Study of analytical solution of the thermal conductivity equation considering relaxation phenomena under the third class boundary conditions

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Abstract. Applying the variable separation method, an exact analytical solution is found for thermal conductivity equation, established based on the modified Fourier's law with thermal flux relaxation and temperature gradient under Newton's boundary conditions (BC). An alternative method is proposed for modeling the transient heat conduction in the plate washed at the boundaries by a liquid or gas, which differs from the traditional one in the absence of the need to determine the heat transfer coefficients from the criterial heat transfer equations, instead of which it is proposed to introduce summands into the classical heat conductivity equation. The summands should take into account the relaxation nature of solids, the features of heat transfer at the boundary, acceleration of heat flux and temperature gradient during the process.

1. Introduction

When formulating the boundary value problem of heat conduction in a solid, the classical parabolic equation of heat conduction, time and boundary conditions [1–3] are most often used. If the process of interest to the researcher occurs under Derichlet's, Newmann's or conjugating boundary conditions, then there are no difficulties in setting these conditions. However, if heat exchange at the boundaries of a solid occurs as a result of contact with a liquid or gas, it becomes necessary to determine the heat transfer factor, which depends on a large number of factors (body geometry, velocity and direction of fluid movement, thermophysical properties of the liquid and the body).

In the case of steady-state heat exchange, the criterion equations and similarity criteria are used to determine the heat transfer factor [1–3]:

1) based on the known velocity of the medium, its properties and the geometric parameters of the solid body, the mode of the fluid motion (laminar, transitional or turbulent) is determined, i.e. Reynolds criterion is determined \( \text{Re} = \frac{wl}{\nu} \), where \( w \) is the medium flow velocity, \( m/s \); \( l \) – determining body size, \( m \); \( \nu \) – kinematic viscosity coefficient of liquid or gas, \( m^2/s \);

2) depending on the fluid movement mode, an empirical formula is selected to determine the \( \text{Nu} \) criterion:
for free convection \( \text{Nu} = A(Gr_{f} \Pr_{f})^{0.25} \left(\frac{Pr_{f}}{Pr_{w}}\right) \),

for forced convection \( \text{Nu} = B \text{Re}^{n} \left(\frac{Pr_{f}}{Pr_{w}}\right)^{0.25} \),

where \( A, B, m, n, l \) are empirical coefficients, whose values depend on the surface type and the heat carrier mode of movement;

3) the required heat transfer factor \( \alpha \) is found from the formula for determining the Nusselt criterion \( \text{Nu} = 2l/(\lambda \alpha) \).

If the heat exchange between the liquid and the wall proceeds under non-stationary conditions, it is difficult to determine the time-varying heat transfer factor, more often it is found from the solution of the inverse problem of heat conductivity based on empirical data subject to body temperatures during heat exchange [1 – 3].

However, the studies carried out in this work suggest that in order to simulate the temperature change in a solid body under Newton's boundary conditions, there is no need to determine the heat transfer factor – to describe the process of heat conductivity, it is sufficient to take into account the relaxation properties of the material specific to the heat transfer process under consideration. For this purpose, the classical Fourier's law is presented in a modified form, taking into account the time variation of the heat flux and the temperature gradient. We should note that in the classical Fourier's law

\[ q = -\lambda \frac{\partial \Theta}{\partial x} \]

effect \( q \) – heat flux and cause \( \frac{\partial q}{\partial t} \) – temperature gradient are not separated in time. That is, with a change in the cause, the effect occurs instantly. However, the transfer of thermal energy does not occur instantaneously, but over a certain period of time \( \tau \). The neglect of relaxation phenomena in heat transfer problems results in the well-known paradoxes: infinite heat flux at the point of setting Dirichlet's boundary conditions, infinite speed of heat propagation [3–13].

The problem of paradoxes of the theory of heat conductivity can be solved by introducing the relaxation corrections in the Fourier's law [14–24]:

\[ q = -\lambda \frac{\partial \Theta}{\partial x} - \tau_1 \frac{\partial q}{\partial t} - \lambda \tau_2 \frac{\partial^2 \Theta}{\partial x \partial \tau}, \]  

(1)

where \( \tau_1, \tau_2 \) are the relaxation coefficients of the heat flux and temperature gradient.

The expediency of introducing additional terms into the Fourier's law formula is proved based on the hypothesis of finite rates of heat and mass diffusion, is confirmed by the coincidence of the model presented in this paper with the Onsager system of differential equations, and is provided in [3 – 6].

2. Mathematical formulation of the problem

In order to consider and analyze the features of heat exchange under Newton's boundary conditions, we will find a solution to the problem of the form

\[ \frac{\partial \Theta(\xi, Fo)}{\partial Fo} + Fo \frac{\partial^2 \Theta(\xi, Fo)}{\partial \xi^2} = \frac{\partial^2 \Theta(\xi, Fo)}{\partial \xi^2} + Fo \frac{\partial^2 \Theta(\xi, Fo)}{\partial \xi^2 \partial Fo}; \]  

(2)

\( (Fo > 0; \ 0 < \xi < 1) \)

\[ \Theta(\xi, 0) = 1; \]  

(3)
\[ \frac{\partial \Theta(x,0)}{\partial F_o} = 0; \]  
\[ \frac{\partial \Theta(0,F_o)}{\partial \xi} = 0; \]  
\[ \frac{\partial \Theta(1,F_o)}{\partial \xi} + B_i \Theta(1,F_o) = 0, \]  
where \( \Theta = \frac{t - t_m}{t_0 - t_m}; \) \( \xi = \frac{x}{\delta}; \) \( F_o = \frac{\alpha \tau}{\delta^2}; \) \( F_{o1} = \frac{\alpha \tau}{\delta^2}; \) \( F_{o2} = \frac{\alpha \tau}{\delta^2}; \) \( B_i = \frac{a \delta}{\lambda}; \) \( F_o \) – Fourier number; \( B_i \) – Biot number; \( t \) – temperature; \( F_{o1}, F_{o2} \) – dimensionless factors of relaxation; \( \tau \) – time; \( \Theta \) – nondimensional temperature; \( x \) – coordinate; \( t_0 \) – initial temperature; \( \delta \) – plate thickness; \( \alpha \) – heat-transfer coefficient; \( t_m \) – medium temperature; \( \lambda \) – thermal conductivity ratio; \( \xi \) – relative coordinate; \( \alpha \) – thermal diffusivity.

To simplify the solution to problem (2) – (6), the inertia of the heat flux and temperature gradient in Newton's boundary condition is neglected. We also note that at some large values of the heat conductivity factor \( \alpha \) (or at small values), \( \lambda \) this assumption will virtually not affect the resulting solution.

3. Accurate analytical solution

The solution to problem (2) – (6) is accepted in the form

\[ \Theta(\xi,F_o) = \varphi(F_o)\psi(\xi). \]  

Substitute (7) in (2), we find

\[ F_o \frac{d^2 \varphi}{dF_o^2} + \left( v^2 F_{o2} + 1 \right) \frac{d \varphi}{dF_o} + v^2 \varphi = 0; \]  
\[ \frac{d^2 \psi}{d\xi^2} + v^2 \psi = 0, \]  
where \( v \) – eigenvalues.

Substituting (7) in (5), (6), we obtain boundary conditions for the equation (9)

\[ \frac{d \psi(0)}{d \xi} = 0; \]  
\[ \frac{d \psi(1)}{d \xi} + B_i \psi(1) = 0. \]  

The solution to Sturm-Liouville problem (9) – (11) is accepted in the form

\[ \psi(\xi) = \cos(v \xi). \]  

Ratio (12) accurately satisfies the equation (9) and limiting condition (10). Substituting (12) in (11), we obtain

\[ \left[ -v \sin(v \xi) + B_i \cos(v \xi) \right]_{\xi=1} = 0. \]  

Hence, we obtain the following transcendental equation

\[ ctg v = \frac{v}{B_i}. \]
The obtained equation for any $\text{Bi}$ is satisfied only for definite discrete values of eigenvalues $\nu$. For example, for $\text{Bi} \to \infty$ we obtain

$$
\nu_k = (2k-1)\pi / 2. \quad (k = 1, \infty)
$$

Values $\nu_k$, obtained using formula (15), completely coincide with eigenvalues of Sturm-Liouville problem under Derichlet's boundary condition. For other values $\text{Bi}$ equation (14) can be solved by numerical or graphical techniques.

The homogeneous differential equation (8) has the following characteristic equation

$$
2 \nu^2 - 1 \pi^2 = 0. \quad (k = 1, \infty)
$$

If $D = (\nu^2 + 1)^2 - 4\nu_0 \nu^2 > 0$, then characteristic equation (16) has 2 real roots $\zeta_{1k}$ and $\zeta_{2k}$

$$
\zeta_{ik} = -\nu^2 \nu_0 - 1 \pm \sqrt{(\nu^2 + 1)^2 - 4\nu_0 \nu^2} ; \quad (i = 1, 2; k = 1, \infty)
$$

Taking into account the found $\zeta_{1k}$ and $\zeta_{2k}$ the equation (8) solution will be

$$
\varphi_k(\nu \nu_0) = C_{1k} \exp(\zeta_{1k} \nu) + C_{2k} \exp(\zeta_{2k} \nu),
$$

where $C_{jk} (j = 1, 2; k = 1, \infty)$ – unknown integration constants, which are determined from the initial conditions (3), (4).

Substituting (12) and (18) in (7) we obtain

$$
\Theta_k(\nu, \nu_0) = [C_{1k} \exp(\zeta_{1k} \nu) + C_{2k} \exp(\zeta_{2k} \nu)] \cos(\nu \xi). \quad (k = 1, \infty)
$$

Each particular solution (19) exactly satisfies the governing differential equation (2), and boundary conditions (5) and (6). To satisfy the solution to the initial conditions (3), (4), we compose the sum of particular solutions

$$
\frac{d^2 \varphi(\nu \nu_0)}{d \nu^2} + \mu^2 \varphi(\nu \nu_0) = 0,
$$

Having substituted (20) in (4), we obtain

$$
C_{1k} = -C_{2k} \zeta_{2k} / \zeta_{1k}.
$$

Having substituted (20) in (3), taking into account (21) we find

$$
\sum_{k=1}^{\infty} C_{2k} \left(1 - \zeta_{2k} / \zeta_{1k}\right) \cos(\nu \xi) = 1.
$$

Relation (22) represents the resolution of identity in a Fourier series in terms of eigenfunctions of the Sturm – Liouville problem (9) – (11). Let us multiply equation (22) by $\cos(\nu \xi)$ by and integrate the resulting relation in the range from $\xi = 0$ to $\xi = 1$

$$
\int_0^1 \sum_{k=1}^{\infty} C_{2k} \left(1 - \zeta_{2k} / \zeta_{1k}\right) \cos(\nu \xi) - \cos(\nu \xi) d\xi = 0.
$$

Relation (23) can be normalized to the following type due to cosines orthogonality
Integrating equation (24) we find

\[ C_{2k} = \frac{\sin \nu_k}{(1 - z_{2k} / z_{1k})[\nu_k / 2 + 0.25\sin(2\nu_k)]}, \quad (k = \overline{1, \infty}) \]  \hspace{1cm} (25)

As a result of finding the unknown coefficients \( C_{1k} \) and \( C_{2k} \) \( (k = \overline{1, \infty}) \) the exact analytical solution of the boundary value problem (2) – (6) is found using formula (20).

If the discriminant of relation (17) \( D = (\nu^2Fo_0 + 1)^2 - 4Fo_0\nu^2 < 0 \), we will have 2 complex roots

\[ z_{1k} = \gamma + i\beta; \quad z_{2k} = \gamma - i\beta, \quad \text{where} \quad i = \sqrt{-1}; \quad \beta = \sqrt{(\nu^2Fo_0 + 1)^2 - 4Fo_0\nu^2 / 2Fo_1}; \quad \gamma = (-\nu^2Fo_0 - 1) / 2Fo_1. \]

Particular solutions of equation (8) will be

\[ \varphi_{ik} = \exp(\gamma + i\beta)Fo; \quad \varphi_{2k} = \exp(\gamma - i\beta)Fo. \]  \hspace{1cm} (26)

We write the general integral of equation (8) subject to the particular solutions

\[ \varphi_k(Fo) = C_{1k} \exp[(\gamma + i\beta)Fo] + C_{2k} \exp[(\gamma - i\beta)Fo], \]  \hspace{1cm} (27)

where \( C_j \) \( (j = 1, 2; \quad k = \overline{1, \infty}) \) – unknown integration constants.

Relation (27) can be rewritten as follows

\[ \varphi_k(Fo) = \exp(\gamma Fo)[C_{1k} \exp(i\beta Fo) + C_{2k} \exp(-i\beta Fo)]. \]  \hspace{1cm} (28)

The occurring complex parts of equation (16) roots, using Euler's formulas, are included in the integration constants. Using Euler's formulas, \( \exp(is) = \cos(s) + i\sin(s) \); \( \exp(-is) = \cos(s) - i\sin(s) \), relation (28) can be normalized to the form

\[ \varphi_k(Fo) = \exp(\gamma Fo)[C_{1k} \cos(\beta Fo) + i\sin(\beta Fo)] + C_{2k} \cos(\beta Fo) - i\sin(\beta Fo)]. \]  \hspace{1cm} (29)

Relation (29) subject to designations \( C_{1k} + C_{2k} = B_{1k}; \quad i(C_{2k} - C_{1k}) = B_{2k} \) will be

\[ \varphi_k(Fo) = \exp(\gamma Fo)[B_{1k} \cos(\beta Fo) - B_{2k} \sin(\beta Fo)]. \]  \hspace{1cm} (30)

Having substituted (12) and (30) in (7) and totted up sum of particular solutions, we find

\[ \Theta(\xi, Fo) = \sum_{k=1}^{\infty} \left[ \exp(\gamma Fo)[B_{1k} \cos(\beta Fo) - B_{2k} \sin(\beta Fo)] \right] \cos\left( r \frac{\pi \xi}{2} \right). \]  \hspace{1cm} (31)

To determine constants \( B_{1k} \) and \( B_{2k} \) we use initial data (3), (4). Having substituted (31) in (4), we obtain
\[
\sum_{k=1}^{\infty} (\gamma B_{ik} - \beta B_{2k}) \cos \left( \frac{\pi \xi}{2} \right) = 0.
\]

Hence we find
\[
B_{2k} = \gamma B_{ik} / \beta.
\] (32)

Having substituted (31) in (3), subject to (32) we obtain
\[
\sum_{k=1}^{\infty} B_{ik} \cos \left( \frac{\pi r}{2} \right) = 1.
\] (33)

Having multiplied equation (33) by \( \cos \left( \frac{j \pi \xi}{2} \right) \) \((j = 2k - 1)\) and integrated it within the range from 0 to 1, we obtain
\[
\int_{0}^{1} \sum_{k=1}^{\infty} B_{ik} \cos \left( \frac{\pi r}{2} \right) \cos \left( \frac{j \pi \xi}{2} \right) d\xi = \int_{0}^{1} \cos \left( \frac{j \pi \xi}{2} \right) d\xi.
\] (34)

Taking into account cosines orthogonality, the expression (34) will take the form
\[
\int_{0}^{1} \sum_{k=1}^{\infty} B_{ik} \cos^{2} \left( \frac{\pi r}{2} \right) d\xi - \int_{0}^{1} \cos \left( \frac{\pi \xi}{2} \right) d\xi = 0.
\] (35)

When calculating integrals in (35), we obtain
\[
B_{ik} = \frac{4}{\pi r}.
\] (36)

After constants \( B_{ik} \) and \( B_{2k} \) are determined, solution to the problem is found from (31).

Verification of solutions (20) and (31) showed that all equations of problem (2) – (6) are satisfied exactly.

4. Findings and discussion

The calculation data using formulas (20) and (31) allow us to infer that for \( Bi \to \infty \) the solution found coincides with the exact solution of this problem under Derichlet's boundary conditions [3]. For small values \( Fo_1, Fo_2 \), when the second terms in equation (2) become close to zero, the results of the found solution using formula (20) coincide with the solution of the corresponding classical parabolic heat conductivity equation under Newton's boundary conditions [3].

The assumption that the change in the temperature function inside the body under consideration under Newton's boundary conditions can be described using the heat conductivity equation including relaxation terms was put forward in [7], however, this hypothesis was confirmed only in the course of the studies performed in this work.

It can be seen from the graphs in figure 1, 2, the difference between the calculations performed in accordance with the classical method taking into account the heat transfer factors with the calculation in accordance with the proposed method taking into account the relaxation properties of the body (bypassing the consideration of the heat transfer factors) does not exceed 4%. Thus, an alternative method for modeling changes in temperature fields in a body is proposed. This method differs from the classical one in that the intensity of heat exchange with the environment is taken into account not by the magnitude of the heat transfer factors, but by the temperature of the medium and the body relaxation coefficients, excluding from consideration the heat transfer factors for which there are no formulas of their time dependence.
Figure 1. Temperature fields in the plate under Derichlet's and Newton's BC. — — — the solution for the Newton's BC using formula (31), $\text{Bi} = 1$; $\text{Fo}_1 = \text{Fo}_2 = 1$; — — — the solution for Derichlet's BC from [5].

Figure 2. Temperature fields in the plate under Derichlet's and Newton's BC. — — — the solution for the Newton's BC using formula (31), $\text{Bi} = 10$; $\text{Fo}_1 = \text{Fo}_2 = 0$; — — — the solution for Derichlet's BC from [5].

Let's consider the option when one of the relaxation factors is virtually equal to zero $\text{Fo}_2 = 10^{-7}$ (figures 3 -- 5). In case the relaxation factor $\text{Fo}_1$ also takes the values close to zero ($\text{Fo}_1 < 10^{-7}$), then the solution (31) will coincide with the solution of classical parabolic equation under Newton's boundary conditions. As the number increases $\text{Fo}_1$ for the same number, Bi a kink appears on the
temperature curves, the front of which moves along the coordinate $\xi$ in time (figure 3). Both direct and backward temperature waves are detected here. A kink on a direct wave has a peculiarity in that the temperature lines make a certain angle with the line of the initial temperature of the plate $\Theta = 1$. In the case of the classical parabolic equation of heat conductivity for the moments of time when the heat wave front has not yet reached the middle of the plate, the temperature lines are tangent to the line of the initial condition $\Theta(\xi, 0) = 1$.

**Figure 3.** Temperature change. $Bi = 1; \quad Fo_1 = 3; \quad Fo_2 = 10^{-7}; \quad n = 1000 –$ series terms number (31).

**Figure 4.** Temperature change. $Bi = 1; \quad Fo_1 = 3; \quad Fo_2 = 10^{-7}; \quad n = 200 –$ series terms number (31).
Figure 5. Temperature change. Bi = 10⁶; Fo₁ = 3; Fo₂ = 10⁻⁷; n = 1000 – series terms number (31).

Figure 6 shows the temperature curves in the plate at Bi = 10 subject to relaxation phenomena for Fo₁ = Fo₂ = 3. In contrast to Figure 3, where the temperature changes are jump-like and shifts to the region of negative values, in this case, the temperature curves take the form of almost horizontal lines, and the heat transfer process proceeds much more slowly than without taking into account the local nonequilibrium at Fo₁ = Fo₂ = 0.

Figure 6. Temperature fields in the plate under Newton’s boundary conditions subject to the relaxation phenomena. Bi = 10; Fo₁ = 3; Fo₂ = 3; n = 100 – series terms number (31).
5. Conclusions
An exact analytical solution is obtained to the boundary value problem of thermal conductivity equation subject to the relaxation phenomena under Newton's boundary conditions.

It is shown that the change in the temperature fields in the body under the boundary conditions of convective heat exchange can be described by the thermal conductivity equation, taking into account a reiterated lagging.

At thermal power plants, industrial facilities, in the aerospace industry, the problem of determining the heat transfer coefficient, which depends on a large number of factors, can be solved by identifying the phenomenological coefficients in the mathematical model presented.

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