A question in the axiomatic approach to Quantum Mechanics

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Abstract

The classical Hilbert space formulation of the axioms of Quantum Mechanics appears to leave open the question whether the Hermitian operators which are associated with the observables of a finite non-relativistic quantum system are uniquely determined.

1. Introduction

For simplicity, we shall only consider the basic axiomatic formulation of the non-relativistic Quantum Mechanics of a finite systems in which the operators $A$ associated with the observables

(1) are everywhere defined bounded Hermitian operators on a given Hilbert space $H$ which corresponds to the quantum system,

(2) have a countable orthonormal complete set of eigenvectors $\alpha_n \in H$, with $n \in \mathbb{N}$,

(3) with the corresponding eigenvalues $A_n \in \mathbb{R}$, with $n \in \mathbb{N}$, pair-wise different.

Such a basic framework, Gillespie, is still capable to point to the question of our concern. Here we recall the relevant axioms on Measurement, Gillespie [pp. 49-58].
Given a moment of time $t \in \mathbb{R}$, let us assume that immediately prior to that moment, the quantum system has been in the state

\begin{equation}
\psi_{t-0} \in H
\end{equation}

Further, let us assume given a certain physical observable $\mathcal{A}$ with the corresponding Hermitian operator

\begin{equation}
A \in \mathcal{L}(A)
\end{equation}

under assumptions (1) - (3), where $\mathcal{L}(A)$ denotes the set of all bounded linear operators on $H$.

Then according to one of the axioms, Gillespie [p. 49], the strongest predictive statement we can make about the result of the measurement following the observation $\mathcal{A}$ of the system effectuated at time $t$ is given by the sequence of relations

\begin{equation}
\text{Prob}(A_n) = | < \alpha_n, \psi_{t-0} > |^2, \quad n \in \mathbb{N}
\end{equation}

Another axiom, Gillespie [p. 58], states that immediately after the moment $t$ when the observation $\mathcal{A}$ was effectuated on the system, the state of the system will collapse to the state

\begin{equation}
\psi_{t+0} = \alpha_n
\end{equation}

where $n \in \mathbb{N}$ is precisely the index of the eigenvalue $A_n$ which happened to be realized by the given measurement following the observation $\mathcal{A}$ was effectuated on the system at time $t$. And in view of the assumption (3), this index $n$ is well defined.

2. A first formulation of the question

We recapitulate. Given a physical observable $\mathcal{A}$, then the association

\begin{equation}
\mathcal{A} \mapsto A \in \mathcal{L}(H)
\end{equation}

under the assumptions (1) - (3), is only required to satisfy (6) and (7).
We note that the operator $A$ is uniquely determined by its eigenvectors and eigenvalues in (2) and (3), respectively. Further, in view of the collapse in (7), it follows that the eigenvectors in (2) are uniquely determined by the observable $A$.

Here it is important to note that, far as the eigenvalues (3) are concerned, the only condition they are supposed to satisfy is (6). However, due to the presence of probabilities in (6), it is not certain that the eigenvalues (3) will also be uniquely determined by that relation.

Let us, therefore, consider relation (6) more carefully about its possible precise meaning.

First, (6) does not refer directly to the actual value of any of the eigenvalues $A_n$, but only to the probability of the occurrence in measurement. Second, the relation (6) cannot in general be perfectly be confirmed or verified in effective practical experiments, since that would involve the performance of very large numbers, if not in fact, of infinitely many measurements. Third, an effective practical measurement cannot in general give the exact value of any of the eigenvalues $A_n$, unless $A_n$ happens to be a moderate size integer, which is not a typical situation.

Therefore, the

**Question:**

To the extent that (6) cannot in general be rigorously confirmed, is the association in (8) nevertheless one-to-one?

In other words, since in general the confirmation of (6) can only be of the form

$$|\text{Prob}(A_n) - |<\alpha_n, \psi_{t-0}|^2| \leq \epsilon_n, \quad n \in \mathbb{N}$$

where $\epsilon_n > 0$ may depend on $\alpha_n$ and $\psi_{t-0}$, can nevertheless the eigenvalues (3) be determined uniquely?
And to aggravate the situation, it should also be noted that (7) itself cannot in general be perfectly confirmed by effective practical experiments. Indeed, it is one of the fundamental assumptions of Quantum Mechanics, expressed explicitly in one of the axioms, that no state of any quantum system can be directly observable. Consequently, since (7) is a statement about such states, it can only be confirmed indirectly, namely, via results of measurements on the corresponding observables $A$.

And this bring us back to the above questions related to (6), (8) and (9).

3. Comments

There exists a certain awareness about the possibility that the association in (8) may fail to be one-to-one. For instance, in Davies [p. 66], we find:

"Although the choice of the operators is not unique, they must comply with the commutation relations and bear the same functional relationship to each other as the corresponding classical quantities (the correspondence principle). In practice the choice of $r$ and $-ihv$ for position and momentum is conventional. Most other operators follow from these."

Two points can be noted in this respect. First, the correspondence principle bring in additional constraints on the association in (8), in case this association may indeed happen to fail to be one-to-one when considered in terms of (6) and (7) alone. Second, a further constraint to be noted is as follows. Assume that in (8) we have the one-to-many association
for certain given observables $\mathcal{A}$ and $\mathcal{B}$. Then we shall have to require that

$$A_1, B_1 \text{ commute} \iff A_2, B_2 \text{ commute}$$

(11)

and in addition, also the relations, Gillespie [p. 67]

$$\Delta A_i \cdot \Delta B_i \geq |c_i|/2 > 0, \quad i = 1, 2$$

(12)

if one of the pairs $A_i, B_i$, with $i = 1, 2$, does not commute, and instead, it satisfies the relation

$$A_iB_i - B_iA_i = c_iI$$

(13)

where $c_i \in \mathbb{C}$, $c_i \neq 0$ and $I$ is the identity operator on $H$. We recall that $\Delta A_i$ and $\Delta B_i$ denote, Gillespie [p. 55], the uncertainties in the operators $A_i$ and $B_i$, respectively.

### 4. Reformulation of the Question

In view of section 3, we can reformulate the question in section 2, as follows.

Assume that, according to the correspondence principle, we identify the set of quantum observables with the set

$$\mathcal{H}(H)$$

(14)

of Hermitian operators on the given Hilbert space $H$. In other words, every Hermitian operator $A \in \mathcal{H}(H)$ is labelled by a unique observable
\( \mathcal{A} \), and through this labelling all observables will become the label of a certain Hermitian operator. Thus we assume the existence of a one-to-one mapping

\[
(15) \quad A \leftrightarrow \mathcal{A}
\]

The only condition required on the mapping in (15) is to satisfy all the axioms of Quantum Mechanics as formulated, for instance, in Gillespie. When it comes to the eigenvalues and eigenvectors of the respective operators \( A \), that will mean that (6) and (7) alone must hold.

And then we are led to the

**Reformulated Question** :

Is it possible to *relabel* in (15) the Hermitian operators according to

\[
(16) \quad A' \leftrightarrow \mathcal{A}
\]

in such a way that

*) the relations (6) and (7) still hold

**) the correspondence principle, and in particular (11), (12) and (13) also hold

***) the operator \( A' \) in (15) need *not* be a unitary transformation of \( A \) in (15) ?

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References

[1] Davies P C W : Quantum Mechanics. Routledge & Kegan Paul, London, 1987

[2] Gillespie D T : A Quantum Mechanics Primer. Billing, London, 1973