SEMISTABLE POINTS WITH RESPECT TO REAL FORMS

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Abstract. We consider actions of real Lie subgroups $G$ of complex reductive Lie groups on Kählerian spaces. Our main result is the openness of the set of semistable points with respect to a momentum map and the action of $G$.

1. Introduction

Throughout this article we consider a compact Lie group $U$ and a Lie subgroup $G$ of the complexified group $U^{\mathbb{C}}$. The corresponding Lie algebras will be denoted by German letters. We will assume that $G$ is compatible with the Cartan decomposition $U^{\mathbb{C}} = U \exp i \mathfrak{u}$ of $U^{\mathbb{C}}$. This means that we have a diffeomorphism $K \times \mathfrak{p} \to G$, $(k, \xi) \mapsto k \exp \xi$, where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i \mathfrak{u}$.

Let $Z$ be a complex space endowed with a holomorphic action of $U^{\mathbb{C}}$ and a $U$-invariant Kählerian structure $\omega$. For simplicity we will assume that $\omega$ is smooth when restricted to any smooth submanifold of $Z$. We also assume that there is a continuous $U$-equivariant momentum map $\mu: Z \to \mathfrak{u}^*$ which is smooth on every smooth submanifold of $Z$. In particular, this means that the restriction of $\omega$ to a smooth $U^{\mathbb{C}}$-stable submanifold $M$ of $Z$ is given by a Kählerian form in the usual sense. Hence $M$ is a Riemannian manifold. For $\xi \in \mathfrak{u}$ we have the function $\mu^\xi: M \to \mathbb{R}$, $\mu^\xi(z) := \mu(z)(\xi)$, with gradient $\nabla(\mu^\xi)(z) = d \bigg|_0 \exp(it\xi) \cdot z = (i\xi)_Z(z) = J\xi_Z(z)$. Here $J$ denotes the complex structure on the tangent bundle $TM$ and $\xi_Z$ is the vector field on $Z$ corresponding to the action of the one parameter group $t \mapsto \exp(t\xi)$.

For a subspace $\mathfrak{m}$ of $\mathfrak{u}$ we have the map $\mu_\mathfrak{m}: Z \to \mathfrak{m}^*$ which is given by composing $\mu$ with the adjoint of the inclusion map $\mathfrak{m} \to \mathfrak{u}$. We call $\mu_\mathfrak{m}$ the restriction of $\mu$ to $\mathfrak{m}$. For a subset $Q$ of $Z$ we set $\mathcal{S}_G(Q) := \{z \in Z; \overline{G \cdot z \cap Q} \neq \emptyset\}$. Here $\overline{G \cdot z}$ denotes the closure of $G \cdot z$ in $Z$. For $\beta \in i\mathfrak{p}^*$ let $\mathcal{M}_\mathfrak{ip}(\beta) := \mu_\mathfrak{ip}^{-1}(\beta)$ and since $\beta = 0$ plays a prominent role we set $\mathcal{M}_\mathfrak{ip} := \mathcal{M}_\mathfrak{ip}(0)$.

The set of semi-stable points of $Z$ with respect to $\mu_\mathfrak{ip}$ and the $G$-action is by definition the set $\mathcal{S}_G(\mathcal{M}_\mathfrak{ip})$. In [HS05] it is shown that $\mathcal{S}_G(\mathcal{M}_\mathfrak{ip})$ is open when $U$ is abelian. Here we give a proof of a more general result.

Theorem 1.1. For every $\beta \in i\mathfrak{p}^*$ the set $\mathcal{S}_G(\mathcal{M}_\mathfrak{ip}(\beta))$ is open in $Z$.

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2. Outline of the proof

First we consider the case $\beta = 0$. It is sufficient to show that every point $x \in M_{ip}$ has an open neighborhood $V(x)$ of $x$ in $Z$ such that $V(x) \subset S_G(M_{ip})$, since in this case $S_G(M_{ip}) = G \cdot V$ is open in $Z$ where $V$ denotes the union of the sets $V(x)$.

Let $\| \cdot \| : u^* \to \mathbb{R}$ denote a $U$-invariant norm function and let $\eta_{ip} : Z \to \mathbb{R}, \eta_{ip}(z) = \frac{1}{2}\|\mu_{ip}(z)\|^2$. Using convexity properties of the momentum map with respect to special commutative subgroups (see section 4), we will show the existence of $G$-stable neighborhoods $\Omega$ of points in $M_{ip}$ such that for some $r > 0$ the sets $V := \Omega \cap \eta_{ip}^{-1}(0, r)$ are relatively compact in $Z$.

For simplicity assume now that $Z$ is smooth. The flow $\psi$ of the gradient vector field $\nabla \eta_{ip}$ is tangent to the $G$-orbits. Moreover for $z \in V$ the integral curve $t \mapsto \psi_z(t)$ exists for all negative $t$ and is contained in $V$. If $t_n \to -\infty$, then possibly after going over to a subsequence a limit point $z_0 = \lim \psi_z(t_n)$ exists. We have $z_0 \in V \cap G \cdot z$ and $z_0$ is a critical point of $\eta_{ip}$. It then remains to show that $V$ can be chosen such that $M_{ip} \cap V$ equals the set of critical point of $\eta_{ip}$ which are contained in $V$.

The case where $\beta$ is arbitrary is done by shifting the momentum map. The usual procedure is to endow the coadjoint orbit $O := U \cdot \beta$ with the structure of a Kählerian manifold such that $\alpha \mapsto -\alpha$ defines a momentum map on $O$. Since $O$ is compact the $U$-action extends to a holomorphic $U^C$-action $(g, \alpha) \mapsto g \cdot \alpha$, i.e., $O = U \cdot \beta = U^C \cdot \beta$ holds. Using the shifting procedure means to replace $Z$ by $Z \times O$ on which $(z, \alpha) \mapsto \mu(z) - \alpha$ defines a momentum map $\mu$ and study the set of semistable points with respect to $\mu$ as well as its relation to $S_G(M_{ip}(\beta))$. In fact in the case where $G = U^C$ this gives rather directly Theorem 1.1. In the general case one has to take into account the basic observation that $G \cdot \beta = K \cdot \beta$ holds for $\beta \in ip^*$.

3. The structure of compatible subgroups

As before let $U$ be a compact Lie group, $U^C$ its complexification and let $G = K \exp p$ be a compatible subgroup of $U^C$.

If $G = K \exp(p)$ is not closed in $U^C$, then its topological closure $\overline{G} = K \exp(p)$ is a closed Lie-subgroup of $U^C$ which is compatible with the Cartan decomposition of $U^C$. Moreover since $\overline{G}z = Gz$ we have $S_G(M_{ip}(\beta)) = S_{\overline{G}}(M_{ip}(\beta))$ for $\beta \in ip^*$. This implies that for the proof of Theorem 1.1 we may assume that $G$ is a compatible closed subgroup of $U^C$. The main advantage of $G$ being closed is that $K$ is a compact subgroup of $U$ and that for a maximal subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ we have $G = KAK$ for $A := \exp \mathfrak{a}$.

The remaining part of this section is not used until section 7 where we need precise information about the coadjoint action of $U$. The main input there is the fact that the coadjoint orbit of $U$ through a point $\beta \in ip^*$ is in a natural way a complex $U^C$-homogeneous manifold $O$ such that the orbit $G \cdot \beta$ equals $K \cdot \beta$.

To begin with recall that the group $U^C$ has a unique structure of an affine algebraic group. Its algebra of regular functions consists of the $U$-finite holomorphic functions on $U^C$. If $G$ is any subgroup of $U^C$, then its algebraic (respectively analytic) Zariski closure is the smallest closed algebraic (respectively analytic) subvariety of $U^C$ which contains $G$. The Zariski closure is automatically an algebraic (respectively analytic) subgroup of $U^C$. Moreover in the compatible case the notion of algebraic and analytic Zariski closure of $G$ coincide. The simplest way to see this is to use the fact that any closed complex
subgroup of $U^C$ which is stable with respect to the Cartan involution $\Theta: U^C \to U^C$, $\Theta(u \exp i\xi) := u \exp(-i\xi)$ where $u \in U$ and $\xi \in u$ is the complexification of its $\Theta$-fixed points. A slightly more precise statement is the following simple

Lemma 3.1. Let $G = K \exp \mathfrak{p}$ be a connected compatible Lie subgroup of $U^C$ and let $U_0$ be the smallest closed subgroup of $U$ which contains $\exp(\mathfrak{k} + i\mathfrak{p})$. Then the Zariski closure of $G$ in $U^C$ equals $(U_0)^C = U_0 \exp(iu_0)$. In particular $G$ is compatible with the Cartan decomposition of $(U_0)^C$. □

Using Lemma 3.1 we may assume in the proof of Theorem 1.1 that $U^C$ coincides with the Zariski closure of $G$. The proof of the following statement is left to the reader.

Lemma 3.2. Let $G$ be a connected Zariski-dense Lie subgroup of $U^C$ and $N$ a connected complex Lie subgroup of $U^C$ which contains $G$. Then every connected normal complex subgroup of $N$ is a normal subgroup of $U^C$. In particular $N$ is normalized by $U^C$. □

Remark. If $U$ is semisimple, then $N = U^C$.

An application of the lemma shows that the connected complex Lie subgroup $G^C$ of $U^C$ with Lie algebra $\mathfrak{g}^C := \mathfrak{g} + i\mathfrak{g}$ is a normal complex Lie subgroup of $U^C$ and also that the connected complex Lie subgroup with Lie algebra $\mathfrak{g} \cap i\mathfrak{g}$ is normalized by $U^C$. Consequently $\mathfrak{g}^C$ and $\mathfrak{g} \cap i\mathfrak{g}$ are ideals in $u^C$.

On the level of Lie algebras the structure of $G$ is described by the following

Proposition 3.3. Let $G$ be a connected compatible Zariski-dense Lie subgroup of $U^C$. Then there exist ideals $u_0$ and $u_1$ in $u$, such that $\mathfrak{g} = \mathfrak{g}_0 + u_1^C$ where $\mathfrak{g}_0 \subset u_0^C$ is a compatible real form of $u_0^C$, i.e. $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ with $\mathfrak{k}_0 \subset u_0$, $\mathfrak{p}_0 \subset iu_0$ and $u_0 = \mathfrak{k}_0 + i\mathfrak{p}_0$.

Proof. Let $u_1 := \mathfrak{k}_\cap i\mathfrak{p}$. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \subset \mathfrak{u} \oplus i\mathfrak{u}$, the equality $u_1^C = (\mathfrak{k} \cap i\mathfrak{p}) \oplus i(\mathfrak{k} \cap i\mathfrak{p}) = \mathfrak{g} \cap i\mathfrak{g}$ holds. Since $G$ is assumed to be connected and Zariski-dense in $U^C$, $u_1^C = \mathfrak{g} \cap i\mathfrak{g}$ is an ideal in $u^C$. Consequently $u_1$ is an ideal in $u$.

We fix a $U$-invariant positive inner product $\langle \ , \rangle$ on $u$. Note that $\langle [\xi, \eta_1], \eta_2 \rangle = -\langle \eta_1, [\xi, \eta_2] \rangle$ holds for every $\xi, \eta, \eta_2 \in u$.

Let $\mathfrak{k}_0 := \mathfrak{k} \cap u_1^C$ and $\mathfrak{p}_0 := \mathfrak{p} \cap iu_1^C$. Defining $u_0 := \mathfrak{k}_0 + i\mathfrak{p}_0$ the sum is direct by construction and $u_0$ is an ideal in $u$ since $\langle \ , \rangle$ is assumed to be $U$-invariant, $u_1$ is an ideal in $u$ and $\mathfrak{g}^C$ is an ideal in $u^C$. For this decompose $\xi \in \mathfrak{g}$ as $\xi = \xi_1 + \xi_{u_1^C} + \xi_{iu_1^C}$ with respect to $u^C = u_1^C \oplus u_1^C \oplus iu_1^C$. Then $\xi_{u_1^C} + \xi_{iu_1^C} \in \mathfrak{g} \cap (u_1^C \oplus iu_1^C) = \mathfrak{k}_0 \oplus \mathfrak{p}_0 = \mathfrak{g}_0$. □

If $U$ is semisimple and $G$ is connected, then Proposition 3.3 extends to the group level. Of course there is also a version for a general compact group $U$ but this is not needed here.

Proposition 3.4. Let $U$ be semisimple and let $G$ be a connected compatible Zariski-dense Lie subgroup of $U^C$. Then there exist compact connected Lie subgroups $U_0$ and $U_1$ of $U$ which centralize each other, such that

1. $U^C = U_0^C \cdot U_1^C$ and the intersection $U_0^C \cap U_1^C$ is a finite subgroup of the center of $U$,
2. $G = G_0 \cdot U_1^C$, where $G_0$ is a real form of $U_0^C$ which is compatible with the Cartan decomposition, i.e. $G_0 = K_0 \exp(\mathfrak{p}_0)$ where $K_0$ is a Lie subgroup of $U_0$ and $\mathfrak{p}_0$ is a $K_0$-stable subspace of $iu_0$. 
Proof. Since \( u \) is semisimple, it decomposes uniquely into a sum of simple ideals. Therefore \( u_0 \) and \( u_1 \) as defined in the proof of Proposition 3.3 are semisimple Lie algebras and consequently the corresponding subgroups \( U_0 \) and \( U_1 \) of \( U \) are compact. They centralize each other since \( u_0 \) and \( u_1 \) are ideals. The first statement follows since \( G \) is Zariski-dense in \( U^* \) and \( u_0 \cap u_1 = \{0\} \).

Defining \( K_0 := U_0 \cap K \), Proposition 3.3 implies the second statement. \( \square \)

4. Compatible commutative subgroups

Let \( a \) be a maximal Lie subalgebra of \( g \) such that \( a \subset p \). Since \([p, p] \subset k\), the Lie algebra \( a \) is commutative. Let \( A := \exp a \) denote the corresponding commutative Lie subgroup of \( G \). Note that \( A \) is compatible with the Cartan decomposition of \( U^* \) and let \( \mu_{ia} : Z \rightarrow i_a^* \) be the restriction of \( \mu \) to \( i_a \). The set \( S_a(M_{ia}) \) where \( M_{ia} := \mu^{-1}_{ia}(0) \) is open in \( Z \) ([HS05]).

For \( \xi \in u^* \) let \( \xi_z \) denote the vector field corresponding to the one parameter group \( (t, z) \mapsto \exp t\xi \cdot z \), i.e. \( \xi_z(z) = \frac{d}{dt}\bigg|_{0} \exp(t\xi) \cdot z \). The Lie algebra of the isotropy group \( A_z = \{ a \in A; \ a \cdot z = z \} \) of \( A \) at \( z \in Z \) is given by \( a_z := \{ \xi \in a; \ \xi_z(z) = 0 \} \). Let \( i_a^* = \{ \beta \in i_a^*; \ \beta|_{i_a z} = 0 \} \) denote the annihilator of \( i_a z \) in \( i_a^* \).

Proposition 4.1. The image \( \mu_{ia}(A \cdot z) \) of \( A \cdot z \) in \( i_a^* \) is an open convex subset of the affine subspace \( \mu_{ia}(z) + i_a^* \) and the map \( \mu_{ia} : A \cdot z \rightarrow \mu_{ia}(A \cdot z) \) is a diffeomorphism.

Proof. By definition of \( \mu_{ia} \) we have

\[ d\mu_{ia}(v)(\xi) = \omega(\xi, y) \]

for all \( y \in A \cdot z \), \( v \in T_y(A \cdot z) \) and \( \xi \in i_a \). Since \( a_y = a_z \) holds for all \( y \in A z \), we conclude \( \mu_{ia}(A \cdot z) \subset \mu_{ia}(z) + i_a^* \). From the identification \( T_y(A \cdot z) \cong a/a_y \) and \( \ker(d\mu_{ia}|A \cdot z) = 0 \) it follows that \( \mu_{ia} \) maps \( A \cdot z \) locally diffeomorphically onto an open subset of \( \mu_{ia}(z) + i_a^* \). Injectivity is proved in [HS05], Lemma 5.4.

In order to see that \( \mu_{ia}(A \cdot z) \) is convex one may proceed as in [HH96]. In fact if \( \exp i a \subset U \) would be compact, then the result would directly follow from the convexity result in [HH96]. For the convenience of the reader we recall the argument in our situation.

Let \( T := \exp i a \) and denote the Lie algebra of \( T \) by \( t \). Let \( T^c = \exp (t + it) \) denote the complexification of the compact torus \( T \). For \( \beta_0 \) and \( \beta_1 \) in \( \mu_{ia}(A \cdot z) \) we choose \( x_j \in A \cdot z \) such that \( \mu_{ia}(x_j) = \beta_j \) holds. Let \( \hat{\beta}_j := \mu_t(x_j) \). The set \( X := T^c \cdot z \) is an injectively immersed complex submanifold of \( Z \) if we endow it with the complex structure induced by the natural identification of \( T^c/T^c_z \) with \( T^c \cdot z \). The Kählerian structure on \( Z \) restricts in the sense of pulling back to \( X \) and \( \mu_t|X \) is a momentum map on \( X \).

Now consider the shifted momentum map \( \tilde{\mu}_j := \mu_t|X - \hat{\beta}_j \). From [HS05], Theorem 10.3, we get the existence of a strictly plurisubharmonic exhaustion function \( \rho_j \) on \( X \), such that \( \tilde{\mu}_j \) is associated to \( \rho_j \), i.e., \( \tilde{\mu}_j(x)(\xi) = \frac{d}{dt}\bigg|_{s} \rho_j(\exp(it\xi) \cdot x) \) for every \( \xi \in t \) and \( x \in L \). Let \( s \in [0, 1] \). We have to show, that \((1-s)\beta_0 + s\beta_1 \in \mu_{ia}(A z)\). Consider

\[ \hat{\mu}_s := (1-s)\hat{\mu}_0 + s\hat{\mu}_1 = \mu_t|X - ((1-s)\beta_0 + s\beta_1) \]

This defines a momentum map on \( X \) and it is associated to the strictly plurisubharmonic exhaustion function \( \rho_s := (1-s)\rho_0 + s\rho_1 \).

Since \( z \) is contained in the zero-set of the shifted momentum map \( \mu_t|X - \mu_t(z) \), the isotropy \((T^c)_z \) is compatible with the Cartan decomposition ([HS05], Lemma 5.5), i.e.,
equals \((T_z)^C = T_z \exp(it_z)\). The orbit \(A \cdot z \cong A/A_z\) is closed in \(T^C/T_z^C \cong T^C \cdot z\). This implies that \(p_s|_{A.z}\) is an exhaustion and therefore attains its minimal value at some point \(z_0 \in A \cdot z\). Then \(\hat{\mu}_s(z_0) = 0\) which implies that \((1-s)\beta_0 + s\beta_1 \in \mu_{ia}(A \cdot z)\). \(\square\)

Let \(m\) be a linear subspace of \(u\) and \(\eta_m := \frac{1}{2}\|\mu_m\|^2\) where \(\|\| : u^* \to \mathbb{R}\) denotes a \(U\)-invariant norm function. We identify \(m\) and \(m^*\) by the corresponding inner product \(\langle , \rangle\) on \(u\). Since \(d\eta_m(z)(v) = \langle d\mu_m(z)(v), \mu_m(z) \rangle = \omega((\mu_m(z))_Z(z), v)\), we have \(\text{grad}(\eta_m)(z) = (i\mu_m(z))_Z(z)\) with respect to the Riemannian structure induced by \(\omega\) on smooth \(U^C\)-stable submanifolds of \(Z\). Note that \(z \mapsto (i\mu_m(z))_Z(z)\) is a globally defined vector field on \(Z\). By abuse of notation, we denote it by \(\text{grad}(\eta_m)\) and call it the gradient vector field of \(\eta_m\).

The set \(C_m := \{z \in Z; \text{grad}(\eta_m)(z) = 0\}\) is called the set of critical points of \(\eta_m\).

For \(m = ia\) we have

**Lemma 4.2.** \(S_A(M_{ia}) \cap C_{ia} = M_{ia}\).

**Proof.** The inclusion \(M_{ia} \subset S_A(M_{ia}) \cap C_{ia}\) is trivial.

Let \(z \in S_A(M_{ia}) \cap C_{ia}\). From Proposition 4.1 we know that \(\mu_{ia}(Az) \subset \mu_{ia}(z) + ia_\infty^2\). Since \(z \in S_A(M_{ia})\), we have \(0 \in \mu_{ia}(Az) \subset \mu_{ia}(z) + ia_\infty^2\), so \(\mu_{ia}(Az) \subset ia_\infty^2\). Then \(0 = \text{grad}(\eta_a)(z) = (i\mu_{ia}(z))_Z(z)\), i.e. \(i\mu_{ia}(z) \in a_\infty^2\) implies \(\mu_{ia}(z) = 0\). \(\square\)

For \(r > 0\) let \(\Delta_r(\eta_{ia}) := \eta_{ia}^{-1}([0, r))\).

**Proposition 4.3.** Let \(C\) be a compact subset of \(M_{ia}\) and let \(W\) be an open neighborhood of \(C\) in \(Z\). Then there exist an \(A\)-stable open set \(\Omega\) in \(Z\) and an \(r > 0\) such that \(C \subset \Omega \cap \Delta_r(\eta_{ia}) \subset W\) holds.

**Proof.** In the proof we use the results in [HS05] freely.

Since \(A\) is commutative \(S_A(M_{ia})\) is open in \(Z\) and we may assume that \(Z = S_A(M_{ia})\) holds. In particular the topological Hilbert quotient \(\pi: Z \to Z/A\) exists. Since the maximal compact subgroup of \(A\) is trivial the map \(\pi \circ i: M_{ia} \to Z/A\) where \(i: M_{ia} \to Z\) is the inclusion is a homeomorphism. If we identify \(Z/A\) with \(M_{ia}\) the quotient map \(\pi: Z \to M_{ia}\) is given by \(z \mapsto \overline{z}\in M_{ia}\) and the \(r\)-fiber trough \(q \in M_{ia}\) is given by \(\pi^{-1}(q) = S_A(q) = \{z \in Z; q \in A \cdot z\}\).

We may assume that \(W\) is relatively compact. Let \(V_{M_{ia}}\) be a relatively compact open neighborhood of \(C\) in \(W \cap M_{ia}\) and let \(r^*\) be the minimal value of \(\eta_{ia}\) restricted to the compact set \(\partial W \cap S_A(V_{M_{ia}})\). Note that \(r^* > 0\). For \(r < r^*\) let \(V_r := W \cap \Delta_r(\eta_{ia}) \subset S_A(V_{M_{ia}})\). We claim that \(V_r = A \cdot V_r \cap \Delta_r(\eta_{ia})\) holds. For this we have to show that \(A \cdot z \cap \Delta_r(\eta_{ia}) \subset V_r\) for every \(z \in V_r\).

Proposition 4.1 implies that \(A \cdot z \cap \Delta_r(\eta_{ia})\) is connected. By definition of \(r\), it does not intersect the boundary of \(W\). Since \(z \in W\), we conclude \(A \cdot z \cap \Delta_r(\eta_{ia}) \subset W\) and therefore \(A \cdot z \cap \Delta_r(\eta_{ia}) \subset V_r\).

Setting \(\Omega := A \cdot V_r\), the proposition follows. \(\square\)

5. Neighborhoods of the zero fiber

Let \(G = K \exp p\) be a closed compatible Lie subgroup of \(U^C\) and \(a\) a maximal subalgebra of \(p\). We have \(G = KAK\) or equivalently \(p = K \cdot a\) with respect to the adjoint action of \(K\) on \(p\).
Recall that $\mathcal{C}_{ip}$ is by definition the set of zeros of the vector field $z \mapsto (i\mu_{ip}(z))Z(z)$ where $ip$ and $ip^*$ are identified by a $U$-invariant positive inner product on $u$. A result analogous to Lemma 4.2 holds for $G$. More precisely, for $X := \bigcap_{k \in K} k \cdot \mathcal{S}_A(\mathcal{M}_{ia})$ we have

**Lemma 5.1.** $X \cap \mathcal{C}_{ip} = \mathcal{M}_{ip}$. 

*Proof.* We have $\mathcal{M}_{ip} \subseteq X \cap \mathcal{C}_{ip}$ since $\mathcal{M}_{ip}$ is $K$-invariant. Let $z \in X \cap \mathcal{C}_{ip}$. There exists a $k \in K$ with $\mu_{ip}(k \cdot z) = k \cdot \mu_{ip}(z) \in \mathfrak{a}^*$. Then $z' := k \cdot z \in X \in \mathcal{S}_A(\mathcal{M}_{ia})$ since $X$ is $K$-invariant and $z' \in \mathcal{C}_{ip}$ since $\mu_{ip}(z') = \mu_{ia}(z')$ and $z' \in \mathcal{C}_{ip}$. Lemma 4.2 implies $z' \in \mathcal{M}_{ia}$. We conclude $z' \in X \cap \mathcal{M}_{ip}$. Invariance of $\mathcal{M}_{ip}$ with respect to $K$ implies $z \in X \cap \mathcal{M}_{ip}$. □

For $r > 0$ let $\Delta_r(\eta_{ip}) := \eta_{ip}^{-1}((0, r))$. Now we give a generalization of Proposition 4.3 to the non-commutative case.

**Proposition 5.2.** Let $C$ be a compact $K$-stable subset of $\mathcal{M}_{ip}$ and let $W$ be an open neighborhood of $C$ in $Z$. Then there exist a $G$-stable open set $\Omega$ in $Z$ and an $r > 0$ such that $C \subseteq \Omega \cap \Delta_r(\eta_{ip}) \subseteq W$ holds.

*Proof.* We may assume that $W$ is $K$-stable. From Proposition 4.3 we get the existence of an $A$-invariant open neighborhood $\Omega_A$ of $C$ and an $r > 0$, such that $\Omega_A \cap \Delta_r(\eta_{ia}) \subseteq W$ holds. Let $V$ be a $K$-stable neighborhood of $C$ in $\Omega_A$. Then $A \cdot V \cap \Delta_r(\eta_{ia}) \subseteq W$. Let $\Omega := G \cdot V$. Since $\eta_{ia}(z) \leq \eta_{ip}(z)$ we conclude

$$
\Omega \cap \Delta_r(\eta_{ip}) = K \cdot A \cdot V \cap \Delta_r(\eta_{ip}) = K \cdot (A \cdot V \cap \Delta_r(\eta_{ia})) \subseteq K \cdot (A \cdot V \cap \Delta_r(\eta_{ia})) \subseteq K \cdot W = W.
$$

□

**Remark.** The proof uses $\Delta_r(\eta_{ip}) \subseteq \Delta_r(\eta_{ia})$. More precisely it can be shown that $\Delta_r(\eta_{ip}) = \bigcap_{k \in K} k \cdot \Delta_r(\eta_{ia})$ holds.

6. The set of semistable points is open

Recall that grad($\eta_{ip}$) denotes the vector field $z \mapsto (i\mu_{ip}(z))Z(z)$. We have grad($\eta_{ip}$)($z$) $\in T_z(G \cdot z)$, i.e., it is tangent to the $G$-orbits. Since the momentum map is assumed to be smooth on submanifolds of $Z$, the gradient vector field is smooth on $U^C$-stable submanifolds. Let $\psi_z : I_z \to Z$, $t \mapsto \psi_z(t)$ denote integral curve of grad($\eta_{ip}$) with $\psi_z(0) = z$. Here we assume the interval $I_z$ to be maximal.

**Theorem 6.1.** The set $\mathcal{S}_G(\mathcal{M}_{ip})$ is open in $Z$.

*Proof.* We may assume that $G$ is closed in $U^C$. Let $x \in \mathcal{M}_{ip}$. It is sufficient to show that an open neighborhood $V$ of $x$ is contained in $\mathcal{S}_G(\mathcal{M}_{ip})$. By Proposition 5.2 there exists a $G$-stable open neighborhood $\Omega$ of $x$, such that for some $r > 0$ the set $\Omega \cap \Delta_r(\eta_{ip})$ is compact and contained in $X = \bigcap_{k \in K} k \cdot \mathcal{S}_A(\mathcal{M}_{ia})$. Let $V := \Omega \cap \Delta_r(\eta_{ip})$. By definition of $\psi$ we have $\psi_z(t) \in G \cdot z \cap \Delta_r(\eta_{ip})$ for all $z \in \Delta_r(\eta_{ip})$ and all $t \leq 0$ in the domain of definition of $\psi_z$. Let $s_z \in [-\infty, 0)$ be minimal with $(s_z, 0] \subseteq I_z$. Note that $s_z = -\infty$ for $z \in \overline{\nabla}$ if $Z$ is smooth.

Let $Z = R_0 \cup R_1 \cup \ldots \cup R_l$ be the stratification of $Z$ into smooth submanifolds, i.e. $R_0$ is the set of smooth points in $Z$ and $R_{l+1}$ is the set of smooth points in $Z \setminus (R_0 \cup \cdots \cup R_l)$. 

Let $z_0 \in V$ and let $t_n \leq 0$ be a decreasing sequence which converges to $s_{z_0}$. After possibly replacing $(t_n)$ by a subsequence, the limit $z_1 := \lim_{n \to \infty} \psi_{z_0}(t_n)$ exists. By induction we get a sequence $(z_n)$. If $z_n \in R_j$ and $s_{z_n} \neq -\infty$, then $z_{n+1} \in R_{j+1}$ for an $l > 0$ since $R_j$ is smooth and $U^C$-invariant. Therefore there exists an $n_0$ with $s_{z_{n_0}} = -\infty$. To simplify notation, we assume $n_0 = 0$.

Let $t_0 < 0$. Then $\psi_{z_1}(t_0) = \lim_{n \to \infty} \psi_{z_0}(t_n + t_0)$ and consequently

$$\eta_{i\beta}(\psi_{z_1}(t_0)) = \lim_{n \to \infty} \eta_{i\beta}(\psi_{z_0}(t_n + t_0)) = \lim_{n \to \infty} \eta_{i\beta}(\psi_{z_0}(t_n)) = \eta_{i\beta}(z_1).$$

Hence $\psi_{z_1}(t_0) = z_1$ and $z_1 \in C_{i\beta}$. Lemma 5.1 implies that $z_1 \in C_{i\beta} \cap V \cap G \cdot z_0 \subset C_{i\beta} \cap X \cap G \cdot z_0 = M_{i\beta} \cap G \cdot z_0$. This shows $V \subset S_G(M_{i\beta})$.

\[\square\]

7. Shifting

In this section we consider the case of a general $\mu_{i\beta}$-fiber $M_{i\beta}(\beta)$. The first step is to shift the momentum map relative to the coadjoint orbit $U \cdot \beta$. The relevant properties are summarized in

**Lemma 7.1.** Let $\beta \in u^*$ and let $O := U \cdot \beta$ be the coadjoint orbit of $\beta$. Moreover let $\xi \in u$ denote the dual vector of $\beta$ with respect to a $U$-invariant negative definite inner product on $u$. Let $Q := \{g \in U^C; \lim_{t \to -\infty} \exp(it\xi)g \exp(-it\xi) \text{ exists in } U^C\}$. Then the following holds

1. $Q$ is a parabolic subgroup of $U^C$ with Lie algebra $q = z^C(\xi) \oplus r$ where $z^C(\xi)$ denotes the centralizer of $\xi$ in $u^C$ and $r$ denotes the sum of eigenspaces of $\text{ad}(-i\xi)$ with negative eigenvalues.
2. The $U$-equivariant map $\iota: O \to U^C/Q$, $u \cdot \beta \mapsto uQ$ is a real analytic isomorphism.
3. The $U$-invariant symplectic structure $\omega_O(\zeta_0(\alpha), \eta_0(\alpha)) := -\alpha([\zeta, \eta])$ for $\alpha \in O$ and $\zeta, \eta \in u$ is a Kählerian structure with respect to the complex structure on $O$ induced by $\iota$. The map $O \to u^*$, $\eta \mapsto -\eta$ defines a momentum map on $O$.

**Lemma 7.1** says that $O = U \cdot \beta$ is a complex manifold such that $U^C$ acts holomorphically and transitively, i.e., $U \cdot \beta = U^C \cdot \beta$ holds. The following observation is crucial for our application of the shifting procedure.

**Proposition 7.2.** $G \cdot \beta = K \cdot \beta$ for $\beta \in i\beta^*$.

**Proof.** As a first step we replace $U^C$ by its image $\text{Ad}(U)^C$ under the adjoint representation. The image $\text{Ad}(G)$ is compatible with the Cartan decomposition of $\text{Ad}(U)^C$. Then $U$ is semisimple.

Assuming in addition that $G$ is Zariski-dense in $U^C$ (Lemma 3.1) and connected, we may apply Proposition 3.4. It yields that $U$ is the product of two compact subgroups $U_0$ and $U_1$ which centralize each other, such that $G = G_0 \cdot U_1^C$. We decompose $\beta \in i\beta^*$ with respect to this decomposition, i.e. $\beta = \beta_0 + \beta_1$ where $\beta_0 \in i\beta_0^* = i(p \cap iu_0)^*$ and $\beta_1 \in iu_1^*$. The coadjoint orbit $O = U \cdot \beta$ can be biholomorphically and $U$-equivariantly identified with $U_0 \cdot \beta_0 \times U_1 \cdot \beta_1$. We have $U^C \cdot \beta_1 = U_1 \cdot \beta_1$ by construction, so we may assume $U = U_0$.

Let $\sigma: u^C \to u^C$ be the anti-holomorphic involution of Lie algebras defined by $\sigma|g = id_g$ and $\sigma|\bar{g} = -id_{\bar{g}}$. Since $\beta \in i\beta^*$, the Lie algebra $q$ of $Q$ is stable with respect to $\sigma$.\[\square\]
Therefore it defines an antiholomorphic involution on the tangent space $T_{\beta}(U \cdot \beta) \cong \mathfrak{u}^C/\mathfrak{q}$. For a subspace $\mathfrak{m} \subset \mathfrak{u}^C$ define $\mathfrak{m} \cdot \beta := \{ \xi_O(\beta); \xi \in \mathfrak{m} \}$. We have $T_{\beta}(U \cdot \beta) = \mathfrak{u}^C \cdot \beta = \mathfrak{k} \cdot \beta + i\mathfrak{p} \cdot \beta = i\mathfrak{k} \cdot \beta + \mathfrak{p} \cdot \beta$. Since $\sigma| (g \cdot \beta) = \text{id}_{g \cdot \beta}$ and $\sigma| (ig \cdot \beta) = - \text{id}_{ig \cdot \beta}$, it follows that $\mathfrak{k} \cdot \beta = \mathfrak{p} \cdot \beta$. Therefore $T_{\beta}(G \cdot \beta) = T_{\beta}(K \cdot \beta)$ and we conclude $G \cdot \beta = K \cdot \beta$. □

Now we can prove the main result of this paper. For the convenience of the reader we repeat the statement.

**Theorem 1.1.** For every $\beta \in i\mathfrak{p}^*$ the set $S_G(\mathcal{M}_{i\mathfrak{p}}(\beta))$ is open in $Z$.

**Proof.** The Kählerian structures on $Z$ and $O$ induce a Kählerian structure on the product $Z \times O$. It follows that $\tilde{\mu}: Z \times O \to \mathfrak{u}^*$, $\tilde{\mu}(z, \alpha) = \mu(z) - \alpha$ is a $U$-equivariant momentum map on $Z \times O$ with respect to this product Kählerian structure and the diagonal action of $U$. Moreover $S_G(\tilde{\mathcal{M}}_{i\mathfrak{p}})$, where $\tilde{\mathcal{M}}_{i\mathfrak{p}} := \tilde{\mu}^{-1}(0)$ is open in $Z \times O$ and therefore its intersection $Z(\beta) \subset Z \times O$ with $Z \times G \cdot \beta$ is open as well. The projection $p: Z \times G \cdot \beta \to Z$ is an open map. Since $G \cdot \beta$ is a $K$-orbit, $p$ maps the open $G$-stable set $Z(\beta)$ onto $S_G(\mathcal{M}_{i\mathfrak{p}}(\beta))$. □

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