PALEY-WIENER-SCHWARTZ NEARLY PARSEVAL FRAMES
AND BESOV SPACES ON NONCOMPACT SYMMETRIC
SPACES

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ABSTRACT. Let $X$ be a symmetric space of the noncompact type. The goal of
the paper is to construct in the space $L^2(X)$ nearly Parseval frames consist-
ing of functions which simultaneously belong to Paley-Wiener spaces and to
Schwartz space on $X$. We call them Paley-Wiener-Schwartz frames in $L^2(X)$.
These frames are used to characterize a family of Besov spaces on $X$. As a
part of our construction we develop on $X$ the so-called average Shannon-type
sampling.

1. Introduction

Wavelet systems and frames which build up of bandlimited functions with a
strong localization on the space became very popular in theoretical and applied
analysis on Euclidean spaces. Frames with similar properties were recently con-
structed in non-Euclidean settings in $L^2$-spaces on spheres [19], compact Riemann-
ian manifolds [12]-[15], certain metric-measure spaces [6]. In [15] Parseval ban-
dimited and localized frames were developed on compact homogeneous manifolds.
Bandlimited ($\equiv$ Paley-Wiener) frames in Paley-Wiener spaces on noncompact man-
ifolds of bounded geometry (in particular on noncompact symmetric spaces) were
constructed in [24]-[30], [10], [11], [5], [7], [8]. In [22], [23] bandlimited frames in
Paley-Wiener spaces (in subelliptic framework) were constructed on stratified Lie
groups. Bandlimited and localized nearly Parseval frames on domains in $\mathbb{R}^n$ with
smooth boundaries can be found in [31].

The goal of our development is to construct Paley-Wiener localized (Schwartz)
frames in $L^2$ spaces on symmetric manifolds of the noncompact type. It should be
noted that such manifolds do not satisfy requirements of the paper [6].

A Riemannian symmetric space of the noncompact type is a Riemannian man-
ifold $X$ of the form $X = G/K$ where $G$ is a connected semisimple Lie group with
finite center and $K$ is a maximal compact subgroup of $G$. The most known examples
of such spaces are the real, complex, and quaternionic hyperbolic spaces.

Let $f$ be a function in the space $L^2(X, d\mu(x)) = L^2(X)$, where $X$ is a symmetric
space of noncompact type and $dx$ is an invariant measure. The notations $Ff = \hat{f}$
will be used for the Helgason-Fourier transform of $f$. The Helgason-Fourier
transform $\hat{f}$ can be treated as a function on $\mathbb{R}^n \times B$ where $B$ is a certain compact
homogeneous manifold and $n$ is the rank of $X$. Moreover, $\hat{f}$ belongs to the space
$L^2 (\mathbb{R}^n \times B; |c(\lambda)|^{-2}d\lambda db)$, where $c(\lambda)$ is the Harish-Chandra’s function, $d\lambda$
is the

Euclidean measure and $db$ is the normalized invariant measure on $B$. The notation $\Pi_{[\omega_1, \omega_2]} \subset \mathbb{R}^n \times B$, $0 < \omega_1 < \omega_2$, will be used for the set of all points $(\lambda, b) \in \mathbb{R}^n \times B$ for which $\omega_1 \leq \sqrt{(\lambda, \lambda)} \leq \omega_2$, where $(\cdot, \cdot)$ is the Killing form. In particular the notation $\Pi_{[0, \omega]} \subset \mathbb{R}^n \times B$ will be used for $\Pi_{[0, \omega]}$.

The Paley-Wiener space $PW_{[\omega_1, \omega_2]}(X)$, $0 < \omega_1 < \omega_2$, is defined as the set of all functions in $L_2(X)$ whose Helgason-Fourier transform has support in $\Pi_{[\omega_1, \omega_2]}$ and belongs to the space $\Lambda_\omega = L_2\left(\Pi_{[\omega_1, \omega_2]}; |c(\lambda)|^{-2} d\lambda db\right)$. In particular, $PW_\omega(X)$ will be used for $PW_{[0, \omega]}(X)$.

Using the $K$-invariant distance on $X$ to the "origin" (see formula (2.2) below) one can introduce a notion of an $L_2$-Schwartz space $S^2(X)$ (see Definition 3 below) which was considered by M. Eguchi [9]. A theorem of M. Eguchi [9] states that if a function $f$ is in $C_0^\infty(\mathbb{R}^n \times B)$ and satisfies certain symmetry conditions (see (2.13) below) then its inverse Helgason-Fourier transform is a function in $S^2(X)$. A refinement of this result was given by N. B. Andersen in [1] (see Theorems 3.3 and 3.4 below).

Although some facts that we discuss in this development already appeared in our previous papers the main result about existence of Paley-Wiener-Schwartz nearly Parseval frames in $L_2(X)$ is completely new. Here is a formulation of our main theorem.

**Theorem 1.1.** Suppose that $X$ is a Riemannian symmetric spaces of the noncompact type. For every $0 < \delta < 1$ there exists a countable family of functions $\{\Theta_{j, \gamma}\}$ such that

1. Every function $\Theta_{j, \gamma}$ is bandlimited to $[2^j-1, 2^{j+1}]$ in the sense that $\Theta_{j, \gamma} \in PW_{[2^{j-1}, 2^{j+1}]}(X)$.
2. Every function $\Theta_{j, \gamma}$ belongs to $S^2(X)$.
3. $\{\Theta_{j, \gamma}\}$ is a frame in $L_2(X)$ with constants $1 - \delta$ and $1 + \delta$, i.e.

$$(1 - \delta)\|f\|^2 \leq \sum_{j \in \mathbb{N}} \sum_{\gamma} |(f, \Theta_{j, \gamma})|^2 \leq (1 + \delta)\|f\|^2, \quad f \in L_2(X).$$

In section 2 we summarize basic facts about harmonic analysis on Riemannian symmetric spaces of the noncompact type. In subsection 2.2 we prove a covering Lemma for Riemannian manifolds of bounded geometry whose Ricci curvature is bounded from below. In section 3 we introduce Paley-Wiener spaces $PW_\omega(X)$, $\omega > 0$. In section 4 we develop average sampling and almost Parseval frames in Paley-Wiener spaces on Riemannian manifolds. The main result is obtained in section 5 where we construct nearly Parseval Paley-Wiener-Schwartz frames in $L_2(X)$. In the last section we characterize Besov spaces $B_{2,q}^\omega(X)$ in terms of coefficients with respect to constructed frames. The entire scale of Besov spaces will be considered in a separate paper.

2. **Harmonic analysis on Riemannian symmetric spaces of the noncompact type**

2.1. **Riemannian symmetric spaces of the noncompact type.** A Riemannian symmetric space of the noncompact type is a Riemannian manifold $X$ of the form $X = G/K$ where $G$ is a connected semisimple Lie group with finite center and $K$ is a maximal compact subgroup of $G$. The Lie algebras of the groups $G$ and $K$ will be denoted respectively as $\mathfrak{g}$ and $\mathfrak{k}$. The group $G$ acts on $X$ by left translations.
If \( e \) is the identity in \( G \) then the base point \( eK \) is denoted by \( 0 \). Every such \( G \) admits Iwasawa decomposition \( G = NAK \), where the nilpotent Lie group \( N \) and the abelian group \( A \) have Lie algebras \( n \) and \( a \) respectively. Correspondingly \( g = n \oplus a \oplus k \). The dimension of \( a \) is known as the rank of \( X \). The letter \( M \) is usually used to denote the centralizer of \( A \) in \( K \) and the letter \( B \) is commonly used for the homogeneous space \( K/M \).

The Killing form on \( g \) induces an \( \text{Ad}K \)-invariant inner product on \( n \oplus a \oplus k \) which generates a \( G \)-invariant Riemannian metric on \( X \). With this metric \( X = G/K \) becomes a Riemannian globally symmetric space of the noncompact type.

Let \( a^* \) be the real dual of \( a \) and \( W \) be the Weyl's group. Let \( \Sigma \) be the set of restricted roots, and \( \Sigma^+ \) will be the set of all positive roots. The notation \( a^+ \) has the following meaning

\[
a^+ = \{ H \in a | \alpha(H) > 0, \alpha \in \Sigma^+ \}
\]

and is known as positive Weyl's chamber. Let \( \rho \in a^* \) is defined in a way that \( 2\rho \) is the sum of all positive restricted roots. The Killing form \( <,> \) on \( a \) defines a metric on \( a \). By duality it defines an inner product on \( a^* \).

We denote by \( a^*_- \) the set of \( \lambda \in a^* \), whose dual belongs to \( a^+ \). According to Iwasawa decomposition for every \( g \in G \) there exist a unique \( A(g) \), \( H(g) \in a \) such that

\[
g = n \exp A(g)k = k \exp H(g)n, \quad k \in K, \quad n \in N, \quad A(g) = -H(g^{-1}),
\]

where \( \exp : a \to A \) is the exponential map of the Lie algebra \( a \) to Lie group \( A \). On the direct product \( X \times B \) we introduce function with values in \( a \) using the formula

\[
A(x, b) = A(u^{-1}g)
\]

where \( x = gK, g \in G, b = uX, u \in K \).

According to Cartan decomposition every element \( g \) of \( G \) has representation \( g = k_1 \exp(H)k_2 \), where \( H \) belongs to the closure of \( \exp a^+ \). The norm of an \( g \) in \( G \) is introduced as

\[
|g| = |k_1 \exp(H)k_2| = \|H\|.
\]

It is the \( K \)-invariant geodesic distance on \( X \) of \( gK \) to \( eK \).

2.2. A covering Lemma for Riemannian manifolds of bounded geometry whose Ricci curvature is bounded from below. Let \( X \), \( \dim X = d \), be a connected \( C^\infty \)-smooth Riemannian manifold with a \((2,0)\) metric tensor \( g \) that defines an inner product on every tangent space \( T_x(X), x \in X \). The corresponding Riemannian distance \( d \) on \( X \) and the Riemannian measure \( d\mu(x) \) on \( X \) are given by

\[
d(x, y) = \inf \int_a^b \sqrt{g \left( \frac{dx}{dt}, \frac{dx}{dt} \right)} \, dt, \quad d\mu(x) = \sqrt{|\det(g_{ij})|} \, dx,
\]

where the infimum is taken over all \( C^1 \)-curves \( \alpha : [a, b] \to X, \alpha(a) = x, \alpha(b) = y \), the \( \{g_{ij}\} \) are the components of the tensor \( g \) in a local coordinate system and \( dx \) is the Lebesgue’s measure in \( R^d \). Let \( \exp_p : T_x(X) \to X \) be the exponential geodesic map i. e. \( \exp_p(u) = \gamma(1), u \in T_x(X) \), where \( \gamma(t) \) is the geodesic starting at \( x \) with the initial vector \( u \) : \( \gamma(0) = x, \frac{dx}{dt}(0) = u \). We denote by \( \text{inj} \) the largest real number \( r \) such that \( \exp_x \) is a diffeomorphism of a suitable open neighborhood of \( 0 \) in \( T_xX \) onto \( B(x, r) \), for all sufficiently small \( r \) and \( x \in X \). Thus for every choice
of an orthonormal basis (with respect to the inner product defined by $g$) of $T_x(X)$ the exponential map $\exp$ defines a coordinate system on $B(x, r)$ which is called geodesic. The volume of the ball $B(x, r)$ will be denoted by $|B(x, r)|$. Throughout the paper we will consider only geodesic coordinate systems.

A Riemannian symmetric space $X$ equipped with an invariant metric has bounded geometry which means that

(a) $X$ is complete and connected;
(b) the injectivity radius $\text{inj}(X)$ is positive;
(c) for any $r \leq \text{inj}(X)$, and for every two canonical coordinate systems $\varphi_x : T_x(X) \to B(x, r), \varphi_y : T_y(X) \to B(x, r)$, the following inequalities holds true:

$$\sup_{x \in B(x, r)/B(y, r)} \| \varphi_x^{-1} \varphi_y \| \leq C(r, k).$$

The Ricci curvature $\text{Ric}$ of $X$ is bounded from below, i.e.

$$\text{Ric} \geq -kg, \ k \geq 0$$

According to the Bishop-Gromov Comparison Theorem this fact implies the so-called local doubling property: for any $0 < \sigma < \lambda < r < \text{inj}(X)$:

$$|B(x, \lambda)| \leq (\lambda/\sigma)^d e^{(kr(d-1))^{1/2}} |B(x, \sigma)|, \ d = \dim X.$$

We will need the following lemma which was proved in [25].

**Lemma 2.1.** If $X$ is a Riemannian manifold of bounded geometry and its Ricci curvature is bounded from below then there exists a natural $N_X$ such that for any $0 < r < \text{inj}(X)$ there exists a set of points $X_r = \{x_i\}$ with the following properties

1. the balls $B(x_i, r/4)$ are disjoint,
2. the balls $B(x_i, r/2)$ form a cover of $X$,
3. the height of the cover by the balls $B(x, r)$ is not greater than $N_X$.

**Proof.** Assumptions (a)-(c) imply that there exist constants $a, b > 0$ such that

$$a \leq \frac{|B(x, r)|}{|B(y, r)|} \leq b, x, y \in X, r < \text{inj}(X),$$

where $\text{inj}(X)$ is the injectivity radius of the manifold. Let us choose a family of disjoint balls $B(x_i, r/4)$ such that there is no ball $B(x, r/4), x \in X$, which has empty intersections with all balls from our family. Then the family $B(x_i, r/2)$ is a cover of $X$. Every ball from the family $\{B(x_i, r)\}$ having non-empty intersection with a particular ball $B(x_j, r)$ is contained in the ball $B(x_j, 3r)$. Since any two balls from the family $\{B(x_i, r/4)\}$ are disjoint, the inequalities (2.5) and (2.4) give the following estimate for the multiplicity $N$ of the covering $\{B(x_i, r)\}$:

$$N \leq \frac{\sup_{y \in X} |B(y, 3r)|}{\inf_{x \in X} |B(x, r/4)|} \leq C(X)b12^d = N_X, \ d = \dim X.$$

Thus the lemma is proved. \hfill $\square$

**Definition 1.** Every set of point $X_r = \{x_i\}$ that satisfies conditions of Lemma 2.1 will be called a $r$-lattice.

To construct Sobolev spaces $H^k(X)$, $k \in \mathbb{N}$, we fix a $\lambda$-lattice $X_\lambda = \{y_\nu\}$, $0 < \lambda < \text{inj}(X)$ and introduce a partition of unity $\varphi_\nu$ that is subordinate to the family $\{B(y_\nu, \lambda/2)\}$ and has the following properties:

1. $\varphi_\nu \in C^\infty_0 B(y_\nu, \lambda/2)$,
(2) $\sup_x \sup_{|\alpha| \leq k} |\varphi^{(\alpha)}(x)| \leq C(k)$, where $C(k)$ is independent on $\nu$ for every $k$ in geodesic coordinates.

The exponential map $\exp_{y_{\nu}} : T_{y_{\nu}} M \to M$ is a diffeomorphism of a ball $B_{T_{y_{\nu}}}(0, r) \subset T_{y_{\nu}} M$ with center 0 and of radius $r$ on a ball $B(y_{\nu}, r)$ in Riemannian metric on $M$ (assuming that $r > 0$ is sufficiently small). If $M$ is homogeneous and a metric is invariant then every ball $B(y_{\nu}, r)$ is a translation on a single ball. In this case there exist two constants $c_1, C_1$ such that for any ball $B(x, r)$ with $x \in B(y_{\nu}, r)$ and $\rho < r$ one has

$$B_{T_{y_{\nu}}}(\exp_{y_{\nu}}^{-1}(x), c_1 r) \subset \exp_{y_{\nu}}^{-1} (B(x, r) \cap B(y_{\nu}, r)) \subset B_{T_{y_{\nu}}}(\exp_{y_{\nu}}^{-1}(x), C_1 r).$$

Note that for a Riemannian measure $d\mu(x)$ and a locally integrable function $F$ on $U \subset M$ the integral of $F$ over $U$ is defined as follows

$$\int_U F(x) d\mu(x) = \int_{\exp_{y_{\nu}}^{-1}(U)} F \circ \exp_{y_{\nu}}(x_1, \ldots, x_d) \sqrt{\det(g_{ij})} \, dx_1 \ldots dx_d,$$

where $g_{ij} = g(\partial_i, \partial_j)$, and $g$ is the Riemann inner product in tangent space. By choosing a basis $\partial_1, \ldots, \partial_d$, which is orthonormal with respect to $g$ we obtain $|\det(g_{ij})| = 1$.

We introduce the Sobolev space $H^k(X), k \in \mathbb{N}$, as the completion of $C^0_0(X)$ with respect to the norm

$$\| f \|_{H^k(X)} = \left( \sum_{\nu} \| \varphi_{\nu} f \|_{H^k(B(y_{\nu}, \lambda/2))}^2 \right)^{1/2},$$

where

$$\| \varphi_{\nu} f \|_{H^k(B(y_{\nu}, \lambda/2))}^2 = \sum_{1 \leq |\alpha| \leq k} \| \partial^{(\alpha)} \varphi_{\nu} f \|_{L^2(B(y_{\nu}, \lambda/2))}^2$$

and all partial derivatives are computed in a fixed canonical coordinate system $\exp_{y_{\nu}}$ on $B(y_{\nu}, \lambda/2)$.

**Remark 2.2.** A geodesic coordinate system $\exp_y^{-1}$ depends on the choice of a basis in the tangent space $T_y, y \in X$. We assume that such basis is fixed and orthonormal for every $y = y_{\nu} \in X_r = \{y_{\nu}\}$.

The Laplace-Beltrami which is given in a local coordinate system by the formula

$$\Delta f = \sum_{m,k} \frac{1}{\sqrt{\det(g_{ij})}} \partial_m \left( \sqrt{\det(g_{ij})} g^{mk} \partial_k f \right),$$

where $g_{ij}$ are components of the metric tensor, $\det(g_{ij})$ is the determinant of the matrix $(g_{ij})$, $g^{mk}$ components of the matrix inverse to $(g_{ij})$. It is known that the operator $(-\Delta)$ is a self-adjoint positive definite operator in the corresponding space $L_2(X, d\mu(x))$, where $d\mu(x)$ is the $G$-invariant measure. The regularity Theorem for the Laplace-Beltrami operator $\Delta$ states that domains of the powers $(-\Delta)^{s/2}$ coincide with the Sobolev spaces $H^s(X)$ and the norm $\|f\|_{L_2(X)}$ is equivalent to the graph norm $\|f\| + \|(-\Delta)^{s/2} f\|$ (see [24], Sec. 7.4.5.) Moreover, since the operator $\Delta$ is invertible in $L_2(X)$ the Sobolev norm is also equivalent to the norm $\|(-\Delta)^{s/2} f\|$. 
2.3. Helgason-Fourier transform on Riemannian symmetric spaces of the noncompact type. For every \( f \in C_0^\infty(X) \) the Helgason-Fourier transform is defined by the formula

\[
\hat{f}(\lambda, b) = \int_X f(x)e^{-(i\lambda + \rho)(A(x, b))}dx,
\]

where \( \lambda \in a^* \), \( b \in B = K/X \), and \( dx \) is a \( G \)-invariant measure on \( X \). This integral can also be expressed as an integral over the group \( G \). Namely, if \( b = uX, u \in K \), then

\[
\hat{f}(\lambda, b) = \int_G f(gK)e^{-(i\lambda + \rho)(A(u^{-1}g))}dg.
\]

The invariant measure on \( X \) can be normalized so that the following inversion formula holds for \( f \in C_0^\infty(X) \)

\[
f(x) = w^{-1} \int_{a^* \times B} \hat{f}(\lambda, b)e^{(i\lambda + \rho)(A(x, b))}|c(\lambda)|^{-2}d\lambda db,
\]

where \( w \) is the order of the Weyl’s group and \( c(\lambda) \) is the Harish-Chandra’s function, \( d\lambda \) is the Euclidean measure on \( a^* \) and \( db \) is the normalized \( K \)-invariant measure on \( B \). This transform can be extended to an isomorphism between the spaces \( L_2(X, d\mu(x)) \) and \( L_2(a^*_+ \times B, |c(\lambda)|^{-2}d\lambda db) \) and the Parseval’s formula holds true

\[
\int_X f_1(x)f_2(x)d\mu(x) = \int_{a^*_+ \times B} \hat{f}_1(\lambda, b)\hat{f}_2(\lambda, b)c(\lambda)|^{-2}d\lambda db
\]

which implies the Plancherel’s formula

\[
\|f\| = \left( \int_{a^*_+ \times B} |\hat{f}(\lambda, b)|^2|c(\lambda)|^{-2}d\lambda db \right)^{1/2}.
\]

Let \( \Delta \) be the Laplace-Beltrami operator of the \( G \)-invariant Riemannian structure on \( X \). It is known that the following formula holds

\[
\widehat{\Delta f}(\lambda, b) = -\left(\|\lambda\|^2 + \|\rho\|^2\right)\hat{f}(\lambda, b), f \in C_0^\infty(X),
\]

where \( \|\lambda\|^2 = \langle \lambda, \lambda \rangle, \|\rho\|^2 = \langle \rho, \rho \rangle \), \( \langle \cdot, \cdot \rangle \) is the Killing form.

2.4. A Paley-Wiener Theorem on \( X \). A function \( \phi(\lambda, b) \) in \( C^\infty(a^*_+ \times B) \), holomorphic in \( \lambda \), is called a holomorphic function of uniform exponential type \( \sigma \), if there exists a constant \( \sigma \geq 0 \), such that, for each \( N \in \mathbb{N} \) one has

\[
\sup_{(\lambda, b) \in a^*_+ \times B} e^{-\sigma|3\lambda|(1 + |\lambda|)^N}|\phi(\lambda, b)| < \infty.
\]

The space of all holomorphic functions of uniform exponential type \( \sigma \) will be denoted \( \mathcal{H}_\sigma(a^*_+ \times B) \) and

\[
\mathcal{H}(a^*_+ \times B) = \bigcup_{\sigma > 0} \mathcal{H}_\sigma(a^*_+ \times B).
\]

One also need the space \( \mathcal{H}(a^*_+ \times B)^W \) of all functions \( \phi \in \mathcal{H}(a^*_+ \times B) \) that satisfy the following property

\[
\int_B e^{(iw\lambda + \rho)(A(x, b))}\phi(w\lambda, b)db = \int_B e^{(i\lambda + \rho)(A(x, b))}\phi(\lambda, b)db,
\]

for all \( w \in W \) and all \( \lambda \in a^*_+ \), \( x \in X \).

The following analog of the Paley-Wiener Theorem is known.
Theorem 2.3. The Helgason-Fourier transform \( (2.10) \) is a bijection of \( C^\infty_0(X) \) onto the space \( \mathcal{H}(a_+^\infty \times B)^W \) and the inverse of this bijection can be expressed as

\[
(2.14) \quad f(x) = \int_{a_+^\infty \times B} \hat{f}(\lambda, b) e^{(i\lambda + \rho)(A(x,b))} |c(\lambda)|^{-2} d\lambda db.
\]

In particular, \( \hat{f} \) belongs to the space \( \mathcal{H}_\sigma(a_+^\infty \times B)^W \) if and only if the support of \( f \) is in the ball \( B_\sigma \). Here \( B_\sigma \) is the ball in invariant metric on \( X \) whose radius is \( \sigma \) and center is \( eK \).

3. Paley-Wiener spaces \( PW_\omega(X) \)

**Definition 2.** We will say that \( f \in L^2(X, d\mu(x)) \) belongs to the class \( PW_\omega(X) \) if its Helgason-Fourier transform \( \hat{f} \in L^2(a_+^\infty \times B) \) has compact support in the sense that \( \hat{f}(\lambda, b) = 0 \) a.e. for \( \|\lambda\| > \omega \). Such functions will be also called \( \omega \)-band limited.

Using the spectral resolution of identity \( P_\lambda \) we define the unitary group of operators by the formula

\[
e^{it\Delta} f = \int_0^\infty e^{i\tau} dP_\tau f, \quad f \in L^2(X), \quad t \in \mathbb{R}.
\]

Let us introduce the operator

\[
(3.1) \quad R_\Delta^\sigma f = \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{i\frac{\pi}{2}(k-1/2)} \Delta f, \quad f \in L^2(X), \quad \sigma > 0.
\]

Since \( \|e^{it\Delta} f\| = \|f\| \) and

\[
(3.2) \quad \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(k-1/2)^2} = \sigma,
\]

the series in \( (3.1) \) is convergent and it shows that \( R_\Delta^\sigma \) is a bounded operator in \( L^2(X) \) with the norm \( \sigma \):

\[
(3.3) \quad \|R_\Delta^\sigma f\| \leq \sigma \|f\|, \quad f \in L^2(X).
\]

The next theorem contains generalizations of several results from the classical harmonic analysis (in particular the Paley-Wiener theorem) and it follows essentially from our more general results in [24]-[31] (see also [1], [20]).

**Theorem 3.1.** Let \( f \in L^2(X) \). Then the following statements are equivalent:

1. \( f \in PW_\omega(X) \);
2. \( f \in C^\infty(X) = \bigcap_{k=1}^{\infty} H^k(X) \), and for all \( s \in \mathbb{R}_+ \) the following Bernstein inequality holds:

\[
(3.4) \quad \|\Delta^s f\| \leq (\omega^2 + \|\rho\|^2)^s \|f\|;
\]

3. \( f \in C^\infty(X) \) and the following Riesz interpolation formula holds

\[
(3.5) \quad (i\Delta)^n f = \left( R_\Delta^{\omega^2+\|\rho\|^2} \right)^n f, \quad n \in \mathbb{N};
\]

4. For every \( g \in L^2(X) \) the function \( t \mapsto \langle e^{it\Delta} f, g \rangle, t \in \mathbb{R} \), is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type \( \omega^2 + \|\rho\|^2 \).
(5) The abstract-valued function \( t \mapsto e^{it\Delta}f \) is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type \( \omega^2 + ||\rho||^2 \).

(6) A function \( f \in L_2(X) \) belongs to the space \( PW_\omega(X) \), \( 0 < \omega_f < \infty \), if and only if \( f \) belongs to the set \( C^\infty(X) \), the limit
\[
\lim_{k \to \infty} \|\Delta^k f\|^{1/k}
\]
exists and
\[
\lim_{k \to \infty} \|\Delta^k f\|^{1/k} = \omega_f^2 + ||\rho||^2.
\]

(7) A function \( f \in L_2(X) \) belongs to \( PW_\omega(X) \) if and only if \( f \in C^\infty(X) \) and the upper bound
\[
\sup_{k \in \mathbb{N}} \left( (\omega^2 + ||\rho||^2)^{-k} \|\Delta^k f\| \right) < \infty
\]
is finite.

(8) A function \( f \in L_2(X) \) belongs to \( PW_\omega(X) \) if and only if \( f \in C^\infty(X) \) and
\[
\lim_{k \to \infty} \|\Delta^k f\|^{1/k} = \omega^2 + ||\rho||^2 < \infty.
\]
In this case \( \omega = \omega_f \).

(9) The solution \( u(t), t \in \mathbb{R}^1 \), of the Cauchy problem
\[
\frac{\partial u(t)}{\partial t} = \Delta u(t), u(0) = f, i = \sqrt{-1},
\]
has a holomorphic extension \( u(z) \) to the complex plane \( \mathbb{C} \) satisfying
\[
\|u(z)\|_{L_2(X)} \leq e^{(\omega^2 + ||\rho||^2)|z|} \|f\|_{L_2(X)}.
\]

Now we are going to prove the following density result which shows that for every \( \omega > 0 \) the subspace \( PW_\omega(X) \) contains "many" functions.

**Theorem 3.2.** For every \( \omega > 0 \) and every open set \( V \subset X \) if a function \( f \in C^\infty_0(V) \) is orthogonal to all functions in \( PW_\omega(X) \) then \( f \) is zero.

**Proof.** Assume that \( f \in C^\infty_0(V) \) is a such function and extend it by zero outside of \( V \). By the Paley-Wiener Theorem and Parseval’s formula the transform \( \hat{f}(\lambda, b) \) is in \( C^\infty(\mathfrak{a}^*_+ \times \mathcal{B}) \) and holomorphic in \( \lambda \) and at the same time should be orthogonal to all functions in \( L_2(\Pi(0, \omega) \times B; |e(\lambda)|^{-2}d\lambda db) \), where \( \Pi(0, \omega) = \{ \lambda \in \mathfrak{a}^*_+ : ||\lambda|| \leq \omega \} \).
It implies that \( \hat{f} \) is zero. The theorem is proved. \( \square \)

3.1. On decay of Paley-Wiener functions. In this section we closely follow Andersen [1]. Let us introduce the following spherical function
\[
\varphi_0(g) = \int_K e^{\mu A(k^{-1}g)}dk = \int_K e^{\mu H(g^{-1}k)}dk.
\]

**Definition 3.** The \( L_2 \)-Schwartz space \( S^2(X) \) is introduced as the space of all \( f \in C^\infty(X) \) such that
\[
\sup_{x \in X} (1 + |x|)^N \varphi_0(x)^{-1} |Df(x)| \leq \infty, \quad N \in \mathbb{N} \cup 0,
\]
for all \( D \in U(g) \), where \( U(g) \) is the universal enveloping algebra of \( g \). Here \( |x| = |g| \), for \( x = gK \in X \) where \( | \cdot | \) is defined in (2.3).
The space $S^2(X)$ can also be characterized as the space of all functions for which

$$(1 + |g|)^N Df(g) \in L_2(X), \quad N \in \mathbb{N} \cup 0,$$

for all $D \in U(g)$, where $|x| = |g|$, for $x = gK \in X$.

Let $C_0^\infty(a^* \times B)^W$ be a subspace of functions in $C_0^\infty(a^* \times B)$ that satisfy the symmetry condition (2.13) for all $w \in W$, $\lambda \in a^*$, $x \in X$.

The following fact is proved by M. Eguchi in [9], Theorem 4.1.1.

**Theorem 3.3.** The inverse Helgason-Fourier transform (2.14) maps $C_0^\infty(a^* \times B)^W$ into Schwartz space $S^2(X)$.

In order to give a more detailed statement we will introduce several notations.

**Definition 4.** The space $PW_S_\omega(X)$ is defined as the set of all functions $f$ in $PW_\omega(X)$ such that for all natural $m, n$

$$(1 + |x|)^m \Delta^n f(x) \in L_2(X),$$

where $|x| = |g|$, for $x = gK \in X$. The space $PW_S(X)$ is defined as the union

$$\bigcup_{\omega > 0} PW_{S_\omega}(X).$$

We will also need the function subspace $C_\omega^\infty(a^* \times B)$ which is defined as the set of all functions $f$ in $C_0^\infty(a^* \times B)$ for which

$$\sup_{(\lambda, b) \in \text{supp} f} ||\lambda|| = \omega.$$

The space $C_\omega^\infty(a^* \times B)^W$ is a subspace of functions in $C_\omega^\infty(a^* \times B)$ that satisfy symmetry condition (2.13) for all $w \in W$, $\lambda \in a^*$, $x \in X$.

The following theorem was proved by N. B. Andersen in [1], Theorem 5.7.

**Theorem 3.4.** The inverse Helgason-Fourier transform is a bijection of $C_\omega^\infty(a^* \times B)^W$ onto $PW_{S}(X)$, mapping $C_\omega^\infty(a^* \times B)^W$ onto $PW_{S_\omega}(X)$.

4. Average sampling and almost Parseval frames in Paley-Wiener spaces on Riemannian manifolds

Let $X_r = \{x_k\}$ be a $r$-lattice and $\{B(x_k, r)\}$ be an associated family of balls that satisfy only properties (1) and (2) of the Lemma 2.1. We define

$$U_1 = B(x_1, r/2) \setminus \bigcup_{i \neq 1} B(x_i, r/4),$$

and

$$U_k = B(x_k, r/2) \setminus \left( \bigcup_{j < k} U_j \cup \bigcup_{i \neq k} B(x_i, r/4) \right).$$

One can verify the following.

**Lemma 4.1.** The sets $\{U_k\}$ form a disjoint measurable cover of $X$ and

$$(4.1) \quad B(x_k, r/4) \subset U_k \subset B(x_k, r/2)$$

With every $U_k$ we associate a function $\psi_k \in C_0^\infty(U_k)$ and we will always assume that for every $k$ it is not identical zero and

$$(4.2) \quad 0 \leq \psi_k \leq 1,$$

We introduce the following family $\Psi = \{\Psi_k\}$ of functionals $\Psi_k$ on $L_2(X)$:

$$\Psi_k(F) = \frac{1}{|U_k|\psi_k} \int_{U_k} F(x)\psi_k(x) d\mu(x) = \frac{1}{|U_k|\psi_k} \int_{exp^{-1}(U_k)} F(expy_w(x))\psi_k(expy_w(x)) dx,$$
where \( x = (x_1, \ldots, x_d), \) \( dx = dx_1 \ldots dx_d \) and
\[
|U_k| \psi_k = \int_{\exp^{-1}U_k} \psi_k(\exp_{y_k}x)dx = \int_{U_k} \psi_k(x)d\mu(x).
\]

Our local Poincare-type inequality is the following [23].

**Lemma 4.2.** For \( m > d/2 \) there exist constants \( C = C(X, m) > 0, \) \( r(X, m) > 0, \) such that for any \( r \)-lattice \( X_r \) with \( r < r(X, m) \) and any associated functional \( \Psi_k \) the following inequality holds true for \( f \in H^m(X) \):
\[
(4.3) \quad \| (\varphi_{\nu} f) - \Psi_k((\varphi_{\nu} f)) \|_{L_2(U_k)}^2 \leq C(X, m) \sum_{1 \leq |\alpha| \leq m} r^{2|\alpha|} \| \delta^\alpha (\varphi_{\nu} f) \|_{L_2(B(x_k, r))}^2,
\]
where for \( \alpha = (\alpha_1, \ldots, \alpha_d) \) \( \partial^\alpha f = \partial^{\alpha_1} \ldots \partial^{\alpha_d} f \) is a partial derivative of order \( |\alpha| \) in a fixed geodesic coordinate system \( \exp_{y_k} \) in \( B(y, \lambda) \) (see [23]).

**Proof.** It is obvious that the following relations hold:
\[
B_{T_{y_k}}(x_k, r/4) = \exp_{y_k}^{-1}B(x_k, r/4) \subset \exp_{y_k}^{-1}U_k \subset \exp_{y_k}^{-1}B(x_k, r) = B_{T_{y_k}}(x_k, r)
\]

For every smooth \( f \) and all \( y = (y_1, \ldots, y_d) \in \exp_{y_k}^{-1}U_k \subset B(x_k, r/2), x = (x_1, \ldots, x_d) \in B(x_k, r/2), \) we have the following
\[
\varphi_{\nu} f(\exp_{y_k}x) = \varphi_{\nu} f(\exp_{y_k}y) + \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{|\alpha|!} \partial^\alpha \varphi_{\nu} f(\exp_{y_k}y)(x-y)^\alpha + \frac{1}{|\alpha|!} \int_0^t t^{m-1} \partial^\alpha \varphi_{\nu} f(\exp_{y_k}(y+t\vartheta))\vartheta^\alpha dt,
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_d), \) \( \alpha! = \alpha_1! \ldots \alpha_d!, \) \( (x - y)^\alpha = (x_1 - y_1)^{\alpha_1} \ldots (x_d - y_d)^{\alpha_d}, \)
\( \eta = ||x - y||, \) \( \vartheta = (x - y)/\eta. \)

We multiply this inequality by \( \psi_k(\exp_{y_k}y) \) and integrate over \( \exp_{y_k}^{-1}U_k \subset B_{T_{y_k}}(x_k, r) \) with respect to \( d\mu(y). \) It gives
\[
\varphi_{\nu} f(\exp_{y_k}x) - \Psi_k(\varphi_{\nu} f) = |U_k|^{-1} \psi_k \int_{\exp_{y_k}^{-1}U_k} \left( \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{|\alpha|!} \partial^\alpha \varphi_{\nu} f(\exp_{y_k}y)(x-y)^\alpha \right) \psi_k(\exp_{y_k}y)dy + |U_k|^{-1} \psi_k \int_{\exp_{y_k}^{-1}U_k} \left( \sum_{|\alpha| = m} \frac{1}{(m-1)!} \int_0^t t^{m-1} \partial^\alpha \varphi_{\nu} f(\exp_{y_k}(y+t\vartheta))\vartheta^\alpha dt \right) \psi_k(\exp_{y_k}y)dy,
\]
where
\[
\Psi_k(\varphi_{\nu} f \circ \exp_{y_k}) = \int_{\exp_{y_k}^{-1}U_k} \varphi_{\nu} f \circ \exp_{y_k}(x) \psi_k(\exp_{y_k}(x))dx = \int_{U_k} \varphi_{\nu} f(x) \psi_k(x)d\mu(x) = \Psi_k(\varphi_{\nu} f).
\]

Then since \( \psi_k \geq 0 \)
\[
|\varphi_{\nu} f(\exp_{y_k}x) - \Psi_k(\varphi_{\nu} f)| \leq |U_k|^{-1} \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{|\alpha|!} \int_{\exp_{y_k}^{-1}U_k} |\partial^\alpha \varphi_{\nu} f(\exp_{y_k}y)(x-y)^\alpha| \psi_k(\exp_{y_k}y)dy +
\]
We square this inequality and integrate over $exp_{y_v}^{-1}U_k$:

$$\|\varphi_v f - \Psi_k(\varphi_v f)\|_{L^2(U_k)}^2 = \|\varphi_v f(y_v) - \Psi_k(\varphi_v f)\|_{L^2(exp_{y_v}^{-1}U_k)}^2 \leq C(m)\|U_k\|_{L^2}^{-2} \sum_{1 \leq |\alpha| \leq m-1} \int_{exp_{y_v}^{-1}U_k} \left( \int_{exp_{y_v}^{-1}U_k} |\partial^\alpha \varphi_v f(y_v) (x-y)^\alpha| \psi_k(y_v) dy \right)^2 dx +$$

$$C(m)\|U_k\|_{L^2}^{-2} \sum_{|\alpha|=m} \int_{exp_{y_v}^{-1}U_k} \left( \int_{exp_{y_v}^{-1}U_k} \int_0^\eta t^{m-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)^\alpha| \psi_k(y_v)dy \right)^2 dx =$$

(4.5) \[ I + II. \]

Since $exp_{y_v}^{-1}U_k \subset B_{r/2}(x_k, r)$, $x, y \in U$, one has $x - y \in B(x_k, r)$, and an application of the Schwartz inequality gives

$$\int_{exp_{y_v}^{-1}U_k} \left| \partial^\alpha \varphi_v f(y_v) (x-y)^\alpha \right| \psi_k(y_v) dy \leq \rho^{|\alpha|} \|U_k\|_{L^2}^{-1/2} \|\partial^\alpha \varphi_v f \circ exp_{y_v} \|_{L^2(U_k)}.$$ 

After all we obtain

(4.6) \[ I \leq C(X, m) \sum_{1 \leq |\alpha| \leq m-1} \rho^{2|\alpha|} \|\partial^\alpha \varphi_v f \|^2_{L^2(B(x_k, r))}. \]

Another application of the Schwartz inequality gives for $|\alpha| = m$

$$\left( \int_{exp_{y_v}^{-1}U_k} \left. \int_0^\eta t^{m-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)^\alpha| \psi_k(y_v) dy \right| dx \right)^2 \leq$$

$$\left( \int_{exp_{y_v}^{-1}U_k} \psi_k(y_v) dy \right) \left( \int_{exp_{y_v}^{-1}U_k} \int_0^\eta t^{m-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)^\alpha| \psi_k(y_v) dy \right)^2 =$$

$$|U_k|_{\psi_k} \left( \int_{exp_{y_v}^{-1}U_k} \int_0^\eta t^{m-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)^\alpha| \psi_k(y_v) dy \right)^2.$$

By the same Schwartz inequality using the assumption $m > d/2$ one can obtain the following estimate

$$\left( \int_0^\eta t^{m-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)^\alpha| \psi_k(y_v) dy \right)^2 \leq C\eta^{2m-d} \int_0^\eta t^{d-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)|^2 dt.$$

Next,

$$\left( \int_{exp_{y_v}^{-1}U_k} \left. \int_0^\eta t^{m-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)^\alpha| \psi_k(y_v) dy \right| dx \right)^2 \leq$$

$$C \left( \int_{exp_{y_v}^{-1}U_k} \eta^{2m-d} \int_0^\eta t^{d-1} |\partial^\alpha \varphi_v f(y_v) (y_v + t\vartheta)|^2 dt \psi_k(y_v) dy \right)$$

Thus, we have

$$II \leq$$
We will need the inequality (4.9) below. One has for all $\psi$ is not identical to zero. Our Lemma 4.3. (compare to [24], [25]).

\begin{align*}
C \int \frac{r}{2} t^{-d-1} (\int_0^r \eta^{2m-d} |\partial^\alpha \varphi_{\nu} f(x) dx|) \psi_k d\mu(x) & \leq C \int \frac{r}{2} t^{-d-1} (\int_0^r \eta^{2m-d} |\partial^\alpha \varphi_{\nu} f(x) dx|) \psi_k d\mu(x) \\
& \leq \left( 1 + \frac{\delta}{3} \right) \|f\|^2 \\
& \leq 2 \|A\| \|f\|^2 + \|A - B\| \|f\|^2, \quad \|A - B\| \leq \alpha^{-1} \|B\|^2, \\
& \alpha > 0.
\end{align*}

We have

\begin{align*}
(1 - \alpha) |\varphi_{\nu} f|^2 & \leq (1 - \alpha) |\varphi_{\nu} f - \Psi_k(\varphi_{\nu} f)|^2 + |\Psi_k(\varphi_{\nu} f)|^2, \\
& \leq \frac{1}{\alpha} |A - B|^2 + |B|^2, \quad 0 < \alpha < 1.
\end{align*}

We introduce the following set of functionals

$$A_k(f) = \sqrt{|U_k|} \Psi_k(f) = \sqrt{|U_k|} \Psi_k \int_{U_k} f(x) \psi_k d\mu(x),$$

where $|U_k| = \int_{U_k} d\mu(x)$, and

$$|U_k| \psi_k = \int_{U_k} \psi_k(x) d\mu(x), \quad \psi_k \in C_0^\infty(U_k),$$

where $\psi_k$ is not identical to zero. Our global Poincare-type inequality is the following (compare to [24], [25]).

\textbf{Lemma 4.3.} For any $0 < \delta < 1$ and $m > d/2$ there exist constants $c = c(X), C = C(X, m)$, such that the following inequality holds true for any $r$-lattice with $r < c\delta$

\begin{equation}
(1 - 2\delta/3) \|f\|^2 \leq \sum_k |A_k(f)|^2 + C \delta^{-1} \|\Delta^{m/2} f\|^2 \quad \text{for all } f \in H^m(X).
\end{equation}

\textbf{Proof.} We will need the inequality (4.9) below. One has for all $\alpha > 0$

$$|A|^2 = |A - B|^2 + 2|A - B| |B|, \quad 2|A - B| |B| \leq \alpha^{-1} |A - B|^2 + \alpha |B|^2,$$

which imply the inequality

$$(1 + \alpha)^{-1} |A|^2 \leq \alpha^{-1} |A - B|^2 + |B|^2, \quad \alpha > 0.$$

If, in addition, $0 < \alpha < 1$, then one has

$$|A|^2 \leq \frac{1}{\alpha} |A - B|^2 + |B|^2, \quad 0 < \alpha < 1.$$

We have

$$\frac{1}{\alpha} |\varphi_{\nu} f|^2 \leq \frac{1}{\alpha} |\varphi_{\nu} f - \Psi_k(\varphi_{\nu} f)|^2 + |\Psi_k(\varphi_{\nu} f)|^2,$$

and since $U_k$ form a disjoint cover of $X$ we obtain

\begin{align*}
(1 - \alpha) \|\varphi_{\nu} f\|^2_{L^2(B(y_\nu, \lambda/2))} & \leq \sum_k (1 - \alpha) \|\varphi_{\nu} f\|^2_{L^2(U_k)} \\
& \leq \alpha^{-1} \sum_k \|\varphi_{\nu} f - \Psi_k(\varphi_{\nu} f)\|^2_{L^2(U_k)} + \sum_k |U_k| \|\Psi_k(\varphi_{\nu} f)\|^2, \quad |U_k| = \int_{U_k} d\mu(x).
\end{align*}
For $\alpha = \delta/3$ by using (4.3) we have
\[
(1-\delta/3)\|\varphi f\|_{L^2_2(B(y,\lambda/2))}^2 \leq \sum_{k} |U_k| |\Psi_k(\varphi f)|^2 + 3\delta^{-1} \sum_{k} \|\varphi f - \Psi_k(\varphi f)\|_{L^2_2(U_k)}^2 \\
\sum_{k} |U_k| |\varphi f|^2 + 3C\delta^{-1} \sum_{k=1}^{\lfloor \alpha \rfloor} \sum_{\nu \leq \alpha \leq m} r^{2|\alpha|} \|\varphi f\|_{L^2_2(B(x_k,\nu\cdot r))}^2 \\
\sum_{k} |U_k| |\varphi f|^2 + 3CNX \delta^{-1} \sum_{\nu \leq \alpha \leq m} r^{2|\alpha|} \|\varphi f\|_{L^2_2((B(y,\lambda/2)))}^2,
\]
and summation over $\nu$ gives
\[
(1-\delta/3)\|f\|_{L^2_2(X)}^2 = (1-\delta/3) \sum_{\nu} \|\varphi f\|_{L^2_2(B(y,\lambda/2))}^2 \\
\sum_{\nu} \sum_{k} |U_k| |\Psi_k(\varphi f)|^2 + 3C\delta^{-1} \sum_{\nu \leq \alpha \leq m} \sum_{j=1}^{m} r^{2j} \|f\|_{H^j(X)}^2 \\
\sum_{\nu} \sum_{k} |U_k| |\varphi f|^2 + 3CNX \delta^{-1} \sum_{\nu \leq \alpha \leq m} \sum_{j=1}^{m} r^{2j} \|f\|_{H^j(X)}^2
\]

The regularity theorem for the elliptic second-order differential operator $\Delta$ shows that for all $j \leq m$ there exists a $b = b(X, m)$ such that
\[
\|f\|_{H^j(X)}^2 \leq b \left( \|f\|_{L^2_2(X)}^2 + \|\Delta^{j/2} f\|_{L^2_2(X)}^2 \right), \quad f \in \mathcal{D}(\Delta^{m/2}), \quad b = b(X, m).
\]
Together with the following interpolation inequality which holds for general self-adjoint operators
\[
r^{2j} \|\Delta^{j/2} f\|_{L^2_2(X)}^2 \leq 4a^{m-j} r^{2m} \|\Delta^{m/2} f\|_{L^2_2(X)}^2 + ca^{-j} \|f\|_{L^2_2(X)}^2, \quad c = c(X, m),
\]
for any $a, r > 0, 0 \leq j \leq m$, it implies that there exists a constant $C'' = C''(X, m)$ such that the next inequality holds true
\[
(1-\delta/3)\|f\|_{L^2_2(X)}^2 \leq \sum_{\nu} \sum_{k} |U_k| |\Psi_k(\varphi f)|^2 +
C'' \left( r^{2\delta^{-1}} \|f\|_{L^2_2(X)}^2 + r^{2m\delta^{-1}} \|\Delta^{m/2} f\|_{L^2_2(X)}^2 + a^{-1} \|f\|_{L^2_2(X)}^2 \right),
\]
where $m > d/2$. By choosing $a = (6C''/\delta) > 1$ we obtain, that there exists a constant $C''' = C'''(X, m)$ such that for any $0 < \delta < 1$ and $r > 0$
\[
(1-\delta/2)\|f\|_{L^2_2(X)}^2 \leq \sum_{\nu} \sum_{k} |U_k| |\Psi_k(\varphi f)|^2 + C''' \left( r^{2\delta^{-1}} \|f\|_{L^2_2(X)}^2 + r^{2m\delta^{-1}} \|\Delta^{m/2} f\|_{L^2_2(X)}^2 \right).
\]
The last inequality shows, that if for a given $0 < \delta < 1$ the value of $r$ is choosen such that
\[
r < c\delta, \quad c = \frac{1}{\sqrt{6C'''}, \quad C''' = C'''(X, m),
\]
then we obtain for a $m > d/2$
\[
(1-2\delta/3)\|f\|_{L^2_2(X)}^2 \leq \sum_{\nu} \sum_{k} |U_k| |\Psi_k(\varphi f)|^2 + C''' \delta^{-1} r^{2m} \|\Delta^{m/2} f\|_{L^2_2(X)}^2,
\]
where $|U_k| = \int_{U_k} d\mu(x)$. Lemma is proved.
Definition 5. We introduce the following functions θν,k ∈ C∞(X)

\[ \theta_{\nu,k} = \frac{\sqrt{U_k}}{|U_k|\psi_k} \varphi_{\nu}. \]

Thus for any \( f \in L_2(X) \) we have

\[ \langle f, \theta_{\nu,k} \rangle = \frac{\sqrt{U_k}}{|U_k|\psi_k} \int_{U_k} f \varphi_{\nu} d\mu(x). \]

Theorem 4.4. There exists a constant \( a_0 = a_0(X) \) such that, if for a given \( 0 < \delta < 1 \) and an \( \omega > 0 \) one has

\[ r < a_0 \delta^{1/d} (\omega^2 + \|\rho\|_2^2)^{-1/2}, \]

and the weight functions \( \psi_k \) chosen in a way that

\[ 1 \leq \sup_k \frac{|U_k|}{|U_k|\psi_k} \leq 1 + \delta, \]

then for the corresponding set of functions \( \theta_{\nu,k} \) defined in (4.13) the following inequalities hold

\[ (1 - \delta) \|f\|_{L_2(X)}^2 \leq \sum_{\nu} \sum_k |\langle f, \theta_{\nu,k} \rangle|^2 \leq (1 + \delta) \|f\|_{L_2(X)}^2, \]

or

\[ (1 - \delta) \|f\|_{L_2(X)}^2 \leq \sum_k |U_k| |\Psi_k(\varphi_{\nu} f)|^2 \leq (1 + \delta) \|f\|_{L_2(X)}^2, \]

where \( 0 < \delta < 1, \ f \in PW_\omega(X) \).

Remark 4.5. Note that functions \( \theta_{\nu,k} \) do not belong to \( PW_\omega(X) \). The theorem actually says that projections \( \theta_{\nu,\omega,k} \) of these functions onto \( PW_\omega(X) \) form an almost Parseval frame in \( PW_\omega(X) \).

Proof. By using the Schwartz inequality we obtain for \( f \in L_2(X) \)

\[ \sum_{\nu} \sum_k |U_k| |\Psi_k(\varphi_{\nu} f)|^2 = \sum_{\nu} \sum_k \frac{|U_k|}{|U_k|\psi_k} \left| \int_{U_k} \psi_k \varphi_{\nu} f d\mu(x) \right|^2 \leq \]

\[ \sum_{\nu} \sum_k \frac{|U_k|}{|U_k|\psi_k} \int_{U_k} |\varphi_{\nu} f|^2 d\mu(x) \leq (1 + \delta) \sum_{\nu} \int_{B(y_{\nu},\lambda)} |\varphi_{\nu} f|^2 d\mu(x) = (1 + \delta) \|f\|_{L_2(X)}^2, \]

where we used the assumption (5.5). According to the previous lemma, there exist \( c = c(X), \ C = C(X) \) such that for any \( 0 < \delta < 1 \) and any \( \rho < c\delta \)

\[ (1 - 2\delta/3) \|f\|_{L_2(X)}^2 \leq \sum_k |U_k| |\Psi_k(\varphi_{\nu} f)|^2 + Cr^{2d}\delta^{-1} \|\Delta^{d/2} f\|_{L_2(X)}^2. \]

Notice, that if \( f \in PW_\omega(X) \), then the Bernstein inequality holds

\[ \|\Delta^{d/2} f\|_{L_2(X)}^2 \leq (\omega^2 + \|\rho\|_2^2)^d \|f\|_{L_2(X)}^2. \]

Inequalities (4.18) and (4.19) show that for a certain \( a_0 = a_0(X) \), if

\[ r < a_0 \delta^{1/d} (\omega^2 + \|\rho\|_2^2)^{-1/2}, \]
then
\begin{equation}
(1 - \delta)\|f\|^2_{L^2(X)} \leq \sum_k |U_k|\|\Psi_k(\varphi)f\|^2 \leq (1 + \delta)\|f\|^2_{L^2(X)},
\end{equation}
where \(0 < \delta < 1\), \(f \in PW_\omega(X)\). Theorem is proved.

5. Nearly Parseval Paley-Wiener frames on \(X = G/K\)

5.1. Paley-Wiener almost Parseval frames on \(X = G/K\). Let \(g \in C^\infty(\mathbb{R}_+^\infty)\) be a monotonic function such that \(\text{supp } g \subset [0, 2]\), and \(g(s) = 1\) for \(s \in [0, 1]\), \(0 \leq g(s) \leq 1, s > 0\). Setting \(Q(s) = g(s) - g(2s)\) implies that \(0 \leq Q(s) \leq 1, s \in \text{supp } Q \subset [2^{-1}, 2]\). Clearly, \(\text{supp } Q(2^{-j}s) \subset [2^{-1}, 2^{j+1}], j \geq 1\). For the functions
\begin{equation}
F_0(\lambda, b) = \sqrt{g(|\lambda|)} \otimes 1_B, \quad F_j(\lambda, b) = \sqrt{Q(2^{-j}||\lambda||)} \otimes 1_B, \quad j \geq 1,
\end{equation}
one has
\begin{equation}
\sum_{j \geq 0} F_j^2(\lambda, b) = 1_{\mathbb{R}^n \times B}.
\end{equation}
By using the fact the Helgason-Fourier transform \(F\) and its inverse \(F^{-1}\) are isomorphisms between the spaces \(L^2(X, d\mu(x)) = L^2(X)\) and \(L^2(a_+^\infty \times B, |c(\lambda)|^{-2}d\lambda db)\) and by using the Paseval’s formula we can introduce the following self-adjoint bounded operator for every smooth compactly supported function \(\Phi\) in \(C^\infty_0(\mathbb{R}^n \times B)\)
\begin{equation}
\Phi(\Delta)f = F^{-1}(\Phi(\lambda, b)f(\lambda, b))
\end{equation}
which maps \(L^2(X)\) onto \(PW_{[\omega_1, \omega_2]}(X)\) if \(\text{supp } \Phi \subset [\omega_1, \omega_2]\).

We are using this definition in the case \(F_j^2 = \Phi:\)
\begin{equation}
F_j^2(\Delta)f = F^{-1}_j(\Phi(\lambda, b)f(\lambda, b)).
\end{equation}
Taking inner product with \(f\) we obtain
\begin{equation}
\|F_j(\Delta)f\|^2 = \langle F_j^2(\Delta)f, f \rangle
\end{equation}
and then \(\text{sum}\) gives
\begin{equation}
\|f\|^2 = \sum_{j \geq 0} \langle F_j^2(\Delta)f, f \rangle = \sum_{j \geq 0} \|F_j(\Delta)f\|^2.
\end{equation}

Let \(\theta_{\nu, k}\) be the same as in Definition 5.1 and \(\theta_{\nu, j, k} \in PW_{2j+1}(X)\) be their orthogonal projections on \(PW_{2j+1}(X)\).

According to Theorem 5.4 for a fixed \(0 < \delta < 1\) there exists a constant \(a_0 = a_0(X)\) such that if for \(\omega_j = 2^{j+1}\) and
\begin{equation}
r_j = a_0\delta^{1/d}(\omega^2 + ||\rho||^2)^{-1/2} = a_0\delta^{1/d}(2^{2j+2} + ||\rho||^2)^{-1/2}, \quad j \in \mathbb{N} \cup 0,
\end{equation}
and the weight functions \(\psi_{j, k}\) chosen in a way that
\begin{equation}
1 \leq \sup_{j, k} \frac{|U_{j, k}|}{|U_{j, k}|\psi_{j, k}} \leq 1 + \delta,
\end{equation}
then the set of functions \(\theta_{\nu, j, k} \in PW_{2j+1}(X)\) form a frame in \(PW_{2j+1}(X)\) and
\begin{equation}
(1 - \delta)\|f\|^2 \leq \sum_{\nu} \sum_k |\langle f, \theta_{\nu, j, k} \rangle|^2 \leq (1 + \delta)\|f\|^2,
\end{equation}
where $0 < \delta < 1$, $f \in PW_{2^{j+1}}(X)$. Since $F_j(\Delta)f \in PW_{2^{j+1}}(X)$ we can apply (5.3), (5.4), (5.6) to obtain

$$(5.7) \quad (1 - \delta) \|F_j(\Delta)f\|^2 \leq \sum_{\nu} \sum_k |\langle F_j(\Delta)f, \theta_{\nu,j,k}\rangle|^2 \leq (1 + \delta) \|F_j(\Delta)f\|^2.$$ 

Since operator $F_j(\Delta)$ is self-adjoint we obtain (via 5.4), that for the functions

$$(5.8) \quad \Theta_{\nu,j,k} = F_j(\Delta)\theta_{\nu,j,k}$$ 

which belong to $PW_{[2^{j-1}, 2^{j+1}]}(X)$, the following frame inequalities hold

$$(5.9) \quad (1 - \delta)\|f\|^2 \leq \sum_{j \geq 0} \sum_{\nu} \sum_k |\langle f, \Theta_{\nu,j,k}\rangle|^2 \leq (1 + \delta)\|f\|^2, \quad f \in L_2(X).$$

5.2. Space localization of functions $\Theta_{\nu,j,k}$. One has

$$(5.10) \quad \Theta_{\nu,j,k} = F_j(\Delta)\theta_{\nu,j,k} = \sqrt{\frac{U_k}{|U_k|}}\mathcal{F}^{-1}(F_j(\lambda, b)\mathcal{F}\psi_{j,k}\varphi_{\nu}(\lambda, b))$$

Since $\psi_{j,k}$ and $\varphi_{\nu}$ belong to $C_0^\infty(X)$ the Paley-Wiener Theorem for the Helgason-Fourier transform shows that $\mathcal{F}\psi_{j,k}\varphi_{\nu}$ belongs to $\mathcal{H}(\mathbb{R}^n \times \mathcal{B})^W$. Note that $F_j$ belongs to $C_0^\infty(\mathbb{R}^n \times \mathcal{B})$, radial in $\lambda$, and independent on $b \in \mathcal{B}$. In other words the function $F_j(\lambda, b)\mathcal{F}\psi_{j,k}\varphi_{\nu}$ belongs to $C_0^\infty(\mathbb{R}^n \times \mathcal{B})^W$ and by the Eguchi Theorem 3.3 the function $\mathcal{F}^{-1}(F_j(\lambda, b)\mathcal{F}\psi_{j,k}\varphi_{\nu})$ belongs to the Schwartz space $S^2(X)$.

To summarize, we proved the frame inequalities (5.9) for any $0 < \delta < 1$, where every function $\Theta_{\nu,j,k}$ belongs to $PW_{[2^{j-2}, 2^{j+2}]}(X) \cap S^2(X)$. The Theorem 1.1 is proved.

6. Besov spaces

We are going to remind a few basic facts from the theory of interpolation and approximation spaces spaces [3, 4, 18].

Let $E$ be a linear space. A quasi-norm $\| \cdot \|_E$ on $E$ is a real-valued function on $E$ such that for any $f, f_1, f_2 \in E$ the following holds true

1. $\|f\|_E \geq 0$;
2. $\|f\|_E = 0 \iff f = 0$;
3. $\|f - g\|_E = \|f\|_E$;
4. $\|f_1 + f_2\|_E \leq CE(\|f_1\|_E + \|f_2\|_E), C_E > 1$.

We say that two quasi-normed linear spaces $E$ and $F$ form a pair, if they are linear subspaces of a linear space $\mathcal{A}$ and the conditions $\|f_k - g\|_E \rightarrow 0$, and $\|f_k - h\|_F \rightarrow 0$, $f_k, g, h \in \mathcal{A}$, imply equality $g = h$. For a such pair $E, F$ one can construct a new quasi-normed linear space $E \cap F$ with quasi-norm

$$\|f\|_{E \cap F} = \max (\|f\|_E, \|f\|_F)$$

and another one $E + F$ with the quasi-norm

$$\|f\|_{E + F} = \inf_{f = f_0 + f_1, f_0 \in E, f_1 \in F} (\|f_0\|_E + \|f_1\|_F).$$
All quasi-normed spaces $H$ for which $E \cap F \subset H \subset E+F$ are called intermediate between $E$ and $F$. A group homomorphism $T : E \to F$ is called bounded if
\[\|T\| = \sup_{f \in E, f \neq 0} \|Tf\|_F/\|f\|_E < \infty.\]

One says that an intermediate quasi-normed linear space $H$ interpolates between $E$ and $F$ if every bounded homomorphism $T : E + F \to E + F$ which is a bounded homomorphism of $E$ into $E$ and a bounded homomorphism of $F$ into $F$ is also a bounded homomorphism of $H$ into $H$.

On $E + F$ one considers the so-called Peetre’s $K$-functional
\[(6.1) \quad K(f, t) = K(f, t, E, F) = \inf_{f = f_0 + f_1, f_0 \in E, f_1 \in F} (\|f_0\|_E + \|f_1\|_F).\]

The quasi-normed linear space $(E, F)_{\theta, q}^K, 0 < \theta < 1, 0 < q \leq \infty$, or $0 \leq \theta \leq 1, q = \infty$, is introduced as a set of elements $f$ in $E + F$ for which
\[(6.2) \quad \|f\|_{\theta, q} = \left( \int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{1/q}.\]

It turns out that $(E, F)_{\theta, q}^K, 0 < \theta < 1, 0 < q \leq \infty$, or $0 \leq \theta \leq 1, q = \infty$, with the quasi-norm $(6.2)$ interpolates between $E$ and $F$.

Let us introduce another functional on $E + F$, where $E$ and $F$ form a pair of quasi-normed linear spaces
\[(E, F) = \inf_{g \in F, \|g\|_F \leq t} \|f - g\|_E.\]

**Definition 6.** The approximation space $\mathcal{E}_{\alpha, q}(E, F), 0 < \alpha < \infty, 0 < q \leq \infty$ is a quasi-normed linear spaces of all $f \in E + F$ with the following quasi-norm
\[(6.3) \quad \left( \int_0^\infty (t^\alpha \mathcal{E}(f, t))^q \frac{dt}{t} \right)^{1/q}.\]

**Theorem 6.1.** Suppose that $\mathcal{T} \subset F \subset E$ are quasi-normed linear spaces and $E$ and $F$ are complete.

If there exist $C > 0$ and $\beta > 0$ such that for any $f \in F$ the following Jackson-type inequality is verified
\[(6.4) \quad t^\beta \mathcal{E}(t, f, E, F) \leq C \|f\|_F, t > 0,
\]
then the following embedding holds true
\[(6.5) \quad (E, F)_{\theta, q}^K \subset \mathcal{E}_{\theta\beta, q}(E, \mathcal{T}), 0 < \theta < 1, 0 < q \leq \infty.\]

If there exist $C > 0$ and $\beta > 0$ such that for any $f \in \mathcal{T}$ the following Bernstein-type inequality holds
\[(6.6) \quad \|f\|_F \leq C \|f\|_{\mathcal{T}}^\beta \|f\|_E
\]
then
\[(6.7) \quad \mathcal{E}_{\theta\beta, q}(E, \mathcal{T}) \subset (F, F)_{\theta, q}^K, 0 < \theta < 1, 0 < q \leq \infty.\]

Now we return to the situation on $X$. The inhomogeneous Besov space $\mathbb{B}_{2,q}^\alpha(X)$ is introduced as an interpolation space between the Hilbert space $L_2(X)$ and Sobolev space $H^r(X)$ where $r$ can be any natural number such that $0 < \alpha < r, 1 \leq q \leq \infty$. Namely, we have
\[\mathbb{B}_{2,q}^\alpha(X) = (L_2(X), H^r(X))_{\theta, q}^K, 0 < \theta = \alpha/r < 1, 1 \leq q \leq \infty.\]
where $K$ is the Peetre’s interpolation functor.

We introduce a notion of best approximation

$$
\mathcal{E}(f, \omega) = \inf_{g \in PW_\omega(X)} \| f - g \|_{L^2(X)}.
$$

Our goal is to apply Theorem 6.1 in the situation where $E$ is the linear space $L^2(X)$ with its regular norm, $F$ is the Sobolev space $H^r(X)$, with the graph norm $(I + \Delta)^{r/2}$, and $T = PW_\omega(X)$ which is equipped with the quasi-norm

$$
\| f \|_T = \sup \{ |\lambda| : (\lambda, b) \in \text{supp} \hat{f} \}.
$$

By using Plancherel Theorem it is easy to verify a generalization of the Bernstein inequality for Paley-Wiener functions $f \in PW_\omega(X)$:

$$
\| \Delta^r f \|_{L^2(X)} \leq \omega^r \| f \|_{L^2(X)}, \quad r \in \mathbb{R}^+,
$$

and an analog of the Jackson inequality:

$$
\mathcal{E}(f, \omega) \leq \omega^{-r} \| \Delta^r f \|_{L^2(X)}, \quad r \in \mathbb{R}^+.
$$

These two inequalities and Theorem 6.1 imply the following result (compare to [21], [28]-[30]).

**Theorem 6.2.** The norm of the Besov space $B^{\alpha, q}_{2, q}(X)$, $\alpha > 0$, $1 \leq q \leq \infty$ is equivalent to the following norm

$$
\| f \|_{L^2(X)} + \left( \sum_{k=0}^{\infty} \left( 2^{k\alpha} \mathcal{E}(f, 2^k) \right)^q \right)^{1/q}.
$$

Let function $F_j$ be the same as in subsection 5.1 and

$$
F_j(\Delta) : L^2(X) \to PW_{2^{j-1}, 2^{j+1}}(X), \quad \| F_j(\Delta) \| \leq 1.
$$

**Theorem 6.3.** The norm of the Besov space $B^{\alpha, q}_{2, q}(X)$ for $\alpha > 0$, $1 \leq q \leq \infty$ is equivalent to

$$
\left( \sum_{j=0}^{\infty} \left( 2^{j\alpha} \| F_j(\Delta) f \|_{L^2(X)} \right)^q \right)^{1/q},
$$

with the standard modifications for $q = \infty$.

**Proof.** In the same notations as above the following version of Calderón decomposition holds:

$$
\sum_{j \in \mathbb{N}} F_j(\Delta) f = f, \quad f \in L^2(X).
$$

We obviously have

$$
\mathcal{E}(f, 2^k) \leq \sum_{j > k} \| F_j(\Delta) f \|_{L^2(X)}.
$$

By using the discrete Hardy inequality [4] we obtain the estimate

$$
\| f \| + \left( \sum_{k=0}^{\infty} \left( 2^{k\alpha} \mathcal{E}(f, 2^k) \right)^q \right)^{1/q} \leq C \left( \sum_{j=0}^{\infty} \left( 2^{j\alpha} \| F_j(\Delta) f \|_{L^2(X)} \right)^q \right)^{1/q}
$$

Conversely, for any $g \in PW_{2^{j-1}}(X)$ we have

$$
\| F_j(\Delta) f \|_{L^2(X)} = \| F_j(\Delta) (f - g) \|_{L^2(X)} \leq \| f - g \|_{L^2(X)}.$$
It gives the inequality
\[ \| F_j(\Delta) f \|_{L^2(X)} \leq E(f, 2^{j-1}), \]
which shows that the inequality opposite to (6.12) holds. This completes the proof. \[\square\]

**Theorem 6.4.** The norm of the Besov space \( B^{\alpha}_{2,q}(X) \) for \( \alpha > 0, 1 \leq q \leq \infty \) is equivalent to
\[
\left( \sum_{j=0}^{\infty} 2^{j\alpha q} \left( \sum_{\nu,k} |\langle f, \Theta_{\nu,j,k} \rangle|^2 \right)^{q/2} \right)^{1/q},
\]
with the standard modifications for \( q = \infty \).

**Proof.** According to (5.7) we have
\[
(6.14) \quad (1 - \delta) \| F_j(\Delta) f \|_{L^2(X)}^2 \leq \sum_{\nu,k} |\langle F_j(\Delta) f, \theta_{\nu,j,k} \rangle|^2 \leq (1 + \delta) \| F_j(\Delta) f \|_{L^2(X)}^2,
\]
where \( F_j(\Delta) f \in PW_{2^{j+1}}(X) \). Since \( \Theta_{\nu,j,k} = F_j(\Delta) \theta_{\nu,j,k} \) we obtain for any \( f \in L^2(X) \)
\[
\frac{1}{1 + \delta} \sum_{\nu,k} |\langle f, \Theta_{\nu,j,k} \rangle|^2 \leq \| F_j(\Delta) f \|_{L^2(X)}^2 \leq \frac{1}{1 - \delta} \sum_{\nu,k} |\langle f, \Theta_{\nu,j,k} \rangle|^2.
\]
Theorem is proved. \[\square\]

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