Quasi exactly solvable matrix Schrödinger operators.

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Abstract

Two families of quasi exactly solvable $2 \times 2$ matrix Schrödinger operators are constructed. The first one is based on a polynomial matrix potential and depends on three parameters. The second is a one-parameter generalisation of the scalar Lamé equation. The relationship between these operators and QES Hamiltonians already considered in the literature is pointed out.

1 Introduction

In a recent paper [1] a classification of $2 \times 2$ matrix quasi exactly solvable (QES) Schrödinger operators in one spatial dimension is attempted. This problem was first addressed in [2] and further developped in [3] and [4]. Here we consider a suitable class of finite dimensional vector spaces of polynomials in a real variable and we construct two families of operators preserving sub-classes of these vector spaces. The first family is related to one of the cases treated in [1]; the second, which is not considered in [1], generalizes an equation considered in [3],[5]. The two corresponding QES equations respectively constitute “coupled channel” generalisations of the anharmonic and Lamé QES scalar equations.
Following the basic idea of QES operators [3, 4] we consider the finite dimensional vector space of couples of polynomials of given degree \( n \) and \( m \) in a real variable \( x \). We slightly generalize this vector space by setting

\[
\mathcal{V} = P \begin{pmatrix} \mathcal{P}(n) \\ \mathcal{P}(m) \end{pmatrix}
\]

(1)

where \( \mathcal{P}(n) \) denotes the set of real polynomials of degree at most \( n \) in \( x \) while \( P \) is a fixed invertible \( 2 \times 2 \) matrix operator; \( P \) can be interpreted as a change of basis in the vector space \( \mathcal{P}(n) \oplus \mathcal{P}(m) \). With such an interpretation it is reasonable to choose \( P \) of the form

\[
P = \begin{pmatrix} 1 & P_{12} \\ 0 & 1 \end{pmatrix} \quad \text{(resp. } P = \begin{pmatrix} 1 & 0 \\ P_{21} & 1 \end{pmatrix} \text{)}
\]

(2)

for \( n \leq m \) (resp. \( m \leq n \)). In this paper, we limit ourselves to scalar operators \( P_{12} \) (or \( P_{21} \)) of the form

\[
P_{12} = \kappa_0 \frac{\partial}{\partial x} + \kappa_1 + \kappa_2 x \frac{\partial}{\partial x} + \kappa_3 x
\]

(3)

where \( \kappa_j \) are constants. The vector space defined in Eq. (6) of [4] can be set in the form (1) with \( \kappa_0 = 1, \kappa_{1,2,3} = 0 \).

\section{Polynomial potential}

We consider an operator of the form

\[
H(y) = -\frac{d^2}{dy^2} I_2 + M_6(y)
\]

(4)

where \( M_6(y) \) is a \( 2 \times 2 \) hermitian matrix whose entries are even polynomials of degree at most six in \( y \). This operator is a natural generalisation of the famous QES anharmonic oscillator [3, 4] to \( 2 \times 2 \) matrix operator. After the standard “gauge transformation” of \( H(y) \) with a factor

\[
\phi(y) = y^\epsilon \exp \left\{-\frac{P_2}{2} y^4 + p_1 y^2 \right\}
\]

(5)
and the change of variable \( x = y^2 \), the equivalent operator \( \hat{H}(x) \) can be computed:

\[
\hat{H}(x) = \phi^{-1}(x)H(y)\phi(x) \big|_{y = \sqrt{x}}
\]  

Then we pose the problem: what is the most general choice of \( M_6 \) such that \( \hat{H}(x) \) preserves a vector space of the form (1),(2),(3)?

The following solutions are obtained after straightforward calculations (we exclude the case where \( M_6(y) \) is diagonal since it corresponds to a direct sum of two scalar QES operators):

\[
\epsilon = 0 \text{ or } 1 \quad , \quad n = m - 2 \quad , \quad \kappa_1 = \kappa_2 = \kappa_3 = 0 .
\]

The corresponding potentials \( M_6 \) has the form

\[
M_6(y) = \{4p_2^2y^6 + 8p_1p_2y^4 + (4p_1^2 - 8mp_2 + 2(1 - 2\epsilon)p_2)y^2\}I_2
\]

\[
+ (8p_2y^2 + 4p_1)\sigma_3 - 8mp_2\kappa_0\sigma_1
\]

where \( \sigma_1, \sigma_3 \) are the Pauli matrices, \( p_2, p_1, \kappa_0 \) are free real parameters and \( m \) is an integer. In particular, the non diagonal term is parametrized by an arbitrary constant which cannot be suppressed because of the \( y \)-dependent term proportional to \( \sigma_3 \). If the parameter \( \epsilon \) is choosen as an arbitrary real number, then the potential \( M_6 \) has a supplementary term of the form \( \epsilon(\epsilon - 1)/y^2 \).

When the parameters of the case 1 of Ref. [1] are choosen so that the potential is a polynomial matrix (i.e. \( \alpha_2 = \alpha_0 = 0, \alpha_1 = 1, \beta_0 = 1/2 \) or \( 3/2 \) in Eq. (34) of [1]) the potential reduces to the matrix \( M_6(y) \) above. The way of obtaining this result here is slightly different because the method starts from the natural vector space \( P(m) \oplus P(n) \). The more elaborated QES operators obtained in [1] can also be produced by our technique but this is not aimed in this note.

Let us also point out that the “gauge factor \( U \)” considered in [4] is limited to be a function of the variable \( x \) and, therefore, is not supposed to contain any derivative operator like our operator \( P \) (see (3)). This explains that the QES polynomial potential (8) was not found in [4]. With the restriction that \( U \) is a function of \( x \) only,
these authors correctly reach the conclusion that Hamiltonian preserving a space like $\mathcal{P}(n) \oplus \mathcal{P}(m)$ with $|n - m| > 1$ are essentially diagonal, in contrast with the present operator related to the case $n - m = 2$.

### 3 Application to N-body hamiltonians

By using the idea of the previous section, a matrix version of the QES many-body problem of Ref. [8] can be constructed. Let us consider the Calogero Hamiltonian [9] (we note it $H_{\text{cal}}$) supplemented by a matrix-valued potential $V^*$:

$$H = H_{\text{cal}} + V^* = \frac{1}{2} \sum_{j=1}^{N} \left[-\frac{\partial^2}{\partial x_j^2} + x_j^2\right] + \sum_{j<i} \frac{\nu(\nu - 1)}{(x_j - x_i)^2} + V^* \quad (9)$$

Along with [8] we assume $V^*$ to depend only on the variable $\tau$:

$$\tau \equiv \sum_{j<i} (x_j - Y)(x_i - Y), \quad Y \equiv \sum_{j=1}^{N} x_j \quad (10)$$

and we look for eigenfunctions of the Hamiltonian (9) of the form

$$\Psi(x) = \psi_0(x) \tau^\epsilon \exp\left\{-\frac{p_2}{2} \tau^4 + p_1 \tau^2\right\} \phi(\tau) \quad (11)$$

where $\psi_0$ denotes the ground state of the standard Calogero system:

$$\psi_0(x) = \prod_{i<j} |x_i - x_j|^{\nu} \exp(-X^2/2), \quad X^2 \equiv \sum_{j=1}^{N} x_j^2 \quad (12)$$

while $\phi(\tau)$ represents a couple of polynomials in $\tau$.

After a standard algebra, the operator acting on $\phi(\tau)$ can be isolated:

$$h \equiv \tau \frac{\partial^2}{\partial \tau^2} + (4\tau + 2b) \frac{\partial}{\partial \tau} + V^* \quad (13)$$

and it can be shown that this operator preserves the space (1) (again with $n = m - 2$, $\kappa_1 = \kappa_2 = \kappa_3 = 0$) provided $V^*$ is of the form

$$V^*(\tau) = -p_2^2 \tau^3 + 2p_2(1 - p_1) \tau^2 + (a - 2p_2\sigma_3) \tau + (1 - p_1)\sigma_3 + \frac{\gamma}{\tau} + 2m\kappa_0 \sigma_1 \quad (14)$$
with the definitions

\[ a \equiv p_1(2 - p_1) + p_2(2m + 3\epsilon - 1 + b) \]
\[ b \equiv \frac{1}{2}(1 + \nu N)(N - 1) \]
\[ \gamma \equiv 2\epsilon(\epsilon - 1 + b) \]

As a consequence, (9),(14) constitutes a QES matrix extension (labelled by the parameters \( p_1, p_2, \epsilon \)) of the exactly solvable Calogero hamiltonian.

4 Lamé type potential.

As a second example, we consider the family of operators

\[ H(z) = -\frac{d^2}{dz^2} + \begin{bmatrix}
  Ak^2\text{sn}^2 + \delta(1 + k^2)/2 & 2\theta kcn \text{dn} \\
  2\theta kcn \text{dn} & Ck^2\text{sn}^2 - \delta(1 + k^2)/2
\end{bmatrix} \]

(16)

where \( A, C, \delta, \theta \) are constants while \( \text{sn}, \text{cn}, \text{dn} \) respectively abbreviate the Jacobi elliptic functions of argument \( z \) and modulus \( k \).

\[ \text{sn}(z,k) , \quad \text{cn}(z,k) , \quad \text{dn}(z,k) \]

These functions are periodic with period \( 4K(k), 4K(k), 2K(k) \) respectively (\( K(k) \) is the complete elliptic integral of the first type). The above hamiltonian is therefore to be considered on the Hilbert space of periodic functions on \([0, 4K(k)]\). For completeness, we mention the properties of the Jacobi functions which are needed in the calculations

\[ \text{cn}^2 + \text{sn}^2 = 1 \]
\[ \text{dn}^2 + k^2\text{sn}^2 = 1 \]

\[ \frac{d}{dz}\text{sn} = \text{cn} \text{dn} \quad \frac{d}{dz}\text{cn} = -\text{sn} \text{dn} \quad \frac{d}{dz}\text{dn} = -k^2\text{sn} \text{cn} \]

The relevant change of variable which eliminates the transcendental functions \( \text{sn}, \text{cn}, \text{dn} \) from (16) in favor of algebraic expressions is (for \( k \) fixed)

\[ x = \text{sn}^2(z, k) \]

In particular the second derivative term in (16) becomes

\[ \frac{d^2}{dz^2} = 4x(1 - x)(1 - k^2x) \frac{d^2}{dx} + 2(3k^2x^2 - 2(1 + k^2)x + 1) \frac{d}{dx} \]
Several possibilities of extracting prefactors then lead to equivalent forms of (10), say $\hat{H}(x)$, which are matrix operators build with the derivative $d/dx$ and polynomial coefficients in $x$. The requirement that $\hat{H}(x)$ preserves a space of the form (1) leads to two possible sets of values for $A, C, \theta$ (we do not consider the case $\theta = 0$ since it corresponds to two decoupled scalar Lamé equations).

**Case 1**

\[ A = 4m^2 + 6m + 3 - \delta \]
\[ C = 4m^2 + 6m + 3 + \delta \]
\[ \theta = \frac{1}{2}[(4m + 3)^2 - \delta^2]^{\frac{1}{2}} \]

The parameter $\delta$ remains free, and also $k$ which fixes the period of the potential. Four invariant spaces are available. In order to present them we conveniently define

\[ R_1 = \frac{4m + 3 - \delta}{4m + 3 + \delta}, \quad (22) \]

We have then

\[ V_1 = \begin{pmatrix} 1 & 0 \\ 0 & \text{cn} \text{ dn} \end{pmatrix} \begin{pmatrix} 1 & \kappa x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P(m) \\ P(m) \end{pmatrix}, \quad \kappa^2 = k^2 R_1 \quad (23) \]

\[ V_2 = \begin{pmatrix} \text{cn} \text{ dn} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa x & 1 \end{pmatrix} \begin{pmatrix} P(m) \\ P(m) \end{pmatrix}, \quad \kappa^2 = \frac{k^2}{R_1} \quad (24) \]

\[ V_3 = \begin{pmatrix} \text{sn} \text{ cn} & 0 \\ 0 & \text{sn} \text{ cn} \end{pmatrix} \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P(m - 1) \\ P(m) \end{pmatrix}, \quad \kappa^2 = k^2 R_1 \quad (25) \]

\[ V_4 = \begin{pmatrix} \text{sn} \text{ dn} & 0 \\ 0 & \text{sn} \text{ cn} \end{pmatrix} \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P(m - 1) \\ P(m) \end{pmatrix}, \quad \kappa^2 = R_1/k^2 \quad (26) \]

**Case 2**

\[ A = 4m^2 + 2m + 1 - \delta \]
\[ C = 4m^2 + 2m + 1 + \delta \]
\[ \theta = \frac{1}{2}[(4m + 1)^2 - \delta^2]^{\frac{1}{2}} \]

The associated invariant vector spaces read, defining $R_2 = (4m + 1 - \delta)/(4m + 1 + \delta)$,
The operator (16) was studied in [3],[5] for $\delta = 1$. For this particular value of $\delta$, the corresponding eigenvalue equation \( H \psi = \omega^2 \psi \) determines the normal modes of the sphaleron classical solution [11] in the Abelian Higgs model in 1+1 dimension. It therefore plays a crucial role in the understanding of the instabilities of the sphaleron in this model.

The above results demonstrate that the remarkable algebraic properties of the Lamé equation [10] also hold for the operator (16), irrespectively of the value of $\delta$. The associated eigenvalue equation therefore constitutes a (one parameter) $2 \times 2$ matrix equation analog of the scalar Lamé equation.

5 Generalization

The kind of operators presented in Sect. 3 can be generalized to matrix potentials of the form

\[
H(z) = -\frac{d^2}{dz^2} + \begin{bmatrix}
V_1(s^2) & \theta s_{\alpha_1} c_{\alpha_2} d_{\alpha_3} \\
\theta c_{\alpha_1} c_{\alpha_2} d_{\alpha_3} & V_2(s^2)
\end{bmatrix}
\]

(31)

where $V_1, V_2$ are polynomials, $\theta$ is a constant and $\alpha_j$ are non-negative integers.

The similarity transformation

\[
\hat{H}(x) = U^{-1}(z)H(z)U(z) \quad U(z) = \text{diag}(s_{\beta_1} c_{\beta_2} d_{\beta_3}, s_{\gamma_1} c_{\gamma_2} d_{\gamma_3})
\]

(32)
sets the operator \((31)\) into a form with polynomial coefficients in the variable \(x = \text{sn}^2\) provided

- \(\beta_j, \gamma_j = 0 \text{ or } 1, \quad j = 1, 2, 3\)

- \(\alpha_j \pm (\beta_j - \gamma_j) = \text{non-negative even integer}, \quad j = 1, 2, 3.\)

After making a choice of \(\alpha_j, \beta_j, \gamma_j\) satisfying the above conditions, the possible forms of \(V_1, V_2\) and of \(P, m, n\) in Eq. \((1)\) have to be determined in order for \(H(z)\) to be QES.

Taking \(k = 0\), the standard trigonometric functions are recovered:

\[
\text{sn}(z, 0) = \sin z, \quad \text{cn}(z, 0) = \cos z, \quad \text{dn}(z, 0) = 1. \quad (33)
\]

The periodic potential below, which is exactly solvable \([12]\), furnishes a particular example of this type

\[
V(z) = \frac{1}{\cos (z) \sin (z) \sin^2 (z)}.
\quad (34)
\]

It determines the normal modes about some static solutions of the Goldstone model in 1+1 dimensions \([12]\).

### 6 Concluding remarks

The examples of operators presented above give evidences of the difficulty to classify the coupled-channel (or matrix) QES Schrodinger equations. The way of constructing the QES potential \(M_6\) in Sect. 2 further provides a clear link between the approaches \([1]\) and \([4]\) to this mathematical problem; we hope that this note will motivate further investigations of it.
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