Abstract. As one would anticipate from the realization of the $q$-commutators by difference operators, the states of maximum localization are smeared $\delta$-functions depending on $q$.

1. Introduction.

The Heisenberg uncertainty rules, depending on the standard commutators between conjugate observables, will of course be modified if the commutators are deformed. This question, which has been previously studied by A. Kempf,\(^1\) will be further examined here and illustrated for the $q$-harmonic oscillator.

2. $q$-Oscillator.

In the Fock representation the familiar equations of the $q$-oscillator are

\begin{align}
  a\bar{a} - q\bar{a}a &= \Delta \\
  H &= \frac{1}{2}(aa + \bar{a}\bar{a})
\end{align}

where we consider two realizations of (2.1) which we shall express in terms of eigenstates $|n\rangle$ of $H$:

(a) \hspace{1cm} a|n\rangle = \langle n \rangle^{1/2}|n - 1\rangle

(b) \hspace{1cm} a|n\rangle = [n]^{1/2}|n - 1\rangle
\[\bar{a}|n\rangle = [n + 1]^{1/2}|n + 1\rangle \quad (2.8)\]
\[H|n\rangle = \frac{1}{2}([n] + [n + 1]) \quad (2.9)\]
\[\Delta|n\rangle = q^{-n}|n\rangle \quad (2.10)\]

where
\[\langle n \rangle = \frac{q^n - 1}{q - 1} \quad (2.11)\]
\[[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad (2.12)\]

Set \(A = \begin{pmatrix} a & \bar{a} \end{pmatrix}\). Then Eq. (2.1) may be written as either
\[A^t \epsilon A = q^{-1/2} \quad \text{if} \quad \Delta = 1 \quad (2.13)\]
or
\[A^t \epsilon A = q^{-\left(\frac{a+1}{2}\right)} \quad \text{if} \quad \Delta = q^{-n} \quad (2.14)\]

where
\[\epsilon = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad (2.15)\]

The \(\epsilon\) matrix is invariant under transformation \(T\) belonging to \(SU_q(2)\):
\[T^t \epsilon T = T \epsilon T^t = \epsilon \quad (2.16)\]

Therefore any new vector \(X\) defined by
\[A = TX \quad (2.17)\]

will satisfy
\[X^t \epsilon X = \Delta \quad (2.18)\]

The transformation from Fock observables to configuration space may be accomplished by choosing
\[X = \begin{pmatrix} D \\ x \end{pmatrix} \quad p = \frac{\hbar}{i} D \quad (2.19)\]
Then
\[qxp - px = i\hbar \Delta \quad (2.20)\]
If $\Delta = 1$, then (2.20) requires

$$D^q = \frac{1}{x} (\theta)^q = \frac{1}{x} \frac{q^\theta - 1}{q - 1}.$$ \hfill (2.21)

Here $\theta$ is the dilatation operator

$$\theta = x \frac{d}{dx}.$$ \hfill (2.22)

If $\Delta = q^{-n}$

$$D = \frac{1}{x} [\theta]^q = \frac{1}{x} \frac{q^\theta - q^{-\theta}}{q - q^{-1}}.$$ \hfill (2.23)

By (2.21)

$$D^q f(x) = \frac{f(qx) - f(x)}{(q - 1)x}.$$ \hfill (2.24)

By (2.23)

$$D f(x) = \frac{f(qx) - f(q_1 x)}{(q - q_1) x}$$ \hfill (2.25)

where

$$q_1 = q^{-1}.$$ 

Although the difference operators $D^q$ and $D$ convey the same idea, it turns out that $D$ is the proper choice in constructing the Fourier transform between configuration and momentum space. (In previous work$^2$ we used the form (2.24) corresponding to $\langle \ \rangle$). We shall now use (2.25) or $[ \ ]$. Then

$$a = \alpha D + \beta x$$ \hfill (2.26)

$$\bar{a} = -q_1 \beta D + \bar{\alpha} x$$ \hfill (2.27)

where

$$T = \begin{pmatrix} \alpha & \beta \\ -q_1 \bar{\beta} & \bar{\alpha} \end{pmatrix}$$ \hfill (2.28)

and $D$ is given by (2.23).

3. Configuration Space Representation of Fock States.

These are obtained by the Fock ladder operators.$^3$ The ground state was previously determined by

$$a^q |0\rangle = (\beta x + \alpha D^q)|0\rangle = 0$$ \hfill (3.1)
and the higher states by the raising operators. We illustrate the different procedures required by \( D^q \) and \( D \) by displaying the ground states in both cases. The solution of (3.1) for \( \langle x|0 \rangle = \psi_0^q(x) \) is
\[
\psi_0^q(x) = \prod_{o}^{\infty} (1 + K q^2)^{-1} \psi^q(0), \quad q < 1
\] (3.2a)
where
\[
K = (1 - q) \alpha^{-1} \beta x^2.
\] (3.2b)

Let us now obtain the ground state \( \psi_o(x) \) corresponding to \( D \) by solving
\[
(\beta x + \alpha D) \psi_o(x) = 0
\] (3.3)
where \( D \) is given by (2.23). Then
\[
\psi_o(qx) - \psi_o(q_1 x) = K' x^2 \psi_o(x)
\] (3.4)
where
\[
K' = (q_1 - q) \alpha^{-1} \beta.
\] (3.5)

The difference equation (3.4) relates \( \psi_o(x) \) at three distinct points. To reduce this equation to the previous case where there are only two distinct points define
\[
\varphi(qx) = \psi_o(qx)/\psi_o(x).
\] (3.6)

Then by (3.4)
\[
\varphi(qx) - \varphi(q_1 x) = K' x^2
\] (3.7)
or
\[
\varphi(x) = \varphi(q^2 x) - K' q^2 x^2.
\] (3.8)

By iterating this relation one obtains
\[
\varphi(qx) = \varphi(q^{2n+1} x) - K' q^4 x^2 q^{4n} - 1 \quad q^n - 1.
\] (3.9)
Letting \( n \to \infty \) one finds
\[
\varphi(qx) = \varphi(0) - \frac{K' q^4 x^2}{1 - q^4}
\] (3.10)
since $q < 1$. As $\varphi(0) = 1$, one may by (3.6) rewrite (3.10) in the following form

$$
\psi_o(qx) = \left[1 - \frac{K'q^4x^2}{1 - q^4}\right]\psi_o(x) .
$$

(3.11)

Again by iteration $\psi_o(x)$ may be found as the infinite product

$$
\psi_o(x) = \prod_{o} (1 - Lq^{2s})^{-1}\psi_o(0)
$$

(3.12)

where

$$
L = \frac{K'q^4x^2}{1 - q^4}
= \frac{q^3x^2}{1 + q^2}\alpha^{-1}\beta .
$$

(3.13)

Both (3.2) and (3.12) are formally convergent for all $x$. Since these series actually lie in the $SU_q(2)$ algebra, however, their interpretation as state functions must depend on supplementary rules for their numerical valuation, as discussed elsewhere.$^{2,3}$

By applying the raising operator the complete set of excited states may be found. In a similar way the complete set of momentum states may be found. The ground states are $q$-Gaussian and all states vanish at $x = \infty. ^{2}$

4. Alternative Transition to Configuration Space.

If the position and momentum variables are defined by the standard expressions$^{1}$ one has

$$
x = L(\bar{a} + a)
$$

$$
p = iK(\bar{a} - a) .
$$

(4.1)

If $a$ and $\bar{a}$ were hermitian conjugate then $x$ and $p$ would be separately hermitian. Since $q \neq 1$, however, $a$ and $\bar{a}$ are not hermitian conjugate, and $x$ and $p$ are not hermitian. The transformation (4.1) does not preserve the $q$-commutator. The usual commutator is

$$
(x, p) = i\hbar \left[\Delta + (q - 1)\frac{1}{4}\left(\frac{x^2}{L^2} + \frac{p^2}{K^2}\right)\right]
$$

(4.2)

if one sets

$$
LK = \frac{\hbar q + 1}{2} .
$$

(4.3)

The corresponding uncertainty relations are$^{1}$

$$
\left(\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle\right)^{1/2} \geq \frac{\hbar^2}{4} \left[\Delta + (q - 1)\left(\frac{\langle x^2\rangle}{4L^2} + \frac{\langle p^2\rangle}{4K^2}\right)\right] .
$$

(4.4)
Since $x$ and $p$ are not the usual hermitian observables supplementary rules of interpretation would also be necessary here.

The Fock commutators (2.1) imply the failure of translational invariance and the modified uncertainty relations as exemplified in (4.4), if the transition to configuration space is made by the standard relations (4.1). Failure of translational invariance and deformed uncertainty relations must always appear, however, in whatever way the transition from (2.1) to configuration space is made. Let us therefore examine this question if $x$ and $p$ are defined by (2.26) and (2.27). To do this we shall next introduce a $q$-Fourier transform.

5. $q$-Fourier Transforms.

One may define “eigenstates” of momentum by

$$\frac{\hbar}{i} D_x \psi_p(x) = p\psi_p(x) .$$  \hfill (5.1)

Then

$$\psi_p(x) = \sum \frac{(ipx/\hbar)^n}{[n]!} .$$  \hfill (5.2)

If $D^q_x$ instead of $D_x$ had appeared in (5.1), then $\langle n \rangle$ would have replaced $[n]$ in (5.2). The series (5.2) is convergent for large $x$. The corresponding $\langle n \rangle$ series is not convergent for large $x$; for this reason the symmetric derivative is required.

In standard quantum mechanics one transforms from configuration to momentum space by writing the wave function in $x$-space as a sum over momentum eigenstates. The corresponding procedure here is

$$\psi(x) = \sum_p \psi_p(x) \varphi(p) .$$  \hfill (5.3)

In Dirac notation

$$\psi_p(x) = \langle x | p \rangle$$  \hfill (5.4)

and (5.3) becomes

$$\langle x | \rangle = \sum_p \langle x | p \rangle \langle p | \rangle .$$  \hfill (5.5)

Here we choose $\Sigma_p$ to be the $q$-integral. Then

$$\psi(x) = \int_{-\infty}^{\infty} \mathcal{E}(ipx) \varphi(p) dq_p$$  \hfill (5.6)
where we have set
\[ \langle x|p \rangle = \mathcal{E}(ipx) \]  \hspace{1cm} (5.7)

Then (5.6) is the $q$-Fourier decomposition of $\psi(x)$. In order that the series for the kernel $\mathcal{E}(ipx)$ be convergent for all $x$ it is necessary to use (5.2) instead of the $\langle n \rangle$ series.

To do integration by parts we need the Leibniz rule for $D_x$, namely:
\[ D_x[f(x)g(x)] = f(qx)D_xg(x) + g(q_1x)D_xf(x) \]
or
\[ = f(q_1x)D_xg(x) + g(qx)D_xf(x) \]  \hspace{1cm} (5.8)

Let us test (5.6) by computing
\[ a_x\psi = (\alpha D_x + \beta x) \int_{-\infty}^{\infty} \mathcal{E}(ipx)\varphi(p)dqp \]  \hspace{1cm} (5.9)
\[ = \int_{-\infty}^{\infty} (\alpha ip\varphi(p) + \beta \mathcal{E}(ipx)\varphi(p)dqp \]  \hspace{1cm} (5.10)
where we have used
\[ D_x\mathcal{E}(ipx) = ip\mathcal{E}(ipx) \]  \hspace{1cm} (5.11)

Note
\[ \int_{-\infty}^{\infty} D_p(\mathcal{E}(ipx)\varphi(p))dqp = (\mathcal{E}(ipx)\varphi(p))_{-\infty}^{\infty} \]  \hspace{1cm} (5.12)
\[ = 0 \]  \hspace{1cm} (5.13)

since $\int_q$ is inverse to $D$ and since
\[ \varphi(\infty) = \varphi(-\infty) = 0 \]  \hspace{1cm} (5.14)

By (5.13) and (5.8)
\[ \int_{-\infty}^{\infty} (\mathcal{E}(iq_1px)D_p\varphi(p) + \varphi(qp)D_p\mathcal{E}(ipx))dqp = 0 \]  \hspace{1cm} (5.15)

Corresponding to (5.11) we have
\[ D_p\mathcal{E}(ipx) = i\alpha \mathcal{E}(ipx) \]  \hspace{1cm} (5.16)
By (5.15) and (5.16) we have
\[ \int_{-\infty}^{\infty} \varphi(qp)ix\mathcal{E}(ipx)dq = -\int_{-\infty}^{\infty} \mathcal{E}(iq1px)D_p\varphi(p)dq . \quad (5.17) \]

Set
\[ p' = qp \]
\[ x' = q_1x . \quad (5.18) \]

Then
\[ \int_{-\infty}^{\infty} \varphi(p')ix'\mathcal{E}(ip'x')dq' = -\int_{-\infty}^{\infty} \mathcal{E}(ipx')D_p\varphi(p)dq \]
\[ \text{or} \]
\[ \int_{-\infty}^{\infty} \varphi(p)ix\mathcal{E}(ipx)dq = -\int_{-\infty}^{\infty} \mathcal{E}(ipx)D_p\varphi(p)dq . \quad (5.20) \]

By (5.10) and (5.20)
\[ a_x\psi = \int_{-\infty}^{\infty} \mathcal{E}(ipx)i(\alpha p + \beta D_p)\varphi dq \]
\[ \text{or} \]
\[ a_x\psi = \int_{-\infty}^{\infty} \mathcal{E}(ipx)a_p\varphi dq \]
where
\[ a_p = i(\alpha p + \beta D_p) . \quad (5.23) \]

Let us next check the following ansatz for the reverse transformation
\[ \varphi(p) = \int_{-\infty}^{\infty} \mathcal{E}(-ipx)\psi(x)dq. \quad (5.24) \]

Then
\[ a_p\varphi(p) = \int_{-\infty}^{\infty} (i\alpha p + \beta x)\mathcal{E}(-ippx)\psi(x)dqK . \quad (5.25) \]

Again the right-hand side may be transformed to give
\[ a_p\varphi(p) = \int_{-\infty}^{\infty} \mathcal{E}(-ipx)a_x\psi(x)dq . \quad (5.26) \]

By (5.22) and (5.26) the ground states in \( x \) and \( p \) space correspond. Thus
\[ a_x\psi_0(x) = 0 \quad \leftrightarrow \quad a_p\varphi_0(p) = 0 . \quad (5.27) \]
The argument may be completed with the aid of the raising operators.

6. The $q$-Delta Function.

In Dirac notation, Eqs. (5.6) and (5.24) may be written

$$\langle x | \rangle = \int \langle x | p \rangle d_q p \langle p | \rangle$$

(6.1)

$$\langle p | \rangle = \int \langle p | x \rangle d_q x \langle x | \rangle$$

(6.2)

Combining these two equations we have

$$\langle x | \rangle = \int \langle x | p \rangle d_q p \int \langle p | x' \rangle d_q x' \langle x' | \rangle$$

(6.3)

or

$$\langle x | \rangle = \int \delta_q(x, x')d_q x' \langle x' | \rangle$$

(6.4)

where

$$\delta_q(x, x') = \langle x | x' \rangle_q$$

$$= \int \langle x | p \rangle d_q p \langle p | x' \rangle$$

(6.5)

$$= \int \mathcal{E}(i p x)\mathcal{E}(-i p x')d_q p .$$

To evaluate the integral (6.5) note

$$\int D_p \mathcal{E}(i p x)\mathcal{E}(-i p x')d_q p = (\mathcal{E}(i p x)\mathcal{E}(-i p x'))_{-\infty}^{\infty} .$$

(6.6)

The left side of (6.6) is

$$\int \mathcal{E}(i p x)D_p \mathcal{E}(-i p x') + \mathcal{E}(-i q_1 p x')D_p \mathcal{E}(i p x))d_q p$$

(6.7)

$$= \int \mathcal{E}(i q p x)(-i x')\mathcal{E}(-i p x') + \mathcal{E}(-i q_1 p x')ix\mathcal{E}(i p x))d_q p .$$

Set $p' = q p$ in the first integral on the right side of (6.7). Then (6.7) becomes

$$\int \mathcal{E}(i p' x)(-i x')\mathcal{E}(-i p' q_1 x')q_1 d p' + \int \mathcal{E}(-i q_1 p x')ix\mathcal{E}(i p x))d_q p$$

(6.8)

$$= \int \mathcal{E}(i p x)\mathcal{E}(-i q_1 p x')(-i q_1 x' + i x)d_q p .$$
Set \( q_1x' = x'' \). Then (6.8) becomes

\[
i(x - x'') \int \mathcal{E}(ipx)\mathcal{E}(-ipx'')dqdp .
\]  

(6.9)

Therefore

\[
\int \mathcal{E}(ipx)\mathcal{E}(-ipx'')dqdp = \left( \frac{\mathcal{E}(ipx)\mathcal{E}(-iqpx'')}{i(x - x'')} \right)_\infty
\]

or

\[
\delta_q(x, x') = \lim_{P \to \infty} \left( \frac{\mathcal{E}(iPx)\mathcal{E}(-iqPx') - \mathcal{E}(-iPx)\mathcal{E}(iqPx')}{i(x - x')} \right).
\]  

(6.11)

If \( q = 1 \), \( \delta_q(x, x') \) becomes the Dirac \( \delta \)-function

\[
\delta(x - x') = 2 \lim_{P \to \infty} \left( \frac{\sin Px}{x} \right).
\]  

(6.12)

In this case \( \langle x|x' \rangle \) vanishes unless \( x = x' \). It is also translationally invariant.

If \( q \neq 1 \), \( \delta_q(x, x') \) is not translationally invariant. Set \( x' = 0 \). Then

\[
\langle x|0 \rangle_q = \delta_q(x, 0) = \lim_{P \to \infty} P \left( \frac{\sin qPx}{x} \right).
\]  

(6.13)

Set \( x = 0 \). Then

\[
\langle 0|x' \rangle_q = \delta_q(0, x') = \lim_{P \to \infty} P \left( \frac{\sin(qPx')}{x'} \right).
\]  

(6.14)

The state of maximum localization is therefore a smeared \( \delta \)-function, the \( q \) \( \delta \)-function. This result is consistent with the lattice properties implied by the implementation of the momentum operator by a difference operator. The smeared \( \delta \)-function is also in agreement with the higher uncertainty bound expressed by Eq. (4.4). These results are purely formal, however, and the pair \( (x, p) \) defined by either ((2.26), (2.27)) or (4.1) would require additional rules for their physical interpretation.

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References.

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