The Minimum Edge Compact Spanner Network Design Problem

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Abstract

In this paper we introduce and study the Minimum Edge Compact Spanner (MECS) problem. We prove hardness results related to the problem, design exact and greedy algorithms for solving the problem, and show related experimental results. The MECS problem looks for sparse subgraphs of an input graph, such that the average shortest path distance is preserved to a constant factor. Average distance is a measure of the ease of communication over the network. As a result such problems have applications in areas where one wants to substitute a dense graph with a sparse subgraph while maintaining a low cost of communication.

Keywords: Algorithms, Complexity, Theory, Graph Theory, Integer Programming, Greedy Algorithms

1. Introduction

In this paper we define and study the minimum edge compact spanner (MECS) problem. The MECS problem is based on the idea of average distance or average path length (APL) in a directed or undirected graph.

The average distance, also called the average all-pairs-shortest-path distance or APL, denoted by \( \mu \), is defined as the average of the shortest path distance between all the vertex pairs. This metric can be used to measure efficiency of information flow over a network [\textsuperscript{1}]. The meaning of the APL as a measure of robustness of a communication network, follows from the fact that the shorter the APL, the more robust the network. As a result, the APL is one of the three most robust measures of network topology, along with its clustering coefficient.
and degree distribution [1]. Examples of the use of APL in a network, include the average number of clicks used to go from one website to another and the average number of hops that one might go through, to get in touch with a complete stranger in a social network.

The APL has important implications in network design. In a real network, like the World Wide Web, a short APL facilitates the quick transfer of information from one node to another and hence reduces costs. The efficiency of mass transfer in a metabolic network can be judged by studying its APL [2]. A power grid network will have less losses if its APL is minimized. Most real networks have a very short APL leading to the concept of a small world, where everyone is connected to everyone else through a very short path. As a result, most models of real networks are created with this condition in mind. One of the first models which tried to explain real networks was the random network model [3]. It was later followed by the models of Watts and Strogatz [4] and the random graph model of networks [5]. Still later there were scale-free networks starting with the BA model [6]. All these models had one thing in common: they all predicted networks with very short average path length (APL).

The average path length is different from the diameter of a graph. The diameter is defined as the longest shortest path between any two nodes in a graph [7]. It is easy to see that the APL is bounded above by the diameter. However, in most cases it is much shorter than the diameter. Thus the APL can be used to measure the average performance of the network whereas the diameter is used to measure the worst case behavior. Moreover, the APL is an upper bound to the independence number of a graph [8].

Given a graph $G$, the APL can change if the vertex set or the edges of the graph change. In this work, we assume that the edge set changes, because of edge deletions. Such deletions have the potential to increase the APL of the graph. Our goal is to find subgraphs of the original graph, obtained through the deletion of edges, such that the APL does not increase too much with respect to the APL of the original graph. In essence, we are looking at the following problem: given a graph $G$, we want to find a sparse subgraph $G_s$, such that the average path length of $G_s$ is bounded above by a constant. We choose this constant to be a constant multiple of the average path length of the original graph $G$.

A related problem that has been studied extensively is that of a spanner [9]. Spanners are sparse subgraphs of a graph $G$, such that the shortest path distance between every pair of nodes is preserved up to a given factor. Next we formally define a spanner but before moving forward we note that given a graph $G = (V, E)$, we can define a metric space using the all pairs shortest path distances in the graph. Thus we define a spanner using the underlying metric space. Formally, a spanner is defined as follows:

**Definition 1 (Spanner).** Let $(V, d)$ be a finite metric space. An undirected graph $G = (V, E)$ is a $t$-spanner for $V$ if for every pair of vertices $x, y \in V$ we have that $d_G(x, y) \leq t \cdot d(x, y)$, where $d_G(x, y)$ is the length of the shortest path from $x$ to $y$ in $G$ (where the length of an edge $\{u, v\} \in E$ is $d(u, v)$). $t$ is called
the *stretch* of the spanner.

We note that a spanner preserves the following, to a constant factor, given by the stretch $t$: (1) the shortest path distance between every vertex pair is preserved to a constant factor (2) the diameter of the graph is preserved to a constant factor and (3) the APL is preserved to a constant factor. Thus if we want to preserve the APL of a graph while sparsifying it, we can use any $t$-spanner algorithm for doing the same.

However, for a communication network with a small APL, we can ensure communication efficiency of a subnetwork by bounding its APL by a small constant. We do not need to ensure that all the shortest path distances or the diameter are preserved. Therefore using standard $t$-spanner algorithms for computing such sparse subgraphs is an overreach. This gives us the motivation to define a new problem, that we call the *Minimum Edge Compact Spanner problem*. The goal of this problem is to find sparse subgraphs of the original input graph such that their APL is bounded above by a small constant. In general this constant can be any design parameter of the underlying network. For our purposes, we consider it to be a multiple of the APL of the original network.

Informally, the MECS problem asks the following question: given a graph $G$, is there a *sparse* subgraph $G_s$ such that the APL is preserved up to a constant factor. Formally, let $G = (V, E, W)$ be a graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E \subseteq V \times V$ and a weight function $W : V \times V \to \mathbb{R}$. Although we consider the weights to be real valued, the MIP implementations are not able to handle arbitrary weights on the edges of the graphs. Moreover, we prove that the underlying problem is NP-complete in the simpler case when the underlying weights are integers, which in turn proves the NP-hardness of the original problem. To formalize the settings, let the average distance in the graph $G$ be denoted by $\mu_G$. For the weighted graph we define $\mu_G$ as follows:

**Definition 2.** Let $v_i, v_j \in V$. Let $d(v_i, v_j)$ denote the weight of the shortest path between the vertices $v_i$ and $v_j$. Then the average distance is defined as $\mu_G = \frac{\sum_{i<j} d(v_i, v_j)}{n(n-1)}$, where $n = |V|$.

Let $\mu_{G_s} \geq \mu_G$ be the average distance of the sparse subgraph $G_s$. Now the problem can be stated as follows: given a positive constant $t > 1$ find a subset $E_s \subseteq E$ with the *minimum* number of edges (or minimum total edge weight) such that the average distance $\mu_{G_s}$ of the sub-graph $G_s = (V, E_s, W_s)$, where $W_s$ is the restriction of $W$ to $G_s$, satisfies $\mu_{G_s} \leq t \mu_G$ or more generally $\mu_{G_s} \leq C$ for some constant $C$. Thus the basic optimization problem can be posed as the following integer program:

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E} x_e \cdot W_e \\
\text{subject to} & \quad \mu_{G_s} \leq t \mu_G \\
& \quad x_e \in \{0, 1\}
\end{align*}
\]
The variable $x_e$ is an indicator variable and takes a value of $x_e = 1$ if $e \in E_s$ and $x_e = 0$ otherwise. We note that the condition $\mu_{G_s} \leq t \mu_G$ encodes the requirement that the resulting subgraph needs to be connected. For if not then this condition is violated as in that case $\mu_{G_s} = \infty$. We also note that, more generally we can pose the following problem:

$$\begin{align*}
\text{minimize} & \quad \sum_{e \in E} x_e \cdot W_e \\
\text{subject to} & \quad \mu_{G_s} \leq C \\
& \quad x_e \in \{0, 1\}
\end{align*}$$

$C$ is a constant that can depend on the average path length of the input graph, either in an additive or a multiplicative way. Formally we pose the MECS problem as follows:

**Definition 3 (Minimum Edge Compact Spanner (MECS) Problem).** Given a graph $G = (V, E, W)$ with the average distance $\mu_G$ and a positive constant $t > 1$ find a subset $E_s \subseteq E$ with the minimum number of edges such that the average distance $\mu_{G_s}$ of the graph $G_s = (V, E_s, W_s)$ satisfies $\mu_{G_s} \leq C$ where $C$ is a constant or we may set $C = t \mu_G$ or $C = \mu_G + \delta$ for some small $\delta > 0$. We also note that $W_s$ is a restriction of $W : V \times V \to \mathbb{R}$ to $G_s$.

As mentioned before, we are the first to introduce the MECS problem. Going forward we first review some of the work on related problems and then we establish the complexity of the MECS problem. More precisely, we show that the MECS problem is $NP$-hard. Then we study exact algorithms for the MECS problem based on integer programs. Finally we study greedy algorithms for this problem and compare the performance of the exact algorithms with them.

2. Previous Work

The problem of sparse spanners for general graphs was introduced in \cite{10,11}. It was proved that the problem is $NP$-complete and since then it has been studied extensively. Several different variations of the sparse spanner problem has also been studied and several results on the complexity of such problems is also available. For example, one such variation is the minimum stretch spanning tree problem. In \cite{12} it was proved that for a given graph $G$, the problem of deciding whether $G$ has a tree $t$-spanner is $NP$-complete for any fixed $t \geq 4$ and is linear time solvable for $t = 1, 2$. For the problem of tree $t$-spanners, an $O(\log n)$ approximation algorithm is also known for finding the smallest value of $t$ for which such a spanner exists \cite{13}. The problem of approximability of sparse spanners for general graphs was studied in \cite{14,15,16}.

Among the several different variations of the sparse spanner problem, the ones that we are interested in, relate to the reliability of the underlying sparse spanner. One variation, that has been widely studied, is that of a Fault tolerant spanner. These were first studied by \cite{17}. The first results on fault
tolerant spanners for general graphs was given in [18], which were later improved by [19]. Subsequently fault tolerant spanners have been studied extensively [20, 21, 22] and several different constructions for fault tolerant spanners are known in Euclidean as well as doubling spaces [20, 19]. As in the case of a simple $t$-spanner, we are interested in finding subgraphs that satisfy the $t$-spanner property. However, we have the added constraint that the resulting subgraph should be tolerant to vertex faults, that is the subgraph should still be a $t$-spanner even if a certain predetermined number of vertices fail. Thus if the resulting subgraph is tolerant to $k$ vertex failures, it is called the $k$-fault tolerant $t$-spanner.

Another variation of spanners is called the robust spanner [23]. Here if a subset $S \subseteq V$ of vertices fail, then a super set $S^* \supseteq S$ are affected and the rest of the graph with $V \setminus S^*$ vertices is still a $t$-spanner.

A related class of problems is that of Network Design and have been studied for a long time. The sparse spanner problem is a special instance of a network design problem. The general description of a network design problem is as follows: given a set of nodes (offices, switches), possible links, costs for each link, and either the number of permitted link or node failures between each pair of nodes, we want to design a cost effective communication network. Some of the well known network design problems are the minimum spanning tree problem [24], the Steiner tree problem [25], the survival network design problem (SNDP) [26], the uniform buy-at-bulk network design problem [27], and the traveling salesman problem [24]. Most of these problems attempt to find subgraphs of an input graph with various spanning properties [28, 29, 30].

For example, in the Steiner tree problem, we are given a graph with vertex set $V$ and a set of terminal nodes $T$. The goal is to find a minimum cost tree in the input graph that connects all the terminals. In the survival network design problem (SNDP) we are given, for every pair of vertices $i, j$ in the input graph, an integer $r_{ij}$. The goal is to find a subgraph in which there are at least $r_{ij}$ edge disjoint paths between the vertices $i$ and $j$. This is very similar to the problem of finding a circuit tolerant spanner, the difference being that in the case of circuit tolerant spanners we consider vertex faults whereas in the case of SNDP we consider path faults. Interested readers may refer to [31] for an excellent survey on approximation algorithms for network design problems.

One of the earliest results on the hardness of network design problems was presented in [32]. They proved that the problem of finding a subgraph with minimum APL under weight constraints is $NP$-complete. A study on the worst case behavior of heuristics for this problem was studied by Wong in [33]. Subsequently many variations of network design problems have been studied in the literature. Most of the network design problems are known to be $NP$-complete. A comprehensive survey of approximation algorithms for the variants of the network design problem can be found in [34, 35]. To the best of our knowledge the Minimum Edge Compact Spanner (MECS) network design problem has never been studied before. There are two problems that are close to the one that we study. The first is that of the $\epsilon$-slack spanner [35]. This paper studied the problem of spanners with slack. They are defined as follows:
Definition 4 ($\epsilon$-slack spanner). Suppose that $M = (V,d)$ is a metric. Then $H_\epsilon = (V,E_H)$ is an $\epsilon$-slack spanner for the metric $M = (V,d)$ if for given $0 < \epsilon < 1$, for any vertex $v \in V$ the furthest $(1-\epsilon) \cdot n$ vertices $w$, have the property that $d(v,w) \leq d_H(v,w) \leq D \cdot d(v,w)$ for some constant $D$. There is no constraint on the distances to the nearest $\epsilon \cdot n$ vertices, where $n = |V|

Note that $d_H$ is the shortest path metric on the graph $H_\epsilon$. The resulting spanner uses the furthest $(1-\epsilon) \cdot n$ vertices. The $\epsilon \cdot n$ vertices that are nearby are not required to satisfy the constraint on the path length. In the paper Chan et al. gave a construction for these structures and showed that they distort the average path length by a constant factor. We note from the definition of the problem, that this is not the MECS problem. In fact the MECS problem is a more general version of the $\epsilon$-slack spanner problem. On the other hand the problem studied in [32] looks for subgraphs that minimize the APL subject to constraints on the weight of the underlying graph.

The most common approach for distance-based network design problems which takes into account the distances between nodes in networks is the flow-based method [36, 37, 38]. Other methods based on the MIP based approach include the path based approach [39]. However, one of the major problems of these exact methods for solving $NP$-complete problems is the time it takes to solve problems of moderate sizes. As mentioned before the MECS problem has not been studied before and as such we do not know of any work that studies this problem from the perspective of exact algorithms.

3. Hardness.

Now we are ready to state and prove hardness results for the MECS problem. More precisely, we prove that the MECS problem is $NP$-complete. We prove $NP$-completeness in the simpler case when the edge weights are integers. From the perspective of complexity, it makes sense to just show that even if we restrict ourselves to the simpler case where the edge weights of the underlying graph are integers, the problem is still $NP$-complete. We assume that we are given a graph $G = (V,E,W)$ where $W$ is a function from $E$ to $\mathbb{N}$. Thus, given an edge $e \in E$ we have an integer weight $W(e)$ associated with $e$. As before we denote the average path length (APL) in the graph $G$ as $\mu_G$ and the MECS problem aims to find a subgraph $G_s \subseteq G$ such that $\mu_{G_s} \leq t \cdot \mu_G$ for some $t \geq 1$ and has the minimum total weight among all such graphs. We note that for $t = 1$ we return the graph $G$ as the solution. So we consider the problem for $t > 1$. We also note that the solution to the problem has to be a connected graph. For if the graph is not connected the average distance, by definition, is infinite.

Our proof derives from the construction used by [32] for proving the $NP$-completeness of the classical network design problem where the goal is to minimize the average path length of a graph subject to constraints on its weight. Thus the problem considered in [32] is a complementary one and hence the underlying decision problems take the same form. This gives us the opportunity to use their constructions for proving the $NP$-completeness of the MECS problem.
as well. In some sense we reinvented the constructions and then realized that they have been used before. As a result some of the technicalities of the proofs are a bit different and hence we have decided to present them in detail here.

In order to prove that the MECS problem is \(NP\)-hard we consider the decision version of the problem and show that it is \(NP\)-complete. Let \( t > 1 \) and let us write \( t \cdot \mu_G = c \). Let the spanning subgraph under the constraint of APL be denoted by \( G_s \). Then the decision version of the MECS problem is as follows: Does there exist a subgraph \( G_s = (V, E_s, W_s) \), \( E_s \subseteq E \) and \( W_s \) is a restriction of \( W \) to \( G_s \) such that:

\[
\sum_{e \in E_s} W_e \leq r \text{ and } \mu_{G_s} \leq c, \ c \text{ finite} \quad (1)
\]

Instead of writing \( \mu_{G_s} \) in the above equation, we could simply replace it with the sum of the all pairs shortest path weights. This will have the effect of changing the constant \( c \) to \( C \), where \( c = \frac{C}{n(n-1)} \) where \( n = |V| \).

\[
\sum_{e \in E_s} W_e \leq r \text{ and } \sum_{(u,v):u,v \in V; u \neq v} d(u,v) \leq C, \ C \text{ finite} \quad (2)
\]

It is easy to see that the problem is in \(NP\), for given a graph it is easy to check whether it is a feasible solution of problem \([2]\). Now we proceed to prove that this problem is \(NP\)-complete. We establish the \(NP\)-completeness in two ways. The first one uses a reduction from the Subset Sum \([24]\) problem. This is a standard reduction and the proof is short and concise. However, it still leaves the possibility of having special and simple instances of the problem which are not \(NP\)-hard. This is where our second proof comes in. The second proof shows that the problem is \(NP\)-hard even if we restrict ourselves to finding spanning trees having the MECS property. This is a much more stronger result and goes on to show that the problem is hard even in the simplest of all cases, namely when we are looking for spanning trees. We start with the reduction from Subset Sum \([24]\], which is a known \(NP\)-complete problem. We state the subset sum problem below:

**Definition 5 (Subset Sum Problem).** Given a set \( U \) of integers \( \{a_1, a_2, \ldots, a_k\} \) and a target integer \( b \), the subset sum problem asks the following question: is there a subset \( S \) of \( \{1, 2, \ldots, k\} \) such that \( \sum_{i \in S} a_i = b \)

This problem \([SP13]\) from \([40]\) is known to be \(NP\)-complete.

### 3.1. \(NP\) Completeness

In order to prove \(NP\)-completeness we use a gadget that given an instance of the subset sum problem creates an instance of the MECS feasibility problem \([2]\). Then we show that solving this new problem is equivalent to solving the original subset sum problem.
Figure 1: Illustration of the graphs used for the \( \text{NP} \)-completeness proof of the MECS problem (a) The graph that creates a spoke pattern and is included in every feasible solution, this is the graph \( G_f \) (b) The input to the MECS feasibility solver

**Definition 6 (Construction for Reduction).** Consider an instance of the subset sum problem with the universe \( U = \{a_1, a_2, \ldots, a_k\} \). We note that \( k = O(1) \) and is an input to the problem. We create a graph \( G \) using the following steps.

1. For every \( i \in \{1, 2, \ldots, k\} \) create two nodes \( i, i' \) in \( G \)
2. Add a node \( N \) to the graph such that the vertex set for \( G \) is \( N \cup \{i, i'\}, i \in \{1, 2, \ldots, k\} \)
3. The edge set \( E \) consists of the following edges: for every \( i \in \{1, 2, \ldots, k\} \) add the edges \((i, i'), (N, i), (N, i')\)
4. The weight function is defined as follows: \( W(i, i') = a_i, W(N, i') = a_i, W(N, i) = a_i \)
5. Let \( T = \sum_{i \in \{1, 2, \ldots, k\}} a_i \). We set \( r = 2T + b \) and \( C = 4kT - b \) in problem \( \textbf{[2]} \)

The resulting graph, which is an input to problem \( \textbf{[2]} \) is shown in figure \( \textbf{[11]} \).

We claim that if we are given a feasible solution for problem \( \textbf{[2]} \) on \( G \), then we can get a solution for problem \( \textbf{[5]} \) in polynomial time. In order to do this we define another graph \( G_f \) as follows: \( G_f = (V, E_f, W_f) \) where \( E_f = \{(N, i), (N, i'), i \in \{1, 2, \ldots, k\}\} \) as before and \( W_f \) contains the corresponding weights. Now we are ready to prove the \( \text{NP} \)-completeness of the MECS problem.

We establish this through the following lemmas.

**Lemma 1.** The total edge weight of \( G_f \) is \( 2T \).

**Proof.** The proof is straightforward. The total weight is given by \( \sum_{i \in \{1, 2, 3, \ldots, k\}} (W(N, i) + W(N, i')) \). Using the fact that \( W(N, i) = a_i \), and the fact that \( T = \sum_{i \in \{1, 2, \ldots, k\}} a_i \), the result follows.
The next lemma looks at the sum of the all pairs shortest path weights in the graph $G_f$.

**Lemma 2.** The sum of the all pairs shortest paths weight of $G_f$ is $4kT$.

**Proof.** We prove this using induction. When $k = 1$ the result holds, as in that case we have 3 pairs of nodes and the sum of the shortest path weights is $W(N, 1) + W(N, 1') + W(SP(1, 1')) = a_1 + a_1 + 2 \cdot a_1 = 4 \cdot a_1$. Let us suppose that the result holds for $k = m$. Now we prove that the result holds for $k = m + 1$. Thus we assume that when $k = m$, the sum of the all pairs shortest paths weight is $4 \cdot m \cdot \sum_{i \in \{1, 2, \ldots, m\}} a_i$. Now suppose that $k = m + 1$. This adds two more nodes and two more edges to the existing graph. This in turn means that with every pair of nodes $(i, i')$, $i \in \{1, 2, \ldots, m\}$, another four pairs of shortest path weights are added to the existing total. The weights of these four pairs is $4 \cdot (a_i + a_{m+1})$; $i \in \{1, 2, \ldots, m\}$. Moreover, the total weight of the shortest path distances between the newly added nodes is $4 \cdot a_{m+1}$. Thus the addition of the two new nodes increases, the sum of the all pairs shortest path weights by:

$$4 \cdot \sum_{i \in \{1, 2, \ldots, m\}} (a_i + a_{m+1}) + (4 \cdot a_{m+1})$$

Thus the new sum of the all pairs shortest path weights is:

$$(4 \cdot m) \cdot \left( \sum_{i \in \{1, 2, \ldots, m\}} a_i \right) + 4 \cdot \left( \sum_{i \in \{1, 2, \ldots, m\}} a_i \right) + 4 \cdot (m + 1) \cdot a_{m+1}$$

The above sum comes out to be $4 \cdot (m + 1) \cdot \sum_{i \in \{1, 2, \ldots, m+1\}} a_i$. Thus the proof follows by induction.

In the next lemma we investigate the result of adding the edge $(i, i')$ to the graph $G_f$.

**Lemma 3.** Let $G_f = (V, E_f, W_f)$ where $E_f = \{(N, i), (N, i')\}, i \in \{1, 2, \ldots, k\}$ as before and $W_f$ contains the corresponding weights. The addition of the edge $(i, i')$ to $G_f$ increases the sum of edge weights by $a_i$ and decreases the sum of the all pairs shortest path weight by $a_i$.

**Proof.** The proof follows from the fact that the total edge weight of $G_f$ is $2T$ and the total all pairs shortest paths weight of $G_f$ is $4kT$. Its easy to see that if the edge $(i, i')$ is added to $G_f$, then the total weight increases by $a_i$ (by construction) and the total shortest path weight decreases by $a_i$ because of the fact that the addition of the edge between $(i, i')$ changes the shortest path between $(i, i')$ and decreases the weight of the shortest path from $2 \cdot a_i$ to $a_i$.

The next lemma establishes the fact that any feasible solution of problem $P$ can be assumed to contain $G_f$ as a subgraph.
Figure 2: Example graph used for lemma 4. The graph is almost similar to the spoke graph that we have used before, but one of the spokes, namely that for vertex 2 is broken and instead we have a pattern like the number 7 in its place.

Lemma 4. Define $G_f = (V, E_f, W_f)$ where $E_f = \{(N, i), (N, i')\}, i \in \{1, 2, \ldots, k\}$ and $W_f$ contains the corresponding weights. Any feasible solution of the problem [2] on $G$ can be assumed to contain $G_f$ as a subgraph.

Proof. Let us suppose that the feasible solution to the problem [2] does not contain the graph $G_f$. Then the feasible solution would contain a graph that is different from $G_f$. However the bad news is that there are several different possibilities for such graphs. The good news is that all these possibilities have one thing in common, each unit consisting of the three vertices $i, i', N$ can have only two possible structures apart from the one where $i$ and $i'$ are both connected to $N$. In both these settings $i$ is connected to $i'$ and either $i$ is connected to $N$ or $i'$ is connected to $N$. Thus a feasible solution can have an underlying graph where some of the triplets are arranged to form a spoke like structure and the others are arranged in one of the two possible ways.

In order to see what happens in this setting we consider the simplest variation of the graph shown in figure 1a. The graph is such that there is a vertex pair $i, i'$, such that the edges associated with this vertex pair are $(N, i), (i, i')$ and the rest of the graph is a spoke graph. An example is shown in the figure 2. It is easy to see that for the graph in figure 2, the sum of the edge weights is $2T$. However, the sum of weights of all the pairs shortest paths is $4kT + (k - 1)(2 \cdot a_2)$. Now suppose that a feasible solution $F$ contains this graph as a subgraph. As the solution is feasible we have $\sum_{e \in E} W_e \leq 2T + b$ and $\sum_{(u,v): u,v \in V; u \neq v} d(u, v) \leq 4kT - b$. Now, if the feasible solution contains $G_f$ as a subgraph, then in order to satisfy the constraint on the all pairs shortest path weights, one would need to add $(i, i')$ edges (each of which reduces the
sum of the shortest path weights by \( a_i \) such that the sum of the all pairs shortest path weights becomes less than or equal to \( 4kT - b \). This in turn means that the sum of the edge weights will increase by at least the same amount, namely \( b + (k - 1)(2 \cdot a_2) \). Thus the sum of the edge weights will be at least \( 2T + b + (k - 1)(2 \cdot a_2) \). But this will violate the first constraint thus making the solution infeasible. Thus this creates a contradiction. In particular, more the number of vertices \( i \) with edges of the form \((N, i), (i, i')\), the larger is the sum of the all pairs shortest path weights. Thus we can use the same argument as above to disprove the fact that any feasible solution will create such a subgraph. Hence we can assume that any feasible solution contains the graph \( G_f \) and the proof follows.

**Theorem 5.** The MECS problem is \( NP \)-complete.

**Proof.** Let us consider a feasible solution of the problem such that equality holds for both the inequalities. Let \( G_s \) be the graph corresponding to this solution. \( G_s \) is a subgraph of \( G \) and \( G_f \) is a subgraph of \( G_s \) by lemma \[4\]. Consider the edge set \( E_s \) of \( G_s \) and consider exactly the edges \((i, i')\). For every such edge, set \( S = S \cup \{i\} \). Then it follows that \( \sum_{i \in S} a_i = b \) by the feasibility of the solution and lemma \[3\] and thus we have got a solution of the problem \[5\]. This completes the proof.

Next we prove a stronger result that precludes the possibility of finding a spanning tree that satisfies the constraint on the APL, in polynomial time. In particular we show that the MECS problem is \( NP \)-complete even in the case, where we restrict ourselves to spanning trees of unweighted graphs. Here the problem is as follows: Given an undirected graph \( G = (V, E) \) with average distance \( \mu_G \) and a finite real number \( c \), find a subgraph \( G_s = (V, E_s) \) such that,

\[
|E_s| \leq |V| - 1 \quad \text{and} \quad \mu_{G_s} \leq c, \quad c \text{ finite}
\]  

(3)

We call this the **Edge Compact Spanning Tree Spanner (ECSTS)** Problem. We prove that the ECSTS problem is \( NP \)-complete. The actual reduction for the proof is shown in Appendix A because of space constraints.

4. Exact Algorithms

Going forward we study solutions to the MECS problem by formulating the problem as a mathematical program. More precisely we look at two approaches: the first approach is based on the idea of flows as described in \[36, 37, 38\] and the second approach is based upon formulating the problem using a mixed integer program. It must be noted that, going forward, whenever we mention spanners we mean a solution to the MECS problem, which is based on the notion of the average path length and not spanners as referred to in the standard spanner literature.
4.1. Flow-based Approach

The most common approach for distance-based network design problems, which take into account the distances between nodes in networks, is the flow-based method \([36, 37, 38]\). Namely, let \(f_{st}^{st} \in \{0, 1\}\) for all \(s, t \in V\) and \((i, j) \in E\) denote the flow sent from vertex \(s \in V\) to vertex \(t \in V\) through an edge \((i, j) \in E, s < t\). Assuming that the total amount of flow sent from vertex \(s \in V\) to vertex \(t \in V\) through a spanner \(G_s = (V, E_s)\) is 1, one can deduce that the length of the path that the flow takes from \(s\) to \(t\), in a spanner \(G_s = (V, E_s)\) is \(\sum_{(i,j) \in E} f_{ij}^{st}\); hence, the average path-length of all flows in a spanner is:

\[
\mu_f = \frac{2}{n(n-1)} \sum_{s,t \in V: i<j} \sum_{(i,j) \in E} f_{ij}^{st}
\]

Note that \(\mu_f \geq \mu_s\); however, there always exists a flow \(f = \{f_{ij}^{st} | s, t \in V, (i, j) \in E\}\) such that \(\mu_f = \mu_s\) (a flow which uses only the shortest paths). Then, the problem formulation can be written as follows:

**Problem 7 (flow-based MECS).**

\[
\text{minimize } \sum_{(i,j) \in E} x_{ij} \quad (4a)
\]

subject to

\[
\frac{2}{n(n-1)} \sum_{s,t \in V: i<j} \sum_{(i,j) \in E_s} f_{ij}^{st} \leq t\mu_G \quad (4b)
\]

\[
\sum_{j:(s,j) \in E} f_{sj}^{st} - \sum_{i:(i,s) \in E} f_{is}^{st} \geq 1 \quad \forall s, t \in V, s < t, \quad (4c)
\]

\[
\sum_{i:(i,t) \in E} f_{it}^{st} - \sum_{j:(t,j) \in E} f_{tj}^{st} \geq 1 \quad \forall s, t \in V, s < t, \quad (4d)
\]

\[
\sum_{j:(i,j) \in E} (f_{ij}^{st} - f_{ji}^{st}) = 0 \quad \forall s, t \in V, s < t, \forall i \in V \setminus \{s, t\}, \quad (4e)
\]

\[
f_{ij}^{st} \leq x_{ij} \quad \forall s, t \in V, s < t, \forall (i,j) \in E, \quad (4f)
\]

\[
x_{ij} \in \{0, 1\}, 0 \leq f_{ij}^{st} \leq 1 \quad \forall s, t \in V, s < t, \forall (i,j) \in E. \quad (4g)
\]

In the formulation above, constraint (4d) is the main constraint on average path-length in a spanner, and constraints (4c)-(4f) are the standard flow-balancing constraints. Note that we relax the binary requirement on variables \(f_{ij}^{st}\). It is easy to verify that the formulation is still correct. This flow-based formulation requires \(O(|V|^2|E|)\) variables and constraints.

4.2. Path-based Approach

In this section, we develop a path-based approach, which is similar to the one presented in [39] and is based on introducing new distance-based variables to
compute the average path-length. The main idea is to define path variables for each pair of nodes and enforce constraints on them recursively. We demonstrate that such an approach allows to reduce the number of variables and constraints.

Let \( u_{ij}^{(l)} \) be a binary variable such that \( u_{ij}^{(l)} = 1 \) if and only if there is a path of length at most \( \ell \) between nodes \( i \) and \( j \) in a spanner \( G_s \), where \( i, j \in V, \ell \leq L \) and \( L \leq |V| - 1 \) is an appropriate constant. Let also \( u_{ij}^{(0)} = 0 \) and \( u_{ij}^{(L+1)} = 1 \) for simplicity. In addition, we define \( y_{ikj}^{(l)} \) to be a binary variable such that \( y_{ikj}^{(l)} = 1 \) if there is a path between nodes \( i \) and \( j \) of length at most \( \ell \) in a spanner \( G_s \) which traverses a neighbor \( k \neq j \) of node \( i \) where \( i, j, k \in V, (i, k) \in E \) and \( \ell \leq L \).

Then, the formulation can be written as

**Problem 8 (path-based MECS).**

\[
\text{minimize } \sum_{(i,j) \in E} x_{ij} \quad (5a) \\
\text{subject to} \\
\sum_{\ell=1}^{L+1} \sum_{i,j=1;i<j}^n \ell(u_{ij}^{(\ell)} - u_{ij}^{(\ell-1)}) \leq t_{ij} n(n-1) \quad (5b) \\
u_{ij}^{(1)} = x_{ij}, \quad (i,j) \in E \quad (5c) \\
u_{ij}^{(1)} = 0, \quad (i,j) \notin E \quad (5d) \\
u_{ij}^{(\ell)} \geq u_{ij}^{(\ell-1)}, \quad \forall i,j, \ell \in \{2, \ldots, L\} \quad (5e) \\
u_{ij}^{(\ell)} \leq x_{ij} + \sum_{k\neq j,(i,k)\in E} y_{ikj}^{(\ell)}, \quad \forall (i,j) \in E, \ell \in \{2, \ldots, L\} \quad (5f) \\
u_{ij}^{(\ell)} \geq \frac{1}{\text{deg}_i} \left( x_{ij} + \sum_{k\neq j,(i,k)\in E} y_{ikj}^{(\ell)} \right), \quad \forall (i,j) \in E, \ell \in \{2, \ldots, L\} \quad (5g) \\
u_{ij}^{(\ell)} \leq \sum_{k\neq j,(i,k)\in E} y_{ikj}^{(\ell)}, \quad \forall (i,j) \notin E, \ell \in \{2, \ldots, L\} \quad (5h) \\
u_{ikj}^{(\ell)} \geq \frac{1}{\text{deg}_i} \left( \sum_{k\neq j,(i,k)\in E} y_{ikj}^{(\ell)} \right), \quad \forall (i,j) \notin E, \ell \in \{2, \ldots, L\} \quad (5i) \\
y_{ikj}^{(l)} \leq x_{ik}, y_{ikj}^{(l)} \leq u_{kj}^{(l-1)}, \quad i, j \in V, (i, k) \in E, \ell \in \{2, \ldots, L\} \quad (5j) \\
y_{ikj}^{(l)} \geq x_{ik} + u_{kj}^{(l-1)} - 1, \quad i, j \in V, (i, k) \in E, \ell \in \{2, \ldots, L\} \quad (5k) \\
x_{ij}, u_{ij}^{(l)}, y_{ikj}^{(l)} \in \{0, 1\}, \quad \forall i,j,k \in V, \ell \in \{1, \ldots, L\}. \quad (5l)
\]

In the formulation above constraint \((5b)\) is the main constraint on average distance. Note that if the shortest path length between a pair of nodes \( i, j \in V \) is \( d \), then \( u_{ij}^{(d)} - u_{ij}^{(d-1)} = 1 \) and \( u_{ij}^{(\ell)} - u_{ij}^{(\ell-1)} = 0 \) for \( d \neq \ell \). Constraints \((5f)-(5k)\) recursively model paths variables \( u_{ij}^{(l)} \). Constraints \((5j)-(5k)\) recursively
model additional paths variables \( y_{ikj}^{(\ell)} \). The formulation requires \(|E|\) variables \( x_{ij}, L|V|(|V| - 1)/2 \) variables \( u_{ij}^{(\ell)} \), and \( L|V||E| \) variables \( y_{ikj}^{(\ell)} \).

Note that if the distance between \( i,j \) is greater than \( L \), it is counted as \( L + 1 \) by the formulation above. Hence, to appropriately compute the average distance, we need to use \( L = n - 1 \) as the maximum possible spanner diameter. However, in practice one can expect the spanner diameter to be \( \ln(n) \) as many real-world and randomly graph topologies exhibit a so-called “small-world” property [4, 41]; therefore, in Section 4.4 we develop an exact iterative MIP-based algorithm to solve the problem more efficiently.

4.3. Formulation enhancements

Below, we outline various formulation enhancements which may help to improve the solvers performance.

4.3.1. Redundant constraints and integrality relaxation

Observe that for any \( i < j \in V \)

\[
\sum_{\ell=1}^{L+1} \ell(u_{ij}^{(\ell)} - u_{ij}^{(\ell-1)}) = u_{ij}^{(1)} + 2(u_{ij}^{(2)} - u_{ij}^{(1)}) + \cdots + L(u_{ij}^{(L)} - u_{ij}^{(L-1)}) + (L + 1)(1 - u_{ij}^{(L)}) = L + 1 - \sum_{\ell=1}^{L} u_{ij}^{(\ell)}
\]

therefore making \( w_{ij}^{(\ell)} \) as large as possible will still make the problem feasible and satisfy the main constraint (5b) on the spanner average distance. Therefore, only constraints on upper bound on \( u_{ij}^{(\ell)} \) and \( y_{ikj}^{(\ell)} \) are needed. Note that these constraints are enforced only to make the aforementioned variables equal to zero, hence, these variables do not need to be integral anymore. Only variables \( x_{ij} \) need to be binary, and we need \(|E|\) of them.

4.3.2. Leaf-node considerations

Denote by \( N_1 \) all nodes of \( V \) with degree 1 in \( G \), i.e., \( N_1 = \{ i \in V \mid \deg_G(i) = 1 \} \). Such nodes are also referred to as leaves (or leaf nodes) of \( G \). Let \( n_1 = |N_1| \). Since a spanner cannot have isolated nodes, then the edges connecting leaf node to the network must be included into any spanner.

Thus, one should simply enforce \( x_{ij} = 1 \) for all \( i \in N_1, (i,j) \in E \). However, the computational experiments show that it is more beneficial of not considering leaf nodes and edges going to leaf nodes in the formulation. In this case, the
average distance of a spanner can be computed by the following expression:

$$
\mu = \frac{1}{n(n-1)} \left\{ \sum_{i,j \in V \setminus N_1; i < j} \left( u_{ij}^{(1)} + \sum_{\ell=2}^{L+1} \ell \left( u_{ij}^{(\ell)} - u_{ij}^{(\ell-1)} \right) \right) \right\} + \sum_{i \in V \setminus N_1} v_i + \sum_{i \in V \setminus N_1} 2 \times \frac{v_i(v_i - 1)}{2} + \sum_{i,j \in V \setminus N_1; i < j} (v_i + v_j) \left( 2u_{ij}^{(1)} + \sum_{\ell=2}^{L} (\ell + 1) \left( u_{ij}^{(\ell)} - u_{ij}^{(\ell-1)} \right) \right) + \sum_{i,j \in V \setminus N_1; i < j} (v_i \cdot v_j) \left( 3u_{ij}^{(1)} + \sum_{\ell=2}^{L-1} (\ell + 2) \left( u_{ij}^{(\ell)} - u_{ij}^{(\ell-1)} \right) \right),
$$

which explicitly represents the fact that nodes and edges from \( N_1 \) are not considered. Pre-computed parameter \( v_i, i \in V \setminus N_1 \), is equal to the number of neighbors of \( i \in V \setminus N_1 \) that belong to \( N_1 \), i.e., \( v_i = |N_G(i) \cap N_1| \), where \( N_G(i) = \{ j : (i,j) \in E \} \) is a set of neighbors of node \( i \). The terms in (7b) correspond to the paths (of length 1) from \( i \in N \setminus N_1 \) to \( N_G(i) \cap N_1 \), and paths (of length 2) between nodes in \( N_G(i) \cap N_1 \), respectively. Next, for any \( i,j \in N \setminus N_1, i < j \), the term in (7c) represent all paths between \( i \) and \( N_G(j) \cap N_1 \) as well as \( j \) and \( N_G(i) \cap N_1 \). Similarly, (7d) computes all paths between \( N_G(i) \cap N_1 \) and \( N_G(j) \cap N_1 \).

### 4.3.3. Other inequalities

- **Isolated nodes consideration.** Since the spanner cannot have isolated nodes, the following inequality can be used to enforce that

  $$
  \sum_{j: (i,j) \in E} x_{ij} \geq 1, \ \forall i \in V.
  $$

- **Connectivity consideration.** Since the spanner has to be connected, then it should have at least \( n - 1 \) edges

  $$
  \sum_{(i,j) \in E} x_{ij} \geq n - 1, \ \forall i \in V.
  $$

- **Connectivity Violating Cuts.** Let \( E_c \) be a set of edge cuts, i.e., subsets of edges \( E_\alpha \subset E \), such that for any \( E_\alpha \in E_c \) a graph \( G_\alpha = (V, E \setminus E_\alpha) \) is disconnected. A spanner should have at least one edge from each edge cut \( E_\alpha \in E_c \):

  $$
  \sum_{(i,j) \in E_\alpha} x_{ij} \geq 1, \ \forall E_\alpha \in E_c
  $$
Average Distance Constraint Violating Cuts. Let $E_d$ be a set of subsets of edges $E_\beta \subset E$, such that for any $E_\beta \in E_d$ the average distance of a graph $G_\beta = (V, E \setminus E_\beta)$ is greater than $t\mu_G$. A spanner should have at least one edge from each $E_\beta \in E_d$:

$$\sum_{(i,j) \in E_\beta} x_{ij} \geq 1, \quad \forall E_\beta \in E_d$$  \hspace{1cm} (11)

4.4. Exact MIP-based algorithm

The number of variables and constraints used in Problem 8 is $O(L|V||E|)$, hence the lower the value of $L$, the better solver performance we can expect. In this section, we develop an exact MIP-based algorithms, which is based on the following proposition.

**Proposition 6.** Let $E_s^*$ be the optimal solution of Problem 8 for $L = L_0$ and $G_s = (V, E_s)$. If $diam(G_s) \leq L_0$, then $G_s$ is also an optimal spanner, i.e., $E_s^*$ is the optimal solution of Problem 8 for $L = n - 1$.

**Proof.** For any spanner $G_s = (V, E_s)$ let $[\mu_s]_L = \frac{1}{n(n-1)} \sum_{i,j=1}^n \min(d_{G_s}(i,j), L)$, which can be viewed as a truncated version of the average distance $\mu_s$, where any distance of greater than $L$ in a graph $G_s$ is treated as $L$. In fact, the constraint (5b) of Problem 8 for any given $L$ restricts the truncated average distance of a spanner $[\mu_s]_L$.

Note that $[\mu_s]_L \leq \mu_s$ for any $L = 1, \ldots, n - 1$. Hence, any feasible solution of Problem 8 with $L = n - 1$ is also a feasible solution for Problem 8 for any $L = 1, \ldots, n - 1$. Moreover, if $L \geq diam(G_s)$, then $[\mu_s]_L = \mu_s$. Therefore, the optimal spanner $E_s^*$ is also a feasible solution of Problem 8 with $L = n - 1$, and, hence, is also optimal.

Hence, this problem can be solved sequentially. First, we set $L = diam(G)$ and solve the corresponding MIP. If the diameter of the obtained spanner greater than $L$, we set $L = L + 1$ and solve the problem again until the diameter of the spanner will not be greater than $L$. This technique was implemented in [39] and according to their observations, it allows to substantially reduce the computational time and the amount of variables.

Below is the formal algorithm description.

4.5. Edge-weighted graphs

Here, we show how to generalize Problem 8 for edge-weighted graphs with integer weights. Formally, let $w_{ij} \in \mathbb{Z}_+$ denote a positive integer weight (cost) of edge $(i, j) \in E$ (for simplicity of exposition, we assume $w_{ij} = 0$ for $(i, j) \notin E$ in the corresponding MIP formulation). Note that the integrality weights assumption should not be too restrictive in many real-world applications. Following a similar notation as in Problem 8, define $w_{ij}^{(l)} = 1$ if and only if there exists a path of length at most $l$ in $G_s$, where $\ell = \{1, \ldots, L\}$ and $L \leq (n - 1) \max_{(i,j) \in E} w_{ij}$.

Then Problem 8 is generalized for weighted graphs as follows:
Algorithm 1: Exact MIP-based Algorithm

**Input:** A graph \( G = (V, E) \), and \( t \)

**Output:** Subset \( E_s^* \subseteq E \)

begin

\[
L_0 \leftarrow \text{diam}(G)
\]
\[
E_s^* \leftarrow \text{optimal solution of Problem 8 and } L = L_0
\]
\[
G_s \leftarrow (V, E_s^*)
\]
\[
L_1 \leftarrow \text{diam}(G_s)
\]

while \( L_1 > L_0 \) do

\[
L_0 \leftarrow L_1
\]
\[
E_s^* \leftarrow \text{optimal solution of Problem 8 and } L = L_0
\]
\[
G_s \leftarrow (V, E_s^*)
\]
\[
L_1 \leftarrow \text{diam}(G_s)
\]

end

return \( E_s^* \)

end

Problem 9 (edge-weighted path-based MECS).

\[
\begin{align*}
\text{minimize} & \quad \sum_{(i,j) \in E} x_{ij} \tag{12a} \\
\text{subject to} & \quad \sum_{\ell=1}^{L+1} \sum_{i,j=1; i<j}^n \ell (u_{ij}^{(\ell)} - u_{ij}^{(\ell-1)}) \leq t \mu_G \frac{n(n-1)}{2} \tag{12b} \\
& \quad u_{ij}^{(1)} = x_{ij}, \quad i,j \in V : w_{ij} = 1 \tag{12c} \\
& \quad u_{ij}^{(1)} = 0, \quad i,j \in V : w_{ij} \neq 1 \tag{12d} \\
& \quad u_{ij}^{(\ell)} \geq u_{ij}^{(\ell-1)}, \quad \forall i,j \in V, \ell \in \{2, \ldots, L\} \tag{12e} \\
& \quad u_{ij}^{(\ell)} \leq x_{ij} + \sum_{k:(i,k) \in E} y_{ikj}^{(\ell)}, \quad \forall i,j \in V : w_{ij} \in \{1, \ldots, \ell\}, \ell \in \{2, \ldots, L\} \tag{12f} \\
& \quad u_{ij}^{(\ell)} \geq \sum_{k:(i,k) \in E} y_{ikj}^{(\ell)}, \quad \forall i,j \in V : w_{ij} \notin \{1, \ldots, \ell\}, \ell \in \{2, \ldots, L\} \tag{12g} \\
& \quad y_{ikj}^{(\ell)} \leq x_{ik}, \quad i,j,k \in V : w_{ik} \in \{1, \ldots, \ell - 1\}, \ell \in \{2, \ldots, L\} \tag{12h} \\
& \quad y_{ikj}^{(\ell)} \leq y_{kij}^{(\ell-w_{ik})}, \quad i,j,k \in V : w_{ik} \in \{1, \ldots, \ell - 1\}, \ell \in \{2, \ldots, L\} \tag{12i} \\
& \quad y_{ikj}^{(\ell)} = 0, \quad i,j,k \in V : w_{ik} \geq \ell, \ell \in \{2, \ldots, L\} \tag{12j} \\
& \quad x_{ij}, u_{ij}^{(\ell)}, y_{ikj}^{(\ell)} \in \{0, 1\}, \quad \forall i,j,k \in V, \ell \in \{1, \ldots, L\}. \tag{12k}
\end{align*}
\]

As a final remark, we note that the exact MIP-based Algorithm 1 can be easily generalized for the edge-weighted graphs using Problem 9.
5. Greedy Strategies

As the MECS problem is \(NP\)-complete, exact solutions using MIP do not scale very well and so exact solutions are hard to find on very large graphs. As a result it is important to have algorithms for solving the MECS problem on large graphs, if possible exactly and if that is not possible then at least approximately. In this section, we consider greedy heuristics for the MECS problem. We consider several greedy heuristics and we also prove some properties of the resulting solutions.

5.1. Greedy Algorithms

We start with the simplest greedy algorithm that has several nice properties. The intuition for this algorithm comes from the observation that we can upper bound the APL, if we can find upper bound for the all pairs shortest paths. This is in some sense an overkill. The APL can be bounded above, without explicitly bounding the all pairs shortest paths. However, this is a good starting point for our discussions and hence we start with it. This algorithm was first proposed for computing standard spanners for both weighted and unweighted graphs by Althofar et al. [9]. First we describe the details of the algorithm and then we describe a few properties of the algorithm. As this algorithm is one of the most practical and easy to implement algorithms for path based spanners, we use this algorithm as a baseline for comparison with our algorithms.

**Algorithm 2:** The Greedy Spanner Algorithm

```
Procedure GreedySpanner(G, t)
  Input: G = (V, E, W) the input graph, t the stretch
  Output: G': The MECS Spanner
  G' ← (V, {})
  Sort E in non-decreasing order of weights
  for e = (u, v) ∈ E do
    P(u, v) ← Shortest path between u and v in G'
    if P(u, v) > t · W(e) then Add e to G'
  end
  return G'
```

Going forward we state and prove a few properties of the algorithm. We first prove that the algorithm always gives a feasible solution to the MECS problem. Then we state a few results that follow directly from the properties of the algorithm. Interested readers may refer to [9] for detailed proofs of the same.

**Lemma 7.** Algorithm GreedySpanner always gives a feasible solution to the MECS problem.
Proof. First consider the edges in $G - G'$. Algorithm \[2\] chose to ignore these edges because for each such edge $e = (u, v)$, $P(u, v) \leq \text{stretch} \cdot W(e)$. Consider any two vertices $a, b \in V$ and consider the shortest path $P_G(a, b)$ between them in $G$. Consider the edges $e = (u, v)$ on this path such that $e \in G - G'$. Each such edge is replaced in $G'$ by a path $P_{G'}(u, v)$ such that $P_{G'}(u, v) \leq \text{stretch} \cdot W(e)$ as $\text{stretch} \geq 1$. Thus we have that $P_G(a, b) \leq \text{stretch} \cdot P_{G'}(a, b)$. This is true for all pairs of vertices $a, b \in G$. Thus $\sum_{a, b \in G} P_G(a, b) \leq \text{stretch} \cdot \sum_{a, b \in G'} P_{G'}(a, b)$. As the vertex sets of $G$ and $G'$ are the same, this completes the proof.

The following lemmas make statements about the structure of the solution returned by the greedy spanner algorithm. The first result relates to the structure of the resulting subgraph and the next result gives bounds on the size and weight of the resulting solution. Interested readers may consult \[9\] for detailed proofs of these results. The following lemma states that the solution returned by the algorithm \[2\] contains the minimum spanning tree as a subgraph.

Lemma 8. The graph $G'$ contains the MST of $G$ as a subgraph.

Next we state a theorem that states that the size and weight of the solution obtained from algorithm \[2\] are bounded above and gives the values of the upper bounds. One of the nice properties of the aforementioned algorithm is the fact that the weight of the resulting spanner is bounded above by a constant multiple of the weight of the MST.

Theorem 9. Given a weighted graph $G = (V, E, W)$ and a stretch factor $t > 0$ it is possible to construct a feasible solution to the MECS problem for stretch $2t + 1$ such that the solution $G'$ has the following properties:

1. $\text{Size}(G') < |V||V|^\frac{1}{2t}$ where size counts the number of edges in the graph
2. $\text{Weight}(G') < \text{Weight(MST}(G))(1 + \frac{|V|}{2t})$

Algorithm \[2\] maintains a forest of connected components at any time and at every step adds an edge to merge two connected components. At the end of the algorithm we are left with a single connected component that satisfies the spanner condition. The algorithm works on local information. At each step, it looks at the shortest path length between every pair of vertices. It makes sure that this is bounded above by a constant multiple of the shortest path length between the vertices in the input graph. This in turn preserves the APL. However, this is more than what we need. This algorithm preserves the shortest path length between all the vertex pairs and hence also preserves the original diameter to a constant multiple. We just need the preservation of the average path length and thus this algorithm does more than what we require.

Going forward, we state two greedy heuristics for the MECS problem. Both of them do not use any local information. The first algorithm starts with the original graph and removes edges from it, one by one, making sure that every edge removal preserves the constraint on the APL. The edges are considered
in the decreasing order of weights, thus making this strategy a greedy strategy. The second algorithm mimics the algorithm of Althofar et al. and starts with a forest. At each iteration, an edge is added, merging two components. The edges are considered in increasing order of weights and this process continues as long as the condition on the APL is violated. We state and analyze these algorithms next.

Algorithm 3: The Greedy Removal MECS Algorithm

**Procedure GreedyMECS(G, t)**

- **Input:** G = (V, E, W) the input graph, t the stretch
- **Output:** G: The MECS Spanner
- μ ← average distance in graph G
- Sort E in non-increasing order of weights
- for e ∈ E do
  - GTemp ← G – e
  - μTemp ← average distance of GTemp
  - if μTemp > t · μ then Do not remove e from G else G ← G – e
- return G

Algorithm 4: The Greedy Addition MECS Algorithm

**Procedure GreedyMECSV2(G, t)**

- **Input:** G = (V, E, W) the input graph, t the stretch
- **Output:** G: The MECS Spanner
- G′ ← (V, {})
- Sort E in non-decreasing order of weights
- μ ← average distance in graph G
- for e ∈ E do
  - μG′ ← average distance of G′
  - if μG′ > t · μ then G′ ← G′ ∪ e
- return G′

One implicit assumption of algorithm 3 is that the removal of the edge e from G does not leave the graph disconnected. In implementations, if the removal of the edge e leaves the graph disconnected, then the APL is distorted infinitely and hence the edge is not removed. Next we state a theorem regarding the nature of the solution returned by the greedy algorithms. The proof of the theorem is provided as separate lemmas in Appendix B.

**Theorem 10.** The following statements hold for the solutions returned by the algorithms 3 and 4:

1. Algorithm 3 always returns a feasible solution to the MECS problem.
2. Consider the graph $G'$ returned by the algorithm \[3\]. For any input graph $G$, the graph $G'$ contains the MST of $G$

3. Algorithm \[3\] always gives a feasible solution to the MECS problem

4. Consider the graph $G'$ returned by the algorithm \[4\]. For any input graph $G$, the graph $G'$ contains the MST of $G$

Apart from being an interesting observation, with a nice and concise proof, the above result also helps us to change the algorithm \[4\]. As we know that the solution returned by the algorithm \[4\] will always contain the MST, instead of starting the algorithm from a forest, we can start the algorithm from the MST. Thus we compute the MST of the input graph and then we start adding edges that are not included in the MST, in increasing order of weights. We continue this process until the APL of the resulting graph satisfies the constraint. These observations are also discussed in Appendix B due to space constraints.

6. Experimental Results

In an attempt to assess the performance of the proposed algorithms in comparison with others in the literature, we performed computational experiments using several standard data sets. In this section, we present and discuss our findings from these experiments.

In order to compare the different algorithms and formulations, we implemented and tested our algorithms on several different real life networks. The flow based approach, the path based approach and the MIP based approach were all implemented using FICO Xpress optimization suite \[42\]. The greedy heuristics were implemented using Python \[43\], using the Networkx \[44\] library to implement the standard graph algorithms. All the implementations were tested against the Greedy Spanner algorithm \[2\] of Althofar et al. \[9\], which, as has been shown above, computes a feasible solution to the MECS problem. However, this solution is not always the optimal one.

All the experiments were carried out on a laptop running OSX Mavericks, having 8 GB RAM and an Intel Core i7 processor. In all our implementations we use the initial average path length ($\mu_G$) and the target average path length ($\mu_{G_t}$) as the input along with the input graph. We do not use the stretch factor explicitly and it is computed based on the two inputs. Going forward we first describe the datasets that we used and finally we describe the results of our experiments.

6.1. Datasets

Most of our experiments were done with unweighted graphs, though the underlying algorithms might as well work with weighted graphs. The main reason for not using arbitrary weighted graphs for the experiments was the observation that solutions based on direct optimization methods did not scale very well with weighted graphs. We do however, report the results of our experiments on weighted unit disk graphs \[45\]. The first dataset that we use is the famous
karate club graph [46]. This is a social network graph that was first collected in 1977 and depicts the friendship relations between 34 members of a karate club. We also used our algorithms on the Kreb's network graph. As mentioned before we ran our experiments on Unit disk weighted graphs. These graphs were generated randomly and we used both a weighted as well as an unweighted version of the graph. As our goal is to minimize the total number of edges (or total edge weight) in the MECS spanner, we report this number for the different algorithms. We compare the results from the greedy spanner algorithm against the optimal solutions obtained by our integer programming based approaches as well as against the results from the greedy heuristics.

6.2. Results

The results obtained by our algorithms on the different datasets are described below. As mentioned at the beginning of this section, instead of explicitly using a stretch, we use an increment. Thus if the original APL for the input graph is $\mu$ and the target increment is $\delta$, then the goal is to find a sparse subgraph of the input graph whose APL is at most $\mu + \delta$. In all our experiments we use three values of the increments namely, 0.1, 0.2 and 0.3. Next we describe the results for each of the datasets separately.

6.2.1. Karate Club Graph

The results of running the MIP based algorithms on the Karate Club graph are shown in figure [3]. The original graph has an average path length of $\mu = 2.41$. The figures show the results of running the algorithm with target average path lengths of 2.51, 2.59 and 2.69. We note that when the target APL is 2.51, the optimal number of edges is 48 and the number of edges decreases to 37 when the target is 2.69. The total number of edges in the minimum spanning tree for the graph is 33. The results for this graph are summarized in the Table 1.

Though the results from the algorithms [3] and [4], are not optimal, they still perform better than the greedy spanner algorithm of Althofar et al. We also note that empirically, the size of the spanners returned by the greedy algorithms is at most twice the size of the optimal spanner. We also note that the size of the MST is 33. An interesting thing happens when the increment to the average distance is set to be at 2. Then the algorithm [3] results in a spanner that has the same size as the MST, that is 33. However, the algorithm [4] still gives a spanner of size 67 and the algorithm of Althofar et al. still returns the original graph. The figure [4] shows the results of running the different variations of the greedy algorithms on the Karate club graph for an increment of 0.3.

| Increment | MIP | Greedy1 | Greedy2 | Althofar |
|-----------|-----|---------|---------|-----------|
| 0.1       | 48  | 73      | 72      | 78        |
| 0.2       | 40  | 71      | 67      | 78        |
| 0.3       | 37  | 66      | 67      | 78        |

Table 1: Size of spanners for the karate club graph
Figure 3: Illustration of the (a) Karate Club network ($|V| = 34, |E| = 78$) with average distance $\mu = 2.41$ (b) the optimal compact spanner (thick edges) with average distance $\mu_s = \mu + 0.1$ (c) the optimal compact spanner (thick edges) with average distance $\mu_s = \mu + 0.2$ (d) the optimal compact spanner (thick edges) with average distance $\mu_s = \mu + 0.3$
6.2.2. Unit Disk Graph

Next we present the results for running our algorithms on an artificially generated graph. We generated the Unit Disk Graph following a standard procedure in wireless communication network analysis:

- First we generate a 100x100 box and put 50 “sensors” inside the box
- Set a communication range of 20, thus if the sensors are less than 20 units apart then they can communicate directly

As before we consider the results of running our algorithms for the increments of 0.1, 0.2 and 0.3. The results obtained are shown in the table 2. Its obvious that for small increments, as we have considered, the algorithm of Althofar et al. is not able to sparsify the graph at all. However, the algorithm 3 performs well and for each of the increments it gives us some sparsification of the underlying graph. However, the algorithm based on the addition of edges, 4, does not perform as well as the first for small increments.

| Increment | MIP | Greedy1 | Greedy2 | Althofar |
|-----------|-----|---------|---------|----------|
| 0.1       | 68  | 103     | 107     | 119      |
| 0.2       | 62  | 99      | 106     | 119      |
| 0.3       | 58  | 89      | 106     | 119      |

Table 2: Size of spanners for the unweighted unit disk graph

The table 3 shows the results of running our algorithms on a weighted unit disk graph. The weights on the edges are either 1 or 2 depending on the Euclidean distance between the vertices. The reason for constructing this graph in this particular way was to be able to run the MIP based algorithms on this graph as a matter of comparison with the greedy algorithms. We were not
able to run the MIP based solutions on general weighted graphs of this size. Our main goal in this experiment was to be able to compare the quality of the solutions obtained for weighted graphs, from both the MIP based algorithms and the greedy algorithms. Again it is clear that algorithm 3 outperforms the other two greedy algorithms.

| Increment | MIP  | Greedy1 | Greedy2 | Althofar |
|-----------|------|---------|---------|----------|
| 0.1       | 119  | 169     | 281     | 189      |
| 0.2       | 102  | 159     | 281     | 189      |
| 0.3       | 93   | 153     | 261     | 189      |

Table 3: Weight of spanners for the weighted unit disk graph

The resulting spanner graphs obtained for the unweighted and the weighted versions of the Unit Disk Graph are shown in the Appendix C.

6.2.3. Krebs Network

The results of running our algorithms on the krebs36 network is given in table 4. The size of the original input graph is 153 and the APL is 2.92. The size of the MST for this graph is 61. As before we consider three values for the increments, namely, 0.1, 0.2 and 0.3. Among the greedy algorithms, we observe that the algorithm 3 outperforms the other two greedy algorithms.

| Increment | MIP  | Greedy1 | Greedy2 | Althofar |
|-----------|------|---------|---------|----------|
| 0.1       | 82   | 139     | 149     | 153      |
| 0.2       | 63   | 132     | 149     | 153      |
| 0.3       | 61   | 136     | 149     | 153      |

Table 4: Size of spanners for the unweighted krebs graph

6.2.4. Discussion

We conclude this section with a small explanation of the results that we observe with the greedy algorithms.

Greedy MECS Algorithms. As seen from the results of the experiments above, the algorithm 3 outperforms the algorithm 4 in most of the cases. In order to understand why this happens one needs to look at the way the two algorithms operate. The first one removes edges from the graph whereas the second one adds edges to the graph. However, when removing the edges 3 only removes edges that are relevant. An edge is relevant for removal if the resulting subgraph has an APL that satisfies the MECS criteria. Thus at every iteration 3 removes only the most important edges. On the other hand 4 starts with a graph whose average path length is infinite (namely a set of disconnected points). At each step it adds and edge if the current subgraph violates the MECS criteria. Thus it will go on doing this until the MECS criteria is satisfied. At this point it
will stop the addition of edges. Thus it may end up adding more edges than is required to satisfy the given APL target. The only way around this problem is to add only relevant edges, edges that bring down the APL the most among all the edges that have not been considered till now. One of the advantages of this algorithm is that most often it achieves a APL value that is lower than the target, however at the cost of increasing the size of the resulting spanner. However, that would complicate the implementation more. We postpone such studies to a later paper.

Greedy Spanner Algorithm. We have observed that for unweighted graphs the greedy spanner algorithm does not return a sparse graph for most of our test cases. We have shown before that the solution returned by this algorithm is feasible for the MECS problem, however because of this issue, this algorithm does not present us with a practical option for computing MECS spanners on unweighted graphs. In order to understand why this problem occurs, we need to realize that there is a distinct difference between the magnitude of the stretch factor that is used for path based spanners and the APL based MECS spanners. In our case, we want the stretch to be really small, we don’t want the average path length to increase too much. As a result for most of our experiments the stretch is in the interval \((1, 1 + \epsilon)\) for small \(\epsilon > 0\). For unweighted graphs, this causes a problem, as the path length can only change in integral multiples of unity. Thus depending on the value of the stretch that we use, we would get some sparsification against no sparsification at all. This in turn justifies effort to look for more efficient algorithms for computing near optimal MECS solutions.

7. Conclusions and Future Work

In this paper we have introduced the Minimum Edge Compact Spanner problem and we have shown that the problem is \(NP\)-hard. We have used several MIP formulations of the problem to get the optimal MECS solutions. Moreover we have proposed two greedy algorithms for the MECS problem and have used them on real life networks. We have compared the results obtained from the MIP based solutions and the greedy algorithms.

In the near future, we would like to study the following problems: (1) is it possible to prove that the algorithm [3] gives a constant factor approximation for the MECS problem (2) are there other approximation algorithms for the MECS problem that gives \(O(\log |V|)\)-factor approximations (3) does the MECS problem have a PTAS and a FPTAS (4) if not then what is the hardness of approximating the MECS problem. We would like to consider these problems in the settings of Euclidean graphs, graphs on metric spaces that have nice packing properties like doubling metrics as well as for general graphs (metric spaces with shortest path metric). Average path length is an important parameter for graphs and there are several other similar parameters. We would like to investigate whether similar problems can be solved for these parameters as well.
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Appendix A. NP-Completeness of ECSTS Problem

We prove the \(NP\)-completeness of the ECSTS problem using a reduction from the Exact 3-Cover Problem \[10\]. This problem is defined as follows:

**Definition 10 (Exact 3-Cover Problem).** Let \( T = \{T_1, T_2, \ldots, T_{3t}\} \) be a set of \( 3t \) elements, for some integer \( t \) and let \( S = \{M_1, M_2, \ldots, M_k\} \) be a collection of 3-element subsets of \( T \). The Exact 3-Cover problem asks whether there exists a collection of subsets \( S' \subseteq S \), of disjoint 3-element subsets of \( T \), such that \( \cup(M \in S') = T \).

Mathematically we can write the problem as that of finding a subgraph \( G_s = (V, E_s) \) of \( G = (V, E) \) such that:

\[
|E_s| \leq |V| - 1 \quad \text{and} \quad \mu_{G_s} \leq c, \quad c \text{ finite}
\]

(A.1)

This in turn can be written as:

\[
|E_s| \leq |V| - 1 \quad \text{and} \quad \sum_{(u,v):u,v \in V, \ u \neq v} d(u,v) \leq C, \quad C \text{ finite}
\]

(A.2)

where \( c = \frac{C}{\pi(n-1)} \) where \( n = |V| \) which is exactly the \( 3 \) problem.

The Exact 3-Cover problem is known to be \( NP\)-complete \[40\]. In order to show the \( NP\)-completeness of problem \( 3 \), we start with the construction of a graph. The vertex set of this graph is the union of three disjoint sets of vertices. We show that given a solution for problem \( 3 \) on this graph, we can get a solution to the exact 3-cover problem. This would complete the reduction.
Definition 11 (Construction for Reduction). Let $T = \{T_1, T_2, \ldots, T_{3t}\}$ be a set of $3t$ elements and let $S = \{M_1, M_2, \ldots, M_k\}$ be a collection of 3-element subsets of $T$. We construct a graph $G = (V, E)$. The vertex set $V = P \cup S \cup T$ where $P$ is a new set $\{P_0, P_1, P_2, \ldots, P_r\}$ where $r$ is a constant, defined later. The value of $r$ depends on certain specific spanning trees of the graph $G$ as we will soon see. The edge set of the graph $G$ consists of the following:

- Edges $\{(P_0, P_i) : i = 1, 2, 3, \ldots, r\}$
- Edges $\{(P_0, M_i) : M_i \in S\}$
- Edges $\{(M_i, T_j) : M_i \in S, T_j \in M_i\}$

The resulting graph is shown in figure [A.5a]. We also consider a particular type of spanning tree for the graph $G$. This spanning tree is a feasible solution for the problem [3]. One such tree is shown in the figure [A.5b]. Let us denote the spanning tree in [A.5b] by $G_s$. Now, we define the constants as follows:

- $r = \sigma_{SP}^G(S, S) + \sigma_{SP}^G(S, T) + \sigma_{SP}^G(T, T)$
- $C = \sigma_{SP}^G(P, P) + \sigma_{SP}^G(P, S) + \sigma_{SP}^G(P, T) + r$

Thus $r$ is the sum of the shortest path distances between the vertices of $S$, the vertices of $S$ and $T$, and finally the vertices of $T$ and $T$, in the spanning tree $G_s$. On the other hand, $C$ is the sum of the shortest path distances between the vertices of all these sets, both for intra-set vertex pairs as well as inter-set vertex pairs, computed with respect to the spanning tree $G_s$. This completes the construction. We note that the graph $G_s$ as defined above is an Exact 3-cover for the set $T$.

Now we are ready to prove results that establish the $NP$-completeness of the problem. Thus we state the following result:

**Theorem 11.** The ECSTS problem is $NP$-complete

As stated before, in order to prove this theorem, we show that the Exact 3-Cover problem has a solution, only if $G$ as defined above, contains a subgraph, that is a feasible solution for the problem [3]. We do this through the following lemmas.

**Lemma 12.** Any spanning tree of the graph $G$ must contain the edges $\{(P_0, P_i) : i = 1, 2, 3, \ldots, r\}$.

**Proof.** The proof follows from the construction of the graph $G$. If, for example, the spanning tree does not contain the edge $(P_0, P_i)$ for some $i$, then there is no way to reach the vertex $P_i$ and thus the resulting graph is disconnected and hence not a spanning tree. This contradicts our assumption.
Lemma 13. Any spanning tree of the graph $G$, that is a feasible solution of the problem \[3\] must contain the edges $\{(P_0, M_i) : M_i \in S\}$.

Proof. Let us suppose that there is a feasible solution $G_f$ such that it does not contain the edge $(P_0, M_i)$ for some $i$. Now $\sigma_{SP}^{G_f} = \sigma_{SP}^{G_f}(P, P) + \sigma_{SP}^{G_f}(P, S) + \sigma_{SP}^{G_f}(P, T) + \sigma_{SP}^{G_f}(S, S) + \sigma_{SP}^{G_f}(S, T) + \sigma_{SP}^{G_f}(T, T)$. As each of these sums are non-negative, we have that

$$\sigma_{SP}^{G_f} > \sigma_{SP}^{G_f}(P, P) + \sigma_{SP}^{G_f}(P, S) + \sigma_{SP}^{G_f}(P, T)$$

$$> \sigma_{SP}^{G_f}(P, P) + \sigma_{SP}^{G_f}(P, S) + \sigma_{SP}^{G_f}(P, T) + 2 \cdot (r + 1)$$

$$> C$$

The second inequality follows from the fact that due to the absence of the edge $(P_0, M_i)$ for some $i$, the shortest paths between the vertices of the sets $(P,S)$ and $(S,T)$, changes. After some simple algebra, one can verify that the second line of the inequality holds. Thus the subgraph $G_f$ cannot be a feasible solution for the problem \[3\] and hence the result follows.

Lemma 14. Let $G_f$ be a feasible solution to the problem \[3\]. Then each vertex in $T$ is adjacent to exactly one vertex in $S$.

Proof. First note that the assumption that $G_f$ is a feasible solution to problem \[3\] implies that $G_f$ is a spanning tree for the graph $G$. We prove the result by contradiction. So let there be a vertex $T_j$ such that it is adjacent to two vertices $M_i$ and $M_k$ in $S$.

We also have that $G_f \subseteq G$ where $G$ is defined in the construction \[11\] above. Now by the same construction and the lemma \[13\], there are edges $(P_0, M_i)$ and $(P_0, M_k)$ in the graph $G_f$. Now this creates a problem as $(T_j, M_i), (M_i, P_0), (P_0, M_k), (M_k, T_j)$ is a cycle and hence $G_f$ cannot be a spanning tree and hence it cannot be feasible for the problem \[3\] as we have assumed and this completes the proof.

Lemma 15. Graph $G_s$ used in the construction \[11\] above and depicted in the figure \[A.5\] is a feasible solution of the problem \[3\].

Proof. The proof follows from lemmas \[12\], \[13\] and \[14\].

Now we are ready to prove the main theorem, which for convenience we state here again.

Theorem 16. The ECSTS problem, as defined below, is NP-complete.

$$|E_s| \leq |V| - 1 \text{ and } \mu_{G_s} \leq C \quad \text{(A.3)}$$

Proof. Let $G_f$ be any spanning tree for the graph $G$. As we have seen from the lemmas \[12\], \[13\] and \[14\], the spanning tree $G_f$ has a specific structure.
Let us denote by $n_i$ the number of vertices in $S$ that are adjacent to exactly $i$ vertices in $T$, $i = 0, 1, 2, 3$.

\[
\sigma_{G_p}^{G_j}(T, T) = 4 \left( \frac{3t(3t-1)}{2} \right) - 2 \cdot \sum_{(T_i, T_j) : T_i \neq T_j} (M_k, T_i), (M_k, T_j) \in E_{G_j}, G \in S
\]

\[
= (18 \cdot t^2 - 6 \cdot t) - (2 \cdot n_2 + 6 \cdot n_3)
\]

\[
= (18 \cdot t^2 - 12 \cdot t) + 6 \cdot t - 6 \cdot n_3 - 2 \cdot n_2
\]

\[
= \sigma_{G_p}^{G_j}(T, T) + 6 \cdot (t - n_3) - 2 \cdot n_2
\]

We note that $\left[ \right]$ denotes the number of elements operator. From the above derivation we note that $\sigma_{G_p}^{G_j}(T, T) = \sigma_{G_p}^{G_j}(T, T)$ if and only if $n_3 = t$ and $n_i = 0, i = 1, 2$ and hence $n_0 = s - t$. Now the condition $\sigma_{G_p}^{G_j}(T, T) = \sigma_{G_p}^{G_j}(T, T)$, by definition of $C$ is equivalent to the fact that $\sigma_{G_p}^{G_j} \leq C$ and hence feasibility of the $\left[3\right]$ problem and the condition $n_3 = t$ and $n_i = 0, i = 1, 2$ and $n_0 = s - t$ is equivalent to the existence of an Exact 3-cover. This completes the reduction.

**Appendix B. Lemmas On Greedy Algorithms**

This appendix states and proves some of the properties of the solutions returned by the greedy algorithms.

We also recall that the algorithm $\left[2\right]$ is the standard spanner algorithm and looks at all pairs shortest paths and tries to preserve them to a constant multiple. We also recall the following lemma, which was proved in the paper:

**Lemma 17.** Algorithm GreedySpanner always gives a feasible solution to the MECS problem.

The algorithms $\left[3\right]$ and $\left[4\right]$ are designed specifically for the MECS problem and instead of considering the all pairs shortest paths individually, they consider the APL for making their greedy choice.

In what follows we state as prove four lemmas that establish the claims made in Theorem 3 in the main paper. We state each claim as a lemma in order that its easier to understand and the proof is not too large.

**Lemma 18.** Algorithm $\left[3\right]$ always returns a feasible solution to the MECS problem.

**Proof.** Let us suppose the contrary. Suppose that the algorithm $\left[3\right]$ returns a graph $G'$ that is not a feasible solution for the MECS problem. Then $\mu_{G'} > \text{stretch} \cdot \mu_G$. This means that at some point in the execution of the algorithm $\left[3\right]$ an edge was removed from the input graph $G$ that resulted in the average of the resulting graph to violate the MECS constraint. But this is not possible because the algorithm checks the condition to make sure that this never happens. Thus we have a contradiction. Hence the result follows.
Lemma 19. Algorithm 4 always gives a feasible solution to the MECS problem.

Proof. The proof follows due to the fact that the for loop over the edges will continue to add edges to the graph $G'$ as long as the constraint in the MECS problem is not satisfied. We also note that there is at least one feasible solution, the graph $G$ itself and hence when the algorithm terminates it will return a feasible solution for MECS.

Next we consider two lemmas that describe the structure of the solution returned by the algorithms 3 and 4. More precisely, we claim that the MECS solutions returned by these algorithms will always contain the MST of the underlying graph as a subgraph.

Lemma 20. Consider the graph $G'$ returned by the algorithm 3. For any input graph $G$, the graph $G'$ contains the MST of $G$.

Proof. Let us suppose the contrary, that is let us suppose that the graph $G'$ returned by the algorithm does not contain the MST. Now the algorithm 3 works by removing edges, starting with the heaviest edge first. Thus if the resulting solution does not contain the MST, then it must be the case that at some iteration of the algorithm one of the MST edges are removed. Let us consider the instant at which an MST edge $e = (u, v)$ is removed by algorithm 3. The removal of this edge does not disconnect that graph and hence there is a path $p(u, v)$ between the vertices $u$ and $v$ of the graph. Thus the edge $e$ that was removed is part of a cycle. Moreover, all the edges on the path $p(u, v)$ have weights that are less than the edge $e$. For otherwise, one of these edges would have been removed earlier and so removal of $e$ would have left the graph disconnected.

Now let us consider the working of Kruskal’s algorithm for computing the MST. It would start with the forest of all the vertices of the graph and start adding edges in increasing order of their weights in the process merging two connected components. This in turn means that Kruskal’s algorithm would create the path $p(u, v)$ between the vertices $u$ and $v$ before considering the edge $e = (u, v)$ for addition. At this point, it would not add $e$ to the graph because $p(u, v) + e$ would form a cycle. Thus the MST resulting from a run of Kruskal’s algorithm on $G$, would not contain the edge $e$.

This argument holds for each of the edges removed from the graph $G$ by algorithm 3. Thus the solution returned by the algorithm, namely $G'$ will contain the MST of $G$ as a subgraph. This completes the proof.

Lemma 21. Consider the graph $G'$ returned by the algorithm 4. For any input graph $G$, the graph $G'$ contains the MST of $G$.

Proof. In order to prove this we note that the execution of the algorithm 4 is very similar to the Kruskal’s Algorithm. Both of them start with a forest and then go on adding edges at each step, which results in different components being
merged together. We recall that in case of Kruskal’s algorithm, an edge is added if and only if it does not create a cycle. Let us denote by $G_i' = 1, \ldots, \text{Size}(E)$ the collection of components generated by algorithm [4]. Similarly let us denote by $H_i = 1, \ldots, \text{Size}(E)$ the collection of components generated by Kruskal’s Algorithm for the MST. We now prove that for each $i$ the number of connected components in $H_i$ is same as the number of connected components in $G_i'$ and moreover each component of $H_i$ is the subset of a corresponding component of $G_i'$. Once we have proved this our result is established.

We prove the result by induction on the number of edges. The base case is easy, $H_1$ contains a forest with each vertex in its own connected component. The same also holds for $G_1'$. Then the base case is established. Let us assume that the hypothesis is true for some $i$. Consider that the edge $e = (u, v)$ is considered for the $(i + 1)^{th}$ step. There can be two situations:

**Same Component** If $u$ and $v$ are in the same component of $H_i$, then the edge $e$ will form a cycle. Hence Kruskal’s algorithm will not add the edge and hence $H_{i+1}$ will not contain the edge $e$. Now in $G_i'$ the edge $e$ is in the same component by the inductive hypothesis. This component contains the component $H_i$. Now two things can happen, either the edge $e$ is not added to $G'_{i+1}$ in which case nothing changes and $H_{i+1}$ is still a subset of $G'_{i+1}$. Or else $e$ is added to $G'_{i+1}$ and then also the containment relationship does not change. Hence the result follows by induction.

**Different Components** If $u$ and $v$ are in different components then the edge $e$ does not form a cycle. Thus Kruskal’s algorithm will add the edge and merge the two components of $H_i$ to get a new component of $H_{i+1}$. Now as $u$ and $v$ belong to different components of $H_i$, by the induction hypothesis they also belong to different components of $G_i'$. Thus there is no path between the two vertices $u$ and $v$ and hence the average distance will satisfy the condition $\mu_{G'} > \text{stretch} \ast \mu$ in algorithm [4]. Thus the edge $e$ will be added and the corresponding components merged to form a single component in $G'_{i+1}$. Thus again we have that each component of $H_{i+1}$ is contained in a corresponding component of $G'_{i+1}$ and hence the result follows by induction.

Now as $H_{\text{Size}(E)} = \text{MST}$ and $G'_{\text{Size}(E)} = G'$ our result follows.

Apart from being an interesting observation, with a nice and concise proof, the above result also helps us to change the algorithm [4]. As we know that the solution returned by the algorithm will always contain the MST, instead of starting the algorithm from a forest, we can start the algorithm from the MST. Thus we compute the MST of the input graph and then we start adding edges that are not included in the MST, in increasing order of weights. We continue this process until the APL of the resulting graph satisfies the constraint.
Optimization of Algorithm [4]. We can do a simple optimization to the algorithm [4] in order to get rid of some redundancy in its operation. In order to do that we note that this algorithm considers the edges of the input graph $G$ in the order of edge weights, in a non-decreasing order. Thus it may be the case that when an edge $e = (u, v)$ is being considered for addition to the graph $G'$ by the algorithm, a path $P(u, v)$ already exists between the vertices $u$ and $v$. Thus if it is the case that $\mu_{G'} > t \cdot \mu_G$ and we consider adding the edge $e$ to the graph $G'$ as per algorithm [4], then this addition only makes sense if the following holds for the existing path between the vertices $u$ and $v$ in $G'$: $P(u, v)_{G'} > t \cdot W(e)$, for if not then the path $P(u, v)_{G'}$ would satisfy $P(u, v)_{G'} \leq t \cdot P(u, v)_G$. We know by lemma [7], that if we can ensure this for all pairs of vertices, then we have got a feasible solution to the MECS problem. This whenever $P(u, v)_{G'} \leq t \cdot P(u, v)_G$ and we are adding the edge $e = (u, v)$, we can safely ignore adding the edge and still ensure the upper bound on the average path length. With this small optimization, we can expect to reduce the weight of the resulting solution $G'$. With this change we can state the algorithm as follows:

**Algorithm 5: The Greedy Addition MECS Algorithm (Optimized)**

```plaintext
Procedure GreedyMECSV2(G, t)
    Input: G = (V, E, W) the input graph, t the stretch
    Output: G': The MECS Spanner
    G' ← (V, {})
    Sort E in non-decreasing order of weights
    μ ← average distance in graph G
    for e ∈ E do
        μ_{G'} ← average distance of G'
        if μ_{G'} > t \cdot μ then
            P(u, v) ← Shortest path between u and v in G'
            if $P(u, v) > t \cdot W(e)$ then
                G' ← G' ∪ e
            end
        end
    end
    return G'
```

The output of algorithm [5] has several nice properties as well. We note that the change that we have done to the algorithm [4] prohibits the formation of cycles and hence we can still claim that the resulting solution contains the MST. Thus we can state that:

**Lemma 22.** Consider the graph $G'$ returned by the algorithm [5]. For any input graph $G$, the graph $G'$ contains the MST of $G$.

The proof follows simply by the fact that we are breaking up cycles only and
hence we are not changing the MST that is contained in the solution returned by the algorithm [4]. As a result we omit a detailed proof here.

Appendix C. Solutions for Unit Disk Graph

The following two figures show the results of running the exact MIP based solutions on the unweighted and weighted unit disk graphs respectively. Figure C.6 shows the results for the unweighted graph and figure C.7 show the results for the weighted unit disk graph.

Figure C.6: Illustration of the Unit Disk randomly generated network instance (|V| = 50, |E| = 119, D = 11) with average distance $\mu = 4.86$ and the optimal spanning subgraph (thick edges) with average distance (b) $\mu_s = \mu + 0.1$, (c) $\mu_s = \mu + 0.2$, (d) $\mu_s = \mu + 0.3$. 
Figure C.7: Illustration of the weighted Unit Disk randomly generated network instance on a 100x100 plane ($|V| = 50, |E_w| = 309, D = 15$) with average distance $\mu = 6.1$ and the optimal spanning subgraph (thick edges) with average distance (b) $\mu_s = \mu + 0.1$, (c) $\mu_s = \mu + 0.2$, (d) $\mu_s = \mu + 0.3$. We set $a_{ij} = 1$ if euclidean distance between $i$ and $j$ is less than 12.5 (blue edge), and $a_{ij} = 1$ if euclidean distance between $i$ and $j$ is less than 25 and greater than 12.5 (green edge).