A HAMILTONIAN FORMULATION
OF NONSYMMETRIC GRAVITATIONAL THEORIES

M. A. CLAYTON

Abstract. The dynamics of a class of nonsymmetric gravitational theories is presented in Hamiltonian form. The derivation begins with the first-order action, treating the generalized connection coefficients as the canonical coordinates and the densitized components of the inverse of the fundamental tensor as conjugate momenta. The phase space of the symmetric sector is enlarged compared to the conventional treatments of General Relativity (GR) by a canonical pair that represents the metric density and its conjugate, removable by imposing strongly an associated pair of second class constraints and introducing Dirac brackets. The lapse and shift functions remain undetermined Lagrange multipliers that enforce the diffeomorphism constraints in the standard form of the NGT Hamiltonian. Thus the dimension of the physical constraint surface in the symmetric sector is not enlarged over that of GR. In the antisymmetric sector, all six components of the fundamental tensor contribute conjugate pairs for the massive theory, and the absence of additional constraints gives six configuration space degrees of freedom per spacetime point in the antisymmetric sector. For the original NGT action (or, equivalently, Einstein’s Unified Field Theory), the U(1) invariance of the action is shown to remove one of these antisymmetric sector conjugate pairs through an additional first class constraint, leaving five degrees of freedom. The restriction of the dynamics to GR configurations is considered, as well as the form of the surface terms that arise from the variation of the Hamiltonian. In the resulting Hamiltonian system for the massive theory, singular behavior is found in the relations that determine some of the Lagrange multipliers near GR and certain NGT spacetimes. What this implies about the dynamics of the theory is not clearly understood at this time.

Introduction

The Hamiltonian formulation of GR (viewing the metric and the extrinsic curvature of a spacelike hypersurface evolving in time as a first order system) has become popular in GR in recent years primarily due to various attempts to canonically quantise GR. However the importance of the Hamiltonian picture goes beyond this formal procedure. In any classical field theory, the Hamiltonian formulation clarifies many dynamical issues of the system. The Bergmann-Dirac canonical analysis begins with a classical action, tells one whether it is consistent, and identifies the constraints and allowed choices of gauge. This identifies the physical constraint surface in phase space, the dimension of which gives the number of dynamical degrees of freedom in the system under consideration. The resulting first-order system (Hamilton’s equations) is then ideally suited for numerical investigations of the initial value problem. As little is known of a general nature about solutions of the field equations in NGT and phenomenology tends to rely heavily on the properties of the static spherically symmetric solutions\(^1\), it is hoped that some results of a more general nature will be accessible through Hamiltonian methods, or at the very least numerical investigations.

In the case at hand, one is looking for a description of a class of Nonsymmetric Gravitational Theories (NGT will refer either to this type of theory in general or to the ‘old’, or massless theory, whereas mNGT will refer to the massive theory recently introduced in \(\text{[37, 36, 4]}\) that will tell one which fields in the fundamental tensor are involved in the dynamical evolution of the system, and which (if any) are given by constraints on the initial Cauchy surface or left as freely specifiable Lagrange multipliers. The analysis performed here will treat the action for NGT in first-order form, treating the symmetric and antisymmetric connection coefficients as canonical coordinates, the components of the spatial part of the fundamental tensor showing up as (weakly equivalent to) the conjugate momenta. Hamilton’s equations for the system are then similar to those of GR given in \(\text{[2]}\), with phase space suitably enlarged.

---

\(\text{PACS: 04.50.+h, 11.10.Ef, 04.20.Fy.}\)

\(\text{[46]}\) gives the general form of the spherically symmetric solution within the context of Einstein’s Unified Field Theory, a special case of which was popular in the early works of NGT \(\text{[35]}\); another is the Wyman solution \(\text{[48]}\), recently resurrected for both the ‘old’ \(\text{[6]}\) and massive \(\text{[5]}\) versions of NGT.
to accommodate the antisymmetric sector, and the algebraic compatibility conditions left unimposed. Although it is possible in principle to treat the action as a second-order system (depending only on the fundamental tensor and its derivatives) by generalizing the inversion formula of Tonnelat [45, 44], the algebraic complications inherent in the solution of the generalized compatibility conditions make this approach unfeasible.

The action for mNGT [15, 34, 3] has recently been introduced in order to make the antisymmetric sector well-behaved when considered as a perturbation of a GR background. The necessity of making this shift away from the original formulation of NGT was hinted at in some early works in NGT [29, 22], but made a necessity by the work of Damour, Deser and McCarthy [8, 9] in showing the bad asymptotic behavior of NGT perturbations about asymptotically flat GR backgrounds (reviewed in [1]). The action for mNGT is designed so that the skew perturbation equations have the form of massive Kalb-Ramond theory on a GR background, with additional curvature coupled potential terms. Thus in the case where the GR background is fixed, one finds three propagating degrees of freedom in the skew sector, and three algebraic relations that couple the remaining modes locally to the source (and background curvature), effectively removing what would be negative energy modes had they propagated. The results found here will indicate that in general, all six antisymmetric components of the fundamental tensor exist as independent degrees of freedom in the theory.

The bulk of this paper (Sections 2-4) will be devoted to developing the formalism and recasting the dynamics of NGT in Hamiltonian form. This development is not particularly elegant, as the Hamiltonian is burdened by a large number of Lagrange multipliers, and the familiar patterns of the GR analysis become obscured. Where possible, parallels to the analogous GR analysis have been made, in order to not lose sight of the (in some ways very simple) overall structure. Indeed, although the algebra is sometimes rather messy, the analysis presented here is a relatively straightforward application of the Bergmann-Dirac constraint algorithm.

Upon concluding the constraint analysis, in Section 5 the number of configuration space degrees of freedom per spacetime point in the old version of NGT is shown rigorously to be five, clearing up some confusion that has occurred in the literature [22]. This however is not a new result, as it has long been known in the context of Einstein’s Unified Field Theory [27]. (Their analysis is not altered by the reinterpretation of the skew sector or the introduction of the source coupling into the action that defines NGT [35].) The Cauchy analysis performed in the aforementioned works also determined that the Unified Field Theory (hence also NGT) ‘suffers’ from multiple light cones in that there are three causal boundaries within which different modes of the gravitational system propagate. It is expected that these ‘causal metrics’ will be slightly different for massive NGT, and in fact the considerations of Section 3 would seem to indicate that at least one of these has no pure GR limit (in the sense that as $g_{[AB]} \to 0$, the related causal metric also vanishes). The presence of more than one physical light cone confuses the issue of what constitutes the proper description of a Cauchy surface, and is discussed in more detail in Section 3.1.

1. The Action and Field Equations

The formalism developed in Section 3 of [1] allows one to easily write the action for theories with a non-symmetric ‘metric’ and connection coefficients in a surface adapted coordinate system suitable for the Hamiltonian analysis presented here. All covariant derivatives and curvatures have been defined with respect to a torsion-free connection $\Gamma^A_{BC}$ (which in general is not required to be compatible with any tensor in the theory), whereas what is normally treated as the antisymmetric part of the connection coefficients is considered as an additional tensor $\Lambda^A_{BC}$. Despite the fact that this split introduces more objects into the formalism, the reduction to GR is made slightly more transparent, and allows the antisymmetric part of the torsion-free connection to be identified with the structure constants of a general basis in the standard way. Although the connection will not initially be assumed torsion-free in the action, a tensor of Lagrange multipliers $L^{AB}_C$ will be introduced to ensure vanishing torsion at the level of the field equations. In fact this is not necessary, as one could impose the torsion-free conditions from the outset and remove the appropriate connection coefficients from the action. This in fact is what will be done for most of the torsion-free conditions, as a judicious choice of arbitrary torsion terms added into the action causes the related connection components to disappear altogether. However, there are three conditions that have not been imposed in this way, as it was felt that the formalism was simplified slightly by not doing so (discussed further in Section 3.1).

2If one had instead defined a covariant derivative that was not torsion-free as in [14], one would have to introduce additional fields into the action, and the split between the effects of torsion and those due to the choice of coordinates would be obscured.
The action given in \([16\pi G = c = 1]\) extended by the above mentioned Lagrange multiplier term is then:

\[
S_{NGT} = \int_M d^4x \, E \left[ -g^{AB} R_{AB}^{\text{NS}} - g^{AB} \nabla_{e[A} [W]_{B]} + \frac{3}{2} \alpha g^{(AB)} W_A W_B + \Gamma^A \Lambda_A + \frac{1}{4} m^2 g^{[AB]} [\Lambda_{[AB]} + \Lambda_C^{AB} T^C_{AB}] \right].
\] (1.1)

Each contribution to this will be denoted by corresponding Lagrangians (in order of appearance), each to be separately decomposed in Section 3.1. Each contribution to this will be denoted by corresponding Lagrangians (in order of appearance), each to be separately

where the mass tensor appearing in the last of these is:

\[
\text{the symmetric sector (GR) quantities is unclear. However, once one understands what data is configurable (Cauchy analysis, and the goal of the present line of research is to make some headway towards understanding the dynamics the sort be attempted here. The action (1.1) will be considered as the fundamental starting point of the Hamiltonian

be used in surface decomposed form to check that Hamilton’s equations (as derived from the NGT Hamiltonian

spacetimes will be dealt with exclusively in this work.

The Ricci-like tensor in the action has been split up into two contributions: \(R_{AB}^{\text{NS}} = R_{AB} + R^A_{AB}\). The first is identified as the Ricci tensor (i.e. it reduces to the GR Ricci tensor in the limit of vanishing antisymmetric sector):

\[
R_{AB} = e_C [\Gamma^C_{BA} - \epsilon_B [\Gamma^C_{CA}] - \frac{1}{2} \epsilon_A [\Gamma^C_{BC}] + \frac{1}{2} \epsilon_B [\Gamma^C_{AC}] + \Gamma^E_{BA} \Gamma^C_{CE} - \Gamma^E_{CB} \Gamma^C_{EA} + \Gamma^E_{[AB]} \Gamma^C_{EC}],
\] (1.2a)

and the second contains contributions from the antisymmetric tensor field \(A^A_{BC} (\Lambda_A := \Lambda^B_{AB})\):

\[
R^A_{AB} = \nabla_{e_C} [\Lambda^C_{AB}] + \nabla_{e_A} [\Lambda^A_{EB}] + \Lambda^C_{AD} \Lambda^D_{BC}.
\] (1.2b)

The Ricci tensor is defined from the two independent contractions of the geometric curvature tensor derived in the standard way from the connection that defines parallel transport of the general basis vectors \([3]\) by: \(\nabla_e [e]_B = \Gamma^C_{AB} e_C\). The basis vectors are related in the usual way to a coordinate basis via a vierbein \(e_C = g^{AC} \partial_a \), although the vierbein in this case is not necessarily mapping the coordinate basis into a Lorentz frame. As mentioned above, the torsion tensor:

\[
T^A_{BC} = \delta^A_{[EB}] - \nabla_{e_B} [e]_C - \nabla_{e_C} [e]_B - \epsilon_B [e, C] = \Gamma^A_{BC} - \Gamma^A_{CB} - C^A_{BC},
\] (1.3)

will be constrained to vanish by variation of the Lagrange multipliers \(L^A_{AB}\), allowing one to determine the antisymmetric part of the connection from the vierbeins.

Variation of both the connection coefficients (\(\Gamma^A_{BC}\)) and the antisymmetric tensor (\(\Lambda^A_{BC}\)) will result in the compatibility conditions:

\[
\epsilon^C_{AB} := \nabla_{e_C} [g]^{AB} - g^{CD} \Lambda^A_{CD} - g^{AD} \Lambda^B_{DC} + \frac{2}{3} \delta^{A[B] \delta_D} g^{(DE)} W_E = 0,
\] (1.4)

where equations (4.14) and (4.28) of \([3]\) have been written in terms of the densitized components of the inverse of the fundamental tensor. Note that the last term is precisely what is necessary in order for the compatibility conditions to be consistent with \(\Lambda_A = 0\). The remaining field equations are\(^3\)

\[
\Lambda_A = 0,
\] (1.5a)

\[
I^4 = \frac{1}{2} \alpha g^{(AB)} W_B,
\] (1.5b)

\[
\nabla_{e_B} [s]^{[AB]} = \alpha g^{(AB)} W_B,
\] (1.5c)

\[
R_{AB} = R_{AB}^{\text{NS}} + \nabla_{e_A} [W]_{B]} - \frac{1}{2} \alpha W_A W_B - \frac{1}{4} m^2 M_{AB} = 0,
\] (1.5d)

where the mass tensor appearing in the last of these is:

\[
M_{AB} = g_{[AB]} - g_{CA} g_{DB} [CD] + \frac{1}{2} g_{BA} g_{[CD]} [CD].
\] (1.6)

Neither \([1.4]\) nor any of \([1.3]\) will be used in order to pass to the second-order form of the action; they will instead be used in surface decomposed form to check that Hamilton’s equations (as derived from the NGT Hamiltonian developed in the following section) are equivalent to the Euler-Lagrange equations quoted here.

To date there has been no complete physical interpretation of the antisymmetric structure, nor will anything of the sort be attempted here. The action \([1.1]\) will be considered as the fundamental starting point of the Hamiltonian analysis, and the goal of the present line of research is to make some headway towards understanding the dynamics of the system. Thus in general one is left with many ambiguities as to how to make measurements on the skew sector, and the meaning of the various additional antisymmetric tensors as well as the additional contributions to the symmetric sector (GR) quantities is unclear. However, once one understands what data is configurable (Cauchy

\^3For the translation of the conventions used here for a coordinate basis with that of \([35, 37]\), see Section 1 of \([4]\).

\^4Note that one of \([1.5a]\) and \([1.5c]\) is redundant, as \(I^4\) has been replaced in the compatibility conditions derived directly from the action using \([1.5d]\), to give the form \([1.4]\).
data), the field equations fix the evolution uniquely (up to diffeomorphisms), allowing one to determine the physical implications of the presence of the antisymmetric sector in cases where the interpretation of the symmetric sector is essentially that of GR. This will be the case in the asymptotic region of spacetimes that are asymptotically dominated by the symmetric sector, as well as in perturbative scenarios (discussed in Section 3) where initial data is close to GR configurations.

The next step in the analysis is to introduce the foliation of the spacetime manifold $M$, and decompose all objects into components perpendicular and parallel to the surfaces.

2. Metric and Compatibility in a Surface Compatible Basis

Normally one would assume at this point that spacetime is globally hyperbolic, so that a Cauchy surface $\Sigma_0$ exists in $M$ (a closed achronal set without edge, intersected by any smooth causal curve that is future and past inextendible). Spacetime may then be viewed as the evolution of 3-geometries, that is, $(M, g)$ emerges from the set of field configurations on spacelike hypersurfaces that make up a foliation of spacetime: $\{(\Sigma_t, g_\ell) | t \in I \subset \mathbb{R}^1\}$. Then one may show that diffeomorphically equivalent initial data on $\Sigma_0$ generate spacetimes that are diffeomorphically equivalent, and a physical spacetime is then identified as an equivalence class of such solutions (equivalent up to diffeomorphisms).

For NGT however, a sensible definition of hyperbolicity is somewhat more complicated. In Chapter IV of [31], it is demonstrated that the field equations of Einstein’s Unified Field Theory propagate information along three separate light cones (this also occurs when considering covariant wave equations for particles of spin greater than or equal to one [17]). These cones are defined by symmetric metric tensors, the set of which will be denoted: $\{g_c\}$, and referred to as the causal metrics of NGT. Thus the statement that a vector is timelike (spacelike) with respect to $\{g_c\}$ indicates that it is timelike (spacelike) with respect to all of the causal metrics in $\{g_c\}$. Global hyperbolicity would then require that $\Sigma_0$ be achronal with respect to $\{g_c\}$, and the causal curves would of course also be causal with respect to $\{g_c\}$. This would ensure that none of $\{g_c\}$ would become degenerate anywhere in spacetime, and hence that $\Sigma_0$ is in fact a Cauchy surface. At this stage it is not known whether one can require such a condition on the fundamental tensor [9]. However even if this is not possible, the Hamiltonian system is still relevant for considering the dynamics of the theory locally. One may instead specify initial data on a large enough $S_0 \subset \Sigma_0$ in order that the configuration on $S_t \subset \Sigma_t$ ($S_t$ is a closed achronal set) is determined uniquely. This requires that $\{g_c\}$ is known in order to determine $D^+(S_0)$ or $D^-(S_t)$ (the future (past) Cauchy development or domain of dependance of $S_0$ ($S_t$)), to ensure that $S_0 \subset D^+(S_0)$ or $S_0 \supset D^-(S_t)$. For the purposes of this work, it will suffice to discuss the global problem, as one may restrict the results to in a straightforward manner once the causal metrics of mNGT are known.

2.1. The Surface Adapted Basis. We begin by assuming that there is a time function $t$ that is used to foliate $M$ into hypersurfaces of constant time $\Sigma_t$. This requires that each (3-)surface of constant time, $\Sigma_t$, be spacelike with respect to $\{g_c\}$, and $\nabla_t t = 1$ defines a vector $t$ that is timelike with respect to $\{g_c\}$. This ensures that the degrees of freedom that travel along each of the light cones in $\{g_c\}$ will evolve forward in what has been chosen as time.

In GR, one may then introduce a coordinate basis on the surface and a unit normal, such that the metric is of the form: $g = \theta^\perp \otimes \theta^\perp - \gamma_{\alpha\beta} \theta^\alpha \otimes \theta^\beta$. This effectively reduces the metric to a Riemannian metric on $\Sigma$, and the projection of $dt$ perpendicular and parallel to the surface are the lapse and shift functions respectively [16]. It is the components of this metric on $\Sigma$ that are taken to be the Canonical coordinates in the Hamiltonian approach to GR, and all other spacetime tensors are decomposed onto the surface by considering the components perpendicular and parallel to the surface as separate tensors on $\Sigma$.

On attempting to generalize this to NGT, one finds that the presence of more than one physical metric on $M$ implies that there is no physically well-motivated or natural choice of metric on $\Sigma$ that would play the same role. One may introduce a coordinate basis on the surface in the usual way, but in general each of the metrics in $\{g_c\}$ will

---

5In the Unified Field Theory, $\{g_c\} = \{l, h, \gamma\}$ where $l$ represents $g^{(\mu\nu)}$ and its inverse, $h$ represents $h_{\mu\nu}$ and its inverse, and $\gamma$ and its inverse are defined by $\gamma^{\alpha\beta} = \frac{1}{2}h^{\alpha\beta} - t^\alpha t^\beta$ where $h = \text{det}[h_{\mu\nu}]$ and $g = \text{det}[g_{\mu\nu}]$. These metrics were found to be compatible in the sense that there is a largest and smallest speed of light, and all three metrics merge into the Riemannian metric of GR in the limit of vanishing antisymmetric sector. As the set of causal metrics is not known explicitly for mNGT at this time, strictly speaking one cannot discuss the causal structure of any given spacetime.

6What is commonly referred to as the ‘metric’ of NGT is the tensor determined directly by the field equations (appearing in $[13]$), the determinant of which defines the spacetime volume element in the action, and from which one derives the causal metrics $\{g_c\}$. The presence of antisymmetric components indicates that it is in fact not a metric (in the usual sense) at all, and will therefore be referred to as the ‘fundamental tensor’ (in accordance with Lichnerowicz [27]). Note however that one may consider it as a Hermitian metric in a complex or hyperbolic complex space [4, 15, 23], or in more general spaces [28].
provide a different definition of the unit normal vector \( e_{\perp} \), and the resulting decomposition of tensors onto \( \Sigma \) will be inequivalent. However as this is merely a choice of parameterization, the physical content of the system will be independent of the choice of metric used to define the surface decomposition. In this work, the components of \( g^{(AB)} \) will be used to define the unit normal \( e_{\perp} \), therefore making a particular choice of decomposition:

\[
\begin{align*}
g^{-1} &= e_{\perp} \otimes e_{\perp} + B^a e_{\perp} \otimes e_a - B^a e_a \otimes e_{\perp} - \gamma^{ab} e_a \otimes e_b, \\
\gamma^{-1}_{\text{symm}} &= e_{\perp} \otimes e_{\perp} - \gamma^{(ab)} e_a \otimes e_b,
\end{align*}
\]

(2.1a)

(2.1b)

where ‘symm’ indicates the symmetric part of the inverse of the fundamental tensor. This choice is made in order to simplify as much as possible the form of the action (1.1), as the fundamental tensor appears in the form \( g^{AB} \).

The time vector is decomposed as \( t = t^A e_A = Ne_{\perp} + N^a e_a \), where \( (N, N^a) \) are the lapse (component of \( t \) along \( e_{\perp} \)) and shift (projection of \( t \) on the surface \( \Sigma_t \)). This basis can therefore be written in terms of the coordinate basis \( (\partial_t, \partial_a) \) by: \( e_{\perp} = \frac{1}{N} \partial_t - \frac{N^a}{N} \partial_a \), \( e_a = \partial_a \), and the covector bases by: \( \theta^\perp = Nd\tau, \theta^a = dx^a + N^a dt \). This defines the vierbein through \( \theta^A = E^A_\mu dx^\mu \), the determinant of which is:

\[
E := \det[E^A_\mu] = N.
\]

(2.2)

Note that \( e_a \) is a coordinate basis on \( \Sigma \) and can be therefore be thought of as acting on tensor fields as an ordinary derivative \( \partial_a \). It is also useful to denote surface vectors (not just the components) as \( \vec{\omega} := \omega_a \theta^a \). It is not difficult to check that in the coordinate basis \( (\partial_t, \partial_a) \), \( g^{-1}_{\text{symm}} \) and its inverse take on the usual ADM form [3].

The fundamental tensor is found from the inverse of (2.1a):

\[
g = F \theta^\perp \otimes \theta^\perp - (\alpha_a - \beta_a) \theta^a \otimes \theta^\perp - (\alpha_a + \beta_a) \theta^a \otimes \theta^a - G_{ab} \theta^a \otimes \theta^b,
\]

(2.3a)

where:

\[
F := 1 + G_{(ab)} B^a B^b, \quad \alpha_a := G_{[ab]} B^b, \quad \beta_a := G_{(ab)} B^b,
\]

(2.3b)

from which one finds \( \alpha_a B^a = 0 \). The spatial part of the fundamental tensor \( (G_{ab}) \) is given as the inverse of \( \gamma - B \otimes B \):

\[
(\gamma^{ac} - B^a B^c) G_{cb} = G_{bc}(\gamma^{ac} - B^a B^c) = \delta^a_b.
\]

(2.3c)

Note that invertibility of the fundamental tensor (2.3) is required in order that the volume element in the action be nondegenerate, and this requires that \( G_{ab} \) exist. Given that the fundamental tensor is a nondegenerate solution of the field equations in no way guarantees that \( \{g_{\Sigma}\} \) are all nondegenerate, allowing for the possibility that a regular solution of the field equations has regions in spacetime where NGT perturbations ‘see’ a singular spacetime.

Some useful identities may be derived from (2.3c):

\[
\begin{align*}
\gamma^{[ac]} G_{[cb]} + \gamma^{[ac]} G_{(cb)} + B^a \alpha_b &= 0, \\
\gamma^{[ac]} G_{[cb]} + \gamma^{[ac]} G_{(cb)} - B^a \beta_b &= \delta^a_b,
\end{align*}
\]

(2.4)

Varying \( \sqrt{-g} \) with respect to the densitized components of the inverse of the fundamental tensor results in:

\[
\delta \sqrt{-g} = -\frac{2}{F} \beta_a \delta B^a + \frac{1}{2 - F} G_{ba} \delta \gamma^{ab}.
\]

(2.5)

This will be used order to compute the variation of the Hamiltonian of NGT with respect to the same densitized components, as well as applied as an ordinary derivative relation in the surface-decomposed compatibility conditions in Appendix A. Note that the condition \( F = 2 \) corresponds to the case where \( \sqrt{-g} \) is independent of \( (B^a, \gamma^{ab}) \), and \( F \neq 2 \) will be assumed throughout this work.

The non-vanishing structure constants for this basis are found to be:

\[
\begin{align*}
[e_{\perp}, e_a] &= C_{\perp a} \perp e_{\perp} + C_{\perp a} \perp e_b, \\
C_{\perp a} \perp &= e_a [\ln(N)], \\
C_{\perp a} \perp &= \frac{1}{N} e_a [N^b],
\end{align*}
\]

(2.6)

and if one were to require that the connection \( \Gamma \) be torsion-free using (1.3), the antisymmetric components of the connection would be:

\[
\begin{align*}
\Gamma_{[\perp a]} &= \frac{1}{2} e_a [\ln(N)], \\
\Gamma_{[\perp a]} &= \frac{1}{2N} e_a [N^b], \\
\Gamma_{[ab]} &= 0, \\
\Gamma_{[bc]} &= 0.
\end{align*}
\]

(2.7)
A final useful property of this basis is:
\[ e_b [C^\perp_a b] + e_a [C^\perp_b b] + C^\perp_a b C^\perp_b \perp = 0, \] (2.8)
which can either be derived directly from a contraction of the Jacobi identity, or proved by a brute force insertion of the structure constants (2.6). This will be useful when identifying the diffeomorphism constraints, linking them to the algebraic field equations in Appendix A.2.

In beginning with the components of the inverse of the fundamental tensor in (2.1), the configuration of the system is being described by the degrees of freedom \( N, N^a, B^a, \gamma^{ab} \) (the latter two are included as densities since they will turn up later on as weakly equal to the conjugate momenta in this form). As one can see from the form of (2.3), different parameterizations of the fundamental tensor will have a large effect on the details of the analysis presented here. If one had begun by requiring that the symmetric part of the fundamental tensor took on the ADM form, then the inverse (that appears in the action) would have had nontrivial \( g^{(0a)} \) components, and the analysis would have been more complicated. In NGT, one must be slightly more careful to be consistent in what one is calling the independent degrees of freedom, as the antisymmetric degrees of freedom will mix with the symmetric sector as one moves from spatial components of the fundamental tensor to the spatial components of its inverse.

2.2. Surface Decomposition. Four-dimensional covariant derivatives are decomposed as usual \[16\] into surface covariant derivatives (written as \( \nabla^{(3)}_a \)) whose action on basis vectors in \( T \Sigma \) is given by \( \nabla^{(3)}_a [e]_b = \Gamma^e_{ab} e_c \), and contributions from derivatives off of \( \Sigma \) which will be written in terms of the surface tensors:
\[ \Gamma := \Gamma^\perp, \quad c_a := \Gamma^\perp_{a\perp}, \quad a_a := \Gamma^a_{\perp a}, \quad \sigma^a := \Gamma^a_{\perp \perp}, \]
\[ w^a_b := \Gamma^a_{b\perp}, \quad w^a_a := \Gamma^a_{\perp \perp}, \quad k_{ab} := \Gamma_{ab}, \] (2.9)
(Both \( k \) and \( \Gamma_{ab} \) will initially be considered as nonsymmetric, allowing for the possibility of non-zero torsion. The torsion-free conditions will be imposed for these components very early in the development, and from then on they will be assumed to be symmetric.) In GR most of these are related by algebraic compatibility conditions \( (\sigma^a = \gamma^{ab} a_a, c_a = 0, \Gamma = 0, w^a_a = \gamma^{ac} k_{bc}) \), and the absence of torsion would allow one to further relate \( a \) and \( (u, w) \) to the structure constants. The skew tensor \( \Lambda \):
\[ b_a := \Lambda^a_{\perp a}, \quad j_{ab} := \Lambda^a_{ab}, \quad v^a_b := \Lambda^a_{ab}, \quad \lambda^a_{bc} := \Lambda^a_{bc}, \] (2.10)
the torsion Lagrange multipliers:
\[ L^a := L^a_{\perp a}, \quad L^{ab} := L^a_{\perp b}, \quad L^a_b := L^a_{b\perp}, \quad L^b_{a} := L^b_{\perp a}, \] (2.11)
and the vector fields:
\[ W_A = (W_a, W_A) \text{ and } l^A = (l^a, l^A), \]
are all decomposed in a similar fashion. Traces are denoted as \( v := v^a_a \), and the traceless part: \( v^a_b := v^a_b - 1/3 g_{ab} v^a \). The components of \( \Lambda A \) are given by \( \Lambda \perp = v, \Lambda_a = b_a + \lambda^b_{ab} \), and \( j_{ab} \) is an antisymmetric surface tensor by definition. It also will be useful to define a few combined quantities that will appear:
\[ K_{ab} := k_{ab} + j_{ab}, \] (2.12a)
\[ k^a_b := \frac{1}{2} (\gamma^{ac} K_{bc} + \gamma^{ca} K_{eb}) = \gamma^{[ac]} k_{bc} + \gamma^{[ac]} j_{bc}, \] (2.12b)
\[ j^a_b := \frac{1}{2} (\gamma^{ac} K_{bc} - \gamma^{ca} K_{eb}) = \gamma^{[ac]} k_{bc} + \gamma^{[ac]} j_{bc}, \] (2.12c)
\[ w^{ab} := \frac{1}{2} (\gamma^{[ac]} u^a_c + \gamma^{[bc]} u^b_c + \gamma^{[ac]} v^a_c + \gamma^{[bc]} v^b_c), \] (2.12d)
\[ v^{ab} := \frac{1}{2} (\gamma^{[ac]} u^a_c - \gamma^{[bc]} u^b_c + \gamma^{[ac]} v^a_c - \gamma^{[bc]} v^b_c). \] (2.12e)
Aside from \( K_{ab} \), which will be useful as a notational tool to combine the results of both sectors into one relation, these combinations correspond to symmetric \( (k_{ab}, w^a_b) \) and antisymmetric \( (j_{ab}, v^a_b) \) sector contributions once an index has been ‘raised’ by the spatial fundamental tensor \( \gamma^{ab} \).

2.3. Lie Derivatives. The standard Lie derivatives defined on \( \Sigma \):
\[ L^{(3)}_{\vec{X}} [\vec{Y}] = [\vec{X}, \vec{Y}] = (X^a e_a [Y^b] - Y^a e_a [X^b]) e_b, \quad L^{(3)}_{\vec{X}} [Y^\perp] = X^a e_a [Y^\perp], \]
\[ L^{(3)}_{\vec{X}} [\omega^a] = (X^a e_a [\omega_c] + \omega_a e_c [X^a]) \theta^c, \quad L^{(3)}_{\vec{X}} [\omega_{\perp}] = X^a e_a [\omega_{\perp}], \]
(2.13a)
(2.13b)
(and the standard generalization to higher-order tensors) will be used throughout. Note that the perpendicular components are treated as scalar fields as far as surface defined derivatives are concerned. This is also true of the derivative off of Σ §16:

\[ \partial_{\perp} \equiv \partial_{\perp} \]

where the structure constant takes into account the possibility that the surface basis may ‘move’ on Σ as one moves perpendicularly off the surface. These definitions are also extended to arbitrary tensors in the standard manner.

From the spacetime Lie derivatives of tensor densities:

\[ \mathcal{L}_X[T] = \sqrt{-g} \mathcal{L}_X[T] + T \mathcal{L}_X[\sqrt{-g}] \]
\[ \mathcal{L}_X[\sqrt{-g}] = \frac{1}{2} \sqrt{g} \mathcal{L}_X[g^{BC}] = -\frac{1}{2} \mathcal{L}_X[g^{BC}] \]

and using the defined surface Lie derivatives, one finds:

\[ \partial_{\perp} \equiv \partial_{\perp} \]

Time derivatives of arbitrary tensors or tensor densities are given as in GR by:

\[ \partial_t[T] = \partial_t[T] = \partial_{\perp} + \frac{1}{2} \mathcal{L}_X[\sqrt{-g}] \]

where \( \theta \equiv \theta \) is any tensor density.

One would now like to identify the extrinsic curvature of the surface Σ under consideration. Although there are a variety of equivalent ways of describing it in the context of Riemannian geometry, none of these are equivalent for the type of nonsymmetric theory considered here. One way is to compute the perpendicular component of the parallel transport of a surface vector along the surface (Gauss’ formula §4): \( \theta_{\perp} \equiv \theta_{\perp} \)

where the structure constant takes into account the possibility that the surface basis may ‘move’ on Σ as one moves perpendicularly off the surface. These definitions are also extended to arbitrary tensors in the standard manner.

Although equivalent in GR, none of these definitions coincide in nonsymmetric theories in general. Indeed, since \( \Gamma_{ab} \neq 0 \), the length of the unit normal will not be preserved under (this definition of) parallel transport. One could also identify the extrinsic curvature of Σ with the change of the 3-metric perpendicularly off the surface \( \nabla_{\perp} \equiv \nabla_{\perp} \)

although equivalent in GR, none of these definitions coincide in nonsymmetric theories in general. Indeed, since the compatibility conditions cannot entirely be written in terms of a metric compatible connection, there is no truly natural choice of parallel transport, and any choice of ‘extrinsic curvature’ based upon a covariant derivative is conventional at best. The approach followed here will be the first of those described above, essentially due to convenience. It turns out that with the choice of decomposition made here, \( \Gamma_{ab} \) is a canonical coordinate, and as such plays a more central role than \( u^a \) (which will turn out to be a derived quantity, as it acts as a determined Lagrange multiplier), or the relation derived from the evolution of the spatial metric.

The stage is now set, as one now has the tools necessary to decompose spacetime tensors and covariant derivatives into spatially covariant objects, and identify time derivatives in the action. The configuration space variables have also been chosen to be: \( N, N^a, B^a, \gamma_{ab} \). In Appendix A.1 the compatibility conditions (1.4) are decomposed, and the field equations (1.3) in Appendix A.2. In both cases, algebraic conditions that will determine Lagrange multipliers on Σ have been identified, as well as those responsible for time evolution. The reader is again reminded that all of these will be derived from the Hamiltonian, and are only given here in order to introduce some defined quantities (3, 6, 3) that will help to simplify the presentation of the algebra, and ensure that one is finding results that are in accord with the Lagrangian variational principle.

3. Determining the Hamiltonian

This section is concerned with obtaining the form of the Hamiltonian for NGT. The various terms in the Lagrangian §1 will be decomposed into surface compatible form, and the fields that appear as time derivatives identified. After this, writing down the form of the Hamiltonian turns out to be a rather simple task, however the work is far from
done. The Hamiltonian is riddled with Lagrange multipliers, and the constraint analysis is where one begins to see the full structure of NGT emerge.

3.1. The Surface Decomposed Action. The simplest Lagrangian density terms in (3.1) are decomposed as:

\[
\mathcal{L}_m = -\mathcal{H}_m = \frac{1}{2}m^2 N B^{a}B_{a} + \frac{1}{2}m^2 N \gamma^{[ab]}G_{[ab]},
\]

\[
\mathcal{L}_{i} = N v_{a} + N\gamma^{i}(h_{a} + \gamma_{ab}),
\]

\[
\mathcal{L}_{T} = 2N[L^{a}K_{[ab]} + L^{ab}a_{[b]a} + L^{b}(u_{a} - w_{a} + C_{[a]}b + C_{a}b)],
\]

\[
\mathcal{L}_{W^2} = \frac{1}{2}\alpha N \left[ -\sqrt{-g(W)^2} - \gamma^{[ab]}W_{a}W_{b} \right],
\]

\[
\mathcal{L}_{\nabla W} = N B^{a}[\nabla^{(3)}_{a}[W] - d_{c\perp}[W]_{a} + W(a_{a} - c_{a})] + N \gamma^{[ab]}\nabla^{(3)}_{[a][W]_{b]}},
\]

The first of these defines \( \mathcal{H}_m \), which will be useful in Section 4.3. The remaining contribution \( \mathcal{L}_{W^2} \), will be further split into contributions from each of the symmetric and antisymmetric sectors as \( \mathcal{L}_R \) and \( \mathcal{L}_{R,\perp} \), and further split to give contributions from each component of the tensor (for example, \( \mathcal{L}_{R_{[ab]}} \) corresponds to the contribution from \( N \gamma^{[ab]}R_{[ab]} \)). Note that the fact that \( g^{(\perp a)} = 0 \) in this decomposition implies that there is no contribution from \( \mathcal{L}_{R_{(\perp a)}} \), which is essentially why the choice of surface adapted basis (2.1) was made. Making a choice of parameterization in which \( g^{(\perp a)} \neq 0 \) would have explicitly introduced a term involving \( \partial_{a}B^{a} \) into the action (see (3.1f)). The remaining contributions are:

\[
\mathcal{L}_{R_{\perp\perp}} = -N\sqrt{-g}v_{a}v_{b},
\]

\[
\mathcal{L}_{R_{[ab]}} = N \gamma^{[ab]}[b_{a}b_{b} - j_{ac}v_{b} - j_{bc}v_{a} + \lambda_{ac}\lambda_{bc}],
\]

\[
\mathcal{L}_{R_{[a]a]} = N B^{a}[d_{c\perp}[\lambda]_{ab} - d_{c\perp}[b]_{a} + 2\nabla^{(3)}_{b}[v]_{a} - \nabla^{(3)}_{a}[v] + 2b_{a}b_{b} - 2b_{a}u + 2(a_{b} - c_{b})v_{b} - (a_{a} - c_{a})v - 2u_{c}\lambda_{ba} + 2u_{b}\lambda_{a}],
\]

\[
\mathcal{L}_{R_{[a]b]} = N \gamma^{[ab]}[d_{c\perp}[\lambda]_{ab} + \nabla^{(3)}_{[a][b]} + \nabla^{(3)}_{b}[\lambda]_{[a]} + \nabla^{(3)}_{a}[\lambda]_{b} - j_{ac}u_{b} + j_{bc}u_{a} + (\Gamma + u)j_{ab} - k_{ac}v_{b} + k_{bc}v_{a} + a_{b}b_{a} - a_{b}b_{a} + a_{c}\lambda_{ab}],
\]

\[
\mathcal{L}_{R_{\perp\perp}} = -N\sqrt{-g}[-d_{c\perp}[u] + \nabla^{(3)}_{a}[\sigma]^{a} + \Gamma + \sigma^{a}(a_{a} - 2c_{a}) - u_{b}b_{a}],
\]

\[
\mathcal{L}_{R_{[a]b]} = N \gamma^{[ab]}[R^{(3)}_{[a]b} + d_{c\perp}[k]_{[ab]} - \nabla^{(3)}_{a}[b]_{b} - a_{a}b_{a} + (\Gamma + u)k_{ab} - k_{cb}v_{a} - k_{ac}v_{a}],
\]

where \( R^{(3)}_{[a]b} \) is the surface Ricci tensor given by (1.2a) determined from surface covariant derivatives, and the identity \( R_{[AB]} = 0 \) (see Section IV in [4]) has been used.

The terms that result in time derivatives in the action are then:

\[
-N B^{a}d_{c\perp}[W]_{a} - N B^{a}d_{c\perp}[\lambda]_{ab} + N B^{a}d_{c\perp}[b]_{a} + N \sqrt{-g}d_{c\perp}[u] + N \gamma^{ab}d_{c\perp}[K]_{ab},
\]

and so defining the vector field:

\[
\overrightarrow{W}_{a} = -W_{a} - \lambda_{a}b_{a} + b_{a},
\]

will simplify the remaining analysis. Rewriting the above in terms of the time derivative gives:

\[
B^{a}[d_{c\perp}[W]_{a} + \sqrt{-g}d_{c\perp}[u] + \gamma^{ab}d_{c\perp}[K]_{ab} - B^{a}L^{(3)}_{N}[W]_{a} - \sqrt{-g}L^{(3)}_{N}[u] - \gamma^{ab}L^{(3)}_{N}[K]_{ab},
\]

where:

\[
L^{(3)}_{N}[u] = N^{a}\nabla^{(3)}_{a}[u],
\]

\[
L^{(3)}_{N}[W]_{a} = N^{b}\nabla^{(3)}_{b}[W]_{a} + \overrightarrow{W}_{b}\nabla^{(3)}_{a}[N]_{b},
\]

\[
L^{(3)}_{N}[K]_{ab} = N^{c}\nabla^{(3)}_{c}[K]_{ab} + K_{cb}\nabla^{(3)}_{b}[N]_{c} + K_{ac}\nabla^{(3)}_{b}[N]_{c}.
\]

Although these have been written in surface covariant form, all dependence on the spatial connection coefficients vanish in these. As it is clear that \( \overrightarrow{W}_{a} \) will be a canonical coordinate, (3.2) will henceforth be used to replace \( W_{a} \) everywhere in the remainder of this work.
Gathering together terms in the action that are multiplied by the same component of the fundamental tensor and identifying the time derivatives as above, the Lagrangian density for NGT can now be written:

\[
\mathcal{L}_{\text{NGT}} = -\mathcal{H}_m + B^a d_k [\mathcal{W}_a] + \sqrt{-g} d_k [u] + \gamma^{ab} d_k [K]_{ab} - B^a L_N^{(3)} [\mathcal{W}_a] - \sqrt{-g} L_N^{(3)} [u] - \gamma^{ab} L_N^{(3)} [K]_{ab}
\]

\[
+ N \sqrt{-g} [v + l^a (b_a + \lambda^b_{ab})] + 2L^a (c_a - a_a + C_{1a} \perp) + \frac{1}{2} \alpha (W)^2 - u_b v_a - \nabla^a [\sigma]^a - \Gamma u - \sigma^a (a_a - 2c_a) + u_b u_a^b
\]

\[
+ N B^a [\nabla^a [W] + W (a_a - c_a) - 2 \nabla^a [v]^b_a + \nabla^a [v] - 2b_b u_a + 2b_a u - 2(a_b - c_b) v_a + (a_a - c_a) v + 2u_b \lambda^c_{ba} - 2ja_b a^b]
\]

\[
+ N \gamma^{(ab)} [-\nabla^a [\mathcal{W}] - \nabla^b [\mathcal{W}] + \nabla^c [\lambda]^c_{ab} + 2 \nabla^3 [a] [b] b]
\]

\[
- j_{ac} u^c_b - j_{bc} u^c_a + (\Gamma + u) j_{ab} - k_{ca} v^c_b + k_{cb} v^c_a + a_a b_a - a_b b_a + a_c \lambda^c_{ab}
\]

\[
+ N \gamma^{(ab)} [H_{(ab)}^{(3)} - \nabla^a [a] [b] - a_a a_b + (\Gamma + u) k_{ab} - k_{cb} u^c_a - k_{ac} u^c_b
\]

\[
- \frac{1}{2} \alpha (W_a + \lambda^c_{ac} - b_a) (W_b + \lambda^d_{bd} - b_b) + b_a b_b - j_{ac} v^c_b - j_{bc} v^c_a + \lambda^d_{ac} \lambda^c_{bd}]
\]

Up to this point, the torsion-free conditions have been applied arbitrarily in order to simplify the Lagrangian density as much as possible. At this stage, the fields \( k_{[ab]} \) and \( \Gamma_{[bc]} \) have been dropped along with their associated torsion Lagrange multipliers \( L_{ab} \) and \( L_{ab}^t \), as they would appear only in these torsion Lagrange multiplier terms. From this point onwards, both \( k_{ab} \) and \( \Gamma_{ab} \) will refer to symmetric objects. The Lagrange multiplier term imposing the torsion-free condition:

\[
w^a_b = u^a_b + C_{1b} a^a,
\]

has also been dropped, as once all terms (in the action as well as compatibility conditions and field equations) are written in terms of \( d_{e_1} \), \( u^a_b \) will not appear either, and may be determined by \( \frac{\delta}{\delta u_{ab}} \) once \( u^a_b \) is known. One would also like to impose the torsion conditions to solve for \( a_a \) explicitly, however this would introduce a number of terms involving the structure constants \( C_{1a} \leq \) into the action, compatibility conditions and field equations, and it was felt that avoiding this simplified the algebra. Although there is nothing preventing one from doing this, the presentation is slightly simpler if the torsion condition is not imposed in this case.

### 3.2. Legendre Transform

In the form \( [3.4] \), the fields with conjugate momenta are trivial to identify (these are densities by definition, and will not be written in boldface):

\[
p := \frac{\delta L_{\text{NGT}}}{\delta (d_k u)} = \sqrt{-g},
\]

\[
p^a := \frac{\delta L_{\text{NGT}}}{\delta (d_k \mathcal{W}_a)} = B^a
\]

\[
\pi^{ab} := \frac{\delta L_{\text{NGT}}}{\delta (d_k K_{ab})} = \gamma^{ab},
\]

where the action has been written in terms of the Lagrangian as: \( S_{\text{NGT}} = \int dt L_{\text{NGT}} \). One could in principle treat all fields in the action as canonical coordinates, generating many cyclic momenta and hence many second class constraints that are trivially removed by the introduction of Dirac brackets \( [12] \). Instead the more economical approach of treating any field in the action whose time derivative does not appear anywhere as a Lagrange multiplier field will be followed\(^7\).

The extended Hamiltonian density is written as:

\[
\mathcal{H}_e = \mathcal{H}_0 + N \phi_{ab} (\pi^{ab} - \gamma^{ab}) + 2N \phi_a (p^a - B^a) + N \phi (p - \sqrt{-g}),
\]

where the form of the Lagrange multiplier terms that enforce the algebraic conditions \( [3.6] \) have been written as the reduction of the covariant form: \( E_{AB} (\pi^{A\beta} - g^{A\beta}) \). The basic Hamiltonian density is derived from the Lagrangian

\(^7\)Note that theCanonical coordinates have been taken to be connection components (and \( \mathcal{W}_a \)), resulting in the densitised components of the fundamental tensor appearing as conjugate momenta. By discarding a total time derivative in the action, one could easily arrange for the fundamental tensor degrees of freedom to play their more traditional role as Canonical coordinates.

\(^8\)It is common to include the lapse and shift functions and their conjugate momenta in phase space, particularly if one is attempting to realize all of the generators of the spacetime diffeomorphism group on phase space \( [17] [14] \). This will not be done in this work simply to avoid the unnecessary algebra, as it is a trivial matter to extend the formalism later if necessary.
difficult but rather lengthy, the results will be given as sparsely as possible.

Symmetric phase space with the same dimension as in GR. As the results are unsurprising and the algebra not unduly

which could be used to strongly remove the conjugate pair (\(d_t\) and \(\gamma_{ab}\)).

Additional second class constraints will also be found in the standard form of the Hamiltonian. Two second class constraints will also be found where the Lagrange multipliers have been split up into smaller subsets for later reference.

\[
\mathcal{H}_0 = H_m + \sqrt{-g}L_N^{(3)}[u] + B^a L_N^{(3)}[W_a] + \gamma_{ab} L_N^{(3)}[K]_{ab}
\]

\[
- N\sqrt{-g}[v + t^a(b_a + \lambda^b_{ab}) + 2L^a(c_a - a_a + C_{a})] + \frac{1}{2}\alpha(W)^2 - v^a_{b}v^b_a - \nabla^{(3)}[\sigma]^a - \Gamma u - \sigma^a(a_a - 2c_a) + v^a_{b}v^b_a
\]

\[
- N\mathbf{B}^a [\nabla^{(3)}[W] + W(a_a - c_a) - 2\nabla^{(3)}[v]^a_a + \nabla^{(3)}[v]] - 2b_b v^b_a + 2b_a u - 2(a_b - c_b)v^b_a + (a_a - c_a)v - 2j_{ab}v^b_a + 2u^b_c\lambda^c_{ab}
\]

\[
(3.7b)
\]

Hamiltonians and Hamiltonian densities will be related everywhere by (for example): \(H = \int_\Sigma d^3x\mathcal{H}_s\), where \(H\) will refer to the Hamiltonian, and \(\mathcal{H}\) to the related Hamiltonian density.

The canonical phase space variables that describe the configuration of the system at any time are:

\[
\{(P^I, Q_I)\} = \{(p, u), (p^a, W_a), (\pi^{ab}, K_{ab})\},
\]

and the usual canonical Poisson brackets have been assumed:

\[
\{Q_I(x), Q_J(y)\} = \{P^I(x), P^J(y)\} = 0,
\]

\[
\{Q_I[f^I], P^J[g_J]\} = \int_\Sigma d^3z f^I(z)g_I(z).
\]

The notation \(Q_I[f^I]\) is used to denote the smearing of the object by an appropriately weighted test tensor (eq. \(p^a[f_a] = \int_\Sigma d^3z p^a(z)f_a(z)\)) leaving a scalar on \(\Sigma\), and the sub/superscript \(\Gamma\) indexes the sets in (3.8). The Poisson brackets are standard:

\[
\{F|Q, P, G|Q, P, \}\} = \int_\Sigma d^3z \sum_I \left[ \frac{\delta F}{\delta Q_I(z)} \frac{\delta G}{\delta P^I(z)} - \frac{\delta F}{\delta P^I(z)} \frac{\delta G}{\delta Q_I(z)} \right],
\]

where \((F, G)\) are scalar functions of the canonical coordinates.

The fields \((p, u)\), although symmetric sector fields, do not show up in the standard Hamiltonian formulations of GR as canonical variables. There one has \(u = \gamma_{ab}k_{ab}\), and this is used in the action to either remove the time derivative from \(u\) (second-order action) or write: \(\partial_t u = \gamma_{ab}\partial_t k_{ab} + k_{ab}\partial_t \gamma_{ab}\) (1). The point being that in nonsymmetric theories, one cannot replace all terms in the action by various combinations of the spatial fundamental tensor and extrinsic curvature in a straightforward manner, and it seems one must be content to carry around the 103 Lagrange multipliers:

\[
(N) := \{N, N^a\}, \quad (L) := \{L^a, (t, l^a)\}, \quad (\phi) := \{\phi, \phi_{ab}, \phi_{(ab)}\},
\]

\[
(\gamma) := \{B^a, \gamma_{ab}\}, \quad (\Lambda) := \{W, b_a, v^b_a, \lambda^c_{ab}\}, \quad (\Gamma) := \{\Gamma, c_a, a_a, \sigma^a, u^a_b, u^b_{ab}, \Gamma^c_{ab}\},
\]

where the Lagrange multipliers have been split up into smaller subsets for later reference.

4. Constraint Analysis

Now begins the analysis of the constraints and conditions that follow from variations of the Hamiltonian with respect to the Lagrange multipliers. It will be found that all of the compatibility conditions and field equations are reproduced from the Hamiltonian, leaving \(\{N\}\) as undetermined Lagrange multipliers enforcing the diffeomorphism constraints in the standard form of the Hamiltonian. Two second class constraints will also be found which could be used to strongly remove the conjugate pair \((p, u)\) from the phase space of the theory, leaving the symmetric phase space with the same dimension as in GR. As the results are unsurprising and the algebra not unduly difficult but rather lengthy, the results will be given as sparsely as possible.
4.1. **Constraints Enforced by** \{φ\}, \{L\}, \{Λ\} and \{Γ\}. The Lagrange multipliers \{φ\} are designed to enforce the conjugate momenta conditions (3.6):

\[
\frac{δH_*}{δφ_{ab}} \approx N[π^{ab} - γ^{ab}] \approx 0,
\]

(4.1a)

\[
\frac{δH_*}{δφ} \approx 2N[p^a - B^a] \approx 0,
\]

(4.1b)

\[
\frac{δH_*}{δφ} \approx N[p - √{-g}] := Nψ_1 \approx 0.
\]

(4.1c)

These conditions clearly determine \{γ\}, with one constraint left over: \ψ_1, which has been identified in (4.1d). From \{L\}, the Lagrange multiplier enforcing the remaining torsion constraint from (2.7) gives:

\[
\frac{δH_*}{δL^a} \approx -2N[c_a - a_a + C_{⊥a}] \approx 0,
\]

(4.2)

which will be used to determine \(a_a\). The constraints resulting from the variation of \(l^a\) are:

\[
\frac{δH_*}{δl^a} \approx -Nv \approx 0,
\]

(4.3a)

\[
\frac{δH_*}{δl^a} \approx -N[b_a + λ^{a}_{ab}] \approx 0.
\]

(4.3b)

These are the components of (1.5a), and will be used to determine \(v^b\) and \(λ_{ab}^b\) respectively.

The conditions arising from the variation of \{Λ\} are now calculated, first varying \(H_*\) with respect to \(W\) and then \(v^b\):

\[
\frac{δH_*}{δW} \approx Nc_{⊥[a]} \approx 0,
\]

(4.4a)

\[
\frac{δH_*}{δv^a} \approx -2N[c_{[b]} - \frac{1}{4}c_{[a]}] - Nδ^c_{[a]}[1 - \frac{1}{4}γ(3)]B^c \approx 0.
\]

(4.4b)

Using the result that \(v ≈ 0\) from (4.3a), the trace of the last of these gives \(l \approx \frac{1}{3}γ(3)B^c \approx \frac{2}{3}W\), so that \(c_{[a]} \approx 0\). Next varying \(b_a\) gives:

\[
\frac{δH_*}{δb_a} \approx 2Nc_{[a]} - N[1 - \frac{1}{4}αγ(3)W_{b}] \approx 0,
\]

(4.4c)

and using the above conditions one finds that:

\[
\frac{δH_*}{δλ^a_{ab}} \approx -Nc_{[a]} \approx 0,
\]

(4.4d)

which, combined with the previous result, yields \(l^a \approx \frac{2}{3}γ(3)W_{b}\). At this point, all of the skew sector algebraic compatibility conditions (A.2d) and (A.2d) as well as the field equations (1.5d) have been reproduced.

Varying \{Γ\}, one finds:

\[
\frac{δH_*}{δΓ} \approx Nc_{[a]} =: Nψ_2 \approx 0,
\]

(4.5a)

\[
\frac{δH_*}{δσ^a} \approx -Nc_{[a]} \approx 0,
\]

(4.5b)

\[
\frac{δH_*}{δε^a} \approx -2N[L^a + σ^a + B^b v^a_{b} - \frac{1}{2}B^aW] \approx 0,
\]

(4.5c)

where the first of these yields a constraint (which has been identified as \(ψ_2\) in (4.5a)) discussed further in Section 4.2 and the last is used to determine the Lagrange multiplier \(L^a\) as:

\[
L^a \approx -σ^a - B^b v^a_{b} + \frac{1}{2}B^aW.
\]

(4.5d)

Using these, one computes:

\[
\frac{δH_*}{δa^a} \approx N[c^{(ab)} - c^{(⊥a)}] \approx 0,
\]

(4.5e)
and the trace-free part of the variation with respect to \( u^b_a \) gives the variation with respect to \( u^b_T a \):

\[
\frac{\delta H_*=}{\delta u^b_T a} \approx -2N \left[ \mathcal{C}^{(L_a)}_b - \frac{1}{3} \delta^b_0 \mathcal{C}^{(L_c)}_c \right] \approx 0,
\]

which yields \( \mathcal{C}^{(L_a)}_b \approx 0 \) when the constraint (4.5a) is taken into account. Finally, the variation with respect to the surface connection coefficients yields:

\[
\frac{\delta H_*=}{\delta W^a} \approx -N \mathcal{C}^{(ab)} \approx 0,
\]

and all but \( \mathcal{C}_T \) of the symmetric sector algebraic compatibility conditions ((A.1a), (A.1d) and (A.1f)) have been found, and the Lagrange multiplier \( L^a \) determined by (4.5d).

4.2. **Time Evolution of the Canonical Fields and the Constraints \( \{ \psi \} \)**. Time evolution of the canonical variables is determined as usual (from (3.9) and (3.10)) by Hamilton’s equations:

\[
\partial_t Q_I = \{Q_I, H_* \} \approx \frac{\delta H_*}{\delta Q^I}, \quad \partial_t P^I = \{P^I, H_* \} \approx -\frac{\delta H_*}{\delta Q_I}.
\]

These evolution equations should reproduce the same dynamics as given by (1.3, 1.4, 1.5). It is convenient at this point to show that the canonical momenta evolve in accordance with the dynamical compatibility conditions, and that the evolution of the canonical coordinates also allows one to identify conditions that must be found in order to reproduce the NGT field equations (1.5d). This is why this calculation is performed at this stage, even though strictly speaking it is not part of the constraint analysis.

The functional derivatives of \( H_* \) with respect to the canonical coordinates will be necessary:

\[
\frac{\delta H_*}{\delta u^a} \approx -\nabla^{(3)}_a[N]^a + N(\Gamma - u - 2B^a_0 b_0),
\]

\[
\frac{\delta H_*}{\delta W^a} \approx -B^a \nabla^{(3)}_b[N]^b + B^b \nabla^{(3)}_a[N]^a - N^b \nabla^{(3)}_b[B]^a \\
+ N \left[ \nabla^{(3)}_b[\gamma]^{[ab]} + \gamma^{[ab]} C_{[ab]} + \alpha \gamma^{[ab]} (W_b - 2b_0) \right],
\]

\[
\frac{\delta H_*}{\delta K_{ab}} \approx -\gamma^{[ab]} \nabla^{(3)}_c[N]^c + \gamma^{[ac]} \nabla^{(3)}_b[N]^b + \gamma^{[cb]} \nabla^{(3)}_c[N]^a - N^c \nabla^{(3)}_c[\gamma]^{[ab]} \\
- N \left[ \gamma^{[ab]} (\Gamma + u) - 2u^{[ab]} \right],
\]

\[
\frac{\delta H_*}{\delta J_{ab}} \approx -\gamma^{[ab]} \nabla^{(3)}_c[N]^c + \gamma^{[ac]} \nabla^{(3)}_b[N]^b + \gamma^{[cb]} \nabla^{(3)}_c[N]^a - N^c \nabla^{(3)}_c[\gamma]^{[ab]} \\
+ N \left[ B^a \sigma^b - B^b \sigma^a - \gamma^{[ab]} (\Gamma + u) + 2\sigma^{[ab]} \right].
\]

Time evolution of the conjugate momenta can be shown to be weakly equivalent to the dynamical compatibility conditions (A.1a, A.1b, A.2a, A.2b):

\[
\partial_t p = -\frac{\delta H_*}{\delta u} \approx dt [\sqrt{-g}] - N \mathcal{C}^{(L_a)}_\perp,
\]

\[
\partial_t p^a = -\frac{\delta H_*}{\delta W^a} \approx dt [B]^a - N \mathcal{C}^{(L_a)}_\perp,
\]

\[
\partial_t \pi^{ab} = -\frac{\delta H_*}{\delta K_{ab}} \approx dt [\gamma]^{[ab]} + N \mathcal{C}^{(L_a)}_{\perp},
\]

since in all cases (heuristically) \( \partial_t P \approx dt [\gamma] \), and one is left with the result that \( \mathcal{C}_\perp \approx 0 \) in evolution. Using (A.3) and the evolution of the coordinates:

\[
\partial_t u = \frac{\delta H_*}{\delta p} \approx N\phi,
\]

\[
\partial_t W^a = \frac{\delta H_*}{\delta p^a} \approx 2N\phi_a,
\]

\[
\partial_t K_{ab} = \frac{\delta H_*}{\delta \pi^{ab}} \approx N\phi_{ab},
\]
allows one to define the objects \( \{ \mathcal{R} \} \):

\[
\mathcal{R} := N\phi - L_\kappa^{(3)}[u] - N\mathcal{F}_{\perp \perp} + \frac{1}{2}m^2NM_{\perp \perp},
\]

\[
\mathcal{R}_a := 2N\phi_a - L_\kappa^{(3)}[\mathcal{W}]_a + 2N\mathcal{F}_{[a\perp]} - \frac{1}{2}m^2NM_{[a\perp]},
\]

\[
\mathcal{R}_{ab} := N\phi_{ab} - L_\kappa^{(3)}[K]_{ab} + N\mathcal{F}_{ab} - \frac{1}{2}m^2NM_{ab}.
\]

In order for Hamilton's equations (4.6) to reproduce the dynamical field equations (1.5), one must find that \( \{ \mathcal{R} \} \approx 0 \), ensuring that \( \{ \phi \} \) are correctly determined.

There are two constraints \( \{ \psi \} \) that have appeared from the variations so far (4.1c) and (4.5a) respectively:

\[
\psi_1 \approx p - \sqrt{-g}\left[p^a, \pi^{ab}\right] \approx 0,
\]

\[
\psi_2 \approx pu - \pi^{ab}K_{ab} \approx 0,
\]

where the density has been written as functionally dependent on the other momenta in order to stress that it must be considered as a functional of those fields alone (as in (2.5)) when calculating Poisson brackets. Although all of the Lagrange multiplier variations have not been dealt with yet, the conditions that result from requiring that these constraints are preserved in time will prove useful at this stage. Requiring that \( \psi_1 \) be preserved in time results in:

\[
\partial_t \psi_1 = \{ \psi_1, H_* \} \approx -\frac{\delta H_*}{\delta u} - \frac{2}{2-F}\beta_a\frac{\delta H_*}{\delta \mathcal{W}_a} + \frac{1}{2-F}\frac{\delta H_*}{\delta K_{ab}} \approx -\frac{4}{2-F}N\mathcal{C}_1 \approx 0,
\]

where (2.5) has been used. This directly gives the remaining algebraic compatibility condition (A.4a). The preservation of \( \psi_2 \) gives:

\[
\partial_t \psi_2 = \{ \psi_2, H_* \} \approx \phi p - \pi^{ab}\phi_{ab} - u\frac{\delta H_*}{\delta u} + K_{ab}\frac{\delta H_*}{\delta K_{ab}} \approx -\frac{4}{2-F}\sqrt{-g}\mathfrak{R} + 2N\mathfrak{S}_\perp,
\]

where \( \mathfrak{R} \) is defined in (4.10) and \( \mathfrak{S}_\perp \) by (A.10a).

### 4.3. Variations of \( \{ \gamma \} \) and \( \{ N \} \)

In order to compute the variations with respect to \( \{ \gamma \} \), it is convenient to rewrite the extended Hamiltonian density (3.7a) in the form (\( \{ 3 \} \) is defined in (3.8)):

\[
H_* \sim H_m + \sqrt{-g}(N\mathcal{F}_{\perp \perp} + L_\kappa^{(3)}[u] - N\phi) + B^a(-2N\mathcal{F}_{[a\perp]} + L_\kappa^{(3)}[\mathcal{W}]_a - 2N\phi_a)
\]

\[
+ \gamma^{ab}(-N\mathcal{F}_{ab} + L_\kappa^{(3)}[K]_{ab} - N\phi_{ab}),
\]

where \( \sim \) refers to the fact that some conditions and constraints have been imposed in \( H_* \), provided that the results of the variations are not weakly affected. (Note that this form holds for variations with respect to \( \{ \gamma \} \) only.) The variation of \( H_m \) as defined in (3.1a) can be written (once again making use of (2.5)):

\[
\delta H_m = -\frac{1}{4}Nm^2M_{AB}\delta \mathfrak{S}^{AB}
\]

\[
= -\frac{1}{2}Nm^2(M_{[A\perp]} - \frac{1}{2-F}\beta_aM_{\perp \perp})\delta B^a + \frac{1}{4}Nm^2(M_{ab} - \frac{1}{2-F}G_{ba}M_{\perp \perp})\delta \gamma^{ab},
\]

and one then computes the conditions enforced by \( \{ \gamma \} \) to be (using (4.10)):

\[
\frac{\delta H_*}{\delta B^a} \approx -\frac{1}{4}Nm^2M_{ab}\delta \mathcal{F}^{ab} \approx 0,
\]

\[
\frac{\delta H_*}{\delta \mathfrak{S}^{ab}} \approx -\frac{1}{4}Nm^2(M_{[a\perp]} - \frac{1}{2-F}G_{ba}M_{\perp \perp})\delta \gamma^{ab} \approx 0.
\]

As it stands, the Hamiltonian \( H_* \), is not in the standard form: \( \int^\Sigma(N\mathcal{H} + N^a\mathcal{H}_a) \). It may be put in this form by the removal of the surface term:

\[
E_* := \int_{\partial \Sigma} dS_a[N^a\mathcal{W}_bB^a + 2N^b\mathcal{F}_b - 2NL^a],
\]

so that

\[
H_* = \int_{\Sigma} d^3x(N\mathcal{H} + N^a\mathcal{H}_a) + E_*.
\]
(Further discussion of surface terms will occur in the next section.) One then finds directly the Hamiltonian constraint $\mathcal{H}$:

$$\mathcal{H} := \frac{\delta H_\pi}{\delta N} \approx -2\Phi^1_a \approx 0,$$

(4.19a)

which has been weakly identified with the algebraic field equation (A.10a) by making use of previous results. This constraint, combined with (4.13) and (4.16), establishes that $\{\eta\} \approx 0$. The momentum constraints are:

$$\mathcal{H}_a := \frac{\delta H_\pi}{\delta N^a} = 2B^b\nabla_a^{(3)}[\overline{W}_b] - \overline{W}_a \nabla_a^{(3)}[B]^b + \gamma^{bc} \nabla_a^{(3)}[K]_{bc} + \sqrt{-g} \nabla_a^{(3)}[u] - 2\nabla_b^{(3)}[k]_a^b,$$

$$\approx p^b c_a[\overline{W}_b] - e_b[p^b\overline{W}_a] + \pi^{bc} c_a[K_{bc}] + p e_a[\{u\}] - 2e_b[K]_a^b,$$

$$\approx -2\Phi^1_a \approx 0,$$

(4.19b)

which has been identified with the related algebraic field equations (A.10b). The identity (A.7) has been used in the momentum constraint, there seems to be no relatively simple formulation of the Hamiltonian constraint in terms of the Lagrange multipliers.

One then finds directly the Hamiltonian constraint (4.22a) for each specific situation. As a simple example, one may assume an asymptotically flat spacetime in which all of the antisymmetric functions fall off fast enough as $r \to \infty$ so that they will not contribute any surface terms. Assuming the fall-off of the symmetric sector variables as given in (4.10), one finds that there are no
additional contributions from (4.22). Evaluating the non-vanishing surface terms: \( \int_{\partial \Sigma} dS_a N \sqrt{-g}(\gamma^{bc})\Gamma^a_{bc} - (\gamma^{ab})\Gamma^a_{bc} \), assuming the asymptotically Schwarzschild form given in [10], yields the Schwarzschild mass parameter as expected.

4.5. Completing the Constraint Analysis. It remains to be shown that the Hamiltonian and momentum constraints are preserved under time evolution (they lead to no further constraints), and that they can both be taken to be first class, leaving \( \{ \psi \} \) second class. It would then be possible to define Dirac brackets and impose strongly \( \{ \psi \} \) in order to examine the momentum constraints, it is useful to note (and can be explicitly checked) that they do indeed generate spatial diffeomorphisms on \( \Sigma \):

\[
\{ Q_{t}, H_{a}[f^a] \} \approx \mathcal{E}_{f}^{(3)}[Q_{t}], \quad \{ P^{I}, H_{a}[f^a] \} \approx \mathcal{E}_{f}^{(3)}[P^{I}],
\]

and, as this may obviously be extended to any tensor field on \( \Sigma \) built strictly out of the canonical variables as: \( \{ T, H_{a}[f^a] \} \approx \mathcal{E}_{f}^{(3)}[T] \), one finds (by explicit calculation, or using the above results for the last three):

\[
\begin{align*}
\{ \psi_{1}[f_{1}], \psi_{2}[f_{2}] \} & \approx - \int_{\Sigma} d^{3}z f_{1}f_{2}p^{A} \frac{4}{2 - F}, \\
\{ \psi_{1}[f], H_{a}[f^a] \} & \approx - \int_{\Sigma} d^{3}z \psi_{1} \mathcal{E}_{f}^{(3)}[f] \approx 0, \\
\{ \psi_{2}[f], H_{a}[f^a] \} & \approx - \int_{\Sigma} d^{3}z \psi_{2} \mathcal{E}_{f}^{(3)}[f] \approx 0, \\
\{ H_{a}[f_{1}^{a}], H_{b}[f_{2}^{b}] \} & \approx - \int_{\Sigma} d^{3}z H_{a} \mathcal{E}_{f_{2}}^{(3)}[f_{1}] \approx 0.
\end{align*}
\]

It has also been checked by explicit calculation that the time evolution of the momentum constraint (4.19b) is weakly vanishing \( \partial_{t} H_{a} \approx 0 \).

Thus the only condition required to show that the constraint algebra closes, is that \( \{ \mathcal{H}[M], \mathcal{H}[N] \} \) is some linear combination of the Hamiltonian and momentum constraints, and therefore vanishes weakly. This is extremely difficult to check explicitly since (as noted before) \( \mathcal{H} \) is not given in a simple form in terms of canonical variables alone, but instead depends on Lagrange multipliers which are difficult to solve for in general.

This calculation however, need not be performed explicitly.

Considering the dynamics of NGT within the larger arena of hyperspace (the manifold of all spacelike embeddings [21]), one may consider the surfaces \( \Sigma_{t} \) as embedded in the Riemannian manifold \( (\mathbf{M}, g^{(AB)}) \). A particular foliation defines a path in hyperspace, the tangent to which is the vector \( t \), as defined in Section 2.1. The hypersurfaces \( \Sigma_{t} \) can then be viewed as deformations of an initial hypersurface \( \Sigma_{0} \) along the vector field \( t \), and the Hamiltonian must then generate the evolution of the fields under these hypersurface deformations. This implies that one must be able to write the Hamiltonian in the standard form: \( \int_{\Sigma} d^{3}x (N\mathcal{H} + N^{a}\mathcal{H}_{a}) \) [10]. It can then be proved from the principle of path independence of dynamical evolution [22], that the Hamiltonian closing relations are constrained to take on the form:

\[
\begin{align*}
\{ \mathcal{H}[M], \mathcal{H}[N] \} & = \int_{\Sigma} d^{3}z \mathcal{H}_{a}(\gamma^{(ab)}(M e_{b}[N] - N e_{b}[M])), \\
\{ \mathcal{H}[M], \mathcal{H}_{b}[N^{a}] \} & = - \int_{\Sigma} d^{3}z \mathcal{H}_{a} \mathcal{E}_{N}^{(3)}[M], \\
\{ \mathcal{H}_{a}[M^{a}], \mathcal{H}_{b}[N^{a}] \} & = - \int_{\Sigma} d^{3}z \mathcal{H}_{a} \mathcal{E}_{N}^{(3)}[M]^{a}.
\end{align*}
\]

This principle essentially requires that the data on \( \Sigma_{0} \) evolve uniquely to \( \Sigma_{t} \), regardless of how \( \Sigma_{0} \) is deformed into \( \Sigma_{t} \) (i.e., independently of how one foliates the spacetime in-between), or equivalently, that evolution along different paths in hyperspace (between identical initial and final points) yield identical results. This also ensures that the

9This would be accomplished quite simply by replacing \( u \) strongly everywhere using (4.11b), and considering both \( p \) and \( \sqrt{-g} \) as functions of \( b^{a} \) and \( \eta^{a} \), as determined by (4.11a) and (2.5). Thus \( (p, u) \) would no longer be included in phase space, nor would they appear anywhere in the Hamiltonian.

10Most of these Lagrange multipliers can in fact be ignored in Poisson brackets if \( \mathcal{H} \) is kept in a form very similar to that appearing in \( H_{s} \), as their variations will vanish weakly using the algebraic compatibility conditions. The variation of \( L^{a} \) (or \( c_{a} \) if \( a_{a} \) has been strongly removed from the system) will only vanish if the smearing function is exactly the lapse function \( N \). This is not the case in general, and one would then require the variation of \( b_{a} \) with respect to canonical variables in order to compute Poisson brackets containing \( H_{s} \); a variation that is not easy to compute.
collection of all possible \( \Sigma_t \)'s is can be interpreted as describing different slicings of \(( M, g^{(AB)})\), and is therefore a reflection of the diffeomorphism invariance of spacetime\(^{[3]}\).

The Hamiltonian for NGT is thus found to be of the form \((4.21)\), consistently generating the field equations and compatibility conditions necessary to make Hamilton’s equations \((1.6)\) reproduce the field equations \((1.3, 1.4, 1.3)\) derived from the action \((1.1)\) via the Euler-Lagrange equations. The number of degrees of freedom for mNGT may now be easily computed using the standard algorithm as given by equation \((1.60)\) of \([13]\). (For now it will be assumed that all of the Lagrange multipliers may be solved for, however this issue is discussed further in Section 6.) In this case, one finds that the number of configuration space degrees of freedom per spacetime point is:

\[ \text{# of canonical constraints} \]

that all of the Lagrange multipliers may be solved for, however this issue is discussed further in Section 6. In this case, one finds that the number of configuration space degrees of freedom per spacetime point is: the # of canonical constraints \((\{Q_1\})\) in \((3.8)\) \((13)\) – the # of first class constraints \(\{H, H_a\}\) in \((3.4)\) \((4)\) – \(\frac{1}{2} \times \text{# of second class constraints} \((\{ \psi \})\) in \((4.11)\) \((2)\) = 8. Since all of the constraints exist in the symmetric sector, two of these are the propagating modes of GR, and the remaining six occur in the NGT sector. In the next section, the dynamics of pure GR configurations is reduced to that of GR (with a subtlety), and the limit to old NGT (Einstein’s Unified Field Theory) is discussed.

5. THE REDUCTION TO GR AND OLD NGT

It is worthwhile at this point to make further contact with GR by considering how one reduces NGT to GR configurations, as well as considering the dynamics of old NGT.

5.1. GR. There is, of course, an important surface in phase space on which all of the antisymmetric components of the fundamental tensor are identically zero. If one sets up the system identically on this surface, it should remain on it since the dynamics there are identically those of GR (there is a complication, which will be discussed below). One could easily just eliminate all skew objects from the action, and derive the ADM results directly. In the context of NGT-type theories one would prefer to view the GR dynamics as occurring in the larger phase space of NGT, and view the reduction of dynamics as the imposition (by hand) of a constraint on the initial Cauchy surface that puts all skew Cauchy data weakly to zero. Consistent dynamics on this surface will require that this constraint be preserved in time.

Considering first the case when \(\alpha \neq 3/4\), one sets up fields on the initial surface such that all skew sector Cauchy data \((p^a, \pi^{(ab)}, j_{ab}, \overline{W}_a)\) vanish identically: the algebraic compatibility conditions require that \(v^a_b = 0\) and \(\lambda^b_{bc} = b_a = 0\) \((A.2a)\) \((A.2c)\) \((A.2d)\) respectively). The time evolution conditions \((A.2a)\) \((A.2b)\) \((A.2c)\) \((A.2d)\) \((A.17)\) guarantee that the antisymmetric components of the fundamental tensor \((B^a, \gamma^{(ab)}\) respectively) will remain zero at later times. The skew sector conjugate momenta vanish by the antisymmetric part of \((4.1a)\) \((4.1b)\) \((4.11)\) \((3)\), and remain zero in evolution once the previous results are inserted into \((4.9)\) through the equations defining \(\phi \). The skew sector is thus consistently eliminated from any future dynamics once all skew Cauchy data is set to zero on the initial hypersurface. The symmetric compatibility conditions \((A.3)\) reduce to those of GR, as do the symmetric time evolution equations in \((4.8)\) for \((3)\) \((3)\) \((3)\). The Hamiltonian and momentum constraints \((4.19a), 4.19b)\) then take on the familiar forms \((R^{(3)} = \gamma^{ab} F^{(3)}_{ab})\):

\[
\begin{align*}
\mathcal{H} &\approx k^b_a k^a_b - k^b_a k^a_b - R^{(3)} \approx 2\mathfrak{g}^b_a \approx 0, \\
\mathcal{H}_a &\approx -2\nabla_a [k^b_a - \delta_a^b k] \approx -2\mathfrak{g}^b_a \approx 0.
\end{align*}
\]

(5.1a)

(5.1b)

Thus given \(\alpha \neq 3/4\), one finds that if at any time, all of the antisymmetric sector canonical variables vanish, then the ensuing dynamics of the system is identical of that of GR. The case where \(\alpha = 3/4\) is slightly different, in that when one imposes \((A.2a)\) \((A.2c)\) \((A.2d)\) (and the remaining conditions), \(b_a\) is left undetermined. Then, although one still finds that \(\partial_t \pi^{(ab)} \approx 0\) from \((3.8)\), the evolution of \(p^a\) is determined from \(\partial_t p^a \approx \frac{3}{2} N\pi^{(ab)} b_b\), and is therefore undetermined. Similar behavior occurs with the canonical coordinates \(j_{ab}, \overline{W}_a\), as there are nonvanishing contributions from \(b_a\) to \((3)\). Thus the skew sector only remains trivial provided that \(b\) is chosen to be zero on every surface, and the existence exists that the system may pass through a GR configuration without remaining there, in contrast to the case above\(^{[4]}\).

\(^{[3]}\) In fact, a close inspection of the arguments in \([4]\) adapted to this case, shows that one only knows \((1.29)\) up to a c-number (i.e. independent of the canonical variables). This c-number has been assumed to be zero in this case, as it would not lead to further constraints, but to an inconsistent system of equations. It is generally believed that the action for NGT generates a consistent set of field equations, and conservation laws \((A.11)\) have been derived based on diffeomorphism invariance that imply that this c-number vanishes.

\(^{[4]}\) Note that requiring \(\partial_t p^a \approx \partial_t \pi^{(ab)} \approx 0\) on the initial surface is actually a stronger condition than \(j_{ab}, \overline{W}_a \approx 0\), as it implies that \(b_a \approx 0\) initially.
5.2. Old NGT. The reduction to old NGT (or Einstein’s Unified Field Theory) is accomplished by setting $\alpha = 0$ (setting $m = 0$ will not affect results, and can easily be left nonzero). The algebraic compatibility conditions in this case may be solved for the determined Lagrange multipliers using similar methods to those in [12, 43, 44], with only mild conditions on the components of the fundamental tensor. However (A.3) is now a constraint related to the $U(1)$ invariance that the action will now possess [25]. It is straightforward to show that this constraint is first class (its Poisson bracket with all other constraints is weakly vanishing) and generates $U(1)$ transformations on $\Sigma$:

$$\left\{ \mathbf{W}_a, e_b \right\}[\theta] \approx e_a[\theta].$$

There is then left five configuration space degrees of freedom per spacetime point, consistent with the results for the Unified Field Theory [7] as well as not NGT [4, 22] and [22]. In the last of these, ad hoc constraints are applied in Section 4 so as to remove $W_a$ from the dynamics, guaranteeing a weakly positive-definite Hamiltonian for the linearized theory. These constraints cannot in fact be imposed even in the linearized theory [4], since lack of gauge invariance in the full action [19, 20] manifests itself as a non-conserved source for the skew tensor field, exciting all five modes even at the linearized level. This is essentially equivalent to the results of Damour, Deser and McCarthy [3] who avoid discussing the source terms directly by demonstrating that the same effect is generated through the coupling of the skew field to a GR background.

6. A Closer Look at the Compatibility Conditions

The symmetric sector algebraic constraints are easily solved for in terms of the skew sector Lagrange multipliers and Cauchy data. $\Gamma$ is determined by (A.4a), $e_a$ is determined by (A.4b), $\alpha$ is determined by (A.2c), $\sigma^\alpha$ is determined by (A.1b), $u_{\alpha}^\gamma$ is determined by (A.1c), the trace of which is the constraint $\psi_2$. The relation (A.1f) can be used to uniquely solve for $\lambda_{bc}^\alpha$, provided the inverse of $\sigma^{(ab)}$ exists. In the skew sector, $\nu^\alpha_a$ is determined by (A.2d), where the trace gives (A.3c), determining the Lagrange multiplier $W$. The remaining equations for $\lambda_{bc}^\alpha$ and $b_{\alpha}$ (A.2d), are complicated since one must first replace the symmetric sector Lagrange multipliers, and then solve for the Lagrange multipliers in terms of Cauchy data. The behavior relevant to this sector occurs in (A.3c) alone, leaving the trace of (A.2d) to determine $\lambda_{bc}^\alpha$. (A.3c) can be written as an algebraic relation for $b_{\alpha}$ solely in terms of Cauchy data (using (A.4b) and (A.1d)):

$$p_\mu^\nu \sigma^{\alpha \beta} \approx \frac{\lambda_{bc}^\alpha}{2} B^\mu B^\nu - \frac{1}{2} \lambda_{bc}^\alpha G_{\mu \nu} \gamma_{(cd)} b_b.$$

which will be written as (with obvious definitions):

$$p^{\alpha \beta} b_b \approx \Xi^\alpha.$$

The issue at hand is whether or not the operator $O^{\alpha \beta}$ is invertible in general. Clearly if the skew sector is weak enough and $\alpha \neq 3/4$, then $\gamma^{(ab)}$ will dominate the operator, and $O_{ab}^{-1}$ will exist. More generally there may exist only a subspace on which it is invertible, in which case the component(s) of $b_{\alpha}$ in this space can be solved for, and the other(s) are left undetermined. This will also imply that there are constraints corresponding to any undetermined components of $b_{\alpha}$ of the form: $\Xi^\alpha \approx 0$. One would then need to check that this constraint is preserved in time $\partial_t \Xi^\alpha \approx 0$, in order to determine whether it is possible for the system to remain on this constraint surface. As discussed below in the case where the Cauchy data is dominated by the symmetric sector, it appears more likely that these are momentary configurations that the system may pass through.

For the case of mNGT ($\alpha = 3/4$), one can easily see that the operator $O^{\alpha \beta}$ is not dominated by the symmetric sector, and that in the absence of any antisymmetric components of the fundamental tensor, it disappears altogether. Considering $(B^\mu, \gamma^{ab})$ as a perturbation on $\gamma^{(ab)}$, all quantities may be expanded to lowest order in powers of $(B^\mu, \gamma^{ab})$:

$$\beta_\mu = B_\mu = \gamma^{ab} B^b, \quad \alpha_\mu = -\gamma^{(ab)} B^b, \quad \sqrt{-g} \approx \sqrt{\gamma}, \quad G_{(ab)} = \gamma_{(ab)}, \quad G_{[ab]} = -\gamma_{[ab]} = -\gamma_{(ac)} \gamma^{(bd)} \gamma^{(cd)}.$$

If one assumes that $b_\mu$ is of order $(B^\mu, \gamma^{ab})$ (indicated heuristically by $O(\text{skew})$), then (6.1b) becomes three constraint equations $\Xi^\alpha \approx 0$. These lead to $\partial_t \Xi^\alpha \approx 0$ for consistency, and one has obtained the dynamics of massive a Kalb-Ramond field on a GR background consistent with the linearized treatment of the field equations given in [37, 26, 24]. This is not the general case, as one does not have the freedom to assume the behavior of the Lagrange multipliers, it must instead be derived from the compatibility conditions. In particular, (6.1b) determines $b_\alpha$ in terms of Cauchy
data, and keeping the dominant terms, one must find an inverse for the operator $O^{ab}$:

$$O^{ab} := B^a B^b + \frac{1}{4} \gamma^{[ac]} \gamma^{[cd]} (\gamma^{(db)} - \gamma^{(ab)} \gamma^c + \gamma^a \gamma^b),$$  \hspace{1cm} (6.3)

where in the second form the following have been introduced:

$$\gamma^{[ab]} := \frac{2}{\sqrt{g}} e^{abc} \gamma_c, \quad \gamma_a := \frac{1}{4} \sqrt{\gamma} e_{abc} \gamma^{[bc]}, \quad \gamma_{[ab]} = 2 \sqrt{\gamma} e_{abc} \gamma^c, \quad \gamma^a := \frac{1}{4 \sqrt{\gamma}} e^{abc} \gamma_{[bc]}.$$  \hspace{1cm} (6.4)

The Levi-Civita tensor density of weight $-1$: $\epsilon^{abc}$ and $+1$: $\epsilon^{abc} = \gamma^{(ad)} \gamma^{(be)} \gamma^{(cf)} \epsilon_{def}$ have been used ($\epsilon^{123} = \epsilon_{123} = +1$), and the notation $\tilde{\gamma} \cdot \tilde{\gamma} = \gamma^a \gamma_a$ has been introduced. (Note that these are not quite duals; the extra numerical factor is introduced for convenience.) The inverse of $O$ is found to be:

$$O_{ab}^{-1} = -\frac{1}{\tilde{\gamma} \cdot \tilde{\gamma}} \left[ \gamma_{(ab)} - \frac{1}{\tilde{\gamma} \cdot \tilde{\gamma}} (B_a \gamma_b + \gamma_a B_b) + \frac{\tilde{\gamma} \cdot \tilde{\gamma}}{\tilde{\gamma} \cdot \tilde{\gamma}} \right].$$  \hspace{1cm} (6.5)

(Although the nonperturbative inversion of the operator $O^{ab}$ in (6.1b) may be accomplished using similar techniques, the general result is not particularly enlightening and will not be given here.) If $\tilde{\gamma} \cdot \tilde{\gamma} = 0$ (in which case $\tilde{\gamma} = 0$ if one assumes that $\gamma^{(ab)}$ is nondegenerate), then assuming that $\tilde{B} \neq 0$, one finds two constraints coming from projecting perpendicular to $\tilde{B}$, and one relation for $b_a$ from that parallel to $\tilde{B}$. If $\tilde{\gamma} \neq 0$ and either $\tilde{B} = 0$ or $\tilde{\gamma} \cdot \tilde{B} = 0$ then there is one constraint from projecting parallel to $\tilde{\gamma}$, and two conditions determining $b$ from the parallel component. (The remaining operator when $\tilde{B} = 0$ projects out directions transverse to $\gamma$). It has not been possible so far to find any of these cases which can be maintained in evolution, and so they are expected to be momentary configurations and not surfaces on which the system may evolve consistently.

The solution (6.5) for $b_a$ implies that $b_a \approx O^{-1} \Xi_b$ is not $O$(skew), but in fact $O$(skew$^{-1}$). (This does not happen in Einstein Unified Field Theory (or old NGT) since $\alpha = 0$ leaves a term that allows one to solve for $b_a$ at lowest order.) This in turn shows up in the evolution equations of the antisymmetric sector canonical variables (1.3) and (1.13), causing them to evolve arbitrarily quickly as the skew initial data is made smaller. It is possible to have $\Xi \approx O$(skew$^3$), in which case $b$ will again reduce to $O$(skew), and this behavior is avoided. However, this is a condition on the variable $b$ itself, and since one could easily set up initial data that does not satisfy this condition, it would have to be realized dynamically. At this point it is not known whether this is a reasonable expectation, though it seems unlikely that it would occur for any initial data configuration in which the skew sector is small.\footnote{Note also that although one might expect that $\mathcal{H}$ might impose this constraint on the system due to the presence of terms of the form $\gamma^{(ab)} b_a b_a$ in $\mathcal{H}$, it can be shown that these terms identically cancel once $\alpha = 3/4$ and $\lambda^{bc} = \lambda^{bc}_{\gamma} + \delta^{bc} b_1$ is employed.}

This sort of behavior in the skew sector must be better understood, as the arbitrarily large time derivatives cause one to worry about whether GR spacetimes would be unstable in mNGT. It is also not hard to see that the same situation may occur near NGT spacetimes as well. As a specific example, consider the solution of (6.1a) specialized to spherically symmetric systems. In the general case where both of $B^1, \gamma^{[23]}$ are non-zero, the solution is:

$$b_1 \approx \frac{1}{2} \frac{\gamma^{11} W_1}{(B^1)^2} + \frac{k^2}{B^1}.$$  \hspace{1cm} (6.6)

Thus one finds that in regions of spacetime where $B^1$ is vanishingly small (for example, perturbative situations, or in the asymptotic region of asymptotically flat spacetimes), $b_1$ becomes arbitrarily large, which in turn may cause $(\partial_t p^1, \partial_t W_1, \partial_t j_{23})$ to become very large. Although the Wyman sector solution (3, 8) (which is becoming the basis for most of the phenomenology in mNGT [17, 20, 39, 38], assumes that both $B^1$ and $W_1$ vanish globally, if one considers perturbations of these fields on $\Sigma_0$, the above behavior reappears.

This is essentially the same effect as was found in [13], where the effect of gravitational dynamics on the constraints of various derivative coupled vector fields was studied. It was found that constraints on the vector field may be lost when GR is considered as evolving concurrently with the vector field (as opposed to the vector field evolving on a particular GR background). This manifested itself as an increase in the number of degrees of freedom in the vector field, and singular behavior in the evolution equations when approaching asymptotically flat spacetimes. There one finds no evidence of this when considering the vector field dynamics on a fixed GR background.

This is intimately linked to degenerate solutions of the Euler-Lagrange equations. If one considers linearized perturbations to such a solution, the field equations require that the perturbations vanish, whereas the analysis of the full dynamics near the degenerate solution may result in quite different behavior. This degeneracy also results in the loss of hyperregularity (Chapter 7.4 of [37]) for the Lagrangian system, and equivalence of the Hamiltonian system.
is not guaranteed. However in this case, Hamilton’s equations correctly reproduce the Euler-Lagrange equations provided the system avoids configurations corresponding to these degenerate solutions.

Conclusions

The Hamiltonian formulation of both old (massless) NGT, as well as the massive theory have been given, identifying the former as having five antisymmetric sector configuration space degrees of freedom per spacetime point, and the latter six. The symmetric sector of phase space has been enlarged over that of GR by a single canonical pair representing the fundamental tensor density \( \sqrt{-g} \). The existence of two related second class constraints allows one to remove these additional coordinates and recover the same number of degrees of freedom as GR in the symmetric sector. The reduction of the system to pure GR field configurations has been discussed and the dynamics of GR recovered in the case of massless NGT. In the massive theory, nontrivial dynamics of the antisymmetric sector occurs unless an undetermined Lagrange multiplier is chosen to vanish identically on every spacelike hypersurface.

All fields have been decomposed by making use of a surface basis adapted to \( g^{\langle AB \rangle} \), where the surface has been assumed to be spacelike (and the time vector timelike) with respect to all of the causal metrics of NGT \( \{g_r\} \). Configuration space has been taken to consist of densitized components of the inverse of the fundamental tensor for convenience, broken up into the lapse and shift functions and the surface degrees of freedom: \( (N, N^a, \gamma^{(ab)}, B^a, \gamma^{[ab]} ) \).

A.1. Compatibility Conditions

The symmetric components of (1.4) are found to be:

\[
\mathcal{C}^{(\perp \perp)} = d_{e\perp} [\sqrt{-g}] + \Gamma - u - 2B^a b_a, \tag{A.1a}
\]

\[
\mathcal{C}^{(ab)} = -d_{e\perp} [\gamma]^{(ab)} + \gamma^{(ab)} (\Gamma + u) - 2u^{ab}, \tag{A.1b}
\]

\[
\mathcal{C}^{(a \perp)} = \sigma^a - \gamma^{(ab)} a_b + B^{b} a^b_a + \gamma^{[ab]} b_b, \tag{A.1c}
\]

\[
\mathcal{C}^{(\perp \perp)} = \nabla^{(\perp)} [\sqrt{-g}] + c_a + 2j_{ab} B^b, \tag{A.1d}
\]

\[
\mathcal{C}^{(a \perp)} = u^a b_b - B^a b_b - B^c \lambda^a_{cb} - k^a, \tag{A.1e}
\]

\[
\mathcal{C}^{(ab)} = -\nabla^{(\perp)} [\gamma]^{(ab)} + \gamma^{(ab)} c_c + B^a a^b_c + B^b a^a_c + \gamma^{[ab]} \lambda^b_{dc} + \gamma^{[db]} \lambda^a_{cd}, \tag{A.1f}
\]
and the skew components:
\[
\begin{align*}
C^{[\perp a]}_e &= d_{e_\perp} B^a - B^a u + B^b u^b + \gamma^{[ab]} a_b - (1 + \frac{2}{3} \alpha) \gamma^{(ab)} b_b + \Gamma^a_{\alpha\delta} \gamma^{(ab)} W_b, \\
C^{[a \perp]}_c &= -d_{e_\perp} [\gamma]^{[ab]} - B^a \sigma^b + B^b \sigma^a + \gamma^{[ab]} (\Gamma + \alpha) - 2 v^{ab}, \\
C^{[a \perp]}_b &= \nabla_b^{(3)} B^a - v^a + \frac{3}{2} \alpha \delta^a_b W, \\
C^{(a \perp)}_c &= -\nabla_c^{(3)} [\gamma]^{[ab]} - B^a u^e + B^b u^e + \gamma^{[ab]} c_b + (\Gamma + \alpha) \lambda_{ac} + \gamma^{(ab)} \chi^a_{dc} + \frac{3}{4} \alpha \delta^a_b \gamma^{(dc)} (W_c - 2 b_c).
\end{align*}
\]  

The first two of each of these are dynamical compatibility conditions, and the rest are algebraic conditions that determine Lagrange multipliers. It is useful to have the contractions of a few of these:
\[
\begin{align*}
C^{(\perp a)}_a &= u - k_a, \\
C^{[a \perp]}_b &= -\nabla_b^{(3)} [\gamma]^{[ab]} + \gamma^{[ab]} c_b + B^b v^a - \gamma^{[ab]} b_b + \gamma^{[bc]} \lambda^a_{cb}, \\
C^{[\perp a]}_c &= \nabla_c^{(3)} B^a - \alpha W, \\
C^{[a \perp]}_b &= -\nabla_b^{(3)} [\gamma]^{[ab]} - B^a u + B^b u_b + \gamma^{[ab]} c_b + (\Gamma - 1) \gamma^{(ab)} b_b - \frac{3}{4} \alpha \gamma^{(ab)} W_b.
\end{align*}
\]

Note that there is one further algebraic dynamical compatibility condition buried in the dynamical compatibility conditions. By taking appropriate combinations in order to make use of (2.5) (replacing the variation with a derivative off the surface), one can find:
\[
C_\Gamma := \Gamma - B^a b_a + \frac{1}{6} \alpha \gamma^{(ab)} \beta_a (W_b - 2 b_b),
\]

\[\text{(A.4a)}\]

\[\text{giving the remaining algebraic condition that determines \( \Gamma \). Repeating this last calculation and considering a derivative in the surface yields a condition that determines \( c_a \):}
\[
c_c := c_a + j_{ab} B^b - \frac{1}{6} \alpha \gamma^{(ab)} (W_c - 2 b_c),
\]

\[\text{(A.4b)}\]

although this is not independent of the algebraic conditions already found.

### A.2. Algebraic Field Equations and Conservation Laws.

The decomposition of the field equations (2.5d) yields:
\[
\begin{align*}
\mathcal{R}_{\perp \perp} &= -d_{e_\perp} [u] + 3_{\perp \perp} - \frac{1}{2} m^2 M_{\perp \perp}, \\
\mathcal{R}_{(a \perp)} &= -d_{e_\perp} [j_{ab} B^b] + 3_{(a \perp)} - \frac{1}{2} m^2 M_{(a \perp)}, \\
\mathcal{R}_{[a \perp]} &= \frac{1}{2} d_{e_\perp} [W]_a + 3_{[a \perp]} - \frac{1}{2} m^2 M_{[a \perp]}, \\
\mathcal{R}_{ab} &= d_{e_\perp} [K]_{ab} + 3_{ab} - \frac{1}{2} m^2 M_{ab},
\end{align*}
\]

the second of which has made use of the Jacobi identity (2.8). For completeness, the components of the mass tensor are:
\[
\begin{align*}
M_{\perp \perp} &= F (F - 1) - 2 \gamma^{(ab)} \beta_a \beta_b + \frac{1}{2} F \gamma^{[ab]} G_{[ab]}, \\
M_{(a \perp)} &= -\frac{1}{2} [\gamma]^{[bc]} G_{[bc]} \alpha_a + \beta_c G_{[ab]} \gamma_{[bc]} - G_{(ab)} \gamma^{[bc]}, \\
M_{[a \perp]} &= -(F - 1) G_{(ab)} - \frac{1}{2} G_{[bc]} \gamma^{[cd]} G_{[cd]} + 2 \beta_a \beta_b - 2 \alpha \alpha \alpha_b + \frac{1}{2} \gamma^{[cd]} (G_{cd} G_{bd} - G_{db} G_{cd}), \\
M_{ab} &= (F - 2) G_{[ab]} + \frac{1}{2} G_{[ab]} \gamma^{[cd]} G_{[cd]} + 2 \beta_a \beta_b - 2 \beta_b \alpha_a - \frac{1}{2} \gamma^{[cd]} (G_{cb} G_{ad} - G_{ca} G_{bd}),
\end{align*}
\]

for which a few identities exist:
\[
\begin{align*}
M_{\perp \perp} + \gamma^{ab} M_{ab} &= -2 (F - 1) - \gamma^{[cd]} G_{[cd]}, \\
M_{(a \perp)} + M_{[a \perp]} B^b &= 0, \\
\gamma^{(ab)} M_{(\perp b)} + \gamma^{[ab]} M_{[\perp b]} &= 0.
\end{align*}
\]

\[\text{(A.7a)}\]

\[\text{(A.7b)}\]

\[\text{(A.7c)}\]
The remaining objects in (3.5) \{3\} are:

\[ Z_{\perp} := \nabla^{(3)}_a [\sigma]^a + \Gamma u - u^a b u^b + \sigma^{a}(a_{a} - 2 c_{a}) + v^{a} v^{\dot{a}} - \frac{1}{2} \alpha(W)^2, \]  
(A.8a)

\[ Z_{(a \perp)} := -\nabla^{(3)}_a [u] + \nabla^{(3)}_b [u] a - \nabla^{(3)}_a [B^b b] \]

\[ + u c_{a} - k a b \sigma^{b} - (a_{a} - c_{a}) B^{b} b_{b} + u_{a} (a_{b} - c_{b}) + v^{b} a b_{b} + v^{b} \lambda_{c b} + \frac{1}{2} \alpha(W_{a} - 2 a_{a}), \]  
(A.8b)

\[ Z_{(ab)} := R^{(3)}_{(ab)} - \nabla^{(3)}_a [a] b + (\Gamma + u) k_{a b} - a_{a} a_{b} - k_{c b} u^{c}_{a} - k_{c a} u^{c}_{b} \]

\[ + b_{a} b_{b} - j_{a c} v^{c}_{b} - j_{b c} v^{c}_{a} + \lambda^{c}_{a d} \lambda^{d}_{b c} - \frac{1}{2} \alpha(W_{a} - 2 a_{a})(W_{b} - 2 b_{b}), \]  
(A.8c)

\[ Z_{[a \perp]} := \frac{1}{2} \nabla^{(3)}_a [W] + \frac{1}{2} W (a_{a} - c_{a}) - \nabla^{(3)}_b [v] a_{b} + j_{a b} \sigma^{b} + (c_{b} - a_{b}) v^{b}_{a} + u b_{a} - u^{b}_{a} b_{b} - u^{b} \lambda_{c b}, \]  
(A.8d)

\[ Z_{[ab]} := -\nabla^{(3)}_a [W b] + 2 \nabla^{(3)}_a [b] a + \nabla^{(3)}_c [\lambda]^{c}_{a b} \]

\[ + (\Gamma + u) j_{a b} + a_{c} \lambda_{c b} + a_{d} a_{b} - a_{d} b_{a} - j_{a b} u_{c}^{c} - j_{a c} u_{b}^{c} - k_{c a} v_{b}^{c} + k_{c b} v_{a}^{c}. \]  
(A.8e)

One may construct the tensor density (these are, of course, just particular combinations of the field equations):

\[ \mathcal{G}^{A}_{B} := \frac{1}{2}(g^{AC} \mathcal{G}_{BC} + g^{CA} \mathcal{G}_{CB}) = \frac{1}{2}(g^{AC} R_{BC} + g^{CA} R_{CB} - \delta^{A}_{B} g^{CD} R_{CD}), \]  
(A.9)

where \( \mathcal{G}_{AB} = R_{AB} - \frac{1}{2} g_{AB} R \) is the equivalent of the Einstein tensor for NGT. It is a fairly straightforward calculation to show that the components \( \mathcal{G}^{A}_{\perp} \) and \( \mathcal{G}^{A}_{a} \) are both algebraic field equations:

\[ \mathcal{G}^{A}_{\perp} = \frac{1}{2} [\sqrt{-g} R_{\perp} + \gamma^{ab} R_{ab}] \]

\[ = \frac{1}{2} [\sqrt{-g} 3_{\perp} + \gamma^{ab} 3_{ab} - \frac{1}{2} m^{2} (\sqrt{-g} M_{\perp} + \gamma^{ab} M_{ab})] - \Gamma u + u B^{a} a + j_{a b} B^{a} \sigma^{b} + k^{a} u^{b}_{a} - j^{a} v^{b}_{a}, \]  
(A.10a)

\[ \mathcal{G}^{A}_{a} = \sqrt{-g} R_{(a \perp)} + B^{b} R_{[a b]} \]

\[ = \sqrt{-g} 3_{(a \perp)} + 3_{[a b]} B^{b} + j_{a b} [-\Gamma B^{b} + u^{b} B^{c} + 2 B^{a} B^{b} u_{c} + \gamma^{[b c]} u_{d} - (1 + \frac{1}{3} \alpha) \gamma^{(b c)} u_{d} + \frac{2}{3} \alpha \gamma^{(b c)} W_{c}], \]  
(A.10b)

analogous to the Gauss and Codacci relations respectively in GR. The conservation laws for NGT derived in \[25\] written in a general basis are:

\[ B_{A} := \nabla_{e a} [\mathcal{G}]^{A}_{e} + \frac{1}{2} R_{BC} \nabla_{e a} [g]^{B C} = 0, \]  
(A.11)

guaranteeing the existence of four Bianchi-type identities related to the above algebraic constraints. Note that these identities cannot be used here in this form, as the compatibility conditions have been imposed in their derivation. Although this will not be necessary here, one could in principle derive the form of \( (A.11) \) that would hold without imposing the compatibility conditions, and could therefore be used here to derive the closing relations \( (1.22) \) directly from the action \( (1.3) \); using the assumption of diffeomorphism invariance in a slightly more operational manner.
References

1. Ralph Abraham and Jerrold E. Marsden, *Foundations of mechanics*, 2 ed., Advanced Book Program, Addison-Wesley Pubishing Company, Inc., DonMills, Ontario, 1987.
2. R. Arnowitt, S. Deser, and C. W. Misner, *Dynamical structure and definition of energy in general relativity*, Phys. Rev. 116 (1959), 1322–1330.
3. Y. Choquet-Bruhat, D. DeWitt-Morette, and M. Dillard-Bleik, *Analysis, manifolds and physics*, vol. 1, North Holland, New York, 1981.
4. M. A. Clayton, *Massive NGT and spherically symmetric systems*, J. Math. Phys. 37 (1996), 395–420.
5. N. J. Cornish, *The nonsingular Schwarzschild-like solution to massive nonsymmetric gravity*, UTPT-94-37, 1994.
6. N. J. Cornish and J. W. Moffat, *A non-singular theory of gravity*, Phys. Lett. B 336 (1994), 337–342.
7. A. Crumeyrolle, *Variétés différentiables à structure complexe hyperbolique. Application à la théorie unitaire relativiste des champs*, Riv. Mat. Univ. Parma 8 (1967), 27–53.
8. T. Damour, S. Deser, and J. McCarthy, *Theoretical problems in nonsymmetric gravitational theory*, Phys. Rev. D 45 (1992), R3289–R3291.
9. *Nonsymmetric gravity theories: Inconsistencies and a cure*, Phys. Rev. D 47 (1993), 1541–1556.
10. Bryce S. DeWitt, *Quantum theory of gravity. I. The canonical theory*, Phys. Rev. 160 (1967), 1113–1148.
11. Marcelo Gleiser, Richard Holman, and Nelson P. Neto, *First order formalism for quantum gravity*, Nucl. Phys. B294 (1987), 1164–1179.
12. Jan Govaerts, *Hamiltonian quantisation and constrained dynamics*, Leuven Notes in Mathematical and Theoretical Physics, vol. 4, Series B: Theoretical Particle Physics, Leuven University Press, Leuven, Belgium, 1991.
13. Marc Henneaux and Claudio Teitelboim, *Quantization of gauge systems*, Princeton University Press, Princeton, New Jersey, 1992.
14. V. Hlavatý, *Geometry of Einstein’s unified field theory*, P. Noordhoff Ltd., Groningen, Holland, 1958.
15. J. L. Isenberg and J. M. Nestor, *The effect of a gravitational interaction on classical fields: A Hamiltonian-Dirac analysis*, Ann. Phys. 107 (1977), 56–81.
16. James Isenberg and James Nester, *Canonical gravity*, General Relativity and Gravitation (New York) (A. Held, ed.), vol. I, Plenum Press, 1980.
17. C. J. Isham and K. V. Kuchar, *Representations of spacetime diffeomorphisms. I. Canonical parameterized field theories*, Ann. Phys. 164 (1985), 288–315.
18. *Representations of spacetime diffeomorphisms. II. Canonical geometrodynamics*, Ann. Phys. 164 (1985), 316–333.
19. P. L. Kelly, *Expansions of non-symmetric gravitational theories about a GR background*, Class. Quantum Grav. 8 (1991), 1217–1229.
20. *Expansions of non-symmetric gravitational theories about a GR background*, Class. Quantum Grav. 9 (1992), 1423, Erratum.
21. Karel Kuchar, *Geometry of hyperspace. I*, J. Math. Phys. 17 (1976), 777–791.
22. G. Kunstatter, H. P. Leivo, and P. Savaria, *Dirac constraint analysis of a linearized theory of gravitation*, Class. Quantum Grav. 1 (1984), 7–13.
23. G. Kunstatter, J. W. Moffat, and J. Malzan, *Geometrical interpretation of a generalized theory of gravitation*, J. Math. Phys. 24 (1983), 886–889.
24. G. Kunstatter and R. Yates, *The geometrical structure of a complexified theory of gravitation*, J. Phys. A 14 (1981), 847–854.
25. J. Légaré and J. W. Moffat, *Field equations and conservation laws in the nonsymmetric gravitational theory*, Gen. Rel. Grav. 27 (1995), 761–775.
26. *Geodesic and path motion in the nonsymmetric gravitational theory*, UTPT-95-19, astro-ph/9509032, 1995.
27. A. Lichnerowicz, *Théories relativistes de la gravitation et de l’électromagnétisme*, Collection d’Ouvrages de Mathématiques à l’Usage des Physiciens, Masson et Cie., Paris, 1955.
28. R. B. Mann, *Five theories of gravity*, Class. Quantum Grav. 1 (1984), 561–572.
29. R. B. Mann and J. W. Moffat, *Ghost properties of generalized theories of gravitation*, Phys. Rev. D 26 (1982), 1858–1861.
30. Jerrold E. Marsden and Tudor S. Ratiu, *Introduction to mechanics and symmetry*, Texts in Applied Mathematics, vol. 17, Springer-Verlag, New York, 1994.
31. F. Maurer-Tison, *Aspects mathématiques de la théorie unitaire du champ d’Einstein*, Ann. scient. Éc. Norm. Sup. 76 (1959), 185–269.
32. J. C. McDow and J. W. Moffat, *Consistency of the Cauchy initial value problem in a nonsymmetric theory of gravitation*, J. Math. Phys. 23 (1982), 634–636.
33. Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *Gravitation*, W. H. Freeman and Company, New York, 1973.
34. J. W. Moffat, *A solution of the Cauchy initial value problem in the nonsymmetric theory of gravitation*, J. Math. Phys. 21 (1980), 1798–1801.
35. *Review of the nonsymmetric gravitational theory*, Gravitation: A Banff Summer Institute (New Jersey) (R. Mann and P. Wesson, eds.), World Scientific, 1991.
36. *A new nonsymmetric gravitational theory*, Phys. Lett. B 355 (1995), 447–452.
37. *Nonsymmetric gravitational theory*, J. Math. Phys. 36 (1995), 3722–3732.
38. *Stellar equilibrium and gravitational collapse in the nonsymmetric gravitational theory*, UTPT-95-18, astro-ph/9510024, 1995.
39. J. W. Moffat and I. Yu. Sokolov, *Galaxy dynamics in the nonsymmetric gravitational theory*, UTPT-95-17, astro-ph/9509143, To appear in Phys. Lett. B, 1995.
40. Tullio Regge and Claudio Teitelboim, *Role of surface integrals in the Hamiltonian formulation of general relativity*, Ann. Phys. 88 (1974), 286–318.
41. Claudio Teitelboim, *How commutators of constraints reflect the spacetime structure*, Ann. Phys. 79 (1973), 542–557.
42. The Hamiltonian structure of spacetime, General Relativity and Gravitation (New York) (A. Held, ed.), vol. 1, Plenum Press, 1980.
43. M. A. Tonnelat, Théorie unitaire affine du champ physique, J. Phys. Rad. 12 (1951), 81–88.
44. Einstein’s theory of unified fields, Gordon and Breach, Science Publishers, New York, 1982.
45. Marie-Antoinette Tonnelat, La solution générale des équations d’Einstein $g_{\mu\nu,\rho} = 0$, J. Phys. Rad. 12 (1955), 21–38.
46. J. R. Vanstone, The general static spherically symmetric solution of the ‘weak’ unified field equations, Can. J. Math. 2 (1962), 568–576.
47. Giorgio Velo and Daniel Zwanzinger, Noncausality and other defects of interaction Lagrangians for particles with spin one and higher, Phys. Rev. 188 (1969), 2218–2222.
48. M. Wyman, Unified field theory, Can. J. Math. 2 (1950), 427–439.

Department of Physics, University of Toronto, Toronto, ON, Canada, M5S 1A7
E-mail address: clayton@medb.physics.utoronto.ca