A simple proof for Imnang’s algorithms

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Abstract
In this paper, a simple proof of the convergence of the recent iterative algorithm by relaxed \((u, v)-\)cocoercive mappings due to Imnang (J. Inequal. Appl. 2013:249, 2013) is presented.

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1 Introduction and preliminaries
In this paper, a simple proof for the convergence of an iterative algorithm is presented that improves and refines the original proof.

Suppose that \(C\) is a nonempty closed convex subset of a real normed linear space \(E\) and \(E^*\) is its dual space. Suppose that \(\langle \cdot, \cdot \rangle\) denotes the pairing between \(E\) and \(E^*\). The normalized duality mapping \(J : E \rightarrow E^*\) is defined by

\[ J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\} \]

for each \(x \in E\). Let \(U = \{x \in E : \|x\| = 1\}\). A Banach space \(E\) is called smooth if for all \(x \in U\), there exists a unique functional \(j_x \in E^*\) such that \(\langle x, j_x \rangle = \|x\| \) and \(\|j_x\| = 1\) (see [1]).

Recall that a mapping \(f : C \rightarrow C\) is a contraction on \(C\), if there exists a constant \(\alpha \in (0, 1)\) such that \(\|f(x) - f(y)\| \leq \alpha \|x - y\|\), \(\forall x, y \in C\). We use \(\Pi_C\) to denote the collection of all contractions on \(C\), i.e., \(\Pi_C = \{f : C \rightarrow C\text{ is a contraction}\}\).

For a map \(T\) from \(E\) into itself, we denote by \(\text{Fix}(T) := \{x \in E : x = Tx\}\), the fixed point set of \(T\).

Recall the following well-known concepts:
(1) Suppose that \(C\) is a nonempty closed convex subset of a real Banach space \(E\).

A mapping \(B : C \rightarrow E\) is called relaxed \((u, v)-\)cocoercive [2], if there exist two constants \(u, v > 0\) such that

\[ \langle Bx - By, j(x - y) \rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2, \]

for all \(x, y \in C\) and \(j(x - y) \in J(x - y)\).
Suppose that $C$ is a nonempty closed convex subset of a real Banach space $E$ and $B$ is a self-mapping on $C$. If there exists a positive integer $\alpha$ such that

$$\|Bx - By\| \geq \alpha \|x - y\|$$

for all $x, y \in C$, then $B$ is called $\alpha$-expansive.

**Lemma 1.1** ([2]) Let $C$ be a nonempty closed convex subset of a real 2-uniformly smooth Banach space $X$ with the 2-uniformly smooth constant $K$. Let $Q_C$ be the sunny nonexpansive retraction from $X$ onto $C$ and let $A_i : C \rightarrow X$ be a relaxed $(c_i, d_i)$-cocoercive and $L_i$-Lipschitzian mapping for $i = 1, 2, 3$. Let $G : C \rightarrow C$ be a mapping defined by

$$G(x) = Q_C\left[Q_C\left(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x)\right)
- \lambda_1 A_1 Q_C\left(Q_C(I - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(I - \lambda_3 A_3 x)\right)\right].$$

If $\lambda_i \leq \frac{d_i - c_i L_i^2}{K^2 2 L_i^2}$ for all $i = 1, 2, 3$, then $G : C \rightarrow C$ is nonexpansive.

**Lemma 1.2** ([3, Lemma 2.8]) Suppose that $C$ is a nonempty closed convex subset of a real Banach space $X$ that is 2-uniformly smooth, and the mapping $A : C \rightarrow X$ is relaxed $(c, d)$-cocoercive and $L_A$-Lipschitzian. Then,

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\left(\lambda c L_A^2 - \lambda d + K^2 \lambda^2 L_A^2\right)\|x - y\|^2,$$

where $\lambda > 0$. In particular, when $d > c L_A^2$ and $\lambda \leq \frac{d - c L_A^2}{K^2 L_A^2}$, note $I - \lambda A$ is nonexpansive.

In this paper, using relaxed $(u, v)$-cocoercive mappings, a new proof for the iterative algorithm [2] is presented.

**2 A simple proof for the theorem**

Imnang [2] considered an iterative algorithm for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality. Our argument will rely on the following lemma.

**Lemma 2.1** Suppose that $C$ is a nonempty closed convex subset of a Banach space $E$. Suppose that $A : C \rightarrow E$ is a relaxed $(m, v)$-cocoercive mapping and $\epsilon$-Lipschitz continuous with $v - \epsilon m^2 > 0$. Then, $A$ is a $(v - \epsilon m^2)$-expansive mapping.

**Proof** Since $A$ is $(m, v)$-cocoercive and $\epsilon$-Lipschitz continuous, for each $x, y \in C$ and $j(x - y) \in J(x - y)$, we have that

$$\langle Ax - Ay, j(x - y) \rangle \geq (-m)\|Ax - Ay\|^2 + v\|x - y\|^2$$

$$\geq (-\epsilon m^2)\|x - y\|^2 + v\|x - y\|^2$$

$$= (v - \epsilon m^2)\|x - y\|^2 \geq 0,$$
and hence
\[ \|Ax - Ay\| \geq (\nu - m\varepsilon^2)\|x - y\|, \]
therefore, A is \((\nu - m\varepsilon^2)\)-expansive.

The following theorem is due to Inman [2] that solves the viscosity iterative problem for a new general system of variational inequalities in Banach spaces:

**Theorem 2.2** (i.e., Theorem 3.1, from [2, §3, p.7]) Suppose that X is a Banach space that is uniformly convex and 2-uniformly smooth with the 2-uniformly smooth constant \( K \), \( C \) is a nonempty closed convex subset of X, and \( Q_C \) is a sunny nonexpansive retraction from X onto C. Assume that \( A_i : C \to X \) is relaxed \( (c_i, d_i) \)-cocoercive and \( L_i \)-Lipschitzian with \( 0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2} \) for each \( i = 1, 2, 3 \). Suppose that \( f \) is a contraction mapping with the constant \( \alpha \in (0, 1) \) and \( S : C \to C \), a nonexpansive mapping such that \( \Omega = F(S) \cap F(G) \neq \emptyset \), where G is defined as in Lemma 1.1. Suppose that \( s_1 \in C \) and \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are the following sequences:

\[
\begin{align*}
  z_n &= Q_C(x_n - \lambda_3 A_3 x_n), \\
  y_n &= Q_C(z_n - \lambda_2 A_2 z_n), \\
  x_{n+1} &= a_n f(x_n) + b_n x_n + (1 - a_n - b_n) S Q_C(y_n - \lambda_1 A_1 y_n),
\end{align*}
\]

where \( \{a_n\} \) and \( \{b_n\} \) are two sequences in \((0, 1)\) such that

\( \text{(C1)} \) \( \lim_{n \to \infty} a_n = 0 \) and \( \sum_{n=1}^{\infty} a_n = \infty; \)

\( \text{(C2)} \) \( 0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1. \)

Then, \( \{x_n\} \) converges strongly to \( q \in \Omega \), which solves the following variational inequality:

\[
\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall f \in \Pi_C, p \in \Omega.
\]

**A Simple Proof** Let \( i = 1, 2, 3 \). Consider Theorem 2.2 and the \( L_i \)-Lipschitz continuous and relaxed \( (c_i, d_i) \)-cocoercive mapping \( A_i \) in Theorem 2.2. From the condition that \( 0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2} \), we have that \( 0 < 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2) < 1 \). Note that from Lemma 1.2, we have that \( I - \lambda_i A_i \) is nonexpansive when \( 0 < 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2) \). Then, applying the coefficients \( \alpha_i = 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2) \) in Lemma 1.2 we have that \( I - \lambda_i A_i \) is an \( \alpha_i \)-contraction, for each \( i = 1, 2, 3 \). Also, note that \( Q_C \) is nonexpansive and \( I - \lambda_i A_i \) is an \( \alpha_i \)-contraction, for each \( i = 1, 2, 3 \). Hence, using the proof of [2, Lemma 2.11], we conclude that

\[
\| G(x) - G(y) \| = \| Q_C[Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\
- \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x)] \\
- Q_C[Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y) \\
- \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y)] \| \\
\leq \| Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\
- \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \| \\
- \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \| \\
- \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \|
\]
\begin{align*}
&= \left[ Q_C \left( Q_C (I - \lambda_3 A_3) y - \lambda_2 A_2 Q_C (I - \lambda_3 A_3) y\right) \\
&= \lambda_1 A_1 Q_C \left( Q_C (I - \lambda_3 A_3) y - \lambda_2 A_2 Q_C (I - \lambda_3 A_3) y\right) \right] \\
&= \| (I - \lambda_1 A_1) Q_C (I - \lambda_2 A_2) Q_C (I - \lambda_3 A_3) x \\
&\quad - (I - \lambda_1 A_1) Q_C (I - \lambda_2 A_2) Q_C (I - \lambda_3 A_3) y \| \\
&\leq \alpha_1 \alpha_2 \alpha_3 \| x - y \|,
\end{align*}

and since $0 < \alpha_1 \alpha_2 \alpha_3 < 1$ then $G$ is an $\alpha$-contraction with $\alpha = \alpha_1 \alpha_2 \alpha_3$, hence from Banach’s contraction principle $F(G)$ is a singleton set and hence, $\Omega$ is a singleton set, i.e., there exists an element $p \in X$ such that $\Omega = \{p\}$. Since $(d_i - c_i L_i^2) > 0$, from Lemma 2.1, $A_i$ is $(d_i - c_i L_i^2)$-expansive, i.e.,

$$\| A_i x - A_i y \| \geq (d_i - c_i L_i^2) \| x - y \|, \quad (1)$$

in Theorem 2.2. The authors in [2, p.11] proved (see (3.12) in [2, p.11]) that

$$\lim_n \| A_3 x_n - A_3 p \| = 0, \quad (2)$$

for $x^* = p$. Now, put $x = x_n$ and $y = p$ in (1), and from (1) and (2), we have

$$\lim_n \| x_n - p \| = 0.
$$

Hence, $x_n \to p$. As a result, one of the main claims of Theorem 2.2 is established (note $\Omega = \{p\}$).

Note that the main aims of Theorem 3.1 in [2] are $x_n \to p$ and

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall f \in \Pi C, p \in \Omega.
$$

Next, we show that the main aim of Theorem 3.1 in [2] can be concluded from the relations (3.12) in [2, page 11] and the proof in Theorem 2.2 can be simplified even further using the above. Note that the part of the proof between the relations (3.12) in [2, page 11] to the end of the proof of Theorem 3.1 can be removed from the proof. Indeed, since immediately from (3.12) in [2], we conclude that $x_n \to p$, i.e., the first aim of Theorem 3.1 is concluded.

The second aim of the theorem, i.e.,

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall f \in \Pi C, p \in \Omega,$$

is clear, because $p = q$ ($\Omega = \{p\}$) and $/\{0\} = \{0\}$. Consequently, the relations between (3.12) in [2, page 11] to the end of the proof of Theorem 3.1 in [2, page 11] can be removed. □

3 Discussion

In this paper, a simple proof for the convergence of an algorithm by relaxed $(u, v)$-cocoercive mappings due to Imnang is presented.

4 Conclusion

In this paper, a refinement of the proof of the results due to Imnang is given.
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