On Primordial Magnetic Fields of Electroweak Origin

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Abstract

We consider Vachaspati’s primordial magnetic field which is generated at the electroweak phase transition. Assuming that either the gradients of the Higgs field or, alternatively, the magnetic field itself are stochastic variables with a normal distribution, we find that the resulting magnetic field has an \textit{rms} value in the present-day universe which is fully consistent with what is required for the galactic dynamo mechanism.
1 Introduction

The magnetic fields observed in galaxies (of the order $10^{-6}$ G) can be understood as an amplification by a dynamo effect \[1\] of a weak seed field of order $10^{-18}$ G on a comoving scale of 100 kpc. The intergalactic plasma has a large electrical conductivity, and the magnetohydrodynamical equation

$$\frac{\partial B}{\partial t} = \nabla \times (\mathbf{v} \times B) - \nabla \times (\eta \nabla \times B), \tag{1}$$

where $\eta$ is the inverse electrical conductivity, then implies that the magnetic lines of force are essentially “frozen into” the fluid. The magnetic flux through any contour moving with the plasma is thus constant. The collapse of the plasma into a galaxy enhances the magnetic field (by a factor $\sim 10^4$), whereas the remaining necessary enhancement is due to the differential rotation and turbulent motion of the plasma in the galaxy.

Eq. (1) is homogeneous in $B$, so if initially one has no field it follows that a field can never be generated. This is basically the reason for the need of a seed field. It has often been speculated that the seed field is of primordial origin, which means that it should be explained by features relevant for the early universe. Electromagnetism first occurs when the standard electroweak $SU(2) \otimes U(1)_Y$ theory is broken down to $U(1)_{\text{em}}$. It is therefore particularly attractive that Vachaspati \[2\] has explained the origin of a primordial field in terms of the cosmological boundary condition that all physical quantities should be uncorrelated over distances greater than the horizon distance. Since we start with the group $SU(2) \otimes U(1)_Y$ before the electroweak phase transition, the resulting electromagnetic field can be constructed in a way which is different from the usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The result is \[2\]

$$F_{ij} = -i(V_i^\dagger V_j - V_j^\dagger V_i^\dagger), \quad V_i = \frac{2}{|\phi|} \sqrt{\frac{\sin \theta}{g}} \partial_i \phi, \tag{2}$$

where $\phi$ is the Higgs field. At the electroweak phase transition the correlation length in the broken phase is $\sim 1/m_W$ (assuming that the Higgs mass is comparable to $m_W$). The field $F_{ij}$ is thus constant over a distance $\sim 1/m_W$, but it varies in a random way over larger distances in order to respect causality. The vector $V_i$ is also random, of course. Its variation is due to the fact that the field $\phi$ makes a random walk on the vacuum manifold of $\phi$. The problem then is to estimate the field $F_{ij}$ over a length scale $\sim N/m_W$. If $N = 1$, then it follows that on dimensional grounds $F_{ij} \sim m_W^2 \sim 10^{24}$ G, with probably an uncertainty of $\pm 1$ in the exponent \[2\]. For $N$ large, one should use a statistical argument. In \[2\] it was argued that the gradients are of order $1/\sqrt{N}$, since $\phi$ makes a random walk on the vacuum manifold with $\Delta \phi \sim \sqrt{N}$, and since $\Delta x \sim N$. Thus $V_i$ is, in a root mean square sense, of the order $1/\sqrt{N}$, and hence $F_{ij}$ is of order $1/N$. Taking further into account that the flux in a co-moving circular contour is constant, the field must decrease like $1/a(t)^2$, where $a(t)$ is the scale factor \[2\]. Using the fact that in the early universe $a$ goes like the inverse temperature, the field was then estimated to behave like

$$\langle F_{ij} \rangle_T \sim \frac{T^2}{N} \tag{3}$$

when the temperature of the universe is $T$. For a scale of 100 kpc this leads to $\langle F_{ij} \rangle_{\text{now}} \sim 10^{-30}$ G, which is far too small to explain the galactic fields (unless there exists some large scale amplification mechanism).
In the present paper we shall present a different statistical scenario where the gradient vectors are taken to be the basic stochastic variables. Our considerations have been influenced by the fact that in large scale 3-dimensional computer simulations of the dynamo effect [3] the computer uses a random initial magnetic field configuration. We find that the very interesting expression (2) obtained in [2] can be interpreted statistically in such a way that the mean magnetic field satisfies

\[ \langle F_{ij} \rangle_T = 0, \quad \sqrt{\langle F_{ij}^2 \rangle_T} \sim \frac{T^2}{\sqrt{N}}. \]  

Comparing the root-mean-square value (4) with Eq. (3), we observe that the scaling behavior is weaker by \( \sqrt{N} \). This means that for a scale of 100 kpc

\[ \sqrt{\langle F_{ij}^2 \rangle_{\text{today}}} \sim 10^{-18} \text{G}, \]  

which is very close (if not equal) to the value desired for the dynamo effect.

It should be emphasized that in estimates it is reasonable, from the point of view of the dynamo effect, to calculate the \( \text{rms} \) value (or, more precisely, to estimate the \( \text{rms} \) value of the projection of the random field on the dynamo eigenfunctions) [4], whereas in actual computer simulations the field configuration should be generated by the computer subject to the two conditions (4).

## 2 Discussion of the averaging procedure

Let us now turn to the detailed arguments. We wish to consider random fields walking around in space in a certain number of steps. Thus we replace the continuum by a lattice, where the points are denoted by greek letters \( \alpha, \ldots \). We want to estimate the magnetic field over a \text{linear} scale (which at most is equal to the horizon scale). Thus, we consider a curve consisting of \( N \) steps in the lattice, and we define the mean value

\[ \overline{B} = \frac{1}{N} \sum_{i=1}^{N} B^{\alpha_i}, \]  

where \( B \) is a component of the magnetic field, and where the lattice points \( \alpha_i \) are on the curve.

Now this curve is arbitrary, and we could take any other curve. We therefore define the average \( \langle \ldots \rangle \), which averages over curves spanning an \( N^3 \) lattice, i.e. over all space (this is well defined on a lattice space; one could e.g. take the set of all curves that are parallel to one of the sides of the \( N^3 \) lattice). Then, for example,

\[ \langle \overline{B} \rangle = \frac{1}{N} \langle \sum_{i=1}^{N} B^{\alpha_i} \rangle, \]  

which means that for each curve with \( N \) steps the mean value \( \overline{B} \) is computed, and this is done for a set of curves which span an \( N^3 \)-lattice, and the average is then computed. Therefore \( \langle \overline{B} \rangle \) depends in general on \( N \), but for simplicity of notation we shall leave out the explicit reference to this dependence. We wish to emphasize that the ensemble average \( \langle \ldots \rangle \) takes into account the field value at each lattice point, so that \text{the average is really over the whole lattice volume.}

Similarly, one can define higher moments such as

\[ \langle \overline{B}^2 \rangle = \frac{1}{N^2} \sum_{i,j=1}^{N} \langle B^{\alpha_i}B^{\alpha_j} \rangle, \]  

together with quantities like 
\[ \langle (\mathcal{B} - \langle \mathcal{B} \rangle)^2 \rangle . \]

Note that in (8) the sum is over curves of length \( N \) steps of the non-local quantity \( \langle B^\alpha B^{\alpha j} \rangle \).

### 3 Random Higgs gradients

In this section we shall present our main statistical assumptions. The general point of view is that when one has a random system it is necessary to specify the statistical distribution and also which variable is to be considered as a stochastic variable. These two specifications are the necessary boundary conditions.

In [2] the stochastic variable was taken to be the Higgs field itself which varies over the vacuum manifold. However, it is clear that also the gradient vectors \( V_i \) are stochastic, and in our scenario we assume that these vectors are the relevant stochastic variables. This is because they directly specify whether there is a magnetic field or not, whereas this is only true indirectly for the Higgs field itself. Also, the vectors \( V_i \) are relevant for questions of alignment between neighbouring domains. Thus, a scenario which takes the gradient vectors as the basic stochastic variables is rather natural.

In this scenario the vectors and the resulting magnetic field have only short-range correlations.

We now return to the expression (2) of the magnetic field in terms of the Higgs gradients \( V_i \). It is convenient to split these fields into real and imaginary parts,

\[ V_i(x) = R_i(x) + iI_i(x) , \tag{9} \]

where \( R_i \) and \( I_i \) are real vectors. We consider the system at a fixed time. The cosmological boundary condition is then that \( R_i \) and \( I_i \) are random fields. We now make the following assumptions:

(i) The random fields have a Gaussian distribution. Thus, the mean value of some quantity \( Q \) is given by

\[ \langle Q \rangle = \prod_{\alpha,i} \int \frac{d^3R^\alpha_i}{D} \frac{d^3I^\alpha_i}{D} Q e^{-\lambda(R^\alpha_i - \langle R^\alpha_i \rangle)^2 - \lambda(I^\alpha_i - \langle I^\alpha_i \rangle)^2} , \tag{10} \]

where \( D \) is a normalization factor defined such that \( \langle 1 \rangle = 1 \), and \( \lambda \) is a measure of the inverse width. The quantities \( \bar{R}_i \) and \( \bar{I}_i \) are the mean values of \( R_i \) and \( I_i \) defined along a curve of length \( N \) steps. Thus, eq. (10) is relevant for a 3-dimensional world which is an \( N^3 \) lattice.

(ii) We assume that the mean values are isotropic, i.e. \( \langle \bar{R}_i \rangle = \langle \bar{R}_2 \rangle = \langle \bar{R}_3 \rangle \) and \( \langle \bar{I}_1 \rangle = \langle \bar{I}_2 \rangle = \langle \bar{I}_3 \rangle \).

Assumption (i) is certainly the simplest way of implementing lack of correlation of the gradient vectors over distances compatible with the horizon scale, whereas assumption (ii) is natural as there is no reason to expect any preferred direction.

It should be noted that the distribution (10) factorizes into an \( I \)-part and an \( R \)-part. Thus, for any expectation value consisting of \( I \)'s and \( R \)'s one has factorization,

\[ \langle \bar{R}_{i_1} \cdots \bar{R}_{i_n} \bar{I}^{j_1} \cdots \bar{I}^{j_m} \rangle = \langle \bar{R}_{i_1} \cdots \bar{R}_{i_n} \rangle \langle \bar{I}^{j_1} \cdots \bar{I}^{j_m} \rangle . \tag{11} \]

This property turns out to be very useful in computing the higher moments.
4 The expectation value of the magnetic field

First we consider the expectation value of a component $B_i$ of the magnetic field. From the expression (2) we find that

$$B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk} = -i \varepsilon_{ijk} V_j^k = 2 \varepsilon_{ijk} R_{j} I_{k} .$$

(12)

Thus

$$\mathcal{B}_j = \frac{1}{N} \sum_{i=1}^{N} B_{j}^{\alpha i} = 2 \varepsilon_{jlk} \frac{1}{N} \sum_{i=1}^{N} R_{l}^{\alpha i} I_{k}^{\alpha i} .$$

(13)

Hence

$$\langle \mathcal{B}_j \rangle = \frac{2}{N} \varepsilon_{jlk} \sum_{i=1}^{N} (R_{l}^{\alpha i} - \langle R_{l} \rangle)(I_{k}^{\alpha i} - \langle I_{k} \rangle) + N \langle R_{l} \rangle \langle I_{k} \rangle .$$

(14)

Now, due to the factorization (11), the first term on the right-hand side of the last Eq. (14) vanishes, and hence

$$\langle \mathcal{B}_j \rangle = \frac{2}{N} \varepsilon_{jlk} \langle R_{l} \rangle \langle I_{k} \rangle = \varepsilon_{jlk} \langle R_{l} \rangle \langle I_{k} \rangle = 0 \quad (15)$$

because of the isotropy assumption (ii). Consequently the mean value of the magnetic field vanishes, as announced in the Introduction.

It should be noticed that if we did not assume isotropy, then $\langle \mathcal{B}_j \rangle \neq 0$ in general. If we then use Vachaspati’s argument [2], $\langle R_{i} \rangle$ and $\langle I_{i} \rangle$ behave like $1/\sqrt{N}$, and hence from (15) one would find $\langle B_j \rangle \sim 1/N$. Thus, our method of averaging over curves leads to the same result as found previously, if we do not assume isotropy. However, we believe that isotropy is a natural assumption.

5 The root mean square of the magnetic field

The second order moment is given by

$$\langle \mathcal{B}_i^2 \rangle = \frac{4}{N^2} \sum_{\alpha \beta} \langle R^\alpha R^\beta \cdot I^\alpha I^\beta - R^\alpha I^\beta \cdot I^\alpha R^\beta \rangle$$

$$= \frac{4}{N^2} \sum_{\alpha \beta} \left\{ \langle R^\alpha R^\beta \rangle \langle I^\alpha I^\beta \rangle - \langle R^\alpha R^\beta \rangle \langle I^\alpha I^\beta \rangle \right\} ,$$

(16)

where we used the factorization (11). Now

$$\langle R_{i}^{\alpha} R_{j}^{\beta} \rangle = \prod_{\gamma} \int \frac{d^3 R_{\gamma}}{D} R_{i}^{\alpha} R_{j}^{\beta} e^{-\lambda(R^\gamma - \langle R \rangle)^2}$$

$$= \frac{1}{2\lambda} \delta_{ij} \delta_{\alpha \beta} + \prod_{\gamma} \int \frac{d^3 R_{\gamma}}{D} \left[ \langle R_{i} \rangle R_{j}^{\beta} + \langle R_{j} \rangle R_{i}^{\alpha} - \langle R_{i} \rangle \langle R_{j} \rangle \right] e^{-\lambda(R^\gamma - \langle R \rangle)^2} \quad (17)$$

and similarly for $\langle I_{i}^{\alpha} I_{j}^{\beta} \rangle$. Further we have e.g.

$$\prod_{\gamma} \int \frac{d^3 R_{\gamma}}{D} R_{j}^{\beta} e^{-\lambda(R^\gamma - \langle R \rangle)^2} = \prod_{\gamma} \int \frac{d^3 R_{\gamma}}{D} (R_{j}^{\beta} - \langle R_{j} \rangle) e^{-\lambda(R^\gamma - \langle R \rangle)^2} + \langle R_{j} \rangle ,$$

(18)

\[\text{Because } \langle R_{i}^{\alpha i} - \langle R_{i} \rangle \rangle = \langle I_{k}^{\alpha i} - \langle I_{k} \rangle \rangle = 0 \text{ for symmetry reasons.}\]
i.e., the mean value in a given arbitrary point $\beta$ on the lattice is equal to the mean value computed over all curves. Using (17) and (18) in (16) we get

$$
\langle B_i^2 \rangle = \frac{4}{N^2} \sum_\alpha \left( \frac{3}{2\lambda^2} + \frac{1}{\lambda} (\langle \mathcal{T} \rangle^2 + \langle \mathcal{R} \rangle^2) \right) + \frac{4}{N^2} \sum_{\alpha\beta} \left( \langle \mathcal{R} \rangle^2 \langle \mathcal{T} \rangle^2 - \langle \langle \mathcal{R} \rangle \langle \mathcal{T} \rangle \rangle^2 \right) .
$$

(19)

The first term is $\mathcal{O}(\frac{N}{N^2}) = \mathcal{O}(\frac{1}{N})$. The last term, being the square of the mean value, actually vanishes because of isotropy: If we take

$$
\langle \mathcal{R} \rangle = \frac{1}{\sqrt{3}} (r, r, r) ; \quad \langle \mathcal{T} \rangle = \frac{1}{\sqrt{3}} (c, c, c) ; \quad \langle \mathcal{R} \rangle^2 = r^2 ; \quad \langle \mathcal{T} \rangle^2 = c^2 ,
$$

(20)

then

$$
\langle \mathcal{R} \rangle^2 \langle \mathcal{T} \rangle^2 - \langle \langle \mathcal{R} \rangle \langle \mathcal{T} \rangle \rangle^2 = r^2 c^2 - \left( \frac{1}{3} (3rc) \right)^2 = 0 .
$$

(21)

Thus we conclude that the rms value of the magnetic field is given by

$$
\sqrt{\langle B_i^2 \rangle} = \frac{2}{N} \sqrt{\sum_\alpha \left( \frac{3}{2\lambda^2} + \frac{1}{\lambda} (\langle \mathcal{T} \rangle^2 + \langle \mathcal{R} \rangle^2) \right)} \sim \mathcal{O}(\frac{1}{\sqrt{N}}) ,
$$

(22)

as announced in the Introduction.

The reason for this slow decrease is the fact that isotropy prevents the mean value from entering in $\langle B_i \rangle$ and $\langle B^2 \rangle$, and that the correlations of the gradient vectors are of short range.

6 Another probability distribution

It should be mentioned that the scenario developed in the previous section is by no means unique from the point of view of producing short range correlations. Instead of assuming that the vector $V_i$ has a random distribution one could make the assumption that the magnetic field $B_i$ itself has a random distribution, with the probability

$$
\prod_{i=1}^{N} \prod_{j=1}^{3} e^{-\lambda(B_{\alpha i})^2} \frac{d^3 B_{\alpha i}}{D} ,
$$

(23)

where $D$ is a normalization factor. Here we have assumed that $\langle B_i \rangle = 0$. Then one has

$$
\langle B_i^\alpha B_j^\beta \rangle = \frac{1}{2\lambda} \delta_{ij} \delta^{\alpha\beta}
$$

(24)

and hence

$$
\langle B^2 \rangle = \frac{1}{N^2} \sum_{i,j=1}^{N} (B_{\alpha i} B_{\alpha j})
$$

$$
= \frac{1}{N^2} \sum_{i=1}^{N} \langle (B_{\alpha i})^2 \rangle \sim \mathcal{O}(\frac{1}{N}) ,
$$

(25)

i.e. the same as the previous result. It should be noted that this result appears in spite of the fact that the distribution (23) is very different from the distribution (11) of the vector field, since e.g. (23) contains correlations between the $I_i$ and $R_i$ fields. Also, it should be noticed that the reason for the result (25) is that the magnetic field has only short-range correlations.
The distribution (23) is the one which is usually assumed in solid state physics when dealing with a random magnetic field. In the continuum version it reads

\[ e^{-\lambda \int d^3x B_i(x)^2 DB(x)} \tag{26} \]

and one then has

\[ \langle B_i(x)B_j(y) \rangle = \frac{1}{2\lambda} \delta_{ij} \delta^3(x-y) , \tag{27} \]

where the \( \delta \)-function is assumed to be smeared.

Let us also comment on the case when \( \langle B \rangle \neq 0 \). For the fluctuations we would have

\[ \langle B^2 \rangle = \frac{1}{N^2} \sum_{\alpha,\beta} \langle B^\alpha B^\beta \rangle = \langle B \rangle^2 + \frac{1}{N^2} \sum_{\alpha,\beta} ((B^\alpha - \langle B \rangle)(B^\beta - \langle B \rangle)) \tag{28} \]

With only short range correlations, i.e. with

\[ \langle (B^\alpha - \langle B \rangle)(B^\beta - \langle B \rangle) \rangle = \delta^{\alpha\beta} \langle (B^\alpha - \langle B \rangle)^2 \rangle \tag{29} \]

this gives

\[ \langle \mathcal{B}^2 \rangle = \langle B \rangle^2 + \mathcal{O}(\frac{1}{N}) . \tag{30} \]

Thus, if one has \( \langle B \rangle \sim \mathcal{O}(1/N) \) as in ref. [4], and if the correlations are of short range, then the dominant term is the fluctuations \( \mathcal{O}(1/N) \) in Eq. (30). Therefore, even in this case one has \( \langle \mathcal{B}^2 \rangle \ll \langle B^2 \rangle \), and hence the field should be estimated from the \( \text{rms} \)-value \( \sqrt{\langle B^2 \rangle} \) when \( N \) is large, not from \( \langle B \rangle \), and the \( \text{rms} \)-value is again effectively of order \( 1/\sqrt{N} \).

7 The energy-momentum tensor

We should briefly discuss the consequences of the results described in Sections 4-6 for the energy-momentum tensor \( T_{\mu\nu} \). After a calculation analogous to that in Eqs. (16)-(22) we obtain

\[ \langle \mathcal{B}^i \mathcal{B}_i \rangle = \frac{4}{N} \delta_{il} \left[ \frac{1}{2\lambda^2} \left( \langle \mathcal{T}^2 \rangle + \langle \mathcal{R}^2 \rangle \right) \right] - \frac{2}{N\lambda} \left( \langle \mathcal{T}_i \rangle \langle \mathcal{T}_i \rangle + \langle \mathcal{R}_i \rangle \langle \mathcal{R}_i \rangle \right) . \tag{31} \]

Contraction over \( i \) and \( l \) of course reproduces Eq. (22). We see that the expectation value in Eq. (31) has two terms, an isotropic and an isotropic term. This is also carried over to the energy-momentum tensor

\[ T_{00} = \frac{1}{2} B^2, \quad T_{ii} = \frac{1}{2} \delta_{il} B^2 - B_i B_l . \tag{32} \]

Consider first the case \( \langle \mathcal{T}_i \rangle = \langle \mathcal{R}_i \rangle = 0 \). Then

\[ \langle T_{ii} \rangle = \frac{1}{2} \delta_{il} \langle B^2 \rangle - \langle B_i B_l \rangle = \frac{1}{6} \delta_{il} \langle B^2 \rangle , \tag{33} \]

so that the equation of state is still isotropic: \( p = \frac{1}{3} \rho \). However, in the case where \( \langle \mathcal{T}_i \rangle \) and \( \langle \mathcal{R}_i \rangle \) are non-vanishing we obtain instead of Eq. (32)

\[ \langle T_{ii} \rangle = \frac{1}{N\lambda^2} \delta_{il} + \frac{2}{N\lambda} \left( \langle \mathcal{T}_i \rangle \langle \mathcal{T}_i \rangle + \langle \mathcal{R}_i \rangle \langle \mathcal{R}_i \rangle \right) . \tag{34} \]
Hence in this case \( \langle T_{ii} \rangle \) has an additional anisotropic part.

If the field \( B \) itself is a Gaussian random field, we find that

\[
\langle T_{00} \rangle = \frac{1}{2} \langle (B - \langle B \rangle)^2 \rangle + \frac{1}{2} \langle B \rangle^2 ,
\]

\[
\langle T_{ii} \rangle = \frac{1}{2} \delta_{ii} \left[ \frac{1}{3} \langle (B - \langle B \rangle)^2 \rangle + \langle B \rangle^2 \right] - \langle B_i \rangle \langle B_i \rangle ,
\]

(35)

so in general there also appears an anisotropic part in \( \langle T_{ii} \rangle \). If \( \langle B \rangle \) is \( \mathcal{O}(1/N) \) as in \[2\] the anisotropic part is subdominant because of Eqs. (28) - (30). Thus, to the leading order \( 1/N \langle T_{ii} \rangle \) is isotropic.

In principle the magnetic energy-momentum tensor discussed above contributes to the expansion of the universe. However, in the present universe its effect is extremely small and can safely be ignored. In the early universe this may not be so. Our results show that in principle there may exist non-isotropic stresses of magnetic origin, which in a statistical sense could contribute to produce turbulence in the primeval plasma, giving possibly rise to an early "universal" dynamo effect. Magnetically generated plasma flows might also be important e.g. for the dynamics of the QCD phase transition.

### 8 Consequences of the magnetic field

We assume that at the electroweak scale \( T = T_0 \simeq 100 \text{ GeV} \) the coherence length of the \textit{rms} field is \( \xi_0 \simeq 1/T_0 \) so that in terms of the physical distance \( L \), we have \( N = L/\xi_0 \). The magnetic field is frozen at that time, so that at later times the original coherence length is redshifted by the expansion according to

\[
\xi(t) = \frac{a(t)}{a_0} \xi_0 \quad .
\]

(36)

The frozen–in magnetic field is also redshifted by the expansion of the universe. Thus at later times at the distance scale \( L \),

\[
B_{\text{rms}}(t, L) = B_0 \left( \frac{a_0}{a(t)} \right)^2 \frac{1}{\sqrt{N}} = B_0 \left( \frac{t_0}{t_*} \right)^\frac{3}{2} \left( \frac{t_*}{t} \right) \left( \frac{\xi_0}{L} \right)^\frac{1}{2} ,
\]

(37)

where \( T_0^2 t_0 = 0.301 M_p/\sqrt{g_*(T_0)} \) with \( g_* \) the effective number of degrees of freedom, and \( t_* \simeq 1.4 \times 10^9 (\Omega_0 h^2)^{-2} \) yrs is the time when the universe becomes matter dominated; for definiteness, we shall adopt the the value \( \Omega_0 h^2 = 0.4 \), which is the upper limit allowed by the age of the universe. We shall also assume that \( B_0 \simeq 10^{24} \text{ G} \).

We may easily find from Eq. (37) the size of the cosmological field today, which could have acted as the seed field for the dynamo mechanism. Taking \( t = 1.5 \times 10^{10} \) yrs and \( L = 100 \text{ kpc} \) (corresponding to \( N = 1.0 \times 10^{24} \)), we find that today the cosmological field at the scale of intergalactic distances is

\[
B_{\text{rms}} = 4 \times 10^{-19} \text{ G} \quad .
\]

(38)

This seems to be exactly what is required for the numerical dynamo simulations to produce the observed galactic magnetic fields of the order \( 10^{-6} \text{ G} \). The inherent uncertainties in the estimate (38) are: the value of \( \Omega_0 h^2 \) used for computing \( t_* \); the time at which the magnetic field froze, or \( T_0 \); the actual value of the field \( B_0 \). Therefore one should view (38) as an order–of–magnitude estimate only.
We should also check what other possible cosmological consequences the existence of the random magnetic field, Eq. (37), may have. Let us first note that the energy density $\rho_B$ in the $\text{rms}$ field is very small. In the radiation dominated era we find that the energy density within a horizon volume $V$ is

$$\rho_B = \frac{1}{2V} \int_0^{r_H} d^3r B_{\text{rms}}^2 = \frac{3}{4} B_0^2 \left( \frac{r_H}{T_0} \right)^4 \frac{1}{r_H T}. \tag{39}$$

The horizon distance is $r_H = 2t$ so that $\rho_B \sim T^5/M_P \ll \rho_\gamma$, and the magnetic field contribution to the total energy density is negligible.

In principle, magnetic fields could modify primordial nucleosynthesis. For instance, it has been argued \[5\] that protons actually become heavier than neutrons in a large enough magnetic field. In our case, however, the magnetic field is glued to the charges according to Eq. (1) so that the relative velocity is zero. Note that this prevents the charged particles in the plasma from accelerating by emitting synchrotron radiation, as would happen if there were a constant background field. At the onset of nucleosynthesis, at about $T \simeq 1 \text{ MeV}$, the effect of the $\text{rms}$ field on the weak reaction rates which change protons to neutrons and determine the crucial $n/p$–ratio turns also out to be negligible. Since at $T \simeq 1 \text{ MeV} n \leftrightarrow p$ reactions have scattering lengths of the order of the horizon length, from Eq. (37) we find that the relevant field at that scale is only $B_{\text{rms}} \simeq 1500 \text{ G}$. Also, creating a thermal population of right–handed neutrinos, disastrous for the successful prediction of primordial light element abundances, via scattering of left–handed neutrinos off the magnetic field \[3\] may not be possible in our case because the mean squared length of the field fluctuation is expected to be very short, $\mathcal{O}(1/T)$. This means that the $\nu_L \leftrightarrow \nu_R$ transition probability would be very much suppressed \[7\]. However, this issue can only be settled by detailed dynamical considerations, for example by computer simulations.

Perhaps more interesting is the role of the $\text{rms}$ field at the QCD phase transition \[8\]. It is believed to be of first order, but the details of bubble nucleation depend on the largely unknown dynamical details of QCD. In simplistic nucleation theory one obtains, by comparing the nucleation rate with the Hubble rate, for the size of the critical bubble about 10 fm. The distance $d$ between nucleation centers depends on the amount of supercooling, but with reasonable assumptions $d \gtrsim 10^{-2} \text{ m}$. This is then the scale at which the bubbles of new phase will feel the random background field. We find

$$B_{\text{rms}}(t_{\text{QCD}}, d) \lesssim 1.8 \times 10^9 \text{ G}. \tag{40}$$

Whether this has an effect on the QCD phase transition or not depends to a large extent on whether the quarks are glued to the flux lines. We shall not discuss this issue further.

9 Conclusions

We have shown that if the derivative of the (logarithm of the) Higgs field is a random variable, then Vahaspati’s construction \[2\] leads to a magnetic field in the present day universe which has the right order of magnitude from the point of view of the galactic dynamo mechanism \[1\].

It should also be mentioned that this result is more general, as is clear from Sect. 6. This is because if by some mechanism a Gaussian random magnetic field is generated at the electroweak phase transition (where it is always of order $m_W^2$ over a correlation length on dimensional grounds), then Eq. (25) shows that the $\text{rms}$-value behaves like...
1/\sqrt{N}, if \langle B \rangle = 0. However, even if \langle B \rangle \neq 0, it follows from Eq. (30) that the \textit{rms} value is at least of order 1/\sqrt{N}. This is because either \langle B \rangle^2 is larger than or equal to 1/N, in which case the \textit{rms} value is larger than or equal to 1/\sqrt{N} or \langle B \rangle^2 is less than 1/N, in which case the fluctuation term of order 1/N dominates, and the \textit{rms}-value is of order 1/\sqrt{N}. Thus, if there exists any mechanism which produces a Gaussian random magnetic field at the electroweak phase transition, it will produce a result which is larger than or equal to the field required by the dynamo effect. It therefore appears that there exists a good case for the primordial origin of the observed galactic magnetic fields.

If we had considered a magnetic field performing a random walk in a given volume, or a random magnetic flux through a given surface, we would have obtained the scalings 1/N^{3/2} and 1/N, respectively. Then one still would have had to weight these spatial averages by the statistical distribution, but this would then have induced double counting of the lattice points. Therefore we have considered random walk only along a given curve, together with an ensemble average over all curves spanning a given volume. Therefore our method takes into account the field value at each lattice point, and in this sense the ensemble average is really over the whole lattice volume. Only a detailed dynamical simulation can however tell for sure whether our seed field actually gives rise to the observed galactic magnetic field. A definite prediction of our scenario is that the seed field has \langle B \rangle = 0 by virtue of the averaging procedure, and that only the \textit{rms} field is non-vanishing.

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