Structure and Stability of the 1-Dimensional Mapper

Mathieu Carrière, Steve Oudot

Abstract

Given a continuous function \( f : X \to \mathbb{R} \) and a cover \( \mathcal{I} \) of its image by intervals, the Mapper is the nerve of a refinement of the pullback cover \( f^{-1}(\mathcal{I}) \). Despite its success in applications, little is known about the structure and stability of this construction from a theoretical point of view. As a pixelized version of the Reeb graph of \( f \), it is expected to capture a subset of its features (branches, holes), depending on how the interval cover is positioned with respect to the critical values of the function. Its stability should also depend on this positioning. We propose a theoretical framework that relates the structure of the Mapper to the one of the Reeb graph, making it possible to predict which features will be present and which will be absent in the Mapper given the function and the cover, and for each feature, to quantify its degree of (in-)stability. Using this framework, we can derive guarantees on the structure of the Mapper, on its stability, and on its convergence to the Reeb graph as the granularity of the cover \( \mathcal{I} \) goes to zero.

1 Introduction

Many data sets nowadays come in the form of point clouds with function values attached to the points. Such data may come either from direct measurements (e.g. think of a sensor field measuring some physical quantity like temperature or humidity), or as a byproduct of some data analysis pipeline (e.g. think of a word function in the quantization phase of the bag-of-words model). There is a need for summarizing such data and for uncovering their inherent structure, to enhance further processing steps and to ease interpretation.

One way of characterizing the structure of a scalar field \( f : X \to \mathbb{R} \) is to look at the evolution of the topology of its level sets — i.e. sets of the form \( f^{-1}(\{\alpha\}) \), for \( \alpha \) ranging over \( \mathbb{R} \). This information is summarized in a mathematical object called the Reeb graph of the pair \( (X, f) \), denoted by \( R_f(X) \) and defined as the quotient space obtained by identifying the points of \( X \) that lie in the same connected component of the same level set of \( f \). The Reeb graph is known to be a graph (technically, a multi-graph) when \( X \) is a smooth manifold and \( f \) is a Morse function, or more generally when \( f \) is of Morse type (see Definition 2.1). Moreover, since the map \( f \) is constant over equivalence classes, there is a well-defined quotient map on \( R_f(X) \).

The connection between the topology of the Reeb graph and the one of its originating pair \( (X, f) \) has been the object of much study in the past and is now well understood. It has gained increasing interest in the recent years, with the introduction of persistent homology \([25, 34]\) and its extended version \([18]\). Indeed, the extended persistence diagram of the quotient map \( \tilde{f} \) (see Section 2 for a formal definition) describes the structure of the Reeb graph in the following sense. It takes the form of a multiset of points in the plane, where each point is matched with a feature (branch or hole) of the Reeb graph in a one-to-one manner. Furthermore, the coordinates of the point characterize the span of the feature, that is, the interval of \( \mathbb{R} \) spanned by its image through \( \tilde{f} \). The vertical distance of the point to the diagonal \( \Delta = \{(x, x) \mid x \in \mathbb{R}\} \) measures the length of that interval and thereby quantifies the prominence of the feature. Thus, the persistence diagram plays the role of a “bag-of-features” type descriptor, summarizing the Reeb graph through its list of features together with their spans, and forgetting about the actual layout of those features.

One issue with the Reeb graph is its computation. Indeed, when the pair \( (X, f) \) is known only through a finite set of measurements, the graph can only be approximated within a certain error. Quantifying this error, in particular finding the right metric in which to measure it, has been the object of intense investigation in the recent years. A simple choice is to use the bottleneck distance between the persistence diagram associated to the Reeb graph and the one associated to its approximation \([17]\). This pseudometric treats the Reeb graph
and its approximation as bags of features, and it measures the differences between their respective sets of features. In particular, it is oblivious to the layouts of the features in each of the graphs. Other distances have been proposed recently, to capture a greater part of this layout [11, 5, 20].

Building approximations from finite point samples with scalar values is a problem in its own right. A natural approach is to build a simplicial complex (for instance the Rips complex) on top of the point samples, to serve as a proxy for the underlying continuous space; then, to extend the scalar values at the vertices to a piecewise linear (PL) function over the simplicial complex by linear interpolation; finally, to apply some exact computation algorithm for PL functions. This is the approach advocated by Dey and Wang [23], who rely on the $O(n \log n)$ expected time algorithm of Harvey, Wenger and Wang [26] for the last step. The drawbacks of this approach are:

- Its relative complexity: the Reeb graph computation from the PL function is based on collapses of its simplicial domain that may break the complex structure temporarily and therefore require some repairs.

- Its overall computational cost: here, $n$ is not the number of data points, but the number of vertices, edges and triangles of the Rips complex, which, in principle, can be up to cubic in the number of data points. Indeed, triangles are needed to compute an approximation of the Reeb graph, in the same way as they are to compute 1-dimensional homology.

The Mapper was introduced by Singh, Mémoli and Carlsson [31] as a new mathematical object to summarize the topological structure of a pair $(X, f : X \to \mathbb{R}^d)$. Its construction depends on the choice of a cover $I$ of the image of $f$ by open sets. Pulling back $I$ through $f$ gives an open cover of the domain $X$. This cover may have some elements that are disconnected, so it is refined into a connected cover by splitting each element into its various connected components. Then, the Mapper is defined as the nerve of the connected cover, having one vertex per element, one edge per pair of intersecting elements, and more generally, one $k$-simplex per non-empty $(k+1)$-fold intersection. From a philosophical point of view, the Mapper can be thought of as a pixelized version of the Reeb space, where the resolution is prescribed by the cover $I$. From a practical point of view, its construction from point cloud data is very easy to describe and to implement, using standard graph traversals to detect connected components. Furthermore, it only requires to build the 1-skeleton graph of the Rips complex, whose size scales up at worst quadratically (and not cubically) with the size of the input point cloud.

As a simple alternative to the Reeb space, the Mapper has been the object of much interest by practitioners in the data sciences. It has played a key role in several success stories, such as the identification of a new subgroup of breast cancers [29], or the elaboration of a new classification of player positions in the NBA [1], due to its ability to deal with very general functions and datasets. Meanwhile, it has become the flagship component in the software suite developed by Ayasdi, a data analytics company founded in the late 2000’s whose interest is to promote the use of topological methods in the data sciences. Somewhat surprisingly, despite this success, very little is known to date about the structure of the Mapper and its stability with respect to perturbations of the pair $(X, f)$ or of the cover $I$. Intuitively, when $f$ is scalar, as a pixelized version of the Reeb graph, the Mapper should capture some of its features (branches, holes) and miss others, depending on how the cover $I$ is positioned with respect to the critical values of $f$. How can we formalize this phenomenon? The stability of the structure of the Mapper should also depend on this positioning. How can we quantify it? These are the questions addressed in this article.

Contributions. We draw an explicit connection between the Mapper and the Reeb graph, from which we derive guarantees on the structure of the Mapper and quantities to measure its stability. Specifically:

- The connection happens through an intermediate object, called the MultiNerve Mapper, which we define as the multiverse of the connected pullback cover in the sense of [19]. The Mapper and its MultiNerve variant are related through the usual Nerve-vs-MultiNerve connection (see Lemma 3.4).

---

1In this article we call Mapper the mathematical object, not the algorithm used to build it. Moreover, we focus on the case where the codomain of the function is $\mathbb{R}$. 

2
• Given a pair \((X, f)\) and an interval cover \(I\), we show that the MultiNerve Mapper itself is a Reeb graph, for a perturbed pair \((X', f')\) (Theorem 5.3). Furthermore, we are able to track the changes that occur in the structure of the Reeb graph as we go from the initial pair \((X, f)\) to its perturbed version \((X', f')\). More precisely, we can match the persistence diagrams of the quotient maps \(\tilde{f}\) and \(\tilde{f}'\) with each other (Theorem 5.2), and thus draw a correspondence between the features of the MultiNerve Mapper and the ones of the Reeb graph of \((X, f)\). This correspondence is oblivious to the actual layouts of the features in the two graphs, which in principle could differ.

• The previous connection allows us to derive a signature for the (MultiNerve) Mapper, which takes the form of a persistence diagram. The points in this diagram are in one-to-one correspondence with the features (branches, holes) in the (MultiNerve) Mapper. Thus, like the persistence diagram of the quotient map \(\tilde{f}\) for the Reeb graph, our diagram for the (MultiNerve) Mapper serves as a bag-of-features type descriptor of its structure.

• An interesting property of our descriptor is to be predictable\(^2\) given the persistence diagram of the quotient map \(f\). Indeed, it is obtained from this diagram by removing the points lying in certain staircases that are defined solely from the cover \(I\) and that encode the mutual positioning of the intervals of the cover. Thus, the descriptor for the (MultiNerve) Mapper is a subset of the one for the Reeb graph, which provides theoretical evidence to the intuitive claim that the Mapper is a pixelized version of the Reeb graph. Then, one can easily derive sufficient conditions under which the bag-of-features structure of the Reeb graph is preserved in the (MultiNerve) Mapper, and when it is not, one can easily predict which features are preserved and which ones disappear (Corollary 5.6).

• The staircases also play a role in the stability of the (MultiNerve) Mapper, since they prescribe which features will (dis-)appear as the function \(f\) is perturbed. Stability is then naturally measured by a slightly modified version of the bottleneck distance, in which the staircases play the role of the diagonal. Our stability guarantees (Theorem 6.1) follow easily from the general stability theorem for extended persistence [18]. Similar guarantees hold when the domain \(X\) or the cover \(I\) is perturbed (Theorems 6.3 and 6.4).

• These stability guarantees can be exploited in practice to approximate the descriptors of the Mapper and MultiNerve Mapper from point cloud data. The approach boils down to applying known scalar field analysis techniques [14] then pruning the obtained persistence diagrams using the staircases (Theorem 7.6). The approach becomes more involved if one wants to further guarantee that the approximate signature does correspond to some perturbed Mapper or MultiNerve Mapper (Theorem 7.5).

Our main proof technique consists in progressively perturbing the so-called telescope [7] associated with the pair \((X, f)\), while tracking the corresponding changes in its persistence diagram. To be more specific, we decompose the perturbation into a sequence of elementary perturbations with a predictable effect on the diagram. We believe the introduction of these elementary perturbations and the analysis of their effects on the persistence diagram of the telescope are of an independent interest (see Section 4).

Related work. Reeb graphs can be seen as particular types of skeletonization, when one tries to recover the geometric structure of data with graphs. Several kinds of graph-like geometric structures have been studied within the past few years, such as persistent skeletons [27] or graph-induced simplicial complexes [21]. See [27] for a list of references. As mentioned previously, Reeb graphs are now well understood and have been used in a wide range of applications. Algorithms for their computation have been proposed, as well as metrics for their comparison. We refer the interested reader to the survey [6] and to the introductions of [4] and [5] for a comprehensive list of references. In a recent study, even more structure has been given to the Reeb graphs by categorifying them [20].

A lot of variants of these graphs have also been studied in the last decade to face the common issues that come with the Reeb graphs (complexity and computational cost among others). The Mapper [31] is one

\(^2\) As a byproduct, we also clarify the relationship between the diagram of \(\tilde{f}\) and the one of \(f\) (Theorem 2.9).
of them. Chazal et al. [16] introduced the $\lambda$-Reeb graph, which is another type of Reeb graph pixelization with intervals. It is the quotient space obtained by identifying the points with the transitive closure of the following relation: $x \sim y \Leftrightarrow x, y$ belong to the same level set and $f(x), f(y)$ belong to the same element of a given family of intervals. The computation is easier than for the Reeb graph, and the authors can derive upper bounds on the Gromov-Hausdorff distance between the space and its Reeb or $\lambda$-Reeb graph. However, this is too much asking in general; as a result, the hypothesis made on the space are very strong (it has to be close to a metric graph in the Gromov-Hausdorff distance already). Moreover, taking the transitive closure makes the structure of the output more difficult to interpret.

Joint Contour Nets [8, 10] and Extended Reeb graphs [8] are Mapper-like objects. The former is the Mapper computed with the cover of the codomain given by rounding the function values, while the latter is the Mapper computed from a partition of the domain with no overlap. Both structures are used for scientific purposes (visualization, shape descriptors, to name a few) and algorithms are proposed for their computation. Babu [2] characterized the Mapper with coarsened levelset zigzag persistence modules and showed that, as the lengths of the intervals in the cover $I$ go to zero uniformly, the Mapper of a real-valued function converges to the continuous Reeb graph in the bottleneck distance. Similarly, Munch and Wang [28] recently characterized the Mapper with constructible cosheaves and showed the same type of convergence for both the Joint Contour Net and the Mapper in the so-called interleaving distance [20]. Their result holds in the general case of vector-valued functions. Differently, we restrict the focus to real-valued functions but are able to make non-asymptotic claims (Corollary 5.6).

On another front, Stovner [32] proposed a categorified version of the Mapper, which is seen as a covariant functor from the category of covered topological spaces to the category of simplicial complexes. Dey et al. [22] pointed out the inherent instability of the Mapper and introduced a multiscale variant, called the MultiScale Mapper, which is built by taking the Mapper over a hierarchy of covers of the codomain. They derived a stable signature by considering the persistence diagram of this family. Unfortunately, their construction is hard to relate to the original Mapper. Here we work with the original Mapper directly and answer two open questions from [22], introducing a signature that gives a complete description of the set of features of the Mapper together with a quantification of their stability and a provable way of approximating them from point cloud data.

2 Background

Throughout the paper we work with singular homology with coefficients in the field $\mathbb{Z}_2$, which we omit in our notations for simplicity. We use the term "connected" as a shorthand for "path-connected", and we use "cc" as an abbreviation for "connected component(s)". Given a real-valued function $f$ on a topological space $X$, and an interval $I \subseteq \mathbb{R}$, we denote by $X^I$ the preimage $f^{-1}(I)$. We omit the subscript $f$ in the notation when there is no ambiguity in the function considered.

2.1 Morse-Type Functions

We restrict our focus to the class of real-valued functions called Morse-type. These are generalizations of the classical Morse functions that share some of their properties without having to be differentiable nor defined over a smooth manifold:

Definition 2.1. A continuous real-valued function $f$ on a topological space $X$ is of Morse type if:

(i) There is a finite set $\text{Crit}(f) = \{a_1 < \ldots < a_n\}$, called the set of critical values, s.t. over every open interval $(a_0 = -\infty, a_1), \ldots, (a_i, a_{i+1}), \ldots, (a_n, a_{n+1} = +\infty)$ there is a compact and locally connected space $Y_i$ and a homeomorphism $\mu_i : Y_i \times (a_i, a_{i+1}) \to X^{(a_i, a_{i+1})}$ s.t. $\forall i = 0, \ldots, n, f|_{X^{(a_i, a_{i+1})}} = \pi_2 \circ \mu_i^{-1}$, where $\pi_2$ is the projection onto the second factor;

(ii) $\forall i = 1, \ldots, n-1, \mu_i$ extends to a continuous function $\tilde{\mu}_i : Y_i \times [a_i, a_{i+1}] \to X^{[a_i, a_{i+1}]}$ - similarly $\mu_0$ extends to $\tilde{\mu}_0 : Y_0 \times (-\infty, a_1] \to X^{(-\infty, a_1]}$ and $\mu_n$ extends to $\tilde{\mu}_n : Y_n \times [a_n, +\infty) \to X^{[a_n, +\infty]}$. 

(iii) Each levelset $X^t$ has a finitely-generated homology.

Morse functions are known to be of Morse type while the converse is clearly not true. In fact, Morse-type functions do not have to be differentiable, and their domain does not have to be a smooth manifold nor even a manifold at all. Furthermore, it is possible to find Morse-type functions that are not Morse even though they satisfy the previous assumptions (think of the Gaussian curvature on a torus, for instance).

### 2.2 Extended Persistence

Let $f$ be a real-valued function on a topological space $X$. The family $\{X^{(-\infty,\alpha]}\}_{\alpha \in \mathbb{R}}$ of sublevel sets of $f$ defines a filtration, that is, it is nested w.r.t. inclusion: $X^{(-\infty,\alpha]} \subseteq X^{(-\infty,\beta]}$ for all $\alpha \leq \beta \in \mathbb{R}$. The family $\{X^{[\alpha,+,\infty)}\}_{\alpha \in \mathbb{R}}$ of superlevel sets of $f$ is also nested but in the opposite direction: $X^{[\alpha,+,\infty)} \supseteq X^{[\beta,+,\infty)}$ for all $\alpha \leq \beta \in \mathbb{R}$. We can turn it into a filtration by reversing the real line. Specifically, let $\mathbb{R}^{op} = \{x \mid x \in \mathbb{R}\}$, ordered by $\hat{x} \leq \hat{y} \Leftrightarrow x \geq y$. We index the family of superlevel sets by $\mathbb{R}^{op}$, so now we have a filtration: $\{X^{[\hat{\alpha},+,\infty)}\}_{\hat{\alpha} \in \mathbb{R}^{op}}$, with $X^{[\hat{\alpha},+,\infty)} \subseteq X^{[\hat{\beta},+,\infty)}$ for all $\hat{\alpha} \leq \hat{\beta} \in \mathbb{R}^{op}$.

Extended persistence connects the two filtrations at infinity as follows. Replace each superlevel set $(X, X^{[\hat{\alpha},+,\infty)})$ of superlevel sets of $f$ by the pair of spaces $(\hat{X}, \mathbb{R}^{op})$, with $\mathbb{R}^{op}$ acting as a blocking set. This is called the ordinary part of the filtration, and the subfamily $\{F^\alpha\}_{\alpha \in \mathbb{R}}$ is called the relative part. See Figure 1 for an illustration.

Applying the homology functor $H_*$ to this filtration gives the so-called extended persistence module $\mathbb{V}$ of $f$:

\[
V_\alpha = H_*(F^\alpha) = H_* (X^{(-\infty,\alpha]}) \quad \text{for } \alpha \in \mathbb{R} \\
V_{+\infty} = H_*(F_{+\infty}) = H_* (X^{(-\infty,\infty]}) \\
V_{\hat{\alpha}} = H_*(F_{\hat{\alpha}}) = H_* (X^{[\hat{\alpha},+,\infty)}) \quad \text{for } \hat{\alpha} \in \mathbb{R}^{op},
\]

and where the linear maps between the spaces are induced by the inclusions in the extended filtration.

For functions of Morse type, the extended persistence module can be decomposed as a finite direct sum of half-open interval modules—see e.g. [11]:

\[
\mathbb{V} \simeq \bigoplus_{k=1}^n \mathbb{I}([b_k, d_k]),
\]

where each summand $\mathbb{I}([b_k, d_k])$ is made of copies of the field of coefficients at each index $\alpha \in [b_k, d_k]$, and of copies of the zero space elsewhere, the maps between copies of the field being identities. Each summand represents the lifespan of a homological feature (cc, hole, void, etc.) within the filtration. More precisely, the birth time $b_k$ and death time $d_k$ of the feature are given by the endpoints of the interval. Then, a convenient way to represent the structure of the module is to plot each interval in the decomposition as a point in the extended plane, whose coordinates are given by the endpoints. Such a plot is called the extended persistence diagram of $f$, denoted $\text{Dg}(f)$. The distinction between ordinary and relative parts of the filtration allows to classify the points in $\text{Dg}(f)$ in the following way:
Figure 1: The extended filtration of the height function on a torus. The upper row displays the ordinary part of the filtration while the lower row displays the relative part. The red and blue cycles both correspond to extended points in dimension 1. The point corresponding to the red cycle is located above the diagonal \((d_1 > b_1)\), while the point corresponding to the blue cycle is located below the diagonal \((d_1 > b_1)\).

- points whose coordinates both belong to \(\mathbb{R}\) are called \textit{ordinary} points; they correspond to homological features being born and then dying in the ordinary part of the filtration;
- points whose coordinates both belong to \(\mathbb{R}^{\text{op}}\) are called \textit{relative} points; they correspond to homological features being born and then dying in the relative part of the filtration;
- points whose abscissa belongs to \(\mathbb{R}\) and whose ordinate belongs to \(\mathbb{R}^{\text{op}}\) are called \textit{extended} points; they correspond to homological features being born in the ordinary part and then dying in the relative part of the filtration.

Note that ordinary points lie strictly above the diagonal \(\Delta = \{(x, x) \mid x \in \mathbb{R}\}\) and relative points lie strictly below \(\Delta\), while extended points can be located anywhere, including on \(\Delta\), e.g. cc that lie inside a single critical level – see Section 2.3. It is common to decompose \(Dg(f)\) according to this classification:

\[
Dg(f) = \text{Ord}(f) \sqcup \text{Rel}(f) \sqcup \text{Ext}^+(f) \sqcup \text{Ext}^-(f),
\]

where by convention \(\text{Ext}^+(f)\) includes the extended points located on the diagonal \(\Delta\).

**Persistence measure.** From a persistence module \(\mathbb{V}\) we derive a measure on the set of rectangles in the plane, called the \textit{persistence measure} and denoted \(\mu_{\mathbb{V}}\). Given a rectangle \(R = [a, b] \times [c, d]\) with \(a < b \leq c < d\), we let

\[
\mu_{\mathbb{V}}(R) = r_c^a - r_a^d + r_a^c - r_c^d,
\]

where \(r_y^x\) denotes the rank of the linear map between the vector spaces indexed by \(x, y \in \mathbb{R}^{\text{Ext}}\) in \(\mathbb{V}\). When \(\mathbb{V}\) has a well-defined persistence diagram, \(\mu_{\mathbb{V}}(R)\) equals the total multiplicity of the diagram within the rectangle \(R\) \([11]\).
Stability. An important property of extended persistence diagrams is to be stable in the so-called bottleneck distance $d^\infty_h$. The definition of this distance is based on partial matchings between the diagrams. Given two persistence diagrams $D, D'$, a partial matching between $D$ and $D'$ is a subset $\Gamma$ of $D \times D'$ such that:

$$\forall p \in D, \text{ there is at most one } p' \in D' \text{ s.t. } (p, p') \in \Gamma,$$

$$\forall p' \in D', \text{ there is at most one } p \in D \text{ s.t. } (p, p') \in \Gamma.$$ 

Furthermore, $\Gamma$ must match points of the same type (ordinary, relative, extended) and of the same homological dimension only. The cost of $\Gamma$ is:

$$\text{cost}(\Gamma) = \max \left\{ \max_{p \in D} \delta_D(p), \max_{p' \in D'} \delta_{D'}(p') \right\},$$

where

$$\delta_D(p) = \|p - p'\|_{\infty} \text{ if } \exists p' \in D' \text{ s.t. } (p, p') \in \Gamma \text{ and } d_\infty(p, \Delta) = \inf_{q \in \Delta} \|p - q\|_{\infty} \text{ otherwise},$$

$$\delta_{D'}(p') = \|p - p'\|_{\infty} \text{ if } \exists p \in D \text{ s.t. } (p, p') \in \Gamma \text{ and } d_\infty(p', \Delta) = \inf_{q \in \Delta} \|p' - q\|_{\infty} \text{ otherwise}.$$

Definition 2.2. Let $D, D'$ be two persistence diagrams. The bottleneck distance between $D$ and $D'$ is:

$$d^\infty_h(D, D') = \inf_{\Gamma} \text{cost}(\Gamma),$$

where $\Gamma$ ranges over all partial matchings between $D$ and $D'$.

Note that $d^\infty_h$ is only a pseudo-metric, not a true metric, because points lying on $\Delta$ can be left unmatched at no cost.

Theorem 2.3 (Stability [18]). For any Morse-type functions $f, g : X \to \mathbb{R}$,

$$d^\infty_h(Dg(f), Dg(g)) \leq \|f - g\|_{\infty}.$$

Moreover, as pointed out in [18], the theorem can be strengthened to apply to each subdiagram Ord, Ext$^+$, Ext$^-$, Rel and to each homological dimension individually.

Horizontal and vertical homology. Given a Morse-type function $f : X \to \mathbb{R}$, we define the horizontal homology group of $f$ as the following subgroup of $H_*(X)$:

$$H_\tau^f(X) = \sum_{\alpha \in \mathbb{R}} \text{im}(H_*(X^\alpha) \to H_*(X)),$$

where the homomorphisms are induced by the inclusions $X^\alpha \hookrightarrow X$. In other words, homology classes in $H_\tau^f(X)$ have representatives on which $f$ takes only finitely many different values. Then we define the vertical homology group as the quotient $H_\tau^f(X) = H_*(X)/H_\tau^f(X)$. A horizontal homology class is a class that belongs to $H_\tau^f(X)$. All other classes are called vertical.

Minimal support and thickness of homology classes. Given a homology class $\omega$, we let $\text{Int}(\omega)$ be the family of intervals whose preimages by $f$ contain a representative of $\omega$. Let $I(\omega)$ be the intersection of all the intervals in $\text{Int}(\omega)$. If $I(\omega) \neq \emptyset$, then we call $I(\omega)$ the minimal support of $\omega$ and we define the thickness of $\omega$ as $|I(\omega)| = \text{sup}(I(\omega)) - \text{inf}(I(\omega))$. Otherwise, we have $I(\omega) = \emptyset$ so $\omega$ has no minimal support, and we arbitrarily set the thickness of $\omega$ to zero. As pointed out in [18], a class with zero thickness is necessarily horizontal, even if it has no minimal support. In fact, the entire horizontal homology group is generated by the zero-thickness classes, by definition. However, not all horizontal classes have zero thickness: for instance, the sum of two classes with zero thickness may have a positive thickness.

Bases of $H_*(X)$ can be ranked according to the thickness of their elements. More precisely, we sort the elements of every basis by non-decreasing thickness. Then we define an optimally thin basis as a basis that is minimal with respect to the lexicographic order on the thicknesses of its elements.
Lemma 2.4. In an optimally thin basis, the elements with zero thickness form a basis of $H^*_{\rightarrow f}(X)$.

Proof. Let $B$ be an arbitrary optimally thin basis, and assume that the elements with zero thickness in $B$ span only a proper subspace of the horizontal subgroup (recall that they are located in $H^*_{\rightarrow f}(X)$). Then, we can find a basis $B'$ that is thinner than $B$. Indeed, take for $B'$ a basis of $H^*_{\rightarrow f}(X)$ whose elements all have zero thickness (such a basis exists because $H^*_{\rightarrow f}(X)$ is finite-dimensional and generated by the zero-thickness classes), and complete it to a basis of $H_*(X)$ arbitrarily. Then, $B'$ is thinner than $B$ because it contains more zero-thickness elements. This contradicts the assumption that $B$ is an optimally thin basis. □

Theorem 2.5 (Thin Basis [18]). Let $f : X \rightarrow \mathbb{R}$ be a function of Morse type $3$. In an optimally thin basis of $H^*_*(X)$, the minimal supports of the basis elements with positive thickness are the intervals $[x,y]$ for which $(y,\tilde{x}) \in \text{Ext}_-(f)$.

Theorem 2.6 (Symmetry [18, 7]). Let $f : X \rightarrow \mathbb{R}$ be a continuous function that admits an extended persistence diagram. Then, for any integer $r \geq 0$,

\[
\text{Ord}_r(f) = T \circ R(\text{Rel}_r(-f)),
\]

\[
\text{Ext}_r(f) = R(\text{Ext}_r(-f)),
\]

where $T$ denotes the reflection across the diagonal $(x,y) \mapsto (y,x)$ and $R$ denotes the reflection across the minor diagonal $(x,y) \mapsto (-y,-x)$. Note that $T \circ R$ is the central reflection through the origin: $(x,y) \mapsto (-x,-y)$.

2.3 Reeb Graphs

Definition 2.7. Given a topological space $X$ and a continuous function $f : X \rightarrow \mathbb{R}$, we define the equivalence relation $\sim_f$ between points of $X$ by:

\[
x \sim_f y \iff [f(x) = f(y) \text{ and } x, y \text{ belong to the same cc of } f^{-1}(f(x)) = f^{-1}(f(y))].
\]

The Reeb graph $R_f(X)$ is the quotient space $X/\sim_f$.

As $f$ is constant on equivalence classes, there is an induced quotient map $\hat{f} : R_f(X) \rightarrow \mathbb{R}$ s.t. $f = \hat{f} \circ \pi$, where $\pi$ is the projection $X \rightarrow R_f(X)$ induced by $\sim_f$:

\[
\begin{array}{c}
X \xrightarrow{\pi} R_f(X) \\
\downarrow f \downarrow \hat{f} \\
\mathbb{R}
\end{array}
\]

Figure 2: We consider the height function. Note how the critical points induce changes on the graph.

Note that the original theorem in [18] assumes the domain to be a smooth manifold and the function to be Morse. However, as pointed out by the authors, the theorem holds more generally. In fact, the assumption is used only in the proof of the so-called ‘decomposition lemma’. The first part of that proof, which involves the assumption, can be replaced by a simple reasoning involving a compatible basis of the extended persistence module induced at the homology level.
If \( f \) is a function of Morse type, then the pair \((X, f)\) is an \( \mathbb{R} \)-constructible space in the sense of [20]. This ensures that the Reeb graph is a multigraph, whose nodes are in one-to-one correspondence with the \( \text{cc} \) of the critical level sets of \( f \). We can equip this multigraph with a metric by assigning the length \( l(v_i, v_j) = |f(v_i) - f(v_j)| \) to each edge \((v_i, v_j)\). In the following, the combinatorial version of the Reeb graph is denoted by \( \mathcal{C}(R_f(X)) \).

**Connection to the extended persistence of \( \tilde{f} \).** There is a nice interpretation of \( \text{Dg}(\tilde{f}) \) in terms of the structure of \( R_f(X) \). We refer the reader to [4] and the references therein for a full description as well as formal definitions and statements. Orienting the Reeb graph vertically so \( \tilde{f} \) is the height function, we can see each \( \text{cc} \) of the graph as a trunk with multiple branches (some oriented upwards, others oriented downwards) and holes. Then, one has the following correspondences, where the vertical span of a feature is the span of its image by \( \tilde{f} \):

- The vertical spans of the trunks are given by the points in \( \text{Ext}_{0}^{+} (\tilde{f}) \);
- The vertical spans of the branches that are oriented downwards are given by the points in \( \text{Ord}_{0} (\tilde{f}) \);
- The vertical spans of the branches that are oriented upwards are given by the points in \( \text{Rel}_{1} (\tilde{f}) \);
- The vertical spans of the holes are given by the points in \( \text{Ext}_{1}^{-} (\tilde{f}) \).

The rest of the diagram of \( \tilde{f} \) is empty. These correspondences provide a dictionary to read off the structure of the Reeb graph from the persistence diagram of the quotient map \( \tilde{f} \). Note that it is a bag-of-features type descriptor, taking an inventory of all the features (trunks, branches, holes) together with their vertical spans, but leaving aside the actual layout of the features. As a consequence, it is an incomplete descriptor: two Reeb graphs with the same persistence diagram may not be isomorphic, as illustrated in Figure 3.

![Figure 3](image.png)

Figure 3: Two Reeb graphs with the same set of features but not the same layout.

**Connection to the extended persistence of \( f \).** Dey et al. [23] related the 1-dimensional homology of the Reeb graph to the 1-dimensional vertical homology of the function:

**Proposition 2.8 ([23]).** For any Morse-type function \( f : X \to \mathbb{R} \), there is an isomorphism \( \Phi : H_{1}^{\uparrow f}(X) \to H_{1}(R_{f}(X)) \) such that the thickness of each class in \( H_{1}^{\uparrow f}(X) \) is preserved by \( \Phi \).

This connection can be rephrased in terms of the extended persistence of \( f \) and of its quotient map \( \tilde{f} \), following the intuition that the 1-dimensional persistence diagram of \( \tilde{f} \) is a subset of the one of \( f \), and that the missing points correspond either to the inessential or horizontal 1-dimensional homology generators, or to the homology generators in dimension 2 and above.
Theorem 2.9. Let $X$ be a topological space and $f : X \rightarrow \mathbb{R}$ a function of Morse type. Then, $\text{Dg}(\tilde{f}) \subseteq \text{Dg}(f)$. Furthermore:

\[
\begin{align*}
\text{Dg}_0(\tilde{f}) &= \text{Dg}_0(f) \\
\text{Dg}_1(\tilde{f}) &= \text{Dg}_1(\tilde{f}) \setminus (\text{Ext}_1^+(f) \cup \text{Ord}_1(f)) \\
\text{Dg}_p(\tilde{f}) &= \emptyset \text{ if } p \geq 2
\end{align*}
\]

We provide a proof for completeness, as we have not seen this result stated formally in the literature. The proof is simpler when $\pi$ admits continuous sections, i.e. when there exist continuous maps $\sigma : R_f(X) \rightarrow X$ such that $\pi \circ \sigma = \text{id}_{R_f(X)}$. Below we give the proof under this hypothesis, deferring the general case of Morse-type functions to Appendix A. The hypothesis holds for instance when $X$ is a compact smooth manifold and $f$ is a Morse function, or when $X$ is a simplicial complex and $f$ is piecewise-linear.

Proof of Theorem 2.9. Consider first $\text{Dg}_0(\tilde{f})$. For convenience of notation, for any $\alpha \in \mathbb{R}$ we let $\tilde{X}^{(-\infty, \alpha]} := f^{-1}((-\infty, \alpha])$ and $\tilde{X}^{[\alpha, +\infty)} := f^{-1}([\alpha, +\infty))$. Since $f \circ \pi = f$, we have $\pi(X^{(-\infty, \alpha]}) \subseteq \tilde{X}^{(-\infty, \alpha]}$ and $\pi(X^{[\alpha, +\infty)}) \subseteq \tilde{X}^{[\alpha, +\infty)}$ for all $\alpha \in \mathbb{R}$. Hence, $\pi$ induces a morphism between the extended persistence modules of $f$ and $\tilde{f}$ at the 0-dimensional homology level. Let us show that this morphism is an isomorphism. Since $\pi$ is surjective, this boils down to showing that $x, y$ are connected in $X^{(-\infty, \alpha]}$ iff $x, y$ are connected in $\tilde{X}^{(-\infty, \alpha]}$, and same for the relative part of the filtration:

- If $x, y$ are connected in $X^{(-\infty, \alpha]}$, then $x, y$ are connected in $\tilde{X}^{(-\infty, \alpha]}$ by continuity of $\pi$ and commutativity of (2).
- If $x, y$ are connected in $\tilde{X}^{(-\infty, \alpha]}$, then choose a path $\gamma$ connecting $x$ to $y$. By definition of $\sigma$, we have $\pi \circ \sigma \circ \pi(x) = \pi(y)$, thus $\sigma \circ \pi(x)$ and $\sigma \circ \pi(y)$ lie in the same $\text{cc}$ of $f^{-1}(f(x))$. Let $\gamma_x$ be a path connecting $x$ to $\sigma \circ \pi(x)$. Similarly, let $\gamma_y$ be a path connecting $\sigma \circ \pi(y)$ to $y$. Then, $\gamma_y \circ \sigma(\gamma) \circ \gamma_x$ is a path between $x$ and $y$ in $X^{(-\infty, \alpha]}$.

The same argument applies to the relative part of the filtration. Hence, $\text{Dg}_0(\tilde{f}) = \text{Dg}_0(f)$.

Consider now $\text{Dg}_1(\tilde{f})$. $\text{Ord}_1(\tilde{f})$ is empty since $R_f(X)$ is a multigraph (thus every 1-cycle is essential). In addition, by Theorem 2.6, $\text{Rel}_1(\tilde{f}) = R \circ T(\text{Ord}_0(\tilde{f})) = R \circ T(\text{Ord}_0(f)) = \text{Rel}_1(f)$. Let us now focus on Ext$_1(\tilde{f})$. Let $B$ be an optimally thin basis of $H_1^{\text{rel}}(R_f(X))$. According to Lemma 2.4 and Theorem 2.5, there is a bijection Ext$_1(\tilde{f}) \leftrightarrow B$ such that the endpoints of the minimal supports of the elements of $B$ are the coordinates of the corresponding points in Ext$_1(\tilde{f})$. Since $H_1^{\text{rel}}(R_f(X)) = H_1(R_f(X))$, $B$ is also an optimally thin basis of $H_1(R_f(X))$. Now, the summands corresponding to Ext$_1(\tilde{f})$ in the direct-sum decomposition of the extended persistence module of $\tilde{f}$ correspond to the essential summands in the decomposition of the ordinary persistence module, so we have $|\text{Ext}_1(f)| = \dim H_1(R_f(X)) = |\text{Ext}_1(\tilde{f})|$ for any basis $C$ of $H_1(R_f(X))$. Hence, taking $C = B$, we have a bijection Ext$_1(\tilde{f}) \leftrightarrow B$. In addition, according to Proposition 2.8, there is an isomorphism $\Phi : H_1(R_f(X)) \rightarrow H_1^{\text{rel}}(X)$ that preserves minimal supports and thicknesses, so $\Phi(B)$ is an optimally thin basis of $H_1^{\text{rel}}(X)$. Again, Lemma 2.4 and Theorem 2.5 state that there is a bijection from the elements of $\Phi(B)$ to Ext$_1(\tilde{f})$. Therefore, Ext$_1(\tilde{f}) = \text{Ext}_1(f) \setminus \text{Ext}_1^+(f)$.

Finally, $\text{Dg}_p(\tilde{f}) = \emptyset$ for $p \geq 2$ comes from the fact that, when $f$ is of Morse type, $R_f(X)$ is a multigraph.

\[\square\]

2.4 Covers and Nerves

Let $Z$ be a topological space. A cover of $Z$ is a family $\mathcal{U}$ of subsets of $Z$, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, such that $Z = \bigcup_{\alpha \in A} U_\alpha$. It is open if all its elements are open subspaces of $Z$. It is connected if all its elements are connected subspaces of $Z$. Its nerve is the abstract simplicial complex $\mathcal{N}(\mathcal{U})$ that has one $k$-simplex per
(k + 1)-fold intersection of elements of \( \mathcal{U} \):
\[
\{ \alpha_0, \ldots, \alpha_k \} \in \mathcal{N}(\mathcal{U}) \iff \bigcap_{i=0, \ldots, k} U_{\alpha_i} \neq \emptyset.
\]

Given a subfamily \( V \) of \( \mathcal{U} \), we denote by \( \bigcap V \) the intersection \( \bigcap_{\alpha \in V} V \), and by \( \bigcup V \) the union \( \bigcup_{\alpha \in V} V \). When \( V \) itself is a cover of \( Z \), it is called a subcover of \( \mathcal{U} \). It is proper if it is not equal to \( \mathcal{U} \). Finally, \( \mathcal{U} \) is called minimal if it admits no proper subcover or, equivalently, if it has no element included in the union of the other elements. Given a minimal cover \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in A} \), for every \( \alpha \in A \) we let
\[
\hat{U}_\alpha = U_\alpha \setminus \bigcup_{\alpha' \neq \alpha \in A} U_{\alpha'},
\]
which is the maximal subset of \( U_\alpha \) that has an empty intersection with the other elements of \( \mathcal{U} \). The cc of \( \hat{U}_\alpha \) are called the proper subsets of \( U_\alpha \). \( \mathcal{U} \) is called generic if no proper subset is a singleton.

Consider now the special case where \( Z \) is a subset of \( \mathbb{R} \), equipped with the subspace topology. A subset \( U \subseteq Z \) is an interval of \( Z \) if there is an interval \( I \) of \( \mathbb{R} \) such that \( U = I \cap Z \). Note that \( U \) is open in \( Z \) if and only if \( I \) can be chosen open in \( \mathbb{R} \). A cover \( \mathcal{U} \) of \( Z \) is an interval cover if all its elements are intervals. In this case, \( \text{End}(\mathcal{U}) \) denotes the set of all of the interval endpoints. Finally, the granularity of \( \mathcal{U} \) is the supremum of the lengths of its elements, i.e. it is the quantity \( \sup_{|U| \in \mathcal{U}} |U| \) where \( |U| := \sup(U) - \inf(U) \in \mathbb{R} \cup \{+\infty\} \).

**Lemma 2.10.** If \( \mathcal{U} \) is a minimal open interval cover, then no more than two elements of \( \mathcal{U} \) can intersect at a time.

**Proof.** Assume for a contradiction that there are \( k \geq 3 \) elements of \( \mathcal{U} \): \( U_1, \ldots, U_k \), that have a non-empty common intersection. For every \( i \), fix an open interval \( I_i \) of \( \mathbb{R} \) such that \( U_i = I_i \cap Z \). Up to a reordering of the indices, we can assume without loss of generality that \( I_1 \) has the smallest lower bound and \( I_2 \) has the largest upper bound. Since \( I_1 \cap I_2 \supseteq U_1 \cap U_2 \neq \emptyset \), the remaining intervals satisfy \( I_i \subseteq I_1 \cup I_2 \). In particular, we have \( U_3 = I_3 \cap Z \subseteq (I_1 \cup I_2) \cap Z = (I_1 \cap Z) \cup (I_2 \cap Z) = U_1 \cup U_2 \), so the cover \( \mathcal{U} \) is not minimal.

**Lemma 2.11.** If \( Z = \mathbb{R} \) itself or a compact subset thereof, then any cover \( \mathcal{U} \) of \( Z \) has a minimal subcover.

**Proof.** When \( Z = \mathbb{R} \), the same argument applies to any subset of the form \([-n, n], \ n \in \mathbb{N} \). Then, a simple induction on \( n \) allows us to build a minimal subcover of \( \mathcal{U} \).

From now on, unless otherwise stated, all covers of \( Z \subseteq \mathbb{R} \) will be generic, open, minimal, interval covers (gomic for short). An immediate consequence of Lemma 2.10 is that every element \( U \) of a gomic \( \mathcal{U} \) has exactly one proper subset \( \hat{U} \). More precisely, \( U \) partitions into three subintervals:
\[
U = U_\cap \cup \hat{U} \cup U_\cup, \tag{3}
\]
where \( U_\cap \) is the intersection of \( U \) with the element right below it in the cover (\( U_\cap = \emptyset \) if that element does not exist), and where \( U_\cup \) is the intersection of \( U \) with the element right above it (\( U_\cup = \emptyset \) if that element does not exist).

### 2.5 Mapper

Let \( f : X \to Z \) be a continuous function. Consider a cover \( \mathcal{U} \) of \( \text{im}(f) \), and pull it back to \( X \) via \( f^{-1} \). Then, decompose every \( V_\alpha = f^{-1}(U_\alpha) \subseteq X \) into its cc: \( V_\alpha = \bigcup_{i \in \{1, \ldots, c(\alpha)\}} V_\alpha^i \), where \( c(\alpha) \) is the number of cc of \( V_\alpha \). Then, \( \mathcal{V} = \{ V_\alpha^i \}_{\alpha \in A, i \in \{1, \ldots, c(\alpha)\}} \) is a connected cover of \( X \). It is called the connected pullback cover, and its nerve \( \mathcal{N}(\mathcal{V}) \) is the Mapper.
Definition 2.12. Let $X, Z$ be topological spaces, $f : X \to Z$ a continuous function, $\mathcal{U}$ a cover of $\operatorname{im}(f)$ and $\mathcal{V}$ the associated connected pullback cover. The Mapper of $X$ is $M_f(X, \mathcal{U}) = N(\mathcal{V})$.

See Figure 4 for an illustration. Notice that, when $Z = \mathbb{R}$ and $\mathcal{U}$ is a gomic, the Mapper has a natural 1-dimensional stratification since no more than two intervals can intersect at a time by Lemma 2.10. Hence, in this case, it has the structure of a (possibly infinite) simple graph and therefore has trivial homology in dimension 2 and above.

![Figure 4](image-url)

Figure 4: Example of the Mapper and the MultiNerve Mapper computed on the double torus $T \# T$ with the height function $f$. The cover $\mathcal{I}$ of $\operatorname{im}(f)$ has four intervals (red, green, blue and purple), and the cover of the double torus has five cc (one is blue, one is red, one is purple and the other two are green). The Mapper and the MultiNerve Mapper are displayed on the right.

3 MultiNerve Mapper

3.1 Simplicial Posets and MultiNerves

The notion of simplicial poset was used in [19] to define the multinerve.

Definition 3.1. A simplicial poset is a partially ordered set $(P, \preceq)$, whose elements are called simplices, such that:

(i) $P$ has a least element called 0 such that $\forall p \in P, 0 \preceq p$;

(ii) $\forall p \in P, \exists d \in \mathbb{N}$ such that the lower segment $[0, p] = \{ q \in P \mid q \preceq p \}$ is isomorphic to the set of simplices of the standard $d$-simplex with the inclusion as partial order, where an isomorphism between posets is a bijective and order-preserving function.

Simplicial posets are extensions of simplicial complexes: while every simplicial complex is also a simplicial poset (with inclusion as partial order and $\emptyset$ as least element), the converse is not always true as different simplices may have the same set of vertices. However, these simplices cannot be faces of the same higher-dimensional simplex, otherwise (ii) would be false. See Figure 5 for an example of a simplicial poset that is not a simplicial complex.

Given a cover $\mathcal{U}$ of a topological space $X$, the nerve is extended to a simplicial poset as follows:

Definition 3.2. The multinerve $\mathcal{M}(\mathcal{U})$ is the simplicial poset defined by:

$$\mathcal{M}(\mathcal{U}) = \{ (\{\alpha_0, \ldots, \alpha_k\}, C) \mid \bigcap_{i=0, \ldots, k} U_{\alpha_i} \neq \emptyset \text{ and } C \text{ is a cc of } \bigcap_{i=0, \ldots, k} U_{\alpha_i} \}.$$
simplices. Given two simplices \((F, C), (F', C')\), we say that \((F, C)\) is a face of \((F', C')\) if \((F, C) \preceq (F', C')\). We also denote by \(V(p)\) the set of vertices that are in the lower segment of \(p\).

Given a connected pullback cover \(\mathcal{V}\), we extend the Mapper by using the multinerve \(\mathcal{M}(\mathcal{V})\) instead of \(\mathcal{N}(\mathcal{V})\). This variant will be referred to as the MultiNerve Mapper in the following.

**Definition 3.3.** Let \(X, Z\) be topological spaces, \(f : X \to Z\) a continuous function, \(\mathcal{U}\) a cover of \(\text{im}(f)\) and \(\mathcal{V}\) the associated connected pullback cover. The MultiNerve Mapper of \(X\) is \(\mathbb{M}_f(X, \mathcal{U}) = \mathbb{M}(\mathcal{V})\).

See Figure 4 for an illustration. For the same reasons as Mapper, when \(Z = \mathbb{R}\) and \(\mathcal{U}\) is a gomic of \(\text{im}(f)\), the MultiNerve Mapper is a (possibly infinite) multigraph and therefore has trivial homology in dimension 2 and above. Contrary to the Mapper, it also takes the cc of the intersections into account. As we shall see in Section 5, the MultiNerve Mapper is able to capture the same features as the Mapper, even with coarser gomics, and is more naturally related to the Reeb graph.

### 3.2 Connection to Mapper

The connection between the Mapper and the MultiNerve Mapper is induced by the following connection between nerves and multinerves:

**Lemma 3.4 (19).** Let \(X\) be a topological space and \(\mathcal{U}\) a cover of \(X\). Let \(\pi_1 : (F, C) \to F\) be the projection of the simplices of \(\mathcal{M}(\mathcal{U})\) onto the first coordinate. Then, \(\pi_1(\mathcal{M}(\mathcal{U})) = \mathcal{N}(\mathcal{U})\).

**Corollary 3.5.** Let \(X, Z\) be topological spaces and \(f : X \to Z\) continuous. Let \(\mathcal{U}\) be a cover of \(\text{im}(f)\). Then, \(\mathbb{M}_f(X, \mathcal{U}) = \pi_1(\mathbb{M}_f(X, \mathcal{U}))\).

Thus, when \(Z = \mathbb{R}\) and \(\mathcal{U}\) is a gomic, the Mapper is the simple graph obtained by gluing the edges that have the same endpoints in the MultiNerve Mapper. In this special case it is even possible to embed \(\mathbb{M}_f(X, \mathcal{U})\) as a subcomplex of \(\mathbb{M}_f(X, \mathcal{U})\). Indeed, both objects are (multi-)graphs over the same set of nodes since they are built from the connected pullback cover. Then, it is enough to map each edge of \(\mathbb{M}_f(X, \mathcal{U})\) to one of its copies in \(\mathbb{M}_f(X, \mathcal{U})\), chosen arbitrarily, to get a subcomplex. This mapping serves as a simplicial section for the projection \(\pi_1\), therefore:

**Lemma 3.6.** When \(Z = \mathbb{R}\) and \(\mathcal{U}\) is a gomic, \(\pi_1\) induces a surjective homomorphism in homology.

Note that this is not true in general when \(\mathbb{M}_f(X, \mathcal{U})\) has a higher dimension. See Figure 4 for an example.
Figure 6: The domain is the disk $B^2$, and we consider the identity function $f$, as well as a generic open minimal cover $\mathcal{U}$ with five elements. The MultiNerve Mapper is homeomorphic to the disk $B^2$ and the Mapper is homeomorphic to the sphere $S^2$. Then, $H_2(M_f(B^2, \mathcal{U})) \neq 0$ while $H_2(\overline{M_f}(B^2, \mathcal{U})) = 0$.

4 Telescope

In this section we introduce the telescopes, which are our main objects of study when we relate the structure of the MultiNerve Mapper to the one of the Reeb graph of a perturbation of the pair $(X, f)$. Recall that, given topological spaces $X$ and $A \subseteq Y$ together with a continuous map $f : A \to X$, the adjunction space $X \cup_f Y$ (also denoted $Y \cup_f X$) is the quotient of the disjoint union $X \sqcup Y$ by the equivalence relation induced by the identifications $\{f(a) \sim a\}_{a \in A}$.

**Definition 4.1 (Telescope [7]).** A telescope is an adjunction space of the following form:

$$T = (Y_0 \times \{a_0, a_1\}) \cup_{\psi_0} (X_1 \times \{a_1\}) \cup_{\phi_1} (Y_1 \times \{a_1, a_2\}) \cup_{\psi_1} \ldots \cup_{\phi_n} (Y_n \times \{a_n, a_{n+1}\}),$$

where $a_0 = -\infty$ and $a_{n+1} = +\infty$ by convention and where the $\phi_i : Y_i \times \{a_i\} \to X_i \times \{a_i\}$ and $\psi_i : Y_i \times \{a_{i+1}\} \to X_{i+1} \times \{a_{i+1}\}$ are continuous maps. The $a_i$ are called the critical values of $T$ and their set is denoted by $\text{Crit}(T)$, the $\phi_i$ and $\psi_i$ are called attaching maps, the $Y_i$ are compact and locally connected spaces called the cylinders and the $X_i$ are topological spaces called the critical slices. Moreover, all $Y_i$ and $X_i$ have finitely-generated homology.

A telescope comes equipped with $\pi_1$ and $\pi_2$, which are the projections onto the first factor and second factor respectively. Given any interval $I$, we let $T^I = \pi_1 \circ \pi_2^{-1}(I)$.

4.1 Construction from $X$

A function $f : X \to \mathbb{R}$ of Morse type naturally induces a telescope $T_X$ defined with

- $\text{Crit}(T) = \text{Crit}(f)$,
- $X_i = f^{-1}(a_i)$,
- $Y_i = \pi_1 \circ \mu_i^{-1} \circ f^{-1}(a_i, a_{i+1})$,
- $\phi_i : (y, a_i) \mapsto (\bar{\mu}_i|_{Y_i \times \{a_i\}})(y, a_i)$, $\forall y \in Y_i, \forall i \in \{1, \ldots, n\}$,
- $\psi_i : (y, a_{i+1}) \mapsto (\bar{\mu}_i|_{Y_i \times \{a_{i+1}\}})(y, a_{i+1})$, $\forall y \in Y_i, \forall i \in \{0, \ldots, n-1\}$,

$T_X$ is well-defined thanks to the following Lemma:

**Lemma 4.2.** $\text{im}(\phi_i) \subseteq f^{-1}(a_i) \times \{a_i\}$ and $\text{im}(\psi_i) \subseteq f^{-1}(a_{i+1}) \times \{a_{i+1}\}$. 

14
Proof. Let \((y, a_{i+1}) \in Y_i \times \{a_{i+1}\}\). Consider the sequence \((y, v_n)_{n \in \mathbb{N}},\) for an arbitrary \((v_n)_{n \in \mathbb{N}} \in (a_i, a_{i+1})\) that converges to \(a_{i+1}\). Then, \((f \circ \bar{\mu}_i(y, v_n))_{n \in \mathbb{N}}\) converges to \(f \circ \bar{\mu}_i(y, a_{i+1})\) by continuity of \(f \circ \bar{\mu}_i\). Moreover, for all \(n \in \mathbb{N}\) we have \(f \circ \bar{\mu}_i(y, v_n) = f \circ \mu_i(y, v_n) = v_n\) since \(f|_{f^{-1}(a_i, a_{i+1})} = \pi_2 \circ \mu_i^{-1}\). Therefore, \((f \circ \bar{\mu}_i(y, v_n))_{n \in \mathbb{N}}\) converges also to \(a_{i+1}\). By uniqueness of the limit, we have \(f \circ \bar{\mu}_i(y, a_{i+1}) = a_{i+1}\), meaning that \(\bar{\mu}_i(y, a_{i+1}) \in f^{-1}(a_{i+1})\). Thus, \(\text{im}(\psi_i) \subseteq f^{-1}(a_{i+1}) \times \{a_{i+1}\}\). The same argument applies to show that \(\text{im}(\phi_i) \subseteq f^{-1}(a_i) \times \{a_i\}\). \(\square\)

We now exhibit a homeomorphism between \(X\) and \(T_X\). Let \(\mu : T_X \to X\) be defined by:

\[
\mu(y, z) = \begin{cases} 
  y & \text{if} (y, z) \in X_i \times \{a_i\} \text{ for some } i; \\
  \mu_i(y, z) & \text{if} (y, z) \in Y_i \times (a_i, a_{i+1}) \text{ for some } i.
\end{cases}
\]

The map \(\mu\) is bijective as every \(\mu_i\) is. It is also continuous as every \(\bar{\mu}_i\) is. Since every continuous bijection from a compact space to a Hausdorff space is a homeomorphism (see e.g. Proposition 13.26 in [33]), \(\mu\) defines a homeomorphism between \(T_X\) and \(X\). Moreover, \(\pi_2 = f \circ \mu\) so \(\text{Dg}(f) = \text{Dg}(\pi_2)\). In fact, more can be said about \(\text{Dg}(\pi_2)\) following the following Lemma.

Lemma 4.3. Since \(\phi_i\) and \(\psi_i\) are continuous,

\[
\forall \alpha \in [a_i, a_{i+1}], \ T^{(-\infty, \alpha]} \text{ deform retracts onto } T^{(-\infty, a_i]} \\
\forall \alpha \in (a_{i-1}, a_i], \ T^{[\alpha, +\infty)} \text{ deform retracts onto } T^{[a_i, a_{i+1})}
\]

Thus, the points in the extended persistence diagram are located at the critical values:

Corollary 4.4. \(\text{Dg}(\pi_2) \subseteq \text{Crit}(T) \times \text{Crit}(T)\).

We will now present three kinds of perturbations on telescopes that preserve the MultiNerve Mapper, namely: \textit{Merge, Split, and Shift}. For this we will use \textit{generalized attaching maps}:

\[
\phi_i^a : Y_i \times \{a\} \to X_i \times \{a\}; \quad (y, a) \mapsto (\pi_1 \circ \phi_i(y, a_i), a), \\
\psi_i^a : Y_i \times \{a\} \to X_{i+1} \times \{a\}; \quad (y, a) \mapsto (\pi_1 \circ \psi_i(y, a_{i+1}), a).
\]

4.2 Operations

4.2.1 Merge

Definition 4.5 (Merge). Let \(T\) be a telescope. Let \(a \leq b\). If \([a, b]\) contains at least one critical value, i.e. \(a_{i-1} < a \leq a_i \leq a_j \leq b < a_{j+1}\), then the \textit{Merge on} \(T\) between \(a, b\) is the telescope \(T' = \text{Merge}_{a,b}(T)\) given by:

\[
\ldots (Y_{i-1} \times [a_{i-1}, a_i]) \cup_{\psi_{i-1}} (X_i \times \{a_i\}) \cup_{\phi_i} \ldots \cup_{\psi_{i-1}} (X_j \times \{a_j\}) \cup_{\phi_j} (Y_j \times [a_j, a_{j+1}]) \ldots \\
\downarrow \\
\ldots (Y_{i-1} \times [a_{i-1}, \bar{a}]) \cup_{f_{i-1}} (T^{[a, b]} \times \{\bar{a}\}) \cup_{g_j} (Y_j \times [\bar{a}, a_{j+1}]) \ldots
\]

where \(\bar{a} = \frac{a+b}{2}\), where \(f_{i-1} = \psi_{i-1}^\bar{a}\) if \(a = a_i\) and \(f_{i-1} = \text{id}_{Y_{i-1} \times \{\bar{a}\}}\) otherwise, and where \(g_j = \phi_j^\bar{a}\) if \(b = a_j\) and \(g_j = \text{id}_{Y_j \times \{\bar{a}\}}\) otherwise.

If \([a, b]\) contains no critical value, i.e. \(a_{i-1} < a \leq b < a_i\), then \text{Merge}_{a,b}(T)\) is given by:

\[
\ldots (X_{i-1} \times \{a_i\}) \cup_{\phi_{i-1}} (Y_{i-1} \times [a_{i-1}, a_i]) \cup_{\psi_{i-1}} (X_i \times \{a_i\}) \ldots \\
\downarrow \\
\ldots (X_{i-1} \times \{a_i\}) \cup_{\phi_{i-1}} (Y_{i-1} \times [a_{i-1}, \bar{a}]) \cup_{f_{i-1}} (T^{[a, b]} \times \{\bar{a}\}) \cup_{g_{i-1}} (Y_{i-1} \times [\bar{a}, a_i]) \cup_{\psi_{i-1}} (X_i \times \{a_i\}) \ldots
\]

where \(\bar{a} = \frac{a+b}{2}\), and where \(f_{i-1} = g_{i-1} = \text{id}_{Y_{i-1} \times \{\bar{a}\}}\).
Figure 7: Left: Effect of a Merge. Middle: Effect on the corresponding Reeb graph. Right: Effect on the corresponding extended PD of dimension 1. Points before the Merge are disks while points after the Merge are squares.

See the left and middle panels of Figure 7 for an illustration. Similarly, we define the Merge between $a, b$ on a diagram $D$ as the diagram $\text{Merge}_{a,b}(D)$ given by:

$$\text{Merge}_{a,b}(x, y) = (\bar{x}, \bar{y}) \text{ where } \bar{x} = \begin{cases} x & \text{if } x \notin [a, b] \\ \bar{a} & \text{otherwise} \end{cases} \text{ and similarly for } y.$$  

Points in the strips $x \in [a, b], y \in [a, b]$ are snapped to the lines $x = \bar{a}, y = \bar{a}$. See the right panel of Figure 7. See also the first intermediate points along the trajectories of the red points in Figure 14 for another illustration on extended persistence diagrams. We will prove in Lemma 4.12 that $Dg(\text{Merge}) = \text{Merge}(Dg)$.

4.2.2 Split

Definition 4.6 (Split). Let $T$ be a telescope. Let $a_i \in \text{Crit}(T)$ and $\epsilon$ s.t. $0 \leq \epsilon < \min\{a_{i+1} - a_i, a_i - a_{i-1}\}$. The $\epsilon$-Split on $T$ at $a_i$ is the telescope $T' = \text{Split}_{\epsilon,a_i}(T)$ given by:

$$\Downarrow$$

$$\Downarrow$$

Figure 8: Left: Effect of a Split. Middle: Effect on the corresponding Reeb graph. Right: Effect on the corresponding extended PD of dimension 1. Points before the Split are disks while points after the Split are squares.
See the left and middle panels of Figure 8 for an illustration. Similarly, we define the \( \epsilon \)-Split at \( a_i \) on a diagram \( D \) as the diagram \( \text{Split}_{\epsilon,a_i}(D) \) given by:

\[
\text{Split}_{\epsilon,a_i}(x,y) = (\bar{x}, \bar{y}) \text{ where } \bar{x} = \begin{cases} 
  x & \text{if } x \neq a_i \\
  a_i + \epsilon & \text{if } x = a_i \text{ and } (x,y) \in \text{Rel} \\
  a_i - \epsilon & \text{if } x = a_i \text{ and } (x,y) \notin \text{Rel}
\end{cases} \quad \text{and } \bar{y} = \begin{cases} 
  y & \text{if } y \neq a_i \\
  a_i - \epsilon & \text{if } y = a_i \text{ and } (x,y) \in \text{Ord} \\
  a_i + \epsilon & \text{if } y = a_i \text{ and } (x,y) \notin \text{Ord}
\end{cases}
\]

Points located on the lines \( x, y = a_i \) are snapped to the lines \( x, y = a_i \pm \epsilon \) according to their type. Note that the definition of \( \text{Split}_{\epsilon,a_i}(D) \) assumes implicitly that \( D \) contains no point within the horizontal and vertical bands \([a_i - \epsilon, a_i] \times \mathbb{R}, [a_i, a_i + \epsilon] \times \mathbb{R} \times [a_i - \epsilon, a_i], \) and \( \mathbb{R} \times (a_i, a_i + \epsilon), \) which is the case under the assumptions of Definition 4.6. See the right panel of Figure 8 for an illustration. See also the second intermediate points along the trajectories of the red points in Figure 14 for another illustration on extended persistence diagrams. We will prove in Lemma 4.13 that \( Dg(\text{Split}) = \text{Split}(Dg) \).

Splits create particular critical values called down- and up-forks. Intuitively, Split operations allow to distinguish between all possible types of changes in 0- and 1-dimensional homology of the sublevel and superlevel sets, namely: union of two cc, creation of a cc, destruction of a cc, and separation of a cc. Unions and creations occur at down-forks while separations and destructions occur at up-forks. See Figure 9 for an illustration. We formalize and prove this intuition in Lemma 4.10.

**Definition 4.7.** A critical value \( a_i \in \text{Crit}(T) \) is called an up-fork if \( \psi_{i-1} \) is an homeomorphism, and it is called a down-fork if \( \phi_i \) is an homeomorphism.

Recall that the new attaching maps introduced by the Split are identity maps, hence the following lemma:

**Lemma 4.8.** The new critical values \( a_i - \epsilon \) and \( a_i + \epsilon \) created after a Split are down- and up-forks respectively.

The next lemma is a direct consequence of the existence and continuity of \( \phi_{i}^{-1} \) (resp. \( \psi_{i-1}^{-1} \)) when \( a_i \in \text{Crit}(T) \) is a down-fork (resp. up-fork):

**Lemma 4.9.** If \( a_i \in \text{Crit}(T) \) is an up-fork, then \( T(-\infty,a_i] \) deform retracts onto \( T(-\infty,a_i] \) for all \( \alpha \in (a_i-1, a_i] \). Similarly, if \( a_i \in \text{Crit}(T) \) is a down-fork, then \( T(a_i, +\infty) \) deform retracts onto \( T[a_i, +\infty) \) for all \( \alpha \in [a_i, a_i+1] \).

Now we can prove the previous intuition concerning down- and up-forks correct:

**Lemma 4.10.** Let \( a_i \in \text{Crit}(T) \). If \( a_i \) is an up-fork, then it can only be the birth time of relative cycles and the death time of relative and extended cycles in \( Dg(\pi_2) \). If \( a_i \) is a down-fork, then it can only be the birth time of ordinary and extended cycles and the death time of ordinary cycles in \( Dg(\pi_2) \).

**Proof.** Let \( 0 \leq \epsilon, \epsilon' < \min \{ a_i-1 - a_i, a_i - a_i-1 \} \). Consider the extended persistence module of \( \pi_2 \):

\[
\begin{align*}
\cdots \quad & \rightarrow \quad H_*(T(-\infty,a_i-\epsilon)) \quad \rightarrow \quad H_*(T(-\infty,a_i)) \quad \rightarrow \quad H_*(T(-\infty,a_i+\epsilon)) \quad \rightarrow \quad \cdots \\
\cdots \quad & \rightarrow \quad H_*(T(T[a_i, +\infty)]) \quad \rightarrow \quad H_*(T(T[a_i-\epsilon, +\infty])) \quad \rightarrow \quad \cdots
\end{align*}
\]

Figure 9: Left and right panels display the space before and after the Split respectively. Subsets of \( X_i \) that are colored in red correspond to \( \text{im}(\pi_1 \circ \psi_{i-1}) \) and subsets that are colored in blue correspond to \( \text{im}(\pi_1 \circ \phi_i) \).

17
If $a_i$ is an up-fork, then the composition $H_*(T(-\infty,a_i-\epsilon]) \to H_*(T(-\infty,a_i+\epsilon'])$ is an isomorphism since $T(-\infty,a_i+\epsilon')$ deform retracts onto $T(-\infty,a_i-\epsilon]$ by Lemmas 4.3 and 4.9. As $\epsilon, \epsilon'$ can be chosen arbitrarily small, there cannot be any creation of ordinary or extended cycle at $a_i$. There also cannot be any destruction of ordinary cycle.

Similarly, if $a_i$ is a down-fork, then the composition $H_*(T_{T,a_i+\epsilon',+\infty}) \to H_*(T_{T,a_i-\epsilon,-\infty})$ is an isomorphism since $T_{T,a_i-\epsilon,-\infty}$ deform retracts onto $T_{T,a_i+\epsilon',+\infty}$. Again, there cannot be any destruction of extended or relative cycle at $a_i$. There also cannot be any creation of relative cycle.

**4.2.3 Shift**

**Definition 4.11 (Shift).** Let $T$ be a telescope. Let $a_i \in \text{Crit}(T)$ and $\epsilon$ s.t. $0 \leq |\epsilon| < \min\{a_i+1-a_i, a_i-a_i-1\}$.

The $\epsilon$-Shift on $T$ at $a_i$ is the telescope $T' = \text{Shift}_{\epsilon,a_i}(T)$ given by:

$$
\cdots(Y_{i-1} \times [a_{i-1}, a_i]) \cup \phi_{i-1} (X_i \times \{a_i\}) \cup \phi_i (Y_i \times [a_i, a_{i+1}]) \cdots
$$

$$
\cdots(Y_{i-1} \times [a_{i-1}, a_i+\epsilon]) \cup \psi_{a_i+\epsilon} (X_i \times \{a_i+\epsilon\}) \cup \phi_{a_i+\epsilon} (Y_i \times [a_i+\epsilon, a_{i+1}]) \cdots
$$

See the left and middle panels of Figure 10 for an illustration. Similarly, we define the $\epsilon$-Shift at $a_i$ on a diagram $D$ as the diagram $\text{Shift}_{\epsilon,a_i}(D)$ given by:

$$
\text{Shift}_{\epsilon,a_i}(x,y) = (\bar{x}, \bar{y}) \text{ where } \bar{x} = \begin{cases} 
    x & \text{if } x \neq a_i \\
    a_i + \epsilon & \text{otherwise}
\end{cases}
$$

Points located on the lines $x, y = a_i$ are snapped to the lines $x, y = a_i + \epsilon$. Note that the definition of $\text{Shift}_{\epsilon,a_i}(D)$ assumes implicitly that $D$ contains no point within the horizontal and vertical bands delimited by $a_i$ and $a_i + \epsilon$, which is the case under the assumptions of Definition 4.11. See the right panel of Figure 10 for an illustration. See also the third intermediate points along the trajectories of the red points in Figure 14 for another illustration on extended persistence diagrams. We will prove in Lemma 4.14 that $Dg(\text{Shift}) = \text{Shift}(Dg)$.

**4.3 Invariance Properties**

In this section we characterize the evolution of $Dg(\pi_2)$ and $\overline{M}_{\pi_2}(T,I)$ as we apply the previous perturbations to the telescope.
4.3.1 Merges

**Lemma 4.12.** Let \( a \leq b \) and \( T' = \text{Merge}_{a,b}(T) \). Let \( \pi_2' : T' \to \mathbb{R} \) be the projection onto the second factor. Then, \( Dg(\pi_2') = \text{Merge}_{a,b}(Dg(\pi_2)) \).

**Proof.** We only study the sublevel sets of the functions, which means that we only prove the result for the ordinary part of the diagrams. The proof is symmetric for superlevel sets, leading to the result for \( \text{Ext} \) and \( \text{Rel} \).

Assume \( a_i < a \leq a_i \leq a_j \leq b < a_{j+1} \). Given \( x \leq y \), we let \( \Pi_{x,y} : H_* \left( T'(-\infty,x] \right) \to H_* \left( T'(-\infty,y] \right) \) and \( \Pi'_{x,y} : H_* \left( T'(-\infty,x] \right) \to H_* \left( T'(-\infty,y] \right) \) be the homomorphisms induced by inclusions. Since \( f \) is of Morse type, Lemma 4.3 relates \( \Pi \) to \( \Pi' \) as follows (see Figure 11):

\[
\Pi'_{x,y} = \begin{cases} 
\Pi_{x,y} & \text{if } x, y \notin [a, b] \text{ (green area)} \\
\Pi_{a_{i-1}, y} & \text{if } x \in [a, a_i), y > b \text{ (blue area)} \\
\Pi_{a_{i-1}, a_i} & \text{if } x \in [a, a_i), y \in [a, b] \text{ (orange area)} \\
\Pi_{x, a_j} & \text{if } x < a, y \in [a, b] \text{ (turquoise area)} \\
\text{id}_{Y_{x', y'}} & \text{if } x, y \in [a, b] \text{ (brown area)} \\
\text{id}_{Y_{x}} & \text{if } x, y \in [a, b] \text{ (purple area)} 
\end{cases}
\]

Figure 11: Left: Example of a Merge between \( a, b \). Center: Corresponding areas in the diagrams. Right: Examples of the boxes we use to prove the result (circles represent points before the Merge, squares represent points after the Merge).

The equality between the diagrams follows from these relations and the inclusion-exclusion formula \([1]\). Consider for instance the case where the point \((x, y) \in Dg(\pi_2)\) belongs to the union \(A\) of the pink and the turquoise areas. One can select two abscissae \(x_1 < x < x_2\) and an arbitrarily small \(\epsilon > 0\). Then, the total multiplicity of the corresponding rectangle \(R\) in \(Dg(\pi_2')\) (displayed in the right panel of Figure 11) is given by:

\[
\text{mult}(R) = \text{rank } \Pi'_{x_2, a - \epsilon} - \text{rank } \Pi'_{x_2, b + \epsilon} + \text{rank } \Pi'_{x_1, b + \epsilon} - \text{rank } \Pi'_{x_1, a - \epsilon}.
\]

The first relation in (5) shows that \(R\) has exactly the same multiplicity in \(Dg(\pi_2)\), since all its corners belong to the green area. As this is true for arbitrarily small \(\epsilon > 0\), it means that \(R' = R \cap A\) also has the same multiplicity in \(Dg(\pi_2)\) as in \(Dg(\pi_2')\). Now, if we pick a point inside \(R'\) with an ordinate different than \(\bar{a}\), we can compute its multiplicity in \(Dg(\pi_2')\) by surrounding it with a box included in the turquoise area (if the ordinate is bigger than \(\bar{a}\)) or in the pink area (if it is smaller). Boxes in the turquoise area have multiplicity \(\text{rank } \Pi'_{x_2, y_1} - \text{rank } \Pi'_{x_2, y_2} \) and boxes in the pink area have multiplicity \(\text{rank } \Pi'_{x_1, y_1} - \text{rank } \Pi'_{x_1, y_2} \). Similarly, boxes in the pink area also have multiplicity zero. Thus, all points of \(R'\) in \(Dg(\pi_2')\) have ordinate \(\bar{a}\). Again, as it is true for \(x_1, x_2\) as close to each other as we want, it means that \((x, y)\) is snapped to \((x, \bar{a})\) in \(Dg(\pi_2')\). The treatment of the other areas in the plane is similar.

Now, if \([a, b]\) contains no critical values, then \(\Pi' = \Pi\), so the result is clear. \(\square\)
4.3.2 Splits

**Lemma 4.13.** Let \( a_i \in \text{Crit}(T) \). Let \( 0 < \epsilon < \min\{a_{i+1} - a_i, a_i - a_i-1\} \), \( T' = \text{Split}_{\epsilon,a_i}(T) \) and \( \pi'_2 : T' \to \mathbb{R} \) the projection onto the second factor. Then, \( \text{Dg}(\pi'_2) = \text{Split}_{\epsilon,a_i}(\text{Dg}(\pi_2)) \).

**Proof.** Notice that \( T = \text{Merge}_{a_i-\epsilon,a_i+\epsilon}(T') \). Hence, by Lemma 4.12 we have \( \text{Dg}(\pi_2) = \text{Merge}_{a_i-\epsilon,a_i+\epsilon}(\text{Dg}(\pi'_2)) \). Notice also that \( \pi'_2 \) has no critical value within the open interval \( (a_i - \epsilon, a_i + \epsilon) \), so \( \text{Dg}(\pi'_2) \) has no point within the horizontal and vertical bands \( \mathbb{R} \times (a_i - \epsilon, a_i + \epsilon) \) and \( (a_i - \epsilon, a_i + \epsilon) \times \mathbb{R} \). Finally, Lemma 4.8 ensures that \( a_i + \epsilon, a_i - \epsilon \) are up- and down-forks respectively, so Lemma 4.11 tells us exactly where the preimages of the points of \( \text{Dg}(\pi_2) \) through the Merge are located depending on their type.

4.3.3 Shifts

**Lemma 4.14.** Let \( a_i \in \text{Crit}(T) \), \( \epsilon \in (a_{i-1} - a_i, a_{i+1} - a_i) \), \( T' = \text{Shift}_{\epsilon,a_i}(T) \) and \( \pi'_2 : T' \to \mathbb{R} \) the projection onto the second factor. Then, \( \text{Dg}(\pi'_2) = \text{Shift}_{\epsilon,a_i}(\text{Dg}(\pi_2)) \).

**Proof.** Again, the following relations coming from Lemma 4.3

\[
\Pi'_{x,y} = \begin{cases} 
\Pi_{x,y} \text{ if } x, y \notin (a_{i-1}, a_{i+1}) \text{ (green area)} & \Pi_{a_i,y} \text{ if } x \in [a_i + \epsilon, a_{i+1}), y \geq a_{i+1} \text{ (grey area)} \\
\Pi_{x,a_{i-1}} \text{ if } x \leq a_{i-1}, y \in (a_{i-1}, a_i + \epsilon) \text{ (pink area)} & \Pi_{a_{i-1},y} \text{ if } x \in (a_{i-1}, a_i + \epsilon), y \geq a_{i+1} \text{ (blue area)} \\
\Pi_{x,a_i} \text{ if } x \leq a_{i-1}, y \in [a_i + \epsilon, a_{i+1}) \text{ (turquoise area)} & \text{id}_{a_i}^{\epsilon} \text{ if } x, y \in (a_{i-1}, a_i + \epsilon) \text{ (brown area)} \\
\Pi_{a_{i-1},a_i} \text{ if } x \in (a_{i-1}, a_i + \epsilon), y \in [a_i + \epsilon, a_{i+1}) \text{ (orange area)} & \text{id}_{a_i}^{\epsilon} \text{ if } x, y \in [a_i + \epsilon, a_{i+1}) \text{ (purple area)}
\end{cases}
\]

allow us to prove the result similarly to Lemma 4.12 — see Figure 12. For instance, one can choose a box that intersects the lines \( y = a_i + \epsilon \) and \( y = a_i \), show that the total multiplicity is preserved, then choose another small box that does not intersect \( y = a_i + \epsilon \) inside the first box, and show that its multiplicity is zero.

Figure 12: Left: Example of a Shift at \( a_i \) with \( \epsilon < 0 \). Right: Corresponding areas in the diagrams.

4.3.4 MultiNerve Mapper

Now we provide an invariance result for the MultiNerve Mapper. The result is stated in a way that is adapted to its use in the following sections. The conclusion would still hold under somewhat weaker assumptions.

**Proposition 4.15.** Let \( T \) be a telescope and \( \mathcal{I} \) be a gomic of \( \text{im}(\pi_2) \).

(i) Let \( a \leq b \) s.t. \( a, b \) belong to the same intersection \( I \cap J \) or proper interval \( \bar{I} \). Then, \( \overline{\text{M}}_{\pi_2}(<\text{Merge}_{a,b}(T),\mathcal{I}) \) is isomorphic to \( \overline{\text{M}}_{\pi_2}(\mathcal{T},\mathcal{I}) \).

(ii) Let \( a_i \in \text{Crit}(T) \setminus \text{End}(\mathcal{I}) \), and \( a < a_i < b \) with \( a, b \) consecutive in \( \text{End}(\mathcal{I}) \). If \( a_{i-1} < a < b < a_{i+1} \) and \( 0 < \epsilon < \min\{a_i - a, b - a_i\} \), then \( \overline{\text{M}}_{\pi_2}(\text{Split}_{\epsilon,a_i}(T),\mathcal{I}) \) is isomorphic to \( \overline{\text{M}}_{\pi_2}(T,\mathcal{I}) \).
Lemma 5.1. Let $a_i \in \text{Crit}(T) \setminus \text{End}(I)$, and $b < a_i < c < d$ with $b,c,d$ consecutive in $\text{End}(I)$. If $a_i$ is an up-fork, $(b,c) = I \cap J$ is an intersection, and $c - a_i < \epsilon < \min\{d,a_{i+1}\} - a_i$, then $\overline{M}_{\pi_2}(\text{Shift}_{a_i}(T), I)$ is isomorphic to $\overline{M}_{\pi_2}(T_X, I)$. 

(iv) Let $a_i \in \text{Crit}(T) \setminus \text{End}(I)$, and $a < b < a_i < c$ with $a,b,c$ consecutive in $\text{End}(I)$. If $a_i$ is a down-fork, $(b,c) = I \cap J$ is an intersection, and $\max\{a,a_{i-1}\} - a_i < \epsilon < b - a_i$, then $\overline{M}_{\pi_2}(\text{Shift}_{a_i}(T), I)$ is isomorphic to $\overline{M}_{\pi_2}(T_X, I)$.

Proof. Under the assumptions given by each item, the cc in every intersection $I \cap J$ and in every element $I \in \mathcal{I}$ remain the same after each operation. Indeed, let $I, J \in \mathcal{I}$, $I \cap J \neq \emptyset$:

1. For any intersection $K = I \cap J$ or interval $K = I$, $T^K$ deform retracts onto $(\text{Merge}_{a,b}(T))^K$;

2. For any intersection $K = I \cap J$ or interval $K = I$, $(\text{Split}_{\epsilon,a_i}(T))^K$ deform retracts onto $T^K$;

3. The Shifts move the up-fork to the upper proper interval, and the down-fork to the lower proper interval, which preserves the cc in each of the two intervals as well as in their intersection.

Thus, the MultiNerve Mapper is not changed by any of the three operations.

5 Structure of the MultiNerve Mapper

Let $f : X \to \mathbb{R}$ be of Morse type, and let $\mathcal{I}$ be a gonic of $\text{im}(f)$. Let $T_X$ be the corresponding telescope. In this section, we move out all critical values of the intersection preimages $f^{-1}(I \cap J)$, so that the MultiNerve Mapper and the Reeb graph become isomorphic. For any interval $I \in \mathcal{I}$, let $a_I < b_I$ be the endpoints of its proper subinterval $\bar{I}$, so we have $I = [a_I, b_I]$. For any non-empty intersection $I \cap J$, we fix a subinterval $[a_{I\cap J}, b_{I\cap J}] \subset I \cap J$ such that every critical value within $I \cap J$ falls into $[a_{I\cap J}, b_{I\cap J}]$. We define

$$T'_X := \text{Merge}'_X \circ \text{Shift}_X \circ \text{Split}_X \circ \text{Merge}_X(T_X),$$

where each operation is defined individually as follows:

- **Merge**$_X$ is the composition of all the $\text{Merge}_{a_I,b_I}, I \in \mathcal{I}$, and of all the $\text{Merge}_{a_{I\cap J},b_{I\cap J}}, I,J \in \mathcal{I}$ and $I \cap J \neq \emptyset$. All these functions commute, so their composition is well-defined. The same holds for the following compositions.

- **Split**$_X$ is the composition of all the $\text{Split}_{\epsilon,a}$ with $a$ a critical value after $\text{Merge}_X$ (therefore not an interval endpoint) and $\epsilon > 0$ such that the assumptions of Prop. 4.15 (ii) are satisfied.

- **Shift**$_X$ is the composition of all the $\text{Shift}_{\epsilon,a}$ with $a$ an up-fork critical value after the $\text{Split}_X$ and $\epsilon > 0$ such that the assumptions of Prop. 4.15 (iii) are satisfied, and all of the $\text{Shift}_{\epsilon,a}$ with $a$ a down-fork critical value after the $\text{Split}_X$ and $\epsilon < 0$ such that the assumptions of Prop. 4.15 (iv) are satisfied. After $\text{Shift}_X$ there are no more critical values located in the intersections of consecutive intervals of $\mathcal{I}$.

- **Merge**$'_X$ is the composition of all the $\text{Merge}_{a_I,b_I}, I \in \mathcal{I}$.

See Figure 13 for an illustration of this sequence of transformations. We also let $\pi'_2 : T'_X \to \mathbb{R}$ be the projection onto the second factor. In the following, we identify the pair $(T_X, \pi_2)$ with $(X, f)$ since they are isomorphic in the category of $\mathbb{R}$-constructible spaces. We also rename $\pi'_2$ into $f'$ for convenience. Let $f' : \overline{R}_f(T'_X, \mathcal{I}) \to \mathbb{R}$ be the induced quotient map.

Lemma 5.1. For $T'_X$ defined as in (6), there is a multigraph isomorphism between $\overline{M}_f(T'_X, \mathcal{I})$ and $\overline{M}_f(X, \mathcal{I})$.

Proof. This result follows from Proposition 4.15 and from the choice of the $a_I, b_I, a_{I\cap J}, b_{I\cap J}, \epsilon$ in the definitions of $\text{Merge}_X$, $\text{Split}_X$, $\text{Shift}_X$ and $\text{Merge}'_X$. □
Figure 13: Illustration of the sequence of transformations in (6) on the features located in an interval intersection. For each figure, we display the original space (middle), its Reeb graph (left) and its MultiNerve Mapper (right).

The effect of these transformations on the extended persistence diagram of $f$ is illustrated in Figure 14. There are two grids in this figure: the one with solid lines is defined by the interval endpoints, while the one with dotted lines is defined by the $\bar{a}$. In the following, we use the term cell to designate a rectangle of the first grid. Cells are closed if they correspond to proper subintervals for both coordinates, they are open if they correspond to intersections for both coordinates, and they are neither closed nor open otherwise. Blue and green cells in Figure 14 correspond to squares associated to a proper subinterval (blue) or intersection (green). We can now interpret the effects of the transformations in (6) on the persistence diagram visually:

- **Merge**$_T$ snaps all the points to the second grid by Lemma 4.12.
- **Split**$_T$ moves the points to one of the four possible quarters inside their cell, depending on the point’s
Figure 14: The left panel displays the trajectories of points in Ord (disks above the diagonal) and Rel (disks
below the diagonal) while the right panel displays the trajectories of points in Ext. For both diagrams, the
original point is red, the final point is purple, and intersections and proper intervals are colored in green and
light blue respectively.

• Then, Shift\_ moves the points to a neighboring cell by Lemma 4.14. This neighboring cell is given by
the point’s type (as in the case of Split\_I) and by the coordinates of the point. For instance, an extended
point \((x, y)\) lying in the row of a green cell and in the column of another green cell, has coordinates
that both belong to interval intersections. Then, this point is moved to the upper-left neighboring cell.
Differently, an extended point whose abscissa (resp. ordinate) is in an intersection and whose ordinate
(resp. abscissa) is not, is only moved to the left (resp. upper) cell. The same can be said for ordinary
(resp. relative) points by changing up-left to down-left (resp. up-right). Points whose coordinates both
belong to proper intervals are not moved by Shift\_I, regardless of their type.

• Finally, Merge\_I re-snaps the points to the second grid by Lemma 4.12.

Thus, each point of Dg(\f) can be tracked through the successive operations of (6), and this tracking leads
to the following elementary observations:

1. The points of Ord(\f) or Rel(\f) that end their course on the diagonal after the sequence of operations \(6\)
disappear in Dg(\f'). This is because ordinary and relative points cannot be located on the diagonal.
The rest of the points of Ord(\f) or Rel(\f) are preserved with the same type in Dg(\f').

2. Differently, all the points of Ext(\f) are preserved with the same type (Ext) in Dg(\f'). However,
some of the points of Ext\_I(\f) may end their course on or across the diagonal after the sequence of
operations \(6\), thus switching from Ext\_I(\f) to Ext\_I(\f'). Their corresponding homology classes switch
from vertical to horizontal.

3. All the points lie in the rows and columns of blue cells after Shift\_I. Therefore, the points that end
their course on the diagonal after the sequence of operations \(6\) are the ones located in blue cells after
Shift\_I.

4. Since transfers between cells occur only during Shift\_I, a point \(p\) that is not in a blue or green cell
initially ends up in a blue cell \(B\) after Shift\_I if and only if:
Figure 15: Left: Staircases of the ordinary (above diagonal) and relative (below diagonal) parts of the PD of $\overline{M}_f(X,\mathcal{I})$ and $M_f(X,\mathcal{I})$. Right: Staircases of the extended parts of the PD of $\overline{M}_f(X,\mathcal{I})$ and $M_f(X,\mathcal{I})$ — $Q^T_{E-}$ is in dark grey while $Q^T_{E}$ is the union of $Q^T_{E-}$ with the light grey area.

- $p$ is extended and it is in the down, right, or down-right neighboring cell of $B$ (grey cells in the right diagram of Figure 14), or
- $p$ is ordinary and it is in the up neighboring cell of $B$ (grey cells above the diagonal in the left diagram of Figure 14), or
- $p$ is relative and it is in the down neighboring cell of $B$ (grey cells below the diagonal in the left diagram of Figure 14).

5. Points that belong to a blue or green cell initially are snapped to the diagonal by Merge$\mathcal{I}$. Among them, those that belong to a blue cell stay there until the end, whereas those that belong to a green cell eventually leave it — they end up in a blue cell after Shift$\mathcal{I}$ if they are ordinary or relative, while they end up in a white cell above the diagonal if they are extended.

The outcome of these observations is the following one. Observations 1, 3, 4, 5 imply that the points of Ord($f$) that disappear in Dg($f'$) are the ones located initially in the staircase $Q^T_O$ made of the blue, green and grey areas above the diagonal in the left panel of Figure 14. Similarly, the points of Rel($f$) that disappear in Dg($f'$) are the ones located initially in the staircase $Q^T_R$ made of the blue, green and grey areas below the diagonal in the left panel of Figure 14. Finally, observations 2, 3, 4, 5 imply that the points of Ext$^-$($f$) that switch to Ext$^+$($f'$) are the ones located initially in the staircase $Q^T_{E-}$ made of the blue, green and grey areas below the diagonal in the right panel of Figure 14. The rest of the points of Dg($f$) are preserved (albeit shifted) with the same type (Ord, Rel, Ext$^+$, Ext$^-$) in Dg($f'$).

The staircases are illustrated in Figure 15 and defined formally as follows. Given an interval $I = (a, b)$ (indifferently open, closed or half-open), let $Q^+_I = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq y \leq b\}$ be the half-square above the diagonal, and $Q^-_I = \{(x, y) \in \mathbb{R}^2 \mid a \leq y < x \leq b\}$ the half-square strictly below the diagonal. Decompose now each interval $I \in \mathcal{I}$ as $I = I^- \cup I \cup I^+_I \in \mathcal{I}$, then let $Q^T_O = \bigcup_{I \in \mathcal{I}} Q^+_I \cup Q^-_I$, $Q^T_R = \bigcup_{I \in \mathcal{I}} Q^+_I \cup Q^-_I$, and $Q^T_{E-} = \bigcup_{I \in \mathcal{I}} Q^-_I$. Our first structure theorem is then stated as follows (recall that we have identified the pairs $(X, f)$ and $(T_X, \pi_2)$ and renamed $\pi'_2$ into $f'$):

**Theorem 5.2.** For $T_X$ defined as in [9], for every $p \geq 0$ there is a perfect matching between:

(i) Ord$_p(f')$ and Ord$_p(f) \setminus Q^T_O$
(ii) Rel$_p(f')$ and Rel$_p(f) \setminus Q^T_R$
(iii) Ext$^-_p(f')$ and Ext$^-_p(f) \setminus Q_{E-}^T$
(iv) Ext$^+_p(f')$ and Ext$^+_p(f) \cup (\text{Ext}^-_p(f) \cap Q^-_{E-})$

Then, Theorem 2.9 induces the following perfect matching between the persistence diagrams of the quotient map$^4$.

$^4$Note that Ext$^-_0(f') = \text{Ext}^-_0(f) = \text{Ext}^-_0(f') = \emptyset$ because every essential 0-dimensional feature corresponds to some cc of the domain, and it is born at the minimum function value and killed at the maximum function value over that cc.
Lemma 5.4. Theorem 5.3 is then a consequence of the following result, whose hypothesis is satisfied by the $T'_X$ of $\{6\}$:

**Theorem 5.3.** For $T'_X$ defined as in $\{6\}$, there is a multigraph isomorphism between $\overline{M}_f(X, I)$ and $C(R_f'(T'_X))$.

Recall from Lemma 5.1 that there is a multigraph isomorphism between $\overline{M}_f(X, I)$ and $\overline{M}_f(T'_X, I)$. Theorem 5.3 is then a consequence of the following result, whose hypothesis is satisfied by the $T'_X$ of $\{6\}$:

**Lemma 5.4.** Suppose every proper interval $\tilde{I}$ in the cover $I$ contains exactly one critical value of $f'$, and that the intersections $I \cap J$ contain none. Then, $\overline{M}_f(T'_X, I)$ and $C(R_f'(T'_X))$ are isomorphic.

**Proof.** The nodes of $C(R_f'(T'_X))$ represent the cc of the preimages of all critical values of $f$, while the nodes of $\overline{M}_f(T'_X, I)$ represent the cc of the preimages of all $I \in I$. The hypothesis of the lemma implies that there is exactly one critical value per interval $I \in I$, hence the nodes of $\overline{M}_f(T'_X, I)$ and of $C(R_f'(T'_X))$ are in bijection. Meanwhile, the edges of $C(R_f'(T'_X))$ are given by the cc of the $Y_i \times [a_i, a_{i+1}]$. Since the proper intervals contain one critical value each and the $I \cap J$ contain none, the pullbacks of all intersections of consecutive intervals also span the $Y_i \times [a_i, a_{i+1}]$. Hence, the edges of $\overline{M}_f(T'_X, I)$ are in bijection with the ones of $C(R_f'(T'_X))$. Moreover, their endpoints are defined in both cases by the $\phi_i$ and $\psi_i$. Hence the multigraph isomorphism.

In passing, it is interesting to study the behavior of the MultiNerve Mapper as the hypothesis of the lemma is weakened. For instance:

**Lemma 5.5.** Suppose every interval $I$ in the cover $I$ contains at most one critical value of $f'$. Then, $\overline{M}_f(T'_X, I)$ is obtained from $C(R_f'(T'_X))$ by splitting some vertices into two and by subdividing some edges once.

Thus, the MultiNerve Mapper may no longer be ‘exactly’ isomorphic to the combinatorial Reeb graph (counter-examples are easy to build, by making some of the critical values fall into intersections of intervals in the cover), however it is still isomorphic to it up to vertex splits and edge subdivisions, which are topologically trivial modifications.

**Proof.** The proof is constructive and it proceeds in 3 steps:
1. For every interval $I \in I$ that does not contain a critical value, add a dummy critical value (with identities as connecting maps) in the proper subinterval $\tilde{I}$. The effect on the Mapper is null, while the effect on the Reeb graph is to subdivide once each edge crossing the dummy critical value. At this stage, every interval of $I$ contains exactly one critical value. For simplicity we identify $T'_X$ with the new telescope.
2. For every interval $I \in I$ whose corresponding critical value does not lie in the proper subinterval $\tilde{I}$ but rather in some intersection $I \cap J$ (defined uniquely since $I$ is a gonic), merge $I$ and $J$ into a single interval $I \cup J$. The coarser cover $\tilde{J}$ thus obtained is still a gonic and it has the extra property that every proper interval contains exactly one critical value and every intersection contains none. Then, by Lemma 5.4 the MultiNerve Mapper $\overline{M}_f(T'_X, \tilde{J})$ is isomorphic to the combinatorial Reeb graph $C(R_f'(T'_X))$.
3. There remains to study the differences between $\overline{M}_f(T'_X, I)$ and $\overline{M}_f(T'_X, \tilde{J})$. The only difference between the two covers is that some isolated pairs of intervals $(I, J)$ have been merged because their intersection $I \cap J$ contained a critical value $a_i$. For every such pair, there are as many cc in the preimage $f^{-1}(I)$ as in $f^{-1}(J)$ as in $f^{-1}(I \cup J)$ because $I \cup J$ contains no critical value other than $a_i$. Hence, every vertex of $\overline{M}_f(T'_X, \tilde{J})$ corresponding to a cc of $f^{-1}(I \cup J)$ is split into two in $\overline{M}_f(T'_X, \tilde{J})$. Moreover, the two copies are connected by a single edge, given by the corresponding cc of $f^{-1}(I \cap J)$. Now, assuming without loss of generality that $J$ lies above $I$, we have $(I \cup J)_{\tilde{J}} = J_{\tilde{J}}$, which by assumption contains no critical value, so the connections between the vertex copy corresponding to $f^{-1}(J)$ and the vertices lying above it in $\overline{M}_f(T'_X, \tilde{J})$ are the same as the connections between the original vertex and the vertices lying above it in $\overline{M}_f(T'_X, \tilde{J})$. Similarly, $(I \cup J)_{\tilde{J}} = I_{\tilde{J}}$ contains no critical value by assumption, so the connections between the vertex copy corresponding to $f^{-1}(I)$ and the vertices lying below it in $\overline{M}_f(T'_X, \tilde{J})$ are the same as the connections between the original vertex and the vertices lying below it in $\overline{M}_f(T'_X, \tilde{J})$. 

25
A signature for MultiNerve Mapper. By Theorem 5.3, the MultiNerve Mapper has the same combinatorial structure as the Reeb graph of a perturbed version of the pair \((X, f)\). From a philosophical point of view, this means that MultiNerve Mappers belong to the category of combinatorial Reeb graphs. From a practical point of view, this means that the dictionary introduced in Section 2.3 can be used to describe the structure of the MultiNerve Mapper from the extended persistence diagram of the perturbed quotient function \(\tilde{f}'\). By Equation (7), \(Dg(\tilde{f}')\) is obtained from \(Dg(\tilde{f})\) by removing the points that fall into the various staircases \((Q_E^0, Q_E^-, Q_R^-)\) corresponding to their type. And by Theorem 5.9 \(Dg(\tilde{f})\) itself is obtained from \(Dg_0(f)\) and \(Dg_1(f)\) by removing the points of \(\text{Ext}_I(f)\) and \(\text{Ord}_I(f)\), which correspond to inessential or horizontal classes. This suggests using the off-staircase part of \(Dg(f)\) as a descriptor for the structure of the MultiNerve Mapper:

\[
Dg(M_f(X, I)) = \text{Ord}(\tilde{f}) \setminus Q_E^0 \cup \text{Ext}(\tilde{f}) \setminus Q_E^- \cup \text{Rel}(\tilde{f}) \setminus Q_R^-
\]

\[
= \text{Ord}_0(f) \setminus Q_E^0 \cup (\text{Ext}_I^0(f) \cup \text{Ext}_I^-(f)) \setminus Q_E^- \cup \text{Rel}_I(f) \setminus Q_R^-.
\]  

(8)

We call this signature the persistence diagram of the MultiNerve Mapper. Note that this descriptor is not computed by applying persistence to some function defined on the multinerve, but it is rather a pruned version of the persistence diagram of \(\tilde{f}\). As for Reeb graphs, it serves as a bag-of-features type descriptor of the structure of \(M_f(X, I)\). Moreover, the fact that \(Dg(M_f(X, I)) \subseteq Dg(\tilde{f})\) formalizes the intuition that the MultiNerve Mapper should be viewed as a pixelized version of the Reeb graph, in which some of the features disappear due to the staircases (prescribed by the cover). For instance, in Figure 16 we show a double torus equipped with the height function, together with its associated Reeb graph, MultiNerve Mapper, and Mapper. We also show the corresponding persistence diagrams. In each case, the points in the diagram represent the features of the object: the extended points represent the holes (dimension 1 and above) and the trunks (dimension 0) while the ordinary and relative points represent the branches.

The following convergence result (which is in fact non-asymptotic) is a direct consequence of our previous results:

**Corollary 5.6.** Suppose the granularity of the gomic \(I\) is at most \(\varepsilon\). Then,

\[
Dg(\tilde{f}) \setminus \{(x, y) \mid |y - x| \leq \varepsilon\} \subseteq Dg(M_f(X, I)) \subseteq Dg(\tilde{f}).
\]

Thus, the features (branches, holes) of the Reeb graph that are missing in the MultiNerve Mapper have spans at most \(\varepsilon\). In particular, we have \(d^\infty_0(Dg(M_f(X, I)), Dg(\tilde{f})) \leq \varepsilon/2\). Moreover, the two signatures become equal when \(\varepsilon\) becomes smaller than the smallest vertical distance of the points of \(Dg(\tilde{f})\) to the diagonal. Finally, \(M_f(X, I)\) and \(C(R_f(X))\) themselves become isomorphic up to one-step vertex splits and edge subdivisions (which are topologically trivial modifications) when \(\varepsilon\) becomes smaller than the smallest absolute difference between distinct critical values of \(f\).

Note that building the signature \(Dg(M_f(X, I))\) requires computing the critical values of \(f\) exactly, which may not always be possible. However, as for Reeb graphs, the signature can be approximated efficiently and with theoretical guarantees under mild sampling conditions using existing work on scalar fields analysis, as we will see in Section 4.

**Induced signature for Mapper.** Recall from Lemma 3.6 that the projection \(\pi_1 : M_f(X, I) \to M_f(X, I)\) induces a surjective homomorphism in homology. Thus, the Mapper has a simpler structure than the MultiNerve Mapper. To be more specific, \(\pi_1\) identifies all the edges connecting the same pair of vertices. This eliminates the corresponding holes in \(M_f(X, I)\). Since the two vertices lie in successive intervals of the cover, the corresponding diagram points lie in the following extended staircase (see the staircase \(Q_E^I\) displayed on the right in Figure 15):

\[
Q_E^I = \bigcup_{I \cup J, I \neq J} Q_{E, I \cup J}^I.
\]

Note that Rel(\(\tilde{f}\)) = Rel_1(\(f\)) since Rel_0(\(f\)) is empty. Indeed, no 0-dimensional homology class (cc) can be created in the relative part of the extended filtration of \(f\). We also recall that Ext_\(\varepsilon\)(\(f\)) is empty too.
Figure 16: From left to right: a 2-manifold equipped with the height function; the corresponding Reeb graph, MultiNerve Mapper, and Mapper. For each object, we display the persistence diagrams of dimension 0 (green points), 1 (orange points) and 2 (purple points). Extended points are squares while ordinary and relative points are disks (above and below the diagonal respectively). The staircases are represented with dashed ($Q^2_O$), dotted ($Q^2_{E^-}$), dash-dotted ($Q^2_L$), and dash-dot-dotted ($Q^2_R$) lines. One can see how to go from the persistence diagram of the height function to the one of the quotient map (remove the points in dimension 2 and the points in dimension 1 above the diagonal), then to the one of the MultiNerve Mapper (remove the points inside the staircases corresponding to their type), and finally, to the one of the Mapper (remove the extended points in $Q^2_E$).

The other staircases remain unchanged. Hence the following descriptor:

$$Dg(M_f(X, I)) = \text{Ord}(\tilde{f}) \setminus Q^2_O \cup \text{Ext}(\tilde{f}) \setminus Q^2_E \cup \text{Rel}(\tilde{f}) \setminus Q^2_R = \text{Ord}_0(f) \setminus Q^2_O \cup (\text{Ext}^+_0(f) \cup \text{Ext}^-_1(f)) \setminus Q^2_E \cup \text{Rel}_1(f) \setminus Q^2_R.$$  \hfill (9)

The interpretation of this signature in terms of the structure of the Mapper follows the same rules as for the MultiNerve Mapper and Reeb graph — see again Figure 16. Moreover, the convergence result stated in Corollary 5.6 holds for the Mapper as well.

6 Stability

Intuitively, for a point in the descriptor $Dg(\tilde{M}_f(X, I))$, the $\ell^\infty$-distance to its corresponding staircase\textsuperscript{6} measures the amount by which the function $f$ or the cover $I$ must be perturbed in order to eliminate the corresponding feature (branch, hole) in the MultiNerve Mapper. Conversely, for a point in the Reeb graph’s

\textsuperscript{6}$Q^2_O$, $Q^2_{E^-}$ or $Q^2_R$, depending on the type of the point.
descriptors $D_g(f)$ that is not in the MultiNerve Mapper’s descriptor (i.e. that lies inside its corresponding staircase), the $\ell^\infty$-distance to the boundary of the staircase measures the amount by which $f$ or $I$ must be perturbed in order to create a corresponding feature in the MultiNerve Mapper. Our goal here is to formalize this intuition. For this we adapt the bottleneck distance so that it takes the staircases into account. Our results are stated for the MultiNerve Mapper, they hold the same for the Mapper with the staircase $Q_E^I$-replaced by its extension $Q_E^I$.

### 6.1 Stability w.r.t. perturbations of the function

Let $\Theta$ be a subset of $\mathbb{R}^2$. Given a partial matching $\Gamma$ between two persistence diagrams $D, D'$, the $\Theta$-cost of $\Gamma$ is:

$$
cost_\Theta(\Gamma) = \max \left\{ \max_{p \in D} \delta_D(p), \max_{p' \in D'} \delta_D(p') \right\},$$

where:

$$
\delta_D(p) = \|p - p'\|_\infty \text{ if } \exists p' \in D' \text{ s.t. } (p, p') \in \Gamma \text{ and } d_\infty(p, \Theta) \text{ otherwise},
$$

$$
\delta_D(p') = \|p - p'\|_\infty \text{ if } \exists p \in D \text{ s.t. } (p, p') \in \Gamma \text{ and } d_\infty(p', \Theta) \text{ otherwise}.
$$

The bottleneck distance becomes:

$$
d_{b,\Theta}^\infty(D, D') = \inf_\Gamma \cost_\Theta(\Gamma),
$$

where $\Gamma$ ranges over all partial matchings between $D$ and $D'$. This is again a pseudometric and not a metric. To avoid heavy notations, we write $d_\Theta$ instead of $d_{b,\Theta}^\infty$. Note that the usual bottleneck distance is obtained by taking $\Theta$ to be the diagonal $\Delta = \{(x, x) \mid x \in \mathbb{R}\}$. Given a gomic $I$, we choose different sets $\Theta$ depending on the types of the points in the two diagrams. More precisely, we define the distance between descriptors as follows, where the notation $D_*$ stands for the subdiagram of $D$ of the right type (Ordinary, Extended or Relative):

$$
d_I(D, D') = \max \left\{ d_{Q_O^I}(D_O, D'_O), d_{Q_{E-}^I}(D_E, D'_E), d_{Q_R^I}(D_R, D'_R) \right\}.
$$

**Theorem 6.1.** Given a gomic $I$, for any Morse-type functions $f, g : X \to \mathbb{R}$,

$$
d_I(D_g(\overline{M}_f(X, I)), D_g(\overline{M}_g(X, I))) \leq \|f - g\|_\infty.
$$

The proof of the theorem relies on the following monotonicity property, which is immediate:

**Lemma 6.2.** Let $\Theta \subseteq \Theta' \subseteq \mathbb{R}^2$. Then, $d_{\Theta'}(D, D') \leq d_{\Theta}(D, D')$.

**Proof of Theorem 6.1.** The result is proven by the following sequence of (in)equalities:

$$
d_I(D_g(\overline{M}_f(X, I)), D_g(\overline{M}_g(X, I))) = d_I(D_g(\tilde{f}), D_g(\tilde{g}))
\leq d_\Delta(D_g(\tilde{f}), D_g(\tilde{g}))
\leq d_\Delta(D_g(f), D_g(g))
\leq \|f - g\|_\infty.
$$

The first equality comes from the observation that the points of $D_g(\tilde{f}) \cup D_g(\tilde{g})$ that lie inside their corresponding staircases can be left unmatched and have a zero cost in the matching, so removing them as in [8] does not change the bottleneck cost. The first inequality follows from Lemma 6.2 since the diagonal $\Delta$ is included in each of the staircases. The second inequality follows from Theorem 2.3 and the fact that the matchings only match points of the same type (ordinary, extended, relative) and of the same homological dimension. The last inequality comes from Theorem 2.3. \qed
Figure 17: We compute the MultiNerve Mapper of the height function $f$ on the torus $T$, given a gomic $I$ with two intervals. We also compute the MultiNerve Mapper of a perturbed function $g$ s.t. $\|f - g\|_\infty \leq \delta$. We plot the PDs of $\tilde{f}$ (dark green) and $\tilde{g}$ (purple). Note that the signature of $\tilde{M}_g(T, I)$ is obtained by removing the purple point beneath the diagonal since it belongs to a staircase, while the signature of $\tilde{M}_f(T, I)$ is equal to $Dg(\tilde{f})$. If we used the bottleneck distance to compare the two signatures, their distance would be equal to the distance to the diagonal of the dark green point beneath $\Delta$ (green segment), which can be arbitrarily large, while, using $d_I$, their distance becomes the distance of the same point to the staircase (tiny pink segment), which is bounded by $\delta$.

Note that the classical bottleneck distance $d_\Delta$ is unstable in this context – see Figure 17. The theorem allows us to make some interesting claims. For instance, denoting by $Q^T_p$ the staircase corresponding to the type of a diagram point $p$, the quantity

$$d_I(D, \emptyset) = \max_{p \in D} d_\infty(p, Q^T_p)$$

measures the amount by which the diagram $D$ must be perturbed in the metric $d_I$ in order to bring all its points to the staircase. Hence, by Theorem 6.1 given a pair $(X, f)$, the quantity

$$d_I(Dg(\tilde{M}_f(X, I))), \emptyset) = \max_{p \in Dg(\tilde{M}_f(X, I))} d_\infty(p, Q^T_p)$$

is a lower bound on the amount by which $f$ must be perturbed in the supremum norm in order to remove all the features (branches and holes) from the MultiNerve Mapper. Conversely,

$$\min_{p \in Dg(\tilde{M}_f(X, I))} d_\infty(p, Q^T_p)$$

is a lower bound on the maximum amount of perturbation allowed for $f$ if one wants to preserve all the features in the MultiNerve Mapper. Note that this does not prevent other features from appearing. The quantity that controls those is related to the points of $Dg(\tilde{f})$ (including diagonal points) that lie in the staircases. More precisely, the quantity

$$\min_{p \in Dg(\tilde{f}) \cup \Delta} d_\infty(p, \partial Q^T_p \setminus \Delta) \quad (11)$$

is a lower bound on the maximum amount by which $f$ can be perturbed if one wants to preserve the structure (set of features) of the MultiNerve Mapper. Note that the quantity in $\clubsuit$ is in fact zero since $\partial Q^T_O \setminus \Delta$ and $\partial Q^T_E \setminus \Delta$ come arbitrarily close to the diagonal $\Delta$ (recall Figure 15). This means that, as small as the perturbation of $f$ may be, it can always make new branches appear in the MultiNerve Mapper. However, it will not impact the set of holes if its amplitude is less than

$$\min_{p \in Ext(\tilde{f}) \cup \Delta} d_\infty(p, \partial Q^T_E \setminus \Delta).$$
From this discussion we derive the following rule of thumb: having small overlaps between the intervals of the gonic helps capture more features (branches and holes) of the Reeb graph in the (MultiNerve) Mapper; conversely, having large overlaps helps prevent new holes from appearing in the (MultiNerve) Mapper under small perturbations of the function. This is an important trade-off to consider in applications.

6.2 Stability w.r.t. perturbations of the domain

More generally, we can derive a stability result for perturbations of the pair \((X, f)\), provided we make some extra assumptions on the regularity of the domain and function. Typically, we will assume \(X\) to be a compact Riemannian manifold (or, more generally, a compact length space with curvature bounded above) and \(f\) to be Lipschitz-continuous. To measure the amount of perturbation of the domain we use the concept of correspondence from metric geometry: given another pair \((Y, g)\), a correspondence is a subset \(C\) of the product space \(X \times Y\) such that the canonical projections \(C \to X\) and \(C \to Y\) are surjective. We consider the functional distortion associated with \(C\), which is the quantity:

\[
\epsilon_f(C) = \sup_{(x, y) \in C} |f(x) - g(y)|.
\]

Similarly, writing respectively \(d_X\) and \(d_Y\) for the intrinsic metrics of \(X\) and \(Y\), we consider the metric distortion of \(C\):

\[
\epsilon_m(C) = \sup_{(x, y) \in C, (x', y') \in C} |d_X(x, x') - d_Y(y, y')|.
\]

The Gromov-Hausdorff distance between \(X\) and \(Y\) is then:

\[
d_{GH}(X, Y) = \frac{1}{2} \inf_C \epsilon_m(C),
\]

where \(C\) ranges over all correspondences between \(X\) and \(Y\). Now we can derive a stability guarantee for the descriptors of MultiNerve Mappers in this context, using a variant of Theorem 2.3 proven in [9]:

**Theorem 6.3.** Fix a gomic \(I\). Let \(X\) and \(Y\) be two compact Riemannian manifolds or length spaces with curvature bounded above. Denote by \(\rho(X)\) and \(\rho(Y)\) their respective convexity radii. Let \(f : X \to \mathbb{R}\) and \(g : Y \to \mathbb{R}\) be Lipschitz-continuous Morse-type functions, with Lipschitz constants \(c_f\) and \(c_g\) respectively. Assume \(d_{GH}(X, Y) \leq \frac{1}{10} \min(\rho(X), \rho(Y))\). Then, for any correspondence \(C \in \mathcal{C}(X, Y)\) such that \(\epsilon_m(C) < \frac{1}{10} \min(\rho(X), \rho(Y))\),

\[
d_I(Dg(M_f(X, I)), Dg(M_g(Y, I))) \leq 10(c_f + c_g)\epsilon_m(C) + \epsilon_I(C).
\]

**Proof.** The proof is the same sequence of (in)equalities as for Theorem 6.1, except the last inequality is replaced by \(d_\Delta(Dg(f), Dg(g)) \leq 10(c_f + c_g)\epsilon_m(C) + \epsilon_I(C)\), which comes from Theorem 3.4 in [9].

This result brings about the same discussion as in Section 6.1 with \(f\) replaced by the pair \((X, f)\).

6.3 Stability w.r.t. perturbations of the cover

Let us now fix the pair \((X, f)\) and consider varying gomics. For each choice of gomic, Eqs. 8-9 tell which points of the diagram \(Dg(f)\) end up in the diagram of the (MultiNerve) Mapper and thus participate in its structure. We aim for a quantification of the extent to which this structure may change as the gomic is perturbed. For this we adopt the dual point of view: for any two choices of gomics, we want to use the points of the diagram \(Dg(f)\) to assess the degree by which the gomics differ. This is a reversed situation compared to Section 6.1 where the gomic was fixed and was used to assess the degree by which the persistence diagrams of two functions differed.

\[\text{Note that Theorem 3.4 in [9] is stated only for the ordinary part of the persistence diagrams, however its analysis extends to the full extended filtrations at no extra cost.}\]
The diagram points that discriminate between the two gomics are the ones located in the symmetric difference of the staircases, since they witness that the symmetric difference is non-empty. Moreover, their $\ell^\infty$-distances to the staircase of the other gomic provide a lower bound on the Hausdorff distance between the two staircases and thus quantify the extent to which the two covers differ. We formalize this intuition as follows: given a persistence diagram $D$ and two gomics $\mathcal{I}, \mathcal{J}$, we consider the quantity:

$$d_D(\mathcal{I}, \mathcal{J}) = \max_{* \in \{O, E^-, E\}} \left\{ \sup_{p \in D_* \cap (Q^\mathcal{I}_* \triangle Q^\mathcal{J}_*)} \max \{d_\infty(p, Q^\mathcal{I}_*), d_\infty(p, Q^\mathcal{J}_*)\} \right\},$$

(12)

where $\triangle$ denotes the symmetric difference, where $D_*$ stands for the subdiagram of $D$ of the right type (Ordinary, Extended or Relative), and where we adopt the convention that $\sup_{p \in \emptyset} ...$ is zero instead of infinite. Note that there is always one of the two terms in (12) that is zero since the supremum is taken over all points that lie in the symmetric difference of the staircases. Deriving an upper bound on $d_D(\mathcal{I}, \mathcal{J})$ in terms of the Hausdorff distances between the staircases is straightforward, since the supremum in (12) is taken over points that lie in the symmetric difference between the staircases:

$$d_D(\mathcal{I}, \mathcal{J}) \leq \max_{* \in \{O, E^-, E\}} d^\infty_H(Q^\mathcal{I}_*, Q^\mathcal{J}_*),$$

(13)

where $d^\infty_H$ stands for the Hausdorff distance in the $\ell^\infty$-norm. The connection to the MultiNerve Mapper appears when we take $D$ to be the persistence diagram of the quotient map $\tilde{f}$ defined on the Reeb graph $R_f(X)$. Indeed, we have

$$Dg_*(\tilde{f}) \cap (Q^\mathcal{I}_* \triangle Q^\mathcal{J}_*) = (Dg_*(\tilde{f}) \cap Q^\mathcal{I}_*) \triangle (Dg_*(\tilde{f}) \cap Q^\mathcal{J}_*) = Dg_*(\bar{M}_f(X, \mathcal{I})) \triangle Dg_*(\bar{M}_f(X, \mathcal{J})),
$$

where the second equality follows from the definition of the descriptor of the MultiNerve Mapper given in [8]. Thus, $d_{Dg(\tilde{f})}(\mathcal{I}, \mathcal{J})$ quantifies the proximity of each descriptor to the other staircase. In particular, having $d_{Dg(\tilde{f})}(\mathcal{I}, \mathcal{J}) = 0$ means that there are no diagram points in the symmetric difference, so the two gomics are equivalent from the viewpoint of the structure of the MultiNerve Mapper. Differently, having $d_{Dg(\tilde{f})}(\mathcal{I}, \mathcal{J}) > 0$ means that the structures of the two MultiNerve Mappers differ, and the value of $d_{Dg(\tilde{f})}(\mathcal{I}, \mathcal{J})$ quantifies by how much the covers should be perturbed to make the two multigraphs isomorphic. Furthermore, (13) provides an upper bound on this quantity:

**Theorem 6.4.** Given a Morse-type function $f : X \to \mathbb{R}$, for any gomics $\mathcal{I}, \mathcal{J}$,

$$d_{Dg(\tilde{f})}(\mathcal{I}, \mathcal{J}) \leq \max_{* \in \{O, E^-, E\}} d^\infty_H(Q^\mathcal{I}_*, Q^\mathcal{J}_*),$$

where $\tilde{f}$ is the quotient map defined on the Reeb graph $R_f(X)$.

It is easy to build examples where the upper bound is tight, for instance by placing a diagram point at a corner of one of the staircases\textsuperscript{8}. On the other hand, there are obvious cases where the bound is not tight, for instance we have $d_{Dg(\tilde{f})}(\mathcal{I}, \mathcal{J}) = 0$ as soon as there are no diagram points in the symmetric difference, whereas the symmetric difference itself may not be empty. What the upper bound measures depends on the subdiagram. For instance, for $* = E^-$, we defined $Q^\mathcal{I}_{E^-}$ to be the set $\bigcup_{(a, b) \in \mathcal{I}} \{(x, y) \in \mathbb{R}^2 \mid a \leq y < x \leq b\}$, so $d^\infty_H(Q^\mathcal{I}_{E^-}, Q^\mathcal{J}_{E^-})$ measures the supremum of the differences between the intervals in one cover to their closest interval in the other cover:

$$d^\infty_H(Q^\mathcal{I}_{E^-}, Q^\mathcal{J}_{E^-}) = \max \left\{ \sup_{(a, b) \in \mathcal{I}} \inf_{(c, d) \in \mathcal{J}} \max\{|a - c|, |b - d|\}, \sup_{(c, d) \in \mathcal{J}} \inf_{(a, b) \in \mathcal{I}} \max\{|a - c|, |b - d|\} \right\}.
$$

Similar formulas can be derived for the other subdiagrams.

\textsuperscript{8}Which is easily done by choosing suitable critical values as coordinates for this point.
7 Discrete Case

In this section we discuss the approximation of the (MultiNerve) Mapper and of its descriptor \(^{[5]}\) when the pair \((X,f)\) is known only through a finite set of sample points equipped with function values. Throughout the section, \(X\) is a compact metric space, \(f\) a continuous function, \(\mathcal{I}\) a gonic, and \(P\) a point cloud in \(X\).

7.1 (MultiNerve) Mapper construction

We report two possible constructions for the (MultiNerve) Mapper from the pair \((P,f)\): one is from the original Mapper paper \([31]\), the other is inspired from the graph-induced complex paper \([21]\). Both constructions take a neighborhood graph as input, such as for instance the 1-skeleton graph of the Rips complex, defined in the section, \(X\) a gonic, and \(P\) a point cloud in \(X\).

**Definition 7.1.** Given \(\delta \geq 0\), the Rips complex of \(P\) of parameter \(\delta\) is the simplicial complex \(\text{Rips}_\delta(P)\) defined by:

\[
\{p_0, \ldots, p_k\} \in \text{Rips}_\delta(P) \iff \forall 0 \leq i,j \leq k, \ d_X(p_i, p_j) \leq \delta.
\]

Its 1-skeleton graph is called the Rips graph of parameter \(\delta\) and denoted by \(\text{Rips}_1^\delta(P)\).

Given a choice of neighborhood parameter \(\delta\) and the corresponding Rips graph, the construction from \([31]\) uses the vertices as witnesses for the cc of the pullback cover on \(\text{Rips}_1^\delta(P)\) and for their pairwise intersections. Differently, the construction from \([21]\) uses the edges as witnesses for the cc of the pullback cover on \(\text{Rips}_1^\delta(P)\) and for their pairwise intersections. Thus, both constructions have the same vertex set but potentially different edge sets.

**Vertex-based connectivity.** Given an arbitrary interval \(I\) in \(\mathbb{R}\), the preimage of \(I\) in \(P\) is defined to be \(P \cap X^I\), and its cc are defined to be the cc of the induced subgraph \(\text{Rips}_1^\delta(P \cap X^I)\). Then, the vertices in the (MultiNerve) Mapper are the cc of the preimages of the intervals \(I \in \mathcal{I}\). Given two intersecting intervals \(I,J\) of \(\mathcal{I}\), given a cc \(C_I\) in the preimage of \(I\) and a cc \(C_J\) in the preimage of \(J\), the corresponding vertices are connected by an edge in the Mapper if there is a cc in the preimage of \(I \cap J\) that is contained in both \(C_I\) and \(C_J\); in the MultiNerve Mapper, there are as many copies of this edge as there are cc in the preimage of \(I \cap J\) that are contained in \(C_I \cap C_J\). We denote these two constructions by \(\mathcal{M}_I^\Delta(\text{Rips}_1^\delta(P), \mathcal{I})\) and \(\mathcal{M}_I^\Delta(\text{Rips}_1^\delta(P), \mathcal{I})\) respectively.

**Edge-based connectivity.** The vertex set of the (MultiNerve) Mapper is the same as in the previous construction. Now, for any intersecting intervals \(I,J\) of \(\mathcal{I}\), we redefine the preimage of the intersection \(I \cap J\) to be the subset of \(\text{Rips}_1^\delta(P)\) spanned not only by the points of \(P \cap X^{I\cap J}\) and the graph edges connecting them, but also by the relative interiors of the edges of \(\text{Rips}_1^\delta(P)\) that have one vertex in \(P \cap X^I\) and the other in \(P \cap X^J\). Then, given a cc \(C_I\) in the preimage of \(I\) and a cc \(C_J\) in the preimage of \(J\), we connect the corresponding vertices by an edge in the Mapper if there is a cc of the redefined preimage of \(I \cap J\) that connects \(C_I\) and \(C_J\) in \(\text{Rips}_1^\delta(P)\); in the MultiNerve Mapper, we add as many copies of this edge as there are cc in the redefined preimage of \(I \cap J\) that connect \(C_I\) and \(C_J\). We denote these two constructions by \(\mathcal{M}_I^\Delta(\text{Rips}_1^\delta(P), \mathcal{I})\) and \(\mathcal{M}_I^\Delta(\text{Rips}_1^\delta(P), \mathcal{I})\) respectively.

7.2 Relationships between the constructions

In each of the two constructions detailed above, the Mapper is included in the MultiNerve Mapper by definition. Moreover, the preimages of the intersections in the second construction are supersets of the preimages in the first construction, and two different cc in the same preimage in the first construction cannot be connected in the second construction, therefore the (MultiNerve) Mapper from the first construction is connected in the second construction, therefore the (MultiNerve) Mapper from the first construction is

\[\text{By which we mean that the closure of the cc in Rips}_1^\delta(P)\) contains points from \(C_I\) and from \(C_J\).\]
included in its counterpart from the second construction. Hence the following diagram of inclusions:

\[
\begin{array}{ccc}
\mathcal{M}_f^\bullet(\text{Rips}_n^1(P), \mathcal{I}) & \longrightarrow & \overline{\mathcal{M}}_f^\bullet(\text{Rips}_n^1(P), \mathcal{I}) \\
\downarrow & & \downarrow \\
\mathcal{M}_f^\Delta(\text{Rips}_n^1(P), \mathcal{I}) & \longrightarrow & \overline{\mathcal{M}}_f^\Delta(\text{Rips}_n^1(P), \mathcal{I})
\end{array}
\]  

(14)

The vertical inclusions become equalities when there are no intersection-crossing edges in the Rips graph, defined as follows:

**Definition 7.2.** An edge \([u, v]\) of the Rips graph is interval-crossing if there is an interval \(I \in \mathcal{I}\) such that \(I \subseteq (\min\{f(u), f(v)\}, \max\{f(u), f(v)\})\). It is intersection-crossing if there is a pair of intervals \(I, J \in \mathcal{I}\) such that \(\emptyset \neq I \cap J \subseteq (\min\{f(u), f(v)\}, \max\{f(u), f(v)\})\).

Indeed, in the absence of intersection-crossing edges, each cc in the preimage of an interval intersection in the second construction contains a vertex and therefore deformation-retracts onto the corresponding cc in the first construction. Hence:

**Lemma 7.3.** If there are no intersection-crossing edges in the Rips graph, then \(\mathcal{M}_f^\bullet(\text{Rips}_n^1(P), \mathcal{I}) = \mathcal{M}_f^\Delta(\text{Rips}_n^1(P), \mathcal{I})\) and \(\overline{\mathcal{M}}_f(\text{Rips}_n^1(P), \mathcal{I}) = \overline{\mathcal{M}}_f^\Delta(\text{Rips}_n^1(P), \mathcal{I})\).

See Figure 18 for an example showing the importance of the hypothesis in the lemma.

There is also a third possible construction for the (MultiNerve) Mapper, inspired from the literature on Reeb graphs approximation from point cloud data [23]. Take a geometric realization of the Rips complex \(\text{Rips}_n(P)\) in \(\mathbb{R}^n\), where \(n\) is the number of points in \(P\), then interpolate \(f\) linearly in the relative interiors of the simplices, calling \(\hat{f}\) the resulting PL function. Now, apply the definitions from Section 2.5 and 3 to the pair \((\text{Rips}_n(P), f)\) to define the Mapper and its MultiNerve variant, where \(|\text{Rips}_n(P)|\) denotes the underlying space of the geometric realization of the Rips complex in \(\mathbb{R}^n\), equipped with the subspace topology. Let \(\mathcal{M}_f(|\text{Rips}_n(P)|, \mathcal{I})\) and \(\overline{\mathcal{M}}_f(|\text{Rips}_n(P)|, \mathcal{I})\) be the resulting Mapper and MultiNerve Mapper, respectively.

**Lemma 7.4.** If there are no interval-crossing edges in the Rips graph, then \(\mathcal{M}_f(|\text{Rips}_n(P)|, \mathcal{I})\) is isomorphic to \(\mathcal{M}_f^\Delta(\text{Rips}_n^1(P), \mathcal{I})\) and \(\overline{\mathcal{M}}_f(|\text{Rips}_n(P)|, \mathcal{I})\) is isomorphic to \(\overline{\mathcal{M}}_f^\Delta(\text{Rips}_n^1(P), \mathcal{I})\).

**Proof.** Note that \(\mathcal{M}_f(|\text{Rips}_n(P)|, \mathcal{I})\) and \(\overline{\mathcal{M}}_f(|\text{Rips}_n(P)|, \mathcal{I})\) are the same as \(\mathcal{M}_f(|\text{Rips}_n(P)|, \mathcal{I})\) and \(\overline{\mathcal{M}}_f(|\text{Rips}_n(P)|, \mathcal{I})\) respectively, since only the cc of the preimages of intervals are involved in the construction of the (MultiNerve) Mapper. Hence, for the rest of the proof we set the domain of \(\hat{f}\) to be \(|\text{Rips}_n^1(P)|\). Every cc in the preimage through \(\hat{f}\) of an interval of \(\mathcal{I}\) must contain a vertex, therefore it deformation-retracts onto the corresponding preimage through \(f\). Hence the vertex sets of the aforementioned simplicial posets are the same. Every cc in the preimage through \(\hat{f}\) of an interval intersection \(I \cap J\) either contains a vertex, in which case it deformation-retracts onto the corresponding preimage through \(f\) in the vertex-based connectivity, or it does not contain any vertex, in which case the edge of the Rips graph that contains the cc creates an edge in the (MultiNerve) Mapper in the edge-based connectivity.

Again, see Figure 18 for an example showing the importance of the hypothesis in the lemma.
Figure 18: We study a Rips complex \( \text{Rips}^\delta \) with two cc, with the height function covered by three intervals. We display the preimages of the intervals and their intersections for \( \hat{M}_f(|\text{Rips}^\delta|, I) \), \( \hat{M}_f(\text{Rips}^1, I) \) and \( \hat{M}_f(\text{Rips}^1, I) \). The edges of the right cc are intersection-crossing but not interval-crossing, so \( \hat{M}_f(\text{Rips}^1, I) \) recovers it correctly while \( \hat{M}_f(\text{Rips}^1, I) \) fails to. The edges of the left cc are interval-crossing, so both \( \hat{M}_f(\text{Rips}^1, I) \) and \( \hat{M}_f(\text{Rips}^1, I) \) fail to recover the cc.
7.3 Relationships between the signatures

The following diagram summarizes the relationships between the various (MultiNerve) Mapper constructions introduced in the paper:

\[
\begin{array}{c}
(X, f) \\
\downarrow \\
(R_f(X), \hat{f}) \\
\downarrow \\
(M_f(X, \mathcal{I})) \\
\downarrow \\
(M_f(\text{Ext}_X, \mathcal{I}))
\end{array}
\quad
\begin{array}{c}
(|\text{Rips}_δ(P)|, \hat{f}) \\
\downarrow \\
(\text{R}_f(\text{Rips}_δ(P)), \hat{f}) \\
\downarrow \\
(M_\hat{f}(\text{Rips}_δ(P)), \mathcal{I}) \\
\downarrow \\
(M_\hat{f}(\text{Rips}_δ(P)), \mathcal{I})
\end{array}
\]

(15)

The vertical arrows between the first and second rows are provided by Theorem 2.9. The ones between the second, third and fourth rows are given by Eqs. 6 and 7.4. The dotted horizontal arrows are provided by Lemmas 7.3 and 7.4. Finally, the dashed horizontal arrow is a consequence of Theorem 2 of [14] in 0-dimensional homology and of Theorem 4.6 of [23] in 1-dimensional homology. We then derive the following approximation guarantee:

**Theorem 7.5.** Fix a gonic \(I\). Let \(X\) be a submanifold of \(\mathbb{R}^d\) with positive reach \(r(X)\) and convexity radius \(\rho(X)\). Let \(f : X \to \mathbb{R}\) be a Lipschitz-continuous Morse-type function, with Lipschitz constant \(c\). Let \(P \subseteq X\) be such that every point of \(X\) lies within distance \(\epsilon\) of \(P\), for some \(\epsilon < \min\{\frac{1}{16} r(X), \frac{1}{16} \rho(X), \frac{s}{8}, \frac{s}{8}\}\), where \(s > 0\) is the minimum distance of the points of Ext\(_1\)(\(f\)) to the diagonal \(\Delta\). Then, for any choice of neighborhood parameter \(\delta\) within the range \([2\epsilon, \min\{\frac{1}{8} r(X), \frac{1}{8} \rho(X), \frac{s}{8}\}]\),

\[
d_{\mathcal{I}} \left( \text{Dg}(M_f(X, \mathcal{I})), \text{Dg}(M_f(\text{Rips}_δ(P)), \mathcal{I}) \right) \leq 2c\delta.
\]

If furthermore there are no interval-crossing edges, then \(M_f(\text{Rips}_δ(P)), \mathcal{I}\) is isomorphic to \(M_\hat{f}(\text{Rips}_δ(P), \mathcal{I})\).

If there are no intersection-crossing edges either, then \(M_f(\text{Rips}_δ(P)), \mathcal{I}\) is isomorphic to \(M_\hat{f}(\text{Rips}_δ(P), \mathcal{I})\).

The same result holds for \(M_f(\text{Ext}_1(\text{Rips}_δ(P)), \mathcal{I})\), \(M_\hat{f}(\text{Ext}_1(\text{Rips}_δ(P)), \mathcal{I})\) and \(M_\hat{f}(\text{Rips}_δ(P), \mathcal{I})\), provided \(d_{\mathcal{I}}\) is replaced by the bottleneck distance with the appropriate extended staircase \(Q_{\mathcal{E}}^k\). Thus, we can construct discrete (MultiNerve) Mappers whose signatures approximate the ones of the corresponding continuous structures \(M_f(X, \mathcal{I})\) and \(M_\hat{f}(X, \mathcal{I})\).

In some situations, one is merely interested in approximating the signatures of \(M_f(X, \mathcal{I})\) and \(M_\hat{f}(X, \mathcal{I})\) without actually building corresponding discrete (MultiNerve) Mappers. In such cases, one can simply apply the scalar fields analysis approach of [14] to approximate \(\text{Dg}(f)\), then remove the points from (Ext\(_1\)(\(f\)) \cup Ord\(_1\)(\(f\))) as well as the points lying in their corresponding staircases, to get an approximation of the signatures:

**Theorem 7.6.** Fix a gonic \(I\). Let \(X\) be a compact Riemannian manifold or length space with curvature bounded above. Denote by \(\rho(X)\) its convexity radius. Let \(f : X \to \mathbb{R}\) be a Lipschitz-continuous Morse-type function, with Lipschitz constant \(c\). Let \(P \subseteq X\) be such that every point of \(X\) lies within distance \(\epsilon\) of \(P\), for some \(\epsilon < \frac{1}{8} \rho(X)\). Then, for any choice of neighborhood parameter \(\delta\) within the range \([2\epsilon, \frac{1}{8} \rho(X)],\)

---

10Like Theorem 3.4 in [9], Theorem 2 in [14] is stated only for the ordinary part of the persistence diagrams but its proof extends to the full extended filtrations at no extra cost. Note also that it is stated for a nested pair of Rips complexes, however, as pointed out in [14], in 0-dimensional homology a single Rips graph is sufficient for the theorem to hold.
the diagram $D$ computed by the algorithm of [14], once pruned by removing the points of the Ext\textsuperscript{1} and Ord\textsubscript{1} subdiagrams as well as the points located in the staircase corresponding to their type, approximates the signature of $\mathbb{M}_f(X, I)$ as follows (where $D'$ denotes the pruned diagram):

$$d_I(Dg(\mathbb{M}_f(X, I)), D') \leq 2c\delta.$$ 

The same bound applies for the approximation of $Dg(M_f(X, I))$, provided the staircase $Q_I^{E-}$ is replaced by its extended version $Q_I^{E}$ in the definitions of $d_I$ and $D'$.

Note that this result holds much more generally than Theorem 7.5, however there may be no discrete (MultiNerve) Mapper construction associated with the approximate diagram $D'$.

8 Conclusion

Here we proposed a theoretical framework for the analysis of the structure and stability of the Mapper. In particular, we showed that the MultiNerve version of the Mapper is nothing but the Reeb graph of a perturbed pair $(X, f)$. From this we derived a bag-of-features type signature for the (MultiNerve) Mapper, which gives a complete description of the features present in the graph (trunks, branches, holes). We related this signature to the persistence of the Reeb graph, then to the persistence of the pair $(X, f)$ itself, using various staircases induced by the interval cover $I$. This allowed us to determine the structure of the Mapper from the corresponding persistence diagrams. We also gave stability guarantees to the signature, with respect to perturbations of the function, of its domain, or of the interval cover. These guarantees make it possible to predict the structural changes that may occur in the Mapper under such perturbations. Finally, we discussed the construction of the Mapper and the approximation of its signature from point cloud data, relating the structure of the discrete Mapper to the one of its continuous version.

We plan to investigate two directions for future work. Firstly, a natural extension of our analysis in the discrete setting is the computation of convergence rates, confidence regions and the use of subsampling methods for the estimation and convergence of the signatures of the (MultiNerve) Mapper to the ones of the Reeb graph. The estimation and convergence of the Mapper itself to the Reeb graph is also a question that we will investigate. We plan to achieve this using the framework developed recently for persistence diagrams [12, 13, 15], since the signatures of the (MultiNerve) Mapper are obtained merely by intersecting these diagrams with staircases.

The second direction that we will work out deals with the use of the (MultiNerve) Mapper as a signature for shapes. Is it possible to adapt the existing graph kernels to the (MultiNerve) Mapper so as to be able to use the (MultiNerve) Mapper in a Machine Learning framework for e.g. shape comparison?

An important open question concerns the distance used to measure the stability of the Mapper. As pointed out in the introduction, the bottleneck distance and its adaptation to our context are both oblivious to the actual layout of the features in the graph. Other distances were proposed recently to capture a greater part of this layout in Reeb graphs [14, 15, 20]. We believe our framework can be adapted to be used with these distances. In particular, under a suitable choice of metric for the (MultiNerve) Mapper, the isomorphism ‘up to vertex splits and edge subdivisions’ worked out in our Lemma 5.5 should induce an interleaving in the sense of [20] between the (MultiNerve) Mapper and the Reeb graph.

Another open question concerns vector-valued functions. Indeed, the Mapper’s definition is not bound to the codomain being $\mathbb{R}$ in any fundamental way. However, the stratification of the domain $X$ by the critical level sets in the general setting is less well understood. Chattopadhyay et al. [10] studied the local configuration of the Reeb space around singular values for bivariate functions. In a different context, assuming $X$ is a combinatorial manifold and $f$ is piecewise linear, Edelsbrunner et al. [24] showed that the corresponding Reeb space is a stratified space, and they characterized its coarsest stratification. We believe part of our framework could be generalized to these settings, leading to convergence and stability results that would refine the current state of knowledge [28].

Finally, let us recall that our Theorem 5.3 raised a philosophical question, namely: how different from Reeb graphs are the Mappers? From the categorical viewpoint introduced in [20], the question becomes
whether the transformation that sends a Reeb graph to its Mapper can be described as an endofunctor on the Reeb category.

Acknowledgements. This work was supported by ERC grant Gudhi (ERC-2013-ADG-339025) and by ANR project TopData (ANR-13-BS01-0008).

References

[1] M. Alagappan. From 5 to 13: Redefining the Positions in Basketball. MIT Sloan Sports Analytics Conference, 2012.

[2] A. Babu. Zigzag Coarsenings, Mapper Stability and Gene-network Analyses. PhD Thesis, 2013.

[3] V. Barra and S. Biasotti. 3D Shape Retrieval and Classification using Multiple Kernel Learning on Extended Reeb graphs. The Visual Computer, 30(11):1247–1259, 2014.

[4] U. Bauer, X. Ge, and Y. Wang. Measuring Distance Between Reeb Graphs. In Proceedings 30th Symposium Computational Geometry, pages 464–473, 2014.

[5] U. Bauer, E. Munch, and Y. Wang. Strong Equivalence of the Interleaving and Functional Distortion Metrics for Reeb Graphs. In Proceedings 31st Symposium Computational Geometry, 2015.

[6] S. Biasotti, D. Giorgi, M. Spagnuolo, and B. Falcidieno. Reeb Graphs for Shape Analysis and Applications. Theoretical Computer Science, 392(1-3):5–22, 2008.

[7] G. Carlsson, V. de Silva, and D. Morozov. Zigzag Persistent Homology and Real-valued Functions. In Proceedings 25th Symposium Computational Geometry, pages 247–256, 2009.

[8] H. Carr and D. Duke. Joint Contour Nets. IEEE Transaction Visual Computer Graphics, 20(8):1100–1113, 2014.

[9] M. Carrière, S. Oudot, and M. Ovsjanikov. Stable Topological Signatures for Points on 3D Shapes. In Proceedings 13th Symposium Geometry Processing, 2015.

[10] A. Chattopadhyay, H. Carr, D. Duke, Z. Geng, and O. Saeki. Multivariate Topology Simplification. arXiv 1509.04465, 2015.

[11] F. Chazal, V. de Silva, M. Glisse, and S. Oudot. The Structure and Stability of Persistence Modules. arXiv 1207.3674, 2012.

[12] F. Chazal, B. T. Fasy, F. Lecci, B. Michel, A. Rinaldo, and L. Wasserman. Subsampling Methods for Persistent Homology. arXiv 1406.1901, 2014.

[13] F. Chazal, M. Glisse, C. Labruère, and B. Michel. Convergence rates for persistence diagram estimation in Topological Data Analysis. In Proceedings 31st International Conference on Machine Learning, pages 163–171, 2014.

[14] F. Chazal, L. Guibas, S. Oudot, and P. Skraba. Analysis of scalar fields over point cloud data. In Proceedings 20th Symposium Discrete Algorithm, pages 1021–1030, 2009.

[15] F. Chazal, P. Massart, and B. Michel. Rates of Convergence for Robust Geometric Inference. arXiv 1505.07602, 2015.

[16] F. Chazal and J. Sun. Gromov-Hausdorff Approximation of Filament Structure using Reeb-type Graph. In Proceedings 30th Symposium Computational Geometry, 2014.
[17] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Stability of persistence diagrams. *Discrete Computational Geometry*, 37(1):103–120, 2007.

[18] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Extending persistence using Poincaré and Lefschetz duality. *Foundations Computational Mathematics*, 9(1):79–103, 2009.

[19] É. Colin de Verdière, G. Ginot, and X. Goaoc. Multinerves and Helly numbers of acyclic families. In *Proceedings 28th Symposium Computational Geometry*, pages 209–218, 2012.

[20] V. de Silva, E. Munch, and A. Patel. Categorified Reeb Graphs. arXiv 1501.04147, 2015.

[21] T. Dey, F. Fan, and Y. Wang. Graph Induced Complex on Point Data. In *Proceedings 29th Symposium Computational Geometry*, pages 107–116, 2013.

[22] T. Dey, F. Mémoli, and Y. Wang. Mutscale Mapper: A Framework for Topological Summarization of Data and Maps. arXiv 1504.03763, 2015. Accepted to SODA 2016.

[23] T. Dey and Y. Wang. Reeb graphs: Approximation and persistence. *Discrete Computational Geometry*, 49(1):46–73, 2013.

[24] H. Edelsbrunner, J. Harer, and A. Patel. Reeb Spaces of Piecewise Linear Mappings. In *Proceedings 24th Symposium Computational Geometry*, pages 242–250, 2008.

[25] H. Edelsbrunner, D. Letscher, and A. Zomorodian. Topological Persistence and Simplification. *Discrete Computational Geometry*, 28:511–533, 2002.

[26] W. Harvey, Y. Wang, and R. Wenger. A randomized O(m log m) time algorithm for computing Reeb graphs of arbitrary simplicial complexes. In *Proceedings 26th Symposium Computational Geometry*, pages 267–276, 2010.

[27] V. Kurlin. A one-dimensional homologically persistent skeleton of an unstructured point cloud in any metric space. In *Proceedings 13th Symposium Geometry Processing*, 2015.

[28] E. Munch and B. Wang. Reeb Space Approximations with Guarantees. Extended abstract, 2015.

[29] M. Nicolau, A. Levine, and G. Carlsson. Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival. *Proceedings National Academy Science*, 108(17):7265–7270, 2011.

[30] G. Reeb. Sur les points singuliers d’une forme de pfaff complètement intégrable ou d’une fonction numérique. *Compte Rendu Académie Science Paris*, 222:847–849, 1946.

[31] G. Singh, F. Mémoli, and G. Carlsson. Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition. In *Symposium Point Based Graphics*, 2007.

[32] R. B. Stovner. On the Mapper Algorithm. Master Thesis, 2012.

[33] W. A. Sutherland. *Introduction to Metric and Topological Spaces*. Oxford University Press, 2009.

[34] A. Zomorodian and G. Carlsson. Computing persistent homology. *Discrete Computational Geometry*, 33(2):249–274, 2005.
A Proof of Theorem 2.9

The result in dimension 1 and above is proven using the same sequence of arguments as in Section 2.3. Only the dimension 0 changes.

Let \( 0 < \epsilon < \frac{1}{2} \) and \( \min_{k=1,\ldots,n-1} (a_{k+1} - a_k) \). The idea of the proof is to replace the right inverse of the projection \( \pi : X \to R_f(X) \) by a continuous map \( \sigma : R_f(X) \to X \) such that the composition \( \pi \circ \sigma \) is homotopic to the identity of \( R_f(X) \). In order to make our new \( \sigma \) compatible with the function \( f \), we need to perturb \( f \) to some other function \( g \) whose extended persistence diagram will be proven to be equal to the one of \( f \) in the first part of the proof.

Let \( g : X \to \mathbb{R} \) be defined by:

\[
\forall x \in X, \ g(x) = \begin{cases} 
 f(x) & \text{if } \min_{k=1,\ldots,n} |f(x) - a_k| > 2\epsilon \\
 a_i & \text{otherwise, where } i = \arg\min_k |f(x) - a_k| 
\end{cases}
\]

As \( g \) is constant on equivalence classes of \( \sim_R \), there is an induced quotient map \( \tilde{g} : R_f(X) \to \mathbb{R} \).

\[
\forall \alpha \in \mathbb{R}, \tilde{\alpha} \in \mathbb{R}^{op}, \text{we let } F_\alpha := X_f(-\infty,\alpha], \ G_\alpha := X_f(-\infty,\alpha], \ \alpha := (X, X_f(\tilde{\alpha}, +\infty)) \text{, and } G^{\alpha} := (X, X^{\tilde{\alpha}}, +\infty)).
\]

Similarly on the Reeb graph: \( \tilde{F}_\alpha := \tilde{f}^{-1}((-\infty, \alpha]), \ \tilde{G}_\alpha := \tilde{g}^{-1}((-\infty, \alpha]), \ \tilde{F}^{\alpha} := (R_f(X), \tilde{f}^{-1}((\tilde{\alpha}, +\infty))), \ \tilde{G}^{\alpha} := (R_f(X), \tilde{g}^{-1}((\tilde{\alpha}, +\infty))). \)

**Claim.** \( \text{Dg}(f) = \text{Dg}(g) \) and \( \text{Dg} (\tilde{f}) = \text{Dg} (\tilde{g}) \).

**Proof.** One has the following equalities:

- If \( \min_{i=1,\ldots,n} |\alpha - a_i| > 2\epsilon \), \( G_\alpha = F_\alpha \), \( G^{\alpha} = F^{\alpha} \) and \( \tilde{G}_\alpha = \tilde{F}_\alpha \), \( \tilde{G}^{\alpha} = \tilde{F}^{\alpha} \).

- If \( a_i - 2\epsilon \leq \alpha < a_i \), \( G_\alpha = F_{a_i-2\epsilon} \) and \( \tilde{G}_\alpha = \tilde{F}_{a_i-2\epsilon} \).

- If \( a_i \leq \alpha \leq a_i + 2\epsilon \), \( G_\alpha = F_{a_i+2\epsilon} \) and \( \tilde{G}_\alpha = \tilde{F}_{a_i+2\epsilon} \).

- If \( a_i - 2\epsilon \leq \alpha \leq a_i \), \( G^{\alpha} = F^{a_i-2\epsilon} \) and \( \tilde{G}^{\alpha} = \tilde{F}^{a_i-2\epsilon} \).

- If \( a_i < \alpha \leq a_i + 2\epsilon \), \( G^{\alpha} = F^{a_i+2\epsilon} \) and \( \tilde{G}^{\alpha} = \tilde{F}^{a_i+2\epsilon} \).

Whenever \( \min_{i=1,\ldots,n} |\alpha - a_i| > 2\epsilon \), the equality between \( F_\alpha \) and \( G_\alpha \) and between \( F^{\alpha} \) and \( G^{\alpha} \) ensure that \( H_\ast(F_\alpha) \) and \( H_\ast(G_\alpha) \) are isomorphic, as well as \( H_\ast(X, F_\alpha) \) and \( H_\ast(X, G_\alpha) \). Otherwise, either \( G_\alpha \subseteq F_\alpha \) (if \( a_i - 2\epsilon \leq \alpha < a_i \)) or \( F_\alpha \subseteq G_\alpha \) (if \( a_i \leq \alpha \leq a_i + 2\epsilon \)). Similarly, either \( G^{\alpha} \subseteq F^{\alpha} \) (if \( a_i < \alpha \leq a_i + 2\epsilon \)) or \( F^{\alpha} \subseteq G^{\alpha} \) (if \( a_i - 2\epsilon \leq \alpha \leq a_i \)). These inclusions allow to build the commutative diagram displayed in Figure 19 where all arrows are morphisms induced by canonical inclusions.

![Figure 19](image-url)

**Figure 19:** We assume that \( 0 < \epsilon_1, \epsilon_2 \leq 2\epsilon \). The exact same diagram can be built for \( \tilde{f} \) and \( \tilde{g} \).

However, the bigger space always deform-retracts onto the smaller space, thus the transversal morphisms in Figure 19 are also isomorphisms, and the two persistence modules are isomorphic. The same argument applies to \( f \) and \( g \).

□

39
we let \( \pi \) an arc of the Reeb graph to the path with the same color in \( X \). Figure 20: The left panel displays the Reeb graph and the right panel displays the space \( \sigma \) sends the arc \([ \tilde{q}_1, \cdots, \tilde{q}_1 ] \) to \([ \tilde{s}_1, \cdots, \tilde{s}_1 ] \) and \([ q_1, \cdots, q_1 ] \) to \([ s_1, \cdots, s_1 ] \). Finally, \( \sigma \) also monotonically reparametrizes the arcs \([ \tilde{p}_1, \cdots, \tilde{p}_1 ] \) and \([ q_1, \cdots, q_1 ] \). Let \( \text{param}_i^- : [a_i - 2\epsilon, a_i - \epsilon] \rightarrow [a_i - 2\epsilon, a_i] \) and \( \text{param}_i^+ : [a_i - 2\epsilon, a_i + \epsilon] \rightarrow [a_i - 2\epsilon, a_i + 2\epsilon] \) be these reparametrizations. Again, see Figure 20.

More formally, let \( x \in X \) and assume that \( a_i \leq f(x) \leq a_{i+1} \) and that \( \pi(x) \) belongs to the \( l \)-th edge of the Reeb graph between these two critical values. Then:

- \( \sigma \circ \pi(x) = \mu_i(y_i, f(x)) \) if \( a_i + 2\epsilon \leq f(x) \leq a_{i+1} - 2\epsilon \);
- \( \sigma \circ \pi(x) = \mu_i(y_i, f(x)) \) if \( a_i + \epsilon \leq f(x) \leq a_i + 2\epsilon \);
- \( \sigma \circ \pi(x) = \mu_i(y_i, \text{param}_i^+ \circ f(x)) \) if \( a_{i+1} - 2\epsilon \leq f(x) \leq a_{i+1} - \epsilon \);
- \( \sigma \circ \pi(x) = \gamma_i^{l+}(f(x)) \) if \( a_i \leq f(x) \leq a_i + \epsilon \);
- \( \sigma \circ \pi(x) = \gamma_i^{l+1,-}(f(x)) \) if \( a_{i+1} - \epsilon \leq f(x) \leq a_{i+1} \).

40
By construction we have $g \circ \sigma = \tilde{g}$ and $\tilde{g} \circ \pi = g$ (note that this is not true for $f$). In particular, we have $\pi(G_\alpha) \subseteq \tilde{G}_\alpha$ and $\pi(G^\alpha) \subseteq \tilde{G}^\alpha$ for all $\alpha \in \mathbb{R}$. Hence, $\pi$ induces a morphism between the extended persistence modules of $g$ and $\tilde{g}$ at the 0-dimensional homology level. Let us show that this morphism is an isomorphism. Since $\pi$ is surjective, this boils down to showing that $x, y$ are connected in $G_\alpha$ iff $\pi(x), \pi(y)$ are connected in $\tilde{G}_\alpha$, and same for the relative part of the filtration:

- If $x, y$ are connected in $G_\alpha$, then so are $\pi(x), \pi(y)$ in $\tilde{G}_\alpha$, by continuity of $\pi$ and the fact that $\tilde{g} \circ \pi = g$.

- If $\pi(x), \pi(y)$ are connected in $\tilde{G}_\alpha$, then choose a path $\gamma$ connecting $\pi(x)$ and $\pi(y)$. Now by definition of $\sigma$, there exists a path $\gamma_x$ connecting $x$ and $\sigma \circ \pi(x)$ in $G_\alpha$. Indeed, $\sigma$ can send $\pi(x)$ to five different locations in $G_\alpha$ according to the value of $f(x)$, as seen above. Assume $f(x) \notin \text{Crit}(f)$. Since there is a path $\tilde{\gamma}$ between $x$ and $\mu_i(y_i, f(x))$, one can always find a path $\gamma_x$ between $x$ and $\sigma \circ \pi(x)$ in $G_\alpha$ with an appropriate combination of $\tilde{\gamma}, \mu_i(y_i^+, \cdot)$ and $\gamma_i^{(i+,+)/(i+1,-)}$. Now, assume $f(x) \in \text{Crit}(f)$, and let $u_i = \pi(x)$. Then $\sigma(u_i)$ and $x$ both belong to $\pi^{-1}(u_i)$, so they belong to the same cc of the $g^{-1}(g(x))$ and one can find a path between them in $G_\alpha$. Similarly, there exists a path $\gamma_y$ connecting $\sigma \circ \pi(y)$ and $y$ in $G_\alpha$. Then $\gamma_y \circ \sigma(\gamma) \circ \gamma_x$ is a path between $x$ and $y$ in $G_\alpha$ by continuity of $\sigma$ and the fact that $g \circ \sigma = \tilde{g}$. So $x, y$ are connected in $G_\alpha$.

The same argument applies to the relative part of the filtration. Hence, $D_{g_0}(\tilde{g}) = D_{g_0}(g)$, which completes the proof.