Efficient Generation of Low Autocorrelation Binary Sequences

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Abstract—Simple and efficient algorithm based on heuristic search by shotgun hill climbing to construct binary sequences with small peak sidelobe levels (PSL) is suggested. The algorithm is applied for generation of binary sequences of lengths between 106 and 300. Improvements are obtained in almost half of the considered lengths while for the rest of the lengths, binary sequences with the same PSL values as reported in the state-of-the-art publications are found.

Index Terms—Aperiodic Autocorrelation Function, Binary Sequences, Peak Sidelobe Level (PSL), Shotgun Hill Climbing

I. INTRODUCTION

SEQUENCES with low autocorrelation functions are necessary for a variety of signal and information processing applications. For example, in pulse codes-based compression for radars and sonars such sequences are used in order to obtain high resolution. The shifts of sequences with low autocorrelation can be also used for better synchronization purposes or to identify users in multi-user systems. Due to their big practical importance, these sequences have been widely studied and various methods for constructing sequences with small values of the autocorrelation are developed.

The binary sequences of low autocorrelation are of special interest and some of the well known such sequences are the Barker codes [11], M-sequences [2], Gold codes [3], Kasami codes [4], Weil sequences [5], Legendre sets [6] and others (see [7][8]). Barker sequences are known to have the best autocorrelation properties, but the longest sequence is of length 13. M-sequences, Gold codes and Kasami sequences have ideal periodic autocorrelation functions but have no constraints on the sidelobes of their aperiodic autocorrelation functions. As summarized in [9], during the years a variety of analytical constructions and computer search methods are developed in order to construct binary sequences with relatively small or minimal PSL. By an exhaustive search the minimum values of the PSL, for \( n \leq 40 \) [10], \( n \leq 48 \) [11], \( n = 64 \) [12], \( n \leq 68 \) [13], \( n \leq 74 \) [14], \( n \leq 80 \) [15], \( n \leq 82 \) [16] and \( n \leq 84 \) [17] are obtained. The best currently known values for PSL, for \( 85 \leq n \leq 105 \) are published in [18], and for \( n \geq 106 \) in [19]. In this work we suggest an efficient and easy to implement heuristic algorithm and, as an illustration of its effectiveness, we apply it for generation of binary sequences with lengths between 106 and 300. The generated by us sequences are better, in terms of PSL values, than a significant part of those obtained in [19] ones. Our algorithm can also be used for generation of sequences with lengths greater than 300.

II. PRELIMINARIES

Let \( B = (b_0, b_1, \cdots, b_{n-1}) \) be a binary sequence of length \( n > 1 \), where \( b_i \in \{-1, 1\}, 0 \leq i \leq n - 1 \). The aperiodic autocorrelation function (AACF) of \( B \) is given by

\[
C_u(B) = \sum_{j=0}^{n-u-1} b_j b_{j+u}, \quad \text{for } u \in \{0, 1, \cdots, n-1\}.
\]

We will note that the AACF is originally defined in the interval \( \{-n+1, -n+2, \cdots, -2, -1, 0, 1, 2, \cdots, n-1\} \). As the AACF is an even function with \( C_u(B) = -C_{-u}(B) \), we will consider it for the interval \( \{0, 1, \cdots, n-1\} \) only. The \( C_0(B) \) is called mainlobe and the rest \( C_u(B) \) for \( u \in \{1, \cdots, n-1\} \) are called sidelobe levels. We define the peak sidelobe level (PSL) [20] of \( B \) as

\[
PSL(B) = \max_{0 \leq u < n} |C_u(B)|.
\]

The value of the PSL can also be represented in decibels

\[
PSL_{dB}(B) = 20 \log \left( \frac{PSL(B)}{n} \right).
\]

Another important measure of an AACF is the merit factor (MF), which gives the ratio of the energy of the mainlobe level to the energy of sidelobe levels, i.e.

\[
MF(B) = \frac{C_0(B)}{2 \sum_{u=1}^{n-1} |C_u(B)|^2}.
\]

III. THE FITNESS FUNCTION CHOICE

Since our goal is to lower the PSL of a given binary sequence, i.e. to lower the value of PSL(B), it makes sense to simultaneously lower the values of each \( C_u(B) \), for \( u \in \{1, \cdots, n-1\} \). By making this observation, we define the following fitness function:

\[
F(B) = \left( \sum_{u=1}^{n-1} |C_u(B)|^P \right)^P = \sum_{u=1}^{n-1} \left( \sum_{j=0}^{n-u-1} |b_j b_{j+u}| \right)^P,
\]

where \( P \) is the magnitude of the fitness function, i.e. the higher the magnitude is the higher the fitness function intolerance to large absolute values of \( C_u(B) \)’s will be. We made experiments with various values of \( P \) and the best results were obtained for values in the interval [3, 5]. Lower values of \( P \) makes the fitness function too tolerant to higher
absolute values of the PSLs $C_w(B)$, while higher values of $P$ are heavily populating the heuristic topology with local minimums. We have fixed the magnitude $P$ of the fitness function to 4.

IV. HEURISTIC SEARCH OF BINARY SEQUENCES WITH SMALL PSL BY SHOTGUN HILL CLIMBING

Let’s denote the $i$-th position of a binary sequence $B$ of length $n$ as $b_i$. Flipping the $i$-th position of $B$ is to interchange $b_i$ with $\overline{b_i}$. By the neighborhood of the binary sequence $B$, denoted by $N(B)$, we define the set of all binary sequences constructed from $B$ by making a single flip in $B$.

The optimization process takes as input the length of the binary sequence $n$, the fitness function $F$, the threshold value $t$, the two integers $h_{\text{min}}$ and $h_{\text{max}}$ defining the flipping allowance interval, and the goal $G$ which is the desired final PSL value to be reached.

At the beginning we generate a random binary sequence $B$ of length $n$. Then, by searching the neighborhood of $B$, we look for a better binary sequence, i.e. a binary sequence with smaller fitness value. If some $X$ out of the neighbors of $B$ has PSL equal to $G$ we output $X$ and quit. If during the search of the neighborhood no better binary sequence is found, we are stuck in some local minimum $B’$. In order to escape the local minimum we flip $h$ randomly chosen elements of $B’$, where $h \in [h_{\text{min}}, h_{\text{max}}]$. We will call such try a quake. In the case when $t$ consecutive quakes are not sufficient to escape the local minimum, we start the process from the beginning by randomly generating new binary sequence, i.e. the shotgun hill climbing approach. The algorithm stops when a binary sequence with the searched value of the PSL is found or when the preliminary defined number of restarts is reached. The pseudo code of the shotgun hill climbing (SHC) algorithm is given in Algorithm 1.

Evidently, the fitness function is the critical resource demanding routine of the algorithm. However, its complexity is comparable to the complexity of the binary sequence PSL calculation itself. The additional negligible overheat is caused by the calculation of the sum of all the $P$-powered mainlobes.

V. THE HYPER-PARAMETERS CHOICE

The parameter $h_{\text{min}}$ should be tolerant to possible optimizations involving any small number of flips. Having this in mind and without any restrictions, we choose $h_{\text{min}} = 1$. On the other hand, fixing a value of the parameter $h_{\text{max}}$ is a trade-off between accuracy and flexibility - smaller values of $h_{\text{max}}$ will decrease the algorithm chances to escape from a given local minimum, while higher values of $h_{\text{max}}$ will greatly defocus the climbing routines (for example, hoping from hill A to another hill B, before reaching the local minimum of A). During our experiments, we have fixed the value of $h_{\text{max}}$ as $\lfloor \sqrt{n} \rfloor$, where $n$ is the length of the starting binary sequence.

Another important parameter is the threshold value $t$. Choosing a small value of $t$ allows us to restart the process of searching a binary sequence with low PSL value and, instead of loosing more time in trying to escape the current local minimum we have stuck at, we reinitialize the searching procedure by starting from the beginning.

We have tried different meta-heuristic strategies like, for example, simulated annealing method and tabu search. However, it appears that regularly reinitializing the current state of the algorithm, i.e. the core concept of the shotgun hill climbing method, is a more productive strategy to utilize than the aforementioned ones. Evidently, the initial state does matter and by having a low value of $t$ we increase our chances to reinitialize the algorithm from a highly-competitive candidate. During our experiments, we have used a threshold of $t = 10^3$.

VI. RESULTS

We present in Tables I II III and IV the obtained by Algorithm 1 results for binary sequences of lengths from 106
to 300. The second column contains the best known by us value of the PSL for the corresponding length. In the third column we present the best value of the PSL obtained by the Algorithm 1 and in the fourth the corresponding sequence with this value of the PSL. The sequences are given in a hexadecimal format where \(1\)'s are replaced by zeros and the leading \(1\)'s are omitted. For example, the binary sequence of length \(11\) \(B = (-1,-1,1,1,-1,-1,1,1,1,1,1)\) is given by \(b_7\). The decoding procedure requires the length of the binary sequence. The corresponding values of the PSL in decibels and of the merit factor are calculated and given in the fifth and sixth columns respectively.

We improve the PSL values for 95 from the included 195 lengths. The remaining 100 binary sequences have the same values of the PSL as the currently known best ones. Furthermore, all of them are unique and unpublished before.

The suggested in this work algorithm is highly parallelizable so that a multicore architecture can be fully utilized. It is implemented on Python on a single mid-range computer with an octa-core CPU. During our experiments, the time required to reach a given PSL goal was between few minutes to several hours. Furthermore, with each instance of the algorithm, we repeatedly reached binary sequences with lower or same PSL than the -table代理商的结果。
### TABLE III

| Old | New | Binary sequence in HEX | db | NF |
|-----|-----|------------------------|----|----|
| 254 | 12 | 11 | 9e75e4101c087f8d4af54b3139f43e6554 | -27.37 | 4.264 |
| 255 | 12 | 11 | 797b0c9505d54560e43f45224f42 | -27.37 | 4.116 |
| 256 | 12 | 11 | 5f4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 257 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 258 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 259 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 260 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 261 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 262 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |

### TABLE IV

| Old | New | Binary sequence in HEX | db | NF |
|-----|-----|------------------------|----|----|
| 254 | 12 | 11 | 9e75e4101c087f8d4af54b3139f43e6554 | -27.37 | 4.264 |
| 255 | 12 | 11 | 797b0c9505d54560e43f45224f42 | -27.37 | 4.116 |
| 256 | 12 | 11 | 5f4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 257 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 258 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 259 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 260 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 261 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 262 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 263 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 264 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 265 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 266 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |
| 267 | 12 | 11 | 5e4d7f49b428a42b4b53e834d330c62c2 | -27.405 | 3.962 |

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**Notes:**
- Old and New columns represent the original and updated binary sequences, respectively.
- NF represents the number of frames.
- The values in the 'db' column indicate the difference in decibel values.
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