REPLY TO PARIS’S COMMENTS ON EXACTIFICATION OF
STIRLING’S APPROXIMATION FOR THE LOGARITHM OF THE
GAMMA FUNCTION

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Abstract. In a recent paper [1] Paris has made several comments concerning the author’s recent work on the exactification of Stirling’s approximation for the logarithm of the gamma function, ln Γ(z). Despite acknowledging that the calculations in Ref. [2] are basically correct, he claims that there is no need for the concept of regularisation when determining values of ln Γ(z) from its complete asymptotic expansion. Here it is shown that he has already applied the concept at the beginning of the analysis in Ref. [1]. Next he claims that the definition used in Ref. [2] for the Stokes multiplier in the subdominant part of a complete expansion, which is responsible for demonstrating that the Stokes phenomenon is discontinuous rather than a smooth transition, is not correct. It is shown that the Stokes multiplier used in Ref. [2] is entirely consistent with Berry’s description of the conventional view of the Stokes phenomenon in his famous work where he developed the smoothing view [15]. Furthermore, it is pointed out that Paris has not substantiated the smoothing view by at least reproducing or improving the results in Table 7 of Ref. [2], which not only confirm the jump discontinuous nature of the Stokes phenomenon, but also provide accurate values of ln Γ(z) to 30 figures, regardless of the size of the variable or whether the truncation is optimal or not. Finally, the issue of computational expediency of MB-regularised values over Borel-summed values is discussed since Paris is critical of the Borel-summed forms being used in computations.

Keywords: Asymptotic series, Asymptotic form, Borel summation, Cauchy integral, Complete asymptotic expansion, Conditional convergence, Conventional view, Discontinuity, Divergence, Exactification, Gamma function, Mellin-Barnes regularisation, Regularisation, Remainder, Smoothing, Stokes line, Stokes multiplier, Stokes phenomenon, Stokes sector, Stirling’s approximation

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1. Introduction

Recently Paris [1] has made several claims concerning a recent work [2] which deals with the exactification of Stirling’s approximation for the first time ever since its discovery almost three centuries ago. Exactification means that a complete asymptotic expansion for a function/integral has undergone regularisation so that it is able to provide exact values of the original function for all values, including arguments or phases, of the power variable. Rather than dealing with the gamma function, Γ(z), the work concentrated on Stirling’s approximation being applied to the more complex and multivalued logarithm of

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the gamma function, viz. \( \ln \Gamma(z) \). The purpose of this article is to refute the more important claims made by Paris, whilst at the same time elucidating the reader on fundamental aspects not presented in Ref. [2] in order that they may gain a better understanding of it.

Before we can address Paris’s claims specifically, we need to present background or preliminary material that will allow the reader to decide on their own as to who is correct. Ref. [2] begins with a derivation of the complete form of Stirling’s approximation for \( \ln \Gamma(z) \), which is given as Eq. (12). This result is

\[
\ln \Gamma(z) - \left( z - \frac{1}{2} \right) \ln z + z - \frac{1}{2} \ln(2\pi) \equiv 2z \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} \frac{\Gamma(2k) \zeta(2k)}{(2\pi z)^{2k}}.
\]

Usually, the asymptotic series on the rhs of the above statement, which is denoted by \( S(z) \) in Ref. [2], is neglected and hence, the remaining terms on the lhs become the most familiar form of Stirling’s approximation. On the few occasions that the series on the rhs is included, only the first few terms are usually given, while the remaining terms are neglected. That is, the asymptotic series when displayed is often truncated at a few terms. The remainder is often denoted by \( + \cdots \) or by the appearance of the Landau gauge symbol, viz. \( O() \). When this occurs, it is known as a standard Poincaré asymptotic expansion. However, in order to obtain exact values of \( \ln \Gamma(z) \) for any value of \( z \), one requires not only all the terms in the above series, but also exponentially subdominant exponential terms that are generally discarded in standard Poincaré asymptotics. Such terms are said to lie beyond all orders of a standard asymptotic expansion [3]. The analysis resulting in (11) was not intended to be overly original, but to ensure that there were no exponentially subdominant terms in Stirling’s approximation so that exactification could be achieved. In short, the derivation of the above statement was complete.

The above statement possesses an equivalence symbol instead of an equals sign. This is because the infinite series on the rhs is divergent in certain sectors of the complex plane and conditionally convergent for the remaining sectors. In the latter case the equivalence symbol can be replaced by the more restrictive equals sign, but this is unnecessary since much of the analysis in Ref. [2] is carried out with the equivalence symbol. In those sectors where the rhs is divergent, the finite values obtained on the lhs cannot possibly equal the values obtained from the infinite series on the rhs. Therefore, the lhs is equivalent to the rhs, but it also means that the divergence must be tamed. The origin of the divergence is due to an impropriety in the method used to obtain the asymptotic series in the first place. That is, all asymptotic methods are improper as described in great detail in Refs. [4] and [5]. Moreover, the divergence results in an infinity in the remainder of the asymptotic series. As a consequence, for the rhs of the above statement or equivalence to agree with the finite lhs, the infinity in the remainder must be removed in an appropriate manner, enabling the series to become summable. This process is known as regularisation. Therefore, provided there is no truncation or the remainder has not been neglected, all asymptotic series must be regularised in order to yield a finite value. If they are truncated, then one is dealing with an approximation, which depending upon the value of \( |z| \) can be extremely accurate, but it can never equal the finite value on the lhs. For other values the approximation can be highly inaccurate, which means that one has ventured outside the range of applicability of the expansion. Hence, exactification means that there are no ranges of applicability. That is, large or small values of the power variable are no longer necessary. Thus, exactification can only be achieved with complete asymptotic expansions with all component asymptotic series undergoing regularisation.
Once the above equivalence is derived, Ref. [2] proceeds to regularise $S(z)$ in order to obtain exact values of $\ln \Gamma(z)$ for all values, including arguments, of $z$. To accomplish this, the Dirichlet series form for the Riemann zeta function is introduced and the order of the summations in the resulting equivalence are interchanged. As a consequence, the asymptotic series on the rhs or $S(z)$ can be expressed as an infinite sum of Type I terminants. Terminants were first introduced by Dingle in his remarkable, but grossly underappreciated, book on asymptotic expansions [6] because he noticed that many of the late terms in the asymptotic expansions of the special functions of mathematical physics could be approximated by them. Basically, he surmised that by studying the behaviour of the two types of terminants, one could come up with very accurate approximations for the special functions of mathematical physics. It is still an open question whether this can be achieved mainly because although the late terms become more accurate as an approximation to the asymptotic expansion, the truncated part may already be too large as discussed in Ref. [7]. Nevertheless, if a Type I terminant can be regularised, then it follows that the rhs of the above equivalence can be regularised by introducing the regularised value for each terminant in $S(z)$. This is basically the approach taken in the exactification of Stirling’s approximation. The only problem with this approach is that because an infinite sum is involved, the regularised value of $S(z)$ may also become infinite, which is why the analysis in Ref. [2] represents a departure from the analysis of asymptotic expansions with only one terminant as carried out in Ref. [4]. Thus, the theory in the latter reference, which advances the concepts and methods in Dingels’s book, was applied to $S(z)$. Once this was done, the resulting infinite series could be regularised as explained later in this article, thereby enabling exact values of $\ln \Gamma(z)$ to be calculated from its asymptotic forms.

As explained in Ref. [4], there are at present two methods that are used widely for regularising divergent series: (1) Borel summation and (2) Mellin-Barnes (MB) regularisation. A third technique based on the Euler-Maclaurin summation formula is discussed in Ref. [8], but it has yet to be studied in detail. Moreover, logarithmically divergent series need to be treated with special care [9,10], although from Lemma 2.2 in Ref. [2], one can see that the logarithmic power series can be regularised by using the regularised value of the geometric series, which also represents a crucial step in Borel summation. Since there is a plethora of techniques that can be used evaluate an integral or solve a differential equation, there is more than likely to be a host of methods of regularising divergent series or asymptotic expansions, although the exactification of Stirling’s approximation can be accomplished by Borel summation and MB regularisation. Thus, there is no need to consider other methods of regularisation, which may not be the case for other asymptotic problems.

The first example of exactification of a complete asymptotic expansion occurred in 1993 when T. Taucher and I carried out an extensive numerical investigation of the complete asymptotic expansion for the exponential series

\[ S_3(a) = \sum_{n=1}^{\infty} \exp(-a n^3), \]

which was given by

\[
S_3(a) \equiv \sum_{k=0}^{\infty} (-1)^{k+1} a^{2k+1} \frac{\Gamma(6k+4)}{\Gamma(2k+2)} \zeta(6k+4) + \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{6}) \Gamma(\frac{2}{3})} \sum_{n=1}^{\infty} \frac{e^{-\sqrt{2z}}}{(6\pi na)^{1/4}} \times \sum_{k=0}^{\infty} \frac{\Gamma(k+1/6) \Gamma(k+5/6)}{(4\sqrt{z})^k \Gamma(k+1/2)} \cos \left( \sqrt{2z} - \frac{\pi}{8} - \frac{3k\pi}{4} \right).
\]

In the above equivalence $z = 2(n\pi/3)^3 a^{-1}$, while $\zeta(s)$ represents the Riemann zeta function. This presentation differs from the original derivation in that the equivalence symbol has
replaced an equals sign. However, since the numerical investigation in Ref. [11] was concerned with positive real values of $a$, where both series are deemed to be conditionally convergent, one could use an equals sign in the analysis. In standard Poincaré asymptotics the above statement is expressed as

$$S_3(a) \sim -\frac{2\zeta(4)}{(2\pi)^4} \frac{\Gamma(4)}{\Gamma(2)} a + \frac{2\zeta(10)}{(2\pi)^{10}} \frac{\Gamma(10)}{\Gamma(4)} a^3 + O(a^5).$$

(3)

It would also be called a small $a$-expansion truncated at the third order. The second series in Equivalence (2) is referred to as being exponentially subdominant because the factor of $\exp\left(-\sqrt{2\pi} z\right)$ vanishes as $a \to 0$. These terms are the ones that are said to lie beyond all orders of the first or dominant asymptotic series [3]. Nevertheless, all these terms are required if one wishes to obtain exact values of $S_3(a)$ from Equivalence (2).

Both series in Equivalence (2) diverge when $\pi/2 < |\arg z| < \pi$, but not in the positive half of the complex plane. Despite the fact that they are conditionally convergent there, the coefficients diverge rapidly for values of $a$ greater than unity, which makes the standard asymptotic form given by Approximation (3) an inaccurate approximation when it is truncated. Moreover, even though both asymptotic series do not possess an infinity in their remainder, they can still undergo regularisation except that now there is no need to remove an infinity. Initially, our investigation concentrated on the Borel summation of both series, but it was found that the forms obtained via this method were not amenable to fast computation. So, we devised an alternative method of regularisation, which cast the remainders of both series in the form of MB integrals. This enabled us to exploit the rapid exponential decay of the gamma function in the integrands along the imaginary axis. Consequently, we were able to obtain exact values of both series for values of $a$ ranging from 0.01 to 10 to incredible accuracy, in some cases as high as 65 figures. All the results from the MB-regularisation of Equivalence (2) were eventually presented in Secs. 7 and 8 of Ref. [11] together with a comparison of the first step in hyperasymptotics as described in Refs. [13]-[14]. The latter approach was found to be far more inferior, especially for the larger values of $a$. Despite this remarkable achievement in that for first time ever the exact values of a function had been obtained from its complete asymptotic expansion, the study has either gone largely unnoticed or even worse, been badly misunderstood.

As discussed in the preface of Ref. [11], in order to be able to extend the analysis to the complex plane, it was necessary that a complete understanding of the Stokes phenomenon would be needed, which entailed not only determining where the lines of discontinuity occurred, but also the size of the subdominant jumps in the asymptotic expansion. A major accomplishment towards this goal occurred with the publication of Ref. [4], but it concentrated only on asymptotic series that could be Borel-summed, namely both types of generalised terminants. In order to verify these results, the MB-regularised forms were also derived. Nevertheless, the advent of Ref. [4] meant that it was now possible to consider the exactification of Stirling’s approximation, which as mentioned previously, involves an infinite number of terminants. Thus, Ref. [2] represents the next stage in the development of improved methods for handling asymptotic expansions over the past two decades.

### 2. Regularisation

Now that the preliminary or background information has been presented, we can now address the comments made by Paris in Ref. [1]. The first major issue in these comments concerns Stokes smoothing or rather the Berry smoothing of the Stokes phenomenon since it is clear from Stokes’s 1858 publication [16] that he believed that discontinuities arose.
in what he described as the arbitrary constants of asymptotic expansions when they were extended over the complex plane. The second claim made by Paris is that there is no need for regularisation to appear in the analysis of asymptotic expansions. We shall discuss the second claim first because it will be helpful when addressing the first claim.

Paris [1] expresses the main asymptotic series in the Stirling approximation as

\[ \Omega(z) = \sum_{k=1}^{N-1} \frac{B_{2k}}{2k(2k-1)} + R_N(z), \]

where the remainder is given by

\[ R_N(z) = \sum_{k=1}^{\infty} \frac{1}{k} \left( e^{2\pi i k z} T_\nu(2\pi i k z) - e^{-2\pi i k z} T_\nu(-2\pi i k z) \right), \]

\[ T_\nu(z) \text{ is defined in terms of the incomplete gamma function as} \]

\[ T_\nu(z) = \frac{e^{\pi i \nu}}{2\pi i} \Gamma(\nu) \Gamma(1 - \nu, z), \]

and \( \nu = 2N - 1. \) As stated in Ref. [1], this result was first derived in Paris and Wood [17] via an MB integral approach. It was derived by a totally different approach based on Borel summation in Ref. [2], where it appears as Eq. (83) and is expressed as

\[ R^{SS}_N(z) = \frac{\Gamma(2N - 1)}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n} \left( e^{-2\pi n z i} \Gamma(2 - 2N, -2\pi n z i) \right) - e^{2\pi n z i} \Gamma(2 - 2N, 2\pi n z i) \].

In fact, the awkward presentation in Ref. [17] makes it difficult to see that both Eqs. (5) and (7) are identical. Nevertheless, this reference will be inserted in Ref. [2]. It should be noted that Ref. [17] was never used in Ref. [2] because from the outset Equivalence (11) was accompanied by the \( \sim \) symbol instead of the equivalence symbol. That is, it already began by employing standard Poincaré asymptotics, which set it on a collision course with Ref. [2] as will become evident here.

Ref. [1] begins with a major inconsistency between (1.1) and (1.3). The first statement is written as a complete asymptotic expansion, where the \( \sim \) symbol implies divergence, but then the finite terms are represented by the following statement:

\[ \Omega(z) := \log \Gamma(z) - (z - 1/2) \log z + z - (1/2) \log 2\pi. \]

At no stage does Paris discuss the meaning of the := symbol, which looks like another version of the equivalence symbol. In fact, this symbol does not appear throughout the remainder of Ref. [1]. Nevertheless, let us accept the premise that \( \Omega(z) \) is finite according to Eq. (5) and that as a result it always has a finite remainder for any value of \( N. \) This means in turn that it does not need to be regularised as claimed by Paris.

Where Paris has missed the need for regularisation is in the introduction of his so-called terminant function \( T(z) \) in Eq. (5). As explained in Ch. 21 of Dingle’s book, terminants are divergent asymptotic series whose coefficients are given by the gamma function. For example, Dingle shows by Borel summation that a Type I terminant can be expressed as

\[ \sum_{k=n}^{\infty} \frac{\Gamma(k + \alpha + 1)}{(-x)^k} = \frac{\Gamma(n + \alpha + 1)}{(-x)^n} \Lambda_{n+\alpha}(x), \quad |\arg x| < \pi, \]
where

\[ \Lambda_s(x) = \frac{1}{\Gamma(n+1)} \int_0^\infty \frac{t^x e^{-t}}{1 + t/x} \, dt , \]

The integral in Eq. (10) can be written in terms of the incomplete gamma function by using No. 3.383(10) in Ref. [18]. Then one finds that the above result reduces to

\[ \sum_{k=n}^{\infty} \frac{\Gamma(k + \alpha + 1)}{(-x)^k} = (-1)^n x^{\alpha - 1} e^{x} \Gamma(n + \alpha) \Gamma(1 - n - \alpha, x) , \]

where, in addition to the condition on \( \text{arg} \, x \), \( \Re(n + \alpha) > 0 \).

Eq. (11) also gives the impression that one does not need regularisation as Paris claims for his terminant function. Unfortunately, nothing could be further from the truth. When \( |\text{arg} \, x| > \pi/2 \), all we have to do is consider \( \Re(-x) \gg \Im(-x) \) for the terminant, in which case the positive real part of \((-x)^k\) dominates the negative value produced by the imaginary part of \((-x)^k\). Consequently, all the terms in the terminant will be positive and thus, the series will become divergent. That is, an infinity exists in the remainder. Yet the rhs of Eq. (11) is finite. In fact, the “sleight of hand” in Eq. (11) has occurred as a result of Borel summation.

Basically, Borel summation consists of a few steps. The first is that the gamma function on the lhs is replaced by its integral representation and then the order of the summation and integration are interchanged. When this is done, one arrives at

\[ \sum_{k=n}^{\infty} \frac{\Gamma(k + \alpha + 1)}{(-x)^k} = \left( \frac{1}{x} \right)^n \int_0^\infty e^{-t} t^{n+\alpha} \sum_{k=0}^{\infty} \left( \frac{-t}{x} \right)^k \, dt . \]

Hence, the terminant has been expressed in terms of the geometric series. If we replace the geometric series by \( 1/(1 + t/x) \), then we obtain Eq. (11). However, a major problem arises because according to p. 19 of Ref. [19], the geometric series is absolutely convergent for \( |t/x| < 1 \) and divergent for \( |t/x| \geq 1 \). Since \( t \) ranges from zero to infinity, this substitution cannot be made for all values of \( t \). In short, “Eq.” (11) is simply not valid.

The situation is even more complex because the geometric series is not always divergent outside the its radius of absolute convergence. To see this more clearly, let us replace \(-t/x\) in the geometric series by \( z \) for the time being. Then we can use the material in Ch. 4 of Ref. [4] or in Ref. [5], where the series is expressed as

\[ \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} \frac{\Gamma(k + 1) \cdot z^k}{k!} = \lim_{p \to \infty} \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_0^p e^{-t} t^{k} \, dt . \]

Since the integral in Eq. (13) is finite, technically, we can interchange the order of the summation and integration. In reality, an impropriety has occurred when we do this, which will be discussed in more detail shortly. For now, interchanging the summation and integration yields

\[ \sum_{k=0}^{\infty} z^k = \lim_{p \to \infty} \int_0^p e^{-t} \sum_{k=0}^{\infty} \frac{(zt)^k}{k!} \, dt = \lim_{p \to \infty} \int_0^p e^{-t(1-z)} \, dt \]

\[ = \lim_{p \to \infty} \left[ \frac{e^{-p(1-z)}}{z - 1} + \frac{1}{1 - z} \right] . \]

For \( \Re z < 1 \), the first term of the last member of the above equation vanishes and the geometric series yields a finite value of \( 1/(1 - z) \). Therefore, we see that the same value is obtained for the series when \( \Re z < 1 \) as when \( z \) lies in the circle of absolute convergence.
given by $|z| < 1$ or the unit disk. According to the definition on p. 18 of Ref. [19], this means that the series is conditionally convergent for $\Re z < 1$ and $|z| > 1$. For $\Re z > 1$, however, the first term in the last member of Eq. (14) is infinite. Since regularisation is the process of removing the infinity in the remainder of a divergent series, we remove or neglect the first term of the last member of Eq. (14). As a consequence, we are left with a finite part that equals $1/(1 - z)$, which is known as the regularised value. Hence, for all complex values of $z$ except $\Re z = 1$, we have

$$
\sum_{k=0}^{\infty} z^k \begin{cases} 
\equiv 1/(1 - z) , & \Re z > 1 , \\
= 1/(1 - z) , & \Re z < 1. 
\end{cases}
$$

(15)

At the barrier of $\Re z = 1$, the situation appears to be unclear. For $z = 1$ the last member of Eq. (14) vanishes, which is consistent with removing the infinity from $1/(1 - z)$. For other values of $\Re z = 1$, the last member of Eq. (14) is clearly undefined. This is to be expected as this vertical line represents the border between the domains of convergence and divergence for the geometric series. Nevertheless, because the finite value remains the same to the right and to the left of the barrier at $\Re z = 1$ and in keeping with the fact that regularisation is effectively the removal of the first term on the rhs of Eq. (14), we take $1/(1 - z)$ to be the finite or regularised value when $\Re z = 1$. Hence, Equivalence (15) becomes

$$
\sum_{k=0}^{\infty} z^k \begin{cases} 
\equiv 1/(1 - z) , & \Re z \geq 1 , \\
= 1/(1 - z) , & \Re z < 1. 
\end{cases}
$$

(16)

Frequently, the values of $z$ over which an asymptotic series is convergent and those over which it is divergent are not known. So, in these cases it is better to replace the equals sign by the less stringent equivalence symbol on the understanding that we may be dealing with a series that is absolutely or conditionally convergent for some values of $z$. As a result, the above statement can be succinctly expressed as

$$
\sum_{k=N}^{\infty} z^k = z^N \sum_{k=0}^{\infty} z^k \equiv \frac{z^N}{1 - z}.
$$

(17)

Such a statement is no longer an equation, but an equivalence statement or equivalence for short. Moreover, it can be extended to the binomialial series as explained in Ch. 4 of Ref. [4]. It should also be noted that the above result is only applicable when the form for the regularised value of the divergent series is identical to the form of the limiting value of the absolutely convergent series. Later in Ch. 4 of Ref. [4] the regularisation of the hypergeometric series $\sum F_1(a + 1, b + 1; a + b + 2 - x; 1)$ is discussed, which is found to possess a the regularised value for $\Re x > 0$ that is different from the limit when the series is convergent for $\Re x < 0$. Hence, in this instance the two forms must be kept separate from each other as in (16).

Now let us see what happens when we replace $z$ by $-t/x$. According to the above analysis, the geometric series in Eq. (12) is divergent whenever $\Re (-t/x) > 1$. As $t$ ranges from zero to infinity there will always be values where it is divergent provided $\Re x < 0$. For these values of $t$ and $x$ we require the regularised value of the geometric series because the series is outside the barrier of convergence. Rather than determine the exact regions where the convergent and divergent regions apply, we simply introduce Equivalence (17).
into Eq. (12). Then we obtain

$$
\sum_{k=n}^{\infty} \frac{\Gamma(k+\alpha+1)}{(-x)^k} = \left(-\frac{1}{x}\right)^n \int_0^\infty \frac{t^{n+\alpha} e^{-t}}{(1+t/x)} \, dt ,
$$

for $|\arg x| \neq \pi$. In other words, we have obtained Dingle’s result given by Eq. (9) here, although with one major difference. That is, we now have an equivalence symbol instead of an equals sign, which informs us that the series may be divergent for certain values of $x$. Even when it is convergent, the series may only be conditionally convergent, which is equally troublesome to treat. Therefore, we have seen that regularisation is crucial for handling terminants contrary to Paris’s comments. It should also be noted that if the steps in Borel summation are reversed, then an asymptotic expansion is obtained. That is, if we wish to obtain an asymptotic series from the convergent integral on the rhs of “Eq.” (11), then all we need to do is expand the denominator as a power series. This method of deriving an asymptotic series is known as the method of expanding most of the exponential and is discussed on p. 113 of Ref. [6]. Like all asymptotic methods it has an impropriety, which arises from using a power series expansion outside its radius of absolute convergence. The divergence in a series expansion obtained by employing a standard technique such as Laplace’s method or the method of steepest descent is therefore an indication that something improper has occurred. Regularisation represents the necessary corrective measure for rendering the values of the original function. Hence, this is why it is necessary that the complete Stirling approximation must be regularised to yield the exact values of $\ln \Gamma(z)$ in Ref. [2].

Since we have seen that terminants can become divergent, the question now becomes: Why does Paris not need it in his analysis in Ref. [1]? The answer to this question is that he has not used Borel summation to obtain the Borel-summed forms given in Ref. [2]. Rather he has used MB integrals in his analysis. In fact, unbeknownst to him he has invoked MB-regularisation, which is evident when one examines his “Eq”. (2.1), but before we can do this, the reader needs to understand what MB regularisation is. So, let us consider the general series given by

$$
S_I(N, z) = \sum_{k=N}^{\infty} f(k)(-z)^k ,
$$

where $N$ is referred to as the truncation parameter.

Basically, MB regularisation applies whenever the function $f(k)$ in Eq. (19) possesses the following properties: (1) as $L \to \infty$, $|f(s)| = O(\exp(-\epsilon_1 L))$ for $s = c+iL$, and $|f(s)| = O(\exp(-\epsilon_2 L))$ for $s = c-iL$, where $\epsilon_1, \epsilon_2 > 0$, (2) $-\pi < \theta = \arg z < \pi$, (3) there exists a real number $c$ such that the poles of $\Gamma(N-s)$ lie to the right of the line $N-1 < c = \Re s < N$ in the complex plane and that the poles of $f(s)\Gamma(s+1-N)\Gamma(N-s)$ is single-valued to the right of the line.

Consider the following integral along the imaginary axis given by

$$
I = (-1)^N \int_{c-i\infty}^{c+i\infty} z^s f(s)\Gamma(1+s-N)\Gamma(N-s) \, ds ,
$$

where the cut-off $c$ is given above. Eq. (20) is basically an MB integral passing through $c$ when the line contour intersects the real axis. The first two properties of $f(s)$ are required for ensuring that the modulus of the integrand in the above integral decays exponentially.
at the endpoints. That is,

\[ \left| \frac{z^s f(s)}{e^{-i\pi s} - e^{i\pi s}} \right|_{s=\pm iL} \approx |z|^c e^{\mp L\theta} e^{-\pi L |f(c \pm iL)|}. \]  

In particular, the upper limit given by \( s=c+i\infty \) decays exponentially for all values of \( \theta \) in the principal branch provided \( f(c+iL) = O(\exp(-\epsilon_1 L)) \) as \( L \to \infty \) and \( \epsilon_1 > 0 \). Similarly, the lower limit decays exponentially provided \( |f(c-iL)| = O(\exp(-\epsilon_2 L)) \) as \( L \to \infty \) and \( \epsilon_2 > 0 \). Furthermore, the integrand of the above integral is single-valued because \( \arg z \) has been confined to the principal branch of the complex plane from the second property. Since the integrand is single-valued to the right of the line \( \Re s = c \), we can close the line contour integral to the right and apply Cauchy’s residue theorem. Hence, we obtain

\[ \int_{c-i\infty}^{c+i\infty} z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds + \int_C z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds \]

\[ = 2\pi i \sum \text{Res} \{z^s f(s) \Gamma(1 + s - N) \Gamma(N - s)\}, \]

where the contour \( C \) represents the great arc contour integral closing the limits of the MB integral, while \( \sum \text{Res} \{f(s)\} \) denotes the sum of the residues of \( f(s) \) in the region bounded by the line \( \Re s = c \) and the arc contour.

In accordance with the third property of \( f(s) \) only the poles of \( \Gamma(N - s) \) lie in the region to the right of the line contour. Therefore, the residues in Eq. \( (22) \) are the simple poles of the gamma function for all positive integers greater than or equal to \( N \). Hence, Eq. \( (22) \) reduces to

\[ \int_{c-i\infty}^{c+i\infty} z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds + \int_C z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds \]

\[ = 2\pi i (-1)^N \sum_{k=N}^{\infty} (-z)^k f(k). \]

It should be noted that for the case when the series on the rhs of Eq. \( (23) \) is convergent, we have two cases; either the MB integral on the lhs and the integral along the great arc are both convergent or they are both divergent. Whichever of the two cases applies depends on the magnitude of \( \arg z \). For example, consider the MB integral for the exponential function, which is

\[ \exp(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} \, ds. \]

The above equation appears on p. 261 of Ref. \[21\] except that \( s \) has been replaced by \(-s\). According to Copson, this result is valid only for \( |\arg z| < \pi / 2 \). When evaluating the MB integral numerically by using the NIntegrate routine in Mathematica \[22\], one obtains values of \( \exp(-z) \) very quickly for \( |\arg z| < \pi / 4 \), but as \( \arg z \) approaches \( \pi / 2 \), the evaluation becomes very slow, indicating that convergence problems are arising. For \( |\arg z| > \pi / 2 \), the MB integral no longer converges. This means that we have encountered the second case where the integral along the great arc is also divergent. Hence, we have seen that the integral along the great arc does not always vanish even when the series on the rhs converges.
For the case where the complex series is divergent, Eq. (24) can be written as

\[
\sum_{k=N}^{\infty} (-z)^k f(k) - \frac{(-1)^N}{2\pi i} \int_C z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds
\]

(25)

\[
= \frac{(-1)^N}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_{N-1<c=\text{Re}s<N} z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds.
\]

If the MB integral on the rhs of Eq. (25) is defined, i.e., it yields a definite value, then the integral along the great arc must also be divergent or else the divergence in the series cannot be removed. By removing the integral in this situation, we are effectively regularising the series. As noted in Ref. [11], this is somewhat analogous to evaluating the Hadamard finite or regularised part of a divergent integral [23]. So, let us consider regularising the series. As noted in Ref. [11], this is somewhat analogous to evaluating series cannot be removed. By removing the integral in this situation, we are effectively removing the infinity in accordance with the process of regularisation.

Therefore, irrespective of whether the contour integral along the great arc yields infinity or vanishes, we arrive at the following equivalence statement:

\[
S_1(N, z) = \sum_{k=N}^{\infty} f(k)(-z)^k \equiv \frac{(-1)^N}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_{N-1<c=\text{Re}s<N} z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds.
\]

(29)

We can simplify the above result by introducing again the reflection formula for the gamma function or Eq. (27). Then we arrive at

\[
S_1(N, z) = \sum_{k=N}^{\infty} f(k)(-z)^k \equiv \int_{c-i\infty}^{c+i\infty} \int_{N-1<c=\text{Re}s<N} z^s f(s) \Gamma(1 + s - N) \Gamma(N - s) \, ds.
\]

(30)

The conditions on \(z^s f(s)\) as \(s \to c \pm i\infty\) are required to ensure that the integral in Eq. (30) is convergent. They have been included more for the sake of completeness rather than utility because the above result has to be integrated numerically in order to obtain a
Hence, \( f(k) = \Gamma(2k-1)\zeta(2k)/2(2\pi)^{2k}z \). Then according to Eq. (25), we obtain
\[
\sum_{k=N}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s-1)\zeta(2s)}{(2\pi z)^{2s-1} \sin(s\pi)} ds + \frac{1}{2\pi i} \int_{C} \frac{\Gamma(2s-1)\zeta(2s)}{(2\pi z)^{2s-1} \sin(s\pi)} ds.
\]
Note that the lower limit has been adjusted. This is required in order that the singularities of the integrand other than those for \( \Gamma(N-s) \) lie to the left of the line contour. Specifically, the line contour needs to lie to the right of the pole at \( s = 1/2 \) in \( \Gamma(2s-1) \), while the truncation parameter \( N \) must be greater than zero. We now make a change of variable by putting \( s = (s' + 1)/2 \) in the MB integral on the rhs of the above equation. This yields
\[
\sum_{k=N}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} = \frac{1}{4\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma(s')\zeta(s' + 1)}{(2\pi z)^{s'} \sin((s' + 1)\pi/2)} ds' + \frac{1}{2\pi i} \int_{C} \frac{\Gamma(2s-1)\zeta(2s)}{(2\pi z)^{2s-1} \sin(s\pi)} ds,
\]
where \( \text{Max}[0, 2N-3] < c' = \Re s' < 2N-1 \). By introducing the reflection formula for the Riemann zeta function as given on p. 269 of Ref. [19] or No. 9.535(3) of Ref. [18], which is
\[
2^{1-s} \Gamma(s) \zeta(s) \cos\left(\frac{s\pi}{2}\right) = \pi^s \zeta(1-s),
\]
we obtain
\[
\sum_{k=N}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{z^{-s} \zeta(-s)}{(2\pi z)^{s} \sin(s\pi)} ds + \frac{1}{2\pi i} \int_{C} \frac{\Gamma(2s-1)\zeta(2s)}{(2\pi z)^{2s-1} \sin(s\pi)} ds,
\]
where the prime superscripts have been dropped. For \( N = 1 \), Eq. (35) reduces to
\[
\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{z^{-s} \zeta(-s)}{(2\pi z)^{s} \sin(s\pi)} ds + \frac{1}{2\pi i} \int_{C} \frac{\Gamma(2s-1)\zeta(2s)}{(2\pi z)^{2s-1} \sin(s\pi)} ds.
\]
This is identical to the result with which Paris begins his analysis in Ref. [1] except for the contour integral along the great arc. The problem is that the contour along the great arc does not always vanish and is divergent for certain values of \( z \). When this occurs, it

\[\text{definite value. If the integral on the rhs Eq. (30) is divergent, then numerical integration will fail.}\]

Now we let \( f(k) = |B_{2k}| z/(2k(2k-1)) \) and \( z = 1/z^2 \) in \( S_1(N, z) \) as given by Eq. (19). In order to determine the MB integral involving this \( f(k) \), we have to continue the Bernoulli numbers analytically to complex values. This is done by expressing the Bernoulli numbers in terms of integer values of the Riemann zeta function and analytically continuing the latter to complex values. From No. 9.616 of Ref. [18], we have
\[
B_{2k} = \frac{(-1)^{k-1}(2k)!}{2^{k-1}\pi^{2k}} \zeta(2k).
\]
must be reflected in the series on the lhs becoming divergent. So let us examine the series on the lhs.

By introducing Eq. (31) into the lhs of Eq. (32) and replacing the Riemann zeta function by its Dirichlet series form, we find that

$$L(N, z) = \sum_{k=N}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} = z \sum_{n=1}^{\infty} \sum_{k=N}^{\infty} \frac{(-1)^k \Gamma(2k-1)}{(2\pi n)^{2k}}.$$  

(37)

The inner series looks familiar indeed. Except for the factor of 2 inside the gamma function, it is basically a Type I generalised terminant, which we studied earlier. In Ref. [4] such terminants are referred to as generalised terminants because of the factor of 2. Specifically, a Type I generalised terminant is defined as

$$S_{p,q}^I(N, z^\beta) = \sum_{k=N}^{\infty} \Gamma(pk + q)(-z^\beta)^k.$$  

(38)

Therefore, the inner series in Eq. (37) can be represented by $S_{2-1}^I(N, -(1/2\pi n)^2)$. Let us now introduce the integral representation for the gamma function into Eq. (38) and interchange the order summation and integration. As presented on p. 156 of Ref. [4], we find that

$$S_{p,q}^I(N, z^\beta) = \int_0^\infty e^{-t} \sum_{k=N}^{\infty} (-z^\beta t^p)^k.$$  

(39)

For the above integral to be always convergent, we require that $\Re(pN + q) > 0$. Nevertheless, we have arrived at the geometric series, which has already been analysed here. In fact, from Equivalence (16) we know that

$$\sum_{k=N}^{\infty} (-z^\beta t^p)^k \begin{cases} (-z^\beta t^p)^N / (1 + z^\beta t^p), & \Re(-z^\beta t^p) \geq 1, \\ (-z^\beta t^p)^N / (1 + z^\beta t^p), & \Re(-z^\beta t^p) < 1. \end{cases}$$  

(40)

For the asymptotic series of the $\ln \Gamma(z)$ or $L(N, z)$, it has already been stated that $\beta = 2$ and $p = 2$. Therefore, the series is divergent or possesses an infinity, whenever $\Re(-z^2 t^2) > 0$. So there is no doubt that the integral along the great arc must also be infinite for these values of $z$ or else Eq. (35) is nonsense. Hence, MB regularisation involves the removal of the contour along the great arc, which can either be zero or infinity. Consequently, Eq. (39) can be written as

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{z^{-s} \zeta(-s)ds}{\sin(s\pi)} ds.$$  

(41)

So, we have seen that the result with which Paris began his analysis in Ref. [1] has already been regularised. More importantly, it is simply false to say that there is no need for regularisation to appear in an asymptotic series as Paris claims in Ref. [1]. All asymptotic series possess regions/sectors where they are either divergent or conditionally convergent because the method used to determine them is improper as explained in Ref. [4]. Regularisation is necessary for correcting the divergence resulting from the application of an asymptotic method. On the other hand, we have seen that an asymptotic series is not divergent everywhere. It has regions where it is convergent, although such series are conditionally convergent rather than possessing a region of absolute convergence like the geometric and binomial series. Furthermore, the above statement is not complete because
the values of arg $z$ over which it is valid have not been stated. This leads to the concept of an asymptotic form, which is discussed in the following section.

3. THE STOKES PHENOMENON AND SMOOTHING

There seems to be a line of thinking amongst the proponents of smoothing of the Stokes phenomenon that the interpretation of the Stokes phenomenon given in Refs. [2] and [4] is not the same as theirs and that as a consequence, although the calculations in these references are correct as stated by Paris [1], we are not talking about the same phenomenon. This section aims to refute such a suggestion.

Before discussing the specific comments made by Paris concerning smoothing, let us consider the following paragraphs from Berry’s famous paper on the asymptotic smoothing of Stokes’s discontinuities [15]. There it is written:

The conventional view (Stokes 19864) is powerfully (and unconventionally) argued by Dingle in I. It asserts that the change in $S$ is discontinuous and localized at the Stokes line: on one side, $S$ takes a value, $S_-$ say; on the other, $S = S_- + 1$; on the line itself, $S = S_- + 1/2$. For the example (3) the intuition behind this view is illustrated by figure 1, which shows how the steepest-descent contours of the integral $(\Im t^2 = \Im Z^2)$ change discontinuously across the Stokes line, suddenly bringing in the subdominant contribution from the stationary point at $t = 0$ ($S_- = 0$ in this case). It is worth repeating ......

From the context it is clear that Stokes is referring to asymptotic series interpreted by truncation near their least term. My aim here is to dispel Stokes’s mist and show that his discontinuity is an artefact of poor resolution: with the appropriate magnification, $S$ changes smoothly.

From the above paragraphs it is obvious that we are not just talking about a minor difference in interpretation, but a completely different view or understanding of the Stokes phenomenon with major ramifications to asymptotics. Moreover, Berry is aiming to demonstrate that since Stokes and Dingle are the main proponents of the conventional view, both of them are wrong. In fact, Paris refers to the dramatic change in the field of asymptotics as a result of Berry’s paper in Ch. 6 of his book [24]. As we shall see, this view is very disturbing because if it is true, then it would be impossible to obtain exact values of a function from its complete asymptotic expansion as has been accomplished in Refs. [2], [4], [9]-[11] and [20].

For the benefit of the reader, the smoothing effect takes place near what are known today as Stokes lines. These are lines where an asymptotic series attains maximal dominance over all other terms in the asymptotic expansion, which appears as Rule B in Ch. 1 of Dingle’s book [6]. Basically, this means that Stokes lines or rays are determined by those phases in which the terms in an asymptotic series are homogeneous in phase and all of the same sign (Rule A). The regions or sectors bordered by Stokes lines are referred to as Stokes sectors. In terms of the Type I terminant given in “Eqs.” (9) and (11), the Stokes lines occur at $\arg x = (2k + 1)\pi$, where $k$ is an integer because all the terms in the asymptotic series are all positive real numbers with no complex or imaginary parts at all. Since they occur virtually outside the principal branch, Dingle does not study the Stokes phenomenon in relation to Type I terminants in Ch. 22 of Ref. [6]. Instead, he applies the conventional view to Type II terminants, which are identical to Type I terminants except the factor of $(-1)^k$ is missing. For these series the Stokes lines occur at $\arg x = 2k\pi$ (including zero), well within the principal branch of the complex plane. In actual fact, this is a shortcoming in his great book because often an asymptotic series is in powers of $x^\beta$, where $\beta > 1$. Then the conditions for the Stokes lines become either
arg $z = (2k + 1)\pi/\beta$ for a generalised Type I terminant or arg $z = 2k\pi/\beta$ for a generalised Type II terminant. Hence, there can be many Stokes lines lying in the principal branch, which is usually taken to be $(-\pi, \pi]$. Because of this, all phases and sectors are studied for both types of generalised terminants in Ref. [4].

Not long after Berry’s paper appeared, Olver [25] claimed that he had “rigorously” proven that the smoothing did occur near a Stokes line and that Stokes had got it wrong after all. Basically, Olver re-expressed Berry’s form for the Stokes multiplier of the sub-dominant asymptotic solution of the one-dimensional Helmholtz equation, viz.

$$S_n(F) = \frac{1}{2} - \frac{i}{2\pi} P \int_0^\infty dt \frac{t^{n-\beta}}{1-t} e^{F(1-t)},$$

as

$$S_n = -\frac{1}{2\pi i} \int_{-1}^1 d\tau \frac{e^{-A(\tau-\text{ln}(1+\tau))} (1+\tau)^\mu e^{-iB\tau}}{\tau},$$

where $2z = -A - iB$, $n-1 = A + \mu$, $A$ is large, real and positive, $n$ is an integer and $B$ and $\mu$ are real. From Laplace’s method, which is a variant of the method of expanding most of the exponential mentioned earlier, one expects as $A \to \infty$ that the major contribution to the above integral comes from the neighbourhood around $\tau = 0$. Hence, Olver expands the first exponential in the above integral in powers of $\tau$ and extends the lower limit of the integral to $-\infty$. In so doing, he eventually obtains

$$S_n \sim \frac{1}{2} + \text{erf} \left( \frac{B}{\sqrt{2A}} \right) + i \frac{e^{-B^2/2A}}{\sqrt{2\pi A}} \left( \mu + \frac{1}{3} - \frac{B^2}{3A} \right) \left( \frac{1}{A} - \frac{B^2}{A^2} \right) \times \left( \frac{\mu^3}{3} - \frac{\mu^2}{2} + \frac{\mu}{6} + \cdots \right) - \frac{B}{2A} \frac{e^{-B^2/2A}}{\sqrt{2\pi A}} (\mu - \mu^2 + \cdots).$$

Hence, we see the emergence of the error function appearing in the Stokes multiplier. It represents the leading order term of complicated asymptotic expansion, which at best is conditionally convergent, but whose terms diverge in magnitude eventually. This is what is referred to as rigorous mathematics. All that has happened here is that more than the first few orders have been obtained via standard Poincaré asymptotics with the remaining terms denoted by $+ \cdots$. On the other hand, not one asymptotic result has been truncated in Ref. [2]. The only instance of truncation was to a convergent series in order to expedite the calculations and even then, it was after $10^5$ terms in the series. For a more detailed description of the terms missing in the above result, the reader should consult Ch. 6 and the appendix in Ref. [4]. So, it appears that Berry’s smoothing of the Stokes phenomenon is an “artefact of poor resolution” rather than the other way around with Stokes mist.

What analysts such as Berry, Olver and Paris fail to realise is that Dingle came up with an explanation of the Stokes phenomenon when discussing Borel summation of Type II terminants in Ch. 22 of his book. There he finds that

$$\sum_{k=n}^{\infty} \frac{\Gamma(k+\alpha+1)}{x^k} \equiv \frac{\Gamma(n+\alpha+1)}{x^n} \bar{\Lambda}_{n+\alpha}(-x), \quad \arg x = 0,$$

where

$$\bar{\Lambda}_s(-x) = \frac{1}{\Gamma(s+1)} P \int_0^\infty \frac{t^s e^{-t}}{1-t/x} dt,$$

and $P$ denotes that the Cauchy principal value must be taken. In the above we have replaced the equals sign in Dingle’s “Eq.” (16) by an equivalence symbol in accordance with the discussion on regularisation in the previous section. The above result is actually
an asymptotic form because it indicates the values of $x$ for which it is valid. Moreover, the result applies only to a Stokes line since according to the conventional view there is a jump discontinuity as $x$ moves either above or below the positive real axis. So, what is this jump discontinuity? Well, if we move to the next section in Dingle’s book, then we see the jump discontinuities emerge due to semi-circular residues of the integral in Eq. (14). So, basically we are interpreting the integral as a Cauchy integral with a line contour along the positive real axis, but as $x$ moves either above or below the positive axis, we have to take into account the semi-residues of the Cauchy integral. Dingle states for arg $x > 0$, the semi-residue is evaluated in an anticlockwise direction, while for arg $x < 0$, it is taken in a clockwise direction. As a consequence, he finds that

$$\Lambda_s(-x) = \begin{cases} 
\Lambda_s(-x) + i\pi x^{s+1}e^{-x}/\Gamma(s+1), & 0 < \text{arg} x < 2\pi, \\
\Lambda_s(-x) - i\pi x^{s+1}e^{-x}/\Gamma(s+1), & -2\pi < \text{arg} x < 0.
\end{cases}$$

In the above results $\Lambda_s(-x)$ is given by Eq. (10). In addition, the above results are virtually identical to the Plemelj relations discussed on p. 414 of Ref. [26].

We can summarise Dingle’s results by expressing them as

$$\sum_{k=n}^{\infty} \frac{\Gamma(k + \alpha + 1)}{x^k} = \frac{\Gamma(n + \alpha + 1)}{x^n} \Lambda_{n + \alpha}(-x) + \frac{2\pi i x^{s+1}e^{-x}}{\Gamma(n + \alpha + 1)} S,$$

where we take the Cauchy principal value when evaluating $\Lambda_{n + \alpha}(-x)$ for $\text{arg} x = 0$ and the factor $S$ multiplying the full residue is given by

$$S = \begin{cases} 
1/2, & 0 < \text{arg} x < 2\pi, \\
0, & \text{arg} x = 0, \\
-1/2, & -2\pi < \text{arg} x < 0.
\end{cases}$$

This indeed looks familiar. It is in fact the conventional view of the Stokes multiplier as described by Berry at the beginning of this section with $S_- = -1/2$. Magic! So, Dingle has provided us with the key to the Stokes phenomenon; we simply interpret the integrals obtained by Borel summation as Cauchy integrals and then investigate their singular behaviour. Moreover, from these results we see that it is necessary that the values of arg $z$ need to be specified for an asymptotic series. The combination of the asymptotic expansion with the values of arg $z$ over which it is valid gives rise to asymptotic forms.

As indicated earlier, the asymptotic series in the Stirling’s approximation for $\ln \Gamma(z)$ involves Type I terminants, which are not discussed in Ch. 22 of Dingle’s book [6]. For this situation we require the material in Ch. 10.1 of Ref. [4], which studies Borel summation of generalised Type I terminants given by Eq. (38). In fact, Eq. (38) is extended by introducing a factor of $\exp(-2li\pi)$ with $z^\beta$. Thus, Borel summation of a generalised Type I terminant yields

$$S^I_{p,q}(N, z^\beta e^{-2li\pi}) \equiv (-1)^N p^{-1} z^{\beta(N-1)} \int_C \frac{s^{N+q/p-1} e^{-s^{1/p}}}{s - (-z^{-\beta} e^{2li\pi})} ds.$$

If we let

$$f(s) = z^{\beta(N-1)} s^{N+q/p-1} \exp(-s^{1/p})/p,$$

then the rhs in the above result can be regarded as a Cauchy integral whose contour $C$ is the line contour along the positive real axis. Furthermore, it possesses a singularity at $s = -z^{-\beta}$. Although $\exp(2li\pi)$ is equal to unity for all values of $l$, it was found in Ch. 7 of Ref. [4] to have an effect on the MB-regularised value of $S^I_{p,q}(N, z^\beta e^{-2li\pi})$. 

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In particular, the difference between the MB-regularised values of $S_{p,q}^I(N, z^\beta e^{-2(l-1)\pi i})$ and $S_{p,q}^I(N, z^\beta e^{-2(l-1)\pi i})$ was found to be given by

$$
\Delta S_{p,q}^I(N, z^\beta) = S_{p,q}^I(N, z^\beta e^{-2(l-1)\pi i}) - S_{p,q}^I(N, z^\beta e^{-2(l-1)\pi i})
$$

(52)

$$
\equiv \int_{c^+}^{c-\infty} z^{\beta s} \Gamma(ps + q) e^{-(2l-1)i\pi s} \, ds,
$$

where the lower bound on the offset $c$ is adjusted to ensure that the poles for the MB integral remain to the right of $-q/p$, if it should be greater than $N-1$, in accordance with the conditions on $f(k)$ in Eq. (19). By making the change of variable, $y = ps + q$, we find that Equivalence (52) becomes

$$
\Delta S_{p,q}^I(N, z^\beta) \equiv p^{-1}(z^\beta e^{(2l-1)\pi i})^{q/p} \int_{c^+}^{c-\infty} (z^\beta e^{(2l-1)\pi i})^{-y/p} \Gamma(y) \, dy
$$

The MB integral can be regarded as the inverse Mellin transform of $\exp(-x)$ with $x = z^{-\beta/p} \exp((2l-1)i\pi/p)$ in Eq. (24). Consequently, the above equivalence reduces to

$$
\Delta S_{p,q}^I(N, z^\beta) \equiv \frac{2\pi i}{p} z^{-\beta q/p} e^{(2l-1)\pi \pi i/p} \exp(-z^{-\beta/p} e^{(2l-1)\pi i/p}).
$$

(53)

From p. 412 of Ref. [20] we know that the Cauchy integral on the rhs of Equivalence (50) develops jump discontinuities as $-z^{-\beta} e^{2li\pi}$ moves across the line contour. This means that while $z^{-\beta} e^{2li\pi}$ is located in a branch of the complex plane, say $(2j-1)\pi < \arg(z^{-\beta} e^{2li\pi}) < (2j+1)\pi$, the regularised value is given by the Cauchy integral, but once $z^{-\beta} e^{2li\pi}$ moves outside of this branch, it acquires extra terms or else the regularised value would be the same for all Stokes sectors. Therefore, the regularised value of $S_{p,q}^I(N, z^\beta e^{-2li\pi})$ cannot be represented solely by a Cauchy integral. As a result, another problem emerges. We need to determine the specific Stokes sector over which the Cauchy integral is the sole contribution to the regularised value. This appears to be arbitrary, much like selecting a principal branch in the complex plane. So, a primary Stokes sector must be nominated. This is taken to be the $j = l = 0$ branch of the complex plane. Consequently, the Cauchy integral on the rhs of Equivalence (50) represents the regularised value only for $-\pi/\beta < \arg z < \pi/\beta$. That is, the $j = l = 0$ branch reduces to the principal branch of the complex plane when $\beta = 1$, thereby yielding the regularised value of the first type of terminant as given on p. 406 of Dingle’s book [6]. Hence, for $l = 0$ we arrive at

$$
S_{p,q}^I(N, z^\beta) \equiv (-1)^N p^{-1} z^{\beta(N-1)} \int_{c^+}^{c-\infty} \frac{s^{N+q/p-1} e^{-s/p}}{s - (-z^{-\beta})} \, ds,
$$

(55)

which is only valid for $-\pi/\beta < \arg z < \pi/\beta$.

We are now in a position to determine the jump discontinuities that apply over the other or secondary Stokes sectors in the complex plane. First, we note that the Cauchy integral in Equivalence (50) is singular whenever $\arg z^{-\beta} = (2j+1)\pi$, and $j$ is an integer. As $z^{-\beta} \exp((2l-1)i\pi)$ crosses from the primary sector Stokes sector to the adjacent sector or $j = l + 1$ branch of the complex plane, $-z^{-\beta} e^{2li\pi}$ moves from below the line contour or positive real axis to above the axis. During this transition the Cauchy integral becomes undefined when $-z^{-\beta} e^{2li\pi}$ is situated on the positive real axis. To evaluate the residue, let us consider an infinitesimal circle around the pole at $s = -z^{-\beta} \exp(-2li\pi)$. At this stage, we shall not be concerned with whether we are considering a complete rotation or a semi-circular rotation around the pole. Nor will we be concerned with the direction of the indentation. Hence, we shall assume that the infinitesimal indentation begins at an
angle, $\gamma_1$ in the complex plane, and ends at another angle, $\gamma_2$. Then we find that the contribution from the pole from the Cauchy integral in Equivalence (50) is given by

$$I^l = ip^{-1}(-1)^N z^{\beta(N-1)} \lim_{t \to 0} \int_{\gamma_1}^{\gamma_2} \left(z^{-\beta} e^{i(2l-1)t\pi/p}\right)^{N+q/p-1} \times \exp\left(-z^{-\beta/p} e^{i(l-1)t\pi/p}\right) \, d\gamma = i(-1)^N \Delta \gamma f(s)\big|_{s=-z^{-\beta} \exp((2l-1)it\pi/p)},$$

(56)

where $\Delta \gamma = \gamma_2 - \gamma_1$. Therefore, we have seen that $f(s)$ or Eq. (51) with $s = -z^{-\beta} \exp((2l-1)it\pi)$ represents the residue of the Cauchy integral. Moreover, if $\Delta \gamma = 2\pi$, which corresponds to a complete rotation in a clockwise direction, then we see that the residue of the Cauchy integral given by Equivalence (50) is identical to the difference of the regularised values for $S_{p,q}^l(N, z^{\beta} e^{-2it\pi})$ and $S_{p,q}^l(N, z^{\beta} e^{-2l(l-1)it\pi})$ or Equivalence (54). Therefore, the above result confirms the remarkable insight made by Dingle on p. 412 of Ref. [6] that the jump discontinuity due to crossing Stokes sectors is dependent upon the singular behaviour of the Cauchy integral that emerges from the introduction of the regularised value of the geometric series during Borel summation.

By putting $l=1$ in Equivalence (54), we arrive at

$$S_{p,q}^l(N, z^{\beta} e^{-2it\pi}) - S_{p,q}^l(N, z^{\beta}) = \frac{2\pi i}{p} z^{-\beta q/p} e^{i\pi q/p} \exp\left(-z^{-\beta/p} e^{i\pi q/p}\right).$$

(57)

Since the above equivalence is valid only for $-\pi/\beta < \arg z < \pi/\beta$, we can replace $S_{p,q}^l(N, z^{\beta})$ by its regularised value since the Cauchy integral is also valid over this sector. Hence, Equivalence (57) becomes

$$S_{p,q}^l(N, z^{\beta} e^{-2it\pi}) \equiv (-1)^N p^{-1} z^{\beta(N-1)} \int_C \frac{s^{N+q/p-1} e^{-s^{1/p}}}{s - (-z^{-\beta})} \, ds$$

$$+ \frac{2\pi i}{p} z^{-\beta q/p} e^{i\pi q/p} \exp\left(-z^{-\beta/p} e^{i\pi q/p}\right).$$

(58)

Now we replace $z \exp(-2it\pi/\beta)$ by $z_*$, which yields

$$S_{p,q}^l(N, z^{\beta}_*) \equiv (-1)^N p^{-1} z^{\beta(N-1)} \int_C \frac{s^{N+q/p-1} e^{-s^{1/p}}}{s + z^{-\beta}} \, ds$$

$$+ \frac{2\pi i}{p} z^{-\beta q/p} e^{i\pi q/p} \exp\left(-z^{-\beta/p} e^{i\pi q/p}\right),$$

(59)

where $-3\pi/\beta < \arg z_* < -\pi/\beta$ and $-\pi/\beta < \arg z < \pi/\beta$. The terms on the rhs of Equivalence (59) have been left in terms of $z$ to emphasise the fact that when they are evaluated, they are done so in the primary Stokes sector. That is, the regularised values of the series on the lhs of Equivalence (59) for values of $z_*$ lying in the sector of $(-3\pi/\beta, -\pi/\beta)$ are determined by evaluating the terms on the rhs for the corresponding values of $z_*$ lying in the primary Stokes sector. This anomaly arises from the fact that if software packages such as Mathematica are used to carry out calculations of the rhs of the above equivalence in determining regularised value of the Type I generalised terminant outside the primary sector, then they will only do so for values of the complex variable lying in the principal branch of the complex plane. That is, in order to obtain regularised value outside the primary sector we need to evaluate forms where the complex variable lies inside it.

To make Equivalence (59) appear less awkward, we replace $z_*$ and $z$, respectively by $z$ and $z_1$, where the latter is defined as $z_1 = z \exp(2i\pi/\beta)$. Then the above equivalence can
be re-written as

$$\begin{align*}
S^I_{p,q}(N, z^\beta) &\equiv (-1)^N p^{-1} z_1^{\beta(N-1)} \int_C \frac{s^{N+q/p-1} e^{s1/p}}{s - (-z_1^{-\beta})} \, ds \\
&+ \frac{2\pi i}{p} z_1^{-\beta q/p} e^{i\pi q/p} \exp\left(-z_1^{-\beta/p} e^{i\pi/p}\right),
\end{align*}$$

(60)

where $-3\pi/\beta < \arg z < -\pi/\beta$. It is this result that is used to derive the regularised value of $S^I_{p,q}(N, z^\beta)$ for any Stokes sector as a result of the continuous rotation of $z^\beta$ in the complex plane.

So far, we have only been concerned with Stokes sectors, but the behaviour on the Stokes lines, where the singularity of the Cauchy integral is situated, is also important. This issue has been completely overlooked by Paris’s analysis in Ref. [1]. To obtain the Borel-summed regularised value of $S^I_{p,q}(N, z^\beta)$ on the Stokes line given by $\arg z = -\pi/\beta$, we need to invoke Rule 8a presented in Ch. 3 of Ref. [3]. The rules presented in this reference are virtually a re-expression of those appearing in Ch. 1 of Dingle’s book [6].

Rule 8a states that the asymptotic form along a Stokes line is the average of the asymptotic forms in the adjoining Stokes sectors except that the principal value must be evaluated.

Although there is no statement about the Cauchy principal value in Dingle’s rules, “Eq.” (62) demonstrates that it is implied. Therefore, the Borel-summed regularised value of a Type I generalised terminant for arg $z = -\pi/\beta$ is given by

$$\begin{align*}
S^I_{p,q}(N, z) &\equiv -p^{-1} |z|^\beta(N-1) P \int_0^\infty \frac{t^{N+q/p-1} e^{-t1/p}}{t - |z|^{-\beta}} \, dt \\
&+ \frac{\pi i}{p} |z|^{-\beta q/p} \exp\left(-|z|^{-\beta/p}\right) .
\end{align*}$$

(61)

From this result we see that the regularised value is composed of the regularised value in the primary Stokes sector except that now the principal value of the Cauchy integral must be evaluated and only the semi-residue contribution is taken in a clockwise direction ($\Delta \gamma = \pi$) of the $l=1$ version of Eq. (60).

If we put $l=2$ in Equivalence (52), then with the aid of Equivalence (58) we find that

$$S^I_{p,q}(N, z^\beta e^{-4i\pi}) - S^I_{p,q}(N, z^\beta e^{-2i\pi}) = \frac{2\pi i}{p} z^{-\beta q/p} e^{3i\pi q/p} \exp\left(-z^{-\beta/p} e^{3i\pi q/p}\right) .$$

(62)

We can replace $S^I_{p,q}(N, z^\beta e^{-2i\pi})$ by introducing Equivalence (58) into the above result. This yields

$$\begin{align*}
S^I_{p,q}(N, z^\beta e^{-4i\pi}) &\equiv (-1)^N p^{-1} z_1^{\beta(N-1)} \int_C \frac{s^{N+q/p-1} e^{-s1/p}}{s - (-z_1^{-\beta})} \, ds + \frac{2\pi i}{p} z^{-\beta q/p} \\
&\times e^{3i\pi q/p} \exp\left(-z^{-\beta/p} e^{3i\pi q/p}\right) + \frac{2\pi i}{p} z^{-\beta q/p} e^{i\pi q/p} \exp\left(-z^{-\beta/p} e^{i\pi q/p}\right).
\end{align*}$$

(63)

Next we replace $z \exp(-4i\pi/\beta)$ by $z$ on the lhs, while on the rhs $z$ is replaced by $z_2$, the latter now being equal to $z \exp(4i\pi/\beta)$. Hence, we arrive at

$$\begin{align*}
S^I_{p,q}(N, z^\beta) &\equiv (-1)^N p^{-1} z_2^{\beta(N-1)} \int_C \frac{s^{N+q/p-1} e^{-s1/p}}{s + z_2^{-\beta}} \, ds + \frac{2\pi i}{p} z_2^{-\beta q/p} \\
&\times \sum_{j=1}^2 e^{i(2j-1)q\pi/p} \exp\left(-z_2^{-\beta/p} e^{i(2j-1)\pi/p}\right),
\end{align*}$$

(64)
where $-(2(2)+1)\pi/\beta < \arg z < -(2(2)-1)\pi/\beta$ or $-5\pi/\beta < \arg z < -3\pi/\beta$. When $z$ lies on the Stokes line that borders the $l=1$ and $l=2$ sectors, i.e. where $\arg z = -3\pi/\beta$, all we need to do is average Equivalences (63) and (64) and take the Cauchy principal value of the resulting contour integral. Before we can average the two equivalences we must replace $z_1$ in Equivalence (63) by $z\exp(2i\pi/\beta)$ and $z_2$ in Equivalence (63) by $z\exp(4i\pi/\beta)$ so that $z$ is the same variable in both equivalences. Then taking the average of the two modified equivalences and setting $\arg z$ equal to $-3\pi/\beta$, we find that the regularised value of the Type I generalised terminant is given by

$$S^I_{p,q}(N,z^\beta) \equiv -p^{-1}|z|^\beta(N-1)P \int_0^\infty \frac{t^{N+q/p-1}e^{-t/\beta}}{t-|z|^{-\beta}} \, dt + \frac{2\pi i}{p} |z|^{-\beta q/p} e^{2i\pi q/p} \left\{ \exp\left(-|z|^{-\beta/p} e^{2i\pi q/p} \right) + \frac{\pi i}{p} |z|^{-\beta q/p} \exp\left(-|z|^{-\beta/p} \right) \right\} .$$

A pattern is now emerging that will allow us to determine the regularised value of a Type I generalised terminant for the Stokes sectors due to clockwise rotations of $z$. We simply replace the upper limit 2 by $M$ in the sum on the rhs of Equivalence (64). As a consequence, we find that the regularised value of $S^I_{p,q}(N,z^\beta)$ for $-(2M+1)\pi/\beta < \arg z < -(2M-1)\pi/\beta$ is given by

$$S^I_{p,q}(N,z^\beta) \equiv (-1)^N z_M^\beta(N-1)P^{-1} \int_0^\infty \frac{t^{N+q/p-1}e^{-t/\beta}}{t+z_M^{-\beta}} \, dt + \frac{2\pi i}{p} z_M^{-\beta q/p} \sum_{j=1}^M e^{i(2j-1)\pi q/p} \exp\left(-z_M^{-\beta/p} e^{i(2j-1)\pi q/p} \right) ,$$

where $z_M = z\exp(2Mi\pi/\beta)$. This equivalence, which appears as (32) in the beginning of the proof to Thm. 2.1 in Ref. [2], represents the base or platform for the exactification of Stirling’s approximation for positive values of $\arg z$. Note that negative values of $\arg z$ correspond to positive values of $\arg z$ in Stirling’s approximation since the asymptotic series in the latter, viz. $S(z)$, is in inverse powers of $z$. On the Stokes line given by $\arg z = -(2M+1)\pi/\beta$, one takes the average of the regularised values for the two adjacent Stokes sectors bordered by the line and replaces the Cauchy integral by its principal value. After a little algebra one finds that the regularised value of the Type I generalised terminant on the Stokes line simplifies to

$$S^I_{p,q}(N,z^\beta) \equiv -|z|^\beta(N-1)P^{-1} \int_0^\infty \frac{t^{N+q/p-1}e^{-t/\beta}}{t-|z|^{-\beta}} \, dt + \frac{2\pi i}{p} |z|^{-\beta q/p} \sum_{j=1}^M e^{2ij\pi q/p} \exp\left(-|z|^{-\beta/p} e^{2ij\pi q/p} \right) + \frac{\pi i}{p} |z|^{-\beta q/p} \exp\left(-|z|^{-\beta/p} \right) .$$

This equivalence appears as (45) in the proof of Thm. 2.1 in Ref. [2].

For negative values of $\arg z^{-\beta}$ in the complex plane, the rotations of $z^{-\beta}$ are clockwise. To obtain the regularised value of a generalised Type I terminant for this case, we put $l=0$ in Equivalences (62)-[54], which yields

$$S^I_{p,q}(N,z^\beta e^{2i\pi}) - S^I_{p,q}(N,z^\beta) \equiv -\frac{2\pi i}{p} z^{-\beta q/p} \sum_{j=1}^M e^{2ij\pi q/p} \exp\left(-|z|^{-\beta/p} e^{i\pi q/p} \right) .$$
Hence, we see that the regularised value for the lower Stokes sector given by $S'_{p,q}(N, z^β)$ is related to the regularised value for the Stokes sector immediately above, viz. $S'_{p,q}(N, z^β \exp(2iπ))$, plus the residue contribution of the Cauchy integral taken in a clockwise direction. In fact, the regularised values of all lower Stokes sectors are related to the regularised values for the Stokes sectors immediately above them plus the residue contributions of the Cauchy integral taken in a clockwise direction. Again, this is consistent with the conventional view of the Stokes phenomenon. Furthermore, the above result represents the complex conjugate of Equivalence (57). Since Equivalence (68) is only valid over the primary Stokes sector, i.e. for $-π/β < \arg z < π/β$, we can introduce the regularised value of $S'_{p,q}(N, z^β)$ from Equivalence (55) into it. Then by replacing $z \exp(2iπ/β)$ in the resulting equivalence with $z$, one obtains

$$S'_{p,q}(N, z) \equiv (-1)^N p^{-1} z^{β(N-1)} \int_{C} ds \frac{s^{N+q/p-1}}{s - (-z^{-1})} e^{-s^{1/p}}$$

(69)

$$- \frac{2πi}{p} z^{-βq/p} e^{-iβq/p} \exp\left(-z^{-1}_1/ e^{-iπ/p}\right),$$

where $π/β < \arg z < 3π/β$ and $z^{-1}_1 = z \exp(-2iπ/β)$.

For the Stokes line of $\arg z = π/β$, we first re-write the rhs of Equivalence (69) in terms of $z$. Then we average the resulting equivalence with Equivalence (55). Next we take the Cauchy principal value of the resulting integral in accordance with Rule 8a in Ch. 3 of the primary Stokes sector, i.e. for the complex conjugate of Equivalence (57). Since Equivalence (68) is only valid over the conventional view of the Stokes phenomenon. Furthermore, the above result represents the Cauchy integral taken in a clockwise direction. Again, this is consistent with the values for the Stokes sectors immediately above them plus the residue contributions of the Cauchy integral taken in a clockwise direction. Setting $z$ equal to $z \exp(-4iπ/β)$ and introducing the appropriate version of Equivalence (50), one eventually obtains

$$S'_{p,q}(N, z^β e^{4iπ}) - S'_{p,q}(N, z^β e^{2iπ}) \equiv -\frac{2πi}{p} z^{-βq/p} e^{-3iπ/β} \exp\left(-z^{-1}_1/ e^{-3iπ/β}\right).$$

(71)

We can express $S'_{p,q}(N, z^β \exp(2iπ))$ in terms of $S'_{p,q}(N, z^β)$ by putting $l=0$ in Equivalences (52) and (54). This gives

$$S'_{p,q}(N, z^β e^{4iπ}) \equiv S'_{p,q}(N, z^β) - \frac{2πi}{p} z^{-βq/p} e^{-3iπ/β} \exp\left(-z^{-1}_1/ e^{-3iπ/β}\right).$$

(72)

where $(2(2)-1)π/β < \arg z < (2(2)+1)π/β$. Setting $z$ equal to $z \exp(-4iπ/β)$ and introducing the appropriate version of Equivalence (50), one eventually obtains

$$S'_{p,q}(N, z^β) \equiv (-1)^N p^{-1} z^{β(N-1)} \int_{C} ds \frac{s^{N+q/p-1}}{s + z^{-2}_1} e^{-s^{1/p}}\ exp\left(-z^{-1}_2/ e^{-iπ/β}\right)$$

(73)

$$\times \sum_{j=1}^{2} e^{-i(2j-1)π/β} \exp\left(-z^{-1}_2/ e^{-i(2j-1)π/β}\right).$$

20
which, as expected, is the complex conjugate of Equivalence (64). For \( \arg z = 3\pi / \beta \), we first express the rhs’s of Equivalences (63) and (66) in terms of \( z \) rather than \( z_{-1} \) and \( z_{-2} \). Then we average the resulting equivalences and evaluate the Cauchy principal value of the resulting contour integral. Alternatively, we can take the complex conjugate of Equivalence (65). Hence, we find that

\[
S_{p,q}^I(N, z^\beta) \equiv -p^{-1} |z|^{\beta(N-1)} P \int_0^\infty \frac{t^{N+q/p-1} e^{-t^{1/p}}}{t - |z|^{-\beta}} dt - \frac{2\pi i}{p} |z|^{-\beta q/p} \exp(-|z|^{-\beta/p}) \exp(-|z|^{-\beta/p}) .
\]

(74)

From the above we see that a similar pattern is emerging as in the case of the anti-clockwise rotations of \( z^{-\beta} \). Therefore, by extending Equivalences (63) and (74) we are able to derive general forms for the Borel-summed regularised values of a Type I generalised determinant for any Stokes sector or line, where \( \arg z > 0 \). In particular, the generalisation of Equivalence (63) to \( (2M+1)\pi / \beta < \arg z < (2M+1)\pi / \beta \), where \( M > 0 \), can be carried out simply by replacing 2 with \( M \). Thus, we arrive at

\[
S_{p,q}^I(N, z^\beta) \equiv (-1)^N z_{-M}^{\beta(N-1)} p^{-1} \int_0^\infty \frac{t^{N+q/p-1} e^{-t^{1/p}}}{t + z_{-M}^{-\beta}} dt - \frac{2\pi i}{p} z_{-M}^{-\beta q/p} \sum_{j=1}^M e^{-(2j-1)q\pi/p} \exp(-z_{-M}^{-\beta/p} e^{-(2j-1)\pi/p}) .
\]

(75)

In the above result \( z_{-M} = z \exp(-2Mi\pi / \beta) \). Hence, we see that Equivalence (75) represents the complex conjugate of Equivalence (63). This is essentially the equivalence given as (41) in the proof of Thm. 2.1 of Ref. [2]. For the Stokes line where \( \arg z = (2M+1)\pi / \beta \), the generalisation of Equivalence (74) yields

\[
S_{p,q}^I(N, z^\beta) \equiv -|z|^{\beta(N-1)} p^{-1} \int_0^\infty \frac{t^{N+q/p-1} e^{-t^{1/p}}}{t - |z|^{-\beta}} dt - \frac{2\pi i}{p} |z|^{-\beta q/p} \sum_{j=1}^M e^{-2ij\pi/p} \exp(-|z|^{-\beta/p} e^{-2ij\pi/p}) \exp(-|z|^{-\beta/p}) .
\]

(76)

which represents the complex conjugate of Equivalence (67). This equivalence appears as the lower-signed version of (45) in the proof of Thm. 2.1 of Ref. [2].

At no stage in the preceding analysis has there been a requirement to introduce the concept of smoothing in the vicinity of a Stokes line. To observe the behaviour near a Stokes line, let us consider the vicinity of the line where \( \arg z = -(2M + 1)\pi / \beta \). For \( -(2M + 1)\pi / \beta < \arg z < -(2M)\pi / \beta \), Equivalence (65) yields

\[
S_{p,q}^I(N, z^\beta) \equiv (-1)^N z_{-M}^{\beta(N-1)} p^{-1} \int_0^\infty \frac{t^{N+q/p-1} e^{-t^{1/p}}}{t + z_{-M}^{-\beta}} dt + \frac{2\pi i}{p} z_{-M}^{-\beta q/p} \sum_{j=0}^{M-1} e^{(2j+1)q\pi/p} \exp(-z_{-M}^{-\beta/p} e^{(2j+1)i\pi/p}) .
\]

(77)
while for the next lower Stokes sector or $-(2M + 3)\pi/\beta < \arg z < -(2M + 1)\pi/\beta$, it gives

$$S_{p,q}^I(N, z^\beta) \equiv (-1)^N z^{\beta(N-1)}p^{-1} \int_0^\infty \frac{t^{N+q/p-1} e^{-t^{1/p}}}{t + z^{-\beta}} \, dt + \frac{2\pi i}{p} z^{-\beta q/p}$$

$$\times \sum_{j=0}^M e^{(2j+1)iq\pi/p} \exp\left(-z^{-\beta/p} e^{(2j+1)i\pi/p}\right).$$

(78)

For $\arg z = -(2M + 1)\pi/\beta$, Equivalence (67) yields

$$S_{p,q}^I(N, z^\beta) \equiv (-1)^N z^{\beta(N-1)}p^{-1} \int_0^\infty \frac{t^{N+q/p-1} e^{-t^{1/p}}}{t + z^{-\beta}} \, dt + \frac{2\pi i}{p} z^{-\beta q/p}$$

$$\times \sum_{j=0}^{M-1} e^{-(2j+1)iq\pi/p} \exp\left(-z^{-\beta/p} e^{-(2j+1)i\pi/p}\right) + \frac{\pi i}{p} z^{-\beta q/p} e^{-(2M+1)iq\pi/p}$$

$$\times \exp\left(-z^{-\beta/p} e^{-(2M+1)i\pi/p}\right).$$

(79)

If we let

$$F(z) = (-1)^N z^{\beta(N-1)}p^{-1} \int_0^\infty \frac{t^{N+q/p-1} e^{-t^{1/p}}}{t + z^{-\beta}} \, dt + \frac{2\pi i}{p} z^{-\beta q/p}$$

$$\times \sum_{j=0}^{M-1} e^{(2j+1)iq\pi/p} \exp\left(-z^{-\beta/p} e^{(2j+1)i\pi/p}\right),$$

(80)

where it is understood that the Cauchy principal of integral is evaluated on the Stokes line, then Equivalences (77)-(79) can be expressed as

$$S_{p,q}^I(N, z^\beta) \equiv F(z) + 2\pi i S z^{-\beta q/p} e^{-(2M+1)iq\pi/p} \exp\left(-z^{-\beta/p} e^{-(2M+1)i\pi/p}\right),$$

with the multiplier $S$ of the subdominant exponential term being given by

$$S = \begin{cases} 0, & -(2M + 1)\pi/\beta < \arg z < -(2M - 1)\pi/\beta, \\ 1/2, & \arg z = -(2M + 1)\pi/\beta, \\ 1, & -(2M + 3)\pi/\beta < \arg z < -(2M + 1)\pi/\beta. \end{cases}$$

(82)

This is simply the conventional view as described by Berry in Ref. [15] and presented at the beginning of this section. In this instance it is clear that $S_+ = 0$ for a Type I generalised terminant when the primary Stokes sector is given by $|\arg z| < \pi/\beta$. Now let us turn our attention to the asymptotic series $S(z)$ appearing in the complete version of Stirling’s “approximation” for $\ln \Gamma(z)$. This series in its entirety, i.e. without any truncation whatsoever, is found in Ref. [2] to be

$$S(z) = z \sum_{k=1}^\infty \frac{(-1)^k}{(2z)^{2k}} \Gamma(2k - 1) c_k(1),$$

(83)

where the cosecant polynomials for unity [8] can be expressed in terms of the Bernoulli numbers as

$$c_k(1) = \frac{(-1)^k}{(2k)!} 2^{2k} B_{2k}.$$

(84)
Later in Ref. [2] the truncation parameter $N$ is introduced with the Dirichlet series form for the Riemann zeta function. Hence, the series becomes

$$S(z) = z \sum_{k=1}^{N-1} \frac{(-1)^k}{(2z)^{2k}} \Gamma(2k-1) \ c_k(1) - 2z \sum_{n=1}^{\infty} S^{I}_{2n-1} \ (N, (1/2n\pi z)^2) .$$

The first term on the rhs of Eq. (85) is basically Paris’s $\Omega(z)$, while the second term is the frequently neglected remainder that is either divergent or conditionally convergent. Moreover, we see that the second term represents an infinite sum of Type I generalised terminants where $p=2$, $q=v-1$, $z=1/2n\pi z$ and $\beta=2$. By introducing these values into Equivalence (77), one obtains

$$S^{I}_{2n-1} \ (N, (1/2n\pi z)^2) \equiv \frac{(-1)^N}{(2n\pi z)^2N-2} \int_{0}^{\infty} \frac{y^{2N-2} e^{-y}}{y^2 + 4n^2\pi^2 z^2} \ dy - \frac{1}{2nz} \sum_{j=1}^{M} (-1)^{M-j} \exp \left(-2(-1)^{M-j} n\pi z \right).$$

Consequently, the asymptotic series for $\ln \Gamma(z)$ becomes

$$S(z) \equiv z \sum_{k=1}^{N-1} \frac{(-1)^k}{(2z)^{2k}} \Gamma(2k-1) \ c_k(1) - 2 \left( - \frac{1}{4\pi^2 z^2} \right)^N z \sum_{n=1}^{\infty} \frac{1}{n^{2N-2}} \times \int_{0}^{\infty} \frac{e^{-y} y^{2N-2}}{((y/2\pi z)^2 + n^2)} \ dy + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{M} (-1)^{M-j} \exp \left( 2(-1)^{M-j} n\pi z \right).$$

Equivalences (86) and (87) appear respectively as (33) and (34) in Ref. [2]. For those Stokes lines where $\arg (1/2n\pi z) = -(M + 1/2)\pi$ or $\theta = \arg z = (M + 1/2)\pi$, the introduction of Equivalence (79) into Eq. (85) yields

$$S(z) \equiv z \sum_{k=1}^{N-1} \frac{(-1)^k}{(2z)^{2k}} \Gamma(2k-1) \ c_k(1) + 2 \left( - \frac{1}{4\pi^2 z^2} \right)^N z \sum_{n=1}^{\infty} \frac{1}{n^{2N-2}} \times P \int_{0}^{\infty} \frac{e^{-y} y^{2N-2}}{y^2 - 4n^2\pi^2 |z|^2} \ dy - ie^{\theta} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{M} (-1)^{j} \exp \left( 2(-1)^{j} n\pi |z| \right)$$

$$- ie^{\theta} \sum_{n=1}^{\infty} \frac{1}{2n} \exp \left( -2n\pi |z| \right).$$

Since each generalised Type I terminant can be expressed in terms of a Stokes multiplier accompanied by an exponential term that is subdominant in the vicinity of a Stokes line, it follows that $S(z)$ can be expressed in terms of such a multiplier and an infinite series of subdominant exponential terms as given by the last term on the rhs of the above equivalence. If we let

$$G(z) = z \sum_{k=1}^{N-1} \frac{(-1)^k}{(2z)^{2k}} \Gamma(2k-1) \ c_k(1) - 2 \left( - \frac{1}{4\pi^2 z^2} \right)^N z \sum_{n=1}^{\infty} \frac{1}{n^{2N-2}} \times \int_{0}^{\infty} \frac{e^{-y} y^{2N-2}}{((y/2\pi z)^2 + n^2)} \ dy + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{M} (-1)^{M-j} \exp \left( 2(-1)^{M-j} n\pi z \right),$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n^{2N-2}} \times \int_{0}^{\infty} \frac{e^{-y} y^{2N-2}}{((y/2\pi z)^2 + n^2)} \ dy + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{M} (-1)^{j} \exp \left( 2(-1)^{j} n\pi |z| \right) - \sum_{n=1}^{\infty} \frac{1}{2n} \exp \left( -2n\pi |z| \right).$$
where it is understood that the Cauchy principal value is evaluated at the Stokes line, then $S(z)$ can be written as

\begin{equation}
S(z) \equiv G(z) + (-1)^M S \sum_{n=1}^{\infty} e^{2(-1)^M n \pi z n},
\end{equation}

with the Stokes multiplier given by Eq. (82). The above result is only partially regularised because the series on the rhs, which arises from the infinite number of singularities situated on the Stokes line, can become divergent. However, the series can be regularised by using Lemma 2.2 in Ref. [2], which proves that

\begin{equation}
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k \begin{cases}
\equiv \ln(1 + z) & , \quad \Re z \leq -1 , \\
= \ln(1 + z) & , \quad \Re z > -1 .
\end{cases}
\end{equation}

Then Equivalence (90) becomes

\begin{equation}
 S(z) \equiv G(z) + (-1)^M S \ln \left( 1 - e^{2(-1)^M \pi z} \right).
\end{equation}

For the Stokes line where $M = 0$, viz. $\arg z = \pi/2$, the above result reduces to

\begin{equation}
S(z) \equiv G(z) - \frac{1}{2} \ln \left( 1 - e^{2 \pi i z} \right).
\end{equation}

The second term on the rhs of both the above equivalences is referred to as the Stokes discontinuity term, denoted by $SD^+(z)$, in Ref. [2] and appears as Eq. (78) there. It is subsequently used in the numerical study of the Stokes phenomenon in Sec. 3. It is also the expression which Paris claims is not the correct interpretation of the Stokes phenomenon. Yes, it is not the interpretation according to the smoothing view espoused by Berry and Olver, but it is, nonetheless, totally consistent with the conventional view of the Stokes phenomenon as stated in Ref. [2].

To summarise the preceding analysis, we have seen that the subdominant exponential term in the Borel-summed regularised value of a generalised terminant emerges as a result of a singularity being situated on a Stokes line. In the case of the asymptotic series $S(z)$ for $\ln \Gamma(z)$, there is an infinite number of generalised terminants with coinciding Stokes lines. As a consequence, each Stokes line for $S(z)$ possesses an infinite number of singularities, albeit located at different positions. This produces an infinite number of subdominant exponential terms, each accompanied by the same Stokes multiplier. The multiplier is entirely consistent with the conventional view of the Stokes phenomenon. The interesting property of the resulting sum of the residues due to the singularities is that it need not necessarily be absolutely convergent and thus, may require regularisation. It is for this reason that the exactification of Stirling’s approximation for $\ln \Gamma(z)$ is a challenging problem in asymptotics, which is discussed in the introduction of Ref. [2]. Despite this, no one can be certain that the above analysis is indeed correct until an effective numerical study has been carried out. After all, we have seen that Olver claimed that his derivation of a smoothed Stokes multiplier, viz. “Eq.” (39), was based on rigorous mathematics. Yet he did not provide an indication as to just how accurate his result is and for what range of values it is valid. In actual fact, the notion of a proof does not really apply here because as we shall soon see, we are talking about encountering discontinuities arising from singularities in Cauchy integrals and introducing an approach or method to handle them as they occur.

If anyone possesses an incorrect interpretation of the Stokes multiplier, then it is surely Paris himself, who seems to believe that the Stokes multiplier should only multiply the
leading subdominant exponential in a complete asymptotic expansion. Neither Stokes nor Dingle were concerned with the leading exponential in a complete asymptotic expansion, although it must be said that they did not study situations where a Stokes line possesses an infinite number of singularities. In addition, the Berry/Olver derivation of Stokes smoothing presented earlier does not isolate the leading exponential in the subdominant part of an asymptotic solution at the expense of the other terms. Despite this, it is not incorrect to isolate or factor out the leading exponential from all the other subdominant exponential terms and then to refer to all the remaining terms as a multiplier, even though it is not what Dingle and Berry originally had in mind.

The problem with Paris’s approach like that of Berry and Olver is that in order to come up with improved results on standard Poincaré asymptotics is that they have been forced to truncate at some stage in their analyses. As a result, we see the standard asymptotic constructs such as the $\sim$ symbol and $+ \cdots$, appear in Paris’s results, which are given by (3.1) to (3.4) in Ref. [1]. Thus, Paris’s version of the Stokes multiplier is given as

$$S(\theta) \sim \frac{1}{2} + \frac{1}{2} \text{erf}\left(\frac{c(\theta + \pi/2) \sqrt{\pi|z|}}{2\pi^{\frac{3}{2}}|z|^\frac{1}{2}}\right) - \frac{C_0 e^{-2\pi\gamma|z|}}{2\pi^{\frac{3}{2}}|z|^\frac{1}{2}} i,$$

where $\gamma = 1 + i \exp(i\theta)$, $C_0 = B_0 \exp(-2\pi\omega|z|) + \exp(i\omega\nu)/(1 + \exp(-i\omega))$,

$$B_0 = \frac{e^{-i\omega\alpha}}{1 - e^{-i\omega}} + \frac{1}{c(\theta + \pi/2)} i,$$

$\alpha = 2N_0 - 1 - 2\pi|z|$, $\nu = 2N_0 - 1$, $N_0$ is the optimal point of truncation and

$$c(\theta + \pi/2) = \omega + \frac{1}{6} i\omega^2 - \frac{1}{36} \omega^3 + \frac{1}{270} i\omega^4 + \cdots.$$

Consequently, the remainder is expressed as

$$R_{N_0}(z) \approx e^{2\pi iz} T_\nu(2\pi iz) - e^{-2\pi iz} T_\nu(-2\pi iz) \sim e^{2\pi iz} S(\theta).$$

In essence, Paris’s “refined version” of $S(\theta)$ is no different from the result given by (82) in Ref. [2] except that there is now an extra imaginary term. This term was neglected in Ref. [2] as an unnecessary complication since it is not possible to obtain hyperasymptotic values of the Stokes multiplier, e.g. to 30 figures, near the Stokes line at $\arg z = \pi/2$, anyway. In fact, the imaginary part raises another problem because imaginary parts for the multiplier simply do not appear in the conventional view of the Stokes phenomenon. So, if the smoothing view is correct, then it means that extra terms need to be included in the conventional view in order to obtain exact values of $\ln \Gamma(z)$ from its asymptotic forms. Note that since this analysis employs symbols such as $\sim$ and $+ \cdots$, it results in the usual drawbacks associated with standard Poincaré asymptotics, viz. vagueness and limited range of applicability. As discussed in the prologue to Ref. [3], these two drawbacks are responsible for giving the discipline a bad name. On the other hand, none of these symbols has been introduced in the derivation of the Borel-summed and MB-regularised asymptotic forms given in Ref. [2].

The whole point about the numerical study in Ref. [2] is to determine those values of $\arg z$ near the Stokes line, where Paris’s form for the Stokes multiplier deviates the most from the conventional view. Specifically, these values are determined by plotting the real part of Approximation (94) as in Figs. 1 of Refs. [1] and [2]. According to Berry, Olver and Paris, one should not obtain exact values of $\ln \Gamma(z)$ for these values of $\arg z$ via the conventional view of the Stokes phenomenon. As can be seen from the figures, the smoothed Stokes multiplier as given by Approximation (94) is closer to $1/2$ than being close to $0$ or $1$ as dictated by the conventional view. Hence, it stands to reason that
if Stokes smoothing is correct, then the conventional view cannot possibly give accurate values of $\ln \Gamma(z)$ in the vicinity of the Stokes line at $\arg z = \pi/2$.

Table 1 here or Table 7 in Ref. [2] presents the results obtained from programming the Borel-summed regularised results for $\ln \Gamma(z)$ as a Mathematica module [22], which appear as Program 2 in the appendix of Ref. [2]. Specifically, the module calculates $\ln \Gamma(z)$ using the following asymptotic forms:

$$(98) \quad \ln \Gamma(z) = \begin{cases} F(z) + TS_N(z) + R_{SS}^N(z) + SD_{1SS,U}^N(z) , & \pi/2 < \theta \leq \pi , \\ F(z) + TS_N(z) + R_{SS}^N(z) , & -\pi/2 < \theta < \pi/2 . \end{cases}$$

In the above equation $F(z)$ represents the standard form of Stirling’s approximation, i.e.,

$$(99) \quad F(z) = \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) ,$$

and the truncated series denoted by $TS_N(z)$ is given by

$$(100) \quad TS_N(z) = z \sum_{k=1}^{N-1} \frac{(-1)^k}{(2z)^{2k}} \Gamma(2k - 1) c_k(1).$$

Eq. (100) is basically Paris’s $\Omega(z)$. In addition, the Borel-summed remainder term denoted by $R^SS_N(z)$ was evaluated via

$$(101) \quad R^SS_N(z) = 2 \frac{(-1)^{N+1} z}{(2\pi z)^{2N-2}} \sum_{n=1}^{\infty} \frac{1}{n^{2N-2}} \int_0^\infty dy \frac{y^{2N-2} e^{-y}}{(y^2 + 4\pi^2 n^2 z^2)} ,$$

while the final term $SD_{1SS,U}^N(z)$ is the Stokes continuity term given by Eq. (93). That is,

$$(102) \quad SD_{1SS}^N(z) = - \ln \left( 1 - e^{2\pi zi} \right).$$

In order to be consistent with the “Stokes smoothing” view, a relatively large value of $|z|$ was chosen to execute the module, viz. $|z| = 3$, which has an optimal point of truncation, $N_0$, approximately equal to 10. This, however, leads to another problem. At the present stage one does not know what form the smoothed multiplier takes for small values of $|z|$. Hence, for small values of $z$, one cannot possibly obtain exact values of $\ln \Gamma(z)$ according to the smoothing view. Presumably, small $|z|$ implies that there is no optimal point of truncation, but there is no quantification from Paris on this issue. Nevertheless, no such restriction applies to any of the Borel-summed and MB-regularised asymptotic forms given here or in Ref. [2]. That is, they are equally valid for $|z| > 1$ and for $|z| \leq 1$. In addition, it should be stressed that Table 1 only presents a small sample of the results from the numerical investigation, in which numerous values of $\delta$, where $\theta = (1/2 + \delta)\pi$, were considered. The values of $\delta$ in the table were chosen because Stokes smoothing is expected to exhibit the greatest deviation from the step-function postulated in the conventional view of the Stokes phenomenon. Those very close to the Stokes line are given by $|\delta| \leq 1/100$ in the table. In particular, those just below the Stokes line correspond to $\arg z$ equal to 0.49\pi, 0.499\pi, 0.4999\pi and 0.49995\pi, while for those just above it correspond to $\arg z$ equal to 0.51\pi, 0.501\pi, 0.5001\pi and 0.50005\pi. Besides using different values of $\delta$ in the study, the module was also run for numerous values of the truncation parameter $N$.

For each positive value of $\delta$ in Table 1 there are three rows of values, while for each negative value there are only two rows. This is because the Stokes discontinuity term is zero for negative values of $\delta$ as indicated by Eq. (93). The first row for each value of
\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\( \delta \) & Method & Value \\
\hline
1/10 & LogGamma[z] & -5.1085546405054331385771175 - 2.4350486413361839587613036i \\
& \text{SD}_{1}^{SS,U}(z) & -0.0000000146924137960847328 + 0.0000000072492097835477097i \\
Top & & -5.1085546405054331385771175 - 2.4350486413361839587613036i \\
Bottom & & -0.0000000146924137960847328 + 0.0000000072492097835477097i \\
\hline
-1/10 & LogGamma[z] & -3.115677061285581062960250 + 0.7915271748617878611398393i \\
& \text{SD}_{1}^{SS,U}(z) & 0.000000005454380883397577 + 0.0000000036684566186113983i \\
Top & & -3.115677061285581062960250 + 0.7915271748617878611398393i \\
Bottom & & 0.000000005454380883397577 + 0.0000000036684566186113983i \\
\hline
1/100 & LogGamma[z] & -4.44480783601992947676721 - 0.6842653947061931557949761i \\
& \text{SD}_{1}^{SS,U}(z) & 0.0000000065016016472424544 - 0.0000000003854594562814978i \\
Top & & -4.44480783601992947676721 - 0.6842653947061931557949761i \\
Bottom & & 0.0000000065016016472424544 - 0.0000000003854594562814978i \\
\hline
-1/100 & LogGamma[z] & -4.35317575755916134008805 - 0.5338516610090575526159669i \\
& \text{SD}_{1}^{SS,U}(z) & 0.000000006512385125175715 - 0.000000000192824558002624i \\
Top & & -4.35317575755916134008805 - 0.5338516610090575526159669i \\
Bottom & & 0.000000006512385125175715 - 0.000000000192824558002624i \\
\hline
1/1000 & LogGamma[z] & -4.34380060288973596127763 - 0.51983352796876654012142i \\
& \text{SD}_{1}^{SS,U}(z) & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
Top & & -4.34380060288973596127763 - 0.51983352796876654012142i \\
Bottom & & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
\hline
-1/1000 & LogGamma[z] & -4.342234406517926897501879 - 0.5166268788967352139359494i \\
& \text{SD}_{1}^{SS,U}(z) & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
Top & & -4.342234406517926897501879 - 0.5166268788967352139359494i \\
Bottom & & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
\hline
1/2000 & LogGamma[z] & -4.34380060288973596127763 - 0.51983352796876654012142i \\
& \text{SD}_{1}^{SS,U}(z) & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
Top & & -4.34380060288973596127763 - 0.51983352796876654012142i \\
Bottom & & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
\hline
-1/2000 & LogGamma[z] & -4.342234406517926897501879 - 0.5166268788967352139359494i \\
& \text{SD}_{1}^{SS,U}(z) & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
Top & & -4.342234406517926897501879 - 0.5166268788967352139359494i \\
Bottom & & 0.0000000065123040290213875 - 0.0000000003854594562814978i \\
\hline
\end{tabular}
\caption{Evaluation of \( \ln \Gamma(3 \exp(i(1/2 + \delta)\pi)) \) via Eq. (70) for various values of \( \delta \)}
\end{table}

\( \delta \) represents the value obtained by using the LogGamma routine in Mathematica and is denoted by the row with LogGamma[z] in the Method column. Depending upon whether \( \delta \) is positive or not, the second row presents the Stokes discontinuity term according to the conventional view of the Stokes phenomenon. In general, this term was found to possess real and imaginary parts of the order of \( 10^{-8} \) or a couple of orders lower. That is, the Stokes discontinuity term is very small and would be neglected in standard Poincaré asymptotics, which is due to choosing a relatively large value of \(|z|\). Even though the Stokes discontinuity term is small, it is still necessary in order to give the correct values of \( \ln \Gamma(z) \) for the hyperasymptotic calculation to thirty figures. The next value for each value of \( \delta \) is labelled either Top or Bottom in the Method column corresponding to whether the top or bottom asymptotic form in Eq. (98) has been used to calculate \( \ln \Gamma(z) \). It should also be noted that the values of the truncated sum, the regularised value of the remainder and the Stirling approximation were all evaluated in the Mathematica module separately, but are not displayed here due to limited space.

All the calculations carried out in the study yielded the value of \( \ln \Gamma(z) \) to the hyperasymptotic accuracy as indicated in the table except when \(|\delta|\) was extremely small, e.g. for \( \delta \leq 10^{-5} \). Then the NIntegrate routine in Mathematica experiences convergence problems because the numerical integration of the remainder \( R_{SSN}^{N}(z) \) is too close to the singularities lying on the Stokes line. For example, when \( \delta = 10^{-5} \), the module prints
out a value of $\ln \Gamma(z)$ that agrees with the actual value to 25 decimal places for the real part, but in the case of the imaginary part the results only agree to 18 decimal places. The module, however, does alert the user of the convergence problems that the NIntegrate routine experiences. Although this calculation is not presented in the table, it still represents a degree of success since the imaginary part of the Stokes discontinuity term is of the order of $10^{-12}$. That is, the Stokes discontinuity term had to be correct to the first six decimal places in order to yield the value of the imaginary part of $\ln \Gamma(z)$ for this very small value of $\delta$.

With the exception of the first value of $\delta$, which reflects the situation as the error function begins to veer away from the step-function of the conventional view, we expect for all other values of $\delta$ that the real part of the smoothed Stokes multiplier given by Approximation (94) to be close to $1/2$ according to Figs. 1 in Refs. [1] and [2]. Note that Fig. 1 in Ref. [1] gives the Stokes multiplier for $|z| = 8$, which has an optimal point of truncation that is approximately equal to 26. As a consequence, the transition is far more rapid than in Fig. 1 of Ref. [2]. Whilst Paris presents values of the Stokes multiplier for arg $z$ ranging form 0.325 to 0.750 at intervals of 0.25, he does not give the values for the Stokes multiplier in the vicinity of the Stokes line as in Table I. Nevertheless, for arg $z = 0.475$, he obtains a value of $0.2894310 - 0.0182669 i$ for his smoothed multiplier, while for arg $z = 0.525$, he obtains a value of $0.7105689 - 0.0182669 i$. Although these are not representative of the situation in Table I and moreover, are nowhere near the accuracy needed to conduct the hyperasymptotic investigation in Table I they indicate that the Stokes discontinuity term should be at least 30 percent less than the figures in the table for $\delta > 0$ and that thirty percent of the Stokes discontinuity needs to be added to the values in which $\delta < 0$ according to the smoothing view of the Stokes phenomenon. That is, the top asymptotic form in Eq. (98) with about half the Stokes discontinuity term should be a far more accurate approximation to the actual value of $\ln \Gamma(z)$. However, we see the opposite. The first asymptotic form yields the exact value of $\ln \Gamma(z)$ for all values of $\delta$ greater than zero despite the fact that the Stokes discontinuity term has no effect on the first nine decimal places. For $\delta < 0$, according to the smoothing view the bottom asymptotic form in Eq. (98) should not yield exact values of $\ln \Gamma(z)$ because it is missing about half the Stokes discontinuity term. Once again, we observe the opposite; the bottom asymptotic form yields exact values of $\ln \Gamma(z)$ for all negative values of $\delta$ in the table. Thus, it is evident that there is no smoothing of the Stokes phenomenon occurring in the vicinity of the Stokes line at $\theta = \pi/2$ or else it would not been possible to give the exact values of $\ln \Gamma(z)$ from Eq. (98).

From the results in the table we see that by using the conventional view of the Stokes multiplier we are able to obtain thirty figure accuracy for $\ln \Gamma(z)$ in the vicinity of the Stokes line. Moreover, we could have considered more decimal places, if this was really necessary. To achieve higher levels of accuracy all one has to do is alter the working precision plus the precision and accuracy goals in the Mathematica module. This will come at the expense of the computing time. Despite this, has Paris done the same in Ref. [1]? If the smoothing point of view for the Stokes phenomenon is indeed correct, then it should not only be vastly superior to the conventional view by yielding more accurate values of $\ln \Gamma(z)$, it should also expose where the errors or deficiencies occur in the conventional view. After all, there is simply no point in offering an alternative view to the mathematical community if it is unable to provide an improvement on the existing view/approach. Despite all the aggrandisement coming from Paris, we do not see any numerical evidence demonstrating how smoothing is able to match or even provide more accurate values of $\ln \Gamma(z)$ in the vicinity of a Stokes line. All we see in Paris’s comments is
a table giving the Stokes multiplier (both real and imaginary parts) to six decimal figures for a much larger value of $|z|$ than in Table 1, presumably because he was unable to obtain this level of accuracy for the Stokes multiplier for $|z|=3$. In addition, the analysis in Ref. 2 was conducted for numerous values of the truncation parameter $N$ far away from the optimal point whereas Paris’s computation require an optimal point of truncation. This means that his approach is useless for $|z|<1$, a limitation that should not apply.

The issues mentioned in the preceding paragraph are the ones that Paris needs to address before attacking the results of $\ln \Gamma(z)$. Until his smoothing approach is able to provide hypersymptotic values of a special function irrespective of whether the variable is large or small and without the need for optimal truncation, there is no point adopting his smoothing approach to asymptotics, which is not only vague and limited, but also very unwieldy. Therefore, the challenge to Paris is that he should at least match the results of Table 1 before he can claim that the smoothing concept is the correct alternative to the conventional view of the Stokes phenomenon. The particularly annoying aspect of Ref. 1 is that he chose a value of $|z| = 8$ when he knew full well that $|z|=3$ was chosen in Ref. 2 because it had a significantly lower optimal point of truncation. Nevertheless, even for $|z|=8$, he does not present a numerical study as his Table 1 where he has used Eq. (97) to determine hypersymptotic values of $\ln \Gamma(z)$.

At the end of Sec. 2 in Ref. 1 Paris criticises the numerical computations with the Borel-summed results or Eq. (98) carried out in Sec. 3 of Ref. 2. There he states that his computations using Eq. (7) only took a fraction of a second to compute compared with the hours taken when the Borel-summed remainder or the second term on the rhs of Equivalence (87) is computed. However, the first computation of $\ln \Gamma(z)$ in Ref. 2 is performed with Eq. (4) under the same conditions as those for the Borel-summed remainder given by Eq. (101). There it is stated that the calculations using the incomplete gamma function in Mathematica took just as long as those for the Borel-summed remainder. It was also stated that the calculations could be sped up dramatically for the computations involving the incomplete gamma function by wrapping the terms being calculated in the module by using the Mathematica construct, N[expr,50]. Then the calculations take only a minute to perform, but it can affect the accuracy of the final result for $\ln \Gamma(z)$, particularly when the remainder begins to diverge, which is far more rapid and earlier for small values of $|z|$. The primary consideration in the calculations was to maintain the same accuracy whether $|z|$ was large or small without the need to change the program or module in the process. As for Paris’s computations, these are not the same as those reported in Ref. 2, which are concerned with obtaining $\ln \Gamma(z)$ to 50 decimal places for $|z|=3$ and $|z|=1/10$. For these calculations it was necessary to consider much higher values of $k$ than those given by Paris in Table 1 of Ref. 1. Furthermore, it is stated later in the paper that the fastest method of computing $\ln \Gamma(z)$ to hypersymptotic accuracy is by using the MB-regularised asymptotic forms, which are presented in Sec. 4 of Ref. 2 and investigated numerically in Sec. 5. The MB-regularised results were obtained in only a few seconds.

Paris has also overlooked the fact that the Borel-summed asymptotic forms are required in order to provide a crucial comparison or check on the MB-regularised results, especially when the argument of variable $z$ lies outside the principal branch. This is because the LogGamma routine in Mathematica cannot be used to check the values of the special function for such values of arg $z$. The issue of considering the asymptotic behaviour outside the principal branch is conveniently passed over by Paris as is the issue of evaluating $\ln \Gamma(z)$ from his asymptotic forms on a Stokes line.
In the introduction to Ref. [2] it is stated that computational expediency was the reason why MB regularisation was first devised in Ref. [11], although it should be pointed out this is not always the case as discussed in Ref. [9]. In relation to \( \ln \Gamma(z) \) the superiority of the MB-regularised asymptotic forms is primarily due to the fact that the Borel-summed asymptotic forms possess an infinite sum over all positive integers with an intractable denominator, e.g., \( H(l, z) = \sum_{n=1}^{\infty} 1/n^l(n^2 + z^2) \). The corresponding sum in the MB-regularised asymptotic forms is replaced by the Riemann zeta function, while in the case of Eq. (7), the integration has been replaced by the incomplete gamma function. So, it is really a tribute to the programming team at Wolfram Research for developing powerful routines such as Zeta[s] and Gamma[N,z], which are capable of calculating these special functions to incredible accuracy extremely quickly. If it were not for the Zeta routine, then the zeta function in the MB-regularised asymptotic forms for \( \ln \Gamma(z) \) would have to be replaced by its Dirichlet series form and we would be on the same level as the Borel-summed asymptotic forms since we would be required to evaluate a similar number of integrals in an infinite sum given by Eq. (98) to achieve the hyperasymptotic accuracy of Table 1. Moreover, if there were no Gamma routine in Mathematica, then it would become a formidable exercise to evaluate Eq. (7) extremely quickly. Alternatively, if someone were to devise a special routine that could calculate \( H(l, z) \) for all values of \( z \), then the calculations via Eq. (101) would be greatly accelerated. Nevertheless, it should be emphasised that exactification of the asymptotic forms for \( \ln \Gamma(z) \) was the main aim of Ref. [2], while computational expediency represented only a secondary issue. After all, the work was concerned with developing the discipline of asymptotics, not the field of numerical mathematics.

4. Conclusion

In this paper we have refuted the two main claims made by Paris [1] concerning Ref. [2]. These are that there is no need for the concept of regularisation to be employed in the asymptotics of \( \ln \Gamma(z) \) and that the “Stokes smoothing” view originated by Berry [15], and subsequently supported by Olver [25], represents the correct view for interpreting asymptotic behaviour near a Stokes line. In regard to the first claim, we have seen that unbeknownst to him, Paris has actually employed the concept of regularisation to arrive at the forms that he derives for the remainder \( R_N(z) \) given by Eqs. (4)- (7) here. With regard to the second claim, we have seen that the Borel-summed regularised values of \( \ln \Gamma(z) \) in Ref. [2] are based entirely on the conventional view of the Stokes phenomenon, which holds that the Stokes multiplier of the subdominant part of a complete asymptotic expansion behaves as a step-function at a Stokes line. Moreover, Paris has not provided a hyperasymptotic study to demonstrate that the smoothing view is indeed superior to the conventional view. Therefore, the challenge to Paris is that he should demonstrate that his asymptotic results in Ref. [1] at least reproduce or even better still improve upon the results in Table [1] which represent the results from a hyperasymptotic analysis of \( \ln \Gamma(z) \) in Ref. [2] based on the conventional view of the Stokes phenomenon. Only then can he claim that smoothing represents a viable concept in asymptotics. After all, if the smoothing approach is indeed correct and the conventional view false, then this should become evident when one wishes to obtain hyperasymptotic values of a function from its asymptotic forms not only near, but also far away from a Stokes line, regardless of whether optimal truncation has been invoked or not. In addition, it should also be able to handle all values of \( |z| \), not just \( |z| \gg 1 \) as in Ref. [1].
[1] R.B. Paris, Comments on Exactification of Stirling’s approximation for the logarithm of the gamma function, arXiv:1406.1320, 2014.
[2] V. Kowalenko, Exactification of Stirling’s Approximation for the Logarithm of the Gamma Function, arXiv:1404.2705, 2014.
[3] H. Segur, S.Tanveer and H. Levine (eds.), Asymptotics beyond All Orders (Plenum Press, New York, 1991).
[4] V. Kowalenko, The Stokes Phenomenon, Borel Summation and Mellin-Barnes Regularisation, Bentham ebooks, http://www.bentham.org 2009.
[5] V. Kowalenko, Towards a theory of divergent series and its importance to asymptotics, in Recent Research Developments in Physics, Vol. 2, (Transworld Research Network, Trivandrum, India, 2001), 17-68.
[6] R.B. Dingle, Asymptotic Expansions: Their Derivation and Interpretation, (Academic Press, London, 1973).
[7] V. Kowalenko and A.A. Rawlinson, Mellin-Barnes regularization, Borel summation and the Bender-Wu asymptotics for the anharmonic oscillator, J. Phys. A, 31, L663-670 (1998).
[8] V. Kowalenko, Applications of the Cosecant and Related Numbers, Acta Appl. Math. 114 15-134 (2011), DOI:10.1007/s10440-011-9604-z.
[9] V. Kowalenko, Euler and Divergent Series, Eur. J. of Pure and Appl. Math. (2011) 4, pp. 370-423.
[10] V. Kowalenko, Euler and Divergent Mathematics, BestThinking Science, articlepermalink/1255?tab=article&title=euler-and-divergent-mathematics, 2011.
[11] V. Kowalenko, N.E. Frankel, M.L. Glasser and T. Taucher, Generalised Euler-Jacobi inversion formula and asymptotics beyond all orders, London Mathematical Society Lecture Note 214, Cambridge University Press, Cambridge, 1995.
[12] M.V. Berry and C.J. Howls, Hyperasymptotics, Proc. Roy. Soc. Lond. A 430, 653-668 (1990).
[13] M.V. Berry and C.J. Howls, Hyperasymptotics for integrals with saddles, Proc. Roy. Soc. Lond. A 434, 657-675 (1991).
[14] M.V. Berry, Asymptotics, Superasymptotics, Hyperasymptotics in Asymptotics beyond All Orders, H. Segur, S. Tanveer and H. Levine (Eds.), (Plenum Press, New York, 1991), 1-9.
[15] M.V. Berry, Uniform asymptotic smoothing of Stokes’s discontinuities, Proc. Roy. Soc. Lond. A 422, 7-21 (1989).
[16] G.G. Stokes, On the discontinuity of arbitrary constants which appear in divergent developments, in Collected Mathematical and Physical Papers, Vol. 4, (Cambridge University Press, Cambridge, 1904), 77-109.
[17] R.B. Paris and A.D. Wood, Exponentially-improved asymptotic for the gamma function, J. Comput. Appl. Math. 41, 135-143 (1992).
[18] I.S. Gradshteyn/I.M. Ryzhik, Alan Jeffrey (Ed.), Table of Integrals, Series and Products, Fifth Ed., (Academic Press, London, 1994).
[19] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Fourth Ed., (Cambridge University Press, Cambridge, 1973), p. 252.
[20] V. Kowalenko, Exactification of the Asymptotics for Bessel and Hankel Functions, Appl. Math. and Comp. (2002) 133, 487-518 (2002).
[21] E.T. Copson, An Introduction to the Theory of Functions of a Complex Variable, (Clarendon Press, Oxford, 1976).
[22] S. Wolfram, Mathematica- A System for Doing Mathematics by Computer, (Addision-Wesley, Reading, 1992).
[23] M.J. Lighthill, Fourier Analysis and Generalised Functions, Student’s Edition. (Cambridge University Press, Cambridge, 1975).
[24] R.B. Paris and D. Kaminski, Asymptotics and Mellin-Barnes Integrals, (Cambridge University Press, Cambridge, 2001).
[25] F.W.J. Olver, On Stokes’ phenomenon and converging factors, in Proc. Int. Conf. on Asymptotic and Computational Analysis, (R. Wong, ed.) (Marcel-Dekker, New York, 1990), pp. 329-355.
[26] G.F. Carrier, M. Krook and C.E. Pearson, Functions of a complex variable, (McGraw-Hill, New York, 1966).
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