The surface density of holographic entropy

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On the basis of postulates for the holographic description of gravity and the introduction of entropic force, for static sources we derive the universal law: the entropy of a holographic screen is equal to quarter of its area in the Planck system of units.

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I. INTRODUCTION

Following the thermodynamic interpretation of black hole horizon in general relativity [1–11] and the holographic principle [12–14] Verlinde [15] and Padmanabhan [16] (both independently) have shown that Newton’s gravity appears by introducing several postulates about holographic screens$^1$. These holographic screens generate both the space and the entropic force, which is equivalent to the gravitational acceleration. The entropic force is treated as the gradient of entropy, that emerges, when the probe particle is approaching the screen. In accordance with the holographic principle, a theory in the three-dimensional space can be described in terms of theory on the two-dimensional surface covering the three-dimensional space. Thus, one dimension of the space is “holographically emerged” and the information about particles inside the surface is encoded on the surface. Three-dimensional space is considered as a union of the holographic screens, which are characterized by temperature and entropy. The microscopic modes hidden on the screen are chaotically distributed in the way that one can tell about thermodynamic system “bits on the surface”. In the Newton’s mechanics with the simplest case of point source, the screen is an equipotential surface, and in the general case the screen corresponds to a surface of constant acceleration for free falling bodies. Such a surface is ascribed to the constant Unruh temperature $^{19}$.

The following postulates are introduced in $^{15}$:

1. The displacement $\delta x$ of the particle with mass $m$ in the emergent holographic direction perpendicularly to the screen in its vicinity changes the screen entropy $S$ according to the law

$$\delta S = 2\pi k_B \frac{mc}{\hbar} \delta x. \quad (1)$$

2. The number of the degrees of freedom (bits) are proportional to the screen area $A$ and equal to the number of area cells in the Planck system of units

$$N = \frac{A c^3}{G \hbar}. \quad (2)$$

3. The gravitational energy is divided evenly over these degrees of freedom, i.e. the equipartition distribution of bits takes place.

As shown the gravitational force arises due to the entropy gradient, i.e. the gravity is the entropic force$^3$,

$$F \delta x = T \delta S. \quad (3)$$

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$^1$ Developments of these arguments can be found in $^{17, 18}$.

$^2$ Here and throughout the paper, we assume that $k_B$ is the Boltzmann constant, $G$ is the Newton gravitational constant, $\hbar$ is the Planck constant, $c$ is the speed of light.

$^3$ The ideas offered by Verlinde in $^{15}$ have generated wide discussions and investigations, wherefrom we can mention applications to cosmology $^{20–32}$, entropic corrections to the Newton’s law of gravity $^{33–38}$, interpretations in terms of Rindler horizon $^{39–42}$, study of specific gravitational systems $^{43–45}$, relations with Loop Quantum Gravity $^{46–50}$ and a specific microscopic justification of bits on the surface $^{51}$. 
Similar ideas were independently developed by Padmanabhan \[16\]. So, in \[50\] he has introduced the definition of entropy \[S = \frac{1}{8\pi G} \int \sqrt{-\mathbf{g}} d^{4}x \nabla_i a^i\] for a static spacetime in terms of proper acceleration \(a^i\) for a comoving observer (here \(\nabla_i\) is a covariant derivative). Such definition is based on the statement that the entropy of black hole is proportional to the area of horizon with an appropriate coefficient especially fixed. In \[16\] Padmanabhan has introduced the equipartition law for the horizon degrees of freedom\(^4\).

On the other hand, variants for the internal structure of black holes as suggested by the string theory results in the unique relation between the horizon area of a black hole and its entropy \(S = k_B \ln \Omega\), calculated over the number of equipartitional microscopic states \(\Omega\) with identical quantum numbers of the black hole for exterior observer (some charges and the mass) \[51\], namely,

\[S = \frac{1}{4} k_B c^3 G \hbar A,\]  

(4)

where \(A\) is the horizon area. Note that the dimensionless coefficient \(\frac{1}{4}\) is universal but not trivial result of a calculation according to Cardy formula \[52, 53\] in the conformal field theory. This formula is asymptotically counting for the amount of states with given invariants in Virasoro algebra (a central charge and a quantum number of generator for null index modes). It is surprising, since \textit{a priori} one could not expect any preferable value for this factor! Similarly, in papers, wherein solutions describing black holes are matched with conformal symmetry algebras, the same Cardy formula gives this factor for the entropy proportional to the area (see review in \[54–57\]). Note also that the thermodynamic interpretation of black hole “equation of state”, i.e. a relation between the black hole mass and its horizon area, angular momentum and charges gives the same coefficient if the horizon is assigned to the Hawking temperature. In this respect, the natural question does arise: why do microscopically different theories give the same simple dependence of entropy on the horizon area? \[58\].

It is interesting that in the Verlinde’s article \[15\] the event horizon can be considered as the holographic screen. Therefore, the relation between the horizon area and its entropy gives rise to the relation between the screen area and its entropy. Thus, we can more universally consider the question about the entropy density on the holographic screen for static systems using the postulates for the holographic screens and entropic force.

Here we establish the area law for the holographic entropy

\[\frac{dS}{dA} = \frac{1}{4} k_B c^3 G \hbar,\]  

(5)

starting from universal principles describing the gravitational force as the entropic force as formulated by Verlinde \[15\].

II. THE ENTROPY DENSITY

Let us consider nonrelativistic particle with kinetic and gravitational potential energies denoted by \(K\) and \(U\), respectively. The total energy \(\tilde{E}\) changes due to velocity and potential variations (due to both a particle displacement and a potential change because of a source variation)

\[d\tilde{E} = dK + dU.\]

The change \(dU\) is associated with the gravitational forces, which arise because of holographic screens possessed entropy. Then, it is caused by a transfer of energy from the holographic screen to the particle. Therefore, the holographic screen energy \(E\) changes due to the screen entropy variation and exchange by the energy, i.e. the work of gravitational force, so that

\[dE = TdS - dU.\]  

(6)

\(^4\) Note that logical arguments and postulates used by Verlinde and Padmanabhan are clearly different (Verlinde’s statements are more simple and, hence, more speculative, while Padmanabhan’s formulation is more technical and specific), but they lead to identical relations between relevant physical quantities.
According to the Hamilton equations for the motion of a particle
\[ \delta U = \frac{\partial U}{\partial x} \delta x = -F \delta x. \]

The introduction of the entropic force acting on the particle in accordance with the postulate for the change of holographic screen entropy under the particle displacement\(^5\) means that

\[ F \delta x = T \delta S, \quad \Rightarrow \quad \delta U = -T \delta S, \]  

(7)

where \( \delta S \) is the variation of holographic screen entropy\(^6\).

In the thermodynamic equilibrium there is the surface density of entropy \( \frac{dS}{dA} \), which can depend on the screen temperature, hence, the entropy change is associated with the area variation

\[ \delta S = \frac{dS}{dA} \delta A, \]  

(8)

so that

\[ \delta U = -T \frac{dS}{dA} \delta A. \]  

(9)

Thus,

\[ dE = T dS + T \frac{dS}{dA} dA. \]  

(10)

In this formula the differential of entropy corresponds to an arbitrary variation, while the differential of area is strictly given by a small element of the surface area on the specific holographic screen. In addition, we can ascribe the second term in (10) to the work of surface tension. In this respect, one can talk on the surface tension origin of entropic force in the case of gravity.

Then, according to (10) we find the surface density of entropy related with the surface density of energy by

\[ \frac{dE}{dA} = 2T \frac{dS}{dA}. \]  

(11)

We emphasize that physical meaning of (11) is clear, namely, the energy density is determined by two terms: the contribution of the entropy density and the work performed, when the surface of holographic screen is changed\(^7\), whereas these contributions are equal to each other. In other words,

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\(^5\) The position of the particle is an external parameter for the holographic screen as the thermodynamical system. The entropy depends on this parameter according to the postulate, so that in the thermodynamical equilibrium the entropy extreme can be ensured by introduction of the work of the entropic force. Indeed, the entropy as a function of energy \( E \) and the external parameter being the position of particle \( x \) can be written in the form \( S = S(E + U, x) \), and the entropy is extremal, when \( \frac{\partial S}{\partial x} = 0 \), i.e. there is the thermodynamic equilibrium, and \( U = 0 \). However, the postulate about the dependence of entropy on the particle position \( \frac{\partial S}{\partial x} \neq 0 \) agrees with the thermodynamic equilibrium if only the entropy is extremal, hence,

\[ \frac{dS}{dx} = \frac{\partial S}{\partial x} \frac{\partial x}{\partial x} = \frac{\partial S}{\partial E} \frac{\partial U}{\partial E} = 0. \]

Therefore, from the formulas \( \frac{\partial U}{\partial x} = -F \) and \( \frac{\partial S}{\partial x} = \frac{1}{T} \) we find the relation for the entropic force \( F \delta x = T \delta S \). If we pass from the system of the holographic screen and the particle to the open thermodynamic system of the holographic screen, which can get an external transferred energy due to the work related to the variation of the screen area, for example, then for the total entropy differential \( S = S(E + U, x) \) we obtain

\[ dS = \frac{\partial S}{\partial E} (dE + dU) \quad \Rightarrow \quad dE = T dS - dU. \]

That is equivalent to the derivation of (11) given above.

\(^6\) Emphasize the following feature of gravity: When a particle is freely falling down to the horizon, the gravitational forces make a positive work, while the gravitational mass of system increases from the viewpoint of an observer outside the horizon, i.e. \( \delta E > 0 \) at \( F \delta x = -\delta U > 0 \).

\(^7\) This conclusion is analogous to the determination of energy density \( \epsilon \) for a gas, when the relation \( dE = T dS - p dV \) straightforwardly yields \( \epsilon = \frac{dE}{dV} = T \frac{dS}{dV} - p \), where \( p \) is the pressure.
considering the part of the holographic screen with a larger area corresponds to an increase of the energy of thermodynamic subsystem, and it is accompanied by both the increase of the subsystem entropy and making the positive work in order to increase the subsystem area.

Thus, the equipartition distribution of modes on the holographic screen, i.e. \( E = \frac{1}{2} k_B T N \), in the differential form gives

\[
\frac{dE}{dA} = \frac{1}{2} k_B T \frac{c^3}{G \hbar},
\]

and it leads to the constant value of the surface density of entropy

\[
\frac{dS}{dA} = \frac{1}{4} k_B c^3 \frac{T}{G \hbar}.
\]  

(12)

Note that in the derivation above we have not used the explicit definition of temperature for the holographic screen and the postulate on the entropy change under the particle displacement near the screen. These quantities become significant only for the calculation of the explicit dependence of gravitational forces on the distance.

The calculated constant surface density of entropy for the holographic screen allows us to determine the equality of the black hole entropy to a quarter of its horizon area in the Planck system of units. Note that there is the difference between a holographic screen and a black hole horizon. The holographic screen of gravitating system with a given mass can be arbitrarily posed and its area is variable independent of the system mass, while the black hole horizon is determined by its mass and other charges, and the horizon area can be evaluated according to “the state equation”, i.e. the relation of the black hole mass to its charges and the horizon area. “The state equation” determines another approach to the variation of area. At the equipartition distribution of bits on the holographic screen we can select a part of the screen and further relate it with an appropriate part of the gravitating energy. In contrast, the variation of the black hole horizon area has another meaning, since it determines the variation of the gravitating mass. However, such difference does not affect the validity of law for the surface density of entropy (12). In other words, in the case of black hole horizon, the area variation at the fixed temperature in formula (10) should be set to zero \( dA = 0 \), and one should substitute the actual relation \( S = \frac{1}{4} k_B c^3 A_H \), where \( A_H \) is the horizon area.

III. CONCLUSION

Thus, in the framework of the equipartition postulate and the entropic interpretation of gravitational force we have deduced the universal law for the surface density of entropy for the holographic screen without setting any microscopic structure of the screen. It means that such a consideration is completely thermodynamic. The holographic screen can particularly be associated with the black hole horizon, then it becomes the event horizon, and one can observe its thermodynamic properties caused by the microscopic structure of the black hole in the form of Hawking radiation, for example. However, the holographic nature of the horizon leads to the simple relation of the entropy to the horizon area independently of the microstructure of the theory.

It is worth noting that formula (12) holds in the static system and for the static holographic screens. So, a transformation to another inertial moving coordinate system without acceleration and inertia forces leads to that the screen should be deformed due to the Lorentz contraction and its area is kinematically changed. However, the screen entropy cannot be changed by this transformation because the number of bits on the screen cannot be changed in this process (see also the paper by Morozov [59]). Therefore, it is necessary to introduce a correction in the second order of velocity to the constant \( \frac{1}{4} k_B c^3 \) in the moving frame.

In the way of derivation offered here, we do not see any obstacle to generalize the considered nonrelativistic consideration up to the case of general relativity for static gravitational fields in the way realized in [12].

Finally, we note that relation (12) was independently obtained by Padmanabhan in [50], by using the special definition of entropy in terms of the gravitational acceleration and temperature as mentioned in the Introduction. Then, he derived the expression for the total energy in the form \( E_H = 2 T H S_H \) for the black holes. Combining this relation with the equipartition distribution [11, 16, 50] \( E = \frac{1}{2} T A_H \frac{k_B c^3}{G \hbar} \), one can elementary find (12) for the fixed equation of state. In a different way, the relation between the entropy and area was found in [60] for the event horizon, only.
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