Notes on the holographic Lifshitz theory

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ABSTRACT

In the Lifshitz black brane geometry of the Einstein-Maxwell-dilaton theory, we clarify thermodynamics and show that the dual theory following the gauge/gravity duality is described by a non-relativistic medium with an equation state parameter $z/2$. In this background, the binding energy and drag force of particles or monopoles are investigated with an F1- or a D1-string respectively. Moreover, the electric DC conductivity carried by an impurity or the same matter fluctuation is holographically investigated in the hydrodynamic limit. Depending on the charge carrier, the DC conductivity shows a totally different behavior.
1 Introduction

The AdS/CFT correspondence is a very useful and fascinating tool for understanding the strongly interacting system [1]. In the last decade, it has been widely used in studying some universal properties of Quantum Chromodynamics (QCD) or the condensed matter system in the strong coupling regime [2, 3, 4, 5, 6, 7]. The asymptotic $\text{AdS}$ geometry plays an important role in such investigations because its dual theory is described by a conformal field theory (CFT) corresponding to the bulk isometry. Can we generalize the AdS/CFT correspondence to the non-$\text{AdS}$ geometry? It is a very interesting and important question for understanding the non-conformal and non-relativistic physics through the holographic methods [8]-[17]. In this paper, we will study, via a dual Lifshitz black brane geometry, some interesting physical quantities of the non-relativistic Lifshitz theory.

Following the gauge/gravity duality it was shown that the Einstein-dilaton theory with a Liouville potential corresponds to a relativistic non-conformal theory [18, 19]. In addition, it was also found that the DC conductivity of the dual system can show different behaviors depending on what kind of vector fluctuation is turned on. If a vector fluctuation is not coupled to the background dilaton, the corresponding DC conductivity in a 2+1-dimensional relativistic non-conformal theory is temperature independent, while it can have a nontrivial temperature-dependence for the vector fluctuation coupled to dilaton. These facts were also checked by using the membrane paradigm [20]. In the similar setup together with a background gauge field and without a Liouville potential, the exact gravity solution was known as a Lifshitz geometry [8, 9]. Although the Lifshitz geometry has different scaling in the temporal and spatial coordinates, the generalized scaling symmetry, the so called hyperscaling symmetry, is still preserved. Due to such a nontrivial scaling, it has been believed that a Lifshitz field theory appears as the dual theory of the Lifshitz geometry. In particular, when the dynamical exponent is 2, the corresponding dual theory becomes non-relativistic.

In the last decade, there were many attempts to understand the strongly interacting quark-gluon plasma through the holographic studies on the binding energy of a heavy quarkonium, drag force and hydrodynamics, etc [21, 22, 23, 24, 25]. In these studies, the asymptotic $\text{AdS}$ space was used as the dual geometry, which means that the dual theory is a relativistic conformal theory. In this paper, we will investigate the binding energy and drag force for fundamental particles and monopoles of the non-relativistic theory. In the relativistic conformal theory dual to the asymptotic $\text{AdS}$ geometry, since a coupling constant is trivial, there is no distinct between particles and monopoles. On the other hand, in the Lifshitz geometry there exists a nontrivial dilaton profile corresponding to the coupling of the dual theory [8, 9, 10, 11]. So the motions of particles and monopoles are distinguished, which is also true in the non-conformal theory [20]. We investigate the effect of the non-relativity on the binding energy and drag force of the Lifshitz theory.

Recently, the DC conductivity and superconductivity in the non-relativistic theory were widely studied by turning on the bulk gauge fluctuation without a nontrivial dilaton coupling [26, 27, 28, 29, 30, 31, 32, 33, 34]. In this paper, we further investigate the thermodynamic properties of the Lifshitz theory and the DC conductivity with a nontrivial dilaton coupling, which provides more information for the charge carrier. In the membrane paradigm [35, 36], since the DC conductivity is determined only by information at the black brane horizon, the result is independent of the asymptotic behavior of the solution. As a result, if the bulk
gauge coupling described by one parameter is fixed, the DC conductivity is uniquely determined in the membrane paradigm. Actually, the story of the Kubo formula is totally different because the asymptotic behavior of the solution plays an important role in determining the hydrodynamic transport coefficients \[37, 38, 39\]. If a new vector fluctuation different from the background vector field, which we call an impurity, is turned on and its bulk gauge coupling associated with the background dilaton is described by a parameter $\gamma$, the resulting DC conductivity is consistent with the membrane paradigm result for $\gamma < 1$. For $\gamma \geq 1$, especially for $\gamma = 2$ we are interested in, the DC conductivity is totally different from the previous case in $\gamma < 1$. The reason is the following. The asymptotic solution of the bulk fluctuation has two independent parts. Usually, one is called a non-normalizable mode and the other is a normalizable mode (in some situations, there are only two normalizable modes \[3\]). For $\gamma \geq 1$, the roles of two modes are exchanged. After imposing a natural boundary condition, the resulting conductivity for $\gamma = 2$ becomes zero. This result is inconsistent with that of the membrane paradigm because the membrane paradigm does not care about the asymptotic behavior of the solution. We also investigate the DC conductivity of the same vector fluctuation in the zero momentum limit. In this case, the parameter of the bulk gauge coupling should be identified with an intrinsic parameter $\lambda$. If we concentrate on the case of $z = 2$, $\lambda$ reduces to 2 and the vector fluctuation should be coupled to the shear mode of the metric fluctuation. Although the metric fluctuation does not affect on the vector fluctuation near the horizon, the asymptotic solution is dramatically changed by the metric fluctuation. This provides a different DC conductivity from the result of an impurity. We find that the DC conductivity carried by the same vector fluctuation has a different temperature dependence from that carried by an impurity.

The rest of the paper is organized as follows: In Sec. 2, we will represent the Lifshitz black brane solution, which shows the hyperscaling symmetry manifestly, and explain its thermodynamics corresponding to the non-relativistic Lifshitz field theory with an equation of state parameter $w = z/2$. In Sec. 3, the binding energies of particles and monopoles in the holographic Lifshitz theory are studied. In Sec. 4, we will investigate the DC conductivity in the zero momentum limit and show that the conductivities of an impurity and the same vector fluctuation have totally different behaviors due to the mixing with the shear modes of the metric fluctuation. Finally, we will finish this work with some concluding remarks.

2 Thermodynamic properties

There exist many scale-invariant field theories without the Lorentz invariance near the critical points \[5, 7\]. One of such examples is the Lifshitz theory 

$$S[\chi] = \int d^3 x \left[ (\partial_t \chi)^2 - K (\nabla^2 \chi)^2 \right],$$  \hspace{1cm} (1)$$

which describes a fixed line parameterized by $K$ with a dynamical exponent $z = 2$ \[10\]. Following the gauge/gravity duality, such a non-relativistic theory can be mapped to the Lifshitz geometry as a dual gravity. There are several models describing a holographic Lifshitz theory. One of them is the gravity theory including higher form fields \[8\] or a massive gauge theory including a non-dynamical scalar field,
which were widely investigated by many authors [10]. Another example appears as a geometric solution of the Einstein-Maxwell-dilaton theory. In this paper we will concentrate on the latter case.

The action for the Einstein-Maxwell-dilaton theory with a negative cosmological constant $\Lambda$ is given by

$$S_{EMd} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda \phi} F_{\mu \nu} F^{\mu \nu} \right).$$

(2)

The equations of motion are given by [29]

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + g_{\mu \nu} \Lambda = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu \nu} (\partial_\phi)^2 + \frac{1}{2} e^{\lambda \phi} F_{\mu \lambda} F^{\lambda \nu} - \frac{1}{8} g_{\mu \nu} e^{\lambda \phi} F^2;$$

(3)

$$\partial_\mu (\sqrt{-g} \partial^\mu \phi) = \frac{\lambda}{4} \sqrt{-g} e^{\lambda \phi} F^2;$$

(4)

$$0 = \partial_\mu (\sqrt{-g} e^{\lambda \phi} F^{\mu \nu});$$

(5)

and allow the following black brane geometry

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 (dx^2 + dy^2),$$

$$\phi(r) = -\frac{4}{\lambda} \log r,$$

$$F_{rt} = \partial_r A_t = q r^{z+1},$$

(6)

with

$$f(r) = 1 - \frac{r_h^{z+2}}{r^{z+2}},$$

$$\lambda = \frac{2}{\sqrt{z-1}},$$

$$q = \sqrt{2(z-1)(z+2)},$$

$$\Lambda = \frac{(z+1)(z+2)}{2},$$

(7)

where $r_h$ implies the black brane horizon. It is worthwhile noting that the Lifshitz black brane is not a charged but uncharged solution because the charge $q$ is not a free parameter describing a hair of the black brane. In other words, once the intrinsic parameters of the theory, $\Lambda$ and $\lambda$, are fixed, the dynamical exponent $z$ and the charge $q$ are automatically determined by the above relations. For $z = 1$, the above geometry reduces to an ordinary AdS black brane.

Before investigating the thermodynamics of the black brane, let us recall the scaling behaviors of all variables. In order to preserve the scaling symmetry, when $r$ scales as $\Omega r$, other variables scale like

$$r_h \to \Omega r_h, \quad t \to \Omega^{-1} t, \quad x \to \Omega^{-1} x, \quad y \to \Omega^{-1} y,$$

$$e^\phi \to \Omega^{-4/\lambda} e^\phi, \quad q \to \Omega^0 q, \quad F_{rt} \to \Omega^{z+1} F_{rt}, \quad A_t \to \Omega^{z+2} A_t.$$

(8)

Then, the action [2] is invariant under the above scale transformations. The time component gauge field satisfying the above scaling is given by

$$A_t = \frac{q}{z+2} (r^{z+2} - r_h^{z+2}),$$

(9)

3
where the last term corresponds to a new integration constant and we choose a special value such that the norm of $A_t$ is regular even at the black brane horizon. The energy and temperature of the system scale as the inverse of time, $E \to \Omega^z E$ and $T \to \Omega^z T$.

After expanding $r$ near the horizon and requiring that there is no conical singularity, the Hawking temperature is determined to be

$$T = \frac{z + 2}{4\pi} r_h^z,$$

which gives the correct scaling behavior previously mentioned. The Bekenstein-Hawking entropy becomes

$$S = \frac{V_2}{4G} r_h^z,$$

where $V_2$ implies a spatial volume of the boundary space and scales like $V_2 \to \Omega^{-2} V_2$. Therefore, the Bekenstein-Hawking entropy is invariant under the above scaling (8).

Using the first law of thermodynamics together with the above Hawking temperature and Bekenstein-Hawking entropy, the internal energy $E$ and the free energy $F$ are given by

$$E = \frac{V_2}{8\pi G} r_h^{z+2},$$
$$F = -\frac{zV_2}{16\pi G} r_h^{z+2}.$$
(12)

Using the definition of pressure $P = -\partial F/\partial V_2$, we can easily evaluate the equation of state parameter of the Lifshitz black brane

$$w = \frac{PV_2}{E} = \frac{z}{2}.$$  

This implies, according to the gauge/gravity duality, that the dual theory is not conformal except the AdS case with the dynamical exponent $z = 1$. Note that in other model described by a massive gauge theory with a non-dynamical scalar field, it was shown that the equation of state parameter is given by 1 for $z = 2$, which is consistent with (13).

The specific heat of the system becomes in terms of temperature

$$C_v = \frac{V_2}{2zG} \left(\frac{4\pi}{z + 2}\right)^{\frac{2}{z}} T^2.$$  

(14)

It is always positive for $z > 0$, so the Lifshitz theory dual to the Lifshitz black brane is thermodynamically stable. In the zero temperature limit $r_h \to 0$, the internal and free energies in (12) become zero. Comparing them with the results at finite temperature, since the free energy at finite temperature is always negative for $z > 0$, the Lifshitz black brane is more preferable than the zero temperature geometry. This fact implies that there is no Hawking-Page transition. A similar situation also occurs in the relativistic non-conformal theory [19].

In [10], only the $z = 2$ case has been considered and the black brane factor $f(r)$ is slightly different from the present one.
3 Binding energies in the Lifshitz theory

Following the gauge/gravity duality, the dual theory of a Lifshitz geometry is described by the Lifshitz-type field theory, especially the non-relativistic one for $z = 2$. In this section, we will investigate the binding energies of particles and monopoles in such a Lifshitz theory. In the 5-dimensional asymptotic AdS geometry, the binding energy of heavy quarkonium has been investigated. Especially for the pure AdS geometry dual to the relativistic conformal theory at zero temperature, the binding energy of heavy quarkonium is inversely proportional to the inter-distance between two quarks due to the conformal symmetry [21, 22, 23]. So it is interesting to ask how the binding energy is modified in the non-relativistic or non-conformal medium [20].

Let us first consider the binding energy of fundamental particles like quarks, which can be described by a fundamental string ending on the infinite boundary in the bottom-up model. In order to write the action of a fundamental string, we should take a string frame in which the string action is defined [40, 41, 42]. Since the Lifshitz geometry includes a nontrivial dilaton profile, the metric in the string frame becomes

$$ds^2_{\text{string}} = e^\frac{\phi}{2} ds^2_{\text{Einstein}},$$

where the Einstein frame metric is given by (5). Then, the Nambu-Gotto action of a fundamental string is

$$S_{F1} = \frac{1}{2\pi \alpha'} \int d^2 \sigma \sqrt{-\det \left( g_{\mu \nu} \frac{\partial x^\mu}{\partial \sigma^{\alpha}} \frac{\partial x^\nu}{\partial \sigma^{\beta}} \right)},$$

where $g_{\mu \nu}$ is the metric in the string frame. Choosing the static gauge

$$\tau = t, \quad \sigma = x^1 = x, \quad x^2 = x^3 = 0 \quad \text{and} \quad r = r(x),$$

and assuming that the end points of string are located at $x = \pm l/2$ and $r = \infty$, the string action reduces to

$$S_{F1} = \frac{T_t}{2\pi \alpha'} \int_{-l/2}^{l/2} dx \ e^\frac{\phi}{2} r^{z-1} \sqrt{r^2 + r^4 f(r)},$$

where $T_t$ means a time interval. If we regard $x$ as a time, the conserved Hamiltonian reads

$$H = -\frac{T}{2\pi \alpha'} r^{\delta-1} \frac{r^4 f(r)}{\sqrt{r^2 + r^4 f(r)}},$$

where $\delta = z - \sqrt{z - 1}$. The binding of two particles is described by an $U$-shape string whose ends describe two fundamental particles in the dual theory, so there is a tip or turning point whose position is denoted by $r_*$. At the tip satisfying $\dot{r}_* = 0$ the conserved Hamiltonian is simply

$$H = -\frac{T}{2\pi \alpha'} r_*^{\delta+1} \sqrt{f(r_*)}.$$

Comparing the above two Hamiltonians, the inter-distance of particles $l$ can be represented by $r_*$

$$l = 2 \frac{\sqrt{f(1)}}{r_*} \int_1^\infty dy \frac{1}{y^2 \sqrt{f(y)} \sqrt{y^{2\alpha+2} f(y) - f(1)}}.$$

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where \( y = \frac{r}{r^*} \) and \( f(y) = 1 - \frac{y^{\delta + 2}}{y^{2 + 2\delta}} \). By taking the analogy to the AdS/CFT correspondence, the energy of two particles becomes

\[
E \equiv \frac{S_{F1}}{T_i} = \frac{r^\delta}{\pi \alpha'} \int_1^{\infty} dy \frac{y^{2\delta} \sqrt{f(y)}}{\sqrt{y^{2\delta + 2}}} f(y) - f(1),
\]

which diverges at \( y \to \infty \). Note that the above energy is the unrenormalized one because of the infinite mass of particles. In order to renormalize it we should subtract the mass of two free particles described by two straight strings, which can be parameterized as

\[
\tau = t, \quad \sigma = r, \quad \text{and} \quad x^1 = \pm \frac{l}{2}.
\]

Then, the energy of two straight strings from (16) becomes in terms of \( y \)-integral

\[
E_{ct} = \frac{r^\delta}{\pi \alpha'} \int_{y_h}^{\infty} dy \frac{y^{\delta - 1}}{\sqrt{y^{2\delta + 2} f(y) - f(1)}},
\]

where \( y_h = \frac{r_h}{r^*} \). This counter term exactly cancels the divergence of the unrenormalized energy and corresponds to the mass of two free particles. As a result, the renormalized binding energy of a pair of particles is given by

\[
V = E - E_{ct}.
\]

In the short inter-distance limit \( (r_h \ll r^*) \), the inter-distance and the binding energy of a pair of particles up to \( r_h^{\delta + 2} \) order are given by

\[
\begin{align*}
  l &= \frac{A_0}{r^*} + A_1 \frac{r_h^{\delta + 2}}{r^*^{\delta + 3}}, \\
  V &= B_0 r^\delta + B_1 r^\delta + B_2 \frac{r_h^{\delta + 2}}{r^*^{\delta + 2 - \delta}},
\end{align*}
\]

where

\[
A_0 = \frac{\sqrt{\pi}}{1 + \frac{\delta}{\Gamma(\frac{2 + \delta}{2 + 2\delta})}},
\]

\[
A_1 = \frac{\sqrt{\pi}}{1 + \frac{\delta}{\Gamma(\frac{2 + \delta}{2 + 2\delta})}} \left[ \frac{\Gamma(\frac{2 + \delta}{2 + 2\delta})}{\frac{1}{2 + 2\delta}} + \frac{\Gamma(\frac{4 + \delta + \delta}{2 + 2\delta})}{\frac{5 + \delta + 2\delta}{2 + 2\delta}} - \Gamma(\frac{6 + \delta + 2\delta}{2 + 2\delta}) \right],
\]

\[
B_0 = -\frac{1}{\sqrt{\pi \alpha' \delta}} \Gamma(\frac{2 + \delta}{2 + 2\delta}),
\]

\[
B_1 = \frac{1}{\pi \alpha' \delta},
\]

\[
B_2 = -\frac{1}{\sqrt{\pi \alpha'}} \left[ \frac{\Gamma(\frac{4 + \delta + \delta}{2 + 2\delta})}{2(1 + \delta) \Gamma(\frac{3 + \delta}{2 + 2\delta})} + \frac{\Gamma(\frac{2 + \delta}{2 + 2\delta})}{\Gamma(\frac{1}{2 + 2\delta})} \right].
\]
For representing the binding energy in terms of the inter-distance, we set

\[ r_\ast = \frac{A_0}{l} (1 + g), \]  

(29)

where \( g \) is a small function of \( l \) and \( r_h \). Inserting (29) into (26) gives rise to

\[ g = \frac{A_1}{A_0^{z+3}} l^{z+2} r_h^{z+2}. \]  

(30)

So the binding energy can be rewritten as

\[ V = \frac{A_0^\delta B_0}{l^\delta} + B_1 l^\delta + \left( \frac{\delta B_0}{A_0^{z+3-\delta}} + \frac{B_2}{A_0^{z+2-\delta}} \right) l^{z+2-\delta} r_h^{z+2}. \]  

(31)

Here, the first term is the zero temperature result, where \( g = 0 \). As expected, the binding energy of the conformal theory (\( z = 1 \)) is inversely proportional to the inter-distance of particles, \( V_0 \sim 1/l \). For a general Lifshitz theory, the zero temperature binding energy usually depends on the dynamical exponent. Interestingly, for the non-relativistic case (\( z = 2 \)) the zero temperature binding energy gives the same result as the conformal case. In other words, the zero temperature binding energies for \( z = 1 \) and \( z = 2 \) are described by the same result

\[ V_0(z = 1) = V_0(z = 2) = \frac{1}{2\sqrt{\pi\alpha'}} \left[ \Gamma \left( \frac{1}{4} \right) - \sqrt{\pi} \Gamma \left( \frac{3}{4} \right) \right] \Gamma \left( \frac{3}{4} \right) \Gamma \left( \frac{5}{4} \right) \frac{1}{l}. \]  

(32)

The second term in (31) is a function of temperature only and independent of the inter-distance of particles. Since this term is originated from the counter term in (24), it may depend on the renormalization scheme. Finally, the third term represents the leading thermal correction which is independent of the renormalization scheme and proportional to

\[ V_{th} \sim l^{2+\sqrt{z-1}} T^{\frac{z+2}{2}}, \]  

(33)

where (10) is used. This result shows that in the short inter-distance limit the thermal correction of the conformal theory is \( V_{th} \sim l^2 T^3 \), whereas it is proportional to \( V_{th} \sim l^3 T^2 \) for the non-relativistic Lifshitz theory (\( z = 2 \)). Therefore, though the zero temperature binding energies for \( z = 1 \) and \( z = 2 \) are same, those two systems have usually different thermal corrections.

Similarly, we can also investigate the binding energy of a pair of monopoles which in string theory can be described by a \( D1 \)-string instead of an \( F1 \)-string. The action for a \( D1 \)-string in the string frame is given by

\[ S_{D1} = \frac{1}{2\pi\alpha'} \int d^2\sigma e^{-\phi} \sqrt{-\det \left( g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^\alpha} \frac{\partial x^\nu}{\partial \sigma^\beta} \right)}, \]

\[ = \frac{T}{2\pi\alpha'} \int_{-l/2}^{l/2} dx e^{-\frac{\phi}{2}} r^{z-1} \sqrt{r^2 + r^4 f(r)}, \]  

(34)

where the same gauge (17) is used. Assuming that a \( D1 \)-string connecting a monopole and an anti-monopole has a tip at \( r_\ast \) and applying the same method used in the \( F1 \)-string case, the inter-distance and the
The renormalized binding energy of monopoles becomes
\[ l = \frac{2\sqrt{f(1)}}{r} \int_{1}^{\infty} dy \frac{1}{y^2 \sqrt{f(y) y^{2\eta+2} f(y) - f(1)}}, \]
\[ V = \frac{r^{2\eta}}{\pi \alpha'} \left( \int_{1}^{\infty} dy \frac{y^{2\eta} \sqrt{f(y)}}{\sqrt{y^{2\eta+2} f(y) - f(1)}} - \int_{y_h}^{\infty} dy y^{\eta-1} \right), \tag{35} \]
where \( \eta = z + \sqrt{z-1} \). Note that the above resulting formula is the same as the fundamental string case if replacing \( \eta \) by \( \delta \). The zero temperature binding energy is proportional to
\[ V_0 \sim \frac{1}{l z + \sqrt{z-1}}. \tag{36} \]
For \( z = 1 \), there is no distinction between particles and monopoles because of the absence of a nontrivial dilaton. Therefore, the binding energy of monopoles is the same as the one of particles. The binding energy of monopoles in the non-relativistic Lifshitz theory \( (z = 2) \) leads to
\[ V_0(z = 2) = -\frac{8\pi}{3\alpha'} \left( \frac{\Gamma \left( \frac{5}{8} \right)}{\Gamma \left( \frac{1}{8} \right)} \right)^4 \frac{1}{l^3}, \tag{37} \]
which is steeper than the binding energy of particles. Ignoring the renormalization scheme dependent term, the first thermal correction becomes
\[ V_{th} \sim l^{2-\sqrt{z-1}} T^{\pm\frac{2}{z}}. \tag{38} \]
which is proportional to \( l T^2 \) for \( z = 2 \).

In Fig 1, we numerically plot the binding energies, \[22\] and \[35\], for \( z = 1 \) and \( z = 2 \) at a given temperature. The result shows that the binding energy of the non-relativistic Lifshitz theory is stronger than the one of the conformal theory although the binding energies of particles at zero temperature are same. So the bound states of particles and monopoles are dissolved more easily in the relativistic conformal theory than the non-relativistic one. We also see that the bound state of the monopoles can be more easily dissolved in the non-relativistic theory than the particle bound state.

### 4 Drag force

Now, let us consider the drag force of the non-relativistic Lifshitz theory. Due to the nontrivial dilaton field, the string action is \[40, 41, 42\]
\[ S = \frac{1}{2\pi \alpha'} \int d^2 \sigma \ e^{\zeta \phi/2} \sqrt{-\det h_{\alpha\beta}}, \tag{39} \]
where \( h_{\alpha\beta} \) is the induced metric on the string worldsheet
\[ h_{\alpha\beta} = g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu, \tag{40} \]
and \( \zeta \) is +1 for a particle or −1 for a monopole. In order to describe the drag force, we take into account a string moving in the \( x \)-direction. The worldsheet current carried by a string leads to
\[ P_\mu^{\alpha} = -\frac{e^{\zeta \phi/2}}{2\pi \alpha'} g_{\mu\nu} \partial^\alpha X^\nu. \tag{41} \]
Figure 1: The binding energies at a given temperature $T = 2.387$, where the black brane horizon is located at $r = 10$ for $z = 1$ or $r = 2.7386$ for $z = 2$. The solid line represents the binding energies of particles and monopole for $z = 1$. The dashed and dotted ones show the binding energies of particles and monopoles respectively in the non-relativistic Lifshitz theory $z = 2$.

For convenience, we set $2\pi\alpha' = 1$ and consider the $z = 2$ case from now on. Under the following ansatz

$$
\tau = t, \quad \sigma = r, \quad \text{and} \quad x = vt + \tilde{x}(r),
$$

(42)

the action is reduced to

$$
S = \int d^2\sigma \, e^{\xi\phi/2} \sqrt{r^2 - \frac{v^2}{f(r)} + r^6 f(r) \, \tilde{x}'^2}.
$$

(43)

If we regard $r$ as a time coordinate, the conserved momentum becomes

$$
\Pi_x = e^{\xi\phi/2} \frac{r^6 f(r) \, \tilde{x}'}{\sqrt{r^2 - \frac{v^2}{f(r)} + r^6 f(r) \, \tilde{x}'^2}}.
$$

(44)

If rewriting $\tilde{x}'$ in terms of $\Pi_x$

$$
\tilde{x}' = \frac{\Pi_x \sqrt{r^2 f(r) - v^2}}{r^3 f(r) \sqrt{e^{\xi\phi} r^6 f(r) - \Pi_x^2}},
$$

(45)

there exists a critical position $r^*$

$$
r^* = \frac{\sqrt{v^2 + \sqrt{v^4 + 4r^4_h}}}{\sqrt{2}},
$$

(46)

where the inside of the square root in the numerator changes its sign. For a well-defined $\tilde{x}'$, the inside of the square root in the denominator also changes the sign at this critical position, which determines the conserved momentum to be

$$
\Pi_x = r^*^{1-\xi} \sqrt{r^*_4 - r^4_h}.
$$

(47)
Using the worldsheet current in (41), the drag force is given by
\[ \frac{dp_1}{dt} = -\sqrt{-\det h_{\alpha\beta}} \, e^{\zeta \phi/2} g_{xx} h^{rr'} \bar{x}' . \]  
(48)

Near the asymptotic boundary \((r \to \infty)\), it reduces in terms of the physical quantities to
\[ \frac{dp_1}{dt} = -\frac{p_1}{\sqrt{m^2 + p_1^2}} \left( \frac{p_1^2 + \sqrt{p_1^4 + 4\pi^2(m^2 + p_1^2)^2 T^2}}{2(m^2 + p_1^2)} \right)^{1-\zeta/2} , \]  
(49)

where \(m\) is the mass of a particle \((\zeta = 1)\) or monopole \((\zeta = -1)\), and \(T\) is the Hawking temperature given in (10) for \(z = 2\).

For the ultra-relativistic case \((p_1 \gg m)\), the momentum decreases linearly in time
\[ p_1 = p_1^0 - \left( \frac{1 + \sqrt{1 + 4\pi^2 T^2}}{2} \right)^{1-\zeta/2} t , \]  
(50)

where \(p_1^0\) denotes the momentum at \(t = 0\). In the large temperature limit, the decreasing rate is proportional to \(T^{1/2}\) for a particle and \(T^{3/2}\) for a monopole. In the non-relativistic case \((m \gg p_1)\), the momentum exponentially decays
\[ p_1 = p_1^0 e^{-\Gamma t} , \]  
(51)

where the decaying rate \(\Gamma\) is given by
\[ \Gamma = \frac{\sqrt{\pi T}}{m} \quad \text{for a particle}, \]
\[ = \frac{(\pi T)^{3/2}}{m} \quad \text{for a monopole}. \]  
(52)

5 Linear response in the zero momentum limit

In order to understand macroscopic properties of a Lifshitz theory, it is useful to investigate the linear response of the various fluctuations. Here, we will concentrate on the electric properties of a Lifshitz theory by turning on the various U(1) vector field fluctuations. Since a Lifshitz geometry includes a nontrivial dilaton profile, a Maxwell term describing a vector fluctuation can have a nontrivial \(r\)-dependent gauge coupling
\[ S_M = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \, \frac{e^{\gamma \phi}}{4} f_{\mu\nu} f^{\mu\nu} , \]  
(53)

where we insert \(\frac{1}{16\pi G}\) for later convenience. In the above, \(f_{\mu\nu}\) implies the field strength of a vector fluctuation and \(\gamma\) is an arbitrary constant. Then, the \(r\)-dependent bulk gauge coupling is given by
\[ 1 \, e^{\gamma \phi} \]  
(54)

It is worth to notice that if a vector fluctuation is the same U(1) gauge field as the background one, \(\gamma\) should be \(\lambda\). From now on, we denote such a vector fluctuation as \(a_\mu\). If a different U(1) vector fluctuation
is turned on, we represent it as $b_\mu$ and in this case $\gamma$ can have any other value because $b_\mu$ is independent of the background gauge field $A_\mu$. Since introducing a new gauge fluctuation $b_\mu$ corresponds to inserting an impurity to the dual Lifshitz theory, we simply call $b_\mu$ an impurity. As will be seen, those two different vector fluctuations provide totally different macroscopic properties. The reason is the following. Since an impurity is nothing to do with the background gauge field, there is no mixing with the metric fluctuations in the linear response theory [29, 43]. For the same vector fluctuation $a_\mu$, it should be coupled to the metric fluctuation through the background gauge field $A_t$ which changes the conductivity dramatically. From now on, we concentrate on the $z = 2$ case, which provides an interesting example for a non-relativistic Lifshitz theory, and take the zero momentum limit because the DC conductivity is well-defined even in this limit.

### 5.1 DC conductivity in the membrane paradigm

At finite temperature, the dual geometry can be mapped to the previous Lifshitz black brane solution (6) with $z = 2$. In order to investigate the macroscopic properties of this non-relativistic Lifshitz theory, we first turn on an impurity $b_\mu$. As mentioned before, since it is decoupled from the metric fluctuations at quadratic order, an impurity is governed by

$$S_{fluc} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \frac{e^{\gamma\phi}}{4} f_{\mu\nu} f^{\mu\nu},$$

(55)

where $f_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$. Before investigating the hydrodynamics of the non-relativistic Lifshitz theory by using the Kubo formula, we first summarize the result of an alternative way, the so called membrane paradigm. In the membrane paradigm, the DC conductivity of the dual theory can be represented by the bulk quantities defined on the black brane horizon. The resulting form for $z = 2$ is [20, 35, 36]

$$\sigma_{DC} = \frac{e^{\gamma\phi}}{16\pi G} \sqrt{\frac{g}{g_{tt} g_{rr}}} g^{tt} \bigg|_{r_h} = \frac{1}{16\pi^{\gamma+1} G} \frac{1}{T^\gamma}. \quad (56)$$

Note that in the membrane paradigm the details of the solution are not important when determining the DC conductivity. The membrane paradigm result implies that the DC conductivity carried by an impurity decreases with temperature for $\gamma > 0$ and is independent of temperature for $\gamma = 0$.

### 5.2 DC conductivity carried by an impurity

Now, let us investigate the DC conductivity carried by an impurity with the Kubo formula where the details of the solution play an important role in determining the DC conductivity. The transverse mode of an impurity, $b_i$ ($i = x$ or $y$), is governed by

$$0 = \partial_\mu \left( \sqrt{-g} e^{\gamma\phi} g^{\mu\rho} g^{i\sigma} F_{\rho\sigma} \right). \quad (57)$$

For $z = 2$ and in the zero momentum limit, under the following Fourier mode expansion

$$b_i(t, r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} b_i(\omega, r), \quad (58)$$
the governing equation simply reduces to
\[ 0 = b_i'' + \left( \frac{3 - 2\gamma}{r} + \frac{f'}{f} \right) b_i' + \frac{\omega^2}{r^6 f^2} b_i. \]  

(59)

At the horizon, \( b_i \) has two independent solutions
\[ b_i(r) = c_1 f^{\pm \nu}, \]  

(60)

with \( \nu = i \frac{\omega}{4r_h} \), where \( c_1 \) is an appropriate normalization constant and the minus or plus sign satisfies the incoming or outgoing boundary condition at the horizon. After choosing an incoming solution, the solution of (59) in the hydrodynamic limit (\( \omega \ll T \)) can be perturbatively expanded to
\[ b_i(r) = f^{-\nu} [G_0(r) + \omega G_1(r)] + \mathcal{O}(\omega^2). \]  

(61)

In this hydrodynamic expansion, \( G_0(r), G_1(r) \) and all higher order terms should be regular at the horizon which we call a regularity condition. In addition, the above perturbative solution should be reduced to (60) at the horizon, so \( G_0(r) \) should be a normalization constant \( c_1 \) at the horizon and at the same time the other terms, \( G_1(r_h) \) and higher order terms, should vanish. We call such a constraint a vanishing condition. Using these two conditions, the solutions in (61) can be exactly determined up to one integration constant

\[ G_0(r) = c_1, \]  

(62)

\[ G_1(r) = c_3 \frac{i c_1 [4 \log r - \log(r^4 - r_h^4)]}{4r_h^2}, \]  

\[ -c_4 \left[ 2 F_1 \left( 1 + \gamma, 1, 2 + \gamma, -\frac{r_+^2}{r^2} \right) + 2 F_1 \left( 1 + \gamma, 1, 2 + \gamma, \frac{r_-^2}{r^2} \right) \right] r^{2+2\gamma}, \]  

(63)

with

\[ c_3 = i c_1 \left[ -PG \left( 0, 1 + \frac{\gamma}{2} \right) + PG \left( 0, \frac{1+\gamma}{2} \right) + 2 \left\{ EG - \log 2 + PG \left( 0, 1 + \gamma \right) \right\} \right], \]  

(64)

\[ c_4 = -ic_1 r_h^{-2\gamma}, \]  

(65)

where \( PG \) and \( EG \) mean the poly gamma and Euler gamma function respectively. In order to determine the remaining integration constant \( c_1 \), we should impose another boundary condition. At the asymptotic boundary, the vector fluctuation \( b_i \) has the following asymptotic expansion
\[ b_i(r) = b_1 + b_2 r^{2\gamma-2}, \]  

(66)

where \( b_1 \) (or \( b_2 \)) is a constant determined by the asymptotic boundary condition.

5.2.1 For \( \gamma < 1 \)

If \( \gamma \) is smaller than 1, the leading behavior of \( b_i(r) \) is determined by the first term \( b_1 \). According to the usual gauge/gravity duality, the first coefficient corresponds to the source while the second describes the vacuum
expectation value (vev) of the dual operator. In this case, it is natural to impose the Dirichlet boundary condition like

$$\text{b}_0 \equiv \lim_{r_0 \to \infty} \text{b}_i(r_0), \quad (67)$$

where $r_0$ implies an appropriate UV cutoff of the dual theory and $b_0$ corresponds to the boundary value of $b_i$ which is equal to $b_1$ for $\gamma < 1$. Comparing the asymptotic expansion of the perturbative solution (61) with the above boundary condition (67), the remaining integration constant $c_1$ can be rewritten in terms of the boundary value $b_0$ as

$$c_1 = \frac{8\sqrt{\pi} h b_0}{8\sqrt{\pi} h + \omega [HN \left(\frac{\gamma}{2}\right) - HN \left(-\frac{1+\gamma}{2}\right) - 2HN(\gamma) + O(1/r^5)]}, \quad (68)$$

where $HN$ means a harmonic number.

The boundary action corresponding to the on-shell action of (55) is given by

$$S_B = -\frac{1}{16\pi G} \int_{r=r_0} d^3x \sqrt{-g} e^{\gamma \phi} g^{rr} g^{ii} b_i b_i' \approx -\frac{1}{16\pi G} \int d^3x r_h^{-2\gamma} r_0^{3-2\gamma} b_0 b_i'. \quad (69)$$

This result shows that the finite contributions to the boundary action can come from $b_i' \sim r_0^{-3+2\gamma}$ when $r_0 \to \infty$. Since the asymptotic expansion of $b_i'$ from (61) has

$$b_i' = -\frac{ic_1 \omega}{r_0^{3-2\gamma}} + O\left(\frac{1}{r^5}\right), \quad (70)$$

the current-current correlation function associated with the imaginary part of the retarded Green function [38, 39] results in

$$\langle J_i J_i \rangle = \frac{i\omega}{16\pi G r_h^{2\gamma}} + O(\omega^2), \quad (71)$$

where (68) is used. Finally, the DC conductivity from the Kubo formula reads

$$\sigma_{DC} \equiv \lim_{\omega \to 0} \frac{\langle J_i J_i \rangle}{i\omega} = \frac{1}{16\pi G} \frac{1}{T^{\gamma}}, \quad (72)$$

which is consistent with the result of the membrane paradigm [56].

5.2.2 For $\gamma \geq 1$

Let us take into account the case with $\gamma \geq 1$. In this case, the DC conductivity of an impurity shows a totally different behavior compared with the previous one because the interpretation of the asymptotic solution is changed. From now on, we will concentrate on the case with $\gamma = 2$ for later comparison with the DC conductivity associated with the same U(1) vector fluctuation $a_i$.

Similar to [61], the perturbative expansion of solution in the zero momentum limit is given by

$$b_i(r) = f^{-\nu} \left[ G_0(r) + \omega G_1(r) + \omega^2 G_2(r) \right] + O(\omega^3), \quad (73)$$
where $\nu = i\frac{\omega}{4r_h^2}$. In this case, $\omega^2 G_2(r)$ is important to determine the DC conductivity unlike the previous case. The solutions, $G_0(r)$ and $G_1(r)$, satisfying the regularity and vanishing condition at the horizon are

$$
G_0(r) = c_1,
$$

$$
G_1(r) = -\left[\frac{i}{2r_h^4}r^2 - \frac{\pi + 2i(1 - \log 2)}{4r_h^2} + \frac{i^2 r^2 \log r - i^2 r^2 \log (r^2 + r_h^2)}{2r_h^4}\right]c_1.
$$

(74)

After inserting these two solutions into (59), we can find the analytic form of $G_2(r)$ which has the following expansion near the horizon

$$
G_2(r) = \frac{12 c_6 r_h^6 - c_1 (6 \log 2 - 10 - 3\pi i)}{48 r h^4} \log (r - r_h) + c_5 + \frac{r_h^2 (2 + \pi i - \log r_h)}{4} c_6
$$

$$
+ \frac{5\pi^2 + 8 (9 - 10 \log 2 + 3(\log 2)^2) + 12\pi i (3 + 2 \log 2)}{4 r_h^2} c_1
$$

$$
+ \frac{(-20 + 42\pi i + 60 \log 2) \log r_h + 48(\log r_h)^2}{96 r_h^4} c_1 + O(r - r_h).
$$

(75)

Again, imposing the regularity and vanishing condition at the horizon, $c_5$ and $c_6$ are fixed to be

$$
c_5 = -\left[\frac{32 + 4\pi i + 11\pi^2 - 56 \log 2 + 36\pi i \log 2 + 24(\log 2)^2}{96 r_h^4} + \frac{48(\pi i + \log 2) \log r_h + (\log r_h)^2}{96 r_h^4}\right]c_1,
$$

$$
c_6 = \frac{6 \log 2 - 10 - 3\pi i}{12 r_h^6} c_1.
$$

(76)

Before calculating the conductivity, it is worth to note that for $\gamma \geq 1$ the second terms in (66) is more dominant when determining the asymptotic behavior of an impurity. This implies that the previous Dirichlet boundary condition in (67) can not fix $b_1$, so we need to change the asymptotic boundary condition. A natural choice is choosing the second coefficient $b_2$ as a source rather than the first and then fixing it by an appropriate boundary condition. Following this strategy, the appropriate asymptotic boundary condition for $\gamma \geq 1$ should be

$$
b_0 = \lim_{r_0 \to \infty} \frac{b_i(r_0)}{r_0^{\gamma - 2}}.
$$

(77)

Especially, for $\gamma = 2$ the boundary condition reduces to

$$
b_0 = \lim_{r \to \infty} \frac{b_i(r_0)}{r_0^2}.
$$

(78)

Using (74) and (76) together with the exact solution for $G_2(r)$, we can easily find the asymptotic expansion of $b_i(r)$ up to $\omega^2$ and comparing it with the boundary condition in (78) determines $c_1$ in terms of $b_0$

$$
c_1 = \omega \left[\frac{24 i r^4}{12 r_h^2 + i \omega (6 \log 2 - 10 - 3\pi i)}\right] b_0.
$$

(79)

At first glance, it looks extraordinary because $c_1$ is proportional to $\omega^{-1}$. However, near the horizon it still becomes a solution whose normalization constant is proportional to $\omega^{-1}$.

Since the boundary action for $\gamma = 2$ is given by

$$
S_B = -\frac{1}{16\pi G} \int d^3x \ r_0 \ b_0 \ b_i',
$$

(80)
only \( b'_i \sim r_0^{-1} \) can provide the finite contribution to the boundary action. If \( \frac{\partial}{\partial r} ( f^{-\nu} G_0 ) \) or \( \frac{\partial}{\partial r} ( f^{-\nu} \omega G_1 ) \) contains such a term, the DC conductivity in the zero frequency limit diverges by \( \omega^{-2} \) or \( \omega^{-1} \) respectively because \( c_1 \sim 1/\omega \). This fact says that the finite contribution to the DC conductivity is determined by not \( \omega G_1 \) but \( \omega^2 G_2 \), as mentioned previously. In the asymptotic region \( (r_0 \to \infty) \), the expansion of \( b'_i \) has the following form

\[
b'_i = \#r_0 + \frac{\#}{r_0^3} + \mathcal{O} \left( \frac{1}{r_0^5} \right), \tag{81}
\]

where \( \# \) implies a certain number. This result shows that \( b'_i \) has no term proportional to \( r_0^{-1} \) so that the finite part of the resulting boundary action becomes zero. Consequently, the DC conductivity carried by an impurity for \( \gamma = 2 \) vanishes

\[
\sigma_{DC} = 0 \quad \text{for} \quad \gamma = 2. \tag{82}
\]

This result is totally different from the result obtained by the membrane paradigm because the physical interpretation of the asymptotic solution is changed for \( \gamma \geq 1 \) and the membrane paradigm does not care about the asymptotic behavior of the solution.

### 5.3 Conductivity carried by the same vector fluctuation

Now, consider the same U(1) vector fluctuation \( a_i \) instead of an impurity. Generally, if there exists a background gauge field, the transverse mode of the same U(1) vector fluctuation should be coupled to the shear mode of the metric fluctuations \[24\]. Therefore, in order to study the linear response of such an U(1) gauge field, we should also consider the shear mode. The equations governing the transverse and shear modes can be derived from the following fluctuation action

\[
S_{\text{fluc}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - \frac{1}{4} e^\phi f_{\mu\nu} f^{\mu\nu} \right), \tag{83}
\]

where \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \). If we rewrite the shear and transverse mode as the Fourier mode expansion

\[
g'_i(t, \omega) = \int \frac{d\omega}{2\pi} e^{-i\omega t} g'_i(\omega, r), \\
a_i(t, \omega) = \int \frac{d\omega}{2\pi} e^{-i\omega t} a_i(\omega, r), \tag{84}
\]

the governing equations for shear modes reduce to

\[
0 = g''_i + \frac{q}{r^{5-z}} a_i, \tag{85}
\]

\[
0 = g'''_i + \frac{(5-z)}{r} g''_i + \frac{q}{r^{5-z}} a'_i, \tag{86}
\]

where \( g'_i \) and \( a_i \) imply \( g'_i(\omega, r) \) and \( a_i(\omega, r) \). The first equation \[85\] is a constraint which automatically satisfies the second equation \[86\]. The equation governing the transverse modes leads to

\[
0 = a''_i + \frac{rf' + (z-3)f}{r f} a'_i + \frac{qr^3-z}{f} g''_i + \frac{w^2}{r^{2+2z} f^2} a_i. \tag{87}
\]
Inserting the constraint into (87), the decoupled differential equation of the transverse mode becomes

\[ 0 = a_i'' + \frac{r f'}{r f} a_i' + \left( \frac{w^2}{r^{2z+2} f^2} - \frac{q^2}{r^2 f} \right) a_i. \]  

(88)

At the horizon, due to vanishing of \( f \), \( a_i \) should have the following two independent solutions up to overall normalization

\[ a_i \sim f^{\pm i \frac{\omega}{4r_h}}, \]  

(89)

where the plus or minus sign again implies the outgoing or incoming mode. In the hydrodynamic limit \( (\omega \ll 1) \), after taking only the incoming part, the near horizon solution of \( a_i \) can be expanded into

\[ a_i = f^{-i \frac{\omega}{4r_h}} \left[ G_0(r) + \omega G_1(r) \right] + O(\omega^2), \]  

(90)

where \( G_0(r) \) and \( G_1(r) \) should be regular functions at the black brane horizon. After substituting the perturbative expansion form to (88), we can solve it order by order. For \( z = 2 \), \( G_0 \) has the following exact solution at \( \omega^0 \) order

\[ G_0 = c_1 \left( r^4 + r_h^4 \right) - \frac{c_2}{8r_h^4} \log \left( \frac{r^2}{r_h^2} \right) + \frac{c_2 \left( r^4 + r_h^4 \right)}{8r_h^6} \arctanh \left( \frac{r^2}{r_h^2} \right), \]  

(91)

where \( c_1 \) and \( c_2 \) are two integration constants. Imposing the regularity condition to \( G_0 \) at the horizon, \( c_2 = 0 \) because \( \arctanh \left( \frac{r^2}{r_h^2} \right) \) diverges. Consequently,

\[ G_0 = c_1 \left( r^4 + r_h^4 \right). \]  

(92)

Using this result, at the next order of \( \omega \) the solution of \( G_1 \) is given by

\[ G_1 = c_3 \left( r^4 + r_h^4 \right) - \frac{c_4 r^2}{8r_h^6} \frac{\log r}{r^2} + \frac{(r^4 + r_h^4)(c_4 + 4c_1 r_h^4)}{16r_h^6} \log \left( \frac{r^2}{r_h^2} \right) \]  

\[ - \frac{(r^4 + r_h^4)(c_4 - 4c_1 r_h^4) \log (r^2 - r_h^2)}{16r_h^6}. \]  

(93)

Since the last term diverges at the horizon, the regularity condition determines

\[ c_4 = i4c_1 r_h^4. \]  

(94)

In addition, the vanishing condition fixes the rest integration constant as

\[ c_3 = - \frac{ic_1 (2 \log 2 - 1)}{4r_h^2}. \]  

(95)

Now, let us consider the asymptotic behavior of the solution. Unlike the impurity case, the asymptotic behavior of \( a_i \) is totally different from that of \( b_i \) due to the nontrivial mixing with the shear mode. From (88), the asymptotic behavior for \( z = 2 \) is governed by

\[ 0 = a_i'' - \frac{1}{r} a_i' - \frac{8}{r^2} a_i, \]  

(96)
where the last term is originated from the shear mode. For an impurity, there is no such term because an impurity does not couple to the shear mode. This new term changes the asymptotic behavior of $a_i$ dramatically and provides a different conductivity. At the asymptotic boundary, $a_i$ has the following perturbative solution

$$a_i = a_1 r^4 + \frac{a_2}{r^2},$$  \hfill (97)

where $a_1$ is a constant to be determined by the asymptotic boundary condition. One can identify the coefficients of the non-normalizable and normalizable mode, $a_1$ and $a_2$, with the source and expectation value of the dual operator respectively. In order to fix the boundary value of $a_i$, we impose the following boundary condition at the asymptotic boundary

$$a_0 \equiv \lim_{r \to \infty} \frac{a_i(r)}{r^4}.$$  \hfill (98)

Then, the integration constant $c_1$ is fixed in terms of the boundary value $a_0$ to be

$$c_1 = \frac{4 r_h^2 a_0}{4 r_h^2 - i \omega (2 \log 2 - 1)}.$$  \hfill (99)

From the action (83), the boundary action of the same gauge fluctuation becomes

$$S_B = -\frac{1}{16 \pi G} \int d^3 x \sqrt{-g} \ e^{\lambda \phi} g^{rr} g^{ii} a_i \partial_r a_i$$

$$= -\frac{1}{16 \pi G} \int d^3 x \ r_0^3 a_0 \ \partial_r a_i,$$  \hfill (100)

where we can see that the finite part of the retarded Green function comes from $\partial_r a_i \sim r_0^{-3}$. When ignoring the divergent parts corresponding to the contact terms, the retarded Green function of the transverse mode becomes

$$\langle J^i J^i \rangle = \frac{i \omega r_0^4}{12 \pi G}.$$  \hfill (101)

Therefore, the DC conductivity carried by $a_i$ in the non-relativistic field theory becomes

$$\sigma_{DC} = \frac{\pi}{12 G} T^2.$$  \hfill (102)

This result shows that the DC conductivity of the same $U(1)$ gauge fluctuation is totally different from the DC conductivity carried by an impurity. The DC conductivity carried by the same vector fluctuation increases with temperature due to the shear mode contribution, whereas one carried by an impurity decreases for $\gamma < 1$ or vanishes for $\gamma = 2$.

Using the above solution of the transverse mode $a_i$, we can easily find the solution of the shear mode $g_t^i$ from (85)

$$g_t^i(r) = -\int_{r_b}^r d\tilde{r} \ \frac{q}{\tilde{r}^3} \ a_i(\tilde{r}) + c,$$  \hfill (103)

where $c$ is a new integration constant which does not play any role in evaluating the retarded Green function of the shear mode. Since the leading term of $a_i$ is proportional to $a_1 r^4$ at the asymptotic region, the leading
behavior of the shear mode is given by \( qa_1 r^2 / 2 \). In order to determine the retarded Green function of the shear mode, we should impose another boundary condition to \( g_i^t \)

\[
g_0 \equiv \lim_{r \to \infty} \frac{g_i^t(r)}{r^2} = -\frac{qc_1}{8r_h^2} \left[ 4r_h^2 - i\omega (2 \log 2 - 1) \right], \tag{104}
\]

Then we can rewrite \( c_1 \), in terms of the boundary value \( g_0 \), as

\[
c_1 = -\frac{8r_h^2 g_0}{q \left[ 4r_h^4 - i\omega (2 \log 2 - 1) \right]}. \tag{105}
\]

The boundary action generated from the two derivative term of the shear mode, which is the two point function of the momentum operators, leads to

\[
S_B = \frac{1}{16\pi G} \int d^3x \, r_0^5 g_0 \, g_i^t'. \tag{106}
\]

After discarding the contact terms, the finite part of the retarded Green function comes from \( g_i^t' \sim r^{-5} \), but actually such a term is absent in \( g_i^t' \). This fact implies that the retarded Green function of the momentum \( T_i^t \) dual to \( g_i^t \) vanishes

\[
\langle T_i^t \rangle = 0. \tag{107}
\]

In order to understand this result, we need to consider the non-zero momentum case. If we turn on the momentum \( k \) in the \( y \)-direction, the retarded Green function of the momentum operator is usually proportional to \( k^2 \) \cite{19, 38, 39}

\[
\langle T_x^t \rangle \sim \frac{k^2}{\omega - iDk^2}, \tag{108}
\]

where \( D \) is the momentum diffusion constant. Therefore, in the zero momentum limit \( k = 0 \), the result in \( \text{(107)} \) seems to be natural.

### 6 Discussion

In this paper, we have investigated the thermodynamics of the Lifshitz black brane, which has a hyperscaling symmetry. After finding the Lifshitz black brane thermodynamics, we identify it with thermodynamics of the dual Lifshitz theory following the gauge/gravity duality. This procedure shows that the dual Lifshitz field theory has an equation of state parameter \( z/2 \). In addition, we showed that the dual Lifshitz theory is always thermodynamically stable for \( z > 0 \). In the Lifshitz theory at finite temperature, the binding energy and drag force of particles and monopoles were investigated. The non-relativity of the Lifshitz theory makes the binding energy of particles and monopoles stronger than the relativistic conformal case. Related to the momentum dissipation of a particle and monopole, we showed that for a particle and monopole moving ultra-relativistically the momentum linearly decreases in time whereas it exponentially decreases for a non-relativistic motion.

We also studied on the hydrodynamics of the dual Lifshitz theory in the zero momentum limit. If we turn on a new gauge field corresponding to an impurity, it does not couple to the metric fluctuation at
the linear level. The DC conductivity carried by an impurity was investigated by two different ways. One is the membrane paradigm in which only information near the black brane horizon is required. In the membrane paradigm, the DC conductivity is given by \(\frac{1}{16\pi\gamma+1}G\frac{1}{T}\gamma\), which shows that the DC conductivity decreases with temperature for \(\gamma > 0\). The other is to study the perturbative expansion of the bulk solution in the hydrodynamic limit, in which the asymptotic form of the solution plays an important role in determining the hydrodynamic transport coefficient unlike the membrane paradigm. We found that depending on the magnitude of a parameter describing the bulk gauge coupling, the roles of a source and operator are exchanged. The DC conductivity in the Kubo formula for \(\gamma < 1\) is consistent with the membrane paradigm result, while the DC conductivity for \(\gamma = 2\) shows totally different behavior. For \(\gamma = 2\), due to the exchange of roles between the source and operator, the resulting DC conductivity vanishes.

We further studied the DC conductivity associated with the same vector fluctuation as the background gauge field. Unlike an impurity with \(\gamma = 2\), the resulting DC conductivity carried by the same vector fluctuation is not zero because it is coupled to the metric fluctuation and the metric fluctuation changes its asymptotic behavior dramatically. After imposing the asymptotic boundary condition correctly, we found that the resulting DC conductivity is proportional to the square of temperature for \(z = 2\). As a result, depending on the charge carrier the DC conductivity can have different temperature dependence. For example, the DC conductivity carried by an impurity becomes zero for \(\gamma = 2\) whereas one carried by the same vector fluctuation shows a nontrivial temperature dependence, \(\sigma_{DC} = \frac{\pi}{12G}T^2\) for \(z = 2\).

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