Origin of the cosmological constant

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Abstract The observed value of the cosmological constant corresponds to a time scale that is very close to the current conformal age of the universe. Here we show that this is not a coincidence but is caused by a periodic boundary condition, which only manifests itself when the metric is represented in Euclidian spacetime. The circular property of the metric in Euclidian spacetime becomes an exponential evolution (de Sitter or $\Lambda$ term) in ordinary spacetime. The value of $\Lambda$ then gets uniquely linked to the period in Euclidian conformal time, which corresponds to the conformal age of the universe. Without the use of any free model parameters we predict the value of the dimensionless parameter $\Omega_\Lambda$ to be 67.2%, which is within $2\sigma$ of the value derived from CMB observations.

Keywords Dark energy · Cosmology: theory · Gravitation · Early universe · Physical data and processes

1 Introduction

The cosmological constant was introduced by Einstein (1917) to allow for the possibility of a static universe but was dismissed as a mistake after the expansion of the universe was discovered. Weinberg (1987) showed that the magnitude of the cosmological constant must be very small to permit our existence as observers. This anthropic argument led to speculations about the existence of parallel universes with a variety of values of the cosmological constant. The accelerated expansion of the universe that was discovered through the use of supernovae type Ia as standard candles (Riess et al. 1998; Perlmutter et al. 1999) could only be modeled in terms of Friedmann-Lemaître cosmology by using the cosmological constant as a free parameter chosen to fit the observations.

While the cosmological constant $\Lambda$ is very useful as a model fitting parameter, its physical nature has remained enigmatic. If it is interpreted as a physical field with an energy density $\rho_\Lambda$, then the circumstance that $\rho_\Lambda$ does not seem to depend on redshift or look-back time leads to a major conceptual problem. If its value remains constant with the evolution of the universe, then it would be an extraordinary coincidence that its value happens to be of the same order of magnitude as the current matter density $\rho_M$. In the past its role would have been insignificant, while in the future it would completely dominate and drive an exponential expansion of the universe. This would violate the Copernican principle, which says that we are not privileged observers.

The aim of the present paper is to show that the accelerated expansion of the universe is not caused by a new physical field but is induced by a boundary condition, which only manifests itself when the metric is expressed in terms of Euclidian spacetime. While this leads to a $\Lambda$ parameter that is constant across the 4D spacetime of the observable universe, its magnitude depends on the age of the universe such that the value of $\rho_\Lambda$ remains of the same order as the energy density of matter plus radiation throughout cosmic history. The cosmic coincidence problem then disappears. Without the use of any free parameters we derive a value of the cosmological constant parameter, $\Omega_\Lambda = 67.2\%$, which agrees within $2\sigma$ with the value obtained from observations (Planck Collaboration et al. 2018).
2 Time scale introduced by the cosmological constant

The Einstein equation with cosmological constant $\Lambda$ can be written in the form

$$R_{\mu\nu} - Ag_{\mu\nu} = \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right).$$

(1)

Here we adopt the standard sign convention of Misner et al. (1973), with $(- + + +)$ for the spacetime signature and a plus sign in front of the right-hand side.

In the weak-field approximation $R_{\mu\nu} \approx -\frac{1}{2} \partial^2 g_{\mu\nu}$ if we adopt the harmonic gauge. $a^2 \equiv -(1/c^2) \partial^2/\partial t^2 + \nabla^2$ is the d’Alembertian operator $\square^2$. Equation (1) then describes gravitational waves that propagate at the speed of light.

In the Fourier domain the operator $\partial^2/\partial t^2$ corresponds to a frequency squared, $\omega^2$, which defines a time scale $2\pi/\omega$. The combination $(1/c^2) \partial^2 g_{\mu\nu}/\partial t^2 - Ag_{\mu\nu}$ for the left-hand side of Eq. (1) shows that $\Lambda$ corresponds to a frequency scale $\omega_A$ and time scale $t_A$ defined by

$$\Lambda \equiv \frac{1}{2} (\omega_A/c)^2 \equiv 2 \left[ \pi/(ct_A) \right]^2.$$  

(2)

In standard cosmological treatments $A$ in Eq. (1) is generally moved to the right-hand side, where it formally appears as an equivalent mass-energy density $\rho_A$, which in modeling is represented by the dimensionless parameter $\Omega_A$. These parameters are defined by

$$\rho_A \equiv \Omega_A \rho_c \equiv \frac{c^2 A}{8\pi G},$$  

(3)

where we have introduced the critical density

$$\rho_c = \frac{3}{8\pi Gt_H^2},$$  

(4)

here expressed in terms of the Hubble time $t_H = 1/H$, where $H$ is the Hubble constant. $\rho_c$ represents the mean mass density in a Friedmann universe with zero spatial curvature, which defines the boundary between open and closed model universes.

Combining Eqs. (2)–(4) we obtain

$$\frac{t_A}{t_H} = \frac{2\pi}{\sqrt{6\Omega_A}}.$$  

(5)

Inserting the observational value of $\Omega_A \approx 0.685$ determined by Planck Collaboration et al. (2018), we find $t_A/t_H \approx 3.10$ which, as will be shown below, implies a value of $t_A$ that is very close to the current conformal age $\eta_u$ of the universe. The distance $r_u = c\eta_u$ represents the radius of the causal or particle horizon, and $\eta_u$ is the time that it would take for a photon to travel this distance if the universe would stop expanding. It is substantially longer than the normal age of the universe, because the spatial points from which light is emitted continually recede from us due to the cosmic expansion.

Instead of the temporal operator $\partial^2/\partial t^2$ we could have used the same arguments for the corresponding spatial operators and would have found that the observed value of $A$ corresponds to a spatial scale $r_A$ that is very close to the magnitude of $r_u$. While the discussion does not depend on the choice of temporal or spatial coordinates, we choose to refer to time scales $t_A$ and $\eta_u$ for convenience. This should become clearer when we develop the concepts in terms of the Robertson-Walker metric in the next section.

The circumstance that the observationally determined value of $t_A$ nearly equals the conformal age $\eta_u$ would be an astounding coincidence unless there is an underlying physical reason for it. Since $t_A \sim 1/\sqrt{\Lambda}$ and $\eta_u$ increases with the age of the universe, the coincidence would only apply to our particular epoch and make us be highly privileged observers, unless $A$ would scale with $1/\eta_u^2$ and thereby conserve the apparent “coincidence” for all cosmic epochs. In the following two sections we will see how this is achieved.

3 Boundary condition in Euclidian spacetime as the origin of the cosmological constant

Our aim here is to show that the cosmological constant has its origin in a boundary condition that ties its value to the conformal age of the universe. This boundary condition is hidden from view in the ordinary spacetime representation, but becomes exposed when the treatment is extended to Euclidian spacetime. When applied to the event horizon of black holes, the Euclidian approach offers a direct and elegant route to the derivation of the expression for the temperature of Hawking radiation, as first shown by Gibbons and Perry (1976). While our objective is to apply it to the Robertson-Walker metric for the cosmological case, it is helpful to begin with a look at the black hole case, to allow us to appreciate the underlying physics with its differences and similarities.

3.1 Euclidian route to the Hawking temperature

The Schwarzschild metric is given by

$$ds^2 = -(1 - r_s/r)c^2 dr^2 + (1 - r_s/r)^{-1} dr^2 + r^2 d\Omega,$$  

(6)

where

$$r_s = \frac{2GM}{c^2},$$  

(7)

is the radius of the event horizon, and

$$d\Omega = d\theta^2 + \sin^2 \theta d\phi^2$$  

(8)

is the surface element on the unit sphere.
Let us now perform a Wick rotation in the complex plane to transform ordinary time \( t \) to Euclidian time \( t_E = it \), to make time space like, thereby changing the signature of the metric from \((-++++)\) to \((++++)\). In the immediate neighborhood of the event horizon, where \( |r - r_s| \ll r_s \), the metric can be written as

\[
ds^2 = [(r - r_s)/r_s]c^2 dt_E^2 + [(r - r_s)/r_s]^{-1} dr^2 + r_s^2 d\Omega.
\]  
(9)

Following the treatment in Zee (2010), we now convert the metric near the horizon to the form

\[
ds^2 = R^2 d\alpha^2 + dR^2 + r_s^2 d\Omega,
\]  
(10)

where \( r \) and \( t_E \) have been replaced by two different coordinates, a distance parameter \( R \) and an angular coordinate \( \alpha \). The way to obtain this form is to first define \( R \) and a fiducial parameter \( \gamma \), which later will be disposed of, through \((r - r_s)/r_s \equiv \gamma^2 R^2 \). Since \( dr/dR = 2Rr_s\gamma^2 \), we find that the second term on the right-hand side of Eq. (9) is \((r - r_s)/r_s)^{-1} dt_E^2 = (2r_s\gamma)^2 dR^2 \), which becomes \(dR^2\) as required by Eq. (10) provided that we choose \( \gamma = 1/(2r_s) \). This finally gives us the desired form of Eq. (10) for the metric, if we define \( d\alpha \) through

\[
d\alpha \equiv \gamma c dt_E = \frac{c^3}{4GM} dr_E.
\]  
(11)

Equation (10) describes a 4D space that has an \( E^2 \otimes S^2 \) representation, i.e., it is a direct product between a Euclidian plane \( (E^2, \text{represented by the first two terms in Eq. (10)), which describe the line element in polar coordinates in the Euclidian plane}) \) and a 2-sphere \( (S^2) \), represented by the surface element \( d\Omega \) of the unit sphere.

Because \( \alpha \) is an angular coordinate with period \( 2\pi \) in the Euclidian plane, Euclidian time has a periodicity of \( \Delta t_E = 2\pi/(c\gamma) \) according to Eq. (11). The metric thus returns to the same state after the \( \Delta t_E \) interval. According to Euclidian field theory, which among other areas has important applications in condensed matter physics (cf. McCoy 1994), the transition amplitude between two states of a quantum field in Euclidian spacetime, when the initial and final states are the same, becomes the partition function of statistical mechanics in ordinary spacetime. In particular this implies that across the Euclidian time interval \( \Delta t_E = i \Delta t \) the oscillating phase factor \( \exp(i\omega \Delta t_E) = \exp(-i\omega \Delta t) \) can be identified in ordinary spacetime with the Boltzmann factor \( \exp(-\hbar \omega/(k_BT)) \).

This identification establishes a link between Euclidian field theory and thermodynamics, allowing us to assign a temperature to the event horizon of the black hole, as a consequence of the periodic boundary condition for the metric in Euclidian spacetime. With Eq. (11) we find

\[
T = \frac{\hbar}{k_B|\Delta t|} = \frac{\gamma\hbar c}{2\pi k_B} = \frac{\hbar c^3}{8\pi GMk_B}.
\]  
(12)

This agrees exactly with the expression for the Hawking temperature that has been derived by other, independent methods.

### 3.2 Euclidian route to the cosmological boundary condition

The Robertson-Walker metric in the case of zero spatial curvature is given by

\[
ds^2 = -c^2 dt^2 + a(t)^2 (dr^2 + r^2 d\Omega),
\]  
(13)

where \( a(t) \) is the scale factor and \( r \) is the comoving distance. Let us now introduce Euclidian conformal time \( \tau \) through the definition

\[
d\tau \equiv icd\eta \equiv icdt/a,
\]  
(14)

where \( \eta \) is ordinary conformal time. Note that the speed of light \( c \) has been incorporated in this definition, so that \( \tau \) has the dimension of space. The metric now acquires the conformal Euclidian form

\[
ds^2 = a(\tau)^2 (dt^2 + dr^2 + r^2 d\Omega).
\]  
(15)

According to Eq. (15) all components of the conformal metric are proportional to \( a(\tau)^2 \). Because monotonous, e.g. exponential, behavior in ordinary time becomes circular behavior in Euclidian time, we expect the Euclidian spacetime metric to have periodic boundary conditions and develop resonances, in particular since Euclidian time \( \tau \) is bounded between \( \tau = 0 \) (Big Bang) and \( \tau = \tau_u = ic\eta_u = ir_u \), where \( \eta_u \) is the conformal age of the observable universe and \( r_u \) the radius of the causal (particle) horizon at the given epoch.

The existence of a metric resonance with respect to Euclidian time implies that

\[
a(\tau)^2 \sim e^{i\alpha \omega/c},
\]  
(16)

where \( \omega \) is the resonant frequency that characterizes the periodicity. Because Euclidian time is circular and bounded, and the circle should close by the available finite time line \( \tau_u \) after an angle of \( 2\pi \) for reasons of topological consistency (as clarified more below in terms of the corresponding metric structure with polar angle \( \alpha \) in Eqs. (21) and (22)), the resonant frequency needs to be

\[
\omega/c = 2\pi/\tau_u.
\]  
(17)

Note that the metric components are represented by the square of the scale factor. Therefore the unsquared scale factor \( a(\tau) \) has an oscillating phase factor with resonant frequency \( \omega/2 \).
We will now introduce new coordinates \( R \) and \( \alpha \), with the aim of converting our metric in Eq. (15) to a form resembling that of Eq. (10) for the black hole case. With the definition

\[
R \equiv a^2
\]

and Eq. (16) we find

\[
dR = i \frac{\omega}{c} R d\tau.
\]

Next we define the angular coordinate \( \alpha \) through

\[
da \equiv i \frac{\omega}{c} d\tau.
\]

It may at first give the impression that we are dealing with imaginary angles, but this is not the case as can be seen if we insert the expression of Eq. (17) for \( \omega \) and recall that \( r_u = ir_u \). We then find that

\[
da = 2\pi dr/ru = 2\pi d\eta/\eta_u.
\]

With these definitions the Robertson-Walker metric can be brought to the form

\[
ds^2 = \frac{1}{R} \left( \frac{r_u}{2\pi} \right)^2 \left( R^2 d\alpha^2 + dR^2 + a^2 R^2 d\Omega^2 \right).
\]

Like in the black hole case Eq. (22) describes a 4D space with an \( E^2 \otimes S^2 \) representation, a direct product between a Euclidian plane and a 2-sphere (which now includes an additional conformal factor). It obeys a periodic boundary condition, because the angular coordinate \( \alpha \) returns the metric to its initial state after an angular interval of \( 2\pi \), which according to Eq. (21) corresponds to a conformal time interval \( \eta_u \).

If the frequency \( \omega \) were smaller than the value given by Eq. (17), then all angles in the Euclidian \( E^2 \) plane would not be covered by the available time line, and if instead \( \omega \) were larger, parts of the plane would be multiply covered. Only the choice of Eq. (17) ensures unique, exact coverage.

Note that while the metric returns to its original state after one revolution of \( 2\pi \), the phase factor for the unsquared scale factor \( a \) requires two revolutions. In this respect it behaves like a spinor in Euclidian spacetime.

In Sect. 3.5 we will show that \( \omega \), which represents a resonance in Euclidian spacetime, generates a cosmological constant \( \Lambda \) that is proportional to \( 1/\eta_u^2 \), when we convert back to ordinary time. The expression for \( \Lambda \) is identical to that of Eq. (2) if we identify the time interval \( t_A \) with the conformal age \( \eta_u \) of the universe. This raises a non-trivial conceptual issue that will be addressed in the next subsection. How can \( \Lambda \) remain constant across all of the observable universe at a given epoch while being a function of the age of the universe?

### 3.3 Global constraint versus physical field

In distance—redshift space observers are by definition always located at redshift \( z = 0 \), although their location in ordinary space may be entirely arbitrary. The choice of observer always defines the zero point of the redshift scale and the age of the universe. All local physics that any observer can ever deal with takes place at \( z = 0 \). In contrast, information about non-local objects (with \( z > 0 \)) can only be inferred from their measured spectral properties via a cosmological model.

The model used for the inference is based on Einstein’s theory of gravity, which describes the relation between the physical fields, represented by the energy-momentum tensor \( T_{\mu\nu} \), and the geometry of spacetime, which is expressed as a function \( G_{\mu\nu} \) of the metric \( g_{\mu\nu} \). The functional form of \( G_{\mu\nu} \) is governed by the requirement that \( T_{\mu\nu} \) of the physical fields must be conserved. A cosmological constant term \( \Lambda g_{\mu\nu} \) can be added to the Einstein equation without being affected by the conservation constraint, because it is divergence free as long as \( \Lambda \) is a true constant. It is therefore absent from the energy-momentum conservation equation, which determines the properties of the physical fields.

In our present theory \( \Lambda \) is not a physical field but expresses a global cyclic property or resonance of the metric. As a consequence the magnitude of \( \Lambda \) is tied to the finite age \( \eta_u \) of the universe. While \( \Lambda \) is a function of \( \eta_u \), it does not depend on look-back time, which is a quantity that is inferred via a cosmological model. Because the constant parameter \( \Lambda \) is an ingredient of the model used, look-back time depends on \( \Lambda \) but does not influence its value.

Our metric resonance has some superficial similarity to the Casimir effect, according to which the energy density of the quantum vacuum is changed as a result of boundary conditions for the vacuum electromagnetic modes inside a volume bounded by electrically conducting parallel plates. The change of the vacuum energy occurs because the boundary constraints only allow the existence of a discrete set of modes. The effect is not a function of position along the dimension perpendicular to the plates, it applies globally to the space between the plates. Similarly, in our metric resonance case, the resonant frequency induced by the boundary condition is constant and representative of the whole 4D cavity of the observable universe at epoch \( \eta_u \). When in the Casimir case the plate separation is changed, the vacuum energy within the entire volume between the plates changes without generating spatial gradients. Similarly, in our metric case when the value of \( \eta_u \) changes (the universe grows older), the resonant frequency (which governs the magnitude of \( \Lambda \)) changes, but its value is independent of 4D position within the observable universe at epoch \( \eta_u \).

Instead of the Casimir effect we could also use a violin string as an example. The discrete frequencies that it generates because of the boundary conditions are not localized...
along the string but represent a global property of the string. If we would stretch the string, the frequencies would change but only be functions of the string length, without spatial gradients.

In spite of this analogy, the origin of $\Lambda$ is fundamentally different. It is not an expression of some form of cosmic Casimir effect, in particular because the nature of the boundary conditions is different. The Casimir effect and the violin string are subject to Dirichlet boundary conditions, which constrain the oscillating amplitudes to vanish at the boundaries. When the end points are clamped down, the string length corresponds to half a wavelength for the fundamental mode, an angular interval of $\pi$. In contrast the metric has a resonance because Euclidian conformal time with a temporal string of length of $\eta_u$ obeys periodic boundary conditions with fundamental angular period $2\pi$. There is no physical justification for applying Dirichlet boundary conditions to $\eta_u$. Such boundary conditions can also be ruled out on the grounds that a temporal string length of $\pi$ (half a wave length) would lead to a $\Lambda$ in discord with the observed value. The observational constraints require a resonant frequency that corresponds to a string length of $2\pi$, not $\pi$.

### 3.4 Classical and quantum aspects

Our explanation of the origin of the cosmological constant has so far only made use of classical physics, at least in the sense that Planck’s constant $\hbar$ does not appear in the expression for $\Lambda$. When using Euclidian spacetime to derive the expression for the Hawking temperature of black holes in Sect. 3.1, $\hbar$ appeared because of the identification made between the oscillating factor in Euclidian spacetime and the Boltzmann factor in fundamental spacetime. Similarly, if $\omega_u$ is the resonant frequency of Eq. (17) that is associated with the finite Euclidian conformal time string $\tau_u = \hbar c/\eta_u$, then we can make the identification of $\exp(i\omega_u \tau_u/c) = \exp(-\omega_u \tau_u)$ with $\exp[-\hbar\omega_u/(k_B T_u)]$ to obtain the temperature $T_u$ that is associated with $\eta_u$.

Equation (12) and $T_u = \hbar/(k_B \eta_u)$ can in fact be seen as direct consequences of Heisenberg’s uncertainty principle $\Delta E \Delta t \approx \hbar/2$, if we identify $\Delta E$ with $k_B T_u/2$ and $\Delta t$ with $\eta_u$. The inverse scaling between $T_u$ and $\eta_u$ allows us to easily scale $T_u$ back to the Planck era. In units of the Planck time $5.4 \times 10^{-44}$ s the current conformal age of the universe is approximately $10^{61}$. This implies that $T_u$ in the Planck era was approximately $10^{-29} \times 10^{61} = 10^{32}$ K, in agreement with the value of $m_p c^2/k_B$ that is generally defined as the Planck temperature, where $m_p = (\hbar c/G)^{1/2} \approx 22$ µg is the Planck mass.

### 3.5 From periodic boundary condition to $\Lambda$

Although the temperature concept or Planck’s constant do not enter in the expression for $\Lambda$, these concepts are needed to estimate the amplitude of the resonance in order to show that the weak-field approximation can safely be used except in the immediate neighborhood of the Planck era. Within classical physics we have no principle that can be used to constrain the amplitude. In the previous subsection we described two ways in which quantum physics determines the mode amplitude: (1) via the relation between Euclidian field theory and statistical mechanics, and (2) through application of the Heisenberg uncertainty principle to the finite time string $\eta_u$. The two ways of dealing with this issue lead to the same result.

The temperature of $T_u \approx 10^{-29}$ K that was shown to be associated with our resonant frequency $\omega_u$ is insignificant in comparison with the CMB temperature and completely irrelevant to the present evolution of the universe. The related mode energy $\hbar \omega_u$, which like $T_u$ scales with $1/\eta_u$, has a present value that is $10^{-61}$ when expressed in units of the Planck energy. While insignificant at present, this incredibly tiny number increases as we go back in time to become unity in the Planck era. It represents the relative amplitude by which the metric fluctuates and can be interpreted in the Newtonian limit as a potential energy. As long as this amplitude is $\ll 1$, we are in the regime where the weak-field approximation for the Einstein equation is valid. This weak-field criterion is satisfied for all times except for the non-linear regime in the vicinity of the Planck era when the universe is younger than about $10^{-41}$ s (at which time the resonant amplitude is about 0.005).

Because the Euclidian spacetime metric obeys a periodic boundary condition with a resonant frequency $\omega_u = 2\pi c/|\tau_u| = 2\pi/\eta_u$, it acquires the behavior of a harmonic oscillator with equation $\partial^2 g_{\mu\nu}/\partial \tau^2 + (\omega_u/c)^2 g_{\mu\nu} = \text{source terms}$ (the physical fields that are represented by $T_{\mu\nu}$). We recall that the $\tau$ coordinate is the conformal Euclidian time that was defined in Eq. (14).

At first glance it may appear strange that the metric could have a circular behavior, governed by a phase factor $\exp(i\omega_u \tau/c)$ that returns the metric to its initial value after a certain interval. It does not at all seem to resemble the universe that we live in. This confusion gets resolved if we recall that the circular behavior refers to Euclidian time, which is not the time that we experience. When we convert the conformal Euclidian time back to the Planck era, the decaying solution can be excluded. An empty universe would experience an exponential de Sitter expansion,
but because physical fields also exist, the \( \Lambda \) contribution is just one of the terms that govern the behavior.

When converting back from Euclidian conformal time \( \tau \) to ordinary conformal time \( \eta \), the relative signs of the two terms in the oscillator equation changes (because of the squaring of imaginary \( i \)), and we get \( \partial^2 g_{\mu\nu}/\partial \eta^2 - \omega_u^2 g_{\mu\nu} = \) source terms. Because of the sign change this is no longer an oscillator equation with a periodic solution.

To relate \( \omega_u^2 \) to \( \Lambda \) we recall from Sect. 2 that in the weak-field approximation the left-hand side of the Einstein equation becomes \( R_{\mu\nu} - Ag_{\mu\nu} \approx \left[1/(2c^2)\right] \partial^2 g_{\mu\nu}/\partial t^2 - Ag_{\mu\nu} \), if we would ignore the expansion scale factor \( a(t) \) represent ordinary time. This expression preserves its form in an expanding universe with scale factor \( a(t) \). If we instead would replace time \( t \) use conformal time \( \eta \) and redefine the metric so that \( g_{\mu\nu} = a(\eta)^2 g_{\mu\nu,\text{non exp}} \), where \( g_{\mu\nu,\text{non exp}} \) refers to the metric in the non-expanding case (with \( a(\eta) = 1 \)). Multiplying all terms of the equation by \( 2c^2 \), the left-hand side becomes \( \partial^2 g_{\mu\nu}/\partial \eta^2 - 2c^2Ag_{\mu\nu} \). Comparing with our corresponding expression that was derived from the metric resonance \( \omega_u \), we find that

\[
\Lambda = \frac{1}{2} (\omega_u/c)^2 = 2 \left( \frac{\pi}{c\eta_u} \right)^2,
\]

which is identical to Eq. (2) if we replace \( t_A \) by \( \eta_u \).

### 4 Unique solution for the value of \( \Omega_A \)

The most convenient way to parametrize \( \Lambda \) for modeling purposes is in terms of the dimensionless parameter \( \Omega_A \), which was defined in Eq. (3). It represents the fraction of the critical density \( \rho_c \), that is in the form of \( \rho_A \), the vacuum energy density version of the cosmological constant. Using Eqs. (3) and (23), we get

\[
\Omega_A = \frac{2}{3} \left( \frac{\pi}{x_u(\Omega_A)} \right)^2.
\]

The dimensionless function \( x_u \) represents the conformal age of the universe in units of the Hubble time:

\[
x_u \equiv \eta_u/t_H.
\]

Because it is a function of \( \Omega_A \), as indicated explicitly in Eq. (24) and as will be brought out in the expressions below, \( \Omega_A \) gets uniquely defined by Eq. (24).

From the definition of conformal time we obtain

\[
x_u = \frac{1}{t_H} \int_0^{\eta_u} \frac{dt}{a} = \int_0^{\infty} \frac{dz}{E(z)},
\]

where \( \eta_u \) is the age of the universe in ordinary time units and \( z \) is the redshift, while

\[
E(z) = \left[ \Omega_M(1+z)^3 + \Omega_R(1+z)^4 + \Omega_A \right]^{1/2}
\]

and

\[
\Omega_M = 1 - (\Omega_R + \Omega_A)
\]

with our assumption of zero spatial curvature. \( \Omega_M \) represents the matter density (including dark matter) in units of \( \rho_c \). The corresponding density parameter for radiation is

\[
\Omega_R = \frac{u_R}{\rho_c c^2}.
\]

The radiation energy density \( u_R \) is the sum of the contributions from the photon and neutrino backgrounds:

\[
u_R = aT^4 \left[ 1 + \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} N_v \right] = 1.681aT^4
\]

(cf. Peebles 1993). The number of neutrino families \( N_v = 3 \), \( T = 2.725 \text{ K} \) is the measured temperature of the cosmic microwave background, and \( aT \) is Stefan’s constant.

The above set of equations are sufficient to allow us to solve Eq. (24), which gives us a unique value for \( \Omega_A \), namely \( 67.2\% \). This is within 2\% of the most recent value of \( 68.5 \pm 0.7\% \) that has been derived from observations (Planck Collaboration et al. 2018).

The dependence of the solution on the current value of the CMB temperature is small. If we would neglect the radiation contribution \( \Omega_R \) altogether we would get \( \Omega_A = 66.3\% \), which only differs by 3\% from the observed value.

The situation is however quite different if we instead would disregard the matter contribution \( \Omega_M \) as we do for the radiation-dominated era of the early universe. In this case \( \Omega_A \) is as high as 93.1\%.

In standard cosmology \( \Omega_M \), \( \Omega_A \), and the spatial curvature can be neglected in the early universe. The scale factor \( a(t) \) is then governed by the Friedmann solution for a radiation-dominated universe, for which the ratio \( t_u/t_H \) between the age of the universe and the Hubble time is 0.5. The cosmology that follows from our solution of Eq. (24) is however quite different for the early universe, since \( \Omega_A \) is 93.1\% and therefore highly significant. Calculating \( t_u/t_H \) from

\[
\frac{t_u}{t_H} = \frac{1}{9} \int_0^{\infty} \frac{dz}{E(z)}
\]

gives us 1.044 in this case, which implies an expansion rate in the early universe that is faster than that of the standard Friedman case by the factor 1.044/0.5 \( \approx 2.09 \). The faster expansion rate has implications for our interpretation and modelling of the physical processes in the early universe.

This has consequences for the values of the model parameters that are needed to fit the observables, like the baryon density parameter \( \Omega_B \) needed for the BBN (Big Bang nucleosynthesis) predictions to match the observed abundances.
of the light elements, or for the array of other parameters needed to fit the observed CMB spectrum.

The presence of a cosmological constant implies the existence of both a time scale and a spatial scale. While we have identified the time scale as the conformal age \( \eta_u \) of the universe, the corresponding spatial scale is \( r_u = c \eta_u \), which represents the radius of the causal or particle horizon at the given epoch. Equation (23) shows how \( \Lambda \) can be represented directly in terms of this spatial scale: \( \Lambda = 2(\pi/r_u)^2 \). The magnitude of \( \Lambda \) thus always tracks the size of the horizon. It is just a matter of convenience whether we choose to refer to this size in terms of temporal or spatial units (as long as we use conformal coordinates).

The tracking \( \Lambda \) term will have no significant direct influence on structure formation provided that the considered spatial and temporal scales are much smaller than the horizon scale. For this reason the effect on galaxy formation should be minor. However, the formation of the large-scale structures and all other physical processes take place in an expanding spacetime arena that is governed by the evolving scale factor \( a(t) \), which is different from that of standard cosmology. To avoid confusion with the non-local concept of look-back time we need to remember that the time \( t \) in \( a(t) \) is the local, dynamical time scale that is experienced by a comoving observer, and which represents the age of the universe for that observer. This means that \( t \equiv t_u \), or that \( \eta \equiv \eta_u \) if we use conformal coordinates. The expansion rate, \( \dot{a}(t) \), is driven by three sources: the energy densities of matter and radiation, and the \( \Lambda \) term. Because \( \Lambda \) represents a non-local effect that has its origin in a global constraint and is therefore expressed by a global integral, Eq. (26), which is used in Eq. (24), we have to solve an integro-differential equation to obtain the solution for \( a(t) \). This solution provides the framework for the applications of the theory, in particular for the tests when confronting it with observational data.

5 Conclusions

As the value of the cosmological constant is tied to the age of the universe in our theory, there is no cosmic coincidence problem, no violation of the Copernican principle. The contribution from the \( \Lambda \) term to the cosmic expansion rate has always been of the same order of magnitude as the contribution from the physical fields (matter plus radiation), and it will remain so in the future. Therefore we are not privileged observers.

The periodic boundary condition for the metric, which is the origin of the \( \Lambda \) term, only manifests itself within the Euclidian spacetime representation. The expression for the Hawking temperature that is obtained with the Euclidian spacetime approach is identical to the corresponding expression obtained by entirely independent methods, e.g. by Hawking (1974) in his discovery paper. This agreement supports the validity of the approach, in spite of the absence of observational verifications of the Hawking temperature. In our cosmological case the Euclidian spacetime technique gets additional validation by predicting a value for \( \Lambda \) that closely agrees with the observed value.

The circular, periodic property of the metric in Euclidian spacetime becomes an exponentially evolving property in ordinary spacetime, a de Sitter evolution, which is tempered by the comparable contributions from the physical fields in a way that preserves the zero spatial curvature of the large-scale metric.

Without the use of any free fitting parameters we have derived a value of \( \Omega_\Lambda = 67.2\% \) that is within 2\( \sigma \) of the observed value, an agreement that can hardly be dismissed as a mere “coincidence”. The uniqueness of our solution for \( \Omega_\Lambda \) without the use of free parameters implies that there does not exist any parallel universes with other values of the cosmological constant. Nature did not have a choice, because logical consistency excludes alternative possibilities.

Our theory implies an expansion history, governed by the scale factor \( a(t) \), which is significantly different from that of standard cosmology. The different expansion rate provides the framework to be used when the theory is to be tested by comparing its predictions with the various observational constraints.

As the \( \Lambda \) term has remained significant throughout the earlier history of the universe and therefore has led to a different expansion rate, there will be implications for the interpretation of events that took place in the early universe, like structure formation, the processes that generated the CMB spectrum, or the BBN (Big Bang nucleosynthesis). One such implication was explored in Stenflo (2019), where it was shown that the different expansion rate in the present theory would need to be compensated for by significantly raising the baryon density parameter \( \Omega_B \) in order to preserve agreement between the BBN predictions and the observed deuterium abundance, which in turn may have consequences for our understanding of the nature of dark matter. Similarly other cosmological parameters will need adjustments because of the changed expansion rate in order to maintain agreement with the powerful CMB constraints. These are some of the issues that will need to be addressed in future work.

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