Multiplicty estimate for solutions of extended Ramanujan’s system.

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September 2, 2011

Abstract

We establish a new multiplicity lemma for solutions of a differential system extending Ramanujan’s classical differential relations. This result can be useful in the study of arithmetic properties of values of Riemann zeta function at odd positive integers (Nesterenko, 2011).

1 Introduction

In what follows we denote by \( \sigma_k(n) \), \( k \in \mathbb{Z} \), \( n \in \mathbb{N} \) the sum of \( k \)th powers of divisors of \( n \):

\[
\sigma_k(n) := \sum_{d \mid n} d^k.
\]

In this paper we consider the following sets of functions. First of all, the Eisenstein series

\[
E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)z^n, \quad k \in \mathbb{N},
\]

where \( B_{2k} \) are Bernoulli numbers. Also we consider

\[
g_{u,v}(z) := \sum_{n=1}^{\infty} n^u \sigma_{-v}(n)z^n, \quad 0 \leq u < v, \quad u, v \in \mathbb{N}.
\]

It is well-known that functions \( E_2, E_4 \) and \( E_6 \) are algebraically independent over \( \mathbb{C}(z) \) and all the other functions \( E_{2k}, k \geq 4 \) can be expressed in terms of \( E_4 \) and \( E_6 \) (see for instance [6]). More precisely, for all \( k \geq 4 \) there exists a polynomial \( A_k \in \mathbb{C}[X,Y] \) such that

\[
E_{2k}(z) = A_k(E_4(z), E_6(z)).
\]

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These polynomials $A_k(X,Y)$, $k \geq 4$ contain only monomials $M$ of bi-degrees $(\deg_X M, \deg_Y M)$ satisfying $2\deg_X M + 3\deg_Y M = k$.

In 2010 P.Kozlov proved (see [2], page 2) that for any fixed $m \in \mathbb{N}$ all the functions

$$E_2(z), E_4(z), E_6(z), g_{u,v}(z), \quad 0 \leq u < v \leq m$$

are algebraically independent over $\mathbb{C}(z)$.

The functions (2) satisfy the following system of differential equations [2]. Denote $\delta := \frac{d}{dz}$. Then

$$\delta E_2 = \frac{1}{12} (E_2^2 - E_4), \delta E_4 = \frac{1}{3} (E_2 E_4 - E_6), \delta E_6 = \frac{1}{2} (E_2 E_6 - E_4^2)$$

and for any odd $v \geq 3$

$$\delta g_{u,v}(z) = g_{u+1,v}(z), \quad 0 \leq u < v - 1,$$

$$\delta g_{v-1,v}(z) = B_{2v+2} \frac{A_{u+1}(E_4(z), E_6(z)) - 1}{2v + 2}.$$ (4)

In the case $v = 1$ one has

$$\delta g_{0,1}(z) = \frac{1}{24} (1 - E_2(z)).$$ (5)

Yu.Nesterenko [2] showed that values of functions $g_{u,v}(z)$ are closely related to the values of the Riemann zeta function $\zeta$ at odd positive integers. In particular, $\zeta(4n+3) \in \mathbb{Q}(E_2(i), g_{0,4n+3}(i))$ [2]. Whereas the system [3], [4], [5] for functions $E_2$, $E_4$, $E_6$, $(g_{u,v})_{0 \leq u < v \leq m}$, $m \in \mathbb{N}$, is quite a simple extension of the system [3], and in the case of the system [3] Nesterenko [4] established an optimal algebraic independence result for its solutions [1], one may hope that this approach will lead to some results concerning algebraic independence of values of $\zeta$ at positive integral odd points. On this way, an important stage is a multiplicity lemma for the functions in question.

In this paper we adopt the method from [1] and [3] Chapter 10 to establish (for any fixed odd $m \geq 3$) a multiplicity lemma for the whole set of functions $E_2$, $E_4$, $E_6$, $(g_{u,v})$, $0 \leq u < v \leq m$, see Theorem 2.1 below.

## 2 Multiplicity Lemma

Let $m \in \mathbb{N}$ be a fixed positive odd integer. We introduce the following notation:

$$R := \mathbb{C}[X_0, X_1, X_2, X_3, Y_{0,1}, Y_{0,3}, Y_{1,3}, Y_{2,3}, \ldots, Y_{m-1,m}].$$
Theorem 2.1 Let $m \geq 1$ be an odd integer. For all non-zero $P \in R$ there exists a constant $C$ depending on $m$ only such that

$$\text{ord}_{z=0} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), g_{0,3}(z), \ldots, g_{0,m}(z), \ldots, g_{m-1,m}(z)) \leq C \left( \text{deg}_{X_0} P + 1 \right) \left( \frac{m-1}{2} \right)^{2+3},$$

(6)

where $\deg_{E_g} P$ denotes the total degree of $P$ in the variables $X_1, X_2, X_3, Y_{0,1}, \ldots, Y_{m-1,m}$, i.e. all the variables appearing in $R$ but $X_0$.

Remark 2.2 The exponent $(\frac{m-1}{2})^2 + 3$ in the r.h.s. of (6) equals the number of functions different than $z$ in the l.h.s. of (6) and also the transcendence degree of $R$ over $\mathbb{C}(z)$. Hence Theorem 2.1 provides multiplicity estimate with the optimal exponent.

In the sequel we denote

$$D_0 := z \frac{d}{dz} + \frac{1}{12} (X_1^2 - X_2) \frac{d}{dX_1} + \frac{1}{3} (X_1 X_2 - X_3) \frac{d}{dX_2} + \frac{1}{2} (X_1 X_3 - X_2^2) \frac{d}{dX_3},$$

$$D_1 := \frac{1}{24} (1 - X_2) \frac{d}{dY_{0,1}},$$

$$D_v := \sum_{k=0}^{v-2} Y_{k+1,v} \frac{d}{dY_{k,v}} + B_{v+1} \frac{A_{v+1}(X_2, X_3) - 1}{2v + 2} \frac{d}{dY_{v-1,v}}, \quad v = 3, 5, \ldots, m$$

and

$$D := D_0 + \sum_{k=0}^{(m-1)/2} D_{2k+1}. \quad (7)$$

The differential operator $D$ satisfies

$$DP(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) = z \frac{d}{dz} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)).$$

(8)

We deduce Theorem 2.1 using Nesterenko’s conditional Multiplicity Lemma (Theorem 1.1, Chapter 10 [3]). This result deals with differential system of the following type:

$$f_i'(z) = \frac{A_i(z, f)}{A_0(z, f)}, \quad i = 1, \ldots, n,$$

(9)

where $A_i(z, X_1, \ldots, X_n) \in \mathbb{C}[z, X_1, \ldots, X_n]$ for $i = 0, ..., n$ (we suppose that $A_0$ is a non-zero polynomial).

Remark 2.3 It is easy to see that system $(3), (4), (5)$ is of the type $(9)$.  

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One associates to the system (9) the differential operator
\[ D_A = A_0(z, X_1, \ldots, X_n) \frac{\partial}{\partial z} + \sum_{i=1}^{n} A_i(z, X_1, \ldots, X_n) \frac{\partial}{\partial X_i}. \] (10)

In our case (i.e. the case of the system (9)) this formula gives exactly the differential operator \( D \) as defined in (7).

**Theorem 2.4 (Nesterenko)** Suppose that functions
\[ \mathcal{f} = (f_1(z), \ldots, f_n(z)) \in \mathbb{C}[[z]]^n \]
are analytic at the point \( z = 0 \) and form a solution of the system (9). If there exists a constant \( K_0 \) such that every \( D \)-stable prime ideal \( \mathcal{P} \subset \mathbb{C}[X'_1, X_1, \ldots, X_n] \), \( \mathcal{P} \neq (0) \), satisfies
\[ \min_{P \in \mathcal{P}} \text{ord}_{z=0} P(z, \mathcal{f}) \leq K_0, \] (11)
then there exists a constant \( K_1 > 0 \) such that for any polynomial \( P \in \mathbb{C}[X'_1, X_1, \ldots, X_n] \), \( P \neq 0 \), the following inequality holds
\[ \text{ord}_{z=0} (P(z, \mathcal{f})) \leq K_1 (\deg X'_1 P + 1)(\deg X_1 P + 1)^n. \] (12)

To apply Theorem 2.4 it is sufficient to prove Proposition 2.5 here below.

**Proposition 2.5** If \( \mathcal{P} \) is a prime ideal of
\[ R = \mathbb{C}[z, X_1, X_2, X_3, Y_{0,1}, \ldots, Y_{m-1,m}] \]
with \( DP \subset \mathcal{P} \), then either \( z \in \mathcal{P} \) or \( \Delta = X_3^2 - X_2^3 \in \mathcal{P} \).

**Proof of Theorem 2.4 modulo Proposition 2.5**. If we have the result announced in Proposition 2.5 then any prime \( D \)-stable ideal \( \mathcal{P} \) contains the polynomial
\[ \Theta := z\Delta = z \left( X_3^2 - X_2^3 \right). \] (13)
In this case we have obviously
\[ \min_{P \in \mathcal{P}} \text{ord}_{z=0} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) \leq \text{ord}_{z=0} \Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)), \]
The quantity \( K_0 := \text{ord}_{z=0} \Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) \) is an absolute constant, in particular independent of \( \mathcal{P} \) (because \( \Theta \) is just a concrete polynomial). Also, the quantity \( K_0 \) is finite, because all the functions \( z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z) \) are algebraically independent over \( \mathbb{C} \) and for this reason no polynomial vanishes on this set (i.e. in particular, \( \Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) \) is a non-zero function, analytic at \( z = 0 \)).

To prove Proposition 2.5 we describe at first principal \( D \)-stable ideals of \( R \).
Lemma 2.6  There exists only two \( D \)-invariant principal prime ideals of \( R \), namely, the ideals generated by \( z \) and \( \Delta \).

**Proof.** Suppose that \( A \in R \) is any irreducible polynomial with the property that \( A|DA \). Thus

\[ DA = AB, \quad B \in R. \tag{14} \]

We readily verify with the definition of \( D \) that \( \deg_Y DA \leq \deg_Y A \) and \( \deg_z DA \leq \deg_z A \), hence (14) implies \( B \in \mathbb{C}[X_1, X_2, X_3] \).

For any \( F \in R \) we define the weight of \( F \) as

\[ \phi(F) := \deg_t F(z, tX_1, t^2X_2, t^3X_3, t^{2m+2}Y). \]

Then \( \phi \) satisfies the following properties:

1. For any \( F \in R \)
   \[ \phi(DF) \leq \phi(F) + 1. \]

2. For any \( F, G \in R \)
   \[ \phi(FG) = \phi(F) + \phi(G). \]

These properties together with (14) imply

\[ \phi(A) + \phi(B) = \phi(DA) \leq \phi(A) + 1, \]

hence \( \phi(B) \leq 1 \). Thus \( B \in \mathbb{C}[X_1] \), \( \deg B \leq 1 \), i.e. \( B = aX_1 + b \), \( a, b \in \mathbb{C}[z] \) and

\[ DA = (aX_1 + b) A. \tag{15} \]

Also \( \deg_z A + \deg_z B = \deg_z DA \leq \deg_z A \), hence \( a, b \in \mathbb{C} \).

Now we consider another weight \( \phi_2 : R \to \mathbb{Z} \). For any \( F \in R \), we denote

\[ \phi_2(F) := \deg_t F(z, tX_1, t^2X_2, t^3X_3, t^{-4}Y_{0,1}, \ldots, t^{-4m}Y_{0,m}, t^{-4m+4}Y_{1,m}, \ldots, t^{-4}Y_{m-1,m}) \]

(i.e. we assign to the variable \( Y_{u,v} \) the weight \( \phi_2(Y_{u,v}) := 4(u-v) \)). Let \( C \) be the sum of monomials of \( A \) with minimal weight \( \phi_2 \). If we compare the sum of the monomials of weight \( \phi_2(C) \) on both sides of (15) and use the definition of \( D \) we obtain

\[ z \frac{d}{dz} C = bC \tag{16} \]

(indeed, for any monomial \( M \) and any differential operator \( D_v, v = 1, 3, 5, \ldots, m \), all the non-zero monomials of \( D_v(M) \) have weight \( \phi_2 \) strictly bigger than \( \phi_2(M) \), also the only term in \( D_0 \) that does not increase \( \phi_2 \) is \( z \frac{d}{dz} \), hence (16)). Comparing the coefficients on the both sides of (16) we conclude \( b = \deg_z C \), in particular \( b \in \mathbb{Z} \).
Substituting $X_1 = E_2(z)$, $X_2 = E_4(z)$, $X_3 = E_6(z)$, $Y_{u,v} = g_{u,v}(z)$, $0 \leq u < v \leq m$ in (13) we obtain

$$(aE_2(z) + b) A(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) = DA(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)).$$

(17)

Let

$$A(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) = cz^M + \text{(terms of order } > M),$$

c \neq 0, be the (first term of the) Taylor series of $A(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z))$. In view of the property we have

$$DA(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) = cMz^M + \text{(terms of order } > M).$$

Using the Taylor series for $E_2$, (11), notably the fact that $E_2(z) = 1 + \text{terms of order } > 1$, we readily deduce from (17)

$$(a + b)cz^M + \text{(terms of order } > M) = cMz^M + \text{(terms of order } > M).$$

Comparing coefficients with $z^M$ in the l.h.s. and in the r.h.s. of (17) and simplifying out $c$ we readily deduce $a + b = M$. We have already established $b \in \mathbb{N}$. Obviously, $M \in \mathbb{N}$ (as it is a degree in a Taylor series). We conclude $a \in \mathbb{Z}$.

So we have established that coefficients $a, b$ involved in (15) are in fact integers.

Note that

$$D(\Delta^{-a}z^{-b}) = (-aX_1 - b) \Delta^{-a}z^{-b}.$$  

(18)

We denote

$$S(z, E_2, E_4, E_6, g_{0,v}, \ldots, g_{v-1,v}) := A(z, E_2, E_4, E_6, g_{0,v}, \ldots, g_{v-1,v}) \Delta^{-a}z^{-b}.$$  

(19)

Applying the differential operator $D$ to the r.h.s. of (19) and using (15), (18) we find out

$$DS = 0.$$  

Using (8) on the latter equality we conclude

$$\frac{d}{dz} S(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) = 0,$$

hence

$$S(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) \in \mathbb{C}.$$

Recall that functions $z, E_2, E_4, E_6, g_{0,v}, \ldots, g_{v-1,v}$ are all algebraically independent over $\mathbb{C}$, see [2] page 2. For this reason we deduce $S[X_0, X_1, X_2, X_3, X_4] \in \mathbb{C}$ and thereby

$$A = \Delta^a z^b.$$
If we suppose that \( A \) is irreducible, we obtain immediately that either \((a, b) = (1, 0)\) or \((a, b) = (0, 1)\).

**Proof of Proposition 2.5** We consider the following nested sequence of rings

\[
\begin{align*}
\mathbb{C}[z, X] & \subset \mathbb{C}[z, X, Y_{0,1}, Y_{1,3}, Y_{2,3}] \subset \mathbb{C}[z, Y_{0,1}, Y_{0,3}, Y_{1,3}, Y_{2,3}] \\
& \quad \subset \cdots \subset \mathbb{C}[z, \ldots, Y_{m-3, m-2}, Y_{m-1, m}] \\
& \quad \subset \cdots \subset \mathbb{C}[z, \ldots, Y_{m, m}, \ldots, Y_{m-1, m}] = R.
\end{align*}
\]

We readily verify with the definition of \( D \) that every term \( R_i \) appearing in the chain (20) satisfies \( DR_i \subset R_i \).

Let \( \mathcal{P} \subset R \) be a prime ideal of \( R \) satisfying \( D\mathcal{P} \subset \mathcal{P} \). If \( \mathcal{P} \cap \mathbb{C}[z, X] \neq \{0\} \), it contains a polynomial \( z\Delta \) as shown in [4][Theorem 1.4]. So everything is proved in this case. We suppose henceforth \( \mathcal{P} \cap \mathbb{C}[z, X] = \{0\} \).

We proceed with recurrence. As we suppose \( \mathcal{P} \neq \{0\} \) and \( \mathcal{P} \cap \mathbb{C}[z, X] = \{0\} \), we find in the chain (20) at some step an extension of rings \( R_i \subset R_{i+1} \) satisfying \( \mathcal{P} \cap R_i = \{0\} \) and \( \mathcal{P} \cap R_{i+1} \neq \{0\} \). In this case the ideal (of the ring \( R_{i+1} \)) \( \mathcal{P} \cap R_{i+1} \neq \{0\} \) is a principal one, because we add exactly one variable at each step in the chain (20), i.e. \( \text{tr.deg}_{R_i} R_{i+1} = 1 \). Hence \( \mathcal{P} \cap R_{i+1} \) is a \( D \)-stable principal ideal (of the ring \( R_{i+1} \), and also this ideal generates a principal \( D \)-stable ideal of the ring \( R \), because \( D \)-stability of a principal ideal means exactly the condition \( Q \mid DQ \) on a generator of the ideal). We deduce with Lemma 2.6 that \( z\Delta \in \mathcal{P} \cap R_{i+1} \subset \mathcal{P} \), Q.E.D.

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