Intriguing structures in iterative maps motivated by N-body problem

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Abstract. Conventional approaches to modeling any system try to incorporate increasingly realistic features into the model, thereby making it more and more complex. An opposite approach seeks to build simpler and simpler conceptual models capable of capturing some observed features of a system. This trend began with Lorenz, who simplified models of the atmosphere to obtain the Lorenz model consisting of a system of only three equations. Despite the simplicity of these equations, this system displayed surprisingly rich properties, and has been used as a conceptual model in diverse disciplines. Poincare maps help study ordinary differential equations from a qualitative perspective. Several investigators like Henon and Feigenbaum followed this simplification approach. Instead of investigating Poincare maps of realistic systems, they, along with several others, investigated simple maps for their own sake. Despite lack of realism, this approach proved to be very fruitful. A map, as simple as the Logistic map, became an important conceptual modeling paradigm. It provided a tool for understanding bifurcation routes to chaos, which were verified experimentally through various experiments in diverse fields. Coupled map lattices (CML) help explore partial differential equations (PDE). Further simplification led to the introduction of Cellular Automata (CA). These fields continue to be explored with vigor and have given rise to a rich body of knowledge, conceptually useful over a wide spectrum of disciplines. In this paper, we follow the simplification approach for modeling the N-body problem. N-body simulations, say in Gravitation, give rise to filamentary structures. Such structures are observed in the actual observed Galactic distribution. The mechanism for creation of such structures is not well understood. We present a simple iterative dynamical model, motivated by the N-body problem, which, though unrealistic, produces such filamentary structures. This model also exhibits a variety of intriguing structures. Attempts to understand these structures may lead to useful insights similar to those provided by investigations in maps, CML, CA etc.
1. Introduction

Conventional approaches to modeling any system try to incorporate increasingly realistic features into the model, thereby making it more and more complex. This approach, though useful in some ways, suffers from some drawbacks also. The cost of modeling in terms of time and money becomes a limiting factor. More important, the capacity to draw useful insights about the system becomes limited. Massive data becomes available, but getting useful manageable mental models, comprehensible by the human brain, becomes increasingly difficult.

During the last half-century an opposite trend in modeling has been advocated. This approach seeks to build simpler and simpler conceptual models capable of capturing some observed features of a system. This trend began with Lorenz, who simplified his models of the atmosphere to simpler and simpler sets of equations eventually ending with the now celebrated Lorenz model consisting of a system of only three equations [1]. Despite their simplicity, these equations displayed immensely rich properties, which despite half a century of extensive exploration, have not yet been fully revealed. Lorenz equations have been used as a conceptual model in diverse disciplines [2-20].

Poincare introduced the idea of studying ordinary differential equations from a qualitative perspective. For this purpose he introduced Poincare maps. A number of investigators like Henon [21] and Feigenbaum [22] pursued the simplification doctrine; instead of investigating Poincare maps of realistic systems, they, along with several others, investigated simple maps for their own sake. This approach was very fruitful. Logistic map became an important conceptual modeling paradigm. Once again, despite its simplicity, Logistic map offered a bewildering richness. It provided a tool for understanding bifurcation route to chaos. This route was verified experimentally through various experiments in diverse fields [23].

Just as maps were introduced to explore qualitative properties of ordinary differential equations (ODE), coupled map lattices (CML) [24] were introduced for exploring the properties of partial differential equations (PDE). Further simplification led to the introduction of Cellular Automata [25]. All these fields continue to be explored with vigor and have given rise to a rich body of knowledge, conceptually useful over a wide spectrum of disciplines.

To the best of our knowledge this simplification approach has not been applied to N-body simulation [26]. N-body simulations, say in Gravitation, give rise to filamentary structures. This is also observed in the actual observed Galactic distribution [27]. In this paper, we present a simple iterative dynamical model which produces such filamentary structures. This model exhibits a variety of intriguing structures. We believe that an attempt to understand these structures will lead to insights similar to those provided by investigations in maps, CML, CA etc.

2. A simplified model of N-body simulation

The problem of N-body simulation in 3-dimensions consists of solving a set of 6N first order ODEs. Numerical simulation requires replacement of continuous time by discrete time. Our first simplification replaces 6N ODE’s by a set of N maps. The i\textsuperscript{th} map gives the position of the i\textsuperscript{th} particle at time t+1 as a function of the positions of all the particles at time t. Let $\vec{R}_i(t)$ denote the position of the i\textsuperscript{th} particle at time t. Let \( \Delta \vec{R}_i = \vec{R}_i(t+1) - \vec{R}_i(t) \) denote the change in the position of the i\textsuperscript{th} particle from time t to time t+1. In the next simplification, we assume that this change is due to the sum of influences from all the other particles, so that \( \Delta \vec{R}_i = \sum_{j \neq i} \Delta \vec{R}_j \), where \( \Delta \vec{R}_j = \vec{f}(\vec{R}_j - \vec{R}_i) \). A map that we have investigated is: \( \vec{f}(\vec{r}) = \frac{\alpha \vec{r}}{r^2} \). We choose the value \( \alpha = 0.001 \). In this model the changes in the
positions of the particle are made synchronously. In the next stage of simplification, we make the changes in position asynchronously. This reduces computation time. As further simplification, we pick a particle randomly and change its position according to the above rule.

Next, we make further departure from realistic modeling. Instead of picking a particle and changing its position by summing the effects of all other particles, we choose a particle and change the positions of all other particles according to a specified map. If particle \( i \) is chosen, the positions of all the other particles change according to the rule \( \Delta \vec{R}_j = -\vec{f}(\vec{R}_j - \vec{R}_i) \forall j \neq i \). The \( i \)th particle can be chosen randomly.

To avoid the singularity at \( r = 0 \), we modify \( \vec{f}(\vec{r}) \) so that \( \vec{f}(\vec{r}) = \frac{\alpha \vec{r}}{r^2 + \beta^2} \). We choose \( N = 100000 \) uniformly distributed random points inside a circle of radius 1. Some of the intriguing structures observed after \( m = 1000 \) iterations for \( \alpha = 0.001 \) and different values of \( \beta \) are shown in figures 1 - 8.

These figures show that for small values of \( \beta \), as \( \beta \) is increased, the observed structure becomes increasingly filamentary and sparse. For \( \beta = 0.018 \) the structure has a complex quasi 1-dimensional form. Dramatic changes take place in the observed structure for higher values of \( \beta \). For \( \beta = 0.019 \) we find that all the particles converge to the vertices of an equilateral triangle. For \( \beta = 0.020 \) we find a 1-dimensional curve. For \( \beta = 0.022 \) a nearly circular structure is observed. For \( \beta = 0.023 \) and higher values, we always find that all the particles converge to a single point.
Keeping $\beta$ fixed at 0.019, we increased the number of iterations. We found that the points remain on the vertices of an equilateral triangle; the length of the side does not change with increase in the number of iterations. For $\beta$ between 0.019 and 0.021, we find that on increasing the number of iterations sufficiently, the structure eventually converges to the vertices of an equilateral triangle, whose size does not change on increasing the number of iterations. Another intriguing observation is that this convergence is much faster if the number of points $N$ is a multiple of 3 compared to when it is not. In one dimension, however, the number of iterations before convergence tends to decrease as the number of points $N$ is increased.

For $\beta = 0.018$ and below, the structures did not converge even as the number of iterations was increased to 250000. For $\beta = 0.022$ the structure remained nearly circular even for $m$ as large as 17000.

3 Analysis of the model

For $N = 2$, the dynamics of the separation between the points can be analyzed exactly in terms of the map $f$. As the only property of interest is the distance between the two points, negative values of $R$ have the same meaning as positive values. The separation $R'$ at time $(t+1)$ is determined by the separation $R$ at time $t$ by

$$R' = \left| R - \frac{\alpha R}{R^2 + \beta^2} \right| = F(R)$$

This map is shown for $\beta = 0.0200$ in Figure 9.
The map has two fixed points at $R = 0$ and $R = \tilde{R} = \left(\frac{\alpha}{2} - \beta^2\right)^{1/2}$ and zeros at $R = 0$ and $R = R_0 = (\alpha - \beta^2)^{1/2}$. For $R < R_0$, the derivative of the map is given by

$$F'(R) = \frac{\alpha}{R^2 + \beta^2} - \frac{2\alpha R^2}{(R^2 + \beta^2)^2} - 1$$

(2)

At $R = 0$, the slope of $F$ is given by

$$F'(0) = \frac{\alpha}{\beta^2} - 1$$

(3)

The magnitude of this slope becomes less than 1 for $\beta > \left(\frac{\alpha}{2}\right)^{1/2} \simeq 0.0224$. Hence for $\beta$ greater than this critical value, $R = 0$ is stable and the separation between the two particles converges to zero. This explains the observation that for $\beta = 0.023$ and above, for any value of $N$, all the particles converge to a single point.

At $R = \tilde{R}$, the slope is given by

$$F'(\tilde{R}) = \frac{8\beta^2}{\alpha} - 3$$

(4)

Therefore, $R = \tilde{R}$ is stable for $-1 < \frac{8\beta^2}{\alpha} - 3 < 1$, i.e., for $(\frac{\alpha}{4})^{1/2} \simeq 0.0158 < \beta < (\frac{\alpha}{4})^{1/2} \simeq 0.0224$. Clearly for this range of $\beta$, the separation between the two particles will converge to $\tilde{R}$.

For larger $N$, we can be sure that if all the particles lie on the vertices of a triangle of side $\tilde{R}$, then this configuration will not change shape or size on further iteration. However, can the stability of $F$
at $\tilde{R}$ guarantee that the N–particle system will converge to the vertices of a triangle of side $\tilde{R}$? The answer is negative, as we find empirically that for $N > 3$, the system converges only for $\beta > \beta_c$, where $\beta_c \approx (\frac{a}{4})^{1/2}$. This observation has been verified for different values of $a$. In order to simplify the analysis, we consider the analogous one-dimensional system with $N = 3$. The state, with two particles having the same coordinate and the third particle at a distance $\tilde{R}$ from them, is an invariant state. We examine the local stability of this state. Without loss of generality we assume that in this state the coordinates of the three particles are $x_1 = 0$ and $x_2 = x_3 = \tilde{R}$. In the perturbed state, we take the coordinates of the three particles to be $x_1 = \epsilon_1$, $x_2 = \tilde{R} + \epsilon_2$ and $x_3 = \tilde{R} + \epsilon_3$, where $\epsilon_i$ are infinitesimals.

If $P_1$ gets chosen as the particle which changes the positions of the other particles, then the new values of the perturbations are given by

$$
\begin{bmatrix}
\epsilon'_1 \\
\epsilon'_2 \\
\epsilon'_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
-1 & F'(\tilde{R}) & F'(\tilde{R}) \\
1 & -F'(\tilde{R}) & 0
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{bmatrix}
$$

(5)

This may be written as

$$
\epsilon' = J^{(1)} \epsilon
$$

(6)

The eigenvalues of $J^{(1)}$ are:

$$
\lambda_1^{(1)} = 1, \quad \lambda_2^{(1)} = F'(\tilde{R}), \quad \lambda_3^{(1)} = F'(\tilde{R})
$$

(7)

with corresponding eigenvectors

$$
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
$$

(8)

If $P_2$ gets chosen as the particle which changes the positions of the other particles, then the new values of the perturbations are given by

$$
\begin{bmatrix}
\epsilon'_1 \\
\epsilon'_2 \\
\epsilon'_3
\end{bmatrix} =
\begin{bmatrix}
F'(\tilde{R}) & 1 & -F'(\tilde{R}) \\
0 & 1 & 0 \\
0 & 1-F'(0) & F'(0)
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{bmatrix}
$$

(9)

which may be written as

$$
\epsilon' = J^{(2)} \epsilon
$$

(10)

The eigenvalues of $J^{(2)}$ are:

$$
\lambda_1^{(2)} = 1, \quad \lambda_2^{(2)} = F'(\tilde{R}), \quad \lambda_3^{(2)} = F'(0)
$$

(11)

with corresponding eigenvectors

$$
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
$$

(12)

If $P_3$ gets chosen as the particle which changes the positions of the other particles, then the new values of the perturbations are given by
\[
\begin{bmatrix}
\varepsilon_1' \\
\varepsilon_2' \\
\varepsilon_3'
\end{bmatrix} = \begin{bmatrix}
F'(\tilde{R}) & 0 & 1 - F'(\tilde{R}) \\
0 & F(0) & 1 - F'(0) \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3
\end{bmatrix}
\]

which may be written as
\[
\varepsilon' = J^{(3)} \varepsilon
\]

The eigenvalues of \(J^{(3)}\) are:
\[
\lambda_1^{(3)} = 1, \quad \lambda_2^{(3)} = F'(\tilde{R}), \quad \lambda_3^{(3)} = F'(0)
\]

with corresponding eigenvectors
\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

It is easy to see that the separation between the points \(P_2\) and \(P_3\) grows by a factor \(\left|F'(\tilde{R})\right|\) each time the point \(P_1\) gets chosen and by a factor \(\left|F'(0)\right|\) each time the point chosen is \(P_2\) or \(P_3\). Since the points \(P_i\) are chosen with equal probability, it follows that the separation between \(P_2\) and \(P_3\) will have an overall tendency to decrease if
\[
\left|F'(\tilde{R})\right|\left|F'(0)\right|^2 < 1
\]

This inequality yields
\[
\left\{ \frac{8\beta^2}{\alpha} - 3 \right\} \left\{ \frac{\alpha}{\beta^2} - 1 \right\}^2 < 1
\]

and holds for \(\beta > \beta_c = \left(\frac{2}{3}\right)^{1/3}\), where \(s\) is the real root of the equation
\[
3s^3 - 14s^2 + 18s - 8 = 0
\]

This equation has a root \(s \approx 2.93\). Although the critical value of \(s\) has been obtained for \(N = 3\) in the one dimensional case, it is remarkably close to the empirically observed value 3 for the two dimensional case with \(N\) greater than or equal to 3.

4. Conclusion

We obtained a family of iterative dynamical systems motivated by the N-body problem. This family exhibits several very intriguing properties which need to be explained. One needs to understand how filamentary structures arise, how nearly circular structures arise before all the points converge to the vertices of an equilateral triangle, how the conditions of stability can be obtained in 2 dimensions. In 3 dimensions all the points converge to the vertices of a tetrahedron, but before this convergence they settle on a nearly spherical surface. It is hoped that the type of dynamical system introduced here will produce a body of knowledge as rich as those produced by Logistic and other maps, Coupled Map Lattices, Cellular Automata etc.

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