THE GAUSS-BONNET-CHERN MASS OF CONFORMALLY FLAT MANIFOLDS

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ABSTRACT. In this paper we show positive mass theorems and Penrose type inequalities for the Gauss-Bonnet-Chern mass, which was introduced recently in [20], for asymptotically flat CF manifolds and its rigidity.

1. INTRODUCTION

Recently motivated by the Einstein-Gauss-Bonnet theory [9, 28] and the pure Lovelock theory [33, 15], we introduced in [20] (and [21]) the Gauss-Bonnet-Chern mass by using the Gauss-Bonnet curvature

\begin{equation}
L_k := \frac{1}{2^k} \delta_{i_1 j_2 \ldots i_{2k - 1} i_{2k}} \delta_{j_1 j_2 \ldots j_{2k - 1} j_{2k}} R_{i_1 j_2 \ldots i_{2k - 1} i_{2k}} R_{j_1 j_2 \ldots j_{2k - 1} j_{2k}} - R_{i_1 j_2 \ldots i_{2k - 1} i_{2k}} R_{j_1 j_2 \ldots j_{2k - 1} j_{2k}}.
\end{equation}

When \( k = 1 \), \( L_1 \) is just the scalar curvature \( R \). When \( k = 2 \), it is the (second) so-called the Gauss-Bonnet curvature

\[
L_2 = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4 R_{\mu \nu} R^{\mu \nu} + R^2,
\]

which appeared at the first time in the paper of Lanczos [29] in 1938. For general \( k \) it is the Euler integrand in the Gauss-Bonnet-Chern theorem [13, 14] if \( n = 2k \) and is therefore called the dimensional continued Euler density in physics if \( k < n/2 \). Here \( n \) is the dimension. In this paper we are interested in the case \( k < n/2 \). The Gauss-Bonnet-Chern mass introduced in [20] is defined

\begin{equation}
m_k = m_{GBC} = c(n, k) \lim_{r \to \infty} \int_{S_r} P^{ijlm}_{(k)} \partial_m g_{ij} \nu_l dS,
\end{equation}

with

\[
c(n, k) = \frac{(n - 2k)!}{2^{k-1}(n - 1)! \omega_{n-1}},
\]

where \( \omega_{n-1} \) is the volume of \( (n - 1) \)-dimensional standard unit sphere and \( S_r \) is the Euclidean coordinate sphere, \( dS \) is the volume element on \( S_r \) induced by the Euclidean metric, \( \nu \) is the outward unit normal to \( S_r \) in \( \mathbb{R}^n \) and the derivative is the ordinary partial derivative. Here the tensor \( P_{(k)}^{ijlm} \) is decided by the decomposition

\begin{equation}
L_k = P_{(k)}^{ijlm} R_{ijlm}.
\end{equation}

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In this paper we use the Einstein summation convention. The tensor \( P_{(k)} \) has a crucial property of divergence-free, which guarantees the Gauss-Bonnet-Chern mass is well-defined and is a geometric invariant, under a suitable decay condition. See Section 2 below or \([20]\). When \( k = 1 \),

\[
P_{ijlm}^{(1)} = \frac{1}{2} (g^{il} g^{jm} - g^{im} g^{jl}),
\]

and \( m_1 \) is just the ADM mass introduced by Arnowitt, Deser, and Misner \([1]\) for asymptotically flat Riemannian manifolds. For a similar mass see also \([30]\).

A complete manifold \((M^n, g)\) is said to be an asymptotically flat (AF) of order \( \tau \) (with one end) if there is a compact \( K \) such that \( M \setminus K \) is diffeomorphic to \( \mathbb{R}^n \setminus B_R(0) \) for some \( R > 0 \) and in the standard coordinates in \( \mathbb{R}^n \), the metric \( g \) has the following expansion

\[
g_{ij} = \delta_{ij} + \sigma_{ij},
\]

with

\[
|\sigma_{ij}| + r|\partial \sigma_{ij}| + r^2|\partial^2 \sigma_{ij}| = O(r^{-\tau}),
\]

where \( r \) and \( \partial \) denote the Euclidean distance and the standard derivative operator on \( \mathbb{R}^n \) respectively. The condition for the welldefinedness of the Gauss-Bonnet-Chern mass is

\[
\tau > \frac{n - 2k}{k + 1},
\]

and \( L_k \) is integrable over \( M \). In this case, the Gauss-Bonnet-Chern mass is a geometric invariant, which is a generalization of the work of Bartnik for the ADM mass \( m_1 \) \([2]\).

The positive mass theorem for the ADM mass \( m_{ADM} = m_1 \), which plays an important role in differential geometry, was proved by Schoen and Yau \([35]\) for \( 3 \leq n \leq 7 \) and by Witten for general spin manifolds. See also \([31, 32]\). Its refinement, the Penrose inequality, was proved by Huisken-Ilmanen \([24]\) and Bray \([3]\) for \( n = 3 \) and Bray-Lee \([7]\) for \( n \leq 7 \). Recently there are many interesting works on special, but interesting classes of asymptotically flat manifolds. In \([28]\) Lam showed the positive mass theorem and the Penrose inequality for asymptotically flat graphs in \( \mathbb{R}^{n+1} \) by using an elementary, but elegant proof. See also the generalizations of Lam’s work in \([16, 17, 25, 26]\). The Penrose type inequality is proved for conformally flat manifolds by Freire-Schwartz \([18]\), Jauregui \([27]\) and Schwartz \([36]\) by using the relation between mass and the capacity. This relation was used already in the proof of Penrose inequality in \([3]\). For this relation, see also \([5, 8]\). It is interesting to see that there is a deep relation between the ADM mass and various geometric objects.

We are interested in generalizing the above results to our Gauss-Bonnet-Chern \( m_{GBC} = m_k \) \((k \geq 2)\). Motivated by the work of Lam \([25]\), we showed a positive mass theorem and an optimal Penrose inequality when \( M \) is an asymptotically flat graphs in \( \mathbb{R}^{n+1} \) in \([20]\). This Penrose inequality establishes a relationship between the mass \( m_{GBC} \) and more geometric objects \([20]\). In this paper we are interested in studying \( m_{GBC} \) mass on conformally flat manifolds.

A conformally flat manifold with or without boundary, CF manifold for short, is a manifold \((M^n, g) = (\mathbb{R}^n/\Omega, e^{-2u}\delta)\), where \( \delta \) is the canonical Euclidean metric on \( \mathbb{R}^n \), \( \Omega \) is a smooth bounded (possibly empty, not necessarily connected) open set and \( u \) is smooth. A CF manifold \((M^n, g)\) is called an asymptotically flat CF manifold of decay order \( \tau \) if

\[
|u| + |x| |\nabla u| + |x|^2 |\nabla^2 u| = O(|x|^{-\tau}).
\]
In this paper we always assume that $k < \frac{n}{2}$, $\tau > \frac{n-2k}{k+1}$ and $L_k$ is integrable.

First we have a positive mass theorem.

**Theorem 1.1.** Let $(\mathcal{M}^n, g) = (\mathbb{R}^n, e^{-2u}\delta)$ be an asymptotically flat CF manifold. Assume further that $L_j(g) \geq 0$ for any $j \leq k$. Then the mass $m_{GBC} \geq 0$. Moreover, equality holds if and only if $u \equiv 0$, i.e., $\mathcal{M}$ is the Euclidean space.

The condition $L_j(g) \geq 0$ for any $j \leq k$ here is equivalent to $g \in \Gamma_k$, which will be discussed in Section 2 below. A similar result was announced by Li-Nguyen in [30].

For the Gauss-Bonnet-Chern mass, $m_{2j+1}$ has different behavior with $m_{2j}$. The former behaves like the ADM mass $m_1$ and the latter like $m_2$. For $k$ even, we have also a positive mass theorem for metrics in a non-positive cone.

**Theorem 1.2.** Let $k$ be even and $(\mathcal{M}^n, g) = (\mathbb{R}^n, e^{-2u}\delta)$ be an asymptotically flat CF manifold. Assume $(-1)^j L_j \geq 0$ for any $j \leq k$. Then the mass $m_{GBC} \geq 0$. Moreover, equality holds if and only if $u \equiv 0$, i.e., $\mathcal{M}$ is the Euclidean space.

Theorem 1.1 and Theorem 1.2 provide a support for our conjecture on the positivity of the Gauss-Bonnet-Chern mass in [20]. Furthermore, from our proof we have a Penrose type inequality.

**Theorem 1.3.** Let $(\mathcal{M}^n, g) = (\mathbb{R}^n \setminus \Omega, e^{-2u}\delta)$ be an asymptotically flat CF manifold. Assume that $\Omega$ is convex, $\partial \mathcal{M} = (\Omega, e^{-2u}\delta)$ is a horizon of $(\mathcal{M}, g)$ and $u$ is constant on $\partial \Omega$. Assume further that $L_j(g) \geq 0$ for any $j \leq k$. Then we have Penrose type inequalities

\begin{equation}
(1.6) \quad m_k \geq \left( \frac{\partial \Omega}{\omega_{n-1}} \right)^{\frac{n-2k}{n-1}}.
\end{equation}

Moreover, if $k \geq 2$, we have the following strengthened Penrose type inequality

\begin{equation}
(1.7) \quad m_k \geq \left( \frac{\int_{\partial \Omega} R}{(n-1)(n-2)\omega_{n-1}} \right)^{\frac{n-2k}{n-3}},
\end{equation}

where $R$ is the scalar curvature of $\partial \Omega$ as a hypersurface in $\mathbb{R}^n$.

The assumptions on the boundary $\partial \Omega$ can be reduced by the result of Guan-Li [23] and the results could be slightly strengthened. For more details see Section 4 below. Unlike the Penrose inequality obtained in [20], this Penrose inequality is not optimal. Our Penrose inequality is motivated by the work of Jauregui in [27], who obtained (1.6) for $k = 1$. The idea is to express the mass via various integral identities.

The rest of the paper is organized as follows. In Section 2 we recall the definitions of the Gauss-Bonnet curvature $L_k$ and the $\sigma_k$-scalar curvature and their relationship when the underlying manifolds are locally conformally flat. In Section 3 we prove the positive mass theorems, Theorem 1.1 and Theorem 1.2. Theorem 1.3 is proved in Section 4.
2. The Gauss-Bonnet curvatures and the $\sigma_k$-scalar curvatures

We recall the definition of generalized $k$-th Gauss-Bonnet curvature

\begin{equation}
L_k := \frac{1}{2k} \delta_{i_1 j_2 \cdots j_{2k-1} i_{2k}} \ R_{i_1 i_2 \cdots j_{2k-1} i_{2k}} R_{i_{2k-1} i_{2k}} j_{2k-1} j_{2k},
\end{equation}

Here the generalized Kronecker delta is defined by

$$
\delta^{i_1 j_2 \cdots j_r}_{i_1 j_2 \cdots j_r} = \det \left( \begin{array}{cccc}
\delta_{i_1 i_2} & \delta_{i_1 i_3} & \cdots & \delta_{i_1 i_r} \\
\delta_{i_2 i_1} & \delta_{i_2 i_3} & \cdots & \delta_{i_2 i_r} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{i_r i_1} & \delta_{i_r i_2} & \cdots & \delta_{i_r i_r}
\end{array} \right).
$$

When $k = 2$, we can write

\begin{equation}
L_2 = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4 R_{\mu \nu} R^{\mu \nu} + 2
\end{equation}

\begin{align*}
&= |W|^2 + \frac{n - 3}{n - 2} \left( \frac{n}{n - 1} R^2 - 4 |Ric|^2 \right) \\
&= |W|^2 + 8(n - 2)(n - 3) \sigma_2(A_g) \\
&= R_{ijkl} P_{ijkl}^{(2)},
\end{align*}

where

\begin{equation}
P_{ijkl}^{(2)} = R_{ijkl} + R_{ij} g_{kl} - R_{ij} g_{kl} - R_{ij} g_{kl} - R_{ij} g_{kl} + \frac{1}{2} R (g_{ik} g_{jl} - g_{il} g_{jk}),
\end{equation}

$W$ denotes the Weyl tensor, $Ric$ the Ricci tensor, $R$ the scalar curvature and

$$
A_g := \frac{1}{n - 2} \left( Ric - \frac{R}{2(n - 1)} g \right),
$$

the Schouten tensor. $P_{(2)}$ is the divergence-free part of the Riemann curvature tensor $Riem$. For the general $L_k$-curvature, the corresponding $P_{(k)}$ curvature is

\begin{equation}
P_{stlm}^{(k)} := \frac{1}{2k} \delta_{i_1 j_2 \cdots j_{2k-3} i_{2k-2} st} \ R_{i_1 i_2 \cdots j_{2k-3} j_{2k-2}} i_{2k-1} j_{2k} g_{j_{2k-2} j_{2k-1}} g_{j_{2k} l}.
\end{equation}

Recall that $L_k = P_{stlm}^{(k)} R_{ijkl}$ and the tensor $P_{(k)}$ has the following crucial property.

**Proposition 2.1.** The tensor $P_{(k)}$ has the same symmetry and anti-symmetry as the Riemann curvature tensor and satisfies

$$
\nabla_i P_{ijlm}^{(k)} = 0.
$$

**Proof.** The case $k = 1$ is trivial. We have proved the $k = 2$ case in [20]. For the general case, it follows from the symmetry of the Riemann curvature tensor and the differential Bianchi identity. We skip the proof here. \qed

Now we consider the case that $(\mathcal{M}^n, g)$ is a conformally flat manifold of dimension $n \geq 5$. Namely, $(\mathcal{M}^n, g) = (\mathbb{R}^n, e^{-2\psi} \delta)$, where $\delta$ is the canonical Euclidean metric on $\mathbb{R}^n$. In this case, the curvature $L_k$ is just the $\sigma_k$-scalar curvature (up to a multiple constant), which was considered by Viaclovsky in [37] and has been intensively studied in the $\sigma_k$ Yamabe problem.
For the convenience of the reader, we recall some basic properties on the elementary symmetric functions (see for example [22, 11, 37]). For $1 \leq k \leq n$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, the $k$-th elementary symmetric function is defined as

$$\sigma_k(\lambda) := \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$ 

The definition can be extended to symmetric matrices. For a symmetric matrix $B$, denote $\lambda(B) = (\lambda_1(B), \cdots, \lambda_n(B))$ be the eigenvalues of $B$. We set

$$\sigma_k(B) := \sigma_k(\lambda(B)).$$

We define also $\sigma_0(B) = 1$. Let $I$ be the identity matrix. Then we have for any $t \in \mathbb{R}$,

$$\sigma_n(I + tB) = \det(I + tB) = \sum_{i=0}^n \sigma_i(B) t^i.$$ 

We recall the definition of the Garding cone: for $1 \leq k \leq n$, let $\Gamma_k^+$ (resp. $\Gamma_k$) is a cone in $\mathbb{R}^n$ determined by

$$\Gamma_k^+ = \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0 \}.$$ 

(resp. $\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) \geq 0, \cdots, \sigma_k(\lambda) \geq 0 \}$).

A symmetric matrix $B$ is called belong to $\Gamma_k^+$ (resp. $\Gamma_k$) if $\lambda(B) \in \Gamma_k^+$ (resp. $\lambda(B) \in \Gamma_k$). The $k$-th Newton transformation is defined as follows

$$ (T_k)^i_j(B) := \frac{\partial \sigma_{k+1}}{\partial b^i_j(B)}, $$

where $B = (b^i_j)$. If there is no confusion, we omit the index $k$. We recall some basic properties about $\sigma_k$ and $T$.

$$\sigma_k(B) = \frac{1}{k!} s_j^{i_1 \cdots i_k} b_j^{i_1} \cdots b_j^{i_k} = \frac{1}{k} \text{tr}(T_{k-1}B),$$

$$ (T_k)^i_j(B) = \frac{1}{k!} s_j^{i_1 \cdots i_k} b_j^{i_1} \cdots b_j^{i_k} $$

$$ = \sum_{i=0}^k \sigma_{k-i}(B)(-B)^i = \sigma_k(B)I - \sigma_{k-1}(B)B + \cdots + (-1)^k B^k. $$

It is well-known that $\sigma_k^{1/k}$ is concave in $\Gamma_k$, which implies that

$$\sigma_k(A + B) \geq \sigma_k(A) + \sigma_k(B), \quad \text{for any } A, B \in \Gamma_k.$$ 

The $\sigma_k$-scalar curvature $\sigma_k(g)$ is defined in [37] by

$$\sigma_k(g) := \sigma_k(g^{-1}A_g),$$

where $A_g$ is the Schouten tensor of $g$. 
Proposition 2.2. Let \((M^n, g)\) be a locally conformally flat metric of dimension \(n\). Assume \(2k < n\). Then
\[
L_k = 2^k k! \frac{(n-k)!}{(n-2k)!} \sigma_k(g).
\]
\[
(2.9)
\]

Proof. We recall the decomposition of the Riemann curvature tensor
\[
Riem = W + A \otimes g.
\]

As \(W \equiv 0\), we have
\[
R_{i_1j_1}^{i_2j_2} = A_{i_1}{}^{i_1} \delta_{i_2}{}^{j_2} + \delta_{i_1}{}^{i_1} A_{i_2}{}^{j_2} - A_{i_1}{}^{j_2} \delta_{i_2}{}^{i_1} - \delta_{i_1}{}^{j_2} A_{i_2}{}^{i_1}.
\]
\[
(2.10)
\]

It follows that
\[
L_k = 2^k \delta_{j_1j_2...j_{2k-1}j_{2k}}^{i_1i_2...i_{2k-1}i_{2k}} A_{i_1}{}^{i_1} \delta_{i_2}{}^{j_2} ... A_{i_{2k-1}}{}^{j_{2k-1}} \delta_{i_{2k}}{}^{j_{2k}}
\]
\[
= 2^k(n-k) \cdot ... \cdot (n-2k+1) \delta_{j_1j_2...j_{2k-1}j_{2k}}^{i_1i_2...i_{2k-1}i_{2k}} A_{i_1}{}^{j_1} ... A_{i_{2k-1}}{}^{j_{2k-1}}
\]
\[
= 2^k k! (n-k) \cdot ... \cdot (n-2k+1) \sigma_k(A).
\]

Here we use the facts
\[
\delta^{i_1i_2...i_{2k-1}i_{2k}}_{j_1j_2...j_{2k-1}j_{2k}} A_{i_1}{}^{i_1} \delta_{i_2}{}^{j_2} = \delta^{i_1i_2...i_{2k-1}i_{2k}}_{j_1j_2...j_{2k-1}j_{2k}} \delta_{i_1}{}^{j_1} A_{i_2}{}^{j_2}
\]
\[
= -\delta^{i_1i_2...i_{2k-1}i_{2k}}_{j_1j_2...j_{2k-1}j_{2k}} A_{i_1}{}^{j_1} \delta_{i_2}{}^{i_1} = -\delta^{i_1i_2...i_{2k-1}i_{2k}}_{j_1j_2...j_{2k-1}j_{2k}} \delta_{i_1}{}^{i_1} A_{i_2}{}^{j_2}.
\]

For \(k = \frac{n}{2}\) see [37]. Another important property will be the following.

Proposition 2.3. (see [37]) Let \((M^n, g)\) be a locally conformally flat manifold of dimension \(n\). Then \(T_{k-1}(A)\) is divergence-free.

Without the conformal flatness Proposition 2.3 still holds for \(k = 2\), i.e., \(T_1\) is divergence-free, which was proved in [37].

3. Positive Mass Theorem for CF Manifolds and Rigidity

In this section we prove Theorem 1.1 and Theorem 1.2. For the proof we need one more well-known property.

Proposition 3.1. Let \(u : \mathbb{R}^n \to \mathbb{R}\) be some smooth function. Denote \(D^2 u = (u_{ij})\) be the hessian matrix of \(u\) with respect to Euclidean metric. Then \(T_k(D^2 u)\) is divergence-free, that is,
\[
\partial_i (T_k^{ij}(D^2 u)) = \partial_j T_k^{ij}(D^2 u) = 0.
\]

Remark 3.2. Note that in Proposition 3.1 the divergence-free is with respect to the standard euclidean metric \(\delta\) and in Proposition 2.3 the divergence-free is with respect to the metric \(g = e^{-2u} \delta\).

For an asymptotically flat CF manifold, we first have an equivalent form of Gauss-Bonnet-Chen mass defined by (1.2). By (1.3), (2.10) together with Proposition 2.1 we have
\[
L_k = 4P_{(k)}^{ij} A_{ij} g_{jm} = 4P_{(k)}^{jj} A_{ij} e^{-2u}.
\]
On the other hand, from (2.6) and (2.9) we have
\[ L_k = 2^k (k-1)! \frac{(n-k)!}{(n-2k)!} (T_{k-1}(A))^{ij} A_{ij}. \]

For the Gauss-Bonnet-Chern mass (1.2) we have
\[ m_k := \frac{(n-2k)!}{2^{k-1}(n-1)!} \omega_{n-1} \lim_{r \to \infty} \int_{S_r} \frac{P^{ijlm}}{(k-1)! (n-k)!} \partial_g g_{ij} \nu_l dS \]
\[ = \frac{(n-2k)!}{2^{k-1}(n-1)!} \omega_{n-1} \lim_{r \to \infty} \int_{S_r} -2e^{-2u} P^{ijl}_{(k)} u_i \nu_l dS. \]

Combining all together, we thus obtain the following equivalent form of (1.2),
\[ m_k = \lim_{r \to \infty} \frac{(k-1)!(n-k)!}{(n-1)!} \omega_{n-1} \int_{S_r} (T_{k-1}(A))^{ij} u_j \nu_l dS. \]

This formula would be useful in the computation of the Gauss-Bonnet-Chern mass. Now we start to prove Theorem 1.1.

**Proof of Theorem 1.1** Since \( g = e^{-2u} \delta \), a direct computation gives
\[ \text{Ric} = (n-2)(D^2 u + \frac{1}{n-2}(\Delta u) \delta + du \otimes du - |\nabla u|^2), \]
\[ R = e^{2u}(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2), \]
which imply
\[ A_g := \frac{1}{n-2} \left( \text{Ric} - \frac{R_g}{2(n-1)} \right) = D^2 u - \frac{|\nabla u|^2}{2} I + du \otimes du. \]

Here \( \nabla \) and \( \Delta \) are operators with respect to the Euclidean metric \( \delta \) and \( D^2 \) are the Hessian operator. Since
\[ T_{k-1}(D^2 u) = T_{k-1}(A) + O(|x|^{-k+2}), \]
which follows from (1.5) and (2.7), we have by (3.1)
\[ m_k = \lim_{r \to \infty} \frac{(k-1)!(n-k)!}{(n-1)!} \omega_{n-1} \int_{S_r} (T_{k-1}(D^2 u))^{ij} u_j \nu_l dS. \]

Applying Proposition 3.1 and Green’s formula, we obtain
\[ \int_{S_r} (T_{k-1}(D^2 u))^{ij} u_j \nu_l dS = \int_{B_r} (T_{k-1}(D^2 u))^{ij} u_j dx = k \int_{B_r} \sigma_k(D^2 u) dx. \]

Now, we write
\[ D^2 u = A + \frac{|\nabla u|^2}{2} I - du \otimes du. \]

It is crucial to see that the matrix \( \frac{|\nabla u|^2}{2} I - du \otimes du \) has one eigenvalue \( -\frac{|\nabla u|^2}{2} \) and \( n-1 \) eigenvalues \( \frac{|\nabla u|^2}{2} \). Therefore, \( B := \frac{|\nabla u|^2}{2} I - du \otimes du \in \Gamma_k^+ \) for \( k < n/2 \), for
\[ \sigma_j(B) = \frac{(n-1)!(n-2j)!}{2^j j!(n-j)!} |\nabla u|^{2j} \] for any \( j \leq k < n/2 \).
Remark 3.3. In the above proof, the calculations before (3.7) are with respect to the Euclidean metric \(g\), namely, which implies from (2.7) and (3.1)

This yields the positivity of the mass (3.7)

Finally, we infer

Thus, a direct calculation leads to

8 YUXIN GE, GUOFANG WANG, AND JIE WU

Proof of Theorem 1.2. Let \(v := e^u\). Thus, the conformal metric is written as \(g = v^{-2}\delta\). For such a representation of the metric, the Schouten tensor (3.2) can be written as

Let \(\alpha \in \mathbb{R}\) be some sufficiently negative number to be fixed later. As in the proof of Theorem 1.1, it follows from the decay condition (1.5) of \(u\) that

which implies from (2.7) and (3.1)

Thus, a direct calculation leads to

On the other hand, it follows from Proposition 3.1 that \((T_{k-1}(D^2v))^{ij}_j = 0\) and also

\((T_{k-1}(D^2v))^{ij}v_j = k\sigma_k(D^2v)\).
Therefore, we have
\[ m_k = \frac{k!(n-k)!}{(n-1)! \omega_{n-1}} \int_{\mathbb{R}^n} v^\alpha \sigma_k(D^2v) dx \]
\[ + \frac{(k-1)!(n-k)! \alpha}{(n-1)! \omega_{n-1}} \int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-1}(D^2v))^{ij} v_i v_j dx. \]

We will try to write the integral of the right hand in terms of \( \sigma_i(D^2v) \) and \( |\nabla v|^{2i} \), then in terms of \( \sigma_i(A) \) and \( |\nabla v|^{2i} \) for \( 0 \leq i \leq k \).

Directly from the definition of the Newton tensor, we know
\[ T_i(D^2v) = \sigma_i(D^2v) I - T_{i-1}(D^2v) D^2v = \sigma_i(D^2v) I - D^2v T_{i-1}(D^2v). \]

It follows, together with the partial integration
\[ \int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-1}(D^2v))^{ij} v_i v_j dx \]
\[ = \int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2 - \int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-2}(D^2v))^{ij} v_i v_j \]
\[ = \int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2 - \frac{1}{2} \int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-2}(D^2v))^{ij} |\nabla v|^2 v_i v_j \]
\[ + \frac{1}{2} \int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-2}(D^2v))^{ij} |\nabla v|^2 v_i v_j + \frac{1}{2} \int_{\mathbb{R}^n} v^{\alpha-2}(T_{k-2}(D^2v))^{ij} |\nabla v|^2 v_i v_j \]
\[ = \frac{k + 1}{2} \int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2 + \frac{\alpha - 1}{2} \int_{\mathbb{R}^n} v^{\alpha-2}(T_{k-2}(D^2v))^{ij} |\nabla v|^2 v_i v_j. \]

More generally, we have the following claim.

**Claim.** For all \( 1 \leq l \leq k - 2 \), we have
\[ \int_{\mathbb{R}^n} v^{\alpha-1-l}(T_{k-1-l}(D^2v))^{ij} |\nabla v|^{2l} v_i v_j \]
\[ = \frac{k + l + 1}{2(l + 1)} \int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)} \]
\[ + \frac{\alpha - l - 1}{2(l + 1)} \int_{\mathbb{R}^n} v^{\alpha-2-l}(T_{k-2-l}(D^2v))^{ij} |\nabla v|^{2(l+1)} v_i v_j. \]

As above we have
\[ \int_{\mathbb{R}^n} v^{\alpha-1-l}(T_{k-1-l}(D^2v))^{ij} |\nabla v|^{2l} v_i v_j dx \]
\[ = \int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)} - \int_{\mathbb{R}^n} v^{\alpha-1-l}(T_{k-2-l}(D^2v))^{ij} |\nabla v|^{2l} v_i v_j v_i \]
\[ = \int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)} - \frac{1}{2} \int_{\mathbb{R}^n} v^{\alpha-1-l}|\nabla v|^{2l}(T_{k-2-l}(D^2v))^{ij} |\nabla v|^2 v_i v_i. \]
On the other hand, we have
\[
-\frac{1}{2} \int_{\mathbb{R}^n} v^{\alpha - 1 - l} |\nabla v|^{2l} (T_{k-2-l}(D^2v))^{ij} (|\nabla v|^2)_{ji} v_i \\
= \frac{\alpha - 1 - l}{2} \int_{\mathbb{R}^n} v^{\alpha - 2 - l} (T_{k-2-l}(D^2v))^{ij} |\nabla v|^{2(l+1)} v_i v_j \\
+ \frac{k - 1 - l}{2} \int_{\mathbb{R}^n} v^{\alpha - 1 - l} \sigma_{k-1-l}(D^2v) |\nabla v|^{2(l+1)} \\
+ \frac{l}{2} \int_{\mathbb{R}^n} v^{\alpha - 1 - l} (T_{k-2-l}(D^2v))^{ij} |\nabla v|^{2l} (|\nabla v|^2)_{ji} v_i, 
\]
which implies
\[
-\frac{1}{2} \int_{\mathbb{R}^n} v^{\alpha - 1 - l} |\nabla v|^{2l} (T_{k-2-l}(D^2v))^{ij} (|\nabla v|^2)_{ji} v_i \\
= \frac{\alpha - 1 - l}{2(l+1)} \int_{\mathbb{R}^n} v^{\alpha - 2 - l} (T_{k-2-l}(D^2v))^{ij} |\nabla v|^{2(l+1)} v_i v_j \\
+ \frac{k - 1 - l}{2(l+1)} \int_{\mathbb{R}^n} v^{\alpha - 1 - l} \sigma_{k-1-l}(D^2v) |\nabla v|^{2(l+1)}. 
\]
Going back to (3.10), the desired claim yields. Hence, we have
\[
\int_{\mathbb{R}^n} v^{\alpha - 1} (T_{k-1}(D^2v))^{ij} v_i v_j dx \\
= \frac{k + 1}{2} \int_{\mathbb{R}^n} v^{\alpha - 1} \sigma_{k-1}(D^2v) |\nabla v|^2 + \frac{(\alpha - 1) \cdots (\alpha - k + 1)}{2^{k-1}(k-1)!} \int_{\mathbb{R}^n} v^{\alpha - k} |\nabla v|^{2k} \\
+ \sum_{l=2}^{k-1} \frac{(\alpha - 1) \cdots (\alpha - l + 1)(k + l)}{2^l l!} \int_{\mathbb{R}^n} v^{\alpha - l} |\nabla v|^{2l} \sigma_{k-l}(D^2v). 
\]
Finally, we infer
\[
\frac{(n - 1)! \omega_{n-1}}{(k-1)!(n-k)!} m_k \\
= k \int_{\mathbb{R}^n} v^n \sigma_k(D^2v) dx + \frac{(k+1)\alpha}{2} \int_{\mathbb{R}^n} v^{\alpha - 1} \sigma_{k-1}(D^2v) |\nabla v|^2 \\
+ \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{2^{k-1}(k-1)!} \int_{\mathbb{R}^n} v^{\alpha - k} |\nabla v|^{2k} \\
+ \sum_{l=2}^{k-1} \frac{\alpha(\alpha - 1) \cdots (\alpha - l + 1)(k + l)}{2^l l!} \int_{\mathbb{R}^n} v^{\alpha - l} |\nabla v|^{2l} \sigma_{k-l}(D^2v). 
\]
(3.11)
Now we want to write $m_k$ in terms of $\sigma_l(A)$ and $|\nabla v|^{2l}$. Recall
\[
D^2v = vA + \frac{|\nabla v|^2 I}{2v},
\]
so that for all $1 \leq l \leq k$ we have
\[
\sigma_l(D^2v) = v^l \sigma_l(A + \frac{|\nabla v|^2 I}{2v^2}) = v^l \sum_{j=0}^{l} C_{n-j}^{l-j} \sigma_j(A) \left( \frac{|\nabla v|^2}{2v^2} \right)^{l-j},
\]
where $C_{n-j}^{k-j} = \frac{(n-j)!}{(n-k)!(k-j)!}$. From (3.11), we deduce

$$
\begin{align*}
\frac{(n-1)! \omega_{n-1}}{(k-1)!(n-k)!} m_k &= k \int_{\mathbb{R}^n} v^{\alpha+k} \sum_{j=0}^{k-1} C_{n-j}^{k-j} \sigma_j(A) \left( \frac{|\nabla v|^2}{2v^2} \right)^{k-j} \\
&\quad + (k+1) \alpha \int_{\mathbb{R}^n} v^{\alpha+k} \sum_{j=0}^{k-1} C_{n-j}^{k-1-j} \sigma_j(A) \left( \frac{|\nabla v|^2}{2v^2} \right)^{k-j} \\
&\quad + \frac{2\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{(k-1)!} \int_{\mathbb{R}^n} v^{\alpha+k} \left( \frac{|\nabla v|^2}{2v^2} \right)^{k} \\
&\quad + \sum_{l=2}^{k-1} \sum_{j=0}^{k-l} \frac{\alpha(\alpha - 1) \cdots (\alpha - l + 1)(k+l)}{l!} \int_{\mathbb{R}^n} v^{\alpha+k} C_{n-j}^{k-l-j} \sigma_j(A) \left( \frac{|\nabla v|^2}{2v^2} \right)^{k-j} \\
&= \int_{\mathbb{R}^n} v^{\alpha+k} \sum_{j=0}^{k} P_{k-j}(\alpha) \sigma_j(A) \left( \frac{|\nabla v|^2}{2v^2} \right)^{k-j}.
\end{align*}
$$

Here for all $0 \leq j \leq k$, $P_j(\alpha)$ is a polynomial of degree $j$ in $\alpha$ with a leading coefficient equal to $k$ when $j = 0$, to $k + 1$ when $j = 1$, to $\frac{2k-j}{(k-j)!}$ when $2 \leq j \leq k - 1$ and to $\frac{2}{(k-1)!}$ when $j = k$. Therefore, we can choose sufficiently negative number $\alpha < 0$ such that $(-1)^j P_j(\alpha) > 0$ for all $0 \leq j \leq k$. By the assumptions $(-1)^j L_j \geq 0$ for all $1 \leq j \leq k$, which are equivalent to $(-1)^j \sigma_j(A) \geq 0$, we have

$$P_{k-j}(\alpha) \sigma_j(A) = (-1)^{k-j} P_{k-j}(-1)^j \sigma_j(A) \geq 0,$$

i.e., each term on the right hand side in the last inequality is non-negative. This gives $m_k \geq 0$. Here we need that $k$ is even. Moreover, if $m_k = 0$, we have $\nabla v = 0$, and hence $v$ is a constant 1 and $\mathcal{M}$ is the standard euclidean space. We finish the proof. \qed

4. Penrose type inequality

Let $(\mathcal{M}^n, g) = (\mathbb{R}^n \setminus \Omega, e^{-2u} \delta)$ be now a CF manifold, where $\Omega$ is a bounded domain such that each connected component of $\Omega$ is star-shaped such that the second fundamental form of the boundary $\partial \Omega$ is in the cone $\Gamma_k^+(\partial \Omega)$. As before, we assume $2k < n$, $g \in \Gamma_k$, $L_k$ integrable and $u$ satisfies the decay condition at the infinity

$$|u| + |x| |\nabla u| + |x|^2 |\nabla^2 u| = O(|x|^{-\tau}),$$

with $\tau > \frac{n-2k}{k+1}$. First, we assume $\Omega$ has just one connected component.

**Theorem 4.1.** Let $(\mathcal{M}, g) = (\mathbb{R}^n \setminus \Omega, e^{-2u} \delta)$ satisfy the above assumptions. Assume, in addition, that $\partial \mathcal{M}$ is a horizon on $(\mathcal{M}, g)$ (i.e. $\partial \mathcal{M} = \partial \Omega \subset \mathcal{M}$ is minimal) and $u$ is constant on $\partial \Omega$. 

Then we have the following Penrose type inequality

\[ m_k \geq \frac{(n-2k)!}{2^k(n-1)!\omega_{n-1}} \int_{\mathcal{M}} e^{(n-2k)u} L_k(g) dvol_g \]

\[ + \frac{n-2k}{2^k} \int_{\mathcal{M}} e^{(n-2k)u} |\nabla u|_g^{2k} dvol_g + \left( \frac{1}{\omega_{n-1}} \right)^{(n-2k)/n-3}. \]

(4.1)

Moreover, if we assume the second fundamental form of \( \partial \Omega \) is in the cone \( \Gamma_{2k-1} \) \( (k \geq 2) \), we have

\[ m_k \geq \frac{(n-2k)!}{2^k(n-1)!\omega_{n-1}} \int_{\mathcal{M}} e^{(n-2k)u} L_k(g) dvol_g \]

\[ + \frac{n-2k}{2^k} \int_{\mathcal{M}} e^{(n-2k)u} |\nabla u|_g^{2k} dvol_g + \left( \frac{\int_{\partial \Omega} R}{(n-1)(n-2)\omega_{n-1}} \right)^{(n-2k)/n-3}. \]

(4.2)

Here \( R \) is the scalar curvature of \( \partial \Omega \) as a hypersurface in \( \mathbb{R}^n \).

Proof. Applying Proposition 3.1 and Green’s formula, we obtain

\[ \int_{S_r} (T_{k-1}(D^2u))^{ij} u_j \nu_i dS - \int_{\partial \Omega} (T_{k-1}(D^2u))^{ij} u_j \nu_i dS = k \int_{B_r \setminus \Omega} \sigma_k(D^2u) dx, \]

for large \( r > 0 \). The argument given in the proof of Theorem 1.1 together with (3.5) to (3.8), implies

\[ m_k \geq \frac{(n-2k)!}{2^k(n-1)!\omega_{n-1}} \int_{\mathcal{M}} e^{(n-2k)u} L_k(g) dvol_g \]

\[ + \frac{n-2k}{2^k} \int_{\mathcal{M}} e^{(n-2k)u} |\nabla u|_g^{2k} dvol_g \]

\[ + \frac{(k-1)!(n-2k)!}{(n-1)!\omega_{n-1}} \int_{\partial \Omega} (T_{k-1}(D^2u))^{ij} u_j \nu_i dS. \]

(4.4)

Recall \( \nu \) is the normal vector pointing to the infinity. Since \( \partial \mathcal{M} \) is a horizon of \( \mathcal{M} \), the mean curvature of \( \partial \mathcal{M} \) is equal to zero at the boundary. We denote \( H \) the mean curvature of \( \partial \Omega \) in \( \mathbb{R}^n \). As \( g \) is a conformal metric, the mean curvature of \( \partial \mathcal{M} \) is equal to \( e^u(H - (n-1)(\nabla u, \nu)) \). Therefore, on the boundary \( \partial \Omega \) we have

\[ H - (n-1)(\nabla u, \nu) = 0. \]

(4.5)

In particular, \( (\nabla u, \nu) > 0 \) on the boundary, since we assume the second fundamental form \( L \) is in the cone \( \Gamma_{k-1}^+ (\partial \Omega) \). On the other hand, from the non-negativity of the scalar curvature, we have

\[ \Delta u \geq 0. \]
Hence, by the Maximum principle, we deduce $u \leq 0$ in $\Omega$. For all $x \in \partial \Omega$, we split $T_x \mathbb{R}^n = T_x \partial \Omega \oplus \mathbb{R} \nu$ as the sum of tangential part and normal part. Let $e_\beta (1 \leq \beta \leq n-1)$ a basis of $\partial \Omega$ and $e_n = \nu$. And Let $B = (D^2 u(e_i, e_j))_{1 \leq i, j \leq n}$ be the Hessian matrix and $B' = (D^2 u(e_\alpha, e_\beta))_{1 \leq \alpha, \beta \leq n-1}$ the first $(n-1) \times (n-1)$ block in $B$. Recall that $u$ is a constant on the boundary $\partial \Omega$. We have for all $1 \leq \alpha, \beta \leq n-1$

$$D^2 u(e_\alpha, e_\beta) = \langle \nabla u, \nu \rangle L(e_\alpha, e_\beta),$$

where $L$ is the second fundamental form with respect to the normal vector $-\nu$. Hence, we can compute

$$(T_{k-1}(D^2 u))^{ij} u_j \nu_i = \langle \nabla u, \nu \rangle \frac{\partial \sigma_k(B)}{\partial \theta_{nm}} = \langle \nabla u, \nu \rangle \sigma_{k-1}(B').$$

Here we have used the fact $\nabla u = 0$ on the boundary. Gathering (4.5) to (4.7), we deduce

$$(T_{k-1}(D^2 u))^{ij} u_j \nu_i = \langle \nabla u, \nu \rangle^k \sigma_{k-1}(L) = \frac{1}{(n-1)^k} \sigma_1(L)^k \sigma_{k-1}(L).$$

Recall that in the Garding cone $\Gamma_m^+$, we have the Newton-MacLaurin inequalities,

$$\frac{\sigma_{m-1} \sigma_{m+1}}{\sigma_m^2} \leq \frac{m(n-m-1)}{(m+1)(n-m)},$$

$$\frac{\sigma_1 \sigma_{m-1}}{\sigma_m} \geq \frac{m(n-1)}{n-m}.$$  

(4.10)

We have

$$T_{k-1}(D^2 u))^{ij} u_j \nu_i \geq \left( \frac{(k-1)!}{(n-1) \cdots (n-k+1)} \right)^{k-1} \sigma_{k-1}(L)^{2k-1}.$$  

From the Hölder inequality and the Aleksandrov-Fenchel inequality (see [34], [23] and [12] for example), we have

$$\int_{\partial \Omega} (T_{k-1}(D^2 u))^{ij} u_j \nu_i dS \geq \left( \frac{(k-1)!}{(n-1) \cdots (n-k+1)} \right)^{k-1} \int_{\partial \Omega} \sigma_{k-1}(L)^{2k-1}$$

$$\geq \left( \frac{(k-1)!}{(n-1) \cdots (n-k+1)} \right)^{k-1} \left( \int_{\partial \Omega} \sigma_{k-1}(L)^{2k-1} \right)^{\frac{1}{k-1}} dS$$

$$\geq \left( \frac{(n-1)!}{k-1} \right)^{\frac{2k-1}{n-1} \omega_{n-1}} \left( \int_{\partial \Omega} \sigma_{k-1}(L)^{\frac{2k-1}{k-1}} dS \right)^{\frac{n-k}{n-1}}.$$  

Going back to (4.14), we get the desired inequality (4.1). Now, assume $L \in \Gamma_{2k-1}$, it follows from the Newton-MacLaurin inequality that

$$\frac{1}{(n-1)^k} \sigma_1(L)^k \sigma_{k-1}(L) \geq \frac{(2k-1)!(n-2k)!}{(k-1)!(n-k)!} \sigma_{2k-1}(L).$$
Hence, again by the Aleksandrov-Fenchel inequality, we get
\[
\int_{\partial \Omega} (T_{k-1}(D^2 u))_{ij} u_j dS \geq \frac{(2k-1)! (n-2k)!}{(k-1)! (n-k)!} \int_{\partial \Omega} \sigma_{2k-1}(L) \geq \frac{(n-1)!}{(k-1)! (n-k)!} \omega_{n-1}^{2k-3} \left( \int_{\partial \Omega} \frac{2\sigma_2(L)}{(n-1)(n-2)} \right)^{\frac{n-2k}{n-3}}.
\]
In view of (4.4), we prove inequality (4.2) and finish the proof. \(\square\)

**Remark 4.2.** In (4.2), the scalar curvature \(R\) could be replaced by other high order curvature tensor of order small than \(k\) which establishes a relationship between the mass \(m_{GBC}\) and more geometric objects.

**Remark 4.3.** We remark that when \(k = 1\), our mass \(m_1 = m_{ADM}\). In this case the Penrose inequality in Theorem 4.1 is
\[
m_1 \geq \left( \frac{\omega^{n-3}_{n-1}}{\omega_{n-1}} \right)^{\frac{n-3}{2}},
\]
which was already proved in [27]. In fact, our Penrose inequality is motivated by his work. Note that we have taken a different test function comparing with the paper [27].

Let \(\Omega_i\) be the components of \(\Omega\), \(i = 1, \cdots, l\), and let \(\Sigma_i = \partial \Omega_i\). If we assume that each \(\Sigma_i\) is a horizon, we have the following

**Corollary 4.4.** With the same condition of Theorem 4.1 and the additional condition that each \(\Sigma_i\) is a horizon Then we have the following Penrose type inequality
\[
m_k \geq \frac{(n-2k)!}{2^k(n-1)! \omega_{n-1}} \int_{\mathcal{M}} e^{(n-2k)u} L_k(g) dv_{g} + \frac{n-2k}{2^k} \int_{\mathcal{M}} e^{(n-2k)u} |\nabla u|_{g}^{2k} dv_{g} + \sum_{i=1}^{l} \left( \frac{\omega_{n-1}}{\omega_{n-1}} \right)^{\frac{n-2k}{n-3}} \left( \frac{\omega_{n-1}}{\omega_{n-1}} \right)^{\frac{n-2k}{n-3}}.
\]
Moreover, if we assume the second fundamental form of \(\partial \Omega\) is in the cone \(\Gamma_{2k-1}\) \((k \geq 2)\), we have
\[
m_k \geq \frac{(n-2k)!}{2^k(n-1)! \omega_{n-1}} \int_{\mathcal{M}} e^{(n-2k)u} L_k(g) dv_{g} + \frac{n-2k}{2^k} \int_{\mathcal{M}} e^{(n-2k)u} |\nabla u|_{g}^{2k} dv_{g} + \sum_{i=1}^{l} \left( \frac{\omega_{n-1}}{\omega_{n-1}} \right)^{\frac{n-2k}{n-3}} \left( \frac{\omega_{n-1}}{\omega_{n-1}} \right)^{\frac{n-2k}{n-3}}.
\]
Here \(R\) is the scalar curvature of \(\partial \Omega\) as a hypersurface in \(\mathbb{R}^n\).
Example 4.5. \( (\mathcal{M}^n = I \times S^{n-1}, g) \) with coordinates \((\rho, \theta)\), general Schwarschild metrics are given
\[
g_{\text{Sch}}^k = (1 - \frac{2m}{\rho^2 - 2})^{-1} d\rho^2 + \rho^2 d\Theta^2,
\]
where \( d\Theta^2 \) is the round metric in \( S^{n-1} \), \( m \in \mathbb{R} \) is the “total mass” of corresponding black hole solutions in the Lovelock gravity \([15, 10]\). When \( k = 1 \) we recover the Schwarzschild solutions of the Einstein gravity.

Motivated by the Schwarzschild solutions, the above metrics also have the following form of conformally flat which is more convenient for computation \([10]\).
\[
g_{\text{Sch}}^k = (1 - \frac{2m}{\rho^2 - 2})^{-1} d\rho^2 + \rho^2 d\Theta^2 = (1 + \frac{m}{2r^2})^{\frac{4k}{n-2k}} (dr^2 + r^2 d\Theta^2).
\]

For this metric the Gauss-Bonnet-Chern mass \( m_k = m^k \) (one can check it by \((4.11)\) below) and the black hole (i.e. the horizon) \( \Sigma = \partial \Omega = \{ r = r_0 = \left( \frac{m}{2} \right)^{\frac{k}{n-2k}} \} \) and its area is
\[
|\Sigma| = \omega_{n-1} r_0^{n-1},
\]
hence
\[
m_k = m_k = \frac{2(n-2k)}{n-1} k
\]
\[
= 2k \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k}{n-1}} = \frac{1}{2k} \left( \frac{|\Sigma|_{g_{\text{Sch}}^k}}{\omega_{n-1}} \right)^{\frac{n-2k}{n-1}}.
\]

We remark that the Penrose inequality in Theorem 1.3 is not optimal, since in Theorem 1.3 the area of \( \Sigma \) is computed with the Euclidean metric \( \delta \), not with the metric \( g = e^{-2u} \delta \) itself. In general, if \( (\mathcal{M}^n, g) \) is spherically symmetric, we have the following result.

Proposition 4.6. Suppose \( (\mathcal{M}^n, g) \) is asymptotically flat CF manifold with \( g = e^{-2u(r)} \delta \), i.e., \( (\mathcal{M}^n, g) \) is spherically symmetric, then
\[
(4.11) \quad m_k = \lim_{r \to \infty} \frac{1}{\omega_{n-1}} \int_{S_r} (\frac{u_r}{r})^k dS_r.
\]
If \( k \) is even, we always have \( m_k \geq 0 \).

Proof. We adopt the equivalent form \((3.3)\) to calculate the Gauss-Bonnet-Chern mass. Denote the radial derivative of \( u \) by \( u_r \triangleq \frac{\partial u}{\partial r} \). We consider \( \Omega = B_r \) being the ball centered at the origin with radius equal to \( r \). Thus \( \Omega \) can be seen as a level set of \( u \) which enable us to use the formulae in the proof of Theorem 4.1. Let \( (e_1, \cdots, e_{n-1}) \) be an orthonormal basis of tangent plane on the boundary \( \partial \Omega \). It follows from \((4.6)\) that for all \( 1 \leq \alpha, \beta \leq n - 1 \), we have
\[
D^2u(e_\alpha, e_\beta) = \frac{u_r}{r} \delta_{\alpha\beta}
\]
since the second fundamental form on \( \partial \Omega = S_r \) is equal to \( \frac{1}{r} I \) where \( I \) is the identity map. By \((4.7)\) we have
\[
T_k-1(D^2u)^{ij} u_i u_j = \frac{(n - 1) \cdots (n - k + 1)}{(k - 1)! r^{k-1}} u_r^k.
\]
Going back to (3.3), we get the desired result (4.11).

References

[1] R. Arnowitt, S. Deser and C. W. Misner, Coordinate invariance and energy expressions in general relativity, Phys. Rev. (2) 122 (1961), 997–1006.
[2] R. Bartnik, The mass of an asymptotically flat manifold, Comm. Pure Appl. Math., 34 (1986) 661–693.
[3] H. L. Bray, Proof of the Riemannian Penrose inequality using the positive mass theorem, J. Differential Geom. 59 (2001), no. 2, 177-267.
[4] H. L. Bray, On the positive mass, Penrose, an ZAS inequalities in general dimension, Surveys in Geometric Analysis and Relativity, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, (2011).
[5] H.L. Bray and K. Iga, Superharmonic functions in Rn and the Penrose inequality in general relativity, Comm. Anal. Geom. 10 (2002), no. 5, 999–1016.
[6] H.L. Bray and J.L. Jauregui, A geometric theory of zero area singularities in general relativity, (2009), arXiv0909.0522.
[7] H. L. Bray and D. A. Lee, On the Riemannian Penrose inequality in dimensions less than eight, Duke Math. J. 148 (2009), no. 1, 81–106.
[8] H. L. Bray, P. Miao, On the capacity of surfaces in manifolds with nonnegative scalar curvature, Invent. Math. 172 (2008), no. 3, 459–475.
[9] D.G. Boulware, S Deser, Physical Review Letters, 55 (1985) 2656
[10] R.G. Cai, N. Ohta, Black holes in pure Lovelock gravities, Phys. Rev. D 74 (2006), 064001.
[11] S.-Y. A. Chang and F. Hang, Integral identities and Minkowski type inequalities involving schouten tensor, Int. Math. Res. Not. IMRN 2007, no. 11, Art. ID rnm037, 17 pp.
[12] S.-Y. A. Chang and Y.Wang, On Aleksandrov-Fenchel inequalities for k-convex domains, Milan J. Math., 79 (2011), no. 1, 13–38.
[13] S. S. Chern, A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds, Ann. of Math. (2) 45 (1944), 747–752.
[14] S. S. Chern, On the curvatura integra in a Riemannian manifold, Ann. of Math. (2) 46 (1945). 674–684.
[15] J. Crisóstomo, R. Troncoso, J. Zanelli, Black hole scan, Phys. Rev. D, (3) 62 (2000), no.8, 084013.
[16] L. L. de Lima and F. Girão, The ADM mass of asymptotically flat hypersurfaces, arXiv:1105.5474 to appear in Trans. AMS.
[17] L. L. de Lima and F. Girão, A rigidity result for the graph case of the Penrose inequality, arXiv:1205.1132 (2012).
[18] A. Freire and F. Schwartz, Masscapacity inequalities for conformally flat manifolds with boundary, arXiv:1107.1407.
[19] L. Garding, An inequality for hyperbolic polynomials, J. Math. Mech. 8 (1959), 957-965.
[20] Y. Ge, G. Wang and J. Wu, A new mass for asymptotically flat manifolds, arXiv:1211.3645.
[21] Y. Ge, G. Wang and J. Wu, A positive mass theorem in the Einstein-Gauss-Bonnet theory, arXiv:1211.7305.
[22] P. Guan, Curvature measures, isoperimetric type inequalities and fully nonlinear PDEs, Lecture notes.
[23] P. Guan and J. Li, The quermassintegral inequalities for k-convex starshaped domains, Adv. Math. 221 (2009), no. 5, 1725–1732.
[24] G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, J. Differential Geom. 59 (2001), no. 3, 353–437.
[25] L.-H. Huang and D. Wu, Hypersurfaces with nonnegative scalar curvature, arXiv:1102.5749.
[26] L.-H. Huang and D. Wu, The equality case of the Penrose inequality for asymptotically flat graphs, arXiv:1205.2061.
[27] J. Jauregui, Penrose-type inequalities with a Euclidean background, arXiv:1108.4042 (2011).
[28] M.-K. G. Lam, The graph cases of the Riemannian positive mass and Penrose inequality in all dimensions, arXiv:1012.4256.
[29] C. Lanczos, A remarkable property of the Riemann-Christoffel tensor in four dimensions Ann. of Math. (2) 39 (1938), no. 4, 842–850.
[30] Y. Li and L. Nguyen, A generalized mass involving higher order symmetric function of the curvature tensor, arXiv:1211.3676.

[31] J. Lohkamp, Scalar curvature and hammocks, Math. Ann. 313 (1999), no. 3, 385–407.

[32] J. Lohkamp, The Higher Dimensional Positive Mass Theorem I, arXiv:0608795v1.

[33] D. Lovelock, The Einstein tensor and its generalizations J. Math. Phys. 12 (1971) 498–501.

[34] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge University Press, Cambridge, (1993).

[35] R. Schoen and S. T. Yau, On the proof of the positive mass conjecture in general relativity, Comm. Math. Phys. 65 (1979), 45–76.

[36] F. Schwartz, A volumetric Penrose inequality for conformally flat manifolds, Ann. Henri Poincaré, 12 (2011), 67–76.

[37] J. Viaclovsky, Conformal geometry, contact geometry and the calculus of variations, Duke J. Math. 101 (2000), no. 2, 283–316.

[38] J. T. Wheeler, Symmetric Solutions to the Maximally Gauss-Bonnet Extended Einstein Equations, Nucl. Phys., B 268 (1986) 737.

[39] E. Witten, A new proof of the Positive Energy Theorem, Commun. Math. Phys. 80 (1981), 381–402.

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