CAUCHY-DAVENPORT TYPE THEOREMS FOR SEMIGROUPS

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Abstract. Assume that $A = (A,+)$ is a (possibly non-commutative) semigroup. For $Z \subseteq A$ we define $Z^\times := Z \cap A^\times$, where $A^\times$ is the set of the units of $A$, and $\gamma(Z) := \sup_{z_0 \in Z^\times} \inf_{z \in Z} \inf_{z_0 \neq z} \text{ord}(z - z_0)$. The paper investigates properties of $\gamma(\cdot)$ to show the following extension of the Cauchy-Davenport theorem: If $A$ is cancellative and $X, Y \subseteq A$, then $|X + Y| \geq \min(\gamma(X+Y), |X| + |Y| - 1)$. This implies a generalization of Kemperman’s inequality for torsion-free groups and a strengthening of another extension of the same Cauchy-Davenport theorem, where $\gamma(X+Y)$ in the above is replaced by the minimal order of the non-trivial subsemigroups of $A$. The latter was first proved by Károlyi in the case of finite groups, based on the Feit-Thompson theorem, and then by Hamidoune for an arbitrary group, based on the isoperimetric method. Here, we present a self-contained, combinatorial proof of Hamidoune’s result. Finally, we discuss aspects of a conjecture that, if true, could provide a unified picture of many more Cauchy-Davenport type theorems.

1. Introduction

The weaker are our assumptions, the larger is the number of problems that we can solve: This is the naive philosophy at the heart of the present paper, concerned as it is with the generalization of aspects of the additive theory of groups to broader and more abstract contexts, in continuation to the work initiated by the author in [19], whose focus is on semigroups and which should be consulted by the reader for prerequisites, as well as for the notation and terminology used but not defined here (most of the relevant definitions will be, however, included below for the sake of exposition).

More specifically, we shall prove an extension (see Theorem 10) of the (classical) Cauchy-Davenport theorem (namely, Theorem 1) to semigroups, whence we derive as an almost immediate consequence a significant strengthening (see Corollary 12) of an addition theorem for groups now commonly attributed to Y. O. Hamidoune.

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and G. Károlyi (to wit, Theorem 11). In fact, a proof of this latter result was first published by G. Károlyi in 2005 for the special case of finite groups [11], based on the structure theory of group extensions, by reduction to finite solvable groups in the light of the Feit-Thompson theorem. In the same paper (p. 242), Károlyi reports a more “elementary” proof of the general statement (for an arbitrary group), which was communicated to him by Hamidoune during the peer-review process of [11]. Hamidoune’s approach depends on a generalization of an addition theorem by L. Shatrowsky and is ultimately built upon the isoperimetric method (see [9] and references therein). However, Károlyi himself has pointed out to the author, as recently as July 2013, that an alternative and even “simpler” approach comes from a Kneser-type result due to J. E. Olson [15, Theorem 2]. Yet another argument along the same lines was suggested by I. Ruzsa to the author in a private communication in mid-June 2013.

On these premises, it is worth remarking from the outset that our proof of Theorem 10, and consequently of Corollary 12, is basically a transformation proof, close in the spirit to Olson’s approach and as “elementary” as many other combinatorial proofs available in the literature; in particular, it is self-contained and does not depend at all on the Feit-Thompson theorem or Hamidoune’s isoperimetric method.

In addition to the above, we present and discuss aspects of a conjecture (namely, Conjecture 8) which, if true, would further improve most of the results in the paper and provide a unified picture of many more theorems of Cauchy-Davenport type, including the ones proved in [19]. In all of this, a key role is played by properties of the so-called Cauchy-Davenport constants of a set, which also are defined and investigated in the manuscript.

Organization. In Section 2, we fix some preliminaries. In Section 3 we recall some basic definitions on semigroups and accordingly extend fundamental group-theoretical concepts such as the notions of order and difference set. In Section 4 we define the lower and the upper Cauchy-Davenport constant, give a partial overview of the existing literature on theorems of Cauchy-Davenport type (with a particular emphasis on those that we are going to strengthen or generalize), and state our main results and a related conjecture. Finally, Section 5 introduces the Cauchy-Davenport transforms and contains intermediate results used in Section 6 for the proof of the principal theorem (namely, Theorem 10).

2. Preliminaries and notation

The purpose of this brief section is to review basic definitions and fix some general terminology and notation. However, we make first a short digression into set theory, which may sound unusual to practitioners, but is necessary in view of future developments.
We use as a foundation the Tarski-Grothendieck set theory, shortly TG. Alternatives are possible, but this issue exceeds the scope of the paper, and we can pass over it without worrying much. We just mention that we choose to work in TG rather than, say, in ZFC (the classical Zermelo-Fraenkel set theory with the axiom of choice), motivated by the fact that we will be concerned, in the sequel of this work, with objects like the “class of all structures of a certain type”, which would make no sense in ZFC, essentially because the latter does not allow for anything like the “class of all sets”. With this in mind, we fix once and for all an uncountable Grothendieck universe $\Omega$, and refer to the elements of $\Omega$ as $\Omega$-sets, or simply sets, and to an arbitrary set in the ontology of TG as a class, a family, or a collection.

We rely occasionally on the positive integers $\mathbb{N}^+ := \{1, 2, \ldots\}$, the natural numbers $\mathbb{N} := \mathbb{N}^+ \cup \{0\}$, the real numbers $\mathbb{R}$, the positive real numbers $\mathbb{R}^+$, and the non-negative real numbers $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$, each of these sets being endowed, unless otherwise specified, with its ordinary addition $+$, multiplication $\cdot$ and total order $\leq$.

We extend as usual the operations and the order of $\mathbb{R}_0^+$ to $\mathbb{R}_0^+ \cup \{\infty\}$, by adjoining a “point at infinity”, namely an element $\infty \notin \Omega$ (in fact, we assume $\infty = \Omega$), and by taking $a + \infty := \infty + a := \infty$ and $a \leq \infty$ for $a \in \mathbb{R}_0^+ \cup \{\infty\}$, $b \cdot \infty := \infty \cdot b := \infty$ for $b \in \mathbb{R}^+ \cup \{\infty\}$, and $0 \cdot \infty := \infty \cdot 0 := 0$. Finally, we use $|X|$ for the counting measure of a set $X$ (this is just the number of elements of $X$ when $X$ is finite), accordingly interpreting $|\cdot|$ as a map from $\Omega$ to $\mathbb{R}_0^+ \cup \{\infty\}$.

At several points throughout the paper, we will use without explicit mention the elementary fact that if $A \subseteq B \subseteq \mathbb{R}_0^+ \cup \{\infty\}$ then $\inf(B) \leq \inf(A)$ and $\sup(A) \leq \sup(B)$, with the common convention that the supremum of the empty set if 0 and its infimum is $\infty$.

3. Semigroups and sumsets

Semigroups are a natural framework for developing large parts of theories traditionally presented in much less general contexts. Not only this can suggest new directions of research and shed light on questions primarily focused on groups (as we are going to show), but it also makes methods and results otherwise restricted to “richer settings” potentially applicable to larger classes of problems that could even be intractable by other means.

Here, a semigroup is a pair $A = (A, +)$ consisting of a (possibly empty) set $A$, referred to as the carrier of $A$, and an associative binary operation $+$ on $A$ (unless otherwise specified, all semigroups considered below are written additively, but are possibly non-commutative).

Given subsets $X, Y$ of $A$, we define as usual the subset, relative to $A$, of the pair $(X, Y)$ as the set $X + Y := \{x + y : x \in X, y \in Y\}$, which is written as $x + Y$ (respectively, $X + y$) if $X = \{x\}$ (respectively, $Y = \{y\}$). Furthermore, we extend
the notion of difference set by
\[ X - Y := \{ z \in A : (z + Y) \cap X \neq \emptyset \} \quad \text{and} \quad -X + Y := \{ z \in A : (X + z) \cap Y \neq \emptyset \}. \]

Expressions of the form \( Z_1 + \cdots + Z_n \) or \( \sum_{i=1}^n Z_i \), involving one or more summands, as well as \(-x + Y\) and \(X - y\) for \(x, y \in A\) are defined in a similar way; in particular, we use \( nZ \) for \( Z_1 + \cdots + Z_n \) if the \( Z_i \) are all equal to the same set \( Z \).

We say that \( A \) is unital, or a monoid, if there exists \( 0 \in A \) such that \( z + 0 = 0 + z = z \) for all \( z \); when this is the case, \( 0 \) is unique and called the identity of \( A \). Then, we let \( \mathbb{A}^\times \) be the set of units of \( A \), so that \( \mathbb{A}^\times = \emptyset \) if \( A \) is not a monoid. In this respect, we recall that, if \( A \) is unital with identity \( 0 \), a unit of \( A \) is an element \( z \) for which there exists \( \tilde{z} \), provably unique and called the inverse of \( z \) in \( A \), such that \( z + \tilde{z} = \tilde{z} + z = 0 \).

Given \( Z \subseteq A \), we write \( Z^\times \) in place of \( Z \cap \mathbb{A}^\times \) (if there is no likelihood of confusion) and \( \langle Z \rangle_A \) for the smallest subsemigroup of \( A \) containing \( Z \). Note that \( A \) is a group if and only if \( A^\times = A \). Then, if \( Z = \{ z \} \), we use \( \langle z \rangle_A \) instead of \( \langle \{ z \} \rangle \) and \( \text{ord}_A(z) \) for the order of \( z \) in \( A \), that is we let \( \text{ord}_A(z) := |\langle z \rangle_A| \), so generalizing the notion of order for the elements of a group. Here and later, the subscript ‘\( A \)’ may be omitted from the notation if \( A \) is clear from the context. Finally, an element \( z \in A \) is called cancellable (in \( A \)) if \( x + z = y + z \) or \( z + x = z + y \) for \( x, y \in A \) implies \( x = y \) (see [1, Section 1.2.2]), and \( A \) is said cancellative if its elements are all cancellable.

Based on these premises, we assume for the rest of the paper that \( A = (A, +) \) is a fixed, arbitrary semigroup (unless differently specified), and let \( 0 \) be the identity of the sometimes called conditional unitization of \( A \), herein denoted by \( A(0) \) and simply referred to as the unitization of \( A \): If \( A \) is not unital, \( A(0) \) is the pair \( (A \cup \{ A \}, +) \), where + is, by an abuse of notation, the unique extension of + to a binary operation on \( A \cup \{ A \} \) for which \( A \) serves as an identity (note that \( A \notin A \), so loosely speaking we are just adjoining a distinguished element to \( A \) and extending the structure of \( A \) in such a way that the outcome is a monoid whose identity is the adjoined element); otherwise \( A(0) := A \) (cf. [10, p. 2]). Then, for a subset \( S \) of \( A \) we write \( p_A(S) \) for \( \inf_{z \in S} \text{ord}_A(0)(z) \), which is simply denoted by \( p(S) \) if no ambiguity can arise.

### 4. Cauchy-Davenport Type Theorems

Sumsets in (mostly commutative) groups have been intensively investigated for several years (see [17] for a recent survey), and interesting results have been also obtained in the case of commutative and cancellative monoids [8] (in A. Geroldinger’s work these are simply termed monoids). The present paper aims to extend aspects of the theory to the much more general setting of possibly non-commutative or non-cancellative semigroups.
Historically, the first significant achievement in the field is probably the Cauchy-Davenport theorem, originally established by A.-L. Cauchy \[3\] in 1813, and independently rediscovered by H. Davenport \[6, 7\] more than a century later:

**Theorem 1** ((Cauchy-Davenport theorem)). Let \(A\) be a group of prime order \(p\) and \(X, Y\) non-empty subsets of \(A\). Then, \(|X + Y| \geq \min(p, |X| + |Y| - 1)\).

The result has been the subject of numerous papers, and received many different proofs, each favoring alternative points of view and eventually leading to progress on analogous questions. The theorem applies especially to the additive group of the integers modulo a prime. Extensions to composite moduli have been given by several authors, and notably by I. Chowla \[5, Theorem 1\] and S. S. Pillai \[16\]. These results, used by Chowla and Pillai in relation to Waring’s problem, have been further sharpened by the author in \[19\], where they appear as Theorems 2 and 3, respectively, and the following is established:

**Theorem 2.** Fix an integer \(m \geq 1\). For a non-empty \(Z \subseteq \mathbb{Z}/m\mathbb{Z}\), we let

\[
\delta_Z := \min \max_{z_0 \in Z} \gcd(m, z - z_0)
\]

if \(|Z| \geq 2\), and \(\delta_Z := 1\) otherwise. Then, for all non-empty subsets \(X, Y\) of \(\mathbb{Z}/m\mathbb{Z}\), it holds

\[
|X + Y| \geq \min(\delta^{-1}m, |X| + |Y| - 1),
\]

where \(\delta := \min(\delta_X, \delta_Y)\). In particular, \(|X + Y| \geq \min(m, |X| + |Y| - 1)\) if there exists \(y_0 \in Y\) such that \(m\) is prime with \(y - y_0\) for each \(y \in Y \setminus \{y_0\}\) (or dually with \(X\) in place of \(Y\)).

This is a strengthening of \[16\] and contains \[5, Theorem 1\] as a special case (for, it is enough to assume in the above statement that the identity of \(\mathbb{Z}/m\mathbb{Z}\) belongs to \(Y\) and \(\gcd(m, y) = 1\) for each non-zero \(y \in Y\)). Theorem 2 is, in fact, a rewording of Corollary 15 in \[19\], where it is obtained as a straightforward consequence of Theorem 7 below, for which we need first the following definition:

**Definition 3.** For an arbitrary subset \(X\) of \(A\), we let

\[
\gamma_A(X) := \sup_{x_0 \in X} \inf_{x \not\in X} \text{ord}(x - x_0)
\]

Then, given \(n \in \mathbb{N}^+\) and \(X_1, \ldots, X_n \subseteq A\), we define

\[
\gamma_A(X_1, \ldots, X_n) := \max_{i=1, \ldots, n} \gamma_A(X_i)
\]

and call \(\gamma_A(X_1, \ldots, X_n)\) the Cauchy-Davenport constant of \((X_1, \ldots, X_n)\) relative to \(A\) (again, the subscript ‘\(A\)’ may be omitted from the notation if there is no likelihood of confusion).
This should be contrasted with Definition 7 in [19, Section 2], where \( \gamma(\cdot) \) is written as \( \omega(\cdot) \). Any pair of subsets of \( A \) has a well-defined Cauchy-Davenport constant (relative to \( A \)), and it is interesting to compare it with other “structural parameters”, as in the following:

**Lemma 4.** Let \( X, Y \) be subsets of \( A \) and assume that \( A \) is cancellative and \( X^* + Y^* \) is non-empty. Then, \( \gamma(X, Y) \geq \min(\gamma(X), \gamma(Y)) \geq \gamma(X + Y) \geq p(A) \).

Lemma 4 is proved by the end of Section 5 and applies, on the level of groups, to any pair of non-empty subsets, while the following basic example suggests that the statement is rather sharp:

**Example 5.** Fix an integer \( m \) and prime numbers \( p, q \) with \( 2 \leq m < p < q \), and set \( n := m \cdot p \cdot q, X := \{ mk \pmod{n} : k = 0, \ldots, p - 1 \} \) and \( Y := \{ mk \pmod{n} : k = 1, \ldots, p \} \). We have \( |X + Y| = 2p, \gamma(X) = \gamma(Y) = p \cdot q \) and \( \gamma(X + Y) = q \), while \( p(\mathbb{Z}/n\mathbb{Z}) \) is the smallest prime divisor of \( m \), with the result that \( p(\mathbb{Z}/n\mathbb{Z}) < \gamma(X + Y) < \min(\gamma(X), \gamma(Y)) = \gamma(X, Y) \), and indeed \( p(\mathbb{Z}/n\mathbb{Z}) \) is “much” smaller than \( \gamma(X + Y) \) if \( q \) is “much” larger than \( m \), and similarly \( \gamma(X + Y) \) is “much” smaller than \( \gamma(X, Y) \) if \( p \) is “much” larger than 2.

The following proposition shows that the Cauchy-Davenport constant of a set is invariant under translation by units. This is proved in Section 5 too and, while fundamental for the proof of our main results, may be of independent interest.

**Proposition 6.** Suppose that \( A \) is a monoid and pick \( z \in A^* \) and \( X \subseteq A \). Then, \( \gamma(X) = \gamma(X + z) = \gamma(z + X) \).

Note that, for \( X \subseteq A \), it holds \( \gamma(X) = 0 \) if \( X^* \) is empty, and then \( \gamma(\cdot) \) provides no significant information on the additive structure of \( A \). However, this is not the case, e.g., when \( X \neq \emptyset \) and \( A \) is a group (to the effect that \( X^* = X \) and the Cauchy-Davenport constant of \( X \) assume a slightly simpler form), which is the “moral basis” for the next non-trivial bounds.

**Theorem 7.** Suppose that \( A \) is cancellative and let \( X, Y \) be non-empty finite subsets of \( A \) such that \( \langle Y \rangle \) is commutative. Then, \( |X + Y| \geq \min(\gamma(Y), |X| + |Y| - 1) \).

The result appears as Theorem 8 in [19], where it is proved by an appropriate generalization of the same kind of transformation used by Davenport in his proof of Theorem 1 (note the asymmetry of the statement), and the following looks plausible:

**Conjecture 8.** Let \( n \) be a positive integer and \( X_1, \ldots, X_n \) non-empty subsets of \( A \). If \( A \) is cancellative, then \( |X_1 + \cdots + X_n| \geq \min(\gamma(X_1, \ldots, X_n), |X_1| + \cdots + |X_n| + 1 - n) \).
We do not have a proof of the conjecture (not even in the case of two summands), which can however be confirmed in some special case (see, in particular, Corollary 14 below, or consider Theorem 7 in the case when $A$ is commutative) and would provide, if it were true, a comprehensive generalization of about all the extensions of the Cauchy-Davenport theorem reviewed through this section. Incidentally, the next example shows that the assumption of cancellativity, or a surrogate of it, is critical and somewhat necessary:

**Example 9.** Let $X$ and $Y$ be non-empty disjoint sets with $|X| < \infty$ and denote by $(F_X, \cdot_X)$ and $(F_Y, \cdot_Y)$, respectively, the free abelian groups on $X$ and $Y$. For a fixed element $e \notin F_X \cup F_Y$, we define a binary operation $\cdot$ on $F := F_X \cup F_Y \cup \{e\}$ by taking $u \cdot v := u \cdot_X v$ for $u, v \in F_X$, $u \cdot v := u \cdot_Y v$ for $u, v \in F_Y$, and $u \cdot v := e$ otherwise.

It is routine to check that $\cdot$ is associative, so we write $F$ for the unitization of $(F, \cdot)$ and $1$ for the identity of $F$. Then, taking $Z := Y \cup \{1\}$ gives $\gamma_F(Z) = \infty$ and $X \cdot Z = X \cup \{e\}$, to the effect that $|X \cdot Z| < |X| + |Z| - 1 \leq \gamma_F(X, Z)$, namely $|X \cdot Z| < \min(\gamma_F(X, Z), |X| + |Z| - 1)$, and the right-hand side can be made arbitrarily larger than the left-hand side.

Nevertheless, we can prove the following theorem, which in fact represents the main contribution of the present paper:

**Theorem 10.** Let $X, Y$ be subsets of $A$ and suppose that $A$ is cancellative. Then, $|X + Y| \geq \min(\gamma(X + Y), |X| + |Y| - 1)$.

At this point, it is worth comparing Theorems 7 and 10. On the one hand, the latter is “much stronger” than the former, for it does no longer depend on commutativity (which leads, by the way, to a perfectly symmetric statement). Yet on the other hand, the former is “much stronger” than the latter, since for subsets $X$ and $Y$ of $A$ we are now replacing $\gamma(X, Y)$ in Theorem 7 with $\gamma(X + Y)$, and it has been already observed in the above that this means, in general, a notably weaker bound.

The whole seems to suggest that a common generalization of the two theorems should be possible, and gives another (indirect) motivation to believe that Conjecture 8 can be true. Let it be as it may, Theorem 10 is already strong enough to allow for a significant strengthening of the following result, as implied by Lemma 4 and Example 5:

**Theorem 11** ((Hamidoune-Károlyi theorem)). If $A$ is a group and $X, Y$ are non-empty subsets of $A$, then $|X + Y| \geq \min(p(A), |X| + |Y| - 1)$.

As mentioned in the introduction, this was first proved by Károlyi in [11] in the particular case of finite groups, based on the structure theory of group extensions, by reduction to the case of finite solvable groups in the light of the celebrated Feit-Thompson theorem. The full theorem was then established by Hamidoune.
through the isoperimetric method [11, p. 242]. In contrast, our proof of Theorem
11 is purely combinatorial, and it comes as a trivial consequence of Theorem 10
in view of Lemma 4. Specifically, we have the following:

**Corollary 12.** Pick \( n \in \mathbb{N}^+ \) and subsets \( X_1, \ldots, X_n \) of \( A \) such that \( X_1^+ + \cdots + X_n^+ \neq \emptyset \). If \( A \) is cancellative, then \(|X_1 + \cdots + X_n| \geq \min(p(A), |X_1| + \cdots + |X_n| + 1 - n)\).

Another result from the literature that is meaningful in relation to the present
work is due to J. H. B. Kemperman [12], and reads as follows:

**Theorem 13** (Kemperman’s inequality). Let \( A \) be a group, and let \( X, Y \) be non-empty subsets of \( A \). Suppose that every element of \( A \setminus \{0\} \) has order \( \geq |X| + |Y| - 1 \) in \( A \) (e.g., this is the case when \( A \) is torsion-free). Then, \(|X + Y| \geq |X| + |Y| - 1\).

Remarkably, [12] is focused on cancellative semigroups (there simply called
semigroups), and it is precisely in this framework that Kemperman establishes a
series of results, mostly related to the number of different representations of an
element in a sumset, that eventually lead to Theorem 13. In fact, the result is
generalized by the following corollary, whose proof is straightforward by Corollary
12 (we may omit the details).

**Corollary 14.** Given \( n \in \mathbb{N}^+ \), let \( X_1, \ldots, X_n \) be subsets of \( A \) such that \( X_1^+ + \cdots + X_n^+ \neq \emptyset \). Define \( \kappa := |X_1| + \cdots + |X_n| + 1 - n \) and assume \( \text{ord}(x) \geq \kappa \) for every \( x \in A \setminus \{0\} \). If \( A \) is cancellative, then \(|X_1 + \cdots + X_n| \geq \kappa\).

For the rest, some earlier contributions by other authors to the additive theory
of semigroups are due to J. Cilleruelo, Y. O. Hamidoune and O. Serra, who proved
in [4] a Cauchy-Davenport theorem for acyclic monoids (these are termed acyclic
semigroups in [4], but they are, in fact, monoids in our terminology), and it could
be quite interesting to find a common pattern among their result and the ones in
this paper. The question was already raised in [19], where it was also observed that
one of the main problems with this idea is actually represented by the fact that
acyclic monoids in [4] are not cancellative, which has served as a basic motivation
for making the results of Section 5, as the readers will see, completely independent
from the assumption of cancellativity.

**Remark 15.** Incidentally, we want to point out here that condition M1 in the
definition of an acyclic semigroup \( M = (M, \cdot) \) in [4, p. 100], namely “\( y \cdot x = x \) implies \( y = 1 \) for every \( x \in M \)” (we write 1 for the identity of \( M \)), needs to be
fixed in some way, since otherwise taking \( M \) to be the unitization of a non-empty
left-zero semigroup \( (N, \cdot) \), where \( x \cdot y := x \) for all \( x, y \in N \), yields a counterexample
to the statement that “If \( M \) is an acyclic semigroup and \( 1 \in S \), where \( S \) is a finite
subset of \( M \), “then the only finite directed cycles in the Cayley graph \( \text{Cay}(M, S) \)
are the loops”: This is first mentioned in the second paragraph of Section 2.
in the cited paper (p. 100), and represents a fundamental ingredient for most of its results. The reader may think that the problem is just a typographical error, and suggest to substitute condition M1 with its “dual”, that is “$x \cdot y = x$ implies $y = 1$, for every $x \in M$.” In fact, this fixes the issue with the Cayley graphs of $M$, but Lemma 1 (p. 100), which is equally essential in many proofs, breaks down completely (for a concrete counterexample, consider the monoid obtained by reversing the multiplication of $M$ in the previous counterexample). Luckily enough, there are at least two possible simple workarounds: The first is to assume that $M$ is commutative, the second to turn condition M1 into a “self-dual” axiom, i.e. to replace it with “$x \cdot y = x$ or $y \cdot x = x$ implies $y = 1$, for every $x \in M$.”

Theorem 10 and Corollary 12 are proved in Section 6, and it is perhaps worth repeating once more that their proofs is at least as elementary as many others in the additive theory of groups. (In fact, the arguments are loosely inspired by ideas originally used by Kemperman in [12] in relation to Theorem 13.) In particular, we do not rely on either the Feit-Thompson theorem or the isoperimetric method.

Remark 16. A couple of things are worth mentioning before proceeding. While every commutative cancellative semigroup embeds as a subsemigroup into a group (as it follows from the standard construction of the group of fractions of a commutative monoid; see [1, Section I.2.4]), nothing similar is true in the non-commutative setting. This is related to a well-known question in the theory of semigroups, first answered by A. I. Mal’cev in [13], and served as an original “precondition” for [19] and the present paper, in that it shows that the study of sumsets in cancellative semigroups cannot be systematically reduced, in the absence of commutativity, to the case of groups (at the very least, not in any obvious way).

On the other hand, it is true that every semigroup can be embedded into a monoid (through the unitization process explained at the beginning of the section), to the effect that, for the specific purposes of the manuscript, we could have assumed in most of our statements that the “ambient” is a monoid rather than a semigroup. However, we did differently for the assumption is not really necessary, and it seems more appropriate to develop as much as possible of the material with no regard to the presence of an identity (e.g., because this is better suited for the kind of generalizations outlined in the above).

5. Preparations

Throughout, we collect basic results to be used later in Section 6 to prove Theorem 10 and Corollary 12. Some proofs are quite simple (and thus omitted without further explanation), but we have no standard reference to anything similar in the context of semigroups, so we include them here for completeness. Notice
that, even though Theorem 10, say, refers to cancellative semigroups, most of the results presented in the section do not depend on the cancellativity of the “ambient”. While this makes no serious difference from the point of view of readability, it seems interesting in itself - and it is the author’s hope that the material can eventually help to find a proof of Conjecture 8 or to further refine it.

**Proposition 17.** Pick \( n \in \mathbb{N}^+ \) and subsets \( X_1, Y_1, \ldots, X_n, Y_n \) of \( A \) such that \( X_i \subseteq Y_i \) for each \( i \). Then, \( \sum_{i=1}^n X_i \subseteq \sum_{i=1}^n Y_i \), and hence \( |\sum_{i=1}^n X_i| \leq |\sum_{i=1}^n Y_i| \).

In spite of being so trivial, the next estimate is often useful (cf. [18, Lemma 2.1, p. 54]).

**Proposition 18.** Given \( n \in \mathbb{N}^+ \) and \( X_1, \ldots, X_n \subseteq A \), it holds \( |\sum_{i=1}^n X_i| \leq \prod_{i=1}^n |X_i| \).

Let \( X, Y \subseteq A \). No matter whether or not \( A \) is cancellative, nothing similar to Proposition 18 applies, in general, to the difference set \( X - Y \), which can be infinite even if both of \( X \) and \( Y \) are not. On another hand, it follows from the same proposition that, in the presence of cancellativity, the cardinality of \( X + Y \) is preserved under translation, namely \( |z + X + Y| = |X + Y + z| = |X + Y| \) for every \( z \in A \). This is an interesting point in common with the case of groups, but a significant difference is that, in the context of semigroups (even when unital), the above invariance property cannot be used, at least in general, to “normalize” either of \( X \) or \( Y \) in such a way as to contain some distinguished element of \( A \). However, we will see in a while that things continue to work fine when \( A \) is a monoid and sets are shifted by units.

**Lemma 19.** Let \( A \) be a monoid, \( X \) a subset of \( A \), and \( z \) a unit of \( A \) with inverse \( \tilde{z} \). Then, \( X - z = X + \tilde{z} \) and \( -z + X = \tilde{z} + X \), and in addition \( | -z + X| = |X - z| = |X| \).

**Proof.** We only prove that \( X - z = X + \tilde{z} \) and \( |X - z| = |X| \), as the other identities are established in a quite similar way. For, notice that \( w \in X - z \) if and only if there exists \( x \in X \) such that \( w + z = x \), which in turn is equivalent to \( x + \tilde{z} = (w + z) + \tilde{z} = w \), namely \( w \in X + \tilde{z} \). This gives one half of the claim, and in order to conclude it is now sufficient to observe that the function \( A \rightarrow A : \xi \mapsto \xi + \tilde{z} \) is bijective. \( \blacksquare \)

**Remark 20.** There is a subtleness in Definition 3 which we have “overlooked” so far, but should be noticed before proceeding further. For, suppose that \( A \) is a monoid and pick \( x, y \in A \). In principle, \( x - y \) and \( -y + x \) are not elements of \( A \): In fact, they are (difference) sets, and no other meaningful interpretation is possible a priori. However, if \( y \) is a unit of \( A \) and \( \tilde{y} \) is the inverse of \( y \), then \( x - y = \{ x + \tilde{y} \} \) and \( -y + x = \{ \tilde{y} + x \} \) by Lemma 19, and we are allowed to identify \( x - y \) with \( x + \tilde{y} \) and \( -y + x \) with \( \tilde{y} + x \), which will turn to be useful in various places.
Remark 21. Considering that units are cancellable elements, Lemma 19 can be (partially) generalized as follows: If $X \subseteq A$ and $z \in A$ is cancellable, then $|z + X| = |X + z| = |X|$ (this is straightforward, because both of the functions $A \to A : x \mapsto x + z$ and $A \to A : x \mapsto z + x$ are bijective).

Lemma 22. Suppose $A$ is a monoid. Pick $n \in \mathbb{N}^+$ and $z_0, \ldots, z_n \in A^\times$, and let $X_1, \ldots, X_n$ be subsets of $A$. Then, $|\sum_{i=1}^n X_i| = |\sum_{i=1}^n (z_{i-1} + X_i - z_i)|$.

Proof. Let $\tilde{z}_i$ be, for $i = 0, \ldots, n$, the inverse of $z_i$ in $A$, and set $X := \sum_{i=1}^n X_i$ for economy of notation. Lemma 19 gives $\sum_{i=1}^n (z_{i-1} + X_i - z_i) = \sum_{i=1}^n (z_{i-1} + X_i + \tilde{z}_i) = z_0 + X + z_n$, and then another application of the same proposition yields $|X| = |z_0 + X + z_n|$.

We let $A^{op}$ be the dual (or opposite) semigroup of $A$, namely the pair $(A, +_{op})$ where $+_{op}$ is the binary operation $A \times A \to A : (x, y) \mapsto y + x$; cf. [1, Section I.1.1, Definition 2].

Proposition 23. Given $n \in \mathbb{N}^+$, let $X$ and $X_1, \ldots, X_n$ be subsets of $A$, and pick $z \in A$. Then, $X_1 + \cdots + X_n = X_n +_{op} \cdots +_{op} X_1$ and $\operatorname{ord}(z) = \operatorname{ord}_{A^{op}}(z)$.

Here and later, to express that a statement follows as a more or less direct consequence of Proposition 23, we will simply say that it is true “by duality”. This is useful for it often allows, for instance, to simplify a proof to the extent of cutting by half its length, as in the following lemma, which generalizes an analogous, well-known property of groups:

Lemma 24. Pick $x, y \in A$ and suppose that at least one of $x$ or $y$ is cancellable. Then, $\operatorname{ord}(x + y) = \operatorname{ord}(y + x)$.

Proof. By duality, there is no loss of generality in assuming, as we do, that $y$ is cancellable. Further, it suffices to prove that $\operatorname{ord}(x + y) \leq \operatorname{ord}(y + x)$, since then the desired conclusion will follow from the fact that, on the one hand,

$$\operatorname{ord}(y + x) = \operatorname{ord}(x +_{op} y) = \operatorname{ord}_{A^{op}}(x +_{op} y) \leq \operatorname{ord}_{A^{op}}(y +_{op} x) = \operatorname{ord}(y +_{op} x) = \operatorname{ord}(x + y),$$

and on the other hand, $y$ is cancellable in $A$ if and only if it is cancellable in $A^{op}$. Now, the claimed inequality is obvious if $\operatorname{ord}(y + x)$ is infinite. Otherwise, there exist $n, k \in \mathbb{N}^+$ with $k < n$ such that $\operatorname{ord}(x + y) = n$ and

$$\underbrace{(y + x) + \cdots + (y + x)}_{k \text{ times}} = \underbrace{(x + y) + \cdots + (x + y)}_{n+1 \text{ times}}.$$

So, by adding $y$ to the right of both sides and using associativity to rearrange how the terms in the resulting expression are grouped we get

$$\underbrace{(x + y) + \cdots + (x + y)}_{k \text{ times}} = \underbrace{(x + y) + \cdots + (x + y)}_{n+1 \text{ times}}.$$
Since $y$ is cancellable, it then follows that 

\[(x + y) + \cdots + (x + y) = (x + y) + \cdots + (x + y),\]

which ultimately gives that $\text{ord}(x + y) \leq n = \text{ord}(y + x)$. Thus, the proof is complete. 

**Proposition 25.** Let $X$ be a subset of $A$. Then, $\gamma(X) = \gamma_{A^\text{op}}(X)$. 

**Proof.** Let $i$ be the map $A^\times \to A^\times$ sending a unit of $A$ to its inverse, and define $i_{\text{op}}$ in a similar way by replacing $A$ with its dual. An element $x_0 \in A$ is a unit in $A$ if and only if it is also a unit in $A_{\text{op}}$, and $\bar{x}_0 \in A$ is the inverse of $x_0$ in $A$ if and only if it is also the inverse of $x_0$ in $A_{\text{op}}$. Thus, $A^\times = (A_{\text{op}})^\times$, $X^\times := X \cap A^\times = X \cap (A_{\text{op}})^\times$ and $i = i_{\text{op}}$, to the effect that

\[\gamma(X) = \sup_{x_0 \in X^\times} \inf_{x \neq x_0 \in X} \text{ord}(x + i(x_0)) \quad \text{and} \quad \gamma_{A^\text{op}}(X) = \sup_{x_0 \in X^\times} \inf_{x \neq x_0 \in X} \text{ord}_{A^\text{op}}(i(x_0) + x),\]

where we use Lemma 19 to express the Cauchy-Davenport constant of $X$ relative to either of $A$ and $A_{\text{op}}$ only in the terms of $i$. But any unit in a monoid is cancellable, to the effect that for all $x_0 \in X^\times$ and $x \in A$ we get, again by Proposition 23 and in the light of Lemma 24, that

\[\text{ord}_{A^\text{op}}(i(x_0) + x) = |i(x_0) + x|_{A^\text{op}} = |i(x_0) + x| = \text{ord}(i(x_0) + x) = \text{ord}(x + i(x_0)).\]

And this, together with the above, is enough to conclude. 

For an integer $n \geq 1$, we define a *Cauchy-Davenport transform* of $A$ of order $n$, here simply called an $n$-transform (of $A$) if no confusion can arise, to be any tuple $T = (T_1, \ldots, T_n)$ of functions on the powerset of $A$, herein denoted by $\mathcal{P}(A)$, with the property that

\[
0 < \left| \sum_{i=1}^{n} T_i(X_i) \right| \leq \left| \sum_{i=1}^{n} X_i \right| \quad \text{and} \quad \sum_{i=1}^{n} |X_i| \leq \sum_{i=1}^{n} |T_i(X_i)|,
\]

as well as

\[\gamma(X_1 + \cdots + X_n) \leq \gamma(T_1(X_1) + \cdots + T_n(X_n))\]

for all non-empty $X_1, \ldots, X_n \in \mathcal{P}(A)$. Furthermore, we say that $T$ is an *invariant* $n$-transform if the weak inequalities in (1) and (2) are, in fact, equalities. An interesting case in this respect is when each of the $T_i$ is a *unital shift*, namely a function of the form $\mathcal{P}(A) \to \mathcal{P}(A): X \to z_l + X + z_r$ for which $z_l$ and $z_r$ are units of $A$, since then $T$ is invariant. This follows from the next results, where we use, among the other things, that if $A$ is a monoid and $z \in A^\times$ then $(X + Y) - z = X + (Y - z)$ and $-(z + X) + Y = -z + (X + Y)$ for all $X, Y \subseteq A$ (as it follows from Lemma 19), with the result that we can drop the parentheses without worries and write, e.g., $X + Y - z$ for $(X + Y) - z$ and $-z + X + Y$ in place of $(-z + X) + Y$. 


Lemma 26. If \( n \in \mathbb{N}^+ \) and \( X_1, \ldots, X_n \subseteq A \), then \( X_1^\times + \cdots + X_n^\times \subseteq (X_1 + \cdots + X_n)^\times \), and the inclusion is, in fact, an equality if \( A \) is cancellative.

Proof. The assertion is obvious for \( n = 1 \), so it is clearly enough to prove it for \( n = 2 \), since then the conclusion follows by induction. So, let \( X \) and \( Y \) be fixed subsets of \( A \).

Suppose first that \( z \in X^\times + Y^\times \) (which means, in particular, that \( A \) is a monoid), i.e. there exist \( x \in X^\times \) and \( y \in Y^\times \) such that \( z = x + y \). If \( \tilde{x} \) is inverse of \( x \) (in \( A \)) and \( \tilde{y} \) is the inverse of \( y \), then it is immediate to see that \( \tilde{y} + \tilde{x} \) is the inverse of \( x + y \), and hence \( x + y \in (X + Y)^\times \). It follows that \( X^\times + Y^\times \subseteq (X + Y)^\times \).

As for the other inclusion, assume now that \( A \) is cancellative and pick \( z \in (X + Y)^\times \). We have to show that \( z \in X^\times + Y^\times \). For, let \( \tilde{z} \) be the inverse of \( z \) in \( A \), and pick \( x \in X \) and \( y \in Y \) such that \( z = x + y \). We define \( \tilde{x} := y + \tilde{z} \) and \( \tilde{y} := \tilde{z} + x \). It is straightforward to check that \( x + \tilde{x} = (x + y) + \tilde{z} = 0 \) and \( \tilde{y} + y = \tilde{z} + (x + y) = 0 \). Also, \( (\tilde{x} + x) + y = y + \tilde{z} + (x + y) = y \) and \( x + (y + \tilde{y}) = (x + y) + \tilde{z} + x = x \), from which we get, by cancellativity, \( \tilde{x} + x = y + \tilde{y} = 0 \). This implies that \( z \) belongs to \( X^\times + Y^\times \), and so we are done.

Remark 27. As a byproduct of the proof of Lemma 26, we get that: If \( x_1, \ldots, x_n \in A^\times \) \((n \in \mathbb{N}^+)\) and \( \tilde{x}_i \) is the inverse of \( x_i \), then \( \tilde{x}_n + \cdots + \tilde{x}_1 \) is the inverse of \( x_1 + \cdots + x_n \). This is a standard fact about groups, which goes through verbatim for monoids; see [1, Section I.2.4, Corollary 1]. We mention it here because it is used below.

Lemma 28. Let \( A \) be a monoid, and pick \( z \in A^\times \) and \( X \subseteq A \). Then, \( \gamma(X) \leq \gamma(X + z) \).

Proof. By Lemma 26, we have \( X^\times + z \subseteq (X + z)^\times \), to the effect that

\[
\gamma(X + z) = \sup_{w_0 \in (X + z)^\times} \inf_{w_0 \neq w \in X + z} \inf_{w_0 \in X^\times + z} \inf_{w_0 \neq w \in X + z} \text{ord}(w - w_0) \geq \sup_{w_0 \in X^\times + z} \inf_{w_0 \neq w \in X + z} \text{ord}(w - w_0).
\]

But \( w \in X + z \) if and only if there exists \( x \in X \) such that \( w = x + z \), and in particular \( w \in X^\times + z \) if and only if \( x \in X^\times \). Also, given \( x_0 \in X^\times \) and \( x \in X \), it holds \( x + z = x_0 + z \) if and only if \( x = x_0 \). As a consequence, it is immediate from (3) and Remark 27 that

\[
\gamma(X + z) \geq \sup_{x_0 \in X \times x_0 + z \neq w \in X + z} \inf_{x_0 \in X \times x_0 \neq w \in X} \text{ord}(w + \tilde{z} - x_0) = \sup_{x_0 \in X \times x_0 \neq w \in X} \inf_{x_0 \in X \times x_0 + z \neq w \in X + z} \text{ord}(x - x_0) = \gamma(X),
\]

where \( \tilde{z} \) is the inverse of \( z \) in \( A \). Thus, our proof is complete.

Proof of Proposition 6. Let \( \tilde{z} \) denote the inverse of \( z \) in \( A \). Lemma 28 yields \( \gamma(X) \leq \gamma(X + z) \leq \gamma((X + z) + \tilde{z}) \), whence \( \gamma(X) = \gamma(X + z) \). Then, we observe that, on the one hand, Proposition 25, together with the fact that \( A \) is the dual of \( A^{op} \), implies \( \gamma(X) = \gamma_{A^{op}}(X) \) and \( \gamma_{A^{op}}(X +_{op} z) = \gamma(X +_{op} z) = \gamma(z + X) \), and on the other hand, it follows from the above that \( \gamma_{A^{op}}(X) = \gamma_{A^{op}}(X +_{op} z) \). This gives \( \gamma(X) = \gamma(z + X) \) and completes our proof.
Corollary 29. Let \( \mathbb{A} \) be a monoid, and for a fixed integer \( n \geq 1 \) pick \( X_1, \ldots, X_n \subseteq A \) and \( z_0, \ldots, z_n \in A^X \). For each \( i = 1, \ldots, n \) denote by \( T_i \) the map \( \mathcal{P}(A) \to \mathcal{P}(A) : X \to z_{i-1} + X - z_i \). Then, \((T_1, \ldots, T_n)\) is an invariant \( n \)-transform and 
\[ \gamma(T_i(X_i)) = \gamma(X_i) \] for each \( i \).

Proof. By construction, it holds \( \sum_{i=1}^n T_i(X_i) = z_0 + (X_1 + \cdots + X_n) + z_n \). Then, we get by Lemma 22 that \( |X_1| = |T_1(X_1)|, \ldots, |X_n| = |T_n(X_n)| \) and \( |\sum_{i=1}^n X_i| = |\sum_{i=1}^n T_i(X_i)| \), while Proposition 6 implies \( \gamma(X_1) = \gamma(T_1(X_1)), \ldots, \gamma(X_n) = \gamma(T_n(X_n)) \) and \( \gamma(X_1 + \cdots + X_n) = \gamma(T_1(X_1) + \cdots + T_n(X_n)) \). By putting all together, the claim follows immediately. \[ \blacksquare \]

Corollary 30. Assume that \( \mathbb{A} \) is a monoid. Fix an integer \( n \geq 1 \) and let \( X_1, \ldots, X_n \) be subsets of \( A \) such that \( X_1^\times + \cdots + X_n^\times \neq \emptyset \). There then exists an invariant transform \( T = (T_1, \ldots, T_n) \) such that \( 0 \in \bigcap_{i=1}^n T_i(X_i) \). Moreover, if \( \mathbb{A} \) is cancellative and \( X_1^\times + \cdots + X_n^\times \) is finite, then \( T \) can be chosen in such a way that
\[ \gamma(T_1(X_1) + \cdots + T_n(X_n)) = \min_{0 \neq w \in T_1(X_1) + \cdots + T_n(X_n)} \text{ord}(w). \]

Proof. For each \( i = 1, \ldots, n \) pick \( x_i \in X_i^\times \), using that \( X_1^\times + \cdots + X_n^\times \) is non-empty (and hence \( X_i^\times \neq \emptyset \)), and let \( T_i \) be the function \( \mathcal{P}(A) \to \mathcal{P}(A) : X \mapsto z_{i-1} + X - z_i \), where \( z_0 := 0 \) and \( z_i := x_1 + \cdots + x_i = z_{i-1} + x_i \). Then, it is clear that \( 0 \in \bigcap_{i=1}^n T_i(X_i) \), while Corollary 29 entails that \((T_1, \ldots, T_n)\) is an invariant \( n \)-transform. This proves the first part of the claim.

As for the rest, assume in what follows that \( \mathbb{A} \) is cancellative and \( X_1^\times + \cdots + X_n^\times \) is finite. Then, letting \( Z := X_1 + \cdots + X_n \) for brevity yields, by Proposition 26, that \( X_1^\times + \cdots + X_n^\times = Z^\times \), so there exist \( \bar{x}_1 \in X_1, \ldots, \bar{x}_n \in X_n \) such that
\[ \gamma(Z) = \min_{\bar{z} \neq z \in Z} \text{ord}(z - \bar{z}), \]
where \( \bar{z} := \bar{x}_1 + \cdots + \bar{x}_n \) and we are using that a supremum taken over a non-empty finite set is, in fact, a maximum. It follows from the above that we can build an invariant \( n \)-transform \( \tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_n) \) such that \( 0 \in \bigcap_{i=1}^n \tilde{T}_i(X_i) \) and \( \sum_{i=1}^n \tilde{T}_i(X_i) = Z - \bar{z} \), to the effect that
\[ \gamma(Z) = \gamma(Z - \bar{z}) \geq \min_{0 \neq w \in Z - \bar{z}} \text{ord}(w) = \min_{\bar{z} \neq z \in Z} \text{ord}(z - \bar{z}), \]
by the invariance of \( \tilde{T} \) and the fact that, on the one hand, \( 0 \in Z - \bar{z} \) and, on the other hand, \( w \in Z - \bar{z} \) if and only if \( w = z - \bar{z} \) for some \( z \in Z \). Together with (5), this ultimately leads to \( \gamma(Z - \bar{z}) = \min_{z \neq z \in Z} \text{ord}(w) \), and thus to (4). \[ \blacksquare \]

We conclude the section with a proof of Lemma 4:

Proof of Lemma 4. By duality, it is enough to prove that \( \gamma(Y) \geq \gamma(X + Y) \geq p(A) \), since all the rest is more or less trivial from our definitions. For, pick
\[ z_0 \in (X + Y)^\times \] using that, on the one hand, \((X + Y)^\times = X^\times + Y^\times\) by Proposition 26 and the cancellativity of \(A\), and on the other hand, \(X^\times + Y^\times\) is non-empty by the standing assumptions. There then exist \(x_0 \in X^\times\) and \(y_0 \in Y^\times\) such that \(z_0 = x_0 + y_0\), and it is immediate from Remark 27 that, for all \(y \in A\),

\[ \langle x_0 + y - z_0 \rangle = x_0 + \langle y - y_0 \rangle - x_0, \]

which, together with Lemma 22, gives \(\text{ord}(y - y_0) = \text{ord}(x_0 + y - z_0)\). Thus, considering that, for \(y \in A\), it holds \(x_0 + y = z_0\) if and only if \(y = y_0\), it follows that

\[
\inf_{y_0 \neq y \in Y} \text{ord}(y - y_0) = \inf_{y_0 \neq y \in Y} \text{ord}(x_0 + y - z_0) \geq \inf_{z_0 \neq z \in X + Y} \text{ord}(z - z_0) \geq p(A),
\]

and this in turn implies the claim by taking the supremum over the units of \(X + Y\).

6. The proof of the main theorem

At long last, we are ready to prove the central contributions of the paper. We start with the following:

**Proof of Theorem 10.** The claim is obvious if \((X + Y)^\times = \emptyset\), so suppose for the remainder of the proof that \((X + Y)^\times\) is non-empty (which, among the other things, yields that \(A\) is a monoid), and set \(\kappa := |X + Y|\), while noticing that, by Lemma 26, both of \(X^\times\) and \(Y^\times\) are non-empty, and so, by Proposition 17 and Lemma 19, we have

\[
\kappa \geq \max(|X|, |Y|) \geq \min(|X|, |Y|) \geq 1.
\]

The statement is still trivial if \(\kappa = \infty\) (respectively, \(\kappa = 1\)), since then either of \(X\) or \(Y\) is infinite (respectively, both of \(X\) and \(Y\) are singletons), and hence \(|X + Y| = |X| + |Y| - 1\) by (6). Thus, we assume in what follows that \(\kappa\) is a positive integer and argue by strong induction on \(\kappa\). To wit, we suppose by contradiction that \(\kappa < \min(\gamma(X + Y), |X| + |Y| - 1)\). Based on the above, this ultimately means that

\[
2 \leq \kappa < \infty, \quad 2 \leq |X|, |Y| < \infty, \quad \kappa < \gamma(X + Y) \quad \text{and} \quad \kappa \leq |X| + |Y| - 2.
\]

More specifically, there is no loss of generality in assuming, as we do, that \((X, Y)\) is a “minimax counterexample” to the claim, by which we mean that if \((\tilde{X}, \tilde{Y})\) is another pair of subsets of \(A\) with \(\tilde{X}^\times + \tilde{Y}^\times \neq \emptyset\) and \(|\tilde{X} + \tilde{Y}| < \min(\gamma(\tilde{X} + \tilde{Y}), |\tilde{X}| + |\tilde{Y}| - 1)\) then either \(\kappa < |\tilde{X} + \tilde{Y}|\) or \(\kappa = |\tilde{X} + \tilde{Y}|\) and at least one of the following conditions holds:

\[
\begin{align*}
(8) \quad & (i) \ |\tilde{X}| + |\tilde{Y}| < |X| + |Y|; \quad (ii) \ |\tilde{X}| + |\tilde{Y}| = |X| + |Y| \quad \text{and} \quad |\tilde{X}| \leq |X|. 
\end{align*}
\]
This makes sense because if $\bar{X}, \bar{Y} \subseteq A$, $\bar{X}^+ + \bar{Y}^+ \neq \emptyset$ and $\kappa = |\bar{X} + \bar{Y}|$ then $\bar{X}^+$ and $\bar{Y}^+$ are non-empty and we get, as before with (6), that

$$|\bar{X}| \leq |\bar{X}| + |\bar{Y}| \leq 2 \cdot \max(|\bar{X}|, |\bar{Y}|) \leq 2 \cdot |\bar{X} + \bar{Y}| = 2\kappa < \infty.$$  

Finally, in the light of Corollary 30, we may also assume without restriction of generality, up to an invariant 2-transform, that

(9) \[ 0 \in X \cap Y \text{ and } \gamma(X + Y) = \min_{0 \neq z \in X + Y} \text{ord}(z). \]

Then, both of $X$ and $Y$ are subsets of $X + Y$, and we get by the inclusion-exclusion principle that $\kappa \geq |X| + |Y| - |X \cap Y|$, which gives, together with (7), that $X \cap Y$ has at least one element different from 0, i.e., $|X \cap Y| \geq 2$. On these premises, we prove the following intermediate claim (from here on, we set $Z := X \cap Y$ for notational convenience):

**Claim.** There exists $n \in \mathbb{N}^+$ such that $X + nZ + Y \nsubseteq X + Y$, but $X + kZ + Y \subseteq X + Y$ for each $k = 0, \ldots, n - 1$, with the convention that $0Z := \{0\}$.

**Proof of the claim.** Assume by contradiction that $X + nZ + Y \subseteq X + Y$ for all $n \in \mathbb{N}^+$. Then, we get from $\langle Z \rangle = \bigcup_{n=1}^\infty nZ$ that $X + \langle Z \rangle + Y \subseteq X + Y$, which implies by (9) that $\langle Z \rangle = 0 + \langle Z \rangle + 0 \subseteq X + Y$. Then, using that $|Z| \geq 2$ to guarantee that $\{0\} \nsubseteq Z \subseteq X + Y$, it follows from Proposition 17 and the same equation (9) that

$$\kappa \geq |\langle Z \rangle| \geq \max_{0 \neq z \in Z} \text{ord}(z) \geq \min_{0 \neq z \in Z} \text{ord}(z) \geq \min_{0 \neq z \in X + Y} \text{ord}(z) = \gamma(X + Y).$$

This is, however, absurd, for it is in contradiction to (7), and we are done. \qed

So, let $n$ be as in the above claim and fix, for the remainder of the proof, an element $\bar{z} \in nZ$ such that $X + \bar{z} + Y \nsubseteq X + Y$, which exists by construction since otherwise we would have $X + nZ + Y \subseteq X + Y$, that is a contradiction. Consequently, observe that

(10) \[ (X + \bar{z}) \cup (\bar{z} + Y) \subseteq X + Y. \]

In fact, $\bar{z}$ being an element of $nZ$ entails that there exist $z_1, \ldots, z_n \in Z$ such that $\bar{z} = z_1 + \cdots + z_n$, whence we get that both of $X + \bar{z}$ and $\bar{z} + Y$ are contained in $X + (n - 1)Z + Y$. But $X + (n - 1)Z + Y$ is, again by construction, a subset of $X + Y$, so (10) is proved. With this in hand, let us now introduce the sets

(11) \[ X_0 := \{x \in X : x + \bar{z} + Y \nsubseteq X + Y\} \text{ and } Y_0 := \{y \in Y : X + \bar{z} + y \nsubseteq X + Y\}. \]

It is then clear that $X$ (respectively, $Y$) is disjoint from $X_0 + \bar{z}$ (respectively, from $\bar{z} + Y_0$). In addition, since $X + \bar{z} + Y \nsubseteq X + Y$, it is also immediate that $X_0$ and $Y_0$ are both non-empty. Finally, it follows from (10) that 0 is not an element of either $X_0$ or $Y_0$. To sum it up,

(12) \[ X_0 \neq \emptyset \neq Y_0, \ 0 \notin X_0 \cup Y_0 \text{ and } (X_0 + \bar{z}) \cap X = (\bar{z} + Y_0) \cap Y = \emptyset. \]
Now, let \( n_X := |X_0| \) and \( n_Y := |Y_0| \). By Remark 21 and the cancellativity of \( A \), we have
\[
(13) \quad |X_0 + \vec{z}| = |X_0| = n_X \quad \text{and} \quad |\vec{z} + Y_0| = |Y_0| = n_Y,
\]
which naturally leads to distinguish between the following two cases:

**Case 1**: \( n_X \geq n_Y \). We form \( \bar{X} \) as the union of \( X \) and \( X_0 + \vec{z} \) and \( Y \) as the relative complement of \( Y_0 \) in \( Y \). First, note that \( 0 \in \bar{X}^\times \cap Y^\times \) by \( (12) \).

Secondly, pick \( x \in \bar{X} \) and \( y \in Y \) and set \( z := x + y \). If \( x \in X \), then obviously \( z \in X + Y \); otherwise, by the construction of \( X \) and \( Y \), we get \( x \in X_0 + \vec{z} \subseteq X + \vec{z} \) and \( y \notin Y_0 \), to the effect that \( x + y \notin X + Y \). Therefore, we see that \( X + Y \) is a non-empty subset of \( X + Y \) with \( 0 \notin \bar{X} + \bar{Y} \), so on the one hand \( |\bar{X} + \bar{Y}| \leq \kappa \) and on the other hand we have by \( (9) \) that
\[
\gamma(X + Y) \leq \inf_{0 \notin z \in X + Y} \text{ord}(z) \leq \gamma(\bar{X} + \bar{Y}).
\]

Furthermore, \( (12) \) and \( (13) \) give that \( |\bar{X}| = |X| + |X_0 + \vec{z}| = |X| + n_X > |X| \) and \( |\bar{Y}| = |Y| - |Y_0| = |Y| - n_Y \), so \( |\bar{X}| + |\bar{Y}| = |X| + |Y| + n_X - n_Y \geq |X| + |Y| \).

**Case 2**: \( n_X < n_Y \). We set \( \tilde{X} := X \setminus X_0 \) and \( Y := (\vec{z} + Y_0) \cup Y \). Then, by repeating (except for obvious modifications) the same reasoning as in the previous case, we get again that \( 0 \in \tilde{X}^\times \cap Y^\times \) and \( \tilde{X} + \tilde{Y} \subseteq X + Y \), with the result that \( |\tilde{X} + \tilde{Y}| \leq \kappa \) and \( \gamma(X + Y) \leq \gamma(\tilde{X} + \tilde{Y}) \). In addition, it follows from \( (12) \) and \( (13) \) that \( |\tilde{X}| = |X| - |X_0| = |X| - n_X \) and \( |\tilde{Y}| = |Y| + |\vec{z} + Y_0| = |Y| + n_Y \); whence \( |\tilde{X}| + |\tilde{Y}| = |X| + |Y| + n_Y - n_X > |X| + |Y| \).

So in both cases, we end up with an absurd, for we find subsets \( \tilde{X} \) and \( \tilde{Y} \) of \( A \) that contradict the “minimaximality” of \( (X, Y) \) as it is expressed by \( (8) \).

Remarkably, “many” pieces of the above proof of Theorem 10 do not critically depend on the cancellativity of the ambient, while others can be adapted to the case where \( \gamma(X + Y) \) is replaced by \( \gamma(X, Y) \), which is one of our strongest motivations for believing that Conjecture 8 should be ultimately true.

**Proof of Corollary 12.** The claim is obvious if \( n = 1 \). Thus, assume in what follows that \( n \) is an integer \( \geq 2 \) and the assertion is true for all sumsets of the form \( Y_1 + \cdots + Y_{n-1} \) with \( Y_1^\times + \cdots + Y_{n-1}^\times \neq \emptyset \). Based on these premises, we get by Theorem 10 that
\[
|X_1 + \cdots + X_n| \geq \min(\gamma(X_1 + \cdots + X_n), |X_1 + \cdots + X_{n-1}| + |X_n| - 1),
\]
which in turn implies, by Lemma 4, that
\[
(14) \quad |X_1 + \cdots + X_n| \geq \min(p(A), |X_1 + \cdots + X_{n-1}| + |X_n| - 1).
\]
But we know from Proposition 26 that $X_1^\times + \cdots + X_{n-1}^\times \neq \emptyset$, so the inductive hypothesis gives

$$|X_1 + \cdots + X_{n-1}| \geq \min(p(A), |X_1| + \cdots + |X_{n-1}| + 2 - n),$$

which, together with (14), yields the desired conclusion by induction. ■

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