Ergodic properties of quasi-Markovian generalized Langevin equations with configuration dependent noise

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Abstract. We discuss the ergodic properties of quasi-Markovian stochastic differential equations, providing general conditions that ensure existence and uniqueness of a smooth invariant distribution. The main new result is an ergodicity condition for the generalized Langevin equation with configuration-dependent noise.

Keywords: generalized Langevin equation, heat-bath, quasi-Markovian model, sampling, molecular dynamics, ergodicity, non-equilibrium

1 Introduction

Generalized Langevin equations (GLE) arise from model reduction and have many applications such as chromosome segmentation in e coli [30], sampling of molecular systems [57,49,55], atom-surface scattering [10], anomalous diffusion in fluids [21], modeling of polymer melts [35], and the modelling of coarse grained particle dynamics [17,34]. The GLE is a non-Markovian formulation, meaning that the evolution of the current state depends not only on the state itself but on the state history. The system is typically formulated with memory terms describing friction with the environment and stochastic forcing. The presence of memory complicates both the analysis of the equation and its numerical solution. In this article, we recall the derivation of the GLE as a Mori-Zwanzig reduction of large system to model the dynamics of a subset of the variables. We consider the ergodicity of the equation (existence of a unique invariant distribution and exponential convergence of the associated semi group in a suitably weighted $L^\infty$ space), providing conditions for its validity in case the coefficients of friction and noise depend directly on the reduced position variables.

1.1 The generalised Langevin equation

Consider the situation of an open system exchanging energy with a heat bath. If there is no strong time scale separation between the dynamics of the heat
bath and the explicitly modelled degrees of freedom, the exchange of energy between these two systems is not well modelled by a Markovian process, i.e.,
dynamic observables such as transport coefficients and first passage times cannot
be expected to be well reproduced by a thermostat model which relies on a
Markovian approximation of the heat bath. For example, if we consider a dis-
tinguished particle surrounded by solvent particles of approximately the same
mass, a reduced model where the interaction between the distinguished particle
and the solvent particles is replaced by a simple Langevin equation would lead
to a poor approximation of the dynamics of the distinguished particle. In such
modelling situations it is necessary to explicitly incorporate memory effects, i.e.,
non-Markovian random forces and history dependent dissipation. The frame-
work in which such models are typically formulated is that of the generalized
Langevin equation. In this article we consider two different types of generalized
Langevin equations, both of which can be viewed as non-Markovian thermostat
models.

Let $\Omega_q \in \{\mathbb{R}^n, \mathbb{T}^n\}$, where $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ denotes the $n$-dimensional standard
torus.\(^3\) We first consider a generalized Langevin equation of the form
\[
\dot{q} = M^{-1}p, \\
\dot{p} = F(q) - \int_0^t K(t - s)M^{-1}p(s)ds + \eta(t).
\]
where the dynamic variables $q \in \Omega_q, p \in \mathbb{R}^n$ denote the configuration variables
and conjugate momenta of a Hamiltonian system with energy function
\[
H(q, p) = U(q) + \frac{1}{2}p^T M^{-1}p,
\]
where the mass tensor $M \in \mathbb{R}^{n \times n}$ is required to be symmetric positive definite
and $U \in C^\infty(\Omega_q, \mathbb{R})$ is a smooth potential function so that $F = -\nabla U$ constitutes
a conservative force. $K : [0, \infty) \to \mathbb{R}^{n \times n}$ is a matrix-valued function of $t$, which
is referred to as the memory kernel, and $\eta$ is a stationary Gaussian process
taking values in $\mathbb{R}^n$ and which (in equilibrium) is assumed to be statistically
independent of $q$ and $p$. We refer to $\eta$ as the noise process or random force. We
further assume that a fluctuation-dissipation relation between the random force $\eta$
and the memory kernel holds so that
(i) the random force $\eta$ is unbiased, i.e.,
\[
\mathbb{E}[\eta(t)] = 0,
\]
for all $t \in [0, \infty)$.
(ii) the auto-covariance function of the random force and the memory kernel $K$
coincide up to a constant prefactor, i.e.,
\[
\mathbb{E}[\eta(s + t)\eta^\top(s)] = \beta^{-1}K(t), \quad \beta > 0,
\]
\(^3\) The assumption that configurations are restricted to the torus eliminates several
technical complications and is motivated by the frequent applications of GLEs in
molecular modelling.
where the constant $\beta > 0$ corresponds to the inverse temperature of the system under consideration.

**Position dependent memory kernels and non-conservative forces.** To broaden the range of applications for our model, we also consider instances of the generalized Langevin equation where

(i) the force $F$ is allowed to be non-conservative, i.e., it does not necessarily correspond to the gradient of a potential function,
(ii) the random force is a non-stationary process.

More specifically, we consider the case where the strength of the random force depends on the value of the configurational variable $q$, i.e.,

\[
\dot{q}(t) = M^{-1}p(t), \\
\dot{p}(t) = F(q(t)) - \tilde{K}(q,t) \ast p + \tilde{\eta}(t).
\]

where $F \in C^\infty(\Omega_q, \mathbb{R}^n)$ is a smooth vector field, and the random force $\tilde{\eta}$ is assumed to be of the form

\[
\tilde{\eta}(t) = g^T(q(t))\eta(t),
\]

with $\eta$ again satisfying (i) and (ii) and the convolution term, $\tilde{K}(q,t) \ast p$, is of the form

\[
\tilde{K}(q,t) \ast p = g^T(q(t)) \int_0^t K(t-s)g(q(s))p(s)ds
\]

with $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n \times n})$ and $K$ as specified above.

The generic form of the GLE can be derived using a Mori-Zwanzig projection of the combined Hamiltonian dynamics of an explicit heat bath representation and the system of interest [62,63,41]. In what follows, we briefly outline the Mori-Zwanzig formalism in a simplified setup following the presentation in [17]. We will then consider the particular case of the Kac-Zwanzig model and demonstrate how the above instances of the GLE can be derived from this model.

### 1.2 Formal derivation of the generalized Langevin equation via Mori-Zwanzig projection

Consider an ordinary differential equation of the form

\[
\dot{u} = f(u, v), \\
\dot{v} = g(u, v),
\]

subject to the initial condition

\[(u(0), v(0)) = (u_0, v_0),\]

(4)

(5)
where \( f, g \) are smooth functions, i.e., \( f \in C^\infty(\mathbb{R}^{n_u \times n_v}, \mathbb{R}^{n_u}), g \in C^\infty(\mathbb{R}^{n_u \times n_v}, \mathbb{R}^{n_v}) \), with \( n_u, n_v \) being positive integers. Also, assume that there is a probability measure \( \mu(\text{d}u, \text{d}v) = \rho(u, v)\text{d}u\text{d}v \) with smooth density \( \rho \in C^\infty(\mathbb{R}^{n_u \times n_v}, [0, \infty)) \), which can be associated with a stationary state \( \frac{d}{dt}\eta \) of the system (4). Consider now the projection operator \( \mathcal{P} \), which maps observables \( w(\cdot, \cdot) \) onto the conditional expectation \( \mathcal{P}u \mapsto E_\mu[w(u, v) | v] \), i.e.,

\[
(\mathcal{P}w)(u) = \frac{\int_{\mathbb{R}^{n_u}} \rho(u, v)w(u, v)\text{d}u\text{d}v}{\int_{\mathbb{R}^{n_v}} \rho(u, v)\text{d}u\text{d}v}.
\]

The Mori-Zwanzig projection formalism allows to recast the system (4) as an integro-differential equation (IDE) of the generic form

\[
\dot{u}(t) = \tilde{f}(u(t)) + \int_0^t K(u(t-s), s)\text{d}s + \eta(u(0), v(0), t),
\]

where \( \tilde{f} = \mathcal{P}f \), \( K : \mathbb{R}^{n_u} \times [0, \infty) \rightarrow \mathbb{R}^{n_u} \) is a memory kernel, and \( \eta \) is a function of the initial values of \( u, v \) and the time variable \( t \). It is important to note that while \( \eta \) depends on the initial condition of both \( u \) and \( v \) in (4), the remaining terms in the IDE (6) only depend explicitly on the dynamic variable \( u \). Similarly as in the stochastic IDEs (1) and (3), the convolution term in (6) can, under appropriate conditions on \( f, g \), be considered as a dissipation term. Likewise, under the assumptions that \( u, v \) are initialized randomly according to \( \mu \), the term \( \eta(u(0), v(0), t) \) in (6) can be interpreted as a random force.

A particularly well studied case is the situation where the functions \( f \) and \( g \) are such that \( (f^T, g^T)^T \) is a Hamiltonian vector field and (4) corresponds to the equation of motion of a Hamiltonian system. In this case a natural choice for \( \mu \) is the Gibbs-Boltzmann distribution associated with the Hamiltonian. This choice of \( \mu \) allows us to interpret the degrees of freedom represented by the dynamical variable \( v \) as a heat bath or energy reservoir. For example, let \( u = (q, p) \in \mathbb{R}^{2n}, \ v = (\dot{q}, \dot{p}) \in \mathbb{R}^{2m} \) with \( 2n = n_u, 2m = n_v \). We may consider the case where \( f \) and \( g \) are derived from the Hamiltonian

\[
H(q, p, \dot{q}, \dot{p}) = V(q) + \frac{1}{2}p^T M p + V_c(q, \dot{q}) + V_h(q) + \frac{1}{2} \dot{p}^T \dot{M} \dot{p},
\]

where \( V, V_c, V_h \) are smooth potential functions such that \( V + V_c + V_h \) is confining and \( M \in \mathbb{R}^{n \times n}, \dot{M} \in \mathbb{R}^{m \times m} \) are symmetric positive definite matrices. In view of (6) the variables \( (q, p) \) correspond to the explicitly resolved part of the system; the variables \( (\dot{q}, \dot{p}) \) correspond to the part of the system which is “projected out” and is replaced by the dissipation term and the fluctuation term, thus it functions as the heat bath in the reduced model. The coupling between heat bath and explicitly resolved degrees of freedom is encoded in the form of the coupling potential \( V_c \), and the statistical properties of the heat bath are both

\footnote{in the sense that \( L\rho = 0 \), with \( L \) being the Liouville operator associated with (4).}
determined by the form of the mass matrix $\tilde{M}$ and the form of the potential $V_h$.

Let $P$ denote the projection $(u, v) \mapsto u$. The first step in the derivation of the IDE (6) is to rewrite the first line in (4) as

$$
\dot{u}(t) = (\mathcal{P}f)(P(u(t), v(t))) + [f(u(t), v(t)) - (\mathcal{P}f)(P(u(t), v(t)))].
$$

(8)

Obviously, the first term in (8) corresponds exactly to $\bar{f}(u(t))$ in (6). Let

$$
L = f(u, v) \cdot \nabla u + g(u, v) \cdot \nabla v
$$

denote the Liouville operator associated with (4). Noting that $L(P(u, v)) = f(u, v)$, the term in the square brackets in (6) can be rewritten in semi-group notation as

$$
f(u(t), v(t)) - (\mathcal{P}f)(P(u(t), v(t))) = e^{tL}(I - P)Lf(u(0), v(0))
$$

(9)

where $e^{tL}$ denotes the flow-map operator associated with the solution of (4), which is defined so that $e^{tL}w(u(0), v(0)) = w(u(t), v(t))$. The integro-differential form (6) then follows by applying the operator identity

$$
e^{tL} = \int_0^t e^{(t-s)L} \mathcal{P}Le^{s(I-P)L}ds + e^{t(I-P)L},
$$

which is known as Dyson’s formula [42], to the last line in (9) yielding

$$
e^{tL}(I - P)Lf(u(0), v(0)) = \int_0^t e^{(t-s)L} \mathcal{P}Le^{s(I-P)L}(I - P)Lf(u(0), v(0))ds
$$

$$
+ e^{t(I-P)L}(I - P)Lf(u(0), v(0)),
$$

(10)

where the second term on the right hand side can be identified with $\eta$ in (6), and the first term in (10) corresponds to the integral term in (6). The form of the last term in (10) suggests that $\eta$ can be formally written as the solution of a differential equation

$$
\frac{\partial}{\partial t} \eta(u(0), v(0), t) = (I - \mathcal{P})L\eta(u(0), v(0), t),
$$

$$
\eta(u(0), v(0), 0) = f(u(0), v(0)) - \mathcal{P}f(u(0)),
$$

(11)

which is commonly referred to as the orthogonal dynamics equation [8,17].

A couple of remarks are in order. First, we reiterate that the above calculations are purely formal, i.e., the above expressions for the memory kernel $K$ and the fluctuation term $\eta$ in general do not possess a closed form solution and are therefore often considered as intractable. Moreover, the well-posedness of the
orthogonal dynamics equation (11) is not obvious and care needs to be taken regarding the existence of solutions and the interpretation of the differential operator \( L \) therein. We refer here to [16] for a rigorous treatment of this equation.

We also mention that the above choice of the projection operator \( P \) as a linear operator which maps functions of \((u, v)\) into the space of functions of \( u \) constitutes a special case of the Mori-Zwanzig formalism. More general forms of the projection operator \( P \) can be considered within the Mori-Zwanzig formalism. For example, the Mori-Zwanzig formalism can be used to derive an IDE for the dynamics of reaction coordinates (collective variables). The corresponding projection operator \( P \) is typically nonlinear in these cases, which can drastically complicate the derivation and the form of the IDE. For a more general presentation of the Mori-Zwanzig projection formalism we refer to the above mentioned papers [8,17] and the references therein as well as the original papers by Mori [41] and Zwanzig [62,63]. In particular the latter paper by Zwanzig considers nonlinear forms of the projection operator \( P \).

Secondly, we point out that in order to derive the stochastic IDEs (1) and (3) an additional step is required. While (1) and (3) are of the form of a stochastic IDE, i.e., they are IDEs driven by a (non-Markovian) stochastic process, the equation (6) constitutes an IDE with random initial data, i.e., the system follows a deterministic trajectory after initialization. In the physics literature it is common, in the situation where \( f, g \) define a Hamiltonian vector field, to establish equivalence of these systems by virtue of an averaging argument which is considered valid when the system is in equilibrium and \( n_v \) is sufficiently large (see e.g. [27]). Drawing a mathematically rigorous connection between (6) and a stochastic IDE which resembles the form of (1) or (3) requires substantial work. As we discuss in the section below, weak convergence as \( n_v \to \infty \) of the trajectory of \( u \) on finite time intervals to the solution of a stochastic integro-differential has been shown in [29,28] for instances of the Ford-Kac model.

The Ford-Kac model We consider the Mori-Zwanzig projection formalism in the situation where the ODE (4) corresponds to the equation of motion derived from the Hamiltonian (7). We already mentioned above that the memory kernel \( K \) and the fluctuation term in the IDE (6) in general do not possess a closed form solution. A notable exception, however, is the situation of a linearly coupled harmonic heat bath, e.g.,

\[
V_c(q, \dot{q}) = q^T \Omega_c \dot{q},
\]

(12)

with \( \Omega_c \in \mathbb{R}^{n \times m} \), and

\[
V_h(\dot{q}) = \frac{1}{2} \dot{q}^T \Omega_h \dot{q},
\]

(13)

with \( \Omega_h \in \mathbb{R}^{m \times m} \) being a symmetric positive (semi-)definite matrix. Under this choice of the potential functions \( V_c \) and \( V_h \), the equation of motion associated
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The system (14) was first studied in [15] and is commonly referred to as Ford-Kac model. Integrating the 3rd and 4th line of (14) we obtain

\[
\left( \begin{array}{c}
\tilde{q}(t) \\
\tilde{p}(t)
\end{array} \right) = e^{tR} \left( \begin{array}{c}
\tilde{q}(0) \\
\tilde{p}(0)
\end{array} \right) + \int_0^t e^{(t-s)R} \left( \begin{array}{c}
0 \\
\Omega_c^T q(s)
\end{array} \right) ds,
\]

(15)

where by \( R \in \mathbb{R}^{2m \times 2m} \) we denote the matrix

\[
R = \left( \begin{array}{cc}
0 & \tilde{M}^{-1} \\
-\Omega_h & 0
\end{array} \right).
\]

Partial integration of the integral term in (15) yields

\[
\left( \begin{array}{c}
\tilde{q}(t) \\
\tilde{p}(t)
\end{array} \right) = e^{tR} \left( \begin{array}{c}
\tilde{q}(0) \\
\tilde{p}(0)
\end{array} \right) + R^{-1} e^{tR} \left( \begin{array}{c}
0 \\
\Omega_c^T q(t)
\end{array} \right) - R^{-1} e^{tR} \left( \begin{array}{c}
0 \\
\Omega_c^T p(s)
\end{array} \right) ds.
\]

Substituting \( \tilde{q} \) in the 2nd line by this expression we obtain an IDE of the form [16] with the deterministic vector field \( \bar{f} \) being of the form

\[
\bar{f}(q,p) = \left( \begin{array}{c}
M^{-1} p \\
-\nabla_q V(q) - \Omega_c \Omega_h \Omega_c^T q
\end{array} \right),
\]

the memory kernel \( K \) being of the form

\[
K(p(t-s),s) = - \left( \begin{array}{cc}
0 & 0 \\
0 & \Omega_c^{-1}
\end{array} \right) e^{(t-s)R} \left( \begin{array}{c}
0 \\
\Omega_c^T p(s)
\end{array} \right),
\]

(16)

and the fluctuation term being of the form

\[
\eta(\tilde{q}(0),\tilde{p}(0),q(0),t) = e^{tR} \left( \begin{array}{c}
\tilde{q}(0) \\
\tilde{p}(0)
\end{array} \right) - R^{-1} e^{tR} \left( \begin{array}{c}
0 \\
\Omega_c^T q(0)
\end{array} \right).
\]

(17)

The thermodynamic limit of the Ford-Kac model A detailed analysis of the thermodynamic limit \( m \to \infty \) of an instance of the Ford-Kac model can be found in [29]; see also [28,17]. The Hamiltonian of the system considered in [29] comprises a single distinguished particle of unit mass, which is subject to an external force associated with the confining potential function \( U \in C^\infty(\mathbb{R},\mathbb{R}) \). The heat bath is modeled by \( m \) particles. Each of the heat bath particles is attached by a linear spring to the distinguished particle. The heat bath particles are not
subject to any additional force apart from the coupling force. The corresponding Hamiltonian can be written as

\[ H(q, p, \tilde{q}, \tilde{p}) = \frac{1}{2}p^2 + U(q) + \frac{1}{2} \sum_{j=1}^{m} \frac{\tilde{p}^2_j}{\tilde{m}_j} + \frac{1}{2} \sum_{j=1}^{m} k_j (\tilde{q}_j - q) \]

where \( k_j > 0 \) corresponds to the stiffness constant of the spring attached to the \( j \)-th heat bath particle and \( \tilde{m}_j > 0 \) is the mass of the \( j \)-th heat bath particle.

For this system one finds that the terms (16) and (17) take a particular simple form, so that the corresponding IDE can be written as

\[ \dot{q} = p, \]
\[ \dot{p} = -\partial_q U(q) - \int_0^t K^{(m)} (t-s)p(s)ds + \eta^{(m)} (\tilde{q}_i, \tilde{p}_i, t), \]

where the memory kernel is of the form

\[ K^{(m)} (t) = \sum_{i=1}^{m} k_i \cos(\omega_i t), \]

and the fluctuation term is of the form

\[ \eta^{(m)} (\tilde{q}_i, \tilde{p}_i, t) = \sum_{i=1}^{m} \sqrt{\frac{k_i}{\beta}} (\tilde{q}_i(0) \cos(\omega_i t) + \tilde{p}_i(0) \sin(\omega_i t)), \]

with \( \omega_i = \sqrt{k_i / \tilde{m}_i} \). If the initial conditions of the heat bath particles are assumed to be distributed according to the Gibbs-measure associated with (18) and the statistical distribution of the values of \( k_j \) and \( \tilde{m}_j \) are controlled in a certain way as \( m \to \infty \), it can been shown that for any finite \( T > 0 \) the trajectories of the solution of (19) converges weakly within the interval \([0, T]\) to the solution of a stochastic IDE of the form (1); for a precise statement see [29, Theorem 4.1].

The Kac-Zwanzig model The Kac-Zwanzig model (see [63]) is a generalization of the Ford-Kac model, the heat bath is still harmonic, i.e., \( V_h \) is of the form (13), but the coupling potential is such that the coupling force is linear in \( \tilde{q} \) but non-linear in \( q \), i.e.,

\[ V_c(q, \tilde{q}) = G(q) \tilde{q}, \]

where \( G \in C^2(\mathbb{R}^n, \mathbb{R}^{n \times m}) \). For such a system a closed form solution of the terms in the Mori-Zwanzig projection (6) can still be derived (see [63] or [19] for a detailed derivation). However, unlike in the situation of the Ford-Kac model the

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5 One easily verifies that this Hamiltonian corresponds to a parametrization of (17) as \( M = 1, \tilde{M} = \text{diag}(\tilde{m}_1, \ldots, \tilde{m}_m), \quad V(q) = U(q) + \frac{1}{2} \sum_{i=1}^{m} k_i q^2, \quad V_c(q, \tilde{q}) = \sum_{i=1}^{m} k_i q \tilde{q}_i, \quad V_h(\tilde{q}) = \frac{1}{2} \sum_{i=1}^{m} k_i \tilde{q}_i^2. \)
closed form solution of the memory kernel $K$ and the fluctuation term $\eta$ are functions of $q$. This observation motivates to consider GLEs of the form (3). Instances of (3) which are derived from such a Kac-Zwanzig heat bath model can be found for example in [27,57,44,45].

1.3 Derivation of the generalized Langevin equation from an infinite dimensional heat-bath

The rather cumbersome step of deriving a stochastic IDE from the IDE (6) in the limit $n \rightarrow \infty$ can be elegantly circumvented by coupling the variable $u = (q, p)$ to a heat bath which is of the form of an infinite dimensional Hamiltonian system. In what follows we outline this alternative approach of deriving the stochastic IDE (1) closely following the presentations in [51] and [48, Chapter 8]. We emphasize that the presentation is far from being self-contained and we refer for a detailed presentation to the above mentioned references.

A natural choice for an infinite dimensional heat-bath model is the linear wave equation

\[
\frac{\partial}{\partial t} \phi = \pi,
\]

\[
\frac{\partial}{\partial t} \pi = \left( \frac{\partial}{\partial x} \right)^2 \phi,
\]

which can be considered as an infinite-dimensional Hamiltonian system with associated Hamiltonian

\[
H_b(\phi, \pi) = \int_\mathbb{R} |\partial_x \phi(x)|^2 + |\pi(x)|^2 \, dx,
\]

It can be shown that solutions of the wave equation (20) can be considered as elements of a Hilbert space $\mathcal{H}$, whose inner product induces a norm which corresponds to the Hamiltonian $H_b(\phi, \pi)$. The coupling of the wave equation with the variable $u = (q, p)$ is modeled by a density function $\rho$ so that the integral term

\[
q \int_\mathbb{R} \rho(x) \partial_x \phi(x) \, dx,
\]

can be interpreted as a coupling potential. Together with the Hamiltonian

\[
H(q, p) = \frac{1}{2} p^2 + V(q), \quad (q, p) \in \mathbb{R}^2,
\]

where $V \in C^\infty(\mathbb{R}, \mathbb{R})$ is assumed to be a confining potential, the combined Hamiltonian of the whole system then takes the form

\[
\mathcal{H}(q, p, \phi, \pi) = \frac{1}{2} p^2 + V(q) + H_b(\phi, \pi) + q \int_\mathbb{R} \rho(x) \partial_x \phi(x) \, dx.
\]
From the Hamiltonian (23) the equation of motions of the combined system can be derived. Under the assumption that the initial states of $\phi, \pi$ are distributed according to the Gibbs measure $\mu$ associated with the Hamiltonian $H$, it can then be shown that the solution of the equation of motion can be written in the form of the stochastic IDE (1) with

$$U(q) = V(q) + \frac{\lambda}{2} q^2,$$

and

$$K(t) = \int_{\mathbb{R}} |\hat{\rho}(k)|^2 e^{ik} dk,$$

where $\hat{\rho}(k)$ denotes the Fourier transform of $\rho$ and $\lambda = \int_{\mathbb{R}} |\rho(x)|^2 dx$. Generalized Langevin equations derived from a Hamiltonian of the form (or similar to) (23) have been extensively studied in [23,24,25]. Likewise, the (non-equilibrium) models by Rey-Bellet and coworkers (see e.g. [14,15,50,51]) are derived from a Hamiltonian similar to (23).

1.4 Organization of the paper

In this article we focus on instances of the GLEs (1) and (30), which can be represented in an extended phase space as an Itô diffusion process. We refer to such GLEs as quasi-Markovian generalized Langevin equations (QGLE). In this chapter we introduce the general form of such SDE representations and discuss their asymptotic properties. The remainder of the article is organized as follows. In Section 2 we derive Markovian representations for the GLE (1) and (30), respectively. In Section 2.3 we briefly review previous results from the literature on the Markovian representation and approximation of generalized Langevin equations. In Section 3 we derive results regarding the ergodic properties of the quasi-Markovian representations.

2 Markovian representation of generalized Langevin equations with configuration independent noise

In this section we derive a Markovian representation of the GLEs introduced in Section 1. We start with an Itô diffusion process of the form

$$\dot{q} = M^{-1} p,$$

$$\dot{p} = F(q) - \Gamma_{1,1}(q) M^{-1} p - \Gamma_{1,2}(q) s + \beta^{-1/2} \tilde{\Sigma}_1(q) \dot{W},$$

$$\dot{s} = -\Gamma_{2,1}(q) p - \Gamma_{2,2}(q) s + \beta^{-1/2} \tilde{\Sigma}_2(q) \dot{W},$$

with $(q(0), p(0), s(0)) \sim \mu_0$.\footnote{Note that such a measure can indeed be explicitly constructed on $\mathbb{R}^2 \times \mathcal{H}$. This follows since the Hamiltonian $H$ is a quadratic functional in $\phi, \pi$, which means that the Gibbs measure conditioned on the values of $q, p$ can be considered as a Gaussian measure on the Hilbert space $\mathcal{H}$. A comprehensive review of Gaussian measures on Hilbert spaces can for example be found in [55, Appendix C].}
where $\mathbf{M}, \mathbf{F}, \beta$ are as previously defined and

(i) the auxiliary variable $s(t)$ takes values in $\mathbb{R}^m$ with $m \geq n$,

(ii) $\mathbf{W} = [W_i]_{1 \leq i \leq n+m}$ is a vector of $(n+m)$ independent Gaussian white-noise, i.e., $\dot{W}_i \sim \mathcal{N}(0,1)$ and $\mathbb{E}[\dot{W}_i(t) \dot{W}_j(s)] = \delta_{ij} \delta(t-s)$.

(iii) $\tilde{T}_{i,j}, \tilde{\Sigma}_i, i = 1,2$ are matrix valued functions so that for $m \geq n$,

$$
\tilde{T} = \begin{pmatrix} \tilde{T}_{1,1} & \tilde{T}_{1,2} \\
\tilde{T}_{2,1} & \tilde{T}_{2,2} \end{pmatrix} \in C^\infty \left( \Omega_q, \mathbb{R}^{(n+m) \times (n+m)} \right),
$$

and

$$
\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_{1,1} & \tilde{\Sigma}_{1,2} \\
\tilde{\Sigma}_{2,1} & \tilde{\Sigma}_{2,2} \end{pmatrix} = \begin{pmatrix} \tilde{\Sigma}_1 \\
\tilde{\Sigma}_2 \end{pmatrix} \in C^\infty \left( \Omega_q, \mathbb{R}^{(n+m) \times (n+m)} \right),
$$

i.e.,

$$
\tilde{T}_{1,1} \in C^\infty (\Omega_q, \mathbb{R}^{n \times n}), \quad \tilde{T}^T_{2,1}, \tilde{T}_{1,2} \in (\Omega_q, \mathbb{R}^{n \times m}), \quad \tilde{T}_{2,2} \in C^\infty (\Omega_q, \mathbb{R}^{m \times m}),
$$

and

$$
\tilde{\Sigma}_1 \in C^\infty (\Omega_q, \mathbb{R}^{n \times (n+m)}), \quad \tilde{\Sigma}_2 \in C^\infty (\Omega_q, \mathbb{R}^{m \times (n+m)}).
$$

(iv) The probability measure $\mu_0$ is such that $(q(0), p(0), s(0))$ has finite first and second moments. In particular,

$$
\int_{\Omega_q \times \mathbb{R}^{n+m}} ||q||^2_2 + ||p||^2_2 + ||s||^2_2 \mu_0(dq, dp, ds) < \infty.
$$

**Notation.** In the sequel, we write $x^T := (q^T, p^T, s^T)$, as well as $z^T := (p^T, s^T)$ as shorthand notation for the phase and auxiliary variables, and we use $\Omega_x := \Omega_q \times \Omega_p \times \Omega_s$, and $\Omega_z := \Omega_p \times \Omega_s$, where $\Omega_p = \mathbb{R}^n, \Omega_s = \mathbb{R}^m$, as shorthand notation for the corresponding domains. With some abuse of notation we also denote points in $\Omega_x, \Omega_z, \Omega_q, \Omega_p, \Omega_s$ by $x, z, q, p, s$, respectively.

**Associated generator.** We denote the generator of (24) by

$$
\tilde{L}_{\text{GLE}} = \mathcal{L}_H + \tilde{L}_O,
$$

where $\mathcal{L}_H$ and $\tilde{L}_O$, considered as differential operators on the core $C^\infty (\Omega_x, \mathbb{R})$, have the form

$$
\mathcal{L}_H := \mathbf{F}(q) \cdot \nabla_p + \mathbf{M}^{-1} p \cdot \nabla_q,
$$

and

$$
\tilde{L}_O := -\mathbf{\Gamma}(q) \left( \frac{1}{2} \mathbf{M}^{-1} p \right) \cdot \nabla_z + \frac{1}{2} \tilde{\Sigma}(q) \tilde{\Sigma}^T(q) : \nabla_z^2,
$$

where

$$
\tilde{\Sigma}(q) \tilde{\Sigma}^T(q) : \nabla_z^2 = \sum_{i=1}^M \sum_{j=1}^M \left[ \tilde{\Sigma}(q) \tilde{\Sigma}^T(q) \right]_{i,j} \partial_{z_i} \partial_{z_j}, \quad M = n + m.
$$
Derivation of the associated stochastic IDE. In what follows we relate the system \(24\) to a non-Markovian stochastic IDE. Consider the following convolution functional
\[
\tilde{K}_F(q, t) * p = \tilde{\Gamma}_{1,1}(q(t))M^{-1}p(t) - \tilde{\Gamma}_{1,2}(q(t)) \int_t^s \exp \left(- \int_s^t \tilde{\Gamma}_{2,2}(q(r)) dr\right) \tilde{\Gamma}_{2,1}(q(s))M^{-1}p(s)ds,
\]
and a random force of the form
\[
\tilde{\eta}(t) = \tilde{\eta}_w(t) + \tilde{\eta}_c(t),
\]
where
\[
\tilde{\eta}_w(t) := \beta^{-1/2} \tilde{\Sigma}_1(q(t))\tilde{W}(t),
\]
and
\[
\tilde{\eta}_c(t) := -\tilde{\Gamma}_{1,2}(q(t))\eta_c(t),
\]
with \(\eta_c\) being the solution of the linear SDE
\[
\eta_c(t) = -\tilde{\Gamma}_{1,2}(q(t))\eta_c(t) + \beta^{-1/2} \tilde{\Sigma}_2(q(t))\tilde{W}(t), \quad \eta_c(0) = s(0).
\]
As shown in the following proposition, under this assumption, the SDE \(24\) can be rewritten as a stochastic IDE of the form
\[
\dot{q}(t) = M^{-1}p(t), \quad \dot{p}(t) = F(q(t)) - \tilde{K}_F(q, t) * p + \tilde{\eta}(t).\tag{30}
\]

**Proposition 1.** If a (weak) solution of \((q(t), p(t), s(t))\) of \(24\) exists for all times \(t \geq 0\), the SDE \(24\) can be rewritten in the form \(30\).

**Proof.** The solution for \(s\) in \(24\) can be written as
\[
s(t) = \Phi(t, 0, q)s(0) - \int_0^t \Phi(t, s, q)\tilde{\Gamma}_{2,1}(q(s))M^{-1}p(s)ds + \int_0^t \Phi(t, s, q)\tilde{\Sigma}_2(q(s))d\tilde{W}(s),\tag{31}
\]
with
\[
\Phi(t, s, q) = \exp \left(- \int_s^t \tilde{\Gamma}_{2,2}(q(r)) dr\right).\tag{32}
\]
Substituting \(s(t)\) in the second equation of \(24\) by the right hand side of \(31\) we obtain
\[
p(t) = F(q(t)) - \tilde{\Gamma}_{1,1}(q(t))M^{-1}p(t) + \tilde{\Gamma}_{1,2}(q(t)) \int_0^t \Phi(t, s, q)\tilde{\Gamma}_{2,1}(q(s))M^{-1}p(s)ds - \tilde{\Gamma}_{1,2}(q(t))\Phi(t, 0, q)s(0)
\]
\[
- \tilde{\Gamma}_{1,2}(q(t)) \int_0^t \Phi(t, s, q)\tilde{\Sigma}_2(q(s))d\tilde{W}(s) + \tilde{\Sigma}_1(q(t))d\tilde{W}(t).
\]
As the solution of (29), \( \eta_c(t) \) can be written as

\[
\eta_c(t) = \Phi(t, 0, q)s(0) - \tilde{\Gamma}_{1,2}(q(t)) \int_0^t \Phi(t, s, q) \tilde{\Sigma}_2(q(s))dW(s),
\]

and we find:

\[
\dot{p}(t) = F(q(t)) - \tilde{K}_F(q, t) \ast p + \tilde{\eta}_w(t) - \tilde{\Gamma}_{1,2}(q(t))\eta_c(t)
\]

\[
= F(q(t)) - \tilde{K}_F(q, t) \ast p + \tilde{\eta}(t).
\]

\[\square\]

**Example 1** (Quasi-Markovian GLE with constant coefficients). If we consider the case where \( \tilde{\Gamma} \) and \( \tilde{\Sigma} \) are constant, i.e., \( \tilde{\Gamma} \equiv \Gamma \) and \( \tilde{\Sigma} \equiv \Sigma \) with \( \Gamma, \Sigma \in \mathbb{R}^{(n+m) \times (n+m)} \), one finds that the convolution term simplifies to

\[
\tilde{K}_F(q, t) \ast p = -\Gamma_{1,1}M^{-1}p(t) + \int_0^t \Gamma_{1,2}e^{-\Gamma_{2,2}(t-s)}\Gamma_{2,1}M^{-1}p(s)ds
\]

and the noise terms become

\[
\tilde{\eta}_w(t) = \Sigma_1 \tilde{W}(t), \quad \tilde{\eta}_c(t) = -\Gamma_{1,2}e^{-\Gamma_{2,2}t} s(0) - \Gamma_{1,2} \int_0^t e^{-\Gamma_{2,2}(t-s)} \Sigma_2 dW(s),
\]

so that the stochastic IDE (30) resembles the form of the GLE (1) with

\[
K(t) = \delta(t)\Gamma_{1,1} + \Gamma_{1,2}e^{-\Gamma_{2,2}t}\Gamma_{2,1}.
\]

**Example 2** (Quasi-Markovian GLE with position dependent noise strength). If we consider the case where \( \tilde{\Gamma}_{2,2} \) and \( \tilde{\Sigma}_{2,2} \) are constant, i.e., \( \tilde{\Gamma}_{2,2} \equiv \Gamma_{2,2} \) and \( \tilde{\Sigma}_{2,2} \equiv \Sigma_{2,2} \) with \( \tilde{\Gamma}, \tilde{\Sigma} \in \mathbb{R}^{m \times m} \), the convolution term simplifies to

\[
\tilde{K}_F(q, t) \ast p = \tilde{\Gamma}_{1,2}(q(t)) \int_0^t e^{-\Gamma_{2,2}(t-s)} \tilde{\Gamma}_{2,1}(q(s))M^{-1}p(s)ds,
\]

and the random force terms \( \tilde{\eta}_w \) and \( \tilde{\eta}_c \) become

\[
\tilde{\eta}_w(t) = \tilde{\Sigma}_1(q(t))\tilde{W}(t),
\]

and

\[
\tilde{\eta}_c(t) = -\tilde{\Gamma}_{1,2}(q(t))e^{-\Gamma_{2,2}t} s(0) - \tilde{\Gamma}_{1,2}(q(t)) \int_0^t e^{-\Gamma_{2,2}(t-s)} \tilde{\Sigma}_2(q(s))dW(s),
\]

so that for \( m = n \) and \( \tilde{\Gamma}_{1,2} = -\tilde{\Gamma}_{2,1} \), \( \tilde{\Sigma}_{1,2} = \tilde{\Sigma}_{2,1} = 0 \), the stochastic IDE (30) resembles the form of the GLE (3) with \( K(t) = e^{-\Gamma_{2,2}t} \).
Remark 1 (Existence of solutions of (24)). A sufficient condition for (24) to possess a unique strong solution $x(t)$ for all times $t \geq 0$, is that the right hand side of the SDE (24) is Lipschitz in $q, p, s$. Provided that the initial state $\mu_0$ is as specified in (iv), it directly follows by standard existence and uniqueness results for SDEs (see e.g. [46, Theorem 5.2.1.]) that for any $T > 0$ there exists a unique strong solution $x(t), t \in [0, T]$ of (24), which is continuous in $t$ and

$$E \left[ \int_0^T \|x(t)\|_2^2 dt \right] < \infty.$$  

Since $F, \tilde{\Gamma}, \tilde{\Sigma}$ are assumed to be smooth the Lipschitz condition is obviously satisfied for $\Omega_q = \mathbb{T}^n$. Similarly, for an unbounded configurational domain, i.e., $\Omega_q = \mathbb{R}^n$, it is the Lipschitz condition for the right hand side of (24) follows directly if the spectra of $\tilde{\Gamma}(q)$ and $\tilde{\Sigma}(q)$ are uniformly bounded in $q$ and $F$ to satisfy certain asymptotic growths conditions (e.g., Assumption 3). We also note that the existence of suitable Lyapunov functions as derived in, e.g., Lemma 4 is sufficient (see e.g. [2]) to ensure the existence of a weak solution $(x(t))_{t \geq 0}$ under less strict asymptotic growth conditions on the force $F$.

2.1 Fluctuation-dissipation relation for quasi-Markovian generalized Langevin equations

The following assumption can be understood as a fluctuation dissipation relation for the SDE (24):

**Assumption 1** There exists a symmetric positive definite matrix $Q \in \mathbb{R}^{m \times m}$ such that for all $q \in \Omega_q$,

$$\tilde{\Gamma}(q) \begin{pmatrix} I_n & 0 \\ 0 & Q \end{pmatrix} + \begin{pmatrix} I_n & 0 \\ 0 & Q \end{pmatrix} \tilde{\Sigma}(q) = \tilde{\Sigma}(q) \tilde{\Sigma}(q)^T.$$  

(38)

As shown in Proposition 2, below, for a quasi-Markovian GLE with constant coefficients (see Example 1), Assumption 1 implies that the random force is stationary with covariance function $K$ as defined in (34).

**Proposition 2.** Let as in Example 4 $\tilde{\Gamma}$ and $\tilde{\Sigma}$ be constant, i.e., $\tilde{\Gamma} \equiv \Gamma$ and $\tilde{\Sigma} \equiv \Sigma$ with $\Gamma, \Sigma \in \mathbb{R}^{(n+m)\times(n+m)}$. If Assumption 4 is satisfied and $\mu_0$ such that $s(0) \sim \mathcal{N}(0,Q)$, where $Q \in \mathbb{R}^{m \times m}$ as specified in Assumption 4, then $\tilde{\eta}$ is a stationary Gaussian process with vanishing expectation and covariance function $K$ as defined in (34).

**Proof.** Let

$$G(r) = G_{1,2} \int_0^r e^{-\Gamma_{2,2}(r-s)} \Sigma_2 dW(s).$$
Without loss of generality we assume that \( t \geq t' \), and we find that the covariance of \( \tilde{\eta} \) is indeed of the form \( (34) \):

\[
E[\tilde{\eta}(t)\tilde{\eta}^T(t')]=E\left[\Sigma_t W(t)W(t')^T\Sigma_t^T - E[G(t)G^T(t')]\right]
+ E\left[\Gamma_{1,2} e^{-\Gamma_{2,2}(s(0)s(0))^T e^{-\Gamma_{2,2}(t')}} \right]
+ E\left[\left(\Gamma_{1,2} \int_0^{t'} e^{-\Gamma_{2,2}(t-s)} \Sigma_2 dW(s) \right) G^T(t')\right]
= \delta(t-t')(\Gamma_{1,1} + \Gamma_{1,1}') - \Gamma_{1,2} e^{-\Gamma_{2,2}(t-t')}(\Gamma_{2,1} + Q \Gamma_{1,2}')
+ \Gamma_{1,2} e^{-\Gamma_{2,2}Q} e^{-\Gamma_{2,2}'(t-t')} \Gamma_{1,2}'
+ \Gamma_{1,2} \int_0^{t'} e^{-\Gamma_{2,2}(t-s)} (\Gamma_{2,1} + Q \Gamma_{2,1}') e^{-\Gamma_{2,2}'(t-s)} \Gamma_{2,1}' ds
= \delta(t-t')(\Gamma_{1,1} + \Gamma_{1,1}') - \Gamma_{1,2} e^{-\Gamma_{2,2}(t-t')} \Gamma_{2,1}',
\]

where expectations are taken over both \( \mu_0 \) and the path measure of the Wiener process \( W \). The last equality follows by partial integration of the integral term.

In the absence of a white-noise component in the random force, i.e., \( \tilde{\Gamma}_{1,1}, \tilde{\Sigma}_{1,1} \equiv 0 \), together with the requirement of \( \tilde{\Gamma}(q) \) to be stable for all \( q \in \Omega_q \), Assumption \( \square \) imposes a constraint on the form \( \tilde{\Gamma}_{1,2} \) and \( \tilde{\Gamma}_{2,1} \) as shown in the following Lemma \( \square \).

**Lemma 1.** Let \( \tilde{\Gamma}, \tilde{\Sigma}, Q \) be such that the conditions of Proposition \( \square \) are satisfied. \( \tilde{\Gamma}_{1,1} \equiv 0 \) implies

\[
\forall q \in \Omega_q : \quad \tilde{\Gamma}_{1,2}(q) Q = -\tilde{\Gamma}_{2,1}'(q). \tag{39}
\]

**Proof.** Writing \( (38) \) in terms of the sub-blocks of \( \tilde{\Gamma} \) we find

\[
\begin{pmatrix}
0 & \tilde{\Gamma}_{1,2}(q) Q + \tilde{\Gamma}_{2,1}'(q) \\
Q \tilde{\Gamma}_{2,2}(q) + \tilde{\Gamma}_{2,1}(q) \tilde{\Gamma}_{2,2}(q) Q + Q \tilde{\Gamma}_{2,2}'(q)
\end{pmatrix} = \tilde{\Sigma}(q) \tilde{\Sigma}^T(q). \tag{40}
\]

By Lemma \( \square \) (iii) it follows that the left hand side of \( (40) \) is a positive semi-definite matrix for all \( q \in \Omega_q \) if and only if \( (39) \) holds. \( \square \)

**Equilibrium generalized Langevin equation** In the particular case of a conservative force, i.e., \( F = -\nabla U \), one can easily derive a closed form solution for an invariant measure of the SDE \( (24) \) if Assumption \( \square \) holds:

**Proposition 3.** Let \( F = -\nabla U \), and let Assumption \( \square \) hold. The SDE \( (24) \) conserves the probability measure \( \mu_{Q,\beta} \) with density

\[
\rho_{Q,\beta}(x) \propto e^{-\beta|U(q)\rangle + \frac{1}{2} p^T M^{-1} p + \frac{1}{2} s^T Q^{-1} s}. \tag{41}
\]

**Proof.** The statement follows by inspection of the stationary Fokker-Planck equation associated with the SDE \( (24) \). \( \square \)
2.2 Non-equilibrium quasi-Markovian generalized Langevin equations without fluctuation-dissipation relation

In general one might also consider instances of [24], where a fluctuation dissipation relation in the form of Assumption 1 does not hold. Such situations might appear in the modelling of temperature gradients or swarming/flocking phenomena; see, e.g., [54] for Markovian variants of such models. For example, one may consider an instance of [24], where \( \Gamma \) and \( \Sigma \) are of the form

\[
\hat{\Gamma} = \begin{pmatrix}
\hat{\Gamma}_{1,1}^{(1)} & \hat{\Gamma}_{1,2}^{(1)} & \hat{\Gamma}_{1,2}^{(2)} \\
\hat{\Gamma}_{2,1}^{(1)} & \hat{\Gamma}_{2,2}^{(1)} & 0 \\
0 & 0 & \hat{\Gamma}_{2,2}^{(2)}
\end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix}
\hat{\Sigma}_{1,1}^{(2)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \hat{\Sigma}_{2,2}^{(2)}
\end{pmatrix},
\]

where

\[
\hat{\Gamma}_{1,1}^{(1)}, \hat{\Sigma}_{1,1}^{(1)} \in C^\infty(\Omega, \mathbb{R}^{n \times n}), \quad \hat{\Gamma}_{1,2}^{(1)}, \hat{\Gamma}_{2,1}^{(1)} \in C^\infty(\Omega, \mathbb{R}^{n \times \tilde{m}}),
\]

and

\[
\hat{\Gamma}_{2,2}^{(1)}, \hat{\Gamma}_{2,2}^{(2)}, \hat{\Sigma}_{2,2}^{(2)} \in C^\infty(\Omega, \mathbb{R}^{\tilde{m} \times \tilde{m}}),
\]

with \( \tilde{m} \in \mathbb{N} \) such that \( m = 2\tilde{m} \) and \( \tilde{m} \geq n \). One can easily verify that in the view of the corresponding non-Markovian form (30), the coefficients \( \hat{\Gamma}_{i,j}^{(1)} \), \( 1 \leq i, j \leq 2 \) determine the statistical properties of the dissipation, i.e., the form of the convolution functional \( \hat{K}_{\hat{\Gamma}}(q, t) * p \), and the coefficients \( \hat{\Gamma}_{1,2}^{(2)}, \hat{\Gamma}_{2,2}^{(2)} \) and \( \hat{\Sigma}_{1,1}^{(2)}, \hat{\Sigma}_{2,2}^{(2)} \) determine the statistical properties of the random force \( \hat{\eta} \). As a simple example we can mention the case where the coefficients \( \hat{\Gamma}_{i,j}^{(k)} \) and \( \hat{\Sigma}_{i,j}^{(k)} \) are constant, i.e.,

\[
\hat{\Gamma}_{1,1}^{(1)}, \hat{\Sigma}_{1,1}^{(1)} \in \mathbb{R}^{n \times n}, \quad \hat{\Gamma}_{1,2}^{(1)}, \hat{\Gamma}_{2,1}^{(1)} \in \mathbb{R}^{n \times m}, \quad \hat{\Gamma}_{1,2}^{(2)}, \hat{\Gamma}_{2,2}^{(2)}, \hat{\Sigma}_{2,2}^{(2)} \in \mathbb{R}^{m \times m},
\]

with \( \hat{\Sigma}_{2,2}^{(2)} = \hat{\Gamma}_{2,2}^{(2)} + \left( \hat{\Gamma}_{2,2}^{(2)} \right)^T \). Under suitable conditions on these matrices (compare with the respective conditions stated in the preceding sections), it can then be easily shown that the SDE (24) can be rewritten as

\[
\dot{q} = M^{-1}p,
\]

\[
\dot{p} = F(q) - \int_0^t K_1(t-s)p(s)ds + \hat{\eta},
\]

where

\[
K_1(t) = \delta(t)\hat{\Gamma}_{1,1}^{(1)} - \hat{\Gamma}_{1,2}^{(1)}e^{-t\hat{\Gamma}_{2,2}^{(1)}},
\]

and \( \hat{\eta} \) is a stationary Gaussian process with covariance function \( K_2 \) of the form

\[
K_2(t) = 2\delta(t)\hat{\Sigma}_{1,1}^{(2)} + \hat{\Gamma}_{1,2}^{(2)}e^{-t\hat{\Gamma}_{2,2}^{(2)}}\left( \hat{\Gamma}_{1,2}^{(2)} \right)^T.
\]
2.3 Markovian representations of the GLE in the literature

In the special case of $\tilde{\Gamma}, \tilde{\Sigma}$ being constant (see Example 1), the Markovian representation (24) is of similar generality to that presented in [7, 31] and the steps in the derivation are essentially the same (see also [48, Chapter 8]). Likewise, a derivation of a Markovian representation of the form (24) can for example be found in a slightly less general setup in [36]. We point out that besides the above mentioned generic frameworks, there are many Markovian representations of the GLE mentioned in the literature which are derived in the context of a particular physical model or application. For example, the Markovian representations of the GLE derived in [9, 1, 28, 51] can be considered as special instances of the SDE (24) with constant coefficients $\tilde{\Gamma}, \tilde{\Sigma}$. Similarly, some of the non-equilibrium models studied in [14, 12, 13, 50, 51] can be represented in the form of (24) with constant coefficients $\tilde{\Gamma}, \tilde{\Sigma}$. Markovian representations of the GLE with position dependent memory kernels, which can be viewed as instances of the SDE (24) can be found in [27, 44, 45, 35].

Sufficient condition for the existence of a Markovian representation

Let $\eta$ be a real-valued stationary Gaussian process with vanishing mean and covariance function $K \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, i.e.,

$$\forall s, t \in \mathbb{R}, \ E[\eta(t)] = 0, \ K(t) = E[\eta(s + t)\eta(s)].$$

We denote by $\hat{\mu}_K$ the spectral measure of $K$, i.e.,

$$K(t) = \int_{\mathbb{R}} e^{ikt} d\hat{\mu}_K(k).$$

Note that the existence of the spectral measure is a direct consequence of the following proposition, which is an adapted (and simplified) version of what is commonly referred to as Bochner’s theorem.

**Proposition 4.** A complex-valued function $C$ with domain $\mathbb{R}$ is the covariance function of a continuous weakly stationary random process on $\mathbb{R}^n$ with finite first and second moments, if and only if it can be represented as

$$C(t) = \int_{\mathbb{R}} e^{ikt} d\mu(k)$$

where $\mu$ is a positive finite measure.

The above Proposition 4 is a simplified version of [56]. For a proof of the theorem we refer to any standard text book in Fourier analysis, such as [53, Chapter 1].

\footnote{A stochastic process $(X(t))_{t \in \mathbb{R}}$ with associated covariance function $C$ is said to be weakly stationary if $E[X(t)] = E[X(t + s)] = 0$ and $C(0, s) = C(t, t + s)$ for all $t, s \in \mathbb{R}$. Since Gaussian processes are fully characterized by the mean and covariance function, a Gaussian processes is weakly stationary if and only if it is stationary.}
Assume that $\tilde{\rho}_K$ possesses a density with respect to the Lebesgue measure, i.e.,

$$\tilde{\mu}_K(dk) = \tilde{\rho}_K(k)dk.$$  

It has been observed in [50] (see also [14,51] for similar results), that $(\tilde{\rho}_K(k))^{-1}$ being of the form of a polynomial implies that $\eta$ can be rewritten as a Markov process in an extended phase space. This can be seen as a consequence of the following criteria for Markovianity:

**Proposition 5.** If $p(k) = \sum_{m=1}^{\infty} c_m (-ik)^m$ is a polynomial with real coefficients and roots in upper half plane then the Gaussian process with spectral density $|p(k)|^{-2}$ is the solution of the stochastic differential equation

$$p \left( -i \frac{d}{dt} \right) \eta(t) dt = dW(t)$$

The above proposition is quoted from [51]. A simple and self-contained proof is also provided in this reference. For a more comprehensive discussion, we refer to [11].

As detailed in [51] the inverse density $(\tilde{\rho}_K(k))^{-1}$ being a polynomial indeed implies the applicability of Proposition 5, as positivity of the measure $\tilde{\mu}_K$ is follows from Bochner’s theorem. Therefore $\tilde{\rho}_K$ must be a positive function, i.e., a positive polynomial of even degree, which in turn implies the existence of a suitable polynomial $p(k) = \sum_{m=1}^{\infty} c_m (-ik)^m$ with properties as stated in Proposition 5.

Proposition 5 has been used extensively in [14,13,50,51] to derive finite dimensional Markovian representations of heat bath models of the form (23). Similarly, Proposition 5 can also be used to derive suitable distributions for the spring constants and the heat bath particle masses in the Ford-Kac model which ensure that in the thermodynamic limit the path of the distinguished particle converges weakly to the solution of a stochastic IDE which can be represented in a Markovian form; see [29,28,17].

3 Ergodicity properties

In this section we provide criteria for geometric ergodicity for the Markovian representations of the GLE introduced in the previous section, i.e., we show under certain conditions that there exists a unique probability measure with smooth density $\mu(dx) = \rho(x)dx$, such that

$$\exists \kappa > 0, C > 0, \forall \varphi \in L^\infty_{K,\mu}, \|E_{\mu} \varphi - e^{t\tilde{L}_{\text{GLE}}} \varphi\|_{L^\infty_{K,\mu}} \leq Ce^{-\kappa t}\|E_{\mu} \varphi - \varphi\|_{L^\infty_{K,\mu}},$$  

and

$$\int_{\Omega_x} K(x) \mu(dx) < \infty,$$  

(47)
Generalized Langevin equation

\[ E_{\mu} \varphi := \int \varphi(x) \mu(dx), \]

(48)

\[ \mathcal{L}_{\text{GLE}} \] as defined in (25), and \( K \in C^2(\Omega_\varphi[1, \infty)) \) is a suitable Lyapunov function. In particular, if \( F = -\nabla U \) and Assumption 1 holds, then

\[ \mu(dx) = \mu_{Q, \beta}(dx), \]

where \( \mu_{Q, \beta} \) is as defined in Proposition 3.

All results are derived using standard Lyapunov techniques, which we sum-
marize in appendix B. That is, we show that (i) the minorization condition (Assumption 5) is satisfied and (ii) a suitable Lyapunov function exists which satisfies Assumption 4 (or more generally the existence of a suitable class of Lyapunov functions of which each instance satisfies Assumption 4). We treat the cases \( \Omega_q = T^n \) and \( \Omega_q = \mathbb{R}^n \) separately. In the situation \( \Omega_q = \mathbb{R}^n \), we show geometric ergodicity for the case of constant coefficients, i.e., \( \bar{\Gamma} \equiv \Gamma \), and \( \bar{\Sigma} \equiv \Sigma \), which in the non-Markovian form (30) corresponds to the situation of a stationary random force. For the case of a bounded domain \( \Omega_q = T^n \) we can show geometric ergodicity also for the case where \( \bar{\Gamma} \) and \( \bar{\Sigma} \) are not constant in \( q \), i.e., the random force, \( \bar{\eta} \), in the corresponding non-Markovian form (30) is non-stationary. In order to simplify presentation we assume for the remainder of this article \( M = I_n \).

3.1 Summary of main results

Let in the sequel \( g(x) = \Theta(f(x)) \) indicate that the function \( f \) is bounded both above and below by \( g \) asymptotically as \( x \to \infty \), i.e., there exist \( c_1, c_2 > 0 \) and \( \hat{x} \geq 0 \), such that \( c_1 g(x) \leq f(x) \leq c_2 g(x) \) for all \( x \geq \hat{x} \).

Results for stationary noise We first present results for the constant coef-
ficient case, i.e., \( \bar{\Gamma} \equiv \Gamma \), and \( \bar{\Sigma} \equiv \Sigma \). Let for the remainder of this subsection \( \Gamma, \Sigma \) be such that

(i) \( -\Gamma \) is stable

(ii) the SDE (24) satisfies the parabolic Hörmander condition both in the presence of the force term \( \nabla U \) and also for the case \( U \equiv 0 \). We provide algebraic conditions on \( \Gamma, \Sigma \) which imply the parabolic Hörmander condition in Section 3.2

(iii) Assumption 1 is satisfied so that for \( F = -\nabla U \) the measure \( \mu_{Q, \beta}(dx) = \rho_{Q, \beta}(x)dx \) with \( \rho_{Q, \beta} \) as defined in (41) is an invariant measure of (24).

Theorem 1. Let \( \Omega_q = T^n \), and \( \bar{\Gamma}, \bar{\Sigma} \) as specified above. There is a unique invariant measure \( \mu \) such that for any \( l \in \mathbb{N} \) there exists \( \mathcal{K}_l \in C^\infty(T^n \times \mathbb{R}^{n+m}) \) with

\[ \mathcal{K}_l(q, p, s) = \Theta(\|z\|^{2l}), \text{ as } \|z\| \to \infty, \quad z = \left( \begin{array}{c} p \\ s \end{array} \right), \]
so that (46) and (47) hold for $K = K_1$. In particular, if $F = -\nabla U$, then $\mu = \mu_{Q, \beta}$.

Proof. The validity of the minorization condition follows from Lemma 3. The existence of a suitable class of Lyapunov functions is shown in Lemma 2. \qed

In the case of an unbounded configurational domain, i.e., $\Omega_q = \mathbb{R}^n$, we require an additional assumption on the force $F$ in order to construct a suitable class of Lyapunov functions.

**Assumption 2** There exists a potential function $V \in C^2(\Omega_q, \mathbb{R})$ with the following properties

(i) there exists $G \in \mathbb{R}$ such that

$$\langle q, F(q) \rangle \leq -\langle q, \nabla_q V(q) \rangle + G.$$  

for all $q \in \Omega_q$.

(ii) the potential function is bounded from below, i.e., there exists $u_{\min} > -\infty$ such that

$$\forall q \in \Omega_q, V(q) \geq u_{\min}.$$

(iii) there exist constants $D, E > 0$ and $F \in \mathbb{R}$ such that

$$\forall q \in \Omega_q, \langle q, \nabla_q V(q) \rangle \geq DV(q) + E\|q\|^2_2 + F.$$  

(49)

**Theorem 2.** Let $\Omega_q = \mathbb{R}^n$, $F$ satisfies Assumption 2 $\bar{F}, \bar{\Sigma}$ as specified above with $\text{rank}(\Sigma) = n + m$ and $\text{rank}(\Gamma_{1,1}) = n$. There is a unique invariant measure $\mu$ such that for any $l \in \mathbb{N}$ there exists $K_l \in C^\infty(\mathbb{R}^{2n+m}, [1, \infty))$ with

$$K_l(x) = \Theta(\|x\|^{2l}), \quad \text{as } \|x\| \to \infty,$$

such that (46) and (47) hold for $K = K_l$. In particular, if $F = -\nabla U$, then $\mu = \mu_{Q, \beta}$.

Proof. The validity of a minorization condition follows from Lemma 5. The existence of a suitable class of Lyapunov functions is shown in Lemma 4. \qed

The above theorem covers instances of the GLE with a non-degenerated white noise component. In order to derive geometric ergodicity for GLEs without a white noise component, i.e. $\Gamma_{1,1} = 0$, we require the force $F$ to satisfy the following assumption:

**Assumption 3** Let the force $F$ be such that

$$F(q) = F_1(q) + F_2(q),$$

where $F_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is uniformly bounded in $\Omega_q$, i.e.,

$$\sup_{q \in \Omega_q} \|F_1(q)\|_\infty < \infty$$

and

$$F_2(q) = Hq,$$

with $H \in \mathbb{R}^{n \times n}$ being a positive definite matrix, i.e., $\min \sigma(H) = \lambda_H > 0$. 


Remark 2. Assumption 3 implies that there is $H > 0$ and $r \in \mathbb{R}$ so that
\[ |\langle g, F(q) \rangle| \leq H |\langle g, q \rangle| + r, \]
for all $q, g \in \mathbb{R}^n$. Moreover, if both Assumption 3 and Assumption 2 hold, then it is easy to see that the potential function $V$ in Assumption 2 is of the form of a perturbed quadratic potential function in the following sense:
\[ V(q) = V_1(q) + V_2(q), \]
where $V_1 \in C^\infty(\mathbb{R}^n, \mathbb{R})$ has bounded derivatives and
\[ V_2(q) = \frac{1}{2} q^T H q. \]

The following theorem provides a sufficient condition for geometric ergodicity of (24) for constant coefficients and $\Gamma_{1,1} = 0$.

**Theorem 3.** Let $\Omega_q = \mathbb{R}^n$, $F$ satisfies Assumption 3 and Assumption 2, and $\tilde{\Gamma}, \tilde{\Sigma}$ as specified above with $\Gamma_{1,1} = 0$. There exists a unique probability measure $\mu(dx)$ such that for any $l \in \mathbb{N}$ there exists $K_l \in C^\infty(\mathbb{R}^{2n+m}, [0, \infty))$ with
\[ K_l(x) = \Theta(||x||^2), \quad \text{as } ||x|| \to \infty, \]
such that (46) and (47) hold for $K = K_l, \mathcal{L} = \mathcal{L}_{\text{GLE}}$. In particular, if $F = -\nabla U$, then $\mu = \mu Q, \beta$.

**Proof.** The validity of the minorization condition follows from Lemma 6. The existence of a suitable class of Lyapunov functions is shown in Lemma 4. \qed

**Results for non-stationary noise** For the case of a periodic configurational domain $\Omega_q = \mathbb{T}^n$ we show geometric ergodicity for the SDE (24) for the general case where $\tilde{\Gamma}$ and $\tilde{\Sigma}$ may not be constant. We focus on the case
\[ \tilde{\Gamma}(\cdot) = \begin{pmatrix} 0 & \tilde{\Gamma}_{1,2}(\cdot) \\ \tilde{\Gamma}_{2,1}(\cdot) & \tilde{\Gamma}_{2,2}(\cdot) \end{pmatrix} \in C^\infty(\Omega_q, \mathbb{R}^{2n \times 2n}), \]
where all non vanishing sub-blocks are assumed to be invertible, i.e.,
\[ \tilde{\Gamma}_{1,2}(q), \tilde{\Gamma}_{2,1}(q), \tilde{\Gamma}_{2,2}(q), \tilde{\Sigma}_{2,2}(q) \in \text{GL}_n(\mathbb{R}), \]
for all $q \in \Omega_q$, where by $\text{GL}_n(\mathbb{R}) \subset \mathbb{R}^{n \times n}$ we denote the set of all invertible $n \times n$-matrices with real valued coefficients. Furthermore, we assume that $-\tilde{\Gamma}(q)$ is a stable matrix for all $q \in \Omega_q$ and that $\tilde{\Gamma}, \tilde{\Sigma}$ are such that Assumption 1 is satisfied, i.e., since $\tilde{\Gamma}_{1,1} \equiv 0$, it follows by Lemma 3 that
\[ \forall q \in \Omega_q, \quad \tilde{\Gamma}_{1,2}(q) = -Q \tilde{\Gamma}_{2,1}(q), \quad (50) \]
holds. Moreover we assume
\begin{equation}
\exists C \in \mathbb{R}^{(n+m) \times (n+m)} \text{ s.p.d., } \forall q \in \Omega_q : \ \tilde{\Gamma}(q)C + C\tilde{\Gamma}^T(q) \text{ s.p.d.} \tag{51}
\end{equation}

We expect that our result can be easily extended to more general forms of $\tilde{\Gamma}$, i.e., to the case where $\tilde{\Gamma}(q) \in \mathbb{R}^{m \times m}$ with $m \neq n$; see item (iv). We also point out that the case $\tilde{\Gamma}_{1,1} \neq 0$ would not cause any additional difficulties in the proof of the result as long as the identity (50) holds. (See e.g. [54] for ergodicity results for under-damped Langevin equation with non-constant coefficients.)

**Theorem 4.** Let $\Omega_q = T^n$. Under the assumptions on $\tilde{\Gamma}$ and $\tilde{\Sigma}$ described in the preceding paragraph, there is a unique invariant measure $\mu$ such that there exists for any $l \in \mathbb{N}$ a function $K_l \in C^\infty(T^n \times \mathbb{R}^{2n}, [1, \infty))$ with

$$K_l(q, p, s) = \Theta(\|z\|^2), \quad as \ |z| \to \infty, \ z = \begin{pmatrix} p \\ s \end{pmatrix},$$

such that (46) and (47) hold for $K = K_l$. In particular, if $F = -\nabla U$, then $\mu = \mu_{Q, \beta}$.

**Proof.** The validity of the minorization condition follows from Lemma 8. The existence of a suitable class of Lyapunov functions is shown in Lemma 11. $\square$

We provide a simple example of an instance of (24), which satisfies the condition of Theorem 4:

**Example 3.** Let $m = n = 1$ and let $\Omega_q = T$. Consider the matrix-valued functions $\tilde{\Gamma}, \tilde{\Sigma}$ defined by

$$\tilde{\Gamma}(q) = \begin{pmatrix} 0 \\ (2 + \cos(2\pi q)) \\ 1 \end{pmatrix}, \quad \tilde{\Sigma}(q) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Obviously, a valid choice for $Q$ in Proposition 3 is

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Moreover,

$$C = \begin{pmatrix} 19/18 & -(1/6) \\ -(1/6) & 1 \end{pmatrix}.$$ 

satisfies (51). This follows by virtue of Lemma 12. We provide a plot of the eigenvalues of the matrix

$$R(q) = \tilde{\Gamma}(q)C + C\tilde{\Gamma}^T(q),$$ 

as a function of $q$ in Figure 1.

---

8 We use s.p.d. as the abbreviation for symmetric positive definite
Central limit theorem for quasi-Markovian GLE dynamics. A direct consequence of the geometric ergodicity of the dynamics (24) is the validity of a central limit theorem for certain observables. Define the projection operator
\[ \Pi \phi = \phi - E_\mu \phi, \]
and let \( L_{K,0}^\infty := \Pi L_K^\infty \subset L_K^\infty \), be the subspace of \( L_K^\infty \) which is comprised of observables with vanishing mean. Denote by \( \| \cdot \|_{B(L_K^\infty)} \) the operator norm
\[ \| A \|_{B(L_K^\infty)} := \sup_{\phi \in L_K^\infty} \frac{\| A\phi \|_{L_K^\infty}}{\| \phi \|_{L_K^\infty}}, \]
induced by the norm \( \| \cdot \|_{L_K^\infty} \) for operators \( A : L_K^\infty \to L_K^\infty \). The validity of (46) immediately implies the inequality
\[ \| \Pi e^t \tilde{L}_{\text{GLE}} \|_{B(L_K^\infty)} \leq C e^{t\kappa}, \tag{53} \]
By [33, Proposition 2.1], \( \tilde{L}_{\text{GLE}} \) considered as an operator on \( L_{K,0}^\infty \) is invertible with bounded spectrum. By [3] this implies a central limit theorem for observables contained in \( \phi \in L_K^\infty \) as summarized in the following corollary 1.

**Corollary 1.** Let the conditions of one of the Theorems 1 to 4 be satisfied and let \( K_l \) for \( l \in \mathbb{N} \) be a suitable Lyapunov function as specified therein. The spectrum of \( \tilde{L}_{\text{GLE}}^{-1} \Pi \) is bounded in \( \| \cdot \|_{B(L_K^\infty)} \), i.e.,
\[ \| \tilde{L}_{\text{GLE}}^{-1} \Pi \|_{B(L_K^\infty)} \leq C_l \frac{1}{\kappa_l}, \tag{54} \]
where \( C_l, \kappa_l > 0 \) are such that (46) holds for \( K = K_l, \kappa = \kappa_l, C = C_l \). In particular, a central limit theorem holds for the solution of (24), i.e.,
\[ T^{-1/2} \int_0^T [E_\mu \phi - \phi(x(t))]^2 \, dt \sim \mathcal{N}(0, \sigma_\phi^2), \text{ as } T \to \infty, \tag{55} \]
for any \( \phi \in L_K^\infty \), where \( \mu \) denotes the unique invariant measure of \( x \) and
\[ \sigma_\phi^2 = -2 \int \left( \tilde{L}_{\text{GLE}}^{-1} \Pi \phi(x) \right) \Pi \phi(x) \mu(dx). \]
Notes on Theorems 1 to 4

(i) The Lyapunov-based techniques on which the proofs of our ergodicity results rely have been studied in the context of stochastic differential equations (see \[39, 59, 37, 2\]) as well as in the context of discrete time Markov chains (see e.g. \[20, 38, 40, 18\]). In particular, we mention the application of these techniques to prove geometric ergodicity of solutions of the under-damped Langevin equation in \[59, 37, 2\]. As discussed in Section 2, the structure of the SDE (24) resembles the structure of the under-damped Langevin equation and it is therefore not surprising that also the structure of the Lyapunov functions constructed in the proofs of \[37\] resemble the structure of the Lyapunov functions presented in the latter two references.

(ii) In \[47\] the authors construct a Lyapunov function for a Markovian reformulation of the GLE with conservative force which in the representation (24) corresponds to the case where \( \tilde{\Gamma}, \tilde{\Sigma} \) are constant with \( \tilde{\Gamma} \equiv \Gamma \) such that \( \Gamma_{1,1} = 0 \) and \( \Gamma_{1,2}, \Gamma_{2,1}, \Gamma_{2,2} \in \mathbb{R}^{n \times n} \) are diagonal matrices. In the same article exponential convergence of the law towards a unique invariant distribution \( \mu \) in relative entropy and exponential decay estimates for the semi-group operator \( e^{t \tilde{L}_{\text{GLE}}} \) in weighted Sobolev space \( H^1(\mu) \) using hypocoercivity techniques by \[60\].

(iii) Ergodic properties of non-equilibrium systems which have a similar structure as the QGLE models considered here have been studied in a series of papers \([14, 13, 12, 52, 50]\). These systems consist of a chain of a finite number of oscillators whose ends are coupled to two heat baths. In a simplified version these systems can be written as

\[
\begin{align*}
\dot{r}_1 &= -\gamma_1 r_1 + \lambda_1 p_1 + \sqrt{2\beta^{-1}} \gamma_1 W_1, \\
\dot{q}_1 &= p_1, \\
\dot{p}_1 &= -\partial_{q_1} U(q) - \lambda_1 r_1, \\
\dot{q}_i &= p_i, \quad i = 2, 3, \ldots, n - 1, \\
\dot{p}_i &= -\partial_{q_i} U(q), \quad i = 2, 3, \ldots, n - 1, \\
\dot{q}_n &= p_n, \\
\dot{p}_n &= -\partial_{q_n} U(q) - \lambda_2 r_2, \\
\dot{r}_2 &= -\gamma_2 r_2 + \lambda_2 p_n + \sqrt{2\beta^{-1}} \gamma_2 W_2,
\end{align*}
\]

where

\[ U(q) = U_1(q_1) + U_n(q_n) + \sum_{i=2}^{n} \tilde{U}(q_i - q_{i-1}), \]

with \( U_1, U_2, \tilde{U} \in C^\infty(\mathbb{R}, \mathbb{R}), \gamma_i > 0, \lambda_i > 0 \) for \( i = 1, 2 \), and \( W_1, W_2 \) are two independent Wiener processes taking values in \( \mathbb{R} \). Under certain conditions on the potential functions \( U_1, U_n \) and \( \tilde{U} \), the existence of an invariant measure (stationary non-equilibrium state) has been shown in \[14\]. Uniqueness
conditions where derived in [13,12], and exponential convergence to the invariant state was shown in [52] (see also the review paper [51] and [4]. In the latter reference slightly more general heat bath models are considered than above in (56)). Exponential convergence towards a unique invariant measure is proven in [52] by showing the existence of a suitable Lyapunov function and by showing hypoellipticity and controllability in the sense of Assumption 7. The construction of a suitable control in the proof provided therein relies on $\tilde{U}$ being strictly convex. We expect that the techniques which are used in [52] to prove the existence of a suitable Lyapunov function and the controllability of the SDE can be extended/modified to prove geometric ergodicity of GLEs which can be represented in the form (24) with constant coefficients. In fact it has been demonstrated in [51] that controllability in the sense of Assumption 7 of a system consisting of a chain of oscillators which are coupled to a single heat bath, can be proven by the same techniques as used in [52].

(iv) We expect that Theorem 4 can be generalized to cover instances of (24), where $\tilde{\Gamma}$ is of a form such that in the non-Markovian reformulation (30) the memory kernel is of the form

$$K_{\tilde{\Gamma}}(q,t) = \tilde{\Gamma}_{1,1}(q)\delta(t) - \sum_{i=1}^{K} \tilde{\Gamma}_{1,2}^{(i)}(q)e^{-t\tilde{\Gamma}_{2,2}^{(i)}}\tilde{\Gamma}_{2,1}^{(i)}(q), \quad K \in \mathbb{N},$$

where each $\tilde{\Gamma}^{(i)}$,

$$\tilde{\Gamma}^{(i)}(q) = \begin{pmatrix} 0 & \tilde{\Gamma}_{1,2}^{(i)}(q) \\ \tilde{\Gamma}_{2,1}^{(i)}(q) & \tilde{\Gamma}_{2,2}^{(i)}(q) \end{pmatrix}$$

satisfies the same conditions as $\Gamma$ in Theorem 4.

3.2 Conditions for hypoellipticity

Consider the case of constant coefficients in (24), i.e., $\tilde{\Gamma} \equiv \Gamma, \tilde{\Sigma} \equiv \Sigma$. In this subsection we provide criteria in the form of algebraic conditions on $\Gamma$ and $\Sigma$ which ensure that (24) satisfies the parabolic Hörmander condition, which by Theorem 6 implies that the differential operators

$$L_{\text{GLE}}, \quad L_{\text{GLE}}^\dagger, \quad \partial_t - L_{\text{GLE}}, \quad \partial_t - L_{\text{GLE}}^\dagger,$$

are hypoelliptic. Let in the following Proposition 6 $\Sigma_i, 1 \leq i \leq n + m$ denote the column vectors of $\Sigma$, i.e.,

$$\Sigma = [\Sigma_1, \ldots, \Sigma_{n+m}] \in \mathbb{R}^{(n+m) \times (n+m)}.$$

Proposition 6. Let $\tilde{\Gamma} \equiv \Gamma \in \mathbb{R}^{(n+m) \times (n+m)}$ such that $-\Gamma$ is stable and $\tilde{\Sigma} \equiv \Sigma \in \mathbb{R}^{(n+m) \times (n+m)}$. Any of the following conditions is sufficient for (24) to satisfy the parabolic Hörmander condition.
(i) \( F = Hq + h \), where \( H \in \mathbb{R}^{n \times n}, h \in \mathbb{R} \), and for all \( q \in \Omega_q \)

\[
\mathbb{R}^{2n+m} = \text{lin} \left( \left\{ S^k \left( \frac{0}{\Sigma_i} \right) : k \in \mathbb{N}, \ 1 \leq i \leq n + m \right\} \right), \tag{57}
\]

where

\[ S := - \begin{pmatrix} 0 & -I_n & 0 \\ H \Gamma_{1,1} & \Gamma_{1,2} \\ 0 & \Gamma_{2,1} & \Gamma_{2,2} \end{pmatrix} \in \mathbb{R}^{(2n+m) \times (2n+m)}. \]

(ii) \( \mathbb{R}^{n+m} = \text{lin} \left( \bigcup_{1 \leq i \leq n+m} \left\{ \Gamma^k \Sigma_i : k \leq k_i \right\} \right), \tag{58} \]

where \( k_i, 1 \leq i \leq n + m \) are defined as

\[ k_i := \text{arg max}_{k \in \mathbb{N}} S_0^k \left( \frac{0}{\Sigma_i} \right) \in \{0\} \times \mathbb{R}^{n+m}, \tag{59} \]

with

\[ S_0 := - \begin{pmatrix} 0 & -I_n & 0 \\ H \Gamma_{1,1} & \Gamma_{1,2} \\ 0 & \Gamma_{2,1} & \Gamma_{2,2} \end{pmatrix} \in \mathbb{R}^{(2n+m) \times (2n+m)}. \]

(iii) \( \text{rank} (\Sigma_{2,2}) = m \), and \( \text{rank} (\Gamma_{1,2}) = n \).

**Proof.** In the view of Theorem 6, the coefficients \( b_i \) are

\[ b_0(x) = -G \left( \begin{pmatrix} -F(q) \\ z \end{pmatrix} \right), \]

and

\[ b_i = \beta^{-\frac{i}{2}} \begin{pmatrix} 0 \\ \Sigma_i \end{pmatrix} \in \mathbb{R}^{2n+m}, 1 \leq i \leq n + m, \]

with \( G \in \mathbb{R}^{(2n+m) \times (2n+m)} \) as defined in (70). Since for \( i > 0 \) the coefficients \( b_i \) are constant in \( x \), we find \([b_i, b_j] = 0\) and \([b_0, b_i] = -\nabla_x b_0 b_i\) for \( i, j > 0 \), where \( \nabla_x b_0 \) denotes the Jacobian of \( b_0 \), i.e.,

\[ \nabla_x b_0 = - \begin{pmatrix} 0 & -I_n & 0 \\ -\nabla F(q) \Gamma_{1,1} & \Gamma_{1,2} \\ 0 & \Gamma_{2,1} & \Gamma_{2,2} \end{pmatrix}, \]

and \( \nabla F(q) \) denotes the Jacobian of the force \( F \). Therefore,

\[ \mathcal{V}_1 = \{- \nabla_x b_0 v : v \in \mathcal{V}_0\} \cup \mathcal{V}_0, \tag{60} \]

where

\[ \left\{ \begin{pmatrix} 0 \\ \Sigma_i \end{pmatrix} \right\}_{i=1}^{n+m}, \]
In the case of (i) it follows that \( \nabla_x b_0(x) = S \). In particular, since \( \nabla_x b_0 \) is constant in \( x \), (62) generalizes to
\[
\mathcal{V}_{i+1} = \{ Sv : v \in \mathcal{V}_i \} \cup \mathcal{V}_i, \quad i \in \mathbb{N}, \tag{61}
\]
Since \( \mathcal{V}_i \) consists only of constant functions, we have \( \text{lin}(\mathcal{V}_i(x)) \equiv \text{lin}(\mathcal{V}_i) \) for all \( x \in \Omega_x, i \in \mathbb{N} \), thus (62) implies the condition (57).

Regarding (ii): Let \( k_{\max} = \max_{1 \leq i \leq n+m} k_i \), \( k_i \) being as defined in (59) together with (58) ensures that there is \( \tilde{\mathcal{V}} \subset \mathcal{V}_{k_{\max}} \) such that all elements in \( \tilde{\mathcal{V}} \) are constant and
\[
\text{lin}(\tilde{\mathcal{V}}) \equiv \left( \begin{array}{c} 0 \\ \mathbb{R}^{n+m} \end{array} \right).
\]
Therefore,
\[
\mathcal{V}_{k_{\max}+1} \supset \left\{ -\nabla b_0 v : v \in \tilde{\mathcal{V}} \right\} \cup \tilde{\mathcal{V}},
\]
thus for all \( x \in \Omega_x \)
\[
\text{lin}(\mathcal{V}_{k_{\max}+1}(x)) = \text{lin}\left( \left\{ -\nabla b_0(x) v(x) : v \in \tilde{\mathcal{V}} \right\} \cup \tilde{\mathcal{V}}(x) \right) = \mathbb{R}^{2n+m},
\]
where the latter equivalence is due to the fact that
\[
\text{lin}(\left\{ -\nabla b_0(x)v : v \in B \right\} \cup B) = \mathbb{R}^{2n+m},
\]
for all \( x \in \Omega_x \) and any basis \( B \subset \mathbb{R}^{2n+m} \) of \( \{0\} \times \mathbb{R}^{n+m} \).

Regarding (iii): Since \( \text{lin}(\Sigma_{x,2}) = \mathbb{R}^m \) and \( \text{rank}(\Gamma_{1,2}) = n \) it follows that
\[
\{0\} \times \mathbb{R}^{n+m} = \text{lin}\left( \left\{ \left( \begin{array}{c} 0 \\ \Gamma_{\Sigma_i} \end{array} \right), 1 \leq i \leq n+m \right\} \right),
\]
thus the result follows by (ii).

\[\square\]

### 3.3 Technical lemmas required in the proofs of ergodicity of (24) with non-stationary random force

In this subsection we provide the necessary technical lemmas to which we refer in the proofs of Theorems 1 to 3, thus in the remainder of this subsection we assume \( \tilde{\Gamma} \equiv \Gamma, \tilde{\Sigma} \equiv \Sigma \). We begin by showing the existence of a class of suitable Lyapunov functions in the case of a bounded configurational domain, i.e., \( \Omega_q = \mathbb{T}^n \).

**Lemma 2.** Let \( \Omega_q = \mathbb{T}^n, -\Gamma \in \mathbb{R}^{(n+m) \times (n+m)} \) stable, then
\[
\mathcal{K}(q,p,s) = (z^T C z)^l + 1, \quad l \in \mathbb{N},
\]
where \( C \in \mathbb{R}^{n+m} \) is a symmetric positive definite matrix such that \( \Gamma^T C + C \Gamma \) is positive definite, defines a family of Lyapunov functions for the differential operator \( \tilde{L}_{\text{GLE}}, \) i.e., for each \( l \in \mathbb{N} \) there exist constants \( a_l > 0, b_l \in \mathbb{R} \), such that for \( \mathcal{L} = \tilde{L}_{\text{GLE}}, \mathcal{K} = \mathcal{K}_l \), Assumption 4 holds with \( a = a_l, b = b_l \).
Proof. We show the existence of suitable constants \( \tilde{a}_l, \tilde{b}_l \) so that the inequality (90) is satisfied for \( K = K_l := K_l - 1 \), and \( L = L_{\text{GLE}}, a = \tilde{a}_l, b = \tilde{b}_l \), which directly implies the statement of Lemma 2 for \( a_l = \tilde{a}_l \) and \( b_l = \tilde{b}_l + \tilde{a}_l \). - \( \Gamma \) being a stable matrix ensures that there indeed exists a symmetric positive definite matrix \( C \) such that \( \Gamma^T C + C \Gamma \) is positive definite. Without loss of generality let \( \min \sigma(C) = 1 \), so that

\[
\|z\|^2 \leq z^T Cz = K_1(x) - 1. \tag{63}
\]

Furthermore,

\[
\lambda = \sup_{z \in \Omega, \|z\| = 1} \frac{z^T (\Gamma^T C + C \Gamma) z}{z^T Cz},
\]

so that

\[
2z^T \Gamma^T Cz \geq \lambda z^T Cz = \lambda (K_1(x) - 1). \tag{64}
\]

We first consider the case \( l = 1 \):

\[
(L_H + L_O)\tilde{K}_1(x) = [2p^T C_{1,1} + 2s^T C_{1,2} - p^T] F(q) - 2z^T \Gamma^T Cz + \beta^{-1} \sum_{i,j} [C \Sigma \Sigma^T C]_{i,j}
\leq c_1 \|z\| - 2z^T \Gamma^T Cz + \beta^{-1} \sum_{i,j} [C \Sigma \Sigma^T C]_{i,j}
\leq \frac{c_1}{\epsilon_1} + \epsilon_1 \|z\|^2 - 2z^T \Gamma^T Cz + \beta^{-1} \sum_{i,j} [C \Sigma \Sigma^T C]_{i,j},
\]

where

\[
c_1 = \max_{q \in \Omega} \frac{[2p^T C_{1,1} + 2s^T C_{1,2} - p^T] F(q)}{\|z\|}.
\]

Thus, by (63) and (64),

\[
(L_H + L_O)\tilde{K}_1(x) \leq \frac{c_1}{\epsilon_1} + \epsilon_1 \tilde{K}_1(x) - \lambda \tilde{K}_1(x) + \beta^{-1} \sum_{i,j} [C \Sigma \Sigma^T C]_{i,j}
= -\tilde{a}_1 \tilde{K}_1(x) + \tilde{b}_1,
\]

with

\[
\tilde{a}_1 := (\lambda - \epsilon_1), \quad \tilde{b}_1 := \frac{c_1}{\epsilon_1} + (\lambda + \epsilon_1) + \beta^{-1} \sum_{i,j} [C \Sigma \Sigma^T C]_{i,j},
\]

so that \( \tilde{a}_1 > 0 \) for sufficiently small \( \epsilon_1 > 0 \).

For \( l > 1 \) we find:

\[
(L_H + L_O)\tilde{K}_l(x) = l \tilde{K}_{l-1}(x) \left[ L_H \tilde{K}_1(x) + \left( - (\Gamma z) \cdot \nabla z \tilde{K}_1(x) \right) + \beta^{-1} \sum_{i,j} [\Sigma \Sigma^T C]_{i,j} \right]
+ 2l(l - 1) \beta^{-1} z^T C \Sigma \Sigma^T C z \tilde{K}_{l-2}(x) \tag{65}
\]
Let
\[ \tilde{\lambda} := \sup_{x \in \Omega_x, \|z\|_2 = 1} \left( \frac{z^T C \Sigma \Sigma^T C z}{K_1(x)} \right), \]
so that
\[ \forall x \in \Omega_x, \ z^T C \Sigma \Sigma^T C z K_{i-2}(x) \leq \tilde{\lambda} K_{i-1}(x). \]
Thus, with
\[ c_l := \min \left( 0, -\beta^{-1} \sum_{i,j} C \Sigma \Sigma^T C \right)_{i,j} + \beta^{-1} \sum_{i,j} \left( \Sigma \Sigma^T C \right)_{i,j} + 2(l-1)\beta^{-1}\tilde{\lambda}, \]
we find
\[ (\mathcal{L}_H + \mathcal{L}_O) K_i(x) \leq l K_{i-1}(x) \left( (\mathcal{L}_H + \mathcal{L}_O) K_1(x) + c_l \right) \]
\[ \leq l K_{i-1}(x) \left( -\tilde{a}_i K_1(x) + \tilde{b}_i + c_l \right) \]
\[ \leq l \left( -\tilde{a}_i K_i(x) + \frac{\tilde{b}_i + c_l}{\tilde{c}_i^{-1}} + \epsilon_l K_i(x) \right) = -\tilde{a}_i K_i(x) + \tilde{b}_i, \]
with
\[ \tilde{a}_i := l(\tilde{a}_1 - \epsilon_l), \quad \tilde{b}_i := l\frac{\tilde{b}_1 + c_l}{\tilde{c}_i^{-1}}, \]
where \( \epsilon_l > 0 \) sufficiently small so that \( \tilde{a}_1 > 0 \).

We next show the existence of a minorization condition in the case of \( \Omega_q = \mathbb{T}^n \). The idea of the proof is to decompose the diffusion process into an Ornstein-Uhlenbeck process and a bounded remainder term, which then allows to conclude the existence of a minorizing measure by virtue of the fact that the solution of Fokker-Planck equation associated with the Ornstein-Uhlenbeck process is a non-degenerate Gaussian at all times \( t > 0 \) and thus has full support. The idea of this approach is borrowed from \[32\] where it was used to show the minorization condition for a discretized version of the under-damped Langevin equation. Other applications of this technique can be found in \[49,26\].

**Lemma 3.** Let \( \Omega_q = \mathbb{T}^n \). If \( \Gamma \in \mathbb{R}^{(n+m)\times(n+m)} \) and \( \Sigma \in \mathbb{R}^{(n+m)\times(n+m)} \) are as in Theorem 4 then Assumption 5 (minorization condition) holds for the SDE (24).

**Proof (Proof of Lemma 3).** Let \( q(0) = q_0 \) and \( z(0) = z_0 \) with
\[ (q_0, z_0) \in \Omega_q \times C_r, \]
where
\[ C_r = \{ z \in \Omega_z : \|z\| < r \}, \]
for arbitrary but fixed \( r > 0 \).
We can write the solution of (24) as
\[ z(t) = z_0 + D_z(t) + \mathcal{G}_z(t), \quad q(t) = q_0 + D_q(t) + \mathcal{G}_q(t), \]
with
\[ D_z(t) = \int_0^t e^{-(t-s)} \Gamma \left( F(q(s)) \right) ds, \quad \mathcal{G}_z(t) = \int_0^t e^{-(t-s)} \Sigma dW(s), \]
and
\[ D_q(t) = \int_0^t \Pi_p D_z(s) ds, \quad \mathcal{G}_q(t) = \int_0^t \Pi_p \mathcal{G}_z(s) ds. \]

The variables \( \mathcal{G}_q(t) \) and \( \mathcal{G}_z(t) \) are correlated and Gaussian, i.e.,
\[ \begin{pmatrix} \mathcal{G}_q(t) \\ \mathcal{G}_z(t) \end{pmatrix} \sim \mathcal{N}(\mu_t, \Sigma_t), \]
with some \( \mu_t \in \Omega_p \) and \( \Sigma_t \in \mathbb{R}^{(2n+m) \times (2n+m)} \). More specifically, \( \tilde{z}(t) = z(0) + \mathcal{G}_z(t) \) and \( q(0) + \mathcal{G}_q(t) \) corresponds to the solution of the linear SDE
\[ \begin{align*}
\dot{\tilde{q}} &= \tilde{p}, \\
\dot{\tilde{z}} &= -\Gamma \tilde{z} + \Sigma \dot{W},
\end{align*} \tag{68} \]
where \( \tilde{z}(t) = (\tilde{p}(t), \tilde{s}(t)) \in \Omega_p \times \Omega_s \). The law of \( \tilde{q}(t), \tilde{z}(t) \) has full support for all \( t > 0 \), provided that the covariance matrix \( \Sigma_t \) is invertible. This is indeed the case since \( \Gamma \) and \( \Sigma \) are required to be such that (24) satisfies the parabolic Hörmander condition. Therefore also (68) satisfy the parabolic Hörmander condition. By Theorem 6 it follows that the law of \( (\tilde{q}(t), \tilde{z}(t)) \) has a density with respect to the Lebesgue measure for any \( t > 0 \), which rules out the possibility of \( \Sigma_t \) being singular.

Let \( C \in \mathbb{R}^{(n+m) \times (n+m)} \) be symmetric positive definite such that \( \Gamma C + C \Gamma^T \) is positive definite as well, and consider the norm \( \| \cdot \|_C \),
\[ \| \cdot \|_C := z^T C z, \quad z \in \mathbb{R}^{n+m}. \]

The increment \( D_z(t) \) is uniformly bounded since
\[ \| D_z(t) \|_C \leq \| \Gamma^{-1} \|_{B(C)} \| F \|_{L^\infty} < \infty, \]
where
\[ \| \Gamma^{-1} \|_{B(C)} := \max_{v \in \mathbb{R}^{2n}} \frac{\| \Gamma^{-1} v \|_C}{\| v \|_C} = \frac{1}{2} \min \sigma \left( \Gamma^T C + C \Gamma \right), \]
denotes the operator norm of \( \Gamma^{-1} \) induced by \( \| \cdot \|_C \). It follows, that also \( D_q(t) \) is bounded since
\[ \| D_q(t) \| \leq t \| D_z(t) \|_C < \infty. \]
Let $\mu_{x_0,t}$ denote the law of $(q(t), z(t))$ and $\rho_{x_0,t}$ the associated density. For fixed $t > 0$, the terms $\mathcal{G}_q(t)$ and $\mathcal{G}_z(t)$ are bounded and the law of $(q(0) + \mathcal{G}_q(t), z(0) + \mathcal{G}_z(t))$ has full support, it follows that the law $\mu_{x_0,t}$ of the superposition of these two random variables has full support, thus $\rho_{x_0,t} \in C(\Omega_x, \mathbb{R}_+)$. Now define

$$
\rho(x) := \min_{x_0 \in \mathcal{E}} \rho_{x_0,t}(x).
$$

By construction the associated probability measure satisfies the properties of $\nu$ in Assumption 5.

We next consider the case $\Omega_q = \mathbb{R}^n$. The following Lemma 4 shows the existence of a suitable class of Lyapunov functions.

**Lemma 4.** Let $\Omega_q = \mathbb{R}^n$. If

(i) $-\Gamma \in \mathbb{R}^{(n+m) \times (n+m)}$ is a stable matrix and $\Sigma \in \mathbb{R}^{(n+m) \times (n+m)}$ such that

$$
\Gamma_2,2Q + Q \Gamma_2,2^T
$$

is positive definite with $Q$ as specified in Assumption 7

(ii) the force $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfies Assumption 2.

Furthermore, if either

(iii) $\Gamma_{1,1}$ is positive definite,

or

(iv) the force $F$ satisfies Assumption 3

then

$$
\mathcal{K}_l(q, p, s) = \left( x^T C_{A,B} x + \|q\|_2^2 + 2(p, q) + BD(V(q) - u_{\min}) + 1 \right)^l, \ l \in \mathbb{N},
$$

where

$$
C_{A,B} = \begin{pmatrix} B I_n & \Gamma_{2,1} \Gamma_{2,1}^T \end{pmatrix} \in \mathbb{R}^{(n+m) \times (n+m)},
$$

is a symmetric positive definite matrix for suitably chosen scalars $A, B > 0$, and $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$ as specified in Assumption 2 defines a family of Lyapunov functions for the differential operator $L_{\text{GLE}}$, i.e., for each $l \in \mathbb{N}$ there exist constants $a_l > 0$, $b_l \in \mathbb{R}$, such that for $\mathcal{L} = L_{\text{GLE}}, \mathcal{K} = \mathcal{K}_l$, Assumption 4 holds for $a = a_l, b = b_l$.

**Proof.** Rewriting $\mathcal{K}_l$ as

$$
\mathcal{K}_l(q, p, s) = \left( z^T C_{A,B} z + BD(V(q) - u_{\min}) + 1 \right)^l, \ l \in \mathbb{N},
$$
where
\[
\hat{C}_{A,B} = \begin{pmatrix}
I_n & I_n & 0 \\
I_n & BI_n & AT_{2,1}^T \\
0 & AT_{2,1} & BQ^{-1}
\end{pmatrix} \in \mathbb{R}^{(n+m) \times (n+m)},
\]
we find by successive application of Lemma 12 that for any \( A' > 0 \) there exists \( B' > 0 \) so that for \( A = A' \) and \( B \geq B' \) the matrix \( \hat{C}_{A,B} \) is positive definite and thus \( \mathcal{K}_l \geq 1 \) and \( \mathcal{K}_l(x) \to \infty \) as \( \|x\| \to \infty \). We first consider the case \( l = 1 \).

Define
\[
G := \begin{pmatrix}
0 & -I_n \\
I_n & I_{1,1} & I_{1,2} \\
0 & I_{2,1} & I_{2,2}
\end{pmatrix} \in \mathbb{R}^{(2n+m) \times (2n+m)},
\]
and
\[
\tilde{Q} := \begin{pmatrix}
I_n & 0 \\
0 & Q
\end{pmatrix},
\]
we find
\[
\mathcal{L}_{GLE} \mathcal{K}_1(x) = -\left( -[F(q)]^T, p^T, s^T \right) G^T \hat{C}_{A,B} x + DBI_n p \cdot \nabla q V(q)
\]
\[
+ \frac{\beta^{-1}}{2} \nabla z \cdot \left( \Sigma \Sigma^T \nabla z (z\tilde{Q}^{-1}z) \right),
\]
with
\[
G^T C = -\begin{pmatrix}
I_n & BI_n \\
-\Gamma_{1,1} & B\Gamma_{1,1} + AT_{2,1}^T \Gamma_{2,1} \\
\Gamma_{1,2} & B\Gamma_{1,2} + B\Gamma_{1,2}^T
\end{pmatrix}.
\]
Hence, by virtue of (49) and Assumption 2 (i),
\[
\mathcal{L}_{GLE} \mathcal{K}_1(x) \leq
\]
\[
-x^T \begin{pmatrix}
EI_n \\
-I_n + \Gamma_{1,1} & -I_n + BI_n + AT_{2,1}^T \Gamma_{2,1} & 0 \\
\Gamma_{1,2} & B\Gamma_{1,2} + B\Gamma_{1,2}^T
\end{pmatrix} x
\]
\[
=: R_{A,B}
\]
\[
- A \nabla q V(q)^T T_{2,1}^T s + F + \frac{\beta^{-1}}{2} \sum_{i,j} \tilde{Q}^{-1} \Sigma \Sigma^T \tilde{Q}^{-1}_{i,j}.
\]

In order to show the existence of constants \( a_1 \) and \( b_1 \) such that the respective Lyapunov inequality satisfied, one needs to show that the right hand side of the above inequality (71) can be bounded from above by a negative definite quadratic form.
Case $\text{rank}(\Gamma_{1,1}) = n$: Let $A = 0$. In this case it is sufficient to show that the symmetric part

$$\tilde{R}^S_{A,B} = \frac{1}{2} \left( \tilde{R}_{A,B} + \tilde{R}^T_{A,B} \right)$$

of $\tilde{R}_{A,B}$ is positive definite. The lower right block

$$\left[ \tilde{R}^S_{A,B} \right]_{(n+1):(2n+m),(n+1):(2n+m)} = -I_n + \frac{B}{2} \left( \Gamma \tilde{Q} + \tilde{Q} \Gamma^T \right) \in \mathbb{R}^{(n+m) \times (n+m)},$$

of $\tilde{R}^S_{0,B}$ is positive definite for sufficiently large $B > 0$. In particular

$$\min \sigma \left( \left[ \tilde{R}^S_{A,B} \right]_{(n+1):(2n+m),(n+1):(2n+m)} \right) = O(B),$$

as $B \to \infty$. Thus, by virtue of Lemma \ref{lemma:positive_definite} for $E > 0$ there is $B' > 0$ so that $\tilde{R}^S_{0,B}$ is indeed positive definite for all $B \geq B'$.

Case $\Gamma_{1,1} = 0$: If Assumption \ref{assumption:3} holds, then by Remark \ref{remark:2} this implies that there is $H > 0$ and $h \in \mathbb{R}$ so that

$$|\langle g, F(q) \rangle| \leq H |\langle g, q \rangle| + h,$$

Therefore, it is sufficient to show that there are constants $A, B, E$ so that the function

$$\varphi(x) = \max \left( -x^T \tilde{R}_{A,B} x - A H q^T \Gamma_{2,1}^T s, \ -x^T \tilde{R}_{A,B} x + A H q^T \Gamma_{2,1}^T s \right)$$

$$= \max_{i=1,2} -x^T \tilde{R}^{(i)}_{A,B,E} x, \quad (72)$$

can be bounded from above by a negative definite quadratic form. This means that we have to show that for suitable constants $A, B, E > 0$ the symmetric part of the matrix

$$\tilde{R}^{(i)}_{A,B,E} = \begin{pmatrix} EI_n & 0 \\ -I_n - I_n + A \Gamma_{2,1}^T \Gamma_{2,1} & (\Gamma_{2,1}^T)^{-1} A H \Gamma_{2,1} \\
-\Gamma_{2,1}^T & A \Gamma_{2,1} \Gamma_{2,2} \end{pmatrix},$$

is positive definite for $i \in \{0, 1\}$. (Note that we used $\Gamma_{2,1}^T - Q^{-1} \Gamma_{2,1} = 0$ in the derivation of the form of $\tilde{R}^{(i)}_{A,B,E}$.) Since $\Gamma_{2,1}^T \Gamma_{2,1}$ is positive definite we can choose $A$ sufficiently large so that $-I_n + A \Gamma_{2,1}^T \Gamma_{2,1}$ is positive definite. The positive definiteness of the symmetric part of $\tilde{R}^{(i)}_{A,B,E}, i \in \{0, 1\}$ follows for sufficiently large $B > 0$ and $E > 0$ by successive application of Lemma \ref{lemma:positive_definite}.
For $l > 1$ we find:

$$
(\mathcal{L}_H + \mathcal{L}_O)K_l(x) = lK_{l-1}(x)\mathcal{L}_H K_1(x) + lK_{l-1}(x)(-\Gamma^Tz \cdot \nabla_z K_1(x))
$$
$$
+ \frac{l}{2} \beta^{-1} \nabla_z \cdot (\Sigma \Sigma^T \nabla_z K_1(x)K_{l-1}(x))
$$
$$
= -lK_{l-1}(x)(z^T \Gamma^T Q z) + l\beta^{-1} \sum_{i,j} \left[ \Sigma \Sigma^T \hat{Q} \right] i,j K_{l-1}(x)
$$
$$
+ 2l(l-1)\beta^{-1} z^T \hat{Q} \Sigma \Sigma^T \hat{Q} z K_{l-2}(x)
$$
$$
\leq -lK_{l-1}(x) ((\mathcal{L}_H + \mathcal{L}_O)K_1(x) + c_2)
$$
$$
\leq lK_{l-1}(x) (-a_1 K_1(x) + b_1 + c_2)
$$
$$
\leq l \left( -a_1 K_i(x) + \frac{b_1 + c_2}{\epsilon_i^{-1}} + \epsilon_i K_i \right) = -a_i K_i(x) + b_i
$$

with

$$
c_2 = -\beta^{-1} \sum_{i,j} \left[ \hat{Q} \Sigma \Sigma^T \hat{Q} \right] i,j + \beta^{-1} \sum_{i,j} \left[ \Sigma \Sigma^T \hat{Q} \right] i,j
$$

and

$$
a_i := l(a_1 - \epsilon_i), \ b_i := \frac{b_1 + c_2}{\epsilon_i^{-1}}
$$

where $\epsilon_i > 0$ sufficiently small so that $a_i > 0$. \hfill \Box

We mention that Assumption 2 is commonly also required for the construction of suitable Lyapunov functions in the case of the underdamped Langevin equation if $\Omega_q$ is unbounded. Assumption 3 an additional constraint on the potential function $U$, which is not required in the case of the underdamped Langevin equation. It is therefore not surprising that this assumption can be dropped if the noise process $\eta$ in the GLE contains a nondegenerate white noise component.

If $\Sigma$ has full rank the minorization can be demonstrated using a simple control argument.

**Lemma 5.** Let $\Omega_q = \mathbb{R}^n$. If $\text{rank}(\Sigma) = n+m$, then (24) satisfies a minorization condition (Assumption 2).

**Proof.** Note that by Proposition 6 (ii) $\text{rank}(\Sigma) = n+m$ immediately implies that the SDE satisfies the parabolic Hormander condition. Since $\Sigma$ is invertible, we can easily solve the associated control problem which then by Lemma 13 implies that a minorization condition is satisfied. The proof of the existence of a suitable control is essentially the same as in the case of the under-damped Langevin equation (see e.g. [37]): Let $T > 0$ and $(q^-, p^-, s^-), (q^+, p^+, s^+) \in \mathbb{R}^{2n+m}$. We need to show that there exists $u \in L^1([0,T], \mathbb{R}^m)$, solving the control
problem
\[ \dot{q} = p, \]
\[ \dot{p} = F(q) - \Gamma_{1,1}p + \Gamma_{1,2}s + \Sigma_1u, \]  \hspace{1cm} (74)
\[ \dot{s} = -\Gamma_{2,1}p + \Gamma_{2,2}s + \Sigma_2u, \]
subject to
\[ (q(0), p(0), s(0)) = (q^-, p^-, s^-), \quad (q(T), p(T), s(T)) = (q^+, p^+, s^+). \]

It is easy to verify that there exists a smooth path \( \tilde{q} \in C^2([0,T], \mathbb{R}^n) \) and \( \tilde{s} \in C^2([0,T], \mathbb{R}^m) \) such that
\[ (\tilde{q}(0), \dot{\tilde{q}}(0)) = (q^-, p^-), \quad (\tilde{q}(T), \dot{\tilde{q}}(T)) = (q^+, p^+), \]
and
\[ \tilde{s}(0) = s^-, \quad \tilde{s}(T) = s^+. \]

Rewrite (74) as a second order differential equation in \( q \) and \( s \):
\[ \ddot{q} = -\nabla_q U(q) - \Gamma_{1,2} \dot{q} - \Gamma_{1,2}s + \Sigma_1u, \]
\[ \dot{s} = -\Gamma_{2,1}q - \Gamma_{2,2}s + \Sigma_2u, \]
thus,
\[ u(t) = \Sigma^{-1} \left( \tilde{q}(t) + \nabla_q U(\tilde{q}(t)) + \Gamma_{1,1} \dot{\tilde{q}}(t) + \Gamma_{1,2} \dot{s}(t) \right), \]  \hspace{1cm} (75)
is a solution of (74).

The following Lemma 6 shows that the minorization condition is satisfied in the case of a GLE with unbounded configurational domain and \( \Gamma_{1,1} = 0 \).

**Lemma 6.** Under the same conditions as Theorem 3 it follows that Assumption 5 is satisfied for (24).

**Proof.** By Assumption 3 the force \( F \) can be decomposed as
\[ F(q) = F_1(q) + F_2(q), \]
where \( \|F_1(q)\|_\infty \) is uniformly bounded in \( q \in \mathbb{R} \) and
\[ F_2(q) = Hq, \]
with \( H \in \mathbb{R}^{n \times n} \) being a positive definite matrix. Consider the dynamics
\[ \ddot{q}^a = p^a, \]
\[ p^a = -Hq^a - \Gamma_{1,2}s^a, \]
\[ \dot{q}^a = -\Gamma_{2,1}p^a - \Gamma_{2,2}s^a + \frac{\beta^{-1}}{2} \Sigma_2 \dot{W}, \]  \hspace{1cm} (76)
with \( (q^a(0), p^a(0), s^a(0)) = x_0, \)
where \( x_0 \in \mathbb{R}^{2n+m} \). The solution of (76) is Gaussian hence
\[
\mu_t^a(dx) = \mathcal{N}(dx; \mu_t, \mathcal{V}_t),
\]
where \( \mu_t \in \mathbb{R}^{2n+m} \) and \( \mathcal{V}_t \in \mathbb{R}^{(2n+m) \times (2n+m)} \). Moreover, by Proposition 6, (iii), the SDE (76) is hypoelliptic, hence \( \mathcal{V}_t \) is non-singular for all \( t > 0 \). As a consequence
\[
\text{supp}(\mu_t^a) = \Omega_x.
\]
Moreover we notice that
\[
F_1(q) = u(q) \Sigma_2,
\]
with
\[
u(q) = -F_1(q)I_{n,m} \Sigma_2^{-1},
\]
where \( I_{n,m} = (I_n, 0) \in \mathbb{R}^{n \times m} \).

Using Lemma 10 it follows by the same chain of arguments as in the proof of Lemma 9, that \( u \) satisfies Novikov’s condition and by virtue of Girsanov’s theorem the support of the law \( \mu_t \) of the solution of (24) with initial condition \( x(0) = x_0 \) coincides with the law of \( \mu_{x_0,t}^a \), i.e., \( \text{supp}(\mu_t) = \Omega_x \). Let \( \mu_{x_0,t}(dx) = \rho_{x_0,t}(x)dx \). As in the proof of Lemma 3 we can construct a minorising measure \( \eta(dx) = \rho(x)dx \), as
\[
\rho(x) := \min_{x_0 \in C_r} \rho_{x_0,t}(x),
\]
where \( C_r \subset \mathbb{R}^{2n+m} \) is a sufficiently large compact set. \( \square \)

Lemma 7. Let
\[
\hat{\mathcal{K}}_\theta(x) = e^{\frac{\theta}{2} \mathcal{K}_1(x)}, \ l = 1,
\]
with \( \mathcal{K}_1 \) as defined in (69). Under the same conditions as in Lemma 6 and provided that Assumption 4 holds for \( \mathcal{L} = \mathcal{L}_{GILE} \), \( \mathcal{K} = \mathcal{K}_1 \), then also \( \hat{\mathcal{K}}_\theta \) satisfies Assumption 4 for \( \mathcal{L} = \mathcal{L}_{GILE} \) and sufficiently small \( \theta > 0 \).

Proof. A simple calculation shows
\[
\mathcal{L}_{GILE}\hat{\mathcal{K}}_\theta(x) = \left( \theta \mathcal{L}_{GILE}\mathcal{K}_1(x) + \frac{\beta - 1}{2} \left( \theta \sum_{i,j} \left[ (Q - I_{n,m}) \tilde{C} \right]_{i,j} + \theta^2 z^T \tilde{C} z \right) \right) \hat{\mathcal{K}}_\theta(x),
\]
with
\[
\tilde{C} = \tilde{Q}^{-1} \Sigma \Sigma^T \tilde{Q}^{-1}.
\]
From Lemma 6 we know \( \mathcal{L}_{GILE}\mathcal{K}_1(x) = \Theta \left( -\|x\|^2 \right) \), thus
\[
\mathcal{L}_{GILE}\hat{\mathcal{K}}_\theta(x) = \left( -\Theta \left( \theta \|x\|^2 \right) + \Theta \left( (1 + \theta) \|z\| \right) + \Theta \left( \theta^2 \|z\|^2 \right) \right) \hat{\mathcal{K}}_\theta(x),
\]
thus for sufficiently small \( \theta > 0 \) and suitable \( b \in \mathbb{R} \),
\[
\mathcal{L}_{GILE}\hat{\mathcal{K}}_\theta(x) < -\hat{\mathcal{K}}_\theta(x) + b.
\]
\( \square \)
3.4 Technical lemmas required in the proofs of ergodicity of (24) with non-stationary random force

We first show that under the assumptions of Theorem 4 a minorization condition is satisfied for (24). For \( r > 0 \) let in the following \( C_r := \{(q, p, s) : \|p, s\|_2 < r\} \).

**Lemma 8.** Let \( \Omega_q = \mathbb{T}^n \) and \( \tilde{\Gamma}_{1,2}, \tilde{\Gamma}_{2,1}, \tilde{\Gamma}_{2,2}, \tilde{\Sigma}_2 \in C^\infty(\Omega_q, \text{GL}_n(\mathbb{R})) \), such that \(-\tilde{\Gamma}(q)\) is stable for all \( q \in \Omega_q \). Let \( r > 0 \) and \( x_0 \in C_r \). For any \( t > 0 \) the law \( \mu_{\tilde{x}_0} = e^{t\tilde{\Gamma}} \delta_{x_0} \) of the solution \( x(t) \) of (24) with initial condition \( x(t) = x_0 \) has full support. In particular, Assumption \( \tilde{\Gamma} \) (minorization condition) holds.

**Proof.** Let \( x_0 = (q_0, p_0, s_0) \in C_r \) and \( \tilde{x}_0 = (q_0, p_0, g_0) \) with \( g_0 = \tilde{\Gamma}_{1,2}(q_0)s_0 \). Consider the following cascade of modifications of (24):

\[
\begin{align*}
\tilde{q}^c &= p^c, \\
\tilde{p}^c &= F(q^c) - g^c \\
\tilde{g}^c &= \sum_{i=1}^n p^c_i \left( \partial_q \tilde{\Gamma}_{1,2}(q^c) \right) q^c_i - \tilde{\Gamma}_{1,2}(q^c) \tilde{\Gamma}_{2,1}(q^c) p^c \\
&\quad - \tilde{\Gamma}_{1,2}(q^c) \tilde{\Gamma}_{2,2}(q^c) \tilde{\Gamma}_{1,2}^{-1}(q^c) g^c + \tilde{\Gamma}_{1,2}(q^c) \tilde{\Sigma}_2(q^c) W_t, \\
\text{with} \quad (q^c(0), p^c(0), g^c(0)) &= \tilde{x}_0,
\end{align*}
\]  

(77)

and

\[
\begin{align*}
\tilde{q}^b &= p^b, \\
\tilde{p}^b &= F(q^b) - g^b, \\
\tilde{g}^b &= p^b - g^b + \tilde{\Gamma}_{1,2}(q) \tilde{\Sigma}_2(q) W_t, \\
\text{with} \quad (q^b(0), p^b(0), g^b(0)) &= \tilde{x}_0,
\end{align*}
\]  

(78)

and

\[
\begin{align*}
\tilde{q}^a &= p^a, \\
\tilde{p}^a &= F(q^a) - g^a, \\
\tilde{g}^a &= p^a - g^a + W, \\
\text{with} \quad (q^a(0), p^a(0), g^a(0)) &= \tilde{x}_0.
\end{align*}
\]  

(79)

Let \( \mu_t^a, \mu_t^b, \mu_t^c \) denote the law of the solution of (79), (78) and (77), respectively. We show that for any \( t > 0 \)

(i) \( \text{supp}(\mu_t^a) = \Omega_x \),
(ii) \( \text{supp}(\mu_t^b) = \text{supp}(\mu_t^c) \),
(iii) \( \text{supp}(\mu_t^c) = \text{supp}(\mu_t^b) \),
(iv) \( \text{supp}(\mu_t) = \text{supp}(\mu_t^f) \),

which then immediately implies that \( \text{supp}(\mu_t) = \Omega_x \) for \( t > 0 \) and the minorization condition follows by the same arguments as in the proof of Lemma 3.
Regarding (i): the system (79) satisfies the condition of Lemma 3, hence for sufficiently large \( t' > 0 \) the law of (79) at times \( t \geq t' \) has full support.

Regarding (ii): since \( \tilde{\Gamma}_1(q)\tilde{\Sigma}_2(q) \) is invertible, the controllability properties of (78) are identical to the controllability properties of (79), hence as a consequence of the Strook-Varadhan support theorem [58] the law of (78) and the law of (79) at time \( t' \) coincide. In particular, together with (i), \( \text{supp}(\mu^c_t) = \text{supp}(\mu^b_t) = \Omega_x \).

Regarding (iii): We show this using Theorem 7 (Girsanov’s theorem). The difference of the drift terms in (78) and (77) can be written as

\[
\tilde{\Gamma}_{1,2}(q')\tilde{\Sigma}_{2}(q')u(q, p, g)
\]

with \( u(q, p, g) \) as defined in (82). By Lemma 9 the function \( u \) satisfies Novikov’s condition (98), which means that Theorem 7 (Girsanov’s theorem) is applicable and it follows that the support of the solution of (78) at \( t' \) coincides with the support of the solution of (77) at \( t' \), i.e., \( \text{supp}(\mu^c_t) = \text{supp}(\mu^b_t) = \Omega_x \).

Regarding (iv): We first note that since (i)-(iii) holds, it trivially follows that \( \mu^c_t(\Omega_x) = \Omega_x \). Applying the change of variables \( s = \tilde{\Gamma}_{1,2}^{-1}(q)g \) to (77) we obtain (24), which means that \( \mu_t \) is the push-forward of \( \mu^c_t \) under the map,

\[
f: \begin{pmatrix} q \\ p \\ g \end{pmatrix} \mapsto \begin{pmatrix} q \\ p \\ \tilde{\Gamma}_{1,2}^{-1}(q)g \end{pmatrix}
\]

i.e.,

\[
\mu_t(A) = f(\mu^c_t)(A) = \mu^c_t \left( f^{-1}(A) \right), \quad A \in \mathcal{B}(\Omega_x).
\]

Since \( f \) is a smooth one-to-one mapping, in particular surjective, and \( \text{supp}(\mu^c_t) = \Omega_x \) we have

\[
\text{supp}(\mu_t) = \text{supp} \left( f(\mu^c_t) \right) = \Omega_x.
\]

The following Lemma 9 shows that Novikov’s condition is satisfied for the function \( u \) required for the application of Girsanov’s theorem in the above proof of Lemma 8.

**Lemma 9.** Let \( \Omega_q = \mathbb{T}^n \) and \( \tilde{\Gamma} \) and \( \tilde{\Sigma} \) as in Lemma 8. Define

\[
u_1(q, p, g) = \left( \tilde{\Gamma}_{1,2}(q)\tilde{\Sigma}_2(q) \right)^{-1} \left( \tilde{\Gamma}_{1,2}(q)\tilde{\Gamma}_{2,1}(q)p - p - \tilde{\Gamma}_{1,2}(q)\tilde{\Gamma}_{2,2}(q)\tilde{\Gamma}_{1,2}^{-1}(q)g + g \right)
\]

\[
= G(q) \begin{pmatrix} p \\ g \end{pmatrix}
\]

with

\[
G(q) := \left( \tilde{\Gamma}_{1,2}(q)\tilde{\Sigma}_2(q) \right)^{-1} \left( \tilde{\Gamma}_{1,2}(q)\tilde{\Gamma}_{2,1}(q) - I_n - \tilde{\Gamma}_{1,2}(q)\tilde{\Gamma}_{2,2}(q)\tilde{\Gamma}_{1,2}^{-1}(q) + I_n \right) \in \mathbb{R}^{n \times 2n},
\]
The function
\[ u(q, p, g) = u_1(q, p, g) + u_2(q, p, g) \] (82)
satisfies Novikov’s condition \[ 98 \].

**Proof (Proof of Lemma 9).** Since by Jensen’s inequality, thus it is sufficient to show that Novikov’s condition holds for \( u_1 \) and \( u_2 \). We only show Novikov’s condition explicitly for \( u_1 \).

Since \( \bar{\Gamma}_{1,2}, \bar{\Gamma}_{2,1}, \bar{\Gamma}_{2,2} \) and \( \bar{\Sigma}_2 \) are smooth functions of \( q \) and since \( \Omega_q \) is compact the spectrum of \( G^T(q)G(q) \) is uniformly bounded from above in \( q \), hence there is \( \lambda_{\text{max}} > 0 \) such that
\[
\lambda_{\text{max}}^2(\|p\|^2 + \|g\|^2) \geq (p^T, g^T)G^T(q)G(q)\left(\begin{array}{c} p \\ g \end{array}\right) = \|u_1(q, p, g)\|^2,
\] (83)
and therefore
\[
E \left[ \exp(\int_0^T \|u_1(q(t), p(t), g(t))\|dt) \right] \leq E \left[ \exp(\int_0^T \lambda_{\text{max}}^2(\|p(t)\|^2 + \|g(t)\|^2)dt) \right],
\]
for any \( T > 0 \). Let \( \epsilon < 2\theta/\lambda_{\text{max}}^2 \), with \( \theta = \theta/\lambda_{\text{max}} \) and \( \theta > 0, \lambda_{\text{max}} \) as defined in Lemma 10. We find
\[
\exp(\int_0^T \lambda_{\text{max}}^2(\|p(t)\|^2 + \|g(t)\|^2)dt) = \exp\left(\frac{1}{\epsilon} \int_0^T \epsilon \lambda_{\text{max}}^2(\|p(t)\|^2 + \|g(t)\|^2)dt\right)
\]
\[
\leq \frac{1}{\epsilon} \int_0^T \exp(\epsilon \lambda_{\text{max}}^2(\|p(t)\|^2 + \|g(t)\|^2))dt,
\]
by Jensen’s inequality, thus
\[
E \left[ \exp(\int_0^T \|u_1(q(t), p(t), g(t))\|dt) \right] \leq E \left[ \frac{1}{\epsilon} \int_0^T \exp(\epsilon \lambda_{\text{max}}^2(\|p(t)\|^2 + \|g(t)\|^2))dt \right]
\]
\[
= \frac{1}{\epsilon} \int_0^T E \left[ \exp(\epsilon \lambda_{\text{max}}^2(\|p(t)\|^2 + \|g(t)\|^2)) \right] dt,
\]
\[9\] The respective proof for \( u_2 \) is essentially the same with the only difference that in \[83\] we need to bound \( \|u_2\|^2 \) by a term proportional to \( \|p\|^2 + \|g\|^2 \) instead of bounding \( u_2 \) by a term which is proportional to \( \|p\|^2 + \|g\|^2 \) as we do in the proof for \( u_1 \). By choosing \( l = 2 \) in \[84\] the remaining steps of the proof are then exactly the same as for \( u_1 \).
by Tonelli’s theorem. Let for $\alpha > 0$,

$$K_{\alpha} := K_{\alpha,l}, \quad l = 1,$$

(84)

with $K_{\alpha,l}$ as defined in (86). Using

$$\exp(\epsilon \lambda_{\max}^2(\|p\|^2 + \|g\|^2)) \leq K_\theta(z),$$

(85)

we conclude using Lemma 10, (87)

$$\frac{1}{\epsilon} \int_0^T E \left[ \exp(\epsilon \lambda_{\max}^2(\|p(t)\|^2 + \|g(t)\|^2)) \right] dt \leq \frac{1}{\epsilon} \int_0^T E \left[ K_\theta(z(t)) \right] dt \leq \frac{1}{\epsilon} \int_0^T e^{-t} K_\theta(p_0, \bar{\Gamma}_{1,2}(q_0)g_0) + b(1 - e^{-t}) dt < \infty,$$

with $b > 0$ as specified in Lemma 10.

Let $\Omega_q = \mathbb{T}^n$ and $\bar{\Gamma}$ and $\bar{\Sigma}$ as in Lemma 8 and let $C \in \mathbb{R}^{2n \times 2n}$

$$\min \sigma(C) = 1,$$

be a symmetric positive definite matrix such that

$$\bar{\Gamma}^T(q)C + C\bar{\Gamma}(q),$$

is positive definite for all $q \in \Omega_q$. For $\alpha > 0$ and $l \in \mathbb{N}$ define

$$K_{\alpha,l}(p, s) = e^{\frac{\alpha}{2}(z^T C z)^l}.$$

(86)

There exists $\theta > 0$ such that Assumption 2 is satisfied with $K = K_{\theta,l}$ and $L = \tilde{L}_{\text{GLE}}$. Moreover, for $\tilde{\theta} = \theta/\lambda_{\max}$ with

$$\lambda_{\max} := \max_{q \in \Omega_q} \left\{|\lambda| \mid \lambda \in \sigma \left( \bar{\Gamma}_{1,2}^{-1}(q) \right) \right\},$$

the expectation of $K_{\theta,l}$ as function of the solution $(q^c, p^c, g^c)$ of (77) can be bounded as

$$E \left[ K_{\theta,l}(p^c, g^c) \mid (p^c(0), g^c(0)) = (p_0, g_0) \right] \leq e^{-t} K_{\theta,l}(p_0, \bar{\Gamma}_{1,2}(q_0)g_0) + b(1 - e^{-t}) + c(l, t),$$

(87)

where $b > 0$ as above and $c(l, t)$ is a finite nonnegative constant which depends on $l$ and $t$ with $c(l, t) = 0$ for $l = 1$ and all $t \geq 0$.

Proof. We recall that the generator of (24) is of the form

$$\tilde{L}_{\text{GLE}} = F(q) \cdot \nabla_p + p \cdot \nabla_q - \bar{\Gamma}(q)z \cdot \nabla_z + \frac{1}{2} \bar{\Sigma}(q) \bar{\Sigma}^T(q) : \nabla_z^2,$$
We show the result only for the case \( l = 1 \). For \( l > 1 \) the result follows by induction. Let \( \mathcal{K}_0 = \mathcal{K}_{\theta, 1} \). Applying the generator on \( \mathcal{K}_\theta \) we obtain

\[
\mathcal{L}\mathcal{K}_\theta(p, s) = (\theta F(q) \cdot (C_{1,1} p + C_{1,2} s)) \mathcal{K}_\theta(p, s) \\
+ \left( -\theta \tilde{\Gamma}(q) z \cdot C z + \frac{1}{2} \left( \theta \mathrm{tr} \left( \tilde{\Sigma}(q) \tilde{\Sigma}^T(q) C \right) \right) \right) \mathcal{K}_\theta(p, s) \\
= \left( -\theta \|z\|^2 + \Theta \left( (1 + \theta)\|z\| \right) \right) \mathcal{K}_\theta(p, s) \\
< -\mathcal{L}\mathcal{K}_\theta(p, s) + b
\]

for sufficiently small \( \theta > 0 \) and sufficiently large \( b > 0 \). Consequently, for \( \tilde{\theta} = \theta/\lambda_{\max} \), we obtain

\[
\mathbb{E} \left[ \mathcal{K}_\tilde{\theta}(p^\ast(t), g^\ast(t)) \ | \ (p^\ast(0), g^\ast(0)) = (p_0, g_0) \right] \\
= \mathbb{E} \left[ \mathcal{K}_\tilde{\theta}(p(t), \tilde{\Gamma}^{-1}(q(t)) s(t)) \ | \ (p(0), s(0)) = (p_0, \tilde{\Gamma}^{-1}(q_0) g_0) \right] \\
\leq \mathbb{E} \left[ \mathcal{K}_\tilde{\theta}(\tilde{\lambda}_{\max} p(t), \tilde{\lambda}_{\max} s(t)) \ | \ (p(0), s(0)) = (p_0, \tilde{\Gamma}^{-1}(q_0) g_0) \right] \\
\leq \mathbb{E} \left[ \mathcal{K}_\tilde{\theta}(p(t), s(t)) \ | \ (p(0), s(0)) = (p_0, \tilde{\Gamma}^{-1}(q_0) g_0) \right] \\
\leq e^{-t} \mathcal{K}_\tilde{\theta}(p_0, \tilde{\Gamma}^{-1}(q_0) g_0) + b(1 - e^{-t}).
\]

The last Lemma 11 of this section provides conditions for the existence of suitable Lyapunov functions with polynomial growth for \( \mathcal{K}_{\mathcal{L}\mathcal{G}\mathcal{L}} \).

**Lemma 11.** Let \( \Omega_q = \mathbb{T}^n \), \( -\Gamma \in \mathbb{R}^{(m+n) \times (n+m)} \) stable, and \( U \in \mathcal{C}^\infty(\mathbb{T}^n, \mathbb{R}) \). Moreover, assume that (51) holds and let \( C \) be as specified therein.

\[
\mathcal{K}_l(q, p, s) = (z^T C z + U(q) - U_{\min} + 1)^l, \ l \in \mathbb{N},
\]

defines a family of Lyapunov functions for the differential operator \( \mathcal{L}_{\mathcal{G}\mathcal{L}} \), i.e., for each \( l \in \mathbb{N} \) there exist constants \( a_l > 0 \), \( b_l \in \mathbb{R} \), such that for \( \mathcal{L} = \mathcal{L}_{\mathcal{G}\mathcal{L}} \), \( \mathcal{K} = \mathcal{K}_l \), Assumption 4 holds for \( a = a_l, b = b_l \).

**Proof.** The proof is very similar to the proof Lemma 2. The existence of a suitable matrix \( C \) as specified in (51) allows to extend all arguments in that proof with only some very small adaptations. For this reason we skip a detailed proof here.

\[\square\]

### 4 Conclusion

In this article we have presented an integrated perspective on ergodic properties of the generalized Langevin equation, for systems that can be written in the quasi-Markovian form. Although the GLE was well studied in the case of constant friction and damping and for conservative forces, our results indicate that these can often be extended to nonequilibrium models with non-gradient forces and non-constant friction and noise, thus providing a foundation for using GLEs in a much broader range of applications.
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A Auxiliary material on linear algebra

The following Lemma 12 is repeatedly used in the proofs of Lemma 1 and Lemma 4 as well as in Example 3 to show the positive (semi-)definiteness of symmetric matrices.

**Lemma 12.** Let $A$ be a symmetric block structured matrix of the form

$$A := \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{1,2}^T & A_{2,2} \end{pmatrix} \in \mathbb{R}^{n+m \times n+m}$$

(i) If $A_{2,2}$ is positive definite, then $A$ is positive (semi-)definite if and only if

$$A_{1,1} - A_{1,2} A_{2,2}^{-1} A_{1,2}^T$$

is positive (semi-)definite.

(ii) If $A_{1,1}$ is positive definite, then $A$ is positive (semi-)definite if and only if

$$A_{2,2} - A_{1,2}^T A_{1,1}^{-1} A_{1,2}$$

is positive (semi-)definite.
(iii) Let $A_{2,2}^g$ denote a generalised inverse of $A_{2,2}$, i.e., $A_{2,2}^g$ is a $m \times m$ matrix which satisfies

$$A_{2,2} A_{2,2}^g A_{2,2} = A_{2,2}.$$ 

The matrix $A$ is positive semi-definite if and only if the matrices $A_{2,2}$ and $A_{1,1} - A_{1,2} A_{2,2}^g A_{1,2}^T$ are positive semi-definite, and

$$(I - A_{2,2} A_{2,2}^g) A_{1,2}^T = 0,$$

i.e., the span of the column vectors of $A_{1,2}$ is contained in the span of the column vectors of $A_{1,1}$.

Proof. The statements (i) and (ii) follow from Theorem 1.12 in [61]. Statement (iii) corresponds to Theorem 1.20 in the same reference. \(\Box\)

\[\text{B Auxiliary material on stochastic analysis}\]

In this section we provide a brief overview of the general framework used in the ergodicity proofs and derivation of convergence rate in Section 3. For a comprehensive overview we refer to the review articles [37,2,33].

Consider an SDE defined on the domain $\Omega_x = \mathbb{T}^{n_1} \times \mathbb{R}^{n_2}, n = n_1 + n_2 \in \mathbb{N}$ which is of the form

$$dX = a(X) dt + b(X) dW, \quad X(0) \sim \mu_0, \quad (88)$$

with smooth coefficients $a \in C^\infty(\Omega_x, \mathbb{R}^n), b = [b_i]_{1 \leq i \leq n} \in C^\infty(\Omega_x, \mathbb{R}^{n \times n})$, and initial distribution $\mu_0$. In order to simplify the presentation we further assume that the diffusion coefficient $b$ is such that the Itô and Stratonovich interpretation of (88) coincide, i.e.,

$$\nabla \cdot (b b^T) - b \nabla \cdot b^T \equiv 0.$$

Let further $\mathcal{L}$ denote the associated infinitesimal generator of (88), i.e.,

$$\mathcal{L} = a(X) \nabla + b(X) : \nabla^2, \quad (89)$$

when considered as an operator on the core $C^\infty(\Omega_x, \mathbb{R})$, and let $\mathcal{L}^\dagger$ denote the formal adjoint of $\mathcal{L}$, i.e., the Fokker-Planck operator associated with the SDE (88). Furthermore, let $e^{t\mathcal{L}}, e^{t\mathcal{L}^\dagger}$ denote the associated semi group operators of $\mathcal{L}$, and $\mathcal{L}^\dagger$, respectively, i.e.,

$$\forall \varphi \in C^\infty(\Omega_x, \mathbb{R}) : e^{t\mathcal{L}} \varphi(x) = E[\varphi(X(t)) \mid X(0) = x], \quad (10)$$

for (Lebesgue-)almost all $x \in \mathbb{R}^n$, and

$$\int (e^{t\mathcal{L}} \varphi)(x) \mu_0(dx) = \int \varphi(x) \left(e^{t\mathcal{L}^\dagger} \mu_0\right)(dx).$$

\(10\) The expectation is taken with respect to the Brownian motion $W$. 
Throughout this article we use Lyapunov function techniques to show (geometric) ergodicity of SDEs of the generic form (88). More specifically, we follow the standard recipe for proofs of exponential convergences of the semigroup operator $e^{tL}$ in weighted $L^\infty$ spaces as outlined, e.g., in [39, 37, 2, 33], that is we show that a suitable Lyapunov condition (Assumption 4) and a minorization condition (Assumption 5) are satisfied:

**Assumption 4 (Infinitesimal Lyapunov condition)** There is a function $K \in C^\infty(\Omega_\omega, [1, \infty))$ with $\lim_{\|x\| \to \infty} K(x) = \infty$, and real numbers $a \in (0, \infty), b \in \mathbb{R}$ such that,
\[ LK \leq -aK + b. \] (90)

**Assumption 5 (Minorization condition)** For some $t' > 0$ there exists a constant $\eta \in (0, 1)$ and a probability measure $\nu$ such that
\[ \inf_{x \in C} e^{t'L} \delta_x(dy) \geq \eta \nu(dy) \]
where $C = \{x \in \Omega_\omega : K(x) \leq K_{\max}\}$ for some $K_{\max} > 1 + 2b/a$, where $a, b$ are the same constants as in (90).

If the above assumptions are satisfied, then the following theorem, which follows from the arguments in [33] (see also the other above mentioned references), allows to derive exponential decay estimates in the respective weighted $L^\infty$ space associated with the Lyapunov function $K$.

**Theorem 5 ([33]).** Let Assumption 4 and Assumption 5 hold. The solution of the SDE (88) admits a unique invariant probability measure $\pi$ such that

(i) there exist positive constant $\lambda, \tilde{C}$ so that for any $\varphi \in L^\infty(\Omega_\omega)$
\[ \left\| e^{tL} \varphi - \mathbb{E}_\pi \varphi \right\|_{L^\infty} \leq \tilde{C} e^{-t\lambda} \left\| \varphi - \mathbb{E}_\pi \varphi \right\|_{L^\infty}. \] (91)

(ii)
\[ \int_{\Omega_\omega} K d\pi < \infty. \] (92)

In the main body of this article we use Theorem 5 to derive exponential decay estimates of the form (46) in Theorems 1 to 4. In these theorems Assumption 4 can be directly shown to hold by explicitly constructing a suitable Lyapunov function $K$ satisfying (90) (see Lemmas 2, 4 and 11). A very common way to show Assumption 5 is by showing (i) that the transition kernel associated with the SDE (88) is smooth as specified in Assumption 6 and (ii) that the SDE (88) is controllable as specified in Assumption 7. By virtue Lemma 13 it then follows that a minorization condition holds.

**Assumption 6** For any $t > 0$ the transition kernel associated with the SDE (88) possesses a density $p_t(x, y)$, i.e.,
\[ \forall x \in \Omega_\omega : (e^{tL} \delta_x)(A) = \int_A p_t(x, y) dy, \ A \subset \Omega_\omega, \ A \text{ measurable}. \]
and $p_t(x, y)$ is jointly continuous in $(x, y) \in \Omega_\omega \times \Omega_\omega$. 
Assumption 7 There is a $t_{\text{max}} > 0$ so that for any $x^-, x^+ \in \Omega_x$, there is a $t > 0$, with $t \leq t_{\text{max}}$, so that the control problem

$$\dot{X} = a(\hat{X}) + b(\hat{X})u,$$

subject to

$$\hat{X}(0) = x^-, \text{ and } \hat{X}(t) = x^+,$$

has a smooth solution $u \in C^1([0,t_{\text{max}}], \Omega_x)$.

Lemma 13 ([37]). If Assumption 6 and Assumption 7 are satisfied, then also Assumption 5 holds.

Assumption 6 follows directly if the operator $\partial_t - L^\dagger$ is hypoelliptic (see e.g. [248], for a precise definition of hypoellipticity). A common way to establish hypoellipticity of a differential operators is via Hörmander’s theorem ([22], Theorem 22.2.1, on page 353). The following theorem is an adaption of Hörmander’s theorem to the parabolic differential operator $\partial_t - L^\dagger$:

Theorem 6. Let $a$ and $b$ be the drift coefficient and the diffusion coefficient of the SDE (88), respectively. Let $b_0 := a$. Iteratively define a collection of vector fields by

$$\mathcal{V}_0 = \{b_i : i \geq 1\}, \quad \mathcal{V}_{k+1} = \mathcal{V}_k \cup \{[v, b_i] : v \in \mathcal{V}_k, 0 \leq i \leq n\}. \quad (94)$$

where

$$[X, Y] = (\nabla Y)X - (\nabla X)Y,$$

denotes the commutator of vector fields $X, Y \in C^\infty(\Omega_x, \mathbb{R}^n)$ and $(\nabla X), (\nabla Y)$ their Jacobian matrices. If

$$\forall x \in \mathbb{R}^n, \quad \text{lin} \left\{ v(x) : v \in \bigcup_{k \in \mathbb{N}} \mathcal{V}_k \right\} = \mathbb{R}^n, \quad (95)$$

then the operators $\partial_t - L^\dagger$ is hypoelliptic.

We use Lemma 13 in the proof of Lemma 5 in Theorem 2. For some instances of (24) it is not easy to construct a suitable control $u$ such that Assumption 7 is satisfied. In these cases we either show a minorization condition by explicitly constructing the minorizing measure $\nu$ in Assumption 5 if the right hand side of (24) can be decomposed into a linear and a bounded part (see Theorem 1), or by inferring the existence of a suitable minorizing measure by showing that the support of the SDE under consideration is equivalent to the support of another SDE satisfying a minorization condition via Girsanov’s theorem (Lemmas 6 and 8).

Girsanov’s theorem provides conditions under which the path measures of two Itô processes are mutually absolutely continuous, which in particular implies that at any time $t \geq 0$ the laws of these Itô processes are equivalent. We will use Girsanov’s theorem in Section 3 in order to proof the minorization condition for instances of the GLE which in a Markovian representation possess coefficients which depend on the configurational variable. Here we provide a version of Girsanov’s theorem which is adapted to Itô-diffusion processes.
Theorem 7 (Girsanov’s theorem, [46]). Consider the two Itô diffusion processes

\[
\begin{align*}
dX(t) &= a_x(X)dt + b(X)dW(t); \quad X(0) = x_0, \quad (96) \\
dY(t) &= a_y(Y)dt + b(Y)dW(t); \quad Y(0) = x_0, \quad (97)
\end{align*}
\]

where \( x_0 \in \Omega_x \), \( W \) is a standard Wiener process in \( \mathbb{R}^n \), and \( a_x, a_y : \Omega_x \to \mathbb{R}^n \) and \( b : \Omega_x \to \mathbb{R}^{n \times m} \), \( m \in \mathbb{N} \), are such that there exist unique strong solutions \( X, Y \) for \( (96) \) and \( (97) \), respectively. If there is a function \( u \in C(\Omega_x, \mathbb{R}^n) \) such that

\[
a_x - a_y = bu
\]

and \( u \) satisfies Novikov’s condition

\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \| u(X(t)) \|^2 ds \right) \right] < \infty. \quad (98)
\]

then the path measures of \( X \) and \( Y \) on any finite time interval are equivalent. In particular, the support of the law of \( X(t) \) and the support of the law of \( Y(t) \) coincide for any \( t > 0 \).