D-branes of A-type, their deformations, and Morse cobordism of A-branes on Calabi-Yau 3-folds under a split attractor flow:
Donaldson/Alexander-Hilden-Lozano-Montesinos-Thurston/
Hurwitz/Denef-Joyce meeting Polchinski-Grothendieck

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Abstract

In [L-Y5] (D(6): arXiv:1003.1178 [math.SG]) we introduced the notion of Azumaya \(C^\infty\)-manifolds with a fundamental module and morphisms therefrom to a complex manifold. In the current sequel, we use this notion to give a prototypical definition of supersymmetric D-branes of A-type (i.e. A-branes) – in an appropriate region of the Wilson’s theory-space of string theory – as special Lagrangian morphisms with a unitary, minimally flat connection-with-singularity. This merges Donaldson’s picture of special Lagrangian submanifolds (1999) and the Polchinski-Grothendieck Ansatz for D-branes in a Calabi-Yau space (Sec. 2.1). The Higgsing/un-Higgsing and the large- vs. small-brane wrapping of A-branes in string theory can be achieved via deformations of such morphisms (Sec. 2.2 and Sec. 2.3). For the case of Calabi-Yau 3-folds, classical results of Alexander (1920), Hilden (1974) and Montesinos (1976), Thurston (1982), and Hilden-Lozano-Montesinos (1983) on 3-manifolds branched-covering \(S^3\) implies that any embedded special Lagrangian submanifold with a complex vector bundle with a unitary flat connection on a Calabi-Yau 3-fold is the image of a special Lagrangian morphism from an Azumaya 3-sphere with a fundamental module, with a unitary minimally flat connection. This suggests a genus-like expansion of the path-integral of D3-branes in type IIB string theory compactified on Calabi-Yau 3-folds that resembles the genus expansion of the path-integral of strings (Sec. 2.4.2). Similarly, for the path-integral of D2-branes and M2-branes respectively. In Sec. 3, we use the technical results of Joyce (2002-2003) on desingularizations of special Lagrangian submanifolds with conical singularities to explain how supersymmetric D3-branes thus defined can be driven and re-assemble under a reverse split attractor flow at a point on the wall of marginal stability in Type IIB superstring theory compactified on varying Calabi-Yau 3-folds, studied by Denef (2001). This last section is to be read alongside the works [De3] (arXiv:hep-th/0107152) of Denef and [Joy3: V] (arXiv:math.DG/0303272) of Joyce. To cover the basic type of deformations of morphisms from Azumaya spaces in this note and its sequel, we discuss in Sec. 1 Morse cobordisms of manifolds and their promotion to Morse cobordisms of Azumaya manifolds with a fundamental module, and of morphisms from Azumaya manifolds to complex manifolds. The notion of cone of special Lagrangian cycles in a Calabi-Yau manifold is brought out in Sec. 2.4.1 for further study. A summary of the needed results from Joyce is given in the appendix.

Key words: connected sum, Morse cobordism; D-brane of A-type, Polchinski-Grothendieck Ansatz; Azumaya manifold, special Lagrangian morphism, unitary minimally flat connection; Higgsing/un-Higgsing, brane-wrapping; cone of special Lagrangian cycles, genus-like expansion; special Lagrangian submanifold, smoothing; assembling/disassembling of D-branes, central charge, split attractor flow.

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0. Introduction and outline.

In [L-Y5] (D(6)), we

- ([L-Y5: Sec. 2.4]) reviewed – based on the previous parts [L-Y1] (D(1)), [L-L-Y-S] (D(2)), [L-Y2] (D(3)), [L-Y3] (D(4)), and [L-Y4] (D(5)) of the project – how D-branes in string theory can be re-understood via the following ansatz and a string/D-brane-theory-oriented, Grothendieck-motivated notion of morphisms from Azumaya-type noncommutative spaces whose local functions rings are modeled in the ansatz:

**Polchinski-Grothendieck Ansatz [Azumaya-type noncommutativity on D-brane]**. The world-volume of a D-brane carries a noncommutative structure locally associated to a function ring of the form $M_r(R)$, where $r \in \mathbb{Z}_{\geq 1}$ and $M_r(R)$ is the $r \times r$ matrix ring over $R$. 

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1. **Mathematical and String-Theoretical Remark.** (Partial history and conceptual overview from earlier parts of the project.) It should be noted that this ansatz already appeared explicitly in the work of Pei-Ming Ho and Yong-Shi Wu, [H-W] (1996), arXiv:hep-th/9611233, as part of their definition of ‘D-branes as quantum space’ ([H-W: Sec. 5]). The significance of their setting seems overlooked by the stringy community and also by us in the brewing years. Ten years later, in early spring 2007, with a better understanding of Grothendieck’s work on modern algebraic geometry, an intense re-thinking/understanding Polchinski’s TASI 1996 lectures and textbook [Po] (1998), and numerous mixed/entangled influences from string theorists (cf. [L-Y1: footnote 1 and footnote 19] (D(1))) and their works (cf. [L-Y1: references] (D(1)) and [L-Y5: references] (D(6))), we unexpectedly re-picked up this thread and realized the possible variations and potentially far-reaching consequences of this ansatz. There are numerous themes yet to be understood along the line.

In retrospect, the main obstruction on the mathematical side to reveal the power of this ansatz is the lack of a “correct” notion of “morphisms” from spaces with such a structure sheaf. The standard noncommutative geometers’ approach, either by assigning a topological space to such type of algebras or by studying their category of modules via Morita equivalence, obscures the richness of this geometry and block its ability to describe D-branes correctly. On the string-theory side, the reason of its being overlooked as a fundamental nature of D-branes (already at the classical, non-supersymmetric, not necessarily stable situation) could be that its appearance (explicitly or hiddenly) in literature is constantly accompanied by something else, notably under a supersymmetry setting, as a quantum space, appearance of another type of noncommutativity structure, and/or with a nontrivial $B$-field background. That makes its role in its own right hidden behind a thick veil since all these additional structures are themselves important, much-more-discussed-and-favored subjects. To make things even worse, whether it’s the space-time that gets noncommutatized matrically or the D-brane itself when D-branes are stacked divide the literature and many highly cited stringy works favor the former. To our best understanding, these two seemingly dual aspects are not tradable to each other. If there is any (world-volume)-vs.-(space-time) type duality between the two, it can only be at best partial and only in some reduced situations. Furthermore,

- mathematically: even if the space-time itself does get noncommutatized somehow, only a noncommutative probe can detect that (after a correct notion of ‘morphism’ is defined; see below),
- physically: when the space-time is treated as an emerged/derived notion and one sees some change (in particular, noncommutativity) to “it”, the first question one should ask is not what happens to the “space-time”, but rather, what happens to the probe!

This makes the emergence of Azumaya-type noncommutativity on (stacked) D-branes an unavoidable path to take. Different string theorists working in D-branes may have come across this ansatz in/on their own way/path with or without our (or even their own) knowing.

With the significance of this ansatz being pointed out and emphasized, now comes the immediate technical issue: one needs also a matching notion of “maps” from D-branes to a space-time. This re-started project began with the re-attempt to understand D-branes by working out one such notion that can match stringy behavior of D-branes as “seen” by open strings while incorporating this ansatz into it. From our current point of view, it is the notion of:

- a “morphism” as an equivalence classes of gluing systems of ring-homomorphisms – as in Grothendieck’s theory of schemes – but without assigning an underlying topological space nor maps between topological spaces even if the latter can be assigned contravariantly functionally in the commutative specialization that unlocks unexpectedly, mathematically unorthodoxically, and yet stringy correctly the power of this ansatz, rendering many D-brane phenomena as its consequences. In particular, the main body of an Azumaya space...
- ([L-Y5: Sec. 2.2]) discussed the four aspects of morphisms from Azumaya schemes with a fundamental module to a (commutative) projective scheme, and

- ([L-Y5: Sec. 3.1]) introduced the notion of Azumaya $C^\infty$-manifolds and morphisms therefrom to a complex manifold.

(See loc. cit. for more details.) In the current sequel, we use the latter notion to give

- a prototypical definition of supersymmetric D-branes of A-type (i.e. A-branes) – in an appropriate region of the Wilson’s theory-space of string theory – as special Lagrangian morphisms with a unitary minimally flat connection-with-singularity.

This merges Donaldson’s picture of special Lagrangian submanifolds (1999) as a special class of maps and the Polchinski-Grothendieck Ansatz for D-branes on a Calabi-Yau space (Sec. 2.1). The phenomena of

- Higgsing/un-Higgsing of Chan-Paton sheaves on A-branes and of the large- vs. small-brane wrapping of A-branes on a cycle in a Calabi-Yau space in superstring theory

$(X, \mathcal{O}_X^\text{Az})$ is the fuzzy noncommutative cloud $\mathcal{O}_X^\text{Az}$, not $X$. The latter should be taken as only auxiliary. (The role of $X$ is only to compensate the limitedness of what localizations of (unital, associative) noncommutative rings can provide in the current situation.) And it is only through a morphism $\varphi : (X, \mathcal{O}_X^\text{Az}) \to (Y, \mathcal{O}_Y)$ defined by $\varphi : \mathcal{O}_Y \to \mathcal{O}_X^\text{Az}$ in the above sense, without a continuous map $X \to Y$, that the Azumaya cloud $\mathcal{O}_X^\text{Az}$ may “condense” into an image-object that is more familiar-looking in the target-space-(time) geometry $Y$, particularly when $Y$ is commutative.

The name of the ansatz given in this project reflects how it is derived ([L-Y1: Sec. 2] D(1)) and is meant to bring it to the forefront due to its importance as a fundamental source of the master nature of D-branes, cf. [L-L-S-Y] (D(2)), [L-Y2] (D(3)), [L-Y3] (D(4)), [L-Y4] (D(5)), and [L-Y5] (D(6)). It indicates how it naturally arises via the fusion of two thoughts - one from the string-theory side and the other from the algebraic geometry side, with both re-writing/revolutionizing their own field -. An additional hidden reason we chose this name when writing [L-Y1] (D(1)), spring 2007, is that we already foresaw at that time that several pivotal existing stringy works on D-branes can be re-done via this ansatz – not surprising at all for a project on D-branes that had to take a decade just to make the first step – and the name would help us assign more easily a subtitle to its sequels, e.g. ‘Douglas-Moore vs. Polchinski-Grothendieck’ for [L-Y2] D(3) and the subtitle of the current note, for comparison/contrast with and fusion of existing works. On the other hand, our naming is destined not to be perfect, despite our fixing to this name after consulting another string-theorist who himself has also contributed a lot to many topics in string theory including branes before and after 1995 (cf. [L-Y2: footnote 1] (D(3))), one may as well, if one wishes, name/call it directly Azumaya-Type Noncommutativity Ansatz for D-Branes since this is exactly what it says. (One may also attempt a thorough survey of related history on both the mathematics and the string-theory side of independent works that either exhibit or hint strongly at this ansatz to produce a complete name: ??-Gelfand-??-Grothendieck-??-??-Ho-??-Polchinski-??-Witten-??-Wu-?? Ansatz. For the moment, we choose to focus on works that remain ahead and leave the naming issue to future/other historians/researchers in this field.) As this ansatz is solely a consequence of open strings (alone!), we believe that everything related to D-branes in a geometric phase always has to do with this ansatz and morphisms from such noncommutative spaces one way or another.

Having said all this, as already pointed out in [L-Y1] (D(1)), while this ansatz (and the correct notion of morphisms) gives us an alternative gateway to access D-branes in their own right, it says only a beginning, lowest level structure thereon (albeit a very rich one) and provides

1. a ground (sheaf of local) function ring(s) with all other fields thereupon realized as (local sections of) its modules and

2. a basis to define the notion of morphisms from a D-brane (world-volume) as one would for a particle (world-line) and a string (world-sheet) in an (either commutative or noncommutative) space-time.

The full structure on and complication of D-branes go much beyond this. For us, D-branes in various contexts, including those in a geometric phase – as in this project – and other in a non-geometric phase – e.g. as boundary conditions/states in a $d = 2$ boundary conformal field theory – remain a largely mysterious yet so amazing object that we have still a lot to learn about.

We thank Pei-Ming for explaining the related points in [H-W] with comments on our work and for discussions on and illuminations of D-branes and M-branes in late spring, 2010.
can be achieved/reali zed via deformations of such special Lagrangian morphisms (Sec. 2.2 and Sec. 2.3).

For the case of Calabi-Yau 3-folds, a classical result on 3-manifolds branched-covering $S^3$, beginning with James Alexander (1920), completed and refined by Hugh Hilden (1974) and José Montesinos (1976) in the form with a universal degree-bound 3 and by William Thurston (1982) and Hilden, María Lozano and Montesinos (1983) in the form of universal links and universal knots respectively implies then, in particular, that

- any embedded special Lagrangian submanifold (without singularity) with a complex vector bundle with a unitary flat connection in a Calabi-Yau 3-fold is the image of a special Lagrangian morphism $\varphi$ from an Azumaya 3-sphere $(S^3,\mathcal{A},\mathcal{E})$ with a fundamental module, with a unitary minimally flat connection $\nabla$.

This singles out special Lagrangian morphisms from Azumaya 3-spheres $(S^3,\mathcal{A},\mathcal{E})$ with a fundamental module, with a minimally flat connection-with-singularity, as the most basic, seed-like D3-branes of A-type in a Calabi-Yau 3-fold. Thus, replacing maps from strings moving along in space-time in string theory by maps from Azumaya 3-spheres $S^3,\mathcal{A}$ suggests

- a genus-like expansion of the path-integral of D3-branes in type IIB string theory compactified on Calabi-Yau 3-folds that resembles the genus expansion of the path-integral of strings

(Sec. 2.4.2). Similarly, for the path-integral of D2-branes in type IIA string theory compactified on Calabi-Yau 3-folds and the path-integral of M2-branes in M-theory compactified on Joyce/$G_2$ 7-manifolds.

In Sec. 3, we use the technical results of Dominic Joyce (2002-2003) on desingularizations of special Lagrangian submanifolds with conical singularities to explain

- how supersymmetric D3-branes thus defined can be driven and re-assemble under a reverse split attractor flow at a point on the wall of marginal stability in Type IIB superstring theory compactified on varying Calabi-Yau 3-folds,

studied by Frederik Denef (2001). This last section is to be read alongside the works [De3] (arXiv:hep-th/0107152) of Denef and [Joy3: V] (arXiv:math.DG/0303272) of Joyce.

To cover the basic type of deformations of morphisms from Azumaya spaces in this note and its sequel, we discuss in Sec. 1 Morse cobordisms of manifolds, their promotion to Morse cobordisms of Azumaya manifolds with a fundamental module, and of morphisms therefrom to complex manifolds. The notion of cone of special Lagrangian cycles of a Calabi-Yau manifold – as a special-Lagrangian analogue to Mori cone of curves of a smooth projective variety – is also brought out in Sec. 2.4.1 for further study. A summary of the needed results of Joyce is given in the appendix A. 1.

Readers are suggested to go through [De3] (resp. [Joy3: V]; [L-Y5] (D(6))) first to get a feel of split attractor flow (resp. desingularization of a special Lagrangian submanifold with conical singularities in a Calabi-Yau manifold; Azumaya geometry and morphisms from an Azumaya space) before reading the current note.

**Convention.** Standard notations\(^2\), terminology, operations, and facts in (1) physics aspects

\(^2\)Apology: With a project that incorporates/merges many things from various well-established mathematical and stringy disciplines and also to take into account notations from earlier parts of the project, we find it more and more difficult to keep the terminology/notations/symbols distinct for different objects. However, what a terminology/notation/symbol means is usually immediately clear either from the context or from the additional label/superscript/subscript to that symbol. Listed in Convention are a few essential ones used in this note, each of which is almost already carved into a stone in its own field. We decide that it is better/more meaningful to get used to them rather than to try to make any change merely for the consistency of notations in a note.
of D-branes; (2) (commutative) algebraic geometry/stacks; (3) complex geometry; (4) symplectic/calibrated geometry; (5) sheaves on manifolds; (6) Hodge theory; (7) surgery and topological cobordism theory can be found respectively in (1) [Po], [Joh]; (2) [Hart]/[L-MB]; (3) [G-H]; (4) [McD-S]/[Ha-L], [Harv], [McL]; (5) [K-S], [Dim]; (6) [Vo]; (7) [Mi1], [Hir].

· A real manifold of dimension $n$ is called an $n$-manifold while a complex manifold of complex dimension $n$ is called an $n$-fold.

· For a smooth/$C^\infty$-manifold $X$,
  - $\mathcal{O}_X$ is the sheaf of $C^\infty$-functions on $X$,
  - $\mathcal{O}_{X,C} = \mathcal{O}_X^\infty \otimes \mathbb{C}$ is the sheaf of complex-valued $C^\infty$-functions on $X$.

· For a complex manifold $Y$,
  - $\mathcal{O}_Y$ is the sheaf of holomorphic functions on $Y$,
  - $\mathcal{O}_{Y,C} = \mathcal{O}_Y^\infty \otimes \mathbb{C}$ is the sheaf of complex-valued $C^\infty$-functions on $Y$.

· $n$-fold $M$ as an $n$-dimensional complex manifold vs. $n$-fold (branched-)covering space $L \to N$ as a (branched-)covering map of degree $n$.

· A (real) ‘singular $C^\infty$-manifold’ or a (real) ‘$C^\infty$-manifold with singularities’ $M$ means a topological space that can be stratified locally finitely into a union of (real) $C^\infty$-manifolds in such a way that it contains an open dense (possibly disconnected) stratum that is a (possibly disconnected) $C^\infty$-manifold (of uniform dimension if disconnected). For such $M$, let $U$ be the interior of the intersection of maximal open $C^\infty$-manifold-subsets of $M$. Then, $\mathcal{O}_M$ is the sheaf of continuous functions on $M$ that is $C^\infty$ on $U$ and $\mathcal{O}_{M,\infty} := \mathcal{O}_M \otimes \mathbb{C}$ its complexification.

· $D^n$ is $n$-dimensional (closed) disk/ball, $S^n$ is $n$-dimensional sphere, and $T^n$ or $T_n$ is $n$-dimensional torus; all as real smooth manifolds. Particularly, $3$-disk $D^3$ vs. D3-brane $X$.

· A Calabi-Yau $n$-fold $Y$ (with a specified holomorphic $n$-form) is denoted in full by $(Y,J,\omega,\Omega)$, where $J$ is the complex structure on $Y$, $\omega$ is the Kähler class of the underlying Ricci-flat metric on $Y$, $\Omega$ is a holomorphic $n$-form on $Y$ that satisfies the identity $\omega^n/n! = (-1)^{(n-1)/2}(i/2)^n\Omega \wedge \bar{\Omega} \ (= \text{vol}_M)$.

Thus, $\Omega$ is uniquely determined by $(J,\omega)$, up to a phase factor.

· $D$-brane vs. disk vs. $D$-module.

· The word ‘relative’ has two different meanings: (1) with respect to a subset, e.g. the relative cohomology $H^\ast(M,S;\mathbb{R})$, vs. (2) with respect to the base of a family, e.g. the relative cotangent sheaf $\Omega_{X/S}$. For the current work, it is (2) that most often appears.

· Graph $\Gamma$ vs. the global-section functor $\Gamma(\cdot)$.

· The sheaf $\Omega_Y^p$ of holomorphic $p$-forms on a complex manifold $Y$ vs. a holomorphic $n$-form $\Omega$ on a Calabi-Yau $n$-fold $Y$ vs. a (smooth or holomorphic) section $\Omega$ on a holomorphic line bundle $\mathcal{H}^{n,0}$ on the moduli space $M$ of complex deformations of $Y$.

· (1) $i = \sqrt{-1}$ as an index vs. $i$ as an inclusion/embedding; (2) $\alpha$ as a phase or a phase function vs. $\alpha$ as an index.
The canonical line bundle $K_Y$ of a complex manifold $Y$ vs. a (local) Kähler potential $K$ of a Kähler manifold $Y$ (particularly in SUSY QFT and stringy literatures, in which the latter $K$ is also related to the kinetic term in a supersymmetric action/Lagrangian density).

A Kähler metric $g$ on a complex manifold $Y$ is usually also denoted by its associated Kähler 2-form $\omega$ on $Y$.

$N$ that counts the number of sets of minimal supersymmetries in each space-time dimension (e.g. $d = 10, N = 2 \Rightarrow 32$ supercharges) vs. $N$ as a manifold vs. $N$ as a tubular neighborhood vs. $N_{Z/Y}$ as a normal bundle of $Z$ in $Y$ (vs. $N$ as an unspecified (usually large) number (usually integer) as in the ‘large $N$ limit’).

rank of a locally-free sheaf/vector bundle vs. rank of an algebra vs. rank of a Lie group.

$Z$ as a subscheme/submanifold/cycle/chain vs. $Z$ as a central charge (vs. $Z$ as a partition function in QFT and string theory.)

Outline.

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Appendix.

A.1 Desingularizations of immersed special Lagrangian submanifolds with transverse intersections and their moduli space à la Joyce.
1 Morphisms from Azumaya manifolds with a fundamental module in a Morse family.

In studying deformation problems in algebraic geometry, there is the notion of flatness that characterizes a “good family” of objects in question. In the category $C^\infty$-manifolds (without singularity), one would very much like to have such a notion as well. For this note, we take a Morse family from manifold/cobordism theory to play the role of a flat family in algebraic geometry. The issue of deformations of A-branes studied in Sec. 2 and Sec. 3 of this note will be based on such families. In Sec. 1.1, we extend the standard notion of a Morse family over an interval in $\mathbb{R}$ to one over a general base $S \subset \mathbb{R}^l$ for some $l$. In Sec. 1.2, we give/review the definition of morphisms from Azumaya manifolds to a complex manifold based on [L-Y5: 3.1] (D(6)) and extend it to the case of a Morse family of Azumaya manifolds.

1.1 Morse family of manifolds with singularities.

Definition/Fact 1.1.1. [handlebody decomposition, Mores function, singularity of Morse type]. ([G-St], [Ki1], and [Mi],) Associated to a handlebody decomposition

$$M(0) = D^{n+1}(= D^0 \times D^{n+1}) \subset M(1) \subset \cdots \subset M(k-1) \subset M(k) = M$$

of a smooth $(n+1)$-manifold $M$, where $M(i)$ is obtained from $M(i-1)$ by attaching a $k_i$-handle $D^{k_i} \times D^{(n+1)-k_i}$ via a smooth embedding $f_i : \partial D^{k_i} \times D^{(n+1)-k_i} \to \partial M(i)$ plus a smoothing after gluing via $f_i$, (in notation, $M(i) = M(i-1) \cup f_i (D^{k_i} \times D^{(n+1)-k_i})$, there is a Morse function

$$h : M \to \mathbb{R}$$

with critical values $t_0 < t_1 < \cdots < t_{k-1} < t_k$ and non-degenerate critical points $p_i \in M$ with $h(p_i) = t_i$ and index $(p_i) = k_i$. $p_i$ corresponds the center $(0,0) \in D^{k_i} \times D^{(n+1)-k_i}$, with $D^{k_i} \times \{0\}$ the descending manifold and $\{0\} \times D^{(n+1)-k_i}$ the ascending manifold of $h$. The singular $n$-manifold $M_t := h^{-1}(t)$ arises from degenerating a smooth $n$-manifold $M := h^{-1}(t) \simeq \partial M_i$, $t \in (t_{i-1}, t_i)$, and then deform to another smooth $n$-manifold $M_t := h^{-1}(t') \simeq \partial M_i$, $t' \in (t_i, t_{i+1})$, by:

- For $k_i = 0$, $M_t \simeq M_t \amalg \{p_i\}$, and then deform to $M_t \simeq M_t \amalg S^n$.
- For $k_i = 1$, $M_t \simeq M_t$ with two points $\{p_-, p_+\}$ identified (to $p_i$), and then deformed to $M_t \simeq$ the (self-)connected sum of $M_t$ at $\{p_-, p_+\}$.
- For $2 \leq k_i \leq n-1$, $M_t \simeq M_t$ with an embedded $S^{k_i-1}$ (whose tubular neighborhood $\nu_{M_t}(S^{k_i-1}) \simeq S^{k_i-1} \times D^{n-(k_i-1)}$) shrunk to a point (i.e. $p_i$), and then deform to $M_t$ via evolving $p_i$ to an $S^{n-k_i}$. This corresponds to a surgery of $M_t$ along $S^{k_i-1}$ by removing $\nu_{M_t}(S^{k_i-1})$ and then filling in $D^{k_i} \times S^{n-k_i}$ via the isomorphisms $\partial(\nu_{M_t}(S^{k_i-1})) \simeq S^{k_i-1} \times S^{n-k_i} \simeq \partial(D^{k_i} \times S^{n-k_i})$.
- For $k_i = n$, $M_t \simeq M_t$ with a two-sided embedded $S^{n-1}$ shrunk to a point (i.e. $p_i$), and then deform to $M_t$ by compactifying $M_t - \{p_i\}$ via filling in points $p_-, p_+$.
- For $k_i = n+1$, $M_t \simeq N \amalg S^n$, $M_t \simeq N \amalg \{p_i\}$, and $M_t \simeq N$ for a smooth $n$-manifold $N$.

Situations $k_i = l$ and $k_i = (n+1) - l$ are reverse to each other, and Situations $k_i = 0$, 1, $n$, $n+1$ are the only ones that may change the number of connected components of $M_t$. For convenience, we will call these singular manifolds $M_t$ that appear as a singular fiber of a Morse function $h$ a manifold with Morse-type singularities.
For the purpose of this note, we define the following version of the notion of a Morse family that extends the notion of Morse function slightly:

**Definition 1.1.2. [Morse family].** Let $S$ be an open domain in some $\mathbb{R}^l$. A smooth map $\pi : X \to S$ from a smooth manifold $X$ to $S$ is said to be a Morse family of manifolds with singularities over $S$ (in short, a Morse family over $S$) if the following conditions are satisfied:

- $\pi$ is a surjective;
- for all $s \in S$, there exists a smooth embedded curve $\gamma : (-\varepsilon, \varepsilon) \to S$, $\varepsilon > 0$, with $\gamma(0) = s$ such that
  
  (1) the total space $\gamma^*X := (-\varepsilon, \varepsilon) \times_S X \simeq \pi^{-1}(\gamma((-\varepsilon, \varepsilon)))$ of the pull-back family is a smooth manifold,
  
  (2) the pull-back map $\gamma^*\pi : \gamma^*X \to (-\varepsilon, \varepsilon)$ is a Morse function on $\gamma^*X$.

**Remark 1.1.3. [topologists’ definition].** A complete general definition of higher dimensional Morse families requires a study of the classification of stable singularities of smooth maps that allows enforced merging of simple nondegenerate singularities on fibers of $\pi : X \to S$ when $\dim \mathbb{R}S > 1$. The very restrictive definition we use here is tailored to the situation of the current note. It corresponds to the case when no such merging occurs. Note also that it is important that in the above definition, the fiber $X_s := \pi^{-1}(s)$ of $\pi$ over $s \in S$ is allowed to be disconnected.

**Example 1.1.4. [from a manifold with Morse-type singularities].** Let $M_0$ be a $n$-manifold with Morse-type singularities $\{p_1, \ldots, p_k\}$ via pinching a smooth $n$-manifold $M_-$ along a disjoint collection of spheres $S^{l_i}$, $0 \leq l_i \leq n - 1$, with tubular neighborhood $\nu_M(S^{l_i})$ a trivial $D^{n-l_i}$-bundle, $i = 1, \ldots, k$. Topologically, a tubular neighborhood $\nu_{M_0}(p_i)$ of $p_i$ in $M_0$ is homeomorphic to a (real) cone over $S^{l_i} \times S^{n-l_i-1}$. Denote the interval $(-\varepsilon, \varepsilon)$ by $I_\varepsilon$. Shrinking the disk bundles if necessary, we may assume that $\nu_M(S^{l_i}), i = 1, \ldots, k$, are disjoint from each other. Then $M_0$ is realizable as $h^{-1}(0)$ of a Morse function

$$h : X \to I_\varepsilon$$

with $h^{-1}(s) \simeq M_-$, for $s \in (-\varepsilon, 0)$, and $h^{-1}(s) \simeq M_+$, for $s \in (0, \varepsilon)$, where $M_+$ is a $n$-manifold obtained from $M_-$ by a surgery

$$\left(M_- - \coprod_{i=1}^k \nu_{M_-}(S^{l_i})\right) \cup \coprod_{i=1}^k g_i \coprod_{i=1}^k (D^{l_i+1} \times S^{n-l_i-1}),$$

with $g_i : \partial(D^{l_i+1} \times S^{n-l_i-1}) \to \partial P_m := \partial(\nu_M(S^{l_i})) \simeq S^{l_i} \times S^{n-l_i-1}$ (with orientations taken into account). There is a special Morse family $\pi[k] : M_0[k] \to I_\varepsilon[k] := I_\varepsilon^k$ associated to $M_0$ that arises from a “prolonged/expanded realization of $h$” as follows:

1. By construction, one has an embedding over $I_\varepsilon$:

$$I_\varepsilon \times \left(M_- - \coprod_{i=1}^k \nu_{M_-}(S^{l_i})\right) \xrightarrow{pr_1} X,$$

where $pr_1$ is the projection map to the first factor. Let

$$X - I_\varepsilon \times \left(M_- - \prod_{i=1}^k \nu_{M_-}(S^{l_i})\right) = \prod_{i=1}^k K_i,$$

where $K_i$ is the connected component that contains $p_i \in M_0 = h^{-1}(0) \subset X$. By construction, $K_i$ is a manifold over $I_\varepsilon$ with boundary $(I_\varepsilon \times (S^{l_i} \times S^{n-l_i-1}))/I_\varepsilon$. 

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(2) Let \( H_i[k] = I^{i-1}_\varepsilon \times \{0\} \times I^{n-i}_\varepsilon \subset I_\varepsilon[k] \) be the \( i \)-th coordinate hyperplane of \( I_\varepsilon[k] \). Consider the manifold with boundary
\[
I_\varepsilon[k] \times M_- - \coprod_{i=1}^k H_i[k] \times I_\varepsilon \times \nu_{M_-}(S^l_i)
\]
over \( I_\varepsilon[k] \) and the filling
\[
M_0[k] = \left( I_\varepsilon[k] \times M_- - \coprod_{i=1}^k H_i[k] \times I_\varepsilon \times \nu_{M_-}(S^l_i) \right) \bigcup \coprod_{i=1}^k (H_i[k] \times K_i).
\]
where \( f_i \) is the built-in isomorphism
\[

\begin{array}{ccc}
I_\varepsilon[k] & \xrightarrow{f_i} & H_i[k] \times \partial K_i \\
\downarrow & & \downarrow \\
& I_\varepsilon[k] & \xrightarrow{H_i[k] \times \partial K_i} H_i[k] \times I_\varepsilon \times \nu_{M_-}(S^l_i)
\end{array}
\]
Here, as manifolds over \( I_\varepsilon[k] \), the \( I_\varepsilon \)-factor of \( H_i[k] \times I_\varepsilon \times \nu_{M_-}(S^l_i) \) is mapped to the \( i \)-th \( I_\varepsilon \)-factor of \( I_\varepsilon[k] \) by the identity map, and the \( K_i \)-factor of \( H_i[k] \times K_i \) is mapped to the \( i \)-th \( I_\varepsilon \)-factor of \( I_\varepsilon[k] \) by the restriction \( h|_{K_i} \). Then, since \( M_0[k] \) is obtained from manifolds and gluing morphisms over \( I_\varepsilon[k] \), there is a built-in morphism of smooth manifolds
\[
\pi[k] : M_0[k] \to I_\varepsilon[k].
\]
Furthermore, since \( I_\varepsilon[k] \times M_- - \coprod_{i=1}^k H_i[k] \times I_\varepsilon \times \nu_{M_-}(S^l_i) \to I_\varepsilon[k] \) is a submersion and \( H_i[k] \times K_i \to I_\varepsilon[k] \) is a Morse family, \( \pi[k] \) defines a Morse family. By construction, the set of critical points of \( \pi[k] \) is given by \( \coprod_{i=1}^k (H_i[k] \times \{p_i\}) \), which is contained in \( \coprod_{i=1}^k (H_i[k] \times K_i) \).

\[ \square \]

Remark 1.1.5. [relative handlebody attachment]. The above construction of \( \pi[k] : M_0[k] \to I_\varepsilon[k] \) is equivalent to a step-by-step relative-handlebody attachment, beginning with \( M_- \times (-\varepsilon,0] \), that avoids the singularities in the fibers of the previous family at each step.

For the conceptual appeal and convenience, we will borrow a terminology from complex algebraic geometry to define:

Definition 1.1.6. [expanded deformation space]. The \( M_0[k] \) constructed in Example 1.1.4 will be called an expanded Morse family of smoothings of the manifold \( M_0 \) with Morse-type singularities and the base \( I_\varepsilon[k] \) an expanded deformation space of \( M_0 \).

Example 1.1.7. [expanded deformation space for connected sum]. In particular, when all \( l_i = 0 \), \( i = 1, \ldots, k \), in Example 1.1.4, \( M_0 \) is the singular manifold obtained from identifying each pair of points \( \{p_{i-}, p_{i+}\} \) (\( S^0_c = \partial D^1 \)), \( i = 1, \ldots, k \), in a possibly disconnected \( n \)-manifold \( M_- \), and \( M_+ \) is the (self-)connected sum of \( M_- \) at each \( \{p_{i-}, p_{i+}\} \). The construction gives then an expanded deformation space \( S = I_\varepsilon[k] + M_0 \) with an expanded Morse family \( \pi[k] : M_0[k] \to I_\varepsilon[k] \) of smoothings.

Remark 1.1.8. [mixed real-complex version]. In the passing, we remark that one may extend Definition 1.1.2 slightly by requiring instead:
...for all \( s \in S \), there exists either a smooth embedded curve \( \gamma_1 : (-\varepsilon, \varepsilon) \to S \), \( \varepsilon > 0 \), with \( \gamma_1(0) = s \) such that

1. the total space \( \gamma_1^*X := (-\varepsilon, \varepsilon) \times S \times X \simeq \pi^{-1}(\gamma_1((-\varepsilon, \varepsilon))) \) of the pull-back family is a smooth manifold and

2. the pull-back map \( \gamma_1^* : \gamma_1^*X \to (-\varepsilon, \varepsilon) \) is a Morse function on \( \gamma_1^*X \), or a smooth embedded 2-disk \( \gamma_2 : \Delta_2^2 = \{ z \in \mathbb{C} : |z| < \varepsilon \}, \varepsilon > 0 \), with \( \gamma_2(0) = s \) such that

1'. the total space \( \gamma_2^*X := \Delta_2^2 \times S \times X \simeq \pi^{-1}(\gamma_2(\Delta_2^2)) \) of the pull-back family is a smooth manifold and

2'. the pull-back map \( \gamma_2^* : \gamma_2^*X \to \Delta_2^2 \) is locally modelled on a complex Morse function on \( \gamma_2^*X \).

Examples of such families include families of nodal curves in complex algebraic geometry, families of complex surfaces with \( A_1 \)-singularities, and conifold degenerations of Calabi-Yau 3-folds.

### 1.2 Morphisms from Azumaya manifolds with a fundamental module in a Morse family.

**Morphisms from Azumaya manifolds with a fundamental module to a complex manifold.**

(Cf. [L-Y1: Sec. 1, Sec. 2] (D(1)); [L-L-S-Y: Sec. 1, Sec. 2.1, Sec. 2.2] (D(2)); [L-Y5: Sec. 2.1, Sec. 2.2] (D(6)).) Given a \((C^\infty)\)-manifold \( X \), a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \), and a complex manifold \( Y \), Let \( \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \) be the sheaf of \( \mathcal{O}_X \)-module endomorphisms of \( \mathcal{E} \). It is an Azumaya algebra over \( \mathcal{O}_X \). A morphism

\[
\varphi : (X^{\text{Az}}, \mathcal{E}) := (X, \mathcal{O}^{\text{Az}}_X := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}), \mathcal{E}) \to Y
\]

from the Azumaya manifold with a fundamental module \((X^{\text{Az}}, \mathcal{E})\) to \( Y \) is by definition an equivalence class, in notation

\[
\varphi^\sharp : \mathcal{O}^{\infty}_{Y, \mathbb{C}} \longrightarrow \mathcal{O}^{\text{Az}}_X
\]

of \( \mathbb{C} \)-algebra homomorphisms from a gluing system of \( \mathbb{C} \)-algebras associated to \( \mathcal{O}^{\infty}_{Y, \mathbb{C}} \) to a fine-enough (with respect to the \( C^\infty \)-topology on \( X \)) gluing system of Azumaya algebras over \( \mathcal{O}_X \) associated to \( \mathcal{O}^{\text{Az}}_X \).\(^3\) In general, there is no map from \( X \) to \( Y \) directly. However, the \( \mathcal{O}_X \)-algebra \( \mathcal{A}_\varphi \) generated by the image \( \mathbb{C} \)-algebras of \( \varphi^\sharp \) under \( \mathcal{O}_X \) defines a \( C^\infty \)-manifold-with-singularity \( X_\varphi \) – the surrogate of \( X^{\text{Az}} \) associated to \( \varphi \) – with structure sheaf \( \mathcal{A}_\varphi \) and with the underlying topology \( X_\varphi \) canonically embedded in \( X \times Y \). By construction, \( \mathcal{E} \) has a tautological \( \mathcal{A}_\varphi \)-module structure, denoted by \( \mathcal{A}_\varphi \mathcal{E} =: \mathcal{E}_\varphi \). In summary and in a re-packaged form:

**Definition 1.2.1. [morphism from Azumaya manifold].** Given an Azumaya manifold with a fundamental module \((X^{\text{Az}}, \mathcal{E})\) and a complex manifold \( Y \), a morphism \( \varphi : (X^{\text{Az}}, \mathcal{E}) \to Y \) is

\(^3\) *String-Theoretical Remark.* This fundamental picture is what makes our definition of morphisms from an Azumaya space with a fundamental module linked with D-branes in string theory. It retraces how the Polchinski-Grothendieck/Azumaya-Type Noncommutativity Ansatz for D-branes appears; cf. [L-Y1: Sec. 2.2] (D(1)).
given by the following data:

\[ \xymatrix{ E \ar[dd]_{\pi_{\varphi}} \ar[rr]^{f_{\varphi}} && Y, \ar[dd] \cr (X_{\varphi}, A_{\varphi}) \ar[rr]_{\pi_{\varphi}} && X } \]

where

- \( X_{\varphi} \) is a \( C^\infty \)-manifold-with singularity that contains an open dense manifold-subset \( V_{\varphi} \) such that \( \pi_{\varphi}|_{V_{\varphi}} : V_{\varphi} \to V := \pi(V_{\varphi}) \) is a covering map of finite order;
- \( A_{\varphi} \) is an \( \mathcal{O}_{X_{\varphi}} \)-algebra and \( (\pi_{\varphi}, f_{\varphi}) : (X_{\varphi}, A_{\varphi}) \to X \times Y \) is an embedding as a map of ringed topological spaces over \( X \), with \( X \times Y \) as a smooth manifold fibered over \( X \) with an analytic structure along fibers \( Y \);
- as an \( A_{\varphi} \)-module, the support of \( E_{\varphi} \) is \( (X_{\varphi}, A_{\varphi}) \) (i.e. there exists no local section \( a \) of \( A_{\varphi} \) on some open set \( U \) of \( X_{\varphi} \) such that \( a \cdot (E_{\varphi}|_U) = 0 \));
- \( \pi_{\varphi*} E_{\varphi} = E \).

A morphism (or arrow) \( \varphi_1 \to \varphi_2 \) is the data \( (h, \tilde{h}, \tilde{\tilde{h}}) \), where \( h : X_1 \to X_2 \) is a diffeomorphism, \( \tilde{h} : (X_{1, \varphi_1}, A_{\varphi_1}) \to (X_{2, \varphi_2}, A_{\varphi_2}) \) is an isomorphism of ringed topological space that lifts \( h \), and \( \tilde{\tilde{h}} : \tilde{h}^* E_{\varphi_2} \to E_{\varphi_1} \) is an \( A_{\varphi_1} \)-module isomorphism such that the following diagram commutes:

\[ \xymatrix{ E_{\varphi_1} \ar[dd] \ar[rr]_{f_{\varphi_1}} && Y, \ar[dd] \cr (X_{1, \varphi_1}, A_{\varphi_1}) \ar[rr]_{\pi_{\varphi_1}} && X_1 \cr E_{\varphi_2} \ar[rr]_{f_{\varphi_2}} && Y \cr (X_{2, \varphi_2}, A_{\varphi_2}) \ar[rr]_{\pi_{\varphi_2}} && X_2 } \]

Remark 1.2.2. [other aspects of morphisms from Azumaya manifolds with a fundamental module].

The way in which an arrow \( \varphi_1 \to \varphi_2 \) between two morphisms is defined says that a morphism \( \varphi : (X, E) \to Y \), with \( E \) of rank \( r \), is the same as a morphism

\[ \phi : X \to \mathfrak{M}_{r}^{0\text{-eff}}(Y), \]

where \( \mathfrak{M}_{r}^{0\text{-eff}}(Y) \) is the stack of 0-dimensional \( \mathcal{O}_Y \)-modules of length \( r \). \( \mathfrak{M}_{r}^{0\text{-eff}}(Y) \) admits a representation-theoretical atlas from the Douady space of 0-dimensional, length \( r \) quotients of \( \mathcal{O}_Y^{\text{pr}} \):

\[ \text{Quot}_{\text{Douady}}^{H^0}(\mathcal{O}_Y^{\text{pr}}, r) := \{ \mathcal{O}_Y^{\text{pr}} \to \overline{E} \to 0, \text{ length} \overline{E} = r, H^0(\mathcal{O}_Y^{\text{pr}}) \to H^0(\overline{E}) \to 0 \} \]
This is a $GL_r(\mathbb{C})$-space. In terms of this, $\phi$, and hence $\varphi$, is encoded in a map

$$\tilde{\phi} : P_X \rightarrow Quot^0_{Douady}(\mathcal{O}_{Y}^{r}, r)$$

of $GL_r(\mathbb{C})$-spaces, where $P_X$ is a principal $GL_r(\mathbb{C})$-bundle over $X$. See [L-Y5: Sec. 2.2] D(6) for the analogue in the realm of projective algebraic geometry, in which Douady spaces are replaced by Grothendieck’s Quot-schemes.

**Morphisms from a Morse family of Azumaya manifolds with a fundamental module.**

**Definition 1.2.3.** [Morse family of morphisms]. Let $X_S$ be a Morse family over $S$, $\mathcal{E}_S$ be a locally free $\mathcal{O}_{X_S}$-module of finite rank, and $Y_S$ be an $S$-family of complex manifolds over $S$, $S \subset \mathbb{R}^l$ for some $l$. Denote the fiber of $(X_S, \mathcal{E}_S, Y_S)$ at an $s \in S$ by $(X_s, \mathcal{E}_s, Y_s)$. Then, a morphism $\varphi_S : (X_s^Az, \mathcal{E}_S) \rightarrow Y_S$ in the sense of Definition 1.2.1 that takes $(X_s^Az, \mathcal{E}_s)$ to $Y_s$ is called a Morse family of morphisms (from Azumaya manifolds with Morse type singularities with a fundamental module to complex manifolds) over $S$.

### 2 Supersymmetric D-branes of A-type and their deformations: Donaldson meeting Polchinski-Grothendieck.

The notion of morphisms from an Azumaya manifold with a fundamental module to a target space(-time) gives a basic tool/language to study D-branes mathematically in their geometric phase. For D-branes in the space(-time) that preserve part of the supersymmetry in either the related $d = 2$ field theory on the open superstring world-sheet(-with-boundary) (cf. [H-I-V], [O-O-Y]; see also [A-L-Z], [L-Z], and [M-P-R]) or the ($d = 10$ or a lower-dimensional effective) supergravity theory with branes (cf. [B-B-St] and [M-M-M-S]), there are constraints on the morphisms and the gauge field on the fundamental module one needs to add to the notion of morphisms above. Depending on what supersymmetry remains, what regime/location in the Wilson’s theory-space of string theory we are in/at, and what other fields either on the background space-time and on the D-branes world-volume itself are brought into play, these additional constraints may take different mathematical forms. For the current note, we address D-branes of A-type in the sense of [O-O-Y] and [B-B-St] in the regime where the string coupling constant $g_s$ is small, the energy scale on the D-branes field theory and the related ambient supergravity theory is low, and the background $B$-field is set to zero. The definition of A-branes in Sec. 2.1 along the line of the Polchinski-Grothendieck Ansatz, with the above considerations taken into account, can be thought of as an extension of Donaldson’s viewpoint on special Lagrangian submanifolds in Calabi-Yau spaces – as a subspace of maps in a space of all maps [Don] – by promoting the domain of maps from a(n ordinary) manifold to an Azumaya manifold with a fundamental module. (Cf. [L-Y5: Sec. 3] (D(6)).) Through deformations of such morphisms, the most basic phenomena of D-branes, Higgsing/un-Higgsing and large- vs. small-brane wrapping, can be reproduced; Sec. 2.2 and Sec. 2.3. Two immediate themes are listed in Sec. 2.4 as guiding questions for further study.

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\[\text{We thank Andrew Strominger and Cumrun Vafa for consultation on the absoluteness/variation of the definition of A-branes. It remains to us a challenging question as how (the working mathematical definition for) A-branes should vary/interpolate when one moves around in the Wilson's theory-space of string theory.}\]
2.1 Supersymmetric D-branes of A-type as morphisms from Azumaya manifolds with a fundamental module.

It’s well-known that an isomorphism class of complex vector bundles-with-flat-connection \((E, \nabla)\) of rank \(r\) over a (real) smooth manifold \(M\) is given by a conjugacy class of group representations \(\rho: \pi_1(M) \to U(r)\). However, from the lesson of D-branes of B-type (e.g. [G-Sh] and [D-K-S]) and from the aspects of morphisms \(\varphi: (X^{\text{Az}}, E) \to Y\) from an Azumaya manifold to a commutative target-space, the surrogate \(X_\varphi\) of \(\varphi\) in general has a scheme-like structure that contains nilpotent elements in its sheaf of local function rings. See also [L-Y5: Remark 4.2.5] (D(6)) for how this may be encoded in symplectic geometry. Thus, to understand A-brane in full, we begin with the notion of flat connections on a coherent sheaf on a scheme and a \(C^\infty\) version of this, and then give a prototypical definition of A-branes guided particularly by [B-B-St], [O-O-Y], and [H-I-V].

Connections on a quasi-coherent sheaf on a scheme and its flatness.

(Cf. [Be], [Bj], [Ka], and [Ko] (but without assuming smoothness); and [Ei], [Mat].)

**Definition 2.1.1. [connection, curvature, and flatness].** Let \(Z\) be a scheme over a base \(T\) and \(\mathcal{F}\) be a quasi-coherent sheaf of \(\mathcal{O}_Z\)-modules. Recall the canonical \(T\)-differential \(d: \mathcal{O}_Z \to \Omega_{Z/T}^1\). An \(T\)-connection \(\nabla\) on \(\mathcal{F}\) is a homomorphism of \(\mathcal{O}_T\)-modules

\[
\nabla: \mathcal{F} \to \Omega_{Z/T} \otimes_{\mathcal{O}_Z} \mathcal{F}
\]

such that

\[
\nabla(fs) = df \otimes s + f \nabla s ,
\]

for functions \(f\) of \(Z\) and sections \(s\) of \(\mathcal{F}\) on the same open subset of \(Z\). \(\nabla\) extends to a homomorphism of \(\mathcal{O}_T\)-modules

\[
\nabla: \Omega_{Z/T}^i \otimes_{\mathcal{O}_Z} \mathcal{F} \to \Omega_{Z/T}^{i+1} \otimes_{\mathcal{O}_Z} \mathcal{F}
\]

by

\[
\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \wedge \nabla s .
\]

In particular, one has

\[
\mathcal{F} \xrightarrow{\nabla} \Omega_{Z/T} \otimes_{\mathcal{O}_Z} \mathcal{F} \xrightarrow{\nabla} \Omega_{Z/T}^2 \otimes_{\mathcal{O}_Z} \mathcal{F} .
\]

The \(\mathcal{O}_T\)-module homomorphism from the composition turns out to be an \(\mathcal{O}_Z\)-module homomorphism and, hence, defines an \(\text{End}_{\mathcal{O}_Z}(\mathcal{F})\)-valued 2-form \(R \in \Gamma(\text{End}_{\mathcal{O}_Z}(\mathcal{F}) \otimes_{\mathcal{O}_Z} \Omega_{Z/T}^2)\) on \(Z\). \(R\) is called the curvature 2-form of \(\nabla\). We say that \(\nabla\) is flat if \(R = 0\).

**Remark 2.1.2. [existence/non-existence of (flat) connection].** In general, a coherent or quasi-coherent \(\mathcal{O}_Z\)-module \(\mathcal{F}\) on a scheme \(Z\) may not admit a connection\(^5\). When it does, it may not admit a flat connection. The difference \(\nabla_1 - \nabla_2\) of two connections \(\nabla_1\) and \(\nabla_2\) on \(\mathcal{F}\) is an \(\mathcal{O}_Z\)-module homomorphism \(\mathcal{F} \to \Omega_Z \otimes_{\mathcal{O}_Z} \mathcal{F}\). In particular, the structure sheaf \(\mathcal{O}_Z\) of \(Z\) (over \(\mathbb{C}\)) admits a connection, given by the canonical differential \(d: \mathcal{O}_Z \to \Omega_Z\), which is also flat. A connection on \(\mathcal{O}_Z\) is thus of the form \(d + h\), where \(h: \mathcal{O}_Z \to \Omega_Z\) is an arbitrary \(\mathcal{O}_Z\)-module homomorphism (i.e. \(h \in \Gamma(\Omega_Z)\)).

\(^5\)Even for a coherent sheaf \(\mathcal{F}\) on a 0-dimensional/punctual scheme \(Z/\mathbb{C}\), the existence of a connection already puts a highly nontrivial constraint on \(\mathcal{F}\), let alone a flat connection.
Remark 2.1.3. [flat connection and $\mathcal{D}$-module]. ([Be].) Let $\Theta_{Z/T}$ (resp. $\Omega^\vee_{Z/T}$) be the sheaf of $T$-derivations on $\mathcal{O}_Z$ and $\mathcal{D}_{Z/T}$ be the $\mathcal{O}_Z$-algebra of differential operators on $Z/T$ generated by $\Theta_{Z/T}$. When $Z$ is smooth over $T$, a flat connection $\nabla$ on $\mathcal{F}$ realizes $\mathcal{F}$ as a $\mathcal{D}_{Z/T}$-module via

$$(\partial_1 \cdots \partial_l)s := \nabla_{\partial_1} \cdots \nabla_{\partial_l}s,$$

where $s$ is a local section of $\mathcal{F}$ and $\partial_1, \ldots, \partial_l$ are commuting local sections of $\Theta_{Z/T}$, all on the same open set of $Z$.

Remark 2.1.4. [lifting and descent of flat connection]. Given a finite morphism $\pi : Z_1 \to Z_2$ with a coherent $\mathcal{O}_{Z_1}$-module $\mathcal{F}_1$ on $Z_1$, let $\mathcal{F}_2 := \pi_* \mathcal{F}_1$. Then, in general, a flat connection on $\mathcal{F}_2$ does not lift to a flat connection on $\mathcal{F}_1$ (even if we assume in addition that $\mathcal{F}_1$ is flat over $Z_2$); nor is a flat connection on $\mathcal{F}_1$ descend to a flat connection on $\mathcal{F}_2$.

However, such lifting and descent do exist in a special case in the analytic category:

**Lemma/Definition 2.1.5. [lifting and descent of flat connection under proper étale morphism].** Let $\pi : Z_1 \to Z_2$ be a proper étale morphism (of schemes of finite type over $\mathbb{C}$), $\mathcal{F}_i$, $i = 1, 2$, be a locally-free coherent $\mathcal{O}_{Z_i}$-module with $\mathcal{F}_2 = \pi_* \mathcal{F}_1$. Then a flat connection on $\mathcal{F}_1$ descends to a flat connection on $\mathcal{F}_2$. $\pi$ determines a direct-sum decomposition $\mathcal{F}_2|_U = \oplus_j \mathcal{F}_{2,U}^{(j)}$ on small enough open sets $U$ in the analytic topology such that $\pi^{-1}(U) \to U$ is a disjoint union of biholomorphic maps. $\nabla$ is said to be $\pi$-admissible if $\mathcal{F}_{2,U}^{(j)}$ is invariant under $\nabla|_U$ for all such $(U, j)$. In terms of this, a $\pi$-admissible flat connection on $\mathcal{F}_2$ lifts to a flat connection on $\mathcal{F}_1$, analytically locally via the canonical isomorphism $\mathcal{F}_1|_{\pi^{-1}(U)} \simeq \oplus_j \mathcal{F}_{2,U}^{(j)}$ as $\mathcal{O}_U$-modules.

Connections on a coherent sheaf on a scheme and its flatness - $C^\infty$ version.

We now give a $C^\infty$-version of the previous theme. Let $M$ be a $(C^\infty)$-manifold, $\mathcal{F}$ be a sheaf of finitely presentable $\mathcal{O}_{M,\mathbb{C}}$-modules on $M$, and $\mathcal{A}$ be a sheaf of commutative $\mathcal{O}_{M,\mathbb{C}}$-algebra that is finitely generated as an $\mathcal{O}_{M,\mathbb{C}}$-module and acts on $\mathcal{F}$, rendering $\mathcal{F}$ an $\mathcal{A}$-module as well.

**Definition 2.1.6. [differential and derivation on $\mathcal{A}$, the sheaf $\Omega_\mathcal{A}$ and $\Theta_\mathcal{A}$].** The sheaf of differentials on $\mathcal{A}$ (over $\mathbb{C}$) is the sheaf of $\mathcal{O}_{M,\mathbb{C}}$-modules on $M$ that is associated to the presheaf

$$U \mapsto \Omega_{\mathcal{A}(U)} := \text{Span}_{\mathcal{A}(U)}\{df : f \in \mathcal{A}(U)\}/\sim,$$

where $\text{Span}_{\mathcal{A}(U)}\{df : f \in \mathcal{A}(U)\}$ is the $\mathcal{A}(U)$-module generated by the set $\{df : f \in \mathcal{A}(U)\}$ and $\sim$ is the equivalence relation on $\text{Span}_{\mathcal{A}(U)}\{df : f \in \mathcal{A}(U)\}$ generated by relators:

- (C-linearity) $d(af + bf') = a df - b df'$ for $a, b \in \mathbb{C}$ and $f, f' \in \mathcal{A}(U)$,
- (Leibniz rule) $d(ff') = f'(df) + df'$ for $f, f' \in \mathcal{A}(U)$,
- $df + f' dg = dg$ whenever $f + f' = g$ in $\mathcal{A}(U)$,
- $df dg = dg$ whenever $f = g$ in $\mathcal{A}(U)$.

$\Omega_\mathcal{A}$ is tautologically a sheaf of $\mathcal{A}$-modules. The sheaf $\Theta_\mathcal{A}$ of $\mathbb{C}$-derivations on $\mathcal{A}$ is defined to be the dual sheaf $\text{Hom}_{\mathcal{A}}(\Omega_\mathcal{A}, \mathcal{A})$ of $\Omega_\mathcal{A}$ as an $\mathcal{A}$-module. By taking the anti-symmetric tensor products over $\mathcal{A}$, one has also $\Omega_{\mathcal{A}}^i := \Lambda^i \Omega_{\mathcal{A}}$, $i \in \mathbb{Z}_{\geq 0}$, with $\Omega_{\mathcal{A}}^0 := \mathcal{A}$ and $\Omega_{\mathcal{A}}^1 = \Omega_{\mathcal{A}}$.

---

The language in this note remains manifold-'n'-scheme direct. It is particularly tailored to fit the situation of the (commutative) surrogate of a morphism from an Azumaya manifold to a commutative target-space. One should finally bring in the notion of $C^\infty$-schemes (cf. [Joy4] and references therein) for the completeness of language to study A-branes in the current setting.
Definition 2.1.7. [connection on \( F \) as \( A \)-module, curvature, and minimal flatness].

Denote by \( 
\mathcal{A}F 
\) the sheaf \( F \) as an \( A \)-module. A \((C-)\)connection \( \nabla \) on \( \mathcal{A}F \) is a homomorphism of \( C \)-modules

\[
\nabla : \mathcal{A}F \rightarrow \Omega_A \otimes_A \mathcal{F}
\]

such that

\[
\nabla(fs) = df \otimes s + f \nabla s
\]

for sections \( f \) of \( A \) and sections \( s \) of \( F \) on the same open subset of \( M \). \( \nabla \) extends to a homomorphism of \( A \)-modules

\[
\nabla : \Omega^i_A \otimes_A \mathcal{F} \rightarrow \Omega^{i+1}_A \otimes_A \mathcal{F}
\]

by

\[
\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \wedge \nabla s.
\]

In particular, one has

\[
\mathcal{F} \xrightarrow{\nabla} \Omega_A \otimes_A \mathcal{F} \xrightarrow{\nabla} \Omega^2_A \otimes_A \mathcal{F}.
\]

The \( C \)-module homomorphism from the composition turns out to be an \( A \)-module homomorphism and, hence, defines an \( \text{End}_A(F) \)-valued 2-form \( R \in \Gamma(\text{End}_A(F) \otimes_A \Omega^2_A) \) on \( A \). \( R \) is called the curvature 2-form of \( \nabla \). We say that \( \nabla \) is flat if \( R = 0 \). Suppose that there is a \((C^\infty, C)\)manifold \( M' \) over \( M \) with an \( \mathcal{O}_{M',C} \)-algebra homomorphism \( j^\#: A \rightarrow \mathcal{O}_{M',C} \). This defines an \( A \)-module homomorphism \( j^* : \Omega_A \otimes_A \mathcal{O}_{M',C} \rightarrow \Omega_{M',C} \). We say that \( \nabla \) is flat along \((M', j^\#)\) (or along \( M' \) when \( j^\# \) is understood), in notation \( R_{(M', j)} = 0 \) (or \( R_{M'} = 0 \)), if \( j^* R = 0 \). Here, \( \Omega_{M',C} := \Omega_{\mathcal{O}_{M',C}} = \Omega_{M'} \otimes C \) is the complexified cotangent sheaf of \( M' \). For convenience, we’ll call such \( \nabla \) also a minimally flat connection, with \((M', j^\#)\) being kept implicit.

Remark 2.1.8. [meaning of these structures – why we set them as above]. With the notation and the situation in Definition 2.1.7, when \( j^\# \) is a \( \mathcal{O}_{M',C} \)-algebra quotient, one should think of \( A \) as the manifold \( M' \) with an extension of its standard (complexified) structure sheaf \( \mathcal{O}_{M',C} \), as a \((C^\infty, C)\)-manifold, extended to \( A \) by nilpotents elements. For our application to D-branes, such nilpotent structure is meant to encode an infinitesimal data of how a collection of D-branes of A-type in a space(-time) collide to form a single D-brane supported on \( M' \).

In case there is also an \( \mathcal{O}_{M', C} \)-algebra homomorphism \( \mathcal{O}_{M', C} \rightarrow A \) such that the composition \( \mathcal{O}_{M', C} \rightarrow A \xrightarrow{j^\#} \mathcal{O}_{M', C} \) is the identity map, \( \mathcal{A}F \) is pushed forward to \( M' \), becoming an \( \mathcal{O}_{M',C} \)-module. The notion of a connection on the \( \mathcal{A} \)-module \( \mathcal{A}F \) is then meant to be a connection on \( F \), as a sheaf on \( M' \), that commutes with these nilpotent linear operators on \( F \) since these nilpotent elements \( f \) is meant to correspond to infinitesimal transverse directions to \( M' \) of various order (and hence the evaluation of \( df \) from such nilpotent \( f \) on \( \Theta_{M', C} \) is meant to be zero). In view of this, it is very natural to consider connections \( \nabla \) on \( \mathcal{A}F \) that are flat only along \( M' \), rather than all over \( A \), since these are flat connections on \( F \) on \( M' \) that are compatible with the nilpotent linear operators on \( F \) encoded in \( A \).

Similar existence/non-existence of lifting and descent statements as in the previous theme hold in the current category. In particular, Lemma/Definition 2.1.9. [lifting and descent of flat connection under covering map].

Let \( \pi : M_1 \rightarrow M_2 \) be a finite covering map of \( C^\infty \)-manifolds, \( F_i, i = 1, 2 \), be locally-free \( \mathcal{O}_{M_i,C} \)-module of finite rank on \( M_i \) with \( F_2 = \pi_* F_1 \). Then a flat connection on \( F_1 \) descends to a flat

\footnote{See [G-Sh] of Gómez and Sharpe and [D-K-S] of Donagi, Katz, and Sharpe for a related discussion in the case of B-branes. The same behavior should also happen for A-branes from the viewpoint of morphisms from Azumaya manifolds since the Azumaya structure sheaf contains nilpotent elements.}

\footnote{Note that in algebraic geometry an \( \acute{e}tale \) morphism may not be proper; however in algebraic topology a \textit{covering map} is by definition always proper: for all \( p \in M_2 \), there exists an open neighborhood \( U \ni p \) in \( M_2 \) such that \( \pi : \pi^{-1}(U) \rightarrow U \) is a disjoint union of diffeomorphisms (in the \( C^\infty \) category). Cf. [Hart] vs. [Sp].}
connection on $\mathcal{F}_2$. $\pi$ determines a direct-sum decomposition $\mathcal{F}_2|_U = \oplus_j \mathcal{F}^{(j)}_{2,U}$ on contractible open sets $U \subset M_2$. A connection $\nabla$ on $\mathcal{F}_2$ is said to be $\pi$-admissible if $\mathcal{F}^{(j)}_{2,U}$ is invariant under $\nabla|_U$ for all such $(U,j)$. In terms of this, a $\pi$-admissible flat connection on $\mathcal{F}_2$ lifts to a flat connection on $\mathcal{F}_1$, via the canonical $\mathcal{O}_U$-module isomorphism $\mathcal{F}_1|_{\pi^{-1}(U)} \simeq \oplus_j \mathcal{F}^{(j)}_{2,U}$.

**G-reduced flat connections with respect to a covering.**

Let $c: \hat{M} \to M$ be a covering map of finite degree $\hat{d}$ between two (not-necessarily-compact) manifolds, $\hat{\mathcal{E}}$ be a locally free $\mathcal{O}_{\hat{M},c}$-module of finite rank $\hat{r}$, and $\mathcal{E} := c_*\hat{\mathcal{E}}$, which is a locally free $\mathcal{O}_{M,c}$-module of rank $r = \hat{r}\hat{d}$.

**Definition 2.1.10. [G-reduced flat connection].** Let $G \subset GL_r(\mathbb{C})$ be a subgroup of the complex general linear group. A flat connection $\hat{\nabla}$ on $\hat{\mathcal{E}}$ is said to be $G$-reduced with respect to $c$ if its descent $\nabla$ on $\mathcal{E}$ (cf. Lemma/Definition 2.1.9) has holonomy in $G$ (i.e. the holonomy group of $\nabla$ on $\mathcal{E}$ at each $p \in M$ is a conjugate of $G$ in $GL_r(\mathbb{C})$).

**Example 2.1.11. [U(r)-reduced flat connection w.r.t. c].** Take $\hat{\mathcal{E}}$ to be the sheaf of $(C^\infty)$-sections of a complex Hermitian vector bundle $\hat{E}$ on $\hat{M}$ of rank $\hat{r}$ with a compatible flat connection $\hat{\nabla}$ (so that the parallel transports are isometries of the Hermitian fibers of $\hat{E}$). I.e. $(\hat{E},\hat{\nabla})$ is the descent, via a representation

$$\hat{\rho} : \pi_1(\hat{M}) \to U(\hat{r})$$

(after a base-point $\hat{*} \in \hat{M}$ is specified implicitly for each connected component of $\hat{M}$ so that $* := c(\hat{*}) \in M$ are all identical), of a trivialized trivial Hermitian $\mathbb{C}^r$-bundle on the universal covering space $\tilde{M}$ of $M$. The push-forward $\mathcal{E} = c_*\hat{\mathcal{E}}$ is now the sheaf of sections of a vector bundle $E$ of rank $r$ on $M$ with a Hermitian structure and a flat connection $\nabla$ canonically induced from from $(\hat{E},\hat{\nabla})$ via $c$. Furthermore, by construction, the parallel transports determined by $\nabla$ are isometries of fibers of $E$. It follows that $\nabla$ is a flat $U(r)$-connection on $E$.

**Supersymmetric D-branes of A-type (i.e. A-branes).**

**Definition 2.1.12. [special Lagrangian submanifold with a phase (factor)].** Let $Y = (Y,J,\omega,\Omega)$ be a Calabi-Yau manifold and $\theta \in [0,2\pi)$ (or $(-\pi,\pi]$ by convention) be a constant. Then a special Lagrangian submanifold $L$ with respect to the calibration $Re(e^{-i\theta}\Omega)$ is called a special Lagrangian submanifold with a phase factor $e^{i\theta}$ in $Y$. In particular, $\Omega|_L = e^{i\theta}vol_L$ on $L$, where $vol_L$ is the volume-form on $L$ induced by the Kähler metric $\omega$.

**Definition 2.1.13. [connection-with-singularity/singular connection].** Given a $(C^\infty)$-manifold-with-singularity $M$ and a finitely presented $\mathcal{O}_{M,c}$-module $F$. By a connection-with-singularity (or singular connection), we mean a connection $\nabla$ on $F|_U$ for some open dense manifold-subset $U \subset M$. Flatness and holonomy of $\nabla$ are defined as flatness and holonomy of $\nabla$ on $F|_U$.

9 **Terminology.** We will also call this factor directly a phase to synchronize with some stringy literatures, though it is also standard to leave the latter term for $\theta$ alone.
Definition-Prototype 2.1.14. [A-brane (with unitary minimally flat singular connection) on Calabi-Yau space]. Let \(Y = (Y,J,\omega,\Omega)\) be a Calabi-Yau \(n\)-fold. A D-brane of A-type (i.e. A-brane) with a phase factor \(e^{i\theta}\) on \(Y\) (in the regime of the Wilson’s theory-space of string theory specified at the beginning of this section) is a morphism

\[
\varphi : (X^\mathbb{A}, \mathcal{E}) \longrightarrow Y
\]

together with constraints and data encoded in the following diagram:

\[
\begin{array}{ccc}
(X_\varphi, A_\varphi) & \xrightarrow{f_\varphi} & L \subset Y, \\
\downarrow \pi_\varphi & & \downarrow \pi_\varphi \\
X & & \\
\end{array}
\]

where the following Properties (1) – (5) hold:

1. is the surrogate and the related maps and sheaves associated to \(\varphi\).

2. The underlying \(n\)-manifold \(X\) in \((X^\mathbb{A}, \mathcal{E})\) is oriented; the underlying singular \(n\)-manifold \(X_\varphi\) in \((X_\varphi, A_\varphi)\) is equipped with the induced orientation from that of \(X\) via \(\pi_\varphi\).

3. \(L = \text{Im}\,\varphi\) is a special Lagrangian singular submanifold with a phase factor \(e^{i\theta}\) in \(Y\).

4. There exists an open dense submanifold \(V_\varphi \subset X_\varphi\) such that
   
   (4.1) \(V := \pi_\varphi(V_\varphi)\) is an open dense submanifold of \(X\);
   \(\pi_\varphi|_{V_\varphi} : V_\varphi \rightarrow V\) is a covering map,

   (4.2) \(f_\varphi|_{V_\varphi} : V_\varphi \rightarrow Y\) is an immersion, and

   (4.3) \(f_\varphi|_{V_\varphi} : V_\varphi \rightarrow L\) is orientation-preserving.

   Note that there are then built-in \(O_{V_\varphi, \mathbb{C}}\)-algebra homomorphisms

   \(O_{V_\varphi, \mathbb{C}} \longrightarrow A_\varphi|_{V_\varphi} \longrightarrow O_{V_\varphi, \mathbb{C}}\)

   with the composition being the identity map.

5. Let \(r\) be the rank of \(\mathcal{E}\); \(\nabla\) is a singular connection on \(A_\varphi \mathcal{E}_\varphi\) that is defined on and is flat along an open dense subset of \(V_\varphi \subset X_\varphi\) in Item (4) with holonomy, when descends to \(V \subset X\), in a subgroup of \(GL_r(\mathbb{C})\) that is isomorphic to the unitary group \(U(r)\).
On the mathematical side, we will call the above data a *special Lagrangian morphism with a unitary minimally flat connection-with-singularity* \(^{10}\) and denote it collectively by \((\varphi, \nabla)\).

A morphism (or arrow) \((\varphi_1, \nabla) \to (\varphi_2, \nabla)\) is a morphism/arrow \(\varphi_1 \to \varphi_2\), encoded by the data \((h, \tilde{h}, \tilde{h})\) in Definition 1.2.1, that satisfies the additional condition that the \(A_{\varphi_1}\)-module isomorphism \(\tilde{h}^*E_{\varphi_2} \to E_{\varphi_1}\) takes \(\tilde{h}^*\nabla_2\) to \(\nabla_1\) over an open dense subset of \(X_1\).

**Remark 2.1.15.** [from A-brane to constructible sheaf and perverse sheaf]. The pair \((A_{\varphi}, E_{\varphi}, \nabla)\) reminds one very strongly of constructible sheaves and, hence, perverse sheaves on a stratified manifold-with-singularity. It would be very interesting if such a link/correspondence can truly be built naturally functorially.

**Example 2.1.16.** [simple A-brane (with unitary flat connection-with-singularity)]. A special class of A-branes in the sense of Definition 2.1.14 with \(A_{\varphi}|_{V_\varphi} = O_{V_\varphi, C}\) can be constructed from the following data \(((c, f), (\hat{E}, \hat{\nabla}))\):

\[
\begin{array}{c}
\hat{E} \\
\downarrow \\
\hat{X} \\
\downarrow \\
(c, f) \\
\downarrow \\
\hat{X} \times Y \\
\downarrow pr_1 \\
Y,
\end{array}
\]

where

- \(Y = (Y, J, \omega, \Omega)\) is a Calabi-Yau \(n\)-fold;
- \(X, \hat{X}\) are closed oriented \(n\)-manifold and \(c : \hat{X} \to X\) is an orientation-preserving branched covering map of finite degree \(\hat{d}\) over a codimension-2 submanifold;
- \(f : \hat{X} \to Y\) is a smooth map that is an immersion on an open dense subset \(\hat{V} \subset \hat{X}\) such that \(f|_{\hat{V}}\) defines a special Lagrangian submanifold of phase factor \(e^{i\theta}\) on \(Y\); (thus, \(f : \hat{X} \to Y\) defines a special Lagrangian submanifold-with-singularity with a phase factor \(e^{i\theta}\) in \(Y\));
- \((\hat{E}, \hat{\nabla})\) is a locally free \(O_{\hat{X}, \mathbb{C}}\)-module of finite rank \(\hat{r}\) with a flat \(U(\hat{r})\)-connection.

---

\(^{10}\)String-Theoretical Remark [pure D-brane vs. D-brane smearing]. The gauge field \(A\) (i.e. connection \(\nabla\)) on the Chan-Paton module/sheaf of a D-brane is the most fundamental field thereupon besides fields that govern the deformations of the brane (i.e. \(\varphi\) in our setting). In general, when the gauge field strength \(F_A\) (i.e. curvature) of \(A\) is nonzero, \(A\) can couple, via \(F_A\), with the Ramond-Ramond fields \(C\) on target the space-time and serves as a source/charge for \(C\) as if there are lower-dimensional D-branes that are smeared along the D-brane one begins with. (See, e.g., [Doug] (1995) for an early discussion, [D-M: Sec. 2.1] (2007) for the case of supersymmetric D-branes of B-type, and [Joh: Chapter 9] (2003) for a related highlight/review.) From this aspect, a D-brane with a gauge field that is flat (i.e. \(F_A = 0\)) is special in the sense that it is *pure* without being mixed implicitly/effectively with lower-dimensional D-branes. In our situation, such flat connection \(\nabla\) is defined only on an open dense manifold-subset of \(X_\varphi\) and can have singularity around the singular locus \(X_{\varphi, \text{sing}}\) of the surrogate \(X_\varphi\) of \(\varphi\). \(\nabla\) may still have non-trivial holonomy/monodromy for a small meridian circle around \(X_{\varphi, \text{sing}}\). In other words, the curvature of \(\nabla\) now can be a Lie-algebra-valued distribution-like 2-form and be supported on \(X_{\varphi, \text{sing}}\). When this happens, it indicates that lower-dimensional D-branes are smeared effectively along \(X_{\varphi, \text{sing}}\).
From this data, one can recover the underlying special Lagrangian morphism with a minimally flat connection-with-singularity \((\varphi: (X^{\mathbb{A}}, \mathcal{E}) \to Y, \nabla)\) as follows:

- **[domain]** \((X^{\mathbb{A}}, \mathcal{E}) = (X, \mathcal{O}_{X^{\mathbb{A}}} = \text{End}_{\mathcal{O}_{x,c}}(\mathcal{E}), \mathcal{E} := c_{*}\hat{\mathcal{E}})\);

  as a \(\mathcal{O}_{X,c}\)-module, \(\mathcal{E}\) has rank \(r = \hat{r}d\).

- **[surrogate]** \(X_{\varphi} = (c, f)(\hat{X}) = (c \times f)(\hat{X}) \subset X \times Y\) and \(\mathcal{A}_{\varphi} = \mathcal{O}_{X_{\varphi},c}\) as a smooth manifold with singularities, which acts tautologically on \(\mathcal{E}\) since \((c, f)_{*}\hat{\mathcal{E}} = \mathcal{O}_{X_{\varphi},c} \mathcal{E}\).

- **[maps]** \(\pi_{\varphi} = pr_{1}|_{X_{\varphi}}: X_{\varphi} \to X\) and \(f_{\varphi} = pr_{2}|_{X_{\varphi}}: X_{\varphi} \to Y\).

- **[minimally flat connection-with-singularity]** From the given generic covering/immersion property of \((c, f)\), there exists an open dense submanifold \(V_{\varphi} \subset X_{\varphi}\) such that

  - \(f_{\varphi}|_{V_{\varphi}}: V_{\varphi} \to Y\) is a special Lagrangian immersion of phase factor \(e^{i\theta}\);

  - \(V := \pi_{\varphi}(V_{\varphi}) \subset X\) (resp. \(\hat{V} := (c, f)^{-1}(V_{\varphi}) \subset \hat{X}\)) is an open dense submanifold with the property that the three maps, \(c|_{\hat{V}}, (c, f)|_{\hat{V}},\) and \(\pi_{\varphi}|_{V_{\varphi}}\), in the commutative diagram

    $$
    \begin{align*}
    \hat{V} &\subset \hat{X} \\
    (c, f)|_{\hat{V}} &\downarrow \\
    V_{\varphi} &\subset X_{\varphi} \\
    \pi_{\varphi}|_{V_{\varphi}} &\downarrow \\
    V &\subset X
    \end{align*}
    $$

  are all covering maps;

  - \(V_{\varphi} = \pi_{\varphi}^{-1}(V)\).

In particular,

\[(c, f) : (c, f)^{-1}(U') \to U' \quad (\text{resp.} \quad \pi_{\varphi} : \pi_{\varphi}^{-1}(U) \to U, \quad c : c^{-1}(U) \to U)\]

is a disjoint unions of diffeomorphisms for contractible open sets \(U \subset V\) (resp. \(U' \subset V_{\varphi}, U \subset V\)). It follows from Lemma/Definition 2.1.9 that

- the flat connection \(\nabla\) on \(\hat{\mathcal{E}}\) descends to a flat connection-with-singularity \(\nabla\) on \(\mathcal{O}_{X_{\varphi},c} \mathcal{E}\).

The holonomy of \(\nabla\), when descends to \(X\), lies \(U(r)\) by considering \(c|_{\hat{V}}\) and Example 2.1.11.

In this way, we obtain all the data that is needed to describes a morphism \(\varphi\) from an Azumaya manifold with a fundamental module to \(Y\), with a unitary minimally flat (singular) connection \(\nabla\). In this note, we’ll use such simple A-branes \((\varphi, \nabla)\) – in the sense that their image \(\varphi(X^{\mathbb{A}})\) in the Calabi-Yau space \(Y\) has no nilpotent structure along it on an open dense subset – to illustrate some well-known D-brane behaviors in string theory.
2.2 Higgsing/un-Higgsing of A-branes via deformations of morphisms.

The Higgsing/un-Higgsing behavior of the gauge symmetry on D-branes in string theory arises, in the current setting, from deformations of morphisms from Azumaya spaces with a fundamental module. We have already seen this in [L-Y1: Sec. 4] (D(1)), [L-Y2: Sec. 2 and Figure 2-1] (D(3)), and [L-Y4: Example 5.1.11] (D(5)) (cf. [L-Y5: Figure 2-1-1 and caption] (D(6))) for the case of supersymmetric D-branes of B-type (i.e. B-branes) in various contexts. For A-branes, it follows from the work of McLean [McL] that such a behavior natural occurs whenever the target special Lagrangian submanifold $L \subset Y$ in a Calabi-Yau manifold admits a finite covering $\tilde{L} \to L$ with the first Betti number strictly increased: $b^1(\tilde{L}) > b^1(L)$. (Here, $\tilde{L}$ is allowed to be disconnected even when $L$ is connected.) The work of Joyce [Joy3] allows one to extend such result to $L$ with isolated conical singularities as well. The following example on a complex Calabi-Yau torus is motivated by the example of Denef in [De3: Sec. 6.1]. It illustrates a Higgsing (resp. un-Higgsing) behavior of A-branes that involves also an assembling (resp. disassembling) of a collection of “small branes” into a “large brane” (resp. a “large brane” into a collection of “small branes”) in the process. Some necessary background from the work of Joyce [Joy3] and notations to understand this example are summarized in Appendix A.1.

Example 2.2.1. [Higgsing and un-Higgsing of A-branes under a Morse cobordism]. We explain the construction in five steps.

(a) A necklace of special Lagrangian submanifolds. Let $\mathbb{C}^3$ be the complex 3-space with coordinates $(z^1, z^2, z^3)$, the standard flat Kähler structure $\frac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3)$, and the holomorphic 3-form $dz^1 \wedge dz^2 \wedge dz^3$; $C_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be the complex 1-torus of modulus $\tau$ with $\text{Im}\,\tau > 0$; and

$$ Y = C_{\tau_1} \times C_{\tau_2} \times C_{\tau_3}, \quad \text{with } \tau_1\tau_2\tau_3 \in \mathbb{R}_{<0} \text{ and } (\tau_1 - 1)(\tau_2 - 1)(\tau_3 - 1) \in \mathbb{R}_{>0}, $$

be a product complex 3-torus equipped with the complex structure $J$, the flat metric with the Kähler form $\omega$ and the holomorphic 3-form $\Omega$, all from the descent as a quotient space of $\mathbb{C}^3$. We will denote $(Y, J, \omega, \Omega)$ also simply by $Y$. The quotient $C_{\tau_i} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_i)$ specifies an isomorphism $H_1(C_{\tau_i}; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}\tau_i$ as $\mathbb{Z}$-modules. Let $(\alpha_i, \beta_i)$ be the basis of $H_1(C_{\tau_i}; \mathbb{Z})$ that corresponds $(1, \tau_i)$; cf. [De3: Sec. 6.1, Figure 9]. One has from the Kühnö formula that

$$ H_3(Y; \mathbb{Z}) = \oplus_{j_1+j_2+j_3=3} H_{j_1}(C_{\tau_1}; \mathbb{Z}) \times H_{j_2}(C_{\tau_2}; \mathbb{Z}) \times H_{j_3}(C_{\tau_3}; \mathbb{Z}). $$

Consider the following three embedded special Lagrangian submanifolds in $Y$ from special Lagrangian 3-planes in $\mathbb{C}^3$, with the orientation specified by the restriction of $\text{Re}\,\Omega:$

| sL in $Y$ | lifting in $\mathbb{C}^3$ | $[\cdot] \in H_3(Y; \mathbb{Z})$ |
|-----------|--------------------------|-----------------|
| $L_1$     | $\text{Re}\,\mathbb{C}^3 = \mathbb{R}(1,0,0) + \mathbb{R}(0,1,0) + \mathbb{R}(0,0,1)$ | $\alpha_1 \times \alpha_2 \times \alpha_3$ |
| $L_2$     | $-\mathbb{R}(\tau_1, 0, 0) - \mathbb{R}(0, \tau_2, 0) - \mathbb{R}(0,0, \tau_3)$ | $-\beta_1 \times \beta_2 \times \beta_3$ |
| $L_3$     | $\left(\frac{1}{\tau_1}, \frac{1}{\tau_2}, \frac{1}{\tau_3}\right) + \mathbb{R}(\tau_1 - 1, 0, 0) + \mathbb{R}(0, \tau_2 - 1, 0) + \mathbb{R}(0,0, \tau_3 - 1)$ | $(\beta_1 - \alpha_1) \times (\beta_2 - \alpha_2) \times (\beta_3 - \alpha_3)$ |

It follows from the constraints $\tau_1\tau_2\tau_3 \in \mathbb{R}_{<0}, \ (\tau_1 - 1)(\tau_2 - 1)(\tau_3 - 1) \in \mathbb{R}_{>0}$ and, hence, $(\frac{\tau_1-1}{\tau_1} \frac{\tau_2-1}{\tau_2} \frac{\tau_3-1}{\tau_3}) \in \mathbb{R}_{<0}$ (note $\text{Im}\,\tau > 0$ implies $\text{Im}\,(-1) > 0$ and $\text{Im}\,(\frac{1}{\tau}) > 0$) that:

- the sum of the characteristic angles from $L_1$ to $L_2$ (resp. from $L_1$ to $L_3$, from $L_2$ to $L_3$) is $\pi$ (resp. $2\pi$, $\pi$).

$L_i$ and $L_j$, $i \neq j$, intersect transversely at exactly one point. The oriented intersection numbers of $L_1$, $L_2$, and $L_3$ at their intersection point are given by

$$ L_1 \cdot L_2 = -L_2 \cdot L_1 = +1, \quad L_2 \cdot L_3 = -L_3 \cdot L_2 = +1, \quad L_3 \cdot L_1 = -L_1 \cdot L_3 = +1. $$

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Then, the family \( f : L := L_1 \amalg L_2 \amalg L_3 \to L_1 \cup L_2 \cup L_3 \subset Y \) with isolated transverse intersections and let \( x^+_i := f^{-1}(y_{ij}) \cap L_i \) and \( x^-_i := f^{-1}(y_{ij}) \cap L_j \). Then, the sum of the characteristic angles from \( f_*T_{x^+_i}L \) to \( f_*T_{x^-_i}L \) at \( y_{ij} \) is \( \pi \). Recall [Joy3: V. Sec. 9.2, Theorem 9.7] (cf. Theorem A.1.4 in Appendix A.1) and consider the following linear system with constraints:

\[
A_{12} - A_{31} = 0, \quad A_{23} - A_{12} = 0, \quad A_{31} - A_{23} = 0, \quad \text{with} \quad A_{12}, A_{23}, A_{31} > 0.
\]

This has solutions:

\[
A_{12} = A_{23} = A_{31} > 0.
\]

It follows that the special Lagrangian submanifold \( L_1 \cup L_2 \cup L_3 \subset Y \) with conical singularities \( \{y_{12}, y_{23}, y_{31}\} \) is smoothable:

- There exists a smooth family of special Lagrangian embeddings

\[
f^{(t)} : L^{(t)} \to Y, \quad t \in (0, \varepsilon) \quad \text{for some} \quad \varepsilon > 0
\]

such that \( L^{(t)} \simeq \) the (self-)connected sum \( N \) of \( L \) at the pairs of points \( (x^+_{12}, x^-_{12}), (x^+_{23}, x^-_{23}), \) and \( (x^+_{31}, x^-_{31}) \) and that \( f^{(t)} \to f \) in the sense of currents as \( t \to 0 \).

Since each \( L_i \) is diffeomorphic to the real 3-torus \( T^3 \), one has

\[
b^1(L_1 \amalg L_2 \amalg L_3) = 9 \quad \text{and} \quad b^1(L_1 \cup L_2 \cup L_3) = b^1(N) = 10.
\]

(c) A family of A-branes on \( Y \) that wrap \( f^{(t)}(L^{(t)}), t \in [0, \varepsilon) \.

Let \( \pi_{(-\varepsilon, \varepsilon)} : X_{(-\varepsilon, \varepsilon)} \to (-\varepsilon, \varepsilon) \) be a Morse family of 3-manifolds with singularities, with

\[
X_t := \pi_{(-\varepsilon, \varepsilon)}^{-1}(t) \simeq \begin{cases} 
L = L_1 \amalg L_2 \amalg L_3 & \text{for} \ t \in (-\varepsilon, 0), \\
L_1 \cup L_2 \cup L_3 & \text{for} \ t = 0, \\
L^{(t)} & \text{for} \ t \in (0, \varepsilon).
\end{cases}
\]

Then, the family \( f^{(t)}, t \in (0, \varepsilon) \), and \( f \) together define a continuous map

\[
\begin{align*}
X_{(-\varepsilon, \varepsilon)} \xrightarrow{F} (-\varepsilon, \varepsilon) \times Y, \\
\xrightarrow{\pi_{(-\varepsilon, \varepsilon)}} (-\varepsilon, \varepsilon) \\
\xrightarrow{pr_1} (-\varepsilon, \varepsilon), \\
F(t, \cdot) = \begin{cases} 
 f(\cdot) & \text{for} \ t \in (-\varepsilon, 0), \\
 Id_{L_1 \cup L_2 \cup L_3}(\cdot) & \text{for} \ t = 0, \\
 f^{(t)}(\cdot) & \text{for} \ t \in (0, \varepsilon),
\end{cases}
\end{align*}
\]

over \( (-\varepsilon, \varepsilon) \) that is smooth on \( \pi_{(-\varepsilon, \varepsilon)}^{-1}((-\varepsilon, 0) \cup (0, \varepsilon)) \).

Let \( c \) in

\[
(\hat{\mathcal{E}}_{(-\varepsilon, \varepsilon)}, \hat{\mathcal{V}}_{(-\varepsilon, \varepsilon)})
\]

\[
\xrightarrow{\hat{\pi}_{(-\varepsilon, \varepsilon)}} (-\varepsilon, \varepsilon) \\
\xrightarrow{\hat{X}_{(-\varepsilon, \varepsilon)}} X_{(-\varepsilon, \varepsilon)} \\
\xrightarrow{c} X_{(-\varepsilon, \varepsilon)}
\]

\[
(\hat{\mathcal{E}}_{(-\varepsilon, \varepsilon)}, \hat{\mathcal{V}}_{(-\varepsilon, \varepsilon)})
\]

\[
\xrightarrow{\hat{\pi}_{(-\varepsilon, \varepsilon)}} (-\varepsilon, \varepsilon) \\
\xrightarrow{\hat{X}_{(-\varepsilon, \varepsilon)}} X_{(-\varepsilon, \varepsilon)} \\
\xrightarrow{c} X_{(-\varepsilon, \varepsilon)}
\]
be a covering map over \((-\varepsilon, \varepsilon)\) of (finite) degree \(r > 1\) and \((\hat{E}_{(-\varepsilon, \varepsilon)}, \hat{\nabla}_{(-\varepsilon, \varepsilon)})\) be a complex line bundle on \(\hat{X}_{(-\varepsilon, \varepsilon)}\), with a \(U(1)\) flat connection. Then one has the following diagram of maps

\[
\begin{array}{ccc}
(\hat{E}_{(-\varepsilon, \varepsilon)}, \hat{\nabla}_{(-\varepsilon, \varepsilon)}) & \rightarrow & (c,F) \\
\hat{X}_{(-\varepsilon, \varepsilon)} & \rightarrow & \hat{X}_{(-\varepsilon, \varepsilon)} \times Y \\
\downarrow & & \downarrow & \\
\hat{X}_{(-\varepsilon, \varepsilon)} \times \pi_{(-\varepsilon, \varepsilon) \times (F, c)} & \rightarrow & (-\varepsilon, \varepsilon) \times Y \\
\end{array}
\]

in which \(\text{Im}(c, F \circ c) = \text{Graph}(F)\) with multiplicity \(r\). Let

\[E_{(-\varepsilon, \varepsilon)} := c_{\ast} \hat{E}_{(-\varepsilon, \varepsilon)} = pr_{1 \ast} \circ (c, F \circ c)_{\ast} \hat{E}_{(-\varepsilon, \varepsilon)} \cdot \]

The diagram defines a \(C\)-algebra homomorphism \(\varphi_{(-\varepsilon, \varepsilon)}^{\hat{E}} : O_{(-\varepsilon, \varepsilon) \times Y, c} \rightarrow \text{End}O_{X_{(-\varepsilon, \varepsilon)} \times \hat{E}_{(-\varepsilon, \varepsilon)}}\) and, hence, a morphism \(\varphi_{(-\varepsilon, \varepsilon)} : (X_{\hat{A}}^{\varepsilon}, E_{(-\varepsilon, \varepsilon)}) \rightarrow (-\varepsilon, \varepsilon) \times Y\) over \((-\varepsilon, \varepsilon)\), with a unitary minimally flat connection-with-singularity. We will think of the latter also interchangeably as a family of morphisms \(\varphi_t : (X_{\hat{A}}^{\varepsilon}, E_t) \rightarrow Y\;|\;t \in (-\varepsilon, \varepsilon)\), with a unitary minimally flat connection-with-singularity.

**Lemma 2.2.1.(d) [first Betti number].** Let \(\hat{X}_t := \hat{\pi}_{(-\varepsilon, \varepsilon)}^{-1}(t)\) for \(t \in (-\varepsilon, \varepsilon)\) and \(\Gamma\) be the dual graph of \(\hat{X}_0\) with the number of vertices \(\Gamma_{(0)}\) and the number of edges \(\Gamma_{(1)} = 3r\). Then, \(b^1(\hat{X}_t) = 3|\Gamma_{(0)}|\), for \(t \in (-\varepsilon, 0)\), and \(b^1(\hat{X}_t) = 3r + 2|\Gamma_{(0)}| + 1\), for \(t \in [0, \varepsilon)\). In particular, \(b^1(\hat{X}_t) \geq 3r + 7 > b^1(L_1 \cup L_2 \cup L_3) = b^1(N) = 10\) for \(t \in [0, \varepsilon)\) since \(r > 1\) and \(3 \leq |\Gamma_{(0)}| \leq 3r\).

Here, recall that the dual graph \(\Gamma\) of \(\hat{X}_0\) has one vertex \(v_i\) for each manifold-component \(M_i \simeq \mathbb{T}^3\) of \(\hat{X}_0\) and one edge \(e_{ij}\) connecting \(v_i\) and \(v_j\) for each intersection point in \(M_i \cap M_j\); \(\Gamma_{(0)}\) is the set of vertices of \(\Gamma\) and \(\Gamma_{(1)}\) is the set of edges of \(\Gamma\).

**Proof.** That \(b^1(\hat{X}_t) = 3|\Gamma_{(0)}|\) for \(t \in (-\varepsilon, 0)\) follows from the fact that the only finite-order covering space of \(\mathbb{T}^3\) is homeomorphic to \(\mathbb{T}^3\). That \(b^1(\hat{X}_t) = b^1(\hat{X}_0)\) for \(t \in (0, \varepsilon)\) follows from the fact that \(\hat{X}_0\) is topologically obtained from \(\hat{X}_t, t \in (0, \varepsilon)\), by pinching a disjoint union of two-sided embedded \(S^2\)’s and, hence, the fundamental groups \(\pi_1(\hat{X}_t) \simeq \pi_1(\hat{X}_0)\) for \(t \in (0, \varepsilon)\). It remains to compute \(b^1(\hat{X}_0)\). Which follows from the following basic facts:

- the exact sequence of groups

\[
1 \rightarrow \pi_1(\bigvee_{v_i \in \Gamma_{(0)}} M_i) \rightarrow \pi_1(\hat{X}_0) \rightarrow \pi_1(\Gamma) \rightarrow 1
\]

where \(\bigvee_{v_i \in \Gamma_{(0)}} M_i\) is the bouquet of \(\{M_i : v_i \in \Gamma_{(0)}\}\) following a(ny) spanning tree of \(\Gamma\),

- \(\pi_1(\bigvee_{v_i \in \Gamma_{(0)}} M_i)\) is isomorphic to the free product of \(\Gamma_{(0)}\)-many copies of \((\pi_1(M_i) \simeq \pi_1(\mathbb{T}^3) \simeq \mathbb{Z}^3)\),

- \(\pi_1(\Gamma)\) is isomorphic to the free group on \(1 - \chi(\Gamma) = 1 - |\Gamma_{(0)}| + 3r\) generators, and
H_1(\bullet; \mathbb{Z}) is the abelianization of \pi_1(\bullet).

This concludes the proof.

(e) The dimension of deformation spaces and Higgsing/un-Higgsing. Let \hat{f}_t := F(\cdot, t) \circ c(\cdot, t) : \hat{X}_t \to Y for \epsilon \in (-\epsilon, \epsilon). For \epsilon \in (-\epsilon, 0), \hat{X}_t is smooth. The deformation space \mathcal{M}_t^{sLag} of special Lagrangian immersions from \hat{X}_t to Y is thus a manifold of dimension \|b^1(\hat{X}_t) = 3|\Gamma(0)|\| around [\hat{f}_t]. For \epsilon = 0, \hat{X}_t is a union of \{\hat{f}_0\}-many T^2-components. The deformation space \mathcal{M}_0^{sLag} of special Lagrangian immersions from \hat{X}_0 to Y contains thus a manifold of dimension 3|\Gamma(0)| around [\hat{f}_0]. For \epsilon \in (0, \epsilon), \hat{X}_t is smooth again. The deformation space \mathcal{M}_t^{sLag} of special Lagrangian immersions from \hat{X}_t to Y is thus a manifold of dimension \|b^1(\hat{X}_t) = 3r + 2|\Gamma(0)| + 1\| around [\hat{f}_t]. As \|b^1(\hat{X}_t)\| is locally constant on \{\hat{f}_t : t \in (-\epsilon, 0)\} \cup \{\hat{f}_t : t \in (0, \epsilon)\}, the collection \{\mathcal{M}_t^{sLag} : t \in (-\epsilon, \epsilon)\} forms a topological space \mathcal{M}_{(-\epsilon, \epsilon)} over (-\epsilon, \epsilon), with the topology from the topology of \hat{X}_{(-\epsilon, \epsilon)} and the topology on the space of maps in question in the sense of currents. By construction, \mathcal{M}_{(-\epsilon, \epsilon)}/(-\epsilon, \epsilon) contains \{\hat{f}_t : t \in (-\epsilon, \epsilon)\} as a (continuous) section \sigma that is smooth on \{\hat{f}_t : t \in (-\epsilon, 0)\} \cup \{\hat{f}_t : t \in (0, \epsilon)\}. Furthermore, there is a submanifold in \mathcal{M}_{(-\epsilon, \epsilon)} of relative dimension \#\sigma > 10 = \|b^1(L_1 \cup L_2 \cup L_3) = b^1(\delta)\| over \{0, \epsilon\} that contains \{\hat{f}_t : t \in (0, \epsilon)\}. It follows that one can perturb \sigma to another section \sigma' - representing a new family \{\hat{f}'_t =: \hat{X}_t \to Y | t \in (-\epsilon, \epsilon)\} of special Lagrangian maps - that remains continuous, is identical to \sigma on \{\hat{f}_t : t \in (0, \epsilon)\}, and is smooth on \{\hat{f}_t : t \in (0, \epsilon)\} such that the following condition holds:

- For all \epsilon \in (0, \epsilon), in the local parameterization of \mathcal{M}_t^{sLag} around [\hat{f}_t] by a (finite-dimensional) submanifold in the (infinite-dimensional) Banach manifold of graphs in T^\ast \hat{X}_t \simeq \mathcal{N}_\hat{f}_t of closed 1-forms on \hat{X}_t under a Sobolev norm, the closed 1-form \hat{\xi}'_t on \hat{X}_t associated to \hat{f}'_t is not the pull-back of a closed 1-form on \hat{X}_t under the covering map \sigma. Here, \mathcal{N}_\hat{f}_t \subset \hat{f}_t T^\ast \hat{X}_t Y is the pull-back of \hat{X}_t in Y along \hat{f}_t.

In particular, for example, \hat{f}'_t : \hat{X}_t \to Y does not factor through a special Lagrangian map from \hat{X}_t to Y for \epsilon \in (0, \epsilon). As a result, for \epsilon \in (0, \epsilon), the overlapped sheets of \sigma : \hat{X}_t \to X_t under \hat{f}_t are now separated by \hat{f}'_t and the image Chan-Paton module \varphi'_t E_t of the associated new family of morphisms \{\varphi'_t : (X_t^A, E_t) \to Y | t \in (-\epsilon, \epsilon)\} from Azumaya spaces with a fundamental module to Y exhibits now a Higgsing (resp. un-Higgsing) phenomenon as \epsilon moves away from 0 for \epsilon \in (0, \epsilon) (resp. as \epsilon moves to 0 for \epsilon \in (0, \epsilon)). Note that the data of the unitary minimally flat connection-with-singularity that accompanies the deformed family of morphisms \varphi'_t still comes from the U(1) flat connection \hat{\nabla}^{(-\epsilon, \epsilon)} on E_{(-\epsilon, \epsilon)} over \hat{X}_{(-\epsilon, \epsilon)}. This concludes the example.

In this example, we fix the Calabi-Yau 3-fold Y in question. In Sec. 3.2, we will see that such Higgsing/un-Higgsing behavior of D-branes – as morphisms from Azumaya spaces with a fundamental module – mixed with assembling/disassembling of branes can also occur when the D-brane is driven to deform alongside with the deformation of the complex structures on Y along an attractor flow.

2.3 Large- vs. small-brane wrapping via deformations of morphisms.

The long vs. short string wrapping behavior of matrix-strings in string theory (e.g., [D-V-V], [Joh: Sec. 16.3.3], [M-S]) generalizes to a large- vs. small-brane wrapping behavior of D-branes. Such phenomenon can be produced in our context via morphisms from an Azumaya
manifold/scheme (with a fundamental module) and their deformations. We explain first two basic local differential topological operations for such a purpose in the case of A-branes and then how transitions between large-brane wrapping and small-brane wrapping can be realized via deformations of morphisms. A simplest example that distinguishes this nature of A-branes and a related question are given in the end.

Gluing of manifolds with singularities along codimension-1 loci.

**Definition 2.3.1. [irreducible component].** Let \( M \) be a manifold with singularities with the smooth locus \( M_{\text{smooth}} \) and the singular locus \( M_{\text{singular}} := M - M_{\text{smooth}} \). Then \( M_{\text{smooth}} \) is a disjoint union \( \bigsqcup_i U_i \) of smooth manifolds \( U_i \). The closure \( \overline{U}_i \subset M \) of each \( U_i \) in \( M \) is called an irreducible component of \( M \).

Given a (possibly disconnected) oriented manifold \( M \) with singularities, let \( Z = \bigsqcup_{i=1}^l Z_i \subset M_{\text{smooth}} \) be a (disconnected) codimension-1 compact smooth oriented embedded submanifold-with-boundary in \( M \) and \( U = \bigsqcup_{i=1}^l U_i \subset M_{\text{smooth}} \) be a manifold-neighborhood of \( Z \) in \( M \) with fixed diffeomorphisms \( (U_i, Z_i) \sim (U_0, Z_0) \) for \( i = 1, \ldots, l \) and for some smooth submanifold-with-(smooth-)boundary \( Z_0 \) in a smooth manifold \( U_0 \). Let \( DZ := Z^+ \cup_{\partial Z} Z^- \) be the doubling of \( Z \) along \( \partial Z \), (here, \( Z^+ = Z \) and \( Z^- \) is \( Z \) with the orientation reversed), and \( \hat{M} \) be the oriented manifold-with-boundary with singularities, obtained from the tautological compactification of \( M - Z \) by \( DZ \). By construction, \( DZ \) constitutes now some boundary components of \( \hat{M} \), with the induced orientation, and there is an orientation-reversing involution \( DZ \to DZ \) that leaves \( \partial Z \subset DZ \) fixed and descends to the identity map on \( Z_0 \). Let \( g : Z^+ \to Z^- \) be an orientation-reversing diffeomorphism that descends to the identity map on \( Z_0 \). (Caution that, in general, this is not a diffeomorphism on \( DZ \).) Let \( M' \) be the quotient manifold with singularities by the equivalence relation generated by \( z \sim g(z) \) for \( z \in DZ \subset \hat{M} \). Then it follows from the construction that:

1. The \( DZ \)-boundary of \( \hat{M} \) is closed up to a compact embedded submanifold-with-singularity \( Z' \subset M' \) that consists of only manifold-points in \( M' \).

2. \( M' \) has a natural smooth structure along \( Z' \) in terms of which \( Z' \subset M'_{\text{smooth}} \) and the tautological inclusion \( M - Z \hookrightarrow M' \) is a smooth embedding.

3. Let \( U' \) be the open submanifold in \( M' \) that arises from \( U' \subset M \). Then the construction defines a branched covering map \( \pi_0 : U' \to U_0 \) of degree \( l \) with branch locus given by the codimension-2 submanifold \( \partial Z_0 \) in \( U_0 \). Let \( \sigma \in \text{Sym}_l \) be the permutation of elements of \( \{1, \cdots, l\} \) defined by \( g(Z_i^+) = Z_{\sigma(i)}^- \). Up to an inner automorphism of \( \text{Sym}_l \) on \( \text{Sym}_l \), the monodromy of \( \pi_0 \) is given by \( \sigma \). The number of components in the cyclic decomposition of \( \sigma \) gives then the number of the connected components of \( U' \).

4. Let \( M = \bigsqcup_{j=1}^k M_j \) be the decomposition of \( M \) by its irreducible components. Then \( g \) induces an equivalence relation on \( \{1, \cdots, k\} \) by generated by \( j \sim j' \) if \( g(Z_i^+) = Z_{i'}^- \) for some \( Z_i^+ \subset M_j \) with \( Z_i^- \subset M_{j'} \). The number of irreducible components \( M' \) is then the number of equivalence classes in \( \{1, \cdots, k\} \). In general, this is smaller than \( k \).

All these are direct generalizations from the case of Riemann surfaces, possibly bordered or with nodes.

One can put the above construction also into a locally generically constant family of manifolds-with-singularities over an interval \((-\varepsilon, \varepsilon)\) for some \( \varepsilon > 0 \), as follows: (continuing the notation from the previous discussion)
Let $\mathcal{M} := \text{the quotient manifold-with-singularities } M/\sim$, where $\sim$ is an equivalence relation on $M$, generated by $m_1 \sim m_2$ if $m_1, m_2 \in Z$ and $g(m_1) = g(m_2)$ where $m_1, m_2 \in Z$. Here, $m_1^\pm$ is the lifting of $m_1$ to $Z^\pm$ in $DZ$. Let $Z := Z/\sim$ with the tautological embedding $Z \hookrightarrow \mathcal{M}$. Then, there are tautological surjections

$$h_- : M \to \mathcal{M} \quad \text{and} \quad h_+ : M' \to \mathcal{M}$$

that restrict to the built-in identity maps $M - Z = \mathcal{M} - Z = M' - Z'$.

Define the sought-for family over $(-\varepsilon, \varepsilon)$ by

$$\mathcal{M}_{(-\varepsilon, \varepsilon)} := (M \times (-\varepsilon, 0])_{-} \cup \mathcal{M} \cup (M' \times [0, \varepsilon)),\]$$

where $h_- \times \{0\} = h_- : M \times \{0\} \to \mathcal{M}$ and $h_+ \times \{0\} = h_+ : M' \times \{0\} \to \mathcal{M}$. The built-in projection $\pi_{(-\varepsilon, \varepsilon)} : \mathcal{M}_{(-\varepsilon, \varepsilon)} \to (-\varepsilon, \varepsilon)$ has $\mathcal{M}_t := \pi_{(-\varepsilon, \varepsilon)}(t) = M \times \{t\}$ for $t \in (-\varepsilon, 0)$; $\mathcal{M}$ for $t = 0$; and $M' \times \{t\}$ for $t \in (0, \varepsilon)$.

Note that, by construction, there is also a built-in map $\mathcal{M}_{(-\varepsilon, \varepsilon)} \to \mathcal{M} = \mathcal{M}_0$.

**Creation of codimension-1 incidence loci via isotopies.**

Recall first the following theorem, stated with an adaptation to our situation:

**Theorem 2.3.2. [isotopy of disk].** [Hir: Sec. 8.3, Theorem 3.1]. Let $f_1$ and $f_2 : \mathbb{D}^s \to \mathbb{R}^n$, $s \leq n$, be smooth embeddings. When $s = n$, assume also that $f_1$ and $f_2$ are both orientation-preserving. Note that, as $\mathbb{D}^s$ is compact in our notation, $f_1(\mathbb{D}^s)$ and $f_2(\mathbb{D}^s)$ are contained in a compact subset of $\mathbb{R}^n$. Then, $f_1$ and $f_2$ are isotopic. Furthermore, an isotopy between $f_1$ and $f_2$ can be realized by a diffeotopy of $\mathbb{R}^n$ with compact support.

The same proof of Theorem 2.3.2 gives indeed another form of Theorem 2.3.2:

**Theorem 2.3.3. [creation of codimension-0 incidence locus via confined isotopy].**

Given two orientation-preserving smooth embeddings $f_1$ and $f_2 : \mathbb{D}^n \to \mathbb{R}^n$ with $f_1(0) = f_2(0) = 0$, there exists an orientation-preserving smooth embedding $f_2^{(t)}$, $t \in [0, 1]$, of $f_2 := f_2^{(0)}$ such that

- $f_2^{(t)} : \mathbb{D}^n \to \mathbb{R}^n$ are (orientation-preserving) smooth embeddings with $f_2^{(t)}(0) = 0$ and $f_2^{(t)}(0) = f_2$ on a neighborhood of $\partial \mathbb{D}^n \subset \mathbb{D}^n$, for all $t \in [0, 1]$,

- there exists a small neighborhood $U$ of $0 \in \mathbb{D}^n$, with the closure $\overline{U}$ contained in the interior of $\mathbb{D}^n$, such that $f_2^{(t)}|_U = f_1|_U$.

Furthermore, such an isotopy of $f_2$ can be realized by a diffeotopy of $\mathbb{R}^n$ with support contained in $f_2(\mathbb{D}^n)$.

For a finite collection of orientation-preserving smooth embeddings $f_1, \cdots, f_l : \mathbb{D}^n \to \mathbb{R}^n$ with $f_i(0) = 0$, one can keep $f_1$ fixed and perform the above isotopy to $f_2, \cdots, f_l$ one by one so that

- $f_1^{(t)} : \mathbb{D}^n \to \mathbb{R}^n$ are (orientation-preserving) smooth embeddings with $f_1^{(t)}(0) = 0$ and $f_1^{(t)}(0) = f_1$ on a neighborhood of $\partial \mathbb{D}^n \subset \mathbb{D}^n$, for $i = 1, \ldots, l$ and $t \in [0, 1]$,

- there exists a small neighborhood $U$ of $0 \in \mathbb{D}^n$, with $\overline{U}$ contained in the interior of $\mathbb{D}^n$, such that $f_1^{(t)}|_U = \cdots = f_l^{(t)}|_U$. 

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It follows then:

**Theorem 2.3.4.** [creation of codimension-1 incidence locus via confined isotopy].
Given a finite collection of orientation-preserving smooth embeddings \( f_1, \ldots, f_l : D^n \to \mathbb{R}^n \) with \( f_i(0) = 0 \) for all \( i \), there exists a neighborhood \( U \) of \( 0 \in D^n \), with \( \overline{U} \) contained in the interior of \( D^n \), for which the following holds: For any codimension-1 compact embedded submanifold-with-boundary \( Z \subset U \), there exist isotopies \( f_2^{(t)}, \ldots, f_l^{(t)}, t \in [0, 1] \), of \( f_1 = f_2^{(0)}, \ldots, f_l = f_l^{(0)} \) respectively such that

\[
\cdot f_i^{(t)} : D^n \to \mathbb{R}^n \text{ are (orientation-preserving) smooth embeddings with } f_i^{(0)}(0) = 0 \text{ and } f_i^{(t)} = f_i \text{ on a neighborhood of } \partial D^n \subset D^n, \text{ for } i = 1, \ldots, l \text{ and } t \in [0, 1],
\]

\[
\cdot f_i^{(1)}|_Z = \cdots = f_l^{(1)}|_Z.
\]

Finally, the following lemma says that the condition \( f_i(0) = 0 \) can be achieved under a minimally required assumption: (Which is indeed a special case of Theorem 2.3.2, with \( s = 0 \).)

**Lemma 2.3.5.** [adjustment of center via confined isotopy]. Let \( f : D^n \to \mathbb{R}^n \) be a (smooth) embedding such that \( f(D^n) \) contains \( 0 \in \mathbb{R}^n \) in its interior. Then there exists an isotopy \( f^{(t)}, t \in [0, 1] \), of \( f = f^{(0)} \) that is identical to \( f \), for all \( t \), on a neighborhood of \( \partial D^n \) in \( D^n \) and has the property that \( f^{(1)}(0) = 0 \).

**Proof.** Under the assumption, there exists an embedded smooth path \( \gamma \) that connects \( 0 \in \mathbb{R}^n \) and \( f(0) \) and is contained in the interior of \( f(D^n) \). Let \( U \) be an (arbitrarily small) neighborhood of \( \gamma \) in \( \mathbb{R}^n \) with \( \overline{U} \) the interior of \( f(D^n) \). The flow on \( \mathbb{R}^n \) given by a smooth vector field that is supported in \( U \), parallel to the tangent direction of \( \gamma \), and is nowhere zero along \( \gamma \) can be adjusted to provide such an isotopy of \( f_0 \) under the composition of \( f_0 \) with such a flow on \( \mathbb{R}^n \). Such a vector field is easily constructed using a partition of unity.

Transitions between large-brane wrapping and small-brane wrapping via deformations of morphisms.

Let \( Y \) be a Calabi-Yau \( n \)-fold, \( \mathcal{E} \) a complex vector bundle of rank \( r \) on \( X \), and

\[
( \varphi : (X^A, \mathcal{E}) \to Y, \nabla )
\]

be a special Lagrangian morphism with a unitary minimally flat connection-with-singularity, given by \( \varphi^s : \mathcal{O}^\infty_X \to \mathcal{O}^A_X : \text{End}_{\mathcal{O}_X}(\mathcal{E}) \) with the associated surrogate

\[
(\mathcal{E}_\varphi, \nabla)
\]

\[
\xymatrix{ X_\varphi \ar[r]^{f_\varphi} & L \subset Y \\
X \ar[u]^{\pi_\varphi} }
\]

Suppose that:

[Assumption.] There exists an embedding \( D^n \hookrightarrow X \) such that

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(1) \( \pi^{-1}_\varphi(D^n) \) contains a disjoint union \( V = \bigsqcup_{i=1}^l D^n_{(i)} \) of connected components that satisfy: \( f_\varphi|_{D^n_{(i)}} \), \( i = 1, \ldots, l \), are (orientation-preserving) embeddings with \( \bigcap_{i=1}^l f_\varphi(D^n_{(i)}) \neq \emptyset \);

(2) \( (E_\varphi|_V, \nabla_V) \simeq (O_{V;\mathbb{C}} \otimes \mathcal{C}'\mathcal{r}', \nabla_0) \) for some \( r' < r \), where \( \nabla_0 \) is the flat connection associated to the built-in trivialization of \( O_{V;\mathbb{C}} \otimes \mathcal{C}'\mathcal{r}' \).

For the following construction, we will assume that \( r' = 1 \) for simplicity of notation. Once the \( r' = 1 \) case is understood, one can then recover the \( r' > 1 \) case by taking \((\cdot) \otimes \mathbb{C}r'\) to the bundles/sheaves constructed in the \( r' = 1 \) case.

Applying the confined isotopy and local gluing discussed in the previous two themes to \( V \subset X_\varphi \) over \( D^n \subset X \) with the codimension-1 embedded submanifold-with-boundary \( Z \) there taken to be, for example, a small smoothly embedded \((n - 1)\)-disk in \( D^n \), one obtains a \((-\varepsilon, \varepsilon)\)-family of local deformations of the pair \((\varphi, \nabla)\):

\[
\left( \hat{\mathcal{E}}_{(-\varepsilon, \varepsilon)}, \hat{\nabla}_{(-\varepsilon, \varepsilon)} \right)
\]

where

- \( \hat{\mathcal{X}}_{(-\varepsilon, \varepsilon)} \simeq ((-\varepsilon, \varepsilon) \times (X_\varphi - V)) \cup_\emptyset \mathcal{V}_{(-\varepsilon, \varepsilon)} \) where \( \mathcal{V}_{(-\varepsilon, \varepsilon)} \) is \( \mathcal{M}_{(-\varepsilon, \varepsilon)} \) in the construction in the first theme, with \( M = V \);

- \( (\hat{\mathcal{E}}_{(-\varepsilon, \varepsilon)}, \hat{\nabla}_{(-\varepsilon, \varepsilon)}) \) is the extension of \(((-\varepsilon, \varepsilon) \times (\varphi|_{X_\varphi - V}, \nabla|_{X_\varphi - V}) \) over \( \mathcal{V}_{(-\varepsilon, \varepsilon)} \) by trivial complex line bundles with a trivial (flat) connection;

- \( \hat{\pi}_{(-\varepsilon, \varepsilon)} \) and \( \hat{f}_{(-\varepsilon, \varepsilon)} \) are the built-in morphisms over \((-\varepsilon, \varepsilon)\) in the construction; note that \( \hat{\pi}_{(-\varepsilon, \varepsilon)}|_{\mathcal{V}_{(-\varepsilon, \varepsilon)}} : \mathcal{V}_{(-\varepsilon, \varepsilon)} \to (-\varepsilon, \varepsilon) \times D^n \subset (-\varepsilon, \varepsilon) \times X \) has constant degree \( l \), counted with multiplicity, with \( \mathcal{V}_{(-\varepsilon, 0)} \to (-\varepsilon, 0) \times D^n \) a covering map and \( \mathcal{V}_{(0, \varepsilon)} \to (0, \varepsilon) \times D^n \) a branched covering map with branch locus \((0, \varepsilon) \times \partial Z\).

\(^{11}\)Condition (2) can be loosened/generalized by introducing the notion of disks with a multiplicity and bundles/sheaves with a filter of subbundles/subsheaves, and allowing the rank of \( E_\varphi|_V \) to vary on different connected components of \( V \); cf. [L-Y5: Sec. 4.2, Theme: ‘The generically filtered structure on the Chan-Paton bundle over a special Lagrangian cycle on a Calabi-Yau torus.’] (D(6)). Since this is a separate issue for D-branes, here, to make the presentation simple, we take all the multiplicity of disks to be 1 and the filter to be trivial.
From the above data, one obtains two families

\[ (\hat{\pi}, \hat{f})_i, \hat{\mathcal{E}}_i, \nabla^i \]

\[ (\pi, f)(\hat{X}_i) \xrightarrow{\pi_f} I \times L \subset I \times Y , \]

where \( I = (-\varepsilon, 0) \) or \((0, \varepsilon)\), of special Lagrangian morphisms from Azumaya spaces with a fundamental module, with a unitary minimally flat connection, and a transition between them via the family with \( I = (-\varepsilon, \varepsilon) \).

**Remark 2.3.6. [large- vs. small-brane wrapping]**. Given a Calabi-Yau \( n \)-fold \( Y \) and a special Lagrangian morphism with a unitary minimally flat connection-with-singularity

\[ (\varphi : (X^\delta, \mathcal{E}) \rightarrow Y , \nabla) \]

specified by \( \varphi^\delta : \mathcal{O}^\infty_{Y, \mathbb{C}} \rightarrow \mathcal{O}^\delta_X := \text{End}_{\mathcal{O}_{X, \mathbb{C}}} (\mathcal{E}) \) with the data on the associated surrogate

\[ (\mathcal{E}_\varphi, \nabla) \]

\[ X_\varphi \xrightarrow{f_\varphi} L \subset Y \]

\[ \pi_\varphi \]

\[ X \]

that satisfies the beginning assumption in the construction above. Then, the above construction deforms \( \varphi \) via locally deforming and gluing different sheets of \( X_\varphi \) over \( X \). When the starting collection \( \pi_\varphi^{-1}(D^0) \) contains disks in different irreducible components of \( X_\varphi \), the procedure in general lead then to \( \varphi' : (X^\delta, \mathcal{E}') \rightarrow Y \) with \( X_\varphi' \) containing an irreducible component \( X^0_\varphi \) with larger volume such that both \( f_{\varphi'}|_{X^0_\varphi} \rightarrow L = \text{Im}(\varphi) = \text{Im}(\varphi') \) and \( \pi_{\varphi'}|_{X^0_\varphi} \rightarrow X \) have larger degree since volume (of branes) and degree (of maps) add under the construction. This gives rise, thus, to the phenomenon of and a transition between a “small-brane wrapping” and a “large-brane wrapping” of a special Lagrangian cycle \( L \) in \( Y \) in superstring theory.

Below is a simplest example that illustrates this particularly behavior of D-branes distinctly. It is also an example that resembles long- vs. short-string wrapping most directly.

**Example 2.3.7. [large- vs. small-brane wrapping on special Lagrangian 3-sphere]**. Let \( X = S^3 \) be oriented, \( \pi' : X' = S^3 \rightarrow X \) be an orientation-preserving branched covering of \( S^3 \) on itself, \( f' : X' \rightarrow L = S^3 \) be an orientation-preserving diffeomorphism, \( \mathcal{E}' \) be a complex line bundle over \( X' \). Since \( \mathcal{E}' \) is isomorphic to \( \mathcal{O}_{S^3, \mathbb{C}} \), we endow \( \mathcal{E}' \) with the trivial connection \( \nabla' = 0 \) under such an isomorphism. It follows that \( \nabla' \) induces a unitary flat connection on \( \pi'_* \mathcal{E}' \), with singularity on the branch locus of \( \pi' \), by endowing \( \mathcal{E}' \simeq \mathcal{O}_{S^3, \mathbb{C}} \) with the standard Hermitian metric, for which \( d \) is \( U(1) \)-flat. The map \((\pi', f') : X' \rightarrow X \times L \) defines now an embedding. Recall that the tangent bundle of an orientable close 3-manifold is always trivial. In particular, let \( \chi_t : L \rightarrow L, t \in \mathbb{R} \), be a flow on \( L \) generated by a nowhere-zero smooth vector field on \( L \); for example, a Hopf flow on \( L = S^3 \). Consider the map \( f := \prod_{t=0}^{d-1} \chi_{t\delta} \circ f' : \Pi^d X' \rightarrow L \) for some \( \delta > 0 \). Let \((X^\delta, \mathcal{E}) = (S^3, \mathcal{E}) := (S^3, \mathcal{O}_{S^3} = \text{End}_{\mathcal{O}_{S^3, \mathbb{C}}} (\mathcal{E}), \mathcal{E} = \oplus^d \pi'_* \mathcal{E}') \) be an Azumaya 3-sphere...
with a fundamental module. Let \( L \) be realized as an embedded special Lagrangian 3-sphere in a Calabi-Yau 3-fold \( Y \). Assume that \( \delta \) is small enough so that \( (\pi := \Pi^d\pi', f) : \Pi^dX' \to X \times L \) is an embedding. Then \(( (\pi, f), (E', \nabla')) \) defines a special Lagrangian morphism \( \varphi : (X^A, \mathcal{E}_-) \to Y \), with \( X_\varphi = \Pi^dX' = \Pi^dS^3 \), \( \pi_\varphi = \pi \), and \( f_\varphi = f \), image \( \text{Im} \varphi = L \) with a multiplicity \( d \), and \( \mathcal{E}_\varphi \simeq \mathcal{O}_{X_\varphi, \mathbb{C}} \), and a unitarily minimally flat connection \( \nabla \) on \( \mathcal{E}_\varphi \) that is isomorphic to \( d \) on \( \mathcal{O}_{X_\varphi, \mathbb{C}} \). In particular, \( X_\varphi \) has \( d \)-many components with each mapped to \( L \) by the degree-1 restriction of \( f_\varphi \) (i.e. with each component wrapping \( L \) once).

Now deform \( \varphi \) by setting \( f_t := \bigsqcup_{i=0}^{d-1} \chi_{-\text{tid}} \circ f' \), \( t \in [-1, 0] \). Then \( f_{-1} = f \) and for \( t \in [-1, 0) \), the associated special Lagrangian morphism \( \varphi_t : (X^A, \mathcal{E}) \to Y \) has \( X_{\varphi_t} = \Pi^dX' \), \( \pi_{\varphi_t} = \pi \), \( f_{\varphi_t} = f_t \), and image \( \text{Im} \varphi_t = L \) with a multiplicity \( d \). For \( t = 0 \), all components of \( X_\varphi \) are deformed to coincide and become a single-component \( X_{\varphi_0} \simeq X' \) with multiplicity \( d \) indicated by the rank \( d \) of \( \oplus^dE' \simeq \mathcal{O}_{X'_\varphi, \mathbb{C}} \otimes \mathbb{C}^d \) on \( X' \). Let \( Z \simeq \mathbb{D}^2 \) be an embedded 2-disc in \( X = S^3 \) and perform a gluing construction in this subsection, with \( \sigma \in \text{Sym}_d \), say, to be transitive, to obtain an orientation-preserving \( d \)-fold branched covering \( g : X'' \simeq S^3 \to X_\varphi \). The composition \( f'' \) of \( X'' \xrightarrow{g} X_{\varphi_0} \xrightarrow{f_{\varphi_0}} L \) then is also a \( d \)-fold orientation-preserving branched covering. A “large brane” (i.e. \( X'' \)) is thus formed from gluing “small branes” (i.e. \( X_\varphi = \Pi^dS^3 \)) and it wraps \( L \) now via \( f'' \) of degree \( d \). Let \( \pi'' : X'' \to X \) be the built-in orientation-preserving branched covering map, \( \mathcal{E}'' \simeq \mathcal{O}_{X''\varphi, \mathbb{C}} \), and \( \mathcal{E}_+ = \pi''_+ \mathcal{E}'' \). By deforming \( f'' : X'' \to L \), for example, via the geodesic flow along \( f \) (e.g. from the induced Riemann metric on \( L \) as a submanifold in \( Y \)) governed by a smooth section \( \xi \) of \( f''^*T_{\mathbb{D}} L \) that is non-zero except at the singular locus of \( f'' \), one can obtain a family of smooth maps \( (\pi'', f'') : X'' \to X \times L, t \in (0, 1] \), with each an orientation-preserving embedding on an open dense subset of \( X'' \). This defines thus a family of special Lagrangian morphisms \( \varphi_t : (X^A, \mathcal{E}_+) \to Y, t \in (0, 1], \) whose image remains \( L \) but which now involve large-brane wrapping \( f_{\varphi_t} : X_{\varphi_t} \to L \) on \( L \).

Note that behind the above two-part construction, one over \([-1, 0] \) and the other over \([0, 1] \), is a family data over \( I = [-1, 1] \), as in the beginning of the theme:

\[
\begin{array}{ccc}
\hat{\mathcal{E}}_I & \xrightarrow{\hat{\varphi}} & \hat{\nabla}'_I \\
\hat{X}_I & \xrightarrow{(\hat{\pi}_I, \hat{f}_I)} & (I \times X) \times I (I \times Y) \\
I \times X & \xrightarrow{pr_1} & I \times Y \\
\end{array}
\]

with \( (\hat{\mathcal{E}}_I, \hat{\nabla}'_I) \simeq (\mathcal{O}_{X_I, \mathbb{C}}, d) \).

We pose a question here before leaving this subsection\(^{12}\).

**Definition 2.3.8. [immersion in codimension 1].** Let \( M, N \) be smooth manifolds. A smooth map \( f : M \to N \) is called an immersion in codimension 1 if there exists an open dense submanifold \( M^0 \subset M \) with the (Hausdorff) codimension of \( Z := M - M^0 \geq 2 \) such that \( f|_{M^0} : M^0 \to N \) is an immersion. \( Z \) is called an exceptional locus of \( f \).

\(^{12}\)We thank Yng-Ing Lee and Wenxuan Lu for some discussions on this problem.
Question 2.3.9. [de-rigidification via large-brane wrapping].

- Does there exist a smooth special Lagrangian map \( f : L \to S^3 \subset Y \) that is an immersion (presumably of sufficiently high degree) in codimension 1 such that there exists a family of deformations of \( f =: f_0 \) into smooth special Lagrangian maps \( f_t : L \to Y, \ t \in [0, \epsilon) \) for some \( \epsilon > 0 \), with \( f_t(L) \neq S^3 \) for \( t \in (0, \epsilon) \) ?

2.4 Remarks/Questions/Conjectures.

Two themes that immediately arise from the previous discussion are given here. Each deserves a study in its own right. The first theme is also relevant to Sec. 3.

2.4.1 Cones of special Lagrangian cycles.

Given a Calabi-Yau \( n \)-fold \( Y = (Y, J, \omega, \Omega) \), let \( \alpha = [L] \in H_n(Y; \mathbb{Z}) \) be a homology class that is representable by a special Lagrangian submanifold \( L \). Then

\[ |Re\Omega| \cdot \alpha > 0 \quad \text{and, hence,} \quad |Re\Omega| \cdot (-\alpha) < 0. \]

This implies that \(-\alpha \in H_n(Y; \mathbb{Z})\) cannot be represented by any special Lagrangian cycle (or current). Furthermore, if \( \alpha = [L_1] \) and \( \beta = [L_2] \) are two classes that are representable by special Lagrangian submanifolds, then \( \alpha + \beta \) is representable by the special Lagrangian cycle \( L_1 + L_2 \).

This gives a foundation for the following definition:

**Definition-Prototype 2.4.1.1. [cone of special Lagrangian cycles].** The following cone

\[ C^{sL}(Y) := \left\{ \sum_{i \in I} a_i[L_i] : |I| < \infty, \ a_i \in \mathbb{R}_{\geq 0}, \ L_i \text{ a special Lagrangian submanifold-with-singularity} \right\} \]

in \( H_n(Y; \mathbb{R}) \) is called the *cone of special Lagrangian cycles* of the Calabi-Yau \( n \)-fold \( Y \). With \( \mathbb{R}_{\geq 0} \) replaced by \( \mathbb{Q}_{\geq 0} \), one can also define \( C^{sL}_\mathbb{Q}(Y) \).

This is only a prototypical definition as there are various enhancements/refinements to it:

- The above definition is based on the underlying choice of *equivalence relation of special Lagrangian cycles*: \( L_1 \sim L_2 \) if \( [L_1] = [L_2] \) in \( H_n(Y; \mathbb{Z}) \). One can use other finer equivalence relations, for example, via Lagrangian or special Lagrangian cobordisms.

- One may specify more specifically the *singularities* allowed in special Lagrangian cycles or currents. In particular, let

\[ C^{sL}(Y)^0 := \left\{ \sum_{i \in I} a_i[L_i] : |I| < \infty, \ a_i \in \mathbb{R}_{\geq 0}, \ L_i \text{ an immersed special Lagrangian submanifold} \right\} \subset C^{sL}(Y). \]

Then, it follows from the immersed version of Theorem A.1.2 that \( C^{sL}(\bullet)^0 \) is locally constant in the dual Hodge bundle \( \mathcal{H}^\vee \) over the moduli space \( \mathcal{M} \) of smooth deformations of \( Y \).

With cones of special Lagrangian cycles to complex deformations as Mori cones ([Ko-M]) to Kähler deformations in mind, two major questions are then:
2.4.1 Question 2.4.1.2. [cone of special Lagrangian cycles].

1. [structure of cone].
   Structure of $C^{sL}(Y)$ (resp. $C^{sL}(Y)^0$, $\overline{C^{sL}(Y)}$, $\overline{C^{sL}(Y)^0}$, ...), existence of extremal rays, ..., etc.?

2. [role in deformation]. How does the structure of $C^{sL}(Y)$ (resp. $C^{sL}(Y)^0$, ...) relate to the vanishing cycle of $Y$ when the complex structure of $Y$ is deformed to singularity?

2.4.2 A genus-like expansion of the path-integral of lower-dimensional branes: Alexander-Hilden-Lozano-Montesinos-Thurston/Hurwitz meeting Polchinski-Grothendieck.

Not all D3-brane topologies are equal from the viewpoint of Azumaya geometry. This suggests a genus-like expansion of the path-integral of D3-branes in type IIB string theory. Similarly for D2-branes in type IIA string theory and for M2-branes in M-theory.

Fundamental D3-branes from the viewpoint of Azumaya geometry: Alexander-Hilden-Lozano-Montesinos-Thurston meeting Polchinski-Grothendieck.

Recall the following classical theorems in the study of 3-manifold topology:

**Theorem 2.4.2.1. [branched covering].** (Alexander [Al], 1920.) Any closed, connected, orientable 3-manifold is realizable as a branched covering of $S^3$.

**Theorem 2.4.2.2. [3-fold enough].** (Hilden [Hil] and Montesinos [Mon], 1976.) Any closed, connected, orientable 3-manifold is realizable as a 3-fold (i.e. degree-3) irregular branched covering of $S^3$ with the branch locus in $S^3$ a knot.

**Theorem 2.4.2.3. [universal link].** (Thurston [Thu], 1982.) There exists a (6-component) link $L^1$ in $S^3$ such that any closed, connected, orientable 3-manifold is realizable as a branched covering of $S^3$ that is branched only over $L^1$.

**Theorem 2.4.2.4. [universal knot].** (Hilden-Lozano-Montesinos [H-L-M], 1985.) There exists a knot $K^1$ in $S^3$ such that any closed, connected, orientable 3-manifold is realizable as a branched covering of $S^3$ that is branched only over $K^1$.

What the fundamental theorems of Alexander-Hilden-Lozano-Montesinos-Thurston mean to D3-branes along the line of the Polchinski-Grothendieck Ansatz is, in particular, that:

**Theorem 2.4.2.5. [S$^{3,\text{Az}}$ and fundamental D3-brane].** Let $Y$ be a Calabi-Yau 3-fold and $L^3$ be a finite union of compact smooth special Lagrangian submanifolds in $Y$, each of which is generically an embedding. Then there exists a morphism $\varphi : S^{3,\text{Az}} \to Y$ from an Azumaya 3-sphere such that the image $\varphi(S^{3,\text{Az}})$ of $\varphi$ is exactly $L^3$. Furthermore, one can require that the rank of the fundamental module $E$ of $S^{3,\text{Az}}$ be 3 · (number of irreducible components of $L^3$). Or one may require that $\pi_\varphi : S^3_\varphi \to S^3$ be a branched-covering map over a universal knot or a universal link in $S^3$.

---

13Notation. In this subsubsection, we will use: (only here)
- $L^1$ to denote a link (i.e. possibly disconnected, embedded, 1-dimensional submanifold) of $S^3$;
- $K^1$ to denote a knot (i.e. connected, embedded, 1-dimensional submanifold) of $S^3$; and
- $L^3$ to denote a special Lagrangian (3-dimensional) submanifold in a Calabi-Yau 3-fold.
This specifies morphisms from \((S^3, \mathcal{A}_e, \mathcal{E})\) as most fundamental D3-branes from the viewpoint of Azumaya geometry.

**A genus-like expansion of the path-integral of D3-branes.**

In understanding the path-integral for higher-dimensional objects, the usual sum-over-Feynman-diagrams in the quantum field theory of (point-like) particles is replaced by

1. a sum over the space of all the topologies of the (Wicked-rotated) D3-brane world-volume of the extended objects and
2. an integration over the space of maps from each topology.

Step (1) has already imposed a very challenging difficulty to understanding the path-integral even in a physicist’s way. In general, one needs to specify these topologies by hand. See, e.g., [B-P] for a recent study.

Recall how closed strings interact to give rise to Riemann surfaces as (Wick-rotated) closed-string world-sheets (cf. [G-S-W: Sec. 1.4, Sec. 3.3]) and the path-integral of closed strings (cf. [Po: vol. I, Sec. 3.1, Sec. 3.2]). Consider the replacements

- string \(\rightarrow\) D3-brane,
- ordinary maps \(\rightarrow\) morphisms from Azumaya manifolds into space-time,

with the same generic interaction assumption in string theory:

- [generic interaction assumption]. Interactions of branes in a space-time happen one at a time at a point with respect to some local equal-time slicing of space-time.

Then, Theorem 2.4.2.5 suggests that

\[
Z_{D3}(Y) = \sum_{\text{fundamental } 4\text{-manifolds } X} \int_{\{\text{morphisms } \varphi: (X^{\mathcal{A}_e}, \mathcal{E}) \rightarrow Y\}} \mathcal{D}\varphi \int_{\{\text{other fields } \cdots \}} \mathcal{D}(\text{other fields } \cdots) e^{-S(\varphi, \cdots)},
\]

where

- \(Z_{D3}(Y)\) is the path-integral of D3-branes on a target-space(-time) \(Y\),
- the set \(\{\text{fundamental 4-manifolds } X\}\) consists of all homeomorphism classes of closed orientable 4-manifolds \(X\) that are obtained from (self-)connected sums of finite disjoint unions \(\Pi^*(S^3 \times S^1)\) of \(S^3 \times S^1\),
- the ‘other fields \(\cdots\)’ here includes gauge fields on the branes, realized as connections-with-singularities on \((X_{\varphi}, \mathcal{E}_\varphi)\),
- \(S(\varphi, \cdots)\) is the action of the (Wicked-rotated) D3-brane theory.

The 4-manifolds \(X\) involved here in the path-integral are now direct generalization of closed orientable 2-manifolds in the case of strings in the sense that closed orientable 2-manifolds can be obtained as direct sums of finite disjoint unions of \(S^1 \times S^1\). This motivates a conjecture:

**Conjecture 2.4.2.6. [genus-like expansion of path-integral for D3-branes].** There exists a built-in natural genus-like expansion for the path-integral \(Z_{D3}(Y)\) of a perturbative D3-brane theory.
D2-branes and M2-branes: Hurwitz meeting Polchinski-Grothendieck.

The same reasoning suggests a similar conjecture for D2-branes and M2-branes respectively. Denote by \( S^{2,\text{Az}} \) for some \((S^{2,\text{Az}},\mathcal{E})\) with \( \mathcal{E} \) unspecified. Recall that any Riemann surface branched-covers \( S^2 \).

**Theorem 2.4.2.7.** \([S^{2,\text{Az}}\text{ and fundamental D2/M2-brane}]\). Let \( Y \) be a Calabi-Yau 3-fold or Joyce/G2 7-manifold and \( C \) be a finite union of compact complex smooth curves immersed in \( Y \), each of which is generically an embedding. Then there exists a morphism \( \varphi: S^{2,\text{Az}} \to Y \) from an Azumaya 2-sphere such that the image \( \varphi(S^{2,\text{Az}}) \) of \( \varphi \) is exactly \( C \).

3-manifolds that are obtained from a (self-)connected sum of a finite disjoint union \( \bigoplus(S^2 \times S^1) \) of \( S^2 \times S^1 \) now play a special role.

**Conjecture 2.4.2.8.** \([\text{genus-like expansion of path-integral for D2/M2-branes}]\). There exists a built-in natural genus-like expansion for the path-integral \( Z_{D2}(Y) \) of a perturbative D2-brane theory. Similarly, for the path-integral \( Z_{M2}(Y) \) of a perturbative M2-brane theory.

## 3 Morse cobordisms of A-branes on Calabi-Yau 3-folds under a reverse split attractor flow at a wall of marginal stability: Denef-Joyce meeting Polchinski-Grothendieck.

In this section, we study how A-branes, in the sense of Definition-Prototype 2.1.14, deform under an attractor flow at the wall of marginal stability on a moduli space of complex structures on a Calabi-Yau 3-fold. We first set up notations and recap a result of Denef ([De3]) in Sec. 3.1 and then use the results of Joyce ([Joy3]) to deal with the technical issue of smoothing a special Lagrangian submanifold with transverse self-intersections. Together, this gives us a Morse cobordism of A-branes on Calabi-Yau 3-folds under a reverse split attractor flow at a wall of marginal stability.

### 3.1 Evolution of \( \text{Im}(e^{-i\alpha_1 \Omega}) \) along a \( \Gamma \)-attractor flow trajectory à la Denef.

The following discussion follows works [De3] and [De1] of Denef, with some background from [Gr], [Str], [Ti], and [To] to set up notations and with a mild convention change to fit calibrated geometry ([Ha-L]).

**Basic setup, notations, facts, and identities.**

Let \((Y,\omega,J)\) be a (smooth) Calabi-Yau 3-fold and \( \mathcal{M} := \mathcal{M}^{\text{smooth}}_{\text{complex}} \) be the moduli space of complex deformations of \( Y \). Let \( h^{p,q} := \dim_{\mathbb{C}} H^{p,q}_J(X;\mathbb{C}) \); then \( \mathcal{M} \) is a complex manifold of dimension \( h^{2,1} \), (cf. [Ti] and [To]). For the purpose of this note, we will assume (by taking either a smaller \( \mathcal{M} \) or a covering of it) that an universal Calabi-Yau 3-fold \( \pi: \mathcal{Y} \to \mathcal{M} \) over \( \mathcal{M} \) exists, with a relative polarization \([\omega]_{\mathcal{Y}/\mathcal{M}}\) from the Kähler class \([\omega]\) on \( Y \). Let \( \mathcal{H} := R^{3,1}_*\mathcal{C}_Y \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M}} \) be the Hodge bundle on \( \mathcal{M} \), where \( \mathcal{C}_Y \) is the constant sheaf on \( \mathcal{Y} \) associated to \( \mathbb{C} \). \( \mathcal{H} \) is holomorphic of rank \( \dim_{\mathbb{C}} H^3(Y,\mathbb{C}) \). The complex symplectic product on \( \langle \cdot, \cdot \rangle \) on \( H^3(Y,\mathbb{C}) \),

---

\(^{14}\)We thank Pei-Ming Ho for discussions and for drawing our attention to recent new developments in the study of multiple M2-branes.
defined by \( \langle \alpha, \beta \rangle := \int_T \alpha \wedge \beta \), induces a complex symplectic product, still denoted by \( \langle \cdot, \cdot \rangle \), on fibers of \( \mathcal{H} \). Let \( \mathcal{H} = \mathcal{H}^{3,0} + \mathcal{H}^{2,1} + \mathcal{H}^{1,2} + \mathcal{H}^{0,3} \) be the Hodge decomposition of \( \mathcal{H} \). In general, \( \mathcal{H}^{p,q} \) is only a smooth complex vector bundle on \( \mathcal{M} \). However, let \( F^p \mathcal{H} := \bigoplus_{q \geq p} \mathcal{H}^{q,p-q} \), \( p = 0, 1, 2, 3 \). Then \( F^p \mathcal{H} \) are holomorphic on \( \mathcal{M} \), and \( F^* \mathcal{H} : \mathcal{H} = F^0 \mathcal{H} \supset F^1 \mathcal{H} \supset F^2 \mathcal{H} \supset F^3 \mathcal{H} = \mathcal{H}^{3,0} \) gives the Hodge filtration of \( \mathcal{H} \). In particular, \( \mathcal{H}^{3,0} \) is a holomorphic (complex) line bundle on \( \mathcal{M} \) while \( \mathcal{H}^{0,3} = \overline{\mathcal{H}^{3,0}} \) is an anti-holomorphic line bundle on \( \mathcal{M} \).

The inclusion \( i : \mathbb{Z} \subset \mathbb{C} \) as abelian groups induces an inclusion \( i : R^3 \pi_* \mathbb{Z} \subset \mathcal{H} \), which induces a flat connection, namely the Gauss-Manin connection \( \nabla \), on \( \mathcal{H} \). In terms of the canonical (up to a monodromy effect) local trivialization of \( \mathcal{H} \) from \( i(R^3 \pi_* \mathbb{Z}) \), \( \nabla \) is simply the differential \( d = \partial + \bar{\partial} \) with respect to complex local coordinates \( t = (t^a) \) (or more completely, \( t = (t^a, \bar{t}^\alpha) \)), on \( \mathcal{M} \). \( \nabla \) preserves neither the decomposition \( \bigoplus_{p=0}^3 \mathcal{H}^{p,3-p} \) nor the filtration \( F^* \mathcal{H} \) of \( \mathcal{H} \). However, it has the property that \( \nabla F^p \mathcal{H} \subset F^{p-1} \mathcal{H} \otimes \Omega^1_\mathcal{M} \), cf. [Gr].

In terms of the Hodge decomposition of \( \mathcal{H} \) on \( \mathcal{M} \), the Weil-Petersson metric \( \omega_{WP} \) on \( \mathcal{M} \), defined through the Ricci flat metric on each fiber \( Y_t := \pi^{-1}(t) \) of \( \pi : \mathcal{Y} \rightarrow \mathcal{M} \) determined by \( [\omega]_\mathcal{Y}/\mathcal{M} \) (cf. [Yau]), has a Kähler potential \( K_{WP} \) that can be expressed locally purely topologically as a smooth real-valued function

\[
K = K(t) := K_{WP}(t) = -\log \left( \int_{Y_t} \Omega(0)(t) \wedge \overline{\Omega(0)(t)} \right) + \log \left( 8 \cdot \text{vol}(Y) \right)
\]
on \( \mathcal{M} \), where \( \Omega(0) = \Omega(0)(t) \) is a local holomorphic section of \( \mathcal{H}^{3,0} \), (cf. [Ti] and [To]). Define

\[
\Omega = \Omega(t) = e^{K(t)/2} \Omega(0)(t)
\]

so that \( \Omega(t) \), as a holomorphic 3-form on \( Y_t \), satisfies the normalization condition\(^{15}\)

\[
\frac{i}{2^3} \Omega(t) \wedge \overline{\Omega(t)} = \frac{1}{3!} (\omega_{\mathcal{Y}/\mathcal{M}}|_{Y_t}) \wedge (\omega_{\mathcal{Y}/\mathcal{M}}|_{Y_t}) \wedge (\omega_{\mathcal{Y}/\mathcal{M}}|_{Y_t}).
\]

Note that \( \Omega \) is now only a smooth local section of \( \mathcal{H}^{3,0} \).

On a local chart \( U \subset \mathcal{M} \) on which \( \mathcal{H} \) is canonically trivialized via the Gauss-Manin connection \( \nabla \) and \( \Omega(0) \), and, hence, \( \Omega \) are defined, one has the following basic identities and notations: (Below, \( t = (t_a, \bar{t}_\alpha) \) is the local coordinates on \( U \) and \( \nabla = d \) under the trivialization of \( \mathcal{H}|_U \).)

\(1\) \( \frac{\partial}{\partial t_a} \Omega = \left( -\frac{1}{2} \frac{\partial}{\partial t_a} K \right) \Omega + \chi_a \) with \( \chi_a \) a section of \( \mathcal{H}^{2,1} \). \( \chi_a \) satisfies \( \int_{Y_t} \chi_a \wedge \overline{\chi_b} = 8 \cdot \text{vol}(Y) \cdot \delta_{ab} \), where \( g_{ab} = \frac{\partial}{\partial t_a} \frac{\partial}{\partial t_b} K \) is the Weil-Petersson metric on \( U \).

\[
\chi_a \quad \text{ satisfies } \int_{Y_t} \chi_a \wedge \overline{\chi_b} = 8 \cdot \text{vol}(Y) \cdot \delta_{ab}.
\]

We adopt the notation from special geometry (cf. [Str]) to denote \( D_a := \frac{\partial}{\partial t_a} + \frac{1}{2} \frac{\partial}{\partial \bar{t}_\alpha} K \); e.g., the \( (2,1) \)-component \( \chi_a \) of \( \frac{\partial}{\partial t_a} \Omega \) is then denoted by \( D_a \Omega \) and \( D_a Z_T := \int_T D_a \Omega = \int_T \chi_a \).

Similarly, \( D_a := \frac{\partial}{\partial t_a} + \frac{1}{2} \frac{\partial}{\partial \bar{t}_\alpha} K \) and for \( D_a \Omega \) and \( D_a Z_T \) by taking the complex conjugate.

\(2\) A (real) harmonic 3-form \( \Xi \) on \( Y_t \) has a Hodge decomposition given by

\[
\Xi = \frac{1}{8 \cdot \text{vol}(Y)} \left( i \tilde{Z}(\Xi) \Omega - i g^{ab} D_b \tilde{Z}(\Xi) D_a \Xi + i g^{ba} D_b \tilde{Z}(\Xi) \tilde{D}_b \Xi - i Z(\Xi) \Omega \right),
\]

where \( Z(\Xi) \) (resp. \( \tilde{Z}(\Xi) \)) denotes \( \int_{Y_t} \Xi \wedge \Omega(t) \) (resp. \( \int_{Y_t} \Xi \wedge \overline{\Omega}(t) \)).

They follow from direct computations, the holomorphicity (resp. anti-holomorphicity) of \( \Omega(0) \) (resp. \( \overline{\Omega(0)} \)) on \( U \), and the variation property of \( \Omega(0) \) (e.g. [Ti: Lemma 7.2]).

\(^{15}\)This normalization condition fits more naturally into the standard setting of calibrated geometry and is used, for example, in work of Joyce [Joy3]. Under this normalization, an orientable (real) 3-dimensional submanifold \( L \) in \( Y_t \) has volume \( \text{vol}(L) \geq |Z(L)| \) and the equality holds when \( L \) is a special Lagrangian submanifold. In particular, \( \Omega \) here = \( \sqrt{8 \cdot \text{vol}(Y)} \cdot \Omega \) in [De3]).
Well-definedness of $|Z_\Gamma|$ and $e^{-i\alpha r} \Omega$ on $\mathcal{M}$.

By construction, $\Omega$ and, hence, $Z_\Gamma$ are locally well-defined only up to a same smooth unit-modular complex-valued function. Thus, both $|Z_\Gamma|$ and $e^{-i\alpha r} \Omega$ are well-defined locally. They automatically become globally well-defined on $\mathcal{M}$ respectively as a positive function and as a smooth section of $\mathcal{H}^{3,0}$. It follows that $e^{-i\alpha r} \Omega$, $\text{Re}(e^{-i\alpha r} \Omega) := \frac{1}{2} (e^{-i\alpha r} \Omega + e^{i\alpha r} \bar{\Omega})$, and $\text{Im}(e^{-i\alpha r} \Omega) := \frac{i}{2} (e^{-i\alpha r} \Omega - e^{i\alpha r} \bar{\Omega})$ are all well-defined, as smooth sections of $\mathcal{H}$ on $\mathcal{M}$.

Evolution of $\text{Im}(e^{-i\alpha r} \Omega)$ along a $\Gamma$-attractor flow trajectory.

A trajectory of a flow on $\mathcal{M}$ gives an embedding $\gamma : I \hookrightarrow \mathcal{M}$, where $I$ is an interval in $\mathbb{R}_+$. There exists thus a neighborhood $\mathcal{N}$ of $\gamma(I)$ in $\mathcal{M}$ that is contractible. The restriction $\mathcal{H}|_\mathcal{N}$ of the Hodge bundle to $\mathcal{N}$ becomes canonically trivialized via the Gauss-Manin connection $\nabla$ on $\mathcal{H}$, through which all fibers of $\mathcal{H}$ along a trajectory of a flow can be identified. With such an identification and with the preparation from the previous two themes, one can now express $\text{Im}(e^{-i\alpha r} \Omega)$ along a $\Gamma$-attractor flow trajectory explicitly:

**Proposition 3.1.1. [Im$(e^{-i\alpha r} \Omega)$ along $\Gamma$-attractor flow].** ([De3: Sec. 3.1, Eq. (3.5).]) With the canonical trivialization of the Hodge bundle along and letting $\Gamma^\vee$ be the Harmonic 3-form representing the Poincaré dual $\in H^3(Y; \mathbb{R})$ of $\Gamma$, then the 3-form $\text{Im}(e^{-i\alpha r} \Omega)$ on Calabi-Yau 3-folds along a trajectory of the attractor flow on $\mathcal{M}$ associated to $Z_\Gamma$ is given by

$$\text{Im} \left( e^{-i\alpha r(\mu)} \Omega(\mu) \right) = (-4 \text{vol}(Y) \cdot \mu \tau(\mu)) \Gamma^\vee + \frac{\mu}{\mu_0} \text{Im} \left( e^{-i\alpha r(\mu_0)} \Omega(\mu_0) \right),$$

with $\tau(\mu) = -\int_{\mu_0}^\mu \frac{d\mu'}{|Z_\Gamma(\mu')|^2}$, for $\mu, \mu_0 \in \mathbb{R}_{>0}$.

**Proof.** With notations from previous themes in this subsection, let $\mathcal{U} := \{U_\beta\}_\beta$ be a locally finite good atlas on $\mathcal{M}$ over which $\mathcal{H}^{3,0}$ becomes trivial (as a holomorphic bundle) and one can introduce a nowhere-zero holomorphic section $\Omega^{(0)}_\beta$ of $\mathcal{H}^{3,0}|_{U_\beta}$ on $U_\beta$ for each $\beta$. Consider a $U_\beta := U$ and trivialize $\mathcal{H}|_U$ via the flat connection $\nabla$ on $\mathcal{H}$. Let $\Omega^{(0)} = \Omega^{(0)}_{\beta}$. Recall the Kähler potential $\tilde{K} := K_\beta$ on $U_\beta$ constructed from $\Omega^{(0)}$ and the smooth section $\Omega := \Omega_\beta$ of $\mathcal{H}^{3,0}|_{U_\beta}$ from a normalization of $\Omega^{(0)}_\beta$. Let $Z = Z(t) := Z_\Gamma(t) = \int_\gamma \Omega(t) = \int_{\gamma_1} \Gamma^\vee \wedge \Omega(t)$ be the central charge function on $U$ associated to $\Gamma$ and $\alpha = \alpha(t) := \alpha(t)$ be the phase function on $U$ associated to $Z$. An attractor flow trajectory $\gamma(t) = t(\mu) \cdot \mu \in \mathbb{R}_{>0}$, on $U$ associated to $Z$ satisfies the flow equation

$$\mu \frac{\partial}{\partial \mu} := \gamma_* \left( \frac{\partial}{\partial \mu} \right) = (d \log |Z|^2)^\sim$$

$$= \left( g^{ab} \frac{\partial}{\partial t^a} \log |Z|^2 \right) \frac{\partial}{\partial t^a} + \left( g^{ba} \frac{\partial}{\partial t^b} \log |Z|^2 \right) \frac{\partial}{\partial t^a}$$

$$= \left( g^{ab} D_a Z \right) \frac{\partial}{\partial t^a} + \left( g^{ba} \frac{D_a Z}{Z} \right) \frac{\partial}{\partial t^a},$$

restricted to the image of $\gamma$,

where $(\cdot)^\sim$ is the metrical equivalent vector field to a 1-form $(\cdot)$ on $U$, with respect to the Weil-Petersson metric $g = \sum_{a,b} g_{ab} dt^a \otimes dt^b$ (previously denoted by its associated 2-form $\omega_{WP}$). On

\[16\] I.e. all the intersections $U_{\beta_1 \cdots \beta_l} := U_{\beta_1} \cap \cdots \cap U_{\beta_l}, l \in \mathbb{N}$, are contractible.
the other hand,
\[
d\left(\text{Im}\left(e^{-i\alpha}\Omega\right)\right) = \frac{1}{2i}\left(e^{-i\alpha}D_a\Omega - \frac{e^{-i\alpha}\Omega + e^{i\alpha}\Omega}{2} D_aZ\right)dt^a + \frac{1}{2i}\left(-e^{i\alpha}D_a\tilde{\Omega} + \frac{e^{-i\alpha}\Omega + e^{i\alpha}\Omega}{2} D_a\tilde{Z}\right)dt^a.
\]

Thus, along a flow trajectory,
\[
\frac{d}{d\mu}\text{Im}\left(e^{-i\alpha}\Omega\right) = d\left(\text{Im}\left(e^{-i\alpha}\Omega\right)\right) \left(\frac{\partial}{\partial \mu}\right)
= \frac{1}{2i}g^{ab}\left(e^{-i\alpha}D_a\Omega - \frac{e^{-i\alpha}\Omega + e^{i\alpha}\Omega}{2} D_aZ\right)\frac{\partial}{\partial \mu} D_b\tilde{Z}
+ \frac{1}{2i}g^{ba}\left(-e^{i\alpha}D_a\tilde{\Omega} + \frac{e^{-i\alpha}\Omega + e^{i\alpha}\Omega}{2} D_a\tilde{Z}\right)\frac{\partial}{\partial \mu} D_bZ
= \frac{1}{2i|Z|}\left(g^{ab}D_a\Omega D_b\tilde{Z} - g^{ba}D_a\tilde{\Omega} D_bZ\right) + \frac{\text{Re}\left(e^{-i\alpha}\Omega\right)}{2|Z|^2}\left(-g^{ab}D_aZ D_b\tilde{Z} + g^{ba}D_a\tilde{Z} D_bZ\right);
\]
the first summand
\[
= \frac{1}{2|Z|}\left(-ig^{ab}D_a\Omega D_b\tilde{Z} + ig^{ba}D_a\tilde{\Omega} D_bZ\right)
= \frac{1}{2|Z|}\left(8\text{vol}(Y)\cdot\Gamma^\vee - i\Omega\tilde{Z} + i\tilde{\Omega}Z\right)
= \frac{4\text{vol}(Y)\cdot\Gamma^\vee}{|Z|} + \text{Im}\left(e^{-i\alpha}\Omega\right);
\]
the second summand
\[
= -\frac{\text{Re}\left(e^{-i\alpha}\Omega\right)}{2|Z|^2}\left(8\text{vol}(Y)\cdot\int_{Y_t} \Gamma^\vee \land \Gamma^\vee + 2\text{Im}|Z|^2\right)
= 0,
\]
where the Hodge decomposition identity applied to \(\Gamma^\vee\) is used.

In full notation, one thus obtains an ordinary differential equation (ODE) in variable \(\mu\):
\[
\mu \frac{d}{d\mu}\text{Im}\left(e^{-i\alpha(\mu)}\Omega(\mu)\right) - \text{Im}\left(e^{-i\alpha(\mu)}\Omega(\mu)\right) = \frac{4\text{vol}(Y)\cdot\Gamma^\vee}{|Z(\mu)|}.
\]
This can be solved by the Leibniz’ rule and the separation-of-variable technique in ODE to give
\[
\frac{1}{\mu}\text{Im}\left(e^{-i\alpha(\mu)}\Omega(\mu)\right) = \left(-4\text{vol}(Y)\cdot\tau_{\mu_0}(\mu)\right)\Gamma^\vee + \frac{1}{\mu_0}\text{Im}\left(e^{-i\alpha(\mu_0)}\Omega(\mu_0)\right),
\]
on \(U = U_j\), where
\[
\tau_{\mu_0}(\mu) = -\int_{\mu_0}^{\mu} \frac{d\mu'}{|Z(\mu')|\mu'^2}.
\]
Since
\[
\tau_{\mu_0}(\mu_1) + \tau_{\mu_1}(\mu_2) = \tau_{\mu_0}(\mu_2)
\]
and the above local solution for \(\text{Im}\left(e^{-i\alpha(\mu)}\Omega(\mu)\right)\) from local computations is independent of all the local choices made, the expression glues under the transition between local charts and remains valid along and throughout an attractor flow trajectory in \(\mathcal{M}\).

This completes the proof. \(\square\)
Remark 3.1.2. [constant trajectory]. A constant trajectory $\gamma$ of a $\Gamma$-attractor flow occurs at $t \in M$ where $Z(t) \neq 0$ and $(D_b Z)(t) = (\bar{D_b} \bar{Z})(t) = 0$. In this case, \( \text{Im}(e^{-i\alpha(\mu)\Omega(\mu)}) \) takes the constant value $-\frac{4 \text{vol}(Y) \Gamma^\vee}{|Z(t)|}$ as $\mu$ runs in $\mathbb{R}_{>0}$.

3.2 Morse cobordisms of A-branes on Calabi-Yau 3-folds under a reverse split attractor flow at a wall of marginal stability.

We derive first a topological smoothability criterion along a $\Gamma$-attractor flow for a special Lagrangian submanifold $L$ in the class $\Gamma$, with only transverse normal crossing singularities in a Calabi-Yau 3-fold, by a fusion of [De3] of Denef and [Joy3: IV and V] of Joyce and then use it as a tool to understand Morse cobordisms of A-branes under a (reverse) split attractor flow. We'll assume in this subsection that all the attractor flow trajectories in question are nonconstant.

**Topological smoothability criterion along attractor flow: Denef meeting Joyce.**

The result of Denef [De3], recapped as Proposition 3.1.1 in Sec. 3.1, gives us a topologically necessary starting point to consider deformations of special Lagrangian submanifolds, with a phase lined up with that of central charge, on Calabi-Yau 3-folds along an attractor flow:

**Lemma 3.2.1. [vanishing of topological obstruction along attractor flow].** Let

\[
\begin{align*}
\cdot &\ \mathcal{F} \text{ be a domain in } M, \\
\cdot &\ \{ Y^s := (Y, J^s, \omega^s, \Omega^s) : s \in \mathcal{F} \} \text{ be a family of smooth Calabi-Yau 3-folds with the family of underlying complex manifolds } (Y, J^s) \text{ specified by } \mathcal{F}, \text{ the K"ahler classes } [\omega^s] \in H^2(Y, \mathbb{R}) \text{ fixed, and the normalization } \frac{1}{3} \omega^s \wedge \omega^s \wedge \omega^s = \frac{i}{2} \Omega^s \wedge \overline{\Omega^s}; \\
\cdot &\ \Gamma \in H_3(Y; \mathbb{Z}), Z_\Gamma \text{ be the central charge function on } \mathcal{F} \text{ associated to } \Gamma, \text{ defined by } Z_\Gamma(s) = [\Omega^s] \cdot \Gamma = \int Y \Omega^s; \ \alpha_\Gamma = \text{Arg}(Z_\Gamma/|Z_\Gamma|) \text{ be the phase-angle function on } \mathcal{F} \text{ associated to } Z_\Gamma, \text{ defined modulo } 2\pi; \\
\cdot &\ s_0 \in \mathcal{F} \text{ with } Z_\Gamma(s_0) \neq 0, \ f : L \rightarrow Y^{s_0} \text{ be a special Lagrangian submanifold (possibly with singularities) with phase } e^{i\alpha_\Gamma(s_0)} \text{ in } Y^{s_0} \text{ and with } f_*([L]) = \Gamma; \\
\cdot &\ \gamma : I \rightarrow \mathcal{F}, I \text{ an interval in } \mathbb{R}_+ \text{, be a trajectory of the attractor flow associated to } Z_\Gamma, \text{ with } \gamma(\mu_0) = s_0.
\end{align*}
\]

Then
\[
[\text{Im}(e^{-i\alpha_\Gamma(\mu)\Omega(s(\mu))})] \cdot f_*([L]) = 0, \quad \text{for all } \mu \in I.
\]

**Proof.** This follows immediately from Proposition 3.1.1 since, under the situation given,
\[
[\Gamma^\vee] \cdot \Gamma = 0 = [\text{Im}(e^{-i\alpha_\Gamma(\mu_0)\Omega(s(\mu_0))})] \cdot f_*([L]).
\]

Continuing the situation of Lemma 3.2.1, assume further that
\[
f = f_1 \cup \cdots \cup f_q : L = L_1 \cup \cdots \cup L_q \rightarrow Y^{s_0}
\]
is an immersed special Lagrangian submanifolds (with phase $e^{i\alpha_\Gamma(s_0)}$) with only transverse intersections and with each $L_j$ connected. (In particular, all $L_j$’s are smooth.) Here, $\cup$ is a disjoint union.
Remark 3.2.2. [location at wall of marginal stability]. Let \( \Gamma_j = f_\ast([L_j]) = f_{j_\ast}([L_j]) \) and \( Z_{\Gamma_j} \) be the associated complex central-charge function on \( \cF \), defined by \( Z_{\Gamma_j}(s) := [\Omega^j] \cdot \Gamma_j = \int_{L_j} f_\ast \Omega^j \).
Assume that \( Z_{\Gamma_j} \neq 0 \), for all \( j \), and let \( e^{i\omega_{\Gamma_j}} := Z_{\Gamma_j}/|Z_{\Gamma_j}| \) be the associated phase function on \( \cF \). Then since \( f_j = f_{L_j}, f_j^* \text{Im}(e^{-i\omega_{\Gamma_j}(s_0)}\Omega^{s_0}) = 0 \) and \( f_j^* \text{Re}(e^{-i\omega_{\Gamma_j}(s_0)}\Omega^{s_0}) \) coincides with the volume-form on \( L_j \) induced by the metric \( f_j^*\omega^j \) for all \( j \). It follows that \( e^{-i\omega_{\Gamma_j}(s_0)} \cdot Z_{\Gamma_j}(s_0) > 0 \), for all \( j \), and, hence (in case that \( q > 1 \)), \( s_0 \) lies in the stratum of the wall \( \Pi_{2s_0}^j \) of marginal stability in \( \cF \) on which all the phase angles \( \omega_{\Gamma_j}(s_0), j = 1, \ldots, q \), and \( \omega_{\Gamma_j}(s_0) \) are equal.

Assumption 3.2.3. [connectivity]. Without loss of generality and for the simplicity of presentation, one may assume that \( f(L) \) is connected since otherwise one simply applies the discussion below component by component and then adjusts the related constants that appear in the estimates/inequalities used so that all the corresponding constants become equal and work simultaneously for all components.

Denote by \( \{y_1, \ldots, y_n\} \subset Y^{s_0} \) the set of points in \( Y^{s_0} \) at which \( f \) meets transversely. Let \( f^{-1}(y_j) = \{p_j^+, p_j^-\} \), where \( \pm \) is determined by the angle condition that the sum of the characteristic angles \( \Pi \) from \( f_\ast T_{p_j^+} L \) to \( f_\ast T_{p_j^-} L \) as a pair of special Lagrangian subspaces with phase \( e^{i\omega_{\Gamma_j}(s_0)} \) in \( T_{y_j} Y^{s_0} := (T_{y_j} Y, J^{s_0}|_{y_j}, \omega^{s_0}|_{y_j}, \Omega^{s_0}|_{y_j}) \) is equal to \( \pi \).

Definition 3.2.4. [dual graph of decomposition]. The dual graph \( \Xi_{f=f_1 \cup \cdots \cup f_q} \) associated to the decomposition \( f : L = L_1 \cup \cdots \cup L_q \to Y^{s_0} \) is a directed graph with

- a vertex \( v_j \) for each \( L_j, j = 1, \ldots, q \), and
- a directed edge (i.e. arrow) \( e_l \) for each \( y_l, l = 1, \ldots, n \), with initial end-point (i.e. tail) \( v_j \) and terminal end-point (i.e. head) \( v_k \) if \( p_l^+ \in L_j \) and \( p_l^- \in L_k \).

The set of vertices (resp. edges) of \( \Xi_{f=f_1 \cup \cdots \cup f_q} \) will be denoted by \( \Xi_{f=f_1 \cup \cdots \cup f_q}^{(0)} \) (resp. \( \Xi_{f=f_1 \cup \cdots \cup f_q}^{(1)} \)).

Remark 3.2.5. [transverse special Lagrangian intersection in CY3: direction = sign]. In \( \mathbb{C}^3 \) with coordinates \((z^1, z^2, z^3) = (x^1 + iy^1, x^2 + iy^2, x^3 + iy^3)\), the standard Hermitian metric \( ds^2 = dz^1 \otimes d\bar{z}^1 + dz^2 \otimes d\bar{z}^2 + dz^3 \otimes d\bar{z}^3 \) and holomorphic 3-form \( \Omega = dz^1 \wedge dz^2 \wedge dz^3 \), a pair of special Lagrangian submanifold with respect to the standard Kähler form \( \omega = -\frac{i}{2} \text{Im}(ds^2) = \frac{i}{2}(dz^1 \wedge d\bar{z}^2 + d\bar{z}^2 \wedge dz^3 + dz^3 \wedge d\bar{z}^3) \) and the calibration \( Re \Omega \) can be put into the canonical form

\[
\Pi^0 = \{(x^1, x^2, x^3) : x^1, x^2, x^3 \in \mathbb{R}\}, \quad \Pi^{(\phi_1, \phi_2, \phi_3)} = \{(e^{i\phi_1}x^1, e^{i\phi_2}x^2, e^{i\phi_3}x^3) : x^1, x^2, x^3 \in \mathbb{R}\}
\]

under the \( SU(3) \)-action on \( (\mathbb{C}^3, ds^2) \), with \( 0 < \phi_1, \phi_2, \phi_3 < \pi, \phi_1 + \phi_2 + \phi_3 = \pi \) or \( 2\pi \); (cf. Appendix A.1). One can check that, with the built-in orientation on \( \mathbb{C}^3 \) and the orientation on \( \Pi^0 \) and \( \Pi^{(\phi_1, \phi_2, \phi_3)} \) determined by \( Re \Omega \) taken into account, the (oriented) intersection

\[
\Pi^0 \cdot \Pi^{(\phi_1, \phi_2, \phi_3)} = -\Pi^{(\phi_1, \phi_2, \phi_3)} \cdot \Pi^0 = \begin{cases} +1 & \text{if } \phi_1 + \phi_2 + \phi_3 = \pi, \\ -1 & \text{if } \phi_1 + \phi_2 + \phi_3 = 2\pi. \end{cases}
\]

It follows that in Definition 3.2.4, \( \Xi_{f=f_1 \cup \cdots \cup f_q} \) has an arrow \( e_l \) from \( v_j \) to \( v_k \) if and only if the (oriented) intersection \( L_j \cdot L_k \) takes value \( +1 \) at \( y_l \); and, hence one may equivalently take the latter condition as the rule to assign arrows in \( \Xi_{f=f_1 \cup \cdots \cup f_q} \).

In particular, \( \Xi_{f=f_1 \cup \cdots \cup f_q} \), which appears naturally in Joyce’ setting as defined above via characteristic angles (cf. [Joy3: V. ‘graphical method’ in the end of Sec. 9.2]), is directly related to the quiver underlying the effective \( d = 1, N = 4 \) supersymmetric quantum mechanics associated to the D-brane configuration in \( Y^{s_0} \) specified in part by \( f \), which is more naturally defined via the sign at the intersections; cf. [De4: Sec. 3] and [D-M: Sec’s. 4.1 and 4.2].

---

17 Here we use the convention in [Joy3: V, Sec. 6.4, Example 6.11], cf. Appendix A.1.
For the convenience of the application of Joyce’ result to the case of attractor flows, we recall [Joy3: V, Sec. 9.3, Theorem 9.8] (cf. Appendix A.1, Theorem A.1.5) in three steps below, with notations and some settings adapted to our situation.

**Definition 3.2.6. [Joyce’ criterion: smoothability by deforming complex structure].** For $A_1, \cdots, A_n > 0$, let $G_{(A_1, \cdots, A_n)}$ be the set of $(s, t) \in F \times (0, 1)$ such that

$$f^* [\text{Im} (e^{-i\alpha(s)} \Omega^s)] \cdot [L_j] = t^3 \cdot \left( \sum_{k \in \{1, \ldots, n\}, p_k^j \in L_j} A_k - \sum_{k' \in \{1, \ldots, n\}, p_{k'}^j \in L_j} A_{k'} \right),$$

for $j = 1, \ldots, q$. When $G_{(A_1, \cdots, A_n)} \neq \emptyset$ and contains a neighborhood of $(s_0, 0)$ in $F \times (0, 1)$, we say that the tuple $(A_1, \cdots, A_n)$ satisfies Joyce’ criterion (of smoothing the special Lagrangian submanifold-with-singularity $f(L)$ with a phase via deforming complex structures).

**Definition 3.2.7. [admissible subset].** For $(A_1, \cdots, A_n)$ that satisfies the Joyce’ criterion above, $\epsilon \in (0, 1)$, $\kappa > 1$, and $C > 0$, define

$$G_{(A_1, \cdots, A_n)}^{\epsilon, \kappa, C} = \{(s, t) \in G_{(A_1, \cdots, A_n)} : t \in (0, \epsilon], |s - s_0| \leq C t^{\kappa + 3/2} \}$$

and call it an admissible subset in $F \times (0, 1)$ (for smoothing $L$).

Here, a local coordinate chart is introduced in a neighborhood of $s_0 \in F$ and $|s - s_0|$ is defined in terms of this chart as a subset in some $\mathbb{R}^{n'}$.

**Theorem 3.2.8. [desingularization in a family of Calabi-Yau’s].** ([Joy3: V, Sec. 9.3, Theorem 9.8]; cf. Appendix A.1: Theorem A.1.5 and Observation A.1.6.) Recall the family \{ $Y^s := (Y, J^s, \omega^s, \Omega^s) : s \in F$ \} of smooth Calabi-Yau 3-folds and the special Lagrangian immersion

$$f = f_1 \cup \cdots \cup f_q : L = L_1 \cup \cdots \cup L_q \to Y^s$$

with phase $e^{i\alpha(s_0)}$. Continuing the situation under study, let $N$ be the (oriented) connected sum of (the disjoint union) $L_1 \cup \cdots \cup L_q$ with itself at the pairs of points $(p_k^s, p_k^-)$ for $k = 1, \ldots, n$. Note that under Assumption 3.2.3, $N$ is connected. Note also that since $[\omega^s] \in H^2(Y; \mathbb{R})$ is fixed, treating $f$ as a map to $Y$ (and hence to all $Y^s$), one has $f^*[\omega^s] = f^*[\omega^{s_0}] = 0$ in $H^2(L; \mathbb{R})$ for all $s \in F$. Suppose that the $n$-tuple $(A_1, \cdots, A_n)$ satisfies Joyce’ criterion in Definition 3.2.6. Then, there exist constants $\epsilon \in (0, 1)$, $\kappa > 1$, and $C > 0$, and a smooth family of maps

$$\{ f^{s,t} : L^{s,t} := \tilde{N}^{s,t} \to Y^s \mid (s, t) \in G_{(A_1, \cdots, A_n)}^{\epsilon, \kappa, C} \}$$

such that

- $\tilde{N}^{s,t}$ is a compact smooth manifold diffeomorphic to $N$, constructed by gluing a Lawlor neck $C^{-1/\kappa + 3/2} t \cdot L_{k, A_k}$ into $f(L)$ at $y_k$ for $k = 1, \ldots, n$;
- $f^{s,t} : L^{s,t} \to Y^s$ is an embedded special Lagrangian submanifold with phase $e^{i\alpha(s)}$;
- in the sense of currents, $f^{s,t} \to f$ as $(s, t) \to (s_0, 0)$.

Furthermore, $\kappa > 1$ can be chosen to be arbitrarily close to 1.

Let us now specialize to what happens along the trajectory $\gamma : I \to F$ of the $\Gamma$-attractor flow with $\gamma(\mu_0) = s_0$. For simplicity of notations, we will denote $e^{i\alpha(s)}$ by $e^{i\alpha}$ and $s = \gamma(\mu)$ by...
s = s(μ). Recall that $\text{Im} \left( e^{-iα(μ)}Ω^s(μ) \right) = (-4 \text{vol}(Y^{s_0}) \cdot μ \tau(μ)) \Gamma^\vee + \frac{dμ}{μ} \text{Im} \left( e^{-iα(μ_0)}Ω^s(μ_0) \right)$ along $γ$, where $τ(μ) = -\int_{μ_0}^{μ} \frac{dμ'}{Z_{τ}(s(μ'))|μ|^2}$, from Proposition 3.1.1. It follows that

$$\left( f^*[\text{Im}(e^{-iα(μ)}Ω^s(μ))] \cdot [L_j] \right)_{j=1}^q = (-4 \text{vol}(Y^{s_0}) \cdot μ \tau(μ)) \cdot \left( \langle Γ, f_*[L_j] \rangle \right)_{j=1}^q \in (-4 \text{vol}(Y^{s_0}) \cdot μ \tau(μ)) \cdot \mathbb{Z}^q$$

since $f^*[\text{Im}(e^{-iα(μ_0)}Ω^s_0)] \cdot [L_j] = 0$ for $j = 1, \ldots, q$. Here, $(\cdot, \cdot) : H_3(Y;\mathbb{Z}) \times H_3(Y;\mathbb{Z}) \to \mathbb{Z}$ is the (symplectic) intersection form on $Y$. Note that

$$-4 \text{vol}(Y^{s_0}) \cdot μ \tau(μ) \begin{cases} > 0 & \text{for } μ > μ_0, \\ = 0 & \text{for } μ = μ_0, \\ < 0 & \text{for } μ < μ_0. \end{cases}$$

A comparison of this with Joyce' criterion in Definition 3.2.6 leads one immediately to the following definition:

**Definition 3.2.9. [topological criterion of smoothing].** For $A_1, \ldots, A_n > 0$, we say that the tuple $(A_1, \ldots, A_n)$ satisfies the topological criterion of smoothing (the special Lagrangian submanifold-with-singularity $f(L)$ with a phase via deforming complex structures) if it satisfies

$$\left( \langle Γ, f_*[L_j] \rangle \right)_{j=1}^q = \left( \sum_{k \in \{1, \ldots, n\}, p_k^i ∈ L_j} A_k - \sum_{k' \in \{1, \ldots, n\}, p_{k'}^i ∈ L_j} A_k' \right)_{j=1}^q$$

This means that $(A_1, \ldots, A_n)$ is a positive solution to a system of inhomogeneous linear equations whose homogeneous/degree-1 part comes solely from the dual graph $Ξ_{f=∑_{j=1}^q f_j}$ and whose constant/degree-0 part is given by $(\langle Γ, f_*[L_j] \rangle)_{j=1}^q$. The following lemma is immediate:

**Lemma 3.2.10. [topological criterion ⇒ Joyce' criterion].** If $(A_1, \ldots, A_n)$ with positive entries satisfies the topological criterion of smoothing, then it satisfies Joyce' criterion.

With the above preparations and as a consequence of Proposition 3.1.1 from Denef and Theorem A.1.5 from Joyce, one can now show that:

**Proposition 3.2.11. [smoothing along attractor flow].** Continuing the situation under study, let $(A_1, \ldots, A_n)$, with positive entries, satisfy the topological criterion of smoothing. Then, there exist constants $δ, \epsilon ∈ (0, 1)$, $κ > 1$, and $C > 0$, and a smooth family of maps

$$\left\{ f^{s,t} : L^{s,t} := N^{s,t} → Y^s \ \big| \ (s,t) ∈ G^{ε,κ,C}_{(A_1, \ldots, A_n)} \right\}$$

as in Theorem A.1.5 such that

- the restriction $γ : (μ_0, μ_0 + δ) ⊂ I → F$ of the $Γ$-attractor flow trajectory $γ : I → F$ lifts to $\tilde{γ} : (μ_0, μ_0 + δ) → G^{ε,κ,C}_{(A_1, \ldots, A_n)} ⊂ F × (0, 1)$ with $pr_1 ∘ \tilde{γ} = γ|_{(μ_0, μ_0 + δ)}$, where $pr_1 : F × (0, 1) → F$ is the projection map.

In other words, Joyce' construction smoothes $f(L) ⊂ Y^{s_0}$ into $Y^{s(μ)}$ along the $Γ$-attractor flow trajectory $γ$ with $μ ≥ μ_0$. 

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Definition 3.2.7 translates in the current situation to an admissibility condition of the form
\[ \kappa > 0 \text{ in A.1.5}, \]
and restriction
\[ \gamma \in \text{the requirement that the system is solvable only on the (}\mu > \mu_0\text{-side of the flow } \gamma, \text{ in which case} \]
\[ \mu = \mu(t) \quad \text{with } |\mu - \mu_0| = O(t^3) \quad \text{and inverse } \quad t = t(\mu) \]
for \( \mu \in (\mu_0, \mu_0 + \delta) \subset I \) for some \( \delta > 0 \).

The further admissibility condition \( |s - s_0| \leq C' t^{\kappa + 3/2} \) with \( \kappa' > 1 \) and \( C' > 0 \) in Definition 3.2.7 translates in the current situation to an admissibility condition of the form
\[ |\mu - \mu_0| \leq C'' t^{\kappa'' + 3/2} \] with \( \kappa'' > 1 \) and \( C'' > 0 \). Recall now Observation A.1.6 that in Theorem A.1.5, \( \kappa > 1 \) can be chosen to be arbitrarily close to 1 and \( C > 0 \) can be adjusted. It follows that for \( \epsilon > 0, \kappa > 1, \) and \( C > 0 \) in Theorem A.1.5 with \( 3 > \kappa + 3/2 \), and a re-choice of \( \delta > 0 \) to be even smaller if necessary, one has
\[ (\mu(t), t) = (\mu, t(\mu)) \in G_{\epsilon, \kappa, C}^{(A_1, \ldots, A_n)} \quad \text{for } \mu \in (\mu_0, \mu_0 + \delta). \]

The map \( \bar{\gamma} : (\mu_0, \mu_0 + \delta) \rightarrow G_{\epsilon, \kappa, C}^{(A_1, \ldots, A_n)} \) defined by \( \mu \mapsto (\mu, t(\mu)) \) gives then a lifting of the restriction \( \gamma|_{(\mu_0, \mu_0 + \delta)} \) of \( \gamma \). This concludes the proof.

\[ \square \]

Remark 3.2.12. [decay of BPS states]. Recall that \( A_k > 0, k = 1, \ldots, n, \) are the moduli of Lawlor necks with fixed asymptotic special Lagrangian planes and \( t > 0 \) is the scaling factor, both involved in the gluing-rectifying construction. Their positivity has thus a geometric meaning and, hence, the construction of smoothing \( f(L) \) is “directional”. As special Lagrangian cycles support supersymmetric D-brane configurations of A-type, reverse of this direction, i.e. \( \mu \) crossing the wall \( \Pi_{MS}^L \) from \( \mu_0^+ \geq \mu_0 \) to \( \mu_0 \leq \mu_0, \) in which the underlying special Lagrangian cycles morph from a smooth one to one with several components indicates then a decay of a BPS state. Cf. Remark 3.2.2.
Morse cobordisms of A-branes under a (reverse) split attractor flow:
Denef-Joyce meeting Polchinski-Grothendieck.

Let
\[ f = f_1 \cup \cdots \cup f_q : L = L_1 \cup \cdots \cup L_q \to Y \]
be a special Lagrangian immersion with phase \( e^{i\alpha} \) in a Calabi-Yau 3-fold \( Y \), where \( \Gamma = f_*[L] \in H_3(Y;\mathbb{Z}) \), with only isolated transverse intersections as in the situation of the previous theme. Continuing the notations there, assume that \( f \) satisfies the topological criterion of smoothing in Definition 3.2.9 and let
\[ f^\mu := f^{s(\mu),t(\mu)} : L^\mu := L^{s(\mu),t(\mu)} \to Y^\mu := Y^{s(\mu)}, \quad \mu \in (\mu_0, \mu_0 + \delta) \subset \mathbb{R}_{>0} \]
be a smoothing of \( f := f^{\mu_0} = f^{s(\mu_0),t(\mu_0)} = f^{s_0,0} \) along the \( \Gamma \)-attractor flow \( \gamma = \gamma(\mu) \) with \( \gamma(\mu_0) = s_0 \), following Joyce’ construction. Assume that \( \delta < \mu_0 \). Let
\[ \pi(\mu_0 - \delta, \mu_0 + \delta) : X(\mu_0 - \delta, \mu_0 + \delta) \to (\mu_0 - \delta, \mu_0 + \delta) \subset \mathbb{R}_{>0} \]
be a Morse family, as constructed in the same way as in Example 2.2.1(c), such that
\[ \pi^\mu(\mu_0 - \delta, \mu_0 + \delta)(\mu) := X_\mu = \begin{cases} \Pi_{i=1}^q L_i & \text{for } \mu \in (\mu_0 - \delta, \mu_0), \\ f(L) & \text{for } \mu = \mu_0, \\ L^\mu \simeq N & \text{for } \mu \in (\mu_0, \mu_0 + \delta). \end{cases} \]

Take a smaller \( \delta > 0 \) if necessary; let \( \Gamma_i := f_*[L_i] \in H_3(Y;\mathbb{Z}) \) and \( \gamma_i : (\mu_0 - \delta, \mu_0) \to \mathcal{F} \) be the \( \Gamma_i \)-attractor flow with \( \gamma_i(\mu_0) = s_0 \). Let
\[ p(\mu_0 - \delta, \mu_0 + \delta) : Y(\mu_0 - \delta, \mu_0 + \delta) \to (\mu_0 - \delta, \mu_0 + \delta), \]
where
\[ p^\mu(\mu_0 - \delta, \mu_0 + \delta)(\mu) := Y_\mu = \begin{cases} \Pi_{i=1}^q Y_i^\mu & \text{for } \mu \in (\mu_0 - \delta, \mu_0), \\ Y = Y^{\mu_0} & \text{for } \mu = \mu_0, \\ Y^\mu & \text{for } \mu \in (\mu_0, \mu_0 + \delta), \end{cases} \]
be the universal Calabi-Yau 3-fold over the union of attractor flow trajectories
\[ \left( \bigcup_{i=1}^q \gamma_i (\mu_0 - \delta, \mu_0) \right) \bigcup \gamma (\mu_0, \mu_0 + \delta) \subset \mathcal{F}. \]

Take even a smaller \( \delta > 0 \) if necessary, recall Theorem A.1.2 in Appendix A. 1, and let
\[ f^\mu_i : L^\mu_i \to Y^\mu_i, \quad \mu \in (\mu_0 - \delta, \mu_0) \]
be a smooth family of immersed special Lagrangian submanifolds, with \( L^\mu_i = L_i \), from Theorem A.1.2. Denote the inclusion \( f(L) \hookrightarrow Y \) by \( f \). Then, one can extend the family \( f^\mu \),

---

\textit{Mathematical Convention vs. String-Theoretical Convention.} This union is called a \textit{split attractor flow trajectory} in \( \mathcal{F} \). In mathematical convention, flow follows the direction of increasing \( \mu \) (i.e. the direction of the vector field on \( \mathcal{F} \) that defines the flow) while in string-theoretical convention for the attractor, the flow follows the direction of decreasing \( \mu \). In this note, we follow the mathematical convention so far as that is more natural for the purpose to address desingularization, instead of bend-and-break, of a special Lagrangian submanifold with singularities. However, the term ‘split attractor flow’ in stringy literature, as follows the stringy convention, is fixed to correspond to the direction \( \Gamma \to \Gamma_1 + \cdots + \Gamma_q \). For that reason, we have to call the natural direction in our question, \( \Gamma_1 + \cdots + \Gamma_q \to \Gamma \), a reverse split attractor flow.
\[ \mu \in [\mu_0, \mu_0 + \delta), \text{ to a family of immersed special Lagrangian submanifolds over } (\mu_0 - \delta, \mu_0 + \delta): \]

\[
\begin{array}{cccc}
X_{(\mu_0 - \delta, \mu_0 + \delta)} & \xrightarrow{f_{(\mu_0 - \delta, \mu_0 + \delta)}} & Y_{(\mu_0 - \delta, \mu_0 + \delta)} \\
\pi_{(\mu_0 - \delta, \mu_0 + \delta)} & \downarrow & \pi_{(\mu_0 - \delta, \mu_0 + \delta)} \\
(\mu_0 - \delta, \mu_0 + \delta) & \xrightarrow{p_{(\mu_0 - \delta, \mu_0 + \delta)}} & (\mu_0 - \delta, \mu_0 + \delta)
\end{array}
\]

with
\[
f_{\mu} := f_{(\mu_0 - \delta, \mu_0 + \delta)}|_{\mu \in (\mu_0 - \delta, \mu_0 + \delta)} : X_{\mu} \longrightarrow Y_{\mu},\]
such that
\[
f_{\mu} = \begin{cases} 
\Pi^q_{i=1} f_i^\mu & \text{for } \mu \in (\mu_0 - \delta, \mu_0), \\
\hat{f} & \text{for } \mu = \mu_0, \\
\hat{f}^\mu & \text{for } \mu \in (\mu_0, \mu_0 + \delta).
\end{cases}
\]

Consider the following data
\[
(\hat{E}_{(\mu_0 - \delta, \mu_0 + \delta)}, \hat{\nabla}_{(\mu_0 - \delta, \mu_0 + \delta)})
\]

similar to Example 2.1.16, where

- all the maps in the commutative diagram are maps over \((\mu_0 - \delta, \mu_0 + \delta)\);
- \(\hat{c}_{(\mu_0 - \delta, \mu_0 + \delta)} : \hat{X}_{(\mu_0 - \delta, \mu_0 + \delta)} \rightarrow X_{(\mu_0 - \delta, \mu_0 + \delta)}\) is a covering map of finite degree \(\hat{d}\);
- \(\hat{f}_{(\mu_0 - \delta, \mu_0 + \delta)} : \hat{X}_{(\mu_0 - \delta, \mu_0 + \delta)} \rightarrow Y_{(\mu_0 - \delta, \mu_0 + \delta)}\) is the composition \(f_{(\mu_0 - \delta, \mu_0 + \delta)} \circ \hat{c}\);
- \(pr_1\) and \(pr_2\) are the built-in projection maps from the fibered product;
- \((\hat{E}_{(\mu_0 - \delta, \mu_0 + \delta)}, \hat{\nabla}_{(\mu_0 - \delta, \mu_0 + \delta)})\) is a locally free \(\mathcal{O}_{X_{(\mu_0 - \delta, \mu_0 + \delta)}}^{\hat{\nabla}}\)-module of finite rank \(\hat{r}\) with a flat \(U(\hat{r})\)-connection.
Let $E_{(\mu_0 - \delta, \mu_0 + \delta)} := \hat{\mathcal{E}}_{(\mu_0 - \delta, \mu_0 + \delta)} \ast \hat{\mathcal{E}}_{(\mu_0 - \delta, \mu_0 + \delta)}$, which is a locally free $O_{X_{(\mu_0 - \delta, \mu_0 + \delta)}}$-module of rank $r = \hat{d}_r$. Then, following the construction of Example 2.1.16, one obtains a Morse cobordism family of morphisms

$$\varphi_{(\mu_0 - \delta, \mu_0 + \delta)} : (X_{(\mu_0 - \delta, \mu_0 + \delta)}, E_{(\mu_0 - \delta, \mu_0 + \delta)}) \rightarrow Y_{(\mu_0 - \delta, \mu_0 + \delta)}$$

over $(\mu_0 - \delta, \mu_0 + \delta)$, with a $U(\tau)$ minimally flat connection-with-singularity $\nabla^{(\mu_0 - \delta, \mu_0 + \delta)}$ on the surrogate $(X_{\varphi_{(\mu_0 - \delta, \mu_0 + \delta)}}, \mathcal{E}_{\varphi_{(\mu_0 - \delta, \mu_0 + \delta)}})$ associated to $\varphi_{(\mu_0 - \delta, \mu_0 + \delta)}$. This gives an example of Morse cobordisms of A-branes under the (reverse) split attractor flow in the sense of Morse family of morphisms from Azumaya manifolds with additional data. Once having $\varphi_{(\mu_0 - \delta, \mu_0 + \delta)}$, one can deform $\varphi_{(\mu_0 - \delta, \mu_0 + \delta)}$ as in Example 2.2.1 to obtain more Morse cobordisms of A-branes under the (reverse) split attractor flow.

**Remark 3.2.13.** [splitting/gluing) flat connection under (reverse) split attractor flow] Given

$$\hat{\pi}_{(\mu_0 - \delta, \mu_0 + \delta)} : \hat{X}_{(\mu_0 - \delta, \mu_0 + \delta)} \rightarrow (\mu_0 - \delta, \mu_0 + \delta)$$

via the composition $\pi_{(\mu_0 - \delta, \mu_0 + \delta)} \circ \hat{\mathcal{E}}_{(\mu_0 - \delta, \mu_0 + \delta)}$ in the above discussion, let $\hat{X}_\mu := \hat{\pi}_{(\mu_0 - \delta, \mu_0 + \delta)}^{-1}(\mu)$ be the fiber over $\mu \in (\mu_0 - \delta, \mu_0 + \delta)$. Then, these fibers are of three homeomorphism types:

$$\hat{X}_\mu \simeq \begin{cases} 
\hat{N} & \text{a covering of } N, \text{ the smoothing of } f(L) \\
\hat{f}(L) & \text{a covering of } f(L), \\
\Pi_{i=1}^q \hat{L}_i & \text{a covering of } L_i. 
\end{cases}$$

It follows that

- $f(\hat{L})$ (resp. $\hat{N}$) is a (self-)bouquet (resp. (self-)connected sum) of $\Pi_{i=1}^q \hat{L}_i$,

- $f(\hat{L})$ is obtained from $\hat{N}$ by pinching a finite collection of two-collared embedded $S^2$'s in $\hat{N}$.

Let

- $\Gamma_{\hat{f}(L)}$ be the dual graph of the manifold-with-Morse-type-singularity $\hat{f}(L)$, with one vertex $v_j$ for each connected component, denoted by $\hat{L}_{v_j}$, of $\Pi_{i=1}^q \hat{L}_i$ and one edge $e_{jj'}$ connecting $v_j$ and $v_{j'}$ for each intersection of the irreducible components in $\hat{f}(L)$ associated to $L_{v_j}$ and $L_{v_{j'}}$ respectively;

- $\Gamma_{\hat{f}(L)}^{(0)}$ be the set of vertices of $\Gamma_{\hat{f}(L)}$; and

- $\Gamma_{\hat{f}(L)}^{(1)}$ be the set of edges of $\Gamma_{\hat{f}(L)}$.

Then, with an implicit base point on each connected component chosen,

$$\pi_1(\hat{N}) = \pi_1(\hat{f}(L))$$

---

$^{19}$C.-H.L. would like to thank Frederik Denef for several discussions on this in spring, 2010. Readers are referred to [Sta], [He], and [Ja] for some of the terminologies used in this theme and for basics on relations between fundamental groups and 3-manifolds.
(i.e. there is a canonical isomorphism between the two under the built-in pinching-map \( \hat{N} \to f(L) \) that takes the base-point of the former to that of the latter) fits into an exact sequence of groups

\[
1 \to \pi_1 \left( \bigvee_{v_j \in \Gamma_f(0)} \hat{L}_{v_j} \right) \to \pi_1(f(L)) \to \pi_1(\Gamma_f(L)) \to 1
\]

where \( \bigvee_{v_j \in \Gamma_f(0)} \hat{L}_{v_j} \) is the bouquet of \( \{ \hat{L}_{v_j} : v_j \in \Gamma_f(0) \} \) following any spanning tree of \( \Gamma_f(L) \).

Its fundamental group is isomorphic to the free-product \( \ast_{v_j \in \Gamma_f(0)} \pi_1(\hat{L}_{v_j}) \) of the groups \( \pi_1(\hat{L}_{v_j}) \), \( v_j \in \Gamma_f(L) \). Let \( \text{Rep}(\bullet, U(\hat{r})) \) be the representation variety of the group \( \bullet \) on \( U(\hat{r}) \) (without modding out the \( U(\hat{r}) \)-action from the \( U(\hat{r}) \)-action on itself by conjugation). Then, the above sequence and isomorphisms induce a morphism

\[
\text{Rep}(\pi_1(\hat{N}), U(\hat{r})) \to \times_{v_j \in \Gamma_f(0)} \text{Rep}(\pi_1(L_{v_j}), U(\hat{r}))
\]

via pulling back a representation. In general, this map is neither injective nor surjective, and can have positive-dimensional fibers. This implies that in the construction of a Morse cobordism of A-branes,

- the choice of the isomorphic class of \((\hat{E}_{(\mu_0, \mu_0 + \delta)}, \nabla^{(\mu_0, \mu_0 + \delta)})\) determines the isomorphic class of the whole \((\hat{E}_{(\mu_0 - \delta, \mu_0 + \delta)}, \nabla^{(\mu_0 - \delta, \mu_0 + \delta)})\);

- but the choice of the isomorphic class of \((\hat{E}_{(\mu_0 - \delta, \mu_0)}, \nabla^{(\mu_0 - \delta, \mu_0)})\) does not determine the isomorphic class of the whole \((\hat{E}_{(\mu_0 - \delta, \mu_0 + \delta)}, \nabla^{(\mu_0 - \delta, \mu_0 + \delta)})\).

In particular, for the simplest class of A-branes that are realized as embedded special Lagrangian submanifolds \((L, V, \nabla)\) with a vector bundle and \(U(\hat{r})\) flat connection on the Calabi-Yau 3-fold \(Y\), assume that the \([L]\)-attractor flow \(\gamma\) bends-and-breaks \(L\) to a union \(L_1 \cup \cdots \cup L_q\) of embedded special Lagrangian submanifolds with only isolated transverse intersections. Then, (in the \(\mu\) decreasing direction, cf. footnote 18) \((V, \nabla)\) can be driven along and bend-and-break accordingly to a collection \((L_i, V_i, \nabla_i), i = 1, \ldots, q\) of A-branes of smaller volume, each of which may continue to flow along its associated \([L_i]\)-attractor flow. However, in the opposite (\(\mu\) increasing) direction, when one try to assemble a collection \((L_i, V_i, \nabla_i), i = 1, \ldots, q\), in this simplest class to a single A-brane \((L, V, \nabla)\) with \([L] = [L_1] + \cdots + [L_q]\), then, even if \(L\) can be constructed, there remains an ambiguity on the choices of \((V, \nabla)\) on \(L\).
Appendix.

A.1 Desingularizations of immersed special Lagrangian submanifolds with transverse intersections and their moduli space à la Joyce.

Calibrated geometry background and results from [Joy3] that are needed for the current work are collected here. Joyce’ results work for the more general almost Calabi-Yau m-folds with $m > 2$. Here, we only state his results in the special case: Calabi-Yau m-folds, with $m > 2$, (and with a slight modification of notations to fit the main contents of the notes). See also [Bu], [Ch], [Lee], and [Mar] for related study/results.

Generalization of McLean [McL] to a family.

Definition A.1.1. [family moduli space]. ([Joy3: II. Sec. 2.3, Definition 2.12].) Let $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ be a smooth family of deformations of a Calabi-Yau m-fold $(M, J, \omega, \Omega)$, with base space $\mathcal{F} \subset \mathbb{R}^d$, and $N$ be a compact special Lagrangian m-manifold in $(M, J, \omega, \Omega)$. Define the moduli space $\mathcal{M}_N^F$ of deformations of $N$ in the family $\mathcal{F}$ to be the set of pairs $(s, \hat{N})$ for which $s \in \mathcal{F}$ and $\hat{N}$ is a compact special Lagrangian m-manifold in $(M, J^s, \omega^s, \Omega^s)$ which is diffeomorphic to $N$ and isotopic to $N$ in $M$. Define a projection $\pi^F : \mathcal{M}_N^F \to \mathcal{F}$ by $\pi^F(s, \hat{N}) = s$. Then $\mathcal{M}_N^F$ has a natural topology and $\pi^F$ is continuous.

Theorem A.1.2. [deformation in family]. ([Joy3: II. Sec. 2.3, Theorem 2.13] and [Mar: Sec. 3.2, Theorem 3.21].) Let $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ be a smooth family of deformations of a Calabi-Yau m-fold $(M, J, \omega, \Omega)$, with base space $\mathcal{F} \subset \mathbb{R}^d$. Suppose $N$ is a compact special Lagrangian m-manifold in $(M, J, \omega, \Omega)$ with $[\omega^s|_N] = 0$ in $H^2(N; \mathbb{R})$ and $[\text{Im}\Omega^s|_N] = 0$ in $H^m(N; \mathbb{R})$ for all $s \in \mathcal{F}$. Let $\mathcal{M}_N^F$ be the moduli space $\mathcal{M}_N^F$ of deformations of $N$ in the family $\mathcal{F}$ and $\pi^F : \mathcal{M}_N^F \to \mathcal{F}$ the natural projection.

Then $\mathcal{M}_N^F$ is a smooth manifold of dimension $d + b^1(N)$ and $\pi^F : \mathcal{M}_N^F \to \mathcal{F}$ is a smooth submersion. For small $s \in \mathcal{F}$, the moduli space $\mathcal{M}_N^s := (\pi^F)^{-1}(s)$ of deformations of $N$ in $(M, J^s, \omega^s, \Omega^s)$ is a nonempty smooth manifold of dimension $b^1(N)$, with $\mathcal{M}_N^0 = \mathcal{M}_N$.

Transverse pair $(\Pi^+, \Pi^-)$ of special Lagrangian m-planes in $\mathbb{C}^m$.

([Joy3: V: Sec. 9.1]; see also [Bu: Sec. 1] and [Lee: Sec. 2].) Let $(\Pi^+, \Pi^-)$ be a pair of special Lagrangian m-planes $\simeq \mathbb{R}^m$ in $\mathbb{C}^m$ that intersect transversely, i.e. $\Pi^+ \cap \Pi^- = \{0\}$. Then, there exists $B \in SU(m)$ and $\phi_1, \cdots, \phi_m \in (0, \pi)$ such that

$$B(\Pi^+) = \Pi^0 \quad \text{and} \quad B(\Pi^-) = \Pi^\phi,$$

where

$$\Pi^0 = \{(x_1, \cdots, x_m) : x_j \in \mathbb{R}\} \quad \text{and} \quad \Pi^\phi = \{(e^{i\phi_1}x_1, \cdots, e^{i\phi_m}x_m) : x_j \in \mathbb{R}\}.$$ 

Furthermore, $\phi_1, \cdots, \phi_m$ are unique up to order – and hence become unique under the assumption $0 < \phi_1 \leq \cdots \leq \phi_m < \pi$ – and $\phi_1 + \cdots + \phi_m = k\pi$ for some $k \in \{1, \cdots, m - 1\}$.

Definition A.1.3. [characteristic angles and type]. The unique angles $0 < \phi_1 \leq \cdots \leq \phi_m < \pi$ is called the characteristic angles of the pair $(\Pi^+, \Pi^-)$ and $k$ is called the type of the pair $(\Pi^+, \Pi^-)$ at their intersection 0.
Note that if the characteristic angles and type of \((\Pi^+, \Pi^-)\) at 0 are \(0 < \phi_1 \leq \cdots \leq \phi_m < \pi\) and \(k\) respectively, then the characteristic angles and type of \((\Pi^-, \Pi^+)\) at 0 are \(0 < \pi - \phi_m \leq \cdots \leq \pi - \phi_1 < \pi\) and \(m - k\) respectively.

**Lawlor necks** \(L^{\phi,A}\) in \(\mathbb{C}^m\).

([Harv: pp. 139-144] and [La]; cf. [Joy3: V. Example 6.11] and also [Bu], [Lee], and [Mar].)

(a) **Lawlor necks** \(L^{\phi,A}\). Let \(m > 2\) and \(a_1, \ldots, a_m > 0\). Define polynomial \(P\) by

\[
P(x) = \frac{(1 + a_1x^2) \cdots (1 + a_mx^2) - 1}{x^2}
\]

and real numbers \(\phi_1, \ldots, \phi_m\) and \(A\) by

\[
\phi_k = a_k \int_{-\infty}^{\infty} \frac{dx}{(1 + a_kx^2)\sqrt{P(x)}} \quad \text{and} \quad A = \omega_m (a_1 \cdots a_m)^{-\frac{1}{2}},
\]

where \(\omega_m\) is the volume of the unit sphere in \(\mathbb{R}^m\). Note that

\[
\phi_k \in (0, \pi) \quad \text{with} \quad \phi_1 + \cdots + \phi_m = \pi, \quad A > 0,
\]

and that the correspondence \((a_1, \ldots, a_m) \mapsto (\phi_1, \ldots, \phi_m, A)\), with the conditions above, is one-to-one. Define functions \(z_k : \mathbb{R} \to \mathbb{C}, k = 1, \ldots, m,\) by

\[
z_k(y) = e^{i\psi_k(y)} \sqrt{a_k^{-1} + y^2}, \quad \text{where} \quad \psi_k(y) = a_k \int_{-\infty}^{y} \frac{dx}{(1 + a_kx^2)\sqrt{P(x)}}.
\]

Write \(\phi = (\phi_1, \ldots, \phi_m)\) and define a submanifold \(L^{\phi,A}\) in \(\mathbb{C}^m\) by

\[
L^{\phi,A} = \{(z_1(y)x_1, \ldots, z_m(y)x_m) : y \in \mathbb{R}, x_k \in \mathbb{R}, x_1^2 + \cdots + x_m^2 = 1\}.
\]

Then \(L^{\phi,A}\) is a closed embedded special Lagrangian submanifold in \(\mathbb{C}^m\) that is diffeomorphic to \(S^{m-1} \times \mathbb{R}\). It intersects each \(\Pi^\psi(y), y \in \mathbb{R}\), along an ellipsoid, where \(\psi(y) = (\psi_1(y), \ldots, \psi_m(y))\).

In terms of [Joy3: I. Definition 7.1], \(L^{\phi,A}\) is asymptotically conical, with rate \(2 - m\) and cone the type-1 transverse pair \(\Pi^0 \cup \Pi^\phi\) of special Lagrangian planes. Furthermore, for \(t > 0\), let \(t : \mathbb{C}^m \to \mathbb{C}^m\) be the associated dilation map (i.e. multiplication by \(t\)). Then \(t \cdot L^{\phi,A} = L^{\phi,tm,A}\).

**Desingularizations of immersed special Lagrangian submanifolds with transverse intersections.**

**Theorem A.1.4. [desingularization in a Calabi-Yau].** ([Joy3: V. Sec. 9.2, Theorem 9.7].)

Let

- \((M, J, \omega, \Omega)\) be a Calabi-Yau \(m\)-fold with \(m > 2\),
- \(t : X \to M\) be a compact, immersed special Lagrangian \(m\)-manifold in \(M\),
- \(x_1, \ldots, x_n \in M\) be transverse self-intersection points of \(X\) with type 1,
- \(x_i^\pm \in X\) be the pair of points in \(t^{-1}(x_i)\) such that the type of the pair \((t_\ast T_{x_1^+}X, t_\ast T_{x_1^-}X)\) at \(0 \in T_{x_i}M\) is 1, and
- \(X_1, \ldots, X_q\), where \(q = b^0(X)\), be the connected components of \(X\).
Suppose $A_1, \ldots, A_n > 0$ satisfy
\[ \sum_{i \in \{1, \ldots, n\}, x_i^+ \in X_k} A_i - \sum_{i \in \{1, \ldots, n\}, x_i^- \in X_k} A_i = 0 \quad \text{for all } k = 1, \ldots, q. \]

Let $N$ be the (oriented multiple self-)connected sum of $X$ at the pairs of points $\{(x_i^+, x_i^-)\}_{i=1}^n$. Suppose that $N$ is connected. Then

- there exist an $\epsilon > 0$ and a smooth family $\{\iota^t : \tilde{N} \to M \, | \, t \in (0, \epsilon)\}$ of compact, immersed special Lagrangian $m$-manifolds in $(M, J, \omega, \Omega)$, with $\tilde{N}$ diffeomorphic to $N$, such that $\iota^t$ is constructed by gluing a Lawlor neck $t \cdot L_{x^+ \iota^t, A_i}^{\pm} (= L_{x^+ \iota^t, A_i}^{\pm, \iota^t})$ into $\iota$ at $x_i$ for $i = 1, \ldots, n$.

In the sense of currents, $\iota^t \to \iota$ as $t \to 0$. If $x_1, \ldots, x_n$ are the only self-intersection points of $\iota$, then $\iota^t$ is an embedding.

In general, it can happen that there is no smoothing for a compact immersed special Lagrangian submanifold at its transverse self-intersection point for a fixed Calabi-Yau manifold. In this case, Joyce proves another result:

**Theorem A.1.5. [desingularization in a family of Calabi-Yau’s].** ([Joy3: V. Sec. 9.3, Theorem 9.8.]) Let

- $(M, J, \omega, \Omega)$ be a Calabi-Yau $m$-fold with $m > 2$,
- $\iota : X \to M$ be a compact, immersed special Lagrangian $m$-manifold in $M$,
- $x_1, \ldots, x_n \in M$ be transverse self-intersection points of $X$ with type 1,
- $x_i^\pm \in X$ be the pair of points in $\iota^{-1}(x_i)$ such that the type of the pair $(\iota_* T_{x_i^+} X, \iota_* T_{x_i^-} X)$ at $0 \in T_{x_i} M$ is 1, and
- $X_1, \ldots, X_q$, where $q = b^0(X)$, be the connected components of $X$.

Let $N$ be the (oriented multiple self-)connected sum of $X$ at the pairs of points $\{(x_i^+, x_i^-)\}_{i=1}^n$. Suppose that $N$ is connected.

Suppose $\{(M, J^s, \omega^s, \Omega^s) : s \in F\}$ is a smooth family of deformations of $(M, J, \omega, \Omega)$ with $\iota^s(\omega^s) = 0$ in $H^2(X; \mathbb{R})$ for all $s \in F$. Let $A_1, \ldots, A_n > 0$. Define $G \subset F \times (0, 1)$ to be the subset of $(s, t) \in F \times (0, 1)$ with

\[ [\text{Im} \Omega^s] \cdot [X_k] = t^m \cdot \left( \sum_{i \in \{1, \ldots, n\}, x_i^+ \in X_k} A_i - \sum_{i \in \{1, \ldots, n\}, x_i^- \in X_k} A_i \right) \quad \text{for all } k = 1, \ldots, q. \]

Then there exist $\epsilon \in (0, 1)$, $\kappa > 1$ and a smooth family

\[ \{ \iota^{s, t} : \tilde{N}^{s, t} \to M \, | \, (s, t) \in G \}, \quad t \in (0, \epsilon], \quad |s| \leq t^{s + m/2} \}
\]

such that

- $\iota^{s, t} : \tilde{N}^{s, t} \to (M, J^s, \omega^s, \Omega^s)$ is a compact, nonsingular special Lagrangian $m$-manifold in $(M, J^s, \omega^s, \Omega^s)$, with $\tilde{N}^{s, t}$ diffeomorphic to $N$, constructed by gluing a Lawlor neck $t \cdot L_{x^+ \iota^{s, t}, A_i}^{\pm, \iota^{s, t}} (= L_{x^+ \iota^{s, t}, A_i}^{\pm, \iota^{s, t}})$ into $\iota$ at $x_i$ for $i = 1, \ldots, n$.

In the sense of currents, $\iota^{s, t} \to \iota$ as $s, t \to 0$. If $x_1, \ldots, x_n$ are the only self-intersection points of $\iota$, then $\iota^{s, t}$ is embedded.
The following observation is needed to apply Joyce’ theorem to our situation:

**Observation A.1.6. [restriction $|s| \leq t^{k+m/2}$].** (Cf. [Joy3: IV. Remark after Theorem 7.9].) In Theorem A.1.5, the restriction $|s| \leq t^{k+m/2}$ can be replaced by any restriction of the form:

$$|s| \leq C t^{k+m/2}$$

for some constant $C > 0$

with the glued Lawlor neck $t \cdot L^{A_i} = L^{t^{m/A_i}}$ replaced by $C^{-1/(k+m/2)} t \cdot L^{t^{A_i}}$ in the construction. Furthermore, $k > 1$ can be chosen to be arbitrarily close to 1.

**Proof.** The first statement follows from a reparameterization of the parameter space with coordinates $(s, t)$ in the procedure of constructing a family of special Lagrangian submanifolds in Calabi-Yau manifolds, parameterized by $(s, t)$. For the second statement, recall (cf. [Joy3: IV, proof of Theorem 7.9]) that the constraint $|s| \leq t^{k+m/2}$ with $k > 1$ becomes part of a sufficient condition for bounds\(^\text{20}\)

$$
\begin{align*}
\|\varepsilon^{s,t}\|_{L^{2m/(m+2)}} &\leq A_2 t^{k+m/2}, \\
\|\varepsilon^{s,t}\|_{C^0} &\leq A_2 t^{k-1}, \\
\|d\varepsilon^{s,t}\|_{L^{2m}} &\leq A_2 t^{k-3/2}, \\
\|\pi_{W^{s,t}}(\varepsilon^{s,t})\|_{L^1} &\leq A_2 t^{k+m-1}
\end{align*}
$$

to hold if $k > 1$ is a solution to the following system of inequalities\(^\text{21}\) ([Joy3: IV: Sec. 7.3, Eq’ns (99), (100), (101), (102))):

$$
\begin{align*}
-\tau(1 + m/2) + \tau(\mu_i - 2) &\geq \kappa + m/2, \\
\tau(1 + m/2) + (1 - \tau)(2 - \lambda_i) &\geq \kappa + m/2, \\
\tau(\mu_i - 2) &\geq \kappa - 1, \\
(1 - \tau)(2 - \lambda_i) &\geq \kappa - 1, \\
-\tau/2 + \tau(\mu_i - 2) &\geq \kappa - 3/2, \\
-\tau/2 + (1 - \tau)(2 - \lambda_i) &\geq \kappa - 3/2, \\
(m + 1)\tau &\geq \kappa + m - 1, \\
i = 1, \ldots, n,
\end{align*}
$$

where $\mu_i \in (2, 3)$ can be chosen to be arbitrarily close to 2 (cf. [Joy3: I, Sec. 5.3, Theorem 5.5]), $\lambda_i = 2 - m$ in the current situation, and $0 < \max_{i=1,\ldots,n}\{\frac{m}{m+1}, \frac{m+2}{2\mu_i+m-2}\} < \tau < 1$.

Since $\mu_i$ can be chosen to be arbitrarily close to 2, $\tau$ can be chosen to be arbitrarily close to 1 as well. Let $\tau = 1 - \delta_0$, $\mu_i = 2 + \delta_i$, and note that $\lambda_i < \frac{1}{2}(2 - m)$. Then, the above system of inequalities has nonempty solution for $k > 1$ if and only if $\delta_0, \delta_i, i = 1, \ldots, n$, satisfy inequalities

$$
\delta_i > (1 + \frac{m}{2}) \frac{\delta_0}{1 - \delta_0}, \quad \text{and} \quad 0 < \delta_0 < \frac{1}{m + 1}.
$$

Note that the latter system does have nonempty positive solutions for $\delta_0, \delta_1, \ldots, \delta_n$ and that these solutions can be chosen to be arbitrarily close to 0. For any such solution $(\delta_0, \delta_1, \ldots, \delta_n)$, any $\kappa$ that satisfies

$$
1 < \kappa \leq 1 + \min_{i=1,\ldots,n} \left\{ (1 - \delta_0)\delta_i - \delta_0(1 + \frac{m}{2}), \delta_0(\frac{m}{2} - 1), 1 - \delta_0(m + 1) \right\}
$$

\(^{20}\)Here, $\varepsilon^{s,t}$ is a smooth function on a Lagrangian submanifold $N^{s,t}$ constructed via gluing Lawlor necks to $X$ in ways parameterized by $(s, t)$. It measures how close $N^{s,t}$ is to being special Lagrangian. $W^{s,t} \simeq \mathbb{R}^d$ is a finite dimensional subspace of $C^\infty(N^{s,t})$ that approximates the $\mathbb{R}$-span of eigenfunctions with small eigenvalues of a second-order elliptic operator arising from the special Lagrangian condition, and $\pi_{W^{s,t}} : L^2(N^{s,t}) \to W^{s,t}$ is the projection onto using the $L^2$-inner product. As we won’t need the precise expression of these for this note and to give a complete definition requires three pages of related definitions, we refer readers directly to [Joy3: IV, Sec. 7.1, Definitions 7.1 and 7.2] (resp. [Joy3: IV, Sec. 7.1, Definition 7.3]; [Joy3: III, Sec. 5.2, Theorem 5.3]; [Joy3: I, Sec. 2.2]) for details of $\varepsilon^{s,t}$ (resp. $W^{s,t}$ and $\pi_{W^{s,t}}$; constant $A_2$; the various Banach/Sobolev spaces $C^k$, $L^p$, and norms $\| \cdot \|_{C^k}, \| \cdot \|_{L^p}$).

\(^{21}\)Here, specified to the current situation we need, $\mu_i$ (resp. $\lambda_i$) is the rate of the conical special Lagrangian submanifold $X$ at $x_i$ (resp. Lawlor neck $L^{A_i}$) and $\tau$ is a parameter that governs in part how the Lawlor necks $L^{A_i}$ are glued into $X' := X - \{x_1, \ldots, x_n\}$ in the construction of $N^{s,t}$. See [Joy3: I, Sec. 3.3, Definition 3.6] (resp. [Joy3: I, Sec. 7, Definition 7.1]; [Joy3: IV, Sec. 7.1, Definition 7.2]) for details.
lies in the solution set of the above system of inequality for $\kappa$. In particular, $\kappa > 1$ can be chosen to be arbitrarily close to 1.

\[\square\]

**The immersed version of Joyce’ results.**

([Joy3: I. Sec. 1 and II. Sec. 8.1].) Nearly all the results of Joyce [Joy3] generalize immediately to immersed special Lagrangian submanifolds with conical singularities, with only superficial change, due to the following version of Lagrangian Neighborhood Theorem:

**Theorem A.1.7. [immersed Lagrangian].** Let $(M, \omega)$ be a symplectic manifold and $f : L \to M$ be a compact immersed Lagrangian submanifold. Then there exist a neighborhood $U \subset T^*L$ of the zero-section and an immersion $\phi : U \to M$ such that $\phi|_L = f$ and $\phi^*\omega = \omega_{can}$, where $\omega_{can}$ is the canonical symplectic form on $T^*L$. 

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