Abstract

Stabilizer codes form an important class of quantum error correcting codes [10, 2, 5] which have an elegant theory, efficient error detection, and many known examples. Constructing stabilizer codes of length \( n \) is equivalent to constructing subspaces of \( \mathbb{F}_p^n \times \mathbb{F}_p^n \) which are isotropic under the symplectic bilinear form defined by \( \langle (a, b), (c, d) \rangle = a^T d - b^T c \) [10, 12, 1]. As a result, many, but not all, ideas from the theory of classical error correction can be translated to quantum error correction [12, 7]. One of the main theoretical contribution of this article is to study stabilizer codes starting with a different symplectic form.

In this paper, we concentrate on cyclic codes. Modifying the symplectic form allows us to generalize the previous known construction for linear cyclic stabilizer codes [9, 8], and in the process, circumvent some of the Galois theoretic no-go results proved there. More importantly, this tweak in the symplectic form allows us to make use of well known error correcting algorithms for cyclic codes to give efficient quantum error correcting algorithms. Cyclicity of error correcting codes is a basis dependent property. Our codes are no more cyclic when they are derived using the standard symplectic forms. Hence this change of perspective is crucial from the point of view of designing efficient decoding algorithm for these family of codes. In this context, recall that for general codes, efficient decoding algorithms do not exist if some widely believed complexity theoretic assumptions are true.

Keywords and phrases Quantum Error Correction, Stabilizer codes, Linear Codes, Symplectic form

1 Introduction

Classical error correcting codes have been instrumental in various areas, not just in communication and data storage systems but even in complexity and cryptography. In the quantum setting, the major technique to construct error correcting codes is through stabilizers — on which there exists a substantial body of research [18, 6, 17, 2, 4, 5, 12].

The theory of quantum information is usually formulated using Hilbert spaces. Nonetheless, a stabilizer code of block length \( n \) over the \( p \)-ary alphabet (for some prime \( p \)) can be uniquely identified with a linear subspace \( C \) of the space \( \mathbb{F}_p^{2n} \) over the finite field
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This subspace essentially determines all the important properties of the code like its distance and dimension (Theorem 3) and hence stabilizer codes can be seen as classical linear codes of twice the block length. However, for quantum stabilizer codes, the associated subspace \( C \) should be *isotropic*: for any two vectors \( u = (a, b) \) and \( v = (c, d) \) of \( C \), the symplectic linear form \( (u, v) = a^T d - b^T c \) should vanish \([5, \text{Section II}][12, \text{Section IV}]\). Therefore, constructing quantum stabilizer codes boils down to constructing an isotropic subspace of \( \mathbb{F}_p^{2n} \) (Theorem 3). This additional condition of isotropy is what differentiates quantum codes from classical codes and often turns out to be a hindrance in transferring results from classical error correction to the quantum world.

Our main theoretical contribution is to rethink the role played by the form \( \langle \cdot, \cdot \rangle \) which arose due to our choice of the Weyl operators as the basis for quantum errors. The symplectic form \( \langle \cdot, \cdot \rangle \) captures the commutation relation between these Weyl operators. For odd characteristic fields, any full rank anti-symmetric bilinear form can be used as our starting point. Any such form is given by \( \langle u, v \rangle_A = u^T A v \) for some full rank skew-symmetric matrix \( A \) and arises from the commutation relation of an appropriate set of Weyl operators. The main idea of this paper is to generalize the study of stabilizer codes by choosing a different symplectic form as the starting point. Concrete constructions of quantum stabilizer codes in this paper fall in the subclass called *linear stabilizer codes* which includes the celebrated Laflamme code \([13]\).

The examples of codes that we study in detail are cyclic codes under a modified symplectic form. This modification of symplectic form enables us to generalize the previous known construction of linear cyclic codes by Dutta et. al. \([9]\). This generalization will be important if we can bound its distance and provide an error correcting algorithm. Significantly, the cyclic nature of these generalized codes allows us to design efficient (i.e. polynomial time in the block length) decoding algorithm (Section 4) along the lines of Dutta et.al. \([8, 9]\) which in turn uses the celebrated Berlekamp-Massey-Welch \([3, 15, 19]\) algorithm for classical cyclic codes. This cyclic property of the codes would not be apparent under the standard symplectic forms and these error correcting algorithms are therefore not evident if we were stuck to the standard symplectic form. We believe this is important as efficient decoding even for general classical codes are intractable \([11]\).

For the distance, the freedom in the choice of the symplectic form needs to be carefully balanced with the ability to give meaningful bounds to the distance of the code. As opposed to the codes constructed using the standard symplectic form, the (joint) Hamming weight of the code no longer measures the distance in our case. This makes the choice of the underlying symplectic form rather delicate. By restricting the symplectic form to be of a particular kind, we can still give meaningful bounds on the distance of these codes. Though, we lose a factor of 2 as compared to the codes constructed by Dutta et. al. \([9]\).

This generalization also allows us to construct codes for \((n, p)\) pairs for which no linear cyclic stabilizer codes can be constructed. In particular, for block lengths \( n \) that divides \( p^t + 1 \) for some odd \( t \), Dutta et.al. \([9]\ \text{Corollary IV.5}] \) (see also the Ph.D thesis \([8]\)) proved that there are no linear cyclic stabilizer codes. This impossibility arise due to the Galois theoretical restrictions imposed on certain ideals due to isotropy condition arising from the symplectic form \( \langle \cdot, \cdot \rangle \). By modifying the underlying form, we are able to circumvent this barrier (Section 3.3).  

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\(^2\) The case of characteristic 2 can also be handled in a similar way.
2 Preliminaries

In the quantum setting, a finite dimensional Hilbert space $\mathcal{H}$ plays the role of the alphabet. A quantum block code $C$ of length $n$ is just a subspace of the tensor product $\mathcal{H}^\otimes n$. From now on, we assume that the alphabet space $\mathcal{H}$ has a prime dimension $p$. When $p = 2$, the Hilbert space $\mathcal{H}$ is the space of qubits. We fix an orthonormal basis $\{ |x\rangle | x \in \mathbb{F}_p \}$ for $\mathcal{H}$.

This is analogous to picking $\mathbb{F}_p$ as the alphabet set in the classical case. Having picked such a basis, a natural basis for the space $\mathcal{H}^\otimes n$ is given by the set $\{|x\rangle \otimes \cdots \otimes |x\rangle\}$, where $|x\rangle$ denotes the state $|x_1\rangle \otimes \cdots \otimes |x_n\rangle$, with $x_i \in \mathbb{F}_p$ is the $i$-th component of $x$.

For any $a$ and $b$ in $\mathbb{F}_p^n$, define the unitary operators

$$U_a|x\rangle = |x + a\rangle \quad \text{and} \quad V_b|y\rangle = \omega^{b^T y}|y\rangle,$$

where $\omega$ is some fixed primitive $p$-th root of unity. These operators are called the Weyl operators and are used to model errors in the quantum setting: $U_a$ corresponds to the bit flips in the classical setting and $V_b$ is the phase flip. The set of all Weyl operators $U_aV_b$ forms the basis of the operator space $\mathcal{B}(\mathcal{H}^\otimes n)$. It follows from the general theory of quantum mechanics that any quantum error in transmission can essentially be modelled using the Weyl operators. In particular, the group generated by these operators are what we call the error group.

**Definition 1.** Let $p$ be an odd prime. The error group $\mathcal{E}$ associated with the block length $n$ is the group of all operators of the from $\zeta U_aV_b$, where $\zeta$ is a $p$-th root of unity and $a$ and $b$ are elements of $\mathbb{F}_p^n$.

When the characteristic $p$ is 2, the error group is similar, except that the scalar factor $\zeta$ is allowed to vary over all the 4-th roots of unity, $\{ \pm 1, \pm i \}$.

2.1 Stabilizer Code

Stabilizer codes are subspaces that are fixed by some subset $\mathcal{S}$ of the error group $\mathcal{E}$. More precisely, for any subset $\mathcal{S}$, the subspace

$$C_\mathcal{S} = \{ |\psi\rangle \in \mathcal{H}^\otimes n | \forall S \in \mathcal{S} \quad S|\psi\rangle = |\psi\rangle \}$$

is called the stabilizer code associated with the subset $\mathcal{S}$. First introduced by Gottesman [10] for the binary alphabet and subsequently generalized [17, 4, 12], the class of stabilizer codes plays a role analogous to the role played by linear codes in the classical setting. The following theorem specifies the conditions under which the code $C_\mathcal{S}$ is non-trivial, i.e., it has non-zero dimension.

**Theorem 2.** [10] For a subset $\mathcal{S}$ of $\mathcal{E}$, the associated stabilizer code $C_\mathcal{S}$ is non-trivial if and only if

1. $\mathcal{S}$ forms an Abelian subgroup of the error group $\mathcal{E}$.
2. The operator $\zeta I$ does not belong to $\mathcal{S}$ for any nontrivial root of unity $\zeta$. 


A subgroup $S$ satisfying the above conditions is called a stabilizer subgroup of the error group.

The centralizer $\mathcal{F}$ is the set of all operators in $\mathcal{E}$ that commute with all the operators of $S$. It determines the error correcting properties of the code $C_S$: an error in $S$ does not affect the code space whereas an error in $\mathcal{E} \setminus \mathcal{F}$ leaves a non-trivial phase on every vector in $C_S$ and hence can be detected. It is precisely the errors in $\mathcal{F} \setminus S$ that modifies a vector in $C_S$ to another vector in $C_S$ and hence cannot be detected [10]. Thus the error correcting parameters, like the distance of the code, depend on the centralizer.

Finding a stabilizer subgroup can be reduced to a problem of designing special subspaces of $\mathbb{F}_p^n \times \mathbb{F}_p^n$. Given two vectors $u = (a, b)$ and $v = (c, d)$ in $\mathbb{F}_p^n \times \mathbb{F}_p^n$, define the symplectic inner product, $(u, v)$, as the scalar $a^T d - b^T c$. A subspace $S$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ is called isotropic if and only if for any two vectors $u$ and $v$ in $S$, $(u, v) = 0$. From the following theorem, designing stabilizer subgroup is essentially equivalent to constructing isotropic subspace.

**Theorem 3.** [2, 3, 4]
1. Let $S$ be a stabilizer subgroup of the error group then the subset

   $$ S = \{ (a, b) \mid \zeta U_a V_b \in S \} $$

   is isotropic.

2. Let $S$ be any isotropic subset of $\mathbb{F}_p^n \times \mathbb{F}_p^n$, then the subgroup

   $$ S = \{ \rho(a, b) U_a V_b \mid (a, b) \in S \subseteq \mathbb{F}_p^n \times \mathbb{F}_p^n \} $$

   forms a stabilizer subgroup. In the above expression $\rho(a, b)$ is $\omega^{a^T b}$ if $p \neq 2$ and $i^{a^T b}$ if $p = 2$.

The above theorem follows from the fact that for two vectors $u = (a, b)$ and $v = (c, d)$, the Weyl operators $W_\vec{u} = U_a V_b$ and $W_\vec{v} = U_c V_d$ commute if and only if the symplectic inner product $(u, v) = 0$. In view of the above theorem, from now on, stabilizer codes will be characterized by the associated isotropic subspaces $S$. We also define the centralizer subspace which corresponds to the centralizer subgroup $\mathcal{F}$.

**Definition 4 (Centralizer subspace).** Let $S$ be any subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$. The centralizer subspace of $S$, denoted by $\mathcal{F}$, is the subspace of all vectors $u$ such that $(u, x) = 0$ for all $x$ in $S$.

The Hamming weight of an error measures the number of bits that the error corrupts in the classical setting. For quantum errors, the joint weight is the corresponding measure.

**Definition 5.** Let $u = (a, b)$ be any vector in the vector space $\mathbb{F}_p^n \times \mathbb{F}_p^n$. The joint weight $\text{wt}(u)$ is defined as the number of indices $1 \leq i \leq n$ such that the pair $\langle a_i, b_i \rangle$ is not $(0, 0)$. The joint weight of a subset $S$, $\text{wt}(S)$, is the minimum of the joint weights of elements in $S \setminus 0$.

We summarize the error correcting properties of the stabilizer code in the following theorem [5, 6].

**Theorem 6.** Let $S$ be an isotropic subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ and let $C_S$ be the associated stabilizer code. Then

1. The dimension of $S$ as a vector space over $\mathbb{F}_p$ is $n - k$ for some non-negative integer $k$.
   The dimension of $C_S$, as a Hilbert space is $p^k$. 


2. If every element of $S \setminus S$ has joint weight at least $d$, then the associated code $C_S$ can detect up to $d - 1$ errors and correct up to $\left\lfloor \frac{d+1}{2} \right\rfloor$ errors.

Often it is easier to lower bound the distance of the code $C_S$ by the joint weight $\text{wt}(S)$. This is known as the pure distance of the code.

### 3 Modifying the symplectic form

The isotropy condition associated with the symplectic form $(\cdot, \cdot)$ is essentially the only challenge that prevents us from lifting constructions of classical linear codes to quantum stabilizer codes (Theorem 3). This symplectic form sometimes even leads to certain Galois theoretic no-go results [4, 8]. This is what motivates us to modify the symplectic form and circumvent such impossibility theorems.

Let $p$ be an odd prime and let $A$ be any $2n \times 2n$ skew-symmetric matrix of full rank with entries in $F_p$. By appropriate relabelling of the Weyl operators, the theory of stabilizer codes can be built where the underlying symplectic form is given by $(u, v)_A = u^T A v$. This is because, there is always a basis transformation $C$ of $F_p^n \times F_p^n$ such that $(\cdot, \cdot)_C^T AC$ is the standard symplectic $(\cdot, \cdot)$ [4, Chapter XV, Corollary 8.2]. The theory of codes further needs to be built out of the Weyl operators $W_C$ instead of the standard Weyl operators $W_A$. A similar change of symplectic form can be done in the case when $p = 2$ as well.

The joint Hamming weight of the vector $(a, b)$ measures the number of indices corrupted by the error $U_a V_b$. If we restrict our attention to symplectic forms given by matrices

$$
\begin{pmatrix}
0 & \sigma \\
-\sigma^T & 0
\end{pmatrix}
$$

for some $n \times n$ permutation matrix $\sigma$, as opposed to general forms, a variant of joint Hamming weight would serve the purpose of measuring errors – the weight of $(a, b)$ in the modified setting should be the joint Hamming weight of $(a, \sigma b)$, i.e. permute the second component before computing weight. Furthermore, if the permutation $\sigma$ in the above matrix is also an involution, i.e. $\sigma^T = \sigma$, the associated symplectic form simplifies further: for vectors $u = (a, b)$ and $v = (c, d)$, we define the $\sigma$-symplectic inner product as follows:

$$(u, v)_\sigma = a^T \sigma d - b^T \sigma c.$$

The notion of isotropy and centralizer can now be formalized in this new setting.

**Definition 7.** A subspace $S$ of $F_p^n \times F_p^n$ is called a $\sigma$-isotropic subspace if for all $u$ and $v \in S$, $(u, v)_\sigma = 0$.

For any subspace $S$ of $F_p^n \times F_p^n$ the $\sigma$-centralizer $S^\sigma$ is the subspace of all vectors $x$ in $F_p^n \times F_p^n$ such that $(x, u)_\sigma = 0$ for all $u$ in $S$.

We have the following result that connects standard isotropy and $\sigma$-isotropy.

**Lemma 8.** For any subset $S$ of $F_p^n \times F_p^n$, let $S^\sigma$ denote the set of all elements $(a, \sigma b)$ such that $(a, b) \in S$, then.

1. $S$ is $\sigma$-isotropic if and only if $S^\sigma$ is isotropic.
2. $S$ is a $\sigma$-centralizer of $S$ if and only if $S^\sigma$ is a centralizer of $S^\sigma$.

**Proof.** Since $\sigma$ is an involution we have $\sigma^T = \sigma$. The proof then follows from the identity

$$(\langle a, b \rangle, \langle c, d \rangle)_{\sigma^T} = (\langle a, \sigma b \rangle, \langle c, \sigma d \rangle).$$
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In view of Theorem 8 and the previous lemma, it follows that constructing stabilizer codes is equivalent to constructing \( \sigma \)-isotropic subspaces of \( \mathbb{F}_p^n \times \mathbb{F}_p^n \). We have the following variant of Theorem 6 for \( \sigma \)-isotropic sets.

> **Theorem 9.** Let \( S \) be a \( \sigma \)-isotropic subspace of \( \mathbb{F}_p^n \times \mathbb{F}_p^n \) with \( \sigma \)-centralizer \( \mathcal{S} \) then

1. The dimension of \( S \) as a vector space over \( \mathbb{F}_p^n \times \mathbb{F}_p^n \) is at most \( n-k \). Using \( S \), we can construct a stabilizer code of dimension \( p^k \).
2. Suppose, every element of \( \mathcal{S} \setminus S \) has joint weight at least \( d \), then the associated stabilizer code has joint weight \( \left\lfloor \frac{d+1}{2} \right\rfloor \) and correct up to to \( \left\lfloor \frac{d}{2} \right\rfloor \) errors.

**Proof.** From Lemma 8, we have \( S^\sigma \) is isotropic and its centralizer is \( \mathcal{S}' \). Notice that the dimension of the space \( S^\sigma \) and \( S \) are equal as the map \( (a, b) \mapsto (a, \sigma b) \) is a permutation on the \( 2n \) indices. The stabilizer code required in part 1 is just the stabilizer code associated with the isotropic set \( S^\sigma \).

Consider any element \( (a, b) \) and let \( A \) denote the indices \( i \) such that \( a_i \neq 0 \). Similarly let \( B \) denote the set of indices \( j \) such that \( b_j \neq 0 \). Then the joint weight of \( (a, b) \) is the cardinality of \( A \cup B \). The joint weight of \( (a, \sigma b) \) is at least the maximum of the cardinalities of \( A \) and \( B \) and hence is at least \( \left\lfloor \frac{d+1}{2} \right\rfloor \). It follows that \( \mathcal{S}' \setminus S^\sigma \) has at least weight \( \left\lfloor \frac{d+1}{2} \right\rfloor \).

Using Theorem 6 we get the necessary result.

Notice that the previous Theorem is more or less equivalent to Theorem 6 except for a factor of 2 that we lost in the distance of the code. Theoretically this is the best bound that we can derive. However, the actual joint weight of \( \mathcal{S}' \setminus S^\sigma \) could even be higher than that of \( \mathcal{S} \setminus S \).

### 3.1 Cyclic codes

We fix a finite field \( \mathbb{F}_p \) as the alphabet set and a block length \( n \) that is co-prime to \( p \). Consider the right shift operator \( N \) that maps a vector \( a = (a_0, \ldots, a_{n-1}) \) to its right shift \( (a_{n-1}, a_0, \ldots, a_{n-2}) \). A classical code \( C \) is cyclic if for all \( a \) in \( C \) its right shift \( Na \) is also in \( C \). It turns out that the right generalization of this notion is simultaneously cyclic.

> **Definition 10.** A subset \( S \) of \( \mathbb{F}_p^n \times \mathbb{F}_p^n \) is simultaneously cyclic if for all \( (a, b) \) in \( S \), its simultaneous shift \( (Na, Nb) \) is also in \( S \).

A quantum stabilizer code is cyclic if the associated isotropic set \( S \) is simultaneously cyclic [9, III.2]. It turns out that the centralizer \( \mathcal{S} \) is also simultaneously cyclic and as in the case of classical cyclic codes can be seen as ideals over an appropriate cyclotomic ring. In the more general setting of \( \sigma \)-isotropic sets, for a simultaneously cyclic subspace \( S \), its centralizer \( \mathcal{S} \) need not be simultaneously cyclic and hence the theory of cyclotomic rings that we use will not be applicable any more. If we further restrict to involution \( \sigma_m \) of the form \( i \mapsto mi \) modulo \( n \) for some \( m \), we get back all the nice properties that we are accustomed to in the classical setting. Fix such a \( \sigma_m \) for the rest of the article. Notice, \( \sigma_m \) being an involution means \( m \) is a square root of \( 1 \) mod \( n \). It is easy to see that the shift operator \( N \) and \( \sigma_m \) satisfy the commutation relation:

\[
N \sigma_m = \sigma_m N^m
\]  

The following theorem more or less follows directly.
Theorem 11. Let $S$ be a $\sigma_m$-isotropic, simultaneously cyclic subspace of $F_p^n \times F_p^n$. Then its $\sigma_m$-centralizer $\overline{S}$ is also simultaneously cyclic.

Consider the cyclotomic ring $R = F_p[X] / (X^n - 1)$. As in the classical case, representing a vector $a$ as the polynomial $a(X) = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} \in R$ provides an elegant mechanism to work with cyclic codes. For example, the cyclic shift of a vector in $F_p^n$ is equivalent to multiplication by $X$ in the ring $R$. The following theorem expresses the $\sigma_m$-isotropy condition in terms of these polynomial representations.

Theorem 12. Let $S$ be a simultaneously cyclic subspace of $F_p^n \times F_p^n$. $S$ is a $\sigma_m$-isotropic subspace if and only if for any two elements $u = (a, b)$ and $v = (c, d)$ in $S$, the corresponding polynomials satisfy the identity:

$$a(X)d(X^{-m}) - b(X)c(X^{-m}) = 0 \mod X^n - 1 \tag{4}$$

Proof. The constant coefficient of the polynomial on the left hand side of Equation (4) is equivalent to the $\sigma_m$-isotropy condition. So the polynomial condition implies that the subspace $S$ is $\sigma_m$-isotropic.

For the converse, notice that $a(X^m) = (\sigma_m a)(X)$. Hence it is sufficient to prove the identity,

$$a(X)(\sigma_m d)(X^{-1}) - b(X)(\sigma_m c)(X^{-1}) = 0 \mod X^n - 1,$$

for every $(a, b), (c, d) \in \mathcal{S}$. The coefficient of $X^1$ on the left is,

$$a^T(N^i[\sigma_m]d - b^T(N^i[\sigma_m]c).$$

Using the Equation (3) repeatedly, the coefficient simplifies to,

$$a^T(\sigma_m N^{mi})d - b^T(\sigma_m N^{mi})c,$$

which is equal to $(\eta, (N^{mi}c, N^{mi}d))_{\sigma_m}$.

Since $S$ is $\sigma_m$-isotropic and simultaneously cyclic, this coefficient is 0.

3.2 Linear codes

Let $p$ be a prime. It is well known that the finite field $F_p$ has a unique quadratic extension $F_{p^2}$. Such an extension is essentially the field $F_p(\eta)$ consisting of all elements of the form $a + \eta b$, where $\eta$ is the root of some irreducible quadratic polynomial $\mu(Y) = Y^2 - c_1 Y - c_0$.

The encoding $(a, b) \rightarrow a + \eta b$ gives an encoding of $F_p^n \times F_p^n$ to $F_{p^2}^n$. We fix such an element for the rest of the section. Stabilizer codes associated to isotropic $F_{p^2}$-vector spaces are called linear stabilizer codes.

For an $F_p$ subspace $C$ of $F_p^n \times F_p^n$, the necessary and sufficient condition for it to be an $F_{p^2}$-subspace under the above encoding is that it should be closed under multiplication by $\eta$. Since $\eta^2 = c_0 + c_1 \eta$, $C$ is $F_{p^2}$-linear iff for all pair $(a, b)$ in $C$, the pair $(c_0 b, a + c_1 b)$ also belongs to $C$. We have the following theorem on $\sigma$-centralizers.

Theorem 13. Let $\sigma$ be an involution in $S_n$ and $S$ be a $\sigma$-isotropic subspace. Let $\overline{S}$ be the corresponding $\sigma$-centralizer subspace. Then $S$ is $F_{p^2}$-linear implies $\overline{S}$ is $F_{p^2}$-linear.
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**Proof.** Consider any arbitrary element \( \mathbf{v} = (x, y) \) in \( S \). By the definition of \( \sigma \)-centralizer, for any \( \mathbf{u} = (a, b) \) in \( S \), we have \( \langle \mathbf{u}, \mathbf{v} \rangle_\sigma = 0 \). In addition, we also have

\[
\langle \eta \mathbf{u}, \mathbf{v} \rangle_\sigma = c_0 \mathbf{b}^T \sigma \mathbf{y} - a^T \sigma \mathbf{x} - c_1 \mathbf{b}^T \sigma \mathbf{x} = 0.
\] (5)

due to the linearity of \( S \). Now consider \( \langle \mathbf{u}, \eta \mathbf{v} \rangle_\sigma \). We have:

\[
\langle \mathbf{u}, \eta \mathbf{v} \rangle_\sigma = a^T \sigma (x + c_1 y) - b^T \sigma (c_0 y) = a^T \sigma x + c_1 a^T \sigma y - c_0 b^T \sigma y = -c_1 b^T \sigma x + c_1 a^T \sigma y \quad \text{from (5)}
\]

This proves that \( \eta \mathbf{v} \) is in \( S \) and hence \( S \) is a \( \mathbb{F}_p^2 \)-subspace. \( \iff \)

The above theorem is true for any involution \( \sigma \). In addition, if the involution is \( \sigma_m \), using Theorem 11 we have

**Theorem 14.** Let \( S \) be a simultaneously cyclic, \( \sigma_m \)-isotropic and \( \mathbb{F}_p^2 \)-linear subspace. Then its \( \sigma_m \)-centralizer \( \overline{S} \) is simultaneously cyclic and \( \mathbb{F}_p^2 \)-linear subspace.

We now look at \( \mathbb{F}_p^2 \)-linear, \( \sigma_m \)-isotropic, and simultaneously cyclic subspaces. Recall that we encoded a vector \( a \) as the polynomial \( \sum a_i X^i \) in the ring \( R \). In this case, we encode pairs of polynomials in \( R \times \mathbb{R} \) as elements of \( R(\eta) = \mathbb{F}_p^2[X]/(X^n - 1) \). A \( \mathbb{F}_p^2 \)-linear simultaneously cyclic subspace has to be an ideal of \( R(\eta) \) and hence should have a generating polynomial. A consequence of previous theorem is that its centralizer would also be an ideal and have a generating polynomial.

### 3.3 Linear stabilizer codes from \( \sigma_m \)-isotropic sets

An \( \mathbb{F}_p^2 \)-linear simultaneously cyclic subspace is equivalent to an ideal of \( R(\eta) \). Define a triplet \( (n, p, m) \) to be **good** if \( n \mid p^t + m \) for some \( t \) and \( m^2 = 1 \mod n \). The following theorem characterizes \( \sigma_m \)-isotropic ideals of \( R(\eta) \) for good triplets.

**Theorem 15.** Let \( n \mid p^t + m \), where \( m \) is a square root of 1 mod \( n \). An ideal \( S \) of \( R(\eta) \) is \( \sigma_m \)-isotropic if and only if \( S \) is generated by the product of two polynomials \( g(X)h(X, \eta) \) which satisfy the following conditions.

1. \( g(X) \) is a factor of \( X^n - 1 \) in \( \mathbb{F}_p \) which includes all the odd irreducible factors.
2. \( h(X, \eta) \) is such that for any \( r(X, \eta) \) which is a factor of \( (X^n - 1)/g(X) \) over \( \mathbb{F}_p(\eta) \) exactly one of \( r(X, \eta) \) or its conjugate \( r(X, \eta)^{\eta} \) divides \( h(X, \eta) \).

**Hence the ideal will be non-trivial only if \( t \) is even.**

**Proof.** Refer to the section 6.1 of the appendix. \( \iff \)

Dutta et. al. [9] also proved that if \( n \mid p^t + 1 \) for some odd \( t \) then there are no linear cyclic stabilizer codes. This will be the case when the order of \( p \) in \( \mathbb{Z}_n \) is 2! (for some odd \( t \)) and \( p^t = -1 \mod n \). However, Thm. 15 allows us to construct \( \sigma_{-1} \)-isotropic ideals for such \( (n, p) \) pairs.
For \( m \neq -1 \), the \( \sigma_m \)-isotropy condition as polynomials is given by
\[
a(X)d(X)^{p^t} - b(X)c(X)^{p^t} \mod X^n - 1
\]
(6)
The \( p^t \) powers that occur in the above condition leads to certain Galois theoretic situations that makes such ideals trivial when \( t \) is odd. The \( \sigma_{-1} \)-isotropy condition on the other hand is much simpler
\[
a(X)d(X) - b(X)c(X) \mod X^n - 1.
\]
This is the reason why we are able to construct linear cyclic stabilizer codes when we modify the symplectic condition to the bilinear form \( \langle \cdot, \cdot \rangle_{\sigma_{-1}} \).

Our results also gives other variants: If \( p \) has order \( 4t \) and the quantity \( -m = p^{2t} \) is a square root different from \( \pm 1 \), by considering \( \sigma_m \)-isotropic sets we get examples of codes that were not considered above. Such non-trivial square roots exists when the block length \( n \) is composite. With such variants, we may be able to prove better lower bounds than the one mentioned here (Theorem 9).

When the order of \( p \) in \( \mathbb{Z}_n \) is odd, all the factors of \( X^n - 1 \) have odd degree. Therefore, we do not have any non-trivial \( \sigma_m \)-isotropic ideals.

To summarize:
- Prime \( p \) has order \( 4t \) and \( p^{2t} = -1 \) then construct codes based on the work of Dutta et al. [9, 8].
- Prime \( p \) has even order, construct codes based on any one of the non-trivial \( \sigma_{-1} \)-isotropic ideals using Theorem 15.
- Prime \( p \) has odd order in \( \mathbb{Z}_n \), our strategy fails.

In the constructions that we have sketched above, we need a characterization of the \( \sigma \)-centralizer \( \mathfrak{S} \) if we need some handle on the error correction properties. We have the following proposition (proof in section 6.2 of the appendix).

**Proposition 16.** Consider a \( \sigma \)-isotropic ideal \( S \) whose generating polynomial is \( q(X)h(X, \eta) \) as in Theorem 15. The \( \sigma \)-centralizer \( \mathfrak{S} \) of \( S \) is given by the ideal generated by \( h(X, \eta) \).

## 4 BCH distance and decoding

For the generators of cyclic codes, we now define the BCH-distance.

**Definition 17.** Let \( f(X) \) be any factor of \( X^n - 1 \) over a field \( \mathbb{F}_q \). The BCH distance of \( f(X) \) is the maximum \( l \) such that there exists a sequence of \( l-1 \) consecutive powers \( \beta, \beta^2, \ldots, \beta^{l-1} \), all of which are roots of \( f(X) \) for some primitive \( n \)-th root of unity \( \beta \).

The BCH distance of \( f(X) \) gives a lower bound on the distance of the associated ideal as a code. Notice that for the codes constructed in the previous section, the centralizer as an ideal over \( \mathbb{R}[\eta] \) has generator \( h(X, \eta) \) and the Hamming distance of this code is its joint weight in the quantum setting. Using Theorem 9, we have the following proposition.

**Proposition 18.** Let \( S \) be a linear cyclic stabilizer code generated using one of Theorems 15 and let \( q(X)h(X, \eta) \) be its generator polynomial. If the polynomial \( h(X, \eta) \) has BCH distance \( d \), then the associated stabilizer code has distance at-least \( \left\lceil \frac{d+1}{2} \right\rceil \).

For a classical cyclic code, the celebrated Berlekamp-Massey-Welch [3, 15, 19] algorithm gives an efficient error correction procedure. We reformulate this result for our use in the decoding of quantum cyclic codes.
Stabilizer codes from modified symplectic form

**Theorem 19** (Berlekamp-Massey-Welch [3 15 19]). Let \( f(X) \) be any factor of \( X^n - 1 \) over a finite field \( \mathbb{F}_q \) with BCH distance at least \( 2t + 1 \). Let \( e(X) \) be any unknown polynomial with Hamming weight at most \( t \). There exists an efficient algorithm, that takes as input any \( r(X) = e(X) \mod f(X) \) and outputs \( e(X) \).

Let \( S \) be any \( \sigma_m \)-isotropic cyclic code that we constructed in the previous section. We fix some notation for this section. Recall, \( S^{\sigma_m} \) is the space of all elements \( (a, \sigma_m(b), \mathbf{where} (a, b) \mathbf{in} S \). This set forms an isotropic subset under the standard symplectic form (Lemma 8). The corresponding stabilizer group \( S' \) consists of Weyl operators \( W_{a,b}' = \rho(a, \sigma_m(b))U_aV_{\sigma_m,b} \).

We would recover one coefficient at a time. Recall that the sent message \( |\psi\rangle = U_{e_1}V_{\sigma_m,e_2}|\psi\rangle \) for some unknown \( e_1 \) and \( e_2 \) in \( \mathbb{F}_p^* \). It is sufficient to find \( e_1 \) and \( e_2 \) to recover the actual message: given \( e_1 \) and \( e_2 \) we apply the operator \( V_{\sigma_m,e_2}^\dagger U_{e_1} \) to \( |\psi\rangle \). The following proposition plays an important role in finding \( e_1 \) and \( e_2 \) efficiently.

**Proposition 20.** For any \((a, b) \in S \), we can efficiently compute the polynomial \( a(X)e_2(X^{-m}) - b(X)e_1(X^{-m}) \mod X^n - 1 \)

**Proof.** We need to compute the following polynomial.

\[
a(x)e_2(X^{-m}) - b(x)e_1(X^{-m}) = \sum_{i=0}^{n-1} \left( (N^i a, N^i b), (e_1, e_2) \right)_{\sigma_m} X^{mi} \mod X^n - 1 \tag{7}
\]

We would recover one coefficient at a time. Recall that the sent message \( |\psi\rangle \) is stabilized by \( W_{a,b} \), and it is easy to verify the commutation relation

\[
W_{a,b}'U_{e_1}V_{\sigma_m,e_2} = \omega((a, b), (e_1, e_2))_{\sigma_m} U_{e_1}V_{\sigma_m,e_2} W_{a,b}'.
\]

Hence the received vector \( |\psi\rangle = U_{e_1}V_{\sigma_m,e_2}|\phi\rangle \) is an eigen vector of \( W_{a,b}' \) with eigen value \( \omega((a, b), (e_1, e_2))_{\sigma_m} \). Using the phase estimation algorithm [10, 5.2], we extract the inner product \( (N^i a, N^i b), (e_1, e_2) \) without modifying the received state \( |\phi\rangle \). This recovers the constant term of the polynomial.

We repeat the above procedure with the Weyl Operator \( W_{N^i a, N^i b}' \) to compute the coefficient \( \left( (N^i a, N^i b), (e_1, e_2) \right)_{\sigma_m} \). Each of these phase estimations gives us an additional coefficient of the polynomial. This allows us to recover the polynomial in Equation (7) after \( n \) phase estimations.

Notice that if the pair \((e_1, \sigma_m e_2)\) is of joint weight less than \( \tau \), the polynomial \( e_1 + \eta e_2 \) as a polynomial in \( \mathcal{R}[\eta] \) will have at most \( 2\tau \) non-zero coefficients. The main idea is to use Proposition 20 to recover the polynomial \( e_1 + \eta e_2 \) modulo the generator polynomial \( h(X, \eta) \).

Then using the classical Berlekamp-Massey-Welch algorithm we recover \( e_1 + \eta e_2 \). This is formalized in the following proposition.

**Theorem 21.** Let \( S \) be \( \sigma_m \)-isotropic ideal of \( \mathcal{R}[\eta] \). Let \( g(X), h(X, \eta) \) be the polynomial satisfying the properties in Theorem 15. Let \( h(X, \eta) \) be of BCH distance \( 4\tau + 1 \). There exists an efficient quantum algorithm that corrects errors of joint weight \( \tau \)

**Proof.** From the proof of Theorem 15 (Section 6.1), we know that there exists a polynomial \( a(X) \in \mathcal{R} \) such that \( g(X) + \eta a(X) \) belongs to the ideal \( S \). By abusing the notation, let \( (g, a) \in S \). For \((e_1, e_2) \in \mathbb{F}_p^n \times \mathbb{F}_p^n \),

\[
r'(X) = g(X)e_2(X^{-m}) - a(X)g(X)e_1(X^{-m})
\]
By Proposition 20 we can compute $r(X)$ efficiently.

Divide both side by $g(X)$ and then take modulo $h(X, \eta)$. From the proof of Theorem 15 (Section 6.1 Proposition 25) we know that $a(X) = \eta^p \mod h(X, \eta)$. Thus,

$$r(X) = e_2(X^{-m}) - \eta^p e_1(X^{-m}) \mod h(X, \eta)$$

If joint weight $wt(e_1, \sigma_m e_2)$ is at most $\tau$ then the joint weight of $wt(e_1, e_2)$ is at most $2\tau$. Notice that the joint weight $wt(\sigma_m e_1, \sigma_m e_2)$ is equal to the joint weight $wt(e_1, e_2)$. Thus, we could use Berlekamp-Massey-Welch algorithm (Theorem 19) to compute $e_1, e_2$.

4.1 Explicit Examples

The Table 1 shows the codes constructed over $\mathbb{F}_2$ based on Theorem 15. Fix a primitive $n$-th root of unity $\beta$. The table uses the following notation: $g_i$ (respectively $h_i$) is the irreducible factor of $X^n - 1$ over the field $\mathbb{F}_2$ (respectively $\mathbb{F}_{2^2}$) with $\beta^i$ as one of its roots.

| $n$ | $k$ | Factors | Consecutive Root of $h(X, \eta)$ | Theorem 15 Detect | Brute Force Detect | Theorem 15 Correct | Brute Force Correct |
|-----|-----|---------|---------------------------------|-------------------|-------------------|-------------------|-------------------|
| 5   | 1   | $g_0$   | $h_2$                           | 1                 | 0                 | 1                 | 0                 |
| 9   | 1   | $g_0$   | $h_3 h_6$                       | 1                 | 0                 | 1                 | 0                 |
| 11  | 1   | $g_0$   | $h_1$                           | 1                 | 0                 | 2                 | 1                 |
| 13  | 1   | $g_0$   | $h_2$                           | $\beta^3, \beta^4, \beta^5$ | 1   | 2                 | 4                 | 2                 |
| 15  | 1   | $g_0 h_5 h_6 h_7$ | $\beta^4, \beta^9, \beta^{10}$ | 2                 | 1                 | 3                 | 1                 |
| 15  | 5   | $g_0 g_7$ | $h_1 h_5 h_6$ | $\beta^4, \beta^5, \beta^6$ | 1   | 0                 | 2                 | 1                 |
| 15  | 9   | $g_0 g_3 g_7$ | $h_1 h_5$ | $\beta^4, \beta^5$ | 1   | 0                 | 1                 | 0                 |
| 17  | 1   | $g_0$   | $h_3 h_6$                       | $\beta^3, \beta^4, \beta^5$ | 3   | 1                 | 5                 | 2                 |
| 19  | 1   | $g_0$   | $h_1$                           | $\beta^4, \beta^5, \beta^6$ | 2   | 1                 | 3                 | 1                 |
| 21  | 13  | $g_0 g_1 g_2 g_0$ | $h_1$ | $\beta^3, \beta^5$ | 1   | 0                 | 1                 | 0                 |
| 25  | 1   | $g_0$   | $h_1 h_5$                       | $\beta^4, \beta^5, \beta^6$ | 1   | 0                 | 2                 | 1                 |
| 27  | 7   | $g_0 g_3$ | $h_1 h_9$ | $\beta^3, \beta^{10}$ | 1   | 0                 | 1                 | 0                 |
| 29  | 1   | $g_0$   | $h_1$                           | $\beta^4, \beta^5, \beta^6$ | 2   | 1                 | 3                 | 1                 |

5 Conclusion

The main theme of this article is to construct stabilizer codes based on alternate symplectic forms. Any two (full rank) symplectic forms are equivalent in the sense that the associated Weyl operators form a basis set for the error space and hence can mathematically model all quantum operations on the relevant Hilbert space. Modifying the symplectic form therefore is clearly not just restricted to cyclic codes. However, if we need to get meaningful bounds on the distance, these changes need to be balanced carefully. In the context of cyclic linear stabilizer code, symplectic forms of the kind $\langle \cdot, \cdot \rangle_{\sigma_m}$ were the only ones that gave us enough control to carry out our constructions and get nontrivial distance bounds at the same time. A future line of work would be to extend some of the ideas here to general stabilizer codes. We believe would lead to some interesting examples of quantum codes.

The equivalence of symplectic forms means that our construction could as well be carried out by considering the set $S^{\sigma_m}$ under the standard isotropy condition. However, notice that
the set $S^m$ as opposed to $S$ is not cyclic and hence the efficient decoding algorithms that we have will not be apparent in the setting of the standard symplectic forms. The reason for this anomaly is that properties like distance and cyclicity are not preserved under a basis change. Therefore, visualizing this code as the subspace $S$ as opposed to $S^m$ is crucial. This is what sets the codes apart from other constructions of codes for similar lengths. In general, decoding is an intractable problem even for classical codes.

Modification of the symplectic condition is not without a cost; we lose a factor of 2 in lower bounding the distance. Experimental constructions given in this article demonstrates cases where we do not incur this loss. However, for the efficient decoding algorithm, we cannot avoid this sacrifice in the distance.

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Appendix

We will set up some preliminaries before proceeding with the proofs.

For a given polynomial $g(X)$ in $\mathcal{R}$, the polynomial $g(X^{-m})$ plays an important role in the $\sigma_m$-isotropy condition. If there exists $t$ such that $n \mid p^t + m$, then $X^{-m} = X^{p^t}$. (If $m = -1$, then we will choose $t = 0$). Therefore, for any polynomial $a \in \mathcal{R}$,

$$a(X^{-m}) = a(X)^{p^t} \mod X^n - 1. \quad (8)$$

The $\sigma_m$-isotropy (Equation 4) becomes

$$a(X)d(X)^{p^t} - b(X)c(X)^{p^t}$$

For a given simultaneously cyclic subspace $S$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$, let’s define the following two set.

$$G = \{a \mid \text{exists } b \text{ such that } (a, b) \in S\} \quad (9)$$

$$H = \{b \mid (0, b) \in S\} \quad (10)$$

As $S$ is simultaneously cyclic, both $G$, $H$ are cyclic code in $\mathbb{F}_p^n$. As seen earlier, they could be thought of as an ideal in $\mathcal{R}$. Thus they are generated by a factor of $X^n - 1$. Let $g(X)$ and $h(X)$ be the factors of $X^n - 1$ that generates $G$ and $H$ as the ideals of $\mathcal{R}$ respectively.

It is easy to see that any simultaneously cyclic subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ could be be described by three polynomials over $\mathcal{R}$.

> **Lemma 22.** Every simultaneously cyclic subspace $S$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ can be described by three polynomials $g(X)$, $h(X)$ which are factor of $X^n - 1$ over $\mathbb{F}_p$ and $f(X) \in \mathcal{R}$ such that $(g, f) \in S$ and $(0, h) \in S$.

In Lemma 23 if $h(X) = 0 \mod X^n - 1$ then such a subspace is known as uniquely cyclic.

> **Lemma 23.** Let $m = -1$ or $n$ divides $p^t + m$. Let $S$ be a simultaneously cyclic, $\sigma_m$-isotropic and $\mathbb{F}_p^n$ linear subspace then $S$ is uniquely cyclic.

**Proof.** The proof of this lemma is similar to the proof of Lemma IV.2 in [9]. For completeness we state it here.

The Lemma 22 states that $S$ could be expressed by three polynomials $g(X)$, $f(X)$, and $h(X)$. The polynomials $g(X)$, $h(X)$ are factors of $X^n - 1$ and $f(X)$ is a polynomial in $\mathcal{R}$ such that the elements $(g, f)$, $(0, h)$ are in $S$.

To show that $S$ is an uniquely cyclic subspace we need to show that $h(X)$ is a multiple of $X^n - 1$. We would show this in two steps. First, we would show $g(X) \mid h(X)$. This proof is similar as in Lemma IV.2 [9] (5.11). Now we would show $\frac{X^n - 1}{g(X)} \mid h(X)$. 

[9] Lloyd R Welch and Elwyn R Berlekamp. Error correction for algebraic block codes, December 30 1986. US Patent 4,633,470.
The elements \((g, h)\), \((0, h)\) belongs to \(S\) and it is \(\sigma_m\)-isotropic. Thus, from the \(\sigma_m\)-isotropy condition \([4]\) between \((g, f)\) and \((0, h)\), we have the following.

\[
\begin{align*}
g(X^{-m})h(X) &\equiv 0 \mod X^n - 1 \\
g(X)^{\sigma_1}h(X) &\equiv 0 \mod X^n - 1 \\
h(X) &\equiv 0 \mod \frac{X^n - 1}{g(X)}
\end{align*}
\]

Thus \(\frac{X^n - 1}{g(X)} \mid h(X)\).

\[\textbf{6.1 Proof of Theorem} 15\]

We would characterize the \(\sigma_m\)-isotropic ideal \(S\) of \(\mathcal{R}(\eta)\). From Lemma \([22]\) and Lemma \([23]\) we know that such an ideal \(S\) could be expressed by two polynomials \(g(X)\) and \(f(X)\). The polynomials \(g(X)\) is a factor of \(X^n - 1\) and \(f(X)\) is a polynomial in \(\mathcal{R}\) such that \((g, f) \in S\).

Any element of \(S\) can be expressed as \(b(X)g(X) + \eta b(X)f(X)\), for some \(b(X)\) in \(\mathcal{R}\). Since, we have \((g(X) + \eta f(X))\) in \(S\), by \(\mathbb{F}_p(\eta)\) linearity, we have \(\eta(g(X) + \eta f(X))\) in \(S\). There exists a polynomial \(a(X) \in \mathcal{R}\) such that we get the following.

\[
\eta(g(X) + \eta f(X)) = a(X)(g(X) + \eta f(X)) \mod X^n - 1 \tag{11}
\]

Compare the coefficient of \(\eta\) in Equation \(11\) we have the following.

\[
f(X) = c_0^{-1} a(X) g(X) \mod X^n - 1 \quad \tag{12}
\]

\[
m(a(X)) = 0 \mod \frac{X^n - 1}{g(X)} \quad \tag{13}
\]

Recall that \(\eta\) is a root of an irreducible quadratic polynomial \(\mu(Y) = Y^2 - c_1 Y - c_0\). Let \(r(X)\) be any irreducible factor \(r(X)\) of \(\frac{X^n - 1}{g(X)}\). Then Equation \(13\) implies that \(a(X)\) mod \(r(X)\) is a root of \(\mu(X)\). An immediate consequence of this is the following proposition.

\[\textbf{Proposition 24.} \ g(X) \text{ contains all the odd degree factors.}\]

**Proof.** The field extension \(\mathbb{F}_p[X]/r(X)\) contains the root of the polynomial \(\mu(X)\). Thus it contains \(\mathbb{F}_p^2\) as a subfield. This implies \(r(X)\) has to be of even degree. \(\square\)

By Proposition \([24]\) we know that \(r(X)\) is of even degree. Thus each \(r(X)\) factorizes as \(r'(X, \eta) r''(X, \eta)^p\) over \(\mathbb{F}_p^2\).

\[\textbf{Proposition 25.} \ r'(X, \eta) \mid h(X, \eta) \text{ if and only if } r'(X, \eta)^p + h(X, \eta)\]

**Sketch.** It could be shown that \(h(X, \eta) = gcd\left(\frac{X^n - 1}{g(X)}, 1 + \eta c_0^{-1} a(X)\right)\). By Equation \(13\) we know \(a(X)\) mod \(r'(X, \eta)\) is either \(\eta\) or \(\eta^p\). We could also show that \(a(X) = \eta^p\) mod \(r'(X, \eta)\) if and only if \(a(X) = \eta\) mod \(r'(X, \eta)^p\). Thus \(r'(X, \eta) \mid h(X, \eta)\) if and only if \(a(X, \eta) = \eta^p\) mod \(r'(X, \eta)\). The details of the proof could be derived from Theorem 5.15 \([8]\) Chapter 5. \(\square\)

When \(m = -1\), \(t\) is the order of \(p\) in \(\mathbb{Z}_n\). However, when the order of \(p\) in \(\mathbb{Z}_n\) is odd there are no even degree factors of \(X^n - 1\) over \(\mathbb{F}_p^2\). From Proposition \([24]\) we have \(g(X)\) is
a multiple of \(X^n - 1\). Thus for ideal to be non-trivial \(t\) has to be even. This completes the proof for the case when \(m = -1\). However, for \(m \neq -1\) a little more is required.

As the \(S\) is \(\sigma_m\)-isotropic subspace, the \(\sigma_m\)-isotropy condition of Equation [4] for \((g, f)\) with itself yields the following.

\[
g(X)f(X^{-m}) = f(X)g(X^{-m}) \quad \text{mod } X^n - 1
\]
\[
g(X)a(X^{-m})g(X^{-m}) = a(X)g(X)g(X^{-m}) \quad \text{mod } X^n - 1
\]

From Equation [8] we have,

\[
a(X)^{p}g(X)^{p+1} = a(X)g(X)^{p+1} \quad \text{mod } X^n - 1
\]
\[
a(X)^{p} = a(X) \quad \text{mod } \frac{X^n - 1}{g(X)}
\]

Equation [14] is trivially satisfied. Hence does not impose any further constraints in such a case. However, for \(m = -1\) assuming \(t\) to be odd, the Equation [13] and Equation [14] leads to contradiction. Details could be derived from Theorem 5.15 [8, Chapter 5]. Thus \(t\) has to be even.

This completes the proof of Theorem [15]

### 6.2 Proof of Proposition [16]

Let \(S\) be a \(\sigma_m\)-isotropic ideal of \(R(\eta)\) as in Theorem [15]. We need to show that \(h(X, \eta)\) maps to the \(\sigma_m\)-centralizer of \(S\). At first, we compute the size of \(S\). This would determine the size of \(\sigma_m\)-centralizer \(S\).

By Lemma [23] we know that \(S\) is expressed by two polynomials \(g(X)\) and \(h(X, \eta)\). From Section [6.1] we know that there exists a polynomial \(a(X) \in R\) and \(g(X)\) is a factor of \(X^n - 1\) over \(F_p\) such that \(g(X) + \eta a(X) g(X)\) generates \(S\). The following proposition formalizes the dimension of the \(S\) in terms of the polynomial \(g(X)\).

> **Proposition 26.** [8, Theorem 5.24][7] Theorem V.7] The dimension of the \(S\) as a subspace is \(n - \deg g(X)\).

Let \(a(X) \in R\) be the polynomial as in the proof of Theorem [15] (Section [6.1] Equation [11]). For such a fixed polynomial \(a(X)\), let \(Z\) be the set defined as follows.

\[
Z = \left\{ (u, v) \in R \left| \begin{array}{c}
u u(X) \in R, \\
u v(X) \in F_p[X] \left/ \left( X^n - 1 \right) \right. \\
u' v(X) = v(X) \left( X^n - 1 \right) g(X) \right) \right\}
\]

> **Proposition 27.** The set \(Z\) is the \(\sigma_m\)-centralizer of \(S\).

**Proof.** First, we need to show that any element \((u, v) \in Z\) satisfies the \(\sigma_m\)-isotropy condition with every element of \(S\). It is enough to show that the element \((u, v)\) is \(\sigma_m\)-isotropic with \((g, ag)\). From the polynomial form of the \(\sigma_m\)-isotropy condition [4], we have

\[
u u(X) a(X^{-m}) g(X^{-m}) = \left( u(X) a(X) + t(X) \frac{X^n - 1}{g(X)} \right) g(X^{-m}) \quad \text{mod } X^n - 1
\]
From Equation 8
\[ u(X)a(X)^p g(X)^p = \left( u(X)a(X) + t(X) \frac{X^n - 1}{g(X)} \right) g(X)^p \mod X^n - 1 \]
\[ u(X)a(X)^p g(X)^p = u(X)a(X)g(X)^p \mod X^n - 1 \]
\[ u(X)a(X)^p = u(X)a(X) \mod \frac{X^n - 1}{g(X)} \]

Since \( a(X)^p = a(X) \mod (X^n - 1)/g(X) \) (from Section 6.1), the above equation is satisfied.

The cardinality of the set \( Z \) is \( p^{n+\deg g} \) which is same as the cardinality of \( \sigma_m \)-centralizer \( S \).

Hence, we have shown that \( Z \) is the centralizer of \( S \).

Now we need to show \( \sigma_m \)-centralizer \( S \) maps to \( h(X, \eta) \). Let \( (u, v) \in \mathbb{F}_p^n \times \mathbb{F}_p^n \) maps to an element \( u(X) + c_0^{-1} \eta v(X) \in \mathcal{R}(\eta) \). It is easy to see that the joint weight of an element remains unchanged under this mapping. Any element of \( (u, v) \in Z \) maps to \( u(X) \left( 1 + c_0^{-1} \eta a(X) \right) + c_0^{-1} \eta \left( t(X) \left( \frac{X^n - 1}{g(X)} \right) \right) \). As \( h(X, \eta) = \gcd \left( \frac{X^n - 1}{g(X)}, 1 + \eta c_0^{-1} a(X) \right) \) (from Section 6.1), \( h(X, \eta) \) divides it. This completes the proof.