AN OBSTRUCTION TO A KNOT BEING DEFORM-SPUN
VIA ALEXANDER POLYNOMIALS

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Abstract. We show that if a co-dimension two knot is deform-spun from a
lower-dimensional co-dimension 2 knot, there are constraints on the Alexander
polynomials. In particular this shows, for all $n$, that not all co-dimension 2
knots in $S^n$ are deform-spun from knots in $S^{n-1}$.

In co-dimension 2 knot theory [6], typically the term ‘$n$-knot’ denotes a mani-
fold pair $(S^{n+2}, K)$ where $K$ is the image of a smooth embedding $f: S^n \to S^{n+2}.$
An $n$-ball pair is a pair $(D^{n+2}, J)$ where $J$ is the image of a smooth embedding
$f: D^n \to D^{n+2}$ such that $f^{-1}(\partial D^{n+2}) = \partial D^n$. Every $n$-knot $K$ is isotopic to
a union $(S^{n+2}, K) = (D^{n+2}, J) \cup_\partial (D^{n+2}, D^n)$ for some unique isotopy class of
$n$-ball pair $(D^{n+2}, J)$, provided we consider $K$ to be oriented. Let $\text{Diff}(D^{n+2}, J)$
denote the group of diffeomorphisms of an $n$-ball pair $(D^{n+2}, J)$. That is, $f \in \text{Diff}(D^{n+2}, J)$
means that $f$ is a diffeomorphism of $D^{n+2}$ which restricts to the identity on
$\partial D^{n+2} = S^{n+1}$, is isotopic to the identity (rel boundary) as a diffeomorphism
of $D^{n+1}$, and $f$ preserves $J$. $f(J) = J$. We say an $n$-knot $(S^{n+2}, K)$ is deform-
spun from an $(n-1)$-knot $(S^{n+1}, K') = (D^{n+1}, J') \cup_\partial (D^{n+1}, D^{n-1})$ if there exists
$g \in \text{Diff}(D^{n+1}, J')$ such that the pair $((D^{n+1}, J') \times_S S^1) \cup_\partial ((S^n, S^{n-1}) \times D^2)$
is diffeomorphic to the pair $(S^{n+2}, K)$. Here $(D^{n+1}, J') \times_S S^1$ is the bundle over $S^1$
with fibre $(D^{n+1}, J')$ and monodromy given by $g$ (i.e., $(D^{n+1}, J') \times_S S^1 =
((D^{n+1}, J') \times \mathbb{R})/\mathbb{Z}$, where $\mathbb{Z}$ acts diagonally) on $(D^{n+1}, J')$ and as the group of
universal covering transformations for $\mathbb{R} \to S^1$.

To picture a deform-spun knot, let $g_t$ be a null-isotopy of $g$; i.e., $g_0 = g$, $g_1 =
Id_{D^{n+1}}$ and $g_t$ is a diffeomorphism of $D^{n+1}$ which restricts to the identity on $\partial D^{n+2}$
for all $0 \leq t \leq 1$. Consider $S^{n+2}$ to be the union of a great $n$-sphere $S^n$ and a
disjoint trivial vector bundle over $S^1$. Identify this trivial vector bundle over $S^1$
with $S^1 \times \text{int}(D^{n+1})$ and identify $S^1$ with $\mathbb{R}/\mathbb{Z}$. We assume that the inclusion
$S^1 \times \text{int}(D^{n+1}) \to S^{n+2}$ extends to a map $S^1 \times D^{n+1} \to S^{n+2}$ such that the
restriction $S^1 \times S^n \to S^{n+2}$ factors as projection onto the great sphere $S^n$ followed
by the inclusion $S^n \to S^{n+2}$. Then the set $\{(t, x) \in S^1 \times \text{int}(D^{n+1}) : x = g_t(p), p \in \text{int}(J')\}$ is a subset of $S^{n+2}$ whose closure is an $n$-knot. This is the deform-spun knot; see Figure 1.

The main observation of this paper is that if $K$ is an $n$-knot, deform-spun from an $(n-1)$-knot $K'$, then there is a relationship between the Alexander modules of $K$ and $K'$ which give rise to constraints on the Alexander polynomials $\Delta_1, \cdots, \Delta_n$ of $K$.

**Theorem 0.1.** Let $K$ be an $n$-knot which is deform-spun. Then there exist polynomials $q_i \in \Lambda = \mathbb{Q}[t^{\pm 1}] = \mathbb{Q}[\mathbb{Z}]$ for $i = 0, 1, \cdots, n$ which satisfy $q_{i+1}q_i = \Delta_{i+1}$ ($q_0 = q_n = 1$) and $q_{n-i} = \overline{q_i}$ for all $i$, where we use the convention $\overline{q_i}(t) = q_i(t^{-1})$.

An elementary consequence of this theorem is that for each $n \geq 2$, not every $n$-knot is deform-spun from an $(n-1)$-knot. This follows from the work of Levine [4] who gave a characterization of the Alexander modules of co-dimension 2-knots. In particular Levine shows that an $n$-knot has Alexander polynomials $\Delta_1, \cdots, \Delta_n \in \Lambda$ which satisfy the relations $\Delta_i(1) \neq 0$, $\overline{\Delta_i} = \Delta_{n-i}$ for all $i$. Moreover, these relations are complete in the sense that given any $n$ polynomials which satisfy these relations, there is an $n$-knot which has the specified Alexander polynomials. The case $n = 2$ has a particularly simple example. Theorem 0.1 states that if $K$ is deform-spun, then $\Delta_1 = \Delta_1$, yet there are 2-knots such that $\Delta_1$ is not symmetric. See example 10 of Fox’s Quick Trip [2], which describes a 2-knot such that $\Delta_1(t) = 2t - 1$.

Litherland’s deform-spinning construction has its origin in papers of Fox and Zeeman. Fox’s ‘Rolling’ [3] paper gave a heuristic outline of the notion eventually called deform-spinning, as a graphing process from a ‘relative 2-dimensional braid group’ which nowadays is frequently called the fundamental group of the space of knots, or (in a slightly different setting) the mapping class group of the knot complement [1]. Zeeman proved that the complements of co-dimension two $n$-twist-spin knots fibre over $S^1$ provided $n \neq 0$ [8]. Litherland [7] went on to formulate a general situation where a deform-spin knot complements fibre over $S^1$. Specifically, Litherland proved that if the diffeomorphism $g : (D^{n+1}, J') \to (D^{n+1}, J')$ preserves
a Seifert surface for the knot \((S^{n+1}, K')\) corresponding to the \((n - 1)\)-disc pair \((D^{n+1}, J')\), then the deform-spun knot associated to the diffeomorphism \(M \circ g : (D^{n+1}, J') \rightarrow (D^{n+1}, J')\) has a complement which fibres over \(S^1\), provided \(M : (D^{n+1}, J') \rightarrow (D^{n+1}, J')\) is a non-zero power of the meridional Dehn twist about \(J'\).

This paper was largely motivated by a result in ‘high’ co-dimension knot theory. In the paper [1] the first author gave a new proof of Haefliger’s theorem, where the monoid of isotopy classes of smooth embeddings of \(S^j\) in \(S^n\) is a group, provided \(n - j > 2\). The heart of the proof is showing that if \(n - j > 2\), then every knot \((S^n, K)\) (where \(K \simeq S^j\)) is deform-spun from a lower-dimensional knot \((S^{n-1}, K')\), where \(K' \simeq S^{j-1}\). Moreover, all knots \((S^n, K)\) are \(i\)-fold deform-spun for \(i = 2(n - j) - 4\), in the sense that one obtains \((S^n, K)\) to be iterating the deform-spinning process \(i\) times. So in a sense this paper represents an investigation of the extreme case \(n - j = 2\). A second motivation is the observation that frequently the groups \(\pi_0\text{Diff}(D^j, J')(\text{a 1-ball pair})\) are quite large [1], in the sense that their classifying spaces all have the homotopy-type of finite-dimensional manifolds, but the dimension of these manifolds can be arbitrarily large. So there are many ways to construct 2-knots by deform-spinning a 1-knot. As far as the authors know, this paper represents the first-known obstructions to knots being deform-spun.

1. Asymmetry obstruction

Given a co-dimension 2-knot \(K\) in \(S^{n+2}\), the complement of the knot, \(C_K\), is a homology \(S^1\). Let \(\tilde{C}_K\) denote the universal abelian cover of \(C_K\), i.e., the cover corresponding to the kernel of the abelianization map \(\pi_1 C_K \rightarrow \mathbb{Z}\), and consider \(H_i(\tilde{C}_K; \mathbb{Q})\) to be a module over the group-ring of covering transformations \(\Lambda = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]\). This is called the \(i\)-th Alexander module of \(K\). \(H_i(\tilde{C}_K; \mathbb{Q})\) is a finitely-generated torsion \(\Lambda\)-module for each \(i\), so \(H_i(\tilde{C}_K; \mathbb{Q}) \simeq \bigoplus_j \Lambda/p_j\) for some collection of polynomials \(p_j\). The product of these polynomials \(\prod_j p_j\) is called the \(\tilde{C}_K\)-Alexander polynomial of \(K\), or the order ideal of the \(i\)-th Alexander module \(H_i(\tilde{C}_K; \mathbb{Q})\), denoted \(\Delta_i\). In general, the order ideal of a finitely generated torsion \(\Lambda\)-module \(M\) will be denoted \(\Delta_M\). A theorem of Levine’s [4] is that Poincaré Duality combined with the Universal Coefficient Theorem induces an isomorphism \(H_i(\tilde{C}_K; \mathbb{Q}) \simeq \text{Ext}_\Lambda(H_{n+1-i}(\tilde{C}_K; \mathbb{Q}), \mathbb{Q})\). Here, if \(M\) is a \(\Lambda\)-module, \(\overline{M}\) denotes the conjugate \(\Lambda\)-module. This is a module whose underlying \(\mathbb{Q}\)-vector space is \(M\), but where action of the generator \(t\) on \(\overline{M}\) is defined as the action of \(t^{-1}\) on \(M\). Thus, the only Alexander polynomials of \(K\) which can be non-trivial are \(\Delta_1, \cdots, \Delta_n\), and they satisfy the relation \(\overline{\Delta_i} = \Delta_{n+1-i}\) for all \(i\).

We collect some elementary results about \(\Lambda\)-modules that will be of use in the proof of Theorem 1.1. To state the lemma, let \(\mathbb{Q}(\Lambda)\) denote the field of fractions of \(\Lambda\), i.e., the field which consists of rational Laurent polynomials.

**Lemma 1.1.** (a) (see [6], 7.2.7) Given a short exact sequence of finitely generated torsion \(\Lambda\)-modules

\[0 \rightarrow H_1 \rightarrow H \rightarrow H_2 \rightarrow 0,\]

the order ideals satisfy \(\Delta_{H_1}\Delta_{H_2} = \Delta_H\).
(b) (see [3], Proposition 4.1) Let $H$ be a finitely-generated torsion $\Lambda$-module. There is a natural isomorphism of $\Lambda$-modules
\[ \text{Ext}_\Lambda(H, \Lambda) \simeq \text{Hom}_\Lambda(H, \mathbb{Q}/\Lambda). \]

(c) With the same setup as (b), there is a natural isomorphism of $\mathbb{Q}$-vector spaces
\[ \text{Hom}_\Lambda(H, \mathbb{Q}/\Lambda) \simeq \text{Hom}_\mathbb{Q}(H, \mathbb{Q}), \]
where we interpret $\Lambda \subset \mathbb{Q}(\Lambda)$ as the rational Laurent polynomials with denominator 1.

(d) Let $g : H \to H$ be a $\Lambda$-linear map, where $H$ is a finitely-generated torsion $\Lambda$-module. Let $g^* : \text{Ext}_\Lambda(H, \Lambda) \to \text{Ext}_\Lambda(H, \Lambda)$ be the Ext-dual of $g$. Then $\ker(g)$ and $\ker(g^*)$ have the same order ideals.

Proof of item (c). Consider a rational polynomial $\frac{p}{q} \in \mathbb{Q}(\Lambda)$. The division algorithm allows us to write $p = sq + r$ for Laurent polynomials $s, r \in \Lambda$, where $r \in \mathbb{Q}[t]$ and $\text{deg}(r) < \text{deg}(q)$. To ensure that $r$ is unique, we demand that $\text{GCD}(p, q) = 1$, $q \in \mathbb{Q}[t]$ and the constant coefficient of $q$ is 1. Define a function $\mathbb{Q}(\Lambda)/\Lambda \to \mathbb{Q}$ by sending $\frac{p}{q}$ to the constant coefficient of $r$. The composition with this map is a $\mathbb{Q}$-linear homomorphism, $\text{Hom}_\mathbb{Q}(H, \mathbb{Q}/\Lambda) \to \text{Hom}_\mathbb{Q}(H, \mathbb{Q})$, which is natural and respects connect-sum decompositions of the domain $H$. Thus to verify that it is an isomorphism, we need only check it on a torsion $\Lambda$-module with one generator:
\[ \text{Hom}_\Lambda(\Lambda/p, \mathbb{Q}/\Lambda) \to \text{Hom}_\mathbb{Q}(\Lambda/p, \mathbb{Q}). \]

In this case the target space has dimension $\text{deg}(p)$, the basis given by the dual basis to the polynomials $t^i$ for $0 \leq i < \text{deg}(p)$. The domain also has dimension $\text{deg}(p)$, with basis given by homomorphisms that send 1 to $t^i/p$ where $0 \leq i < \text{deg}(p)$. Hence the map is a bijection between these basis vectors.

To prove item (d), consider the ‘prime factorization’ of $H$. Let $P \subset \Lambda$ be the prime factors of the order ideal $\Delta_H$. Given $p \in P$ let $H_p \subset H$ be the sub-module of elements of $H$ killed by a power of $p$; thus $\bigoplus_{p \in P} H_p \simeq H$. Since $g$ must respect the splitting, we have maps $g_p$ such that
\[ g = \bigoplus_{p \in P} g_p : H_p \to H_p. \]

Thus,
\[ \Delta_{\ker(g)} = \prod_{p \in P} \Delta_{\ker(g_p)}. \]

Let $d_p \in \mathbb{Z}$ be defined so that $\Delta_{\ker(g_p)} = p^{d_p}$. By part (c), $g$ and $g^*$ can be thought of as the $\text{Hom}_\mathbb{Q}(\cdot, \mathbb{Q})$-duals of each other. Thus $\ker(g)$ and $\ker(g^*)$ have the same dimension as $\mathbb{Q}$-vector spaces, and so $\text{dim}_\mathbb{Q}(\ker(g_p)) = \text{deg}(p)d_p$ and $\Delta_{\ker(g_p)}$ is determined by the rank of $\ker(g_p)$ as a $\mathbb{Q}$-vector space. Hence $\ker(g)$ and $\ker(g^*)$ have the same order ideals.

Remark. Although they have the same order ideals, in general the two kernels are not isomorphic as $\Lambda$-modules. An example is given by $g : \Lambda/p \oplus \Lambda/p^2 \to \Lambda/p \oplus \Lambda/p^2$ defined by $g(a, b) = (0, pa)$. In this case, $\ker(g) \simeq \Lambda/p^2$, while $\ker(g^*) \simeq \bigoplus_2 \Lambda/p$.

Proof of Theorem 0.1. Let $C_K$ be the complement of an open tubular neighbourhood of $K \subset S^{n+2}$, and let $C_{K'}$ be the complement of an open tubular neighbourhood of $K' \subset S^{n+1}$. As in the introduction, let $g : (D^{n+1}, J') \to (D^{n+1}, J')$
be the diffeomorphism for the deform-spinning construction of $K$ from $K'$, so we can isotope $g$ so that it preserves a regular neighbourhood of $J' \cup S^n$. Therefore $g$ restricts to a diffeomorphism of $C_{K'}$ (which we can think of as the complement of an open regular neighbourhood of $S^n \cup J'$ in $D^{n+1}$), giving a diffeomorphism
\[ C_K \simeq (\partial C_{K'} \times S^1) \cup_{\nu S^1} (\nu S^1 \times D^2), \]
where $\nu S^1$ is a trivial $D^{n-1}$-bundle over $S^1$ (a meridian of $\partial C_{K'}$). The decomposition lifts to the universal abelian covering space, giving the isomorphism $H_1(C_K; \mathbb{Q}) \simeq coker(I - g\ast)$ and short exact sequences
\[ 0 \to coker(g\ast - I) \to H_i(\tilde{C}_K; \mathbb{Q}) \to ker(g\ast - I) \to 0, \quad i > 1, \]
with $g\ast : H_i(\tilde{C}_K; \mathbb{Q}) \to H_i(\tilde{C}_K; \mathbb{Q})$ the induced map coming from $\tilde{g} : \tilde{C}_K \to \tilde{C}_K'$. Let $q_i$ be the order ideal of $coker(g\ast - I)$.

The map $g\ast - I : H_i(\tilde{C}_K; \mathbb{Q}) \to H_i(\tilde{C}_K'; \mathbb{Q})$ gives rise to a canonical short exact sequence
\[ 0 \to ker(g\ast - I) \to H_i(\tilde{C}_K'; \mathbb{Q}) \to img(g\ast - I) \to 0 \]
and the inclusion $img(g\ast - I) \to H_i(\tilde{C}_K'; \mathbb{Q})$ to another,
\[ 0 \to img(g\ast - I) \to H_i(\tilde{C}_K'; \mathbb{Q}) \to coker(g\ast - I) \to 0. \]
Lemma 4.1(a) applied to our short exact sequences tells us that $\Delta = q_gq_{i-1}$.

We now reconsider the proof of the symmetry of the Alexander polynomial of a knot in $S^3$ [5, 6] or, more precisely, the isomorphism $\overline{H}_1(C_K'; \mathbb{Q}) \simeq H_{n-i}(\tilde{C}_K'; \mathbb{Q})$ derived from Poincaré Duality [4], paying special attention to naturality with respect to diffeomorphisms $g \in Diff(C_{K'})$, with an eye towards proving the symmetry conditions $y_{n-i} = y_i$.

1. $H_i(\tilde{C}_K'; \mathbb{Q}) \simeq H_i(\tilde{C}_K', \partial; \mathbb{Q})$: This is a natural isomorphism coming from the long exact sequence of a pair.
2. $H_i(\tilde{C}_K', \partial; \mathbb{Q}) \simeq H^{n+1-i}(\tilde{C}_K'; \mathbb{Q})$: This is the Poincaré duality isomorphism; it is also natural, although it reverses arrows [4].
3. $H^{n+1-i}(\tilde{C}_K'; \mathbb{Q}) \simeq Ext\Lambda(H_{n-i}(\tilde{C}_K'; \mathbb{Q}), \Lambda)$: This is a natural isomorphism coming from the universal coefficient theorem [4].
4. $Ext\Lambda(H_{n-i}(\tilde{C}_K'; \mathbb{Q}), \Lambda) \simeq H_{n-i}(\tilde{C}_K'; \mathbb{Q})$: This last result uses the fact that both modules have a square presentation matrix, with one being the transpose of the other. Since $\Lambda$ is a principal ideal domain, the presentation matrices are equivalent to the same diagonal matrices. This isomorphism is not natural.

Thus we have a non-natural isomorphism $H_i(\tilde{C}_K'; \mathbb{Q}) \simeq H_{n-i}(\tilde{C}_K'; \mathbb{Q})$. The natural part of the isomorphism can be expressed by the commutative diagram

$$\begin{array}{ccc}
H_i(\tilde{C}_K) & \xrightarrow{g\ast} & H_i(\tilde{C}_K', \partial) \\
\downarrow{g\ast} & & \downarrow{g\ast} \\
H_i(\tilde{C}_K) & \xrightarrow{PD} & H^{n+1-i}(\tilde{C}_K) \\
& & \uparrow{g\ast} \\
& & Ext\Lambda(H_{n-i}(\tilde{C}_K), \Lambda) \\
& & \uparrow{(g\ast)^*} \\
& & Ext\Lambda(H_{n-i}(\tilde{C}_K), \Lambda) \\
& & \uparrow{(g\ast)^*}
\end{array}$$

This gives us an isomorphism of $\Lambda$-modules $ker(I - g\ast) \simeq ker(I - (g\ast)^{-1}_{(n-i)\ast})$, so
\[ ker(I - g\ast) \simeq ker(I - (g\ast)^{-1}_{(n-i)\ast}) = ker(I - (g\ast)^{-1}_{(n-i)\ast}). \]
Lemma 1.1(d) tells us that $\ker(I - (g_{(n-i)*})^*)$ and $\ker(I - g_{(n-i)*})$ have the same order ideals. Thus, $\mathcal{Q}_i = q_{n-i}$. □

2. Comments and questions

Levine [4] has a complete characterization of the Alexander modules of co-dimension two knots. A natural question would be, Could one further derive other obstructions to deform-spinning from the Alexander modules of knots? The primary aspect of Levine’s work that we’ve neglected is the $\mathbb{Z}$-torsion submodule of $H_i(\widetilde{C}_K; \mathbb{Z})$. Simple experiments show that when $K \subset S^{n+2}$ is deform-spun from a knot $K' \subset S^{n+1}$, the Alexander modules of $K$ can have $\mathbb{Z}$-torsion, even when the Alexander modules of $K'$ do not. Moreover, twist-spinning suffices to produce many such examples. So any torsion obstructions to deform-spinning, if they exist, would likely be fairly subtle.

In co-dimension larger than two, deform-spinning is the boundary map in the pseudo-isotopy long exact sequence for embedding spaces and diffeomorphism groups [1]. Moreover, Cerf’s Pseudoisotopy Theorem states that in the case of diffeomorphism groups of discs, this map is onto, provided the dimension of the disc is 6 or larger. So one might expect an analogy.

Question 2.1. Is there a simple characterization of deform-spun co-dimension two knots $K \subset S^{n+2}$ (provided $n$ is large)?

One would certainly expect more obstructions to deform-spinning than the ones in this paper. For example, let $K_1$ and $K_2$ be two otherwise unrelated 2-knots such that $\Delta_{K_1}(t) = 2 - t$ and $\Delta_{K_2}(t) = 2t - 1$. Their connect sum has Alexander polynomial $\Delta_{K_1 \# K_2}(t) = -2t^2 + 3t - 2$, which is symmetric, but we have no reason to expect that $K_1 \# K_2$ is deform-spun.

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