Microscopic model for relativistic hydrodynamics of ideal plasmas

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Abstract. Relativistic hydrodynamics of classic plasmas is derived from the microscopic model in the limit of ideal plasmas, while the microscopic description assumes the level of dynamics of individual particles and the kinetic models look as the macroscopic models in this approach. The chain of equations is constructed step by step starting from the concentration evolution. It happens that the energy density and the momentum density do not appear at such approach, but new relativistic hydrodynamic variables appear in the model. These variables has no nonrelativistic analogs, but they are reduced to the concentration, the particle current, the pressure (the flux of the particle current) if relativistic effects are dropped. These variables are reduced to functions of the concentration, the particle current, the pressure if the thermal velocities are dropped in compare with the relativistic velocity field. Final equations are presented in the monopole limit of the mean-field (the self-consistent field) approximation. Hence, the contributions of the electric dipole moment, magnetic dipole moment, electric quadrupole moment, etc., of the macroscopically infinitesimal element of volume appearing in derived equations are dropped.

1 Introduction

The relativistic plasmas are found in the astrophysical objects and in the laboratory experiments on the laser-surface interaction and thermonuclear devises. The role of the relativistic effects is also important for the quantum plasmas [1–10]. The degenerate electrons demonstrate the noticeable quantum effects in the relativistic regime [11–14]. Therefore, the full relativistic model requires the quantum effects as well. However, this paper is focused on the classical plasmas and the relativistic temperature effects.

The hydrodynamic description of relativistic plasmas with the relativistic temperatures (so at least one species has relativistic temperature) is under consideration, so the temperature (in the energy units) of electrons (like the lightest particle in the plasma) is comparable with the rest energy of the electron $m_e c^2$, where $m_e$ is the mass of electron, and $c$ is the speed of light. One of simplest derivations of the relativistic hydrodynamics of the ideal liquid starts from the energy-momentum tensor in the rest frame [15,16]:

$$T^{\alpha \beta} = (\epsilon + p) u^\alpha u^\beta - pg^{\alpha \beta},$$

where $\epsilon$ and the isotropic pressure $p$ are combined into the enthalpy $w = \epsilon + p$, and the following components of the four-velocity field in the rest frame $u^0 = 1$, $u^i = 0$. Here and below the Greek letters denote the four dimensions $\alpha = 0, 1, 2, 3$, and the Latin letters denote three-dimensional (spatial) indexes $i = 1, 2, 3$.

Knowledge of the energy-momentum tensor leads to the following equation of motion [16] (see also [19])

$$\partial_\beta T^{\alpha \beta} = 0. \tag{1}$$

If we consider the charged particles interacting with the electromagnetic field, we need to extend Eq. (1)

$$\partial_\beta T^{\alpha \beta} = qnF^{\alpha \beta} w^\beta, \tag{2}$$

where $q$ is the charge of particle, $n$ is the concentration, and $F^{\alpha \beta}$ is the tensor of electromagnetic field.

Here the rest frame is defined for the macroscopically motionless fluid. However, the rest frame can be defined in the presence of the local macroscopic flows or/and turbulence.

Next step is generalization of the presented in the rest frame hydrodynamic equations to the arbitrary inertial frame. It includes the Lorentz transformation for the representation of hydrodynamic equations (or the energy-momentum tensor) in the arbitrary frame, where the arbitrary frame moves with the velocity $u^i$ relatively the rest frame.

Apparently, the velocity of the frame $u^i$ is not related to the local flows which can exist in the fluid and described by the velocity field. Moreover, it is essential to point out that velocity $u^i$ is a constant. It does not
depend on the position, while the velocity field of the fluid is a function of space. The following remark is in order. The application of the described method locally cannot be done, since the Lorentz transformation is a global transformation. Described transformation is useful for the study of beams or global flows with nonrelativistic temperatures. It can be assumed that the energy density and the pressure in the energy-momentum tensor correspond to the relativistic temperatures via suitable equations of state. Since the Lorentz transformation makes the motion of the whole system or the motion of the observer relatively medium and does not related to the appearance of the local flows.

The energy-momentum tensor $T^{\alpha\beta}$ is constructed of all possible tensors and vectors in the isotropic fluid which are the Kronecker symbol $\delta^{\alpha\beta}$ and the velocity vector $v^\alpha$. However, the presented below analysis of the relativistically hot plasmas demonstrates presence of another nontrivial four-vector. Hence, the energy-momentum tensor will be reconsidered after the introduction of our model.

Presented in this paper equations are derived under the assumption of an ideal (dissipationless) plasma. Real plasmas exhibit some level of dissipation, particularly at small scales. Hence, the fully ideal description is invalid. Nevertheless, it is still meaningful to consider the ideal description as relevant time scales that are sufficiently short compared to the dissipation timescale. Particularly, Refs. [17,18] shows that the ideal plasma dynamics preserves generalized connections between the “fluid” plasma elements and the magnetic and vorticity field lines. These magnetofluid connections play a pivotal role in governing the plasma dynamics by preventing evolutions that do not preserve the magnetofluid connectivity.

Refs. [20–28] present development and application of relativistic hydrodynamics of neutral fluids, where the translational motion and viscous effects are considered in the relativistic regimes. Reference [29] describes dissipative effects in relativistic charged fluids. These models are justified phenomenologically or from the kinetic model. However, both the hydrodynamic and kinetic models require justification from the microscopic point of view, where the motion of individual particles is considered. Moreover, the explicit form of the averaging on the physically infinitesimal volume is required as well. This items are considered in this paper.

The comments described above show some problems of the hydrodynamic model constructed phenomenologically on the macroscopic scale. Anyway, present level of knowledge requires some microscopic justification of the macroscopic models. Let us specify that the microscopic description assumes the level of dynamics of individual particles, while the kinetic models appear to be as the macroscopic models in this approach. Sometimes authors refer to the kinetic theory as the microscopic model. Here, we mean that the microscopic level is the scale, where motion of each particle is distinguishable (the scale of electrons and protons for the hydrogen plasmas). From this point of view, the kinetics is the macroscopic method of description formulated in the six-dimensional space of coordinates and momentum. In our derivation, we avoid derivation of the kinetic model as an intermediate stage. We directly derive the hydrodynamics from the microscopic motion of particles.

This paper is organized as follows. In Sect. 2, derivation of the relativistic hydrodynamic model based on exact microscopic motion of particles is shown. The basic definitions and method of derivation are demonstrated. In Sect. 3, the suggested relativistic hydrodynamic model is represented in terms of the velocity field. In Sect. 4, method of derivation of equations of state is described to make truncation of the set of equations. In Sect. 5, four-vector notations are presented for the novel structure of hydrodynamic equations. In Sect. 6, the microscopic structure of the energy-momentum tensor appearing via the microscopic derivation is demonstrated and discussed. In Sect. 7, the covariant form of hydrodynamic equations on the example of the energy-momentum evolution is derived within suggested method. In Sect. 8, the dispersion dependence of high-frequency longitudinal plane waves in the relativistically hot isotropic plasmas is presented. In Sect. 9, the zero-temperature limit of the model is found to demonstrate its agreement with well-known results for the relativistic beams in the low temperature plasmas. In Sect. 10, brief discussion of the plane waves in the relativistically hot magnetized plasmas is given. In Sect. 11, a brief summary of obtained results is presented.

2 Model

Our goal is the derivation of the macroscopic hydrodynamic equations for the relativistic plasmas with the relativistically large temperatures. We are going to trace exact microscopic evolution of particles obeying the relativistic modification of the Newton equations of motion.

2.1 Basic definitions

We start our derivation with the definition of the concentration of particles. In classical physics, the particle is modeled as the point-like object. Hence, its mathematical representation is the delta function. (One particle in the zero volume space gives infinite particle number density in one point and zero density in other point.) Therefore, the microscopic concentration of the classical system of particles is the sum of delta functions:

$$n_{\text{mic}}(\mathbf{r}, t) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i(t)), \quad (3)$$

where the subindex mic refers to the microscopic definition of the concentration of particles. Function $\mathbf{r}_i(t)$ is the radius vector of $i$-th particle. Its evolution happens in accordance with the Newton equation of motion,
where the interaction with all surrounding particles is included. Presented here microscopic definition of concentration evolves in accordance with exact microscopic evolution of particles. So, if one wants to find its evolution, one needs to solve set of Newton equations of motion. Our goal is to construct approximate macroscopic model to capture main features of the relativistic plasmas.

However, if we consider the macroscopic theory, we need to introduce the macroscopically infinitesimal element of volume \( \Delta \) and present the number of particles in this element of volume

\[
n(\mathbf{r}, t) \equiv n_{\text{mac}}(\mathbf{r}, t) = \frac{1}{\Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)).
\]

The presented integral does not directly related to the simplified point-like structure of particles. It counts number of particles in the chosen vicinity \( \Delta \) of the chosen point \( \mathbf{r} \) since the integral of the delta function is equal to one if the particle is in the vicinity and the integral of the delta function is equal to zero if the particle is outside of the vicinity. Equation (4) can be used to the center of mass of finite size objects. However, for the point-like objects, Eq. (4) is equivalent to microscopic concentration (3).

The macroscopic concentration can be rewritten in different form which can provide more physical inside of the presented definition (4):

\[
n(\mathbf{r}, t) = \frac{1}{m\Delta} \sum_{i=1}^{N(\mathbf{r}, t)} m_i,
\]

where all \( m_i \) are equal to each other. (The concentration is defined for each species in plasmas.) Function \( N(\mathbf{r}, t) \) is the number of particles in the \( \Delta \)-vicinity of point \( \mathbf{r} \) at the fixed moment in time \( t \). These definitions (4) and (5) are equivalent to each other, but (5) less useful for the derivation of the continuity equation since the unknown function \( N(\mathbf{r}, t) \) is present in the upper limit of summation. The integral operator in Eq. (4) counts the number of particles \( N(\mathbf{r}, t) \) following their exact motion in accordance with the trajectory of each particle \( \mathbf{r}_i(t) \). Equation (5) can be rewritten with no application of mass of particles \( m_i \) via the count of units: \( n(\mathbf{r}, t) = \sum_{i=1}^{N(\mathbf{r}, t)} \mathbf{1}_i/\Delta = N(\mathbf{r}, t)/\Delta \).

At initial step, we have no conditions on volume \( \Delta \). If we need to present a macroscopic theory, we should have \( \Delta \) large enough so \( \Delta \)-vicinity of each point of space contains macroscopically large number of particles. So, \( \Delta \)-volume can play the role of the macroscopically infinitesimal element of space. On the other hand, we can show transition from Eq. (4) to Eq. (3) at \( \Delta \to 0 \). If the point of space \( \mathbf{r} \) contains a particle (or several point-like particles), we have \( n(\mathbf{r}, t) = 1/\Delta \to 0 \to \infty \). If the point of space \( \mathbf{r} \) does not contain any particle we find \( n(\mathbf{r}, t) = \lim_{\Delta \to 0}(0/\Delta) = 0 \). So, we have \( N \) delta functions giving \( N \) infinite values at points \( \mathbf{r} = \mathbf{r}_i(t) \) for \( i \in [1, N] \) and zero values in other points. So, we have same distribution as in Eq. (3).

A possible candidate for the concentration definition is

\[
\hat{n}(\mathbf{r}, t) = \frac{1}{\Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} \gamma_i \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)),
\]

where \( \gamma_i = (1 - v_i^2(t)/c^2)^{-1/2} \) with \( v_i(t) = d\mathbf{r}_i(t)/dt \) the velocity of \( i \)th particle. However, function \( \hat{n} \) is proportional to the energy density \( \hat{\varepsilon}(\mathbf{r}, t)/mec^2 \), and hence it obviously does not satisfy the continuity equation, where \( m \) is the mass of particle of the species under consideration and \( c \) is the speed of light.

Presented method is the three-dimensional reduction of the method of microscopic derivation of the relativistic kinetics [30].

2.2 Continuity equation

Analysis of the concentration dynamics can be obtained without the equation of motion since it is defined by the kinematic effects

\[
\partial_t n(\mathbf{r}, t) = \frac{1}{\Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} \partial_t \delta(\mathbf{r} + \xi - \mathbf{r}_i(t))
\]

\[
= \frac{1}{\Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} (-\mathbf{v}_i(t)) \cdot \partial_\xi \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)).
\]

The derivative on the space variable \( \partial_\xi \) can be placed outside of the integral in the last term. Hence, the time evolution of the concentration leads to the appearance of the particles current:

\[
j(\mathbf{r}, t) = \frac{1}{\Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} \mathbf{v}_i(t) \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)).
\]

Summing up presented results as the well-known continuity equation:

\[
\partial_t n + \nabla \cdot \mathbf{j} = 0.
\]

There is no statistical averaging in the described prescription. Moreover, it is not necessary to use the statistics while we trace the microscopic motion itself. However, some short notations (which can remind statistical physics) are in order. For example, the particle current
appears as the action of operator

\[ \langle \ldots \rangle \equiv \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \]  

(11)
on the velocities of particles. Therefore, the particle current can be written in a short form: \( \mathbf{j} = \langle \mathbf{v}_i \rangle \).

The nonrelativistic hydrodynamics is considered with in this method in Ref. [31]. It is also briefly discussed in Ref. [32].

### 2.3 Equation for the current evolution

Derivation of the other equations including the current evolution equation requires equation of motion for each particle to trace their exact motion in terms of collective (macroscopic) variables. We use the relativistic Newton equations in terms of the velocity evolution (or the acceleration caused by the interaction) [15] (see section 17):

\[ \mathbf{v}_i = \frac{e_i}{m_i} \sqrt{1 - \frac{v_i^2}{c^2}} \left[ \mathbf{E}_i + \frac{1}{c} [\mathbf{v}_i \times \mathbf{B}_i] - \frac{1}{c^2} \mathbf{v}_i \cdot \mathbf{E}_i \right], \]  

(12)

where \( \mathbf{v}_i = \mathbf{v}_i(t), \mathbf{E}_i = \mathbf{E}(\mathbf{r}_i(t), t) \) and \( \mathbf{B}_i = \mathbf{B}(\mathbf{r}_i(t), t) \) are the electric and magnetic fields in the point \( \mathbf{r}_i(t) \) and at time \( t \) acting on \( i \)th particle. Here, we also have \( m_i \) is the mass of \( i \)-th particle, and \( q_i \) is the charge of \( i \)-th particle.

We consider the evolution of the particle current \( \mathbf{j}(\mathbf{r}, t) \). To this end, we calculate the time derivative of the current:

\[ \partial_t j^a(\mathbf{r}, t) = \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} v_i^a(t) \partial_t \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \]

\[ + \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} \partial_i^a(t) \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \]  

(13)

We present the presented equation introducing the flux of the particle current \( \Pi^{ab} \) and the force field \( F^a \)

\[ \partial_t j^a + \partial_i \Pi^{ab} = F^a, \]  

(14)

where

\[ \Pi^{ab} = \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} v_i^a(t) v_i^b(t) \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)), \]  

(15)

and

\[ F^a = \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} v_i^a(t) \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \]  

(16)

Strictly speaking, the introduced force field \( F^a \) is not exactly the force field, since the force causes the change of momentum \( \mathbf{p}_i = \mathbf{F}_i \), while \( F^a \) is the acceleration of particles given by Eq. (12) in terms of the electromagnetic field acting on the particle.

Next, we substitute the acceleration from Eq. (12) in the force field \( F^a \) (16) and find the expression containing new hydrodynamic functions

\[ F^a = \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{e_i}{m_i \gamma_i} E^a(r + \xi - r_i(t)) \]

\[ + \frac{1}{c^2} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{e_i}{m_i \gamma_i} v_i^b B^c(r + \xi - r_i(t)) \]

\[ - \frac{1}{c^2} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{e_i}{m_i \gamma_i} v_i^a v_i^b E^c(r + \xi - r_i(t)) \].  

(17)

In Eq. (16), we apply the replacement of coordinate of particles in the argument of the electromagnetic field \( \mathbf{r}_i(t) \) on \( \mathbf{r} + \xi \) using the \( \delta \)-function \( \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \).

We consider the monopole approximation of the electric and magnetic fields \( E^a(r + \xi, t) \approx E^a(r, t), \) and \( B^a(r + \xi, t) \approx B^a(r, t) \). It corresponds to the mean-field (self-consistent field) approximation traditionally applied at the study of the plasmas. This approximation is possible if the electric and magnetic fields have small change on the scale of \( \Delta \)-vicinity. For instance, if the particle is under the action of the electromagnetic wave its wavelength \( \lambda \) should be large in compare with the radius of the vicinity: \( \lambda \gg \sqrt{\Delta} \).

The mean-field approximation simplify the force field to the following form

\[ F^a = E^a(r, t) \cdot \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{e_i}{m_i \gamma_i} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \]

\[ + \frac{1}{c^2} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{e_i}{m_i \gamma_i} v_i^b B^c(r + \xi - r_i(t)) \]

\[ - \frac{1}{c^2} E^a(r, t) \cdot \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{e_i}{m_i \gamma_i} v_i^a v_i^b E^c(r + \xi - r_i(t)). \]  

(18)

This procedure gives the mean-field (the self-consistent) approximation.

Evolution of the particle current leads to three new hydrodynamic variables

\[ \Gamma = \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{1}{\gamma_i} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)), \]  

(19)

\[ \Theta^a = \frac{1}{\Delta} \int \Delta \, d\xi \sum_{i=1}^{N} \frac{1}{\gamma_i} v_i^a \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)), \]  

(20)
and
\[ \Xi^{ab} = \frac{1}{\Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} \frac{1}{\gamma_i} v_i^a v_i^b \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)). \] (21)

Overall, we find the following equation
\[ \partial_t j^a + \partial_a \Pi^{ab} = \frac{e}{m} \left( \Gamma \mathcal{E}^a + \frac{1}{c^2} \Xi^{ab} \mathcal{B}^c - \frac{1}{c^2} \Xi^{ab} \mathcal{E}^b \right), \] (22)

which is the particle current evolution equation. It is one of the generalizations of the nonrelativistic Euler equation for the velocity field on the relativistic regime.

A brief presentation of this model is given in short report [33] Some steps toward present model are made in Ref. [34].

### 2.4 New relativistic hydrodynamic variables

Relativistic equation for the particle current evolution gives three new hydrodynamic functions \( \Gamma, \Theta^a, \) and \( \Xi^{ab}. \) These hydrodynamic functions do not exist in the nonrelativistic limit. The particle current \( j^a \) coincides with the momentum density in the nonrelativistic limit. Moreover, the hydrodynamic Gamma function \( \Gamma \) tends to the concentration \( n \) in the nonrelativistic limit. The hydrodynamic Theta function \( \Theta^a \) has same relativistic limit as the current of particles \( j^a \) and the momentum density. The hydrodynamic Xi function has same form in the relativistic regime as the flux of the particle current \( \Pi^{ab} \) (15) and the momentum flux (containing the pressure).

We have two different functions in the relativistic hydrodynamics: the current of particles and the momentum density. Hence, we need to choose which of them should be included in the model. The answer on this question follows from the Maxwell equations, where the electromagnetic field is caused by the concentration and the current of particles. Therefore, we consider the momentum density as a function which is unnecessary for application in our model.

Hydrodynamic Gamma function \( \Gamma \) is the “average” reverse relativistic gamma factor \( \Gamma = \langle \frac{1}{\gamma_i} \rangle \) (see Eq. (19)). Hydrodynamic Theta vector function \( \Theta^a \) is the current of the reverse gamma factor. So, vector field \( \Theta^a \) is the current of the scalar function \( \Gamma: \Theta^a = \langle v_i^a / \gamma_i \rangle \) (see Eq. (20)). Next, the tensor field \( \Xi^{ab} \) is the flux or current of the vector field \( \Theta^a: \Xi^{ab} = \langle v_i^a v_i^b / \gamma_i \rangle \) (see Eq. 21).

### 2.5 Equations evolution for the new relativistic hydrodynamic variables

For the further development of the relativistic hydrodynamic equations consider the evolution of the hydrodynamic Gamma and Theta functions.

It is well known that the application of nonrelativistic hydrodynamic composed of the continuity and Euler equations gives incorrect coefficient for the thermal contributions in the dispersion dependencies in compare with the kinetic results [35,36]. However, the extension of the hydrodynamic model including the pressure evolution equation allows to improve the results [13,35–37]. Here, we try to create a relativistic minimal coupling hydrodynamic model, which is based on the evolution of the scalar and vector functions \( n, j^a, \Gamma, \Theta^a, \) where the tensor fields of the higher tensor rank (like \( \Pi^{ab} \) and \( \Xi^{ab} \)) are expressed using equations of state.

#### 2.5.1 Equation of the Gamma function evolution

We consider the temporal evolution of the Gamma function according to the described method (by the calculation of the time derivative of the definition of required function)

\[ \partial_t \Gamma + \partial_a \Theta^a = -\frac{1}{c^2 \Delta} \int_{\Delta} d\xi \sum_{i=1}^{N} \frac{1}{\gamma_i} v_i^a v_i^a \delta_i, \] (23)

where the second term in equation comes from differentiating of the delta function, the last term appears from differentiating of the reverse gamma factor, \( \delta_i = \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)) \) is the short representation of the delta function. Moreover, let us mention a technical detail that the gamma factor is in the denominator in the last term of Eq. (23).

We consider the right-hand side of Eq. (23) using the relativistic Newton equation (12)

\[ \langle \frac{1}{\gamma_i} v_i^a v_i^a \rangle = \frac{e}{m} \langle v_i^a \mathcal{E}_i^a \rangle - \frac{1}{c^2} \frac{e}{m} \langle v_i^a \mathcal{E}_i^2 \rangle, \] (24)

where we use that \( \gamma_i v_i^a v_i^a = \gamma_i \frac{e}{m} (\mathcal{E}_i^a - \frac{1}{c^2} v_i^a (v_i \cdot \mathbf{E})) = \gamma_i \frac{e}{m} (v_i \cdot \mathbf{E}) (1 - \frac{1}{c^2} v_i^2) \). It can be rewritten via the gamma factor \( \gamma_i \) as \( \frac{e}{m} (v_i \cdot \mathbf{E}) \), but it gives no further usage.

Ones again, we consider the monopole approximation of the electric field \( \mathcal{E}_i^a(\mathbf{r} + \xi, t) \approx \mathcal{E}_i^a(\mathbf{r}, t) \), which basically gives the mean-field approximation, and find

\[ \langle v_i^a \mathcal{E}_i^a \rangle - \frac{1}{c^2} \langle v_i^a \mathcal{E}_i^2 \rangle = \mathcal{E}_i^a(\mathbf{r}, t) \left( j^a - \frac{1}{c^2} Q^a \right), \] (25)

where

\[ Q^a = \langle v_i^a v_i^2 \rangle \] (26)

is the flux vector of the velocity square which coincides (proportional) with the kinetic energy current in the nonrelativistic regime.

Finally, we present the evolution equation for the Gamma function

\[ \partial_t \Gamma + \partial_a \Theta^a = -\frac{1}{c^2} \mathbf{E} \left( j - \frac{1}{c^2} \mathbf{Q} \right), \] (27)
where we see that the Theta function is the current of the Gamma function (as it is mentioned above judging on the structure of the definitions). Hence, the set of equation partially closes itself. So, evolution of Gamma function leads to functions $\gamma^a$, $\Theta^a$, which are introduced above, and single new function $Q^a$.

2.5.2 Equation for the Theta function evolution

Next equation which appears in the developing model is the evolution equation for the hydrodynamic Theta function.

Differentiating function (20) with respect to time, we find the required equation. Derivative of function (20) leads to three terms: the derivative of the velocity, the derivative of the delta function, and the derivative of the reverse gamma factor.

The derivative of the delta function leads to the flux of current $\Pi^a$, which is presented on the structure of the definitions). Hence, the set of equations is almost closing itself one again. One additional function appears at this step $\Sigma^{ab}$ (21) along with function $Q^a$ (26) obtained at the derivation of Eq. (27). Therefore, we can try to stop at this step and truncate the set of equations.

2.5.3 Xi function evolution

The particle current evolution Eq. (22) suggests that we need evolution equation for the Xi function $\Xi^{ab}$. However, it is necessary to create a limited set of equations. Hence, at this stage of the model development find an equation of state for the Xi function $\Xi^{ab}$ (21).

2.6 Equations of electromagnetic field

The mean-field electromagnetic field $E$ and $B$ appearing in Eqs. (22), (27), and (28) satisfies the Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad (30)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}, \quad (31)$$

$$\nabla \cdot \mathbf{E} = 4\pi \sum_{s=e,i} e_s n_s, \quad (32)$$

and

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} + 4\pi \sum_{s=e,i} e_s \mathbf{j}_s. \quad (33)$$

Full set of hydrodynamic equations is developed for each species, while the superposition of all particles appears as the source of the electromagnetic field. Hence, the sum of species “s” is presented in Eqs. (32) and (33). As an example, we show the summation on two species: the electrons and ions.

Quantum analog of this concept is suggested in Ref. [38]. The many-particle quantum hydrodynamics is developed for the number of physical systems [13,39,40].

3 Velocity field in equations for the relativistic hydrodynamics

We introduce the velocity field as the ratio between the particle current and the concentration $n$. Our goal is to find the contribution of the velocity field in other hydrodynamic functions. The velocity field is the local average velocity, where average means the arithmetical average. It is the average velocity of all particle in the delta vicinity $\Delta$ of point $r$. It allows us to split the velocity of each particle on the average velocity $v$ and the deviation from the velocity field $\mathbf{u}_i$. This deviation is caused by the difference of velocities of particles related to the thermal effects: $\mathbf{v}_i = \mathbf{v} + \mathbf{u}_i$. Function $\mathbf{u}_i$ can be interpreted as the local thermal velocity of $i$-th particle. We can combine definition of the current (9) and definition of the velocity field. It leads to the fact that the average of the thermal velocity is equal to zero $\langle \mathbf{u}_i \rangle = 0$.

Next, we substitute the decomposition of the velocity $\mathbf{v}_i = \mathbf{v} + \mathbf{u}_i$ in the definitions of the hydrodynamic functions to find required representations. The first function under consideration is the flux of current

$$\Pi^{ab} = \langle v^a_i u^b_i \rangle = n v^a v^b + p^{ab}. \quad (34)$$

It includes the flux of current of particles on the thermal velocities $p^{ab} = \langle u^a_i u^b_i \rangle$, which is an analog of pressure, but it is a different function. The terms linear on the
thermal velocity are equal to zero. We consider the representation of hydrodynamic Theta function

$$\Theta^a = \langle \frac{v^a}{\gamma_i} \rangle = \Gamma v^a + t^a, \quad (35)$$

where $t^a = \langle \frac{u^a_i u^b_i u^c_i u^d_i}{\gamma_i} \rangle$, with

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{v^2 + 2v_i u^i + u^2}{c^2}}} \quad (36)$$

Function $t^a$ can be called the thermal part of the hydrodynamic Theta function. However, function $t^a$ also contains the velocity field in nonadditive form. Same is true for the hydrodynamic Gamma function $\Gamma$ since “averaged” reverse gamma factor contains both the velocity field and the thermal velocity. We find the structure of the hydrodynamic Xi function $\Xi^{ab}$ in terms of the velocity field:

$$\Xi^{ab} = \langle \frac{v^a v^b}{\gamma_i} \rangle = \Gamma v^a v^b + v^a t^b + t^a v^b + t^{ab}, \quad (37)$$

where $t^{ab} = \langle \frac{u^a_i u^b_i}{\gamma_i} \rangle$.

Representation of major functions is given above. It allows to give representation of the continuity equation and the particle current evolution equation. However, the hydrodynamic Gamma function evolution equation contains vector $Q^a$. Moreover, the hydrodynamic Theta function evolution equation contains vector $Q^a$ and the partial trace of tensor $L^{abcd}$. So, we need to consider the contribution of the velocity field in these functions.

We show the structure of presented functions:

$$Q^a = \langle u^a_i u^b_i u^c_i \rangle = nv^a v^b v^c + v^a p^b + p^a v^b + q^a, \quad (38)$$

where

$$q^a = \langle u^a_i u^b_i u^c_i \rangle, \quad (39)$$

and

$$L^{abcd} = \langle u^a_i u^b_i u^c_i u^d_i \rangle = nv^a v^b v^c v^d + [v, v]^a, b, c, d + [v, q]^a, b, c, d + M^{abcd}. \quad (40)$$

We have rather huge expression for tensor $L^{abcd}$ in terms of the velocity field. Therefore, we introduce the following notations including permutations of similar terms

$$[v, q]^{a, b, c, d} \equiv v^a v^b v^c v^d + v^a v^b q^{c, d} + v^a v^b p^{c, d} + v^a v^b q^{c, d} + v^a v^b q^{c, d}, \quad (41)$$

and

$$[v, v, p]^{a, b, c, d} \equiv v^a v^b p^{c, d} + v^a v^b p^{c, d}$$

Above, we also introduce the third- and fourth-rank tensors composed of the thermal velocities:

$$q^{abc} = \langle u^a_i u^b_i u^c_i \rangle, \quad (43)$$

and

$$M^{abcd} = \langle u^a_i u^b_i u^c_i u^d_i \rangle. \quad (44)$$

Equation for the hydrodynamic Theta function evolution contains the partial trace of tensor $L^{abcd}$, which has the following form:

$$L^{abcd} = nv^a v^b v^c + v^a v^b p^{c} + 2v^a v^c v^b + 2v^b v^c p^{ab} + v^2 p^{ab} + v^a q^b + v^b q^a + 2v^c q^{abc} + M^{abcd} \quad (45)$$

### 3.1 Intermediate form of the suggested hydrodynamic model

The representation of the hydrodynamic functions via the velocity is given. Therefore, we can represent the hydrodynamic Eqs. (10), (13), (27), and (28) via the velocity field.

The continuity equation has the traditional form:

$$\partial_t n + \nabla \cdot (nv) = 0. \quad (46)$$

Equation of the velocity field evolution:

$$n(\partial_t + (\mathbf{v} \cdot \nabla))v^a + \partial_t p^{ab} = \frac{\epsilon}{m} \Gamma E^a + \frac{\epsilon}{mc^2} \Gamma e^{abc} v^b B^c + \frac{\epsilon}{mc^2} (\Gamma v^a v^b + v^a t^b + t^a v^b + t^{ab}) E^b \quad (47)$$

has familiar left-hand side, while the right-hand side (at least first three terms) has an intuitively understandable structure, but new (untraditional) functions define the right-hand side.

Tensor $p^{ab}$ looks like the pressure. However, the traditional pressure is the momentum transmitted through the element of surface during the unit of time. While $p^{ab}$ is the current of particles transmitted through the element of surface during the unit of time.

Next, we find equation for the hydrodynamic Gamma function

$$\partial_t \Gamma + \partial_b (v^b \Gamma) + \partial_b q^b = -\frac{\epsilon}{mc^2} \left[ nv v^b E_b + \frac{1}{c^2} E^b (nv v^2 + v_b p^{bc} + 2p^{bc} v^c + q_b) \right] \quad (48)$$

Finally, we present the hydrodynamic Theta function evolution equation

$$\partial_t t^a + \partial_b (t^a v^b) + t^b \partial_b v^a + \partial_b t^{ab}$$
These equations have been presented in Refs. of hydrodynamic equations for the relativistic plasmas. Functions values can be obtained via the equilibrium distribution and dynamic models of classical fluids and plasmas. Required equations of state. This situation is usual for the hydrodynamic models of classical fluids and plasmas. However, we have not developed extended model, it is planned as higher rank tensors can give this information. However, the approximate application of the equations for the relativistic plasmas. These definitions are based on the exact microscopic motion of particles which is unknown. However, the approximate application of the equations for the higher rank tensors can give this information. However, we have not developed extended model, it is planned as the future work. Here we go another way and neglect the physical picture demonstrated during derivation. So, some formal technic is applied to get the required equations of state. This situation is usual for the hydrodynamic models of classical fluids and plasmas. Required values can be obtained via the equilibrium distribution functions \( f_0(p) \), where \( p \) is the module of momentum.

For instance the Gamma function is the average of the reverse of the relativistic factor \( \gamma \). Hence, the equilibrium Gamma function can be calculated as \( \Gamma_0 = \int \gamma^{-1}(p) f_0(p) dp \), where \( \gamma(p) = \sqrt{p^2 + m^2c^2}/mc \). Here we make substitution of concepts. We use the distribution function instead of the arithmetic average on the Delta-vicinity.

4 Truncation

The truncation requires functions of state for several functions including \( \Pi^{\cdots} \) (reducing to \( p^{ab} \)), \( \Sigma^{ab} \) (reducing to \( p^{ab} \)), \( Q^{\cdots} \) (reducing to \( q^{ab} \)), \( L^{abcd} \) (reducing to \( M^{abcd} \)). They should be presented via functions \( n, v^a, \Gamma, \Gamma^a \).

Explicit microscopic definitions like (4), (19), (20), (21) are not useful for calculation of equilibrium values of considering functions or derivations of equations of state. These definitions are based on the exact microscopic motion of particles which is unknown. However, the approximate application of the equations for the higher rank tensors can give this information. However, we have not developed extended model, it is planned as the future work. Here we go another way and neglect the physical picture demonstrated during derivation. So, some formal technic is applied to get the required equations of state. This situation is usual for the hydrodynamic models of classical fluids and plasmas. Required values can be obtained via the equilibrium distribution functions \( f_0(p) \), where \( p \) is the module of momentum.

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4.1 Equations of state: rest frame

Calculate required expressions using relativistic equilibrium distribution function presented in the rest frame

\[
f_{0r}(p) = Z e^{-\epsilon/T},
\]

where the subindex 0 refers to the equilibrium state, the subindex \( r \) refers to the rest frame,

\[
Z = \frac{n}{4\pi m^2c^2T K_2(\frac{mc^2}{T})},
\]

where \( T \) is the equilibrium temperature in the energy units, \( p \) is the momentum, \( K_2(b) \) is the second-order Macdonald function, and \( \epsilon = \sqrt{m^2c^4 + p^2c^2} \). The rest frame is the inertial frame, where the fluid is macroscopically motionless or it has minimal energy. The energy-momentum tensor presented in the Introduction is constructed in the first regime for the macroscopically motionless fluid. Same assumption is made for the distribution function (50). The useful application of the rest frame includes the assumption that the deviations from the macroscopically motionless state are relatively small. So, we can consider the medium as the average background, where all structures and flows can be viewed as processes happening on this background. Hence, we introduce some equilibrium-like background for the large amplitude nonlinear processes. The rest frame can be easily associated with this background. If we have infinite number of the large amplitude structures overlapping each over, we have no distinguishable background. In this case, the rest frame can be found as the inertial frame, where system has minimal energy. However, this formal step does not give simple physical picture to include in calculations of the distribution function or the energy-momentum tensor. The Macdonald functions \( K_\mu(b) \) have the following definition (the integral form):

\[
K_\mu(b) = \sqrt{\pi b^\mu} \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_{-\infty}^{+\infty} (t^2 - 1)^{\mu-1/2} e^{-bt} dt,
\]

where \( \text{Re} \mu > -1/2 \) and \( \text{Re} b > 0 \).

For instance, the Macdonald functions \( K_\mu(b) \) allow to calculate analytically the equilibrium value of the average reverse \( \gamma \)-factor: \( \Gamma_0 = m^2c^2T K_1(b) \), where \( b = mc^2/T \).

Presented analysis provides the following equations of state \( p^{ab} = U^p n \delta^{ab} \), \( v^{ab} = U^t n \delta^{ab} \), \( q^{ab} = 0 \), \( M^{abcd} = (U^z_M)^3 n I_0^{abcd} \), \( I_0^{abcd} = \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \), and \( U_p, U_t, U_M, U_z \) are constants.

After the application of the equations of state for the high-rank tensors introduced in the model, we obtain the following set of truncated hydrodynamic equations. The continuity equation:

\[
\partial_t n + \nabla \cdot (n v) = 0.
\]
Equation of the velocity field evolution:

\[ n(\partial_t + (\mathbf{v} \cdot \nabla))\mathbf{v} + U_p^2 \nabla n = \frac{e}{m} \Gamma \mathbf{E} \]

\[ + \frac{e}{mc} \Gamma [\mathbf{v} \times \mathbf{B}] + \frac{e}{mc} [\mathbf{t} \times \mathbf{B}] - \frac{e}{mc^2} \left( \Gamma (\mathbf{v} \cdot \mathbf{E}) + \mathbf{v} (\mathbf{t} \cdot \mathbf{E}) + t (\mathbf{v} \cdot \mathbf{E}) + U_t^2 n \mathbf{E} \right). \]  

The Gamma function evolution equation

\[ \partial_t \Gamma + \nabla \cdot (\mathbf{v} \Gamma) + \nabla \cdot \mathbf{t} = -\frac{e}{mc^2} \left[ n \mathbf{v} \cdot \mathbf{E} - \frac{1}{c^2} (n \mathbf{v} \cdot \mathbf{E}) \mathbf{v}^2 + 5U_p^2 n \mathbf{v}^2 \right]. \]  

The Theta function evolution equation

\[ \partial_t \Theta + \nabla \cdot (\mathbf{v} \Theta) + \nabla \cdot \mathbf{t} = -\frac{e}{mc^2} \left[ n \mathbf{v} \cdot \mathbf{E} - \frac{1}{c^2} (n \mathbf{v} \cdot \mathbf{E}) \mathbf{v}^2 + 5U_p^2 n \mathbf{v}^2 \right]. \]

Equations (53–56) are coupled to the Maxwell Eqs. (30–33). Three different characteristic velocities \( U_p, U_t, \) and \( U_M \) are calculated in the following way.

The application of the isotropic distribution function leads to diagonal form of tensors \( p^{ab} \) and \( \rho^{ab} \). The “diagonal” form is obtained for tensor \( M^{abcd} \) as well: \( M^{abcd} = M_0 (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc})/3 \). The equilibrium expressions for functions \( p, \rho, \mathbf{q}, \) and \( \mathbf{M} \) are used as the equations of state for the nonequilibrium functions. Approximate calculation of functions \( p_{ab}, \rho_{ab}, \mathbf{q}, \) and \( \mathbf{M} \) gives the following representations:

\[ p_{ab} = c^2 \delta_{ab} \mathcal{Z} f_1(\beta)/3, \quad \rho_{ab} = c^2 \delta_{ab} \mathcal{Z} f_2(\beta)/3 \]

\[ M_{ab} = c^4 (\delta_{ab} \delta^{cd} + \delta_{ac} \delta^{bd} + \delta_{ad} \delta^{bc}) \mathcal{Z} f_3(\beta)/15, \]  

where \( \beta = mc^2/T, \mathcal{Z} = 4\pi Z (mc)^3 = n\beta K_2^{-1}(\beta), \)

\[ f_1(\beta) = \int_1^{+\infty} \frac{dx}{x^2} (x^2 - 1)^{3/2} e^{-\beta x}, \]  

\[ f_2(\beta) = \int_1^{+\infty} \frac{dx}{x^2} (x^2 - 1)^{3/2} e^{-\beta x}, \]

and

\[ f_3(\beta) = \int_1^{+\infty} \frac{dx}{x^3} (x^2 - 1)^{5/2} e^{-\beta x}. \]

Functions \( f_1(\beta), f_2(\beta) \) and \( f_3(\beta) \) are calculated numerically below for the chosen values of temperatures. For each function describing the thermal we introduce corresponding velocity \( \delta p = U_p^2 \delta n, \delta t = U_t^2 \delta n, \delta M = U_M \delta n \).

The energy evolution equation, see zero-component (or time-component) of Eq. (71) below, leads to the temperature evolution equation. We consider the collisionless plasmas. The electromagnetic field leads to the change of the temperature in the nonlinear regime. It allows us to consider constant temperature in our model. It gives restriction on the applicability of the suggested isothermal model to the nonlinear effects.

Application of distribution function (50) suggests that we find the final form of equations in the rest frame. Therefore, it is necessary to specify how the rest frame defined for the macroscopically motionless fluid should be defined. Moreover, it is essential due to the presence of turbulence, which is important for most applications. It leads to the presence of large, inhomogeneous velocity fields. This explanation is combined of several parts. We start the derivation of hydrodynamic equations in the arbitrary inertial frame and consider the individual motion of interacting particles. It gives us a chain of equations. In order to find a closed set of equations useful from some applications, we need to make truncation of the set of equations. This is second part of this work. We focus of the regime of nonrelativistic macroscopic velocity field and other fluxes, but this regime include the high temperatures close to \( mc^2 \). The distribution function is used to calculate terms which have thermal nature. In this approximate calculation, we assume that macroscopic fluxes are small in comparison with the thermal motion. Therefore, in the zeroth order on the small corrections, we can consider the distribution function (thermal motion) of the approximately macroscopically motionless fluid. This approach does not require to choose single rest frame, but a number of inertial frames which have nonrelativistic relative velocities. However, we need to identify these preferred reference frames. It is made via the definition of current in Eq. (9), which is proportional to the velocity field. Let us specify this statement. Choosing different inertial frames, we obtain various velocity fields. Preferred reference frames are those, where the velocity field appears to be small in comparison with the speed of light. If there is no inertial frame with described properties, it means that the presented model cannot be applied. Otherwise, it can be used in the chosen inertial frame.

4.2 Equations of state: arbitrary inertial frame

For the calculation of equations of state, we have used the relativistic Maxwellian distribution function in the
rest frame. This approach narrows down the variety of physical scenario, where the suggested model can be applied. Let us describe some backgrounds for the further generalizations. First, we can use the relativistic Maxwellian distribution in the arbitrary frame:

\[ f_0(p) = \hat{Z} e^{U_\alpha p^\alpha / T}, \quad (60) \]

where \( p^\alpha = p^0 + cA^\alpha \) is the canonical momentum, \( p^\alpha = \{\epsilon/c, p\} \) is the four-momentum, \( A^\alpha = \{\phi, A\} \) is the four-potential of the electromagnetic field, \( U^\alpha = \{-c, \mathbf{v}\} \) (while \( U^\alpha = \{c, \mathbf{v}\} \)) is the hydrodynamic four-velocity field giving the four-current \( j^\alpha = nU^\alpha \), \( \mathbf{v} \) is the hydrodynamic velocity field, and

\[ \hat{Z} = \frac{n}{4\pi m^2 c T K_2(mc_T)} e^{\phi/T}. \quad (61) \]

with \( A^0 = \Phi \) The hydrodynamic four-velocity field entering the distribution function along with the concentration presented in \( \hat{Z} \).

More general distribution functions are used in the literature for the derivation of hydrodynamics from the kinetic model. For instance, generalized prefactor in front of exponential function is discussed in Ref. [44], where the prefactor is constructed of several hydrodynamic functions in the covariant form. This function is aimed to cover the gyrokinetic effects in the relativistic plasmas.

5 On the four-vector notation for the suggested structure of hydrodynamic equations

At the phenomenological derivations of the macroscopic equations of motion, we need to use some fundamental principles which allows to avoid some misinterpretations. One of such fundamental principles in relativistic physics is the covariant form of equations. However, sometimes one needs to neglect the covariant form of equations in order to find an appropriate macroscopic in a particular frame. One of the most important examples is the quantum field theory at the finite temperatures, where there is chosen reference frame bound to the thermostat.

In our derivation, we are not bound to particular inertial frame. We start our derivation in the arbitrary inertial frame. However, afterwards we work in this microscopically fixed frame in order to make transition to the macroscopic scale. Systematic application of the equations correctly describing the microscopic motion allows us to obtain the correct macroscopic equations.

We use microscopic equation of motion presented in noncovariant form (12). Consequently, the obtained macroscopic equations appear in noncovariant notations. Moreover, presented equations are found in the three-vector notations. Nevertheless, we can use these equations for the analysis of relativistic effects working in the single inertial frame.

Introduced above hydrodynamic functions are presented in the three-vector notations. However, they can be combined as components of the four-tensors:

\[ \Pi^\alpha\beta = \left( \begin{array}{cc} nc^2 & \Pi_{\alpha\beta} \\ \Pi_{\beta\alpha} & \Pi_{\gamma\gamma} \end{array} \right), \quad (62) \]

and

\[ \Gamma^{\alpha\beta} = \left( \begin{array}{cc} \Gamma^{bc} & \Gamma^{bc} \\ \Gamma^{cb} & \Gamma^{cb} \end{array} \right). \quad (63) \]

These are analogs of the energy-momentum tensor which is described below at the analysis of the four-momentum density evolution.

6 On a possible structure of the energy-momentum tensor

The presented method of the microscopic derivation allows to derive the energy-momentum four-vector and derive equation for its evolution providing the energy-momentum tensor. This equation is not a part of suggested hydrodynamic equations because the found structure of hydrodynamic equations is a consequence revealed by the concentration evolution.

Relativistic hydrodynamic allows several generalization of concentration and velocity field. Such as the particle current and the momentum density coincide in nonrelativistic limit, but they are different functions in general case.

Equations (46–48) show that the energy and momentum do not appear as a part of evolution of concentration and current, while the concentration and current are relevant since they exist as sources of field in the Maxwell equations.

However, we show the energy-momentum \( \{\epsilon/c, p^\alpha\} \) evolution to demonstrate agreement between the presented method and relativistic hydrodynamics presented in other works. Let us to point out the general matrix structure of the energy-momentum tensor together with its microscopic definitions:

\[ T^{\alpha\beta} = \left( \begin{array}{cc} \epsilon & p^h_c \\ p^c_T & T^{ab} \end{array} \right) = \langle n_4^\alpha P^\beta_i \rangle. \quad (64) \]

to the best of my knowledge all relativistic hydrodynamic models are based on the momentum balance equation. Therefore, we present the microscopic definition of the momentum density

\[ p^\alpha(r, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} p_i^\alpha(t) \delta(r + \xi - r_i(t)), \quad (65) \]

where \( p_i^\alpha(t) = \{\epsilon_i/c, p_i\} = \{mc\gamma_i, mc\gamma_i\mathbf{v}_i\} \).
Next, we consider the time evolution of the momentum density via the calculation of the time derivative of the presented function

$$\frac{\partial t p^a}{\partial t} = \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} \left( q_i E^a + \frac{q_i}{c} e^{abc} v^b i_1^c \delta_{i} \right)$$

$$- \frac{\partial_b}{\partial t} \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} p_{i}^a v_{i}^b \delta_{i}, \quad (66)$$

where $\delta_{i} = \delta(r + \xi - r_{i}(t))$.

In the mean-field approximation, Eq. (66) reduces to the following equation:

$$\frac{\partial t p^a}{\partial t} + \frac{\partial_b}{\partial t} (m v_{i}^a v_{i}^b \xi_{i}) = qn E^a + \frac{q}{c} e^{abc} v^b B^c. \quad (67)$$

The left-hand side of Eq. (71) can be rewritten via the energy-momentum tensor in the four-vector. Therefore, the right-hand side should be rewritten in some notations via the tensor of electromagnetic field. Equation (71) requires two equations of state. First, one needs to get rid of the momentum density $p^a$ presenting it in terms of the velocity field to make it in agreement with the right-hand side of momentum evolution Eq. (71) (which is represented via velocity), the Maxwell equations, and the continuity equation.

Moreover, the momentum flux $(v_{i}^a v_{i}^b \xi_{i})$ (containing the pressure tensor) requires an equation of state as the function of concentration and the velocity field. Any hydrodynamic model requires one or several equations of state.

Our model shows that in the relativistic case there are three generalizations of the Euler equation. They are equations for particle current (velocity field) $j \sim v$, t-vector function $t$, and the momentum $p$. All of them have same nonrelativistic limit.

The microscopic energy-momentum tensor (64) is derived as a part of the momentum balance Eq. (71). However, we show above that the relativistic hydrodynamic can be contracted of several scalars $n$, $\Gamma$, $t$, $p$, $M_0$ and two vector fields $v$ and $t$. It gives the following four-vector notations $w^a = \{c, v\}$ and $\Gamma^a = \{\Gamma, t\}$. Hence, an attempt of the microscopic contraction of the energy-momentum tensor $T_{\alpha \beta}$ discussed in Introduction can be generalized up to account of term proportional to $\Gamma^a \Gamma^\beta$ along with $w^a w^\beta$ and $\delta^\alpha_\beta$. However, our further discussion demonstrates no necessity to extend the energy-momentum tensor, but the introduction of novel tensors $\Pi^\alpha_\beta$ (62) and $\Gamma^\alpha_\beta$ (63).

7 Energy-momentum tensor based on covariant form of equations for the microscopic motion of individual particles

We start this section with the presentation of the equation of motion of the single particle in covariant form

$$m \ddot{\xi}^\mu = q F_{\mu \nu} \dot{\xi}^\nu, \quad (68)$$

where $\xi^\mu = \{ct, r\}$ is the four-coordinate, $\dot{\xi}^\mu = d\xi/ds$, $ds = dt \sqrt{1 - \nabla^2 / c^2}$, $F_{\mu \nu}$ is the electromagnetic field tensor.

We need to consider systems of many particles, so we need to present evolution equation for each particle

$$m_s \ddot{\xi}^\mu_i = q_s F_{\mu \nu} \dot{\xi}^\nu_i, \quad (69)$$

where $m_s$ and $q_s$ are the mass and the charge of $i$-th particle, which belong to species $s$, $\xi^\mu_i = \{ct, r_i(t)\}$ is the four-coordinate of $i$-th particle, $\dot{\xi}^\mu_i = d\xi^\mu_i / ds_i$, $ds_i = dt \sqrt{1 - \nabla_i^2 (t) / c^2}$, $F_{\mu \nu}^s$ is the electromagnetic field tensor for the field acting on the $i$-th particle. Function $\xi^\mu_i$ gives us the four-momentum $p^a_i = m_s \xi^\mu_i$.

We start derivation of the evolution equations for the energy-momentum tensor with the definition of this function for species $s$

$$T_{\mu \nu}^s = \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} \frac{p_{i}^\mu (t) p_{i}^\nu (t)}{p_{i}^r (t)} \delta(r + \xi - r_i(t)), \quad (70)$$

Next, we consider its evolution

$$\frac{\partial_t T_{\mu \nu}^s}{\partial t} = \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} \left[ \frac{\partial_b p_{i}^\mu (t)}{\partial t} \right] \delta(r + \xi - r_i(t))$$

$$= \frac{q_s}{m_s c} \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} \frac{p_{i}^\mu}{p_{i}^r} F_{\mu \nu}^s (r_i, t) \delta(r + \xi - r_i(t))$$

$$\approx \frac{q_s}{m_s c} F_{\mu \nu}^s (r, t) j_\nu, \quad (71)$$

where we used $F_{\mu \nu}^s (r, t) = F_{\mu \nu}^s (r + \xi, t) \approx F_{\mu \nu}^s (r, t)$. The right-hand side of Eq. (71) contains the four-current $j_\nu$ which also exists in the Maxwell equations $\partial_\nu F_{\mu \nu} = 4\pi j_\mu / c$. It shows necessity to present an equation of state to connect the four-momentum with the four-current to get a closed set of hydrodynamic equations. The model (10), (13), (27), and (28) or (53)–(56) is constructed to deal with this problem. Hence, this paper gives an alternative way to solve this problem.

An equation of state for the momentum density in the form of $n v$ exists in the literature, where $h$ is the enthalpy. One of the reasons for the development of the model presented in this manuscript is an attempt to avoid this equation of state. The reason is following. First, if we consider nonrelativistic system, we have consequence of functions like concentration (zero moment of the distribution function on velocity), velocity field (proportional to the momentum density and to the first moment of the distribution function), the kinetic pressure (proportional to the second moment of the distribution function), etc. We can continue this chain up to required order before truncation with further application of some equations of state. Second, if we consider the relativistic regime, we have more branches for extending of the chain of equations. Set of moments of
the distribution function on velocity is not equivalent to the set of moments of the distribution function on the momentum. It is well known that the four-momentum is considered to construct the set of hydrodynamic equations. It is useful for the neutral fluids. However, the plasmas require to use the concentration (or the charge density) and the velocity field (proportional to the electric current). Hence, we need to make partial truncation on the level of vector function. I mean to approximately present the momentum density via the velocity field like \( \pi \). Hence, we lose some information on the first step. In our approach, we chose set of functions related to average of different degrees of velocity. So we include concentration and velocity field as sources of the electromagnetic field. We do not use the momentum density. Moreover, we present equation of state for the momentum density. It is shown to specify the energy conservation in our model. Also, we show that the four-momentum can be considered in our approach, but it is not necessary to include it in the final form of the closed set of equations.

8 Longitudinal one-dimensional waves in the relativistic isotropic plasmas

Let us demonstrate the application of the developed relativistic hydrodynamic model (53)–(56) on the relatively simple (but fundamentally important) examples. We start our illustration with the high-frequency longitudinal waves in the isotropic plasmas. We focus on the regime, where plasma is assumed macroscopically motionless, but this plasmas has the relativistic temperature \( T \sim mc^2 \). The temperature is proportional to the trace of pressure, which is a part of the momentum flux tensor \( T^{ab} = \langle u^a u^b \rangle \). Hence, the temperature can be larger then \( mc^2 \) due to the relativistic \( \gamma \)-factor in the definition of \( T^{ab} \).

Considering system is described by nonzero equilibrium concentration \( n_0 \) and the equilibrium average \( \gamma \)-factor \( \Gamma_0 = n_0 K_1(b) / K_2(b) \). The equilibrium velocity field \( v_0 \), vector \( t_0 \), and the electric field \( E_0 \) are equal to zero. The equilibrium magnetic field is equal to zero and perturbations of the magnetic field are absent.

We present the corresponding linearized equations set which follows from Eqs. (53–56) for the linear approximation on the small amplitude perturbations:

\[
\begin{align*}
\partial_t \delta n + n_0 \partial_x \delta v_x &= 0, \quad (72) \\
n_0 \partial_t \delta v_x + \partial_x \delta p &= \frac{e}{m} \Gamma_0 \delta E_x - \frac{e}{mc^2} t_0 \delta E^x, \quad (73) \\
\partial_t \delta \Gamma + \Gamma_0 \partial_x \delta v_x + \partial_x \delta t_x &= \frac{e}{mc^2} \frac{q_0}{c^2} \delta E^x, \quad (74)
\end{align*}
\]

and

\[
\begin{align*}
\partial_t \delta t_x + \partial_x \delta t - \frac{\Gamma_0}{n_0} \partial_x \delta p &= \frac{e}{m} \frac{\Gamma_0}{n_0} \delta E_x \\
&= \frac{e}{m} n_0 \delta E_x - \frac{e}{mc^2} p_0 \delta E_x + 2 \frac{e}{mc^2} p_0 \partial_x \delta E_x. \quad (75)
\end{align*}
\]

Solve Eqs. (72) and (73) to get \( \delta n \) and \( \delta v_x \) as functions of the electric field perturbations \( \delta E_x \) and find the following expressions:

\[
\begin{align*}
\delta v_x &= \omega \frac{e}{m n_0} \frac{\Gamma_0}{\omega^2 - (U_p^2 k^2)} \delta E_x, \quad (76) \\
\delta n &= i k \frac{e}{m} \frac{\Gamma_0}{\omega^2 - (U_p^2 k^2)} \delta E_x. \quad (77)
\end{align*}
\]

Functions \( \delta \Gamma \) and \( \delta t_x \) can be also found as functions of the electric field perturbations \( \delta E_x \) from Eqs. (74) and (75), but these functions go nowhere. They do not affect the evolution of the concentration \( \delta n \) and the velocity field \( \delta v_x \) in the linear approximation. They do not contribute in the equations of field either. The nonlinear evolution should be affected by these functions. This conclusion is correct for the isotropic medium. More complex picture would appear at the presence of the flows and/or the magnetic field.

We substitute the concentration (77) in the Poisson equation \( \nabla \cdot e \pi = 4 \pi \rho \), where \( \rho \) is the charge density of the plasmas. In our case, it goes to \( \partial_x E_x = 4 \pi e \delta n \). It gives the dispersion dependence of the relativistic Langmuir waves:

\[
\omega^2 = \omega_{Le}^2 \frac{\Gamma_0}{n_0} - \frac{1}{c^2} t_0 + U_p^2 k^2. \quad (78)
\]

The Langmuir wave dispersion dependence (78) requires three equations of state, while whole set of linearized equations includes more unknown functions approximately calculated for the truncation.

Necessary equations of state are demonstrated above, where it is shown that function \( t_0 \) is represented via the characteristic velocity \( U_i \). However, we want to discuss briefly its more general structure for different values of the velocity field \( \mathbf{v} \) contributing in \( t_0 \) in the general case.

Function \( \dot{i} \) has the following structure in the arbitrary regime, where the thermal velocity \( \mathbf{u_t} \) and the velocity field \( \mathbf{v} \) are relativistic and comparable to each other

\[
\dot{i} = \left< \frac{u_t^2 u_t^2}{\gamma_i} \right> = \left< u_t^2 u_t^2 \sqrt{1 - \frac{\mathbf{v}^2 + 2 \mathbf{u_t} \cdot \mathbf{v} + \mathbf{u_t}^2}{c^2}} \right>. \quad (79)
\]

Function \( \dot{i} \) reduces to the flux of particle current if we deal with the relativistic beam (beam of electrons for instance), where the thermal effects are negligible

\[
\dot{i} = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \cdot p. \quad (80)
\]
where the velocity field of the beam $v$ is also incorporated.

However, the opposite limit is more interesting. If the thermal effects dominate over the velocity field existing in the relativistically hot beamless plasmas function $I$ combines of fluxless part of thermal effects dominate over the velocity field existing
\[ \tilde{I} \approx \left\langle u_i^2 u_i^2 \left( 1 - \frac{u_i^2}{c^2} \right) \right\rangle - \frac{1}{c^2} \left\langle u_i^2 u_i^2 v_i^4 \frac{1}{\sqrt{1 - u_i^2 / c^2}} \right\rangle. \]
\[ \text{(81)} \]

For the further analysis of dispersion dependence (78), we consider the main term in the expansion (81). Hence, we completely neglect the contribution of the velocity field in this function.

Finally, we have the following dispersion dependence for the longitudinal waves in the macroscopically motionless isotropic plasmas:
\[ \omega^2 = \omega_{Le}^2 \left( \frac{K_1}{K_2} - \frac{U_r^2}{c^2} \right) + U_r^2 k^2. \]
\[ \text{(82)} \]

Fraction $\frac{K_1}{K_2}$ goes to 1 at the small temperatures (so the argument $b = mc^2/T$ goes to infinity), since the argument of the Macdonald functions goes to infinity.

Numerical analysis of dispersion dependence (82) can be found in Ref. [33]. Nevertheless, we want to point out that additional multiplier in front of the square of the Langmuir frequency is below unit. Hence, the relativistic temperature effects reduce the minimal frequency of the minimal frequency of the longitudinal wave below $\omega_{Le}$. This decrease can be of several orders.

8.1 On the causality of suggested model

Let us consider the problem of causality in suggested model. To this end, we present an estimation of the group velocity for the Langmuir wave, which is most fundamental wave phenomenon in plasmas. Here, we demonstrate that the group velocity is smaller than the speed of light. We consider $v_g = d\omega/dk = U_r^2 k/\omega$. In the limit $k \to 0$ ($k \to \infty$), we obtain $v_g = 0$ ($v_g = U_r$). It also requires an estimation for the characteristic velocity $U_r = c\sqrt{f_1(\beta)/(3K_2(\beta))}$. Numerical analysis demonstrates that this characteristic velocity $U_r$ is below the speed of light $U_r < c$. We conclude that the spectrum (82) shows no causality problem. We made same conclusion on the suggested hydrodynamic model.

9 Relativistic beams in the nonrelativistic plasmas

Major goal for the development of the presented hydrodynamic model is the study of plasmas with the large temperatures of order of $m_e c^2$ and above. However, for justification of the model, it is useful to consider relativistic effects in the absence of the temperature effects. This regime corresponds to the propagation of the relativistic beam through the plasmas. Let us consider the high-frequency perturbations in the macroscopically motionless plasmas through which the electron beam propagates. The beam is characterized by the equilibrium concentration $n_{eb} \ll n_{de}$. The equilibrium velocity of beam is $v_0 = \{v_0, 0, 0\}$. We consider propagation of excitations parallel to the direction of the beam propagation $k = \{k, 0, 0\}$. Under described conditions, the set of hydrodynamic Eqs. (53–56) used for each species gives the following dispersion equation:

\[ 1 - \frac{\omega_{Le}^2}{\omega^2 - U_r^2 k^2} \frac{\omega_{Lb}^2}{\gamma_0^2 (\omega - kv_0)^2} = 0, \]
\[ \text{(83)} \]

where $\gamma_0 = \sqrt{1 - v_0^2 / c^2}$, and $\omega_{Lb}^2 = 4\pi e^2 n_{eb}/m_e$. Presented cold regime corresponds to equation found in Refs. [45–49].

10 Plane waves in the relativistically hot magnetized plasmas

10.1 Linearly polarized waves propagating parallel to the magnetic field

Similarly to Sect. 7, we consider the plasmas which are macroscopically motionless in the equilibrium state. However, here we consider the plasmas placed in the uniform constant magnetic field. The equilibrium state is characterized by the constant nonzero values for the concentration $n_0$, the equilibrium average $\gamma$-factor $\Gamma_0$, and the magnetic field $B_0 = B_0 \hat{z}$.

We consider waves propagating parallel to the external magnetic field $k = \{0, 0, k\} \parallel B_0$ with the electric field perturbations parallel to the external magnetic field $\delta E = \{0, 0, E_z\} \parallel B_0$ as well. We focus on the small amplitude excitations and consider they linear dynamic. In this regime, the set of hydrodynamic Eqs. (53–56) reduces to the following couple of equations:

\[ - i \omega n_0 \delta v_{ez} = \frac{q_e \Gamma_0}{m} \delta E_z - \frac{q_e}{mc^2} U_t^2 n_0 \delta E_z, \]
\[ \text{(84)} \]

and

\[ (\omega^2 - k^2 c^2) \delta E_z + 4\pi e \omega q_e n_{oe} \delta v_{ez} = 0. \]
\[ \text{(85)} \]

Equations (84) and (85) give the following dispersion dependence:

\[ \omega^2 = k^2 c^2 + \omega_{Le}^2 \left( \frac{K_1}{K_2} - \frac{U_r^2}{c^2} \right). \]
\[ \text{(86)} \]
It gives same result as the electromagnetic one-dimensional waves in the relativistic isotropic plasmas. Equation (86) shows that the transverse waves include same reduction of the Langmuir frequency as it happens for the longitudinal waves (82).

11 Conclusion

Hydrodynamic model for the relativistically hot ideal plasmas has been developed as a chain of equations. Derivation of equations is performed for functions which appear in previous equations. For example, derivation of the continuity equation leads to the current of particles. Next, the equation for evolution of the current has been derived. The particle current evolution contains four novel functions. Two of them are the three-scalar and three-vector. Therefore, equations for the evolution of these functions have been found. So, no functions is introduced ad hoc. Therefore, no energy-momentum tensor is involved. Instead of it, two four-tensors relatively to equations for two four-vectors have been found. They are the four-particle current, which also exists in the Maxwell equations as the source of field, and the four-Gamma vector, which is the average of the reverse gamma factor, and the average of the thermal velocity divided by the relativistic gamma factor. These variables are chosen for a simple reason. The evolution of four-particle current leads to appearance of four-Gamma vector, while the evolution of four-Gamma vector produces the four-particle current. Therefore, the set of equations tries to close itself. Few new functions similar to the pressure appear either. Therefore, the set of equations is not completely closed. Consequently, the chain of equations can be continued. Or, the chain can be truncated at this stage forming an example of minimal coupling model for relativistic plasmas.

Relativistic plasmas show particular interest in the regime of the relativistically hot plasmas. Consequently, details of evolution in the momentum space are essential. However, the application of kinetic equations can be rather complicates. Therefore, we need as much of higher moments as it is possible to cover the reach kinetic behavior.

Presented model does not cover high tensor-dimensional hydrodynamic functions, but consider two variations of four-vectors. There is the third variation of four-vector. It is the four-momentum. However, it is not appear in obtained equations.

Derivation of hydrodynamic equations starts with the definition of concentration, where the integral operator explicitly averages the microscopic dynamics over the macroscopically small volume. The truncation procedure is discussed after derivation of basic equations.

The closed set of equations is applied for study of fundamental collective phenomena in relativistic plasmas. The dispersion dependence of Langmuir waves is obtained. The dispersion equation for the cold relativistic beam propagating through macroscopically motionless relativistically hot plasmas is demonstrated. The cold regimes, particularly, the beam propagation, shows agreement with the well-known models of relativistic plasmas motion, while the thermal effects, particularly in the magnetic field, give new information about plasma dynamics in hydrodynamic regime. (These effects are discussed in earlier papers.)

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Data Availability Statement This manuscript has associated data in a data repository.[Author comment: Data sharing is not applicable to this article as no new data were created or analyzed in this study, which is a purely theoretical one.]

Appendix: Role of relativistic temperatures in definition of hydrodynamic functions

Here we present a citation from p.2 (before eq. 5) of Ref. [21] “For a relativistic system, the mass density $\rho(t, x)$ is not a good degree of freedom because it does not account for kinetic energy that may become sizable for motions close to the speed of light. Instead, it is useful to replace it by the total energy density $\epsilon(t, x)$, which reduces to $\rho$ in the nonrelativistic limit.” It is true as the relation of functions, but the mass density $\rho(t, x)$ (or, more precisely, the charge density $\left(\frac{q}{m}\right)\rho(t, x)$) is the source of electromagnetic field in the Maxwell’s equations. So, it cannot be eliminated from the model (see Eq. 71 and the comment after it).

Equation 6 of Ref. [21] being true for the single particle or cold fluid cannot be so easily justified for the hot fluids since the coordinate and time are independent variables in any field theory including the hydrodynamics. Therefore, values $dx/dt$ and $dx^\mu/dt$ have no physical meaning. They have no relation to the velocity field. Equations 5 and 6 of Ref. [21] give background for the Lorentz covariant theory, but it is inconsistent for the hot fluids since it contradicts the microscopic picture. Since the introduced in Ref. [21] four-velocity looks inconsistent, we have same conclusion about the energy-momentum tensor presented by equation (8) Ref. [21]. It is possible that same equations can be justified in proper way.

Theories of relativistic viscous hydrodynamics [27] like “Hydrodynamics as a gradient expansion,” “Conformal viscous hydrodynamics,” “Nonconformal hydrodynamics,” etc., are presented on the macroscopic level, so they required microscopic justification like one presented in this paper.

References [20,22-25] follow the same phenomenological picture as Ref. [21]. As an example, on page 57 of Ref. [24] authors consider “Effective Action Formulation” instead of some microscopic picture. It is major difference from work presented here. Moreover, on page 62 of Ref. [24], the chapter 3 entitled “Microscopic The-
ory Background” is presented. However, notion “Microscopic” refers to the kinetic theory which is constructed in simplified macroscopic way, as it is demonstrated by equations (3.1) and (3.2). The microscopic picture is not so simple in area of high-energy physics. However, we can expect that the quantum-relativistic hydrodynamics discussed in Refs. [23–25] should have nonquantum relativistic limit. Strict derivation based on individual relativistic motion of classic particles is addressed here. So, problem unveiled here are expected to appear in the quantum regime. Quantum relativistic regime is properly addressed in Refs. [38–40,50] (see also simplified approach in Ref. [51]). These are two limits of quantum-relativistic hydrodynamics.

Ref. [28] is entitled “Derivation of transient relativistic fluid dynamics from the Boltzmann equation.” Being focused on the microscopic justification of hydrodynamic model, we should to point out that any kinetic model also requires some microscopic justification. So, derivation from the kinetic model does not answer the formulated problem.

Let us also comment on the method of description of distributed mediums presented in Ref. [19]. Particularly, we focus on chapter II, where eq. 1 on page 22 gives the method of averaging of physical quantities. On the first step, authors used unspecified distribution function to average microscopic concentration of particles. It creates a logical gap in analysis since the evolution determined by the interaction of particles is hidden via the probabilistic approach. In particular, let us comment on the method of consideration of time derivatives of macroscopic function presented with eq. 2 on page 22 in Ref. [19]. Authors suggest that the time derivative acts on one of two functions under integral, but this mathematically incorrect step is not justified from any physical requirements. Single reason to use such method of calculation of the time derivative is an attempt to obtain well-known equations from definition given by eq. 1. In contrast with Ref. [19], we use specific method of averaging on the space volume and consider the straightforward derivation of macroscopic equations using the microscopic dynamics of individual particles.

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