On the space-time localization and vacuum energy of scalar quantum fields

Vasileios I. Kiosses

1QSTAR, INO-CNR, Largo Enrico Fermi 2, I-50125 Firenze, Italy

By parameterizing the vacuum energy of Klein-Gordon theory in terms of a potential energy, a new representation of scalar fields is obtained in which a set of general space-time operators is defined, bypassing the Wightman-Pauli objection. A study on the space-time defined by the spectrums of these space-time operators reveals that the potential energy causes the non-inertial motion of field’s excitations. Interestingly, the association of vacuum energy with potential energy establishes the former as a space-time dependent quantity. An observer near the horizon (i.e the locus where the position square is equal to time square) measures a diverging vacuum energy, while at infinity sees the vacuum energy redshifted to zero and thus to coincide with the normal-ordered ground state energy.

I. INTRODUCTION

A massive, scalar field is known to be described by the Klein-Gordon theory. Even though this description is part of the extensively verified physical theory of Standard model, it encounters certain difficulties. Space-time localization or, in other words, the definition of general space-time operators is unquestionably one of them. The concept of space-time localization in the context of quantum field theory has been a challenging issue since the advent of the latter. And yet, a definite answer is still lacking [1–9]. Indeed, there is no unique way to describe localization of relativistic particles, even when the particles are unambiguously defined. In standard formalism of quantum mechanics, time is never treated as an operator, and therefore has a entirely different description from that of space position [10, 11]. It was Pauli [12], who first pointed out that a construction of a general time operator in quantum mechanics is impossible. Since time translations are generated by the Hamiltonian, any legitimate general time operator, as conjugate to the Hamiltonian, should translate energy arbitrarily. Thus, the definition of time operator contradicts the fact that any realistic Hamiltonian has a spectrum bounded from below. The same argument was generalised by Wightman in relativistic case (where space and time are merged into space-time) for position-time operators [13]. The standard textbook answer to Pauli-Wightman objection is that space-time position in quantum field theory cannot be represented by an operator and is just a parameter, external to the theory [12].

Another difficulty is the infinite energy of the ground state or vacuum energy as it is known. Even though the ground state demonstrates that the energy is bounded below, an interesting, and for the same reason disturbing, phenomenon takes place in Klein-Gordon theory (which is common in all standard quantum field theories), the ground state has infinite energy. The real problem is not the infinite space [1], but the energy density contributed by high frequency (momentum) modes. This is a short distance infinity known by the name ultra-violet divergence. This divergence arises because it was assumed that this theory holds to arbitrarily short distance scales, corresponding to arbitrarily high energies. Something that obviously is not true.

The standard way to deal with this problem, is to accept the theory only up to a certain ultraviolet momentum $p_{\text{max}}$ and cut off the very high momentum modes. In the absence of gravity this finite energy has no effect. The values that it can take is completely arbitrary. We could add any constant to the potential, and thus to the Hamiltonian, without changing the theory. Traditionally, this energy is discarded by the process of “normal-ordering”.

However, the vacuum fluctuations are very real, having measurable impact in standard physics, see Casimir effect [15]. On top of that, gravity does exist, and the actual value of the vacuum energy has important consequences. It turns out that the energy density of the vacuum measures the cosmological constant. The cosmological constant problem, the discrepancy between the theoretical (from calculations of vacuum energy) and observational values of the cosmological constant, is not related to the fact that quantum theory supply a system with a huge amount of vacuum energy, since this contribution can be renormalized away, the problem is that there is no reason the resulting number to be zero [11].

Furthermore, it would be a misleading picture to consider ground state as an inert entity. As it is clearly manifested by its name, Klein-Gordon field theory is a theory of fields, not a theory of particles, but an interpretation of its quantum states in terms of particles naturally arises when quantum fields interact with other quantum systems. Nevertheless, the notion of particles in quantum field theory has certain limitations, in some cases to the extend that a useful particle representation of states does not, in general, exist. Exactly this arbitrariness in defining particles renders quantum field theory’s ground state unstable in particle creation [17]. This effect has been already studied in case of quantum field theory in a box of finite volume [14].
various space-time backgrounds: non-inertial motion i.e. Rindler space-time (Unruh effect \[19\]), black hole space-time (Hawking effect \[20\]) and time-dependent gravitational fields (particle production in expanding cosmological space-time \[21\]).

While, in principle, the ground state, as it is, could have solved the problem of space-time localization of quantum particles, in the sense that the infinite vacuum energy practically renders Hamiltonian spectrum unbound from below, the fact that the current field theories cannot handle vacuum energy put forward Pauli-Wightman objection setting us incapable of defining space-time operators and therefore forcing us to relocate position \(x\) and time \(t\) to mere parameters. From the discussion above, one can conclude that with regard to the aforementioned issues, the present description of Klein-Gordon field is no satisfactory and a new context able to theorize the arbitrariness of vacuum energy in terms of a new potential in order to adequately define space-time operator, while remaining consistent with the known laws of physics, would be desirable. In this work we show that there exists a way which accomplishes this end. The method consists of attaching to the momentum space of the free Klein-Gordon field an extra degree of freedom, interpreted as potential energy, associated it to the vacuum energy, and then introducing, in addition to the standard equation of motion of the field (i.e. Klein-Gordon equation) a second equation in the extended momentum space of the field. As we will show, the physical meaning of the new equation is the energy transfer between the free energy \(E_p\) (or momentum \(p\)) and the newly-established potential energy. Practically speaking, it can be interpreted as a new type of quantum fields responsible for the quantization of space-time.

More specifically we shall show that:

\(a\). For a real, scalar, massive field exists a representation of Klein-Gordon theory in which, in addition to the energy operator (Hamiltonian) and position operator expressed in the standard form, a new set of position and time hermitian operators is determined which comply with the language of second quantization. The corresponding space-time eigenvalues are proportional to the wave numbers of the new field and the eigenvalues of the number operator in the old representation. On account of these operators, the space-time localization of field excitations is not, any more, external to the theory, through parameters, as standard approach supports \[13\], but on some internal degrees of freedom of the field operators.

\(b\). In the new representation of Klein-Gordon fields there is another ground state than the standard one; the new ground state has the property that its energy is not the same everywhere, it depends on the eigenvalues of the new position-time operators. More precisely, to a local observer the vacuum energy appears always the same, non-zero, but a distant observer sees something else, since its vacuum energy tends to zero.

c. From a comparison between the Hamiltonians of the new and the old representations, it is shown that the creation and annihilation operators in the new theory are related to those of the old one through a Bogoliubov transformations. The new creation-annihilation operators appear to be explicitly space-time dependent quantities. Based on Bogoliubov transformations, we show that the spectrum of Klein-Gordon particles corresponding to Bogoliubov transformations is actually a thermal spectrum with temperature intimately related to the space observables. Namely, the ground state of the old representation is a space-dependent thermal state in the new representation.

d. From the kinematic study of the space-time localized quantum particles further insight on the ground state energy difference between the two representations of Klein-Gordon fields is obtained. It is found that the temperature of the thermal fluctuations of the ground state is proportional to the proper acceleration and the number of particles (i.e. eigenvalue of number operator). In other words, the vacuum energy in standard approach in the new representation turns out to be the energy of the non-inertial motion ascribed to the quantum field excitations. An interesting detail about this result is that the product of proper acceleration and number of particles parameterizes the amount of vacuum energy and not proper acceleration alone.

For simplicity, we consider the quantum theory of real Klein-Gordon field in two dimensions, with metric signature \((+, -)\), but our results can easily be generalised to four dimensions (see \[VI\]). Furthermore, the units are chosen such that \(c = \hbar = 1\), unless specified otherwise.

\section{II. Space-Time Localization of Multi-Particle States}

A massive, scalar field \(\Phi\) is known to be described by the Klein-Gordon equation

\[ (\partial_t^2 - \partial_x^2 + m^2)\Phi(t, x) = 0. \]  

\(x, t\) are the coordinates of \((1+1)\) space-time where the field is defined. One can (second-)quantize the field by forming the field operator \(\Phi(t, x)\) in the usual way and expanding in the normal mode solutions \(\phi_p (t, x) = e^{-i E_p t - p x} / \sqrt{4 \pi E_p}\) of Klein-Gordon equation \[\text{(1)}\].

\[ \Phi(t, x) = \int dp \left( a_p \phi_p (t, x) + a_p^\dagger \phi_p (t, x) \right). \]  

For each value of momentum \(p\), it corresponds the energy eigenvalue \(E_p = \sqrt{p^2 + m^2}\). The associated ground state \(|0\rangle\) is defined by

\[ a_p |0\rangle = 0 \quad \langle 0 | a_p^\dagger = 0, \quad \forall p \]  

where \(a_p\) and \(a_p^\dagger\) are the standard annihilation and creation operators of the theory, respectively. The Hamilto-
nian of the Klein-Gordon field $\Phi$ reads
\[ H_n = \int dp \, E_p \left( a_p^\dagger a_p + \frac{1}{2} \left( E_p, a_p^\dagger \right) \right). \tag{4} \]

Let us start by considering the momentum space of the Klein-Gordon field $\Phi$ to have as many as two independent degrees of freedom, which can characterize any point in this space. We choose the one to be the momentum $p$ of the Klein-Gordon particles and the second to be some abstractly, for now, defined energy $E$ (remember that $c = 1$ so energy and momentum share the same units). We work with a specific Klein-Gordon field, so there is no restriction on the values that the coordinates $E, p$ can take, the domain of both $E, p$ is the set of all real numbers.

Looking for a way to integrate the energy $E$ to the free Klein-Gordon field $\Phi$, we add an interaction between the momentum $p$ and $E$ in a form of a novel field. Let us assume that the momentum space of the Klein-Gordon field, as described above, has its own fields in the sense that each point in momentum space is associated with a continuous field variable $\tilde{\Psi}(E, p)$, which satisfies the differential equation:
\[ \left( \hbar^2 \partial^2_E - \hbar^2 \partial^2_p - \frac{1}{\kappa^2} \right) \tilde{\Psi}(E, p) = 0. \tag{5} \]

The motivation to introduce the equation in this form will be clarified in section IV. At the moment let us explain the terms that constitute this equation. First we should notice that temporarily we recover the reduced Planck constant $\hbar$ to avoid confusion in our analysis below. $\kappa \in \mathbb{R}$ is just a field parameter. Later we will see that it is related to temperature and proper acceleration of Klein-Gordon particles. The differential operators $\hbar^2 \partial^2_p, \hbar^2 \partial^2_E$ are the square of the linear differential operators $i\hbar \partial_p, -i\hbar \partial_E$, respectively.

To understand the physical meaning of these operators we need first to solve equation (4). Mathematically speaking, Eq. (5) is recognized as a classical wave equation including an extra term, so its solutions are plane waves of the form
\[ u(E, p) = e^{i(kx - \omega t)} = \frac{e^{i(\tilde{k}x - \tilde{\omega} t)}}{\sqrt{4\pi \tilde{\omega}}}. \tag{6} \]

Factor $1 / \sqrt{4\pi \tilde{\omega}}$ was inserted for later convenience. Plane waves $u$ are solutions to eq. (5) in the same sense in which $\phi$ are solutions to Klein-Gordon equation (1). But there is an essential difference between the two cases, which reflects the difference between the fields $\Phi$ and $\tilde{\Psi}$: while $\phi$ is defined in space-time $x - t$, $u$ is defined in momentum space $p - E$. Accordingly, $t, x$ are the coordinate variables for $\phi$, whereas $p, E_p$ are the components of the wave vector with $E_p$ to obey the dispersion relation $E_p^2 = p^2 + m^2^2$.

The coordinates at $u$, instead, are the energy $E$ and momentum $p$, while $(\tilde{k}, \tilde{\omega})$ are the components of a wave vector with $\tilde{\omega}$ to satisfy the dispersion relation
\[ \tilde{\omega}^2 = \tilde{k}^2 + \frac{1}{\hbar^2 \kappa^2}. \tag{7} \]

Recalling the standard definition (which applies to plane waves $\phi_p$), a plane wave is a function of space and time coordinates, $x, t$, which is proportional to
\[ \text{plane wave} = Ae^{i(kx - \omega t)}, \tag{8} \]

(where $k$ is the ordinary angular wavenumber and $\omega$ is the ordinary angular frequency). Apparently, according to this definition, it is rather opaque our interpretation of (6) as plane wave. In case of matter plane wave $\phi$, this has been clarified by the de-Broglie hypothesis
\[ E_p = \hbar \omega \quad \text{and} \quad p = \hbar k. \tag{9} \]

The same hypothesis does not apply onto $u$, since the energy-momentum vector is already present in the argument. We need an analogous hypothesis to associate the new wave vector $(\tilde{k}, \tilde{\omega})$ with the space-time coordinates, which is lacking. To this end acting the differential operators $i\hbar \partial_p$ and $-i\hbar \partial_E$ on the plane wave $u$ we get
\[ i\hbar \partial_p u = \hbar \tilde{\omega} u \quad \text{and} \quad -i\hbar \partial_E u = \hbar \tilde{k} u. \tag{10} \]

We postulate that the momentum derivative $i\hbar \partial_p$ is the position operator and therefore we theorize the association of frequency $\tilde{\omega}$ to a position $x_t$:
\[ x_t = \hbar \tilde{\omega}. \tag{11} \]

Similarly, we postulate that the energy derivative $-i\hbar \partial_E$ corresponds to a time operator and thus $k$ is associated to time $t$
\[ t = \hbar \tilde{k}. \tag{12} \]

Applying eqs (1) and (12) on (6) we finally get
\[ u(E, p) = \frac{e^{i(\tilde{k}x - \tilde{\omega} t)}}{\sqrt{4\pi \tilde{\omega} / \hbar}}. \tag{13} \]

Evidently, if matter plane waves $\phi_p$ ($\hbar = 1$) represent a free particle that carries momentum $p$ and energy $E_p$,

\[ ^2 \text{In momentum representation of quantum mechanics position operator has exactly this form.} \]
then $u$ above, according to de-Broglie formulas, represents a particle with energy $-E$ and momentum $-p$. The potential energy $E$ is justified because the particle is not anymore free - in section IV we will demonstrate that its motion is non-inertial. The position $x_t$ of the particle is restricted now to the spacetime hyperboloid

$$ x_t^2 - t^2 = \frac{1}{\kappa^2}. \quad (14) $$

Hyperboloid (14) is nothing else than the dispersion relation (7) after making use of (11) (12).

The analysis above suggests that the determination of position and time of a quantum particle turns into the eigenvalue problems:

$$ ih\partial_t u = x_t u, \quad \text{and} \quad -ih\partial_x u = t u, \quad (15) $$

where $x_t$ are the position eigenvalues and $t$ are the time eigenvalues.

Equations (15) say that the position operator $ih\partial_t$ is the generator of momentum change and the time operator $-ih\partial_x$ is the generator of (potential) energy change. More specifically, if we consider the plane wave $u(E, p + q)$ and expand it into power series, we obtain

$$ u(E, p + q) = u(E, p) + q\partial_x u(E, p) + \frac{1}{2} (q\partial_x)^2 u(E, p) + \ldots $$

$$ = e^{-\frac{x_t}{\kappa} \cdot -ih\partial_t} u(E, p) $$

$$ = e^{-\frac{x_t}{\kappa} \cdot q \cdot x_t} u(E, p). \quad (16) $$

Similarly, the plane wave $u(E - e, p)$ expanded into power series gives

$$ u(E - e, p) = u(E, p) - e\partial_x u(E, p) + \frac{1}{2} (-e\partial_x)^2 u(E, p) + \ldots $$

$$ = e^{-\frac{x_t}{\kappa} \cdot (-ih\partial_x)} u(E, p) $$

$$ = e^{-\frac{x_t}{\kappa} \cdot e \cdot x_t} u(E, p). \quad (17) $$

The unitary operators $U_{x_t}(q) := e^{-\frac{x_t}{\kappa} \cdot -ih\partial_t}$ and $U_t(e) := e^{-\frac{x_t}{\kappa} \cdot (-ih\partial_x)}$ built by the position and time operators, respectively, responsible for the transformation of $u(E, p)$, do not commute since position $x_t$ and time $t$, are not independent degrees of freedom, see eq. (14). Their non-commutativity will be demonstrated later in terms of creation and annihilation operators. For now we present the relation

$$ u(E, p \pm q) = u(E - f(q, \kappa), p), \quad (18) $$

which is easily derived combining eqs. (16) (17) with (14), assuming the function

$$ f(q, \kappa) = \pm q \sqrt{t^2 + \frac{1}{\kappa^2}}, \quad (19) $$

The double sign in front of $q$ covers the fact that solving (14) for $x_t$ we get $x_t = \pm \sqrt{t^2 + 1/\kappa^2}$.

Remember that we have justified the introduction of the new field equation (15), arguing that in this way the actual momentum $p$ of Klein-Gordon particles is integrated with the conjectural potential energy $E$. Eq. (18) explicitly verifies that this is the case. The plane wave of a particle with potential energy $E$ and momentum $p$ increased by $q$ is the same with the plane wave of a particle with the initial momentum $p$ and potential energy decreased by a function $q$ and parameter $\kappa$. So, one can infer that the physical meaning of $\tilde{\Psi}$ is the energy transfer between potential energy $E$ and free energy $E_p$ ("kinetical" in the sense that depends on momentum, $E_p = E_p(p)$).

It is important to underline the fact that we have identified the eigenvalues of time operator $-ih\partial_t$ as the time parameter which appears in standard Klein-Gordon field equation, by using the same letter, $t$, in both cases. Doing this we assert that the time that appears in Klein-Gordon equation (11) is not anymore a parameter but the spectrum of an operator. We cannot argue the same regarding the position parameter $x$ and position eigenvalue $x_t$ for two reasons, both rooted in relation (14). Whereas $x$ and $t$ are independent degrees of freedom, $x_t$ and $t$ are not. Secondly, in contrast to $x$ which can take any value from real line, $x_t$ is restricted to the spacetime hyperboloid (14).

From now on, natural units comprising $\hbar = 1$ (in addition to $c = 1$) are used allowing time $t$, wavenumber $k$, length $x_t$ and angular frequency $\omega$ to be used interchangeably. Thus, henceforward we recognize as dispersion relation the equation (14), and as plane wave the relation

$$ u_t(E, p) = e^{i(t(E - x_t) p)} \sqrt{4\pi x_t}. \quad (20) $$

To write down the most general solution of (9), we need to construct a complete, orthonormal set of plane waves (modes) in terms of which any solution may be expressed. But to make sense of orthonormal we have to define an inner product on the space of solutions to the equation (9). The appropriate inner product is expressed as an integral over a constant-momentum curve $C$,

$$ (u, v) := i \int_C dE \left( u^* \partial_t v - v \partial_t u^* \right). \quad (21) $$

The functions $u$ and $v$ are plane waves of the form of Eq. (20). It is easily verified that the plane waves set up an orthonormal set under this product,

$$ (u_{t'}, u_t) = \delta(t - t') $$

$$ (u_{t'}^*, u_t^*) = -\delta(t - t') $$

$$ (u_{t'}, u_t^*) = 0. \quad (22) $$

Note that the inner product (21) is not positive definite. From (22) we can choose the positive-frequency mode as $u_t$ and, consequently, the negative-frequency one as $u_t^*$. Based on this inner product, $\tilde{\Psi}$ can be expressed as a Fourier expansion of these normal modes

$$ \tilde{\Psi}(E, p) = \int dt \left( c_t u_t(E, p) + c_t^* u_t^*(E, p) \right). \quad (23) $$
with the Fourier coefficients to be defined by
\[ c_t = (x_t, \tilde{\psi}) \quad \text{and} \quad c_t^\dagger = -(x_t^*, \tilde{\psi}). \tag{24} \]

The field \( \tilde{\psi} \) can be quantized according to the rules of second quantization by promoting \( c_t, c_t^\dagger \) to annihilation and creation operators, respectively, and satisfying the usual commutation relation for the raising and lowering operators,
\[ [c_t, c_t^\dagger] = \delta(t - t'). \tag{25} \]

Due to Eq. (28), the commutation relations in coordinate space are equivalent to the commutation relations in momentum space. Here, index \( p \) stands for a curve in momentum space of constant \( p \). This relation implies that operators at equal momentum commute everywhere except at coincident potential-energy points.

Creation operators \( c_t^\dagger \) and annihilation operators \( c_t \) following the general machinery of second quantization should be operator-valued distributions. Furthermore, since \( \tilde{\psi}(E, p) \) lives in momentum space of the Klein-Gordon theory, its Fourier coefficients, the quantities \( c_t, c_t^\dagger \), should live in coordinate space \( t - x_t \), or the one-dimensional space \( t \) due to \([13]\). Remember that we have identified this \( t \) with the time parameter appearing in the field operator \( \Phi(t, x) \), which, again according to second quantization, is an operator-valued distribution.

Thus, we infer that the time annihilation and creation operators can be represented by the projections of Klein-Gordon field operator, \( \Phi(t, x) \) and \( \Phi^\dagger(t, x) \) respectively, for fixed value of position:
\[ \Phi(t, x) := \Phi(t, x_0)|_{x_0 \to x} = c_t \tag{27} \]

and
\[ \Phi^\dagger(t, x) := \Phi^\dagger(t, x_0)|_{x_0 \to x} = c_t^\dagger. \tag{28} \]

Apparently, since \( x \) does not appear on the solutions to \[\text{[27]}\], we assert that \([27]\) hold for every \( x \). The index notation is used to distinguish the parameter \( x \) from the real variable \( t \).

Another argument that supports the connection between the time creation/annihilation operators and Klein-Gordon field operator is the following. Using equation (4) for \( \tilde{\psi} \) and the fact that \( u_t \) is solution to the same differential equation one immediately verifies, through an integration by parts, that \( \Phi_x(t) \) and \( \Phi^\dagger_x(t) \) are constant in momentum \( p \). This is consistent with the fact that the Klein-Gordon field operator in standard framework is independent of momentum since according to \([2] \) \( \Phi(t, x) \) is in superposition of all the possible momentum eigenstates.

Operators \( \Phi(t, x) \) and \( \Phi^\dagger(t, x) \) acquire a dual role. As Klein-Gordon field operators, from expansion \([2]\), they create Klein-Gordon particles as superposition of momentum eigenstates, at specific space-time position \((t, x)\). At the same time, as Fourier coefficients in the expansion of the wave operator \( \tilde{\psi}(E, p) \), updated to
\[ \tilde{\psi}(E, p) = \int dt \left( \Phi_x(t) u_t(E, p) + \Phi_x^\dagger(t) u_t^*(E, p) \right). \tag{29} \]
they have the function to create and annihilate time eigenstates \( u_t \) and \( u_t^* \). Mathematically, this dual role is translated into two commutation relations that \( \Phi \) should satisfy. With respect to the first case, the equal-time standard canonical commutation relation
\[ [\Phi_x(t), \partial_x \Phi_x(x')] = i\delta(x - x'), \tag{30} \]
and in connection with the second case, the commutation relation
\[ [\Phi_x(t), \Phi^\dagger_x(t')] = \delta(t - t'). \tag{31} \]
which is typical for any set of creation and annihilation operators.

A key physical question is what are the observable quantities defined by this new field theory? In standard matter fields (like Klein-Gordon, Dirac), the simplest and most important such object is the overall Hamiltonian, which represents the total energy of the system. Passing to the new field \( \tilde{\psi} \), the same formalism can be maintained by considering the total “Hamiltonian”
\[ X = \int dt \ x_t \ \Phi^\dagger_x(t) \ \Phi_x(t), \tag{32} \]
which has dimensions of length, \( x_t = \sqrt{t^2 + 1/\kappa^2} \). In addition to the Hamiltonian, from our theory it follows also the operator
\[ T = \int dt \ t \ \Phi^\dagger_x(t) \ \Phi_x(t), \tag{33} \]
which has dimensions of time, see also \([22]\). For more details on the derivation see Appendix.

The operators \( T \equiv T^0 \) and \( X \equiv T^1 \) commute and
\[ [T^\mu, \Phi^\dagger_x(t)] = t^\mu \Phi^\dagger_x(t), \]
\[ [T^\mu, \Phi_x(t)] = -t^\mu \Phi_x(t), \mu = 0, 1 \tag{34} \]
where \( t^\mu := (t, x_t) \), showing that \( \Phi^\dagger_x(t) \) (\( \Phi_x(t) \)) acting on a state adds (subtracts) a time interval \( t \) and length \( x_t \).

Our physical intuition leads us to generate the states starting from the ground state \( |0\rangle \), defined by
\[ \Phi_x(t) |0\rangle = 0. \tag{35} \]
and proceed with the excited states \( \Phi^\dagger_x(t) |0\rangle \) for \( \forall t \), interpreted as one-particle states (the name will become clear below). To construct the full Hilbert space of states generated by multi-particle states of the form
\[ |n\rangle = (\Phi^\dagger_x(t))^n |0\rangle, \tag{36} \]
we use the standard mathematical procedure of linear superposition and Cauchy’s completion. The resulting space is a Fock space.

The interpretation as multi-particle states becomes apparent once we apply the expansion \([21]\) on \([30]\),

\[
|N\rangle = \int dp_1 \ldots dp_N \phi^\dagger_{p_1}(t, x) \ldots \phi^\dagger_{p_N}(t, x) \, |p_1, \ldots, p_N\rangle,
\]

where \(|p_1, \ldots, p_N\rangle := a^\dagger_{p_1} \ldots a^\dagger_{p_N} |0\rangle\). We see that \(|N\rangle\) is a superposition of \(N\)-particle Klein-Gordon momentum eigenstates and thus can be interpreted as \(N\) massive Klein-Gordon particles.

In this framework, particle states are well defined entities and conserved in the sense that the number operator

\[
N = \int dt \Phi_x^\dagger(t) \Phi_x(t)
\]

commutes with operators \(T^\mu\). Considering the action of \(N\) and \(T^\mu\) on the state \(|N\rangle\), we find

\[
N |N\rangle = N |N\rangle \quad \text{and} \quad T^\mu |N\rangle = t^\mu (N |N\rangle).
\]

So we verify that \(|N\rangle\) is eigenstate of \(X\) with eigenvalue \(X_N = x_N N\) and simultaneously eigenstate of operator \(T\) with eigenvalue \(T_N = t_N N\). Since \(T^\mu\) is the space-time operator, and \(N\) is the number of particles, the interpretation of states \(|N\rangle\) as eigenstates of \(T^\mu\) must be the spacetime localization of particle states as a function of particle number. If a single particle state, \(|1\rangle\), is localized in space-time position \((t, x)\), then a two-particle state, \(|2\rangle\), is localized in position \((2t, 2x)\), and a \(N\)-particle state, \(|N\rangle\), is localized in spacetime position \((Nt, Nx)\). Quantum states, which correspond to quantum particles of different number, cannot be ascribed to the same spacetime position. This is a remarkable result regarding the localization of Klein-Gordon particle states, which has important implications in the structure of space-time as described by the eigenvalues of space-time operators \(T^\mu\) (see \([18]\)). Notice that with the eigenvalues of \(T^\mu\) in case of one-particle states (i.e. \(N = 1\)) we take back classical space-time, made of space-time events (i.e. the space-time position of single quantum particles). But, we also get linear combinations of these particles (see eq. \([37]\)), which makes no sense in the context of classical space-time whatsoever. From now on, we shall refer to the space-time described by the eigenvalues of \(T^\mu\) as quantum space-time. In sections \([33]\) and \([34]\) we work with \(N = 1\) space-time coordinates \(t, x_1\), while in \([35]\) with general quantum space-time \(X_N = T_N\).

### III. MODIFIED HAMILTONIAN AND VACUUM ENERGY

It should become clear that the space-time localization of Klein-Gordon field states, attained in the previous section, was due to the new representation of Klein-Gordon field, \(\tilde{\Psi}(E, p)\), we have introduced. However, in this new representation, Klein-Gordon field is not free anymore, as have already mentioned, since in addition to the standard free energy \(E_p\), it carries also the potential energy \(E\). Apparently this extra energy renders Hamiltonian \(H_a\), Eq. \([4]\), inadequate to fully describe the quantum field, since it does not represent anymore the total energy of the system.

Our aim in this section is to derive the new Hamiltonian operator and then to calculate the ground state energy. The strategy we are going to pursue is the following. Since both representations refer to the same Klein-Gordon field, it should be possible the two field operators, i.e. \(\Phi(t, x)\) and Fourier transform of \(\tilde{\Psi}(E, p)\), to be expanded in terms of the same plane waves reducing their difference to the difference between their corresponding creation/annihilation operators. Because the actual difference between the two representantions is the potential \(E\), the relation between the two sets of creation/annihilation operators and the relation between the corresponding vacua have to depend on \(E\). We shall now show that these relations exist and indeed depend explicitly on potential energy \(E\).

The Fourier transform of \(\tilde{\Psi}(E, p)\) from \([27]\) reads

\[
\Psi(t, x) = \Phi_x(t) |u_E(p)\rangle + \Phi^\dagger_x(t) |u^*_E(p)\rangle,
\]

where \(\Phi(t, x)\) and \(\Phi^\dagger(t, x)\) are, respectively, the field and the field operator, \(\Phi(t, x) = \frac{1}{\sqrt{2\pi}} \int dp \Phi^\dagger(t, x, p) \phi^\dagger_p\), \(\Phi(t, x) = \frac{1}{\sqrt{2\pi}} \int dp \Phi(t, x, p) \phi_p\).

On account of \([40]\), \(\Psi(t, x)\) is a solution to Klein-Gordon equation and therefore it can be expanded in terms of the standard momentum eigenstates (wave planes) \(\phi_p, \phi^\dagger_p\):

\[
\Psi(t, x) = \int dp \left( b_p \phi_p(t, x) + b^\dagger_p \phi^\dagger_p(t, x) \right),
\]

with the operator coefficients, \(b_p = \langle \phi_p, \Psi \rangle_{KG}\) and \(b^\dagger_p = -\langle \phi^\dagger_p, \Psi \rangle_{KG}\), to be interpreted, as usual, as annihilation and creation operators with respect to the set of modes \(\{\phi_p, \phi^\dagger_p\}\). In the definition of creation and annihilation operators is used the Klein-Gordon inner product

\[
\langle \phi, \phi \rangle_{KG} := i \int dx (\phi^* \partial_t \theta - \theta \partial_t \phi^*).
\]

One can guess (or compute) the form of the observables associated to field operator \(\Psi\). He just needs to replace \(\partial_t\) with \(b_p\) to obtain the Hamiltonian

\[
H_b = \int dp \, E_p \left( b^\dagger_p b_p + \frac{1}{2} |b_p, b^\dagger_p| \right),
\]

and the momentum operator

\[
P_b = \int dp \, p \, b^\dagger_p b_p.
\]

As in the standard way, the vacuum state \(|0_b\rangle\), is defined by

\[
b_p |0_b\rangle = 0 \quad |0_b\rangle b^\dagger_p = 0, \quad \forall p.
\]
Unavoidably, this definition of vacuum is accompanied by the typical problem that characterizes all quantum field theories. The ground state energy is infinite, the integral
\[ \langle 0_b | H_b | 0_b \rangle = \frac{1}{2} \int dp \; E_p \langle 0_b | [b_p, b_p^\dagger] | 0_b \rangle \] (46)
is strongly divergent because \([b_p, b_p^\dagger] = \delta(0)\). However, we can get rid off this divergence by employing the standard process of normal-ordering:
\[ : \frac{1}{2} [b_p, b_p^\dagger] : = 0. \] (47)
Below we show that the condition of normal ordering is equivalent to setting the potential energy zero, \(E = 0\).

The single states \(|p\rangle \equiv b_p^\dagger |0_b\rangle\) carries momentum \(p\) and energy \(E_p = \sqrt{p^2 + m^2}\), and for that reason we interpret them as Klein-Gordon massive particle. Under the normal-order, these particle states are well-defined entities and conserved in the sense that the number operator
\[ N_b = \int dp \; b_p^\dagger b_p \] (48)
commutes with the Hamiltonian : \(H_b :\). It is trivial to verify that the states \((b_p^\dagger)^n |0_b\rangle\) are eigenstates of : \(H_b :\) and \(P_b\) (unlike the case of energy, no normal-ordering for momentum is required). The modes \(p\) and \(-p\) compensate each other leaving vacuum translationally invariant) with eigenvalues \(E_{\pm} = E_p \mp P_b\) and \(E_{\pm} = p \mp P_b\), respectively. Similarly, the state \(|p_1, \ldots, p_n\rangle \equiv b_{p_1}^\dagger \ldots b_{p_n}^\dagger |0_b\rangle\) has momentum \(p_1 + \ldots + p_n\) and energy \(E_{p_1} + \ldots + E_{p_n}\) and thus refers to \(n\) Klein-Gordon massive particles.

Quantum states \(\{|p_1, \ldots, p_n\rangle, \ldots\}\) differ from the quantum states \(\{|p\rangle, \ldots, |p_1, \ldots, p_n\rangle, \ldots\}\), related to free field operators \(a_p, a_p^\dagger\), simply because the vacuum states defined by the free annihilation operators \(a_p\) (i.e. \(|0\rangle\)), and the new annihilation operators \(b_p\) (i.e. \(|0_b\rangle\)), are distinct. To show this, recall that we have reasoned before the new operators \(b_p, b_p^\dagger\) should be possible to be expressed in terms of the free-field operators \(a_p, a_p^\dagger\). Furthermore, because of (11) combined with (2), we should expect \(b_p, b_p^\dagger\) to be functions of time eigenfunctions \(u_t(E, p)\). Indeed, the relations between \(b_p, b_p^\dagger\) and \(a_p, a_p^\dagger\), derived from equations (2) and (11), are found to be
\[ b_p = u_t(E, p) \left( a_p \left[ 1 - \frac{E}{2E_p} \right] \right. \] (49)
\[ + a_p^\dagger e^{-2i(tx + xp)} \left[ 1 + \frac{E}{2E_p} \right] \) \]
and
\[ b_p^\dagger = u_t^*(E, p) \left( a_p^\dagger \left[ 1 - \frac{E}{2E_p} \right] \right) \] (50)
\[ + a_p e^{2i(tx + xp)} \left[ 1 + \frac{E}{2E_p} \right] \) \]
For this derivation we have introduced the total energy \(H = E_p - E\), the total position \(\chi = x - x_t\) and applied to \(u_t\). Also, since \(\Phi\) is Hermitian, we have considered that \(a_p = a_p^\dagger\) and \(a_{-p} = a_{-p}^\dagger\).

Transformations (49,50) mix the free-field annihilation and creation operators, so they actually are Bogolubov transformations, which correspond to particle creation. Calculating the expectation value of the number operator \(b_p^\dagger b_p\) for the ground state of the free theory \(|0\rangle\) we find
\[ \langle 0 | b_p^\dagger b_p | 0 \rangle = |u_t|^2 \left[ 1 + 4 \cos^2(tE_p) \left( \frac{E}{2E_p} + \frac{E^2}{4E_p^2} \right) \right] \]
\[ = \frac{1}{4\pi\kappa_t} \left[ 1 + 4 \cos^2(tE_p) \left( \frac{E}{2E_p} + \frac{E^2}{4E_p^2} \right) \right]. \] (52)
The number of Klein-Gordon particles with momentum \(p\) and free energy \(E_p\), as defined in the new representation expressed by the operators \(b_p^\dagger\) and \(b_p\), is not zero in the ground state of free theory. Of course, recalling that the new system is not any more free, this result is justified and expected. The unanticipated and remarkable new element is that, regardless of the value that the potential energy \(E\) takes, this number is an explicitly space-time dependent quantity. For high frequency position modes, i.e. \(x_t \to \infty\) (and thus, for constant \(\kappa, t \to \infty\)) the expected number of particles tends to zero, \(\langle 0 | b_p^\dagger b_p | 0 \rangle \to 0\), recovering the free model, normal-ordered, ground state. Of course, the same holds for the expectation value of the general number operator (as superposition of momentum modes):
\[ \langle 0 | N_b | 0 \rangle = \int dp \; \langle 0 | b_p^\dagger b_p | 0 \rangle \] for \(x_t \to \infty\) or \(t \to \infty\) (53)

The maximum number of particles the ground state can accommodate, with respect to the space-time position, is given by
\[ \langle 0 | N_b^{\text{max}} | 0 \rangle = \frac{\kappa}{4\pi}, \] for \(E = 0\), (54)
which corresponds to \(t = 0\) and \(x_t = \frac{1}{\kappa}\). Apparently, from (52), we see that the potential energy \(E\) contributes to the augmentation of this number.

Interestingly, the condition imposed by the constraint \(E = 0\) is related to the process of normal-ordering, (17),

\[ \frac{1}{2} (b_p b_p + b_p b_p^\dagger) = \frac{1}{2} (b_p^\dagger b_p + b_p b_p^\dagger) = \delta(0) \]

Another version of normal-ordering is : \(\frac{1}{2} (b_p b_p + b_p b_p^\dagger) = \delta(0)\), which is equivalent to (47).
we employed for the new creation/annihilation operators, to avoid the divergence of ground state energy. The commutation relations for $b_p$ and $b_p^\dagger$ due to transformations \cite{59,60} can be written as

$$[b_p, b_p^\dagger] = -\frac{E}{4\pi x_t} \cos^2 \left(\frac{t E_p}{E_p}\right) [a_p, a_p^\dagger].$$  \hspace{1cm} (55)

This relation implies that the condition $E = 0$ is equivalent to the normal-ordering condition \cite{177}.

We can calculate now explicitly the expectation value of the energy of the new Hamiltonian $H_b$ in the ground state of the free theory,

$$\langle 0 | H_b | 0 \rangle = \int dp \ E_p \left( \langle 0 | b_p^\dagger b_p | 0 \rangle + \frac{1}{2} \langle 0 | [b_p, b_p^\dagger] | 0 \rangle \right)$$

$$= \frac{1}{4\pi x_t} \int dp \left( E_p + \frac{E^2 \cos^2 \left(\frac{t E_p}{E_p}\right)}{E_p} \delta(0) \right)$$  \hspace{1cm} (56)

where we have used the relations \cite{62} and \cite{65}. Delta function $\delta(0)$ is established from the commutation relation $[a_p, a_p^\dagger]$. The resulting integral can be computed by changing the variable from momentum $p$ to (free) energy $E_p$, giving the result

$$\langle 0 | H_b | 0 \rangle = \frac{1}{4\pi x_t} \int_m^\infty dE_p \frac{E_p^2 + E^2 \cos^2 \left(\frac{t E_p}{E_p}\right)}{\sqrt{E_p^2 - m^2}}$$

$$= \frac{1}{8\pi x_t} \left[ pE_p + \left( m^2 + \frac{L E^2}{2\pi} \right) \tanh^{-1} \left( \frac{E_p}{p} \right) \right.$$

$$+ K_0(i2\pi m) \left. \right] \infty_m, \hspace{1cm} (57)$$

where by $\tanh^{-1}(E_p/p)$ is implemented the principal value of inverse function of the hyperbolic tangent. $K_0$ are the modified Bessel functions of the second kind of zero order.

The factor $\frac{1}{2\pi}$ in front of potential energy $E$ was introduced in order to extract the $\delta(0)$ divergence. We have considered the theory in a 1-dimensional box of length $L$. Imposing periodic boundary conditions on the field and taking then the limit $L \to \infty$, we get

$$2\pi \delta(0) = \lim_{L \to \infty} \int_{-L/2}^{L/2} dx = L.$$  \hspace{1cm} (59)

It must be mentioned that the length of the box is related to the free field position parameter $x$ and not to the position eigenvalue $x_\ell$. This is reasonable since the delta function arises from the commutation relation of free field operators $a_p, a_p^\dagger$.

With the energy on the free theory ground state expressed by \cite{65}, we see that, even in the new representation of scalar fields, infinity is still there. The infinity problem remains even if we turn to computing the energy density (i.e. $\langle 0 | H_b | 0 \rangle / L$) instead of the total energy. But our new approach to describe scalar, massive fields has added two important elements in quantum field theory that are new. The first is that the residual energy of the ground state is not the same everywhere. The residual energy of an observer located at space position $x_\ell < \infty$ does not coincide with the corresponding energy of a distant observer, $x_\ell \to \infty$, since in the latter case holds

$$\lim_{x_\ell \to \infty} \langle 0 | H_b | 0 \rangle = 0.$$  \hspace{1cm} (60)

This feature will be analyzed further in the following section.

The second element is that the origin of high frequency infinity or ultra-violet divergence (i.e. the $E_p \to \infty$, and thus $p \to \infty$, limit in the expression \cite{55}) we encounter in the ground state is not hidden anymore. This divergence arises due to the time eigenfunctions $u_\kappa$, the plane wave solutions of field $\tilde{\Psi}$, and disappear when these time eigenfunctions are not properly defined. As we have noticed in the previous section, $u_\kappa$ has been recognized as the mean through which the energy transfer between potential energy $E$ and momentum/free energy occurs, see eqs \cite{18,19}. According to these relations, for $x_\ell < \infty$ there will be always a well-defined matter plane wave representing a particle with momentum (and thus free energy), even for particles having only potential energy $E$,

$$u(E - f(q, \kappa), 0) = u(E, \pm q). \hspace{1cm} (61)$$

This means that even in the vacuum state of the free theory, where the momentum and free energy should be zero, it is not, because $u_\kappa$ always fetch momentum from the repository of potential energy $E$.

This correspondence fades away for $x_\ell \to \infty$. The momentum transfer ceases to work for $x_\ell \to \infty$, because in this limit, the arbitrary function $f(q, \kappa)$, \cite{19}, tends to infinity, and \cite{61} reduces to identity $u(E, 0) = u(-\infty, 0)$. This becomes apparent considering that

$$f(q, \kappa) \to \infty \equiv \lim_{\kappa \to 0} f(q, \kappa), \hspace{1cm} (62)$$

which is the only possible option (apart from $q \to \infty$, but $q$ is not parameter of the theory). But in case of $\kappa = 0$, the definition of $\tilde{\Psi}$ through the differential equation \cite{51} is not applied. Thus, in the limit $x_\ell \to \infty$ the mechanism of momentum transfer from potential energy repository ceases to work and a vacuum state with zero momentum can be defined.

**IV. THE THERMAL BEHAVIOR OF GROUND STATE**

So far, we have seen that, in the new representation of Klein-Gordon fields, when an observer looks at distant regions of space-time (i.e. $t \to \infty$ and $x_\ell \to \infty$), those regions appear to contain less energy, than the region where the observer resides.

The analogy with stimulated emission suggests that a quantum field in the new representation should exhibits spontaneous energy emission. This radiation will,
of course, come in the form of the corresponding field quanta, i.e. Klein-Gordon particles. In this section we will demonstrate that the radiation emerges in a steady flux with a thermal spectrum at a temperature which is a function of the field parameter $\kappa$, $T = T(\kappa)$. The field parameter $\kappa$ has been already seen to play critical role in determination of residual energy (i.e. we have seen that for $\kappa = 0$, or $x_i \to \infty$, the ground energy $\langle 0 | H_b | 0 \rangle$ becomes zero, while for $\kappa \neq 0$, or $x_i < \infty$, $\langle 0 | H_b | 0 \rangle$ is infinite), thus the result presented here is consistent with the context of the previous section.

In order to show the thermal behavior of the ground state, we work with the Klein Gordon field in its standard, free form $\Phi(t, x)$ and in its new representation $\Psi(t, x)$ with the only difference being that here it is assumed to be massless, in order to simplify the calculations. The approach presented below is based on [23] where the thermal effect was computed by means of the field correlation function.

Starting with the free theory and field operator $\Phi(t, x)$, consider the two-time correlation function $\langle \Phi(t, 0) \Phi(t + \tau, 0) \rangle$ at a specific point in space for the field in thermal equilibrium at temperature $T$, i.e. the mode whose frequency is $E_p$ has $n(E_p)$ particles on average

$$\langle a_p^\dagger a_p \rangle = \delta(p' - p) n(E_p), \quad n(E_p) = \left( e^{E_p/T} - 1 \right)^{-1}. \quad (63)$$

From this it follows that

$$\langle \Phi(t, 0) \Phi(t + \tau, 0) \rangle = \frac{1}{\pi} (\pi T)^2 \text{csch}^2(\pi T \tau) \quad (64)$$

We proceed with $\Psi(t, x)$, considering the correlation function $\langle \Psi(t_1, x_1) \Psi(t_2, x_2) \rangle$ in the vacuum state $|0_b \rangle$. In this case it holds $\langle b_p b_{p'} \rangle = \langle b_p^\dagger b_{p'}^\dagger \rangle = \langle b_p^\dagger b_{p'} \rangle = 0$ and $\langle b_p b_{p'} \rangle = \delta(p - p')$. Furthermore, the space position is not anymore independent of time. From the total position relation $\chi = x - x_t$, the correlation function at a specific point $x = 0$ reduces to $\langle \Psi(t_1, x_{t1}) \Psi(t_2, x_{t2}) \rangle_b$. Therefore, we obtain

$$\langle \Psi(t_1, x_{t1}) \Psi(t_2, x_{t2}) \rangle_b = \frac{1}{\pi} \frac{1}{\Delta x^2 - \Delta t^2}. \quad (65)$$

with $\Delta t = t_2 - t_1$ and $\Delta x = x_{t2} - x_{t1}$. By construction, the relation $x_2^2 - t^2 = 1/\kappa^2$ holds. As this represents hyperbolic curves, we can use hyperbolic functions and set

$$x_{t1} = \frac{1}{\kappa} \cosh(\kappa \rho_{t1}) \quad i = 1, 2, \quad (66)$$

$$t_i = \frac{1}{\kappa} \sinh(\kappa \rho_{ti})$$

with $\rho_{ti}$ a parameter. One can calculate the difference $\Delta \chi^2 - \Delta t^2$ by making use of Eqs. (66)

$$\Delta \chi^2 - \Delta t^2 = -\frac{4}{\kappa^2} \sinh^2 \left( \frac{\kappa (\rho_{2} - \rho_{1})}{2} \right). \quad (67)$$

Thus the correlation function $\langle \Psi(t_1, x_{t1}) \Psi(t_2, x_{t2}) \rangle_b$ in the vacuum of the new representation of Klein-Gordon field takes the form

$$\langle \Psi(t_1, x_{t1}) \Psi(t_2, x_{t2}) \rangle_b = -\kappa^2 \frac{4\pi}{\pi^2} \text{csch}^2 \left( \frac{\kappa (\rho_{2} - \rho_{1})}{2} \right) \quad (68)$$

which is equivalent to the thermal-field correlation function [22] with temperature

$$T = \frac{\kappa}{2\pi}. \quad (69)$$

The meaning of this result is that an observer living on the quantum space-time described by the coordinates $x_t, t$ will perceive the Minkowski vacuum (the ground state of the free-field theory) as a thermal state with temperature given by eq. (69). The field parameter $\kappa$ is just the inverse of the quantity $\sqrt{x_t^2 - t^2}$, hence, as $\kappa \to 0$ this temperature is redshifted to zero. So, an observer at infinity sees only the zero temperature vacuum - the vacuum which coincides with the ground state in the standard field theory after applying the normal-ordering. As “quantum horizon” $x_t = \pm t$ (or $\kappa \to \infty$) is approached, an observer on quantum spacetime $x_t - t$ feels a diverging temperature, and thus a ground state with diverging energy density (the standard field theory ground state without the renormalization of normal-ordering.)

So when an observer looks at space-time infinity, those regions appear “cooler” than the region where the observer resides. Thinking analogously, an observer at infinity will consider the region near the quantum horizon $x_t = \pm t$ as “warmer” than his/her own region. As a result, the observer at infinity feels a flow of blackbody thermal radiation coming from that warmer region.

It is tempting to remark the similarity between this mechanism and the fundamental process behind Hawking radiation [20] and Unruh radiation [19]. As a matter of fact one can claim that, in case of classical space-time $x - t$, the arguments given for the structure of the vacuum near a black hole horizon (Hawking effect) and the Minkowski vacuum near an acceleration horizon in flat space-time (Unruh effect) applies equally well on space-time $x_t - t$, to the vacuum state near the quantum horizon $x_t = \pm t$.

V. NON-INERTIAL MOTION OF QUANTUM EXCITATIONS

The quantum space-time we have discussed until now was described by the $N = 1$ coordinates $T_1 \equiv t$ and $X_1 \equiv x_t$. However, the general coordinates of quantum space-time are $T_\gamma, X_\gamma$. Here we shall analyse the implications of this generalization. In particular, we will see that space-time described by $T_\gamma, X_\gamma$ is, by definition, relativistic, in the sense that its structure satisfies both postulates of special relativity: relativity principle and a universal speed limit. More specifically, we derive the identity $X_\gamma^2$
Fourier transform \( \tilde{x} \) \( t \) the relative velocity \(| s \) sider are operator on two different space-time points: \( \Phi \) proper acceleration is defined, quantized is that the corre-
se [24]). A consequence of being the space-time, where
on Unruh effect on quantum space-time our effect proper acceleration is determined on “quan-
the same event as measured by two observers in relative
the field operator \( \Phi \) is the same, we claim that the two
in each coordinate system. Making use of the dispersion
relation (14) we get
\[ O: \quad S^2 := X^2 - T_y^2 = \frac{N^2}{\kappa^2}. \]
\[ O': \quad S'^2 := X'^2 - T'^2 = \frac{N^2}{\kappa'^2}. \]
This result is really interesting. It demonstrates that, although the position and time of the event differ for measurements made by different observers, the spacetime interval of the event from the origin in each observer’s coordinate system is the same, provided of course that the number of particles are the same in each frame, which is the case we examine here. Recall that \( \kappa \) is the same in two equations. Hence, a new reading of eqs.(71) should be the following
\[ S^2 = S'^2, \quad \forall \kappa, N \]
which implies the identity
\[ X^2 - T_y^2 = X'^2 - T'^2. \]
In classical mechanics these identities are of fundamental importance since they summarize the two postulates of special relativity [22]. In spite of the traditional and conceptually very convenient use of light signals in the derivation of these identities, in our case the derivation is quite independent of the existence of light signals, or actually of any real-world effect that travels at the speed of light. We could say that it stems from the fact that all spacetime field pairs \( (x_i, t) \), \( (x'_i, t'_i) \), \( \cdots \) are emanated from a single field \( \tilde{\Psi} \), or more practically expressed, share the same field parameter \( \kappa \). This statement becomes more clear rewriting the dispersion relation (14) as
\[ x_i^2 - t^2 = \frac{1}{\kappa^2}, \quad \forall x_i, t. \]
Thus, working with fields \( \tilde{\Psi} \), special relativity as the logi-
cal consequence of two postulates, i.e. the relativity prin-
ciple and the universality of a speed limit are not some-
thing that you have to put in by hand. Rather, it is a
consequence of the framework.
It is straightforward from the identity (73) to show that the space-time coordinates \( T_y \), \( X_y \) are related to the space-time coordinates \( T'_y \), \( X'_y \) by the transformations
\[ X'_y = \gamma(X_y - \textbf{w} T_y), \quad T'_y = \gamma(T_y - \textbf{w} X_y) \]
where \( \textbf{w} \) is the relative velocity between the two observers, \( \mathcal{O} \) and \( \mathcal{O}' \). To construct transformations (75), we have...
introduced the quantity \( \gamma = (1 - u^2)^{-1/2} \). Evidently, \( \frac{\alpha}{\gamma} \) have the form of classical Lorentz transformations.

If we consider two spacetime events, \( A \) and \( B \), then \( \Delta X^a, \Delta T^b \) denote the finite coordinate differences \( X^a_B - X^a_A, T^b_A - T^b_B \). In that case, by successively replacing the coordinates \( A \) and \( B \) into \( (77) \) and subtracting, we get the transformation

\[
\Delta X^a_b = \gamma (\Delta X^a - w \Delta T^b), \\
\Delta T^b_a = \gamma (\Delta T^a - w \Delta X^b). 
\]

(76)

If, in place of differences, we are forming differentials, we obtain identical transformations with the above but in the differentials:

\[
dX^a_b = \gamma (dT^a - w dX^b), \\
dT^b_a = \gamma (dX^a - w dT^b). 
\]

(77)

That being said, it follows that together with \( (72) \) it must also hold

\[
\Delta S^2 = \Delta S'^2, \quad \forall \kappa, \mathbb{N} 
\]

(78)

and

\[
dS^2 = dS'^2, \quad \forall \kappa, \mathbb{N}. 
\]

(79)

where \( \Delta S^2 = \Delta X^a_B - \Delta X^a_A, dS^2 = dX^a_B - dX^a_A \) and \( \Delta S'^2 = \Delta X^b_B - \Delta X^b_A \) and \( dS'^2 = dX^b_B - dX^b_A \).

Let us consider once again the two observers \( O \) and \( O' \), but this time we put the group of \( \mathbb{N} \) particles to move non-uniformly relative to both frames. The path that the particles follows can be considered as a succession of the aforementioned spacetime events. Position and time differentials allow us to calculate the velocities \( v \) and \( v' \), and the accelerations \( g \) and \( g' \) of the particles in \( O \) and \( O' \), respectively. They are simply defined as

\[
O: \quad v := \frac{dx^a}{dt^a}, \quad g := \frac{d^2 x^a}{dt^a} \\
O': \quad v' := \frac{dx^b}{dt^b}, \quad g' := \frac{d^2 x^b}{dt^b}. 
\]

(80)

Substituting from \( (77) \) into \( (81) \), and considering, in particular, the rectilinear motion, which in one dimension reads \( w = v \), yields the acceleration transformation formula:

\[
\alpha = \gamma^2 g. 
\]

(82)

where we have defined the proper acceleration \( \alpha \) of the \( \mathbb{N} \) particles as that which is measured in its rest-reference frame (in our case \( \alpha = g' \)). Noticing that the right-hand side of \( (82) \) is equivalent to \( d(\gamma w)/dT^b \) and integrating twice, setting the constant of integration equal to zero in both cases, yields the following equation

\[
X^a_B - T^a_A = \frac{1}{\alpha^2}. 
\]

(83)

This equation represents an hyperbolic path in the coordinate system of observer \( O \). It describes the uniformly accelerated motion of \( \mathbb{N} \) particles, and it is of particular importance because when one analyzes the hyperbolic path from the perspective of transformations \( (77) \), due to the invariance \( (73) \), eq. \( (83) \) translates into itself in any reference frame \( O' \), that is \( X^a_{B'} - T^a_{A'} = \frac{1}{\alpha^2} \).

From eq. \( (83) \), expressing the multi-particle space-time coordinates as products of one particle coordinates times the number of particles we get

\[
x^2 - t^2 = \frac{1}{\alpha^2 \mathbb{N}^2}. 
\]

(84)

Comparing \( (83) \) with equation \( (12) \) we find that the field parameter \( \kappa \) is proportional to the proper acceleration \( \alpha \) and the number of particles \( \mathbb{N} \):

\[
\kappa = \pm \alpha \mathbb{N}. 
\]

(85)

Substituting this result on eq. \( (80) \) (keeping the plus sign) we finally obtain:

\[
T = \frac{\alpha \mathbb{N}}{2\pi}. 
\]

(86)

In the previous section, we have seen that the vacuum energy of the free-theory, in the new representation, can be parameterized by a temperature, using a suitable thermal state as the ground state. Eq. \( (83) \), together with eq. \( (55) \), uncovers another aspect of this view. The vacuum energy of the free-field can be parameterized by including proper-acceleration to its excitations or quantum particles. According to eq. \( (85) \) the value of the proper acceleration depends on the new field parameter \( \kappa \) and the eigenvalue of the number operator \( \mathbb{N} \), see \( (88) \). If none of the quantum particles have acceleration, i.e. \( \alpha = 0 \), which corresponds to \( \kappa = 0 \), then the standard Klein-Gordon description with the arbitrarily huge amount of vacuum energy applies. On the other hand, if there really exists quantum particles with proper acceleration on space-time \( x_t, t \), then the value of acceleration itself, \( \alpha = |\kappa/\mathbb{N}| \), parameterizes the amount of vacuum energy:

\[
\lim_{\alpha \to 0} \langle 0 | H_b | 0 \rangle = 0 \\
\lim_{\alpha \to \infty} \langle 0 | H_b | 0 \rangle = \infty \quad \text{for} \quad \mathbb{N} = \text{const.} 
\]

(87)

Interestingly, due to \( (83) \), the parametrization of vacuum energy in terms of particles acceleration turns into parameterization in terms of space-time position. An observer at infinity \( x_t, \rightarrow \infty \), since \( \alpha \to 0 \), sees the ground state to coincide with the normal-ordered free-theory ground state. While, near the horizon \( x_t = \pm \tau \), because \( \alpha \to \infty \), the observer measures a diverging vacuum energy.

VI. IN LIEU OF CONCLUSIONS

In this work we have proposed a new representation of scalar fields (chosen as the simplest among QFTs for
our arguments to be more clear) in an attempt to tackle two structural problems of the current theory: the lack of space-time operators and infinite vacuum energy. It turns out that in the context that our new description provides, the two problems are interrelated, and thus the same treatment appears to be adequate to cope with both of them. Put simply, the treatment we refer to is the new field \( \tilde{\Psi} \), which together with the free Klein-Gordon field \( \Phi \) results in the new, more general, representation of scalar fields.

Since this new field lives in the extended momentum space of Klein-Gordon field, unavoidably the two fields interact with each other. However, this interaction cannot be described by introducing non-linear terms in the free-fields Hamiltonian that couple different Fourier modes of the fields as done in standard interacting field theory. This is due to the fact that the new field \( \tilde{\Psi} \) cannot be considered independent of a standard field (it is really physically meaningless to define \( \tilde{\Psi} \) without specifying first system whose momentum space is employed), in other words, it has no free form. In addition, the field itself was introduced, in the first place, to exchange potential energy for momentum (or free energy). Thus, \( \tilde{\Psi} \) should be conceived as a more advanced version of interaction term in free field Hamiltonian.

From that perspective, someone could argue that the field parameter \( \kappa \) plays the role of the coupling constant. The motivation for this statement is based on eq. (85) and is as follows. A general property that distinguishes free field theories from interacting field theories is that in the former case the particle number is conserved, while in the latter not. Eq. (85), rearranged in the form \( N = \pm \kappa / \alpha \), suggests that the number of particles in the ground state of the free theory is zero when the field parameter \( \kappa \) is zero, as it should be in a free theory. In compliance with this, for \( \kappa = 0 \) the definition of the field \( \tilde{\Psi} \) is not valid, so there is no interaction term, as reported above. For \( \kappa \neq 0 \) (the interaction field is properly defined) the exact values of \( N \) depend on the value of \( \kappa \) multiplied by the inverse of proper acceleration \( \alpha \). Unfortunately, even though our approach is not perturbative, we are not avoiding the divergence in the limit \( \kappa \to \infty \), where \( N = \infty \).

In this work, we have considered the field \( \tilde{\Psi} \) in 1 + 1 momentum space, \( (E, p) \) and, as consequence, an 1 + 1-dimensional quantum space-time, \( (T_\kappa, X_\kappa) \), emerged. However, our argument can be easily generalised to physical dimensions: \( (T_\kappa := \sqrt{N} t, X_\kappa := \sqrt{N} x^i, Y_\kappa := \sqrt{N} y^j, Z_\kappa := \sqrt{N} x^2) \) respectively. All is needed is to extend differential equation (85) to

\[
\left( \hbar^2 \frac{\partial^2}{\partial E^2} - \frac{1}{\kappa^2} \right) \tilde{\Psi}(E, \vec{p}) = 0, \quad (88)
\]

with normal mode solutions modified to

\[
u_t(E, \vec{p}) = \frac{e^{i(t E - \vec{x} \cdot \vec{p})}}{\sqrt{4\pi |\vec{x}|}}. \quad (89)
\]

Where \( \vec{p} = (p_1, p_2, p_3) \) and \( \vec{x} = (x^1, x^2, x^3) \). From this, every part of our analysis can be obtained.

The problems we address in this work are not confined only to Klein-Gordon fields. The same problems are met in every field theory. In this sense, our approach presented here, which aims to deal with the difficulties of Klein-Gordon theory, can be equally well applied to the rest of quantum field theories. The implementation of our approach to Dirac fields is left for a later communication.

ACKNOWLEDGMENTS

I acknowledge financial support from Instituto Nazionale di Ottica - Consiglio Nazionale delle Ricerche (CNR-INO).

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The differential equation \( \frac{\partial^2 \tilde{\Psi}}{\partial t^2} - \frac{1}{\kappa^2} \tilde{\Psi} = 0 \) is a wave equation with an extra term. In 2-dimensional spaces with coordinates \((x, t)\), we know that each solution of the d’Alembert wave equation \( \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) F(t, x) = 0 \) is a function in spatial part, \( x \), and \( C^\infty \)-dependent on the temporal part, \( t \), as a parameter. Of course, \( F(t, x) \) can also be regarded as a function in \( t \) that is \( C^\infty \)-dependent on \( x \) as a parameter. While the set of solutions to the Klein-Gordon equation \( (A2) \) are parameterized according to the first case, this form of parameterization for the function \( \tilde{\Psi}(E, p) \) leads the obtained frequency to acquire an imaginary part, resulting in an unstable (exponential) growth. Thus, we choose to associate with \( \tilde{\Psi}(E, p) \) the family of functions

\[
\tilde{\Psi}_p(E) = \tilde{\Psi}(E, p')|_{p' = p}, \tag{A1}
\]

which are \( C^\infty \)-dependent on \( p \) and satisfy the differential equation

\[
\frac{\partial^2 \tilde{\Psi}_p}{\partial E^2} = \left( \frac{\partial^2}{\partial E^2} - \frac{1}{\kappa^2} \right) \tilde{\Psi}_p(E). \tag{A2}
\]

The Lagrange density of this equation can be constructed by inverting the Euler-Lagrange equation and is given by

\[
\tilde{L} = \frac{1}{2} \left( \left( \frac{\partial \tilde{\Psi}}{\partial E} \right)^2 - \left( \frac{\partial \tilde{\Psi}}{\partial p} \right)^2 + \frac{1}{\kappa^2} \tilde{\Psi}^2 \right). \tag{A3}
\]

Due to the chosen parameterization of \( \tilde{\Psi}_p(E) \), its uniqueness as solution of \( (A2) \) is satisfied when the initial data \( \left( \tilde{\Psi}, \frac{\partial \tilde{\Psi}}{\partial p} \right) \) are selected on a hypersurface of constant \( p \). So far the field configuration is based only on the equation \( (A2) \), thus the shape of the fields \( \tilde{\Psi}_p(E) \) is supposed not to change under translation in (momentum) space. Therefore, from the homogeneity of the (momentum) space, Noether’s theorem provides us with a conserved current given by

\[
C_{ij} = \frac{\partial \tilde{L}}{\partial (\partial_i \tilde{\Psi})} \partial_j \tilde{\Psi} - \eta_{ij} \tilde{L}, \quad \text{with} \ {i, j} = \{E, p\}. \tag{A4}
\]

Above, we adopt the metric signature \((+,-)\). Particularly in our case, for the Lagrangian density \( (A3) \), the quantities that are “momentum” independent are

\[
C_{pp} = \frac{1}{2} \left( \left( \frac{\partial \tilde{\Psi}}{\partial E} \right)^2 + \left( \frac{\partial \tilde{\Psi}}{\partial p} \right)^2 + \frac{1}{\kappa^2} \tilde{\Psi}^2 \right) \tag{A5}
\]

\[
C_{pE} = - \left( \frac{\partial \tilde{\Psi}}{\partial E} \right) \left( \frac{\partial \tilde{\Psi}}{\partial p} \right) \tag{A6}
\]

Translating this to the Hamiltonian description is a straightforward procedure. The conjugate momentum for \( \tilde{\Psi} \) is defined as

\[
\Pi(E) = \partial_p \tilde{\Psi}_p(E), \tag{A7}
\]

thus leading to the Hamiltonian

\[
X = \frac{1}{2} \int dE \left( \left( \frac{\partial \tilde{\Psi}}{\partial E} \right)^2 + \Pi^2 + \frac{1}{\kappa^2} \tilde{\Psi}^2 \right), \tag{A8}
\]
in agreement with $c_{pp}$.

After we have defined the creation and annihilation operators, we can then express the Hamiltonian in terms of those operators

$$X = \int dt \mathbf{x}_t \left( \Phi_x^\dagger(t) \Phi_x(t) + \frac{1}{2} [\Phi_x(t), \Phi_x^\dagger(t)] \right)$$  \hspace{1cm} (A9)

The second term is proportional to $\delta(0)$, an infinite c-number. It is the sum over all modes of the zero-point “positions” $\mathbf{x}_t/2$. We cannot avoid the existence of this term, since our treatment resembles that of harmonic oscillator and the infinite c-number term is the field analogue of the harmonic oscillator zero-point energy. In case of real Klein-Gordon fields, which we investigate here, this term is identically zero. However, it should be addressed the general case in which this term survives.

By similar logic applied to the Hamiltonian $X$, we can construct an operator corresponding to the one showed in Eq. (A6)

$$T = -\int dE \tilde{\Pi} \partial_t \tilde{\Psi} = \int dt t \Phi_x^\dagger(t) \Phi_x(t)$$  \hspace{1cm} (A10)

Unlike the position case, no $\delta(0)$ term appears here. The modes $t$ and $-t$ compensate each other, so the vacuum is translationally invariant.