New cases of logarithmic equivalence of Welschinger and Gromov-Witten invariants

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Dedicated to V. I. Arnol’d, at his 70th birthday anniversary.

Abstract

We consider $\mathbb{P}^1 \times \mathbb{P}^1$ equipped with the complex conjugation $(x, y) \mapsto (\bar{y}, \bar{x})$ and blown up in at most two real, or two complex conjugate, points. For these four surfaces we prove the logarithmic equivalence of Welschinger and Gromov-Witten invariants.

1 Introduction

Welschinger invariants \cite{9, 10} applied to unnodal Del Pezzo surfaces bound from below the number of real rational curves in a given linear system which pass through a real generic collection of points. In our previous papers \cite{1, 2}, using the methods

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of tropical enumerative geometry developed by G. Mikhalkin and E. Shustin, we studied the toric unnodal Del Pezzo surfaces with tautological real structure and showed that for these surfaces Welschinger and Gromov-Witten invariants are equivalent in the logarithmic scale, if all or almost all fixed points in the generic collection are real. Here we continue such an asymptotic study of Welschinger invariants and consider non-tautological real structures on toric unnodal Del Pezzo surfaces. Up to isomorphisms respecting the real structure, there are only five toric unnodal Del Pezzo surfaces with a non-tautological real structure and non-empty real part. One is obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) equipped with the standard (tautological) complex conjugation by blowing up two complex conjugate points, and the four others are obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) equipped with the complex conjugation \((x, y) \mapsto (\bar{y}, \bar{x})\) by blowing up at most two real, or two complex conjugate, points.

We look at collections of real points on any of the four latter surfaces, apply the tropical formula elaborated in [8] to the multiples \( nD \) of a real ample divisor \( D \) on such a surface \( \Sigma \), and prove that Welschinger and Gromov-Witten invariants, \( W_{\Sigma, nD} \) and \( GW_{\Sigma, nD} \), are equivalent in the logarithmic scale: \( \log W_{\Sigma, nD} = \log GW_{\Sigma, nD} + O(n) \). Recall that, as is shown in [2, 3], \( \log GW_{\Sigma, nD} = (c_1(\Sigma) \cdot D) n \log n + O(n) \).

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2 Combinatorial bound

As toric surfaces, the four real Del Pezzo surfaces, \( S^2, S_{1,0}^2, S_{2,0}^2, \) and \( S_{0,2}^2 \), we deal with are associated with the following convex lattice polygons in \( \mathbb{R}^2 \) (see Figure [1]):

- squares with vertices \((0, 0), (d, 0), (0, d), (d, d)\), where \( d \geq 1 \);
- pentagons with vertices \((d, d), (0, d), (0, d_1), (d_1, 0), (d, 0)\), where \( 1 \leq d_1 < d \);
- hexagons with vertices \((0, d_1), (d_1, 0), (d, d - d_2), (d - d_2, d), (0, d)\), where \( 1 \leq d_2 \leq d_1 < d \);
- and hexagons with vertices \((0, 0), (d - d_1, 0), (d, d_1), (d, d), (d_1, d), (0, d - d_1)\), where \( 1 \leq d_1 < d \).

For the toric surface \( \Sigma \) associated with such a polygon \( \Delta \), the real structure we study is the involution which acts in the principal orbit \((\mathbb{C}^*)^2 \subset \Sigma\) by \( \text{Conj}(x, y) = (\bar{y}, \bar{x}) \). Its natural lift to the ample line bundle \( \mathcal{L}_\Delta \) generated by monomials \( x^i y^j \), \((i, j) \in \Delta \cap \mathbb{Z}^2\), acts by \( \text{Conj}_*(a_i x^i y^j) = \overline{a_j} x^j y^i \), \((i, j) \in \Delta\), and thus gives rise to the reflection of \( \Delta \) with respect to the bisectrix \( \mathcal{B} \) of the positive quadrant. Denote by \( D_\Delta \) (or simply \( D \)) an ample divisor which defines \( \mathcal{L}_\Delta \).
The goal of this section is to deduce from [8], Theorem 1.1, a lower bound for the Welschinger invariant $W_{\Sigma, D}$. To this end, we introduce the following objects.

For each integer point belonging to the boundary of $\Delta$, trace the straight line through this point and its image under the reflection with respect to $B$. The union of all the traced lines cuts $B \cap \Delta$ in certain segments; denote their number by $m$. Identify $B \cap \Delta$ with the segment $[0, m] \subset \mathbb{R}$ in such a way that the intersection points of $B \cap \Delta$ with the traced lines are mapped to the integer points of $[0, m]$. To each integer point $i \in [0, m]$ associate a non-negative integer number $\sigma(i)$ equal to the integer length of the intersection of the corresponding straight line with $\Delta$.

A finite multi-set of closed intervals in $\mathbb{R}$ is called a $\Delta$-proper system (or simply proper system) if

- each interval is contained in $[0, m]$ and has integer endpoints (intervals reduced to a point are allowed),
- the total number of intervals is $|\partial \Delta| - m - 1$, where $|\partial \Delta|$ is the integer length of the boundary of $\Delta$,
- for any integer $i \in [0, m]$, the number of intervals containing $i$ is equal to $\sigma(i)$.

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1In [8], this invariant is denoted by $W_0(\Sigma, \mathcal{L})$, where $\mathcal{L} = \mathcal{L}_\Delta$. 

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Figure 1: Polygons defining $S^2$, $S^2_{1,0}$, $S^2_{2,0}$, and $S^2_{0,2}$
Given a $\Delta$-proper system, consider the disjoint union $g'$ of the intervals of the system, and complete $g'$ to a graph $g$ introducing $m$ additional vertices indexed by the half-integer points $i + 1/2, i = 0, \ldots, m - 1$, and additional edges connecting each point $i + 1/2$ with all the right endpoints $i$ and all the left endpoints $i + 1$ of the intervals in $g'$.

A $\Delta$-proper system is called admissible, if its graph $g$ is a tree. An admissible $\Delta$-proper system is marked, if it is equipped with a marking which associates to each interval $I$ of the system an integer point of $I$; the latter point is called marked.

The following statement is an immediate consequence of [8], Theorem 1.1.

**Lemma 1.** Let $\Delta$ be one of the polygons shown in Figure [1] and $\Sigma$ the toric surface associated with $\Delta$ and equipped with the real structure Conj (described above). Then, the Welschinger invariant $W_{\Sigma, D\Delta}$ is greater than or equal to the number of marked admissible $\Delta$-proper systems.

### 3 Logarithmic asymptotics

#### 3.1 Main theorem

**Theorem 1.** Let $\Sigma$ be one of the real surfaces $S^2$, $S^2_{1,0}$, $S^2_{2,0}$, and $S^2_{0,2}$. For any real ample divisor $D$ on $\Sigma$, it holds

$$\log W_{\Sigma, nD} = (c_1(\Sigma) \cdot D)n \log n + O(n).$$

In particular,

$$\lim_{n \to \infty} \frac{\log W_{\Sigma, nD}}{\log GW_{\Sigma, nD}} = 1,$$

where $GW_{\Sigma, nD}$ is the genus zero Gromov-Witten invariant.

Since $W_{\Sigma, nD} \leq GW_{\Sigma, nD}$ and $GW_{\Sigma, nD} = (c_1(\Sigma) \cdot D) n \log n + O(n)$, to prove Theorem 1 it is sufficient to prove the lower bound $W_{\Sigma, nD} \geq (c_1(\Sigma) \cdot D) n \log n + O(n)$. Due to Lemma 1 and the identity $|\partial \Delta| = c_1(\Sigma) \cdot D$, the latter lower bound would follow from the inequality

$$\log S_{n\Delta} \geq |\partial \Delta| \cdot n \log n + O(n),$$

where $S_{n\Delta}$ is the number of marked admissible $n\Delta$-proper systems. This inequality is proved in Sections 3.3, 3.4, 3.5, and 3.6 where each of the surfaces $S^2$, $S^2_{1,0}$, $S^2_{2,0}$, and $S^2_{0,2}$ is treated separately.

#### 3.2 Admissibility

Let $\Gamma$ be a finite set of disjoint horizontal segments with integer endpoints in $\mathbb{R}^2$ (degenerated segments are allowed). For any vertical strip $b = \{i \leq x \leq i + 1\}$,
where $i$ is an integer, denote by $\Gamma_L^b$ (respectively, $\Gamma_R^b$) the subset of $\Gamma$ formed by the segments whose right endpoint belongs to $x = i$ (respectively, left endpoint belongs to $x = i + 1$).

**Lemma 2** Assume that $\Gamma$ can be represented as the disjoint union of two subsets $\Gamma_L$ and $\Gamma_R$ satisfying the following properties:

(i) for any vertical strip $b = \{i \leq x \leq i + 1\}$ such that $i$ is an integer, the union of $\Gamma_R \cap \Gamma_R^b$ and $\Gamma_L \cap \Gamma_L^b$ contains at most one element,

(ii) if the union of $\Gamma_R \cap \Gamma_R^b$ and $\Gamma_L \cap \Gamma_L^b$ contains an element $s$, no element of $\Gamma_L \cup \Gamma_R^b$ lies below $s$;

(iii) there exists exactly one vertical strip $b = \{i \leq x \leq i + 1\}$ such that $i$ is an integer, at least one of the sets $\Gamma_L^b$ and $\Gamma_R^b$ is nonempty, and the union of $\Gamma_R \cap \Gamma_R^b$ and $\Gamma_L \cap \Gamma_L^b$ is empty.

If the projections of segments of $\Gamma$ on the horizontal axis form a proper system, then this proper system is admissible.

**Proof.** For a proper system as in the lemma, identify $\Gamma$ with the disjoint union $g'$ of the intervals of the system, and consider the graph $g$ as in Section 2. Orient the segments of $\Gamma_L$ to the left, the segments of $\Gamma_R$ to the right, and orient each additional edge of $g$ by extending the orientation of the adjacent horizontal segment. The conditions (i) and (ii) give a deformation retraction of $g$ to a finite set of vertices, and the condition (iii) guarantees that the latter set has only one element. \qed

### 3.3 Case $\Sigma = \mathbb{S}^2$

Let $\Delta$ be the square shown in Figure 1(a). In this case, the required inequality reads as

$$\log S_{n\Delta} \geq 4dn \log n + O(n).$$

(4)

To construct an appropriate number of marked admissible $n\Delta$-proper systems, consider the triangle $T(n, d)$ with vertices $(1, nd - 1)$, $(nd, 0)$, and $(2nd - 1, nd - 1)$ (see Figure 2(a)). At each integer level $y = j$, $0 \leq j \leq nd - 1$ consider the maximal horizontal segment contained in $T(n, d)$. If $j \neq 0$, make a hole in the considered segment by removing an open unit interval with integer endpoints. This perforation procedure gives rise to a set of $2nd - 1$ horizontal segments whose projections form an $n\Delta$-proper system.

Inscribe in $T(n, d)$ a sequence of maximal size rectangles $R_i$ satisfying the following properties: each rectangle is symmetric with respect to the vertical line $x = nd$, and the length of horizontal edges of each rectangle is twice the length of its vertical
Figure 2: First steps in the construction of admissible systems for $S^2$

edges (see Figure 2(b)). The right upper vertices $(x_i, y_i)$, $i \geq 1$ of these rectangles are given by

$$x_1 = nd + \left\lfloor \frac{nd - 1}{2} \right\rfloor, \quad y_1 = nd - 1, \quad y_{i+1} = y_i - \left\lfloor \frac{y_i}{2} \right\rfloor, \quad x_{i+1} = nd + y_{i+1} - \left\lfloor \frac{y_{i+1}}{2} \right\rfloor.$$ 

Let $k$ be the number of rectangles. Notice that $y_k = 2$, and put $y_{k+1} = y_k - \left\lfloor \frac{y_k}{2} \right\rfloor = 1$.

Restrict the choice of holes in the perforation procedure in the following way:

- all the holes are contained in the half-plane $x \geq nd$,
- for any integer $1 \leq i \leq k$ all the holes at the levels $y_{i+1} + 1 \leq y \leq y_i$ are contained in $R_i$,
- for any integer $1 \leq i \leq k - 1$ no two holes at the levels $y_{i+1} + 1 \leq y \leq y_i$ have the same projection on the horizontal axis.

The set of segments obtained via such a perforation procedure is called a *perforated $(n, d)$-collection*. The number $M(n, d)$ of perforated $(n, d)$-collections is equal to

$$(y_1 - y_2)! (y_2 - y_3)! \ldots (y_k - y_{k+1})!.$$ 

According to the Stirling formula,

$$\log M(n, d) = (y_1 - y_2) + (y_2 - y_3) + \ldots + (y_k - y_{k+1})) \log n + O(n) = dn \log n + O(n).$$

For any perforated $(n, d)$-collection $c$ and any permutations $\sigma_1, \ldots, \sigma_{k-1}$, where $\sigma_i$, $i = 1, \ldots, k - 1$, is a permutation of $\{y_{i+1} + 1, y_{i+1} + 2, \ldots, y_i\}$, consider the set of segments $c_{\sigma_1, \ldots, \sigma_{k-1}}$ obtained from $c$ in the following way: for each integer $1 \leq i \leq k - 1$, cut along the vertical line $x = x_i$ the segments of $c$ lying on the levels $y_{i+1} + 1 \leq y \leq y_i$ and intersecting the line $x = x_i$, permute according to $\sigma_i$ the right-hand parts of the segments we have cut, and glue the adjacent parts in order to form new segments (see Figure 3). The set $c_{\sigma_1, \ldots, \sigma_{k-1}}$ is called a *permuted*
perforated \((n, d)\)-collection. It consists of the point \((nd, 0)\) and two segments at each integer level \(1 \leq y \leq nd - 1\).

The number \(\bar{M}(n, d)\) of the permuted perforated \((n, d)\)-collections \(c_{\sigma_1, \ldots, \sigma_{k-1}}\), where \(c\) runs over all the perforated \((n, d)\)-collections and \(\sigma_i, i = 1, \ldots, k - 1\), runs over all the permutations of \(\{y_{i+1} + 1, y_{i+1} + 2, \ldots y_i\}\), is equal to

\[
M(n, d)(y_1 - y_2)!(y_2 - y_3)!(y_k - y_{k+1})!
\]

Thus, \(\log \bar{M}(n, d) = 2dn \log n + O(n)\).

The projection on the horizontal axis of any permuted perforated \((n, d)\)-collection is an \(n\Delta\)-proper system. The restriction imposed above on the choice of holes guarantees that the projection of all the permuted perforated \((n, d)\)-collections produces \(\bar{M}(n, d)\) pairwise distinct \(n\Delta\)-proper systems. All the resulting systems are admissible as it follows from Lemma \(\ref{lem:admissible}\) applied to any permuted perforated \((n, d)\)-collection represented as the disjoint union of the segments lying on the left-hand side of the holes (the subset \(\Gamma_R\)) and the segments lying on the right-hand side of the holes (the subset \(\Gamma_L\)).

Mark each of \(\bar{M}(n, d)\) admissible \(n\Delta\)-proper systems as above in such a way that

- no marked point of the projection of a segment at level 1 does coincide with the point \(nd\),

- for any integer \(i\) between 1 and \(k\), the marked points of the projections of segments at any level \(y_{i+1} + 1 \leq y \leq y_i\) are placed outside of the projection of \(R_i\).

For each system, this can be done in \(((y_1 - y_2)!(y_2 - y_3)!(y_k - y_{k+1})!)^2\) different ways. Thus, the logarithm of the number of obtained marked admissible \(n\Delta\)-proper systems is \(4dn \log n + O(n)\). This proves Theorem \(\ref{thm:admissible}\) in the case \(\Sigma = S^2\).
3.4 Case $\Sigma = S_{1,0}^2$

Let $\Delta$ be the pentagon shown in Figure 1(b). In this case, the required inequality (3) reads as

$$\log S_{n\Delta} \geq (4d - d_1)n \log n + O(n). \quad (5)$$

To construct an appropriate number of marked admissible $n\Delta$-proper systems, consider the quadrangle $Q(n, d, d_1)$ with vertices 

$$(0, nd - 1), (0, n(d - d_1)), (n(d - d_1), 0), \text{ and } (n(2d - d_1) - 1, nd - 1),$$

(see Figure 1(a)). For the triangle $T(n, d - d_1) \subset Q(n, d, d_1)$ use the construction described in Section 3.3. To complete the resulting permuted perforated $(n, d - d_1)$-collections, we proceed in the following way.

Consider the up-right staircase $E$ formed by squares of size $n \times n$ such that $E$ starts at the middle point $(n(d - d_1), n(d - d_1) - 1)$ of the upper side of $T(n, d - d_1)$ (see Figure 1(b)). At each integer level $y = j, n(d - d_1) \leq j \leq nd - 1$ consider the maximal horizontal segment contained in $Q(n, d, d_1)$, and use the perforation procedure (that is, make a hole in each segment considered) choosing holes in such a way that all these holes are contained in $E$, no hole is taken on the lower sides of the squares forming $E$, and no two holes have the same projection on the horizontal axis. This gives $(n!)^{d_1}$ sets of segments. For any of these sets and any permuted perforated $(n, d - d_1)$-collection, their union is called a perforated $(n, d, d_1)$-collection.

The projection to the horizontal axis of any perforated $(n, d, d_1)$-collection is an $n\Delta$-proper system. Due to Lemma 2, any resulting $n\Delta$-proper system is admissible. For any such system, there are at least $(nd_1)!(nd_1)!$ choices of marking for the projections of segments lying above $T(n, d - d_1)$. Thus, the logarithm of the number of marked admissible $n\Delta$-proper systems is at least

$$4(d - d_1)n \log n + O(n) + d_1 \log n! + 2 \log(n d_1)! = (4d - d_1)n \log n + O(n).$$
This proves Theorem 1 in the case \( \Sigma = S^2_{1,0} \).

### 3.5 Case \( \Sigma = S^2_{2,0} \)

Let \( \Delta \) be the hexagon shown in Figure 5(c). In this case, the required inequality (3) reads as

\[
\log S_{n\Delta} \geq (4d - d_1 - d_2)n \log n + O(n) .
\]

To construct an appropriate number of marked admissible \( n\Delta \)-proper systems, consider the pentagon \( P(n, d, d_1, d_2) \) with vertices

\[
(0, nd - 1), \ (0, n(d - d_1)), \ (n(d - d_1), 0), \\
(n(2d - d_1 - d_2), n(d - d_2)), \text{ and } (n(2d - d_1 - d_2), nd - 1),
\]

(see Figure 5(a)). For the quadrangle \( Q(n, d - d_2, d - d_1) \subset P(n, d, d_1, d_2) \) use the construction described in Section 3.4. To complete the resulting perforated \( (n, d - d_2, d_1 - d_2) \)-collections, we proceed in the following way.

The remaining part of \( P(n, d, d_1, d_2) \) is formed by a horizontal strip of height 1 and a rectangle of width \( n(2d - d_1 - d_2) \) and height \( nd_2 - 1 \), see Figure 5(a). Consider an up-right staircase

- starting at a point \( (x_0, n(d - d_2) - 1) \) with \( x_0 \geq [n(2d - d_1 - d_2)/4] \),
- ending at a point \( (x_1, nd - 1) \) with \( x_1 \leq [3n(2d - d_1 - d_2)/4] - 1 \),
- and formed by rectangles such that each rectangle is of width 1 and of positive height smaller than or equal to \( a = \left\lfloor \frac{2d}{2d - d_1 - d_2} \right\rfloor + 1 \).
Figure 6: Construction of admissible systems for $S_{0,2}$

(see Figure 5(b)). At each integer level $y = j$, $n(d - d_2) \leq j \leq nd - 1$ consider the maximal horizontal segment contained in $P(n, d, d_1, d_2)$, and use the perforation procedure choosing holes in the rectangles of the staircase in such a way that no hole is taken on the lower sides of the rectangles. For any perforated $(n, d - d_2, d_1 - d_2)$-collection, its union with the constructed set of segments is called a **perforated** $(n, d, d_1, d_2)$-collection.

The projection to the horizontal axis of any perforated $(n, d, d_1, d_2)$-collection is an $n\Delta$-proper system. Due to Lemma 2, any resulting $n\Delta$-proper system is admissible. For any such system, there are at least $\left[ \frac{n(2d - d_1 - d_2)}{4} \right]^{2nd_2} (a!)^{-2nd_2}$ choices of marking for the projections of segments lying above $Q(n, d - d_2, d_1 - d_2)$. Thus, the logarithm of the number of marked admissible $n\Delta$-proper systems is at least

$$(4(d - d_2) - (d_1 - d_2))n \log n + O(n) = (4d - d_1 - d_2)n \log n + O(n).$$

This proves Theorem 1 in the case $\Sigma = S_{0,2}$.

### 3.6 Case $\Sigma = S_{0,2}$

Let $\Delta$ be the hexagon shown in Figure 1(d). In this case, the required inequality (3) reads as

$$\log S_{n\Delta} \geq (4d - 2d_1)n \log n + O(n).$$

To construct an appropriate number of marked admissible $n\Delta$-proper systems, consider the trapeze $K(n, d, d_1)$ with vertices

$$(1, n(d - d_1) - 1), (n(d - d_1), 0), (nd, 0), \text{ and } (n(2d - d_1) - 1, n(d - d_1) - 1),$$

$$(n(2d - d_1) - 1, n(d - d_1) - 1)$$

...
Consider the sequence \( C \) of up-right staircases formed by squares of size \( n \times n \) such that

- all the staircases of \( C \) are contained in the vertical strip
  \[
  \mathcal{B} = \{ n(d - d_1) \leq x \leq nd \},
  \]
- each staircase starts at the level \( y = -1 \),
- each staircase ends at the upper side of \( K(n, d, d_1) \), the only possible exception being the last staircase,
- the first staircase starts at the point \((-1, n(d - d_1))\),
- for each staircase, except the first one, the vertical line where the staircase starts coincides with the vertical line where the preceding staircase ends,

(see Figure 6(a)).

At each integer level \( y = j, 0 \leq j \leq n(d - d_1) - 1 \) consider the maximal horizontal segment contained in \( \mathcal{B} \), and use the perforation procedure (this time we authorize several holes at the same level) by choosing holes in such a way that all these holes are contained in \( C \), no hole is taken on the lower sides of the squares forming the staircases, and there is exactly one hole in each integer vertical strip \( i \leq x \leq i + 1 \) contained in \( \mathcal{B} \). This gives \( (n!)^{d_1} \) sets of segments.

Pick a permuted perforated \( (n, d - d_1) \)-collection \( \pi \) in \( T(n, d - d_1) \), cut \( \pi \) along the vertical line \( x = n(d - d_1) \), keep the left half of \( \pi \) at its place and shift the right half by the vector \((nd_1, 0)\). The result of gluing of the obtained collection with a set of segments constructed in \( \mathcal{B} \) as described above is called a perforated \( K(n, d, d_1) \)-collection. The projection to the horizontal axis of any perforated \( K(n, d, d_1) \)-collection is a \( n\Delta \)-proper system.

Any resulting \( n\Delta \)-proper system is admissible. Indeed, let \( \gamma \) be a perforated \( K(n, d, d_1) \)-collection. Identifying \( \gamma \) with the disjoint union \( g' \) of the intervals of the projection of \( \gamma \) to the horizontal axis, consider the graph \( g \) as in Section 2. In each integer vertical strip \( i \leq x \leq i + 1 \) contained in \( \mathcal{B} \), there is exactly one pair of additional edges of \( g \), and this pair fill up the only hole in \( i \leq x \leq i + 1 \). Once the holes in \( \mathcal{B} \) are filled up, Lemma 2 applies. This proves the admissibility of the projection of \( \gamma \).

Consider a perforated \( K(n, d, d_1) \)-collection \( \gamma \) obtained by gluing of a permuted perforated \( (n, d - d_1) \)-collection \( \pi \) with a set of segments constructed in \( \mathcal{B} \) as above. Any marking of the projection of \( \pi \) can be extended to a marking of the projection of \( \gamma \) via a choice of an integer point on each segment entering under a staircase. The latter choice can be done in at least \( (nb_1)! \ldots (nb_k)! \) ways, where \( b_1, \ldots, b_k \) are the
numbers of stairs in the staircases (in fact, \( b_1 = \ldots = b_k - 1 \)). Thus, the logarithm of the number of marked admissible \( n \Delta \)-proper systems is at least

\[
4(d - d_1)n \log n + O(n) + d_1 \log n! + n(b_1 + \ldots + b_k) \log n + O(n)
\]

\[
= (4d - 2d_1)n \log n + O(n).
\]

This proves Theorem [1] in the case \( \Sigma = \mathbb{S}^2_{0,2} \).

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