HEEGNER POINTS ON HIJIKATA–PIZER–SHEMANSKE CURVES

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Abstract. We study Heegner points on elliptic curves, or more generally modular abelian varieties, coming from uniformization by Shimura curves attached to a rather general type of quaternionic orders closely related to those introduced by Hijikata–Pizer–Shemanske in the 80’s. We address several questions arising from the Birch and Swinnerton-Dyer (BSD) conjecture in this general context. In particular, under mild technical conditions, we show the existence of non-torsion Heegner points on elliptic curves in all situations in which the BSD conjecture predicts their existence.

Introduction

This work grows on an attempt to study Shimura curves and Heegner points in arithmetically interesting situations which at present are poorly understood.

To motivate our study, let us present an example first. Suppose that $E/Q$ is an elliptic curve of conductor $p^aq$, where $p$ and $q$ are two distinct odd primes, and let $K$ be an imaginary quadratic field in which $p$ is ramified and $q$ is inert. In this situation one can not construct rational points on $E(K)$ using parametrizations by modular curves $X_0(N)$, because the classical Heegner hypothesis fails.

However, in this scenario it may perfectly be the case that the sign of the functional equation satisfied by $L(E/K, s)$ be $-1$ (indeed, this holds under certain arithmetic conditions on the local root numbers of the functional equation of $L(E/K, s)$ at $p$ and $q$ which are made precise below). If in addition the order of vanishing of $L(E/K, s)$ at $s = 1$ is one, then the Birch and Swinnerton-Dyer conjecture predicts the existence of a rational point in $E(K)$ generating $E(K) \otimes \mathbb{Q}$. One expects to construct points of infinite order in $E(K)$ using parametrizations by Shimura curves $X_U$ attached to the quaternion algebra of discriminant equal to $pq$ and open compact subgroups $U$ of $(B \otimes \mathbb{Z})^\times$; note that $U$ can not be maximal at $p$ because the conductor of the elliptic curve is divisible by $p^2$.

Nevertheless, in this setting one still has a rich theory of local quaternionic orders whose level is divisible by $p^2$ and optimal embeddings of quadratic orders of $K$ into them; this theory has been developed by Pizer [Piz80], and then extended in greater generality by Hijikata–Pizer–Shemanske in [HPS89a]. Shimura curves attached to these orders play an important role in what follows, motivating the title of this paper.

One of the main motivations that led us to work on this project is that these curves can be $p$-adically uniformized by the $p$-adic rigid analytic space corresponding to the first (abelian) covering of the Drinfel’d tower over the $p$-adic upper half plane $H_p$. This rigid analytic space has an explicit description (see [Tei90]) which can be used to study $p$-adic aspects of Heegner points, including their connection to Iwasawa theory and $p$-adic $L$-functions, as in the case where the elliptic curve has multiplicative reduction and can be uniformized by Drinfel’d upper half plane (cf. [BD98, BD99, BD99]).

It would also be highly interesting to extend the theory of Stark–Heegner points in this context (starting with the foundational paper [Dar01], and developed in [BD07, BD09, Das05, HDD07, Gre09, LRV12, LRV13, LV14]). Such a generalization is however not straight-forward, essentially because the jacobian varieties of the Shimura curves referred to above have additive reduction at $p$ (as opposed to having toric reduction, which is a crucial feature in the above approaches). We regard this as an exciting obstacle to overcome rather than a forbidding difficulty, and this note aims to settle the first step towards this program that we hope to pursue in the near future.

In any case let us stress from the beginning that, independently of any eventual applications of our work building on the $p$-adic uniformization of elliptic curves, in this article we address much more general questions related to the existence of Heegner points than those coming from the specific example described above (and, therefore, use much more general orders than those introduced in [Piz80] and [HPS89a]).

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Indeed, many of our arguments involving Euler systems and applications to the conjecture of Birch and Swinnerton-Dyer rely on the following apparently naive question:

**Question.** Given an elliptic curve $E/\mathbb{Q}$, an imaginary quadratic field $K$ and an anticyclotomic character $\chi$ of $\text{Gal}(K^{ab}/K)$ factoring through the ring class field $H_c$ of conductor $c \geq 1$ of $K$, under which conditions there exist non-trivial Heegner points in $E(H_c)$?

Let us now describe in more detail the main results of this article and their implications to the conjecture of Birch and Swinnerton-Dyer.

As above, let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$, $K$ be an imaginary quadratic field and

$$\chi : G_K = \text{Gal}(K/K) \to \mathbb{C}^\times$$

be a character of finite order. We assume throughout that $\chi$ is anticyclotomic, meaning that $\chi(\tau \sigma \tau^{-1}) = \chi^{-1}(\sigma)$ for any $\sigma \in G_K$ and $\tau \in G_\mathbb{Q} \setminus G_K$. The abelian extension cut out by $\chi$ is then a ring class field associated to some order $R_c$ in $K$ of conductor $c = c(\chi) \geq 1$. Let $H_c$ denote the corresponding abelian extension, determined by the isomorphism $\text{Gal}(H_c/K) \simeq \text{Pic}(R_c)$ induced by the Artin map.

Let $L(E/K,\chi, s)$ denote the Rankin L-series associated to the twist of $E/K$ by $\chi$. Since $\chi$ is anticyclotomic, the motive associated to $L(E/K,\chi, s)$ is Kummer self-dual and this implies that the global root number $\varepsilon(E/K,\chi)$ of $L(E/K,\chi, s)$ is either $+1$ or $-1$. Assume for the remainder of the article that $\varepsilon(E/K,\chi) = -1$,

hence in particular $L(E/K,\chi, s)$ vanishes at the central critical point $s = 1$. Define

$$\{E(H_c) \otimes \mathbb{C}\}^\chi := \{x \in E(H_c) \otimes \mathbb{C} : \sigma(x) = \chi(\sigma)x, \forall \sigma \in \text{Gal}(H_c/K)\}.$$

In this situation, the Galois equivariant version of the Birch–Swinnerton-Dyer conjecture predicts that

$$L'(E/K,\chi, 1) \neq 0 \iff \dim_{\mathbb{C}}(E(H_c) \otimes \mathbb{C})^\chi = 1.$$

In this paper we are concerned with the implication “$\Rightarrow$”. The well-known strategy to prove this implication, after Kolyvagin’s fundamental work [Ko88], is to exploit Euler systems of Heegner points on $E$ arising from Heegner points on modular curves $X_0(N)$, via a parametrization

$$\pi_E : X_0(N) \to E$$

whose existence is a consequence of Wiles’s Modularity Theorem [Wi95, TW93, BCDT01]. More generally, in order to enlarge the source of rational points on elliptic curves, one can use uniformizations of our fixed elliptic curve $E$ by Jacobians $\text{Jac}(X_U)$ of Shimura curves $X_U$ associated to indefinite quaternion algebras $B/\mathbb{Q}$ and compact open subgroups $U \subseteq B_\infty^+ = (B \otimes \hat{\mathbb{Z}})^\times$.

When $U = R_\infty = (R \otimes \hat{\mathbb{Z}})^\times$ for some order $R$ of $B$, we simply write $X_R$ for $X_{R, \infty}$; also, to emphasize the role of $B$, we will sometimes write $X_{B_\infty}^U$ and $X_{B_\infty}$ for $X_U$ and $X_{B_\infty}$, respectively.

Section 1 is devoted to introduce an explicit family of special orders $R$ in $B$ which shall play a central role in our work. These orders are determined by local data at the primes of bad reduction of the elliptic curve $E$, following classical work of Hijikata, Pizer and Shemanske that apparently did not receive the attention it justly deserved.

In section 2 we study the Shimura curves $X_R$ associated to the above mentioned Hijikata–Pizer–Shemanske orders and work out explicitly the Jacquet–Langlands correspondence for these curves, which allows us to dispose of a rich source of modular parametrizations of the elliptic curve $E$. For any integer $c$, there is a (possibly empty) collection of distinguished points on $X_R$, called Heegner points of conductor $c$. The set of Heegner points of conductor $c$ on $X_R$ is in natural correspondence with the set of conjugacy classes of optimal embeddings of the quadratic order $R_c$ in the quaternion order $R$, and we denote it $\text{Heeg}(R, K, c)$. We say that a point in $E(H_c)$ is a Heegner point of conductor $c$ associated to the order $R$ if it is the image of a Heegner point of conductor $c$ in $X_R$ for some uniformization map

$$\pi_E : \text{Jac}(X_R) \to E$$

defined over $\mathbb{Q}$. The corresponding set of Heegner points is denoted $\text{Heeg}_E(R, K, c)$.

In section 3 we perform a careful and detailed analysis of the rather delicate and involved theory of optimal embeddings of quadratic orders into Hijikata–Pizer–Shemanske orders. Combining all together this allows to prove the main result of this article. A slightly simplified version of this result is the following. The main virtue of the statement below with respect to previous results available in the literature is that it is both general (removing nearly all unnecessary hypothesis on divisibility and congruence relations among $N$, $D$ and $c$) and precise (pointing out to a completely explicit Shimura curve).
Theorem A. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$ not divisible neither by $2^3$ nor by $3^4$. Let $K$ be an imaginary quadratic field of discriminant $-D$ and $\chi$ be an arbitrary anticyclotomic character of conductor $c \geq 1$. Assume that $\varepsilon(E/K, \chi) = -1$. Then

1. there exists an explicit Hijikata–Pizer–Shemanske order $\mathcal{R} = \mathcal{R}(E, K, \chi)$ for which the set of Heegner points $\text{Heeg}_E(\mathcal{R}, K, c)$ in $E(H_\mathcal{R})$ is non-empty.

2. If $L'(E/K, \chi, 1) \neq 0$ and $E$ does not acquire CM over any imaginary quadratic field contained in $H_\mathcal{R}$, then $\dim_{\mathbb{C}}(E(H_\mathcal{R}) \otimes \mathbb{C})^\chi = 1$.

This theorem is proved in the last section of the article, where we also provide a more general statement for elliptic curves and prove a similar but weaker result (Theorem 4.15) for modular abelian varieties; we close the paper with a refined conjecture on the existence of Heegner points on modular abelian varieties.

Theorem A follows from Theorem 4.14 below, where the condition that $2^3$ and $3^4$ do not divide $N$ is considerably relaxed into the much weaker (but more involved) Assumption 4.9. Namely, we can also prove the above theorem in all cases where $3^5$ divides $N$; there is only one case with $3^4$ dividing $N$ exactly that we cannot treat with our arguments (it is a case in which $3$ is inert in $K$ and the $p$-power conductor of $\chi$ is equal to 1). In addition, if $\text{val}_3(N) \geq 4$ is even, then the newform $f \in S_2(\Gamma_0(N))$ associated to $E$ by modularity is a twist of a newform $f' \in S_2(\Gamma_1(N'))$ for some level $N'$ with $\text{val}_3(N') < \text{val}_3(N)$ (Theorem 3.9) and in these cases it seems possible to investigate the existence of Heegner points with different methods (namely, prove the existence of Heegner points for the abelian variety corresponding to $f'$ and then twisting back to show the existence of Heegner points on the elliptic curve). Therefore, if we assume that the form is primitive, in the sense that it is not a twist of a form of lower level, then we can exclude form our discussion all cases in which $\text{val}_3(N)$ is even and greater or equal than 4. Finally, if we further assume the condition that the 2-component $r_2 \in \mathbb{Z}$ of the automorphic representation $\tau_f$ associated with $f$ has minimal conductor among all its twists by quasi-characters of $\mathbb{Q}_2^\times$, then the only cases we need to exclude in Theorem A above are the following, where $\Delta$ is the discriminant of the quaternion algebra:

(Missing Cases) \[ \text{val}_3(N) = 4, 3 \nmid \Delta, 3 \text{ is inert in } K \text{ and } \text{val}_3(c) = 1; \]
\[ \text{val}_2(N) \geq 3, 2 \nmid \Delta \text{ and } 2 \text{ is ramified in } K. \]

See Corollary 4.19 for a complete statement. However, let us stress that these missing cases, even if they seem to be isolated in the case of elliptic curves, they are not rare at all in the more general context of modular abelian varieties associated with modular forms of level $\Gamma_0(Mp^r)$ with $p$ any prime number and $r$ arbitrarily large.

Statement (2) in the above theorem follows from (1) and well-known Kolyvagin type arguments which are spelled out in detail in [Nek07]. Namely, given

- a parametrization of the elliptic curve $E$ by a Shimura curve $X_{B,U}$,
- a Heegner point $x$ in $X_{B,U}(K_{ab})$, rational over a subfield $K(x) \subseteq K_{ab}$, and
- a character $\chi$ factoring through $\text{Gal}(K(x)/K)$.

Nekovár shows that if the special value of the derivative of the complex $L$-function at $s = 1$ is nonzero, then the dimension of the $\mathbb{C}$-vector space $(E(K(x)) \otimes \mathbb{C})^\chi$ is equal to 1, provided that $E$ does not acquire CM over any imaginary quadratic field contained in $K(x)^{\ker(\chi)}$.

In some sense, our Theorem A reverses the logical order of the result in [Nek07], starting with a character of a given conductor and asking for a Heegner point rational over the subextension of $K_{ab}$ cut out by that character. Therefore, the whole focus of our work is on statement (1) of the above theorem. More precisely, this work grows out from a systematic study of existence conditions for Heegner points in all scenarios in which the BSD conjecture predicts the existence of a non-zero element in $(E(H_\mathcal{R}) \otimes \mathbb{C})^\chi$. To understand the flavour of this work, it is therefore important to stress that we do not require any condition on the triplet $(N, D, c)$, besides the above restrictions at 2 and 3 (cf. also Assumption 4.10).

Quite surprisingly, the interplay between local root numbers, non-vanishing of the first derivative of the $L$-function and the theory of optimal embeddings shows that these conditions match perfectly and, in all relevant cases, Heegner points do exist.

Our approach consists in three steps. Firstly, we need to find a suitable candidate Shimura curve $X_{\mathcal{R}}$ equipped with a non-constant map $\text{Jac}(X_{\mathcal{R}}) \to E$. For this, we need to specify the ramification set of the quaternion algebra $B$, which is prescribed as usual in terms of local root numbers of $\varepsilon(E/K, \chi)$ and $\eta_K(-1)$, where $\eta_K$ is the quadratic character associated with $K/\mathbb{Q}$ (see for example [YZZ13]).
Secondly, we need to specify a convenient order $\mathcal{R}$, and this is an application of a fine version of the Jacquet–Langlands theory, using works of Pizer, Chen and others.

Thirdly, having fixed our Shimura curve $X_{\mathcal{R}}$, we need to prove the existence of Heegner points of conductor $c$, and this follows, as hinted above, from a careful study of relations between local root numbers and optimal embeddings. The main new ingredient in this context is the adaptation of the theory of local orders in quaternion division algebras developed by Hijikata–Pizer–Schrembske.

In closing this introduction, we would like to mention three interesting papers that have recently seen the light, two of them by Kohen–Pacetti [KPi16a, KPi16b] and one by Cai–Chen–Liu [CCL16], addressing similar but non-overlapping problems in the case of Cartan level structure. As opposed to these works, we rather focus on the case where the modular parametrization of the elliptic curve is given by a Shimura curve associated with $\lambda$.

Here, $[g,f]$ denotes the point on $X_U$ corresponding to a pair $(g,f) \in \hat{B}^\times \times \text{Hom}(\mathbb{C},\mathcal{M}_2(\mathbb{R}))$. It is immediate to check that $\lambda(b)$ defines the identity on $X_U$ if and only if $b \in U\mathbb{Q}^\times$. The group $\text{Aut}^{\text{mod}}(X_U)$ of modular automorphisms on $X_U$ is then defined to be the group of all the automorphisms obtained in this way, so that

$$\text{Aut}^{\text{mod}}(X_U) := U\mathbb{Q}^\times \backslash N(U).$$
If \( U = \hat{S} \) is the group of units in the profinite completion of some order \( S \subseteq \mathcal{O} \), then we shall write \( X_S := X_{\hat{S}}/\mathbb{Q} \) for the Shimura curve associated with the order \( S \). In this case, the set of geometric connected components of \( X_S \) is identified with the class group \( \text{Pic}(S) \) of \( S \).

**Remark 1.1.** The most common setting in the literature is when \( S = \mathcal{S}_{N^*} \) is an Eichler order of level \( N^* \) in \( \mathcal{O} \), where \( N^* \geq 1 \) is an integer prime to \( N^- := \Delta(B) \). In this case, the Shimura curve \( X_{N^*, N^-} := X_S/\mathbb{Q} \) associated with \( S \) is not only connected but also geometrically connected, and its group \( \text{Aut} \mod(G_{N^*, N^-}) \) of modular automorphisms is the group of Atkin-Lehner involutions, which are indexed by the positive divisors of \( N^* N^- \). Further, \( X_{N^*, N^-}/\mathbb{Q} \) is the coarse moduli space classifying abelian surfaces with quaternionic multiplication by \( \mathcal{O} \) and \( N^* \)-level structure.

When \( N^- = 1 \) (so that \( B \) is the split quaternion algebra \( M_2(\mathbb{Q}) \)), a case which we exclude in this paper, the Shimura curve \( X_{N^*, 1}/\mathbb{Q} \) is the affine modular curve \( Y_0(N^*) \) obtained as a quotient of the upper half plane by the congruence subgroup \( \Gamma_0(N^*) \), whose compactification by adding finitely many cusps is the usual modular curve \( X_0(N^*)/\mathbb{Q} \).

In this article, we will be working with Shimura curves associated with certain suborders of \( \mathcal{O} \) which are not Eichler orders, but rather with more general orders that for example might have non-trivial level at primes dividing \( \Delta(B) \). The special class of quaternion order we shall be dealing with is described in the next section.

### 1.2. Choice of quaternion orders

Let \( p \) be a rational prime and let \( B_p \) be a quaternion algebra over \( \mathbb{Q}_p \). The object of this section is introducing several families of local quaternion orders in \( B_p \) which in turn will give rise to a fauna of Shimura curves that will serve as the appropriate host of the Heegner systems we aim to construct.

Assume first that \( B_p = D_p \) is the unique (up to isomorphism) quaternion division algebra over \( \mathbb{Q}_p \), and let \( \mathcal{O}_p \) be the unique maximal order in \( D_p \). If \( L_p \) is a quadratic extension of \( \mathbb{Q}_p \) and \( \nu \geq 1 \) is an integer, one can define the (local) quaternion order

\[
R_{\nu}(L_p) = \mathcal{O}_{L_p} + \pi_{\nu}^{\nu-1} \mathcal{O}_p,
\]

where \( \mathcal{O}_{L_p} \) denotes the ring of integers of \( L_p \) and \( \pi_p \) is a uniformizer element in \( \mathcal{O}_p \). Such local orders are studied in detail by Hijikata, Pizer and Shemanske in [HPS89a]. Notice that \( R_1(L_p) \) coincides with the maximal order \( \mathcal{O}_p \), regardless of the choice of \( L_p \). Further, if \( L'_p \) is another quadratic extension of \( \mathbb{Q}_p \) with \( L_p \simeq L'_p \), then \( R_\nu(L_p) \) and \( R_\nu(L'_p) \) are conjugated by an element in \( D_p^\times \). For \( \nu \geq 2 \), the order \( R_\nu(L_p) \) is characterized as the unique order in \( D_p \) containing \( \mathcal{O}_{L_p} \) and \( \pi_{\nu-1}^{\nu-1} \mathcal{O}_p \) but not containing \( \pi_{\nu-2} \mathcal{O}_p \).

**Remark 1.2.** If \( p \) is odd and \( L_p \) is the unique unramified quadratic extension, then \( R_{2\nu+1}(L_p) = R_{2\nu+2}(L_p) \) for every \( \nu \geq 0 \), thus one can think of the orders \( R_\nu(L_p) \) as being indexed by odd positive integers. These orders were studied in [Piz74], where they are called orders of level \( p^{2\nu+1} \). When \( p = 2 \) or \( L_p \) is ramified, then \( R_{\nu+1}(L_p) \subsetneq R_{\nu}(L_p) \) for every \( \nu \geq 1 \), and the order \( R_{\nu}(L_p) \) depends in general on the choice of \( L_p \). However, \( R_2(L_p) \) is independent of \( L_p \), and therefore one can speak of the unique order of level \( p^2 \) in \( D_p \) (cf. [Piz80]).

Assume now that \( B_p = M_2(\mathbb{Q}_p) \) is the split quaternion algebra over \( \mathbb{Q}_p \). In this algebra the order \( M_2(\mathbb{Z}_p) \) is maximal and it is the only one up to conjugation by elements in \( GL_2(\mathbb{Q}_p) \). Below we introduce, for each positive integer, two different \( GL_2(\mathbb{Z}_p) \)-conjugacy classes of suborders in \( M_2(\mathbb{Z}_p) \), which therefore define two different \( GL_2(\mathbb{Q}_p) \)-conjugacy classes of orders in \( M_2(\mathbb{Q}_p) \). Let \( \nu \geq 1 \) be an integer.

- The subring of \( M_2(\mathbb{Z}_p) \) consisting of those matrices \(
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

in \( M_2(\mathbb{Z}_p) \) such that \( p^\nu | c \) is commonly referred to as the *standard* Eichler order of level \( p^\nu \) in \( M_2(\mathbb{Z}_p) \). An Eichler order of level \( p^\nu \) is then any order in \( M_2(\mathbb{Q}_p) \) which is conjugated to the standard one. We shall denote any of them by \( R_{Eic}^\nu \), whenever only its conjugacy class is relevant in the discussion.

- Let \( Q_{p^\nu} \) denote the unique unramified quadratic extension of \( \mathbb{Q}_p \), and \( \mathcal{O} = \mathbb{Z}_{p^\nu} \) be its valuation ring. Then \( \mathcal{O}/p^\nu \mathcal{O} \) is a finite, free, commutative \((\mathbb{Z}/p^\nu\mathbb{Z})\)-algebra of rank 2 with unit discriminant. In particular, the choice of a basis for \( \mathcal{O}/p^\nu \mathcal{O} \) gives an embedding of \( (\mathcal{O}/p^\nu \mathcal{O})^\times \) into \( GL_2(\mathbb{Z}/p^\nu\mathbb{Z}) \). Its image \( C_{\mathcal{O}}(p^\nu) \) is then well-defined up to conjugation. The inverse image of \( C_{\mathcal{O}}(p^\nu) \cup \{ (0 1) \} \) under the reduction modulo \( p^\nu \) map \( M_2(\mathbb{Z}_p) \to M_2(\mathbb{Z}/p^\nu\mathbb{Z}) \) is an order of \( M_2(\mathbb{Z}_p) \), commonly referred to as a *non-split Cartan order* of level \( p^\nu \). We shall denote any of the orders arising in this way simply by \( R_{\nu}^{\text{Cart}} \), at any time that it is only the conjugacy class that matters in the discussion.
Now let $B/Q$ be an indefinite quaternion algebra of discriminant $\Delta = \Delta(B)$ as before.

**Definition 1.3.** Let $N_{\text{Eic}} \geq 1$ and $N_{\text{Car}} \geq 1$ be such that $(N_{\text{Eic}}, N_{\text{Car}}) = 1$ and $(N_{\text{Eic}} \cdot N_{\text{Car}}, \Delta) = 1$. For each prime $p \mid N_{\text{Eic}} \cdot N_{\text{Car}}$, set $n_p$ to be the $p$-adic valuation of $N_{\text{Eic}} \cdot N_{\text{Car}}$. For each prime $p \mid \Delta$, choose an integer $n_p \geq 1$ and a quadratic extension $\mathbb{Q}_p$. An order $\mathcal{R}$ in $B$ is said to be of type $T = (N_{\text{Eic}}; N_{\text{Car}}; \{(L_p, n_p)\}_{p \mid \Delta})$ if the following conditions are satisfied:

1. If $p \nmid N_{\text{Eic}}N_{\text{Car}}$, then $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order in $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{M}_2(\mathbb{Q}_p)$.
2. If $p \mid N_{\text{Eic}}$, then $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is conjugate to $R^\text{Eic}_p$ in $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{M}_2(\mathbb{Q}_p)$.
3. If $p \mid N_{\text{Car}}$, then $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is conjugate to $R^\text{Car}_p$ in $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{M}_2(\mathbb{Q}_p)$.
4. For every $p \mid \Delta$, $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong R_{n_p}(L_p)$ in $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D_p$.

Fix for the rest of this section an order $\mathcal{R}$ in $B$ of type $T = (N_{\text{Eic}}; N_{\text{Car}}; \{(L_p, n_p)\}_{p \mid \Delta})$ as in Definition 1.3. Define the level of $\mathcal{R}$ to be the integer $N_{\mathcal{R}} := N_{\text{Eic}} \cdot N_{\text{Car}}^2 \cdot N_{\Delta}$, where we put $N_{\Delta} := \prod_{p \mid \Delta} p^{n_p}$. If $n_p = 1$ for every $p \mid \Delta$, we will sometimes refer to $\mathcal{R}$ as a Cartan–Eichler order of type $(N_{\text{Eic}}; N_{\text{Car}})$ (and level $N_{\text{Eic}} \cdot N_{\text{Car}}^2$).

Associated with $\mathcal{R}$, we have the Shimura curve $X_{\mathcal{R}}/\mathbb{Q}$ defined as in the previous paragraph. The Shimura curve $X_{\mathcal{R}}$ is projective and smooth over $\mathbb{Q}$, but in general it is not geometrically connected. The reduced norm on $\mathcal{R}^\times$ is locally surjective onto $\mathbb{Z}_p^\times$ at every prime $\ell \mid \Delta$ (both Eichler and Cartan orders in indefinite rational quaternion algebras have class number one), but however the reduced norm on the local orders $R_{n_p}(L_p)$ is not necessarily surjective onto $\mathbb{Z}_p^\times$ when restricted to the invertible elements. Despite of this, it is easy to see that $[\mathbb{Z}_p^\times : n(R_{n_p}(L_p)^\times)]$ is either 1 or 2. Thus if we set

$$C := \{p \mid \Delta \text{ prime: } n(R_{n_p}(L_p)^\times) \neq \mathbb{Z}_p^\times\},$$

then the number of connected components of $X_{\mathcal{R}} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ is $2^{|C|}$. If $\Delta$ is odd, or if $n_\infty \leq 1$ in case that $\Delta$ is even, it follows from [HPS89a, Theorem 3.11] that

$$C = \{p \mid \Delta \text{ prime: } n_p > 1, L_p \text{ ramified}\}.$$

The behaviour at $p = 2$ is a bit more involved, but one still has a characterization of whether $n(R_{n_2}(L_2)^\times)$ has index 1 or 2 in $\mathbb{Z}_2^\times$ (see [HPS89a, Theorem 3.11, 3) and 4]) for details). Furthermore, if $\Delta$ is odd, the connected components of $X_{\mathcal{R}} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ are defined over a polyquadratic extension: the number field obtained as the composium of the quadratic extensions $\mathbb{Q}(\sqrt{p^\pm})$ for $p \in C$, where $p^\pm = (\pm p^2)^{\frac{1}{p}}$.

**Example 1.4.** Suppose $\Delta = pq$ with $p$ and $q$ distinct odd primes, and let $L_p$ be a quadratic ramified extension of $\mathbb{Q}_p$. Consider an order $\mathcal{R}$ of type $(M;1;\{(L_p, 2),(L_q, 1)\})$ and level $N = Mq^2$. As noticed in the above remark, this order does not depend on the choice of $L_p$. The Shimura curve $X_{\mathcal{R}}/\mathbb{Q}$ has two geometric connected components defined over the quadratic field $\mathbb{Q}(\sqrt{p^\pm})$, and they are conjugated by the Galois action (in particular, they are isomorphic over $\mathbb{Q}(\sqrt{p^\pm})$). There is a unique Eichler order $\mathcal{S}$ containing $\mathcal{R}$, and the morphism of Shimura curves $X_{\mathcal{R}} \to X_\mathcal{S}$ induced by the inclusion $\mathcal{R}^\times \subset \mathcal{S}^\times$ is cyclic of degree $p + 1$. Modular cusp forms in $S_2(\Gamma_0(N))$ which are $N/M$-new and not principal series at $p$ lift via the Jacquet–Langlands correspondence to quaternionic modular forms on the Shimura curve $X_{\mathcal{R}}$ (see below).

2. **MODULAR FORMS AND THE JACQUET–LANGLANDS CORRESPONDENCE**

We fix throughout this section an indefinite quaternion algebra $B$ of discriminant $\Delta$ and an order $\mathcal{R}$ of $B$ of type $T = (N_{\text{Eic}}; N_{\text{Car}}; \{(L_p, n_p)\}_{p \mid \Delta})$ and level $N_{\mathcal{R}} = N_{\text{Eic}}N_{\text{Car}}^2N_{\Delta}$.

2.1. **Cusp forms with respect to $\mathcal{R}$**. We identify the Lie algebra of left invariant differential operators on $B^\infty := (B \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong GL_2(\mathbb{R})$ with $M_2(\mathbb{C})$, and define the differential operators

$$X_\infty = \begin{pmatrix} 1 & \sqrt{-1} \\ -1 & -\sqrt{-1} \end{pmatrix}, \quad \overline{X}_\infty = \begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix}, \quad W_\infty = \frac{1}{2} \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

**Definition 2.1.** Let $k$ be an integer. A cusp form of wight $k$ with respect to $\mathcal{R}$ is a function

$$f : (B \otimes_{\mathbb{Q}} \mathbb{A}_Q)^\times \to \mathbb{C}$$

satisfying the following properties:

1. If $g \in (B \otimes_{\mathbb{Q}} \mathbb{A}_Q)^\times$, then the function $GL_2(\mathbb{R}) \to \mathbb{C}$ given by $x \mapsto f(xg)$ is of $C^\infty$-class and satisfies $W_\infty f = (k/2) f, \quad \overline{X}_\infty f = 0$;
(2) for every \( \gamma \in B^\times \) and every \( u \in \mathcal{R}^\times \times \mathbb{R}^\times, f(ug\gamma) = f(g) \).

The \( \mathbb{C} \)-vector space of all cusp forms of weight \( k \) with respect to \( \mathcal{R} \) will be denoted \( S_k(\mathcal{R}) \).

The product \( \prod_{p \mid \Delta} B_p^\times \) acts on the space \( S_k(\mathcal{R}) \) by left translation, and through this action one can decompose \( S_k(\mathcal{R}) \) into the direct sum of subspaces on which \( \prod_{p \mid \Delta} B_p^\times \) acts through some admissible representation (with trivial central character). More precisely, suppose that for each \( p \mid \Delta \) we are given an irreducible admissible representation (with trivial central character) \( \rho_p \) of \( B_p^\times \) whose restriction to \( \mathcal{R}_p^\times \) is trivial (i.e., \( \mathcal{R}_p^\times \subseteq \ker(\rho_p) \)). Define \( \rho := \otimes_{p \mid \Delta} \rho_p \), regarded as a representation of \( \prod_{p \mid \Delta} B_p^\times \). Since the representations \( \rho_p \) are finite-dimensional, the integer \( d_\rho := \dim(\rho) = \prod_{p \mid \Delta} \dim(\rho_p) \) is well-defined.

**Definition 2.2.** Let \( k \) be an integer, and \( \rho \) be a representation as above. A **cusp form of weight \( k \) with respect to \((\mathcal{R}, \rho)\)** is a function

\[
\begin{aligned}
f : (B \otimes \mathbb{Q} \mathcal{A}_Q)^\times & \rightarrow \mathbb{C}^{d_\rho},
\end{aligned}
\]
satisfying the following conditions, for every \( g \in (B \otimes \mathbb{Q} \mathcal{A}_Q)^\times \):

1. for every \( \gamma \in B^\times \), \( f(g\gamma) = f(g) \);
2. for every \( b \in \prod_{p \mid \Delta} B_p^\times \), \( f(bg) = \rho(b)f(g) \);
3. for every prime \( \ell \mid \Delta \) and \( u \in \mathcal{R}_\ell^\times \), \( f(ug) = f(g) \);
4. the function \( GL_2(\mathbb{R}) \rightarrow \mathbb{C}^{d_\rho} \) given by \( x \mapsto f(xg) \) is of \( C^\infty \)-class and satisfies \( W_\infty f = (k/2)f, \ X_\infty f = 0 \);
5. for every \( z \in \hat{\mathbb{Q}}^\times \times \mathbb{R}^\times \), \( f(gz) = f(g) \).

We write \( S_k(\mathcal{R}, \rho) \) for the \( \mathbb{C} \)-vector space of cusp forms of weight \( k \) with respect to \((\mathcal{R}, \rho)\).

The \( \mathbb{C} \)-vector spaces \( S_k(\mathcal{R}, \rho) \) enjoy the following multiplicity one property:

**Proposition 2.3** (cf. Prop. 2.14 in [Hid81]). *If two forms in \( S_k(\mathcal{R}, \rho) \) are common eigenforms of the Hecke operators \( T_\ell \) for all primes \( \ell \mid \Delta \) with same eigenvalues, then they differ only by a constant factor.*

The subspace of \( S_k(\mathcal{R}) \) on which \( \prod_{p \mid \Delta} B_p^\times \) acts through an admissible representation \( \rho \) as above is isomorphic to \( S_k(\mathcal{R}, \rho)^{d_\rho} \), hence one deduces that

\[
(3) \quad S_k(\mathcal{R}) \simeq \bigoplus_{\rho} S_k(\mathcal{R}, \rho)^{d_\rho},
\]
where \( \rho \) ranges over the representations \( \rho = \otimes_{p \mid \Delta} \rho_p \) as above, satisfying \( \mathcal{R}_p^\times \subseteq \ker(\rho_p) \).

**Remark 2.4.** The automorphic approach sketched before is related to the more classical point of view as follows. Let \( h = h(\mathcal{R}) \) denote the class number of \( \mathcal{R} \) and choose elements \( a_i \in \hat{B}^\times, i = 1, \ldots, h \), such that

\[
\hat{B}^\times \subseteq \bigcap_{i=1}^h \mathcal{R}_i^\times a_i B^\times.
\]

Consider the discrete subgroups of \( SL_2(\mathbb{R}) \) defined by

\[
\Gamma_i := B_i^\times \cap a_i^{-1} \mathcal{R}_i^\times, \quad (i = 1, \ldots, h),
\]
where \( B_i^\times \) is the subgroup of units of positive reduced norm (we may write \( B_i^\times = B^\times \cap GL_2^\times(\mathbb{R}) \) using our identification of \( B \otimes \mathbb{R} \) with \( M_2(\mathbb{R}) \)). If we denote by \( S_k(\Gamma_i) \) the \( \mathbb{C} \)-vector space of cusp forms of weight \( k \) with respect to the group \( \Gamma_i \), then there is an isomorphism of complex vector spaces:

\[
\bigcap_{i=1}^h S_k(\Gamma_i) \xrightarrow{\sim} S_k(\mathcal{R}).
\]

**2.2. Jacquet–Langlands.** The space \( S_k(\mathcal{R}) \) can be equipped with a standard action of Hecke operators and Atkin–Lehner involutions, described for example in [Hid81]. We have the following version of the Jacquet–Langlands correspondence:

**Theorem 2.5** (cf. Prop. 2.12 in [Hid81]). *There is a Hecke equivariant injection of \( \mathbb{C} \)-vector spaces\)

\[
S_k(\mathcal{R}, \rho) \hookrightarrow S_k(\Gamma_1 \cap N^2 \mathcal{N}_p, \mathbb{N}_p),
\]

where \( N_p \) is the conductor of \( \rho \).
Combining Theorem 2.4 with (3) we can embed the space $S_k(\mathcal{R})$ into a space of classical modular cusp forms

$$JL : S_k(\mathcal{R}) \longrightarrow \bigoplus_{\rho} S_k(\text{N}_{\text{Eic}} N_{\text{Car}}^2 N_{\rho})^{d_{\rho}}.$$  

The multiplicities $d_{\rho}$ can be described explicitly: cf. [Car84, §5].

**Example 2.6.** Suppose $p \mid \Delta$ is an odd prime. The quaternion algebra $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is equipped with a natural decreasing filtration $O_p^\times (i)$ defined by setting

$$O_p^\times (0) = O_p^\times \quad \text{and} \quad O_p^\times (i) = 1 + \pi_p^i O_p^\times,$$

where $O_p$ is the unique maximal order in $B_p$ and $\pi_p$ is a local uniformizer. If $\rho_p$ is an admissible irreducible representation of $B_p^\times$, then its conductor is by definition $p^{n+1}$, where $n \geq 0$ is the smallest integer such that $O_p^\times (n)$ lies in the kernel of $\rho_p$. In particular, observe that the conductor is at least $p$.

Thus, if $p^2 \nmid N_{\mathbb{R}}$ then $R_p^\times$ is precisely the group of units in the local maximal order at $p$, thus the admissible irreducible representations $\rho_p$ we are concerned with all have conductor $p$. If $p^2 \nmid N_{\mathbb{R}}$ and $p^3 \nmid N_{\mathbb{Q}}$, we have $O_p^\times (1) \subseteq R_p^\times \subseteq O_p^\times$, and therefore the conductor of the admissible irreducible representations $\rho_p$ might be either $p$ or $p^2$.

For each prime $p \mid \Delta$, the dimension of $p$ is determined by its minimal conductor, which by definition is the smallest conductor of the representations $\rho_p \otimes \chi$, as $\chi$ ranges over the characters of $\mathbb{Q}_p^\times$. Let $N_{\rho_p}^{\min}$ be the minimal conductor of $\rho_p$. By [Car84, §5], if $N_{\rho_p}^{\min} = p^a$ with $a \in \{1, 2\}$, then $d_{\rho_p} = a$.

The above arguments give us a Hecke equivariant inclusion of $S_k(\mathbb{R})$ into a direct sum of spaces of classical modular cusp forms. In order to circumvent the problem of explicitly determining the multiplicities $d_{\rho_p}$ we use Proposition 2.7 below, which benefits from an explicit version of Eichler trace formula due to Hijikata, Pizer and Shimanskii.

For the reader’s convenience, we recall the classification of Jacquet–Langlands lifts given in [HPS89b], and from now on we focus on the weight 2 case. So let $f \in S_2(\Gamma_0(N_f))$ be a weight 2 modular cusp form, and assume that $N_f = p^s M$ for some prime $p$ and integers $s, M \geq 1$, with $p \mid M$. Suppose $\phi$ is a Jacquet–Langlands lift of $f$ which is realized on the definite quaternion algebra $B^{(p)}$ of discriminant $p$.

We want to determine the level of $\phi$, by which we mean the local $p$-type of the order $\mathcal{R}$ of $B^{(p)}$ used to define its level structure. Such local order is of the form $R_n(L)$, for some positive integer $n$ and quadratic extension $L/\mathbb{Q}_p$, and it is determined as follows:

1. if $p$ is odd:
   (a) $s$ odd: $L$ is unramified and $n = s$ ([HPS89b, Theorem 8.5]).
   (b) $s$ even: $L$ is ramified (any of the two ramified extensions) and $n = s$ ([HPS89b, Proposition 8.8 Case D]).
2. if $p = 2$:
   (a) $s = 1$: $L$ is the unramified quadratic extension of $\mathbb{Q}_2$ and $n = 1$ ([HPS89b, Proposition 8.8 Case C]).
   (b) $s$ odd, $s \geq 3$: $L$ is unramified and $n = s$ ([HPS89b, Theorem 8.5]).
   (c) $s = 2$: $L = \mathbb{Q}_2(\sqrt{5})$ or $L = \mathbb{Q}_2(\sqrt{7})$ and $n = 2$ ([HPS89b, Proposition 8.8 Case F Eq. (8.17)]).
   (d) $s$ even, $s \geq 4$: In this case, $f$ is a twist by a non-trivial character of conductor $2^{s/2}$ of one of the previous cases ([HPS89b, Theorem 3.9]).

**Proposition 2.7.** Let $f \in S_2(\Gamma_0(N_f))$ be a newform and fix a set $\Sigma$ of even cardinality consisting of primes $\ell \mid N_f$ such that the local admissible representation $\pi_{f,\ell}$ of $\text{GL}_2(\mathbb{Q}_\ell)$ attached to $f$ is square-integrable. Let $B/\mathbb{Q}$ be the indefinite quaternion algebra of discriminant $\Delta = \prod_{\ell \in \Sigma} \ell$, and $N_{\text{Eic}}, N_{\text{Car}}$ be positive integers such that $(N_{\text{Eic}}, N_{\text{Car}}) = 1$, $N_{\text{Eic}} N_{\text{Car}}^2 = N_f$ and $(N_{\text{Eic}}, N_{\text{Car}}, \Delta) = 1$.

Then there exists an order $\mathcal{R}_{\text{min}} \subset B$ of type $T_{\text{min}} = (N_{\text{Eic}}, N_{\text{Car}}; \{(L_p, \nu_p)\}_{p | \Delta})$ such that $f$ lifts to a quaternionic modular form on $S_2(\mathcal{R}_{\text{min}})$ having the same Hecke eigenvalues for the Hecke operators $T_\ell$ at primes $\ell \nmid N_f$. Further, for each prime $p \mid \Delta$ the data $(L_p, \nu_p)$ depends only on $\text{val}_p(N_f)$.

The subscript ‘min’ in $\mathcal{R}_{\text{min}}$ refers to the minimal level for primes dividing $\Delta$, determined by the classification explained above. We note that if $\nu_p$ is odd then $L_p$ is unramified, and if $\nu_p$ is even then $L_p$ is ramified.

**Proof.** Since $\pi$ is square integrable at all primes in $\Sigma$, [Gel75, Theorem 10.2] implies the existence of an automorphic form $\pi'$ on the algebraic group of invertible elements of the indefinite quaternion algebra $B$.
as in the statement such that \( \pi'_e \simeq \pi_e \) for all primes \( e \). To specify the order \( \mathcal{R} \) we need to describe \( \pi'_e \) at every prime \( e \). For primes \( e \nmid \N_{\text{Car}} \Delta \) the assertion is obvious. Fix a prime \( p \mid \Delta \) and let \( B^{(p)} \) be the definite quaternion algebra of discriminant \( p \). Then Eichler’s trace formula in [HS89] shows the existence of an automorphic form \( \pi^{(p)} \) for \( B^{(p)} \) attached to a specific order \( \mathcal{R}^{(p)} \) of type \( (L, \nu(L)) \) depending only on \( \val_p(N_f) \) with \( \pi'_p \simeq \pi^{(p)}_p \) (we have sketched above the recipe for \( L \) and \( \nu(L) \)). Finally, for primes dividing \( \N_{\text{Car}} \) a similar argument works using this time the trace formula in [Che98, Sec. 6] (the proof in [Che98] only works for \( p \neq 2 \), but one can check that it can be extended to the case \( p = 2 \)).

\[ \square \]

**Remark 2.8.** Let \( N_{\min} \) be the level of the order \( \mathcal{R}_{\min} \) in the proposition. Then observe that \( N_{\min} \) divides \( N_f \). Even more, we have \( \val_p(N_{\min}) = \val_p(N_f) \) for every odd prime \( p \). And in case that \( N_f \) is even, then \( \val_2(N_{\min}) < \val_2(N_f) \) implies that \( 2 \mid \Delta \) and \( \val_2(N_f) \) is even and at least 4.

**Example 2.9.** It follows from [Piz80] that, given a primitive (in the sense of [Piz80] Definition 8.6]) new cuspidal eigenform \( f \) of level \( \Gamma_0(p^2M) \) as in the proposition with \( p \mid M \) an odd prime, the subspace of new forms in \( S_2(\mathcal{R}) \) having the same system of Hecke eigenvalues as \( f \) (at primes outside \( N_f = p^2M \)) is two-dimensional. So we have a “multiplicity 2 phenomenon” as expected from the Example 2.6 and Hecke-equivariant monomorphism \( J_L \) in (4).

2.3. **Modular parametrizations.** Let \( J_{\mathcal{R}}/\mathbb{Q} \) denote the Jacobian variety of \( X_{\mathcal{R}} \). It is a (principally polarized) abelian variety defined over \( \mathbb{Q} \), of dimension equal to the genus of \( X_{\mathcal{R}} \). Since \( X_{\mathcal{R}} \) is not in general geometrically connected, it follows that \( J_{\mathcal{R}} \) might not be absolutely simple. Recall the following:

**Definition 2.10.** An abelian variety \( A/\mathbb{Q} \) is said to be modular if there exists a normalized newform \( f = \sum_{n \geq 1} a_nq^n \) of weight 2 and level \( \Gamma_0(N_f) \) for some \( N_f \geq 1 \) such that

\[
L(A, s) = \prod_{\sigma,F \rightarrow \mathbb{Q}} L(f^\sigma, s),
\]

where \( F \) stands for the number field generated by the Fourier coefficients of \( f \), \( \sigma \) ranges over the embeddings of \( F \) into an algebraic closure of \( \mathbb{Q} \) and \( f^\sigma = \sum_{n \geq 1} \sigma(a_n)q^n \).

**Proposition 2.11.** Suppose that \( A/\mathbb{Q} \) is a modular abelian variety associated with a modular form \( f = J_L(\phi) \) for some \( \phi \in S_2(\mathcal{R}) \). Let \( I_\phi \subset \mathbb{T} \) be the kernel of the ring homomorphism \( \mathbb{T} \rightarrow \mathbb{Z} \) determined by the system of Hecke eigenvalues of \( \phi \). Then the quotient abelian variety \( A_\phi := J_{\mathcal{R}}/I_\phi J_{\mathcal{R}} \) is isogenous to \( A' \) for some \( r \geq 1 \).

**Proof.** Let \( \ell \nmid N_f \) be a prime and \( \varrho : G_\mathbb{Q} \rightarrow \text{Aut}(\text{Tate}(A) \otimes \mathbb{Q}_\ell) \) be the 2-dimensional \( \ell \)-adic Galois representation arising from the natural action of \( G_\mathbb{Q} \) on the \( \ell \)-adic Tate module \( \text{Tate}(A) \) of \( A \). Similarly, let \( \theta : G_\mathbb{Q} \rightarrow \text{Aut}(\text{Tate}(A_\phi) \otimes \mathbb{Q}_\ell) \) be the \( \ell \)-adic Galois representation attached to \( A_\phi \). The Eichler–Shimura relations (proved in the required generality in [Nek]) imply that \( \theta(\sigma) \) is annihilated by the characteristic polynomial of \( \rho(\sigma) \) for every \( \sigma \in G_\mathbb{Q} \). Then the Boston–Lenstra–Ribet Theorem [BLR91] implies that \( \text{Tate}(A_\phi) \otimes \mathbb{Q}_\ell \) is isogenous to a direct sum of \( r \) copies of \( \text{Tate}(A) \otimes \mathbb{Q}_\ell \) for some \( r \geq 1 \). Finally, Faltings’ Isogeny Theorem implies that \( A_\phi \) is isogenous to \( r \) copies of \( A \). \( \square \)

**Example 2.12.** Suppose that \( \mathcal{R} \) is of type \( (M; 1; \{ (L_p, 2), (L_\ell, 1) \}) \) and level \( \N := \N_{\mathcal{R}} = Mp^2q^2 \) with \( p \) and \( q \) distinct odd primes as in Example 2.4. Let \( F = \mathbb{Q}(\sqrt{P^3}) \). Then

\[
J_{\mathcal{R}} \times_{\mathbb{Q}} F \sim J_{\mathcal{R}, 1} \times J_{\mathcal{R}, 2},
\]

where \( J_{\mathcal{R}, i}/F \) is the Jacobian variety of \( X_{\mathcal{R}, i} \). Let \( \mathbb{S}_2(\Gamma_0(N)) \) be the subspace of \( S_2(\Gamma_0(N)) \) consisting of primitive newforms. From Example 2.4 we know that there is a 2-to-1 Hecke-equivariant morphism of \( \mathbb{C} \)-vector spaces

\[
\mathbb{S}_2(\mathcal{R}) \rightarrow \mathbb{S}_2(\Gamma_0(N)),
\]

where \( \mathbb{S}_2(\mathcal{R}) \) is the subspace of modular forms \( \phi \in \mathbb{S}_2(\mathcal{R}) \) such that \( J_L(\phi) \in \mathbb{S}_2(\Gamma_0(N)) \). By a slight abuse of notation we continue to denote by \( J_L \) this morphism.

Fix \( f \in \mathbb{S}_2(\Gamma_0(N)) \) and assume that the Fourier coefficients of \( f \) belong to \( \mathbb{Z} \). Then the abelian variety associated with \( f \) is an elliptic curve \( E \) of conductor \( Mp^2q^2 \). Let \( \phi \in \mathbb{S}_2(\mathcal{R}) \) be such that \( J_L(\phi) = f \) (we have two linearly independent possible choices). The space \( \mathbb{S}_2(\mathcal{R}) \) of weight 2 modular forms for \( \mathcal{R} \) is identified with \( H^0(X_{\mathcal{R}}, \Omega^1) \), which in turn is identified with the tangent space at the identity \( T_0(J_{\mathcal{R}}) \) of \( J_{\mathcal{R}} \). The subspace \( \mathbb{S}_2(\mathcal{R}) \) corresponds then to a subspace of \( H^0(X_{\mathcal{R}}, \Omega^1) \), and hence to the tangent space \( T_0(J_{\mathcal{R}}) \) of an abelian subvariety \( J_{\mathcal{R}} \) of \( J_{\mathcal{R}} \). The space of modular forms \( \mathbb{S}_2(\mathcal{R}) \) has rank 2 over
3. Heegner points

3.1. Optimal embeddings. As in previous sections, $B$ denotes an indefinite rational quaternion algebra of discriminant $\Delta = \Delta(B)$. We fix an order $\mathcal{R}$ and a quadratic field $K$. For each positive integer $c$ write $R_c$ for the (unique) order of conductor $c$ in $K$, $R_1$ being the full ring of integers of $K$.

**Definition 3.1.** Let $c$ be a positive integer. An embedding from $K$ to $B$, i.e. a $\mathbb{Q}$-algebra homomorphism $f : K \to B$, is said to be optimal with respect to $\mathcal{R}/R_c$ if the equality
\[
f(K) \cap \mathcal{R} = R_c\]
holds. Since $f$ is determined by its restriction to $R_c$, we also speak of optimal embeddings of $R_c$ into $\mathcal{R}$.

Two optimal embeddings $f, f'$ of $R_c$ into $\mathcal{R}$ will be considered to be equivalent if they are conjugate one to each other by an element in $\mathcal{R}$. The set of $\mathcal{R}$-conjugacy classes of optimal embeddings of $R_c$ into $\mathcal{R}$ will be denoted $\text{Emb}^\text{op}(R_c, \mathcal{R})$. We are interested in computing the integer
\[v(R_c, \mathcal{R}) = |\text{Emb}^\text{op}(R_c, \mathcal{R})|,\]
and in particular in knowing whether the set $\text{Emb}^\text{op}(R_c, \mathcal{R})$ is empty or not.

Suppose now that $\mathcal{R}$ is of type $T = (N_{\text{Eic}}; \nu, \{\nu(p)\}_{p \mid \Delta})$ and level $N_{\mathcal{R}} = N_{\text{Eic}}N_{\text{Car}}^2N_\Delta$. Recall that the class number $h(\mathcal{R})$ of the order $\mathcal{R}$ is $2^{[\mathcal{L}]}$, where $\mathcal{L}$ is the set introduced in (2). Although the class number of $\mathcal{R}$ is therefore not trivial in general, the lemma below asserts that the type number of $\mathcal{R}$ is always trivial, which amounts to saying that all orders in $B$ of the same type $T$ are conjugate one to each other.

**Lemma 3.2.** The type number of orders of a fixed type $T$ is 1.

**Proof.** Fix a type $T = (N_{\text{Eic}}; N_{\text{Car}}; \{\nu(p)\}_{p \mid \Delta})$ as in Definition 3.3 and let $\mathcal{R}$ and $\mathcal{R}'$ be two orders of type $T$ in $B$. First of all, notice that if $\mathcal{R}$ (resp. $\mathcal{R}'$) is a suborder of a unique Cartan–Eichler order $\mathcal{S}$ (resp. $\mathcal{S}'$) of level $N_{\text{Eic}}N_{\text{Car}}^2$. Namely, the order which is locally equal to $\mathcal{R}$ (resp. $\mathcal{R}'$) at every prime $p \mid \Delta$ and locally maximal at primes $p \mid \Delta$, hence of type $(N_{\text{Eic}}; N_{\text{Car}}; \{\nu(p)\}_{p \mid \Delta})$. Conversely, it is clear by construction that $\mathcal{R}$ (resp. $\mathcal{R}'$) is the unique suborder of type $T$ of the Cartan–Eichler order $\mathcal{S}$ (resp. $\mathcal{S}'$). The lemma now follows from the fact that the type number of Cartan–Eichler orders in $B$ is 1, so that $\mathcal{S}$ and $\mathcal{S}'$ are conjugate. By the above observation, this immediately implies that $\mathcal{R}$ and $\mathcal{R}'$ are conjugate as well.

By virtue of the above lemma, the number $v(R_c, \mathcal{R})$ can be expressed essentially as a product of local contributions that can be explicitly computed. Indeed, proceeding as in the proof of the ‘trace formula’ in [Vig80, Ch. III, 5.C] (cf. especially Theorems 5.11 and 5.11 bis, or [Brz89] for Eichler orders, we have that
\[
v(R_c, \mathcal{R}) = \frac{h(R_c)}{h(\mathcal{R})} \prod_{\ell \mid N_{\text{Eic}}} v_\ell(R_c, \mathcal{R}),\]
where $h(R_c)$ (resp. $h(\mathcal{R})$) is the class number of the quadratic order $R_c$ (resp. of $\mathcal{R}$), the product ranges over all rational primes and, for each $\ell$, $v_\ell(R_c, \mathcal{R})$ denotes the number of local optimal embeddings of $R_c \otimes \mathbb{Z}_\ell$ into $\mathcal{R} \otimes \mathbb{Z}_\ell$ modulo conjugation by $(\mathcal{R} \otimes \mathbb{Z}_\ell)^*$. These local contributions are 1 for every prime $\ell \mid N$. The number of local optimal embeddings is determined in 3.2 below. Here we give the following:

**Example 3.3.** Assume that $N_{\text{Car}} = 1$, $\Delta$ is odd and $\nu_p \leq 2$ for all $p \mid \Delta$. Then
\[
v(R_c, \mathcal{R}) = \frac{h(R_c)}{h(\mathcal{R})} \prod_{\ell \mid N_{\text{Eic}}} \left(1 + \left\{\frac{R_c}{\ell}\right\} \prod_{\nu(q) = 1} \left(1 - \left\{\frac{R_c}{\ell}\right\}\right) \prod_{\nu(p) = 2} v_p(R_c, \mathcal{R}),\right)
\]
where for primes $p \mid \Delta$ with $\nu_p = 2$,
\[
v_p(R_c, \mathcal{R}) = \begin{cases} 2 & \text{if } p \mid c \text{ and } p \text{ is inert in } K, \\ p + 1 & \text{if } p \mid c \text{ and } p \text{ ramifies in } K, \\ 0 & \text{otherwise.} \end{cases} \]
3.2. Local optimal embeddings. For the reader’s convenience, we reproduce in this subsection the criteria for the existence of local optimal embeddings of orders in quadratic fields into quaternion orders in the Eichler, Cartan and division cases.

3.2.1. Eichler case. Let \( p \) be a prime, \( K/\mathbb{Q}_p \) be a quadratic separable field, and \( \mathcal{O}_m \subseteq K \) be the order in \( K \) of conductor \( p^m \). Let also \( M_2(\mathbb{Q}_p) \) be the split quaternion algebra over \( \mathbb{Q}_p \) and \( R_{Eic}^n \) be the standard Eichler order of level \( p^n \) in \( M_2(\mathbb{Q}_p) \). Write \( h(m,n) \) for the number of (equivalence classes of) optimal embeddings of \( \mathcal{O}_m \) into \( R_{Eic}^n \). For later reference, we state the following lemma.

**Lemma 3.4.** If \( K \) is the split quadratic \( \mathbb{Q}_p \)-algebra, then \( \mathcal{O}_m \) can be optimally embedded in \( R_{Eic}^n \) for every \( m \geq 0 \). That is, \( h(m,n) \neq 0 \) for every \( m \geq 0 \).

Next we assume that \( K/\mathbb{Q}_p \) is a quadratic field extension with valuation ring \( \mathcal{O} \), and again for each \( m \geq 1 \) let \( \mathcal{O}_m \) be the order of conductor \( p^m \) in \( K \). Recall that the Eichler symbol is defined as follows:

\[
\begin{cases}
\mathcal{O}_m/p & = \\
-1 & \text{if } m = 0 \text{ and } K/\mathbb{Q}_p \text{ is unramified}; \\
0 & \text{if } m = 0 \text{ and } K/\mathbb{Q}_p \text{ is ramified}; \\
1 & \text{if } m \geq 1.
\end{cases}
\]

It is well known ([Hij74], [Vig80]) that if \( n = 0 \) then \( h(m,n) = 1 \), and for \( n = 1 \) one has \( h(m,n) = 1 + \left\{ \frac{m}{p} \right\} \). Thus, in particular, every quadratic order \( \mathcal{O}_m \) can be optimally embedded in the maximal Eichler order unless \( m = 0 \) and \( K/\mathbb{Q}_p \) is unramified, the only case when \( h(m,1) = 0 \). More generally (see [Brz91], Corollary 1.6):

**Lemma 3.5.**

1. If \( K/\mathbb{Q}_p \) is unramified, then \( h(m,n) \neq 0 \) if and only if \( m \geq n/2 \).

2. If \( K/\mathbb{Q}_p \) is ramified, then \( h(m,n) = 0 \) if and only if \( m \geq (n-1)/2 \).

3.2.2. Cartan Case. Let \( p \) be a prime, \( K = \mathbb{Q}_{p^2} \) be the unramified quadratic extension of \( \mathbb{Q}_p \) and \( \mathcal{O} = \mathbb{Z}_{p^2} \) be its valuation ring. As above, for \( m \geq 1 \) write \( \mathcal{O}_m \) for the order of conductor \( p^m \) in \( K \). From the very definition of non-split Cartan orders, we have the following lemma, which we state for later reference:

**Lemma 3.6.** Let \( R_{Car}^n \) be a non-split Cartan order of level \( p^n \) in \( M_2(\mathbb{Q}_p) \). Then \( \mathcal{O} \) can be optimally embedded in \( R_{Car}^n \). For \( m > 1 \), the order \( \mathcal{O}_m \) does embed in \( R_{Car}^n \), but not optimally.

3.2.3. Division case. References: [HPS89a]. Let \( p \) be a prime, and \( D_p \) be the unique division quaternion algebra over \( \mathbb{Q}_p \). As above, write \( R_n(L) \) for the local order in \( D_p \) associated to the choice of an integer \( n \geq 1 \) and a quadratic extension \( L/\mathbb{Q}_p \). Let \( K/\mathbb{Q}_p \) be a quadratic field extension, and \( \mathcal{O}_m \) denote the order of conductor \( p^m \) in \( K \) as before. Recall that \( h(m,n) \) denotes the number of equivalence classes of optimal embeddings of \( \mathcal{O}_m \) into \( R_n(L) \).

It might be useful first to recall the notation used in [HPS89a] for the symbols \( t(L) \) and \( \mu(L,L') \). For any quadratic field extension \( L/\mathbb{Q}_p \):

- \( t(L) = -1 \) means \( L \) unramified;
- \( t(L) = 0 \) means \( L \) ramified and \( p \neq 2 \);
- \( t(L) = 1 \) means \( p = 2 \) and \( L = \mathbb{Q}_p(\sqrt{3}) \) or \( L = \mathbb{Q}_p(\sqrt{7}) \);
- \( t(L) = 2 \) means \( p = 2 \) and \( L = \mathbb{Q}_p(\sqrt{2}) \), \( L = \mathbb{Q}_p(\sqrt{6}) \), \( L = \mathbb{Q}_p(\sqrt{10}) \) or \( L = \mathbb{Q}_p(\sqrt{14}) \).

And for any pair of quadratic field extensions \( (L,L') \) of \( \mathbb{Q}_p \) having discriminants \( \Delta(L) \) and \( \Delta(L') \) we have:

- \( \mu(L,L') = \mu(L',L) \) (Theorem 3.10 A (iii) of [HPS89a]);
- If \( \Delta(L) = \Delta(L') \) (which is the case if \( L \simeq L' \)) then \( \mu(L,L') = \infty \);
- If \( t(L) = -1 \) and \( t(L') \neq \Delta(L') \) then \( \mu(L,L') = 1 \);
- If \( t(L) = 0 \) and \( t(L') = 0 \) and \( \Delta(L) \neq \Delta(L') \) then \( \mu(L,L') = 2 \);
- If \( t(L) = 1 \) and \( t(L') = 1 \) and \( \Delta(L) \neq \Delta(L') \) then \( \mu(L,L') = 3 \);
- If \( t(L) = 1 \) and \( t(L') = 2 \) then \( \mu(L,L') = 3 \).
If $t(L) = 2$, $t(L') = 2$ and $\Delta(L) \neq \Delta(L')$ then $\mu(L, L') = 5$.

The criteria for the existence of optimal embeddings then reads as follows:

(1) $p$ odd:

(a) $n = 2\varrho + 1$ odd, $K$ unramified, $L$ unramified: if $m \leq \varrho$. In particular, if $R_{n}(L)$ is maximal and $O_{n}$ is not maximal (i.e. $m > 0$ and $n = 0$) then $h(m, n, L) = 0$.

(b) $n = 2\varrho + 1$ odd, $K$ ramified, $L$ unramified: if and only if $m = \varrho$. In particular, if $R_{n}(L)$ is maximal and $O_{n}$ is not maximal (i.e. $m > 0$ and $n = 0$) then $h(m, n, L) = 0$.

(c) $n = 2\varrho$ even. $K$ unramified, $L$ ramified: if and only if $m = \varrho$.

(d) $n = 2\varrho$ even. $K$ ramified, $L$ ramified and $K \nmid L$: if and only if $m = \varrho - 1$.

(e) $n = 2\varrho$ even, $K$ ramified, $L$ ramified and $K \nmid L$: if and only if $m < \varrho - 1$.

(2) $p = 2$:

(a) $n = 1$, $K$ ramified or unramified, $L$ unramified: if and only if $m = 0$.

(b) $n = 2\varrho$, $K$ unramified, $L = \mathbb{Q}_{2}(\sqrt{3})$ or $L = \mathbb{Q}_{2}(\sqrt{7})$: if and only if $m = \varrho$.

(c) $n = 2\varrho$, $K = \mathbb{Q}_{2}(\sqrt{3})$ or $K = \mathbb{Q}_{2}(\sqrt{7})$, $L = \mathbb{Q}_{2}(\sqrt{3})$ or $L = \mathbb{Q}_{2}(\sqrt{7})$ and $K \nmid L$: if and only if $m = \varrho$.

(d) $n = 2\varrho$, $K = \mathbb{Q}_{2}(\sqrt{3})$ or $K = \mathbb{Q}_{2}(\sqrt{7})$ and $K \nmid L$: if and only if $m = \varrho$.

(e) $n < 2\varrho$, $K = \mathbb{Q}_{2}(\sqrt{3})$ or $K = \mathbb{Q}_{2}(\sqrt{7})$, $L = \mathbb{Q}_{2}(\sqrt{3})$ or $L = \mathbb{Q}_{2}(\sqrt{7})$: this is the case if $t(L) = 1$, $t(K) = 1$ and $\Delta(L) \neq \Delta(K)$ and if and only if $m = 1$.

(f) $n = 2\varrho + 1$ odd, $n \geq 3$, $K$ unramified and $L$ unramified: if and only if $m = \varrho$.

(g) $n = 2\varrho + 1$ odd, $n \geq 3$, $K$ ramified and $L$ unramified: if and only if $m = \varrho$.

(h) $n = 2\varrho$, $K$ unramified and $L$ ramified. Then $t(L) = 1$, $t(K) = 1$, and if and only if $m = \varrho$.

(i) $n = 2\varrho$, $K = \mathbb{Q}_{2}(\sqrt{3})$ or $K = \mathbb{Q}_{2}(\sqrt{7})$, $L = \mathbb{Q}_{2}(\sqrt{2})$, $L = \mathbb{Q}_{2}(\sqrt{6})$, $L = \mathbb{Q}_{2}(\sqrt{10})$ or $L = \mathbb{Q}_{2}(\sqrt{14})$. Then $t(L) = 2$, $t(K) = 3$, $h(m, n, L) = 0$ and if and only if $m = \varrho - 1$.

(j) $n = 2\varrho$, $K = \mathbb{Q}_{2}(\sqrt{3})$, $K = \mathbb{Q}_{2}(\sqrt{7})$, $K = \mathbb{Q}_{2}(\sqrt{10})$ or $K = \mathbb{Q}_{2}(\sqrt{14})$, $L = \mathbb{Q}_{2}(\sqrt{2})$, $L = \mathbb{Q}_{2}(\sqrt{6})$, $L = \mathbb{Q}_{2}(\sqrt{10})$ or $L = \mathbb{Q}_{2}(\sqrt{14})$ and $K \nmid L$. Then $t(L) = 2$, $t(K) = 2$, $\mu(L, K) = 5$, $h(m, n, L) = 0$ and if and only if $m = \varrho - 1$ or $m = \varrho - 2$.

(k) $n = 2\varrho$, $K = \mathbb{Q}_{2}(\sqrt{3})$, $K = \mathbb{Q}_{2}(\sqrt{7})$, $K = \mathbb{Q}_{2}(\sqrt{10})$ or $K = \mathbb{Q}_{2}(\sqrt{14})$, and $K \nmid L$. Then $t(L) = 2$, $t(K) = 2$, $\mu(L, K) = 5$, $h(m, n, L) = 0$ and if and only if $m = \varrho - 1$.

3.3. Heegner points. Let $U$ be any open compact subgroup of $\hat{B}^\times$, and assume that $K$ is an imaginary quadratic field. There is a natural map

$$\hat{B}^\times \times \text{Hom}(K, B) \rightarrow \left( U \setminus \hat{B}^\times \times \text{Hom}(\mathbb{C}, M_{2}(\mathbb{R})) \right) / B^\times = X_{\mathbb{R}}(\mathbb{C})$$

obtained by extending scalars (i.e., tensoring with $\mathbb{R}$). Notice that the left-hand side can certainly be the empty set, as Hom$(K, B)$ is empty if $K$ does not embed into $B$. We shall assume that this is not the case in the discussion below. If $(g, f) \in \hat{B}^\times \times \text{Hom}(K, B)$, write $[g, f]$ for its image in $X_{\mathbb{R}}(\mathbb{C})$. Points in the image of this map are called Heegner points; the set of such Heegner points is denoted Heeg$(U, K)$.

For each positive integer $c$, continue to denote by $R_{c}$ the order of conductor $c$ in $K$ and let $\mathcal{R}$ be an order of $B$.

**Definition 3.7.** A point $x \in X_{\mathbb{R}}$ is called a Heegner point of conductor $c$ associated to $K$ if $x = [g, f]$ for some pair $(g, f) \in \hat{B}^\times \times \text{Hom}(K, B)$ such that

$$f(K) \cap g^{-1}R_{c}g = f(R_{c}).$$

This last condition means that $f$ is an optimal embedding of $R_{c}$ into the order $g^{-1}R_{c}g \cap B$. We shall denote by Heeg$(\mathcal{R}, K, c)$ the set of Heegner points of conductor $c$ associated to $K$ in $X_{\mathbb{R}}$.

Recall that the set of geometrically connected components of the Shimura curve $X_{\mathbb{R}}$ is in bijection with $\mathcal{R}^\times \setminus B^\times / B^\times$, and hence with the class group Pic($\mathcal{R}$) of $\mathcal{R}$. In particular, the number of geometric
connected components coincides with the class number $h(R)$. Fix representatives $I_j$ for the distinct $h(R)$ ideal classes in $\text{Pic}(R)$, and let $a_j \in \hat{B}^\times$ be the corresponding representatives in $\hat{R}^\times \setminus \hat{B}^\times / B^\times$. It is then clear that every Heegner point in $\text{Heeg}(R, K, c)$ can be represented by a pair of the form $(a_j, f)$, for a unique $j \in \{1, \ldots, h(R)\}$ and some optimal embedding $f$ from $R_c$ into the order $a_j^{-1}Ra_j \cap B$. Further, two pairs $(a_j, f)$ and $(a_j, g)$ represent the same Heegner point if and only if the embeddings $f$ and $g$ are $\hat{R}^\times$-conjugate. Therefore, we have the following identity relating Heegner points on $X_R$ attached to $R_c$ and optimal embeddings of $R_c$ into $\hat{R}$:

$$|\text{Heeg}(R, K, c)| = h(R) |\text{Emb}^{op}(R_c, \mathbb{R})| = h(R) v(R_c, R),$$

thus applying (5) we find:

**Proposition 3.8.** The number of Heegner points on $X_R$ attached to $R_c$ is $h(R_c) \prod v(R_c, R)$.

3.4. Galois action and fields of rationality. Keep the same notations as above, and assume that $R_c$ embeds optimally in $\hat{R}$, so that Heegner points with respect to $R_c$ do exist on $X_R$. The reciprocity law, cf. [Del71, 3.9], [Mil90, II.5.1] (with a sign corrected [Mil92, 1.10]), asserts that CM($\hat{R}^\times$) is the ring class field of conductor $c$, asse rts that CM($\hat{R}^\times$) on CM($\hat{R}^\times$) $K$ into $\hat{R}^-\text{conjugate}$. Therefore, we have the following identity relating Heegner points on $X_R$.

**Proposition 3.9.** With notations as above, $\text{Heeg}(R, K, c) \subseteq X_R(H_c)$, and the action of $\text{Gal}(H_c/K)$ on the set of Heegner points $\text{Heeg}(R, K, c)$ is described by the rule in (4).

4. Applications

4.1. Gross–Zagier formula. We briefly review the general form of Gross–Zagier formula in [YZZ13] for modular abelian varieties. Let $B/\mathbb{Q}$ be an indefinite quaternion algebra of discriminant $\Delta$. If $U_1 \subseteq U_2$ are open compact subgroups of $\hat{B}^\times$, then we have a canonical projection map $\pi_{U_1, U_2} : X_{U_1} \twoheadrightarrow X_{U_2}$, and one may consider the projective limit

$$X = \lim_{\supseteq U} X_U,$$

and let $J := \text{Jac}(X)$ denote the Jacobian variety of $X$.

**Definition 4.1.** A simple abelian variety $A/\mathbb{Q}$ is said to be uniformized by $X$ if there exists a surjective morphism $J \twoheadrightarrow A$ defined over $\mathbb{Q}$.

Let $A/\mathbb{Q}$ be a simple abelian variety uniformized by $X$ and fix $U$ such that there is a surjective morphism $J_U := \text{Jac}(X_U) \twoheadrightarrow A$ defined over $\mathbb{Q}$. Let $\xi_U$ be the normalized Hodge class in $X_U$ and define

$$\pi_A := \lim_{\supseteq U} \text{Hom}_U^0(X_U, A),$$

where $\text{Hom}_U^0(X_U, A)$ denotes morphisms of $\text{Hom}(X_U, A) \otimes \mathbb{Q}$ defined by using the Hodge class $\xi_U$ as a base point. Since, by the universal property of Jacobians, every morphism $X_U \twoheadrightarrow A$ factors through $J_U$, we also have

$$\pi_A := \lim_{\supseteq U} \text{Hom}_U^0(J_U, A),$$
where $\text{Hom}^0_{\mathcal{O}_v}(J_U, A) := \text{Hom}(J_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$. For any $\varphi \in \pi_A$ and any point $P \in X_U(H)$, where $H/\mathbb{Q}$ is a field extension, we then see that $P(\varphi) := \varphi(P) \in A(H)$.

Let $K/\mathbb{Q}$ be an imaginary quadratic field and assume there exists an embedding $\psi : K \hookrightarrow B$; this is equivalent to say that all primes dividing $\Delta$ are inert or ramified in $A$. Define $X^{K^{*}}$ to be the subscheme of $X$, defined over $\mathbb{Q}$, consisting of fixed points under the canonical action by left translation of $\psi : K^{*} \hookrightarrow B^{*}$. The subscheme $X^{K^{*}}$ is independent up to translation of the choice of $\psi$. We will often omit the reference to $\psi$, viewing $K$ simply as a subfield of $B$. Recall that the theory of complex multiplication shows that every point in $X^{K^{*}}(\mathbb{Q})$ is defined over $K^{ab}$, the maximal abelian extension of $K$, and that the Galois action is given by the left translation under the reciprocity map. Fix a point $P \in X^{K^{*}}(K^{ab})$. This amounts to choose a point $P_U$ for all open compact subgroups $U$, satisfying the condition that $\pi_{U_1, U_2}(P_{U_1}) = P_{U_2}$.

Let $dr$ denote the Haar measure of $\text{Gal}(K^{ab}/K)$ of total mass equal to 1 and fix a finite order character $\chi : \text{Gal}(K^{ab}/K) \to \mathbb{F}_{\chi}$, where $F_{\chi} = \mathbb{Q}(\chi)$ is the finite field extension of $\mathbb{Q}$ generated by the values of $\chi$. Define

$$P_{\chi}(\varphi) := \int_{\text{Gal}(K^{ab}/K)} \varphi(P^\sigma) \otimes \chi(\sigma) d\tau.$$ 

This is an element in $A(K^{ab}) \otimes M F_{\chi}$, where $M = \text{End}_{\mathbb{Q}}(A) := \text{End}_{\mathbb{Q}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. This element can be essentially written as a finite sum: suppose that $P = (P_U)_U$, and each $P_U$ is defined over the abelian extension $H_U$ of $K$. Suppose that $\chi$ factors through $\text{Gal}(H_U/K)$ for some $U$. Then the $F_{\chi}$-subspace of $A(H_U) \otimes F_{\chi}$ spanned by $P_{\chi}(\varphi)$ and

$$\sum_{\sigma \in \text{Gal}(H_U/K)} \varphi(P)^\sigma \otimes \chi(\sigma)$$

are the same. We also note that $P_{\chi}(\varphi)$ belongs to $(A(H_U) \otimes \mathbb{Z}) \chi$.

Let $\eta_K$ be the quadratic character of the extension $K/\mathbb{Q}$. Suppose that $\chi$ satisfies the self-duality condition $\omega_A \cdot \chi|_{\mathbb{A}_K^{*}} = 1$, where $(\cdot)|_{\mathbb{A}_K^{*}}$ means restriction of the character $(\cdot)$ to the idele group $\mathbb{A}_K^{*}$ and $\omega_A$ is the central character of the automorphic representation $\pi_A$. We assume for simplicity that $\omega_A$ is trivial, and therefore $\chi|_{\mathbb{A}_K^{*}} = 1$. For any place $v$ of $\mathbb{Q}$, let $\varepsilon(1/2, \pi_{A,v}, \chi_v) \in \{ \pm 1 \}$ be the sign of the functional equation with respect to its center of symmetry $s = 1/2$ of the local representation $\pi_{A,v} \otimes \chi_v$. Define the set

$$\Sigma(A, \chi) = \{ v \text{ place of } \mathbb{Q} : \varepsilon(1/2, \pi_{A,v}, \chi_v) \neq \eta_K, v(-1) \}.$$

**Proposition 4.2.** The real place $\infty$ belongs to the set $\Sigma(A, \chi)$, and every finite prime $p \in \Sigma(A, \chi)$ divides the conductor of $A$.

**Proof.** According to [CV07] Section 1, the real place $\infty$ belongs to the set $\Sigma(A, \chi)$ if $\chi_{\infty} = 1$ and $\pi_{A,\infty}$ is the holomorphic discrete series (of weight at least 2). The first condition is true by our assumptions, while the second one holds because $\pi_A$ is the automorphic representation attached to an abelian variety. On the other hand, also from loc. cit. we know that if $p$ is a finite prime in the set $\Sigma(A, \chi)$, then $\pi_{A,p}$ is either special or supercuspidal, and therefore $p$ must divide the conductor of $A$. \hfill $\Box$

**Remark 4.3.** If $p$ is a finite prime belonging to $\Sigma(A, \chi)$, one also knows that $K_p := K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a field. In particular, if $B$ is an indefinite quaternion algebra whose ramification set is supported in $\Sigma(A, \chi)$, then $K$ splits $B$ (i.e. $K$ embeds as a maximal subfield of $B$).

Let $\varepsilon(1/2, \pi_{A,v}, \chi_v)$ be the sign of the functional equation with respect to its center of symmetry $s = 1/2$ of the global representation $\pi_A \otimes \chi$. Then

$$\varepsilon(1/2, \pi_{A,v}, \chi_v) = (-1)^{|\Sigma(A, \chi)|}.$$

Recall our assumption that the central character $\omega_A$ of $\pi_A$ is trivial, and let now $\chi$ be a character of $\text{Gal}(K^{ab}/K)$. Suppose that $\chi$ factors through $\text{Gal}(H_{c}/K)$ where $H_{c}$ is the ring class field of conductor $c$; if there is no $c' \mid c$ such that $\chi$ factors through $\text{Gal}(H_{c'}/K)$, we say that $\chi$ has conductor $c$; if $\chi$ factors through $\text{Gal}(H_{c}/K)$, then the conductor of $\chi$ divides $c$.

Suppose we have a character $\chi$ of conductor dividing the positive integer $c$ and a Heegner point $P_c$ of conductor $c$ in $X(H_c)$. For any $\varphi \in \pi_A$ define

$$P_{\chi, c, \varphi} := \sum_{\sigma \in \text{Gal}(H_{c}/K)} \varphi(P_c^\sigma) \otimes \chi(\sigma).$$
If $|\Sigma(A, \chi)|$ is odd, by [Y4/Z18] Theorem 1.3.1] one can choose $\varphi$ such that
\begin{equation}
\left(\text{7}\right) \quad P_c^\chi \neq 0 \text{ in } (A(H_c) \otimes \mathbb{Z} \mathbb{C})^\chi \iff L'(\pi_A, \chi, 1/2) \neq 0.
\end{equation}
From now on, we fix such a $\varphi$ and write simply $P_c^\chi$ for $P_c^\chi_{\varphi}$.

4.2. Euler systems and BSD conjecture. Before discussing our applications to the BSD conjecture, we recall the following result, which in this general form is due to Nekovár [Nek07].

**Theorem 4.4** (Nekovár). Suppose that $A/\mathbb{Q}$ is a modular abelian variety of dimension $d$. Fix an imaginary quadratic field $K$ and an anticyclotomic character $\chi$ factoring through $H_c$ for some integer $c \geq 1$ such that the cardinality of $\Sigma(A, \chi)$ is odd. Let $B$ be the indefinite quaternion algebra of discriminant equal to the product of finite primes in $\Sigma(A, \chi)$. Assume that $A$ does not acquire CM over any imaginary quadratic field contained in $H_c$, and that there exists

1. an order $R$ of $B$ with an uniformization $J_R = \text{Jac}(X_R) \to A$ defined over $\mathbb{Q}$, and
2. a Heegner point $P_c \in X_K(H_c)$.

Then the following implication holds:

$$L'(\pi_A, \chi, 1/2) \neq 0 \implies \dim_{\mathbb{C}} (A(H_c) \otimes \mathbb{Z} \mathbb{C})^\chi = d.$$ 

We first observe that if the ramification set of the quaternion algebra $B$ coincides with $\Sigma(A, \chi) - \{\infty\}$, then there always exists a uniformization $J_U = \text{Jac}(X_U) \to A$ for some open compact subgroup $U$ of $B^\times$; so in (1) we are asking that this $U$ is associated with an order $R$.

**Theorem 4.5.** Fix the following objects:

1. a modular abelian variety $A/\mathbb{Q}$ of dimension $d$ and conductor $N^d$,
2. an imaginary quadratic field $K$ and
3. an anticyclotomic character $\chi$ factoring through the ring class field $H_c$ of $K$ of conductor $c \geq 1$ such that the cardinality of $\Sigma(A, \chi)$ is odd.

Let $B$ denote the indefinite quaternion algebra of discriminant $\Delta$ equal to the product of all the finite primes in $\Sigma(A, \chi)$. Then there exists an order $R$ of type $T = (N_{\text{Eic}}; N_{\text{Car}}; \{(L_p, \nu_p')\}_{p|\Delta})$ in $B$ and a Heegner point $P_c' \in X_R(H_c)$ with $c | c'$ such that:

1. $A$ is uniformized by $X_R$, hence there is a surjective morphism $J_R \to A$ defined over $\mathbb{Q}$;
2. $N$ divides the level $N_{\text{Eic}} \cdot N_{\text{Car}}^2 \cdot \prod_{p|\Delta} p^{\nu_p}$ of $R$;
3. $c$ divides $c'$.

**Proof.** The problem is local, being equivalent to the existence of optimal local embeddings for all primes $\ell$. Fix the order $R_{\text{min}}$ of type $T_{\text{min}} = (N_{\text{Eic}}; N_{\text{Car}}; \{(L_p, \nu_p')\}_{p|\Delta})$ and level $N_{\text{min}} = N_{\text{Eic}} N_{\text{Car}}^2 \cdot \Delta$ as in the proof of Proposition 2.2.3, choosing the integers $N_{\text{Eic}}$ and $N_{\text{Car}}$ such that $N_{\text{Car}}$ is divisible only by primes $p$ which are inert in $K$ and do not divide $c$.

For primes $p | N_{\text{Eic}} N_{\text{Car}}$ which are split in $K$, one knows that the set of local optimal embeddings of the required form is non-empty (cf. Lemma 3.4).

Fix until the end of the proof a prime $p | N_{\text{min}}$, which is inert or ramified in $K$. Let $m$ be the $p$-adic valuation of $c$ and set $n := \text{val}_p(N_{\text{min}})$. If $p$ divides $N_{\text{Car}}$, then we can apply Lemma 3.4 and show that the maximal order $R_c \otimes \mathbb{Z} \mathbb{Z}_p$ embeds optimally into $R_{\text{min}} \otimes \mathbb{Z}_p$. So suppose from now on that $p$ does not divide $N_{\text{Car}}$.

Suppose first that $p \notin \Sigma(A, \chi)$. If $m \geq n/2$ (unramified case) or $m \geq (n - 1)/2$ (ramified case) then Lemma 3.5 shows that the set of local optimal embeddings of $R_c \otimes \mathbb{Z}_p$ into $R_{\text{min}} \otimes \mathbb{Z}_p$ is non-empty. If these conditions do not hold, replacing $m$ by $m'$ such that $m' \geq n/2$ (unramified case) or $m' \geq (n - 1)/2$ (ramified case) then the local order $R_c \otimes \mathbb{Z}_p$ with $c' = c \cdot p^{m'-m}$ embeds optimally into $R_{\text{min}} \otimes \mathbb{Z}_p$.

Suppose now that $p \in \Sigma(A, \chi)$. Take any pair $(m', n')$ satisfying the following condition:

- If $n$ is odd, then $n' = 2m' + 1$.
- If $n$ is even, then $n' = 2m'$ if $p$ is inert in $K$ whereas $n' = 2(m' + 1)$ if $p$ ramifies in $K$.

Choose also the pair $(m', n')$ so that $m' \geq m$ and $n' \geq n$. Comparing with the results recalled in [3.2.3] we see that the set of optimal embeddings of the local quadratic order of conductor $p^{n'}$ into the local quaternion order $R_{n'}(L_p) \subseteq R_{\text{min}} \otimes \mathbb{Z}_p$ of type $(L_p, n')$ is non-zero.

**Corollary 4.6.** Let $A/\mathbb{Q}$, $K$ and $\chi$ be as in the previous theorem. If $A$ does not acquire CM over any imaginary quadratic field contained in $H_c$ and $L'(\pi_A, \chi, 1/2) \neq 0$, then $\dim_{\mathbb{C}} (A(H_c) \otimes \mathbb{Z} \mathbb{C})^\chi = d$. 


Proof. Let $R$ and $c'$ be as in the statement of Theorem 4.3 and apply Theorem 4.3 viewing $\chi$ as a character of $\text{Gal}(H_c/K)$ via the canonical projection $\text{Gal}(H_c/K) \to \text{Gal}(H_c/K)$.

\section{Proof of Theorem A.} Theorems 4.4 and 4.5, although giving an Euler System which is sufficient for the proof of our main result in the introduction, are not completely satisfying in the sense that they are not effective in the computation of the order $R$. Suppose we are in the situation of the theorem, so that we are given a modular abelian variety $A/Q$, an imaginary quadratic field $K$ and an anticyclotomic character $\chi$ of conductor $c$ such that $\Sigma(A, \chi)$ has odd cardinality. Then we would like to have Heegner points in $R_{\text{min}}$ for a choice of minimal parametrization $J_{R_{\text{min}}} \to A$ described in Proposition 2.4 or at least of level $R$ with $R \subseteq R_{\text{min}}$. We begin by discussing a couple of examples.

Example 4.7. Let $A = E$ be an elliptic curve of conductor $N = p^2q$, with $q$ and $p$ odd distinct primes both inert in $K$. Assume that the automorphic representation $\pi_E$ attached to $E$ is supercuspidal at $p$. Let $B$ be the quaternion algebra of discriminant $pq$, $R = R_{\text{min}}$ be the Hijikata–Pizer–Schemanske order $R = R_{\text{min}}$ of level $N = p^2q$ (and type $1; 1; \{(L_p, 2), (L_q, 1)\}$), for any choice of quadratic ramified extension $L_p/Q_p$, and let $X_R$ be its associated Shimura curve. Note that $K$ factors into $B$ because both $p$ and $q$ are inert in $K$.

Consider first the case of the trivial character $1$. Then $\varepsilon_p(E/K, 1) = 1_p(-1) = +1$ (see (5.1.5)), and therefore $p \notin \Sigma(E, 1)$. So, if the sign of the functional equation of $E/K$ is $-1$ then $\Sigma(E, 1) = \{\infty\}$ (because any prime in $\Sigma(E, 1)$ must divide $N$). Note also that, even if $f$ admits a JL lift to $X_R$, there are no Heegner points of conductor $1$ in this curve (cf. (3.2.3) case (1c)).

For a non-trivial character $\chi$ of conductor $p$, again by case (1c) in (3.2.3) we see that in this case Heegner points of conductor $c = p$ do exist in $X_R$. Therefore, if $\Sigma(E, \chi)$ is odd, then the set of Heegner points of conductor $c = p$ in $X_R$ is non-empty.

Finally, consider the case of non-trivial conductor $p^n$ with $m \geq 2$. By (1m83) p. 1299 we know that $\varepsilon_p(E/K, \chi) = +1$, and therefore, as in the case of the trivial character, if the sign of the functional equation of $E/K$ is $-1$ then $\Sigma(E, 1) = \{\infty\}$. Also note that, again as in the case of the trivial character, there are no Heegner points of conductor $p^n$ in $X_R$.

Example 4.8. As in the above example, let $A = E$ be an elliptic curve of conductor $N = p^2q$, with $q$ and $p$ odd distinct primes and suppose that $q$ is inert and $p$ is ramified in $K$. Identify the Weil–Deligne group $W_{Q_p}$ of $Q_p$ with $Q_p^\times$ via the reciprocity map $v_{Q_p}$, normalized in such a way that $v_{Q_p}(x)$ acts on $\mathbb{F}_{p^2}$ by the character $x \mapsto x^{q}$, where $|\cdot| = |\cdot|_{Q_p}$ is the $p$-adic absolute value satisfying $|p| = p^{-1}$. Assume that the automorphic representation $\pi_E$ attached to $E$ is supercuspidal at $p$, and write it as $\pi_E = \text{Ind}_{W_{K_p}}^{W_{Q_p}}(\psi)$ where $F/Q_p$ is a quadratic extension with associated character $\eta : W_{Q_p} \to \mathbb{C}^\times$ is a quasi-character factoring through the norm map; then we have $\eta \psi = |\cdot|^{-1}$ as quasi-characters of $Q_p^\times$. The above conditions force $\psi$ to have conductor equal to $1$, $p \equiv 3 \pmod{4}$ and $\psi_{|Z_p^\times} = \eta$ (see Cor. 3.1)).

Consider the quaternion algebra $B$ of discriminant $pq$, the Hijikata–Pizer–Schemanske order $R$ of level $p^2q$ as in the previous example and its associated Shimura curve $X_R$. Again $\pi_E$ admits a Jacquet–Langlands lift to $X_R$. Take a character $\chi$ of conductor $p^m$, $m \geq 1$. Then $\varepsilon_p(E/K, \chi) = 1$ if $m \geq 4$ by (1m83) p. 1299. If $p \equiv 3 \pmod{4}$, then $q_{K,p}(-1) = -1$ and therefore $p \not\in \Sigma(E, \chi)$. Assuming that $\Sigma(E, \chi)$ is odd, we see that $q \in \Sigma(E, \chi)$ too. However, there are no Heegner points of conductor $p^m$ with $m \geq 1$ in $X_R$.

The above examples motivate our discussion below, leading to the proof of (a slightly refined version of) Theorem A in the Introduction.

Fix for the rest of the article an elliptic curve $E/Q$ of conductor $N$, an imaginary quadratic field $K$ of discriminant $-D$ and a ring class character $\chi$ of conductor $c$ of $K$. Let $\Delta$ be the product of the finite primes in $\Sigma(A, \chi)$, which is assumed to have odd cardinality, and let $B$ be the quaternion algebra of discriminant $\Delta$. Fix also $R := R_{\text{min}}$ to be the minimal order of type $T_{\text{min}} = (N_{\text{Eic}}, N_{\text{Car}}; \{(L_p, v_p)\})$ as in Proposition 2.4, on which the Jacquet–Langlands lift to $B$ of the newform $f \in S_2(\Gamma_0(N))$ associated with $E$ is realized, and let $N_R = N_{\text{Eic}}N_{\text{Car}}^2\Delta$ be its level. We can further assume the (coprime) integers $N_{\text{Eic}}$ and $N_{\text{Car}}$ satisfy that, for every prime $p$ dividing $N_{\text{Eic}}N_{\text{Car}}$,

$$p \mid N_{\text{Car}} \text{ if and only if } p \text{ is inert in } K, \text{ val}_p(N) \text{ is even and } p \nmid c.$$

From now on, we shall make the following assumption on $N$. Observe that under the hypothesis that neither $2^3$ nor $3^4$ divide $N$, as in the statement of Theorem A, the assumption below is obviously satisfied.

Assumption 4.9. Let $f \in S_2(\Gamma_0(N))$ be the newform attached to the elliptic curve $E$ by modularity. With the above notations, the following holds:
(1) If $2^3 \mid N_{Eic}$, then either 2 splits in $K$ or $\text{val}_2(N_{Eic})$ is odd and 2 is inert in $K$.
(2) If $\Delta$ is even and $2^3 \mid N$, then 2 is inert in $K$, and if in addition $\pi_E$ is supercuspidal at 2 then $\text{val}_2(N)$ is odd.
(3) The 2-component $\pi_{f,2}$ of the automorphic representation attached to $f$ has minimal Artin conductor among its twists by quasi-characters of $Q^*_2$.
(4) If $\text{val}_3(N_{Eic}) = 4$ and 3 is inert in $K$, then $\text{val}_3(c) \neq 1$.

Remark 4.10. If $f \in S_2(\Gamma_0(N))$ and $\text{val}_2(N)$ is even and greater or equal than 4, then $f$ is a twist of a modular form of lower level by [HPSS94, Thm. 3.9]. Conditions (1) and (2) in the above assumption rule out these cases; in fact, one expects to treat these cases via different methods. If $f = g \otimes \chi$, then one expects to construct points on the modular abelian varieties attached to $g$, and then, using twisting techniques, to construct points on the elliptic curve. It seems possible that condition (3) can be treated by similar considerations.

Write also $R_c$ for the order of conductor $c$ in $K$ of conductor $c$ as usual. Our goal now is to investigate under which conditions $R_c$ embeds optimally into $R_{min}$. And in those cases where this does not happen, we must find a suitable suborder of $R_{min}$ such that $R_c$ does optimally embed into it.

The problem is clearly local, and it suffices to study it at those primes dividing $N$. So fix from now on a prime $p \mid N$, and set the following notations. We write $m := \text{val}_p(c)$ for the $p$-adic valuation of $c$, and $n := \text{val}_p(N_K)$ for that of $N_K$. By the discussion prior to Proposition 2.7 observe that $n$ coincides with $\text{val}_p(N)$ if $p$ is odd (cf. also Remark 2.8). And if $p = 2$, under Assumption 1.9, the only cases with $n \neq \text{val}_2(N)$, and for which this inequality is relevant in our discussion, are those in Lemma 4.11 below. In view of this, except for Lemma 4.11 we may use that $n = \text{val}_p(N)$ without further explicit mention. Then we denote by $\mathcal{E}_p(m,n)$ the set of (local) optimal embeddings of $R_c \otimes Z_p$ into $R \otimes Z_p$. Recall that the conditions that characterize the non-emptiness of $\mathcal{E}_p(m,n)$, in each of the possible cases, have been collected in Section 2.

If $p$ is not split in $K$, then we write $\chi_p$ for the component of $\chi$ at the unique prime of $K$ above $p$. In that case, notice that $\chi = \chi_p$.

If the prime $p$ divides $N_{Eic}\Delta$ and $n' \geq n$ is an integer, then we define a suborder $R'$ of $R$ in the following way. If $p \mid N_{Eic}$, then we define $R'$ to be locally equal to $R$ at every prime distinct from $p$, and that $R' \otimes Z_p$ is the Eichler suborder of level $p^{n'}$ (hence index $p^{n' - n}$) in $R \otimes Z_p$. In particular, $\text{val}_p(N_{Eic}) = n'$. And if $p \mid \Delta$, then $R'$ is obtained from $R$ by replacing the local data $(L_p, n)$ at $p$ in the type of $R$ by the data $(L_p, n')$. Besides, given an integer $m' \geq m$, we denote by $R_c'$ the suborder of $R_c$ of conductor $c' = cp^{m' - m}$. Finally, given two integers $m' \geq m$ and $n' \geq n$, we will simply denote by $\mathcal{E}_p(m,n')$ the set of (local) optimal embeddings of $R_c' \otimes Z_p$ into $R' \otimes Z_p$.

First we consider the case where $p$ does not belong to the set $\Sigma(E, \chi)$, so that $B$ is split at $p$.

Lemma 4.11. If $p \notin \Sigma(E, \chi)$, then $\mathcal{E}_p(m,n) \neq \emptyset$.

Proof. First observe that if $p \notin \Sigma(E, \chi)$ then $p$ divides $N_{Eic}N_{Car}$. Having said this, notice that if $p \mid N_{Car}$ then $\mathcal{E}_p(m,n)$ is non-empty by Lemma 3.6 (because if $p \mid N_{Car}$ then $m = 0$). So we assume for the rest of the proof that $p$ divides $N_{Eic}$. By our choice of $N_{Eic}$ and $N_{Car}^2$, we shall distinguish three cases:

1. $p$ is split in $K$;
2. $p$ is inert or ramified in $K$ and $n$ is odd;
3. $p$ is inert or ramified in $K$, $n$ is even and $p \mid c$.

If $p$ is in case (1), then Lemma 3.4 implies that $\mathcal{E}_p(m,n)$ is non-empty. Suppose that $p$ is in case (2), and assume first that $p \mid \Delta$, which is the only possible value if $p \geq 5$. If $p$ ramifies in $K$, then $\mathcal{E}_p(m,n) \neq \emptyset$ by part (ii) in Lemma 3.5. If $p$ is inert in $K$, then we split the discussion according to whether $p \mid c$ or $p \mid c'$. In the former case, $\mathcal{E}_p(E/K, \chi) = \mathcal{E}_p(E/K, 1) = -1$ but $\eta_{K,p}(-1) = 1$, thus $p$ should be in $\Sigma(E, \chi)$, contradicting our hypotheses. And in the latter case, we have $m \geq 1$ and therefore $2m \geq n = 1$, hence Lemma 5.7 shows that $\mathcal{E}_p(m,n) \neq \emptyset$. Thus we are left with the cases where $p = 2$ or 3 and $n = \text{val}_p(N) > 1$ is odd.

- If $p = 3$, then $n$ can be either 3 or 5. Then $\pi_{E,3}$ is supercuspidal induced from a quasicharacter $\psi$ of conductor $n - 1$ of a ramified quadratic extension $F_3$ of $Q_3$. If 3 is inert in $K$, we know on the one hand by Lemma 5.7 that $\mathcal{E}_3(m,n) \neq \emptyset$ if and only if $m \geq n/2$, hence if and only if $m > (n - 1)/2$. On the other hand, being 3 inert in $K$ the assumption that $3 \notin \Sigma(E, \chi)$ tells us that $\mathcal{E}_3(E/K, \chi) = 1$, and by [Tun83, Prop. 2.8] this holds if and only if $m > (n - 1)/2 = 1$. Thus it follows that $\mathcal{E}_3(m,n) \neq \emptyset$. Now suppose that 3 ramifies in $K$. Then $\eta_{K,3}(-1) = -1$, hence
Lemma 4.12. Let $E/K,\chi$ be the automorphic representation attached to $E \to \mathbb{Q}$ with conductor $2$ and discriminant valuation $2$. Then $\varphi(\psi_{\chi})$ is supercuspidal induced from a quasicharacter of conductor $2$ of $\mathbb{Q}_2$. If $\eta_{K,2}(-1) = -1$, then $\varepsilon_3(E/K,\chi) = -1$. By Assumption 4.9 i), the case $\varepsilon_3(E/K,\chi) = -1$ implies that $m \geq (n-1)/2$. We next show that $\varepsilon_3(E/K,\chi) = -1$ implies that $m \geq (n-1)/2$. If $F_3 = K_3$, [Tun83] Cor. 1.9.1] implies that $\varepsilon_3(E/K,\chi) = -1$. If $m = c(\chi_3) < c(\psi) = n-1$, we would have $c(\psi_{\chi_3}) = c(\psi) = n-1$. Since $n$ is odd, in both cases we would have $\varepsilon_3(E/K,\chi) = 1$, thus it must be $m > n-1 > (n-1)/2$ and therefore $\mathcal{E}_3(m,n) \neq \emptyset$. If in contrast $F_3 = K_3$, we can apply [Tun83] Prop. 1.10] instead, which tells us that $\varepsilon_3(E/K,\chi) = -1$ is the only possible case with $E$ that if $m = c(\chi_3) < c(\psi) = n-1$, we would have $c(\psi_{\chi_3}) = c(\psi) = n-1$. Since $n$ is odd, in both cases we would have $\varepsilon_3(E/K,\chi) = 1$, thus it must be $m > n-1 > (n-1)/2$, and we conclude that $\mathcal{E}_3(m,n) \neq \emptyset$.

If $p = 2$, then $n$ can be either $3$, $5$ or $7$. And by Assumption 4.9 i), we may suppose that $2$ is inert in $K$, so that $2 \notin \Sigma(E,\chi)$ implies that $\varepsilon_2(E/K,\chi) = 1$.

Suppose first that $n = 3$. Then [Tun83] Prop. 3.7] implies that $m \geq 2$, and then by part (i) of Lemma 3.3 we deduce that $\mathcal{E}_2(m,n) \neq \emptyset$.

Suppose now that $n = 5$. Now $\pi_{E,2}$ is supercuspidal induced from a quasicharacter of conductor $3$ on a ramified extension of $\mathbb{Q}_2$ with discriminant valuation $2$. If the conductor of $\chi$ were $m < 3$, then [Tun83] Cor. 1.9.1] would imply that $\varepsilon_2(E/K,\chi) = -1$, thus we deduce that $m \geq 3$. And then by part (i) of Lemma 3.3 we conclude that $\mathcal{E}_2(m,n) \neq \emptyset$.

If $n = 7$, then $\pi_{E,2}$ is supercuspidal of exceptional type, and its conductor is minimal with respect to twist. Then $\varepsilon_2(E/K,\chi) = 1$ implies, by [Tun83] Lemma 3.2], that $m \geq 4$. But then we deduce that $\mathcal{E}_2(m,n) \neq \emptyset$ thanks to Lemma 3.3 part (i) (here, and here only, we use condition iii) in Assumption 4.9).

Finally, suppose that $p$ is in case (3). Again let us start with the case $n = 2$, which is the only possible case if $p \geq 5$. Since $m = \text{val}_p(c) \geq 1$, we see that $2m \geq n$, hence Lemma 3.5 implies that $\mathcal{E}_p(m,n) \neq \emptyset$, regardless $p$ is inert or ramified in $K$. We are again left with the cases where $p = 2$ or $3$ and $n = \text{val}_p(N) > 2$ is even.

By Assumption 4.9 i), the case $p = 2$ does not arise, so we assume that $p = 3$. Then the only possible case is $n = 4$. In this case $\pi_{E,3}$ is supercuspidal induced from a quasicharacter of conductor $2$ of the unramified quadratic extension of $\mathbb{Q}_2$. If $3$ is ramified in $K$, then $\eta_{K,2}(-1) = -1$, and therefore $\varepsilon_3(E/K,\chi) = -1$. By [Tun83] Cor. 1.9.1], if $m = 1$ then we would have $\varepsilon_3(E/K,\chi) = 1$, hence we see that $m \geq 2$. On the other hand, part (ii) in Lemma 3.5] tells us that $\mathcal{E}_3(m,n) \neq \emptyset$ if and only if $m \geq 2$. Thus we deduce that indeed $\mathcal{E}_3(m,n) \neq \emptyset$.

If in contrast $3$ is inert, then Assumption 4.9 iii) implies that $m \geq 2$, and by part (i) in Lemma 3.3 we conclude that $\mathcal{E}_3(m,n) \neq \emptyset$.

Next we will deal with the case that $p \in \Sigma(E,\chi)$, or equivalently $p \mid \Delta$. This means that $\varepsilon_p(E/K,\chi) = -\eta_{K,p}(-1)$. So if $p$ is odd, then

$$\varepsilon_p(E/K,\chi) = \begin{cases} -1 & \text{if } p \text{ is inert in } K, \\ -1 & \text{if } p \text{ is ramified in } K \text{ and } p \equiv 1 \mod 4, \\ 1 & \text{if } p \text{ is ramified in } K \text{ and } p \equiv 3 \mod 4. \end{cases}$$

Let $\pi_E$ be the automorphic representation attached to $E$, and $\pi_{E,p}$ be its $p$-th component. The (exponent of the) conductor of $\pi_{E,p}$ is $\text{val}_p(N)$.

We will split our discussion into distinct lemmas, to distinguish between the cases $p \text{ is supercuspidal or Steinberg. If } \pi_{E,p} \text{ is supercuspidal, then it is well-known that } \text{val}_p(N) \geq 2$. For $p \geq 5$ this means that $\text{val}_p(N) = 2$, whereas for $p = 3$ (resp. $p = 2$) we have $2 \leq \text{val}_p(N) \leq 5$ (resp. $2 \leq \text{val}_2(N) \leq 8$). Besides, if $\pi_{E,p}$ is Steinberg, then $\text{val}_p(N)$ can only be $1$ or $2$ if $p$ is odd, whereas if $p = 2$ then $\text{val}_2(N) \in \{1, 4, 6\}$. However, the reader should keep in mind that under Assumption 4.9 some of the previous cases with $p = 2$ do not appear in our discussion.

Lemma 4.12. If $p \in \Sigma(E,\chi)$, $\pi_E$ is supercuspidal at $p$ and $p$ is inert in $K$ then there exists $n' \geq n$ such that $\mathcal{E}_p(m,n') \neq \emptyset$.

Proof. The assumptions $p \in \Sigma(E,\chi)$ and $p$ inert in $K$ imply that $\varepsilon_p(E/K,\chi) = -1$. Suppose first that $p$ is odd. We have the following cases:

1. $n = 2$. If $m = 0$, then $\varepsilon_p(E/K,\chi) = 1$ by [Del73] (5.5.1)], so we may assume that $m \geq 1$. But then defining $n' := 2m \geq n$ we conclude by case (1c) in [3.2.3] that $\mathcal{E}_p(m,n') \neq \emptyset$. 


(2) $p = 3$ and $n > 2$. In this case $n = \text{val}_p(N) = 2$. As above, if $m = 0$ then $\varepsilon_p(E/K, \chi) = 1$, hence it must be $n \geq 1$. Letting $n' := 2m + n > n$ we have $E_p(m, n') \neq \emptyset$.

Suppose now that $n = 4$. In this case, $\pi_{E,3}$ is induced from a quasicharacter $\psi$ of conductor $2$ of the unramified quadratic extension of $\mathbb{Q}_3$. By [Tun83, Prop. 1.7], we know that $\varepsilon_3(E/K, \chi) = -1$ if and only if $m \leq (n - 1)/2$. Thus we conclude that $E_3(m, n') \neq \emptyset$.

Now we assume that $p = 2$. Again we can split the discussion into cases.

i) First suppose $n = \text{val}_p(N) = 2$. As above, if $m = 0$ then $\varepsilon_p(E/K, \chi) = 1$, hence it must be $n \geq 1$. Letting $n' := 2m + n$ case (2b) now ensures that $E_p(m, n') \neq \emptyset$.

ii) If $\text{val}_p(N) > 2$, then Assumption [14] ii) implies that $n$ is odd. If $n = 3$, on the one hand by [Tun83, Prop. 3.7] we have that $\varepsilon_2(E/K, \chi) = -1$ if and only if $m \leq 1$. And on the other hand, case (2f) in §3.2.3 tells us that $E_2(m, n) \neq \emptyset$ if and only if $m \leq 1$, thus we conclude that $E_2(m, n') \neq \emptyset$. If $n$ is either 5 or 7, again according to §3.2.3 case (2f) we see that if $m \leq (n - 1)/2$ then $E_2(m, n) \neq \emptyset$. If not, defining $n' := 2m + 1$ we will have $E_2(m, n') \neq \emptyset$.

This concludes the proof.

Lemma 4.13. If $p \in \Sigma(E, \chi)$, $\pi_E$ is supercuspidal at $p$ and $p$ is ramified in $K$ then there exists $n' \geq n$ such that $E_p(m, n') \neq \emptyset$.

Proof. As in the previous lemma, we assume first that $p$ is odd. We have the following cases:

1. Suppose $n = \text{val}_p(N) = 2$. If $m = 0$, we deduce from §3.2.3 cases (1d) or (1e)) that $E_p(m, n) \neq \emptyset$.

2. Suppose that $p = 3$ and $n = \text{val}_p(N) = 3$. In this case, $3 \leq n \leq 5$. Assume first that $n = 4$. In this case, the quadratic extension $L_3/\mathbb{Q}_3$ is ramified. Up to replacing $L_3$ by the other quadratic ramified extension, we might assume that $K_3 \neq L_3$. Then from case (1e) in §3.2.3 we see that $E_p(m, n') \neq \emptyset$ if and only if $m \leq 1$. If $m > 1$, then we take $n' := 2m + 1$, and again case (1e) in §3.2.3 tells us that $E_p(m, n') \neq \emptyset$.

Now we deal with the case $p = 2$. By Assumption [17] if it were $\text{val}_p(N) > 2$ then $\text{ord}_p(n') = 2$ should be inert in $K$, thus we only need to consider the case $n = \text{val}_2(N) = 2$. By cases (2c), (2d) or (2e) in §3.2.3 we have that $E_p(m, n) \neq \emptyset$ if and only if $m = (n - 1)/2$. Provided that $n \geq (n - 1)/2$, defining $n' := 2m + 1 + n$ we obtain $E_p(m, n') \neq \emptyset$ as we want. Thus we need to prove that $m \geq (n - 1)/2$. Since $3$ is ramified, we have $\eta_{K,3}(-1) = -1$, and therefore the hypothesis that $3$ belongs to $\Sigma(E, \chi)$ implies that $\varepsilon_3(E/K, \chi) = 1$. Besides, we know that $\pi_{E,3}$ is induced from a quasicharacter of conductor $n - 1$ of a ramified quadratic extension of $\mathbb{Q}_3$. By [Tun83, Prop. 2.8], $\varepsilon_3(E/K, \chi) = 1$ then implies that $m \geq (n - 1)/2$ as we wanted.

Next we consider the Steinberg case. Write $\pi_{E,p} = \text{Sp}_2 \otimes \psi$ where $\psi : W_{Q_p}^{ab} \to \mathbb{C}^*$ is a quadratic character. By [Tun83, Prop. 1.7], we know that $\varepsilon(E, \chi) = -1$ if and only if $\chi_p^{-1} = \psi \circ \text{Nr}$, where $x \mapsto \text{Nr}(x)$ is the norm map from $K_p = K \otimes_{\mathbb{Q}} Q_p$ to $Q_p$.

Lemma 4.14. If $p \in \Sigma(E, \chi)$, $\pi_E$ is Steinberg at $p$ and $p$ is inert in $K$ then there exists $n' \geq n$ such that $E_p(m, n') \neq \emptyset$.

Proof. By the above discussion, $p \in \Sigma(E, \chi)$ if and only if $\chi_p^{-1} = \psi \circ \text{Nr}$. We assume first that $p$ is odd, so that $n = \text{val}_p(N)$ can be either 1 or 2. We split the discussion into subcases:
(1) $n = 1$. Comparing with \([3.2.3](1a)\), we see that $\mathcal{E}_p(m, n) \neq \emptyset$ if and only if $m = 0$. On the other hand, $\psi$ is unramified and therefore if $p \in \Sigma(E, \chi)$ then $m = 0$.

(2) $n = 2$. Looking now at \([3.2.3](1c)\), $\mathcal{E}_p(m, n) \neq \emptyset$ if and only if $m = 1$. On the other hand, $\psi$ is ramified with conductor equal to 1, and therefore if $p \in \Sigma(E, \chi)$ then $m = 1$.

Assume now that $p = 2$. Then val$_2(N) \in \{1, 4, 6\}$.

i) If $n = \text{val}_2(N) = 1$, the character $\psi$ is unramified, and then since $2 \in \Sigma(E, \chi)$ we deduce that $m = 0$. On the other hand, by case (2a) in \([3.2.3]\) we also have $\mathcal{E}_p(m, n) \neq \emptyset$ if and only if $m = 0$. Thus $\mathcal{E}_p(m, n) \neq \emptyset$ as we want.

ii) If val$_2(N) = 4$, then $\psi$ is ramified with conductor 2. Since $\chi_p^{-1} = \psi \circ \text{Nr}$, it follows that also $m = 0$. By the discussion before Proposition \([2.7]\), $n = \text{val}_2(N_{R'})$ can be either 1, 2 or 3. If $n$ is either 1 or 3, defining $n' := 5 \geq n$ we see by case (2f) in \([3.2.3]\) that $\mathcal{E}_2(m, n') \neq \emptyset$. And if $n = 2$, by case (2b) in \([3.2.3]\) we conclude that for $n' := 2m + 4 \geq n$ we have $\mathcal{E}_2(m, n') \neq \emptyset$.

iii) If val$_2(N) = 6$, then $\psi$ is ramified with conductor 3. Similarly as before, we deduce that $\chi_p$ has another conductor $m = 3$. From the discussion before Proposition \([2.7]\), we know that the possible values for $n = \text{val}_2(N_{R'})$ are $1, 2, 3, 4$ or 5. When $n$ is either 1, 3 or 5, $L_2$ is the unramified quadratic extension of $\mathbb{Q}_2$, and defining $n' := 2m + 1 = 7 \geq n$, case (2f) in \([3.2.3]\) implies that $\mathcal{E}_2(m, n') \neq \emptyset$. Finally, if $n = 2$ or $n = 4$, then we may define $n' := 2m + 6 \geq n$ and now case (2b) shows that $\mathcal{E}_2(m, n') \neq \emptyset$.

This concludes the proof. \[\square\]

**Lemma 4.15.** If $p \in \Sigma(E, \chi)$, $\pi_E$ is Steinberg at $p$ and $p$ is ramified in $K$ then there exists $n' \geq n$ such that $\mathcal{E}_p(m, n') \neq \emptyset$.

**Proof.** Suppose first that $p$ is odd, so that $n = \text{val}_p(N)$ is either 1 or 2. By the above discussion, $p \in \Sigma(E, \chi)$ if and only if either $\chi_p^{-1} = \psi \circ \text{Nr}$ and $p \equiv 1$ mod 4, or $\chi_p^{-1} \neq \psi \circ \text{Nr}$ and $p \equiv 3$ mod 4. We have the following cases:

1. $n = 1$. If $m = 0$, by case (1b) in \([3.2.3]\) we see that $\mathcal{E}_p(m, n) \neq \emptyset$. Otherwise, we can take $n' := 2m + 1$, and again case (1b) in \([3.2.3]\) implies $\mathcal{E}_p(m, n') \neq \emptyset$.

2. $n = 2$. Again, for $m = 0$ we have $\mathcal{E}_p(m, n) \neq \emptyset$ by case (1d) or (1e) in \([3.2.3]\). If instead $m > 0$, then we consider $n' := 2(m + 1)$ and by applying \([3.2.3]\), case (1d) or (1e), we see that $\mathcal{E}_p(m, n') \neq \emptyset$.

Now assume that $p = 2$. Similarly as in the previous lemma, by Assumption \([1.9]\) we only need to deal with the case $n = \text{val}_2(N) = 1$. Then notice that $\psi$ is unramified. On the other hand, now the hypothesis that 2 belongs to $\Sigma(E, \chi)$ implies that either $\eta_{R, 2}(-1) = 1$ and $\chi_p^{-1} = \psi \circ \text{Nr}$ or $\eta_{R, 2}(-1) = -1$ and $\chi_p^{-1} \neq \psi \circ \text{Nr}$. Having this into account, if $\eta_{R, 2}(-1) = 1$ then the equality $\chi_p^{-1} = \psi \circ \text{Nr}$ implies that $m = 0$. By case (2a) in \([3.2.3]\) it thus follows that $\mathcal{E}_p(m, n) \neq \emptyset$. And if $\eta_{R, 2}(-1) = -1$, it could be the case that $m > 0$. But in any case, defining $n' := 2m + 1 \geq n = 1$ case (2g) in \([3.2.3]\) implies that $\mathcal{E}_p(m, n') \neq \emptyset$. \[\square\]

Combining the above lemmas, we obtain the following:

**Theorem 4.16.** Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$, $K$ be an imaginary quadratic field and $\chi$ be an anticyclotomic character of conductor $c$. Suppose that the set $\Sigma(E, \chi)$ has odd cardinality, so that $\varepsilon(E/K, \chi) = -1$ and hence $L(E/K, \chi, 1) = 0$. If Assumption \([1.9]\) holds, then the set of Heegner points in $E(H_c)$ is non-empty. And if further $E$ does not acquire CM over any imaginary quadratic field contained in $H_c$ and $L'(E/K, \chi, 1) \neq 0$, then $\dim_{\mathbb{C}}(E(H_c) \otimes \mathbb{C})^N = 1$.

**Proof.** Let $B$ be the indefinite quaternion algebra ramified exactly at the finite primes in $\Sigma(E, \chi)$, and let $\mathcal{R}_{\text{min}}$ be the order in $B$ from Proposition \([2.4]\). The above lemmas together imply that there is a suborder $\mathcal{R}$ of $\mathcal{R}_{\text{min}}$ such that the set of Heegner points of conductor $c$ in $X_R(H_c)$ is non-empty. The Jacobian of $X_R$ uniformizes $E$ as well, hence the set of Heegner points of conductor $c$ in $E(H_c)$ is non-empty. By Theorem \([4.11]\) if $E$ does not acquire CM over any imaginary quadratic field contained in $H_c$, then $\dim_{\mathbb{C}}(E(H_c) \otimes \mathbb{C})^N = 1$. \[\square\]

We state now the above result in more restrictive, but maybe more attractive form, introducing a couple of definitions.

**Definition 4.17.** We say that a form $f \in S_2(\Gamma_0(N))$ is **primitive** if $f \neq g \otimes \chi$ for any Dirichlet character $\chi$ and any $g \in S_2(\Gamma_0(M))$ with $M | N$ and $M \neq N$ (see Piz80 Def. 8.6 for a similar terminology).
Definition 4.18. Let \( p \) be a prime. We say that \( f \in S_2(\Gamma_0(N)) \) has \( p \)-minimal Artin conductor if the \( p \)-component \( \pi_{f,p} \) of the automorphic representation \( \pi_f \) attached to \( f \) has minimal conductor among its twists by quasi-characters of \( \mathbb{Q}_p^* \); in other words, if we write \( a(\pi_f) \) for the Artin conductor of the automorphic representation \( \pi_f \), we require that \( a(\pi_f) \leq a(\pi_f \otimes \chi) \) for all quasi-characters \( \chi \) of \( \mathbb{Q}_p^* \).

Corollary 4.19. Let \( E/\mathbb{Q} \) be an elliptic curve of conductor \( N \), \( K \) be an imaginary quadratic field and \( \chi \) be an anticyclotomic character of conductor \( c \). Suppose that the set \( \Sigma(E, \chi) \) has odd cardinality. If the following conditions hold:

1. \( f \) is a primitive form;
2. \( f \) has 2-minimal Artin conductor;
3. If \( \text{val}_2(N_{\text{Eic}}) = 4 \) and \( 3 \) is inert in \( K \), then \( \text{val}_2(c) \neq 1 \);
4. If \( \text{val}_2(N_{\text{Eic}}) \geq 3 \), then \( 2 \) is not ramified in \( K \);

then the set of Heegner points in \( E(H) \) is non-empty. If further \( E \) does not acquire CM over any imaginary quadratic field contained in \( H_c \) and \( L(E/K, \chi, 1) \neq 0 \), then \( \dim_{\mathbb{Q}}(E(H_c) \otimes \mathbb{C}) \chi = 1 \).

Proof. We only need to remark that conditions i) and ii) in Assumption [1.4] are satisfied if we ask that \( f \) is primitive by [HPS89b, Thm. 3.9].\( \square \)

4.4. Final remarks.

It might be interesting to discuss how to extend the above theorem to the general case of abelian varieties. One can easily show that if \( A/\mathbb{Q} \) is a modular variety of dimension \( d \) and conductor \( N^d \), and no prime divides \( N \) to a power greater than 3, then the argument for elliptic curves developed in the previous section also works for abelian varieties. However, it is easy to construct examples in which we do not have Heegner points in any cover of \( X_{\text{Ram}} \) if we allow the conductor of \( A \) to be divisible by arbitrary powers of \( p \) if we just consider orders of type \( (N_{\text{Eic}}; N_{\text{Car}}; \{\mathcal{L}_p, \mathcal{R}_p\}) \), as the following examples show:

Example 4.20. Let \( A/\mathbb{Q} \) be a modular abelian variety of conductor \( N^d \), and suppose \( N = p^aq \) with \( p \) and \( q \) distinct odd primes and \( n = 2q + 1 \) an odd integer. Let \( \chi \) be a character of conductor \( p^m \) with \( m \geq 1 \). Suppose that \( p \) is ramified in \( q \) and \( q \) is inert in \( K \). Then \( q \in \Sigma(A, \chi) \), and \( p \in \Sigma(A, \chi) \) if and only if
\[
\varepsilon_p(A, \chi) = \eta_{K,p}(-1) = (-1)^{(p-1)/2}.
\]
Now assume that \( n \) is minimal among the conductor of all twists of \( \pi_{A,p} \). In this case [Tun83, Prop. 3.2] shows that if \( m \leq n - 1 \) then \( \varepsilon_p(A, \chi) = -1 \), so if \( p \equiv 3 \mod 4 \) then \( p \in \Sigma(A, \chi) \). If \( m < (n-1)/2 \), then comparing with [1.2.3] we see that there are no Heegner points of conductor \( p^m \) in any cover of \( X_{\text{Ram}} \) associated with an order as in Definition [1.2].

Example 4.21. As in the above example, let \( A/\mathbb{Q} \) be a modular abelian variety of conductor \( N^d \), and suppose now that \( N = p^aq \) with \( p \) and \( q \) two odd primes and \( n = 2q \geq 4 \) an even integer. Let \( \chi \) be a character of conductor \( p^m \) with \( m \geq 1 \). Suppose first that \( p \in \Sigma(A, \chi) \), so \( \varepsilon_p(A, \chi) = -1 \), and \( q \) is inert in \( K \), so \( q \in \Sigma(A, \chi) \). Consider the quaternion algebra \( B \) of discriminant \( pq \) and the order \( R_{\text{min}} \) of \( B \) and form the corresponding Shimura curve \( X_{\text{Ram}} \). From [1.2.3] we see that if \( m < n/2 \), then there are no Heegner points of conductor \( p^m \) in any covering of \( X_{\text{Ram}} \) associated with special orders as in Definition [1.2]. Secondly, suppose \( p \notin \Sigma(A, \chi) \), so \( \varepsilon_p(A, \chi) = +1 \) and \( q \) is split in \( K \), so \( q \notin \Sigma(A, \chi) \). In this case, if \( m < n/2 \) then again there are no Heegner points of conductor \( p^m \) in any covering of the Shimura curve \( X_{\text{Ram}} \) associated with Eichler orders (which, in this case, correspond to modular curves and usual congruence subgroups).

As we may see from the above examples, it seems to us that that one should introduce more general type of orders to find other sources of Heegner points defined over the predicted ring class field.

Conjecture 4.22. Let \( A/\mathbb{Q} \) be a modular abelian variety, \( K \) be an imaginary quadratic field, \( \chi \) be an anticyclotomic character factoring through the ring class field \( H_c \) of \( c \geq 1 \), and suppose that the cardinality of \( \Sigma(A, \chi) \) is odd. Let \( B \) denote the indefinite quaternion algebra of discriminant \( \Delta \) equal to the product of the finite primes in \( \Sigma(A, \chi) \), and \( f \) be the newform associated with \( A \). Suppose that \( f \) is primitive and has \( p \)-minimal Artin conductor, for all primes \( p \). Then, there exists an open compact subgroup \( U \) in \( B^* \) equipped with a surjective morphism \( J_U \rightarrow A \) and such that the set of Heegner points in \( X_U(H_c) \) is non-empty.

As a variant of the above conjecture, one can ask if we can take \( U = \hat{\mathbb{R}}^* \) for some global order \( \mathbb{R} \) in \( B \). This conjecture is inspired by Corollary [4.19] we only point out that the relevant part in this conjecture
is to show the existence of suitable open compact subgroups (non necessarily arising from global orders) so that we have a good understanding of rationality questions of points arising from embeddings $\mathbf{K} \hookrightarrow B$. This would allow us to solve cases as \cite{BD07,DKP16}. 

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