Multi-quark matrix elements in the proton and three gluon exchange for exclusive $\eta_c$ production in photon-proton diffractive scattering

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Exclusive production of a $\eta_c$ pseudo-scalar meson in $\gamma^{(*)} + p \rightarrow \eta_c + p$ scattering at high energies involves a $C$-odd exchange in the $t$-channel. We formulate the description of this process within the high-energy framework of eikonal dipole scattering. We obtain expressions for the light-cone wave function of the $\eta_c$ required in this framework as well as for the $C$-odd amplitudes due to exchange of a single photon, of a photon plus two gluons, and of three gluons. We relate these amplitudes to correlators of the $+$ component of the quark current in the light-cone wave function of the proton. For high transverse momenta these correlators correspond to Generalized Parton Distributions (GPDs) given by diagrams where all exchanged gauge bosons attach to a single quark in the proton. Diagrams involving multi-quark matrix elements potentially sensitive to correlations, screen infrared singularities. Moreover, they are numerically important for configurations where the exchanged bosons nearly share the total momentum transfer.

Using two simple models for the three quark Fock state of the proton at $x \approx 0.1$, we find that single photon exchange dominates for $|t| < 1.5$ GeV$^2$. Here, the quark GPD could be measured cleanly in $\gamma^{(*)} + p \rightarrow \eta_c + p$ via single photon exchange. For higher momentum transfer three gluon ("Odderon") exchange is dominant.

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I. INTRODUCTION

The existence of a color singlet three-gluon exchange with negative C-parity in QCD at high energies was established long ago [1]. Such an exchange could provide some understanding of the difference of particle and anti-particle cross sections and of the violation of the Pomeronchuk theorem in high-energy scattering. An “Odderon” exchange has been proposed nearly 50 years ago within the framework of Regge theory to explain the different cross sections in C-conjugate channels [2]. For a review of the theory and experimental searches for the Odderon up until the year 2003 we refer to ref. [3].

The TOTEM collaboration at the CERN-LHC has recently measured the differential cross section for pp elastic scattering at √s = 2.76 GeV [4]. They observe a significant difference to the data by the D0 collaboration for p ¯p scattering at √s = 1.96 GeV [5]. Assuming that the difference in energy is negligible they conclude that these results provide evidence for a color singlet 3-gluon exchange. Even though these measurements are very exciting, the data does not quite correspond to a kinematic regime where perturbative QCD may be reliable.

Exclusive production of pseudo-scalar ηc mesons in (virtual) photon - proton scattering has been highlighted [6–8] as the cleanest channel for discovery of C-odd three gluon (“Odderon”) exchange. Here, the large mass of the c-quark ensures that (at high energy) the process corresponds to scattering of a small dipole of transverse extent much less than the QCD color neutralization scale, from the proton. The focus in these papers was on ηc production at rather high energies and small parton momentum fractions x, at HERA. However, the searches at HERA did not observe exclusive ηc production. The cross-section for this process is small, in fact our estimates below are substantially lower yet than old predictions from the literature [7, 8]. Thus, such searches for C-odd three gluon exchange at a future high-luminosity Electron Ion Collider (EIC) would be more promising.

Exclusive measurements in photon-proton scattering offer the opportunity to extract fundamental nonperturbative QCD physics contained in the light cone wave function of the proton [16, 17] and its Generalized Parton Distributions (GPDs) [18–24]. Specifically, these processes in fact involve correlators of multiple “+” currents evaluated as matrix elements between multi-parton states [25]. Here, we illustrate explicitly how they probe multi-parton correlations in the proton. We recover the description in terms of a GPD when the transverse momenta of the exchanged gluons or quarks are large and generic.

II. SETUP

The light cone wave function of an unpolarized on-shell proton with four-momentum ⃗Pμ = (P+, P−, ⃗P⊥) is written as [16, 17]

\[ |P⟩ = \frac{1}{\sqrt{6}} \int \frac{dx_1dx_2dx_3}{\sqrt{x_1x_2x_3}} \delta(1 - x_1 - x_2 - x_3) \int \frac{d^2k_1d^2k_2d^2k_3}{(16\pi^3)^3} 16\pi^3δ(⃗k_1 + ⃗k_2 + ⃗k_3) \]

\[ \times ψ_3(p_1, p_2, p_3) \sum_{i_1,i_2,i_3} \epsilon_{i_1i_2i_3}|p_1, i_1; p_2, i_2; p_3, i_3⟩ . \]  

(1)

The n-parton Fock space amplitudes ψ_n(p_1, ⋯, p_n) are universal and process independent. They encode the non-perturbative structure of hadrons. Here, we have restricted ourselves to the valence quark Fock state, assuming that the process probes parton momentum fractions of order x ∼ 0.1 or greater. The three on-shell quark momenta are specified by their lightcone momenta p_μ^v = x_i P^μ and their transverse momenta ⃗p_i = x_i P⊥ + ⃗k_i. Colors and flavors of the quarks are denoted by i_1,2,3 and f_1,2,3, respectively. In eq. (1) we have assumed that the Fock space amplitude ψ_3(p_1, p_2, p_3) is flavor blind and we omit helicity quantum numbers as they play no role in our analysis. The amplitude ψ_3 in Eq. (1) is symmetric under exchange of any two of the quarks, and is normalized according to

\[ \int dx_1dx_2dx_3 \delta(1 - x_1 - x_2 - x_3) \int \frac{d^2k_1d^2k_2d^2k_3}{(16\pi^3)^3} 16\pi^3δ(⃗k_1 + ⃗k_2 + ⃗k_3)|ψ_3|^2 = 1 . \]  

(2)

This corresponds to the proton wave function normalization

\[ ⟨K|P⟩ = 16\pi^3 P^+ δ(P^+ - K^+) δ(P^Z - K^⊥) . \]  

(3)

1 Also see ref. [9] for a calculation of γ p → η_c p using Regge theory and effective Odderon-proton and Odderon-η_c vertices. Moreover, one could search for Odderon exchange also through exclusive vector meson production in proton-proton scattering [10].

2 On the other hand, successful fits of exclusive J/Ψ production at HERA energies (for example ref. [11]) can be interpreted to provide evidence for two-gluon Pomeron exchange [12, 13] supplemented by QCD high-energy evolution [14].
Below, we neglect plus momentum transfer so that $\xi = (K^+ - P^+)/P^+ \to 0$. This approximation is valid at high energies and $|t| \gg |t_{\text{min}}|$. Accordingly, the light cone momentum of the produced meson is close to that of the incoming photon.

For numerical estimates we shall employ the “harmonic oscillator” and “power law” model wave functions of Brodsky and Schlumpf [26],

$$\psi_{\text{H.O.}}(k_1, k_2, k_3) = N_{\text{H.O.}} \exp(-\mathcal{M}^2/2\beta^2) , \quad \psi_{\text{Power}}(k_1, k_2, k_3) = N_{\text{Power}}(1 + \mathcal{M}^2/\beta^2)^{-p} .$$  

The invariant mass $\mathcal{M}$ of the configuration is given by

$$\mathcal{M}^2 = \sum_{i=1}^{3} \frac{k_{i}^2 + m^2}{x_i} .$$  

$\beta$ determines the color neutralization scale and the typical transverse momentum of quarks in the proton. The parameters $\beta$ and $m^2$ were obtained in ref. [27] by fitting to electroweak parameters of the baryon octet: $m = 0.26$ GeV, $\beta = 0.55$ GeV for $\psi_{\text{H.O.}}$ and $m = 0.263$ GeV, $\beta = 0.607$ GeV, $p = 3.5$ for $\psi_{\text{Power}}$. The normalization constants $N_{\text{H.O.}}$ and $N_{\text{Power}}$ are obtained from the normalization condition (2). Other models and parameter sets can be found in refs. [28–32].

Following ref. [25] we introduce the charge density operators corresponding to the light cone plus component of the quark currents

$$\rho(x_k, \vec{k}) = \sum_{f,i} \int \frac{dx_q}{\sqrt{x_q(x_q + x_k)}} \int \frac{d^2q}{16\pi^3} b_{q,i,f}^\dagger b_{q+i,f} x_{k1, f} \sum_{f,i} \int \frac{dx_q}{x_q} \int \frac{d^2q}{16\pi^3} b_{q+i,f}^\dagger b_{q+i+} x_{k2, f} ,$$  

(6)

$$\rho^a(x_k, \vec{k}) = \sum_{f,i,j} \int \frac{dx_q}{\sqrt{x_q(x_q + x_k)}} \int \frac{d^2q}{16\pi^3} b_{q,i,f}^\dagger b_{q+q,f} (t^a)_{ij} x_{k1, f} \sum_{f,i,j} \int \frac{dx_q}{x_q} \int \frac{d^2q}{16\pi^3} b_{q+q,f}^\dagger b_{q+q+} (t^a)_{ij} .$$  

(7)

These equations define the densities of electric and color charge, respectively; factors of $e$ and $g$ will be attached in eqs. (8) below. $b_{q,i,f}^\dagger$ and $b_{q,i,f}$ denote creation and annihilation operators for quarks with plus momentum $q^+ = x_q P^+$, transverse momentum $\vec{q}$, color $i$, and flavor $f$.

In what follows we shall neglect longitudinal momentum transfer to the quarks and use the kinematic approximation where $x_k \sim 0.1 \ll 1$. This allows us to simplify the color charge operators as indicated above. Moreover, we will assume that the scattering of a energetic $c\bar{c}$ dipole from the valence charges in the proton is eikonal, to first approximation; see eq. (19) below. Kinematic finite-$x$ corrections are suppressed by powers of the light cone momentum $P^+$. Of course, for quantitative comparisons to future experiments it will be important to quantify these corrections.

The charge densities are the sources for the static electromagnetic and color fields in covariant gauge,

$$\int dx^- A^+(x^+, \vec{k}) = \frac{e}{k^2} \rho(x_k = 0, \vec{k}) , \quad \int dx^- A^a(x^+, \vec{k}) = \frac{g}{k^2} \rho^a(x_k = 0, \vec{k}) .$$  

(8)

The physical picture of representing the quarks with large light cone momenta as static ($x^+$ independent) color charge densities sourcing soft gluon fields was introduced by McLerran and Venugopalan [33]. In their effective theory, however, $\rho^a(x_k = 0, \vec{k})$ corresponds to a classical color charge vector describing a large ensemble of quarks in a high-dimensional representation of color-$SU(3)$. Here, instead, the operator $\rho^a(x_k = 0, \vec{k})$ acts on single quarks and color charge correlators will be evaluated over the light cone wave function of the proton.

### III. CORRELATORS OF CHARGE DENSITY OPERATORS IN THE PROTON

In this section we provide expressions for correlators of various (electric and color) charge density operators in the proton. Some of these have been considered long before, especially for forward $K_T \to 0$ scattering (see, for example ref. [34]). Here we are interested in non-forward matrix elements for single photon, two gluon, photon plus two gluon, and three gluon exchanges. In particular, we follow the approach of ref. [25] to relate these matrix elements explicitly to the light cone wave function of the proton, and to analyze their GPD limits.
Consider first the expectation value of the electric charge density operator in the proton, in the kinematic $x \ll 1$ limit described above. A straightforward calculation yields

\begin{equation}
\langle \rho(\vec{q}) \rangle_{K_T} = \sum_f e_f \int dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3) \times \int \frac{d^2p_1 d^2p_2 d^2p_3}{(16\pi^3)^2} \delta(p_1 + p_2 + p_3) \psi_3^*(p_1 + (1 - x_1)\vec{K}_T, p_2 - x_2\vec{K}_T, p_3 - x_3\vec{K}_T) \psi_3(p_1, p_2, p_3) \equiv f(K_T) \sum_f e_f .
\end{equation}

Here, $e_f$ denotes the fractional electric charge of quark $f = (u, u, d)$, so that $\sum_f e_f = 1$. For a lighter notation we omit the arguments $x_1, x_2$ and $x_3$ of the Fock amplitudes.

The function $f(K_T)$ is a Generalized Parton Distribution (GPD) of quarks in the proton at $x \ll 1$, as it corresponds to the non-forward matrix element of the plus component of the quark current between single quark states. The photon probe can only attach to one quark at a time and so this matrix element does not probe correlations among the quarks. Also, for a single probe its transverse momentum is equal to minus the recoil momentum of the proton, $\vec{q} = -\vec{K}_T$. For $K_T \to 0$ the wave function normalization in eq. (2) implies $f(0) = 1$.

Next, we consider the correlator of two color charge density operators which enters the amplitude for $C$-even two-gluon exchange. It is given by [25],

\begin{equation}
\langle \rho^{ab}(\vec{q}) \rho^{b}(-\vec{q} - \vec{K}_T) \rangle_{K_T} = \frac{1}{2} \delta^{ab} \int dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3) \times \int \frac{d^2p_1 d^2p_2 d^2p_3}{(16\pi^3)^2} \delta(p_1 + p_2 + p_3) \left[ \psi_3^*(p_1 + (1 - x_1)\vec{K}_T, p_2 - x_2\vec{K}_T, p_3 - x_3\vec{K}_T) \right. \\
\left. - \psi_3^*(p_1 - \vec{q} + (1 - x_1)\vec{K}_T, p_2 - \vec{K}_T, p_3 - x_3\vec{K}_T) \right] \psi_3(p_1, p_2, p_3) \equiv \frac{1}{2} \delta^{ab} G(\vec{q}, -\vec{q} - \vec{K}_T) .
\end{equation}

$G(\vec{q}, -\vec{q} - \vec{K}_T)$ is real because the bound state wave functions $\psi_3$ are real. It is invariant under a simultaneous rotation of both $\vec{q}$ and $\vec{K}_T$ by the same angle and is also symmetric under exchange of its two arguments, $G(\vec{q}, -\vec{q} - \vec{K}_T) = G(-\vec{q} - \vec{K}_T, \vec{q})$. This color charge correlator vanishes when $\vec{q} \to 0$ or $\vec{q}_2 = -\vec{q} - \vec{K}_T \to 0$ which expresses the color charge neutrality of the proton. On the other hand, for large $|\vec{q}_1|, |\vec{q}_2|$, and large $|\vec{q} - \vec{q}_2|$, it is dominated by the first term in the square brackets of eq. (10). That is the “one-body” diagram where both gluon probes attach to the same quark. In this GPD limit,

\begin{equation}
G(\vec{q}, \vec{q}_2) \to f(K_T) \quad \text{for} \quad q^2, \vec{q}_2^2, (\vec{q} - \vec{q}_2)^2/4 \gg \Lambda_{\text{eff}}^2 .
\end{equation}

Here, $\Lambda_{\text{eff}} \sim \beta$ is a soft scale encoded in the light cone wave function of the proton, eqs. (4). Note that this approximation does not necessarily require large $K_T$. However, at high momentum transfer, the contribution from the diagram where the gluons couple to different quarks is important not only for $\vec{q} \to 0$ or $\vec{q}_2 \to 0$ but also when $\vec{q} \sim \vec{q}_2 \sim -\vec{K}_T/2 \gg \Lambda_{\text{eff}}$. Here, the second term in eq. (10) is actually much greater than the first one due to contributions from configurations where the two active quarks have large light cone momenta: at $K_T \gg \Lambda_{\text{eff}}$ there is a large mismatch of the arguments of $\psi_3$ and $\psi_3$ in the first term of eq. (10) while there is strong overlap in the second term when $\vec{q} \sim -\vec{K}_T/2$ and $x_1 \sim x_2 \sim 0.5$. We shall see below that restricting to the single quark matrix element leads to a too rapid fall-off of $d\sigma^{J/\Psi}/dt$ with $|t|$.

We now move on to the correlator of one electric charge with two color charge densities which is relevant for $C$-odd

\begin{footnote}{(\cdots)}{K_T}\text{ corresponds to }\langle K | \cdots | P \rangle\text{ stripped of the }\delta\text{-functions expressing conservation of transverse and plus momentum such as }\langle K | \rho(\vec{q}) | P \rangle = 16\pi^3 P_+ \delta(P_+ - K^+) \delta(K_T + \vec{q}) \langle \rho(\vec{q}) \rangle_{K_T}^{\text{eff}}\text{ if we set }\vec{P}_f = 0\text{ for the incoming proton.}
\end{footnote}

\begin{footnote}{(\cdots)}{\psi_3 = \psi_{\text{power}}\text{ and large momentum transfer }K_T,\text{ it dominates by a power of }q^2/K_T^2\text{ when }q^2 \gg K_T^2.}
\end{footnote}
exchange of a photon and two gluons. It is given by
\[
\left\langle \rho^a(\vec{q_1}) \rho^b(\vec{q_2}) \rho^c(\vec{q_3}) \right\rangle_{K_T} = \frac{1}{6} \delta^{ab} \sum_f e_f \int dx_1dx_2dx_3 \delta(1-x_1-x_2-x_3) \int \frac{d^2p_1d^2p_2d^2p_3}{(16\pi^3)^2} \delta(\vec{p_1} + \vec{p_2} + \vec{p_3}) \left[ \psi_3(\vec{p_1} + (1-x_1)\vec{K}_\perp, \vec{p_2} - x_2\vec{K}_\perp, \vec{p_3} - x_3\vec{K}_\perp) 
+ 2\psi_3^*(\vec{p_1} + \vec{q_2} + \vec{q_3} + (1-x_1)\vec{K}_\perp, \vec{p_2} - \vec{q_2} - \vec{q_3} - x_2\vec{K}_\perp, \vec{p_3} - x_3\vec{K}_\perp) 
- \psi_3^*(\vec{p_1} + \vec{q_2} + (1-x_1)\vec{K}_\perp, \vec{p_2} - \vec{q_2} - x_2\vec{K}_\perp, \vec{p_3} - x_3\vec{K}_\perp) 
- \psi_3^*(\vec{p_1} + \vec{q_2} + (1-x_1)\vec{K}_\perp, \vec{p_2} - \vec{q_2} - x_2\vec{K}_\perp, \vec{p_3} - x_3\vec{K}_\perp) 
\right] \psi_3(\vec{p_1}, \vec{p_2}, \vec{p_3})
\]
\(\equiv \frac{1}{2} \delta^{ab} \left( \frac{1}{3} \sum_f e_f \right) H(-\vec{q_2} - \vec{q_3} - K_T, \vec{q_2}, \vec{q_3}) \).  

(13)

This correlator vanishes when the transverse momentum of either one of the attaching gluons vanishes, \(\vec{q_2} = 0\) or \(\vec{q_3} = 0\). When the transverse momentum of the photon vanishes, \(\vec{q_1} \equiv -\vec{q_2} - \vec{q_3} - K_T = 0\) then it reduces to the correlator of two color charge operators from eq. (10) times \(\sum_f e_f = 1\).

For high transverse photon and gluon momenta, the first term in eq. (13) dominates and
\[
H(\vec{q_1}, \vec{q_2}, \vec{q_3}) \rightarrow f(K_T) \quad \text{(for } \vec{q_1}^2, \vec{q_2}^2, \vec{q_3}^2, (\vec{q_1} + \frac{1}{3}K_T)^2 \gg \Lambda^2_{\text{ch}})\).
\(\quad \text{(15)}\)

This is the GPD limit where both gluons and the photon attach to the same quark in the proton. Note, however, that this approximation does not apply when the exchanged bosons share the total momentum transfer \(\vec{K}_T\) in approximately equal proportions.

Finally, we recall the correlator of three color charge densities obtained in ref. [25],
\[
\langle \rho^a(\vec{q_1}) \rho^b(\vec{q_2}) \rho^c(\vec{-q_1-q_2-K_T}) \rangle_{K_T} = \frac{1}{4} \delta^{abc} \int dx_1dx_2dx_3 \delta(1-x_1-x_2-x_3) \int \frac{d^2p_1d^2p_2d^2p_3}{(16\pi^3)^2} \delta(\vec{p_1} + \vec{p_2} + \vec{p_3}) \left[ \psi_3^*(\vec{p_1} + (1-x_1)\vec{K}_\perp, \vec{p_2} - x_2\vec{K}_\perp, \vec{p_3} - x_3\vec{K}_\perp) 
- \psi_3^*(\vec{p_1} + \vec{q_2} + (1-x_1)\vec{K}_\perp, \vec{p_2} - \vec{q_2} - x_2\vec{K}_\perp, \vec{p_3} - x_3\vec{K}_\perp) 
- \psi_3^*(\vec{p_1} + \vec{q_2} + (1-x_1)\vec{K}_\perp, \vec{p_2} - \vec{q_2} - x_2\vec{K}_\perp, \vec{p_3} - x_3\vec{K}_\perp) 
+ 2\psi_3^*(\vec{p_1} - \vec{q_1} - x_1\vec{K}_\perp, \vec{p_2} + \vec{q_1} + \vec{q_2} - (1-x_2)\vec{K}_\perp, \vec{p_3} - \vec{q_2} - x_3\vec{K}_\perp) 
\right] \psi_3(\vec{p_1}, \vec{p_2}, \vec{p_3})
\]
\(\equiv \frac{1}{4} \delta^{abc} G_O(\vec{q_1}, \vec{q_2}, -\vec{q_1} - \vec{q_2} - K_T) \).
\(\quad \text{(16)}\)

(We have redefined the normalization of \(G_O\) as compared to ref. [25] by a factor of \(N_c/4\) for convenience.) Here, on the r.h.s. we have written only the \(C\)-odd contribution which is symmetric under the exchange of the color indices \(a, b,\) and \(c\) since this is the only piece that couples to a dipole. \(G_O\) is invariant under a simultaneous rotation of the three transverse momentum vectors \(\vec{q_1}, \vec{q_2}, \vec{K}_T\) and under permutations of its arguments. When either one of the three momenta \(\vec{q_1}, \vec{q_2},\) or \(\vec{q_3} \equiv -\vec{q_1} - \vec{q_2} - \vec{K}_T\) is zero then \(G_O\) vanishes. Once again one recovers the GPD corresponding to the matrix element of the plus current between single quark states in the limit of large transverse momenta of the gluon probes,
\[
G_O(\vec{q_1}, \vec{q_2}, \vec{q_3}) \rightarrow f(K_T) \quad \text{(for } \vec{q_1}^2, \vec{q_2}^2, \vec{q_3}^2, (\vec{q_1} + \frac{1}{3}K_T)^2 \gg \Lambda^2_{\text{ch}})\).
\(\quad \text{(18)}\)

This approximation again requires not only large transverse gluon momenta but also that the total momentum transfer is not being (approximately) shared equally.
In closing this section we note that in our expressions for the color charge correlators \( \langle \rho^a(\vec{q}) \rho^b(-\vec{q} - \vec{K}_T) \rangle_{\vec{K}_T} \) etc. we have ignored the appropriate gauge links connecting the sources [35–37]. These gauge links account for soft multiple scattering while we retain only two or three “hard” gluon exchanges with the target. In this paper we shall be interested mainly in the limit \( K_T^2 \gg \Lambda_{\text{eff}}^2 \).

**IV. DIPOLE SCATTERING AMPLITUDE**

The invariant amplitude \( \mathcal{T} \) for elastic scattering of the \( \bar{c}c \) pair off the fields in the target proton can be expressed as

\[
\mathcal{T}(\vec{r}, \vec{b}_\perp; \vec{K}_\perp) = 2N_c \left[ 1 - \frac{1}{N_c} \text{tr} \left\langle U \left( \frac{\vec{b} + \vec{r}}{2} \right) U^\dagger \left( \frac{\vec{b} - \vec{r}}{2} \right) \right\rangle \right],
\]

\[
\mathcal{T}(\vec{r}, \vec{K}_\perp) = \int d^2b e^{i\vec{b} \cdot \vec{K}_T} \mathcal{T}(\vec{r}, \vec{b}_\perp; \vec{K}_\perp).
\]

At \( K_T = 0 \) eq. (19) is related to the so-called dipole gluon distribution evaluated in covariant gauge [35]. Here, \( U \) (and \( U^\dagger \)) are lightlike Wilson lines representing the eikonal scattering of the dipole of size \( \vec{r} \) at impact parameter \( \vec{b} \).

Two of the diagrams that contribute to the \( \mathcal{C} \)-odd part of this amplitude are shown in fig. 1.

![Diagram](attachment:image.png)

**FIG. 1.** Two of the diagrams that contribute to the production of a pseudoscalar \( \eta_c \) meson via \( \mathcal{C} \)-odd 3-gluon exchange. The diagram on the left involves a matrix element between single quark states in the proton while the diagram on the right involves a matrix element in a three quark state sensitive to multi-quark correlations.

To account for photon exchange to the scattering amplitude (19) we use Wilson lines in the combined color and electromagnetic fields:

\[
U^\dagger(\vec{x}_T) = \mathcal{P} e^{i \int d\vec{x}_T \left[ gA^+(x_-, \vec{x}_T) \epsilon^{+} + e_Q \epsilon A^+(x_-, \vec{x}_T) \right]}.
\]

Here, \( e_Q = 2/3 \) is the fractional charge of the \( c \)-quark.

In what follows we expand \( \mathcal{T}(\vec{r}, \vec{K}_\perp) \) up to first order in the electromagnetic field, and up to third order in the color field. The relation of such a weak field expansion to a resummation of kinematic twists in Wandzura-Wilczek type approximations has been elucidated recently in ref. [36]. Indeed, we do not expand the scattering amplitude about small dipole size \( r \) or small momentum exchange \( K_T \). However, as already indicated at the end of the previous section, we do neglect the resummation of multiple soft scattering. For a proton target and \( x \sim 0.1 \) the weak field limit should provide a reasonable first approximation.

Expanding to first order in the fields we obtain the amplitude for single photon exchange,

\[
\mathcal{T}_\gamma(\vec{r}, \vec{K}_\perp) = 16\pi N_c \alpha e_Q \sum_f e_f \frac{f(K_T)}{K_T^2} \sin \left( \frac{\vec{r} \cdot \vec{K}_T}{2} \right),
\]

with \( f(K_T) \) from eq. (9). Note that the exchanged photon is off shell as it possesses only transverse but no light cone momentum.
The contribution at second order in the color field $gA^{+a}$ corresponds to $C$-even two gluon ("Pomeron") exchange [25],

$$T_{gg}(\vec{r}, \vec{K}_\perp) = -4(4\pi\alpha_s)^2 N_c C_F \int q \frac{1}{(q - \frac{1}{2} \vec{K}_T)^2 (\vec{q} + \frac{1}{2} \vec{K}_T)^2} \left( \cos (\vec{r} \cdot \vec{q}) - \cos \left( \frac{\vec{r} \cdot \vec{K}_T}{2} \right) \right)$$

$$\times G \left( \vec{q} - \frac{1}{2} \vec{K}_T, -\vec{q} - \frac{1}{2} \vec{K}_T \right).$$

(23)

We use the shorthand notation $\int_q = \int d^2q/(2\pi)^2$.

$T_{gg}(\vec{r}, \vec{K}_\perp)$ is even under a sign flip of either $\vec{r}$ or $\vec{K}_T$. For forward scattering of a small dipole, $T_{gg}(\vec{r}, \vec{K}_\perp) \simeq \frac{1}{4} (4\pi\alpha_s)^2 N_c C_F r^2 \log \frac{1}{r \Lambda_{\text{eff}}}$, for $r \Lambda_{\text{eff}} \ll 1$.

(24)

exhibits the well-known "color transparency" effect. The logarithm in the previous expression arises because the transverse momenta of the exchanged gluons are distributed from $\Lambda_{\text{eff}}$ to the hard scale $r^{-1}$ according to $dq^2/q^2$.

At second order in $gA^{+a}$ and first order in $eA^+$ we have

$$T_{gg}(\vec{r}, \vec{K}_\perp) = \frac{1}{2} e Q 4\pi\alpha (4\pi\alpha_s)^2 (N_c^2 - 1) \left( \frac{1}{3} \sum f \right) \int_{q_1, q_2} \frac{1}{q_1^2} \frac{1}{q_2^2} \frac{1}{q_3^2} G_{O}(\vec{q}_1, \vec{q}_2, \vec{q}_3) \left[ \sin \left( \vec{q}_1 \cdot \vec{r} + \frac{1}{2} \vec{K}_T \cdot r \right) \right.$$

$$+ \sin \left( \vec{q}_2 \cdot \vec{r} + \frac{1}{2} \vec{K}_T \cdot r \right) - \sin \left( \vec{q}_3 \cdot \vec{r} + \frac{1}{2} \vec{K}_T \cdot r \right) \left] \right.$$

$$\times H(-\vec{q}_2 - \vec{q}_3 - \vec{K}_T, \vec{q}_2, \vec{q}_3),$$

(25)

with $H$ as given in eq. (13). The integrand does not exhibit any infrared divergences at $\vec{q}_2 = 0$ or $\vec{q}_3 = 0$ or $\vec{q}_2 + \vec{q}_3 + \vec{K}_T = 0$. $T_{gg}(\vec{r}, \vec{K}_\perp)$ is odd under a sign flip of either $\vec{r}$ or $\vec{K}_T$.

Finally, at third order in $gA^{+a}$ we have the following scattering amplitude for $C$-odd three gluon exchange\(^5\) [25]:

$$T_{ggg}(\vec{r}, \vec{K}_\perp) = \frac{5}{3} (4\pi\alpha_s)^3 \int_{q_1, q_2} \frac{1}{q_1^2} \frac{1}{q_2^2} \frac{1}{q_3^2} G_{O}(\vec{q}_1, \vec{q}_2, \vec{q}_3) \left[ \sin \left( \frac{1}{2} \vec{r} \cdot \vec{K}_T \right) - \frac{1}{3} \sin \left( \frac{1}{2} \vec{r} \cdot \vec{K}_T \right) \right].$$

(26)

Here, $\vec{q}_3 \equiv -\vec{q}_1 - \vec{q}_2 - \vec{K}_T$. $T_{ggg}(\vec{r}, \vec{K}_\perp)$ is also odd under a sign flip of either $\vec{r}$ or $\vec{K}_T$. The relation of the Odderon amplitude to the $T$-odd gluon GTMDs and GPDs in the gluon dominated regime of very small $x$ has been worked out in ref. [37].

In the limit of nearly forward scattering of a small dipole the Odderon behaves differently than the Pomeron. The second term in the equation above, for example, is

$$\sim \sin \left( \frac{1}{2} \vec{r} \cdot \vec{K}_T \right) \int_{q_1, q_2} \frac{1}{q_1^2} \frac{1}{q_2^2} \frac{1}{q_3^2} G_{O}(\vec{q}_1, \vec{q}_2, \vec{q}_3).$$

(27)

The integral in this expression is independent of the hard scale $1/r$ set by the size of the dipole. For nearly forward scattering all three exchanged gluons will have transverse momenta of order of the soft color neutralization scale $\Lambda_{\text{eff}}$ because the integrands drop faster than $d^2q_i/q_i^2$. On the other hand, if one requires a large momentum transfer $K_T^2$ then $q_2$ in eq. (26) runs from $\Lambda_{\text{eff}}$ to $K_T$ while $q_1$ extends from $\Lambda_{\text{eff}}$ to $\min(K_T, 1/r)$. Hence, for large $K_T$ but small $r$, with $r \cdot K_T \sim 1$ or greater, and neglecting the contribution from $\vec{q}_1 \sim \vec{q}_2 \sim \vec{q}_3 \sim -\vec{K}_T/3$, the leading logarithmic contribution to $T_{ggg}$ is

$$T_{ggg}(\vec{r}, \vec{K}_\perp) \simeq \frac{10}{9} (4\pi\alpha_s)^3 \sin \left( \frac{1}{2} \vec{r} \cdot \vec{K}_T \right) f(K_T) \frac{\log(K_T/\Lambda_{\text{eff}}) \log(1/r \Lambda_{\text{eff}})}{(2\pi)^2 K_T^2}. $$

(28)

This expression again involves the same GPD $f(K_T)$ encountered above.

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\(^5\) Ref. [25] denotes this amplitude $iO(\vec{r}, \vec{K}_\perp)$. Our expression in eq. (26) includes a factor of $-2N_c$ omitted in ref. [25] in the step from their eqs. (74) to (76).
We have indicated in the previous section that the correlators of multiple charge density operators in the proton reduce to the expectation value of a single such operator (a GPD) when the transverse momenta of the attached gauge bosons are large and not close to each other. In that regime, the correlators are dominated by the diagram where all gauge bosons couple to the same quark. We shall illustrate the importance of the diagrams involving multi-quark matrix elements as follows. In eqs. (10) and (16) for the two and three gluon exchange matrix elements, respectively, we drop all but the first contribution in the square brackets; which is the diagram where all exchanged gluons couple to a single quark. The respective correlators at fixed total momentum transfer \( K_T \) are then independent of the transverse momenta of the attached gluons. Hence, \( G \left( \vec{q} - \frac{1}{2} \vec{K}_T, -\vec{q} - \frac{1}{2} \vec{K}_T \right) \rightarrow f(K_T) \) in eq. (23), and \( G_O(\vec{q}_1, \vec{q}_2, \vec{q}_3) \rightarrow f(K_T) \) in eq. (26). However, now the contributions to the integrals in eqs. (23, 26) are no longer cut off at soft transverse momentum. To restore color screening at low transverse momentum we therefore introduce by hand cutoffs of the form

\[
\left( 1 - e^{-\frac{(\vec{q} - \vec{K}_T/2)^2}{2\Lambda^2}} \right) \left( 1 - e^{-\frac{(\vec{q} + \vec{K}_T/2)^2}{2\Lambda^2}} \right) , \quad \left( 1 - e^{-\frac{q^2}{2\Lambda^2}} \right) \left( 1 - e^{-\frac{q^2}{2\Lambda^2}} \right)
\]

in eqs. (23, 26). In particular, for three gluon exchange such ad hoc cutoffs are unavoidable if one restricts to the one-body diagram as \( T_{ggg} \) is otherwise infrared divergent.

We shall use \( \Lambda = 0.1 \) GeV for numerical estimates. We have not attempted to “fine tune” the cutoff \( \Lambda \) to the color neutralization scale encoded in the light cone wave function of the proton. However, we have checked that imposing such a cutoff on the complete set of diagrams does not affect the cross sections much. Below we show the numerical accuracy of these “single quark + cutoff” approximations for the \( J/\Psi \) and \( \eta_c \) cross sections.

V. EXCLUSIVE \( J/\Psi \) AND \( \eta_c \) PRODUCTION IN \( \gamma^*(p) \rightarrow p \) SCATTERING

A. Light cone wave function of the \( \eta_c \)

Before discussing the amplitude for exclusive meson production in \( \gamma^* p \rightarrow M p \) we need to derive the light cone wave function of the \( \eta_c \) required in the dipole scattering approach.

We take the spinor part as \( i\gamma^5 \) sandwiched between \( \bar{u}_h(z, \vec{k}_T) \) and \( v_h(1 - z, -\vec{k}_T) \) spinors, in the transverse rest frame of the \( \eta_c \). This is multiplied by a phenomenological scalar wave function \( \phi^P(k_T, z) \) like for the \( J/\Psi \) meson, c.f. appendix A:

\[
\Psi_{\eta_c h h}^c(\vec{k}_T, z) = -i \frac{\bar{u}_h(z, \vec{k}_T)}{\sqrt{z}} \gamma^5 \frac{v_h(1 - z, -\vec{k}_T)}{\sqrt{1 - z}} \phi^P(k_T, z).
\]

(30)

Here, \( z, \vec{k}_T \) and \( h \) denote the light cone momentum fraction, the transverse momentum, and the helicity of the c quark; \( 1 - z, -\vec{k}_T \) and \( \bar{h} \) those of the \( \bar{c} \) anti-quark.

The spinor matrix element can be computed using the expressions summarized in refs. [16, 39]:

\[
-\bar{u}_h(z, \vec{k}_T)\gamma^5 v_h(1 - z, -\vec{k}_T) = \frac{1}{\sqrt{z(1 - z)}} \left[ -k_T^L \delta_{h - h} - k_T^R \delta_{h + h} + m_c (\delta_{h + h} - \delta_{h - h}) \right].
\]

(31)

\( m_c \) denotes the mass of the charm quark which we take as 1.4 GeV [11]. Here, the transverse momenta are written in complex representation as \( k_T^{R,L} = k^1 \pm ik^2 \). Written as operators in coordinate representation, \( k_T^{R,L} \rightarrow \pm i e^{\pm i \phi_r} \partial_r \). Then

\[
\Psi_{\eta_c h h}^c(r, z) = \frac{1}{z(1 - z)} \left[ i \left( \delta_{h + h} e^{-i\phi_r} - \delta_{h - h} e^{i\phi_r} \right) \partial_r - m_c (\delta_{h + h} - \delta_{h - h}) \right] \phi^P(r, z).
\]

(32)

The scalar part \( \phi^P(r, z) \) of the pseudoscalar meson wavefunction has to be modeled. In order to fix \( \phi^P(r, z) \) we adopt a simple approach and use the same “Boosted Gaussian” functional form of the scalar function as for the vector meson [11, 40–42]:

\[
\phi^P(r, z) = N_p z (1 - z) \exp \left( \frac{-m_{\pi}^2 R_p^2}{8z(1 - z)} - \frac{2z(1 - z)r^2}{R_p^2} + \frac{m_{\pi}^2 R_p^2}{2} \right).
\]

(33)

---

6 As already mentioned, here we restrict to the valence quark component of the proton wave function. For a discussion of correlations among quarks in the proton at \( x < 0.1 \) see, for example, ref. [38].

7 We do not discuss \( T_{ggg} \) in this context as its contribution turns out to be very small.
To fix the parameters $\mathcal{R}_P^2$ and $\mathcal{N}_P$ we impose the normalization condition

$$1 = N_c \sum_{h,h} \int d^2r \int_0^1 \frac{dz}{4\pi} |\Psi_{hh}^n(r,z,Q^2)|^2,$$

which, after substitution of (32) becomes:

$$1 = \frac{N_c}{2\pi} \sum_{h,h} \int d^2r \int_0^1 \frac{dz}{z^2(1-z)^2} \left[ (\partial_r \phi^P)^2 + (m_c \phi^P)^2 \right].$$  \hspace{1cm} (35)

We have assumed that $\phi^P(r,z)$ is real.

A second constraint on the wave function arises from the requirement that it matches the coupling to the axial-vector current,

$$\langle 0 | \bar{c}(0) \gamma^\mu \gamma_5 c(0) | P \rangle = i f_P P^\mu.$$ \hspace{1cm} (36)

The meson state with momentum $P^\mu$ is written as

$$| P \rangle = \frac{\bar{N}_P}{\sqrt{N_c}} \sum_{h,h,i} \int \frac{dx d^2 k_T}{16\pi^3 \sqrt{z(1-x)}} \Psi_{hh}^c(x,k_T) |x, k_T - x P, h, i; 1 - x, -k_T - (1 - x) P, 0, i\rangle.$$ \hspace{1cm} (37)

Here, $i = r, g, b$ denotes color and $h, \bar{h}$ are the helicities of the $c$-quarks. The normalization condition (34) requires $\bar{N}_P = \sqrt{N_c}$ in order to make sure that

$$\langle K | P \rangle = 16\pi^3 P^+ \delta(P^+ - K^+) \delta(P_T - K_T).$$ \hspace{1cm} (38)

We now introduce quark and anti-quark creation and annihilation operators through the bare field expansions \cite{16}

$$c_i(x^\mu) = \int \frac{dx_p d^2 p_T}{16\pi^3 x_p} \sum_h \left[ b_{hi}(p) u_h(p) e^{-ip \cdot r} + d_{hi}^\dagger(p) v_h(p) e^{ip \cdot r} \right],$$ \hspace{1cm} (39)

$$\bar{c}_i(x^\mu) = \int \frac{dx_p d^2 p_T}{16\pi^3 x_p} \sum_h \left[ b_{hi}^\dagger(p) u_h(p) e^{ip \cdot r} + d_{hi}(p) v_h(p) e^{-ip \cdot r} \right].$$ \hspace{1cm} (40)

These satisfy the anti-commutation relations

$$\left\{ b_{hi}(p), b_{h'i'}^\dagger(k) \right\} = \left\{ d_{hi}(p), d_{h'i'}^\dagger(k) \right\} = 16\pi^3 p^+ \delta(p^+ - k^+) \delta(p_T - k_T) \delta_{ii'} \delta_{hh'}.$$ \hspace{1cm} (41)

A straightforward calculation of the + component of eq. (36), using again the spinor matrix elements summarized in ref. \cite{39}, now leads to

$$f_P = \bar{N}_P \sqrt{N_c} \int \frac{dz}{\pi} \frac{m_c}{z(1-z)} \Phi^P(z, r = 0)$$ \hspace{1cm} (42)

The leading order decay rate of the $\eta_c$ to two photons is related to $f_P$ through (see, for example, ref. \cite{43})

$$\Gamma_{\eta_c \to \gamma\gamma} = 4\pi e^4 \alpha^2 \frac{f_P^2}{M_{\eta_c}}.$$ \hspace{1cm} (43)

Experimentally \cite{44},

$$M_{\eta_c} = 2.984 \text{ GeV}, \quad \Gamma_{\eta_c \to \gamma\gamma} = 5.04 \times 10^{-6} \text{ GeV}.$$ \hspace{1cm} (44)

Eqs. (43, 44) give $f_P = 0.337$ GeV. We employ $m_c = 1.4$ GeV both for the $\eta_c$ and $J/\Psi$ wave functions. The two constraints (34) and (42) determine the two parameters of the $\eta_c$ wave function to be $\mathcal{N}_P = 0.547$ and $\mathcal{R}_P = 2.48 \text{ GeV}^{-2}$. As expected, these values are close to those for the $J/\Psi$ meson given in the appendix.
The products of photon and $\eta_c$ wave functions are
\begin{equation}
\left(\Psi^{\eta_c}_c\right)^* \Psi^\gamma_L h = i \frac{m_c e_c e Q}{\pi} K_0(\epsilon r) \phi^P(z, r),
\end{equation}
\begin{equation}
\left(\Psi^{\eta_c}_c\right)^* \Psi^\gamma_{T,\lambda=\pm} h \frac{m_c e e}{2\pi z(1-z)} m_c e^{i\lambda\phi_r} \times \left\{ K_0(\epsilon r) \partial_r \phi^P + \epsilon K_1(\epsilon r) \phi^P \left[ z \delta_{\lambda h,+} - \delta_{\lambda h,-} + (1-z) \delta_{\lambda h,-} - \delta_{\lambda h,+} \right] \right\},
\end{equation}
where $\epsilon \equiv \sqrt{z(1-z)Q^2 + m_c^2}$. Summing the amplitude over quark helicities gives
\begin{equation}
\left(\Psi^{\eta_c}_c\right)^* \Psi^\gamma_L (\epsilon r, z, Q^2) = 0,
\end{equation}
\begin{equation}
\left(\Psi^{\eta_c}_c\right)^* \Psi^\gamma_{T,\lambda=\pm} (\epsilon r, z, Q^2) = -\frac{\sqrt{2} e e}{2\pi z(1-z)} m_c e^{i\lambda\phi_r} \times \left\{ K_0(\epsilon r) \partial_r \phi^P + \epsilon K_1(\epsilon r) \phi^P \right\}.
\end{equation}

The transverse amplitude corresponds to an average over $\lambda = \pm$,
\begin{equation}
\left(\Psi^{\eta_c}_c\right)^* \Psi^\gamma_T (\epsilon r, z, Q^2) = -\frac{\sqrt{2} e e}{2\pi z(1-z)} m_c \cos(\phi_r) \times \left\{ K_0(\epsilon r) \partial_r \phi^P + \epsilon K_1(\epsilon r) \phi^P \right\}.
\end{equation}

### B. Amplitudes and cross sections for exclusive $J/\Psi$ and $\eta_c$ production

The amplitude for exclusive production of a $J/\Psi$ meson is given by
\begin{equation}
A_{T,L}^{\gamma \rightarrow J/\Psi p}(Q^2, K_T) = i \int \frac{dz}{4\pi} \left( \Psi_T, \Psi_{J/\Psi} \right)_{T,L}(\epsilon r, z, Q^2) e^{-i \frac{\pi(1-2z)}{4} r^2 K^2 T} T_{gg}(\epsilon r, K^2_T).
\end{equation}

This is independent of the direction of the transverse momentum transfer $K_T$ and satisfies
\begin{equation}
\left[A_{\lambda}^{\gamma \rightarrow J/\Psi p}(Q^2, K_T)\right] = -A_{\lambda}^{\gamma \rightarrow J/\Psi p}(Q^2, -K_T) = -A_{\lambda}^{\gamma \rightarrow J/\Psi p}(Q^2, -K_T),
\end{equation}
since it does not depend on the sign of $\lambda$ either. This last expression verifies the analyticity property of the S-matrix, $S = 1 + A_{\lambda}(K_T)$, for elastic scattering.

In Eq. (50), $\Psi_{J/\Psi}$ and $\Psi_{J/\Psi}$ denote the $J/\Psi$ and virtual photon light cone wave functions (for longitudinal or transverse polarization); their product is summed over the helicities of the $c$ and $\bar{c}$ quarks. We use the expressions given in ref. [11] for numerical estimates, see appendix A.

$T_{gg}(\epsilon r, K^2_T)$ is independent of $x$ due to us neglecting QCD evolution. On physical grounds, the resulting amplitude $A_{T,L}^{\gamma \rightarrow J/\Psi p}$ should provide a first approximation for a collision energy such that $x \sim 0.1$. Greater $x$ are not accessible due to our assumption of eikonal scattering with negligible longitudinal momentum transfer. In fact, kinematic finite $x$ corrections may be substantial even at $x \sim 0.1$ and should be accounted for in the future. From $x = (Q^2 + M^2_{J/\Psi})/(2pq) = (Q^2 + M^2_{J/\Psi})/(2m_p E_{\gamma})$ we estimate that $E_{\gamma} \simeq 25 - 50$ GeV (on a fixed target), or $W^2 \simeq 50 - 100$ GeV$^2$. For such energy, and $Q^2 \lesssim 1$ GeV$^2$, the minimal value of $|t|$ due to longitudinal momentum exchange is
\begin{equation}
-t_{\text{min}} = \left[ m_p Q^2 + M^2_{J/\Psi} \right]^2 W^2 = \left[ m_p \frac{Q^2 + M^2_{J/\Psi}}{2m_p E_{\gamma} - Q^2 + m_p^2} \right]^2 \lesssim 0.05 \text{ GeV}^2.
\end{equation}

Since we neglect longitudinal momentum transfer, we restrict to $|t - t_{\text{min}}| > 0.1$ GeV$^2$, and our values for $t$ should be understood as corresponding to $t - t_{\text{min}}$.

In the high energy limit the differential cross section is given by [11]
\begin{equation}
\frac{d\sigma}{dt} = \frac{1}{16\pi} \sum_{T,L} A_{T,L}^{\gamma \rightarrow J/\Psi p}^2.
\end{equation}

On the r.h.s. of this equation, the squared amplitude can be evaluated for arbitrary direction of $K_T$ as it is invariant under rotations.
FIG. 2. The differential cross section for exclusive \( J/\Psi \) production. The bands indicate the variation due to the two proton wave functions used here, and also cover the range \( Q^2 < 0.5 \text{ GeV}^2 \). The flatter curve is two gluon exchange with up to two quarks in the proton, while the steeper curve corresponds to the \( c\bar{c} \) dipole scattering from a single quark in the proton. This figure is for a \( \gamma^{(*)} - p \) collision energy of approximately \( W \approx 7 - 10 \text{ GeV} \).

In Fig. 2 we plot the resulting cross section for \( J/\Psi \) production. Fitting by an exponential fall-off over the range \(-0.5 \text{ GeV}^2 > t-t_{\text{min}} > -1 \text{ GeV}^2 \) we obtain a slope of \( B \approx 3 \text{ GeV}^{-2} \) which is close to data at comparable energies [45]. On the other hand, a fit with a (squared) dipole form factor [46], \( d\sigma/dt \sim (1 - (t - t_{\text{min}})/m_g^2)^{-4} \), can be performed over the entire range \(-0.1 > t-t_{\text{min}} > -1 \text{ GeV}^2 \). It results in \( m_g^2 \approx 0.6 \text{ GeV}^2 \), which is close to \( m_p^2 \) but somewhat smaller than \( m_Q^2 \approx 1 \text{ GeV}^2 \) suggested in ref. [46].

Our expressions for the scattering amplitude have been obtained in a fixed coupling approximation. Assuming \( \alpha_s = 0.35 \), the integral of \( d\sigma/dt \) over \( |t-t_{\text{min}}| > 0.1 \text{ GeV}^2 \) is \( \sigma \approx 2.0 \text{ nb} \); it increases by about a factor of 1.7 if we extrapolate the integral all the way down to \( t = t_{\text{min}} \). Then the differential cross section at \( t = t_{\text{min}} \) is \( d\sigma/dt \approx 20 \text{ nb/GeV}^2 \).

The extrapolation to \( t = t_{\text{min}} \) of course suffers from some uncertainty due to neglecting longitudinal momentum transfer. Also, there are uncertainties as to the values of \( \alpha_s \) and \( m_Q \). Lastly, for more accurate results one should account for the real part of the amplitude, too (see, for example, refs [11, 40, 41, 47, 48]). However, \( d\sigma/dt \approx 20 \text{ nb/GeV}^2 \) is not very far from previous estimates for the \( J/\Psi \) cross section at \( W \approx 7 - 10 \text{ GeV} \) [48]. The \( J/\Psi \) cross section scales with the coupling as \( \sim \alpha_s^2 \).

Fig. 2 also shows the cross section obtained from only the “one body” diagram where both exchanged gluons couple to the same quark in the proton. This is a fair approximation to within a factor of about 2 for \( |t-t_{\text{min}}| \lesssim 1 \text{ GeV}^2 \). The figure illustrates the effect on the slope of \( d\sigma/dt \) of the diagrams where the \( c\bar{c} \) dipole scatters from multiple quarks in the proton\(^8\). They lead to a harder slope due to the contribution from configurations where the exchanged gluons have similar transverse momenta.

The amplitude for elastic exclusive production of a \( \eta_c \) meson is given by

\[
A^\gamma p \rightarrow \eta_c p(Q^2, \vec{K}_T) = i \int d^2r \int_0^{1/4\pi} \frac{dz}{(\Psi^\dagger (\Psi^\dagger)^*) (\vec{r}, z, Q^2)} e^{-i(1-z)\vec{r} \cdot \vec{K}_T} (T_\gamma + T_{\eta gg} + T_{\eta g g})(\vec{r}, -\vec{K}_T) .
\]  

(54)

Here, \( \lambda = \pm \) denotes the polarization of the transverse photon, the longitudinal photon does not contribute. The product \( \Psi^\dagger (\Psi^\dagger)^* \) changes sign under \( \vec{r} \rightarrow -\vec{r} \) and therefore

\[
A^\gamma p \rightarrow \eta_c p(Q^2, \vec{K}_T) = -A^\gamma p \rightarrow \eta_c p(Q^2, -\vec{K}_T) .
\]  

(55)

\(^8\) Charm production in photon scattering off multiple quarks has also been advocated in an unrelated setting for the \( x \rightarrow 1 \) threshold region [49].
The complex conjugate amplitude is
\[
\left[ A_\lambda^{\gamma^* p \rightarrow n_c p} (Q^2, \vec{K}_T) \right]^* = A_\lambda^{\gamma^* p \rightarrow n_c p} (Q^2, -\vec{K}_T). \tag{56}
\]

This confirms that the S-matrix, \( S(\vec{K}_T, \lambda) = 1 + A(\vec{K}_T, \lambda) \), exhibits the proper analytical properties for elastic scattering. We note that \( \sum_\lambda |A_\lambda^{\gamma^* p \rightarrow n_c p}|^2 \) is invariant under rotations of \( \vec{K}_T \).

FIG. 3. The differential cross section for exclusive \( n_c \) production. The bands indicate the variation due to the two proton wave functions used here, and also cover the range \( Q^2 < 0.5 \text{ GeV}^2 \). This figure is for a \( \gamma^* p \) collision energy of approximately \( W \simeq 7 - 10 \text{ GeV} \). Left: the cross section due to three gluon exchange alone. The flatter curve accounts for all diagrams while the steeper curve corresponds to the \( \bar{c}c \) dipole scattering from a single quark in the proton. Right: individual contributions due to single photon, photon plus two gluon, and three gluon exchanges, and the complete \( n_c \) cross section summed (at the amplitude level) over all these exchanges.

In fig. 3 we present our results for exclusive production of \( n_c \) mesons. We extend all curves down to \( K_T = 0 \) in order to show that the differential cross sections due to \( \gamma gg \) or \( ggg \) exchange vanish in this limit. However, as mentioned above, we do neglect longitudinal momentum transfer and so we do not expect that these curves are reliable for \( t \simeq t_{\text{min}} \).

The left panel of fig. 3 shows the cross section due to three gluon exchange. We compare the full cross section which includes matrix elements between two and three quark states to the one-body approximation with ad hoc infrared cutoff (29). They exhibit very different behavior for \( |t| \lesssim 1.5 \text{ GeV}^2 \). It is interesting to note that the sum of all three-gluon-exchange diagrams achieves its maximum at rather hard \( |t| \gtrsim 1 \text{ GeV}^2 \). The dominant contribution corresponds to approximate sharing of the momentum transfer among the three gluons, \( \vec{q}_i \sim -\vec{K}_T / 3 \). At the same time, gluons with transverse momentum less than \( \Lambda_{\text{eff}} \) are screened so that the cross section below \( |t| \sim 1 \text{ GeV}^2 \) decreases with \( |t| \).

On the other hand, the \( n_c \) cross section at \( |t| \lesssim 1.5 \text{ GeV}^2 \) is anyhow dominated by single photon exchange as seen in fig. 3 on the right. This appears to present the clearest opportunity for measuring the quark GPD \( f(K_T) \) in this process. The photon plus two gluon exchange is negligible over the entire range of \( |t| \). “Odderon” exchange dominates for \( |t| \gtrsim 1.5 \text{ GeV}^2 \). Hence, discovery of the three gluon QCD exchange in this process requires measurements at fairly large momentum transfer. Since the dominant dipole scale for charmonium production is about \( r \sim 1 \text{ GeV}^{-1} \), at large \( K_T \) one requires the dipole scattering amplitude to all orders in \( \vec{r} \cdot \vec{K}_T \). Our predictions for the \( n_c \) cross section due to three gluon exchange in \( \gamma - p \) scattering are substantially lower than earlier estimates\(^9\) [7, 8]. Their cross section peaks at \( \approx 10 - 25 \text{ pb/GeV}^2 \) at \( |t| \approx 0.5 \text{ GeV}^2 \). In contrast, we find that the cross section due to three gluon exchange increases with the momentum transfer up until \( |t| \sim 1 \text{ GeV}^2 \), where its

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\(^9\) Those papers aimed at much higher HERA energies. However, their numerical estimates did not account for small-\( x \) evolution of the Odderon and exhibit no energy dependence (for \( W \) well above threshold).
magnitude is about 50 fb/GeV$^2$, and then decreases rather slowly for $|t|$ up to 3 GeV$^2$. Aside from using a different approach for the three gluon exchange amplitude which here is related to the light cone wave function of the proton, and perhaps more realistic models for the wave function of the $\eta_c$, we note the following differences. Refs. [7, 8] employ a coupling of the exchanged gluons to the valence quarks of the proton of $\alpha_s = 1$. We use the same $\alpha_s = 0.35$ obtained from the cross section for $J/\Psi$ production for all gluon-quark vertices$^{10}$. This amounts to a suppression factor of $0.35^3 = 0.04$. Furthermore, we find that the “non-relativistic approximation” which sets the momentum fractions of the charm quarks in the $\eta_c$ to 1/2 overestimates the cross section by as much as a factor of 4 (this agrees with the findings in ref. [50]). We also use a more up to date value of $\Gamma(\eta_c \rightarrow \gamma \gamma) = 5$ keV [44] to normalize the $\eta_c$ wave function, and this is smaller than the value used in refs. [7, 8] by a factor of 1.4. Lastly, as the maximum of the differential cross section is shifted to higher $|t|$ it is natural that its magnitude would be substantially lower.

VI. SUMMARY

Exclusive $\eta_c$ production in $\gamma - p$ scattering could provide clean evidence for the semi-hard Odderon in QCD. Moreover, this process would also provide valuable insight into the light cone wave function of the proton which determines the coupling to both Pomeron and Odderon simultaneously [25]. Accordingly, we first applied our approach to $J/\Psi$ production, and from the magnitude of the cross section, $\sigma_{J/\Psi} \sim \alpha_s^4$, we determined the effective quark-gluon coupling, $\alpha_s \approx 0.35$. We also obtain a reasonable slope of $d\sigma/dt$ without having to tune any parameters. Most importantly, we find that it is a fair approximation to consider $J/\Psi$ production (up to intermediate $|t| \approx 1$ GeV$^2$) as due to two gluon exchange with single quarks in the proton. This involves the matrix element of a product of two color charge densities between single quark states and can be expressed in terms of a Generalized Parton Distribution (GPD).

C-parity even states like the $\eta_c$ can be produced only via C-odd exchanges such as a single photon, a photon and two gluons, three gluons etc. Exchange of a single photon dominates at low transverse momentum $K_T$ while the contribution from the exchange of a photon and two gluons is very small for any $K_T$. Our analysis of three gluon exchange indicates that diagrams involving coupling of the three gluons to multiple quarks are important not only for the cancellation of infrared divergences due to color neutrality of the proton. Rather, such many-body contributions are also numerically very important for $|t| \lesssim 1$ GeV$^2$ (however, single photon exchange dominates over three gluon exchange in that regime) and for $|t| \gg 1$ GeV$^2$. At high momentum transfer $K_T^2 = |t|$ one may not expand the dipole scattering amplitude in powers of $\vec{r} \cdot \vec{K}_T$.

A rather interesting outcome of our numerical analysis using the proton light cone wave functions by Brodsky and Schlumpf [26] is that the differential cross section for $\gamma(p) \rightarrow \eta_c(p)$ via three gluon exchange achieves its maximum for $|t| \approx 1 - 3$ GeV$^2$. This is remarkable since older estimates from the literature [7, 8] using proton impact factors dominated by the scale $m_c^2$ located the peak at a much lower $|t| \approx 0.5$ GeV$^2$. A cross section on the order of (at least) tens to a hundred fb/GeV$^2$ at $|t| > 1.5$ GeV$^2$ would represent good evidence for C-odd three gluon exchange. On the other hand, the best opportunity to measure the quark GPD in this process is through single photon exchange at $|t| \lesssim 1.5$ GeV$^2$.

We estimate the cross section for $\eta_c$ production at $|t| = 1.5 - 3$ GeV$^2$ to be 30-150 fb/GeV$^2$, for $\alpha_s = 0.35$ and photon-proton collision energy $W \sim 7 - 10$ GeV. Experimental detection therefore requires very high luminosities. For other values of the coupling, the $J/\Psi$ and $\eta_c$ cross sections scale like $\alpha_s^4$ and $\alpha_s^6$, respectively.

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Figure 1 has been prepared with Jaxodraw [51].

$^{10}$ F. E. Low estimated more than 40 years ago that $\alpha_s \approx \frac{1}{4}$ for two-gluon “Pomeron” exchange [12], so our qualitative extraction of $\alpha_s$ from the expected $J/\Psi$ cross section at $x \sim 0.1$ is not a new result.
Appendix A: Photon and $J/\psi$ light cone wave functions

This appendix presents the expressions for the light-cone wave functions of the photon and $J/\psi$ meson used here. These have been discussed in many papers [11, 40, 46, 52]; we follow the approach of ref. [11]. (For more recent numerical solutions of quarkonium light cone wave functions see [53].)

The wave function of a longitudinally polarized virtual photon is given by [11]

$$\Psi_{hh,\lambda=0}^\gamma(\vec{r}, z, Q^2) = e_{e_ e} \delta_{h,-\hbar} \frac{K_0(\epsilon r)}{2\pi} 2Qz(1-z)\ . \ (A1)$$

For transverse polarization,

$$\Psi_{hh,\lambda=\pm}^\gamma(\vec{r}, z, Q^2) = \lambda e_{e_ e} \sqrt{2} \left\{ i e^{i\phi_ r} [z \delta_{h,\pm} \delta_{\hbar,-\lambda} - (1-z) \delta_{h,-\lambda} \delta_{\hbar,\lambda}] \partial_\perp + m_Q \delta_{h,\lambda} \delta_{\hbar,\lambda} \right\} \frac{K_0(\epsilon r)}{2\pi} . \ (A2)$$

In the above expressions $K_0$ is a modified Bessel function of the second kind, $\phi_ r$ is the azimuthal angle of the $\vec{r}$ vector, and $\epsilon^2 \equiv z(1-z)Q^2 + m_\gamma^2$. Note that in eqs. (A1) and (A2) we do not include a factor of $\sqrt{N_c}$ as in ref. [11] because we have explicitly included this factor in our expression for the scattering amplitude. Ref. [40] also includes a factor of $1/\sqrt{4\pi}$ in the photon (and meson) wave function which we write explicitly in eqs. (50, 54) for the amplitudes.

For the vector meson we employ the following model wave functions [11, 40, 50, 52]:

$$\Psi_{hh,\lambda=0}^V(r, z) = \delta_{h,-\hbar} \frac{m_V^2 - \nabla_\perp^2}{m_V z(1-z)} \phi_ l^V(r, z), \ (A3)$$

and

$$\Psi_{hh,\lambda=\pm}^V(r, z) = \pm \frac{\sqrt{2}}{z(1-z)} \left\{ i e^{i\phi_ r} [z \delta_{h,\pm} \delta_{h,\mp} - (1-z) \delta_{h,\mp} \delta_{h,\pm}] \partial_\perp + m_f \delta_{h,\pm} \delta_{h,\mp} \right\} \phi_ r^V(r, z). \ (A4)$$

Several phenomenological models of the scalar part are available in the literature. Here, we use the “Boosted Gaussian” model [11, 40–42]

$$\phi_ {T, L}^V(r, z) = N_{T, L} z(1-z) \exp \left( -\frac{m_V^2 R_c^2}{8z(1-z)} - \frac{2z(1-z)^2 r^2 + m_V^2 R_c^2}{2} \right) . \ (A5)$$

The parameters have been obtained in ref. [11] from the normalization condition and from the electronic decay width. They are $M_{J/\psi} = 3.097$ GeV, $m_c = 1.4$ GeV, $N_T = 0.578$, $N_L = 0.575$, $R_c = 2.3$ GeV$^{-2}$.

The product of photon and meson wave functions summed over quark helicities, are given by

$$\left( \Psi_{\lambda, \lambda'}^V \right)^* \Psi_{\lambda, \lambda'}^\gamma(\vec{r}, z, Q^2) \equiv \sum_{h, \hbar = \pm} \frac{\Psi_{hh,\lambda}^V(\vec{r}, z) \Psi_{hh,\lambda'}^\gamma(\vec{r}, z, Q^2)}{Q^2} . \ (A6)$$

For a $J/\Psi$ meson, after averaging over $\lambda = \lambda' = +$ and $\lambda = \lambda' = -$, one obtains explicitly [11]

$$\left( \Psi_{J/\psi}^V \right)^* \Psi_{J/\psi}^\gamma_T (\vec{r}, z, Q^2) = e_ e e_ e \frac{1}{\pi z(1-z)} \left\{ m_J^2 K_0(\epsilon r) \phi_ T^V(r, z) - [z^2 + (1-z)^2] e_1 K_1(\epsilon r) \partial_\perp \phi_ T^V(r, z) \right\} , \ (A7)$$

$$\left( \Psi_{J/\psi}^V \right)^* \Psi_{J/\psi}^\gamma_L (\vec{r}, z, Q^2) = e_ e e_ e \frac{1}{\pi} 2Qz(1-z) K_0(\epsilon r) \left[ M_{J/\psi} \phi_ L^V(r, z) + \frac{m_c^2 - \nabla_\perp^2}{M_{J/\psi} z(1-z)} \phi_ T^V(r, z) \right] . \ (A8)$$

In eq. (A7) the polarizations $\lambda = \lambda'$ of the photon and the $J/\psi$ are equal; the off-diagonal overlap for $\lambda = -\lambda'$ is proportional to $e^{iz \phi_ r}$ and gives no contribution to the amplitude (50) for $J/\psi$ production as $T_{gg}(\vec{r}, \vec{K}_T)$ is invariant under a simultaneous rotation of $\vec{r}$ and $\vec{K}_T$. Also, the overlaps $\left( \Psi_{L}^V \right)^* \Psi_{T}^\gamma$ and $\left( \Psi_{T}^V \right)^* \Psi_{L}^\gamma$ do not vanish either but change sign under $\vec{r} \rightarrow -\vec{r}$ and so do not contribute to eq. (50).

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