Non-Parametric Analyses of Log-Periodic Precursors to Financial Crashes

Wei-Xing Zhou ¹ and Didier Sornette ²³

1. Institute of Geophysics and Planetary Physics, University of California, Los Angeles, CA 90095
2. Department of Earth and Space Sciences, University of California, Los Angeles, CA 90095
3. Laboratoire de Physique de la Matière Condensée, CNRS UMR 6622 and Université de Nice-Sophia Antipolis, 06108 Nice Cedex 2, France

Abstract

We apply two non-parametric methods to test further the hypothesis that log-periodicity characterizes the detrended price trajectory of large financial indices prior to financial crashes or strong corrections. The analysis using the so-called \((H,q)\)-derivative is applied to seven time series ending with the October 1987 crash, the October 1997 correction and the April 2000 crash of the Dow Jones Industrial Average (DJIA), the Standard & Poor 500 and Nasdaq indices. The Hilbert transform is applied to two detrended price time series in terms of the \(\ln(t_c - t)\) variable, where \(t_c\) is the time of the crash. Taking all results together, we find strong evidence for a universal fundamental log-frequency \(f = 1.02 \pm 0.05\) corresponding to the scaling ratio \(\lambda = 2.67 \pm 0.12\). These values are in very good agreement with those obtained in past works with different parametric techniques.

1 Introduction

Several recent analyses [12, 18, 27, 28, 24, 23] have shown that large financial crashes are “outliers” or “kings” [22] with statistically different properties than the rest of the population of price returns. A proposed mechanism for understanding this “king” effect involves collective interactions between agents leading to a cascade of amplifications, which may result from several origins [21, 19, 20, 30, 29, 23, 10].

Associated with this “king” effect, log-periodic patterns [26] in financial price time series have been found to be precursory signatures of large crashes or large drawdowns [32, 3, 33, 34, 2, 15, 1, 17, 1]. (see also [27, 23] and references therein). Most of the analyses have been of a parametric nature, based on formulas involving combinations of powers laws and of log-periodic functions of the type \(\cos[\omega \ln(t_c - t)]\) [31, 13, 14], where \(t_c\) is a “critical” time at or close to the crash. A first attempt at providing a non-parametric test of log-periodicity in financial time series was based on a spectral Lomb analysis [25] of the oscillatory residuals to a preliminary power law fit to the time series [13, 24, 27].

Here, we propose to extend the search for non-parametric signatures of log-periodicity. For this, we introduce two non-parametric techniques and apply them to seven financial time series culminating in a crash or a strong correction. The first non-parametric method is called the generalized \(q\)-analysis or \((H,q)\)-analysis and was introduced in Ref. [37] in the analogous context of the detection of log-periodic oscillations in precursory signals to material failures. The second non-parametric method uses the Hilbert transform to construct the local cumulative phase of the oscillatory component in the time series. By the new lights provided by these analyses that are cast
on the difficult empirical problem of detecting reliable precursory patterns to financial crashes or to strong corrections, we hope to provide what we believe are strong additional evidence for the reality of log-periodicity in price trajectories. Compared with a similar analysis of material failures using the \((H, q)\)-analysis \[37\], we find that log-periodicity seems to be a significantly stronger signal in financial time series than it is for rupture.

2 The generalized \((H, q)\)-analysis

2.1 Definitions

The so-called \(q\)-analysis is based on the concept of a \(q\)-derivative, the inverse of the Jackson \(q\)-integral \[11\], which is the natural tool \[3, 4\] for describing discrete scale invariance \[26\]. The \(q\)-derivative of an arbitrary function \(f(x)\) for any \(q \in (0, 1) \cup (1, \infty)\) is defined as

\[
D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \tag{1}
\]

where \(x\) is the distance to the critical point. If the signal is a price \(p(t)\) function to time \(t\), then \(x = t_c - t\), where \(t_c\) is the critical time, that is, the time of the culmination of the bubble, which is in general close to the time of the crash \[20, 21\]. The case \(q \to 1\) recovers the normal definition of derivative. It follows that

\[
D_{1/q} f(x) = D_q f(x/q). \tag{2}
\]

It is thus enough to study \(D_q f(x)\) for \(q \in (0, 1)\) to derive its values for all \(q\)'s.

The necessary and sufficient condition for a function \(f(x)\) to be homogeneous of order \(\psi\) is

\[
D_q f(x) = \frac{q^\psi - 1}{q - 1} f(x). \tag{3}
\]

This expression suggests the introduction of a generalized \(q\)-derivative that we call \((H, q)\)-derivative, such that the dependence in \(x\) of \(D^H_q f(x)\) disappears for homogeneous functions, for the choice \(H = \psi\). Consider therefore the following definition

\[
D^H_q f(x) \triangleq \frac{f(x) - f(qx)}{(1-q)x}^H, \tag{4}
\]

such that \(D^H_q f(x) \equiv 1\) recovers the standard \(q\)-derivative \(D_q f(x)\). For a power law function \(f(x) = Bx^m\), \(D^H_q \equiv m [Bx^m] = B(1 - q^m)/(1 - q)^m\) is constant. For a statistically homogeneous function \(f(x) \equiv Bx^m\), \(D^H_q \equiv 1\) is constant, where the symbol \(\equiv\) means equivalence in distribution.

The generalized \((H, q)\)-derivative has two control parameters: the discrete scale factor \(q\) devised to characterize the log-periodic structure and the exponent \(H\) introduced to account for a possible power law dependence, i.e., to correct for the existence of trends in log-log plots.

Let us consider the following log-periodic function of \(x = (t_c - t)/\tau\):

\[
y(x) = A - B \tau^\beta x^\beta + C \tau^\beta x^\beta \cos (\omega \ln x), \tag{5}
\]

where the phase in the cosine \(\phi = \omega \ln \tau\) has been scaled away by a change of time scale. In the following, we shall refer to \(\omega\) as the angular log-frequency and shall also use the log-frequency defined as

\[
f = \frac{\omega}{2\pi}. \tag{6}
\]
The \((H, q)\)-derivative of \(y(x)\) is

\[
D_q^H y(x) = x^{\beta - H}[B' + C'g(x)],
\]

where \(B' = -\frac{B\gamma}{(1-q)^H}\), \(C' = \frac{C\gamma}{(1-q)^H}\) and

\[
g(x) = \cos(\omega \ln x) - q^\beta \cos(\omega \ln qx) = C_1 \cos(\omega \ln x) + C_2 \sin(\omega \ln x),
\]

where \(C_1 = [1 - q^\beta \cos(\omega \ln q)] > 0\) and \(C_2 = \sin(\omega \ln q)\). In principle, we could optimize the value \(q\) to adapt it to the angular log-frequency \(\omega\) to ensure \(C_1 = 0\), so that \(g(x)\) becomes a pure sinus: a study of the phase of the log-periodic oscillations as a function of \(q\) can in principle get access to \(\omega\). We have not tried this approach as it seems very sensitive to noise.

Varying \(\beta\), we find that the analysis in terms of the \((H, q)\)-derivative of \(y(x)\) given by (5) gives essentially the same results for the highest peaks \(P_N(H, q)\) and their associated log-frequencies \(f(H, q)\) when varying \(\beta\) at fixed \(\omega = 2\pi f\) but gives very different and volatile results when varying \(\omega\) at fixed \(\beta\). This shows the value of the \((H, q)\)-analysis which is a robust detector of the log-frequency but not of the power law trend.

2.2 Effect of errors in the critical time \(t_c\)

An important issue is the determination of the critical time \(t_c\) entering in the definition of the scaling parameter \(x = t_c - t\), such that the time series is (discretely) scale invariant with respect to scale transformation performed on \(x\). Indeed, according to the theory of rational bubbles and crashes [20, 21, 30], the crash is not a certain deterministic event but is described by its hazard rate \(h(t)\), which is the probability per unit time that the market crashes conditioned on the fact that it has not crashed yet. In this framework, the critical \(t_c\) is the end of the bubble and coincides with the peak of the crash hazard rate. In practice, this means that the time of the crash, if it occurs, is expected to be close to and slightly smaller than \(t_c\). Thus, knowing the time of the crash does not provide a perfect measurement of \(t_c\). Therefore, there is a fundamental and unavoidable uncertainty in the determination of the value of \(t_c\) entering the definition of \(x\). This uncertainty is a severe limitation for parametric methods of detections of log-periodicity. In contrast, our previous work [57] has shown that the \((H, q)\)-derivative is rather insensitive to an error in the estimation of \(t_c\) up to times of the order of the error made on the determination of \(t_c\) (see figure 2 of [57]). Technically, this can be seen from the following derivation.

An error in \(t_c\) amounts to a spurious translation of the coordinate system \((x, y)\) away from the origin \((0, 0)\) to a new origin \((x_0, y_0)\). This defined the following coordinate transformation \((x, y) \rightarrow (\bar{x}, \bar{y})\):

\[
\begin{align*}
    x & \rightarrow \bar{x} = x - x_0 \\
    y & \rightarrow \bar{y} = y - y_0,
\end{align*}
\]

where \(y_0 = y(x_0)\). According to this transform, \(\bar{y}(\bar{x}) = y(x) - y_0 = y(\bar{x} + x_0) - y_0\) and \(\bar{y}(q\bar{x}) = y(q\bar{x} + x_0) - y_0 = y(q\bar{x} - qx_0 + x_0)\). Thus, the \((H, q)\)-derivative is

\[
D_q^H \bar{y}(\bar{x}) = \frac{B\bar{q}^\beta - B + C\cos(\omega \ln x) - C\bar{q}^\beta \cos[\omega \ln(q\bar{x})]}{(1-q)^H(x - x_0)^{H-m}}
\]

where \(\bar{q} = q + \frac{(1-q)x_0}{x}\).

Consider two values \(x_{n+1} > x_n\) of the control parameter \(x\) corresponding to successive local extrema of the log-periodic function, such that \(\cos[\omega \ln(q_nx_n)] = \cos[\omega \ln(q_{n+1}x_{n+1})]\), that is,

\[
\ln(q_nx_n) - \ln(q_{n+1}x_{n+1}) = 2k\pi/\omega.
\]
Therefore,
\[
\frac{x_{n+1} - qx_0 + x_0}{x_n - qx_0 + x_0} = e^{2k\pi}/\omega
\]
(13)
For \(x_{n+1} > x_n \gg x_0\), we obtain
\[
\ln x_{n+1} - \ln x_n = 2k\pi/\omega
\]
(14)
that is, the translational error in the \(x\) variable has a negligible impact on the determination of the angular log-frequency \(\omega\) defined by (5) and recovered by the measure (14), as long as the analysis is not too close to the critical point. In contrast, consider for instance \(x_1 < x_2\) comparable to \(x_0\) and which satisfy Eq. (14) with \(k = 1\). In the translated coordinate system, the phase difference between these two points of the term \(Cq^3 \cos[\omega \ln(qx)]\) in the \((H, q)\)-derivative of \(y(x)\) given by (11) is given by
\[
\Delta \phi = \omega \ln \frac{x_2 - qx_0 + x_0}{x_1 - qx_0 + x_0}.
\]
(15)
Since \(0 < q < 1\), \(0 < \Delta \phi < 2\pi\), which implies that the “period” near \(x_0\) is stretched by the translation of \(x\) to the uncorrect critical value. In this case, the log-periodic oscillations disappear for \(x\) of the order of and less than \(x_0\) (see figure 2 of [37]).

Suppose that the log-periodic oscillations for times \(t_c - t < t_c - t_{\text{max}}\) are not well-determined or unobservable. Then, for a given \(q\), the \((H, q)\)-derivative is not well-determined for
\[
t > t_c - \left(t_c - t_{\text{max}}\right)/q.
\]
(16)
The smaller \(q\) is, the larger is the time interval in which the \((H, q)\)-derivative is not determined. This explains our results below that small \(q\)’s do not give good results and tend to over-emphasize slow log-periodic oscillations, hence the existence of the so-called and spurious B platform at very low log-frequency. The existence of spurious low log-frequencies is a general phenomenon which may derive from a variety of origins [9].

In the sequel, we shall not attempt to optimize our analysis of \(t_c\) and shall fix \(t_c\) to the time of the crash or strong correction ending each bubble, keeping in mind that one should thus expect a deterioration of the searched signal close to the end of the time series. In practice, we have not found this to be a limitation for the present purpose of establishing the existence of the log-periodic patterns. However, the situation is not as bright when attempting to use this analysis in a prediction mode, that is, by scanning \(t_c\) with an incomplete time series ending earlier than the crash, as discussed in [37].

### 2.3 Procedure and selection criteria

Our analysis of a given time series consists in the following steps.

1. Read the critical time \(t_c\) as given by the time of the crash or of the strong correction and transform the time series \(p(t)\) into a function \(y(x) = p(t)\) with \(x = t_c - t\).

2. Fix \(q\) and \(H\) to some arbitrary value and construct the corresponding \((H, q)\)-derivative \(D^H_q p(\tau)\) of the function \(y(x)\).

3. Perform a spectral Lomb analysis [8, 25] of \(D^H_q p(\tau)\) in order to identify the presence of possible periodic oscillations: the most significant oscillations occur at those log-frequencies that make the spectrum \(P_N(H, q)\) maximum.

4. Change \(q\) and \(H\) and redo the analysis. In this work, we scan \(H \in (-1, 1)\) with spacing 0.1 and \(q \in (0, 1)\) with spacing 0.05.
Having identified the strongest peak in the spectrum $P_N(H, q)$ and its associated log-frequency, we study their dependence as a function of $H$ and $q$.

Let us illustrate our procedure on the daily evolution of the Dow Jones Industrial Average from 02-Jan-1980 to the “Black Monday” on 19-Oct-1987, shown in Fig. 1.

In Figs. 2, 3 and 4, we present typical plots of the generalized $q$-derivative and its Lomb periodogram for the Dow Jones 1987 Crash. Figure 2 shows the evolution as a function of $\ln(t_c - t)$ of the generalized $q$-derivative (left panel) and their Lomb periodograms (right panel) for fixed $q = 0.65$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The log-periodic structures of $D_q^H p(t)$ in plots (a), (c) and (e) are clearly visible. The amplitude of the log-periodic oscillations decreases with increasing $H$ for fixed $q$. The envelopes of the oscillations can be fitted well by the expression

$$D_m = \left( B' \pm C' \sqrt{C_1^2 + C_2^2} \right) \left( x_m \right)^{\beta - H}$$

(17)

giving the local maxima of the log-periodic function $D(t)$. We observe a peak at $f_1 = 1.06$ in (b), $f_1 = 1.06$ in (d) and $f_1 = 0.94$ in (f). We also see harmonics $f_2 = 1.94$ in (d) and $f_2 = 1.90$ in (f). Figure 3 is for $q = 0.9$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The Lomb peaks in plots (b) and (d) are very high with log-frequencies $f_2 = 2.02$ and 1.96 that may be interpreted as the second harmonics $2f$ of the frequency found in figure 2. We find a peak at $f_1 = 1.15$ in (d) and peaks at $f_1 = 1.01$, $f_2 = 1.87$ and $f_4 = 4.06$, which makes more plausible the existence of harmonics of a fundamental log-periodic component in the signal. Figure 4 is for $q = 0.3$. The log-periodic structures of $D_q^H p(t)$ in plots (a), (c) and (e) are much less obvious visually than in previous figures. Nevertheless, the Lomb spectral analysis shown in panel (b) shows that the log-periodic structure in (a) is significant at two log-frequencies $f_1 = 1.22$ and $f_2 = 1.92$. We note that the almost equal strength of the two overlapping spectral peaks at these two frequencies is probably causing a distortion in their estimation.

In panels (d) and (f) of figure 4, the highest peaks occur at a significantly smaller log-frequency which is found close to 0.32. This log-frequency, which corresponds to approximately 1 to 1.5 full log-periodic oscillation in the data, can usually be associated with noise rather than a genuine signal. Indeed, log-periodicity appears generically when dealing with power laws in the presence of some kind of cumulative noise [9] or with noise with long-range correlations [36]. The most probable log-frequency resulting from noise is given by [36]

$$f_{\text{mp}} = \frac{1.5}{\ln(t_c - t_{\text{min}}) - \ln(t_c - t_{\text{max}})},$$

(18)

where $t_{\text{min}}$ and $t_{\text{max}}$ are respectively the times defining the beginning and the end of the time series. The value 0.32 of the highest peaks in panels (d) and (f) is compatible with $f_{\text{mp}}$ given by [18]. We should therefore exclude from our statistics those pairs $(H, q)$ whose spectra are dominated by the spurious low log-frequency $f_{\text{mp}}$ resulting from noise.

In addition, we should also discard from our statistics the pairs $(H, q)$ whose log-frequencies $f$ of the largest spectral peak are very large, because a large log-frequency corresponds to many oscillations which most often can be associated with high-frequency noise. In Ref. [16], log-periodic components with $2\pi f \geq 14$ were discarded for this reason.

Finally, let us note that panels (d) and (f) of figure 4 also contains secondary peaks at $f_1 = 1.22$ and $f_2 = 1.92$ in (d) and $f_1 = 1.33$ and $f_2 = 1.86$ in (f) which are not far from the log-frequencies previously estimated.
3 Log-periodicity of Wall Street Crashes

We now exploit the procedure described in section 2.3 to analyze seven financial time series ending at

1. the Dow Jones Industrial Average, October 1987 Crash,
2. the Dow Jones Industrial Average, October 1997 strong correction,
3. the S&P 500 Index, October 1987 crash,
4. the S&P 500 Index, October 1997 strong correction,
5. the Nasdaq Index, October 1987 crash,
6. the Nasdaq Index, October 1997 strong correction, and
7. the Nasdaq Index, April 2000 crash.

In the spirit of R. Feynman, we now describe extensively the evidence we have accumulated on these seven time series, in particular by presenting all the evidence that we think relevant, so that the reader can judge for him- or her-self about the validity and robustness of the results.

3.1 Time series ending at the Dow Jones October 1987 Crash

The log-periodic patterns characterizing the bubble phase preceding this crash has been studied parametrically in \([32, 33, 19, 20, 21]\). We synthesize the information obtained by our new non-parametric analyses such as those shown in Figs. 2, 3 and 4 for all pairs \((H, q)\) by plotting in figure 5 the log-frequency \(f\) of the largest spectral peak as a function of an index counting the pairs \((H, q)\). Since we scan \(H \in (-1, 1)\) with spacing 0.1 and \(q \in (0, 1)\) with spacing 0.05, this corresponds to 324 pairs.

- cluster \(P_1\) centered on \(f_1 = 1.04 \pm 0.11\) seems to be associated with a fundamental log-frequency in the data;
- cluster \(P_2\) centered on \(f_2 = 1.98 \pm 0.07\) can be interpreted as the frequency 2\(f_1\) of the second harmonics to the fundamental frequency \(f_1\);
- cluster \(B\) centered on \(f_{mp} \approx 0.3\) can be attributed to the spurious log-frequency defined in (18).

Fig. 6 presents a more detailed description of the log-frequency \(f(H, q)\) of the most significant peak in each Lomb periodogram of the \((H, q)\)-derivative for the Dow Jones 1987 crash, by plotting \(f(H, q)\) as a function of \(H\) and \(q\). The clusters of figure 5 now corresponds to “platforms”. It is convenient to classify all pairs of \((H, q)\) into three classes based on the geometric shape of \(f(H, q)\): W (wedge or wall), P (platform) and B (bottom of valley or basin). In figure 6, we can clearly distinguish platforms \(P_1\) and \(P_2\) centered on \(f_1 = 1.04 \pm 0.11\) and \(f_2 = 1.98 \pm 0.07\). Two wedges

\[\text{1R. Feynman, “Surely you’re joking, Mr. Feynman”, Norton, 1985, page 341: “If you’re doing an experiment, you should report everything that you think might make it invalid – not only what you think is right about it; other causes that could possibly explain your results; ... Details that could throw doubt on your interpretation must be given, if you know them. You must do the best you can — if you know anything at all wrong, or possibly wrong — to explain it. If you make a theory, for example, and advertise it, or put it out, then you must also put down all the facts that disagree with it, as well as those that agree with it. ... In summary, the idea is try to give ALL of the information to help others to judge the value of your contribution; not just the information that leads to judgment in one particular direction or another.”}\]
are also indicated in Fig. 6 as $W_1$ and $W_2$ but their log-frequencies are not distinguishable from those of $P_2$. The spurious cluster $B$ is also clearly visible.

In order to compare the relative values of different $(H, q)$ pairs, it is also instructive to construct in Fig. 7 the spectral height $P_N(H, q)$ of the highest peak of each Lomb periodogram for each $(H, q)$-derivative as a function of $H$ and $q$. By matching Fig. 6 with Fig. 7, we verify that the two parts of $B$ have quite high Lomb peaks, while $W_1$ and $W_2$ in between correspond to valleys in Fig. 7. In Fig. 6, pairs $(H, q)$ with $q > 0.5$ lead to either $f_1$ or to its harmonic $f_2$. Small values of $q$ thus give a weaker signal for log-periodicity. This property will be seen to apply in all other time series discussed below. We also note that on average, for pairs $(H, q)$ with $q > 0.5$, the largest Lomb spectral peak increases with decreasing $H$. This is also illustrated in Figs. 2, 3 and 4.

Define an optimal pair $(\hat{H}, \hat{q})$ satisfying that $P_N(\hat{H}, \hat{q})$ is the maximum of $P_N(H, q)$. Figure 8 shows on the left vertical coordinate and with the thin line with squares a sectional cut of Fig. 7 for $H = -0.7$, that is, plots the highest peak $P_N(\hat{H}, q)$ as a function of $q$. The right vertical coordinate and the thick line marked with circles correspond to a sectional cut of Fig. 6 for the same value $H = -0.7$ and gives the log-frequency of the highest peak as a function of $q$. The strongest spectral power is found for $q = 0.6$ for which the log-frequency is $f_1$.

3.2 Time series ending with the Dow Jones October 1997 strong correction

The log-periodic patterns decorating the price trajectory of the bubble phase preceding this strong correction have been already studied parametrically in [7, 34, 29].

Figure 9 shows the daily evolution of the Dow Jones Industrial Average from 02-Jan-19890 to 19-Oct-1997. On Monday, September 27th, the Dow Jones index suffered from a strong loss of more than 7% that ended a strong bull regime of strongly appreciating prices. Many observers believed on this day that it was the starting point of a crash. The next day, a rally of +5% alleviated fears and started a market phase that lasted approximately 3 months in which the market remained essentially flat. The results of the analysis of the $(H, q)$-derivative of this time series shown in figures 10 to 16 are found to be very similar to those reported previously for the Dow Jones time series of ending with the October 1987 crash. Two novel features appear. First, figure 13 shows the existence of an additional cluster $P_{1/2}$ at $f_{1/2} = 0.59 \pm 0.04$, that it is tempting to interpret as the first sub-harmonic of the fundamental log-frequency $f_1$. Second, for $q = 0.9$, we observe in figure 11 several harmonics beyond the second harmonics, suggesting a very strong log-periodicity with more complex nonlinearity that for the October 1987 crash.

3.3 Time series ending with the S&P 500 Index, October 1987 crash

Figure 17 shows the Daily evolution of the S&P 500 Index from 02-Jan-1980 to the “Black Monday” on 19-Oct-1987, which started the crash of October 1987. The time series is obviously not independent of the Dow Jones time series shown in figure 1 as the S&P 500 Index contains the 30 blue chip stocks constituting the Dow Jones index. However, it is influenced by the behavior of 470 other stocks and we think it useful to carry out a detailed analysis of this time series as well, to compare with the Dow Jones time series and develop a sense about how different realization of the noise and of the signal may modify the log-periodicity.

The results of the analysis of the $(H, q)$-derivative of this time series shown in figures 18 to 23 are found to be very similar to those reported previously in the two previous time series. The same fundamental log-frequency $f_1$ and its harmonics are observed. The only really novel feature is the appearance of what is probably a spurious “wedge” cluster $W$ with log-frequency $f_W = 1.74 \pm 0.07$ shown in figure 21.
3.4 Time series ending with the S&P 500 Index, October 1997 strong correction

Figure 24 shows the daily evolution of the S&P 500 Index from 02-Jan-1990 to 27-Oct-1997. Similarly to the comparison of the Dow Jones time series ending in October 1987 and 1997, we think it useful to compare the S&P 500 time series ending in October 1987 and 1997, to test for the robustness of the log-periodic signal with respect to modifications and/or alterations brought by different realization of the noise and of the signal. Figures 25 to 29 gives the results of the analysis of the \((H, q)\)-derivative of this time series. The results are highly consistent with those obtained in the previous three time series.

3.5 Time series ending with the Nasdaq Index, October 1987 crash

Figure 30 shows the daily evolution of the Nasdaq Index from 11-Oct-1984 to the “Black Monday” on 19-Oct-1987, which started the crash. Figures 31 to 36 show the results of our analysis, which essentially recovers similar results as for the previous time series. The main difference is the weaker evidence of clustering into three clusters \(P_1\) with \(f_1 = 0.88 \pm 0.04\), \(P_2\) with \(f_2 = 1.45 \pm 0.06\) and \(B\) with \(f_B \approx 0.21\) (which corresponds as in other previously analyzed time series to approximately one oscillation).

3.6 Time series ending with the Nasdaq Index, October 1997 strong correction

Figure 37 shows the daily evolution of the Nasdaq Index from 02-Jan-1990 to 27-Oct-1997, at which a strong correction occurred. Figures 38 to 42 show the results of our analysis of this time series, which are very similar to previous ones.

3.7 Time series ending with the Nasdaq Index, April 2000 crash

Figure 43 shows the time evolution of the daily quotation of the Nasdaq Index from 03-Mar-1997 to 14-Apr-2000. This crash corresponds to the burst of the so-called “new economy” bubbles that has spurred in the few preceding years. We refer to [15, 29] for a detailed discussion of this crash and a thorough analysis of its log-periodic signature using a parametric approach as well as the non-parametric Lomb spectral analysis of the residuals to the fit of the Nasdaq index time series to a power law. Figures 44 to 48 show the results of our analysis, which are very similar to previous time series.

4 Summary and discussion of the \((H, q)\)-analysis

By comparing the log-frequency \(f(H, q)\) associated with the highest peak \(P_N(H, q)\) of the Lomb spectrum of the \((H, q)\)-derivative for the seven financial indices reported above, we find striking similarities, in particular in the shape and values of the plateaux and clusters of log-frequencies. Some of these similarities can be attributed to the fact that all these time series are not independent, especially those covering the same time spans which use indices with overlapping stocks. However, the found universality of the fundamental log-frequency is quite remarkable as it seems

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\(^2\)The DJIA is an index of 30 “blue-chip” US stocks. It is the oldest continuing US market index. It is called an “average” because it was originally computed by adding up stock prices and dividing by the number of stocks (the very first average price of industrial stocks, on May 26, 1896, was 40.94) and should ideally represent a correct measure of the state of the economy. The methodology remains the same today, but the divisor has been changed to preserve historical continuity. The standard & Poor 500 index is a capitalization weighted average of 500 of the major stocks on the US market. Thus the S&P 500 includes the 30 stocks constituting the Dow Jones index and it is natural to expect strong correlations between the two, especially at times of strong herding as in speculative bubbles.
Analysis of log-periodicity using the Hilbert transform

Table 1: Synthesis of the results of the analysis of seven price time series using the \((H, \hat{q})\)-derivative. The periods of the analyzed data sets are: from 02-Jan-1980 to 19-Oct-1987 for the Dow Jones and S&P 500 indices, from 11-Oct-1984 to 19-Oct-1987 for the Nasdaq index, from 02-Jan-1990 to 27-Oct-1997 for the Dow Jones, S&P 500 and Nasdaq Indices, and from 03-Mar-1997 to 14-Apr-2000 for the Nasdaq index. The columns \(f_1\), \(f_2\), \(f_3\) and \(f_{1/2}\) are respectively the fundamental log-frequency, its second and third harmonics and its first sub-harmonic. The columns \((H, \hat{q})\), \(\hat{f}\) and \(P_N\) are the optimal pairs and their corresponding log-frequency and their Lomb peak height.

| Crashes      | \(f_1\)       | \(f_2\)       | \(f_3\)       | \(f_{1/2}\)   | \((H, \hat{q})\) | \(\hat{f}\) | \(P_N\) |
|--------------|---------------|---------------|---------------|---------------|-----------------|-------------|---------|
| Dow Jones 87 | 1.04 ± 0.11   | 1.98 ± 0.07   | /             | /             | \((-0.7, 0.60)\) | 0.96        | 580     |
| Dow Jones 97 | 1.04 ± 0.05   | 2.34          | /             | 0.59 ± 0.04   | \((-0.8, 0.60)\) | 1.06        | 503     |
| S&P 500 87   | 1.00 ± 0.11   | 1.97 ± 0.06   | /             | /             | \((-0.3, 0.80)\) | 1.97        | 516     |
| S&P 500 97   | 1.06 ± 0.04   | /             | /             | 0.61 ± 0.05   | \((-0.5, 0.65)\) | 1.06        | 652     |
| Nasdaq 87    | 0.88 ± 0.04   | 1.45 ± 0.06   | /             | /             | \((-0.4, 0.50)\) | 0.87        | 286     |
| Nasdaq 97    | 1.11 ± 0.04   | /             | 2.90 ± 0.03   | 0.51 ± 0.07   | \((-0.4, 0.65)\) | 1.12        | 599     |
| Nasdaq 00    | 0.98 ± 0.04   | /             | /             | 0.48 ± 0.13   | \((-0.5, 0.20)\) | 1.04        | 186     |

We summarize in Table 1 all the results presented in previous figures by listing the most significant log-frequencies for each of the seven time series. Apart from the Nasdaq time series ending on Oct. 1987, all the other 6 time series give a “fundamental” log-frequency which is remarkably universal and compatible with a universal value close to 1. This value corresponds to a preferred scaling ratio between successive maxima of the oscillations equal to \(\lambda = e^{1/f} \approx 2.7\). One can also observe the presence of harmonics and of the first sub-harmonic of this fundamental log-frequency, which reinforce the evidence for the existence of a discrete scaling invariance: indeed, the sub-harmonic corresponds to \(\lambda^2\), that is, to two steps of the discrete hierarchy; as for the harmonics, any nonlinearity in the price dynamics will produce them from a fundamental log-frequency. Taking into account of the log-frequencies \(f_1\), \(f_2\) and \(f_3\), excluding the Nasdaq 1987 time series, and assuming that all these log-frequencies derive from a universal fundamental value \(f\), our best estimate for \(f\) is then \(f = 1.02 \pm 0.05\). The corresponding preferred scaling ratio is then \(\lambda = 2.67 \pm 0.12\), which is in excellent agreement with that from other analyses [33, 27]. The standard deviation of \(\lambda\) is evaluated by the formulae \(\sigma_\lambda = e^{1/f} \sigma_f / f^2\). Our analysis also suggest that the most sensitive range for \(q\) lies in the interval \(q \in [0.6, 0.85]\).

The Nasdaq 87 time series seems to belong to a different class with a significantly smaller fundamental frequency \(f_1\). This may be due to the fact that the Nasdaq was more like an “emerging” market before the October 1987 Crash, and thus shared the property of having larger fluctuations, as found in a recent systematic study of other emerging markets [17].

5 Analysis of log-periodicity using the Hilbert transform
5.1 Definition and useful properties of the Hilbert transform

The Hilbert transform (HT) of a signal \( x(t) \) is defined by the equation

\[
\hat{x}(s) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(t)}{s - t} dt,
\]

(19)

where the integral is the Cauchy principal value. The reconstruction formula

\[
x(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(s)}{t - s} ds
\]

(20)
defines the inverse HT. Expression (19) shows that \( \hat{x}(s) = (1/\pi t) \otimes x(t) \) is the convolution of \((1/\pi t)\) with \(x(t)\), which by taking the Fourier transform implies that

\[
\hat{X}(\nu) = -j \text{sign}(\nu) X(\nu),
\]

(21)

where \(j^2 = -1\) and \(X(\nu)\) (respectively \(\hat{X}(\nu)\)) is the Fourier transform of \(x(t)\) (respectively \(\hat{x}(s)\)). This means that the HT produces a \(-\pi/2\) radian phase shift for the positive frequency components of the input \(x(t)\), while the amplitude does not change. A signal \(x(t)\) and its HT \(\hat{x}(s)\) have the same amplitude spectrum, the same autocorrelation function, they are orthogonal and the HT of \(\hat{x}(s)\) is \(-x(t)\).

The HT is thus an ideal tool for detecting an arbitrary phase function \(P(t)\) in a signal of the form \(x(t) = A(t) \cos(\theta(t)) + n(t)\), where \(n(t)\) is a noise with small amplitude \((n \ll A)\), and \(A(t)\) is a slowly varying amplitude modulation. The phase \(\theta(t)\) is in principle recovered by the formula

\[
\theta(t) = \tan^{-1}[\hat{x}(t)/x(t)].
\]

(22)

However, the phase obtained in this way consists in saw-tooth peaks that drop sharply from \(\pi\) down to \(-\pi\). It is thus necessary to unwrap the phase to obtain a smooth continuous phase.

An example is given in figure 49 which shows the function

\[
x(t) = \epsilon(t) + \begin{cases} 
2 \sin(2\pi x) - 4, & 0 < x \leq 3, \\
\sin(4\pi x), & 3 < x \leq 10, \\
3 \sin(2\pi x), & 10 < x \leq 20.
\end{cases}
\]

(23)

where \(\epsilon(t)\) is normally distributed, which consists of a high-frequency burst in a low-frequency background. The unwrapped phase \(\theta(t)\) is plotted in Fig. 50. The abrupt translation of \(x(t)\) at \(t = 0.3\) leads to an incorrect determination of the phase in this interval, and the corresponding distortion also spreads in the high-frequency interval approximately up to \(t = 5.5\). Otherwise, the abrupt change of frequency is correctly detected at \(t = 10\) notwithstanding the abrupt change in amplitude.

5.2 Hilbert transform to financial time series

Before applying the HT transform, we prepare the data (financial time series) by transforming the time variable \(t\) into \(x \equiv \ln(t_c - t)\), where \(t_c\) is fixed as in the \((H, q)\)-analysis to the time of the realized crash. The new variable \(\tau\) is such that log-periodic oscillations in the variable \(t\) become regular periodic oscillations in the variable \(x\). We then fit the price time series as a linear function of \(x\). The residuals of this fit for the two S&P 500 price time series ending with the October 1987 and the October 1997 correction are shown in figures 51 and 52. We then apply the Hilbert transform to these residuals and obtain the corresponding unwrapped phase in figures 53 and 54. Apart from jumps and distortions at the boundaries due to fast oscillations and rapid changes of amplitudes, the
unwrapped phases provide a reliable determination of the log-frequency in the vicinity of \( f = 1 \), in agreement with the results obtained above with the \((H, q)\)-analysis.

As an attempt to improve the HT, we first smooth the financial time series in two steps, by first performing a spline interpolation with 100 evenly spaced sampling points in \( \ln(t_c - t) \) and then applying the Savitzky-Golay filter with centered window of width 21 and six-order polynomials. Such smoothing will alleviate some of the problems in the HT coming from fast-varying amplitudes. The resulting residuals (that is, filtered versions of the curves shown in figures 51 and 52) are given in figures 55 and 56. The corresponding unwrapped phases obtained by taking the HT of these data are shown in figures 57 and 58. The slope of these unwrapped phases provide a good estimation of the log-periodic frequency close to 1, in very good agreement with the \((H, q)\) analysis.

6 Conclusion

In summary, we have applied two non-parametric analyses to test the hypothesis that log-periodicity is present in financial time series preceding crash or strong corrections. The \((H, q)\)-derivative and the Hilbert transform performed on the detrended price in the \( \ln(t_c - t) \) variable are both consistent with the existence of a strong log-periodic signal with log-frequency \( f \approx 1 \).

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Figure 1: The daily evolution of the Dow Jones Industrial Average from 02-Jan-1980 to the “Black Monday” on 19-Oct-1987.
Figure 2: Time dependence in the variable $\ln(t_c - t)$ of the $(H,q)$-derivative (left panels) for the Dow Jones time series shown in figure 1 and their Lomb spectra (right panels) for fixed $q = 0.65$ and varying $H$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The log-periodic structures of $D_q^H p(t)$ in plots (a), (c) and (e) is visible to the naked eye. The amplitude of the log-periodic oscillations decreases with increasing $H$ for fixed $q$. We observe peaks at $f_1 = 1.06$ in (b), at $f_1 = 1.06$ in (d) and at $f_1 = 0.94$ in (f). We also find harmonics $f_2 = 1.94$ in (d) and $f_2 = 1.90$ in (f). Recall the definition (5,6) of the log-frequency.
Figure 3: Same as Fig. 2 but for $q = 0.9$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The peaks of the Lomb spectra in plots (b) and (d) are very high with log-frequencies $f_2 = 2.02$ and $1.96$ that can be interpreted as harmonics $2f$ of the log-frequencies observed in Fig. 2. We also see a peak at $f_1 = 1.15$ in (d) and peaks at $f_1 = 1.01$, $f_2 = 1.87$ and $f_4 = 4.06$ in (f), suggesting the observation of several levels a discrete hierarchy.
Figure 4: Same as Fig. 2 but for $q = 0.3$. The log-periodic structures of $D^H_q p(t)$ in plots (a), (c) and (e) are much weaker than in previous figures. Plot (b) illustrates that the log-periodic structure in (a) is nevertheless very significant with the log-frequencies $f_1 = 1.22$ and $f_2 = 1.92$ corresponding to the highest peaks of the spectrum. In the plots (d) and (f), we also observe secondary peaks at $f_1 = 1.22$ and $f_2 = 1.92$ in (d) and $f_1 = 1.33$ and $f_2 = 1.86$ in (f) that are consistent with previous values. However, the highest peak is in both cases found for a very low log-frequency $f_{mp} = 0.32$, that can be shown to result from the trend induced by this choice of $q$. See text for an explanation.
Figure 5: For the Dow Jones time series shown in figure [1], we report all the log-frequencies corresponding to the highest peak of the Lomb spectrum obtained for each of the pair $(H, q)$ that have been systematically sampled with $H \in ] - 1, 1 [$ with spacing 0.1 and with $q \in ] 0, 1 [$ with spacing 0.05, providing a total of $18 \times 18$ “best” log-frequencies. The abscissa is counting the 324 pairs with an arbitrary choice of indexing. Three clusters are clearly delineated: cluster $P_1$ with $f_1 = 1.04 \pm 0.11$, cluster $P_2$ with $f_2 = 1.98 \pm 0.07$ and cluster $B$ with a log-frequency close to 0.3.
Figure 6: Dependence of the log-frequency $f(H, q)$ of the most significant peak in each Lomb spectrum of the $(H, q)$-derivative for the Dow Jones financial time series ending at the October 1987 crash. Two platforms $P_1$ and $P_2$ with $f_1 = 1.04 \pm 0.11$ and $f_2 = 1.98 \pm 0.07$ can be observed.
Figure 7: Dependence of the highest peak $P_N(H, q)$ in each Lomb spectrum of the $(H, q)$-derivative for the Dow Jones time series ending at the October 1987 crash.
Figure 8: Right vertical axis and thin line marked with squares: dependence as a function of $q$ for fixed $H = -0.7$ of the highest peak $P_N(H, q)$ of the $(H, q)$-derivative of the Dow Jones time series ending at the October 1987 crash. Left vertical axis and thick line marked with circles: dependence as a function of $q$ for fixed $H = -0.7$ of the log-frequency associated with these highest spectral peaks.
Figure 9: The daily evolution of the Dow Jones Industrial Average from 02-Jan-1990 to 27-Oct-1997.
Figure 10: The evolution of the generalized $q$-derivative (left panels) for the Dow Jones time series ending at the October 1997 strong correction and their Lomb spectra (right panels) for fixed $q = 0.7$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The highest peaks in the right panels are: (b) $f_1 = 0.96$ and $f_2 = 2.34$; (d) $f_1 = 1.05$ and $f_2 = 2.31$; and (f) $f_1 = 1.04$ and $f_2 = 2.21$. 
Figure 11: Same as Fig. 10 for $q = 0.9$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The log-frequencies and the associated harmonics are remarkably well-defined in the Lomb spectra: (b) $f_1 = 1.10$, $f_2 = 2.36$, $f_3 = 3.18$, $f_4 = 4.21$, $f_5 = 5.74$, $f_6 = 6.64$ and $f_7 = 7.21$; (d) $f_1 = 1.12$, $f_2 = 2.35$, $f_3 = 3.21$, $f_4 = 4.28$, $f_5 = 5.19$, $f_6 = 5.87$ and $f_7 = 7.22$; and (f) $f_1 = 1.10$, $f_2 = 2.25$, $f_3 = 3.19$ and $f_4 = 3.91$. The amplitude of the log-periodic oscillations decreases with increasing $H$ for fixed $q$. 
Figure 12: Same as Fig. 10 for $q = 0.4$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The log-periodic oscillations are ambiguous and the extracted log-frequencies in (d) and (f) are close to the spurious log-frequency $f_{mp}$. We can also see peaks at $f_{1/2} = 0.52$ in (b), $f_{1/2} = 0.61$ in (d) and $f_{1/2} = 0.65$ in (f), which may be probably interpreted as sub-harmonics of the fundamental log-frequency $f_1$. 


Figure 13: Same as figure 5 for the Dow Jones time series ending at the October 1997 strong correction. Four clusters $P_1$ with $f_1 = 1.04 \pm 0.05$, $P_2$ with $f_2 \equiv 2.34$, $P_{1/2}$ with $f_{1/2} = 0.59 \pm 0.04$ and $B$ are clearly visible. The small log-frequency $f_B = 0.21$ of cluster $B$ corresponds to only one oscillation in the $(H, q)$-derivative.

Figure 14: Dependence of the log-frequency $f(H, q)$ of the most significant peak in each Lomb periodogram of the $(H, q)$-derivative for the Dow Jones time series ending with the October 1997 strong correction. The two platforms $P_1$ and $P_2$ have log-frequencies of $f_1 = 1.04 \pm 0.05$ and its $2f$ harmonic $f_2 \equiv 2.34$, while $P_{1/2}$ shows probably what can be interpreted as the subharmonic $f_{1/2} = 0.59 \pm 0.04$. 
Figure 15: Dependence of the highest peak $P_N(H, q)$ of each Lomb periodogram of the $(H, q)$-derivative for the Dow Jones time series ending with the October 1997 strong correction.
Figure 16: Dependence as a function of $q$ for fixed $H = -0.8$ of the highest spectral peak $P_N(\hat{H}, q)$ of the $(H, q)$-derivative, shown on the left vertical axis with thin line marked with squares and of the associated log-frequencies $f(\hat{H}, q)$ on the right vertical axis with the thick line marked with circles, for the Dow Jones time series ending with the October 1997 strong correction.
Figure 17: Daily evolution of the S&P 500 Index from 02-Jan-1980 to the “Black Monday” on 19-Oct-1987.
Figure 18: Left panels: evolution of the generalized $q$-derivative for the S&P 500 time series ending with the October 1987 Crash and their Lomb periodograms (right panel) for fixed $q = 0.6$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The log-periodic structures of $D_q^H p(t)$ in plots (a), (c) and (e) are clearly visible. The amplitude of the log-periodic oscillations decreases with increasing $H$ for fixed $q$. The envelops of the oscillations can be fitted very well with Eq. [17]. We see Lomb peaks at $f_1 = 1.03$ in (b), $f_1 = 1.06$ in (d) and $f_1 = 0.91$ in (f) which are very significant. We can also observe the harmonics $f_2 = 1.65$ in (b) which is reddened by the highest peak and $f_2 = 1.92$ in (d).
Figure 19: Same as Fig. 18 for $q = 0.85$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The spectral peaks in panels (b) and (d) are very high with log-frequencies $f_2 = 2.03$ and 1.97 corresponding to the second harmonics of $f_1$. We can also see a peak at $f_1 = 1.14$ in (d) and peaks at $f_1 = 0.99$, $f_2 = 1.90$, $f_3 = 3.10$ and $f_4 = 4.10$ in (f) that suggest a strong log-periodicity with its several harmonics.
Figure 20: Same as figure 5 for the S&P 500 time series ending at the October 1987 crash. Four clusters can be observed: $P_1$ with $f_1 = 1.00 \pm 0.11$, $P_2$ with $f_2 = 1.97 \pm 0.06$ and $B$ with $f_B = 0.21$ probably associated with the spurious log-frequency $f_{mp}$; the fourth “wedge” cluster $W$ has log-frequency $f_W = 1.74 \pm 0.07$.

Figure 21: Dependence of the log-frequency $f(H, q)$ of the most significant peak in each Lomb spectrum of each $(H, q)$-derivative for the S&P 500 time series ending with the October 1987 crash. We observe two platforms $P_1$ and $P_2$ with $f_1 = 1.00 \pm 0.11$ and $f_2 = 1.97 \pm 0.06$ respectively.
Figure 22: Dependence of the highest peak $P_N(H,q)$ in each Lomb spectrum of the $(H,q)$-derivative for the S&P 500 time series ending with the October 1987 crash.
Figure 23: Dependence as a function of $q$ for fixed $H = -0.3$ of the highest peak $P_N(\hat{H}, q)$ shown as the thin line marked with squares with scales on the left vertical axis and of the associated log-frequencies $f(\hat{H}, q)$ shown as the thick line marked with circles with scales on the right vertical axis, for the S&P 500 time series ending with the October 1987 crash.
Figure 24: Daily evolution of the S&P 500 Index from 02-Jan-1990 to 27-Oct-1997.
Figure 25: Evolution of the \((H, q)\)-derivative (left panel) for the S&P 500 time series ending with the 1997 strong correction and their Lomb spectra (right panel) for fixed \(q = 0.65\): (a-b) \(H = -0.9\); (c-d) \(H = 0.1\); and (e-f) \(H = 0.9\). The log-periodic structures of \(D^H_q p(t)\) in plots (a), (c) and (e) are clearly visible. The amplitude of the log-periodic oscillations decreases with increasing \(H\) for fixed \(q\). The envelops of the oscillations can be fitted very well with Eq. (17). We see Lomb peaks at \(f_1 = 1.07\) in (b), \(f_1 = 1.06\) in (d) and \(f_1 = 1.06\) in (f), which are very significant. We can also distinguish the sub-harmonics in the right panels.
Figure 26: Same as figure 5 for the S&P 500 time series ending at the October 1997 strong correction. Three clusters can be observed: $P_1$ with $f_1 = 1.06 \pm 0.04$, $P_{1/2}$ with $f_{1/2} = 0.61 \pm 0.05$ and $B$ with $f_B = 0.21$.

Figure 27: Dependence of the log-frequency $f(H, q)$ of the most significant peak in each Lomb periodogram of the $(H, q)$-derivative for the S&P 500 time series ending with the October 1997 strong correction. We see clearly two platforms $P_1$ and $P_{1/2}$ with $f_1 = 1.06 \pm 0.04$ and $f_{1/2} = 0.61 \pm 0.05$. 
Figure 28: Dependence of highest peak $P_N(H, q)$ in each Lomb periodogram of the $(H, q)$-derivative for the S&P 500 time series ending with the October 1997 strong correction.
Figure 29: Dependence with $q$ for $H = -0.5$ of the highest spectral peak $P_N(\hat{H}, q)$ shown as the thin line marked with squares with left vertical scales and of the associated log-frequencies $f(\hat{H}, q)$ shown as the thick line marked with circles with right vertical axis, for the S&P 500 time series ending with the October 1997 strong correction.
Figure 30: Daily evolution of the Nasdaq Index from 11-Oct-1984 to the “Black Monday” on 19-Oct-1987.
Figure 31: The time evolution of the generalized $q$-derivative (left panels) for the Nasdaq index time series ending with the October 1987 Crash and their Lomb spectra (right panels) for fixed $q = 0.5$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. Three log-periodic oscillations of $D^H_q p(t)$ in plots (c) and (e) are clearly visible, while (a) exhibits a strong trend. The amplitude of the log-periodic oscillations decreases with increasing $H$ for fixed $q$. The envelops of the oscillations can be fitted very well with Eq. (17). In the right panels, we observe peaks at the following log-frequencies: $f_1 = 0.85$ in (b), $f_1 = 0.88$ in (d) and $f_1 = 0.84$ in (f).
Figure 32: Same as Fig. 31 for \( q = 0.9 \); (a-b) \( H = -0.9 \); (c-d) \( H = 0.1 \); and (e-f) \( H = 0.9 \). The log-periodic structures of \( D_{q}^{H}p(t) \) in plots (a), (c) and (e) are clearly visible. The two highest Lomb peaks lie at \( f_{2} = 1.60 \) in (b), \( f_{2} = 1.39 \) in (d) and \( f_{2} = 1.37 \) in (f).
Figure 33: Same as figure 5 for the Nasdaq time series ending at the October 1987 crash. Three clusters can be observed: $P_1$ with $f_1 = 0.88 \pm 0.04$, $P_2$ with $f_2 = 1.45 \pm 0.06$ and $B$ with $f_B \approx 0.21$ (which corresponds as in other previously analyzed time series to approximately one oscillation). Compared to previous time series, the clusters are less clearly defined with a continuous range of log-frequencies joining clusters $P_1$ and $P_2$. 
Figure 34: Dependence of the log-frequency \( f(H, q) \) of the most significant peak in each Lomb periodogram of the \((H, q)\)-derivative for the Nasdaq index time series ending at the October 1987 crash. One can observe two types of platforms \( P_1 \) and \( P_2 \) with \( f_1 = 0.88 \pm 0.04 \) and \( f_2 = 1.45 \pm 0.06 \).
Figure 35: Dependence of the highest peak $P_N(H, q)$ in each Lomb periodogram of the $(H, q)$-derivative for the Nasdaq index time series ending at the October 1987 crash.
Figure 36: Dependence with $q$ for fixed $H = 0.4$ of the highest peak $P_N(\hat{H}, q)$ of the Lomb spectrum (thin line marked with squares) with scales given by the left vertical axis and of the associated log-frequencies $f(\hat{H}, q)$ shown as the thick line marked with circles with scales given by the right vertical axis, for the Nasdaq index time series ending with the October 1987 crash.
Figure 37: Daily evolution of the Nasdaq Index from 02-Jan-1990 to 27-Oct-1997.
Figure 38: Time evolution of the generalized $q$-derivative (left panels) for the Nasdaq index ending with the October 1997 correction and their Lomb periodograms (right panels) for fixed $q = 0.65$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. The Lomb peaks at $f_1 = 1.14$ in (b), $f_1 = 1.13$ in (d) and $f_1 = 1.10$ in (f) are very significant.
Figure 39: Same as figure 5 for the Nasdaq time series ending at the October 1997 correction. Several clusters can be observed: $P_1$ with log-frequency $f_1 = 1.11 \pm 0.04$, $P_{1/2}$ with $f_{1/2} = 0.51 \pm 0.07$, $W + P_3$ with $f_3 = 2.90 \pm 0.03$ and $B$ with $f_B = 0.21$. See figure 40 for a better definition of the clusters.
Figure 40: Dependence of the log-frequency $f(H,q)$ of the most significant peak in each Lomb periodogram of the $(H,q)$-derivative for the Nasdaq index time series ending with the October 1997 crash. One can identify several structures: platform $P_1$ with log-frequency $f_1 = 1.11 \pm 0.04$, $P_{1/2}$ with $f_{1/2} = 0.51 \pm 0.07$, $W + P_3$ with $f_3 = 2.90 \pm 0.03$. 
Figure 41: Dependence of the highest peak $P_N(H, q)$ in each Lomb periodogram of the $(H, q)$-derivative for the Nasdaq 1997 crash.
Figure 42: Dependence with $q$ for fixed $H = -0.4$ of the highest peak $P_N(\hat{H}, q)$ shown as a thin line marked with squares with left vertical axis and of the associated log-frequencies $f(\hat{H}, q)$ shown as the thick line marked with circles with right vertical axis, for the Nasdaq index time series ending at the October 1997 correction.
Figure 43: Daily evolution of the Nasdaq Index from 03-Mar-1997 to 14-Apr-2000.
Figure 44: Time evolution of the generalized $q$-derivative (left panels) for the Nasdaq index time series ending with the April 2000 crash and their Lomb periodograms (right panels) for fixed $q = 0.7$: (a-b) $H = -0.9$; (c-d) $H = 0.1$; and (e-f) $H = 0.9$. 
Figure 45: Same as figure [5] for the Nasdaq time series ending at the April 2000 crash. Several clusters can be observed: $P_1$ with log-frequencies $f_1 = 0.98 \pm 0.04$, $P_{1/2}$ with $f_{1/2} = 0.48 \pm 0.13$ and $B$ with $f_B = 0.25$. Two small clusters are also observed whose significance is not obvious.

Figure 46: Dependence of the log-frequency $f(H,q)$ of the most significant peak in each Lomb periodogram of the $(H,q)$-derivative for the Nasdaq index time series ending with the April 2000 crash. The two main plateaux are $P_1$ with log-frequency $f_1 = 0.98 \pm 0.04$, $P_{1/2}$ with $f_{1/2} = 0.48 \pm 0.13$. 
Figure 47: Dependence of the highest peak $P_N(H, q)$ in each Lomb periodogram of the $(H, q)$-derivative for the Nasdaq index time series ending with the April 2000 crash.
Figure 48: Dependence with $q$ for fixed $H = -0.5$ of the highest peak $P_N(\hat{H}, q)$ shown as the thin line marked with squares with the left vertical axis and of the associated log-frequencies $f(\hat{H}, q)$ shown as the thick line marked with circles with the right vertical axis, for the Nasdaq index time series ending with the April 2000 crash.
Figure 49: An example of function (23) exhibiting a combination of an abrupt translation with phase velocity and amplitude changes.

Figure 50: Unwrapped phase of the function shown in Fig. [49].
Figure 51: Difference between the S&P 500 price time series ending with the October 1987 crash and the fit with $A - B \ln(t_c - t)$, where $A$ and $B$ are two constants, as a function of $\ln(t_c - t)$. $t_c$ is the time of the crash.

Figure 52: Same as figure 51 for the S&P 500 price time series ending with the October 1997 correction.
Figure 53: Unwrapped phase $\theta$ extracted from the residuals in Fig. 51 using the Hilbert transform. The slopes of the three linear segments are respectively the log-frequencies $f = 0.9$, $f = 0.9$ and $f = 1.9$. This is in agreement with the existence of a fundamental log-frequency and its harmonics. The jumps in the phase are due to the fast oscillations of small amplitudes in the interval $\ln(t_c - t) \in (5, 7)$.

Figure 54: Unwrapped phase $\theta$ extracted from the residuals shown in Fig. 52 using the Hilbert transform. Two parallel linear segments are indicated by the straight lines whose slopes $\approx 1.0$ give the value of the log-frequency. The jumps are caused by the fast fluctuations in the second oscillation in Fig. 52 near $\ln(t_c - t) = 5.7$. The fluctuations in the fourth oscillation for $\ln(t_c - t) > 7.4$ in Fig. 52 completely spoil the linear dependence of the unwrapped phase.
Figure 55: The residuals for the October 1987 crash time of the S&P 500 index from the smoothened data using spline interpolation and then Savitzky-Golay filter. The 100 interpolation points are evenly sampled in $\ln(t_c - t)$. 
Figure 56: Same as Fig. 55 but for the October 1997 correction of the S&P 500 index.

Figure 57: The unwrapped phase $\theta$ extracted from the residuals in Fig. 55 using the Hilbert transform. Small oscillations within one oscillation lead to the undulations in the unwrapped phase. The two straight lines define log-frequencies equal respectively to $f = 0.85$ and $f = 1.8$. 
Figure 58: The unwrapped phase $\theta$ extracted from the residuals in Fig. using the Hilbert transform. The main part of the phase can be fitted to a straight line of slope giving the log-periodic frequency $\tilde{f} = 1.0$. Small oscillations within one oscillation lead to the undulations in the unwrapped phase.