Yet another breakdown point notion: EFSBP
Illustrated at scale-shape models

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Abstract The breakdown point in its different variants is one of the central notions to quantify the global robustness of a procedure. We propose a simple supplementary variant which is useful in situations where we have no obvious or only partial equivariance: Extending the Donoho and Huber (The notion of breakdown point, Wadsworth, Belmont, 1983) Finite Sample Breakdown Point, we propose the Expected Finite Sample Breakdown Point to produce less configuration-dependent values while still preserving the finite sample aspect of the former definition. We apply this notion for joint estimation of scale and shape (with only scale-equivariance available), exemplified for generalized Pareto, generalized extreme value, Weibull, and Gamma distributions. In these settings, we are interested in highly-robust, easy-to-compute initial estimators; to this end we study Pickands-type and Location-Dispersion-type estimators and compute their respective breakdown points.

Keywords Global robustness · Finite sample breakdown point · Partial equivariance · Scale-shape parametric family · LD estimator

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1 Introduction

In an industrial project to compute robust variants of OpVar, i.e., the regulatory capital as required in Basel II (2006) for a bank to cover its operational risk, we came across the problem of determining the (finite sample) breakdown point of certain considered procedures. Here operational risk is by definition “the risk of direct or indirect loss resulting from inadequate or failed internal processes, people and systems or from external events.”

These extremal events, as motivated by the Pickands-Balkema-de Haan Extreme Value Theorem (see Balkema and de Haan 1974; Pickands 1975) suggest the use of the generalized Pareto distribution (GPD) for modeling in this context. In an intermediate step this modeling involves estimation of the scale and shape parameters of this distribution. To this end, several robust procedures have been proposed in the literature, see Ruckdeschel and Horbenko (2010) for a more detailed discussion.

One of the quantities to judge robustness of a procedure is the breakdown point (see Definition 3.1). In particular, we are interested in the finite sample version FSBP of this notion to be able to quantify the degree of protection a procedure provides in the estimation at an actual (finite) set of observations.

It turns out that for our purposes the original definition has some drawbacks, as it depends strongly on the configuration of the actual sample. To get rid of the dependence on possibly highly improbable sample configurations while still preserving the aspect of a finite sample, we propose an expected FSBP, $EFSBP$, i.e., to integrate out the FSBP with respect to the ideal distribution.

We illustrate the usefulness of this new concept for scale-shape models by means of two types of robust estimators, quantile-type estimators (Pickands Estimator $PE$) and robust Location-Dispersion (LD) estimators as introduced by Marazzi and Ruffieux (1999); for the latter type we study estimators based on the median for the location part and several robust scale estimators for the dispersion part: a (new) asymmetric version of the median of absolute deviations $kMAD$, as well as $Qn$ and $Sn$ from Rousseeuw and Croux (1993)—combined to $MedkMAD$, $MedQn$, and $MedSn$, respectively.

These estimators are meant to be used as initial estimators with acceptable to good global robustness properties for (more efficient) robust estimators afterwards. In particular, they can be computed without the need of additional (robust, consistent) initial estimators, which precludes otherwise promising alternatives like Minimum Distance estimators, for which we could have read off asymptotic breakdown point values as high as half the optimal value from Donoho and Liu (1988). We have also excluded the method-of-median approach of Peng and Welsh (2001), because in contrast to PE and MedkMAD, $MedQn$, and $MedSn$, for this estimator in the GPD and GEVD case, no explicit calculations are possible. We have studied this approach in another paper, though (Ruckdeschel and Horbenko (2010)), and empirically found that in the GPD case its breakdown behavior is worse than the one of MedkMAD and MedQn.

Our paper is organized as follows: In Sect. 2, we list our reference examples for scale-shape models, i.e.; the generalized Pareto, the generalized extreme value, the Weibull, and the Gamma distribution, as well as the Gross Error model which we use to capture deviations from the ideal model. In Sect. 3, we recall the standard definitions of the asymptotic and finite sample breakdown points ABP and FSBP and introduce
the new concept of EFSBP in Definition 3.2. Section 4 then defines the considered estimators, i.e.; quantile-type estimators PE, and LD estimators MedkMAD, MedQn, MedSn. At these estimators, we demonstrate our new breakdown point notion in Sect. 5, giving analytic formulae for FSBP, ABP, and EFSBP in Propositions 5.1, 5.2, and 5.3, together with some numerical evaluations of EFSBP at some reference situation and with simulation-based evaluations. Proofs for our results are gathered in Appendix.

Remark 1.1 This paper is a part of the PhD thesis of the second author; a preliminary version of it may be found in Ruckdeschel and Horbenko (2010).

2 Model setting

For notions of invariance of statistical models and equivariance of estimators we refer to Eaton (1989): Given a measurable space \((\Omega, \mathcal{B})\), a family of probability measures \(\mathcal{P}\) defined on \(\mathcal{B}\) is a statistical model.

Notationally, we use the same symbol for the cumulative distribution function (c.d.f.) and the probability measure; we write \(F(x - 0)\) to denote left and, correspondingly, \(+0\) for right limits, and \(F^{-}\) to denote the right continuous quantile function given by \(F^{-}(s) = \inf\{t \in \mathbb{R}: F(t) \geq s\}\).

Definition 1 Suppose a group \(G\) acts measurably on \(\Omega\). Model \(\mathcal{P}\) is called \(G\)-invariant iff for each \(P \in \mathcal{P}\), the image probability \(gP\) of \(P\) under group action \(g\) stays in \(\mathcal{P}\).

For simplicity, we assume that \(g(P_1) = g(P_2)\) implies \(P_1 = P_2\) for any two elements of \(\mathcal{P}\). In a \(G\)-invariant parametric model \(\mathcal{P} = \{P_\theta | \theta \in \Theta\}\), where \(\Theta\) is the parameter space, group \(G\) induces an isomorphic group \(\tilde{G}\), acting on the parameter space with the identification \(g(P_\theta) = P_{\tilde{g}(\theta)}\). In this situation, a point estimator \(t\) mapping \(\Omega\) to \(\Theta\) is equivariant iff \(t(g(x)) = \tilde{g}(t(x))\).

2.1 Generalized Pareto distribution and other scale-shape families

We illustrate our concepts at scale-shape models; our reference example is the three-parameter generalized Pareto distribution (GPD) which has c.d.f. and density

\[
F_\theta(x) = 1 - \left(1 + \xi \frac{x - \mu}{\beta}\right)^{-\frac{1}{\xi}}, \quad f_\theta(x) = \frac{1}{\beta} \left(1 + \xi \frac{x - \mu}{\beta}\right)^{-\frac{1}{\xi} - 1}
\]

(2.1)

where \(x \geq \mu\) for \(\xi \geq 0\), and \(\mu < x \leq \mu - \frac{\beta}{\xi}\) if \(\xi < 0\). It has parameter \(\theta = (\xi, \beta, \mu)^T\), for location \(\mu\), scale \(\beta > 0\) and shape \(\xi\). Special cases of GPDs are the uniform \((\xi = -1\), the exponential \((\xi = 0, \mu = 0\), and Pareto \((\xi > 0, \beta = 1\) distributions. We limit ourselves to the case of known location \(\mu = 0\) and unknown scale and shape here and abbreviate the pair \((\beta, \xi)\) by \(\vartheta\), i.e.; we are concerned with joint estimation of \(\vartheta = (\beta, \xi)\) only.
Other scale-shape families for which our considerations apply mutatis mutandis are the generalized extreme value distribution (GEVD) given by its c.d.f.

\[ F_\theta(x) = \exp \left( - \left( 1 + \xi \frac{x - \mu}{\beta} \right)^{-\frac{1}{\xi}} \right) I_{(-\frac{\beta}{\xi} + \mu, \infty)}(x) \]  

(2.2)

the Weibull distribution with density

\[ f_\theta(x) = \frac{\xi}{\beta} \left( \frac{x}{\beta} \right)^{\xi-1} \exp\left(-\left(\frac{x}{\beta}\right)^\xi\right) I_{(0, \infty)}(x) \]  

(2.3)

and the Gamma distribution with density

\[ f_\theta(x) = \frac{x^{\xi-1}}{\beta^\xi \Gamma(\xi)} \exp\left(-\left(\frac{x}{\beta}\right)\right) I_{(0, \infty)}(x) \]  

(2.4)

For the Weibull and Gamma case we require \( \xi > 0 \), whereas in the GEVD case the same distinction applies as in the GPD case.

Reparametrization In the Weibull family, passage to the log-observations transforms this model into a location-scale model with the standard Gumbel as central distribution. This approach has been taken by Boudt et al. (2011), and allows them to recur to the rich theory (both classical and robust) available for location-scale models.

In both GPD and GEVD, a similar approach is possible, once instead of \( \mu \) we use \( \tilde{\mu} = \mu \xi - \beta \), so that in this setting we get

\[ 1 + \xi \frac{x - \mu}{\beta} = \xi \frac{x - \tilde{\mu}}{\beta} \]  

(2.5)

In the GPD case, this leads to a location-scale model with the standard Exponential as central distribution. This parametrization is used for two-parameter Pareto distribution, e.g. in Brazauskas and Serfling (2000). Two issues, however, are bought with this approach: First, knowledge of \( \mu \) is not the same as knowledge of \( \tilde{\mu} \), so our original setting where \( \mu \) was assumed known does not carry over easily. Second, the corresponding transformed model about the Exponential distribution is not smooth—\( L_2 \)-differentiable to be precise. The reason for this is essentially that observations around the left endpoint of the distribution carry overwhelmingly much information about the location parameter. As a consequence, usual optimality theory no longer is available, and in the ideal model setting there are estimators which are consistent at faster rates than the usual \( 1/\sqrt{n} \). On the other side, this high accuracy requires to base inference essentially completely on the minimal observations which makes these procedures extremely prone to outliers. Robustifications avoid this problem, but still, due to the lack of smoothness no optimality theory is available. For this reason, we stick to the original parametrization.
Our reference model  In the sequel, we use the reference values $\beta = 1$ and $\xi = 0.7$ for all our scale-shape models; in case of the GPD this amounts to moderately fat tails which reflects well the situation we met in our application to OpVar.

In-/equivariance  The reduced model enjoys a certain invariance: with an included scale component, it remains invariant under scale transformations $s_\beta(x) = \beta x$ of the observations. Using the matrix $d_\beta = \text{diag}(\beta, 1)$, this invariance is reflected by a corresponding notion of equivariance of estimators, i.e.; an estimator $S$ for $\vartheta = (\beta, \xi)$ is called scale-equivariant if

$$S(\beta x_1, \ldots, \beta x_n) = d_\beta S(x_1, \ldots, x_n) \quad (2.6)$$

For the shape parameter $\xi$, there is no obvious such invariance, entailing a dependence of estimator properties like robustness on this parameter.

2.2 Gross error model

Extending the ideal model setting, Robust Statistics defines suitable distributional neighborhoods about this ideal model. In this paper, we limit ourselves to the Gross Error Model, i.e.; as neighborhoods, we use the sets of all distributions $F^{\text{re}}$ representable as

$$F^{\text{re}} = (1 - \varepsilon) F^{\text{id}} + \varepsilon F^{\text{di}} \quad (2.7)$$

for some given size or radius $\varepsilon > 0$, where $F^{\text{id}}$ is the underlying ideal distribution and $F^{\text{di}}$ some arbitrary, unknown, and uncontrollable contaminating distribution.

3 Global robustness: the breakdown point

In this paper we focus on the Breakdown Point as a global measure of robustness, specifying the reliability of a procedure under massive deviations from the ideal model. In the gross error model (2.7), it gives the largest radius $\varepsilon$ at which the estimator still produces meaningful results.

In standard literature on Robust Statistics, there are two notions of breakdown point—the asymptotic (functional) breakdown point (ABP) and the finite sample breakdown point (FSBP) introduced in Hampel (1968) and Donoho and Huber (1983), respectively:

Definition 3.1  (a) (Hampel et al. 1986, 2.2 Definition 1) The asymptotic breakdown point (ABP) $\varepsilon^*$ of the sequence of estimators $T_n$ for parameter $\theta \in \Theta$ at probability $F$ is given by

$$\varepsilon^* := \sup \left\{ \varepsilon \in (0, 1] \mid \text{there is a compact set } K_\varepsilon \subset \Theta \text{ s.t.} \right\}$$

$$\pi(F, G) < \varepsilon \implies G([T_n \in K_\varepsilon]) \overset{n \to \infty}{\longrightarrow} 1 \quad (3.1)$$
where \( \pi \) is Prokhorov distance.

(b) (Hampel et al. 1986, 2.2 Definition 2) The finite sample breakdown point (FSBP) \( \varepsilon_n^* \) of the estimator \( T_n \) at the sample \((x_1, \ldots, x_n)\) is given by

\[
\varepsilon_n^*(T_n; x_1, \ldots, x_n) := \frac{1}{n} \max \left\{ m; \max_{i_1, \ldots, i_m} \sup_{y_1, \ldots, y_m} |T_n(z_1, \ldots, z_n)| < \infty \right\}, \tag{3.2}
\]

where the sample \((z_1, \ldots, z_n)\) is obtained by replacing the data points \(x_{i_1}, \ldots, x_{i_m}\) by arbitrary values \(y_1, \ldots, y_m\).

Note that \( \varepsilon_n^* \) from (3.2) is by \(1/n\) smaller than the Donoho and Huber (1983) FSBP.

Definition 3.1 (b) does not cover the scale case, where we must take into account the possibility of implosion as well: As noted by an anonymous referee, otherwise one could achieve arbitrarily high breakdown points by choosing estimators based on two very low quantiles, which of course would not be stable at all—an argument valid in the location-scale case as well. A remedy for the scale parameter is given by the log-transformation as mentioned in He (2005), i.e.;

\[
\varepsilon_n^*(T_n; x_1, \ldots, x_n) := \frac{1}{n} \max \left\{ m; \max_{i_1, \ldots, i_m} \sup_{y_1, \ldots, y_m} |\log(T_n(z_1, \ldots, z_n))| < \infty \right\}, \tag{3.3}
\]

Breakdown and partial invariance

By arguments given in Davies and Gather (2005), a certain equivariance of the considered estimator under a suitable group of transformations is required to obtain meaningful upper bounds for the breakdown point. In our scale-shape models, however, as indicated in Sect. 2.1, we canonically only have scale invariance. This lack of complete equivariance does not invalidate the cited authors’ considerations, but rather these can be extended to also cover this partial invariance:

While due to the lack of shape-equivariance, we conjecture that similar defective constructions, which produce breakdown points arbitrarily close to 1 in the AR(1) case [as mentioned in Genton and Lucas (2005)], should be feasible in the pure shape case as well, in the joint scale-shape case, imposing scale-equivariance, we do obtain sensible upper bounds as such constructions are eliminated by this (partial) equivariance.

In particular, as the scale model is a submodel of our scale-shape model, the corresponding upper bounds for the maximal breakdown point among all scale-equivariant estimators from Davies and Gather (2005, Thms. 3.1,3.2) remain valid in our setting without change. Hence, in the sequel, we restrict ourselves to scale-equivariant estimators. In particular, following Davies and Gather (2007, sect. 4.2), we note that with \( n_0 \) being the highest frequency of a single data point in the original sample,

\[
\varepsilon_n^* \leq \frac{(n - n_0 - 1)_+}{2} / n \tag{3.4}
\]

(adapted to (3.2)) among all scale-equivariant estimators.

Breakdown and restricted parameter space

In the GPD and GEVD families, there are two canonical parameter spaces for \( \xi \): Either one does not impose any restriction,
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\[ \xi \in \mathbb{R} \]—which could be seen as “natural” there, or one restricts \( \xi \) to be positive (which is the only possibility for the Weibull and Gamma case).

In the GPD and GEVD case, \( \xi = 0 \) is a discontinuity as to the statistical properties of the model, comparable to parameter values \( \pm 1 \) in the AR(1) model. While GPD and GEVD for \( \xi < 0 \) have compact support, in the AR(1) model \( \pm 1 \) mark the border of stationarity. In both cases, the discontinuity only becomes visible when passing to sequences of observations, in our case when motivating GPD and GEVD by asymptotic arguments, i.e.; by the Pickands–Balkema-de Haan and Fisher-Tippet-Gnedenko Extreme Value Theorems. To this end we need a uniformity over sets of quantiles which gets lost when passing over the value \( \xi = 0 \). In particular, shape in the GPD and GEVD models decides to which domain of attraction belongs the underlying distribution in the corresponding Extreme Value Limit Theorems. In both the scale-shape and the AR(1) case, it is hence well debatable to restrict the parameter space accordingly, see Genton and Lucas (2005) and the rejoinder in Davies and Gather (2005, p. 1033). E.g.; we are mainly interested in the case when \( \xi > 0 \), which corresponds to heavy-tailed GPD / GEVD, and an estimate \( \xi \leq 0 \) would lead to drastic under-estimation of the corresponding operational risk.

In the sequel, for the GPD and GEVD cases, we hence consider both situations: with and without restriction on the parameter space, i.e.; that \( \xi > 0 \) or \( \xi \in \mathbb{R} \).

Similar arguments could be carried out in case of shape estimation in the Weibull case, where \( 0 < \xi < 1 \) corresponds to heavy-tailed, \( \xi \geq 1 \) to light-tailed distributions; we do not pursue this further here.

**Breakdown and finite samples** As for our purposes, reliability at finite samples is of primary interest, we will focus on the FSBP.

For deciding upon which procedure to take before having made observations, in particular for ranking procedures in a simulation study, the FSBP from Definition 3.1 (b) has some drawbacks: It is deliberately probability-free and based on an actual sample \((x_1, \ldots, x_n)\), which we assume from the ideal situation for the moment. Hence its value depends on the configuration of this sample. This is desirable when checking safety of a procedure at an actual data set, but also entails that for the estimators considered in this paper, a generally valid value for FSBP does not exist, and the only possible universal lower bound will be the minimal possible value of 0; and even if we made a sample-wise restriction, banning such samples from the application of the estimator, we would have other ones to come up with an FSBP of \( 1/n \) and so forth. This does not reflect the situation to be expected in the ideal model, though. Hence, we follow the general spirit of robustness to tie robustness concepts to a central ideal probability model—compare Definition 3.1 (a): To get rid of the dependence on possibly highly improbable sample configurations leading to an overly small FSBP, but still preserving the aspect of a finite sample, we propose an expected FSBP:

**Definition 3.2** For an estimator \( T \) with FSBP \( \varepsilon^*_n = \varepsilon^*_n(T; X_1, \ldots, X_n) \), we define the expected FSBP or EFSBP as

\[
\tilde{\varepsilon}^*_n(T) := E \varepsilon^*_n(T; X_1, \ldots, X_n) \quad (3.5)
\]
where expectation is evaluated in the ideal model.

At some places, if existent, for a sequence $T$ of estimators $T_n$, we also consider the limit

$$\bar{\varepsilon}^*(T) := \lim_{n \to \infty} \bar{\varepsilon}^*_n(T_n) \quad (3.6)$$

and which, for brevity, we also call EFSBP where unambiguous.

Admittedly, the evaluation of the expectation in (3.5) in general assumes knowledge of the parameter, but some vague prior information could be used to restrict the range of the plausible parameter values, say to $\xi \in (0.5; 2)$, and take the worst behavior of $\bar{\varepsilon}^*_n(T)$ on this range to base our decisions on, compare, e.g. Fig. 2.

Weighted by their (ideal) occurrence probability, by this definition, improbable sample configurations of the ideal sample—before contamination—are smoothed out in EFSBP; we still cannot exclude these configurations, but usually by corresponding Chebyshev-type inequalities for growing sample size $n$ these will occur with decreasing probability and $\bar{\varepsilon}^*_n(T)$ will concentrate about $\bar{\varepsilon}^*_n$. Hence, in practice, without extra knowledge, a priori, the user can rely on being protected against up to $\bar{\varepsilon}^*_n(T)n$ outliers on average; i.e., although there may be (rare) cases where we have considerably less protection, these cases are balanced by corresponding cases with considerably stronger protection.

By averaging, EFSBP is closer again to the ABP of Hampel (1968), but preserves the finite sample aspect of FSBP. In the examples, we will show that this aspect is non-negligible, and that for sample sizes about 40, the ABP will still be somewhat misleading (see Table 2 and Fig. 3 below), while at the same time, as mentioned, FSBP will be way too pessimistic. By dominated convergence though, the limit of EFSBP will coincide with the ABP whenever the FSBP converges to the ABP.

Small values of $\varepsilon^*_n$ for particular samples do not only occur in the models discussed here: In the one-dimensional normal scale model, we can already have FSBP of 0 for the median of absolute deviations MAD for large enough values of $n_0$ as introduced before (3.4). Such events (and similarly extraneous sample configurations), however, occur with probability 0 in a continuous setting. Otherwise, in situations where a FSBP of 0 could occur with positive probability in the ideal model, necessarily we have mass points violating the standard smoothness assumptions usually required in scale models: the corresponding Fisher information of scale would be infinite then, compare Ruckdeschel and Rieder (2010), and one may then rather question the use of MAD. In our case, this is somewhat different, as without arbitrary restrictions on the sample space, samples with FSBP of 0 can occur with small but positive ideal probability (see $p_0$ in Table 2), although our model remains smooth (and Fisher information finite).

4 Robust estimators types

We illustrate the concept of FSBP in our scale-shape models for Pickands-type and LD-type estimators, as defined in the sequel.
4.1 Pickands estimator

**Pickands estimator (PE)** for GPD is a special case of the Elementary Percentile Method (EPM) as discussed by Castillo and Hadi (1997) for GPD. Such estimators are based on the empirical quantiles, in our case, we follow Pickands (1975) and use the empirical 50 and 75% quantiles $\hat{Q}_2$ and $\hat{Q}_3$. Pickands estimators for $\xi$ and $\beta$ in GPD model then are defined as

$$\hat{\xi} = \frac{1}{\log(2)} \log \frac{\hat{Q}_3 - \hat{Q}_2}{\hat{Q}_2}, \quad \hat{\beta} = \hat{\xi} \frac{\hat{Q}_2^2}{\hat{Q}_3 - 2\hat{Q}_2}$$

(4.1)

where we see that for $\hat{\beta} > 0$ we have to require $\hat{Q}_3 > 2\hat{Q}_2$, in which case $\hat{\xi} > 0$ automatically. Apparently PE is equivariant in the sense of (2.6).

For GEVD, analogue estimates can be obtained by

$$\hat{\xi} = \left\{ \xi \in \mathbb{R} \mid \frac{\hat{Q}_3 - \hat{Q}_2}{\hat{Q}_2} = q_0(\xi) \right\}, \quad q_0(\xi) = \frac{\log(4/3)^{\hat{\xi}} - \log(2)^{-\hat{\xi}}}{\log(2)^{-\hat{\xi}} - 1},$$

(4.2)

$$\hat{\beta} = \hat{\xi} \frac{\hat{Q}_2^2}{\hat{Q}_3 - 2\hat{Q}_2} \frac{\log(4/3)^{-\hat{\xi}} + 1 - 2\log(2)^{-\hat{\xi}}}{\log(2)^{-2\hat{\xi}} + 1 - 2\log(2)^{-\hat{\xi}}}$$

(4.3)

where $q_0$ is obviously smooth, and, if plotted, easily seen to be strictly isotone, compare Fig. 1; in particular, $\hat{\xi} > 0$ iff $\hat{Q}_3 > \hat{Q}_2(1 + q_0(0)) \doteq 3.39\hat{Q}_2$, and $\hat{\beta} > 0$ iff $\hat{Q}_3 > 2\hat{Q}_2$. 

![Graph of q0(ξ) for different values of ξ; note the logarithmic y-scale](image-url)
In the Weibull model, Boudt et al. (2011) have shown Pickands (quantile) estimators to have an explicit representation as

$$
\hat{\xi} = \frac{f_{1,1}^{-1}(3/4) - f_{1,1}^{-1}(1/2)}{\log(\hat{Q}_3) - \log(\hat{Q}_2)}, \quad \hat{\beta} = \hat{Q}_2/(-\log(1/2))^{1/\hat{\xi}}
$$

(4.4)

where $f_{1,1}(\alpha) = \log(-\log(1 - \alpha))$.

For the Gamma distribution the quantile estimates have no closed solutions, so the matching of empirical and theoretical quantiles is to be done numerically by root solving procedures.

4.2 MedkMAD and other LD estimators

Location-Dispersion estimators, introduced by Marazzi and Ruffieux (1999), match empirical location and dispersion measures of data against their population counterparts to get the estimates of model parameters, and are applicable for asymmetric location-scale (Lognormal), as well as in scale-shape models (GPD, Pareto, Weibull, Gamma).

Let $\theta = (\alpha, \sigma)$ be a parameter vector, $F_n, F_{\alpha,\sigma}$ empirical and model distribution functions, $m(F_n), s(F_n), m(F_{\alpha,\sigma}), s(F_{\alpha,\sigma})$ corresponding empirical and model location and dispersion, then LD estimators $(\hat{\alpha}, \hat{\sigma})$ are solutions of

1. $\hat{\sigma} m(F_{0,1}) + \hat{\alpha} = m(F_n), \quad \hat{\sigma} s(F_{0,1}) = s(F_n)$

when $\alpha$ is a location parameter,

2. $\hat{\sigma} m(F_{\hat{\alpha},1}) = m(F_n), \quad \hat{\sigma} s(F_{\hat{\alpha},1}) = s(F_n)$

when $\alpha$ is a shape parameter.

Efficiency and robustness of these estimators depend on the choice of $m(\cdot)$ and $s(\cdot)$, and, of course, on the respective parametric model. Mean and standard deviation are classical measures for location and dispersion, respectively. Robust alternatives are median, trimmed mean—for location, IQR, MAD, trimmed MAD, Sn, Qn—for dispersion. In addition, for asymmetric distributions, we propose a new dispersion measure, namely kMAD. Table 1 displays different variations for LD estimators with increasing efficiency together with corresponding references.

Definitions of some particular LD estimators

Empirical median $\hat{m} = \hat{m}_n$ and median of absolute deviations $\hat{M} = \hat{M}_n$ are well known for their high breakdown point, jointly achieving the highest possible asymptotic breakdown point of 50% among all affine equivariant estimators at symmetric, continuous univariate distributions.

Hence it is plausible to define an estimator for $\xi$ and $\beta$, matching $\hat{m}$ and $\hat{M}$ against their population counterparts $m$ and $M$ within a scale-shape model. It turns out that the mapping $(\beta, \xi) \mapsto (m, M)(F_\theta)$ is indeed a Diffeomorphism, hence for sufficiently
Table 1  LD estimators and literature of using for scale-shape models

| Location  | Dispersion     | Location/dispersion             |
|-----------|----------------|---------------------------------|
| Median    | IQR (Interquantile Range) | Marazzi and Ruffieux (1999) (Gamma, Weibull) |
| Median    | MAD (Median of Absolute Deviations) | Boudt et al. (2011) (Weibull) |
| Trimmed mean | Trimmed M(ean)AD | Marazzi and Ruffieux (1999) (Gamma, Weibull) |
| Median    | kMAD           | Ruckdeschel and Horbenko (2010) (GPD) |
| Median    | Sn             | —                               |
| Median    | Qn             | Boudt et al. (2011) (Weibull)   |

1 Unchecked credit given to Olive (2008) in the cited reference

large sample size $n$, we can solve the implicit equations for $\beta$ and $\xi$ to obtain the MedMAD estimator.

More efficient estimators for dispersion than MAD, but with same breakdown point of 50% at continuous distributions, and in particular suitable for asymmetric distributions, have been proposed in Rousseeuw and Croux (1993) as $M = Q_n$ and $M = S_n$. In this context, $Q_n = \{ |x_i - x_j|; \ i < j \}_{(k)}$, $k = \left( \frac{h}{2} \right) \approx \left( \frac{n}{2} \right)/4$, $h = \left\lfloor \frac{n}{2} \right\rfloor + 1$, while $S_n = \text{med}_i \{ \text{med}_j |x_i - x_j| \}$ where in case of discrepancies, the inner median is to be taken as hi-med, the outer as lo-med, where lo-med$(F) = F(1/2)$, and hi-med$(F) = F^-(1/2 + 0)$. The resulting LD estimators are named MedQn and MedSn, respectively.

Note that for asymmetric $G$, the functionals $S(G) = \text{med}_X \text{med}_Y |X - Y|$, $X, Y \sim G$ and $Q(G) = \inf \{ s > 0; \int G(t + d^{-1}s)dG(t) \geq 5/8 \}$ involve expensive, careful numerical calculations, in particular for the heavy-tailed GPD and GEVD cases.

In the GEVD and GPD case, due to their considerable skewness to the right, one can improve the MedMAD estimator considerably, using a dispersion functional that takes this skewness into account: For a distribution $F$ on $\mathbb{R}$ with median $m$ let us define for $k > 0$

$$k\text{MAD}(F, k) := \inf \{ t > 0 \mid F(m + kt) - F(m - t) \geq 1/2 \} \quad (4.5)$$

i.e.; kMAD only searches among the class of intervals about the median $m$ with covering probability 50%, where the part right to $m$ is $k$ times longer than the one left to $m$ and returns the shortest of these. In our case, $k$ would be chosen to be a suitable number larger than 1, and $k = 1$ would reproduce the MAD. Apparently, whenever $F$ is continuous, kMAD preserves the ABP of the MAD of 50%, i.e.; covering both the explosion and implosion case.
Computation of LD estimators  Each of our dispersion estimators $S_n, Q_n,$ and $k\text{MAD}$ is scale-equivariant, and the same also holds for the respective population counterparts, as well as for any fixed quantile, in particular for the median; hence denoting the dispersion functional by $s,$ both the quotient $q(\xi) := s(\beta, \xi)/m(\beta, \xi)$ and its empirical counterpart $\hat{q}_n(q_k, \hat{q}_k; n$ for MedkMAD) are scale-free; so we have reduced the problem by one dimension. In the sequel we also write $q_k, \hat{q}_k; n$ for $S_n$ and $Q_n,$ where $k$ is then simply void. Assuming continuity and monotonicity, we obtain an estimator for $\xi$ given by $\hat{\xi}_n = q_k^{-1}(\hat{q}_{n,k}).$

A corresponding estimator for $\beta$ for each of the variants $k\text{MAD}, S_n,$ and $Q_n,$ is then simply given by

$$\hat{\beta}_n = \hat{m}/m(1, \hat{\xi}_n) \quad (4.6)$$

In particular, by construction all LD estimators are equivariant in the sense of (2.6).

Continuity and Monotonicity of $q$ as a function in $\xi$ ensure existence and uniqueness of the implicitly defined estimator for $\xi$.

Continuity of $q_k$ in $\xi$ for all our scale-shape models, i.e.; GPD, GEVD, Gamma, and Weibull and all our dispersion functionals $k\text{MAD}(k), S$ and $Q$ is straightforward, even for the limit cases $\xi \to 0.$

Monotonicity of $q_k,$ though, is not so obvious from the analytic terms, but the plots of function $\xi \mapsto q(\xi)$ for dispersions $k\text{MAD}, S_n,$ and $Q_n,$ in Fig. 4 indicate strict monotonicity for each of the dispersions and the GPD, Gamma, and Weibull cases, while for the GEVD case, $q$ is bitone with maximum $\bar{q}_k$ taken in $\xi_0 > 0.$ To obtain consistent estimators in this case, we restrict ourselves to the range left or right to $\xi_0$ containing $\xi = 0.$

Restriction(s) of solvability domain Besides this restriction of the range of $\xi$ in the GEVD case, we conclude, that in the GPD and in GEVD cases, for each of the dispersions, our restriction to $\xi > 0$ implies a restriction of the solvability domain for $q_k(\xi)$ within the set of admissible values of $\xi$:

$$q_k(\xi) \geq \lim_{\xi \to 0} q_k(\xi) =: \check{q}_k > 0 \quad (4.7)$$

while in the Weibull and Gamma case, $\check{q}_k$ can be taken as 0.

The following lemma gives us yet other restrictions:

**Lemma 4.1** Let $s$ the functional version to any of the scale estimators $S_n, Q_n,$ and $k\text{MAD}$ (for any $k > 0$). Let $G$ be a distribution on $\mathbb{R}$ such that $-\infty < x_0 = \sup\{x: G(x) = 0\},$ i.e.; with finite left endpoint. Then with $m = G^{-}(1/2 + 0),$ the hi-med of $G,$

$$s(G) \leq m - x_0 =: s_0 \quad (4.8)$$

with equality iff

(kMAD) $G((m; m + ks_0)) = 0.$
Yet another breakdown point notion: EFSBP

\[ G(x + 2s_0 - 0) - G(x) < 1/2 \text{ for each } x \geq x_0. \]

\[ G(m) = 1/2, G(x_0) = 0. \]

Consequently, as \( x_0 = 0 \), in the GPD, Gamma, and Weibull case, 

\[ q_k(\xi) < 1 \quad \forall \xi \]

and, the same relation in the ideal model also holds sample-wise, i.e.;

\[ \hat{q}_{k,n} < 1 =: \bar{q}_k \]

in each sample (from the ideal model distribution) where

1. (kMAD) at least one observation in \((\hat{m}; \hat{m} + k(\hat{m} - X_{(1)}))\).
2. (Sn) at least one interval of length shorter than \(2(\hat{m} - X_{(1)})\) containing more than \([n/2] + 1\) observations.
3. (Qn) all observations finite.

Hence, for the LD estimators, we have to find the unique zero \( \hat{\xi}_n \) of 

\[ H_k(\xi) = q_k(\xi) - \hat{q}_{n,k} \]

in the interval \((\hat{q}_k; \bar{q}_k)\) which can easily be solved with a standard univariate root-finding tool like \texttt{uniroot} in \texttt{R} (R Development Core Team 2011).

Producing breakdown Clearly, in the GPD case, we could drive \( \hat{q}_{k,n} \) to values larger than 1 by modifying observations in the original sample to values smaller than \( x_0 \). These values would then be identifiable as outliers without error then, and we could cancel them from the sample. Instead we only consider contaminations by values larger than \( x_0 \) (which could also have been produced in the ideal model).

On first glance, values of \( \hat{q}_{k,n} \) outside \((\hat{q}_k; \bar{q}_k)\) would make for a “definition breakdown”, but if, for \( \hat{s}_n \) the respective scale estimator, \( \hat{s}_n \to \hat{m} \), this entails \( \hat{\xi}_n \to \infty \) in the GPD case and \( \hat{\xi}_n \to 0 \) in the Gamma and Weibull case. Hence we can produce a breakdown in the original sense by modifying an original sample such that \( \hat{s}_n \to \hat{m} \).

5 Calculation of (E)FSBP for Pickands and LD estimators

In some of our scale-shape models and for some of our estimators we have analytic expressions for the different breakdown point notions.

5.1 Pickands estimator

Proposition 5.1 (Breakdown for PE) In the GPD, GEVD, Weibull, and Gamma cases, an upper bound for FSBP of PE is given by 25%, which also invariably is the FSBP in the Weibull case. In the GPD case, no matter if \( \xi \in \mathbb{R} \) or \( \xi > 0 \), and in the unrestricted GEVD case, i.e.; \( \xi \in \mathbb{R} \), FSBP is given by

\[ \varepsilon^* = \hat{N}_0^0 / n, \quad \text{for} \quad \hat{N}_0^0 := \# \{X_i | 2 \hat{Q}_2 \leq X_i \leq \hat{Q}_3\}. \]
The ABP then is given by
\[ \bar{\varepsilon}^* = \varepsilon^* = P_\theta(2Q_2 < X_1 \leq Q_3) \tag{5.2} \]
which in the GPD case is just \( \bar{\varepsilon}^* = (2^{1+1/\xi} - 1)^{-1/\xi} - 1/4 \), and, in the GEVD case, \( \bar{\varepsilon}^* = 3/4 - \exp \left( - (2 \log(2)^{1/\xi} - 1)^{-1/\xi} \right) \). In the restricted GEVD case, where \( \xi > 0 \),
\[ \varepsilon_n^* = \tilde{N}_n^0 / n, \quad \text{for} \quad \tilde{N}_n^0 := \#\{ X_i | q_0(0) \hat{Q}_2 \leq X_i \leq \hat{Q}_3 \}. \tag{5.3} \]

The ABP then is given by
\[ \bar{\varepsilon}^* = \varepsilon^* = P_\theta(q_0(0)Q_2 < X_1 \leq Q_3). \tag{5.4} \]

For \( \xi = 0.7 \), we obtain \( \bar{\varepsilon}^* \doteq 6.42\% \) in the GPD case, and in the GEVD case, \( \bar{\varepsilon}^* \doteq 15.42\% \) in the unrestricted case, and \( \bar{\varepsilon}^* \doteq 6.13\% \) in the restricted case. For the figures for \( \bar{\varepsilon}_n^* \), for \( n = 40, 100, 1,000 \) in the GPD, GEVD, and Weibull case, see Table 3, where we make use of Proposition 5.3 below. In the Gamma case, the situation is more involved, and we skip computation of the actual breakdown points.

5.2 LD estimators

The FSBPs of 50% of the median and the dispersion estimators obviously form an upper bound for the FSBP of the LD estimators, implying that you could at least drive one of the parameters \( \beta \) and \( \xi \) to \( \infty \). However, similarly to regression based estimators for the Weibull case of Boudt et al. (2011), breakdown is not only entailed by moving mass to 0 or \( \infty \), and the actual breakdown points of the LD estimators are smaller; for the MedMAD, we come up with some explicit expressions, while for the MedSn and MedQn we have to recur to simulations, see subsect. 5.5.

**Proposition 5.2** (Breakdown for MedMAD) In the GPD, Weibull, and Gamma cases, the FSBP of MedMAD is given by

\[ \varepsilon_n^* = \begin{cases} \hat{N}_n'/n & \text{Weibull; Gamma; GPD, unrestr. case, i.e.; } \xi \in \mathbb{R} \\ \min(\hat{N}_n', \hat{N}_n'')/n & \text{GPD, restr. case, i.e.; } \xi > 0 \end{cases} \tag{5.5} \]
\[ \hat{N}_n' := \#\{ X_i | \hat{m} < X_i \leq (k + 1)\hat{m} \}, \quad \hat{N}_n'' := \lceil n/2 \rceil - \#\{ X_i | (1 - \hat{q}_k)\hat{m} < X_i < (k\hat{q}_k + 1)\hat{m} \}. \tag{5.6} \]

The ABP in this case is given by \( \bar{\varepsilon}^* = \bar{\varepsilon}' \) for the unrestricted and \( \bar{\varepsilon}^* = \min(\bar{\varepsilon}', \bar{\varepsilon}'') \) for the restricted case where
\[ \bar{\varepsilon}' = F_\theta((k + 1)m) - 1/2, \quad \bar{\varepsilon}'' = 1/2 - F_\theta((k\hat{q}_k + 1)m) + F_\theta((1 - \hat{q}_k)m). \tag{5.8} \]

At \( k = 10 \) and \( \xi = 0.7 \), we obtain \( \bar{\varepsilon}^* \doteq 44.75\% \) (GPD; \( \xi \in \mathbb{R} \)), 11.87% (GPD; \( \xi > 0 \)), 49.47% (Gamma), and 47.56% (Weibull). For further figures for \( \varepsilon_n^*, \bar{\varepsilon}_n^*, \bar{\varepsilon}, \) see Table 3,
Yet another breakdown point notion: EFSBP

Fig. 2  $\varepsilon^*$ (MedkMAD$_{10}$; GPD$_{\theta} = (1, \xi))$ for different $\xi$ with or without restriction $\xi > 0$

where again we make use of Proposition 5.3. In particular, contrary to Boudt et al. (2011), not only is our FSBP varying sample-wise in these cases, but also do ABP and EFSBP depend on $\xi$. A plot of the dependency $\xi \mapsto \varepsilon^*$ (MedkMAD$_{10}$; GPD($\xi$)) is displayed in Fig. 2.

5.3 Calculation of EFSBP

To obtain actual values of EFSBP, we have the following proposition.

Proposition 5.3 Consider $\hat{N}_n^0$, $\hat{N}_n'$, $\hat{N}_n''$ as defined in (5.1), (5.6), (5.7) and write $\bar{F}$ for $1 - F$. Then for $n \geq 3$,

(a) Setting $i_1 = \lfloor n/2 \rfloor$, $i_2 = \lceil 3n/4 \rceil$, and abbreviating $2F^{-1}(u)$ by $t_2$, we obtain for $l \in \{1, \ldots, i_2 - i_1 - 1\}$

$$P(\hat{N}_n^0 = l) = n \sum_{l=0}^{n-i_2} \int_{0}^{1} \int_{i_1 - 1}^{n-i_1-1} \int_{i_2-l-1}^{n-i_2+l+1} u^{i_1-1}(F(t_2) - u)^{i_2-l-1} \bar{F}(t_2)^{n-i_2-l} du$$

(5.9)

and

$$P(\hat{N}_n^0 = 0) = n \sum_{l=0}^{n-i_2} \int_{0}^{1} \int_{i_1 - 1}^{n-i_1-1} \int_{i_2+l}^{n-i_2-l} u^{i_1-1}(F(t_2) - u)^{i_2+l} \bar{F}(t_2)^{n-i_2-l} du$$

(5.10)

The case of $\tilde{N}_n^0$ is obtained from (5.9), (5.10) replacing $t_2$ by $t_q := q_0(0)F^{-1}(u)$. 
The dependency of EFSBP on \( n \) is visualized in Fig. 3. We see a saw-tooth like oscillation which is explained by the use of finite sample quantiles in Proposition 5.3. In particular there are considerable deviations from ABP for moderate sample sizes.

5.4 Illustration: usefulness of EFSBP

The expressions given in Propositions 5.1, 5.2, and 5.3 illustrate that in both the Pickands and LD estimator case, even starting from an ideal sample, the “usual” sample-wise fluctuations of FSBP = \( \hat{N}_n/n \) are considerable. Moreover, Proposition 5.3 shows that we even have a positive, although very small ideal probability

\[
p_0 := P^X(\hat{N}_n = 0) > 0
\]
for breakdown already in the ideal model. Now, on the event \{\hat{N}_n = 0\}, \varepsilon_n^* = 0, so no universal non-trivial lower bound can be given for the FSBP in both the Pickands and LD estimator case. As the figures in Table 2 below illustrate, however, such an event will hardly ever occur provided only moderately small sample sizes, and the same goes for similarly small realizations of \(\hat{N}_n\), so these cases, as motivated in the
introduction of EFSBP, are not representative, indeed. To grasp the difference between $\tilde{\varepsilon}_n^*$ and $\varepsilon^*$, we consider the following Hoeffding-type lemma for empirical quantiles

Lemma 5.4 (a) Let $0 < \delta < 1/2$ and $t \in \mathbb{R}$ and for given $\alpha \in (0, 1)$ and c.d.f. $F$, let $q = F^-(\alpha)$, and $\hat{q}_n = \hat{F}_n^-(\alpha)$. Assume that $F$ is differentiable in $q$ with density $f(q) > 0$. Then with $t_n = tn^{-1/2+\delta}$, for $n$ large enough,

$$P(|\hat{q}_n - q| \geq t_n) \leq \exp(-2f(q)^2n^\delta) \quad (5.14)$$

(b) Let $a_i \neq 0$, $\alpha_i \in (0, 1), \alpha_1 \neq \alpha_2$, $i = 1, 2$ be given as well as c.d.f. $F$; assume $F$ differentiable in $a_iq_i$, $i = 1, 2$. Then under the assumptions of (a) for $q_i,$
for \( \hat I_n = (a_1 \hat q_{1,n}, a_2 \hat q_{2,n}) \) and \( I = (a_1 q_1; a_2 q_2) \), we have for \( n \) large enough,

\[
P^X(\hat I_n) = P^X(I) + O(n^{-1/2+\delta/2}).
\]

To illustrate the size of the \( O(n^{-1/2+\delta/2}) \)-term, let us also determine the upper \( p_1 \)-quantile of \( \varepsilon^*_n \) for \( p_1 = 0.95^{0.001} \), i.e.; the minimal number \( q_1 \), such that with probability 0.95 we will not see realizations with \( \varepsilon^*_n < q_1 \) in 10,000 runs of sample size \( n \).

**Evaluations for PE and MedkMAD** Using the actual distribution of \( \hat N_n \) given in Proposition 5.3, in Table 2, for Pickands (PE) and MedkMAD, \( k = 10 \) we determine \( \tilde \varepsilon^*_n, p_0 \) and \( q_1 \) for \( n = 40, 100, 1,000 \) in the GPD (with and without restriction to \( \xi > 0 \)), Gamma, and Weibull cases, each with \( \xi = 0.7 \). The Gamma case is skipped, though, in the PE case for lack of explicit formulae. Apparently \( \tilde \varepsilon^*_n \) is quickly converging in \( n \), so \( \tilde \varepsilon^* \) gives indeed a useful bound on average.

According to the values of \( p_0 \), breakdown in the ideal model will hardly ever happen for PE for \( n \geq 1,000 \), and for MedkMAD for \( n \geq 100 \), and only rarely for \( n \geq 40 \).

The values for \( q_1 \) demonstrate that in a simulation study at the GPD with \( \xi = 0.7 \) with 10,000 runs of sample size upto \( n = 1,000 \), we will probably see breakdowns for PE, as well as for the MedkMAD restricted to \( \xi > 0 \). Contrary to this, as long as we have no more outliers than 8, 30, 411 for sample sizes \( n = 40, 100, 1,000 \), we will not see a breakdown for MedkMAD in the unrestricted case; in the Gamma case with same shape we obtain 9, 38, 476, and in the Weibull 7, 32, 442; analogue figures for PE at the Weibull with \( \xi = 0.7 \) are 10, 25, 250.

We may interpret the values of \( \tilde \varepsilon_n \) as follows: Before having made any observations, at the GPD at \( \xi = 0.7 \), using PE, one may be confident to be protected against 3 outliers for sample size 40, 7 for sample size 100, and 65 for sample size 1,000, while for MedkMAD, the corresponding figures are 17, 43, and 447 in the unrestricted case and 5, 12, and 118 when restricted to \( \xi > 0 \); calculations in the Gamma and Weibull cases give comparable numbers.

5.5 Breakdown calculations in the remaining cases: simulational approach

For the breakdown point of MedQn and MedSn, as well as for MedkMAD in the GEVD case, there are no analytical expressions, so we calculate them using simulations.

More precisely, for each of the estimators MedkMAD \((k = 10)\), MedQn, MedSn, PE, and each of the ideal distributional settings GPD, GEVD, Weibull, and Gamma (each at \( \vartheta = (1, 0.7) \)), we produced \( M = 10,000 \) runs of sample sizes \( n = 40, 100, 1,000 \) and noted the number of alterations needed to move \( \hat q_{k,n} \) to \( \tilde q \), and in a second round, starting from the same runs of ideal observations, for GPD and GEVD, the minimal number of alterations needed to move \( \hat q_{k,n} \) to \( \tilde q_k \), respectively the minimum of these two rounds. In the cases where explicit formulae are available this gives us a possibility to cross-check our results. Some small discrepancies should arise though, as we use the default median in R, R Development Core Team (2011), i.e.; \((\text{hi-med} + \text{lo-med})/2\) for even sample size, while Proposition 5.3 below is limited to hi-med. For actual simulated values for \( \tilde \varepsilon^*_n \), see Table 3.
### Table 3  Simulated EFSBP in % with CLT-based 95%-confidence interval (CI) for \( \theta = (\xi = 0.7, \beta = 1) \); number of runs is 10,000

| Model          | Med-Sn | Med-Qn | Med-kMAD | Med-kMAD\textsubscript{10} | ±CI   | PE   | ±CI   |
|----------------|--------|--------|----------|----------------------------|-------|------|-------|
| GPD \( \xi \in \mathbb{R} \)  | 34.69  | 0.33   | 43.74    | 0.09                       | 44.68 | 0.13 | 5.94  | 0.10 |
| GPD \( \xi > 0 \)          | 8.78   | 0.18   | 23.44    | 0.21                       | 10.65 | 0.07 | 5.94  | 0.10 |
| GEVD \( \xi \in \mathbb{R} \) | 6.99   | 0.21   | 5.89     | 0.21                       | 13.38 | 0.24 | 14.85 | 0.13 |
| GEVD \( \xi > 0 \)          | 6.99   | 0.21   | 5.89     | 0.21                       | 4.75  | 0.13 | 7.87  | 0.16 |
| Weibull               | 37.63  | 0.34   | 40.32    | 0.11                       | 47.31 | 0.02 | 25.00*| 0.00*|
| Gamma                 | 34.55  | 0.32   | 41.97    | 0.10                       | 49.17 | 0.02 | n.a.  | –    |
| GPD \( \xi \in \mathbb{R} \)  | 23.55  | 0.21   | 47.51    | 0.04                       | 44.73 | 0.09 | 6.12  | 0.07 |
| GPD \( \xi > 0 \)          | 12.44  | 0.16   | 18.42    | 0.16                       | 11.32 | 0.05 | 6.12  | 0.07 |
| GEVD \( \xi \in \mathbb{R} \) | 3.25   | 0.09   | 2.88     | 0.09                       | 8.86  | 0.14 | 15.01 | 0.09 |
| GEVD \( \xi > 0 \)          | 3.25   | 0.09   | 2.88     | 0.09                       | 6.32  | 0.11 | 6.71  | 0.05 |
| Weibull               | 26.58  | 0.30   | 45.12    | 0.05                       | 47.41 | 0.02 | 25.00*| 0.00*|
| Gamma                 | 25.42  | 0.21   | 45.90    | 0.04                       | 49.35 | 0.02 | n.a.  | –    |
| GPD \( \xi \in \mathbb{R} \)  | 21.86  | 0.03   | 49.75    | 0.00                       | 44.75 | 0.03 | 6.38  | 0.03 |
| GPD \( \xi > 0 \)          | 14.99  | 0.13   | 16.06    | 0.02                       | 11.82 | 0.02 | 6.37  | 0.03 |
| GEVD \( \xi \in \mathbb{R} \) | 1.06   | 0.03   | 1.27     | 0.03                       | 7.25  | 0.05 | 15.39 | 0.04 |
| GEVD \( \xi > 0 \)          | 1.06   | 0.03   | 1.27     | 0.03                       | 7.22  | 0.05 | 6.20  | 0.08 |
| Weibull               | 19.77  | 0.03   | 49.01    | 0.01                       | 47.55 | 0.01 | 25.00*| 0.00*|
| Gamma                 | 24.13  | 0.04   | 49.16    | 0.01                       | 49.46 | 0.01 | n.a.  | –    |

* Theoretical values, n.a. not available, in these cases, 25% is an upper bound

### 6 Conclusion

This article provides a new measure for global robustness of an estimator at finite samples, i.e.; EFSBP, a variant of the finite sample breakdown point which is particularly useful in situations where we have only partial equivariance and no non-trivial, universal lower bounds for FSBP are available. This variant comes closer to the (sample-free) ABP while still retaining the finite sample aspect of FSBP.

We have illustrated this measure at a set of scale-shape models, applying it to LD and Pickands/Quantile-type estimators meant for high-breakdown initial estimators to be enhanced in efficiency by reweighting afterwards.

Although kMAD, Qn, and Sn all share the same breakdown properties in the location-scale setting, where they are defined, the corresponding LD estimators in the considered scale-shape models exhibit a differentiated breakdown behavior, and there is not one single best estimator.

In the unrestricted GEVD case, the easy-to-compute Pickands-type estimator turned out to have the highest breakdown point among all considered estimators, while in the setting restricted to \( \xi > 0 \), from sample size 100, MedkMAD becomes superior. In all other situations, the best estimator is either MedkMAD or MedQn. In the
unrestricted and restricted GPD case MedQn performs best, with MedkMAD close in the unrestricted case for \( n = 40 \). In the Weibull and Gamma cases MedkMAD performs best, except for the Weibull at \( n = 1,000 \) where MedQn is best, but with MedkMAD close by. For deciding between MedkMAD and MedQn in cases where their breakdown points are similar though, one also should take into account computational costs as well, which so far clearly favors MedkMAD.

Appendix

Proof to Lemma 4.1: For any \( k > 0 \), \( G(m + ks_0) - G(m - s_0 - 0) = G(m + ks_0) \geq 1/2 \), so \( s_0 \geq k \text{MAD}(G, k) \). For \( x \geq x_0 \) and \( Y \sim G \), let \( g_G(x) = \text{med}_x(|Y - x|) = \inf\{s \geq 0: G(s + x) - G(x - s - 0) \geq 1/2\} \). But \( G(s_0 + x) - G(x - s_0 - 0) = G(s_0 + x) \) for \( x \leq m \), so \( g_G(x) \leq s_0 \) for \( x \leq m \), and hence, as \( \{x \leq m\} \subset \{G(x) \leq s_0\} \), \( S(G) = \inf\{t \geq 0: P(g_G(x) \leq t) \geq 1/2\} \leq s_0 \). Finally, for \( X, Y \sim G \), stoch. indep. \( Q(G) = \inf\{s: P(|X - Y| \leq s) \geq 1/4\} \leq s_0 \), as

\[
P(|X - Y| \leq s_0) = \int_{[x_0; m]} G(x + s_0) - G(x - s_0 - 0) G(dx) \\
\geq \int_{[x_0; m]} G(x + s_0) G(dx) \geq \int_{[x_0; m]} \frac{1}{2} G(dx) \geq \frac{1}{4} \quad (A.1)
\]

Assume \( s(G) = s_0 \). In case of kMAD this happens iff \( G(m + ks_0) = 1/2 \), or, equivalently, \( G((m; m + ks_0)) = 0 \). In case of Sn, \( S(G) = s_0 \) iff \( P(g_G(X) > s) \geq 1/2 \) for all \( s < s_0 \), or, equivalently, \( P(x: G((x - s_0; x + s_0)) < 1/2) \geq 1/2 \). But \( x - s_0 < x_0 \) whenever \( x < m \), so \( G((x - s_0; x + s_0)) = G(x + s_0) \geq G(m) = 1/2 \). Hence \( S(G) = s_0 \) iff \( G((x - s_0; x + s_0)) < 1/2 \) for all \( x \geq m \), or, equivalently, iff \( G(x + 2s_0 - 0) - G(x) < 1/2 \) for \( x \geq x_0 \). In case of Qn, \( S(G) = s_0 \) iff the inequalities in (A.1) are equalities, i.e.; iff \( G([x_0; m]) = 1/2 = G(m + s_0) \), and \( \int_{[m; \infty)} G(x + s_0) - G(x - s_0 - 0) G(dx) = 0 \). The last integral is 0 iff \( G((m; \infty)) = 0 \), so that altogether, \( S(G) = s_0 \) iff \( G(m) = G((\infty)) = 1/2 \). \( \square \)

Proof to Proposition 5.1: For all models, i.e.; GPD, GEVD, Weibull, and Gamma, we can render the scale estimator arbitrarily large for \( \hat{Q}_3 \) sufficiently large, so \( \varepsilon_n^* \leq 1/4 \).

In case of GPD and GEVD, \( \hat{\beta} < 0 \) once \( \hat{Q}_3 \leq 2\hat{Q}_2 \), which certainly happens if, in an ideally distributed sample, we replace all observations \( X_i, 2\hat{Q}_2 \leq X_i \leq \hat{Q}_3 \) by \( \hat{Q}_2 \), entailing (5.1). Appealing to Lemma 5.4, up to an event of probability \( O(\exp(-cn^3)) \) for some \( c > 0 \),

\[
\varepsilon_n^* = \tilde{\varepsilon}^* + O_p\left(n^{-1/2+\delta/2}\right) \quad (A.2)
\]

As (4.4) gives valid values for \( \xi \) and \( \beta \) for any values of \( \hat{Q}_3 \) and \( \hat{Q}_2 \), in the Weibull case, we cannot lower the upper bound of 25%, i.e.; \( \lim_{n} \tilde{\varepsilon}^* = \varepsilon^* = 1/4 \). \( \square \)

Proof to Proposition 5.2: As we have seen in the considerations in Sect. 4.2 on producing breakdown, we only can solve (uniquely) for \( \xi \) and \( \beta \) as long as the quotient

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\( \hat{q}_{k:n} \) falls into (\( \bar{q}_k, \tilde{q}_k \)); case-by-case considerations indeed show that by driving \( \hat{q}_{k:n} \) to either \( \bar{q}_k \) (in case of GPD and GEVD) or \( \tilde{q}_k \) (in all cases) produces breakdown, that is, breakdown could be achieved by either moving all \( \hat{N}_n \) observations from (5.6) for which \( \tilde{m} < X_i \leq \tilde{m} + \hat{M}_k \) to \((k+1)\tilde{m}\) (entailing \( \hat{q}_{k:n} \approx 1 \)) or by moving a number of \( \hat{N}_n' \) observations (as defined in (5.7)) to the interval \([1 - \hat{q}_k\hat{m}, (k\hat{q}_k + 1)\tilde{m}]\) up to the point that it contains \( n/2 \) observations (entailing \( \hat{q}_{k:n} < \bar{q}_k \)). The actual FSBP is then given by the alternative needing to move less observations. The terms for ABP follow with the usual LLN argument.

\[ G(s, t) = n \int_{-\infty}^{s} f(x) (n-1)_{i-1} F(x)^{i-1} \sum_{k=2}^{n-1-i} (n-i-1)_{k-2}^2 (F(t) - F(x))^{k-2} \tilde{F}(t)^{n-1-k} \ dx \]

Hence

\[ P(\hat{N}_n' \geq l) = P(X_{[n/2+l+1]:n} \leq (k+1)X_{[(n/2+1):n]}) \]

\[ = n \int_{0}^{l} \int_{0}^{n/2} \sum_{k=l}^{n/2-1} \left( n/2-1 \right)_{k-2}^2 \left( F(t_k) - u \right)^{k-2} \tilde{F}(t_k)^{n/2-1-k} \ du \]

and (5.11) follows by taking differences. Cases (5.9) and (5.12) follow similarly.

Proof to Lemma 5.4: We note that \( \{\hat{q}_n \leq t\} = \{\sum_i I(X_i \leq t) \geq n\alpha\} \). Hence with Hoeffding’s inequality, Hoeffding (1963), \( P(|\hat{q}_n - q| \geq t_n) \leq 2 \exp(-2n(F(t_n + q) - \alpha)^2) \) and (a) follows from \( F(t_n + q) - \alpha = f(q) t_n + o(t_n) \). For (b), note that \( P(\hat{I}_n \leq I) \leq E|F(\hat{q}_{1,n} - \alpha_1| + E|F(\hat{q}_{2,n} - \alpha_2| \). Hence, for large enough \( n \), \( P(\hat{I}_n \leq I) \leq 2f(a_1 q_1)\alpha_1|E|\hat{q}_{1,n} - q_1| + 2f(a_2 q_2)\alpha_2|E|\hat{q}_{2,n} - q_2| \) and, applying that for a random variable \( Z \) taking values in \([0, 1]\), for \( t \leq (0, 1), 0 \leq E Z \leq t + \int_{t}^{1} P(X > t), \) so by Mill’s ratio, \( P(\hat{I}_n \leq I) \leq 2t + \sum_i \exp(-2nt^2 f(q_i)^2)/(2nt f(q_i)^2) \). Plugging in \( t = n^{-1/2+\delta} \), we obtain (b).

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