ON STATE-DEPENDENT SWEEPING PROCESS 
IN BANACH SPACES

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Abstract. In this paper we prove, in a separable reflexive uniformly smooth 
Banach space, the existence of solutions of a perturbed first order differential 
inclusion governed by the proximal normal cone to a moving set depending on 
the time and on the state. The perturbation is assumed to be separately upper 
semicontinuous.

1. Introduction. The existence of solutions for sweeping processes has been stud-
ied by many authors since the pioneering work by J.J. Moreau in the 70’s (see [29]). 
He expressed that sweeping process by the following evolution differential inclusion
\[-\dot{u}(t) \in N_{C(t)}(u(t)), \text{ a.e. } t \in [0, T]; \quad u(0) = u_0 \in C(0),\]
where \(C(t)\) is a closed convex set in a Hilbert space \(H\) and \(N_{C(t)}(.)\) is the normal cone to \(C(t)\) in the sense of convex analysis, see also [30] and [31]. Then, some 
contributions in the context of nonconvex sets \(C(t)\) were given in a series of papers, 
see for instance [4, 17, 18, 34, 35, 36].

Concerning the perturbed sweeping process, namely, the differential inclusion
\[-\dot{u}(t) \in N_{C(t)}(u(t)) + F(t, u(t)), \text{ a.e. } t \in [0, T]; \quad u(0) = u_0 \in C(0),\]
a lot of work has been done in the finite dimensional setting, we refer for example to [10, 11, 17, 34] and the references therein.

In the infinite Hilbert space, the authors in [9, 20, 21, 34, 3] showed the existence 
of solutions of (1) when the sets \(C(t)\) are prox-regular, this property got around the 
convexity of these sets and it is well adapted to the resolution of sweeping processes.

The study of such differential inclusions was motivated by its various applications 
to mechanical problems. See [28, 29, 30, 31].

Recently, the differential inclusion (1), where the sets in the normal cone de-
pend on the time and the state, has been studied by many authors motivated by 
the applications of this type of sweeping processes to parabolic quasi variational 
inequalities and modelisation of 2D and 3D quasistatic evolution problems with

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friction, see [27]. The first work in the convex case was done in the thesis of Chraibi in the finite dimensional space $\mathbb{R}^3$ (see [15]), and in a Hilbert space by Kunze and Marques in [27]. Then, in the case when $C(t, x)$ belongs to the more general class of prox-regular sets, some results were established with various type of perturbations in finite dimensional and Hilbert spaces, see [14, 12, 22, 2, 32, 23]. The same problem with the class of subsmooth sets which strictly contains the class of prox-regular sets, was established in [24] with an upper semicontinuous perturbation in Hilbert spaces, and in [25] with a mixed semicontinuous perturbation in a finite dimensional space. Recently, some existence results for this kind of problems with positively $\alpha$-far sets were proved by Jourani and Vilches in [26].

In the framework of separable reflexive uniformly smooth Banach spaces, a detailed study of the inclusion (1) has been made by Bernicot and Venel in [6] where the process is governed by the proximal normal cone.

For first order sweeping processes in $p$-uniformly smooth and $q$-uniformly convex Banach spaces, we refer the reader to [7] for a time dependent sweeping process, to [8] for a time and state dependent non-perturbed sweeping process and to [1] for the perturbed process.

Our aim in this paper is to prove the result in [6] by considering the state dependent process. In fact, we are interested in the existence of solutions, in a separable reflexive uniformly smooth Banach space $E$, of the following differential inclusion

$$
(P_F) \begin{cases}
\dot{u}(t) \in -N_C(t, u(t))(u(t)) + F(t, u(t)), \text{ a.e. } t \in [0, T]; \\
u(t) \in C(t, u(t)), \forall t \in [0, T]; \\
u(0) = u_0 = C(0, u_0),
\end{cases}
$$

where $C : [0, T] \times E \mapsto E$ is a nonempty closed and $r$-prox-regular valued set-valued mapping, and $F : [0, T] \times E \mapsto E$ a separately scalarly upper semicontinuous set-valued mapping with nonempty convex weakly compact values not necessarily bounded. The proof of our result uses ideas from [6] and [24].

The organization of this paper is as follows. In the next section we give some definitions and notation needed in the sequel and in section 3 we present our main theorem.

2. Preliminaries and notation. From now on $(E, \|\cdot\|)$ is a Banach space, $E'$ is its topological dual and $\langle \cdot, \cdot \rangle$ their duality product. $B_E(x, r)$ and $B_E(x, r)$ are respectively the closed and the open ball of $E$ of center $x \in E$ and radius $r > 0$ while $\overline{B}_E$ and $B_E$ are the closed and the open unit ball and $S_E$ is the unit sphere of $E$. We denote by $B(E)$ the Borel tribe on $E$.

Let $I := [0, T]$ $(T > 0)$. We denote by $C_E(I)$ the Banach space of all continuous mappings $u : [0, T] \rightarrow E$, endowed with the sup-norm $\|\cdot\|_C$. We also denote by $C(I)$ the $\sigma$-algebra of Lebesgue measurable subsets of $I$. $(L^1_k(I), \|\cdot\|_1)$ is the quotient Banach space of Lebesgue-Bochner integrable $E$-valued mappings and $(L^\infty_k(I), \|\cdot\|_\infty)$ is the quotient Banach space of essentially bounded $E$-valued mappings.

We said that a mapping $u : I \rightarrow E$ is absolutely continuous if there is a mapping $v \in L^1_k(I)$ such that $u(t) = u(0) + \int_0^t v(s) ds$, $\forall t \in I$, in this case $v = \dot{u}$ a.e.

For $A \subset E$, $co(A)$ denotes the convex hull of $A$ and $\overline{co}(A)$ its closed convex hull. It is well known that if $K$ is a nonempty subset of $E$, then

$$
\overline{co}(K) = \{x \in E/ \forall x' \in E'; \langle x', x \rangle \leq \delta^*(x', K)\},
$$

(2)
where $\delta^*(x', K)$ denotes the support function associated with $K$, that is

$$\delta^*(x', K) = \sup_{y \in K} \langle x', y \rangle \forall x' \in E'.$$

Let $A, B$ be two subsets of $E$, the Hausdorff distance between $A$ and $B$ is defined by

$$\mathcal{H}(A, B) = \sup(e(A, B), e(B, A)),$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{x \in B} \|a - x\|$.

Let $x$ be a point in $A \subset E$. We recall (see [16]) that the proximal normal cone of $A$ at $x$ is defined by

$$N_A^P(x) = \{\xi \in E : \exists \alpha > 0 \text{ s.t. } x \in PA(x + \alpha \xi)\},$$

where

$$PA(u) := \{y \in A : d(u, A) := \|u - y\| \}.$$

We will adapt the notation $N_A(.)$ instead of $N_A^P(.)$.

We refer to [5] Lemma 1.1 for the following geometric lemma.

**Lemma 2.1.** Let $E$ be a Banach space, $A$ be a closed subset of $E$ and $s > 0$. Then for $x \in A$ and $v \in E$ such that $x \in PA(x + sv)$ we have $x \in PA(x + \lambda sv)$ for all $\lambda \in (0, 1)$.

**Proof.** Let $u \in E$ and $z \in PA(u)$, then we have $z \in PA(u + t(z - u))$, for all $t \in (0, 1)$. Indeed, for all $t \in (0, 1)$, put $u_t = u + t(z - u)$. We have

$$\|u_t - z\| = \|u + t(z - u) - z\| = (1 - t)\|u - z\| = (1 - t)d(u, A).$$

On the other hand, for any $y \in A$

$$\|u_t - y\| = \|u + t(z - u) - y\| \geq \|u - y\| - t\|u_z - y\| \geq d(u, A) - t\|u - y\| = (1 - t)d(u, A) = \|u_t - z\|.$$

Hence $z \in PA(u_t)$. Now, Let $x \in E$ and $v \in E$ such that $x \in PA(x + sv)$, by what precedes if we take $t = 1 - \lambda$ we obtain

$$x \in PA(x + sv + (1 - \lambda)(x - x - sv)) = PA(x + \lambda sv).$$

This finishes the proof of the Lemma. \qed

In the following we give the definition and some important consequences of the prox-regularity needed in the sequel. For the proof and more details we refer the reader to [5] and [33].

**Definition 2.2.** Let $A$ be a closed subset of $E$ and $r > 0$. The set $A$ is said to be $r$-prox-regular if for all $x \in A$ and $v \in N_A(x) \backslash \{0\}$

$$B_E(x + r \frac{v}{\|v\|}, r) \cap A = \emptyset.$$

**Proposition 2.3.** Let $A$ be a nonempty closed and $r$-prox-regular subset of $E$ and $x \in A$. Then we have the following assertions.

(i) For all $x \in E$ with $d(x, A) < r$, the projection of $x$ onto $A$ is well-defined and continuous, that is, $PA(x)$ is a nonempty singleton subset of $E$;

(ii) if $v = PA(u)$ then, $v = PA(v + \frac{u - v}{\|u - v\|}).$

Next we recall some properties of the geometric structure of Banach spaces and we refer the reader to [19] for more details.
Definition 2.4. The Banach space \((E, \| \cdot \|)\) is said to be uniformly smooth if its norm is uniformly Fréchet differentiable away of 0, it means that the limit

$$\lim_{t \downarrow 0} \frac{\|x_0 + th\| - \|x_0\|}{t}$$

exists uniformly with respect to \(h, x_0 \in S_E\).

Proposition 2.5. Let \(E\) be a uniformly smooth Banach space and \(p \in (1, \infty)\) be an exponent. The function \(x \mapsto \|x\|^p\) is \(C^1\) over the whole space \(E\).

Definition 2.6. For \(E\) a uniformly smooth Banach space and \(p \in (1, \infty)\), we denote

$$J_p(x) := \frac{1}{p} \langle \nabla \| \cdot \|^p \rangle(x) \in E'.$$

Definition 2.7. (see [6]) Let \(I\) be an interval of \(\mathbb{R}\). A separable reflexive uniformly smooth Banach space \(E\) is said to be “\(I\)-smoothly weakly compact” for an exponent \(p \in (1, \infty)\) if for all bounded sequence \((x_n)_n\) of \(L^\infty_p(I)\), we can extract a subsequence \((y_n)_n\) weakly converging to a point \(y \in L^\infty_p(I)\) such that for all \(z \in L^\infty_p(I)\) and \(\phi \in L^1_p(I)\),

$$\lim_{n \to \infty} \int_I \langle J_p(z(t) + y_n(t)) - J_p(y_n(t)), y_n(t) \rangle \phi(t) \, dt = \int_I \langle J_p(z(t) + y(t)) - J_p(y(t)), y(t) \rangle \phi(t) \, dt. \quad (3)$$

Proposition 2.8. All separable Hilbert space \(H\) is \(I\)-smoothly weakly compact for \(p = 2\).

The following proposition describes a useful property of weak continuity of the projection operator. For the proof we refer the reader to [6].

Proposition 2.9. Let \((E, \| \cdot \|)\) be a separable, reflexive and uniformly smooth Banach space. Let \(C_\alpha, C : I \to E\) be set-valued mappings taking nonempty closed values and satisfying

$$\sup_{t \in I} \mathcal{H}(C_\alpha(t), C(t)) \to n \to \infty 0.$$

We assume that for an exponent \(p \in [2, \infty)\) and a bounded sequence \((v_n)_n\) of \(L^\infty_p(I)\), we can extract a subsequence \((v_{k(n)})_n\) weakly converging to a point \(v \in L^\infty_p(I)\) such that for all \(z \in L^\infty_p(I)\) and \(\phi \in L^1_p(I)\),

$$\lim_{n \to \infty} \int_I \langle J_p(z(t) + v_{k(n)}(t)) - J_p(v_{k(n)}(t)), v_{k(n)}(t) \rangle \phi(t) \, dt = \int_I \langle J_p(z(t) + v(t)) - J_p(v(t)), v(t) \rangle \phi(t) \, dt.$$

Then the projection \(P_{C_\alpha(t)}\) is weakly continuous in \(L^\infty_p(I)\) (relatively to the directions given by the sequence \((v_n)_n\) in the following sense: for all \(s > 0\) and for any bounded sequence \((u_n)_n\) of \(L^\infty_p(I)\) satisfying

$$\begin{cases} u_n \to u \text{ in } L^\infty_p(I); \\ u_n(t) \in P_{C_\alpha(t)}(u_n(t) + sv_n(t)), \ a.e. \ t \in I \end{cases}$$

one has

$$u(t) \in P_{C_\alpha(t)}(u(t) + sv(t)), \ a.e. \ t \in I.$$
3. **Main result.** Now, we are able to prove our main existence theorem.

**Theorem 3.1.** Let $I = [0,T]$ ($T > 0$) and $E$ be a separable, reflexive, uniformly smooth Banach space which is $I$-smoothly weakly compact for an exponent $p \in [2, \infty)$. Let $F : I \times E \mapsto E$ be a set-valued mapping with nonempty convex weakly compact values such that

(H$_1$) $F$ is scalarly $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable, that is for each $e \in E$, the scalar function $\delta^e(e, F(\cdot, \cdot))$ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable;

(H$_2$) for each $t \in I$, $F(t, \cdot)$ is scalarly measurable and scalarly upper semicontinuous, that is for each $e \in E$, the scalar function $\delta^e(e, F(t, \cdot))$ is upper semicontinuous on $E$.

**Proof.**

*Step 1.* Let $n_0 \in \mathbb{N}^*$ such that

$$
\frac{T}{n_0} \left( \frac{k_1 + m(1 + k_2)}{1 - k_2} \right) \leq \frac{r}{2}.
$$

For all $n \geq n_0$, consider the partition $\{[t_{n,i}], I_{n,i}\} 0 \leq i \leq n - 1$, of the time-interval $I = [0,T]$, where $I_{n,i} = (t_{n,i}, t_{n,i+1}]$, $t_{n,i} = ih_n$ for $0 \leq i \leq n$ and $h_n = \frac{T}{n}$.

For each $(t,x) \in I \times E$, let $f(t,x) = P_{F(t,x)}(0)$ \forall $(t,x) \in I \times E$.

Since $F$ is scalarly measurable and has convex weakly compact values and $H$ is separable, then for all $x \in E$, the mapping $t \mapsto f(t,x)$ is Lebesgue measurable (see [13]), and by (4) it is Lebesgue-integrable.

We start by constructing, for each $n \geq n_0$, a finite sequence $\{u_{n,i} : i = 0, \ldots, n\}$ such that $u_{n,0} = u_0 \in C(0, u_0)$:

$$
\begin{align*}
    u_{n,i+1} &= P_{C(t_{n,i+1}, u_{n,i})} \left( u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds \right); \\
    \|u_{n,i+1} - u_{n,i}\| &\leq h_n \frac{2m + k_1}{1 - k_2}.
\end{align*}
$$
We proceed by induction. Set \( u_{n,0} = u_0 \in C(0, u_0) \). Since the constants \( m \) and \( k_1 \) satisfy relation (6), \( F \) satisfies (4), \( C \) satisfies (5) and the sets \( C(t, x) \) are assumed to be \( r \)-prox-regular, we have

\[
d(u_{n,0} - \int_{t_{n,0}}^{t_{n,1}} f(s, u_{n,0}) \, ds, C(t_{n,1}, u_{n,0}))
\]

\[
\leq d(u_{n,0}, C(t_{n,1}, u_{n,0})) + \int_{t_{n,0}}^{t_{n,1}} \|f(s, u_{n,0})\| \, ds
\]

\[
= |d(u_{n,0}, C(t_{n,1}, u_{n,0})) - d(u_{n,0}, C(t_{n,0}, u_{n,0}))| + \int_{t_{n,0}}^{t_{n,1}} \|f(s, u_{n,0})\| \, ds
\]

\[
\leq k_1|t_{n,1} - t_{n,0}| + h_n m = h_n(m + k_1) \leq h_n(m + k_1) \leq \frac{r}{2} < r.
\]

By Proposition 2.3, we have that \( PC_{C(t_{n,1}, u_{n,0})}(u_{n,0} - \int_{t_{n,0}}^{t_{n,1}} f(s, u_{n,0}) \, ds) \) is a nonempty single-valued set. Then we define the point \( u_{n,1} \in C(t_{n,1}, u_{n,0}) \) by

\[
u_{n,1} = PC_{C(t_{n,1}, u_{n,0})}(u_{n,0} - \int_{t_{n,0}}^{t_{n,1}} f(s, u_{n,0}) \, ds).
\]

Furthermore, we have

\[
\|u_{n,1} - u_{n,0}\|
\]

\[
= \|PC_{C(t_{n,1}, u_{n,0})}(u_{n,0} - \int_{t_{n,0}}^{t_{n,1}} f(s, u_{n,0}) \, ds) - u_{n,0}\|
\]

\[
\leq \|PC_{C(t_{n,1}, u_{n,0})}(u_{n,0} - \int_{t_{n,0}}^{t_{n,1}} f(s, u_{n,0}) \, ds) - (u_{n,0} - \int_{t_{n,0}}^{t_{n,1}} f(s, u_{n,0}) \, ds)\|
\]

\[
+ \int_{t_{n,0}}^{t_{n,1}} \|f(s, u_{n,0})\| \, ds
\]

\[
= d(u_{n,0} - \int_{t_{n,0}}^{t_{n,1}} f(s, u_{n,0}) \, ds, C(t_{n,1}, u_{n,0})) + \int_{t_{n,0}}^{t_{n,1}} \|f(s, u_{n,0})\| \, ds
\]

\[
\leq d(u_{n,0}, C(t_{n,1}, u_{n,0})) + 2 \int_{t_{n,0}}^{t_{n,1}} \|f(s, u_{n,0})\| \, ds
\]

\[
= |d(u_{n,0}, C(t_{n,1}, u_{n,0})) - d(u_{n,0}, C(t_{n,0}, u_{n,0}))| + 2 \int_{t_{n,0}}^{t_{n,1}} \|f(s, u_{n,0})\| \, ds
\]

\[
\leq k_1|t_{n,1} - t_{n,0}| + 2mh_n = h_n(2m + k_1) < h_n\frac{2m + k_1}{1 - k_2}
\]

since \( 0 \leq k_2 < 1 \). Clearly, the conditions in the induction are satisfied at the step \( i=0 \).

Suppose that, for \( i \in \{1, \ldots, n-1\} \), the sequence \( \{u_{n,j} : j = 0, \ldots, i\} \) is well defined satisfying (7) and (8). We will define \( u_{n,i+1} \). Since by (7), \( u_{n,1} \in C(t_{n,1}, u_{n,1-1}) \), we have

\[
d(u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds, C(t_{n,i+1}, u_{n,i}))
\]

\[
\leq d(u_{n,i}, C(t_{n,i+1}, u_{n,i})) + \int_{t_{n,i}}^{t_{n,i+1}} \|f(s, u_{n,i})\| \, ds
\]
Then we define the point $u_{n,i+1}$ in $C(t_{n,i+1}, u_{n,i})$, by

$$u_{n,i+1} = P_{C(t_{n,i+1}, u_{n,i})}(u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds).$$

Furthermore, we have

$$\|u_{n,i+1} - u_{n,i}\| = \|P_{C(t_{n,i+1}, u_{n,i})}(u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds) - u_{n,i}\| \leq \|P_{C(t_{n,i+1}, u_{n,i})}(u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds) - (u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds)\| + \int_{t_{n,i}}^{t_{n,i+1}} \|f(s, u_{n,i})\| \, ds \leq d(u_{n,i}, C(t_{n,i+1}, u_{n,i})) + 2 \int_{t_{n,i}}^{t_{n,i+1}} \|f(s, u_{n,i})\| \, ds \leq k_1 |t_{n,i+1} - t_{n,i}| + k_2 \|u_{n,i} - u_{n,i-1}\| + 2mh_n \leq h_n(2m + k_1) + k_2 h_n \frac{2m + k_1}{1 - k_2} = h_n \frac{2m + k_1}{1 - k_2}.$$

Hence the finite sequence $\{u_{n,i} : i = 0, ..., n\}$ is well defined satisfying (7) and (8).

Observe that relation (7) and Proposition 2.3 give

$$u_{n,i+1} = P_{C(t_{n,i+1}, u_{n,i})}\left(u_{n,i+1} + \frac{u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds - u_{n,i+1}}{\|u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds - u_{n,i+1}\|}\right). \quad (9)$$

**Step 2.** For every $t \in I_{n,i}$, with $i = 0, ..., n - 1$, we set

$$u_n(t) = \left(\frac{t}{h_n} - i\right) (u_{n,i+1} - u_{n,i}) + \int_{t_{n,i}}^{t} f(s, u_{n,i}) \, ds + u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds$$

$$z_n(t) = f(t, u_{n,i}).$$
It is clear that \( u_n \) is a continuous mapping, that \( u_n(t_{n,i+1}) = u_{n,i+1} \) and that for almost every \( t \in I_{n,i} \)
\[
\dot{u}_n(t) = \frac{1}{h_n} (u_{n,i+1} - u_{n,i}) + \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds - f(t, u_{n,i}).
\]
For almost every \( t \in I_{n,i} \) we put
\[
\Delta_n(t) = \dot{u}_n(t) + z_n(t) = \frac{1}{h_n} (u_{n,i+1} - u_{n,i}) + \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds.
\]
First, let us check that \( \Delta_n(t) \) is a bounded vector. Using relation (8) we have for almost every \( t \in I_{n,i} \)
\[
\|\Delta_n(t)\| \leq \frac{1}{h_n} \|u_{n,i+1} - u_{n,i}\| + m
\leq \frac{2m + k_1}{1 - k_2} + m = \frac{k_1 + m(3 - k_2)}{1 - k_2} =: M.
\]
That is,
\[
\|\Delta_n(t)\| \leq M \quad \text{a.e. } t \in I_{n,i}. \tag{10}
\]
Next, consider the vector \( v = u_{n,i} - \int_{t_{n,i}}^{t_{n,i+1}} f(s, u_{n,i}) \, ds \), and put for every \( t \in I_{n,i} \)
\( C_n(t) = C(t_{n,i+1}, u_{n,i}) \). Since \( C \) has r-prox-regular values, relations (9) and (7) give
\[
u_{n,i+1} = P_{C_n(t)} \left( P_{C_n(t)}(v) - r \frac{P_{C_n(t)}(v) - v}{\|P_{C_n(t)}(v) - v\|} \right). \tag{11}
\]
Observe that by relation (10), we have \( \|P_{C_n(t)}(v) - v\| \leq M \). Then, applying Lemma 2.1 to relation (11), with \( \lambda = \frac{1}{\pi_n M \|P_{C_n(t)}(v) - v\|} \), we get
\[
u_{n,i+1} \in P_{C_n(t)} \left( P_{C_n(t)}(v) - r \frac{P_{C_n(t)}(v) - v}{h_n M} \right)
= P_{C_n(t)} \left( P_{C_n(t)}(v) - \frac{r}{h_n M} (P_{C_n(t)}(v) - v) \right)
= P_{C_n(t)} \left( P_{C_n(t)}(v) - \frac{r}{M} \Delta_n(t) \right),
\]
that is,
\[
u_{n,i+1} \in P_{C_n(t)} \left( u_{n,i+1} - \frac{r}{M} \Delta_n(t) \right) \quad \text{a.e. on } I_{n,i}. \tag{12}
\]
**Step 3.** Existence of limit mapping. We will prove the convergence of the sequence \((u_n(.)) \subset C_E(I)\).
By relations (4) and (10), we have for almost every \( t \in I \),
\[
\|\dot{u}_n(t)\| \leq \|\Delta_n(t)\| + \|z_n(t)\| \leq M + m =: \alpha. \tag{13}
\]
This shows that \((\dot{u}_n(.))\) is uniformly bounded by \( \alpha \). So \((u_n(.))\) is a bounded sequence of \( C_E(I) \) since for every \( t \in I \)
\[
\|u_n(t)\| \leq \|u_0\| + \int_0^t \|\dot{u}_n(s)\| \, ds \leq \|u_0\| + T \alpha =: \beta,
\]
and it is clear that \((u_n(.))\) is equicontinuous. Let us prove that for every fixed \( t \),
the sequence \((u_n(t))_{n \geq n_0}\) is relatively compact.
Set for every \( t \in I_{n,i}, \ \theta_n(t) = t_{n,i+1}, \ \delta_n(t) = t_{n,i}, \) and observe that
\[
\lim_{n \to \infty} |\delta_n(t) - t| = \lim_{n \to \infty} (t - t_{n,i}) \leq \lim_{n \to \infty} (t_{n,i+1} - t_{n,i}) = \lim_{n \to \infty} \frac{T}{n} = 0,
\]
that is, \( \lim_{n \to \infty} \delta_n(t) = t. \) By the same calculus we have \( \lim_{n \to \infty} \theta_n(t) = t. \) Then relation (7) and the definition of \( u_n \) show that
\[
u_n(\theta_n(t)) \in C(\theta_n(t), u_n(\delta_n(t))) \ \forall t \in I.
\]
This last relation with (14) implies that
\[
(u_n(\theta_n(t))) \subset C(I \times \beta \mathbb{B}_E) \bigcap \beta \mathbb{B}_E.
\]
Then, hypothesis (ii) ensures the relative compactness of the sequence \( (u_n(\theta_n(t))) \).
But, since for every \( t \in I \)
\[
\|u_n(\theta_n(t)) - u_n(t)\| \leq \alpha(|\theta_n(t) - t| \to 0 \ \text{as} \ n \to +\infty,
\]
we have that the sequence \( (u_n(t))_{n \geq n_0} \) is also relatively compact. By Ascoli-Arzelà theorem, we get that \( (u_n) \) is relatively compact. By extracting a subsequence (that we do not relabel), we conclude that \( (u_n) \) converges uniformly to some mapping \( u \in C(I \times \beta \mathbb{B}_E) \).
In the following, remember that \( z_n(t) = f(t, u_n(\delta_n(t))) \in F(t, u_n(\delta_n(t))) \) for all \( t \in I \), and let us prove the convergence of the sequences \( (z_n(.)) \) and \( (u_n(.)) \) (we use techniques from [13]). We have for all \( t \in I \)
\[
\lim_{n \to \infty} \|u_n(\delta_n(t)) - u(t)\| \leq \lim_{n \to \infty} \left( |\|u_n(\delta_n(t)) - u_n(t)\| + \|u_n(t) - u(t)\| \right)
\]
\[
\leq \lim_{n \to \infty} (\alpha|t - \delta_n(t)| + \|u_n(t) - u(t)\|),
\]
that is, \( \lim_{n \to \infty} \|u_n(\delta_n(t)) - u(t)\| = 0. \) The convergence of the sequence \( (u_n(\theta_n(.)))_n \) to \( u(.) \) is also obtained.
Now, by relation (4), we deduce that \( (z_n(.))_n \) is a bounded sequence in \( L^\infty_E(I) \), then we can extract a subsequence, still denoted \( (z_n(.)) \), that converges \( \sigma(L^\infty_E, L^1_{E'}) \) to some mapping \( z(.) \) in \( L^\infty_E(I) = (L^1_{E'}(I))' \) since \( E \) is reflexive, i.e. for all \( \zeta(.) \in L^1_{E'}(I) \), we have
\[
\lim_{n \to \infty} \langle z_n(.), \zeta(.) \rangle = \langle z(.), \zeta(.) \rangle.
\]
As \( L^\infty_E(I) \subset L^1_{E'}(I) \), by relation (16) we deduce that for all \( \zeta(.) \in L^\infty_E(I), \)
\[
\lim_{n \to \infty} \langle z_n(.), \zeta(.) \rangle = \langle z(.), \zeta(.) \rangle,
\]
that is, \( (z_n(.)) \) converges \( \sigma(L^1_{E'}, L^\infty_E) \) to \( z(.) \) in \( L^1_{E'}(I) \), so Mazur’s Lemma ensures that for a.e. \( t \in I \), there exists a sequence \( (\xi_n(.)) \) (where \( \xi_n(.) \) is a convex combination of \( \{z_k(.), k \geq n\} \) which converges to \( z(.) \) in \( L^1_{E'}(I) \). We can extract from the sequence \( (\xi_n(.)) \) a subsequence which converges a.e. to \( z(.) \). Then,
\[
z(t) \in \overline{\{\xi_n(t), \ n \in \mathbb{N}\}} \bigcap \bigcup_{n \in \mathbb{N}} \{\xi_n(t)\}, \ \text{a.e.} \ t \in I,
\]
and so,
\[
z(t) \in \bigcap_{n \in \mathbb{N}} \overline{\{z_k(t), \ k \geq n\}}, \ \text{a.e.} \ t \in I.
\]
Set
\[
A_n = \{z_k(t), \ k \geq n\}.
\]
Then, by relation (2), we obtain for all \( x' \in E' \),
\[
\langle x', z(t) \rangle \leq \delta^*(x', A_n), \quad \forall n \in \mathbb{N}
\]
\[= \sup_{k \geq n} \langle x', z_k(t) \rangle, \quad \forall n \in \mathbb{N}, \]
that is,
\[
\langle x', z(t) \rangle \leq \inf_{n \in \mathbb{N}} \sup_{k \geq n} \langle x', z_k(t) \rangle
\]
\[= \lim_{n \to \infty} \sup_{k \geq n} \langle x', z_k(t) \rangle \leq \lim_{n \to \infty} \sup_{k \geq n} \delta^* \left( x', F(t, u_n(\delta_n(t))) \right), \]
and by the scalar upper semicontinuity of \( F \) compact values and
\[
\text{Since by (13), that (} \dot{u_n}(.) \text{) is absolutely continuous, and hence } w(.) \text{ and that }
\]
\[\forall x' \in E', \exists \text{ a sequence exists since } E \text{ is separable}, \text{ by Castaing-Valadier ([13] Prop. III. 35) we obtain}
\]
\[z(t) \in F(t, u(t)), \text{ a.e. } t \in I. \] (17)
On the other hand, we see by the relation (13), that \((\dot{u}_n(.)_n)\) is bounded in \( L_1^\infty (I) \), up to a subsequence, we may suppose that \((\dot{u}_n(.)_n)\) weakly* converges in \( L_1^\infty (I) \) to some mapping \( w(.) \) and that \( w(.) = \dot{u}(.) \). Indeed, for all \( y \in L_1^\infty (I) \),
\[\lim_{n \to \infty} \langle \dot{u}_n(.), y(.) \rangle = \langle w(.), y(.) \rangle, \]
i.e.,
\[
\lim_{n \to \infty} \int_0^t \langle \dot{u}_n(s), y(s) \rangle ds = \int_0^t \langle w(s), y(s) \rangle ds,
\]
in particular for \( y(.) = 1_{[0, t]}(.) e_j \), with \( t \in I \), \( 1_{[0, t]} \) the characteristic function of the interval \([0, t]\), and \( (e_j) \) a sequence of the space \( E' \) which separates the points of \( E \) (such a sequence exists since \( E \) is separable), then we obtain
\[
\langle \lim_{n \to \infty} \int_0^t \dot{u}_n(s) ds, e_j \rangle = \langle \int_0^t w(s) ds, e_j \rangle, \quad \forall j,
\]
which ensures,
\[
\lim_{n \to \infty} \int_0^t \dot{u}_n(s) ds = \int_0^t w(s) ds.
\]
As \( (u_n(.)) \) is a sequence of absolutely continuous mappings, we have the following equality
\[
\lim_{n \to \infty} (u_n(t) - u_n(0)) = \lim_{n \to \infty} \int_0^t \dot{u}_n(s) ds = \int_0^t w(s) ds,
\]
then
\[u(t) = u(0) + \int_0^t w(s) ds,
\]
so \( u(.) \) is absolutely continuous, and hence \( w(.) = \dot{u}(.) \).

Observe again, that for all \( t \in I \), we have by (5)
\[
\mathcal{H} \left( C_n(t), C(t, u(t)) \right) = \mathcal{H} \left( C(\theta_n(t), u_n(\delta_n(t))), C(t, u(t)) \right)
\]
\[\leq k_1|\theta_n(t) - t| + k_2\|u_n(\delta_n(t)) - u(t)\| \to 0. \] (18)
Furthermore, for every $t \in I$, we have by using (15) and (5)
\[
\begin{align*}
d(u(t), C(t, u(t))) &= |d(u(t), C(t, u(t))) - d(u_n(\theta_n(t)), C(\theta_n(t), u_n(\delta_n(t))))| \\
&\leq \|u_n(\theta_n(t)) - u(t)\| + k_2 \|u_n(\delta_n(t)) - u(t)\|,
\end{align*}
\]
the expression on the right-hand side of the inequality goes to 0 as $n \to \infty$. Consequently, $u(t) \in C(t, u(t))$ for all $t \in I$ since $C(t, u(t))$ is a closed set.

Let us prove now that for almost every $t \in I$
\[
u(t) \in P_{C(t, u(t))}\left(u(t) - \frac{r}{M}(\dot{u}(t) + z(t))\right).
\]
Set $r' = \frac{r}{M}$. We have $\Delta_n(t) = \dot{u}_n(t) + z_n(t)$ and by the arguments given above we know that $(\Delta_n(\cdot))_n$ weakly*-converges in $L_{\infty}^F(I)$ to $\dot{u}(\cdot) + z(\cdot) =: \Delta(\cdot)$. Since $E$ satisfies the property “I-smoothly weakly compact”, then we can apply relation (3) to the sequence $(r'\Delta_n(\cdot))_n$ to obtain for all $y \in L_{\infty}^F(I)$ and all $\phi \in L_b^1(I)$,
\[
\lim_{n \to \infty} \int_I \langle J_p(y(t) - r'\Delta_n(t)) - J_p(-r'\Delta_n(t)), \Delta_n(t)\rangle \phi(t) dt
= \int_I \langle J_p(y(t) - r'\Delta(t)) - J_p(-r'\Delta(t)), \Delta(t)\rangle \phi(t) dt.
\]
By (12) we know that for a.e. $t \in I$
\[
u_n(\theta_n(t)) \in P_{C_n(t)}(u_n(\theta_n(t)) - r'\Delta_n(t)),
\]
and since the sequence $(u_n(\theta_n(\cdot)))_n$ strongly converges in $L_{\infty}^F(I)$ to $u(\cdot)$ (because it is bounded and uniformly converges to $u$), using relation (18), we conclude by Proposition 2.9, that for almost every $t \in I$
\[
u(t) \in P_{C(t, u(t))}\left(u(t) - r'\Delta(t)\right),
\]
that is, $-\Delta(t) \in N_{C(t, u(t))}(u(t))$ (see the definition of the proximal normal cone), or equivalently
\[-\dot{u}(t) - z(t) \in N_{C(t, u(t))}(u(t)), \text{ a.e. } t \in I,
\]
and by (17) we get
\[-\dot{u}(t) \in N_{C(t, u(t))}(u(t)) + F(t, u(t)), \text{ a.e. } t \in I
\]
with $u(0) = 0$, that is, our problem $\mathcal{P}_F$ has at least a Lipschitz solution $u(\cdot)$. Furthermore, by (13)
\[
\|\dot{u}(t)\| \leq \frac{2m(2 - k_2) + k_1}{1 - k_2} \text{ a.e. } t \in I.
\]
The proof of our theorem is then complete. \qed

If $F$ is a single-valued mapping Theorem 3.1 reads as

**Corollary 3.2.** Let $I = [0, T]$ $(T > 0)$ and $E$ be a separable, reflexive, uniformly smooth Banach space which is I-smoothly weakly compact for an exponent $p \in [2, \infty)$. Let $f : I \times E \to E$ be a Carathéodory mapping, that is for each $x \in E$ $f(\cdot, x)$ is Lebesgue-measurable and for each $t \in I$ $f(t, \cdot)$ is continuous on $E$. Furthermore, we suppose that for some real constant $m \geq 0$
\[
\|f(t, x)\| \leq m, \forall (t, x) \in I \times E.
\]
Let \( r > 0 \) and let \( C : I \times E \to E \) be as in Theorem 3.1. Then for any \( u_0 \in C(0, u_0) \), the differential inclusion

\[
\begin{align*}
    u(0) &= u_0; \\
    u(t) &\in C(t, u(t)), \quad \forall t \in I; \\
    -\dot{u}(t) &\in N_C(t, u(t)) + f(t, u(t)), \quad a.e. t \in I,
\end{align*}
\]

has a Lipschitz solution \( u : I \to E \).

4. Comments. (1) Our main theorem is new even when the sets in the process are assumed to be convex.

(2) In [27], [14], [12] and [32], the authors used in their proofs the implicit algorithm

\[
u_{n,i} = P_{C(t_n, u_{n,i})}(u_{n,i-1})
\]

thanks to Darbo or Schauder fixed point theorems applied to the continuous mapping \( v \mapsto P_{C(t,v)}(u) \). The proof of the continuity of this mapping uses the Euclidean or Hilbert structure of the space. In our study, we have not been able to prove this continuity in the context of Banach spaces, that is an open question that attracts our interest.

(3) In the proof of our main theorem it was clear that the inequality \( k_2 < 1 \) was necessary to obtain our result, and this inequality cannot be relaxed. In fact, in [27] the authors showed by concrete examples that this kind of problems may have no solution when \( k_2 \geq 1 \).

(4) In the vein of [27], [1] and others, let us consider for example \( E = L^p(Z) \) which is a separable, reflexive, uniformly smooth Banach space and I-smoothly weakly compact for \( p \geq 2 \) (see [6]), and let \( J : E \to E' \) the duality mapping of \( E \). If we consider \( C \) and \( F \) as in Theorem 3.1 and we consider the problem of finding an absolutely continuous mapping \( u : I \to E \) such that there exists a measurable mapping \( g : I \to E \) satisfying for almost every \( t \in I \), \( u(t) \in C(t, u(t)) \), \( g(t) \in F(t, u(t)) \), \( u(0) = u_0 \in C(0, u_0) \) and

\[
\langle \dot{u}(t) + g(t), J(\dot{u}(t) + g(t)) - J(\dot{u}(t) + g(t) + w - u(t)) \rangle \\
\leq \langle w - u(t), J(\dot{u}(t) + g(t) + w - u(t)) \rangle
\]

for all \( w \in C(t, u(t)) \). Then we can show that this problem can be rewritten as our differential inclusion \((P_F)\). Consequently, Theorem 3.1 ensures the existence of a Lipschitz solution to the latter.

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