Uniqueness of real closure $*$ of regular rings

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Abstract

In this paper we give a characterisation of real closure $*$ of regular rings, which is quite similar to the characterisation of real closure $*$ of Baer regular rings seen in [4]. We also characterize Baer-ness of regular rings using near-open maps. The last part of this work will concentrate on classifying the real closure $*$ of Baer and non-Baer regular rings (upto isomorphisms) using continuous sections of the support map, we construct a topology on this set for the Baer case. For the case of non-Baer regular rings, it will be shown that almost no information of the ring structure of the Baer hull is necessary in order to study the real and prime spectra of the Baer hull. We shall make use of the absolutes of Hausdorff spaces in order to give a construction of the spectra of the Baer hulls of regular rings. Finally we give example of a Baer regular ring that is not rationally complete.

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Henceforth, when we say regular ring, we mean a von Neumann regular ring. When we say ring, we usually mean commutative unitary partially ordered ring. Poring is a ring $A$ that has a partial ordering $A^+$. We assume that the reader is familiar with the notations used in [3] and [4]. However for completeness, here are a list of notations that may be used.

Notation. Let $A$ be a ring and $x \in A$

- If $A$ is a poring then Sper $A$ is the topological space (Harrison Topology) consisting of prime cones containing $A^+$

- $E(A) := \{ e \in A : e^2 = e \}$ is the set of the idempotents of $A$

- $B(A)$ is the Baer hull of $A$, if $A$ is a poring with partial ordering $A^+$ then we use the partial ordering

$$B(A)^+ := \{ \sum_{i=1}^n b_i^2 a_i : n \in \mathbb{N}, b_i \in B(A), a_i \in A^+ \text{ for } i = 1, \ldots, n \}$$

for $B(A)$

- $Q(A)$ will be the complete ring of quotients of $A$. If $A$ is a poring with partial ordering $A^+$, then we use the partial ordering

$$Q(A)^+ := \{ \sum_{i=1}^n x_i a_i : n \in \mathbb{N}, x_i \in Q(A), a_i \in A^+ \text{ for } i = 1, \ldots, n \}$$

for $Q(A)$

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Theorem made by Storrer (where the unlabeled maps in the commutative diagram above are all canonical maps. quotients of rings, I suggest following official call the ∼ in (Storrer’s Satz) Let
Theorem 1. Q
φ
image of φ
We may also write
CRings commute (in the category Satz above is constructed. For completeness, we shall write down how the monomorphism in the Storrer’s Satz is above is constructed.

Conjecture 2. For completeness, we shall write down how the monomorphism in the Storrer’s Satz above is constructed.

For any ring A, there is a ring monomorphism A → Q(A) (see [7] §2.3 Proposition 6 p.40). We may also write
\[ Q(A) = \bigcup_{D \preceq A} \text{Hom}_A(D, A)/\sim_A \]
where \( \sim_A \) is a specific equivalence relation and \( D \preceq A \) means that \( D \) is a dense ideal of \( A \). For readers unfamiliar with the terminology and concept used in the study of the complete ring of quotients of rings, I suggest [5] §1 and [7] §2.3 and §2.4 p.36-46 as reference.

Henceforth, for any ring A and for any \( \phi \in \bigcup \text{Hom}_A(D, A) \) we write \( [\phi]_A \) to mean the canonical image of \( \phi \) in \( Q(A) \). Now we are ready to make the construction. Let A and B satisfy the condition of the Storrer’s Satz. Let \( \phi : D \to A \) be a module morphism with \( D \) a dense ideal of \( A \). Storrer showed the following

1. There is a maximal family \( \{d_i\}_i \subset D \) such that \( \oplus_i d_i A \) is a direct sum and is dense in \( A \)
2. \( \overline{D} := \oplus_i d_i B \) is then a direct sum and is dense in \( B \)
3. We then associate \([\phi]_A \text{ to } [\bar{\phi}]_B\) where
\[
\bar{\phi} := \bigoplus I \phi_i : T \to B
\]
with \(\phi_i : d_i B \to B\) defined by \(\phi_i(d_i) := \phi(d_i) \in A \subset B\). This association turns out to be not only a well-defined function between \(Q(A)\) and \(Q(B)\), but also a ring monomorphism satisfying the Storrer’s Satz above.

There is another result by Raphael which I have made use in [4] and I will also make constant use of it hereafter. The result I shall call Raphael’s Lemma whose proof is a combination of proofs found (but not formally stated) in [12] Lemma 1.14, Proposition 1.16 and Remark 1.17.

**Lemma 3.** (Raphael’s Lemma) If \(A\) is a regular Baer ring and \(B\) is a regular ring which is an essential extension of \(A\) then \(B\) is also Baer and we have a canonical homeomorphism
\[
\phi : \text{Spec } B \to \text{Spec } A \quad p \mapsto p \cap A
\]
whose inverse is
\[
\phi^{-1} : \text{Spec } A \to \text{Spec } B \quad q \mapsto qB
\]

**Lemma 4.** Let \(A\) be a real regular ring and let \(C\) be a real closure * of \(A\), then
1. \(C\) can be regarded as a real closure * of \(B(A)\)
2. The spectral map \(\text{Spec } C \to \text{Spec } B(A)\) induced from 1. is a homeomorphism.

**Proof.** By Storrer’s Satz, we have the following commutative diagram of rings
\[
\begin{array}{ccc}
A & \subseteq & Q(A) \\
\uparrow & & \downarrow \\
C & \subseteq & Q(C)
\end{array}
\]
We can thus regard all the given rings as subrings of \(Q(C)\). By Theorem 15 of [3] we know that \(C\) is Baer, thus by Proposition 2 in [3] \(C\) contains all the idempotents of \(Q(C)\). Specifically, \(C\) contains \(A\) and all the idempotents of \(Q(A)\). But \(A\) and the idempotents of \(Q(A)\) together generate \(B(A)\). Therefore \(B(A)\) may indeed be regarded as a subring of \(C\). We originally had \(B(A)^+\) constructed in such a way that it is the partial ordering of \(B(A)\) which is the weakest extension of \(A^+\) (see [2] §1.3 p.34-35). Thus \(C^+ \cap B(A) \supset B(A)^+ \supset A^+\) and therefore \(B(A)\) can in fact be regarded as a subring of \(C\). We thus have the following extension of porings
\[
A \hookrightarrow B(A) \hookrightarrow C
\]
we also know that \(C\) is an integral and essential extension of \(A\) meaning that it is also an integral and essential extension of \(B(A)\). \(C\) being real closed * implies that \(C\) is indeed a real closure * of \(B(A)\). By Raphael’s Lemma, \(\text{Spec } C \to \text{Spec } B(A)\) is a homeomorphism.

**Lemma 5.** Let \(A\) be a real regular ring and let \(B, C\) be two real closure * of \(A\) such that they are not \(A\)-isomorphic. Then there exists \(p \in \text{Spec } B(A)\) such that
\[
B/pB \not\cong_{A/pA} C/pC
\]
Proof. Set $X := \text{Spec } A$, $Y := \text{Spec } B$ and $Z := \text{Spec } C$. By Lemma 4, we regard $B(A)$ as a subpor-
ing of both $B$ and $C$ and we know then that $\text{Spec } B$ and $\text{Spec } C$ are (canonically) homeomorphic to $\text{Spec } B(A)$. By Theorem 8 in [4] there is an $x \in X$ such that

for all $y_x \in Y$ and $z_x \in Z$ that lie over $x$ (i.e. $y_x \cap A = z_x \cap A = x$) we get

$$B/y_x \not\sim A/x C/z_x$$

Fix an $x \in X$ with the above property and choose $y_x \in Y$ lying over $x$ (this can be done, since the spectral map $Y \to X$ is a surjective one, see for instance [12] Lemma 1.14). Now consider $p := y_x \cap B(A) \in \text{Spec } B(A)$ then $pB \in \text{Spec } B$ and $pC \in \text{Spec } C$ (by Raphael’s Lemma) that lie over $x$ and so by Property $\star$

$$B/pB \not\sim A/x C/pC$$

\[\square\]

Definition. Let $f : X \to Y$ be a function between topological spaces $X$ and $Y$. This function will be called a near open (or near-open) function (German: fast offene Abbildung) iff for all nonempty opens set $U \subset X$ there exists a nonempty open set $V \subset Y$ such that $V \subset f(U)$

Example.

1. Let $\mathbb{R}$ be the real numbers endowed with the usual Euclidean topology. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then this function is a continuous function that is near open however it is not open, because for instance $f((-1, 1)) = [0, 1)$.

2. As will be seen in Theorem 7, if $A$ is a von Neumann regular ring that is not Baer, then the canonical map $\text{Spec } B(A) \to \text{Spec } A$ is a continuous near open map between Stone spaces that is not open.

Lemma 6. Let $A$ be a von Neumann regular ring and let $B$ be an overring of $A$. Set

$$\phi : \text{Spec } B \to \text{Spec } A \quad p \mapsto p \cap A$$

Then for any $a \in A$ we have the identity

$$\phi(D_B(a)) = D_A(a)$$

Proof. Suppose $a \in A$.

"$\subset$" Let $p \in \text{Spec } B$ and suppose $a \not\in p$, then clearly $a \not\in p \cap A$. In other words $\phi(p) \in D_A(a)$.

"$\supset$" Let $q \in \text{Spec } A$ and let $a \not\in q$, then by [12] Lemma 1.14 there exists a $p \in \text{Spec } B$ such that $\phi(p) = q$. If $a \in p$ then $a \in p \cap A = q = \phi(p)$ is a contradiction, thus $a \not\in p$. So there is a $p \in D_B(a)$ such that $\phi(p) = q$. \[\square\]

Theorem 7. Let $A$ be a von Neumann regular ring, then the canonical map

$$\phi : \text{Spec } B(A) \longrightarrow \text{Spec } A$$

is a near open surjection. Moreover $\phi$ is open iff $A = B(A)$ (i.e. $A$ is Baer).

Proof. Suppose $U \subset \text{Spec } B(A)$ is a nonempty open set. Without loss of generality we may assume

$$U = D_B(A)(x)$$

for some $x \in B(A) \setminus \{0\}$.
Now because $B(A)$ is a ring of quotients of $A$ (see for instance the last paragraph of [5] p.8) there exists a $y \in A$ such that $xy \in A \setminus \{0\}$ (this is because $A$ is semiprime and commutative, see [5] Theorem following Lemma 1.5). We also then have
\[ D_{B(A)}(x) \supset D_{B(A)}(xy) \]
Using the above equation and the preceding Lemma we obtain
\[ \phi(D_{B(A)}(x)) \supset D_A(xy) \]
and therefore $\phi$ is near open. $\phi$ is a surjection because of [12] Lemma 1.14.

Now we prove the last statement of the Theorem, the proof that follows is by Niels Schwartz.

If $A$ is Baer then $A = B(A)$ and so $\phi$ is a homeomorphism, thus an open map. If $A$ is not Baer then Spec $A$ is not extremally disconnected (see Prop. 2.1 [9]), suppose then that $\phi$ is open. Since Spec $A$ is not extremally disconnected, there exists an open set $U \subseteq$ Spec $A$ such that $\overline{U}$ (i.e. the topological closure of $U$ in Spec $A$) is not open in Spec $A$. Because Spec $B(A)$ is extremally disconnected $\overline{\phi^{-1}(U)}$ (closure in Spec $B(A)$) is clopen, but because $\phi$ is a continuous surjection, Spec $B(A)$ is compact and Spec $A$ is Hausdorff we the following result from basic general topology
\[ \phi(\overline{\phi^{-1}(U)}) = \overline{U} \subseteq \text{Spec } A \]
And because we assumed $\phi$ is open, the above equation implies that $\overline{U}$ is open, which is a contradiction.

\[ \square \]

**Theorem 8.** Let $A$ be a real regular ring, then $A$ has no unique real closure $*$ if and only if there exists an $x \in A$ such that
\[ [\text{supp}_A P(x) \cap \text{supp}_A P(-x)]^\circ \neq \emptyset \]

**Proof.** "$\Rightarrow$" Theorem 14 of [4] states the same thing as this Proposition, however it was assumed there that $A$ is Baer and no mention of near openness is made. However in the sufficiency condition of the said Theorem there was no implementation of $A$ being Baer. Thus we need only prove the necessity for this Proposition.

"$\Rightarrow$" We almost use the same method of proof as seen in Theorem 14 [4]. Let $C_1, C_2$ be two real closure $*$ of $A$ such that they are not $A$-isomorphic.

Throughout the proof let $i = 1, 2$. By Lemma 4 we may regard $B(A)$ as a subporing of $C_i$ and denote
\[ \nu_i : \text{Spec } C_i \sim \rightarrow \text{Spec } B(A) \]
to be the canonical spectral map (it is a homeomorphism by Raphael’s Lemma).

Now, by Lemma 5, there exists a $p \in B(A)$ such that
\[ C_1/pC_1 \not\cong_{A/p \cap A} C_2/pC_2 \]
We observe that $C_1/pC_1$ is a real closed field (as $C_1$ is real closed $*$, therefore has factor fields that are real closed. See [3] Theorem 15), and is algebraic over the field $A/p \cap A$. Thus there are $\alpha_1, \alpha_2 \in \text{Sper } A$ such that
\[ \text{supp}(\alpha_1) = \text{supp}(\alpha_2) = p \cap A \]
and
\[ \rho(\alpha_i) \cong_{A/p \cap A} C_i/pC_i \quad i = 1, 2 \]
We also have the following commutative diagram of topological (spectral) spaces

\[
\begin{array}{ccc}
\text{Sper } A & \xrightarrow{\phi_i} & \text{Spec } B(A) \\
\downarrow \text{supp}_A & & \\
\text{Spec } A & \xrightarrow{\psi} & \\
\end{array}
\]

where

\[
\phi_i := \mu_i \circ \text{supp}_{C_i}^{-1} \circ \nu_i^{-1}
\]

with

\[
\mu_i : \text{Sper } C_i \to \text{Sper } A \quad \alpha \mapsto \alpha \cap A
\]

Note that \(\text{supp}_{C_i}\) and \(\nu_i\) are homeomorphisms (\(C_i\) is a real closed ring too, see [3]), therefore \(\phi_i\) is indeed well-defined.

Now let \(x \in \alpha_1 \setminus \alpha_2\) then

\[
p \in \text{supp}_A P(x) \cap \text{supp}_A P(-x)
\]

and

\[
\text{supp}_A P(x) \supset \psi \phi_1^{-1} P(x)
\]

\[
\text{supp}_A P(-x) \supset \psi \phi_2^{-1} P(-x)
\]

so

\[
\text{supp}_A P(x) \cap \text{supp}_A P(-x) \supset \psi \phi_1^{-1} P(x) \cap \psi \phi_2^{-1} P(-x) \supset \psi (\phi_1^{-1} P(x) \cap \phi_2^{-1} P(-x))
\]

but

\[
p \in \phi_1^{-1} P(x) \cap \phi_2^{-1} P(-x)
\]

because

\[
\phi_1(p) = \alpha_1 \in P(x)
\]

and

\[
\phi_2(p) = \alpha_2 \in P(-x)
\]

Therefore \(\phi_1^{-1} P(x) \cap \phi_2^{-1} P(-x)\) is a nonempty open set in \(\text{Spec } B(A)\).

Now by Theorem 7 \(\psi\) is near open, therefore there exists a nonempty open set \(U \subset \text{Spec } A\) such that

\[
U \subset \psi (\phi_1^{-1} P(x) \cap \phi_2^{-1} P(-x))
\]

Hence

\[
[\text{supp}_A P(x) \cap \text{supp}_A P(-x)]^\circ \neq \emptyset
\]

**Definition.** Let \(A\) be a poring. By a section of \(\text{supp} : \text{Spec } A \to \text{Sper } A\), we mean a map

\[
s : \text{Spec } A \to \text{Spec } A
\]

such that \(\text{supp} \circ s = \text{id}_{\text{Spec } A}\), where for any set \(X\) by \(\text{id}_X\) we mean the identity map

\[
\text{id}_X : X \to X \quad x \mapsto x
\]

**Theorem 9.** Let \(A\) be a real Baer regular ring, then there is a one to one correspondence between the set of all real closure \(\ast\) of \(A\) identified up to \(A\)-isomorphisms and the set of continuous sections of \(\text{supp}_A\).
Proof. Set
\[ S := \{ s : \text{Spec } A \to \text{Sper } A : s \text{ is continuous and } \text{supp} \circ s = \text{id}_{\text{Spec } A} \} \]
and \( C := \mathcal{D}/\equiv_A \)

We now attempt to define a bijection \( \Phi : S \to C \). Let \( s \in S \), since \( s \) is a section of \( \text{supp} \) (i.e. \( \text{supp} \circ s = \text{id}_{\text{Spec } A} \)) we know then that \( A \) can be considered as a subring of

\[ B := \prod_{p \in \text{Spec } A} \rho(s(p)) \]

\( B \) is a real closed ring (see Remark 1 [3]), and therefore \( \text{supp}_B \) is a homeomorphism. We thus have the following commutative diagram of spectral spaces

\[ \begin{array}{ccc}
\text{Spec } A & \phi & \text{Sper } B \cong \text{Spec } B \\
\text{Sper } A & \downarrow \text{supp} & \\
\text{Spec } A & \psi & \\
\end{array} \]

where \( \phi \) and \( \psi \) are canonical maps.

Now for any set \( Z \subset \text{Spec } B \), define , as is usual in algebraic geometry,

\[ I_B(Z) := \bigcap_{p \in Z} p \]

Define \( X := s(\text{Spec } A) \) and observe then that

\[ I_B(\phi^{-1}(X)) \cap A = \bigcap_{p \in \phi^{-1}(X)} p \cap A = \bigcap_{p \in \phi^{-1}(X)} \phi(p) \]

\[ = \bigcap_{p \in \phi^{-1}(X)} \text{supp}(\phi(p)) = \bigcap_{q \in \text{supp}_B(\phi^{-1}(X))} q \]

\[ = \bigcap_{q \in \text{supp}(X)} q = (0) \]

the last row of the equation is because \( \phi \) is surjective and that \( \text{supp}(X) = \text{supp}(s(\text{Spec } A)) = \text{Spec } A \).

We may therefore, by Zorn’s Lemma, choose an ideal \( I \subseteq B \) such that \( I \subseteq B \) \( \phi^{-1}(X) \subset I \) and

\[ A \longrightarrow B \longrightarrow B/I \]

is an essential extension of \( A \). Set \( Y := \text{Spec } B/I \cong \text{Spec } B/I \) (Because \( B/I \) is real closed, see [3] Remark 1), we then have the following commutative diagram

\[ \begin{array}{ccc}
\text{Spec } A & \xleftarrow{\pi} & Y \\
\text{Sper } A & \downarrow \gamma & \\
\text{Spec } A & \xleftarrow{\psi} & \\
\end{array} \]

where \( \pi \) and \( \gamma \) (\( \gamma \) being a homeomorphism by Raphael’s Lemma) are canonical maps. Define

\[ s' : \text{Spec } A \longrightarrow X \quad s'(p) := s(p) \forall p \in \text{Spec } A \]

We now claim . . .
Claim 1: \( s' \circ \text{supp} \mid X = \text{id}_X \)

We know \( s' \circ \text{supp} \circ s' = s' \)

and we know that \( s' \) is bijective (as \( s \) is a section and therefore injective). So we may compose the right side by \( s'^{-1} \) and we get the desired identity!

Claim 2: \( \pi(Y) = X \)

Since \( I \supset \mathcal{I}_B(\phi^{-1}(X)) \) and since we know that \( \phi^{-1}(X) \) is closed (this is because \( \phi \) is continuous and \( s \) is a continuous map between a compact space and a Hausdorff space, and so \( X = s(\text{Spec } A) \) and \( \phi^{-1}(X) \) are closed) in \( \text{Spec } B \), we then know that

\[
Y \cong \mathcal{V}_B(I) \subset \mathcal{V}_B(\mathcal{I}_B(\phi^{-1}(X))) = \phi^{-1}(X)
\]

This implies that

\[
\pi(Y) = \phi(\mathcal{V}_B(I)) \subset \phi(\phi^{-1}(X)) \subset X
\]

Therefore \( \pi(Y) \subset X \) and so by Claim 1 we get

\[
s \circ \gamma = s \circ \text{supp} \circ \pi = \pi
\]

In other words we have the commutative diagram

\[
\begin{array}{ccc}
\text{Sper } A & \xleftarrow{\pi} & Y \\
s \downarrow & & \downarrow \gamma \\
\text{Spec } A & \xleftarrow{s} & \text{Y}
\end{array}
\]

but \( s \circ \gamma(Y) = \pi(Y) = s(\text{Spec } A) = X \) (because \( \gamma \) is a homeomorphism and thus a surjection).

Now define

\[
\Phi(s) := \text{ic}(A, B/I) / \cong_A
\]

we need yet to show that \( \Phi \) defined in this way for any \( s \in S \) is . . .

Claim 3: well-defined

In other words we need to show that for \( s \in S \), \( \Phi(s) \) is in \( \mathcal{C} \) and is independent of the choice of \( I \) (as constructed above). Let \( B \) and \( I \trianglelefteq B \) be as constructed above. Because \( B/I \) is a von Neumann regular ring that is essential over the Baer ring \( A \), \( B/I \) is Baer and real closed (by Raphael’s Lemma and Remark 1 in [3]). Therefore \( B/I \) is a real closed * ring (by [3] Theorem 15). And so by [4] Proposition 6 \( \text{ic}(A, B/I) \in \mathcal{D} \). This proves that \( \Phi(s) \in \mathcal{C} \).

Now suppose that \( I_1, I_2 \) are two ideals in \( B \) such that

\[
I_1, I_2 \supset \mathcal{I}_B(\phi^{-1}(X))
\]

and such that \( B/I_1, B/I_2 \) are essential extensions of \( A \). We show that

\[
\text{ic}(A, B/I_1) \cong_A \text{ic}(A, B/I_1)
\]

Let \( i = 1, 2 \) and define \( C_i := \text{ic}(A, B/I_i) \). We then have the following commutative diagram of porings

\[
\begin{array}{ccc}
A & \xrightarrow{B/I_i} & B/I_i \\
\downarrow & & \downarrow \\
C_i
\end{array}
\]
with all the maps being canonical injections (whose spectral maps on their prime spectra are all homeomorphic). Now suppose that \( p \in \text{Spec} \ A \) then there is a unique \( p_i \in \text{Spec} \ B/I_i \) such that \( p_i \cap A = p \) (in fact \( p_i = p(B/I_i) \) by Raphael's Lemma). Now according to the commutative diagram in Claim 2, we have the following commutative diagram of spectral spaces

\[
\begin{array}{ccc}
\text{Spec} \ A & \xrightarrow{s} & \text{Sper} B/I_i \cong \text{Spec} B/I_i \\
\uparrow \pi_i & & \uparrow \gamma_i \\
\text{Spec} \ A & & i = 1, 2
\end{array}
\]

where \( \pi_i, \gamma_i \) are canonical maps. Therefore \( s \gamma_i(p_i) = \pi(p_i) = s(p) \) and so because \( C_i/pC_i \) and \( B/p_i \) are real closed fields we obtain

\[
C_i/pC_i = \text{ic}(A/p, B/p_i) \cong_{A/p} \rho(s(p))
\]

and this is valid for all \( p \in \text{Spec} \ A \). Thus by Theorem 8 of [4]

\[
C_1 \cong_A C_2
\]

Claim 4: injective

Let \( s, t \in \mathcal{S} \). Suppose also that \( \Phi(s) = \Phi(t) \). Let \( C \in \mathcal{D} \) such that

\[
\Phi(s) = \Phi(t) = C/ \cong_A
\]

as we have seen in Claim 3, we know that for all \( p \in \text{Spec} \ A \) one has

\[
\rho(s(p)) \cong_{A/p} C/pC \cong_{A/p} \rho(t(p))
\]

thus one concludes at once that for all \( p \in \text{Spec} \ A \) one has \( s(p) = t(p) \) and therefore \( s = t \)

Claim 4: surjective

Let \( C \in \mathcal{D} \), one then has the following commutative diagram of spectral spaces

\[
\begin{array}{ccc}
\text{Sper} \ A & \xrightarrow{\pi} & \text{Sper} \ C \cong \text{Spec} \ C \\
\supp \downarrow & & \downarrow \gamma \\
\text{Spec} \ A & &
\end{array}
\]

where \( \gamma \) and \( \pi \) are canonical maps. So here, for any \( q \in \text{Spec} \ C \) (because \( C \) is real closed) we have the identity

\[
\rho(\pi(q)) \cong_{A/q \cap A} C/q
\]

Now define

\[
s : \text{Spec} \ A \to \text{Sper} \ A \quad s(p) := \pi(\gamma^{-1}(p)) \quad \forall p \in \text{Spec} \ A
\]

We show first that \( s \in \mathcal{S} \). For all \( p \in \text{Spec} \ A \) we get

\[
\supp \circ s(p) = \supp \circ \pi(\gamma^{-1}(p)) = \gamma \gamma^{-1}(p) = p
\]

Thus \( \supp \circ s = \text{id}_{\text{Spec} \ A} \) (i.e. \( s \) is indeed a section of \( \supp \)). Because both \( \pi \) and \( \gamma^{-1} \) are continuous maps we see then that \( s \) is a continuous map.

We now show that \( \Phi(s) = C/ \cong_A \). Let \( C' \in \mathcal{D} \) such that \( C'/ \cong_A = \Phi(s) \). But from Claim 3 we have seen that for any \( p \in \text{Spec} \ A \) we have

\[
C'/pC' \cong_{A/p} \rho(s(p)) = \rho(\pi(\gamma^{-1}(p))) \cong_{A/p} C/pC
\]

One then uses Theorem 8 of [4] to claim that \( C \cong_A C' \).
Proposition 10. Let $A$ be a real von Neumann regular ring, then a section of $\text{supp}_A$ is a homeomorphism onto its image iff it is continuous.

Proof. The proof is quite straightforward. One side of the equivalence is trivial. Because $\text{Spec } A$ and $\text{Sper } A$ are compact and Hausdorff then the image of a continuous section of $\text{supp}$ is closed compact and Hausdorff in $\text{Sper } A$. The same reasoning tells us that the section brings closed sets to closed set in the image (because it is a continuous map from a compact space to a Hausdorff space). The section being injective is thus a homeomorphism onto its image. 

Let $A$ be a real Baer regular ring, and set $X := \text{Spec } A$ and $Y := \text{Sper } A$. In [8] Chapter I there is a beautiful treatise on the different topologies that the set of continuous functions from $X$ to $Y$ may have (and by which practical application may be applied on these topologies). We shall denote the set of continuous functions from $X$ and $Y$ as $C(X, Y)$ for now. $C(X, Y)$ may have the so called point convergence topology which is simply the topology relative to the Tychonoff product topology of $Y^X$. A finer topology would be the compact-open topology (see [8] p.4). As is shown in Theorem 1.1.3 of [8] most reasonable topologies of $C(X, Y)$ contain the point convergence topology. So if we show that a subset of $C(X, Y)$ is closed with respect to the point convergence topology then it is automatically closed in these other topologies of $C(X, Y)$ (namely those induced by closed networks on $X$, for terminologies and further reading the reader is advised to consult [8] Chapter I).

Below is a Lemma that is proven by K.P. Hart (with a bit of rewording by me) in the sci.math newsgroup during one of our discussion regarding the set of continuous sections of a continuous map.

Lemma 11. (K.P. Hart, 12.2007) Given a surjective continuous function between T1 topological spaces, say $\pi : Y \rightarrow X$, the set of continuous sections of $\pi$ is closed in $C(X, Y)$ (i.e. set of continuous functions from $X$ to $Y$) with the point convergence topology.

Proof. Let

$$F_x := \{ f \in Y^X : f(x) \in \pi^{-1}(x) \}$$

then this set is obviously closed (with the point convergence topology) in $Y^X$ and the set of continuous sections of $\pi$ can be written as the intersection

$$\bigcap_{x \in X} F_x \cap C(X, Y)$$

and this is also obviously closed relative to $C(X, Y)$. 

Corollary 12. Let $A$ be a real Baer von Neumann regular ring, then the set of real closure $*$ of $A$ identified up to $A$-isomorphism form a Hausdorff topological space and can be identified as a closed subspace of $C(X, Y)$ with the point convergence topology (and thus also in other finer topologies induced by closed networks on $X$ as defined in [8] p.3, this fact is due to Theorem 1.1.3 of [8])

Proof. Because of Theorem 9, we may identify the set of real closure $*$ of $A$ with the set of continuous sections of $\text{supp}_A$. Set $X := \text{Spec } A$ and $Y := \text{Sper } A$ and write $C(X, Y)$ to be the set of continuous functions from $X$ to $Y$ and use the above Lemma substituting $\pi$ with $\text{supp}_A$.

During the investigation of von Neumann regular rings, I made many use of the Baer hull of the ring. It was therefore natural to ask the question whether the Baer hull and the complete ring of quotients of such rings coincide. The example below shows that one may indeed have a Baer von Neumann regular ring that is not rationally complete.

Example. Let $K$ be a real field (say $\mathbb{R}$). Also define a ring

$$R := \prod_{x \in K} K_x \quad K_x := K \quad \forall x \in K$$
with canonical (componentwise) addition and multiplication. We may also from now on regard $K$ as a subring $R$ by taking the canonical monomorphism

$$K \hookrightarrow R \quad k \mapsto \{k_x | x \in K, k_x = k\}$$

We now define a subring of $R$

$$A := \{\sum_{i=1}^{n} e_i x_i : e_i \in E(R), n \in \mathbb{N}\}$$

We shall now give some facts regarding $A$ with sketches of their proof

Claim 1 For any $a \in A$ we claim that we may write $a$ as

$$a = \sum_{i=1}^{n} e_i x_i$$

with $x_i \in K$ and $e_i \in E(R)$ for $i = 1, \ldots, n$ and the $e_i$’s satisfy the fact that they have pairwise disjoint supports. In other words

$$\{x \in K : e_i(x) \neq 0\} \cap \{x \in K : e_j(x) \neq 0\} = \emptyset \quad i, j = 1, \ldots, n \quad i \neq j$$

To show this, we first write $a$ as $\sum_{i=1}^{m} f_i y_i$ for some $y_i \in K$ and $f_i \in E(R)$ (by definition of $A$). Now we define $S$ to be the powerset of $\{1, \ldots, m\}$ without the emptyset and for any $S \in S$ set

$$e_S := \prod_{j \in S} f_j \prod_{k \notin S} (1 - f_k)$$

and

$$X_S := \{x \in X : f_S(x) \neq 0\}$$

Then one shows that for any $S, T \in S$ such that $S \neq T$ we get $S \neq T$ and we have the identity

$$a = \sum_{i=1}^{m} f_i y_i = \sum_{S \in S} e_S \sum_{j \in S} y_j$$

Thus we may write $a$ as a linear combination (with $K$ as the scalar) of $2^n - 1$ idempotents with disjoint support.

Claim 2 One checks that $A$ is a proper subring of $R$. To check that $A$ is strictly contained in $R$, one need to only show that the element $r \in R$ defined by $r(x) = x$ is not in $A$. To do this we note a fact that $r$ can never be written as linear combination of idempotents of $R$ with disjoint supports, and then we make use of Claim 1.

Claim 3 We now claim that $A$ is in fact von Neumann regular. Let $a \in A \setminus \{0\}$, then we may write $a$ as

$$a = \sum_{i=1}^{n} e_i x_i \quad x_i \in K \setminus \{0\}, e_i \in E(R) \setminus \{0\}$$

with $e_i$’s having pairwise disjoint supports. Then define $a' \in A$ by $a' = \sum e_i x_i^{-1}$, one easily sees that $a'$ is the quasi-inverse of $a$, i.e. $a a' = a$. Because $a$ was an arbitrary nonzero element of $A$, we have proven that any element of $A$ has a quasi-inverse and so the ring is von Neumann regular.

Claim 4 $R$ is a rational extension of $A$ and $A$ is a Baer proper subring of $R$. $R$ is obviously a rationally complete ring (its the product of fields). And if $r \in R \setminus \{0\}$ then one can easily multiply it by an idempotent with finite support to have an element in $A$. So $R$ is a rational extension of $A$ which is rationally complete and thus the complete ring of quotients of $A$ is $R$. $A$ also has all the idempotents of $R$, thus $A$ is Baer by Mewborn’s Proposition (see [3] Proposition 2).
Notation. Let $A$ be a poring and $\alpha \in \text{Sper } A$, then we write $A(\alpha)$ to mean the real field $\text{Quot}(A/\text{supp}(\alpha))$ with the canonical partial ordering corresponding to $\alpha$ (i.e. $\alpha/\text{supp}(\alpha) \subset A(\alpha)^+$). The real closed field (upto $A/\text{supp}(\alpha)$-isomorphism) which is a real field extension of $A(\alpha)$ will then be denoted as $\rho(A(\alpha))$. We formerly used $\rho(\alpha)$ to denote this, but there is a good reason why we use $\rho(A(\alpha))$ instead. Firstly $\rho$ was the symbol first used to mean the real closure (in the sense of Niels Schwartz) functor, and $\rho(A(\alpha))$ is indeed the real closure of $A(\alpha)$. Therefore we reduce confusion here (since $\rho(\alpha)$ used previously had nothing to do with the real closure functor $\rho$). Secondly, sometimes it is important for us to specify the ring involved and $\rho(A(\alpha))$ does show us that we are dealing with the poring $A$. So, we shall henceforth make use of this notation.

**Theorem 13.** Let $A$ be a real von Neumann regular ring and consider the pullback

$$
\begin{array}{ccc}
\text{Sper } A \times_{\text{Spec } A} \text{Spec } B(A) & \longrightarrow & \text{Sper } A \\
\downarrow & & \downarrow \text{supp } A \\
\text{Spec } B(A) & \longrightarrow & \text{Spec } A \\
\phi_p & & \\
\end{array}
$$

with $\phi_p$ being the canonical map (i.e. $\phi_p(\tilde{p}) := \tilde{p} \cap A$ for all $\tilde{p} \in \text{Spec } B(A)$). It turns out then that the fiber product

$$
\text{Sper } A \times_{\text{Spec } A} \text{Spec } B(A)
$$

is (canonically) homeomorphic to $\text{Sper } B(A)$.

**Proof.** Abbreviate $B := B(A)$, set $X := \text{Sper } A \times_{\text{Spec } A} \text{Spec } B$ and name the projection of the pullback by

$$
\pi_A : X \longrightarrow \text{Sper } A
$$

and

$$
\pi_B : X \longrightarrow \text{Spec } B
$$

Then we have the following commutative diagram in spectral spaces

$$
\begin{array}{ccc}
\text{Sper } B & \longrightarrow & \text{Sper } A \\
\downarrow \text{supp } B & & \downarrow \text{supp } A \\
\text{Spec } B & \longrightarrow & \text{Spec } A \\
\phi_p & & \\
\end{array}
$$

where $\phi_r$ is the canonical map. By the universal property of the pullback there is a unique continuous map $\psi : \text{Sper } B \rightarrow X$ such that the diagram below commutes

$$
\begin{array}{ccc}
\text{Sper } B & \longrightarrow & \text{Sper } A \\
\downarrow \text{supp } B & & \downarrow \text{supp } A \\
\text{Spec } B & \longrightarrow & \text{Spec } A \\
\phi_r & & \\
\end{array}
$$

This Theorem claims that $\psi$ is in fact a homeomorphism.

Observe that because $B$ is an integral poring extension of $A$, one has for any $\hat{\alpha} \in \text{Sper } B$ the identity

$$
\rho(A(\alpha)) \cong_{A/\text{supp}(\alpha)} \rho(B(\hat{\alpha}))
$$
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(see also Lemma 2(i) in [4]) where \( \alpha := \tilde{\alpha} \cap A = \phi_r(\tilde{\alpha}) \). Now we show that ...

\( \psi \) is injective

Let \( \tilde{\alpha}, \tilde{\beta} \in \text{Sper} B \) such that \( \psi(\tilde{\alpha}) = \psi(\tilde{\beta}) =: x \) for some \( x \in X \). Then

\[
\pi_A(x) = \tilde{\alpha} \cap A = \tilde{\beta} \cap A =: \alpha \in \text{Sper} A
\]

for some \( \alpha \in \text{Sper} A \), this implies that

\[
\rho(B(\tilde{\beta})) \cong_{A/\text{supp}(\alpha)} \rho(A(\alpha)) \cong_{A/\text{supp}(\alpha)} \rho(B(\tilde{\alpha}))
\]

Also

\[
\pi_B(x) = \text{supp}_B(\tilde{\alpha}) = \text{supp}_B(\tilde{\beta}) = \tilde{p} \in \text{Spec} B
\]

for some \( \tilde{p} \in \text{Spec} B \). But the prime cone \( \tilde{\alpha} \) of \( B \) can be regarded also as the pair

\[
(\rho(B(\tilde{\alpha})), \text{supp}_B(\tilde{\alpha})) = (\rho(\alpha), \tilde{p}) = (\rho(B(\tilde{\beta})), \text{supp}_B(\tilde{\beta}))
\]

see for instance [6] §3 or Proposition 1.3 in [1] so in fact \( \tilde{\alpha} = \tilde{\beta} \).

\( \psi \) is surjective

We may regard the elements of \( X \) as pairs of the form \((\alpha, \tilde{p}) \in \text{Sper} A \times \text{Spec} B\) such that \( \text{supp}_A(\alpha) = \tilde{p} \cap A = \phi_r(\tilde{p}) \). Thus let \( (\alpha, \tilde{p}) \in X \) and let the prime cone of \( B \) associated to the pair \((\rho(A(\alpha)), \tilde{p})\) be denoted by \( \tilde{\alpha} \), in fact specifically

\[
\tilde{\alpha} = \{ b \in B : b \text{ mod } \tilde{p} \in \rho(A(\alpha))^+ \}
\]

(see for instance remark in [6] after Satz 1, p.108). Then \( \text{supp}_B(\tilde{\alpha}) = \tilde{p} \) and \( \tilde{\alpha} \cap A = \phi_r(\tilde{\alpha}) = \alpha \) and therefore (by the definition of \( X \)) we get \( \psi(\tilde{\alpha}) = (\alpha, \tilde{p}) \).

It is easy to see that \( X \) is also a Stone space, therefore we have a continuous bijection \( \psi \) between a compact space \( \text{Sper} B \) and a Hausdorff space \( X \). This bijection is therefore also a closed map and thus a homeomorphism.

**Definition.** Let \( X \) be a topological space with topology \( T \) then

1. An **open filter**, \( \mathcal{U} \) on \( X \) is a subset of \( T \) which is also a filter (with the usual containment as partial ordering)

2. Similarly one defines an **open ultrafilter** on \( X \)

Below is a construction of absolutes of Hausdorff space as implemented by Porter and Woods in [11] §6.6 and in [10] §3.1.

**Construction 14.** (Iliadis absolutes) Let \( X \) be a Hausdorff space with topology \( T \). It is shown in [11] §6.6(d) that if \( \mathcal{U} \) is an open ultrafilter on \( X \) one has

\[
\bigcap_{U \in \mathcal{U}} U \neq \emptyset \Leftrightarrow \exists x \in X \Rightarrow \bigcap_{U \in \mathcal{U}} U = \{x\}
\]

The **Gleason space** of \( X \), denoted \( \theta X \), consists of the set of all open ultrafilters on \( X \) equipped with a topology generated by the open basis consisting of the sets of the form

\[
\{ \mathcal{U} \in \theta X : U \in \mathcal{U} \}
\]

\( U \in T \)

The **Iliadis absolute or absolute** of \( X \) is defined by

\[
\mathcal{E}X := \{ \mathcal{U} \in \theta X : \bigcap_{U \in \mathcal{U}} U \neq \emptyset \}
\]
and it is equipped with the subspace topology of $\theta X$. It is shown in [11] §6.6(e) that $E X$ is Stone and extremally disconnected.

There is a surjection from $E X$ to $X$, which we shall call the projection of the absolute of $X$ and denote it by $\pi_X$ which is defined by

$$\pi_X : E X \rightarrow X \quad \pi(U) := \bigcap_{U \in U} U \quad \forall U \in E X$$

It is shown in [11] §6.6(e)(6) that $X$ is regular (as topological space) iff $\pi_X$ is continuous. In particular if $X$ is Stone then $\pi_X$ is a continuous map.

**Definition.** A function $f : X \rightarrow Y$ between two topological spaces is called an irreducible surjection iff the function is continuous, surjective, closed and for any proper closed set $C \subseteq X$ we have $f(C) \subseteq Y$.

The above definition can be found in [11] 6.5(a). However, when discussing about a function having the property in the above definition we always accompany the word irreducible with the word surjection in order to avoid confusion (because "irreducible" is very frequently used in mathematics and could mean many different things).

**Lemma 15.** Let $A$ be a von Neumann regular ring, then the canonical map $\phi : \text{Spec} B(A) \rightarrow \text{Spec} A$ is an irreducible surjection.

**Proof.** That $\phi$ is continuous and closed is clear (because $\text{Spec} A$ is Hausdorff and $\text{Spec} B(A)$ is compact), it is also clearly surjective (see for instance [12] Lemma 1.14). Suppose now that there is a closed set $C \subseteq \text{Spec} B(A)$ such that $\phi(C) = \text{Spec} A$. Without loss of generality we may assume $C$ to be of the form $V_{B(A)}(b)$ for some $b \in B(A) \setminus \{0\}$ (Spec $B(A)$ \(C\) is open, so there is a nonempty basic open set contained in it). Now because $B(A)$ is a rational extension of $A$, there is an $a \in A$ such that $ba \in A \setminus \{0\}$. We know by Lemma 6 that $\phi(V_{B(A)}(ab)) = V_A(ab)$ (because we have a regular ring, we can express $V_{B(A)}(ab) = D_{B(A)}(x)$ for some $x \in A$) and so

$$V_{B(A)}(b) \subset V_{B(A)}(ab) \Rightarrow \text{Spec} A = \phi(V_{B(A)}(b)) \subset \phi(V_{B(A)}(ab)) = V_A(ab) = \text{Spec} A \Rightarrow ab = 0$$

which is a contradiction. \(\square\)

Because the above Lemma only uses the fact that $B(A)$ is a rational extension of $A$, we can use the same proof to show

**Corollary 16.** Let $A$ be a von Neumann regular ring and let $B$ be a rational extension of $A$, then the canonical map

$$\text{Spec} B \rightarrow \text{Spec} A$$

is an irreducible surjection.

**Proposition 17.** If $A$ is a von Neumann regular ring, then there is a homeomorphism

$$\psi : E \text{Spec} A \rightarrow \text{Spec} B(A)$$

such that $\phi \circ \psi = \pi_{\text{Spec} A}$, where

$$\phi : \text{Spec} B(A) \rightarrow \text{Spec} A$$

is just the canonical map.

**Proof.** This is an immediate consequence of the above Lemma, the fact that $\text{Spec} A$ is a regular space (since it is Stone), [11] §6.1(a) and [11] §4.8(h)(3). \(\square\)
So we do see that no direct information of the ring structure of $B(A)$ (for a real regular ring $A$) is necessary to obtain information about the topological space $\operatorname{Spec} B(A)$ and $\operatorname{Sper} B(A)$, the only information we needed for these topological spaces were those of $\operatorname{Spec} A$ and $\operatorname{Sper} A$.

Now we try to classify the real closure $\ast$ of an arbitrary real von Neumann regular ring. One may expect a combination of Lemma 4 and a modification of Theorem 9, however the result is rather more complicated than just that. We may indeed argue that the set of real closure $\ast$ of a real regular ring, say $A$, is the same as the set of real closure $\ast$ of its Baer hull (from Lemma 4). But we are dealing here with the sets with an equivalence relation that identify the real closure $\ast$ of $A$ upto $A$-isomorphisms. And we are not aware whether $B(A)$-isomorphism and $A$-isomorphism of the real closure $\ast$ are equivalent. What has been just discussed is best illustrated by the following Proposition (and its proof).

**Proposition 18.** Let $A$ be a real regular ring and set $B$ to be the Baer hull of $A$. Define now the following.

1. $S := \{s : \operatorname{Spec} B \to \operatorname{Sper} B : s$ is a continuous section of $\operatorname{supp}_B\}$
2. $C := \{C : C$ is a real closure $\ast$ of $A\} = \{C : C$ is a real closure $\ast$ of $B\}$
3. $\pi_r : \operatorname{Sper} B \to \operatorname{Sper} A$ $\bar{a} \mapsto \bar{a} \cap A$
4. an equivalence relation $\sim$ on $S$

$s \sim t \iff \operatorname{supp}_A(\pi_r(s(\operatorname{Spec} B)) \cap \pi_r(t(\operatorname{Spec} B))) = \operatorname{Spec} A$ $(s, t \in S)$

Then there is a bijection

$$\Phi : S/\sim \to C/\cong_A$$

**Proof.** Define first $\Phi_B : S \to C/\cong_B$ to be the bijection between the continuous sections of $\operatorname{supp}_B$ and the real closure $\ast$ of $B$ upto $B$-isomorphism as shown in Theorem 9. Now for any $s \in S$ set $C_s$ to be any chosen ring in $C$ such that $\Phi_B(s) = C_s/\cong_B$ (throughout, as we are dealing with $A$-isomorphisms our proof will be independent of the choice of this $C_s$ for any $s$).

We now need to first show that $\sim$ is actually an equivalence relation on $S$. The only difficult problem actually lies on proving transitivity. We claim that

$s \sim t \iff C_s \cong_A C_t$ $(s, t \in S)$

(independent of the choices of $C_s$ and $C_t$) and if we show this then we have also shown that $\sim$ is an equivalence relation. Let $s, t \in S$ then

$$s \sim t$$

$$\forall p \in \operatorname{Spec} A \exists \tilde{p}_s, \tilde{p}_t \in \operatorname{Spec} B$ and $\alpha \in \operatorname{Sper} A \Rightarrow$$

$$\tilde{p}_s \cap A = \tilde{p}_t \cap A = \operatorname{supp}_A(\alpha) = p$$

and $s(\tilde{p}_s) \cap A = t(\tilde{p}_t) \cap A = \alpha$

$$\iff$$

$$\forall p \in \operatorname{Spec} A \exists \tilde{p}_s, \tilde{p}_t \in \operatorname{Spec} B$ and $\alpha \in \operatorname{Sper} A \Rightarrow$$

$$\tilde{p}_s \cap A = \tilde{p}_t \cap A = \operatorname{supp}_A(\alpha) = p$$

and $C_s/\tilde{p}_s C_s \cong_{A/p} C_t/\tilde{p}_t C_t \cong_{A/p} \rho(B(s(\tilde{p}_s))) \cong_{A/p} \rho(B(t(\tilde{p}_t))) \cong_{A/p} \rho(A(\alpha))$

$$\iff$$

$$C_s \cong_A C_t$$

Now for any $s \in S$ let us denote $\bar{s}$ to be the image of $s$ in $S/\sim$. For such an $s$ we now set

$$\Phi(\bar{s}) = C_s/\cong_A$$

and we show that $\Phi$ defined in such way is ...
well-defined. Let $s, t \in S$ and $s \sim t$. Then by our previous claim this is equivalent to $C_s \cong_A C_t$. And thus $\Phi$ is indeed well-defined.

Injectivity is almost clear, because if $\Phi(s) = \Phi(t)$ for some $s, t \in S$ then by construction of $\Phi$ we get $C_s \cong_A C_t$ which by our very first claim implies that $s \sim t$.

Surjectivity is due to the fact that for any real closure $\ast$ of $A$ say $C$, there is an $s \in S$ such that $\Phi_B(s) = C/\cong_B$. And we thus have $\Phi(s) = C/\cong_A$.

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