Boundary Terms and Three-Point Functions: An AdS/CFT Puzzle Resolved

Krzysztof Pilch

University of Southern California

Supergravity at 40
GGI, Florence
October 26, 2016

joint work with
Dan Freedman, Silviu Pufu and Nick Warner
arXiv: 1611.xxxxx

Research supported by DOE grant DE-SC0011687
$\mathcal{N} = 8, \ d = 4$ Supergravity

Bosonic sector

- 1 graviton, $e_\mu^\alpha$
- 28 gauge fields, $A_\mu^{IJ}$
- $35_v \oplus 35_c$ scalar fields, $\phi_{ijkl}$

Fermionic sector

- 8$_s$ gravitini, $\psi_\mu^i / \psi_{\mu i}$
- 56$_s$ spins 1/2 fields, $\chi^{ijk} / \chi_{ijk}$
  
  $\gamma^5 \chi^{ijk} = \chi^{ijk}$, etc.

The scalar 56-bein in the symmetric gauge is

$$\mathcal{V} \equiv \begin{pmatrix} u_{ij}^{IJ} & v_{ij}^{IJ} \\ v_{ij}^{IJ} & u_{ij}^{IJ} \end{pmatrix} = \exp \left( -\frac{1}{\sqrt{2}} \phi_{ijkl} \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \phi_{ijkl} \\ \frac{1}{\sqrt{2}} \phi_{ijkl} & 0 \end{pmatrix} \right) \in \frac{\mathbb{E}_{7(7)}}{\mathrm{SU}(8)}$$

where

$$\phi_{ijkl} = (\phi_{ijkl})^*, \quad \phi_{ijkl} = \frac{1}{24} \epsilon_{ijklmnpq} \phi^{mnpq}$$

We will work with the asymptotic expansion ($r \to \infty$)

$$\phi^{ijkl}(r, \vec{x}) = e^{-r/L} \phi^{(1)}_{ijkl}(\vec{x}) + e^{-2r/L} \phi^{(2)}_{ijkl}(\vec{x}) + \ldots ,$$

$$\phi^{(n)}_{ijkl}(\vec{x}) = \alpha^{ijkl}_{(n)}(\vec{x}) - i \beta^{ijkl}_{(n)}(\vec{x}).$$

where $\alpha^{ijkl}$ / $\beta^{ijkl}$ are the scalars / pseudoscalars.
The Scalar Potential

The scalar potential is

\[ \mathcal{P}(\phi) = -\left( \frac{3}{4} |A_1^{ij}|^2 - \frac{1}{24} |A_2^{ijkl}|^2 \right) \]

where the \( A \)-tensors have the following expansions \[ \text{[de Wit-Nicolai '82]} \]

\[ A_1^{(ij)} = (1 + \frac{1}{192} |\phi|^2) \delta^{ij} + \frac{\sqrt{2}}{96} \phi^{ikmn} \phi_{mpq} \phi^{pqkj} + O(\phi^4) \]

\[ A_2^{l[ijk]} = -\frac{\sqrt{2}}{2} \left( 1 + \frac{1}{144} |\phi|^2 \right) \phi^{ijkl} - \frac{3}{8} \phi_{mnl[i} \phi^{jk]mn} + \frac{\sqrt{2}}{16} \phi_{lpqr} \phi^{pqs[i} \phi^{jk]rs} + O(\phi^4) \]

and \( |\phi|^2 = \phi_{ijkl} \phi^{ijkl} \). But then

\[ |A_1^{ij}|^2 = 8 + \frac{1}{12} |\phi|^2 - \frac{\sqrt{2}}{96} (\phi^{ijkl} \phi_{klmn} \phi^{mnij} + \text{c.c.}) + O(\phi^4) \]

\[ |A_2^{l[ijk]}|^2 = \frac{1}{2} |\phi|^2 - \frac{3\sqrt{2}}{16} (\phi^{ijkl} \phi_{klmn} \phi^{mnij} + \text{c.c.}) + O(\phi^4) \]

Note that \( 4 \times 96 = 24 \times 16 \), hence

\[ \mathcal{P}(\phi) = -6 - \frac{1}{24} |\phi|^2 + O(\phi^4) \]

has no cubic terms in its expansion! Hence **THE PUZZLE.**
Comment

For maximal supergravities in $d = 4, 5$ and $7$, there is a truncation of the potential to the $\text{SL}(N, \mathbb{R})/\text{SO}(N)$ sector with $N = 8, 6$ and $5$, respectively,

$$
P = -\frac{1}{2} \left[ \left( \sum_{i=1}^{N} X_i \right)^2 - 2 \sum_{i=1}^{N} X_i^2 \right]
$$

where [Cvetič-Gubser-Lü-Pope ’99]

$$
X_i = \exp\left( -\frac{1}{2} \vec{b}_i \cdot \vec{\phi} \right), \quad \vec{b}_i = \text{weights of } N \text{ of } \text{SL}(N, \mathbb{R})
$$

and $\varphi^1, \ldots, \varphi^{N-1}$ are canonically normalized scalar fields. Then

$$
P \propto (N^2 - 2N) + (2N - 4)(x_1 + \ldots + x_N) + (N - 4)(x_1^2 + \ldots + x_N^2) + (x_1 + \ldots + x_N)^2
$$

$$
+ \left( \frac{N}{3} - \frac{8}{3} \right)(x_1^3 + \ldots + x_N^3) + (x_1 + \ldots + x_N)(x_1^2 + \ldots + x_N^2)
$$

$$
+ \ldots
$$

where

$$
x_i = -\frac{1}{2} \vec{b}_i \cdot \vec{\phi}, \quad x_1 + \ldots + x_N = 0
$$

The cubic term vanishes only for $d = 4$. 
DZF’s Bogomolny Type Argument

In DZF’s talk, the supersymmetric boundary counterterm was given by the superpotential, $W$, of $\mathcal{N} = 1$ supergravity

$$S_{\text{s-ct}} = -\frac{1}{4\pi G_4} \int d^3 x \ e^{3r_0} \ e^{K/2} |W|$$

It can be derived by a Bogomolny type argument in $\mathcal{N} = 1$ supergravity. [Freedman-Pufu ’13]

It can be derived by a Bogomolny type argument in $\mathcal{N} = 1$ supergravity. [Skenderis-Townsend ’99]

▶ Assume a domain wall background metric

$$ds^2 = e^{2A(r)} (dx_m dx^m) + dr^2, \quad z^\alpha = z^\alpha(r), \quad \bar{z}^\alpha = \bar{z}^\alpha(r)$$

▶ Rewrite the supergravity action as a sum of squares + boundary terms:

$$\left| \frac{dz^\alpha}{dr} - e^{K/2} \sqrt{\frac{W}{\bar{W}} \ K}\overline{\nabla z^\alpha} \right|^2, \quad \left| \frac{dA}{dr} - e^{K/2} |W| \right|^2$$

and $S_{\text{boundary}} = -S_{\text{s-ct}}$.

• Can we apply the same type argument to the full $\mathcal{N} = 8$ supergravity?

• How would it work with no $W$?
$\mathcal{N} = 8$ “Bogomolny Argument”

$\mathcal{N} = 8$ “Bogomolny Argument”

- Take the Poincaré invariant domain wall metric

$$ds^2 = e^{2A(r)}(-dx_0^2 + dx_1^2 + dx_2^2) + dr^2$$

- Set the vector fields, $A_{\mu}^{IJ} = 0$.

- But, keep the scalar fields, $\phi^{ijkl}(\vec{x}, r)$, arbitrary.

The bosonic action, modulo the [Gibbons-Hawking] boundary term, is

$$S_B = \int d^4x \ e^{3A} \left[ \frac{1}{96} A_{\mu}^{ijkl} A^{\mu}_{ijkl} + \frac{3}{4} g^2 |A_{1}^{ij}|^2 - \frac{1}{24} g^2 |A_{2}^{ijkl}|^2 \right]$$

where $A_{\mu}^{ijkl} = \partial_{\mu} \phi^{ijkl} + \mathcal{O}(\phi^3)$.

Hints:

- In $\mathcal{N} = 1$ truncations, $e^K|W|^2$ is an eigenvalue of $(A_{1}^{ik} A_{1kj})$.

- For $\mathcal{N} = 1, 2, 4$ domain wall solutions, the BPS equations are

$$\delta \psi_a^i = A' \gamma^3 \epsilon^i + \sqrt{2} g A_{1}^{ij} \epsilon_j = 0$$

$$\delta \chi^{ijk} = -A_r^{ijkl} \gamma^3 \epsilon_l - 2 g A_{2}^{ijkl} \epsilon_l = 0$$

and imply an algebraic constraint, $\gamma^3 \epsilon^i = X^{ij} \epsilon_j$, $X^{ik} X_{jk} = \Pi^{i}_{j}$.  

[Ahn-Woo '00, Pope-Warner '04, Bobev-KP-Warner '14, ...]
Some Elementary Algebra

We all know that

- A hermitian matrix $H$ can be diagonalized by a unitary transformation, $U$,

\[ H = U \Lambda U^\dagger \]

- A real, symmetric matrix, $A$, can be diagonalized by an orthogonal congruence, $O$,

\[ A = O \Lambda O^T \]

What if $A$ is symmetric but complex?
Some Elementary Algebra

We all know that

- A hermitian matrix $H$ can be diagonalized by a unitary transformation, $U$,
  \[ H = U \Lambda U^\dagger \]

- A real, symmetric matrix, $A$, can be diagonalized by an orthogonal congruence, $O$,
  \[ A = O \Lambda O^T \]

What if $A$ is symmetric but complex?

- A complex, symmetric matrix, $A$, can be diagonalized by a unitary congruence, $S$,
  \[ A = SDS^T, \quad D \geq 0 \]

  where
  \[ AA^\dagger = SD^2S^\dagger \]  
  \[ \text{[Autonne '1915, Takagi '25]} \]

Now, let’s apply this to the symmetric matrix $(A_{ij}^1)$ of $\mathcal{N} = 8$ supergravity.

* For a different use of AT-factorization in supergravity, see [Kodama-Nozawa '15].
\( \mathcal{N} = 8 \) “Bogomolny Argument”

Start with the AT-factorization and define

\[
A_1^{ij} = (SDS^T)^{ij}, \quad (S^i_j) \in SU(8)
\]
\[
X^{ij} = (SS^T)^{ij} \quad \implies \quad X^{ij} = X^{ji}, \quad (X^{ij}) \in SU(8)
\]

Then

\[
e^{3A} \left[ 3(A')^2 + \frac{3}{4} g^2 \left| A_1^{ij} \right|^2 \right] = \frac{3}{8} e^{3A} \left| A' X_{ij} - \sqrt{2} g A_1^{ij} \right|^2
\]

\[
+ \frac{3}{4 \sqrt{2}} g A' e^{3A} \left[ X_{ij} A_1^{ij} + X^{ij} A_1^{ij} \right]
\]

\[
e^{3A} \left[ -\frac{1}{96} A_r^{ijkl} A_r^{ijkl} - \frac{g^2}{24} \left| A_2^{ijkl} \right|^2 \right] = -\frac{1}{96} e^{3A} \left| A_r^{ijkl} + 2 g X^{im} A_2^{ijkl} \right|^2
\]

\[
+ \frac{g}{48} e^{3A} \left[ A_r^{ijkl} X^{im} A_2^{ijkl} + A_r^{ijkl} X_{lm} A_2^{ijkl} \right]
\]

Using

\[
D_\mu A_1^{ij} = \frac{1}{12 \sqrt{2}} \left( A_{2i}^{klm} A_\mu^{jklm} + A_{2j}^{klm} A_\mu^{iklm} \right) \quad [\text{de Wit-Nicolai '82}]
\]

the cross-terms can be rewritten as a boundary term

\[
( \ldots ) = \frac{g}{2 \sqrt{2}} \frac{\partial}{\partial r} \text{Tr} \left[ e^{3A} D \right] = \frac{g}{2 \sqrt{2}} \frac{\partial}{\partial r} \text{Tr} \left[ e^{3A} \sqrt{A_1 A_1^\dagger} \right]
\]
The $\mathcal{N} = 8$ Boundary Counterterm

\[ S_{\text{s-ct}} = -\frac{1}{4L} \int d^3x \, e^{3r_0/L} \, \text{Tr} \, \sqrt{A_1 A_1^\dagger} \]

\[ = \int d^3x \, e^{3r_0/L} \left[ -\frac{2}{L} - \frac{1}{96L} \phi_{ijkl} \phi_{ijkl}^\dagger + \frac{1}{384\sqrt{2}L} (\phi_{ijkl} \phi_{ijmn} \phi_{klmn} + \text{c.c.}) + \ldots \right]. \]

Both the divergent and finite terms in $S_B$ are cancelled at the boundary:

\[ S_B + S_{\text{s-ct}} = \int d^3xdr \, e^{3A} \left[ \frac{3}{8} \left| A' X_{ij} - \frac{1}{L} A_{1ij} \right|^2 - \frac{1}{96} \left| A_r \right|_{ijkl}^2 + \frac{\sqrt{2}}{L} X_{im} A_{2m} {^j_{kl}}^l \right]^2 - \frac{1}{96} g^{ab} A_a \phi_{ijkl} \phi_{ijkl} \right] \]

where

\[ A \sim \frac{r}{L} + O(e^{-2r/L}), \quad \phi_{ijkl} \sim O(e^{-r/L}) \]

\[ g^{ab} \sim O(e^{-2r/L}), \quad \phi_{ijkl} \sim O(e^{-r/L}) \]

\[ L = \frac{1}{\sqrt{2} g} \]
The $\mathcal{N} = 8$ Boundary Counterterm

- The cubic counterterm is purely scalar ($\phi = \alpha + i\beta$)

\[
\frac{1}{384\sqrt{2}} L \left( \phi_{ijkl} \phi_{ijmn} \phi_{klmn} + \text{c.c.} \right) = \frac{\sqrt{2}}{384} L \alpha_{ijkl} \alpha_{klmn} \alpha_{mnij}
\]

which is a consequence of SO(8) identities, e.g., [de Wit '79]

\[
\begin{align*}
\alpha_{mn[ij} \alpha_{kl]mn} & , \\
\beta_{mn[ij} \beta_{kl]mn} & ,
\end{align*}
\]

self-dual, anti-self-dual

and/or the SO(8) branching rules

\[
\begin{align*}
35_i \otimes 35_i & \rightarrow 1 + 35_i + \ldots , \\
35_i \otimes 35_j & \rightarrow 35_k + \ldots
\end{align*}
\]

- In the $\mathcal{N} = 1$ theories in DZF's talk, that can be obtained by a consistent truncation to either the $U(1)^3$ or $SU(3) \times U(1)^2$ invariant sectors, the natural counterterm (single field $z = A + iB$)

\[
e^{K/2}(W + \overline{W}) \propto \ldots + (A^3 - 3AB^2)
\]

is a representative of a family of counterterms allowed by the lower symmetry. They are all supersymmetric and have the same scalar cubic term!
Boundary Sources and $\mathcal{N} = 8$ Supersymmetry

\[ \tilde{S}_{\text{ren}} = S_{\text{bulk}} + S_{\text{s-ct}} + S_{\chi-ct} + S_L \]

$S_{\text{bulk}}$ is the bulk action of $\mathcal{N} = 8$ gauged supergravity.

$S_{\text{s-ct}}$ is the scalar counterterm introduced above.

$S_{\chi-ct} = \frac{1}{24} \int d^3 x \ e^{-3r_0} \left[ \bar{\chi}^{ijk} \chi^{ijk} + \text{c.c.} \right]$ is the spin-1/2 counterterm.

$S_L = \frac{1}{48} \int d^3 x \ \mathcal{A}^{ijkl}(\vec{x}) \alpha_{(1)}^{ijkl}(\vec{x})$

\[ \mathcal{A}^{ijkl}(\vec{x}) = - \lim_{r \to \infty} e^{-r/L} \Pi^{ijkl}(\vec{x}, r) \]

\[ = - \frac{1}{L} \left[ \alpha_{(2)}^{ijkl}(\vec{x}) + \frac{3}{4\sqrt{2}} \alpha_{(1)}^{mn[ij}(\vec{x}) \alpha_{(1)}^{k]lmn}(\vec{x}) \right] \]

is the conjugate of the scalar source, $\alpha_{(1)}^{ijkl}$.

The Legendre trasformed action $\tilde{S}_{\text{ren}}$ is on-shell invariant under the $\mathcal{N} = 8$ superconformal symmetry generated by the AdS$_4$ Killing spinors, $e^i/\epsilon_i$,

\[ e^i(r, \vec{x}) = e^{r/2L} \zeta_+^i(\vec{x}) + e^{-r/2L} \zeta_-^i \]

\[ \gamma^5 \zeta_\pm^i = \zeta_\pm^i, \quad \gamma^3 \zeta_\pm^i = \pm \zeta_\pm^i, \quad \slashed{\partial} \zeta_+^i = -\frac{3}{L} \zeta_-^i \]
Define
\[
\Xi^{ijk} = \frac{1}{2} \left( \chi^{ijk} - \gamma^3 \chi^{ijk} \right), \quad \Upsilon^{ijk} = \frac{1}{2} \left( \chi^{ijk} + \gamma^3 \chi^{ijk} \right)
\]
\[
\gamma^3 \Xi^{ijk} = -\Xi^{ijk}, \quad \gamma^3 \Upsilon^{ijk} = \Upsilon^{ijk}
\]
\[
\Xi^{ijk} = e^{-3r/2L} \Xi_{(3/2)}^{ijk} + \ldots, \quad \Upsilon^{ijk} = e^{-3r/2L} \Upsilon_{(3/2)}^{ijk} + \ldots
\]

and rewrite the supersymmetry transformations for the boundary fields

\[
\delta \alpha_{(1)}^{ijkl} = 8 \bar{\zeta} \left[ \gamma^{(3/2)}_{(3/2)} jkl \right] + \ldots
\]
\[
\delta \beta_{(1)}^{ijkl} = -8i \bar{\zeta} \left[ \Xi^{(3/2)}_{(3/2)} jkl \right] + \ldots
\]
\[
\delta \alpha_{(2)}^{ijkl} = 8 \left( \bar{\zeta} \left[ \Xi^{(3/2)}_{(3/2)} jkl \right] + \bar{\zeta} \left[ \gamma_{(5/2)}^{(5/2)} jkl \right] \right) + \ldots
\]
\[
\delta \beta_{(2)}^{ijkl} = -8i \left( \bar{\zeta} \left[ \gamma^{(3/2)}_{(3/2)} jkl \right] + \bar{\zeta} \left[ \Xi^{(5/2)}_{(5/2)} jkl \right] \right) + \ldots
\]
\[
\delta \Xi_{(3/2)}^{ijkl} = -\frac{2i}{L} \beta_{(1)}^{ijkl} \bar{\zeta} - \frac{1}{L} \left[ \alpha_{(2)}^{ijkl} + \frac{3}{4\sqrt{2}} \alpha_{(1)}^{mn[ij} \alpha_{(1)}^{k]lmn} \right.
\]
\[
\left. + \frac{3}{4\sqrt{2}} \beta_{(1)}^{mn[ij} \beta_{(1)}^{k]lmn} - iL \gamma^3 \partial \beta_{(1)}^{ijkl} \right] \bar{\zeta} + l
\]
\[
\delta \gamma_{(3/2)}^{ijkl} = \frac{2}{L} \alpha_{(1)}^{ijkl} \bar{\zeta} - \frac{i}{L} \left[ -\beta_{(2)}^{ijkl} + \frac{3}{4\sqrt{2}} \alpha_{(1)}^{mn[ij} \beta_{(1)}^{k]lmn} \right.
\]
\[
\left. - \frac{3}{4\sqrt{2}} \beta_{(1)}^{mn[ij} \alpha_{(1)}^{k]lmn} - iL \gamma^3 \partial \alpha_{(1)}^{ijkl} \right] \bar{\zeta} + l
\]
Highlights

- The sources \((\mathcal{A}_{ijkl}(\vec{x}), \beta_{(1)}^{}_{ijkl}(\vec{x}), \Xi_{(3/2)}^{ijk}(\vec{x}))\) form a closed multiplet on-shell. We need to use the spin-1/2 EOMs, e.g.,
  \[
  \gamma_{(5/2)}^{ijk} = L\partial\Xi_{(3/2)}^{ijk} - \frac{1}{12\sqrt{2}} \eta_{ijklpqrlm} \left( \alpha_{(1)}^{npqr} \gamma_{(3/2)}^{lmn} - i \beta_{(1)}^{npqr} \Xi_{(3/2)}^{lmn} \right)
  \]

- The boundary terms in \(\delta S_{\text{bulk}}\) in can be quickly determined from
  \[
  \delta L_{\text{bulk}} = \bar{V}_i \epsilon^i + \bar{X}^\mu \epsilon^i + \text{c.c.}
  \]
  and then using the bulk invariance. The result is
  \[
  \delta S_{\text{bulk}} = \int d^3 x \ e^{3r_0/L} \left[ -\frac{1}{6} A^3_{ijkl} \bar{\epsilon}_i \chi_{jkl} - \frac{1}{12} \delta \bar{\chi}_{jkl} \gamma^3 \chi^{jkl} + \text{c.c.} \right]
  \]

- Using radiality
  \[
  \int d^3 x \ e^{3r_0/L} \left[ \delta \bar{\chi}_{jkl} \gamma^3 \chi^{jkl} \right] = \int d^3 x \ \left[ \delta \bar{\Xi}_{(3/2)}^{ijk} \gamma_{(3/2)}^{ijk} + \delta \bar{\gamma}_{(3/2)}^{ijk} \Xi_{(3/2)}^{ijk} + O(e^{-r_0/L}) \right]
  \]
  It combines with \(\delta S_{\chi-\text{ct}}\). Both vanish when \(\text{sources} = 0\).

- For \(\text{sources} = 0\), using the “Bogomolny estimate”
  \[
  \delta \tilde{S}_{\text{ren}} = \delta S_{\text{bulk}} + \delta S_{\text{s-ct}} = \int d^3 x \ O(e^{-r_0/L}) \rightarrow 0
  \]

- For \(\text{sources} \neq 0\),
  \[
  \delta \tilde{S}_{\text{ren}} = \int d^3 x \left[ -\frac{1}{3} \frac{\partial}{\partial x^a} \left( \alpha_{(1)}^{ijk} \bar{\zeta}_+ i \gamma^a \Xi_{(3/2)}^{jkl} \right) + O(e^{-r_0/L}) \right] \rightarrow 0
  \]
The Correlators

We want to use the AdS/CFT to compute the 2- and 3-point functions for \( \Delta = 1 \) operators \( O_{IJ}(\vec{x}) \) in ABJM theory. Heuristically,

\[
O_{IJ} = \text{Tr} \left[ X_I X_J - \frac{1}{8} \delta_{IJ} X_K X_K \right]
\]

It is more natural to work with the symmetric tensor representation of \( 35_v \) – change from the \( \text{SU}(8) \) to \( \text{SL}(8, \mathbb{R}) \) basis

\[
A^{IJ} = \frac{1}{96} (\Gamma_{IK})^{ij} (\Gamma_{JK})^{kl} \alpha^{ijkl}
\]

The renormalized action for the scalars continued to the Euclidean signature reads

\[
S_{\text{ren}} = \frac{1}{\kappa^2} \int d^4 x \sqrt{g} \left[ \frac{1}{4} \partial_\mu A^{IJ} \partial^\mu A^{IJ} - \frac{1}{2} A^{IJ} A^{IJ} \right]
\]

\[
+ \frac{1}{\kappa^2} \int d^3 x e^{3r_0} \left[ \frac{1}{4} A^{IJ} A^{IJ} - \frac{1}{6\sqrt{2}} A^{IJ} A^{JK} A^{KI} \right] + O(A^4)
\]

where \( \kappa^2 = 1/8\pi G_4 \) and \( L = 1 \). Near the boundary

\[
A^{IJ}(r, \vec{x}) = e^{-r} A_{(1)}^{IJ}(\vec{x}) + e^{-2r} A_{(2)}^{IJ}(\vec{x}) + \cdots
\]
The bulk fields with Dirichlet boundary data \( A_{(1)}^{IJ}(\vec{x}) \) are constructed using the usual bulk-boundary propagator

\[
A^{IJ}(r, \vec{x}) = \int d^3y \ K_2(r, \vec{x}; \vec{y}) A_{(1)}^{IJ}(\vec{y}), \quad K_2(r, \vec{x}; \vec{y}) \equiv \frac{1}{\pi^2} \frac{e^{-2r}}{(e^{-2r} + |\vec{x} - \vec{y}|^2)^2}
\]

Substitute into the action

\[
S_{\text{on-shell}}[A_{(1)}^{IJ}] = -\frac{1}{4\kappa^2} \int d^3x \ d^3y \ \frac{A_{(1)}^{IJ}(\vec{x}) A_{(1)}^{IJ}(\vec{y})}{\pi^2|\vec{x} - \vec{y}|^4}
\]

\[
- \frac{1}{6\sqrt{2}\kappa^2} \int d^3x \ A_{(1)}^{IJ}(\vec{x}) A^{JK}(\vec{x}) A^{KL}(\vec{x}) + O(A^4_{(1)})
\]

This would suffice if \( O_{IJ}(\vec{x}) \) had \( \Delta = 2 \). For \( \Delta = 1 \) and alternate quantization we must perform the Legendre transform \( \text{[Klebanov-Witten '99]} \)

\[
\tilde{S}_{\text{on-shell}}[\mathcal{A}^{IJ}] = S_{\text{on-shell}}[A_{(1)}^{IJ}] + \frac{1}{2\kappa^2} \int d^3x \ \mathcal{A}^{IJ}(\vec{x}) A_{(1)}^{IJ}(\vec{x})
\]

computed after extremizing the right hand side with respect to \( A_{(1)}^{IJ}(\vec{x}) \),

\[
\mathcal{A}^{IJ}(\vec{x}) = -\delta S_{\text{on-shell}}[A_1] \delta A_1(\vec{x}) = \frac{1}{\pi^2} \int d^3y \ \frac{A_{(1)}^{IJ}(\vec{y})}{|\vec{x} - \vec{y}|^4} - \frac{1}{\sqrt{2}} A_{(1)}^{JK}(\vec{x}) A_{(1)}^{IJ}(\vec{x}) + O(A^3_{(1)})
\]

This must be solved for \( A_{(1)}^{IJ} \) in terms of \( \mathcal{A}^{IJ}(\vec{x}) \).
\[[\mathcal{A}^{IJ}(\vec{x})] = \frac{1}{\pi^2} \int d^3 y \frac{A_{(1)}^{IJ}(\vec{y})}{|\vec{x} - \vec{y}|^4} - \frac{1}{\sqrt{2}} A_{(1)}^K(J)(\vec{x}) A_{(1)}^I(K)(\vec{x}) + O(A^3_{(1)})\]

Convolute with $1/2\pi^2|\vec{z} - \vec{x}|^2$ and use

\[
\int d^3 x \frac{1}{2\pi^2|\vec{z} - \vec{x}|^2} \frac{1}{\pi^2|\vec{x} - \vec{y}|^4} = -\delta^{(3)}(\vec{z} - \vec{y})
\]

shown by formal Fourier transform or better by holographic regularization. Then

\[
A_{(1)}^{IJ}(\vec{x}) = -\int d^3 y \frac{\mathcal{A}^{IJ}(\vec{y})}{2\pi^2|\vec{x} - \vec{y}|^2} - \frac{1}{(2\pi)^3} \int d^3 y d^3 z \frac{\mathcal{A}^{K(I}(\vec{y})\mathcal{A}^{J)K}(\vec{z})}{|\vec{x} - \vec{y}||\vec{y} - \vec{z}||\vec{x} - \vec{z}|} + O(\mathcal{A}^3)
\]

and

\[
\tilde{S}_{on-shell}[\mathcal{A}^{IJ}] = -\frac{1}{8\pi^2\kappa^2} \int d^3 x d^3 y \frac{\mathcal{A}^{IJ}(\vec{x})\mathcal{A}^{IJ}(\vec{y})}{|\vec{x} - \vec{y}|^2}
\]

\[
+ \frac{1}{48\sqrt{2}\pi^3\kappa^2 L} \int d^3 x d^3 y d^3 z \frac{\mathcal{A}^{IJ}(\vec{x})\mathcal{A}^{JK}(\vec{y})\mathcal{A}^{KI}(\vec{z})}{|\vec{x} - \vec{y}||\vec{y} - \vec{z}||\vec{x} - \vec{z}|} + O(\mathcal{A}^4)
\]

Use $-\tilde{S}_{on-shell}[\mathcal{A}^{IJ}]$ to compute connected correlators of $\mathcal{O}_{IJ}(\vec{x})$. 

16
The Result

\[ \tilde{S}_{\text{on-shell}}[\mathcal{A}^{IJ}] = -\frac{1}{8\pi^2\kappa^2} \int d^3x \, d^3y \, \frac{\mathcal{A}^{IJ}(\vec{x})\mathcal{A}^{IJ}(\vec{y})}{|\vec{x} - \vec{y}|^2} \]

\[ + \frac{1}{48\sqrt{2}\pi^3\kappa^2 L} \int d^3x \, d^3y \, d^3z \, \frac{\mathcal{A}^{IJ}(\vec{x})\mathcal{A}^{JK}(\vec{y})\mathcal{A}^{KI}(\vec{z})}{|\vec{x} - \vec{y}||\vec{y} - \vec{z}||\vec{x} - \vec{z}|} + O(\mathcal{A}^4) \]

The normalization of \( \mathcal{A}^{IJ}(\vec{x}) \) vs the field theory sources is

source for \( \mathcal{O}_{IJ}(\vec{x}) = \frac{C}{L} \mathcal{A}^{IJ}(\vec{x}) \)

Adjusting for this normalization (no sum)

\[ \langle \mathcal{O}_{IJ}(\vec{x}_1)\mathcal{O}_{IJ}(\vec{x}_2) \rangle = \frac{C_2}{|\vec{x}_1 - \vec{x}_2|^2}, \]

\[ \langle \mathcal{O}_{IJ}(\vec{x}_1)\mathcal{O}_{JK}(\vec{x}_2)\mathcal{O}_{KI}(\vec{x}_3) \rangle = \frac{C_3}{|\vec{x}_1 - \vec{x}_2||\vec{x}_1 - \vec{x}_3||\vec{x}_2 - \vec{x}_3|} \]

\[ C_2 = \frac{L^2}{16\pi^3 G_4 C^2}, \quad C_3 = -\frac{L^2}{64\sqrt{2}\pi^4 G_4 C^3} \]

The normalization independent ratio

\[ \frac{C_3^2}{C_2^3} = \frac{\pi G_4}{2L^2} \]

This should be reproduced by a field theory calculation in ABJM.
The Field Theory Calculation

[Jafferis ’10], [Jafferis-Klebanov-Pufu-Safdi ’11]
[Closset-Dumitrescu-Festuccia-Komargodski-Seiberg ’12]

In any $\mathcal{N} = 8$ SCFT in three dimensions, we have two point functions for canonically normalized energy momentum tensor

$$
\langle T_{\mu\nu}(\vec{x}) T_{\rho\sigma}(0) \rangle = \frac{c_T}{64} \left( P_{\mu\rho} P_{\nu\sigma} + P_{\nu\rho} P_{\mu\sigma} - P_{\mu\nu} P_{\rho\sigma} \right) \frac{1}{16\pi^2 |\vec{x}|^2}
$$

where $P_{\mu\nu} \equiv \eta_{\mu\nu} \partial^\lambda \partial_\lambda - \partial_\mu \partial_\nu$ and the SO(8) R-symmetry current

$$
\langle j^\mu_{IJ}(\vec{x}) j^\nu_{KL}(0) \rangle = \frac{c_T}{64} \left( \delta_{IK} \delta_{JL} - \delta_{IL} \delta_{JK} \right) P^{\mu\nu} \frac{1}{16\pi^2 |\vec{x}|^2}
$$

where $c_T$ is a constant that depends on the theory.

By conformal and SO(8) invariance, for $\Delta = 1$ scalar operators $O_{IJ}(\vec{x})$ in 35$_{\nu}$ of SO(8)

$$
\langle O_{IJ}(\vec{x}_1) O_{IJ}(\vec{x}_2) \rangle = \frac{c_2}{|\vec{x}_1 - \vec{x}_2|^2},
$$

$$
\langle O_{IJ}(\vec{x}_1) O_{JK}(\vec{x}_2) O_{KI}(\vec{x}_3) \rangle = \frac{c_3}{|\vec{x}_1 - \vec{x}_2| |\vec{x}_1 - \vec{x}_3| |\vec{x}_2 - \vec{x}_3|}
$$

By evaluating those correlators using supersymmetric localization for special choices of operators determined by the branching from $\mathcal{N} = 8$ to $\mathcal{N} = 2$, one finds

$$
c_2 = \frac{c_T}{16(4\pi)^2}, \quad c_3 = \frac{c_T}{16} \frac{1}{(4\pi)^3}, \quad \frac{c_3^2}{c_2} = \frac{16}{c_T}
$$
The Comparison

- From $\mathcal{N} = 8$ supergravity
  \[ \frac{C_3^2}{C_2^3} = \frac{\pi G_4}{2 L^2} \]

- From $\mathcal{N} = 8$ SCFT
  \[ \frac{c_3^2}{c_2^3} = \frac{16}{c_T} \]

- For an $\mathcal{N} = 8$ SCFT with a holographic dual, $c_T$ is a universal function of $L$ and $G_4$,
  \[ c_T = \frac{32 L^2}{\pi G_4} \]  
  [Chester-Lee-Pufu-Yacoby '14]

- The normalization of the sources can be fixed using the 2-point function
  \[ c_2 = C_2 \implies C = -\frac{1}{\sqrt{2}} \implies c_3 = C_3 \]

  and the 3-point functions agree!
Conclusions

- Our puzzle has been solved.
- We have a new precision test of $\text{AdS}_4/\text{CFT}_3$.
- The 3-point correlators of $\Delta = 1$ scalar operators $\mathcal{O}(\vec{x})$ arise from a finite boundary counterterm in the renormalized supergravity action.
  - This may be generic for 3d SCFTs with holographic duals – the 3-point functions computed from Witten diagrams with the bulk vertex $AAA$ or $A\partial_\mu A\partial_\mu A$ diverge when $d \to 3$ and $\Delta \to 1$.

  [Freedman, Mathur, Matusis, Rastelli ’99]

- The relevant counterterm can be obtained by a Bogomolny type argument and/or by requiring supersymmetry of the Legendre transformed renormalized on-shell action.
- The use of Legendre transform and alternate quantization has been clarified in an explicit example.
- The importance of boundary (counter-)terms have been appreciated since the early days of AdS/CFT, see, e.g.,

  [Henningson-Sfetsos ’98], [Mueck-Viswanathan ’98], [Arutyunov-Frolov ’99],
  [Henneaux ’99], . . .,
  [Bianchi-Freedman-Skenderis ’01-’02], . . .,
  [Belyaev-van Nieuwenhuizen ’08], [Grumiller-van Nieuwenhuizen ’08], . . .,
  [Andrianopoli-D’Auria ’14], . . .