Entanglement from the vacuum

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We explore the entanglement of the vacuum of a relativistic field by letting a pair of causally disconnected probes interact with the field. We find that, even when the probes are initially non-entangled, they can wind up to a final entangled state. This shows that entanglement persists between disconnected regions in the vacuum. However the probe entanglement, unlike correlations, vanishes once the regions become sufficiently separated. The relation between entropy, correlations and entanglement is discussed.

Keywords: entanglement, entropy, vacuum state, entanglement probes.
1 Introduction

The Hilbert space of two subsystems contains a subclass of entangled states that manifest unique quantum mechanical properties. As was highlighted by Bell [1], the correlations between observables measured separately on each subsystem can be “stronger” than the correlations predicted by any local classical models.

Having the causal structure and locality built in, relativistic field theory offers a natural framework for investigating entanglement. Here we will consider the entanglement of the relativistic vacuum state, which as we shall shortly recall, has a role in both the Hawking black-hole radiation [2] and Unruh acceleration radiation effects [3]. It is known that field observables at space-like separated points in vacuum are correlated. For massless fields in 3+1-D these correlations decay with the distance, $L$, between two points as $1/L^2$. These correlations by themselves however do not imply the existence of quantum entanglement, because they can in principle arise as classical correlations. However, a number of studies provide evidence that the vacuum is indeed entangled [3][4][5]. In the Rindler quantization, one spans the Hilbert space of a free field by direct products of Rindler particle number states $|n, 1\rangle$ and $|n, 2\rangle$ with non-vanishing support confined within the two complementary space-like separated wedges $x < -|t|$ and $x > t$, respectively. It then turns out [3] that the Minkowski vacuum state can be expressed as an entangled Einstein-Podolsky-Rosen (EPR) like state $\sim \sum_n \alpha^n |n, 1\rangle |n, 2\rangle$ for each mode. However does entanglement persist when the regions are separated by a finite distance? We will examine this question, but also emphasize that in this case of separated regions the relation between entanglement and entropy breaks down.
In a somewhat different framework, of algebraic quantum field theory, it has been argued that indeed local field observables in arbitrary two space-like separated regions are entangled. However this method as well as that used in Ref. assumes exact analyticity and cannot be applied in the presence of a cutoff.

In the present work we consider a gedanken-experiment for probing entanglement which is not sensitive to a short scale cutoff. It involves a pair of probes, point-like two-level systems, which couple for a finite duration with the field. The process takes place in two causally disconnected regions. Since the probes are taken to be initially non-entangled, and since entanglement cannot be produced locally, we will regard the presence of entanglement in the final state of the probes as a (lower bound) measure for vacuum entanglement.

2 Entropy, Correlations and Entanglement

We begin with a short review of the relation between entropy, correlations and entanglement. Consider a division of a system into two sets of commuting degrees of freedom whose combined Hilbert space can be described by the direct product of Hilbert spaces $\mathcal{H}_1 \times \mathcal{H}_2$. The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ that act on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively and thus commute: $[\mathcal{O}_1, \mathcal{O}_2] = 0$. In a relativistic theory, the division of space to two space-like separated regions, implies by causality commutativity of local observables, and the above Hilbert space structure follows.

We can then distinguish between the two different cases: a) The system is in a pure state. and b) the system is in a mixture of pure states, described by a density operator. The latter situation can arise, for instance, if our system constitutes a
sub-system of a larger system whose state is pure. This will indeed be the case in our model, as well as that for two separated regions in vacuum which do not cover the full space.

Consider first pure states, and for simplicity take a pair of two level (spin-half) systems. The Hilbert space contains pure states like

\[ |\psi\rangle = a|\uparrow\rangle_1|\uparrow\rangle_2 + b|\downarrow\rangle_1|\downarrow\rangle_2 \]

which we call entangled whenever both \(a\) and \(b\) are non-zero. More generally, a state is entangled if no local unitary transformation can convert the state into a single direct product like \(|\psi_1\rangle|\psi_2\rangle\). We observe that entanglement exists if and only if there are correlations between local observables:

\[ \langle \sigma_1 \sigma_2 \rangle \neq \langle \sigma_1 \rangle \langle \sigma_2 \rangle \]

But are these correlations classical or quantum? This question was answered by Bell and subsequent work. It was shown that for any entangled pure state, one can construct an inequality involving two operator correlations, which are satisfied in a local classical model but are violated by quantum mechanics.

The existence of correlations means that some information is stored in the combined state and cannot be traced by inspecting one half of the system. A subsystem then behaves as a mixture. The von-Neumann (or Shannon) entropy

\[ S_1 = -Tr_1 \rho \ln \rho \]

where \( \rho = Tr_2 |\psi\rangle\langle\psi| \), indicates this lack of knowledge when probing a sub-system. For pure states, \( S_1 = S_2 \) is non-vanishing if and only if the state is entangled. It therefore comes to us as no surprise that entropy can be viewed as a quantitative measure of entanglement. The surprise is perhaps that for an ensemble of identical states, it is in fact a unique measure of entanglement [6, 7].
We see that for the case of pure states, entropy, correlations, entanglement (and in a qualitative sense Bell inequalities), are equivalent descriptions of the same physical phenomena.

The situation differs dramatically in the mixed case. First how do we define entanglement of a mixed state? We will define entanglement by saying when a density operator is not entangled. A density operator, $\rho_{12}$, is not entangled if we can find a basis which entails a separable form:

$$\rho_{12} = \sum_i p_i \rho_{1i} \rho_{2i}$$

(3)

with $\rho_1$ and $\rho_2$ as local density operators, and $p_i > 0 \sum p_i = 1$. We note that $\rho_{12}$ does exhibit non-trivial correlations since we have $\langle O_1 O_2 \rangle = \sum p_i \langle O_1 \rangle_i \langle O_2 \rangle_i$, but nevertheless the density operator is not entangled, and describes classical correlations. Likewise, we note that the entropy function, $S = -\sum p_i \ln p_i$, does not vanish for a non-entangled mixed density operator.

To exemplify this, consider the following class of density operators:

$$\rho = \frac{1-x}{4} I + x |\psi \rangle \langle \psi|$$

(4)

where $|\psi \rangle = 1/\sqrt{2}(| \uparrow \rangle | \downarrow \rangle - | \downarrow \rangle | \uparrow \rangle)$ is the EPR-Bohm state. For any $x \neq 0$ the above density exhibits correlations. However it can be shown that only for $x > 1/3$, it is entangled. It is also interesting to note that, the simple relation between non-locality described by Bell inequalities and entanglement breaks down. In the above case for $x < 1/\sqrt{2}$ there is no violation, though the state is entangled.

We see that for the mixed case entropy and correlations are no longer equivalent to entanglement. Thus, correlations do not necessarily imply entanglement, and the
von-Neumann entropy is not an appropriate measure of entanglement. We will further discuss a possible relation of this issue with the entanglement interpretation [9] of the Bekenstein black-hole entropy [10] in the last section.

3 Probing Vacuum Entanglement

To model the field we shall consider a massless relativistic scalar field $\phi(\vec{x}, t)$ in 3 spatial dimensions, driven by the usual massless free Hamiltonian. We assume that initially the field is in its ground state. The probe systems will be modelled by a pair of localized (infinitely massive) two-level systems with energy gap $\Omega$. The interaction between the systems and the field will be time dependant but otherwise linear

$$H_{\text{int}} = H_A + H_B$$

$$= \epsilon_A(\tau)(e^{-i\Omega \tau} \sigma_A^+ + e^{+i\Omega \tau} \sigma_A^-)\phi(x_A(\tau), t)$$

$$+ \epsilon_B(\tau')(e^{-i\Omega' \tau'} \sigma_B^+ + e^{+i\Omega' \tau'} \sigma_B^-)\phi(x_B(\tau'), t)$$

(5)

where $\tau$ and $\tau'$ denotes the proper time of the probes. We shall take the coupling functions $\epsilon_A(\tau)$ and $\epsilon_B(\tau)$, as non-zero during a finite time $T$ to keep the two probes causally disconnected.

entanglement. for a time

The simplest set up which takes care of this causality restriction involves a pair of uniformly accelerated detectors that follow the trajectories

$$x_A = -L/2 \cosh(2\tau/L) \quad t_A = L/2 \sinh(2\tau/l)$$

(6)

$$x_B = L/2 \cosh(2\tau'/L) \quad t_B = L/2 \sinh(2\tau'/l)$$

(7)
Obviously the probes are confined to two causally disconnected regions; A is confined
to the space-time region $-x < |t|$ and $B$ to $x < |t|$.

Since the interaction takes place in two causally disconnected disconnected re-
gions, the field operators in $H_A$ and $H_B$ commute, and $[H_A, H_B] = 0$ Therefore, the
evolution operator for the system factorizes, and may be expressed in the interaction picture as a direct product

$$U = e^{-i \int (H_A(\tau) + H_B(\tau'))d\tau d\tau'} = e^{-i \int H_A(\tau)d\tau} \times e^{-i \int H_B(\tau')d\tau'}$$

(8)

We note that this ensures that $U$ does not generate entanglement between degrees of
freedom at the two causally disconnected regions $x > |t|$ and $x < -|t|$.

Suppose that the probes are initially in their ground states. Hence the initial
state with the field and probes is $|\Psi_i \rangle = | \downarrow_A \rangle | \downarrow_B \rangle | 0 \rangle$. Expanding $U$ to second order
in the coupling functions $\epsilon_i$ ($i = A, B$) and using the notation

$$\Phi^\pm_i = \int d\tau \epsilon_i(\tau)e^{\pm i\Omega_{\tau}\phi_i(x_i(\tau), t)}$$

(9)

we obtain

$$|\Psi_f \rangle = \left[ (1 - \Phi^-_A \Phi^+_A + \Phi^-_B \Phi^+_B) | \downarrow\downarrow \rangle - \Phi^+_A \Phi^-_B | \uparrow\uparrow \rangle \right] | 0 \rangle + O(\epsilon^3)$$

(10)

The first term above describes processes where the initial state of the probes is un-
changed. The second term describes two types of processes, either an emission of two
quanta, or an exchange of a single quanta between the the probes. The final state
of the field in this case is $|X_{AB} \rangle \equiv \Phi^+_A \Phi^+_B | 0 \rangle$. Finally, the last two terms describe an
emission of one quantum either by probe $A$ or $B$. In this case the final state of the field is $|E_A\rangle \equiv \Phi^+_A|0\rangle$, or $|E_B\rangle \equiv \Phi^+_B|0\rangle$, respectively.

Tracing over the field degree’s of freedom we obtain to the lowest order

$$
\rho = 
\begin{pmatrix}
1 - C & -\langle X_{AB}|0 \rangle & 0 & 0 \\
-\langle 0|X_{AB}\rangle & |X_{AB}|^2 & 0 & 0 \\
0 & 0 & |E_A|^2 & \langle E_B|E_A\rangle \\
0 & 0 & \langle E_A|E_B\rangle & |E_B|^2
\end{pmatrix}
$$

(11)

where $C = 2\text{Re}\langle 0|T(\Phi^+_A\Phi^+_A + \Phi^-_B\Phi^+_B)|0\rangle$, $|X_{AB}|^2 = \langle X_{AB}|X_{AB}\rangle$, and we used the basis $\{|i\rangle, |j\rangle\} = \{\downarrow\downarrow, \uparrow\uparrow, \downarrow\uparrow, \uparrow\downarrow\}$.

We noted two types of off-diagonal matrices elements. The projection of the exchange amplitude on the vacuum, $\langle 0|X_{AB}\rangle$ acts as to maintain coherence between the $|\downarrow_A\downarrow_B\rangle$ and the $|\uparrow_A\uparrow_B\rangle$ atom states. The product $|\langle E_A|E_B\rangle$ acts to maintain coherence between the $|\downarrow_A\uparrow_B\rangle$ and $|\uparrow_A\downarrow_B\rangle$ states. It is the magnitude of these off-diagonal terms compared to the magnitude of the diagonal (decoherence) terms which determine if the density operator is entangled.

For our case of a $2 \times 2$ system, it is known that necessary \cite{11} and sufficient \cite{12} condition for the density operator to be non-separable (and therefore entangled) is that the partial transpose of $\rho$ is negative. Denoting $\rho_{ij,kl}$ as the matrix elements with respect to the basis $|i\rangle|j\rangle$, the partial partial transposition takes $\rho_{ij,kl} \rightarrow \rho_{il,kj}$. In our case we obtain the following two conditions for non-separability.

$$
|\langle 0|X_{AB}\rangle|^2 > |E_A|^2|E_B|^2
$$

(12)
and

\[ |\langle E_B|E_A\rangle|^2 > |X_{AB}|^2 \]  \hspace{1cm} (13)

When either of these conditions is satisfied \( \rho \) is entangled.

The first inequality, (12), amounts to the requirement that the exchange process, (which leaves the field in a vacuum state) is more probable than single quanta emissions which reduces the coherence of the atoms. In this case the main contribution comes from states like \(| \downarrow_A \downarrow_B \rangle + \alpha | \uparrow_A \uparrow_B \rangle\). Considering the second inequality (13), we note that \( \langle E_A|E_B \rangle \) measures the distinguishability of the quanta emitted by either atom \( A \) or \( B \). Hence the second inequality demands that this inner product is larger than the probability \( \approx |X_{AB}|^2 \) of emitting two quanta. When the second condition is met, the main contribution comes from states like \(| \downarrow_A \uparrow_B \rangle + \beta | \uparrow_A \downarrow_B \rangle\).

Let us evaluate explicitly the emission and exchange terms in the first inequality. The emission term reads

\[ |E_A|^2 = \int d\tau_A \int d\tau'_A e^{-i\Omega(\tau'_A - \tau_A)} D^+(A', A) \]  \hspace{1cm} (14)

and the exchange term

\[ \langle 0|X_{AB} \rangle = \int d\tau_A \int d\tau_B e^{i\Omega(\tau_A + \tau_B)} D^+(A, B) \]  \hspace{1cm} (15)

where \( D^+(x', x) = \langle 0|\phi(x', t')\phi(x, t)|0 \rangle = -\frac{1}{4\pi^2}((t' - t - i\epsilon)^2 - (\vec{x}' - \vec{x})^2) \) is the Wightman function. Substituting \( x(\tau) \) and \( t(\tau) \) one gets \[13\]

\[ D^+(A', A) = -\frac{1}{4\pi^2 L^2 \sinh^2[(\tau'_A - \tau_A - i\epsilon)/L]} \]  \hspace{1cm} (16)

and when the points are on different trajectories

\[ D^+(A, B) = \frac{1}{4\pi^2 L^2 \cosh^2[(\tau_B - \tau_A - i\epsilon)/L]} \]  \hspace{1cm} (17)
The integral (14), for the emission probability, can be performed by complexifying \( \tau_A - \tau_A' \) to a plane and closing the contour in the lower complex plane. This picks up the poles \( \tau_A - \tau_A' = i\epsilon + i\pi n L \) with \( n = -1, -2... \). On the other hand, the contour for the exchange integral (15) should be closed on the upper half plane. This picks up the contributions at \( (\tau_A + \tau_B) = i\epsilon + i\pi (n + \frac{1}{2}) L \) with \( n = 0, 1, 2... \). The ratio between the two terms is then

\[
\frac{|\langle 0 | X_{AB} \rangle|}{|E_A|^2} = \frac{e^{-\pi \Omega L/2} \sum_{n=0}^{\infty} e^{-\pi n \Omega L}}{\sum_{n=1}^{\infty} e^{-\pi n \Omega L}} = e^{\pi \Omega L/2}
\]

Therefore (12) is always satisfied. Unlike the previous stationary case, this ratio can become arbitrarily large, while \( X_{AB} \) and \( E_A \) become exponentially small. The reason for that is that for the hyperbolic trajectories we can have \( \tau \Omega \to \infty \) while keeping \( \Omega L \) finite. By increasing \( L \), the emission probability decreases like \( |E_A|^2 \sim e^{-\pi \Omega L} \). However \( \langle 0 | X_{AB} \rangle \sim e^{-\pi \Omega L/2} \) decreases slower, hence the ratio (18) increases exponentially. It can be shown that similar conclusions follow if we switch the interaction only for a finite duration as long as \( \tau \gg 1/\Omega \) is satisfied.

## 4 Inertial Probes

Does the above result have to do with the special effect of acceleration? To check this we reconsider the problem for the case of a pair of two stationary probes which are switched on and off for a duration \( T < L/c \) where \( L \) is the separation.

Specializing to the case \( |\Psi_i\rangle = |↓_A \downarrow_B\rangle|0\rangle \), substituting \( \phi(x,t) \), and integrating over time eq. (12) can be re-expressed as

\[
\int_0^{\infty} \frac{d\omega}{L} \sin(\omega L)\bar{\epsilon}(\omega - \Omega)\bar{\epsilon}(\omega + \Omega) > \int_0^{\infty} \omega d\omega |\bar{\epsilon}(\Omega + \omega)|^2
\]
Figure 1: The ratio $X/E^2$, with $L = T = 1$, as a function of the energy gap $\Omega$.

where $\tilde{\epsilon}(\omega)$ is the Fourier transform of $\epsilon(t) = \epsilon_i(t)$.

The right hand side in the above inequality is independent of $L$ and tends to zero as $\Omega T \to \infty$. The left hand side depends on both $T$ and $L$ and decays like $\sim 1/L^2$ for $L > T$. However for $\Omega L$ not too big, $\tilde{\epsilon}(\omega - \Omega)$ has a sharp peak near $\omega = \Omega$, which enhances the exchange amplitude. This suggests that there may exist a finite window of frequencies around some $\Omega^{-1} \sim T \sim L$, where (12) can be satisfied.

The following plots exhibit the ratio $X_{AB}/E_A$, for the window function (with $T = 1$)

$$
\epsilon(t) = \begin{cases} 
\cos^2(\pi t), & \text{for } |t| \leq 1/2 \\
0, & \text{for } |t| > 1/2 
\end{cases}
$$

as a function of the energy gap $\Omega$ and the separation $L$ between the atoms.

It follows from Fig. 1. that Eq. (12) is satisfied for $8 < \Omega < 11$. For a different distance $L$, one has to employ atoms with appropriate $\Omega = O(1/L)$. It follows from
Figure 2: The ratio $X/E^2$, with $T = 1$ and $\Omega = 9.5$, as a function of the distance.

Fig. 2 that the spatial region where entanglement persists, extends up to $L/T < 1.1$. This implies that the maximal space-like separation between the pair of spacetime regions that affect the probes can be extended up to $L - T \sim 0.1L$. This result suggests that the probed must be "contained" in the space-like range of a single coherent vacuum fluctuation. Finally we note that the window functions $\epsilon(\omega)$ act as cutoff functions. Hence our result is not sensitive to a cutoff.

5 Discussion

We conclude with several comments. We have shown how quantum correlation, or entanglement, can be extracted to another physical system. To some extent, such quantum correlations are exploited in the process of black hole pair creation, but there we have no access to the quanta emitted into the black hole. Similarly, in the
Unruh effect, the event of thermalization of the detector is correlated with a flux emitted into the other Rindler wedge [14]. In the present process we can control the regions which we probe, and demonstrate that entanglement persists even between non-complementary wedges, i.e., when the regions are separated by a finite distance. Nevertheless when the separation becomes too large the extracted entanglement drops to zero while the classical type of correlations do not.

We have stressed that entropy is no longer a good measure of entanglement once the state is not pure, when for instance the two regions are separated. The black-hole naturally divides space to interior and exterior complementary regions, with a combined pure state. The von-Neumann entropy, coincides in this case with the entanglement measure. How do we renormalize this entanglement entropy? A naive cutoff seems unjustified, because this will also truncate the ultra-high modes which are needed in Hawking’s derivation of black-hole radiation. On the other hand if we would slightly separate between the interior and exterior regions (effectively done here by introducing physical probes) entropy becomes indeed finite, but would no longer be a measure of entanglement but rather of the classical correlations.

Finally, we note that by transforming vacuum entanglement to pairs of probes or atom-like system, vacuum entanglement becomes a physical operational quantity. It has been recently shown that by acting locally on an ensemble of such generated pairs, one can ”purify” the amount of entanglement and while reducing the number of pairs, approach gradually to a perfect pure EPR-Bohm pair [15]. The resulting EPR-Bohm pairs are useful for quantum processes such as teleportation [16] or dense coding [17] which are not possible with the aid of classical correlation alone.
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