FINITE PRESENTATION, THE LOCAL LIFTING PROPERTY, AND LOCAL APPROXIMATION PROPERTIES OF OPERATOR MODULES

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Abstract. We introduce notions of finite presentation and co-exactness which serve as qualitative and quantitative analogues of finite-dimensionality for operator modules over completely contractive Banach algebras. With these notions we begin the development of a local theory of operator modules by introducing analogues of the local lifting property, nuclearity, and semi-discreteness. For a large class of operator modules we prove that the local lifting property is equivalent to flatness, generalizing the operator space result of Kye and Ruan [30]. We pursue applications to abstract harmonic analysis, where, for a locally compact quantum group $G$, we show that $L^1(G)$-nuclearity of $LUC(G)$ and $L^1(G)$-semi-discreteness of $L^\infty(\hat{G})$ are both equivalent to co-amenability of $G$. We establish the equivalence between $A(G)$-injectivity of $G\hat{\rtimes} M$, $A(G)$-semi-discreteness of $G\hat{\rtimes} M$, and amenability of $W^*$-dynamical systems $(M, G, \alpha)$ with $M$ injective. We end with remarks on future directions.

1. Introduction

The local theory of operator spaces, like its Banach space counterpart, aims to study global properties of a space through the structure of its finite-dimensional subspaces. In this context, local properties often translate into homological properties of functors on associated categories. For instance, exactness of a $C^*$-algebra $A$, defined by exactness of the functor $A \otimes (-)$, is equivalent to approximate factorizations of an inclusion $A \subseteq B(H)$ through full matrix algebras. In fact, some of the deepest theorems in operator algebras concern the equivalence between local and homological characterizations of a certain property. The equivalence of injectivity and approximate finite-dimensionality for von Neumann algebras [11] being a prominent example.

In the setting of operator modules, the author’s recent work [12–15] characterizes important properties of a locally compact quantum group $G$ in terms of homological properties of various operator modules over the convolution algebra $L^1(G)$ (or its dual). For instance, $G$ is amenable if and only if the dual von Neumann algebra $L^\infty(\hat{G})$ is injective over $L^1(\hat{G})$ [12, Theorem 5.1]. A natural question is whether one can characterize this quantum group injectivity in terms of a “local property” inside a relevant module category, analogous to semi-discreteness. Having local characterizations of such homological properties should prove beneficial for the future development of harmonic analysis on locally compact quantum groups.

Motivated by the above (and related) question(s), we introduce a suitable generalization of finite-dimensionality to the context of operator modules through a notion of (topological) finite presentation. Analogous to the purely algebraic setting, we define (topological)
finite presentation through (topologically) exact sequences of matricially free modules, a notion we also introduce. After establishing basic properties and examples, we show, among other things, that every operator module in $A\text{mod}$ is a direct limit of topologically finitely presented ones.

Our notion of matricial freeness also leads naturally to an operator module analogue of co-exactness [44, §4.5], which measures the completely bounded Banach–Mazur distance from a module to a quotient of a finitely generated matrically free module. For a large class of finitely presented modules $E \in A\text{mod}$, we show that $E$ is $\lambda$-co-exact if and only if for any family $(X_i)_{i \in I}$ in $\text{mod} A$, and any free ultrafilter $\mathcal{U}$ on $I$,

$$\left( \prod_{i \in I} X_i/\mathcal{U} \right) \hat{\otimes}_A E \cong_{\lambda} \prod_{i \in I} (X_i \hat{\otimes}_A E)/\mathcal{U},$$

where $\hat{\otimes}_A$ is the projective module tensor product (see Theorem 5.6). This is an operator module generalization of Dong’s characterization of co-exactness [17, Theorem 2.2], which itself is a dual analogue of Pisier’s ultraproduct characterization of exactness [38, Proposition 6].

Through finite presentation we introduce an operator module analogue of the local lifting property. We prove, for a large class of completely contractive Banach algebras $A$, that flatness of an operator module $X \in A\text{mod}$ is equivalent to the local lifting property (see Theorem 6.6). This generalizes the corresponding result for operator spaces by Kye and Ruan [30, Theorem 5.5]. Our class includes any $A$ which is the predual of a von Neumann algebra, e.g. $A = L^1(\mathbb{G})$. Thus, as a corollary, a locally compact quantum group $\mathbb{G}$ is amenable if and only if $L^1(\hat{\mathbb{G}})$ has the local lifting property in $L^1(\mathbb{G})\text{mod}$ (see Corollary 6.9), a result which is new even for groups.

We then initiate an investigation into local approximation properties for operator modules, introducing analogues of nuclearity and semi-discreteness. Any nuclear module necessarily has the weak expectation property in the sense of [6], and any semi-discrete module is necessarily injective. Specializing to the setting of locally compact quantum groups, we establish:

**Theorem 1.1.** Let $\mathbb{G}$ be a locally compact quantum group. The following conditions are equivalent:

1. $\mathbb{G}$ is co-amenable;
2. $L^\infty(\mathbb{G})$ is semi-discrete in $\text{mod} L^1(\mathbb{G})_1$;
3. $L^\infty(\mathbb{G})$ is semi-discrete in $L^1(\mathbb{G})\text{mod}_1$;
4. $\text{LUC}(\mathbb{G}) = \{ L^\infty(\mathbb{G}) \ast L^1(\mathbb{G}) \}$ is nuclear in $L^1(\mathbb{G})\text{mod}_1$;
5. $\text{RUC}(\mathbb{G}) = \{ L^1(\mathbb{G}) \ast L^\infty(\mathbb{G}) \}$ is nuclear in $\text{mod} L^1(\mathbb{G})_1$;
6. The inclusion $C_0(\mathbb{G}) \subset C_0(\mathbb{G})^{**}$ is nuclear in $\text{mod} L^1(\mathbb{G})_1$;
7. The inclusion $C_0(\mathbb{G}) \subset C_0(\mathbb{G})^{**}$ is nuclear in $L^1(\mathbb{G})\text{mod}_1$.

In the setting of $W^*$-dynamical systems $(M, G, \alpha)$ with $M$ is injective (as a von Neumann algebra), we show the equivalence between $A(G)$-injectivity of $G \bar{\rtimes} M$, $A(G)$-semi-discreteness of $G \bar{\rtimes} M$, and amenability of $(M, G, \alpha)$, where $A(G)$ is the Fourier algebra of $G$ acting on $G \bar{\rtimes} M$ through the dual co-action.

The paper is structured as follows. Section 2 contains the necessary preliminaries from the homology of operator modules and the theory of locally compact quantum groups. Section 3 contains the definition and existence of matricially free modules. Section 4 is devoted
to our notion of (topological) finite presentation, its basic properties and examples. Section 5 concerns our generalization of co-exactness, and the aforementioned ultraproduct characterization. Section 6 introduces the local lifting property (LLP) for modules and establishes the equivalence with flatness. Section 7 contains the definitions of nuclearity and semi-discreteness, along with basic properties and examples arising from abstract harmonic analysis. Section 8 contains the proof of the equivalence between $A(G)$-injectivity and $A(G)$-semi-discreteness for crossed products of $W^*$-dynamical systems.

Several natural lines of investigation are suggested by this work. We therefore finish with concluding remarks on future directions.

2. Preliminaries

2.1. Categorical Notions in $A\text{mod}$. Throughout this paper, unless otherwise specified, $A$ is a completely contractive Banach algebra. An operator space $X$ is a left operator $A$-module if it is a left Banach $A$-module such that the module map $m_X : A \hat{\otimes} X \to X$ is completely contractive, where $\hat{\otimes}$ denotes the operator space projective tensor product. We say that $X$ is faithful if for every non-zero $x \in X$, there is $a \in A$ such that $a \cdot x \neq 0$, and we say that $X$ is essential if $\langle A \cdot X \rangle = X$, where $\langle \cdot \rangle$ denotes the closed linear span. We denote by $A\text{mod}$ (respectively, $A\text{mod}_1$) the category of left operator $A$-modules with morphisms given by completely bounded (respectively, contractive) module homomorphisms. Given $X,Y$ in $A\text{mod}$, we write the space of morphisms from $X$ to $Y$ as $\text{Hom}(X,Y) := CB(A,X,Y)$. Right operator $A$-modules are defined similarly, and we denote the resulting categories by $\text{mod} A$ and $\text{mod}_1 A$. In $\text{mod} A$ we have $\text{Hom}(X,Y) = CB(X,Y)_A$. When $A = \mathbb{C}$ we recover the category $\text{Op}$ of operator spaces and completely bounded mappings. If $X \in A\text{mod}$ or $\text{mod} A$ is a dual operator space such that the action of each $a \in A$ is weak* continuous, we say that $X$ is a dual operator $A$-module.

We often restrict attention to $A\text{mod}$, the corresponding notions in $\text{mod} A$ naturally carrying over.

The following terminology will be used throughout the paper. A morphism $\varphi : X \to Y$ in $A\text{mod}$ is:

- a complete $\lambda$-embedding, often denoted $\hookrightarrow_\lambda$, if $||[x_{i,j}]|| \leq \lambda ||\varphi(x_{i,j})||$ for all $[x_{i,j}] \in M_n(X), n \in \mathbb{N}$;
- a complete $\lambda$-quotient map, often denoted $\twoheadrightarrow_\lambda$, if it maps $M_n(X)_{\|\| < \lambda}$ onto $M_n(Y)_{\|\| < 1}$ for every $n \in \mathbb{N}$;
- a $\lambda$-strict morphism if $\varphi(X)$ is closed and the induced map $\overline{\varphi} : X/\text{Ker}(\varphi) \to \varphi(X)$ satisfies $\|\overline{\varphi}^{-1}\|_{cb} \leq \lambda$.

Let $X \in \text{mod} A$ and $Y \in A\text{mod}$. The $A$-module tensor product of $X$ and $Y$ is the quotient space $X \hat{\otimes}_A Y := X \hat{\otimes} Y/N$, where

$$N = \langle x \cdot a \otimes y - x \otimes a \cdot y \mid x \in X, y \in Y, a \in A \rangle,$$

and, again, $\langle \cdot \rangle$ denotes the closed linear span. It follows that

$$CB(X,Y^*)_A \cong N^\perp \cong (X \hat{\otimes}_A Y)^* \cong CB_A(Y,X^*).$$

The identification $A_+ = A \oplus_1 \mathbb{C}$ turns the unitization of $A$ into a unital completely contractive Banach algebra, and any $X \in A\text{mod}$ becomes an operator $A_+$-module via the extended action

$$(a, \lambda) \cdot x = a \cdot x + \lambda x, \quad a \in A, \lambda \in \mathbb{C}, x \in X.$$
Let $X \in \text{Op}$, and let $\mathcal{F} : \text{A mod} \to \text{Op}$ denote the forgetful functor. For $Y$ in $\text{A mod}$, the map

$$\text{Hom}(A_+ \hat{\otimes} X, Y) \ni \varphi \mapsto \varphi|_{\hat{\otimes} X} \in \mathcal{CB}(X, \mathcal{F}(Y))$$

is a complete contraction with completely contractive inverse

$$\mathcal{CB}(X, \mathcal{F}(Y)) \ni \psi \mapsto m_Y^+ \circ (\text{id}_{A_+} \otimes \psi) \in \text{Hom}(A_+ \hat{\otimes} X, Y).$$

Thus, $\text{Hom}(A_+ \hat{\otimes} X, Y) \cong \mathcal{CB}(X, \mathcal{F}(Y))$ completely isometrically in $\text{Op}$.

Let $\lambda \geq 1$. $X \in \text{A mod}$ is $\lambda$-projective if for every $Y, Z \in \text{A mod}$, every complete quotient morphism $q : Y \to Z$, every morphism $\varphi : X \to Z$, and every $\varepsilon > 0$, there exists a morphism $\tilde{\varphi}_\varepsilon : X \to Y$ such that $\|\tilde{\varphi}_\varepsilon\|_{cb} < \lambda \|\varphi\|_{cb} + \varepsilon$ and $q \circ \tilde{\varphi}_\varepsilon = \varphi$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
Y & \xrightarrow{q} & Z \\
\downarrow \tilde{\varphi}_\varepsilon & & \\
X & \xrightarrow{\varphi} & Z
\end{array}$$

For example, $A_+ \hat{\otimes} T_n$ is $1$-projective for any $n \in \mathbb{N}$, where $T_n$ is the space of $n \times n$ trace class operators.

For any $X \in \text{A mod}$, there is a canonical completely isometric morphism $\Delta_X^+ : X \hookrightarrow \mathcal{CB}(A_+, X)$ given by

$$\Delta_X^+(x)(a) = a \cdot x, \quad x \in X, \ a \in A_+,$$

where the left $A$-module structure on $\mathcal{CB}(A_+, X)$ is defined by

$$(a \cdot \varphi)(b) = \varphi(ba), \quad a \in A, \ \varphi \in \mathcal{CB}(A_+, X), \ b \in A_+.$$

Given $\lambda \geq 1$, $X \in \text{A mod}$ is $\lambda$-injective if for every $Y, Z \in \text{A mod}$, every completely isometric morphism $i : Y \hookrightarrow Z$, and every morphism $\varphi : Y \to X$, there exists a morphism $\tilde{\varphi} : Z \to X$ such that $\|\tilde{\varphi}\|_{cb} \leq \lambda \|\varphi\|_{cb}$ and $\tilde{\varphi} \circ i = \varphi$, that is, the following diagram commutes:

$$\begin{array}{ccc}
Z & \xrightarrow{i} & Y \\
\uparrow \tilde{\varphi} & & \\
X & \xrightarrow{\varphi} & X
\end{array}$$

A module $X \in \text{A mod}$ $\lambda$-flat if its dual $X^*$ is $\lambda$-injective in $\text{mod} A$ with respect to the canonical module structure given by

$$\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle, \quad f \in X^*, \ x \in X, \ a \in A.$$ 

It is well-known that $X \in \text{A mod}$ is $\lambda$-flat if and only if for every 1-exact sequence

$$0 \to Y \hookrightarrow Z \to Z/Y \to 0,$$

in $\text{mod} A$ the sequence

$$0 \to Y \hat{\otimes} A X \hookrightarrow \lambda Z \hat{\otimes} A X \to Z/Y \hat{\otimes} A X \to 0$$

is exact, and the second arrow is a complete $\lambda$-embedding. For example, $A_+ \hat{\otimes} \mathcal{T}(H)$ is 1-flat for any Hilbert space $H$.

Limits and co-limits in $\text{A mod}$ are defined analogously to the Banach space setting (see [10, §I.1], for example). Given a family $(X_i)_{i \in I}$ in $\text{A mod}$, their product $\prod_{i \in I} X_i$ is the $\ell^\infty$
direct sum of operator spaces, with diagonal module structure:

\[ a \cdot (x_i) = (a \cdot x_i), \quad (x_i) \in \prod_{i \in I} X_i, \quad a \in A. \]

It is defined uniquely through the following universal property: given \( Y \in A_{\text{mod}} \) and a uniformly completely bounded family of morphisms \( \varphi : X \to Y \), with \( \| \varphi \|_{cb} \leq \lambda \), there is a unique morphism \( \varphi : Y \to \prod_{i \in I} X_i \) with \( \| \varphi \|_{cb} \leq \lambda \) such that \( \pi_i \circ \varphi = \varphi_i \) for all \( i \), where \( \pi_i \) is the canonical projection onto the \( X_i \) factor.

The coproduct \( \bigoplus_{i \in I} X_i \) of \( (X_i)_{i \in I} \) is the \( \ell^1 \) direct sum of operator spaces, with diagonal module structure:

\[ a \cdot (x_i) = (a \cdot x_i), \quad (x_i) \in \bigoplus_{i \in I} X_i, \quad a \in A. \]

It is defined uniquely through the following universal property: given \( Y \in A_{\text{mod}} \) and a uniformly completely bounded family of morphisms \( \varphi_i : X_i \to Y \) with \( \| \varphi_i \|_{cb} \leq \lambda \) there is a unique morphism \( \varphi : \sum_{i \in I} X_i \to Y \) such that \( \| \varphi \|_{cb} \leq \lambda \) and \( \varphi \circ \iota_i = \varphi_i \) for all \( i \), where \( \iota_i \) is the canonical inclusion of the \( X_i \) factor.

An inductive system \( (X_i, \varphi_{j,i})_{i,j \in I} \) in \( A_{\text{mod}} \) is a directed family \( (X_i)_{i \in I} \) of modules together with a family of morphisms \( \varphi_{j,i} : X_i \to X_j \) for \( i \leq j \), satisfying \( \varphi_{i,i} = \text{id}_{X_i} \) and \( \varphi_{k,j} \circ \varphi_{j,i} = \varphi_{k,i} \) whenever \( k \geq j \geq i \). An operator module \( X \in A_{\text{mod}} \) together with a family of morphisms \( \varphi_i : X_i \to X \) in \( A_{\text{mod}} \) is a direct limit of the inductive system \( ((X_i), \varphi_{j,i}) \), if the universal property, illustrated by

\[
\begin{array}{ccc}
  X_i & \xrightarrow{\varphi} & X \\
    \downarrow \varphi_{j,i} & & \downarrow \psi \\
  X_j & \xrightarrow{\varphi_j} & Y
\end{array}
\]

is satisfied. That is, for every \( Y \in A_{\text{mod}} \) and uniformly completely bounded family of morphisms \( \psi_i : X_i \to Y \) with \( \| \psi_i \|_{cb} \leq \lambda \) making the above diagrams commute, there exists a unique morphism \( \psi : \prod_{i \in I} X_i \to Y \) such that \( \| \psi \|_{cb} \leq \lambda \) and \( \psi \circ \varphi_i = \psi_i \) for each \( i \). We denote \( X \) by \( \lim_i X_i \). Analogous to the Banach space setting [10], it coincides with the quotient of \( \bigoplus_{i \in I} X_i \) by the closed submodule generated by \( \varphi_i(x_i) - \varphi_j(\varphi_{j,i}(x_i)) \), for all \( x_i \in X_i \), and \( i \leq j \). We can also identify \( \lim_i X_i \) with the subspace \( \pi_{\infty}(X_{\infty}) \) of the asymptotic product \( \prod_{i \in I} X_i / \sum_{i \in I} X_i \), where \( \sum_{i \in I} X_i \) is the \( c_0 \)-direct sum,

\[ \pi_{\infty} : \lim_{i \in I} X_i \to \prod_{i \in I} X_i / \sum_{i \in I} X_i \]

is the canonical quotient map, and \( X_{\infty} \) is the submodule

\[ \{ (x_i) \in \prod_i X_i \mid \exists i_0 x_j = \varphi_{j,i}(x_i), \ j \geq i \geq i_0 \}. \]

For sequential inductive systems in \( \text{Op} \) with completely isometric connecting maps, this fact is proven in [20, §1.3]. Although well-known in the \( C^* \)-context, we include a proof for completeness.
Proposition 2.1. Let \((X_i, \varphi_{j,i})_{i,j \in I}\) be an inductive system in \(\text{A mod}_1\). Then \(\lim_{\pi} X_i \cong \pi_\infty(X_\infty)\), completely isometrically.

Proof. The standard argument shows that

\[
\| (\pi_\infty)_n (x_i) \| = \limsup_i \| x_i \|, \quad (x_i) \in M_n \left( \prod_{i \in I} X_i \right), \quad n \in \mathbb{N}.
\]

(See [20, Proposition A.6.1] for the case of sequences of Banach spaces.)

For each \(i \in I\), let \(\varphi_i : X_i \to \pi_\infty(X_\infty)\) be given by \(\varphi_i(x_i) = \varphi_{j,i}(x_i)\), if \(j \geq i\) and \(\varphi_i(x_i) = 0\) otherwise. Clearly, \((\varphi_i(x_i)) \in X_\infty\) as for every \(k \geq j \geq i\),

\[
\| \varphi_i(x_i) \| = \| \pi_\infty((\varphi_i(x_i))) \| = \limsup \| \varphi_i(x_i) \| = \limsup \| \varphi_{j,i}(x_i) \| \leq \| x_i \|.
\]

The same argument shows that \(\varphi_i\) is completely contractive, and thus a morphism in \(\text{A mod}_1\).

We show that \(\pi_\infty(X_\infty)\) together with the family \((\varphi_i)\) satisfies the universal property of the direct limit in \(\text{A mod}_1\). To that end, let \(Y \in \text{A mod}_1\) and \(\psi : X \to Y\) be a family of morphisms in \(\text{A mod}_1\) for which \(\psi_j \circ \varphi_{j,i} = \psi_i\) for all \(i \leq j\). First define \(\psi : X_\infty \to Y\) by \(\psi((x_i)) = \psi_{i_0}(x_{i_0})\). This definition is independent of the \(i_0\) chosen: if \(j_0 \in I\) such that \(x_j = \varphi_{j,i}(x_i)\) for all \(j \geq i \geq j_0\), pick \(k_0 \geq i_0, j_0\). Then

\[
\psi_{i_0}(x_{i_0}) = \psi_{k_0}(\varphi_{k_0,i_0}(x_{i_0})) = \psi_{k_0}(x_{k_0}) = \psi_{k_0}(\varphi_{k_0,j_0}(x_{j_0})) = \psi_{j_0}(x_{j_0}).
\]

Also, if \(x = (x_i)\) and \(y = (y_i)\) belong to \(X_\infty\) and \(\pi_\infty(x) = \pi_\infty(y)\), then for every \(\varepsilon > 0\), there exists \(i_\varepsilon\) such that \(\| x_i - y_i \| < \varepsilon\) for \(i \geq i_\varepsilon\). Pick \(i_0 \geq i_\varepsilon\) for which \(\varphi_{j,i}(x_i) = x_j\) and \(\varphi_{j,i}(y_i) = y_j\) for all \(j \geq i \geq i_0\). Then

\[
\| \psi(x) - \psi(y) \| = \| \psi_{i_0}(x_{i_0}) - \psi_{i_0}(y_{i_0}) \| \leq \| x_{i_0} - y_{i_0} \| < \varepsilon.
\]

Hence, \(\psi(x) = \psi(y)\), so \(\psi\) induces a well defined \(A\)-module map \(\tilde{\psi} : \pi_\infty(X_\infty) \to Y\). Finally,

\[
\| \tilde{\psi}(\pi_\infty((x_i))) \| = \| \psi_{i_0}(x_{i_0}) \| \leq \sup_{i \geq i_0} \| x_i \|,
\]

for any \(i_0\) satisfying \(x_j = \varphi_{j,i}(x_i)\), \(j \geq i \geq i_0\). Hence,

\[
\| \tilde{\psi}(\pi_\infty((x_i))) \| \leq \limsup_i \| x_i \| = \| \pi_\infty((x_i)) \|.
\]

A similar argument shows that \(\tilde{\psi}\) is completely contractive. Hence \(\tilde{\psi}\) extends to a morphism \(\pi_\infty(X_\infty) \to Y\) in \(\text{A mod}_1\) satisfying

\[
\tilde{\psi}(\varphi_i(x_i)) = \psi_j(\varphi_j(x_j)) = \psi_j(\varphi_{j,i}(x_i)) = \psi_i(x_i).
\]

Uniqueness is obvious. \(\square\)

We also require the following standard result. The categorical Banach space argument from [10, 1.86] generalizes verbatim as the universal property of \(\hat{\otimes}_A\) implies that

\[
\text{CB}(Y \hat{\otimes}_A X, Z) \cong \varphi \mapsto (x \mapsto (y \mapsto \varphi(y \otimes_A x)) \in \text{Hom}(X, \text{CB}(Y, Z)),
\]

is a completely isometric isomorphism for any \(X \in \text{A mod}_1\), \(Y \in \text{mod} A_1\) and \(Z \in \text{Op}_1\).

Proposition 2.2. Let \((X_i, \varphi_{j,i})_{i,j \in I}\) be an inductive system in \(\text{A mod}_1\). Then for any \(Y \in \text{mod} A_1\), \(X \hat{\otimes}_A \lim_{\pi} X_i \cong \lim_{\pi} (Y \hat{\otimes}_A X_i)\) completely isometrically.
A sequence

\[ X \longrightarrow Y \longrightarrow Z \]

is \( A \text{ mod} \) is (topologically) exact if the image of the first morphism is (norm dense in) the kernel of the second. We require the following lemma concerning topologically exact sequences.

**Lemma 2.3.** Let \( A \) be a completely contractive Banach algebra, and let

\[ X \xrightarrow{\psi} Y \xrightarrow{q} Z \]

be a topologically exact sequence in \( A \text{ mod} \) with \( q \) a complete \( \lambda \)-quotient map for some \( \lambda \geq 1 \). Then for any \( W \in A \text{ mod} \), the sequence

\[ \text{Hom}(Z, W) \longrightarrow \text{Hom}(Y, W) \longrightarrow \text{Hom}(X, W) \]

is exact in \( \text{Op} \), and for any \( V \in \text{mod} A \), the sequence

\[ V \hat{\otimes}_A X \longrightarrow V \hat{\otimes}_A Y \longrightarrow V \hat{\otimes}_A Z \]

is topologically exact in \( \text{Op} \).

**Proof.** Let \( \tilde{q} : Y/\psi(X) \cong_{cb} Z \) be the induced complete isomorphism. Any morphism

\[ \varphi \in \text{Ker}(\text{Hom}(Y, W) \to \text{Hom}(X, W)) \]

satisfies \( \varphi \circ \psi = 0 \). Thus, \( \varphi \) induces a morphism \( \tilde{\varphi} \in \text{Hom}(Y/\psi(X), W) \). Then \( \tilde{\varphi} \circ \tilde{q}^{-1} \in \text{Hom}(Z, W) \) and \( \tilde{\varphi} \circ \tilde{q}^{-1}(q(y)) = \varphi(y) \) for all \( y \in Y \), meaning \( \varphi \in \text{Im}(\text{Hom}(Z, W) \to \text{Hom}(Y, W)) \). It follows that (2) is exact. Owing to the fact that a sequence in \( \text{Op} \) is topologically exact whenever its dual sequence is topologically \( w^* \)-exact (by the Hahn-Banach theorem), (3) is topologically exact by exactness of (2) with \( W = V^* \). \( \square \)

2.2. **Locally Compact Quantum Groups.** A locally compact quantum group is a quadruple \( \mathbb{G} = (L^\infty(\mathbb{G}), \Gamma, h_L, h_R) \), where \( L^\infty(\mathbb{G}) \) is a Hopf-von Neumann algebra with co-product \( \Gamma : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \hat{\otimes} L^\infty(\mathbb{G}) \), and \( h_L \) and \( h_R \) are fixed left and right Haar weights on \( L^\infty(\mathbb{G}) \), respectively [28, 29].

For every locally compact quantum group \( \mathbb{G} \), there exists a left fundamental unitary operator \( W \) on \( L^2(\mathbb{G}, h_L) \otimes L^2(\mathbb{G}, h_L) \) and a right fundamental unitary operator \( V \) on \( L^2(\mathbb{G}, h_R) \otimes L^2(\mathbb{G}, h_R) \) implementing the co-product \( \Gamma \) via

\[ \Gamma(x) = W^*(1 \otimes x)W = V(x \otimes 1)V^*, \quad x \in L^\infty(\mathbb{G}). \]

Both unitaries satisfy the pentagonal relation; that is,

\[ W_{12}W_{13}W_{23} = W_{23}W_{12} \quad \text{and} \quad V_{12}V_{13}V_{23} = V_{23}V_{12}. \]

For simplicity we write \( L^2(\mathbb{G}) \) for \( L^2(\mathbb{G}, h_L) \) throughout the paper.

Let \( L^1(\mathbb{G}) \) denote the predual of \( L^\infty(\mathbb{G}) \). Then the pre-adjoint of \( \Gamma \) induces an associative completely contractive multiplication on \( L^1(\mathbb{G}) \), defined by

\[ \ast : L^1(\mathbb{G}) \hat{\otimes} L^1(\mathbb{G}) \ni f \otimes g \mapsto f \ast g = \Gamma_*(f \otimes g) \in L^1(\mathbb{G}). \]

The canonical \( L^1(\mathbb{G}) \)-bimodule structure on \( L^\infty(\mathbb{G}) \) is given by

\[ f \ast x = (\text{id} \otimes f)\Gamma(x) \quad \text{and} \quad x \ast f = (f \otimes \text{id})\Gamma(x). \]
for \( x \in L^\infty(G) \), and \( f \in L^1(G) \). A left invariant mean on \( L^\infty(G) \) is a state \( m \in L^\infty(G)^* \) satisfying
\[
\langle m, x \ast f \rangle = \langle f, 1 \rangle \langle m, x \rangle, \quad x \in L^\infty(G), \ f \in L^1(G).
\]
Right and two-sided invariant means are defined similarly. A locally compact quantum group \( G \) is amenable if there exists a left invariant mean on \( L^\infty(G) \). It is known that \( G \) is amenable if and only if there exists a right (equivalently, two-sided) invariant mean. \( G \) is co-amenable if \( L^1(G) \) has a bounded left (equivalently, right or two-sided) approximate identity [7, Theorem 3.1].

For general \( G \), the left regular representation \( \lambda : L^1(G) \to B(L^2(G)) \) is defined by
\[
\lambda(f) = (f \otimes \text{id})(W), \quad f \in L^1(G),
\]
and is an injective, completely contractive homomorphism from \( L^1(G) \) into \( B(L^2(G)) \). Then \( L^\infty(\hat{G}) := \{ \lambda(f) : f \in L^1(G) \}'' \) is the von Neumann algebra associated with the dual quantum group \( \hat{G} \). Analogously, we have the right regular representation \( \rho : L^1(G) \to B(L^2(G)) \) defined by
\[
\rho(f) = (\text{id} \otimes f)(V), \quad f \in L^1(G),
\]
which is also an injective, completely contractive homomorphism from \( L^1(G) \) into \( B(L^2(G)) \). Then \( L^\infty(\hat{G}') := \{ \rho(f) : f \in L^1(G) \}'' \) is the von Neumann algebra associated to the quantum group \( \hat{G}' \).

If \( G \) is a locally compact group, we let \( G_a = (L^\infty(G), \Gamma_a, h_L, h_R) \) denote the commutative quantum group associated with the commutative von Neumann algebra \( L^\infty(G) \), where the co-product is given by \( \Gamma_a(f)(s,t) = f(st) \), and \( h_L \) and \( h_R \) are integration with respect to a left and right Haar measure, respectively. The fundamental unitaries in this case are given by
\[
W_a \xi(s,t) = \xi(s, s^{-1} t), \quad V_a \xi(s,t) = \xi(st, t) \Delta(t)^{1/2}, \quad \xi \in L^2(G \times G).
\]
The dual \( \hat{G}_a \) of \( G_a \) is the co-commutative quantum group \( G_s = (VN(G), \Gamma_s, h) \), where \( VN(G) \) is the left group von Neumann algebra with co-product \( \Gamma_s(\lambda(t)) = \lambda(t) \otimes \lambda(t) \), and \( h \) is Haagerup’s Plancherel weight. Then \( L^1(G_a) \) is the usual group convolution algebra \( L^1(G) \), and \( L^1(G_s) \) is the Fourier algebra \( A(G) \).

For general \( G \), we let \( C_0(G) := \bar{\lambda(L^1(\hat{G}))} \) denote the reduced quantum group \( C^* \)-algebra of \( G \). The operator dual \( M(G) := C_0(G)^* \) is a completely contractive Banach algebra containing \( L^1(G) \) as a norm closed two-sided ideal via the map \( L^1(G) \ni f \mapsto f|_{C_0(G)} \in M(G) \). The co-product satisfies \( \Gamma(C_0(G)) \subseteq M(C_0(G) \otimes^\Lambda C_0(G)) \), where \( M(C_0(G) \otimes^\Lambda C_0(G)) \) is the multiplier algebra of the \( C^* \)-algebra \( C_0(G) \otimes^\Lambda C_0(G) \), and \( \otimes^\Lambda \) is the injective operator space tensor product (which coincides with the minimal \( C^* \)-tensor norm on \( C^* \)-algebras).

We let \( C_0^u(G) \) be the universal quantum group \( C^* \)-algebra of \( G \) [27], and denote the canonical surjective \( * \)-homomorphism onto \( C_0(G) \) by \( \Lambda_G : C_0^u(G) \to C_0(G) \). The space \( C_0^u(G)^\ast \) then has the structure of a unital completely contractive Banach algebra such that the map \( L^1(G) \to C_0^u(G)^* \) given by the composition of the inclusion \( L^1(G) \subseteq M(G) \) and \( \Lambda_G^* : M(G) \to C_0(G)^* \) is a completely isometric homomorphism, and it follows that \( L^1(G) \) is a norm closed two-sided ideal in \( C_0^u(G)^* \) [27, Proposition 8.3].

An element \( \tilde{b} \in L^\infty(\hat{G}') \) is a completely bounded right multiplier of \( L^1(G) \) if \( \rho(f) \tilde{b} \in \rho(L^1(G)) \) for all \( f \in L^1(G) \) and the induced map
\[
m^\tilde{b} : L^1(G) \ni f \mapsto \rho^{-1}(\rho(f) \tilde{b}) \in L^1(G)
\]
is completely bounded on $L^1(\mathbb{G})$. We let $M^r_{cb}(L^1(\mathbb{G}))$ denote the space of completely bounded right multipliers of $L^1(\mathbb{G})$, which is a completely contractive Banach algebra with respect to the norm

$$\| [b_{ij}] \|_{M_n(M^r_{cb}(L^1(\mathbb{G})))} = \| [m^r_{bij}] \|_{cb}.$$ 

Completely bounded left multipliers are defined analogously and we denote by $M^l_{cb}(L^1(\mathbb{G}))$ the corresponding completely contractive Banach algebra.

Given $b \in M^r_{cb}(L^1(\mathbb{G}))$, the adjoint $\Theta^r(b) := (m^r_b)^*$ defines a normal completely bounded right $L^1(\mathbb{G})$-module map on $L^\infty(\mathbb{G})$. When $b = \rho(f)$, for some $f \in L^1(\mathbb{G})$, the map $\Theta^r(\rho(f))$ is nothing but the convolution action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$, that is,

$$\Theta^r(\rho(f))(x) = f \ast x = (\text{id} \otimes f)\Gamma(x), \quad x \in L^\infty(\mathbb{G}).$$

Moreover, the map

$$\Theta^r : M^r_{cb}(L^1(\mathbb{G})) \cong CB^r_{L^1(\mathbb{G})}(L^\infty(\mathbb{G})) = CB_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G}))$$

induces a completely isometric isomorphism of completely contractive Banach algebras [26]. The second equality follows from [26, Proposition 4.1]. Since $L^1(\mathbb{G})$ is a right ideal in $C_0^\infty(\mathbb{G})^*$, any $\mu \in C_0^\infty(\mathbb{G})^*$ defines a completely bounded left $L^1(\mathbb{G})$-module map on $L^1(\mathbb{G})$ by right multiplication, and therefore by duality, a completely bounded multiplier in $CB^r_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$. The resulting map will be denoted $\Theta^r(\mu)$ for simplicity.

### 3. Matricially Free Modules

A notion of free operator module was introduced by Helemskii [24] following the categorical approach to free objects (see [24, Definition 2.10], [33, Chapter III]). In the case of $\text{Op}$, free objects are defined with respect to the functor

$$V \ni \text{Op} \mapsto \prod_{n \in \mathbb{N}} \text{Ball}(M_n(V)) \in \text{Set}. \tag{5}$$

Although this is a very natural notion, with the infinite product in (5) corresponding to the completely bounded nature of morphisms in $\text{Op}$, free operator spaces in this sense are always infinite dimensional, e.g., the free operator space over the singleton $I = \{i_0\}$ is $\bigoplus_{n=1}^\infty T_n$ (see [24, Theorem 5.9]).

In this section we introduce a related, but different notion of freeness for operator modules based on a matricial version of the well-studied notion from Banach space theory (see e.g., [37]). Our matricially free modules over a finite set are actually finitely generated in the usual sense.

It is well-known that the free objects of the category $\text{Ban}_1$ of Banach spaces and contractive linear maps are of the form $\ell^1(I)$ for a non-empty set $I$. Indeed, $f : I \ni i \mapsto e_i \in \text{Ball}(\ell^1(I))$ is a function such that for any $Y \in \text{Ban}_1$ and function $g : I \to \text{Ball}(Y)$, there exists a unique morphism $\varphi : X \to Y$ satisfying $\varphi \circ f = g$. Viewing $g \in \text{Ball}(\ell^\infty(I,Y))$, when $Y$ is an operator space, the natural matricial analogue of $\text{Ball}(\ell^\infty(I,Y))$ is the unit ball of $M_I(Y)$. This analogy motivates the following

**Definition 3.1.** Let $I$ be a non-empty set and $X \in A_{\text{mod}}$.

- A function $f : I \times I \to X$ is matricially bounded if $\| [f(i,j)] \|_{M_I(X)} < \infty$ and matricially contractive if $\| [f(i,j)] \|_{M_I(X)} \leq 1$.
- $X$ is matricially free over $I$ relative to the matricial contraction $f$, or simply matricially free if $I$ and $f$ are clear from context, if:
Lemma 3.2. Let $M$ be a matricially free module. The following lemma is included for completeness. For a Hilbert space $H$, let $\mathbf{Op}$ and $\mathbf{mod}$ denote the lattice of orthogonal projections on $H$.

Proposition 3.3. Let $A$ be a completely contractive Banach algebra and let $I$ be a non-empty set. Then there exists a matricially free operator module $X \in \mathbf{mod}$ over $I$, unique up to completely isometric isomorphism.

Proof. Let $X(I \times I)$ be the (algebraically) free $A_+$-module over $I \times I$, and for $n \in \mathbb{N}$, let $S_n = \{ f : I \times I \to M_n(A_+^*) \mid f$ is a matricial contraction$\}$. Note that $X(I \times I)$ is a vector space under the action of $\mathbb{C}I \subseteq A_+$. Every $f \in S_n$ possesses a unique $A_+$-module extension $\tilde{f} : X(I \times I) \to M_n(A_+^*)$. For $x \in X(I \times I)$, define

$$\|x\| = \sup\{\|\tilde{f}(x)\| \mid f \in S_n, \ n \in \mathbb{N}\}.$$ 

That $\|\cdot\|$ is indeed a norm follows exactly as in [37, Theorem 1]. Let $X$ denote the completion of the normed space $(X(I \times I), \|\cdot\|)$. By definition of the norm, $X$ becomes a Banach $A$-module and any $f \in S_n$ extends to a contractive morphism $\tilde{f} : X \to M_n(A_+^*)$. Now, given
$x = [x_{k,l}] \in M_n(X)$, define

$$||x||_n = \sup\{||\tilde{f}(x_{k,l})|| \mid f \in S_r, \ r \in \mathbb{N}\}.$$ 

Then one easily verifies that $||\cdot||_n$ yields an operator space matrix norm on $X$. Moreover, given $[a_{i,j}] \in M_n(A)$, we have

$$||[a_{i,j} \cdot x_{k,l}]||_{M_{mn}(X)} = \sup\{||\tilde{f}(a_{i,j} \cdot x_{k,l})|| \mid f \in S_r, \ m \in \mathbb{N}\}$$

$$= \sup\{||[a_{i,j} \cdot \tilde{f}(x_{k,l})]|| \mid f \in S_r, \ m \in \mathbb{N}\}$$

$$\leq ||[a_{i,j}]|| \sup\{||\tilde{f}(x_{k,l})|| \mid f \in S_r, \ m \in \mathbb{N}\}$$

$$= ||[a_{i,j}]|| ||x_{k,l}||.$$ 

Hence, $X \in A_{\text{mod}}$.

We now verify freeness. Let $f_I : I \times I \ni (i, j) \mapsto e_{i,j} \in X(I \times I) \subseteq X$ be the canonical function, let $Y \in A_{\text{mod}}$ and $g : I \times I \rightarrow Y$ be matricially contractive. We may suppose that $Y \subseteq B(H)$ for some Hilbert space $H$, in which case the composition of this inclusion and the canonical embedding yield a complete isometry $\Delta_+ : Y \rightarrow CB(A_+, B(H))$. Then

$$\text{Ad}(p) \circ \Delta_+ \circ g : I \times I \rightarrow CB(A_+, B(pH)) \cong M_{\dim(pH)}(A_+^*)$$

is a matricial contraction for any finite-rank projection $p \in \mathcal{P}(H)$. If $\tilde{g} : X(I \times I) \rightarrow Y$ denotes the unique $A_+$-module extension of $g$, then for any $x \in X(I \times I)$, Lemma 3.2 implies that

$$||\tilde{g}(x)|| = ||\Delta_+ \circ \tilde{g}(x)||_{CB} = \sup\{||\text{Ad}(p) \circ \Delta_+ \circ \tilde{g}(x)|| \mid p \in \mathcal{P}(H), \ dim(pH) < \infty\} \leq ||x||.$$ 

Hence $\tilde{g}$ extends uniquely to a contractive morphism $\tilde{g} : X \rightarrow Y$. Another application of Lemma 3.2 shows that $\tilde{g}$ is completely contractive. By construction $\tilde{g} \circ f_I = g$, and $X$ is the closed linear span of $\{f_I(i, j) \mid i, j \in I\}$. Hence $X$ is matricially free.

Suppose $X'$ is another matricially free operator module over $I$ via the matricial contraction $f : I \times I \rightarrow X'$. Then, on the one hand, $f$ extends uniquely to a completely contractive morphism $\tilde{f} : X \rightarrow X'$ such that $\tilde{f} \circ f_I = f$. On the other hand, $f_I$ extends uniquely to a completely contractive morphism $\tilde{f}_I : X' \rightarrow X$ such that $\tilde{f}_I \circ f_I = f_I$. By uniqueness of $\tilde{f}$ and density of $\{f_I(i, j) \mid i, j \in I\}$ in $X$, it follows that $\tilde{f}_I \circ \tilde{f} = \text{id}_X$. Similarly, $\tilde{f} \circ \tilde{f}_I = \text{id}_X$, so that $X \cong X'$ completely isometrically in $A_{\text{mod}}$. \hfill \Box

**Example 3.4.** When $I = [n] := \{1, 2, ..., n\}$, the matricially free operator module over $I$ is $A_+ \overline{\otimes} T_n = T_n(A_+)$. To see this, first note that the map

$[n] \times [n] \ni (i, j) \mapsto 1 \otimes e_{i,j} \in A_+ \overline{\otimes} T_n$

is a matricial contraction as

$$||[1 \otimes e_{i,j}]||_{M_n(A_+ \overline{\otimes} T_n)} = ||[e_{i,j}]||_{M_n(T_n)}$$

$$= \sup\{||[e_{i,j}]||_{M_n(M_n)} \mid [A_{k,l}] \in \text{Ball}(M_n(M_n))\}$$

$$= \sup\{\sup\{||[A_{k,l}]||_{M_n(M_n)} \mid [A_{k,l}] \in \text{Ball}(M_n(M_n))\}$$

$$= 1,$$

where the third equality uses the canonical scalar pairing

$$\langle \cdot, \cdot \rangle : T_n \times M_n \ni (\rho, a) = \sum_{i,j=1}^n \rho_{i,j} a_{i,j} \in \mathbb{C}.$$
Next, given \( Y \in \mathcal{A} \mod \) and a matricial contraction \( g : [n] \times [n] \to Y \), define

\[
(6) \quad \varphi_g : T_n(A_+) \ni [a_{k,l}] \mapsto \sum_{k,l=1}^{n} a_{k,l} \cdot g(k,l) \in Y.
\]

It follows that \( \varphi_g([a_{k,l}]) = \varphi([a_{k,l} \otimes [g(k,l)]] \), where \( \varphi : T_n(A_+) \otimes M_n(Y) \to Y \) is the complete contraction given by the composition

\[
T_n(A_+) \hat{\otimes} M_n(Y) = (A_+ \hat{\otimes} T_n) \hat{\otimes} (M_n \otimes \mathbb{C}) \to Y
\]

Thus, \( \varphi_g \) is a completely contractive morphism satisfying \( \varphi_g(1 \otimes e_{i,j}) = g(i,j) \), \( i,j \in [n] \). Uniqueness is clear. Hence, \( A_+ \hat{\otimes} T_n \) is matricially free.

When \( A \) is unital, we use \( A \) instead of its (unconditional) unitization \( A_+ \) in the above construction/example. In particular, when \( A = \mathbb{C} \), we obtain a notion of matricially free operator space, which differs from Helemskii’s construction of freeness in [24].

Our main interest in defining free operator modules is towards a notion of finite presentation. We therefore pursue this line of interest and postpone further analysis of matricial freeness to future work. There is however a natural dual notion to matricial freeness which we record here that will underlie the approximation properties in section 7.

**Definition 3.5.** Let \( A \) be a completely contractive Banach algebra. A module \( X \in \mathcal{A} \mod \) is co-free if it is of the form \( \mathcal{CB}(A_+, \mathcal{B}(H)) \) for some Hilbert space \( H \). When \( H \) is finite-dimensional we say that \( X \) is finitely co-generated. Similar definitions applies to left modules.

**4. Finitely Presented Modules**

A right module \( M \) over a unital ring \( R \) is finitely presented if and only if there is an exact sequence

\[
R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0.
\]

Based on this algebraic formulation, we define finitely presented operator modules through exact sequences of finitely generated matricially free modules.

**Definition 4.1.** Let \( A \) be a completely contractive Banach algebra. An object \( E \) in \( \mathcal{A} \mod \) is finitely presented (respectively, topologically finitely presented) if there is an exact (respectively, topologically exact) sequence in \( \mathcal{A} \mod \)

\[
(7) \quad T_n(A_+) \longrightarrow T_n(A_+) \varphi \longrightarrow E,
\]

where \( \varphi \) is a complete \( \lambda \)-quotient map for some \( \lambda \geq 1 \).

Several remarks are in order.

**Remark 4.2.**
When $A$ is unital, we replace $A^+$ with $A$ in Definition 4.1.

(2) The (topological) exactness of (7) in $A \text{mod}_1$ means that all arrows are completely contractive $A$-module maps, and that the image of the first arrow is (norm dense in) the kernel of the second.

(3) Clearly, finite presentation implies topological finite presentation. Note that the topological adjective in the latter refers to it “relations” and not its algebraic generation. A more accurate choice of terminology would be finitely generated and topologically finitely related. However, for simplicity, we stick with the chosen terminology.

(4) One is tempted to use exact sequences of the form

$T_m(A^+) \longrightarrow T_n(A^+) \longrightarrow E \longrightarrow 0$

in Definition 4.1. However, contrary to the Banach space category, a surjective morphism need not be a complete $\lambda$-quotient map, so this property would still have to be specified. We find the concise sequence (7) appropriate for our needs.

(5) It is not always possible to take $\lambda = 1$ in Definition 4.1. See section 5 for a detailed analysis.

We now present elementary examples, fixing a completely contractive Banach algebra $A$.

**Example 4.3.** Let $F$ be a finite-dimensional operator space with $n = \dim(F)$. Pick a bijective contraction $\ell_1^n \to F$, which is automatically a complete contraction when $\ell_1^n$ is equipped with its max operator space structure. Composing this complete contraction with the canonical complete quotient map $T_n \to \max\ell_1^n$, we obtain a complete contraction $\varphi : T_n \to F$, which, by finite-dimensionality, is a complete $\lambda$-quotient map for some $\lambda$ (take $\lambda = n$ for instance). Since $\hat{\otimes}$ preserves $\lambda$-quotients, it follows that $\id \otimes \varphi : A^+ \hat{\otimes} T_n \to A^+ \hat{\otimes} F$ is a $\lambda$-quotient map. The kernel $K$ of $\varphi$ is a finite-dimensional operator space, so as above one can find a complete contraction $T_m \to K$. The resulting amplification generates the exact sequence

$T_m(A^+) \longrightarrow T_n(A^+) \longrightarrow A^+ \hat{\otimes} F,$

so $A^+ \hat{\otimes} F$ is finitely presented. In particular, when $A = \mathbb{C}$, any finite dimensional operator space $F$ is finitely presented (under the convention from Remark 4.2). Since the converse is obvious, an operator space is finitely presented if and only if it is finite dimensional.

**Example 4.4.** Let $P$ be $\lambda$-projective in $A \text{mod}$ and finitely generated in the sense that there exists a complete $\mu$-quotient $q : T_n(A^+) \to P$ for some $n \in \mathbb{N}$. Then by projectivity, for every $\varepsilon > 0$ there exists $\varphi : P \to T_n(A^+)$ for which $q \circ \varphi = \id_P$ and $\|\varphi\|_{cb} < \lambda + \varepsilon$. Then $\varphi \circ q : T_n(A^+) \to T_n(A^+)$ is a projection onto $\varphi(P)$, and $\psi := \id_{T_n(A^+)} - \varphi \circ q$ is a projection onto $\Ker(\varphi \circ q) = \Ker(q)$. Normalizing $\psi$, we obtain the exact sequence in $A \text{mod}_1$:

$T_n(A^+) \overset{\psi/\|\psi\|_{cb}}{\longrightarrow} T_n(A^+) \overset{q}{\longrightarrow} P.$

Hence, $P$ is finitely presented.

The following reformulation of finite presentation is natural and will be useful in section 6.

**Proposition 4.5.** Let $A$ be a completely contractive Banach algebra. Then $E \in A \text{mod}$ is (topologically) finitely presented if and only if there exist finite-dimensional operator spaces
$F_1, F_2$ and a (topologically) exact sequence

\[ (8) \quad A_+ \otimes F_1 \overset{\varphi}{\longrightarrow} A_+ \otimes F_2 \overset{\psi}{\longrightarrow} E \]

in $A \text{mod}_1$ where $\varphi$ is a complete quotient map.

**Proof.** Suppose $E$ is topologically finitely presented. Then there exists $\lambda \geq 1$, $n, m \in \mathbb{N}$ and a topologically exact sequence

\[ A_+ \otimes T_m \overset{\psi}{\longrightarrow} A_+ \otimes T_n \overset{\varphi}{\longrightarrow} E, \]

with $\psi$ and $\varphi$ completely contractive morphisms and $\varphi$ a complete $\lambda$-quotient map. Note that $m^+_E : A_+ \otimes E \to E$ is a complete quotient map, so that $m^+_E : A_+ \otimes E/K \cong E$ completely isometrically, where $K$ is the Kernel of $m^+_E$. Let $q$ denote the quotient map $A_+ \otimes E \to A_+ \otimes E/K$.

Define $F := \varphi(1 \otimes T_n) \subseteq E$. Then $F$ is a finite-dimensional subspace of $E$. We will show that the restriction $m^+_E : A_+ \otimes F \to E$ induces a complete isometry

\[ m^+_E : A_+ \otimes F/(K \cap A_+ \otimes F) \cong E. \]

First, given $e \in E$, pick $u \in A_+ \otimes E$ such that $m^+_E(q(u)) = e$, and hence $\|e\| = \|q(u)\|$. Given $\varepsilon > 0$, by [20, Theorem 10.2.1] there exist $\alpha \in M_{1,\infty \times \infty}$, $a \in K_\infty(A_+)$, $x \in K_\infty(E)$ and $\beta \in M_{\infty \times 1}$ for which $u = \alpha(a \otimes x)\beta$ and $\|\alpha\|a\|x\|\|\beta\| < \|u\| + \varepsilon$. Without loss of generality we may assume $\|\alpha\| = \|a\| = \|\beta\| = 1$ and $\|x\| < \|u\| + \varepsilon$. Since $\varphi$ is a complete $\lambda$-quotient map, there exists $y \in K_\infty(A_+ \otimes T_n)$ such that $\varphi(y) = x$ and $\|y\| < \lambda(\|u\| + \varepsilon)$.

Writing $y = [y_{k,l}]$, for each $k, l$, we have $y_{k,l} = \sum_{r,s=1}^n b_{k,l}^{r,s} \otimes e_{r,s}$, where $e_{r,s}$ is the canonical matrix unit in $T_n$. Writing $e_{r,s}^* \in T_n^*$ for the dual vector to $e_{r,s}$, we have $b_{k,l}^{r,s} = (\text{id}_{A_+} \otimes e_{r,s}^*)(y_{k,l})$, and since $\|\text{id}_{A_+} \otimes e_{r,s}^*\|_{cb} = 1$, it follows that for each $r, s$,

\[ b_{k,l}^{r,s} = [b_{k,l}^{r,s}] = (\text{id}_{A_+} \otimes e_{r,s}^*)([y_{k,l}]) \in K_\infty(A_+) \]

with $\|[b_{k,l}^{r,s}]\| < \lambda(\|u\| + \varepsilon)$. Therefore $\alpha(a \otimes b_{k,l}^{r,s})\beta \in A_+ \otimes A_+$ for each $r, s$, implying $a_{r,s} := m_{A_+}(\alpha(a \otimes b_{k,l}^{r,s})\beta) \in A_+$ as $A_+$ is a completely contractive Banach algebra. Hence,

\[ q(u) = q\left(\sum_{i,j,k,l} \alpha_{i,k}(a_{i,j} \otimes \varphi(y_{k,l}))\beta_{j,l}\right) \]

\[ = q\left(\sum_{i,j,k,l} \alpha_{i,k}(a_{i,j} \otimes \left(\sum_{r,s=1}^n b_{k,l}^{r,s} \cdot \varphi(1 \otimes e_{r,s})\right)\beta_{j,l}\right) \]

\[ = q\left(\sum_{r,s=1}^n \sum_{i,j,k,l} \alpha_{i,k}(a_{i,j} \otimes b_{k,l}^{r,s} \cdot \varphi(1 \otimes e_{r,s}))\beta_{j,l}\right) \]

\[ = q\left(\sum_{r,s=1}^n \sum_{i,j,k,l} \alpha_{i,k}(a_{i,j} \cdot b_{k,l}^{r,s} \otimes \varphi(1 \otimes e_{r,s}))\beta_{j,l}\right) \]

\[ = q\left(\sum_{r,s=1}^n m_{A_+}(\alpha(a \otimes b_{k,l}^{r,s})\beta) \otimes \varphi(1 \otimes e_{r,s})\right) \]

\[ = q\left(\sum_{r,s=1}^n a^{r,s} \otimes \varphi(1 \otimes e_{r,s})\right) \]
where $v = \sum_{r,s=1}^n a^{r,s} \otimes \varphi(1 \otimes e_{r,s}) \in A_+ \hat{\otimes} F$. It follows that $m_+^+$ induces an isometric isomorphism $A_+ \hat{\otimes} F/(K \cap A_+ \hat{\otimes} F) \cong E$. Using the amplified representation [20, Theorem 10.2.1], a similar argument shows that $m_+^+$ induces a complete isometry $A_+ \hat{\otimes} F/(K \cap A_+ \hat{\otimes} F) \cong E$.

Next, letting $i : T_n \ni \rho \mapsto 1 \otimes \rho \in A^+ \hat{\otimes} T_2$, one easily sees that $\varphi = m_+^+ \circ (\text{id} \otimes \varphi \circ i)$ and that $(\text{id} \otimes \varphi \circ i) : A_+ \hat{\otimes} T_n \rightarrow A_+ \hat{\otimes} F$ is surjective. It follows that

$$\psi(A_+ \hat{\otimes} T_m) = \text{Ker}(\varphi) = ((\text{id} \otimes \varphi \circ i))^{-1}(A_+ \hat{\otimes} F \cap K).$$

Hence, the map $\tilde{\psi} : A_+ \hat{\otimes} T_m \rightarrow A_+ \hat{\otimes} F$ given by $(\text{id} \otimes \varphi) \circ (\text{id} \otimes i) \circ \psi$ is a completely contractive morphism mapping onto a dense subspace of $A_+ \hat{\otimes} F \cap K$. Whence,

$$A_+ \hat{\otimes} T_m \xrightarrow{\tilde{\psi}} A_+ \hat{\otimes} F \xrightarrow{m_+^+} E$$

is topologically exact in $A \text{mod}_1$ and $m_+^+$ is a complete quotient map.

Conversely, suppose we have a topologically exact sequence in $A \text{mod}_1$ of the form

$$A_+ \hat{\otimes} F_1 \xrightarrow{\psi} A_+ \hat{\otimes} F_2 \xrightarrow{\varphi} E,$$

where $\varphi$ is a complete quotient map. Taking a surjective complete contraction $T_{m_1} \rightarrow F_1$, amplifying with $\text{id}_{A_+}$, and composing with $\psi$ yields a morphism $\psi_1 : A_+ \hat{\otimes} T_{m_1} \rightarrow A_+ \hat{\otimes} F_2$ whose image is dense in $\text{Ker}(\varphi)$.

There exists a complete $\lambda$-quotient map $\varphi_1 : T_n \rightarrow F_2$ for some $\lambda$ (depending on the operator space structure of $F_2$). By definition of the projective norm, its amplification $\text{id}_{A_+} \otimes \varphi_1 : A_+ \hat{\otimes} T_n \rightarrow A_+ \hat{\otimes} F_2$ is also a complete $\lambda$-quotient map. By 1-projectivity of $A_+ \hat{\otimes} T_{m_1}$ in $A \text{mod}$, there exists a completely bounded $A$-module map $\tilde{\psi}_1 : A_+ \hat{\otimes} T_{m_1} \rightarrow A_+ \hat{\otimes} T_n$ making the following diagram commute

$$\begin{array}{ccc}
A_+ \hat{\otimes} T_{m_1} & \xrightarrow{\psi_1} & A_+ \hat{\otimes} F_2 \\
\downarrow{\text{id} \otimes \varphi_1} & & \downarrow{\psi} \\
A_+ \hat{\otimes} T_n & \xrightarrow{\tilde{\psi}_1} & A_+ \hat{\otimes} T_m \\
\end{array}$$

Then $\text{Ker}(\varphi) = \psi_1(A_+ \hat{\otimes} T_{m_1}) = (\text{id} \otimes \varphi_1)(\tilde{\psi}_1(A_+ \hat{\otimes} T_{m_1})).$ It follows that

$$\text{Ker}(\varphi \circ (\text{id} \otimes \varphi_1)) = \psi_1(A_+ \hat{\otimes} T_{m_1}) + A_+ \hat{\otimes} \text{Ker}(\varphi_1),$$

where we have used the fact that $\text{Ker}(\text{id} \otimes \varphi_1) = A_+ \hat{\otimes} \text{Ker}(\varphi_1)$ which follows from a cb-analogue of [20, Proposition 7.1.7], together with the complete $\lambda$-quotient property of $(\text{id} \otimes \varphi_1)$. By finite-dimensionality of $\text{Ker}(\varphi_1)$, there exists a complete $\mu$-quotient map $\psi_2 : T_{m_2} \rightarrow \text{Ker}(\varphi_1)$. Then $A_+ \hat{\otimes} \text{Ker}(\varphi_1) = (\text{id}_{A_+} \otimes \psi_2)(A_+ \hat{\otimes} T_{m_2})$, and composing

$$\tilde{\psi}_1 \otimes (\text{id}_{A_+} \otimes \psi_2) : A_+ \hat{\otimes} T_{m_1} \otimes_1 A_+ \hat{\otimes} T_{m_2} = A_+ \hat{\otimes} (T_{m_1} \oplus_1 T_{m_2}) \rightarrow A_+ \hat{\otimes} T_n$$

with the amplified conditional expectation $\text{id}_{A_+} \hat{\otimes} T_{m_1} \oplus_1 T_{m_2} \rightarrow \text{id}_{A_+} \hat{\otimes} (T_{m_1} \oplus_1 T_{m_2}),$ we obtain a completely bounded morphism $\tilde{\psi} : A_+ \hat{\otimes} T_m \rightarrow A_+ \hat{\otimes} T_n$ whose image is dense in $\text{Ker}(\varphi \circ (\text{id} \otimes \varphi_1))$, where $m = m_1 + m_2$. Upon normalization we may assume $\|\tilde{\psi}\|_{cb} \leq 1$, so that

$$A_+ \hat{\otimes} T_m \xrightarrow{\tilde{\psi}} A_+ \hat{\otimes} T_n \xrightarrow{\varphi \circ (\text{id} \otimes \varphi_1)} E,$$
is topologically exact in $A\text{mod}_1$ with $\varphi \circ (\text{id} \otimes \varphi_1)$ a complete $\lambda$-quotient map. Hence, $E$ is topologically finitely presented.

The equivalence for finitely presented modules follows similarly (with the closures removed). □

We now show that the category $A\text{mod}$ is locally finitely presented in the sense that any module $X \in A\text{mod}$ is a direct limit of topologically finitely presented ones, adapting the argument from [40, Lemma 5.39].

Let $X, Y \in A\text{mod}$ with $Y$ a closed submodule of $X$. Let $S$ be a set of closed submodules of $X$ partially ordered by inclusion and $T$ be a set of closed submodules of $Y$ partially ordered by inclusion. Following [40] we say that the quadruple $(X, Y, S, T)$ forms an $(X, Y)$-system if

1. $S$ and $T$ are directed sets;
2. $X = \bigcup_{S \in S} S$ and $Y = \bigcup_{T \in T} T$;
3. for each $T \in T$, there exists $S \in S$ such that $S \supseteq T$.

Any $(X, Y)$-system gives rise to a canonical directed set

$$I := I(X, Y, S, T) = \{(S, T) \in S \times T \mid S \supseteq T\},$$

where $(S_1, T_1) \leq (S_2, T_2)$ if $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$.

**Proposition 4.6.** Let $A$ be a completely contractive Banach algebra and let $(X, Y, S, T)$ be an $(X, Y)$-system in $A\text{mod}$. Then $\lim_{I}(S, T) \cong X/Y$ completely isometrically.

**Proof.** We show that $X/Y$ satisfies the universal property (1) in $A\text{mod}_1$. To this end, let $Z \in A\text{mod}_1$ and let $(f_{S,T})_{(S,T) \in I}$ be a family of morphisms $S/T \rightarrow Z$ such that the following diagram commutes for all $(S, T), (S', T') \in I:

$$
\begin{array}{ccc}
S/T & \xrightarrow{f_{S,T}} & Z \\
\downarrow & & \downarrow \\
S'/T' & \xrightarrow{f_{S',T'}} & \\
\end{array}
$$

Given $x \in \bigcup_{S \in S} S$, we have $x \in S$ for some $S \in S$. If $T \in T$, there exists $S' \in S$ such that $S' \supseteq T$. Since $S$ is directed, pick $S'' \in S$ with $S'' \supseteq S, S'$. Then $(S'', T) \in I$ with $x \in S''$. Define $\varphi(x + y) = f_{S'', T}(x + T)$. To see that $\varphi$ is well-defined, first note that the definition is independent of the chosen $f_{S'', T}$ for which $(S'', T) \in I$ and $x \in S''$ by commutativity of the above diagrams. Moreover, if $S, S' \in S$, $x \in S$ and $x' \in S'$ with $x - x' \in Y$, for every $\varepsilon > 0$, there exists $T \in T$ and $b \in T$ such that $\|(x - x') - b\|_Y < \varepsilon$. There exists $S'' \in S$ with $S'' \supseteq S, S', T$, so that $(S'', T) \in I$ and

$$\|\varphi(x + y) - \varphi(x' + y)\| = \|f_{S'', T}((x - x') + T)\| \leq \|(x - x') + T\| < \varepsilon.$$ 

By a similar argument as above it follows that $\|\varphi(x + Y)\| \leq \|x + T\|$ for all $T \in T$ such that $T \subseteq S$ for some $S \in S$ with $x \in S$. Since $S$ is directed and $Y = \bigcup_{T \in T} T$, it follows that $\|\varphi(x + Y)\| \leq \|x + Y\|$. Thus, $\varphi$ extends to a well-defined contractive morphism $X/Y \rightarrow Z$. It follows that $\varphi$ is actually completely contractive as each $f_{S,T}$ is so. Uniqueness is clear. □
**Theorem 4.7.** Let $A$ be a completely contractive Banach algebra. Every $X \in A\text{mod}_1$ is a direct limit of topologically finitely presented modules.

**Proof.** Let $m^+ : A_+ \hat{T} X \to X$ be the extended multiplication map. It follows that

$$K := \text{Ker}(m^+) = \langle a \cdot b \otimes x - a \otimes b \cdot x \mid a, b \in A_+, \ x \in X \rangle,$$

and $X = A_+ \hat{T} X/K$. For finite-dimensional subspaces $C1 \subseteq E \subseteq A_+$ and $F \subseteq X$, define $X_{E,F} = \langle E \cdot F \rangle$ and $K_{E,F} \subseteq A_+ \hat{T} X_{E,F}$ by

$$K_{E,F} = \langle a \cdot b \otimes x - a \otimes b \cdot x \mid a \in A_+, \ b \in E, \ x \in F \rangle.$$

Since $X_{E,F}$ is finite-dimensional, we may view $A_+ \hat{T} X_{E,F}$ as a closed submodule of $A_+ \hat{T} X$. The corresponding sets $\mathcal{S} := \{A_+ \hat{T} X_{E,F}\}$ and $\mathcal{T} := \{K_{E,F}\}$ are directed under the canonical partial order $(E, F) \leq (E', F')$ iff $E \subseteq E'$ and $F \subseteq F'$. Clearly $A_+ \hat{T} X = \bigcup_{E,F} A_+ \hat{T} X_{E,F}$ and $K = \bigcup_{E,F} K_{E,F}$. By construction, each $K_{E,F} \subseteq A_+ \hat{T} X_{E,F}$ so that $(A_+ \hat{T} X, K, \mathcal{S}, \mathcal{T})$ forms an $(X,Y)$-system. Hence, by Proposition 4.6

$$X = A_+ \hat{T} X/K \cong \lim_{E,F} A_+ \hat{T} X_{E,F}/K_{E,F}.$$

Next, observe that the map

$$m_{A_+} \otimes \text{id}_X - \text{id}_{A_+} \otimes m^+_X : A_+ \hat{T} A_+ \hat{T} X \to A_+ \hat{T} X$$

restricts to a completely bounded morphism $A_+ \hat{T} E \hat{T} F \to K_{E,F}$ with dense range. The normalized map has the same image, so we obtain a morphism $A_+ \hat{T} E \hat{T} F \to K_{E,F}$ in $A \text{mod}_1$ with dense range. The resulting sequence in $A \text{mod}_1$

$$A_+ \hat{T} (E \hat{T} F) \longrightarrow A_+ \hat{T} X_{E,F} \longrightarrow A_+ \hat{T} X_{E,F}/K_{E,F},$$

is topologically exact. Hence, $A_+ \hat{T} X_{E,F}/K_{E,F}$ is topologically finitely presented by Proposition 4.5.

\[\square\]

It is not clear whether one can remove the word “topologically” from the statement of Theorem 4.7, in general. The difficulty stems from the fact that morphisms from finitely presented modules need not be strict when $A$ is infinite dimensional.

**Remark 4.8.** There is a notion of finite presentation in any co-complete category, defined through the preservation of direct limits through the functor $\text{Hom}(E, (\cdot))$ (see, e.g., [1, Definition 1.1]). There is an obvious operator module analogue of this notion. One can show that any topologically finitely presented module satisfies this property, and that the notions are equivalent when $A$ is finite dimensional. We omit the (somewhat lengthy) details.

5. **Co-exact modules**

Any finite-dimensional normed space $E$ is isomorphic to a subspace of some $\ell^\infty_n$, as well as a quotient of some $\ell^1_n$. Moreover, for every $\varepsilon > 0$, there exists a subspace $S$ of some $\ell^\infty_n$ and a quotient $Q$ of $\ell^1_m$ such that $d(E, S) < 1 + \varepsilon$ and $d(E, Q) < 1 + \varepsilon$, where $d$ is the Banach–Mazur distance. The operator space analogues of these properties are false, and their failure is measured by the notions of *exactness* [38] and *co-exactness* [44, §4.5].
A finite-dimensional operator space $E$ is \( \lambda \)-exact if for every \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \) and a subspace \( S \subseteq M_n \) such that \( d_{cb}(E, S) < \lambda + \varepsilon \), where \( d_{cb} \) is the completely bounded Banach–Mazur distance. For example, \( \max \ell_1 \) and \( T_3 \) are not \( 1 \)-exact. Dually, a finite-dimensional operator space \( E \) is \( \lambda \)-co-exact if for every \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \) and a quotient \( Q \) of \( T_n \) such that \( d_{cb}(E, Q) < \lambda + \varepsilon \). Clearly, \( E \) is \( \lambda \)-co-exact if and only if \( E^* \) is \( \lambda \)-exact. It is known that \( \lambda \)-co-exactness is equivalent to the \( \lambda \)-OLLP of [35] for finite-dimensional operator spaces (see [35, Theorem 2.5]).

Pisier showed that a finite-dimensional operator space \( E \) is \( \lambda \)-exact if and only if for any family \( (X_i)_{i \in I} \) of operator spaces and any free ultrafilter \( \mathcal{U} \) on \( I \),

\[
\bigotimes_{i \in I} X_i / \mathcal{U} \cong \prod_{i \in I} (E \otimes X_i) / \mathcal{U}
\]

[38, Proposition 6]. Dually, it was shown by Dong that a finite-dimensional operator space \( E \) is \( 1 \)-co-exact if and only if for any family \( (X_i)_{i \in I} \) of operator spaces and any free ultrafilter \( \mathcal{U} \) on \( I \),

\[
E \hat{\otimes} \prod_{i \in I} X_i / \mathcal{U} \cong \prod_{i \in I} (E \hat{\otimes} X_i) / \mathcal{U}
\]

(10)

completely isometrically [17, Theorem 2.2]. The goal of this section is to show that a similar phenomena to (10) persists at the level of operator modules. We remark that Dong’s argument from [17], which factors through Pisier’s characterization of exactness and results from [35], does not readily generalize to the module setting.

**Definition 5.1.** Let \( A \) be a completely contractive Banach algebra, \( E \in A \text{ mod} \) and \( \lambda \geq 1 \). \( E \) is \( \lambda \)-co-exact if for every \( \varepsilon > 0 \) there exists \( n \in \mathbb{N} \) and a complete quotient \( Q \) of \( T_n(A_+) \) with \( d_{cb}(Q, E) < \lambda + \varepsilon \).

**Remark 5.2.** There is an analogous notion of \( \lambda \)-exact operator modules using submodules of finitely co-generated co-free modules of the form \( M_n(A^*_+) \), \( n \in \mathbb{N} \). A detailed investigation of this (and related) notion(s) will appear in forthcoming work. See the outlook section for more details.

In what follows we adopt the notation from [20, §10.3] surrounding ultraproducts. By [20, Proposition 10.3.2] it follows that for any family \( (X_i)_{i \in I} \) in \( \text{mod} A \), and any free ultrafilter \( \mathcal{U} \) on \( I \), the ultraproduct \( \prod_{i \in I} X_i / \mathcal{U} \in \text{mod} A \) via

\[
\pi_{\mathcal{U}}((x_i)) \cdot a = \pi_{\mathcal{U}}((x_i \cdot a)), \quad (x_i) \in \prod_{i \in I} X_i, \ a \in A.
\]

Given \( Y \in A \text{ mod} \), the canonical map

\[
\left( \prod_{i \in I} X_i / \mathcal{U} \right) \hat{\otimes}_A Y \ni \pi_{\mathcal{U}}((x_i)) \otimes_A y \mapsto \pi_{\mathcal{U}}(x_i \otimes_A y) \in \prod_{i \in I} (X_i \hat{\otimes}_A Y) / \mathcal{U}
\]

is completely contractive by the universal property of \( \hat{\otimes}_A \).

In preparation for the lemma below, note that any finite-dimensional operator space \( F \) is a complete quotient of \( T_\infty = \mathcal{T}(\ell^2) \). Indeed, as \( F \) is separable there exists a Banach space quotient map \( \ell^1 \to F \). Equipping \( \ell^1 \) with its max operator space structure, this map becomes a complete quotient map \( \max \ell^1 \to F \). Composing this with the canonical conditional expectation \( T_\infty \to \max \ell^1 \) gives the desired mapping.
Lemma 5.3. Let $F$ be a $d$-dimensional operator space, and let $q : T_∞ \to F$ be a complete quotient map. For any $0 < ε < 1/2$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d_{cb}(F, T_n / \text{Ker}(q_n)) \leq d^2 \frac{(1 + ε)^2}{(1 - ε)},$$

where $q_n = q|_{T_n} : T_n \to F$.

Proof. Let $\{x_1, ..., x_d\}$ be a normalized Auberach basis of $F$ with dual basis $\{x_1^*, ..., x_d^*\}$. Given $0 < ε < 1/2$, pick $y_1, ..., y_d \in (T_∞)_{\|\cdot\| < 1 + ε}$ for which $q(y_k) = x_k$, $k = 1, ..., d$. There exists $n_0 \in \mathbb{N}$ such that $\|P_n y_k P_n - y_k\| < ε/d$ for all $n \geq n_0$, where $P_n(\cdot)P_n$ is the natural compression onto the $n^{th}$ block. Define $p_n : F \to T_n$ by $p_n(x_k) = P_n y_k P_n$. Then for any $x = \sum_{k=1}^d \langle x_k^*, x \rangle x_k \in F$, we have

$$\|p_n(x)\| \leq \sum_{k=1}^d \|\langle x_k^*, x \rangle\| \|y_k\| \leq (1 + ε)d \|x\|,$$

implying $\|p_n\| \leq d(1 + ε)$. Moreover,

$$\|q(p_n(x)) - x\| \leq \sum_{k=1}^d \|\langle x_k^*, x \rangle\| \|q(p_n(x)) - x_k\| \leq \|x\| \sum_{k=1}^d \|P_n y_k P_n - y_k\| < ε \|x\|,$$

Then, as in the proof of [19, Proposition 5.3], since $ε < 1/2$,

$$\|x\| - \|q(p_n(x))\| \leq \|x - q(p_n(x))\| < \frac{1}{2} \|x\| \Rightarrow \|q(p_n(x))\| \geq \frac{1}{2} \|x\|.$$

Hence, $q \circ p_n : F \to F$ is injective and therefore bijective by finite-dimensionality of $F$. It follows that $q_n = q|_{T_n} : T_n \to F$ is surjective, and therefore $\tilde{q}_n : T_n / \text{Ker}(q_n) \simeq F$. Moreover, for each $x \in F_{\|\cdot\| = 1}$ there exists a unique $y_x \in F$ satisfying $\tilde{q}_n^{-1}(x) = p_n(y_x) + \text{Ker}(q_n)$. Then

$$\|y_x\| - 1 = \|y_x\| - \|x\| \leq \|y_x - x\| = \|y_x - q_n(p_n(y_x))\| < ε \|y_x\|,$$

so that $\|y_x\| < 1/(1 - ε)$. Thus,

$$\|\tilde{q}_n^{-1}(x)\| = \|p_n(y_x) + \text{Ker}(q_n)\| \leq \|p_n(y_x)\| \leq d(1 + ε) \frac{(1 + ε)}{(1 - ε)},$$

implying $\|\tilde{q}_n^{-1}\| \leq d(1 + ε)/(1 - ε)$. By [20, Corollary 2.2.4] we have $\|\tilde{q}_n^{-1}\|_{cb} \leq d^2(1 + ε)/(1 - ε)$. \qed

Let $I$ be a set and $U$ be a free ultrafilter on $I$. It is well-known that the ultrapower functor $\text{Ban}_I : X \mapsto X^U \in \text{Ban}_I$ is exact [9, Lemma 2.2.8]. We require a generalization of this fact in the operator module setting.

Lemma 5.4. Let $A$ be a completely contractive Banach algebra. Let $I$ be a set and $U$ be a free ultrafilter on $I$. Suppose that for each $i \in I$, the sequence

$$X_i \xrightarrow{\psi_i} Y_i \xrightarrow{\varphi_i} Z_i,$$

is exact in $A \text{mod}$ with $(\psi_i)$, and $(\varphi_i)$ uniformly completely bounded, $\psi_i$ a $λ$-strict morphism and $\varphi_i$ a $μ$-strict morphism for each $i$. Then the ultraproduct sequence

$$\prod_{i \in I} X_i/U \xrightarrow{(\psi_i)_U} \prod_{i \in I} Y_i/U \xrightarrow{(\varphi_i)_U} \prod_{i \in I} Z_i/U,$$

is exact.
Proof. If \( \pi_U(y_i) \in \text{Ker}((\varphi_i)_U) \), then \( \lim_{i \in U} \| \varphi_i(y_i) \| = 0 \). Pick a net \((i_\alpha) \subseteq I\) converging to \( U \) in the Stone-Čech compactification \( \beta I \). Then for each \( \delta > 0 \), there exists \( \alpha_0 \) such that
\[
\| \varphi_{i_\alpha}(y_{i_\alpha}) \| < \frac{\delta}{\mu}, \quad \alpha \geq \alpha_0.
\]
Since each \( \varphi_i \) is a \( \mu \)-strict morphism
\[
\| y_{i_\alpha} + \text{Ker}(\varphi_{i_\alpha}) \| \leq \mu \| \varphi_{i_\alpha}(y_{i_\alpha}) \| < \delta, \quad \alpha \geq \alpha_0.
\]
Pick \( k_{i_\alpha} \in \text{Ker}(\varphi_{i_\alpha}) \) for which \( \| y_{i_\alpha} - k_{i_\alpha} \| < \delta \). Since
\[
\text{Ker}(\varphi_{i_\alpha}) = \text{Im}(\psi_{i_\alpha}) \cong X_{i_\alpha}/\text{Ker}(\psi_{i_\alpha}),
\]
there exists \( x_{i_\alpha} \in X_{i_\alpha} \) for which \( \psi_{i_\alpha}(x_{i_\alpha}) = k_{i_\alpha} \) and
\[
\| x_{i_\alpha} + \text{Ker}(\psi_{i_\alpha}) \| \leq \lambda \| k_{i_\alpha} \| < \lambda(\delta + \sup_i \| y_i \|), \quad \alpha \geq \alpha_0.
\]
Thus, setting \( x_i = x_{i_\alpha} + h_{i_\alpha} \) for suitable elements \( h_{i_\alpha} \in \text{Ker}(\psi_{i_\alpha}) \), and setting \( x_i = 0 \) whenever \( i \neq i_\alpha \), \( \alpha \geq \alpha_0 \), we obtain a bounded family \( (x_i) \in \prod X_i \) for which
\[
\| ((\psi_i)_U(\pi_U((x_i))) - \pi_U((y_i))) \| < \delta.
\]
Since \( \delta > 0 \) was arbitrary, it follows that \( \pi_U((y_i)) \in \text{Im}((\psi_i)_U) \). \( \square \)

Definition 5.5. Let \( A \) be a completely contractive Banach algebra. A module \( E \in \text{A mod} \) is \textit{finitely generated} if there exists a complete \( \lambda \)-quotient morphism \( T_n(A_+) \to \lambda E \) for some \( n \in \mathbb{N} \) and \( \lambda \geq 1 \).

Theorem 5.6. Let \( A \) be a completely contractive Banach algebra, \( E \in \text{A mod} \) be finitely generated, and \( \lambda \geq 1 \). Consider the following conditions:

1. for any family \( (X_i)_{i \in I} \) in \( \text{mod} A \), and any free ultrafilter \( U \) on \( I \), the canonical map
   \[
   \Phi_E : \left( \prod_{i \in I} X_i/U \right) \hat{\otimes}_A E \to \prod_{i \in I} (X_i \hat{\otimes}_A E)/U
   \]
   is a complete isomorphism with \( \| \Phi_E^{-1} \|_{cb} \leq \lambda \);
2. for any family \( (X_i)_{i \in I} \) in \( \text{mod} A \), and any free ultrafilter \( U \) on \( I \), the canonical map
   \[
   \Delta : \prod_{i \in I} \text{Hom}(E, X_i)/U \to \lambda \text{Hom}(E, \prod_{i \in I} X_i/U)
   \]
   is a complete \( \lambda \)-embedding;
3. \( E \in \text{A mod} \) is \( \lambda \)-co-exact.

Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3). If, in addition, \( E \) is projective, the conditions are equivalent.

Proof. (1) \( \Rightarrow \) (2): Let \( I \) be a set, \( (X_i) \) be a family in \( \text{mod} A \) and \( U \) be a free ultrafilter on \( I \). By assumption, the canonical map
\[
\Phi_E : \left( \prod_{i \in I} X_i^*/U \right) \hat{\otimes}_A E \to \prod_{i \in I} (X_i^* \hat{\otimes}_A E)/U
\]
is a complete isomorphism with \( \| \Phi_E^{-1} \|_{cb} \leq \lambda \). Let
\[
T : \text{Hom}(E, \prod_{i \in I} X_i/U) \to \left( \prod_{i \in I} (X_i^* \hat{\otimes}_A E)/U \right)^*
\]


denote the following composition

\[ \operatorname{Hom}(E, \prod_{i \in I} X_i/\mathcal{U}) \hookrightarrow \operatorname{Hom}(E, \prod_{i \in I} X_i^{**}/\mathcal{U}) \]

\[ \hookrightarrow \operatorname{Hom}(E, \left(\prod_{i \in I} X_i/\mathcal{U}\right)^*) \]

\[ \cong \left(\prod_{i \in I} X_i^*/\mathcal{U}\right) \widehat{\otimes} A E \]

\[ \xrightarrow{\left(\Phi_E^{-1}\right)^*} \left(\prod_{i \in I} (X_i^* \widehat{\otimes} A E)/\mathcal{U}\right)^*. \]

Then \( ||T||_{cb} \leq \lambda. \)

By [20, Proposition 10.3.2] and [20, Corollary 10.3.4], the composition

\[ I : \prod_{i \in I} \operatorname{Hom}(E, X_i)/\mathcal{U} \hookrightarrow \prod_{i \in I} \operatorname{Hom}(E, X_i^{**})/\mathcal{U} = \prod_{i \in I} (X_i^* \widehat{\otimes} A E)^*/\mathcal{U} \hookrightarrow \left(\prod_{i \in I} (X_i^* \widehat{\otimes} A E)/\mathcal{U}\right)^* \]

is a complete isometry. The following diagram is easily seen to commute

\[ \prod_{i \in I} \operatorname{Hom}(E, X_i)/\mathcal{U} \xrightarrow{I} \left(\prod_{i \in I} (X_i^* \widehat{\otimes} A E)/\mathcal{U}\right)^* \]

\[ \xrightarrow{\Delta} \operatorname{Hom}(E, \prod_{i \in I} X_i/\mathcal{U}) \xrightarrow{T} \left(\prod_{i \in I} (X_i^* \widehat{\otimes} A E)/\mathcal{U}\right)^*, \]

where \( \Delta \) denotes (right) composition with the canonical embedding \( \Delta : E \hookrightarrow E' = \prod_{i \in I} E/\mathcal{U} \) (see [20, Lemma 10.3.1]). Then for any \( ([\varphi_i^{k,l}]_{i\in I}) \in M_n\left( \prod_{i \in I} \operatorname{Hom}(E, X_i)/\mathcal{U} \right) \), we have

\[ ||([\varphi_i^{k,l}]_{i\in I})|| = ||(I([\varphi_i^{k,l}]_{i\in I}))|| = ||T(\Delta(\varphi_i^{k,l}))|| \leq \lambda ||\Delta(\varphi_i^{k,l})||. \]

Hence, \( \Delta \) is a complete \( \lambda \)-embedding.

(2) \( \Rightarrow \) (3): Since \( E \) finitely generated, it follows from the proof of Proposition 4.5 that there is a finite-dimensional operator space \( F \) and a complete quotient morphism \( \varphi : A_+ \widehat{\otimes} F \to E \).

Pick a complete quotient map \( q : T_\infty \to F \), and let \( q_n := q|_{T_n} : T_n \to F \). Fix some \( 0 < \varepsilon < 1/2 \). By Lemma 5.3, there exists \( n_0 \in \mathbb{N} \) such that \( ||\tilde{q}_n^{-1}||_{cb} \leq d^2(1+\varepsilon)/(1-\varepsilon) \leq 3d^2 \) for all \( n \geq n_0 \).

Let \( K_n = \ker(\varphi \circ (\text{id}_{A_+} \otimes q_n)) \) and \( Q_n = (A_+ \widehat{\otimes} T_n)/K_n \). Since \( K_n = (\text{id}_{A_+} \otimes q_n)^{-1}(\ker(\varphi)) \), the standard argument shows that

\[ \varphi \circ (\text{id}_{A_+} \otimes q_n) : A_+ \widehat{\otimes} T_n \to E \]

induces a complete isomorphism \( \varphi_n : Q_n \cong E \) with \( ||\varphi_n^{-1}|| \leq 3d^2 \) for all \( n \geq n_0 \).

Put \( \varphi_n = 0 \) (and \( \varphi_n^{-1} = 0 \)) for all \( n < n_0 \), and let \( \mathcal{U} \) be a free ultrafilter on \( \mathbb{N} \). Since the \( \varphi_n \) have uniformly completely bounded inverses asymptotically,

\[ (\varphi_n)_{\mathcal{U}} : \prod_{n \in \mathbb{N}} Q_n/\mathcal{U} \cong \prod_{n \in \mathbb{N}} E/\mathcal{U} \]
is a complete isomorphism. By (3), it we have
\[ \lim_{n \to \infty} \| \varphi_n^{-1} \|_{cb} = \| \pi_U((\varphi_n^{-1})) \| \leq \lambda \| \Delta((\varphi_n)_{U})^{-1}) \|_{cb} = \lambda \| (\varphi_n)_{U}^{-1} \circ \Delta \|_{cb}. \]
However,
\[ (\varphi_n)_{U}^{-1} \circ \Delta : E \to \prod_{n \in \mathbb{N}} Q_n/U \]
is a contraction: given \( x \in E_{\| \cdot \| < 1} \), pick \( y \in (A_+ \hat{\otimes} T_{\infty})_{\| \cdot \| < 1} \) for which
\[ x = \varphi((\text{id}_{A_+} \otimes q)(y)) = \lim_n \varphi((\text{id}_{A_+} \otimes q_n)(\text{Ad}(1_{A_+} \otimes P_n)(y))) = \lim_n \varphi((\text{Ad}(1_{A_+} \otimes P_n)(y)). \]
Then \( \pi_U((\text{Ad}(1_{A_+} \otimes P_n)(y) + K_n)) \) lies in the unit ball of \( \prod_{n \in \mathbb{N}} Q_n/U \) and
\[ \Delta(x) = (\varphi_n)_{U}((\text{Ad}(1_{A_+} \otimes P_n)(y))) \]
It follows that \( (\varphi_n)_{U}^{-1}(\Delta(x)) = \pi_U((\text{Ad}(1_{A_+} \otimes P_n)(y) + K_n)) \) has norm less than 1. A similar argument shows that \( (\varphi_n)_{U}^{-1} \circ \Delta \) is completely contractive. Thus, for every \( \varepsilon > 0 \)
\[ \lim_{n \to \infty} \| \varphi_n^{-1} \|_{cb} < \lambda + \varepsilon, \]
so there exists \( n \in \mathbb{N} \) for which \( \| \varphi_n^{-1} \| < \lambda + \varepsilon \), and \( E \) is \( \lambda \)-co-exact.

(3) \( \Rightarrow \) (1): Now, suppose that \( E \) is \( \lambda \)-co-exact and projective. Then by an argument similar to Example 4.4, for every \( \varepsilon > 0 \) there is an exact sequence in \( A_{\text{mod}} \)
\[ T_n(A_{+}) \xrightarrow{\psi} T_n(A_{+}) \xrightarrow{\varphi} E, \]
for which \( \psi \) is a completely bounded projection onto \( \text{Ker}(\varphi) \) and \( \varphi \) is a \( (\lambda + \varepsilon) \)-complete quotient map.

Observe that for any \( n \in \mathbb{N} \),
\[ \left( \prod_{i \in I} X_i/U \right) \hat{\otimes}_A (A_+ \hat{\otimes} T_n) = \left( \left( \prod_{i \in I} X_i/U \right) \hat{\otimes}_A A_+ \right) \hat{\otimes} T_n \]
\[ \cong \left( \prod_{i \in I} X_i/U \right) \hat{\otimes} T_n \]
\[ \cong \prod_{i \in I} (X_i \hat{\otimes} T_n)/U \quad \text{(by \cite[Lemma 10.3.8]{20})} \]
\[ \cong \prod_{i \in I} (X_i \hat{\otimes}_A (A_+ \hat{\otimes} T_n))/U. \]

We therefore obtain the commutative diagram
(11)
\[ \begin{array}{c}
\left( \prod_{i \in I} X_i/U \right) \hat{\otimes}_A T_n(A_{+}) \xrightarrow{\text{id} \otimes \psi} \left( \prod_{i \in I} X_i/U \right) \hat{\otimes}_A T_n(A_{+}) \xrightarrow{\text{id} \otimes \varphi} \left( \prod_{i \in I} X_i/U \right) \hat{\otimes}_A E \\
\prod_{i \in I} (X_i \hat{\otimes}_A T_n(A_{+}))/U \xrightarrow{} \prod_{i \in I} (X_i \hat{\otimes}_A T_n(A_{+}))/U \xrightarrow{} \prod_{i \in I} (X_i \hat{\otimes}_A E)/U.
\end{array} \]
Since \( (\text{id}_{X_i} \otimes \varphi) \) is a \( (\lambda + \varepsilon) \)-complete quotient map for each \( i \), it follows from the proof of \cite[Proposition 10.3.2]{20} that the second arrow in the bottom row is also a \( (\lambda + \varepsilon) \)-complete
quotient map. Hence, so too is \( \Phi \). Showing that \( \Phi \) is injective will prove that

\[
d_{cb}\left( \prod_{i \in I} X_i/U, \prod_{i \in I} (X_i \widehat{\otimes} A E)/U \right) \leq \lambda + \epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, the claim would follow. To that end, Lemma 2.3 implies the sequence

\[
X_i \widehat{\otimes} A T_n(A_+) \overset{\text{id}_{X_i \otimes A} \psi}{\longrightarrow} X_i \widehat{\otimes} A T_n(A_+) \overset{\text{id}_{X_i \otimes A} \varphi}{\longrightarrow} X_i \widehat{\otimes} A E,
\]

is topologically exact for each \( i \). Moreover, as \( \psi \) is a completely bounded projection, the sequence is exact as \( (\text{id}_{X_i} \otimes A) \) is a completely bounded projection onto its closed range. Moreover, as a completely bounded projection, \( (\text{id}_{X_i} \otimes A) \) is a \( 1 \)-strict morphism for each \( i \). It then follows from Lemma 5.4 that the bottom row of (11) is exact. The usual diagram chasing entails the injectivity of \( \Phi \).

\[ \square \]

**Remark 5.7.** Inspection of the proof shows that (3) \( \Rightarrow \) (1) is valid for any finitely presented module \( E \in A \text{mod} \) for which there is an exact sequence

\[
T_m(A_+) \overset{\psi}{\longrightarrow} T_n(A_+) \overset{\varphi}{\longrightarrow} E,
\]

in \( A \text{mod} \) with \( \varphi \) a complete \( \lambda \)-quotient map and \( \psi \) “stably \( \mu \)-strict” in the sense that the amplified morphism \( \text{id}_{X} \otimes A \psi : X \widehat{\otimes} A T_m(A_+) \to X \widehat{\otimes} A T_n(A_+) \) is \( \mu \)-strict for any \( X \in \text{mod} A \). Projectivity is a natural assumption that ensures this feature.

6. The Local Lifting Property

An operator space \( X \) has the \( \lambda \)-local lifting property (\( \lambda \)-LLP) if given any operator spaces \( Z \subseteq Y \) and a complete contraction \( \varphi : X \to Y/Z \), for every finite-dimensional subspace \( E \subseteq X \) and \( \epsilon > 0 \), there exists a lifting \( \tilde{\varphi} : E \to Y \) with \( \|\tilde{\varphi}\|_{cb} < \lambda + \epsilon \) making the following diagram commute:

\[
\begin{array}{ccc}
E & \longrightarrow & X \\
\downarrow & & \downarrow \varphi \\
& \longrightarrow & Y/Z.
\end{array}
\]

This property was studied by Kye and Ruan in [30], where they showed, among other things, that \( X \) has the \( \lambda \)-LLP if and only if \( X^* \) is \( \lambda \)-injective [30, Theorem 5.5]. It is also known that \( X \) has the \( 1 \)-LLP if and only if for every \( 1 \)-exact sequence

\[
0 \to Y \leftarrow Z \to Z/Y \to 0,
\]

in \( \text{Op} \) the sequence

\[
0 \to X \hat{\otimes} Y \leftarrow X \hat{\otimes} Z \to X \hat{\otimes} Z/Y \to 0
\]

is \( 1 \)-exact [18, Theorem 3.4]. Thus, the \( 1 \)-LLP is equivalent to \( 1 \)-exactness of the functor \( X \hat{\otimes} (\cdot) : \text{Op} \to \text{Op} \), in other words, the \( 1 \)-flatness of \( X \) in \( \mathbb{C} \text{mod} \). Given these results and the well-known duality between flatness and injectivity in \( A \text{mod} \), it is natural to investigate an operator module analogue of the local lifting property and its relation to flatness.

In the purely algebraic setting, it is well-known that a right module \( M \) over a unital ring \( R \) is flat in \( \text{mod} R \) if and only if for any epimorphism \( \varphi : N \to M \) and any finitely presented module \( E \), any homomorphism \( \psi : E \to M \) can be lifted to some \( \tilde{\psi} : E \to N \) (see, e.g.,
Flatness in \( \text{mod} \, R \) is therefore equivalent to a “local” lifting type property, where finite presentation is playing the role of locality in the category \( \text{mod} \, R \). This result motivated our notion of finite presentation as well as:

**Definition 6.1.** Let \( A \) be a completely contractive Banach algebra, and \( \lambda \geq 1 \). A module \( X \in A \text{mod} \) has the \( \lambda \)-local lifting property (\( \lambda \)-LLP) if given any \( Y \in A \text{mod} \) and a complete quotient map \( q \in \text{Hom}(Y, X) \), for every topologically finitely presented \( E \in A \text{mod} \), every morphism \( \varphi \in \text{Hom}(E, X) \) and every \( \varepsilon > 0 \), there exists a morphism \( \bar{\varphi} : E \to Y \) such that \( \| \bar{\varphi} \|_{cb} < \lambda \| \varphi \|_{cb} + \varepsilon \) and \( q \circ \bar{\varphi} = \varphi \), that is, the following diagram commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
\downarrow & & \downarrow \\
E & \xrightarrow{\varphi} & X \\
\end{array}
\]

An operator space \( X \) has the \( \lambda \)-LLP in \( \mathbb{C} \text{mod} \) in the sense of Definition 6.1 if and only if it has the operator space \( \lambda \)-LLP by [30, Theorem 3.2]. Hence, Definition 6.1 is a bona fide generalization of the operator space local lifting property.

Clearly, any \( \lambda \)-projective module \( X \in A \text{mod} \) satisfies the \( \lambda \)-LLP. A necessary condition for the \( \lambda \)-LLP is \( \lambda \)-flatness, as we now show. Due to the additional quotient structure from the definition of the module projective tensor product \( \hat{\otimes}_A \) (when \( A \neq \mathbb{C} \)), the operator space argument in [30, Proposition 5.1] does not readily generalize. Instead, we use direct limit arguments and appeal to Theorem 4.7.

**Proposition 6.2.** Let \( A \) be a completely contractive Banach algebra, and \( \lambda \geq 1 \). If \( X \in A \text{mod} \) has the \( \lambda \)-LLP then it is \( \lambda \)-flat.

**Proof.** Let \( Y \subseteq Z \) in \( \text{mod} \, A \), and write \( i \) for the inclusion map. We show that \( (i \otimes \text{id}_X) : Y \hat{\otimes}_A X \to Z \hat{\otimes}_A X \) is a complete \( \lambda \)-embedding, which will entail the \( \lambda \)-flatness of \( X \).

First, pick a Hilbert space \( H \) and an operator space complete quotient map \( q_0 : \mathcal{T}(H) \to X \) ([8, Corollary 3.2]). Then

\[
q := m_X^+ \circ (\text{id}_A^+ \otimes q_0) : A^+ \hat{\otimes} \mathcal{T}(H) \to X
\]

is a complete quotient morphism. Throughout the proof we denote \( A^+ \hat{\otimes} \mathcal{T}(H) \) by \( F \), to emphasize its 1-flatness.

By Theorem 4.7, \( X \cong \lim \rightarrow E_i \) completely isometrically for an inductive system \( (E_i, \varphi_{j,i})_{i,j \in I} \) in \( A \text{mod}_1 \) where each \( E_i \) is topologically finitely presented. Since \( X \) has the \( \lambda \)-LLP, for each \( i \in I \) and every \( \varepsilon > 0 \) there exists a morphism \( \bar{\varphi}_i : E_i \to F \) such that \( \| \bar{\varphi}_i \|_{cb} < \lambda + \varepsilon \) and the following diagram commutes:

\[
\begin{array}{ccc}
F & \xrightarrow{q} & X \\
\downarrow & & \downarrow \\
E_i & \xrightarrow{\varphi_i} & X \\
\end{array}
\]

where \( \varphi_i : E_i \to X \) is the canonical morphism. Then for \( u \in M_n(Y \hat{\otimes}_A E_i) \),

\[
\| (\text{id} \otimes \varphi_i)_n(u) \|_{M_n(Y \hat{\otimes}_A X)} = \| (\text{id} \otimes q \circ \bar{\varphi}_i)_n(u) \|_{M_n(Y \hat{\otimes}_A X)} \\
\leq \| (\text{id} \otimes \bar{\varphi}_i)_n(u) \|_{M_n(Y \hat{\otimes}_A F)} \\
= \| (\text{id} \otimes \bar{\varphi}_i)_n((i \otimes \text{id})(u)) \|_{M_n(Z \hat{\otimes}_A F)} \\
\leq (\lambda + \varepsilon) \| (i \otimes \text{id})(u) \|_{M_n(Z \hat{\otimes}_A E_i)};
\]
where the second equality follows from 1-flatness of $F$. Since $\varepsilon > 0$ was arbitrary,
\[ \| (\text{id} \otimes \varphi_i)_n (u) \|_{M_n (Y \hat{\otimes} A X)} \leq \lambda \| (\text{id} \otimes (u)) \|_{M_n (Z \hat{\otimes} A E_i)}. \]
Write $Y \hat{\otimes}^Z A X$ for the closure of $(i \otimes \text{id}_X) (Y \hat{\otimes} A X)$ in $Z \hat{\otimes} A X$, and similarly for each $Y \hat{\otimes}^Z A E_i$. By above, the map $(\text{id} \otimes \varphi_i)$ extends to a morphism
\[ (\text{id} \otimes \varphi_i) : Y \hat{\otimes}^Z A E_i \to Y \hat{\otimes} A X \]
in $A \text{mod}$ with $\| (\text{id} \otimes \varphi_i) \|_{cb} \leq \lambda$. Moreover, for each $i,j \in I$ the following diagram commutes

\[
\begin{array}{ccc}
Y \hat{\otimes}^Z A E_i & \overset{(\text{id} \otimes \varphi_i)}{\longrightarrow} & Y \hat{\otimes} A X \\
\downarrow & & \downarrow \\
Y \hat{\otimes}^Z A E_j & \overset{(\text{id} \otimes \varphi_j)}{\longrightarrow} & Y \hat{\otimes} A X
\end{array}
\]

(This is easily checked on the image $(i \otimes \text{id})(Y \hat{\otimes} A E_i)$, so the result follows by density.) By the universal property of direct limits, there is a unique morphism $\varphi : \lim_i Y \hat{\otimes}^Z A E_i \to Y \hat{\otimes} A X$ with $\| \varphi \|_{cb} \leq \lambda$ satisfying
\[ \varphi((i \otimes \text{id}) (\text{id} \otimes \varphi_i)(u_i)) = (i \otimes \varphi_i)(u_i), \quad u_i \in Y \hat{\otimes} A E_i, \ i \in I. \]
Since each $Y \hat{\otimes}^Z A E_i \hookrightarrow Z \hat{\otimes} A E_i$ completely isometrically, it follows from the representation in Proposition 2.1 that the induced mapping between the direct limits
\[ \lim_i Y \hat{\otimes}^Z A E_i \hookrightarrow \lim_i Z \hat{\otimes} A E_i \]
is completely isometric. Furthermore, by Proposition 2.2 $\lim_i Z \hat{\otimes} A E_i \cong Z \hat{\otimes} A X$, canonically. It follows that $\lim_i Y \hat{\otimes}^Z A E_i = Y \hat{\otimes} A X$, canonically. Under this identification, $\varphi : Y \hat{\otimes}^Z A X \to Y \hat{\otimes} A X$ is a left inverse to $(i \otimes \text{id})$ with $\| \varphi \|_{cb} \leq \lambda$. Hence, $(i \otimes \text{id})$ is a complete $\lambda$-embedding.

We now establish a partial converse to Proposition 6.2 under the assumption that $A_* \hat{\otimes} \mathcal{T}(H)$ is locally reflexive for any Hilbert space $H$. By [21], this condition is satisfied whenever $A$ is the predual of a von Neumann algebra, e.g., $A = L^1(\mathbb{G})$ for any locally compact quantum group $\mathbb{G}$. In what follows we use the notation $\simeq_b$ to mean isomorphic in $\text{Ban}$ (i.e., not necessarily completely isomorphic in $\text{Op}$).

For $X, Y \in A \text{mod}$ there is a canonical complete contraction
\[ \Phi : X^* \hat{\otimes} A Y \ni f \otimes_A y \mapsto (\varphi \mapsto \langle f, \varphi(y) \rangle) \in \text{Hom}(Y, X)^*. \]
When $A = \mathbb{C}$ and $Y = E$ is finite dimensional, then $\text{Hom}(E, X) = E^* \otimes^\vee X$ and the map $\Phi$ is the canonical isomorphism
\[ X^* \hat{\otimes} E = X^* \hat{\otimes} E^{**} \simeq_b (E^* \otimes^\vee X)^*, \]
which appears in the study of local reflexivity. Such an isomorphism holds for general $A$ whenever $E$ is topologically finitely presented, as we now show. This may be seen as an operator module analogue of (a special case of) [40, Lemma 9.71].
Proposition 6.3. Let $A$ be a completely contractive Banach algebra, and let $E \in A\text{mod}$ be topologically finitely presented. Then $X^* \hat{\otimes}_A E \simeq_b \text{Hom}(E, X)^*$ for all $X$ in $A\text{mod}$.

Proof. First suppose that $E = A_+ \hat{\otimes} T_n$ for some $n \in \mathbb{N}$. Then

$$X^* \hat{\otimes}_A E = X^* \hat{\otimes}_A (A_+ \hat{\otimes} T_n) = (X^* \hat{\otimes}_A A_+) \hat{\otimes} T_n \cong X^* \hat{\otimes} T_n = (X \otimes^\vee M_n)^* = \mathcal{CB}(T_n, X)^* \cong \text{Hom}(E, X)^*,$$

completely isometrically. Now, suppose that we have a topologically exact sequence in $A\text{mod}_1$

$$A_+ \hat{\otimes} T_m \longrightarrow A_+ \hat{\otimes} T_n \longrightarrow \varphi \longrightarrow E,$$

with $\varphi$ a complete $\lambda$-quotient map for some $\lambda \geq 1$. Then by Lemma 2.3

$$\text{Hom}(E, X) \longleftarrow \text{Hom}(A_+ \hat{\otimes} T_n, X) \longrightarrow \text{Hom}(A_+ \hat{\otimes} T_m, X)$$

is exact in $\text{Op}$. Since $\mathbb{C}$ is injective in $\text{Op}$, the sequence

$$\text{Hom}(A_+ \hat{\otimes} T_n, X)^* \longrightarrow \text{Hom}(A_+ \hat{\otimes} T_n, X)^* \longrightarrow \text{Hom}(E, X)^*$$

is also exact. Consider the commutative diagram

$$(12) \quad \begin{array}{ccc}
X^* \hat{\otimes}_A (A_+ \hat{\otimes} T_m) & \longrightarrow & X^* \hat{\otimes}_A (A_+ \hat{\otimes} T_n) \\
\downarrow & & \downarrow \Phi \\
\text{Hom}(A_+ \hat{\otimes} T_m, X)^* & \longrightarrow & \text{Hom}(A_+ \hat{\otimes} T_n, X)^* \longrightarrow \text{Hom}(E, X)^*.
\end{array}$$

Since the bottom row is exact, the first two vertical arrows are complete isometries, and the second arrows in each row are surjective, the usual diagram chasing implies that $\Phi$ is bijective, and thus a bounded isomorphism by the inverse mapping theorem. \qed

Proposition 6.3 suggests a notion of $\lambda$-local reflexivity for modules $X \in A\text{mod}$: every completely contractive morphism $\varphi : E \to X^{**}$ from a topologically finitely presented module $E$ can be approximated in the point weak* topology by a net $\varphi_i : E \to X$ satisfying $\|\varphi_i\|_{cb} \leq \lambda$. Equivalently, the isomorphism

$$\Phi : X^* \hat{\otimes}_A E \cong_b \text{Hom}(E, X)^*$$

satisfies $\|\Phi^{-1}\| \leq \lambda$ for every topologically finitely presented $E \in A\text{mod}$. However, this notion coincides with $\lambda$-local reflexivity in $\text{Op}$.

Proposition 6.4. Let $A$ be a completely contractive Banach algebra, and $\lambda \geq 1$. Then $X \in A\text{mod}$ is $\lambda$-locally reflexive in $A\text{mod}$ (in the sense above) if and only if it is $\lambda$-locally reflexive in $\text{Op}$.

Proof. If $X$ is $\lambda$-locally reflexive in $A\text{mod}$, then for every finite-dimensional $E \in \text{Op}$, $A_+ \hat{\otimes} E$ is finitely presented, so we have

$$X^* \hat{\otimes} E \cong X^* \hat{\otimes}_A (A_+ \hat{\otimes} E) \cong_b \text{Hom}(A_+ \hat{\otimes} E, X)^* \cong \mathcal{CB}(E, X)^* \cong (E^* \otimes^\vee X)^*,$$

with the norm of the inverse bounded by $\lambda$. Thus, $X$ is $\lambda$-locally reflexive in $\text{Op}$.

Conversely, suppose $X$ is $\lambda$-locally reflexive in $\text{Op}$. Then for any finite-dimensional operator space $F$,

$$X^* \hat{\otimes}_A (A_+ \hat{\otimes} F) \cong X^* \hat{\otimes} F \cong_b \mathcal{CB}(F, X)^* \cong \text{Hom}(A_+ \hat{\otimes} F, X)^*,$$
with the norm of the inverse bounded by $\lambda$. If $E \in A\mathsf{mod}$ is topologically finitely presented, then by Proposition 4.5 there is a topologically exact sequence

$$A_+ \hat{\otimes} F_1 \longrightarrow A_+ \hat{\otimes} F_2 \overset{\varphi}{\longrightarrow} E$$

in $A\mathsf{mod}_1$ with $\varphi$ a complete quotient map. Building the corresponding diagram as in (12), it follows that the first two vertical arrows have inverses bounded by $\lambda$, and the second horizontal arrows in each row are complete quotient maps. It follows that

$$X^* \hat{\otimes}_A E \simeq_b \text{Hom}(E, X)^*$$

with inverse bounded by $\lambda$. □

The previous two results allow for an operator module extension of (the first part of) [30, Proposition 5.4]. Contrary to Proposition 6.2, the operator space argument from [30] extends more or less verbatim.

**Proposition 6.5.** Let $Y, Z \in A\mathsf{mod}$, $Z \subseteq Y$ and $Y$ be locally reflexive in $\text{Op}$. If $Z \perp$ is $\lambda$-completely complemented as a submodule of $Y^*$, then

1. $Y/Z$ is $\lambda$-locally reflexive, and
2. the canonical map

$$T : \text{Hom}(E, Y)/\text{Hom}(E, Z) \to \text{Hom}(E, Y/Z)$$

satisfies $\|T^{-1}\| \leq \lambda$ for every topologically finitely presented $E \in A\mathsf{mod}$.

**Proof.** Let $E \in A\mathsf{mod}$ be topologically finitely presented. Let $q : Y \twoheadrightarrow Y/Z$ and

$$Q : \text{Hom}(E, Y) \to \text{Hom}(E, Y)/\text{Hom}(E, Z)$$

be the canonical quotient maps, so that $\text{Hom}(E(q)) = T \circ Q$, where $\text{Hom}(E(\cdot))$ is the standard functor. Write $i$ for the inclusion $Z \perp \subseteq Y^*$, and let $Z \perp \hat{\otimes}_A Y^*$ denote the closed subspace generated by $(i \otimes \text{id})(Z \perp \hat{\otimes}_A E)$. We will show that the following diagram

\begin{equation}
\begin{array}{ccccccccc}
\text{Hom}(E, Y/Z)^* & \overset{T^*}{\longrightarrow} & \text{Hom}(E, Y)^*/(\text{Hom}(E, Y)/\text{Hom}(E, Z))^* & \overset{Q^*}{\longrightarrow} & \text{Hom}(E, Y)^* \\
\Phi_1 \downarrow & & \Phi \downarrow & & \Phi \downarrow \\
Z \perp \hat{\otimes}_A E & \overset{(i \otimes \text{id})}{\longrightarrow} & Z \perp \hat{\otimes}_A Y^* & \longrightarrow & Y^* \hat{\otimes}_A E
\end{array}
\end{equation}

commutes, where $\Phi_2$ is the restriction of $\Phi$ to $Z \perp \hat{\otimes}_A Y^*$. For any $f \in Z \perp$, $x \in E$, and $\varphi \in \text{Hom}(E, Y)$, we have

$$\langle \text{Hom}(E(q))^* \circ \Phi_1(f \otimes_A x), \varphi \rangle = \langle \Phi_1(f \otimes_A x), q \circ \varphi \rangle = \langle f, q(\varphi(x)) \rangle$$

$$= \langle q^*(f), \varphi(x) \rangle = \langle \Phi(q^*(f) \otimes_A x), \varphi \rangle$$

$$= \langle \Phi((i \otimes \text{id})(f \otimes_A x)), \varphi \rangle.$$

Hence, the outer rectangle in (13) commutes, that is $\text{Hom}(E(q))^* \circ \Phi_1 = \Phi \circ (i \otimes \text{id})$. Since $Q^*$ is just the inclusion $\text{Hom}(E, Z) \perp \subseteq \text{Hom}(E, Y)^*$, and $\text{Hom}(E(q))^* = Q^* \circ T^*$, it follows that $T^* \circ \Phi_1 = \Phi_2 \circ (i \otimes \text{id})$. Hence, the entire diagram (13) commutes.
Now, since $Z^\perp$ is a $\lambda$-completely complemented submodule of $Y^*$, $\| (i \otimes \text{id})^{-1} \| \leq \lambda$, and since $Y$ is locally reflexive, Proposition 6.4 implies that $\| \Phi^{-1} \| \leq 1$. Thus, $\Phi$ is an isometric isomorphism, implying its restriction $\Phi_2$ is an isometric isomorphism onto its range, which is all of $(\text{Hom}(E, Y)/\text{Hom}(E, Z))^*$ by commutativity of (13) and the fact that $Z^\perp$ is completely complemented in $Y^*$. Hence, $\| \Phi_2^{-1} \| \leq 1$, and
\[ \| \Phi_1^{-1} \| = \| (i \otimes \text{id})^{-1} \circ \Phi_2^{-1} \circ T^* \| \leq \| (i \otimes \text{id})^{-1} \| \leq \lambda. \]
Since $E \in \text{A}_{\text{mod}}$ was arbitrary, it follows that $Y/Z$ is $\lambda$-locally reflexive (in either category, by Proposition 6.4). Finally,
\[ \| T^{-1} \| = \| (T^*)^{-1} \| = \| \Phi_1 \circ (i \otimes \text{id})^{-1} \circ \Phi_2^{-1} \| \leq \| (i \otimes \text{id})^{-1} \| \leq \lambda. \]

\[ \square \]

**Theorem 6.6.** Let $A$ be a completely contractive Banach algebra such that $A_+ \hat{\otimes} \mathcal{T}(H)$ is locally reflexive for every Hilbert space $H$, and let $\lambda \geq 1$. Then $X \in \text{A}_{\text{mod}}$ is $\lambda$-flat if and only if $X$ has the $\lambda$-LLP.

**Proof.** The reverse direction follows immediately from Proposition 6.2.

Suppose that $X$ is $\lambda$-flat in $\text{A}_{\text{mod}}$, equivalently, $X^*$ is $\lambda$-injective in $\text{mod} \ A$. As in the proof of Proposition 6.2, $X = A_+ \hat{\otimes} \mathcal{T}(H)/Z$ for some Hilbert space $H$ and closed submodule $Z \subseteq A_+ \hat{\otimes} \mathcal{T}(H)$. By assumption, $A_+ \hat{\otimes} \mathcal{T}(H)$ is locally reflexive and $Z^\perp = X^*$ is a $\lambda$-completely complemented submodule of $(A_+ \hat{\otimes} \mathcal{T}(H))^*$. Proposition 6.5 implies that the canonical map
\[ T : \text{Hom}(E, A_+ \hat{\otimes} \mathcal{T}(H))/\text{Hom}(E, Z) \to \text{Hom}(E, X) \]
satisfies $\| T^{-1} \| \leq \lambda$ for any topologically finitely presented $E \in \text{A}_{\text{mod}}$. This implies that $X$ has the $\lambda$-LLP with respect to any quotient of the form $A_+ \hat{\otimes} \mathcal{T}(H) \to X$. Since any module in $\text{A}_{\text{mod}}$ is a complete quotient of some $A_+ \hat{\otimes} \mathcal{T}(H)$, it follows that $X$ has the $\lambda$-LLP. \[ \square \]

**Remark 6.7.** Local type characterizations of (relatively) flat Banach modules were studied by Aristov in [4]. Among other things, he showed that flatness of a Banach $A$-module $X$ is equivalent to a different type of local lifting property, namely with respect to finite-dimensional subspaces of $X$ of the form $E = \text{span}\{a_i \cdot x_j \mid i = 1, \ldots, m, \ j = 1, \ldots, n\}$ with $a_1, \ldots, a_m \in A_+$ and $x_1, \ldots, x_n \in X$ (see [4, Theorem 2.1]). However, the morphisms involved in this property are only assumed to be approximate $A$-module maps, and therefore live outside $\text{A}_{\text{mod}}$. His work inspired us to seek a local characterization of flatness within $\text{A}_{\text{mod}}$. Theorem 6.6 achieves this and shows that our concept of topological finite presentation provides a suitable notion of locality in this context.

**Corollary 6.8.** Let $A$ be a completely contractive Banach algebra such that $A_+ \hat{\otimes} \mathcal{T}(H)$ is locally reflexive for every Hilbert space $H$, and $\lambda \geq 1$. If $X \in \text{A}_{\text{mod}}$ is $\lambda$-flat and topologically finitely presented then it is $\lambda$-projective.

**Proof.** Simply apply Theorem 6.6 together with the fact that $X$ is topologically finitely presented. \[ \square \]

As previously mentioned, the local reflexivity assumption in Theorem 6.6 is satisfied whenever $A$ is the predual of a von Neumann algebra by [21] (and [20, Theorem 7.2.4]). In particular, when $A = L^1(\mathbb{G})$ for a locally compact quantum group $\mathbb{G}$. Combining Theorem 6.6 with [12, Theorem 5.1], we obtain a local characterization of flatness of $L^1(\mathbb{G})$ in $L^1(\mathbb{G})_{\text{mod}}$, which is new even for groups.
Corollary 6.9. Let $\mathbb{G}$ be a locally compact quantum group. Then $\hat{\mathbb{G}}$ is amenable if and only if $L^1(\mathbb{G})$ has the 1-LLP in $L^1(\mathbb{G})$ mod.

Thus, for a locally compact group $G$, $L^1(G)$ always has the 1-LLP in $L^1(G)$ mod, and $A(G)$ has the 1-LLP in $A(G)$ mod if and only if $G$ is amenable.

7. Nuclearity and Semi-discreteness

An operator space $X$ is *nuclear* if there exist diagrams of complete contractions

\[
\begin{array}{ccc}
M_{n_\alpha} & \rightarrow & Y \\
\downarrow r_{\alpha} & & \downarrow s_{\alpha} \\
X & \rightarrow & X \\
\end{array}
\]

which approximately commute in the point-norm topology [20, §14.6]. In other words, the identity map on $X$ approximately factorizes through finitely co-generated co-free operator spaces. From this categorical perspective, the following definition is natural.

**Definition 7.1.** Let $A$ be a completely contractive Banach algebra, and $X,Y \in A_{\text{mod}}$.

- A morphism $\varphi \in \text{Hom}(X,Y)$ is *nuclear* (or *A-nuclear*) if there exist diagrams of morphisms

\[
\begin{array}{ccc}
M_{n_\alpha}(A^+) & \rightarrow & Y \\
\downarrow r_{\alpha} & & \downarrow s_{\alpha} \\
X & \rightarrow & X \\
\text{id}_X & \rightarrow & \varphi \\
\end{array}
\]

which approximately commute in the point-norm topology.

- An operator module $X \in A_{\text{mod}}$ is *nuclear* (or *A-nuclear*) if $\text{id}_X \in \text{Hom}(X,X)$ is nuclear.

Similar definitions apply to right modules and their morphisms. As always, when $A$ is unital, we use $A$ for $A_+$ above.

Recently, a generalization of the weak expectation property was introduced in [6] for operator modules over completely contractive Banach algebras. An object $X \in A_{\text{mod}}$ has the *weak expectation property* ($A$-WEP) if for any completely isometric morphism $\kappa : X \hookrightarrow Y$ there exists a morphism $\psi : Y \rightarrow X^{**}$ such that $\psi \circ \kappa = i_X$, where $i_X : X \hookrightarrow X^{**}$ is the canonical inclusion. Any nuclear module has $A$-WEP, a consequence of:

**Proposition 7.2.** Let $A$ be a completely contractive Banach algebra, and $X \in A_{\text{mod}}$. If the inclusion $i_X : X \hookrightarrow X^{**}$ is nuclear then $X$ has the $A$-WEP. In particular, nuclearity implies the $\text{WEP}$ in $A_{\text{mod}}$.

**Proof.** Pick diagrams of morphisms

\[
\begin{array}{ccc}
M_{n_\alpha}(A^+) & \rightarrow & X^{**} \\
\downarrow r_{\alpha} & & \downarrow s_{\alpha} \\
X & \rightarrow & X^{**} \\
\text{i}_X & \rightarrow & \text{id}_{X^{**}} \\
\end{array}
\]
Proposition 7.4. Let $\Delta_+ : X \hookrightarrow CB(A_+, B(H))$ be the completely isometric composition of this inclusion with the canonical embedding $\triangleleft X \hookrightarrow CB(A_+, B(H))$. By injectivity of $M_{n_0}(A_+^*)$, each $r_\alpha$ extends to a morphism $\tilde{r}_\alpha : CB(A_+, B(H)) \to M_{n_0}(A_+^*)$, for which $\tilde{r}_\alpha \circ \kappa = r_\alpha$. Let $\varphi \in CB(CB(A_+, B(H)), X^{**})$ be a weak*-cluster point of $(s_\alpha \circ \tilde{r}_\alpha)$. Then $\varphi \circ \kappa = i_X$. Hence, the inclusion $X \subseteq X^{**}$ factors through the injective module $CB(A_+, B(H))$, so $X$ has the A-WEP by (the left version of) [6, Theorem 3.4(6)]. □

We have the analogous notion and result for dual modules.

Definition 7.3. Let $A$ be a completely contractive Banach algebra, and $X, Y \in A\text{mod}_1$ be dual modules.

- A normal morphism $\varphi \in \text{Hom}(X, Y)$ is weakly nuclear (or weakly A-nuclear) if there exist diagrams of normal morphisms

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow{r_\alpha} & & \downarrow{s_\alpha} \\
M_{n_0}(A_+^*) & & \\
\end{array}
\]

which approximately commute in the point-weak* topology.

- A dual module $X \in A\text{mod}_1$ is semi-discrete (or A-semi-discrete) if $\text{id}_X \in \text{Hom}(X, X)$ is weakly nuclear.

Similar definitions apply to dual right modules and their normal morphisms.

As expected, any semi-discrete module is injective.

Proposition 7.4. Let $A$ be a completely contractive Banach algebra and $X \in A\text{mod}_1$ be a dual module. If $X$ is semi-discrete then it is injective in $A\text{mod}_1$.

Proof. Pick diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow{r_\alpha} & & \downarrow{s_\alpha} \\
M_{n_0}(A_+^*) & & \\
\end{array}
\]

of normal morphisms in $A\text{mod}_1$ which approximately commute in the point-weak* topology. Also, pick an embedding $X \hookrightarrow CB(A_+, B(H))$ in $A\text{mod}_1$ for some Hilbert space $H$ (as above). By injectivity of $M_{n_0}(A_+^*)$, each $r_\alpha$ extends to a morphism $\tilde{r}_\alpha : CB(A_+, B(H)) \to M_{n_0}(A_+^*)$. Let $\Phi \in \text{Hom}(CB(A_+, B(H)), X)$ be weak*-cluster point of $(s_\alpha \circ \tilde{r}_\alpha)$. Then $\Phi$ is a module projection of norm 1, witnessing the injectivity of $X$. □

Clearly, any module of the form $M_n(A_+^*)$ is both $A$-nuclear and A-semi-discrete. Another simple class of examples follows from

Proposition 7.5. Let $A$ be a completely contractive Banach algebra. If $A$ has a contractive left (respectively, right) approximate identity, then $\langle A \cdot A^* \rangle$ (respectively, $\langle A^* \cdot A \rangle$) is nuclear and $A^*$ is semi-discrete in $\text{mod}_A$ (respectively, $A\text{mod}_1$).

Proof. Viewing $A_+$ in $A\text{mod}_1$, the right module structure on $A_+^* = A^* \oplus_\infty \mathbb{C}$ is given by

$$(x, \lambda) \cdot a = (x \cdot a, \langle x, a \rangle), \quad a \in A, \; x \in A^*, \; \lambda \in \mathbb{C}.$$
It follows that the projection \( p : A_+^* \ni (x, \lambda) \mapsto x \in A^* \) is a morphism in \( \mathbf{mod} \ A_1 \).

Let \( (a_\alpha) \) be a contractive left approximate identity for \( A \). Then we obtain diagrams of morphisms in \( \mathbf{mod} \ A_1 \)

\[
\begin{array}{ccc}
A_+^* & \xrightarrow{r_\alpha} & (A \cdot A^*) \\
\downarrow{s_\alpha} & & \downarrow{id} \\
(A \cdot A^*) & \rightarrow & (A \cdot A^*)
\end{array}
\]

where \( r_\alpha(x) = (a_\alpha \cdot x, \langle x, a_\alpha \rangle) \), and \( s_\alpha(x, \lambda) = l(a_\alpha) \circ p(x, \lambda) = a_\alpha \cdot x \). The composition \( s_\alpha \circ r_\alpha(x) = a_\alpha^2 \cdot x \rightarrow x \) for all \( x \in (A \cdot A^*) \). Hence, \( (A \cdot A^*) \) is nuclear in \( \mathbf{mod} \ A_1 \). Similarly, \( s_\alpha \circ r_\alpha(x) = a_\alpha^2 \cdot x \rightarrow x \) weak* for all \( x \in A^* \).

The argument for the right case is identical. \( \square \)

Letting \( A = L^1(G) \) in Proposition 7.5 for a co-amenable locally compact quantum group \( G \), we see that both \( \text{LUC}(G) = \langle L^\infty(G) \ast L^1(G) \rangle \) and \( \text{RUC}(G) = \langle L^1(G) \ast L^\infty(G) \rangle \) are \( L^1(G) \)-nuclear, and \( L^\infty(G) \) is \( L^1(G) \)-semi-discrete. In particular, for any locally compact group \( G \), \( \text{LUC}(G) \) and \( \text{RUC}(G) \) are \( L^1(G) \)-nuclear and \( L^\infty(G) \) is \( L^1(G) \)-semi-discrete. Dually, for an amenable locally compact group \( G \), \( \text{UCB}(\hat{G}) \) is \( A(G) \)-nuclear and \( V(N)(G) \) is \( A(G) \)-semi-discrete, where \( \text{UCB}(\hat{G}) = \langle A(G) \cdot V(N)(G) \rangle \) is the space of uniformly continuous linear functionals on \( A(G) \), introduced by Granirer [22]. When \( G \) is discrete, \( \text{UCB}(\hat{G}) = C_0^\ast(G) \), so \( C_0^\ast(G) \) is \( A(G) \)-nuclear for any discrete amenable group \( G \).

We now establish the converse of Proposition 7.5 for any locally compact quantum group. The next two results are helpful in this regard.

**Lemma 7.6.** Let \( X \in \mathbf{mod} \ A_1 \) be faithful. Any morphism \( A_+^* \rightarrow X \) in \( \mathbf{mod} \ A_1 \) is of the form \( \varphi \circ p \), where \( p : A_+^* \rightarrow A^* \) is the canonical projection and \( \varphi \in \text{Hom}(A^*, X) \). If, in addition, \( X \) is a dual module and \( A_+^* \rightarrow X \) is normal, the resulting morphism \( \varphi \in \text{Hom}(A^*, X) \) is normal.

**Proof.** Let \( \psi \in \text{Hom}(A_+^*, X) \). Since \( \psi \) is \( \mathbb{C} \)-linear we can write

\[
\psi(x, \lambda) = \psi(x, 0) + \psi(0, \lambda) = \varphi(x) + \lambda x_\psi,
\]

where \( \varphi = \psi|_{A^*_{\lambda \neq 0}} \in \mathcal{CB}(A^*, X) \) and \( \psi|_{0 \in \mathbb{C}} = \mathcal{CB}(\mathbb{C}, X) \) is uniquely determined by \( x_\psi \in X \). Then for every \( \lambda \in \mathbb{C} \) and \( a \in A \),

\[
0 = \psi((0, \lambda) \cdot a) = \psi(0, \lambda) \cdot a = \lambda x_\psi \cdot a.
\]

Since \( X \) is faithful, this forces \( x_\psi = 0 \). Thus, \( \psi(x, \lambda) = \varphi(x) = \varphi \circ p(x, \lambda) \), and \( \varphi \) is necessarily a morphism.

The normality statement is immediate. \( \square \)

Recall that a locally compact quantum group \( G \) has the approximation property if there exists a net \( (\hat{f}_l) \) in \( L^1(\hat{G}) \) such that \( \hat{\Theta}'(\hat{f}_l) \) converges to \( \text{id}_{L^\infty(\hat{G})} \) in the stable point-weak* topology.

**Proposition 7.7.** Let \( G \) be a locally compact quantum group. Consider the following conditions:

1. \( G \) is co-amenable;
2. \( C_0(G) \) has the WEP in \( \mathbf{mod} L^1(G)_1 \);
3. \( \hat{G} \) is amenable.
Then (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3). When \(\mathcal{G}\) is co-commutative, or if \(\hat{\mathcal{G}}\) has the approximation property, the conditions are equivalent.

**Proof.** (1) \(\Rightarrow\) (2): By Proposition 7.5, (1) implies that \(\text{RUC}(\mathcal{G}) = \langle L^1(\mathcal{G}) \ast L^\infty(\mathcal{G}) \rangle\) is nuclear in \(\text{mod} \ L^1(\mathcal{G})_1\). Since

\[
\text{C}_0(\mathcal{G}) \subseteq \text{RUC}(\mathcal{G}) \subseteq M(\text{C}_0(\mathcal{G})) \subseteq \text{C}_0(\mathcal{G})^{**}
\]

[41], it follows that the inclusion \(\text{C}_0(\mathcal{G}) \hookrightarrow \text{C}_0(\mathcal{G})^{**}\) is nuclear, so Proposition 7.2 entails (2).

(2) \(\Rightarrow\) (3): If \(\text{C}_0(\mathcal{G})\) has the WEP in \(\text{mod} \ L^1(\mathcal{G})_1\), then there exists a morphism \(\varphi : B(L^2(\mathcal{G})) \to \text{C}_0(\mathcal{G})^{**}\) such that \(\varphi|_{\text{C}_0(\mathcal{G})} = i_{\text{C}_0(\mathcal{G})}\). Then \(\varphi\) is a weak expectation, and hence, by [6, Lemma 2.3], a completely positive \(M(\text{C}_0(\mathcal{G}))\)-bimodule map. Composing with the canonical morphism \(\text{C}_0(\mathcal{G})^{**} \to L^\infty(\mathcal{G})\), we obtain a completely positive \(L^1(\mathcal{G})\)-morphism \(\varphi : B(L^2(\mathcal{G})) \to L^\infty(\mathcal{G})\) which is the identity on \(\text{C}_0(\mathcal{G})\) and is also an \(M(\text{C}_0(\mathcal{G}))\)-bimodule map. The \(L^1(\mathcal{G})\)-module property implies that \(\varphi(L^\infty(\hat{\mathcal{G}})) \subseteq \mathbb{C}_1\). In particular, \(\varphi(1) = \lambda 1\) for some \(\lambda \geq 0\). But then complete positivity implies

\[
\lambda = \|\lambda 1\| = \|\varphi(1)\| = \|\varphi\|_{cb} = 1,
\]

so \(\varphi\) is unital. Since the left fundamental unitary of \(\hat{\mathcal{G}}\)

\[
\hat{W} \in M(\text{C}_0(\hat{\mathcal{G}}) \otimes \text{C}_0(\mathcal{G})) \subseteq L^\infty(\hat{\mathcal{G}}) \overline{\otimes} B(L^2(\mathcal{G})),
\]

it follows (as in [34, Theorem 3.2]) that

\[
(f \otimes \text{id})(\text{id} \otimes \varphi)(\hat{W}) = \varphi((f \otimes \text{id})(\hat{W})) = (f \otimes \text{id})(\hat{W}), \quad f \in L^1(\hat{\mathcal{G}}),
\]

as \(\varphi\) is unital and \((f \otimes \text{id})(\hat{W}) \in M(\text{C}_0(\mathcal{G}))\). Hence, \((\text{id} \otimes \varphi)(\hat{W}) = \hat{W}\), and by unitarity, \(\hat{W}\) is in the multiplicative domain of \((\text{id} \otimes \varphi)\). The bimodule property of completely positive maps over their multiplicative domains implies that

\[
\varphi((\hat{f} \otimes \text{id})(\hat{W}^*(1 \otimes T)\hat{W})) = (f \otimes \text{id})(\text{id} \otimes \varphi)((\hat{W}^*(1 \otimes T)\hat{W}))
\]

\[
= (\hat{f} \otimes \text{id})(\hat{W}^*(1 \otimes \varphi(T))\hat{W}),
\]

for all \(T \in B(L^2(\mathcal{G}))\) and \(\hat{f} \in L^1(\hat{\mathcal{G}})\). Thus, \(\varphi|_{L^\infty(\hat{\mathcal{G}})} : L^\infty(\hat{\mathcal{G}}) \to \mathbb{C}\) satisfies

\[
\varphi|_{L^\infty(\hat{\mathcal{G}})}(\hat{x} \ast \hat{f}) = \varphi|_{L^\infty(\hat{\mathcal{G}})}(\hat{x}) \ast \hat{f}, \quad \hat{x} \in L^\infty(\hat{\mathcal{G}}), \quad \hat{f} \in L^1(\hat{\mathcal{G}}),
\]

so is a left invariant mean, and \(\hat{\mathcal{G}}\) is amenable.

When \(\mathcal{G}\) is co-commutative, the equivalence of (1) and (3) is simply an application of Leptin’s theorem [32]. When \(\hat{\mathcal{G}}\) has the approximation property, the result follows from [14, Corollary 7.4]. \(\square\)

**Theorem 7.8.** Let \(\mathcal{G}\) be a locally compact quantum group. The following conditions are equivalent:

1. \(\mathcal{G}\) is co-amenable;
2. \(L^\infty(\mathcal{G})\) is semi-discrete in \(\text{mod} \ L^1(\mathcal{G})_1\);
3. \(L^\infty(\mathcal{G})\) is semi-discrete in \(L^1(\mathcal{G})\) \(\text{mod}_1\);
4. \(\text{LUC}(\mathcal{G})\) is nuclear in \(L^1(\mathcal{G})\) \(\text{mod}_1\);
5. \(\text{RUC}(\mathcal{G})\) is nuclear in \(\text{mod} L^1(\mathcal{G})_1\);
6. The inclusion \(\text{C}_0(\mathcal{G}) \hookrightarrow \text{C}_0(\mathcal{G})^{**}\) is nuclear in \(\text{mod} L^1(\mathcal{G})_1\);
7. The inclusion \(\text{C}_0(\mathcal{G}) \hookrightarrow \text{C}_0(\mathcal{G})^{**}\) is nuclear in \(L^1(\mathcal{G})\) \(\text{mod}_1\).
Proof. (1) ⇒ (2) and (1) ⇒ (3) follow from Proposition 7.5.

(2) ⇒ (1): Suppose that \( L^\infty(G) \) is semi-discrete in \( \text{mod} \ L^1(G)_1 \). Then there exist diagrams of normal morphisms in \( \text{mod} \ L^1(G)_1 \)

\[
\begin{array}{ccc}
M_{n_a}(L^\infty(G) \oplus_\infty \mathbb{C}) & \xrightarrow{r_\alpha} & \text{id} \\
L^\infty(G) & \xrightarrow{\alpha} & L^\infty(G),
\end{array}
\]

which approximately commute in the point-weak* topology. Also, by Proposition 7.4, \( L^\infty(G) \) is injective in \( \text{mod} \ L^1(G)_1 \). Hence, combining [14, Theorem 7.2] with the canonical isometric isomorphism \( M^1_b(L^1(G)) \cong M^r_b(L^1(G)) \) (see, e.g., [25, 4.19]), we see that \( M^r_b(L^1(G)) \cong C^0_u(G)^* \) isometrically. Note that

\[ r_\alpha \in \mathcal{CB}_L^r(L^\infty(G), M_{n_a}(L^\infty(G) \oplus_\infty \mathbb{C})) = M_{n_a}(\mathcal{CB}_L^r(L^\infty(G), L^\infty(G) \oplus_\infty \mathbb{C})), \]

so we may write \( r_\alpha = [r^\alpha_{i,j}] \), with each \( r^\alpha_{i,j} \in \mathcal{CB}_L^r(L^\infty(G), L^\infty(G) \oplus_\infty \mathbb{C}) \). Letting \( p_1 \) and \( p_2 \) be the canonical projections from \( L^\infty(G) \oplus_\infty \mathbb{C} \) onto its first and second summands, for each \( i, j \) it follows that

\[ p_1 \circ r^\alpha_{i,j} \in \mathcal{CB}_L^r(L^\infty(G), L^\infty(G)) \cong M^r_b(L^1(G)) \cong C^0_u(G)^*, \]

(we used the fact that \( p_1 \) is a normal morphism) and \( p_2 \circ r^\alpha_{i,j} \in (L^\infty(G))^* = L^1(G) \), so there exist \( \mu^\alpha_{i,j} \in C^0_u(G)^* \) and \( f^\alpha_{i,j} \in L^1(G) \) for which

\[ r^\alpha_{i,j}(x) = (\Theta^r(\mu^\alpha_{i,j})(x), \langle x, f^\alpha_{i,j} \rangle), \quad x \in L^\infty(G). \]

The \( L^1(G) \)-module property implies that

\[
(\Theta^r(\mu^\alpha_{i,j})(x) \ast g, \langle f^\alpha_{i,j} \ast x, g \rangle) = (\Theta^r(\mu^\alpha_{i,j})(x \ast g), \langle x \ast g, f^\alpha_{i,j} \rangle)
\]

\[ = r^\alpha_{i,j}(x \ast g) \]

\[ = r^\alpha_{i,j}(x) \ast g \]

\[ = (\Theta^r(\mu^\alpha_{i,j})(x), \langle x, f^\alpha_{i,j} \rangle) \ast g \]

\[ = (\Theta^r(\mu^\alpha_{i,j})(x) \ast g, (\Theta^r(\mu^\alpha_{i,j})(x), g)) \]

for all \( g \in L^1(G) \). Hence,

\[ \Theta^r(\mu^\alpha_{i,j})(x) = f^\alpha_{i,j} \ast x = \Theta^r(f^\alpha_{i,j})(x), \quad x \in L^\infty(G), \]

which forces \( \mu^\alpha_{i,j} = f^\alpha_{i,j} \in L^1(G) \) by injectivity of \( \Theta^r \). In summary,

\[ r_\alpha(x) = [(f^\alpha_{i,j} \ast x, \langle f^\alpha_{i,j}, x \rangle)], \]

with \( f^\alpha_{i,j} \in L^1(G) \).

Next, observe that \( s_\alpha \) satisfies

\[ s_\alpha([y_{i,j}]) = \sum_{i,j=1}^{n_a} s^\alpha_{i,j}(y_{i,j}), \quad [y_{i,j}] \in M_{n_a}(L^\infty(G) \oplus_\infty \mathbb{C}), \]

for some collection of morphisms \( s^\alpha_{i,j} : L^\infty(G) \oplus_\infty \mathbb{C} \rightarrow L^\infty(G) \). Since \( L^\infty(G) \) is faithful, Lemma 7.6 implies that \( s^\alpha_{i,j} = \psi^\alpha_{i,j} \circ p_1 \) for normal morphisms \( \psi^\alpha_{i,j} \in \text{Hom}(L^\infty(G)) \). Thus,
where the first equality uses $\nu_{i,j}^\alpha \in C_0^u(\mathbb{G})^*$. The composition $s_\alpha \circ r_\alpha$ then satisfies
\[
s_\alpha \circ r_\alpha(x) = \sum_{i,j=1}^{n_\alpha} \Theta^r(\nu_{i,j}^\alpha)(f_{i,j}^\alpha \star x) = \sum_{i,j=1}^{n_\alpha} (\nu_{i,j}^\alpha \star f_{i,j}^\alpha) \star x = f_\alpha \star x, \quad x \in L^\infty(\mathbb{G})
\]
where $f_\alpha = \sum_{i,j=1}^{n_\alpha} \nu_{i,j}^\alpha \star f_{i,j} \in L^1(\mathbb{G})$ as $L^1(\mathbb{G})$ is an ideal in $C_u(\mathbb{G})^*$. Moreover, the isometry $M_{cb}(L^1(\mathbb{G})) \cong C_u(\mathbb{G})^*$ implies
\[
\|f_\alpha\|_{L^1(\mathbb{G})} = \|\Theta^r(f_\alpha)\|_{cb} = \|s_\alpha \circ r_\alpha\|_{cb} \leq 1,
\]
for all $\alpha$. Since $f_\alpha \star x \to x$ weak* for all $x \in L^\infty(\mathbb{G})$, the standard convexity argument yields a bounded right approximate identity for $L^1(\mathbb{G})$, which entails the co-amenability of $\mathbb{G}$.

(3) $\Rightarrow$ (1) follows similarly using left multipliers.

(1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) follow from Proposition 7.5.

(5) $\Rightarrow$ (1) is a $C^*$-analogue of (2) $\Rightarrow$ (1), using the $L^1(\mathbb{G})$-WEP of $C_0(\mathbb{G})$ in lieu of $L^1(\mathbb{G})$-injectivity of $L^\infty(\mathbb{G})$ to utilize the multiplier representation. If $RUC(\mathbb{G})$ is nuclear in $\text{mod} \ L^1(\mathbb{G})_1$ then there exist diagrams of morphisms in $\text{mod} \ L^1(\mathbb{G})_1$

\[
\begin{array}{ccc}
M_{na}(L^\infty(\mathbb{G}) \oplus_\infty \mathbb{C}) & & \text{id} \\
RUC(\mathbb{G}) \xrightarrow{r_\alpha} & & \xrightarrow{s_\alpha} \text{RUC}(\mathbb{G})
\end{array}
\]

which approximately commute in the point-norm topology. Since $C_0(\mathbb{G}) \subseteq RUC(\mathbb{G}) \subseteq M(C_0(\mathbb{G})) \subseteq C_0(\mathbb{G})^{**}$ (see [41]), restricting the morphisms $r_\alpha$ to $C_0(\mathbb{G})$ shows that the inclusion $C_0(\mathbb{G}) \hookrightarrow C_0(\mathbb{G})^{**}$ is nuclear. Thus, $C_0(\mathbb{G})$ has the $L^1(\mathbb{G})$-WEP by Proposition 7.2 and therefore $\widehat{\mathbb{G}}$ is amenable by Proposition 7.7. Combining [14, Theorem 7.2] with the canonical isometry $M_{cb}(L^1(\mathbb{G})) \cong M'_{cb}(L^1(\mathbb{G}))$ we see that $M'_{cb}(L^1(\mathbb{G})) \cong C_0^u(\mathbb{G})^*$ isometrically. Note that
\[
r_\alpha|_{C_0(\mathbb{G})} \in CB_{L^1(\mathbb{G})}(C_0(\mathbb{G}), M_{na}(L^\infty(\mathbb{G}) \oplus_\infty \mathbb{C})) = M_{na}(CB_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G}) \oplus_\infty \mathbb{C})) ,
\]
so we may write $r_\alpha = [r_{i,j}^\alpha]$, with each $r_{i,j}^\alpha \in CB_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G}) \oplus_\infty \mathbb{C})$. Letting $p_1$ and $p_2$ be the canonical projections from $L^\infty(\mathbb{G}) \oplus_\infty \mathbb{C}$ onto its first and second summands, for each $i,j$ it follows that
\[
p_1 \circ r_{i,j}^\alpha \in CB_{L^1(\mathbb{G})}(C_0(\mathbb{G}), L^\infty(\mathbb{G})) \cong M'_{cb}(L^1(\mathbb{G})) \cong C_0^u(\mathbb{G})^*,
\]
where the first equality uses [26, Proposition 4.1]. Also, $p_2 \circ r_{i,j}^\alpha|_{C_0(\mathbb{G})} \in C_0(\mathbb{G})^* = M(\mathbb{G})$, so there exist $\mu_{i,j}^\alpha \in C_0^u(\mathbb{G})^*$ and $\nu_{i,j}^\alpha \in M(\mathbb{G})$ for which
\[
r_{i,j}^\alpha(x) = (\Theta^r(\mu_{i,j}^\alpha)(x), \langle x, \nu_{i,j}^\alpha \rangle), \quad x \in C_0(\mathbb{G}).
\]
As in the proof of (2) $\Rightarrow$ (1), the $L^1(\mathbb{G})$-module property implies that
\[
(\Theta^r(\mu_{i,j}^\alpha)(x) \star g, \langle \nu_{i,j}^\alpha \star x, g \rangle) = (\Theta^r(\mu_{i,j}^\alpha)(x) \star g, \langle \Theta^r(\nu_{i,j}^\alpha)(x), g \rangle)
\]
for all $g \in L^1(\mathbb{G})$, which forces $\Theta^r(\mu_{i,j}^\alpha)(x) = \nu_{i,j}^\alpha \star x$ for all $x \in C_0(\mathbb{G})$ and therefore $\mu_{i,j}^\alpha = \nu_{i,j}^\alpha \in M(\mathbb{G})$ by injectivity of $\Theta^r$. In summary,
\[
r_\alpha(x) = [(\nu_{i,j}^\alpha \star x, \langle \nu_{i,j}^\alpha, x \rangle)],
\]
with $\nu_{i,j}^\alpha \in M(\mathbb{G})$. 

\[\text{(cont.)}\]
Next, observe that \( s_\alpha \) satisfies
\[
s_\alpha([y_{i,j}]) = \sum_{i,j=1}^{n_\alpha} s_{i,j}^\alpha(y_{i,j}), \quad [y_{i,j}] \in M_{n_\alpha}(L^\infty(G) \oplus \infty \mathbb{C}),
\]
for some collection of morphisms \( s_{i,j}^\alpha : L^\infty(G) \oplus \infty \mathbb{C} \to L^\infty(G) \). Since \( L^\infty(G) \) is faithful, Lemma 7.6 implies that \( s_{i,j}^\alpha = \psi_{i,j}^\alpha \circ p_1 \) for morphisms \( \psi_{i,j}^\alpha \in \text{Hom}(L^\infty(G)) \). Thus, \( \psi_{i,j}^\alpha |_{C_0(G)} = \Theta^r(\mu_{i,j}^\alpha) \) for some \( \mu_{i,j}^\alpha \in C_0^u(G)^* \). The composition \( s_\alpha \circ r_\alpha \) then satisfies
\[
s_\alpha \circ r_\alpha(x) = \sum_{i,j=1}^{n_\alpha} \Theta^r(\mu_{i,j}^\alpha)(\nu_{i,j}^\alpha \star x) = \sum_{i,j=1}^{n_\alpha} (\mu_{i,j}^\alpha \star \nu_{i,j}^\alpha) \star x = \nu_\alpha \star x, \quad x \in C_0(G),
\]
where \( \nu_\alpha = \sum_{i,j=1}^{n_\alpha} \mu_{i,j}^\alpha \star \nu_{i,j}^\alpha \in M(G) \). Moreover,
\[
\|\nu_\alpha\|_{M(G)} = \|\Theta^r(\nu_\alpha)\|_{cb} = \|s_\alpha \circ r_\alpha\|_{cb} \leq 1,
\]
fors all \( \alpha \). Since \( \nu_\alpha \star x \to x \) in norm for all \( x \in C_0(G) \), the net \( (\nu_\alpha) \) clusters to a right identity for \( M(G) \), which entails the co-amenability of \( G \) (see the end of the proof of [12, Theorem 5.12] for details).

(4) \( \Rightarrow \) (1) is proved similarly as (5) \( \Rightarrow \) (1), using left multipliers.

(1) \( \Rightarrow \) (6) follows from the implication (1) \( \Rightarrow \) (4) and the fact that (4) \( \Rightarrow \) (6), as noted above.

(6) \( \Rightarrow \) (1): If \( C_0(G) \hookrightarrow C_0(G)^{**} \) is nuclear in \( \text{mod} L^1(G)_1 \) then there exist diagrams of morphisms in \( \text{mod} L^1(G)_1 \)
\[
\begin{array}{ccc}
M_{n_\alpha}(L^\infty(G) \oplus \infty \mathbb{C}) & \xrightarrow{s_\alpha} & C_0(G)^{**}, \\
C_0(G) & \xrightarrow{id} & C_0(G)^{**} \quad \text{and} \quad \text{id} & \xrightarrow{s_\alpha} & C_0(G)^{**},
\end{array}
\]
that approximately commute in the point norm topology. Then \( C_0(G) \) has the \( L^1(G) \)-WEP by Proposition 7.2 and therefore \( \hat{G} \) is amenable by Proposition 7.7. Composing \( s_\alpha \) with the canonical morphism \( \pi : C_0(G)^{**} \to L^\infty(G) \), the resulting diagrams
\[
\begin{array}{ccc}
M_{n_\alpha}(L^\infty(G) \oplus \infty \mathbb{C}) & \xrightarrow{r_\alpha} & \pi \circ s_\alpha \quad \text{and} \quad \pi \circ s_\alpha \quad \xrightarrow{r_\alpha} & L^\infty(G),
\end{array}
\]
approximately commute in the point norm topology. An argument similar to the proof of (4) \( \Rightarrow \) (1) establishes the co-amenability of \( G \).

Finally, the equivalence (1) \( \iff \) (7) is similar to (1) \( \iff \) (6).

\[\square\]

**Remark 7.9.** Combining Theorem 7.8 with [12, Theorem 5.1], we see that the long-standing open problem on the equivalence between co-amenability of a locally compact quantum group \( G \) and amenability of \( \hat{G} \) [43] is the problem of equivalence between injectivity and semi-discreteness of \( L^\infty(G) \) in \( \text{mod} L^1(G)_1 \).

**Remark 7.10.** It is natural to wonder whether \( L^1(G) \)-nuclearity of \( C_0(G) \) is related to co-amenability of \( G \). When \( G \) is compact, meaning \( C_0(G) \) is unital, this is indeed the case...
as \( \text{LUC}(G) = \text{RUC}(G) = C_0(G) \). If \( G \) is not compact and \( C_0(G) \) is nuclear, then \( \hat{G} \) is amenable so that \( M'_b(L^1(G)) \cong C'_b(G)^* \), and any \( \varphi \in \text{Hom}(C_0(G), L^\infty(G)) \) decomposes into a sum of completely positive morphisms. However, there is no non-zero completely positive \( L^1(G) \)-morphism from \( L^\infty(G) \) into \( C_0(G) \), since the image of 1 in \( C_0(G) \) would necessarily be a fixed point for the \( L^1(G) \)-action on \( L^\infty(G) \), forcing it to lie in \( \mathbb{C}1 \).

**Remark 7.11.** There are notions of relative nuclearity/semi-discreteness for inclusions of \( C^*-\)/von Neumann algebras using the language of correspondences [3, 23, 39]. These notions may be viewed as analogues of nuclearity and semi-discreteness in categories of Hilbert modules over the respective algebras. For instance, if \( B \subseteq A \) is a unital inclusion of \( C^* \)-algebras with conditional expectation \( A \to B \), then the pair \((B, A)\) is strongly relatively nuclear in the sense of [23] if, roughly speaking, the identity map on \( A \) can be approximated by certain finite sums of \( B \)-module maps which factor through \( M_n(B) \) in a manner which utilizes the canonical Hilbert \( B \)-module structure on \( A \) induced from the conditional expectation. Viewing \( A \in B \text{mod} \), this module notion of nuclearity is different from that in Definition 7.1, which takes place in a different category, and in which the identity approximately factors through \( M_n(B^*) \). For a concrete distinction, the pair \((A, A)\) is always strongly relatively nuclear for any unital \( C^* \)-algebra \( A \). If a \( C^* \)-algebra \( A \) is nuclear in \( A \text{mod}_1 \) then it necessarily has Lance’s weak expectation property by Proposition 7.2 and [6, Corollary 3.10]. Similar remarks apply to relative semi-discreteness in the sense of [3].

8. **Equivalence of injectivity and semi-discreteness for crossed products**

In this section we establish the equivalence between \( A(G) \)-injectivity of \( G \wr \hat{M} \), \( A(G) \)-semi-discreteness of \( G \wr \hat{M} \), and amenability of \( W^* \)-dynamical systems \((M, G, \alpha)\) with \( M \) injective. The proof relies on a recent Herz-Schur multiplier characterization of amenable actions [5, Theorem 3.13] together with a generalized construction from (the proof of) [16, Theorem 3.5] to produce a specific \( A(G) \)-module left inverse to the dual co-action which lies in the point weak* closure of \emph{normal} \( A(G) \)-module maps. We begin with the necessary tools from dynamical systems.

A \( W^* \)-dynamical system \((M, G, \alpha)\) consists of a von Neumann algebra \( M \) endowed with a homomorphism \( \alpha : G \to \text{Aut}(M) \) of a locally compact group \( G \) such that for each \( x \in M \), the map \( G \ni s \mapsto \alpha_s(x) \in M \) is weak* continuous. We let \( M_c \) denote the unital \( C^* \)-subalgebra consisting of those \( x \in M \) for which \( s \mapsto \alpha_s(x) \) is norm continuous. By [36, Lemma 7.5.1], \( M_c \) is weak* dense in \( M \).

Every \( W^* \)-dynamical system induces a normal \( G \)-equivariant injective \( * \)-homomorphism \( \alpha : M \to L^\infty(G) \hat{\otimes} M \) via

\[
\alpha(x)(s) = \alpha_s^{-1}(x), \quad x \in M, \ s \in G,
\]

and a corresponding right \( L^1(G) \)-module structure on \( M \) [42, 18.6]. The crossed product of \( M \) by \( G \), denoted \( G \wr \hat{M} \), is the von Neumann subalgebra of \( \mathcal{B}(L^2(G)) \hat{\otimes} M \) generated by \( \alpha(M) \) and \( VN(G) \otimes 1 \).

The system \((M, G, \alpha)\) admits a dual co-action

\[
\hat{\alpha} : G \hat{\wr} M \to VN(G) \hat{\otimes} (G \hat{\wr} M)
\]

of \( VN(G) \) on the crossed product, given by

\[
\hat{\alpha}(X) = (\hat{W}^* \otimes 1)(1 \otimes X)(\hat{W} \otimes 1), \quad X \in G \hat{\wr} M,
\]
where $\hat{W}$ is the left fundamental unitary of $V N(G)$. On the generators we have $\hat{\alpha}(\hat{x} \otimes 1) = (\hat{W}^* (1 \otimes \hat{x}) \hat{W}) \otimes 1$, $\hat{x} \in VN(G)$ and $\hat{\alpha}(\alpha(x)) = 1 \otimes \alpha(x)$, $x \in M$. This co-action yields a canonical right operator $A(G)$-module structure on the crossed product via

$$X \cdot u = (u \otimes \text{id})\hat{\alpha}(X), \quad X \in G\hat{\kappa}M, \ u \in A(G).$$

A $C^*$-dynamical system $(A, G, \alpha)$ consists of a $C^*$-algebra endowed with a homomorphism $\alpha : G \to \text{Aut}(A)$ of a locally compact group $G$ such that for each $a \in A$, the map $G \ni s \mapsto \alpha_s(a) \in A$ is norm continuous.

A covariant representation $(\pi, \sigma)$ of $(A, G, \alpha)$ consists of a representation $\pi : A \to B(H)$ and a unitary representation $\sigma : G \to B(H)$ such that $\pi(\alpha_s(a)) = \sigma_s \pi(a) \sigma_s^{-1}$ for all $a \in A$, $s \in G$. Given a covariant representation $(\pi, \sigma)$, we let

$$(\pi \times \sigma)(f) = \int_G \pi(f(t))\sigma_t \ dt, \quad f \in C_c(G, A).$$

The full crossed product $G \ltimes_f A$ is the completion of $C_c(G, A)$ in the norm

$$\|f\| = \sup_{(\pi, \sigma)} \|(\pi \times \sigma)(f)\|$$

where the sup is taken over all covariant representations $(\pi, \sigma)$ of $(A, G, \alpha)$.

Let $A \subseteq B(H)$ be a faithful non-degenerate representation of $A$. Then $(\alpha, \lambda \otimes 1)$ is a covariant representation on $L^2(G, H)$, where

$$\alpha(a)\xi(t) = \alpha_{t^{-1}}(a)\xi(t), \ (\lambda \otimes 1)(s)\xi(t) = \xi(s^{-1}t), \ \xi \in L^2(G, H).$$

The reduced crossed product $G \ltimes A$ is defined to be the norm closure of $(\alpha \times (\lambda \otimes 1))(C_c(G, A))$. This definition is independent of the faithful non-degenerate representation $A \subseteq B(H)$. We often abbreviate $\alpha \times (\lambda \otimes 1)$ as $\alpha \times \lambda$.

Analogous to the group setting, dual spaces of crossed products can be identified with certain $A^*$-valued functions on $G$. We review aspects of this theory below and refer the reader to [36, Chapters 7.6, 7.7] for details.

For each $C^*$-dynamical system $(A, G, \alpha)$ there is a universal covariant representation $(\pi, \sigma)$ such that

$$G \ltimes_f A \subseteq C^*(\pi(A) \cup \sigma(G)) \subseteq M(G \ltimes_f A).$$

Each functional $\varphi \in (G \ltimes_f A)^*$ then defines a function $u : G \to A^*$ by

(15) $$\langle u(s), a \rangle = \varphi(\pi(a)\sigma_s), \quad a \in A, \ s \in G.$$ 

Let $B(G \ltimes_f A)$ denote the resulting space of $A^*$-valued functions on $G$. An element $u \in B(G \ltimes_f A)$ is positive definite if it arises from a positive linear functional $\varphi$ as above. We let $A(G \ltimes_f A)$ denote the subspace of $B(G \ltimes_f A)$ whose associated functionals $\varphi$ are of the form

$$\varphi(x) = \sum_{n=1}^{\infty} \langle \xi_n, (\alpha \times \lambda)(x)\eta_n \rangle, \quad x \in G \ltimes_f A,$$

for sequences $(\xi_n)$ and $(\eta_n)$ in $L^2(G, H)$ with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$. Then $A(G \ltimes_f A)$ is a norm closed subspace of $(G \ltimes_f A)^*$ which can be identified with $((G \ltimes A)^*)^\prime$.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system. We let $L^2(G, A)$ be the right Hilbert $A$-module given by the completion of $C_c(G, A)$ under $\|\xi\| = \|\langle \xi, \xi \rangle\|_A^{1/2}$, where

$$\langle \xi, \zeta \rangle = \int_G \xi(s)^* \zeta(s) \ ds, \quad \xi \cdot a(s) = \xi(s)a, \quad \xi, \zeta \in C_c(G, A), \ a \in A.$$
We let \( \lambda_s \otimes \alpha_t \in B(L^2(G, A)) \) denote the isometry
\[
(\lambda_s \otimes \alpha_t)\xi(r) = \alpha_t(\xi(s^{-1}r)), \quad \xi \in C_c(G, A), \ s, t \in G.
\]
By left invariance of the Haar measure and continuity of the action it follows that
\[
\langle (\lambda_s \otimes \alpha_t) \xi, (\lambda_s \otimes \alpha_t) \zeta \rangle = \alpha_t(\langle \xi, \zeta \rangle), \quad \xi, \zeta \in L^2(G, A), \ s, t \in G.
\]

**Proposition 8.1.** Let \((M, G, \alpha)\) be a W*-dynamical system, and let \(\xi \in C_c(G, Z(M)_c)\). The function \(h : G \times G \ni (s, t) \mapsto \langle \xi, (\lambda_s \otimes \alpha_t) \xi \rangle \) defines a normal completely positive \(A(G)\)-module map \(\Phi_h : VN(G) \otimes (G \bowtie M) \to G \bowtie M\) satisfying \(\|\Phi_h\|_{cb} = \|\langle \xi, \xi \rangle\|\) and
\[
(\Phi_h)_* : A(G \ltimes_f M_c) \ni u \mapsto (1 \otimes u) \cdot h \in A((G \times G) \ltimes_f (\mathbb{C} \otimes M_c)).
\]

*Proof.* We first construct a map at the level of the Fourier–Stieltjes space \(B(G \ltimes_f M_c)\). To that end, fix \(u \in B(G \ltimes_f M_c)^+\). We show that \((1 \otimes u) \cdot h \in B((G \times G) \ltimes_f (\mathbb{C} \otimes M_c))^+\), where \((\mathbb{C} \otimes M_c, G \times G, \text{tr} \otimes \alpha)\) is the tensor product system with the trivial action on \(\mathbb{C}\), and the operation \(\cdot\) is the pointwise action of \(M_c\) on its dual. Given \((s_1, t_1), \ldots, (s_n, t_n) \in G \times G\) and \(1 \otimes x_1, \ldots, 1 \otimes x_n\) in \(\mathbb{C} \otimes M_c\), we have
\[
\sum_{j,k=1}^n \langle ((1 \otimes u) \cdot h)(s_j^{-1}s_k, t_j^{-1}t_k), (\text{tr} \otimes \alpha)(s_j^{-1}t_j^{-1})(1 \otimes x_j^*x_k) \rangle
\]
\[
= \sum_{j,k=1}^n \langle u(t_j^{-1}t_k) \cdot h(s_j^{-1}s_k, t_j^{-1}t_k), \alpha_{t_j^{-1}t_k}^{-1}(x_j^*x_k) \rangle
\]
\[
= \sum_{j,k=1}^n \langle u(t_j^{-1}t_k), h(s_j^{-1}s_k, t_j^{-1}t_k)\alpha_{t_j^{-1}t_k}^{-1}(x_j^*x_k) \rangle
\]
\[
= \sum_{j,k=1}^n \langle u(t_j^{-1}t_k), \langle \xi, \lambda_{s_j^{-1}s_k} \otimes \alpha_{t_j^{-1}t_k}^{-1} \xi \rangle \alpha_{t_j^{-1}t_k}^{-1}(x_j^*x_k) \rangle
\]
\[
= \sum_{j,k=1}^n \langle u(t_j^{-1}t_k), \alpha_{t_j^{-1}t_k}^{-1}((\lambda_{s_j} \otimes \alpha_{t_j} \xi, \lambda_{s_k} \otimes \alpha_{t_k} \xi) x_j^*x_k) \rangle.
\]
Since \(\xi\) takes values in \(Z(M)_c\), it follows that
\[
\langle \lambda_{s_j} \otimes \alpha_{t_j} \xi, \lambda_{s_k} \otimes \alpha_{t_k} \xi \rangle x_j^*x_k = \int_G (\lambda_{s_j} \otimes \alpha_{t_j} \xi(r))^* x_j^*x_k (\lambda_{s_k} \otimes \alpha_{t_k} \xi)(r) \ dr.
\]
Hence,
\[
\sum_{j,k=1}^n \langle ((1 \otimes u) \cdot h)(s_j^{-1}s_k, t_j^{-1}t_k), (\text{tr} \otimes \alpha)(s_j^{-1}t_j^{-1})(1 \otimes x_j^*x_k) \rangle
\]
\[
= \int_G \sum_{j,k=1}^n \langle u(t_j^{-1}t_k), \alpha_{t_j^{-1}}^{-1}((\lambda_{s_j} \otimes \alpha_{t_j} \xi(r))^* x_j^*x_k (\lambda_{s_k} \otimes \alpha_{t_k} \xi)(r)) \rangle
\]
\[
= \int_G \sum_{j,k=1}^n \varphi_u \left(\alpha_{t_j^{-1}}^{-1}((\lambda_{s_j} \otimes \alpha_{t_j} \xi(r))^* x_j^*x_k (\lambda_{s_k} \otimes \alpha_{t_k} \xi)(r)) \sigma(t_j^{-1}) \sigma(t_k)\right)
\]
\[
= \int_G \varphi_u \left(\left(\sum_{j=1}^k x_j^*(\lambda_{s_j} \otimes \alpha_{t_j} \xi(r)) \sigma(t_j)\right)^* \left(\sum_{k=1}^k x_k (\lambda_{s_k} \otimes \alpha_{t_k} \xi(r)) \sigma(t_k)\right)\right).
\]
≥ 0.

It follows from [36, Proposition 7.6.8] (applied to the $C^*$-dynamical system $(\mathbb{C} \otimes M_c, G \times G, \text{tr} \otimes \alpha)$) that $(1 \otimes u) \cdot h \in B((G \times G) \rtimes_f (\mathbb{C} \otimes M_c))^+$. In particular, we obtain a well defined linear map

$$h : B(G \rtimes_f M_c) \ni u \mapsto (1 \otimes u) \cdot h \in B((G \times G) \rtimes_f (\mathbb{C} \otimes M_c))$$

through the Jordan decomposition. Since $(M_n(\mathbb{C}) \otimes M_c, G, \text{id}_{M_n} \otimes \alpha)$ and $(M_n(\mathbb{C}) \otimes \mathbb{C} \otimes M_c, G \times G, \text{id}_{M_n} \otimes \text{tr} \otimes \alpha)$ are $C^*$-dynamical systems satisfying $M_n(\mathbb{C}) \otimes (G \rtimes_f M_c) \cong G \rtimes_f (M_n(\mathbb{C}) \otimes \mathbb{C} \otimes M_c)$ and

$$M_n(\mathbb{C}) \otimes ((G \times G) \rtimes_f (\mathbb{C} \otimes M_c)) \cong (G \times G) \rtimes_f (M_n(\mathbb{C}) \otimes \mathbb{C} \otimes M_c)$$

canonical (by [45, Lemma 2.75]), and since [36, Proposition 7.6.8] applies to any $C^*$-dynamical system, the matricial analogue of the above argument together with the previous identifications show that the linear map

$$h : B(G \rtimes_f M_c) \ni u \mapsto (1 \otimes u) \cdot h \in B((G \times G) \rtimes_f (\mathbb{C} \otimes M_c))$$

is completely positive. Moreover, since the left marginal of $h$ is compactly supported, if $u$ is compactly supported, then so is $(1 \otimes u) \cdot h$. Since compactly supported elements of $B((G \times G) \rtimes_f (\mathbb{C} \otimes M_c))^+$ lie in $A((G \times G) \rtimes_f (\mathbb{C} \otimes M_c))^+$ [36, Lemma 7.7.6], it follows that $h$ induces a completely positive map

$$h : A(G \rtimes_f M_c) \ni u \mapsto (1 \otimes u) \cdot h \in A((G \times G) \rtimes_f (\mathbb{C} \otimes M_c)).$$

Since $A((G \times G) \rtimes_f (\mathbb{C} \otimes M_c))^* \cong ((G \times G) \rtimes (\mathbb{C} \otimes M_c))'' \cong VN(G)\overline{\otimes}(G\tilde{\otimes} M)$, we obtain a completely positive $A(G)$-module map

$$\Phi_h := h^* : VN(G)\overline{\otimes}(G\tilde{\otimes} M) \to G\tilde{\otimes} M,$$

where the module structure on the domain is on the left leg. Moreover,

$$\langle \Phi_h(1), u \rangle = \langle 1 \otimes 1, (1 \otimes u) \cdot h \rangle = \langle 1, u(e)h(e, e) \rangle = \langle h(e, e), u(e) \rangle = \langle \alpha(h(e, e)), u \rangle$$

for all $u \in A(G \rtimes_f M_c)$, so it follows that

$$\|\Phi_h\|_{cb} = \|\Phi_h(1)\| = \|\alpha(h(e, e))\| = \|h(e, e)\| = \|\xi, \xi\|.$$  

\[\Box\]

**Theorem 8.2.** Let $(M, G, \alpha)$ be a $W^*$-dynamical system with $M$ injective in $\mathbb{C} \text{mod}$. Then the following are equivalent.

1. $(M, G, \alpha)$ is amenable;
2. $G\tilde{\otimes} M$ is $A(G)$-semi-discrete;
3. $G\tilde{\otimes} M$ is $A(G)$-injective.

**Proof.** (1) $\Rightarrow$ (2): By the proof of [5, Theorem 3.13], there exists a net $(\xi_i) \in C_c(G, Z(M)_c)$ whose corresponding Herz-Schur multipliers $h_i(s) = \langle \xi_i, (\lambda_s \otimes \alpha_s)\xi_i \rangle$ satisfy $h_i(e) = \langle \xi_i, \xi_i \rangle = 1$ for all $i$ and $S_i \to \text{id}_{G\tilde{\otimes} M}$ point weak*, where $S_i$ are the normal completely positive $A(G)$-module maps on $G\tilde{\otimes} M$ determined by

$$S_i((\alpha \otimes \lambda)(f)) = \int_G \alpha(h(s)f(s))(\lambda_s \otimes 1) \, ds, \quad f \in C_c(G, M_c).$$

(Note the slightly different notation from [5].)
Let $h_i$ also denote the function

$$G \times G \ni (s, t) \mapsto \langle \xi_i, (\lambda_s \otimes \alpha_t) \xi_i \rangle \in Z(M)_c,$$

and let $\Phi_{h_i} : VN(G) \overline{\otimes} (G \bar{\rtimes} M) \to G \bar{\rtimes} M$ be the associated completely positive $A(G)$-module map from Proposition 8.1. Since $\|\langle \xi_i, \xi_i \rangle\| \leq 1$, each $\Phi_{h_i}$ is completely contractive. We claim that $\Phi_{h_i} \circ \hat{\alpha} = S_i$, where $\hat{\alpha} : G \bar{\rtimes} M \to VN(G) \overline{\otimes} (G \bar{\rtimes} M)$ is the dual co-action. To this end, let $f \in C_c(G, M)$. Then for any $u \in A(G \ltimes_f M)$, we have

$$\langle \Phi_{h_i}(\hat{\alpha}(\alpha \times \lambda(f))), u \rangle = \langle \hat{\alpha}((\alpha \times \lambda(f)), ((1 \otimes u) \cdot h_i)) \rangle$$

$$= \int_G \langle (1 \otimes \alpha(f(s)))(\lambda_s \otimes \lambda_s \otimes 1) \rangle ds, \langle (1 \otimes u) \cdot h_i \rangle$$

$$= \int_G \langle f(s), u(s) \cdot h_i(s, s) \rangle$$

$$= \int_G \langle h_i(s) f(s), u(s) \rangle$$

$$= \int_G \langle \alpha(h_i(s) f(s))(\lambda_s \otimes 1), u \rangle$$

$$= \langle S_i((\alpha \times \lambda(f)), u) \rangle.$$

Normality then establishes the claim.

Now, since $M$ is injective and $(M, G, \alpha)$ is amenable, $G \bar{\rtimes} M$ is an injective von Neumann algebra [2, Proposition 3.12]. Pick diagrams

$$\begin{array}{ccc}
G \bar{\rtimes} M & \xrightarrow{\id} & G \bar{\rtimes} M \\
\xrightarrow{r_j} & & \xleftarrow{s_j}
\end{array}$$

which approximately commute in the point-weak* topology. We then obtain diagrams

$$\begin{array}{ccc}
(VN(G) \oplus \mathbb{C}1) \overline{\otimes} (G \bar{\rtimes} M) & \xrightarrow{\id \otimes r_j} & M_{n_j}(VN(G) \oplus \mathbb{C}1) & \xrightarrow{\id \otimes s_j} & (VN(G) \oplus \mathbb{C}1) \overline{\otimes} (G \bar{\rtimes} M) \\
\xrightarrow{\Delta_+} & & \xrightarrow{\Phi_{h_i} \circ (p_1 \otimes \id)} & & \xrightarrow{\Phi_{h_i} \circ (p_1 \otimes \id)}
\end{array}$$

of morphisms in $\text{mod} A(G)_1$. Since $(p_1 \otimes \id) \circ \Delta_+ = \hat{\alpha}$ (as is easily verified), $\Phi_{h_i} \circ \hat{\alpha} = S_i$ for each $i$, and $S_i \to \id_{G \bar{\rtimes} M}$ point weak*, it follows that the diagrams approximately commute in the point-weak* topology after the appropriate iterated limit (first in $j$ then in $i$). Hence, $G \bar{\rtimes} M$ is $A(G)$-semi-discrete.

$(2) \Rightarrow (3)$ follows immediately from Proposition 7.4.

$(3) \Rightarrow (1)$: By [6, Theorem 5.2], $A(G)$-injectivity of $G \bar{\rtimes} M$ together with injectivity of $M$ implies that $(M, G, \alpha)$ is amenable. 

9. Outlook

Several natural lines of investigation are suggested by this work. First and foremost, the dual notion of topological finite co-presentation, its relation to operator module analogues of (weak*) exactness, and connection to exact quantum groups will appear in forthcoming work.
With the dual notions of finite (co)-presentation at hand, which are suitable analogues of finite dimensionality, one can formulate operator module analogues of more general finite dimensional approximation properties and pursue a Grothendieck type programme in the category of operator modules.

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