Fields interpretable in the free group

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Abstract

We prove that no infinite field is interpretable in the first-order theory of nonabelian free groups.

1 Introduction

Tarski in 1946 asked whether the first-order theory of nonabelian free groups is complete. In seminal works Sela [Sel09] and Kharlampovich-Myasnikov [KM06] answered the question positively. Moreover, Sela proved that this theory is stable [Sel13]. In one of the first papers on the subject, after the solution of Tarski’s problem, Pillay conjectured that no infinite field is interpretable in this theory and he coined it the name "the first-order theory of the free group" [Pil08]. In this paper we confirm the conjecture.

Theorem 1: The first-order theory of the free group does not interpret an infinite field.

Since the first-order theory of the free group does not have the finite cover property [Skl18], it is enough to prove that no infinite field is interpretable in a particular model. More precisely we prove:

Theorem 2: Let $F$ be a nonabelian free group and $X$ be a set interpretable in $F$. Then either $X$ is internal to a finite set of centralizers or it cannot be given definably the structure of an abelian group.

In [BS19], in the course of proving that no infinite field is definable in the first-order theory of the free group, we proved that centralizers (of nontrivial elements) are one-based, hence no set internal to a family of centralizers can define an infinite field. This latter argument completes the proof of the main theorem of this paper.

As one might expect a crucial tool for passing from the nondefinability to the noninterpretability of infinite fields is some form of elimination of imaginaries. The first-order theory of the free group (weakly) eliminates imaginaries after adding some sorts [Sel].

Interestingly the first-order theory of the free group is $n$-ample for all $n$ [OT12, Skl15], hence our result yields the first example of an ample theory that interprets an infinite group but not an infinite field. An example of an ample theory that does not interpret an infinite field had been constructed in [Eva03]. The latter theory does not interpret an infinite group, but still forking independence is nontrivial.

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2 Overview

In the first-order theory of the free group any first-order formula is equivalent to a boolean combination of $\forall \exists$-formulas. Despite the quantifier elimination it is hard to understand these basic definable sets. Still some special classes of formulas are easy to understand and even enlightening. Here is an argument that allows us to gain some intuition.

**Proposition 1** (The noncommutativity argument): Let $X := \phi(\bar{x})$ be a definable set over $\mathbb{F}_n$. Suppose $\phi(\mathbb{F}_n) \neq \phi(\mathbb{F}_{n+1})$. Then $X$ cannot be given definably the structure of an abelian group.

**Proof.** Suppose, for the sake of contradiction, that $(X, \diamond)$ (where $\circ := \psi(\bar{x}, \bar{y}, \bar{z})$ restricts to a group operation on $X$) is an abelian group. We work in $\mathbb{F}_\omega$. Let $\bar{a} \in X$ be a tuple that $X$ gains in $\mathbb{F}_{n+1}$, then $\bar{a} \in \mathbb{F}_{n+1} \setminus \mathbb{F}_n$. We fix a tuple of words $\bar{a}(x_1, \ldots, x_{n+1})$ in variables $x_1, \ldots, x_{n+1}$, such that $\bar{a}(e_1, \ldots, e_{n+1}) = \bar{a}$. It is easy to see that $\bar{a}' := \bar{a}(e_1, \ldots, e_n, e_{n+2})$ belongs to $X$ and $\bar{a}' \in \mathbb{F}_{n+2} \setminus \mathbb{F}_{n+1}$. We then have that $\bar{a} \circ \bar{a}'$ is a tuple in $\mathbb{F}_{n+2} \setminus \mathbb{F}_{n+1}$, if not then $\bar{a}'$ is a tuple in $\mathbb{F}_{n+1}$, a contradiction. So, let $\bar{a} \circ \bar{a}' = \bar{w}(e_1, \ldots, e_{n+1}, e_{n+2})$. We consider the automorphism, $f \in \text{Aut}(\mathbb{F}_{n+2}/\mathbb{F}_n)$, exchanging $e_{n+1}$ with $e_{n+2}$, it is clear that $f(\bar{a}) = \bar{a}'$, $f(\bar{a}') = \bar{a}$ and $f$ fixes $\circ$. Thus, we have $f(\bar{a}) \circ f(\bar{a}') = \bar{a}' \circ \bar{a}$ and as $(X, \circ)$ is abelian we have $\bar{a}' \circ \bar{a} = \bar{w}(e_1, \ldots, e_{n+1}, e_{n+2})$. But, $f(\bar{a}) \circ f(\bar{a}') = f(\bar{w}(e_1, \ldots, e_{n+1}, e_{n+2})) = \bar{w}(e_1, \ldots, e_{n+2}, e_{n+1})$ and as $\bar{w}(e_1, \ldots, e_{n+1}, e_{n+2})$ is in $\mathbb{F}_{n+2} \setminus \mathbb{F}_{n+1}$ we have that $\bar{w}(e_1, \ldots, e_{n+2}, e_{n+1}) \neq \bar{w}(e_1, \ldots, e_{n+1}, e_{n+2})$, a contradiction. 

Of course when we talk about interpretable sets we also have to deal with imaginary sorts. Sela [Sel] proved a (weak) elimination of imaginaries up to adding some sorts: A sort for conjugation, a family of sorts for cosets of centralizers and a family of sorts for double cosets of centralizers (see Section 3).

There are two major predicaments in generalizing the above argument. First, there is a class of formulas that define abelian groups, namely centralizers of nontrivial elements. For this class of formulas we need to find a different argument. Second, even for formulas that gain an element, like in the above proposition, the noncommutativity argument fails for conjugacy classes. Consider, for a simple counterexample, that the sum is given by the word $e_1e_2$, then exchanging $e_1$ with $e_2$ gives $e_2e_1$ which is in the same conjugacy class as $e_1e_2$. In this latter case we can fix the problem by taking more than two summands. Permuting the primitive elements that define the different summands will increase the number of conjugacy classes of the resulting sum and thus obtain a contradiction (see [PPST14]).

The way we deal with centralizers is by proving that they are pure groups, i.e. any definable subset of a cartesian power $C_{\mathbb{F}}(a) \times C_{\mathbb{F}}(a) \times \ldots \times C_{\mathbb{F}}(a)$ is definable by multiplication alone (see [BS19]). By a theorem of Wagner [Wag04], the latter result takes care of all interpretable sets internal to a family of centralizers.

**Strategy:** We first replace the interpretable set by its *Diophantine Envelope*. The Diophantine Envelope is a finite family of *towers*. To every tower we can assign a Diophantine definable set.

The Diophantine Envelope has the following properties. The union of sets assigned to the Diophantine Envelope contain the original set and moreover for particular sequences of elements, called *test sequences*, one may decide whether they eventually belong to the set or not. At this point the proof splits. If the only used part of the towers is their *abelian pouch*,

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then the set is internal to a finite set of centralizers hence it cannot be the domain of a field. If not, then we can proceed applying the noncommutativity argument as described below.

From a tower in the Diophantine Envelope, we construct an $N$-multiplet of towers for which we can decide which test sequences belong to the $N$ cartesian product of the original set. The $N$-multiplet of towers still has the structure of a tower and it can be seen as a star of groups with $N$ rays.

We next apply the implicit function theorem on the $N$-multiplet of towers and the formula that defines the $N$-summation. The implicit function theorem gives an element in the $N$-multiplet of towers that its image under certain test sequences is the value of the $N$-summation. Permuting the rays of the $N$-multiplet of towers is the analogue of permuting the primitive elements in the noncommutativity argument. Hence choosing $N$ large enough we get enough different values for the $N$-summation and this finally shows that the interpretable set cannot be given an abelian group structure.

Paper structure: In Section 3 we explain some model theoretic concepts, namely imaginaries and their relative elimination in the first-order theory of the free group.

In Section 4 we define a special type of graph of groups that we call a star of groups because the underlying graph is a star. We prove theorems about the basic imaginary sorts in stars of groups. In particular we show how the equivalence classes change as we permute the rays.

Section 5 contains the construction of towers. Towers, their multiplets and closures are the main tools to partially understand definable sets.

In Section 6 we define the notion of a test sequence. We state Sela’s Diophantine Envelope theorem and generalizations of the implicit function theorem.

Finally, in the last section we prove the main theorem of the paper as well as some special cases that are free of certain technicalities.

3 Some model theory

3.1 Imaginaries

We fix a first order structure $\mathcal{M}$ and we are interested in the collection of definable sets in $\mathcal{M}$, i.e. all subsets of some cartesian power of $\mathcal{M}$ which are the solution sets of first order formulas. We motivate the definition of imaginaries with the following question: Suppose $X$ is a definable set in $\mathcal{M}$, is there a canonical way to define $X$, i.e. is there a tuple $\bar{b}$ and a formula $\psi(\bar{x}, \bar{y})$ such that $\psi(\mathcal{M}, \bar{b}) = X$ but for any other $\bar{b}' \neq \bar{b}$, $\psi(\mathcal{M}, \bar{b}') \neq X$?

To give a positive answer to the above mentioned question one has to move to a multi-sorted expansion of $\mathcal{M}$ called $\mathcal{M}^{eq}$. This expansion is constructed from $\mathcal{M}$ by adding a new sort for each $\emptyset$-definable equivalence relation, $E(\bar{x}, \bar{y})$, together with a class function $f_E : M^n \rightarrow M_E$, where $M_E$ (the domain of the new sort corresponding to $E$) is the set of all $E$-equivalence classes. The elements in these new sorts are called imaginaries. In $\mathcal{M}^{eq}$, one can assign to each definable set a canonical parameter in the sense discussed above. Moreover, every formula in the multi-sorted language corresponds to a formula in the original language in the following sense.

Fact 3.1: Let $\phi(x_1, x_2, \ldots, x_k)$ be a formula in the multi-sorted language $\mathcal{L}^{eq}$, where $x_i$ is a variable of the sort $S_{E_i}$. Then there exists an $\mathcal{L}$-formula $\psi(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k)$ such that:

$$\mathcal{M}^{eq} \models \forall \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k (\phi(f_{E_1}(\bar{y}_1), f_{E_2}(\bar{y}_2), \ldots, f_{E_k}(\bar{y}_k)) \leftrightarrow \psi(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k))$$
A set is interpretable in $\mathcal{M}$ if it is definable up to a definable equivalence relation. Actually, understanding interpretable sets in $\mathcal{M}$ is equivalent to understanding definable sets in $\mathcal{M}^{eq}$.

**Fact 3.2:** A set is interpretable in $\mathcal{M}$ if and only if it is definable in $\mathcal{M}^{eq}$.

We now specialize to the first order theory of the free group. Following Sela [Sel] we define some families of equivalence relations that will be used to eliminate the rest.

**Definition 3.3:** Let $\mathbb{F}$ be a non abelian free group. The following equivalence relations in $\mathbb{F}$ are called basic.

1. $E_1(a, b)$ if there is $g \in \mathbb{F}$ such that $a^g = b$. (conjugation)
2. $E_2_m([(a_1, b_1), (a_2, b_2)])$ if either $b_1 = b_2 = 1$ or $b_1 \neq 1$ and $C_{\mathbb{F}}(b_1) = C_{\mathbb{F}}(b_2) = \langle b \rangle$ and $a_1^{-1}a_2 \in \langle b^m \rangle$. ($m$-coset)
3. $E_{3,m,n}([(a_1, b_1, c_1), (a_2, b_2, c_2)])$ if either $a_1 = a_2 = 1$ or $c_1 = c_2 = 1$ or $a_1, c_1 \neq 1$ and $C_{\mathbb{F}}(a_1) = C_{\mathbb{F}}(a_2) = \langle a \rangle$ and $C_{\mathbb{F}}(c_1) = C_{\mathbb{F}}(c_2) = \langle c \rangle$ and there is $\gamma \in \langle a^m \rangle$ and $\epsilon \in \langle c^n \rangle$ such that $\gamma b_1 \epsilon = b_2$. ($m,n$-double coset)

We denote a class that corresponds to $E_1$ by $[x]_1$, classes that correspond to $E_{2, m}$ by $[(x, y)]_{2^m}^m$ and finally classes that correspond to $E_{3, m, n}$ by $[m, (x, y, z)]_{3^m}^n$. Of course, the equivalence relation that corresponds to conjugation makes sense in any group. We remark that the $m$-coset and $m,n$-double coset equivalence relations make sense in any commutative transitive group. In particular, they define equivalence relations in any limit group.

Sela proved the following theorem concerning imaginaries in nonabelian free groups (see [Sel] Theorem 4.4).

**Theorem 3.4:** Let $\mathbb{F}$ be a nonabelian free group. Let $E(\bar{x}, \bar{y})$ be a definable equivalence relation in $\mathbb{F}$, with $|\bar{x}| = m$. Then there exist $k, \ell < \omega$ and a definable relation:

$$R_E \subseteq \mathbb{F}^m \times \mathbb{F}^k \times S_1(\mathbb{F}) \times \ldots \times S_\ell(\mathbb{F})$$

such that:

1. each $S_i(\mathbb{F})$ is one of the basic sorts;
2. for each $\bar{a} \in \mathbb{F}^m$, $|R_E(\bar{a}, \bar{z})|$ is uniformly bounded (i.e. the bound does not depend on $\bar{a}$);
3. $\forall \bar{z}(R_E(\bar{a}, \bar{z}) \leftrightarrow R_E(\bar{b}, \bar{z}))$ if and only if $E(\bar{a}, \bar{b})$.

We will occasionally denote by $R_E(x, \bar{z})$ the same relation as in the theorem above, but with $x$ a variable in the sort $S_E$. Formally the latter relation is defined by $R_E(f_E(\bar{x}), \bar{z})$.

From this point on when we refer to variables or elements in the basic sorts we will mean the basic sorts of Definition 3.1, together with the real sort.

## 4 Stars of groups

In this section we recount results about a particular type of graph of groups, i.e. graph of groups in which the underlying graph is a star (see Figure 4). Equivalently, we amalgamate a (finite) family of groups $\{G_i\}_{i \in I}$ over a common subgroup $A$, and denote it by $G = *_A \{G_i\}_{i \in I}$. We call the resulting group a star of groups and each $G_i$, $i \in I$, a factor subgroup. Moreover, we call the cardinality of the index set $I$, the number of rays of the star of groups. Finally, for the sake of clarity we will abuse notation and identify $A$ with its images in the $G_i$’s under the defining embeddings $f_i : A \to G_i$. 

4
4.1 Canonical Forms

We define the notion of a word in reduced form in a star of groups.

**Definition 4.1** (reduced forms): Let $G = \ast_{A} \{G_i\}_{i \in I}$ be a star of groups. A word $a g_1 g_2 \ldots g_n$ is reduced if

- $a \in A$;
- $g_j \in \bigcup \{G_i \setminus A\}_{i \in I}$, for each $j \leq n$;
- no consecutive $g_i$'s are in the same factor subgroup.

To each reduced word $a g_1 g_2 \ldots g_n$ we can assign its length $L(a g_1 g_2 \ldots g_n) = n$.

**Fact 4.2:** Every element $g$ of a star of groups, $G = \ast_{A} \{G_i\}_{i \in I}$, can be represented by a reduced word, i.e. $g = a \cdot g_1 \cdot g_2 \cdot \ldots \cdot g_n$.

Furthermore, the representation is not unique, but the length of every representation is the same.

**Remark 4.3:** The representation can be made unique if we fix a transversal for each $G_i$ with respect to (the image) of $A$. For the purpose of this paper it will be enough to observe that every reduced word representative of an element of $G$ has identical sequence of factor subgroups, i.e. if $g = a g_1 g_2 \ldots g_n = b h_1 h_2 \ldots h_n$, then for each $i \leq n$ $h_i$ and $g_i$ belong to the same factor subgroup.

When dealing with conjugacy classes it will be convenient to have a similar form to work with.

**Definition 4.4** (Cyclically reduced forms): A reduced word $a g_1 g_2 \ldots g_n$ in a star of groups is cyclically reduced, if $g_1, g_n$ are in different factor subgroups unless $n = 1$.

**Fact 4.5:** Every element of a star of groups, $G = \ast_{A} \{G_i\}_{i \in I}$, is conjugate to (the representative of) a cyclically reduced word.

Furthermore, suppose $b$ and $c$ are cyclically reduced and conjugate in $G$. Then:
• If $b$ belongs to $A$, then $c$ belongs to some factor subgroup $G_i$, $i \in I$, and there is a sequence $b, a_1, a_2, \ldots, a_m, c$, where each $a_i$ is in $A$ and $a_{i+1}$ is conjugate to $a_i$, for $i < m$, in some factor subgroup.

• if $b$ belongs to some factor subgroup but not in $A$, i.e. $b \in G_i \setminus A$, for some $i \in I$, then $c$ belongs to the same factor and they are conjugates in this factor.

• if $b = ab_1b_2\ldots b_n = b_1b_2\ldots b_n$ with $L(ab_1b_2\ldots b_n) > 1$, then $c$ can be obtained by cyclically permuting $b_1, b_2, \ldots, b_n$ and then conjugating by an element of $A$.

4.2 Permuting the rays

For this subsection we fix a star of groups $G = \ast_A \{G_i\}_{i \in I}$, for which the factor subgroups are all isomorphic, and are all amalgamated over the same subgroup. We will call $G$ a star of isomorphic groups.

We will first study the orbits of cyclically reduced words under the action induced by permuting the rays of a star of groups. Let $G$ be a star of isomorphic groups with $n$ rays and let $X$ be the set of cyclically reduced words of length $\geq 1$. Then the group of permutations $S_n$ acts on $X$ as follows. We consider the natural action of $S_n$ on the set of $n$ rays, each element $\sigma \in S_n$ induces a permutation of the factor groups $\{G_i\}_{i \leq n}$ and sends a cyclically reduced word $ag_1g_2\ldots g_m$ to a cyclically reduced word $ah_1h_2\ldots h_m$ in the following way: if $g_i$ belongs to the factor $G_j$, then $h_i$ belongs to the factor $G_{\sigma(j)}$ and it is the image of $g_i$ under the isomorphism between $G_j$ and $G_{\sigma(j)}$. It is immediate, by properties of permutation groups, that the word $ah_1h_2\ldots h_m$ is cyclically reduced (and of length $m \geq 1$). The action is free on the subset $X_f$ of $X$ that consists of cyclically reduced words that every of the $n$ factor subgroups has a representative in $g_1g_2\ldots g_m$. As a matter of fact a stronger property holds for $g \in X_f$, if $\sigma(g) = aga^{-1}$ for some $a \in A$, then $\sigma$ must be the trivial permutation and $a = 1$.

Lemma 4.6: Let $G$ be a star of isomorphic groups with $n$ rays. Let $g = ag_1'g_2'\ldots g_m' = g_1g_2\ldots g_m$ be a cyclically reduced word of length $\geq 1$. Then the orbit $S_n\cdot g$ contains at least $\lceil \frac{n}{2} \rceil$ conjugacy classes.

Proof. We may assume that $n$ is large enough.

We first prove the result for cyclically reduced words in which all factor subgroups have a representative in $g_1g_2\ldots g_m$.

We will use permutations of order 2. In particular we consider the following set of $\lfloor n/2 \rfloor$ transpositions (in cycle notation) $\Sigma := \{(1,2), (3,4), \ldots, (n-1, n)\}$. In order to prove the result it is enough to prove that for any three transpositions $\sigma, \tau, \delta$ in $\Sigma$, the set $\{g, \sigma g, \tau g, \delta g\}$ contains at least two conjugacy classes. For the latter, since permutations act on the set of conjugacy classes, it is enough to prove that the set $\{g, \sigma \tau g, \sigma \delta g\}$ contains at least two conjugacy classes. We note that all permutations in $\Sigma$ commute, hence for any $\sigma, \tau, \delta$ in $\Sigma$, the permutations $\sigma \tau$ and $\sigma \delta$ still have order 2. In addition, there exists $i \leq n$, such that $i$ is fixed by $\sigma \tau$, but moved by $\sigma \delta$.

We assume, for the sake of contradiction, that there exist $\mu_1, \mu_2$ as described above such that $[\mu_1]_1 = [\mu_2]_1 = [g]_1$, where $[x]_1$ denotes the conjugacy class of $x$. Then, since $g$ has length $\geq 1$, Fact 4.5 implies that there exist cyclic permutations $\sigma_1, \sigma_2 \in \langle (12\ldots m) \rangle$ such that $\mu_1 \cdot g = ag_{\sigma(1)}g_{\sigma(2)}\ldots g_{\sigma(m)}a^{-1}$ and $\mu_2 \cdot g = bg_{\sigma(2)}g_{\sigma(3)}\ldots g_{\sigma(m)}b^{-1}$ for some $a, b \in A$. Moreover, $\mu_1' \cdot g = ag_{\sigma(1)}g_{\sigma(2)}\ldots g_{\sigma(m)}a^{-1}$ and $\mu_2' \cdot g = bg_{\sigma(2)}g_{\sigma(3)}\ldots g_{\sigma(m)}b^{-1}$ Now, since
\( \mu_i^{o(\sigma_i)} \) fixes \( g \) (up to conjugation by an element of \( A \)) , we must have that \( o(\sigma_i) \) is even. But, for any two cyclic permutations \( \sigma, \tau \) in \( \langle (12 \ldots m) \rangle \), there exist \( k, l \) not both even, such that \( \sigma^k = \tau^l \). Applying the latter fact to \( \sigma_1, \sigma_2 \), and without loss of generality, we get \( \sigma_1^k = \sigma_2^l \) with \( k \) odd.

Consider \( \mu_1^j.g = a_{\sigma_1^j(1)}g_{\sigma_1^j(2)} \ldots g_{\sigma_1^j(m)}a^{-1} \) and \( \mu_2^j.g = b_{\sigma_2^j(1)}g_{\sigma_2^j(2)} \ldots g_{\sigma_2^j(m)}b^{-1} \). Since, all factor subgroups are represented in the cyclically reduced word \( g_1g_2 \ldots g_m \), there exists some \( i \leq m \) such that \( \mu_1 \) moves \( g_i \), but \( \mu_2 \) fixes \( g_i \). Now, since \( k \) is odd, \( g_{\sigma_1^j(i)} \) and \( g_i \) do not belong to the same factor subgroup, but \( g_{\sigma_2^j(i)} \) and \( g_i \) do belong to the same factor subgroup, a contradiction.

When a factor subgroup is not represented in the cyclically reduced word \( g_1g_2 \ldots g_m \) we can do better. Assume, that \( k \) out of \( n \) factor subgroups are not represented. Without loss of generality, we may assume that \( G_1, \ldots, G_k \) are not represented. Then, we may consider the \( (n-k) \cdot k \)-many transpositions, \( \{(i,k+j) \mid 1 \leq i \leq k, 1 \leq j \leq n-k\} \). Any cyclically reduced word of the above type has as many images (up to conjugation) under those transpositions and the result follows.

We next prove similar results for cosets and double cosets, we assume that the fundamental group of the star of groups is a commutative transitive group so the basic equivalence relations make sense.

**Lemma 4.7:** Let \( G \) be a star of isomorphic groups with \( n \) rays. Suppose \( C_G(a) \subseteq A \), for any non-trivial element \( a \) of the common subgroup \( A \). Let \( g = c_{g_1}g_2 \ldots g_l = g_1g_2 \ldots g_l \) and \( h = dl_1h_2 \ldots h_m = h_1h_2 \ldots h_m \neq 1 \) be reduced words whose sum of lengths is \( \geq 1 \). Then the orbit \( S_n.(g,h) \) contains at least \( \left\lceil \frac{n-2}{2} \right\rceil \cdot k \)-cosets \( [(x,y)]_k^k \), for any \( k < \omega \).

**Proof.** We may assume that \( n \) is large enough.

We first assume that \( h = h_1h_2 \ldots h_m \) has length \( \geq 1 \). If \( h \) has length \( 1 \) and belongs to the factor subgroup \( G_i \), then the \( (n-1) \)-many transpositions \( (i,j) \) move \( h \) to \( n-1 \) pairwise non-commuting images. Hence we may assume that the length of \( h \) is strictly greater than \( 1 \). We will show that at least \( \left\lceil \frac{n-2}{2} \right\rceil \) images of \( h \) under \( S_n \) do not pairwise commute. Consider the \( \lfloor m/2 \rfloor \)-th and \( \lceil m/2 \rceil + 1 \)-th elements in the reduced word sequence of \( h \) and assume they belong to the factor subgroups \( G_i \) and \( G_j \) (since they are consecutive elements we must have \( i \neq j \)). We split the set \( \{1, 2, \ldots, n\} \setminus \{i, j\} \) in two equal size sets (by subtracting one more element if necessary) \( I \) and \( J \) and we choose a correspondence \( f : I \rightarrow J \). Let \( \Sigma \) be the set of \( \left\lceil \frac{n-2}{2} \right\rceil \)-many permutations of the form \( (i,i')(j,j') \) for \( (i',j') \in f \). We show that, for any \( \sigma, \tau \in \Sigma \), the elements \( \sigma.h, \tau.h \) do not commute. We consider the possible reduced word representatives for the element \( \sigma.h \cdot \tau.h \):

\[
\sigma.h_1 \sigma.h_2 \ldots \sigma.h_{\lfloor m/2 \rfloor} \sigma.h_{\lceil m/2 \rceil} \ldots \sigma.h_m \cdot \tau.h_1 \tau.h_2 \ldots \tau.h_{\lfloor m/2 \rfloor-1} \tau.h_{\lceil m/2 \rceil} \ldots \tau.h_m
\]

We claim that any reduced word representing the product \( \sigma.h_{\lfloor m/2 \rfloor} \sigma.h_{\lceil m/2 \rceil} \ldots \sigma.h_m \cdot \tau.h_1 \tau.h_2 \ldots \tau.h_{\lfloor m/2 \rfloor-1} \tau.h_{\lceil m/2 \rceil} \) starts with an element in the same factor subgroup as its first element \( \sigma.h_{\lceil m/2 \rceil} \) and ends with an element in the same factor subgroup as its last element \( \tau.h_{\lceil m/2 \rceil} \). We see this as follows:

Starting from the center of the word we consider the product \( \sigma.h_m \cdot \tau.h_1 \), then either \( \sigma.h_m, \tau.h_1 \) belong to the same factor subgroup and in addition their product belongs to \( A \) or our claim holds. Hence we may assume that we are in the first case and \( \sigma.h_m \cdot \tau.h_1 = a \) for some \( a \in A \).
Since multiplying an element $g$ of length 1 with an element in the common subgroup $A$ does not change neither the length nor the factor subgroup of $g$, we may continue applying the same argument to the product $\sigma.h_{m-1} \cdot \tau.ah_2$ which lies in the center of the product $\sigma.h_{[m/2]} \cdot \sigma.h_{[m/2]+1} \cdot \cdots \cdot \sigma.h_{m-1} \cdot \tau.ah_2 \cdots \tau.h_{[m/2]} - 1 \tau.h_{[m/2]}$. At each step, either the product of the two elements at the center of the word belongs to $A$ and the length is reduced by 1 or our claim holds. Finally, we arrive at a product $\sigma.h_{[m/2]} \cdot \sigma.h_{[m/2]+1} \cdot \cdots \cdot \sigma.h_{[m/2]}$, if $m$ is even and $\sigma.h_{[m/2]} \cdot \tau.bh_{[m/2]}$ if $m$ is odd for some $b \in A$. Since $\sigma.h_{[m/2]} \cdot \sigma.h_{[m/2]} \cdot \cdots \cdot \sigma.h_{[m/2]}$ belong to different factor subgroups, any reduced word representing the above products satisfies our claim.

As a consequence, the first $[m/2]$ elements in the sequence of the reduced word that corresponds to $\sigma.h \cdot \sigma.h$ are $\sigma.h \sigma.h \cdot \cdots \cdot \sigma.h_{[m/2]}$, while the first $[m/2]$ elements in the sequence of the reduced word that corresponds to $\tau.h \cdot \tau.h$ are $\tau.h \tau.h \cdot \cdots \cdot \tau.h_{[m/2]}$. In particular the $[m/2]$-th elements of the sequences do not belong to the same factor subgroup and this implies that $\sigma.h, \tau.h$ do not commute. We conclude that, in this case, the couple $(g, h)$ has at least $\lfloor \frac{n}{2} \rfloor$-coset images under the action of $S_n$.

We next assume that $h$ is nontrivial of length 0, hence it belongs to $A$. In this case, $g = g_1g_2 \cdots g_l$ must have length $\geq 1$. We note that, since $h \in A$ and $C_G(h) \subseteq A$, the couples $(g, h), (g', h)$ belong to the same $k$-coset only if $g' = g \cdot a$, for some $a \in A$. Suppose that $g_1$ belongs to the factor subgroup $G_i$ and we consider the set $\Sigma$ of all $n$ permutations of the form $(i, j)$ (including the identity). Then for any $\sigma, \tau$ in $\Sigma$ we have that $\sigma.g \neq \tau.g \cdot a$ for any $a \in A$. Hence $(g, h)$ has at least $n$ $k$-coset images under the action of $S_n$ and this concludes the proof.

Identical arguments show the following result for double cosets.

**Lemma 4.8:** Let $G$ be a star of isomorphic groups with $n$ rays. Suppose $C_G(a) \subseteq A$, for any nontrivial element $a$ of the common subgroup $A$. Let $u = au_1u_2 \cdots u_1u_2 \cdots u_p \neq 1$, $g = b_1g_2 \cdots g_l = g_1g_2 \cdots g_l$ and $h = c_1h_2 \cdots h_m = h_1h_2 \cdots h_m \neq 1$ be reduced words whose sum of lengths is $\geq 1$. Then the orbit $S_n(\{u, g, h\})$ contains at least $\lfloor \frac{n-2}{2} \rfloor$ $(q, k)$-double-cosets $q[(x, y, z)]_{\frac{k}{3}}$, for any $q, k < \omega$.

## 5 Towers

A limit group $L$ is a finitely generated group for which there exists a sequence of morphisms, $(h_n)_{n<\omega} : L \to \mathbb{F}$, such that for every $g \in L$, $h_n(g) \neq 1$ for all but finitely many $n$. We call a sequence such as $(h_n)_{n<\omega}$, a witnessing sequence.

In this section we define a special subclass of limit groups namely groups that have the structure of a tower. Towers played an important role in the proof of the elementary equivalence of nonabelian free groups. Notably they have been used in order to generalize Merzlyakov’s theorem [Sel03, KM05], which is the conceptual basis of the proof of the $\forall \exists$-equivalence of nonabelian free groups.

A tower is built recursively by adding floors to a given ground floor, that consists of a nonabelian free group. There are two types of floors, surface floors and abelian floors. The corresponding notion in the work of Kharlampovich-Myasnikov is the notion of an NTQ group, i.e. the coordinate group of a nondegenerate triangular quasiquadratic system of equations (see [KM98, Definition 9]).
Towers can be thought of as groups equipped with a decoration. The decoration consists of the nonabelian free group of the ground floor, the additional floors, and finally the way each floor is added to the already constructed tower. To any such tower we may assign a closure. The closure of a tower is obtained by augmenting the abelian floors of the original tower. It is still a tower with the same number and type of floors which are added in the same way as in the original tower, moreover it contains the original tower as a subgroup. When no abelian floors take part in the construction of a tower, then we call it hyperbolic and it coincides with any of its closures.

Finally, we will define multiplets of a tower, namely we will add identical floors on the same basis multiple times. The end product is still a tower and in addition it can be seen as a star of groups.

5.1 The construction of a tower

We assume some familiarity with Bass-Serre theory [Ser83]. We start by defining the notion of a surface floor.

Definition 5.1 (Surface-type vertex): A vertex $v$ of a graph of groups $\Gamma$ is called a surface-type vertex if the following conditions hold:

- the group $G_v$ carried by $v$ is the fundamental group of a compact surface $\Sigma$ (usually with boundary), with Euler characteristic $\chi(\Sigma) < 0$;
- incident edge groups are maximal boundary subgroups of $\pi_1(\Sigma)$, and this induces a bijection between the set of incident edges and the set of boundary components of $\Sigma$.

Definition 5.2 (Exceptional surfaces): Four hyperbolic surfaces with $\chi(\Sigma) = -1$ are considered exceptional because their mapping class group is “too small” (they do not carry pseudo-Anosov diffeomorphisms): the thrice-punctured sphere, the twice-punctured projective plane, the once-punctured Klein bottle, and the closed non-orientable surface of genus 3.

Definition 5.3 (Centered splitting): A centered splitting of $G$ is a graph of groups decomposition $G = \pi_1(\Gamma)$ such that the vertices of $\Gamma$ are $v, v_1, \ldots, v_m$, with $m \geq 1$, where $v$ is surface-type and every edge joins $v$ to some $v_i$ (see Figure 2).

The vertex $v$ is called the central vertex of $\Gamma$. The vertices $v_1, \ldots, v_n$ are the bottom vertices, and we denote by $H_i$ the group carried by $v_i$. The base of $\Gamma$ is the abstract free product $H = H_1 \ast \cdots \ast H_m$.

The centered splitting $\Gamma$ is non-exceptional if the surface $\Sigma$ is non-exceptional.

![Figure 2: A centered splitting](image-url)
Definition 5.4 (Surface floor): Let $G$ be a group and $H$ be a nonabelian subgroup of $G$. Then $G$ has the structure of a surface floor over $H$, if the following conditions hold:

- the group $G$ admits a non-exceptional centered splitting with base $H$;
- there exists a retraction $r : G \to H$ that sends the group carried by the central vertex of $\Gamma$ to a non abelian image.

An abelian floor is defined in a similar way.

Definition 5.5: Let $G$ be a group and $H$ be a subgroup of $G$. Then $G$ has the structure of an abelian floor over $H$, if $G$ admits a splitting as an amalgamated free product $H \ast_E (E \oplus \mathbb{Z}^m)$, where $E$ is a maximal abelian subgroup of $H$ and $\mathbb{Z}^m$ is a free abelian group of rank $m$ (see Figure 3).

We call $E$ the peg of the abelian floor.

![Figure 3: An abelian floor](image)

We can now define towers.

Definition 5.6: A group $G$ has the structure of a tower (of height $m$) if there exists a nonabelian free group $F$ and a sequence $G = G^m > G^{m-1} > \ldots > G^0 = F$ such that for each $i$, $0 \leq i < m$, either $G^{i+1}$ is a surface floor or an abelian floor over $G^i$.

![Figure 4: A tower of height 4.](image)

The next lemma follows from the definition of a constructible limit group.
Lemma 5.7: If $G$ has the structure of a tower, then $G$ is a limit group.

It will be useful to collect the information witnessing that a group $G$ has the structure of a tower. Thus we define:

Definition 5.8: Suppose $G$ has the structure of a tower (of height $m$). Then the tower $T(G,F)$ is the group $G = G^m$ together with the following collection of data:

$((G(G^1,G^0), r_1), (G(G^2,G^1), r_2), \ldots, (G(G^m,G^{m-1}), r_m))$

where the splitting $G(G^{i+1},G^i)$ is the splitting that witnesses that $G^{i+1}$ has the structure of a surface or abelian floor over $G^i$.

A tower in which no abelian floor occurs in its construction is called a hyperbolic tower (or regular NTQ group in the terminology of Kharlampovich-Myasnikov) and it is actually a hyperbolic group. For the rest of the paper we follow the convention:

Convention: Suppose $T(G,F)$ is a tower. Let $\{E_j\}_{j \in J}$ be the collection of pegs that correspond to the abelian floors that occur along the construction of the tower. Then:

• no two pegs are conjugate;
• if a peg can be conjugated into the ground floor $F$, then it is a subgroup of $F$;
• the abelian floors that correspond to pegs that belong to $F$ always come first in the construction of the tower.

Definition 5.9 (Abelian Pouch): Let $T(G,F)$ be a tower and $E_1, E_2, \ldots, E_n$ be the pegs that belong to the ground floor $F$. Then the fundamental group of the graph of group $\Gamma$, with vertices $u, u_1, u_2, \ldots, u_n$ where $u$ carries the group $F$, each $u_i$ carries the free abelian group $E_i \oplus \mathbb{Z}^{m_i}$ and every edge joins $u_i$ to $u$ and carries the group $E_i$ (see Figure 5), is called the abelian pouch of the tower.

When no such pegs exist the abelian pouch is just the ground floor $F$.

Figure 5: The abelian pouch
5.2 Multiplets of towers

We fix a tower $T(G, \mathcal{F})$ and we construct a new one by adding multiple times the floors of $T$ on the ground floor $\mathcal{F}$. The group that corresponds to the end product of this construction splits as a star of groups where the common subgroup is the abelian pouch of $T$. When the abelian pouch is the ground floor itself the factor subgroups are all isomorphic to $G$.

**Definition 5.10:** Let $T(G, \mathcal{F})$ be a tower of height $m$ whose abelian pouch is $\mathcal{F}$. Then the $N$-multiplet of $T$, $T_N := T \# T \# \ldots \# T$ is the tower constructed as follows:

- The ground floor of $T_N$ is $\mathcal{F}$, and moreover $T_N$ has $m \cdot N$ many floors;
- the first $m$-floors of $T_N$ are identical to the floors of $T$;
- the floors $im + 1$ to $(i + 1)m$, for $1 \leq i < N$, are identical to the floors of $T$, only glued on $T_i$.

The group that corresponds to the $N$-multiplet of the above $T(G, \mathcal{F})$ is the star of groups $G_N = \ast_{\mathcal{F}} \{ G_i \}_{i \leq N}$ with each $G_i$ isomorphic to $G$.

![Figure 6: The $N$-multiplet of $T$](image)

When the abelian pouch of the tower is not just the ground floor, i.e. there exists some peg of an abelian floor attached to the ground floor, then the construction of an $N$-multiplet is slightly different. One has to take into account that when we attach the pegs of the abelian pouch for the second, third or $N$-th time the edge groups are not maximal abelian anymore, hence we attach them all together in the beginning.

**Definition 5.11 (Multiplets of Towers):** Let $T(G, \mathcal{F})$ be a tower of height $m$ and $E_1, E_2, \ldots, E_n$ be the pegs that belong to the ground floor $\mathcal{F}$. Let $E_i \oplus \mathbb{Z}^{m_i}$ be the abelian floor that corresponds to the $i$-th peg. Then the $N$-multiplet of $T$, $T_N := T \# T \# \ldots \# T$ is the tower constructed as follows:
• The ground floor of $T_N$ is $F$, and moreover $T_N$ has $m + (m - n) \cdot (N - 1)$ many floors;
• the first $n$ floors are $E_i \oplus \mathbb{Z}^{m_i} \oplus \mathbb{Z}^{m_i} \oplus \ldots \oplus \mathbb{Z}^{m_i}$ (adding $N$-times the summand $\mathbb{Z}^{m_i}$), for $i \leq n$, where the pegs $E_i$ are attached on the ground floor as in the tower $T$;
• the floors from $n + 1$ to $m$ are identical to the floors $n + 1$ to $m$ of $T$, attached to the first summand $\mathbb{Z}^{m_i}$, when appropriate;
• the floors $im + 1$ to $(i + 1)m - n$, for $1 \leq i < N$ are identical to the floors $n + 1$ to $m$ of $T$, attached to the $i + 1$-th summand $\mathbb{Z}^{m_i}$, when appropriate.

The group that corresponds to the $N$-multiplet of $T(G, F)$ is still a star of groups, but now the common subgroup is the tower of the first $n$ floors of $T_N$ and each factor is isomorphic to the tower of the first $m$ floors of $T_N$.

5.3 Closures of towers

In this subsection we define the notion of a closure of a tower.

Definition 5.12 (Free Abelian Closure): Let $\mathbb{Z} \oplus \mathbb{Z}^m, \mathbb{Z} \oplus A^m$ be free abelian groups of rank $m + 1$. Then an embedding $f: \mathbb{Z} \oplus \mathbb{Z}^m \to \mathbb{Z} \oplus A^m$ is called a closure if it satisfies the following properties:

• $\mathbb{Z}$ is sent onto $\mathbb{Z}$;
• the matrix $A_f$ that corresponds to the map $f$ is invertible.

The last condition equivalently states that $f(\mathbb{Z} \oplus \mathbb{Z}^m)$ has finite index in $\mathbb{Z} \oplus A^m$. A way to identify $\mathbb{Z} \oplus \mathbb{Z}^m$ as a subgroup of $\mathbb{Z} \oplus A^m$ with respect to the closure $f$, is by starting with the group $\mathbb{Z} \oplus \mathbb{Z}^m \oplus A^m$ and add the relations $a = f(a)$ for every $a \in \mathbb{Z}^m$. We call the latter group the group closure of $\mathbb{Z} \oplus \mathbb{Z}^m$ with respect to $f$.

Now, the closure of a tower is the tower equipped with closure embeddings for each abelian floor of the tower.

Definition 5.13 (Tower Closure): Let $\mathcal{T}(G, F)$ be a tower and $E_1, E_2, \ldots, E_n$ be the pegs of the abelian floors. Let $E_i \oplus \mathbb{Z}^{m_i}$ be the abelian floor that corresponds to the $i$-th peg and $f_i: E_i \oplus \mathbb{Z}^{m_i} \to E_i \oplus A^{m_i}$ be a closure. Then the closure of $\mathcal{T}(G, F)$, $\text{Cl}(\mathcal{T}(G, F))$, with respect to $\{f_i\}_{i \leq n}$ is defined as follows:

• The ground and surface floors of $\text{Cl}(\mathcal{T})$ are identical to the floors in $\mathcal{T}$;
• the peg $E_i$ of each abelian floor $E_i \oplus \mathbb{Z}^{m_i}$ is attached as in $\mathcal{T}$ and the abelian floor is the group closure of $E_i \oplus \mathbb{Z}^{m_i}$ with respect to $f_i$.

As groups a tower is a subgroup of any of its closures and as abstract groups a tower is isomorphic to any of its closures.

The point of introducing closures of towers is twofold. First, when applying Merzlyakov’s theorem in order to find a formal solution, we need to move to a closure to ensure its existence (see Fact 6.8) and second closures allow us to choose morphisms of the original tower that map certain elements in the solution sets of first-order formulas (see Fact 6.7).
Example 5.14: Let $T(G,F)$ be a tower with one abelian floor over $F = \langle e_1, e_2 \rangle$. Let $Z \oplus Z := \langle z, z_1 \mid [z, z_1] \rangle$ be the abelian floor and suppose its peg is attached to the maximal abelian subgroup $\langle e_1^2 e_2^3 \rangle$ of $F$. Let $Z \oplus Z \oplus A := \langle z, z_1, a \mid [z, a] = [z, z_1] = [z_1, a] = 1, z_1 = za^4 \rangle$ be the group closure with respect to the closure $f : Z \oplus Z \rightarrow Z \oplus A$ given by $f(z) = z$ and $f(z_1) = za^4$.

A morphism $h : T \rightarrow F$ with $f \mid F = Id$ extends to the given closure if and only if $h(z_1)$ is sent to $(e_1^2 e_2^3)^{1+4k}$ for some integer $k$.

Generalizing the example above a closure of a tower determines for every abelian floor $Z \oplus Z^m$ a coset of the corresponding free abelian group $Z^m$ so that only when a morphism gives values inside this coset it extends to the closure.

**Remark 5.15:** Let $T(G,F)$ be a tower and $Cl(T(G,F))$ be a closure. We will denote the group that corresponds to $Cl(T)$ by $Cl(G)$. Then $G$ is a subgroup of $Cl(G)$ and as abstract groups $G$ and $Cl(G)$ are isomorphic.

6 Test sequences, Envelopes and Implicit Function Theorems

6.1 Test sequences and their properties

We fix a tower $T(G,F)$. Then a test sequence for $T$ is a sequence of morphisms $(h_n)_{n<\omega} : G \rightarrow F$ that satisfy certain combinatorial properties that depend on the structure of the tower (see [Sel03, p. 222]). The values of test sequences are quite flexible as long as in the limit action one can still see the tower structure.

**Fact 6.1** (Surface floor limit action): Let $T(G,F)$ be a tower and $(h_n)_{n<\omega} : G \rightarrow F$ be a test sequence for $T$.

Suppose $G^{i+1}$ is a surface floor over $G^i := G_1^i \ast \ldots \ast G_m^i$, witnessed by the graph of groups $A(G^{i+1}, G^i)$, and $(h_n \mid G^{i+1})_{n<\omega}$ is the restriction of $(h_n)_{n<\omega}$ to that of $T$.

Then, any subsequence of $(h_n \mid G^{i+1})_{n<\omega}$ that converges, induces a faithful action of $G^{i+1}$ on a based real tree $(Y, \ast)$, with the following properties:

1. $G^{i+1} \curvearrowright Y$ decomposes as a graph of actions $(G^{i+1} \curvearrowright T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$, with the action $G^{i+1} \curvearrowright T$ being identical to $A(G^{i+1}, G^i)$;
2. if $v$ is not a surface type vertex then $Y_v$ is a point stabilized by the corresponding $G_j^i$ for some $j \leq m$;
3. if $u$ is the surface type vertex, then $Stab_G(u) = \pi_1(\Sigma_{g,l})$ and the action $Stab_G(u) \curvearrowright Y_u$ is a surface type action coming from $\pi_1(\Sigma_{g,l})$;

**Fact 6.2** (Abelian floor limit action): Let $T(G,F)$ be a tower and $(h_n)_{n<\omega} : G \rightarrow F$ be a test sequence for $T$.

Suppose $G^{i+1} = G^i \ast_A (A \oplus Z)$ is an abelian floor and $(h_n \mid G^{i+1})_{n<\omega}$ is the restriction of $(h_n)_{n<\omega}$ to that floor of $T$.

Then any subsequence of $(h_n \mid G^{i+1})_{n<\omega}$ that converges, induces a faithful action of $G^{i+1}$ on a based real tree $(Y, \ast)$, with the following properties:

1. the action of $G^{i+1}$ on $Y$, $G^{i+1} \curvearrowright Y$, decomposes as a graph of actions $(G^{i+1} \curvearrowright T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$;
2. the Bass-Serre presentation for $G^{i+1} \curvearrowright T$, $(T_1 = T_0, T_0)$, is a segment $(u, v)$;
3. \( \text{Stab}_G(u) := A \oplus \mathbb{Z} \cap Y_u \) is a simplicial type action, its Bass-Serre presentation, \((Y_u^1, Y_u^0, t_e)\) consists of a segment \(Y_u^1 := (a, b)\) whose stabilizer is \(A\), a point \(Y_u^0 = a\) whose stabilizer is \(A\) and a Bass-Serre element \(t_e\) which is \(z\);

4. \( Y_u \) is a point and \( \text{Stab}_G(v) = G_i \);

5. the edge \((u, v)\) is stabilized by \(A\).

We now state lemmas that we will use to move from different basic equivalence classes in a tower, to different classes of their images under a test sequence. Recall that since towers are limit groups all basic equivalence relations make sense in them as well.

**Lemma 6.3:** Let \( \mathcal{T}(G, F) \) be a tower and \((f_n)_{n<\omega} : G \to F\) be a test sequence for \(\mathcal{T}\). Let \(g, h\) be non-conjugate elements in \(G\), then \(f_n(g), f_n(h)\) are not conjugate in \(F\) for all but finitely many \(n < \omega\).

**Proof.** We give a proof for a hyperbolic tower \(\mathcal{T}\). Suppose, for a contradiction, that \(f_n(g), f_n(h)\) are conjugate in \(F\) for infinitely many \(n < \omega\). The refined sequence consisting of these values is still a test sequence, which we still denote by \((f_n)_{n<\omega}\). Hence we have \(\mathcal{F} \models \exists z (f_n(g) = f_n(h)^z)\).

In particular there exists a formal solution \(w\) in \(G\) such that \(\mathcal{F} \models f_n(g) = f_n(h)^{f_n(w)}\). Therefore, since \(G\) is a limit group, and every test sequence is in particular a witnessing sequence we get that \(g, h\) are conjugate in \(G\), a contradiction.

If \(\mathcal{T}\) is not hyperbolic, then \(g, h\) are conjugate in a closure of \(\mathcal{T}\). In this case, a normal form argument shows that \(g, h\) are already conjugate in \(G\).

Since we can express the equality of \([(g, h)]^m_3\) and \([(g, h, z)]^m_3\) classes by Diophantine first-order formulas we obtain the same result for these classes.

**Lemma 6.4:** Let \( \mathcal{T}(G, F) \) be a tower and \((f_n)_{n<\omega} : G \to F\) be a test sequence for \(\mathcal{T}\). Let \((g_1, h_1), (g_2, h_2)\) be different \(m\)-cosets in \(G\), then \((f_n(g_1), f_n(h_1)), (f_n(g_2), f_n(h_2))\) are different \(m\)-cosets in \(F\) for all but finitely many \(n < \omega\).

**Lemma 6.5:** Let \( \mathcal{T}(G, F) \) be a tower and \((f_n)_{n<\omega} : G \to F\) be a test sequence for \(\mathcal{T}\). Let \((g_1, h_1), (g_2, h_2)\) be different \(m\)-cosets in \(G\), then \((f_n(g_1), f_n(h_1)), (f_n(g_2), f_n(h_2))\) are different \(m\)-cosets in \(F\) for all but finitely many \(n < \omega\).

6.2 Diophantine Envelopes

In this subsection we will connect solutions of first-order formulas with test sequences of towers. The principle is that for a nonempty first-order formula \(\phi(\bar{x})\) there are always a tower \(\mathcal{T}(G, F)\), a designated tuple (denoted by the same letters \(\bar{x}\)) in \(G\), and a test sequence \((h_n)_{n<\omega} : G \to F\) such that \(F \models \phi(h_n(\bar{x}))\). Sela \cite{Sel} proved that for every nonempty first-order formula, there exist finitely many such towers which in addition contain the set. Since the definable sets that correspond to the towers are Diophantine, he called them the Diophantine Envelope.

**Fact 6.6** (Diophantine envelope): Let \(\phi(\bar{x}, \bar{a})\) be a nonempty first-order formula over \(\mathbb{F}\). Then there exist finitely many towers, \(\{(\mathcal{T}(G_i, F))_{i \leq k}\}\) with \(G_i := \langle \bar{u}_i, \bar{x}, \bar{a} \mid \Sigma_i(\bar{u}_i, \bar{x}, \bar{a}) \rangle\), that we call the Diophantine Envelope of \(\phi(\bar{x}, \bar{a})\), with the following properties:

(i) \(F \models \phi(\bar{x}, \bar{a}) \rightarrow \exists \bar{u}_1, \ldots, \bar{u}_k (\bigvee_{i=1}^k \Sigma_i(\bar{u}_i, \bar{x}, \bar{a}) = 1)\);

(ii) for each \(i \leq k\), there exists a test sequence, \((h_n)_{n<\omega} : G_i \to F\) for \(\mathcal{T}\) such that \(F \models \phi(h_n(\bar{x}), \bar{a})\).
Moreover, one can decide whether certain test sequences belong to a definable set using two levels of closures.

**Fact 6.7**: Let \( \phi(\vec{x}, \vec{a}) \) be a first order formula over \( \mathbb{F} \). Let \( T(G, \mathbb{F}) \) where \( G := \langle \vec{u}, \vec{x}, \vec{a} | \Sigma(\vec{u}, \vec{x}, \vec{a}) \rangle \) be a tower over \( \mathbb{F} \). Suppose there exists a test sequence, \( (h_n)_{n<\omega} : G \to \mathbb{F} \) for \( T(G, \mathbb{F}) \), such that \( \mathbb{F} \models \phi(h_n(\vec{x}), \vec{a}) \).

Then there exist:

- Finitely many closures, \( T_1 := Cl_1(T), \ldots, T_k := Cl_k(T) \) of \( T \);
- for each \( i \leq k \), finitely many closures, \( Cl_1(T_i), \ldots, Cl_{m_i}(T_i) \).

So that \( (h_n)_{n<\omega} \) except finitely many morphisms can be decomposed in finitely many (disjoint) subsequences \( (h_n^i)_{n<\omega} \), for \( i \leq q \), with the following properties:

- each subsequence extends to a nonempty subset of the first level of closures \( T_i \) and does not extend to the complement of this subset, i.e. each morphism of the subsequence extends to all closures of the subset and does not extend to some closure that belongs to the complement;

- for each subsequence either none of the morphisms extends to any of the second level of closures \( Cl_{m_i}(T_i) \) of the closure \( T_i \) it originally extended or each morphism of the subsequence extends to all closures of a nonempty subset of the second level closures of \( T_i \).

Moreover, for any test sequence, \( (g_n)_{n<\omega} : G \to \mathbb{F} \), that the above two conditions hold, there exists \( n_0 < \omega \) such that \( \mathbb{F} \models \phi(g_n(\vec{x}), \vec{a}) \) for all \( n > n_0 \).

### 6.3 Implicit function theorems

Implicit function theorems or extended Merzlyakov theorems form the basis for proving that the \( \forall \exists \)-theories of nonabelian free groups are equal. We give an example of such a type of theorem.

**Fact 6.8** (Extended Merzlyakov Theorem): Let \( T(G, \mathbb{F}) \) be a tower where \( G := \langle \vec{x}, \vec{v}, \vec{a} \rangle \), \( (h_n)_{n<\omega} : G \to \mathbb{F} \) be a test sequence for \( T \), and \( \Sigma(\vec{x}, \vec{y}) = 1 \) be a system of equations over \( \mathbb{F} \). Suppose \( \mathbb{F} \models \Sigma(h_n(\vec{x}), \vec{c}_n) = 1 \) for some \( (\vec{c}_n)_{n<\omega} \) in \( \mathbb{F} \).

Then there exists a closure of \( T, Cl(T) \) where \( Cl(G) := \langle \vec{x}, \vec{v}, \vec{z}, \vec{a} \rangle \), and a tuple of words \( \vec{w} = \vec{w}(\vec{x}, \vec{v}, \vec{z}, \vec{a}) \) such that for any morphism \( h : Cl(G) \to \mathbb{F} \)

\[
\mathbb{F} \models \Sigma(h(\vec{x}), h(\vec{w})) = 1
\]

The tuple \( \vec{w} \) is called a formal solution because it gives a formal way to find \( \vec{c} \) for a tuple \( \vec{b} = h(\vec{x}) \) when \( h : Cl(G) \to \mathbb{F} \) such that \( \Sigma(\vec{b}, \vec{c}) = 1 \) is satisfied.

Implicit function theorems can be generalized allowing at the place of a system of equations \( \Sigma(\vec{x}, \vec{y}) = 1 \) a first-order formula \( \phi(\vec{x}, \vec{y}) \) as long as \( \mathbb{F} \models \forall \exists^<\omega \exists^<\omega \phi(\vec{x}, \vec{y}) \) (see [BS19, Theorem 6.34]).

**Theorem 6.9**: Let \( \vec{x} \) be a tuple of variables in sort \( E \), and \( \vec{y} = (y_1, y_2, \ldots, y_k) \) be a tuple of variables in the basic sorts. Let \( \mathbb{F}^eq \models \forall \exists^<\omega \exists^<\omega \phi(\vec{x}, \vec{y}) \) and suppose there exists a test sequence \( (h_n)_{n<\omega} : G \to \mathbb{F} \) for the tower \( T := T(G, \mathbb{F}) \) and a sequence of tuples in the basic sorts \( (\vec{c}_n)_{n<\omega} \) such that \( \mathbb{F}^eq \models \phi([h_n(\vec{x})]_{E}, \vec{c}_n) \) for all \( n < \omega \).
Then there exist finitely many closures, \( Cl_1(T), \ldots, Cl_m(T) \), of \( T \) and for each closure a tuple of elements \( \bar{w} = (\bar{w}_1, \bar{w}_2, \ldots , \bar{w}_k) \) such that every test sequence \((g_n)_{n<\omega}\) of \( T \) that its images extend to solutions of \( \phi(\bar{x}, \bar{y}) \) is covered by one of the closures in the following sense: it extends to a test sequence for \( Cl_i(T) \), \((g_n)_{n<\omega} : Cl_i(G) \to F\) such that:

\[
F^{eq} \models \phi([G_n(\bar{x})]_{E}, [G_n(\bar{w}_1)]_{E_1}, \ldots , [G_n(\bar{w}_k)]_{E_k}) \text{ for all but finitely many } n.
\]

Since a closure of a hyperbolic tower coincides with the tower itself we get the following corollary.

**Corollary 6.10:** Let \( \bar{x} \) be a tuple of variables in sort \( E \), and \( \bar{y} = (y_1, y_2, \ldots , y_k) \) be a tuple of variables in the basic sorts. Let \( F^{eq} \models \forall \bar{x} \exists^{<\infty} \bar{y} \phi(\bar{x}, \bar{y}) \). Suppose there exists a test sequence \((h_n)_{n<\omega} : G \to F\) for a hyperbolic tower \( T := T(G, F) \) and a sequence of tuples in the basic sorts \((\bar{c}_n)_{n<\omega}\) such that \( F^{eq} \models \phi([h_n(\bar{y})]_{E}, \bar{c}_n) \) for all \( n < \omega \).

Then there exists a tuple of elements \( \bar{w} = (\bar{w}_1, \bar{w}_2, \ldots , \bar{w}_k) \) in \( G \) such that for any test sequence for \( T \), \((g_n)_{n<\omega} : G \to F\):

\[
F^{eq} \models \phi([g_n(\bar{y})]_{E}, [g_n(\bar{w}_1)]_{E_1}, \ldots , [g_n(\bar{w}_k)]_{E_k}) \text{ for all but finitely many } n.
\]

## 7 Main Proof

This section contains the proof of the main result, namely no infinite field is interpretable in the first-order theory of the free group. We tackle some special cases before moving to the full proof. In particular, we consider the abelian case and the hyperbolic case.

In the abelian case we prove that when an infinite interpretable set is contained in the abelian pouch of its envelope, it cannot be the domain of a field. This case is actually an essential part of the main proof, as it deals with the situation where the noncommutativity argument cannot be applied. In the hyperbolic case we prove that when the Diophantine envelope of an interpretable set contains a hyperbolic tower, then it cannot be the domain of an abelian group.

### 7.1 Abelian case

We first tackle the case where the interpretable set is contained in the images of the abelian pouches of its Diophantine Envelope. We recall that centralizers of nontrivial elements are pure groups \([RS10]\). In particular:

**Fact 7.1:** Let \( a \) be a nontrivial element of \( F \). Then \( C_F(a) \) is one-based.

**Definition 7.2:** Let \( T(G, F) \) be a tower and \( \bar{g} \) be a tuple in the basic sorts. Then we say that \( \bar{g} \) depends of the abelian pouch of \( T \) if each basic class in \( \bar{g} \) has a representative in the abelian pouch of \( T \).

For example, if \( g = g_1g_2g_3 \), where \( g_1 \) is a conjugacy class and \((g_2, g_3)\) is an \( m \)-coset class, then \( \bar{g} \) depends on the abelian pouch of \( T \) if \( g_1 \) can be conjugated in the abelian pouch and \([[(g_2, g_3)]_T^n] = [[(a_2, a_3)]_T^n] \) for some elements \( a_2, a_3 \) in the abelian pouch.

**Proposition 7.3:** Let \( x \) be a variable in sort \( S_E \) and \( \phi(x) \) be a first-order formula in \( L^{eq} \). We consider the formula \( \theta(\bar{z}) := \exists \bar{x} (\phi(x) \land R_E(x, \bar{z})) \), where \( \bar{z} \) is a tuple in the basic sorts, and
suppose for the corresponding splitting of the tuple of variables in the real sort \( \bar{y} = \bar{y}_1 \bar{y}_2 \ldots \bar{y}_\ell \) we get:

\[
\mathbb{F}^{eq} \models \forall \bar{y}(\theta(f_{E_1}(\bar{y}_1), f_{E_2}(\bar{y}_2), \ldots, f_{E_n}(\bar{y}_\ell)) \leftrightarrow \psi(\bar{y}))
\]

Let \( T_i(G_i, \mathbb{F}), \ldots, T_n(G_n, \mathbb{F}) \) be the towers in the Diophantine Envelope of \( \psi(\bar{x}) \).

Suppose, for each \( i \leq n \), the tuple of elements \( \bar{y}_i \) in \( G_i \) that are mapped in the definable set \( \psi(\bar{y}) \), i.e. \( \mathbb{F} \models \psi(h(\bar{y}_i)) \) for some \( h : G_i \to \mathbb{F} \), depends on the abelian pouch of \( T_i \).

Then \( \phi(x) \) cannot define an infinite field in \( \mathbb{F}^{eq} \).

**Proof.** We will show that \( \phi(x) \) is internal to a finite set of centralizers. Since, by Fact \[7.1\] centralizers are one-based, we get that \( \phi(x) \) is one-based \cite{Wag04} and consequently it cannot be the domain of an infinite field.

We fix a tower \( T_i(G_i, \mathbb{F}) = T(G, \mathbb{F}) \) and the tuple of elements \( \bar{y}_i = \bar{g} \). We may see the abelian pouch of \( T \) as a graph of groups, where we amalgamate free abelian groups on the nonabelian free group \( \mathbb{F} = \langle \bar{a} \rangle \). By our hypothesis each element \( w \) in the tuple \( \bar{g} \) can be put in the following form \( w = c_1(\bar{a}) d_1 c_2(\bar{a}) \ldots d_m c_{m+1}(\bar{a}) \), where the \( c_i \)'s belong to \( \mathbb{F} \) (and are possibly trivial) and the \( d_i \)'s in some abelian floor (but not in the peg). Any morphism from \( G \) to \( \mathbb{F} \) (that its restriction on \( \mathbb{F} \) is the identity) fixes the \( c_i \)'s and sends the \( d_i \)'s in powers of the corresponding pegs \( E_i = \langle e_i \rangle \). In each element \( w \) of \( \bar{g} \) we consider the \( d_i \)'s as variables and we define the formula \( B(\bar{y}) := \exists d_1, \ldots, d_m (\land [d_i, e_i] = 1 \lor \land y_i = w_i(\bar{a}, d_1, \ldots, d_m)) \).

For each tower \( T_i \) we construct a formula \( B_i(\bar{y}) \). We consequently have:

\[
\psi(\bar{y}) \models B_1(\bar{y}) \lor \ldots \lor B_n(\bar{y})
\]

In particular, there are \( n \) tuples of words \( \bar{w}_i(\bar{a}, d_1, \ldots, d_i) \) and a fixed finite family of centralizers such that for every solution \( \bar{b} \) of \( \psi(\bar{y}) \) there exist \( s_1, \ldots, s_i \) in the family of centralizers so that \( \bar{b} \) is one of the \( \bar{w}_i(\bar{a}, s_1, \ldots, s_i) \) for \( i \leq n \). In particular \( \psi(\bar{y}) \) is internal to a finite set of centralizers and consequently \( \phi(x) \) is.

---

### 7.2 Hyperbolic case

In this subsection we tackle the case of definable sets that their Diophantine envelope contains a hyperbolic tower.

**Proposition 7.4:** Let \( X := \phi(x) \) be a definable set in sort \( S_E \). Let \( T(G, \mathbb{F}) \) be a hyperbolic tower and \( \bar{x}, \bar{y}, \bar{a} \) a generating set for \( G \). Suppose there exists a test sequence for \( T, (h_n)_{n<\omega} : G \to \mathbb{F} \), such that \( \mathbb{F}^{eq} \models \phi(f_E(h_n(\bar{x}))) \), for all \( n < \omega \).

Then \( X \) cannot be given definably the structure of an abelian group.

**Proof.** Suppose, for the sake of contradiction, that \( X \) can be given definably an abelian group structure and \( \psi(x, y, z) \) be the formula that defines the graph of the abelian group operation on \( X \). By Theorem 3.4, the relation \( R_E \) gives a bounded number, say \( m \), of \( \ell \)-tuples in the basic sorts for any element in the sort \( S_E \).

We take an \( N \)-multiplet, \( T_N \), of the tower \( T \), where \( N \) is a number such that \( \lceil \frac{N/2}{\ell} \rceil \) is larger than \( m \ell + 1 \). We consider the formula \( \psi_N(x_1, x_2, \ldots, x_N, z) \) which defines the \( N \)-summation of elements in \( X \). We apply Theorem 6.10 to \( T_N \) and the formula \( \theta(x_1, x_2, \ldots, x_N, z_1, z_2, \ldots, z_\ell) := \exists z(\psi_N(x_1, x_2, \ldots, x_N, z) \land R_E(z, z_1, \ldots, z_\ell)) \). Hence we obtain a formal solution, which is a
We consider the formula:

\[
\mathbb{F}_\text{eq} \models \theta([g_n(\bar{x}_1)]_E, \ldots, [g_n(\bar{x}_N)]_E, [g_n(\bar{w}_1)]_{E_{i_1}}, \ldots, [g_n(\bar{w}_\ell)]_{E_{i_\ell}})
\]

for all but finitely many \(n\).

We first prove that in \(\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_\ell)\) there exists at least one \(\bar{w}_i\) such that its \(E_{i_\ell}\)-equivalence class does not have a representative in \(\mathbb{F}\). We assume not, hence for any test sequence of \(T_N\), the \(N\)-summation of the respective values is the same. We recall that \([h_n(\bar{x})]_E\) is infinite and actually any subsequence of \((h_n)_{n<\omega}\) is still a test sequence for \(T\). Hence, we may assume for some \(n\) large enough the images of the original sequence and the subsequence belong to different \(E\)-classes. We may substitute the values of the elements of the first tower in \(T_N\) with the values of the subsequence. Since the rest of the summands get the same values this is a contradiction.

Finally, we see the \(N\)-multilet tower as a star of groups with \(N\) rays. Permuting the values of the test sequences corresponds to permuting the rays hence we obtain at least \(m \cdot \ell + 1\) different equivalence classes for each basic equivalence class. But permuting the test sequences permutes the values of the \(N\)-summands and by commutativity it fixes their sum. This is a contradiction since a fixed sum value corresponds to at most \(m \cdot \ell\) many tuples, while for \(n\) large enough by Lemmas 6.3, 6.4, 6.5 we have strictly more than that. \(\square\)

7.3 General proof

In this final subsection we will prove the main result of the paper, namely no infinite field is interpretable in the first-order theory of the free group.

Theorem 7.5: The first-order theory of the free group does not interpret an infinite field.

Proof. Suppose, for a contradiction, that it does. Let \(\phi(x)\) be a formula in the language \(\mathcal{L}_{\text{eq}}\), in a single variable \(x\) of sort \(S_E\), that defines the domain of an infinite field. Consider the formula

\[
\theta(\bar{z}) := \exists x(\phi(x) \land R_E(x, \bar{z})),
\]

where \(\bar{z} = \ell\) and for each \([a]_E\) there exist \(m\)-many \(\ell\)-tuples in \(R_E(a, \bar{z})\). Let \(\psi(\bar{y})\) be a formula in the language \(\mathcal{L}\), where \(\bar{y}\) splits in \(\bar{y} = \bar{y}_1 \bar{y}_2 \ldots \bar{y}_\ell\), such that:

\[
\mathbb{F}_{\text{eq}} \models \forall \bar{y}(\psi(\bar{y}_1 \bar{y}_2 \ldots \bar{y}_\ell) \iff \theta(f_E(\bar{y}_1), \ldots, f_E(\bar{y}_\ell)))
\]

By Fact 6.6 there exist finitely many towers \(T_1(G_1, \mathbb{F}), \ldots, T_k(G_k, \mathbb{F})\), the Diophantine Envelope of \(\psi(\bar{y})\), that cover the formula \(\psi(\bar{y})\) and for each tower \(T_i(G_i, \mathbb{F}) = T(G, \mathbb{F})\) there exists a test sequence, \((h_n)_{n<\omega} : G \to \mathbb{F}\), such that \(\mathbb{F} \models \psi(h_n(\bar{y}))\), for all \(n < \omega\). We may assume, by Proposition 7.3, that \(\bar{y}\) does not depend on the abelian pouch of \(T\). We fix \(N\) such that \([\lfloor N/2 \rfloor] \) is larger than \(m \cdot \ell + 1\). Since \(\phi(x)\) is the domain of an infinite field, we consider the formula \(\oplus_N(x_1, x_2, \ldots, x_N, z)\) in \(\mathcal{L}_{\text{eq}}\) that defines \(N\)-summation on elements of \(\phi(x)\), and we consider the formula:

\[
\theta_N(\bar{z}_1, \ldots, \bar{z}_N, \bar{z}) := \exists x_1, x_2, \ldots, x_N, z(\oplus_N(x_1, x_2, \ldots, x_N, z) \land R_E(x_i, \bar{z}_i) \land R_E(z, \bar{z}))
\]

We take an \(N\)-multilet, \(T_N\), of the tower \(T\), and we apply Theorem 6.2 to \(T_N\), the formula \(\theta_N\) and all test sequences of \(T_N\) that restrict to test sequences in each of the \(N\) towers and moreover the values of their images belong to the \(N\) cartesian product of \(\phi(x)\). From the finite set of closures in which formal solutions for \(\bar{z}\) exist, we may use a diagonal argument.
and choose a closure and a test sequence such that when we permute the values of the test sequence it still extends to the closure of each tower (of the N many).

We first prove that the obtained formal solution does not depend on the abelian pouch of the closure of $T_N$. Suppose it does, then since $\bar{z}_i$ does not depend on the abelian pouch, we may refine the part of the test sequence that restricts to the first tower (without changing the values of the sequence on the abelian pouch). As a consequence, the original test sequence and the refinement will give different values to the first summand while keeping the same values for the rest of the $N - 1$ summands. Since the formal solution belongs to the abelian pouch (whose values have been kept the same) we obtain a contradiction.

We may now proceed as in Proposition 7.4. We view the closure of $T_N$ as a star of groups over the closure of the abelian pouch. Since, the formal solution does not depend on the abelian pouch, permuting the values of the test sequence corresponds to permuting the rays of the star of groups, hence we obtain at least $m \cdot \ell + 1$ different equivalence classes for each basic equivalence class. But, permuting the test sequences permutes the values of the $N$-summands and by commutativity it fixes their sum. This is a contradiction since a fixed sum value corresponds to at most $m \cdot \ell$ many tuples, while for $n$ large enough by Lemmas 6.3, 6.4, 6.5 we have strictly more than that.

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