Convex relaxation for optimal fixture layout design

Zhen Zhong, Shancong Mou, Jeffrey H. Hunt & Jianjun Shi

To cite this article: Zhen Zhong, Shancong Mou, Jeffrey H. Hunt & Jianjun Shi (2023) Convex relaxation for optimal fixture layout design, IISE Transactions, 55:7, 746-754, DOI: 10.1080/24725854.2022.2102272

To link to this article: https://doi.org/10.1080/24725854.2022.2102272

Published online: 24 Aug 2022.

Article views: 309

View related articles

View Crossmark data

Citing articles: 1 View citing articles
Convex relaxation for optimal fixture layout design

Zhen Zhonga, Shancong Moua, Jeffrey H. Huntb, and Jianjun Shi'a

aH. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, USA; bThe Boeing Company, El Segundo, CA, USA

ABSTRACT
This article proposes a general fixture layout design framework that directly integrates the system equation with the convex relaxation method. Note that the optimal fixture design problem is a large-scale combinatorial optimization problem; we relax it to a convex Semi-Definite Programming (SDP) problem by adopting sparse learning and SDP relaxation techniques. It can be solved efficiently by existing convex optimization algorithms and thus generates a near-optimal fixture layout. A real case study in the half-to-half fuselage assembly process indicates the superiority of our proposed algorithm compared to the current industry practice and state-of-art methods.

1. Introduction

Composite materials are widely used in the aircraft industry due to their superior properties including high strength-to-weight ratio, corrosion resistance, and high durability (Jones, 1998). An aircraft fuselage consists of thin sheets of composite materials of a large area, which makes them compliant parts (Megson, 2016). Due to the compliant property, the deformation and dimensional variation induced by the load due to gravity will significantly influence the quality of the assembly process. When the deformation is larger than the engineering specification, a shape adjustment method has to be adopted to adjust the shape of the fuselage, which will not only introduce residual stress into the subassembly, but also increase the cycling time (Yue et al., 2018; Du et al., 2019). Therefore, the optimal design of the fixture layout for a fuselage is important for reducing initial deformation, thus reducing the assembly cycling time and the residual stress.

Fixtures are widely used in a variety of manufacturing systems to hold the parts and thus control their locations and orientations (Wang et al., 2010). Fixture design can be divided into several stages including setup planning, fixture planning, unit design, and verification (Wang et al., 2010). In this article, we focus on optimal fixture layout design in the fixture planning stage for compliant sheet parts.

In current industrial practice, Computer-aided fixture design methods can be categorized into three categories: rule-based methods, case-based reasoning methods, and heuristic-based methods (Boyle et al., 2011). The first two categories belong to non-optimized layout planning methods (Boyle et al., 2011) which also highly rely on the re-use of experiential knowledge (Nee, 1991; Kumar and Nee, 1995; Joneja and Chang, 1999; Zhang and Lin, 1999; Gologlu, 2004; Wang and Rong, 2008). For the heuristic-based method, Finite Element Analysis (FEA) tools are typically utilized to calculate the shape and deformation of the workpiece or subassembly with a given fixture layout. Then, a heuristic method is applied to search in the design space to find the fixture layout that minimizes the workpiece deformations. The commonly used heuristic methods include the Genetic Algorithm and Pseudo-gradient techniques (Edward, 1998; Li and Melkote, 1999; Vallapuzha et al., 2002; Bazaraa et al., 2013). However, those methods all require a large number of expensive FEA replications, which is time-consuming and can become computationally prohibitive, especially in high precision modelling where a large number of mesh nodes is required. Recently, Du et al. (2021) proposed a method to tackle the optimal fixture layout design problem for compliant parts in ship assembly problems. To avoid intensive implementations of FEA software, an integer programming problem was formulated using the stiffness matrix exported from the FEA software. It is a hard combinatorial optimization problem, and the heuristic (Simulated Annealing (SA)) algorithm was adopted to solve it. In essence, this method is equivalent to running FEA software repeatedly and selecting the best result from multiple replications. As a result, the benefit of direct problem modelling using its stiffness matrix is not fully utilized and their approach still belongs to heuristic methods.

In this article, we proposed a general fixture layout design framework (called SECR) which directly integrates the system equation (the SE in SECR) from the FEA software to formulate optimization problems, and the convex relaxation (the CR in SECR) techniques are adopted to solve it. To the best knowledge of the
authors, this is the first non-heuristic-based general optimal fixture design framework.

There are two steps in the proposed SECR framework: In the first step, the system equation is exported from the FEA software, which describes the relationship between the linear shape deformation and forces applied on the part. This will serve as the foundation of our problem formulation. A detailed illustration of the system equation will be shown in Section 2.1.

The second step is to formulate and solve the optimization problem. In a fixture design problem, finding optimal fixture locating point locations by using system equations is still intrinsically combinatorial. Convex relaxation is a powerful method to handle such problems. There are several famous convex relaxation techniques such as sparse learning (Beck et al., 2012; Sojoudi and Lavaei, 2014) techniques. In particular, sparse learning relaxation is commonly used in transforming the combinatorial problem into a convex problem and has been proven to be effective in many engineering applications, such as compress sensing (Donoho, 2006) and optimal actuator placement (Du et al., 2019). SDP relaxation is usually applied to tackle quadratic nonlinearities; it has been adopted in many areas such as graph theory (Goemans and Williamson, 1995), approximation theory (Sojoudi and Lavaei, 2014), and power systems (Beck et al., 2012). Compared with existing heuristic-based methods, our contribution is to derive a relaxation of the optimal fixture design problem by reformulating it into a convex SDP problem by using sparse learning relaxation and SDP relaxation techniques. The effectiveness of the convex relaxation algorithm is validated by a real case study in the half-to-half fuselage assembly process.

The remainder of this article is organized as follows. Section 2 provides a detailed illustration of our convex relaxation-based optimal fixture design framework. A case study is provided to validate the performance of our proposed method in Section 3. Finally, conclusions are presented in Section 4.

2. SECR framework

In this section, we first elaborate on the FEA-based process model (Zhong et al., 2022). Then, we illustrate the general formulation for the optimal fixture design in Section 2.2. Finally, the detailed derivation of convex relaxation is provided in Section 2.3. In the convex relaxation, we utilize the sparsity property of fixture force which can further improve the computational efficiency and scalability. This makes it possible for general fixture design problems to be considered in high precision modelling, as the number of the nodes is not a concern in this method. By formulating the optimal fixture design problem in a convex manner, we can efficiently obtain an optimal solution by using the CVX software (Grant et al., 2008).

2.1 FEA-based process model

For an assembly process that can be modelled in FEA software under a static force and small linear deformation assumptions (Kohnke, 2013), the relationship between linear elastic deformation and the force can be described by the following Equation (1):

\[
KU = F_g + F_r,
\]

where, \( K \in \mathbb{R}^{6N \times 6N} \) is the global stiffness matrix; \( U = [u_1; \ldots; u_N] \in \mathbb{R}^{6N} \) is the nodal displacement vector assembled by aggregating the displacement vector \( u_i = [u_i^x, u_i^y, u_i^z, \omega_i^x, \omega_i^y, \omega_i^z]^T \in \mathbb{R}^{6 \times 1}, \quad i \in \{1, \ldots, N\}, \) on each mesh node. Here, we denote the total number of mesh nodes as \( N \) and a set containing all mesh nodes as \( N \). Moreover, we use \( F_r \in \mathbb{R}^{6N} \) and \( F_g \in \mathbb{R}^{6N} \) to represent the load exerted at the fixture locating points by the fixture and gravity respectively. Vectors \( F_r \) and \( F_g \) are defined similarly as force vectors: \( F_r = [f_{r1}; \ldots; f_{rN}] \in \mathbb{R}^{6N} \) and \( F_g = [f_{g1}; \ldots; f_{gN}] \in \mathbb{R}^{6N} \), where \( f_{gi} = [f_i^g; f_i^g; f_i^g; f_i^g; f_i^g; f_i^g]^T \in \mathbb{R}^{6}, \quad i \in \{1, \ldots, N\}, \) is the gravity-induced load vector and \( f_{ri} = [r_i^g; r_i^g; r_i^g; r_i^g; r_i^g; r_i^g]^T \in \mathbb{R}^{6}, \quad i \in \{1, \ldots, N\}, \) is the fixture-induced load vector on each mesh node. Specifically, the first three elements of \( f_{gi} \), \( [f_i^g; f_i^g; f_i^g]^T \), denote the three-dimensional load-induced force of the \( i \)th mesh node; and the last three elements of \( f_{gi} \), \( [r_i^g; r_i^g; r_i^g]^T \), denote the three-dimensional load-induced torque of the \( i \)th mesh node. \( f_r \) is defined similarly as \( f_{gi} \) for its elements. Without loss of generality, we assume fixtures can only constrain linear displacement, i.e., \( [u_i^x, u_i^y, u_i^z]^T = 0 \), when the \( i \)th mesh node is used as a fixture locating point. If a fixture can constrain both linear and angular displacement, then, \( [u_i^x, u_i^y, u_i^z, \omega_i^x, \omega_i^y, \omega_i^z]^T = 0 \), when the \( i \)th mesh node is used as a fixture locating point.

The stiffness matrix \( K \) and force \( F_g \) can be directly exported from the FEA software. To apply our algorithm, we assume that at least three locating points that make the structure stable are pre-specified. Once those three fixtures locating points are fixed, the fuselage has a unique deformation under gravity. Then, we follow the common practice in FEA (Kohnke, 2013) to remove the corresponding rows and columns of those three pre-specified fixture mesh nodes in \( K \) to obtain an invertible stiffness matrix \( K^* \in \mathbb{R}^{6(N-3) \times 6(N-3)} \), where \( N_1 = N - 3 \). Denote the remaining set of nodes as \( N^*_1 \). Similarly, by removing the corresponding rows, we can obtain \( U^* \), \( F_g^* \) and \( F_r^* \), respectively. Then, by conducting basic linear algebra operations, we obtain:

\[
K^* U^* = F_g^* + F_r^*.
\]
Since $K^*$ is an invertible matrix, we use $A^*$ to denote the inverse matrix of $K^*$:

$$U^* = A^* \left(F^*_g + F^*_r\right)$$

### 2.2 Original formulation for optimal fixture design

For a compliant part, we adopt the “N-2-1” locating principle in the fixture design (Cai et al., 1996). Denote the set of potential fixture locating points as $N_{PT} \subseteq N_1$ and its cardinality as $N_{PT}$. We aim to find the set of $N_f$ fixture locating points, denoted as $N_{f_i}$ out of $N_{PT}$ potential fixture locating points to achieve minimum total deformation $\delta^2$, i.e.,

$$\delta^2 = (U^*)^T W U^*,$$

where $U^*$ is the shape deformation vector; $W$ is a diagonal matrix with only ones on locations of the linear displacement $[u_x, u_y, u_z]^T$ of mesh nodes that we are interested in, and all zeros otherwise, i.e.,

$$\text{diag}(W) = [0, 0, 0, \ldots, 1, 1, 0, 0, 0, \ldots, 0, 0, 0]^T,$$

where $\text{diag}(W)$ returns the diagonal elements of matrix $W$.

In reality, the following physical constraints are considered in the fixture design optimization:

1. Notice that compared with the number of nodes, the number of fixture locating points is usually much smaller. Therefore, $F^*_g$ is a sparse vector whose nonzero elements can only appear in potential fixture locating points. Let $N_{PT}$ represent the complementary set of $N_{PT}$, i.e., $N_{PT} = N_1 \setminus N_{PT}$. Let the fixture-induced force vector at node $i$ be $f^*_i = [f^*_{x_i}, f^*_{y_i}, f^*_{z_i}]^T$, i.e., $f^*_i = 0, \forall i \in N_{PT}$. Let $F^*_{PT} = [f^*_{ri_1} : \ldots : f^*_{ri_{N_f}}]$ T, where $i_1 \in N_{PT}, l \in \{1, \ldots, N_{PT}\}$. Notice that each element in $F^*_{PT}$ has its corresponding index in $F^*_r$. In order to link the indices of elements between $F^*_{PT}$ and $F^*_r$, we define the following mapping: the $i$th element in $F^*_{PT}$ is the $(i_{(floor(i/3)+1)} - 1) \times 6 + (i \mod 3)$th element in $F^*_r$, where $i_k \in N_{PT}, k \in \{1, \ldots, N_{PT}\}$, floor$(i/3)$ denotes the quotient and $i \mod 3$ denotes the remainder. Define this mapping as $m(l) = (i_{(floor(l/3)+1)} - 1) \times 6 + (l \mod 3)$. 

2. Let $\tau^*_i$ be the fixture-induced torque vector on mesh node $i$. Note that fixtures only exert force and no torque on the part, $\tau^*_i$ on all mesh nodes should be zero, i.e., $[\tau^*_{x_i}, \tau^*_{y_i}, \tau^*_{z_i}]^T = 0, \quad i \in N_1$. 

3. Note that the fixture locating point will restrict the deviation of the corresponding mesh node to be zero, i.e., $[u^*_x, u^*_y, u^*_z]^T = 0, \quad i \in N_f$. For non-fixture location points, $[f^*_{x_i}, f^*_{y_i}, f^*_{z_i}]^T = 0, \quad i \in N_{PT} \setminus N_f$. Here, $N_f$ represents the non-fixture location points. Then, for any given points $i$, we will always have $u^*_{x_i} f^*_{x_i} = 0, \quad i \in \{1, \ldots, N_f\}, \quad a \in \{x, y, z\}$. Moreover, since $\tau_i = 0, \quad i \in N_{PT}$, we can further write this constraint as $U^* (l) F^*_i (l) = 0, \quad j = \{1, \ldots, 6N_f\}$. If we focus on potential fixture locations, we have $U^* (m(l)) F^*_{PT} (l) = 0, \quad l \in \{1, \ldots, 3N_{PT}\}$. 

4. We further define the magnitudes of fixture-induced force vector on all potential fixture locations as $F^*_i = \left(\|F^*_i\|_2 : \ldots : \|F^*_n\|_2\right)^T$. When we have $n_a$ fixture locating points, there are only $n_a$ fixture force vectors with nonzero elements. Thus, we have $\|F^*_i\|_0 = n_a$, where $\| \cdot \|_0$ represents the number of nonzero entries in a vector.

Finally, the problem can be formulated as

$$\min_{F^*_{PT}} (U^*)^T W U^*$$

subject to: $U^* = A^* (F^*_g + F^*_r)$

$$f^*_i = 0, \quad i \in N_{PT}$$

$$\tau^*_i = 0, \quad i \in N_1$$

$$\left[u^*_x, u^*_y, u^*_z\right]^T = 0, \quad i \in N_f$$

$$\tau^*_i = \left[\tau^*_{x_i}, \tau^*_{y_i}, \tau^*_{z_i}\right]^T, \quad f^*_i = \left[f^*_{x_i}, f^*_{y_i}, f^*_{z_i}\right]^T, \quad s^* = \left[s^*_1, s^*_2\right]^T$$

$$i \in N_1 \quad \text{and} \quad F^*_r = \left[f^*_{ri_1} : \ldots : f^*_{ri_{N_f}}\right]^T$$

$$F^*_i = \left(\|F^*_i\|_2 : \ldots : \|F^*_n\|_2\right)^T$$

$$\|F^*_i\|_0 = n_a$$

$$U^* (m(l)) F^*_{PT} (l) = 0, \quad l \in \{1, \ldots, 3N_{PT}\}$$

2.3 Convex formulation for optimal fixture design

The selection of the set of $N_f$ fixture locating points out of mesh nodes can be formulated as an integer programming problem. Previous research demonstrated that traditional integer programming methods are hard to apply (Du et al., 2021) in fixture design problems, due to the huge number of mesh nodes or too large a search space. Therefore, heuristic methods are prevalent in this field. To avoid the drawback of heuristic methods, we propose a convex relaxation method to solve problem (3). The main challenge is to find a good convex relaxation of the following non-convex constraints: (a) the $l_0$-norm constraint (3g), and (b) complementary slackness constraint (3h).

(a) Convex relaxation of $l_0$-norm

To handle the $l_0$-norm constraint (3g), we follow the procedure of Du et al. (2019) and transform the optimization problem (3) into an optimization problem with group lasso
penalty:
\[
\min_{F_{iPT}} \left( U^* \right)^T W U^* + \lambda \| F_i^* \|_1
\]  
subject to: Constraints (3a) - (3e), (3h), (3i)

(b) Convex relaxation of complementary slackness constraint
Finding a good convex relaxation of the complementary slackness constraint (3h) is tricky. There are three main steps in relaxing this constraint: (i) we first exploit the linear relationship between \( U^* \) and \( F_{iPT}^* \) to transform the complementary slackness constraint into a quadratic equality constraint; (ii) then, we further transform this problem into an SDP problem with rank-1 constraint; (iii) finally, we relax this rank-1 constraint by penalizing its nuclear norm and thereby transform the problem (4) into a convex problem. The relaxation steps are presented as follows:

(i) Transforming constraint (3h) into a quadratic equality constraint: Since the linear relationship between \( U^* \) and \( F_{iPT}^* \) is known by constraint (3a), the quadratic equality constraint can be obtained by substituting \( U^* \) with \( F_{iPT}^* \). Notice that either substituting \( U^* \) with \( F_{iPT}^* \) or substituting \( F_{iPT}^* \) with \( U^* \) are mathematically equivalent actions. However, we choose the first option to utilize the sparsity property of \( F_{iPT}^* \), which significantly improves the computational efficiency and makes the method scalable. Adopting this approach, the scale of the SDP problem is irrelevant to the number of nodes, which can be seen in the following derivations. After substituting \( F_{iPT}^* \) with \( U^* \), (3h) can be written as

\[
\sum_{j=1}^{6N_i} A^*(m(l), j) F_{iPT}^*(j) F_{iPT}^*(l) + \sum_{k(l) \in [1, ..., 3N_{PT}]} A^*(m(l), m(k)) F_{iPT}^*(m(k)) F_{iPT}^*(m(l)) = 0, l \in \{1, ..., 3N_{PT}\}, (3h)
\]

(ii) Transforming the quadratic equality constraint into an SDP problem with a rank-1 constraint: The constraint (3h) is still nonconvex. To deal with this quadratic equality constraint, we define a new matrix \( S_1 = F_{iPT}^* F_{iPT}^* \). Then we have \( S_1(l, k) = F_{iPT}^*(l) F_{iPT}^*(k) = F_{iPT}^*(m(l)) F_{iPT}^*(m(k)) \) and constraint (3h) is equivalent to

\[
\sum_{j=1}^{6N_i} A^*(m(l), j) F_{iPT}^*(j) F_{iPT}^*(l) \\
+ \sum_{k(l) \in [1, ..., 3N_{PT}]} A^*(m(l), m(k)) S_1(l, k) = 0, l \in \{1, ..., 3N_{PT}\}, (3h')
\]

Now, constraint (3h') is a linear constraint. However, constraint \( S_1 = F_{iPT}^* F_{iPT}^* \) is still nonconvex, which makes solving problem (4) intractable.

To solve this problem, an idea is to reformulate problem (4) into an SDP problem. Next, the following theorem establishes the equivalence between the original problem (4) and an SDP problem with rank-1 constraint.

**Theorem 1.** If there is an \( l \in \{1, ..., 3N_{PT}\} \) such that \( F_{iPT}^*(l) \sum_{j=1}^{6N_i} A^*(m(l), j) F_{iPT}^*(j) \neq 0 \), then problem (4) is equivalent to the problem (5) below:

\[
\min_{F_{iPT}, S_1} \left( U^* \right)^T W U^* + \lambda \| F_i^* \|_1
\]

subject to: Constraints (3a) - (3e), (3i), (3h')

\[
S_i - F_{iPT}^* F_{iPT}^* \geq 0
\]

\[
\text{rank}(S_1) = 1 \quad \text{(5)}
\]

The proof of Theorem 1 is given in the Appendix. The assumption in Theorem 1 is easy to check. We propose two approaches: (i) before solving the problem: to balance the gravity load, at least one element in potential fixture locations is nonzero (the location is unknown). In this case, if \( \sum_{j=1}^{6N_i} A^*(m(l), j) F_{iPT}^*(j) \neq 0, \forall l \in \{1, ..., 3N_{PT}\} \), then the assumption is valid. (ii) after solving the problem: the locations (l) of nonzero elements in potential fixture locations are known. For this case if at least one l value allows \( F_{iPT}^*(l) \sum_{j=1}^{6N_i} A^*(m(l), j) F_{iPT}^*(j) \neq 0 \), then the assumption is valid.

Notice that the dimension of this SDP problem is only related to the number of potential locating points for fixtures, which is usually small and independent of the number of mesh nodes. There are some existing convex relaxation methods proposed to deal with constraint (5), i.e., nuclear norm relaxation method (Zhang et al., 2012; Mou et al., 2020). Before dealing with constraint (5), we can write constraint (5o) more compactly. To achieve this, we define

\[
S_2 = \begin{bmatrix} S_1 & F_{iPT}^* F_{iPT}^* \\ F_{iPT}^* F_{iPT}^* & 1 \end{bmatrix}
\]

and constraint (5o) is transformed into:

\[
S_2 \geq 0
\]

\[
S_2 = \begin{bmatrix} S_1 & F_{iPT}^* F_{iPT}^* \\ F_{iPT}^* F_{iPT}^* & 1 \end{bmatrix}
\]

(iii) SDP relaxation by applying nuclear norm penalty: By applying the nuclear norm relaxation method, we can remove the constraint (5r) by adding the nuclear norm penalty into the objective function. Finally, we transform problem (5) into the problem (6) below:

\[
\min_{F_{iPT}, S_2} \left( U^* \right)^T W U^* + \lambda \| F_i^* \|_1 + \mu \| S_2 \|_{*}
\]

subject to: Constraints (3a) - (3e), (3i), (3h’)

\[
f_i^* = 0, i \in \bar{N}_{PT}
\]

\[
\tau_i^* = 0, i \in N_1
\]

\[
[u_i^*, u_i^*, u_i^*] = 0, i \in N_f
\]

\[
\tau_i^* = \left[ \tau_i, \tau_{i^*}, \tau_{i^*}, \tau_{i^*} \right], \quad f_i^* = \left[ f_{i^*}, f_{i^*}, f_{i^*}, f_{i^*} \right]^T, \quad f_{i^*} = \left[ \tau_i; f_i^* \right]
\]

\[
i \in N_1 \text{ and } F_i^* = \left[ f_{i^*}, \ldots, f_{i^*} \right]
\]

\[
F_i^* = \left[ \| f_i^* \|_2, \ldots, \| f_i^* \|_2 \right]^T
\]
The rank of $k$ algorithm proposed in Du et al. (2019). Here is the general idea: if the tuning parameter $\lambda$ is too large, we will have $\|F_1\|_0 < n_a$, then we should decrease the $\lambda$ value; when the tuning parameter $\lambda$ is too small, we will have $\|F_1\|_0 > n_a$, then we should increase the $\lambda$ value. The selection of the tuning parameter $\mu$ is based on the rank of the $S_2$ matrix, when the rank of $S_2$ is not equal to 1, we should increase the value of $\mu$ until that the constraint $\text{rank}(S_2) = 1$ is satisfied.

\[
\begin{align*}
\sum_{j=1}^{6N_l} A^*(m(l), j)F_j(j)F_{iPT}(l) \\
+ \sum_{l \in \{1, \ldots, 3N_{PT}\}} A^*(m(l), m(k))S_1(l, k) \\
= 0, \quad l \in \{1, \ldots, 3N_{PT}\} \\
F_{iPT} = [f_{i1}^r; \ldots; f_{ini}^r; \ldots; f_{iN_{PT}}^r], \quad i \in N_{PT}, l \in \{1, \ldots, N_{PT}\} \\
S_2 \geq 0 \\
S_2 = \begin{bmatrix} S_1 & F_{iPT}^T \end{bmatrix} \\
\end{align*}
\]

(6h)

Problem (6) is the final problem formulation, which is a typical SDP problem and can be solved efficiently by using the CVX software. We find the $\lambda$ value by using a binary search algorithm proposed in Du et al. (2019). Here is the general idea: if the tuning parameter $\lambda$ is too large, we will have $\|F_1\|_0 < n_a$, then we should decrease the $\lambda$ value; when the tuning parameter $\lambda$ is too small, we will have $\|F_1\|_0 > n_a$, then we should increase the $\lambda$ value. The selection of the tuning parameter $\mu$ is based on the rank of the $S_2$ matrix, when the rank of $S_2$ is not equal to 1, we should increase the value of $\mu$ until that the constraint $\text{rank}(S_2) = 1$ is satisfied.

3. Case study

In this case study, we will use a half-to-half fuselage assembly process to demonstrate the proposed SECR framework. In the fuselage assembly process, gravity-induced deformation is directly related to assembly precision. Different fixture layouts will result in a maximum shape deformation induced by the gravity load varying from less than 0.01 inches to larger than 10 inches. For example, we have two different sets of fixtures in Figures 1(a) and (b). For the fixture layout shown in Figure 1(a), the maximum total deformation is 0.02 inches, and the maximum residual stress is 611.3 psi. Whereas for the fixture layout in Figure 1(b), the maximum total deformation is 0.13 inches, and the maximum residual stress is 1497.3 psi. From Figure 1, we can see that the fixture layout will significantly influence the precision of the half fuselage assembly process. However, both layouts cannot meet the engineering specifications, since ultra-high precision assembly is vital for large-scale aircraft production. Without a well-designed fixture layout, it is challenging to achieve such high precision during assembly operations. A proper fixture layout should be able to compensate for the gravity-induced shape deformation. In this application, our objective is to find the optimal locations for a given number of fixture locating points such that the gravity-induced deformation is minimized.

3.1 Comparison study of the optimally designed fixture with the current industrial practice

In the proposed SECR method, we assume that there are 30 locations for the potential fixture locating points and three pre-specified locations for the fixture locating points, which are shown in Figure 2. In Figure 2, the red points and the green points represent the mesh nodes and locations for potential fixture locating points, respectively. We checked that the assumptions in Theorem 1 hold. The tuning parameters are selected as $\lambda = 5$ and $\mu = 10$.

To show the superiority of the proposed method, we first conduct a comparison among the results of the proposed method and the current industry practice. The fixture layout corresponding to the proposed method is shown in Figure 3(a), with the fixture layout of current industry practice being shown in Figure 3(b). With the same number of fixture locating points (e.g., both have $N_f = 8$), the SECR method can achieve a maximum total deformation of 0.014 inches whereas the maximum total deformation of current industry practice is 0.27 inches. This result indicates that with the same number of fixture locating points, our algorithm can significantly reduce the maximum total deformation by optimizing the locations of the fixture points.

3.2 Comparison of the optimal design with heuristic-based design

We also compare the result of the proposed method with the heuristic-based DSMSO method (Du et al., 2021). The method in Du et al. (2021) first loads the stiffness matrix from the FEA simulation platform and an integer programming problem aiming at minimizing the maximum deformation is formulated. Then, SA is adopted to solve this integer programming problem.

We conduct the following two comparisons:

1. We run the DSMSO method for 300 replications with the same amount of computational time as our method. By
using the same settings as above, the maximum total deformation of our fixture layout is 0.0112 inches. The best scenario maximum total deformation of the DSMOSO method among those 300 replications is 0.0159 inches. The comparison results are documented in Figure 4. In Figure 4, the x-axis represents the replications whereas the y-axis represents the maximum total deformation. The upper curve represents the result of the DSMOSO method, and the lower curve stands for the result of the proposed SECR method. The result indicates that the proposed method outperforms the state-of-the-art DSMOSO method.

2. We run the DSMOSO method for 1000 replications with the same SA stopping criteria proposed by Du et al. (2021). As a result, each SA replication has 1044 FEA runs and takes 2016 seconds on average, which is approximately 30 times the SECR running time. Figure 5 shows the histogram of the maximum total deformation of those 1000 DSMOSO replications.

The x-axis represents the maximum total deformation and the y-axis represents the number of DSMOSO replication results in a specific maximum total deformation range. It approximates the distribution of the DSMOSO result. Among those 1000 DSMOSO replications, only 7% achieve the same (or slightly better) result than that of the SECR method. This means, on average, 14.3 DSMOSO replications (8 hours) are needed to achieve a similar result as the SECR method which only takes 69 seconds. Moreover, the SECR result can serve as a warm start for the DSMOSO method, which will significantly reduce the computation time and improve the solution quality.

We also tried to use the solver Gurobi to solve the problem (3). However, the solver does not make any progress in 10 hours. Therefore, we terminated it.

Another advantage of the proposed method is its computational efficiency. For heuristic-based methods, a large linear system has to be solved repeatedly and there is no guarantee of the optimality of its solution. However, by using our proposed methods, we can obtain the results by simply solving a small-scale SDP problem, which is much more computationally efficient. For comparison, we also document the computational time for a different number of potential fixture locations in Table 1. As can be seen from the table, the computational time of solving the SDP problem will increase significantly as the scale (number of potential fixture locations) increases, since the complexity of solving such an SDP problem per iteration is \( O(N_{PT}^3) \). This shows the necessity of adopting the sparsity property to reduce the problem scale in the proposed SECR method. In
real practice, by simply choosing 30–40 potential fixture locations, we can already obtain a result within tolerance.

4. Conclusion

This article proposed a novel SECR framework for general fixture layout design. Since the optimal fixture layout design is generally a large-scale combinatorial optimization problem, convex-relaxation techniques, especially sparse-learning and SDP relaxation techniques, are adopted to formulate the optimal fixture design problem in a convex manner. The proposed framework is computationally efficient and scalable. The solution of this convex problem implies a near-optimal fixture layout, and this convex problem can be solved efficiently by using existing convex optimization algorithms such as interior-point methods, ADMM, etc.

To validate the effectiveness of the proposed method, we conducted a real case study in a half-to-half fuselage assembly process. In the case study, we compare the fixture layout generated by our algorithm with the current industrial practice and the DSMSO method. The result shows that the fixture layout obtained by the proposed SECR method outperforms both the current fixture used in a real assembly process and the fixture layout generated by the DSMSO method in terms of maximum total deformation.

Our proposed SECR algorithm was developed with the assumptions of small linear deformations and static forces, which can be applied to broad applications beyond fuselage fixture layout design. Because:

- The static force can be static manufacturing or assembly forces instead of gravity.

The SECR framework can also be generalized to broad settings by considering either the relaxation of the small linear deformation assumption or the static force assumption.

Nomenclature

\[ \mathcal{N} \] a set contains all mesh node
\[ N \] size of set \( \mathcal{N} \)
\[ K \] global stiffness matrix
\[ u^*_k \] the linear displacement of the \( i \)th mesh node in direction \( k \), where \( k = x, y, z \)
\[ \omega^*_k \] the angular displacement of the \( i \)th mesh node in direction \( k \), where \( k = x, y, z \)
\[ u_i \] nodal displacement vector on the \( i \)th mesh node defined as \[ u_i = [u_i^*, u_i^*, u_i^*, \omega_i^*, \omega_i^*, \omega_i^*]^T, \ i \in \{1,..,N\} \]
\[ U \] nodal displacement vector of all mesh nodes defined as \[ U = [u_1; \ldots; u_N] \]
\[ f_{g_k} \] gravity-induced force on the \( i \)th mesh node in direction \( k \), where \( k = x, y, z \)
\[ r_{g_k} \] gravity-induced torque on the \( i \)th mesh node in direction \( k \), where \( k = x, y, z \)
\[ f_{g_i} \] gravity-induced load vector on the \( i \)th mesh node defined as \[ f_{g_i} = [f_{g_{x_i}} f_{g_{y_i}} f_{g_{z_i}}]^T, \ i \in \{1,..,N\} \]
\[ F_{g} \] gravity-induced load vector of all mesh nodes defined as \[ F_{g} = [f_{g_1} \ldots f_{g_N}] \]
\[ f_{r_{k}} \] fixture locating points induced force on the \( i \)th mesh node in direction \( k \), where \( k = x, y, z \)
\[ r_{r_{k}} \] fixture locating points induced torque on the \( i \)th mesh node in direction \( k \), where \( k = x, y, z \)
\[ f_{r_i} \] gravity-induced load vector on the \( i \)th mesh node defined as \[ f_{r_i} = [f_{r_{x_i}} f_{r_{y_i}} f_{r_{z_i}}]^T, \ i \in \{1,..,N\} \]
\[ F_{r} \] fixture locating points-induced load on all mesh nodes defined as \[ F_{r} = [f_{r_1} \ldots f_{r_N}] \]
\[ N_1 \] a set contain all mesh nodes beside three fixture locating points
\[ N_1 \] size of set \( N_1 \)
\[ K^* \] stiffness matrix removing the corresponding rows and columns of fixture locating points
\[ U^* \] nodal displacement vector after removing the corresponding rows of prespecified fixture locating points
\[ F_{g}^* \] gravity-induced load vector of all mesh nodes after removing the corresponding rows of pre-specified fixture locating points
\[ f_{r_i}^* \] fixture-induced 3-dimensional force vector on the \( i \)th mesh node defined as \[ f_{r_i}^* = [f_{r_{x_i}} f_{r_{y_i}} f_{r_{z_i}}]^T, \ i \in \{1,..,N\} \]
\[ r_{r_i}^* \] fixture-induced 3-dimensional torque vector on the \( i \)th mesh node defined as \[ r_{r_i}^* = [r_{r_{x_i}} r_{r_{y_i}} r_{r_{z_i}}]^T, \ i \in \{1,..,N\} \]
\[ F_{r}^* \] fixture-induced load vector on all mesh nodes after removing the corresponding rows of prespecified fixture locating points defined as \[ F_{r}^* = [f_{r_1}^* \ldots f_{r_N}^*] \]
\[ W \] diagonal matrix with ones and all other elements are zeros
\[ \delta^2 \] total deviation
\[ N_{PT}^\prime \] a set of potential fixture locations
\[ N_{PT} \] size of \( N_{PT}^\prime \)
\[ \bar{N}_{PT}^\prime \] complementary set of locations of potential fixture points
\[ \bar{N}_{PT} \] the set of fixture locating points locations
\[ \bar{N}_f \] represents the non-fixture location points

Figure 5. Histogram of the maximum total deformation of DSMSO replications.

Table 1. The computational time of SECR for different number of potential fixture locations.

| Number of potential fixture locations | Computational time (seconds) |
|--------------------------------------|-----------------------------|
| 20                                   | 29                          |
| 30                                   | 69                          |
| 42                                   | 358                         |
| 56                                   | 1661                        |
$F_i$ \quad \text{the magnitudes of fixture-induced force vector defined as $[|f_{i1}|, \ldots, |f_{ik}|]^T$} \\
$n_d$ \quad \text{number of available fixtures locating points} \\
$S_o$ \quad \text{$F_i(F_i)^T$} \\
$F_{PT}$ \quad \text{$[f_{n_1}, \ldots, f_{n_k}, \ldots, f_{n_{PT}}]$} \\
$S_i$ \quad \text{$F_{PT}(F_{PT})^T$} \\
$S_2$ \quad \text{$[S_i, (F_{PT})^T F_{PT}^T] / (F_{PT})^T 1$} \\
$\lambda, \mu$ \quad \text{tuning parameters}

**Funding**

The work is supported by the Strategic University Partnership between the Boeing Company and the Georgia Institute of Technology (Funder ID: 10.13039/1000000003).

**Data availability statement**

Due to the nature of this research, participants of this study did not agree for their data to be shared publicly, so supporting data is not available.

**Notes on contributors**

Zhen Zhong received a BS degree in electrical engineering from the University of Science and Technology of China, Hefei, Anhui, China, in 2017. He is currently pursuing a PhD degree with H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, USA. His research interest is focused on the process control in semiconductor manufacturing.

Shancong Mou received a BS degree in energy and power engineering from Xi’an Jiaotong University, Xi’an, China, in 2017. He is currently pursuing a PhD degree with the School of Industrial and Systems Engineering, Georgia Tech, Atlanta, GA, USA. His research interests include data analytics for monitoring, control, and diagnostics of complex engineering systems. Mr. Mou is also a member of the Institute of Industrial and Systems Engineers (IISE) and the Institute for Operations Research and the Management Sciences (INFORMS).

Jeffrey H. Hunt received a BS degree in physics from the Massachusetts Institute of Technology, Cambridge, MA, USA, in 1979, a MA degree in physics and a PhD degree in physics from the University of California at Berkeley, Berkeley, CA, USA, in 1981 and 1988, respectively. He is currently a Principal Scientist and a Senior Technical Fellow with Boeing Company, El Segundo, CA, USA. His career has included physics-based projects in condensed matter physics, quantum information sciences, surface science, and nonlinear optics and work on diverse applications, including both in defense sciences and commercial air and space technologies. He has published over 30 articles, three books, and two encyclopedia articles on condensed matter sciences. He holds 61 U.S. patents. His main research areas are wide scientific and technical challenges in aviation and aerospace industry, with particular applications in condensed matter sciences and nonlinear optics, composite aircraft assembly, and so on. Dr. Hunt is a fellow of the American Physical Society and the Optical Society of America.

Jianjun Shi (https://sites.gatech.edu/jianjun-shi/) received BS and MS degrees in automation from the Beijing Institute of Technology in 1984 and 1987, respectively, and a PhD degree in mechanical engineering from the University of Michigan in 1992. Currently, Dr. Shi is the Carolyn J. Stewart Chair and Professor at the Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology. His research interests include the fusion of advanced statistical and domain knowledge to develop methodologies for modeling, monitoring, diagnosis, and control for complex manufacturing systems. Dr. Shi is a Fellow of four professional societies, including ASME, IISE, INFORMS, and SME, an elected member of the International Statistics Institute (ISI), a life member of ASA, an Academician of the International Academy for Quality (IAQ), and a member of National Academy of Engineers (NAE).

**ORCID**

Zhen Zhong http://orcid.org/0000-0002-0653-3890
Shancong Mou http://orcid.org/0000-0003-2481-3308

**References**

Bazaraa, M.S., Sherali, H.D. and Shetty, C.M. (2013) *Nonlinear Programming: Theory and Algorithms*, John Wiley & Sons, New York, NY.

Beck, A., Drori, Y. and Teboulle, M. (2012) A new semidefinite programming relaxation scheme for a class of quadratic matrix problems. *Operations Research Letters*, 40(4), 298–302.

Boyle, I., Rong, Y. and Brown, D.C. (2011) A review and analysis of current computer-aided fixture design approaches. *Robotics and Computer-Integrated Manufacturing*, 27(1), 1–12.

Cai, W., Hu, S.J. and Yuan, J. (1996) Deformable sheet metal fixturing: Principles, algorithms, and simulations. *ASME Transactions, Journal of Manufacturing Science and Engineering*, 118(3), 318–324.

Donoho, D.L. (2006) Compressed sensing. *IEEE Transactions on Information Theory*, 52(4), 1289–1306.

Du, J., Liu, C., Liu, J., Zhang, Y. and Shi, J. (2021) Optimal design of fixture layout for compliant part with application in ship curved panel assembly. *ASME Transactions, Journal of Manufacturing Science and Engineering*, 143(6), 061007.

Du, J., Yue, X., Hunt, J.H. and Shi, J. (2019) Optimal placement of actuators via sparse learning for composite fuselage shape control. *ASME Transactions, Journal of Manufacturing Science and Engineering*, 141(10), 101004.

Edward, C. (1998) Fast support layout optimization. *International Journal of Machine Tools and Manufacture*, 38(10-11), 1221–1239.

Goemans, M.X. and Williamson, D.P. (1995) Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the ACM (JACM)*, 42(6), 1115–1145.

Gologlu, C. (2004) Machine capability and fixturing constraints-imposed automatic machining set-ups generation. *Journal of Materials Processing Technology*, 148(1), 83–92.

Grant, M., Boyd, S. and Ye, Y. (2008) Cvx: Matlab software for disciplined convex programming. Available at http://cvxr.com/cvx (accessed 18 August 2022).

Joneja, A. and Chang, T.-C. (1999) Setup and fixture planning in automated panel assembly. *ASME Transactions, Journal of Manufacturing Science and Engineering*, 121(3), 653–665.

Jones, R.M. (1998) *Mechanics of Composite Materials*, Chapman & Hall/CRC Press, Boca Raton, FL.

Kohnke, P. (2013) *Ansys Mechanical APDL. Theory Reference*, ANSYS Inc. Canonsburg, PA.

Kumar, A.S. and Nee, A. (1995) Framework for a variant fixture design system using case-based reasoning technique. *American Society of Mechanical Engineers, Manufacturing Engineering Division*, MED 2-1: pp. 763–775.

Li, B. and Melkote, S.N. (1999) Improved workpiece location accuracy through fixture layout optimization. *International Journal of Machine Tools and Manufacture*, 39(6), 871–883.

Megson, T.H.G. (2016) *Aircraft Structures for Engineering Students*, Butterworth-Heinemann, Oxford, United Kingdom.
Mou, S., Wang, A., Zhang, C. and Shi, J. (2021) Additive tensor decomposition considering structural data information. IEEE Transactions on Automation Science and Engineering (Early Access).
Nee, A. (1991) A framework for an object-rule-based automated fixture design system. CIRP Annals, 40(1), 147–151.
Sojoudi, S. and Lavaei, J. (2014) Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure. SIAM Journal on Optimization, 24(4), 1746–1778.
Vallapuzha, S., Edward, C., Choudhuri, S. and Khetan, R.P. (2002) An investigation into the use of spatial coordinates for the genetic algorithm based solution of the fixture layout optimization problem. International Journal of Machine Tools and Manufacture, 42(2), 265–275.
Wang, H. and Rong, Y.K. (2008) Case based reasoning method for computer aided welding fixture design. Computer-Aided Design, 40(12), 1121–1132.
Wang, H., Rong, Y.K., Li, H. and Shaun, P. (2010) Computer aided fixture design: Recent research and trends. Computer-Aided Design, 42(12), 1085–1094.
Yue, X., Wen, Y., Hunt, J.H. and Shi, J. (2018) Surrogate model-based control considering uncertainties for composite fuselage assembly. ASME Transactions, Journal of Manufacturing Science and Engineering, 140(4), 041017.
Zhang, D., Hu, Y., Ye, J., Li, X. and He, X. (2012) Matrix completion by truncated nuclear regularization, in Proceedings of the 2012 IEEE Conference on Computer Vision and Pattern Recognition, IEEE Press, Piscataway, NJ, pp. 2192–2199.
Zhang, H.-C. and Lin, E. (1999) A hybrid-graph approach for automated setup planning in Capp. Robotics and Computer-Integrated Manufacturing, 15(1), 89–100.
Zhong, Z., Mou, S., Hunt, J. and Shi, J. (2022) FEA model based cautious automatic optimal shape control for fuselage assembly. ASME Transactions, Journal of Manufacturing Science and Engineering, 144(8), 081009.

Appendix. Proof of Theorem 1

Proof: Since (5) is the convex relaxation of (4), the solution of (4) is also a feasible solution of (5). We will now prove that (5) implies (4):

(i) Since \( \text{rank}(S_l) = 1 \), \( S_l \) can be written as \( S_l = vv^T \), where \( v \in \mathbb{R}^{3N_{PT}} \). Constraint (5o) can be written as \( vv^T - F_{PT}^T(F_{PT})^T \geq 0 \).

First, we show that \( v \) is parallel to \( F_{PT} \), i.e., there exists a \( \rho \in \mathbb{R} \), such that \( v = \rho F_{PT} \).

Suppose that for all \( \rho \in \mathbb{R} \), \( v \neq \rho F_{PT} \), then there always exists a vector \( a \), such that \( a^T v = 0 \) and \( a^T F_{PT} \neq 0 \). Using the same \( a \), we have:

\[
a^T \left( vv^T - F_{PT}^T(F_{PT})^T \right) a = a^T vv^T a - a^T F_{PT}^T(F_{PT})^T a = - \left( a^T F_{PT} \right)^2
\]

which contradicts with \( vv^T - F_{PT}^T(F_{PT})^T \geq 0 \). Therefore, \( v = \rho F_{PT} \).

Then, we prove \( \rho^2 = 1 \). Substitute \( v = \rho F_{PT} \) into constraint (3h'), we have:

\[
F_{PT}^T(l) \sum_{j=1}^{6N_{PT}} A^*(i,j) F_{g}^*(j) + \rho^2 F_{PT}^T(l) \sum_{j=1}^{6N_{PT}} A^*(m(l),j) F_{g}^*(j) + \sum_{k \in \{1, \ldots, 3N_{PT} \}} A^*(m(l),k) F_{PT}^T(k) = 0.
\]

which is equivalent to

\[
(1 - \rho^2) F_{PT}^T(l) \sum_{j=1}^{6N_{PT}} A^*(i,j) F_{g}^*(j) + \rho^2 F_{PT}^T(l) \sum_{j=1}^{6N_{PT}} A^*(m(l),j) F_{g}^*(j) + \sum_{k \in \{1, \ldots, 3N_{PT} \}} A^*(m(l),k) F_{PT}^T(k) = 0.
\]

Since

\[
\sum_{j=1}^{6N_{PT}} A^*(i,j) F_{g}^*(j) + \sum_{k \in \{1, \ldots, 3N_{PT} \}} A^*(m(l),k) F_{PT}^T(k) = U^*(m(l)),
\]

we have

\[
(1 - \rho^2) F_{PT}^T(l) \sum_{j=1}^{6N_{PT}} A^*(i,j) F_{g}^*(j) + \rho^2 F_{PT}^T(l) U^*(m(l)) = 0.
\]

Since \( F_{PT}^T(l) U^*(m(l)) = 0 \), we have

\[
(1 - \rho^2) F_{PT}^T(l) \sum_{j=1}^{6N_{PT}} A^*(i,j) F_{g}^*(j) = \rho^2 F_{PT}^T(l) U^*(m(l)) = 0.
\]

According to the assumption that there is \( l \in \{1, \ldots, 3N_{PT} \} \) such that \( F_{PT}^T(l) \sum_{j=1}^{6N_{PT}} A^*(i,j) F_{g}^*(j) \neq 0 \), we conclude that \( 1 - \rho^2 = 0 \). Finally, the proof is complete. \( \square \)