Interacting Dark Energy and the Cosmic Coincidence Problem

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The introduction of an interaction for dark energy to the standard cosmology offers a potential solution to the cosmic coincidence problem. We examine the conditions on the dark energy density that must be satisfied for this scenario to be realized. Under some general conditions we find a stable attractor for the evolution of the Universe in the future. Holographic conjectures for the dark energy offer some specific examples of models with the desired properties.

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I. INTRODUCTION

The observation that the Universe appears to be accelerating has been one of the biggest surprises in both the astronomy and particle physics communities. This acceleration can be accounted for by adding a cosmological constant in Einstein’s equations. While particle physicists never had a definite reason for setting the cosmological constant to zero, there was always the view that like all small or zero quantities, there must be a symmetry enforcing it. The observation that the size of the cosmological constant is neither zero nor at one of the natural scales like the Planck length or the scale of supersymmetry breaking has focused attention on explaining the dark energy component.

There are at least two requirements for any model of dark energy: (1) one must obtain in a natural way the observed size of the energy density, $\rho \sim 3 \times 10^{-3} \text{eV}^4$. Many models attempt to relate this to the observed coincidence of this scale with the scale $H_0^2 M_{Pl}^2$, where $H_0$ is the Hubble constant at the current epoch and $M_{Pl}$ is the Planck mass. (2) One must also obtain an equation of state $w$ which is similar to that of a cosmological constant at least in the recent past, namely $w \approx -1$. In fact, the most recent data from the Wilkinson Microwave Anisotropy Probe (WMAP) satellite and supernova and sky surveys have constrained the equation of state $-1.4 < w < -0.8$ at the 95% confidence level for dark energy with a constant $w$.

An interesting approach to the dark matter problem that has arisen from recent advances in understanding string theory and black hole physics is to employ a holographic principle. Here the motivation is the recognition that the observed size of the energy density at the present epoch seems to be consistent with taking the geometric average of the two scales $H_0$ and $M_{Pl}$. Early attempts to relate the dark energy to the Hubble scale or the particle horizon did not give acceptable solutions in detail that were consistent with the observations.

The evolution of the dark energy density also depends on its precise nature. If the dark energy is like a cosmological constant with equation of state $w = -1$, the dark energy density is constant and within the expanding Universe ultimately the dark energy density comes to dominate over the matter density. In this scenario the transition from matter domination to dark energy domination is rapid, and the fact that experimental observations seem to be consistent with sizeable amounts of both matter and dark energy indicates that we are currently residing in this transition period. The cosmic coincidence problem is the statement that it is unlikely that the current epoch coincides with this rapid transition period.

A possible solution is to assume that there is some mechanism that is converting dark energy into matter. Cosmic antifriction forces were introduced in Ref. 1 as a possible solution to the cosmic coincidence problem. If these forces are present one can obtain a fixed ratio of matter density to dark energy density as the final state of the Universe and hence solve the coincidence problem. The resulting cosmology is described by Friedmann equations. The basic scenario is quite simple: the natural tendency of a cosmological component with a more negative equation of state to dominate at large times the energy density of the Universe is compensated by the decay of the dark energy component into the other component(s). At large times an equilibrium solution can develop for which the various components can coexist. The interpretation is then that we are living in or close to this equilibrium thus eliminating the cosmic coincidence problem.

Frictional and antifrictional forces we envision here were introduced for a single component in Refs. 1, 2. The particular case where the dark energy component is assumed to have a constant equation of state was treated in Ref. 3. These efforts were preceded by searches for attractor solutions involving scalar fields 4, 5. Recently these kind of scenarios have been employed in models with brane-bulk energy exchange 6. Some attempts have been made to explain the size of the dark energy density on the basis of holographic ideas 7, and lead to the consideration of the Hubble horizon as the holographic scale 8, 9. A holographic model based on the future event horizon was examined in Ref. 10. Danielsson studied the effects of a transplanckian backreaction as
a particular source for the holographic dark energy\[11\]. More recently an emphasis on the case of interacting dark energy has been discussed in Refs. \[12\, 13\, 14\].

Ultimately one needs to appeal to the observational data to constrain these ideas. In this direction Wang, et al. have begun a more detailed comparison of the predictions of a holographic universe with interaction with the observational data\[15\]. In addition to the overall development of the densities of the cosmological components there are other observables which are sensitive to the modified scenarios\[3\, 16\].

II. FRAMEWORK OF INTERACTING DARK ENERGY

We assume two component equations for dark energy and matter,

\[
\begin{align*}
\dot{\rho}_\Lambda + 3H(1 + w_\Lambda)\rho_\Lambda &= -Q, \\
\dot{\rho}_m + 3H(1 + w_m)\rho_m &= Q.
\end{align*}
\]

The equality of $Q$ in the two equations guarantees the overall conservation of the energy-momentum tensor, and positive $Q$ can be interpreted as a transfer from the dark energy component to the matter component. Presumably this arises from some microscopic mechanism, but we do not specify one here. The quantity $\rho_m$ is the matter component for which one usually takes $w_m = 0$, but we temporarily retain a nonzero equation of state so as to retain the generality of an two interacting components.

One can define effective equations of state,

\[
\begin{align*}
\dot{\rho}_\Lambda + 3H(1 + w_\Lambda^{\text{eff}})\rho_\Lambda &= 0, \\
\dot{\rho}_m + 3H(1 + w_m^{\text{eff}})\rho_m &= 0.
\end{align*}
\]

Define the ratio $r = \rho_m / \rho_\Lambda$ and the rate $\Gamma = Q/\rho_\Lambda$, then the effective equations of state are expressed in terms of the native equations of state and a ratio of rates as

\[
\begin{align*}
w_\Lambda^{\text{eff}} &= w_\Lambda + \frac{\Gamma}{3H}, & w_m^{\text{eff}} &= w_m - \frac{1}{r} \frac{\Gamma}{3H}. \tag{3}
\end{align*}
\]

The time evolution of the ratio $r$ is then

\[
\dot{r} = 3Hr \left[ w_\Lambda - w_m + \frac{1 + r}{r} \frac{\Gamma}{3H} \right] = 3Hr \left[ w_\Lambda^{\text{eff}} - w_m^{\text{eff}} \right]. \tag{4}
\]

The general behavior of solutions can be ascertained from this equation. The stable points in the evolution are obtained when either $r = 0$ or when $w_\Lambda^{\text{eff}} = w_m^{\text{eff}}$. If $w_\Lambda^{\text{eff}} < w_m^{\text{eff}}$ then the vanishing dark energy density ($r = 0$) solution will apply in the infinite past while a constant nonzero ratio of dark energy to matter will apply in the infinite future with equal effective equations of state.

Using the definitions

\[
\begin{align*}
\Omega_\Lambda = \frac{8\pi \rho_\Lambda}{3M_p^2 H^2}, & \quad \Omega_m = \frac{8\pi \rho_m}{3M_p^2 H^2}, \tag{5}
\end{align*}
\]

which satisfy a Friedmann equation

\[
\Omega_\Lambda + \Omega_m = 1, \tag{6}
\]

one can convert to the physical parameters

\[
r = \frac{1 - \Omega_\Lambda}{\Omega_\Lambda}, \quad \dot{r} = -\frac{\dot{\Omega}_\Lambda}{\Omega_\Lambda^2}. \tag{7}
\]

One can write the differential equation in the suggestive form using $\dot{\Omega}_\Lambda = H \frac{d\Omega_\Lambda}{dx}$,

\[
\frac{d\Omega_\Lambda}{dx} = -3\Omega_\Lambda(1 - \Omega_\Lambda) \left[ w_\Lambda^{\text{eff}} - w_m^{\text{eff}} \right]. \tag{8}
\]

where $x = \ln(a/a_0)$. The utility of the evolution equation in this form is that it allows one to understand in a general way the possible solutions for $\Omega_\Lambda$. Firstly if one turns off the interaction ($\Gamma = 0$) one recovers the usual evolution equation with the effective equations of state replaced with the usual equations of state $w_\Lambda$ and $w_m$, (we will call these the native equations of state). For constant $w_\Lambda$ and $w_m$ and $w_\Lambda < w_m$ then the dark energy evolution proceeds from the fixed points of the differential equation (zeros of the right hand side) in the usual way from $\Omega_\Lambda = 0$ to $\Omega_\Lambda = 1$ in the usual way. When an interaction is present a more interesting solution may occur where the evolution of $\Omega_\Lambda$ approaches a condition where instead $w_\Lambda^{\text{eff}} = w_m^{\text{eff}}$ at which $\Omega_\Lambda$ approaches a fixed asymptotic value for large time less than 1. For a properly chosen interaction this might constitute a solution to the cosmic coincidence problem.

For illustration one can also consider the (redundant) differential equation for the matter density,

\[
\begin{align*}
\frac{d\Omega_\Lambda}{dx} &= -3\Omega_\Lambda \Omega_m \left[ w_\Lambda^{\text{eff}} - w_m^{\text{eff}} \right], \\
\frac{d\Omega_m}{dx} &= -3\Omega_m \Omega_\Lambda \left[ w_m^{\text{eff}} - w_\Lambda^{\text{eff}} \right]. \tag{9}
\end{align*}
\]

An initial condition $(\Omega_m, \Omega_\Lambda) = (1, 0)$ is inevitably driven to an asymptotic solution for which $w_m^{\text{eff}} = w_\Lambda^{\text{eff}}$ and $\Omega_m$ and $\Omega_\Lambda$ approach their limiting (equal) value. This behavior is displayed for a specific solution\[13\] shown in Fig. 1.

Two physical assumptions must be provided to determine the evolution of the parameters of the universe. A holographic condition on the dark energy $\rho_\Lambda$ will determine by itself the effective equation of state $w_\Lambda^{\text{eff}}$. A specific specification for the source $Q$ will determine the effective matter equation of state $w_m^{\text{eff}}$. Alternatively one of these conditions can be replaced by the assumption that $w_\Lambda$ is constant.

One typically obtains an equation for the evolution of the dark energy density of the form

\[
\frac{d\Omega_\Lambda}{dx} = f(\Omega_\Lambda). \tag{10}
\]
FIG. 1: Evolution of $\Omega_\Lambda$ (solid) and the effective equations of state, $w_\text{eff}$ (dashed, bottom) and $w_m$ (dashed, top). The horizontal axis is $x = \ln(a/a_0)$ for an arbitrary $a_0$. The future event horizon (FH) is assumed to be the length scale for determining $w_\text{eff}$, while the interaction is given by $\Gamma/3H = b^2/\Omega$ with $b^2 = 0.2$.

The evolution of $\Omega_\Lambda$ with respect to $a$ or $x$ is qualitatively determined by the fixed points of the differential equation in Eq. (10) which are determined by the condition $f(\Omega_\Lambda) = 0$. Examining the equations in Eqs. (7), it would appear that this condition would require that $\dot{\rho} = 0$ and then that $w_\text{eff} = w_m$ via Eq. (4). However, as shown in Ref. [13] if one of the fixed points occurs for the physically interesting value of $\Omega_\Lambda = 0$, then one can have the Universe with close to vanishing dark energy simultaneously with $w_\text{eff} \neq w_m$. In the remainder of this paper we explore the general conditions that give rise to this type of equilibrium solution.

### III. Dimensional Analysis

First define a length scale $L_\Lambda$ via

$$L_\Lambda = \frac{c}{H \sqrt{\Omega_\Lambda}}.$$  

where the constant $c$ represents an order one constant [17]. Specific assumptions might associate the length scale with a physical length such as the Hubble horizon, the particle horizon, the future event horizon, or perhaps some other parameter. The common feature of any realistic choice is that the length scale at the present epoch should be of order the size of the observable Universe so as to obtain the appropriate size of the dark energy density consistent with experimental observations. One has

$$L_\Lambda = \frac{c}{H \sqrt{\Omega_\Lambda}}.$$  

There need to be two constraints imposed to solve the set of equations. We have three choices: 1) holographic principle on $\rho_\Lambda$, 2) suppose a form for $Q$ or equivalently $\Gamma$, and 3) suppose a form for the equation of state $w_\Lambda$. The equation linking them is obtained from Eq. (11),

$$\Gamma = 3H(-1 - w_\Lambda) + 2\frac{\dot{\Lambda}}{\Lambda}$$  

(13)

This implies the generic scale of the decay rate $\Gamma$ is the same order as $H$. In the particular case $w_\Lambda = -1$ Eq. (13) reduces to

$$\Gamma = 2\frac{\dot{\Lambda}}{\Lambda}$$  

(14)

which connects the sign of $Q$ to the sign of $\frac{\dot{\Lambda}}{\Lambda}$. In the case that the interaction vanishes, $\Gamma = 0$, one returns to the condition of a cosmological constant with constant energy density (and therefore constant $\Lambda$). On the other hand the interaction can either increase or decrease $\rho_\Lambda$ depending on the sign of the interaction. Henceforth we will refer to the rate $\frac{\dot{\Lambda}}{\Lambda}$ as the holographic rate, but it should be understood that it is an equivalent description for the time development of the dark energy density independent of any assumption based on the holographic principle.

Since the dark energy density scales as $\rho_\Lambda \propto \frac{1}{L^3_\Lambda}$, one has

$$\rho_\Lambda = -\frac{2\dot{\Lambda}}{L_\Lambda}.$$  

(15)

Comparing to Eq. (2) one obtains the relationship between the effective equation of state and the length scale

$$w_\text{eff} = -1 + \frac{2}{3H} \frac{\dot{\Lambda}}{L_\Lambda}.$$  

(16)

This equation recasts the scaling exhibited by the dark energy in terms of the ratio of the two rates, $L_\Lambda/L_\Lambda$ and $H$. When there is no interaction the same equation applies for the native equation of state $w_\Lambda$. The familiar case of a constant dark energy density simply corresponds to the statement that the length scale does not change, $\dot{\Lambda}/L_\Lambda = 0$. More generally Eq. (16) shows that an assumption for $L_\Lambda$ based on some holographic principle determines the form of the effective equation of state even in the presence of an interaction. If one applies Eq. (12) one has

$$\frac{\dot{L_\Lambda}}{L_\Lambda} = -\frac{\dot{H}}{H} - \frac{1}{2} \frac{\ddot{\Omega_\Lambda}}{\Omega_\Lambda}.$$  

(17)

Inserting this into Eq. (16) and using the definition for the deceleration parameter $q = -\ddot{a}/\dot{a}^2$ and $\ddot{H}/H^2 = -1 - q$, one obtains

$$w_\text{eff} = -\frac{1}{3} + 2q - \frac{1}{3H} \frac{\dot{\Omega_\Lambda}}{\Omega_\Lambda}.$$  

(18)
Asymptotically (i.e. for large $x$) the last term vanishes, so that
\[ w_A^{\text{eff}} \to \frac{1}{3} + \frac{2}{3}q = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}. \] (19)

Finally using Eq. (19) one obtains
\[ \frac{d\Omega_L}{dx} = -2\Omega_L \left[ \frac{\dot{H}}{H^2} + \frac{1}{H} \frac{\dot{L}_A}{L_A} \right] \] (20)
which indicates that the differential equation (right hand side) has a zero for $\Omega_L = 0$ provided the term in brackets does not vanish like $1/\Omega_L$.

The resulting interpretation of the possible evolutions of the Universe becomes straightforward using Eq. (20). The derivative of $\Omega_L$ has fixed points at $\Omega_L = 0$ and at some larger value where the condition
\[ \frac{\dot{H}}{H^2} + \frac{1}{H} \frac{\dot{L}_A}{L_A} = 0, \] (21)
is satisfied. For large times the Universe approaches this solution where the effective equations of states for matter and the dark energy become equal. At early times the dark energy is repelled from the $\Omega_L = 0$ and grows monotonically to a positive value. For the case $\dot{L}_A/L_A > 0$ one observes that $\dot{H}/H^2 = -1 - q$ must be negative for asymptotically large times.

IV. EXAMPLES

A holographic condition based on relating $L_A$ to the Hubble scale or to the particle horizon leads to conflict with observation. In the former case the ratio of the dark energy density to matter density is constant. In the latter case the equation of state of the dark energy is greater than $-1/3$. Taking the length scale to be the Hubble horizon $L_A = RH_H = 1/H$ which implies
\[ \rho_L = \frac{3\Lambda^2 M_p^2 H^2}{8\pi}, \] (22)
so that $c^2 = \Omega_L$ is a constant. This yields
\[ w_A^{\text{eff}} = -\frac{1}{3} + \frac{2}{3}q, \] (23)
so that the asymptotic solution obtained in Eq. (19) is identically satisfied.

Setting the holographic length scale equal to the particle horizon gives
\[ R_{PH} = a \int_0^t \frac{dt}{a} = a \int_0^a \frac{da}{Ha^2}. \] (24)
while setting it equal to the future event horizon gives
\[ R_{FH} = a \int_t^{\infty} \frac{dt}{a} = a \int_a^{\infty} \frac{da}{Ha^2}. \] (25)

If the length scale is set equal to the Hubble horizon, the particle horizon, or the future horizon, then the rates go like
\[ \frac{1}{H} \frac{\dot{L}_A}{L_A} = -\frac{\dot{H}}{H^2} \] (HH),
\[ = 1 + \frac{\sqrt{\Omega_L}}{c} \] (PH),
\[ = 1 - \frac{\sqrt{\Omega_L}}{c} \] (FH),
respectively. Finally for a cosmological constant one has $L_A/L_A = 0$. These conditions yield the following expressions for the effective dark energy equation of state
\[ w_A^{\text{eff}} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} \] (HH),
\[ = -\frac{1}{3} + \frac{2}{3} \frac{\sqrt{\rho_L}}{c} \] (PH),
\[ = -\frac{1}{3} - \frac{2}{3} \frac{\sqrt{\rho_L}}{c} \] (FH), (27)
and $w_A^{\text{eff}} = -1$ for a cosmological constant.

The case of the Hubble horizon is particularly simple: From Eq. (17) one has that $\Omega_L = 0$, so that the ratio of dark energy to matter is constant. This is independent of the existence of a interaction $\Gamma$. If the interaction is absent, then $w_{m}^{\text{eff}} = 0$ and $\dot{H}/H^2 = -3/2$ identically. Including an interaction these generalize to
\[ w_A^{\text{eff}} = w_{m}^{\text{eff}} = -\frac{\Gamma}{3H} \frac{\Omega_L}{1 - \Omega_L} = -\frac{\Gamma}{3H} \frac{1}{r}. \] (28)

One particular mechanism for determining the holographic condition is to assume the dark energy density is created out of the vacuum as the Universe expands. The component is created via the Bogolubov modes which result from the expansion of the Universe and the variation of the vacuum in such a situation. One has
\[ \rho_L = \frac{3}{2\pi a^4} \int a r^3 H^2(x), \] (29)
where $A$ is some mass of order the Planck scale. Translating this into the varying scale $L_A$, one obtains
\[ L_A = a^2 \left[ \int a r^3 H^2(x) \right]^{-1/2}, \] (30)
where an overall constant that arises in this definition is absorbed into the definition of the order one constant coefficient $c$. One calculates
\[ \frac{1}{H} \frac{\dot{L}_A}{L_A} = 2 - \frac{1}{2} H^2 L_A^2 = 2 - \frac{1}{2} \frac{c^2}{2 \Lambda_L}. \] (31)

The effective equation of state then follows from Eq. (28). In this formulation the behavior near $\Omega_L = 0$ results from lower momentum modes which were created at earlier times when $H$ was larger.
The asymptotic value for the effective equation of state should approach something near that of a cosmological constant to be consistent with observations. Having accumulated the results for the previous examples it is easy to conclude that the future horizon [10] yields an acceptable effective equation of state near $-1$, while the particle horizon has an effective equation of state which is too large. Generally one wants a length scale rate change that becomes small compared to the Hubble expansion $\dot{L}_A/L_A < H$ so that the effective equation of state approaches $-1$ as in Eq. (10). The future horizon and other holographic conditions that result in a decreasing value for $L_A/L_A$ as a function of increasing $\Omega_A$ have the desired properties for an accelerating Universe.

V. INTERACTIONS

The coupled equations in Eq. (1) defines an interaction between the dark energy and matter components. Given a specific rate $\Gamma$ as a function of $H$ and $\Omega_A$, one can study the behavior of the evolution equations with a specific behavior on the energy density. The ratio of rates is taken to be a function of $\Omega_A$

$$\frac{\Gamma}{3H} = b^2 g(\Omega_A), \quad (32)$$

where the dimensionless constant $b^2$ is included to facilitate comparison with previous works. In particular Ref. [13] assumed an interaction of this form with $g(\Omega_A) = 1/\Omega_A$. One then obtains (we assume henceforth that $w_m = 0$)

$$w_{\text{eff}}^{\text{m}} = -\frac{\Gamma}{3H} \frac{\Omega_A}{1 - \Omega_A}. \quad (33)$$

So the effective equation of state $w_{\text{eff}}^{\text{m}}$ for the matter component depends on the interaction only. Together with the fact that the effective equation of state $w_{\text{eff}}^{\text{A}}$ depends only on the assumption for the physical condition setting the scale $L_A$, allows for a more general analysis of possible cases that give the desired equilibrium solution. Using Eqs. (4) and (16) one obtains

$$\dot{r} = 3Hr \left[-1 + \frac{2}{3H} \frac{\dot{L}_A}{L_A} + \frac{\Gamma}{3H} \frac{\Omega_A}{1 - \Omega_A} \right]. \quad (34)$$

Using Eq. (17) to convert this into an equation involving only $\Omega_A$, one obtains

$$\frac{d\Omega_A}{dx} = 3\Omega_A(1 - \Omega_A) \left[1 - \frac{2}{3H} \frac{\dot{L}_A}{L_A} - \frac{\Gamma}{3H} \frac{\Omega_A}{1 - \Omega_A} \right]. \quad (35)$$

This expression is quite general, so it facilitates a more general understanding of the physical conditions required for the holographic conditions determining $L_A$ and the interaction $\Gamma$. If these conditions have a functional expression in terms of $\Omega_A$, then Eq. (35) represents a differential equation that can be solved. This is the case for the holographic conditions arising from the particle and future horizons as well as the form of the interactions assumed in this paper. Furthermore the differential equation involves the effective equations of state in a simple way. The first two terms in brackets are $-w_{\text{eff}}^{\text{A}}$ while the last term is $w_{\text{eff}}^{\text{m}}$. Together with Eq. (20) it determines the acceleration of the Universe through $H/H^2 = 1 - g$. One has

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(1 - \Omega_A) - \frac{\Omega_A}{H} \frac{\dot{L}_A}{L_A} + \frac{\Gamma}{2H} \Omega_A. \quad (36)$$

A complete specification of the problems involves the specification of two ratios of rates as in Eq. (13). A generic choice of definition for $\dot{L}_A/L_A$ and $\Gamma$ will determine the native equation of state $w_A$ which will then vary with time. Alternatively one can choose a constant $w_A$ and $L_A/L_A$ in which case the interaction rate is determined. We examine these cases in more detail in the remainder of this section.

A. Constant equation of state

The first case we shall examine is the one where the native equation of state for dark energy is constant. For this purpose take the equation involving the three rates, Eq. (13), and eliminate the interaction term in Eq. (35) in favor of the other two rates. One obtains

$$\frac{d\Omega_A}{dx} = 3\Omega_A \left[1 + w_A \Omega_A - \frac{2}{3H} \frac{\dot{L}_A}{L_A} \right]. \quad (37)$$

This equation is valid irregardless of whether $w_A$ is constant in time, but we are presently interested in the constant case. Provided the holographic rate does not have a singularity for $\Omega_A$, the right hand side has the required zero at $\Omega_A = 0$, and for $0 < \Omega_A < 1$ the first two terms in brackets is positive provided $w_A \Omega_A > -1$. For both the case of the holographic rate based on the particle horizon and future event horizon, there is another fixed point at positive $\Omega_A$ (see Eq. (26)).

Using Eqs. (16) and (37) one obtains the simple result

$$\frac{d\Omega_A}{dx} = 3\Omega_A \left[w_A \Omega_A - w_{\text{eff}}^{\text{A}} \right], \quad (38)$$

which shows that the fixed point occurs when $w_{\text{eff}}^{\text{A}} = w_A \Omega_A$. As examples take $w_A = -1$ and $c = 1$. Then the attractive fixed point occurs at $\Omega_A = \frac{1}{c}$ for the particle horizon (PH) case and at $\Omega_A = 1$ for the future event horizon (FH) case, and from Eq. (27) these correspond to $w_{\text{eff}}^{\text{A}} = -\frac{1}{c}$ and $-1$ respectively. More generally Eq. (38) implies that $w_{\text{eff}}^{\text{A}} \leq w_A$.

In the case the scale $L_A$ is identified with the Hubble scale $1/H$, the differential equation vanishes and one has the relation

$$w_{\text{eff}}^{\text{A}} = w_A \Omega_A. \quad (39)$$
If one further assumes a constant $w_{\Lambda}$ then the interaction $\Gamma$ is determined, and $w_{m}^{\text{eff}}$ is simply proportional to the constant $\Omega_{\Lambda}$. The interaction rate is then proportional to $\Omega_{m}$ as shown in Ref. [18],

$$\frac{\Gamma}{3H} = -w_{\Lambda}(1 - \Omega_{\Lambda}). \quad (40)$$

So this holographic condition corresponds to the one in which the differential equation is identically zero and equilibrium between the dark energy density and matter density holds for all times.

**B. Assumptions about the holographic and interaction rates**

The inclusion of an interaction term for the dark energy component can be accommodated most easily by defining an effective equation of state $w_{m}^{\text{eff}}$ which differs from the native equation of state $w_{m}$. By virtue of its definition it satisfies the same equation that the native equation of state satisfies in the noninteraction case, namely Eq. (18). If $0 < \Lambda_{A} < H$ then $-1 < w_{m}^{\text{eff}} < -1/3$ and the universe accelerates. The effective equation of state must lie between that of a cosmological constant ($w = -1$) and the curvature component ($w = -1/3$).

While there is a certain appeal in taking the future event horizon as the length scale for a holographic principle, it is clear from the evolution equation for the dark energy that an equilibrium fixed point solution can occur for a wider variety of assumptions.

In order to examine the conditions on the interaction rate consider the cases

$$\frac{\Gamma}{3H} = \frac{b^2}{\Omega_{\Lambda}}, \quad (41)$$

for some exponent $n$. One then obtains

$$w_{m}^{\text{eff}} = -\frac{b^2}{(1 - \Omega_{\Lambda})\Omega_{\Lambda}^{n-1}}. \quad (42)$$

Figure 2 shows the functional dependence of $w_{m}^{\text{eff}}$ and $w_{m}^{\text{eff}}$ versus $\Omega_{\Lambda}$ for various choices of the holographic condition and interaction. An initial condition of $\Omega_{\Lambda} = 0$ will evolve to the right until $w_{m}^{\text{eff}} = w_{m}^{\text{eff}}$ which represents the asymptotic solution. It is clear that an interaction of the form $\Gamma/3H = b^2/\Omega_{\Lambda}^n$ with $n \leq 1$ can give an acceptable result with a equilibrium balance between $\Omega_{\Lambda}$ and $\Omega_{m} = 1 - \Omega_{\Lambda}$ provided the holographic condition involves the future event horizon.

For $n > 1$ the effective equation of state $w_{m}^{\text{eff}}$ diverges for small $\Omega_{\Lambda}$, and in fact is less than $w_{m}^{\text{eff}}$ so does not yield an acceptable solution. We note that for $n < 1$ one has $w_{m}^{\text{eff}} = 0$ for $\Omega_{\Lambda} = 0$ which may be needed for better agreement with the observational data. An explicit example is plotted in Fig. 3 for which the interaction is assumed to be of the form in Eq. (41) with $n = 1/2$. In fact this is the typical behavior for all $n < 1$.

**VI. MULTICOMPONENT GENERALIZATION**

An obvious generalization of the model considered here is the multicomponent case

$$\dot{\rho}_{\Lambda} + 3H(1 + w_{\Lambda})\rho_{\Lambda} = -Q, \quad \dot{\rho}_{i} + 3H(1 + w_{i})\rho_{i} = Q_{i}, \quad (43)$$

subject to the constraint $\sum_{i} Q_{i} = Q$. The evolution equation for the dark energy component is

$$\frac{d\Omega_{\Lambda}}{dx} = -3\Omega_{\Lambda} \sum_{i} \Omega_{i} \left[w_{i}^{\text{eff}} - w_{i}^{\text{eff}} \right] \quad (44)$$

where

$$w_{i}^{\text{eff}} = w_{i} - \frac{1}{r_{i}^{2}} \frac{\Gamma_{i}}{3H}, \quad r_{i} = \frac{\rho_{i}}{\rho_{\Lambda}}, \quad \sum_{i} \Omega_{i} = 1 - \Omega_{\Lambda}, \quad (45)$$

are quantities defined for each component other than the dark energy. The generic solution that occurs for typical cases where the effective equations of state are monotonic functions of $\Omega_{\Lambda}$ is that two of the components come to
equilibrium with the remaining components being diluted away by the expansion of the Universe. This case is of course realized if one adds a radiation component for example.

VII. SUMMARY

This work presents a unifying and general treatment of the physical assumptions that have been made for models with an interacting dark energy component in the Universe. The discussion in the literature has typically involved specifications for a holographic principle, a choice for an interaction between dark matter and dark energy, or the assumption of constant equations of state. A specification for any two of the three choices determines the third. In particular a generic specification of a holographic principle and an interaction gives an effective equation of state that is not constant. The existence of a future fixed point for the dark energy of the Universe can be easily established without a numerical calculation by analyzing Eq. \( \frac{\rho}{\rho_0} = \frac{w_0}{w_0 + \frac{1}{3} - 2q/3} \) with appropriate assumptions for the relative size of the rate for the Universe expansion \( H \), the rate for the evolution of the dark energy density \( L_\Lambda / L_\Lambda \), and the interaction rate \( \Gamma \). Furthermore if the Universe in the present epoch is at or near this fixed point, a condition on the parameters involved in these physical assumptions can be derived by the vanishing of this differential equation. The evolution of a Universe with vanishing dark energy can be understood in a qualitative fashion if the interaction rate and the holographic rate or plotted versus as shown in Fig. 2, and the asymptotic future state of the Universe is identified as the balance between two competing factors: the natural tendency for dark energy to dominate over matter as the Universe expands versus the decay of dark energy into matter. The solutions are characterized by the condition \( w_\Lambda < w_m \) for which the late time (fixed point) solution is \( w_\Lambda = w_m = -1/3 + 2q/3 \).

The resulting equilibrium between dark energy and matter offers a possible solution to the cosmic coincidence problem. For example, the viability of holographic models using the future horizon rather than the particle horizon is clearly a result of the effective equation of state decreasing with \( \Omega_\Lambda \). However it should be clear that the existence of an equilibrium solution is more general than the holographic principle in terms of the future horizon. Detailed quantitative comparison of interacting models with present and future observational data should be able to further delineate between them.

The interacting models have been generalized to the case involving more than two interacting components. The resulting evolution equation were shown to have a universal form in terms of the effective equations of state.

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Alternatively one can define the length scale to have some constant value and let the parameter $c$ vary \[18\]. This approach is useful for investigating the relationship of the dark energy density with the holographic bound.

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\[17\] Note that the quantity $\dot{H}/H$ is not another independent rate, but is determined in terms of the fundamental rates $H$, $L_\Lambda/L_\Lambda$, and $\Gamma$.