Nonlinear Modeling and Simulation of Thermal Effects in Microcantilever Resonators Dynamic

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Abstract. Thermal dependency of material characteristics in micro electromechanical systems strongly affects their performance, design, and control. Hence, it is essential to understand and model that in MEMS devices to optimize their designs. A thermal phenomenon introduces two main effects: damping due to internal friction, and softening due to Young modulus-temperature relation. Based on some reported theoretical and experimental results, we model the thermal phenomena and use two Lorentzian functions to describe the restoring and damping forces caused by thermal phenomena. In order to emphasize the thermal effects, a nonlinear model of the MEMS, by considering capacitor nonlinearity, have been used. The response of the system is developed by employing multiple time scales perturbation method on nondimensionalized form of equations. Frequency response, resonant frequency and peak amplitude are examined for variation of dynamic parameters involved.

Keywords: MEMS dynamics; Thermoelastic; Microcantilever; Microresonators; Nonlinear Modeling; Thermal damping; Thermal Relaxation; Capacitor Sensors.

1. Introduction
Some of the sources of quality factor decreasing are considered extrinsic. However the intrinsic sources of energy loss play significant role in attainable quality factor. The purpose of this paper is to investigate the mathematical modeling of thermal effects in dynamic behavior and sensitivity analysis of microresonators. Temperature dependent properties of the microbeam material play a significant role in affecting the design and application of Microsystems utilizing a microbeam or microcantilever resonator [1-4]. The effects of thermal phenomena [5-8] are modeled as an increase in damping [9] and decrease in stiffness rates [9], both as Lorentzian function of excitation frequency. The steady state response frequency-amplitude dependency of system will be derived utilizing the multiple time scale perturbation method by considering the nonlinearity of the actuated force. The developed analytic equation describing the frequency response of the system around resonance can be utilized to explain the dynamic of the system, as well as resonant frequency and peak amplitude.
2. Modelling of Microresonators Considering Thermoelastic Effects

The thermoelastic effects will be investigated in two parts, the first one is the thermal damping which is the energy dissipation mechanism [9] and the other one is thermal relaxation which affects the rigidity of material [10].

Thermoelastic damping is proportional to frequency; hence, when the principal natural frequency moves up while the size of devices decreases, the thermoelastic damping becomes more significant. Thermal energy dissipation is caused by irreversible heat flow across the thickness of the microcantilever as it oscillates.

For simulating the damping force corresponding to thermal damping Jazar et al. [9] introduce a frequency dependent force

\[ f_{td} = c_T \cdot L \left( \frac{\omega^2}{\omega_0^2} \right) \dot{z} \],

where \( c_T \) defines the thermal damping per unit length of the microbeam which depends on geometric and material properties of the microbeam and must be determined experimentally, \( \omega_0 \) is the natural frequency, \( \omega \) is frequency of vibration, \( \dot{z} \) is velocity in system with one degree of freedom, and \( L \) is Lorentzian function as defined above.

Since the warming of the microbeam material is Lorentzian frequency dependent, the effect of stiffness softening of the microbeam is also a frequency dependent characteristic. So, we present a negative softening function to define this behavior. More specifically, a negative restoring force with stiffness as a Lorentzian function of excitation frequency [10]

\[ f_{ts} = -k_T \cdot \frac{\omega / \omega_0}{1 + (\omega / \omega_0)} \cdot w = -k_T \cdot L \left( \frac{\omega^2}{\omega_0^2} \right) \cdot w 
\]

determines the drop in linear rigidity stiffness force, \( EI \left( \frac{\partial^4 w}{\partial x^4} \right) \). The breaking frequency of the thermal stiffness softening is also at the fundamental resonance frequency. The softening stiffness coefficient per unit length, \( k_T \), which depends on geometric parameters and material properties of the microbeam must be determined experimentally.

The microresonator is composed of a beam resonator, a ground plane underneath in contact with the beam, and one (or more) capacitive transducer electrode(s). The one dimensional electrostatic force, \( f_e \), between two electrodes is

\[ f_e = \frac{\varepsilon_0 A (v - v_p)^2}{2(d - w)^2} 
\]

where, \( \varepsilon_0 = 8.85 \times 10^{-12} \text{ As/Vm} \) is permittivity in vacuum, \( A \) is the area of the microplate, \( w = w(x,t) \) is the lateral displacement of the microbeam, and The electric load composed of a DC polarization voltage, \( v_p \), and an AC actuating voltage, \( v = v_c \sin(\omega t) \) [11].

The equation describing lateral vibrations of the microbeam can be summarized and simplified to the following equation when the beam’s geometry is uniform.

Fig. 1. A microcantilever model of microresonators.
\[
\frac{\partial^2 y}{\partial t^2} + a_1 \frac{\partial y}{\partial t} + a_2 \frac{\partial^2 y}{\partial z^2} + a_6 \frac{r}{1 + r^2} \frac{\partial y}{\partial t} - a_7 \frac{r}{1 + r^2} y = \frac{\varepsilon_0 A (v - v_p)^2}{2(1 - y^2)}
\]

where the parameters are \(n\) is a constant depending on mode shape of the microbeam.

\[
\tau = \omega_L t, \quad \omega_L = \frac{n^2 E I}{L^2 \rho}, \quad z = \frac{x}{L}, \quad \frac{\partial y}{\partial t} = \omega_L \frac{y}{d}, \quad r = \frac{a_1}{a_2},
\]

\[
a_6 = \frac{\varepsilon_0 A L^4}{2n^2 d^3 E I}, \quad a_7 = \frac{c_L L^2}{n \sqrt{\rho E I}}, \quad a_8 = \frac{k_T L^4}{n^2 E I}
\]

We apply a separation solution \(y = Y(\tau, \phi(z))\) where the spatial function \(\phi(z) = \cos(\frac{\pi z}{2})\) is called mode shape function Then the required differential equation for the temporal function \(Y(\tau)\) related to a microcantilever, would be

\[
Y' + \left( h + a_6 \frac{r}{1 + r^2} \right) Y' + \left( 1 - a_7 \frac{r}{1 + r^2} \right) Y = \frac{1}{(1 - Y)^2} \left[ (\alpha + \beta) + 2\sqrt{2} \alpha \beta \sin(\tau) - \beta \cos(\tau) \right]
\]

where, \(h = a_2, \quad \alpha = a_1 v_p^2, \quad \sqrt{2} \alpha \beta = a_1 v_p v_i, \quad \beta = \frac{a_2}{2} v_i^2\).

The third order expansion (Taylor series) will be used to model the electrostatic force on microresonator.

Applying the multiple time scales method produces \(Y = a(\tau) \cos(r \tau + \gamma(\tau))\), and the following coupled equations yields

\[
a' = \frac{1}{2(1 + r^2)} \left( -\sqrt{2} (1 + r^2) \sqrt{\beta \alpha \cos(\sigma \tau - \gamma)} + a(-ra_6 - (1 + r^2)) \right)
\]

\[
Y' - \sigma = \frac{1}{2(1 + r^2)a} \left( -\sqrt{2} \beta \alpha \sin(\sigma \tau - \gamma) + a(-ra_7 + (1 + r^2)(-9\sqrt{2} \beta \alpha \sin(\sigma \tau - \gamma) + \right.
\]

\[
-\gamma a_6 - 2(\alpha + \beta)(1 + 6a_2) + \beta \cos(2\sigma \tau - 2\gamma)(1 + 8a_2^2)) \right) - \sigma
\]

where, \(\sigma = r - 1\).

Assuming \(a'\) and \(Y' - \sigma\) remain zero in steady state response. (\(Y' - \sigma\) is argument of sinusoidal term and must invariant in time when \(\tau \to \infty\).) Eliminating \(Y(\tau)\) and assuming \(0 \leq a \leq 1\) provide a relationship between the parameter of the system to have a periodic steady state response with frequency \(r\) give the equation that shows the \(a\) as a function of \(h, a_6, a_7, \beta, \alpha, r\).

From Nonlinear modeling the amplitude equation is:

\[
(4096a^6(1+9a^2+12a^4)(1+10a^2+24a^4)(1+r^2)(a^2+1)(a+15a^2+54a^4)-2\beta(1+4a^2)(1+8a^2)^2(1+r^2))
\]

\[
(-\alpha+128\alpha^4(3a+\beta)+2a^2(-5\alpha+2\beta+4a)\alpha^4(3\alpha+48\beta+64\alpha^2)+4a^2(1+8a^2)(a+1)^3+\beta^4((32a_h h r)(1+r^2))
\]

\[
(a+8a^2)^4+16a^2 r^2(a+8a^2)^4(1+r^2)^2(16h^2(a+8a^2)^4-\alpha(1+9a^2+12a^4)^2(-\alpha+12\beta a^4+2(4\beta-9a)\alpha^2+(-8l \alpha+128\beta \alpha)^4+1(1+r^2)(\alpha+128(3\alpha+\beta)\alpha^2+2(5\alpha+2\beta+4a)\alpha^2+3(3\alpha+48\beta+64\alpha^2)(4\alpha+1+8a^2)\alpha^2)^4+4a_h h r (1+r^2)^2)
\]

\[
((1+r^2)(\alpha+128(3\alpha+\beta)\alpha^2+2(5\alpha+2\beta+4a)\alpha^2+3(3\alpha+48\beta+64\alpha^2)a^4)+4a_h h r (1+r^2)^2) \]
\[(1+r^2)(-\alpha+128(3\alpha+\beta)x^2+2(-5\alpha+2\beta+4\sigma)x^3+(3\alpha+48\beta+64\sigma)x^4)4ra_\gamma(1+8x^2y^2)) = 0 \quad (8)\]

and from linear modeling the amplitude is:

\[
a = \frac{(1 + r^2)^2}{(1+r^2)^2(2r+\alpha-1)+\beta)\Gamma(2r+\alpha-1)+\beta)} + ra_\gamma(2h(1+r^2)) + 4r(1 + r^2)^2(\alpha+\beta-1)r_\gamma + r^2 a_\gamma^2 \quad (9).
\]

The accuracy of above equations depend on how much Amplitude is greater than zero.

3. Results

Eq.(8,9) describes the frequency behavior of the microresonator indicating that its dynamics governs by polarization voltage parameter \(\alpha\), alternative excitation voltage parameters \(\beta\), damping parameter \(h\), the excitation frequency ratio \(r\), as well as the thermal damping and stiffness parameters \(a_6\), and \(a_7\). The nominal values of a sample microcantilever to analyze the dynamic behavior of the MEMS are...
$$m = 1 \times 10^{-11} \; kg, \; \alpha = 0.0000553125 \nu^2, A = 200 \mu m \times 50 \mu m, \; \beta = 0.0002765625 \nu^2, \; k = 1 \; N/m, \; c = 1 \times 10^{-8} \; N s/m, \; d = 2.0 \; \mu m \; [9,10,12,13].$$

Fig. 1. depicts the effect of variation of voltage for a set of parameters. The amplitude of steady state oscillation increases by increasing the polarization voltage. No thermal effect is shown in Fig. 2(a). where \(a_6\) and \(a_7\) are set to zero. Thermal damping increases the damping of the system near resonance and hence diminishes the resonance amplitudes but has almost no effect on the off-resonance responses as shown in Fig. 2(b). Thermal relaxation appears by nonzero \(a_7\) which made the microbeam softer. Softening of the microbeam shifts the resonance frequency to lower values.

Increasing damping ratio diminishes the amplitude of the oscillation as expected. The picture shows the peak Amplitude is not increasing function of \(\beta\) and it increases by increasing \(\beta\) to somewhere then it reduces. (Because of that the intersection can be seen in the Fig. 3(b).) Also \(A_F\) is nonlinear increasing function of both polarization and excitation voltages.

As illustrated in Fig. 4(a),(b) The resonance frequency is monotonically decreasing function of increasing both polarization and excitation voltages. The behavior of resonance shifting looks linear with variation of both voltages. The behavior of resonant frequency is not linear non-monotonic when damping is varied.

**4. Conclusion**

We have modeled the thermal phenomena by considering the nonlinearity from actuated force in the nondimensionalized equation. The thermal phenomena have been translated to effective forces per unit length of the vibrating microbeam. Thermal properties of microbeam contribute into damping system due to warming and heat energy dissipation called “thermal damping”, and into restoring system due to material heat softening called “temperature relaxation”.

The achieved equation solved in primary resonance by utilizing the multiple time scales perturbation method and receives to two differential equations are able to give amplitude respect time response. The differential equations solved in steady state then plotting the frequency response of the system by varying the effective dynamic parameters and the effects of them were discussed.

Temperature relaxation reduces the peak amplitude little, while Thermal damping has a reduction effect on peak amplitude with a dominant effect. In addition, temperature relaxation shows a significant effect on shift of resonance frequency to lower values. Resonance shifting is a very important phenomenon especially in resonator-based sensors. It seems to be the most effective source of errors in resonant-sensors which are designed based on constant stiffness assumption.

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