The Cauchy problem for a Schrödinger - Korteweg - de Vries system with rough data

Hartmut Pecher
Fachbereich Mathematik und Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20
D-42097 Wuppertal
Germany
e-mail Hartmut.Pecher@math.uni-wuppertal.de

Abstract
The Cauchy problem for a coupled system of the Schrödinger and the KdV equation is shown to be globally well-posed for data with infinite energy. The proof uses refined bilinear Strichartz type estimates and the I-method introduced by Colliander, Keel, Staffilani, Takaoka, and Tao.

0 Introduction
Consider the Cauchy problem

\begin{align*}
    iu_t + u_{xx} &= \alpha vu + \beta |u|^2 u \\
    v_t + vv_x + v_{xxx} &= \gamma (|u|^2)_x \\
    u(x,0) &= u_0(x), \quad v(x,0) = v_0(x)
\end{align*}

where \( u \) is a complex-valued und \( v \) a real-valued function defined for \((x,t) \in \mathbb{R} \times \mathbb{R}^+\) and \( \alpha, \beta, \gamma \in \mathbb{R}\).

In the theory of capillary-gravity waves the interaction of a short and a long wave was modelled by such a coupled system of a Schrödinger and a KdV type equation (cf. Kawahara et al. [16], Funakoshi and Oikawa [11]). It also appears in plasma physics (cf. Nishikawa et al. [19]) modelling the interaction of the Langmuir and ion-acoustic waves.

The system [1, 2, 3] was considered by M. Tsutsumi [22] who showed that for \((u_0, v_0) \in H^{m+\frac{1}{2}}(\mathbb{R}) \times H^m(\mathbb{R}), m = 1, 2, 3, \ldots\), the problem is locally well-posed and for \( \alpha \gamma > 0 \) also globally well-posed by using the conservation laws. Later, Bekiranov, Ogawa and Ponce [1], using the Fourier restriction norm method, lowered down the regularity assumptions on the data and proved local well-posedness for \((u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})\) for any \( s \geq 0\).

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Because our main aim is to consider the global problem which requires to use the conservation laws (cf. [21], [25], [26] below) leading to an a-priori bound of the $H^1$-norms of $u$ and $v$, if $\alpha \beta > 0$, we assume $u_0$ and $v_0$ to belong to the same Sobolev space $H^s(\mathbb{R})$. Then we are able to show local well-posedness for any $s > 0$ (cf. Theorem 2.1) which implies especially global well-posedness in energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, if $\alpha \gamma > 0$. A global well-posedness result in this space was proven before by Guo and Miao [13] already. Moreover we are able to show global-wellposedness for less regular data, namely $u_0, v_0 \in H^s(\mathbb{R})$ with $s > 3/5$ (if $\beta = 0$) and $s > 2/3$ (if $\beta \neq 0$). We use the Fourier restriction norm method and especially the I-method introduced by Colliander, Keel, Staffilani, Takaoka and Tao ([3], [4], [5], [6], [7], [8], [9]). It was successfully applied to the (2+1)- and (3+1)-dimensional Schrödinger equation, and to the KdV and modified KdV equation with sometimes optimal results. In all these cases a scaling invariance was used which in our situation does not hold. Similar results were also given for the Klein - Gordon - Schrödinger system by Tzirakis [23] and for the (1+1)-dimensional Zakharov system by the author [20].

The paper is organized as follows. In section 1 we give various bilinear Strichartz type estimates for the nonlinearities in the solution spaces $H$ energy space method and especially the I-method introduced by Colliander, Keel, Staffilani, Takaoka and Tao ([3], [4], [5], [6], [7], [8], [9]). It was successfully applied to the (2+1)- and (3+1)-dimensional Schrödinger equation, to the (1+1)-dimensional derivative Schrödinger equation, and to the KdV and modified KdV equation with sometimes optimal results. In all these cases a scaling invariance was used which in our situation does not hold. Similar results were also given for the Klein-Gordon-Schrödinger system by Tzirakis [23] and for the (1+1)-dimensional Zakharov system by the author [20].

For a given time interval $I$ we define $\|u\|_{X^{s,b}(I)} := \inf_{\tilde{u}} \|\tilde{u}\|_{X^{s,b}}$ and similarly $\|v\|_{Y^{s,b}(I)}$. The refined bilinear estimates are partly new or variants of known versions. In section 2 we formulate the local existence theorem and a variant of it for the modified system of differential equations after application of the operator $I$ to the original one ([1], [2], [3]) giving a precise lower bound for the local existence time $T$ in terms of the norm of the data. This operator $I$, which gave the method its name, is defined as follows: $I = I_N$ for given $s < 1$ and $N >> 1$ is defined by $I_N f(\xi) := m_N(\xi) \tilde{f}(\xi)$. Here $m_N(\xi)$ is a smooth, radially symmetric and nonincreasing function of $|\xi|$, defined by $m_N(\xi) = 1$ for $|\xi| \leq N$ and $m_N(\xi) = \frac{(N}{|\xi|})^{1-s}$ for $|\xi| \geq 2N$. Dropping $N$ from the notation we have $I : H^s \to H^1$ is a smoothing operator with $\|u\|_{X^{m,b}} \leq \|Iu\|_{X^{m+1-s,b}} \leq cN^{1-s} \|u\|_{X^{m,b}}$ and similarly for $Y^{s,b}$. In section 3 we consider the conserved quantities $L(u,v)$ and $E(u,v)$ (cf. [25], [26]) and their modified versions $L(Iu,Iv)$ and $E(Iu,Iv)$ which are no longer conserved, but it is possible to control their growth in time, because here some sort of cancellation helps. As is typical for the I-method we then in sections 4 and 5 consider in detail this increment of $E(Iu,Iv)$ and $L(Iu,Iv)$, respectively, which is shown to be small for small time intervals and
large \( N \). Important tools here are the refined bilinear Strichartz estimates of section 1, especially a new estimate for the product of a Schrödinger and a KdV part. These estimates also allow to control the growth of the corresponding norms of the solution during its time evolution. One iterates in section 6 the local existence theorem with time steps of equal length. To achieve this one has to make the process uniform which can be done if \( s \) is close enough to 1.

We collect some elementary facts about the spaces \( X^{m,b} \) (and analogously \( Y^{m,b} \)).

The following interpolation property is well-known:

\[
X^{(1-\Theta)m_0+\Theta m_1, (1-\Theta)b_0+\Theta b_1} = (X^{m_0,b_0}, X^{m_1,b_1})_{\Theta} \quad \text{for } \Theta \in [0,1].
\]

If \( u \) is a solution of \( iu_t + \frac{\partial^2}{\partial x^2} u = 0 \) with \( u(0) = f \) and \( \psi \) is a cutoff function in \( C_0^\infty(\mathbb{R}) \) with \( \text{supp} \psi \subset (-2,2) \), \( \psi \equiv 1 \) on \([-1,1]\), \( \psi(t) = \psi(-t) \), \( \psi(t) \geq 0 \), \( \psi_\delta(t) := \psi(\frac{t}{\delta}) \), \( 0 < \delta \leq 1 \), we have for \( b > 0 \):

\[
\| \psi_1 u \|_{X^{m,b}} \leq c \| f \|_{H^m}.
\]

If \( v \) is a solution of the problem \( iv_t + \partial_x^2 v = F \), \( v(0) = 0 \), we have for \( b' + 1 \geq b \geq 0 \geq b' > -1/2 \)

\[
\| \psi_\delta v \|_{X^{m,b}} \leq c \delta^{1+b-b'} \| F \|_{X^{m,b'}}
\]

(for a proof cf. \[12\], Lemma 2.1).

Finally, if \( 1/2 > b > b' \geq 0 \), \( m \in \mathbb{R} \), we have the embedding

\[
\| f \|_{X^{m,b'}([0,\delta])} \leq c \delta^{b-b'} \| f \|_{X^{m,b}([0,\delta])}.
\]

(4)

For the convenience of the reader we repeat the proof of \[14\], Lemma 1.10. The claimed estimate is an immediate consequence of the following

**Lemma 0.1** For \( 1/2 > b > b' \geq 0 \), \( 0 < \delta \leq 1 \), \( m \in \mathbb{R} \) the following estimate holds:

\[
\| \psi_\delta f \|_{X^{m,b'}} \leq c \delta^{b-b'} \| f \|_{X^{m,b}}.
\]

**Proof:** The following Sobolev multiplication rule holds:

\[
\| fg \|_{H^{b'}_t} \leq c \| f \|_{H^{1-(b-b')}_t} \| g \|_{H^b_t}.
\]

This rule follows easily by the Leibniz rule for fractional derivatives, using \( J^s := \mathcal{F}^{-1}(\tau^s \mathcal{F}) \):

\[
\| fg \|_{H^{b'}_t} \leq c(\| (J^{b'} f)g \|_{L^2_t} + \| (J^{b'} g)\|_{L^2_t})
\]

\[
\leq c(\| J^{b'} f \|_{L^p_t} \| g \|_{L^q_t} + \| f \|_{L^p_t} \| J^{b'} g \|_{L^q_t})
\]

with \( \frac{1}{p} = b \), \( \frac{1}{p} = \frac{1}{2} - b \), \( \frac{1}{q} = b - b' \), \( \frac{1}{q} = \frac{1}{2} - (b - b') \). Sobolev’s embedding theorem gives the claimed result. Consequently we get

\[
\| \psi_\delta g \|_{H^{b'}_t} \leq c \| \psi \|_{H^{1-(b-b')}_t} \| g \|_{H^b_t} \leq c \delta^{b-b'} \| g \|_{H^b_t},
\]

and thus

\[
\| \psi_\delta f \|_{X^{m,b'}} = \| e^{-it\partial_x^2} \psi_\delta f \|_{H^m_t \otimes H^{b'}_t} \leq c \delta^{b-b'} \| e^{-it\partial_x^2} f \|_{H^m_t \otimes H^b_t}
\]

\[
= c \delta^{b-b'} \| f \|_{X^{m,b}}.
\]
Fundamental are the following linear Strichartz type estimates for the Schrödinger equation (cf. e.g. [12], Lemma 2.4):

$$\|e^{it\partial^2_x} \psi\|_{L^q_t(I, L^r_x(\mathbb{R}))} \leq C\|\psi\|_{L^2_x(\mathbb{R})}$$

and

$$\|u\|_{L^q_t(I, L^r_x(\mathbb{R}))} \leq C\|u\|_{X^{0,\frac{1}{2}+}(I)},$$

if $0 \leq \frac{2}{q} = \frac{1}{2} - \frac{1}{r}$, especially

$$\|u\|_{L^q_{xt}} \leq C\|u\|_{X^{0,\frac{1}{2}+}},$$

which by interpolation with the trivial case $\|u\|_{L^1_{xt}} = \|u\|_{X^{0,0}}$ gives:

$$\|u\|_{L^q_{xt}} \leq C\|u\|_{X^{0,\frac{1}{2}+}},$$

if $2 < p \leq 6$.

For the KdV (Airy) equation we have (cf. e.g. [17], Theorem 2.4):

$$\|e^{-it\partial^2_x} \psi\|_{L^6_{xt}} \leq C\|\psi\|_{L^6_x},$$

and thus $\|v\|_{L^6_{xt}} \leq C\|v\|_{Y^{0,\frac{1}{2}+}}$.

We use the notation $(\lambda) := (1 + \lambda^2)^{1/2}$. Let $a \pm$ denote a number slightly larger (resp., smaller) than $a$.

## 1 Bilinear Strichartz type estimates

**Lemma 1.1** If $s \geq 0$, $b > \frac{1}{2}$, $b' \geq \max(\frac{1}{2} - \frac{s}{3}, 0)$, the following estimate holds:

$$\|\partial_x(v_1v_2)\|_{Y^{-s,-b'}} \leq C\|v_1\|_{Y^{s,b}}\|v_2\|_{Y^{s,b}}.$$  

(5)

**Proof:** With $\xi_i \in \text{ supp } \hat{v}_i$ ($i = 1, 2$) we first consider the case $|\xi_1| \gg |\xi_2|$ (or similarly $|\xi_2| \gg |\xi_1|$). In this case $|\xi| := |\xi_1 + \xi_2| \sim |\xi_1| \sim |\xi_1 + \xi_2|^{1/2} |\xi_1 - \xi_2|^{1/2},$ and we use (5) to conclude

$$\|\partial_x(v_1v_2)\|_{L^2_{xt}} \leq C\|v_1\|_{Y^{0,b}}\|v_2\|_{Y^{0,b}}$$

which immediately implies

$$\|\partial_x(v_1v_2)\|_{Y^{s,0}} \leq C\|v_1\|_{Y^{s,b}}\|v_2\|_{Y^{s,b}}.$$}

Next we consider the case $|\xi_1| \sim |\xi_2|$. This implies $|\xi| \leq |\xi_1|, |\xi_2|$. We have to show

$$\left\| \frac{\xi(\xi)^s}{(\tau - \xi^3)^{b'}} \int \int \frac{f_1(\xi_1, \tau_1)}{(\tau_1 - \xi_1^{3b}\xi_1^s)} \cdot \frac{f_2(\xi_2, \tau_2)}{(\tau_2 - \xi_2^{3b}\xi_2^s)} d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1}} \leq C\|f_1\|_{L^2_{\xi_1}}\|f_2\|_{L^2_{\xi_2}},$$

where $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$. Using the Schwarz method this is implied by

$$\sup_{\xi, \tau} \frac{|\xi| (\xi)^s}{(\tau - \xi^3)^{b'}} \left( \int \int \frac{d\tau_1 d\xi_1}{(\tau_1 - \xi_1^{3b}(\tau_2 - \xi_2^{3b})^{2b}(\xi_1^{2s}(\xi_2^{2s}))} \right)^{1/2} \leq C.$$
Using (the proof of) [18], Lemma 2.4, the l.h.s. is bounded by
\[ c \sup_{\xi,\tau} \frac{|\xi|}{(\xi^s(\tau - \xi^3))^b} \left( \int \int \frac{d\tau_1 d\xi_1}{(\tau_1 - \xi^3_1)^{2b}(\tau_2 - \xi^3_2)^{2b}} \right)^{\frac{1}{2}} \]

\[ \leq c \sup_{\xi,\tau} \frac{|\xi|}{(\xi^s(\tau - \xi^3))^b} \left( 4\tau - \xi^3 \right)^{\frac{1}{2}}. \]

This can easily seen to be finite, if \( \frac{3}{2} - s - 3b' \leq 0 \) and \( b' \geq 0 \), which is assumed.

**Lemma 1.2**
1. If \( s \geq 0 \), \( b > 1/2 \), \( b' = \max(\frac{1}{2}, \frac{1}{b} - s) \), we have
\[ \| \partial_x(u_1 u_2) \|_{Y_s, -b'} \leq c \| u_1 \|_{X_s, b} \| u_2 \|_{X_s, b}. \]

(6)

2. If \( b, b' > 1/2 \), we have
\[ \| \partial_x(u_1 u_2) \|_{Y_0, -b'} \leq c \| u_1 \|_{X_0, b} \| u_2 \|_{X_0, b}. \]

(7)

**Proof:** First we consider 2., which is proved along the lines of [1], Lemma 3.2. With the restriction \( \xi = \xi_1 + \xi_2 \), \( \tau = \tau_1 + \tau_2 \), \( \sigma = \tau - \xi^3 \), \( \sigma_1 = \tau_1 - \xi^3_1 \), \( \sigma_2 = \tau_2 + \xi^3_2 \), we have to show
\[ \int \frac{|\xi|g(\xi, \tau)}{(\sigma)^b} \cdot f_1(\xi_1, \tau_1) \cdot f_2(\xi_2, \tau_2) \, d\tau_1 d\xi_1 \, d\tau \, d\xi \leq c \| g \|_{L^2_\sigma} \| f_1 \|_{L^2_\tau} \| f_2 \|_{L^2_\tau}. \]

Using Schwarz’ method this is implied, if one of the following estimates holds in each of the regions into which we are going to split our domain:

\[ \sup_{\tau, \xi} \frac{|\xi|}{(\sigma)^b} \left( \int \int \frac{d\xi_1 d\tau_1}{(\sigma_1)^{2b}(\sigma_2)^{2b}} \right)^{\frac{1}{2}} \leq c, \]

\[ \sup_{\tau_1, \xi_1} \frac{1}{(\sigma_1)^b} \left( \int \int \frac{|\xi_1|^2 d\xi d\tau}{(\sigma)^{2b}(\sigma_2)^{2b}} \right)^{\frac{1}{2}} \leq c, \]

\[ \sup_{\tau_2, \xi_2} \frac{1}{(\sigma_2)^b} \left( \int \int \frac{|\xi_2|^2 d\xi_1 d\tau_1}{(\sigma)^{2b}(\sigma_1)^{2b}} \right)^{\frac{1}{2}} \leq c. \]

Here the integrations w.r. to \( \tau \) and \( \tau_1 \) can be done using [1], Lemma 2.5, and lead to the following sufficient conditions, using \( b > 1/2 \) and w.l.o.g. \( b' \leq b \):

\[ \sup_{\tau, \xi} \frac{|\xi|}{(\sigma)^b} \left( \int \frac{d\xi_1}{(\tau + (\xi - \xi_1)^2 - \xi^3_1)^{2b}} \right)^{\frac{1}{2}} \leq c, \]

(8)

\[ \sup_{\tau_1, \xi_1} \frac{1}{(\sigma_1)^b} \left( \int \frac{|\xi_1|^2 d\xi}{(\xi_1^3 - \tau_1 + (\xi - \xi_1)^2)^{2b}} \right)^{\frac{1}{2}} \leq c, \]

(9)

\[ \sup_{\tau_2, \xi_2} \frac{1}{(\sigma_2)^b} \left( \int \frac{|\xi_2|^2 d\xi_1}{(\xi_1^3 + \tau_2 - (\xi_1 + \xi_2)^3)^{2b}} \right)^{\frac{1}{2}} \leq c. \]

(10)

Now we split our domain \((\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4\) as follows: \( \mathbb{R}^4 = A \cup B \cup C \), where \( A = \{|\xi| \leq 10\} \), \( B = \{|\xi| > 10, |3\xi^2 - 2(\xi - \xi_1)| \geq \xi^3\} \), \( C = \{|\xi| > 10, |\xi^2 + \)
\[ \xi - 2\xi_1 \geq \frac{1}{2} \xi^2 \].

We remark here (for later reference) that the condition in \( C \) is especially fulfilled, if \( |\xi_2| > |\xi_1| \) (or \( |\xi_1| > |\xi_2| \)) and \( |\xi| > 10 \).

That we in fact have \( \mathbb{R}^4 = A \cup B \cup C \) can be seen as follows: if both conditions defining \( B \) and \( C \) are violated, i.e. \( 3\xi^2 - 2(\xi - \xi_1) < \xi^2 \) and \( |\xi^2 + \xi - 2\xi_1| < \frac{1}{2} \xi^2 \), then we have

\[ |2\xi_1| = |(2\xi_1 - 2\xi + 3\xi^2) + 2\xi - 2\xi^2| > 3\xi^2 - 2|\xi| - \xi^2 = 2\xi^2 - 2|\xi| \]

and

\[ |2\xi_1| = |(2\xi_1 - \xi - \xi^2) + \xi + \xi^2| < \frac{1}{2} \xi^2 + |\xi| + \xi^2 = \frac{3}{2} \xi^2 + |\xi| . \]

Thus

\[ 2\xi^2 - 2|\xi| < \frac{3}{2} \xi^2 + |\xi| \iff \frac{1}{2} \xi^2 < 3|\xi| \iff |\xi| < 6 , \]

thus we are in \( A \) in this case.

For the region \( A \) we trivially have (9).

For the region \( B \) we again prove (9). The change of variables \( \eta := \xi^3 - \tau_1 + (\xi - \xi_1)^2 \)
gives \( d\xi = \frac{d\eta}{3\xi^2 + 2(\xi - \xi_1)} \). Using the definition of \( B \) we thus estimate the l.h.s. of (9) by

\[ \sup_{\tau_1, \xi_1} \frac{1}{(\tau_1 - \xi^2)^b} \left( \int \frac{d\eta}{(\eta)^{2b'}} \right)^{\frac{1}{2}} \leq c , \]

provided \( b \geq 0 \), \( b' > 1/2 \).

For the region \( C \) we use the algebraic inequality

\[ |\sigma_1| + |\sigma_2| \geq |(\tau - \xi^3) - (\tau - \xi^3) - (\tau_2 + \xi^2)| \]

\[ = | - \xi^3 + \xi^2 - \xi^2| = | - \xi^3 + \xi^2 - (\xi - \xi_1)^2| \]

\[ = | - \xi^3 - \xi^2 + 2\xi_1| = |\xi||\xi^2 + \xi - 2\xi_1| \geq \frac{1}{2} |\xi|^3 , \]

which leads to 3 cases, depending on which one of the 3 terms on the l.h.s. is dominant.

If \(|\sigma|\) is dominant, we prove (9). Its l.h.s. is estimated for \( b > 1/2 \) and \( b' > 1/6 \) by

\[ c \sup_{\tau, \xi} |\xi|^{1 - 3b'} \left( \int \frac{d\xi}{(\tau + \xi^2 - 2\xi_1)^{2b}} \right)^{\frac{1}{2}} \leq c \sup_{\xi} |\xi|^{\frac{1}{2} - 3b'} \leq c . \]

If \(|\sigma_1|\) is dominant, we estimate the l.h.s. of (9) for \( b > 1/2 \) and \( b' \geq 0 \) by

\[ c \sup_{\tau_1, \xi_1} \left( \int \frac{d\xi}{(\xi^2 + 2\xi_1 + \tau) - (\xi - \xi_1)^2)^{2b}} \right)^{\frac{1}{2}} \leq c . \]

If \(|\sigma_2|\) is dominant, we estimate the l.h.s. of (9) by (1), Lemma 2.5 for \( b > 1/2 \) and \( b' > 1/6 \) :

\[ c \sup_{\tau_2, \xi_2} \left( \int \frac{d\xi_1}{(\xi^2 + 2\xi_1 + \tau_2 - (\xi_1 + \xi_2)^3)^{2b'}} \right)^{\frac{1}{2}} \leq c . \]
This proves claim 2. Next we prove claim 1. In the regions $|\xi_1| >> |\xi_2|$ or $|\xi_2| >> |\xi_1|$ we can immediately use our considerations above for the regions $A$ and $C$ and get for $\beta' > 1/6$, $\beta > 1/2$:

$$||\partial_x (u_1 u_2)||_{Y^0, -\beta'} \leq c||u_1||_{X^{0, b}}||u_2||_{X^{0, b}},$$

and consequently for $s \geq 0$:

$$||\partial_x (u_1 u_2)||_{Y^s, -\beta'} \leq c||u_1||_{X^{s, b}}||u_2||_{X^{s, b}}.$$

In the region $|\xi_1| \sim |\xi_2|$ we interpolate our estimate 2. with the following estimate of Lemma 3.2:

$$||\partial_x (u_1 u_2)||_{Y^{l, 0}} \leq c||u_1||_{X^{l, b}}||u_2||_{X^{l, b}}, \quad (11)$$

and get for $0 \leq l \leq 1/2$:

$$||\partial_x (u_1 u_2)||_{Y^{l, -\beta', -\beta}} \leq c||u_1||_{X^{l, b}}||u_2||_{X^{l, b}}.$$  

This implies, using $|\xi| \leq |\xi_1|, |\xi_2|$ and $\beta' > \frac{1}{2} - s$, for $0 \leq s \leq \frac{1}{2}$:

$$||\partial_x (u_1 u_2)||_{Y^{s, -\beta'}} \leq c||J^s \partial_x (u_1 u_2)||_{Y^{s, -\beta'}} \leq c||\partial_x (J^s u_1 J^s u_2)||_{Y^{s, -\beta'}} \leq c||u_1||_{X^{s, b}}||u_2||_{X^{s, b}}.$$  

If $s > 1/2$ we estimate similarly for $\beta' = 0$ using $\Pi$.

This completes the proof of Lemma 1.2.

**Lemma 1.3** If $\tilde{u}(\xi_1, t)$ and $\tilde{v}(\xi_2, t)$ are supported in the region $|\xi_2|^2 >> |\xi_1|$, we have the following inequality for $\beta > 1/2$:

$$||uD_x v||_{L^2_{xt}} \leq c||u||_{X^{0, b}}||v||_{Y^{0, b}}. \quad (12)$$

Here $u$ on the l.h.s. can be replaced by $\pi$.

**Proof:** a) If $|\xi_2|$ is bounded, this follows directly from Strichartz’ estimates:

$$||uD_x v||_{L^4_{xt}} \leq c||u||_{L^4_{xt}}||D_x v||_{L^4_{xt}} \leq c||u||_{X^{0, b}}||D_x v||_{Y^{0, b}} \leq c||u||_{X^{0, b}}||v||_{Y^{0, b}}.$$  

b) It suffices to prove for functions $\tilde{u}_0(\xi_1)$ and $\tilde{v}_0(\xi_2)$ with support in $\{|\xi_2|^2 >> |\xi_1|, |\xi_2| >> 1\}$:

$$||e^{it\partial_x^2} u_0 \tilde{u}_0(\xi_1) \tilde{v}_0(\xi_2) D_x v_0||_{L^2_{xt}} \leq c||u_0||_{L^4_{xt}}||v_0||_{L^2_{xt}}.$$  

We have

$$||e^{it\partial_x^2} u_0 \tilde{u}_0(\xi_1) \tilde{v}_0(\xi_2) D_x v_0||_{L^2_{xt}} = \int d\xi dt \int_k d\xi_2 d\eta_2 e^{-it(\xi_1^2 + \xi_2^2 - 2\eta_2)} \xi_2 \eta_2 \tilde{u}_0(\xi_1) \tilde{u}_0(\eta_1) \tilde{v}_0(\xi_2) \tilde{v}_0(\eta_2).$$  

$$= \int d\xi \int_k d\xi_2 d\eta_2 \delta(P(\eta_2)) \xi_2 \eta_2 \tilde{u}_0(\xi_1) \tilde{u}_0(\eta_1) \tilde{v}_0(\xi_2) \tilde{v}_0(\eta_2).$$
Here * denotes integration over $\xi = \xi_1 + \xi_2 = \eta_1 + \eta_2$, $|\xi_2|^2 \gg |\xi_1|$, $|\eta_2|^2 \gg |\eta_1|$, $|\xi_2|, |\eta_2| > 1$, and

$$P(\eta_2) := \eta_1^2 + \eta_2^2 - \xi_1^2 - \xi_2^2 = (\xi - \eta)\eta^2 + \eta_2^3 - (\xi - \xi_2)^2 - \xi_2^3$$

$$= (\eta_2 - \xi_2)(\eta_2^2 + \eta_2(1 + \xi_2) - 2\xi + \xi_2 + \xi_2^2).$$

Now $P(\eta_2)$ has the root $\eta_2 = \xi_2$ and no further real zero, because

$$\eta_2^2 + \eta_2(1 + \xi_2) - 2\xi + \xi_2 + \xi_2^2 = (\eta_2 + \frac{1 + \xi_2}{2})^2 + \frac{3}{4}\xi_2^2 - \frac{3}{2}\xi_2 - 2\xi_1 - \frac{1}{4} > 0$$

for $|\xi_2|^2 \gg |\xi_1|, |\xi_2| \gg 1$. Moreover $P'(\eta_2) = 3\eta_2^2 - 2(\xi - \eta_2)$. Thus for $|\xi_2|^2 \gg |\xi_1|$ we get $|P'(\xi_2)| = 3\xi_2^2 - 2(\xi - \xi_2) = |3\xi_2^2 - 2\xi_1| \geq \xi_2^2$. Now we use the well-known identity $\delta(P(\eta_2)) = \frac{\delta(\eta_2 - \xi_2)}{|P'(\xi_2)|}$ and estimate as follows:

$$\|e^{it\partial_x^2} u_0 e^{-it\partial_x^2} D_x v_0\|_{L_x^2}^2 \leq c \int d\xi \int d\xi_2 d\eta_2 \frac{\delta(\eta_2 - \xi_2)}{\xi_2^2} |\xi_2| \eta_2 |\xi_2 \eta_2| |\xi_2 \eta_0(\xi_1)| \xi_2 \eta_0(\xi_1) \xi_0(\xi_2) \eta_0(\eta_2)| = c \|u_0\|_{L_x^2}^2 \|v_0\|_{L_x^2}^2.$$

Further bilinear and trilinear estimates used in the sequel are the following:

$$\|uv\|_{X^{s,0}} \leq c \|u\|_{X^{s,b}} \|v\|_{Y^{s,b}}$$  \hspace{1cm} (13)

for $s \geq 0$, $b > 1/2$, which follows easily for $s = 0$ by Strichartz’ estimates, and thus for any $s \geq 0$. This holds true for $u$ replaced by $v$ on the l.h.s.

Similarly one gets:

$$\|u^2u\|_{X^{s,0}} \leq c \|u\|_{X^{s,b}}^3$$  \hspace{1cm} (14)

for $s \geq 0$, $b > 1/2$.

Defining

$$\mathcal{F}(u,v)(\xi,\tau) := \int_\sigma d\xi_1 d\tau_1 |\xi_1 - \xi_2|^{\alpha} \tilde{u}(\xi_1,\tau_1) \tilde{v}(\xi_2,\tau_2),$$

where * denotes integration over $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$, we have by [15], Corollary 3.2 for $b > 1/2$:

$$\|D_x^{1/2} \mathcal{I}_x^{1/2}(v_1, v_2)\|_{L_x^{2_t}} \leq c \|v_1\|_{Y^{0,b}} \|v_2\|_{Y^{0,b}}.$$  \hspace{1cm} (15)

Moreover we have

$$\|\mathcal{I}_x^{-1/2}(u_1, u_2)\|_{L_x^{2_t}} \leq c \|u_1\|_{X^{0,b}} \|u_2\|_{X^{0,b}}$$  \hspace{1cm} (16)

for $b > 1/2$, cf. [14], Lemma 4.2 or [3], (the proof of) Lemma 7.1, and especially

$$\|(D_x^{\beta/2} u_1)u_2\|_{L_x^{2_t}} \leq c \|u_1\|_{X^{0,b}} \|u_2\|_{X^{0,b}},$$  \hspace{1cm} (17)

if $|\xi_1| \geq \beta|\xi_2|$ for $\xi_j \in \text{supp} \hat{u}_j$ ($j = 1, 2$), where $\beta > 1$. The last inequality remains true for $u_1$ and/or $u_2$ replaced by $\overline{u}_1$ and/or $\overline{u}_2$ on the l.h.s. Also

$$\|D_x^{\beta/2}(u_1 \overline{u}_2)\|_{L_x^{2_t}} \leq c \|u_1\|_{X^{0,b}} \|u_2\|_{X^{0,b}}$$  \hspace{1cm} (18)
for \( b > 1/2 \) (cf. [2], Lemma 3.2), and
\[
\|v_1v_2\|_{L^2_t} \leq c\|v_1\|_{Y^{-\frac{1}{2},\frac{1}{2}}} \|v_2\|_{Y^{\frac{1}{2},\frac{1}{2}}}
\]  
(cf. [2], Prop. 6.2). Finally
\[
\|\partial_x(v_1v_2)\|_{X^{0,-\frac{1}{2}}} \leq c\|v_1\|_{Y^{-\frac{3}{4},\frac{1}{2}}} \|v_2\|_{Y^{-\frac{1}{4},\frac{1}{2}}}
\]
(20)
if \( \hat{v}_i \) are supported outside \( |\xi| \leq 1 \) and \( \gamma_1 + \gamma_2 > -\frac{3}{4} ; \gamma_1, \gamma_2 > -\frac{1}{5} \) (cf. [4], Lemma 1).

## 2 Local well-posedness

Consider the Cauchy problem (1), (2), (3), where \( \alpha, \beta, \gamma \in \mathbb{R} \). Using the Fourier restriction norm method it is not difficult to prove the following local well-posedness result using the estimates in chapter 1.

**Theorem 2.1** For any \( (u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \) and \( s > 0 \) there exists \( b > 1/2 \) and \( \delta = \delta(\|u_0\|_{H^s}, \|v_0\|_{H^s}) > 0 \), such that (1), (2), (3) has a unique solution satisfying \( (u, v) \in X^{s,b}[0, \delta] \times Y^{s,b}[0, \delta] \) and \( (u, v) \in C^3([0, \delta], H^s(\mathbb{R}) \times H^s(\mathbb{R})) \). This solution depends continuously on the data \( (u_0, v_0) \).

**Proof:** One constructs a fixed point of the mapping \( S = (S_0, S_1) \) induced by the corresponding integral equations:
\[
S_0(u, v) := e^{it\partial_x^2}u_0 - i\int_0^t e^{i(t-s)\partial_x^2}[\alpha u(s)v(s) + \beta |u(s)|^2u(s)] \, ds,
\]
and
\[
S_1(u, v) := e^{-it\partial_x^2}v_0 - \int_0^t e^{-(t-s)\partial_x^2}v(s)\partial_x^3v(s) - \gamma \partial_x(|u(s)|^2) \, ds.
\]
In order to estimate the nonlinear terms we use [13], Lemma 1.1 and Lemma 1.2 which allows to choose \( b' < 1/2 \), if \( s > 0 \). Standard arguments using some of the facts given in the introduction imply the claimed result.

In order to apply the I-operator we also need a variant of this local result. Applying the I-operator to the system (1), (2), (3) we get
\[
iIu_t + Iu_{xx} = \alpha I(uv) + \beta I(|u|^2u), \tag{21}
\]
\[
Iv_t + Iv_{xx} + Iv_{xxx} = \gamma I(|u|^2)_x, \tag{22}
\]
\[
Iu(0) = Iu_0, \quad Iv(0) = Iv_0 \tag{23}
\]

**Proposition 2.1** For any \( (u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \) and \( s \geq 1/3 \) there exists \( \delta \leq 1 \) and \( \delta \sim (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^{-\frac{1}{3}} \), if \( \beta \neq 0 \), and \( \delta \sim (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^{-\frac{3}{4}} \), if \( \beta = 0 \), such that system (1), (2), (3) has a unique local solution in the time interval \([0, \delta]\) with the property (dropping from now on \([0, \delta]\) from the notation):
\[
\|Iu\|_{X^{1,b}} + \|Iv\|_{Y^{1,b}} \leq c(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}),
\]
where \( b = \frac{1}{2} + \).

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Proof: We want to construct a fixed point of the mapping $\tilde{S} = (\tilde{S}_0, \tilde{S}_1)$ induced by the integral equations belonging to the system \([21], [22], [23]\):

\[
\begin{align*}
\tilde{S}_0(Iu, Iv) &:= e^{i\delta t^2} Iu \, 0 - i \int_0^t e^{i(t-s)\delta^2} [\alpha I(u(s)v(s)) + \beta I(|u(s)|^2 u(s))] \, ds, \\
\tilde{S}_1(Iu, Iv) &:= e^{-i\delta^2 t} Iv \, 0 - \int_0^t e^{-(t-s)\delta^2} [I(v(s)\partial_x v(s)) - \gamma I(\partial_x (|u(s)|^2))] \, ds.
\end{align*}
\]

The estimates for the nonlinearities in the previous proof carry over to corresponding estimates including the $I$-operator by the interpolation lemma of \([3]\), namely:

\[
\begin{align*}
\|I(uv)\|_{X^{1,0}} &\leq c \|Iu\|_{X^{1,0}} \|Iv\|_{Y^{1,0}}, \\
\|I(|u|^2 u)\|_{X^{1,0}} &\leq c \|Iu\|_{Y^{1,0}}, \\
\|I(v\partial_x v)\|_{Y^{1,0}} &\leq c \|Iv\|_{Y^{1,0}}^{b'} \text{ for } b' \geq \max(1 - \frac{s}{3}, 0), \text{ especially for } b' \geq \frac{1}{6}, \\
\|I\partial_x (|u|^2)\|_{Y^{1,0}} &\leq c \|Iu\|_{Y^{1,0}}^{b'} \text{ for } b' > \max(\frac{1}{6}, \frac{1}{2} - s) = \frac{1}{6}, \text{ if } s \geq \frac{1}{3}.
\end{align*}
\]

This implies

\[
\begin{align*}
\|\tilde{S}_0(Iu, Iv)\|_{X^{1,0}} &\leq c \|Iu_0\|_{H^1} + c\alpha \|Iu\|_{X^{1,0}} \|Iv\|_{Y^{1,0}} \delta^{\frac{5}{6}} + c\beta \|Iu\|_{Y^{1,0}}^{\delta^{\frac{5}{6}}}, \\
\|\tilde{S}_1(Iu, Iv)\|_{Y^{1,0}} &\leq c \|Iv_0\|_{H^1} + c \|Iv\|_{Y^{1,0}} \delta^{\frac{5}{6}} + c\gamma \|Iu\|_{X^{1,0}} \delta^{\frac{5}{6}}.
\end{align*}
\]

This gives the desired bounds, provided $c\beta \delta^{\frac{5}{6}} (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^2 < 1$ and $c\delta^{\frac{5}{6}} (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}) < 1$. Thus our claimed choice of $\delta$ is possible.

### 3 Conserved and almost conserved quantities

Our system \([11], [22], [3]\) has the following conserved quantities (cf. \([22]\)):

\[
\begin{align*}
M &:= \|u\|, \\
L(u, v) &:= \alpha \|v\|^2 + 2\gamma \int \text{Im}(u\overline{v}) \, dx, \\
E(u, v) &:= \alpha \gamma \int |v|u|^2 \, dx + \gamma \|u_x\|^2 + \frac{\alpha}{2} \|v_x\|^2 - \frac{\alpha}{6} \int |v|^3 \, dx + \frac{\beta \gamma}{2} \int |u|^4 \, dx.
\end{align*}
\]

Assume from now on $\alpha \gamma > 0$.

These conservation laws imply a-priori bounds of the $H^1$ - norm of the solutions $u$ and $v$ as follows: concerning $L$ we immediately get

\[
|L(u, v)| \leq c(\|v\|^2 + M \|u_x\|)
\]

and

\[
\|v\|^2 \leq c(\|L\| + M \|u_x\|).
\]

Concerning $E$ we have by Gagliardo-Nirenberg

\[
\int |v|^3 \, dx \leq c \|v\|^\frac{5}{2} \|v_x\|^\frac{3}{2} \leq c(\|L\|^\frac{3}{2} \|v_x\|^\frac{3}{2} + M \|u_x\|^\frac{3}{2} \|v_x\|^\frac{3}{2}) \leq c M \|v\|^\frac{3}{2} + c M \|u_x\|^\frac{3}{2} \leq c M \|L\|^\frac{3}{2} + (\|v_x\|^2 + \|u_x\|^2) + c M^S
\]

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and
\[
\int |u|^4 \, dx \leq \|u\|^3 \|u_x\| = M^3 \|u_x\| \leq \epsilon \|u_x\|^2 + cM^6
\]
as well as
\[
\int |vu|^2 \, dx \leq \|v\| \|u\|^2 \|u_x\| \leq c(\|L\|^\frac{5}{3} \|u\|^2 \|u_x\| + M^2 \|u_x\|)
\]
\[
\leq c(\|L\|^\frac{5}{3} M^2 + M^4) + \epsilon \|u_x\|^2 \leq c(\|L\|^\frac{5}{3} + \frac{10M^4}{3} + M^4) + \epsilon \|u_x\|^2.
\]
This implies
\[
|\gamma\| \|u_x\|^2 + \frac{\alpha}{2} \|v_x\|^2 \leq |E| + c(\|L\|^\frac{5}{3} + M^8 + 1) + \epsilon (\|u_x\|^2 + \|v_x\|^2)
\]
and therefore
\[
\|u_x\|^2 + \|v_x\|^2 \leq c(|E| + \|L\|^\frac{5}{3} + M^8 + 1).
\]
(29)
Similarly we also get
\[
|E| \leq c(\|u_x\|^2 + \|v_x\|^2 + \|L\|^\frac{5}{3} + M^8 + 1).
\]
(30)
The bounds (27) and (30) imply
\[
|E| \leq c(\|u_x\|^2 + \|v_x\|^2 + \|v\|^\frac{10}{3} + \frac{5}{3} \|u_x\|^\frac{5}{3} + M^8 + 1)
\]
\[
\leq c(\|u_x\|^2 + \|v_x\|^2 + \|v\|^\frac{4}{3} + M^{10} + 1).
\]
(31)
Finally, by (28) and (29) we also have
\[
\|v\|^2 \leq c(\|L\| + M(\|E\|^\frac{1}{3} + \|L\|^\frac{5}{3} + M^4 + 1)) \leq c(\|L\| + M|E|^\frac{1}{3} + M^6 + 1)
\]
(32)
and thus
\[
\|u\|^2_{H^1} + \|v\|^2_{H^1} \leq c(|E| + \|L\|^\frac{5}{3} + M^8 + 1).
\]
(33)
These estimates imply an a-priori bound for the $H^1$ - norms of $u$ and $v$ for any data with finite energy $E$, finite $L$ and finite $\|u_0\|$. This is the case for $H^1$ - data $u_0$ and $v_0$. Thus we have from our local result (Theorem 2.1):

**Theorem 3.1** For data $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and $\alpha \gamma > 0$ there exists $b > 1/2$ such that (1), (2), 3 has a unique global solution $(u, v) \in X^{1,b} \times Y^{1,b}$ with $(u, v) \in C^0(\mathbb{R}^+, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$.

A crucial role is played by the modified functionals $E(Iu, Iv)$ and $L(Iu, Iv)$, which are "almost" conserved, i.e. their growth in time is controllable. Using the modified system (21), (22), (23), an elementary calculation shows
\[
\frac{d}{dt} E(Iu, Iv)
\]
\[
= \alpha \left[ \int (I(vv_x) - Iv Iv_x v_{xx}) \, dx + \frac{1}{2} \int (Iv)^2 (I(vv_x) - Iv Iv_x) \, dx \right]
\]
\[
+ 2 \beta \gamma Im \int (I(|u|^2 u)_x - ((Iu)^2 I\pi)_x) I\pi_x \, dx
\]
We have to show

$$+\alpha\gamma\left[\int |Iu|^2(IvIv_x - I(uv_x))dx + \int (|Iu|^2 - I(|u|^2))IvIv_x dx \right. \\
+ \int I_{vxx}(|Iu|^2_x - I(|u|^2)_x)dx - 2Im \int I_{u}(IvIv_x - (\overline{Iv})_x)dx \right]$$

(34)

$$\quad + \alpha^2 \int (|Iu|^2)_x - (|Iu|^2)_x |Iu|^2 dx + 2\alpha^2 \gamma Im \int (Iv-I\mu v)IvIu dx$$

$$\quad + 2\beta^2 \gamma Im \int (Iv-I\mu v)^2(|Iu|^2 - (\overline{Iu})^2(Iv-I\mu v))dx$$

$$\quad - 2\alpha \beta \gamma Im \int IvIu(I(|u|^2 - Iu(I\mu v)^2)dx + Im \int (Iu)^2 I\mu (Iv-I\mu v)dx$$

$$=: \sum_{j=1}^{12} I_j.$$ 

Similarly

$$\frac{d}{dt} L(Iu, Iv) = 2\alpha \int Iv(IvIv_x - I(uv_x))dx$$

$$+ 2\alpha \gamma \left[ \int Iv(I(|u|^2)_x - (|Iu|^2)_x)dx + 2Re \int I\mu_x(IvIu - I(uv))dx \right]$$

$$+ 2\beta \gamma Re \int ((Iu)^2 I\mu - I(u^2 I\mu))I\mu_x dx$$

(35)

$$=: \sum_{j=1}^{4} J_j.$$ 

4 Estimates for the modified energy functional

We need exact control of the increment of the modified energy.

Proposition 4.1 If \((u, v)\) is a solution of (1), (2), (3) on \([0, \delta]\) in the sense of Proposition 2.1 then the following estimate holds for \(N \geq 1\) and \(s > 1/2\) :

$$|E(Iu(\delta), Iv(\delta)) - E(Iu(0), Iv(0))|$$

$$\leq c(\langle N^{-1+\delta^{s-\frac{1}{2}}} + N^{-\frac{s}{2}} \rangle^3 |Iu|_4^3 + |Iv|_4^3_{\alpha+1} + |Iv|_4^3_{\beta+1})$$

$$+ N^{-2+\langle |Iu|_4^4 + |Iv|_4^4_{\alpha+1} + N^{-3+\langle |Iu|_4^4 + |Iv|_4^4_{\alpha+1} + |Iv|_4^4_{\beta+1} + |Iv|_4^4_{\beta+1}}. \right]$$

Proof: Integrating (34) over \(t \in [0, \delta]\) we have to estimate the various terms on the r.h.s. Here and in the sequel we assume w.l.o.g. the Fourier transforms of all the functions to be nonnegative and ignore the appearance of complex conjugates if this is irrelevant for the argument. We use dyadic decompositions w.r. to the frequencies \(|\xi_j| \sim N_j = 2^k (k = 0, 1, 2, ...)\) in many places, so that we need extra factors \(N_j^{\alpha-}\) everywhere in order to sum the dyadic pieces.

Estimate of \(I_1\): We have to show

$$\int_0^\delta \int_s |m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)| |\xi_1 + \xi_2| |\mu|^{\xi_1}(\xi_1, t)|\xi_2| |\mu|^{\xi_2}(\xi_2, t)|\xi_3| |\mu|^{\xi_3}(\xi_3, t)dt$$

$$\leq c(\langle N^{-1+\delta^{s-\frac{1}{2}}} + N^{-\frac{s}{2}} \rangle^3 \prod_{i=1}^{3} |v_i|_y^1_{\alpha+1}.$$ 

(36)
Thus the integral is bounded by use of (15) and (4):

$$
\left| \frac{m(\xi_1 + \xi_2) - m(\xi_2)}{m(\xi_2)} \right| \leq c \frac{\|v\|}{\|\xi\|} \leq c \frac{N_1}{N_2}.
$$

Thus the integral is bounded by use of (15) and (4):

$$
\int_0^\delta \left| \xi_1 + \xi_2 \right| \left| \xi_1 - \xi_2 \right| \frac{\hat{\nu}_1(\xi_1, t)}{\xi_2} \hat{\nu}_2(\xi_2, t) \hat{\nu}_3(\xi_3, t) \right| d\xi dt
\leq c \frac{N_1}{N_2} \left\| D_\xi \hat{\nu}_1(\xi_1, t) \right\|_{L^2_t} \left\| D_\xi \hat{\nu}_2(\xi_2, t) \right\|_{L^2_t} \left\| D_\xi \hat{\nu}_3(\xi_3, t) \right\|_{L^2_t}
\leq c \frac{N_1}{N_2} \frac{1}{\delta} \sum_{i=1}^{3} \|v_i\|_{L^1_y} \leq c N^{-1} N_{\max}^{0-} \frac{1}{\delta} \sum_{i=1}^{3} \|v_i\|_{L^1_y}.
$$

Case 2: \(|\xi_1| < |\xi_2|, |\xi_1|, |\xi_2| \geq N\).
Here we avoid dyadic decompositions and estimate the multiplier by

$$
\frac{c}{m(\xi_1)} \leq c \left( \frac{N_1}{N_2} \right)^{\frac{1}{2}}.
$$

Similarly as in case 1 we control the integral by

$$
N^{-\frac{1}{2}} \int_0^\delta \left| \xi_1 + \xi_2 \right| \left| \xi_1 - \xi_2 \right| \frac{\hat{\nu}_1(\xi_1, t)}{\xi_2} \hat{\nu}_2(\xi_2, t) \hat{\nu}_3(\xi_3, t) \right| d\xi dt
\leq c N^{-\frac{1}{2}} \left\| D_\xi \hat{\nu}_1 \right\|_{L^2_y} \left\| D_\xi \hat{\nu}_2 \right\|_{L^2_y} \left\| D_\xi \hat{\nu}_3 \right\|_{L^2_y} \leq c N^{-1} \frac{1}{\delta} \sum_{i=1}^{3} \|v_i\|_{L^1_y}.
$$

Similarly we treat the case \(|\xi_1| > |\xi_2|\), so that it remains to consider

Case 3: \(|\xi_1| \sim |\xi_2| \geq N\).

The multiplier is bounded by \(c N^{-1}\) and thus we get using (20):

$$
c N^{-1} \left\| D_\xi (v_1 D_\xi v_2) \right\|_{L^2_y} \leq c N^{-1} \frac{1}{\delta} \sum_{i=1}^{3} \|v_i\|_{L^1_y} \leq c N^{-\frac{1}{2}} N_{\max}^{0-} \sum_{i=1}^{3} \|v_i\|_{L^1_y}.
$$

Estimate of \(I_2\): It is sufficient to show

$$
\int_0^\delta \left| \frac{m(\xi_3 + \xi_4) - m(\xi_3) m(\xi_4)}{m(\xi_3) m(\xi_4)} \right| \hat{\nu}_1(\xi_1, t) \hat{\nu}_2(\xi_2, t) \hat{\nu}_3(\xi_3, t) \right| d\xi dt
\leq c N^{-2+} \sum_{i=1}^{4} \|v_i\|_{L^1_y}.
$$

At least two of the \(N_i\) are \(\geq c N\). Assume w.l.o.g. \(N_1 \geq N_2 \geq N_3\) such that \(N_1 \geq c N\). Then we get by use of (19) the bound

$$
c N_{\max}^{-1} \left\| v_1 v_2 \right\|_{L^2_y} \left\| v_3 D_\xi v_4 \right\|_{L^2_t}.
$$
\[
\leq cN_{\max}N^{-1}\|v_1\|_{Y^{-\frac{2}{3},+}}\|v_2\|_{Y^{\frac{2}{3},+}}\|v_3\|_{Y^{\frac{2}{3},+}}\|D_xv_4\|_{Y^{-\frac{2}{3},+}}
\]

\[
\leq cN_{\max}N^{-1}N_1^{-\frac{3}{2}}\langle N_2 \rangle^{-\frac{3}{2}}\langle N_3 \rangle^{-\frac{3}{2}}\langle N_4 \rangle^{-\frac{3}{2}}\prod_{i=1}^{4}\|v_i\|_{Y^{-1,\frac{1}{3},+}}
\]

\[
\leq cN^{-2}+N_{\max}^{0-}\prod_{i=1}^{4}\|v_i\|_{Y^{-1,\frac{1}{3},+}}.
\]

**Estimate of \( I_3 \):** We have to show

\[
\int_{\delta}^{\delta} \int_{\sigma} m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3) \left| \prod_{i=1}^{4}\frac{\|v_i\|_{Y^{-1,\frac{1}{3},+}}}{m(\xi_1)} \right| \times 
\]

\[
\times \tilde{u}_1(\xi_1,t)\tilde{u}_2(\xi_2,t)\tilde{u}_3(\xi_3,t)\xi_1^2\tilde{u}_4(\xi_4,t)d\xi dt \leq cN^{-2}+\prod_{i=1}^{4}\|u_i\|_{X^{1,\frac{1}{3},+}}.
\]

**Case 1:** \(|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4| \geq cN\).
We get the bound by use of Strichartz:

\[
c(N_1/N)^{\frac{3}{2}}\|u_1\|_{L^4_{t,x}}\|u_2\|_{L^4_{t,x}}\|u_3\|_{L^4_{t,x}}\|D_x^2u_4\|_{L^4_{t,x}}
\]

\[
\leq c(N_1/N)^{\frac{3}{2}}(N_1N_2N_3)^{-1}N_4\prod_{i=1}^{4}\|u_i\|_{X^{1,\frac{1}{3},+}}
\]

\[
\leq cN^{-2}+N_{\max}^{0-}\prod_{i=1}^{4}\|u_i\|_{X^{1,\frac{1}{3},+}}.
\]

**Case 2:** Two of the frequencies are \( \geq cN \), the most difficult case is \( |\xi_4| \geq cN \) and, say, \(|\xi_1| \sim |\xi_4| \), \(|\xi_2|,|\xi_3|<<|\xi_1|,|\xi_4|\).

a. \(|\xi_2| \geq N \) (or similarly \(|\xi_3| \geq N \)).
In this case the multiplier is bounded by \( c\langle (N_3/N)^{\frac{3}{2}} \rangle \langle (N_2/N)^{\frac{3}{2}} \rangle \), and using (17) we get the bound

\[
c\langle (N_3/N)^{\frac{3}{2}} \rangle \langle (N_2/N)^{\frac{3}{2}} \rangle\|u_1u_2\|_{L^2_{t,x}}\|u_3D_x^2u_4\|_{L^2_{t,x}}
\]

\[
\leq c\langle (N_3/N)^{\frac{3}{2}} \rangle \langle (N_2/N)^{\frac{3}{2}} \rangle\|D_x^{\frac{1}{3}}u_1\|_{X^{0,\frac{1}{3},+}}\|u_2\|_{X^{0,\frac{1}{3},+}}\|u_3\|_{X^{0,\frac{1}{3},+}}\|D_x^2u_4\|_{X^{0,\frac{1}{3},+}}
\]

\[
\leq c\langle (N_3/N)^{\frac{3}{2}} \rangle \langle (N_2/N)^{\frac{3}{2}} \rangle N_1^{-\frac{3}{2}}N_2^{-1}\langle N_3 \rangle^{-1}N_4\prod_{i=1}^{4}\|u_i\|_{X^{1,\frac{1}{3},+}}
\]

\[
\leq cN^{-2}+N_{\max}^{0-}\prod_{i=1}^{4}\|u_i\|_{X^{1,\frac{1}{3},+}}.
\]

b. \(|\xi_2|,|\xi_3| \leq N\).
By use of the mean value theorem the multiplier is bounded by

\[
\left| \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)}{m(\xi_1)} \right| = \left| \frac{(\nabla m)(\xi_1) \cdot (\xi_2 + \xi_3)}{m(\xi_1)} \right| \leq \frac{N_2 + N_3}{N_1}
\]

and exactly as in case a the claimed estimate follows.

**Case 3:** Three of the frequencies are equivalent, say \(|\xi_1| \sim |\xi_2| \sim |\xi_4| \geq cN\), \(|\xi_3|<<|\xi_1|,|\xi_2|,|\xi_4|\). Then two of the large frequencies have different sign, say
\( \xi_1 \) and \( \xi_2 \), so that \( |\xi_1 - \xi_2|^{1/2} \sim |\xi_1|^{1/2} \) (the other cases are treated similarly). We get the bound, using (10) and (17):

\[
c(N_1/N)^{\frac{1}{2}}(N_2/N)^{\frac{1}{2}} \langle (N_3/N)^{\frac{1}{2}} \rangle \|u_1 u_2\|_{L^2_{xt}} \|u_3 D_x^2 u_4\|_{L^2_{xt}} \\
\leq c(N_1/N)^{\frac{1}{2}}(N_2/N)^{\frac{1}{2}} \|D_x^{-\frac{3}{2}} u_1\|_{L^2_{xt}} \|D_x^2 u_4\|_{L^2_{xt}} \\
\leq c(N_1/N)^{\frac{1}{2}}(N_2/N)^{\frac{1}{2}} \langle (N_3/N)^{\frac{1}{2}} \rangle \times \\
\quad \times \|D_x^{-\frac{3}{2}} u_1\|_{X^0, \frac{1}{4}} \|u_2\|_{X^0, \frac{1}{4}} \|u_3\|_{X^0, \frac{1}{4}} \|D_x^2 u_4\|_{X^0, \frac{1}{4}} \\
\leq c(N_1/N)^{\frac{1}{2}}(N_2/N)^{\frac{1}{2}} \langle (N_3/N)^{\frac{1}{2}} \rangle N_1^{-\frac{3}{2}} N_2^{-1} \langle N_3 \rangle^{-1} N_4^{\frac{1}{4}} \prod_{i=1}^{4} \|u_i\|_{X^1, \frac{1}{4}} \\
\leq cN^{-2+} N_{max}^0 \prod_{i=1}^{4} \|u_i\|_{X^1, \frac{1}{4}} .
\]

Estimate of I_4: We have to show

\[
\int_0^\delta \left[ \int_\ast \left| m(\xi_1 + \xi_2) - m(\xi_1) m(\xi_2) \right| \tilde{v}_1(\xi_1, t) \langle t \rangle \tilde{v}_2(\xi_2, t) \tilde{v}_3(\xi_3, t) \tilde{v}_4(\xi_4, t) d\xi dt \right] \\
\leq cN^{-2+} \|u_1\|_{X^1, \frac{1}{4}} \|u_2\|_{X^1, \frac{1}{4}} \|v_1\|_{Y^1, \frac{1}{4}} \|v_2\|_{Y^1, \frac{1}{4}} .
\]

We bound the multiplier by \( cN_{max} N^{-1} \). We have \( |\xi_1| \geq N \) or \( |\xi_2| \geq N \). Assume the more difficult case \( |\xi_2| \geq N \).

Case 1: Exactly two of the \( N_i \) are \( \geq cN \), thus w.l.o.g. \( N_3 \ll N_2 \). Using Lemma L3 and Strichartz we get the bound

\[
c N_{max} N^{-1} \|D_x v_2\|_{L^4_{xt}} \|v_1\|_{L^4_{xt}} \|u_4\|_{L^4_{xt}} \\
\leq c N_{max} N^{-1} \|v_2\|_{Y^1, \frac{1}{4}} \|u_3\|_{X^0, \frac{1}{4}} \|v_1\|_{Y^1, \frac{1}{4}} \|u_4\|_{X^0, \frac{1}{4}} \\
\leq c N_{max} N^{-1} (N_1 N_2 N_3 N_4)^{-1} \prod_{i=3}^{4} \|u_i\|_{X^1, \frac{1}{4}} \|v_i\|_{Y^1, \frac{1}{4}} \\
\leq c N^{-2+} N_{max}^0 \prod_{i=3}^{4} \|u_i\|_{X^1, \frac{1}{4}} \|v_i\|_{Y^1, \frac{1}{4}} .
\]

Case 2: At least three of the \( N_i \) are \( \geq cN \).

In this case Strichartz directly gives the bound

\[
c \left( \frac{N_{max}}{N} \right)^{1-\epsilon} \|v_1\|_{L^4_{xt}} \|D_x v_2\|_{L^4_{xt}} \|u_3\|_{L^4_{xt}} \|u_4\|_{L^4_{xt}} \\
\leq c \left( \frac{N_{max}}{N} \right)^{1-\epsilon} (N_1 N_2 N_4)^{-1} \prod_{i=3}^{4} \|u_i\|_{X^1, \frac{1}{4}} \|v_i\|_{Y^1, \frac{1}{4}} \\
\leq c N^{-2+} N_{max}^0 \prod_{i=1}^{4} \|u_i\|_{X^1, \frac{1}{4}} \|v_i\|_{Y^1, \frac{1}{4}} .
\]

Estimate of I_5: We have to show

\[
\int_0^\delta \left[ \int_\ast \left| m(\xi_1 + \xi_2) - m(\xi_1) m(\xi_2) \right| \tilde{v}_1(\xi_1, t) \langle t \rangle \tilde{v}_2(\xi_2, t) \tilde{v}_3(\xi_3, t) \tilde{v}_4(\xi_4, t) d\xi dt \right] \\
\leq cN^{-2+} \|u_1\|_{X^1, \frac{1}{4}} \|u_2\|_{X^1, \frac{1}{4}} \|v_1\|_{Y^1, \frac{1}{4}} \|v_2\|_{Y^1, \frac{1}{4}} .
\]
This can be shown similarly as the previous case.

**Estimate of $I_6$:** We want to show

\[
\int_0^\delta \int_* \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| |\xi_1 + \xi_2| \eta(\xi_1, t) \overline{u_2}(\xi_2, t) \xi_3^2 \overline{v_3}(\xi_3, t) d\xi dt \\
\leq c N^{-1 + \frac{1}{2}} \delta \frac{1}{2} \left| u_1 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| u_2 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| v_3 \right|_{Y^\frac{1}{2}, Y^\frac{1}{2}}.
\]

**Case 1:** $|\xi_3| \sim |\xi_1| \geq c N$, $|\xi_2| << |\xi_1|, |\xi_3|$ (thus $|\xi_1 + \xi_2| \sim |\xi_1|$).

The multiplier is bounded by $c |\frac{(\nabla m)(\xi_1, \xi_2)}{m(\xi_1)}| \leq c \frac{N}{N^2}$ . This implies by Lemma 1.3 the bound

\[
c N^2 N^{-1}\left| D_x u_1 \right|_{L^2_t} \left| u_2 D^2 v_3 \right|_{L^2_t} \\
\leq c N^2 N^{-1}\left| D_x u_1 \right|_{L^2_t} \left| u_2 \right|_{X^\frac{1}{2}, Y^\frac{1}{2} + N^2} \left| v_3 \right|_{Y^\frac{1}{2}, Y^\frac{1}{2}} \\
\leq c N^{-1 + N^2} \delta \frac{1}{2} \left| u_1 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| u_2 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| v_3 \right|_{Y^\frac{1}{2}, Y^\frac{1}{2}}.
\]

**Case 2:** $|\xi_1| \sim |\xi_2| \geq c N$, $|\xi_3| >> |\xi_2|$ , thus $|\xi_2 + \xi_1| \leq c |\xi_1|$ .

We get the bound by Lemma 1.3

\[
c \left( \frac{N}{N^2} \right)^{1 - \epsilon} \left| D_x u_1 \right|_{L^2_t} \left| u_2 D^2 v_3 \right|_{L^2_t} \\
\leq c \left( \frac{N}{N^2} \right)^{1 - \epsilon} \left| D_x u_1 \right|_{L^2_t} \left| u_2 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| D_x v_3 \right|_{Y^\frac{1}{2}, Y^\frac{1}{2}} \\
\leq c \left( \frac{N}{N^2} \right)^{1 - \epsilon} \delta \frac{1}{2} \left| u_1 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| u_2 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| v_3 \right|_{Y^\frac{1}{2}, Y^\frac{1}{2}} \\
\leq c N^{-1 + N^2} \delta \frac{1}{2} \left| u_1 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| u_2 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| v_3 \right|_{Y^\frac{1}{2}, Y^\frac{1}{2}}.
\]

**Case 3:** $|\xi_1| \sim |\xi_2| \geq c N$, $|\xi_3|^2 \leq c |\xi_1|^2 \sim c |\xi_2|^2$ .

Using $|\xi_1 + \xi_2| |\xi_3|^2 \leq c |\xi_1 + \xi_2|^2 |\xi_1| |\xi_3|$ and the multiplier bound $c \left( \frac{N}{N^2} \right)^{1 - \epsilon}$ we estimate the integral by use of Lemma 1.3:

\[
c \left( \frac{N}{N^2} \right)^{1 - \epsilon} \left| D_x \left( D_x u_1 \overline{u_2} \right) \right|_{L^2_t} \left| D_x v_3 \right|_{L^2_t} \\
\leq c \left( \frac{N}{N^2} \right)^{1 - \epsilon} \left| D_x u_1 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| u_2 \right|_{X^\frac{1}{2}, Y^\frac{1}{2}} \left| D_x v_3 \right|_{L^2_t}
\]
\[
\leq \frac{c(N^2)^{1-\epsilon}N^{-\frac{1}{2}}}{N}\|u_1\|_{X^1,\frac{1}{8}+} + \|u_2\|_{X^1,\frac{1}{8}+} + \|v_3\|_{Y^1,\frac{1}{4}+}
\leq cN^{-1} + \frac{N^{0-\frac{1}{2}}}{\max} - \|u_1\|_{X^1,\frac{1}{8}+} + \|u_2\|_{X^1,\frac{1}{8}+} + \|v_3\|_{Y^1,\frac{1}{4}+}.
\]

Estimate of $I_7$: We show
\[
\int_0^\delta \int_s \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| \left| \xi_1 + \xi_2|\hat{u}_1(\xi_1,t)\hat{u}_2(\xi_2,t)\right|\hat{u}_3(\xi_3,t) d\xi dt
\leq cN^{-1} \frac{\delta^{\frac{1}{2}}}{N}\|u_1\|_{X^1,\frac{1}{8}+} + \|v_2\|_{Y^1,\frac{1}{4}+} + \|u_3\|_{X^1,\frac{1}{8}+}.
\]

Case 1: $|\xi_1| > |\xi_2| \geq cN$.
Using no dyadic decomposition and the multiplier bound $c\left(\frac{|\xi_1|}{N}\right)^{\frac{1}{2}}$ we get the bound by Strichartz:
\[
cN^{-1} \frac{\delta^{\frac{1}{2}}}{N}\|u_1\|_{X^1,\frac{1}{8}+} + \|v_2\|_{Y^1,\frac{1}{4}+} + \|u_3\|_{X^1,\frac{1}{8}+}.
\]

Case 2: $|\xi_1| \sim |\xi_2| \geq cN$.
Again using no dyadic decomposition and the multiplier bound $c\left(\frac{|\xi_1|}{N}\right)^{\frac{1}{2}}$ we get similarly as in case 1 the bound
\[
cN^{-1} \|D_x u_1\|_{L^4_t} \|D_x v_2\|_{L^4_t} \|D_x u_3\|_{L^2_t} \leq cN^{-1} \frac{\delta^{\frac{1}{2}}}{N}\|u_1\|_{X^1,\frac{1}{8}+} + \|v_2\|_{Y^1,\frac{1}{4}+} + \|u_3\|_{X^1,\frac{1}{8}+}.
\]

Case 3: $|\xi_1| \geq cN, |\xi_2| \leq N$, thus $\|\xi_1\| \sim N_{max}$.
We bound the multiplier by $c\frac{N_2}{N_1}$ and get by Strichartz an integral bound
\[
cN^{-1} \frac{N_2}{N_1} \|D_x u_1\|_{L^4_t} \|D_x v_2\|_{L^4_t} \|D_x u_3\|_{L^2_t} \leq cN^{-1} \frac{\delta^{\frac{1}{2}}}{N}\|u_1\|_{X^1,\frac{1}{8}+} + \|v_2\|_{Y^1,\frac{1}{4}+} + \|u_3\|_{X^1,\frac{1}{8}+}.
\]

The remaining cases are handled similarly by exchanging the roles of $\xi_1$ and $\xi_2$.

Estimate of $I_8$: We want to show
\[
\int_0^\delta \int_s \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| \left| \xi_1 + \xi_2\prod_{i=1}^4 \hat{a}_i(\xi_i,t)\right| d\xi dt \leq cN^{-2+} \|u_i\|_{X^1,\frac{1}{8}+}.
\]

Case 1: At least three of the $|\xi_i|$ are $\geq cN, |\xi_1| \geq |\xi_2|$ w.l.o.g.
Estimate the multiplier by $cN_{max}N^{-1}$ and use Strichartz to control the integral by
\[
cN_{max}N^{-1} \frac{4}{i=2} \left\| u_i \right\|_{L^4_t} \|D_x u_1\|_{L^4_t} \leq cN_{max}N^{-1} \frac{4}{i=1} \left\| u_i \right\|_{X^1,\frac{1}{8}+} \leq cN^{-2+} \frac{N^{0-\min}}{\max} \frac{4}{i=1} \left\| u_i \right\|_{X^1,\frac{1}{8}+}.
\]
Case 2: Exactly two of the $|\xi_i|$ are $\geq cN$, the others $<< N$, e.g. $|\xi_1| \sim |\xi_2| \geq cN$ and $|\xi_3|, |\xi_4| << N$.

Estimate the multiplier by $cN_{\text{max}}N^{-1}$ and use (17) to bound the integral by

$$cN_{\text{max}}N^{-1}\left|(D_x u_1)u_2||u_2 u_4||_{L^2_t||u_2 u_4||} \right| \leq cN_{\text{max}}N^{-1}\left|(D_x u_1)u_2||u_2 u_4||_{X^{\frac{3}{4}+}}\right| + \left|D_x^2 u_1||u_3||_{X^{\frac{3}{4}+}} + \left|D_x^2 u_1||u_4||_{X^{\frac{3}{4}+}} \right|

\leq cN_{\text{max}}N^{-1}N_1^{\frac{1}{2}}N_2^{\frac{1}{2}} \prod_{i=1}^4 \left|u_i||X^{\frac{1}{4}+}\right| \leq cN^{-2}N_{\text{max}}^{-1}\prod_{i=1}^4 \left|u_i||X^{\frac{1}{4}+}\right| .

Estimate of $I_9$: We control the integral

$$\int_0^\delta \int_s^t \frac{|m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} \left|\hat{u}_1(\xi_1, t)\hat{u}_2(\xi_2, t)\hat{u}_3(\xi_3, t)\hat{u}_4(\xi_4, t)\right| dt dt$$

by Strichartz, using the fact that at least two of the $|\xi_i|$ are $\geq cN$, by

$$cN_{\text{max}}N^{-1}\left|u_1||L^\infty_t||v_2||L^2_t||u_3||L^2_t||v_4||L^2_t \right| \leq cN_{\text{max}}N^{-1}(N_1N_2N_3N_4)^{-1}\left|u_1||X^{\frac{3}{4}+}||v_2||Y^{\frac{1}{4}+}||u_3||X^{\frac{3}{4}+}||v_4||Y^{\frac{1}{4}+} \right| \leq cN^{-2}N_{\text{max}}^{-1}\left|u_1||X^{\frac{1}{4}+}||v_2||Y^{\frac{1}{4}+}||u_3||X^{\frac{1}{4}+}||v_4||Y^{\frac{1}{4}+} .

Estimate of $I_{10}$: We want to show

$$\int_0^\delta \int_s^t \frac{|m(\xi_4 + \xi_5 + \xi_6) - m(\xi_4)m(\xi_5)m(\xi_6)|}{m(\xi_4)m(\xi_5)m(\xi_6)} \left|\hat{u}_1(\xi_1, t)\hat{u}_2(\xi_2, t)\hat{u}_3(\xi_3, t)\hat{u}_4(\xi_4, t)\right| dt dt \leq cN^{-3}N_{\text{max}}^{-1}\prod_{i=1}^6 \left|u_i||X^{\frac{1}{4}+} \right| .

Case 1: At least three of the $|\xi_i|$ are $\geq cN$.

The multiplier is bounded by $c(N_{\text{max}}^{-\frac{3}{2}})$, so that an application of Strichartz gives an integral bound

$$c(N_{\text{max}}^{-\frac{3}{2}})\prod_{i=1}^6 \left|u_i||L^\infty_t \right| \leq c(N_{\text{max}}^{-\frac{3}{2}})\prod_{i=1}^6 N_i^{-\frac{1}{2}} \prod_{i=1}^6 \left|u_i||X^{\frac{1}{4}+} \right| \leq cN^{-3}N_{\text{max}}^{-1}\prod_{i=1}^6 \left|u_i||X^{\frac{1}{4}+} \right| .

Case 2: Exactly two of the $|\xi_i|$ are $\geq cN$, the others $<< N$, e.g. $|\xi_5|, |\xi_6| \geq cN$.

Then the multiplier is bounded by $c(N_{\text{max}}^{-\frac{3}{2}})$, and the integral, using (17), by

$$c(N_{\text{max}}^{-\frac{3}{2}})\prod_{i=1}^6 \left|u_i||L^\infty_t u_5||L^2_t u_2||L^2_t u_3||L^\infty_t u_4||L^\infty_t \right| \leq c(N_{\text{max}}^{-\frac{3}{2}})\prod_{i=1}^6 N_i^{-\frac{1}{2}} \prod_{i=1}^6 \left|u_i||X^{\frac{1}{4}+} \right| \times \left|D_x^{\frac{3}{4}} u_5||u_2||X^{\frac{1}{4}+} \right| \times \left|D_x^{\frac{3}{4}} u_6||X^{\frac{1}{4}+} \right| \times \prod_{i=1}^6 \left|u_i||X^{\frac{1}{4}+} \right| \leq cN^{-3}N_{\text{max}}^{-1}\prod_{i=1}^6 \left|u_i||X^{\frac{1}{4}+} \right| .
Estimate of $I_{11}$: We have to show
\[
\int_0^\delta \int \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)}{m(\xi_1)m(\xi_2)m(\xi_3)} \prod a_i(\xi_i, t)\bar{v}_5(\xi_5, t) \, d\xi dt 
\leq cN^{-3+} \prod_{i=1}^4 \|u_i\|_{X^{1,1/2}} \|v_5\|_{Y^{1,1/2}}.
\]

Case 1: At least three of the $|\xi_i|$ are $\geq cN$.
In this case we bound the multiplier by $c(N_{\text{max}}^{-1})^{\frac{3}{2}}$ and use Strichartz to control the integral by
\[
c(N_{\text{max}}^{-1})^{\frac{3}{2}} \prod_{i=1}^4 \|u_i\|_{L_x^\infty} \|v_5\|_{L_x^\infty} \leq c(N_{\text{max}}^{-1})^{\frac{3}{2}} \prod_{i=1}^4 \|u_i\|_{X^{1,1/2}} \|v_5\|_{Y^{1,1/2}} 
\leq cN^{-3} N_{\text{max}}^{0-} \prod_{i=1}^4 \|u_i\|_{X^{1,1/2}} \|v_5\|_{Y^{1,1/2}}.
\]

Case 2: Exactly two of the $|\xi_i|$ are $\geq cN$, the others $<< N$.
(a) $|\xi_1|, |\xi_2| \geq cN$.
The multiplier is bounded by $c(N_{\text{max}}^{-1})^{\frac{3}{2}} (N_{\text{max}}^{-1})^{\frac{3}{2}}$, and the integral, using (17), by
\[
c(N_{\text{max}}^{-1})^{\frac{3}{2}} (N_{\text{max}}^{-1})^{\frac{3}{2}} \|u_1 u_3\|_{L_x^2} \|u_2 u_4\|_{L_x^2} \|v_5\|_{L_x^\infty} 
\leq c(N_{\text{max}}^{-1})^{\frac{3}{2}} (N_{\text{max}}^{-1})^{\frac{3}{2}} \|D_x^{-\frac{1}{4}} u_1\|_{X^{0,1/2}} \|u_3\|_{X^{0,1/2}} \|D_x^{-\frac{1}{4}} u_2\|_{X^{0,1/2}} \|u_4\|_{X^{0,1/2}} \|v_5\|_{L_x^{2+}} 
\leq c(N_{\text{max}}^{-1})^{\frac{3}{2}} (N_{\text{max}}^{-1})^{\frac{3}{2}} N_{\text{max}}^{-\frac{3}{2}} \prod_{i=1}^4 \|u_i\|_{X^{1,1/2}} \|v_5\|_{Y^{1,1/2}} 
\leq cN^{-3} N_{\text{max}}^{0-} \prod_{i=1}^4 \|u_i\|_{X^{1,1/2}} \|v_5\|_{Y^{1,1/2}}.
\]
(b) $|\xi_1|, |\xi_5| \geq cN$.
Estimate the multiplier crudely by a constant and use Strichartz and Lemma 13 to bound the integral by
\[
c \prod_{i=1}^3 \|u_i\|_{L_x^2} \|u_4 v_5\|_{L_x^2} \leq c \prod_{i=1}^4 \|u_i\|_{X^{0,1/2}} \|D_x^{-1} v_5\|_{Y^{0,1/2}} 
\leq c N_{\text{max}}^{-1} N_{\text{max}}^{-2} \prod_{i=1}^4 \|u_i\|_{X^{1,1/2}} \|v_5\|_{Y^{1,1/2}} \leq cN^{-3} N_{\text{max}}^{0-} \prod_{i=1}^4 \|u_i\|_{X^{1,1/2}} \|v_5\|_{Y^{1,1/2}}.
\]
The remaining cases are similar.

Estimate of $I_{12}$: similarly as $I_{11}$.

5 Estimates for the modified L - functional

We also need control over the increment of the modified L - functional.
Proposition 5.1 If \((u, v)\) is a solution of \((\mathcal{J}), (\mathcal{Z}), (\mathcal{A})\) on \([0, \delta]\) in the sense of Proposition 2.1, then the following estimate holds for \(N \geq 1\) and \(s > 1/2\):

\[
|L(Iu(\delta), Iv(\delta)) - L(Iu(0), Iv(0))| \\
\leq c \left[ N^{-2+\frac{3}{s}} \left( \|Iu\|_{X^1, \frac{s}{2}+}^3 + \|Iv\|_{Y^1, \frac{s}{2}+}^3 \right) + N^{-3+\frac{4}{s}} \right].
\]

**Proof:** We use (35) and argue similarly as in the previous proposition.

**Estimate of** \(J_1\): We have to show

\[
\int_0^\delta \int_s \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} |\tilde{v}_1(\xi_1, t)| |\tilde{v}_2(\xi_2, t)| \tilde{v}_3(\xi_3, t) d\xi dt \\
\leq cN^{-2+\frac{3}{s}} \prod_{i=1}^3 \|v_i\|_{Y^1, \frac{s}{2}+}.
\]

**Case 1:** \(|\xi_1| < |\xi_2| \sim |\xi_3| \geq cN\).

a. \(|\xi_1| \leq N\).

The multiplier is controlled by \(c\left| \frac{(\nabla m)(\xi_1)\xi_2}{m(\xi_2)} \right| \leq cN^2\) and the integral, using (35), by

\[
cN^2 \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| |\tilde{v}_1(\xi_1, t)| |\tilde{v}_2(\xi_2, t)| \tilde{v}_3(\xi_3, t) d\xi dt \\
\leq cN^2 \left[ N^{-2+\frac{3}{s}} \|v_1\|_{X^1, \frac{s}{2}+}^3 + N^{-3+\frac{4}{s}} \right].
\]

b. \(|\xi_1| \geq N\).

Using the multiplier bound \(c\left| \frac{(\nabla m)(\xi_1)\xi_2}{m(\xi_2)} \right| \) and estimating as in case a we get the same.

Similarly we treat the case \(|\xi_1| > |\xi_2|\).

**Case 2:** \(|\xi_1| \sim |\xi_2| \geq cN\), \(|\xi_3| \ll |\xi_1|, |\xi_2|\).

This gives the bound, using (35):

\[
cN^2 \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| |\tilde{v}_1(\xi_1, t)| |\tilde{v}_2(\xi_2, t)| \tilde{v}_3(\xi_3, t) d\xi dt \\
\leq cN^2 \left[ N^{-2+\frac{3}{s}} \|v_1\|_{X^1, \frac{s}{2}+}^3 + N^{-3+\frac{4}{s}} \right].
\]

**Case 3:** \(|\xi_1| \sim |\xi_2| \sim |\xi_3| \geq cN\).

By Strichartz we get the bound

\[
cN^2 \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| |\tilde{v}_1(\xi_1, t)| |\tilde{v}_2(\xi_2, t)| \tilde{v}_3(\xi_3, t) d\xi dt \\
\leq cN^2 \left[ N^{-2+\frac{3}{s}} \|v_1\|_{X^1, \frac{s}{2}+}^3 + N^{-3+\frac{4}{s}} \right].
\]

**Estimate of** \(J_2\): We want to show

\[
\int_0^\delta \int_s \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} |\xi_1 + \xi_2| \tilde{u}_1(\xi_1, t)|\tilde{u}_2(\xi_2, t)| \tilde{v}_3(\xi_3, t) d\xi dt \\
\leq cN^{-2+\frac{3}{s}} \|u_1\|_{X^1, \frac{s}{2}+} \|u_2\|_{X^1, \frac{s}{2}+} \|v_3\|_{Y^1, \frac{s}{2}+}.
\]
We estimate the multiplier by \( c(\frac{N}{N^2})^{\frac{1}{2}} \) and the rest of the integral using Lemma 3.4 by

\[
\|D_xu_1\|_{L_{xt}^2} \|u_2v_3\|_{L_{xt}^2} \leq c\|D_xu_1\|_{L_{xt}^2} \|u_2\|_{X^0,\frac{1}{2}+} \|D_x^{-1}v_3\|_{Y^0,\frac{1}{2}+} \\
\leq cN^{-\frac{1}{2}}\delta^{-\frac{1}{2}} \|u_1\|_{X^1,\frac{1}{2}+} \|u_2\|_{X^1,\frac{1}{2}+} \|v_3\|_{Y^1,\frac{1}{2}+}.
\]

This gives the desired estimate.

**Case 2:** \( |\xi_1| \sim |\xi_2| \gtrsim cN \).

By Strichartz we get the bound

\[
cN^{-1}\|u_1\|_{L_{xt}^1} \|u_2\|_{L_{xt}^1} \|D_xv_3\|_{L_{xt}^2} \\
\leq cN^{-1}N^{-1}N^{-1}\delta^{-\frac{1}{2}} \|u_1\|_{X^1,\frac{1}{2}+} \|u_2\|_{X^1,\frac{1}{2}+} \|v_3\|_{Y^1,\frac{1}{2}+} \\
\leq cN^{-4}N_{\max}^{-\frac{1}{2}} \|u_1\|_{X^1,\frac{1}{2}+} \|u_2\|_{X^1,\frac{1}{2}+} \|v_3\|_{Y^1,\frac{1}{2}+}.
\]

### Estimate of \( J_3 \):

We have to show

\[
\int_0^\delta \int \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} |\hat{u}_1(\xi_1,t)\hat{v}_2(\xi_2,t)||\xi_3|\hat{u}_3(\xi_3,t) \, d\xi \, dt \\
\leq cN^{-2} \|\hat{u}_1\|_{X^1,-\frac{1}{2}+} \|\hat{v}_2\|_{Y^1,-\frac{1}{2}+} \|\hat{u}_3\|_{Y^1,-\frac{1}{2}+}.
\]

A typical case is \( |\xi_3| \sim |\xi_1| \gtrsim cN \), \( |\xi_2| \ll |\xi_1|, |\xi_3| \). Estimating the multiplier by \( c\frac{N}{N^2} \), if \( |\xi_2| \leq N \), and by \( c(\frac{N}{N^2})^{\frac{1}{2}} \), if \( |\xi_2| \gtrsim cN \), we get by Strichartz the following bound for the rest of the integral

\[
\|\hat{u}_1\|_{L_{xt}^\delta} \|\hat{v}_2\|_{L_{xt}^\delta} \|D_xu_3\|_{L_{xt}^2} \leq N^{-1/2}N^{-1}\delta^{-\frac{1}{2}} \|\hat{u}_1\|_{X^1,\frac{1}{2}+} \|\hat{v}_2\|_{Y^1,\frac{1}{2}+} \|\hat{u}_3\|_{Y^1,\frac{1}{2}+},
\]

which gives the claimed estimate. The other cases are treated similarly.

### Estimate of \( J_4 \):

The desired estimate is

\[
\int_0^\delta \int \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)}{m(\xi_1)m(\xi_2)m(\xi_3)} \prod_{i=1}^4 |\hat{u}_i(\xi_i, t)||\xi_i| \, d\xi \, dt \\
\leq cN^{-3} \prod_{i=1}^4 \|\hat{u}_i\|_{X^1,\frac{1}{2}+}.
\]

**Case 1:** \( N_1, N_2, N_3 \gtrsim cN \).

Estimate the multiplier by \( c(\frac{N_1N_2N_3}{N})^{\frac{1}{2}} \) and the rest of the integral using Strichartz by

\[
\prod_{i=1}^3 \|\hat{u}_i\|_{L_{xt}^\delta} \|D_xu_4\|_{L_{xt}^2} \leq c(N_1N_2N_3)^{-1} \prod_{i=1}^4 \|\hat{u}_i\|_{X^1,\frac{1}{2}+}.
\]

This gives the desired bound.

**Case 2:** \( N_1 \sim N_2 \gtrsim cN \), \( N_3, N_4 \ll N_1, N_2 \).
This gives the bound by (17):
\[
c\left(\frac{N_1}{N}\right)\frac{\frac{1}{2}}{\left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}}} ||u_1 u_3||_{L^2_t} ||u_2 D_x u_4||_{L^2_t} \\
\leq c\left(\frac{N_1}{N}\right)\frac{\frac{1}{2}}{\left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}}} \left(\int D_x^{-\frac{1}{2}} u_1 \right) \left(\int D_x^{-\frac{1}{2}} u_2 \right) \left(\int D_x u_4 \right) \\
\leq c\left(\frac{N_1}{N}\right)\frac{\frac{1}{2}}{\left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}}} \left(\int N_1 N_2 \right) \left(\int N_3 \right)^{-\frac{1}{2}} \prod_{i=1}^4 \left(\int u_i \right) \\
\leq cN^{-3+\frac{N_0-\frac{4}{3}}{N_\text{max}}} \prod_{i=1}^4 \left(\int u_i \right). 
\]

Case 3: \( N_1 \sim N_4 \geq cN \), \( N_2, N_3 << N_1, N_4 \).

a. \( N_2, N_3 \leq N \).
The multiplier is bounded by \( c\left(\frac{\sum_{m}^{n}(\xi_i)}{m(\xi_i)}\right) (\xi_2 + \xi_3) \leq c\frac{N_2+N_3}{N_1} \), and thus by (17) we get the bound
\[
c\left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \left(\frac{N_1}{N}\right)^{-\frac{1}{2}} ||u_1 u_2||_{L^2_t} ||u_3 D_x u_4||_{L^2_t} \\
\leq c\left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \left(\frac{N_1}{N}\right)^{-\frac{1}{2}} \left(\int u_1 \right) \left(\int u_2 \right) \left(\int u_3 \right) \left(\int D_x^\frac{1}{2} u_4 \right) \\
\leq \frac{N_2+N_3}{N_1} N_1^{-\frac{3}{2}} \left(\int N_2 \right)^{-1} \left(\int N_3 \right)^{-1} N_4 \prod_{i=1}^4 \left(\int u_i \right) \\
\leq cN^{-3+\frac{N_0-\frac{4}{3}}{N_\text{max}}} \prod_{i=1}^4 \left(\int u_i \right). 
\]

b. \( N_2 \geq N \) (similarly \( N_3 \geq N \)).
The estimate is similar to case a, but the multiplier bound is \( c\left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \left(\frac{N_1}{N}\right)^{-\frac{1}{2}} \).
This leads to the same bound as in a.

Case 4: \( N_1 \sim N_2 \sim N_4 \geq cN \), \( N_3 \ll N_1, N_2, N_4 \).
Because of \( \sum_{i=1}^{4} \xi_i = 0 \), two of the large frequencies have different sign, say, \( \xi_2 \) and \( \xi_4 \). Thus \( |\xi_1|^{\frac{1}{2}} \leq |\xi_2 - \xi_4|^{\frac{1}{2}} \), and we get the bound for the integral, using (10) and (17):
\[
c\left(\frac{N_1}{N}\right)^{\frac{1}{2}} \left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}} ||u_2 D_x u_4||_{L^2_t} ||u_1 u_3||_{L^2_t} \\
\leq c\left(\frac{N_1}{N}\right)^{\frac{1}{2}} \left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \left(\int u_2 \right) \left(\int D_x^\frac{1}{2} u_4 \right) \left(\int u_1 u_3 \right) \\
\leq c\left(\frac{N_1}{N}\right)^{\frac{1}{2}} \left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \left(\int u_2 \right) \left(\int D_x^\frac{1}{2} u_4 \right) \left(\int u_1 u_3 \right) \\
\leq c\left(\frac{N_1}{N}\right)^{\frac{1}{2}} \left(\frac{N_2}{N}\right)^{\frac{1}{2}} \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \left(\int N_2^{-\frac{1}{2}} N_4 \right)^{-1} \left(\int N_1 \right)^{-\frac{3}{2}} \prod_{i=1}^4 \left(\int u_i \right) \\
\leq cN^{-3+\frac{N_0-\frac{4}{3}}{N_\text{max}}} \prod_{i=1}^4 \left(\int u_i \right). 
\]
6 The global existence result

Theorem 6.1 Let $1 > s > 3/5$, if $\beta = 0$, and $1 > s > 2/3$, if $\beta \neq 0$, and $\alpha \gamma > 0$. The system \((1), (2), (3)\) has a unique global solution for data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$. More precisely, for any $T > 0$ there exists $b > 1/2$ and a unique solution $(u, v) \in X^{sb}[0, T] \times Y^{sb}[0, T]$ with $(u, v) \in C^0([0, T], H^s(\mathbb{R}) \times H^s(\mathbb{R}))$.

Proof: The data satisfy the estimates $\|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 \leq cN^{2(1-s)}$ and $\|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 \leq cN^{1-s}$. These bounds imply by (32) and (33): $|L(Iu_0, Iv_0)| \leq \tau N^{1-s}$ and $|E(Iu_0, Iv_0)| \leq \tau N^{2(1-s)}$, and any such bounds for $L$ and $E$ imply by (32) and (33): $\|Iu_0\|_{L^2}^2 \leq M^2$, $\|Iv_0\|_{L^2}^2 \leq cN^{1-s}$, $\|Iu_0\|_{H^1}^2 + \|Iv_0\|_{H^1}^2 \leq \tilde{c}N^{2(1-s)}$ with $\tilde{c} = \tilde{c}(\tau)$.

We use our local existence theorem on $[0, \delta]$, where $\delta \sim N^{-4(1-s)+}$, if $\beta \neq 0$, and $\delta \sim N^{-3(1-s)+}$, if $\beta = 0$, and conclude

$$\|Iu\|_{X^{\frac{1}{2}}[0, \delta]} + \|Iv\|_{Y^{\frac{1}{2}}[0, \delta]} \leq c_0(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}) \leq c_1N^{1-s},$$

where $c_1 = c_1(\tau, M)$. In order to reapply the local existence theorem with intervals of equal length we need a uniform bound of the $H^1$-norms of the solution at time $t = \delta$ and $t = 2\delta$ etc. This follows from uniform control over $|E|$ and $|L|$ by (37). The increment of $E$ is controlled by Proposition [5.1] and (37) as follows:

$$|E(Iu(\delta), Iv(\delta)) - E(Iu_0, Iv_0)| \leq c_2[(N^{-1+\frac{1}{2}s} + N^{-\frac{1}{2}+})N^{3(1-s)} + N^{-2+}N^{4(1-s)} + N^{-3+}N^{6(1-s)}],$$

where $c_2 = c_2(\tau, M)$.

The number of iteration steps to reach the given time $T$ is $T\delta^{-1}$. This means that in order to give a uniform bound of the energy of the iterated solutions by $2\tau N^{2(1-s)}$, the following condition has to be fulfilled:

$$c_2[(N^{-1+\frac{1}{2}s} + N^{-\frac{1}{2}+})N^{3(1-s)} + N^{-2+}N^{4(1-s)} + N^{-3+}N^{6(1-s)}]T\delta^{-1} < \tau N^{2(1-s)},$$

where $c_2 = c_2(2\tau, 2M)$ (recall that the initial energy is bounded by $\tau N^{2(1-s)}$).

Similarly, the increment of $L$ is controlled by Proposition [5.1] and (37):

$$|L(Iu(\delta), Iv(\delta)) - L(Iu_0, Iv_0)| \leq c_2[N^{-2+}\frac{1}{2}s - N^{-3+}N^{4(1-s)}].$$

Thus, similarly as for $E$, in order to give a uniform bound of $L$ by $2\tau N^{1-s}$, the following condition has to be fulfilled:

$$c_2(N^{-2+}\frac{1}{2}s - N^{-3+}N^{4(1-s)})T\delta^{-1} < \tau N^{1-s}.$$  \hspace{1cm} (39)

If the inequalities (38) and (39) are satisfied, the uniform control of $|E|$ and $|L|$ implies by (32) and (33) uniform control

$$\|v(t)\|_{L^2} \leq cN^{\frac{1}{2}s} \quad \text{and} \quad \|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq cN^{1-s}.$$  

Now, using the definition of $\delta$ above, (38) can be fulfilled for a sufficiently large $N$, provided the following conditions hold:
a. in the case $\beta \neq 0$:

$$-\frac{7}{4} + 3(1 - s) + 4(1 - s) + 3(1 - s) < 2(1 - s) \iff s > 13/20$$

and

$$-\frac{2}{4} - 2(1 - s) + 3(1 - s) + 4(1 - s) < 2(1 - s) \iff s > 1/2$$

b. in the case $\beta = 0$:

$$-\frac{3}{4} + 2(1 - s) + 3(1 - s) + 3(1 - s) < 2(1 - s) \iff s > 3/5$$

and

$$-\frac{7}{4} + 3(1 - s) + 3(1 - s) < 2(1 - s) \iff s > 9/16$$

Similarly, (39) is fulfilled for $N$ sufficiently large, provided

a. in the case $\beta \neq 0$:

$$-\frac{7}{4} + 3(1 - s) + 4(1 - s) + 3(1 - s) < 2(1 - s) \iff s > 13/20$$

and

$$-\frac{2}{4} - 2(1 - s) + 3(1 - s) + 4(1 - s) < 2(1 - s) \iff s > 1/2$$

b. in the case $\beta = 0$:

$$-\frac{7}{4} + 3(1 - s) + 3(1 - s) < 2(1 - s) \iff s > 3/5$$

and

$$-\frac{2}{4} - 2(1 - s) + 3(1 - s) + 4(1 - s) < 2(1 - s) \iff s > 1/2$$

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