Finite Size Universe or Perfect Squash Problem

Ludwik Turko
Institute of Theoretical Physics, University of Wroclaw,
Pl. Maksy Borna 9, 50-204 Wroclaw, Poland
(Dated: April 21, 2004)

We give a physical notion to all self-adjoint extensions of the operator $\frac{id}{dx}$ in the finite interval. It appears that these extensions realize different non-unitary equivalent representations of CCR and are related to the momentum operator viewed from different inertial systems. This leads to the generalization of Galilei equivalence principle and gives a new insight into quantum correspondence rule. It is possible to get transformation laws of wave function under Galilei transformation for any scalar potential. This generalizes mass superselection rule. There is also given a new and general interpretation of a momentum representation of wave function. It appears that consistent treatment of this problem leads to the time-dependent interactions and to the abrupt switching-off of the interaction.

PACS numbers: 03.65.-w, 03.65.Ca, 03.65.Db, 03.65.Ge, 02.30.Tb

I. INTRODUCTION

A square well potential, although this is the simplest analytically solvable quantum model, can be used as a tool to investigate more involved quantum peculiarities. It was used recently for such different phenomena as quantum fractals [1, 2], quantum chaos [3] or wave-function revivals [4, 5]. It can also be used as an approximation to experimentally realized semiconductor quantum well lasers or micromaser cavities with atomic rubidium [6, 7].

Schrödinger equation with a square well potential can also be considered as a model of a quantum squash. An infinite well corresponds to perfectly rigid and perfectly resistant side walls. A finite square well potential corresponds to perfectly rigid but not perfectly resistant side walls — a high energy squash ball breaks through the wall. A physicist is here like a passive player. He or she (=(s)he) can use a racket only as a measurement apparatus — to register the energy or the momentum of the ball.

A simplicity of the model may be misleading. A closer inspection (see e.g. [8, 9, 10]) shows that the infinite potential well has mathematical traps which, when neglected, lead to contradictions or misinterpreted results.

The aim of this paper is to study physical consequences of different self-adjoint extensions of the “momentum” operator for a quantum squash. The “momentum” means here the differential operator $-i\hbar\nabla$. In the case of square integrable functions on $\mathbb{R}^n$, ($n = 1, 2, 3$) this operator is self-adjoint — so it is interpreted as the momentum operator. A situation is much more involved for a particle in a box. There are infinitely many self-adjoint extensions of the “momentum” operator. Different extensions correspond to different boundary conditions of functions from the domain of the operator and they have different spectra. A question arises which of these extensions is the physical momentum operator and what are physical notions of other self-adjoint extensions of the operator $-i\hbar\nabla$?

It will be shown that all those self-adjoint extensions have physical meanings. They are closely related to the Galilei transformed reference frames moving with different velocities with respect to the primary frame. The primary frame is chosen as the frame with a time-independent potential. This means that our squash play does not move on the squash field. When (s)he changes this passive strategy and starts to run (with a constant velocity of course) then (s)he observes shifted momenta of the squash ball. This picture is in a perfect agreement with a physical “classical” intuition. It appears also that when (s)he solves corresponding Schrödinger equation then a transformed wave function behaves according to projective representations of the Galilei group.

A rôle of projective representations of the Galilei group is well established since long [11, 12, 13, 14, 15, 16]. All results were then obtained with Galilei-invariant potentials. For one particle this was equivalent to a free-particle case. It appears that basic results, the Bargmann superselection rule including, can be reproduced for any potential.

Finally, we are going to clarify the momentum representation puzzle. Let us consider a squash player confined to the finite region bounded by perfectly rigid and perfectly resistant side walls. For the player this squash-room...
is like a finite Universe. The spectrum of the momentum operator is discrete in this Universe. Since according to basic rules of quantum mechanics the only possible results of the momentum measurement are eigenvalues of the corresponding observable the momentum distribution should be discrete one. However, there is a common procedure (see e.g. [17, 18]) to take the Fourier integral transform of the wave function. This Fourier transform is interpreted as the momentum representation. This inconsistency was also observed in [8] but authors didn’t push the problem further.

One can show that both momentum representations have well established physical interpretations, although both describe different physical situations. The Fourier integral transformation of the wave function is simply related to the abrupt switch-off of the potential. As the infinite square well potential can be used as a model for the perfect squash so the Fourier integral of the harmonic oscillator wave function can be used for the quantum sling theory.

We begin by consideration of the notion of momentum distributions. It appears that a momentum distribution of the wave function understood as a Fourier transform is directly related to the solution of Schrödinger equation with a time-dependent interaction. The Fourier transform \( \Phi(p, t) \) of the wave function \( \Psi(r, t) \) is the probability amplitude to measure at time \( t > 0 \) the momentum \( p \) when an interaction was turned off at \( t = 0 \). This gives also a new insight into David-Goliath fight, as is presented in Section II B.

In Section III some necessary mathematical preliminaries are given. These are related to self-adjoint extensions of differential operators \( id/dx \) and \( d^2/dx^2 \). This material does not pretend to give a new insight into the problem, but collects some mathematical facts not always known to the physical community. An analysis of stationary solutions of the infinite square well potential is given as an example in Section III A.

Section IV deals with a physical interpretation of self-adjoint extensions of the operator \( -id/dx \) in the Hilbert space of square integrable functions on a finite interval. First an analysis of the notion of the quantum momentum observable is performed. An operator can be identified with the physical momentum only if it transformsunder Galilei transformation similarly to the classical momentum. This assumption allows to add physics to all self-adjoint extensions of the operator \( -id/dx \). These extensions correspond to momenta seen by moving observers from different inertial systems. It will be shown that those different extensions realize different non-unitary equivalent representations of Canonical Commutation Relations.

A natural problem which arises at that moment is to find how different moving observers see quantum mechanics from their systems. It is well known since papers of Bargmann, Inönü and Wigner [11, 12] that free Schrödinger equation is Galilei invariant provided that wave function transforms under a projective representation of the Galilei group. Section V deals with this problem and generalizes a concept of Galilean covariance to any scalar potential. Now, a problem of the momentum distribution is reexamined. Solutions of the infinite potential well are taken as examples. It appears that momentum distributions are more tricky as it seemed before. In particular, a mathematical identity

\[
\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})
\]

is not so obvious in a quantum word. This is explained in Section V C.

Final conclusions are given in Section VI.

II. MOMENTUM DISTRIBUTIONS

It is a common knowledge that there is the discrete energy spectrum of a quantum particle placed in an infinite square well potential. A one-dimensional potential of the form

\[
U(x) = \begin{cases}
0, & \text{for } 0 \leq x \leq a, \\
\infty, & \text{for } x \text{ everywhere else},
\end{cases}
\]

with boundary conditions

\[
\psi(0) = \psi(a) = 0,
\]

leads to the solutions

\[
\psi_N(x) = \begin{cases}
\sqrt{\frac{2}{a}} \sin \frac{N\pi}{a} x, & \text{for } 0 \leq x \leq a, \\
0, & \text{for } x \text{ everywhere else},
\end{cases}
\]

where \( N \) is an arbitrary positive integer.
Finite Size Universe or Perfect Squash Problem v.2

Corresponding energy levels are

\[ E_N = \frac{\pi^2 \hbar^2}{2ma^2 N^2}. \]  

(4)

Let us consider the Fourier integral of the wave function

\[ \tilde{\psi}_N(k) = \frac{1}{\sqrt{2\pi}} \int_0^a dx \psi_N(x) e^{-ikx}. \]  

(5)

One gets

\[ \tilde{\psi}_N(k) = -\sqrt{\frac{\pi a}{2}} \frac{2N}{a^2 k^2 - N^2 \pi^2} e^{-\frac{iak}{2}} \left\{ \begin{array}{ll} \sin \frac{2N}{a} k, & \text{for } N \text{ even,} \\ \cos \frac{2N}{a} k, & \text{for } N \text{ odd.} \end{array} \right. \]  

(6)

This mathematical expression is usually (see e.g. [17, 18] and a lot of other textbooks) interpreted as the physical momentum (with \( p = \hbar k \)) representation of the wave function. According to such an interpretation the probability distribution of the measurement of the momentum yielding a result between \( p \) and \( p + dp \) is

\[ P_N(p) = \frac{4\pi a h^3 N^2}{(a^2 p^2 - h^2 N^2 \pi^2)^2} \left\{ \begin{array}{ll} \sin^2 \frac{2N}{a} p, & \text{for } N \text{ even,} \\ \cos^2 \frac{2N}{a} p, & \text{for } N \text{ odd.} \end{array} \right. \]  

(7)

This gives an average value of the momentum equal to zero, and an average value of the squared momentum is

\[ \langle p^2 \rangle_N = \int_{-\infty}^{+\infty} dp \, p^2 P_N(p) = \frac{N^2 \pi^2 \hbar^2}{a^2}. \]  

(8)

This is in agreement (in an average) with (4). This is, however, not an answer for the question about the squash ball momentum. Besides that, there is a question about the energy conservation: how is it possible to get any value of the momentum in the state with a given value of the energy (4)?

For the player in his/her finite Universe \( 0 \leq x \leq a \) the only allowed values of a momentum are those which are eigenvalues of the corresponding self-adjoint observable. Using the trivial identity

\[ \psi_N(x) = \sqrt{\frac{2}{a}} \sin \frac{N \pi}{a} x = \frac{1}{2i} \sqrt{\frac{2}{a}} \left( e^{i\frac{N \pi}{a} x} - e^{-i\frac{N \pi}{a} x} \right), \]  

(9)

one gets a simple conclusion that allowed values of momenta are \( \pm \frac{N \pi \hbar}{a} \). This is of course in a perfect agreement (not only in an average) with (4).

Such kind of contradictions led recently to conclusion [8] that the Fourier integral “is just a mathematically equivalent version of the same object, not the momentum representation wave function.”

As we’ll see later, the using of Eq. (9) as a plane waves superposition is an oversimplification of the problem. There is, however, a surprisingly simple answer to a question about the physical notion of Eq. (6).

A. General momentum distribution

Let us consider a time-dependent hamiltonian

\[ \hat{H} = \begin{cases} -\frac{\hbar^2}{2m} \Delta + U(\vec{r}), & \text{for } t \leq 0, \\ -\frac{\hbar^2}{2m} \Delta, & \text{for } t > 0. \end{cases} \]  

(10)

Let \( \psi \) be any solution of the Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + U(\vec{r}) \psi. \]

A general form of the free Schrödinger equation is a wave packet

\[ \int d^3 \bar{p} \, g(\bar{p}) e^{-i\sum_{\alpha} \frac{p_{\alpha}^2}{2m}} e^{i\vec{p} \cdot \vec{r}}. \]
A function
\[
\Psi(\vec{r}, t) = \begin{cases} 
\psi(\vec{r}, t), & \text{for } t \leq 0, \\
\int d^3 p \, g(\vec{p}) \, e^{-i \frac{p^2}{2m} t} e^{i \vec{p} \cdot \vec{r}}, & \text{for } t > 0.
\end{cases}
\]
(11)
is a solution of the Schrödinger equation
\[
i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi,
\]
and the wave function \( \Psi(\vec{r}, t) \) is continuous at \( t = 0 \). This continuity condition gives
\[
\psi(\vec{r}, 0) = \int d^3 p \, g(\vec{p}) \, e^{i \frac{p \cdot \vec{r}}{\hbar}}.
\]
(12)
If a function \( \psi \) is a stationary solution then
\[
\psi_E(\vec{r}, t) = \phi_E(\vec{r}) \, e^{-i \frac{E}{\hbar} t},
\]
with \( \phi_E \) satisfying a stationary Schrödinger equation
\[
-\frac{\hbar^2}{2m} \Delta \phi_E + U(\vec{r}) \phi_E = E \phi_E.
\]
Eq. (12) gives now
\[
\phi_E(\vec{r}) = \int d^3 p \, g(\vec{p}) \, e^{i \frac{p \cdot \vec{r}}{\hbar}}.
\]
(13)
So we have got a general interpretation of the Fourier transform of a wave function. This gives the momentum distribution of a particle which was influenced by a potential and at time \( t = 0 \) was suddenly freed. There is no question here about the energy conservation because of the time-dependency of the hamiltonian (10).

Let us take as an example a well known Biblical story

B. How Goliath was defeated by David

David’s sling can be considered as a two-dimensional quantum rotator with a potential
\[
U(\vec{r}) = \frac{1}{2} m \omega^2 (x^2 + y^2).
\]
Stationary solutions corresponding to the energy
\[
E_{n_1, n_2} = \hbar \omega (n_1 + n_2 + 1),
\]
are given by
\[
\phi_{n_1, n_2}(x, y) = C_{n_1, n_2} e^{-\frac{m \omega}{\hbar} (x^2 + y^2)} H_{n_1} \left( x \sqrt{\frac{m \omega}{\hbar}} \right) H_{n_2} \left( y \sqrt{\frac{m \omega}{\hbar}} \right).
\]
(14)
If Goliath were hit directly by a stone still on a cord then he would absorb an impact energy \( E_{n_1, n_2} \). But if a stone was freed from the sling then its momentum distribution was given by the Fourier transform of the function (14). So the probability distribution to have a stone with a momentum between \( p \) and \( p + dp \) is proportional to
\[
e^{-\frac{p_x^2 + p_y^2}{m \omega \hbar}} \left( \frac{p_x}{\sqrt{m \omega \hbar}} \right)^2 H_{n_1}^2 \left( \frac{p_x}{\sqrt{m \omega \hbar}} \right) \left( \frac{p_y}{\sqrt{m \omega \hbar}} \right)^2.
\]
(15)
The corresponding impact energy is \( \frac{p^2}{2m} \), in general different from \( E_{n_1, n_2} \). It is easy to check that it is more probable to get the impact energy lower than \( E_{n_1, n_2} \). However there is finite, although exponentially decreasing probability, that a high momentum stone would be thrown. One should notice that such an effect is impossible for a classical (i.e. not-quantum) sling. An exponentially small probability was not a problem in the considered case taking into account David’s Protector. The crucial point was here a quantum nature of the sling.
III. MATHEMATICAL PRELIMINARIES

A cornerstone of quantum mechanics is a precise mathematical interpretation to the notion of observables. To each observable there corresponds a self-adjoint operator in the Hilbert state $\[19\]$. For unbounded symmetric operators there was a nontrivial problem to find all self-adjoint extensions but it was solved long ago $\[20, 21, 22\]$. To give a careful mathematical definition of operators related to observables it is not a matter of a mathematical pedantry. Even in the simplest case of one dimensional infinite square well a lack of precision leads to obvious paradoxes $\[9\]$.

Let us consider a differential operator $-i\frac{d}{dx}$ in the Hilbert space $L^2(0,a)$. Since:

$$\int_0^a dx f\frac{dg}{dx} = \bar{f}g|_0^a - \int_0^a dx \frac{df}{dx}g,$$

there are infinitely many self-adjoint extensions of the operator $-i d/dx$. These extensions are parameterized by a continuous parameter $\sigma \in [0, 2\pi)$ and are defined on domains $D_\sigma = \{ f: f(a) = e^{i\sigma}f(0) \}$. (16)

Corresponding eigenvalues are

$$\lambda^{(\sigma)}_n = \frac{\sigma}{a} + \frac{2\pi n}{a},$$

and normalized eigenfunctions

$$f^{(\sigma)}_n(x) = \begin{cases} \frac{1}{\sqrt{a}} e^{i\frac{\sigma}{2\pi}x}, & \text{for } 0 \leq x \leq a \\ 0, & \text{for } x \text{ everywhere else} \end{cases}$$

(18)

where $n = 0, \pm 1, \pm 2 \ldots$

Self-adjoint operators

$$\hat{p}(\sigma) = -i\hbar \frac{d}{dx},$$

defined on the domains $D_\sigma$ will henceforth be called $\sigma$-momentum operator. Standard solutions of the infinite potential well take as the “physical momentum” the operator $\hat{p}(0)$ and other extensions are simply rejected. We are going to show that other $\sigma$-momenta have also well established physical meaning.

To consider the energy operator one should look for a self-adjoint extension of the operator $d^2/dx^2$. Here the situation is more involved. It was shown $\[23, 24\]$ that domains of self-adjoint extensions are given by a set of boundary conditions

$$\begin{align*}
\alpha_{11} f(0) + \beta_{11} f(a) - \alpha_{12} f'(0) - \beta_{12} f'(a) &= 0, \\
\alpha_{21} f(0) + \beta_{21} f(a) - \alpha_{22} f'(0) - \beta_{22} f'(a) &= 0,
\end{align*}$$

(19a, 19b)

with coefficients $\alpha_{ij}$ and $\beta_{kl}$ satisfying

$$\begin{align*}
\alpha_{11}\delta_{12} - \alpha_{12}\delta_{11} &= \beta_{11}\delta_{12} - \beta_{12}\delta_{11}, \\
\alpha_{21}\delta_{22} - \alpha_{22}\delta_{21} &= \beta_{21}\delta_{22} - \beta_{22}\delta_{21}.
\end{align*}$$

(20a, 20b)

In the case of the infinite potential well $\[11\]$ a natural choice is to impose on the wave functions boundary conditions

$$f(0) = f(a) = 0,$$

(21a)

which are consistent with the continuity of the wave function. This choice corresponds to coefficients $\alpha_{ij}$ and $\beta_{kl}$

$$\begin{align*}
\alpha_{11} &= 1, & \beta_{11} &= -1, & \alpha_{12} &= 0, & \beta_{12} &= 0, \\
\alpha_{21} &= 1, & \alpha_{22} &= 0, & \beta_{21} &= 0, & \beta_{22} &= 0.
\end{align*}$$

(21b, 21c)
This means that functions satisfying boundary conditions form a domain \( D_{\Pi} \) of the self-adjoint extension of the operator \( d^2/dx^2 \). In the case of a particle on a circle a natural choice is to impose on the wave functions boundary conditions

\[
f(0) = f(a), \quad f'(0) = f'(a). \tag{22a}
\]

This choice corresponds to coefficients \( \alpha_{ij} \) and \( \beta_{kl} \)

\[
\begin{align*}
\alpha_{11} &= 1, & \beta_{11} &= -1, & \alpha_{12} &= 0, & \beta_{12} &= 0, \\
\alpha_{21} &= 0, & \alpha_{22} &= 1, & \beta_{21} &= 0, & \beta_{22} &= -1. 
\end{align*}
\tag{22b}\tag{22c}
\]

It is remarkable that the intersection of all admissible domains of \( \sigma \)-momenta is

\[
\bigcap_{\sigma} D_{\sigma} = \{ f : f(a) = f(b) = 0 \} = D_{\Pi}. \tag{23}
\]

This property makes the extension exceptional, at least from the point of view of momentum operators. A kinetic term \( d^2/dx^2 \) with this domain is well defined in (not on!) domains of all \( \sigma \)-momenta.

\( D_{\Pi} \) is a dense set in the Hilbert space \( L_2(0, a) \) as the domain of a self-adjoint operator. This set is too small, however, to define on it a self-adjoint extension of the operator \( id/dx \). But the property together with the general theorem allows to write

**Theorem.** Any function from the domain of a self-adjoint operator \( A \) can be expanded in an uniformly convergent series of eigenfunctions of this operator.

**Corollary 1.** Any energy eigenfunction can be expanded in an uniformly convergent series of eigenfunctions of any \( \sigma \)-momentum.

We have also

**Corollary 2.** \( \sigma \)-momentum eigenfunctions cannot be represented as uniformly convergent series of energy eigenfunctions.

Both corollaries can be stated as follows

*In the infinite potential well \( \sigma \)-momentum representations of stationary states are always uniformly convergent. Energy representations of \( \sigma \)-momentum eigenfunctions are not uniformly convergent."

Let us make a mathematical exercise to calculate

**A. \( \sigma \)-momentum representation of stationary states**

We have

\[
\sqrt{\frac{2}{a}} \sin \frac{N\pi x}{a} = e^{i\pi x} \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{+\infty} c_n(\sigma)e^{i\frac{2\pi n}{a} x}. \tag{24}
\]

Coefficients \( c_n(\sigma) \) are given here as

\[
\frac{\sqrt{2}}{a} \int_0^a dx e^{-i\frac{\pi x}{a} + \frac{2\pi n}{a} x} \sin \frac{\pi N x}{a} = \pi N \sqrt{2} \frac{e^{-i\sigma}(-1)^N - 1}{(\sigma + 2\pi n)^2 - \pi^2 N^2}. \tag{25}
\]

It is convenient to discuss cases of even and odd \( N \) separately.

If \( N = 2r \) we can write

\[
c_n(\sigma) = -4i\pi r \sqrt{2} e^{-i\frac{\sigma}{2}} \frac{\sin \frac{\pi}{2}}{(\sigma + 2\pi n)^2 - 4\pi^2 r^2}, \tag{26}
\]
A special care is needed when the nominator of this expression is equal to zero. For \( \sigma = 0 \) one gets then

\[
c_n(0) = \frac{1}{i\sqrt{2}} \begin{cases} 
1, & \text{for } n = r, \\
-1, & \text{for } n = -r, \\
0, & \text{in other cases}.
\end{cases}
\]  

(27)

After substitution to Eq. (24) this gives a consistency check

\[
\psi_{2r}(x) = \frac{1}{2i} \sqrt{\frac{2}{a}} \left( e^{i \frac{2\pi n}{a} x} - e^{-i \frac{2\pi n}{a} x} \right).
\]  

(28)

For nonzero \( \sigma \) we have

\[
\psi_{2r}(x) = -4\pi i r \sqrt{\frac{2}{a}} e^{-i \frac{\pi}{4}} e^{i \frac{\pi}{2} x} \sin \frac{\sigma}{2} \sum_{n=-\infty}^{+\infty} \frac{e^{i \frac{2\pi n}{a} x}}{(\sigma + 2\pi n)^2 - \pi^2 (2r + 1)^2}.
\]  

(29)

If \( N = 2r + 1 \) we can write Eq. (25) as

\[
c_n(\sigma) = -2\pi(2r + 1) \sqrt{\frac{2}{a}} e^{-i \frac{\pi}{4}} e^{i \frac{\pi}{2} x} \cos \frac{\sigma}{2} \sum_{n=-\infty}^{+\infty} \frac{e^{i \frac{2\pi n}{a} x}}{(\sigma + 2\pi n)^2 - \pi^2 (2r + 1)^2}.
\]  

(30)

For \( \sigma = \pi \) one gets similarly like in Eqs (27)

\[
c_n(\pi) = \frac{1}{i\sqrt{2}} \begin{cases} 
1, & \text{for } n = r, \\
-1, & \text{for } n = -r - 1, \\
0, & \text{in other cases}.
\end{cases}
\]  

(31)

This gives, similarly like in Eq. (28),

\[
\psi_{2r+1}(x) = \frac{1}{2i} \sqrt{\frac{2}{a}} \left( e^{i \frac{2\pi n}{a} x} - e^{-i \frac{2\pi n}{a} x} \right).
\]  

(32)

For \( \sigma \neq \pi \) we have

\[
\psi_{2r+1}(x) = -2\pi(2r + 1) \sqrt{\frac{2}{a}} e^{-i \frac{\pi}{4}} e^{i \frac{\pi}{2} x} \cos \frac{\sigma}{2} \sum_{n=-\infty}^{+\infty} \frac{e^{i \frac{2\pi n}{a} x}}{(\sigma + 2\pi n)^2 - \pi^2 (2r + 1)^2}.
\]  

(33)

All these mathematical expansions from Eqs (28), (29), (32), and (33) would have physical meaning with a satisfactory physical interpretation of \( \sigma \)-momenta. This will be done in the next section. It should be now noted that the choice \( \sigma = 0 \) gives expansions of the potential well stationary states into momentum eigenfunctions. The momentum spectrum is given then by Eq. (17) with \( \sigma = 0 \). An elusively simple equation (9) is not always a momentum expansion because \( N\pi/a \) are allowed momenta only for even \( N \). Only in such cases a stationary state can be visualized as the superposition of two waves with opposite momenta. For odd \( N \) the momentum expansion is

\[
\psi_{2r+1}(x) = -2\pi(2r + 1) \sqrt{\frac{2}{a}} e^{-i \frac{\pi}{4}} e^{i \frac{\pi}{2} x} \sum_{n=-\infty}^{+\infty} \frac{e^{i \frac{2\pi n}{a} x}}{4\pi^2 n^2 - \pi^2 (2r + 1)^2},
\]  

(34)

with a much richer structure.

More detailed analysis of this problem will be performed in Section V C.

**IV. MOMENTUM SEEN FROM THE MOVING REFERENCE FRAME**

We are going to find physical meaning of different self-adjoint extensions of the operator \(-id/dx\). It is a standard procedure to identify this operator with the translation generator. It is not enough, however, to relate this to the physical momentum. The same differential operator can be also related to a component of the angular momentum even for the same boundary conditions.
Let us consider as an example the operator $-i\hbar d/dx$ in the Hilbert space $L^2(0,2\pi)$ defined on the domain

$$D_0 = \{ f : f(2\pi) = f(0) \}.$$  

A spectrum of this operator

$$\lambda_n^{(0)} = n\hbar ,$$

and normalized eigenfunctions

$$f_n^{(0)}(x) = \frac{1}{\sqrt{2\pi}} e^{inx},$$

are the same both for the momentum in the interval $(0,2\pi)$ as for the third component of the angular momentum when the variable $x \in (0,2\pi)$ is interpreted as an angular variable.

To get a momentum operator proper transformation properties are needed, specific for the corresponding classical variable. Let us consider two coordinate systems $O(x,t)$ and $O'(\zeta,\tau)$ related by the Galilei transformation

$$x = \zeta + V \tau ; \quad t = \tau .$$

The following discussion is based on the “passive point of view” when the same system is observed by different observers $A$ in $O$ and $A'$ in $O'$ having different relations to the system.

Let $-i\hbar d/dx$ be a momentum operator in the system $O$ and let $f_\lambda$ be an eigenfunction of the momentum operator associated with the eigenvalue $\lambda$. The momentum operator should fulfill the following conditions:

1. a momentum operator has the same structure in all inertial systems i.e.

$$-i\hbar d/d\zeta$$

is a momentum operator in the coordinate system $O'$,  

2. a physical state with a defined momentum in one inertial system has a definite momentum in any inertial system i.e.

$$f_\lambda$$

is transformed into $\tilde{f}_\lambda$:

$$-i\hbar \frac{d}{d\zeta} f_\lambda = \tilde{\lambda} \tilde{f}_\lambda ,$$

3. eigenvalues of the momentum operator transform under the Galilei transformation like their classical counterparts i.e.

$$\tilde{\lambda} = \lambda - mV .$$

We make an ansatz

$$\tilde{f}_\lambda(\zeta,\tau) = e^{ig(\zeta,\tau)} f(\zeta + V \tau) .$$

This gives

$$-i\hbar \frac{d}{d\zeta} = h \frac{dg}{d\zeta} e^{ig} f - ihe^{ig} \frac{df_\lambda}{dx}$$

$$= h \frac{dg}{d\zeta} f + \lambda \tilde{f} = \tilde{\lambda} \tilde{f} .$$

The correspondence rule gives

$$h \frac{dg}{d\zeta} + \lambda = \lambda - mV .$$

A general solution has a form

$$g(\zeta,\tau) = -\frac{mV}{h} \zeta + T(\tau) ,$$
where $T$ is an arbitrary function of the variable $\tau$.

Starting from the consistency conditions (39) we have obtained a general transformation rule for momentum eigenfunctions under the Galilei transformation

$$\tilde{f}_\lambda(\zeta, \tau) = e^{-i\left(\frac{mV}{\hbar}\zeta - T(\tau)\right)} f_\lambda(\zeta + V\tau).$$

Because of properties of self-adjoint operators this rule gives transformation rules for any vector from the Hilbert space.

If the momentum operator has a point spectrum, then its eigenfunctions form a base in a Hilbert space. This is a case for the $L^2_{(0,a)}$ space. Any element $u$ of this space can be expanded as

$$u(x) = \sum_\lambda c_\lambda \tilde{f}_\lambda(x).$$

(45)

It follows from this that an observer from the Galilei transformed reference frame (38) sees this vector as

$$\tilde{u}(\zeta, \tau) = e^{-i\left(\frac{mV}{\hbar}\zeta - T(\tau)\right)} \sum_\lambda c_\lambda f_\lambda(\zeta + V\tau).$$

(46)

A function $T$ is fixed by subsidiary conditions fulfilled by the function $u$. It will be shown in Section V that for one particle Schrödinger equation a function $T(\tau)$ has a form $-mV^2 \tau/2$.

A result (46) can be easily generalized to the case of a continuous spectrum of the momentum operator. That is a standard mathematical procedure [21, 22, 26] equivalent to the replacement of the sum in Eq. (45) by the Fourier integral.

For the moment we restrict ourselves to

### A. Momentum inside the infinite potential well

The momentum observable “at rest” is $-i\hbar d/dx$ with the domain

$$D_0 = \{ f : f(a) = f(0) \}.$$

(47)

Boundary conditions in the moving reference frame $O'(\zeta, \tau)$ are given at points $\zeta = -V\tau$ and $\zeta = -V\tau + a$. Using transformations rules given by Eqs (41) and (43) one gets the boundary conditions for the function $\tilde{f}$

$$\tilde{f}(-V\tau + a, \tau) = e^{-i\frac{mV}{\hbar}a} \tilde{f}(-V\tau, \tau).$$

(48)

This gives us a direct interpretation of $\sigma$-momenta:

$$\sigma$$-momentum it is the momentum observable measured by the observer moving with a velocity $V$ such that

$$\sigma = -\frac{mV}{\hbar}a \mod 2\pi.$$

(49)

To make the $\sigma$-momentum a real quantum mechanical momentum one should check the validity of Canonical Commutation Relations (CCR) of a momentum and a position operator in our system. In general, a problem of CCR on the finite interval is far from being obvious [27]. As opposed to the entire real line case where both position $\hat{X}$ and momentum $\hat{P}$ operators are unbounded, here the operator $\hat{X}$ is bounded in the Hilbert space $L_2(0,a)$ whereas the operator $\hat{P} = -i\hbar \partial$ is unbounded. This leads to technical troubles related to the domain $D([\hat{X}, \hat{P}])$ of the commutator $[\hat{X}, \hat{P}]$. The domain where CCR are fulfilled is

$$D(\hat{P}\hat{X}) \cap D(\hat{X}\hat{P}),$$

(50)

where $D(\hat{P}\hat{X})$ and $D(\hat{X}\hat{P})$ are domains of operator products $\hat{P}\hat{X}$ and $\hat{X}\hat{P}$ correspondingly.

The domain $D(\hat{P}\hat{X})$ is

$$D(\hat{P}\hat{X}) = \{ f : (\hat{X}f) \in D(\hat{P}) \},$$

(51)
and the domain of the product $\hat{X}\hat{P}$ is equal here to the domain of the momentum operator because of the boundedness of the position operator.

For the $\sigma$-momentum $\hat{p}_\sigma$ the domain $D_\sigma$ is given by (11). Then
\[ D([\hat{X}, \hat{p}_\sigma]) = \{ f : (\hat{X} f)(a) = e^{\sigma} (\hat{X} f)(0) \} \cap \{ f : f(a) = e^{\sigma} f(0) \} = \{ f : f(a) = f(b) = 0 \}. \] (52)

So CCR are realized on the dense domain in $L^2(0, a)$. This domain does not depend on the $\sigma$-realization of the momentum operator and coincides with the domain $D_\Pi$ of the energy operator.

Different $\sigma$-momenta, as corresponding to unitary non-equivalent projective representation of the Galilei group, correspond to different unitary non-equivalent representations of CCR although all are realized on the same dense domain $D_\Pi$.

Coming back to our quantum squash model: a player running with the velocity $V$ sees a squash ball having $\sigma$-momentum $\sigma$ is given here by Eq. (19). H(is)er momentum eigenfunctions take on the form
\[ \tilde{f}_n(\zeta, \tau) = e^{-i\frac{\pi n}{a}} e^{i\frac{\pi n}{a} (\zeta + V \tau)} = e^{\frac{\pi n}{a} (-mV) \zeta} e^{i\frac{\pi n}{a} V \tau}. \] (53)

It is also interesting to look for solutions of the infinite potential well observed by a running player. This will be the subject of the next section.

V. SCHRODINGER EQUATION SEEN FROM THE MOVING REFERENCE FRAME

Let us consider a particle subjected to the influence of a time-dependent potential $U$. In the coordinate system $O(x, t)$ the Schrödinger equation takes on the form
\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U(x, t) \Psi. \] (54)

This equation is obviously not Galilei-invariant unless the potential is a trivial constant. A typical procedure is to investigate physical consequences of the symmetry group starting from the symmetry-invariant equations. In the case of the Galilei (or Poincaré) group this leads to a free particle wave function realizing a unitary representation of the group. For the Galilei group and the Schrödinger equation one gets [11, 12, 13, 14] that only nontrivial projective representations are physical realizations of the symmetry.

We are going to consider a more general approach based on the equivalence of all inertial coordinate systems. This Galilean equivalence principle demands that all laws of physics take the same form in different frames connected by the Galilei (or Poincaré for relativistic theory) transformations. We derive from the postulates that “the Galilei transformation is true” and “the Schrödinger equation is true” the transformation law of wave function for any scalar potential. So we do, what Galilei would do, “if Galilei had know quantum mechanics” [23].

An observer in the reference frame $O'(\zeta, \tau)$ sees the potential $U$ as
\[ \tilde{U}(\zeta, \tau) = U(\zeta + V \tau, \tau). \] (55)

It is assumed here that the potential is a scalar with respect to the Galilei transformation [33]. The equivalence principle demands that a wave function $\tilde{\Psi}(\zeta, \tau)$ viewed by an observer $A'$ in the coordinate system $O'$ satisfies the Schrödinger equation
\[ i\hbar \frac{\partial \tilde{\Psi}}{\partial \tau} = -\frac{\hbar^2}{2m} \frac{\partial^2 \tilde{\Psi}}{\partial \zeta^2} + \tilde{U}(\zeta, \tau) \tilde{\Psi}. \] (56)

An ansatz
\[ \tilde{\Psi}(\zeta, \tau) = e^{iu(\zeta, \tau)} \Psi(\zeta + V \tau, t), \] (57)
gives
\[ \frac{\partial \tilde{\Psi}}{\partial \tau} = ie^{iu} \frac{\partial \Psi}{\partial \tau} + e^{iu} \frac{\partial^2 \Psi}{\partial \zeta^2} V + e^{iu} \frac{\partial \Psi}{\partial \zeta}, \]
\[ \frac{\partial \tilde{\Psi}}{\partial \zeta} = ie^{iu} \frac{\partial \Psi}{\partial \zeta} + e^{iu} \frac{\partial^2 \Psi}{\partial \zeta^2} V, \]
\[ \frac{\partial^2 \tilde{\Psi}}{\partial \zeta^2} = ie^{iu} \frac{\partial^2 \Psi}{\partial \zeta^2} V - e^{iu} \left( \frac{\partial \Psi}{\partial \zeta} \right)^2 + 2ie^{iu} \frac{\partial \Psi}{\partial \zeta} \frac{\partial \Psi}{\partial \zeta} + e^{iu} \frac{\partial^2 \Psi}{\partial \zeta^2}. \]
We see that Eq. (56) is fulfilled if
\[ i\hbar \frac{\partial \Psi}{\partial x} V = -\frac{\hbar^2}{2m} 2i \frac{\partial u}{\partial \zeta} \frac{\partial \Psi}{\partial x}, \] (58a)
and
\[ -\hbar \frac{\partial u}{\partial \tau} = \frac{\hbar^2}{2m} \frac{m^2}{\hbar^2} V^2. \] (58b)

A solution of Eqs (58) takes on the form
\[ u(\zeta, \tau) = -\frac{m}{\hbar} V \zeta - \frac{mV^2}{2\hbar} \tau + C(V). \] (59)

So wave functions in different inertial reference frames connected by the Galilei transformation (38) are connected (up to the constant phase factor \( e^{iC} \)) by the relation
\[ \tilde{\Psi}(\zeta, \tau) = e^{-\frac{i}{\hbar}(mV \zeta + \frac{mV^2}{2\hbar})} \Psi(x, t). \] (60)

We have got the same factor as obtained by Bargmann [12] for a free Schrödinger particle subjected to the Galilei transformation. This factor leads to the mass-superselection rule what is mathematically due to the fact that projective (ray) representations of the Galilei group are not unitary equivalent to the usual representations [13].

### A. Stationary states seen from the moving reference frame

Let us consider now a particle in a static potential \( U(x) \) described by a stationary wave function
\[ \Psi_n(x, t) = \psi_n(x) e^{-\frac{i}{\hbar} E_n t}, \] (61)
\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi_n}{dx^2} + U(x) \psi_n(x) = E_n \psi_n(x). \] (62)

According to the general rule (60), this state when viewed by a moving observer from the reference frame \( O'(\zeta, \tau) \) is described by a wave function
\[ \tilde{\Psi}_n(\zeta, \tau) = e^{-\frac{i}{\hbar}(mV \zeta + \frac{mV^2}{2\hbar})} \psi_n(x, t). \] (63)

One should note that this is not an energy eigenstate. This follows, at least formally, from the fact that Galilei transformed potential \( \tilde{U}(\zeta, \tau) \) is now time dependent. The energy, however, is still conserved. To check this let us calculate an average value of the energy for the state described by the wave function (63). It is given as
\[ \langle E \rangle_n = i\hbar \int d\zeta \bar{\tilde{\Psi}}_n^*(\zeta, \tau) \frac{\partial}{\partial \tau} \tilde{\Psi}_n(\zeta, \tau). \] (64)

Taking into account that solutions of Eq. (62) are real one obtains then
\[ \langle E \rangle_n = E_n + \frac{mV^2}{2}. \] (65)

This result seems to be surprising even in the simplest case of a free particle with the momentum \( p \). Taking into account that the energy \( E = p^2/2m \) and the momentum is Galilei transformed to \( p - mV \) one should expect in Eq. (65) a subsidiary term of the form \( -pV \). However, this is not the case. Energy \( p^2/2m \) is “produced” by a particle with the momentum \( \pm p \). A real stationary state is a superposition of two waves corresponding to opposite momenta \( \pm p \). They are Galilei transformed to \( p \mp mV \) correspondingly and give contributions to the energy \( p^2/2m + mV^2/2 \mp pV \). These are additive contributions to the total energy so terms \( \pm pv \) cancel each other. This remark gives a perfect agreement of Eq. (65) with our classical intuition although, as we’ll see later, does not always agree with a quantum reality.
B. Infinite potential well seen from the moving reference frame

When the potential well (1) is observed from the moving reference frame $O'$ it is seen as

$$U(\zeta + V\tau) = \begin{cases} 0; & \text{if } -V\tau \leq \zeta \leq a - V\tau, \\ \infty; & \text{if } \zeta \notin (-V\tau, a - V\tau). \end{cases}$$  \hspace{1cm} (66)$$

The wave function satisfies boundary conditions

$$\tilde{\Psi}(-V\tau, \tau) = \tilde{\Psi}(a - V\tau, \tau) = 0.$$  \hspace{1cm} (67)

The solution \[\text{(63)}\] has now a form

$$\tilde{\Psi}_N(\zeta, \tau) = \begin{cases} \sqrt{\frac{2}{a}} e^{-\frac{1}{2}mV\zeta} \sin \frac{N\pi}{a} (\zeta + V\tau) e^{-\frac{1}{\hbar}(\frac{2a^2}{ma^2} N^2 + \frac{m}{2})} \tau; & \text{if } -V\tau \leq \zeta \leq a - V\tau, \\ 0; & \text{if } \zeta \notin (-V\tau, a - V\tau). \end{cases}$$  \hspace{1cm} (68)

where $N = 1, 2, \ldots$

This solution can be written in the region $-V\tau \leq \zeta \leq a - V\tau$ as a superposition of two plain waves

$$\tilde{\Psi}_N(\zeta, \tau) = \psi_{(+),N}(\zeta, \tau) - \psi_{(-),N}(\zeta, \tau)$$  \hspace{1cm} (69a)

where

$$\psi_{(+),N}(\zeta, \tau) = \frac{1}{2i} \sqrt{\frac{2}{a}} e^{\frac{1}{\hbar}(\frac{2a^2}{ma^2} - mV) \zeta} e^{-\frac{1}{2ma}(\frac{na}{\hbar} - mV)^2 \tau},$$  \hspace{1cm} (69b)

$$\psi_{(-),N}(\zeta, \tau) = \frac{1}{2i} \sqrt{\frac{2}{a}} e^{-\frac{1}{\hbar}(\frac{2a^2}{ma^2} + mV) \zeta} e^{-\frac{1}{2ma}(\frac{na}{\hbar} + mV)^2 \tau}.$$  \hspace{1cm} (69c)

This decomposition confirms our semiclassical understanding of the quantum problem. Alas, our semiclassical understanding does not quite agree with the mathematics behind the scene.

C. Moving observer measures momentum in the well

A measurement of an observable is mathematically equivalent to the spectral decomposition of the wave function into corresponding eigenfunctions. A general mathematical decomposition into momentum eigenfunctions was done in Section III A. A physical problem: “what are possible momenta measured by a moving observer in the infinite potential well?” will be solved when $h(is)er$ wave function $\tilde{\Psi}_N(\zeta, \tau)$ \[\text{(68)}\] will be expanded into $h(is)er$ momentum eigenfunctions $\tilde{f}_n(\zeta, \tau)$ \[\text{(68)}\]

$$\tilde{\Psi}_N(\zeta, \tau) = \sum_{n=-\infty}^{+\infty} c_{n}^{(N)}(\tau) \tilde{f}_n(\zeta, \tau).$$  \hspace{1cm} (70)

Coefficients $c_{n}^{(N)}$ are calculated as

$$c_{n}^{(N)}(\tau) = \sqrt{\frac{2}{a}} e^{-\frac{1}{\hbar}(\frac{2a^2}{ma^2} N^2 + \frac{m}{2})} \tau \int_{-V\tau}^{-V\tau+\alpha} d\zeta \sin \frac{N\pi}{a} (\zeta + V\tau) e^{-\frac{2\alpha}{\hbar}(\zeta + V\tau)}.$$  \hspace{1cm} (71)

This gives for even $N$ a simple expression, consistent with a semiclassical approach

$$\tilde{\Psi}_N(\zeta, \tau) = \frac{1}{2i} \sqrt{\frac{2}{a}} e^{\frac{1}{\hbar}(\frac{2a^2}{ma^2} - mV) \zeta} e^{-\frac{1}{2\hbar}(V - \frac{n\alpha}{\hbar})^2 \tau} - e^{\frac{1}{\hbar}(\frac{2a^2}{ma^2} + mV) \zeta} e^{-\frac{1}{2\hbar}(\frac{n\alpha}{\hbar} + mV)^2 \tau}$$  \hspace{1cm} (72)
The case of odd \( N \) is more involved and the momentum expansion \( \Psi_N \) takes on the form

\[
\Psi_N(\zeta, \tau) = \frac{2}{\pi N} e^{-i mV \zeta} e^{-\frac{1}{2} \left( \frac{2\zeta^2}{2na^2} + \frac{mV^2}{2} \right) \tau} e^{-i mV \zeta} e^{-\frac{1}{2} \left( \frac{2\zeta^2}{2na^2} + \frac{mV^2}{2} \right) \tau} \sum_{n=1}^{+\infty} \frac{1}{4a^2 - N^2} \left[ e^{i \frac{2\pi}{a} V (\zeta + V \tau)} + e^{-i \frac{2\pi}{a} V (\zeta + V \tau)} \right],
\]

(73a)

This can be also written as

\[
\Psi_N(\zeta, \tau) = \frac{4N}{\pi} \sqrt{2} e^{-i \frac{2\pi}{a} V (N^2 - 4n^2) \tau} \frac{e^{\frac{i 2mV}{2na^2} N^2 - 4n^2}}{N^2 - 4n^2} e^{\frac{i 2mV}{a} (\zeta + V \tau)} e^{-i \frac{2\pi}{a} V (\zeta + V \tau)},
\]

(73b)

where contributions from stationary plane waves states are explicitly selected. We shall call from this time on a “stationary plane wave state” a plane wave with an explicit time dependence, \( i.e. \), a function of the form

\[
e^{-\frac{2\pi}{a} V t} \Psi \tilde{\Psi}.
\]

(74)

There is a striking difference in the behavior of odd- and even- \( N \) states \( \Psi_N \). Any even \( N \) state is a superposition of two stationary plane waves states, while an odd- \( N \) state cannot be represented as a superposition of stationary plane wave states. An underlying mechanism is the same which made difference between Eqs (28) and (34) — allowed momenta in the infinite well are not always the same as formal arguments of the energy eigenfunctions.

This can be changed with a change of boundary conditions \( \Psi (2) \). If they are replaced by periodic-type conditions

\[
\psi(0) = \psi(a), \quad \psi'(0) = \psi'(a),
\]

(75)

then eigenfunctions of the Hamiltonian are

\[
\sqrt{\frac{2}{a}} \sin \frac{2N\pi}{a} x, \quad \sqrt{\frac{2}{a}} \cos \frac{2N\pi}{a} x.
\]

(76)

We see that for such boundary conditions allowed momenta in the infinite well are always the same as formal arguments of energy eigenfunctions and any state \( \Psi(16) \) is a superposition of two plane waves with the definite momenta. So, in a sense, a situation of periodic boundary conditions is “better” than for boundaries.

It is easy to create a ”worse” situation. To this end it is enough to take antiperiodic-type boundary conditions

\[
\psi(0) = -\psi(a), \quad \psi'(0) = -\psi'(a).
\]

(77)

Then eigenfunctions of the Hamiltonian are

\[
\sqrt{\frac{2}{a}} \sin \frac{(2N + 1)\pi}{a} x, \quad \sqrt{\frac{2}{a}} \cos \frac{(2N + 1)\pi}{a} x.
\]

(78)

For such boundary conditions allowed momenta in the infinite well are never the same as formal arguments of energy eigenfunctions and there is no state \( \Psi(78) \) as a superposition of two plane waves with the definite momenta.

VI. CONCLUSIONS

We performed a careful analysis of the notion of an observable related to the physical momentum. For the beginning one should identify a self-adjoint operator connected to this notion. In quantum mechanics defined on a finite interval a situation is more complicated than in the case of an unrestricted theory on \( \mathbb{R} \) because there is a continuum \( \Pi(10) \) of self-adjoint extensions of the differential operator \( id/dx \).

If you want to go beyond an argument that \(-i\hbar d/dx\) is a physical momentum when it is assigned to the letter “\( p \)”, and it is an angular momentum when assigned to the sign “\( l_z \)”, then appropriate transformation properties must be taken into account. In a similar manner, three numbers can be a finite three-elements set or...
components of three dimensional vector. A choice depends on assumed transformation properties with respect to rotations. In the momentum case, transformation properties are given (in a nonrelativistic approach) by the Galilei group as was done by conditions (39). All this together, supplemented with the equivalence of all inertial coordinate systems, led to different realizations of the momentum operator in the finite volume — in Section V C. Those different realizations have different spectra. This is quite obvious from the physical point of view.

Such an approach, based on a generalized correspondence principle (39) gave a physical interpretation of all self-adjoint extensions of the operator \(-i\hbar d/dx\). It was also shown that those different extensions realize different non-unitary equivalent representations of CCR on the universal dense domain.

Obtained results can be generalized for a three dimensional rectangular box and the momentum operator \(-i\hbar \nabla\).

Results of Section V show that important physical properties, related to transformation laws of wave functions, can be obtained under much weaker assumption than it was done in the past. A condition of Galilei invariance, widely used to obtain mass superselection rule, is replaced by (generalized) Galilei equivalence principle. This allows to go beyond a free particle theory and gives results also for an arbitrary scalar potential.

The transformation law (63) can be treated as a realization of different self-adjoint extensions of the Hamiltonian. This is clearly visible in the infinite potential well where Eqs (19) and (20) give different self-adjoint extensions of the operator \(d^2/dx^2\). However, a situation here is not such simple as in the case of momentum operators. A structure of self-adjoint extensions of the operator \(d^2/dx^2\) is much richer than in the case of the momentum operator. Only a part of those extensions can be related to the Galilei transformations and these are done by Eq. (68).

Results related to the momentum distribution, obtained in Sections III and V C need some comments. We have shown that the Fourier integral of the stationary wave function is directly related to the time-dependent dynamics given by Eq. (10). This gives a direct interpretation of that, what is usually called “momentum representation of the wave function” or “wave function in momentum space”. This interpretation is different from that what is usually found in textbooks. A simple statement that a wave function in momentum space

\[
\Phi(\vec{p}, t) = \int d^3r \Psi(\vec{r}, t) e^{-i \vec{p} \cdot \vec{r}},
\]

is a probability amplitude to measure the momentum \(\vec{p}\) at time \(t\) is simply not true! It sometimes reasonable, for technical reasons, to use a momentum representation of the wave function but only because of its mathematical equivalence to the wave function.

There is an exception in the “finite Universe” with dynamics defined on a finite interval. Let us introduce here two notions of momentum representation. The first is analogous to the previous one. You take the Fourier integral and you obtain a dynamics as in Eq. (10). This means that at \(t = 0\) all impenetrable walls vanish and you are left with a free particle. Such a situation is used in the HBT effect [29], originally invented to determine the dimensions of distant astronomical objects. This method is widely used in high energy hadronic interaction to obtain an information about the geometric properties of the source. Multi-pion and photon spectra provide precise information about reaction space time geometry in hadron-hadron and heavy ion collisions [30, 31].

Another concept of momentum representation — let call it “momentum distribution” — means to expand a wave function into stationary plane wave states (74). It was shown in Eq. (72) that this was possible for even-\(N\) states and impossible for odd-\(N\) states.

Such a momentum distribution in the “infinite Universe” would mean that

\[
\Psi(\vec{r}, t) = \int d^3p \Phi(\vec{p}, t) e^{i \vec{p} \cdot \vec{r}} e^{-i \frac{p^2}{2m} t},
\]

what is in general not possible.

Acknowledgments

This paper is dedicated to Professor Jan Lopuszański on his 80th birthday. Author is indebted to Professor Lopuszański for his whole life attitude which is a good example of wisdom, courage, and a sense of humor.

I am grateful to R. Olkiewicz for interesting and inspiring discussions.

This work is partially supported by the Polish Committee for Scientific Research under contract KBN 2 P03B 069 25.
[1] M. V. Berry, J. Phys. A 29, 6617 (1996).
[2] D. Wójcik, I. Białynicki-Birula, and K. Życzkowski, Phys. Rev. Lett. 55, 5022 (2000), quant-ph/0005060.
[3] B. Hu, B. Li, J. Liu, and Y. Gu, Phys. Rev. Lett. 82, 4224 (1999), chao-dyn/9903006.
[4] D. L. Aronstein and C. R. Stroud, Jr., Phys. Rev. A 55, 4526 (1997).
[5] R. W. Robinett, Am. J. Phys. 68, 410 (2000).
[6] J. T. Verdeyen, Laser Electronics (Prentice Hall, Upper Saddle River, NJ, 1995), 3rd ed.
[7] Y. Arakawa and A. Yariv, IEEE Journal of Quantum Electronics QE-22, 1887 (1986).
[8] J.-P. Antoine, J.-P. Gazeau, P. Monceau, J. R. Klauder, and K. A. Penson, J. Math. Phys. 42, 2349 (2001), math-ph/0012044.
[9] G. Bonneau, J. Faraut, and G. Valent, Am. J. Phys. 69, 322 (2001), quant-ph/0103153.
[10] P. Garbaczewski and W. Karwowski, Impenetrable barriers and canonical quantization (2001), math-ph/0104010.
[11] E. Inönü and E. P. Wigner, Nuovo Cimento 9, 705 (1952).
[12] V. Bargmann, Ann. Math. 59, 1 (1954).
[13] M. Hamermesh, Ann. Phys., N. Y. 9, 518 (1960).
[14] J.-M. Levy-Leblond, Journ. Math. Phys. 4, 776 (1963).
[15] J.-M. Levy-Leblond, Riv. Nuovo Cimento 4, 99 (1974).
[16] D. Giulini, Ann. of Phys. 249, 222 (1996).
[17] L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Nonrelativistic Theory (Pergamon, Oxford, 1977), 3rd ed.
[18] C. Cohen-Tannoudji, B. Diu, and F. Laloé, Quantum Mechanics, vol. 1 (John Wiley and Sons, New York - London - Sydney - Toronto, 1977).
[19] J. von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, 1955), Mathematische Grundlagen der Quantenmechanik (Julius Springer, Berlin, 1932).
[20] J. von Neumann, Math. Ann. 102, 49 (1929).
[21] N. Dunford and J. T. Schwartz, Linear Operators (Interscience Publishers, New York - London, 1958).
[22] M. A. Naimark, Linear Differential Operators (Nauka, Moscow, 1969), 2nd ed., In Russian.
[23] M. A. Naimark, chap. V §18.2, in 22 (1969).
[24] N. Dunford and J. T. Schwartz, Self Adjoint Operators in Hilbert Space, chap. 12, vol. 2 of 21 (1963).
[25] M. A. Naimark, chap. III §9, in 22 (1969).
[26] F. Riesz and B. Sz.-Nagy, Leçons d’Analyse Fonctionnelle (Akadémiai Kiadó, Budapest, 1972).
[27] G. Lassner, G. A. Lassner, and C. Trapani, J. Math. Phys. 28, 174 (1987).
[28] F. A. Kämpferr, Concepts in Quantum Mechanics (Academic Press, New York and London, 1965), Appendix 7.
[29] R. Hanbury Brown and R. Q. Twiss, Nature (London) 178, 1046 (1956).
[30] U. A. Wiedemann and U. W. Heinz, Phys. Rept. 319, 145 (1999), nucl-th/9901094.
[31] R. M. Weiner, Phys. Rept. 327, 249 (2000), hep-ph/9904389.