Secant varieties and successive minima

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Let $X$ be a semi-stable arithmetic surface over the spectrum $S$ of the ring of integers in a number field $K$. We assume that the generic fiber $X_K$ is geometrically irreducible and has positive genus. Consider a line bundle $L$ on $X$, which is non-negative on every vertical fiber and equipped with an admissible metric (in the sense of Arakelov [A]) of positive curvature. In [S] we have shown that any non-trivial extension class of $L$ by the trivial line bundle is such that its $L^2$-norm is bounded away from zero.

In this paper we refine this analysis by showing that all successive minima of the euclidean lattice $\text{Ext}(L, \mathcal{O}_X)$ admit an explicit lower bound, which can be computed in terms of the Arakelov intersection theory on $X$ (Th. 4). By Serre duality, this result gives also an upper bound for the successive minima of the global sections over $X$ of the tensor product of $L$ with the relative dualizing sheaf $\omega_{X/S}$ of $X$ over $S$.

Our method to prove this result is a continuation of [S], Th. 2. The new ingredient is a study of the secant varieties $\Sigma_d$, $d > 0$, of the curve $X_K$, when this curve is embedded into a projective space by the sections of $L \otimes \omega_{X/S}$. We consider the maximal dimension of a projective subspace contained in $\Sigma_d$. When $\deg(L) > 2d + 2$, a result due to C. Voisin asserts that this dimension is as small as possible (equal to $d - 1$, Th. 1).

The study of the projective subspaces of the secant variety $\Sigma_d$ is related to our initial question for the following reasons. On one hand, the first $k$ successive minimal vectors of $\text{Ext}(L, \mathcal{O}_X)$ span a rank $k$ sublattice; if $k$ is big enough, this sublattice is not contained in the secant variety. On the other hand, if we know that an extension class $e$ does not lie in $\Sigma_d(K)$ for a large value of $d$, the proof of [S], Th. 2, gives a large lower bound for the $L^2$-norm of $e$.

The paper is organized as follows. In Section 1 we compute the degree of the secant varieties and we give Voisin’s result on secant varieties of curves (Th. 1). We deduce from it a geometric result on extension classes in Theorem 2. In Section 2 we propose another approach to linear subspaces in secant varieties of curves. Though weaker than Theorem 1, it might be of independent interest, especially because of Theorem 3, due to A. Granville. Granville’s result computes the maximal length of a sequence of binomial coefficients, in a given line of Pascal’s triangle, which have a common divisor. The answer turns out to be related to the distribution of primes among positive integers.
We start dealing with arithmetic surfaces in Section 3. We prove a “transfer theorem” for successive minima of dual lattices (Prop. 4), and we extend a result of Szpiro on the minimal height of algebraic points on $X$ (Prop. 5). We then prove our main theorem on successive minima over $X$ (Th. 4). We conclude with some explicit estimates when $L$ is a high of $\omega_{X/S}$ (Cor. 1 and 2).

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1. Secant varieties

1.1. Let $K$ be an algebraically closed field of characteristic zero, $C$ a smooth projective connected curve of genus $g$ over $K$, $\omega_C$ the canonical line bundle on $C$, and $L$ a line bundle of degree $m = \deg(L) > 2$ over $C$. Denote by

$$\mathbb{P} = \mathbb{P}(H^0(C, L \otimes \omega_C)^*)$$

the projective space defined by sections of $L \otimes \omega_C$ on $C$, and let

$$j : C \to \mathbb{P}$$

be the projective embedding defined by these sections. For any positive integer $d > 0$, we let $\Sigma_d \subset \mathbb{P}$ be the secant varieties of $d$-uples of points on $\mathbb{P}$, i.e.

$$\Sigma_d = \bigcup_{\deg(D)=d} \overline{D},$$

where $\overline{D} \subset \mathbb{P}$ is the linear span of the effective divisor $D$ of degree $d$ over $C$ ([ACGH], [B]). We first compute the degree of this projective variety.

**Proposition 1.** Assume $2d \leq \dim \mathbb{P} + 1$. Then $\Sigma_d$ has dimension $2d - 1$ and the degree of $\Sigma_d$ in $\mathbb{P}$ is

$$\deg(\Sigma_d) = D(g, m, d) = \sum_{\alpha=0}^{\inf(d,g)} \binom{m + g - 1 - d - \alpha}{d - \alpha} \binom{g}{\alpha}.$$

1.2. **Proof of Proposition 1.** Let $M = L \otimes \omega_C$, let $C_d$ be the symmetric product of $d$ copies of $C$ and let $\Delta \subset C \times C_d$ be the universal divisor ([ACGH] p. 339). If $\pi_1 : C \times C_d \to C$ and $\pi_2 : C \times C_d \to C_d$ are the projections, we let

$$E_L = \pi_2^*(\mathcal{O}_\Delta \otimes \pi_1^* M).$$
This is a vector bundle of rank \( d \) on \( C_d \) and the canonical map
\[
H^0(C, M) \otimes \mathcal{O}_{C_d} \to E_L
\]
induces a morphism
\[
\alpha : \mathbb{P}(E_L^*) \to \mathbb{P}
\]
the image of which is \( \Sigma_d \) (op.cit. p. 340).

We first notice that \( \alpha \) is a birational isomorphism onto \( \Sigma_d \). Indeed, by the general position theorem, [ACGH] p. 109, a general hyperplane \( H \) in \( \mathbb{P} \) cuts \( C \) in exactly \( \deg(M) = m + 2g - 2 \) points, any \( N \) of which are independent, where
\[
N := \dim(\mathbb{P}) + 1 = m + g - 1.
\]
In particular, \( d \) points in \( H \cap C \) span a linear subspace \( Q \subset \mathbb{P} \) of dimension \( d - 1 \), which meets \( C \) in exactly \( d \) points. A general point on \( Q \) has thus only one preimage by \( \alpha \).

It follows that \( \Sigma_d \) has dimension \( 2d - 1 \) and the degree of \( \Sigma_d \) in \( \mathbb{P} \) coincides with the integer \( \deg(\xi^{2d-1}) \) where
\[
\xi = c_1(\alpha^*(\mathcal{O}(1))) \in \mathrm{CH}^1(\mathbb{P}(E_L^*)).
\]
The direct image of \( \xi^{2d-1} \) by the projection
\[
\mathbb{P}(E_L^*) \to C_d
\]
is the top Segre class of \( E_L^* \) on \( C_d \). Therefore \( \deg(\xi^{2d-1}) \) is the coefficient of \( t^d \) in the formal power series in one variable
\[
s_t(E_L^*) = c_t(E_L^*)^{-1}.
\]
The total Chern series
\[
c_t(E_L^*) = \sum_{i \geq 0} t^i c_i(E_L^*) = c_{-t}(E_L)
\]
is computed in [ACGH], Lemma 2.5, p. 340. Fix a point \( P_0 \in C(K) \) and denote by \( x \) the class of \( P_0 + C_{d-1} \) in \( \mathrm{CH}^1(C_d) \), and by \( \theta \in \mathrm{CH}^1(C_d) \) the restriction to \( C_d \) of the class of the theta divisor on the Jacobian of \( C \). We know from loc.cit. that
\[
c_t(E_L^*) = (1 + xt)^{-A} e^{-t\theta/(1+xt)}
\]
where
\[
A = -d + \deg(L \otimes \omega_X) - g + 1 = m + g - 1 - d > 0.
\]
Therefore
\[ \deg(\Sigma_d) = [c_t \left( E^*_L \right)^{-1}]_{t_d} = \sum_{\alpha=0}^{\text{Inf}(d,g)} \left( A - \alpha \right) \frac{\theta^\alpha}{\alpha!} \sum_{\alpha=0}^{\text{Inf}(d,g)} \left( A - \alpha \right) \left( g \right) \]

by [ACGH], p. 343, last two lines. This proves Proposition 1. q.e.d.

1.3. By definition, \( \Sigma_d \) is a union of linear subspaces of \( \mathbb{P} \) of dimension \( d - 1 \). We are interested in the dimensions of linear subspaces of \( \mathbb{P} \) contained in \( \Sigma_d \).

**Theorem 1.** (C. Voisin [V]) Assume \( m > 2d + 2 \). Then, there is no linear subspace contained in \( \Sigma_d \) of dimension bigger than \( d - 1 \). Furthermore, the only linear subspaces of dimension \( d - 1 \) are those spanned by effective divisors of degree \( d \) on \( C \).

C. Voisin conjectures that the hypothesis \( m > 2d \) is enough for Theorem 1 to hold. When \( m = 2d + 1 \) or \( m = 2d + 2 \), we have a weaker bound than Theorem 1:

**Proposition 2.**

If \( m = 2d + 1 \) or \( m = 2d + 2 \), any linear subspace contained in \( \Sigma_d \) has dimension at most \( 2d - g - 1 \).

1.4. Proof of Proposition 2. By induction on \( d \), we may consider a linear space \( Q \subset \Sigma_d \) such that \( Q \not\subset \Sigma_{d-1} \). The morphism
\[ \alpha : \mathbb{P} \left( E^*_L \right) \to \mathbb{P} \]
is an embedding when restricted to \( \Sigma_d - \Sigma_{d-1} \) ([B] Lemma 1.2 a)). Therefore the closure \( Y \) in \( \mathbb{P} \left( E^*_L \right) \) of \( \alpha^{-1}(Q - \Sigma_{d-1}) \) is birationally equivalent to \( Q \). In particular, its Albanese variety is trivial. Let \( \text{Pic}^{(d)}(C) \) be the component of the Picard variety of \( C \) parametrizing line bundles of degree \( d \), and \( \sigma : C_d \to \text{Pic}^{(d)}(C) \) the canonical map. The composite map
\[ Y \to \mathbb{P} \left( E^*_L \right) \xrightarrow{\pi} C_d \xrightarrow{\sigma} \text{Pic}^{(d)}(C) \]

must be constant since \( Alb(Y) \) is trivial. Let \( x_0 \in \text{Pic}^{(d)}(C) \) be the image of \( Y \). We denote by \( S \) (resp. \( T \)) the inverse of \( x_0 \) by the map \( \sigma \) (resp. \( \sigma \circ \pi \)). Since \( \text{deg}_K(M) > 2g - 2 \), \( \sigma \) is a projective bundle, hence \( S \simeq \mathbb{P}^{d-g} \) ([ACGH], VII, Prop. 2.1), and the map \( \pi : T \to S \) is also a projective bundle, of relative dimension \( d - 1 \). It follows that the dimension of \( Q \) is at most \( 2d - g - 1 \).

q.e.d.

1.5. We keep the notation of § 1.1. Let \( L^{-1} \) be the dual of \( L \) and
\[ e \in \text{Ext}^1(L, \mathcal{O}_C) = H^1(C, L^{-1}) \]
any extension class of $L$ by the trivial bundle. There exists a rank two vector bundle $E$ over $C$ and an exact sequence of coherent sheaves

$$0 \to O_C \to E \to L \to 0$$

which is classified by $e$. This extension is uniquely determined by $e$ up to isomorphism.

Recall that $E$ is called semi-stable (resp. stable) when any invertible subsheaf $M \subset E$ satisfies

$$\deg (M) \leq \deg (E)/2 = m/2$$

(resp. $\deg (M) < m/2$).

**Theorem 2.** Let $d$ be a positive integer and $V$ a $K$-vector space of $H^1(C, L^{-1})$.

i) If $m > 2d + 2$ and $\dim_K(V) > d$ there exists a non-trivial extension class $e \in V$ with the following property: given any line bundle $M \subset V$, one has

$$\deg (M) < m - d.$$ 

ii) When $m = 2d + 1$ (resp. $m = 2d + 2$) and $\dim_K(V) > 2d - g$ there exists a non-trivial $e \in V$ such that $E$ is stable (resp. semi-stable).

**1.6. Proof of Theorem 2.** Let $E$ be a non-trivial extension of $L$ by $O_C$ as in (1) and let $M \subset E$ be a line bundle of positive degree. Then the map $M \to L$ is non-trivial (otherwise $M \subset O_C$), therefore there exists an effective divisor $D$ on $C$ such that the map $M \to L$ induces an isomorphism

$$M \sim \to L(-D).$$

When restricted to $L(-D)$ the extension (1) becomes trivial, i.e. $e$ lies in the kernel of the map

$$H^1(X, L^{-1}) \to H^1(X, L(-D)^{-1}).$$

The exact sequence of sheaves on $C$

$$0 \to L^{-1} \to M^{-1} \to M^{-1} \otimes O(D) \to 0$$

gives an exact sequence of cohomology groups

$$H^0(C, M^{-1} \otimes O(D)) \xrightarrow{\partial_D} H^1(C, L^{-1}) \to H^1(C, M^{-1}).$$

Therefore $e$ lies in the image of $\partial_D$. But the map $\partial_D$ coincides with the restriction of

$$\alpha : \mathbb{P}(E^*_L) \to \mathbb{P}$$
to the fiber at $D \in C_d$ of the projection map $\mathbb{P}(E^*_L) \to C_d$ (see [ACGH], p. 340, or [B], Observation 2, p. 451). In other words, if $d = \deg(D) = \deg(L) - \deg(M)$, the class of $e$ in $\mathbb{P}$ must lie in the secant variety $\Sigma_d$. By Theorem 1 we know that if $m > 2d + 2$ and $\dim(V) > d$, there exists a non-zero $e \in V$ whose class in $\mathbb{P} = \mathbb{P}(H^1(C, L^{-1}))$ (Serre duality) is not in $\Sigma_d$. This proves the first assertion in Theorem 2.

When $m = 2d+1$ (resp. $m = 2d+2$) we get from Proposition 1, under the hypothesis in ii), that $\deg(M) < d + 1$, hence

$$\deg(M) < \deg(E)/2$$

(resp. $\deg(M) \leq d + 1 = \deg(E)/2$). Therefore $E$ is stable (resp. semi-stable).

q.e.d.

2. On the divisibility of binomial coefficients

2.1. For any integer $n > 0$, let $b(n) \geq 0$ be the smallest integer $b$ such that the set of binomial coefficients $\binom{n}{m}$, where $b < m < n - b$, has a (non-trivial) common divisor.

**Theorem 3.** (A. Granville) The integer $b(n)$ is the smallest integer of the form $n - p^k$, where $p^k$ is a prime power less or equal to $n$.

2.2. Let $c(n)$ be the smallest integer of the form $n - p^k$, with $p^k \leq n$ a prime power. The inequality $b(n) \leq c(n)$ is not hard to prove. First, when $n = p^k$ is itself a prime power, it is clear that $b(n) = 0$. Indeed (we follow here a suggestion of Tamvakis), for any $m > 0$, with $m < p^k$, the number

$$\binom{p^k - 1}{m} = \frac{(p^k - 1)(p^k - 2) \ldots (p^k - m)}{m!}$$

is congruent to $(-1)^m$ modulo $p$. Therefore

$$\binom{p^k}{m} = \binom{p^k - 1}{m} + \binom{p^k - 1}{m-1}$$

is divisible by $p$.

Now, given any integer $n > 1$, we can find a prime power $p^k$ such that

$$\frac{n}{2} < p^k \leq n$$

(this fact is due to Chebyshev, see (6) below for a stronger statement). Let $m$ be any integer such that

$$n - p^k < m \leq p^k.$$
We have

\[
\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}.
\]

If \( n \) is not a prime power, both \( m \) and \( m-1 \) are in the interval \([n-1-p^k,p^k]i\) and \( p^k \leq n-1 \). If we fix the prime power \( p^k \) and proceed by induction on \( n \), with \( p^k \leq n < 2p^k \) (starting with the case \( n = p^k \) treated above) we conclude that both \( \binom{n-1}{m} \) and \( \binom{n-1}{m-1} \) are divisible by \( p \), therefore \( p \) divides \( \binom{n}{m} \) as soon as \( n - p^k < m \leq p^k \). This implies

\[
b(n) \leq n - p^k.
\]

Therefore \( b(n) \leq c(n) \).

2.3. It is more difficult to check that \( b(n) = c(n) \). For each prime \( p \) let \( b_p \) be the largest integer \( b \leq n/2 \) such that \( p \) does not divide \( \binom{n}{b} \). We have

\[
b(n) = \inf_{p \text{ prime}} b_p.
\]

Now write \( n \) and \( m \) in base \( p \):

\[
n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_0
\]

\[
m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_0
\]

with \( 0 \leq m_i, n_i \leq p - 1 \) and \( n_k > 0 \) (we assume \( m \leq n \)). Kummer showed that \( p \) divides \( \binom{n}{m} \) if and only if \( n_i < m_i \) for some \( i \) ([R] p.23-24). In particular, given any integer \( r \) such that \( 0 \leq r \leq n_k \), \( p \) does not divide \( \binom{n}{r p^k} \).

Assume \( n_k \geq 2 \) and let \( r \) be the integral part of \( n_k/2 \). We have

\[
r \geq 1 \quad \text{and} \quad r \geq (n_k - 1)/2,
\]

therefore

\[
n < (n_k + 1) p^k \leq 2(r + 1) p^k \leq 4r p^k.
\]

On the other hand

\[
b_p \geq r p^k.
\]

Therefore

\[
b_p > n/4.
\]
As we shall see below (cf. (6)), for any integer \( n \geq 2 \) there exists a prime \( p \) such that \( 3n/4 \leq p \leq n \). By Kummer’s criterion recalled above, we know that \( p \) divides \( \binom{n}{m} \) when \( n - p < m < p \). Therefore

\[
b_p \leq n - p \leq n/4,
\]

which contradicts (3).

So, when computing \( b_p \), we can assume that \( n_k = 1 \). This implies that \( n < 2p^k \). When \( n - p^k < m < p^k \) there exists \( i < k \) with \( m_i > n_i \). Therefore, by Kummer’s criterion (or by 2.2), \( p \) divides \( \binom{n}{m} \). On the other hand \( p \) does not divide \( \binom{n}{n - p^k} \), again by Kummer’s criterion. Therefore

\[
b_p = n - p^k
\]

hence \( b(n) = c(n) \). This ends the proof of Theorem 1. q.e.d.

2.4. Given two real valued functions \( f(n) \) and \( g(n) \) on positive integers (or real numbers), we shall write \( f(n) \ll g(n) \) when there exists a positive constant \( C > 0 \) such that, for all \( n, f(n) \leq Cg(n) \).

According to [ B-H ], there exists a positive constant \( C \) such that, for any integer \( n > 1 \), there is a prime number \( p \) with \( p \leq n \) and

\[
p \geq n - Cn^{0.535}.
\]

This implies

\[
b(n) \ll n^{0.535}.
\]

The Riemann hypothesis is known to imply

\[
b(n) \ll \sqrt{n} \log(n)
\]

([ I ] Th. 12.10).

It was shown by Nagura [N] that if \( x \geq 25 \) there exists a prime \( p \) with

\[
x < p < 6x/5.
\]

Given any integer \( n \geq 30 \), write \( n = 6y + r \) with \( 0 \leq r < 6 \). If we apply the previous inequality to \( x = 5y \) we get that there exists a prime \( p \) such that

\[
(5n/6) - 4 \leq p < n.
\]
This implies in particular

(6) \[ b(n) \leq n/4 \]

when \( n \geq 30 \). The inequality (6) can be checked directly when \( 2 \leq n \leq 30 \).

2.5. From Theorem 3 one can also derive estimates on the sum of the first \( n \) values of the function \( b \) when \( n \) gets large. Indeed, if \( p_1, p_2, p_3, \ldots \) is the list of prime numbers in increasing order, one has

\[
\sum_{j=1}^{n} b(j) \leq \sum_{p_k \leq n} \sum_{p_{k+1}}^{p_k+1} b(j) \\
\leq \sum_{p_k \leq n} \left( \sum_{j=0}^{p_{k+1}-p_k} j \right) \\
= \sum_{p_k \leq n} (p_{k+1} - p_k)(p_{k+1} - p_k - 1)/2 \\
\leq \frac{1}{2} \sum_{p_k \leq n} (p_{k+1} - p_k)^2 \\
\ll_{\varepsilon} n^{23/18 + \varepsilon},
\]

where the symbol \( \ll_{\varepsilon} \) means that the constant involved depends on \( \varepsilon \) (but not on \( n \)), and the last inequality is a result due to D.R. Heath-Brown ([I] Th. 12.17, (12.117)). So we have

(7) \[ \sum_{j=1}^{n} b(j) \ll_{\varepsilon} n^{23/18 + \varepsilon}. \]

A result of Selberg (cf. [I] p. 349) says that the Riemann hypothesis implies that

\[ \sum_{j=1}^{n} b(j) \ll n \log(n)^3. \]

2.6. The following result is weaker than C.Voisin’s Theorem 1. Still, because of the facts described in §2.1 to 2.5, it puts strong constraints on the dimension of linear subspaces contained in secant varieties. We use the notation of §1.1.

**Proposition 3** If \( m > 2d \) and

\[ b(m + g - 1 - d) \leq m + 2g - 1 - 2d, \]
\[ \varepsilon(d) < b(m + g - 1 - d). \]

2.7. Proof of Proposition 3. From Theorem 3 we get

\[ b(n) \leq b(n - 1) + 1 \]

therefore the quantity \( b(m + g - 1 - d) + d - 1 \) is an increasing function of \( d \). Therefore, by induction on \( d \), to prove Proposition 3, we may consider a linear space \( Q \subset \Sigma_d \) such that \( Q \not\subset \Sigma_{d-1} \) and \( \dim Q = d - 1 + \varepsilon(d) \). As in the proof of Proposition 1, we let \( Y \) be the closure in \( \mathbb{P}(E_L^*) \) of \( \alpha^{-1}(Q - \Sigma_{d-1}) \) and \( x_0 \in \text{Pic}^{(d)}(C) \) be the image of \( Y \). Denote by \( S \) (resp. \( T \)) the inverse of \( x_0 \) by the map \( \sigma \) (resp. \( \sigma \circ \pi \)). Since \( \deg_K(M) > 2g - 2 \), \( \sigma \) is a projective bundle, hence \( S \simeq \mathbb{P}^{d-g} \) ([ACGH], VII, Prop. 2.1). Let \( x \in \text{CH}^1(S) \) be the class of any hyperplane. The map \( \pi: T \to S \) is also a projective bundle, of relative dimension \( d - 1 \). We let \( \xi_0 \in \text{CH}^1(\mathbb{P}) \) be the class of an hyperplane, hence \( \xi = \alpha^*(\xi_0) \in \text{CH}^1(T) \) is a generator of the ring \( \text{CH}^*(T) \) over \( \text{CH}^*(S) \). In particular, \( x \) and \( \xi \) span the ring \( \text{CH}^*(T) \).

From [ACGH], Lemma 2.5, p. 340, one can compute the Segre classes of \( E_L^* \) restricted to \( S \). If we let

\[ s_t(E_L^*) = \sum_{i \geq 0} s_i(E_L^*) t^i \]

we get (see § 1.2)

\[ s_t(E_L^*)|_S = (1 + xt)^{m+g-1-d}. \]

Therefore, for any \( i \geq 0 \),

\[ \pi_*(\xi^{d-1+i}) = s_i(E_L^*)|_S = \binom{m + g - 1 - d}{i} x^i. \]

Since \( \alpha \) restricted to \( Y \) is a birational isomorphism of \( Y \) with the linear subspace \( Q \) in \( \mathbb{P} \), we have

\[ [Y] \cdot \xi^{d-1+\varepsilon(d)} = 1, \]

where \([Y]\) is the class of \( Y \) in the appropriate Chow group of \( T \), namely \( \text{CH}^k(T) \) with

\[ k = 2d - 1 - g - (d - 1 + \varepsilon(d)) = d - g - \varepsilon(d). \]

Let us write \([Y]\) in terms of our basis of \( \text{CH}^*(T) \):

\[ [Y] = \sum_{i+j=d-g-\varepsilon(d)} n_i x^i \xi^j. \]
From (8) and (9) we get

\[ 1 = \sum_{i+j=d-g-\varepsilon(d)} n_i x^i \xi^{j+d-1+\varepsilon(d)} = \sum_{i+j=d-g-\varepsilon(d)} n_i \left( \frac{m+g-1-d}{j+\varepsilon(d)} \right). \]

Therefore the binomial coefficients \( \binom{m+g-1-d}{b} \) are prime to each other when

\[ \varepsilon(d) \leq b \leq d-g. \]

Let \( A = m + g - 1 - d \) and \( n = A - d + g = m + 2g - 1 - 2d \). By the definition of \( b(A) \) there are two possibilities:

i) \( n < \varepsilon(d) \) and \( n < b(A) \)

ii) \( \varepsilon(d) \leq n \) and \( \varepsilon(d) < b(A) \).

Therefore, if \( b(A) \leq n \) we must have

\[ \varepsilon(d) < b(A). \]

q.e.d.

3. Successive minima on arithmetic surfaces

3.1. Let \( K \) be a number field, \( \mathcal{O}_K \) its ring of integers and \( S = \text{Spec}(\mathcal{O}_K) \). Consider an hermitian vector bundle \( \mathbf{V} \) of rank \( N+1 \) over \( S \), i.e. a finitely generated projective \( \mathcal{O}_K \)-module \( V \) and, for every complex embedding \( \sigma : K \rightarrow \mathbb{C} \), an hermitian scalar product on the complex vector space \( V_{\sigma} = V \otimes_{\mathcal{O}_K} \mathbb{C} \) defined by \( \sigma \); furthermore these scalar products are invariant by complex conjugation. We denote by \( \widehat{\text{deg}(\mathbf{V})} \in \mathbb{R} \) the arithmetic degree of \( \mathbf{V} \) (see [BGS] (2.1.11), (2.1.15)).

Given any positive integer \( p \), \( 1 \leq p \leq N+1 \), we let \( \lambda_p(\mathbf{V}) \) be the \( p \)-th minimum of \( \mathbf{V} \), i.e. the infimum of the set of real numbers \( \lambda \) such that there exist \( p \) vectors \( e_i \in V \), \( 1 \leq i \leq p \), which are linearly independent in \( V \otimes_{\mathcal{O}_K} K \) and such that, for any \( i \) and any complex embedding \( \sigma : K \rightarrow \mathbb{C} \), \( \log \|e_i\|_{\sigma} \leq \lambda \). Furthermore, let \( \ell_p(\mathbf{V}) \) be the minimal projective height of \( \mathbb{P}(F) \subset \mathbb{P}(V) \) when \( F \) is a subbundle of rank \( p \) in \( V \).

Let \( V^* = \text{Hom}(V, \mathcal{O}_K) \) be the dual of the module \( V \). We equip \( V^* \) with the metric dual to the one chosen on \( V \). Finally, if \( B_n \) is the euclidean volume of the unit ball in \( \mathbb{R}^n \), let \( r_1 \) (resp. \( r_2 \)) be the number of real (resp. complex) places of \( K \), and let \( \Delta_K \) be the absolute discriminant of \( K \). We introduce the following constant

\[ C(N, K) = (N+1)(r_1 + r_2) \log(2) + (N+1)(\log|\Delta_K|)/2 - r_1 \log B_N - r_2 \log B_{2N+2}. \]
**Proposition 4.** Given any $p$ with $1 \leq p \leq N + 1$, the following inequalities hold:

$$\ell_{N+1-p}(\mathbf{V}^*) \leq [K : \mathbb{Q}] \sum_{j=1}^{p} \lambda_j(\mathbf{V}) \leq C(N, K) + \ell_{N+1-p}(\mathbf{V}^*) .$$

### 3.2. Proof of Proposition 4.

Since $\hat{\deg}(\mathbf{V}^*) = -\hat{\deg}(\mathbf{V})$, it follows from [BGS] (4.1.3) that

$$\ell_p(\mathbf{V}) = \ell_{N+1-p}(\mathbf{V}^*) + \hat{\deg}(\mathbf{V}) .$$

From [BGS] Theorem 5.2.4, this implies

$$\ell_{N+1-p}(\mathbf{V}^*) = \ell_p(\mathbf{V}) - \hat{\deg}(\mathbf{V}) \leq [K : \mathbb{Q}] \sum_{j=1}^{p} \lambda_j(\mathbf{V}) .$$

From [BGS] (5.2.14) and (5.2.15) (where $\lambda_j(\mathbf{V})$ is denoted $\lambda'_j$), and from the result of Bombieri and Vaaler [Bo-Va], we get

$$[K : \mathbb{Q}] \sum_{j=1}^{p} \lambda_j(\mathbf{V}) \leq C(N, K) - \hat{\deg}(\mathbf{V}) - \sum_{j=p+1}^{N+1} \lambda_j(\mathbf{V})$$

$$\leq C(N, K) - \hat{\deg}(\mathbf{V}) + \ell_p(\mathbf{V}) = C(N, K) + \ell_{N+1-p}(\mathbf{V}^*) .$$

This concludes the proof of Proposition 4. q.e.d.

### 3.3. We keep the notation of section 3.1. Let $\mathbb{P} = \text{Proj}(\text{Sym}(\mathbf{V}^*))$ be the projective space of $\mathbf{V}$ and $\mathbb{P}_K$ its generic fiber. Consider a closed subvariety $\Sigma \subset \mathbb{P}_K$ of projective degree $D = \deg(\Sigma)$. Let

$$\lambda_{\max}(\mathbf{V}) = \lambda_{N+1}(\mathbf{V})$$

be the last successive minimum of $\mathbf{V}$.

**Proposition 5.** There exists a non zero vector $v \in \mathbf{V}$ such that, for all $\sigma : K \hookrightarrow \mathbb{C}$,

$$\log \|v\|_\sigma \leq \lambda_{\max}(\mathbf{V}) + \log(D(N + 1)) ,$$

and such that the point $[v] \in \mathbb{P}(K)$ does not lie in $\Sigma(K)$.

**Proof of Proposition 5.** Choose a basis $\{e_1, \ldots, e_{N+1}\}$ of $\mathbf{V} \otimes K$ such that $e_i \in \mathbf{V}$ and

$$\log \|e_i\| = \lambda_i$$

for all $i = 1, \ldots, N+1$. Using this basis, we can identify $\mathbb{P}$ with $\mathbb{P}^N$. 


We first notice that there exists an homogeneous polynomial $F(X_1, \ldots, X_{N+1})$ of degree $D$ such that $\Sigma$ lies in the zero set of $F$. Indeed we can find a linear projection $\pi : \mathbb{P}^N \rightarrow \mathbb{P}^M$ such that $\pi(\Sigma)$ is an hypersurface of degree $D$ in $\mathbb{P}^M$. If $\tilde{F} = 0$ is an equation for $\pi(\Sigma)$, we can take $F = \tilde{F} \circ \pi$.

Now let $\Phi$ be the finite set of vectors of the form $v = \sum_i n_i e_i$ where $n_i \in \mathbb{Z}$ and $0 \leq n_i \leq D$ for all $i = 1, \ldots, N+1$. Assume that $F(v) = 0$ for any $v \in \Phi$. Let us write

$$F(X) = \sum_\alpha r_\alpha X^\alpha,$$

where $\alpha$ runs over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_{N+1}) \in \mathbb{N}^{N+1}$ of degree $\alpha_1 + \cdots + \alpha_{N+1} = D$ and $X^\alpha = X_1^{\alpha_1} \cdots X_{N+1}^{\alpha_{N+1}}$. We get

$$\sum_\alpha r_\alpha n_\alpha = 0,$$

where $n_\alpha = n_1^{\alpha_1} \cdots n_{N+1}^{\alpha_{N+1}}$ for all $(n_i) \in \mathbb{Z}^{N+1}$ such that $0 \leq n_i \leq D$ for all $i = 1, \ldots, N+1$. We can view (10) as a system of linear equations in the variables $r_\alpha$, with coefficients $n_\alpha$. Its determinant is the determinant of the tensor product of $N+1$ Vandermonde $(D+1) \times (D+1)$ square matrices of the form $(n^j)$, with $j = 0, 1, \ldots, D$ and $0 \leq n \leq D$. Up to sign, this determinant is

$$\prod_{n \neq m} (n - m)^{N+1},$$

where $n, m$ run over all integers such that $0 \leq n, m \leq D$. Since this determinant does not vanish we get a contradiction.

Therefore there exists $v \in \Phi$ such that $F(v) \neq 0$, hence $[v] \notin \Sigma$. Now, for all $\sigma : K \hookrightarrow \mathbb{C}$, we have

$$\log \|v\|_\sigma = \log \left\| \sum_i n_i e_i \right\|_\sigma \leq \max_i \log \|e_i\| + \log (N+1) + \log (D) = \lambda_{\max} (\mathcal{V}) + \log (D(N+1)).$$

q.e.d.

3.4. Let $K$ be a number field and $S = \text{Spec} (\mathcal{O}_K)$. From now on, $f : X \rightarrow S$ will denote a semi-stable curve over $S$ with generic fiber a geometrically irreductible curve $X_K$ of genus $g \geq 2$. Let $\omega_{X/S}$ be the relative dualizing sheaf, and $\mu$ the Kähler form introduced by Arakelov in [A], § 4. We endow $\omega_{X/S}$ with the same metric as in [A].
Let $\mathcal{L}$ be an hermitian line bundle on $X$ such that the curvature of $\mathcal{L}$ is a positive multiple of $\mu$, i.e.

$$c_1(\mathcal{L}) = m \mu$$

where $m = \deg_K(L)$ is a positive integer. For any algebraic point $P$ on $X_K$, consider its normalized height $h_{\mathcal{L}}(P) \in \mathbb{R}$, which is defined as follows. Let $K'$ be the field of definition of $P$, $\mathcal{O}_{K'}$ its ring of integers, and $s : \text{Spec}(\mathcal{O}_{K'}) \to X$ the section defined by $P$ (using the valuative criterion for properness). Then

$$h_{\mathcal{L}}(P) = \widehat{\deg}(s^* \mathcal{L})/[K' : K].$$

When $P$ varies, $h_{\mathcal{L}}(P)$ is bounded below and we let

$$e(\mathcal{L}) = \inf_P h_{\mathcal{L}}(P).$$

Given two hermitian line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $X$ we write

$$\mathcal{L}_1 \cdot \mathcal{L}_2 = f_*(\bar{c}_1(\mathcal{L}_1) \cdot \bar{c}_1(\mathcal{L}_2)) \in \mathbb{R}$$

the arithmetic intersection number of their first Chern classes. The following result is due to Szpiro when $\mathcal{L} = \omega_{X/S}$ ([E], Th. 5.1).)

**Proposition 6.** The following inequality holds:

$$e(\mathcal{L}) \geq \frac{g \mathcal{L}^2}{2m} - \frac{\mathcal{L} \cdot \omega_{X/S}}{2} + \frac{m \omega^2_{X/S}}{8g}. \tag{11}$$

**3.5. Proof of Proposition 6.** Let $P$ be an algebraic point on $X_K$, defined over $K'$. After extending scalars from $K$ to $K'$ (this multiplies both the height of $P$ and the right hand side of (11) by $[K' : K]$), we may assume that $K = K'$. Let $s : S \to X$ be the section defined by $P$ and $D$ the Zariski closure of $\{P\}$ in $X$. We endow $\mathcal{O}(D)$ with its canonical admissible metric $[A]$ and write $\mathcal{D}$ instead of $\mathcal{O}(D)$. If we apply the Hodge index theorem as in [S], (33), we get the inequality

$$\mathcal{L}^2 - 2m \mathcal{L} \mathcal{D} + m^2 \mathcal{D}^2 \leq 0.$$ 

The adjunction formula $[A]$

$$\mathcal{D}^2 = -\omega_{X/S} \cdot \mathcal{D}$$

implies

$$\mathcal{L}^2 \leq (2m \mathcal{L} + m^2 \omega_{X/S}) \cdot \mathcal{D}$$
i.e., if $\overline{M} = \mathcal{L} \otimes^2 \omega_X^m$, 

$$\left( \frac{\overline{M} - m \omega_X}{2} \right)^2 \leq m \overline{M} \cdot \mathcal{D}.$$ 

Since $m' = \deg_K(M) = 2mg$, we get

$$h_{\overline{M}}(P) \geq \frac{1}{2g m'} \left( \frac{2g \overline{M} - m' \omega_X}{2} \right)^2$$

hence

$$e(\overline{M}) \geq \frac{g \overline{M}^2}{2m'} - \frac{\overline{M} \cdot \omega_X}{2} + \frac{m' \omega_X^2}{8g}.$$ 

Now the inequality (11) is true for $\mathcal{L}$ if and only if it is true for a positive power of $\mathcal{L}$, and such a power can be written

$$\mathcal{L}^\otimes k = \mathcal{L'}^\otimes^2 \otimes \omega_X^{\deg_K(L')}$$,

where $\mathcal{L'}$ satisfies the same positivity assertion as $\mathcal{L}$. It follows from (12) that (11) is true.

3.6. We keep the notation of § 2.4 and we also assume that the degree of the restriction of $L$ to any component of a fiber of $X$ over $S$ is nonnegative. Consider the group

$$H^1(X, L^{-1}) = \text{Ext}(L, \mathcal{O}_X)$$

of extensions of $L$ by the trivial line bundle. This is a finitely generated projective $\mathcal{O}_K$-module. We equip $H^1(X, L^{-1}) \otimes \mathbb{C}$ with the $L^2$-metric defined by $\mu$ and the chosen metric on $L$. The Serre duality isomorphism

$$H^1(X, L^{-1}) \simeq H^0(X, L \otimes \omega_{X/S})^*$$

is compatible with the $L^2$-metrics on both cohomology groups. Since $H^0(X, L^{-1}) = 0$, the Riemann-Roch theorem reads

$$\text{rk} H^1(X, L^{-1}) = m + g - 1.$$ 

For any integer $k = 1, \ldots, m + g - 1$, we let $\lambda_k(\mathcal{L})$ be the $k$-th successive minimum of the metrized $\mathcal{O}_K$-lattice $H^1(X, L^{-1})$, and we let 

$$\mu_k(\mathcal{L}) = \ell_{m+g-1-k}(H^0(X, L \otimes \omega_{X/S})).$$
**Theorem 4.**

i) Let \( k > 1 \) be such that \( m > 2k \). Then the following inequalities hold:

\[
\lambda_k(L) + 1 \geq [k(L^2 - 2m \cdot e(L)) + m^2 \cdot e(L) - \log(D(g, m, k-1)(m+g))[K : \mathbb{Q}]/(m^2[K : \mathbb{Q})]
\]

and

\[
\mu_k(L) \geq -C(m + g - 2, K) +
\]

\[
\left[\frac{k(k+1)}{2}(L^2 - 2m \cdot e(L)) + km^2 \cdot e(L) - \sum_{j=1}^{k} \log(D(g, m, j-1)(m+g))[K : \mathbb{Q}]\right]/m^2.
\]

ii) When \( m \) is odd (resp. when \( m \) is even and \( g > 0 \)) we have

\[
\lambda_{m-g-1}(L) + 1 \geq [L^2 - \log(D(g, m, m-g-2)(m+g))[K : \mathbb{Q}] / (2m[K : \mathbb{Q}])
\]

(resp.

\[
\lambda_{m-g}(L) + 1 \geq [L^2 - \log(D(g, m, m-g-1)(m+g))[K : \mathbb{Q}] / (2m[K : \mathbb{Q}])
\]

**Remark.** A lower bound \( \lambda_1(L) \) is given by Theorem 2 in [S]. However the statement of loc.cit. is not correct when \( \deg_K(L) = m = 1 \). In that case, we deduce from [S], Corollary 1, that

\[
L^2 \leq 2(\log \|e\| + 1)[K : \mathbb{Q}],
\]

which is slightly weaker than [S], Th. 2.

**3.7. Proof of Theorem 4.** Let \( \pi: \tilde{X} \rightarrow X \) be a resolution of \( X \). The canonical map

\[
H^1(X, L^{-1}) \rightarrow H^1(\tilde{X}, \pi^*(L^{-1}))
\]

is split injective, and it induces an isometry of hermitian complex vector spaces. Therefore, it is enough to prove Theorem 4 for \( \tilde{X} \) and \( \pi^*(L^{-1}) \), i.e. we can assume that \( X \) is regular. (The same argument will prove Theorem 4 under the assumption that \( X \) is a normal scheme, not necessarily semi-stable over \( S \).)

If \( K \) is an algebraic closure of the field \( K \), the secant variety

\[
\Sigma_d \subset \mathbb{P}(H^1(X, L^{-1})_K)
\]

is defined over \( K \) for any integer \( d \). Its degree is \( D(g, m, d) \).
For any $e \in H^1(X, L^{-1})$ let
\[ \|e\| = \sup_{\sigma} \|e\|_{\sigma} \]
where $\sigma$ runs over all complex embeddings $K \hookrightarrow \mathbb{C}$, and $\|e\|_{\sigma}$ is the corresponding $L^2$-norm.

Choose vectors $e_1, \ldots, e_{m+g-1}$ in $H^1(X, L^{-1})$ such that, for any $j = 1, \ldots, m+g-1$, we have
\[ \|e_j\| = \lambda_j(L) . \]

Given $k$ as in the statement of Theorem 4 i) or $k = m - g$ (resp. $k = m - g - 1$) in case ii), we let $V$ be the $O_K$-module spanned by $e_1, e_2, \ldots, e_k$. By Theorem 1 and Proposition 1, the projective space $\mathbb{P}(V_K)$ does not contain $\Sigma_{k-1}$. Using Proposition 4, we know that there exists an extension class $e \in H^1(X, L^{-1})$ such that $[e] \notin \Sigma_{k-1}$ and
\[
(15) \quad \log \|e\| \leq \lambda_{\max}(V) + \log (D(g, m, k-1)(m+g)) ,
\]
with
\[
(16) \quad \lambda_{\max}(V) = \lambda_k(L) .
\]

Since $e \notin \Sigma_{k-1}$ the corresponding extension $E$ of $L$ by $O_X$ is either semi-stable when restricted to $X_K$ or, in case i), such that there exists a line bundle $M_K$ contained in $E_K$ with
\[ \deg(M_K) = m - d , \]
with $d \geq k$. When $E_K$ is semi-stable, [S] Corollary 1 asserts that
\[
(17) \quad L^2 \leq (2m \log \|e\| + 2)[K : \mathbb{Q}] .
\]

In the remaining case, the inequality (35) in [S] reads
\[ d^2 \mathcal{L}^2 + (m^2 - 2md) d e(L) \leq m^2 [K : \mathbb{Q}] d \log \|e\| + m^2 [K : \mathbb{Q}] / 2 . \]

Since $d \geq k$ and since
\[
(18) \quad 2me(L) \leq \mathcal{L}^2
\]
([Z] Theorem 6.3, valid under our assumption on $L$) we deduce that
\[
(19) \quad k(\mathcal{L}^2 - 2me(L)) + m^2 e(L) \leq m^2 [K : \mathbb{Q}] (\log \|e\| + 1) .
\]

From (15), (16) and (19) it follows that
\[
(20) \quad \lambda_k(L)+1 \geq [k(\mathcal{L}^2 - 2me(L)) + m^2 e(L) - \log(D(g, m, k-1)(m+g))][K : \mathbb{Q}] / (m^2 [K : \mathbb{Q}]) .
\]
On the other hand, (15), (16) and (17) imply

\[(21) \quad \lambda_k(L) + 1 \geq \left[ L^2 - \log(D(g, m, k - 1)(m + g))[K : \mathbb{Q}] / (2m[K : \mathbb{Q}]) \right].\]

This proves ii). In case i), we have to check that the right hand side in (21) is greater or equal to the right hand side in (20). But this follows easily from (18) and \(m > 2k\). This proves (15).

The inequality (16) follows by Serre duality from (15) and Proposition 3. q.e.d.

3.8. Theorem 4 provides a precise lower (resp. upper) bound for the successive minima of \(H^1(X, L^{-1})\) (resp. \(H^0(X, L \otimes \omega_{X/S})\)), since we can combine it with the lower bound for \(e(L)\) given in Prop. 4.

Let us make this more explicit when \(L = \omega^{\otimes n}_{X/S}\), where \(n\) is a large positive integer, hence \(m = 2(g - 1)n\). Assume \(g > 1\). Proposition 4 reads

\[e(L) \geq n \frac{\omega^2_{X/S}}{4g(g - 1)}.\]

Therefore, since the coefficient of \(e(L)\) is non-negative,

\[k(L^2 - 2m e(L)) + m^2 e(L) \geq n^2(k + n) \frac{g - 1}{g} \omega^2_{X/S}.\]

From Theorem 4 i) we get

\[(22) \quad (\lambda_k(L) + 1)[K : \mathbb{Q}] \geq \frac{k + n}{4g(g - 1)} \omega^2_{X/S} - \log(D(g, 2n(g - 1), k - 1)(m + g))[K : \mathbb{Q}] / m^2.\]

We have

\[D(g, 2n(g - 1), k - 1) = \sum_{\alpha=0}^{g} \binom{m + g - 1 - k - \alpha}{k - \alpha} \left( \frac{g}{\alpha} \right) \leq C_1(g)m! ,\]

where \(C_1(g)\) is a positive constant depending on \(g\) only. The Stirling formula implies

\[\log D(g, 2n(g - 1), k - 1) \leq C_2(g)m \log(m).\]

Therefore the second summand in the right hand side of (22) is bounded above by a constant multiple of \(m \log(m)/(2m^2) = \log(m)/2\).

Let \(\omega^2 = \omega^2_{X/S}/[K : \mathbb{Q}]\). We get the following:
Corollary 1. There exist constants $C(g) > 0$ such that, if $\mathcal{L} = \omega_X^{\otimes n}/S, n \geq 1$, $k < (g - 1)n$ and $m = 2(g - 1)n$, the following inequality holds
\[
\lambda_k(\omega_X^{\otimes n}) \geq \frac{n + k}{4g(g - 1)} \omega^2 - C(g) \log(n + 1).
\]

3.9. From Theorem 4 one can also get an explicit lower bound for $\mu_k(\omega_X^{\otimes n})$.

First, since $k \leq m/2$, we get, as in 3.8,
\[
(23) \sum_{j=1}^{k} \log D(g, m, d(j) - 1) \frac{m + g}{m^2} \leq C_3(g) n \log(n + 1).
\]

Similarly, by Stirling’s formula and the known formula for the euclidean volume of the unit ball,
\[
(24) C(m + g - 2, K) = \frac{m + g - 1}{2} \log |\Delta_K| + O(m \log m)[K : \mathbb{Q}].
\]

On the other hand, using Proposition 4 as in 3.8 above, we get
\[
(25) \frac{k(k+1)}{2} (\mathcal{L}^2 - 2m \mathcal{e}(\mathcal{L})) + k m^2 \mathcal{e}(\mathcal{L}) \geq n^2 \frac{g - 1}{g} \left[ \frac{k(k+1)}{2} + kn \right] \omega^2_{X/S}.
\]

If we put (14), (23), (24), (25) together we conclude that:

Corollary 2. There exist constants $C(g) > 0$ such that, if $\mathcal{L} = \omega_X^{\otimes n}, n \geq 1$, we have, when $k < (g - 1)n$,
\[
\mu_k(\mathcal{L}) \geq \frac{1}{4g(g - 1)} \left[ \frac{k(k+1)}{2} + kn \right] \omega^2_{X/S} - (2n + 1) \frac{g - 1}{2} \log |\Delta_K| - C(g)n \log(n + 1).
\]

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