On the stability of precessing superfluid neutron stars

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We discuss a new superfluid instability occurring in the interior of mature neutron stars with implications for freely precessing neutron stars. This short-wavelength instability is similar to the instability which is responsible for the formation of turbulence in superfluid Helium. Its existence raises serious questions about our understanding of neutron star precession and complicates attempts to constrain neutron star interiors using such observations.

Introduction.— Neutron stars tend to be extremely stable rotators, with stability that sometimes rivals that of the best atomic clocks. Yet a growing sample of pulsars exhibit spin irregularities, like glitches and timing noise. They may also be undergoing free precession. From the theory point of view, one might expect precession to be generic. Nevertheless, for reasons still to be understood, compelling evidence for long-period precession has only been found in the timing data of a few pulsars. The best candidate is PSR B1828-11 \cite{1} which exhibits a ∼ 500 d high-quality periodicity, with an amplitude of a few degrees. The paucity of precessing neutron stars is one of the mysteries of pulsar physics. To explain why precession is so rare is difficult. After all, a description of even a modestly realistic neutron star requires the fusion of much of modern theoretical physics. One would need to account for strong gravity, supranuclear density matter, superfluidity/superconductivity and potentially very strong magnetic fields.

In the standard picture of a mature neutron star the bulk of the neutrons are superfluid and rotate by forming a dense array of vortices. Meanwhile the outer core protons are expected to form a type II superconductor, with the magnetic flux carried by fluxtubes. The coupling between these two distinct fluid components is usually assumed to have the same form as in the case of superfluid Helium, see \cite{2} for a recent discussion. However, this model is based on the assumption that the neutron vortex array is (locally) straight. This may not be the case. In a body that undergoes a more complex motion one might expect to find that the vortices get tangled up to form a state of superfluid turbulence. In Helium, the formation of a vortex tangle is assumed to follow the onset of an instability in the vortex array \cite{3}. It has recently been suggested that an analogue of this so-called Donnelly-Glaberson instability may be relevant for neutron stars \cite{4,5,6}. If this is the case, one would expect it to have interesting repercussions for neutron star precession. In this Letter we confirm this expectation by demonstrating that short-wavelength instabilities are generic in precessing superfluid neutron stars.

Plane wave analysis. — Our main objective is to investigate whether analogues of the Donnelly-Glaberson instability are likely to occur in a neutron star interior. Our analysis is based on the standard two-fluid picture, where the superfluid neutrons are dynamically distinguished from a conglomerate of comoving superconducting protons and normal electrons. We will loosely refer to the latter as the “protons” in the following. Variables associated with each fluid will be labelled by $x = \{n, p\}$. The smooth-averaged hydrodynamics of the system is governed by two coupled Euler-type equations, see \cite{2} for more details. In a frame rotating with angular velocity $\Omega$ we have

\begin{align}
D^i_n v^n_i & + \nabla_i \psi_n = 2\epsilon_{ijk} v^n_j \Omega^k + f_{ij}^m f^m_j \\
D^i_p v^n_p & + \nabla_i \psi_p = 2\epsilon_{ijk} v^n_j \Omega^k - f_{ij}^m / x_p + \nu_{ee} \nabla^2 v^n_i
\end{align}

Here the fluid velocities are denoted by $v^n_i$, we have introduced the convective derivatives $D^n_i = \partial_t + v^n_j \nabla_j$ and $x_p = \rho_p / \rho_n$ is the density fraction. The scalars $\psi_n$ are the sums of specific chemical potentials and the gravitational potential \cite{2}. For simplicity, we assume that both fluids are incompressible, i.e. we have $\nabla_i \psi_n = 0$. In the interest of clarity, we also ignore the entrainment effect in this study. A key property of the system is that neutrons and protons are coupled via mutual friction, a force $f_{ij}^m$ mediating the interaction between the quantized neutron vortices and the proton fluid/magnetic fluxtubes. The standard expression for this force is, see \cite{2},

\begin{align}
f_{ij}^m = B \epsilon_{ijk} \hat{\omega}^k_n \omega^j_m w^i_n + B' \epsilon_{ijk} \omega^j_n u^i_{np}
\end{align}

where $w^i_{np} = v^n_i - v^i_p$ and the neutron vorticity is given by $\omega^k_n = \epsilon_{ijk} \nabla_j v^i_p$. A “hat” denotes a unit vector. This form for the mutual friction force results from balancing the Magnus force that acts on the neutron vortices and a resistive “drag” force which represents the interaction between the vortices and the charged fluid \cite{2}. Representing the drag force by a dimensionless coefficient $R$, one finds that

\begin{align}
B = \frac{R}{1 + R^2}, \quad \text{and} \quad B' = \frac{R^2}{1 + R^2}
\end{align}

In the most commonly considered case, the mutual friction arises from scattering of electrons off the vortex's
intrinsic magnetic field. This leads to a relatively weak coupling, with \( \mathcal{R} \approx 4 \times 10^{-4} \Rightarrow B' \approx B^2 \Rightarrow B \ll 1 \) \( \Rightarrow \) \( B' \approx 1 - B^2 \) \( \Rightarrow \) \( B' \approx \mathcal{R} \ll 1 \) \( \Rightarrow \) \( B \ll 1, \quad B' \approx 1 - B^2 \) \( \Rightarrow \) \( B' \approx \mathcal{R} \ll 1 \)

It is, however, not established that it is this limit that applies. Hence, one must also consider the case of strong coupling which follows from taking \( \mathcal{R} \gg 1 \). This translates into

\[
B \ll 1, \quad B' \approx 1 - B^2
\]

The strong coupling limit is relevant if the interaction between neutron vortices and fluxtubes is efficient, if there is a fluxtube cluster associated with each neutron vortex, or if there is significant vortex pinning (in the limit \( \mathcal{R} \to \infty \) the vortices can be considered as perfectly “pinned”).

Returning to the Euler equations, only the proton equation contains a shear viscosity term. This is because the dominant process is expected to be electron-electron scattering. The upshot of this is that the neutron fluid is not directly affected by shear viscosity. The relevant viscosity coefficient \( \nu_{\text{ee}} \) has been estimated in [13].

For a uniform density star with \( M = 1.4M_\odot, R = 10 \, \text{km} \) and \( x_p = 0.1 \) (the canonical values we will use later) we have \( \nu_{\text{ee}} \approx 10^7 \left( T/10^8 \text{K} \right)^{-2} \, \text{cm}^2/\text{s} \).

We consider perturbations of eqns. (1) and (2), only for a background configuration where both fluids rotate rigidly with \( \nu_{\text{st}} = \nu_{\text{ee}} \Omega \hat{n} \). By allowing for an arbitrary orientation of the angular velocity vectors, this configuration can represent the standard free precession modes of a two-fluid star [14, 15]. We then linearise the Euler equations, focussing on short-wavelength motion by making the standard plane-wave decomposition

\[
\delta \nu_{\text{x}} = A_{\text{x}}^i e^{i(\Omega t + k_i x^i)}, \quad A_{\text{x}}^i = \text{constant}
\]

and similarly for all other variables. Since we expect the flow along the background vortex array to play a central role, we carry out the perturbation calculation in the neutron frame. That is, we take \( \Omega = \Omega_{\text{n}} = \Omega_{\text{n}} \hat{n} \). In order to simplify the problem, without any real loss of generality, we only consider waves propagating along the vortices, i.e. \( k^i = k_i \hat{n} \). Then the fact that we have assumed the fluids to be incompressible means that the waves are transverse, \( \hat{n} \cdot A_{\text{x}}^i = 0 \). After some algebra, cf. [16] for a similar analysis, the perturbed versions of (1) and (2) lead to a 4 \times 4 system, the determinant of which provides the dispersion relation for short-wavelength waves. A detailed analysis of the problem will be provided elsewhere. Here we focus on the modes that may become unstable.

Let us first consider the weak drag limit. Then we find a mode with frequency (with viscous corrections of order \( 1/k_{\parallel} \))

\[
\sigma \approx 2\Omega_{\text{n}} + (iB - B') \left( 2\Omega_{\text{n}} - k_{\parallel} w_{\parallel} \right)
\]

Here \( w_{\parallel} \) represents the relative linear flow along the (background) neutron vortex array. In our case we have \( w_{\parallel} = -\hat{n}^i e_{ijk} \Omega_{\text{n}} \hat{n}^k \). In the local analysis we have taken \( w_{\parallel} \) to be constant. Hence it is clear that our analysis is only consistent for short wavelength motion. Anyway, from [5] we see that the system is unstable (Im \( \sigma < 0 \)) if

\[
w_{\parallel} > 2\Omega_{\text{n}}/k_{\parallel}
\]

As discussed in [3], the solution (8) represents inertial waves in the neutron fluid. This instability is the exact analogue of the Donnelly-Glaberson instability in Helium [3], and hence its existence should come as no real surprise. As in Helium, one would expect the onset of the instability to lead to the formation of tangled vortices, reconnection and superfluid turbulence. Since turbulence alters the form of the macroscopic mutual friction force [4, 5], it is not yet clear how the system will evolve once the unstable waves grow to large amplitude.

As far as we are aware, the strong drag problem has not been considered previously. Interestingly, there are unstable modes also in this case. The nature of the instability is, however, more complex. In the strong drag limit, with \( B = 0 \) and \( B' = 1 \), we find a mode with frequency

\[
\sigma \approx \Omega_{\text{n}} \left( 1 - \frac{1}{x_p} \right) + k_{\parallel} w_{\parallel} + i\nu_{\text{ee}} k_{\parallel}^2 \left\{ \frac{\Omega_{\text{n}}^2 (1 + x_p)^2}{x_p} \right. - \frac{2\Omega_{\text{n}} k_{\parallel} w_{\parallel}}{x_p} - \frac{\nu_{\text{ee}} k_{\parallel}^4}{4} - \frac{\left( 1 - x_p \right)^2 \nu_{\text{ee}} k_{\parallel}^2 \Omega_{\text{n}}}{x_p} \left. \right\}^{1/2}
\]

This result clearly shows that there will be unstable waves (representing coupled inertial waves in the neutron/proton fluids). In the inviscid (\( \nu_{\text{ee}} = 0 \)) limit the instability is active provided that

\[
w_{\parallel} > \frac{\Omega_{\text{n}} (1 + x_p)^2}{2k_{\parallel} x_p}
\]

As in the weak drag case, one would expect the onset of this instability to lead to tangled vortices.

**Implications for precessing neutron stars.** — In order to discuss the implications of the above results we need to make contact between our background configuration and the global precession motion. Fortunately, this is straightforward. The precession of a two-component neutron star model, including mutual friction coupling, has already been discussed in [13]. The simplest model consists of two components that rotate rigidly. The neutron component is assumed spherical with moment of inertia \( I_n \). At the same time, the protons (including the crust) are assumed to be slightly deformed in such a way that \( I_p = I_1 = I_2 = I_3 \) and \( I_3 = I_4 + (1 + \epsilon) I_3 \) (in a principal coordinate system where the deformation axis is along \( \hat{x}_3 \)). When perturbed away from alignment of the two rotation axes, \( \Omega_{\text{x}}^n \), the crust precesses with a certain frequency
and observable wobble angle $\theta_w$ (the angle between the deformation axis, $\hat{x}_3$, and the total angular momentum axis)\textsuperscript{12}. The plane-wave analysis is consistent for the precessing system provided that the two rotation vectors $\Omega^i_x$ can be considered fixed. This is true as long as the precession period $P_{\text{pr}}$ is significantly longer than the timescale associated with the local waves. As already mentioned, the wavelength of the waves we consider must also be short enough that $w_{\parallel}$ can be treated as a constant. If these conditions hold then we are simply considering local perturbations of a given precession model. In order the check whether this system is locally stable we only need to work out $w_{\parallel}$ from the precession solution. If an instability is present, then the precession solution must be considered questionable. It certainly cannot be the case that the two components rotate rigidly, a key assumption in the standard analysis\textsuperscript{14}.

Weak drag slow precession. — In the weak drag limit, there exists a long period precession solution that is slowly damped by mutual friction. For this solution we have\textsuperscript{14}

$$P_{\text{pr}} \approx P/\epsilon \quad \text{and} \quad t_d \approx \frac{P}{2\pi B \epsilon I_n}$$

where $P_{\text{pr}}$ is the precession period, $t_d$ is the damping time and $P$ is the rotation period of the star. We then find that

$$w_{\parallel} \approx 2\pi \epsilon \theta_w x_2 / P$$

where $x_2(<R)$ is one of the coordinates associated with crust system. This estimate can be used in\textsuperscript{9} to show that all waves with wavelength ($\lambda = 2\pi / k_{\parallel}$) shorter than

$$\lambda_{\text{max}} = 5 \times 10^{-4} \left( \frac{\theta_w}{1^\circ} \right) \left( \frac{\epsilon}{10^{-8}} \right) \left( \frac{R}{10^6 \text{cm}} \right) \text{ cm}$$

are unstable. However, there must be a short wavelength cut-off for the instability. To make progress it would seem natural to assume that our analysis becomes invalid once the wavelength is so short that the fluid description is no longer relevant. Then it seems reasonable to use something like

$$\lambda_{\text{min}} \approx 100 d_n \approx \left( \frac{P}{1 \text{s}} \right)^{1/2} \text{ cm}$$

where $d_n$ is the intervortex spacing. Since we need to have $\lambda_{\text{max}} > \lambda_{\text{min}}$ in order to argue that the instability is relevant we see that we must have

$$\left( \frac{\theta_w}{1^\circ} \right) > 1900 \left( \frac{P}{1 \text{s}} \right)^{1/2} \left( \frac{\epsilon}{10^{-8}} \right)^{-1} \left( \frac{R}{10^6 \text{cm}} \right)^{-1}$$

What does this result tell us? It suggests that, if the drag is weak, the superfluid instability is unlikely to play a role in slowly spinning systems. For the archetypal precessing PSR B1828-11\textsuperscript{11} the spin period is 0.4 s and in order to have precession with the observed period one would need $\epsilon \approx 10^{-8}$. It is then clear from\textsuperscript{16} that precession with a wobble angle of a few degrees is safely in the stable regime. Nevertheless, the weak drag result is not without interest. Consider for example a millisecond pulsar with a maximally strained crust. From\textsuperscript{10} we see that if the spin period is 1 ms, then precession with $\theta_w$ larger than a degree would be unstable provided that $\epsilon > 6 \times 10^{-7}$. Since the theoretically predicted range for crustal deformations has $\epsilon < 10^{-5}$\textsuperscript{16}, we see that our result puts a constraint on slow precession in very fast spinning neutron stars.

Strong drag fast precession. — In the strong drag limit, the relevant precession solution is such that\textsuperscript{14}

$$P_{\text{pr}} \approx \frac{I_p}{I_n} P \quad \text{and} \quad t_d \approx \frac{P}{2\pi B \epsilon I_n}$$

and we find that

$$w_{\parallel} \approx \frac{2\pi \theta_w x_2}{P} \frac{I_n}{I_p}$$

The (inviscid) instability criterion\textsuperscript{11} then implies that waves with wavelength shorter than

$$\lambda_{\text{max}} = 2 \times 10^5 \left( \frac{\theta_w}{1^\circ} \right) \left( \frac{R}{10^6 \text{cm}} \right) \text{ cm}$$

(we have assumed $I_p/I_n \approx x_p$) will be unstable.

However, as is clear from\textsuperscript{10}, the unstable strong drag modes are affected by viscosity. To unveil the detailed behaviour we have solved the dispersion relation numerically for a range of parameter values. Typical results are shown in Figure 11. This figure shows $\tau_{\text{grow}}$ as a function of $\lambda$ and illustrates how the importance of shear viscosity varies with temperature. The results for core temperatures $T = 10^9$ K and $T = 10^7$ K show a clear transition from a regime where the inviscid approximation to (10) is valid (above $\lambda \approx 20$ cm and $10^4$ cm, respectively) to a short wavelength regime where viscosity alters the result. However, a surprising feature appears as one proceeds towards shorter wavelengths for a fixed temperature. As $k_{\parallel}$ becomes large, it turns out that there is a cancellation of the leading order viscosity terms, cf.\textsuperscript{10}. For short wavelengths, the mode frequencies are instead accurately (with errors of order $1/k_{\parallel}$) described by\textsuperscript{8} with $B = 1$. Hence, for $\lambda \ll \lambda_{\text{max}}$ the short wavelength modes grow on a timescale given by

$$\tau_{\text{grow}} \approx \frac{\lambda}{2\pi B |\epsilon|}$$

For typical parameters, we have

$$\tau_{\text{grow}} > 140 \left( \frac{x_p}{0.1} \right) \left( \frac{\epsilon}{10^{-8}} \right)^{-1} \left( \frac{P}{1 \text{s}} \right) \left( \frac{R}{10^6 \text{cm}} \right) \left( \frac{\lambda}{R} \right) \text{ s}$$
Moreover, in order to have the short-wavelength instability operate (I) for $T = 10^7$ K is highlighted. Finally, the $\lambda < \lambda_{\text{min}}$ region where we assume that the hydrodynamical description fails is shaded.

For consistency the unstable waves need to grow on a timescale that is short compared to the precession period. If we require (say) $\tau_{\text{grow}} < 0.1 P_{\text{pr}}$, then we have

$$\lambda < 70 \left( \frac{R}{10^3} \right)^{-1} \left( \frac{\theta_w}{1^\circ} \right) \text{ cm}$$

The corresponding instability region is indicated by an I in Fig. 1. Moreover, in order to have $\lambda > \lambda_{\text{min}}$ (noting that the short wavelength cut-off remains as in the weak drag case) we must have

$$\mathcal{R} < 7 \times 10^4 \left( \frac{\theta_w}{1^\circ} \right) \left( \frac{P}{1 \text{ s}} \right)^{-1/2}$$

This shows that the short-wavelength instability constrains a wide range of fast precession models. From the results in Fig. 1 it is also clear that there may exist a medium wavelength instability regime (well approximated by (10)). This is relevant for temperatures above $10^7$ K, and could well lead to the fastest growing instability in young neutron stars.

Brief discussion. — In this Letter we have demonstrated that short wavelength superfluid instabilities may operate in freely precessing neutron stars. In the weak drag regime, the instability affects only rapidly spinning stars that have significantly deformed crusts. PSR B1828-11, the currently best candidate precessor, lies well within the stable regime. In contrast, our results have serious implications for systems in the strong drag regime. We predict that these systems will suffer local instabilities, possibly leading to the formation of superfluid turbulence, for a wide range of the relevant parameter space. This calls into question the standard precession model, which is based on two co-existing fluids rotating as solid bodies [14], and any conclusions drawn from it. In particular, one would note Link’s argument [9, 10] that the coupling between vortices and fluxtubes ought to lead to fast precession according to (17). Since this is contradicted by the observed slow precession of PSR B1828-11, Link suggests that our understanding of the neutron star core physics is wrong and that the protons would actually form a type I superconductor (without fluxtubes). Our results add an element of doubt. We have essentially shown that the strong drag fast precession solution may be inconsistent for a neutron star spinning at the rate of PSR B1828-11. If the precessing motion triggers a range of unstable short wavelength waves then the original solid-body assumption that led to (17), cf. [14], cannot hold. The precession problem may thus be more complex than usually assumed, and a consistent description of fast precession must properly include superfluid wave dynamics and potential turbulence.

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