On coherence of quantum operations by using Choi-Jamiołkowski isomorphism

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In quantum information, most information processing processes involve quantum channels. One manifestation of a quantum channel is quantum operation acting on quantum states. The coherence of quantum operations can be considered as a quantum resource, which can be exploited to perform certain quantum tasks. From the viewpoint of Choi-Jamiołkowski isomorphism, we study the coherence of quantum operations in the framework of resource theory. We define the phase-out superoperation and give the operation which transforms the Choi-Jamiołkowski state of a quantum operation to the Choi-Jamiołkowski state of the another quantum operation obtained by using the phase-out superoperation to act on the quantum operation. The set of maximally incoherent superoperations, the set of nonactivating coherent superoperations and the set of de-phase incoherent superoperations are defined and we prove that these sets are closed to compound operation and convex combination of quantum superoperations. Further, we introduce the fidelity coherence measure of quantum operations and obtain the exact form of the fidelity coherence measure of the unitary operations on the single qubit.

Keywords: coherence of operations; Choi-Jamiołkowski isomorphism; the phase-out superoperation; fidelity

I. INTRODUCTION

Quantum coherence is not only a fundamental feature of quantum physics, but also an important research field of quantum information theory. As a kind of special resource, it plays a significant role in quantum thermodynamics [1–4], quantum algorithm [5–8], quantum meteorology [9], etc. Quantifying and applying coherence is a very interesting work. In 2014, Baumgratz et al established a rigorous framework of the coherence resource theory for measuring quantum coherence [10]. After that, the study on the characterization and measure of quantum coherence has been made a great leap forward [11–18].

The framework of the coherence resource theory consists of three ingredients: incoherent states, incoherent operations and coherence measures. Incoherent states are those quantum states which do not possess any coherence resource. Incoherent operations are special quantum operations which can not generate coherence from incoherent states. Coherence measures being used to quantify the coherence of quantum states are functions mapping quantum states to real numbers.

In quantum information theory, most information processing processes involve and rely on quantum channels. One of the manifestation of quantum channel is the quantum operation acting on quantum states. Therefore, some researchers investigated the coherence of quantum operations. The works have been done mainly on characterizing a certain property of quantum operations, such as entanglement of quantum channels [19, 20], channel discrimination [21–23], channel simulation [24–27], quantum memory [28–30], the coherence of quantum operation and others [31–34]. As a matter of fact, the coherence of quantum operations can be regarded as a resource and applied to the corresponding quantum information processing.

Coherence resource theory of quantum operations is also composed of three elements: free operations, free superoperations and coherence measures of quantum operations. Here the superoperation is a map between operations [36–39]. Only free superoperations can transform free operations to free operations. The coherence measures of operations are functions which map quantum operations to real numbers.

Quantifying the coherence of operations is a very important and meaningful work which can enable us to have a deeper understanding of basic physics and also can provide new ideas for quantum information processing. There are two main aspects in the research of the coherence of quantum operations. The first one is to study quantum operations from the perspective that quantum operations change the coherence of quantum states [40–42]. The other one is to analyze quantum operations directly. Coherence resource theory of quantum operations promotes the research strongly [35, 43–49]. Among them, Orzekwa et al studied the coherence of quantum
operations based on entropy coherence and 2-norm coherence of quantum operations [46]. Bera proposed a resource theory framework to quantify the superposition that exists in any quantum evolutions [47]. Xu defined incoherent channels and incoherent superchannels, and established a resource theory for quantifying the coherence of quantum channels [48].

In this paper we investigate the coherence of quantum operations from the viewpoint of Choi-Jamiołkowski isomorphism. We first define the phase-out superoperation and study its properties. Then the three sets of superoperations are defined, and we prove that these three sets are closed to compound operation and convex combination of quantum superoperations. Then we introduce the fidelity coherence measure of quantum operations, and obtain the exact form of the fidelity coherence measure of the unitary operations on the single qubit.

II. PRELIMINARIES

Before we quantify the coherence of quantum operations, an introduction of the concepts and notation that will be used in the subsequent sections of our article is necessary. Let $\mathcal{H}^I$ and $\mathcal{H}^O$ be two $d$-dimensional Hilbert spaces with $\{|i\rangle\}_{i=0}^{d-1}$ and $\{|\alpha\rangle\}_{\alpha=0}^{d-1}$ being the orthonormal bases of $\mathcal{H}^I$ and $\mathcal{H}^O$, respectively. We assume that $\{|i\rangle\}_{i=0}^{d-1}$ and $\{|\alpha\rangle\}_{\alpha=0}^{d-1}$ are fixed and adopt the tensor basis $\{|i\alpha\rangle\}_{i\alpha}$ as the fixed basis when we consider the multipartite system with the Hilbert space $\mathcal{H}^{IO} = \mathcal{H}^I \otimes \mathcal{H}^O$. Let $\mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O)$ be the set of all quantum operations from $\mathcal{H}^I$ to $\mathcal{H}^O$.

A quantum operation is governed in terms of Choi-Jamiołkowski isomorphism $\text{SO}$ [50, 51]. An arbitrary operation $\Phi \in \mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O)$ is fully characterized by a matrix called the Choi-Jamiołkowski matrix of the operation $\Phi$

$$C_\Phi = (\mathbb{I} \otimes \Phi)|\varphi\rangle \langle \varphi|,$$

in $\mathcal{H}^I \otimes \mathcal{H}^O$. Here $|\varphi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ is a maximally entangled state in $d^2$-dimensional Hilbert space $\mathcal{H}^I \otimes \mathcal{H}^I$, and $\mathbb{I}$ is the identity operation. The one-to-one correspondence between $C_\Phi$ and $\Phi$ is given by

$$\Phi(\rho) = \text{tr}_I \left( \rho^T \otimes \mathbb{I} \right) C_\Phi,$$

where $\rho^T$ is the transpose of state $\rho$. The quantum operation $\Phi$ is completely positive and trace preserving if and only if $C_\Phi \geq 0$ and $\text{tr}_O(C_\Phi) = \frac{1}{d}$. When $C_\Phi \geq 0$ and $\text{tr}_O(C_\Phi) = 1$, the Choi-Jamiołkowski matrix $C_\Phi$ can be viewed as a density operator, a Choi-Jamiołkowski quantum state in the compound space $\mathcal{H}^I \otimes \mathcal{H}^O$. Since Choi-Jamiołkowski isomorphism guarantees an equivalence between $\Phi$ and $C_\Phi$, so one can treat quantum operations with the same tools which were normally used to treat quantum states. Further, the study of the coherence of the operation in the set $\mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O)$ of all quantum operations corresponding to Choi-Jamiołkowski states is transformed into the study of coherence of Choi-Jamiołkowski state.

Assume that $\Phi \in \mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O)$. It has been shown that $\Phi$ is an incoherent operation (IO) if [48]

$$C_\Phi = \sum_{i,\alpha=0}^{d-1} \frac{\Phi_{i\alpha\alpha}}{d} |i\alpha\rangle \langle i\alpha|,$$

where $\Phi_{i\alpha\alpha} = d \langle i\alpha| \otimes \Phi(|\varphi\rangle \langle \varphi|) |i\alpha\rangle \rangle = \langle i\alpha| \Phi(|i\rangle \langle i|) |i\alpha\rangle$.

Let $\mathcal{F}$ be the set of all incoherent operations in $\mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O)$, $\mathcal{F}_C$ be the set of all Choi-Jamiołkowski states of incoherent operations. A superoperation $\Omega$ is a map which transforms the operations $\Phi$ to the operation $\Omega(\Phi) = \Phi_1 \circ \Phi \circ \Phi_2$, where $\Phi_1, \Phi_2$ are also quantum operations in $\mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O)$. Let $\mathcal{SO}$ be the set of all superoperators i.e. $\mathcal{SO} = \{\Omega|\Omega : \mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O) \to \mathcal{O}(\mathcal{H}^I \to \mathcal{H}^O)\}$. A superoperation can be understood as a quantum operation that relates input Choi-Jamiołkowski states and output Choi-Jamiołkowski states, with an associated operator-sum representation. That is, for a superoperation $\hat{\Omega}$ relating input $\Phi$ and output $\Lambda$ operations, one may write

$$C_\Lambda = \Omega(C_\Phi) = \sum_n K_n C_\Phi K_n^\dagger = \sum_n p_n C_{\Lambda_n},$$

where

$$p_n = \text{tr} \left( K_n C_\Phi K_n^\dagger \right), \ C_{\Lambda_n} = \frac{K_n C_\Phi K_n^\dagger}{p_n},$$

(5)
and \{K_n\} are the Kraus operators of \(\Omega\) determined by \(\hat{\Omega}\), on the Choi-Jamiołkowski state. Obviously, the operation \(\hat{\Omega}\) has a one-to-one correspondence with the superoperation \(\hat{\Omega}\).

A superoperation \(\hat{\Omega}\) is a maximally incoherent superoperation if for all \(\Phi \in F\) one has \(\hat{\Omega}(\Phi) \in F\). We call \(\hat{\Omega}\) an incoherent superoperation if Kraus operators \{\(K_n\)\} of \(\hat{\Omega}\) determined by \(\hat{\Omega}\) are incoherent for each \(n \geq 1\) [48]. We use \(\mathcal{MISO}\) and \(\mathcal{ISO}\) to denote the set of all maximally incoherent superoperations and the set of all incoherent superoperations, respectively.

A coherence measure \(M\) of quantum operations should satisfy the following conditions [48].

1. Nonnegativity: \(M(\Phi) \geq 0\), for any \(\Phi \in \mathcal{O}(\mathcal{H}^I \rightarrow \mathcal{H}^O)\); and \(M(\Phi) = 0\) if and only if \(\Phi \in F\).
2. Monotonicity: \(M(\hat{\Omega}(\Phi)) \leq M(\Phi)\), for any \(\hat{\Omega} \in \mathcal{ISO}\), \(\Phi \in \mathcal{O}(\mathcal{H}^I \rightarrow \mathcal{H}^O)\).
3. Strong monotonicity: \(\sum_n p_n M(\Lambda_n) \leq M(\Phi)\) for any \(\hat{\Omega} \in \mathcal{ISO}\), with \(\{\Lambda_n\}\) being an incoherent expression of \(\Omega\) corresponding to \(\hat{\Omega}\), \(p_n = \text{tr} (K_n C_\Phi K_n^\dagger)\), and \(C_{\Lambda_n} = K_n C_\Phi K_n^\dagger\).
4. Convexity: \(M(\sum_n p_n \Phi_n) \leq \sum_n p_n M(\Phi_n)\), where \(\{\Phi_n\}\) are quantum operations belonging to \(\mathcal{O}(\mathcal{H}^I \rightarrow \mathcal{H}^O)\), and \(\{p_n\}\) is a probability distribution satisfying \(p_n \geq 0\) and \(\sum_n p_n = 1\).

III. MAIN RESULTS

A. The phase-out superoperation

It is well-know that in Hilbert space there is the phase-out operation, which can eliminate the coherence of the quantum state. Similarly, one can define the phase-out superoperation [48]. Since the incoherent states depend on the choice of the basis of the Hilbert space, so does the phase-out operation.

**Definition 1.** A superoperation \(\hat{\Theta} \in \mathcal{SO}\) is called the phase-out superoperation if \(\hat{\Theta}(\Phi)\) is an incoherent operation for any quantum operation \(\Phi\). The phase-out superoperation is implemented by

\[
\hat{\Theta}(\Phi) = \Delta^O \circ \Phi \circ \Delta^I,
\]

where \(\Delta^O\) and \(\Delta^I\) are the phase-out operations on Hilbert space \(\mathcal{H}^O\) and \(\mathcal{H}^I\) respectively.

According to Definition 1, we have

**Theorem 1.1.** The specific form of the quantum operation \(\Theta\) corresponding to the phase-out superoperation \(\hat{\Theta}\) is

\[
C_{\hat{\Theta}(\Phi)} = \Theta(C_\Phi) = \sum_{i=0}^{d-1} \langle \alpha | C_\Phi | \alpha \rangle \langle \alpha | \alpha \rangle.
\]

**Proof.** For any quantum operation \(\Phi \in \mathcal{O}(\mathcal{H}^I \rightarrow \mathcal{H}^O)\), the Choi-Jamiołkowski state corresponding to \(\Phi\) is

\[
C_\Phi = \sum_{i,j=0}^{d-1} \Phi_{i,j} | i\alpha \rangle \langle j\beta |,
\]

where \(\Phi_{i,j} = d (\langle \alpha | C_\Phi (| i\rangle \langle j |) | \beta \rangle) = \langle \alpha | \Phi (| i\rangle \langle j |) | \beta \rangle\). According to the definition of the phase-out superoperation, we know that the new operation \(\hat{\Theta}(\Phi)\) is an incoherent operation, then the Choi-Jamiołkowski state corresponding to \(\hat{\Theta}(\Phi)\) must be the diagonal in the fixed basis \(\{|i\alpha\rangle\}_{i=0}^{d-1}\), i.e.,

\[
C_{\hat{\Theta}(\Phi)} = \sum_{i=0}^{d-1} \hat{\Theta}(\Phi)_{i,i} | i\alpha \rangle \langle i\alpha | = \sum_{i=0}^{d-1} \hat{\Theta}(\Phi)_{i,i} | i\alpha \rangle \langle i\alpha |,
\]

\[
\hat{\Theta}(\Phi)_{i,i} = \sum_{i=0}^{d-1} \hat{\Theta}(\Phi)_{i,i} | i\alpha \rangle \langle i\alpha |.
\]
where

\[
\Theta(\Phi)_{i\alpha\alpha} = d \langle i\alpha| \mathbb{I} \otimes \Theta(\Phi)(|\varphi\rangle\langle \varphi|)|i\alpha\rangle \\
= \langle i\alpha| \Theta(\Phi)(|i\rangle\langle i|)|i\alpha\rangle \\
= \langle i\alpha| \Delta^O \circ \Phi \circ \Delta^I (|i\rangle\langle i|)|i\alpha\rangle \\
= \langle i\alpha| \Delta^O \circ \Phi (|i\rangle\langle i|)|i\alpha\rangle \\
= \langle i\alpha| \Delta (\sum_{\beta, \gamma} \langle \beta|\Phi (|i\rangle\langle i|)|\beta\rangle \langle \gamma|)|i\alpha\rangle \\
= \langle i\alpha| \Phi (|i\rangle\langle i|)|i\alpha\rangle \\
= \Phi_{i\alpha\alpha}. \tag{10}
\]

Thus,

\[
C_{\Theta(\Phi)} = [\mathbb{I} \otimes (\Delta^O \circ \Phi \circ \Delta^I)] |\varphi\rangle\langle \varphi| \\
= \sum_{i,\alpha=0}^{d-1} \frac{\Phi_{i\alpha\alpha}}{d} |i\alpha\rangle \langle i\alpha|. \tag{11}
\]

Obviously the Kraus operators of \( \Theta \) are \( \{ M_{i\alpha} = |i\alpha\rangle\langle i| \}_{i,\alpha=0}^{d-1} \), because

\[
\Theta(C_{\Phi}) = \sum_{i\alpha} M_{i\alpha} C_{\Phi} M^\dagger_{i\alpha} \\
= \sum_{i\alpha} |i\alpha\rangle \langle i\alpha| \left( \sum_{j,j',a',b'=0}^{d-1} \frac{\Phi_{j'j'a'b'}}{d} |j'a'\rangle \langle j'| \right) |i\alpha\rangle \langle i\alpha| \\
= \sum_{i,\alpha=0}^{d-1} \frac{\Phi_{i\alpha\alpha}}{d} |i\alpha\rangle \langle i\alpha| \\
= \sum_{i,\alpha=0}^{d-1} \langle i\alpha| C_{\Phi} |i\alpha\rangle \langle i\alpha| \\
= C_{\Theta(\Phi)}. \tag{12}
\]

It is easy to check that \( \Theta \) is a resource destroying map [52]. The essence of the quantum operation \( \Theta \) acting on any Choi-Jamiołkowski state is that all the non-diagonal elements in the selected base are completely eliminated and a new Choi-Jamiołkowski state with only diagonal elements is created.

So we get the specific form of the quantum operation \( \Theta \) corresponding to the phase-out superoperation \( \tilde{\Theta} \) is

\[
\Theta(C_{\Phi}) = \sum_{i,\alpha=0}^{d-1} \langle i\alpha| C_{\Phi} |i\alpha\rangle \langle i\alpha| = C_{\tilde{\Theta}(\Phi)}. \tag{13}
\]

This completes the proof of Theorem 1.1.

**Theorem 1.2.** If a quantum operation \( \Phi \in O(\mathcal{H}^I \rightarrow \mathcal{H}^O) \) is a completely positive trace preserving operation (CPTP), then \( \tilde{\Theta}(\Phi) \) is a CPTP.

**Proof.** Obviously, if a quantum operation \( \Phi \in O(\mathcal{H}^I \rightarrow \mathcal{H}^O) \) is a CPTP, then \( \Phi \) satisfies

\[
C_{\Phi} \geq 0, \quad \text{tr}(C_{\Phi}) = \frac{\mathbb{I}}{d},
\]

where \( d \) is the dimension of the Hilbert space \( \mathcal{H}^I \).
It is easy to obtain
\[
\text{tr}_O(C_\Phi) = \text{tr}_O \left( \frac{1}{d} \sum_{i,j=0}^{d-1} |i\rangle \langle j| \otimes \sum_{\alpha,\beta=0}^{d-1} \Phi_{ij\alpha\beta} |\alpha\rangle \langle \beta| \right)
\]
\[
= \left( \frac{1}{d} \sum_{i,j=0}^{d-1} |i\rangle \langle j| \right) \text{tr} \left( \sum_{\alpha,\beta=0}^{d-1} \Phi_{ij\alpha\beta} |\alpha\rangle \langle \beta| \right)
\]
\[
= \sum_{i,j=0}^{d-1} \Phi_{ij\alpha\alpha} \frac{|i\rangle \langle j|}{d}.
\]
(14)

As \(\Phi \in O(H_I \rightarrow H_O)\) is a CPTP, so one must have
\[
\sum_{i,j=0}^{d-1} \Phi_{ij\alpha\alpha} \frac{|i\rangle \langle j|}{d} = I_d.
\]
(15)
The above equation means that
\[
\sum_{\alpha=0}^{d-1} \Phi_{ij\alpha\alpha} = \delta_{ij}.
\]
(16)
The Choi-Jamiołkowski state corresponding to \(\tilde{\Theta}(\Phi)\) is
\[
C_{\tilde{\Theta}(\Phi)} = \sum_{i,a=0}^{d-1} \frac{\tilde{\Theta}(\Phi)_{ia\alpha} |i\alpha\rangle \langle \alpha|}{d} = \sum_{i,a=0}^{d-1} \frac{\Phi_{ija\alpha} |i\alpha\rangle \langle \alpha|}{d}.
\]
(17)
So it is easy to obtain
\[
\text{tr}_O(C_{\tilde{\Theta}(\Phi)}) = \text{tr}_O \left( \frac{1}{d} \sum_{i=0}^{d-1} |i\rangle \langle i| \otimes \sum_{\alpha=0}^{d-1} \tilde{\Theta}(\Phi)_{ia\alpha} |\alpha\rangle \langle \alpha| \right)
\]
\[
= \left( \frac{1}{d} \sum_{i=0}^{d-1} |i\rangle \langle i| \right) \text{tr} \left( \sum_{\alpha=0}^{d-1} \tilde{\Theta}(\Phi)_{ia\alpha} |\alpha\rangle \langle \alpha| \right)
\]
\[
= \sum_{i=0}^{d-1} \frac{\tilde{\Theta}(\Phi)_{ia\alpha} |i\rangle \langle i|}{d} \quad (18)
\]
\[
= \frac{\sum_{i=0}^{d-1} \Phi_{ija\alpha} |i\rangle \langle i|}{d}
\]
\[
= \frac{I_d}{d}.
\]
Thus we have
\[
C_{\tilde{\Theta}(\Phi)} \geq 0, \quad \text{tr}_O(C_{\tilde{\Theta}(\Phi)}) = \frac{I_d}{d},
\]
(19)
which means that \(\tilde{\Theta}(\Phi)\) is a CPTP. Theorem 1.2 has been proved.

**B. The three sets of quantum superoperations**

We now give the three sets of quantum superoperations based on the phase-out superoperation \(\tilde{\Theta}\).

Let \(\tilde{\Omega}\) be a quantum superoperation. Suppose that \(\tilde{\Omega}\) satisfies
\[
\Omega \circ \tilde{\Theta} = \tilde{\Theta} \circ \Omega \circ \tilde{\Theta},
\]
(20)
where \( \circ \) is the composition of superoperations. The compound superoperation \( \hat{\Omega} \circ \Theta \) and \( \Theta \circ \hat{\Omega} \circ \hat{\Theta} \) acting on the any quantum operation \( \Phi \), generate the following quantum operations

\[
\hat{\Omega} \circ \hat{\Theta}(\Phi) = \hat{\Omega}(\hat{\Theta}(\Phi)), \\
\hat{\Theta} \circ \hat{\Omega} \circ \hat{\Theta}(\Phi) = \hat{\Theta}(\hat{\Omega}(\hat{\Theta}(\Phi))).
\]

Since Eq.(20) holds, so any output operation of the superoperation \( \hat{\Omega} \circ \hat{\Theta} \) can be regarded as an output operation of the phase-out superoperation \( \hat{\Theta} \). Thus, all output operations of superoperation \( \hat{\Omega} \circ \hat{\Theta} \) are incoherent operations. In other words, the set of incoherent operations is closed under \( \hat{\Omega} \) if \( \hat{\Omega} \) satisfies Eq.(20). Therefore we call condition Eq.(20) the nongenerating coherent conditions. The superoperations satisfying this condition are called nongenerating coherent superoperations. Usually, nongenerating coherent superoperations is also called maximally incoherent superoperations \([29, 49]\). The set of maximally incoherent superoperations is denoted as \( MISO \).

Next, we consider the following dual form of the nongenerating coherent condition:

\[ \hat{\Theta} \circ \hat{\Omega} = \hat{\Theta} \circ \hat{\Omega} \circ \hat{\Theta}. \]

We call this condition the exchangeable resource condition. The superoperations satisfying this condition are called exchangeable resource superoperations. We also call exchangeable resource superoperations de-phase incoherent superoperations \([49]\). The set of de-phase incoherent superoperations is denoted as \( DISO \).

It is easy to obtain the following conclusions for the sets \( MISO, MISO^* \) and \( DISO \).

1. The relationship between \( MISO, MISO^* \) and \( DISO \) is \( DISO = MISO \cap MISO^* \).

2. For any quantum superoperation \( \hat{\Omega} \), the following relationship exists

\[
\hat{\Theta} \circ \hat{\Omega} \in MISO, \quad \hat{\Theta} \circ \hat{\Omega} \circ \hat{\Theta} \in MISO.
\]

3. If a quantum superoperation \( \hat{\Omega} \in MISO^* \), then there exists the following relationship

\[
\hat{\Theta} \circ \hat{\Omega} \in MISO^*, \quad \hat{\Omega} \circ \hat{\Theta} \in MISO^*.
\]

Furthermore, we also have

**Theorem 2.1.** \( MISO, MISO^* \) and \( DISO \) are closed to compound operation and convex combination of quantum superoperations.

**Proof.** First, we prove that \( MISO \) is closed to the compound operation and convex combination of quantum superoperations.

For any quantum superoperations \( \hat{\Omega}_1, \hat{\Omega}_2 \in MISO \), there are

\[
\hat{\Omega}_1 \circ \hat{\Theta} = \hat{\Theta} \circ \hat{\Omega}_1 \circ \hat{\Theta}, \\
\hat{\Omega}_2 \circ \hat{\Theta} = \hat{\Theta} \circ \hat{\Omega}_2 \circ \hat{\Theta}.
\]

Thus we have

\[
(\hat{\Omega}_1 \circ \hat{\Omega}_2) \circ \hat{\Theta} = \hat{\Theta} \circ (\hat{\Theta} \circ (\hat{\Omega}_1 \circ \hat{\Theta})) \\
= \hat{\Theta} \circ \hat{\Omega}_1 \circ \hat{\Theta} \circ \hat{\Omega}_2 \circ \hat{\Theta} \\
= \hat{\Theta} \circ (\hat{\Omega}_1 \circ \hat{\Omega}_2) \circ \hat{\Theta}.
\]
The above equation means that $\tilde{\Omega}_1 \circ \tilde{\Omega}_2 \in MISO$. Therefore $MISO$ is closed to the compound operation. If there is a quantum superoperation $\tilde{\Omega}$ which satisfies $\tilde{\Omega} = p\tilde{\Omega}_1 + (1 - p)\tilde{\Omega}_2$, where $0 \leq p \leq 1$, then we have

$$
\tilde{\Omega} \circ \tilde{\Theta} = (p\tilde{\Omega}_1 + (1 - p)\tilde{\Omega}_2) \circ \tilde{\Theta} = p\tilde{\Theta} \circ \tilde{\Omega}_1 \circ \tilde{\Theta} + (1 - p)\tilde{\Theta} \circ \tilde{\Omega}_2 \circ \tilde{\Theta} = \tilde{\Theta} \circ (p\tilde{\Omega}_1 + (1 - p)\tilde{\Omega}_2) \circ \tilde{\Theta} = \tilde{\Theta} \circ \tilde{\Omega} \circ \tilde{\Theta}.
$$

Hence $\tilde{\Omega} \in MISO$. That indicates that $MISO$ is closed to the convex combination of quantum superoperations.

Similarly, one can prove that $MISO^*$ and $DISO$ are closed to the compound operation and convex combination of quantum superoperations. Therefore Theorem 2.1 holds.

C. The fidelity coherence measure of operations

How to measure the coherence of operations is very important in the study of quantum coherence resources. The coherence of operations have been measured by the relative entropy coherence measure [47, 48], the $l_1$ norm coherence measure [47, 48] and the robustness coherence measure [27]. Next we will give another measure, the fidelity coherence measure of operations and provide some examples.

**Definition 2.** For any two quantum operations $\Phi, \Lambda \in O(H^I \rightarrow H^O)$, the fidelity $F(\Phi, \Lambda)$ of the operations $\Phi$ and $\Lambda$ is defined as

$$
F(\Phi, \Lambda) = F(C_\Phi, C_\Lambda),
$$

where $C_\Phi$ and $C_\Lambda$ are Choi-Jamiołkowski states corresponding to $\Phi$ and $\Lambda$ respectively, $F(C_\Phi, C_\Lambda)$ is the fidelity of $C_\Phi$ and $C_\Lambda$.

By Ref. [48], we have

**Lemma 1.** If $M$ is a coherence measure for quantum states in the Baumgratz-Cramer-Plenio (BCP) framework [10], then

$$
M(\Phi) \equiv M(C_\Phi), \quad \Phi \in O(H^I \rightarrow H^O)
$$

is a coherence measure for quantum operations.

According to the fidelity of quantum operations and Lemma 1, we can define the fidelity coherence measure of quantum operations as follows.

**Definition 3.** For any quantum operation $\Phi \in O(H^I \rightarrow H^O)$, if the Choi-Jamiołkowski state $C_\Phi$ is a pure state, the fidelity coherence measure of the operation $\Phi$ is defined as

$$
M_f(\Phi) = \min_{C_\Lambda \in \mathcal{C}_\epsilon} \sqrt{1 - F(C_\Phi, C_\Lambda)},
$$

where $F(\rho, \sigma) = (\text{tr}(\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}))^2$ is the Uhlmann fidelity [53], $\mathcal{C}_\epsilon$ is the set of pure incoherent states in the Hilbert space $H^I \otimes H^O$. Then we can define convex-roof extended fidelity coherence measure of the operations for the general case as

$$
M_f(\Phi) = \min_{\{p_n, \Phi_n\}} \sum_n p_n M_f(\Phi_n),
$$

where the minimum is taken over all the ensembles $\{p_n, \Phi_n\}$ realizing $\Phi$, i.e., $\Phi = \sum_n p_n \Phi_n$, for any $n$, $C_{\Phi_n}$ is the pure Choi-Jamiołkowski state corresponding to the operation $\Phi_n$.

For the general unitary operations on a single qubit we have the following result.

**Theorem 3.1.** The fidelity coherence measure of the general form of unitary operations

$$
U = e^{i\alpha} \begin{bmatrix}
    e^{-i\frac{\pi}{2}} & 0 & \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\
    0 & e^{i\frac{\pi}{2}} & \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \\
    e^{-i\frac{\pi}{2}} & 0 & \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \\
    0 & e^{i\frac{\pi}{2}} & \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2}
\end{bmatrix}
$$
Because the Choi-Jamiołkowski state corresponding to the unitary matrix $U$ is

$$
M_f(U) = \min \left\{ \sqrt{1 - \frac{|\cos \frac{\theta}{2}|^2}{2}}, \sqrt{1 - \frac{|\sin \frac{\theta}{2}|^2}{2}} \right\}.
$$

(34)

Here $a, \beta, \gamma$ and $\delta$ are any real numbers.

**Proof.** It is easy to derive that the general form of unitary operation $U$ on a single qubit can be written as

$$
U = \begin{bmatrix}
a & -b \\
e^{i\alpha}b^* & e^{i\alpha}a^*
\end{bmatrix},
$$

(35)

where

$$
a = e^{i(\alpha - \beta + i\delta)}\cos \frac{\gamma}{2}, \quad b = e^{i(\alpha - \beta + i\delta)}\sin \frac{\gamma}{2}.
$$

(36)

The Choi-Jamiołkowski state corresponding to the unitary matrix $U$ is

$$
C_U = (\mathbb{I} \otimes U) \left( \frac{1}{2} \sum_{ij=0}^{1} |ij\rangle \langle jj| \right) (\mathbb{I} \otimes U)^\dagger,
$$

$$
= \frac{1}{2} \begin{bmatrix}
aa^* & e^{-2ia}ab & -ab^* & e^{-2ia}aa \\
e^{2ia}a^*b^* & bb^* & -e^{2ia}b^*b & ab^* \\
-e* b & -e^{-2ia}b^*b & bb^* & -e^{-2ia}ab \\
e^{2ia}a^*a^* & a^*b & -e^{2ia}a^*b^* & aa^*
\end{bmatrix}
$$

(37)

$$
= \frac{1}{\sqrt{2}} \left( a|00\rangle + e^{2ia}b^*|01\rangle - b|10\rangle + e^{2ia}a^*|11\rangle \right) \times \frac{1}{\sqrt{2}} \left( a^*|00\rangle + e^{-2ia}b|01\rangle - b^*|10\rangle + e^{-2ia}a^*|11\rangle \right).
$$

Because the Choi-Jamiołkowski state $C_U$ is a pure state, the fidelity coherence measure of $U$ is

$$
M_f(U) = \min \left\{ \sqrt{1 - \frac{|a|^2}{2}}, \sqrt{1 - \frac{|b|^2}{2}} \right\} = \min \left\{ \sqrt{1 - \frac{|\cos \frac{\theta}{2}|^2}{2}}, \sqrt{1 - \frac{|\sin \frac{\theta}{2}|^2}{2}} \right\}.
$$

(38)

The proof of Theorem 3.1 is completed.

Obviously, we have the following result.

**Corollary 3.2.** For any unitary operation $U$ on a single qubit, the range of values of $M_f(U)$ is $[\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}]$.

**Corollary 3.3.** The fidelity coherence measure $M_f(\Phi_{\text{max}})$ of the the maximally coherent operation $\Phi_{\text{max}} = \sum_{\alpha=0}^{1} \frac{1}{\sqrt{2}} e^{i\theta_{\alpha}}|\alpha\rangle \langle \alpha|$ on a single qubit [48], is largest in the unitary operations on a single qubit. Here $\theta_{\alpha}$ is a real number.

**Proof.** The Choi-Jamiołkowski state corresponding to $\Phi_{\text{max}}$ is

$$
C_{\Phi_{\text{max}}} = \sum_{i,j,\alpha=0}^{1} \frac{1}{4} e^{i(\alpha - \beta)}|i\alpha\rangle \langle j\beta|.
$$

(39)

Because $C_{\Phi_{\text{max}}}$ is a pure state, it is easy to derive that the fidelity coherence measure of $\Phi_{\text{max}}$ is

$$
M_f(\Phi_{\text{max}}) = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}.
$$

(40)

By using Corollary 3.2, we know that the fidelity coherence measure of the the maximally coherent operation is largest in the unitary operations on a single qubit. This ends the proof.
Based on Choi-Jamiolkowski isomorphism we investigated the coherence of quantum operations in the resource theory. We first defined the phase-out superoperation and gave the form of operation which transforms the Choi-Jamiolkowski state of the quantum operation $\Phi$ to the Choi-Jamiolkowski state of the quantum operation $\hat{\Theta}(\Phi)$ obtained by using the phase-out superoperation $\hat{\Theta}$ to act on $\Phi$. By using phase-out superoperation we defined the set of maximally incoherent superoperations, the set of nonactivating coherent superoperations and the set of de-phase incoherent superoperations and proved that these sets are closed to compound operation and convex combination of quantum superoperations. Further, the fidelity coherence measure of quantum operations was introduced. The fidelity coherence measure of the unitary operations on the single qubit has been calculated. We hope the research will lead to a better understanding of the coherence of quantum operations.

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