LETTER TO THE EDITORS IN CHIEF

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This is to comment on the paper [1]. The main purpose of the paper is to point out the supposed errors in our paper [2].

We begin by pointing out some of the more serious flaws in [1].

(1) At the beginning of [1], it is asserted that estimate (108) in [2]:

\[ \|h\|_{L^\infty_{3,\xi}}(x) \leq C \int_{\mathbb{R}} e^{-\beta |x-y|} \|g\|_{L^\infty_{2,\xi}}(y) dy, \quad \beta > 0 \]

may be expressed equivalently as a bound

\[ |R(x,\cdot,\cdot)|_{L^\infty_{2,\xi} \to L^\infty_{3,\xi}} \leq C e^{-\beta |x|}, \quad \beta > 0 \]

on the resolvent kernel \( R(x,\xi,\xi^*) \) for (2).

This interpretation ignores the role of the Mach number. The exponent \( \beta \) depends sensitively on the Mach number of the underlying Maxwellian for the linearized collision operator \( L \). As the Mach number approaches -1, 0, 1, the exponent \( \beta \) vanishes. Therefore, the assertion of the equivalence is incorrect.

(2) Contrary to the statement in [1]: (See [2] for a construction of \( R \)). [2], referred as [2] in [1], does not construct the resolvent operator \( R \). As pointed out above, the resolvent operator does not even exist when the Mach number is 0, -1, or 1. The very expression in [1]:

\[ R = \int_{\mathbb{R}} G dt \]

is divergent when the Mach number is 0, -1, or 1. Such a basic mis-interpretation of the analysis in [2] by [1] leads to wrong conclusions.

(3) In [1], there are places where [2] is quoted too casually. In [1],

\[ S_x b = \int_{\mathbb{R}} \int_{\mathbb{R}^3} G(x,t,\cdot,\xi^*) \xi_{1,*} d\xi_{2,*} dt \]

is supposed to be equation (67) of [2]; while (67) in [2] is

\[ S_x b = \int_{0}^{\infty} G(x,\tau) [\xi^{1+} (1 - \tilde{B}_+) b] d\tau \quad \text{for} \quad x \geq 0. \]

Missing the crucial upwind Euler flux projection \( \tilde{B}_+ \) can be very misleading. The notion \( S_x b \) stated in [1] fails to be a projection operator mapping

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b to \( S_x b \) on the stable manifold. There appears a false statement below in [1]:

In [2, (102)], this is used to estimate

\[
\|
\int_{\mathbb{R}^3} G(x, t, \xi) \tilde{b}(\xi) d\xi dt \|_{L^\infty_{x,t}} \leq C e^{-\beta |x|} \|	ilde{b}\|_{L^\infty_{x,t}},
\]

in effect asserting \( \|\xi^1 \tilde{b}\|_{L^\infty_{x,t}} \leq C \|\tilde{b}\|_{L^\infty_{x,t}}, \) or \( |\xi_1|^{-1} \leq C(1 + |\xi|)^{-1} \) - evidently false for \( \xi_1 \) small. But (4) is the basis for (1), the resolvent estimate underlying the fixed-point construction of [2].

(4) [1] misses some analytical reasoning in [2]. For instance, the convergence rate of \( "x^{-1}\) as \( x \to 0" \) is mentioned in [1] for the "Discussion and open problem." In fact, in [2], \( x^{-1} \) is not the optimal rate, it is simply an intermediate estimate in establishing the linear projection onto the stable manifold for the expression

\[
S_x b = \int_0^\infty G(x, \tau) |\xi^1(1 - \tilde{B}_+)|b| d\tau \quad \text{for} \quad x \geq 0.
\]

Moreover, in the intermediate estimate of the convergence rate \( x^{-1} \), the main point in [2] is for \( x \to \infty; \) and not \( "x \to 0" \), as understood in [1]. The optimal rate is exponential \( e^{-\beta|x|} \), not \( |x|^{-1} \). This is one of the places in [2] with subtle analysis that requires close reading.

We would like to use this occasion to point out some of the key components of the subject of kinetic boundary phenomena and the fluid dynamic aspects of the Boltzmann equation for the benefit of the general reader. Specifically, we mention the following basic facts that are relevant to the present analysis:

I. Boltzmann equation is equivalent to an infinite system of partial differential equations. In particular, the stationary Boltzmann equation is an infinite dimensional dynamical system. There is incomplete information on the continuous spectrum of the operator \(-\xi^1 \partial_x + L\). As a consequence, spectral analysis alone is not sufficient for the study of wave structure for the Boltzmann equation.

A basic tool for [LY] is the explicit construction of the Green’s function in [3]. The global structure of the Green’s function \( G \), obtained in the paper is necessarily rich and its analysis is done in several steps. The first step is the Mixture Lemma to single out the particle-like waves. In spite of incomplete information on the continuous spectrum, these steps allow for study of the structure of the Green’s function.

II. Imbedded in the theory for the Boltzmann equation is the fluid dynamics phenomena. The theory for the Euler equations and Navier–Stokes equations in gas dynamics is therefore basic for the study of the Boltzmann equation. The notion of Mach number plays an essential role in the study of stationary Boltzmann equation.

This corresponds to the above discussion (1) on the dependence of decay rate on the Mach number.

III. To apply the common practice in functional analysis to the Boltzmann equation requires extra caution. Because of the infinite dimensional nature of the Boltzmann equation, some standard operators in functional analytic usage may not exist.

The above discussion (2) on the existence of the resolvent operator \( R \) is a case in point. Even in a situation where the resolvent operator \( R \) does exist, the main task is really to obtain definite rates of convergence of flows on the stable and unstable manifolds to the equilibrium states. Definite exponent in the convergence rate is needed to go from the linear to the full Boltzmann equation. The injection of Proposition 1 in [1] on global norms is misleading to the present discussion. Instead
of framing the situation in global norms, the appropriate tool for such a crucial task is the pointwise estimates for the Green’s function in [3].

IV. The coupling of kinetic boundary layer and fluid dynamics waves results in rich scaling property.

The optimal exponential rate as referred to in the above discussion (4) is obtained using weighted energy estimate, starting with the estimate for the micro part, then for the macro part of $S_x b$. $S_x b$ belongs to all $L_p$.

There is a particularly rich scaling property in the resonance cases when the Mach number is near 0, -1, or 1. In such cases, there is resonance between the Knudsen-type boundary layer and the interior fluid-like waves. As mentioned above, it is true that the exponential convergence rate of $e^{-\beta|x|}$ vanishes in general in the resonance cases. On the other hand, it is recognized in [2] that there are exceptional situations. To study the striking bifurcation phenomena resulting from the resonance, an essential step is the construction of the Bifurcation Manifold and the Sone Manifold. On these manifolds of finite co-dimensions, the exponential decay rate $e^{-\beta|x|}$ does not vanish in the resonance cases. The estimate on the Bifurcation Manifold and the Sone Manifold requires further idea beyond the one just mentioned relating to the discussion (4). Close reading of the subtle analysis involved would enhance the appreciation of the depth and beauty of this ongoing, far from finished, subject.

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