AN EXPLICIT VOLUME FORMULA FOR THE LINK $7_2^3(\alpha, \alpha)$ CONE-MANIFOLDS

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ABSTRACT. We calculate the volume of the $7_2^3$ link cone-manifolds using the Schl"afli formula. As an application, we give the volume of the cyclic coverings branched over the link.

1. Introduction

Let us denote the link complement of $7_2^3$ in Rolfsen’s link table by $X$. Note that it is a hyperbolic knot. Hence by Mostow-Prasad rigidity theorem, $X$ has a unique hyperbolic structure. Let $\rho_\infty$ be the holonomy representation from $\pi_1(X)$ to PSL$(2,\mathbb{C})$ and denote $\rho_\infty(\pi_1(X))$ by $\Gamma$, a Kleinian group. $X$ is a (PSL$(2,\mathbb{C})$, $H^3$)-manifold and can be identified with $H^3/\Gamma$. Thurston’s orbifold theorem guarantees an orbifold, $X(\alpha) = X(\alpha, \alpha)$, with underlying space $S^3$ and with the link $7_2^3$ as the singular locus of the cone-angle $\alpha = 2\pi/k$ for some nonzero integer $k$, can be identified with $H^3/\Gamma'$ for some $\Gamma' \in \text{PSL}(2,\mathbb{C})$; the hyperbolic structure of $X$ is deformed to the hyperbolic structure of $X(\alpha)$. For the intermediate angles whose multiples are not $2\pi$ and not bigger than $\pi$, Kojima [10] showed that the hyperbolic structure of $X(\alpha)$ can be obtained uniquely by deforming nearby orbifold structures. Note that there exists an angle $\alpha_0 \in \left[\frac{2\pi}{3}, \pi\right)$ for the link $7_2^3$ such that $X(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$ [19, 8, 10, 20]. For further knowledge of cone-manifolds a reader can consult [1, 7].

Even though we have wide discussions on orbifolds, it seems to us we have a little in regard to cone-manifolds. Explicit volume formulae for hyperbolic cone-manifolds of knots and links are known a little. The volume formulae for hyperbolic cone-manifolds of the knot $4_1$ [8, 10, 11, 13], the knot $5_2$ [13], the link $5_2^1$ [14], the link $6_2^3$ [17], and the link $6_2^5$ [2] have been computed. In [9] a method of calculating the volumes of two-bridge knot cone-manifolds was introduced but without explicit formulae. In [7, 8], explicit volume formulae of cone-manifolds for the hyperbolic twist knot and for the knot with Conway notation $C(2n, 3)$ are computed. Similar methods are used for computing Chern-Simons invariants of orbifolds for the twist knot and $C(2n, 3)$ knot in [5, 4].

The main purpose of the paper is to find an explicit and efficient volume formula of hyperbolic cone-manifolds for the link $7_2^3$. The following theorem gives the volume formula for $X(\alpha)$.

**Theorem 1.1.** Let $X(\alpha)$, $0 \leq \alpha < \alpha_0$ be the hyperbolic cone-manifold with underlying space $S^3$ and with singular set the link $7_2^3$ of cone-angle $\alpha$. $X(0)$ denotes $X$. Then the volume of $X(\alpha)$ is given by the following formula

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\[
\text{Vol}(X(\alpha)) = \int_{\alpha}^{\pi} 2 \log \left| \frac{A - iV}{A + iV} \right| d\alpha,
\]

where for \(A = \cot \frac{\alpha}{2}, V (\text{Re}(V) \leq 0 \text{ and } \text{Im}(V) \geq 0 \text{ is the largest})\) is a zero of the Riley-Mednykh polynomial \(P = P(V, A)\) for the link \(7^2_3\) given below.

\[
P = 8V^5 + 8A^2V^4 + (8A^4 + 16A^2 - 8)V^3 + (4A^6 + 8A^4 - 12A^2)V^2
+ (A^8 + 4A^6 - 2A^4 - 12A^2 + 1)V - 4A^6 - 8A^4 + 4A^2.
\]

The following corollary gives the hyperbolic volume of the \(k\)-fold strictly-cyclic covering \([12, 18]\) over the link \(7^2_3, M_k(X)\), for \(k \geq 3\).

**Corollary 1.2.** The volume of \(M_k(X)\) is given by the following formula

\[
\text{Vol}(M_k(X))) = k \int_{\frac{\pi}{2\pi}}^{\pi} 2 \log \left| \frac{A - iV}{A + iV} \right| d\alpha,
\]

where for \(A = \cot \frac{\alpha}{2}, V (\text{Re}(V) \leq 0 \text{ and } \text{Im}(V) \geq 0 \text{ is the largest})\) is a zero of the Riley-Mednykh polynomial \(P = P(V, A)\) for the link \(7^2_3\).

In Section 2, we present the fundamental group \(\pi_1(X)\) of \(X\) with slope \(9/16\). In Section 3, we give the defining equation of the representation variety of \(\pi_1(X)\). In Section 4, we compute the longitude of the link \(7^2_3\) using the Pythagorean theorem. And in Section 5, we give the proof of Theorem 1.1 using the Schlafli formula.

2. **Link 7^2_3**

Link \(7^2_3\) is presented in Figure 1. It is the same as \(W_3\) from [2]. The slope of this link is \(7/16\). The link with slope \(9/16\) is the mirror of the link \(7^2_3\). Since the volume of the link with slope \(7/16\) is the same as the volume of link with slope \(9/16\), in the rest of the paper, the link with slope \(9/16\) is used.

The following fundamental group of \(X\) is stated in [2] with slope \(7/16\).

**Proposition 2.1.**

\[
\pi_1(X) = \langle s, t \mid s\ w s^{-1} w^{-1} = 1 \rangle,
\]

where \(w = s^{-1}[s, t]^2[s, t^{-1}]^2\).
3. \((\text{PSL}(2, \mathbb{C}), \mathbb{H}^3)\) Structure of \(X(\alpha)\)

Let \(R = \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))\). Given a set of generators, \(s, t, \) of the fundamental group for \(\pi_1(X)\), we define a set \(R(\pi_1(X)) \subset \text{SL}(2, \mathbb{C})^2 \subset \mathbb{C}^8\) to be the set of all points \((h(s), h(t))\), where \(h\) is a representation of \(\pi_1(X)\) into \(\text{SL}(2, \mathbb{C})\). Since the defining relation of \(\pi_1(X)\) gives the defining equation of \(R(\pi_1(X))\) \([21]\), \(R(\pi_1(X))\) is an affine algebraic set in \(\mathbb{C}^8\). \(R(\pi_1(X))\) is well-defined up to isomorphisms which arise from changing the set of generators. We say elements in \(R\) which differ by conjugations in \(\text{SL}(2, \mathbb{C})\) are equivalent. A point on the variety gives the \((\text{PSL}(2, \mathbb{C}), \mathbb{H}^3)\) structure of \(X(\alpha)\).

Let

\[
    h(s) = \begin{bmatrix}
        \cos \frac{\alpha}{2} & i e^{\frac{\alpha}{2}} \sin \frac{\alpha}{2} \\
        i e^{-\frac{\alpha}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2}
    \end{bmatrix}, \quad h(t) = \begin{bmatrix}
        \cos \frac{\alpha}{2} & i e^{-\frac{\alpha}{2}} \sin \frac{\alpha}{2} \\
        i e^{\frac{\alpha}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2}
    \end{bmatrix}.
\]

Then \(h\) becomes a representation if and only if \(A = \cot \frac{\alpha}{2}\) and \(V = \cosh \rho\) satisfies a polynomial equation \([21] [14]\). We call the defining polynomial of the algebraic set \(\{ (V, A) \}\) as the Riley-Mednykh polynomial for the link \(7_6^3\). Throughout the paper, \(h\) can be sometimes any representation and sometimes the unique hyperbolic representation.

Given the fundamental group of \(X\),

\[\pi_1(X) = \langle s, t \mid sws^{-1}w^{-1} = 1 \rangle,\]

where \(w = s^{-1}[s, t]s[t, s^{-1}]\), let \(S = h(s), \ T = h(t)\) and \(W = h(w)\). Then the trace of \(S\) and the trace of \(T\) are both \(2 \cos \frac{\alpha}{2}\).

**Lemma 3.1.** For \(n \in \text{SL}(2, \mathbb{C})\) which satisfies \(nS = S^{-1}n, nT = T^{-1}n,\) and \(n^2 = -I\),

\[SWS^{-1}W^{-1} = -(SWn)^2.\]

**Proof.**

\[
    (SWn)^2 = SWnSWn = SWS^{-1}n(S^{-1}(STS^{-1}T^{-1})^2(ST^{-1}S^{-1}T)^2)n
    = SWS^{-1}(S(S^{-1}T^{-1}ST)^2(S^{-1}TST^{-1})^2)n^2 = -SWS^{-1}W^{-1}.
\]

\[
\]

From the structure of the algebraic set of \(R(\pi_1(X))\) with coordinates \(h(s)\) and \(h(t)\) we have the defining equation of \(R(\pi_1(X))\). The following theorem is stated in \([2]\) Proposition 4] with slope 7/16.

**Theorem 3.2.** \(h\) is a representation of \(\pi_1(X)\) if \(V\) is a root of the following Riley-Mednykh polynomial \(P = P(V, A)\) which is given below.

\[
P = 8V^5 + 8A^2V^4 + (8A^4 + 16A^2 - 8)V^3 + (4A^6 + 8A^4 - 12A^2)V^2
    + (A^8 + 4A^6 - 2A^4 - 12A^2 + 1)V - 4A^6 - 8A^4 + 4A^2.
\]

**Proof.** Note that \(SWS^{-1}W^{-1} = I\), which gives the defining equations of \(R(\pi_1(X))\), is equivalent to \((SWn)^2 = -I\) in \(\text{SL}(2, \mathbb{C})\) by Lemma 3.1 and \((SWn)^2 = -I\) in \(\text{SL}(2, \mathbb{C})\) is equivalent to \(\text{tr}(SWn) = 0\).
We can find two \( n \)'s in \( \text{SL}(2, \mathbb{C}) \) which satisfies \( nS = S^{-1}n \) and \( n^2 = -I \) by direct computations. The existence and the uniqueness of the isometry (the involution) which is represented by \( n \) are shown in [3, p. 46]. Since two \( n \)'s give the same element in \( \text{PSL}(2, \mathbb{C}) \), we use one of them. Hence, we may assume

\[
n = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},
\]

\[
S = \begin{bmatrix} \cos \frac{\alpha}{2} & i e^{\frac{\alpha}{2}} \sin \frac{\alpha}{2} \\ i e^{-\frac{\alpha}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix},
\]

\[
T = \begin{bmatrix} \cos \frac{\alpha}{2} & i e^{-\frac{\alpha}{2}} \sin \frac{\alpha}{2} \\ i e^{\frac{\alpha}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix}.
\]

Recall that \( P \) is the defining polynomial of the algebraic set \( \{(V, A)\} \) and the defining polynomial of \( R(\pi_1(X)) \) corresponding to our choice of \( h(s) \) and \( h(t) \). By direct computation \( P \) is a factor of \( \text{tr}(SWn) = -4i \sinh \rho(2V^2 + A^4 + 2A^2 - 1)P \). As in [2], \( P \) can not be \( \sinh \rho \) or have only real roots. Also, \( P \) can not have only purely imaginary roots similarly. \( P \) in the theorem is the only factor of \( \text{tr}(SWn) \) which is different from \( \sinh \rho \) and has roots which are not real or purely imaginary. \( P \) is the Riley-Mednykh polynomial. \( \square \)

4. Longitude

Let \( l_s = ws \) and \( l_t = (t^2 - [t, s]s^2[t, s^{-1}]s^2)t \). Then \( l_s \) and \( l_t \) are the longitudes which are null-homologous in \( X \). Let \( L_S = h(l_s) \) and Let \( L_T = h(l_t) \).

**Lemma 4.1.** \( \text{tr}(S^{-1}L_T) = \text{tr}(S) \) and \( \text{tr}(T^{-1}L_S) = \text{tr}(T) \).

**Proof.** Since

\[
S^{-1}L_T = S^{-1}(T^{-1}(TST^{-1}S^{-1}TST^{-1}S^{-1} \cdot TS^{-1}TST^{-1}S^{-1}T^{-1}S)T)
\]

\[
= (T^{-1}S^{-1}TST^{-1}S^{-1}T)(S^{-1})(T^{-1}S^{-1}TST^{-1}S^{-1}T^{-1}S^{-1}T)\text{,}
\]

\[
\text{tr}(S^{-1}L_T) = \text{tr}(S^{-1}) = \text{tr}(S).
\]

The second statement can be obtained in a similar way. \( \square \)

**Definition.** The *complex length* of the longitude \( l \) (\( l_s \) or \( l_t \)) of the link \( \pi_3^{\alpha} \) is the complex number \( \gamma_\alpha \) modulo \( 4\pi \mathbb{Z} \) satisfying

\[
\text{tr}(h(l)) = 2 \cosh \frac{\gamma_\alpha}{2}.
\]

Note that \( l_\alpha = |\text{Re}(\gamma_\alpha)| \) is the real length of the longitude of the cone-manifold \( X(\alpha) \).

By sending common fixed points of \( T \) and \( L_T = h(l_t) \) to 0 and \( \infty \), we have

\[
T = \begin{bmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{bmatrix}, \quad L_T = \begin{bmatrix} e^{\frac{2\pi}{4}} & 0 \\ 0 & e^{-\frac{2\pi}{4}} \end{bmatrix},
\]
and the following normalized line matrices of $T$ (resp. $L_T$) which share the fixed points with $T$ (resp. $L_T$).

$$l(T) \equiv \frac{T - T^{-1}}{2i \sinh \frac{i\alpha}{2}}$$

$$= \frac{1}{i(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}})} \left[ e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}} 0 e^{-\frac{i\alpha}{2}} - e^{\frac{i\alpha}{2}} \right]$$

$$= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

$$l(L_T) \equiv \frac{L_T - L_T^{-1}}{2i \sinh \frac{i\alpha}{2}}$$

$$= \frac{1}{i(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}})} \left[ e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}} 0 e^{-\frac{i\alpha}{2}} - e^{\frac{i\alpha}{2}} \right]$$

$$= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

which give the orientations of axes of $T$ and $L_T$.

Now, we are ready to prove the following theorem which gives Theorem 4.3. Recall that $\gamma_\alpha$ modulo $4\pi\mathbb{Z}$ is the complex length of the longitude $l_s$ or $l_t$ of $X(\alpha)$. The following theorem is a particular case of Proposition 5 from [2].

**Theorem 4.2.** (Pythagorean Theorem) [2] Let $X(\alpha)$ be a hyperbolic cone-manifold and let $\rho$ be the complex distance between the oriented axes $S$ and $T$. Then we have

$$i \cosh \rho = \cot \frac{\alpha}{2} \coth \left( \frac{\gamma_\alpha}{4} \right).$$
Proof.

\[
\cosh \rho = -\frac{\text{tr}(l(S)l(T))}{2} = -\frac{\text{tr}(l(S)l(L_T))}{2} = \frac{\text{tr}((S - S^{-1})(L_T - L_T^{-1}))}{8 \sinh \frac{i \alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(SL_T - S^{-1}L_T - SL_T^{-1} + (L_TS)^{-1})}{8 \sinh \frac{i \alpha}{2} \sinh \frac{\gamma}{2}} = \frac{2(\text{tr}(SL_T) - \text{tr}(S^{-1}L_T))}{8 \sinh \frac{i \alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(S)\text{tr}(L_T) - 2\text{tr}(S^{-1}L_T)}{4 \sinh \frac{i \alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(S)\text{tr}(L_T) - 2\text{tr}(S)}{4 \sinh \frac{i \alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(S)(\text{tr}(L_T) - 2)}{4 \sinh \frac{i \alpha}{2} \sinh \frac{\gamma}{2}} = \frac{2 \cos \frac{\alpha}{2}(2 \cosh \frac{\gamma}{2} - 2)}{4i \sin \frac{\gamma}{4}} = -i \cot \frac{\alpha}{2} \tanh \left(\frac{\gamma}{4}\right).
\]

where the first equality comes from [3, p. 68], the sixth equality comes from the Cayley-Hamilton theorem, and the seventh equality comes from Lemma 4.1. Therefore, we have

\[
i \cosh \rho = \cot \frac{\alpha}{2} \coth \left(\frac{\gamma}{4}\right).
\]

\[\square\]

Pythagorean theorem 4.2 gives the following theorem which relates the eigenvalues of \(h(l)\) and \(V = \cosh \rho\) for \(A = \cot \frac{\alpha}{2}\).

**Theorem 4.3.** Recall that \(l\) is the longitude. By conjugating if necessary, we may assume \(h(l)\) is upper triangular. Let \(L = h(l)_{11}\). Let \(A = \cot \frac{\alpha}{2}\). Then the following formulae show that there is a one to one correspondence between the the eigenvalues of \(h(l)\) and \(V = \cosh \rho\):

\[
iV = A \frac{L - 1}{L + 1} \quad \text{and} \quad L = \frac{A - iV}{A + iV}.
\]
Proof. By Theorem 4.2

\[ iV = i \cosh \rho \]
\[ = \cot \frac{\alpha}{2} \tanh(\frac{\gamma \alpha}{4}) \]
\[ = \cot \frac{\alpha}{2} \sinh(\frac{\gamma \alpha}{4}) \]
\[ = \cot \frac{\alpha}{2} \left( \frac{e^{\frac{\gamma \alpha}{2}} - e^{-\frac{\gamma \alpha}{2}}}{2} \right) \]
\[ = \cot \frac{\alpha}{2} \left( \frac{e^{\frac{\gamma \alpha}{2}} - 1}{2} \right) \]
\[ = \frac{\alpha}{2} \left( e^{\frac{\gamma \alpha}{2}} - 1 \right) \]
\[ = \frac{A}{L} - 1 \]

If we solve the above equation,

\[ iV = A \frac{L - 1}{L + 1} \]

for \( L \), we have

\[ L = \frac{A - iV}{A + iV} \]

\[ \square \]

5. Proof of Theorem 1.1

According to [19, 8, 10, 20], there exists an angle \( \alpha_0 \in [\frac{2\pi}{3}, \pi) \) such that \( X(\alpha) \) is hyperbolic for \( \alpha \in (0, \alpha_0) \), Euclidean for \( \alpha = \alpha_0 \), and spherical for \( \alpha \in (\alpha_0, \pi] \). Denote by \( D(X(\alpha)) \) the discriminant of \( P(V, A) \) over \( V \). Then \( \alpha_0 \) is the only zero of \( D(X(\alpha)) \) in \( [\frac{2\pi}{3}, \pi) \).

From Theorem 4.3, we have the following equality,

\[ |L|^2 = \left| \frac{A - iV}{A + iV} \right|^2 = \frac{|A|^2 + |V|^2 + 2AImV}{|A|^2 + |V|^2 - 2AImV}. \]

For the volume, we choose \( L \) with \( |L| \geq 1 \) and hence we have \( \text{Im}(V) \geq 0 \) by Equality (1). The component of \( V \) with \( \text{Im}(V) \geq 0 \) which becomes real at \( \alpha_0 \) has negative real part. On the geometric component which gives the unique hyperbolic structure, we have the
volume of a hyperbolic cone-manifold $X(\alpha)$ for $0 \leq \alpha < \alpha_0$:

$$\text{Vol}(X(\alpha)) = -\int_{\alpha_0}^{\alpha} 2 \left( \frac{l_\alpha}{2} \right) \, d\alpha$$

$$= -\int_{\alpha_0}^{\alpha} 2 \log |L| \, d\alpha$$

$$= -\int_{\alpha}^{\pi} 2 \log |L| \, d\alpha$$

$$= \int_{\alpha}^{\pi} 2 \log \left| \frac{A-iV}{A+iV} \right| \, d\alpha,$$

where the first equality comes from the Schl"afli formula for cone-manifolds (Theorem 3.20 of \cite{1}), the second equality comes from the fact that $l_\alpha = \left| \text{Re}(\gamma_\alpha) \right|$ is the real length of the one longitude of $X(\alpha)$, the third equality comes from the fact that $\log |L| = 0$ for $\alpha_0 < \alpha \leq \pi$ by Equality (1) since all $V$’s are real for $\alpha_0 < \alpha \leq \pi$, and $\alpha_0 \in \left[ \frac{2\pi}{3}, \pi \right)$ is the zero of the discriminant $D(X(\alpha))$. Numerical calculations give us the following value for $\alpha_0 : \alpha_0 \approx 2.83003$.

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