GLOBAL SOLUTIONS TO THE ISENTROPIC COMpressible Navier-Stokes EQUAtIONS WITH A CLASS OF LARGE INITIAL DATA

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ABSTRACT. In this paper, we consider the global well-posedness problem of the isentropic compressible Navier-Stokes equations in the whole space \( \mathbb{R}^N \) with \( N \geq 2 \). In order to better reflect the characteristics of the dispersion equation, we make full use of the role of the frequency on the integrability and regularity of the solution, and prove that the isentropic compressible Navier-Stokes equations admit global solutions when the initial data are close to a stable equilibrium in the sense of suitable hybrid Besov norm. As a consequence, the initial velocity with arbitrary \( H^{1-1_2}_2 \) norm of potential part \( \mathcal{P}^1 u_0 \) and large highly oscillating are allowed in our results. The proof relies heavily on the dispersive estimates for the system of acoustics, and a careful study of the nonlinear terms.

1. Introduction

The isentropic compressible Navier-Stokes equations are governed by conservation of mass and conservation of momentum:

\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, & t > 0, \ x \in \mathbb{R}^N, \ N \geq 2, \\
\rho(\partial_t u + u \cdot \nabla u) + \nabla P(\rho) - \mu \Delta u - (\lambda + \mu)\nabla \text{div} u = 0,
\end{cases}
\]

where the unknowns \( \rho \) and \( u \) are the density and velocity of the fluid, respectively. \( P = P(\rho) \) is the pressure, which is a smooth function of \( \rho \). The viscous coefficients \( \mu \) and \( \lambda \) are assumed to be constants, satisfying the following physical restrictions:

\[
\mu > 0, \quad 2\mu + N\lambda \geq 0,
\]

with \( N \geq 2 \) the spacial dimension. Clearly, (1.2) implies \( \nu := \lambda + 2\mu > 0 \), which, together with (1.2) ensures the ellipticity for the Lamé operator \( \Delta + (\lambda + \mu)\nabla \text{div} \). Moreover, without loss of generality, we assume that \( \tilde{\rho} = 1 \) and

\[
P'(1) = 1.
\]

There are huge literatures on the well-posedness results of the compressible Navier-Stokes equations. To the best of our knowledge, the local existence and uniqueness of classical solutions are first established in [33] with \( \rho_0 \) bounded away from zero. For the case that the initial density may vanish in open sets, see [12] and [34]. The global classical solutions were first obtained by Matsumura and Nishida [32] for initial data \((\rho_0, u_0)\) close to an equilibrium \((\tilde{\rho}, 0)\) in \( H^3 \times H^2 \), \( \rho > 0 \). Later, by exploiting some smoothing effects of the so-called effective viscous flux \( F := (2\mu + \lambda)\text{div} u - P(\rho) + P(\tilde{\rho})\), Hoff [24] constructed the global weak solutions with discontinuous initial data. For arbitrary initial data and \( \tilde{\rho} = 0 \), the breakthrough was made by Lions [31], where he proved the global existence of weak solutions provided the specific heat ratio \( \gamma \) is appropriately large, for example, \( \gamma \geq 3N/(N + 2), N = 2, 3 \). Later, Feireisl, Novotný and Petzeltový [21] improved Lions’s results to the case \( \gamma > \frac{4}{N} \). If the initial data possess some symmetric properties, Jiang and Zhang [27] obtained the global weak solutions for any \( \gamma > 1 \). Even in the two dimensional case, the uniqueness of weak solutions is still an open problem up to now. For the case of small energy, Huang, Li and
Xin [26] recently established the global existence and uniqueness of classical solutions, which can be regarded as a uniqueness and regularity theory of Lions-Feireisl’s weak solutions.

The common point among all these papers above is that they did not use scaling considerations, which can help us to find solution spaces as large as possible. This approach goes back to the pioneering work by Fujita and Kato [22] for the classical incompressible Navier-Stokes equations:

\[
\begin{align*}
\partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi &= 0, \quad t > 0, \quad x \in \mathbb{R}^N, \quad N \geq 2, \\
\operatorname{div} v &= 0, \\
v|_{t=0} &= v_0.
\end{align*}
\]

(1.4)

The classical incompressible Navier-Stokes equations, the system (1.4), possesses a structure of scaling invariance. Indeed, if \( v \) is a solution of (1.4) on a time interval \([0, T]\) with initial data \( v_0 \), then the vector field \( v_\lambda \) defined by

\[
v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x)
\]

is also a solution of (1.4) on the time interval \([0, \lambda^{-2} T]\) with the initial data \( \lambda v_0(\lambda x) \). There are many works considering the global well-posedness for the classical incompressible Navier-Stokes equations (1.4) in the scaling invariant spaces, like [3, 4, 22, 29, 30] etc. The importance of these results can be illustrated by the following example [4] in three dimensional case: if \( \phi \) is a function in the Schwartz space (1.6), let us introduce the family of divergence free vector fields

\[
\phi_\varepsilon := \varepsilon^{\alpha-1} \sin \left( \frac{3\varepsilon}{\varepsilon^2} \right) (-\partial_2 \phi, \partial_1 \phi, 0).
\]

Then, for small \( \varepsilon \), the size of \( \|\phi_\varepsilon\|_\text{BMO}^{-1} \) is \( \varepsilon^2 \). The result in [30] implies that the classical incompressible Navier-Stokes system (1.4) is global well-posed with the initial data \( v_0 = \phi_\varepsilon \) for sufficient small \( \varepsilon \). If \( \text{Supp} \phi_\varepsilon \subset B(0, R) = \{ \xi \in \mathbb{R}^3, |\xi| \leq R \} \), then \( \text{Supp} \phi_\varepsilon \subset B((0, 0, \frac{1}{\varepsilon}), R) = \{ \xi \in \mathbb{R}^3, |\xi - (0, 0, \frac{1}{\varepsilon})| \leq R \} \). Thus, such class of the initial data \( v_0 = \phi_\varepsilon \) has an interesting property that in the frequency space, it almost concentrates on the high frequency part. We would like to remark that due to the parabolic property of the system (1.4), the high frequency part of the solution can decay very fast. A natural question which arises is: what will happen when the initial data almost concentrate on the low frequency part?

Inspired by this question, let us come back to the isentropic compressible Navier-Stokes equations (1.5). In this case, the first work following the scaling invariant approach was given by Danchin, see [13], who proved the global well-posedness of strong solutions to (1.1) with initial data \((\rho_0, u_0)\) close to a stable equilibrium in

\[
\left( B^\infty_{2,1} \cap B^\infty_2 \right) \times B^\infty_{2,1}.
\]

(1.6)

In fact, (1.1) is not really invariant under the transformation

\[
\begin{align*}
(\rho_0, u_0) &\rightarrow (\rho_0(\lambda x), \lambda u_0(\lambda x)), \\
(\rho(t, x), u(t, x)) &\rightarrow (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x)), \quad \lambda > 0,
\end{align*}
\]

(1.7)

unless we neglect the pressure term \( P = P(\rho) \). That’s why Danchin introduced the hybrid Besov spaces in [13]. Roughly speaking, by careful analysis of behaviors of the following hyperbolic-parabolic system

\[
\begin{align*}
\partial_t b + \Lambda d &= f, \\
\partial_t d - \Delta d - \Lambda b &= g,
\end{align*}
\]

(1.8)

both in low frequency and high frequency parts, Danchin obtained the \( L^2 \)-decay in time for \( \rho - \tilde{\rho} \) in a \( L^2 \) type Besov space, which is the key point to construct global solutions to (1.1). There is an interesting question how to obtain the global well-posedness result with the large initial data in the space \( (1.6) \). Inspired by works about the classical incompressible Navier-Stokes system [3, 4], with the aid of Green matrix of (1.8), Charve and Danchin [5], Chen, Miao and Zhang [11] obtained the
global well-posedness result in the critical $L^p$ framework respectively, i.e., the high frequency part of
the initial data are small in the following Besov space,

$$b_{0H} \in \dot{B}^{\frac{N}{p},1}_{p,1}, \quad u_{0H} \in \dot{B}^{\frac{N}{p},1-1}_{p,1}, \quad b_0 = \rho_0 - 1.$$ 

In this paper,

$$f_L := \sum_{q < 1} f_q, \quad \text{and} \quad f_H := \sum_{q \geq 1} f_q,$$

with $f \in S'$ and $f_q := \hat{f}_q f$. Later, Haspot [23] gave a new proof via the so called effective velocity.
Similar to the incompressible Navier-Stokes system, the results in [5, 11, 23] imply that the isentropic
compressible Navier-Stokes system (1.1) is global well-posed with the highly oscillating initial ve-
locity $u_0 = \phi_\varepsilon$ in (1.5) for $N = 3$, small $\varepsilon$ and some $\alpha$. However, in [5, 11, 23], the low frequency part
of the initial data are small in the following Besov space,

$$(1.9) \quad b_{0L}, u_{0L} \in \dot{B}^{\frac{N}{p}-1}_{2,1}.$$ 

A natural question which arises is: what will happen when the low frequency part of the initial data
are large in (1.9)? Recently, for the large volume viscosity $\lambda$, Danchin and Mucha [19] established the
global solutions to the two dimensional compressible Navier-Stokes equations (1.1) with large initial
velocity and almost constant density.

The aim of this paper is to construct global solutions to the isentropic compressible Navier-Stokes
equations (1.1) when the low frequency part of the initial velocity field is large. For example, if $N = 3$, for
any fixed $\phi \in S$ with $\hat{\phi}$ supported in a compact set, say, $\text{Supp} \hat{\phi} \subset B(0,1)$, the initial data can be
chosen as

$$(1.10) \quad (\rho_0, u_0) := (1, \gamma^p \nabla \phi_l + \hat{\phi}_\varepsilon),$$
in our result, where

$$\phi_l(x) := \phi(lx),$$

and

$$(1.11) \quad \hat{\phi}_\varepsilon := \varepsilon^{\frac{3}{p} - 1} \sin \left(\frac{\lambda \eta}{\varepsilon}\right)(-\partial_2 \hat{\phi}, \partial_1 \hat{\phi}, 0), \quad \text{for some} \quad \hat{\phi} \in S,$$

with some $0 < \varepsilon \ll 1, \beta \geq 0, \varepsilon > 0$, and $p > 3$. Please refer to Remark [12] for more details.

Since (1.11) is not really invariant under the transformation (1.7), one may guess that the Besov
space $\dot{B}^{\frac{N}{p}-1}_{2,1}$ is not a good functional space for the low frequency part of the initial data. By virtue of
the low frequency embedding

$$(1.12) \quad \|\phi\|_{\dot{B}^{s_1}_{p,1}} \leq C\|\phi\|_{\dot{B}^{s_2}_{p,1}}, \quad \text{for all} \quad \phi \in \dot{B}^{s_2}_{p,1}, \text{and} \quad s_1 > s_2,$$

we should consider a class of the initial data that the low frequency part of the initial data $(b_{0L}, \dot{\rho} \cdot u_{0L})$
are small in the Besov space $\dot{B}^{\frac{N}{p}-1+\alpha}_{2,1}$ but large in $\dot{B}^{\frac{N}{p}-1}_{2,1}$. More precisely, we will prove the global
existence and uniqueness of solutions to the isentropic compressible Navier-Stokes system (1.1) with
initial data $(\rho_0, u_0)$ close to a stable equilibrium (1.0), satisfying $(\rho_0 - 1, u_0) \in \mathcal{E}_0$ defined by

$$(1.13) \quad \mathcal{E}_0 := \left\{ (\phi, \varphi) \in S'_h \times S'_h : (\phi_L, \dot{\rho} \cdot \varphi_L) \in \dot{B}^{\frac{N}{p}-1+\alpha}_{2,1}, \phi_H \in \dot{B}^{\frac{N}{p}_{2,1}}, \dot{\rho} \cdot \varphi_H \in \dot{B}^{\frac{N}{p}-1}_{2,1}, \varphi \in \dot{B}^{\frac{N}{p}-1}_{p,1} \right\},$$

with some $\alpha > 0$ and $p > 2$. To simplify the presentation, in the following we denote

$$(1.14) \quad \|b_0, u_0\|_{\mathcal{E}_0} := \|\phi_L\|_{\dot{B}^{\frac{N}{p}-1+\alpha}_{2,1}} + \|\phi_H\|_{\dot{B}^{\frac{N}{p}_{2,1}}} + \|\dot{\rho} \cdot \varphi_L\|_{\dot{B}^{\frac{N}{p}-1}_{2,1}} + \|\varphi\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}.$$ 

We shall construct solutions $(\rho, u)$ to system (1.1) with $(\rho - 1, u)$ lying in the spaces below.

**Definition 1.1.** Let $T > 0$, and $N \geq 2$. 


• For $p > 2$, $\alpha > 0$, denote by $\mathcal{E}^{\frac{N}{2} - \alpha}_p (T)$ the space of functions $(b, u)$ such that

$$(b_L, \mathbb{D}^1 u_L) \in C_T(B^{\frac{N}{2} - 1 + \alpha}_{2,1}) \cap L^1_T(B^{\frac{N}{2} + 1 + \alpha}_{p,1}) \cap L^1_T(B^{\frac{N}{2} + 2\alpha - 1}_{p,1});$$

$b_H \in C_T(B^{\frac{N}{2}}_{2,1}) \cap L^1_T(B^{\frac{N}{2}}_{p,1}), \mathbb{D}^1 u_H \in C_T(B^{\frac{N}{2} - 1}_{2,1}) \cap L^1_T(B^{\frac{N}{2} + 1}_{p,1});$

$$\mathbb{P}u \in C_T(B^{\frac{N}{2} - 1}_{p,1}) \cap L^1_T(B^{\frac{N}{2} + 1}_{p,1}).$$

We shall endow the space with the norm:

$$\|(b, u)\|_{\mathcal{E}^{\frac{N}{2} - \alpha}_p (T)} := \|(b_L, \mathbb{D}^1 u_L)\|_{L^1_T(B^{\frac{N}{2} - 1 + \alpha}_{2,1}) \cap L^1_T(B^{\frac{N}{2} + 1 + \alpha}_{p,1}) \cap L^1_T(B^{\frac{N}{2} + 2\alpha - 1}_{p,1})} + \|b_H\|_{L^1_T(B^{\frac{N}{2}}_{2,1}) \cap L^1_T(B^{\frac{N}{2}}_{p,1})} + \|\mathbb{D}^1 u_H\|_{L^1_T(B^{\frac{N}{2} - 1}_{2,1}) \cap L^1_T(B^{\frac{N}{2} + 1}_{p,1})} + \|\mathbb{P}u\|_{L^1_T(B^{\frac{N}{2} - 1}_{p,1}) \cap L^1_T(B^{\frac{N}{2} + 1}_{p,1})}.$$

• For $p = 2$, denote by $\mathcal{E}^{\frac{N}{2}}_2 (T)$ the space of functions $(b, u)$ such that

$$(b_L, u) \in C_T(B^{\frac{N}{2} - 1}_{2,1}) \cap L^1_T(B^{\frac{N}{2} + 1}_{2,1}), b_H \in C_T(B^{\frac{N}{2}}_{2,1}) \cap L^1_T(B^{\frac{N}{2}}_{2,1}),$$

with

$$\|(b, u)\|_{\mathcal{E}^{\frac{N}{2}}_2 (T)} := \|(b_L, u)\|_{L^1_T(B^{\frac{N}{2} - 1}_{2,1}) \cap L^1_T(B^{\frac{N}{2} + 1}_{2,1})} + \|b_H\|_{L^1_T(B^{\frac{N}{2}}_{2,1}) \cap L^1_T(B^{\frac{N}{2}}_{2,1})}.$$

Indeed, $\mathcal{E}^{\frac{N}{2}}_2 (T)$ is nothing but the space introduced by Danchin in [13].

We use the notation $\mathcal{E}^{\frac{N}{2} - \alpha}_p (\mathcal{E}^{\frac{N}{2}}_2 )$ if $T = \infty$, changing $[0, T]$ into $[0, \infty)$ in the definition above.

The main results are stated as follows.

**Theorem 1.1.** Let

$$2 < p < 4, \quad \text{if} \quad N = 2,$$

$$2 < p \leq 4, \quad \text{if} \quad N = 3,$$

$$2 < p \leq \frac{2N}{N - 2}, \quad \text{if} \quad N \geq 4,$$

and

$$0 < \alpha \leq \frac{N - 1}{2} \left(\frac{1}{2} - \frac{1}{p}\right).$$

Assume that $(\rho_0, u_0)$ satisfies $(\rho_0 - 1, u_0) \in \mathcal{E}_0$. Then there exist two constants $c_0$ and $C_0 > 0$ depending on $N, \mu$ and $\lambda$, such that if

$$\|(\rho_0 - 1, u_0)\|_{\mathcal{E}_0} \leq c_0,$$

then system (1.1) admits a global solution $(\rho, u)$ with $(\rho - 1, u) \in \mathcal{E}^{\frac{N}{2} - \alpha}_p$, satisfying

$$\|(\rho - 1, u)\|_{\mathcal{E}^{\frac{N}{2} - \alpha}_p} \leq C_0 \|(\rho_0 - 1, u_0)\|_{\mathcal{E}_0}.$$

Furthermore, if

$$\text{if} \quad N \geq 3, \quad \text{or} \quad \mathbb{P}u_0 \in B^0_{2,1} \text{ when } N = 2,$$

then the solution is unique.

**Remark 1.1.** For initial data $(\rho_0, u_0)$ with $(\rho_0 - 1, u_0) \in C_T(B^{\frac{N}{2} - 1}_{2,1} \cap B^{\frac{N}{2}}_{2,1}) \times B^{\frac{N}{2} - 1}_{2,1}$ with

$$\|(\rho_0 - 1, u_0)\|_{\left(B^{\frac{N}{2} - 1}_{2,1} \cap B^{\frac{N}{2}}_{2,1}\right) \times B^{\frac{N}{2} - 1}_{2,1}} := R,$$

one easily deduces that

$$\|(b_0, u_0)\|_{\mathcal{E}_0} \leq C2^{-Q\alpha} \left(\|P_{< -Q} b_0\|_{B^0_{2,1}} + \|P_{< -Q} \mathbb{D}^1 u_0\|_{B^0_{2,1}}\right) + C\|P_{\geq -Q} b_0\|_{B^0_{2,1} \cap B^{\frac{N}{2} - 1}_{2,1}}.$$
Theorem 1.2. Let
\[ \begin{cases} 
0 < \alpha < \frac{1}{3}, & \text{if } N = 2, \\
0 < \alpha \leq \frac{1}{4}, & \text{if } N = 3, \\
0 < \alpha \leq \frac{1}{N-1}, & \text{if } N \geq 4.
\end{cases} \]

There exists a constant \( c_1 \) depending on \( N, \mu \) and \( \lambda \), such that for all \((\rho_0, u_0)\) with \((\rho_0 - 1, u_0) \in \left( B_{2,1}^{\frac{N}{N-1}} \cap B_{2,1}^{\frac{N}{N+1}} \right) \times B_{2,1}^{\frac{N}{N-1}} \), and \( Q \in \mathbb{N} \), if
\[
(1.23) \quad 2^{-Q^\alpha} \left( \|P_{\leq Q}(\rho_0 - 1)\|_{B_{2,1}^{\frac{N}{N-1}}} + \|P_{\leq Q}^{\leq 1} u_0\|_{B_{2,1}^{\frac{N}{N-1}}}, Q = \frac{1}{\alpha} \log \left( \frac{2CR}{c_0} \right) + 1,
\]
then the initial data \((\rho_0, u_0)\) satisfy the condition \((1.17)\).

From Theorem 1.1 and Remark 1.1, we easily obtain the following theorem.

**Theorem 1.2.** Let
\[
\begin{cases} 
0 < \alpha < \frac{1}{3}, & \text{if } N = 2, \\
0 < \alpha \leq \frac{1}{4}, & \text{if } N = 3, \\
0 < \alpha \leq \frac{1}{N-1}, & \text{if } N \geq 4.
\end{cases}
\]

There exists a constant \( c_1 \) depending on \( N, \mu \) and \( \lambda \), such that for all \((\rho_0, u_0)\) with \((\rho_0 - 1, u_0) \in \left( B_{2,1}^{\frac{N}{N-1}} \cap B_{2,1}^{\frac{N}{N+1}} \right) \times B_{2,1}^{\frac{N}{N-1}} \), and \( Q \in \mathbb{N} \), if
\[
(1.23) \quad 2^{-Q^\alpha} \left( \|P_{\leq Q}(\rho_0 - 1\|_{B_{2,1}^{\frac{N}{N-1}}} + \|P_{\leq Q}^{\leq 1} u_0\|_{B_{2,1}^{\frac{N}{N-1}}}, \quad \|P_{\leq Q}(\rho_0 - 1\|_{B_{2,1}^{\frac{N}{N-1}}} + \|P_{\leq Q}^{\leq 1} u_0\|_{B_{2,1}^{\frac{N}{N-1}}} \leq c_1,
\]
then system \((1.1)\) admits a unique solution \((\rho, u)\) with \((\rho - 1, u) \in \mathcal{E}^N_{N} \).

**Remark 1.2.** We give some examples of large initial data \((\rho_0, u_0)\) with \((\rho_0 - 1, u_0)\) satisfying \((1.17)\) and \((1.23)\). For the sake of simplicity, we take \( \rho_0 = 1 \). In doing so, we just need to focus on the initial velocity \( u_0 \). More precisely, for any fixed \( \phi \in \mathcal{S} \) with \( \|\nabla \phi\|_{B_{2,1}^{\frac{N}{N-1}}} = R \) and \( \hat{\phi} \) supported in a compact set, say, \( \text{Supp} \hat{\phi} \subset B(0,1) \), let us denote
\[
\phi_l(x) := \phi(lx).
\]

Then
\[
\begin{align*}
&\|\nabla \phi_l\|_{B_{2,1}^{\frac{N}{N-1}}} = \|\nabla \phi\|_{B_{2,1}^{\frac{N}{N-1}}} = R, \\
&\nabla \phi_l(\xi) = l^{1-N}\nabla \phi \left( \frac{\xi}{l} \right), \quad \text{and \ Supp} \nabla \phi_l \subset B(0,l).
\end{align*}
\]
Consequently, for all \( \beta \in [0, \alpha) \), taking \( l > 0 \) and \( Q \in \mathbb{N} \) satisfying
\[
(1.24) \quad \begin{cases} 
l < \frac{1}{4} 2^{-Q}, \\
2^{-\alpha} l^{-\beta} R \leq \frac{c_0}{2},
\end{cases}
\]
we find that
\[
\Delta_q \nabla \phi_l = 0, \quad q \geq -Q,
\]
and
\[
(1.25) \quad \|l^{-\beta} P_{\leq Q}(\nabla \phi_l)\|_{B_{2,1}^{\frac{N}{N-1}+\beta}} \leq \frac{c_0}{2}.
\]

**Example 1.** Let
\[
(1.26) \quad (\rho_0, u_0) := (1, l^{-\beta} \nabla \phi_l),
\]
with $l$ satisfying (1.24). Then from Remark 1.7 and (1.25), it is not difficult to verify that the data given in (1.26) apply to Theorems 1.7 and 1.2. This indicates that our results allow for initial data with large potential part of the initial velocity.

In addition, noticing the smallness restriction in (1.20), initial velocity with highly oscillating is also permitted in Theorem 1.1 as a by-product. For instance, if $N = 3$, the incompressible part of the initial velocity can be given as in (4).

(1.27) \[ \tilde{\phi}_c := \varepsilon^{\frac{1}{p}-1} \sin \left( \frac{x_3}{\varepsilon} \right) (-\partial_1 \tilde{\phi}, \partial_1 \tilde{\phi}, 0), \text{ for some } \tilde{\phi} \in S, \]

with $\varepsilon > 0$, and $p > 3$. Combining (1.26) with (1.27), we can give another example.

Example 2. Let \[ (\rho_0, u_0) := (1, l^\beta \nabla \phi_0 + \tilde{\phi}_c). \]

Then for $N = 3$, the data in (1.28) are applicable to Theorem 1.1

Remark 1.3. Our results can be extended to the case with the high frequency part $(b_{0H}, u_{0H})$ of the initial data lying in some $L^p$-type Besov spaces. We omit the details in this paper to avoid tedious computations.

Remark 1.4. Taking the anisotropy into consideration as in [10] and [36], it is possible to relax the smallness restriction on the divergence free part $\mathbb{P}u_0$ of the initial velocity $u_0$. Please refer to [8, 9] for a recent panorama.

Remark 1.5. Our methods can be used to other related models. Similar results for the incompressible viscoelastic fluids will be given in a forthcoming paper.

It is worth pointing out that we impose neither any symmetrical structure on the initial data, nor largeness assumptions on the viscosity coefficients $\mu$ or $\lambda$. What’s more, our results hold for all dimensional $N \geq 2$. Different from [13], our proof relies not only on the energy estimates for the hyperbolic-parabolic system (1.8), but also on the dispersive properties for the following acoustics

\begin{equation}
\begin{aligned}
\begin{cases}
\partial_t b + \partial_t p + \frac{\partial b}{\varepsilon} = f, \\
\partial_t d - \frac{\partial b}{\varepsilon} = g,
\end{cases}
\end{aligned}
\end{equation}

This method was used before to study the zero Mach number limit problem of the compressible Navier-Stokes equations [20, 14, 18]. It seems that the combination of the energy estimates and Strichartz estimates has never been used to study the global well-posedness problem of the viscous compressible fluids. In this paper, we try to apply this idea to the isentropic compressible Navier-Stokes equations.

Let us now explain how to construct our solution spaces and show the ingredients of the proof. First of all, just as in [13], writing $\rho = 1 + b$, and decomposing $u = \mathbb{P}^\perp u + \mathbb{P}u$, where

\[ \mathbb{P}^\perp := -\nabla (-\Delta)^{-1} \text{div}, \quad \text{and} \quad \mathbb{P} := I - \mathbb{P}^\perp, \]

we reformulate (1.1) as follows:

\begin{equation}
\begin{aligned}
\begin{cases}
\partial_t b + \text{div}\mathbb{P}^\perp u = -\text{div}(bu), \\
\partial_t \mathbb{P}^\perp u - \nu \Delta \mathbb{P}^\perp u + \nabla b = -\mathbb{P}^\perp (u \cdot \nabla u + K(b) \nabla b + \mathcal{I}(b) \mathcal{A} u), \\
(b, \mathbb{P}^\perp u)_{l=0} = (b_0, \mathbb{P}^\perp u_0),
\end{cases}
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\begin{cases}
\partial_t \mathbb{P} u - \mu \Delta \mathbb{P} u = -\mathbb{P} (u \cdot \nabla u + \mathcal{I}(b) \mathcal{A} u), \\
\mathbb{P} u|_{l=0} = \mathbb{P} u_0,
\end{cases}
\end{aligned}
\end{equation}
where $b_0 := \rho_0 - 1$, $I(a) := \frac{1}{1 + \nu}$, and $K(a) := \frac{p'(1+a)}{1 + a} - 1$. For the sake of simplicity, $\nu$ is assumed to be 1 throughout this paper. Moreover, the condition (1.3) ensures that $K(0) = 0$. Let us denote $d := \Lambda^{-1} \text{div} u$. From (1.30), one easily deduces that $(b, d)$ solves

$$
\begin{align*}
\partial_t b + \Lambda d &= -\text{div}(bu), \\
\partial_t d - \Delta d - \Lambda b &= -\Lambda^{-1} \text{div} (u \cdot \nabla u + K(b) \nabla b + I(b) \mathcal{A} u),
\end{align*}
$$

(1.32)

$b, d \in L^\infty([0, T])$ where $\mathcal{A} u := \nabla u - \nabla b$. Since $\text{curl} b \notin L^2$, it is easy to verify that (1.32) is equivalent to (1.30). In the following, we shall use (1.32) to replace (1.30) and do not make a distinction between $\mathcal{P}^\perp u$ and $d$ in the absence of confusion.

Next, in order to show our ideas more clearly, we divide Danchin’s arguments in [13] into the following three parts:

(i): Global estimates for the linearized system of (1.32).

(ii): Commutator estimates for the convection terms.

(iii): Product estimates for other nonlinear terms.

Combining (i) with (ii), Danchin established the global estimates for the paralinearized system of (1.32) with $(b_0, d_0) \in \left( \dot{B}^{\frac{-1}{2}, 1}_{2, 1} \cap \overline{B}^{\frac{1}{2}, 1}_{2, 1} \right) \times \dot{B}^{\frac{1}{2}, 1}_{2, 1}$, see Proposition 10.23 in [1], for example. Then substituting the results in (iii) into the estimates obtained in (i) and (ii) yields the global estimates of $(b, d)$. Our proof follows this line, but aside from part (i), we develop different approaches to deal with parts (ii) and (iii). In particular, the dispersive properties of the system of acoustics (1.29) is taken into consideration. Indeed, for $(b_0, d_0)$ satisfying

$$
(b_0, d_0) \in \left( \dot{B}^{\frac{-1}{2}, \alpha}_{2, 1} \cap \overline{B}^{\frac{1}{2}, \alpha}_{2, 1} \right) \times \left( \dot{B}^{\frac{-1}{2}, 1+\alpha}_{2, 1} \cap \overline{B}^{\frac{1}{2}, 1+\alpha}_{2, 1} \right),
$$

(1.33)

it has been shown in [14] that some Strichartz norms of $(b, d)$ decay algebraically with respect to the Mach number $\epsilon$. In our case, $\epsilon = 1$, we can not expect any decay with respect to the Mach number. Nevertheless, in the low frequency part, we still gain some decay by means of the low frequency embedding:

$$
\|P_{< \Omega}(b, d)\|_{L_p^1(B^{\frac{-1}{2}, \alpha}_{2, 1})} \leq \left\{ \|P_{< \Omega}(b_0, d_0)\|_{B^{\frac{-1}{2}, 1+\alpha}_{2, 1}}, \|P_{< \Omega}(f, g)\|_{L_p^1(B^{\frac{1}{2}, 1+\alpha}_{2, 1})} \right\} \leq 2^{-\alpha Q} \left( \|b_0\|_{B^{\frac{-1}{2}, 1+\alpha}_{2, 1}}, \|P_{< \Omega}(f, g)\|_{L_p^1(B^{\frac{1}{2}, 1+\alpha}_{2, 1})} \right),
$$

(1.34)

with

$$
\alpha > 0, \quad p \geq 2, \quad \frac{2}{r} \leq \min \left\{ 1, (N - 1) \left( \frac{1}{2} - \frac{1}{p} \right) \right\}, \quad (r, p, N) \neq (2, \infty, 3).
$$

(1.35)

This is the basic idea underneath our approach, which leads us to believe that it is possible to construct global solutions to (1.1) with large potential part $\mathcal{P}^\perp u_0$ of initial velocity in $B^{\frac{-1}{2}, 1}_{2, 1}$.

Motivated by (1.34), we just impose the extra regularity on the low frequency part of $(b_0, d_0)$. More precisely,

$$
(b_{0L}, d_{0L}) \in \dot{B}^{\frac{-1}{2}, 1+\alpha}_{2, 1} \times \dot{B}^{\frac{1}{2}, 1+\alpha}_{2, 1}, \quad (b_{0H}, d_{0H}) \in \dot{B}^{\frac{-1}{2}, 1}_{2, 1} \times \dot{B}^{\frac{1}{2}, 1}_{2, 1}.
$$

(1.36)

In order to handle parts (ii) and (iii) under the condition (1.30), we need to compensate the loss of critical norms of $(b, d)$ in the low frequency part. To this end, we set

$$
r = \frac{1}{\alpha}
$$

in (1.34). In this way, $p = 2$ is not permitted in (1.35) any more, otherwise $\alpha = 0$. This explains the condition (1.16) in Theorem 1.1.
On the other hand, the divergence free part $\mathbb{P}u$ of the velocity $u$ satisfies the parabolic system (1.31), and hence possesses no dispersive property at all. Accordingly, it seems that it is reasonable to assume
\begin{equation}
(1.37) \quad \mathbb{P}u_0 \in \dot{B}^{\frac{\alpha}{2} - 1}_{p,1}.
\end{equation}

As a result, by the property of heat equation, the space for $\mathbb{P}u$ should be
\[ L^\infty_T(B^{\frac{\alpha}{2} - 1}_{p,1}) \cap L^1_T(B^{\frac{\alpha}{2} + 1}_{p,1}), \]
and we have to bound the right hand side of (1.31) in $L^1_T(B^{\frac{\alpha}{2} - 1}_{2,1})$. Unfortunately, we do not know how to control $\|\mathbb{P}(\bar{T}_u \nabla \mathbb{P}^4 u)\|_{L^1_T(B^{\frac{\alpha}{2} - 1}_{2,1})}$ since from (1.34) and the property of heat equation, we just have
\begin{equation}
(1.38) \quad \mathbb{P}^4 u \in L^\infty_T(B^{\frac{\alpha}{2} - 1 + \alpha}_{2,1}) \cap L^1_T(B^{\frac{\alpha}{2} + 1 + \alpha}_{2,1}) \cap L^{1+2\alpha}_T(B^{\frac{\alpha}{2} - 1 + 2\alpha}_{p,1}), \quad \text{with} \quad p > 2.
\end{equation}

To overcome this problem, owing to the fact that $\mathbb{P}^4 = 0$, we find that
\begin{equation}
(1.39) \quad \mathbb{P}(\bar{T}_u \cdot \nabla \mathbb{P}^4 u) = [\mathbb{P}, \bar{T}_u^f] \partial_k \mathbb{P}^4 u.
\end{equation}

Then the commutator estimate (Lemma 2.99 in [1]) enables us to bound
\[ \|([\mathbb{P}, \bar{T}_u^f] \partial_k \mathbb{P}^4 u)\|_{L^1_T(B^{\frac{\alpha}{2} + 1}_{2,1})} \leq \|\nabla \mathbb{P}^4 u\|_{L^1_T(B^{\frac{\alpha}{2} - 1}_{2,1})} \|\nabla \mathbb{P}^4 u\|_{L^{1+2\alpha}_T(B^{\frac{\alpha}{2} - 1 + 2\alpha}_{p,1})}, \]
with $p^* := \frac{2p}{p - 2}$, provided
\begin{equation}
(1.40) \quad \frac{N}{p^*} - 2\alpha + 1 \leq 1, \quad \text{i.e.} \quad \alpha \geq \frac{N}{2} \left( \frac{1}{2} - \frac{1}{p} \right),
\end{equation}
which contradicts to (1.35). The above analysis has proved a blind alley if the assumption on $\mathbb{P}u_0$ is given by (1.37). However, if
\begin{equation}
(1.41) \quad \mathbb{P}u_0 \in \dot{B}^{\frac{\alpha}{2} - 1}_{p,1}
\end{equation}
with $p$ the same as in (1.34), the above method to deal with $\mathbb{P}(\bar{T}_u \cdot \nabla \mathbb{P}^4 u)$ works since
\begin{equation}
(1.42) \quad \|([\mathbb{P}, \bar{T}_u^f] \partial_k \mathbb{P}^4 u)\|_{L^1_T(B^{\frac{\alpha}{2} - 1}_{p,1})} \leq \|\nabla \mathbb{P}^4 u\|_{L^1_T(B^{\frac{\alpha}{2} - 1}_{p,1})} \|\nabla \mathbb{P}^4 u\|_{L^{1+2\alpha}_T(B^{\frac{\alpha}{2} - 1 + 2\alpha}_{p,1})},
\end{equation}
holds for all $\alpha > 0$. Combining (1.36) with (1.41), the condition on $(b_0, u_0)$ becomes
\begin{equation}
(1.43) \quad \begin{cases}
(b_{0L}, \mathbb{P}^4 u_{0L}) \in \dot{B}^{\frac{\alpha}{2} - 1 + \alpha}_{2,1}, \\
(b_0H, \mathbb{P}^4 u_{0H}) \in \dot{B}^{\frac{\alpha}{2} - 1}_{2,1}, \\
\mathbb{P}u_0 \in \dot{B}^{\frac{\alpha}{2} - 1}_{p,1}, \quad \text{with} \quad p > 2.
\end{cases}
\end{equation}

This explains the construction of $\mathcal{E}_0$ in (1.13).

The rest part of this paper is organized as follows. In Section 2, we introduce the tools (the Littlewood-Paley decomposition and paradiifferential calculus) and give some product estimates in Besov spaces. In Section 3, we recall some properties of the system of acoustics, transport and heat equations. Section 4 is devoted to the global a priori estimates of system (1.30)–(1.31). The proof of Theorem (1.1) is given in Section 5. In Section 6, we prove Theorem (1.2). Some nonlinear estimates needed in the proof of Theorems (1.1) and (1.2) are put in the Appendix in Section 7.
Notation.

(1) For $f \in \mathcal{S}'$, $Q \in \mathbb{N}$, denote $f_q := \hat{\Delta}_q f$, and

\begin{equation}
\begin{aligned}
P_{<Q} f &:= \sum_{q<Q} f_q, \\
P_{\geq Q} f &:= f - P_{<Q} f = \sum_{q\geq Q} f_q.
\end{aligned}
\end{equation}

In particular,

\begin{equation}
\begin{aligned}
f_L &:= \sum_{q<1} f_q, \\
f_H &:= \sum_{q\geq 1} f_q.
\end{aligned}
\end{equation}

(2) Denote $p^* := \frac{2p}{p-2}$, i.e. $p^* = 1 + \frac{2}{p}$, for $p \geq 2$.

(3) Throughout the paper, $C$ denotes various “harmless” positive constants, and we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. The Functional Tool Box

The results of the present paper rely on the use of a dyadic partition of unity with respect to the Fourier variables, the so-called the Littlewood-Paley decomposition. Let us briefly explain how it may be built on $\mathbb{R}^N$, and the readers may see more details in [1, 6]. Let $(\chi, \varphi)$ be a couple of $C^\infty$ functions satisfying

\begin{equation}
\begin{aligned}
\text{Supp } \chi &\subset \left\{ |\xi| \leq \frac{4}{3} \right\}, \\
\text{Supp } \varphi &\subset \left\{ \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\},
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) &= 1, \\
\sum_{q\in \mathbb{Z}} \varphi(2^{-q}\xi) &= 1, \text{ for } \xi \neq 0.
\end{aligned}
\end{equation}

Set $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}(\varphi_q)$, and $\tilde{h} = \mathcal{F}^{-1}(\chi)$. The dyadic blocks and the low-frequency cutoff operators are defined for all $q \in \mathbb{Z}$ by

\begin{equation}
\hat{\Delta}_q u = \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x-y)dy,
\end{equation}

\begin{equation}
\dot{S}_q u = \chi(2^{-q}D)u = \int_{\mathbb{R}^N} \tilde{h}_q(y)u(x-y)dy.
\end{equation}

Then

\begin{equation}
u = \sum_{q\in \mathbb{Z}} \hat{\Delta}_q u,
\end{equation}

holds for tempered distributions modulo polynomials. As working modulo polynomials is not appropriate for nonlinear problems, we shall restrict our attention to the set $\mathcal{S}'_h$ of tempered distributions $u$ such that

\begin{equation}
\lim_{q \to -\infty} \|\dot{S}_q u\|_{L^\infty} = 0.
\end{equation}

Note that (2.1) holds true whenever $u$ is in $\mathcal{S}'_h$ and that one may write

\begin{equation}
\dot{S}_q u = \sum_{p=-q-1}^{q}\hat{\Delta}_p u.
\end{equation}

Besides, we would like to mention that the Littlewood-Paley decomposition has a nice property of quasi-orthogonality:

\begin{equation}
\hat{\Delta}_p \hat{\Delta}_q u \equiv 0 \text{ if } |p-q| \geq 2 \text{ and } \hat{\Delta}_p (\dot{S}_{q-1} u \hat{\Delta}_q u) \equiv 0 \text{ if } |p-q| \geq 5.
\end{equation}

One can now give the definition of homogeneous Besov spaces.
Definition 2.1. For \( s \in \mathbb{R} \), \((p, r) \in [1, \infty]^2\), and \( u \in \mathcal{S}'(\mathbb{R}^N)\), we set
\[
\|u\|_{B^s_{p,r}} = \left\| \|\Delta^s u\|_{L^r} \right\|_{L^p}.
\]
We then define the spaces \( \dot{B}^s_{p,r} := \{ u \in \mathcal{S}'(\mathbb{R}^N), \|u\|_{B^s_{p,r}} < \infty \} \).

The following lemma describes the way derivatives act on spectrally localized functions.

**Lemma 2.1** (Bernstein’s inequalities). Let \( k \in \mathbb{N} \) and \( 0 < r < R \). There exists a constant \( C \) depending on \( r \), \( R \) and \( d \) such that for all \((a, b) \in [1, \infty]^2\), we have for all \( \lambda > 0 \) and multi-index \( \alpha \)
- If \( \text{Supp} \hat{f} \subset B(0, \lambda R) \), then \( \sup_{\alpha = k} \|\partial^\alpha f\|_{L^r} \leq C^{k+1} \lambda^{k+d\left(\frac{1}{p} + \frac{1}{q}\right)} \|f\|_{L^p} \).
- If \( \text{Supp} \hat{f} \subset C(0, \lambda R) \), then \( C^{-k-1} \lambda^k \|f\|_{L^r} \leq \sup_{\alpha = k} \|\partial^\alpha f\|_{L^r} \leq C^{k+1} \lambda^{k} \|f\|_{L^r} \).

Let us now state some classical properties for the Besov spaces.

**Proposition 2.1.** For all \( s, s_1, s_2 \in \mathbb{R}, 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty \), the following properties hold true:
- If \( p_1 \leq p_2 \) and \( r_1 \leq r_2 \), then \( \dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^s_{p_2,r_2} \).
- If \( s_1 \neq s_2 \) and \( \theta \in (0, 1) \), then \( \dot{B}^{s_1}_{p_1,r_1} \ast \dot{B}^{s_2}_{p_2,r_2} \hookrightarrow \dot{B}^{s_1 + (1-\theta)s_2}_{p_1,r_1} \).
- For any smooth homogeneous of degree \( m \in \mathbb{Z} \) function \( F \) on \( \mathbb{R}^N \setminus \{0\} \), the operator \( F(D) \) maps \( \dot{B}^s_{p,r} \) into \( \dot{B}^{\min s,0}_{p,r} \). 

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Firstly introduced by J. M. Bony in [2], the paraproduct between \( f \) and \( g \) is defined by
\[
\dot{T} f g = \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} f \Delta_q g,
\]
and the remainder is given by
\[
\dot{R}(f, g) = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q f \Delta_q g
\]
with
\[
\dot{\Delta}_q f = (\dot{\Delta}_{q-1} + \Delta_q + \Delta_{q+1}) f.
\]
We have the following so-called Bony’s decomposition:
\[
(2.3) \quad f g = \dot{T} f g + \dot{T} f^* g + \dot{R}(f, g) = \dot{T} f g + \dot{T}^* f,
\]
where \( \dot{T}^* f := \dot{T} f + \dot{R}(f, g) \). The paraproduct \( \dot{T} \) and the remainder \( \dot{R} \) operators satisfy the following continuous properties.

**Proposition 2.2** ([1]). For all \( s \in \mathbb{R}, \sigma \geq 0, \) and \( 1 \leq p, p_1, p_2 \leq \infty \), the paraproduct \( \dot{T} \) is a bilinear, continuous operator from \( B^{-\sigma}_{p_1,1} \times B^{\sigma}_{p_2,1} \) to \( B^{\sigma}_{p,1} \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). The remainder \( \dot{R} \) is bilinear continuous from \( B^{s_1}_{p_1,1} \times B^{s_2}_{p_2,1} \) to \( B^{s_1 + s_2}_{p,1} \) with \( s_1 + s_2 > 0 \), and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \).

In view of (2.3), Proposition 2.2 and Bernstein’s inequalities, one easily deduces the following product estimates. Please find the proof in Appendix.

**Corollary 2.1.** Let \( \rho, p_1, p_2, q_1, q_2 \in [1, \infty], \frac{1}{\rho} \leq \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{\rho} \leq \frac{1}{q_1} + \frac{1}{q_2}, s_1 - \frac{N}{p_1} \leq \min(0, N(\frac{1}{q_2} - \frac{1}{p_2})), s_1 - \frac{N}{q_1} \leq \min(0, N(\frac{1}{q_2} - \frac{1}{p_2})), s_1 + s_2 > N \max(0, \frac{1}{p_1} + \frac{1}{p_2} - 1), s = s_1 + s_2 + N(\frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2}) = \sigma_1 + \sigma_2 + N(\frac{1}{\rho} - \frac{1}{q_1} - \frac{1}{q_2}), \) then there holds
\[
\|uv\|_{B^s_{p,1}} \leq C \|\|uv\|_{B^{s_1}_{p_1,1}} \|uv\|_{B^{s_2}_{p_2,1}} + C \|v\|_{B^{s_1}_{p_1,1}} \|u\|_{B^{s_2}_{p_2,1}}.
\]
In particular,
\[
\|uv\|_{B^{s_1 + s_2}_{p,1}} \leq C \|\|uv\|_{B^{s_1}_{p_1,1}} \|v\|_{B^{s_2}_{p_2,1}},
\]
where \( p \in [1, \infty], r_1, r_2 \leq \frac{N}{p} \) and \( r_1 + r_2 > N \max(0, \frac{2}{p} - 1) \).
The following Proposition will be used to prove the uniqueness of solutions obtained in Theorem 1.1 for $N = 2$.

**Proposition 2.3 ([16]).** Let $p \geq 2$, $s_1 \leq \frac{N}{p}$, $s_2 < \frac{N}{p}$, and $s_1 + s_2 \geq 0$, then
\[
\|uv\|_{\tilde{B}^{s_1+s_2+\frac{N}{p}}_{p,\infty}} \leq C\|u\|_{\tilde{B}^{s_1}_p} \|v\|_{\tilde{B}^{s_2}_p}.
\]

The study of non-stationary PDEs requires spaces of the type $L^p_t(X) = L^p(0, T; X)$ for appropriate Banach spaces $X$. In our case, we expect $X$ to be a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain the bounds in spaces which are not of the type $L^p(0, T; \tilde{B}^{s}_{p,r})$. That naturally leads to the following definition introduced by Chemin and Lerner in [7].

**Definition 2.2.** For $\rho \in [1, +\infty]$, $s \in \mathbb{R}$, and $T \in (0, +\infty)$, we set
\[
\|u\|^r_{\tilde{L}^p_T(B^s_{p,r})} = \left\|2^{qs}\|\Lambda\hat{u}(t)\|_{L^p(\mathbb{R}^n)}\right\|_{L^r(T)}
\]
and denote by $\tilde{L}^p_T(B^s_{p,r})$ the subset of distributions $u \in \mathcal{D}'((0, T), S^s_p(\mathbb{R}^3))$ with finite $\|u\|_{\tilde{L}^p_T(B^s_{p,r})}$ norm. When $T = +\infty$, the index $T$ is omitted. We further denote $\tilde{C}^r_T(B^s_{p,r}) = C([0, T]; B^s_{p,r}) \cap \tilde{L}^\infty_T(B^s_{p,r})$.

**Remark 2.1.** All the properties of continuity for the paraproduct, remainder, and product remain true for the Chemin-Lerner spaces. The exponent $\rho$ just has to behave according to Hölder's inequality for the time variable.

**Remark 2.2.** The spaces $\tilde{L}^p_T(B^s_{p,r})$ can be linked with the classical space $L^p_T(B^s_{p,r})$ via the Minkowski inequality:
\[
\|u\|_{\tilde{L}^p_T(B^s_{p,r})} \leq \|u\|_{L^p_T(B^s_{p,r})} \quad \text{if} \quad r \geq p,
\]
\[
\|u\|_{\tilde{L}^p_T(B^s_{p,r})} \geq \|u\|_{L^p_T(B^s_{p,r})} \quad \text{if} \quad r \leq p.
\]

3. Preliminaries

In this section, we first recall the estimates for the acoustics system ([1.29]), which are very useful in the proof of Theorem 1.1.

**Proposition 3.1 ([14]).** Let $(b, v)$ be a solution of the following system of acoustics:
\[
\begin{aligned}
\partial_t b + \epsilon^{-1} \Lambda v &= f, \\
\partial_t v - \epsilon^{-1} \Lambda b &= g, \\
(b, v)|_{t=0} &= (b_0, v_0).
\end{aligned}
\]
Then, for any $s \in \mathbb{R}$ and $T \in (0, \infty)$, the following estimate holds:
\[
\|(b, v)\|_{\tilde{L}^p_T(B^{s,\frac{N}{p}+\frac{1}{2}}_{p,1})} \leq C\epsilon^\frac{s}{r}\|(b_0, v_0)\|_{B^0_{q,1}} + C\epsilon^{1+\frac{s}{r}}\|(f, g)\|_{\tilde{L}^p_T(B^{s,\frac{N}{p}+\frac{1}{2}}_{p,1})},
\]
with
\[
p \geq 2, \frac{2}{r} \leq \min(1, \gamma(p)), (r, p, N) \neq (2, \infty, 3),
\]
\[
\tilde{p} \geq 2, \frac{2}{\tilde{q}} \leq \min(1, \gamma(\tilde{p})), (\tilde{r}, \tilde{p}, N) \neq (2, \infty, 3),
\]
where $\gamma(q) := (N - 1)(\frac{1}{2} - \frac{1}{q})$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $\frac{1}{r} + \frac{1}{r'} = 1$.

Next, we recall the classical estimates in Besov space for the transport and heat equations (Theorem 3.37, [11]).
Proposition 3.2. Let $\sigma \in (-N \min\{\frac{1}{p}, \frac{1}{r}\}, 1 + \frac{N}{p})$ and $1 \leq p, r \leq +\infty$, or $\sigma = 1 + \frac{N}{p}$ if $r = 1$. Let $v$ be a smooth vector field such that $\nabla v \in L^1_T(\dot{B}_{p,r}^\sigma \cap L^\infty)$, $f_0 \in \dot{B}_{p,r}^\sigma$ and $g \in L^1_T(\dot{B}_{p,r}^\sigma)$. There exists a constant $C$, such that for all solution $f \in L^\infty([0, T]; \dot{B}_{p,r}^\sigma)$ of the equation
\[ \partial_t f + v \cdot \nabla f = g, \quad f|_{t=0} = f_0, \]
we have the following a priori estimate
\[ \|f\|_{L^1_T(\dot{B}_{p,r}^\sigma)} \leq e^{CV(T)} \left( \|f_0\|_{\dot{B}_{p,r}^\sigma} + \int_0^T e^{-CV(t)} \|g(t)\|_{\dot{B}_{p,r}^\sigma} \, dt \right), \]
where $V(t) = \int_0^t \|\nabla v(\tau)\|_{\dot{B}_{p,r}^\sigma} \, d\tau$.

Proposition 3.3. Let $\sigma \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. Assume that $f_0 \in \dot{B}_{p,r}^\sigma$ and $g \in L^1_T(\dot{B}_{p,r}^{\sigma-2+\frac{2}{r}})$. There exists a constant $C$, such that for all solution $f \in L^\infty([0, T]; \dot{B}_{p,r}^\sigma \cap L^1([0, T]; \dot{B}_{p,r}^{\sigma+2})$ of the equation
\[ \partial_t f - v \Delta f = g, \quad f|_{t=0} = f_0, \]
we have the following a priori estimate, for all $p \leq p_1 \leq +\infty$,
\[ \|f\|_{L^p_t(\dot{B}_{p,r}^{\sigma+\frac{2}{p}-1})} \leq C \left( \|f_0\|_{\dot{B}_{p,r}^\sigma} + v^{\frac{2}{p}-1} \|g\|_{L^1_t(\dot{B}_{p,r}^{\sigma-2+\frac{2}{r}})} \right). \]

4. A priori estimates

Before proceeding any further, let us denote
\[ X_L(T) := \|b_0\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}+\alpha})} + \|\mathbb{P}u\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}+\alpha})} \]
\[ + \|\mathbb{P}u\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}+\alpha})} \]
\[ X_H(T) := \|b_0\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}})} + \|\mathbb{P}u\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}})} \]
\[ + \|\mathbb{P}u\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}})} \]
\[ Y_o(T) := \|(b_0, \mathbb{P}u)\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}+\alpha})} \]
\[ W(T) := \|\mathbb{P}u\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}})} \]
\[ + \|\mathbb{P}u\|_{L_T^{\frac{N}{p-1}}(B_{p,1}^{\frac{N}{p-1}})} \]
\[ X(T) := X_L(T) + X_H(T) + Y_o(T) + W(T), \]
\[ X_0^L := \|b_0\|_{B_{2,1}^{\frac{N}{p-1}+\alpha}} + \|\mathbb{P}u\|_{B_{2,1}^{\frac{N}{p-1}+\alpha}}, \]
\[ X_0^H := \|b_0\|_{B_{2,1}^{\frac{N}{p-1}+\alpha}} + \|\mathbb{P}u\|_{B_{2,1}^{\frac{N}{p-1}+\alpha}}, \]
and
\[ W^0 := \|\mathbb{P}u\|_{B_{2,1}^{\frac{N}{p-1}+\alpha}}, \]
\[ X^0 := X_L + X_H + W^0. \]

4.1. Nonlinear estimates. Now we estimate the nonlinear terms one by one as follows.

By virtue of the low frequency embedding
\[ \|P^{<1}\phi\|_{B_{2,1}^{\frac{N}{p-1}}} \leq C\|P^{<1}\phi\|_{B_{2,1}^{\frac{N}{p-1}}}, \]
for all $\phi \in B_{2,1}^{\frac{N}{p-1}}$, and $s_1 > s_2$,
the high frequency embedding
\[ \|P^{\geq1}\phi\|_{B_{2,1}^{\frac{N}{p-1}}} \leq C\|P^{\geq1}\phi\|_{B_{2,1}^{\frac{N}{p-1}}}, \]
for all $\phi \in B_{2,1}^{\frac{N}{p-1}}$, and $s_1 < s_2$,
and Corollary 2.7 we can obtain the following lemma, whose proof will be given in Appendix.

Lemma 4.1. Assume $(b, u) \in \mathcal{E}_{p,\alpha}^{\frac{N}{p}} (T)$ with $(p, \alpha)$ satisfying $1.15$–$1.16$, then we have
\[ \|P^{<1}(b \text{div} u)\|_{L_T^{\frac{N}{p-1}}(B_{2,1}^{\frac{N}{p-1}+\alpha})} \leq CX^2(T), \]
\[ \|P^{<1}(T \text{div} b u)\|_{L_T^{\frac{N}{p-1}}(B_{2,1}^{\frac{N}{p-1}+\alpha})} \leq CX^2(T). \]
and
\begin{equation}
\|P_{<1}(T_u \nabla b)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T).
\end{equation}

Since \( \text{div}(bu) = b\text{div}u + \dot{T}_v b u + \dot{T}_u \nabla b \), from Lemma 4.1 we easily get the following Corollary, which will be used to bound \( Y_\alpha(T) \).

**Corollary 4.1.** Under the conditions in Lemma 4.1 we have
\begin{equation}
\|P_{<1}\text{div}(bu)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T).
\end{equation}

From (4.1)-(4.2), Lemma 2.1 and Proposition 2.2, we can obtain the following lemma, whose proof will be given in Appendix.

**Lemma 4.2.** Under the assumptions in Lemma 4.1 we have
\begin{equation}
\|P_{\geq 1}(b\text{div}u)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T),
\end{equation}
\begin{equation}
\|P_{\geq 1}(\dot{T}_v b u)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T).
\end{equation}

From the low frequency embedding (4.1), Lemma 2.1 Proposition 2.2 Corollary 2.1 Theorem 2.61 in \[\Pi\], and the special structure of \( \text{div}(I(b)A^2 u) \), we could get the following lemma, whose proof will be given in Appendix.

**Lemma 4.3.** Under the assumptions in Lemma 4.1 and
\[ \|b\|_{L^\infty_t(L^\infty_x)} \leq \frac{1}{2}, \]
we have
\begin{equation}
\|P_{<1}\left(I(b)A^2 u\right)\|_{L^2_T(B^N_{2,1})} + \|P_{\geq 1}\left(I(b)A^2 u\right)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T),
\end{equation}
and
\begin{equation}
\|P_{<1}\left(\Lambda^{-1}\text{div}(I(b)A^2 u)\right)\|_{L^2_T(B^N_{2,1})} + \|P_{\geq 1}\left(\Lambda^{-1}\text{div}(I(b)A^2 u)\right)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T).
\end{equation}

Similar, using Lemma 7.1 in the Appendix, we could get the following lemma, whose proof will be given in Appendix.

**Lemma 4.4.** Under the assumptions in Lemma 4.3 we have
\begin{equation}
\|P_{<1}(K(b)\nabla b)\|_{L^2_T(B^N_{2,1})} + \|P_{\geq 1}(K(b)\nabla b)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T).
\end{equation}

In the next two lemmas, we shall estimate the convection term \( \Lambda^{-1}\text{div}(u \cdot \nabla u) \). Here, we distinguish the terms with the potential part \( P^1 u \) from the terms with the divergence free part \( P u \).

**Lemma 4.5.** Under the assumptions in Lemma 4.1 we have
\begin{equation}
\|P_{<1}(P^1_\perp u \cdot \nabla)\|_{L^2_T(B^N_{2,1})} + \|P_{\geq 1}(P^1_\perp u \cdot \nabla)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T),
\end{equation}
\begin{equation}
\|P_{<1}(\dot{T}_v P^1_\perp u \cdot \nabla)\|_{L^2_T(B^N_{2,1})} + \|P_{\geq 1}(\dot{T}_v P^1_\perp u \cdot \nabla)\|_{L^2_T(B^N_{2,1})} \leq CX^2(T).
\end{equation}
Proof. From Bony's decomposition, the low frequency embedding (4.1) and the high frequency embedding (4.2), Proposition 2.2, Corollary 2.1 and Lemma 2.1, we have

\[ |P < 1(\mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P \geq 1(\mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \]

\[ \leq C \left( |T_{\mathbb{P}^u} \cdot \nabla \mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |T_{\mathbb{P}^u} \cdot \nabla \mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |T_{\mathbb{P}^u} \cdot \nabla \mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |T'_{\mathbb{P}^u} \cdot \nabla \mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \right) \]

\[ + |T_{\mathbb{P}^u} \cdot \nabla \mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |T_{\mathbb{P}^u} \cdot \nabla \mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |T_{\mathbb{P}^u} \cdot \nabla \mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \]

\[ \leq C \left( |P^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \right) \]

\[ \leq C \mathbb{X}^2(T), \]

where we have used the facts (7.17) and

\[ \mathcal{L}^+_{T}(B_{R,1}^{\frac{N}{p+1}}) \cap L^+_{T}(B_{R,1}^{\frac{N}{p+1}}) \subset \mathcal{L}^+_{T}(B_{R,1}^{\frac{N}{p+1}}). \]

Moreover, \( T_{\mathbb{P}^u} \cdot \nabla d \) can be bounded in a similar way. This completes the proof of Lemma 4.5.

\[ \mathbf{Lemma \ 4.6.} \quad \text{Under the assumptions in Lemma 4.7, we get} \]

\[ |P < 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P \geq 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \leq C \mathbb{X}^2(T), \]

\[ |P < 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P \geq 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \leq C \mathbb{X}^2(T), \]

\[ |P < 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P \geq 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \leq C \mathbb{X}^2(T), \]

\[ |P < 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P \geq 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \leq C \mathbb{X}^2(T). \]

Proof. From Lemma 2.1 Bony's decomposition, the low frequency embedding (4.1), and Proposition 2.2, we have

\[ |P < 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + |P \geq 1(\Lambda^{-1} \text{div} \mathbb{P}^u \cdot \nabla \mathbb{P}^u)|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \]

\[ \leq C |\mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + C |\mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + C |\mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + C |\mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \]

\[ \leq C |\mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} + C |\mathbb{P}^u|_{L^2(B_{R,1}^{\frac{N}{p+1}})} \]

\[ \leq C \mathbb{X}^2(T), \]

where we have used the fact \( p^* \geq p \) in the third inequality of (4.20). This explains why we need to assume \( p \leq 4 \) in (4.13). Next, using \( \text{div} \mathbb{P}^u = 0 \) and the fact \( u = u_L + u_H \), we can decompose \( \Lambda^{-1} \text{div}(\mathbb{P}^u \cdot \nabla \mathbb{P}^u) \) as follows:

\[ \Lambda^{-1} \text{div}(\mathbb{P}^u \cdot \nabla \mathbb{P}^u) = \Lambda^{-1} \hat{T}_{\mathbb{P}^u} \cdot \mathbb{P}^u \hat{\partial}_k(\mathbb{P}^u) + \Lambda^{-1} \text{div}(\hat{T}_{\mathbb{P}^u} \cdot \mathbb{P}^u) + \Lambda^{-1} \text{div}(\mathbb{P}^u \cdot \nabla \mathbb{P}^u). \]
Then it is easy to see that
\[
\|P_{<1} \left( \Lambda^{-1} T \partial_{(\varpi+\varphi)^2} \partial_k (\mathbb{P} u) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
+\|P_{\geq 1} \left( \Lambda^{-1} T \partial_{(\varpi+\varphi)^2} \partial_k (\mathbb{P} u) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
\leq C \|T \partial_{(\varpi+\varphi)^2} \partial_k (\mathbb{P} u) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
\leq C \|\partial_k (\mathbb{P} u) \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\partial_k (\mathbb{P} u) \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} \\
\leq C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+1\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} \leq CX^2(T),
\]
and
\[
\|P_{<1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
+\|P_{\geq 1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
\leq C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} \leq CX^2(T).
\]

Similar to (4.20) and (7.33), we have
\[
\|P_{<1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u) \cdot \nabla \mathbb{P} u \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} + \|P_{\geq 1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u) \cdot \nabla \mathbb{P} u \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
\leq C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} + C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} \leq CX^2(T).
\]

To bound $\mathbb{P} u \cdot \nabla \mathbb{P} u$, we need to decompose it as follows:
\[
\mathbb{P} u \cdot \nabla \mathbb{P} u = \mathbb{P} u \cdot \nabla \mathbb{P} u_L + \mathbb{P} u \cdot \nabla \mathbb{P} u_H.
\]
Then using the high frequency embedding (4.2), one easily deduces that
\[
\|P_{<1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u \cdot \nabla \mathbb{P} u_L) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} + \|P_{\geq 1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u \cdot \nabla \mathbb{P} u_L) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
\leq C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} + C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} \leq CX^2(T).
\]
In the same manner as (4.20), we are led to
\[
\|P_{<1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u \cdot \nabla \mathbb{P} u_H) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} + \|P_{\geq 1} \left( \Lambda^{-1} \text{div}(\mathbb{P} u \cdot \nabla \mathbb{P} u_H) \right) \|_{L^1_T(B_{2,1}^{\frac{3}{2}+1\alpha})} \\
\leq C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} + C \|\mathbb{P} u \|_{L^\infty_T(B_{\varpi+\varphi}^{\frac{3}{2}+2\alpha})} \|\mathbb{P} u \|_{L^1_T(B_{\varpi+\varphi}^\frac{3}{2})} \leq CX^2(T).
\]
Finally, $\dot{T}_P \cdot \nabla d$ can be bounded in the same way as $\mathbb{P}u \cdot \nabla \mathbb{P}u$. The proof of Lemma 4.6 is completed.

The next two lemmas will be used to bound the nonlinear terms in the equation of the divergence free part $\mathbb{P}u$ of the velocity $\mathbb{P}u \cdot \nabla \mathbb{P}u$.

**Lemma 4.7.** Under the assumptions in Lemma 4.1, we obtain

$$\|\mathbb{P}(u \cdot \nabla u)\|_{L^1_t(B^{-1}_{p,1})} \leq CX^2(T).$$

**Proof.** It is not difficult to verify that

$$\mathbb{P}(u \cdot \nabla u) = \mathbb{P}(\mathbb{P}u \cdot \nabla u) + \mathbb{P}(\mathbb{P}u \cdot \nabla \mathbb{P}u) + \mathbb{P}(\mathbb{P}^\perp u \cdot \nabla \mathbb{P}u).$$

Then using Lemma 2.4 and (2.5) yields

$$\|\mathbb{P}u \cdot \nabla \mathbb{P}u\|_{L^1_t(B^{-1}_{p,1})} \leq C\|\mathbb{P}u \otimes \mathbb{P}u\|_{L^1_t(B^{-1}_{p,1})} \leq C\|\mathbb{P}u\|^2_{L^2_t(B^{N-1}_{p,1})} \leq CX^2(T).$$

Next, in view of (1.39), using Proposition 2.2, Lemma 2.9 in [1] and $\text{div}\mathbb{P}u = 0$, we find that

$$\|\mathbb{P}(\mathbb{P}u \cdot \nabla \mathbb{P}u)\|_{L^1_t(B^{-1}_{p,1})} \leq C\|\mathbb{P}(\mathbb{P}_T \mathbb{P}^\perp u)\|_{L^1_t(B^{-1}_{p,1})} + C\|\mathbb{P}(\mathbb{P}u \cdot \nabla \mathbb{P}u)\|_{L^1_t(B^{-1}_{p,1})} + C\|\mathbb{P}(\mathbb{P}u \cdot \nabla \mathbb{P}u)\|_{L^1_t(B^{-1}_{p,1})}$$

$$\leq C\|\mathbb{P}u\|_{L^{1/2}_t(B^{-1}_{p,1})} \|\nabla \mathbb{P}u\|_{L^{1/2}_t(B^{-1}_{p,1})} \|\mathbb{P}u\|_{L^{1/2}_t(B^{-1}_{p,1})} + C\|\mathbb{P}u\|_{L^{1/2}_t(B^{-1}_{p,1})} \|\nabla \mathbb{P}u\|_{L^{1/2}_t(B^{-1}_{p,1})}$$

$$+ C\|\partial_k \mathbb{P}(\mathbb{P}u)^k\|_{L^1_t(B^{-1}_{p,1})} \leq CX^2(T).$$

Finally, the condition (1.15) on $p$ ensures that $p < 2N$, then it is easy to see that

$$\|\mathbb{P}(\mathbb{P}^\perp u \cdot \nabla \mathbb{P}u)\|_{L^1_t(B^{-1}_{p,1})} \leq C\|\mathbb{P} \mathbb{P}^\perp u\|_{L^1_t(B^{-1}_{p,1})} + C\|\mathbb{P} \mathbb{P}^\perp u\|_{L^1_t(B^{-1}_{p,1})} + C\|\mathbb{P}(\mathbb{P}^\perp u \cdot \nabla \mathbb{P}u)\|_{L^1_t(B^{-1}_{p,1})}$$

$$\leq C\|\mathbb{P}^\perp u\|_{L^1_t(B^{-1}_{p,1})} + C\|\nabla \mathbb{P}u\|_{L^1_t(B^{-1}_{p,1})} \|\mathbb{P}^\perp u\|_{L^1_t(B^{-1}_{p,1})}$$

$$+ C\|\mathbb{P}(\mathbb{P}^\perp u \cdot \nabla \mathbb{P}u)\|_{L^1_t(B^{-1}_{p,1})} \leq CX^2(T).$$

This explains why we need to assume $p < 4$ if $N = 2$. We complete the proof of Lemma 4.7.

**Lemma 4.8.** Under the assumptions in Lemma 4.4, we have

$$\|I(b) \mathcal{A}u\|_{L^1_t(B^{-1}_{p,1})} \leq CX^2(T).$$
Proof. Let us first decompose $u$ as

$$ u = \mathbb{P}u_L + (\mathbb{P}u_H + \mathbb{P}u). $$

Then using Corollary 2.1 with $u = I(b), v = \mathcal{A}\mathbb{P}u_L, p_1 = q_2 = 2, \rho = p_2 = q_1 = p, s_1 = \sigma_2 = \frac{N}{2}, s_2 = \sigma_1 = \frac{N}{p} - 1$, and Theorem 2.61 in [1], we obtain

$$
\|I(b)\mathcal{A}\mathbb{P}u_L\|_{L^1_L(B_{p,1}^{\frac{N}{p}-1})} \\
\leq C\|I(b)\|_{L^\infty_T(B_{1,1}^{\frac{N}{p}})}\|\mathcal{A}\mathbb{P}u_L\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1})} \\
\leq C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p}})}\left(\|\mathbb{P}u_H\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1})} + \|\mathbb{P}u\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1})}\right)
$$

(4.31)

where we have used (7.24), (7.27) and (7.28). Similarly, using (7.25), we arrive at

$$
\|I(b)\mathcal{A}(\mathbb{P}u_H + \mathbb{P}u)\|_{L^1_L(B_{p,1}^{\frac{N}{p}-1})} \\
\leq C\|I(b)\|_{L^\infty_T(B_{1,1}^{\frac{N}{p}})}\|\mathcal{A}(\mathbb{P}u_H + \mathbb{P}u)\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1})} \\
\leq C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p}})}\left(\|\mathbb{P}u_H\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1})} + \|\mathbb{P}u\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1})}\right)
$$

(4.32)

where $X(\sigma,\theta) \subseteq \mathcal{C}(\beta, \gamma, \rho)$ satisfies (I)–(III) and Theorem 2.61 in [1]. This completes the proof of Lemma 4.9.

4.2. Estimates of $X(T)$. Using the above lemmas, we could obtain the Dispersive estimates and Energy estimates as follows.

Step (I): Dispersive estimates.

Lemma 4.9. Let $p$ and $\alpha$ satisfy (1.15) and (1.16), respectively. Assume that $(b, u)$ is a solution to system (1.30)–(1.31) in $E_p^{\frac{N}{p}+\alpha} (T)$ with

$$
\|b\|_{L^\infty_T(L^p)} \leq \frac{1}{2}.
$$

Then we have

$$
Y_\alpha(T) \leq CX_0^0 + CX_\theta(T) + CX^2(T).
$$

(4.33)

Proof. First of all, let us cut off the system (1.32) by using the operator $P_{<\zeta}$. Then applying Proposition 3.1 to the resulting system with $\epsilon = 1, s = \frac{N}{2} - 1 + \alpha, \bar{p} = 2, \bar{r} = \infty$, and $r = \frac{1}{\alpha}$, we arrive at

$$
Y_\alpha(T) \leq C \left(\|(b_{0L}, d_{0L})\|_{B_{2,1}^{\frac{N}{2}+1+\alpha}} + \|P_{<\zeta} \text{div}(bu)\|_{L^1_T(B_{2,1}^{\frac{N}{2}+1+\alpha})} + \|P_{<\zeta} \Delta d\|_{L^1_T(B_{2,1}^{\frac{N}{2}+1+\alpha})}ight)
$$

$$
+ \|P_{<\zeta} \Lambda^{-1}\text{div}(u \cdot \nabla u + K(b) \nabla b + I(b) \mathcal{A}u)\|_{L^1_T(B_{2,1}^{\frac{N}{2}+1+\alpha})}
$$

$$
\leq C \left(X_0^0 + X_\theta(T) + \|P_{<\zeta} \text{div}(bu)\|_{L^1_T(B_{2,1}^{\frac{N}{2}+1+\alpha})}ight)
$$

$$
+ \|P_{<\zeta} \Lambda^{-1}\text{div}(u \cdot \nabla u + K(b) \nabla b + I(b) \mathcal{A}u)\|_{L^1_T(B_{2,1}^{\frac{N}{2}+1+\alpha})}.$$

Combining Corollary 4.1 with Lemmas 4.3–4.6, we find that the estimate (4.33) holds. This completes the proof of Lemma 4.9.

□
Step (II): Energy estimates.

To begin with, let us localize the system \((1.32)\) as follows:

\[
\begin{aligned}
\partial_t b_q + \dot{S}_{q-1} u \cdot \nabla b_q + \Delta d_q &= \tilde{f}_q, \\
\partial_t d_q + \dot{S}_{q-1} u \cdot \nabla d_q - \Delta d_q - \Lambda b_q &= \tilde{g}_q,
\end{aligned}
\]

with

\[
\begin{aligned}
\tilde{f}_q &:= f_q + \left( S_{q-1} u \cdot \nabla b_q - \Delta q T_u \cdot \nabla b \right), \\
\tilde{g}_q &:= g_q + \left( S_{q-1} u \cdot \nabla d_q - \Delta q \dot{T}_u \cdot \nabla d \right),
\end{aligned}
\]

and

\[
\begin{aligned}
f &= -bd\text{div} u - T_{\partial_b} u^k, \\
g &= \dot{T}_u \cdot \nabla d - \Lambda^{-1} \text{div} (u \cdot \nabla u + K(b) \nabla b + I(b) \Delta u).
\end{aligned}
\]

Now, we estimate the low frequency part \(X_L(T)\) and high frequency part \(X_H(T)\) separately.

(i) Estimates of \(X_L(T)\).

Lemma 4.10. Under the conditions in Lemma [4.9] we have

\[
X_L(T) \leq CX_L^0 + CX^2(T).
\]

Proof. Similar to the energy estimates for the isentropic Navier-Stokes equations obtained by Danchin [13], we easily get the following three equalities

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|b_q\|_{L^2}^2 + (\Delta d_q) \|b_q\| &= \frac{1}{2} \int \text{div} \dot{S}_{q-1} u |b_q|^2 + (\tilde{f}_q) |b_q|, \\
\frac{1}{2} \frac{d}{dt} \|d_q\|_{L^2}^2 + \|\nabla d_q\|_{L^2}^2 - (\Lambda b_q) \|d_q\| &= \frac{1}{2} \int \text{div} \dot{S}_{q-1} u |d_q|^2 + (\tilde{g}_q) |d_q|,
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{d}{dt} (d_q |\Lambda b_q|) - (\Lambda b_q) |d_q| + \|\Lambda d_q\|_{L^2}^2 - (\Delta d_q) |b_q|
\end{aligned}
\]

(4.38)

\[
= \int \text{div} \dot{S}_{q-1} u (b_q |d_q| + (\tilde{g}_q) \Lambda b_q) + (\Lambda \tilde{f}_q) |d_q|.
\]

A linear combination of (4.36)–(4.38) yields

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \|b_q\|_{L^2}^2 + \|d_q\|_{L^2}^2 - \frac{1}{4} \|d_q \Lambda b_q\| \right) + \frac{7}{8} \|\Lambda d_q\|_{L^2}^2 + \frac{1}{8} \|\Lambda b_q\|_{L^2}^2 + \frac{1}{8} (\Delta d_q) |b_q|
\end{aligned}
\]

(4.39)

\[
+ (\tilde{g}_q) |d_q| - \frac{1}{8} \Lambda b_q) + (\tilde{f}_q) |b_q| - \frac{1}{8} \Delta d_q) + \frac{1}{8} (\Lambda, \dot{S}_{q-1} u \cdot \nabla) |b_q| |d_q|.
\]

Noting that \(u = u_L + u_H\), it is easy to see that

\[
\|\nabla S_{q-1} u\|_{L^\infty} \leq C \left( 2^{q(2-2\alpha)} \|\nabla S_{q-1} u_L\|_{L^\infty} + \|\nabla S_{q-1} u_H\|_{L^\infty} \right)
\]

(4.40)

\[
\leq C \left( m(u) + \|\nabla u_H\|_{L^\infty} \right),
\]

where

\[
m(u) := \min \left( 2^{q(2-2\alpha)} \|\nabla u_L\|_{B^\alpha_{2q,2}}, \|\nabla u_L\|_{L^\infty} \right).
\]

Then following the proof of Lemma 2.99 in \([1]\), we have

\[
\|\Lambda, \dot{S}_{q-1} u \cdot \nabla) b_q\|_{L^2} \leq C \|\nabla S_{q-1} u\|_{L^\infty} \|\Lambda b_q\|_{L^2}
\]

(4.41)

\[
\leq C \left( m(u) + \|\nabla u_H\|_{L^\infty} \right) \left( 2^{q}\|b_q\|_{L^2} \right).
\]
According to Lemma 7.5 in [14], we arrive at
\[\|\dot{\mathcal{S}}_{q-1} u \cdot \nabla d_q - \dot{\Delta}_q \dot{u} \cdot \nabla d\|_{L^2} \leq \|\dot{\mathcal{S}}_{q-1} u \cdot \nabla d_q - \dot{\Delta}_q \dot{u} \cdot \nabla d\|_{L^2} + \|\dot{\mathcal{S}}_{q-1} u \cdot \nabla d_q - \dot{\Delta}_q \dot{u} \cdot \nabla d\|_{L^2} \leq C (m(u) + \|\nabla u\|_{L^\infty}) \sum_{|q'-q|\leq 4} \|d_{q'}\|_{L^2},\]
(4.43)
and
\[\|\dot{\mathcal{S}}_{q-1} u \cdot \nabla b_q - \dot{\Delta}_q \dot{u} \cdot \nabla b\|_{L^2} \leq C (m(u) + \|\nabla u\|_{L^\infty}) \sum_{|q'-q|\leq 4} \|b_{q'}\|_{L^2} .\]
(4.44)
Then thanks to Bernstein’s inequality, we infer from (4.39)–(4.44) that, for \(q \leq 0\), there holds
\[\|b_q(t)\|_{L^2} + \|\dot{b}_q(t)\|_{L^2} + 2^2 \|b_q\|_{L^1_{t}(L^2)} + 2^2 \|\dot{b}_q\|_{L^1_{t}(L^2)} \leq C \left( \|b_q(0)\|_{L^2} + \|\dot{b}_q(0)\|_{L^2} + \|f_q\|_{L^1_{t}(L^2)} + \|g_q\|_{L^1_{t}(L^2)} \right)\]
(4.45)
\[+ C \int_0^t (m(u) + \|\nabla u\|_{L^\infty}) \sum_{|q'-q|\leq 4} (\|b_{q'}\|_{L^2} + \|d_{q'}\|_{L^2}) dt'.\]
Recalling that
\[X_L(T) = \|\langle b, \mathbb{P}^\perp u \rangle\|_{L^2_{t}((L^\infty_{\infty}(B_{2,1}^{\infty})) \cap L^1_{t}(B_{2,1}^{\infty + 1 + a}))},\]
multiplying \(X_L(T)\) by \(2^{(\frac{\alpha}{2} + 1)(\alpha)}\), and taking sum with respect to \(q\) over \(\{\cdots, -2, -1, 0\}\), we obtain
\[X_L(T) \leq C \left( X_L^0 + \|P_{1f}\|_{L^2_{t}(B_{2,1}^{\infty + 1 + a})} + \|P_{1g}\|_{L^2_{t}(B_{2,1}^{\infty + 1 + a})} \right)\]
(4.46)
\[+ C \int_0^T \sum_{q \leq 0} 2^{q(\frac{\alpha}{2} + 1 + a)}(m(u) + \|\nabla u\|_{L^\infty}) \sum_{|q'-q|\leq 4} (\|\dot{\Delta}_q b, \dot{\Delta}_q d\|_{L^2} ds.\]
Now we go to bound the right hand side of (4.46). First of all, from Lemmas [4.1, 4.3, 4.6] we have
\[\|P_{1f}\|_{L^1_{t}(B_{2,1}^{\infty + 1 + a})} + \|P_{1g}\|_{L^1_{t}(B_{2,1}^{\infty + 1 + a})} \leq CX^2(T).\]
(4.47)
The remaining terms of the right hand side of (4.46) can be bounded as follows. In fact, by virtue of Young’s inequality, Hölder’s inequality and the high frequency embedding (4.2), we are led to
\[\int_0^T \sum_{q \leq 0} 2^{q(\frac{\alpha}{2} + 1 + a)}||\nabla u\|_{L^\infty} \sum_{|q'-q|\leq 4} (\|\dot{\Delta}_q b, \dot{\Delta}_q d\|_{L^2} ds\]
\[\leq \int_0^T \sum_{q \leq 0} 2^{q(\frac{\alpha}{2} + 1 + a)}||\nabla u\|_{L^\infty} \sum_{|q'-q|\leq 4} (\|\dot{\Delta}_q b, \dot{\Delta}_q d\|_{L^2} ds\]
\[+ \int_0^T \sum_{q \leq 0} 2^{q(\frac{\alpha}{2} + 1 + a)}||\nabla u\|_{L^\infty} \sum_{|q'-q|\leq 4} (\|\dot{\Delta}_q b, \dot{\Delta}_q d\|_{L^2} ds\]
\[\leq C \|\nabla u\|_{L^1_{t}(L^\infty)} \left( ||\langle b, d \rangle\|_{L^2_{t}(B_{2,1}^{\infty + 1 + a})} + ||b H\|_{L^2_{t}(B_{2,1}^{\infty})} + ||d H\|_{L^2_{t}(B_{2,1}^{\infty})} \right)\]
\[\leq C \left( ||\mathbb{P}^\perp u\|_{L^1_{t}(L^\infty)} + ||\nabla u\|_{L^1_{t}(L^\infty)} \right) X(T)\]
\[\leq CX^2(T).\]
Similarly, using (4.17), and the interpolation
\[\bar{L}^\infty_{T}(B_{p,1}^{\frac{\alpha}{2} + 1}) \cap L^1_T(B_{p,1}^{\frac{\alpha}{2} + 1}) \subset \bar{L}^\infty_{T}(B_{p,1}^{\frac{\alpha}{2} + 2a + 1}) \subset \bar{L}^\infty_{T}(B_{p,1}^{2a + 1}) ,\]
(4.48)
we have
\[
\int_0^T \sum_{q \leq 0} 2^q \sum_{|q'-q| \leq 4} \|\tilde{\Delta}^q b_L, \tilde{\Delta}^q d_L\|_{L^2} ds
\leq C \|u_L\|_{L^2} \left( \|b_L, d_L\|_{L^2} (B_{p,1}^\infty) \right)
\leq C \left( \|u_L\|_{L^2} (B_{p,1}^\infty) \right)^2.
\]

Finally, according to the following interpolations
\[
(4.49) \quad \tilde{L}_T^\infty (B_{p,1}^{\frac{5}{2},-1}) \cap L_T^1 (B_{p,1}^{\frac{5}{2},+1}) \subset \tilde{L}_T^\infty (B_{p,1}^{\frac{5}{2},-1+\alpha}),
(4.50) \quad \tilde{L}_T^\infty (B_{p,1}^{\frac{5}{2},-1+\alpha}) \cap L_T^1 (B_{p,1}^{\frac{5}{2},+1+\alpha}) \subset \tilde{L}_T^\infty (B_{p,1}^{\frac{5}{2},+1}),
(4.51) \quad \tilde{L}_T^\infty (B_{p,1}^{\frac{5}{2},-1}) \cap L_T^1 (B_{p,1}^{\frac{5}{2},+1}) \subset \tilde{L}_T^\infty (B_{p,1}^{\frac{5}{2},-1+\alpha}),
\]
the low frequency embedding
\[
\| u_L \|_{L_T^2 (B_{p,1}^{\frac{5}{2},+1})} \leq C \| u_L \|_{L_T^2 (B_{p,1}^{\frac{5}{2},-1+\alpha})},
\]
and the high frequency embedding
\[
\| b_H \|_{L_T^2 (B_{p,1}^{\frac{5}{2},-1+\alpha})} \leq C \| b_H \|_{L_T^2 (B_{p,1}^{\frac{5}{2},+1})},
\]
we find that
\[
\int_0^T \sum_{q \leq 0} \|\tilde{\Delta}^q b_L, \tilde{\Delta}^q d_L\|_{L^2} ds
\leq C \|u_L\|_{L^2} \left( \|b_L, d_L\|_{L^2} (B_{p,1}^\infty) + \|\tilde{\Delta}^q b_H, \tilde{\Delta}^q d_H\|_{L^2} \right)
\leq C \left( \|u_L\|_{L^2} (B_{p,1}^\infty) \right)^2.
\]

Combining these estimates with (4.46), we obtain (4.35). The proof of Lemma 4.10 is completed.

(ii) Estimates of \(X_H(T)\).

**Lemma 4.11.** Under the conditions in Lemma 4.9, we have
\[
X_H(T) \leq CX_H^0 + CX^2(T).
\]

**Proof.** To begin with, let us give the \(L^2\) energy estimate for \(\Lambda b_q\),
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda b_q\|_{L^2}^2 + (\Lambda^2 d_q |\Lambda b_q|) = \frac{1}{2} \int \text{div} \tilde{\Lambda} q-1 u |\Lambda b_q|^2 - ([\Lambda, \tilde{\Lambda} q-1 u \cdot \nabla] b_q |\Lambda b_q| + (\Lambda f_q |\Lambda b_q|).
\]
It follows from (4.47), (4.48) and (4.54), that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda b_q\|_{L^2}^2 + 2 ||d_q|_{L^2}^2 - 2 (d_q |\Lambda b_q|) + ||\Lambda d_q|_{L^2}^2 + ||\Lambda b_q|_{L^2}^2 - 2 (d_q |\Lambda b_q|) \right)
\]
\[
= \int \text{div} \tilde{\Lambda} q-1 u \left( \frac{1}{2} |\Lambda b_q|^2 + |d_q|^2 \right) - \int \text{div} \tilde{\Lambda} q-1 u \Lambda b_q d_q
\]
To exhibit the smoothing effect of $u$ in high frequency case, we need the following $L^2$ energy estimate for $d_q$,

$$
\frac{1}{2} \frac{d}{dt} \|d_q\|_{L^2}^2 + \|\Lambda d_q\|_{L^2}^2 = \frac{1}{2} \int \text{div} \hat{S}_{q-1} u |d_q|^2 + \langle \Lambda b_q |d_q \rangle + (\bar{g}_q |d_q|).
$$

Using (4.42) and Lemma 7.5 in [14] again yields

$$
\|\Lambda \left( \hat{S}_{q-1} u \cdot \nabla b_q - \hat{\Lambda}_q \hat{T}_u \cdot \nabla b \right) \|_{L^2} \\
\leq C \|\Lambda \hat{S}_{q-1} u \cdot \nabla b_q \|_{L^2} + C \|\hat{S}_{q-1} u \cdot \nabla \Lambda b_q - \Lambda \hat{\Lambda}_q \hat{T}_u \cdot \nabla b \|_{L^2} \\
\leq C (m(u) + \| \nabla u_H \|_{L^\infty}) \sum_{|q'| - |q| \leq 4} 2^q \|b_{q'}\|_{L^2}.
$$

Taking the advantage of Bernstein’s inequality, we infer from (4.40), (4.43), (4.55) and (4.57) that, for $q \geq 1$, there holds

$$
2^q \|b_q(t)\|_{L^2} + \|d_q(t)\|_{L^2} + 2^q \|b_{q'}\|_{L^2} + \|d_{q'}\|_{L^2} \\
\leq C \left( 2^q \|b_q(0)\|_{L^2} + \|d_q(0)\|_{L^2} + 2^q \|f_q\|_{L^2} + \|g_q\|_{L^2} \right) \\
+ C \int_0^t (m(u) + \| \nabla u_H \|_{L^\infty}) \sum_{|q'| - |q| \leq 4} 2^q \|b_{q'}\|_{L^2} + \|d_{q'}\|_{L^2} dt'.
$$

Similarly, for $q \geq 1$, (4.40), (4.43) and (4.56) imply that

$$
\|d_q(t)\|_{L^2} + 2^q \|d_{q'}\|_{L^2} \\
\leq C \left( \|d_q(0)\|_{L^2} + 2^q \|b_{q'}\|_{L^2} + \|g_q\|_{L^2} \right) \\
+ C \int_0^t (m(u) + \| \nabla u_H \|_{L^\infty}) \sum_{|q'| - |q| \leq 4} 2^q \|b_{q'}\|_{L^2} + \|d_{q'}\|_{L^2} dt'.
$$

Combining these two inequalities, we find that, if $q \geq 1$, there holds

$$
2^q \|b_q(t)\|_{L^2} + \|d_q(t)\|_{L^2} + 2^q \|b_{q'}\|_{L^2} + 2^q \|d_{q'}\|_{L^2} \\
\leq C \left( 2^q \|b_q(0)\|_{L^2} + \|d_q(0)\|_{L^2} + 2^q \|f_q\|_{L^2} + \|g_q\|_{L^2} \right) \\
+ C \int_0^t (m(u) + \| \nabla u_H \|_{L^\infty}) \sum_{|q'| - |q| \leq 4} 2^q \|b_{q'}\|_{L^2} + \|d_{q'}\|_{L^2} dt'.
$$

Multiplying (4.58) by $2^q \frac{q}{2}^{-1}$, and taking sum with respect to $q$ over $\{1, 2, 3, \cdots \}$, we arrive at

$$
X_H(T) \leq C \left( X^0_H + \| P_{\geq 1} f \|_{L^2_1(\mathcal{B}^{\frac{q}{2}}_{2,1})} + \| P_{\geq 1} g \|_{L^2_1(\mathcal{B}^{\frac{q}{2}}_{2,1})} \right) \\
+ C \int_0^T \sum_{q \geq 1} 2^q (\frac{q}{2})^{-1} (m(u) + \| \nabla u_H \|_{L^\infty}) \sum_{|q'| - |q| \leq 4} 2^q \|\hat{\Lambda}_{q'} b\|_{L^2} + \|\hat{\Lambda}_{q'} d\|_{L^2} ds.
$$

Now let us bound the right hand side of (4.59). In fact, we infer from Lemmas [4,2] and [4,4] that

$$
\| P_{\geq 1} f \|_{L^2_1(\mathcal{B}^{\frac{q}{2}}_{2,1})} + \| P_{\geq 1} g \|_{L^2_1(\mathcal{B}^{\frac{q}{2}}_{2,1})} \leq C X^2(T).
$$

The estimates of the last term in (4.59) are a little bit trickier. First of all, using Young’s inequality, Hölder’s inequality, and (7.25) yields

$$
\int_0^T \sum_{q \geq 1} 2^q (\frac{q}{2})^{-1} \| \nabla u_H \|_{L^\infty} \sum_{|q'| - |q| \leq 4} 2^q \|\hat{\Lambda}_{q'} b\|_{L^2} + \|\hat{\Lambda}_{q'} d\|_{L^2} ds \\
\leq \int_0^T \sum_{q \geq 1} 2^q (\frac{q}{2})^{-1} \| \nabla u_H \|_{L^\infty} \sum_{|q'| - |q| \leq 4} 2^q \|\hat{\Lambda}_{q'} b\|_{L^2} + \|\hat{\Lambda}_{q'} d\|_{L^2} ds
$$
Moreover, using (7.17), the following low frequency embedding

\[ \|b_L\|_{L^2 B_{p,1}^{\infty}} \leq C \|b_L\|_{F_T B_{p,1}^{\infty}} \]

and the interpolation (4.48), we find that

\[ \int_0^T \sum_{q \geq 1} 2^{q(\frac{N}{2} - 1)} \|\nabla u_L\|_{L^\infty} \sum_{l' - q \leq 4} 2^{q} \| \dot{\Delta}^q b_L \|_{L^2} ds \]

\[ \leq C \|\nabla u_L\|_{F_T L^2 B_{p,1}^{\infty}} \sum_{l' - q \leq 4} 2^{q} \| \dot{\Delta}^q d_L \|_{L^2} ds \]

Similar to (4.52), we have

\[ \int_0^T \sum_{q \geq 1} 2^{q(\frac{N}{2} - 1)} \|\nabla u_L\|_{L^\infty} \sum_{l' - q \leq 4} 2^{q} \| \Delta^q b_H \|_{L^2} ds \]

Finally, using (4.15) and (4.48) again, we arrive at

\[ \int_0^T \sum_{q \geq 1} 2^{q(\frac{N}{2} + 1 - 2\alpha)} \|\nabla u_L\|_{L^\infty} \sum_{l' - q \leq 4} \| \Delta^q d_H \|_{L^2} ds \]

(3) Estimates of \( W(T) \).

In fact, applying Proposition 3.3 to (1.31), and using Lemmas 4.7 and 4.8, we easily get the following estimate for \( W(T) \).

**Lemma 4.12.** Under the conditions in Lemma 4.9 we have

\[ W(T) \leq CW^0 + CX^2(T). \]

Collecting Lemmas 4.9, 4.12 we conclude that
Proposition 4.1. Let $p$ and $\alpha$ satisfy \((1.15)\) and \((1.16)\), respectively. Assume that \((b, u)\) is a solution to system \((1.30)–(1.31)\) in $E_p^{\frac{N}{2}, \alpha}(T)$ with
\[
\|b\|_{L_p^\infty(L^\infty)} \leq \frac{1}{2}.
\]
Then we have
\[
(4.62) \quad X(T) \leq CX^0 + CX^2(T).
\]

5. Proof of Theorem 1.1

5.1. The global existence. First of all, we construct the approximate solutions to that system \((1.30)–(1.31)\) with smoothing initial data. For the sake of simplicity, we just outline it here (for the details, see e.g. \([1]\) and \([17]\)). To begin with, let us recall the following local existence theorem.

Theorem 5.1 \(([17])\). Let $N \geq 2$. Assume that $\rho_0 - 1 \in B_{2,1}^\infty$ and $u_0 \in B_{2,1}^{\frac{N}{2} - 1}$ with $\rho_0$ bounded away from 0. There exists a positive time $T$ such that system \((1.1)\) has a unique solution $(\rho, u)$ with $\rho$ bounded away from 0,

\[
\rho - 1 \in \overline{C_T(B_{2,1}^\infty)}, \quad \text{and} \quad u \in \overline{C_T(B_{2,1}^{\frac{N}{2} - 1})} \cap L_T^1(B_{2,1}^{\frac{N}{2} + 1}).
\]

Moreover, the solution $(\rho, u)$ can be continued beyond $T$ if the following three conditions hold:

(i) The function $\rho - 1$ belongs to $L_T^\infty(B_{2,1}^\infty)$,
(ii) the function $\rho$ is bounded away from 0,
(iii) $\int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau < \infty$.

Remark 5.1. In addition, we claim that if $\rho_0 - 1 \in B_{2,1}^{\frac{N}{2} - 1 + \alpha}$, then $\rho - 1 \in \overline{C_T(B_{2,1}^{\frac{N}{2} - 1 + \alpha})}$. In fact, using Proposition 3.2 and Corollary 2.1, we have

\[
\|b\|_{L_T^\infty(B_{2,1}^{\frac{N}{2} - 1 + \alpha})} \leq \exp \left\{ C\|\nabla u\|_{L_T^\infty(B_{2,1}^{\frac{N}{2} - 1})} \right\} \left( \|b_0\|_{B_{2,1}^{\frac{N}{2} - 1 + \alpha}} + \int_0^T \|1 + b\|_{L_T^\infty(B_{2,1}^{\frac{N}{2} - 1})} \|\nabla u\|_{L_T^\infty(B_{2,1}^{\frac{N}{2} - 1})} d\tau \right) \leq C.
\]

For initial data $(\rho_0, u_0)$, by embedding, it is easy to see that
\[
(5.1) \quad \|b_0\|_{L^\infty} \leq C \|b_0\|_{B_{2,1}^{\frac{N}{2} - 1}} \leq C \|(b_0, u_0)\|_{E_0}.
\]

Before proceeding any further, let us denote by $\hat{C}$ the maximum of constants 1 and $C$ appearing in Proposition 4.1 and (5.1), and choose $(\rho_0, u_0)$ with $(b_0, u_0)$ so small that
\[
(5.2) \quad \|(b_0, u_0)\|_{E_0} \leq \frac{1}{8 C^2}.
\]

It follows from (5.1) and (5.2) that
\[
(5.3) \quad \|b_0\|_{L^\infty} \leq \frac{1}{8}.
\]

Thanks to Proposition 2.27 in \([1]\), we can find a sequence of functions $\{(b_0^n, u_0^n)\} \subset S \times S$ satisfying
\[
(5.4) \quad \|(b_0^n - b_0, u_0^n - u_0)\|_{E_0} \to 0, \quad \text{as} \quad n \to \infty.
\]
and
\[(5.5) \quad \|b_0^n\|_{L^\infty} \leq \frac{1}{4^n} \quad \text{for all} \quad n \in \mathbb{N}.\]

Then using Theorem 5.1 and Remark 5.1 above, one could obtain a unique local solution \((b^n, u^n)\) to the system \((1.30)-(1.31)\) with smoothing initial data \((b_0^n, u_0^n)\) on the maximal lifespan \([0, T^n]\), satisfying
\[(b^n, u^n) \in \left(\tilde{C}_T(B^{\frac{N}{2}+1+\sigma}_{2,1} \cap \tilde{B}^{\frac{N}{2}}_{2,1}) \times \left(\tilde{C}_T(B^{\frac{N}{2}+1}_{2,1} \cap L^1_T(B^{\frac{N}{2}+1}_{2,1}))\right)\right) \subset \mathcal{E}_{\rho}^{\frac{N}{2}-\alpha}(T), \ \forall \ T \in (0, T^n).

Now define \(T^n_1\) be the supremum of all time \(T' \in [0, T^n]\) such that
\[(5.6) \quad X^n(T') \leq 4\tilde{C}\|(b_0, u_0)\|_{\mathcal{E}_0},
\]
where
\[X^n(T) = \|(b^n_L, \mathbb{P}^\perp u^n_L)\|_{L^\infty_T(B^{\frac{N}{2}+1+\sigma}_{2,1})} + \|(b^n_L, \mathbb{P}^\perp u^n_L)\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})} + \|(\mathbb{P}^\perp u^n_L)\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})} + \|(\mathbb{P}^\perp u^n_L)\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})}.
\]

Combining (5.6) with (5.1)–(5.2), one easily deduces that
\[(5.7) \quad \|b^n\|_{L^\infty(T^n)} \leq \frac{1}{2}.
\]

Then from Proposition 4.1 and (5.4), we find that
\[(5.8) \quad X^n(T^n_1) \leq \tilde{C}\|(b_0^n, u_0^n)\|_{\mathcal{E}_0} + 16\tilde{C}^3\|(b_0, u_0)\|_{\mathcal{E}_0}^2 \leq 2\tilde{C}\|(b_0, u_0)\|_{\mathcal{E}_0} \left(1 + 8\tilde{C}^2\|(b_0, u_0)\|_{\mathcal{E}_0}\right) \leq 3\tilde{C}\|(b_0, u_0)\|_{\mathcal{E}_0},
\]
provided the initial data \((\rho_0, u_0)\) satisfy
\[(5.9) \quad \|(b_0, u_0)\|_{\mathcal{E}_0} \leq \frac{1}{16\tilde{C}^2}.
\]

Thus \(T^n = T^n_1\), and (5.6) holds true on the interval \([0, T^n_1]\) provided \(\|(b_0, u_0)\|_{\mathcal{E}_0} \leq c_0\) with \(c_0 := \frac{1}{16\tilde{C}^2} \). Consequently, (5.7) holds with \(T^n_1\) replaced by \(T^n\), and
\[\|b^n\|_{L^\infty_T(B^{\frac{N}{2}}_{2,1})} + \int_0^{T^n_1} \|\nabla \mathbb{P}^\perp u^n_L\|_{L^\infty} + \|\nabla \mathbb{P}^\perp u^n_H\|_{L^\infty} dt + \left(\int_0^{T^n_1} \|\nabla \mathbb{P}^\perp u^n_H\|_{L^\infty}^2 dt\right)^{\frac{\tilde{\alpha}}{2}} \leq C.
\]

Therefore, using Theorem 5.1 again, we conclude that \(T^n = +\infty\) for all \(n \in \mathbb{N}\). Moreover, for all \(n \in \mathbb{N}\), there holds
\[X^n(T) \leq C_0\|(b_0, u_0)\|_{\mathcal{E}_0}, \ \forall \ T > 0,
\]
with \(C_0 := 4\tilde{C}\). Then, using the compactness arguments similar as that in Chapter 10 of [11], we obtain that \((b^n, u^n)\) weakly converges (up to a subsequence) to some global solution \((\tilde{b}, \tilde{u})\) to the system \((1.30)-(1.31)\) with the initial data \((b_0, u_0)\) satisfying (1.17). Thus, we prove the global existence part of Theorem 1.1.

5.2. The uniqueness when \(N \geq 3\). Next, we will prove the uniqueness part of Theorem 1.1 when \(N \geq 3\). Assume there exist two solutions \((b^1, u^1)\) and \((b^2, u^2)\) for the system \((1.30)-(1.31)\) with the same initial data \((b_0, u_0)\), satisfying the regularity conditions in Theorem 1.1. In order to show that
these two solutions coincide, we shall give some estimates for \((\delta b, \delta u) = (b^2 - b^1, u^2 - u^1)\). It is easy to verify that \((\delta b, \delta u)\) satisfies the following system

\[
\begin{aligned}
\begin{cases}
\partial_t \delta b + u^2 \cdot \nabla \delta b &= -\text{div} \mathbb{P} \partial^+ \delta u - \delta u \cdot \nabla b^1 - \delta b \text{div} u^2 - b^1 \text{div} \delta u, \\
\partial_t \mathbb{P} \partial^+ \delta u - \Delta \mathbb{P} \partial^+ \delta u + \nabla \delta b &= -\mathbb{P} \left( u^2 \cdot \nabla u^2 + K(b^2) \nabla b^2 + I(b^2) \mathcal{A} u^2 - u^1 \cdot \nabla u^1 - K(b^1) \nabla b^1 - I(b^1) \mathcal{A} u^1 \right), \\
\partial_t \mathbb{P} \partial^+ \delta u - \mu \Delta \mathbb{P} \partial^+ \delta u &= -\mathbb{P} \left( u^2 \cdot \nabla u^2 + I(b^2) \mathcal{A} u^2 - u^1 \cdot \nabla u^1 - I(b^1) \mathcal{A} u^1 \right), \\
(\delta b, \delta u)|_{t=0} &= (0, 0).
\end{cases}
\end{aligned}
\]

Following the proof of Proposition 3.2 using Corollary 2.1 and Lemma 2.100 in [1], we have

\[
\|\delta b(t)\|_{B^{\frac{3}{2}}_{2,1}} \leq \int_0^t \left( \|\text{div} \mathbb{P} \partial^+ \delta u\|_{B^{\frac{3}{2}}_{2,1}} + \|\delta u \cdot \nabla b^1\|_{B^{\frac{3}{2}}_{2,1}} + \|\delta b \text{div} u^2\|_{B^{\frac{3}{2}}_{2,1}} + \|b^1 \text{div} \mathbb{P} \partial^+ \delta u\|_{B^{\frac{3}{2}}_{2,1}} \\
+ \frac{1}{2} \|\text{div} u^2\|_{L^2} \|\delta b\|_{B^{\frac{3}{2}}_{2,1}} + \sum_{q \in \mathbb{Z}} 2^{q(\frac{3}{2} - 1)} \|\Delta_q (\delta u, u^2) \cdot \nabla \delta b\|_{L^2} \right) ds
\]

(5.11) \quad C \int_0^t \left( \|\mathbb{P} \partial^+ \delta u\|_{B^{\frac{3}{2}}_{2,1}} \|\mathbb{P} \partial^+ \delta u\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+ \delta u\|_{B^{\frac{3}{2}}_{2,1}} \|\mathbb{P} \partial^+ \delta u\|_{B^{\frac{3}{2}}_{2,1}} + \|\delta b\|_{B^{\frac{3}{2}}_{2,1}} \|u^2\|_{B^{\frac{3}{2}}_{2,1}} \right) ds.

Applying Proposition 3.3 to (5.10)\_2, we find that

\[
\|\mathbb{P} \partial^+ \delta u\|_{L^2_t(B^{\frac{3}{2}}_{2,1})} \leq C \int_0^t \left( \|\nabla \delta b\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+(\delta u \cdot \nabla u^1)\|_{B^{\frac{3}{2}}_{2,1}} + \|K(b^2) \nabla b^2 - K(b^1) \nabla b^1\|_{B^{\frac{3}{2}}_{2,1}} \\
+ \|\mathbb{P}^1(I(b^2) \mathcal{A} \delta u)\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P}^1(I(b^2) - I(b^1)) \mathcal{A} u^1\|_{B^{\frac{3}{2}}_{2,1}} \right) ds.
\]

(5.12) \quad \|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}}

Since

\[
\|\Lambda^{-1} \text{div} (u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}} \leq \|
\]

\[
\|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}} \leq C \left( \|\nabla b\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+(\delta u \cdot \nabla u^1)\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}} \right)
\]

and

\[
\|u^2 \cdot \nabla \mathbb{P} \partial^+ \delta u\|_{B^{\frac{3}{2}}_{2,1}} \leq C \left( \|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+(\delta u \cdot \nabla u^1)\|_{B^{\frac{3}{2}}_{2,1}} + \|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}} \right)
\]

we get

\[
\|\mathbb{P} \partial^+(u^2 \cdot \nabla \delta u)\|_{B^{\frac{3}{2}}_{2,1}}
\]
Similarly, we have
\[
\|\mathcal{P} \delta u \cdot \nabla u^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \leq \|T_{\mathcal{V}} \mathcal{P} \delta u \cdot \nabla u^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} + \|T_{\mathcal{V}} \mathcal{P} \delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} + \|\text{div} \mathcal{R} (\mathcal{P} \delta u, u^1)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \\
\leq C \|u^1\|_{\mathcal{B}_{p,1}^{\frac{\alpha}{p}}} \|\mathcal{P} \delta u\|_{\mathcal{B}_{p,1}^{\frac{\alpha}{p}}},
\]
\[
\|\mathcal{P}^{-\frac{1}{2}} \delta u \cdot \nabla u^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \leq \|T_{\mathcal{F}}^{-\frac{1}{2}} \mathcal{P} \delta u \cdot \nabla u^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} + \|T_{\mathcal{F}} \mathcal{P}^{-\frac{1}{2}} \delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} + \|\mathcal{R} (\mathcal{P}^{-\frac{1}{2}} \delta u, \nabla u^1)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \\
\leq C \|\nabla u^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \|\mathcal{P}^{-\frac{1}{2}} \delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}}.
\]
Thus,
\[
\|\delta u \cdot \nabla u^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \leq C \|u^1\|_{\mathcal{B}_{p,1}^{\frac{\alpha}{p}}} \left( \|\mathcal{P} \delta u\|_{\mathcal{B}_{p,1}^{\frac{\alpha}{p}}} + \|\mathcal{P}^{-\frac{1}{2}} \delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \right).
\]
Using Corollary 2.1 and Theorem 2.61 in [11], we obtain
\[
\|K(b^2) \nabla b^2 - K(b^1) \nabla b^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} = \|\nabla [K(b^2) - K(b^1)]\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \\
\leq C \|\overline{K}(b^2) - \overline{K}(b^1)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} = C \left\| \int_0^1 K(b^1 + \tau(b^2 - b^1)) d\tau db \right\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \\
\leq C \left\| \int_0^1 K(b^1 + \tau(b^2 - b^1)) d\tau \right\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} \|db\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \\
\leq C \|\delta b\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}},
\]
where \(\overline{K}(b) = \int_0^b K(s) ds\). Noting that \(\mathcal{P}^{-\frac{1}{2}} = 0\), in view of Theorem 2.99 in [11] and Corollary 2.1, we are led to
\[
\|\mathcal{P}^{-\frac{1}{2}} [I(b^2) \mathcal{A} d\delta u]\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \\
\leq \left\| \|\mathcal{P}^{-\frac{1}{2}}, I(b^2)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} + \|I(b^2) \mathcal{A} \mathcal{P}^{-\frac{1}{2}} d\delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \right\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \\
\leq C \|\nabla I(b^2)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \|d\delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} + C \|I(b^2)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} \|\mathcal{P} \delta u\|_{\mathcal{B}_{p,1}^{\frac{\alpha}{p}}} + C \|I(b^2)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} \|\mathcal{P}^{-\frac{1}{2}} \delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}}
\leq C \|\mathcal{B}_{2,1}^{\frac{\alpha}{2}} \left( \|\mathcal{P}^{-\frac{1}{2}} \delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} + \|\mathcal{P} \delta u\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} \right)\).
\]
Moreover, using Corollary 2.1 with \(u = I(b^2) - I(b^1), v = \mathcal{A} u^1, \rho = p_1 = q_2 = 2, p_2 = q_1 = p, \) and \(s_1 = \sigma_2 = \frac{\alpha}{2} - 1, s_2 = \sigma_1 = \frac{\alpha}{p} - 1\), we find that
\[
\|I(b^2) - I(b^1)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-2}} \\
\leq C \|I(b^2) - I(b^1)\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \|\mathcal{A} u^1\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \\
\leq C \left\| \int_0^1 I'(b^1 + \tau(b^2 - b^1)) d\tau db \right\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} \|u^1\|_{\mathcal{B}_{p,1}^{\frac{\alpha}{p}}} \\
\leq C \left( 1 + \left\| \int_0^1 I'(b^1 + \tau(b^2 - b^1)) d\tau - 1 \right\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}}} \right) \|\delta b\|_{\mathcal{B}_{2,1}^{\frac{\alpha}{2}-1}} \|u^1\|_{\mathcal{B}_{p,1}^{\frac{\alpha}{p}}} \]
(5.17) \[ \leq C \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{\alpha}{
abla}}_{2,1}} \right) \|	heta(\|b_x\|_{B^{\frac{\alpha}{
abla}}_{2,1}}, \|u\|_{B^{\frac{2\alpha}{
abla}}_{p,1}}) \].

The estimates (5.12)–(5.17) imply that

\[ \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \]
\[ \leq C \int_0^T \left( \|\theta\|_{B^{\frac{\alpha}{
abla}}_{2,1}} \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \right) \left( 1 + \|u\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|\theta(b, b^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} + \|\|\theta(u, u^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|u\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \right) \] ds.

Next, using similar arguments as in the proof of (5.18), Corollary 2.4 and the embedding \( B^{\frac{\alpha}{
abla}}_{2,1} \hookrightarrow B^{\frac{\alpha}{
abla}}_{p,1} \) for \( p > 2 \), we get

\[ \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \]
\[ \leq C \int_0^T \left( \|\theta\|_{B^{\frac{\alpha}{
abla}}_{2,1}} \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \right) \left( 1 + \|u\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \|\|I(b^2)A\theta u\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} + \|\|I(b^2) - I(b^2)A\theta u\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \] ds.

By virtue the interpolation inequality and Hölder’s inequality, we obtain

\[ \|\|\theta(b, b^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \]
\[ \leq C \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ \leq C \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|\theta(b, b^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} + \|\|\theta(u, u^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|u\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \right) \] ds.

By virtue the interpolation inequality and Hölder’s inequality, we obtain

\[ \|\|\theta(b, b^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \]
\[ \leq C \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ \leq C \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|\theta(b, b^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} + \|\|\theta(u, u^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|u\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \right) \] ds.

and

\[ \|\|\theta(b, b^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \]
\[ \leq C \left( 1 + \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|\theta(b, b^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} + \|\|\theta(u, u^2)\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \left( \|\|u\|\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} \right) \left( \|\|D_x^\alpha \theta\|\|_{L^2_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \right) \] ds.

for \( i = 1, 2 \). Combining the above estimates, (1.18), (5.11) and (5.18)–(5.19), choosing \( T = 1 \), we have for all \( t \in [0,1] \)

\[ \|\theta(b, b^2)\|_{L^\infty_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \]
\[ \leq C(1 + X^0) \left( \|\|\theta(b, b^2)\|\|_{L^\infty_t(B^{\frac{2\alpha}{
abla}}_{p,1})} + \|\|\theta(u, u^2)\|\|_{L^\infty_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \right) \]
\[ + \|\|\theta(b, b^2)\|\|_{L^\infty_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \]
\[ \leq C(1 + X^0) \int_0^T \|\theta(b, b^2)\|_{B^{\frac{2\alpha}{
abla}}_{p,1}} ds + C(1 + X^0)X_0 \|\theta(b, b^2)\|_{L^\infty_t(B^{\frac{2\alpha}{
abla}}_{p,1})} \]
Using the estimate (1.18) and Propositions 2.1–2.2, the above three terms can be estimated as follows,

\[ \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{2,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| \}. \]

When \( C X^0 \leq \frac{1}{2} \), the above inequalities reduce to

\[ \| \delta b \| L^2(B_{2,1}^{\frac{\alpha}{2}}) \leq 2 C (1 + X^0) \left( \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{2,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| \right), \]

and

\[ \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{2,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| \right) \]

(5.23)

\[ \leq 2 C (1 + X^0) (t + X_0) \| \delta b \| L^2(B_{2,1}^{\frac{\alpha}{2}}). \]

Substituting (5.22) into (5.23), we are led to

\[ \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{2,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| \right) \]

\[ \leq 4 C^2 (1 + X^0)^2 (t + X_0) \left( \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{2,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| \right) \]

(5.24)

\[ \left( 8 C^2 + 2 \right) (t + X_0) \left( \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{2,1}^{\frac{\alpha}{2}}) \| + \| \| B^\frac{\alpha}{2} \delta u \| L^2(B_{p,1}^{\frac{\alpha}{2}}) \| \right). \]

Accordingly, we conclude that if \( X^0 \leq \frac{1}{4 (8 C^2 + 2)} \), then \( \delta b = \delta u = 0 \) for all \( t \in [0, T] \) with \( T_0 := \frac{1}{4 (8 C^2 + 2)} \).

Repeat this argument on \( [T_0, 2 T_0], [2 T_0, 3 T_0], \ldots \), we can easily prove that \( (b^1, u^1) = (b^2, u^2) \) for all \( t \geq 0 \). The proof of Theorem 1.1 when \( N \geq 3 \) is completed.

5.3. **The uniqueness when** \( N = 2 \). Finally, we prove the uniqueness part of Theorem 1.1 when \( N = 2 \). To this end, we give the following lemma with additional assumption on the initial data.

**Lemma 5.1.** Under the assumptions in Theorem 1.1 and \( N = 2 \), in addition, if \( \mathbb{P} u_0 \in B_{2,1}^0 \), then for all \( T > 0 \), we have

\[ \mathbb{P} u \in \tilde{L}^\infty_T(B_{2,1}^0), \]

with

\[ \| \mathbb{P} u \| L^\infty_T(B_{2,1}^0) \leq C \| \mathbb{P} u_0 \| B_{2,1}^0 + C \left( X^0 \right)^2 \left( 1 + T \frac{\alpha}{2} \right). \]

**Proof.** According to Proposition 3.3, we get

\[ \| \mathbb{P} u \| L^\infty_T(B_{2,1}^0) \leq C \| \mathbb{P} u_0 \| B_{2,1}^0 + C \| (u \cdot \nabla u + I(b) A u) \| L^1_T(B_{2,1}^0). \]

From (4.27), we need to bound \( \| \mathbb{P} (u \cdot \nabla u) \| L^1_T(B_{2,1}^0) \) by the following three terms,

\[ \| \mathbb{P} (u \cdot \nabla u) \| L^1_T(B_{2,1}^0) \leq \| \mathbb{P} (u \cdot \nabla u) \| L^1_T(B_{2,1}^0) + \| \mathbb{P} (u \cdot \nabla \mathbb{P} u) \| L^1_T(B_{2,1}^0) + \| \mathbb{P} (u \cdot \nabla \mathbb{P} u) \| L^1_T(B_{2,1}^0). \]

Using the estimate (1.18) and Propositions 2.1, 2.2, the above three terms can be estimated as follows,

\[ \| \mathbb{P} (u \cdot \nabla u) \| L^1_T(B_{2,1}^0) = \| \text{div} (\mathbb{P} u \otimes \mathbb{P} u) \| L^1_T(B_{2,1}^0) \leq C \| \mathbb{P} u \otimes \mathbb{P} u \| L^1_T(B_{2,1}^0), \]

(5.29)

\[ \leq C \| \mathbb{P} u \| L^1_T(B_{p,1}^{\frac{\alpha}{2}}) \| \mathbb{P} u \| L^1_T(B_{p,1}^{\frac{\alpha}{2}}) \leq C X^2(T) \leq C \left( X^0 \right)^2, \]

\[ \| \mathbb{P} \left( u \cdot \nabla \mathbb{P} u \right) \| L^1_T(B_{2,1}^0), \]
From (5.27), (5.34) and (5.35), we have

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

Using the interpolation inequality, we obtain

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

Consequently, (5.26) holds if

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

Next, using (7.30) and Corollary 2.1, one easily deduces that

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

Using the interpolation inequality, we obtain

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

and

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

The above estimates (5.28)–(5.35) imply that

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

Next, using (7.30) and Corollary 2.1 one easily deduces that

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

From (5.27), (5.34) and (5.35), we have

\[ \parallel u \parallel_{L^2_t(B^{2}_{\gamma,1})} \]

Consequently, (5.26) holds if \( CX^0 \leq \frac{1}{T} \). This completes the proof of Lemma 5.1
Now we turn to prove the uniqueness of solutions for $N = 2$ with the additional assumption that $\mathbb{P}u_0 \in B_{2,1}^0$. In fact, for any fixed $T > 0$, from (5.20)–(5.21), Lemma 3.1 and (1.18), we infer that

$$\|u^2\|_{L^2_t(L^2_{2,1} \cap L^2_{2,1})} \leq \|\mathbb{P}^{-1}u^2\|_{L^2_t(L^2_{2,1} \cap L^2_{2,1})} + \|\mathbb{P}^{-1}u^1\|_{L^2_t(L^2_{2,1} \cap L^2_{2,1})} + \|\mathbb{P}u^1\|_{L^2_t(L^2_{2,1} \cap L^2_{2,1})}
$$

$$\leq C \left(X(T)T\tilde{\tau} + X(T) + \|\mathbb{P}u_0\|_{B_{2,1}^0}^2 + (X^0)^2(1 + T\tilde{\tau})\right)
$$

(5.37)

$$\leq C \left(X^0 + \|\mathbb{P}u_0\|_{B_{2,1}^0}^2 + X^0 \left(1 + X^0\right)(1 + T\tilde{\tau})\right),
$$

for $i = 1, 2$. On this basis, using Propositions 3.2–3.3 and 2.3 the estimate (1.18), we are led to

$$\|\delta b(t)\|_{B_{2,0}^0} \leq e^{C\nu_2(t)} \int_0^t \left(-\Delta u - \delta u \cdot \nabla b - \delta b \nabla u^2 - b^1 \Delta \delta u \|\mathbb{P}u^1\|_{B_{2,0}^0} \right) ds
$$

$$\leq C e^{C\nu_2(t)} \int_0^t \left(\|\delta u\|_{B_{2,1}^0} + \|\delta b\|_{B_{2,1}^0} + \|\delta u\|_{B_{2,1}^0} + \|\delta b\|_{B_{2,0}^0}\right) ds
$$

(5.38)

$$\leq C e^{C\nu_2(t)} \int_0^t \left(1 + X^0\right) \|\delta u\|_{B_{2,1}^0} + \|\delta b\|_{B_{2,0}^0} \right) ds
$$

where $V_2(t) = \int_0^t \|\Delta u^2(\tau)\|_{B_{2,1}^0} d\tau$, and

$$\|\delta u\|_{B_{2,0}^0} \leq \int_0^t \left(\|\Delta u\|_{B_{2,0}^0} + \|\Delta b\|_{B_{2,0}^0} + \|\Delta u \cdot \nabla b\|_{B_{2,0}^0} + \|\Delta b \cdot \nabla u\|_{B_{2,0}^0} + \|\Delta b \cdot \nabla b\|_{B_{2,0}^0} + \|\Delta b \cdot \nabla b\|_{B_{2,0}^0}
$$

$$+ \|\Delta b \cdot \nabla b\|_{B_{2,0}^0} + \|\Delta b \cdot \nabla b\|_{B_{2,0}^0} + \|\Delta b \cdot \nabla b\|_{B_{2,0}^0} + \|\Delta b \cdot \nabla b\|_{B_{2,0}^0}
$$

(5.39)

$$\leq C \left(1 + X^0\right) \int_0^t \left(1 + \|u\|_{B_{2,1}^0} + \|\delta b\|_{B_{2,0}^0}\right) ds,
$$

where we have used the estimates

$$\|\Delta b \cdot \nabla b\|_{B_{2,0}^0} \leq C \left\|\int_0^1 K(b^1 + \tau(b^2 - b^1)) d\tau \delta b\right\|_{B_{2,0}^0}
$$

$$\leq C \left\|\int_0^1 K(b^1 + \tau(b^2 - b^1)) d\tau \delta b\right\|_{B_{2,0}^0}
$$

(5.40)

$$\leq C \left\|\Delta b \cdot \nabla b\|_{B_{2,0}^0},
$$

and

$$\|\Delta b \cdot \nabla b\|_{B_{2,0}^0} = C \left\|\int_0^1 I'(b^1 + \tau(b^2 - b^1)) d\tau \delta b\right\|_{B_{2,0}^0}
$$

$$\leq C \left(1 + \left\|\int_0^1 I'(b^1 + \tau(b^2 - b^1)) d\tau - 1\right\|\right) \|\delta b\|_{B_{2,0}^0}
$$

(5.41)

$$\leq C \left(1 + \|\Delta b \cdot \nabla b\|_{B_{2,0}^0}\right) \|\delta b\|_{B_{2,0}^0}.$$
Choosing $X^0$ and $t \leq \bar{T}$ so small that the first two terms on the right hand side of (5.39) can be absorbed by the left hand side, then (5.39) reduces to

$$
\|\delta u\|_{L^1_t(B^1_{2,∞})} \leq C(X^0, \bar{T}) \int_0^t \left(1 + \|u^t\|_{B^1_{2,∞}}\right) \|\delta b\|_{B^1_{2,∞}} \, ds,
$$

where $C(X^0, \bar{T})$ denotes the various constants depending on $X^0$ and $\bar{T}$. Thanks to (5.37), applying Gronwall’s lemma to (5.38), we find that for all $t \in [0, \bar{T}]$,

$$
\|\delta b\|_{L^1_t(B^1_{2,∞})} \leq C(X^0, \bar{T})\|\delta u\|_{L^1_t(B^1_{2,∞})},
$$

From Proposition 2.8 in [15], we have

$$
\|\delta u\|_{L^1_t(B^1_{2,∞})} \leq C\|\delta u\|_{L^1_t(B^1_{2,∞})} \ln \left(e + \frac{\|\delta u\|_{L^1_t(B^1_{2,∞})} + \|\delta u\|_{L^1_t(B^1_{2,∞})}}{\|\delta u\|_{L^1_t(B^1_{2,∞})}}\right).
$$

Substituting (5.43)–(5.44) into (5.42), we obtain

$$
\|\delta u\|_{L^1_t(B^1_{2,∞})} \leq C \int_0^t \left(1 + \|u^t\|_{B^1_{2,∞}}\right) \|\delta u\|_{L^1_t(B^1_{2,∞})} \ln \left(e + V_3(s)\|\delta u\|_{L^1_t(B^1_{2,∞})}\right) \, ds,
$$

where

$$
V_3(t) := \|\delta u\|_{L^1_t(B^1_{2,∞})} + \|\delta u\|_{L^1_t(B^1_{2,∞})}.
$$

For all $t \in [0, \bar{T}]$, by Hölder’s inequality and interinations, there hold

$$
\|u^t\|_{L^2_t(B^1_{2,∞})} \leq \|\mathbb{P}^{-1} u^t\|_{L^1_t(B^1_{2,∞})} + \|\mathbb{P}^0 u^t\|_{L^1_t(B^1_{2,∞})} + \mathbb{P} u^t\|_{L^1_t(B^1_{2,∞})} \leq \mathbb{C} \|\mathbb{P}^{-1} u^t\|_{L^1_t(B^1_{2,∞})} + \mathbb{C} \|\mathbb{P}^0 u^t\|_{L^1_t(B^1_{2,∞})} + \mathbb{P} u^t\|_{L^1_t(B^1_{2,∞})} \leq C(X^0, \bar{T}),
$$

and

$$
\|u^t\|_{L^2_t(B^1_{2,∞})} \leq \|\mathbb{P}^{-1} u^t\|_{L^1_t(B^1_{2,∞})} + \|\mathbb{P}^0 u^t\|_{L^1_t(B^1_{2,∞})} + \mathbb{P} u^t\|_{L^1_t(B^1_{2,∞})} \leq C \|\mathbb{P}^{-1} u^t\|_{L^1_t(B^1_{2,∞})} + \mathbb{C} \|\mathbb{P}^0 u^t\|_{L^1_t(B^1_{2,∞})} + \mathbb{P} u^t\|_{L^1_t(B^1_{2,∞})} \leq C(X^0, \bar{T}).
$$

These two inequalities imply that

$$
V_3(t) \leq C(X^0, \bar{T}), \quad \text{for all} \quad t \in [0, \bar{T}].
$$

Since

$$
\int_0^1 \frac{ds}{\sqrt{e + V_3(T)s^{-1}}} = +\infty,
$$

Osgood’s lemma implies that $\delta b = \delta u = 0$ on $[0, \bar{T}]$. Standard arguments then yield that $(b^1, u^t) = (b^2, u^t)$ for all $t \geq 0$. The proof of Theorem [1.1] is completed. \hfill \Box

6. Proof of Theorem [1.2]

To simplify the presentation, for $T > 0$, let us denote

$$
Z_L(T) := \left\| (b_L, \mathbb{P}^1 u_L) \right\|_{L^p_t(B^1_{2,∞}) \cap L^q_t(B^{1+q}_{2,∞})}, \quad Z_{L}^0 := \left\| (b_{0,L}, \mathbb{P}^1 u_{0,L}) \right\|_{B_{2,∞}^{1+q}},
$$

$$
H(T) := \left\| \mathbb{P} u \right\|_{L^p_t(B^1_{2,∞}) \cap L^q_t(B^{1+q}_{2,∞})}, \quad H^0 := \left\| \mathbb{P} u \right\|_{B_{2,∞}^{1+q}},
$$

$$
Z(T) := \left\| (b, u) \right\|_{L^q_t(T)} = Z_L(T) + X_H(T) + H(T), \quad Z^0 := Z_{L}^0 + X_H^0 + H^0.
$$
Now we are in a position to prove Theorem 1.2. On the one hand, from (1.19) and the embedding (6.1)
\[ ||\mathcal{P}u_0||_{\mathcal{B}_{p,1}^{\sigma}} \leq C||\mathcal{P}u_0||_{\mathcal{B}_{\sigma}^{\frac{1}{p},1}}, \]
taking \( c_1 \) be any constant not larger than \( \frac{C_1}{2} \), we infer that (1.23) implies (1.17). Consequently, in view of Theorem 1.1 there is a solution \((\rho, u)\) to the Navier-Stokes equations (1.1). Moreover, using (1.19) and (6.1) again, (1.18) reduces to
\[ X(T) \leq C\chi c_1, \quad \text{for all } T > 0. \]
On the other hand, for the same initial data \((\rho_0, u_0)\), owing to Theorem 5.1 there exists a unique local solution \((\rho^*, u^*)\) in \( \mathcal{E}_{\rho}^*(T^*) \), where \( T^* \) is the maximal existence time of \((\rho^*, u^*)\). By using the uniqueness of the solution, we conclude that
\[ (\rho, u) \equiv (\rho^*, u^*), \quad \text{for all } t \in [0, T^*). \]
Next, we go to bound \( Z(T) \) for \( T < T^* \). Since \( X_H(T) \) has been estimated in Lemma 4.11, it suffices to dominate \( H(T) \) and \( Z_L(T) \). To this end, using Proposition 3.3 (7.44) and (7.49) in the Appendix, we easily have
\[ H(T) \leq H^0 + CX(T)Z(T), \quad \text{for all } T < T^*. \]
To bound \( Z_L(T) \), we follow the proof of Lemma 4.10 line by line. Indeed, replacing \( \frac{N}{2} - 1 + \alpha \) by \( \frac{N}{2} - 1 \) in (4.46), and using Lemmas 7.2, 7.4 in the Appendix, it is not difficult to verify that
\[ Z_L(T) \leq Z^0_L + CX(T)Z(T), \quad \text{for all } T < T^*. \]
Now from (4.32), (6.3), (6.4) and the fact that \( X(T) \leq CZ(T) \), we are led to
\[ Z(T) \leq Z^0 + CX(T)Z(T), \quad \text{for all } T < T^*. \]
Combining (6.2) with (6.5), and choosing \( c_1 \) so small that
\[ C^2 C_0 c_1 \leq \frac{1}{2}, \]
we conclude that
\[ Z(T) \leq 2Z^0, \quad \text{for all } T < T^*. \]
This implies that the local solution \((\rho, u)\) can be extended to a global one. The proof of Theorem 1.2 is completed. \( \square \)

7. Appendix

7.1. Proof of Corollary 2.1

From Propositions 2.1, 2.2 using the conditions \( s_1 - \frac{N}{p_1} \leq \min(0, N(\frac{1}{p_2} - \frac{1}{p})), s = s_1 + s_2 \leq N(\frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2}) \) and \( \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \), we have
\[ ||\mathcal{T}_u v||_{\mathcal{B}_{p,1}^{s}} \leq C||u||_{\mathcal{B}_{p,1}^{s-\frac{1}{p},1}} ||v||_{\mathcal{B}_{p,1}^{s-\frac{1}{p},1}}, \]
(7.1) where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), and
\[ ||\mathcal{T}_u v||_{\mathcal{B}_{p,1}^{s}} \leq C||u||_{\mathcal{B}_{p,1}^{s-\frac{1}{p},1}} ||v||_{\mathcal{B}_{p,1}^{s-\frac{1}{p},1}}, \quad (\text{when } p_2 \geq p), \]
(7.2)
Similarly, noting that \( \sigma - \frac{N}{q_1} \leq \min(0, N(\frac{1}{q_2} - \frac{1}{p})), s = \sigma_1 + \sigma_2 \leq N(\frac{1}{p} - \frac{1}{q_1} - \frac{1}{q_2}) \) and \( \frac{1}{p} \leq \frac{1}{q_1} + \frac{1}{q_2} \), we have
\[ ||\mathcal{T}_v u||_{\mathcal{B}_{p,1}^{s}} \leq C||u||_{\mathcal{B}_{p,1}^{s-\frac{1}{p},1}} ||v||_{\mathcal{B}_{p,1}^{s-\frac{1}{p},1}}. \]
Lemma 7.1. Let $r = \frac{1}{q_2} + \frac{1}{q_3}$, and
\[
\|T_\nu u\|_{B^r_{p,1}} \leq C\|v\|_{B^{-q_1,1}}\|v||_{B^{q_2,1}} \quad \text{(when } q_2 \geq p),
\]
(7.3)
where $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}$, and
\[
\|T_\nu u\|_{B^r_{p,1}} \leq C\|v\|_{B^{-q_1,1}}(\frac{1}{q_2} - \frac{1}{p})\|v\|_{B^{q_2,1}} \quad \text{(when } q_2 < p).
\]
(7.4)
Next, from Propositions 2.21 and 2.22, using Meyer’s first linearization, we rewrite (7.7) as
\[
\|\hat{\mathcal{R}}(u, v)\|_{B^r_{p,1}} \leq C\|\hat{\mathcal{R}}(u, v)\|_{B^r_{p,1}} \quad \text{(when } \frac{1}{p_1} + \frac{1}{p_2} \leq 1),
\]
(7.5)
and
\[
\|\hat{\mathcal{R}}(u, v)\|_{B^r_{p,1}} \leq C\|\hat{\mathcal{R}}(u, v)\|_{B^r_{p,1}} \quad \text{(when } \frac{1}{p_1} + \frac{1}{p_2} > 1),
\]
(7.6)
where $1 = \frac{1}{p_1} + \frac{1}{p_2}$. Combining (2.3) with (7.1)–(7.6), we have (2.4). Then, we can easily obtain (2.5) and finish the proof of Corollary 2.1.

7.2. Action of smooth functions. Here we give a variant of Theorem 2.61 in [1], which will be used to deal with the nonlinear term stemming from the pressure $P = \rho(r)$. Let $r \in [1, \infty], s \geq 0$, and $T > 0$. Assume that $u \in L^\infty_T(B^s_{2,1})$ with $u_L \in L^\infty_T(B^s_{2,1})$, $u_H \in L^\infty_T(B^s_{2,1})$, and $K$ is a smooth function on $\mathbb{R}$ which vanishes at $0$. Then there hold
\[
\|K(u)\|_{L^\infty_T(B^s_{2,1})} \leq C(K', \|u\|_{L^\infty_T(B^s_{2,1})}) \left(\|u_L\|_{L^\infty_T(B^s_{2,1})} + \|u_H\|_{L^\infty_T(B^s_{2,1})}\right),
\]
(7.7)
and
\[
\|K(u)\|_{L^\infty_T(B^s_{2,1})} \leq C(K', \|u\|_{L^\infty_T(B^s_{2,1})}) \left(\|u_L\|_{L^\infty_T(B^s_{2,1})} + \|u_H\|_{L^\infty_T(B^s_{2,1})}\right).
\]
(7.8)
Proof. In order to obtain (7.7), we just need to modify the proof of Theorem 2.61 in [1]. For the convenience of readers, we give some details here. First of all, using Meyer’s first linearization method, we rewrite $K(u)$ as
\[
K(u) = \sum_{q' \in \mathbb{Z}} m_{q'} \hat{\Delta}_{q'} u,
\]
(7.9)
where
\[
m_{q'} = \int_0^1 K'(S_{q'} u + \tau \hat{\Delta}_{q'} u) d\tau.
\]
The series in (7.9) converges to $K(u)$ in $L^\infty + L^2$, and $K(u) \in \dot{S}'_{k}$. In view of (7.9), we have
\[
\|K(u)\|_{L^\infty_T(B^s_{2,1})} \leq C \sum_{q < 1} 2^{q}\|\hat{\Delta}_{q} K(u)\|_{L^\infty_T(B^s_{2,1})}
\]
Proof of Lemma 4.1.

\[
\begin{align*}
\leq & \ C \sum_{q<1} \sum_{q' > q} 2^{q' \frac{N}{p} + s} ||\Delta_q(m_q \Delta q u)||_{L^p_q(L^2)} \\
+ & \ C \sum_{q<1} \sum_{q' \leq q} 2^{q' \frac{N}{p} + s} ||\Delta_q(m_q \Delta q u)||_{L^p_q(L^2)} \\
=: & \ I_1 + I_2.
\end{align*}
\]

Using the Hölder’s inequality and the convolution inequality, we have

\[
I_1 \leq \ C \sum_{q<1} \sum_{q' > q} 2^{q' \frac{N}{p} + s} ||m_q||_{L^p_q(L^\infty)} ||\Delta_q u||_{L^p_q(L^2)}
\]

\[
\leq \ C(K', ||u||_{L^p_q(L^\infty)}) \sum_{q<1} \sum_{q' > q} 2^{q' \frac{N}{p} + s} ||\Delta_q u||_{L^p_q(L^2)}
\]

\[
\leq \ C(K', ||u||_{L^p_q(L^\infty)}) \left( \sum_{q<1} \sum_{q' > q} 2^{q' \frac{N}{p} + s} ||\Delta_q u||_{L^p_q(L^2)} + \sum_{q<1} \sum_{q' > q} 2^{q' \frac{N}{p}} ||\Delta_q u_H||_{L^p_q(L^2)} \right)
\]

\[
(7.11)
\]

and

\[
I_2 \leq \ C \sum_{q<1} \sum_{q' \leq q} \left( 2^{q' \frac{N}{p} + \frac{s}{q - q'}} ||\Delta_q(m_q \Delta q u_L)||_{L^p_q(L^2)} + 2^{q' \frac{N}{p}} ||\Delta_q(m_q \Delta q u_H)||_{L^p_q(L^2)} \right)
\]

\[
\leq \ C \sum_{q<1} \sum_{q' \leq q} \left( 2^{q' \frac{N}{p} + \frac{s}{q - q'}} \sum_{|\beta| = \frac{N}{p} + 1} ||\partial^\beta \Delta_q(m_q \Delta q u_L)||_{L^p_q(L^2)} \right)
\]

\[
+ \ C(K', ||u||_{L^p_q(L^\infty)}) \sum_{q<1} \sum_{q' \leq q} \left( 2^{q' \frac{N}{p} + \frac{s}{q - q'}} \sum_{|\beta| = \frac{N}{p} + 1} ||\partial^\beta \Delta_q(m_q \Delta q u_H)||_{L^p_q(L^2)} \right)
\]

\[
(7.12)
\]

The proof of (7.8) can be given in a similar way. This completes the proof of Lemma 7.1 \qed

7.3. Nonlinear estimates. Here, we give the detail proofs of Lemmas 4.1 in Section 4.

Proof of Lemma 4.1

Clearly,

\[
b\text{div} u = b\text{div}^{\mathbb{P}} u = b_L \text{div}^{\mathbb{P}} u_L + b_L \text{div}^{\mathbb{P}} u_L + b_H \text{div}^{\mathbb{P}} u_L + b_H \text{div}^{\mathbb{P}} u_H.
\]

From Corollary 2.1 with \( u = b_L, v = \text{div}^{\mathbb{P}} u_L, \rho = p_2 = q_2 = 2, p_1 = q_1 = p, s_1 = \frac{N}{p} + 2\alpha - 1, s_2 = \frac{N}{p} - \alpha, \sigma_1 = \frac{N}{p} + 2\alpha - 2, \sigma_2 = \frac{N}{p} - \alpha + 1, \) one deduces that

\[
||b_L \text{div}^{\mathbb{P}} u_L||_{L^p(\mathbb{R}^n)} \\
(7.13)
\]

\[
\leq \ C ||b_L||_{L^\frac{p}{2} (\mathbb{R}^{2n-1})} ||\text{div}^{\mathbb{P}} u_L||_{L^\frac{p}{p-2} (\mathbb{R}^{2n-1})} + ||\text{div}^{\mathbb{P}} u_L||_{L^\frac{p}{p-2} (\mathbb{R}^{2n-1})} ||b_L||_{L^\frac{p}{p-2} (\mathbb{R}^{2n-1})}.
\]
Using Corollary [2.1] again with \( u = b_L, v = \text{div}^p u_H, \rho = p_2 = q_2 = p_1 = q_1 = 2, s_1 = \sigma_2 = \frac{N}{2} + \alpha - 1, s_2 = \sigma_1 = \frac{N}{2} \), we have
\[
(7.14) \quad \|b_L \text{div}^p u_H\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} \leq C\|b_L\|_{L^2_T(B_{2,1}^{\frac{N}{2}+\alpha-1})} \|\text{div}^p u_H\|_{L^1_T(B_{2,1}^{\frac{N}{2}})}.
\]
Similarly, taking \( u = b_H, v = \text{div}^p u_L, \rho = p_2 = q_2 = p_1 = q_1 = 2, s_1 = \sigma_2 = \frac{N}{2}, s_2 = \sigma_1 = \frac{N}{2} + \alpha - 1 \) in Corollary [2.1] yields
\[
(7.15) \quad \|b_H \text{div}^p u_L\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} \leq C\|b_H\|_{L^2_T(B_{2,1}^{\frac{N}{2}+\alpha-1})} \|\text{div}^p u_H\|_{L^1_T(B_{2,1}^{\frac{N}{2}})}.
\]
By virtue of the low frequency embedding (4.1) and Corollary [2.1] with \( u = b_H, v = \text{div}^p u_H, \rho = p_2 = q_2 = p_1 = q_1 = 2, s_1 = \sigma_2 = \frac{N}{2}, s_2 = \sigma_1 = \frac{N}{2} - 1 \), we obtain
\[
(7.16) \quad \|P_c(b_H \text{div}^p u_H)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} \leq C\|b_H\|_{L^2_T(B_{2,1}^{\frac{N}{2}+\alpha-1})} \|\text{div}^p u_H\|_{L^1_T(B_{2,1}^{\frac{N}{2}})}.
\]
Then, it follows from the above estimates, and the fact
\[
(7.17) \quad \overline{L^\alpha_T}(B_{2,1}^{\frac{N}{2}+1+\alpha}) \cap L^1_T(B_{2,1}^{\frac{N}{2}+\alpha-1}) \subset \overline{L^\alpha_T}(B_{2,1}^{\frac{N}{2}-\alpha+1}),
\]
that the estimate (4.3) holds.

Next, \( \|P_c(\nabla b_L)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} \) and \( \|P_c(\nabla b_L)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} \) will be bounded as follows. On the one hand, using Proposition [2.2], we are led to
\[
(7.18) \quad \|P_c(\nabla b_L)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} \leq \|P_c(\nabla b_L)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} + C\|\nabla b_L\|_{L^2_T(B_{2,1}^{\frac{N}{2}+1+\alpha})} \|\text{div}^p u\|_{L^1_T(B_{2,1}^{\frac{N}{2}+\alpha-1})} + C\|\nabla b_L\|_{L^2_T(B_{2,1}^{\frac{N}{2}+1+\alpha})} \|\text{div}^p u\|_{L^1_T(B_{2,1}^{\frac{N}{2}+\alpha-1})}.
\]
On the other hand, since (1.15) ensures that \( \frac{N}{p_0} - 1 \leq 0 \), i.e. \( p \leq \frac{2N}{N-2} \), thanks to \( \text{div}^p u = 0 \), we obtain
\[
(7.19) \quad \|P_c(\nabla b_L)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} \leq \|P_c(\nabla b_L)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})} + C\|\nabla b_L\|_{L^2_T(B_{2,1}^{\frac{N}{2}+1+\alpha})} \|\text{div}^p u\|_{L^1_T(B_{2,1}^{\frac{N}{2}+\alpha-1})} + C\|\nabla b_L\|_{L^2_T(B_{2,1}^{\frac{N}{2}+1+\alpha})} \|\text{div}^p u\|_{L^1_T(B_{2,1}^{\frac{N}{2}+\alpha-1})}.
\]
where we have used (4.1)–(4.2). Similarly, owing to (7.17) and the following interpolation,
\[
(7.20) \quad \overline{L^\alpha_T}(B_{2,1}^{\frac{N}{2}-1}) \cap L^1_T(B_{2,1}^{\frac{N}{2}+1}) \subset \overline{L^\alpha_T}(B_{2,1}^{\frac{N}{2}+2\alpha-1}) \subset \overline{L^\alpha_T}(B_{2,1}^{\frac{N}{2}+2\alpha-1}),
\]
we have
\[
(7.21) \quad \|P_c(\nabla b_L)\|_{L^1_t(B_{2,1}^{\frac{N}{2}+1+\alpha})}.
\]
Moreover, with the aid of the following low frequency embedding

Combining the above estimates, we complete the proof of Lemma \[4.1\]

**Proof of Lemma \[4.2\]**

First of all, using the fact

and

we find that

Moreover, with the aid of the following low frequency embedding

we find that

Now using Bony’s decomposition, the high frequency embedding (4.2), Lemma \[2.1\] and Proposition \[2.2\], we are led to

(7.26)
where we have used (7.24)–(7.25), the interpolation

\begin{equation}
(7.27) \quad \bar{L}^\infty_T(B_{2,1}^{\frac{N}{p_0}-1+a}) \cap L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a}) \subset \bar{L}^\infty_T(B_{2,1}^{\frac{N}{p_0}+2a}) \cap L^1_T(B_{2,1}^{\frac{N}{p_0}+2a}),
\end{equation}

and the following low frequency embedding

\begin{equation}
(7.28) \quad \|\|P^\perp u_L\|\|_{\bar{L}^\infty_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \leq C\|\|P^\perp u_L\|\|_{\bar{L}^\infty_T(B_{2,1}^{\frac{N}{p_0}+2a})}.
\end{equation}

Next, noting that \(\frac{N}{p_0} \leq 1\), using Proposition 2.2, we find that

\(\|P_{\geq 1}(\dot{T}_v b) u\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})}\)

\begin{align*}
\leq & \quad \|P_{\geq 1}(\dot{T}_v b \cdot \dot{P} u_L)\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})} + \|P_{\geq 1}(\dot{T}_v b (\dot{P} \cdot u_H + P u))\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})} \\
\leq & \quad C\|P_{\geq 1}(\dot{T}_v b \cdot \dot{P} u_L)\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})} + C\|\nabla b\|_{L^\infty_T(B_{p_0,1}^{\frac{N}{p_0}})} \|\|\dot{P} u_H + P u\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})} \\
\leq & \quad C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}})} \|\|\dot{P} u_L\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})} + C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}})} \|\|\dot{P} u_H\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})} + \|\|P u\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}})} \\
(7.29) & \quad \leq CX^2(T).
\end{align*}

This completes the proof of Lemma 4.2. \(\Box\)

**Proof of Lemma 4.3**

From the low frequency embedding (4.1), Corollary 2.1 and Theorem 2.61 in [1], we infer that

\begin{align*}
\|P_{\leq 1}(I(b)A\dot{P} u)\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} + \|P_{\geq 1}(I(b)A\dot{P} u)\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|I(b)A\dot{P} u\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} + \|I(b)A\dot{P} u_H\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|I(b)\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \|\|A\dot{P} u\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} + C\|I(b)\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \|\|A\dot{P} u_H\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \|\|\dot{P} u\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} + C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \|\|\dot{P} u_H\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
(7.30) & \quad \leq CX^2(T),
\end{align*}

where we have used (7.24)–(7.25), and (7.27)–(7.28). Next, using \(\text{div} P u = 0\), we decompose \(\Lambda^{-1} \text{div}(I(b)A\dot{P} u)\) as follows:

\begin{align*}
\Lambda^{-1} \text{div}(I(b)A\dot{P} u) = \Lambda^{-1}\left(\dot{T}_{\nabla I(b)} A\dot{P} u\right) + \Lambda^{-1} \text{div}\left(\dot{T}_{\nabla A\dot{P} u} I(b)\right) + \Lambda^{-1} \text{div}\left(\dot{R}(I(b), A\dot{P} u)\right).
\end{align*}

Then according to Lemma 2.1, Proposition 2.2, (7.25), and Theorem 2.61 in [1] again, we have

\begin{align*}
\|\Lambda^{-1}\left(\dot{T}_{\nabla I(b)} A\dot{P} u\right)\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|\dot{T}_{\nabla I(b)} A\dot{P} u\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|\nabla I(b)\|_{L^\infty_T(B_{p_0,1}^{\frac{N}{p_0}+1+a})} \|\|A\dot{P} u\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \|\|\dot{P} u\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
(7.31) & \quad \leq CX^2(T),
\end{align*}

and

\begin{align*}
\|\Lambda^{-1} \text{div}\left(\dot{T}_{\nabla A\dot{P} u} I(b)\right)\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|\nabla A\dot{P} u\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \|\|I(b)\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad C\|b\|_{L^\infty_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \|\|A\dot{P} u\|\|_{L^1_T(B_{2,1}^{\frac{N}{p_0}+1+a})} \\
\leq & \quad CX^2(T).
\end{align*}
where

\[ \Theta = \frac{(2 - 4\alpha)p}{(4 + N - 6\alpha)p - 2N}. \]

(7.32)

Finally, we go to bound the remainder term \( \Lambda^{-1} \text{div} \left( \hat{R}(I(b), \mathcal{A} \hat{u}) \right) \). Noting that \( \frac{N}{p} - 1 = 0 \) for \( N = 2 \), we can not use Proposition 2.2 directly if \( N = 2 \). Fortunately, Proposition 2.1 enables us to bound \( \| \Lambda^{-1} \text{div} \left( \hat{R}(I(b), \mathcal{A} \hat{u}) \right) \|_{L^1_\nu(B_{2R}^p)} \) by

\[ \| \Lambda^{-1} \text{div} \left( \hat{R}(I(b), \mathcal{A} \hat{u}) \right) \|_{L^1_\nu(B_{2R}^p)} \]

first. Then Proposition 2.2 is applicable, and we have

\[ \| \Lambda^{-1} \text{div} \left( \hat{R}(I(b), \mathcal{A} \hat{u}) \right) \|_{L^1_\nu(B_{2R}^p)} \]

\[ \leq C \| \hat{R}(I(b), \mathcal{A} \hat{u}) \|_{L^1_\nu(B_{2R}^p)} \]

(7.33)

It follows from (7.31)–(7.33) and the low frequency embedding (4.1) that (4.10) holds. This completes the proof of Lemma 4.3. \( \square \)

**Proof of Lemma 4.4**

Using the low frequency embedding (4.1), high frequency embedding (4.2), and the decomposition \( b = b_L + b_H \), we have

\[ \| P_{<1}(K(b) \nabla b) \|_{L^1_\nu(B_{2R}^p)} + \| P_{\geq 1}(K(b) \nabla b) \|_{L^1_\nu(B_{2R}^p)} \]

\[ \leq C \| K(b) \nabla b_L \|_{L^1_\nu(B_{2R}^p)} + C \| K(b) \nabla b_H \|_{L^1_\nu(B_{2R}^p)} \]

By virtue of Corollary 2.1 (7.25), and Theorem 2.16 in [1], we obtain

\[ \| K(b) \nabla b_H \|_{L^1_\nu(B_{2R}^p)} \]

(7.34)

The remaining term will be divided into three parts.

\[ \| K(b) \nabla b_L \|_{L^1_\nu(B_{2R}^p)} \]

\[ \leq C \| T'_{K(b) \nabla b_L} \|_{L^1_\nu(B_{2R}^p)} + C \| T'_{\nabla b_L} K(b) \|_{L^1_\nu(B_{2R}^p)} + C \| T'_{b_L} K(b) \|_{L^1_\nu(B_{2R}^p)} \]

(7.35)

\[ =: J_1 + J_2 + J_3. \]

To bound \( J_1 \), using the interpolation inequality, we infer that

\[ \| b_L \|_{L^1_\nu(B_{2R}^p)} \leq C \| b_L \|_{L^\theta_\nu(B_{2R}^p)} \| b_L \|_{L^1(B_{2R}^p)} \]

where

\[ \theta = \frac{(2 - 4\alpha)p}{(4 + N - 6\alpha)p - 2N}. \]

(7.36)

\[ \frac{1}{r_1} = \theta \alpha + (1 - \theta)(1 - \alpha), \]

\[ \frac{1}{p_1} = \frac{\theta}{\theta} + \frac{1 - \theta}{2}. \]
The proof of Lemma 4.4 is completed. □

Finally, in view of Lemma 7.1 in the Appendix, we have

\[ \text{Theorem 2.61 in [1]} \]

and we obtain

\[ \|b\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \leq C \left( \|b_L\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} + \|b_H\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \right) \]

(7.37)

where

\[ \frac{1}{r_2} = (1 - \theta)\alpha + \theta(1 - \alpha), \quad \frac{1}{p_2} = \frac{1 - \theta}{p} + \frac{\theta}{2}. \]

Here, \( r_1, r_2, p_1 \) and \( p_2 \) satisfy

(7.38)

\[ \frac{1}{r_1} + \frac{1}{r_2} = 1, \quad \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p}, \quad 2 \leq p_1, p_2 \leq p. \]

Thus, using Proposition 2.2, Theorem 2.61 in [1] and (7.36)–(7.38), we arrive at

\[ J_1 \leq C \|K(b)\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \|\nabla b_L\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \]

(7.39)

\[ \leq C X^2(T). \]

As for \( J_2 \), according to (7.26)–(7.27), the following high frequency embedding

\[ \|K(b)_H\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \leq C \|K(b)_H\|_{L^p_{\infty}(B^s_{p,q})}, \]

and Theorem 2.61 in [1] again, we obtain

\[ J_2 \leq C \|\nabla b_L\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \|K(b)_H\|_{L^p_{\infty}(B^s_{p,q})} \]

(7.40)

\[ \leq C X^2(T). \]

Finally, in view of Lemma 7.1 in the Appendix, we have

(7.41)

\[ \|K(b)_L\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \leq C \left( \|b_L\|_{L^p_{\infty}(B^s_{p,q})} + \|b_H\|_{L^p_{\infty}(B^s_{p,q})} \right). \]

Therefore, using Proposition 2.2 again, one deduces that

\[ J_3 \leq C \|\nabla b_L\|_{L^2_T(L^p_{\infty}(B^s_{p,q}))} \|K(b)_L\|_{L^p_{\infty}(B^s_{p,q})} \]

(7.42)

\[ \leq C X^2(T). \]

The proof of Lemma 4.4 is completed. □
7.4. **Some nonlinear estimates in** $L^1_t(B^\frac{N-1}{2}_2)$. In next three lemmas, we shall bound $\|(f,g)\|_{L^1_t(B^\frac{N-1}{2}_2)}$ in terms of $X(T)Z(T)$, where $(f, g)$ are the nonlinear terms on the right hand side of (4.34).

**Lemma 7.2.** Let $p$ and $\alpha$ satisfy (1.15) and (1.16), respectively. Assume $(b, u) \in E^N(T)$, then we have

\[
\|b\|_{L^1_t(B^\frac{N-1}{2}_2)} + \|\hat{T}_b^{\hat{u}}u\|_{L^1_t(B^\frac{N-1}{2}_2)} \leq CX(T)Z(T),
\]

and

\[
\|u\cdot \nabla u\|_{L^1_t(B^\frac{N-1}{2}_2)} + \|\hat{T}_u\cdot \nabla d\|_{L^1_t(B^\frac{N-1}{2}_2)} \leq CX(T)Z(T).
\]

**Proof.** To begin with, we give the estimate of $\|b\|_{L^1_t(B^\frac{N-1}{2}_2)}$. In fact, using the high frequency embedding

\[
\|b_H\|_{L^1_t(B^{\frac{N-2}{2}}_2)} \leq C\|b_H\|_{L^1_t(B^{\frac{N-1}{2}}_2)} \leq C\|b_H\|_{L^1_t(B^{\frac{N}{2}}_2)},
\]

and the decomposition $b = b_L + b_H$, one deduces that

\[
\|b\|_{L^1_t(B^{\frac{N-2}{2}}_2)} \leq C \left(\|b_L\|_{L^1_t(B^{\frac{N-1}{2}}_2)} + \|b_H\|_{L^1_t(B^{\frac{N}{2}}_2)}\right) \leq CX(T).
\]

Now if $N \geq 3$, using Proposition 2.2, (7.20), 4.15, and (7.45) we have

\[
\|b\|_{L^1_t(B^\frac{N-1}{2}_2)} \\
\leq C \left(\|\hat{T}_b\|_{L^1_t(B^\frac{N-1}{2}_2)} + \|\hat{T}_{\nabla b}\|_{L^1_t(B^\frac{N-1}{2}_2)} + \|\hat{T}_{\nabla b}^\alpha u\|_{L^1_t(B^\frac{N-1}{2}_2)}\right) \\
\leq C \left(\|b\|_{L^1_t(B^{\frac{N-1}{2}}_2)} \|\nabla u\|_{L^\infty_t(B^{\frac{N}{2}}_2)} + \|\|\nabla u\|_{L^\infty_t(B^{\frac{N}{2}}_2)} \|\nabla b\|_{L^\infty_t(B^{\frac{N}{2}}_2)}\right) \\
\leq C \left(\|b\|_{L^1_t(B^{\frac{N-1}{2}}_2)} + \|u\|_{L^\infty_t(B^{\frac{N-2}{2}}_2)} \|\nabla b\|_{L^\infty_t(B^{\frac{N}{2}}_2)}\right) \leq CX(T)Z(T),
\]

and

\[
\|\hat{T}_b^{\hat{u}}u\|_{L^1_t(B^\frac{N-1}{2}_2)} \leq C\|\nabla b\|_{L^1_t(B^{\frac{N-1}{2}}_2)} \|u\|_{L^\infty_t(B^{\frac{N-2}{2}}_2)} \leq CX(T)Z(T).
\]

If $N = 2$, we just need to reestimate $\|\hat{R}(\nabla u, b_L)\|_{L^1_t(B^\frac{N-1}{2}_2)}$ and $\|\hat{R}(\nabla b, u)\|_{L^1_t(B^\frac{N-1}{2}_2)}$. Indeed, they can be treated in the same way as follows:

\[
\|\hat{R}(\nabla u, b_L)\|_{L^1_t(B^\frac{N-1}{2}_2)} + \|\hat{R}(\nabla b, u)\|_{L^1_t(B^\frac{N-1}{2}_2)} \\
\leq C\|\hat{R}(\nabla u, b_L)\|_{L^1_t(B^{\frac{N-1}{2}}_2)} + \|\hat{R}(\nabla b, u)\|_{L^1_t(B^{\frac{N-1}{2}}_2)} \\
\leq C\|\nabla u\|_{L^1_t(B^{\frac{N-1}{2}}_2)} \|\nabla b\|_{L^1_t(B^{\frac{N-1}{2}}_2)} + C\|\nabla b\|_{L^1_t(B^{\frac{N-1}{2}}_2)} \|u\|_{L^1_t(B^{\frac{N}{2}}_2)}
\]
Lemma 7.3. The completion of the proof of Lemma 7.3.

\[ \leq CX(T)Z(T). \]

(7.48)

It follows from (7.46)–(7.48) that (7.43) holds. (7.44) can be obtained similarly since \( b \) and \( u \) lie in the same space \( \frac{1}{L_1^T (B_{p,1}^2)} \). The proof of Lemma 7.2 is completed. \( \square \)

Lemma 7.3. Under the conditions of Lemma 7.2 we have

\[ (7.49) \quad \|I(b)Au\|_{L_1^1 (B_{p,1}^2)} \leq CX(T)Z(T). \]

Proof. Using Corollary 2.1, Lemma 2.1 and (7.25), we are led to

\[ \|I(b)Au\|_{L_1^1 (B_{p,1}^2)} \leq C\|I(b)\|_{L_1^1 (B_{p,1}^2)} \|Au\|_{L_1^1 (B_{p,1}^2)} \leq C\|b\|_{L_1^1 (B_{p,1}^2)} \|u\|_{L_1^1 (B_{p,1}^2)} \leq CX(T)Z(T). \]

This completes the proof of Lemma 7.3. \( \square \)

Finally, we estimate \( \|K(b)\nabla b\|_{L_1^1 (B_{p,1}^2)} \) in a similar manner as Lemma 4.3.

Lemma 7.4. Under the conditions of Lemma 7.2 we have

\[ (7.50) \quad \|K(b)\nabla b\|_{L_1^1 (B_{p,1}^2)} \leq CX(T)Z(T). \]

Proof. First of all, we can use (7.34) to bound \( \|K(b)\nabla b\|_{L_1^1 (B_{p,1}^2)} \). In order to bound \( \|\tilde{I}_K(b)\nabla b\|_{L_1^1 (B_{p,1}^2)} \), using the interpolation inequality, we get

\[ \|b_L\|_{L_1^1 (B_{p,1}^2)} \leq C\|b_L\|_{L_1^1 (B_{p,1}^2)} \|b_L\|_{L_1^1 (B_{p,1}^2)} \|b_H\|_{L_2^2 (B_{p,1}^2)} \]

(7.51)

\[ \leq C \left( \|b_L\|_{L_1^1 (B_{p,1}^2)} \right)^{\theta} \left( \|b_H\|_{L_2^2 (B_{p,1}^2)} \right)^{1-\theta} \]

(7.52)

\[ \leq CX^{1-\tilde{\theta}}(T)Z(\tilde{\theta})(T), \]

(7.53)

where

\[ \tilde{\theta} = \frac{(2 - 4\alpha)p}{(4 + N - 8\alpha)p + 2N}, \quad \frac{1}{\tilde{p}_1} = \tilde{\theta}\alpha + (1 - \tilde{\theta})(1 - \alpha), \quad \frac{1}{\tilde{p}_2} = \frac{\tilde{\theta}}{p} + \frac{1}{2}. \]

(7.54)

and

\[ \|b\|_{L_1^1 (B_{p,1}^2)} \leq \left( \left. \|b_L\|_{L_1^1 (B_{p,1}^2)} \right)^{\theta} \right. \left( \left. \|b_H\|_{L_2^2 (B_{p,1}^2)} \right)^{1-\theta} \]

(7.55)

\[ \leq C \left( \|b_L\|_{L_1^1 (B_{p,1}^2)} \right)^{\theta} \left( \|b_H\|_{L_2^2 (B_{p,1}^2)} \right)^{1-\theta} \]

(7.56)

\[ \leq CX^{1-\tilde{\theta}}(T)Z(\tilde{\theta})(T), \]

(7.57)

where

\[ \frac{1}{\tilde{p}_1} = (1 - \tilde{\theta})\alpha + \tilde{\theta}(1 - \alpha), \quad \frac{1}{\tilde{p}_2} = 1 - \tilde{\theta} + \tilde{\theta}. \]

(7.58)

Here

\[ \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2} = 1, \quad \frac{1}{\tilde{p}_2} = \frac{1}{\tilde{p}_1} + \frac{1}{p}, \quad 2 \leq \tilde{p}_1, \tilde{p}_2 \leq p. \]

(7.59)

Then similar to (7.39), we find that

\[ \|\tilde{I}_K(b)\nabla b\|_{L_1^1 (B_{p,1}^2)} \leq C\|K(b)\|_{L_1^1 (B_{p,1}^2)} \|\nabla b_L\|_{L_1^1 (B_{p,1}^2)} \]

(7.60)
Next, similar to (7.42), if \( N \geq 3 \), we have
\[
\|T_{\nabla b_t} K(b)_L\|_{L^2_T(B^{\frac{N}{2} - 1}_{2,1})} \leq C \|
abla b_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{p,1})} \|K(b)_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{2,1})}
\leq C \|b_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{p,1})} \left( \|b_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{2,1})} + \|b_H\|_{L^2_T(B^{\frac{N}{2}}_{2,1})} \right)
\leq CX(T)Z(T).
\]

If \( N = 2 \), the remainder \( R(\nabla b_L, K(b)_L) \) should be estimated as follows:
\[
\|R(\nabla b_L, K(b)_L)\|_{L^2_T(B^{\frac{N}{2} - 1}_{2,1})} \leq C \|R(\nabla b_L, K(b)_L)\|_{L^2_T(B^{\frac{N}{2} - 1}_{2,1})}
\leq C \|\nabla b_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{p,1})} \|K(b)_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{2,1})}
\leq C \|b_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{p,1})} \left( \|b_L\|_{L^2_T(B^{\frac{N}{2} + 1 - 2\sigma}_{2,1})} + \|b_H\|_{L^2_T(B^{\frac{N}{2}}_{2,1})} \right)
\leq CX(T)Z(T).
\]

Finally, thanks to Lemma 7.1 in the Appendix, we have
\[
\|K(b)_H\|_{L^1_T(B^{\frac{N}{2}}_{2,1})} \leq C \left( \|b_L\|_{L^1_T(B^{\frac{N}{2}}_{2,1})} + \|b_H\|_{L^1_T(B^{\frac{N}{2}}_{2,1})} \right).
\]

Then using Proposition 2.2 and (7.25), we arrive at
\[
\|T_{\nabla b_t} K(b)_H\|_{L^1_T(B^{\frac{N}{2} - 1}_{2,1})} \leq C \|\nabla b_L\|_{L^1_T(B^{\frac{N}{2} - 1}_{2,1})} \|K(b)_H\|_{L^1_T(B^{\frac{N}{2}}_{2,1})}
\leq C \|b_L\|_{L^1_T(B^{\frac{N}{2} - 1}_{2,1})} \left( \|b_L\|_{L^1_T(B^{\frac{N}{2} - 1}_{2,1})} + \|b_H\|_{L^1_T(B^{\frac{N}{2}}_{2,1})} \right)
\leq CX(T)Z(T).
\]

Combining (7.34) with the above estimates yields (7.50). We complete the proof of Lemma 7.4. \( \square \)

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