Two-component plasma in a gravitational field

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In this paper we study a model for the sedimentation equilibrium of a charged colloidal suspension: the two-dimensional two-component plasma in a gravitational field which is exactly solvable at a special value of the reduced inverse temperature $\Gamma = 2$. The density profiles are computed. The heavy particles accumulate at the bottom of the container. If the container is high enough, an excess of light counterions form a cloud floating at some altitude.
I. INTRODUCTION

When colloidal particles are solvated in a polar fluid they usually release counterions and therefore they acquire an electric charge. Then, a solution of heavy charges (the colloidal particles) and light charges (the counterions) is obtained. A model of such charged colloidal suspensions in sedimentation equilibrium based on the local density approximation has been proposed and numerical computations about that model show that the heavy charges accumulate at the bottom of the container while the light particles accumulate at the top. The suspension behaves like a condenser: a vertical electric field is induced into the suspension. Also, several experimental measurements of the densities profiles in such suspensions have been done.

In this paper we consider a simple model of sedimentation equilibrium, taking into account only the Coulomb interactions between the particles and the gravitational force (in particular we neglect excluded volume interactions), which is exactly solvable. The model is a two-dimensional two-component plasma in a gravitational field, that is a system composed of two species of particles with electric charges $\pm q$ and masses $M_{\pm}$. The Coulomb interaction is logarithmic in two-dimensions, and the particles are submitted to a uniform vertical gravitational field.

The two-component plasma model has been solved by Gaudin when the inverse temperature $\beta$ verifies $\Gamma := \beta q^2 = 2$, and much work has been done with that model. Studing the electrical double layer for the two-component plasma, Cornu and Jancovici introduced a very general method for treating the two-component plasma in presence of an external potential. We use here their method to deal with the gravitational field.

In the following section we present the model and recall some results of the method as given in Ref. 3. In section III, we solve the model and compute quantities such as the density profiles and the electric field in the suspension.

II. THE MODEL AND THE GENERAL METHOD OF RESOLUTION

The model is a two-dimensional system of particles of charges $\pm q$ and masses $M_{\pm}$. The position $r$ of a particle is represented by its Cartesian coordinates $(x, y)$. The gravitational field is $g = -gy$. The particles are in a container of height $h$ and infinite
The bottom of the container is at \( y = 0 \). The half-spaces \( y < 0 \) and \( y > h \) are impenetrable to the particles. The interaction of the particles with the gravitational field is

\[
M_+ g \sum_i y_i^+ + M_- g \sum_j y_j^-
\]  
(2.1)

where \( y_i^\pm \) is the altitude of the \( i \)th particle of charge \( \pm q \). Expression (2.1) can be rewritten as

\[
M_{\text{eff}} g \left( \sum_i y_i^+ + \sum_j y_j^- \right) - q E_{\text{eff}} \left( \sum_i y_i^+ - \sum_j y_j^- \right)
\]  
(2.2)

where \( M_{\text{eff}} = \left( M_+ + M_- \right) / 2 \) and \( q E_{\text{eff}} = -(M_+ - M_-) g / 2 \). This shows that the system is equivalent to a system where all the particles have the same mass \( M_{\text{eff}} \) but there is an external electric field \( E_{\text{eff}} \) acting on the particles. For this reason it is useful to define \( k_\pm = \beta M_\pm g \), \( k_0 = (k_+ + k_-) / 2 \) and \( \delta k = k_+ - k_- \).

For \( \Gamma \geq 2 \) a point-particle model is unstable against the collapse of pairs of particles of opposite charge. To prevent this collapse we introduce some short range cutoff by representing the particles as hard discs of diameter \( R \) and obtain results in the small-\( R \) limit. In fact, at \( \Gamma = 2 \), only the one-body density diverges, while the \( n \)-body correlations \( (n \geq 2) \) have well-defined limits as \( R \to 0 \). From now on we shall consider that the temperature is such that \( \Gamma = 2 \). The model is solved in the grand canonical ensemble. It is convenient to introduce position-dependent rescaled fugacities that have inverse length dimensions: \( m_\pm (r) = m_0 \exp(-k_\pm y) \). There are two length scales in this problem: the gravitational length given by \( k_0^{-1} \) and the screening length which in fact is given by the inverse of the rescaled fugacity \( m_0^{-1} \). Here we concentrate on the usual case close to the continuous medium limit where \( m_0 \gg k_0 \).

Using the equivalence between the two-dimensional Coulomb gas at \( \Gamma = 2 \) and the free Dirac field, one can write the grand partition function in the form

\[
\Xi = \det \left[ \left( \phi + m_+ (r) \frac{1 + \sigma_z}{2} + m_- (r) \frac{1 - \sigma_z}{2} \right) \theta^{-1} \right],
\]  
(2.3)

where \( \phi = \sigma_x \partial_x + \sigma_y \partial_y \) is the Dirac operator in two dimensions and \( \sigma_x, \sigma_y \) and \( \sigma_z \) are the Pauli matrices. The particle densities \( \rho_s \) (where \( s = \pm 1 \)) and the truncated two-body densities \( \rho_{ss'}^{(2)T} \) can be obtained from the Green functions \( G_{ss'} \) defined by

\[
\left[ \phi + m_+ (r) \frac{1 + \sigma_z}{2} + m_- (r) \frac{1 - \sigma_z}{2} \right] G = \mathbf{1} \delta (r - r').
\]  
(2.4)
One finds
\[ \rho_s(r) = m_s(r)G_{ss}(r, r), \quad (2.5) \]
and
\[ \rho_{ss}^{(2)T}(r, r') = -m_s(r)m_s(r')G_{ss'}(r, r')G_{s's}(r', r). \quad (2.6) \]

It is convenient to write the fugacities as \( m_{\pm}(r) = m(r)\exp[-(\pm)2V(r)] \) where \( m(r) = m_0\exp[-k_0y] \) and \( V(r) = \partial y/4 \), and to define
\[ g_{ss'}(r, r') = \exp[-sV(r)]G_{ss'}(r, r') \exp[-s'V(r')]. \quad (2.7) \]

Then, in terms of the operators \( a = \partial_x + i\partial_y + \partial_xV + i\partial_yV \) and \( a^\dagger = -\partial_x + i\partial_y + \partial_xV - i\partial_yV \), the functions \( g_{ss'} \) are given by the partial differential equations
\[ \left[ m(r) + a^\dagger (m(r))^{-1}a \right] g_{++}(r, r') = \delta(r - r'), \quad (2.8) \]
\[ \left[ m(r) + a (m(r))^{-1}a^\dagger \right] g_{--}(r, r') = \delta(r - r'), \quad (2.9) \]
and
\[ g_{-+}(r, r') = -[m(r)]^{-1}ag_{++}(r, r'), \quad (2.10) \]
\[ g_{+-}(r, r') = [m(r)]^{-1}a^\dagger g_{--}(r, r'). \quad (2.11) \]

III. CALCULATIONS AND RESULTS

A. The Green functions

Because of the translational invariance along the \( x \) axis we can suppose that \( x' = 0 \). It is useful to work with the Fourier transform \( \hat{g}_{ss'} \) of Green functions \( g_{ss'} \) defined by
\[ g_{ss'}(r, r') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{g}_{ss'}(y, y', k) e^{ikx} dk. \quad (3.1) \]

From equation (2.8) we find that the Fourier transform \( \hat{g}_{++} \) satisfies the ordinary differential equation
\[ \left( (k + \delta k/4)k_0 - (k + \delta k/4)^2 - m_0^2 e^{-2k_0y} \right) \hat{g}_{++}(y, y', k) \]
\[ + k_0\partial_y\hat{g}_{++}(y, y', k) + \partial^2_y\hat{g}_{++}(y, y', k) = -m_0 e^{-k_0y}\delta(y - y'). \quad (3.2) \]
In terms of \( u = \exp[-k_0y], u' = \exp[-k_0y'] \) and \( f_{ss'}(u, u', k) = \hat{g}_{ss'}(y, y', k) \), we have for \( f_{++} \) the equation
\[ \partial^2_u f_{++}(u, u', k) - \left( \nu_+^2 - 1/4 \right) u^{-2} + \left( \frac{m_0}{k_0} \right)^2 \right] f_{++}(u, u', k) = -\frac{m_0}{k_0} \delta(u - u'), \quad (3.3) \]
where $\nu_+^2 = [k - (k_-/2)]^2/k_0^2$. A similar equation is obtained for $f_-$ by replacing $\nu_+^2$ by $\nu_-^2 = [-k - (k_+/2)]^2/k_0^2$. Equation (3.3) can be solved in terms of the modified Bessel functions $I_{\nu_+}$ and $K_{\nu_+}$:

$$f_{++}(u, u', k) = \frac{m_0}{k_0} \sqrt{uu'} \left[ I_{\nu_+}(\frac{m_0}{k_0} u) K_{\nu_+}(\frac{m_0}{k_0} u) + A I_{\nu_+}(\frac{m_0}{k_0} u) + B K_{\nu_+}(\frac{m_0}{k_0} u) \right],$$  

(3.4)

where $u_\leq = \min(u, u')$, $u_\geq = \max(u, u')$, and $A$ and $B$ are constants (with respect to $u$) that depend on the boundary conditions. Note that the sign of $\nu_+$ can be arbitrarily chosen; we choose $\nu_+$ positive. Using equation (2.110) we find

$$f_{--}(u, u', k) = -i \frac{m_0}{k_0} \sqrt{uu'} \left[ A I_{\nu_+ + \sigma}(\frac{m_0}{k_0} u) - B K_{\nu_+ + \sigma}(\frac{m_0}{k_0} u) \right] + \begin{cases} I_{\nu_+ + \sigma}(\frac{m_0}{k_0} u) K_{\nu_+}(\frac{m_0}{k_0} u'), & \text{if } u < u' \\
-I_{\nu_+}(\frac{m_0}{k_0} u') K_{\nu_+ + \sigma}(\frac{m_0}{k_0} u), & \text{if } u' < u \end{cases},$$  

(3.5)

where $\sigma$ is the sign of $k - (k_-/2)$.

For a source point in the allowed region $0 < y' < h$, in the impenetrable region $y < 0$ the particle fugacities vanish and the equations for the Fourier transforms of $G_{++}$ and $G_{--}$ reduce to

$$(k + \partial_y) \hat{G}_{++}(y, y', k) = 0, \quad (k - \partial_y) \hat{G}_{--}(y, y', k) = 0,$$  

(3.6)

with the solution

$$\hat{G}_{++}(y, y', k) = C e^{-k y}, \quad \hat{G}_{--}(y, y', k) = D e^{k y}.$$  

(3.7)

Similar equations hold for $y > h$.

The boundary conditions are that $\hat{g}_{ss'}$ must vanish when $y \to \pm \infty$ and be continuous at $y = 0$ and $y = h$. The boundary conditions when $y \to -\infty$ and equations (3.4) lead to $f_{++}(1, u', k) = 0$ if $k > 0$ and $f_{--}(1, u', k) = 0$ if $k < 0$, while the boundary conditions when $y \to +\infty$ give at $y = h$: $f_{++}(u_h, u', k) = 0$ if $k < 0$ and $f_{--}(u_h, u', k) = 0$ if $k > 0$, where we have defined $u_h = \exp(-k_0 h)$. Using these boundary conditions in (3.4) and (3.3) we find the values of $A$ and $B$ which depend on the sign of $k$ and the sign of $k - k_-/2$. We shall use the following notation: $A_{\sigma'}(u', \nu_+)$ and $B_{\sigma'}(u', \nu_+)$ are the values of $A$ and $B$ when $k$ has sign $\sigma'$ and $k - k_-/2$ has sign $\sigma$. Using this notation we have

$$A_{\nu_+}(u', \nu) = -K_{\nu}(\frac{m_0}{k_0} u') I_{\nu}(\frac{m_0}{k_0} u_h) K_{\nu + \sigma}(\frac{m_0}{k_0} u_h) + K_{\nu}(\frac{m_0}{k_0} u') I_{\nu + \sigma}(\frac{m_0}{k_0} u_h) \left[ K_{\nu}(\frac{m_0}{k_0} u_h) I_{\nu}(\frac{m_0}{k_0} u) + K_{\nu + \sigma}(\frac{m_0}{k_0} u_h) I_{\nu + \sigma}(\frac{m_0}{k_0} u) \right],$$  

(3.8a)
An interesting limit is when the height $r = r(\sigma) = m_0 \frac{u_h}{k_0} + K(\sigma) \frac{u_h}{k_0}$ + $\nu(\sigma) \frac{m_0}{k_0}$, $\nu(\sigma) \frac{m_0}{k_0}$ (3.8b).

$A_\nu(u', \nu) = K(\sigma) \frac{m_0}{k_0} I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) - I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) + I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) + I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) (3.8c).

$B_\nu(u', \nu) = -I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) + I(\nu) \frac{m_0}{k_0} u_h(\nu(\sigma)) m_0(k_0) + I(\nu) \frac{m_0}{k_0} u_h(\nu(\sigma)) m_0(k_0) (3.8d).

Similar calculations can be done for $f_{-\nu}$. Then changing the integral over $k$ in (3.4)

into an integral over $\nu$ we find for the Green functions

$$g_{ss}(r, r') = \frac{m_0}{2\pi} \exp[i k_{-} x / 2] \sqrt{u x}$$

$$\times \left[ 2 \int_{0}^{+\infty} I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) \cos(k_0 x \nu) \, d\nu \right. + \int_{0}^{+\infty} \left[ A_{+}(u', \nu) I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) + B_{+}(u', \nu) K(\sigma) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) \right] e^{i k_0 x \nu} \, d\nu (3.9)$$

$$+ \int_{k_{-}/2k_0}^{+\infty} A_{-}(u', \nu) I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) + B_{-}(u', \nu) K(\sigma) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) \right] - e^{-i k_0 x \nu} \, d\nu .$$

while $g_{-+}$ and $g_{+-}$ are given by (2.10) and (2.11).

An interesting limit is when the height $h$ of the container goes to infinity. One can take the limits of $A_{\sigma}(u', \nu)$ and $B_{\sigma}(u', \nu)$ when $u_h \to 0$ in (3.9). One finds

$$g_{ss}(r, r') = \frac{m_0}{2\pi} \exp[i k_{-} x / 2] \sqrt{u x}$$

$$\times \left[ 2 \int_{0}^{+\infty} I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) \cos(k_0 x \nu) \, d\nu \right. - \int_{0}^{+\infty} I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) \frac{k_{-}(\nu)}{k_{+}(\nu)} e^{i k_0 x \nu} \, d\nu$$

$$+ \int_{k_{-}/2k_0}^{+\infty} I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) K_{\nu-1}(\nu \frac{m_0}{k_0}) e^{-i k_0 x \nu} \, d\nu (3.10)$$

$$+ \int_{0}^{k_{-}/2k_0} \left[ -I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) K_{\nu}(\nu \frac{m_0}{k_0}) \right.$$ 

$$+ \left\{ -I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) + K_{\nu}(\nu \frac{m_0}{k_0} u_h) \right\} K_{\nu}(\nu \frac{m_0}{k_0})$$

$$+ \frac{2 \sin \pi \nu}{\pi} \int_{0}^{+\infty} I(\nu) \frac{m_0}{k_0} u_h(I(\nu) \frac{m_0}{k_0} u_h) \frac{2 \sin \pi \nu}{\pi} e^{-i k_0 x \nu} \, d\nu \right] .$$
B. The charge density

The densities $\rho_s$ are given by equation (2.5). The first integral in (3.9) and in (3.10) diverges for $u' = u$ and must be cut-off as mentioned in section II; an equivalent method is to introduce a cutoff $\nu_{\text{max}}$ in the integral. The other terms are finite when $u < 1$ ($y \neq 0$). The charge density is $\rho = q(\rho_+ - \rho_-)$ and has a finite limit when the cut-off $R$ goes to zero. Figures 1, 2 and 3 show the charge density profile when the heavy particles have charge $+q$, and for $m_0/k_0 = 10$, $k_+ = 1.5k_0$, $k_- = 0.5k_0$ and different heights $h$ of the container ($k_0h = 0.5, 3, \text{ and } 10$ respectively).

When $h$ is of the same order or less than the gravitational length $k_0^{-1}$ we notice a strong accumulation of heavy particles at the bottom of the container and of light particles at the top, while in the middle region the charge density is almost zero. In that zone there is an almost constant electric field. The suspension behaves like a condenser. This was noticed before in other models. Note that we have considered the usual physical case where $m_0 \gg k_0$. In the hypothetical case where $k_0 \gg m_0$ the neutral zone would not exist. We shall assume from now on that $m_0 \gg k_0$.

When $h \gg k_0^{-1}$, there still is a strong accumulation of heavy particles at the bottom of the container but the light particles no longer accumulate on the top of the container. There is an excess of light particles at some altitude that will be shown later to be of order $k_0^{-1}\ln(m_0/k_0)$.

Let us consider the case $h = +\infty$. The charge density can be written as

$$\rho(r) = \frac{q m_0 u^2}{2\pi} \int_{k_-/2k_0}^{k_+/2k_0} \left[ I_\nu\left(\frac{m_0}{k_0} u\right)^2 I_{\nu-1}\left(\frac{m_0}{k_0}\right) \right] d\nu.$$  

First we shall study the charge density near the bottom of the container and at intermediate altitudes. Remember that we are in the case $m_0 \gg k_0$. If $y \ll k_0^{-1}\ln(m_0/k_0)$ and $u > 1/2$ then using the asymptotic expansion of the Bessel functions we have

$$\rho(r) \sim -q \frac{m_0 k_0}{4\pi} \exp \left[ -k_0y - 2\frac{m_0}{k_0}(1 - e^{-k_0y}) \right].$$  

(3.12)
From this expression we notice that for \( y \ll k_0^{-1} \ln(m_0/k_0) \), the function \( q\delta k \rho(y) \) is a decreasing function of \( y \). Furthermore from expression (3.12) we see that the layer of heavy particles has a thickness of order \( m_0^{-1} \), the screening length. Now at high altitudes for \( y \gg k_0^{-1} \ln(m_0/k_0) \), the function \( \rho(y) \) is now an increasing function of \( y \). This shows that at an altitude of order \( k_0^{-1} \ln(m_0/k_0) \) we a local minimum of the function \( q\delta k \rho(y) \).

The electric field in the region where the system is almost neutral \( (m_0^{-1} \ll y \ll k_0^{-1} \ln(m_0/k_0)) \) can be estimated as follows. Remember that the system is equivalent to a system of particles with the same mass in an external electric field \( E_{\text{eff}} = -(M_+ - M_-)g/2q \). The screening properties of this equivalent system will ensure that its total electric field is zero, so the electric field in our system, created by the accumulation of particles at the bottom and top of the container (or at an altitude of order \( k_0^{-1} \ln(m_0/k_0) \) if \( h \gg k_0^{-1} \ln(m_0/k_0) \)), will be equal to \( -E_{\text{eff}} = (M_+ - M_-)g/2q \).

### C. The density profiles

In the almost neutral region we can estimate the individual densities of the particles through the following macroscopic argument. At altitude \( y \) the Coulomb gas pressure \( p \) compensates the gravitational force:

\[
\frac{dp}{dy} = -g(\rho_+ M_+ + \rho_- M_-).
\]  

(3.15)

This equation, the fact that \( \rho_+ = \rho_- \), and the equation of state for the two-dimensional two-component plasma for \( \Gamma \leq 2 \)

\[
\beta p = (\rho_+ + \rho_-) \left( 1 - \frac{\Gamma}{4} \right),
\]

(3.16)
yield the barometric law

\[ \rho_+(y) = \rho_-(y) = \rho_0 \exp \left[ -k_0 \left( 1 - \frac{\Gamma}{4} \right)^{-1} y \right]. \]  

(3.17)

This can be verified in the exact expressions of the densities given by the Green functions. The dominant term in the Green function \( g_{++} \) in that region is

\[
g_{++}(r, r') \sim \frac{m_0}{\pi} \exp[ik_x/2] \nu \int_0^{+\infty} \nu \left( \frac{m_0}{k_0} u_\nu K_\nu \left( \frac{m_0}{k_0} u_\nu \right) \cos \nu k_0 x \right) \, d\nu, \]  

(3.18)

and \( g_{--} \) is the same as \( g_{++} \) except for the phase factor \( \exp[ik_x/2] \) that is changed into \( \exp[-ik_x/2] \), which is in fact irrelevant for the physical quantities. The integral over \( \nu \) can be approximated by a sum over integers by using the Euler–MacLaurin expansion when \( m_0 \gg k_0 \). Then the sum can be performed by using the addition theorems for the Bessel functions. This gives, for the first two terms of the expansion,

\[
g_{++}^b(r, r') \sim \frac{m_0}{\pi} \exp[ik_x/2] e^{-k_0(y+y')/2} \times \left[ K_0 \left( \frac{m_0}{k_0} |e^{k_0(x-y)} - e^{-k_0y}| \right) \right. \]  

\[ + \left. \frac{1}{6} K_0 \left( \frac{m_0}{k_0} \exp[-k_0y] \right) K_0 \left( \frac{m_0}{k_0} \exp[-k_0y'] \right) + \cdots \right] . \]  

(3.19)

As stated in section II, when the diameter \( R \) of the particles goes to 0, the density diverges, so we must cut-off expression (3.19) when \( |r - r'| \to 0 \) replacing \( |r - r'| \) by \( R \) for \( |r - r'| \leq R \). Then the regularized form of the particle density is

\[ \rho_+(r) = \rho_-(r) \sim \frac{m(r)^2}{2\pi} K_0(m(r)R) . \]  

(3.20)

If \( m_0 R \ll \exp(-k_0y) \) then

\[ \rho_+(r) = \rho_-(r) \sim \rho_0 e^{-2k_0y} , \]  

(3.21)

where \( \rho_0 = m_0^2 K_0(m_0 R)/(2\pi) \) is the density when \( g = 0 \). Notice that the expression (3.21) of \( \rho_\pm \) is the same as the one for \( \rho_0 \) except that the fugacity \( m(r) \) depends on the altitude \( y \). We have found a barometric law (3.21) with mass \( M_+ + M_- \) which agrees with (3.17) when \( \Gamma = 2 \).

Far away from the neutral zone, for \( y \gg k_0^{-1} \ln(m_0/k_0) \) the dominant term of \( \rho_s \) in the case \( m_0 \gg k_0 \) is

\[ \rho_s(y) = \frac{m_0^2 u^2}{\pi^2} \int_0^{k_s/2k_0} \sin(\pi \nu) K_\nu \left( \frac{m_0}{k_0} u \right)^2 \, d\nu . \]  

(3.22)
The asymptotic behavior of this integral for $y \to \infty$ is computed in the Appendix. One finds if $k_s \neq 0$
\[
\rho_s(y) \sim \frac{2^{k_s/k_0}}{8\pi^2} \left[ \frac{m_0}{k_0} \right]^{k_s/k_0} k_0 \Gamma(k_{-s}/2k_0)^2 \sin(\pi k_{-s}/2k_0) \frac{e^{-k_s y}}{y}. \tag{3.23}
\]
The density $\rho_s$ decays like $y^{-1} \exp(-\beta M_s g y)$. The argument of the exponential is the same of the one of an ideal neutral gas, but the decay of the density is faster than the one of an ideal gas (because of the factor $y^{-1}$).

In real colloidal suspensions the mass of a counterion is much smaller than the mass of a colloidal particle. If we assume in our model that the negative counterions have mass $M_- = 0$ then the expression (3.10) for the positive colloidal particles is simplified: the last integral in (3.10) vanishes. The dominant term of the profile density of the colloidal particles given by (3.20) is now valid not only in neutral zone but also up to $y \to \infty$. For the counterions the profile density in the neutral zone is still dominated by (3.20) but for $y \gg k_0^{-1} \ln(m_0/k_0)$ expression (3.23) is no longer valid because $k_- = 0$. Instead we have (see the Appendix for details)
\[
\rho_-(y) \sim \frac{1}{4\pi y^2}. \tag{3.24}
\]
We recover an already known result: in this particular case where $k_- = 0$, in the considered region, because of the exponential decay of the density $\rho_+(y)$, one can consider that the system is composed only of particles with charge $-q$ without mass: the system is a one-component plasma without a background. The asymptotic behavior given by (3.24) was found for the solvable model of the one-component plasma without a background. The following macroscopic argument also leads to expression (3.24): the one-component plasma without a background can be seen as a neutralized one-component plasma (where we have added a background charge density $q \rho_-$) immersed in an external charged background $-q \rho_-$. At altitude $y$ the gradient of the thermal pressure of the neutralized plasma compensates the electric force exerted on the external background:
\[
\frac{dp(y)}{dy} = -q \rho_-(y) E(y), \tag{3.25}
\]
where $p(y)$ is the thermal pressure of the neutralized one-component plasma and $E(y)$ is the electric field created by the charge distribution $-q \rho_-(y)$. The Poisson equation
is
\[
\frac{dE(y)}{dy} = -2\pi q \rho_-(y).
\] (3.26)

These two equations together with the equation of state for the neutralized one-component plasma
\[
\beta p = \left(1 - \frac{\Gamma}{4}\right) \rho_-,
\] (3.27)
yield the differential equation for \(\rho_-\)
\[
\left(1 - \frac{\Gamma}{4}\right) \frac{d}{dy} \left[ \frac{1}{\rho_-(y)} \frac{d\rho_-(y)}{dy} \right] = 2\pi \Gamma \rho_-(y),
\] (3.28)
with the solution
\[
\rho_-(y) = \frac{1 - \Gamma/4}{\Gamma \pi (y - y_0)^2} \sim \frac{1 - \Gamma/4}{\Gamma \pi y^2}.
\] (3.29)

which agrees with (3.24) for \(\Gamma = 2\).

Let us now consider that the mass of the counterions is not zero but that \(k_- \ll k_+\).
In that case equation (3.23) becomes for the counterions
\[
\rho_-(y) \sim \frac{k_- e^{-k_+ y}}{4\pi y}.
\] (3.30)

It is interesting to notice that this expression is the same as the one which would be obtained for a one-component plasma without a background but with massive particles.
For a one-component plasma, it is equivalent to consider that the particles have a mass \(M_-\) or that they are in an external electric field \(E = -M_- g/q\) and do not have a mass.
The calculations done in Ref. 7 can be easily extended to the present case and they give the same asymptotic behavior (3.30) for the particle density.

**IV. CONCLUSION**

We have discussed the density profiles in this model of sedimentation equilibrium. Unfortunately our model cannot account for excluded volume interactions which are also important in the sedimentation equilibrium. Nevertheless, for heights of the container of the same order as the gravitational length, we find the same qualitative results as in other model: heavier particles accumulate at the bottom while lighter particles accumulate at the top. For heights of the container much larger than the gravitational length, we find an interesting phenomenon: lighter particles do not accumulate any
longer at the top of the container but there is an excess of them forming a cloud cen-
tered at an altitude of order of the gravitational length times some logarithm. An open
problem is whether such a cloud of light charged particles might be observed, because
we have not considered possible instabilities in the horizontal direction, with the excess
charge going to the lateral walls for instance.

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APPENDIX

We wish to find the asymptotic behavior of the integral
\[ \int_0^{k_s/2k_0} \sin(\pi \nu) K_\nu(m_0 u/k_0)^2 \, d\nu \]
when \( u \to 0 \) (\( y \to \infty \)). We define \( z = m_0 u/k_0 \), \( \alpha = k_s/2k_0 \), and
\[ I(z) = \int_0^\alpha \sin(\pi \nu) K_\nu(z)^2 \, d\nu. \]  
(A.1)

We use the formula
\[ K_\nu(z)^2 = 2 \int_0^\infty K_0(2z \cosh t) \cosh(2\nu t) \, dt, \]  
(A.2)
in equation (A.1) and perform the integral over \( \nu \) to find
\[ I(z) = 2 \int_0^\infty \frac{\pi - \pi \cos(\alpha \pi) \cosh(2\alpha t) + 2t \sin(\alpha \pi) \cosh(2\alpha t)}{\pi^2 + 4t^2} K_0(2z \cosh t) \, dt. \]  
(A.3)

If \( \alpha < 1 \) (ii.e. if \( k_{-s} \neq 0 \)) then doing the change of variable \( s = 2z \cosh t \) in (A.3) one
finds that
\[ \lim_{z \to 0} z^{2\alpha} \ln(1/z) I(z) = 2^{-1} \sin(\alpha \pi) \int_0^\infty s^{2\alpha - 1} K_0(s) \, ds. \]  
(A.4)
The integral over \( s \) can be done and finally we have
\[ I(z) \sim 4\alpha \Gamma(\alpha)^2 \sin(\alpha \pi) \frac{z^{-2\alpha}}{\ln(1/z)}. \]  
(A.5)

This expression is not valid when \( \alpha = 1 \), that is in the case when one of the masses of
the particles is zero. In that case equation (A.3) becomes
\[ I(z) = 4 \int_0^\infty \frac{\pi}{\pi^2 + 4t^2} K_0(2z \cosh t) \, dt. \]  
(A.6)
Then doing again the change of variable $s = 2z \cosh t$ one finds

$$I(z) \sim \frac{\pi}{4} \frac{1}{z^2 \ln(1/z)^2}. \quad (A.7)$$
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FIG 1: The charge density $\rho(y)$ for $m_0/k_0 = 10$, $k_+ = 1.5k_0$, $k_- = 0.5k_0$ and $h = 0.5k_0^{-1}$.
FIG 2: The charge density $\rho(y)$ for $m_0/k_0 = 10$, $k_+ = 1.5k_0$, $k_- = 0.5k_0$ and $h = 3k_0^{-1}$. 
FIG 3: The charge density $\rho(y)$ for $m_0/k_0 = 10$, $k_+ = 1.5k_0$, $k_- = 0.5k_0$ and $h = 10k_0^{-1}$. 