PROJECTIONS OF DEL PEZZO SURFACES AND CALABI–YAU THREEFOLDS

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Abstract. We study the syzygetic structure of projections of del Pezzo surfaces in order to construct singular Calabi–Yau threefolds. By smoothing those threefolds, we obtain new examples of Calabi–Yau threefolds with Picard group of rank 1. We also give an example of type II primitive contraction whose exceptional divisor is the blow-up of the projective plane at a point.

1. Introduction

A Calabi–Yau threefold is an irreducible projective threefold with Gorenstein singularities, trivial canonical class such that the intermediate cohomologies of its structure sheaf are all trivial ($h^i(X, O_X) = 0$ for $0 < i < \dim(X)$). The aim of this work is to extend and complement the results obtained in [K] and [KK]. Considering projections of del Pezzo surfaces, we find new examples of smooth Calabi–Yau threefolds with Picard group of rank 1. Note that these threefolds of low degree are not complete intersection in toric varieties (see [BB]) and are interesting from the point of view of the Picard–Fuchs equations and mirror symmetry (see [ES], [Pr] and [5.7]). In particular it is an open problem to find mirror families of these examples (or at least a candidate for the Picard-Fuchs equation of the mirror). Recall that it is not known whether there are a finite number of families of Calabi–Yau threefolds with Picard group of rank 1.

The first step is to construct Calabi-Yau threefolds with Picard group of rank 2 containing a del Pezzo surface $D'$. Such a Calabi–Yau threefold is obtained as a small resolution of a nodal Calabi–Yau threefold $X'$ containing the projection $\tilde{D}$ of a del Pezzo surface in its anticanonical embedding. The del Pezzo surface $D'$ can then be contracted and the resulting threefold smoothed (by [C]). We denote the Calabi–Yau threefolds obtained by $Y_t$. In comparison with the construction using del Pezzo surfaces in their anticanonical embedding, in this context new technical problems appear. In particular, we cannot use Theorem 2.1 from [K] to find a nodal Calabi–Yau threefold containing the del Pezzo. Indeed, the base locus of hypersurfaces of minimal degree from the ideal of the projected surface contains
not only the surface $\tilde{D}$ but also its maximal multisecant lines (see Proposition 5.2). Instead, we prove Lemma 4.1 that can be useful in a more general context (cf. [DH]). Throughout the paper we deal with the ideal of the projected del Pezzo surfaces using [AK, KP, P]; we hope our methods will be used in this larger context. It turns out that the generators of the ideals of the del Pezzo surfaces considered are closely connected with the number of multisecant lines to those surfaces (analogous relations for projections of rational normal curves were observed in [P]; see also [T]) and can be studied using the geometry of the natural nodal Calabi–Yau threefold $X'$. The number of multisecants is computed using results of Le Barz [LB1, LB2].

From the results of [KK] we compute the Hodge numbers of $Y_t$ (a generic element of $Y$). We also find other important invariants of the threefolds obtained: the degree of the second Chern class, i.e. $c_2 \cdot H$, and the degree of the generator of the Picard group, i.e. $H^3$, where $H$ is the generator of the Picard group of $Y_t$. The results are presented in Table 1; in each case we have $h^{1,1}(Y_t) = 1$.

| No | $\deg D'$ | $X'$ | $\text{sing } X'$ | $\chi(Y_t)$ | $H^3$ | $h^3(H)$ |
|----|----------|-----|-----------------|------------|-------|----------|
| 1  | 6        | $X'_{2,4}$ | 44 ODP         | -92        | 14    | 7        |
| 2  | 6        | $X'_{2,4}$ | 44 ODP         | -94        | 14    | 7        |
| 3  | 6        | $X'_{3,3}$ | 36 ODP         | -76        | 15    | 7        |
| 4  | 6        | $X'_{3,3}$ | 36 ODP         | -78        | 15    | 7        |
| 5  | 7        | $X'_{3,3}$ | 44 ODP         | -60        | 16    | 7        |
| 6  | 8        | $X'_{3,3}$ | 52 ODP         | -44        | 17    | 7        |
| 7  | 7        | $X'_{2,2,2}$ | 37 ODP       | -74        | 19    | 8        |
| 8  | 8        | $X'_{2,2,2}$ | 44 ODP       | -60        | 20    | 8        |
| 9  | 8        | $X'_{2,2,2}$ | 42 ODP       | -50        | 24    | 9        |
| 10 | 8        | $X'_{2,2,2}$ | 36 ODP       | -          | -     | -        |

The Calabi–Yau threefolds labeled 1, 2, 7, 8, 9 are new. Note that Nos. 4, 5 and 6 have the same numerical invariants as the examples constructed by Tonoli in [T]. Nos. 5 and 6 are studied in the first half of Section 5 and Nos. 7 and 8 are discussed at the end of this section. Note also that for Nos. 1, 2, 3 the Calabi–Yau threefold do not have smoothing in $\mathbb{P}^6$ and the generator of their Picard group is not very ample.

The last construction from Table 1 give an example of type II primitive contraction whose exceptional divisor is the blow-up of the projective plane at a point and completes the classification of primitive contractions of type II (see [K, Prob. 2.1]). In this case the image of the contraction is not smoothable.

The methods developed in this paper permit us also to avoid the computer calculations of [K]. The new ingredient here is the use of the results about the restriction of syzygies from [E-P], and the theory of linkages (see [PS, MN]), to compute the
dimension of the linear system giving the contraction and the degree of the image. The use of the computer algebra system Singular [GPS] is, however, needed to compute Gröbner bases in the larger context studied in this paper (useful Singular scrips are available at the end of the paper). To prove several statements, we find using Singular an example where the statements holds, and then use semicontinuity arguments.

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2. Projections of del Pezzo surfaces

We are interested in minimal resolutions of the ideals of generic linear projections of del Pezzo surfaces. Denote by \( \tilde{D} \subset \mathbb{P}^{N-1} \) the projection of \( D \subset \mathbb{P}^N \) from a generic point of \( \mathbb{P}^N \) (in particular \( \tilde{D} \subset \mathbb{P}^{N-2} \) is a projection from a generic line). Denote by \( D_i \subset \mathbb{P}^i \) the del Pezzo surface of degree \( i \) for \( i \leq 7 \), by \( D_8 \) the double Veronese embedding of a quadric and by \( F_1 \) the blow-up of \( \mathbb{P}^2 \) in one point (embedded by the anticanonical system).

Proposition 2.1. (1) The ideal of \( \tilde{F}_1 \) in the homogeneous coordinate ring of \( \mathbb{P}^7 \) is generated by 11 quadrics and one cubic.

(2) The ideal of \( \tilde{D}_8 \subset \mathbb{P}^7 \) is generated by 11 quadrics.

(3) The ideal of \( \tilde{D}_7 \subset \mathbb{P}^6 \) is generated by 6 quadrics and 3 cubic.

Proof. It is well known (see [Ho]) that the minimal free resolutions of the del Pezzo surfaces \( D_6 \subset \mathbb{P}^6 \), \( D_7 \subset \mathbb{P}^7 \), and \( D_8 \subset \mathbb{P}^8 \) are the following:

\[
\begin{align*}
\mathcal{O}_9(-2) & \leftarrow \mathcal{O}^{16}(-3) \leftarrow \mathcal{O}^9(-4) \leftarrow \mathcal{O}(-6) \leftarrow 0 \\
\mathcal{O}^{14}(-2) & \leftarrow \mathcal{O}^{35}(-3) \leftarrow \mathcal{O}^{35}(-4) \leftarrow \mathcal{O}^{14}(-5) \leftarrow \mathcal{O}(-7) \leftarrow 0 \\
\mathcal{O}^{20}(-2) & \leftarrow \mathcal{O}^{64}(-3) \leftarrow \mathcal{O}^{60}(-4) \leftarrow \mathcal{O}^{64}(-5) \leftarrow \mathcal{O}^{20}(-6) \leftarrow \mathcal{O}(-8)
\end{align*}
\]

respectively. Moreover, \( F_1 \) has the same free resolution as \( D_8 \). It follows from [KP] Thm. 1.2 that the projected surfaces have ideals generated by cubics. The number of quadrics in the ideals is computed using [AK] Cor. 3.12. Let us find the number of cubics. The idea is to compute this number for a concrete example and use the Zariski semicontinuity of the graded Betti numbers on the open set where the Hilbert function is maximal (see [BG] Prop. 2.15]. First, from [AK] Cor. 3.12 the Hilbert function is constant for isomorphic projections. Let us consider the example above in the case of \( D_8 \subset \mathbb{P}^8 \) with coordinates \((x, y, z, t, u, v, w, s, m)\). Such a surface is given by the \( 2 \times 2 \) minors of a symmetric \( 4 \times 4 \) matrix whose entries are linear
forms of the nine coordinates. Let us project from the point \((1, 0, \ldots, 0)\) the surface given by the matrix

\[
\begin{pmatrix}
x & y + x & z & t \\
y + x & u & v & w \\
z & v & s & m - x \\
t & w & m - x & s
\end{pmatrix}.
\]

We can calculate by hand (or with Singular) by eliminating the variable \(x\) that the projected surface needs only quadric generators.

The upper bound of the number of cubics in the other cases is obtained analogously by considering a random example. To obtain a lower bound of this number observe that \(\tilde{F}_1\) has one trisecant line (this follows for example from [LB2]). Next in case 3, the system \(|2H - E_1 - E_2|\) on \(D_7\), where \(E_1, E_2\) are exceptional divisors and \(H\) the pull-back of the hyperplane section from \(\mathbb{P}^2\), is a system of rational normal curves of degree 4. This system defines a 3-dimensional family of \(\mathbb{P}^4 \subset \mathbb{P}^7\) (spanned by the rational curves) covering a generic point in \(\mathbb{P}^7\) (since the trisecant planes to \(D_7\) cover all \(\mathbb{P}^7\) and we can find a rational quartic curve passing through 3 generic points of \(D_7\)). Thus we can find one considered \(\mathbb{P}^4\) passing through the center of projection. It is now enough to remind that the projected rational normal curve of degree 4 in \(\mathbb{P}^3\) is a divisor of bi-degree \((1, 3)\) in a smooth quadric \(\mathbb{P}^1 \times \mathbb{P}^1\) thus needs 3 cubic generators.

Remark 2.2. In the same way we can also show that the ideal of \(\tilde{D}_6 \subset \mathbb{P}^5\) is generated by 2 quadrics and 7 cubics.

3. degree 15 Calabi–Yau threefolds

The methods and remarks from this section can also be applied to Calabi–Yau threefolds considered in [K]. Let \(D_6 \subset \mathbb{P}^6\) be an anticanonically embedded del Pezzo surface of degree 6. Denote by \(\hat{D}\) the projection of \(D_6\) into \(\mathbb{P}^5\) from a generic point \(Q\) in \(\mathbb{P}^6\). It follows from [KP Thm. 1.2] that the ideal of \(\hat{D} \subset \mathbb{P}^5\) is generated by cubics. From [K Thm. 2.1], we infer that the generic complete intersection \(X' \subset \mathbb{P}^5\) of two cubics containing \(\hat{D}\) is a nodal Calabi–Yau threefold. Using Chern classes we compute the 36 nodes on \(X'\). Denote by \(S'\) the surface linked via a general cubic to \(\hat{D}\) on \(X'\), i.e. \(S' \in |3H - \hat{D}|\) where \(H\) is the hyperplane section of \(X' \subset \mathbb{P}^5\).

Lemma 3.1. The surface \(S' \subset \mathbb{P}^5\) is smooth and is contained in a quartic that does not contain \(\hat{D}\).

Proof. From [FS Prop. 4.1] we deduce that \(S'\) is smooth. Next we prove that there is a quartic in the ideal of \(S' \subset \mathbb{P}^5\) that is not generated by the three cubics defining \(S' \cup \hat{D}\). From the following standard liaison exact sequence

\[
0 \to \omega_{\hat{D}}(1) \to \mathcal{O}_{S' \cup \hat{D}}(4) \to \mathcal{O}_{S'}(4) \to 0
\]
we infer $h^0(I_{S'}(4)) > h^0(I_{S'\cup\tilde{D}}(4))$ since $\omega_{\tilde{D}} = O_{\tilde{D}}(-1)$.

Let $G'$ be the smooth surface linked to $S'$ via a general quartic on $X' \subset \mathbb{P}^5$. Denote by $X$ the Calabi–Yau threefold obtained by flopping the exceptional curves of the blowing-up of $X'$ along $\tilde{D}$. Let $D'$ and $G$ be the strict transforms on $X$ of $\tilde{D}$ and $G'$ respectively.

**Proposition 3.2.** The image of $X$ under the morphism $\varphi|_G$ is a threefold $Y \subset \mathbb{P}^6$ of degree 15 with one singular point $P$. Moreover, $X' \subset \mathbb{P}^5$ is the projection of $Y$ from $P$.

**Proof.** First, $G \in |H^* + D'|$ where $H^*$ is the pull-back of $H$ on $X$. So $|G|$ is very ample outside $D'$. From Lemma 3.1 we infer that the effective divisors $D' \subset X$ and $G \subset X$ do not have common components. Since $G|_{D'}$ is trivial we obtain $D' \cap G = \emptyset$. So $|G|$ is base-point-free and contracts $D'$ to a point.

To see that $h^0(O_X(G)) = 7$ we need to prove that $G' \subset \mathbb{P}^5$ is linearly normal (cf. [K, Lem. 2.1]). This follows from the fact that $G$ and $\tilde{D}$ are doubly linked so we have $H^1(I_{\tilde{D}}(k)) = H^1(I_{G'}(k+1))$ for $k \in \mathbb{Z}$ (cf. [MN, Cor. 5.11]).

Finally, the projection of $Y$ from $P$ can be seen as the image of $X$ under the linear subsystem of $|D' + H^*|$ of dimension 6 with $D'$ being a fixed component, thus under $|H^*|$.

**Remark 3.3.** The threefold $Y$ is not normal at $P$. We need to take a multiple of $G$ to obtain a primitive contraction (cf. [KK, Lem. 2.5]). However, it is possible that in some cases $Y$ can be smoothed by Calabi–Yau threefolds in $\mathbb{P}^6$. Note that we know that the germ of the cone over a projected del Pezzo surface of degree 6 can be smoothed by taking hyperplane sections of the cone over the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

**Theorem 3.4.** The morphism $\varphi|_{2G}$ gives a primitive contraction with image being a singular Calabi–Yau threefold that is a degeneration two family of Calabi–Yau threefolds with $h^{1,2} = 39$ and $h^{1,2} = 40$ of degree 15. Moreover, the Picard groups of the threefolds obtained are isomorphic to $\mathbb{Z}$.

**Proof.** If we prove that $\varphi|_{2G}$ is a primitive contraction, then from [G] the resulting singular Calabi–Yau threefolds can be smoothed. The problem is to show the normality of the image. Since from [AK, Thm. 3.1] the restriction $\mathcal{O}_{\mathbb{P}^7}(2H) \to \mathcal{O}_{\tilde{D}}(2H)$ is surjective, we can prove this as in the proof of [K, Thm. 2.3].

Let us show that the rank $\rho(X)$ of the Picard group of $X$ is 2 and that this group is torsion free. First, from [AK, Thm. 3.1], we see that the syzygies between the cubics containing $\tilde{D}$ are linear. Let $C \supset \tilde{D}$ be a general smooth cubic in $\mathbb{P}^5$. Arguing as in the proof of [K, Thm. 2.2] we infer that it is enough to prove that the morphism $\pi$ obtained from the system of cubics on $C$ containing $\tilde{D}$ does not contract any divisor to a curve. From [AK, Prop. 3.1] the two-dimensional fibers of
\[\pi\] are planes cutting \(\tilde{D}\) along cubic curves. Such planes are images of trisecant planes to \(D_6\) passing through the center of the projection \(Q\) (from [E-P, Thm. 1.1] there are no trisecant lines to \(D_6\) since this surface is cut out by quadrics). It follows that the dimension of \(S_3\) is 3. Since the system of cubics containing \(\tilde{D}\) is base-point-free on \(\mathbb{P}^5 - \tilde{D}\) we see that there is at most a one-dimensional family of trisecant lines to \(\tilde{D}\), thus there is no divisor on \(C\) that contracts to a curve.

Denote by \(Y \subset \mathbb{P}^N\) the image \(\varphi|_{\mathcal{O}^2}(X)\) and by \(Y_t (t \in \mathbb{C} \text{ in the neighborhood of } 0)\) a generic element of the smoothing family \(Y\) of \(Y\). First from the proof of [KK, Prop. 3.1] we have \(H^2(Y, \mathbb{Z}) \simeq H^2(Y_t, \mathbb{Z})\). We claim that these cohomology groups are torsion free. Indeed, if \(\mathcal{L}\) is a torsion sheaf on \(\varphi(X)\) then \(\varphi^*(\mathcal{L})\) is torsion on \(X\), moreover it is non-zero form the projection formula (\(\varphi(X)\) is normal). It remains to recall that \(H^2(X, \mathbb{Z})\) is torsion free. Next, the image \(T\) of \(G\) on \(Y\) is an ample divisor (\(2T\) is very ample) such that \(T^3 = 15\) and \(h^0(\mathcal{O}(T)) = 7\) (by Proposition 3.2). From the discussion in [Wi, §3] we obtain, for \(t\) sufficiently small, a flat family of ample (and base-point-free) divisors \(T_t \subset Y_t\) such that \(T_0 = T\). It follows that \(T_t^3 = 15\).

Let us compute \(c_2 \cdot T_t\) where \(T_t\) is the generator of \(H^2(Y_t, \mathbb{Z})\). From [K, Lem. 2.2] we can embed \(Y_t\) into \(\mathbb{P}^N\). This embedding is clearly given by the complete linear system \(2T_t\), since \(h^1(Y_t) = 1\) and the embedding \(Y_t \subset \mathbb{P}^N\) is linearly normal (\(h^1(\mathcal{I}_{Y_t|\mathbb{P}^N}(1)) = 0\)). Since \(c_2 \cdot 2T = 108\), we infer \(c_2 \cdot T_t = 54\); this ends the proof.

**Remark 3.5.** The above families have the same invariants as the following Calabi–Yau threefolds:

1. of degree 15 constructed by Tonoli (see [T]);
2. of degree 15 constructed by Lee in [L1] as double cover of a singular Fano threefold.

It would be interesting to know whether the families obtained are exactly the above ones.

**Remark 3.6.** The Calabi–Yau threefold \(X\) is birational to another interesting Calabi–Yau threefold. Consider the blow-up \(Z\) of \(X'\) along \(\tilde{D}\). Then \(Z\) is a Calabi-Yau threefold with Picard group of rank 2 thus admits a second primitive contraction or a fibration. This morphism is given by a multiple of the pull-back of \(T \in |kH - \tilde{D}|\) on \(Z\), for some \(k \in \mathbb{Z}\), and can be studied by liaison methods as above. We obtain in this way many examples of Calabi-Yau threefolds and geometric transitions. This will be discussed elsewhere.

**Example 3.1.** In an analogous way, we can embed \(\tilde{D}\) into a Calabi–Yau threefold which is a complete intersection of a quadric and a quartic. The problem is to prove that in this case we obtain a nodal Calabi–Yau threefold. Then it must
have 44 nodes, thus the resulting Calabi–Yau threefolds of degree 14 have Euler characteristic $-92$ and $-94$. In order to use [K] Thm. 2.1 it remains to prove the following.

**Lemma 3.7.** A generic quadric from the ideal of $\tilde{D}\subset\mathbb{P}^5$ is smooth.

**Proof.** It is enough to prove that the generic singular quadric from the ideal of $D_6\subset\mathbb{P}^6$ is a cone over a smooth quadric. Recall that the ideal of a del Pezzo surface of degree 6 is given by $2\times2$ minors of a $3\times3$ matrix whose entries are linear forms. We find explicitly a cone over a smooth quadric in this ideal. $\square$

**Remark 3.8.** We compute using Singular that the intersection of two generic quadrics from $I\tilde{D}(2)$ is a nodal Fano threefold with 6 nodes. It would be interesting to try to apply our construction to this threefold.

### 4. degree 24 Calabi–Yau threefolds

Let us consider the projections of del Pezzo surfaces of degree 8. There are two of them, the Hirzebruch surface $F_1\subset\mathbb{P}^8$ and $D_8\subset\mathbb{P}^8$. Denote by $\tilde{D}$ and $\tilde{F}$ the projections of $D_8$ and $F_1$ into $\mathbb{P}^7$ from a generic point $Q$ in $\mathbb{P}^8$. We shall embed $\tilde{D}$ and $\tilde{F}$ into nodal complete intersections of four quadrics in $\mathbb{P}^7$. In these cases however more technical problems arise. We need the following lemma.

**Lemma 4.1.** (cf. [DH]) Let $X\subset\mathbb{P}^n$ be a reduced two-dimensional sub-scheme whose ideal is generated by hypersurfaces of degree $d$. Assume that the scheme $X$ has no embedded components, and has exactly one two-dimensional component $X_e$ such that the other components have smaller dimension and intersect $X_e$ transversally. Then the generic complete intersection of $n-2$ hypersurfaces of degree $d$ from the ideal of $X$ is a nodal variety with nodes lying on $X$.

**Proof.** This proof is a generalization of [K] Thm. 2.1. We need only prove that if $X_e$ is a component of codimension $> n-2$ then the intersection of $n-2$ hypersurfaces from $H^0(\mathcal{I}_X(d))$ is non-singular along $X_e$. Let us consider the following diagram:

$\begin{array}{c}
\mathbb{P}^n \\
\uparrow \quad \nearrow \\
\mathbb{P}^N
\end{array}$

where the vertical map is the blow-up of $\mathcal{I}_X$, denoted by $\pi$. The variety $\mathbb{P}^n$ can be seen as the closure of the graph of the morphism given by $(q_0 : \cdots : q_N)$ (here $q_0, \ldots, q_N$ are the degree $d$ generators of the ideal of $X$). The horizontal map $\beta$ is given by the linear system $H^0(\mathcal{O}(d) \otimes \mathcal{I}_X)$. The remaining map is the projection $p: \mathbb{P}^n \times \mathbb{P}^N \rightarrow \mathbb{P}^N$.

The complete intersections $C$ containing $X$ are the pre-images of linear spaces $L_C$ in $\mathbb{P}^N$. A singularity appears on a normal $C$ (in particular generic, see [DH])
at \( q \in X_e - X_c \) iff \( L_C \) intersects the linear space \( p(\pi^{-1}(q)) \) non-transversally. For dimensional reasons a generic \( L_C \) does not contain any such linear space.

Let us consider the points \( q \in X_e \cap X_c \). Locally analytically the blow-up of \( \mathbb{C}^n \) along the union \( S \) of a line and a plane passing through \( 0 \) has \( \mathbb{P}^{n-2} \) as exceptional locus over \( 0 \) with two distinguished coordinates corresponding to quadrics from the ideal of \( S \). Moreover, an analytic germ at \( 0 \) containing \( S \) is smooth if its strict transform on the exceptional divisor is smooth and meets transversally the codimension 2 linear space determined by the two distinguished coordinates. We conclude that the exceptional divisor \( \pi^{-1}(q) \subset \mathbb{P}^N \) is isomorphic to \( \mathbb{P}^{n-2} \) with two distinguished coordinates. From the Bertini theorem a complete intersection corresponding to \( L_C \) is smooth at \( q \) for a generic choice of \( L_C \).

\[ \square \]

Proposition 4.2. The intersection of the 11 quadrics from the homogeneous ideal \( \mathcal{I}_F \subset \mathbb{P}^7 \) defines scheme-theoretically the union of \( \tilde{F} \) and the unique trisecant line to \( \tilde{F} \) that is transversal to \( \tilde{F} \).

Proof. It follows from \cite[Example 5.16]{CR} that there is exactly one trisecant plane to \( F_1 \) (it is transversal to \( \tilde{F} \)) passing through a generic point in \( \mathbb{P}^8 \), so there is a unique trisecant line \( t \) to \( \tilde{F} \).

Each cubic containing \( t \cup \tilde{F} \) is of the form
\[ a_1q_1 + \ldots + a_{11}q_{11} + b \cdot c \]
where \( q_1, \ldots, q_{11} \) are the quadric generators of \( \tilde{F} \), \( c \) the cubic generator, \( a_1, \ldots, a_{11} \) are linear forms, and \( b \) is a constant. It follows that \( b = 0 \) since \( c|_t \neq 0 \). It is enough to prove that the ideal defining \( t \cup \tilde{F} \) is generated by cubics. We need the following generalization of \cite[Thm. 3.1]{HK}.

Lemma 4.3. Let \( X \subset \mathbb{P}^n \) be a non-degenerate projective variety satisfying property \( N_{3,p} \) and \( q \) be a smooth point of \( X \). Suppose that \( \mathcal{I}_X \) is generated by quadrics. Consider the inner projection \( \pi_q : X \to Y \subset \mathbb{P}^{n-1} \). Then the projected variety \( Y \) satisfies property \( N_{3,p-1} \).

Proof. The proof is analogous to the proof of \cite[Thm. 3.1]{HK}. \[ \square \]

To use Lemma 4.3 we need to show that the ideal of \( F_1 \cup L \subset \mathbb{P}^8 \), where \( L \) is a general trisecant plane, is generated by quadrics. Let \( \mathbb{P}^8 \supset H \) be a 3-dimensional linear space containing \( L \). From \cite[Thm. 1.1]{EP} it follows that \( H \) cuts \( F_1 \) in at most four points. Since \( F_1 \) satisfies property \( N_{2,3} \) we deduce from \cite[Thm. 1.2]{EP} that each reducible quadric \( W \) containing \( L \) and the schematic intersection of \( H \) and \( F_1 \), is the restriction of one from the 17-dimensional family \( \Gamma \) of quadrics containing \( F_1 \). It follows that the scheme determined by the quadrics from \( \Gamma \) cannot have other components then \( F_1 \) and \( L \) (there are no embedded components with support contained in \( L \cap F_1 \) since \( H \) can be chosen generically thus this component would be contained in \( L \)). \[ \square \]
From Lemma 4.1 the generic intersection $X'_1$ (resp. $X'_2$) of four quadrics containing $\tilde{D}$ (resp. $\tilde{F}$) is a nodal Calabi–Yau threefold with 42 (resp. 36) nodes on $\tilde{D}$ (resp. $\tilde{F}$). We denote by $X_1$ (resp. $X_2$) the Calabi–Yau threefold obtained by the flopping of the exceptional curves of the blow-up $X''_1 \to X'_1$ (resp. $X''_2 \to X'_2$), and by $D'$ (resp. $F'$) the strict transforms of $\tilde{D}$ (resp. $\tilde{F}$).

**Theorem 4.4.** The Calabi–Yau threefolds $X_1$ and $X_2$ have Picard group of rank 2. There exist primitive contractions $X_1 \to Y_1$ and $X_2 \to Y_2$ with exceptional divisors $D'$ and $F'$ respectively. Moreover, the threefold $Y_1$ can be smoothed by a family of Calabi–Yau threefolds of degree 24 with Picard groups of rank 1 and $h^{1,2} = 26$.

**Proof.** Since the syzygies between the quadric generators of $\tilde{D}$ and $\tilde{F}$ are not generated by linear forms, we cannot compute $\rho(X_1)$ and $\rho(X_2)$ as before. Let us concentrate on $X_1$.

Consider the cone $C$ over $D_8 \subset \mathbb{P}^8$ with vertex $Q$ being the center of the projection. Let $\alpha : \mathbb{P}^8 - C \to \mathbb{P}^10$ be the morphism given by the system of quadrics containing the cone $C$.

**Claim:** The morphism $\alpha$ is an embedding outside the subset $\text{Join}(D_8, C)$ (i.e. the sum of lines joining points on $D_8$ with points on $C$) of codimension $\geq 2$.

We show that the image under the morphism $\alpha$ of a line $l \subset \mathbb{P}^8$ that is not contained in $\text{Join}(D_8, C) \subset \mathbb{P}^8$ is a line or a conic. Let $\beta : \mathbb{P}^8 \to \mathbb{P}^{19}$ be given by the system $H^0(O(2) \otimes I_{D_8})$. From [AR, Prop. 3.1], it is an embedding off $\text{Sec}(D_8)$. By Proposition 4.2 the image $\beta(C)$ is contained in a 9-dimensional linear space $L$ such that $L$ cuts exactly $\beta(C)$ out of the closure of $\beta(\mathbb{P}_8)$. The morphism $\alpha$ can be seen as the composition of $\beta$ with the projection from $L$. From [E-P, Thm. 1.2] the image $\beta(l)$ is either a plane conic disjoint from $\beta(C)$ so also disjoint from $L$, or a line disjoint from $L$ (if $l$ intersect $D_8$). The claim follows since any 10-dimensional linear space containing $L$ meets $\beta(l)$ in zero, one or, if $\beta(l)$ spans a plane that intersects $L$, in two points.

From [RaS, Thm. 6] we obtain $\rho(X_1) = \rho(X_2) = 2$; the other claim follows as before. \qed

**Remark 4.5.** The example with $F_1$ as exceptional locus completes the classification in [K, Thm. 2.5] of exceptional loci of primitive contractions of type II. The image of such contraction is not smoothable.

**Remark 4.6.** Let us project $D_8$ from a point in $\text{Sec}_3(D_8) - \text{Sec}_2(D_8)$. The resulting surface $\tilde{D}$ has then one cubic and 11 quadric generators. Since $\text{Sec}_3(D_8)$ is degenerate, the intersection of the 11 quadrics is the union of $\tilde{D}$ and a plane intersecting $\tilde{D}$ along a cubic. We cannot perform the previous construction in this case since Lemma 4.1 does not work.
5. Other examples

We shall apply our method to construct families of Calabi–Yau threefolds of degrees 16 and 17 with the same invariants as the Tonoli examples. We project isomorphically, from a generic plane $P \subset \mathbb{P}^8$ (resp. line), the del Pezzo surfaces $D_8$ (resp. $D_7$) of degree 8 (resp. 7) into $\mathbb{P}^5$. From [AK, Cor. 4.10] the resulting surfaces $L_P, K_P \subset \mathbb{P}^5$ are 4-regular and 5-regular respectively; we can prove more.

**Lemma 5.1.** The ideal of the surface $K_P \subset \mathbb{P}^5$ is generated by 1 quartic and 13 cubics; moreover, this surface has exactly one quadrisecant line. The ideal defining $L_P \subset \mathbb{P}^5$ needs 7 quartics and 7 cubics generators, and this surface has exactly 7 quadrisecant lines.

**Proof.** The number of quadrisecant lines is obtained using the formula on page 182 of [LB1]. To compute the invariants $d$, $\delta$ and $t$, we use [LB2, p. 59]. For the surface $L_P$ we obtain $d = 20$, $\delta = 10$ and $t = 20$.

We find the generators using Singular [GPS] for a chosen generic example. Now use the semicontinuity knowing that $K_P$ has at least one quartic generator.

Denote by $S_P$ the scheme which is the union of $L_P \subset \mathbb{P}^5$ and its 4-secant lines. From [AK] we obtain $h^0(I_{L_P}(4)) = 45$. Since the quadrisecant lines are disjoint for $P_0$ a randomly chosen point (so from Lemma 5.3 generically) we have $h^0(O_{S_P} \otimes I_{L_P}) = 7$ (computer calculations). Moreover, by calculations in Singular, we obtain $h^0(I_{S_{P_0}}(4)) = 38$. From the exact sequence

$$0 \rightarrow I_{S_P}(4) \rightarrow I_{L_P}(4) \rightarrow O_{S_P} \otimes I_{L_P}(4) \rightarrow 0,$$

we deduce that $h^1(I_{S_{P_0}}(4)) = 0$. Now by the semicontinuity we obtain $h^1(I_{S_P}(4)) = 0$ for a generic $P$, thus $h^0(I_{S_P}(4)) = 38$. Since the cubics from the ideal of $L_P \subset \mathbb{P}^5$ vanish on $S_P$, we deduce that $L_P$ needs 7 quartics generators. We compute with Singular that $L_{P_0}$ is generated by 7 cubics and 7 quartics and conclude with the semicontinuity argument from [BG].

**Proposition 5.2.** The intersection of the cubics from the homogeneous ideal of $K_P \subset \mathbb{P}^5$ (resp. $L_P \subset \mathbb{P}^5$) defines scheme-theoretically the union of $K_P$ and the unique quadrisecant line (resp. the union of $L_P$ and the quadrisecant lines).

**Proof.** We compute with Singular that the intersection of the cubics containing $L_{P_0}$ define $S_{P_0}$ scheme-theoretically. To prove that this holds for a generic $P \in G(2, 8)$ consider the family

$$\mathcal{B} \supset S \rightarrow U \subset G(2, 8),$$

where $\mathcal{B}$ is the natural $\mathbb{P}^5$ bundle over the subset $U \subset G(2, 8)$ such that $S_P \subset \mathbb{P}^5$ is the smooth fiber over $P$.

**Lemma 5.3.** The family $S \rightarrow U$ is flat.
Proof. Let $Q_P$ be the union of the quadrisecant lines and let $Q \to U$ be the natural family. We shall show that the families $Q$ and $Q \cap L$ with fiber $Q_P \cap L_P$ are flat over an open subset. Consider the Hilbert scheme $\text{Hilb}^4(D_P) \simeq \text{Hilb}^4(D)$ and the scheme $\text{Al}^4(\mathbb{P}^5)$ of aligned points on $\mathbb{P}^5$. It is proved in [LB1] that for generic $P \in G(2,8)$ the intersection $\text{Al}^4(\mathbb{P}^5) \cap \text{Hilb}^4(D_P) \subset \text{Hilb}^4(\mathbb{P}^5)$ has degree 7. Moreover, the family

$$L \to U,$$

where $L$ is the natural family obtained from the family $B \supset L \to U$ by taking $\text{Hilb}^4(\mathbb{P}^5)$ of the fibers, is smooth, so in particular flat. Consider the natural smooth fiber bundle of Hilbert schemes of 4 points in a fiber $\mathcal{H} \to U$ obtained from $B \to U$ and its subbundle $A \to U$ such that $A_P$ is equal to $\text{Al}^4(\mathbb{P}^5)$. We have a natural embedding $f : A \to \mathcal{H}$. Consider the pull-back $f^*(D)$ of the flat family $D \to U$. From [H, III Prop. 9.1A] the family $p : f^*(D) \to U$ is flat (this is exactly the family $Q \cap L$). It remains to remark that in our chosen example the fiber of $p$ is smooth. There exists an open $V \subset U$ such that $p^{-1}(V) \to V$ is smooth, thus flat.

We conclude that $Q \cap L$ is a flat family. Moreover, using the natural morphism $\text{Al}^4(\mathbb{P}^5) \to G(1,5)$ we deduce that $Q$ is a flat family.

Since $L \to V$ is a flat family (it is smooth), we deduce from the exact sequence

$$0 \to I_{Q \cap L} \to \mathcal{O}_L \to \mathcal{O}_{Q \cap L} \to 0$$

that $I_{Q \cap L}$ is a flat $\mathcal{O}_V$-module. From the exact sequence

$$0 \to I_Q \to \mathcal{O}_S \to \mathcal{O}_Q \to 0,$$

we deduce that $S \to V$ is flat. □

To use the semicontinuity arguments [BG] it is enough to show that the Hilbert function $HF_P$ is constant for $P \in W$, where $P_0 \supset W \in G(2,8)$ is open. Since the Hilbert polynomial $HP_P$ is constant in flat families and has the same value as the Hilbert function for $k \geq \text{reg}(S_P)$ (see [E, Thm. 4.2]), it is enough to show that $HF_P(k)$ is constant for

$$\text{reg}(S_P) - 1 \geq k \geq 3.$$

We compute with Singular that $\text{reg}(S_{P_0}) = 5$. Since the regularity is an upper-semicontinuous function (because the set where $H^i(I_{S_P}|_{\mathbb{P}^5}(5-i)) = 0$ for $i \geq 0$ is open) it is enough to check that $HF_P(4)$ is constant. This last number is generically equal to 38 as we saw in the proof of Lemma [5.1] □

Remark 5.4. The analogy between the number of quadrisecant lines and the number of quartic generators seems to be a general phenomenon.

Finally let us now denote by $\tilde{D}_7$ and $\tilde{D}_8$ general projections of del Pezzo surfaces of degrees 7 and 8 to $\mathbb{P}^6$. From Proposition [2.1] the ideal of $\tilde{D}_7$ is generated by cubics.
In the same way we deduce that $\tilde{D}_8 \subset \mathbb{P}^6$ is generated by cubics. The following lemma shows (from [K, Thm. 2.1]) that these surfaces are embedded into nodal Calabi–Yau threefolds in $\mathbb{P}^6$ which are complete intersections of two quadrics and a cubic with 37 (resp. 44) nodes.

**Lemma 5.5.** The generic complete intersection of two quadrics containing the surface $\tilde{D}_7$ (resp. $\tilde{D}_8$) is smooth.

**Proof.** We find explicitly using Singular with a random choice of the center of projection, two quadrics containing the projected surface intersecting each other along a smooth threefold. □

**Remark 5.6.** Note that the intersection of three generic quadrics containing $\tilde{D}_7$ (resp. $\tilde{D}_8$) is a nodal Fano threefold with 16 (resp. 20) nodes.

**Remark 5.7.** It is an open problem to find the mirror families of the obtained Calabi–Yau threefolds. The strategy would to find a weak Landau–Ginzburg model following the Batyrev approach: i.e to embed the given Calabi–Yau threefold as a complete intersection in a Fano manifold, then to degenerate the Fano to a toric $T$ with terminal Gorenstein singularities, and finally find the appropriate Laurent polynomial using the generators of the fans of $T$ (see [Pr]). This problem will be considered in a future paper.

6. Appendix

Recall that the ideal of the del Pezzo surface $D_8$ can be described by the 2 by 2 minors of a symmetric 4 by 4 matrix. The following script gives the ideal of a projection of $D_8$ into $\mathbb{P}^5(t, u, v, w, p, q)$:

```plaintext
ring r=0,(x,y,z,t,u,v,w,p,q),dp;
LIB"random.lib";
matrix A2=randommat(4,4,maxideal(1),7);
matrix A1=A2+transpose(A2);
ideal jj=minor(A1,2);
ideal j=eliminate(jj,xyz);
minbase(j);
```

Recall that the ideal of the del Pezzo surface $D_7$ can be described by the 2 by 2 minors of a partially symmetric 4 by 5 matrix. The following script gives the ideal of a projection of $D_7$ into $\mathbb{P}^5(t, u, v, w, p, q)$:

```plaintext
ring r=0,(y,z,t,u,v,w,p,q),dp;
LIB"random.lib";
matrix A2=randommat(5,5,maxideal(1),7);
```
matrix \( A_1 = A_2 + \text{transpose}(A_2) \);
matrix \( A = \text{submat}(A_1,2..4,1..4) \);
ideal \( jj = \text{minor}(A,2) \);
ideal \( j = \text{eliminate}(jj,xy) \);
\( \text{minbase}(j) \);

7. Appendix 1

We present a list of all Calabi-Yau threefolds with Picard group of rank 1 known to the author. In the references column we show a place where we can find more information about this Calabi-Yau threefold. We put ? in the description column when the Calabi–Yau threefold is only conjectured to exists. Moreover \( B \) denotes a appropriated singular Fano threefold.

| \( H^3 \) | \( h^{1,1} \) | \( h^{1,2} \) | \( \chi \) | \( c_2 \cdot H \) | \( \text{dim}[H] \) | Description | Reference |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 61 | -120 | 22 | 2 | \( X_{5,6} \subset P(1,1,2,2,3,3) \) | [KT] |
| 1 | 1 | 145 | -288 | 34 | 3 | \( X_{10} \subset P(1,1,1,2,5) \) | [ES] |
| 2 | 1 | 23 | -447 | 20 | 2 | ? | [ES] |
| 2 | 1 | 149 | -296 | 44 | 4 | \( X_8 \subset P(1,1,1,1,4) \) | [KT] |
| 3 | 1 | 103 | -204 | 42 | 4 | \( X_6 \subset P(1,1,1,1,2) \) | [KT] |
| 4 | 1 | 73 | -144 | 40 | 4 | \( X_{4,4} \subset P(1,1,1,1,2,2) \) | [KT] |
| 4 | 1 | 129 | -256 | 52 | 5 | \( X_{2,6} \subset P(1,1,1,1,1,3) \) | [KT] |
| 5 | 1 | 51 | -100 | 38 | 4 | ? | [ES] |
| 5 | 1 | 101 | -200 | 50 | 5 | \( X_5 \subset P^4 \) | [COGP] |
| 5 | 1 | 156 | -310 | 62 | 6 | ? | [ES] |
| 6 | 1 | 79 | -156 | 32 | 3 | \( X_{3,4} \subset P(1,1,1,1,1,2) \) | [KT] |
| 6 | 1 | 37 | -72 | 36 | 4 | ? | [ES] |
| 6 | 1 | 129 | -256 | 52 | 5 | \( X_{4,6} \subset P(1,1,1,2,2,3) \) | [KT] |
| 7 | 1 | 61 | -120 | 46 | 5 | ? | [ES] |
| 7 | 1 | 79 | -156 | 58 | 6 | ? | [K] |
| 8 | 1 | 5 | -8 | 32 | 4 | ? | [ES] |
| 8 | 1 | 89 | -176 | 56 | 6 | \( X_{2,4} \subset P^5 \) | [LT] |
| 9 | 1 | 73 | -144 | 56 | 6 | \( X_{3,3} \subset P^5 \) | [LT] |
| 10 | 1 | 26 | -50 | 40 | 5 | ? | [ES] |
| 10 | 1 | 10 | -32 | 40 | 5 | ? | [ES] |
| 10 | 1 | 59 | -116 | 52 | 6 | ? | [ES] |
| 10 | 1 | 59 | -116 | 62 | 7 | ? | [K] |
| 12 | 1 | 10 | -32 | 36 | 6 | ? | [ES] |
| 12 | 1 | 31 | -60 | 48 | 6 | ? | [ES] |
| 12 | 1 | 35 | -68 | 48 | 6 | \( X^{21} \rightarrow B \) | [L] |
| 12 | 1 | 61 | -120 | 84 | 9 | ? | [K] |
| $H^3$ | $h_1^{1.1}$ | $h_1^{1.2}$ | $\chi$ | $c_2 \cdot H$ | $\dim[H]$ | Description | Reference |
|-------|-------------|-------------|--------|--------------|------------|-------------|-----------|
| 12    | 1           | 73          | -144   | 60           | 7          | $X_{2,2,3} \subset P^6$ | LT        |
| 13    | 1           | 61          | -120   | 58           | 7          | $5 \times 5$ Pffafian $\subset P^6$ | T         |
| 13    | 1           | 52          | -102   | 82           | 9          | $\mathbb{K}$ | K         |
| 14    | 1           | 43          | -84    | 80           | 9          | $\mathbb{K}$ | K         |
| 14    | 1           | 44          | -86    | 80           | 9          | $\mathbb{K}$ | K         |
| 14    | 1           | 47          | -92    | 56           | 7          | Table I |           |
| 14    | 1           | 48          | -94    | 56           | 7          | Table I |           |
| 14    | 1           | 50          | -98    | 56           | 7          | $7 \times 7$ Pffafian $\subset P^6$ | Rod, Be   |
| 14    | 1           | 49          | -96    | 56           | 7          | $X \to B$ | L         |
| 14    | 1           | 51          | -100   | 56           | 7          | $X \to B$ | L         |
| 15    | 1           | 35          | -68    | 78           | 9          | $\mathbb{K}$ | K         |
| 15    | 1           | 39          | -76    | 54           | 7          | $X \to B$ | L         |
| 15    | 1           | 39          | -76    | 54           | 7          | $\mathbb{L}$ | Table I |
| 15    | 1           | 40          | -78    | 54           | 7          | $\mathbb{L}$ | Table I |
| 15    | 1           | 40          | -78    | 54           | 7          | $\mathbb{L}$ | Table I |
| 15    | 1           | 43          | -84    | 54           | 7          | $T_{0,15} \subset P^6$ | T, Be     |
| 15    | 1           | 76          | -150   | 66           | 8          | $X_{1,1,3} \subset G(2,5)$ | BCKS      |
| 15    | 1           | 31          | -60    | 52           | 7          | Table I |           |
| 16    | 1           | 31          | -60    | 52           | 7          | $T_{0,16} \subset P^6$ | T         |
| 16    | 1           | 37          | -72    | 52           | 7          | $X \to B$ | L         |
| 16    | 1           | 65          | -128   | 64           | 8          | $X_{2,2,2,2} \subset P^7$ | LT        |
| 17    | 1           | 23          | -44    | 50           | 7          | Table I |           |
| 17    | 1           | 23          | -44    | 50           | 7          | $T_{0,17} \subset P^6$ | T         |
| 17    | 1           | 55          | -108   | 62           | 8          | $\mathbb{K}$ | K         |
| 17    | 1           | 55          | -108   | 62           | 8          | $X \to B$ | L, Be     |
| 17    | 1           | 33          | -64    | 50           | 7          | $X \to B$ | L         |
| 18    | 1           | 45          | -88    | 60           | 8          | $\mathbb{K}$ | K         |
| 18    | 1           | 45          | -88    | 60           | 8          | $\mathbb{K}$ | K         |
| 18    | 1           | 43          | -84    | 60           | 8          | $X \to B$ | L         |
| 18    | 1           | 46          | -90    | 60           | 8          | $\mathbb{K}$ | K         |
| 18    | 1           | 47          | -92    | 60           | 8          | $X \to B$ | L         |
| 18 n² | 1           | 54          | -106   |             |            | $\mathbb{K}$ | K         |
| 19    | 1           | 38          | -74    | 58           | 8          | Table I |           |
| 19    | 1           | 39          | -76    | 58           | 8          | $X \to B$ | L         |
| $H^3$ | $h^{1,1}$ | $h^{1,2}$ | $\chi$ | $c_2 \cdot H$ | $\dim |H|$ | Description | Reference |
|-------|----------|----------|------|-------------|---------|-------------|-----------|
| 20    | 1        | 31       | -60  | 56          | 8       |              |           |
| 20    | 1        | 61       | -120 | 68          | 9       |              |           |
| 21    | 1        | 52       | -102 | 66          | 9       | $X_{1,2,2} \subset G(2,5)$ | BCKS      |
| 21    | 1        | 51       | -100 | 66          | 9       | $X \xrightarrow{2:1} B$ | L         |
| 21    | 1        | 53       | -104 | 66          | 9       | $X \xrightarrow{2:1} B$ | L         |
| 22    | 1        | 47       | -92  | 64          | 9       | $X \xrightarrow{2:1} B$ | L         |
| 24    | 1        | 26       | -50  | 60          | 9       |              |           |
| 24    | 1        | 59       | -116 | 72          | 10      | $X_{1,1,1,1,1,1,2} \subset X_{10}$ | ES        |
| 25    | 1        | 51       | -100 | 70          | 10      |              |           |
| 25    | 1        | 51       | -100 | 70          | 10      | $X \xrightarrow{2:1} B$ | L         |
| 28    | 1        | 59       | -116 | 76          | 11      | $X_{1,1,1,1,2} \subset G(2,6)$ | ES        |
| 29    | 1        | 51       | -100 | 74          | 11      | $X \xrightarrow{2:1} B$ | L         |
| 29    | 1        | 53       | -104 | 74          | 11      | $X \xrightarrow{2:1} B$ | L         |
| 30    | 1        | 49       | -96  | 74          | 11      | $X \xrightarrow{2:1} B$ | L         |
| 32    | 1        | 59       | -116 | 80          | 12      | $X_{1,1,2} \subset LG(3,6)$ | ES        |
| 32    | 1        | 59       | -116 | 80          | 12      | $X \xrightarrow{2:1} B$ | L         |
| 33    | 1        | 52       | -102 | 78          | 12      | $3 \times 3$ minors of $5 \times 5$ sym. mat. | KK1       |
| 34    | 1        | 45       | -88  | 76          | 12      | $X \xrightarrow{2:1} B$ | L         |
| 34    | 1        | 49       | -96  | 76          | 12      | $X \xrightarrow{2:1} B$ | L         |
| 35    | 1        | 26       | -50  | ?           | 11      | $3 \times 3$ minors of $5 \times 5$ sym. mat. | KK1       |
| 36    | 1        | 37       | -72  | 72          | 12      | $X_{1,2} \subset X_5$ | ES        |
| 42    | 1        | 50       | -98  | 84          | 14      | $X_{1,1,1,1,1,1,1} \subset G(2,7)$ | BCKS      |
| 42    | 1        | 49       | -96  | 84          | 14      | $X_{1,1,1,1,1,1,1} \subset G(3,6)$ | BCKS      |
| 44    | 1        | 65       | -128 | 92          | 15      | $X_{2,1} \xrightarrow{2:1} A_{2,2}$ | ES        |
| 47    | 1        | 46       | -90  | 86          | 15      | $X_{1,1,1,1} \subset F_1(Q_5)$ | ES        |
| 48    | 1        | 79       | -156 | 96          | 16      | $X_{1,1,1,1} \subset F_1(Q_5)$ | ES        |
| 56    | 1        | 47       | -92  | 92          | 17      | $X_{1,1,1,1} \subset F_1(Q_5)$ | ES        |
| 57    | 1        | 43       | -84  | 90          | 17      | $X_{1,1,1,1} \subset F_1(Q_5)$ | ES        |
| 74    | 1        | 29       | -56  | 92          | 20      | $X_{1,1,1,1} \subset F_1(Q_5)$ | ES        |
| 78    | 1        | 31       | -60  | 96          | 21      | $X_{1,1,1,1} \subset F_1(Q_5)$ | ES        |
| 78    | 1        | 33       | -64  | 96          | 21      | $X_{1,1,1,1} \subset F_1(Q_5)$ | ES        |
| $H^3$ | $h^{1,1}$ | $h^{1,2}$ | $\chi$ | $c_2 \cdot H$ | $\text{dim}[H]$ | Description | Reference |
|-------|-----------|-----------|--------|---------------|---------------|-------------|-----------|
| 79    | 1         | 25        | -48    | 94            | 21            |             | BK        |
| 80    | 1         | 101       | -200   | 128           | 24            |             | BK        |
| 82    | 1         | 36        | -70    | 100           | 22            |             | BK        |
| 83    | 1         | 31        | -60    | 98            | 22            |             | BK        |
| 83    | 1         | 32        | -62    | 98            | 22            |             | BK        |
| 86    | 1         | 41        | -80    | 104           | 23            |             | BK        |
| 87    | 1         | 35        | -68    | 102           | 23            |             | BK        |
| 88    | 1         | 29        | -56    | 100           | 23            |             | BK        |
| 91    | 1         | 40        | -78    | 106           | 24            |             | BK        |
| 92    | 1         | 35        | -68    | 104           | 24            |             | BK        |
| 92    | 1         | 36        | -70    | 104           | 24            |             | BK        |
| 93    | 1         | 29        | -56    | 102           | 24            |             | BK        |
| 96    | 1         | 39        | -76    | 108           | 25            |             | BK        |
| 97    | 1         | 33        | -64    | 106           | 25            |             | BK        |
| 97    | 1         | 34        | -66    | 106           | 25            |             | BK        |
| 97    | 1         | 35        | -68    | 106           | 25            |             | BK        |
| 98    | 1         | 29        | -56    | 104           | 25            |             | BK        |
| 98    | 1         | 30        | -58    | 104           | 25            |             | BK        |
| 98    | 1         | 31        | -60    | 104           | 25            |             | BK        |
| 98    | 1         | 32        | -62    | 104           | 25            |             | BK        |
| 99    | 1         | 28        | -54    | 102           | 25            |             | BK        |
| 102   | 1         | 34        | -66    | 108           | 26            |             | BK        |
| 102   | 1         | 35        | -68    | 108           | 26            |             | BK        |
| 102   | 1         | 38        | -74    | 108           | 26            |             | BK        |
| 103   | 1         | 30        | -58    | 106           | 26            |             | BK        |
| 103   | 1         | 31        | -60    | 106           | 26            |             | BK        |
| 104   | 1         | 28        | -54    | 104           | 26            |             | BK        |
| 107   | 1         | 36        | -70    | 110           | 27            |             | BK        |
| 108   | 1         | 30        | -58    | 108           | 27            |             | BK        |
| 108   | 1         | 31        | -60    | 108           | 27            |             | BK        |
| 108   | 1         | 32        | -62    | 108           | 27            |             | BK        |
| 108   | 1         | 33        | -64    | 108           | 27            |             | BK        |
| 108   | 1         | 129       | -256   | 156           | 31            |             | BK        |
| 112   | 1         | 35        | -68    | 112           | 28            |             | BK        |
| 113   | 1         | 32        | -62    | 110           | 28            |             | BK        |
| 116   | 1         | 41        | -80    | 116           | 29            |             | BK        |
| 117   | 1         | 37        | -72    | 114           | 29            |             | BK        |
| 118   | 1         | 31        | -60    | 112           | 29            |             | BK        |
| $H^3$ | $h^{1,1}$ | $h^{1,2}$ | $\chi$ | $c_2 \cdot H$ | $\dim|H|$ | Description | Reference |
|---|---|---|---|---|---|---|---|
| 118 | 1 | 32 | -62 | 112 | 29 | | BK |
| 123 | 1 | 34 | -66 | 114 | 30 | | BK |
| 124 | 1 | 31 | -60 | 112 | 30 | | BK |
| 136 | 1 | 55 | -108 | 124 | 33 | | BK |
| 144 | 1 | 45 | -88 | 120 | 34 | | BK |
| 144 | 1 | 47 | -92 | 120 | 34 | | BK |
| 152 | 1 | 40 | -76 | 116 | 35 | | BK |
| 168 | 1 | 51 | -100 | 132 | 39 | | BK |
| 168 | 1 | 53 | -104 | 132 | 39 | | BK |
| 176 | 1 | 47 | -92 | 128 | 40 | | BK |
| 200 | 1 | 51 | -100 | 140 | 45 | | BK |
| 232 | 1 | 53 | -104 | 148 | 51 | | BK |
| 432 | 1 | 79 | -156 | 192 | 88 | | BK |
| 648 | 1 | 103 | -204 | 252 | 129 | | BK |
| ? | 1 | 1 | ? | ? | ? | not simply connected | D |
| ? | 1 | 4 | ? | ? | ? | not simply connected | D |

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