Solutions of the Yamabe Equation by Lyapunov–Schmidt Reduction

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Abstract
Given any closed Riemannian manifold \((M, g)\) we use the Lyapunov–Schmidt finite-dimensionnal reduction method and the classical Morse and Lusternick–Schnirelmann theories to prove multiplicity results for positive solutions of a subcritical Yamabe type equation on \((M, g)\). If \((N, h)\) is a closed Riemannian manifold of constant positive scalar curvature we obtain multiplicity results for the Yamabe equation on the Riemannian product \((M \times N, g + \varepsilon^2 h)\), for \(\varepsilon > 0\) small. For example, if \(M\) is a closed Riemann surface of genus \(g\) and \((N, h) = (S^2, g_0)\) is the round 2-sphere, we prove that for \(\varepsilon > 0\) small enough and a generic metric \(g\) on \(M\), the Yamabe equation on \((M \times S^2, g + \varepsilon^2 g_0)\) has at least \(2 + 2g\) solutions.

Keywords Yamabe problem · Elliptic PDE on manifolds · Finite dimensional reduction

Mathematics Subject Classification 35J60 · 58J05 · 35B09

1 Introduction

In [29] Yamabe considered the following question: Let \((M, g)\) be a closed Riemannian manifold of dimension \(n \geq 3\). Is there a metric \(h\) which is conformal to \(g\) and has constant scalar curvature? If we express the conformal metric \(h\) as \(h = u^{\frac{4}{n-2}} g\) for a positive function \(u\), the scalar curvature \(s_h\) of \(h\) is related to the scalar curvature of \(g\).
by

\[-a_n \Delta_g u + s_g u = s_h u^{p_n-1},\]

where \(\Delta_g\) is the Laplacian operator associated with the metric \(g\), \(a_n = \frac{4(n-1)}{(n-2)}\) and \(p_n = \frac{2n}{n-2}\). It follows that the metric \(h\) has constant scalar curvature \(\lambda \in \mathbb{R}\) if and only if \(u\) is a positive solution of the Yamabe equation:

\[-a_n \Delta_g u + s_g u = \lambda u^{p_n-1}. \quad (1)\]

It is easy to check that Eq. (1) is the Euler–Lagrange equation of the Yamabe functional, \(Y_g\), defined by:

\[Y_g(u) = \frac{\int_M \left( a_n |\nabla u|^2 + s_g u^2 \right) d\mu_g}{\left( \int_M u^{p_n} d\mu_g \right)^{\frac{n-2}{n}}} = \frac{\int_M \left( a_n |\nabla u|^2 + s_g u^2 \right) d\mu_g}{\|u\|^2_{p_n}}. \quad (2)\]

If \(E\) denotes the normalized Hilbert–Einstein functional

\[E(g) = \frac{\int M s_g d\mu_g}{Vol(M, g)^{\frac{n-2}{n}}},\]

it follows that \(Y_g(u) = E(u^{\frac{4}{n-2}} g)\).

The Yamabe constant of \(g\) is defined as the infimum of the Yamabe functional \(Y_g:\)

\[Y(M, g) = \inf_{u \in H^1(M) - \{0\}} Y_g(u). \quad (3)\]

A minimizer for the Yamabe constant is therefore a solution of (1) and, moreover, from elliptic theory this must be strictly positive and smooth. Yamabe presented a proof that a minimizer always exists, but his argument contained an error which was pointed out (and fixed under certain conditions) by Trudinger in [27]. Later Aubin [2] and Schoen [25] completed the proof that for any metric \(g\) the infimum of the Yamabe functional is achieved. Therefore there is always at least one (positive) solution to the Yamabe equation (1). If \(Y(M, g) \leq 0\) the solution is unique (up to homothecies). In the case of \(Y(M, g) > 0\) uniqueness in general fails. The sphere \((S^n, g_o)\) with the curvature one metric is a first example of multiplicity of solutions.

The case of the sphere is very special because it has a non-compact family of conformal transformations which induces a non-compact family of solutions to the Yamabe equation. By a result of Obata [20] each metric of constant scalar curvature which is conformal to the round metric on \(S^n\) is obtained as the pull-back of the round metric under a conformal diffeomorphism. Therefore, if \(g_o\) is the round metric over \(S^n\), every solution to (1) is minimizing. But in general, for the positive case there will be
non-minimizing solutions. For instance, Pollack proved in [23] that every conformal class with positive Yamabe constant can be $C^0$-approximated by a conformal class with an arbitrary number of (non-isometric) metrics of constant scalar curvature which are not minimizers. Also, Brendle in [5] constructed high dimensional ($n \geq 52$) examples of Riemannian metrics with a non-compact family of non-minimizing solutions of the Yamabe equation (this was extended to dimension $n \geq 25$ in [6]. In dimension $n \leq 24$ the space of solutions is compact, except for the round sphere [16]).

Another important example was considered by Schoen in [26] (and also by Kobayashi in [14]). In [26] Schoen worked with the product metric on $S^{n-1} \times S^1(L)$ (the circle of radius $L$). He showed that all solutions to (1) are constant along the $(n-1)$-spheres and, therefore, the Yamabe equation reduces to an ordinary differential equation. By a careful analysis of this equation, Schoen proved that there are many non-minimizing solutions if $L$ is large.

Similar to the case of $S^{n-1} \times S^1(L)$, particular interest arises in the study of products of the form $(M \times N, g + \delta h)$, where the constant $\delta > 0$ goes to 0 (or $\infty$). The Yamabe constants of such Riemannian products were studied in [1]. Multiplicity results for the Yamabe equation were obtained in [4,7,8,12,13,21] using bifurcation theory, assuming that the scalar curvatures of $g$ and $h$ are constant.

In this paper we consider the case of Riemannian products were one of the scalar curvatures is not a constant. Let $(M^n, g)$ be any closed Riemannian manifold and $(N^m, h)$ be a Riemannian manifold of constant positive scalar curvature. The function $u : M \to \mathbb{R}_{>0}$ is a solution of the Yamabe equation in $(W, g_\varepsilon) = (M \times N, g + \varepsilon^2 h)$ if it satisfies

$$-a_{n+m} \Delta_g u + \left(s_g + \varepsilon^{-2} s_h\right) u = u^{p_{m+n}-1}.$$  

This is of course equivalent to finding solutions of the equation

$$-a_{n+m} \Delta_g u + \left(s_g + \varepsilon^{-2} s_h\right) u = \varepsilon^{-2} s_h u^{p_{m+n}-1}. \quad (4)$$

Moreover, we can normalize $h$ and assume that $s_h = a_{m+n}$. Then Eq. (4) is equivalent to:

$$-\varepsilon^2 \Delta_g u + \left(\frac{s_g}{a_{m+n}} \varepsilon^2 + 1\right) u = u^{p_{m+n}-1}. \quad (5)$$

We will find solutions of (5) using the Lyapunov–Schmidt reduction technique, which was introduced in [3,10,17], for instance. The same technique was also used by Micheletti and Pistoia in [18] to study the sub-critical equation $-\varepsilon^2 \Delta_g u + u = u^{p-1}$ on a Riemannian manifold. Here we will use a similar approach. We now give a brief description of this method and state the results we have obtained.

Let $H^1_g(M)$ be the Hilbert space $H^1_g(M)$ equipped with the inner product

$$\langle u, v \rangle_\varepsilon = \frac{1}{\varepsilon^n} \left(\varepsilon^2 \int_M \langle \nabla_g u, \nabla_g v \rangle \, d\mu_g + \int_M u \, v \, d\mu_g \right),$$

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and the induced norm
\[ \|u\|_\varepsilon^2 = \frac{1}{\varepsilon^n} \left( \varepsilon^2 \int_M |\nabla g u|^2 \, d\mu_g + \int_M u^2 \, d\mu_g \right). \]

Consider the functional \( J_\varepsilon : H_\varepsilon(M) \to \mathbb{R} \) given by
\[ J_\varepsilon(u) = \varepsilon^{-n} \int_M \left( \frac{1}{2} \varepsilon^2 |\nabla u|^2 + \frac{s_n \varepsilon^2 + a_{m+n}}{2a_{m+n}} u^2 - \frac{1}{p_{m+n}} (u^+)^{p_{m+n}} \right) \, d\mu_g. \]
where \( u^+ = \max\{u, 0\} \). The critical points of the functional \( J_\varepsilon \) are the positive solutions of Eq. (5). Let us consider the map
\[ S_\varepsilon = \nabla J_\varepsilon : H_\varepsilon \to H_\varepsilon. \]
The Yamabe equation (5) is then equivalent to \( S_\varepsilon(u) = 0 \).

Note that \( p_{m+n} < p_n \). From now on we let \( q \in (2, p_n) \). There exists a unique (up to translation) positive finite-energy solution \( U \) of the equation on \( \mathbb{R}^n \)
\[ -\Delta U + U = U^{q-1}. \] (6)
The function \( U \) is radial (around some fixed point). We also consider the linear equation
\[ -\Delta \psi + \psi = (q - 1)U^{q-2} \psi \quad \text{in } \mathbb{R}^n. \]
It is well known that all solutions of above equation are the directional derivatives of \( U \), i.e., the solutions are of the form
\[ \psi^v(z) = \frac{\partial U}{\partial v}(z), \quad v \in \mathbb{R}^n. \]
The function \( U_\varepsilon(x) = U((1/\varepsilon)x) \) is a solution of
\[ -\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{q-1}. \]
Similarly, we have that \( \psi^v_\varepsilon(x) = \psi^v((1/\varepsilon)x) \) solves
\[ -\varepsilon^2 \Delta \psi_\varepsilon + \psi_\varepsilon = (q - 1)U_\varepsilon^{q-2} \psi_\varepsilon. \]
For any \( r > 0 \) let \( \chi_r : \mathbb{R}^n \to [0, 1] \) be a smooth cut-off function such that \( \chi_r(z) = 1 \) if \( z \in B(0, r/2) \), \( \chi_r(z) = 0 \) if \( z \in \mathbb{R}^n \setminus B(0, r) \), \( |\nabla \chi_r(z)| \leq 2/r \) and \(|\nabla^2 \chi_r(z)| \leq 2/r^2 \). Then for any \( x \in M \) and \( r \) smaller than the injectivity radius, using the exponential map \( \exp_x : B(0, r) \to B_\varepsilon(x, r) \), we define
\[ U_{\varepsilon, x}(y) = \begin{cases} U_\varepsilon(\exp_x^{-1}(y)) \chi_r(\exp_x^{-1}(y)) & \text{if } y \in B_\varepsilon(x, r), \\ 0 & \text{otherwise}. \end{cases} \]
We regard $U_{\varepsilon, x}$ as an approximate solution of Eq. (5), and we will try to find an exact solution of the form $u \doteq U_{\varepsilon, x} + \phi$, where $\phi$ is a small perturbation. For that we consider the following subspace of $H_\varepsilon(M)$:

$$K_{\varepsilon, x} = \left\{ W^v_{\varepsilon, x} : v \in \mathbb{R}^n \right\},$$

where

$$W^v_{\varepsilon, x}(y) = \begin{cases} \psi^v_{\varepsilon} \exp^{-1}_x(y) \chi_r \exp^{-1}_x(y) & \text{if } y \in B_r(x, r), \\ 0 & \text{otherwise.} \end{cases}$$

$W^v_{\varepsilon, x}$ is an approximate solution of the linearized equation $S'_\varepsilon(U_{\varepsilon, x})(v) = 0$, and $K_{\varepsilon, x}$ an approximation to the kernel of $S'_\varepsilon(U_{\varepsilon, x})$.

We are going to solve equation (5) modulo $K_{\varepsilon, x}$ for $\phi$ in the orthogonal complement $K_{\varepsilon, x}^\perp$ of $K_{\varepsilon, x}$ in $H_\varepsilon$. In other words, for $\varepsilon > 0$ small and $x \in M$, we will find $\phi_{\varepsilon, x} \in K_{\varepsilon, x}^\perp$ such that

$$\Pi_{\varepsilon, x}^\perp \left\{ S_\varepsilon \left( U_{\varepsilon, x} + \phi_{\varepsilon, x} \right) \right\} = 0,$$

where $\Pi_{\varepsilon, x}^\perp : H_\varepsilon \to K_{\varepsilon, x}^\perp$ is the orthogonal projection. Hence, if for some $x_0 \in M$ we have

$$\Pi_{\varepsilon, x_0} \left\{ S_\varepsilon \left( U_{\varepsilon, x_0} + \phi_{\varepsilon, x_0} \right) \right\} = 0,$$

with $\Pi_{\varepsilon, x} : H_\varepsilon \to K_{\varepsilon, x}$ the orthogonal projection, then $U_{\varepsilon, x_0} + \phi_{\varepsilon, x_0}$ is a solution of Eq. (5). In this way, the problem is reduced to a problem in finite dimensions. This is called the Lyapunov–Schmidt finite-dimensional reduction.

The following theorem is the key result of this paper:

**Theorem 1** There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and for any $x \in M$ there exists a unique $\phi_{\varepsilon, x} \in K_{\varepsilon, x}^\perp$ such that

$$\Pi_{\varepsilon, x}^\perp \left\{ S_\varepsilon \left( U_{\varepsilon, x} + \phi_{\varepsilon, x} \right) \right\} = 0,$$

and $\|\phi_{\varepsilon, x}\|_\varepsilon = O(\varepsilon^2)$. The map $x \in M \mapsto J_\varepsilon(U_{\varepsilon, x} + \phi_{\varepsilon, x})$ is $C^2$, and if $x_0$ is a critical point of this map then $U_{\varepsilon, x_0} + \phi_{\varepsilon, x_0}$ is a positive solution of equation (5).

Let $F_\varepsilon(x) = J_\varepsilon(U_{\varepsilon, x} + \phi_{\varepsilon, x})$. The critical points of this $C^2$ function on $M$ give positive solutions of Eq. (5). This allows to apply the classical results about the number of critical points of functions on closed manifolds.

The most direct application comes from Lusternik-Schnirelmann Theory. Recall that the Lusternik-Schnirelmann category of $M$, $\text{Cat}(M)$, is the minimal integer $k$ such that $M$ can be covered by $k$ subsets, $M \subset M_1 \cup M_2 \ldots \cup M_k$, with $M_i$ closed and contractible in $M$. The classical result of Lusternick–Schnirelmann theory says that any $C^1$ function on a closed manifold $M$ has at least $\text{Cat}(M)$ critical points.

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Therefore, from Theorem 1 (and the discussion above) we can deduce the following result, which was proved by J. Petean in [22] with a different approach:

**Theorem 2** Let \((M,g)\) be any closed Riemannian manifold and \((N,h)\) be a Riemannian manifold of constant positive scalar curvature. There exist \(\varepsilon_0 > 0\) such that for \(0 < \varepsilon < \varepsilon_0\) the Yamabe equation on the Riemannian product \((M \times N, g + \varepsilon^2 h)\) has at least \(\text{Cat}(M)\) solutions which depend only on \(M\).

In [22] J. Petean proves the existence of \(\text{Cat}(M)\) low energy solutions and one higher energy solution. The solutions provided in our theorem have low energy and they are close to the explicit approximate solutions. We also mention that C. Rey and M. Ruiz [24] also applied the Lyapunov–Schmidt reduction technique to construct multipeak high-energy solutions under certain conditions. These seem to be the only known results when the scalar curvature of \(g\) is not a constant.

Further applications can be obtained using Morse Theory. For that we have to consider the expansion of \(F_\varepsilon\) in terms of \(\varepsilon\), to study when \(F_\varepsilon\) is a Morse function. Similar expansions were considered when studying solutions of the equation \(-\varepsilon^2 \Delta_g u + u = u^{p-1}\) on a Riemannian manifold by Micheletti and Pistoia for instance in [18]. Positive solutions of this equation are the critical points of the functional

\[
J_\varepsilon^0(u) = \varepsilon^{-n} \int_M \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p_{m+n}} (u^+)^{p_{m+n}} \right) \, d\mu_g.
\]

Then Micheletti and Pistoia perform the Lyapunov–Schmidt reduction and define the map \(F_\varepsilon^0(x) = J_\varepsilon^0(U_{\varepsilon,x} + \phi_{\varepsilon,x})\) and prove in [18, Lemma 5.1] that we have the following \(C^1\)-uniformly expansion:

\[
F_\varepsilon^0(x) = \alpha - \frac{\varepsilon^2}{6} s_g(x) \int_{\mathbb{R}^n} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^4 \, dz + o(\varepsilon^2),
\]

where \(U = U_{p_{m+n}}\) is the solution of equation (6) with \(q = p_{m+n}\), \(U'\) means the derivative of \(U\) in the radial direction, and \(\alpha = \frac{1}{2} \|U\|_{H^1(\mathbb{R}^n)}^2 - \frac{1}{p} \|U\|_{L^p(\mathbb{R}^n)}^p\).

There is an extra factor in the functional \(J_\varepsilon\) involving \(s_g \varepsilon^2\), which has an effect in the expansion of the function \(F_\varepsilon\). This was considered by C. Rey and M. Ruiz in [24, Lemma 3.3]. They obtain:

\[
F_\varepsilon(x) = \alpha - \frac{\varepsilon^2}{6} s_g(x) \int_{\mathbb{R}^n} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^4 \, dz + \frac{\varepsilon^2}{2} a_{m+n} s_g(x) \int_{\mathbb{R}^n} U(z)^2 \, dz + o(\varepsilon^2)
\]

\[
= \alpha + \beta_{m,n} \frac{\varepsilon^2}{2} s_g(x) + o(\varepsilon^2)
\]

which is \(C^1\)-uniformly with respect to \(x\) when \(\varepsilon\) tends to zero, where

\[
\beta_{m,n} = \frac{1}{a_{m+n}} \int_{\mathbb{R}^n} U(z)^2 \, dz - \frac{1}{3} \int_{\mathbb{R}^n} \left( \frac{U'(|z|)}{|z|} \right)^2 z_1^4 \, dz.
\]
In [24] it is also proved that
\[ \beta_{m,n} = \frac{1}{a_{m+n}} \int_{\mathbb{R}^n} U^2(z) dz - \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U(z)|^2 |z|^2 dz, \]
and that numerical computations show that \( \beta_{m,n} \neq 0 \) if \( m + n \leq 9 \). It is difficult to prove analytically that \( \beta_{m,n} \neq 0 \) in general but in Sect. 6 we will prove it in the case \( m = n = 2 \). Assuming that \( \beta_{m,n} \neq 0 \) and that \( x_0 \) is a nondegenerate critical point of \( s_g \) it is easy to prove, using the previous expansion, that for any \( \delta > 0 \), if \( \varepsilon \) is small enough, then \( F_\varepsilon \) has a critical point in \( B(x_0, \delta) \). It was proved by Micheletti and Pistoia in [19] that for a generic metric (on any closed manifold) all the critical points of its scalar curvature are nondegenerate, i.e. the scalar curvature is a Morse function on the manifold. We can then apply Morse theory. Let \( b_1(M) = \dim(H_1(M, \mathbb{R})) \) and \( b(M) = b_1(M) + \cdots + b_n(M) \). If \( f \) is a Morse function on \( M \) then \( f \) has at least \( b(M) \) critical points. Therefore we obtain:

**Theorem 3** Let \((N, h)\) be a closed Riemannian manifold of dimension \( m \) of constant positive scalar curvature. Let \( M \) be a closed manifold of dimension \( n \). Assume that \( \beta_{m,n} \neq 0 \). For a generic Riemannian metric \( g \) on \( M \) there exist \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \) the Yamabe equation on the Riemannian product \((M \times N, g + \varepsilon^2 h)\) has at least \( b(M) \) positive solutions.

Using that \( \beta_{2,2} \neq 0 \) we have:

**Theorem 4** Let \( g_0 \) be the round metric on the sphere \( S^2 \). Let \( M \) be a closed manifold of dimension 2. For a generic Riemannian metric \( g \) on \( M \) there exist \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \) the Yamabe equation on the Riemannian product \((M \times S^2, g + \varepsilon^2 g_0)\) has at least \( b(M) \) positive solutions.

In case the scalar curvature of \( g \) is constant the expansion of \( F_\varepsilon \) up to order \( \varepsilon^2 \) is constant and to obtain critical points one would need to consider higher order expansions. For the equation \(-\varepsilon^2 \Delta_g u + u = u^{q-1}\) such an expansion was considered for instance by S. Deng, Z. Khemiri and F. Mahmoudi in [9].

In Sects. 2 and 3 we will discuss some preliminary results about the Lyapunov–Schmidt reduction technique and prove some delicate estimates involving the approximate solutions. In Sect. 4 we prove the existence of the appropriate perturbation functions \( \phi_{x, \varepsilon} \), see Proposition 3. In Sect. 5 we complete the proof of Theorem 1. Finally in Sect. 6 we will prove that \( \beta_{2,2} \neq 0 \).

**2 Preliminaries**

**2.1 The Limiting Equation and Its Solution on \( \mathbb{R}^n \)**

Let \( 2 < q < p_n \) (where if \( n = 2 \) then \( p_n = \infty \)). It is well known that there exists a unique (up to translation) positive finite-energy solution \( U \) of the equation
\[ -\Delta U + U = U^{q-1}, \quad \text{in} \ \mathbb{R}^n. \]
The function $U$ is radial (around some chosen point) and it is exponentially decreasing at infinity (see for instance [11]):

$$|U(x)| \leq Ce^{-c|x|} \quad \text{and} \quad |\nabla U(x)| \leq Ce^{-c|x|}.$$  

Consider the functional $E : H^1(\mathbb{R}^n) \rightarrow \mathbb{R}$,

$$E(f) = \int_{\mathbb{R}^n} \left( \frac{1}{2} |\nabla f|^2 + \frac{1}{2} f^2 - \frac{1}{q} (f^+)^q \right) dx,$$

where $f^+(x) := \max\{f(x), 0\}$. Note that $U$ is a critical point of $E$.

For any $\varepsilon > 0$ let

$$E_\varepsilon(f) = \varepsilon^{-n} \int_{\mathbb{R}^n} \left( \frac{\varepsilon^2}{2} |\nabla f|^2 + \frac{1}{2} f^2 - \frac{1}{q} (f^+)^q \right) dx.$$  

The function $U_\varepsilon(x) = U(\frac{x}{\varepsilon})$ is a critical point of $E_\varepsilon$, i.e. a solution of

$$-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{q-1}. \quad (7)$$

Now, let us consider the linear equation

$$-\Delta \psi + \psi = (q - 1)U^{q-2}\psi \quad \text{in} \ \mathbb{R}^n. \quad (8)$$

It is well known that all solutions of Eq. (8) are the directional derivatives of $U$, i.e. the solutions are of the form

$$\psi^v(z) = \frac{\partial U}{\partial v}(z), \ v \in \mathbb{R}^n.$$  

In particular, set $\psi^i \doteq \psi^{e_i}$. Since $U$ is radial, we have that the set $\{\psi^1, \ldots, \psi^n\}$ is orthogonal in $H^1(\mathbb{R}^n)$, i.e.

$$\int_{\mathbb{R}^n} \left\{ \langle \nabla \psi^i, \nabla \psi^j \rangle + \psi^i(z)\psi^j(z) \right\} dz = 0, \quad \text{for} \ i \neq j. \quad (9)$$

For more details see for instance [11,15,28].

2.2 The Setting on a Riemannian Manifold

Let $H_\varepsilon$ be the Hilbert space $H^1_g(M)$ equipped with the inner product

$$\langle u, v \rangle_\varepsilon \doteq \frac{1}{\varepsilon^n} \left( \varepsilon^2 \int_M (\nabla_g u, \nabla_g v) d\mu_g + \int_M uvd\mu_g \right),$$
and the induced norm
\[ \|u\|_{2, \varepsilon} = \frac{1}{\varepsilon^n} \left( \varepsilon^2 \int_M |\nabla u|^2 d\mu_g + \int_M u^2 d\mu_g \right). \]

Let \( L^q_{\varepsilon} \) be the Banach space \( L^q_g(M) \) with the norm
\[ |u|_{q, \varepsilon} = \left( \frac{1}{\varepsilon^n} \int_M |u|^q d\mu_g \right)^{1/q}, \quad u \in L^q_g(M). \]

The standard norm in \( L^q_g(M) \) will be denoted from now on by
\[ |u|_q = \left( \int_M |u|^q d\mu_g \right)^{1/q}, \quad u \in L^q_g(M). \]

**Remark 1** For \( u \in H^1(\mathbb{R}^n) \) we let \( u_\varepsilon(x) = u(\varepsilon^{-1} x) \). For any \( \varepsilon > 0 \) we have
\[ \|u_\varepsilon\|_\varepsilon = \|u\|_{H^1} \quad (10) \]
and
\[ |u_\varepsilon|_{q, \varepsilon} = |u|_q. \quad (11) \]

**Remark 2** For \( q \in (2, p_n) \) if \( n \geq 3 \) or \( q > 2 \) if \( n = 2 \), the embedding \( i_\varepsilon : H_\varepsilon \hookrightarrow L^q_\varepsilon \) is a continuous map. Moreover, one can easily check that there exists a constant \( c \) independent of \( \varepsilon \) such that
\[ |i_\varepsilon(u)|_{q, \varepsilon} \leq c\|u\|_\varepsilon, \quad \text{for any } u \in H_\varepsilon. \]

Let \( q' = \frac{q}{q-1} \), so that \( \frac{1}{q} + \frac{1}{q'} = 1 \). Then, there exists a continuous operator \( i^{\ast}_\varepsilon : L^{q'}_\varepsilon \rightarrow H_\varepsilon \), called the adjoint of \( i_\varepsilon \), such that
\[ \langle i^{\ast}_\varepsilon(v), \varphi \rangle_\varepsilon = \langle v, i_\varepsilon(\varphi) \rangle_\varepsilon = \frac{1}{\varepsilon^n} \int_M v \cdot i_\varepsilon (\varphi), \quad \forall v \in L^{q'}_\varepsilon \text{ and } \forall \varphi \in H_\varepsilon. \]

In order to see this, we notice that for \( v \in L^{q'}_\varepsilon \), the map \( \mathfrak{F}_v : H_\varepsilon \rightarrow \mathbb{R} \), given by
\[ \mathfrak{F}_v(\varphi) = \langle v, i_\varepsilon(\varphi) \rangle, \quad \varphi \in H_\varepsilon, \]
is a continuous functional by the compact embedding \( i_\varepsilon : H_\varepsilon \hookrightarrow L^q_\varepsilon \). By the Riesz representation theorem, there exists \( u_v \in H_\varepsilon \) such that
\[ \mathfrak{F}_v(\varphi) = \langle v, \varphi \rangle_\varepsilon, \quad \forall \varphi \in H_\varepsilon. \quad (12) \]
Therefore, \( i^*_\varepsilon(v) = u_\varepsilon \). Finally, observe that
\[
\|i^*_\varepsilon(v)\|_\varepsilon \leq c|v|_{q',\varepsilon}, \quad \text{for any } v \in L^q_{\varepsilon},
\]
where the constant \( c > 0 \) does not depend on \( \varepsilon > 0 \).

Recall that if \( v \in L^{q'}_{\varepsilon} \), then a function \( u \) is a solution of
\[
\begin{aligned}
-\varepsilon^2 \Delta_g u + u &= v \quad \text{in } M, \\
\end{aligned}
\]
if and only if \( u \in H^1_g(M) \), and it satisfies
\[
\frac{1}{\varepsilon^n} \left( \varepsilon^2 \int_M (\nabla_g u, \nabla_g \varphi) d\mu_g + \int_M u \varphi \, d\mu_g \right) = \frac{1}{\varepsilon^n} \int_M v \cdot i^*_\varepsilon(\varphi) \, d\mu_g, \quad \forall \varphi \in H_\varepsilon.
\]
If we define \( u = i^*_\varepsilon(v) \), with \( v \in L^{q'}_{\varepsilon} \), then \( u \) is a solution of (14). This implies that if \( v \in C^{k,\alpha}(M) \) then \( u \in C^{k+2,\alpha}(M) \).

Now, let \( u \in H_\varepsilon \), then
\[
\frac{1}{\varepsilon^n} \int_M |(u^+)^{q-1}|^{q'} \, d\mu_g \leq \frac{1}{\varepsilon^n} \int_M |u|^q \, d\mu_g = |u|^{q}_{\varepsilon},
\]
Moreover, by Jensen’s inequality
\[
\left| \frac{s_g(x)}{a_{m+n}} \varepsilon^2 u \right|_{q',\varepsilon} \leq c_\alpha \varepsilon^{2+\frac{n}{q'} - \frac{m}{q}} |u|_{q,\varepsilon},
\]
where \( c_\alpha > 0 \) depends only on \( M \). It is easy to see that
\[
2 + \frac{n}{q} - \frac{n}{q'} > 0, \quad \text{since } 2 < q < \frac{2n}{n-2}.
\]

Now, we set \( q = p_{m+n} \). It follows that if \( u \in H_\varepsilon \), then
\[
F(u) = (u^+(x))^{p_{m+n}-1} - \frac{s_g(x)}{a_{m+n}} \varepsilon^2 u(x) \in L^{p_{m+n}}_{\varepsilon}.
\]
We define the operator \( S_\varepsilon : H_\varepsilon \rightarrow H_\varepsilon \) by
\[
S_\varepsilon(u) = u - i^*_\varepsilon(F(u)).
\]

By the Remark 2, \( S_\varepsilon(u) = \nabla J_\varepsilon(u) \), where, as in the Introduction,
\[
J_\varepsilon(u) = \varepsilon^{-n} \int_M \left( \frac{1}{2} \varepsilon^2 |\nabla u|^2 + \frac{s_g \varepsilon^2 + a_{m+n}}{2a_{m+n}} u^2 - \frac{1}{p_{m+n}} (u^+)^{p_{m+n}} \right) d\mu_g.
\]
In particular, $S_\varepsilon(u) = 0$ if and only if $u$ is a critical point of the functional $J_\varepsilon$.

Note also that

$$S'_\varepsilon(u)\varphi = \varphi - i_\varepsilon^* \left( (p_{m+n} - 1)(u^+)^{p_{m+n}-2} \varphi - \frac{S_g(x)}{a_{m+n}} \varepsilon^2 \varphi \right), \quad \varphi \in H_\varepsilon(M). \tag{17}$$

### 3 Approximate Solutions

Let $U$ be the solution of Eq. (6) where $q = p_{m+n}$. For simplicity we will use $p$ to denote $p_{n+m}$. Let

$$U_{\varepsilon,x}(y) = \begin{cases} U_\varepsilon(\exp_{x}^{-1}(y))\chi_{r}(\exp_{x}^{-1}(y)), & \text{if } y \in B_{g}(x, r), \\ 0, & \text{otherwise} \end{cases}. \tag{18}$$

Since $U_\varepsilon$ solves (7), we consider $U_{\varepsilon,x}$ as an approximate solution of Eq. (5). In this section we will prove some estimates related to $U_{\varepsilon,x}$. Similar estimates have been obtained before, see for instance in [18]. We sketch the proofs of the estimates for completeness and to point out the necessary adjustments to handle the extra term $\frac{s_{\varepsilon}^2}{a_{m+n}}u$ in Eq. (5).

The function $U_{\varepsilon,x}$ is an approximate solution in the following sense.

**Lemma 1** There exists an $\varepsilon_0 > 0$ and $C > 0$ such that for every $x \in M$ and every $\varepsilon \in (0, \varepsilon_0)$ we have

$$\|S_\varepsilon(U_{\varepsilon,x})\|_E \leq C \varepsilon^2.$$

**Proof** Observe

$$\|S_\varepsilon(U_{\varepsilon,x})\|_E = \sup_{\|v\|_x = 1} \langle S_\varepsilon(U_{\varepsilon,x}), v \rangle_E.$$

Now

$$\langle S_\varepsilon(U_{\varepsilon,x}), v \rangle_E = \frac{1}{\varepsilon^n} \int_M \left[ \varepsilon^2 \langle \nabla U_{\varepsilon,x}, \nabla v \rangle + \left( 1 + \frac{S_g \varepsilon^2}{a_{m+n}} \right) U_{\varepsilon,x}v - U_{\varepsilon,x}^{p-1}v \right] d\mu_g$$

$$= \frac{1}{\varepsilon^n} \int_M \left( -\varepsilon^2 \Delta U_{\varepsilon,x} + U_{\varepsilon,x} - U_{\varepsilon,x}^{p-1} \right) v d\mu_g + \frac{1}{\varepsilon^n} \int_M \frac{S_g \varepsilon^2}{a_{m+n}} U_{\varepsilon,x}v d\mu_g.$$

On one hand

$$\frac{\varepsilon^2}{\varepsilon^n} \int_M \frac{S_g}{a_{m+n}} U_{\varepsilon,x}v d\mu_g \leq C_1 \frac{\varepsilon^2}{\varepsilon^n} \int_M |U_{\varepsilon,x}v| d\mu_g$$

$$= C_1 \varepsilon^2 |U_{\varepsilon,x}|_{p',\varepsilon} |v|_{p,\varepsilon} \leq C_2 \varepsilon^2 |U_{\varepsilon,x}|_{p',\varepsilon}, \tag{19}$$
using Hölder’s inequality and Remark 2. It follows from the exponential decay of $U$ and change of variables, as in Remark 1, that $\lim_{\varepsilon \to 0} |U_{\varepsilon, x}|_{p', \varepsilon} = |U|_{p'} < \infty$. Therefore there exists $C > 0$ such that

$$\left| \frac{1}{\varepsilon^n} \int_M s_g \varepsilon^2 U_{\varepsilon, x} \, v \, d\mu_g \right| \leq C \varepsilon^2.$$ 

On the other hand, we have by the embedding that

$$\left| \frac{1}{\varepsilon^n} \int_M \left( - \varepsilon^2 \Delta U_{\varepsilon, x} + U_{\varepsilon, x} - U_{\varepsilon, x}^{p-1}\right) v \, d\mu_g \right| \leq | - \varepsilon^2 \Delta U_{\varepsilon, x} + U_{\varepsilon, x} - U_{\varepsilon, x}^{p-1}|_{p', \varepsilon} v_{p, \varepsilon}$$

$$\leq c | - \varepsilon^2 \Delta U_{\varepsilon, x} + U_{\varepsilon, x} - U_{\varepsilon, x}^{p-1}|_{p', \varepsilon}.$$ 

From the proof of Lemma 3.3 in [18], we have that there is positive constant $C$ and $\varepsilon_o > 0$ such that for all $x \in M$ and $\varepsilon \in (0, \varepsilon_o)$ it holds,

$$\left| - \varepsilon^2 \Delta U_{\varepsilon, x} + U_{\varepsilon, x} - U_{\varepsilon, x}^{p-1}\right|_{p', \varepsilon} \leq C \varepsilon^2. \quad (20)$$

This completes the proof of the lemma. \qed

We consider now the kernel of the linearized equation at the approximate solution, $\{ v \in H^1(M) : S'_\varepsilon(U_{\varepsilon, x})(v) = 0 \}$. In order to have information about the kernel we consider $\varepsilon > 0$, $x \in M$, and pick an orthonormal basis of $T_x M$ to identified it with $\mathbb{R}^n$. Using normal coordinates we define the following subspace of $H^1(M)$:

$$K_{\varepsilon, x} = \{ W^v_{\varepsilon, x} : v \in \mathbb{R}^n \},$$

where

$$W^v_{\varepsilon, x}(y) = \begin{cases} \psi^v_{\varepsilon}(\exp_x^{-1}(y)) \chi_r(\exp_x^{-1}(y)) & \text{if } y \in B_g(x, r), \\ 0 & \text{otherwise}, \end{cases} \quad (21)$$

with $\psi_{\varepsilon}(z) = \psi(z) - \varepsilon\bar{z}$ (as in the Introduction). Note that $W^v_{\varepsilon, x}$ depends on the choice of the orthonormal basis but the space itself $K_{\varepsilon, x}$ does not. We will also set $W^i_{\varepsilon, x} \doteq W^e_{\varepsilon, x}$. It is easy to see from (9) and Remark 1 that

$$\langle W^i_{\varepsilon, x}, W^j_{\varepsilon, x} \rangle_\varepsilon \to C, \quad \langle W^i_{\varepsilon, x}, W^j_{\varepsilon, x} \rangle_\varepsilon \to 0 \quad \text{if } i \neq j, \quad \text{as } \varepsilon \to 0, \quad (22)$$

where the constant $C = \int_{\mathbb{R}^n} \langle \nabla \psi^i, \nabla \psi^j \rangle + \psi^j \psi^j \rangle dx > 0$ is independent of $i \in \{ 1, \ldots, n \}$ and $x \in M$.

One can also show the following (details can be found in Lemma 6.1 and Lemma 6.2 in [18]).
Proposition 1 We have that
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \left\| \frac{\partial}{\partial \varepsilon} W^v_{\varepsilon, x} \right\|_{\varepsilon} = 0,
\]
and
\[
\lim_{\varepsilon \to 0} \varepsilon \left( \frac{\partial}{\partial \varepsilon} (U^v_{\varepsilon, x}, W^v_{\varepsilon, x}) \right)_{\varepsilon} = \langle \psi^v, \psi^v \rangle_{H^1} > 0.
\]

The function $W^v_{\varepsilon, x}$ is an approximate solution of the linearized equation in the following sense.

Lemma 2 For any $v \in \mathbb{R}^n$ there exists an $\varepsilon_0 > 0$ and $C > 0$ such that for every $x \in M$ and all $\varepsilon \in (0, \varepsilon_0)$ we have
\[
\| S'_{\varepsilon} (U^v_{\varepsilon, x}, W^v_{\varepsilon, x}) \|_{\varepsilon} \leq C \varepsilon^2 \| v \|.
\]

Proof It is enough to consider the case $v = e_i$. We have
\[
\| S'_{\varepsilon} (U^v_{\varepsilon, x}, W^v_{\varepsilon, x}) \|_{\varepsilon} = \sup_{\| w \|_{\varepsilon} = 1} \langle S'_{\varepsilon} (U^v_{\varepsilon, x}, W^v_{\varepsilon, x}), w \rangle_{\varepsilon}.
\]
Now, we have that
\[
\langle S'_{\varepsilon} (U^v_{\varepsilon, x}, W^v_{\varepsilon, x}), w \rangle_{\varepsilon} = \frac{1}{\varepsilon^n} \int_M \left[ \varepsilon^2 (\nabla W^v_{\varepsilon, x}, \nabla w) + \left( 1 + \frac{s_g \varepsilon^2}{a_{m+n}} \right) W^v_{\varepsilon, x} w ight.
\]
\[
- (p - 1) (U^v_{\varepsilon, x})^{p-2} W^v_{\varepsilon, x} w \] \[d \mu_g \]
\[
= \frac{1}{\varepsilon^n} \int_M \left[ - \varepsilon^2 \Delta W^v_{\varepsilon, x} + W^v_{\varepsilon, x} - (p - 1) (U^v_{\varepsilon, x})^{p-2} W^v_{\varepsilon, x} \right] w \] \[d \mu_g \]
\[
+ \frac{1}{\varepsilon^n} \int_M \frac{s_g \varepsilon^2}{a_{m+n}} W^v_{\varepsilon, x} w \] \[d \mu_g .
\]
Observe that
\[
\frac{\varepsilon^2}{\varepsilon^n} \int_M \frac{s_g}{a_{m+n}} W^v_{\varepsilon, x} w d \mu_g \leq C \frac{\varepsilon^2}{\varepsilon^n} \int_M |W^v_{\varepsilon, x} w| d \mu_g \]
\[
\leq C \varepsilon^2 |W^v_{\varepsilon, x}|_{p', \varepsilon} |w|_{p, \varepsilon} \leq C \varepsilon^2 |W^v_{\varepsilon, x}|_{p', \varepsilon},
\]
by a similar argument as in (19).

It follows from the exponential decay of $\psi^i$ and change of variables that $\lim_{\varepsilon \to 0} |W^v_{\varepsilon, x}|_{p', \varepsilon} = |\psi^i|_{p'}$. We conclude that
\[
\frac{\varepsilon^2}{\varepsilon^n} \int_M \frac{s_g}{a_{m+n}} W^v_{\varepsilon, x} w d \mu_g \leq C \varepsilon^2 .
\]

Moreover, by Remark 2 we have

\[
\left| \frac{1}{\varepsilon^n} \int_M \left( -\varepsilon^2 \Delta W_{\varepsilon,x}^i + W_{\varepsilon,x}^i - (p - 1)(U_{\varepsilon,x})^{p-2} W_{\varepsilon,x}^i \right) w \, d\mu_g \right|
\]

\[
= \left| -\varepsilon^2 \Delta W_{\varepsilon,x}^i + W_{\varepsilon,x}^i - (p - 1)(U_{\varepsilon,x})^{p-2} W_{\varepsilon,x}^i \right|_{p',\varepsilon} \| w \|_{p,\varepsilon}
\]

\[
\leq c \left| -\varepsilon^2 \Delta W_{\varepsilon,x}^i + W_{\varepsilon,x}^i - (p - 1)(U_{\varepsilon,x})^{p-2} W_{\varepsilon,x}^i \right|_{p',\varepsilon} \| w \|_{p,\varepsilon}.
\]

It is shown in Lemma 5.2 of \cite{18} that

\[
\left| -\varepsilon^2 \Delta W_{\varepsilon,x}^i + W_{\varepsilon,x}^i - (p - 1)(U_{\varepsilon,x})^{p-2} W_{\varepsilon,x}^i \right|_{p',\varepsilon} \leq C \varepsilon^2,
\]

Estimate (26) together with (25) finishes the proof of the lemma. \qed

We now solve \( S_{\varepsilon}(u) = 0 \) modulo \( K_{\varepsilon,x} \). We consider the orthogonal complement \( K_{\varepsilon,x}^\perp \) of \( K_{\varepsilon,x} \) in \( H_{\varepsilon} \) and we find \( \phi_{\varepsilon,x} \in K_{\varepsilon,x}^\perp \) such that

\[
\Pi_{\varepsilon,x} \left\{ S_{\varepsilon} \left( U_{\varepsilon,x} + \phi_{\varepsilon,x} \right) \right\} = 0,
\]

where \( \Pi_{\varepsilon,x} : H_{\varepsilon} \rightarrow K_{\varepsilon,x}^\perp \) is the orthogonal projection. In the next section we will show that there exists \( \varepsilon_o = \varepsilon_o(M) > 0 \), such that for every \( x \in M \) and \( \varepsilon \in (0, \varepsilon_o) \), there is a unique \( \phi_{\varepsilon,x} \in K_{\varepsilon,x}^\perp \) that solves Eq. (27). It will remain then to find points \( x \in M \) for which

\[
\Pi_{\varepsilon,x} \left\{ S_{\varepsilon} \left( U_{\varepsilon,x} + \phi_{\varepsilon,x} \right) \right\} = 0,
\]

where \( \Pi_{\varepsilon,x} : H_{\varepsilon} \rightarrow K_{\varepsilon,x} \) is the orthogonal projection.

4 The Finite-Dimensional Reduction

This section is devoted to solve Eq. (27). For \( x \in M \) and \( \varepsilon > 0 \) we consider the linear operator \( L_{\varepsilon,x} : K_{\varepsilon,x}^\perp \rightarrow K_{\varepsilon,x}^\perp \) defined by

\[
L_{\varepsilon,x}(\phi) = \Pi_{\varepsilon,x} \left\{ S'(U_{\varepsilon,x})\phi \right\},
\]

where by (17)

\[
S'(U_{\varepsilon,x})\phi = \phi - i^*_{\varepsilon} \left[ (p - 1)(U_{\varepsilon,x})^{p+m-n-2} \phi - \varepsilon^2 \frac{s_{\varepsilon}}{a_{m+n}} \phi \right].
\]
In the following proposition we show that the bounded operator \( L_{\varepsilon,x} \) satisfies a coercivity estimate for \( \varepsilon > 0 \) small enough, uniformly on \( M \). From this result it follows the invertibility of \( L_{\varepsilon,x} \) for \( \varepsilon > 0 \) small.

**Proposition 2** There exists \( \varepsilon_0 > 0 \) and \( c > 0 \) such that for any point \( x \in M \) and for any \( \varepsilon \in (0, \varepsilon_0) \)

\[
\| L_{\varepsilon,x}(\phi) \|_\varepsilon \geq c \| \phi \|_\varepsilon \quad \text{for all } \phi \in K_{\varepsilon,x}^\perp.
\]

**Proof** Assume the proposition is not true. Then there exists a sequence of positive numbers \( \varepsilon_i \), with \( \lim_{i \to \infty} \varepsilon_i = 0 \), and sequences \( \{x_i\} \subset M \), \( \{\phi_i\} \subset K_{\varepsilon_i,x_i}^\perp \) with \( \| \phi_i \|_{\varepsilon_i} = 1 \), such that \( \| L_{\varepsilon_i,x_i}(\phi_i) \|_{\varepsilon_i} \to 0 \). Moreover, since \( M \) is compact we can assume that there exists \( x \in M \) such that \( x_i \to x \).

\[\Box\]

**Claim** Let \( \omega_i := L_{\varepsilon_i,x_i}(\phi_i) \) and set

\[
\xi_i := S_{\varepsilon_i}(U_{\varepsilon_i,x_i})\phi_i - \omega_i \in K_{\varepsilon_i,x_i}^\perp.
\]

Then,

\[
\| \xi_i \|_{\varepsilon_i} \to 0, \quad \text{as } i \to \infty.
\]

**Proof** *(Proof of Claim 4)* To prove the claim note that for any \( v \in \mathbb{R}^n \),

\[
\langle \xi_i, W_{\varepsilon_i,x_i}^v \rangle_{\varepsilon_i} = \langle S_{\varepsilon_i}(U_{\varepsilon_i,x_i})\phi_i, W_{\varepsilon_i,x_i}^v \rangle_{\varepsilon_i} = \langle \phi_i, S_{\varepsilon_i}(U_{\varepsilon_i,x_i})(W_{\varepsilon_i,x_i}^v)\phi_i \rangle_{\varepsilon_i}.
\]

The claim then follows from Lemma 2.

Now, we have

\[
u_i := \phi_i - \omega_i - \xi_i = \phi_i - S_{\varepsilon_i}(U_{\varepsilon_i,x_i})\phi_i = i_{\varepsilon_i}^* \left( (p - 1)(U_{\varepsilon_i,x_i})^{p-2} \phi_i - \frac{s_g(x)}{a_{m+n}} \varepsilon_i^2 \phi_i \right),
\]

by (17). It follows from Claim 4 that

\[
\| u_i \|_{\varepsilon_i} \to 1.
\]

From Remark 2 and Eq. (30), \( u_i \) solves

\[
- \varepsilon_i^2 \Delta_g u_i + u_i = (p - 1)(U_{\varepsilon_i,x_i})^{p-2} \phi_i - \frac{s_g(x)}{a_{m+n}} \varepsilon_i^2 \phi_i.
\]

Let

\[
u_i := i_{\varepsilon_i}^* \left( (p - 1)(U_{\varepsilon_i,x_i})^{p-2} \phi_i \right) = u_i + i_{\varepsilon_i}^* \left( \frac{s_g(x)}{a_{m+n}} \varepsilon_i^2 \phi_i \right).
\]

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Then \( v_i \) is supported in \( B(x_i, r) \) and
\[
\| v_i \|_{\varepsilon_i} \to 1, \quad \| v_i - \phi_i \|_{\varepsilon_i} \to 0.
\] (33)

Moreover, it solves
\[
- \varepsilon_i^2 \Delta_g v_i + v_i = (p - 1)(U_{\varepsilon_i, x_i})^{p-2} \phi_i. \tag{34}
\]

**Claim** Let
\[
\tilde{v}_i(y) = v_i \left( \exp_{x_i}(\varepsilon_i y) \right), \quad y \in B(0, r/\varepsilon_i) \subset \mathbb{R}^n.
\]

Then,
\[
\tilde{v}_i \to 0 \text{ weakly in } H^1(\mathbb{R}^n) \text{ and strongly in } L^q_{loc}(\mathbb{R}^n), \quad \text{for any } q \in (2, p_n) \text{ if } n \geq 3 \text{ or } q > 2 \text{ if } n = 2. \tag{35}
\]

**Proof** (Proof of Claim 4) Let \( \tilde{v}_{i, \varepsilon_i}(y) = \tilde{v}_i(\varepsilon_i^{-1} y) = v_i \left( \exp_{x_i}(y) \right) \). Observe that
\[
\| \tilde{v}_i \|_{H^1(\mathbb{R}^n)} = \| \tilde{v}_{i, \varepsilon_i} \|_{H_{\varepsilon_i}(\mathbb{R}^n)} \leq C \| v_i \|_{\varepsilon_i} \leq C, \quad \text{for all } i \in \mathbb{N}. \tag{36}
\]

Therefore, by taking a subsequence we can assume that there exists \( \tilde{v} \in H^1(\mathbb{R}^n) \) such that \( \tilde{v}_i \to \tilde{v} \) weakly in \( H^1(\mathbb{R}^n) \), and strongly in \( L^q_{loc}(\mathbb{R}^n) \) for any \( q \in (2, p_n) \) if \( n \geq 3 \) or \( q > 2 \) if \( n = 2 \).

Now, observe that by Claim 4 for \( j = 1, \ldots, n, \)
\[
(W_{\varepsilon_i, x_i}^j, v_i)_{\varepsilon_i} = (W_{\varepsilon_i, x_i}^j, u_i)_{\varepsilon_i} + o(\varepsilon_i) = - (W_{\varepsilon_i, x_i}^j, \varepsilon_i)_{\varepsilon_i} + o(\varepsilon_i) \to 0, \quad \text{as } i \to \infty,
\] (37)

and (by change of variables and the exponential decay of \( \psi^j \))
\[
(W_{\varepsilon_i, x_i}^j, v_i)_{\varepsilon_i} \to \int_{\mathbb{R}^n} \left( \nabla \psi^j \nabla \tilde{v} + \psi^j \tilde{v} \right) dy, \quad \text{as } i \to \infty. \tag{38}
\]

We have from (33) and (34) that \( \tilde{v} \) solves
\[
- \Delta \tilde{v} + \tilde{v} = (p - 1)(U)^{p-2} \tilde{v} \text{ in } \mathbb{R}^n. \tag{39}
\]

Therefore, \( \tilde{v} \in \text{span}\{ \psi^1, \ldots, \psi^n \} \). From Eq.'s (37) and (38), we have that \( \tilde{v} \) is orthogonal to \( \{ \psi^1, \ldots, \psi^n \} \), hence \( \tilde{v} \equiv 0 \), and the claim follows.

Multiplying Eq. 34 by \( v_i \in H_{\varepsilon_i} \), we obtain from (33)
\[
\| v_i \|_{\varepsilon_i}^2 = \frac{1}{\varepsilon_i} \int_M \left\{ (p - 1)(U_{\varepsilon_i, x_i})^{p-2} \right\} v_i \phi_i \to 1. \tag{40}
\]
But, by Claim 4 we have
\[
\frac{1}{\varepsilon_i^n} \int_M \left\{ (p - 1) (U_{\varepsilon_i, x})^{p-2} \right\} v_i \phi_i \rightarrow \int_{\mathbb{R}^n} (p - 1) (U)^{p-2} v^2 = 0. \tag{41}
\]
This is a contradiction, thus proving the proposition.

Now, we write for \( \phi \in K_{\varepsilon, x}^\perp \)
\[
S_{\varepsilon}(U_{\varepsilon, x} + \phi) = S_{\varepsilon}(U_{\varepsilon, x}) + S'_{\varepsilon}(U_{\varepsilon, x}) \phi + \tilde{N}_{\varepsilon, x}(\phi), \tag{42}
\]
where
\[
\tilde{N}_{\varepsilon, x}(\phi) = S_{\varepsilon}(U_{\varepsilon, x} + \phi) - S_{\varepsilon}(U_{\varepsilon, x}) - S'_{\varepsilon}(U_{\varepsilon, x}) \phi
\]
\[
= -i_{*\varepsilon} \left( [(U_{\varepsilon, x} + \phi)^+]^{p-1} - (U_{\varepsilon, x})^{p-1} - (p - 1)(U_{\varepsilon, x})^{p-2} \phi \right).
\]
Applying \( \Pi_{\varepsilon, x}^\perp \) to (42) we see that (27) is equivalent to
\[
L_{\varepsilon, x}(\phi) = N_{\varepsilon, x}(\phi) - \Pi_{\varepsilon, x}^\perp (S_{\varepsilon}(U_{\varepsilon, x})), \tag{43}
\]
where
\[
N_{\varepsilon, x}(\phi) = -\Pi_{\varepsilon, x}^\perp (\tilde{N}_{\varepsilon, x}(\phi)) = \Pi_{\varepsilon, x}^\perp \left[ i_{*\varepsilon} \times \left( [(U_{\varepsilon, x} + \phi)^+]^{p-1} - (U_{\varepsilon, x})^{p-1} - (p - 1)(U_{\varepsilon, x})^{p-2} \phi \right) \right].
\]

We are now ready to prove the main result of this section.

**Proposition 3** There exists an \( \varepsilon_o > 0 \) and \( A > 0 \) such that for any \( x \in M \) and for any \( \varepsilon \in (0, \varepsilon_o) \) there exists a unique \( \phi_{\varepsilon, x} = \phi(\varepsilon, x) \in K_{\varepsilon, x}^\perp \) that solves Eq. (27) with \( \|\phi_{\varepsilon, x}\|_\varepsilon \leq A \). Moreover, there exists a constant \( c_o > 0 \) independent of \( \varepsilon \) such that
\[
\|\phi_{\varepsilon, x}\|_\varepsilon \leq c_o \varepsilon^2, \tag{44}
\]
and \( x \rightarrow \phi_{\varepsilon, x} \) is a \( C^2 \) map.

**Proof** In order to solve Eq. (27), or equivalently Eq. (43), we have to find a fixed point of the operator \( T_{\varepsilon, x} : K_{\varepsilon, x}^\perp \rightarrow K_{\varepsilon, x}^\perp \) given by
\[
T_{\varepsilon, x}(\phi) = L_{\varepsilon, x}^{-1} \left( N_{\varepsilon, x}(\phi) - \Pi_{\varepsilon, x}^\perp (S_{\varepsilon}(U_{\varepsilon, x})) \right).
\]
Now, from Proposition 2 we have that there is a constant \( C > 0 \) such that
\[
\|T_{\varepsilon, x}(\phi)\|_\varepsilon \leq C \left( \|N_{\varepsilon, x}(\phi)\|_\varepsilon + \|\Pi_{\varepsilon, x}^\perp (S_{\varepsilon}(U_{\varepsilon, x}))\|_\varepsilon \right), \quad \forall \phi \in K_{\varepsilon, x}^\perp. \tag{44}
\]
\( \square \)
**Claim** For any $b \in (0, 1)$ there exist constants $a, \varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, if $\phi_1, \phi_2 \in K_{\varepsilon,x}^1$, with $\|\phi_1\|_{\varepsilon}, \|\phi_2\|_{\varepsilon} < a$, then $\|N_{\varepsilon,x}(\phi_1) - N_{\varepsilon,x}(\phi_2)\|_{\varepsilon} \leq b\|\phi_1 - \phi_2\|_{\varepsilon}$.

**Proof** (Proof of Claim 4)

$$N_{\varepsilon,x}(\phi_1) - N_{\varepsilon,x}(\phi_2) = \Pi_{\varepsilon,x}^1\{S_\varepsilon(U_{\varepsilon,x} + \phi_2) - S_\varepsilon(U_{\varepsilon,x} + \phi_1) - S_\varepsilon^1(U_{\varepsilon,x})(\phi_2 - \phi_1)\}$$

Therefore,

$$\|N_{\varepsilon,x}(\phi_1) - N_{\varepsilon,x}(\phi_2)\|_{\varepsilon} \leq \|S_\varepsilon(U_{\varepsilon,x} + \phi_2) - S_\varepsilon(U_{\varepsilon,x} + \phi_1) - S_\varepsilon^1(U_{\varepsilon,x})(\phi_2 - \phi_1)\|_{\varepsilon}$$

$$= \|i_\varepsilon^*\left(\left(U_{\varepsilon,x} + \phi_1\right)^{p-1}\right.$$

$$\left.-\left(U_{\varepsilon,x} + \phi_2\right)^{p-1} + (p-1)U_{\varepsilon,x}^{p-2}(\phi_2 - \phi_1)\right)\|_{\varepsilon}$$

$$\leq c\left|\left(U_{\varepsilon,x} + \phi_1\right)^{p-1} - \left(U_{\varepsilon,x} + \phi_2\right)^{p-1}\right.$$\n
$$\left.- (p-1)(U_{\varepsilon,x})^{p-2}(\phi_1 - \phi_2)\right|_{p',\varepsilon}$$

By the Intermediate Value Theorem, there is a $\lambda \in [0, 1]$ such that

$$(U_{\varepsilon,x} + \phi_1)^{p-1} - (U_{\varepsilon,x} + \phi_2)^{p-1}$$

$$= (p-1)(U_{\varepsilon,x} + \phi_2 + \lambda(\phi_1 - \phi_2))^{p-2}(\phi_1 - \phi_2). \tag{45}$$

Then, we have from Eq. (45) that

$$\left|\left(U_{\varepsilon,x} + \phi_1\right)^{p-1} - \left(U_{\varepsilon,x} + \phi_2\right)^{p-1} - (p-1)(U_{\varepsilon,x})^{p-2}(\phi_1 - \phi_2)\right|_{p',\varepsilon}$$

$$= \left|\left((p-1)(U_{\varepsilon,x} + \phi_2 + \lambda(\phi_1 - \phi_2))^{p-2} - (p-1)(U_{\varepsilon,x})^{p-2}\right)(\phi_1 - \phi_2)\right|_{p',\varepsilon}$$

$$\leq c\left|\left(U_{\varepsilon,x} + \phi_2 + \lambda(\phi_1 - \phi_2)\right)^{p-2} - (U_{\varepsilon,x})^{p-2}\right|_{p,\varepsilon} \|\phi_1 - \phi_2\|_{\varepsilon},$$

by Hölder’s inequality and Remark 2. In order to complete the estimate we need the following elementary observation which appeared in [17, Lemma 2.1]. Let $a > 0$ and $b \in \mathbb{R}$, then

$$||a + b|^\beta - a^\beta| \leq \begin{cases} C(\beta) \min\{|b|^\beta, a^{\beta-1}|b|\} & \text{if } 0 < \beta < 1. \\ C(\beta)(|a|^{\beta-1}|b| + |b|^\beta) & \text{if } \beta \geq 1. \end{cases} \tag{46}$$

Applying (46), we see that for all $v \in H_\varepsilon$

$$\left|\left(U_{\varepsilon,x} + v\right)^{p-2} - (U_{\varepsilon,x})^{p-2}\right| \leq \begin{cases} C(p)|v|^{p-2} & \text{if } 2 < p < 3. \\ C(p)\left(|U_{\varepsilon,x}|^{p-3}|v| + |v|^{p-2}\right) & \text{if } p \geq 3. \end{cases}$$

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Then, it follows that
\[
| (U_{\varepsilon, x} + v)|^{p-2} - (U_{\varepsilon, x})^{p-2} | \leq \begin{cases} 
C(p) |v|^{p-2}_{p, \varepsilon} & \text{if } 2 < p < 3, \\
C(p) \left( |U_{\varepsilon, x}|^{p-3}_{p, \varepsilon} |v|_{p, \varepsilon} + |v|^{p-2}_{p, \varepsilon} \right) & \text{if } p \geq 3.
\end{cases}
\]

Using (48) and Remark 2 we can see that if \( a \) is small enough then
\[
c \left( (U_{\varepsilon, x} + \phi_2 + \lambda(\phi_1 - \phi_2)) \right)^{p-2} - (U_{\varepsilon, x})^{p-2} \right| \leq b,
\]
proving the claim.

In similar fashion we can prove the following claim.

**Claim** For any \( b \in (0, 1) \) there exist constants \( a > 0 \) and \( \varepsilon_o > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_o) \), if \( \| \phi \|_\varepsilon < a \) then \( \| N_{\varepsilon, x}(\phi) \|_\varepsilon \leq b \| \phi \|_\varepsilon \).

**Proof** (Proof of Claim 4)

\[
\| N_{\varepsilon, x}(\phi) \|_\varepsilon = \| \Pi^{\perp}_{\varepsilon} \left( S_\varepsilon(U_{\varepsilon, x} + \phi) - S_\varepsilon(U_{\varepsilon, x}) - S'_{\varepsilon}(U_{\varepsilon, x})(\phi) \right) \|_\varepsilon
\]

\[
= \| \lambda (U_{\varepsilon, x})^{p-1} - (U_{\varepsilon, x} + \phi)^{p-1} + (p - 1)(U_{\varepsilon, x})^{p-2} \|_\varepsilon,
\]

and we can apply the Intermediate Value Theorem and Remark 4.3, as in the proof of Claim 4, to prove the claim.

Now we prove the first statements of the proposition using the claims. Let \( C \) be the constant in (44) and take \( b = \frac{1}{2C} \). Let \( a \) be the constant given by Claim 4 and Claim 4 (the minimum of the two, to be precise). From Lemma 1 and Claim 4 there exists \( \varepsilon_o > 0 \) such that if \( \varepsilon \in (0, \varepsilon_o) \) then \( T_{\varepsilon, x} \) sends the ball of radius \( a \) in \( K_{\varepsilon, x} \) to itself.

If \( \| \phi_1 \|_\varepsilon, \| \phi_2 \|_\varepsilon < a \), we have that
\[
\| T_{\varepsilon, x}(\phi_1) - T_{\varepsilon, x}(\phi_2) \|_\varepsilon \leq C \| N_{\varepsilon, x}(\phi_1) - N_{\varepsilon, x}(\phi_2) \|_\varepsilon \leq \frac{1}{2} \| \phi_1 - \phi_2 \|_\varepsilon.
\]

We see then that \( T_{\varepsilon, x} \) is a contraction in the ball of radius \( a \). It follows that it has a unique fixed point there. The fixed point is obtained for instance as the limit of the sequence \( a_k = T_{\varepsilon, x}^k(0) \). Note that \( \| a_1 \|_\varepsilon \leq C\varepsilon^2 \) by Lemma 1 and then from Claim 4 we have that for all \( k, \| a_k \|_\varepsilon \leq 2C\varepsilon^2. \)

It remains to prove that the map \( x \rightarrow \phi_{\varepsilon, x} \) is \( C^2 \). In order to show this, we apply the Implicit Function Theorem to the \( C^2 \)-function \( G : M \times H_\varepsilon \rightarrow H_\varepsilon \) defined by
\[
G(x, u) = \Pi^{\perp}_{\varepsilon, x} \left\{ S_\varepsilon(U_{\varepsilon, x} + \Pi^{\perp}_{\varepsilon, x} u) \right\} + \Pi_{\varepsilon, x} u.
\]

Observe that \( G(x, \phi_{\varepsilon, x}) = 0 \), and that the derivative \( \frac{\partial G}{\partial u}(x, \phi_{\varepsilon, x}) : H_\varepsilon \rightarrow H_\varepsilon \) is given by
\[
\frac{\partial G}{\partial u}(x, \phi_{\varepsilon, x})(u) = \Pi^{\perp}_{\varepsilon, x} \left\{ S'_{\varepsilon}(U_{\varepsilon, x} + \phi_{\varepsilon, x}) \Pi^{\perp}_{\varepsilon, x} u \right\} + \Pi_{\varepsilon, x} u
\]

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The proof would be done if we show the next claim.

**Claim** For $\varepsilon > 0$ small enough, there is $C > 0$ such that

$$\left\| \frac{\partial G}{\partial u}(x, \phi_{\varepsilon, x})(u) \right\|_\varepsilon \geq C \| u \|_\varepsilon,$$

for every $x \in M$.

**Proof (Proof of Claim 4)** We have that for $c = \frac{1}{\sqrt{2}}$ that

$$\left\| \frac{\partial G}{\partial u}(x, \phi_{\varepsilon, x})(u) \right\|_\varepsilon \geq c \left\| \Pi_{\varepsilon, x} \left\{ S'_\varepsilon(U_{\varepsilon, x} + \phi_{x})\Pi_{\varepsilon, x}^{-1}(u) \right\} \right\|_\varepsilon + c \left\| \Pi_{\varepsilon, x}(u) \right\|_\varepsilon$$

$$= c \left\| \Pi_{\varepsilon, x} \left\{ S'_\varepsilon(U_{\varepsilon, x})\Pi_{\varepsilon, x}^{-1}(u) + S'_\varepsilon(U_{\varepsilon, x} + \phi_{x})\Pi_{\varepsilon, x}^{-1}(u) - S'_\varepsilon(U_{\varepsilon, x})\Pi_{\varepsilon, x}^{-1}(u) \right\} \right\|_\varepsilon + c \left\| \Pi_{\varepsilon, x}(u) \right\|_\varepsilon$$

$$\geq c \left\| \Pi_{\varepsilon, x}(u) \right\|_\varepsilon + c \left\| L_{\varepsilon, x}(\Pi_{\varepsilon, x}^{-1}(u)) \right\|_\varepsilon - c \left\| \Pi_{\varepsilon, x} \left\{ S'_\varepsilon(U_{\varepsilon, x} + \phi_{x})\Pi_{\varepsilon, x}^{-1}(u) - S'_\varepsilon(U_{\varepsilon, x})\Pi_{\varepsilon, x}^{-1}(u) \right\} \right\|_\varepsilon$$

It follows from Proposition 2 that, for another constant $c > 0$, $\left\| L_{\varepsilon, x}(\Pi_{\varepsilon, x}^{-1}(u)) \right\|_\varepsilon \geq c \left\| \Pi_{\varepsilon, x}^{-1}(u) \right\|_\varepsilon$. Then we have that for some constant $C > 0$,

$$c \left\| \Pi_{\varepsilon, x}(u) \right\|_\varepsilon + c \left\| L_{\varepsilon, x}(\Pi_{\varepsilon, x}^{-1}(u)) \right\|_\varepsilon \geq C \| u \|_\varepsilon.$$

Therefore, it only remains to prove that

$$\lim_{\varepsilon \to 0} \left\| \Pi_{\varepsilon, x} \left\{ S'_\varepsilon(U_{\varepsilon, x} + \phi_{x})\Pi_{\varepsilon, x}^{-1}(u) - S'_\varepsilon(U_{\varepsilon, x})\Pi_{\varepsilon, x}^{-1}(u) \right\} \right\|_\varepsilon = 0.$$

But,

$$S'_\varepsilon(U_{\varepsilon, x} + \phi_{x})\Pi_{\varepsilon, x}^{-1}(u) - S'_\varepsilon(U_{\varepsilon, x})\Pi_{\varepsilon, x}^{-1}(u) = (p - 1)i_\varepsilon^p((U_{\varepsilon, x} + \phi_{x}))^{p-2} - (U_{\varepsilon, x})^{p-2}\Pi_{\varepsilon, x}^{-1}(u)).$$

Hence, as in the proof of Claim 4,

$$\left\| \left\{ S'_\varepsilon(U_{\varepsilon, x} + \phi_{x})\Pi_{\varepsilon, x}^{-1}(u) - S'_\varepsilon(U_{\varepsilon, x})\Pi_{\varepsilon, x}^{-1}(u) \right\} \right\|_\varepsilon \leq c\left|((U_{\varepsilon, x} + \phi_{x}))^{p-2} - (U_{\varepsilon, x})^{p-2}\right|\Pi_{\varepsilon, x}^{-1}(u)\right|_{p', \varepsilon}$$

$$\leq c\left|((U_{\varepsilon, x} + \phi_{x}))^{p-2} - (U_{\varepsilon, x})^{p-2}\right|\Pi_{\varepsilon, x}^{-1}(u)\right|_{p, \varepsilon}$$

$$\leq c\left|((U_{\varepsilon, x} + \phi_{x}))^{p-2} - (U_{\varepsilon, x})^{p-2}\right|\Pi_{\varepsilon, x}^{-1}(u)\right|_{p, \varepsilon}$$

Arguing as in the end of the proof of Claim 4 we can see that

$$\lim_{\varepsilon \to 0} \left|((U_{\varepsilon, x} + \phi_{x}))^{p-2} - (U_{\varepsilon, x})^{p-2}\right|_{p, \varepsilon} = 0.$$
thus completing the proof of the claim. □

This finishes the proof of the proposition. □

5 Proof of Theorem 1

Recall that the critical points of the functional $J_\varepsilon : H^1(M) \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \varepsilon^{-n} \int_M \left( \frac{1}{2} \varepsilon^2 \|\nabla u\|^2 + \frac{s_\varepsilon \varepsilon^2 + a_{m+n}}{2a_{m+n}} u^2 - \frac{1}{p} (u^+)^p \right) d\mu_g,$$

are the positive solutions of Eq. (5).

Proposition 3 tells us that there exists $\varepsilon_o > 0$ such that for $\varepsilon \in (0, \varepsilon_o)$ and $x \in M$ there exists a uniquely defined $\phi_{\varepsilon,x} \in K_{\varepsilon,x}^\bot$ such that $U_{\varepsilon,x} + \phi_{\varepsilon,x}$ solves Eq. (27). In order to finish the proof of Theorem 1 we have to establish the following result.

**Proposition 4** There exists $\varepsilon_o > 0$ such that if $\varepsilon \in (0, \varepsilon_o)$ and $x_o \in M$ is a critical point of $F_\varepsilon : M \to \mathbb{R}$, where

$$F_\varepsilon(x) = J_\varepsilon(U_{\varepsilon,x} + \phi_{\varepsilon,x}), \quad (49)$$

then $U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}$ is a positive solution of Eq. (5).

**Proof** Let $x_o \in M$ be a critical point of $F_\varepsilon$ where $\varepsilon > 0$. We need to show that for each $\varphi \in H^1(M)$ one has that

$$\langle S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}), \varphi \rangle_\varepsilon = 0.$$

If $\varphi \in K_{\varepsilon,x_o}^\bot$ then

$$\langle S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}), \varphi \rangle_\varepsilon = \langle \Pi_{\varepsilon,x}^\bot(S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o})), \varphi \rangle_\varepsilon = 0,$$

since $U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}$ solves Eq. (27).

Then it is enough to show that $\langle S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}), \varphi \rangle_\varepsilon = 0$ if $\varphi \in K_{\varepsilon,x_o}$. On the other hand we know that $\langle S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}), \varphi \rangle_\varepsilon = 0$ if $\varphi$ is tangent to the map $x \mapsto V(x) = U_{\varepsilon,x} + \phi_{\varepsilon,x}$ at $x_o$. And since $M$ and $K_{\varepsilon,x_o}$ have the same dimension it is enough to see that the projection $\Pi_{\varepsilon,x_o} \circ D_{x_o} V : T_{x_o} M \to K_{\varepsilon,x_o}$ is injective.

Then to finish the proof it is enough to show that, fixing geodesic coordinates centered at $x_o$, for any $v \in \mathbb{R}^n$

$$\left\langle \frac{\partial}{\partial v}(U_{\varepsilon,x} + \phi_{\varepsilon,x})(x_o), W_{\varepsilon,x_o}^v \right\rangle_\varepsilon \neq 0. \quad (50)$$

Note that $\langle \phi_{\varepsilon,x}, W_{\varepsilon,x}^v \rangle_\varepsilon = 0$. Then
\begin{equation}
\left(\frac{\partial}{\partial v}(\phi_{\varepsilon,x}), W_{\varepsilon,x_0}^v\right)_\varepsilon = -\left(\phi_{\varepsilon,x}, \frac{\partial}{\partial v} W_{\varepsilon,x_0}^v\right)_\varepsilon.
\end{equation}

As we pointed out in (23), we have
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \left\|\frac{\partial}{\partial v} W_{\varepsilon,x_0}^v\right\|_\varepsilon = 0.
\]

Then, it follows from Cauchy-Schwarz inequality and Proposition 3 that
\[
\lim_{\varepsilon \to 0} \left(\frac{\partial}{\partial v}(\phi_{\varepsilon,x}), W_{\varepsilon,x_0}^v\right)_\varepsilon = 0.
\]

From (24),
\[
\lim_{\varepsilon \to 0} \varepsilon \left(\frac{\partial}{\partial v}(U_{\varepsilon,x}), W_{\varepsilon,x_0}^v\right)_\varepsilon = \langle \psi^v, \psi^v \rangle > 0.
\]

Then, for \(\varepsilon > 0\) small enough (50) holds, and the proposition is proved. \(\square\)

6 Analytic Proof That \(\beta_{2,2} \neq 0\)

In [24] Rey and Ruiz numerically checked that \(\beta_{m,n} < 0\) if \(n + m \leq 9\). In this section we prove that \(\beta_{m,n}\) is not equal to zero for values \(m\) and \(n\) such that \(m + n = 4\). Fix \(m, n\) and let
\[
\beta = a_{m+n} \beta_{m,n} = \int_{\mathbb{R}^n} U^2 - \frac{a_{m+n}}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 dz.
\]

Recall that \(p = p_{m+n}\).

Theorem 5 If \(m\) and \(n\) such that \(m + n = 4\), then \(\beta < 0\). If \(n \neq 4\), \(m + n > 4\) we have
\[
\beta = \frac{6 - a_{m+n}}{n(n-4)} \int_{\mathbb{R}^n} \left(\frac{2}{p} \cdot \frac{m}{m+n-4} U^p - U^2\right) |z|^2 dz.
\]

Proof We know that \(U\) satisfies
\[
\Delta U = U - U^{p-1}.
\] (51)

Let us multiply (51) by \(U|z|^2\) and integrate
\[
\int_{\mathbb{R}^n} \left(U^2 - U^p\right) |z|^2 dz = \int_{\mathbb{R}^n} \Delta U \cdot U|z|^2 dz
\]
\[ \begin{align*}
&= - \int_{\mathbb{R}^n} \langle \nabla U, \nabla (U \cdot |z|^2) \rangle \, dz \quad \text{(by the Divergence Theorem)} \\
&= - \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 \, dz - 2 \int_{\mathbb{R}^n} U \langle \nabla U, z \rangle \\
&= - \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 \, dz - \int_{\mathbb{R}^n} \langle \nabla U^2, z \rangle \\
&= - \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 \, dz + n \int_{\mathbb{R}^n} U^2 \, dz
\end{align*} \]

Hence,

\[ n \int_{\mathbb{R}^n} U^2 \, dz = \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 \, dz + \int_{\mathbb{R}^n} (U^2 - U^p) |z|^2 \, dz. \tag{52} \]

It is proved in Lemma 5.5 in [18] that

\[ \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial z_i} \right)^2 \, dz = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla U|^2 \, dz + \int_{\mathbb{R}^n} \left( (1/2)U^2 - (1/p)U^p \right) \, dz \]

And using that (see for instance the proof of Lemma 3.3 in [24])

\[ \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial z_i} \right)^2 \, dz = \int_{\mathbb{R}^n} \left( \frac{U'(|z|)}{|z|} \right)^2 \, dz = \frac{3}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 \, dz, \quad i = 1, \ldots, n, \]

we have

\[ \left( \frac{n-4}{n+2} \right) \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 \, dz = \frac{2}{p} \int_{\mathbb{R}^n} U^p |z|^2 \, dz - \int_{\mathbb{R}^n} U^2 |z|^2 \, dz. \tag{53} \]

Now, observe that by (52)

\[ n \beta = \left( \frac{n+2}{n+2} - a_{m+n} \right) \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 \, dz + \int_{\mathbb{R}^n} (U^2 - U^p) |z|^2 \, dz. \tag{54} \]

Hence, by (53) and (54)

\[ n(n-4) \beta = \frac{1}{p} \cdot (2n + 4 - 2a_{m+n} + (4-n)p) \int_{\mathbb{R}^n} U^p |z|^2 \, dz - (6-a_{m+n}) \int_{\mathbb{R}^n} U^2 |z|^2 \, dz = \frac{4}{p} \cdot \frac{m}{m+n-2} \int_{\mathbb{R}^n} U^p |z|^2 \, dz - (6-a_{m+n}) \int_{\mathbb{R}^n} U^2 |z|^2 \, dz. \]
Notice that if \( n = 3 \), \( m = 1 \) or \( n = m = 2 \), we have \( a_{m+n} = 6 \). Therefore, in these two cases we obtain \( \beta < 0 \).

Finally, if \( n \neq 4 \) and \( m + n > 4 \),

\[
\beta = \frac{6 - a_{m+n}}{n(n-4)} \int_{\mathbb{R}^n} \left( \frac{2}{p} \cdot \frac{m}{(m+n-4)} U^p - U^2 \right) |z|^2 \, dz.
\]

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