Sparse Normal Means Estimation with Sublinear Communication

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Abstract

We consider the problem of sparse normal means estimation in a distributed setting with communication constraints. We assume there are $M$ machines, each holding a $d$-dimensional observation of a $K$-sparse vector $\mu$ corrupted by additive Gaussian noise. A central fusion machine is connected to the $M$ machines in a star topology, and its goal is to estimate the vector $\mu$ with a low communication budget. Previous works have shown that to achieve the centralized minimax rate for the $\ell_2$ risk, the total communication must be high – at least linear in the dimension $d$. This phenomenon occurs, however, at very weak signals. We show that once the signal-to-noise ratio (SNR) is slightly higher, the support of $\mu$ can be correctly recovered with much less communication. Specifically, we present two algorithms for the distributed sparse normal means problem, and prove that above a certain SNR threshold, with high probability, they recover the correct support with total communication that is sublinear in the dimension $d$. Furthermore, the communication decreases exponentially as a function of signal strength. If in addition $KM \ll d$, then with an additional round of sublinear communication, our algorithms achieve the centralized rate for the $\ell_2$ risk. Finally, we present simulations that illustrate the performance of our algorithms in different parameter regimes. Distributed statistical inference, sparse normal mean estimation, sublinear communication, support recovery

1 Introduction

In the past couple of decades, the steady increase in data collection capabilities has lead to a rapid growth in the size of datasets. In many applications, the collected datasets cannot be stored or analyzed on a single machine. This has sparked the development of distributed approaches to machine learning, statistical analysis and data mining. A few examples of this vast body of work are [McDonald et al., 2009, Bekkerman et al., 2011, Duchi et al., 2012, Guha et al., 2012].

One of the most popular distributed schemes is known as one-shot, embarrassingly parallel or split-and-merge. In this scheme, there are $M$ machines, each holding $n$ samples from some

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unknown distribution. The $M$ machines are connected in a star topology to a central node, also called a fusion center or simply the center. The task of the fusion center is to estimate some quantity $\theta$ related to the distribution, with small communication with the $M$ machines. We consider the following one-round variant. First, the center node sends setup messages to the machines (or a subset of them). Then, each machine performs a local computation and sends back its results to the center. Finally, the fusion center forms a global estimator $\hat{\theta}$ based on these messages. A clear advantage of such one-shot schemes is their simplicity and ease of implementation.

Statistical inference in a distributed setting, in particular under communication constraints, raises several fundamental theoretical and practical questions. One question is what is the loss of statistical accuracy incurred by distributed schemes, compared to a centralized setting, whereby a single machine has access to all $Mn$ samples. Various works proposed multi-round communication-efficient schemes and analyzed their accuracy, see for example [Shamir et al., 2014, Zhang and Lin, 2015, Wang et al., 2017, Jordan et al., 2019]. In the context of one-shot schemes, several works analyzed the case where the fusion center simply averages the estimators computed by the individual machines or for robustness, takes their median [Zhang et al., 2013b, Rosenblatt and Nadler, 2016, Minsker, 2019]. In a high dimensional setting where the parameter of interest is a-priori known to be sparse, [Lee et al., 2017] and [Battey et al., 2018] considered a variant where the averaged estimator is further thresholded at the fusion center. A key result in these papers is that in various scenarios and under suitable regularity assumptions, the $\ell_2$ risk of the resulting distributed estimate converges at the same rate as the centralized one, provided that the data is not split across too many machines.

Another important theoretical aspect in distributed learning, are fundamental lower bounds on the required communication between the machines, to attain the same statistical accuracy rates as in a centralized setting, regardless of any specific inference scheme. Several works derived lower bounds on the achievable accuracy under communication as well as memory constraints for a variety of problems [Zhang et al., 2013a, Garg et al., 2014, Steinhardt et al., 2016, Cai and Wei, 2020, Zhu and Lafferty, 2018, Szabo et al., 2020, Acharya et al., 2020]. Lower bounds on the estimation accuracy were also studied for problems involving a sparse quantity, including sparse linear regression, correlation detection and more [Steinhardt and Duchi, 2015, Braverman et al., 2016, Dagan and Shamir, 2018, Han et al., 2018]. A central result of these works is that to achieve the centralized minimax rate for the $\ell_2$ risk, the communication must scale at least linearly in the ambient dimension.

Intuitively, when the task is to estimate a sparse quantity, the communication should increase linearly with its sparsity level, and only logarithmically with the ambient dimension. Indeed, in the context of supervised learning with a goal of minimizing prediction error, [Acharya et al., 2019] showed that in various linear models with a sparse vector, optimal error rates are achievable with total communication logarithmic in the dimension. However, their algorithm is sequential and not compatible with one-shot inference schemes. An interesting question is the following: can problems that involve a sparsity prior admit one-shot algorithms with communication that is sublinear in the ambient dimension?

We consider sparse normal means estimation, which is one of the simplest and most well-studied inference problems with sparsity priors, but in a distributed setting. The parameter of interest is a $K$-sparse vector $\mu \in \mathbb{R}^d$, and each of the $M$ machines has $n$ samples of the form
where the noise vectors $\xi_l$ are i.i.d. $\mathcal{N}(0, \frac{1}{n} I_d)$. The goal is exact recovery of the support of $\mu$ using little communication between the machines and the fusion center. As we discuss in Section 4, if $KM \ll d$ then achieving this goal implies that the vector $\mu$ itself can be estimated with small $\ell_2$ risk.

For this sparse normal means problem, Braverman et al. [2016] and Han et al. [2018] derived communication lower bounds for the $\ell_2$ risk of any estimator, and proved that to achieve the minimax rate, the total communication must be at least $\Omega(d)$. Shamir [2014] proved that for a 1-sparse vector $\mu$ whose non-zero entry has magnitude $O\left(\frac{1}{\sqrt{n \log d}}\right)$, no scheme can accurately estimate its support with high probability using only $o(d)$ total communication. These works paint a pessimistic view, that to achieve the performance of the centralized solution, distributed inference must incur high communication costs.

In contrast, we show that once the signal-to-noise ratio (SNR) is slightly higher, the story is different. The main contribution of our work is to show that at sufficiently high SNRs, the support of $\mu$ can be exactly recovered with communication sublinear in the dimension $d$. Moreover, the total communication decreases exponentially as a function of the SNR.

In further details, we present and analyze the performance of two distributed schemes. Our analysis is non-asymptotic, but the setting we have in mind is of a sparse vector in high dimension, namely $d \gg 1$ and $K \ll d$. For these two schemes we prove the following results. First, assuming that $\mu_{\min}$, a lower bound on the non-zero entries of $|\mu|$, is known to the center and exceeds $\Omega\left(\frac{1}{\sqrt{n}} \cdot \frac{\log \log d}{\sqrt{\log d}}\right)$, then its support can be accurately found with total communication sublinear in $d$. Second, the communication costs of our proposed schemes decrease exponentially as $\mu_{\min}$ increases towards $\sqrt{\frac{2 \log d}{n}}$, at which point the support of $\mu$ may be found by a single machine. Third, perhaps counter-intuitively, the availability of more machines enables exact support recovery using less total communication. Finally, we show that if $KM \ll d$, then an additional single round of communication, also sublinear in $d$, results in an estimator for $\mu$ that achieves the centralized rate for the $\ell_2$ risk. We believe a similar behavior should hold for many other popular statistical learning problems involving estimation of a sparse quantity in a high dimensional setting.

**Paper organization.** Section 2 formalizes the distributed sparse normal means problem. Section 3 presents algorithms for exact support recovery whose total communication costs are sublinear in the ambient dimension $d$, that succeed with high probability for a large range of SNR values. Section 4 discusses the relation between exactly recovering the support of a vector and estimating it with small $\ell_2$ risk, and shows a reduction from the latter to the former with one additional round of sublinear communication. Section 5 elaborates on how our results relate to the lower bounds of Braverman et al. [2016], Han et al. [2018] and Shamir [2014]. Section 6 presents simulations that illustrate our results. All proofs can be found in the appendix.

## 2 Problem setup

Consider a distributed setting of $M$ machines connected in a star topology to a central fusion node. Each machine $i$ has $n$ i.i.d. samples of the form $x_i^{(l)} = \mu + \xi_i^{(l)}$, where the mean vector
\( \mu \in \mathbb{R}^d \) is exactly \( K \)-sparse and the noise is Gaussian, \( \xi_{i}^{(l)} \sim \mathcal{N}(0, \sigma^2 I_d) \). Without any loss of statistical information regarding the unknown vector \( \mu \), the assumption of Gaussian noise implies that each machine \( i \) may compute its normalized empirical mean \( x_i \equiv \frac{1}{n} \sum_l x_{i}^{(l)} = \mu + \xi_i \) where \( \xi_i \equiv \frac{1}{n} \sum \xi_{i}^{(l)} \). The mean \( x_i \) is a sufficient statistic, so we may think of each machine having instead a single sample with an effective noise level \( \frac{\sigma^2}{n} \). For simplicity, we assume the noise level is known. Thus, without loss of generality, we may assume \( \frac{\sigma^2}{n} = 1 \). We consider a one-round communication scheme where the fusion center sends a setup message to each of the machines (or a subset of them), and then each contacted machine sends back its message to the center. We emphasize that in our setup the machines communicate only with the center and not with each other. The goal of the center is to recover the support of \( \mu \) under the constraint that the total communication between the fusion center and the machines (including the setup stage) is bounded by a budget of \( B \) bits.

For simplicity we assume that the sparsity level \( K \in \mathbb{N} \) is known to the fusion center and that \( \mu_j \geq 0 \) for all \( j \in [d] \). However, with slight variations our methods can work when \( K \) is unknown or for vectors \( \mu \) that have both positive and negative entries.

Beyond the assumption that \( \mu \) is exactly \( K \)-sparse we further assume a lower bound \( \mu_{\min} \) on its smallest non-zero coordinate, namely \( \mu_j \geq \mu_{\min} \) for all \( j \in \Omega = \{ i \mid \mu_i > 0 \} \). It will be convenient to use the natural scaling

\[
\mu_{\min} = \sqrt{2r \log (d - K)}.
\] (1)

In what follows the term support recovery with high probability means an ability to correctly estimate \( \Omega \) with a probability tending to one as \( d \to \infty \). We focus on the following question: Given a lower bound on the signal-to-noise ratio (SNR) \( r \), how much communication is sufficient for exact recovery of the support \( \Omega \) of a \( K \)-sparse vector \( \mu \) with high probability?

Let us first discuss what is the interesting regime for the SNR parameter \( r \). Recall that for \( d \gg K \), the maximum of \( d - K \) i.i.d. standard Gaussian random variables is tightly concentrated around \( \sqrt{2 \log (d - K)} \). At a high SNR \( r > 1 \), each individual machine can exactly recover the support set \( \Omega \) with high probability. Hence, it suffices that only one machine sends \( O(K \log d) \) bits to the fusion center. At the other extreme, let \( r < \frac{c}{M} \) for \( 0 < c < 1 \). Here, even in a centralized setting, exact support recovery with high probability is not possible. To see this, note that the empirical mean of all samples is a sufficient statistic, and the effective SNR is \( c < 1 \). Therefore, with probability tending to 1 as \( d \to \infty \), the smallest support entry of the mean is smaller than its largest non-support entry. If the index of \( \mu_{\min} \) is chosen uniformly at random, then any algorithm would fail to recover the support. Hence, the relevant range of SNR ratio values is

\[
\frac{1}{M} < r < 1.
\] (2)

In this range, a single machine cannot individually recover the support with high probability. Yet, as we show next, for a large range of SNR values in the range \( \frac{1}{M} \), exact support recovery by the fusion center is possible with very limited total communication \( o(d) \) bits. Furthermore, as \( r \) increases towards 1, the total communication decays exponentially fast to \( O\left(K \log^{1+c} d\right) \) for an appropriate constant \( c > 0 \).
3 Distributed algorithms for the sparse normal means problem

We present two one-shot algorithms for the distributed sparse normal means problem and derive non-asymptotic bounds on their performance. In both algorithms, the lower bound \( r \) on the SNR is assumed to be known to the center and is used to decide how many machines to communicate with and what messages to send them. We denote the number of contacted machines by \( M_c \). For our analysis below, we assume the total number of machines is sufficiently large, in particular \( M \geq M_c \).

In our first algorithm, denoted Top-\( L \), the center sends a parameter \( L \) to \( M_c \) machines. Each contacted machine \( i \) sends back a message \( y_i \) with the indices of the \( L \) highest coordinates of its sample \( x_i \). Our second algorithm is threshold-based; the center sends a threshold \( t_m \) to \( M_c \) machines, and each contacted machine \( i \) sends back all indices \( j \) with \( x_{i,j} > t_m \). In either of the two algorithms, the center then estimates the support of \( \mu \) by a voting procedure. We prove in Theorems 1 and 2 that under suitable assumptions, and in particular for a sufficiently high SNR, both algorithms can achieve exact support recovery with high probability using \( M_c = \tilde{O} \left( d^{(1 - \sqrt{r})^2} \right) \) machines, where the \( \tilde{O} \) notation hides factors that are polylogarithmic in \( d \). The resulting total communication cost is \( B = \tilde{O} \left( K d^{(1 - \sqrt{r})^2} \right) \) bits, which is sublinear in \( d \), provided that \( K \) is at most polylogarithmic in \( d \) and \( r > \Omega \left( \frac{\log^2 \log d}{\log^2 d} \right) \).

To put our results in context, we illustrate in Figure 1 the different communication regimes...
as a function of SNR $r$ and number of machines $M$ for $K = 1$. As discussed above, if $r < \frac{1}{M}$, then even with infinite communication, exact support recovery with high probability is impossible. The corresponding $(r, M)$ values are in the pink area below the red curve. Shahii [2014] showed that when $r < O \left( \log^{-3} d \right)$, i.e., for $(r, M)$ values in the gray area to the left of the dashed black line, no distributed scheme can recover the support with high probability using $o(d)$ communication. By our Theorems 1 and 2, when $r > \Omega \left( \frac{\log^2 \log d}{\log^2 d} \right)$ and $M \geq O \left( d^{(1-\sqrt{r})/2} \log^c d \right)$ for an appropriate constant $c$, i.e., in the green area, exact recovery is possible using sublinear communication. In the white area, exact support recovery is possible using communication that is at least linear in $d$. An example of a recovery scheme in this range is to send the entire sample (up to a quantization error) from a subset of the machines. It remains an open question whether exact support recovery with sublinear communication is possible for any of the $(r, M)$ values in the white area.

Before presenting the two algorithms, we address the issue of representing a real number by a finite amount of bits. Recall that the scientific binary representation of a number $x \in \mathbb{R}$ consists of a bit representing its sign, and bits $\{b_j\}_{j \in \mathbb{Z}}$, such that $|x| = \sum_{j=-\infty}^{\lfloor \log_2 |x| \rfloor} b_j 2^j$. One can approximate $x$ by truncating its binary representation at a predetermined precision level. Specifically, given two parameters $U, P \in \mathbb{N}$, let the procedure $s = Trunc(x, U, P)$ output a truncated binary representation of $x$ of length $U + P + 2$ such that $s = (\text{sign}(x), b_{-P}, \ldots, b_U)$. Given $s$, let the procedure $\hat{x} = Approx(s, U, P)$ construct an approximation for $x$, given by $\hat{x} = (2\text{sign}(x) - 1) \cdot \sum_{j=-P}^{U} b_j 2^j$. If $U \geq \lfloor \log_2 |x| \rfloor$, then $\hat{x}$ and $x$ consist of the same bits up to the $P$-th bit after the binary dot, and thus the resulting approximation error is bounded by $|\hat{x} - x| < 2^{-P}$. This scheme is a variant of [Szabo et al. 2020, Algorithm 1].

In our analysis we assume that $\mu_{\text{max}} = \max_{j \in \Omega} \mu_j$ is at most polynomial in $d$. Thus taking $U, P = O(\log d)$ ensures that with high probability all quantities of interest are approximated up to polynomial error. In addition, since $P, U$ only depend on $d$ and the bound $\mu_{\text{max}}$, they can be set in advance without communication.

### 3.1 Top-$L$ Algorithm

In the Top-$L$ algorithm, the center uses its knowledge of the parameters $d, M, r, K$ to determine the number of machines $M_c$ to contact, and sends them a parameter $L \in \mathbb{N}$. The $i$-th contacted machine then sends a message $y_i$ consisting of the $L$ indices with the largest coordinates in its stored vector $x_i$. Given the messages $y_1, \ldots, y_M$ and the sparsity level $K$, the fusion center counts how many votes each index received and estimates the support to be the $K$ indices with the highest number of votes. Voting ties can be broken arbitrarily. This scheme is outlined in Algorithm 1. Its total communication cost is $B = O(M_c (L + 1) \log d)$ bits.

**Remark 1.** The above description assumes that the fusion center knows the sparsity level $K$. However, the following simple variant can handle a case where only an upper bound $K_{\text{max}} \geq K$ is known. In this case, the number of contacted machines $M_c$ is determined using $K_{\text{max}}$ instead of $K$, and each contacted machine sends its top $L \geq K_{\text{max}}$ indices to the fusion center. The center then estimates the support as the set of indices that received more votes than a suitable threshold.
Algorithm 1 Top-L

**At the fusion center:**
**Input** dimension $d$, number of machines $M$, SNR $r$, sparsity level $K$, parameter $L$
**Output** setup message $s$

1: calculate $M_c(d,r,M,K,L)$
2: send $s$ containing $L$ to each of the first $M_c$ machines

**At each machine** $i = 1, \ldots, M_c$:
**Input** setup message $s$, sample $x_i$
**Output** message $y_i$ to center

1: sort $x_i$ in descending order, $x_{i,\sigma(i,1)} \geq \cdots \geq x_{i,\sigma(i,d)}$
2: send to the center the $L$ indices with the largest coordinates, $y_i = \{\sigma(i,1), \ldots, \sigma(i,L)\}$

**At the fusion center:**
**Input** messages $y_1, \ldots, y_{M_c}$, sparsity level $K$
**Output** estimated support $\hat{\Omega}$

1: for each coordinate $j \in [d]$, let the set of votes it received be $V_j = \{i \in [M] : j \in y_i\}$
2: sort the indices by descending number of votes, $v_{\pi(1)} \geq \cdots \geq v_{\pi(d)}$ where $v_j = |V_j|
3: return $\hat{\Omega} = \{\pi(1), \ldots, \pi(K)\}$

We prove that for sufficiently high SNR, the Top-L algorithm recovers the exact support of $\mu$ with high probability. We first analyze the case $L = K = 1$ and then extend the analysis for general $L \geq K \geq 1$. The proofs of the theorems stated below appear in Appendix A.1.

To proceed, we define the quantity

\[
M_0(d,r) = \max \left\{ 1, \frac{\sqrt{2\pi e} \left( 2 \left( 1 - \sqrt{r} \right)^2 \log d + 1 \right)}{(1 - \sqrt{r}) \sqrt{2 \log d}} \cdot d^{(1-\sqrt{r})^2} \right\} \cdot 8 \log d \quad . \tag{3}
\]

Notice that for any fixed SNR $r < 1$, $M_0(d,r)$ is sublinear in $d$, and up to polylogarithmic terms it is proportional to $d^{(1-\sqrt{r})^2} < d$. The following theorem provides a support recovery guarantee in the setting $K = L = 1$.

**Theorem 1.A.** Assume $r < 1$ and satisfies $M_0(d,r) \leq M, d$. Then, if the center contacts $M_c = M_0$ machines, the Top-1 algorithm recovers the support of a 1-sparse vector $\mu$ with probability at least $1 - d^{-1} - e^3 d^{-3}$. Its total communication is $B = O(M_0 \log d)$ bits.

Several insights follow from Theorem 1.A. First, recall that for any $r < 1$ no machine can successfully recover the support of $\mu$ on its own. Yet, for $d \gg 1$ and for any fixed $r < 1$, as implied by the theorem, the fusion center can recover the support of $\mu$ by communicating with only $M_0(d,r)$ machines, receiving from each machine a very noisy estimate of the support. Second, as the SNR lower bound $r$ increases towards 1, the algorithm needs fewer machines and thus less communication to succeed with high probability. Moreover, by Eq. (3), $M_0(d,r)$ decreases exponentially fast with $r$. Lastly, for a fixed $r$ the required number of machines
$M_0(d, r)$ increases sublinearly with $d$, and hence the communication cost increases sublinearly with $d$ as well. In fact, even if $r > \Omega \left( \frac{\log^2 \log d}{\log d} \right)$, then the number of contacted machines is $M_c = o \left( \frac{d}{\log d} \right)$, and thus the total communication cost is sublinear in $d$.

Next, we consider the more general case where the unknown vector $\mu$ is exactly sparse with sparsity level at most $K$, and its support is estimated by the Top-$L$ algorithm with parameter $L \geq K$. To this end, we define the auxiliary quantities

$$a = a(K, L, d) = \sqrt{2 \log \frac{d-K}{L-K+1}}, \quad (4)$$

$$b = b(K, L, d, r) = a - \sqrt{2r \log (d-K)}, \quad (5)$$

and the quantity

$$M_{K,L}(d, r) = \left\lceil \max \left\{ 8, \frac{4\sqrt{2\pi}(b^2+1)}{b} \cdot (d-K) \left( \sqrt{1 - \frac{\log (L+1) - \log(d-K)}{\log(d-K)}} - \sqrt{r} \right)^2 \right\} \cdot 8 \log d \right\rceil. \quad (6)$$

The following theorem provides a support recovery guarantee in this setting.

**Theorem 1.B.** Assume $r < 1$ and satisfies $M_{K,L}(d, r) \leq \min \{ M, \frac{d-K}{L} \}$. Then, if the center contacts $M_c = M_{K,L}$ machines, the Top-$L$ algorithm with $K \leq L \leq (d-K)/2$ recovers the support of a $K$-sparse vector $\mu$ with probability at least $1 - Kd^{-1} - e^{3d^{-1}}$ using $B = O \left( M_{K,L}(L+1) \log d \right)$ communication bits.

Note that in a high dimensional setting $d \gg 1$, the Top-$L$ algorithm with $L = K$ incurs a total communication cost of $O \left( K \cdot (d-K)(1-\sqrt{r})^2 \log^{2.5} d \right)$.

### 3.2 Thresholding Algorithm

Our second algorithm is based on thresholding, whereby the fusion center chooses a threshold $t_m = t_m(d, r, M, K)$ and sends (a truncated binary representation of) it to a subset of the machines $M_c = M_c(d, r, M, K) \leq M$. Each contacted machine $i$ sends back all indices $j$ such that $x_{i,j} > t_m$. Similarly to the Top-$L$ algorithm, given the messages $y_1, \ldots, y_M$ and the sparsity level $K$, the fusion center estimates the support as the $K$ indices with the highest number of received votes. Voting ties can be broken arbitrarily. The scheme is outlined in Algorithm[2]. If instead of the sparsity level $K$ only an upper bound on it $K_{\text{max}} \geq K$ is known, and $K_{\text{max}} \ll d$, then the fusion center can set $t_m$ and $M_c$ by approximating $d-K \approx d$. In addition, the center estimates the support as outlined in Remark[1].

The thresholding algorithm has several desirable properties. First, it is simple to implement in a distributed setting. Second, in the centralized setting, thresholding algorithms were shown to be optimal in various aspects (see Section[3] for further details). Third, adjusting the threshold allows for a tradeoff between the number of contacted machines and the expected message length per machine. For example, using a large threshold such as $t_m = \sqrt{2\log d}$ implies that with high probability at most one non-support index is sent to the center from each machine.
Algorithm 2 Thresholding

At the fusion center:
Input dimension $d$, number of machines $M$, SNR $r$, sparsity level $K$
Output setup message $s$
1: calculate $M_c$ and $t_m$
2: send $s = \text{Trunc}(t_m, \lceil \log_2 t_m \rceil, \lceil \log_2 d \rceil)$ to each of the first $M_c$ machines

At each machine $i = 1, \ldots, M_c$:
Input setup message $s$, sample $x_i$
Output message $y_i$
1: construct threshold $\hat{t}_m = \text{Approx}(s, \lceil \log_2 t_m \rceil, \lceil \log_2 d \rceil)$
2: let $y_i = \{j \in [d] : x_{i,j} > \hat{t}_m\}$
3: send $y_i$ to center

At the fusion center:
Input messages $y_1, \ldots, y_{M_c}$, sparsity level $K$
Output estimated support $\hat{\Omega}$
1: for each coordinate $j \in [d]$, let the set of votes it received be $V_j = \{i \in [M] : j \in y_i\}$
2: sort the indices by descending number of votes, $v_{\pi(1)} \geq \cdots \geq v_{\pi(d)}$ where $v_j = |V_j|$  
3: return $\hat{\Omega} = \{\pi(1), \ldots, \pi(K)\}$

If the SNR is sufficiently high such that with high probability for each support index $k \in \Omega$ at least two machines send it, then the center would return the correct support. Notice that if the SNR is high, there may not even be a need to use all machines to recover the support. By the same logic, when the SNR is lower, one can lower the threshold. Of course, this would incur a higher communication cost. Hence, since the fusion center knows both $r$ and $M$, it can set an optimal threshold $t_m$ and send it only to $M_c \leq M$ machines, which ensures exact support recovery with high probability at minimal communication cost (among all possible thresholds).

We formalize these intuitions in three regimes, small, intermediate and large $M$. In each of the three settings we analyze the thresholding algorithm with an appropriate threshold $t_m = t_m(d, r, M)$, where the SNR parameter $r$ and sparsity level $K$ are assumed to be known. The proofs of all theorems stated below appear in Appendix A.2.

First, we consider a threshold $t_m$ that does not increase with the number of machines, and show that exact support recovery with sublinear communication can be achieved by contacting $O(\log d)$ machines.

**Theorem 2.A.** Assume that $d \geq 16$ and $M \geq 16 \log d$. Further assume $\frac{\log 5}{\log(d-K)} < r < 1$. Then, with probability at least $1-(K+1)/d$, the thresholding algorithm with $M_c = \lfloor 16 \log d \rfloor$ and
\[
t_m = \mu_{\min} = \sqrt{2r \log (d-K)}
\]
reverses the support of the $K$-sparse vector $\mu$ using
\[
O ((d-K)^{1-r} r^{-0.5} \log^{1.5} d + K \log^2 d)
\]
communication bits in expectation.

The communication cost (8) is sublinear in \(d\) for all \(r > \frac{2 \log \log d}{\log(d-K)}\) and \(K \ll d/\log^2 d\). As we show in the next theorem, perhaps counter-intuitively, given more than \(O(\log d)\) machines, the fusion center can recover the support using less total communication, by setting a higher threshold.

**Theorem 2.B.** Let \(d \geq 15\) and assume that \(32 \sqrt{e \pi} \log^{1.5} d \leq M \leq d\). Further assume \(r < 1\) and satisfies

\[
 r > \left( \frac{\sqrt{2 \log 5M}}{\sqrt{2 \pi} 4 \log d} - \sqrt{2 \log \frac{M}{32 \sqrt{\pi} \log^{1.5} d}} + \frac{1}{d} \right)^2.
\]

Then, with probability at least \(1 - (K + 1)/d\), the thresholding algorithm with \(M_c = M\) and

\[
t_m = \sqrt{2 r \log (d - K)} + \sqrt{2 \log \frac{M}{32 \sqrt{\pi} \log^{1.5} d}}
\]

recovers the support of the \(K\)-sparse vector \(\mu\) using

\[
O \left( KM \log d + (d - K)^{1-r} e^{-2 \sqrt{r \log (d - K) \log \frac{M}{32 \sqrt{\pi} \log^{1.5} d}}} \log^{2.5} d \right)
\]

communication bits in expectation.

Notice that the threshold \(t_m\) is adjusted to the number of machines, which leads to a possibly non-intuitive dependence of the communication cost on \(M\). Namely, the first term in (11) increases with \(M\) whereas the second term decreases with \(M\). Thus, if \(M\) is too large, then the center can contact only \(M_c \leq M\) machines to reduce the total communication cost. The following theorem shows that by contacting \(M_c = \tilde{O} \left( (d - K)^{1-r} \right)\) machines and setting the threshold \(t_m = \sqrt{2 \log (d - K)}\), the thresholding algorithm achieves a total communication cost of \(\tilde{O} \left( K (d - K)^{1-r} \right)\). It is easy to verify that this choice of \(M_c\) indeed minimizes (11).

**Theorem 2.C.** Assume that \(d - K \geq 20\) and \(\left( \frac{\log 10}{2 \log(d-K)} \right)^2 < r < 1\). Let

\[
M_c = \left[ \frac{8 \sqrt{2 \pi} \left( (1 - \sqrt{r})^2 \ 2 \log (d - K) + 1 \right)}{(1 - \sqrt{r})^2 \sqrt{2 \log (d - K)}} (d - K)^{(1-r)^2} \log d \right],
\]

and assume that \(M \geq M_c\). Then, with probability at least \(1 - (K + 1)/d\), the thresholding algorithm with \(M_c\) machines and \(t_m = \sqrt{2 \log (d - K)}\) recovers the support of the \(K\)-sparse vector \(\mu\) using

\[
O \left( K (d - K)^{(1-r)^2} \log^{2.5} d \right)
\]

communication bits in expectation.
Let us now briefly compare the Top-$L$ and thresholding algorithms, in terms of communication cost and recovery guarantees. By Theorems 1.B and 2.C with appropriately set parameters the algorithms exhibit qualitatively similar performances for high SNR and large number of machines $M$. The main differences between the two algorithms occur when $M$ is small, for example logarithmic in $d$. If the SNR is low, for example $r = O \left( \frac{\log^{-2} d}{\log^2 d} \right)$, then the Top-$K$ algorithm might fail to recover the support, while Theorem 2.A implies that the thresholding algorithm recovers the support at the expense of total communication cost superlinear in $d$. Yet, if the SNR is slightly higher, namely $r = O \left( \frac{\log \log d \log d}{\log^2 d} \right)$, then by Theorems 1.B and 2.A, with high probability both algorithms succeed, and the Top-$K$ algorithm incurs less total communication cost than the thresholding algorithm.

However, the thresholding algorithm is more robust in the following sense. If the sparsity level $K$ is fixed and the center only knows an upper bound on it $K_{\max} = cK$ for $c > 1$, then the Top-$K_{\max}$ algorithm incurs a communication cost that is linear in $c$, while the thresholding algorithm incurs a communication cost that is roughly the same as when $c = 1$.

## 4 Sublinear distributed algorithms with small $\ell_2$ risk

In the previous section we considered distributed estimation of the support of $\mu$. Another common task is to estimate the vector $\mu$ itself, with both small $\ell_2$ risk and low total communication. We show that this can be achieved with only a single additional round of communication. Furthermore, under certain parameter regimes, specifically $KM \ll d$, the resulting estimate achieves the centralized $\ell_2$ risk, with sublinear total communication. The proof of this result is based on the fact that both of our algorithms achieve exact support recovery with high probability. We thus first discuss the relation between support recovery and $\ell_2$ risk, as well as lower bounds for the centralized minimax risk.

### 4.1 On exact support recovery and $\ell_2$ risk

Let us first briefly discuss estimation of $\mu$ in a centralized setting with $M$ samples and noise level $\sigma$. Without any assumptions on the vector $\mu$, the empirical mean $\bar{x} = \frac{1}{M} \sum_i x_i$ is a rate-optimal estimator. When the vector $\mu$ is assumed to be sparse, various works suggested and theoretically analyzed the set of diagonal estimators $O_{\text{diag}}$. An estimator $\hat{\mu} \in O_{\text{diag}}$ has the form $\hat{\mu}_j = a_j(\bar{x}_j) \bar{x}_j$ for all $j \in [d]$, where each $a_j(\cdot)$ is a scalar function. For further details see for example Mallat [1999, Chapter 11].

**Projection oracle risk.** In analyzing the lowest risk achievable in the set $O_{\text{diag}}$, a key notion is the projection oracle risk, defined as the smallest expected $\ell_2$ error of a diagonal projection estimator $\hat{\mu}_{\text{oracle}}$ but with additional prior knowledge of $\mu$, such that $\hat{\mu}_{\text{oracle}} = a_j(\mu_j) \bar{x}_j$ and $a_j \in \{0, 1\}$. It is easy to show that $\hat{\mu}_{\text{oracle}}^2 = \bar{x}_j \cdot 1(|\mu_j| > \sigma/\sqrt{M})$. Its corresponding risk is

$$R_{\text{oracle}}(\mu) = \mathbb{E} \left[ \|\mu - \hat{\mu}_{\text{oracle}}\|^2 \right] = \sum_{j=1}^d \min \left\{ \frac{\sigma^2}{M} \mu_j^2, \mu_j^2 \right\} \leq K\sigma^2 \frac{d}{M}.$$  

(14)
Algorithm 3 Protocol Π

At the fusion center:
Input estimated support set $\hat{\Omega}$
Output setup message $s$
send $s$ containing $\hat{\Omega}$ to each of the $M$ machines

At each machine $i = 1, \ldots, M$:
Input setup message $s$, sample $x_i$, precision parameters $U, P$
Output message $w_i$ to center
1: for each $k \in \hat{\Omega}$, calculate $w_{i,k} = \text{Trunc}(x_{i,k}, U, P)$
2: send to center $w_i = \{w_{i,k} : k \in \hat{\Omega}\}$

At the fusion center:
Input messages $w_1, \ldots, w_M$
Output estimated vector $\hat{\mu}_\Pi$
1: for each $i \in [M]$ and each $k \in \hat{\Omega}$, reconstruct $z_{i,k} = \text{Approx}(w_{i,k}, U, P)$
2: for each $k \in \hat{\Omega}$, calculate the mean $\bar{z}_k = \frac{1}{M} \sum_{i \in [M]} z_{i,k}$
3: return $\hat{\mu}_\Pi$ where $\hat{\mu}_j^\Pi = \bar{z}_j \cdot \mathbb{1} \{ j \in \hat{\Omega}\}$

Note that the projection oracle is not a realizable estimator, as it relies on knowledge of the underlying $\mu$ for support recovery. However, the oracle risk provides a lower bound for the risk of any diagonal estimator. Also note that given a lower bound on the SNR, of the form $\min_{j \in \Omega} |\mu_j| > \sigma/\sqrt{M}$, the oracle risk is $R_{\text{oracle}}(\mu) = K\sigma^2/M$.

Centralized lower bound. Donoho and Johnstone [1994, Theorem 3] proved the following lower bound on the asymptotic minimax rate among all diagonal estimators,

$$
\lim_{d \to \infty} \inf_{\mu \in \mathcal{O}_{\text{diag}}} \sup_{\mu \in \mathbb{R}^d} \frac{\mathbb{E}[\|\hat{\mu} - \mu\|^2]}{\frac{\sigma^2}{M} + R_{\text{oracle}}(\mu)} \frac{1}{2 \log d} = 1.
$$

(15)
Moreover, they proved that thresholding at a suitable level achieves this minimax rate.

In the result above, no assumptions are made neither regarding the sparsity of $\mu$, nor on its SNR or equivalently on $\mu_{\text{min}}$. Indeed, the proof of (15) relies on a construction of vectors $\mu$ with $\log d$ coordinates having values slightly smaller than $\frac{\sigma}{\sqrt{M}}\sqrt{2\log d}$, namely with a low SNR. Thus, it cannot be used as a lower bound for the centralized minimax rate in our setting. In fact, if $\mu$ is $K$-sparse and $\mu_{\text{min}}$ is sufficiently high, then asymptotically as $d \to \infty$ with $KM/d \to 0$, the risk of a suitable thresholding estimator is equal to $R_{\text{oracle}}(\mu) (1 + o(1))$. The reason is that in this case one can achieve exact support recovery with high probability. We now prove a similar result for the distributed setting.
4.2 The \( \ell_2 \) risk of the Top-L and thresholding algorithms

The Top-L and thresholding algorithms described in Section 3 output an estimated support set \( \hat{\Omega} \). As we describe now, using an additional round of communication, the center can also estimate the vector \( \mu \) itself. In particular, we consider the following protocol, denoted \( \Pi \): First, the center sends the indices of \( \hat{\Omega} \) to all \( M \) machines. Then, each machine \( i \) replies with the binary representation \( w_{i,k} = \text{Trunc}(x_{i,k}, U, P) \) for the estimated support coordinates \( k \in \hat{\Omega} \), for appropriately chosen \( U, P = O(\log d) \). The center proceeds to approximate the sample values as \( z_{i,k} = \text{Approx}(w_{i,k}, U, P) \) and calculates the empirical mean \( \bar{z}_k = \frac{1}{M} \sum_{i\in[M]} z_{i,k} \).

Finally, the center estimates \( \mu \) as follows

\[
\hat{\mu}_j^\Pi = \bar{z}_j \cdot 1 \left\{ j \in \hat{\Omega} \right\} .
\]

The scheme is outlined in Algorithm 3.

The following corollary shows that applying \( \Pi \) to the set \( \hat{\Omega} \) computed by one of our algorithms results in near-oracle \( \ell_2 \) risk \( R_{\Pi} = \mathbb{E} \left[ \| \mu - \hat{\mu}^\Pi \|_2^2 \right] \). Its proof appears in Appendix A.3.

**Corollary 1.** Let \( d \geq 5 \). Assume that the conditions of Theorem 1.B hold and let \( \hat{\Omega} \subset [d] \) be the estimate computed by the Top-L algorithm. In addition, assume that \( \mu_{\max} < d^\gamma \) for \( \gamma > 0 \). Then, the \( \ell_2 \) risk of \( \hat{\mu}_j^\Pi \) with precision parameters \( P = \lceil \log_2 d \rceil \) and \( U = \left\lfloor \log_2(d^\gamma + \sqrt{4(\gamma + 1) \log d}) \right\rfloor \) is bounded as follows

\[
R_{\Pi} \leq \frac{K}{M} \left( 1 + d^{-1} + d^{-2} \right) + \frac{2K\mu_{\min}^2}{d}. \tag{16}
\]

The expected total communication cost of \( \Pi \) is \( O(KM \log d) \). Thus, in an asymptotic setting where \( K, M, d \to \infty \) with \( \frac{KM \log d}{d} \to 0 \), the protocol \( \Pi \) has sublinear expected communication cost and its \( \ell_2 \) risk is \( R_{\text{oracle}}(\mu) (1 + o(1)) \).

If we assume that the conditions of either Theorem 2.A, Theorem 2.B or Theorem 2.C hold, then essentially the same proof shows that a two-round algorithm that first estimates the support of \( \mu \) by the respective thresholding algorithm and then applies protocol \( \Pi \) as a second round to estimate the vector \( \mu \) itself can achieve near-oracle \( \ell_2 \) risk with sublinear communication as well.

5 Relation to previous works

In the context of the distributed sparse normal means problem, several works derived communication lower bounds for exact support recovery and for the \( \ell_2 \) risk of any distributed scheme with total communication budget \( B \). We now describe in further detail three closely related previous works and their relation to our results.
5.1 Lower bounds on the $\ell_2$ risk in distributed settings

Braverman et al. [2016, Theorem 4.5] and Han et al. [2018, Theorem 7] derived communication lower bounds for the distributed minimax $\ell_2$ risk of estimating a $K$-sparse vector $\mu$. Their results imply that to achieve the centralized minimax rate, the required total communication by any distributed algorithm must be at least linear in $d$. However, their proof relies on sparse vectors with a very low signal-to-noise ratio. In contrast, in scenarios where the SNR is sufficiently high these bounds do not apply, and as our theoretical analysis reveals, both exact support recovery and rate-optimal $\ell_2$ risk are achievable with sublinear communication, provided that $KM \ll d$.

In more detail, Braverman et al. [2016] considered blackboard communication protocols, where all machines communicate via a public blackboard and the total number of bits that they can write in the transcript is bounded by $B$. Denote the set of estimators whose inputs are blackboard communication protocols by $\mathcal{O}_{bb}$ and the set of all $K$-sparse $d$ dimensional vectors as $S_{d,K}$. Their Theorem 4.5 states that if $d > 2K$, then the $\ell_2$ risk of any distributed estimator in this model is lower bounded by

$$R_{bb} = \inf_{\hat{\mu} \in \mathcal{O}_{bb}} \sup_{\mu \in S_{d,K}} \mathbb{E}[\|\hat{\mu} - \mu\|^2] \geq \Omega \left( \min \left\{ \sigma^2 K, \max \left\{ \sigma K \frac{d}{B}, \sigma K \frac{M}{B} \right\} \right\} \right). \quad (17)$$

Note that if the total communication $B$ is sublinear in $d$, then the above simplifies to $\Omega(\sigma^2 K)$, which is significantly larger than the centralized minimax rate, Eq. (15). The reason is that $R_{bb}$ involves a supremum over all $K$-sparse vectors, without any assumptions on their SNR. Indeed, in their analysis a vector $\mu$ with extremely low SNR is used to prove the bound.

Han et al. [2018] considered a more restricted case of one-shot protocols where each of the $M$ machines has a budget of at most $b$ bits that are sent simultaneously to the center, i.e. $B = Mb$. Denote by $\mathcal{O}_{sim}$ the set of estimators based on such protocols. Their Theorem 7 states that if $d \geq 2K$ and $M \geq \frac{Kd^2 \log(d/K)}{\min(b, d^2)}$, then the risk is lower bounded by

$$R_{sim} = \inf_{\hat{\mu} \in \mathcal{O}_{sim}} \sup_{\mu \in S_{d,K}} \mathbb{E}[\|\hat{\mu} - \mu\|^2] \geq \Omega \left( \frac{\sigma K}{M} \log \frac{d}{d/K} \cdot \max \left\{ \frac{d}{b}, 1 \right\} \right). \quad (18)$$

Two remarks are in place here. First, our protocol $\Pi$ described in Section 4 requires two rounds of two-way communication between the center and the machines instead of one-round of one-way communication from the machines to the center. In addition, during the first round a subset of the machines may not be contacted and thus remain idle. For these reasons our estimator $\hat{\mu}_{\Pi}$ is not in $\mathcal{O}_{sim}$, and thus the above lower bound does not apply to it.

Second, the lower bound (18) does not apply for estimators in $\mathcal{O}_{sim}$ with sublinear communication, since the condition on $M$ translates to requiring $B \geq d$. To show this, notice that if $B < d$ then in particular each machine has a sublinear communication budget, i.e., $b = d^3$ for $0 < \beta < 1$. The requirement on the number of machines then translates to $M \geq Kd^{2-2\beta} \log(d/K)$, and thus the total communication budget is $B = Mb \geq Kd^{2-\beta} \log(d/K)$, which is superlinear in $d$ for all $\beta < 1$. 

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5.2 Lower bound on exact support recovery in a distributed setting

Shamir [2014] proved several lower bounds for distributed estimation problems under communication constraints. The following corollary follows from his Theorem 5, and relates his result to the sparse normal means problem. For completeness we present its proof in Appendix A.4.

**Corollary 2.** Let the dimension $d > 36$ and let the SNR parameter be $r = O \left( \log^{-3} d \right)$. Then the probability of recovering the support of a $1$-sparse vector $\mu$ by any algorithm with communication budget $b$ per machine is upper bounded by

$$\Pr[\text{support recovery}] \leq O \left( \frac{1}{d} + \sqrt{\frac{Mb}{d}} \right).$$

In particular, for any distributed algorithm with total communication budget $B = Mb$, its probability to correctly recover the support of a $1$-sparse vector $\mu$ is bounded by $O(1/d + \sqrt{B/d})$. If the communication budget is sublinear, i.e., $B = o(d)$, then the probability of exact support recovery by any distributed scheme is $o(1)$.

As we proved in Section 3, if the SNR is slightly higher, namely $r > \Omega \left( \frac{\log^3 \log d}{\log^2 d} \right)$, then exact support recovery is possible using sublinear communication. This implies that our results are nearly tight. It would be interesting to study if any distributed scheme can recover the support using sublinear communication for SNR values between the aforementioned lower bound our upper bound and thus close this theoretical gap.

6 Simulations

We present several simulations that illustrate the ability of our algorithms to detect the support of a $K$-sparse $d$-dimensional vector $\mu$ with sublinear communication. In the simulations below, we compare the performance of the Top-$L$ algorithm when $L = K$ (blue), the Top-$L$ algorithm when $L > K$ (red), variant A of the thresholding algorithm which contacts all machines, i.e., $M_c = M$ (orange), and variant B of the thresholding algorithm which limits the number of contacted machines, i.e., $M_c < M$ (purple). See Appendix B for details on optimizing simulation parameters.

Figure 2 depicts the success probabilities and communication costs (on a logarithmic scale) of the aforementioned algorithms as a function of $r$, averaged over 100 noise realizations. We consider three different settings of parameters $M$ and $K$. In all settings the dimension is $d = 2^{15}$ and in the Top-$L$ algorithm with $L > K$ we set $L = 10$. In Setting 1, $M = 2^6$ and $K = 1$; in Setting 2, $M = 2^6$ and $K = 5$; and in Setting 3 $M = 2^{10}$ and $K = 1$.

The vertical black dashed line is calculated by the maximum between the centralized information theoretic lower bound of $1/M$ and the $\log^{-3} d$ lower bound of [Shamir 2014]. Up to multiplicative constants, this line represents the necessary SNR, below which any sublinear algorithm fails with high probability.

In addition, we define a sufficient SNR bound for each algorithm, above which it exactly recovers the support with high probability $1 - O(Kd^{-1})$. The vertical blue and red dashed lines correspond to sufficient SNR bounds for the Top-$K$ and Top-$L$ algorithms, respectively.
Figure 2: The plots on the left depict the success probability of the algorithms from Section 3 as a function of $r$ in Settings 1-3. The plots on the right depict the communication cost of the algorithms on a logarithmic scale as a function of $r$ in these settings. The blue curve corresponds to the Top-$K$ algorithm, the red curve corresponds to the Top-$L$ algorithm, the orange and purple curves correspond to variants A and B of the thresholding algorithm, respectively. The vertical black line is a lower bound on the performance of all algorithms. The colored vertical lines are sufficient SNR bounds for the corresponding algorithms, as described in the main text.

The vertical orange dashed line corresponds to the sufficient SNR bounds for the thresholding algorithms. Note that the bounds depicted by all the aforementioned sufficient SNR lines are conservative, and while they are quite tight in the presented settings, the actual range of SNRs where the algorithms are successful is often larger.

The simulation results reveal several interesting behaviors. First, when the SNR is extremely low, i.e., to the left of the dashed black line, none of our algorithms succeeds with high probability. Second, no algorithm uniformly outperforms the others for all parameter regimes.
At low SNR values, the thresholding algorithms have a higher success probability compared to the Top-$L$ algorithms, but require higher communication costs. Similarly, at low SNR values the success probability of the Top-$L$ algorithm increases with $L$ at the expense of higher communication. At high SNR values, all algorithms succeed with high probability, but the communication costs of the algorithms depend on the parameter settings. For example, the Top-$K$ algorithm can either incur a lower communication cost compared to the thresholding $B$ algorithm (Setting 1), or a higher one (Setting 2), or they can be comparable (Setting 3). In addition, there is a wide range of SNR values for which the communication costs of all algorithms decrease exponentially with $r$ and their total communication costs are sublinear in $d$.

To understand how a higher sparsity level $K$ affects the performance of the algorithms, we compare between Setting 1 and Setting 2. The communication cost of the Top-$K$ algorithm increases linearly with $K$. In contrast, dependence of the communication costs of the thresholding algorithms on $K$ varies with the SNR. Specifically, at low SNR values they are comparable for different values of $K$, but for high SNR values they increase linearly with $K$. This phenomenon is consistent with the higher number of messages containing support indices.

Finally we compare between Setting 1 and Setting 3 to understand how the availability of more machines affects the performance of the algorithms. With more machines, the Top-$L$ algorithms succeed at much lower SNR values, at the expense of higher communication costs. Variant A of the thresholding algorithm has a higher communication cost in Setting 3 compared to Setting 1 since it uses all machines. However, there is still a large range of SNR values where it is smaller than $d$, due to its adaptive threshold. As shown by our proofs, when $M$ is large, variant B of the thresholding algorithm performs similarly to the Top-$K$ algorithm, and they outperform the other algorithms.

### A Proofs

In our proofs we shall use the following well known auxiliary lemmas.

**Lemma 1** (Gaussian tail bounds). For $Z \sim \mathcal{N}(0,1)$ and $t > 0$,
\[
\frac{t}{\sqrt{2\pi(t^2 + 1)}} e^{-t^2/2} \leq \Pr[Z > t] \leq \frac{1}{\sqrt{2\pi t}} e^{-t^2/2}.
\]

If in addition $t \geq 1$,
\[
\Pr[Z > t] \geq \frac{1}{2\sqrt{2\pi t}} e^{-t^2/2}.
\]

A consequence of Eq. (19) is that the maximum of $n-1$ i.i.d. standard normal random variables $Z_1, \ldots, Z_{n-1} \sim \mathcal{N}(0,1)$ is highly concentrated around $\sqrt{2 \log n}$. In particular, by the well known identity $(1 - \frac{1}{n})^{n-1} \geq \frac{1}{e}$, for all $n \geq 2$
\[
\Pr\left[\max_{i \in [n-1]} Z_i > \sqrt{2 \log n}\right] = 1 - \left(1 - \Pr\left[Z_i > \sqrt{2 \log n}\right]\right)^{n-1}
\]
\[
\leq 1 - \left(1 - \frac{1}{n\sqrt{4\pi \log n}}\right)^{n-1} \leq 1 - \left(1 - \frac{1}{n}\right)^{n-1}
\]
\[
\leq 1 - e^{-1},
\]
where the third step follows from $4\pi \log n > 1$.

**Lemma 2 (Chernoff [1952]).** Suppose $X_1, \ldots, X_n$ are i.i.d. Bernoulli random variables and let $X$ denote their sum. Then, for any $\delta \geq 0$,

$$\Pr [X \geq (1 + \delta) \mathbb{E} [X]] \leq e^{-\frac{\delta^2 \mathbb{E}[X]}{2 + \delta}}, \quad (22)$$

and for any $0 \leq \delta \leq 1$,

$$\Pr [X \leq (1 - \delta) \mathbb{E} [X]] \leq e^{-\frac{\delta^2 \mathbb{E}[X]}{2}}. \quad (23)$$

Towards proving the main theorems, we need a few definitions. Denote by $I_{i,k}$ the indicator that machine $i$ sends the index $k$ to the fusion center and let $p_k = \Pr [I_{i,k} = 1]$. Further define the Binomial random variable $v_{\min} \sim \text{Bin} (M_c, p_{\min})$, where $p_{\min}$ is the probability that machine $i$ sends a support index whose nonzero coordinate is $\mu_{\min}$. In addition, we set the threshold $t_c = 4 \log d$.

We conclude this subsection with two useful lemmas. First, we show that if $p_{\min}$ is sufficiently high, then any support index receives a number of votes exceeding $t_c$ with high probability.

**Lemma 3.** If $p_{\min} \geq \frac{2t_c}{M_c}$, then

$$\Pr \left[ \min_{k \in \Omega} v_k < t_c \right] \leq \frac{K}{d}. \quad (26)$$

**Proof.** Let $\delta = 1 - \frac{t_c}{M_c p_{\min}}$. By the Chernoff bound (23),

$$\Pr [v_{\min} < t_c] = \Pr [v_{\min} < M_c p_{\min} (1 - \delta)] \leq \exp \left( -M_c p_{\min} \delta^2 / 2 \right). \quad (24)$$

The assumption $p_{\min} \geq \frac{2t_c}{M_c}$ implies that $\delta > 1/2$ and $\delta M_c p_{\min} > t_c$. Thus

$$\exp \left( -M_c p_{\min} \delta^2 / 2 \right) \leq \exp \left( -t_c / 4 \right) \leq 1/d, \quad (25)$$

where the last inequality follows from $t_c = 4 \log d$. Combining Eqs. (24) and (25) implies that the probability that $v_{\min} < t_c$ is bounded by $1/d$.

Now, fix $k \in \Omega$. By the independence of the noise in the different machines $v_k \sim \text{Bin} (M_c, p_k)$. By definition of $\mu_{\min}$, the coordinate $\mu_k \geq \mu_{\min}$ and thus $p_k \geq p_{\min}$. Therefore,

$$\Pr [v_k < t_c] \leq \Pr [v_{\min} < t_c].$$

Finally, by a union bound,

$$\Pr \left[ \min_{k \in \Omega} v_k < t_c \right] \leq \frac{K}{d}. \quad \square$$

Next, let us consider the non-support coordinates. The following lemma shows that if $p_j$ is sufficiently low for each non-support index $j \notin \Omega$, then no non-support index receives more than $t_c$ votes with high probability.
Lemma 4. If for each \( j \not\in \Omega \), the probability \( p_j \leq \frac{t_c}{5M_c} \), then
\[
\Pr[\max_{j \in \Omega} v_j > t_c] \leq \frac{1}{d}.
\]

Proof. The average number of messages at the fusion center containing index \( j \) is \( E[v_j] = M_c p_j \). Let
\[
\delta = \frac{1}{M_c p_j} (t_c - M_c p_j) = \frac{t_c}{M_c p_j} - 1.
\]
The assumption \( p_j \leq \frac{t_c}{5M_c} \) implies that \( \delta M_c p_j = t_c - M_c p_j \geq \frac{4t_c}{5} \) and \( \delta \geq 4 \), which in turn implies that \( \delta / (2 + \delta) \geq 2/3 \). Note that for each \( j \in [d] \) the random variables \( I_{1,j}, \ldots, I_{M,j} \) are independent. By a Chernoff bound (22),
\[
\Pr[v_j > t_c] = \Pr[v_j > M_c p_j (1 + \delta)] \leq \exp \left( -\frac{\delta}{2 + \delta} \delta M_c p_j \right) \leq \exp \left( -\frac{8t_c}{15} \right). \tag{26}
\]
Since \( t_c = 4 \log d \), the above probability is at most \( d^{-2} \). We conclude by applying a union bound,
\[
\Pr[\max_{j \not\in \Omega} v_j > t_c] \leq (d - k) \Pr[v_j > t_c] \leq 1/d.
\]

A.1 Proof of Theorem 1

We begin by proving Theorem 1.A where \( L = K = 1 \) and then outline the necessary changes in order to prove Theorem 1.B for \( L \geq K \geq 1 \).

For future use, note that by definition of the Top-\( L \) algorithm, the probability that machine \( i \) sends a coordinate \( k \in [d] \) is
\[
p_k = \Pr[\exists j_1, \ldots, j_{d-L} \in [d] \setminus \{k\} : x_{i,k} > x_{i,j_1}, \ldots, x_{i,j_{d-L}}]. \tag{27}
\]

The communication of the Top-\( L \) algorithm is \( B = O( M_c (L + 1) \log d ) \) since the center sends one message to each participating machine indicating \( L < d \), and each of these machines sends back exactly \( L \) indices.

Proof of Theorem 1.A. Without loss of generality, let the support index be \( \Omega = \{1\} \). Thus,
\[
p_1 = \Pr[x_{i,1} > \max_{j > 1} x_{i,j}].
\]
We show that w.h.p. both \( v_1 > t_c \) and \( v_j < t_c \) for all \( j > 1 \).

By the law of total probability and the independence of the random variables \( \xi_{i,j} \),
\[
p_{\min} = \Pr[\sqrt{2r \log d + \xi_{i,1} > \max_{j > 1} \xi_{i,j}}]
\geq \Pr[\sqrt{2r \log d + \xi_{i,1} > \sqrt{2 \log d}} \mid \max_{j > 1} \xi_{i,j} < \sqrt{2 \log d}] \cdot \Pr[\max_{j > 1} \xi_{i,j} < \sqrt{2 \log d}]
\geq \Pr[\xi_{i,1} > (1 - \sqrt{r}) \sqrt{2 \log d}] \cdot \Pr[\max_{j > 1} \xi_{i,j} < \sqrt{2 \log d}].
\]
Recall that the random variables $\xi_{i,j}$ are i.i.d standard Gaussians. By Eq. (21),

$$\Pr \left[ \max_{j > 1} \xi_{i,j} < \sqrt{2 \log d} \right] \geq e^{-1}.$$ 

Therefore, by the Gaussian tail bound (19),

$$p_{\min} \geq e^{-1} \cdot \frac{(1 - \sqrt{r}) \sqrt{2 \log d}}{\sqrt{2 \pi} \left( 2 (1 - \sqrt{r})^2 \log d + 1 \right)} d^{- (1 - \sqrt{r})^2}. \quad (28)$$

Combining Eq. (28) with the bound (3) implies that $p_{\min} \geq \frac{2t_c}{M_c}$, and thus we can apply Lemma 3 and get that $\Pr \left[ v_1 < t_c \right] \leq d^{-1}$.

Now consider a non-support index $j > 1$. By symmetry considerations, the probability that machine $i$ sends $j$ to the center is

$$p_j = \frac{1 - p_1}{d - 1}.$$ 

Recall that by definition of $\mu_{\min}$, the coordinate $\mu_k \geq \mu_{\min}$ and thus $p_1 \geq p_{\min}$. Since for any strictly positive SNR $p_1 > p_{\min} > \frac{1}{\pi}$, it follows that $p_j < \frac{1}{\pi}$ for each $j > 1$. Hence, the expected number of votes for index $j$ is $E[v_j] = M_c p_j < \frac{M_c}{p_j}$. Let $\delta = \frac{t_c}{M_c p_j} - 1$ and note that the assumption $M \leq d$ implies that $M_c p_j \leq 1$ and hence $\delta \geq 4 \log d - 1 > 0$. By the Chernoff bound (22),

$$\Pr \left[ v_j > t_c \right] = \Pr \left[ \sum_{i=1}^{M_c} I_{i,j} > (1 + \delta) M_c p_j \right] \leq e^{-\frac{\delta^2 M_c p_j}{2 + \delta}} = e^{-\frac{4 \log d - M_c p_j}{4 \log d + M_c p_j}} \leq e^{\delta^2 M_c p_j} \leq e^3 d^{-4}.$$ 

By a union bound over all $d - 1$ non-support coordinates,

$$\Pr \left[ \max_{j > 1} v_j > t_c \right] \leq (d - 1) \cdot e^3 d^{-4} \leq e^3 d^{-3}.$$ 

By an additional union bound on the two events, the algorithm outputs the correct support index with probability at least $1 - d^{-1} - e^3 d^{-3}$.

**Proof of Theorem 1.B** The proof is similar to that of Theorem 1.A with the following changes. For any threshold $a \in \mathbb{R}$, the probability $p_k$ that $k \in \Omega$ is sent to the fusion center is lower bounded by

$$p_k \geq \Pr \left[ x_{i,k} > a , \sum_{j \notin \Omega} 1 \{ x_{i,j} > a \} \leq L - K \right]$$

$$= \Pr \left[ \xi_{i,k} > a - \mu_k , \sum_{j \notin \Omega} 1 \{ \xi_{i,j} > a \} \leq L - K \right].$$
Set \( a = a(K, L, d) \) and \( b = b(K, L, d, r) \) by Eqs. (4) and (5) respectively. Recall that \( \xi_{i,j} \) are i.i.d. for all \( i \in [M] \) and \( j \in [d] \), i.e., the two events in the probability above are independent of each other. Combining this with the definition of \( \mu_{\text{min}} \) yields

\[
\mu_{\text{min}} \geq \Pr \left[ Z > b \right] \cdot \Pr \left[ \sum_{j \notin \Omega} 1 \{ Z_j > a \} \leq L - K \right], \tag{29}
\]

where \( Z, Z_j \sim \mathcal{N}(0, 1) \).

We begin by bounding the first term of Eq. (29). If \( b \leq 0 \) then \( \Pr \left[ Z > b \right] \geq 1/2 \). Otherwise, by the Gaussian tail bound (19),

\[
\Pr \left[ Z > b \right] \geq \frac{b}{\sqrt{2\pi} (b^2 + 1)} (d - K)^{-\left(\sqrt{1+\frac{\log(l/K+1)}{\log(d-K)}} - \sqrt{d-K}\right)^2}.
\]

Next, we show that with probability \( \geq \frac{1}{4} \) the number of non-support indices that pass the threshold \( a \) is upper bounded by \( L - K \). Denote by \( p_a \) the probability that a standard normal random variable passes the threshold \( a \), i.e., \( p_a \equiv \Pr [Z_j > a] \). By Eq. (19), \( p_a \) is upper bounded by

\[
p_a \leq \frac{1}{\sqrt{2\pi a}} \cdot \frac{L - K + 1}{d - K}.
\tag{30}
\]

Next, let \( \delta = \frac{L-K+1}{p_a(d-K)} - 1 \). Note that the assumption \( K \leq L < (d-K)/2 \) implies that \( \sqrt{2\pi a} \geq \sqrt{4\pi \log 2} > 1 \), and thus \( \delta > 0 \). By the Chernoff bound (22),

\[
\Pr \left[ \sum_{j \notin \Omega} 1 \{ Z_j > a \} \geq L - K + 1 \right] = \Pr \left[ \sum_{j \notin \Omega} 1 \{ Z_j > a \} \geq (1 + \delta) p_a (d - K) \right]
\leq e^{-\delta^2 p_a (d-K)/2-\delta}
= e^{-\left(\frac{L-K+1}{p_a(d-K)}-1\right)^2 \frac{p_a(d-K)}{1+\frac{L-K+1}{p_a(d-K)}}}
= e^{-\left(\frac{L-K+1-p_a(d-K)}{L-K+1+p_a(d-K)}\right)^2}.
\]

For \( A_1, A_2 > 0 \) the function \( e^{-\frac{(A_1-A_2)^2}{A_1+A_2}} \) is monotonically increasing in \( A_2 \). Letting \( A_1 = L - K + 1 \) and \( A_2 = p_a (d-K) \), we can now apply the upper bound on \( A_2 \) in Eq. (30) to the equation above. Thus the complementary probability, i.e., the second term in Eq. (29), can be lower bounded as follows

\[
\Pr \left[ \sum_{j \notin \Omega} 1 \{ Z_j > a \} \leq L - K \right] \geq 1 - e^{-\left(\frac{L-K+1}{1+\sqrt{2\pi a}}\right)^2}
\geq 1 - e^{-\frac{1}{1+\sqrt{4\pi \log 2}}} \geq \frac{1}{4},
\]

where the second inequality follows from the assumption \( K \leq L < (d-K)/2 \).
By Eq. (6) the probability $p_{\min} \geq 2t_c^c M$ and thus $\Pr[\min_{k \in \Omega} v_k < t_c] \leq Kd^{-1}$ by Lemma 3. Let $W_i \sim Bin(K, p_{\min})$ be a binomial random variable that serves as a lower bound for how many of the support coordinates machine $i$ sends to the center. By the law of total probability and symmetry of the non-support indices, the probability that machine $i$ sends to the center a non-support index $j \notin \Omega$ is

$$p_j \leq \sum_{n=0}^{K} \Pr[i \in V_j \mid W_i = n] \cdot \Pr[W_i = n] = \sum_{n=0}^{K} \frac{L-n}{d-K} \Pr[W_i = n] = \frac{L-Kp_{\min}}{d-K} \leq \frac{L}{d-K}.$$  \hspace{1cm} (31)

Using the requirement $M_{K,L} \leq \frac{d-K}{L}$, the rest of the proof continues in the same manner. \hfill \square

A.2 Proof of Theorem 2

Note that we set the precision parameters $P, U$ such that $t_m - 1/d \leq \hat{t}_m \leq t_m$. By definition of the thresholding algorithm, the probability that machine $i$ sends a support coordinate $k \in \Omega$ is

$$p_k = \Pr[x_{i,k} > \hat{t}_m] \geq \Pr[\xi_{i,k} > t_m - \mu_k].$$  \hspace{1cm} (32)

Thus, for the extreme case $\mu_k = \mu_{\min}$,

$$p_{\min} \geq \Pr[Z > t_m - \mu_{\min}] \geq \Pr[Z > t_m - \mu_{\min}],$$  \hspace{1cm} (33)

for $Z \sim N(0,1)$. For a non-support coordinate $j \notin \Omega$, the Gaussian tail bound \cite{19} implies that

$$p_j = \Pr[\xi_{i,j} > \hat{t}_m] \leq \frac{e^{-\mu^2_{\min}/2}}{\sqrt{2\pi t_m}} \leq e^{t_m/d} \frac{e^{-\mu^2_{\min}/2}}{\sqrt{2\pi (t_m - 1/d)}}.$$  \hspace{1cm} (34)

In terms of communication, each coordinate $j \in [d]$ appears in $M_{c}p_j$ messages on average. In addition, in the setup stage the fusion center sends $M_c$ messages with the truncated threshold $\hat{t}_m$, whose binary representation is $O(\log d)$ bits long. Hence the average total communication is

$$\mathbb{E}[B] = O\left(M_c \log d + \left(\sum_{k \in \Omega} p_k + \sum_{j \notin \Omega} p_j\right) M_c \log d\right) = O\left(M_c \log d + \left(K + \sum_{j \notin \Omega} p_j\right) M_c \log d\right),$$  \hspace{1cm} (35)

where the last step follows from the trivial bound $p_k \leq 1$ for each $k \in \Omega$.

We now proceed to proving the sub-theorems.

Proof of Theorem 2.A  By Eq. (7),

$$p_{\min} \geq \Pr[Z > 0] = \frac{1}{2}.$$  \hspace{1cm} (36)
Since \( t_c = 4 \log d \) and \( M_c = \lceil 16 \log d \rceil \), we have that \( p_{\text{min}} \geq 2t_c/M_c \), and thus \( \Pr[\min_{k \in \Omega} v_k < t_c] \leq \frac{K}{d} \) by Lemma 3. Now fix \( j \notin \Omega \). Due to the assumptions \( d \geq 16 \) and \( r > \frac{1}{\log(d-K)} \), by Eq. (34) we have that

\[
p_j \leq \frac{e^{\sqrt{2r \log(d-K)/d} (d-K)^{-r}}}{\sqrt{2\pi} \left( \sqrt{2r \log(d-K)} - 1/d \right)} \leq \frac{e^{2 \log 5/16}}{10 \left( \sqrt{\pi \log 5} - 1/16 \right)} \leq \frac{t_c}{5M_c}.\]

Applying Lemma 4 yields \( \Pr[\max_{j \in \Omega} |v_j| > t_c] \leq 1/d \).

Finally, the average total communication follows from inserting the expressions for \( p_j \) and \( M_c \) into Eq. (35).

Proving of Theorem 2.18 Note that the bound \( M > \sqrt{e \cdot 32 \sqrt{\pi \log^1.5 d}} \) implies that \( 2 \log \frac{M}{32 \sqrt{\pi \log^1.5 d}} > 1 \). By the expression (10) for \( t_m \) and the Gaussian tail bound (20),

\[
p_{\text{min}} \geq \Pr \left[ Z > \frac{2 \log \frac{M}{32 \sqrt{\pi \log^1.5 d}}}{\sqrt{2 \log \frac{5M}{2 \pi 4 \log d}}} \right] \geq \frac{1}{2 \sqrt{2\pi}} \sqrt{2 \log \frac{5M}{2 \pi 4 \log d}} e^{-\frac{5M}{2 \pi 4 \log d}} \geq t_c \left( \begin{array}{c} \frac{1}{M} \end{array} \right), \quad (37)\]

where the last inequality follows from the upper bound on \( M \). Thus, by Lemma 3, \( \min_{k \in \Omega} v_k < t_c \) with probability at most \( K/d \). Due to Assumption (7), \( t_m \geq \sqrt{2 \log \frac{5M}{2 \pi 4 \log d}} \), and thus by the first inequality of Eq. (34),

\[
p_j \leq \frac{1}{2 \sqrt{2\pi}} \sqrt{2 \log \frac{5M}{2 \pi 4 \log d}} e^{-\frac{5M}{2 \pi 4 \log d}} = \frac{4 \log d}{5M} \leq \frac{t_c}{5M_c}, \quad (38)\]

where the last inequality follows from the definition of \( t_c \) and the condition on \( M \). Thus, by Lemma 4, \( \Pr[\max_{j \in \Omega} |v_j| > t_c] \leq 1/d \).

Towards computing the expected communication of the algorithm, we bound \( p_j \) more carefully using the second inequality of Eq. (34),

\[
p_j \leq \frac{e^{\left( \sqrt{2r \log(d-K)} + \sqrt{2 \log \frac{5M}{32 \sqrt{\pi \log^1.5 d}}} /d \right) (d-K)^{-r}} e^{-2\sqrt{r \log(d-K)} \log \frac{M}{32 \sqrt{\pi \log^1.5 d}}} /d}}{\sqrt{2} \left( \sqrt{2r \log (d-K)} + \sqrt{2 \log \frac{5M}{32 \sqrt{\pi \log^1.5 d}}} - 1/d \right)} \leq \frac{(d-K)^{-r} e^{-2\sqrt{r \log(d-K)} \log \frac{M}{32 \sqrt{\pi \log^1.5 d}}}}{M} \cdot \frac{32 \log 1.5 d}{M}, \quad (39)\]

where the second inequality follows from bounding \( \sqrt{2r \log (d-K)} + \sqrt{2 \log \frac{5M}{32 \sqrt{\pi \log^1.5 d}}} - 1/d > 1 \) and from \( d \geq 15 \), which implies that \( e^{\left( \sqrt{2r \log(d-K)} + \sqrt{2 \log \frac{5M}{32 \sqrt{\pi \log^1.5 d}}} /d \right) < \sqrt{2} \). By inserting Eq. (39) into Eq. (35), the expected communication of the algorithm is

\[
\mathbb{E}[B] = O \left( KM \log d + (d-K) \cdot (d-K)^{-r} e^{-2\sqrt{\log(d-K)} \log \frac{M}{32 \sqrt{\pi \log^1.5 d}}} /d \cdot M \log d \right) \cdot \quad (39)\]

Rearranging completes the proof.\]
**Proof of Theorem 2.3** Recall that $t_m = \sqrt{2 \log (d - K)}$. By the Gaussian tail bound (19) and the definition of $M_c$ in Eq. (12),

$$p_{\min} \geq \Pr \left[ Z > (1 - \sqrt{r}) \sqrt{2 \log (d - K)} \right] \geq \frac{(1 - \sqrt{r}) \sqrt{2 \log (d - K)}}{\sqrt{2\pi} \left( (1 - \sqrt{r})^2 2 \log (d - K) + 1 \right)} (d - K)^{-1} \cdot (1 - \sqrt{r})^2 = \frac{2t_c}{M_c}. \quad (40)$$

Thus by Lemma 3, $\min_{k \in \Omega} v_k < t_c$ with probability at most $K/d$.

Fix a non-support index $j \notin \Omega$. Note that the assumption $d - K \geq 20$ implies that $e^{\sqrt{2 \log (d - K)/d}} < \sqrt{2}$. Thus, by Eq. (34),

$$p_j \leq \frac{e^{\sqrt{2 \log (d - K)/d}}}{\sqrt{2\pi} \left( \sqrt{2 \log (d - K) - 1/d} \right)} (d - K)^{-1} \leq \frac{1}{\sqrt{\pi} \left( \sqrt{2 \log (d - K) - 1/d} \right)} (d - K)^{-1}. \quad (41)$$

It is easy to verify that the aforementioned assumption and the condition $r > \left( \frac{\log 10}{2 \log (d - K)} \right)^2$, or equivalently, $e^{2\sqrt{r}} \geq 10$, imply that $p_j < \frac{t_c}{M_c}$. Thus the desired bound $\Pr[\max_{j \notin \Omega} v_j > t_c] \leq 1/d$ follows from Lemma 4.

By inserting Eq. (41) into Eq. (35), the expected communication of the algorithm is

$$\mathbb{E} [B] = O \left( KM_c \log d + (d - K) \cdot \frac{1}{\sqrt{2\pi} \sqrt{2 \log (d - K)}} (d - K)^{-1} \cdot M_c \log d \right).$$

Using Eq. (12) concludes the proof. \hfill \Box

### A.3 Proof of Corollary 1

We first analyze the total communication cost of $\Pi$. Each machine $i$ sends a message $w_i$ consisting of the truncated binary representations of $x_{i,k}$ for $k \in \Omega$. Recall that the length of each $w_{i,k}$ is $P + U + 2$ bits. Since $P, U = O(\log d)$, the expected total communication cost of $\Pi$ is $O(KM \log d)$.

Let $\hat{\mu}$ be the output of protocol $\Pi$, and recall that $\hat{\mu}_j = \bar{z}_j \cdot 1 \{ j \in \Omega \}$. By linearity of expectation and the law of total probability,

$$\mathbb{E} \left[ \| \mu - \hat{\mu} \|_2^2 \right] = \sum_{j \in [d]} \mathbb{E} \left[ (\mu_j - \hat{\mu}_j)^2 \right] = \sum_{j \in [d]} \left( \mathbb{E} \left[ (\mu_j - \bar{z}_j)^2 \right] \Pr [ j \in \Omega ] + \mu_j^2 \Pr [ j \notin \Omega ] \right). \quad (42)$$

We now bound each of the terms in the RHS.

Fix $j \in [d]$. Since $\mathbb{E} [x_j] = \mu_j$, it follows that

$$\mathbb{E} \left[ (\mu_j - \bar{z}_j)^2 \right] = \mathbb{E} \left[ (\mu_j - \bar{x}_j + \bar{x}_j - \bar{z}_j)^2 \right] = \mathbb{E} \left[ (\mu_j - \bar{x}_j)^2 \right] + \mathbb{E} \left[ (\bar{x}_j - \bar{z}_j)^2 \right].$$

Furthermore, since the noise in different machines is i.i.d.,

$$\mathbb{E} \left[ (\mu_j - \bar{x}_j)^2 \right] = \frac{1}{M} \mathbb{E} \left[ (\mu_j - x_{i,j})^2 \right] = \frac{1}{M},$$

where $x_{i,j}$ is the $j$-th bit of the $i$-th machine's input.
and
\[ \mathbb{E} \left[ (x_j - z_j)^2 \right] = \frac{1}{M} \mathbb{E} \left[ (x_{i,j} - z_{i,j})^2 \right] \]
for any fixed \( i \in [M] \).

We now bound \( \mathbb{E} \left[ (x_{i,j} - z_{i,j})^2 \right] \) for any fixed \( j \in [d] \) and \( i \in [M] \). Since \( x_{i,j} \sim \mathcal{N}(\mu_j, 1) \) and \( z_{i,j} \) is a deterministic function of it, then
\[ \mathbb{E} \left[ (x_{i,j} - z_{i,j})^2 \right] = \int_{-\infty}^{\infty} (x - z(x))^2 \exp \left( -\frac{(x - \mu_j)^2}{2} \right) \frac{1}{\sqrt{2\pi}} dx. \]

If \( x < 2^{U+1} \), then the truncation step of the protocol implies that the remainder is bounded such that \( |x - z| \leq 2^{-P} \). Otherwise, the value \( x \) is higher than the range that is representable using \( U + 1 \) bits before the binary dot, and thus the magnitude \( |x - z| \) can be as large as \( |x| \) itself. Therefore,
\[ \mathbb{E} \left[ (x_{i,j} - z_{i,j})^2 \right] \leq 2^{-P} + 2 \int_{2^{U+1}}^{\infty} x^2 \exp \left( -\frac{(x - \mu_j)^2}{2} \right) \frac{1}{\sqrt{2\pi}} dx. \]

Using integration by parts,
\[ \int_{2^{U+1}}^{\infty} x^2 \exp \left( -\frac{(x - \mu_j)^2}{2} \right) \frac{1}{\sqrt{2\pi}} dx \leq \frac{1}{\sqrt{2\pi}} \left( e^{(2^{U+1} + \mu_j)^2/2} + (1 + \mu_j^2)^\infty \right) \int_{2^{U+1}}^{\infty} \exp \left( -\frac{(x - \mu_j)^2}{2} \right) \frac{1}{\sqrt{2\pi}} dx. \]

By the Gaussian tail bound \([19]\),
\[ \int_{2^{U+1}}^{\infty} x^2 \exp \left( -\frac{(x - \mu_j)^2}{2} \right) \frac{1}{\sqrt{2\pi}} dx \leq \frac{1}{\sqrt{2\pi}} \left( 2^{U+1} + \mu_j + \frac{1 + \mu_j^2}{2^{U+1} - \mu_j} \right) e^{(2^{U+1} - \mu_j)^2/2} \]
\[ \leq \frac{1}{\sqrt{2\pi}} \left( \sqrt{4(\gamma + 1) \log d + 2d^\gamma} + \frac{1 + d^{2\gamma}}{\sqrt{4(\gamma + 1) \log d}} \right) d^{-2(\gamma + 1)} \leq \frac{1}{2d^\gamma} \]

where the second inequality follows from the bound \( \mu_{\text{max}} < d^\gamma \) and the selection \( U \), and the last inequality holds for all \( d \geq 5 \) and \( \gamma \geq 0 \). Finally, Since \( P = \lfloor \log_2 d \rfloor \),
\[ \mathbb{E} \left[ (x_{i,j} - z_{i,j})^2 \right] \leq d^{-2} + d^{-1}. \]

In addition, since \( \mathbb{E} \left[ |\Omega| \right] = K \), the sum \( \sum_{j \in [d]} P \left[ j \in \Omega \right] = K \), and thus the first term in the RHS of Eq. (42) is bounded by \( \frac{K}{M} (1 + d^{-2} + d^{-1}) \).

It remains to prove that for each support index \( k \in \Omega \),
\[ \mu_k^2 P \left[ k \notin \Omega \right] \leq 2\mu_{\text{min}}^2/d. \]

Denote by \( G \) the “good” event that each non-support index \( j \notin \Omega \) receives less than \( t_c = 4 \log d \) votes. Fix \( k \in \Omega \). By the law of total probability,
\[ P \left[ k \notin \Omega \right] \leq \left( P \left[ k \notin \Omega | G \right] P \left[ G \right] + (1 - P \left[ G \right]) \right). \]

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Conditioned on $G$, the index $k \in \hat{\Omega}$ if $v_k > t_c$. The complementary probability can be bounded by Chernoff (23),

$$\Pr \left[ k \notin \hat{\Omega} | G \right] \leq \Pr \left[ v_k < t_c \right] \leq e^{-\frac{1}{2}(M_c p_k - t_c) \left( 1 - \frac{t_c}{M_c p_k} \right)}. $$

Recall that under the conditions of Theorem 1.B, $p_j \leq \frac{t_c}{5M_c}$ for each non-support index $j \notin \Omega$. Therefore $\Pr \left[ G \right] \geq 1 - 1/d$ by Lemma 4. Thus,

$$\mu_k^2 \Pr \left[ k \notin \hat{\Omega} \right] \leq \mu_k^2 \left( e^{-\frac{1}{2}(M_c p_k - t_c) \left( 1 - \frac{t_c}{M_c p_k} \right)} (1 - 1/d) + 1/d \right). $$

(43)

In addition, recall that $p_k$ is defined by Eq. (27) for the Top-$L$ algorithm (or by Eq. (32) for the thresholding algorithm), and decays exponentially with $\mu_k$. Therefore, the right hand side of Eq. (43) is monotonically decreasing in $\mu_k$, and thus upper bounded by

$$\mu_k^2 \Pr \left[ k \notin \hat{\Omega} \right] \leq \mu_k^2 \left( e^{-\frac{1}{2}(M_c p_{\min} - t_c) \left( 1 - \frac{t_c}{M_c p_{\min}} \right)} (1 - 1/d) + 1/d \right).$$

Recall that the assumption $p_{\min} \geq \frac{2t_c}{M_c}$ also holds under the conditions of Theorem 1.B. Thus we can apply Eq. (25) and get the desired bound

$$\mu_k^2 \Pr \left[ k \notin \hat{\Omega} \right] \leq \mu_{\min}^2 \left( \frac{1}{d} (1 - 1/d) + 1/d \right) \leq \frac{2\mu_{\min}^2}{d}.$$

We now turn to proving the last part of the corollary. The lower bound in Condition (2) implies that $\mu_{\min} > 1/\sqrt{M}$. Thus, by Eq. (14), the oracle risk is $R_{\text{oracle}}(\mu) = K/M$. In addition, the upper bound in Condition (2) implies that $\mu_{\min}^2 \leq 2 \log d$. Taking $K, M, d \to \infty$ with $KM \log d \to 0$ yields the desired result.

**A.4 Proof of Corollary 2**

Shamir [2014, Thm 5] considers estimation of a sparse covariance $\tilde{d} \times \tilde{d}$ matrix, where the underlying population covariance has a single non-zero off-diagonal entry. Essentially, it translates to a problem involving a vector $\mu$ of length $d = \tilde{d}(\tilde{d} - 1)/2$ for which all coordinates are zero, except one coordinate with value $\tau$. There are a total of $m$ instances of the form $\mu + \xi_j$ and the assumption is that the empirical mean at each coordinate is sub-Gaussian. The first part of Theorem 5 considers detection of the non-zero entry in a single machine. Relevant to us is the second part of Theorem 5. It assumes that at each of $M$ machines there are $n = \tilde{d}(\tilde{d} - 1) = 2d$ samples. It considers the class of all possible distributed protocols with a budget of $b$ bits per machine, $n = \tilde{d}(\tilde{d} - 1)$ samples per machine and $M = \lceil m/\tilde{d}(\tilde{d} - 1) \rceil$ machines. The key result is that with a signal strength $\tau = O \left( \frac{1}{\sqrt{d \log d}} \right)$, the probability of detecting the non-zero entry is bounded by

$$\Pr[\text{success in support recovery}] \leq O \left( \frac{1}{d} + \sqrt{\frac{Mb}{d^2}} \right) = O \left( \frac{1}{d} + \sqrt{\frac{Mb}{d}} \right).$$

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Let us now translate this result to our notations. Each machine has \( n = O(d) \) samples, which implies that each machine \( i \) may compute its normalized empirical mean \( \bar{x}_i = \frac{1}{\sqrt{n}} \sum_j x_i^{(j)} \), which is a sufficient statistic. Hence, we may now think of each machine having a single sample, with a noise level \( \sigma = 1 \), but with an effective signal strength \( \mu_{\text{min}} = \sqrt{n \tau} = O(\log^{-1} d) \). In terms of the quantity \( r \), via the relation \( \mu_{\text{min}} = \sqrt{2r \log d} \), the effective signal strength is thus \( r = O(\log^{-3} d) \).

\[
\text{B Simulation parameter settings}
\]

For simplicity of the proofs we did not fully optimize the choices of \( M_c \) and thresholds. We outline below the choices used for our simulations in Section 6. In terms of setup message length, in all simulations \( L \) is represented by \( \log L \) bits and \( \hat{t}_m \) is represented with \( U = 2 \) bits before the binary dot and \( P = 3 \) bits after the binary dot.

**Top-\( L \) algorithm.** We define the following random variables that represent bounds on the number of votes that a support coordinate \( k \in \Omega \) receives \( Y_{\text{top}}^s (d, r, K, L) = \text{Bin} \left( M_c, p_{\text{top}}^s \right) \) and on the number of votes that a non-support coordinate \( j \notin \Omega \) receives \( Y_{\text{top}}^n (d, r, K, L) = \text{Bin} \left( M_c, p_{\text{top}}^n \right) \), where \( p_{\text{top}}^s = p_s (d, r, K, L) \) is the probability that \( k \in \Omega \) is sent by machine \( i \), defined in Eq. (29), and \( p_{\text{top}}^n = p_n (d, r, K, L) \) is the probability that \( j \notin \Omega \) is sent by machine \( i \), defined in Eq. (31).

With high probability \( Y_{\text{th}}^n \) does not deviate from its expectation by more than a\( \frac{\log(d-K)}{\log \log(d-K)} \) multiplicative bound. Thus, we set the number of contacted machines as

\[
M_c = \max \left\{ \left\lfloor \frac{1}{p_{\text{top}}^s (d, r, K, L)} \cdot \frac{\log(d-K)}{\log \log(d-K)} \right\rfloor, 1 \right\}.
\]

Intuitively, this selection ensures that the expected number of votes for a fixed support index is equal to the maximal expected number of votes for any non-support index.

In all of our simulations \( \frac{\log(d-K)}{\log \log(d-K)} \cdot \mathbb{E} Y_{\text{top}}^n < 1 \). Hence, the sufficient SNR bound for the Top-\( L \) algorithm (vertical blue/red line) is the minimal \( r \) for which \( \mathbb{E} Y_{\text{top}}^s \geq 2 \), i.e., the support indices have at least 2 votes in expectation while the non-support indices have at most 1.

**Thresholding algorithm.** Similarly to the calculation for the Top-\( L \) algorithm, we define \( Y_{\text{th}}^s (d, r, K, t_m) = \text{Bin} \left( M_c, p_{\text{th}}^s \right) \) and \( Y_{\text{th}}^n (d, r, K, t_m) = \text{Bin} \left( M_c, p_{\text{th}}^n \right) \) as the number of votes for a support coordinate and non-support coordinate respectively, where \( p_{\text{th}}^s \) is by Eq. (33) and \( p_{\text{th}}^n \) is by Eq. (34).

For variant A, given \( r \) and \( M \), we set the number of contacted machines \( M_c = M \) and the threshold \( t_m \) as the highest \( t \) s.t.

\[
\Pr \left[ Y_{\text{th}}^s (d, r, K, t) < \mathbb{E} Y_{\text{th}}^n (d, r, K, t) \frac{\log(d-K)}{\log \log(d-K)} \right] < \frac{1}{d}.
\] (44)
Intuitively, Eq. (44) requires that the probability that the number of votes for a fixed support index is higher than the maximal expected number of votes for any non-support index is lower than $d^{-1}$.

In variant B, the parameters $t_m$ and $M_c$ are set in the following manner. If for $M_c = M$ the threshold $t < \sqrt{2 \log \frac{d-K}{K}}$, then $t_m = t$ as in variant A. Otherwise, we set $t_m = \sqrt{2 \log \frac{d-K}{K}}$ and take the lowest $M_c$ for which Eq. (44) with $t = t_m$ holds.

Let $r_{\min}$ and $t_{\min}$ denote the minimal $r$ value and the corresponding $t$ value for which Eq. (44) holds, respectively. $r_{\min}$ is the sufficient SNR bound for the thresholding algorithm (vertical orange line). Note that when $r < r_{\min}$, there is no value of $t$ for which this Eq. (44) holds. For completeness of the simulations, in this case we set $t = t_{\min}$.

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