On explicit inversion of a subclass of operators with $D$-difference kernels and Weyl theory of the corresponding canonical systems

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Abstract

Explicit inversion formulas for a subclass of integral operators with $D$-difference kernels on a finite interval are obtained. A case of the positive operators is treated in greater detail. An application to the inverse problem to recover canonical system from a Weyl function is given.

MSC(2000) Primary 34A55, 45Q05; Secondary 47B65, 47G10

Keywords: integral operator with difference kernel, operator with $D$-difference kernel, explicit inversion, canonical system, inverse problem, Weyl function.

1 Introduction

Integral operators with difference kernels are important in mathematics and applications and are actively used in the study of numerous homogenious processes. The papers [13] [18] on the inversion of the operators with difference kernels on the semi-axis became classical. Various results and references on the operators with difference kernels on a finite interval or a system of intervals are given in [24] [26]. Interesting explicit results on the inversion of the operators with exponential type difference kernels on a finite interval one can find in [2] [12].
Operators with $D$-difference kernels in $L^2_p(0, l)$, which we shall treat, are bounded operators of the form

$$S_t f = S f = \frac{d}{dx} \int_0^l s(x, t)f(t)dt, \quad s(x, t) = \{s_{ij}(x, t)\}_{i, j=1}^p, \quad (1.1)$$

$$s_{ij}(x, t) = s_{ij}(d_i x - d_j t), \quad s_{ij}(x) \in L^2(-d_j l, d_i l), \quad (1.2)$$

where $D = D^* = \text{diag}\{d_1, d_2, \ldots, d_p\} > 0$ is a fixed $p \times p$ diagonal matrix. The notion of an operator with a $D$-difference kernel is a natural generalization of the operator with a difference kernel, i.e., of the case $D = I_p$, where $I_p$ is the $p \times p$ identity matrix. The class of operators with $D$-difference kernels on a finite interval includes the operators with difference kernels on systems of intervals, which are important, for instance, in elasticity theory, diffraction theory, and the theory of stable processes (see [16] and Chapter 6 in [26]).

Explicit inversion formulas for an interesting subclass of operators with $D$-difference kernels are obtained in Section 2 of this paper using the classical results on semiseparable operators. Note also that the inversion of semiseparable matrices and operators is another interesting and actively developed theory, see [8, 9] and bibliography in [30]. Some further possible applications are connected with the paper [17].

Operator identities for the operators with $D$-difference kernels are discussed in Section 3.

The case of positive and boundedly invertible operators with $D$-difference kernels is treated further in Theorem 4.3 of Section 4. As an application, we solve explicitly in terms of Weyl functions an inverse problem for a subclass of canonical systems. Some results from [28, 29] are developed further in this section too.

We use the standard notations $\mathbb{C}$ and $\mathbb{C}_+$ for the complex plane and upper semi-plane, respectively. By $\{\mathcal{H}_1, \mathcal{H}_2\}$ we denote the class of the bounded linear operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$, and by $\sigma(\beta)$ we denote the spectrum of $\beta$. 

2
2 Inversion of operators with $D$-difference kernels

Consider a self-adjoint operator with $D$-difference kernel

$$S = I + \int_0^t k(x, t) \cdot dt, \quad k(x, t) = \{k_{ij}(x, t)\}_{i,j=1}^p = k_{ij}(d_i x - d_j t), \quad (2.1)$$

where $I$ is the identity operator, the $p \times p$ matrix function $k(x)$ on the right hand side of the second relation in (2.1) is given by the equalities

$$k(x) = \Theta_2 e^{ix\beta} \Theta_1 (x > 0), \quad k(-x) = k(x)^*, \quad (2.2)$$

$\Theta_m (m = 1, 2)$ is an $n \times p$ matrix, and $\beta$ is an $n \times n$ matrix for some integer $n > 0$. Without loss of generality we assume further that

$$d_1 \geq d_2 \geq \ldots \geq d_p > 0. \quad (2.3)$$

Remark 2.1 We suppose that equalities (2.2) hold on $(0, d_1 l)$, and so, according to (2.3), each entry $k_{ij}(x)$ is determined by (2.2) on the interval, which contains $(-d_j l, d_i l)$, i.e., the operator $S$ of the form (2.1) is determined by (2.2).

Introduce the operator

$$E \in \{L^2_p(0, l), L^2(D)\} \quad L^2(D) := L^2(0, d_1 l) \oplus L^2(0, d_2 l) \oplus \ldots \oplus L^2(0, d_p l) \quad (2.4)$$

by the equality $(Ef)_j(z) = f_j(z/d_j)$. We shall denote also by $E$ the corresponding operator from $\{L^2_p(0, l), L^2(0, d_1 l)\}$ with the natural embedding of $L^2(D)$ into $L^2_p(0, d_1 l)$:

$$(Ef)_j(z) = f_j(z/d_j) \quad (0 < z < d_j l), \quad (Ef)_j(z) = 0 \quad (d_j l < z < d_1 l). \quad (2.5)$$

By (2.1) and (2.5) it is easy to see that

$$S = E^{-1} \left( I + \int_0^a \tilde{k}(y, z) \cdot dz \right) E, \quad a := d_1 l, \quad (2.6)$$

$$\tilde{k}(y, z) = \{\tilde{k}_{ij}(y, z)\}_{i,j=1}^p, \quad \tilde{k}_{ij}(y, z) = 0 \quad \text{if } z > d_j l \text{ or } y > d_i l, \quad (2.7)$$

$$\tilde{k}_{ij}(y, z) = \frac{1}{d_j} k_{ij}(y - z) \quad \text{if } 0 < z < d_j l \text{ and } 0 < y < d_i l. \quad (2.8)$$
According to (2.2), (2.7), and (2.8), the operator
\[
\tilde{S} = I + \int_0^a \tilde{k}(y, z) \cdot dz
\] (2.9)
is not an operator with a difference kernel but it is a semiseparable operator. Recall [8] that the integral operator \(\tilde{S}\) of the form (2.9) is called semiseparable, when \(\tilde{k}\) admits representation
\[
\tilde{k}(y, z) = F_1(y)G_1(z) \quad \text{for } y > z, \quad \tilde{k}(y, z) = F_2(y)G_2(z) \quad \text{for } y < z,
\] (2.10)
where \(F_1\) and \(F_2\) are \(p \times n\) matrix functions and \(G_1\) and \(G_2\) are \(n \times p\) matrix functions for some \(n > 0\). It is assumed that the entries of \(F_1\), \(F_2\), \(G_1\), and \(G_2\) are square integrable. When the operator \(\tilde{S}\) is invertible and its kernel \(\tilde{k}\) is given by (2.10), the kernel of the operator \(\tilde{T} = \tilde{S}^{-1}\) is expressed in terms of the \(2n \times 2n\) solution \(U\) of the differential equation
\[
\left( \frac{d}{dy} U \right)(y) = \tilde{J}\tilde{H}(y)U(y), \quad y \geq 0, \quad U(0) = I_{2n},
\] (2.11)
where
\[
\tilde{J}\tilde{H}(y) := B(y)C(y), \quad \tilde{J} = (\tilde{J}^*)^{-1} = \left[ \begin{array}{cc} 0 & -I_p \\ I_p & 0 \end{array} \right].
\] (2.12)
\[
B(y) = \left[ \begin{array}{c} -G_1(y) \\ G_2(y) \end{array} \right], \quad C(y) = \left[ \begin{array}{cc} F_1(y) & F_2(y) \end{array} \right].
\] (2.13)
Namely, we have (see, for instance, [8])
\[
\tilde{T} = \tilde{S}^{-1} = I + \int_0^a \tilde{T}(y, z) \cdot dz,
\] (2.14)
\[
\tilde{T}(y, z) = \begin{cases} C(y)U(y)(I_{2n} - P^\times U(z)^{-1}B(z), & y > z, \\ -C(y)U(y)P^\times U(z)^{-1}B(z), & y < z. \end{cases}
\] (2.15)
Here \(P^\times\) is given in terms of the \(n \times n\) blocks \(U_{21}(a)\) and \(U_{22}(a)\) of \(U(a)\):
\[
P^\times = \left[ \begin{array}{cc} 0 & 0 \\ U_{22}(a)^{-1}U_{21}(a) & I_n \end{array} \right],
\] (2.16)
and the invertibility of $U_{22}(a)$ is a necessary and sufficient condition for the invertibility of $\tilde{S}$.

When the semiseparable operator $\tilde{S}$ is not invertible, its kernel subspace is given by the equality (8, p. 157):\
\[
\text{Ker} \ \tilde{S} = \{ h(y) : h(y) = C(y)U(y) \begin{bmatrix} 0 \\ g \end{bmatrix}, \ U_{22}(a)g = 0 \}. \tag{2.17}
\]

Rewrite $D$ in the form
\[
D = \text{diag}\{ \tilde{d}_1 I_{p_1}, \ldots, \tilde{d}_k I_{p_k} \}, \quad p_1 + \ldots + p_k = p,
\]
\[
\tilde{d}_{j_1} > \tilde{d}_{j_2} > 0 \quad (j_1 < j_2 \leq k), \tag{2.18}
\]
and put
\[
\tilde{d}_{k+1} = 0, \quad P_{k+1} = I_p, \quad P_j = \text{diag}\{ I_{p_1}, \ldots, I_{p_{j-1}}, 0, \ldots, 0 \} \ (2 \leq j \leq k). \tag{2.19}
\]

Then, in view of (2.2), (2.7), (2.8), and (2.13) we have
\[
B(y) = e^{-y\mathcal{A}} \begin{bmatrix} -\Theta_1 \\ \Theta_2 \end{bmatrix} D^{-1} P_j, \quad C(y) = P_j \begin{bmatrix} \Theta_2 & \Theta_1^* \end{bmatrix} e^{y\mathcal{A}}, \tag{2.20}
\]
for
\[
\tilde{d}_{j_1} < y < \tilde{d}_{j_1-1} \quad (2 \leq j \leq k + 1), \quad \mathcal{A} := i \begin{bmatrix} \beta^* & 0 \\ 0 & \beta \end{bmatrix}. \tag{2.21}
\]

**Remark 2.2** By (2.8), (2.17) and (2.20), it is immediate that
\[
\text{Ker} \ \tilde{S} \in \text{Im} E. \tag{2.22}
\]

where Im means image. The integral parts of $S$ and $\tilde{S}$ are compact operators. Hence, if $\tilde{S}$ is not invertible, then $\text{Ker} \ \tilde{S} \neq 0$, and according to (2.22) the subspace $E^{-1}\text{Ker} \ \tilde{S}$ is well defined. In view of (2.7) and (2.8), we have $S E^{-1} \text{Ker} \ \tilde{S} = 0$, i.e., $S$ is not invertible too. It follows from (2.6), (2.14), and (2.15) that if $\tilde{S}$ is invertible, then $S$ is invertible. In other words, $S$ and $\tilde{S}$ are simultaneously invertible.
Next, introduce notations

\[ \mathcal{A}_j^\times = \mathcal{A} + Y_j, \quad Y_j = \begin{bmatrix} -\Theta_1 \\ \Theta_2 \end{bmatrix} D^{-1} P_j \begin{bmatrix} \Theta_2^\ast \\ \Theta_1^\ast \end{bmatrix}. \] (2.23)

For \( 2 \leq j \leq k + 1 \), put

\[ U(y) = e^{-yA} e^{(y-d_j l)A_j^\ast} e^\tilde{d}_j l A U(\tilde{d}_j l) \quad (d_j l \leq y \leq \tilde{d}_j l), \quad U(0) = I_{2n}, \] (2.24)

Now, we are prepared to formulate the inversion theorem.

**Theorem 2.3** Let \( S \) be an operator with the \( D \)-difference kernel, which has the form (2.1), where \( k \) is given by (2.2) and \( D \) satisfies (2.18). Let also \( \det U_{22}(a) \neq 0 \) for \( U \) given by (2.24). Then \( S \) is invertible and its inverse is given by the formula \( S^{-1} = E^{-1} \tilde{\mathcal{T}} E \), where \( E \) is defined by (2.5) and \( \tilde{\mathcal{T}} \) is given by (2.14)-(2.16). The matrix functions \( B \) and \( C \) in (2.15) are given by (2.20) and the \( \tilde{\mathcal{J}} \)-unitary matrix function \( U \) in (2.15) has the form (2.24).

**Proof.** To prove the theorem we need to show that \( U \) of the form (2.24) satisfies (2.11). Then by the properties of the semiseparable operators we shall obtain that \( \tilde{S} \) given by (2.9) is invertible and that \( \tilde{\mathcal{T}} = \tilde{S}^{-1} \) is given by (2.14)-(2.16), (2.24). The formula \( S^{-1} = E^{-1} \tilde{T} E \) will be immediate from (2.6).

By formulas (2.20) and (2.23) it is easy to see that \( U \) of the form (2.24) satisfies equation

\[
\left( \frac{d}{dy} U \right)(y) = e^{-yA} \left( \mathcal{A}_j^\times - \mathcal{A} \right) e^{yA} e^{(y-d_j l)A_j^\ast} e^\tilde{d}_j l A U(\tilde{d}_j l)
\]

\[
= e^{-yA} Y_j e^{yA} U(y) = B(y) C(y) U(y)
\] (2.25)

for \( 0 \leq y \leq a \). Hence, by (2.12) \( U \) satisfies (2.11).

Finally, let us prove that \( U \) is \( \tilde{\mathcal{J}} \)-unitary, i.e., \( U(y)^\ast \tilde{\mathcal{T}} U(y) = \tilde{\mathcal{T}} \). Indeed, according (2.21) we have

\[ \mathcal{A}^\ast = -\tilde{\mathcal{T}} A \mathcal{J}^\ast. \] (2.26)

As we noted in (2.25), the equality \( B(y) C(y) = e^{-yA} Y_j e^{yA} \) is true. Thus, taking into account (2.12), (2.23), and (2.26) we obtain

\[ \tilde{H}(y) = \mathcal{J}^\ast e^{-yA} Y_j e^{yA} = e^{yA^\ast} \mathcal{J}^\ast Y_j e^{yA} \]

\[
= e^{yA^\ast} \begin{bmatrix} \Theta_2 \\ \Theta_1 \end{bmatrix} D^{-1} P_j \begin{bmatrix} \Theta_2^\ast \\ \Theta_1^\ast \end{bmatrix} e^{yA} \geq 0. \] (2.27)
It follows from (2.27) that $\tilde{H}^* = \tilde{H}$. Therefore, formulas (2.11) and (2.12) imply $\frac{d}{dy}(U(y)^*\tilde{J}U(y)) = 0$. Moreover, from $\frac{d}{dy}(U(y)^*\tilde{J}U(y)) = 0$ and $U(0) = I_{2n}$ we get $U(y)^*\tilde{J}U(y) = \tilde{J}$. ■

**Remark 2.4** If $S$ is invertible, then from Theorem 2.3 we derive

$$T = S^{-1} = I + \int_0^t \{T_{ij}(x,t)\}_{i,j=1}^p \cdot \, dt,$$  \quad (2.28)

where for $d_i x > d_j t$ and $e_i = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}$ we have

$$T_{ij}(x,t) = e_i \begin{bmatrix} \Theta_2^* & \Theta_1^* \end{bmatrix} e^{d_i x A} U(d_i x) (I_{2n} - P^x) U(d_j t)^{-1} e^{-d_j t A} \begin{bmatrix} -\Theta_1 \\ \Theta_2 \end{bmatrix} e_j^*,$$

and for $d_i x < d_j t$ we have

$$T_{ij}(x,t) = -e_i \begin{bmatrix} \Theta_2^* & \Theta_1^* \end{bmatrix} e^{d_i x A} U(d_i x) P^x U(d_j t)^{-1} e^{-d_j t A} \begin{bmatrix} -\Theta_1 \\ \Theta_2 \end{bmatrix} e_j^*.$$

### 3 Operator identities for operators with $D$-difference kernels

According to [26] (Ch. 6) a bounded in $L^2_p(0, l)$ operator $S$ with $D$-difference kernel, that is, an operator of the form (1.1), (1.2) satisfies the operator identity

$$AS - SA^* = i\Pi J\Pi^*,$$  \quad (3.1)

where $A_l = A \in \{L^2_p(0, l), L^2_p(0, l)\}$, $\Pi_l = \Pi = [\Phi_1 \quad \Phi_2]$, $\Phi_k \in \{C^p, L^2_p(0, l)\}$, the index "$l"$ is often omitted in our notations, and

$$A = i D \int_0^x \cdot \, dt, \quad \Phi_1 g = Ds(x, 0) g, \quad \Phi_2 g \equiv g.$$  \quad (3.2)

It is said that $A$, $S$, and $\Pi$, which satisfy (3.1), form an $S$-node. Further we assume that $A$ and $\Phi_2$ have the form (3.2). Operator identities play an important role in the study of structured operators [26, 27, 29].
Let us show that not only the operator with the $D$-difference kernel satisfies (3.1) but the inverse statement is also true, i.e., (3.1) implies that $S$ is an operator with a $D$-difference kernel (see also the corresponding statement in Example 1.2, p. 104 [29]). Quite similar to the proof of Theorem 1.3 (26), p. 11), where the case $D = I_p$ was treated, one can prove the following theorem.

**Theorem 3.1** Suppose a bounded operator $T \in \{ L^2_p(0, l), L^2_p(0, l) \}$ satisfies the operator identity

$$TA - A^*T = i \int_0^l Q(x, t) \cdot dt,$$  

(3.3)

where $Q, Q_1,$ and $Q_2$ are $p \times p$, $p \times \hat{p}$, and $\hat{p} \times p$ ($\hat{p} > 0$) matrix-functions, respectively. Then $T$ has the form

$$Tf = \frac{d}{dx} \int_0^l \frac{\partial}{\partial t} \Upsilon(x, t) f(t) dt,$$  

(3.5)

where $\Upsilon(x, t) = \{ \Upsilon_{ij}(x, t) \}_{i,j=1}^p$ is absolutely continuous in $t$, and

$$\Upsilon_{ij}(x, t) := (2d_i d_j)^{-1} \int_{d_i x + d_j t}^{f_{\min}} Q \left( \frac{u + d_i x - d_j t}{2d_i}, \frac{u - d_i x + d_j t}{2d_j} \right) du, \quad (3.6)$$

$$f_{\min} := \min \left( d_i (2l - x) + d_j t, d_i x + d_j (2l - t) \right). \quad (3.7)$$

In fact, Theorem 3.1 is true for a much wider class of functions $Q$ than the one given by (3.4). Similar to Theorem 2.2 in [26], the next theorem is immediate from Theorem 3.1 and equality $\hat{U} A \hat{U} = A^* ((\hat{U}f)(x) = f(l - x)).$

**Theorem 3.2** Suppose $S \in \{ L^2_p(0, l), L^2_p(0, l) \}$ satisfies the operator identity

$$AS - SA^* = i \int_0^l \left( \Phi_1(x) + \hat{\Phi}_1(t) \right) \cdot dt,$$  

where $\Phi_1(x)$ and $\hat{\Phi}_1(t)$ are $p \times p$ matrix functions with the entries from $L^2(0, l)$. Then $S$ is an operator with a $D$-difference kernel, i.e., the operator of the form (1.1), (1.2), and $s(u, 0) = D^{-1}\Phi_1(u)$, $s(0, u) = -D^{-1}\hat{\Phi}_1(u)$. Moreover, when (3.1) holds, that is, $\hat{\Phi}_1(t) = \Phi_1(t)^*$ we have

$$s(x, t) = -D^{-1}s(t, x)^*D, \quad S = S^*.$$  

(3.8)
4 Positive operators $S$ and an inverse problem for canonical system

Operators with $D$-difference kernels are essential for the construction of solutions of an inverse problem for an important subclass of canonical systems \[21\] \[28\] \[29\]. Canonical system is a system of the form

$$
\frac{d}{dx} w(x, \lambda) = i\lambda J H(x) w(x, \lambda), \quad H(x) \geq 0, \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \tag{4.1}
$$

where the Hamiltonian $H$ is a $m \times m$ ($m = 2p$) locally summable matrix function. A Weyl function of the canonical system on the semi-axis $x \geq 0$ is a $p \times p$ matrix function $\varphi(\lambda)$, which is analytic in $\mathbb{C}_+$ and satisfies the condition \[29\]

$$
\int_0^\infty \begin{bmatrix} I_p & i\varphi(\lambda) \end{bmatrix} w(x, \lambda)^* H(x) w(x, \lambda) \begin{bmatrix} I_p \\ -i\varphi(\lambda) \end{bmatrix} dx < \infty, \quad \lambda \in \mathbb{C}_+. \tag{4.2}
$$

The corresponding inverse problem is the problem to recover $H$ or, equivalently, canonical system from the Weyl function. In the case of rational Weyl matrix functions several inverse problems were solved explicitly using a GBDT version of the Bäcklund-Darboux transformation \[5\] \[10\] \[11\] \[20\] \[22\]. (See \[3\] \[7\] \[15\] \[19\] \[22\] \[31\] and references therein for various versions of the Bäcklund-Darboux transformation and commutation methods.) However, taking into account that the positivity of operators $S$ and the application of the inversion formulas for semiseparable operators is of independent interest, we shall use a general scheme \[25\] \[29\] and its modification \[21\] for the inverse problem treated in this section. As a result of the application of the general scheme to rational matrix functions, semiseparable operators appear. Inverse problems for self-adjoint and skew-self-adjoint Dirac-type systems were studied using semiseparable operators in \[1\] and \[6\], respectively.

Consider rational Herglotz $p \times p$ matrix functions $\varphi$. The statement below is immediate from Theorem 5.2 \[11\].

**Proposition 4.1** If $\varphi$ is a rational matrix function such that

$$
\lim_{\lambda \to \infty} \varphi(\lambda) = \frac{i}{2} D, \quad \Im \varphi(\lambda) \geq 0 \quad (\lambda \in \mathbb{C}_+), \tag{4.3}
$$

were $D$ is a $2p \times 2p$ matrix function.
then \( \varphi \) admits a representation (i.e., realization in terms of control theory)

\[
\varphi(\lambda) = \frac{i}{2} D + \Theta_1^*(\beta - \lambda I_n)^{-1} \Theta_2, \tag{4.4}
\]

where \( \Theta_1 \) and \( \Theta_2 \) are \( n \times p \) matrix functions, \( n \) is some positive integer number, and \( n \times n \) matrix \( \beta \) satisfies the matrix identity

\[
\beta^* - \beta = i (\Theta_2 - \Theta_1) D^{-1} (\Theta_2 - \Theta_1)^*. \tag{4.5}
\]

Dirac systems and Weyl matrix functions \( \tilde{\varphi} \), which have the form \( \tilde{\varphi} = 2D^{-\frac{1}{2}} \varphi D^{-\frac{1}{2}} \), were studied in [10]. The next proposition follows from the Step 1 of the proof of Theorem 4.3 [10] (see also [11]).

**Proposition 4.2** Let relations (4.4) and (4.5) hold. Then \( \Im \varphi(\lambda) > 0 \ (\lambda \in \mathbb{C}_+) \) and \( \varphi \) admits Herglotz representation

\[
\varphi(\lambda) = \nu + \int_{-\infty}^{\infty} \left( \frac{1}{z - \lambda} - \frac{z}{1 + z^2} \right) d\tau(z) \quad (\nu = \nu^*), \tag{4.6}
\]

where

\[
\tau(z) = \int_0^z \rho(t) dt + \sum_{z_k < z} \nu_k, \tag{4.7}
\]

numbers \( z_1 < z_2 < \ldots \) are the real eigenvalues of \( \beta \),

\[
\nu_k = \text{res}_{z=z_k} \Theta_2^*(zI_n - \beta)^{-1} \Theta_2 \geq 0, \tag{4.8}
\]

and \( \rho \) is \( p \times p \) rational matrix function:

\[
\rho(t) = \frac{1}{2\pi} \zeta(t)^* D \zeta(t) \geq 0, \quad \zeta(t) := I_p - iD^{-1} (\Theta_2 - \Theta_1)^*(tI_n - \beta)^{-1} \Theta_2. \tag{4.9}
\]

It is easy to see from (3.2) that

\[
(I - zA)^{-1} \Phi_2 = e^{izxD}, \quad \Phi_2^*(I - zA^*)^{-1} f = \int_0^l e^{-ixD} f(x) dx \quad (f \in L^2_p(0, l)). \tag{4.10}
\]

By (4.7)-(4.10) the right-hand side of the equality

\[
S := \int_{-\infty}^{\infty} (I - zA)^{-1} \Phi_2 d\tau(z) \Phi_2^*(I - zA^*)^{-1} \tag{4.11}
\]
weakly converges, and so the equality defines an operator $S$. Moreover, it is easy to see that the inequalities
\[ c(f, f)_{L^2} > (Sf, f)_{L^2} > 0 \] (4.12)
hold for some fixed $c > 0$ and arbitrary $f \neq 0$. (Here $(\cdot, \cdot)_{L^2}$ denotes the scalar product in $L^2_p(0, l)$.) Thus, $S$ is a bounded and positive operator. We shall show that operators $S$ belong to a subclass of operators of the form $(2.1)$, $(2.2)$.

**Theorem 4.3** Let the matrix identity (4.5) hold. Then the operator $S$ given by $(2.1)$ and $(2.2)$ is positive and boundedly invertible.

**Proof.** The theorem is obtained by proving that $S$ of the form $(2.1)$, $(2.2)$ admits representation (4.11).

First, consider $S$ given by (4.11). It can be calculated directly (see also Section 1.1 in [27]) that this operator $S$ satisfies the operator identity (3.1),

\[ \Phi_1 = i \left( \nu - \int_{-\infty}^{\infty} \left( A(I - zA)^{-1} + \frac{z}{1 + z^2} I \right) \Phi_2 d\tau(z) \right). \] (4.13)

Here the operator $\Phi_1$ is an operator of multiplication by the matrix function, which we denote by $\Phi_1(x)$. From the identity (3.1) and Theorem 3.2 it follows that $S$ is an operator with a $D$-difference kernel $s(x, t) = \{s_{ij}(d_i x - d_j t)\}_{i,j=1}^{p}$, and $s(x, 0) = D^{-1} \Phi_1(x)$. Introduce $S = S_l$ and $\Phi_1 = \Phi_{1,l}$ by (4.11) and (4.13), respectively, for all $0 < l < \infty$. Then the kernel $s(x)$ of the integral operators $S_l$ is determined on $\mathbb{R}$ by the equalities

\[ s_{ij}(x) = d_i^{-1} (\Phi_1)_{ij} (x/d_i) \quad (x > 0), \quad s_{ij}(-x) = -\frac{d_j}{d_i} s_{ji}(x). \] (4.14)

For $\varphi$ satisfying (4.6), according to Statement 3 in [21], after the corresponding change of notations we get

\[ \varphi(\lambda) = \lambda \int_0^{\infty} s(x, 0)^* e^{i\lambda x} D^2 dx D^2 = \lambda \int_0^{\infty} e^{i\lambda x} s(x)^* dx D. \] (4.15)

Note that in view of formula (4.13) and Proposition 4.2 we can present $s$ as a sum $s(x) = s_1(x) + s_2(x)$, where the entries of $s_1$ are bounded and the
entries of \( s_2 \) belong \( L^2(0, \infty) \). Finally, we apply Fourier transform to derive from (4.15) the equality

\[
e^{-\eta x}s(x)^* = \frac{1}{2\pi} \text{l.i.m.}_{a \to \infty} \int_{-a}^{a} e^{-i\xi x} \lambda^{-1} \varphi(\lambda) D^{-1} d\xi \quad (\lambda = \xi + i\eta, \quad \eta > 0),
\]

(4.16)

the limit l.i.m. being the limit in \( L^2(0, l) \) \((0 < l < \infty)\). Using (4.4) and (4.16), we obtain

\[
e^{-\eta x}s(x)^* = \frac{1}{2\pi} \text{l.i.m.}_{a \to \infty} \int_{\Gamma_a} e^{-i\xi x} \lambda^{-1} \varphi(\lambda) D^{-1} d\xi \quad (\lambda = \xi + i\eta, \quad \eta > 0),
\]

(4.17)

where \( \Gamma_a \) is a clockwise oriented contour:

\[
\Gamma_a = [-a, a] \cup \{ \xi : |\xi| = a, \Im \xi < 0 \}.
\]

It is easy to see that

\[
\frac{1}{2\pi} \text{l.i.m.}_{a \to \infty} \int_{\Gamma_a} e^{-i\xi x} \lambda^{-1} d\xi = -ie^{-\eta x}.
\]

(4.18)

According to (4.3) we have \( \sigma(\beta) \subset \mathbb{C}_- \), where \( \sigma \) is spectrum. Similar to [6] we turn to zero \( \varepsilon \) in the equality \( \lambda^{-1}(\beta_\varepsilon - \lambda I_n)^{-1} = \beta_\varepsilon^{-1}(\lambda^{-1}I_n + (\beta_\varepsilon - \lambda I_n)^{-1}) \), where \( \det \beta_\varepsilon \neq 0, \|\beta - \beta_\varepsilon\| < \varepsilon \), and thus obtain

\[
\frac{1}{2\pi} \text{l.i.m.}_{a \to \infty} \int_{\Gamma_a} e^{-i\xi x} \lambda^{-1}(\beta - \lambda I_n)^{-1} d\xi = e^{-\eta x} \int_0^x \exp(-iu\beta)du.
\]

(4.19)

Here we take into account that, when the spectrum of some matrix \( \mathcal{K} \) is situated inside the anti-clockwise oriented contour \( \Gamma \) we have

\[
\frac{1}{2\pi} \int_{\Gamma} e^{-i\lambda x}(\lambda I_n - \mathcal{K})^{-1} d\lambda = \exp(-ix\mathcal{K}).
\]

By (4.4) and (4.17)-(4.19) we get

\[
s(x) = \frac{1}{2} I_p + D^{-1} \Theta_2^* \int_0^x \exp(iu\beta^*) du \Theta_1 \quad (x > 0).
\]

(4.20)

It follows from (4.14) that \( s(x) = -D^{-1}s(-x)^*D \) \((x < 0)\), and so according to (4.20) \( s(x) \) is continuously differentiable for \( x \neq 0 \). As the functions \( s_{ij}(x) \)
are continuous at \( x = 0 \) for \( i \neq j \), and \( s_{ii}(+0) - s_{ii}(-0) = 1 \), formulas (1.1) and (1.2) imply (2.1), where \( k(x) = D \left( \frac{d}{dx} s \right)(x) \). Therefore we have
\[
  k(x) = \Theta_2^* \exp(ix\beta^*)\Theta_1 \quad (x > 0), \quad k(x) = k(-x)^*.
\] (4.21)

Now, note that equalities (2.2) and (4.21) coincide. In other words, the operator \( S \), which is considered in the theorem, admits representation (4.11). Hence, by (4.12) this operator is bounded and positive, and so in view of (2.1) and (2.2) it is also boundedly invertible. □

The matrix function \( \tau \) of the form (4.7)-(4.9) and the \( S \)-node given by (3.2), (4.11), and (4.13) satisfy conditions of Theorem 2.4 [29], p. 57. Therefore \( \varphi(\lambda) \) given by (4.6) can be presented as a linear-fractional transformation
\[
  \varphi(\lambda) = i(\mathcal{W}_{11}(\lambda)R_1(\lambda) + \mathcal{W}_{12}(\lambda)R_2(\lambda))(\mathcal{W}_{21}(\lambda)R_1(\lambda) + \mathcal{W}_{22}(\lambda)R_2(\lambda))^{-1},
\] (4.22)
where \( \mathcal{W}_{ij}(\lambda) \) are \( p \times p \) blocks of the matrix function \( \mathcal{W} \),
\[
  \mathcal{W}(\lambda) := W(l, \lambda)^*, \quad W(l, \lambda) = I_{2p} + i\lambda J\Pi^*S^{-1}(I - \lambda A)^{-1}\Pi \quad (4.23)
\]
and \( R_1(\lambda) = R_1(l, \lambda), \quad R_2(\lambda) = R_2(l, \lambda) \) is a pair of \( p \times p \) matrix functions, which are meromorphic in \( \mathbb{C}_+ \) and have property-\( J \), that is,
\[
  R_1(\lambda)^*R_1(\lambda) + R_2(\lambda)^*R_2(\lambda) > 0, \quad \begin{bmatrix} R_1(\lambda)^* & R_2(\lambda)^* \end{bmatrix} J \begin{bmatrix} R_1(\lambda) \\ R_2(\lambda) \end{bmatrix} \geq 0.
\] (4.24)

It is easy to see from (4.23) that \( \lim_{l \to +0} W(l, \lambda) = I_{2p} \) and thus we put \( W(0, \lambda) = I_{2p} \). Now, by Theorem 2.1 from [29], p. 54 the matrix function \( W \) satisfies for \( x \geq 0 \) the equation
\[
  W(x, \lambda) = I_{2p} + i\lambda J \int_0^x (dB_1(r))W(r, \lambda), \quad B_1(r) := \Pi_r^*S_r^{-1}\Pi_r \quad (4.25)
\]
where \( S_r \in \{L_p^2(0, r), L_p^2(0, r)\} \), \( \Pi_r \in \{\mathbb{C}^{2p}, L_p^2(0, r)\} \). As the operators \( S_r \) \((0 < r \leq l < \infty)\) are invertible, the operators \( S_l \) admit triangular factorisation (see [14], p. 184). It follows that \( B_1 \) is differentiable, and we rewrite (4.25) as the canonical system
\[
  \frac{d}{dx} W(x, \lambda) = i\lambda JH(x)W(x, \lambda), \quad (4.26)
\]
\[
  H(x) := \frac{d}{dx} \left( \Pi_x^*S_x^{-1}\Pi_x \right). \quad (4.27)
\]
Moreover, in view of Remark 2.4 the kernel $T(x, t)$ of the integral operator $S_l^{-1}$ is continuous with respect to $x, t, r$ excluding the lines $d_i x = d_j t$. Therefore, for $d_i x \neq d_j t$ ($1 \leq i, j \leq p$) similar to the continuous kernels (14, p.186) we have

$$k(x, r) + T_r(x, r) + \int_0^r k(x, u)T_r(u, r)du = 0, \quad x \leq r \leq l.$$  \hspace{1cm} (4.28)

Introduce an upper triangular operator

$$V_+ = I + \int_x^l T_r(x, r) \cdot dr \in \{L^2_p(0, l)\}. \hspace{1cm} (4.29)$$

According to (4.28) and (4.29) the operator $S_l V_+$ is a lower triangular operator. Hence, the operator $V_+^* S_l V_+$ is a lower triangular operator. On the other hand $V_+^* S_l V_+$ is selfadjoint, and so the integral part of $V_+^* S_l V_+$ equals zero, i.e., $V_+^* S_l V_+ = I$ or equivalently

$$S_l^{-1} = V_+ V_+^*, \quad V_+^* = V_+^* = I + \int_0^x T_x(x, r) \cdot dr.$$ \hspace{1cm} (4.30)

In the second equality above we used formula (4.29) and relation $T_x(r, x)^* = T_x(x, r)$ ($x \geq r$).

**Theorem 4.4** Let $\varphi$ be a rational function, which satisfies (4.3). Then $\varphi$ is a Weyl function of the canonical system (4.26), where the Hamiltonian $H$ has the form

$$H(x) = \gamma(x)^* \gamma(x), \quad \gamma(x) = \left(V_+^*[\Phi_1 \quad \Phi_2]\right)(x) \quad (x \leq l < \infty),$$ \hspace{1cm} (4.31)

and the operator $V_+^*$ is given by (4.30) and is applied columnwise to the matrix functions $\Phi_1(x) = \{d_i s_{ij}(d_i x)\}_{i,j=1}^p$ and $\Phi_2 \equiv I_p$. The matrix function $s(x)$ is given by (4.20) and the matrix function $T_x(x, r)$ in (4.30) is given in Remark 2.4.

**Proof.** It follows from (4.26) that

$$\frac{d}{dx} \left( W(x, \bar{\lambda})^* JW(x, \lambda) \right) = 0,$$ \hspace{1cm} (4.32)

$$\frac{d}{dx} \left( W(x, \lambda)^* JW(x, \lambda) \right) = i(\lambda - \bar{\lambda})W(x, \lambda)^* H(x)W(x, \lambda).$$ \hspace{1cm} (4.33)
In view of (4.33) we obtain
\[
\int_0^l W(x, \lambda)^* H(x) W(x, \lambda) dx = i(\lambda - \lambda)^{-1} \left( W(l, \lambda)^* J W(l, \lambda) - J \right), \tag{4.34}
\]

Note also that according to (4.32) the equality \( W(l, \lambda)^* JW(l, \lambda) = J \) holds, or equivalently
\[
W(l, \lambda)^* = JW(l, \lambda)^{-1} J. \tag{4.35}
\]

By Proposition 4.1 \( \varphi \) admits representation (4.4) and identity (4.5) is valid. So, by Proposition 4.2 \( \varphi \) admits Herglotz representation, where the matrix function \( \tau(t) \) has the form (4.7)-(4.9). Hence, as it was shown above, the representation (4.22) of \( \varphi \), where \( W \) is expressed via the matrizant \( W(l, \lambda) \) satisfies (4.24), is also true. Using (4.35), we rewrite (4.22) in the form
\[
\begin{bmatrix} I_p & -i\varphi(\lambda) \end{bmatrix} = W(l, \lambda)^{-1} J \begin{bmatrix} R_1(\lambda) \\ R_2(\lambda) \end{bmatrix} \left( W_{21}(\lambda) R_1(\lambda) + W_{22}(\lambda) R_2(\lambda) \right)^{-1}. \tag{4.36}
\]

Taking into account (4.24), (4.31), and (4.36) we derive
\[
\int_0^l \begin{bmatrix} I_p & -i\varphi(\lambda)^* \end{bmatrix} W(x, \lambda)^* H(x) W(x, \lambda) \begin{bmatrix} I_p \\ -i\varphi(\lambda) \end{bmatrix} dx \leq i(\lambda - \lambda)^{-1} \\
	imes \begin{bmatrix} I_p & -i\varphi(\lambda)^* \end{bmatrix} J \begin{bmatrix} I_p \\ -i\varphi(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{C}_+. \tag{4.37}
\]

As the right-hand side in the inequality (4.37) does not depend on \( l \) we can substitute \( \infty \) instead of the limit \( l \) of integration in the left-hand side. Hence \( \varphi \) is a Weyl function of the constructed system.

According to the second relation in (4.30) we obtain \( (V_{+, \lambda} f)(x) = (V_{+, \lambda} \tilde{f})(x) \) for \( x \leq l \), where \( \tilde{f} \) is the restriction of \( f \) on the interval \([0, x]\). Therefore, relations (4.27) and (4.30) imply (4.31).

**Corollary 4.5** Let the conditions of Theorem 4.4 hold and let \( \text{det} \beta \neq 0 \). Then we have
\[
\gamma(x) = \left( V_{+, \lambda}^* \left[ \frac{1}{2} D + i\Theta_2(\beta^*)^{-1} \Theta_1 \right] - i\gamma_0(x) \right], \tag{4.38}
\]
where the $s\text{-th}$ row of $\gamma_0 \ (p \geq s \geq 1)$ is given by the equality
\[
e_s \gamma_0(x) = e_s \left( \Theta_2^* e^{i d_s x \beta^*} + [ \Theta_2^* \Theta_1^* ] e^{d_s x A} U(d_s x) \right)
\times \left( P^x U(d_1 x)^{-1} - U(d_s x)^{-1} + I_{2n} - P^x \right) \left[ \begin{array}{c} I_p \\ 0 \end{array} \right] \right) (\beta^*)^{-1} \Theta_1, \quad (4.39)
\]

$U$ in (4.39) is defined by (2.24) after substitution $l = x$, and $P^x$ is defined by (2.16) after substitution $a = d_1 l = d_1 x$.

Proof. By (1.2), (3.2), and (4.20) the equality
\[
e_s [\Phi_1(x) \Phi_2] = e_s \left[ \frac{1}{2} D + i \Theta_2^*(\beta^*)^{-1} \Theta_1 - i \Theta_2^* e^{i d_s x \beta^*} (\beta^*)^{-1} \Theta_1 \right] \left[ \begin{array}{c} I_p \\ 0 \end{array} \right] \quad (4.40)
\]
is true. Using (4.40) and the second equality in (4.31) we obtain (4.38), where
\[
\gamma_0(x) = \left( V_+ \{ e_s \Theta_2^* e^{i d_s x \beta^*} (\beta^*)^{-1} \Theta_1 \}_{s=1}^p \right) (x). \quad (4.41)
\]

From (2.21) it follows that
\[
\Theta_2^* e^{i d_s x \beta^*} = [\Theta_2^* \Theta_1^*] e^{d_s x A} \left[ \begin{array}{c} I_p \\ 0 \end{array} \right]. \quad (4.42)
\]

According to the representation of $V_+^*$ in (4.30), Remark 2.4, formula (4.42) and second relation in (2.23) we get
\[
V_+ \{ e_s \Theta_2^* e^{i d_s x \beta^*} \}_{s=1}^p = \left\{ e_s \Theta_2^* e^{i d_s x \beta^*} \right\}_{s=1}^p + \{ \mathcal{F}_s(x) \mathcal{G}_s(x) \}_{s=1}^p, \quad (4.43)
\]
where
\[
\mathcal{F}_s(x) = e_s \left[ \Theta_2 \Theta_1^* \right] e^{d_s x A} U(d_s x), \quad (4.44)
\]
\[
\mathcal{G}_s(x) = \left( (I_{2n} - P^x) \int_0^{d_s x} U(z)^{-1} e^{-z A} Y(z) e^{z A} dz \right)
\times \left[ \begin{array}{c} I_p \\ 0 \end{array} \right], \quad (4.45)
\]
\[
Y(z) = \sum_{j: d_j > \tilde{d}_m} \frac{1}{d_j} \left[ \begin{array}{c} -\Theta_1 \\ \Theta_2 \end{array} \right] e_j^* e_j \left[ \Theta_2^* \Theta_1^* \right] = Y_m \quad \text{for} \quad \tilde{d}_m x \leq z \leq \tilde{d}_{m-1} x.
\]

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Taking into account (2.25) rewrite (4.45) in the form
\[ G_s(x) = \left( (I_{2n} - P^*)(I_{2n} - U(d_s x)^{-1}) + P^* (U(d_1 x)^{-1} - U(d_s x)^{-1}) \right) \begin{bmatrix} I_p \\ 0 \end{bmatrix}. \] (4.46)

Finally, formulas (4.41), (4.43), (4.44), and (4.46) imply (4.39). ■

In view of Corollary 4.5 to recover \( \gamma \) and Hamiltonian \( H \) we need only to calculate the action of \( V_+^* \) on constant vectors.

The matrix function \( \gamma(x) \), which is recovered in Theorem 4.4, satisfies the equality
\[ \gamma(x) J \gamma(x)^* \equiv D. \] (4.47)

Indeed, by (3.1), the first equality in (4.30), and the second equality in (4.31) we have
\[ V_+^* A (V_+^*)^{-1} - V_+^{-1} A^* V_+ = i \gamma(x) J \int_0^t \gamma(t)^* \cdot dt. \] (4.48)

As \( V_+^* A (V_+^*)^{-1} \) is a lower triangular operator and \( V_+^{-1} A^* V_+ \) is an upper triangular operator, we derive
\[ V_+^* A (V_+^*)^{-1} = i \gamma(x) J \int_0^x \gamma(t)^* \cdot dt. \] (4.49)

Rewrite (4.49) in terms of the kernels of the corresponding integral operators and put \( t = x \) to get (4.47).

As it is stated in the proposition below, equality (4.47) means that we recover canonical systems from the subclass of systems with linear similar matrix functions \( JH(x) \), though (differently from [29], p. 104) the kernel of \( S^{-1} \) is not necessarily continuous.

**Proposition 4.6** Let the conditions of Theorem 4.4 hold. Then \( JH(x) \) is similar to the matrix \( JH_0 \), where
\[ H_0 := \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}. \] (4.50)

**Proof.** Fix \( x \geq 0 \) and denote by \( X \) a \( p \times 2p \) matrix such that it has rank \( p \) and satisfies the equality \( XJ \gamma(x)^* = 0 \). As the maximal \( J \)-nonnegative subspaces
are $p$-dimensional, it easily follows from $\gamma(x)J\gamma(x)^* > 0$ and $XJ\gamma(x)^* = 0$ that $XJX^* < 0$. Then, we have
\[ \tilde{X}J\tilde{X}^* = -I_p, \quad \tilde{X}J\gamma(x)^* = 0 \quad \text{for} \quad \tilde{X} := (-XJX^*)^{-\frac{1}{2}}X. \tag{4.51} \]
Now, put
\[ L := \begin{bmatrix} D^{-\frac{1}{2}}\gamma(x) \\ \tilde{X} \end{bmatrix}. \tag{4.52} \]
By (4.47), (4.51), and (4.52) the equality
\[ L^{-1} = [J\gamma(x)^*D^{-\frac{1}{2}} - J\tilde{X}^*] \tag{4.53} \]
is true. According to (4.50), (4.52), and (4.53) we get $L^{-1}H_0L = J\gamma(x)^*\gamma(x)$. In view of (4.31) the last equality yields $L^{-1}H_0L = JH(x)$. \hfill \Box

Acknowledgement. The work of A.L. Sakhnovich was supported by the Austrian Science Fund (FWF) under Grant no. Y330, and his visit to Mexico was supported by the PIFI grant P/CA-9 2007-14-17. A.L. Sakhnovich is grateful to the Autonomous University of Hidalgo for its hospitality.

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