Superfield Formalism of Stochastic Quantization Method with Field-Dependent Kernels

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ABSTRACT

I consider a Langevin equation with field-dependent kernels and investigate supersymmetry of the stochastic generating functional constructed from the Langevin equation. Moreover I describe the stochastic generating functional in terms of a superfield. In the superfield formalism, it becomes clear that the stochastic quantization method with the field-dependent kernel is equivalent to the path-integral quantization method.
1. Introduction

The stochastic quantization method (SQM) was first proposed by Parisi and Wu as an alternative quantization method in 1981.\cite{1,2} SQM can be applied to gauge theories without the gauge fixing procedure, i.e. without Faddeev-Popov ghost fields. Instead of introducing ghost field, the method produces the same contribution as the path-integral quantization method (PIQM). This fact was already confirmed perturbatively for Yang-Mills fields\cite{3} and for non-Abelian anti-symmetric tensor fields.\cite{4}

SQM has a powerful tool, “kernel”, which, among others, gives new regularization schemes.\cite{5} Kernel is also introduced for system including massless fermion.\cite{6} Moreover, the “field-dependent” kernel is introduced for system including graviton,\cite{7} system with spontaneously broken symmetry,\cite{8} and bottomless systems.\cite{9} On the other hand, it is well known that theories quantized stochastically display supersymmetry\cite{10} and can be described in superfield formalism. So my question is whether SQM with field-dependent kernel has supersymmetry or not. While Ref.\cite{11} showed that stochastic action with field-dependent kernel had a supersymmetry, the action cannot be described in superfield formalism. Besides, the stochastic action is correct only when the field-dependent kernel is a metric included in classical action. In this paper, I show that the generating functional is invariant under two independent super-transformations and can be described in terms of superfield for boson and fermion systems in general.

SQM without kernel is equivalent to PIQM and proof of the equivalence is given with the help of Fokker-Planck equation\cite{12} or superfield formalism.\cite{13,14} The equivalence is, however, not given yet in superfield formalism in case Langevin equation has field-dependent kernel. I remark the latter equivalence in this paper.

This paper is organized as follows. Supersymmetric generating functional for boson system is given in section 2 and for fermion system in section 3. In section 4, the equivalence of SQM with field-dependent kernel to PIQM is remarked and summary is given.
2. Supersymmetric generating functional for boson

I take up a system with variables \( q(x) \) and the classical action \( S(q) \) in \( n \)-dimensional space-time. To quantize the system, I give a Langevin equation with field-dependent kernel \( K(q) \)

\[
\partial_t q(x,t) \equiv \dot{q}(x,t) = -X(x,t) + R(q(x,t))\eta(x,t),
\]

(1)

where \( \eta \) is a white noise field defined by the following correlation

\[
\langle \eta(x,t)\eta(y,s) \rangle_\eta = 2\delta^n(x-y)\delta(t-s),
\]

(2)

\[
\langle f(\eta) \rangle_\eta \equiv \int D\eta \ f(\eta) \exp\{-\int d^n xdt \frac{1}{4}\eta^2(x,t)\}.
\]

\( f(\eta) \) is an arbitrary function of \( \eta \). In this paper, only Stratonovich type calculus is used which allows the Leibnitz rule with respect to stochastic time derivative. Now, let me introduce the stochastic generating functional

\[
Z[j] = \langle e^{\int d^n xdt \ q_\eta(x,t)j(x,t)} \rangle_\eta = \int D\eta \ e^{-\int d^n xdt \{\frac{1}{4}(\dot{\eta} + X)(q) - j(x,t)q_\eta(x,t)\}},
\]

(3)

where \( q_\eta \) is solution of eq.(1). Inserting the right-hand side of

\[
1 = \int Dq \ \delta(R^{-1}(\dot{q} + X) - \eta)\frac{\delta\eta}{\delta q},
\]

I get

\[
Z[j] = \int Dq Det[\frac{\partial}{\partial q}\{R^{-1}(\dot{q} + X(q))\}] \times \exp[-\int d^n xdt \{\frac{1}{4}(\dot{\eta} + X(q))K^{-1}(\dot{\eta} + X(q)) - j(x,t)q_\eta(x,t)\}],
\]

(4)

\[
= \int Dq D\omega D\overline{\omega} Dp \ exp[-\int d^n xdt \{pK^{-1}p - ip(\dot{\omega} + X(q)) - \overline{\omega}R\frac{\partial}{\partial q}\{R^{-1}(\dot{\omega} + X(q))\}\omega - j(x,t)q_\eta(x,t)\}],
\]

where \( \omega, \overline{\omega}, p \) are auxiliary fields. This expression is rather complicated and it is difficult to recognize in eq.(4) whether \( Z[0] \) has supersymmetry or not. In fact, the
stochastic action appearing in eq.(4)

\[
S_{SQM}(q, \omega, \bar{\omega}, p) \equiv \int d^nxd\tau \{ pK^{-1}p - ip(\dot{q} + K\frac{\delta S}{\delta q} - R\frac{\delta R}{\delta q})
- \bar{\omega}R\frac{\partial}{\partial q}(R^{-1}\dot{q} + R\frac{\delta S}{\delta q} - \frac{\delta R}{\delta q})\omega\},
\]

is not invariant under the supersymmetric transformation

\[
\delta q = \epsilon \omega + \bar{\epsilon} \bar{\omega}, \quad \delta \bar{\omega} = -i \epsilon p, \quad \delta \omega = -i \epsilon p - \epsilon \dot{q}, \quad \delta p = i \bar{\omega} \epsilon,
\]

which makes the stochastic action without kernel invariant. Here \(\epsilon, \bar{\epsilon}\) are infinitesimal anticommuting constant parameters. The change of variables

\[
q' = \int dqR^{-1}(q), \quad p' = R(q)p, \quad \bar{\omega}' = \bar{\omega}R(q), \quad \omega' = R^{-1}(q)\omega,
\]

leads to

\[
Z[j] = \int Dq'D\omega'D\bar{\omega}'Dp'exp[- \int dxdt\{ p'^2 - ip'\dot{q}' + \frac{\delta S}{\delta q'}^\prime
- \bar{\omega}'\frac{\partial}{\partial q'}(\dot{q}' + \frac{\delta S}{\delta q'})\omega' + ip' R^{-1} \frac{\delta R}{\delta q'}\omega' + \bar{\omega}'\frac{\partial}{\partial q'}(R^{-1} \frac{\delta R}{\delta q'})\omega' - j(x,t)q(q')\}],
\]

where I assume that the first relation in eq.(7) can be solved for \(K\) in terms of \(q\). \(Z[j]\) can then be rewritten in terms of a superfield \(\Phi'\) as

\[
Z[j] = \int D\Phi' exp[- \int d^2\theta d\tau d^n x\{ \overline{D\theta}\Phi' D\theta\Phi' + L(q(\Phi'))
- \delta^n(0) \ln R(q(\Phi')) - jq(q')\}],
\]

\[
\Phi' \equiv q' + \bar{\theta}\omega' + \bar{\omega}'\theta - i\theta\theta p', \quad D\Phi' \equiv Dq'D\omega'D\bar{\omega}'Dp',
\]

\[
\overline{D\theta} \equiv \partial_\theta, \quad D\theta \equiv \partial_\theta - \theta \partial_\tau,
\]

where \(\theta, \bar{\theta}\) are anticommuting superspace coordinates, \(L(q)\) is Lagrangian density.
\[ \int d^n x L = S, \text{ and } q(\Phi') \equiv q(q')|_{q'=\Phi'}. \] Now it is obvious that the stochastic action

\[ S'_{SQM}(\Phi') = S_{SQM}(q, \omega, \bar{\omega}, p) \]

\[ = \int d^2 \theta d\tau d^n x \{ \nabla_{\theta} \Phi' D_{\theta} \bar{\Phi}' + L(q(\Phi')) - \delta^n(0) \ln R(q(\Phi')) \}, \]

is invariant by operation with supercharges \( Q, \bar{Q} \)

\[ Q \equiv \partial_{\theta}, \quad \bar{Q} \equiv \partial_{\theta} + \bar{\theta} \partial_{\tau}, \quad \{ Q, D_{\sigma} \} = \{ \bar{Q}, D_{\bar{\sigma}} \} = \{ Q, \bar{Q} \} = \{ Q, D_{\theta} \} = \{ \bar{Q}, \bar{D}_{\bar{\theta}} \} = 0, \]

or equivalently under the supertransformation

\[ \delta q' = \bar{\epsilon} \omega' + \bar{\omega}' \epsilon, \quad \delta \bar{\omega}' = -i \epsilon p', \quad \delta \omega' = -i \epsilon p' - \epsilon \dot{q}', \quad \delta p' = i \dot{\omega}' \epsilon. \]

In terms of original variables \( q, \omega, \bar{\omega}, p \), the transformation can be expressed as

\[ \delta q = \bar{\epsilon} \omega + \bar{\omega} K(q) \epsilon, \quad \delta \bar{\omega} = -i \epsilon p' - \bar{\epsilon} \bar{\omega} \frac{\partial R(q)}{\partial q} R^{-1}(q), \]

\[ \delta \omega = -i \epsilon K(q) p - \epsilon \dot{q} + \bar{\omega} \frac{\partial R(q)}{\partial q} R(q) \omega, \]

\[ \delta p = i \bar{\omega} \epsilon + i \bar{\omega} \frac{\partial R(q)}{\partial q} R^{-1}(q) \dot{q} \epsilon - R^{-1}(q) \frac{\partial R(q)}{\partial q} \bar{\omega} \partial_\omega \frac{\partial \omega}{\partial q} R(q) p. \]

Thus the stochastic action with field-dependent kernel is invariant, for any boson system, under the super-transformation. With the help of the generating functional (9), it is shown that the Green functions in SQM are equivalent to those in PIQM as will be remarked in section 4.
3. Fermion case

Next, I show the supersymmetry for the fermion system which is quantized stochastically. In general, as kernel $K(\psi, \bar{\psi})$ for fermion field includes Dirac matrices $\gamma_\mu$, $\sqrt{K}$ does not always exist. So I start with the Langevin equation as \cite{15}

\begin{align}
\dot{\psi}_\alpha(x, t) &= -X_\psi + \frac{1}{2} K_{\alpha\beta} \eta_{2\beta}, \\
\bar{\psi}_\alpha(x, t) &= -X_{\bar{\psi}} + \frac{1}{2} \bar{\eta}_{1\beta} K_{\alpha\beta} + \bar{\eta}_{2\alpha}, \tag{15}
\end{align}

where $\delta/\delta \psi, \delta/\delta \bar{\psi}$ are left derivatives and $\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2$ are anticommuting white noise fields defined as

\begin{align*}
\langle \eta_{1\alpha}(x, t) \bar{\eta}_{1\beta}(x', t') \rangle &= \langle \eta_{2\alpha}(x, t) \bar{\eta}_{2\beta}(x', t') \rangle = 2 \delta_{\alpha\beta} \delta^n(x - x') \delta(t - t'), \tag{17}
\end{align*}

\begin{align*}
\langle f(\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2) \rangle &= \int D\eta_1 D\eta_2 D\bar{\eta}_1 D\bar{\eta}_2 f(\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2) e^{-\frac{1}{2} \int d^n x d\tau (\eta_1 \eta_1 + \eta_2 \eta_2)}. \tag{18}
\end{align*}

The stochastic generating functional for fermion field is defined as

\begin{align}
Z[j, \bar{j}] &= \int D\eta_1 D\eta_2 D\bar{\eta}_1 D\bar{\eta}_2 e^{-\frac{1}{2} \int d^n x d\tau (\eta_1 \eta_1 + \eta_2 \eta_2) - (\bar{j} \psi_\eta + \psi \bar{j})}, \tag{19}
\end{align}

where $\psi_\eta, \bar{\psi}_\eta$ are solutions of eq.(15). The change of variables $(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2) \rightarrow (\psi, \bar{\psi}, \eta'_2 \equiv K \eta_2, \bar{\eta}'_2 \equiv \bar{\eta}_2 K^{-1})$ leads to

\begin{align*}
Z[j, \bar{j}] &= \int D\eta_2 D\bar{\eta}_2 D\psi D\bar{\psi} D\bar{\psi}_1 D\bar{\psi}_2 D\varphi_1 D\varphi_2 D\varphi_1 D\varphi_2 D\tau(K) \exp \left[ - \frac{1}{2} \int d^n x d\tau (\bar{\eta}'_2 \eta'_2 \\
&- \bar{\psi}_2 (\psi + X_\psi) - \frac{1}{2} (\bar{\psi} + X_{\bar{\psi}}) K^{-1} \eta'_2 + (\bar{\psi} + X_{\bar{\psi}}) K^{-1} (\psi + X_\psi) \\
&- (\varphi_1, \varphi_2) (J) \left( \begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right) - (\bar{J} \psi_\eta + \bar{\psi}_\eta j) \right], \tag{20}
\end{align*}
\[
(J)_{\alpha\beta} \equiv \left( \begin{array}{cc} -\frac{\partial}{\partial \psi_\alpha}(\dot{\psi}_\alpha + X_\psi)_\alpha & -\frac{\partial}{\partial \psi_\beta}(\dot{\psi} + X_\psi)_\alpha \\ \frac{\partial}{\partial \psi_\beta}\{((\dot{\psi} + X_\psi)_\chi K^{-1})K_\gamma\}K_\alpha & \frac{\partial}{\partial \psi_\beta}\{((\dot{\psi} + X_\psi)_\chi K^{-1})K_\gamma\} \end{array} \right),
\]

where \( \varphi, \varphi \) are auxiliary fields. After the integration over \( \eta'_2, \eta_2' \),

\[
Z[j, \bar{j}] = \int D\bar{\pi} D\bar{\pi}_2 D\varphi_2 D\varphi_1 D\varphi D\bar{\psi} D\psi
\times \exp \left[ -\int d^n xd\tau \left\{ 2\pi K \pi - i\bar{\pi}(\dot{\psi} + X_\psi) - i(\dot{\bar{\psi}} + X_{\bar{\psi}})\pi \right. \\
- \left. (\varphi_1 \varphi_2) \left( J \begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right) - (\bar{\psi}_\eta + \bar{\varphi}_\eta j) \right\} \right],
\]

(21)

where \( \pi, \bar{\pi} \) are auxiliary fields. As in section 2, the change of variables

\[
\begin{align*}
\psi' &= \psi, & \varphi_1' &= \varphi_1, & \varphi_2' &= \varphi_2 K^{-1}, & \pi' &= \pi, \\
\delta \bar{\psi}' &= \delta \bar{\psi} K^{-1}, & \varphi_2' &= \varphi_2 K^{-1}, & \bar{\varphi}_2' &= K \bar{\varphi}_2, & \pi' &= K \pi,
\end{align*} \tag{22}
\]

leads to the generating functional \( Z[j, \bar{j}] \) written in terms of superfield

\[
Z[j, \bar{j}] = \int D\bar{\Psi} D\Psi' \exp \left[ -\int d^n xd^2 \theta d\tau \{ D_\eta \bar{\Psi} D_\theta \Psi' - D_\theta \bar{\Psi} D_\eta \Psi' \right. \\
+ L(\Psi', \bar{\Psi}') - \frac{1}{2} \delta^n(0) \ln \det K(\Psi', \bar{\Psi}') \left. \} \right],
\]

(23)

\[
\Psi' \equiv \psi' + \bar{\theta} \varphi_1' + \bar{\varphi}_2' \theta - i\bar{\theta} \theta \pi', & & \bar{\Psi}' \equiv \bar{\psi}' + \bar{\theta} \varphi_2' + \bar{\varphi}_1' \theta - i\bar{\theta} \theta \pi',
\]

(24)

where \( L \) is Lagrangian density and \( \bar{\psi}' \) is defined from \( \frac{\partial \bar{\psi}'}{\partial \psi} = K^{-1} \). The supersym-
metry transformation in terms of original fields is

\[ \delta \psi_\alpha = \eta \varphi_1 + K_{\alpha \beta} \overline{\varphi}_2 \epsilon, \quad \delta \varphi_1 = -i \epsilon K_{\alpha \beta} \pi_\beta - \epsilon \dot{\psi}_\alpha, \quad \delta \overline{\varphi}_1 = i \epsilon \pi_\alpha, \quad \delta \pi_\alpha = i \epsilon \dot{\varphi}_\alpha, \]

\[ \delta \psi_2 = \eta \varphi_2 + \overline{\varphi}_1 K_{\alpha \beta} \epsilon, \]

\[ \delta \varphi_2 = -i \epsilon \pi_\beta K_{\beta \alpha} - \epsilon \dot{\psi}_\alpha - \overline{\varphi}_2 (\overline{\varphi}_1 + K \overline{\varphi}_2 \epsilon) \delta \frac{\partial K^{-1}}{\partial \psi_\delta} K_{\beta \alpha} \]

\[ - \overline{\varphi}_2 \gamma (\overline{\varphi}_2 + \overline{\varphi}_1 K \epsilon) \delta \frac{\partial K^{-1}}{\partial \psi_\delta} K_{\beta \alpha}, \]

\[ \delta \overline{\varphi}_2 = i \epsilon \pi_\alpha - (\overline{\varphi}_1 + K \overline{\varphi}_2 \epsilon) \delta K_{\alpha \beta} \overline{\varphi}_2 \gamma - (\overline{\varphi}_2 + \overline{\varphi}_1 K \epsilon) \delta K_{\alpha \beta} \overline{\varphi}_2 \gamma, \]

\[ \delta \pi = i \epsilon \dot{\varphi}_2 + i \epsilon \dot{\psi}_\alpha K_{\alpha \beta} \overline{\varphi}_2 \gamma - (\overline{\varphi}_1 + \overline{\varphi}_2 K \epsilon) \delta K_{\alpha \beta} \overline{\varphi}_2 \gamma \]

\[ - (\overline{\varphi}_2 + \overline{\varphi}_1 K \epsilon) \delta K_{\alpha \beta} \overline{\varphi}_2 \gamma, \]

which is the supersymmetry for \( S_{SQM} \).

4. Summary

I showed that the stochastic generating functional for fermion or boson system, which is constructed from the Langevin equation with field-dependent kernel, has supersymmetry and can be described in terms of superfield.

Further I remarked that \( SQM \) with field-dependent kernel is equivalent to \( PIQM \). The generating functional written in terms of superfield is identical to that of Ref.[14] which is constructed from the Langevin equation without field-dependent kernel with the replacement of \( \Phi \) with \( \Phi' \) and \( L(\Phi) \) with \( L(q(\Phi')) - \ln(R(q(\Phi'))) \). In Ref.[14] the equivalence of \( SQM \) without kernel to \( PIQM \) was proved when \( \frac{\partial^2 L}{\partial q^2} \) is positive. So the equivalence of \( SQM \) with field-dependent kernel to \( PIQM \) is proved in the same way if

\[ \frac{\partial^2 (L - \delta^n(0) \ln R)}{\partial q^2} \]

is positive.
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