We relate two notions of classicality for quantum transformations often arising in popular sub-theories of quantum mechanics: covariance of the Wigner representation of the theory and the existence of a transformation noncontextual ontological model of the theory. We show that covariance implies transformation noncontextuality. The converse holds provided that the underlying ontological model is the one given by the Wigner representation. In addition, we investigate the relationships of covariance and transformation noncontextuality with the existence of a positivity preserving quasiprobability distribution for the transformations of the theory. We conclude that covariance implies transformation noncontextuality, which implies positivity preservation. Therefore the violation of the latter is a stronger notion of nonclassicality than the violation of the former.

In this work we move a step towards such understanding by studying two properties of quantum transformations that are usually associated to classical behaviors and therefore, when broken, to genuinely nonclassical features. These are the covariance of the Wigner function for the transformations allowed in the theory under examination and transformation noncontextuality – i.e. the existence of a transformation noncontextual ontological model for the theory. The former indicates that transformations can be represented as symplectic affine transformations in the phase space (a subset of the permutations in the discrete dimensional case) while the latter means that operationally equivalent experimental procedures for performing a transformation must correspond to the same ontological representation. The main reason for considering covariance a classical feature derives from the Hamiltonian formulation of classical mechanics, where the physical trajectories are represented – necessary and sufficient condition – by symplectic transformations. Hence, covariantly represented quantum transformations can be interpreted as classical trajectories in the phase space. Moreover, covariance characterizes the representation of quantum transformations in classical-like theories such as Spekkens’ toy model. The existence of a transformation noncontextual ontological model is a classical feature because, while holding for classical mechanics, it is incompatible with the statistics of quantum theory, as proven in.

Historically, most research on the role of nonclassical features in quantum computation has focused on properties of preparations and measurements. Instead, we here study properties of quantum transformations. They have recently been shown to play a crucial role in information processing tasks and in encoding nonclassical behaviors of subtheories of quantum mechanics that were previously considered to behave classically, like the single qubit stabilizer theory.

The latter provides one of the main reasons why considering covariance and transformation noncontextuality as connected: they are both broken by the presence of the Hadamard (or the phase) gate. Another result that ignores transformations is due to Spekkens in. He showed that contextuality and negativity of quasiprobability representations are equivalent notions of nonclassicality. Here we use this result – that trivially extends to transformations – for proving theorems 1 and 2.

The letter is structured as follows. We first define Wigner functions, covariance and transformation noncontextuality. The property of covariance, originally defined for unitaries, is extended to more general completely positive trace-preserving (CPTP) maps. Successively, we treat the example of the single qubit stabilizer theory and then prove the results relating covariance and transformation noncontextuality, as well as their relation with the existence of a positivity preserving quasiprobability representation for the transformations of the theory. We conclude by discussing possible generalizations of the results and future avenues.
Wigner functions. Wigner functions are a way of reformulating quantum theory in the phase space \( \mathbb{R}^d \), which is the framework of classical Hamiltonian mechanics, and therefore provide a tool to compare the two theories on the same ground. They are also the most popular example of quasiprobability distributions \([14, 27, 34]\). The latter are linear and invertible maps from operators on Hilbert space to real distributions on a measurable space \( \Lambda \). Quasiprobability representations of quantum mechanics associate i) a real-valued function \( \mu_\rho : \Lambda \rightarrow \mathbb{R} \) to any density operator \( \rho \) such that \( \int d\lambda \mu_\rho(\lambda) = 1 \), ii) a real-valued function \( \xi_{\Pi_k} : \Lambda \rightarrow \mathbb{R} \) to any element \( \Pi_k \) of the POVM \( \{ \Pi_k \} \) such that \( \sum_k \xi_{\Pi_k}(\lambda) = 1 \ \forall \ \lambda \in \Lambda \), and iii) a real-valued matrix \( \Gamma_{\varepsilon} : \Lambda \times \Lambda \rightarrow \mathbb{R} \) to any CPTP map \( \varepsilon \), such that \( \int d\lambda \Gamma_{\varepsilon}(\lambda', \lambda) = 1 \). These distributions provide the same statistics of quantum theory – given by the Born rule – if

\[
p(k|\rho, \varepsilon, \{ \Pi_k \}) = \text{Tr}[\varepsilon(\rho)\Pi_k] = \int d\lambda d\lambda' \xi_{\Pi_k}(\lambda')\Gamma_{\varepsilon}(\lambda', \lambda)\mu_\rho(\lambda), \tag{1}
\]

Quasiprobability distributions are named this way as they behave similarly to probability distributions, with the crucial difference that they can take negative values. The negativity is usually assumed to be a signature of quantumness because it is unavoidable in order to reproduce the statistics of quantum mechanics \([35]\). A quasiprobability distribution, for example \( \mu_\rho(\lambda) \), is non-negative if \( \mu_\rho(\lambda) \geq 0 \ \forall \ \lambda \in \Lambda \). Non-negative quasiprobability representations of quantum mechanics are an important tool to support the epistemic view of quantum theory \([12, 36]\), where quantum states are interpreted as states of knowledge rather than states of reality, and are useful to perform classical simulations of quantum computations \([37]\). We will focus on the following property of quasiprobability distributions.

**Definition 1.** Given a set \( S \) of quantum states \( \rho \) that are non-negatively represented by a quasiprobability distribution \( \mu_\rho(\lambda) \), a transformation \( \varepsilon \) that maps \( \rho \) to \( \rho' = \varepsilon(\rho) \) is positivity preserving if, for every \( \rho \in S \), the quasiprobability distribution \( \mu_{\rho'}(\lambda) \) is non-negative too.

A subtheory of quantum mechanics allows for a positivity preserving quasiprobability representation if there exists a quasiprobability distribution for which all the transformations of the subtheory are positivity preserving.

We are interested in Wigner functions defined over the discrete phase space \( \Lambda = \mathbb{Z}_d^2 \), where the integers \( d, n \) denote the dimensionality of the system and the number of systems, respectively. We follow the definition provided in \([14]\) that includes the mostly used examples of Wigner functions \([27, 38]\).

**Definition 2.** The Wigner function of a quantum state \( \rho \) (and, analogously, of a POVM element \( \Pi_k \)) is defined as

\[
W_\rho^\gamma(\lambda) = \text{Tr}[A^\gamma(\lambda)\rho],
\]

where the phase-point operator is

\[
A^\gamma(\lambda) = \frac{1}{N_\lambda} \sum_{\lambda' \in \Lambda} \chi([\lambda, \lambda'])\hat{W}^\gamma(\lambda'),
\]

where the phase-point operator is

\[
\hat{W}^\gamma(\lambda) = \int d\lambda d\lambda' \xi_{\Pi_k}(\lambda')\Gamma_{\varepsilon}(\lambda', \lambda)\mu_\rho(\lambda).
\]

The normalization \( N_\lambda \) is such that \( \text{Tr}(A^\gamma(\lambda)) = 1 \). The Weyl operators \( \hat{W}^\gamma(\lambda) \) are defined as \( \hat{W}^\gamma(\lambda) = w^\gamma(\lambda)Z(p)X(x) \), where the phase space point is \( \lambda = (x, p) \in \Lambda \). The operators \( X(x), Z(p) \) represent the (generalized) Pauli operators, \( X(x) = \sum_{x' \in \mathbb{Z}_d} |x' - x\rangle\langle x' \rangle \) and \( Z(p) = \sum_{x \in \mathbb{Z}_d} \chi(px) |x\rangle\langle x| \). The functions \( w \) and \( \chi \) associated to a CPTP map \( \varepsilon \) are \( w^\gamma(\lambda) = \varepsilon(\lambda) \), \( \chi(a) = (-1)^a \) in the case of qubits and \( w^\gamma(\lambda) = \chi(-2^{-1} \gamma(\lambda)) \), \( \chi(a) = e^{\pi i a} \) in the case of qudits of odd dimensions. By choosing the function \( \gamma : \Lambda \rightarrow \mathbb{Z}_d \), where \( q \) is an integer, we can specify which Wigner function to consider. The reason why choosing different Wigner functions is that they are non-negative – thus providing a proof of operational equivalence with classical theory – for different subtheories of quantum mechanics. For example \( \gamma = x \cdot p \) gives Gross’ Wigner function \([27, 39]\) for odd dimensional qudits, which non-negatively represents stabilizer quantum mechanics in odd dimensions \([29, 30]\), while \( \gamma = x \cdot p \mod 4 \) gives the Wigner function developed by Gibbons and Wootters \([38, 40]\) for qubits, which non-negatively represents the subtheory of quantum mechanics composed by all the separable eigenstates of Pauli operators, Pauli measurements and transformations between them \([14, 17]\). We will omit the superscript \( \gamma \) in the future in order to soften the notation.

We report the properties of Wigner functions in appendix A. We here mention only the ones that will be used in the following. The phase-point operators form a complete basis of the Hermitian operators in the Hilbert space with respect to the Hilbert–Schmidt inner product, thus obeying Hermitianity, \( A(\lambda) = A(\lambda) \forall \lambda \in \Lambda \), and orthonormality, \( \text{Tr}[A(\lambda)A(\lambda')] = \frac{1}{N_\lambda} \delta_{\lambda, \lambda'} \). This implies that \( \rho = \sum_{\lambda \in \Lambda} A(\lambda)\rho(\lambda) \), where \( \rho \) is an Hermitian operator. For consistency with equation (1), the Wigner function associated to a CPTP map \( \varepsilon \) is \( W_\rho(\lambda', \lambda) = \text{Tr}[\varepsilon(\rho(\lambda))A(\lambda')] \) and it is such that \( \sum_{\lambda \in \Lambda} W_\rho(\lambda', \lambda) = \frac{1}{N_\lambda} \delta_{\lambda, \lambda'} \).

In this work we consider subtheories of quantum mechanics defined by a set \( (S, T, M)_\gamma \) of quantum states, transformations and measurements. The sets \( S \) and \( M \) are composed by the states and measurements, respectively, that are non-negatively represented by the Wigner function specified by \( \gamma \) (this explains the subscript), as in \([14]\). The set of transformations \( T \) is only restricted by the fact that transformations must map between states (and observables) in the subtheory. In this way the possibility of nonclassicality is only confined to transformations. In the following we define and relate properties that further restrict the set \( T \).
Covariance. Following [27], the Wigner function $W_\rho(\lambda)$ is covariant with respect to the unitary gate $U \in \mathcal{T}$ if

$$W_{U \rho U^\dagger}(\lambda) = W_\rho(S\lambda + a) \quad \forall \rho \in \mathcal{S}, \forall \lambda \in \Lambda,$$  

where $S$ is a symplectic transformation, i.e. $S^TJS = J$, where $J = \oplus_{j=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_j$ is the standard invertible matrix used in symplectic geometry, and $a$ is a translation vector. We say that the unitary $U$ is covariantly represented, by the Wigner function $W_\rho(\lambda)$, if equation (4) holds. Equation (4) can also be written as

$$UA(\lambda)U^\dagger = A(S\lambda + a) \quad \forall \lambda \in \Lambda.$$  

A covariantly represented subtheory $(\mathcal{S}, \mathcal{T}, \mathcal{M})$, is a theory where all $T \in \mathcal{T}$ are covariantly represented. We now extend the above definition of covariance to general quantum channels. We recall that a quantum channel is represented by a CPTP map – can in general be written as

$$\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger,$$  

where the Kraus operators $E_k$ satisfy the completeness relation $\sum_k E_k^\dagger E_k = I$. There are infinitely many equivalent Kraus decompositions, and they are all mapped to each other via unitary maps.

Definition 3. A CPTP map $\varepsilon \in \mathcal{T}$ is covariantly represented, by the Wigner function $W_\rho(\lambda)$, if all its decompositions into unitary Kraus operators, i.e. $E_k \in \mathcal{T}$ such that $E_k^\dagger = E_k^{-1}$, have covariantly represented Kraus operators,

$$E_k A(\lambda) E_k^\dagger = A(S\lambda + a_k) \quad \forall \lambda \in \Lambda, \forall k.$$  

The motivation for definition 3 is that covariance is a concept defined for unitaries (see equation 4). It would not apply to irreversible transformations (not even in classical theory).

Transformation Noncontextuality. A natural way of justifying why quantum theory works is to provide an ontological model that reproduces its statistics [12]. An ontological model associates the physical system of the state at a given time – the ontic state – to a point $\lambda$ in a measurable set $\Lambda$, and the experimental procedures – classified in preparations, transformations and measurements – to probability distributions on the ontic space $\Lambda$. A preparation procedure $P$ of a quantum state $\rho$ is represented by a probability distribution $\mu_P(\lambda)$ over the ontic space, $\mu_P : \Lambda \rightarrow \mathbb{R}$ such that $\int \mu_P(\lambda) d\lambda = 1$ and $\mu_P(\lambda) \geq 0 \quad \forall \lambda \in \Lambda$. A transformation procedure $T$ of a CPTP map $\varepsilon$ is represented by a transition matrix $\Gamma_T(\lambda',\lambda)$ over the ontic space, $\Gamma_T : \Lambda \times \Lambda \rightarrow \mathbb{R}$ such that $\int \Gamma_T(\lambda',\lambda) d\lambda' = 1$ and $\Gamma_T(\lambda',\lambda) \geq 0 \quad \forall \lambda, \lambda' \in \Lambda$. A measurement procedure $M$ with associated outcomes $k$ of a POVM $\{\Pi_k\}$ is represented by a set of indicator functions $\{\xi_{M,k}(\lambda)\}$ over the ontic space, $\xi_{M,k} : \Lambda \rightarrow \mathbb{R}$ such that $\sum_k \xi_{M,k}(\lambda) = 1$ and $\xi_{M,k}(\lambda) \geq 0 \quad \forall \lambda \in \Lambda, \forall k$. The ontological model reproduces the predictions of quantum theory according to the law of classical total probability,

$$p(k|P, T, M) = \text{Tr}[\varepsilon(\rho) \Pi_k]$$  

$$= \int d\lambda p(\xi_{M,k}(\lambda)) \Gamma_T(\lambda',\lambda) \mu_P(\lambda).$$  

All the preparation procedures that prepare the state $\rho$ belong to the equivalence class that we denote with $e_P(P)$. Analogous reasoning for the equivalence classes $e_\varepsilon(T)$ associated to the CPTP map $\varepsilon$ and $e_{\{\Pi_k\}}(M)$ associated to the POVM $\{\Pi_k\}$. The idea of the generalized notion of noncontextuality is that operational equivalences – e.g. different Kraus decompositions of a CPTP map – are represented by identical probability distributions on the ontic space.

Definition 4. An ontological model of (a subtheory of) quantum mechanics is transformation noncontextual if

$$\Gamma_T(\lambda',\lambda) = \Gamma_{\varepsilon}(\lambda',\lambda) \quad \forall T \in e_\varepsilon(T), \forall \varepsilon.$$  

The above definition can be analogously extended to preparation noncontextuality and measurement noncontextuality [12],

$$\mu_P(\lambda) = \mu_{e_\varepsilon(P)}(\lambda) \quad \forall P \in e_\varepsilon(P), \forall \rho,$$

$$\xi_{M,k}(\lambda) = \xi_{e_{\{\Pi_k\}}(M)}(\lambda) \quad \forall M \in e_{\{\Pi_k\}}(M), \forall \{\Pi_k\}.$$  

An ontological model is universally noncontextual if it is preparation, transformation and measurement noncontextual. It can be shown that a universally noncontextual ontological model of quantum mechanics is impossible [12] and, in particular, that a transformation noncontextual ontological model of quantum theory is impossible too.

From the definitions of ontological models above and the definitions of quasiprobability representations provided previously, by substituting equations (9) and (10) into equation (8), it is immediate to see that the existence of a non-negative quasiprobability representation that provides the statistics of quantum theory as in equation (1) coincides with the existence of a noncontextual ontological model for it. This result was proven in [33] and here stated also considering transformations.

The single qubit stabilizer theory. The stabilizer theory of one qubit is defined as the subtheory of one qubit quantum mechanics that includes the eigenstates of $X,Y,Z$ Pauli operators, the Clifford unitaries – generated by the Hadamard gate $H$ and the phase gate $P$ – and $X,Y,Z$ Pauli observables. Its states and measurement elements are non-negatively represented by Wootters-Gibbons’ Wigner functions [35], and its transformations
preserve positivity. More precisely, as shown in [10], there are two possible Wigner functions for one qubit stabilizer states that are non-negative and obey the desired properties listed in appendix A. We denote the two Wigner functions with $W_+$ and $W_-$. The corresponding phase-point operators are $A_+(0,0) = \frac{1}{2}(I + X + Y + Z)$, $A_-(0,0) = \frac{1}{2}(I - X + Y - Z)$, and $A_j(0,1) = X A_j(0,0) X^\dagger$, $A_j(0,1) = Y A_j(0,0) Y^\dagger$, $A_j(1,1) = Z A_j(0,0) Z^\dagger$ for $j \in \{+,-\}$.

The 8-state model [31] provides a natural preparation and measurement noncontextual ontological model of the single qubit stabilizer theory. The quantum states (and measurement elements) are represented as uniform probability distributions over an ontic space of dimension 8 and the Clifford transformations are represented by permutations over the ontic space. It can be seen as a model that corresponds to take into account both the Wigner functions $W_+$ and $W_-$, as shown in figure 1.

![Figure 1: Non-negative Wigner functions of one qubit stabilizer states.](image)

Results. We now relate the notions of classicality for quantum transformations defined so far.

Theorem 1. Let us consider the subtheory of quantum mechanics $(\mathcal{S}, \mathcal{T}, \mathcal{M})_\gamma$. If the subtheory is covariantly represented, then the Wigner representation provides a transformation noncontextual ontological model for it. The converse implication holds too.

Proof. Let us consider a transformation $\varepsilon \in \mathcal{T}$.

$\Rightarrow$. Given covariance (equation (7)), the orthonormality of the phase-point operators, and the linearity of the trace, then $\text{Tr}[\varepsilon(A(\lambda))A(\lambda')] = \text{Tr}[\sum_k A(S_k \lambda + a_k)A(\lambda')] = 1/N\lambda \sum_{\lambda', \lambda''} \delta_{\lambda, \lambda + a_k} \lambda' \lambda'' \geq 0$. A non-negative Wigner function for the transformation $\varepsilon$ implies a transformation noncontextual ontological model for it, as we have shown at the end of the section on transformation noncontextuality [33].

$\Leftarrow$. Transformation noncontextuality for the ontological model given by the Wigner representation of the theory coincides with the Wigner function $\text{Tr}[\varepsilon(A(\lambda))A(\lambda')]$ to be non-negative. This implies also the Wigner representations of the unitary Kraus operators $E_k$, $\text{Tr}[E_k A(\lambda) E_k^\dagger A(\lambda')]$, to be non-negative. If this was not the case, we could consider the multiple representations of the $E_k$ to obtain multiple representations of $\varepsilon$ too, which would contradict the original transformation noncontextuality assumption. In order to show that $\varepsilon$ is covariantly represented, let us define $B \equiv E_k A(\lambda) E_k^\dagger$ and, by the fact that the phase-point operators $A(\lambda)$ form a basis for the Hermitian operators in the Hilbert space, let us write it as $B = \sum_{\lambda'} W_{E_k}(\lambda') A(\lambda')$. Successively, notice, from $\text{Tr}[A(\lambda)] = 1$, that $\text{Tr}[B] = \sum_{\lambda'} W_{E_k}(\lambda') = 1$. Moreover, from the orthonormality of the phase-point operators, $\text{Tr}[B^2] = \sum_{\lambda'} W_{E_k}^2(\lambda') = 1$. Therefore, given that $W_{E_k}(\lambda') \geq 0$, $\sum_{\lambda'} W_{E_k}(\lambda') = 1$, and
\( \sum_{\lambda'} W_{E_k}^B(\lambda|\lambda') = 1 \), we conclude that \( W_{E_k}(\lambda|\lambda') = 0 \) \( \forall \lambda' \) apart from one \( \lambda' \), for which \( W_{E_k}(\lambda|\lambda') = 1 \). This means that \( B \) coincides with one of the phase-point operators, \( i.e. \) \( E_k \) is covariantly represented. This is true for every \( k \), thus \( \varepsilon \) is covariantly represented.

Notice, from the theorem above, that while covariance implies transformation noncontextuality (as it is enough to have a transformation noncontextual model to guarantee transformation noncontextuality), the converse implication may not hold in general. In principle, it could be possible to have a transformation noncontextual ontological model, even if the ontological model corresponding to the Wigner representation is transformation contextual.

Let us now state the relationships of covariance and transformation noncontextuality with the existence of a positivity preserving quasiprobability distribution. The proofs of the theorems are contained in appendix B.

Theorem 2. Let us consider the subtheory of quantum mechanics \( (S,T,M) \). If the subtheory is covariantly represented, then it allows for a positivity preserving quasiprobability representation. The converse implication does not hold.

Theorem 3. Let us consider the subtheory of quantum mechanics \( (S,T,M) \). If the subtheory allows for a transformation noncontextual ontological model, then it also allows for a positivity preserving quasiprobability representation. The converse implication does not hold.

Notice that in theorem 3, considering that no properties of Wigner functions are involved, we assume the subtheory to be arbitrary \[42\], and not defined by a choice of Wigner function as we did so far. This allows us to make the theorem more general. The relations found are depicted in figure 2.

**Figure 2: Relationship between three notions of classicality associated to quantum transformations.** The existence of a covariant Wigner function implies the existence of a transformation noncontextual ontological model, which implies the existence of a positivity preserving quasiprobability distribution for a subtheory of quantum mechanics. The existence of a noncontextual ontological model for the subtheory implies covariance if the model is the one provided by the Wigner representation (starred arrow).

**Discussion.** The results obtained regard the relationships between a property of Wigner functions – covariance, a property of ontological models – transformation noncontextuality, and a property of quasiprobability representations – positivity preservation. It would be interesting to extend the notion of covariance to more general quasiprobability representations. However, we argue that a relation between this extended covariance and transformation noncontextuality is not expected to hold. Such covariance would hold in the CSS rebit subtheory studied in \[13\], while transformation noncontextuality is violated \[43\]. Still, the latter is not expected to imply covariance, considering that the property of orthonormality – crucial for proving theorem 1 – does not hold for generic quasiprobability representations.

One extra way to generalize covariance would be to consider the CPTP map \( \varepsilon \) as given by a unitary \( U \) acting on a Hilbert space that describes both the system and the environment, \( \varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger = \text{Tr}_E[U \rho_{SE} U^\dagger] \), where \( \rho_{SE} \) is the state of the system and environment, while \( \rho \) is the state of the system only. Covariance is then defined as \( U A_{SE}(\lambda) U^\dagger = A_{SE}(S_{SE} \lambda + a_{SE}) \), where \( S_{SE} \) and \( a_{SE} \) are a symplectic matrix and a vector acting on the phase space associated to the system and environment, for all the possible unitaries that define \( \varepsilon \). We leave the study of this notion for future research.

Another open question is whether any quasiprobability representation on an overcomplete frame \[35\] (like the one in \[15\] and unlike the Wigner representation) always implies the corresponding model to be transformation contextual. In an overcomplete frame representation the phase-point operators are, by definition, more than the ones needed to form a basis. This is expected to imply multiple distinct representations of each transformation. Showing that only complete frame representations provide noncontextual models would be a first step to ultimately try to prove that any noncontextual ontological model has to correspond to a Wigner representation, thus also generalizing theorem 1.

Our results show that breaking positivity preservation is a stronger notion of nonclassicality than transformation contextuality and non-covariance. With respect to positivity preservation, the qubit stabilizer theory is classical, even if it shows transformation contextuality and breaks covariance. This consideration motivates the study of the physical justifications for considering positivity preservation as a legitimate classical feature. With this work we aim to promote further research, as in \[41\], on the reasons why the contextuality present in qubit stabilizer theory has, computationally, a classical nature.

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[42] We still assume, as stated at the beginning, that the set of states and measurements are classical – i.e. noncontextually represented, thus restricting the possibility of nonclassicality to transformations only.
[43] The two decompositions, $\rho \rightarrow 1/2(\rho + Y\rho Y)$ and $\rho \rightarrow 1/2(X\rho X + Z\rho Z)$, correspond to the same channel in the CSS subtheory of [15], even if any ontological model of the theory represents them as distinct. The author thanks Piers Lillystone for making him aware of this fact.
APPENDIX A

We here report the properties of the Wigner functions defined in definition 2.

- The marginal of a Wigner function on the state $\rho$ behaves as a probability distribution: $\sum_{p \in \mathbb{Z}_d} W_\rho(x, p) = |\langle x | \rho | x \rangle|^2$.

- The Wigner function of many systems in a product state is the tensor product of the Wigner functions of each system, as a consequence of the factorability of $A(\lambda), A(x_0, p_0 \ldots x_{n-1}, p_{n-1}) = A(x_0, p_0) \otimes \cdots \otimes A(x_{n-1}, p_{n-1})$.

- The phase-point operators form a complete basis of the Hermitian operators in the Hilbert space with respect to the Hilbert-Schmidt inner product, thus obeying Hermitianity, $A(\lambda) = A^\dagger(\lambda)$ $\forall \lambda \in \Lambda$, and orthonormality, $\text{Tr}[A(\lambda)A(\lambda')] = \frac{1}{N^2} \delta_{\lambda, \lambda'}$. This implies that $\rho = \sum_{\lambda \in \Lambda} A(\lambda)W_\rho(\lambda)$ and $\sum_{\lambda \in \Lambda} W_\rho(\lambda)W_\sigma(\lambda) = \text{tr}(\rho \sigma)$, where $\rho, \sigma$ are any two Hermitian operators.

- $\sum_{\lambda \in \Lambda} A(\lambda) = I$, thus implying that $\text{Tr}[\rho] = 1 = \sum_{\lambda} W_\rho(\lambda)$.

APPENDIX B

We here report the proofs of theorems 2 and 3.

**Theorem 2.** Let us consider the subtheory of quantum mechanics $(\mathcal{S}, \mathcal{T}, \mathcal{M})_\gamma$. If the subtheory is covariantly represented, then it allows for a positivity preserving quasiprobability representation. The converse implication does not hold.

**Proof of theorem 2** Let us consider the non-negative Wigner function $W_\rho(\lambda)$ and a covariantly represented transformation $\varepsilon$ which maps $\rho$ to $\rho' = \varepsilon(\rho)$ such that $W_{\rho'}(\lambda) = \sum_k W_\rho(S_k \lambda + a_k)$. Since all $W_\rho(S_k \lambda + a_k)$ are non-negative by definition, then also $W_{\rho'}(\lambda)$ is non-negative. The converse implication does not hold, as proven by the counterexample of the single qubit stabilizer theory, where the Hadamard gate maps non-negative states to non-negative states (Wootters-Gibbons’ Wigner function [38, 40]), but it is not covariantly represented.

**Theorem 3.** Let us consider the subtheory of quantum mechanics $(\mathcal{S}, \mathcal{T}, \mathcal{M})$. If the subtheory allows for a transformation noncontextual ontological model, then it also allows for a positivity preserving quasiprobability representation. The converse implication does not hold.

**Proof of theorem 3** From Spekkens’ result [33] extended to transformations, the existence of a transformation noncontextual ontological model to represent any transformation $\varepsilon \in \mathcal{T}$ implies that there exists a non-negative quasiprobability distribution $\Gamma_\varepsilon(\lambda, \lambda') \geq 0$ $\forall \lambda, \lambda' \in \Lambda$. Given a state $\rho \in \mathcal{S}$ which is represented by the non-negative quasiprobability distribution $\mu_\rho(\lambda)$, the state $\rho' = \varepsilon(\rho)$ is also non-negatively represented, as $\mu_{\rho'}(\lambda) = \sum_{\lambda'} \mu_\rho(\lambda')\Gamma_\varepsilon(\lambda, \lambda')$. This proves the first part of the theorem. The converse implication does not hold, as proven by the counterexample of the single qubit stabilizer theory, where the Hadamard gate maps non-negative states to non-negative states (Wootters-Gibbons’ Wigner function [38, 40]), but the theory does not allow for a transformation noncontextual model. The core reason why transformation contextuality, and, equivalently, the unavoidable presence of some negativity in $\Gamma_\varepsilon$, is a weaker notion of nonclassicality than the breaking of positivity preservation is that $\Gamma_\varepsilon$, despite assuming some negative values, can still preserve the positivity between the quasiprobability representations of the states.