots, \mathbf{W}_N) := \|\mathbf{W}_N \cdots \mathbf{W}_1 \mathbf{X} - \mathbf{Y}\|_F^2, \quad N \geq 2,$$

where $\mathbf{W}_i \in \mathbb{R}^{d_i \times d_{i-1}}, d_0 = d_x, d_N = d_y, \mathbf{X} \in \mathbb{R}^{d_x \times m}, \mathbf{Y} \in \mathbb{R}^{d_y \times m}$. For brevity, we denote the induced weight as $\mathbf{W} = \mathbf{W}_N \cdots \mathbf{W}_1 \in \mathbb{R}^{d_y \times d_x}$, and $\mathbf{W}_N = (\mathbf{W}_1, \ldots, \mathbf{W}_N) \in \mathbb{R}^{d_1 \times d_0} \times \cdots \times \mathbb{R}^{d_N \times d_{N-1}}$.

**Notation.** We denote vectors by lowercase bold letters (e.g., $\mathbf{u}, \mathbf{x}$), and matrices by capital bold letters (e.g., $\mathbf{W} = [w_{ij}]$). We set $[a, b] = \{a, \ldots, b\}, [a] = [1, a], \forall a, b \in \mathbb{N}$ if no ambiguity with the closed interval, and $s_i(\mathbf{Z})$ as the $i$-th largest singular of $\mathbf{Z}$. We adopt $\|\cdot\|$ as the standard Euclidean norm ($\ell_2$-norm) for vectors, and $\|\cdot\|_F$ as Frobenius norm for matrices. The convergence of vectors and matrices in this paper is defined under the standard Euclidean norm and Frobenius norm. We use the standard $\mathcal{O}(\cdot), \Omega(\cdot)$ and $\Theta(\cdot)$ notation to hide universal constant factors.

We integrate our assumptions in this paper below:
Assumption 2.1 We assume that the target, data, network and the initialization satisfy:

- Target: $Z := YX^T \in \mathbb{R}^{d_y \times d_x}$ has different nonzero singular values, i.e., $s_1(Z) > \cdots > s_d(Z) > 0$, where $d = \text{rank}(YX^T)$.
- Data: $XX^T = I_{d_x}$.
- Network: $r := \min_{j \leq N} d_j = 1$.
- Initialization: $W_iW_i^T = W_{i+1}W_{i+1}^T, \forall i \in [N - 1]$.

For the target, it is reasonable to assume different nonzero singular values in practice, since matrices with the same singular values have zero Lebesgue measure. The last three assumptions are the same as Eftekhari [12].

The second assumption shows that the data is statistically whitened, which is common in the analysis of linear networks ([2, 6]). The network assumption includes the scalar output case. As mentioned in [12], this case is significant as it corresponds to the popular spiked covariance model in statistics and signal processing ([13, 22, 42, 11, 9]), to name a few. Moreover, the case $r = 1$ appears to be the natural beginning building block for understanding the behavior of trajectory. Finally, the last assumption is the common initialization technique for linear networks, which has appeared in [10, 6, 2, 3, 4].

From $XX^T = I_{d_x}$ in Assumption 2.1, we can simplify the problem as

$$\min_{W_1, \ldots, W_N} L^N(W_1, \ldots, W_N) := \|W_N \cdots W_1 - Z\|_F^2 + \|Y\|_F^2 - \|Z\|_F^2. \quad (1)$$

Hence, we call $Z$ as the target matrix. Moreover, we focus on the standard gradient flow method:

$$W_i(t) := \frac{dW_i(t)}{dt} = -\nabla_{W_i} L^N(W_N(t)), \forall j \in [N], t \geq 0. \quad (2)$$

Under the balanced initialization in Assumption 2.1 i.e., $W_iW_i^T = W_{i+1}W_{i+1}^T, \forall i \in [N - 1]$, we have the induced weight flow of $W(t) := W_N(t) \cdots W_1(t)$ following Arora et al. [3, Theorem 1]:

$$W(t) = -A(W(t)) := -\sum_{j=1}^N (W(t)W(t)^T) \frac{N - j}{N} \nabla L^1(W(t)) (W(t)^TW(t))^\frac{j-1}{N}$$

$$= -\sum_{j=1}^N (W(t)W(t)^T) \frac{N - j}{N} (W(t) - Z) (W(t)^TW(t))^\frac{j-1}{N}. \quad (3)$$

It is known that the induced flow in Eq. (3) admits an analytic singular value decomposition (SVD), see Lemma 1 and Theorem 3 in Arora et al. [4] for example. Since rank($W$) $\leq 1$ from Assumptions 2.1, we can denote the SVD of $W(t)$ as $W(t) = s(t)\mathbf{u}(t)\mathbf{v}(t)^T$ if $W(t) \neq 0$. Here, $s(t), \mathbf{u}(t), \mathbf{v}(t)$ are all analytic functions of $t$. Moreover, $s(t) \in \mathbb{R}, \mathbf{u}(t) \in \mathbb{R}^{d_x}, \mathbf{v}(t) \in \mathbb{R}^{d_y}$, and $\|\mathbf{u}(t)\| = \|\mathbf{v}(t)\| = 1$. Previous work has already shown the variation of these terms:

$$\dot{\mathbf{u}}(t) = s(t)^{1-\frac{2}{N}} \left( I_{d_x} - \mathbf{u}(t)\mathbf{u}(t)^T \right) Z\mathbf{v}(t), \quad (4)$$

$$\dot{\mathbf{v}}(t) = s(t)^{1-\frac{2}{N}} \left( I_{d_y} - \mathbf{v}(t)\mathbf{v}(t)^T \right) Z^\top \mathbf{u}(t), \quad (5)$$

$$\dot{s}(t) = Ns(t)^{2-\frac{2}{N}} \left( \mathbf{u}(t)^\top Z\mathbf{v}(t) - s(t) \right). \quad (6)$$

Readers can discover the derivation of $\dot{s}(t)$ in Arora et al. [3, Theorem 3], and $\dot{\mathbf{u}}(t), \dot{\mathbf{v}}(t)$ in Eftekhari [12, Eq. (139)] or Arora et al. [4, Lemma 2] with some simplification. We also give the derivation of $\dot{\mathbf{u}}(t), \dot{\mathbf{v}}(t)$ in Lemma A.7

To describe the solution obtained by flow, we also need the full SVD of target as $Z = UDV^\top = \sum_{i=1}^d s_i u_i v_i^\top$ with $s_1 > \cdots > s_d > 0$, orthogonal matrices $U = [u_i] \in \mathbb{R}^{d_x \times d_y}$ and $V = [v_i] \in \mathbb{R}^{d_y \times d_x}$, and $D = (\text{diag}(s)) \in \mathbb{R}^{d_x \times d_x}$ where $s = (s_1, \ldots, s_d)^\top \in \mathbb{R}^d$. And the best rank-one approximation matrix $Z_1 = \mathbf{u}(t)\mathbf{v}(t)^T$.
Note that $Z_1$ is the unique solution of problem Eq. (1), because $Z$ has a nontrivial spectral gap by Assumption 2.1 (see [14] Section 1). For brevity, we define $s_k = 0, \forall k > d$. We adopt the projection length of $u(t), v(t)$ to each $u_i$ and $v_i$ as $a_i(t) = u_i^\top u(t), \forall i \in [d_y]$ and $b_j(t) = v_j^\top v(t), \forall j \in [d_x]$. So we have

$$u(t) = \sum_{i=1}^{d_y} a_i(t)u_i, v(t) = \sum_{j=1}^{d_x} b_j(t)v_j, \text{and } \sum_{i=1}^{d_y} a_i^2(t) = \sum_{j=1}^{d_x} b_j^2(t) = 1. \quad (7)$$

Then we get

$$u(t)^\top Z v(t) = \sum_{i=1}^{d_y} \sum_{j=1}^{d_x} a_i(t)b_j(t)u_i^\top Z v_j = \sum_{i=1}^{d_y} a_i(t)b_j(t)u_i^\top Z v_j = \sum_{i=1}^{d_y} s_i a_i(t) b_j(t), \quad (8)$$

$$Z v(t) = \sum_{j=1}^{d_x} b_j(t)Z v_j = \sum_{j=1}^{d_x} s_j b_j(t)u_j, \quad Z^\top u(t) = \sum_{i=1}^{d_y} a_i(t)Z^\top u_i = \sum_{i=1}^{d_y} s_i a_i(t) v_i. \quad \text{where we uses the fact that } Z^\top u_i = 0, \forall i > d, Z v_j = 0, \forall j > d, \text{and } u_i^\top Z v_j = 0, u_i^\top Z v_i = s_i, \forall i \neq j \leq d. \text{ Hence, we have the gradient flow of each item:}$$

$$\dot{a}_i(t) \overset{4}{=} s(t)^{1 - \frac{\omega}{2}} u_i^\top (I_{d_y} - u(t)u(t)^\top) Z v(t) \overset{8}{=} s(t)^{1 - \frac{\omega}{2}} \left(s_i b_i(t) - a_i(t) \sum_{j=1}^{d_y} [s_j a_j(t) b_j(t)] \right), \forall i \in [d_y],$$

$$\dot{b}_j(t) \overset{4}{=} s(t)^{1 - \frac{\omega}{2}} u_i^\top (I_{d_y} - v(t)v(t)^\top) Z^\top u(t) \overset{8}{=} s(t)^{1 - \frac{\omega}{2}} \left(s_i a_i(t) - b_i(t) \sum_{j=1}^{d_x} [s_j a_j(t) b_j(t)] \right), \forall i \in [d_x]. \quad (9)$$

Before we provide our results, we first state several useful invariance during the whole training dynamic as follows, which is crucial to our proofs.

**Proposition 2.2** If not mentioned specifically, we assume $s(0) > 0$. We have the following useful properties:

- 1). If $s(0) > 0$, then $\forall t \geq 0, s(t) > 0$. Otherwise, $s(0) = 0$, then $\forall t \geq 0, s(t) = 0$ (i.e., $W(t) = 0$).
- 2). $u(t)^\top Z v(t)$ is non-decreasing and converges.
- 3). $u(t)^\top Z_1 v(t)$ is non-decreasing and converges.
- 4). For all $t \geq 0$, $a_i(t) + b_i(t)$ has the same sign with $a_i(0) + b_i(0)$, i.e., $a_i(t) + b_i(t)$ is identically zero if $a_i(0) + b_i(0) = 0$, is positive if $a_i(0) + b_i(0) > 0$, and is negative if $a_i(0) + b_i(0) < 0$.
- 5). If for some $k \in [0, d - 1]$, $a_i(0) + b_i(0) = 0, \forall i \in [k]$ (if $k = 0$, then no such assumptions) and $a_{k+1}(0) + b_{k+1}(0) \neq 0$, then $|a_{k+1}(t) + b_{k+1}(t)|$ is non-decreasing, and $\lim_{t \to +\infty} a_{k+1}(t) + b_{k+1}(t)$ exists.

### 3 Convergent Behavior of Trajectories

#### 3.1 Rank-stable Set

By Bah et al. [5] Theorem 5] (Theorem 3.5], ($W_1(t), \ldots, W_N(t)$) always converges to a critical point of $L^N$ as $t \to +\infty$. Hence, we can define $\bar{W} := \lim_{t \to +\infty} W(t)$, and $\bar{s} = \lim_{t \to +\infty} s(t) = \lim_{t \to +\infty} \|W(t)\|_F$. To specify the convergent point, we define rank-$r$ set following [12] as

$$\mathcal{M}_r = \{W : \text{rank}(W) = r\}.$$ 

Furthermore, as mentioned in Eftekhari [12] Lemma 3.3] (Lemma 3.1], we have rank($W(t)$) = rank($W(0)$) = 1, $\forall t \geq 0$ if $W(0) \neq 0$. However, the limit point $\bar{W}(t)$ might not belong to $\mathcal{M}_1$ because $\mathcal{M}_1$ is not closed,
see [12, Lemma 3.4]. To exclude the zero matrix (s = 0) as the limit point of gradient flow, Eftekhari [12] introduced a restricted initialization set:

\[ \mathcal{N}_\alpha(Z) = \{ W : W \overset{\text{SVD}}{=} u \cdot s \cdot v^\top, s > \alpha s_1 - s_2 > 0, u^\top Z_1 v > \alpha s_1 \} \cdot \alpha \in (s_2/s_1, 1] \]

While we find another rank-stable set \( \mathcal{R}_b(Z) \) below with similar rank-stable property shown in Lemma 3.1:

\[ \mathcal{R}_b(Z) = \{ W : W \overset{\text{SVD}}{=} u \cdot s \cdot v^\top, s > b, u^\top Z v > b \}, b > 0 \]

**Lemma 3.1 (Extension of Lemma 3.7 in Eftekhari [12])** Under Assumption 2.1 for gradient flow initialized at \( W(0) \in \mathcal{R}_b(Z) \), the limit point exists and satisfies \( W = \lim_{t \to +\infty} W(t) \in M_1 \).

**Proof:** We only need to prove that \( W(t) \in \mathcal{R}_b(Z), \forall t \geq 0 \). Since \( W(0) \in \mathcal{R}_b(Z) \), then \( u(0)^\top Z v(0) > b \). Thus, we have \( u(t)^\top Z v(t) > b, \forall t \geq 0 \) by 2) in Proposition 2.2. Hence, \( W(t) \) only could leave the region \( \mathcal{R}_b(Z) \) when \( s(T) = b \) for some time \( T > 0 \).

Since \( W(0) \in \mathcal{R}_b(Z) \), then \( s(0) > b \). Thus, we have \( s(T) > 1 \) by 1) in Proposition 2.2. Therefore, we get

\[ \dot{s}(T) = N s(T)^2 - \hat{\sigma} (u(T)^\top Z v(T) - s(T)) = N s(T)^2 - \frac{\hat{\sigma}}{2} (u(T)^\top Z v(T) - b) > 0, \]

which pushes the singular value up and thus pushes the induced flow back into \( \mathcal{R}_b(Z) \). Contradiction! □

**Remark 3.2** Indeed, our rank-stable set \( \mathcal{R}_b(Z) \) is more general than \( \mathcal{N}_\alpha(Z) \). For any \( \alpha \in (s_2/s_1, 1] \), \( W \in \mathcal{N}_\alpha(Z) \) since \( |u^\top (Z - Z_1) v| \leq s_2 \), we get \( u^\top Z v \geq u^\top Z_1 v - s_2 \geq \alpha s_1 - s_2 > 0 \). Hence, we can find some \( b > 0 \) such that \( W \in \mathcal{R}_b(Z) \). Moreover, when \( W = s_2 u_2 v_2^\top \), we can see \( W \in \mathcal{R}_{s_2/2}(Z) \), but \( W \notin \mathcal{N}_\alpha(Z), \forall \alpha \in (s_2/s_1, 1] \), since \( u^\top Z_1 v = 0 \). Additionally, we will see the necessity of our rank-stable set by showing counterexamples in Section 3.4.

Applying the same analysis as Eftekhari [12, Theorem 3.8], we could obtain almost surely convergence to the global minimizer from the initialization in our rank-stable set.

**Theorem 3.3 (Extension of Theorem 3.8 in Eftekhari [12])** Under Assumption 2.1, gradient flow converges to a global minimizer of the original problem Eq. (1) from the initialization in \( W(0) \in \mathcal{R}_b(Z), b > 0 \), outside of a subset with Lebesgue measure zero.

**Proof:** From Lemma 3.1, we already know the convergent point \( W(t) \) is still rank-one. Hence, using the facts shown in Bah et al. [3, Theorem 28] (Theorem A.11) that gradient flows avoid strict saddle points almost surely; and Bah et al. [5, Proposition 33] (Proposition A.6) that \( L^1(W) \) on \( M_1 \) satisfies the strict saddle point property, we could see gradient flow of \( W(t) \) converges to a global minimizer almost surely. □

### 3.2 Global Minimizer Convergent Set

Section 3.1 mainly analyzes the behavior of \( s(t) \) to guarantee the rank of limit point is not degenerated. Though Theorem 3.3 ensures the almost surely convergence to the global minimizer, there still have some bad trajectories which converge to saddle points, such as \( s_i u_i v_i^\top, i \neq 1 \). In this subsection, we move on to give another restricted initialization set to guarantee the global minimizer convergence without excluding a zero measure set. Our strategy mainly adopts singular vector analysis. To analyze the behavior of \( u(t), v(t) \), we need the following lemmas. In the following, we always assume \( W(0) \neq 0 \) to ensure well-defined \( u(t), v(t) \).

**Lemma 3.4** There exists a sequence \( \{ t_n \} \) with \( t_n \to +\infty \), such that \( \lim_{n \to +\infty} \left( I_{d_y} - u(t_n) u(t_n)^\top \right) Z v(t_n) = 0 \), and \( \lim_{n \to +\infty} \left( I_{d_x} - v(t_n) v(t_n)^\top \right) Z^\top u(t_n) = 0 \). More specifically, we have

\[
\begin{align*}
\lim_{n \to +\infty} \left( \sum_{j=1}^d s_j a_j(t_n) b_j(t_n) \right) a_i(t_n) - s_i b_i(t_n) &= 0, \forall i \in [d_y], \\
\lim_{n \to +\infty} \left( \sum_{j=1}^d s_j a_j(t_n) b_j(t_n) \right) b_i(t_n) - s_i a_i(t_n) &= 0, \forall i \in [d_x].
\end{align*}
\]
Furthermore, if there exists \( i_0 \in [d] \), such that \( \lim_{n \to +\infty} a_{i_0}(t_n) + b_{i_0}(t_n) \) exists and is not zero, we could obtain
\[
\lim_{n \to +\infty} u(t_n)^\top Z v(t_n) = \lim_{n \to +\infty} \sum_{j=1}^d s_j a_j(t_n) b_j(t_n) = s_{i_0}. \tag{12}
\]

**Lemma 3.5** Suppose there exists a sequence \( \{t_n\} \) that \( t_n \to +\infty \), and for some \( k < d, a_i(t_n) + b_i(t_n) = 0, \forall i \in [k], n \geq 0. \) Then if \( \lim_{n \to +\infty} \sum_{j=1}^d s_j a_j(t_n) b_j(t_n) = s_{k+1} \), we could obtain
\[
\lim_{n \to +\infty} a_i(t_n) = \lim_{n \to +\infty} b_j(t_n) = 0, \forall i, j \neq k + 1, i \in [d], j \in [d]; \lim_{n \to +\infty} a_{k+1}(t_n) b_{k+1}(t_n) = 1. \tag{13}
\]
That is, \( \lim_{n \to +\infty} u(t_n)v(t_n)^\top = u_{k+1}v_{k+1}^\top. \)

Now we define the global minimizer convergent set as
\[
\mathcal{G}_b(Z) = \{ W : W^{\text{SVD}} = u \cdot s \cdot v^\top, s > b, u^\top Z v > b \}.
\]
The only difference compared to \( \mathcal{R}_b(Z) \) is that we replace \( Z \) to \( Z_1 \).

**Theorem 3.6 (Extension of Theorem 3.8 in Eftekhari [12])** Under Assumption 2.1, gradient flow converges to a global minimizer of the original problem Eq. (1) from the initialization \( W(0) \in \mathcal{G}_b(Z), b > 0. \)

Proof: Our proof is separated into the following steps.

- **Step 1.** Since \( W(0) \in \mathcal{G}_b(Z) \), we have \( s_1 a_1(0) b_1(0) = u(0)^\top (s_1 u_1 v_1^\top) v(0) = u(0)^\top Z_1 v(0) > b. \) By \ref{prop2}, in Proposition 2.2, we obtain \( u(t)^\top Z v(t) = s_1 u(t)^\top u_1 v_1^\top v(t) = s_1 a_1(0) b_1(0) > b > 0, \) leading to \( |a_1(t) + b_1(t)| > 0. \) Additionally, by \ref{prop5} in Proposition 2.2, we get \( \lim_{n \to +\infty} a_i(t) + b_i(t) \geq |a_i(0) + b_i(0)| > 0. \) Thus, applying Lemma 3.4 with \( i_0 = 1, \) Eq. (12) holds, i.e., there exists a sequence \( \{t_n\} \) with \( t_n \to +\infty, \) s.t., \( \lim_{n \to +\infty} u(t_n)^\top Z v(t_n) = \lim_{n \to +\infty} \sum_{i=1}^d s_j a_j(t_n) b_j(t_n) = s_1. \)

- **Step 2.** From **Step 1**, \( \lim_{n \to +\infty} \sum_{i=1}^d s_j a_j(t_n) b_j(t_n) = s_1. \) Then we can employ Lemma 3.5 with \( k = 0, \) showing that \( u(t_n)^\top v(t_n) = u_1 v_1^\top. \)

- **Step 3.** From **Step 1**, we know that \( u(t_n)^\top Z v(t_n) \to s_1 > 0, \) leading to \( \exists \mathbb{N} > 0, u(t_N)^\top Z v(t_N) > 0. \) Moreover, since \( W(0) \in \mathcal{G}_b(Z), \) we have \( s(0) > 0. \) Thus, we have \( s(t_N) > 0 \) by 1) in Proposition 2.2. Hence, \( W(t_N) \in \mathcal{R}_b(Z) \) for some \( b > 0. \) Thus, by Lemma 3.4, we have \( \bar{s} > 0. \)

- **Step 4.** Finally, taking \( t_n \to +\infty \) in Eq. (10), we obtain \( 0 = N \bar{s}^2 - \frac{2}{\bar{s}} (s_1 - \bar{s}). \) While by **Step 3**, \( \bar{s} > 0. \) Hence, \( \bar{s} = s_1. \)

Combining **Step 2** and **Step 4**, we obtain \( W(t_n) \) converges to the global minimizer \( s_1 u_1 v_1^\top. \) Finally, we note that by Theorem 5 in Bah et al. [9], \( W(t) \) converges. Hence, \( W(t) \to s_1 u_1 v_1^\top, \) which is the global minimizer.

**Remark 3.7** We underline that Theorem 3.6 does not need to leave out a zero measure set comparing to Theorem 3.3. Meanwhile, \( \mathcal{G}_b(Z) \) is more general than \( \mathcal{N}_a(Z) \) as well, since we have less constraint for \( u^\top Z v \) in \( \mathcal{G}_b(Z) \). Moreover, we will see the necessity of our global minimizer convergent set by showing counterexamples in Section 3.4.

### 3.3 Convergence Analysis for All Initialization

Now we change our perspective to the whole initialization, which includes the trajectories that converge to saddle points, instead of the global minimizer. The main conclusion is that the convergent point is decided by the indicator of initialization: \( a_i(0) + b_i(0), i \in [d]. \)
Theorem 3.8  Under Assumption 2.1, and assume $s(0) > 0$. (I) If $k \in [0, d - 1], a_i(0) + b_i(0) = 0, \forall i \in [k]$ (if $k = 0$, then no such assumptions), and $a_{k+1}(0) + b_{k+1}(0) \neq 0$, we have $W(t) \rightarrow s_{k+1}u_{k+1}v_{k+1}^T$. (II) Otherwise, i.e., $a_i(0) + b_i(0) = 0, \forall i \in [d]$, then we have $W(t) \rightarrow 0$.

Proof: (I) For the first conclusion, our proof is separated to the following steps.

- **Step 1.** From 5) in Proposition 2.2, we could see $|\lim_{t \rightarrow +\infty} a_{k+1}(t) + b_{k+1}(t)| \geq |a_{k+1}(0) + b_{k+1}(0)| > 0, \forall t \geq 0$. Thus, applying Lemma 3.4 we obtain $\exists \{t_n\}$ with $t_n \rightarrow +\infty$, s.t.,

$$
\lim_{n \rightarrow +\infty} u(t_n)^T Zv(t_n) = \lim_{n \rightarrow +\infty} \sum_{i=1}^d s_j a_j(t_n)b_j(t_n) = s_{k+1}.
$$

- **Step 2.** Hence, by Step 1 and Lemma 3.5, we could obtain $u(t_n)v(t_n)^T \rightarrow u_{k+1}v_{k+1}^T$.

- **Step 3.** By Step 1, $u(t_n)^T Zv(t_n) \rightarrow s_{k+1} > 0$, leading to $\exists N > 0, u(t_N)^T Zv(t_N) > 0$. Moreover, since $s(0) > 0$, we have $s(t_N) > 0$ by 1) in Proposition 2.2. Hence, $W(t_N) \in \mathcal{R}_b(Z)$ for some $b > 0$. Thus, by Lemma 3.1 we have $\bar{s} > 0$.

- **Step 4.** Finally, taking $t_n \rightarrow +\infty$ in Eq. (6), by Step 1, we obtain $0 = N\bar{s}^2 - \bar{s} (s_{k+1} - \bar{s})$. While we have $\bar{s} > 0$ by Step 3. Hence, $\bar{s} = s_{k+1}$.

Combining Step 2 and Step 4, we obtain that $W(t_n)$ converges to $s_{k+1}u_{k+1}v_{k+1}^T$. Finally, we note that by Theorem 5 in Bah et al. [5], $W(t)$ converges. Hence, $W(t) \rightarrow s_{k+1}u_{k+1}v_{k+1}^T$.

(II) For the second conclusion, i.e., assume $a_i(0) + b_i(0) = 0, \forall i \in [d]$. Then by 4) in Proposition 2.2, we obtain $a_i(t) + b_i(t) = 0, \forall i \in [d], t \geq 0$. Hence, we get

$$
u(t)^T Zv(t) = \sum_{j=1}^d s_j a_j(t)b_j(t) = - \sum_{j=1}^d s_j a_j^2(t) \leq 0 \Rightarrow \dot{s}(t) \leq Ns(t)^2 - \frac{s(0)^2}{2 - 2N} \leq -Ns(t) + Ns(t) + s(0)^2 - 2s(0)^2 \geq 2N - 2t.
$$

By solving the ordinary differential equation (ODE) above, we derive that

$$
\frac{N}{2 - 2N}s(t)^2 - 2 - \frac{N}{2 - 2N}s(0)^2 \leq -Nt \Rightarrow s(t)^2 - 2 - s(0)^2 + 2(2N - 2)t \geq 0.
$$

Therefore, we obtain

$$
s(t)^2 - 2 \leq \left[\frac{s(0)^2}{2 - 2N} + (2N - 2)t\right]^{-\frac{N}{2}} \Rightarrow s(t) \leq \lim_{t \rightarrow +\infty} s(t) \leq \lim_{t \rightarrow +\infty} \left[\frac{s(0)^2}{2 - 2N} + (2N - 2)t\right]^{-\frac{N}{2}} = 0.
$$

That is, $W(t) \rightarrow 0$.

Remark 3.9 We note that $s(0) = 0$ indicates that $W(0) = 0$, and $W(t) = 0, \forall t \geq 0$, which is a trivial case. Moreover, we also give a convergence rate for the second conclusion, i.e., the rate for the convergence to the original point.

3.4 Some Intuitive Examples

Previous sections have shown the convergent behavior of arbitrary initialization. To give a better understanding of our results and the training behavior, we list some examples below.

Example 1. If $W(0) = -s(0)u_i v_i^T$ for some $i \in [\min\{d_y, d_x\}]$, we have $\forall t \geq 0, \dot{u}(t) = 0, \dot{v}(t) = 0$ and $u(t)^T Zv(t) = s_i$ from Eqs. (1) and (3). Thus, we obtain the ODE of $s(t)$ as follows

$$
\dot{s}(t) \leq Ns(t)^2 - \frac{s(0)^2}{2 - 2N} (u(t)^T Zv(t) - s(t)) \leq -Ns(t)^2 + s_i + s(t) \leq 0.
$$

So we could obtain $s(t) \rightarrow 0$. We see that $W(t) \rightarrow 0$, which is a rank-unstable trajectory. Thus, the gradient flow of Eq. (1) does not converges to a global minimizer.
Remark. We note that our Theorem 3.8 covers this example by choosing \( k = d \), i.e., the second case in Theorem 3.8. Moreover, we can see \(-s(0)u_i v_i^\top \notin \mathcal{R}_b(Z)\), since \(-u_i^\top Zv_i = -s_i \leq 0\). Thus, we could not further improve our definition of \( \mathcal{R}_b(Z) \).

**Example 2.** If \( W(0) = s(0)u_i v_i^\top \) for some \( i \in [d] \) and \( s(0) > 0 \), then from Eqs. (4) and (5), we obtain \( \dot{u}(t) = 0, \dot{v}(t) = 0 \) and \( u(t)^\top Zv(t) = s_i, \forall t \geq 0 \). Thus, we obtain the ODE of \( s(t) \) as follows:

\[
\dot{s}(t) \triangleq Ns(t)^{2-\frac{1}{2}} \left( u(t)^\top Zv(t) - s(t) \right) = Ns(t)^{2-\frac{1}{2}} \left( s_i - s(t) \right).
\]

Hence, we obtain \( s(t) \to s_i \). Once \( i \neq 1 \), we could see that \( W(t) \to s_1 u_1 v_1^\top \), i.e., the gradient flow of Eq. (1) does not converge to a global minimizer.

Remark. We note that our Theorem 3.8 covers this example by choosing \( k = i \). Moreover, if \( i \neq 1 \), we can see \( s(0)u_i v_i^\top \notin \mathcal{G}_b(Z) \), since \( u_i^\top Z_1 v_i = 0 \). Thus, we could not further improve our definition of \( \mathcal{G}_b(Z) \).

**Example 3.** Bah et al. [5, Remark 42]: If \( Z \succeq 0 \), and \( s(0) > 0, u(0) = -v(0) \). Then from Eqs. (4) and (5), we obtain \( \dot{u}(t) = -\dot{v}(t) \) if \( u(t) = -v(t) \). Hence, we get \( u(t) = -v(t), \forall t \geq 0 \). Thus, we obtain \( \dot{s}(t) \) as follows:

\[
\dot{s}(t) \triangleq Ns(t)^{2-\frac{1}{2}} \left( u(t)^\top Zv(t) - s(t) \right) = Ns(t)^{2-\frac{1}{2}} \left( -u(t)^\top Zu(t) - s(t) \right) \leq -Ns(t)^{3-\frac{1}{2}} \leq 0,
\]

leading to \( s(t) \to 0 \). We could see that \( W(t) \to 0 \), which is a rank-unstable trajectory. Thus, the gradient flow of Eq. (1) does not converge to a global minimizer.

**Remark.** We note that our Theorem 3.8 covers this example by choosing \( k = d \), i.e., the second case in Theorem 3.8. Moreover, we can see \(-s(0)u(0) v(0)^\top \notin \mathcal{R}_b(Z)\), since \(-u(0)^\top Zv(0) \leq 0\). Thus, we could not further improve our definition of \( \mathcal{R}_b(Z) \) as well.

### 4 Convergence Rates to Global Minimizers

We briefly show the specific convergence rates in this section. We note that the proof of Theorem 3.8 has already shown the rate for \( W(t) \to 0 \). Now we consider the rates to the global minimizers under Assumption 2.1, which is common in previous works. That is, from Theorem 3.8, we consider the initialization which satisfies \( a_1(0) + b_1(0) \neq 0 \). Typically, the trajectories can be divided into three stages:

**Stage 1.** For \( t \in [0, t_1] \), where \( t_1 := \inf \{ t : a_1(t)b_1(t) \geq 0 \} < +\infty \), we have \( a_1(t)b_1(t) \leq 0 \), and the rates are

\[
1 - a_1(t)b_1(t) = \mathcal{O} \left( ([N-2]t)^{-\frac{s_1}{2}} \right), \quad s(t) = \Omega \left( ([N-2]t)^{-\frac{N}{2}} \right). \quad [\text{Theorem 4.5}]
\]

**Stage 2.** For \( t \in (t_1, t_2] \), where \( t_2 := \inf \{ t : u(t)^\top Zv(t) \geq s(t) \} \), we have \( a_1(t)b_1(t) > 0, \dot{s}(t) \leq 0 \), and

\[
1 - a_1(t)b_1(t) = \mathcal{O} \left( ([N-2]t)^{-\frac{s_1}{2}} \right), \quad s(t) = \Omega \left( ([N-2]t)^{-\frac{N}{2}} \right). \quad [\text{Theorem 4.7}]
\]

**Stage 3.** For \( t \in (\max \{ t_1, t_2 \}, +\infty) \), we have \( a_1(t)b_1(t) > 0 \) and \( \dot{s}(t) \geq 0 \), and the rates are

\[
1 - a_1(t)b_1(t) = \mathcal{O} \left( e^{-c_1 t} \right), \quad s_1 - s(t) = \mathcal{O} \left( e^{-\min(c_0,c_1)c_1 t} \right). \quad [\text{Theorem 4.9}]
\]

\(^2\text{We only list the case } N \geq 3, \text{ which is more common in practice. The theorem in this section also focus on the case } N \geq 3. \text{ We leave the case of } N = 2 \text{ in Appendix C, where we provide the similar results and convergence rates.}\)
Remark 4.1 We explain other minor cases: 1) If \( t_1 = 0 \), then the Stage 1 vanishes; 2) If \( t_1 \geq t_2 \), then the Stage 2 vanishes; 3) If \( t_2 = +\infty \) [Theorem 4.8], then we have similar rates as Stage 3: \( 1 - a_1(t)b_1(t) = \mathcal{O}(e^{-c_1 t}) \), \( |s(t) - s_1| = \mathcal{O}(e^{-c_2 t}) \). However, this case is not suitable in our framework.

The convergence rates go through a polynomial to a linear rate, which is intuitively correct in practice. Before we give the detail of analysis, we need some preparation in advance. Theorem 4.2 below shows the characteristic through \( t_1 \) and \( t_2 \).

Theorem 4.2 Let \( c_1(t) := a_1(t)b_1(t) \), \( t_1 := \inf \{ t : a_1(t)b_1(t) \geq 0 \} \), \( t_2 := \inf \{ t : u(t)^T Z u(t) \geq s(t) \} \). Then we have (I) \( t_1 < +\infty \) if \( a_1(0) + b_1(0) \neq 0 \), and \( c_1(t) \geq 0 \) for all \( t \geq 0 \); (II) \( \dot{s}(t) \leq 0 \) for \( t \in [0, t_2) \), and \( \dot{s}(t) \geq 0 \) for \( t \in [t_2, +\infty) \).

Remark 4.3 (I) in Theorem 4.2 tells us that the first stage, if exists, only appears a finite time in the beginning. (II) In Theorem 4.2 shows the induced weight norm (\( \|W(t)\|_F = s(t) \)) goes through descending and ascending periods. If the initial induced weight norm starts with descending behavior, then it could descend forever, or it will change to ascending and continue increasing to \( s_1 \). If the initial induced weight norm begins with ascending behavior, then it would increase to \( s_1 \) directly. Such induced weight norm behavior also appears in deep linear networks with the logit loss.

4.1 Convergence Rates of \( s(t) \): Stage 1 and Stage 2

At Stage 1 and Stage 2, we have \( \dot{s}(t) \leq 0 \) from Theorem 4.2. Now we first give a global lower bounds for the singular value \( s(t) \), which may work as a proper lower bound within Stage 1 and Stage 2. Meanwhile, we provide a global upper bound of \( s(t) \).

Theorem 4.4 Assume \( s(0) > 0 \), then we have \( 0 < s(t) < s_0 := \max \{ s_1, s(0) \} \) for all \( t \geq 0 \). Further we have

\[
\text{when } N = 2 : s(t) \geq s(0)e^{-2(s_1 + s_0)t}, \text{i.e., } s(t) = \Omega \left( e^{-2(s_1 + s_0)t} \right); \tag{14}
\]

\[
\text{when } N \geq 3 : s(t) \geq (s_1 + s_0)(N - 2)t + s(0)^{\hat{s} - 1} \frac{N}{s^{\hat{s} - 1}}, \text{i.e., } s(t) = \Omega \left( [(N - 2)t + s(0)^{\hat{s} - 1}]^{\frac{N}{s^{\hat{s} - 1}}} \right). \tag{15}
\]

We note that different lower bounds of \( s(t) \) lead to different rates for the case \( N = 2 \) and \( N \geq 3 \). For brevity, we only give the results for \( N \geq 3 \), and leave the simple case \( N = 2 \) in Appendix C.

4.2 Convergence Rates of \( a_1(t)b_1(t) \): Stage 1

In the case where \( a_1(0)b_1(0) < 0 \), we prove that the case will reduce to the case \( a_1(0)b_1(0) \geq 0 \) in a finite time when \( a_1(0) + b_1(0) \neq 0 \). We further give an upper bound for time staying in Stage 1 and a lower bound of \( a_1(t)b_1(t) \).

Theorem 4.5 Suppose \( N \geq 3 \), \( a_1(0)b_1(0) < 0 \) and \( a_1(0) + b_1(0) \neq 0 \). Then we have

\[
1 - a_1(t)b_1(t) = \mathcal{O}((N - 2)t \cdot s^{\hat{s} - 1}), 0 \leq t \leq t_1,
\]

where \( c_1 = (s_1 + s_0)^{-1} \), and \( s_0 \) inherited from Theorem 4.4. Furthermore, we have the upper bound of \( t_1 \) below:

\[
t_1 \leq \frac{s(0)^{\hat{s} - 1}}{2(s_1 + s_0)(N - 2)} \cdot \left[ \left( \frac{a_1(0) - b_1(0)}{a_1(0) + b_1(0)} \right)^{(s_1 + s_0)(N - 2)} - 1 \right]. \tag{16}
\]

Additionally, we could obtain

\[
a_1(t)b_1(t) = \Omega \left( \frac{(a_1(0) + b_1(0))^2}{N} \right), \text{ if } t \geq \frac{s(0)^{\hat{s} - 1}}{2(s_1 + s_0)(N - 2)} \cdot e \left( \frac{(a_1(0) - b_1(0))}{a_1(0) + b_1(0)} \right)^{(s_1 + s_0)(N - 2)} - 1 \right]. \tag{17}
\]

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Remark 4.6 The upper bound of $t_1$ in Theorem 4.5 shows that if $a_1(0) + b_1(0) \approx 0$, then the Stage 1 would last for a long time according to Eq. (16). Moreover, Theorem 3.8 has already shown that once $a_1(0) + b_1(0) = 0$, the trajectory would not converge to the global minimizer. Hence, our finding in Theorem 4.5 is consistent with Theorem 3.8. Additionally, we also give guarantee of the trajectory to arrive at $G_0(\mathbf{Z})$ for some $b > 0$ from Eq. (17). That is, the trajectory enters in our global minimizer convergent set.

4.3 Convergence Rates of $a_1(t)b_1(t)$: Stage 2

Based on Theorem 4.5, we can see after finite time, we obtain $a_1(t)b_1(t) > 0$, that is, the trajectory enters in the global minimizer convergent set $G_0(\mathbf{Z})$. In the following, we begin with $a_1(0)b_1(0) > 0$ for short. We discover a similar convergence rate in Stage 2.

Theorem 4.7 Assume $N \geq 3$, $a_1(0)b_1(0) > 0$. Then we have

$$1 - a_1(t)b_1(t) = \mathcal{O}\left(\left((N-2)t\right)^{-\frac{4}{N+1}}\right),$$

where $c_2 = \frac{2(s_1-s_2)}{s_1+s_0}$, and $s_0$ inherited from Theorem 4.4.

4.4 Convergence Rates of $a_1(t)b_1(t)$ and $s(t)$: Stage 3

Before we start our analysis in Stage 3, we need to handle the minor case $t_2 := \inf\{t : \mathbf{u}(t)^\top \mathbf{Zv}(t) \geq s(t)\} = +\infty$. We can assume $a_1(0)b_2(0) > 0$ from Stage 1.

Theorem 4.8 Suppose $N \geq 3$, $a_1(0)b_1(0) > 0$ and $t_2 = +\infty$. Then we have

$$1 - a_1(t)b_1(t) = \mathcal{O}\left(e^{-c_3t}\right), \quad |s(t) - s_1| = \mathcal{O}\left(e^{-c_4t}\right),$$

where $c_3 = 2s_1^{1-\frac{2}{N}}(s_1-s_2)$, $c_4 = Ns_1^{2-\frac{2}{N}}$.

Now we turn to the case $t_2 < +\infty$. Additionally, we can assume $s(0) \geq 0$ for short in Stage 3.

Theorem 4.9 Assume $N \geq 3$, $a_1(0)b_1(0) > 0$, and $s(0) \geq 0$. Then we have

$$1 - a_1(t)b_1(t) = \mathcal{O}\left(e^{-c_5t}\right), \quad |s_1 - s(t)| = \mathcal{O}\left(e^{-\min\{c_5,c_6\}t}\right),$$

where $c_5 = 2s(0)^{1-\frac{2}{N}}(s_1-s_2)$, $c_6 = Ns(0)^{2-\frac{2}{N}}$.

As we mentioned before, the difference between the minor case $t_2 = +\infty$ and Stage 3 is the constant above the exponent, and the proofs are similar between these two schemes. Thus, we combine them in a subsection.

Remark 4.10 Though we don’t provide an upper bound of $t_2$ here, we still have a slower global convergence guarantee of $\mathbf{u}(t), \mathbf{v}(t)$ following Stage 2. Moreover, we discover the linear rate in Stage 3 only appears in the late training phase from experiments (see Section 4), and gives high precision guarantee of solution at last. Furthermore, Ettekhari [12] also gave a linear rate in their restricted initialization set. Thus, we mainly focus on the previous stages to highlight that our results cover a larger initialization set.

5 Experiments

In this section, we conduct simple numerical experiments to verify our discovery.
Figure 1: We choose $N = 6, d = 5$ with hidden-layer width $(d_N, \ldots, d_0) = (5, 4, 1, 10, 5, 3, 8)$, and set different $k \in [0, d]$ in Theorem 3.8. The transparent horizontal lines are the singular value of $Z$ in order. Learning rate is $5 \times 10^{-4}$. Legends present $s(t), u(t)^\top Zv(t), u(t)^\top Z_1v(t)$. Here $t$ is the running step in gradient descent.

Different kinds of trajectories. We construct $u(0) = U\alpha_1$ and $v(0) = V\alpha_2$, where $\alpha_1 \in \mathbb{R}^{d_1 \times 1}$ and $\alpha_2 \in \mathbb{R}^{d_2 \times 1}$ have the inverse items until $k$-th dimension, i.e., $(\alpha_1)_i + (\alpha_2)_i = 0, \forall i \in [k], (\alpha_1)_{k+1} + (\alpha_2)_{k+1} \neq 0$, and $k \leq d$. Then we can see $a_i(0) + b_i(0) = u_i(0) + v_i(0) = 0, \forall i \in [k], a_{k+1}(0) + b_{k+1}(0) \neq 0$. After $u(0), v(0)$ decided, we construct $W_{(0)} = h_{k+1}h_{k+1}^\top$ with $\|h_i\| = 1$ and $h_1 = v(0), h_{N+1} = u(0)$ to obtain a balanced initialization $(W_1(0), \ldots, W_N(0))$ and $W(0) = u(0)v(0)^\top$. Finally, we run gradient descent (GD) for the problem (1) with a small learning rate $5 \times 10^{-4}$. The simulations are shown in Figure 1.

As Figure 1 depicts, we discover $u(t)^\top Zv(t), u(t)^\top Z_1v(t)$ are non-decreasing as our Proposition 2.2 shows. Moreover, we could see our construction gives a stuck region around $s_{k+1}$ according to the choice of $k \leq d$. Though our Theorem 3.8 shows that the gradient flow of $W(t)$ would finally converges $s_{k+1}u_{k+1}v_{k+1}^\top$, we find after a period of (long) time, gradient descent can escape from the saddle point around $s_{k+1}u_{k+1}v_{k+1}^\top$, and finally converges to a global minimizer. We consider the numerical error during optimization and unbalanced weight matrix caused by GD may lead to the inconsistent of gradient flow and its discrete version GD. Overall, we describe the possible convergence behavior of all initialization in the ideal setting.

Trajectories converging to the global minimizer. We also plot the trajectory converging to the global minimizer in detail shown Figure 2. To give a more clear variation of stages, we adopt $s(0) = 5$ and a small $u(0)^\top Zv(0)$. As the left figure of Figure 2 shown, $s(t)$ first decreases, then increases. Additionally, the middle figure shows that $s(t)$ decreases and $u(t)^\top Z_1v(t)$ increases with an approximate polynomial rate (noting the log scale in both x-axis and y-axis). Moreover, the middle figure also shows that once $s(t)$ increases,
6 Conclusion

In this work, we have studied the training dynamic of deep linear networks which have a one-neuron layer. Specifically, we focus on the gradient flow methods under the quadratic loss and balanced initialization. We have shown the convergent point of an arbitrary starting point. Moreover, we also give the convergence rates of training trajectories towards the global minimizer by stages. The behavior predicted by our theorems is also observed in numerical experiments. However, the analysis of general linear networks without a one-neuron layer remains a challenging open problem. We hope that our limited view of training trajectories would bring a better understanding of (linear) neural networks.

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A Auxiliary Conclusion

A.1 Previous Results

Lemma A.1 (Lemma 3.3 in Eftekhari [12]) For the induced flow in Eq. (3), we have that \( \text{rank}(W(t)) = \text{rank}(W(0)), \forall t \geq 0 \), provided that \( XX^\top \) is invertible and the network depth \( N \geq 2 \).

Lemma A.2 (Lemma 4 in Arora et al. [4]) Let \( \alpha \geq 1/2 \) and \( g : [0, \infty) \to \mathbb{R} \) be a continuous function. Consider the initial value problem:

\[
s(0) = s_0, \quad \dot{s}(t) = (s^2(t))^\alpha \cdot g(t), \quad \forall t \geq 0,
\]

where \( s_0 \in \mathbb{R} \). Then, as long as it does not diverge to \( \pm \infty \), the solution to this problem \( (s(t)) \) has the same sign as its initial value \( (s_0) \). That is, \( s(t) \) is identically zero if \( s_0 = 0 \), is positive if \( s_0 > 0 \), and is negative if \( s_0 < 0 \).

Theorem A.3 (Theorem 5 in Bah et al. [5]) Assume \( XX^\top \) has full rank. Then the flows \( W_i(t) \) defined by Eq. (2) and \( W(t) \) given by Eq. (3) are defined and bounded for all \( t \geq 0 \) and \( (W_1, \ldots, W_N) \) converges to a critical point of \( L^N \) as \( t \to +\infty \).

Definition A.4 (Definition 27 in Bah et al. [5]) Let \((M, g)\) be a Riemannian manifold with Levi-Civita connection \( \nabla \) and let \( f : M \to \mathbb{R} \) be a twice continuously differentiable function. A critical point \( x_0 \in M \), i.e., \( \nabla f(x_0) = 0 \) is called a strict saddle point if \( \text{Hess} f(x) \) has a negative eigenvalue. We denote the set of all strict saddles of \( f \) by \( X = X(f) \). We say that \( f \) has the strict saddle point property, if all critical points of \( f \) that are not local minima are strict saddle points.

The following theorem shows that flows avoid strict saddle points almost surely (See Section 6.2 in Bah et al. [5] for detail).

Theorem A.5 (Theorem 28 in Bah et al. [5]) Let \( L : M \to \mathbb{R} \) be a \( C^2 \)-function on a second countable finite dimensional Riemannian manifold \((M, g)\), where we assume that \( M \) is of class \( C^2 \) as a manifold and the metric \( g \) is of class \( C^1 \). Assume that \( \phi_t(x_0) \) exists for all \( x_0 \in M \) and all \( t \in [0, +\infty) \). Then the set

\[
S_L := \{ x_0 \in M : \lim_{t \to +\infty} \phi_t(x_0) \in X = X(L) \}
\]

of initial points such that the corresponding flow converges to a strict saddle point of \( L \) has measure zero.

Proposition A.6 (Proposition 33 in Bah et al. [5]) The function \( L^1 \) on \( M_k \) for \( k \leq r \) satisfies the strict saddle point property. More precisely, all critical points of \( L^1 \) on \( M_k \) except for the global minimizers are strict saddle points.
A.2 Auxiliary Lemmas

Lemma A.7 (Dynamic of \( s(t), u(t), v(t) \)) We give the derivation of \( \dot{u}(t), \dot{v}(t), \dot{s}(t) \) shown in the main context in this lemma:

\[
\dot{u}(t) = s(t)^{1-\frac{2}{N}} (I_d - u(t)u(t)^\top) Zv(t), \\
\dot{v}(t) = s(t)^{1-\frac{2}{N}} (I_d - v(t)v(t)^\top) Z^\top u(t), \\
\dot{s}(t) = Ns(t)^{2-\frac{2}{N}} (u(t)^\top Zv(t) - s(t)).
\]

Proof: \( \dot{s}(t) \) directly follows Arora et al. [34] Theorem 3. As for \( \dot{u}(t) \) and \( \dot{v}(t) \), we begin with \( u(t)^\top u(t) = v(t)^\top v(t) = 1 \). Then by taking the derivative of the identities, we get

\[
(18) \quad u(t)^\top \dot{u}(t) = v(t)^\top \dot{v}(t) = 0, \forall t \geq 0.
\]

By taking derivative of both sides of the SVD \( W(t) = s(t)u(t)v(t)^\top \), we also find that

\[
\dot{W}(t) = s(t)\dot{u}(t)v(t)^\top + s(t)u(t)\dot{v}(t)^\top + \dot{s}(t)u(t)v(t)^\top, \forall t \geq 0.
\]

Hence, multiplying \( (I_d - u(t)u(t)^\top) \) and \( v(t) \), we get

\[
s(t)^{-1} (I_d - u(t)u(t)^\top) \dot{W}(t)v(t) = (I_d - u(t)u(t)^\top) \dot{u}(t).
\]

From Eq. (18), we know \( \dot{u}(t) \perp u(t) \). Therefore, we obtain

\[
(19) \quad \dot{u}(t) = s(t)^{-1} (I_d - u(t)u(t)^\top) \dot{W}(t)v(t).
\]

Similarly, we can find that

\[
(20) \quad \dot{v}(t) = s(t)^{-1} (I_d - v(t)v(t)^\top) \dot{W}(t)^\top u(t).
\]

Now we replace \( \dot{W}(t) \) by Eq. (3) and \( W(t) = s(t)u(t)v(t)^\top \):

\[
\dot{W}(t)^\top = -Ns(t)^{2-\frac{2}{N}} u(t)u(t)^\top [W(t) - Z] v(t)v(t)^\top \\
- s(t)^{2-\frac{2}{N}} (I_d - u(t)u(t)^\top) [W(t) - Z] v(t)v(t)^\top \\
- s(t)^{2-\frac{2}{N}} u(t)u(t)^\top [W(t) - Z] (I_d - v(t)v(t)^\top) \\
= -Ns(t)^{1-\frac{2}{N}} (s(t) - u(t)^\top Zv(t)) \dot{W}(t) \\
+ s(t)^{2-\frac{2}{N}} (I_d - u(t)u(t)^\top) Zv(t)v(t)^\top \\
+ s(t)^{2-\frac{2}{N}} u(t)u(t)^\top Z (I_d - v(t)v(t)^\top).
\]

Substituting \( \dot{W}(t) \) back into Eq. (19) and (20), we reach

\[
\dot{u}(t) = s(t)^{1-\frac{2}{N}} (I_d - u(t)u(t)^\top) Zv(t), \quad \dot{v}(t) = s(t)^{1-\frac{2}{N}} (I_d - v(t)v(t)^\top) Z^\top u(t).
\]

\[
\square
\]

Proposition A.8 (Stationary Singular Vector) If for time \( T \geq 0, s(T) > 0, \dot{u}(T) = 0, \dot{v}(T) = 0 \), then

\[
\dot{u}(t) = \dot{v}(t) = 0, \forall t \geq T, \text{ that is, } u(t) = u(T), v(t) = v(T), \forall t \geq T.
\]

Moreover, \( \dot{u}(t) = 0, \dot{v}(t) = 0, \forall t \geq T \), for some \( i \leq d \), or \( u(t) \perp u_i, v(t) \perp v_i, \forall i \leq d \).

\[\text{Lemma A.7 (Dynamic of } s(t), u(t), v(t) \text{)} \]

\[\checkmark\]

\[\checkmark\]
Proof: From $s(T) > 0, \dot{u}(T) = 0, \dot{v}(T) = 0$ and Eqs. (4) and (5), we obtain

$$(I_{d_y} - u(T)u(T)^\top) Zv(T) = 0, \quad (I_{d_x} - v(T)v(T)^\top) Z^\top u(T) = 0. \quad (21)$$

Hence, we could see

$$Zv(T) = u(T)^\top Zv(T) \cdot u(T), \quad Z^{\top} u(T) = v(T)^{\top} Z^{\top} u(T) \cdot v(T), \quad \text{(22)}$$

showing that $Z^\top Zv(T) = [u(T)^\top Zv(T)]^2 \cdot v(T), \quad ZZ^\top u(T) = [u(T)^\top Zv(T)]^2 \cdot u(T)$. Thus, we can see $u(T), v(T)$ are the eigenvectors of $Z^\top Z, ZZ^\top$ with the same eigenvalue $[u(T)^\top Zv(T)]^2$. Therefore, if $u(T)^\top Zv(T) \neq 0$, we obtain $u(T) = \pm u_i, v(T) = \pm v_i$, for some $i \in [d]$. Otherwise, $u(T)^\top Zv(T) = 0$. From Eq. (22), we obtain $Zv(T) = 0, Z^{\top} u(T) = 0$, showing that $u(T) \perp u_i, v(T) \perp v_i, \forall i \in [d]$.

Finally, we note that variation of $s(t)$ can not make $\dot{u}(t), \dot{v}(t)$ become nonzero from time $T$. Specifically,

$$W(T) = -Ns(T)^2 - \frac{\dot{s}}{s} u(T)u(T)^\top |W(T) - Z| v(T)v(T)^\top$$

and

$$= \begin{cases} 
&W(T) - Ns(T)^2 - \frac{\dot{s}}{s} u(T)u(T)^\top |W(T) - Z| v(T)v(T)^\top \\
& \quad - Ns(T)^2 - \frac{\dot{s}}{s} u(T)u(T)^\top |W(T) - Z| (I_{d_x} - v(T)v(T)^\top) \\
& = \begin{pmatrix} 2 \end{pmatrix} \end{cases}$$

Thus, if $\dot{u}(t) \neq 0$ or $\dot{v}(t) \neq 0$, and $s(t) > 0$, by Eqs. (4) and (5), we obtain

$$\| (I_{d_y} - u(t)u(t)^\top) Zv(t) \|^2_2 + \| (I_{d_x} - v(t)v(t)^\top) Z^\top u(t) \|^2_2 > 0. \quad (23)$$

From the derivation of $d(u(T)^\top Zv(T) - s(t))/dt$ and $ds(t)/dt$, we get

$$\frac{d(u(T)^\top Zv(t) - s(t))}{dt} = s(t)^{1 - \frac{\dot{s}}{s}} \left( \| (I_{d_y} - u(t)u(t)^\top) Zv(t) \|^2_2 + \| (I_{d_x} - v(t)v(t)^\top) Z^\top u(t) \|^2_2 - Ns(t) (u(t)^\top Zv(t) - s(t)) \right)$$

$$= s(t)^{1 - \frac{\dot{s}}{s}} \left( \| (I_{d_y} - u(t)u(t)^\top) Zv(t) \|^2_2 + \| (I_{d_x} - v(t)v(t)^\top) Z^\top u(t) \|^2_2 \right) \geq 0,$$

where the second equality uses the assumption $u(t)^\top Zv(t) = s(t)$.

\section{B Missing Proofs}

\subsection{B.1 Proof of Proposition 2.2}

Proof: 1). From Bah et al. [5] Theorem 5 (Theorem A.3), we have $W(t)$ converges. Thus $s(t) = \|W(t)\|$, also converges, and not diverges to infinity. Applying Arora et al. [4] Lemma 4 (Lemma A.2), we can see $s(t)$ obviously preserves the sign of its initial value.
2). \( u(t)^T Z v(t) \) is non-decreasing follows

\[
\frac{du(t)^T Z v(t)}{dt} = \frac{du(t)^T}{dt} Z v(t) + u(t)^T Z \frac{dv(t)}{dt}
\]

\[ (24) \]

Also, for \( s(t)^{1 - \frac{2}{d}} \) \( \left( \| (I_{d_x} - u(t)u(t)^T) Z v(t) \|_2^2 + \| (I_{d_x} - v(t)v(t)^T) Z^T u(t) \|_2^2 \right) \geq 0. \)

Additionally, since \( \| u(t) \| = \| v(t) \| = 1 \), we have \( u(t)^T Z v(t) \leq s_1 \). Hence, \( u(t)^T Z v(t) \) converges.

3). Using Eq. (9), we obtain

\[
\frac{da_1(t)b_1(t)}{dt} = \frac{da_1(t)}{dt} \cdot b_1(t) + a_1(t) \cdot \frac{db_1(t)}{dt} \geq s(t)^{1 - \frac{2}{d}} \left( s_1 b_1^2(t) + s_1 a_1^2(t) - 2s_1 a_1(t)(b_1(t) \sum_{j=1}^{d} |a_j(t)b_j(t)|) \right)
\]

\[
\geq s(t)^{1 - \frac{2}{d}} \left( s_1 b_1^2(t) + s_1 a_1^2(t) - 2s_1 |a_1(t)|b_1(t) \right) = s_1 s(t)^{1 - \frac{2}{d}} (|b_1(t)| - |a_1(t)|)^2 \geq 0,
\]

where the second inequality uses Cauchy inequality:

\[
\left( \sum_{j=1}^{d} a_j(t)b_j(t) \right)^2 \leq \left( \sum_{j=1}^{d} a_j^2(t) \right) \cdot \left( \sum_{j=1}^{d} b_j^2(t) \right) \leq \left( \sum_{j=1}^{d} a_j^2(t) \right) \cdot \left( \sum_{j=1}^{d} b_j^2(t) \right) = 1.
\]

We note that \( s_1 a_1(t)b_1(t) = s_1 \cdot u(t)^T u_1 \cdot v_1^T v(t) = u(t)^T (s_1 u_1 v_1^T) v(t) = u(t)^T Z_1 v(t) \). Hence, we obtain \( u(t)^T Z_1 v(t) \) is non-decreasing. Moreover, since \( \| u(t) \| = \| v(t) \| = 1 \), we have \( u(t)^T Z_1 v(t) \leq s_1 \). Hence, \( u(t)^T Z_1 v(t) \) also converges.

4). Using the derivative in the above, we obtain

\[
\frac{d(a_i(t) + b_i(t))}{dt} \geq s(t)^{1 - \frac{2}{d}} \left[ s_i (b_i(t) + a_i(t)) - (a_i(t) + b_i(t)) \sum_{j=1}^{d} |s_j a_j(t)b_j(t)| \right]
\]

\[ (25) \]

Moreover, \( |a_i(t) + b_i(t)| = |v_i^T u(t) + a_i^T v(t)| \leq 2 \), showing that \( a_i(t) + b_i(t) \) does not diverge to infinity. Hence, by Arora et al. [3] Lemma 4) (Lemma A.2), \( a_i(t) + b_i(t) \) obviously preserves the sign of its initial value.

5). Since \( a_i(0) + b_i(0) = 0 \), we get \( a_i(t) + b_i(t) = 0 \) by 4), i.e.,

\[
a_i(t) = -b_i(t), \forall i \in [k], t \geq 0.
\]

(26)

Now we can bound

\[
\sum_{j=k+1}^{d} s_j a_j(t)b_j(t) \leq s_{k+1} \sum_{j=k+1}^{d} \left| a_j(t)b_j(t) \right| \leq s_{k+1} \sum_{j=k+1}^{d} a_j^2(t) \sum_{j=k+1}^{d} b_j^2(t)
\]

\[ (27) \]

Hence, we obtain

\[
s_{k+1} - \sum_{j=1}^{d} [s_j a_j(t)b_j(t)] \geq s_{k+1} + \sum_{j=1}^{k} s_j a_j^2(t) - \sum_{j=k+1}^{d} [s_j a_j(t)b_j(t)] \]

\[ (28) \]

\[
\geq s_{k+1} \left( 1 + \sum_{j=1}^{k} a_j^2(t) \right) - s_{k+1} \left( 1 - \sum_{j=1}^{k} a_j^2(t) \right) = 2s_{k+1} \sum_{j=1}^{k} a_j^2(t) \geq 0.
\]
Now we consider the gradient of $a_{k+1}(t) + b_{k+1}(t)$:

$$\frac{d(a_{k+1}(t) + b_{k+1}(t))}{dt} s(t)^{1 - \frac{2}{N}} (a_{k+1}(t) + b_{k+1}(t)) = \left( s_{k+1} - \sum_{j=1}^{d} [s_j a_j b_j(t)] \right).$$  \quad (29)

If $a_{k+1}(t) + b_{k+1}(t) > 0$, from Eqs. (28) and (29) we can see $d(a_{k+1}(t) + b_{k+1}(t)) / dt \leq 0$. Thus, $a_{k+1}(t) + b_{k+1}(t)$ is non-decreasing. The case of $a_{k+1}(t) + b_{k+1}(t) < 0$ is similar. Therefore, we get $|a_{k+1}(t) + b_{k+1}(t)|$ is non-decreasing. Since $\|u(t)\| = \|v(t)\| = 1$, we have $|a_{k+1}(t) + b_{k+1}(t)| = |\nu_{k+1} u(t) + v_{k+1} v(t)| \leq 2$. Hence, $|a_{k+1}(t) + b_{k+1}(t)|$ converges. Moreover, we note that from 4), $a_{k+1}(t) + b_{k+1}(t)$ preserves the sign of its initial value, showing that $\lim_{t \to +\infty} a_{k+1}(t) + b_{k+1}(t)$ exists.

\[ \Box \]

\subsection*{B.2 Proof of Lemma 3.4}

Proof: From 1) in Proposition 2.2 and $s(0) \neq 0$, we obtain $s(t) > 0, \forall t > 0$.

\textbf{Case 1.} If $u(t_0)^\top Z v(t_0) > 0$ for some $t_0 \geq 0$, we get $W(t_0) \in \mathcal{H}_b(Z)$ for some $b > 0$. Hence, by Lemma 3.1 we obtain $\ddot{s} > 0$. From Eq. (6), we obtain

$$0 = \lim_{t \to +\infty} N s(t)^{2 - \frac{2}{N}} (u(t)^\top Z v(t) - s(t)).$$

Using $s(t) \to \ddot{s} > 0$ again, we obtain $\lim_{t \to +\infty} u(t)^\top Z v(t)$ exists. Therefore,

$$0 = \lim_{t \to +\infty} \frac{du(t)^\top Z v(t)}{dt} = \lim_{t \to +\infty} s(t)^{1 - \frac{2}{N}} \left[ \left\| (I_{d_y} - u(t)u(t)^\top) Z v(t) \right\|_2^2 + \left\| (I_{d_x} - v(t)v(t)^\top) Z^\top u(t) \right\|_2^2 \right].$$

By $s(t) \to \ddot{s} > 0$, we obtain $(I_{d_y} - u(t)u(t)^\top) Z v(t) \to 0$, and $(I_{d_x} - v(t)v(t)^\top) Z^\top u(t) \to 0$. Thus, we can choose $t_n$ for example.

\textbf{Case 2.} If $u(t)^\top Z v(t) \leq 0, \forall t \geq 0$. Then from Eq. (6), we get $\ddot{s}(t) \leq 0, \forall t \geq 0$. Hence, $s(t) \leq s(0)$. Moreover, by 2) in Proposition 2.2 we have $u(t)^\top Z v(t) \geq u(0)^\top Z v(0)$. Therefore,

$$\dot{s}(t) = N s(t)^{2 - \frac{2}{N}} (u(t)^\top Z v(t) - s(t)) \geq N s(t)^{2 - \frac{2}{N}} (u(0)^\top Z v(0) - s(0)).$$  \quad (30)

Now we denote

$$C(a) := \inf_{t \geq a} \left\{ \left\| (I_{d_y} - u(t)u(t)^\top) Z v(t) \right\|_2^2 + \left\| (I_{d_x} - v(t)v(t)^\top) Z^\top u(t) \right\|_2^2 \right\} \geq 0.$$

In the following, we show that $C(a) = 0, \forall a \geq 0$.

1) If $N = 2$, then we can see $\forall t \geq a$, by $u(t)^\top Z v(t) \leq 0, \forall t \geq 0$,

$$-u(a)^\top Z v(a) \geq u(t)^\top Z v(t) - u(a)^\top Z v(a) = \int_a^t \frac{du(x)^\top Z v(x)}{dx} dx \geq C(a)(t-a).$$

Taking $t \to +\infty$, we obtain $C(a) = 0, \forall a \geq 0$.

2) If $N > 2$, by solving Eq. (30), we get

$$\frac{N}{2-N} s(t)^{\frac{2}{N}-1} - \frac{N}{2-N} s(0)^{\frac{2}{N}-1} \geq N (u(0)^\top Z v(0) - s(0)) t.$$

Therefore, we obtain

$$s(t)^{\frac{2}{N}-1} \leq (N-2)(s(0) - u(0)^\top Z v(0)) t + s(0)^{\frac{2}{N}-1} := A + Bt, A, B > 0.$$

(31)
Then we can see $\forall t \geq a$,

\[
-u(a)^T Z v(a) \geq u(t)^T Z v(t) - u(a)^T Z v(a) = \int_a^t \frac{du(x)^T Z v(x)}{dx} dx
\]

\[
\geq C(a) \int_a^t \frac{1}{A + Bx} dx \geq C(a) \int_a^t \frac{1}{A + Bx} dx = \frac{C(a)}{B} \ln \frac{A + Bt}{A + Ba}.
\]

Taking $t \to +\infty$, we obtain $C(a) = 0, \forall a \geq 0$.

Therefore, combining 1) and 2), we conclude $C(a) = 0, \forall a \geq 0$. Hence, we can find a sequence $\{t_n\}$ with $t_n \to +\infty$, s.t.,

\[
0 = \lim_{n \to +\infty} \| (I_{dp} - u(t_n)u(t_n)^T) \|_2^2 + \| (I_{dx} - v(t_n)v(t_n)^T) Z^T u(t_n) \|_2^2.
\]

Thus, $\lim_{n \to +\infty} (I_{dp} - u(t_n)u(t_n)^T) Z v(t_n) = 0$, and $\lim_{n \to +\infty} (I_{dx} - v(t_n)v(t_n)^T) Z^T u(t_n) = 0$.

Now we adopt the expansion following Eq. (7): $u(t) = \sum_{i=1}^d a_i(t)u_i$, $v(t) = \sum_{i=1}^d b_i(t)v_i$. Thus, we have

\[
u(t)^T Z v(t) = \sum_{j=1}^d s_j a_j(t)b_j(t), Z v(t) = \sum_{i=1}^d s_i b_i(t)u_i, Z^T u(t) = \sum_{i=1}^d s_i a_i(t)v_i.
\]

Therefore, we obtain

\[
(I_{dp} - u(t_n)u(t_n)^T) Z v(t_n) = \sum_{i=1}^d \left[ s_i b_i(t_n) - \left( \sum_{j=1}^d s_j a_j(t_n)b_j(t_n) \right) a_i(t_n) \right] u_i,
\]

where we utilize $s_i = 0, \forall i > d$. Since $\lim_{n \to +\infty} (I_{dp} - u(t_n)u(t_n)^T) Z v(t_n) = 0$, and $u_i$s are orthonormal basis, we obtain Eq. (10). Similarly, we could obtain Eq. (11) by $\lim_{n \to +\infty} (I_{dx} - v(t_n)v(t_n)^T) Z^T u(t_n) = 0$.

Finally, adding the equation in Eq. (10) and Eq. (11), we obtain

\[
\lim_{n \to +\infty} \left( \sum_{j=1}^d s_j a_j(t_n)b_j(t_n) - s_i \right) \left( a_i(t_n) + b_i(t_n) \right) = 0, \forall i \in [d].
\]

Since we have for some $i_0 \in [d]$ that $\lim_{n \to +\infty} a_{i_0}(t_n) + b_{i_0}(t_n)$ exists and not zero. Thus we obtain

\[
\lim_{n \to +\infty} u(t_n)^T Z v(t_n) = \lim_{n \to +\infty} \sum_{j=1}^d s_j a_j(t_n)b_j(t_n) = s_{i_0}.
\]

The proof is finished. $\square$

**B.3 Proof of Lemma 3.5**

Proof: Since $a_i(t_n) + b_i(t_n) = 0, \forall i \in [k], n \geq 0$, we obtain

\[
b_i(t_n) = -a_i(t_n), \forall i \in [k], n \geq 0,
\]

and

\[
\sum_{j=1}^d s_j a_j(t_n)b_j(t_n) = -\sum_{j=1}^k s_j a_j^2(t_n) + \sum_{j=k+1}^d s_j a_j(t_n)b_j(t_n) \leq s_{k+1} \left( 1 - \sum_{j=1}^k a_j^2(t_n) \right).
\]
Taking limit inferior in both sides and noting that \( \lim_{n \to +\infty} \sum_{j=1}^d s_j a_j(t_n) b_j(t_n) = s_{k+1} \), we get
\[
s_{k+1} \leq \liminf_{n \to +\infty} s_{k+1} \left( 1 - \sum_{j=1}^k a_j^2(t_n) \right). \tag{33}
\]

Moreover, naturally we have
\[
\limsup_{n \to +\infty} s_{k+1} \left( 1 - \sum_{j=1}^k a_j^2(t_n) \right) \leq s_{k+1}. \tag{34}
\]

By Eq. (33) and Eq. (34), we obtain \( \lim_{n \to +\infty} \sum_{j=1}^k a_j^2(t_n) = 0 \), showing that
\[
\lim_{n \to +\infty} -b_j(t_n) = \lim_{n \to +\infty} a_j(t_n) = 0, \forall j \in [k]. \tag{35}
\]

Hence, we derive that
\[
\lim_{n \to +\infty} \sum_{j=k+1}^{d} s_j a_j(t_n) b_j(t_n) = \lim_{n \to +\infty} \sum_{j=1}^{d} s_j a_j(t_n) b_j(t_n) - \lim_{n \to +\infty} \sum_{j=1}^{k} s_j a_j(t_n) b_j(t_n) \tag{36}
\]

Using Cauchy inequality, we have
\[
\left[ \sum_{j=k+1}^{d} s_j a_j(t_n) b_j(t_n) \right]^2 \leq \sum_{j=k+1}^{d} s_j^2 a_j^2(t_n) \cdot \sum_{j=k+1}^{d} b_j^2(t_n) \leq \left( \sum_{j=k+1}^{d} s_j^2 a_j^2(t_n) \right) \cdot \left( 1 - \sum_{j=1}^{k} b_j^2(t_n) \right). \tag{37}
\]

Since \( \lim_{n \to +\infty} \sum_{j=1}^{k} b_j^2(t_n) = 0 \), and \( \sum_{j=1}^{d} a_j^2(t_n) = 1 \), we obtain
\[
\lim_{n \to +\infty} \sum_{j=k+1}^{d} s_j a_j(t_n) b_j(t_n) = \lim_{n \to +\infty} \sum_{j=k+1}^{d} s_j^2 a_j^2(t_n). \tag{38}
\]

Noting that \( s_j = 0, \forall j > d \), we get \( 0 \leq \liminf_{n \to +\infty} \sum_{j=k+2}^{d} (s_j^2 - s_{k+1}^2) a_j^2(t_n) \).

However, \( s_j^2 - s_{k+1}^2 < 0, \forall j \geq k + 2 \) and \( a_j^2(t_n) \geq 0 \), showing that \( \limsup_{n \to +\infty} \sum_{j=k+2}^{d} (s_j^2 - s_{k+1}^2) a_j^2(t_n) \leq 0 \).

Hence, we obtain \( \lim_{n \to +\infty} a_j(t_n) = 0, \forall j \geq k + 2 \). The similar analysis holds for \( b_j(t_n) \). Therefore, we obtain
\[
\lim_{n \to +\infty} s_{k+1} a_{k+1}(t_n) b_{k+1}(t_n) = \lim_{n \to +\infty} \sum_{j=1}^{d} s_j a_j(t_n) b_j(t_n) = s_{k+1}.
\]

Finally, we have
\[
u_{k+1}^T u_{k+1} \lim_{n \to +\infty} a_{k+1}(t_n) b_{k+1}(t_n) u_{k+1} = \lim_{n \to +\infty} \sum_{i,j} a_i(t_n) b_j(t_n) u_i v_j^T \lim_{n \to +\infty} u(t_n) v(t_n)^T.
\]

The proof is finished. \( \square \)

### B.4 Proof of Theorem 4.2

Proof: (I) The truth that \( \dot{c}_1(t) \geq 0 \) is direct from 3) in Proposition 2.2. Moreover, from Theorem 3.8 with \( k = 0 \), we have \( c_1(t) = a_1(t) b_1(t) \to 1 \). Thus, we obtain \( r_1 \to +\infty \).
(II) As for $s(t)$, if $t_2 = +\infty$, then $\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) \leq s(t), \forall t \in [0, +\infty)$. Thus by Eq. (40), $\dot{s}(t) \leq 0$ for all $t \geq 0$. Now we consider the remaining case where $t_2 < +\infty$. Since $t_2 = \inf \{ t : \mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) \geq s(t) \}$, we have $\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) \leq s(t)$ when $t \in [0, t_2)$. Thus

$$\dot{s}(t) = Ns(t)^{2-\frac{2}{\sqrt{T}}} (\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) - s(t)) \leq 0, \forall t \in [0, t_2).$$

Now from $t_2 < +\infty$, we get $\mathbf{u}(t_2)^\top \mathbf{Z} \mathbf{v}(t_2) = s(t_2)$. We denote $T = \inf \{ t : \dot{u}(t) = 0, \dot{v}(t) = 0 \}$.

(i) When $T > t_2$. Then for $t \in [t_2, T)$, we have $\dot{u}(t) \neq 0$ or $\dot{v}(t) \neq 0$. Thus, applying lemma A.9, we have

$$\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) = s(t) \Rightarrow d(\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) - s(t))/dt > 0, \forall t \in [t_2, T).$$

By Lin et al. [30, Lemma 10], we obtain $\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) \geq s(t), \forall t \in [t_2, T)$. Hence,

$$\dot{s}(t) =Ns(t)^{2-\frac{2}{\sqrt{T}}} (\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) - s(t)) \geq 0, \forall t \in [t_2, T).$$

And for $t \geq T$, we get $T < +\infty$. We obtain stationary singular vectors from time $T$ by Proposition A.8. Thus, $\mathbf{u}(T)^\top \mathbf{Z} \mathbf{v}(T) = c, \forall t \geq T$ for a constant $c$, which reduce the variation of $s(t)$ as

$$\dot{s}(t) = Ns(t)^{2-\frac{2}{\sqrt{T}}} (c - s(t)), \forall t \geq T.$$

Moreover, since $\mathbf{u}(t)^\top \mathbf{Z} \mathbf{v}(t) \geq s(t), \forall t \in [t_2, T)$, we obtain $c = \mathbf{u}(T)^\top \mathbf{Z} \mathbf{v}(T) \geq s(T)$. Hence, we can see $\dot{s}(t) \geq 0, \forall t \geq T$.

(ii) When $T \leq t_2$, we have $T < +\infty$. We obtain stationary singular vectors from time $T$ by Proposition A.8. Thus, $\mathbf{u}(T)^\top \mathbf{Z} \mathbf{v}(T) = c, \forall t \geq T$ for a constant $c$, which reduce the variation of $s(t)$ as

$$\dot{s}(t) = Ns(t)^{2-\frac{2}{\sqrt{T}}} (c - s(t)), \forall t \geq T.$$

We note that $c = \mathbf{u}(t_2)^\top \mathbf{Z} \mathbf{v}(t_2) = s(t_2)$. Thus $\dot{s}(t) = 0, \forall t \geq t_2$. The proof is finished. \hfill \Box

### B.5 Proof of Theorem 4.4

Proof: We first consider the upper bound. From $s(0) > 0$ and 1) in Proposition 2.2, we have $s(t) > 0$ for all $t \geq 0$. Moreover, we have

$$\dot{s}(t) = Ns(t)^{2-\frac{2}{\sqrt{T}}} \left( \sum_{j=1}^{d} s_j a_j(t)b_j(t) - s(t) \right) \leq Ns(t)^{2-\frac{2}{\sqrt{T}}} (s_1 - s(t)). \tag{38}$$

Let $\tilde{s}(t)$ be the solution of the ODE

$$\dot{s}(t) = N\tilde{s}(t)^{2-\frac{2}{\sqrt{T}}} (s_1 - \tilde{s}(t)), \quad \tilde{s}(0) = s(0).$$

Then we can see $s(t) \leq \tilde{s}(t)$ from Eq. (38).

If $s(0) > s_1$, then $\tilde{s}(t) \leq \tilde{s}(0) = s(0)$, showing that $s(t) \leq \tilde{s}(t) \leq s(0)$. Otherwise, $s(0) \leq s_1$, we get $\tilde{s}(t) \leq s_1$, showing that $s(t) \leq \tilde{s}(t) \leq s_1$. Therefore, we know $s(t) < s_0 := \max\{s_1, s(0)\}$ for all $t \geq 0$.

Now we consider the lower bound. Note that

$$\sum_{j=2}^{d} s_j a_j(t)b_j(t) \leq s_2 \sqrt{(1 - a_1^2(t))(1 - b_1^2(t))} = s_2 \sqrt{a_2^2(t)b_2^2(t) - a_2^2(t) - b_2^2(t) + 1} \leq s_2 \sqrt{a_2^2(t)b_2^2(t) - 2|a_1(t)b_1(t)| + 1} = s_2 (1 - |a_1(t)b_1(t)|), \tag{39}$$

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where we use $|a_1(t)b_1(t)| = |\mathbf{u}_1^T \mathbf{u}(t) \cdot \mathbf{v}_1^T \mathbf{v}(t)| \leq 1$ in the last equality.

\[
\dot{s}(t) - Ns(t)^{2-\frac{2}{N}} \left( \sum_{j=1}^{d} s_j a_j(t)b_j(t) - s(t) \right) \geq Ns(t)^{2-\frac{2}{N}} [s_1 a_1(t)b_1(t) - s_2 (1 - |a_1(t)b_1(t)|)] - s(t) \tag{40}
\]

\[
\geq Ns(t)^{2-\frac{2}{N}} [(s_2 - s_1) |a_1(t)b_1(t)| - s_2 - s(t)] \geq -N(s_1 + s(t))s(t)^{2-\frac{2}{N}} \geq -N(s_1 + s_0)s(t)^{2-\frac{2}{N}},
\]

where the last inequality uses $s(t) < s_0 := \max\{s_1, s(0)\}$, which is proved previously.

When $N = 2$, we solve Eq. (40) and get

\[
\dot{s}(t) \geq -2(s_1 + s_0)s(t) \Rightarrow \frac{d}{dt} (\ln s(t)) \geq -2(s_1 + s_0) \Rightarrow \ln s(t) \geq -2(s_1 + s_0)t \Rightarrow s(t) \geq s(0)e^{-2(s_1+s_0)t}.
\]

When $N \geq 3$, we solve Eq. (40) and get

\[
\frac{N}{2-N} \cdot \frac{d}{dt} \left( \frac{s(t)^{\frac{2}{N}} - 1}{s(t)} \right) \geq -N(s_1 + s_0) \Rightarrow s(t)^{\frac{2}{N}} - s(0)^{\frac{2}{N}} - 1 \leq (s_1 + s_0)(N - 2)t.
\]

Thus, we finally obtain

\[
s(t) \geq \left( (s_1 + s_0)(N - 2)t + s(0)^{\frac{2}{N}} - 1 \right)^{-\frac{N}{N-2}}.
\]

The proof of Eqs. (14) and (15) is finished. \[\square\]

**B.6 Proof of Theorem 4.5**

Proof: Since $a_1(0)b_1(0) < 0, a_1(0) + b_1(0) \neq 0$, without loss of generality, we suppose $a_1(0) > 0, b_1(0) < 0$ and $a_1(0) + b_1(0) > 0$. Note that

\[
\dot{a}_1(t) - \dot{b}_1(t) = s(t)^{1-\frac{2}{N}} (b_1(t) - a_1(t)) \left( s_1 + \sum_{j=1}^{d} [s_j a_j(t)b_j(t)] \right).
\]

By Arora et al. [4] Lemma 4 and $|a_1(t) - b_1(t)| \leq 2$, we get that $a_1(t) - b_1(t)$ preserves the sign of its initial value:

\[
a_1(t) - b_1(t) > 0, \forall t \geq 0. \tag{41}
\]

Moreover, from 5) in Proposition 2.2 and $a_1(0) + b_1(0) > 0$, we obtain

\[
a_1(t) + b_1(t) \geq a_1(0) + b_1(0) > 0, \forall t \geq 0. \tag{42}
\]

Then we have

\[
a_1(t) \geq \frac{a_1(t) + b_1(t)}{2} \geq \frac{a_1(0) + b_1(0)}{2} > 0, \forall t \geq 0, \tag{43}
\]

and

\[
-a_1(t) \leq \frac{b_1(t)}{a_1(t)} \leq \frac{a_1(t) + b_1(t)}{a_1(t)} \geq 0 \Rightarrow -1 < \frac{b_1(t)}{a_1(t)} < 1, \forall t \geq 0. \tag{44}
\]

Furthermore, we can derive that

\[
\frac{d}{dt} \left( \frac{b_1(t)}{a_1(t)} \right) = \frac{\dot{b}_1(t)a_1(t) - \dot{a}_1(t)b_1(t)}{a_1^2(t)} \geq s(t)^{1-\frac{2}{N}} \left( 1 - \left( \frac{b_1(t)}{a_1(t)} \right)^2 \right) \geq 0.
\]

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Solving the above ODE, we obtain
\[
d \left( \ln \sqrt{\frac{a_1(t) + b_1(t)}{a_1(t) - b_1(t)}} \right) / dt \geq s(t)^{1 - \frac{2}{s}} \geq \frac{1}{2(s_1 + s_0)(N - 2)t + s(0)^{\frac{2}{s} - 1}}.
\]
Therefore, we obtain
\[
\ln \sqrt{\frac{a_1(t) + b_1(t)}{a_1(t) - b_1(t)}} - \ln \sqrt{\frac{a_1(0) + b_1(0)}{a_1(0) - b_1(0)}} \geq \int_0^t \frac{dx}{2(s_1 + s_0)(N - 2)x + s(0)^{\frac{2}{s} - 1}} = \frac{1}{2(s_1 + s_0)(N - 2)} \ln \left[ 1 + \frac{2(s_1 + s_0)(N - 2)t}{s(0)^{\frac{2}{s} - 1}} \right].
\]
We hide constants related to initialization, and rewrite the inequality as
\[
\frac{a_1(t) + b_1(t)}{a_1(t) - b_1(t)} \geq C_1(1 + A_1t)^{B_1},
\]
where \( A_1 := \frac{2(s_1 + s_0)(N - 2)}{s(0)^{\frac{2}{s} - 1}} \), \( B_1 := \frac{1}{(s_1 + s_0)(N - 2)} \), \( 1 > C_1 := \frac{a_1(0) + b_1(0)}{a_1(0) - b_1(0)} \geq 0 \). Hence, we obtain
\[
a_1(t)b_1(t) \geq \frac{C_1(1 + A_1t)^{B_1} - 1}{C_1(1 + A_1t)^{B_1} + 1} \cdot a_1^2(t).
\]
Then we can see \( a_1(t)b_1(t) \geq 0 \) provided \( C_1(1 + A_1t)^{B_1} > 1 \), i.e.,
\[
t \geq T_1 = \frac{C_1^{-1/B_1} - 1}{A_1} = \frac{s(0)^{\frac{2}{s} - 1}}{2(s_1 + s_0)(N - 2)} \cdot \left[ \left( \frac{a_1(0) - b_1(0)}{a_1(0) + b_1(0)} \right)^{s_1 + s_0)(N - 2)} - 1 \right].
\]
Therefore, we obtain \( t_1 \leq T_1 \). Moreover, when \( t \leq T_1 \), by \( a_1^2(t) \leq 1 \), we have
\[
a_1(t)b_1(t) \geq \frac{C_1(1 + A_1t)^{B_1} - 1}{C_1(1 + A_1t)^{B_1} + 1}.
\]
That is, \( 1 - a_1(t)b_1(t) = O((1 + A_1t)^{-B_1}) = O(([N - 2]t)^{-\frac{2}{s} - 1}). \)
Additionally, when \( A_1t \geq e \cdot C_1^{-1/B_1} - 1 \), we have
\[
A_1t \geq e \cdot C_1^{-1/B_1} - 1 \geq \left( \frac{1 + B_1}{C_1} \right)^{1/B_1} - 1 \Rightarrow C_1(1 + A_1t)^{B_1} \geq 1 + B_1.
\]
Thus, we get
\[
a_1(t)b_1(t) \geq \frac{C_1(1 + A_1t)^{B_1} - 1}{C_1(1 + A_1t)^{B_1} + 1} \cdot a_1^2(t) \geq \frac{B_1}{2 + B_1} \cdot \frac{(a_1(0) + b_1(0))^2}{4} = \Theta \left( \frac{(a_1(0) + b_1(0))^2}{N} \right) > 0.
\]
\[\square\]
B.7 Proof of Theorem 4.7

Proof: Since \( a_1(0)b_1(0) \geq 0 \), then by 3) in Proposition 2.2, we know \( a_1(t)b_1(t) \geq 0 \) for all \( t \geq 0 \). Now we consider the flow of \( a_1(t)b_1(t) \).

\[
\frac{da_1(t)b_1(t)}{dt} \geq s(t)^{1-\frac{2}{N}} \left( s_1b_1(t)^2 + s_1a_1(t)^2 - 2a_1(t)b_1(t) \sum_{j=1}^{d} [s_ja_j(t)b_j(t)] \right)
\]

\[
\geq s(t)^{1-\frac{2}{N}} \left[ 2s_1a_1(t)b_1(t) - 2a_1(t)b_1(t) (s_1a_1(t)b_1(t) + s_2 (1 - |a_1(t)b_1(t)|)) \right]
\]

\[
= 2a_1(t)b_1(t)s(t)^{1-\frac{2}{N}} \left[ s_1 - s_1a_1(t)b_1(t) - s_2 (1 - a_1(t)b_1(t)) \right]
\]

\[
= 2a_1(t)b_1(t)s(t)^{1-\frac{2}{N}} (s_1 - s_2) (1 - a_1(t)b_1(t)) .
\]

By the lower bound of \( s(t) \) in Theorem 4.4, we obtain

\[
\frac{da_1(t)b_1(t)}{dt} \geq 2a_1(t)b_1(t)s(t)^{1-\frac{2}{N}} (s_1 - s_2) (1 - a_1(t)b_1(t)) \geq \frac{2a_1(t)b_1(t)(s_1 - s_2)(1 - a_1(t)b_1(t))}{(s_0 + s_1)(N-2)t + s(0)^{\frac{2}{N}-1}} .
\]

Denoting \( c_1(t) := a_1(t)b_1(t) \), we get

\[
\ln \frac{c_1(t)}{1 - c_1(t)} - \ln \frac{c_1(0)}{1 - c_1(0)} \geq \frac{2(s_1 - s_2)}{(s_1 + s_0)(N-2)} \ln \left( 1 + \frac{(s_1 + s_0)(N-2)t}{s(0)^{\frac{2}{N}-1}} \right) .
\]

Further we can rewrite the bound as

\[
c_1(t) \geq 1 - \frac{1}{A(1 + B(N-2)t)^{\frac{2}{N}} + 1} ,
\]

where \( A = \frac{c_1(0)}{1 - c_1(0)} > 0 \), \( B = (s_1 + s_0)s(0)^{1-\frac{2}{N}} > 0 \), \( c_2 = \frac{2(s_1 - s_2)}{s_1 + s_0} > 0 \). Then we have \( 1 - a_1(t)b_1(t) = O \left( [(N-2)t]^{-c_2/(N-2)} \right) \). The proof is finished. \( \square \)

B.8 Proof of Theorem 4.8

Proof: When \( t_2 = +\infty \), we have \( u(t)^\top Zv(t) < s(t) , \forall t \geq 0 \). Thus, we obtain

\[
\dot{s}(t) \geq Ns(t)^{2-\frac{2}{N}} (u(t)^\top Zv(t) - s(t)) \leq 0 .
\]

We note that by Theorem 3.8 \( s(t) \rightarrow s_1 \). Thus, we conclude

\[
\lim_{t \to +\infty} s(t) = s_1 .
\]

Then we have

\[
\frac{da_1(t)b_1(t)}{dt} \geq 2a_1(t)b_1(t)s(t)^{1-\frac{2}{N}} (s_1 - s_2) (1 - a_1(t)b_1(t))
\]

\[
\geq 2a_1(t)b_1(t)s_1^{1-\frac{2}{N}} (s_1 - s_2) (1 - a_1(t)b_1(t)) .
\]

Setting \( c_1(t) := a_1(t)b_1(t) \) and solving the ODE in the above, we get

\[
\ln \frac{c_1(t)}{1 - c_1(t)} - \ln \frac{c_1(0)}{1 - c_1(0)} \geq 2s_1^{1-\frac{2}{N}} (s_1 - s_2) t .
\]
We rewrite the bound to

$$1 - c_1(t) \leq \left(1 + \frac{c_1(0)}{1 - c_1(0)} \cdot e^{2s(0)^{1 - \frac{2}{N}}(s_1 - s_2)t}\right)^{-1}, i.e., 1 - c_1(t) = O\left(e^{-c_5 t}\right).$$

To obtain the bound of $s(t)$, we notice that

$$\dot{s}(t) = N s(t)^{2 - \frac{2}{N}} \left(\sum_{j=1}^{d} s_j a_j(t)b_j(t) - s(t)\right) \geq N s(t)^{2 - \frac{2}{N}} (s_1 - s(t)) \leq N s_1^{2 - \frac{2}{N}} (s_1 - s(t)).$$

We can obtain the upper bound of the evolution $s(t)$ as

$$d \left(\ln (s(t) - s_1)\right)/dt \leq -N s_1^{2 - \frac{2}{N}} \Rightarrow s(t) \leq s_1 + (s(0) - s_1) e^{-N s_1^{2 - \frac{2}{N}} t}, i.e., s(t) - s_1 = O\left(e^{-c_5 t}\right).$$

Finally, noting that $s(t) \geq s_1$, we obtain $|s(t) - s_1| = O\left(e^{-c_5 t}\right)$. The proof is finished.

**B.9 Proof of Theorem 4.9**

Proof: By Theorem 4.2 we have $\dot{s}(t) \geq 0, \forall t \geq 0$. Thus, we can lower bound $s(t) \geq s(0) > 0$. Then we obtain

$$\frac{da_1(t)b_1(t)}{dt} \geq 2a_1(t)b_1(t)s(t)^{1 - \frac{2}{N}} (s_1 - s_2) (1 - a_1(t)b_1(t)) \geq 2a_1(t)b_1(t)s(0)^{1 - \frac{2}{N}} (s_1 - s_2) (1 - a_1(t)b_1(t)).$$

Setting $c_1(t) := a_1(t)b_1(t)$ and solving the ODE in the above, we get

$$1 - c_1(t) \leq \left(1 + \frac{c_1(0)}{1 - c_1(0)} \cdot e^{2s(0)^{1 - \frac{2}{N}}(s_1 - s_2)t}\right)^{-1}, i.e., 1 - c_1(t) = O\left(e^{-c_5 t}\right). \tag{51}$$

Next we derive the bound for $s(t)$. We continue from

$$\dot{s}(t) = N s(t)^{2 - \frac{2}{N}} \left(\sum_{j=1}^{d} s_j a_j(t)b_j(t) - s(t)\right) \geq N s(t)^{2 - \frac{2}{N}} [s_1 a_1(t)b_1(t) - s_2 (1 - |a_1(t)b_1(t)|) - s(t)] \geq N s(t)^{2 - \frac{2}{N}} \left[-(s_1 + s_2) (1 + Ae^{c_5 t})^{-1} + s_1 - s(t)\right].$$

Solving the above ODE, we arrive at

$$\frac{d(s(t)e^{c_5 t})}{dt} \geq c_6 e^{c_5 t} \left[s_1 - (s_1 + s_2) (1 + Ae^{c_5 t})^{-1}\right].$$

Hence, we get

$$s(t)e^{c_5 t} - s(0) \geq s_1 \left(e^{c_5 t} - 1\right) - \int_{0}^{t} c_6 e^{c_5 x}(s_1 + s_2) (1 + Ae^{c_5 x})^{-1} \, dx \geq s_1 \left(e^{c_5 t} - 1\right) - e^{c_5 t} \int_{0}^{t} c_6 (s_1 + s_2) (1 + Ae^{c_5 x})^{-1} \, dx = s_1 \left(e^{c_5 t} - 1\right) - \left(s_1 + s_2\right) c_6 e^{c_5 t} \cdot \ln \left(1 + A^{-1} e^{-c_5 t}\right) \geq s_1 \left(e^{c_5 t} - 1\right) - \frac{(s_1 + s_2) c_6 e^{c_5 t}}{Ac_5} \cdot e^{-c_5 t}. $$

Therefore, we obtain

$$s(t) \geq s_1 - (s(0) + s_1)e^{-c_5 t} - \frac{(s_1 + s_2) c_6}{Ac_5} \cdot e^{-c_5 t}. $$
After hiding the constants and noting that $s(t)$ is non-decreasing, we obtain
\[
s_1 - s(t) = \mathcal{O}\left(e^{-\min\{c_5,c_6\}t}\right).
\]
We note that by Theorem 3.8, \( s(t) \to s_1 \). Since $s(t)$ is non-decreasing, we conclude $s(t) \leq \lim_{t \to +\infty} s(t) = s_1$. Then we obtain
\[
|s_1 - s(t)| = \mathcal{O}\left(e^{-\min\{c_5,c_6\}t}\right).
\]
The proof is finished. \(\square\)

C Convergence Rates: \(N = 2\)

We provide convergence rates of the case \(N = 2\) in this section. Corresponding to the case \(N \geq 3\), we list the rates of three stages as below:

**Stage 1.** For \(t \in [0,t_1]\), where \(t_1 := \inf\{t : a_1(t)b_1(t) \geq 0\} < +\infty\), we have \(a_1(t)b_1(t) \leq 0\), and the rates are
\[
1 - a_1(t)b_1(t) = \mathcal{O}(e^{-2t}), \quad s(t) = \Omega\left(e^{-2(s_1 + s_0)t}\right).
\]

**Stage 2.** For \(t \in (t_1,t_2]\), where \(t_2 := \inf\{t : u(t)^T Z v(t) \geq s(t)\}\), we have \(a_1(t)b_1(t) > 0, \dot{s}(t) \leq 0\), and
\[
1 - a_1(t)b_1(t) = \mathcal{O}\left(e^{-2(s_1 - s_2)t}\right), \quad s(t) = \Omega\left(e^{-2(s_1 + s_0)t}\right).
\]

**Stage 3.** For \(t \in (\max\{t_1,t_2\},+\infty)\), we have \(a_1(t)b_1(t) > 0\) and \(\dot{s}(t) \geq 0\), and the rates are
\[
1 - a_1(t)b_1(t) = \mathcal{O}\left(e^{-2(s_1 - s_2)t}\right), \quad |s_1 - s(t)| = \mathcal{O}\left(e^{-\min\{2(s_1 - s_2),2s(0)\}t}\right).\]

We have shown that the definitions of the stages are well defined by Theorem 4.2. The convergence rates of \(s(t)\), i.e. \(s(t) = \Omega\left(e^{-2(s_1 + s_0)t}\right)\) in Stage 1 and Stage 2 are given by Theorem 4.4.

Convergence Rates of \(a_1(t)b_1(t)\): Stage 1

**Theorem C.1** Suppose \(N = 2\), \(a_1(0)b_1(0) < 0\) and \(a_1(0) + b_1(0) \neq 0\). Then we have
\[
1 - a_1(t)b_1(t) = \mathcal{O}(e^{-2t}), 0 \leq t \leq t_1.
\]
Furthermore, we have the upper bound of \(t_1\) below:
\[
t_1 \leq \frac{1}{2} \ln \left| \frac{a_1(0) - b_1(0)}{a_1(0) + b_1(0)} \right|.
\]

(52)

Additionally, we could obtain
\[
a_1(t)b_1(t) = \Omega\left((a_1(0) + b_1(0))^2\right), \quad \text{if} \quad t \geq \frac{1}{2} \ln \left| \frac{2(a_1(0) - b_1(0))}{a_1(0) + b_1(0)} \right|.
\]

(53)

Proof: Since \(a_1(0)b_1(0) < 0, a_1(0) + b_1(0) \neq 0\), without loss of generality, we suppose \(a_1(0) > 0, b_1(0) < 0\) and \(a_1(0) + b_1(0) > 0\). Note that
\[
a_1(t) - b_1(t) \leq s(t)^{1-\frac{d}{2}} (b_1(t) - a_1(t)) \left( s_1 + \sum_{j=1}^{d} |s_j a_j(t)b_j(t)| \right) = (b_1(t) - a_1(t)) \left( s_1 + \sum_{j=1}^{d} |s_j a_j(t)b_j(t)| \right).
\]

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By Arora et al. [4] Lemma 4] and \(|a_1(t) - b_1(t)| \leq 2\), we get that \(a_1(t) - b_1(t)\) preserves the sign of its initial value:

\[
a_1(t) - b_1(t) > 0, \forall t \geq 0. \tag{54}
\]

Moreover, from 5) in Proposition 2.2 and \(a_1(0) + b_1(0) > 0\), we obtain

\[
a_1(t) + b_1(t) \geq a_1(0) + b_1(0) > 0, \forall t \geq 0. \tag{55}
\]

Then we have

\[
a_1(t) \geq \frac{a_1(t) + b_1(t)}{2} \geq \frac{a_1(0) + b_1(0)}{2} > 0, \forall t \geq 0, \tag{56}
\]

and

\[
-a_1(t) < b_1(t) \leq a_1(t), \forall t \geq 0 \quad \Rightarrow \quad -1 < \frac{b_1(t)}{a_1(t)} < 1, \forall t \geq 0. \tag{57}
\]

Furthermore, we can derive that

\[
\frac{d}{dt} \left( \frac{b_1(t)}{a_1(t)} \right) = \frac{\dot{b}_1(t)a_1(t) - \dot{a}_1(t)b_1(t)}{a_1^2(t)} \geq \left( 1 - \left( \frac{b_1(t)}{a_1(t)} \right)^2 \right) > 0.
\]

Then we have

\[
d \left( \ln \left( \frac{a_1(t) + b_1(t)}{a_1(t) - b_1(t)} \right) \right) / dt = 1 \Rightarrow \frac{b_1(t)}{a_1(t)} = \frac{e^{2t} \left[ \frac{a_1(0) + b_1(0)}{a_1(0) - b_1(0)} \right] - 1}{e^{2t} \left[ \frac{a_1(0) + b_1(0)}{a_1(0) - b_1(0)} \right] + 1}. \tag{58}
\]

Thus, we get

\[
a_1(t)b_1(t) = a_1^2(t) \cdot \frac{b_1(t)}{a_1(t)} \geq A_2 e^{2t} - 1 \cdot a_1^2(t), A_2 := \frac{a_1(0) + b_1(0)}{a_1(0) - b_1(0)}. \tag{59}
\]

Then we can see \(a_1(t)b_1(t) \geq 0\) provided \(A_2 e^{2t} \geq 1\), i.e.,

\[
t \geq T_2 := \frac{1}{2} \ln \frac{a_1(0) - b_1(0)}{a_1(0) + b_1(0)}.
\]

Therefore, Eq. (52) is proved. Moreover, when \(t \leq T_2\), by \(a_1^2(t) \leq 1\), we have

\[
a_1(t)b_1(t) \geq \frac{A_2 e^{2t} - 1}{A_2 e^{2t} + 1}.
\]

That is, \(1 - a_1(t)b_1(t) = O(e^{-2t})\).

Additionally, when \(t \geq \frac{1}{2} \ln \frac{2(a_1(0) - b_1(0))}{a_1(0) + b_1(0)}\), we have \(A_2 e^{2t} \geq 2\). Hence, we derive that

\[
a_1(t)b_1(t) \geq \frac{A_2 e^{2t} - 1}{A_2 e^{2t} + 1} \cdot a_1^2(t) \geq \left( \frac{a_1(0) + b_1(0)}{12} \right)^2 = \Theta \left( (a_1(0) + b_1(0))^2 \right) > 0.
\]

Thus, Eq. (53) is proved. \(\square\)

**Convergence Rates of \(a_1(t)b_1(t)\): Stage 2 and Stage 3**

**Theorem C.2** Assume \(N = 2\), \(a_1(0)b_1(0) > 0\). Then we have

\[
1 - a_1(t)b_1(t) = O \left( e^{-2(s_1 - s_2)^t} \right).
\]

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Proof: Since $a_1(0)b_1(0) > 0$, then by 3) in Proposition 3.8, we know $a_1(t)b_1(t) > 0$ for all $t \geq 0$. Now we consider the flow of $a_1(t)b_1(t)$.

$$\frac{da_1(t)b_1(t)}{dt} \geq s(t)^{1-\frac{2}{N}} \left( s_1b_1(t)^2 + s_1a_1(t)^2 - 2a_1(t)b_1(t) \sum_{j=1}^{d} [s_ja_j(t)b_j(t)] \right)$$

Further we can rewrite the bound as

$$\geq s(t)^{1-\frac{2}{N}} \left[ 2s_1a_1(t)b_1(t) - 2a_1(t)b_1(t) (s_1a_1(t)b_1(t) + s_2 (1 - |a_1(t)b_1(t)|)) \right]$$

$$= 2a_1(t)b_1(t) \left[ s_1 - s_1a_1(t)b_1(t) - s_2 (1 - a_1(t)b_1(t)) \right]$$

$$= 2a_1(t)b_1(t) (s_1 - s_2) (1 - a_1(t)b_1(t)).$$

Denoting $c_1(t) := a_1(t)b_1(t)$, by solving the ODE above, we obtain

$$\ln \frac{c_1(t)}{1 - c_1(t)} - \ln \frac{c_1(0)}{1 - c_1(0)} \geq 2(s_1 - s_2)t.$$ 

Further we can rewrite the bound as

$$c_1(t) \geq 1 - \frac{1}{\frac{c_1(0)}{1 - c_1(0)} e^{2(s_1 - s_2)t} + 1}.$$ 

Then we have $1 - a_1(t)b_1(t) = O(e^{-2(s_1-s_2)t})$. The proof is finished. □

**Convergence Rates of $s(t)$: Stage 3**

Similarly, before we start our analysis in Stage 3, we need to handle the minor case $t_2 := \inf \{ t : u(t)^\top Zv(t) \geq s(t) \} = +\infty$.

**Theorem C.3** Suppose $N = 2$, $a_1(0)b_1(0) > 0$ and $t_2 = +\infty$. Then we have

$$|s(t) - s_1| = O(e^{-2s_1t}).$$

Proof: When $t_2 = +\infty$, we have $u(t)^\top Zv(t) < s(t), \forall t \geq 0$. Thus, we obtain

$$\dot{s}(t) \overset{(6)}{=} Ns(t)^{2-\frac{2}{N}} (u(t)^\top Zv(t) - s(t)) = 2s(t) (u(t)^\top Zv(t) - s(t)) \leq 0.$$ 

We note that by Theorem 3.8 $s(t) \to s_1$. Thus, we conclude

$$s(t) \overset{(62)}{\geq} \lim_{t \to +\infty} s(t) = s_1.$$ 

To obtain the bound of $s(t)$, we notice that

$$\dot{s}(t) \overset{(62)}{=} Ns(t)^{2-\frac{2}{N}} \left( \sum_{j=1}^{d} s_ja_j(t)b_j(t) - s(t) \right) \overset{(63)}{\leq} Ns(t)^{2-\frac{2}{N}} (s_1 - s(t))$$

$$\leq Ns_1^{2-\frac{2}{N}} (s_1 - s(t)) = 2s_1 (s_1 - s(t)).$$ 

By solving the ODE above, we can obtain the upper bound of the evolution $s(t)$ as

$$d(\ln (s(t) - s_1)) / dt \leq -2s_1 \Rightarrow s(t) \leq s_1 + (s(0) - s_1)e^{-2s_1t}, i.e., s(t) - s_1 = O(e^{-2s_1t}).$$

Finally, noting that $s(t) \geq s_1$, we obtain $|s(t) - s_1| = O(e^{-2s_1t})$. The proof is finished. □

Now we turn to the case $t_2 < +\infty$. We assume $a_1(0)b_1(0) > 0$ and $\dot{s}(0) \geq 0$ for short in Stage 3.
Theorem C.4 Assume \( N = 2, a_1(0)b_1(0) > 0 \), and \( \dot{s}(0) \geq 0 \). Then we have

\[
|s_1 - s(t)| = \mathcal{O} \left( e^{-\min(2(s_1 - s_2), 2s(0))} \right).
\]

Proof: By Theorem 4.2 we have \( \dot{s}(t) \geq 0, \forall t \geq 0 \). Thus, we can lower bound \( s(t) \geq s(0) > 0 \). Furthermore, we have

\[
\dot{s}(t) \geq Ns(t)^{2-\frac{2}{N}} \left( \sum_{j=1}^{d} s_j a_j(t) b_j(t) - s(t) \right) \geq Ns(t)^{2-\frac{2}{N}} \left[ s_1 a_1(t) b_1(t) - s_2 \left( 1 - |a_1(t)b_1(t)| \right) - s(t) \right] \geq Ns(t)^{2-\frac{2}{N}} \left[ (s_1 + s_2) a_1(t) b_1(t) - s_2 - s(t) \right] \geq 2s(0) \left[ - (s_1 + s_2) \left( 1 + Ae^{2(s_1 - s_2)t} \right)^{-1} + s_1 - s(t) \right],
\]

where \( A = \frac{c_1(0)}{1 - c_1(0)} \). By solving the above ODE, we get

\[
\frac{d(s(t)e^{2s(0)t})}{dt} \geq 2s(0)e^{2s(0)t} \left[ s_1 - (s_1 + s_2) \left( 1 + Ae^{2(s_1 - s_2)t} \right)^{-1} \right].
\]

Hence, we get

\[
s(t)e^{2s(0)t} - s(0) \geq s_1 \left( e^{2s(0)t} - 1 \right) - \int_0^t 2s(0)e^{2s(0)x} (s_1 + s_2) \left( 1 + Ae^{2(s_1 - s_2)x} \right)^{-1} dx
\]

\[
\geq s_1 \left( e^{2s(0)t} - 1 \right) - e^{2s(0)t} \int_0^t 2s(0)(s_1 + s_2) \left( 1 + Ae^{2(s_1 - s_2)x} \right)^{-1} dx
\]

\[
= s_1 \left( e^{2s(0)t} - 1 \right) - \frac{2s(0)(s_1 + s_2)e^{2s(0)t}}{2(s_1 - s_2)} \cdot \ln \left( 1 + A^{-1}e^{2(s_1 - s_2)t} \right)
\]

\[
\geq s_1 \left( e^{2s(0)t} - 1 \right) - \frac{2s(0)(s_1 + s_2)e^{2s(0)t}}{2A(s_1 - s_2)} \cdot e^{-2(s_1 - s_2)t}.
\]

Therefore, we obtain

\[
s(t) \geq s_1 - (s(0) + s_1)e^{-2s(0)t} - \frac{2s(0)(s_1 + s_2)}{2A(s_1 - s_2)} \cdot e^{-2(s_1 - s_2)t}.
\]

After hiding the constants, we obtain

\[
s_1 - s(t) = \mathcal{O} \left( e^{-\min(2(s_1 - s_2), 2s(0))} \right).
\]

We note that by Theorem 3.8 \( s(t) \to s_1 \). Since \( s(t) \) is non-decreasing, we conclude

\[
s(t) \leq \lim_{t \to +\infty} s(t) = s_1.
\]

Then we obtain \( |s_1 - s(t)| = \mathcal{O} \left( e^{-\min(2(s_1 - s_2), 2s(0))t} \right) \). The proof is finished. \( \square \)