Abstract. This paper studies structural aspects of lattice path matroids, a class of transversal matroids that is closed under taking minors and duals. Among the basic topics treated are direct sums, duals, minors, circuits, and connected sets. One of the main results is a characterization of lattice path matroids in terms of fundamental sets, which are special connected sets from which one can recover the paths that define the matroid. We examine some aspects related to key topics in the literature of transversal matroids and we determine the connectivity of lattice path matroids. We also introduce notch matroids, a minor-closed, dual-closed subclass of lattice path matroids, and we find their excluded minors.

1. Introduction

A lattice path matroid is a special type of transversal matroid whose bases can be thought of as lattice paths in the region of the plane delimited by two fixed bounding paths. These matroids, which were introduced and studied from an enumerative perspective in [5], have many attractive structural properties that are not shared by arbitrary transversal matroids; this paper focuses on such properties.

The definition of lattice path matroids is reviewed in Section 2, where we also give some elementary properties of their bases and make some remarks on connectivity and automorphism. Section 3 proves basic results that are used throughout the paper; for example, we show that the class of lattice path matroids is closed under minors, duals, and direct sums, we determine which lattice path matroids are connected, and we describe circuits and connected sets. The next section discusses generalized Catalan matroids, a minor-closed, dual-closed subclass of lattice path matroids that has particularly simple characterizations. Section 4 introduces special connected sets called fundamental sets that we use to characterize lattice path matroids and to show that the bounding paths can be recovered from the matroid.

In Section 5, we describe the maximal presentation of a lattice path matroid, and we use this result to give a geometric description of these matroids as well as a polynomial time algorithm for recognizing lattice path matroids within the class of transversal matroids. We also contrast lattice path matroids with fundamental transversal matroids and bicircular matroids. Section 6 treats higher connectivity. The final section introduces another minor-closed, dual-closed class of lattice path matroids, the notch matroids, and characterizes this class by excluded minors.

We assume familiarity with basic matroid theory (see, e.g., [16, 20]). We follow the notation and terminology of [16], with the following additions. A set \( X \) of a matroid \( M \) is connected if the restriction \( M \mid X \) is connected. A set \( X \) is trivial if \( X \) is independent; otherwise \( X \) is nontrivial. The sets in a collection \( F \) of sets are
Incomparable, or mutually incomparable, if no  at in $F$ contains another  at in $F$. The nullity, $k_j r(x_j)$, of a set $X$ is denoted by $(x_j)$. Recall that a matroid $M$ of rank $r$ is a paving matroid if every  at of rank less than $r$ is trivial.

Most matroids in this paper are transversal matroids (see [6,12,20]). Recall that for a transversal matroid $M$, a presentation of $M$ is a multiset $A = \{D_1, D_2, \ldots, D_k\}$ of subsets of the ground set $E$ ($M$) such that the bases of $M$ are the maximal transversals of $A$. As is justified by the following lemma (see [6]), we always consider presentations of rank-$r$ transversal matroids by set systems of size $r$.

Lemma 1.1. Let $A = \{D_1, D_2, \ldots, D_k\}$ be a presentation of a rank-$r$ transversal matroid $M$. If some basis of $M$ is a transversal of $\{D_{i_1}, D_{i_2}, \ldots, D_{i_r}\}$, with $i_1 < i_2 < \ldots < i_r$, then $\{D_{i_1}, D_{i_2}, \ldots, D_{i_r}\}$ is also a presentation of $M$.

We use $[i; j]$ to denote the interval from $i$ to $j$ of integers, and, sim ilarly, $[i; j)$ to denote the interval from $i$ to $j$ of integers.

## 2. Background

This section begins by reviewing the definition and basic properties of lattice path $m$ matroids from [3]. The notation established in this section is used throughout the paper. Also included are the basic results about matroid connectivity that we use later.

Unless otherwise stated, all lattice paths in this paper start at the point $(0; 0)$ and use steps $E = (1; 0)$ and $N = (0; 1)$, which are called East and North, respectively. Paths are usually represented as words in the alphabet $EN\overline{G}$. We say that a lattice path $P$ has a NE corner at $(i; j)$ if step $i$ of $P$ is North and step $i + 1$ is East. An EN corner at $(i; j)$ can also be specified by the coordinates of the point where the North and East steps meet.

A lattice path $m$ matroid, is up to isomorphism, a $m$ matroid of the type $M \{P; Q\}$ that we now define. Let $P$ and $Q$ be lattice paths from $(0; 0)$ to $(m; r)$ with $P$ never going above $Q$. Let $P$ be the set of all lattice paths from $(0; 0)$ to $(m; r)$ that go neither above $Q$ nor below $P$. For $i$ with $1 \leq i \leq r$, let $N_i$ be the set

\[ N_i = \{j : \text{step } j \text{ is the } i\text{-th North step of some path in } P; Q \} \]

Thus, $N_1; N_2; \ldots; N_r$ is a sequence of intervals in $[n + r]$, and both the left endpoints and the right endpoints from strictly increasing sequences; the left and right endpoints of $N_i$ correspond to the positions of the $i$-th North steps in $Q$ and $P$, respectively. The $m$ matroid $M \{P; Q\}$ is the transversal matroid on the ground set $[m + r]$ that has $\{N_1; N_2; \ldots; N_r\}$ as a presentation. We call $\{N_1; N_2; \ldots; N_r\}$ the standard presentation of $M \{P; Q\}$. Note that $M \{P; Q\}$ has rank $r$ and nullity $m$.

Figure 1 shows a lattice path $m$ matroid of rank 4 and nullity 7. The intervals in the standard presentation are $N_1 = \{1\}, N_2 = \{2; 7\}, N_3 = \{5; 10\}$, and $N_4 = \{6; 11\}$. (Section 5.3 explains how to nd a geometric representation of a lattice path $m$ matroid.)

A feature that enriches the subject of lattice path $m$ matroids is the variety of ways in which these matroids can be viewed. On the one hand, the theory of transversal matroids provides many useful tools. On the other hand, the following theorem from [6] gives an interpretation of the bases that is used as a tool. This theorem was based on the concept of m atroids and provides a means of obtaining descriptions of many concepts (see, e.g., [5], Theorem 5.4) on basis activities.)
For this incidence function and for any element \( x \in [n + r] \), the set \( n(x) \) is an interval in \( [p] \); if \( x < y \), then \( \max n(x) \leq \max n(y) \) and \( \min n(x) \geq \min n(y) \).

An independent set \( I \) in a lattice path matroid \( \{P; Q\} \) is a partial transversal of \( (N_1; N_2; \ldots; N_r) \). Typically there are many ways to match \( I \) with \( N_1; N_2; \ldots; N_r \). The next two results show that one can always be matched in a natural way. The following lemma, which is crucial in the proof of Theorem 2.1, is from [5].

**Lemma 2.3.** Assume \( b_0; b_1; \ldots; b_r \) is a basis of a lattice path matroid \( \{P; Q\} \) with \( b_i < b_{i+1} \leq \xi \). Then \( b_i \in N_i \) for all \( i \).

**Corollary 2.4.** follows by extending the given independent set \( I \) to a basis and applying Lemma 2.3.
Corollary 2.4. Assume \( I \) is an independent set of a lattice path matroid \( M \) with \( f(j) = h(j) \) \( j \) let \( I \) be \( f_1 \); \( a_2 \); \( \ldots \); \( a_n \) with \( a_1 < a_2 < \cdots < j \) and let \( n(I) \) be \( f_1 \); \( b_2 \); \( \ldots \); \( b_n \) with \( b_1 < b_2 < \cdots < j \). Then \( a_j \) is in \( N_{ij} \) for all \( j \) with \( 1 \leq j \leq k \).

We now gather several results on matroid connectivity that are relevant to parts of the paper. The first result \([1] \), Theorem 7.1.16] gives a fundamental link between connectivity and the operation of parallel connection.

Lemma 2.5. If \( M \) is connected and \( M = p \) is the direct sum \( M_1 \oplus M_2 \), then \( M \) is the parallel connection \( P(M_1 \oplus M_2) \) of \( M_1 = M \oplus \emptyset \) and \( M_2 = M \oplus \emptyset \).

In Lemma 2.5, since \( M \) is connected, both \( M_1 \) and \( M_2 \) are connected. Recall that the rank \( R(P(M_1 \oplus M_2)) \) of a parallel connection whose basepoint is not a loop is \( R(M_1) + R(M_2) \). These observations give the following lemma.

Lemma 2.6. If \( M \) is connected, \( X \) is not parallel to any element of \( M \), and \( M \setminus X \) is disconnected, then there is a pair \( A \parallel B \) of nontrivial incomparable connected sets of \( M \) with \( r(A) + r(B) = r(M) + 1 \) and \( A \setminus B = \emptyset \).

The following useful Lemma 2.6 is easy to prove by using separating sets.

Lemma 2.7. Assume that \( X \) is a connected subset of a connected matroid \( M \), that \( X \) is in \( X \), and that \( M \setminus X \) is connected. Then \( M \setminus X \) is connected.

The cyclic sets of a matroid \( M \) (that is, the sets \( F \) for which \( M \setminus F \) has no isomorphisms), together with their ranks, determine the matroid \([2] \), Proposition 2.1]. As we show next, in the loopless case it suffices to consider nontrivial connected sets. Note that nontrivial connected sets are cyclic, but cyclic sets need not be connected. Thus, the next result is a mild generalization of \([2] \), Proposition 2.1], and essentially the same idea proves both results.

Lemma 2.8. The circuits of a loopless matroid \( M \) (and hence \( M \) itself) are determined by the nontrivial connected sets and their ranks.

Proof. Note that if \( C \) is an i-circuit, then \( C \setminus (C) \) is a connected set of rank \( i - 1 \). Thus, the circuits can be recovered inductively as follows: the 2-circuits are the 2-subsets of nontrivial rank-1 sets; the 3-circuits are the 3-subsets of \( E \setminus M \) that contain no 2-circuit and are subsets of connected lines, and so on.

Corollary 2.9. The automorphisms of a loopless matroid are the permutations of the ground set that are rank-preserving bijections of the collection of nontrivial connected sets.

3. Basic Structural Properties of Lattice Path Matroids

This section treats the basic structural properties of lattice path matroids that play key roles throughout this paper. Some of these properties are shared by fewer other classes of matroids; for instance, every nontrivial connected lattice path matroid has a spanning circuit. Other properties, such as the closure of the class of lattice path matroids under minors and duals, while shared by fewer classes of matroids, do not hold for the larger class of transversal matroids. Some of the properties are more technical and their significance will become apparent only later in the paper. The topics treated are fairly diverse, so we divide the material into subsections that focus in the following issues: minors, duals, and direct sum; connectivity and spanning circuits; the structure of circuits and connected sets.
3.1. Minors, Duals, and Direct Sums. The class of transversal matroids, although closed under deletions and direct sum, is closed under neither contractions nor duals. In contrast, we have the following result for lattice path matroids.

Theorem 3.1. The class $L$ is closed under minors, duals, and direct sums.

Proof. Figure 2 illustrates the obvious construction to show that $L$ is closed under direct sums. For closure under duality, note that, from Theorem 2.1, a basis of the dual of $M_{P;Q}$ (i.e., the complement of a basis of $M_{P;Q}$) corresponds to the East steps in a lattice path; the East steps of a lattice path are the North steps of the lattice path obtained by reflecting the entire diagram about the line $y = x$. This idea is illustrated in Figure 3.

For closure under minors, it suffices to consider single-element deletions. Let $x$ be in the lattice path matroid $M = M_{P;Q}$ on $[n + r]$ with standard presentation $(N_1; N_2; \ldots; N_r)$. Note that $(N_1 \cup \{x\}; N_2; \ldots; N_r; x)$ is a presentation of $M_{nx}$; from this presentation, we will obtain one that shows that $M_{nx}$ is a lattice path matroid. Some set $N_i$ is fixed if and only if $x$ is an isthmus of $M$; in this case, discard the empty set $N_i \cup \{x\}$ from the presentation above to obtain the required presentation of $M_{nx}$. Thus, assumption holds if and only if $x$ is an isthmus of $M$. The sets $N_1; N_2; x; \ldots; N_r; x$ are intervals in the induced linear order on $[n + r]$. In only two cases will the least elements or the greatest elements (or both) fail to increase strictly: (a) $x$ is the least element of the interval $N_i$ and $x + 1$ is the least element of $N_{i+1}$, and (b) $x + 1$ and $x$ are the greatest elements of $N_{i+1}$ and $N_i$, respectively. Assume case (a) applies. Any basis of $M_{nx}$ (that is, any basis of $M$ that does not contain $x$) that contains $x + 1$ can, by Lemma 2.3, be matched with $N_1; N_2; \ldots; N_r$, so that $x + 1$ is not matched to $N_{i+1}$. Thus, the set system obtained by replacing $N_{i+1}$ by...
Single-element deletions and contractions can be described in terms of the bounding paths of $M = [P; Q]$ as follows. An isthmus is an element $x$ for which some $N_i$ is a $g_x$; to delete or contract $x$, eliminate the corresponding common North step from both bounding paths. A loop is an element that is in no set $N_i$; to delete or contract a loop, eliminate the corresponding common East step from $P$ and $Q$.

Now assume $x$ is neither a loop nor an isthmus. The upper bounding path for $M$ is formed by deleting from $Q$ the first East step that is at or after step $x$; the lower bounding path for $M$ is formed by deleting from $P$ the last East step that is at or before step $x$. This is shown in Figure 4, where the dashed steps in the middle diagram indicate the steps that bases of $M$ must avoid. Dually, the upper bounding path for the contraction $M = x$ is formed by deleting from $Q$ the last North step that is at or before step $x$; the lower bounding path for $M = x$ is formed by deleting from $P$ the first North step that is at or after step $x$.

Corollary 3.2 treats restrictions of lattice path $m$-matroids to intervals. The lattice path interpretation of this result is illustrated in Figure 5 on page 13.

Corollary 3.2. Let $M$ be the lattice path $m$-matroid $[P; Q]$ on the ground set $[m + r]$. Let $X$ be the initial segment $[i]$ and $Y$ be the final segment $[j + 1; m + r]$ of $[m + r]$. Let the $i$-th step of $Q$ end at the point $(h; k)$ and let the $j$-th step of $P$ end at $(h^0; k^0)$.

(a) The bases of the restriction $M \backslash X$ correspond to the lattice paths that go from $(0; 0)$ to $(h; k)$ and go neither below $P$ nor above $Q$.

(b) The bases of the restriction $M \backslash Y$ correspond to the lattice paths that go from $(h^0; k^0)$ to $(m; r)$ and go neither below $P$ nor above $Q$.

(c) If $h^0 > h$, then the bases of $M \backslash (X \backslash Y)$ correspond to the lattice paths that go from $(h^0; k^0)$ to $(h; k)$ and go neither below $P$ nor above $Q$.

We close this section by noting that although $U_{1,2} \cup U_{1,2} \cup U_{1,2}$ is a lattice path $m$-matroid, its truncation is not transversal. It follows that $L$ is not closed under the following operations: truncation, free extension, and elongation.
32. Connectivity and Spanning C Circuits. We begin with a rare property.

Theorem 3.3. A connected lattice path matroid $M\{P; Q\}$ on at least two elements has a spanning circuit.

Proof. Let $M\{P; Q\}$ have rank $r$, let $N_j$ be $[l_j; g_j]$ for $1 \leq j \leq r$, and let $C$ be the set $f_1; l_1; \ldots; l_j; g_j; g$. Showing that each set $C - x$, for $x$ in $C$, is a basis shows that $C$ is a spanning circuit. That $C$ has rank $r$ and $C$ is a basis is clear. Since $M\{P; Q\}$ is not a direct sum of two matroids, the spanning circuit $x$ must be in $N_i$ for $1 \leq i < r$, from which it follows that each set $C - l_j$, with $1 \leq j < r$, is a basis.

It will be useful to single out the following immediate corollary of Theorem 3.3.

Corollary 3.4. If $X$ is a nontrivial connected lattice path matroid and $M\{X\}$ is a lattice path matroid, then $X$ is a circuit $C$ of $M$.

The next theorem determines which lattice path matroids are connected. One implication follows from the description of direct sums and the other from the construction of the spanning circuit in the proof of Theorem 3.3.

Theorem 3.5. A lattice path matroid $M\{P; Q\}$ of rank $r$ and nullity $n$ is connected if and only if $P$ and $Q$ intersect only at $(0; 0)$ and $(m; r)$.

The parallel connection of two 3-point lines, which has only one spanning circuit, shows that there may be elements of a connected lattice path matroid that are in no spanning circuit. There are several ways to identify the elements of connected lattice path matroids that are in spanning circuits. The next result identifies these elements via the standard presentation.

Theorem 3.6. An element $x$ of a nontrivial connected lattice path matroid $M\{P; Q\}$ of rank $r$ is in a spanning circuit of $M\{P; Q\}$ if and only if $x$ is in at least two of the sets $N_1; N_2; \ldots; N_r$, or $x$ is in $N_1$ or $N_r$.

Proof. Assume $x$ is in $N_i$ and $N_{i+1}$. Let $C$ be $f_1; l_1; \ldots; l_i; x; g_{i+1}; g_{i+2}; \ldots; g_r$ where $N_j$ is $[l_j; g_j]$. By connectivity, we have $l_1, 2 N_1; l_2, 2 N_2; \ldots; l_i, 2 N_i$ and $g_{i+1}, 2 N_{i+1}; g_{i+2}, 2 N_{i+2}; \ldots; g_r, 2 N_r$. An argument like that in the proof of Theorem 3.3 shows that $C$ is a spanning circuit. Similar ideas show that $x$ is in a spanning circuit of $M\{P; Q\}$ if $x$ is in $N_1$ or $N_r$.

Assume $x$ is in $N_i$ with $1 < i < r$. Note that the basepoint is in no spanning circuit of a parallel connection of two matroids of rank two or more, so to complete the proof we need only show that $M\{P; Q\}$ is a parallel connection of two lattice path matroids, each of rank at least two, with basepoint $x$. Thus, by Lemma 3.4, we need to show that $M\{P; Q\} = xN\bar{x}$ is disconnected where $X$ is the set of loops of $M\{P; Q\} = x$. This statement follows from the lattice path description of contraction along with the observations that $N_{i+1}$ contains only elements less than $x$ while $N_i$ contains only elements greater than $x$.

The following characterizations of the elements that are in spanning circuits use structural properties rather than presentations.

Corollary 3.7. Let $x$ be in a nontrivial connected lattice path matroid $M$.

(a) No spanning circuit contains $x$ if and only if $M$ is a parallel connection of two lattice path matroids, each of rank at least two, with basepoint $x$. 
(b) Some spanning circuit contains $x$ if and only if $M = x \cap X$ is connected, where $X$ is the set of loops of $M = x$.

**Proof.** Part (a) follows from the proof of Theorem 3.9. If $x$ is in a spanning circuit $C$ of $M$, then $C \cap x$ is a spanning circuit of $M = x$, so $M = x \cap X$ is connected. Conversely, if $x$ is in no spanning circuit of $M$, then, by part (a), $M$ is a parallel connection, with basepoint $x$, of matroids of rank at least two, so $M = x \cap X$ is disconnected.

### 3.3 Circuits and Connected Flats

Our next goal in this section is to characterize the circuits of lattice path matroids. This is done in Theorem 3.9, the proof of which uses the following well-known elementary result about the circuits of arbitrary transversal matroids. This lemma follows easily from Hall's theorem.

**Lemma 3.8.** Let $n$ be the incidence function of a presentation of a transversal matroid $M$. If $C$ is a rank-$k$ circuit of $M$, then $\langle C \cap x \rangle$ is a connected lattice path circuit for any $x$ in $C$.

**Theorem 3.9.** Let $C = f_{C_0; C_1; \ldots; C_m} g$ be a set in the lattice path matroid $M = x \cap X$. Assume $C_0 < C_1 < \ldots < C_m$. Let $n(C) = f_{l_1; l_2; \ldots; l_m} g$, where $l_1 < l_2 < \ldots < l_m$. Then $C$ is a circuit of $M = x \cap X$ if and only if

1. $s = k$,
2. $C_{j} \cap N_{l_j}$, and
3. $C_{k} \cap N_{l_k}$ for $j$ with $0 < j < k$.

Furthermore, if $C$ is a circuit, then $l_j = l_{j+1}$ for $1 \leq j < k$.

**Proof.** It is immediate to check that if conditions (1)-(4) hold, then $C$ is dependent and every $k$-subset of $C$ is a partial transversal and so is independent; thus $C$ is a circuit. For the converse, assume $C$ is a circuit. Assertion (1) follows from Lemma 3.8, which also gives the equalities $n(C) = f_{C_0; C_1; \ldots; C_m} g$. Since $C_0$ is independent and $n(C) = f_{C_0; C_1; \ldots; C_m} g$, it follows from Corollary 2.3 that $C_j$ is in $N_{l_j}$ for $1 \leq j \leq k$. A similar argument using $C_k$ shows that $C_k$ is in $N_{l_k}$ for $0 < j < k$. This proves assertions (2)-(4). To prove the last assertion, assume there are an $h$ in $n(C)$ with $l_j < h < l_{j+1}$. From statement (4), we have that $C_j$ is in both $N_{l_j}$ and $N_{l_{j+1}}$. The inequalities

$$m \in N_{h_j} \subset m \in N_{l_{j+1}} \cap C_j \subset m \in N_{l_j} \subset m \in N_{h_j}$$

imply that $C_j$ is in $N_h$, which contradicts the assumption that $h$ is not in $n(C)$.

By Lemma 3.8, if $x$ is parallel to some element, then $n(C) = 1$. By property (4) of Theorem 3.9, at most two elements $x$ in a circuit of a lattice path matroid can satisfy the equality $n(C) = 1$. This observation proves the next result.

**Corollary 3.10.** At most two elements in any circuit of a lattice path matroid are in nonsingleton parallel classes.

The following result gives two useful properties of connected flats.

**Theorem 3.11.** Let $M = x \cap X$ have rank $r$ and nullity $m$. Any nontrivial connected at $X$ of $M = x \cap X$ is an interval in $[m + r]$ and $n(X)$ is an interval of $r(X)$ elements in $[r]$. 

Proof. The second assertion follows from Corollary 3.12 and Theorem 3.13. For the rest statement, let \( n(\mathcal{X}) \) be \([s,t]\) and assume \( i < j < k \) with \( i, k \leq 2 \). That \( j \) is in \( X \) follows from the inequalities

\[
\min_{i \leq j \leq k} n(i) \quad \min_{i \leq j \leq k} n(j) \quad \max_{i \leq j \leq k} n(j) \quad \max_{i \leq j \leq k} n(k) \quad t
\]

Theorem 3.11 has many implications for the connected \( k \)-nets of lattice path matroids, of which we mention four.

Corollary 3.12. Assume \( M[\mathcal{P}; \mathcal{Q}] \) has rank \( r \).

(I) For \( 0 < k < r \), there are at most \( k + 1 \) nontrivial connected \( j \)-nets of rank \( r \) for each \( j \) in \( M[\mathcal{P}; \mathcal{Q}] \). In particular, \( M[\mathcal{P}; \mathcal{Q}] \) has at most two connected hyperplanes and at most \( r - 1 \) connected lines.

(ii) A nontrivial connected \( j \)-net of \( M[\mathcal{P}; \mathcal{Q}] \) is covered by at most two connected \( j \)-nets.

(iii) The nontrivial connected \( j \)-nets of \( M[\mathcal{P}; \mathcal{Q}] \) that are not contained in a connected hyperplane \( H \) of \( M[\mathcal{P}; \mathcal{Q}] \) are linearly ordered by inclusion.

(iv) If \( H \) and \( H' \) are connected hyperplanes of \( M[\mathcal{P}; \mathcal{Q}] \), then every nontrivial connected \( j \)-net of \( M[\mathcal{P}; \mathcal{Q}] \) is contained in at least one of \( H \) and \( H' \).

The matroid \( M[\mathbb{E}^2 \to \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N}] \), which is a parallel connection of \( r + 1 \) three-point lines in which elements have been added parallel to the \( \{ \text{points} \} \) and the \( \{ \text{ends} \} \), shows that all upper bounds in parts (i) and (ii) of Corollary 3.12 are optimal.

The next result is another corollary of Theorem 3.13.

Corollary 3.13. Let \( C \) be the circuit \( f_0; c_1; \ldots; c_g \) of \( M[\mathcal{P}; \mathcal{Q}] \) with \( c_0 < c_1 < \cdots < c_g \). If \( x \) is not in \( C \) and \( Z \) is a circuit of \( M[\mathcal{P}; \mathcal{Q}] \) for some subset \( Z \) of \( C \), then \( Z \) is either an initial segment \( f_0; c_1; \ldots; c_g \) or a final segment \( f_0; c_1; \ldots; c_g \).

Proof. The result follows from Lemma 3.16 and this simple corollary of Theorem 3.15. For any proper subset \( X \) of \( C \) that is neither an initial nor final segment of \( C \), the inequality \( h(X) < |X| \) holds.

We conclude this section with a result we will use to show that certain matroids are not lattice path matroids.

Theorem 3.14. Assume a rank-\( r \) matroid \( M \) has two nontrivial connected \( j \)-nets \( X \) and \( X' \) such that

1. \( X \setminus X' \neq \emptyset \);
2. \( r(X) = r(X') \);
3. \( X \setminus X' \) is a proper subset of the ground set \( E(M) \) of \( M \).

Then \( M \) is not a lattice path matroid.

Proof. Assume, to the contrary, that \( M \) is \( M[\mathcal{P}; \mathcal{Q}] \). Fix \( x \in X \setminus X' \) and \( y \in E(M) \setminus \{X \setminus X' \} \). By Theorem 3.11, along with assumptions (1) and (2), up to switching \( X \) and \( X' \) we would have \( n(X) = k \) and \( n(X') = [k; r] \) for some \( k \) and \( k' \) with \( k < k' \). The inequality \( x < y \) would give \( \max_{x \leq y} n(y) = \max_{x \leq y} n(x) = k \), so \( y \) would be in \( \text{cl}(X) \). The inequality \( x < y \) would give \( \min_{x \leq y} k \) and \( y \) would be in \( \text{cl}(X') \). That these conclusions contradict the hypothesis proves the lemma.
Generalized Catalan Matroids

Our next aim is to characterize lattice path matroids; this will be done in Section 4. This section focuses on an important subclass of L that has particularly simple characterizations and many interesting properties.

Definition 4.1. The n-th Catalan matroid $M_n$ is $M \{ E^n N^n; E \in N \}$, A generalized Catalan matroid is, up to isomorphism, a matroid of the form $M \{ E^n N^n; Q \}$.

For generalized Catalan matroids, the notation $M \{ P; Q \}$ is simplified to $M [Q]$. We use $C$ to denote the class of generalized Catalan matroids.

Generalized Catalan matroids have arisen in different contexts with a corresponding variety of names and perspectives. We gather here the references currently known to us. Crapo [2, Section 8] introduced these matroids to show that there are at least $\binom{n}{r}$ nonisomorphic matroids of rank $r$ on $n$ elements. His perspective was rediscovered in [5, Theorem 3.14]; generalized Catalan matroids are precisely the matroids that are obtained from the empty matroid by repeatedly applying the operations of adding an isthmus and forming the free extension (this result is generalized in Theorem 6.7 below). By using "nested" presentations, Welsh [19] proved that Crapo's lower bound on the number of matroids holds within the smaller class of transversal matroids. These matroids arose again in [17] in connection with matroids defined in terms of integer-valued functions on finite sets. They were studied further in [18], where they were called Schubert matroids and shown to have the rapid mixing property. In [18] they were rediscovered and related to shifted complexes, and so acquired the name shifted matroids. The link that was established in [5] between generalized Catalan matroids and an enumerative problem known as the tennis ball problem is used in the techniques used in [19] to solve that problem. In [10], under the name of freedom matroids, general Catalan matroids were used to construct a free algebra of matroids.

Catalan matroids have rich enumerative properties (see [5]). Their name comes from the fact that the number of bases of $M_n$ is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$; several other invariants of $M_n$ are also Catalan numbers. Although there is only one Catalan matroid of each rank, these matroids generate the entire class $C$, in the sense of the following theorem.

Theorem 4.2. The smallest minor-closed class of matroids that contains all Catalan matroids is $C$.

Proof. It follows from the lattice path interpretation of deletion and contraction given after the proof of Theorem 3.1 that $C$ is closed under minors. To see that any generalized Catalan matroid $M [P; Q]$ is a minor of a Catalan matroid, simply insert East and North steps into $Q$ so that the result is a Catalan matroid $M \{ E \in N \}$. From $M \{ E \in N \}$, delete the elements that correspond to the added East steps and contract the elements that correspond to the added North steps; by the lattice path interpretation of these operations, the resulting minor of $M \{ E \in N \}$ is $M [Q]$.

It is easy to see that $C$, in addition to being closed under minors, is closed under duals and (unlike $L$) free extension; therefore $C$ is closed under truncation and elongation. However, $C$ is not closed under direct sums.

By Theorem 3.2, a generalized Catalan matroid with at least two elements is connected if and only if it has neither loops nor isthmuses. The rest of this section focuses mainly on connected generalized Catalan matroids since some results are
s e n t i a l l y no l o s s o f g e n e r a l i t y.

The feature that makes generalized Catalan matroids easy to characterize is the structure of the connected dats, as described in the following lemma.

Lemma 4.3. A sum $E$ of $M_Q$ has rank $r$, nullity $m$, and neither loops nor isthmuses. Let the $EN$ corners of $Q$ be at steps $i_1, i_2, \ldots, i_k$ with $i_1 < i_2 < \ldots < i_k$. The proper nontrivial connected $dats$ of $M_Q$ are the initial segments $[i_1, i_2, \ldots, i_k]$ of $\mathbb{N}$. The rank (resp. nullity) of $[i_1, i_2, \ldots, i_k]$ is the number of North (resp. East) steps among the last $k$ steps of $Q$.

Proof. Let the lemma follow easily once we show that any proper nontrivial connected $dat$ at $F$ of $M_Q$ is an initial segment of $[m + r]$. By Theorem 3.1, $F$ is an interval, say $[u; v]$, in $[m + r]$. By Corollary 4.4, the restriction of $M_Q$ to $[v]$ is $M_{Q,v}$ where $Q_v$ consists of the first $v$ steps of $Q$. Since $v$ is not an isthmus of $M_Q$, $F$ is not an isthmus of $M_Q$, so the $v$-th step of $Q$ must be East. Let $M_{Q,v}$ have rank $k$. Note that $[v, k; v]$ is a spanning circuit of $M_{Q,v}$ that is contained in $F$ and has closure $[v]$. Thus, $F$ is the initial segment $[v]$.

The following result (which is essentially Lemma 2 of [17]) is an immediate corollary of Lemma 4.3.

Corollary 4.4. A connected matroid is a generalized Catalan matroid if and only if its nontrivial connected $dats$ are linearly ordered by inclusion.

The following excluded-minor characterization of $C$ from [17] is not difficult to prove from Corollary 4.4 and the results in Section 8. Let $P_n$ be the truncation $T_n(U_{n,1})$ to rank $n$ of the direct sum of two $n$-circuits. Thus, $P_n$ is the paving matroid of rank $n$ whose only nontrivial proper $dats$ are two disjoint circuit-hyperplanes whose union is the ground set. It follows that $P_n$ is isomorphic to $M[E^n; E^n]$ and, by Corollary 4.4, that $P_n$ is not in $C$.

Theorem 4.5. A matroid is in $C$ if and only if it has no minor isomorphic to $P_n$ for any $n \geq 2$.

5. Fundamental Flats and a Characterization of Lattice Path Matroids

While the structure of the connected $dats$ of arbitrary connected lattice path matroids is not as simple as that for generalized Catalan matroids (Corollary 4.3), this structure is still easy to describe. We analyze this structure in this section and use it to characterize connected lattice path matroids. We also show that if $M[P; Q]$ is connected, then the paths $P$ and $Q$ are determined, up to a 180 rotation, by any matroid isomorphic to $M[P; Q]$. The $dats$ of central interest for these results are those we define now.

Definition 5.1. Let $X$ be a connected $dat$ at of a connected matroid $M$ for which $j > 1$ and $r(X) < r(M)$. We say that $X$ is a fundamental $dat$ at of $M$ if for some spanning circuit $C$ of $M$, the intersection $X \setminus C$ is a basis of $X$.

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Lemma 5.2. Assume $e M \ U P ; Q U$ is connected and has rank $r$ and nullity $m$. Let $X$ be a connected path of $M \ U P ; Q U$ with $K > 1$ and $r(X) < r$. Then $X$ is a fundamental path of $M \ U P ; Q U$ if and only if $X$ is an initial or null segment of $[m + r]$.

Proof. Let $N$ be the initial segment of $[m + r]$. If $X$ is an initial segment of $[m + r]$, then the spanning circuit $C = f_{i_1} l_{j_1}; \ldots; l_{j_r} g_{i_r}; g$; constructed in the proof of Theorem 3.3, has the property that $X \backslash C$ is a basis of $X$. Similarly, for a null segment $X$ of $[m + r]$, a spanning circuit with the required property is $f_{i_1} g_{j_1}; g_{j_2}; \ldots; g_{j_r}$.

Corollary 5.4. For each pair of two chains under inclusion, no set in one chain contains a set in the other chain. Further, for each pair $X; Y$ of incomparable fundamental paths,

(a) If $X \backslash Y \notin \cup$, then $X \cup \{ \ U Y = E ( M ) \}$,
(b) If $r(X) + r(Y) = r(M)$, then $r(X \cup \{ Y \}) = r(M)$.

While a connected lattice path $M$ of rank $r$ has at most $k + 1$ connected paths of rank $r$, the fundamental paths of any given rank.

Theorem 5.5. For generalized Catalan matroids, the fundamental paths are precisely the paths given in Lemma 5.2, so they form one chain under inclusion.

Corollary 5.4. For any lattice path $M$, the fundamental paths of the dual $M$ are the set complements, $E ( M ) = F$, of the fundamental paths $F$ of $M$.

A key observation that follows from Theorem 5.5 is that although which paths are fundamental is independent of the order of the elements that is inherent in any particular lattice path presentation of a lattice path $M$. Such a presentation
m ake s i t e as y t o i de nt i f y t he f undam e nt al
at s. Conve r s e l y ,t he c hai ns of f unda-
mentsal
at sgi ve t heboundi ng pat hs .M or epr e c i s e l y ,t hepaths
Theorem 5.7.
The t he onl y m at r oi ds
M
are f undam ent al
at s. Conne c te d
lat t i ce p at h m at r oi d i n t e r m sofi t
f u ndam e nt al
at s.g i ven t he f oll o wi ng t he or e m ,whi c h i sone of t he m ai n r e s ul t s of t hi s s e c t i o n.

Theorem 5.6. The bounding paths P and Q of a connected lattice path matroid M [P; Q] are determined by the matroid structure, up to a 180° rotation. That is, the only matroids M [P; Q] isomorphic to M [P; Q] are M [P; Q] and M [Q; P].

Theorem 5.5 and its corollaries (including Theorem 5.6) show that a connected lattice path matroid is determined by its fundamental atoms and their ranks. The next several results further develop this idea. The following theorem describes all connected matroids of a connected lattice path matroid in terms of its fundamental atoms.

Theorem 5.7. Let M be the connected lattice path matroid M [P; Q] of rank r and nullity m, and let F_1, F_2, ..., F_k be the chains of fundamental atoms of M. The proper nontrivial connected atoms of M are

- (P) F_1,F_2, ..., F_k,G_1;G_2;...;G_k, being fundamental, are connected. The element 1 is in either F_1 or G_1; we may assume it is in F_1. For part (ii), we use Corollary 5.3 to find a lattice path presentation that shows that F_1 \setminus G_j is connected. Using the notation in that corollary, let X be F_1, so the point (n;k) on Q is (F_1);r(F_1) ; let Y be G_j, so the point (m;k) on P is m \cap G_j; r(G_j). The inequality in part (ii) along with part (c) of Corollary 5.3 give a presentation of M [F_1 \setminus G_j] (illustrated in Figure 5) that, together with the fact that P and Q meet only at (0;0) and (m;r), implies that F_1 \setminus G_j is connected and nontrivial.

Now assume X is a proper nontrivial connected atom. By Theorem 5.1, X is an interval, say [i;v], in [m + 1]. As in the proof of Lemma 5.6, it follows that the u-th step of P and the v-th step of Q are East steps. Since X is a \hat{at} both rX [fu 1g] and rX [fv+1g] exceed rX), so step u 1 of P and step v+1 of Q, if there are such steps, are North steps. From these observations and Theorem 5.3, it follows that X is of the form F_1,G_j, or F_1 \setminus G_j. We need to show that if F_1 \setminus G_j...
is connected, then the inequality \( (G_j) \triangleleft (F_i) \) holds. This inequality follows by viewing \( M \setminus \{ F_1 \} \setminus G_j \) as a restriction of \( M \setminus F_1 \) and using the path presentations of these m-atroids given in Corollary 5.7. Indeed, from the lattice path diagram s (Figure 5) it follows that \( M \setminus \{ F_1 \} \setminus G_j \) is either free or connected, and the latter holds precisely when \( m \ (G_j); r \ (G_j) \) is strictly to the left of \( \{ F_1 \}; r \ (F_1) \).

Lastly, let the connected at \( X \) be \( F_1 \setminus G_j \). From lattice path diagram s, we get
\[
\begin{align*}
r(M) = r(F_1) + r(G_j) + r(X) + r(X)
\end{align*}
\]
from which the last assertion follows.

It follows from Theorem 5.7 that any intersection of connected m-atroids is either a fundamental at or an intersection of two fundamental ats. From this observation and the second paragraph of the proof, it follows that a nonepty intersection of connected m-atroids is either connected or trivial. Despite what the last part of Theorem 5.7 might suggest, it is easy to construct examples in which the fundamental m-atroids of lattice path m-atroids are not modular.

The image, under an automorphism, of a fundamental at of any m-atroid is also fundamental. This observation, Corollary 5.8, and Theorem 5.10 give the following result.

Corollary 5.8. The automorphisms of a connected lattice path m-atroid are the permutations of the ground set that are rank-preserving bijections of the collection of fundamental ats.

The proof of the second main result of this section, Theorem 5.10, uses the following basic notions about ordered sets. A strict partial order is an irre exive, transitive relation. Thus, strict partial orders differ from partial orders only in whether each element is required to be unrelated, or required to be related, to itself. Given a strict partial order \( < \) on \( S \), elements \( x \) and \( y \) of \( S \) are incomparable if neither \( x < y \) nor \( y < x \) holds. Weak orders are strict partial orders in which incomparability is an equivalence relation. Thus, linear orders are weak orders in which the incomparability classes are singletons. Two weak orders \( <_1 \) and \( <_2 \) on \( S \) are compatible if whenever elements \( x \) and \( y \) of \( S \) are comparable in both \( <_1 \) and \( <_2 \), and \( x <_1 y \), then \( x <_2 y \).

Lemma 5.9. Any two compatible weak orders have a common linear extension.

Proof. Let \( <_1 \) and \( <_2 \) be compatible weak orders on \( S \) and let the relation \( < \) on \( S \) be defined as follows: \( x < y \) if either \( x <_1 y \) or \( x <_2 y \). It is easy to check that \( < \) is a weak order. The lemma follows since \( < \), like any strict partial order, can be extended to a linear order.

We now turn to the second main result of the section. This theorem shows that the properties we developed above for the fundamental ats and the connected m-atroids characterize these m-atroids.

Theorem 5.10. A connected m-atroid \( M \) is a lattice path m-atroid if and only if the following properties hold.

(i) The fundamental ats form at most two disjoint chains under inclusion, say \( F_1 \setminus F_2 \) and \( G_1 \setminus G_2 \).
(ii) If \( F_1 \setminus G_j \), then \( F_1 \setminus G_j = E \setminus M \).
(iii) The proper nontrivial connected ats of \( M \) are precisely the following sets:
Proof. By Theorem 5.3, Lemma 4.4 and Corollary 4.5, \( M \) is a generalized Catalan matroid if and only if properties (i)–(iv) hold where there is at most one chain of fundamental bases. By Theorem 5.3 and 5.7, the fundamental bases of a lattice path matroid that is not a generalized Catalan matroid satisfy properties (i)–(iv) with neither chain of fundamental bases being empty. Hence we need only prove the converse in the case that neither chain of fundamental bases is empty.

Assume \( M \) has rank \( r \) and nullity \( m \). To show that \( M \) is a lattice path matroid, we construct lattice pairs \( P \) and \( Q \) and an isomorphism of \( M \) onto \( [P; Q] \). To show that \( P \) stays strictly below \( Q \) except at \((0;0)\) and \((m;r)\), we will use the following statements about fundamental bases:

(A) If \( F_1 \setminus G_j \) is not empty, then \( r(F_1 \setminus G_j) > r \).
(B) If \( F_1 \setminus G_j = \emptyset \), then \( (F_1) + G_j < (F_1 \setminus G_j) \).

To prove statement (A), note that we have the inequality

\[
r(F_1) + r(G_j) = r(F_1 \setminus G_j) + r(F_1 \setminus G_j) \leq r(M) + r(F_1 \setminus G_j)
\]

by sem in cadularity and property (ii). Since \( M \) has no loops, \( r(F_1 \setminus G_j) \) is positive, so the desired inequality follows. To prove statement (B), recall that if \( (F_1) + G_j \) is nondecreasing, i.e., if \( X \subseteq Y \), then \( (X) \subseteq (Y) \). Since \( F_1 \) and \( G_j \) are disjoint, we have \( (F_1) + (G_j) = (F_1 \setminus G_j) \cup \{g_j\} \). By Theorem 5.3, if \( r(F_1) + r(G_j) > r(F_1 \setminus G_j) \), then we have \( (F_1) + r(G_j) < (F_1 \setminus G_j) \). Since \( M \) stays strictly below \( Q \) except at \((0;0)\) and \((m;r)\), we have \( (F_1 \setminus G_j) < (M) \), which gives the desired inequality.

Let lattice paths \( P \) and \( Q \) from \((0;0)\) to \((m;r)\) be given as follows.

(a) The NE corners of \( P \) are at the points \( (G_j) \) for \( j \in [k] \).
(b) The EN corners of \( Q \) are at the points \( (F_i) \) for \( i \in [k] \).

Note that \( P \) stays strictly below \( Q \) except at the endpoints if and only if for every NE corner \((x; y)\) of \( P \) and every EN corner \((x_0; y_0)\) of \( Q \), at least one of the inequalities \( x < x_0 \) or \( y > y_0 \) holds. These inequalities are those in statements (A) and (B), so \( P \) stays strictly below \( Q \) except at \((0;0)\) and \((m;r)\).

To construct an isomorphism of \( M \) onto \([P;Q] \), we define a linear order on \( E(M) \). We use to map \( E(M) \) onto \([n+r], \) the ground set of \( [P;Q] \). We must define two relations \( <_F \) and \( <_G \) on \( E(M) \). Let \( F_{i-1} \) and \( G_{i+1} \) be \( E(M) \). Define \( <_F \) as follows: \( x <_F y \) for \( x, y \in E(M) \) if there is an integer \( j \) with \( x < F_i \) and \( y < F_{i+1} \). Note that \( <_F \) is a weak order whose incomparable classes are \( F_i \) and the set differences \( F_{i+1} \setminus F_i \) for each \( i \). Define \( <_G \) similarly: \( x <_G y \) for \( x, y \in E(M) \) if there is an integer \( j \) with \( x < G_j \) and \( y < G_j \). Note that \( <_G \) is also a weak order and the incomparable classes are \( G_j \) and the differences \( G_{j+1} \setminus G_j \).

Let \( E(M) \) be given by \( \{x_i\} = i \). By construction, \( <_F \) is a bijection of \( E(M) \) onto \([n+r] \) that is a rank-preserving bijection of the fundamental bases of \( M \) onto the fundamental bases of \( [P;Q] \). Furthermore, by assumptions (iii) and...
(iv) and Theorem 5.7 is a rank-preserving bijection of the set of connected
paths of $M$ onto those of $\mathbb{P};Q$. By Lemma 5.3 it follows that is an isomorphism of
$M$ onto $\mathbb{P};Q$; thus, $M$ is a lattice path $M$.

We close this section by giving a pair of six-element matroids that have the same
collection of fundamental paths, yet only one of which is in $L$; thus, conditions (i)
and (ii) in Theorem 5.10 are not enough to characterize lattice path $M$.
The uniform matroid $U_{4,6}$ is a lattice path $M$ with no fundamental paths since the
bounding paths are $P = E \times N^4$ and $Q = N^4 \times E^2$. The prism (the matroid $C_{4,2}$
of Figure 13 on page 26) is not a lattice path $M$ (condition (iii) of Theorem 5.10
fails) and, since it has no spanning circuits, it too has no fundamental paths.

6. Lattice Path $M$ Matroids as Transversal Matroids

The aspects of lattice path $M$ matroids treated in this section relate to important
topics in the theory of transversal matroids. We start by characterizing the set
system $\mathcal{S}$ that are maximal presentations of lattice path $M$ matroids. This result plays
a key role in an algorithm for determining whether a transversal matroid is in $L$.
By combining the result on maximal presentations with Brylawski's unique
representation of transversal matroids, we get a geometric description of lattice path
matroids. We conclude the section by comparing $L$ with the dual-closed class of
foundational transversal matroids and the minor-closed class of bicircular matroids.

6.1. Maximal and Minimal Presentations. Two types of presentations are
of interest in this section. A presentation $A = (A_1; A_2; \ldots; A_r)$ of a transversal
matroid $M$ is minimal if the only presentation $(A_1'; A_2'; \ldots; A_r')$ of $M$ with $A_i'$
contained in $A_i$ for all $i$. The presentation $A$ is maximal if the only presentation
$(A_1'; A_2'; \ldots; A_r')$ of $M$ with $A_i'$ contained in $A_i$ for all $i$. It is well known that
while each transversal matroid has a unique maximal presentation, it typically has
many minimal presentations. (See, e.g., [6, 11].)

Theorem 6.1. Standard presentations of lattice path $M$ matroids are minimal.

**Proof.** Let $(N_1; N_2; \ldots; N_r)$ be the standard presentation of the matroid $M = (P; Q)$
and let $(N_1'; N_2'; \ldots; N_r')$ be any presentation of $M = (P; Q)$ with $N_i' \leq N_i$ for all $i$.
To prove the theorem, we must show the inclusion $N_i' \leq N_i$ for all $i$. Let $x$ be
in $N_i$. Let $B$ consist of the least elements of $N_1; N_2; \ldots; N_i, 1$, the greatest elements
of $N_{i+1}; N_{i+2}; \ldots; N_r$, and $x$. Thus, $B$ is a basis of $M = (P; Q)$. Note that for $B$
to be a transversal of $(N_1; N_2; \ldots; N_r)$, the element $x$ must be in $N_i$, as needed.

With the following result of Bondy [1], we will get a simple description, in terms of
intervals, of the maximal presentation of a lattice path $M$ matroid.

**Lemma 6.2.** Given a presentation $(A_1; A_2; \ldots; A_r)$ of a rank-$r$ transversal matroid
$M$, the maximal presentation of $M$ is $[I_1; I_2; \ldots; I_r]$ where $I_j$ is the set of
isomorphs of the deletion $M - A_j$.

Together with Lemma 6.1, the following result from [11] implies that from any
presentation of a transversal matroid, the maximal presentation can be found in
polynomial time in the size of the ground set. This observation will be important in
the algorithm for recognizing lattice path $M$ matroids among transversal matroids.

**Lemma 6.3.** The maximal size of a matching in a bipartite graph can be found in
polynomial time in the number of vertices.
The discussion below focuses on matroids that have no isthmuses. This restriction is justified by noting that the isthmus uses of a transversal matroid are in all sets in the maximal presentation, and so are easy to deal with.

Let \((N_1; N_2; \ldots ; N_r)\) be the standard presentation of the lattice path matroid \(M = \mathcal{P}(P; Q)\) on \([n + r]\), where \(M\) has no isthmuses. Let \(N_1 = [a; b]\). Theorem 3.9 in [1] implies that each connected component of \(N\) has the union of the sets \(I^+\) and \(I^-\) of isthmuses of the restrictions of \(M\) to \([n + 1; m + r]\) and \([n + 1]\) respectively.

Corollary 3.2 implies that \(I^+\) and \(I^-\) are given as follows:

1. \(I^+_i = f_{g_i + j} : g_i + j\) is the greatest element of \(N_{g_i + j} \); \(j > 0\);

2. \(I^-_i = f_{l_i + j} : l_i + j\) is the least element of \(N_{l_i + j} \); \(j > 0\);

This proves the following theorem.

Theorem 6.4. Let \((N_1; N_2; \ldots ; N_r)\) be the standard presentation of the lattice path matroid \(M = \mathcal{P}(P; Q)\) that has no isthmuses. The maximal presentation of \(M = \mathcal{P}(P; Q)\) is \((N_0; N_2; \ldots ; N_r)\) where \(N_0 = N_1[I^+_1 \setminus I^-_1]\) and \(I^+_1\) and \(I^-_1\) are given by Eqs. (1) and (2).

The sets in the maximal presentation of a lattice path matroid have a simple graphical interpretation, as Figure 6 illustrates. While there are no containment sentences among intervals in the standard presentation, this figure shows that there may be containment sentences (even equalities) among intervals in the maximal presentation.

Theorem 6.5, which characterizes the multisets of intervals in \([n + r]\) that are maximal presentations of lattice path matroids, uses the following notation. For an indexed multiset \((T_1; T_2; \ldots ; T_r)\) of nonempty intervals in \([n + r]\) with \(T_i = [a_i; b_i]\), write \(T_i \leq T_j\) if either \(a_i < a_j\) or \(b_i < b_j\). Thus, two intervals are unrelated if and only if they are equal. For arbitrary multisets of intervals, both \(T_i \leq T_j\) and \(T_j \leq T_i\) may hold; in contrast, if \((T_1; T_2; \ldots ; T_r)\) is the maximal presentation of a lattice path matroid, then \(T_i \leq T_j\) implies that \(T_i \leq T_j\) only for \(i < j\). In this case, let \(d(T_i) = \{i : i < h; a_i = a_h, g_j\}\) and let \(d^0(T_i) = \{j : h < j; b_i = b_j\}\).
Theorem 6.5. A set system \((T_1; T_2; \ldots; T_r)\) of nonempty intervals in \([n + r]\) is the maximal presentation of a rank-1 lattice path matroid on \([n + r]\) that has no isthmuses if and only if

\begin{enumerate}
  \item the relation \((\preceq)\) is a weak order,
  \item for all pairs \(T_i \text{ and } T_j\), neither \(T_i \text{ nor } T_j\) \(T_i \cup T_j\) is 1, and
  \item \(d(T_i) + d(T_j) + 2 \geq \# T_i\) for every \(i\).
\end{enumerate}

Proof. For the maximal presentation of a lattice path matroid \(M \{P; Q\}\) with no isthmuses, properties (i) (ii) follow from Theorem 6.4. For the converse, note that removing from \(T_i\) its least \(d(T_i)\) elements and its greatest \(d(T_i) - 2\) yields the standard presentation of a lattice path matroid that, by property (iii), has no isthmuses and for which \((T_1; T_2; \ldots; T_r)\) is, by Theorem 6.4, the maximal presentation.

6.5. Recognizing Lattice Path Matroids. When treating algorithmic questions about matroids, it is usual to assume that a matroid is given by an independence oracle, that is, a subroutine that outputs, in constant time, whether a subset of the ground set is independent. While there are algorithms that recognize transversal matroids within the class of all matroids (see [7]), Jensen and Korte [13] have shown that there is no polynomial-time algorithm to decide if a matroid is transversal from an independence oracle. The same proof as in [13] shows that there is no such algorithm to decide whether a matroid is a lattice path matroid. Transversal matroids are more conveniently specified by set systems than by independence oracles. This section gives a polynomial-time algorithm that, given a set system, decides whether the corresponding transversal matroid is a lattice path matroid.

We start with some simplifications. A presentation \(A\) of \(M\) can be represented by a bipartite graph \([A]\) in the obvious way [13, Section 1.6]. Therefore, by Lemma 6.3, the isthmuses of a transversal matroid can be identified and deleted in polynomial time. If \(M\) has no isthmuses, then the connected components of \(M\) come from those of \([A]\). These observations and Theorem 6.4 justify focusing on connected transversal matroids. As noted in Section 6.4, the maximal presentation can be found from any presentation in polynomial time, so we focus on maximal presentations.

The key to the recognition algorithm below is to efficiently recover lattice path orderings from the maximal presentation. We begin with some observations that relate these notions. Assume \(A = (A_1; A_2; \ldots; A_r)\) is the maximal presentation of the connected lattice path matroid \(M \{P; Q\}\) on the ground set \([n + r]\) and let \(n\) be the incidence function of \(A\). Let \(C_1; C_2; \ldots; C_k\) be the equivalence classes of the relation \((\preceq)\) on \([n + r]\) in which \(x \text{ and } y\) are related if and only if \(n(x) = n(y)\). Each set \(C_i\) is an interval in \([n + r]\). We may assume that \(C_1; C_2; \ldots; C_k\) are indexed so that \(x_1 < x_2 < \cdots < x_k\) for any elements \(x_1; x_2; \ldots; x_k\) with \(x_i\) in \(C_i\). Since \(M \{P; Q\}\) is connected, we have \(n(C_i) \cap n(C_j) \neq \emptyset\) for \(1 \leq i < k\). Any permutation of \([n + r]\) with \(C_i = C_j\) for \(1 \leq i \leq k\) is clearly an automorphism of \(M \{P; Q\}\), so the linear order \((1) \prec (2) \prec \cdots \prec (n + r)\) is a lattice path order, as is \((n + r) \prec (2) \prec (1)\). Relative to any of these linear orders, the sets in \(A\) are intervals and the properties in Theorem 6.5 hold. These lattice path orderings of \([n + r]\) are essentially equivalent to the orderings \(C_1 \prec C_2 \prec \cdots \prec C_k\) and \(C_k \prec C_{k-1} \prec \cdots \prec C_1\). Observe that \(C_1; C_2; \ldots; C_k\) and \(C_k; C_{k-1}; \ldots; C_1\) are the only permutations \(X_1; X_2; \ldots; X_k\) of \(C_1; C_2; \ldots; C_k\) that satisfy the following property.
Thus, to determine whether a transversal matroid $M$ with a given presentation is a lattice path matroid, carry out the following steps.

1. Detect and delete the isthmuses.
2. Determine the connected components.
3. Find the maximal presentation for each connected component.
4. For each component, find the classes defined above relative to the maximal presentation.
5. For each component, determine whether there is a linear order of these classes that satisfies property (P).
6. If there is such a linear order of these classes for each component, then use the criterion in Theorem 6.6 to determine whether, with respect to any corresponding linear order of a component, the intervals in the maximal presentation of that component are those of a maximal presentation of a lattice path matroid.

If, in step (6), there is no suitable order for some connected component, then $M$ is not a lattice path matroid. If there is such an order for each connected component, then $M$ is a lattice path matroid if and only if step (6) yields only positive results. Each of these steps can be done in polynomial time in the size of the ground set, so we get the following result.

**Theorem 6.6.** Whether a transversal matroid is a lattice path matroid can be determined from any presentation in polynomial time in the size of the ground set.

6.3. A Geometric Description of Lattice Path Matroids. Brylawski [8] (see also [14, Proposition 12.2.26]) gave a geometric description of arbitrary transversal matroids. This section applies his result to lattice path matroids.

Let $M$ be a transversal matroid on the set $x_1; x_2; \ldots; x_n$ with presentation $(A_1; A_2; \ldots; A_t)$. Brylawski showed that $M$ can be realized geometrically as follows. Start with the free matroid $M_0$ on the set $e_1; e_2; \ldots; e_n$ disjoint from $E(M)$. For $i = 1$ to $k$, form $M_i$ from $M_{i-1}$ by taking the principal extension of $M_{i-1}$ defined by the set $c_{i, 1}$, $(e_1; x_i; A_i, g)$, with the element added being $x_i$. The matroid $M$ is $M_k$. The rank-$r$ matroid is transversal if and only if it can be realized by placing the elements freely on the faces of the $r$-simplex.

The next theorem, which is illustrated in Figure 4, shows how lattice path matroids can be constructed by successively adding isthmus and loops, and by taking principal extensions by certain sets. To motivate this result, consider a lattice path matroid $M$ that has rank $r$ and nullity $m$ in which $m + r$ is neither a loop nor an isthmus. Let $I$ be the length of the longest segment of North steps in $P$. By Theorem 6.6, the sets of the maximal presentation of $M$ that contain $m + r$ are the last $l$ (those arising from $N_{r+1}/N_i$). By Brylawski's result, $m + r$ is added freely to the set spanned by $e_r; \ldots; e_{r-1}$ in the notation above; note that this set is also spanned by the last $l$ elements of $[m + r - 1]$, since they are independent in $M$.
Theorem 6.7. A matroid $M$ is a lattice path matroid if and only if the ground set can be written as $fx_1;x_2;\ldots;x_g$ so that each restriction $M_i = M^fx_i;x_2;\ldots;x_g$ is formed from $M_{i-1}$ by either

(i) adding $x_i$ as an isthmus,
(ii) adding $x_i$ as a loop, or
(iii) adding $x_i$ via the principal extension of $M_{i-1}$ generated by the closure of an independent set of the form $fx_i;x_{i+1};\ldots;x_g$ for some $h$ with $h < i$.

6.4. Relation to Other Classes of Transversal Matroids. We have seen that the class of lattice path matroids is closed under taking both minor and duals. While [2] develops a dual-closed, minor-closed class of transversal matroids that properly contains $L$, and while there are in only $m$ any dual-closed, minor-closed classes contained in $L$ (see Sections 4 and 8 for two such classes), few other known classes of transversal matroids are either dual-closed or minor-closed. In this section, we make some remarks about two important classes of transversal matroids, each of which has one of these properties.

Fundamental transversal matroids (called principal transversal matroids in [3]) were introduced by Bondy and Welsh [3] and they play an important role in the study of transversal matroids. A transversal matroid $M$ is a fundamental transversal matroid if it can be represented on the simplex with an element of $M$ at each vertex of the simplex. Thus, transversal matroids are the restrictions of fundamental transversal matroids. While the class $F$ of fundamental transversal matroids is closed under neither deletion nor contraction, it is well-known and not hard to prove that $F$ is dual-closed. The class $F$ is much larger than $L$: Byalwasi [8] showed that there are at order of $c^n$ fundamental transversal matroids on $n$ elements, for some constant $c$; in contrast, $4^n$ is an upper bound on the number of lattice path $m$-matroids on $n$ elements since there are $4^n$ pairs of paths of length $n$ (see [2] for a formula for the number of connected lattice path $m$-matroids). Both $F$ and $L$ contain all transversal matroids of rank two. However, a fundamental transversal matroid of rank three or more cannot have a pair of disjoint connected hyperplanes, but such hyperplanes can occur in lattice path matroids, such as the matroid $P_n = T_n(U_{n,1}U_{n,2})$ of Theorem 4.5. On the other hand, the number of connected hyperplanes of a fundamental transversal matroid, such as the $n$-whirl $W_n$, can exceed two (see Corollary 5.12).

Let us call a matroid bitransversal if both the matroid and its dual are transversal. It is easy to prove that the class of bitransversal matroids is closed under direct sum, free extensions, and free coextensions. Hence by starting with the union of the classes $L$ and $F$, and using these three operations, we can construct a larger class of bitransversal matroids; let $LF$ denote this class. For instance, the free extension $P_n$, $W_n^+$ of $P_n$, $W_n$ is in $LF$ but not in $L$ [8]. There are bitransversal matroids, such as the identically self-dual matroids of [3, Section 4], that are not in $LF$. The problem of characterizing all bitransversal matroids, which was posed by Welsh, currently remains open (see [16, Problem 14.7A]).

Bicircular $m$-matroids [14] form another important class of transversal matroids. The notion of a bicircular $m$-matroid we consider is a mild extension of that in [14] (as originally defined, bicircular $m$-matroids have no loops). A transversal matroid $M$ is bicircular if it has a representation $A$ so that each element of $M$ is in at most two sets in $A$ (counting multiplicity). Thus, bicircular $m$-matroids are the transversal matroids that have a representation on the simplex in which all non-loops are on vertices or
The standard presentation of \( N_E \) has once we show the following statements: if \( Q \) has an \( E \) corner at \( j \) and only if \( Q^0 \) has an \( E \) corner at \( j \); also, \( Q \) has an \( E \) corner at \( j \) and only if \( Q^0 \) has a \( N \) corner at \( j \) 1. Thus, the Lemma states once we show the following statements: if \( Q \) has an \( E \) corner at \( j \) and only if \( Q^0 \) has an \( E \) corner at \( j \). These assertions hold since we can pair the relevant \( E \) and \( N \) steps that share a lattice point, as suggested in Figure 4(b).

**Lemma 7.1.** The number \( k_M \) is invariant under duality, that is, \( k_M = k_M \).

Proof. Recall that the lattice path diagram for the dual of \( M \) \( \{P; Q\} \) is obtained by reflecting the lattice path diagram for \( M \{P; Q\} \) about the line \( y = x \). Figure 3. Equivalently, the dual of \( M \{P; Q\} \) is \( M \{Q^0; P^0\} \), where \( P^0 \) and \( Q^0 \) are obtained from \( P \) and \( Q \) by switching \( E \) and \( N \) steps. Let \( n \) and \( n^0 \) be the incidence functions of the standard presentations of \( M \{P; Q\} \) and \( M \{Q^0; P^0\} \), respectively. Note that \( P \) has a \( N \) corner at \( j \) if and only if \( P^0 \) has an \( E \) corner at \( j \); also, \( Q \) has an \( E \) corner at \( j \) 1 if and only if \( Q^0 \) has a \( N \) corner at \( j \). Thus, the lemma follows once we show the following statements: if \( Q \) has an \( E \) corner at \( j \), then \( j \) \( j \) = \( j \) \( j \) if \( P \) has a \( N \) corner at \( j \), then \( j \) \( j \) = \( j \) \( j \) + 1. These assertions hold since we can pair the relevant \( E \) and \( N \) steps that share a lattice point, as suggested in Figure 4(b).
Thus, since (c) is trivial. The proof of part (b) uses the following easily-verifiable statements:

(i) If the j-th and (j + 1)-st steps of Q are East, then n(j + 1) is either n(j) or n(j) [n in n(]), so we have n(j) = 1 if (j + 1), n(j) = j if (j)j.

(ii) If the j-th step of Q is North, then n(j) = 1 if n([j] = X or X in X) where X is 1 : h 2 n(j)g, so (j)j = 1 if (j)j = j if (j)j.

First assume that steps i; i + 1;:::,h of Q are East and that step h + 1 is North. Thus, Q has an EN corner at h. Statem ents (i) and (ii) give the inequalities

(j)j = j if (j) = k M . Since n(h + 1)j = k M , we have n(j) = k M 1. Finally, if the i-th step of Q is North, a similar application of statement (ii) completes the proof of part (b).

From Lemmas 7.2 and 7.2, the rank of any circuit of M is at least k M . The next lemma follows from this observation and Lemma 7.4. The generalized Catalan matroid M [NE] shows that M can have circuits of rank k M 1.

Lemma 7.3. Any set of k M 1 elements of [n + r] is independent in both M and M . Circuits of M have at least k M elements, as do circuits of M .

We now prove that k M is the connectivity of the lattice path matrix M.
Theorem 7.4. Let $M$ be a connected lattice path matroid of rank $r$ and nullity $m$, say $M[P;Q]$, that is, not uniform. The connectivity $(M)$ of $M$ is $k_M$, where $k_M$ is $m \inf h(j): P$ has a NE corner at $j$ or $Q$ has an EN corner at $j$. Let $X$ and $Y$ be $[j]$ and $[j+1;m+r]$, respectively. Thus, $Y$ is a fundamental cut of $M$. Note that both $X$ and $Y$ have at least $h(j)$ elements. It follows from the path presentations of restrictions given in Corollary 3 that $r(M)$ is $r(Y) + h(j)$, that is, $r(M) + r(Y) = r(M) + k_M - 1$, so $X \subseteq Y$ is an exact $k_M$-separation of $M$. Similarly, if $k_M$ is $h(j)$, where $Q$ has an EN corner at $j$, then $[j]$ and $[j;m+r]$ give an exact $k_M$-separation of $M$.

It remains to show that $M$ has no $h$-separation for any positive integer $h$ less than $k_M$. Let $h$ be such an integer and assume $X$ and $Y$ partition $m+r$, where both $X$ and $Y$ have at least $h$ elements. We need to prove the inequality

$$r(X) + r(Y) - r(M) + h$$

If an element $y$ in $X$ is in the closure of $Y$, and if $X$ has more than $h$ elements, then we have $X \subseteq Y$ and $r(X) + r(Y) = r(X) + r(Y)$. Thus, it suffices to prove inequality (3) when $X \subseteq Y$ is a nontrivial cut of $M$. By Lemmas 7.3 and 7.4, each nontrivial connected component of the restriction $M[Y]$ has at least $h$ elements; with an argument similar to the one above, it follows that if $Y$ is a nontrivial cut of $M$, then we may assume $Y$ is connected.

Now assume $Y$ is a nontrivial connected cut of $M$. If $Y$ is a fundamental cut at $j$, then we may use the result in the first paragraph. If $Y$ is not a fundamental cut at $j$, then by Theorem 5.7, $Y$ is the intersection of two incomparable fundamental cuts, say $Y \subseteq A$ and $Y \subseteq B$ where $A$ and $B$ partition $X$. We may assume $1 \in A$, so $m + r$ is in $B$. Since $A \subseteq Y$ is a fundamental cut at $j$ and $B$ is the complement of $A \subseteq Y$, we have $r(A \subseteq Y) + r(B) = r(M) + k_M - 1$. Thus, since $k_M$ exceeds $h$, to prove inequality (3), it suffices to prove $r(X) + r(Y) = r(A \subseteq Y) + r(B)$, that is,

$$r(A \subseteq B) + r(Y) = r(A \subseteq Y) + r(B)$$

Observe that $r(A \subseteq B)$ is $h(A \subseteq B)$. The inequality $r(A \subseteq B) = h(A \subseteq B)$ is obvious and the inequality $r(A \subseteq B) \geq h(A \subseteq B)$ follows by matching each set $N_i$ for $i$ in $n(A \subseteq B)$, with its $r_i$-set element, which must be in $A$, and each set in $N_i$ for $j$ in $n(B \subseteq A)$, with its last element, which must be in $B$. A similar argument gives the equality $r(B) = h(B)$. From Theorem 5.7, we also have $r(A \subseteq Y) = h(A \subseteq Y)$ and $r(Y) = h(Y)$. Thus, inequality (4) is equivalent to

$$h(A \subseteq B)+ h(Y) \geq h(A \subseteq Y)+ h(B)$$

Note that $h(A \subseteq B) = h(A)+ h(B)$. Substituting this and the analogous formula for $h(A \subseteq Y)$ into inequality (5) and simplifying gives that this inequality is equivalent to the inequality $h(A \subseteq Y) \geq h(A \subseteq B) + h(Y)$ which clearly holds. Thus, inequality (3) holds, as needed to complete the proof.
As the matroid $E_3$ of Figure 14 shows, not every exact $k_M$-separation of a lattice path matroid $M$ has a fundamental as one of the sets.

8. Notch M matroids and their Excluded M inors

There are infinitely many minor-closed, dual-closed classes of transversal matroids within the class of lattice path matroids. One way to define such classes is to impose certain requirements on the bounding paths; for example, the lower bounding path of a generalized Catalan matroid must have the form $E^m N^r$. In this section we introduce the minor-closed, dual-closed class of notch matroids, which is defined by special forms for the bottom bounding path. We relate notch matroids to generalized Catalan matroids via circuit-hyperplane relaxations. The main result is the characterization of notch matroids by excluded minors. We include some remarks on the excluded minors for lattice path matroids.

Definition 8.1. A notch matroid is, up to isomorphism, a lattice path matroid of the form $M \left[ E^m N^r ; Q \right]$ or $M \left[ E^m N^r ; Q \right]$. As Figure 8 illustrates, notch matroids are either in $C$ or their lattice path presentations differ from those of generalized Catalan matroids by the "notch" in the lower right corner. It follows from the lattice path descriptions of $C$ and duals, along with Theorem 5.7, that the class $N$ of notch matroids is minor-closed and dual-closed. Note that $N$, like its subclass $C$, is not closed under direct sums. In contrast to $C$, the class $N$ is not closed under any of the following operations, as can be seen from the matroid $D_3$ of Figure 14: free extension, truncation, and the dual operations. Theorem 8.3 gives a basic property that $N$ shares with $C$. Lemma 8.2. Adding loops and isthmuses to a notch matroid yields a notch matroid.

Note that a connected notch matroid either is in $C$ or has a circuit-hyperplane relaxation in $C$. Not every matroid that has a circuit-hyperplane relaxation in $C$ is a notch matroid; for instance, the matroids $A_3$ and $A_4$ of Figure 14 each have two circuit-hyperplane relaxations that are in $C$, yet neither is a lattice path matroid since condition (ii) of Theorem 5.10 fails. However, we have the following result.

Theorem 8.3. A connected matroid in $C$ is a notch matroid if and only if it has a circuit-hyperplane. Relaxing any circuit-hyperplane of a lattice path matroid yields a generalized Catalan matroid.

Proof. The last $r$ elements of a connected notch matroid $M \left[ E^m N^r ; Q \right]$ obviously form a circuit-hyperplane. For the converse, assume that $H$ is a circuit-hyperplane of the rank-$r$, nullity-$m$ matroid $M = M \left[ P; Q \right]$. Since $H$ is an $r$-circuit of $M$, by Theorem 5.11 the set $n(H)$ is an interval of $r-1$ elements in $[r]$; we may...
Figure 9. The 3-wheel $W_3$ and the 3-whirl $W_3$.

Assume that $n \notin 2 \cdot r$. Since $H$ is a mat, $H$ is an interval of $r$ elements in the ground set $[n + r]$ of $M$, so $[n + r]$ consists of an initial interval, the interval $H$, and an interval $Y$. Since $H$ is a hyperplane, $Y$ must be empty, so $H$ consists of the last $r$ elements of $[n + r]$. From these conclusions, it is immediate that $M$ is a notch matroid. The last assertion follows from part (iii) of Corollary 3.12.

Similar ideas yield the following result.

**Lemma 8.4.** Let $M^0$ be $M \setminus \{a_1\}$, a connected rank-$r$, nullity-$m$ matroid in $C$. If the basis $B$ of $M^0$ is mapped onto the interval $[m + 1; m + r]$ by some automorphism of $M^0$, then there is a unique matroid $M$ in which $B$ is a circuit-hyperplane and from which $M^0$ is obtained by relaxing $B$. Further, $M$ is in $N$.

The following two lemmas will be used heavily in the proof of the excluded-minor characterization of $N$.

**Lemma 8.5.** If $X$ and $Y$ are nontrivial incomparable connected atoms of a notch matroid $M$ that has no isthmuses, then either $X$ or $Y$ is a circuit-hyperplane.

**Proof.** The incomparable atoms $X$ and $Y$ show that $M$ is not in $C$, so $M$ has a circuit-hyperplane, say $H$. Either $X$ or $Y$ must be $H$ since $H$ cannot properly contain either $X$ or $Y$ and, by part (iii) of Corollary 3.12, nontrivial connected atoms that are not contained in $H$ are comparable.

**Lemma 8.6.** Three nontrivial connected atoms $X$, $Y$, and $Z$ of a notch matroid $M$ cannot be mutually incomparable.

**Proof.** We may assume that $M$ has no isthmuses and that $X$ and $Y$ are incomparable. From Lemma 8.5, either $X$ or $Y$, say $X$, is a circuit-hyperplane of $M$. Part (iii) of Corollary 3.12 implies that $Y$ and $Z$ are comparable.

We turn to the excluded-minor characterization of $N$. Let $\text{ex}(N)$ and $\text{ex}(L)$ denote the sets of excluded minors for $N$ and $L$, respectively. We first discuss the matroids in $\text{ex}(N)$ that are not lattice path matroids and are in $\text{ex}(N) \setminus \text{ex}(L)$. In each case, we show that the matroids are not in $L$; it is easy to check that all their proper minors are in $N$, so we omit this part.

Among the self-dual matroids in $\text{ex}(N) \setminus \text{ex}(L)$ are the 3-wheel $W_3$ and the 3-whirl $W_3$, which are shown in Figure 8. Since all 3-point lines of $W_3$ and $W_3$ are fundamental atoms, condition (i) of Theorem 5.11 fails, so $W_3$ and $W_3$ are not in $L$.

For $n \geq 3$, let $A_n$ be the rank-$n$ paving matroid with only two nontrivial hyperplanes, $f_0; a_1; \cdots; a_r$ and $f_0; b_1; \cdots; b_r$, and with only one point, $y$, in neither circuit-hyperplane (Figure 11). The two circuit-hyperplanes violate condition (ii) of Theorem 5.11, so $A_n$ is not in $L$. Note that $A_n$ is self-dual.
We next consider two doubly-indexed families in $\text{ex}(N) \setminus \text{ex}(L)$ that are related by duality; three of these matroids are shown in Figure 11. Let $n$ and $k$ be integers with $2 \leq k \leq n$. Let $B_{n,k}$ be the truncation $T_n(U_{n,1}) \cup U_{n,1+k}$ to rank $n$ of the direct sum of two $n$-circuits and a $k$-circuit. The three disjoint circuits are fundamental to $B_{n,k}$, so condition (i) of Theorem 5.10 shows that $B_{n,k}$ is not in $L$. The dual $C_{n+k,k}$ of $B_{n,k}$ is the rank-$(n+k)$ paving matroid $C_{n+k,k}$ for which the ground set can be partitioned into sets $X; Y; Z$ with $j$ and $j' = k$ so that the only nontrivial hyperplanes are $X \setminus Y, X \setminus Z$, and $Y \setminus Z$.

The remaining matroids in $\text{ex}(N) \setminus \text{ex}(L)$, two of which are shown in Figure 11, form two infinite families that are related by duality. Recall that $M + y$ denotes the free extension of $M$ by the point $y$. For $n \geq 3$, let $D_n$ be the rank-$n$ matroid

$T_n(U_{n,2}) \cup U_{n,2} \cup U_{1,1} + y$.

That $D_n$ is not in $L$ for $n \geq 5$ follows since the two $(n - 1)$-circuits, as well as their union, are fundamental to $D_n$, contrary to condition (i) of Theorem 5.10. In the dual $E_n$ of $D_n$, the element $y$ is parallel to an element $x$, and the deletion $E_n, y$ is a rank-$n$ paving matroid whose only nontrivial hyperplanes are two circuit-hyperplanes that intersect in $x$. (The matroids $D_3$ and $E_3$, which are shown in Figure 11, are lattice path matroids.)

We have proven the easy part of the following theorem; the more substantial part of this result follows from the excluded-minor characterization of notch matroids, which is given in Theorem 8.8.

**Theorem 8.7.** The matroids in $\text{ex}(L) \setminus \text{ex}(N)$ are:

1. the three-wheel $W_3$ and the three-whirl $W_3^3$,
2. $A_n$ for $n \geq 3$, 
La T T I C E PA T H M A T R O I D S 27

Fi gur e 12. The m at r oi ds D 4 and E 4.

Fi gur e 13. Latt i c e path presentat i ons of F 6, G 6, and H 6.

Fi gur e 14. Path presentat i ons and geometric representations of U 1;2 , U 1;2 , U 1;2 , T 2 (U 1;2 , U 1;1 , U 1;1 ) , U 1;2 , D 3, and E 3.

(3) B n;k and C n+k;k for n and k with 2 ≤ k ≤ n, and
(4) D n and E n for n ≤ 4.

W e now turn to the excluded-minor characterization of notch matroids. The excluded minors are those in Theorem 8.7 together with the three types of lattice path matroids illustrated in Figure 13 and the four matroids in Figure 14.

Theorem 8.8. The excluded minors for the class of notch matroids are:
(1) U 1;2 , U 1;2 , U 1;2 , and T 2 (U 1;2 , U 1;1 , U 1;1 ) , U 1;2 ,
(2) the three-wheel, W 3, and the three-whirl, W 3,
(3) A n for n ≤ 3,
(4) B n;k and C n+k;k for n and k with 2 ≤ k ≤ n,
(5) D n for n ≤ 3,
(6) E n for n ≤ 3,
(7) for n ≤ 4, the rank-n matroid F n := T n (U n 2m 1 , U n 2m 1 ).
To make the proof of Theorem 8.8 less verbose, we will use abbreviations such as the following: from Theorem 3.14 applied to $X_1, X_2$, and $y$, we get $M \not\in L$. By this we mean that the matroid $M$ and the sets $X_1$ and $X_2$ satisfy the hypotheses of Theorem 3.14, with the point $y$ showing the validity of the third condition.

Proof of Theorem 8.8. The remarks before Theorem 3.14 show that of the matroids in the theorem, only $D_3, E_3$, and those in items (1) and (7) are in $L$. The presentations of these matroids, illustrated in Figures 13 and 14, make it clear that they are not in $N$. It is easy to check that all proper minors of these matroids are in $N$. Note that $H_n$ is self-dual, and that $F_n$ and $G_n$ are dual to each other.

The proof that Theorem 8.8 gives all excluded minors is intricate, so we first outline the argument. Part (8.8.1) proves that the disconnected excluded minors are $U_{1,2} U_{1,2} U_{1,1} T_2 (U_{1,2} U_{1,1} U_{1,1}) U_{1,2}, F_4, G_2$, and $H_3$. The rest of the proof revolves around three properties a connected excluded minor $M$ may have:

(a) $r(X_1 \cup X_2) < r(M)$ for some nontrivial incomparable connected sets $X_1, X_2$,
(b) $M$ contains three mutually incomparable connected sets,
(c) $M$ has no circuit-hyperplane.

In (8.8.2), we show that if $M$ has property (a), then $M \not\in D_n$ for some $n \geq 3$. Part (8.8.3) gives a key property of all connected excluded minors. In (8.8.4), we show that if property (b) but not (a) holds, then $M$ is one of the matroids in items (2) and (4). Part (8.8.5) shows that if only property (c) holds, then $M$ is one of the matroids in items (6) and (9). If none of the properties holds, then for any mutually incomparable connected sets $X_1, X_2, \ldots, X_k$, we have $k = 2$, and if $k = 2$, then at least one of $X_1$ or $X_2$ is a circuit-hyperplane. Since restrictions to proper subsets of circuit-hyperplanes are free, it follows that relaxing the circuit-hyperplane of such an excluded minor yields a matroid $M'$ in which the connected sets are linearly ordered by inclusion, that is, $M' \not\in C$. The proof of Theorem 8.8 is completed in (8.8.6) by showing that the only rank-$n$ excluded minor that has a circuit-hyperplane relaxation in $C$ is $A_n$.

Throughout the proof, $M$ denotes a rank-$n$ excluded minor for the class of rank-$m$ matroids. By Lemma 8.2, $M$ has neither loops nor isthmuses.

Proof of (8.8.1). A sum $eM$ has at least three components. Each component has a circuit of two or more elements, so $M$ has $U_{1,2} U_{1,2} U_{1,2}$ as a minor, which is itself an excluded minor. Thus, $M \not\in U_{1,2} U_{1,2} U_{1,2}$.

Now assume $eM$ has exactly two components, $M_1$ and $M_2$. Being proper minors of $M$, both $M_1$ and $M_2$ are rank-$m$ matroids. Observe that if $r(M_1) > 2$, then, by Theorem 3.14 and Corollary 3.10, there is an element $x$ for which $M_1 = x$ is connected. Dually, if $r(M_2) > 2$, then $M_2 = y$ is connected for some $y$.

A sum $eM_1 U_{1,2}$. From lattice path presentations and from the statements $M_2 \not\in N$ and $U_{1,2} M_2 \in N$, it follows that $r(M_2)$ and $M_2$ are both at least 2. Similarly, if $r(M_1) > 2$ is a connected minor of $M_2$ for which $r(M_1) > 2$ and $M_1$ are both 2, then $U_{1,2} M_2 \not\in N$. These observations, together with those in the last paragraph, imply that $r(M_2)$ and $M_2$ are both 2. From lattice path presentations, we see
that only two connected lattice path matroids have rank and nullity 2, namely $U_{2,4}$ and $T_2(U_{1,2} \cup U_{1,1} \cup U_{1,1})$, so $M$ is either $H_3$ or $T_2(U_{1,2} \cup U_{1,1} \cup U_{1,1}) \cup U_{1,2}$.

Now assume $M_1 = U_{1,k}$ with $k \geq 3$. Since $M \cong N$, the nullity of $M_2$ is at least 2. Arguments like those in the last paragraph imply that $k = 3$, that $M_2$ is 2, and that $r(M_2) = 1$, therefore $M_2 \cong U_{1,1}$, so $M \cong G_2$.

Finally, if $M_1$ and $M_2$ have rank 2 or greater, then, by the same types of arguments, both $M_1$ and $M_2$ have rank 2 and nullity 1, so $M \cong F_4$.

From now on, we assume $M$ is connected.

2.3.2] If $M$ has nontrivial incomparable connected sets $X_1$ and $X_2$ with $r(X_1 \cup X_2) < n$, then $M \cong D_n$.

Proof of 2.3.2]. Choose such a pair of sets $X_1 \cup X_2$ so that $r(X_1) + r(X_2)$ is as small as possible. Lemma 2.3 applied to $jX_1 \cup X_2$, $X_1$, and $X_2$ implies that either $X_1$ or $X_2$ is a circuit-hyperplane of $M$.

A sum $M \cong jX_1 \cup X_2$ is disconnected. This disconnected notch matroid has neither loops nor isthmuses, so one component, say $X_1$, has rank 1 and the other, $X_2$, has nullity 1; thus, $X_1$ is a parallel class and $X_2$ is a circuit. If $X_1 \cap X_2 = 2$ and $y \neq 2$, then $M$ has only one component, $X_1$, and $X_2$ contradicts Lemma 2.3. If $X_1 \cap X_2 = 2$ and $y = 2$, then $M = z = (X_1)$, and $X_2$ contradicts Lemma 2.3. Thus, $X_1 \cap X_2 = 2$. Since $M$ has neither $B_{2,2}$ nor $U_{1,2} \cup U_{1,2}$, it is a proper minor. Now, $X_1$ and $X_2$ are the only nontrivial parallel classes of $M$. Let $x$ and $y$ be in $E(M)$ and $jX_1 \cup X_2$. By Lemma 2.3, the rank-1 ats $C_{1,x}(X_1)$ and $C_{1,y}(X_2)$ are hyperplanes of $M$, so $r(M)$ is 3. It follows that $M \cong jX_1 \cup X_2$ is connected.

To prove this statement, note that since $M \cong jX_1 \cup X_2$ is connected, and since $X_1$ and $X_2$ are no longer connected, neither $X_1$ nor $X_2$ is a parallel class. If $x$ and $y$ are parallel, then $M$ has only one component, $X_1 \cup X_2$ and $X_1 \cup X_2$ contradicts Lemma 2.3.

For statement (ii), we first show that both $M \cong jX_1 \cup X_2$ are connected for any $x$ in $X_1 \setminus X_2$. If say, $M \cong jX_1 \cup x$ were disconnected, then by Lemma 2.3, there would be nontrivial incomparable connected sets $A$ and $B$ of $M \cong jX_1 \cup x$ with $r(A) + r(B) = r(X_1) + 1$. Since $M$ is simple, $r(X_1) + 1$, so the sets $A$ and $B$ of $M$ would contradict the choice of $X_1$ and $X_2$ as minimising the sum $r(X_1) + 1$. Since $M \cong jX_1 \cup x$ and $M \cong jX_2 \cup x$ are connected, $M \cong jX_1 \cup x$ and $X_2 \cup x$ contradict Lemma 2.3. Thus, $X_1 \setminus X_2$ are disconnected. The connected notch matroids $M \cong jX_1 \cup x$ and $M \cong jX_2 \cup x$ have spanning circuits; this observation and the minimality of $M$ show that $X_1$ and $X_2$ are circuits. For any $x$ in $cl(X_1 \cup X_2)$, the deletion $M \cong jX_1 \cup x$ is connected, so $M \cong jX_1 \cup x$ and $X_2 \cup x$ would violate Lemma 2.3. Thus, $cl(X_1 \cup X_2)$ is $X_1 \setminus X_2$, so statement (iii) holds.
Let $y$ be in $E(M)$. $(X_1 \cup X_2)$. The contraction $M = y$ has neither loops nor isthmuses. By Lemma 8.3, at least one of $c_1 = y(X_1)$ and $c_2 = y(X_2)$ is a circuit-hyperplane of the through matroid $M = y$, so $r(X_1 \cup X_2) = 1$. For $M = y$, $X_1$, and $X_2$ to not contradict Lemma 8.5, $n$ must have an isthmus. From these conclusions, statement (iii) follows.

Assume $e \in C$ is a nonspanning circuit of $M$ $y$ other than $X_1$ and $X_2$. Recall that either $X_1$ or $X_2$, say $X_1$, is a circuit-hyperplane of $M$. Thus, $X_1$ and $cl(C)$ are incomparable and $X_1 \cap C$ spans the difference $X_2 \setminus C$. Note that $M = x$ is connected. That $M = x$, $X_1$, and $cl(C)$ $x$ contradict Lemma 8.5 proves statement (iv). Statement (v) follows since if $X_2 < X_1$, and $z$ is in $X_1$, then $M = z$, $X_1$, $z$, and $X_2$ would contradict Lemma 8.5.

From statements (i) and (v), we have $4$. Assume $x$ and $y$ are in a nonspanning circuit $C$. At least one of $X_1$ and $X_2$ is not contained in $cl(C)$, so we may assume that $X_1$ and $cl(C)$ are incomparable. Let $z$ be in the difference $X_2 \setminus C$ of circuits. Note that $X_1 \setminus Z$ is a connected hyperplane of $M$ since $n = 4$, so $M = x$ is connected. Let $a$ be applied to $M = x$, $X_1$ and $cl(C)$ implies that $cl(C)$ must be a circuit-hyperplane of $M$ since $n = 4$, so $M = x$ is a hyperplane of $M$. Note that $cl(C)$ is either $cl(C)$ or $cl(C)\{z\}$, that is, either $C$ or $C \{z\}$ is called $n = 1$. Thus, if $X_1 \not\in cl(C)$, then $cl(C) = X_2 \setminus xyg$. However, if $cl(C) = X_2 \setminus xyg$ and $w$ is in $X_1$, then $M = w$, $X_1$, $w$, $X_2 \setminus xyg$ contradicts Lemma 8.5. Therefore, $X_2$ and $cl(C)$ are incomparable. By switching $X_1$ and $X_2$ if necessary, we may assume $e \in C \setminus X \in E$; since $r(C) = n = 1$, we have $r(C) \setminus X = n$; however, there are at least two elements, say $a$ and $b$, in $X_2 \setminus cl(C) \setminus X_1$, that is, in $X_2 \setminus cl(C)$, so by Theorem 5.1, $M$ is not a lattice path matroid, contrary to the minimality of $M$. Thus, statement (vi) holds, so $M$ is $D_n$.

8.3.3) If $X$ is a proper nontrivial connected at $of M$ and the element $x$ of $X$ is not parallel to any element, then $X \cdot x$ is connected at $of M = x$.

Proof of 8.3.3. If $\not\exists x$ were a disconnected at $of M = x$, then, by Lemma 8.2 applied to $M$, we would have $r(X_1 \cup X_2) = r(C) < r(M)$ for some $\not\exists x$ connected at $X_1 \cup X_2 \setminus of M \cup X$. Since $X_1$ and $X_2$ would also be ats of $M$, by 8.2, $M$ would be $D_n$. That $D_n$ has no such at $X$ and element $x$ provides the contradiction that proves the result.

8.3.4) If $M$ has three mutually incomparable connected $ats X_1 \cup X_2 \cup X_3$, then $M$ is $W_3$, $W_3^\perp$, $B_{n,k}$, or $C_{n,k}$.

Proof of 8.3.4. The minimality of $M$ and Lemma 8.3 imply that the ground set of $M$ is $X_1 \cup X_2 \cup X_3$ and that any pair $x,y$ of parallel elements can be in only one of $X_1 \cup X_2 \cup X_3$. If an element $x$ were in $X_1 \cup X_2 \cup X_3$, then by 8.3.3, $X_1 \cup X_2 \cup X_3 = x$, $X_2 \cup X_3 = x$, and $X_3 = x$ would contradict Lemma 8.3, so $X_1 \cup X_2 \cup X_3 = x$. Note that $M$ is not $D_n$, so we have $r(X_1 \cup X_2 \cup X_3) = n$ for $\not\exists x,y$ $flij; gk; fji; gk$.

First assume $X_1 \cup X_2 = x$. There are at least two points $x$ and $y$ in $X_2 \cup X_3$, so if $X_1 \cup X_3$ were nonempty, then $M = x$, $X_1 \cup X_3$, and $X_2$ would contradict Theorem 8.1. Thus, $X_1 \cup X_2 = x$. Similarly $X_2 \cup X_3 = y$. The minimality of $M$ implies that $X_1$, $X_2$, and $X_3$ are circuits. Let $flijyk$ be $fji; gk; fji; gk$. Since $r(X_1 \cup X_2) = n$, for any $x$ in $X_1 \cup X_2$ the through matroid $M_{nx}$ has no isthmuses; thus, from Lemma 8.3, either $X_1$ or $X_j$ is a circuit-hyperplane of $M_{nx}$ and so of $M$. It follows that at least two of
Similarly, there is one point in either \( t \) that of the equality \( \text{the third paragraph shows that } X \text{ is a hyperplane. Since } X \text{ is connected, } X \text{ follows that } X = X_3, \text{ as needed.}

3.3.5) If \( M \) has no circuit-hyperplane and is not \( D_n \), then \( M \) is one of \( E_n, F_n, G_n, \text{ or } H_n \).

Proof of 3.3.5. Since \( M \) is not a generalized Catalan matroid, there is a pair \( X_1, X_2 \) of incomparable connected sets. Since \( M \) is not \( D_n \), part 3.2.2 gives the equality \( r(X_1 \setminus X_2) = n \) for any such pair of sets.

A sum \( X \) there are a pair \( X_1, X_2 \), we get \( X \setminus X_1 \setminus X_2 = \emptyset \), so \( X = X \). Since \( X_1 \setminus X_2 \) and \( X_2 \setminus X_3 \) are independent, both \( X \setminus X_1 \) and \( X \setminus X_2 \) are non-py. W ith this, the claim in the third paragraph shows that \( X \) is a hyperplane. Since \( X \), it follows that \( X = X_3 \), as needed.
(i) each element in $X_1 \setminus X_2$ is parallel to another element of $M$.
(ii) $X_1 \setminus X_2$ contains just two elements, say $x$ and $y$, and at least one of $X_1 \setminus X_2$ and $X_2 \setminus X_1$, say $X_1 \setminus x$, is a circuit-hyperplane of $M$.
(iii) $X_2 \setminus x$ is a circuit.
(iv) $\{X_1, x\}$ and $\{X_2, j\}$.
(v) the non-spanning circuits of $M$ are $X_1 \setminus x$, $X_1 \setminus y$, $X_2 \setminus x$, $X_2 \setminus y$, and $fx;gy$.

A sume statement (i) failed for some $x \in X_1 \setminus X_2$. From (8.8.3) and Lemma 5.2, either $X_1 \setminus x$ or $X_2 \setminus x$, say $X_1 \setminus x$, would be a circuit-hyperplane of $M = x$. It follows that $X_1$ would be a circuit-hyperplane of $M$. This contradiction to the hypotheses of (8.5) proves statement (ii). It follows that for each $x \in X_1 \setminus X_2$, the deletion $M \setminus x$ is a connected matroid, so by Lemma 5.2, either $X_1 \setminus x$ or $X_2 \setminus x$, say $X_1 \setminus x$, is a circuit-hyperplane of $M \setminus x$. Since the circuit $X_1 \setminus x$ of $M \setminus x$ cannot contain parallel elements, statement (iii) follows. By (8.8.3) the minor $M \setminus X_2 = x$ is connected, so by part (ii) of Corollary 5.7 there is a spanning circuit $X_2 \setminus y$ of $M \setminus x$ that contains $y$. Lemma 5.2 and the minimum of the excluded minor $M$ imply that $X_2 \setminus x = X_2 \setminus X_1$, so statement (iii) holds. For statement (iv), note that if $X_1 \setminus x > X_2 \setminus x$ and $X_2 \setminus x$, then $M = \{x, X_1 \setminus x\}$, and $cl(X_1 \setminus x)$ is in contrast to Lemma 5.2. Statement (v) follows from part (v) of Corollary 5.7 since each of the notches $M$ of $M \setminus x$ and $M \setminus y$ is a circuit-hyperplane.

Now assume any two incomparable non-trivial connected sets are disjoint. We showed that the union of any two such sets is $E(M)$. Let $X_1 \setminus X_2$ be such sets. It follows that all non-spanning circuits of $M$ span either $M \setminus X_1$ or $M \setminus X_2$, so $M$ is a $T_n(M \setminus X_1, M \setminus X_2)$; also, $M \setminus X_1$ and $M \setminus X_2$ are uniform matroids. If $X_1$ is not a circuit and $x$ is in $X_1$, then $M \setminus x$ is a connected matroid which $X_2$ is not a circuit-hyperplane, so $X_2 \setminus x$ is a circuit-hyperplane of $M \setminus x$; it follows that $M \setminus X_1$ is $U_m 1_{m+1}$. Assume that $X_1$ is a circuit, and so not a hyperplane of $M$; let $x \in X_2$. Note that $X_1$ and $X_2 \setminus x$ are incomparable connected sets of the notched matroid $M \setminus x$, which has no isthmuses. Since $X_2$ is not a circuit-hyperplane of $M$, it follows that $X_2 \setminus x$ cannot be a circuit-hyperplane of $M \setminus x$. Therefore by Lemma 5.3, $X_1$ is a circuit-hyperplane of $M \setminus x$. Thus, $M \setminus X_1$ is $U_m 2_{m+1}$. In this manner, we see that there are, up to switching $X_1$ and $X_2$, three possibilities: $M \setminus X_1$ and $M \setminus X_2$ are both $U_m 2_{m+1}$; $M \setminus X_1$ is $U_m 2_{m+1}$ and $M \setminus X_2$ is $U_m 1_{m+1}$; both $M \setminus X_1$ and $M \setminus X_2$ are $U_m 1_{m+1}$. These possibilities give, respectively, $F_m, H_m$, and $G_n$.

(8.3.6) If relaxing some circuit-hyperplane $C$ of $M$ gives a generalized Catalan matroid $M \setminus 0$, then $M = A_n$.

Proof of (8.3.6). We show that $M = A_n$ by proving the following statements.

(i) There is a non-spanning circuit $C \setminus 6$ of $M$ with $C \setminus C \setminus 6$.
Fix such a circuit $C \setminus 0$ of least cardinality.
(ii) There is at least one element $y \in E(M) \setminus C \setminus \{cl(C)\}$.
(iii) The ground set of $M$ is $C \setminus \{y\}$; also $C \setminus C \setminus 0 = 1$.
(iv) The circuit $C \setminus 0$ is a hyperplane of $M$.
(v) The only non-spanning circuits of $M$ are $C$ and $C \setminus 0$.

Let the chain of proper non-trivial connected sets of $M \setminus 0$ be $X_1, \ldots, X_k$. If $C \setminus X_k$ were empty, then, by Corollary 5.2, there would be an automorphism of
Figure 15. Two more excluded minors for the class of lattice path matroids.

$M^0$ that maps $C$ to a leash segment; by Lemma 8.6 we would get the contradiction that $M$ is a notch matroid. Thus, $C \setminus X_i$ is not empty, which gives statement (i). Among all circuits that intersect $C$, choose $C^0$ with smallest cardinality. The closure $\text{cl}(C^0)$ is one of the connected sets $X_j$, and by the choice of $C^0$, the basis $C$ of $M^0$ is disjoint from $X_i$ for $i < j$. To prove statement (ii) we must show that $C$ does not contain the complement of $X_j$; this is false, then by Corollary 5.3 and Lemma 8.6 we would get, as before, that $M$ is a notch matroid.

By Theorem 3.14, $M \setminus C \setminus C^0 \{y\}$ is not a lattice path matroid. This observation and the minimality of $M$ prove the first part of statement (iii). The second part holds since if $C \setminus C^0 \setminus x$ and $x \setminus C \setminus C^0$, then, by Theorem 8.6, $M = x$ would not be a lattice path matroid. Let $C \setminus C^0$ be $z$.

To prove statement (iv), first note that $M \setminus \text{cl}(C^0)$ is a uniform matroid since, by the choice of $C^0$, any nonspanning circuit $Z$ of $M \setminus \text{cl}(C^0)$ would be disjoint from $C$, which gives the contradiction that the circuit $C^0$ properly contains the circuit $Z$. Since $M \setminus \text{cl}(C^0)$ is a uniform matroid that consists of $C^0$ and a subset of $C$, and since, by statement (ii), any circuit $C^0 \setminus C$ with $C^0 \setminus C$ that intersects $C$ contains just one element of $C$, it follows that $C \setminus \text{cl}(C^0)$ is $x$, so $C^0$ is closed. If $C^0$ is not a hyperplane of $M$, then there is an element $z$ in $C \setminus \text{cl}(C^0) \{y\}$, so $y$ is not in $\text{cl}(C^0) \{z\}$. However, for such a $z$, Theorem 5.11 applied to $M = z$, $C^0 = \{z\}$, $C = z$, and $y$ shows that $M = z$ is not in $L$, contrary to $M$ being an excluded minor for $N$.

Since $C^0$ is a circuit-hyperplane of $M$ and of the generalized Catalan matroid $M^0$, it follows that $C^0$ is the only nonspanning circuit of $M^0$, so $C$ and $C^0$ are the only nonspanning circuits of $M$, as needed to complete the proof.
be extended to a graph of this type in which each face is bounded by at most three edges, and the cycle m-atrioids of such graphs, which are certain parallel connections of 3-point lines, are easily seen to be lattice path m-atrioids.

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