SHARP SOBOLEV REGULARITY OF RESTRICTED X-RAY TRANSFORMS

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Abstract. We study $L^p$-Sobolev regularity estimate for the restricted X-ray transforms generated by nondegenerate curves. Making use of the inductive strategy in the recent work by the authors [23], we establish the sharp $L^p$-regularity estimates for the restricted X-ray transforms in $\mathbb{R}^{d+1}$, $d \geq 3$. This extends the result due to Pramanik and Seeger [29] in $\mathbb{R}^3$ to every dimension.

1. Introduction

Let $\gamma$ be a smooth curve from $I = [-1, 1]$ to $\mathbb{R}^d$. We consider

$$\mathcal{R}f(x, s) = \psi(s) \int f(x + t\gamma(s), t)\chi(t)dt, \quad f \in S(\mathbb{R}^{d+1}),$$

where $\psi$ and $\chi$ are smooth functions supported in the interiors of the intervals $I$ and $[1, 2]$, respectively. The operator $\mathcal{R}f$ is referred to as the restriction of X-ray transform to the line complex generated in the direction $(\gamma(s), 1)$. We say $\gamma$ is nondegenerate if

$$\det(\gamma'(s), \ldots, \gamma^{(d)}(s)) \neq 0, \quad \forall s \in I.$$  \hspace{1cm} (1.1)

The operator $\mathcal{R}f$ is a model case of the general class of restricted X-ray transforms (see [12, 15, 17, 18, 19]). Especially in $\mathbb{R}^3$, under the nondegeneracy assumption (1.1), $\mathcal{R}f$ is a typical example of Fourier integral operators with one-sided fold singularity ([13]). Regularity properties of $\mathcal{R}f$ have been studied in terms of $L^p$ improving and $L^p$ Sobolev regularity estimates. $L^p$ improving property of $\mathcal{R}$ is well understood by now ([14, 16, 25, 24]). The problem was, in fact, considered in a more general framework: $L^p - L^q_{\alpha}$ estimates for $\mathcal{R}$ were studied by some authors (see, for example, [34, 11, 7]) and the estimates on the optimal range of $p, q$ were established except for some endpoint cases. (See also [8, 33, 20, 9, 10] for related results.)

The $L^2 - L^{2/(2d)}$ bound on $\mathcal{R}$ is easy to obtain via $TT^*$ argument and van der Corput’s lemma ([15]) (see also [19, 13] for the sharp $L^2$ Sobolev estimates for general class of operators). Interpolation between this and the trivial $L^\infty$ estimate shows that $\mathcal{R}$ is bounded from $L^p$ to $L^1_{1/(pd)}$ for $p \geq 2$. This is optimal in that $L^p - L^\alpha_p$ estimate fails if $\alpha > 1/(pd)$ (see Proposition 5.1 below). However, when $p < 2$, the sharp $L^p$ regularity estimate is less straightforward. Such estimate was not known until recently. When $d = 2$, the optimal $L^p - L^1_{1/p}$ estimate was established for $1 < p < 4/3$ by Pramanik and Seeger’s conditional result [29] and the sharp decoupling inequality for the cone $\subset \mathbb{R}^3$ due to Bourgain and Demeter [5]. Those

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estimates and interpolation give the sharp $L^p - L^p_{1/(2d+)}$ estimate for $4/3 \leq p < 2$ but the endpoint $L^p - L^p_{1/(2d)}$ estimate remains open. (See Conjecture 1.1 below.) In $\mathbb{R}^3$ the result has been extended to more general operators. In fact, Pramanik and Seeger [31] obtained the sharp $L^p$ regularity estimates for Fourier integral operator with folding canonical relation. Bentsen [4] (also see [3]) extended the result to a class of radon transforms with fold and blowdown singularities.

However, in higher dimensions ($d \geq 3$) the sharp $L^p$ regularity estimate for $\mathcal{R}$ has remained open for $1 < p < 2$. Set $p_d = 2d/(2d - 1)$ and

$$\alpha(p) = \begin{cases} 1 - \frac{1}{p}, & 1 \leq p < p_d, \\ \frac{1}{2d}, & p_d \leq p \leq 2. \end{cases}$$

It is natural to conjecture the following.

**Conjecture 1.1.** Let $d \geq 3$ and $1 < p < 2$. Suppose $\gamma$ is a smooth nondegenerate curve. Then, $\mathcal{R}$ boundedly maps $L^p$ to $L^p_\alpha$ for $\alpha \leq \alpha(p)$.

Failure of $L^p - L^p_\alpha$ boundedness for $\alpha > \alpha(p)$ can be shown by a slight modification of the examples in [29]. (See Proposition 5.1 below.) The following is our main result which verifies the conjecture except for some endpoint cases in every dimension $d \geq 3$.

**Theorem 1.2.** Let $d \geq 3$ and $1 \leq p < p_d$. Suppose $\gamma$ is nondegenerate. Then,

$$\|\mathcal{R}f\|_{L^p_\alpha(\mathbb{R}^{d+1})} \leq C\|f\|_p$$

holds if and only if $\alpha \leq 1 - 1/p$.

When $p \in [p_d, 2)$, interpolation with $L^2 - L^2_{1/(2d)}$ estimate yields (1.2) for $\alpha < \alpha(p)$ but the estimate (1.2) with the endpoint regularity $\alpha = \alpha(p)$, which looks to be a subtle problem, remains open. By a standard scaling argument ([29, 30]) the result in Theorem 1.2 can be extended to the curves of finite type.

A curve $\gamma : I \to \mathbb{R}^d$ is said to be of finite type if there is an $L = L(s)$ such that $\text{span}\{\gamma^{(1)}(s), \ldots, \gamma^{(L)}(s)\} = \mathbb{R}^d$ for each $s \in I$, and the smallest of such $L(s)$ is called the type at $s$. The supremum of the type over $s \in I$ is called the maximal type of $\gamma$ (see, e.g., [30, 21]).

**Corollary 1.3.** Let $d \geq 3$, $1 \leq p < 2$, and $L > d$. Suppose $\gamma$ is a curve of maximal type $L$. Then, $\|\mathcal{R}f\|_{L^p(\mathbb{R}^{d+1})} \to L^p_\alpha(\mathbb{R}^{d+1})$ for $\alpha \leq \min(\alpha(p), 1/(Lp))$ if $p \neq (L + 1)/L$ when $L \geq 2d - 1$, and if $p \in (1, p_d) \cup (2d/L, 2)$ when $d < L < 2d - 1$.

For $p \in [2, \infty]$ it is easy to show the sharp $L^p - L^p_{1/(2L)}$ estimate, which can be shown by using the $L^2 - L^2_{1/(2L)}$ estimate and interpolation in a similar manner as above. Corollary 1.3 and Proposition 5.1 completely settle the problem of the optimal Sobolev regularity estimate for $\mathcal{R}$ if $L \geq 2d - 1$ when $p \neq (L + 1)/L$. However, some endpoint cases remain left open not to mention such estimates for the nondegenerate curve.

In this paper, we make use of the inductive strategy in the recent work of the authors [29], where smoothing properties of the (convolution) averaging operator over curves were studied. Exploiting similarity between $\mathcal{R}^*f$ and the averaging operator, we adapt our previous argument. The main new feature of the current paper is use of the decoupling inequality associated with the conical sets generated
by curves (see Definition 2.15 and Theorem 3.1 below). Compared with our previous work where the averaging operator was decoupled by a class of symbols adjusted to short subcurves, our new decoupling inequality allows us to dispense with some technicality due to the symbols. The decoupling inequality can also be used to simplify the argument in [23].

Organization. In Section 2, we reduce the proof of Theorem 1.2 to obtaining Proposition 2.4. We prove a decoupling inequality associated to a nondegenerate curve (Theorem 3.1) in Section 3 which is crucial for the proof of Proposition 2.4. The proofs of Proposition 2.4 and Theorem 1.2 are given in Section 4 and Section 5, respectively. We discuss the sharpness of the smoothing order \( \alpha \) in Section 5.

Notation. For positive constants \( A,D \), we denote \( A \lesssim D \) if there exists a (independent) constant \( C \) such that \( A \leq CD \), where the constant \( C \) may vary from line to line depending on the context.

2. Estimates with localized frequency

In this section, we reduce the proof of Theorem 1.2 to showing an inductive statement (see Proposition 2.4 below). Afterwards, we obtain some preliminary results which are needed to prove Proposition 2.4.

Let us consider the operator

\[
Rf(x,t) = \chi(t) \int f(x - t\gamma(s), s)\psi(s) ds,
\]

which is the dual operator of \( R \). By duality the estimate (1.2) is equivalent to

\[
\|Rf\|_{L^p(\mathbb{R}^d + 1)} \lesssim \|f\|_{L^{p - 1/p}}, \quad 2d < p < \infty.
\]

(2.1)

For the purpose, we closely follow the line of arguments in our previous paper [23]. So, there is a significant overlap between the current paper and [23]. This can be avoided by omitting some shared details. However, we decide to include them so that the paper is self-contained and more easily accessible.

2.1. Frequency localized estimate. We begin with defining a class of curves in order to prove (2.1) in an inductive manner. For an integer \( 1 \leq L \leq d \), by \( \text{Vol}(v_1, \ldots, v_L) \) we denote the \( L \)-dimensional volume of the parallelepiped generated by vectors \( v_1, \ldots, v_L \in \mathbb{R}^d \).

Definition 2.1. Let \( B \geq 1 \). We say \( \gamma \in \mathfrak{P}(L, B) \) if \( \gamma \in C^{3d + 1}(I) \) satisfies

\[
\max_{s \in I} |\gamma^{(j)}(s)| \leq B, \quad 0 \leq j \leq 3d + 1,
\]

(2.2)

\[
\min_{s \in I} \text{Vol}(\gamma^{(1)}(s), \ldots, \gamma^{(L)}(s)) \geq B^{-1}.
\]

(2.3)

For a smooth function \( a(s,t,\xi) \) on \( I \times [1, 2] \times \mathbb{R}^d \), we define

\[
R[a]f(x,t) = (2\pi)^{-d} \int e^{i(x-t\gamma(s)) \cdot \xi} a(s,t,\xi) \mathcal{F}_x f(\xi, s) ds d\xi.
\]

Here \( \mathcal{F}_x \) denotes Fourier transform in \( x \). Note that \( Rf = R[a]f \) if \( a(s,t,\xi) = \psi(s)\chi(t) \). We prove the estimate (2.1) by induction on \( L \) for \( \gamma \in \mathfrak{P}(L, B) \) under the localized nondegeneracy assumption:

\[
\sum_{\ell=1}^L |(\gamma^{(\ell)}(s), \xi)| \geq B^{-1} |\xi|
\]

(2.4)
which holds if \((s, t, \xi) \in \text{supp } a\) for some \(t\). When \(L < d\), (2.4) can not be true in general even if \(\gamma\) is nondegenerate. However, an appropriate decomposition in the frequency domain makes it possible that (2.4) holds. To do this, we consider a class of symbols \(a\).

**Definition 2.2.** Let \(\mathcal{A}_k = \{\xi \in \mathbb{R}^d : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}\) for \(k \geq 0\), and \(\mathcal{I}_L = \{(j, \alpha) : 0 \leq j \leq 2L, |\alpha| \leq d + L + 2\}\). We say a symbol \(a \in C^{d+L+2}(\mathbb{R}^{d+2})\) is of type \((2^k, L, B)\) if \(\text{supp } a \subset I \times [2^{-1}, 2^2] \times \mathcal{A}_k\),

\[
|\partial_t^j \partial_s^\alpha a(s, t, \xi)| \leq B|\xi|^{-|\alpha|}, \quad (j, \alpha) \in \mathcal{I}_L.
\]

and (2.4) holds on \(\text{supp } a\) \(s, \xi\). Here, as in (2.3), we denote \(\text{supp}_{s, \xi} a = \cup_i \text{supp } a(\cdot, t, \cdot)\). We simply say a statement \(S(s, \xi)\), depending on \(s, \xi\), holds on \(\text{supp } a\) if \(S(s, \xi)\) holds for \(s, \xi \in \text{supp } a\). We also use the same convention with other variables.

The estimate (2.4) (and hence Theorem 1.2) follows from the next theorem via a standard argument using Fefferman-Stein \#-function. See Section 5.1 for details.

**Theorem 2.3.** Suppose that \(\gamma \in \Psi^d(N, B)\) and \(a\) is a symbol of type \((2^k, L, B)\). Then, for \(p > 2L\)

\[
(2.5) \quad \|\mathcal{R}[a]f\|_{L^p(\mathbb{R}^{d+1})} \leq C2^{-\frac{k}{p}}\|f\|_p.
\]

As we mentioned above, we prove Theorem 2.3 by induction on \(L\). Theorem 2.3 with \(L = 1\) is easy to prove. Indeed, setting \(\mathcal{R}f = F_x(\mathcal{R}[a]F_x^{-1}f)\), we note that

\[
\mathcal{R}^* \mathcal{R}f(\xi, s) = \int K(s, s', \xi) f(\xi, s') ds',
\]

where

\[
K(s, s', \xi) = \int e^{it(\gamma(s) - \gamma(s'))} \pi(s, t, \xi) a(s', t, \xi) dt.
\]

Since (2.4) holds with \(L = 1\) on \(\text{supp } a\), integration by parts gives \(|K(s, s', \xi)| \leq C(1 + 2^{k}|s - s'|)^{-2}\). By Young’s convolution inequality it follows that \(\|\mathcal{R}^* \mathcal{R}f\|_2 \leq 2^{-k/2}\|f\|_2\). Thus, we get \(\|\mathcal{R}[a]f\|_2 \leq 2^{-k/2}\|f\|_2\) by Plancherel’s theorem. Interpolation with the trivial estimate \(\|\mathcal{R}[a]f\|_{\infty} \leq \|f\|_{\infty}\) gives (2.5) with \(L = 1\).

Consequently, Theorem 2.3 for \(L \geq 2\) follows from the next proposition (cf. Proposition 2.3).

**Proposition 2.4.** Let \(2 \leq N \leq d\). Suppose Theorem 2.3 holds with \(L = N - 1\). Then, Theorem 2.3 holds with \(L = N\).

We prove the proposition through the rest of this section, Section 3 and 4. Fixing \(2 \leq N \leq d\), we assume that Theorem 2.3 holds with \(L = N - 1\). Additionally, assuming that \(\gamma \in \Psi^d(N, B)\) and \(a\) is of type \((2^k, N, B)\), we prove (2.4) for \(p > 2N\). For the purpose, composing the symbol \(a\), we may further assume that

\[
(2.6) \quad |\gamma^{(N)}(s) \cdot \xi| \geq (2B)^{-1}|\xi|
\]

holds on \(\text{supp } a\). Otherwise, (2.4) holds with \(L = N - 1\), so the hypothesis (Theorem 2.3 with \(L = N - 1\)) yields (2.5) for \(p > 2(N - 1)\).

We prove Proposition 2.4 in Section 4 using the associated decoupling inequality which is obtained in Section 3. The rest of the section is devoted to proving two lemmas (Lemma 2.6 and 2.8) which play crucial roles in proving Proposition 2.4.
2.2. Symbols adapted to $\gamma$. We define a class of symbols adapted to the curve $\gamma$. From now on, we assume that $\delta$ satisfies
\begin{equation}
2^{-k/N} \leq \delta \leq (2^2 B)^{-N}.
\end{equation}

Let $\gamma$ satisfy (2.9) with $L = N - 1$. For $s \in I$, set $V_s^{\gamma, L} = \text{span} \{ \gamma^{(j)}(s) : j = 1, \ldots, \ell \}$. Consider a linear map $L_s^\delta : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given as follow:
\begin{align*}
(\hat{L}_s^\delta)^{j} \gamma^{(j)}(s) &= \delta^{-j} \gamma^{(j)}(s), \quad j = 1, \ldots, N - 1, \\
(\hat{L}_s^\delta)^{v} v &= v, \quad v \in (V_s^{\gamma, N - 1})^\perp.
\end{align*}

We also consider a linear map $L_s^\delta : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ given by
\[L_s^\delta(t, \xi) = (\delta N \tau - \gamma(s) \cdot \hat{L}_s^\delta \xi, \hat{L}_s^\delta \xi), \quad (t, \xi) \in \mathbb{R} \times \mathbb{R}^d.
\]

Denoting $G(s) = (1, \gamma(s))$, we set
\[\Lambda_k(\delta, s) = \bigcap_{0 \leq \ell \leq N - 1} \left\{ (t, \xi) \in \mathbb{R} \times \mathbb{R} : |(G^{(j)}(s), (t, \xi))| \leq B 2^{k+5} \delta^{-j} \right\},
\]

which roughly corresponds to the Fourier support of the operator $\mathcal{R}[a]f$ with $\text{supp}_s \mathcal{R}$ included in an interval centered at $s$ of length about $\delta$. We define a class of symbols associated with $\Lambda_k(\delta, s)$

**Definition 2.5.** Let $s \in (-1, 1)$ and $0 < \delta \leq 1$ such that $I(s, \delta) := [s, -\delta, s + \delta] \subset I$. We denote by $A_k(\delta, s) := A_k(\delta, s, d, N, B, \gamma)$ the set of smooth functions $a$ on $\mathbb{R}^{d+3}$ which satisfy the following:
\begin{align}
\supp a &\subset I(s, \delta) \times [1, 2] \times \Lambda_k(\delta, s), \\
|\partial_t^{\alpha} \partial_{\tau, \xi}^{\alpha} a(s, t, L_s^\delta(t, \xi))| &\leq B |(t, \xi)|^{-[\alpha]}, \quad (j, \alpha) \in \mathcal{I}_N.
\end{align}

It should be noted that there is no $s$-differentiation in (2.9). Here, $\mathcal{I}_N$ is given in Definition 2.2. We set
\begin{equation}
F(T[a]f)(\xi, \tau) = \int e^{-i(t'\tau + \gamma(s) \cdot \xi)} a(s, t', \tau, \xi) dt' F_x f(\xi, s) ds.
\end{equation}

Clearly, $\mathcal{R}[a] = T[a]f$ if $a = a(s, t, \xi)$. The following is an analogue of [23, Lemma 2.7].

**Lemma 2.6.** Let $\tilde{\chi} \in C_0^\infty((2^{-2}, 2^2))$ such that $\tilde{\chi} = 1$ on $[3^{-1}, 3]$. Let $a$ be a smooth function which satisfies (2.8) and (2.9) with $j \leq 2$ and $|\alpha| \leq d + 3$. Then, we have
\begin{equation}
\|T[a]f\|_{L^p(\mathbb{R}^{d+1})} \leq C 3^{1 - \frac{3}{p}} \|f\|_p
\end{equation}
for $p \geq 2$, and
\begin{equation}
\|T[a]f\|_{L^p(\mathbb{R}^{d+1})} \leq C 3^{1 - \frac{1}{p} - N} 2^{-k} \|f\|_p, \quad p > 1.
\end{equation}

**Proof.** Note that $T[a]f(x, t) = \int K[a](s, t, \cdot) \ast \tilde{\chi}(s)(x) ds$ where
\[K[a](s, t, x) = \frac{1}{(2\pi)^{d+1}} \int \int e^{i(x' \tau + i(x-t'\gamma(s)) \cdot \xi)} a(s, t', \tau, \xi) d\xi dt'.
\]

It is easy to show that $|(L_s^\delta)^{-1} L_s^\delta(t, \xi)| \sim |(t, \xi)|$ provided $|s - s_0| \leq \delta$ (cf. [23, Lemma 2.6]). Since (2.9) holds with $j = 0$ and $|\alpha| \leq d + 3$, it follows that
\[\supp a(s, t, 2^k L_s^\delta(t, \xi)) \subset \{(t, \xi) : |(t, \xi)| \lesssim 1\} \quad \text{and} \quad |\partial_{\tau, \xi}^{\alpha} a(s, t, 2^k L_s^\delta(t, \xi))| \lesssim 1, \quad |\alpha| \leq d + 3.
\]
\[ d + 3. \text{ By changing variables } (\tau, \xi) \to 2^k L^d_0(\tau, \xi) \text{ followed by repeated integration by parts, we have} \]
\[ |K[a](s, t, x)| \lesssim \delta \int_{2^{N+1}}^{2^{k+1}} dt' \left(1 + 2^k|\langle \delta^N (t - t'), (\tilde{L}^\delta_{a})^T (x - t\gamma(t))\rangle|^{-d+3} dt' \right). \]
This gives \[ \|K[a](s, t, \cdot)\|_{L^1_1} \lesssim 1. \]
From (2.3), note \[ T[a]f(x, t) = \int_{I(s, \delta)} K[a](s, t, \cdot) * f(\cdot, s)(x) \, ds. \]
Thus, we get \[ \|T[a]f\|_{L^\infty} \lesssim C\delta \|f\|_\infty. \]
Recall (2.11). By translation \( \tau \to \tau - \gamma(s) \cdot \xi \), integration by parts in \( t' \), we see \[ |T[a]f(\xi, \tau)| \lesssim f(1 + |\tau|^{-1})|F_x f(\xi, s)| \, ds. \]
Thus from Plancherel’s Theorem and Hölder’s inequality, we obtain
\[ \|T[a]f\|_2^2 \lesssim \delta \int_{I(s, \delta)} \|F_x f(\cdot, s)\|_2^2 \, ds \lesssim \delta \|f\|_2^2. \]
Therefore, interpolation gives (2.11). To show (2.12), we note from the above estimate for \( K[a](s, t, x) \) that \[ \|(1 - \tilde{\chi}(t)) K[a](s, t, \cdot)\|_{L^1_1} \lesssim \delta(t) = \left(2^{-k}\delta^{-N}|t - 1|^{-1}(1 - \tilde{\chi}(t))\right). \]
By (2.8), using Hölder’s and Young’s convolution inequalities, as before, we see that \[ \|(1 - \tilde{\chi})T[a]f\|_p \] is bounded above by a constant times
\[ \delta^{-1} \int_{I(s, \delta)} \|f(\cdot, s)\|_p \, ds \, dt \lesssim C\delta^{p-1-N}2^{-pk}\|f\|_p. \]
This gives (2.12). \( \square \)

2.3. Rescaling. Let \( I(s, \delta) \subset I \). For \( \gamma \in \mathfrak{M}(N, B) \) we consider a rescaled curve
\[ \gamma^\delta_{s, a}(s) := \delta^{-N}\left(\tilde{L}^\delta_{a}\right)^T(\gamma(s) + s) - \gamma(s)). \]

**Lemma 2.7.** Let \( \gamma \in \mathfrak{M}(N, B) \). If \( 0 < \delta < \delta_* \) for a \( \delta_* \) small enough, \( \gamma^\delta_{s, a} \in \mathfrak{M}(N, 3B) \) and \( \gamma^\delta_{s, a} \in \mathfrak{M}(N - 1, B') \) for some \( B' \).

**Proof.** Taylor series expansion of \( \gamma^{(k)}(s) + s \) at \( s = 0 \) yields
\[ \gamma^{(k)}_{s, a}(s) = \sum_{0 \leq j \leq N-1} \gamma^{(k+j)}(s) \frac{s^j}{j!} + (\tilde{L}^\delta_{a})^T \gamma^{(N)}(s) \frac{s^{N-k}}{(N-k)!} + O(B\delta) \]
for \( 1 \leq \ell \leq N - 1 \) and \( \gamma^{(k)}_{s, a}(s) = (\tilde{L}^\delta_{a})^T \gamma^{(N)}(s) + O(B\delta) \). Writing \( \gamma^{(N)}(s) = v_1 + v_2 \in V^{\gamma_{s, a}}_{N-1} \oplus (V^{\gamma_{s, a}}_{N-1})^\perp \), we have \( (\tilde{L}^\delta_{a})^T \gamma^{(N)}(s) = (\tilde{L}^\delta_{a})^T v_1 + v_2 = v_2 + O(B\delta) \).

Since \( \gamma \in \mathfrak{M}(N, B) \), we see \( \gamma^\delta_{s, a} \in \mathfrak{M}(N, 3B) \) if \( 0 < \delta < \delta_* \) for a sufficiently small \( \delta_* > 0 \). In a similar manner, one can also see that \( \gamma^\delta_{s, a} \in \mathfrak{M}(N - 1, B') \) for some \( B' \).

The following lemma, which is an analogue of [23] Lemma 2.8, is important for our inductive argument. Let us set
\[ \mathcal{R}[\gamma^\delta_{s, a}]f(x, t) = (2\pi)^{-d} \int \int e^{i(x-t\gamma^\delta_{s, a}(s)) \cdot \xi} a(s, t, \xi) F_x f(\xi, s) \, ds \, d\xi. \]

**Lemma 2.8.** Let \( s, \in (-1, 1) \), \( a \in \mathfrak{A}_k(\delta, s), \) and \( \gamma \in \mathfrak{M}(N, B) \). Suppose
\( \sum_{j=1}^{N-1} \delta^j|\gamma^{(j)}(s), \xi| \geq B^{-1}2^k \delta^N \)
(2.13)
for $(s, \xi) \in I(s, \delta) \times \text{supp}_\xi a$. Then, there exist constants $C$, $\tilde{B}$, $\delta_s = \delta_s(B, N, d)$, and $\tilde{f}$ and a symbol $\tilde{a}$ such that

\begin{equation}
\| \tilde{\chi}(t)T[a]f \|_p = \delta^{1-\frac{1}{p}} \| R[\gamma^\delta_s, a] \tilde{f} \|_p
\end{equation}

for $0 < \delta < \delta_s$, $\| f \|_p = \| f \|_p$, $|\partial^j_t \omega^\delta_t a(s, t, \xi)| \leq \tilde{B}|\xi|^{-|\alpha|}$ for $(j, \alpha) \in \mathcal{I}_{N-1}$, and

\begin{equation}
\text{supp} \tilde{\alpha}_s \subset I \times [2^{-2}, 2^2] \times \{ \xi \in \mathbb{R}^d : C^{-1} \delta^{N} 2^k \leq |\xi| \leq C\delta^{N} 2^k \}.
\end{equation}

**Proof.** Let $a_\delta(s, t, \tau, \xi) = a(\delta s + s, t, \tau, \xi)$. By Fourier inversion and (2.10), changing variables $s \to \delta s + s$, $(\tau, \xi) \to (\tau - \gamma(s), \xi)$ gives

\begin{equation}
T[a]f(x, t) = (2\pi)^{-d} \int e^{i(x - t\gamma(s), t, \xi)} b(s, t, \xi) F_x f(\xi, \delta s + s) \, dsd\xi,
\end{equation}

where

\begin{equation}
b(s, t, \xi) = \frac{1}{2\pi} \int e^{it\tau} e^{-i\tau(t + \gamma(\delta s + s) - \gamma(s), \xi)} a_\delta(s, t', t - \gamma(s) \cdot \xi, \xi) \, dt' dt.
\end{equation}

We observe that

\begin{equation}
\tilde{\chi}(t) b(s, t, \delta^{-N} \tilde{L}^\delta_s, \xi) = e^{-it\gamma^\delta_s(\xi)} \tilde{a}(s, t, \xi),
\end{equation}

where

\begin{equation}
\tilde{a}(s, t, \xi) = \frac{1}{2\pi} \int e^{-it(t + \gamma^\delta_s(s) - \gamma(s), \xi)} \tilde{\chi}(t) a_\delta(s, t', t - \gamma^{-N} \tilde{L}^\delta_s (t, \xi)) \, dt' dt.
\end{equation}

It is clear that (2.15) holds for some $C \geq 1$. Since $a \in \mathfrak{A}_k(\delta, s)$, it is not difficult to see $|\partial^j_t \partial^\delta_t \tilde{a}(s, t, \xi)| \leq B|\xi|^{-|\alpha|}$ for $(j, \alpha) \in \mathcal{I}_{N-1}$ (see (2.25) in [23]).

Set $C_p = C_p(\delta) := \delta^{1/p} \text{det } \delta^{-N} \tilde{L}^\delta_s |^{-1/p}$. Let $\tilde{f}$ be given by $F_x \tilde{f}(\xi, s) = C_p F_x f(\delta^{-N} \tilde{L}^\delta_s, \xi, \delta s + s)$, thus $\| \tilde{f} \|_p = \| f \|_p$. Recalling (2.16) and changing variables $\xi \to \delta^{-N} \tilde{L}^\delta_s \xi$, we now have

\begin{equation}
\tilde{\chi}(t) T[a] f(x, t) = \frac{C_p}{(2\pi)^d} \int e^{i(x - t\gamma(s), \delta^{-N} \tilde{L}^\delta_s, \xi)} e^{-it\gamma^\delta_s(\xi)} \tilde{a}(s(t, \xi)) F_x \tilde{f}(\xi, s) \, dsd\xi.
\end{equation}

This gives $\tilde{\chi}(t) T[a] f(x, t) = C_p R[\gamma^\delta_s, \hat{a}] \tilde{f}(y, t)$ where $y = \delta^{-N} (\tilde{L}^\delta_s) \gamma x + t\gamma(s)$.

Therefore, changing variable $x \to \delta^{-N} (\tilde{L}^\delta_s)^{-1} x + t\gamma(s)$, we obtain (2.14).

Combining Lemma 2.8 and the hypothesis (Theorem 2.3) with $L = N - 1$, we obtain the following.

**Corollary 2.9.** Suppose that Theorem 2.3 holds with $L = N - 1$, and $a, \gamma$, and $\delta_s$ are the same as in Lemma 2.8. Then, if $p > 2(N - 1)$, for $0 < \delta < \delta_s$ we have

\begin{equation}
\| \tilde{\chi} T[a] f \|_p \lesssim 2^{-\frac{1}{p}} \| f \|_p.
\end{equation}

**Proof.** By (2.14) and dyadic decomposition (of $\tilde{a}$ in the Fourier side), we have

\begin{equation}
\| \tilde{\chi} T[a] f \|_p \leq C \delta^{1-\frac{1}{p}} \text{sup}_{0 \leq t \leq C} \| R[\gamma^\delta_s, a_t] f_t \|_p,
\end{equation}

for some constant $C$ where $\| f_t \|_p = \| f \|_p$, and $a_t$ are symbols of type $(2j, N - 1, \tilde{B})$ with $C^{-1} 2^k \delta^{N} \leq 2^j \leq C 2^k \delta^{N}$. Once we have this, the proof is straightforward. By Lemma 2.7, $\gamma^\delta_s \in \mathfrak{G}^2(N - 1, B')$ for some $B' > 0$. Since $\| f_t \|_p = \| f \|_p$, applying Theorem 2.3 with $L = N - 1$, we have

\begin{equation}
\| \tilde{\chi} T[a] f \|_p \leq C \sum_{t} \delta^{1-\frac{1}{p}}(2^k \delta^{N})^{-\frac{1}{p}} \| f_t \|_p \lesssim 2^{-\frac{1}{p}} \delta^{1-\frac{N+1}{p}} \| f \|_p.
\end{equation}
for \( p > 2(N - 1) \). Recalling \((2.7)\), we combine this and \((2.12)\) to get the desired bound.

It remains to show \((2.13)\). In fact, after applying Lemma \((2.8)\) we only need to adjust the support of the consequent symbol \( \tilde{a} \) via by moderate decomposition and scaling. We omit details. (See the proof of \((23, \text{Lemma } 2.8)\).)

\[ \square \]

\section{Decoupling inequality for curve}

In this section, we prove the decoupling inequality, which is to be used to decompose the operator \( T[a]f \). In our earlier work \((23)\), the averaging operator was decoupled by making use of decomposition based on a class of symbols which are adjusted to short subcurves. The same approach also works to prove Proposition \((2.4)\). However, instead of following the previous strategy, we directly obtain a decoupling inequality associated with the conic sets

\[ \Lambda_k(\delta, s_i), \quad 1 \leq l \leq L, \]

while \( \{s_1, \ldots, s_L\} \subset I \) is a collection of \( \delta \)-separated points contained in \( I \). More precisely, we have the following.

\begin{theorem}
Let \( 0 < \delta \leq 1 \) and \( S := \{s_1, \ldots, s_L\} \subset I \) be a collection of \( \delta \)-separated points. Then, if \( 2 \leq p \leq N(N + 1) \), for any \( \epsilon > 0 \) there is a constant \( C_\epsilon = C_\epsilon(B) \), independent of \( S \), such that

\begin{equation}
\left\| \sum_{1 \leq l \leq L} \tilde{f}_l \right\|_{L^p(\mathbb{R}^{d+1})} \leq C_\epsilon \delta^{-\epsilon} \left( \sum_{1 \leq l \leq L} \|f_l\|_{L^p(\mathbb{R}^{d+1})}^2 \right)^{1/2}
\end{equation}

holds whenever \( \supp \tilde{f}_l \subset \Lambda_k(\delta, s_i) \).
\end{theorem}

Hölder’s inequality gives \( \| \sum_{1 \leq l \leq L} \tilde{f}_l \|_p \leq C_\delta \delta^{-\epsilon} \delta^{1/p - 1/2} (\sum_{1 \leq l \leq L} \|f_l\|_p)^{1/p} \). Interpolation with the trivial \( L^\infty \rightarrow \ell^\infty L^\infty \) estimate yields the inequality

\begin{equation}
\left\| \sum_{1 \leq l \leq L} \tilde{f}_l \right\|_{L^p(\mathbb{R}^{d+1})} \leq C_\epsilon \delta^{-1/p + \delta^{1/p} \delta^{1/p} + \epsilon} \left( \sum_{1 \leq l \leq L} \|f_l\|_{L^p(\mathbb{R}^{d+1})}^p \right)^{1/p}
\end{equation}

for \( p > 2N \) whenever \( \supp \tilde{f}_l \subset \Lambda_k(\delta, s_i) \).

\subsection{Decoupling inequality for curve}

Fixing \( N \geq 2 \), we now consider the slabs given by an anisotropic neighborhood of the moment curve

\[ \gamma_0(s) := (s, s^2/2!, \ldots, s^{N+1}/(N+1)!). \]

\begin{definition}
Let \( 0 < \delta \leq 1 \) and \( B \geq 1 \). For \( s \in I \), let \( S(s, \delta, B) \) denote the set of \( (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^N \) such that

\[ B^{-1} \leq |(\gamma_0(N+1)^{(j)}(s), (\tau, \xi))| \leq B; \quad |(\gamma_0(j)^{(j)}(s), (\tau, \xi))| \leq \delta^{N+1-j}, \quad j = 1, \ldots, N. \]

We now recall the decoupling inequality for such slabs as above which was shown in \((2)\) (see also \((23)\) Corollary 2.15)).
\end{definition}

\begin{theorem}
Let \( 0 < \delta \leq 1 \) and \( \{s_1, \ldots, s_L\} \subset I \) be a collection of \( \delta \)-separated points contained in \( I \). Denote \( S_l = S(s_l, \delta, B) \). Then, if \( 2 \leq p \leq N(N + 1) \), for any \( \epsilon > 0 \) there is a constant \( C_\epsilon = C_\epsilon(B) \) such that

\[ \left\| \sum_{1 \leq l \leq L} F_l \right\|_{L^p(\mathbb{R}^{N+1})} \leq C_\epsilon \delta^{-\epsilon} \left( \sum_{1 \leq l \leq L} \|F_l\|_{L^p(\mathbb{R}^{N+1})}^2 \right)^{1/2} \]
holds whenever $\text{supp} \hat{f}_1 \subset S_l$.

To show Theorem 3.1, we apply the decoupling inequality after projecting the
sets $\Lambda_0(\delta, s_l)$ to the subspace $V_\mu$ which is spanned by $\{G^{(0)}(\mu), \ldots, G^{(N)}(\mu)\}$. To do so, for $\mu \in I$ we consider a coordinate system $y_\mu = y_\mu(\tau, \xi)$ given by

$$(3.3) \quad y_\mu = (y_\mu^0, \ldots, y_\mu^N) = (\langle G^{(0)}(\mu), (\tau, \xi) \rangle, \ldots, \langle G^{(N)}(\mu), (\tau, \xi) \rangle).$$

Recall that $\gamma \in \mathfrak{D}(N, B)$, so $\text{Vol}(\langle G^{(0)}(\mu), \ldots, G^{(N)}(\mu) \rangle) \geq 1/B$. Let $\delta, \delta'$ be positive numbers satisfying

$$(3.4) \quad 0 < \delta, \delta' \leq \delta^{N/(N+1)} \leq 1.$$ 

Then it is easy to see that

$$(3.5) \quad (\delta')^{\ell+1} \leq \delta^\ell, \quad \ell = 1, \ldots, N.$$ 

The following lemma shows that the projections of the sets $\Lambda_0(\delta, s_l)$ form a reverse $\delta/\delta'$-adapted cover after a proper linear change of variables (cf. [23, Lemma 3.3]) if $s_l$ are contained in an interval of length $\delta'$. Let $D_{\delta}$ denote the $(N+1) \times (N+1)$ diagonal matrix given by

$$D_{\delta} = (\delta^{-N}e_1, \delta^{-N+1}e_2, \ldots, \delta^0e_{N+1}).$$

**Lemma 3.4.** Let $\delta, \delta'$ be positive numbers satisfying (3.3) and $s' \in [\mu - \delta', \mu + \delta']$. Suppose $(\tau, \xi) \in \Lambda_0(\delta, s_l)$. Then we have

$$(3.6) \quad (4B)^{-1} \leq |\langle D_{\delta} y_\mu, \gamma_\circ^{(N+1)} \rangle| \leq 4B,$$

$$(3.7) \quad |\langle D_{\delta} y_\mu, \gamma_\circ^{(j)}(s' - \mu) \rangle| \lesssim B(\delta/\delta')^{N+1-j}, \quad 1 \leq j \leq N.$$ 

**Proof.** Note that (3.6) is clear from (2.6). To prove (3.7), we first note that

$$\langle y_\mu, \gamma_\circ^{(j)}(s) \rangle = (\delta')^{N+1-j} \langle D_{\delta} y_\mu, \gamma_\circ^{(j)}(s/\delta') \rangle.$$ 

Thus, it is sufficient to show that

$$|\langle y_\mu, \gamma_\circ^{(j)}(s' - \mu) \rangle| \lesssim B\delta^{N+1-j}$$

for $1 \leq j \leq N$. Recalling (3.3), we observe

$$\langle y_\mu, \gamma_\circ^{(j)}(s' - \mu) \rangle = \left\langle \sum_{\ell=1}^N G^{(j)}(\mu) \frac{s' - \mu}{(\ell - j + 1)!}, (\tau, \xi) \right\rangle.$$ 

Taylor’s theorem gives

$$\left| G^{(j-1)}(s') - \sum_{\ell=j}^N G^{(\ell)}(\mu) \frac{(s' - \mu)^{\ell-j+1}}{(\ell-j+1)!} \right| \leq B|s' - \mu|^{N-j+2}$$

for $j = 1, \ldots, N$. Since $|s' - \mu| \leq \delta'$ and $(\tau, \xi) \in \Lambda_0(\delta, s_l)$, (3.7) follows by (3.6). \qed

By Lemma 3.4 and Theorem 3.3, we can show that (3.1) holds if a $\delta$-separated set $\{s_1, \ldots, s_L\}$ are contained in an interval of length $\lesssim \delta^{N/(N+1)}$. More precisely, we have the following.

**Lemma 3.5.** Let $0 < \delta \leq 1$ and $\delta \leq \delta' \leq \delta^{N/(N+1)}$. Let $\{s_1, \ldots, s_L\} \subset [\mu - \delta', \mu + \delta']$ be a collection of $\delta$-separated points. Then, if $2 \leq p \leq N(N+1)$, for any $\epsilon > 0$ there is a constant $C_\epsilon = C_\epsilon(B)$ such that (3.1) holds whenever $\text{supp} \hat{f}_l \subset \Lambda_0(\delta, s_l)$. 

Proof: Set $V_\mu = \text{span}\{\gamma'(\mu), \ldots, \gamma^{(N)}(\mu)\}$ and let $\{v_{N+1}, \ldots, v_d\}$ be an orthonormal basis of $V_\mu^\perp$. Recalling that \((2.23)\) holds with $L = N$, we write $\xi = \bar{\xi} + \sum_{j=N+1}^d y_j(\xi)v_j$ for $\bar{\xi} \in V_\mu$. Changing of variables

$$
(\tau, \xi) \rightarrow Y_\mu(\tau, \xi) := (y_\mu(\tau, \xi), y_{N+1}(\xi), \ldots, y_d(\xi))
$$

(see \((3.3)\)), we may work with the coordinate system given by $\{y_\mu, y_{N+1}, \ldots, y_d\}$ instead of $(\tau, \xi)$. We consider the linear map

$$
Y'_\mu(\tau, \xi) \equiv (D_\delta y_\mu(\tau, \xi), y_{N+1}(\xi), \ldots, y_d(\xi)).
$$

Since $\{s_1, \ldots, s_L\} \subset [\mu - \delta', \mu + \delta']$ and $\delta' \leq \delta^{N/(N+1)}$, by Lemma \(3.4\) it follows that

$$
(3.8) \quad Y^{\delta'}_\mu(\Lambda_0(\delta, s_l)) \subset S_l := S\left(\frac{\delta l - \mu}{\delta'}, C\frac{\delta}{\delta'}A B\right) \times \mathbb{R}^{d-N}
$$

for some $C > 0$ depending only on $B$. Applying Theorem \(3.3\) with $\delta$ replaced by $C\delta/\delta'$ and slabs $S_l, 1 \leq l \leq L$, and then using a trivial extension via Minkowski’s inequality, we have

$$
\|\sum_{1 \leq l \leq L} f_l\|_p \leq C\delta^{-\alpha} \left(\sum_{1 \leq l \leq L} \|f_l\|_p^2\right)^{1/2}
$$

for $2 \leq p \leq N(N+1)$ whenever $f_l \subset S_l$. Since the decoupling inequality is invariant under affine changes of variables, by undoing the change of variables $(\tau, \xi) \rightarrow Y_\mu(\tau, \xi)$ and rescaling $(\tau, \xi) \rightarrow 2^{-k}(\tau, \xi)$, we obtain \((3.1)\) whenever $\text{supp } f_l \subset \Lambda_k(\delta, s_l)$. 

\[\square\]

### 3.2. Proof of Theorem \(3.1\)

We now prove Theorem \(3.1\). Let $2 \leq p \leq N(N+1)$. For the purpose, for some $\alpha > 0$ we assume that

$$
\mathfrak{D}(\alpha) \quad \|\sum_{1 \leq l \leq L} f_l\|_{L^p(\mathbb{R}^{d+1})} \leq C\delta^{\alpha} \left(\sum_{1 \leq l \leq L} \|f_l\|_{L^p(\mathbb{R}^{d+1})}^2\right)^{1/2}
$$

holds for $0 < \delta \leq \delta_0 := (2^3B)^{-N-1}$ with a constant $C$, independent of $S$, whenever $\text{supp } f_l \subset \Lambda_k(\delta, s_l), 1 \leq l \leq L$. Of course, $\mathfrak{D}(\alpha)$ holds true if $\alpha \geq 1/2$ by Minkowski’s and Hölder’s inequalities. We set

$$
\delta' = \delta^{N/(N+1)}.
$$

Let us denote $I_\nu, 1 \leq \nu \leq M$, be disjoint intervals of length $\rho \in (2^{-3}\delta', 2^{-2}\delta']$ which partition $I$. Let $s'_\nu$ be a point contained in $I_\nu$ such that $s'_1, \ldots, s'_M$ are separated at least by $2^{-4}\delta'$. We now claim that

$$
(3.9) \quad \Lambda_k(\delta, s_l) \subset \Lambda_k(\delta', s'_{\nu})
$$

if $s_l \in I_\nu$. Indeed, by scaling it is sufficient to show $\Lambda_0(\delta, s_l) \subset \Lambda_0(\delta', s'_{\nu})$. Let $\gamma = \Lambda_0(\delta, s_l)$. Then, it follows that $|\langle G^{(I)}(s_l), (\tau, \xi)\rangle| \leq 2^\gamma B\delta^{1/(N+1)}(\delta')^{N-\ell}$. By Taylor’s theorem we have

$$
\langle G^{(j)}(s'_{\nu}), (\tau, \xi)\rangle = \sum_{\ell=j}^{N-1} \langle G^{(\ell)}(s_l), (\tau, \xi)\rangle \frac{(s'_{\nu} - s_l)^{\ell-j}}{(\ell-j)!} + \mathcal{E},
$$

where $|\mathcal{E}| \leq 2B|s'_{\nu} - s_l|^{N-j}$. Therefore, we see that $(\tau, \xi) \in \Lambda_0(\delta', s'_{\nu})$. 

Let \( \text{supp} \hat{f}_i \subset \Lambda_k(\delta, s_i), 1 \leq i \leq L \). We write \( \sum_{1 \leq i \leq L} f_i = \sum_{1 \leq \nu \leq M} \sum_{s_i \in I_\nu} f_i \).

By (3.9) the Fourier support of \( \sum_{s_i \in I_\nu} f_i \) is included in \( \Lambda_k(\delta', s'_i) \). Since \( s'_i \) are separated by \( 2^{-4} \), \( \mathcal{D}(\alpha) \) implies

\[
\left\| \sum_{1 \leq i \leq L} f_i \right\|_{L^p(\mathbb{R}^{d+1})} \leq C \delta^{-\frac{2N}{N+1}} \left( \sum_{1 \leq \nu \leq M} \left\| \sum_{s_i \in I_\nu} f_i \right\|_{L^p(\mathbb{R}^{d+1})} \right)^{1/2}
\]

for a constant \( C \). Since the length of interval \( I_\nu \) is less than \( \delta^{N/(N+1)} \), by Lemma 3.5 we have \( \left\| \sum_{s_i \in I_\nu} f_i \right\|_p \leq C \delta^{-\epsilon} \langle \sum_{s_i \in I_\nu} \| f_i \|_p^2 \rangle^{1/2} \). Therefore, combining this and the above inequality, we obtain

\[
\left\| \sum_{1 \leq i \leq L} f_i \right\|_{L^p(\mathbb{R}^{d+1})} \leq C \delta^{-\frac{2N}{N+1}} \left( \sum_{1 \leq \nu \leq M} \left\| \sum_{s_i \in I_\nu} f_i \right\|_{L^p(\mathbb{R}^{d+1})} \right)^{1/2}
\]

for a constant \( C_\epsilon \). This establishes the implication \( \mathcal{D}(\alpha) \rightarrow \mathcal{D}(\epsilon + N\alpha/(N+1)) \).

Iteration of this implication suppresses \( \alpha \) arbitrarily small. \( \square \)

4. Proof of Proposition 2.4

In this section, we prove Proposition 2.4 by making use of the decoupling inequality (3.2). As mentioned in Section 2.1 (below Proposition 2.4), in order to prove Proposition 2.4 it suffices to show Theorem 2.3 with \( L = N \). We first reduce the matter to obtaining estimates for \( \mathcal{R}[a_0] \) with a suitable \( a_0 \).

4.1. Reduction. We begin by recalling \( \gamma \in \mathcal{A}^{\delta_2}(N, B) \) and \( a \) is of type \( (2^k, N, B) \). Let \( \delta_* \) be the small number given in Lemma 2.8 and set

(4.1) \( \delta_* = \min\{\delta_*, (2^2 B)^{-N}\} \).

Let \( \beta_0 \in C_0^\infty([-1, 1]) \) such that \( \beta_0 = 1 \) on \([-1/2, 1/2]\). We set

\[
a_N(s, t, \xi) = a(s, t, \xi) \prod_{1 \leq j \leq N-1} \beta_0(100dB2^{-k}\delta^{-N} (\gamma^{(j)}(s), \xi) \).
\]

Clearly, (2.4) holds on \( \text{supp}(a - a_N) \) with \( L = N - 1 \) and \( B \) replaced by \((100dB)^{-1}\delta^N\).

Since \( a \) is of type \( (2^k, N, B) \), it is easy to see \( (a - a_N) \) is a symbol of type \( (2^k, N - 1, B') \) for some \( B' \). Thus, the hypothesis (Theorem 2.3) with \( L = N - 1 \) and \( B = B' \) gives the estimate

\[
\|\mathcal{R}[a - a_N]f\|_p \lesssim 2^{-\frac{k}{p}} \|f\|_p
\]

for \( p > 2(N-1) \). So, we need only to consider \( \mathcal{R}[a_N] \) instead of \( \mathcal{R}[a] \). Furthermore, by a moderate decomposition of \( a_N \) we assume

\[
\text{supp}_s a_N \subset [s - \delta, s + \delta]
\]

for some \( s \in (-1, 1) \). We may assume that \( s = \delta \nu \) for \( \nu \in \mathbb{Z} \).

It is not difficult to see that the contribution of the frequency part \( \{\tau, \xi : |\tau + \gamma(s) \cdot \xi| \geq 2^{k+1}\delta^N, \forall s \in I\} \) is not significant. To see this, let us set

\[
a_0(s, t, \tau, \xi) = a_N(s, t, \xi) \beta_0(\delta_2^{-2N} 2^{-2k} |\tau + \gamma(s) \cdot \xi|^2)
\]

and \( a_1 = a_0 - a_N \). Recalling (2.10), by Fourier inversion we have

\[
\mathcal{R}[a_N]f = \mathcal{T}[a_0]f + \mathcal{T}[a_1]f.
\]

The operator \( \mathcal{T}[a_1] \) is easy to handle. Let us set \( a_i = (-i2^k\delta^N(\tau + \langle \gamma(s), \xi \rangle)^{-1})^i a_1 \). Then, by integration by parts in \( t' \) and (2.10) we see \( \mathcal{T}[a_i] = (2^k\delta^N)^{-i} \mathcal{T}[a] \). Note that \( |\tau + \gamma(s) \cdot \xi| \geq 2^k \delta^N \) on \( \text{supp} a_1 \) and so on \( \text{supp} a \). It is clear that \( a \) satisfies
Then, it follows that
\[(4.4)\]
\[\delta \]
\[\text{may assume} \]
\[\text{Decomposition. For } n \geq 0, \text{ let us set } \delta_n = 2^n 2^{-k/N} \text{ and} \]
\[\delta_n = \delta_n \mathbb{Z} \cap I. \]

We consider
\[\mathfrak{G}_N(s, \tau, \xi) = \sum_{0 \leq j \leq N-1} (2^{-k} |\langle G^{(j)}(s), (\tau, \xi) \rangle|) \Delta_N, \]
by which we can decompose \(a_0\) into the symbols contained in \(\mathfrak{A}_k(\delta_n, s)\) for \(s \in \mathfrak{I}_n\).

Set \(\beta_s = \beta_0 - \beta_0(2^{2N!}). \text{ Note that } \beta_0 + \sum_{n \geq 1} \beta_s(2^{-2N!}) = 1. \)
Let \(\zeta \in C^\infty([-1, 1]) \text{ such that } \sum_{\nu \in \Sigma} \zeta(-\nu) = 1. \text{ We set} \]
\[a^n_0 = a_0 \times \left\{ \begin{array}{ll}
\beta_0(\delta_0 - 2N!) \mathfrak{G}_N(\zeta(\delta_0^{-1} s - \nu), \nu \in \mathfrak{I}_0, n = 0, \\
\beta_s(\delta_n - 2N!) \mathfrak{G}_N(\zeta(\delta_n^{-1} s - \nu), \nu \in \mathfrak{I}_n, n \geq 1.
\end{array} \right. \]
Then, it follows that
\[(4.4) \quad a_0(s, t, \tau, \xi) = \sum_{n \geq 0} \sum_{\nu \in \mathfrak{I}_n} a^n_0(s, t, \tau, \xi). \]

Since \(\delta\) is the fixed constant, it is clear that \(C^{-1} a_0 \in \mathfrak{A}_k(\delta, \delta, s) \text{ for a large constant} \]
\(C > 0. \) So, \(\text{supp } a_0 \subseteq \Lambda_k(\delta, \delta, s) \text{ and } \mathfrak{G}_N \lesssim 1 \text{ for } (\tau, \xi) \subseteq \text{supp } a^n_0. \)
Obviously, we may assume \(\delta_n \lesssim 1\) since \(a^n_0 = 0\) otherwise.

The following tells that \(a^n_0\) is contained in a proper symbol class.

**Lemma 4.1 (cf. [23 Lemma 3.2]).** For \(n \geq 0, \text{ there exists a constant } C \text{ such that} \]
\(C^{-1} a^n_0 \in \mathfrak{A}_k(\delta_n, \delta_n \nu). \)

**Proof:** The condition (2.8) trivially holds for \(a = a^n_0. \text{ So, we only need to show} (2.9) \text{ for } \delta = \delta_n\) and \(s = \delta_n \nu. \)

It is not difficult to see that \(a_0\) satisfies (2.9) (see [23 (3.35)]). So it suffices to show (2.9) for \(\beta_N(\delta_n - 2N!) \mathfrak{G}_N(s, \tau, \xi)\). By Leibniz’s rule, it is enough to prove that
\[(4.5) \quad |\nabla_{\tau, \xi} \delta_n^{N-j} 2^{-k} |\langle G^{(j)}(s), (\tau, \xi) \rangle| \lesssim 2^{-k} \quad \text{for } j = 0, \ldots, d-1. \]
Note that if \(|\delta_n - s| \leq \delta_n, \text{ then} \]
\[(4.6) \quad (L_{\delta_n}^k)^{-1} \mathcal{L}_{\delta_n}^{k}(\tau, \xi) \sim |(\tau, \xi)| \]
(see [23 Lemma 2.6]). Recall that \(\nabla_{\tau, \xi} \langle G^{(j)}(s), \mathcal{L}_{\delta_n}^{k}(\tau, \xi) \rangle = (L_{\delta_n}^k)^j G^{(j)}(s). \)
Thus, by (4.6) we get (4.5). \(\square\)

**4.3. Proof of Proposition 2.4** By the reduction in Section 4.1 it suffices to prove (4.2). Recalling (4.1) and applying the Minkowski inequality, we have
\[
\| T[a_0] f \|_p \leq \sum_{2^{-k/N} \leq \delta_n \leq 1} \| \sum_{\nu \in \mathfrak{I}_n} T[a^n_0] f \|_p. \]
Using Lemma 4.1 one can easily see that \(\text{supp } a^n_0 \subseteq \Lambda_k(\delta_n, \delta_n \nu)\). Thus, we may use the decoupling inequality (3.2). Combining this and the above inequalities gives
\[
\| T[a_0] f \|_p \leq C \sum_{2^{-k/N} \leq \delta_n \leq 1} \delta_n^{-1 + \frac{N+1}{p}} \left( \sum_{\nu \in \mathfrak{I}_n} \| T[a^n_0] f \|_p \right)^{1/p}. \]
for $2N < p < \infty$. Hence, for the estimate (1.2) it suffice to show that

\begin{equation}
(4.7) \quad \|T[a^n_0]f\|_p \lesssim \delta_n^{1 - \frac{N+1}{p}} 2^{-\frac{k}{p}} \|f\|_p, \quad p > 2N.
\end{equation}

Indeed, let $f_\nu(x,s) = \tilde{\zeta}(\delta_n^{-1}s - \nu)f(x,s)$ where $\tilde{\zeta} \in C^\infty([-2,2])$ such that $\tilde{\zeta} = 1$ on supp $\zeta$. From (2.10) we see $T[a^n_\nu]f = T[a^n_0]f_\nu$. Combining this and (4.7), we have

\begin{equation}
(\sum_{\nu \in \mathcal{N}} \|T[a^n_\nu]f\|_p^{p/\nu})^{1/p} \lesssim \delta_n^{1 - \frac{N+1}{p}} 2^{-\frac{k}{p}} \sum_{\nu \in \mathcal{N}} \|f_\nu\|^{p/\nu}\|f\|_p^{p/\nu} \lesssim \delta_n^{1 - \frac{N+1}{p}} 2^{-\frac{k}{p}} \|f\|_p.
\end{equation}

Therefore, taking sum over $n$, we get (1.2), which proves Proposition 2.4.

It remains to prove (4.7). By Lemma 4.1, we have $C^{-1}a^n_\nu \in \mathcal{A}_k(\delta_n, \delta_n \nu)$ for a constant $C > 0$. For $n = 0$, it is easy to show (4.7). Since $\delta_0 = 2^{-k/N}$, applying Lemma 2.6 we get

\begin{equation}
\|T[a^n_0]f\|_p \lesssim \delta_0^{1 - \frac{1}{p}} \|f\|_p = \delta_0^{1 - \frac{N+1}{p}} 2^{-\frac{k}{p}} \|f\|_p, \quad 2 \leq p \leq \infty.
\end{equation}

For $n \geq 1$, we need to decompose $a^n_0$ further. Let us set

\begin{equation}
a^n_{0,1}(s,t,\tau,\xi) = a^n_0(s,t,\tau,\xi)(1 - \delta_0)(10\delta_0^{-2N} |(2^{-k}G(s), (\tau, \xi))|^{2(N-1)})
\end{equation}

and $a^n_{0,2} = a^n_0 - a^n_{0,1}$, so we have $a^n_0 = a^n_{0,1} + a^n_{0,2}$. We note that $C^{-1}a^n_{0,i} \in \mathcal{A}_k(\delta_n, \delta_n \nu)$, $i = 0, 1$ for some $C > 0$. This can be shown by following the proof of Lemma 4.1. So, we omit the detail.

We now decompose $T[a^n_0]f = T[a^n_{0,1}]f + T[a^n_{0,2}]f$. For (4.7), it suffices to show

\begin{equation}
\|T[a^n_{0,1}]f\|_p \leq \delta_n^{1 - \frac{N+1}{p}} 2^{-\frac{k}{p}} \|f\|_p, \quad i = 0, 1,
\end{equation}

for $p > 2N - 2$. It is clear that (2.19) holds with $\delta = \delta_n$, $s = \delta_n \nu$, and some large $B$ on supp $a^n_{0,1}$. By Corollary 2.9 we have (1.8) for $i = 1$ if $p > 2N - 2$. The operator $|T[a^n_{0,1}]f$ can be handled in the same manner as $T[a_1]$ since

\begin{equation}
|\tau + (\gamma(s), \xi)| \lesssim \delta_n^{N} 2^k
\end{equation}

holds on supp $a^n_{0,2}$. We set $a = -i2^k \delta_n^N (\tau + (\gamma(s), \xi))^{-1} \partial_\tau a^n_{0,2}$. Integration by parts in $t'$ and (2.10) yields $T[a^n_{0,2}] = (2^k \delta_n^N)^{-1} T[a]$. Using (4.9) and the fact that $C^{-1}a^n_{0,2} \in \mathcal{A}_k(\delta_n, \delta_n \nu)$ for some $C > 0$, one can easily verify that (2.8) and (2.9) hold for $a$ with $\delta = \delta_n$, $s = \delta_0 \nu$. Thus, by Lemma 2.6 we have

\begin{equation}
\|T[a^n_{0,2}]f\|_p \lesssim \delta_n^{1 - \frac{N+1}{p}} (\delta_n 2^k)^{-1} \|f\|_p \lesssim \delta_n^{1 - \frac{N+1}{p}} 2^{-\frac{k}{p}} \|f\|_p
\end{equation}

for $p \geq 2$, which gives (4.8) for $i = 0$. For the second inequality we use the fact that $\delta_n \geq 2^{-k/N}$.

\section{5. Proof of Theorem 1.2}

We first prove the sufficiency part, that is to say, the estimate (1.2) with $\alpha = 1 - 1/p$ for $1 \leq p < p_d$ by making use of Theorem 2.3.

5.1. \textbf{Proof of the estimate (1.2) with $\alpha = 1-1/p$.} We make use of the argument in [29, 28]. As mentioned before, it suffices to prove (2.1) by duality. Let $P_k$ denote the (Littlewood-Paley projection) operator defined by

\begin{equation}
\mathcal{F}(P_k g)(\xi, \tau) = \beta(2^{-k} |(\xi, \tau)|) \tilde{g}(\xi, \tau), \quad k \geq 1
\end{equation}

where $\beta(x) = x^{-1} 1_{\mathbb{R}^2}(x)$.
for $\beta \in C^\infty_0([1/2, 2])$. Recall that $\beta_0 \in C^\infty_0([-1, 1])$ such that $\beta_0 = 1$ on $[-1/2, 1/2]$ and set $\beta_s(t) = \beta_0(C_0^{1(-2s) - t} - C_0^{2s}t)$. Here $C_0 = 1 + 2 \sup(|\gamma(s)| + |\gamma'(s)| : s \in \text{supp } \psi}$. Let $f_k$ be given by
\[ \hat{f}_k(\xi, u) = \beta_s(2^{-k}|(\xi, u)|) \hat{f}(\xi, u). \]
We claim that
\begin{equation}
(5.1) \quad \left\| \left( \sum_{k \geq 1} |P_k R f|^2 \right)^{1/2} \right\|_p \lesssim \left\| \left( \sum_{k \geq 1} 2^{-\frac{kN}{3}} |f_k|^2 \right)^{1/2} \right\|_p + \|f\|_{L^p_x,M}
\end{equation}
for $p > 2d$ and $M \gg 1$. Then (2.1) follows by the Littlewood-Paley inequality.

Let $\tilde{\beta} = \beta_0(2^{-3} \cdot) - \beta_0(2^3 \cdot)$ Consider the operator $R_k$ given by
\[ F_x(R_k f)(\xi, t) = \tilde{\beta}(|\xi|^{2k})F_x(Rf)(\xi, t), \]
we decompose
\begin{equation}
(5.2) \quad \quad P_k R f = P_k R_k f_k + P_k R_k (f - f_k) + P_k (R - R_k) f.
\end{equation}
In what follows we show that the contributions from the second and third terms are negligible. In fact, for any $M \geq 1$ if $p \geq 1$, we have
\begin{equation}
(5.3) \quad \left\| \left( \sum_{k \geq 1} |P_k R_k (f - f_k)|^2 \right)^{1/2} \right\|_p \lesssim \|f\|_{L^p_x,M}
\end{equation}
and (5.4) below.

To see (5.3), note $F_x(R_k g)(\xi, t') = \int m(\xi, t', u) \hat{g}(\xi, u) \, du$ where
\[ m(\xi, t', u) = (2\pi)^{-1} \chi(t') \tilde{\beta}(|\xi|/2^k) \int e^{i(su - t') \gamma(s) \xi)} \psi(s) \, ds. \]
Since $|\xi, u| \geq C_0 2^{k+5}$ or $|\xi, u| \leq C_0^{-1} 2^{k-5}$ on $\text{supp } F(f - f_k)$, we have $|u| \geq C_0 |\xi|$ if $C_0^{-1} 2^{k-4} \leq |\xi| \leq 2^{k+3}$. Therefore, integration by parts gives
\[ |\partial_{x,u}^a m(\xi, t', u)| \lesssim 2^{-kN} (1 + |(\xi, u)|)^{-N}, \quad (\xi, u) \in \text{supp } F(f - f_k) \]
for any $a$ and $N \geq 1$. Note that $P_k R_k g(x, t) = \int K(x, y, t, s')g(y, s') \, dyds'$ where
\[ K(x, y, t, s') = \frac{1}{(2\pi)^{d+1}} \int e^{ix(y-t') \gamma(s) \xi + is'u \beta}(\frac{|(\xi, \tau)|}{2^k}) m(\xi, t', u) \, d\xi d\tau ds dtd'. \]
Thus if $g = P_j(f - f_k)$, then $|\partial_{x,u}^a m(\xi, t', u)| \lesssim 2^{-kN} 2^{-jN}$ for $(\xi, u) \in \text{supp } F g$ and integration by parts shows
\[ |K(x, y, t, s')| \lesssim 2^{-kN} 2^{-jN} (1 + |x - y| + |s'|)^{-N} (1 + |t|)^{-N}. \]
Decomposing $R_k (f - f_k) = \sum_j R_k P_j (f - f_k)$, we get (5.3) for any $M \geq 1$ and $p \geq 1$.

We now show
\begin{equation}
(5.4) \quad \left\| \left( \sum_{k \geq 1} |P_k (R - R_k) f|^2 \right)^{1/2} \right\|_p \lesssim \|f\|_{L^p_x,M}
\end{equation}
for $p \geq 1$ and $M \geq 1$. We write $F(R f - R_k f)(\xi, \tau) = \int b(s, \xi, \tau) F_x f(\xi, s) \, ds$ where
\[ b(s, \xi, \tau) = \frac{1}{2\pi} \int e^{iat' \gamma(s) \xi - \tau} (1 - \tilde{\beta}(|\xi|/2^k)) \chi(t') \, dl' \psi(s). \]
Since $|\xi| \leq C_0|\alpha|^{-2}$ or $|\xi| \geq 2^{k+2}$ on $\text{supp} F_\sigma (\mathcal{R} f - \mathcal{R} k)$, we have $|\tau| \geq C_0|\xi|$ if $2^{k-1} \leq |\langle \xi, \tau \rangle| \leq 2^{k+1}$. Integration by parts gives $|\partial_x^\alpha b(s, \xi, \tau)| \lesssim 2^{-kN}$ for any $\alpha$ and $N$. Hence,

$$(5.5) \quad \|P_k(\mathcal{R} - \mathcal{R} k)f\|_p \lesssim 2^{-kN}\|f\|_p, \quad p \geq 1$$

for all $N \geq 1$. Since $|\xi| \leq C_0|\alpha|^{-2}$ on $\text{supp} F(P_k(\mathcal{R} - \mathcal{R} k)f)$, similarly as in the proof of (5.3), we have $\|P_k(\mathcal{R} - \mathcal{R} k)f\|_p \lesssim 2^{-jN}\|P_jf\|_p$ for $j \geq k + C'$ for some $C' \geq 1$. The estimate (5.5) gives $\|P_k(\mathcal{R} - \mathcal{R} k)f\|_p \lesssim 2^{-kN}\|P_jf\|_p$ for $j \leq k + C'$. Combining those estimates, we get (5.4).

Therefore, the estimate (5.1) follows if we show

$$\|(\sum_k |P_k\mathcal{R} kf|^2)^{1/2}\|_p \lesssim \|(\sum_k 2^{-2^k}|f|^2)^{1/2}\|_p$$

for $p > 2d$. This can be done by using [28] Theorem 1] and (2.5) (also see [29], [30], [2]). Indeed, let $\tilde{\beta} \in C_c^\infty((1/4, 4))$ such that $\tilde{\beta}\beta = \beta$. Consider the operator $\hat{P}_k$ given by $\hat{F}(\hat{P}_k g)(\xi, \nu) = \hat{\beta}(2^k(|\xi|, |\nu|))\hat{g}(\xi, \nu)$. Note that $P_k \mathcal{R} kf = P_k \hat{P}_k \mathcal{R} kf$.

Let us denote the center of a cube $Q$ by $(x_Q, t_Q)$ and set

$$\mathcal{E}_Q = \{(y, s) : \text{dist}(y - x_Q, t_Q \gamma(I)) \leq 10diam(Q), \ s \in I\}.$$

Since $T_k = \hat{P}_k \mathcal{R} k$ and $\mathcal{E}_Q$ satisfy the assumptions in [28] Theorem 1], by using (2.5) we obtain (5.5). We omit the details.

5.2. Sharpness of smoothing order. In this section, we show upper bounds on the smoothing order $\alpha$ for which $L^p - L^p_\alpha$ estimate for $\mathcal{R} f$ holds when $\gamma$ is of maximal type $L$. In [29] those bounds were obtained for $d = 2$. Modifying the examples in [29], we show the following.

**Proposition 5.1.** Let $d \geq 3$, $L \geq d$, and $1 \leq p \leq \infty$. Let $\psi$ and $\chi$ be nontrivial, nonnegative continuous functions supported in the interiors of $I$ and $[1, 2]$, respectively. Suppose there is an $s$, such that $\psi(s) \neq 0$ and $\gamma$ is of type $L$ at $s$. Then, $\mathcal{R} f$ maps $L^p(\mathbb{R}^{d+1})$ boundedly to $L^p_\alpha(\mathbb{R}^{d+1})$ only if

(i) $\alpha \leq 1 - p^{-1}$,  
(ii) $\alpha \leq (2d)^{-1}$,  
(iii) $\alpha \leq (Lp)^{-1}$.

In particular, the upper bound (i) provides the necessity part of Theorem 1.2.2; thus, the proof Theorem 1.2.2 is completed. We prove the upper bounds (i), (ii), and (iii), separately.

**Proof of (i).** Let $t_0 \in (1, 2)$ such that $\chi(t_0) > 0$. We choose $\zeta \in \mathcal{S}(\mathbb{R}^d)$ such that $\zeta \geq 1$ on $[-1, 1]^d$, supp $\tilde{\zeta} \subset [1/2, 4]^d$, and $\zeta = 1$ on $[1, 2]^d$. Let $\psi_0 \in C_c^\infty((-1, 1))$ satisfy $\psi_0 = 1$ on $[-1/2, 1/2]$. We take

$$f(x, t) = \zeta(\lambda x)\psi_0(\lambda r_0|t - t_0|),$$

where $r_0 = 1 + \sup_{x \in I} |\gamma(s)|$. Note $\mathcal{R} f(x, s) \gtrsim \lambda^{-1}$ if $|x + t_0 \gamma(s)| \leq c\lambda^{-1}$ and $|s - s_0| < c$ for a small constant $c > 0$. Thus, $\|\mathcal{R} f\|_{L_p(\mathbb{R}^{d+1})} \gtrsim \lambda^{-1-d/p}$. Since

$$\mathcal{F}_x (\mathcal{R} f(\cdot, s))(\xi) = \lambda^{-d}\psi_0(\lambda|t - t_0|)\tilde{\zeta}(\lambda^{-1}\xi)e^{i\lambda \gamma(s) \xi}$$

it follows that supp $\mathcal{F}_x (\mathcal{R} f)$ is included in $\{\xi : |\xi| \sim \lambda\}$. Hence, $\|\mathcal{R} f(\cdot, s)\|_{L^p_\alpha(\mathbb{R}^{d+1})} \gtrsim \lambda^{-1-d/p}$, so we have $\|\mathcal{R} f\|_{L^p_\alpha(\mathbb{R}^{d+1})} \gtrsim \lambda^{-1-d/p}$. Since $\|f\|_p \lesssim \lambda^{-(d+1)/p}$, we get $\alpha \leq 1 - 1/p$. \qed
Proof of (ii). Let \( \tilde{I} \subset (-1,1) \) be a nonempty compact interval such that \( (1) \) holds for \( s \in \tilde{I} \). Also, we fix a constant \( \rho \gg 1 \) to be chosen later. Let \( \{s_t\} \subset \tilde{I} \) be a collection of \( \rho^{-1/d} \)-separated points which are as many as \( \mathcal{O}(\rho^{-1/2}) \). Since \( G(s_t), G'(s_t), \ldots, G^{(d-1)}(s_t) \) are linearly independent in \( \mathbb{R}^{d+1} \), there is a unit vector \( \Xi_t \in (\text{span}(G^{(j)}(s_t) : j = 0, 1, \ldots, d-1)) \).

Let \( \phi \in \mathcal{S}(\mathbb{R}^{d+1}) \) such that \( \phi \geq 1 \) on \([-3r_0,3r_0]^{d+1} \) and \( \tilde{\phi} \) is supported in \([-1,1]^{d+1} \) where \( r_0 = 1 + \sup_{s \in \tilde{I}} |\gamma(s)| \). Let \( \varepsilon_t \in \{\pm 1\} \) be independent random variables. We consider

\[
Rf(x,s) = e^{i\lambda \Xi_t \cdot (x,0)} \psi(s) \int \phi(x + t \gamma(s), t)e^{i\lambda \Xi_t \cdot G(s)} \chi(t) \, dt,
\]

we see \( |Rf(x,s)| \geq 1 \) if \( (x,s) \in B_t := [-c,c]^d \times \tilde{I} \). Thus, \( \sum_t \|Rf_t\|^p_{L^p(B_t)} \gtrsim \rho^{-1} \).

Meanwhile, by (3.5), (5.7), and integration by parts in \( t \) we have \( |\sum_t \|Rf_t\|^p_{L^p(B_t)}| \lesssim (1 + \lambda |s_t - s_m|^d)^{-N} \) for any \( N \geq 1 \) if \( m \neq t \) and \( s \in \tilde{I} \). Since \( \{s_t\} \) are \( \rho^{-1/4} \)-separated, it is easy to see

\[
\sum_t \sum_{m \neq t} \|Rf_m\|^p_{L^p(B_t)} \lesssim \sum_t \sum_{m \neq t} (1 + \lambda |s_t - s_m|^d)^{-pN} \lambda^{-1/4} \lesssim \rho^{-pdN}.\]

Therefore, taking \( \rho, N \) sufficiently large, we have \( \|Rf\|_p \gtrsim \rho^{-1} \) for any choice of \( \varepsilon_t \).

By our choice of \( \phi \) it follows that \( R_{\varepsilon_t}(Rf) \) is supported on \( \{\xi : C_1 \lambda \leq |\xi| \leq C_2 \lambda\} \) for some positive constant \( C_1, C_2 \). Thus, \( \|Rf\|_{L^p_{\varepsilon_t}} \gtrsim \lambda^\alpha \|Rf\|_p \). Combining this with the \( L^p-L^p_{\varepsilon_t} \) estimate gives \( \lambda^\alpha \lesssim C \|f\|_p \) for any choice of \( \varepsilon_t \). By Khinchine’s inequality we have \( \mathbb{E}(\|f\|_p^2) \sim \int (\sum_t |f^2|^2)^{\frac{1}{2}} \, dx dt \sim C_\varepsilon \lambda^\frac{\alpha}{2} \). Therefore, we see \( \lambda^\alpha \lesssim \lambda^{\frac{\alpha}{2}} \) and then \( \alpha \lesssim 1/(2d) \) taking \( \lambda \rightarrow \infty \).

\( \square \)

Proof of (iii). Since \( \gamma \) is of type \( L \) at \( s \), by an affine transformation and taking \( \psi \) supported near \( s \), we may assume

\[
\gamma(s+\varepsilon s) = \gamma(s) + (s^a_1 \varphi_1(s), \ldots, s^a_d \varphi_d(s))
\]

for \( 1 \leq a_1 < \cdots < a_d = L \) and smooth functions \( \varphi_j, j = 1, \ldots, d, \) where \( \|\varphi_j - 1/a_j\|_{C^{a_j+1}(I)} \leq c \) for a small constant \( c > 0 \). We may also assume \( s = 0 \) and furthermore \( \gamma(0) = 0 \) by replacing \( f(x,t) \) by \( f(x-t\gamma(0),t) \).

Let \( \phi_1 \in \mathcal{S}(\mathbb{R}) \) such that \( \phi_1 \geq 1 \) on \([-1,1] \), and \( \sup \phi_1 \in [1/2, 4] \) with \( \phi_1 = 1 \) on \([1,2] \). Let \( \psi_0 \in C_\infty([-1,1]) \) with \( \psi_0 = 1 \) on \([-1/2, 1/2] \). We consider

\[
f(x,t) = \prod_{j=1}^{d-1} \psi_0(\lambda^{a_j/L} x_j) \phi_1(\lambda x_d) \chi(t).
\]

Denoting \( \|a\| = \sum_{j=1}^d a_j \), we have \( \|f\|_p \lesssim \lambda^{-\|a\|/(LP)} \). Set \( E_\lambda = \{(x,s) \in \mathbb{R}^d \times I : |x_j| \leq c\lambda^{-a_j/L}, j = 1, \ldots, d, |s| \leq c\lambda^{-1/L}\} \) for a sufficiently small \( c > 0 \). Since \( \gamma(s) = (s^a_1 \varphi_1(s), \ldots, s^a_d \varphi_d(s)) \), \( |(x + t\gamma(s), e_j)| \leq 2^{-1} \lambda^{-a_j/L}, j = 1, \ldots, d, \) for \( (x,s) \in E_\lambda \) and \( t \in [1,2] \). So, \( Rf(x,s) \geq 1 \) for \( (x,s) \in E_\lambda \). This gives \( \|Rf\|_p \gtrsim \lambda^{-\|a\|/(LP)} \gtrsim \lambda^{-\|a\|/(LP)} \). 

\( \square \)
\[\lambda^{-(\|a\|+1)/(Lp)}.\] Since \(\text{supp} F_x(\mathcal{R} f) \subset \{\xi_d : |\xi_d| \sim \lambda\}\), \(\|\mathcal{R} f\|_{L^p} \gtrsim \lambda^{-((\|a\|+1)/(Lp)}\). Therefore, we obtain \(\alpha \leq 1/(Lp)\). \(\square\)

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