BRST and Anti BRST structure of a topological current algebra

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Abstract

The BRST structure of a current satisfying a non abelian affine algebra in two dimensions was studied by Isidro and Ramallo and an \( N = 2 \) Superconformal Algebra was obtained. In this paper, we study the total BRST and anti BRST structure of the topological algebra. We end up with an \( N = 4 \) Superconformal algebra in which the central charge drops out of most of the OPE’s. The price one has to pay is that the no. of operators proliferates tremendously and the algebra becomes infinite dimensional.

1 Introduction

The study of topological quantum field theories has aroused great interest since their inception. These theories possess a nilpotent symmetry \( Q \) satisfying \( Q^2 = 0 \) and the physical observables are the cohomology classes of the operator \( Q \) as it acts in the full Hilbert space. Such theories are therefore also known as cohomological field theories. Since the stress energy tensor \( T_{\alpha\beta} \) is \( Q \) exact, i.e.

\[
T_{\alpha\beta} = [Q, G_{\alpha\beta}] 
\]

(1.1)
correlation functions of the observables are independent of the two dimensional metric \( g_{\alpha\beta} \). The topological theories therefore have a much larger symmetry than conformal theories. However, even in the category of topological quantum field theory, there is a counterpart of conformal invariance, since the special class of models in which \( T_{\alpha\beta} \) is traceless, even before restricting to the \( Q \) cohomology. These are the topological conformal field theories (TCFT’s). In the same way that the conformal theories correspond to critical points in the space of ordinary quantum field theories (QFT’s) the topological CFT’s are the critical points in the space of topological QFT’s. One can consider perturbations of a TCFT by turning on the couplings of the appropriate physical fields. This maintains the topological symmetry, but in general destroys the conformal invariance, so one obtains a parameter family of more general ‘massive’ topological theories. The study of two dimensional TCFT’s has attracted much interest due to its connection with non-critical string theories. A possible way of generating new TCFT’s is to study the BRST structure of different chiral algebras that extend the Virasaro algebra. This method has

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been applied in \cite{8} and \cite{9} to the case of an affine Lie Algebra, whereas in \cite{10}, the analysis was extended to the case of a superconformal current algebra. In \cite{11}, the analysis was extended to the case of an arbitrary affine Lie superalgebra. The topological symmetry of a TCFT is encoded in its topological algebra which is the operator algebra closed by the chiral algebra of the TCFT and the BRST current. It was shown in \cite{8, 10, 11} that the topological algebra of a TCFT possessing a nonabelian current algebra symmetry must include operators of dimensions one, two and three, realizing the so called Kazama algebra \cite{12}. This algebra differs from the standard twisted $N = 2$ SCA as the former includes two dimension three operators and can be regarded as an extension of the latter. The operator algebra of \cite{9} is, however similar to that of \cite{13}. The anti BRST symmetry, introduced in 1976 by Curci and Ferrari \cite{14} and Ojima \cite{15} has been studied extensively by Baulieu et al \cite{16}, Hwang \cite{17} and Perry et al \cite{18} among others. The question that has most often been asked is whether the anti BRST condition $\bar{Q}|\text{phys} > = 0$ yields any further information, in addition to the imposition of the BRST condition $Q|\text{phys} > = 0$. The answer to the above questions have been studied in detail as well \cite{8, 19, 20, 21}. It is the purpose of this paper to investigate the consequence of anti BRST invariance in TCFT’s, in particular, to examine the SCA formed by the generators of the BRST and the anti BRST symmetry, thus extending the algebra found in \cite{8}. The most important feature of our results is that the central charge drops out almost totally, except in the Operator Product Expansion (OPE) of the conformal charges with each other. The price we have to pay for this is that the no. of terms in the algebra increases, for each dimension, and the algebra can, in fact, be shown to be infinite dimensional.

The paper is organized as follows: In Sec.2, we investigate the effects of the BRST and anti BRST transformations on the matter and ghost fields and see that they satisfy the conditions of a topological algebra. We next obtain the algebra of the generators and see that the central charge drops out of most of the equations. In Sec. 3, we see how the algebra becomes infinite dimensional, while in Sec. 4, we present an analysis of the operators of the algebra of order one, upto dimension 3. In Sec. 5, we conclude by summing up the results of our analysis and some additional comments.

## 2 The Topological Algebra

Let us consider a Lie Algebra $\mathfrak{g}$ generated by the Hermitian matrices $T^a$ ($a = 1, \ldots, \dim \mathfrak{g}$) that satisfy the commutation relations

$$[T^a, T^b] = i f^{abc} T^c$$

The Lie Algebra $\mathfrak{g}$ is assumed to be semisimple so that

$$f^{abc} f^{dbc} = c_A \delta^{ad}$$

$c_A$ being the quadratic Casimir of the adjoint representation of the Algebra. A holomorphic current taking values on $\mathfrak{g}$ is an operator $J_a(z)$ with Laurent modes defined by

$$J_a(z) = \sum_{n \in \mathbb{Z}} J^n_a z^{n-1}$$
The OPE of two currents is given by
\[ J_a(z_1)J_b(z_2) = \frac{1}{(z_1 - z_2)^2}k\delta_{ab} + \frac{1}{z_1 - z_2}if^{abc}J_c \] (2.4)

The Sugawara energy momentum tensor for the currents \(J_a\) is given by [23]
\[ T(z) = \frac{1}{(2k + c_A)}J_a(z)J_a(z) \] (2.5)

It is important to mention here that all expressions are assumed to be normal ordered unless specified to the contrary and that only singular terms appear in the OPE's of the operators. It can be checked using standard methods that
\[ T(z_1)J_a(z_2) = J_a(z_2) - \frac{\delta_{ab}}{(z_1 - z_2)} \] (2.6)
\[ T(z_1)T(z_2) = \frac{c}{2(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{(z_1 - z_2)} \] (2.7)

where the central charge in (2.7) is given by
\[ c = \frac{2kdimg}{2k + c_A} \] (2.8)

In order to define a topological theory, we need to define a BRST symmetry. In order to do this, we introduce the Faddeev-Popov ghosts, \(\gamma_a(z_1)\) and \(\rho_a(z_1)\) and anti ghosts \(\bar{\gamma}_a(z_1)\) and \(\bar{\rho}_a(z_2)\):
\[ \gamma_a(z_1)\rho_b(z_2) = -\frac{\delta_{ab}}{(z_1 - z_2)} \] (2.9)
\[ \bar{\gamma}_a(z_1)\bar{\rho}_b(z_2) = -\frac{\delta_{ab}}{(z_1 - z_2)} \] (2.10)

We use the standard expression for the BRST charge \(Q\) (see for example) \[17\] for a constrained Lagrangian \(\mathcal{L}\). In this case, the currents are the constraints, as for example in the gauged WZNW model \[24\]. We introduce the Lagrangian multipliers, \(v_a\) and their conjugate multipliers \(P_b\):
\[ v_a(z_1)P_b(z_2) = \frac{\delta_{ab}}{z_1 - z_2} \] (2.11)

The expression for the BRST charge takes the following form
\[ Q = -\gamma_aJ_a : -\frac{i}{2}f^{abc} \gamma_b\gamma_c : - \bar{P}_a \bar{\rho}_a : \] (2.12)

The expression used above differs from that of \[3\] and \[4\] due to the inclusion of the last term. Our intention was to see if the algebra resulting from the inclusion of the Lagrangian multipliers would be nontrivially different. The answer to the above question
is a resounding yes as we shall see in the following sections. It is easy to check that the
BRST charge introduces the following transformation on $J_a$ and $\gamma_a$

$$
\begin{align*}
\delta J_a &= i f_{abc} : \gamma_b J_c : -k \partial \gamma_a \\
\delta \gamma_a &= i f_{abc} : \gamma_b \gamma_c :
\end{align*}
$$

(2.13) (2.14)

and that

$$
\delta^2 \gamma_a = \delta^2 J_a = 0
$$

(2.15)
due to the Jacobi Identity. The BRST transformation introduced on $\rho_a$ is given by

$$
\delta \rho_a = J_a + i f_{abc} : \gamma_b \rho_c :
$$

(2.16)

Due to the nilpotency of the BRST charge, $\delta^2 \rho$ must vanish. We see that

$$
\delta^2 \rho = -k \partial \gamma_a - \frac{1}{2} f_{abc} f_{bmn} : \gamma_m \gamma_n : \rho_c : + f_{abc} f_{cmm} : \gamma_b \gamma_m \rho_n :=
$$

(2.17)

The last term on the right takes the form

$$
f_{abc} f_{cmm} : \gamma_m \rho_n \gamma_b : - f_{abc} f_{cmm} \partial \gamma_m \delta \rho_n = - f_{abc} f_{cmm} (\gamma_m \gamma_b \rho_n) - \delta \rho_n \partial \gamma_m f_{abc} f_{cmb}(2.18)
$$
on using the identity

$$
[\gamma^n_b : (\gamma_r \rho_s)_m :] = \delta_{sb} \gamma^{n+m}_r
$$

(2.19)

so that the right hand side of (2.17) takes the following form:

$$
- (k + c_A) \partial \gamma_a
$$

(2.20)

while the other terms drop out because of the Jacobi Identity. Hence the BRST transformation is nilpotent on $\rho_a$ only for the critical level $k = -c_A$. The BRST transformation on the antighost fields yield

$$
\begin{align*}
\delta \bar{\rho}_a &= 0; \\
\delta \bar{\gamma}_a &= P_a; \\
\delta P_a &= 0; \\
\delta \bar{v}_a &= \bar{\rho}_a
\end{align*}
$$

(2.21) (2.22) (2.23) (2.24)

and $\delta^2$ is easily seen to vanish on all of them. We now consider the action of the anti
BRST transformation. The anti BRST operator takes the form

$$
\bar{Q} = -\bar{\gamma}_a J_a - \frac{i}{2} f_{abc} \bar{\gamma}_a \bar{\gamma}_b \bar{\rho}_c + P_a \rho_a - i f_{abc} \bar{\gamma}_a \gamma_b \rho_c - i f_{abc} \bar{\gamma}_a \gamma_b \bar{P}_c
$$

(2.25)

We notice the presence of the following currents in the expressions for $Q$ and $\bar{Q}$ :

$$
\begin{align*}
J_{gh}^a &= i f_{abc} \gamma_b \rho_c; \\
J_{agb}^a &= i f_{abc} \bar{\gamma}_b \bar{\rho}_c; \\
J_{Lag}^a &= i f_{abc} \gamma_b \bar{P}_c;
\end{align*}
$$

(2.26) (2.27) (2.28)
Table 1: Levels of the Anti BRST Currents

| Current | Level |
|---------|-------|
| $J_a$   | $-c_A$|
| $f_a^h$ | $c_A$ |
| $f_{ab}^h$ | $c_A$ |
| $J_{Lag}$ | $-c_A$ |

with levels given in Table 1. The anti BRST transformations on the fields $J_a$, $\gamma_a$, $\rho_a$, $\bar{\gamma}_a$, $\bar{\rho}_a$, $v_a$ and $P_a$ are as follows:

\[
\delta J_a = i f_{abc} \bar{\gamma}_b J_c - k \partial \bar{\gamma}_a; \quad (2.29)
\]

\[
\delta \gamma_a = i f_{abc} \bar{\gamma}_b \gamma_c - P_a; \quad (2.30)
\]

\[
\delta \rho_a = i f_{abc} \bar{\gamma}_b \rho_c; \quad (2.31)
\]

\[
\delta \bar{\gamma}_a = \frac{i f_{abc}}{2} \bar{\gamma}_b \bar{\gamma}_c; \quad (2.32)
\]

\[
\delta \bar{\rho}_a = J_a + i f_{abc} \bar{\rho}_b \bar{\rho}_c + i f_{abc} \gamma_b \rho_c + i f_{abc} v_b P_c = \mathcal{T}_a; \quad (2.33)
\]

\[
\delta v_a = -\rho_a + i f_{abc} \bar{\gamma}_b v_c; \quad (2.34)
\]

\[
\delta P_a = i f_{abc} \bar{\gamma}_b P_c; \quad (2.35)
\]

It is easy to check using the Jacobi Identity that $\bar{\delta}$ is nilpotent on all of the fields except for $\bar{\rho}_a$, where it is nilpotent for $k = -c_A$.

Let us look at the algebra formed by $\mathcal{J}_a$ and $\mathcal{T}_a$. After a short calculation, we see that

\[
\mathcal{J}_a(z_1) \mathcal{J}_b(z_2) = \frac{(k + c_A)}{(z_1 - z_2)^2} \delta_{ab} + \frac{i f_{abc} \mathcal{J}_c(z_2)}{(z_1 - z_2)} \quad (2.37)
\]

\[
\mathcal{T}_a(z_1) \mathcal{T}_b(z_2) = \frac{(k + c_A)}{(z_1 - z_2)^2} \delta_{ab} + \frac{i f_{abc} \mathcal{T}_c(z_2)}{(z_1 - z_2)} \quad (2.38)
\]

and for $k = -c_A$, the algebras formed by $(\mathcal{J}_a, \rho_a)$ and $(\mathcal{T}_a, \bar{\rho}_a)$ takes the form of the topological current algebra introduced in [8].

\[
\mathcal{J}_a(z_1) \mathcal{J}_b(z_2) = i f_{abc} \frac{\mathcal{J}_c(z_2)}{z_1 - z_2} \quad (2.39)
\]

\[
\mathcal{J}_a(z_1) \rho_b(z_2) = i f_{abc} \frac{\rho_c(z_2)}{(z_1 - z_2)} \quad (2.40)
\]

\[
\rho_a(z_1) \rho_b(z_2) = 0 \quad (2.41)
\]

and

\[
\mathcal{T}_a(z_1) \mathcal{T}_b(z_2) = i f_{abc} \frac{\mathcal{T}_c(z_1)}{(z_1 - z_2)} \quad (2.42)
\]

\[
\mathcal{T}_a(z_1) \bar{\rho}_b(z_2) = i f_{abc} \frac{\bar{\rho}_c(z_2)}{(z_1 - z_2)} \quad (2.43)
\]

\[
\bar{\rho}_a(z_1) \bar{\rho}_b(z_2) = 0 \quad (2.44)
\]
The OPE between the other elements of the algebra \((\mathcal{J}_a, \rho_a, T_a, \bar{\rho}_a)\) are as follows:

\[
\begin{align*}
T_a(z_1)\rho_b(z_2) &= if_{abc} \frac{\rho_c(z_2)}{(z_1 - z_2)} \\
\mathcal{J}_a(z_1)T_b(z_2) &= if_{abc} \frac{\mathcal{J}_c(z_2)}{(z_1 - z_2)} \\
\mathcal{J}_a(z_1)\bar{\rho}_b(z_2) &= 0 \\
\rho_a(z_1)\bar{\rho}_b(z_2) &= 0
\end{align*}
\]

The absence of a central term in the algebra formed by \((\mathcal{J}_a, \rho_a, T_a, \bar{\rho}_a)\) is noted, as is the fact that \((\mathcal{J}_a, \rho_a)\) constitutes an ideal of the total algebra formed by \((\mathcal{J}_a, \rho_a, T_a, \bar{\rho}_a)\). We now attempt to construct the total energy momentum tensor. Following [8], we consider the BRST variation of

\[
\delta\left[\frac{\rho_a\mathcal{J}_a + v_a\partial\gamma_a}{2k + c_A}\right] = \frac{J_aJ_a}{2k + c_A} + k\frac{\rho_a\partial\gamma_a}{2k + c_A} + \bar{\rho}_a\partial\bar{\gamma}_a + v_a\partial P_a
\]

The right hand side is the energy-momentum tensor of the \((\rho, \gamma)\) system if the coefficient of the \(\rho\partial\gamma\) term is one, which happens for \(k = -c_A\). We note that for this value of \(k\), the total energy momentum tensor \(T\) of the system is obtained. Hence the BRST partner \(G\) of \(T\) is given by

\[
G = -\frac{\rho_a\mathcal{J}_a}{c_A} + v_a\partial\gamma_a
\]

We consider the anti BRST variation of

\[
\bar{G} = \frac{\bar{\rho}_a\mathcal{J}_a + if_{abc}v_b\gamma_c\mathcal{J}_a}{2k + c_A} - v_a\partial\bar{\gamma}_a
\]

We obtain

\[
\begin{align*}
\delta\left[\frac{\bar{\rho}_a\mathcal{J}_a + if_{abc}v_b\gamma_c\mathcal{J}_a}{2k + c_A} - v_a\partial\bar{\gamma}_a\right] &= \frac{J_aJ_a}{2k + c_A} + k\frac{\bar{\rho}_a\partial\bar{\gamma}_a}{2k + c_A} + \rho_a\partial\gamma_a + v_a\partial P_a \\
&+ ikf_{abc}v_b\gamma_c\partial\bar{\gamma}_a - if_{abc}v_b\gamma_c\partial\gamma_a
\end{align*}
\]

This is the total energy momentum tensor for \(k = -c_A\). We see thus at this critical value of \(k\), that \(T\) is both BRST and anti BRST exact, and thus the theory is topological. We find it convenient to specify the combination of terms \(\bar{\rho}_a + if_{abc}v_b\gamma_c\) as \(R_a\). Hence,

\[
\bar{G}_a = -\frac{R_a\mathcal{J}_a}{c_A} - v_a\partial\gamma_a
\]

We now attempt to look at the algebra formed by the generators \(Q, G, \bar{Q}, \bar{G}\), and \(T\). It may be checked after a little calculation that

\[
\begin{align*}
Q(z_1)Q(z_2) &= 0 \\
\bar{Q}(z_1)\bar{Q}(z_2) &= 0 \\
Q(z_1)\bar{Q}(z_2) &= 0
\end{align*}
\]
We also have,

\[
T(z_1)Q(z_2) = \frac{Q(z_2)}{(z_1 - z_2)^2} + \frac{\partial Q(z_2)}{(z_1 - z_2)} \tag{2.57}
\]

\[
T(z_1)\bar{Q}(z_2) = \frac{\bar{Q}(z_2)}{(z_1 - z_2)^2} + \frac{\partial \bar{Q}(z_2)}{(z_1 - z_2)} \tag{2.58}
\]

\[
T(z_1)G(z_2) = \frac{2G(z_1)}{(z_1 - z_2)^2} + \frac{\partial G(z_2)}{(z_1 - z_2)} \tag{2.59}
\]

\[
T(z_1)\bar{G}(z_2) = \frac{2\bar{G}(z_2)}{(z_1 - z_2)^2} + \frac{\partial \bar{G}(z_2)}{(z_1 - z_2)} \tag{2.60}
\]

and

\[
T(z_1)T(z_2) = 2\frac{T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{(z_1 - z_2)} \tag{2.61}
\]

so the total central term of the system vanishes. This is a nontrivial result and we feel it is a consequence of the fact that the full algebra with all the Lagrangian multipliers is being considered. We also have

\[
Q(z_1)G(z_2) = \frac{R_{gh}(z_2)}{(z_1 - z_2)^2} + \frac{T(z_1)}{(z_1 - z_2)} \tag{2.62}
\]

\[
\bar{Q}(z_1)\bar{G}(z_2) = \frac{R_{agh}(z_2)}{(z_1 - z_2)^2} + \frac{T(z_2)}{(z_1 - z_2)} \tag{2.63}
\]

where \(R_{gh}\) and \(R_{agh}\) are defined as

\[
R_{gh} = \rho_a \gamma_a + P_a v_a \tag{2.64}
\]

\[
R_{agh} = \bar{\rho}_a \bar{\gamma}_a + P_a v_a \tag{2.65}
\]

We note that there is no central extension term. This is a departure from [8]. \(R_{gh}\) and \(R_{agh}\) are the ghost no.operators. The total charge \(R\) is defined as

\[
R = R_{gh} - R_{agh} \tag{2.66}
\]

The OPE of \(R_{gh}\) and \(R_{agh}\) with the other operators is as follows:

\[
R_{gh}(z_1)Q(z_2) = \frac{Q(z_2)}{(z_1 - z_2)} \tag{2.67}
\]

\[
R_{agh}(z_1)Q(z_2) = 0 \tag{2.68}
\]

\[
R_{gh}(z_1)\bar{Q}(z_2) = 0 \tag{2.69}
\]

\[
R_{agh}(z_1)\bar{Q}(z_2) = \frac{\bar{Q}(z_2)}{(z_1 - z_2)} \tag{2.70}
\]

\[
R_{gh}(z_1)G(z_2) = \frac{G(z_2)}{(z_1 - z_2)} \tag{2.71}
\]

\[
R_{agh}(z_1)\bar{G}(z_2) = \frac{\bar{G}(z_2)}{(z_1 - z_2)} \tag{2.72}
\]
\[ T(z_1)R_{gh}(z_2) = \frac{R_{gh}(z_2)}{(z_1 - z_2)^2} + \frac{\partial R_{gh}(z_2)}{(z_1 - z_2)} \tag{2.73} \]
\[ T(z_1)R_{agh}(z_2) = \frac{R_{agh}(z_2)}{(z_1 - z_2)^2} + \frac{\partial R_{agh}(z_2)}{(z_1 - z_2)} \tag{2.74} \]

The OPE of \( Q \) with \( \bar{G} \) and \( \bar{Q} \) with \( G \) yields two operators \( K \) and \( \bar{K} \) which were studied by Hwang\[^{[17]}\]. We find that

\[ Q(z_1)\bar{G}(z_2) = -\frac{\bar{K}(z_2)}{(z_1 - z_2)^2} \tag{2.75} \]
\[ \bar{Q}(z_1)G(z_2) = -\frac{K(z_2)}{(z_1 - z_2)^2} \tag{2.76} \]

where

\[ K = -\rho_a\bar{\gamma}_a - i\frac{f}{2}v_{abc}\bar{\gamma}_b\bar{\gamma}_c \tag{2.77} \]
\[ \bar{K} = -\bar{\rho}_a\gamma_a - i\frac{f}{2}v_{abc}\gamma_b\gamma_c \tag{2.78} \]

In \[^{[17]}\], Hwang showed that \( K \) is a solution to

\[ [K, [K, Q]] = 0 \tag{2.79} \]

and is of the same form as (2.77). It is easy to check that

\[ Q(z_1)K(z_2) = \frac{\bar{Q}(z_2)}{z_1 - z_2} \tag{2.80} \]

while

\[ \bar{Q}(z_1)\bar{K}(z_2) = \frac{Q(z_2)}{z_1 - z_2} \tag{2.81} \]

and

\[ Q(z_1)\bar{K}(z_2) = \bar{Q}(z_1)K(z_2) = 0 \tag{2.82} \]
\[ Q(z_1)K(z_2) = 0 \tag{2.83} \]

The operators \( K, \bar{K}, G \) and \( \bar{G} \) have the following OPE:

\[ K(z_1)G(z_2) = 0 \tag{2.84} \]
\[ \bar{K}(z_1)\bar{G}(z_2) = 0 \tag{2.85} \]
\[ K(z_1)\bar{G}(z_2) = \frac{G(z_2)}{z_1 - z_2} + \frac{V_{agh}}{(z_1 - z_2)^2} \tag{2.86} \]
\[ \bar{K}(z_1)G(z_2) = \frac{\bar{G}(z_2)}{z_1 - z_2} - \frac{V_{gh}(z_2)}{(z_1 - z_2)^2} \tag{2.87} \]

where

\[ V_{gh} = v_a\gamma_a \tag{2.88} \]
and
\[ V_{agh} = v_\alpha \bar{\gamma}_\alpha. \] (2.89)

We thus have all the elements of an \( N = 4 \) Superconformal Algebra. However due to the presence of the \( V_{gh} \) and similar terms, it becomes infinite dimensional as we shall see in the next section. We also have

\[
R_{agh}(z_1)G(z_2) = \frac{V_{agh}(z_2)}{(z_1 - z_2)^2} \quad (2.90)
\]

\[
R_{gh}(z_1)G(z_2) = -\frac{V_{gh}(z_2)}{(z_1 - z_2)} \quad (2.91)
\]

The only OPE where the central charge enters are as follows:

\[
R_{gh}(z_1)R_{gh}(z_2) = -\frac{d}{(z_1 - z_2)^2} \quad (2.92)
\]

\[
R(z_1)R(z_2) = -\frac{2d}{(z_1 - z_2)^2} \quad (2.93)
\]

and

\[
K(z_1)\bar{K}(z_2) = \frac{d}{(z_1 - z_2)^2} - \frac{R(z_2)}{z_1 - z_2} \quad (2.94)
\]

\[
Q(z_1)V_{agh}(z_2) = \frac{R_{agh}(z_2)}{z_1 - z_2} + \frac{d}{(z_1 - z_2)^2} \quad (2.95)
\]

\[
\bar{Q}(z_1)V_{gh}(z_2) = -\frac{R_{gh}(z_2)}{(z_1 - z_2)} + \frac{d}{(z_1 - z_2)^2} \quad (2.96)
\]

Also

\[
R_{gh}(z_1)R_{gh}(z_2) = R_{agh}(z_1)R_{agh}(z_2) = 0 \quad (2.97)
\]

The action of \( K \) and \( \bar{K} \) on \( R_{gh} \) and \( R_{agh} \) is as given by

\[
R_{gh}(z_1)K(z_2) = -\frac{K(z_2)}{(z_1 - z_2)} \quad (2.98)
\]

\[
R_{gh}(z_1)\bar{K}(z_2) = \frac{\bar{K}(z_2)}{z_1 - z_2} \quad (2.99)
\]

\[
R_{agh}(z_1)K(z_2) = \frac{K(z_2)}{z_1 - z_2} \quad (2.100)
\]

\[
R_{agh}(z_1)\bar{K}(z_2) = -\frac{\bar{K}(z_2)}{z_1 - z_2} \quad (2.101)
\]

while

\[
K(z_1)K(z_2) = \bar{K}(z_1)\bar{K}(z_2) = 0 \quad (2.102)
\]

We note that the operators \((K, \bar{K}, -R)\) form an \( SU(2) \) algebra with \( \bar{K} \to T^+, \, K \to T^- \), and \( R \to T^3 \). However, \( K \) and \( \bar{K} \) also behave as raising (lowering) operators wrt \( R_{agh}(R_{gh}) \).
respectively. This has interesting consequences in grouping the various terms of the algebra into multiplets wrt $K$, $\bar{K}$ as we shall see in Sec. 4. The OPE of $T$ with $K$ and $\bar{K}$ is as expected according to the rules of conformal field theory:

\[ T(z_1)K(z_2) = \frac{K(z_2)}{(z_1 - z_2)^2} + \frac{\partial K(z_2)}{z_1 - z_2} \]  
\[ T(z_1)\bar{K}(z_2) = \frac{\bar{K}(z_2)}{(z_1 - z_2)^2} + \frac{\partial \bar{K}(z_2)}{z_1 - z_2} \]  

(2.103) (2.104)

### 3 Proliferation of terms in the Algebra

We now see how the algebra proliferates, actually becoming infinite dimensional. Consider

\[ G(z_1)V_{gh}(z_1) = G(z_1)V_{agh}(z_2) = \frac{J_V(z_2)}{z_1 - z_2} \]  

(3.1)

where

\[ J_V(z_1) = \frac{v_a J_a}{c_A} \]  

(3.2)

Other operators of dimension 2 arise as follows. Consider

\[ G(z_1)J_V(z_1) = \frac{P_{gh}(z_2)}{(z_1 - z_2)^2} + \frac{H(z_2)}{z_1 - z_2} \]  

(3.3)

where

\[ P_{gh} = \frac{v_a \rho_a}{c_A} \]  
\[ H = \frac{v_a \rho_a}{c_A} + if_{abc} \frac{v_a \rho_b J_c}{c_A^2} \]  

(3.4) (3.5)

and

\[ \bar{G}(z_1)J_V(z_2) = \frac{P_{agh}(z_2)}{(z_1 - z_2)^2} + \frac{\bar{H}(z_2)}{z_1 - z_2} \]  

(3.6)

where

\[ P_{agh} = \frac{v_a \rho_a}{c_A} \]  
\[ \bar{H} = v_a \partial R_a + if_{abc} \frac{v_a R_b J_c}{c_A^2} \]  

(3.7) (3.8)

Other operators of dimension two, not involving derivatives, arise in the OPE of the following:

\[ V_{gh}(z_1)P_{gh}(z_2) = -\frac{\mathcal{V}(z_2)}{z_1 - z_2} \]  
\[ V_{agh}(z_1)P_{agh}(z_2) = -\frac{\mathcal{V}(z_2)}{z_1 - z_2} \]  
\[ G(z_1)\bar{G}(z_2) = \frac{W''(z_2) - J_V(z_2)}{(z_1 - z_2)^2} - \frac{\partial J_V(z_2)}{z_1 - z_2} + \frac{W'(z_2)}{z_1 - z_2} \]  

(3.9) (3.10) (3.11)
where
\[ W'' = \frac{R_a \rho_a}{c_A} \]  
\[ \mathcal{V} = \frac{v_a v_a}{c_A} \]  
and
\[ W' = \frac{R_a \partial \rho_a}{c_A} + if_{abc} \frac{R_a \rho_b J_c}{c_A} \]  
is a dimension 3 operator. We also note as in [8] that
\[ G(z_1)G(z_2) = \frac{W(z_2)}{z_1 - z_2} \]  
where
\[ W = \frac{\rho_a \partial \rho_a}{c_A} + if_{abc} \frac{\rho_a \rho_b J_c}{c^2_A} \]  
and
\[ \tilde{G}(z_1)\tilde{G}(z_2) = \frac{\tilde{W}(z_2)}{z_1 - z_2} \]  
where
\[ \tilde{W} = \frac{R_a \partial R_a}{c_A} + if_{abc} \frac{R_a R_b J_c}{c^2_A} \]  
Remarkably all the Ws and \( \tilde{W} \)s are closed wrt \( Q \) and \( \tilde{Q} \), i.e.
\[ Q(z_1)W(z_2) = Q(z_1)\tilde{W}(z_2) = Q(z_1)W'(z_2) = 0 \]  
\[ \bar{Q}(z_1)W(z_2) = \bar{Q}(z_2)\bar{W}(z_2) = Q(z_1)W'(z_2) = 0 \]  
We also have the terms \( V, \tilde{V} \), which arise as follows:
\[ Q(z_1)V(z_2) = \frac{W(z_1)}{z_1 - z_2} \]  
\[ \bar{Q}(z_1)\bar{V}(z_2) = \frac{\bar{W}(z_2)}{z_1 - z_2} \]  
where
\[ V = if_{abc} \frac{\rho_a \rho_b \rho_c}{c^2_A} \]  
\[ \tilde{V} = if_{abc} \frac{\bar{\rho}_a \bar{\rho}_b \bar{\rho}_c}{c^2_A} \]  
According to expectation, we also see that
\[ G(z_1)W(z_2) = \frac{V(z_2)}{(z_1 - z_2)^2} + \frac{\partial V(z_2)}{z_1 - z_2} \]  
\[ \bar{G}(z_1)\bar{W}(z_2) = \frac{\bar{V}(z_2)}{(z_1 - z_2)^2} + \frac{\partial \bar{V}(z_2)}{z_1 - z_2} \]
We note also that $P_{agh}$ and $P_{gh}$ are closed wrt $Q(\bar{Q})$ respectively, while

$$Q(z_1)P_{gh}(z_2) = \frac{W''(z_1) + J_V(z_2)}{z_1 - z_2} = \bar{Q}(z_1)P_{agh}(z_2)$$  \hfill (3.27)

and

$$Q(z_1)(W''(z_2) + J_V(z_2)) = \bar{Q}(z_1)(W''(z_2) + J_V(z_2)) = 0$$ \hfill (3.28)

$$G(z_1)(W''(z_2) + J_V(z_2)) = \frac{2P_{gh}(z_2)}{(z_1 - z_2)^2} + \frac{\partial P_{gh}(z_2)}{z_1 - z_2}$$ \hfill (3.29)

$$\bar{G}(z_1)(W''(z_2) + J_V(z_2)) = \frac{2P_{agh}(z_2)}{(z_1 - z_2)^2} + \frac{\partial P_{gh}(z_2)}{z_1 - z_2}$$ \hfill (3.30)

Other operators of dimension 3 arise as follows. Having noted that $K(\bar{K})$ are lowering(raising) operators wrt $R_{gh}$, we note that

$$K(z_1)V(z_2) = \frac{3V_{1/2}(z_2)}{z_1 - z_2} - \frac{P_{agh}(z_2)}{(z_1 - z_2)^2} + \frac{\partial P_{agh}(z_2)}{z_1 - z_2} - \frac{H_1(z_2)}{z_1 - z_2}$$ \hfill (3.31)

where

$$V_{1/2} = i f_{abc} \frac{R_a R_b \rho_c}{3c_A^2}$$ \hfill (3.32)

and

$$H_1 = \frac{\nu_a \partial R_a}{c_A}$$ \hfill (3.33)

Also

$$\bar{K}(z_1)V(z_2) = \frac{3V_{-1/2}(z_2)}{z_1 - z_2} - \frac{P_{gh}(z_2)}{(z_1 - z_2)^2} + \frac{\partial P_{gh}(z_2)}{z_1 - z_2} - \frac{H_1(z_2)}{z_1 - z_2}$$ \hfill (3.34)

where

$$V_{-1/2} = i f_{abc} \frac{R_a \rho_b \rho_c}{3c_A^2}$$ \hfill (3.35)

and

$$H_1 = \frac{\nu_a \partial \rho_a}{c_A}$$ \hfill (3.36)

More dimension 3 operators are obtained if we consider the coefficients of the quadratic terms in $(z_1 - z_2)^{-1}$ in the following:

$$J_V(z_1)\bar{W}(z_2) = -\frac{X_1(z_2)}{(z_1 - z_2)^2}$$ \hfill (3.37)

$$J_V(z_1)W'(z_2) = -\frac{X_0(z_2)}{(z_1 - z_2)^2}$$ \hfill (3.38)

$$J_V(z_1)W(z_2) = -\frac{X_{-1}(z_2)}{(z_1 - z_2)^2}$$ \hfill (3.39)
where

\[ X_1 = \frac{if_{abc} v_a R_b R_c}{c_A^2} \]  
\[ X_0 = \frac{if_{abc} v_a R_b \rho_c}{c_A^2} \]  
\[ X_{-1} = \frac{if_{abc} v_a \rho_b \rho_c}{c_A^2} \]  

Before we proceed further, we see first that the algebra does not close and we end up with operators of higher and higher dimension. We take the example of the following OPE:

\[ J_V(z_1)H(z_2) = -if_{abc} \partial v_a v_b \rho_c (z_2) + if_{abc} \rho_b \rho_c J_d(z_2) \]  
\[ = \frac{X_{1/2}(z_2)}{z_1 - z_2} \]  

We note that \( X_{1/2} \) has dimension 4. If we take the OPE of \( J_V \) with \( X_{1/2} \) again, we get an operator of dimension 5 due to the OPE of the \( J \)'s with each other. This process of taking the OPE of the resulting operator with \( J_V \) can be repeated ad infinitum and each time we get an operator of the next higher dimension due to the fact that the OPE of a \( J \) with a \( J \) gives another \( J \). Hence the algebra does not close and becomes infinite dimensional.

### 3.1 Form of Terms in Higher Dimensions

What form do the operators in higher dimension take? To find out, we consider the OPE of \( X_{1/2} \) with \( J_V \). We get

\[ J_V(z_1)X_{1/2}(z_2) = \frac{X'_{1/2}(z_2)}{z_1 - z_2} + \frac{Y'_{1/2}(z_2)}{z_1 - z_2} \]  

where

\[ X'_{-1/2} = \frac{f_{abc} f_{dec} \partial v_e v_d v_a \rho_b}{c_A^3} \]  
\[ Y'_{-1/2} = \frac{if_{abc} f_{dec} f_{dg f} v_a v_e v_d \rho_b J_g}{c_A^4} \]  

If the group is \( SU(2) \), \( if_{ijk} = \epsilon_{ijk} \) and so, after a little calculation, it turns out that

\[ X'_{-1/2} = \frac{1}{c_A} (P_{gh} \partial V - V(\partial P_{gh} - H)) \]  

while,

\[ Y'_{-1/2} = \frac{1}{c_A} \left( V (H - H_1) \right) \]  

We also note from (3.46) and (3.47) that as we go on to operators of higher and higher dimension, we obtain operators which are tensors of the algebra of successively higher rank. When the highest rank of the algebra has been exhausted, the operators tend to take the form of products of these tensors. As this part of the investigation is highly algebra dependent, we do not pursue it further.
Table 2: Operators in Each Dimension

| Dimension | Operator                                      |
|-----------|-----------------------------------------------|
| 1         | $Q, \bar{Q}, K, \bar{K}, R_{gh}, R_{agh}, V_{gh}, V_{agh}$ |
| 2         | $G, \bar{G}, T, J_Y, P_{gh}P_{agh}, \mathcal{V}, W''$ |
| 3         | $W, W', \bar{W}, H, \bar{H}, X_1, X_0, X_{-1}, V, \bar{V}, V_{1/2}, V_{-1/2}, H_1, \bar{H}_1$ |

3.2 Proliferation of Terms in Each Dimension

So far, we have seen that the algebra is infinite dimensional if one considers operators of all possible dimensions and is highly algebra dependent. However, the no. of operators proliferates in each dimension. Let us group the operators that we have obtained so far in each dimension. These are listed in Table 2.

3.2.1 Action of $K(\bar{K})$ on $\rho$ and $R$

We note the action of $K$ and $\bar{K}$ on $\rho_a$ and $R_a$. It is easy to check that

\[
K(z_1)\rho_a(z_2) = 0 \quad (3.50)
\]

\[
\bar{K}(z_1)R_a(z_2) = 0 \quad (3.51)
\]

\[
K(z_1)R_a(z_2) = \frac{\rho_a(z_2)}{z_1 - z_2} \quad (3.52)
\]

\[
\bar{K}(z_1)\rho_a(z_2) = \frac{R_a(z_2)}{z_1 - z_2} \quad (3.53)
\]

We also see that

\[
\bar{K}(z_1)(\rho_d(z_2)\rho_e(z_2)) = \frac{R_d(z_2)\rho_e(z_2) - R_e(z_2)\rho_d(z_2)}{z_1 - z_2} \quad (3.55)
\]

and

\[
K(z_1)(R_d(z_2)R_e(z_2)) = \frac{\rho_d(z_2)R_e(z_2) - \rho_eR_d(z_2)}{z_1 - z_2} \quad (3.56)
\]

while

\[
\bar{K}(z_1)\partial\rho_d(z_2) = \frac{R_d(z_2)}{(z_1 - z_2)^2} + \frac{\partial R_d(z_2)}{z_1 - z_2} \quad (3.57)
\]

and

\[
K(z_1)\partial R_d(z_2) = \frac{\rho_d(z_2)}{(z_1 - z_2)^2} + \frac{\partial \rho_d(z_2)}{z_1 - z_2} \quad (3.58)
\]

Also

\[
K(z_1)v_a(z_2) = \bar{K}(z_1)v_a(z_2) = 0 \quad (3.59)
\]

Hence all terms of the form $\rho_d\rho_e$ must be accompanied by $R_d\rho_e$ and $R_dR_e$, i.e. each $\rho$ is replaced by an $R$, these being generated by the action of $K$ and $\bar{K}$ on the above. Similarly
all terms of the form $\partial \rho_a$ must be accompanied by $\partial R_d$ (within the same dimension) as well as an extra $R_d$ (in a lower dimension). Finally, as

$$K(z_1)\gamma_a(z_2) = -\frac{\gamma_a(z_2)}{z_1 - z_2}$$

(3.60)

and

$$\bar{K}(z_1)\bar{\gamma}_a(z_2) = -\frac{\gamma_a(z_2)}{z_1 - z_2}$$

(3.61)

while

$$K(z_1)\gamma_a(z_2) = \bar{K}(z_1)\gamma_a(z_2) = 0$$

(3.62)

Hence all terms of the form $\gamma_a \gamma_b$ must be accompanied by $\bar{\gamma}_a \bar{\gamma}_b$ and $\gamma_a \bar{\gamma}_b$ in order to close the algebra.

### 3.2.2 Action of $V_{gh}(V_{agh})$ on $\rho(R)$:

Next, we consider the action of $V_{gh}$ ($V_{agh}$) on a term like $\rho_d \rho_e (R_d R_e)$. We get terms like $(v_d \rho_e - v_e \rho_d)(v_d R_e - v_e R_d)$ which is effectively replacing a $\rho(R)$ by its antisymmetric combination with $v$. Hence all terms of the form $\rho_d \rho_e (R_d R_e)$ are accompanied by terms of the form $v_d \rho_e (v_d R_e)$ in which a $\rho(R)$ is replaced by a $v$.

### 3.2.3 Action of $Q$

$Q$ does not respect the $\rho \leftrightarrow R$ symmetry as

$$Q(z_1)v_a(z_2) = \frac{\bar{\rho}_a(z_2)}{z_1 - z_2}$$

(3.63)

and not $\bar{R}_a$. Also

$$Q(z_1)\bar{\rho}_a(z_2) = 0$$

(3.64)

and

$$Q(z_1)\rho_a(z_2) = \frac{J_a(z_2)}{z_1 - z_2}$$

(3.65)

as seen in Eq. (and

$$Q(z_1)\partial \rho_a(z_2) = \frac{J_a(z_2)}{z_1 - z_2} + \frac{\partial J_a(z_2)}{z_1 - z_2}$$

(3.66)

implying that nonderivative terms in a higher dimension are given rise to by derivative terms in a higher dimension. Hence all operators in a given dimension cannot be obtained by the closure of the algebra in that dimension alone. We now see how the no. of terms escalates. Consider, for example,

$$Q(z_1)H_1(z_2) = \frac{J_V(z_2)}{(z_1 - z_2)^2} + \frac{v_d \partial J_d}{c_A(z_1 - z_2)} + \frac{\rho_a \partial \rho_a}{c_A(z_1 - z_2)} + \frac{if_{abc} v_a \gamma_b \gamma_c}{(z_1 - z_2)^2} + \frac{if_{abc} \partial (\rho_b \gamma_c) v_a}{z_1 - z_2}$$

(3.67)

Hence we end up with a term $f_{abc} v_a \gamma_b \rho_c$ in dimension 2 in addition to the new terms $v_d \partial J_d$, $\bar{\rho}_a \partial \rho_a$ and $if_{abc} \partial (\rho_b \gamma_c) v_b$. The action of $Q$ on $if_{abc} v_a \gamma_b \rho_c$ yields an additional term $v_a \partial \gamma_a$ in dimension 2. However, the action of $Q$ on $v_a \partial \gamma_a$ is as follows:

$$Q(z_1)(v_a \partial \gamma_a(z_2)) = \frac{R_a \partial \gamma_a}{z_1 - z_2} + \frac{if_{abc} v_a \gamma_b \gamma_c}{(z_1 - z_2)^2}$$

(3.68)
so that we end up with an additional term in dimension 1, i.e. \( f_{abc}v_a\gamma_b\gamma_c \). We have not been able to find a general way of inducing all terms, however, a list of some of the terms in dimension 1 is presented in Table 3, accompanied by a similar list of dimension 2 terms in Table 4. Both lists are, by no means exhaustive. A complete analysis and classification of such terms is outside the scope of the present paper.

4 Analysis of our Results

1. In our analysis of operators, we would like to introduce the terms, operators of order one, two, three and so on. Operators of order one are generated by the action of the operators \( Q(\bar{Q}) \) on operators of the same dimension or lower. Operators of order two are generated by the action of \( Q(\bar{Q}) \) on operators of one higher dimension. Operators of order three are generated by the action of \( Q(\bar{Q}) \) on operators of dimension three. The list continues. We confine our discussion only to operators of order 1 with dimensions up to three only. It is interesting to group these operators into multiplets of \((K, \bar{K}, R_{gh})\). This is done in Table 5. The BRST structure of some of these operators is illustrated in Table 6. Their anti BRST structure is illustrated in Table 7.

The doublet structure of the BRST and anti BRST multiplets (i.e. absence of BRST and anti BRST invariant, non-exact operators) is highlighted as is the duality between the BRST and anti BRST structures, in spite of the vastly differing form of the BRST and anti BRST operators.

2. We now look at the algebra of the operators with \( v = P = 0 \). We end up with the operators given in Table 8. In this case, \( R_a = \bar{\rho}_a \) and the algebra is seen to truncate.
Table 5: Multiplets of \((K, \bar{K}, R_{gh})\)

| Dimension | Singlets | Doublets | Triplets | Quadruplets |
|-----------|----------|----------|----------|-------------|
| 1         | \(Q, \bar{Q}\) | \(K, R_{gh}, \bar{K}\) | \(V_{gh}, V_{agh}\) | \(\bar{K}, R_{agh}, K\) |
| 2         | \(W'', J_V\) | \(G, \bar{G}\) | \(P_{gh}, P_{agh}\) | \(H, \bar{H}\) |
| 3         | \(H, \bar{H}\) | \(W, W'', \bar{V}\) | \(V, V_{-1/2}, V_{1/2}, V\) | \(H_1, \bar{H}_1\) |

Table 6: BRST structure of the Operators

| Dimension | Doublets |
|-----------|----------|
| 1         | \((Q, R_{gh})\) \( (R_{agh}, V_{agh})\) \( (\bar{Q}, K)\) \( (\bar{K}, V_{gh})\) |
| 2         | \((W'' + J_V, P_{gh})\) \( (P_{agh}, V)\) \( (\bar{G} + V_{gh}, J_V)\) \( (G \bar{V}_{gh}, W'')\) \( (T + R_{gh}, G)\) |
| 3         | \((W, V)\) \( (W', H)\) \( (\bar{W}, \bar{H})\) \( (\bar{V}, X_1)\) |

Table 7: Anti BRST structure of the Operators

| Dimension | Doublets |
|-----------|----------|
| 1         | \((Q, R_{agh})\) \( (R_{gh}, V_{gh})\) \( (Q, \bar{K})\) \( (K, V_{agh})\) |
| 2         | \((W'' + J_V, P_{agh})\) \( (P_{gh}, V)\) \( (-G + V_{agh}, J_V)\) \( (G - V_{gh}, W'')\) \( (T + R_{agh}, \bar{G})\) |
| 3         | \((\bar{W}, \bar{V})\) \( (-W' + \partial W'' - \partial J_V, \bar{H})\) \( (-W, H)\) \( (V, X_{-1})\) |
Table 8: Operators with $v = P = 0$

| Dimension | Operators |
|-----------|-----------|
| 1         | $Q = -\gamma_a J_a - \frac{if_{abc}}{c_A} \gamma_a \gamma_b \rho_c$  
$\bar{Q} = -\bar{\gamma}_a J_a - \frac{if_{abc}}{2} \bar{\gamma}_a \bar{\gamma}_b \rho_c - if_{abc} \bar{\gamma}_a \gamma_b \rho_c$  
$R_{gh} = \rho_a \gamma_a$  
$R_{agh} = \bar{\rho}_a \bar{\gamma}_a$ |
| 2         | $T = \rho_a \partial \gamma_a + \bar{\rho}_a \partial \bar{\gamma}_a - \frac{J_a J_a}{c_A}$  
$G = -\frac{\rho_a J_a}{c_A}$  
$\bar{G} = -\frac{\bar{\rho}_a J_a}{c_A}$  
$W'' = \bar{\rho}_a \rho_a$ |
| 3         | $W = \frac{\rho_a \partial \bar{\rho}_a}{c_A} + if_{abc} \frac{\rho_a \rho_b J_a}{c_A}$  
$W' = \frac{\bar{\rho}_a \partial \rho_a}{c_A} + if_{abc} \frac{\bar{\rho}_a \bar{\rho}_b J_a}{c_A}$  
$\bar{W} = \frac{\bar{\rho}_a \partial \rho_a}{c_A} + if_{abc} \frac{\rho_a \bar{\rho}_b J_a}{c_A}$  
$V = if_{abc} \frac{\rho_a \rho_b \bar{\rho}_a}{3c_A}$  
$V_{-1/2} = if_{abc} \frac{\bar{\rho}_a \rho_b \bar{\rho}_a}{3c_A}$  
$V_{1/2} = if_{abc} \frac{\rho_a \bar{\rho}_b \bar{\rho}_a}{3c_A}$  
$V = if_{abc} \frac{\bar{\rho}_a \rho_b \bar{\rho}_a}{3c_A}$ |
| 4         | $X_{-1/2} = if_{abc} \frac{\partial \rho_a \rho_b \bar{\rho}_a}{c_A} + \frac{f_{def} f_{afg} \frac{\rho_a \rho_b \bar{\rho}_a J_a}{c_A}}{c_A}$  
$X_{1/2} = if_{abc} \frac{\partial \rho_a \rho_b \bar{\rho}_a}{c_A} + \frac{f_{def} f_{afg} \frac{\rho_a \rho_b \bar{\rho}_a J_a}{c_A}}{c_A}$ |

at dimension 4. $K$ and $\bar{K}$ do not act any more as SU(2) raising and lowering operators and all relevance to antighosts drop out of the BRST operator when $P_a$ is put to zero. The central term reappears in the OPE of $T$ with $T$ as follows:

$$T(z_1)T(z_2) = \frac{-d}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{z_1 - z_2}$$ (4.1)

The OPE of $Q$ with $G$ does not yield the total energy momentum tensor and the central term reappears as follows:

$$Q(z_1)G(z_2) = \frac{d}{(z_1 - z_2)^3} - \frac{J_a J_a(z_2)}{c_A(z_1 - z_2)} + \frac{\rho_a \partial \gamma_a(z_2)}{z_1 - z_2} + \frac{\rho_a \gamma_a(z_2)}{(z_1 - z_2)^2}$$ (4.2)

Clearly if antighosts are to be introduced, the Lagrange multipliers cannot be discarded as they enter into the BRST operator via the term $P_a \rho_a$. If all antighosts are put equal to zero along with the $v$’s and the $P$’s, we recover the algebra of $[\bar{8}]$.

3. It is instructive to compare the above algebra with that of $[\bar{8}]$ and $[\bar{5}]$. In Ref. $[\bar{4}]$, the OPE of $G$ with $G$ is zero, while in $[\bar{8}]$ it is not, being given by Eq.(3.15). however, if one considers the complete expression for the operator $G$ in $[\bar{8}]$, as given in Eq. (2.50), we note that it forms a representation for the SU(2) triplet of operators $(K, \bar{K}, R_{gh})$. This is not so for the operator $G$ in Ref. $[\bar{4}]$, because the term $\frac{1}{2} f_{ijk} \rho_i \rho_j \gamma_k$ transforms differently from the others under the action of $K(\bar{K})$. We were unable to find an expression for $G(\bar{G})$: $(G\bar{G})$ was a multiplet of $(K\bar{K}R_{gh})$.  

18
and $GG = GG = 0$. In that case, the algebra would truncate and many terms would drop out.

4. We now take a look at the BRST and anti BRST currents and how they transform under the action of the principal operators in the algebra i.e. $(Q, G, \bar{Q}, \bar{G}, T, K, \bar{K}, R_{gh}, R_{agh})$. After a little calculation it can be shown that

$$Q(z_1)\rho_a(z_2) = \frac{\mathcal{J}_a(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.3)  
$$Q(z_1)\mathcal{J}_a(z_2) = 0$$  \hspace{1cm} (4.4)  
$$G(z_1)\rho_a(z_2) = 0$$  \hspace{1cm} (4.5)  
$$G(z_1)\mathcal{J}_a(z_2) = \frac{\rho_a(z_2)}{(z_1 - z_2)^2} + \frac{\partial\rho_a(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.6)  
$$\bar{Q}(z_1)\bar{\rho}_a(z_2) = \frac{\mathcal{T}_a(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.7)  
$$\bar{Q}(z_1)\mathcal{T}_a(z_2) = 0$$  \hspace{1cm} (4.8)  
$$\bar{G}(z_1)\bar{\rho}_a(z_2) = 0$$  \hspace{1cm} (4.9)  
$$\bar{G}(z_1)\mathcal{T}_a(z_2) = \frac{\bar{\rho}_a(z_2)}{(z_1 - z_2)^2} + \frac{\partial\bar{\rho}_a(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.10)  
$$Q(z_1)\bar{\rho}_a(z_2) = 0$$  \hspace{1cm} (4.11)  
$$Q(z_1)\mathcal{T}_a(z_2) = 0$$  \hspace{1cm} (4.12)  
$$G(z_1)\bar{\rho}_a(z_2) = \frac{\gamma_c(z_2)}{(z_1 - z_2)^2} + \frac{\partial\gamma_c(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.13)  
$$G(z_1)\mathcal{T}_a(z_2) = \frac{\bar{\mathcal{R}}_a(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.14)  
$$\bar{Q}(z_1)\rho_d(z_2) = \frac{i f_{dac} \bar{\gamma}_a(z_2) \bar{\rho}_c(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.15)  
$$\bar{Q}(z_1)\mathcal{J}_d(z_2) = \frac{i f_{dac} \bar{\gamma}_a(z_2) \mathcal{J}_c(z_2)}{z_1 - z_2} + \frac{i f_{dac} \rho_a(z_2) \mathcal{P}_c(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.16)  
$$\bar{G}(z_1)\rho_d(z_2) = \frac{i f_{dac} \mathcal{J}_a(z_2) \gamma_c(z_2)}{z_1 - z_2} - \frac{\nu_d(z_2)}{(z_1 - z_2)^2} - \frac{\partial\nu_d(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.17)  
$$\bar{G}(z_1)\mathcal{J}_d(z_2) = \frac{i f_{dac} \bar{\rho}_d(z_2)}{(z_1 - z_2)^2} + \frac{\partial\bar{\mathcal{R}}_d(z_2)}{z_1 - z_2} + \frac{i f_{dac} \bar{\mathcal{R}}_a(z_2) \mathcal{J}_c(z_2)}{z_1 - z_2} - \frac{i f_{dac} \partial\nu_a(z_2) \gamma_c(z_2)}{z_1 - z_2}$$  \hspace{1cm} (4.18)  

where

$$\bar{\mathcal{R}}_a = \rho_a + i f_{abc} \nu_b \bar{\gamma}_c$$  \hspace{1cm} (4.19)  

We note that the topological BRST charges $(Q, G)$ and the topological currents $(\rho_a, \mathcal{J}_a)$ are compatible, as are the topological anti BRST charges $(\bar{Q}, \bar{G})$ and the topological anti BRST currents $(\bar{\rho}_a, \mathcal{T}_a)$. The topological anti BRST currents are invariant under the action of the topological BRST charges. However, this does not hold for the BRST currents under the action of the topological anti BRST charges.
as is also demonstrated by the action of $\bar{G}$ on $J_a$ which has terms in both $\bar{\rho}_a$ and $R_a$, in contrast with eq.(4.14) where only $\bar{R}_a$ enters. The reason is probably due to the fact that the anti BRST current is more complete with more terms in order to be compatible with the BRST transformation.

5. We close this section with a comment on the existence of two independent currents that close an affine algebra with levels $k_1$ and $k_2$ respectively as in Ref.\[8\]. The total current $J_a = J^1_a + J^2_a + i\bar{\gamma}_a \rho_a \gamma_a$ and if we assume that $J^1_a$ and $J^2_a$ transform under the BRST symmetry as in Eq.(2.13), we end up as in Ref.\[8\] with the following condition for the sum of the levels $k_1$ and $k_2$:

$$k_1 + k_2 = c_A$$

(4.20)

As in [8], only the sum of the current algebra levels is constrained. Similarly, using Eq.(2.29), we see that the anti BRST variation of the currents restricts the sum of the levels $k_1 + k_2$ to

$$k_1 + k_2 = -c_A$$

(4.21)

in order to achieve nilpotency for the anti BRST transformation. As in Ref. \[8\], denoting $k_1$ by $k$, we get

$$T = \frac{1}{2k + c_A} (J^1_a J^1_a - J^2_a J^2_a) + \rho_a \partial \gamma_a + \bar{\rho}_a \partial \bar{\gamma}_a + v_a \partial P_a$$

(4.22)

where we applied the BRST transformation on

$$G = \frac{1}{2k + c_A} \left[ \rho_a (J^1_a - J^2_a) : + i f_{abc} : v_b \gamma_c (J^1_a - J^2_a) - v_a \partial \gamma_a \right]$$

(4.23)

thus seeing that $T$ is both BRST and anti BRST exact. The currents $J^1_a$ and $J^2_a$ contribute $\frac{2w_{dim}}{2k + c_A}$ and $-\frac{2k + 2c_A}{2k + c_A} \times \text{dim}$ respectively whose sum exactly cancels the central charge of the ghost system. We see that the system offers a description of the gauged WZNW model. As noted in [8], the construction cannot be extended to three or more currents.

5 Discussion and Conclusion

We thus see that the full topological algebra involving ghosts, antighosts and free multipliers becomes indefinitely large for each operator dimension. This is possibly the price one pays for the disappearance of the central term in the OPE of $Q$ with $G$ and $\bar{Q}$ with $\bar{G}$. We still have an $N = 4$ Superconformal Algebra, albeit infinite dimensional. The ghost no.(antighost no.) charges $R_{gh}(R_{agh})$ have been redefined from $\rho_a \gamma_a$ to $\rho_a \gamma_a + P_a v_a$ and $\bar{\rho}_a \bar{\gamma}_a$ to $\bar{\rho}_a \bar{\gamma}_a + P_a v_a$. The $P_a v_a$ term acts as a sort of gauge transformation term. The central term of the affine algebra arises only in the OPE of the $SU(2)$ charges $(K, \bar{K}, R_{gh}, R_{agh})$ with their opposites. The doublet structure of the operators under the action of the BRST and the anti BRST charges is also noted. It is also pointed out that the algebra is finite when the parameters $v_a$ and $P_a$ are put to zero. However, the
central term reappears in the OPE of $T$ with $T$ and that of $Q$ with $G$ and $\bar{Q}$ with $\bar{G}$. Moreover, the energy momentum tensor appearing in the OPE of $Q$ with $G$ and $\bar{Q}$ with $\bar{G}$ is not complete, since the antighost(ghost) contribution is missing. As seen in the last section, the algebra of the topological BRST and the anti BRST charges and the topological BRST and the anti BRST currents $(Q, G, \rho_a, J_a)$ and $(\bar{Q}, \bar{G}, \bar{\rho}_a, \bar{T}_a)$ are compatible. The anti BRST current is also invariant under the action of the BRST charge, however, the same does not hold for the action of the anti BRST charge on the BRST current. The two sets of algebras are thus not compatible with each other. Finally, we note that for two independent affine currents $J_1^a$ and $J_2^a$ with levels $k_1$ and $k_2$ : $k_1 + k_2 = c_A$, the above construction holds true, as the BRST charge contains the sum $(J_1^a + J_2^a)$. Results for the topological algebra obtained above can thus be applied to the gauged WZNW model.

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