A new reconstruction method in integral geometry

V. P. Palamodov
Tel Aviv University

Abstract: A general method for analytic inversion of geometric integral transforms is proposed.

Key words: curve family, generating function, principal value integral, trigonometric polynomial, reconstruction formula

MSC 53C65 44A12 65R10

1 Introduction

There are numerous applications of integral geometry to various problem of image reconstruction, in particular to X-ray, emission, magnetic resonance, wave, acoustic, thermo and photoacoustic, scattering, Doppler tomographies as well as to radar technique and texture analysis. We propose here a general method of analytic inversion formulas for various transformations in integral geometry. Several examples for the case of integral transforms in a plane domain are given.

2 Curves and integrals

Let $X$ and $\Sigma$ be smooth 2-manifolds and $\Phi$ be a smooth real function defined in $X \times \Sigma$ such that $d_x\Phi \neq 0$. For any point $\sigma \in \Sigma$ the set $F(\sigma) = \{x \in X; \Phi(x,\sigma) = 0\}$ is a smooth curve in $X$. We call $\Phi$ generating function of the curve family $\{F(\sigma), \sigma \in \Sigma\}$.

Let $dS$ be the Riemannian area form; we define a Funk-Radon transform generated by $\Phi$ by

$$M_\Phi f(\sigma) = \int \delta(\Phi(x;\sigma)) f dS = \int_{F(\sigma)} \frac{f dS}{d_x\Phi(x;\sigma)}$$

for continuous functions $f$ compactly supported in $X$. The quotient $f dS/d_x\Phi$ denotes an arbitrary 1-form $q$ such that $d_x\Phi \wedge q = f dS$. It is defined up to a term $h d_x\Phi$ where $h$ is a continuous function. An orientation of a curve $F(\sigma)$ is defined by means of the form $d_x\Phi$ and the integral of $q$ over $F(\sigma)$ is uniquely defined. Let $g$ be a Riemannian metric in a manifold $X$; the form $d_g s = \sqrt{g(dx)}$ is the Riemannian line element. We have then

$$M_\Phi f(\sigma) = \int_{F(\sigma)} \frac{f d_g s}{|\nabla_\Phi(x;\sigma)|}$$
where $|\nabla g a| = \sqrt{g(\nabla a)}$ is the Riemannian gradient of a function $a$. Suppose that the gradient factorizes through $X$ and $\Sigma$ that is $|\nabla \Phi (x, \sigma)| = m(x) \mu(\sigma)$ for some positive continuous functions $m$ in $X$ and $\mu$ in $\Sigma$. It follows that data of the Funk-Radon transform is equivalent to data of Riemannian curve integrals:

$$ R_g f(\sigma) = \int_{F(\sigma)} f d_s, \sigma \in \Sigma $$

since $Rf(\sigma) = \mu(\sigma) M_\Phi(mf)(\sigma)$. The reconstruction problem of a function $f$ from Riemannian integrals $R_g f$ is then reduced to inversion of the operator $M_\Phi f$.

### 3 Main theorem

We assume now that $\Sigma = \mathbb{R} \times S^1$ and a generating function is linear in the first argument: $\Phi(x, \lambda, \varphi) = \lambda + \psi(x, \varphi)$, $\lambda \in \mathbb{R}, \varphi \in S^1 = \{0 \leq \varphi < 2\pi\}$. We call this function regular if (i) $d_x \psi \wedge d_x \psi' \neq 0$ in $X \times S^1$ and (ii) there are no conjugated points, that is the equations $\psi(x, \varphi) = \psi(y, \varphi)$ and $\psi'(x, \varphi) = \psi'_y(x, \varphi)$ are fulfilled for no $x \neq y \in X, \varphi \in S^1$.

**Theorem 3.1** Let $\Phi(x, \lambda, \varphi) = \lambda + \psi(x, \varphi)$ be a regular generating function in $X \times \Sigma$ such that the function $\psi$ is analytic in $X$ and the principal value integral

$$ N(x, y) \doteq (P) \int_0^{2\pi} \frac{d\varphi}{(\psi(x, \varphi) - \psi(y, \varphi))^2} \doteq \text{Re} \int \frac{d\varphi}{(\psi(x, \varphi) - \psi(y, \varphi) \pm i0)^2} \quad (1) $$

vanishes for any $x \neq y \in X$. Then the reconstruction formula

$$ f(x) = -\frac{1}{4\pi^2 D(x)} (P) \int_0^{2\pi} \int_{\mathbb{R}} \frac{M_\Phi f(\lambda, \varphi)}{\Phi^2(x, \lambda, \varphi)} d\lambda d\varphi \quad (2) $$

$$ D(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{|\nabla g \psi(x, \varphi)|^2} \quad (3) $$

holds for an arbitrary function $f \in L_2(X)_{\text{comp}}$. The integral (2) converges in $L_2(X)_{\text{loc}}$.

**Remark 1.** A more invariant form of (2) is a reconstruction of the density $t$.

$$ f(x) dS = -\frac{1}{4\pi^2 D(x)} (P) \int_0^{2\pi} \int_{\mathbb{R}} \frac{M_\Phi f(\lambda, \varphi)}{\Phi^2(x, \lambda, \varphi)} d\lambda d\varphi $$

Here the quotient $dS/D(x)$ depends only of the conformal class of the Riemannian metric $g$.

**Remark 2.** G. Beylkin studied "the generalized Radon transform" in $\mathbb{R}^n$ [3] which coincides with our Funk-Radon transform $M_\Phi$ in a Euclidean space (see also the last
implies (3) since $D$ is a constant. Then the right-hand side of (2) is a parametrix. Beylkin’s parametrix gives a high frequency approximation to a solution but not an exact inversion due to limitation of the method of FIO. The explicit form of the nucleus $N(x, y)$ does not appear in Beylkin’s approach but is important in our method. Note that if the nucleus does not vanish it is still a smooth function in the complement to the diagonal, since condition (ii) keeps the number of real zeros of $\psi(x, \varphi) - \psi(y, \varphi)$ constant. Then the right-hand side of (2) is a parametrix.

**Proof of Theorem.** To simplify our arguments we assume that $X$ is an open set in $\mathbb{R}^2$ and $g$ is the Euclidean metric, the area form is denoted $\text{d}x$. Let $q$ be a 1-form in $X$ such that $\text{d}\psi \wedge q = \text{d}x$; we have $Mf(\lambda, \varphi) = \int_{\lambda + \psi = 0} f q$. For an arbitrary $x \in X$ and any function $f$ that vanishes in a neighborhood of $x$ we find

$$\int_X \frac{Mf(\lambda, \varphi)}{\Phi^2(x; \lambda, \varphi)} \text{d}x = \int_{\Sigma^1} \text{d}\varphi \int_{\mathbb{R}} \left( \int_{\lambda + \psi = 0} f(y)q(y) \right) \frac{\text{d}\lambda}{(\lambda + \psi(x, \varphi))^2}$$

$$= - \int_X \left( \int_0^{2\pi} \frac{\text{d}\varphi}{(\psi(x, \varphi) - \psi(y, \varphi))^2} \right) f(y)q(y) \wedge \text{d}\psi(y, \varphi)$$

$$= \int_X N(x, y) f(y) \text{d}y$$

since $\text{d}\lambda = -\text{d}\psi$ as $\lambda + \psi(y, \varphi) = 0$ and $\text{d}\lambda \wedge q = -\text{d}y$ by definition of $q$. It follows that the function $N$ is the off-diagonal nucleus of $I$ and it vanishes since of the assumption. Therefore the nucleus $I(x, y)$ of the operator $I$ (see (3)) is supported in the diagonal and by Lemma 3.3 below we have $I(x, y) = a(x) \delta_x(y)$ for some continuous function $a$. To calculate this function at a point $x_0 \in X$ we apply $I$ to a density $h_\varepsilon$ that is equal to $\text{d}x$ in $\varepsilon$-neighborhood $X_\varepsilon$ of $x_0$ and $h_\varepsilon = 0$ in the complement:

$$Ih_\varepsilon(x_0) = \int_{X_\varepsilon} \text{d}x \int_0^{2\pi} \frac{\text{d}\varphi}{(\psi(x, \varphi) - \psi(x_0, \varphi))^2} = \int_0^{2\pi} \text{d}\varphi \int_{X_\varepsilon} \frac{\text{d}x}{(\psi(x, \varphi) - \psi(x_0, \varphi))^2}$$

**Lemma 3.2** We have for arbitrary $\varphi \in S^1$, $x_0 \in X$ and small $\varepsilon$

$$D(x_0, \varphi) \doteq \int_{X_\varepsilon} \frac{\text{d}x}{(\psi(x, \varphi) - \psi(x_0, \varphi))^2} = \frac{2\pi}{|\nabla \psi(x_0, \varphi)|^2} + O(\varepsilon)$$

where $O(\varepsilon) \leq C\varepsilon$ where $C$ does not depend of $\varphi$.

Integrating over $\varphi$ yields the equation $D(x_0) = -\int |\nabla \psi(x_0, \varphi)|^{-2} \text{d}\varphi + O(\varepsilon)$. This implies (3) since $D(x_0)$ does not depend of $\varepsilon$.

**Proof of Lemma.** Replacing $\psi$ by $\psi/|\nabla \psi(x_0, \varphi)|$ we can assume that $|\nabla \psi(x_0, \varphi)| = 1$. Shift the origin to the point $x_0$ and suppose first that $\psi(x, \varphi)$ is linear in $x$. By an
orthogonal transformation we can assume that \( \psi(x, \varphi) = x_1 \). In this case we have

\[
D(x_0, \varphi) = \text{Re} \int_{X_\varepsilon} \frac{dx}{(x_1 + i0)^2} = -\text{Re} \int_{-\varepsilon}^{\varepsilon} dx_2 \frac{1}{x_1 + i0} \frac{1}{\sqrt{x^2 - x_2^2}}
\]

\[
= -2 \int_{-\varepsilon}^{\varepsilon} \frac{dx_2}{\sqrt{x^2 - x_2^2}} = -\frac{2\pi}{|\nabla \psi(x_0, \varphi)|^2}
\]

In the general case we can write \( \psi^2(x, \varphi) = x_1^2 - \rho(x) \), where \( \rho(x) = O(|x|^3) \) is analytic in 3\( \varepsilon \)-neighborhood of the origin if \( \varepsilon \) is sufficiently small. We have

\[
\frac{1}{\psi^2} = \frac{1}{x_1^2} + \sigma, \quad \sigma = \sum_{n=1}^{\infty} \frac{\rho^n}{x_1^{2n+2}}
\]

and

\[
\int \frac{dx}{(\psi + i0)^2} = \int \frac{dx}{(x_1 + i0)^2} + \int \sigma(x_1 + i0, x_2) dx
\]  \tag{4}

The first integral in the right-hand side is equal to \( 2\pi \) and shall show the second integral is small. The field

\[
v(x) = \left( |x_2| + \sqrt{x^2 - |x|^2}, -\frac{x_2}{|x_2|x_1} \right)
\]

is defined and continuous for \( x \in X_\varepsilon, x_2 \neq 0 \). For any \( t, \ 0 \leq t \leq 1 \) we define 2-chains

\[
V_{t,\pm} = \left\{ z = z(x, t) = \frac{x + itv(x)}{\sqrt{1 - t^2}}, \pm x_2 \geq 0, \ |x| = \varepsilon \right\}
\]

\[
W_t = \left\{ z = \frac{1}{\sqrt{1 - t^2}} \left( x_1 + it \sqrt{\varepsilon^2 - x_1^2}, istx_1 \right), \ |x_1| \leq \varepsilon, \ -1 \leq s \leq 1 \right\}
\]

oriented by the orientation of \( X_\varepsilon \). The boundary of the chain \( V_t = V_{t, +} + V_{t, -} + W_t \) is the image of \( \partial X_\varepsilon \) under the map \( x \mapsto z(x, t) \). The function \( \sigma \) has a holomorphic continuation to the domain \( Z_+ = \{ z \in \mathbb{C}^2, |z| < 3\varepsilon, y_1 > 0 \} \). The form \( d\sigma \) is holomorphic in \( Z_+ \) hence \( d\sigma \wedge dz = 0 \). Consider an open 3-chain \( Z_\delta = \bigcup_{t=\delta}^{1/2} V_t \subset Z \) where \( 0 < \delta \leq 1/2 \). It is contained in \( Z_+ \) since \( y_1 \equiv \text{Im} \ z_1 = t \left( |x_2| + \sqrt{\varepsilon^2 - |x|^2} \right) > 0 \) and \( |v| \leq 2\varepsilon \). By Stokes’ we have

\[
\int_{\partial Z_\delta} \sigma dz = \int_{Z_\delta} d\sigma \wedge dz = 0
\]

The boundary of \( Z_\delta \) is equal to \( V_{1/2} - V_\delta + S + T \) where

\[
S = \left\{ z = \frac{x + itv(x)}{\sqrt{1 - t^2}}, \ |x| = \varepsilon, \ \delta \leq t \leq 1/2 \right\},
\]

\[
T = \left\{ z = \frac{(\varepsilon, ist\varepsilon)}{\sqrt{1 - (st)^2}}, \varepsilon = \pm 1, \ \delta \leq t \leq 1/2, -1 \leq s \leq 1 \right\}
\]
Therefore
\[ \int_{V_{1/2}} \sigma \, dz - \int_{V_5} \sigma \, dx + \int_{S+T} \sigma \, dz = 0 \]

For any point \( z \in S \) we have
\[ z^2 = z_1^2 + z_2^2 = (1 - t^2)^{-1} (|x|^2 + 2it \langle x, v \rangle - t^2 |v|^2) = (1 - t^2)^{-1} (|x|^2 - t^2 |x|^2) = |x|^2 = \varepsilon^2 \]
since \( \varepsilon^2 - |x|^2 = 0 \) and \( \langle x, v \rangle = 0 \) on \( \partial X_\varepsilon \). The same equation holds in \( T \). It follows that the chain \( S + T \) is contained in the complex algebraic curve \( z^2 = \varepsilon^2 \). This implies that the integral of the holomorphic form \( \sigma \, dz \) over \( S + T \) vanishes. This yields for an arbitrary \( \delta > 0 \)
\[ \int_{V_\delta} \delta \sigma \, dz = \int_{V_{1/2}} \sigma \, dz = \frac{\sum_{n=1}^{\infty} \int_{V_{1/2}} \rho^n}{\varepsilon_1^{2n+2}} \]
The left-hand side tends to the second term of (4) as \( \delta \to 0 \). The right-hand side can be estimated from above by the sum
\[ \sum_{n=1}^{\infty} \left| \int_{V_{1/2}} \frac{\rho^n \, dz}{\varepsilon_1^{2n+2}} \right| \leq \int_{V_{1/2}} dS \sum_{n=1}^{\infty} M_{\varepsilon} \max_{z \in V_{1/2}} |z|^{-2n-2} \]
where
\[ M_{\varepsilon} = \max_{z} |\rho| \leq C \varepsilon^3 \]
and \( dS \) is the Euclidean area element. The area of the chain \( V_1 \) is estimated by \( \text{const} \, \varepsilon^2 \) since \( |\nabla v| \) is bounded in \( X_\varepsilon \). For we have any point \( z \in V_1 \)
\[ |z_1|^2 = x_1^2 + y_1^2 \geq \frac{4}{3} \left( x_1^2 + \frac{1}{4} \left( x_2^2 + \varepsilon^2 - |x|^2 \right) \right) \geq \frac{1}{3} \varepsilon^2 \]
This yields
\[ \left| \int_{V_{1/2}} \sigma \, dz \right| \leq C_0 \sum_{n=1}^{\infty} \frac{\varepsilon^2 (3M_{\varepsilon})^n}{\varepsilon^{2(n+1)}} \leq C_0 \sum_{n=1}^{\infty} C^n \varepsilon^n = C_0 \frac{C \varepsilon}{1 - C \varepsilon} = O (\varepsilon) \]
if \( \varepsilon < C^{-2} \). This completes proofs of Lemma 3.2 and Theorem 3.1.

**Lemma 3.3** The integral transform
\[ I f (x) = (P) \int_0^{2\pi} \int_{\mathbb{R}} \frac{M_{\phi} f (\lambda, \varphi)}{\Phi^2 (x; \lambda, \varphi)} \, d\lambda \, d\varphi \]
is a continuous operator \( L_2 (X)_{\text{comp}} \to L_2 (X)_{\text{loc}} \).
Proof of Lemma. The condition (i) can be written in the form $J_{x;\lambda,\varphi} (\Phi) \neq 0$ in $F$ where

$$J_{x;\lambda,\varphi} (\Phi) = \det \begin{pmatrix} \frac{\partial^2 \Phi}{\partial x_1 \partial \lambda} & \frac{\partial^2 \Phi}{\partial x_1 \partial \varphi} & \frac{\partial \Phi}{\partial x_1} \\ \frac{\partial^2 \Phi}{\partial x_2 \partial \lambda} & \frac{\partial^2 \Phi}{\partial x_2 \partial \varphi} & \frac{\partial \Phi}{\partial x_2} \\ \frac{\partial \Phi}{\partial \lambda} & \frac{\partial \Phi}{\partial \varphi} & 0 \end{pmatrix}$$

(6)

$x_1, x_2$ are coordinates in $X$. According to [4], Theorem 25.3.1 the map $M_\Phi : L^2 (X)_{\text{comp}} \to L^2 (\Sigma)_{\text{loc}}$ is a Fourier integral operator of order $-1/2$ since the corresponding Lagrange manifold is locally a graph of a canonical transformation (see details in [10]; the bundles $\Omega^{1/2} (X)$ and $\Omega^{1/2} (\Sigma)$ are trivial in our case). The image of this operator is contained in $L^2 (\Sigma)_{\text{comp}}$ since for an arbitrary function $f \in L^2 (X)_{\text{comp}}$ the support of $M_\Phi f$ is contained in $\Theta = \pi^*_\Sigma \pi^{-1}_X (\text{supp} f)$ where $\pi_X$ and $\pi_\Sigma$ are projections of $F$ to $X$ and $\Sigma$ respectively. The projection $\pi_X$ is obviously proper. The integral transform

$$Sh (x) = \int_\Sigma \frac{h (\lambda, \varphi) \, d\lambda d\varphi}{\Phi^2 (x; \lambda, \varphi)}$$

can be written by means of a Fourier integral operator

$$Sh (x) = -\frac{1}{2} \int_\Sigma \int_\mathbb{R} \exp (it \Phi (x, \lambda, \varphi)) |t| h (\lambda, \varphi) \, dt d\lambda d\varphi$$

with the phase function $\Phi$. The corresponding Lagrange manifold coincides with that of $M_\Phi$ with source and target spaces interchanged. The order of $S$ equals $1/2$ due to the factor $|t|$. It follows that the composition $I = SM_\Phi$ is well defined as a PDO of order $0$. ▶

4 Integrals of rational trigonometric functions

We study now the condition $N (x, y) = 0$. A function

$$t (\varphi) = \sum_{m=0}^k a_m \cos m \varphi + b_m \sin m \varphi$$

is called trigonometric polynomial of order $k$ if $|a_k|^2 + |b_k|^2 > 0$. Any trigonometric polynomial is $2\pi$-periodic and is well-defined and holomorphic in the cylinder $\mathbb{C}/2\pi \mathbb{Z}$. It always has $2k$ zeros in the cylinder. If a polynomial is real the number of real zeros is even.

Lemma 4.1 Let $t (\varphi), s (\varphi)$ be real trigonometric polynomials such that $\deg s < n$ deg $t$ for a natural $n$ and all zeros of $t$ are real and simple. Then

$$(P) \int_0^{2\pi} \frac{s (\varphi)}{t^n (\varphi)} d\varphi = \frac{1}{2} \int_0^{2\pi} \frac{s (\varphi)}{(t (\varphi) + i0)^n} d\varphi + \frac{1}{2} \int_0^{2\pi} \frac{s (\varphi)}{(t (\varphi) - i0)^n} d\varphi = 0$$

(7)
Proof. The function \( r(\zeta) = s(\zeta) t^{-n}(\zeta) \) is meromorphic for \( \zeta = \varphi + i\tau \in \mathbb{C}/2\pi\mathbb{Z} \) and has no nonreal poles since of the assumption. Fix a small positive \( \varepsilon \) we choose a real continuous function \( \lambda = \tau(\varphi) \) defined on the circle \( 0 \leq \varphi \leq 2\pi \) that vanishes except for \( \varepsilon \)-neighborhood of the zero set of \( t \) and

\[
\tau(\varphi) = \text{sgn} \ t'(\alpha) \sqrt{\varepsilon^2 - (\varphi - \alpha)^2}
\]

in \( \varepsilon \)-neighborhood of each zero \( \alpha \) of \( t \). We have \( t(\alpha + i\tau(\alpha)) = t(\alpha) + i0 \) for any zero \( \alpha \) and

\[
\int_0^{2\pi} \frac{s(\varphi)}{(t(\varphi) + i0)^n} d\varphi = \int_0^{2\pi} \frac{s(\zeta)}{t^n(\zeta)} d\zeta, \quad \zeta(\varphi) = \varphi + i\tau(\varphi)
\]

This yields

\[
\int_0^{2\pi} \frac{s(\varphi)}{(t(\varphi) + i0)^n} d\varphi + \int_0^{2\pi} \frac{s(\varphi)}{(t(\varphi) - i0)^n} d\varphi = \int_{\Gamma \cup \Gamma} \frac{s(\zeta)}{t^n(\zeta)} d\zeta
\]

where \( \Gamma = \{\zeta = \zeta(\varphi)\} \). We can write \( \Gamma \cup \Gamma = \Gamma_+ \cup \Gamma_- \) where \( \Gamma_\pm = \{\zeta = \varphi \pm i|\tau(\varphi)|\} \). Finally

\[
\int_{\Gamma \cup \Gamma} \frac{s(\zeta)}{t^n(\zeta)} d\zeta = \int_{\Gamma_+} \frac{s(\zeta)}{t^n(\zeta)} d\zeta + \int_{\Gamma_-} \frac{s(\zeta)}{t^n(\zeta)} d\zeta = 0
\]

since \( t(\varphi + i\tau) \neq 0 \) for \( \tau \neq 0 \) and the function \( st^{-n} \) tends to zero at infinity.

The following formula can be useful for calculation of \( D(x) \):

**Lemma 4.2** If \( t \) and \( s \) are real trigonometric polynomials, \( \deg s < \deg t = k \) and \( t \) has no real zeros, then

\[
\int_0^{2\pi} \frac{s(\varphi)}{t(\varphi)} d\varphi = \text{Re} \left( 2\pi i \sum_{t(\varphi_m) = 0, \text{Im} \varphi_m > 0} \frac{s(\varphi_m)}{t'(\varphi_m)} \right)
\]

where the sum is taken over \( k \) zeros of \( t \) with positive (or negative) imaginary part.

For a proof we apply the Residue theorem for the form \( s(\varphi) d\varphi/t(\varphi) \) in the upper half-cylinder \( \mathbb{C}_+/2\pi\mathbb{Z} \).

5 Radon’s and Funk’s reconstructions

We apply Theorem 3.1 to recover some known and unknown formulas for regular curve families.

**Radon’s formula.** Take a generating function \( \Phi(x; \lambda, \omega) = \lambda - \langle x, e(\varphi) \rangle \) in \( \mathbb{R}^2 \times \Sigma \) where \( e(\varphi) = (\cos \varphi, \sin \varphi) \). The classical formula of Radon-John

\[
f(x) = -\frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^{\infty} \frac{g(\lambda, \varphi) d\lambda d\varphi}{(\lambda - \langle x, e(\varphi) \rangle)^2} \quad (8)
\]

For a proof we apply the Residue theorem for the form \( s(\varphi) d\varphi/t(\varphi) \) in the upper half-cylinder \( \mathbb{C}_+/2\pi\mathbb{Z} \).
coincides in this case with (2) since integral data \( g = R f \) are equal to \( M f (\lambda, \varphi) \) because of \( |\nabla \psi| = 1 \). The equation \( N (x, y) = 0 \) immediately follows from Lemma 4.1. The coefficient (3) is reduced to \(-1/2\pi^2\) due to symmetry \( g (\lambda, \varphi + \pi) = g (\lambda, \varphi) \).

**Funk’s formula** provides reconstruction of an even function \( f \) in the unit sphere \( S^2 \) from integrals of \( f \) over the family \( \Sigma \) of big circles. Take the unit hemisphere \( X = \{ x \in E^3, |x| = 1, x_0 \geq 0 \} \) and a generating function \( \Phi (x; \lambda, \varphi) = \lambda + \psi, \ \psi = \langle y, e (\varphi) \rangle \) where \( y_1 = x_1/x_0, y_2 = x_2/x_0 \) defined in \( X \times \Sigma \). By Lemma 4.1 we have \( N (x, x') = 0 \) for any \( x \neq x' \in X \). Let \( g \) be the standard spherical metric in \( X \). We have

\[
|\nabla_g \psi|^2 = x_0^{-2} \left( 1 + \langle y, e \rangle^2 \right) = \frac{1 + \lambda^2}{x_0^2},
\]

\[
\int_0^{2\pi} \frac{d\varphi}{|\nabla_g \psi|^2} = x_0^2 \int_0^{2\pi} \frac{d\varphi}{1 + y^2 \cos^2 \varphi} = \frac{2\pi x_0^2}{\sqrt{1 + y^2}} = 2\pi x_0^2.
\]

By Theorem 3.1 for any function \( f \in L_2 (X) \)

\[
f (x) = -\frac{1}{4\pi^2} \int_0^{2\pi} \frac{M f (\lambda, \varphi) d\lambda d\varphi}{x_0^3 (\lambda + \langle y, e (\varphi) \rangle)^2}.
\]

Choose coordinates \( \theta, -\pi/2 \leq \theta \leq \pi/2, \varphi \) in the dual sphere \( \Sigma \) so that \( \sigma = (\sin \theta, \cos \theta \cos \varphi, \cos \theta \sin \varphi) \in \Sigma \). We have \( \lambda = \tan \theta, \ \lambda + \langle y, e (\varphi) \rangle = \langle \sigma, x \rangle / x_0 \cos \theta \) and \( |\nabla_g \psi|^{-1} = x_0 \cos \theta \).

\[
M f (\lambda, \varphi) = \int_{F(\lambda, \varphi)} \frac{f d\sigma}{|\nabla_g \psi|} = \cos \theta \int_{F(\lambda, \varphi)} x_0 f d\sigma = \cos \theta R (x_0 f) (\sigma)
\]

We apply (9) to the function \( h (x) = x_0 f (x) \) and rearrange this formula as follows

\[
f (x) = -\frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \frac{R_f (\sigma) \cos \theta d\theta d\varphi}{\langle \sigma, x \rangle^2} = -\frac{1}{2\pi^2} \int_{S_+} \frac{R_f (\sigma) d\sigma}{\langle \sigma, x \rangle^2}.
\]

where \( d\sigma = \cos \theta d\theta d\varphi \) is the area form in the hemisphere \( S_+ \). This formula coincides with original Funk’s result (1) after a partial integration.

### 6 Geodesic transform in Lobachevski plane

Take a generating function \( \Phi (x; \lambda, \varphi) = \lambda + \psi, \ \psi = -2 (|x|^2 + 1)^{-1} \langle x, e (\varphi) \rangle, \ -1 < \lambda < 1 \) in the unit disc \( X = D \). The curves \( F (\lambda, \varphi) \) are geodesics in the Poincaré model of the Lobachevski plane and \( d_g s = 2 (1 - |x|^2)^{-1} ds \) is the hyperbolic metric. We have

\[
|\nabla \psi|^2 = \left( \frac{2}{1 + |x|^2} \right)^2 (1 - \psi^2)
\]
and
\[ Mf(\lambda, \varphi) = (1 - \lambda^2)^{-1/2} \int \frac{1 + |x|^2}{2} f(x) \, ds = (1 - \lambda^2)^{-1/2} \int \frac{1 - |x|^4}{4} f(x) \, ds, \]
where \( ds, ds_g \) means the Euclidean and hyperbolic line element respectively. Further
\[ D(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{|\nabla \psi(x, \varphi)|^2} = \frac{1}{8\pi} (1 + |x|^2)^2 \int_0^{2\pi} \frac{d\varphi}{1 - y^2 \cos^2 \varphi} = \frac{(1 + |x|^2)^3}{4 (1 - |x|^2)} \]
since the integral in the right-hand side is equal to \( 2\pi (1 - |y|^2)^{-1/2}, \ y = 2 |x| (1 - |x|^2)^{-1} \) (e.g. Lemma 4.2). By Theorem 3.1 this yields
\[ f(x) = -\frac{1}{\pi^2} \frac{1 - |x|^2}{(1 + |x|^2)^3} \int \frac{Mf(\lambda, \varphi) \, d\lambda d\varphi}{\Phi^2(x; \lambda, \varphi)} \tag{11} \]
We apply this equation to the function \( f^*(x) = 4 (1 - |x|^4)^{-1} f(x) \) and get
\[ f(x) = -\frac{1}{4\pi^2} (1 - |x|^2)^2 \int_0^{2\pi} \int_{-1}^1 Rf(\lambda, \varphi) (1 - \lambda^2)^{-1/2} d\lambda d\varphi \]
where \( Rf(\lambda, \varphi) = \int f \, ds_g \) and the integral is taken over a geodesic \( F(\lambda, \varphi) \) in Lobachevski plane. This formula is equivalent to Helgason’s theorem [6].

Applying a map
\[ x = \frac{y'}{1 + y_0}, \ y' = (y_1, y_2) \]
we transform the Poincaré model to the Lorentz model in the "upper" sheet hyperboloid
\[ Q_+ = \{(y_0, y) \in E^3, y_0^2 = 1 + y_1^2 + y_2^2, y_0 > 0\} \]
with the metric induced from the Euclidean metric in \( E^3 \). For any central plane \( P \) in \( E^3 \) the hyperbola \( \gamma = P \cap Q_+ \) is a geodesic curve in \( Q_+ \). Vice versa, the image in \( Q_+ \) of an arbitrary geodesic circle \( F(\lambda, \varphi) = \{\lambda = 2 (1 + |x|^2)^{-1} \langle x, e(\varphi) \rangle \} \) is contained in the plane \( P = \{y; \lambda y_0 - \langle e(\varphi), y' \rangle = 0\} \). This plane is orthogonal to a vector \( \xi = (1 - \lambda^2)^{-1/2}(\lambda - e(\varphi)) \) which is contained in the one-sheet hyperboloid \( Q_- = \{\xi \in E^3, \xi_0^2 - \xi_1^2 - \xi_2^2 = -1\} \). Therefore we use notation \( \gamma(\xi) = F(\lambda, \varphi) = P \cap Q_+ \) for a geodesic in \( Q_+ \) and the hyperboloid \( Q_- \) parametrizes all the geodesics in the Lorentz model. We can write \( \Phi(x; \lambda, \varphi) = \lambda - y_0^{-1} \langle y, e(\varphi) \rangle = y_0^{-1} (1 - \lambda^2)^{1/2} \langle \xi, y \rangle \) and have
\[ 1 - |x|^2 = 2 (1 + y_0)^{-1}, \ 1 + |x|^2 = 2 y_0 (1 + y_0)^{-1} \]. The equation (11) is now read as follows
\[ f(y) = -\frac{1}{4\pi^2} \int \int \frac{Rf(\gamma(\xi)) (1 - \lambda^2)^{-3/2} d\lambda d\varphi}{\xi^2} \]
Expressing the projective area form

\[ \omega (\xi) = \xi_0 d\xi_1 d\xi_2 + \xi_1 d\xi_2 d\xi_0 + \xi_2 d\xi_0 d\xi_1 \]

in terms of coordinates \( \lambda \) and \( \varphi \) we get \( \omega (\xi) = (1 - \lambda^2)^{3/2} d\lambda d\varphi \). Finally

\[ f (y) = -\frac{1}{4\pi^2} \int_{Q^+} \frac{Rf (\gamma (\xi)) \omega (\xi)}{\langle \xi, y \rangle^2} \]

which makes complete similarity with (10).

**Remark.** A formula of this form appears in [7], p.96 however with misprinted numerical coefficient.

## 7 Equidistant curves in Lobachevski plane

Let \( X = D \) be the open unit disc and \( F = \{ F (\lambda, \varphi) \} \) be the family of arcs in \( D \) connecting two opposite points \( e (\varphi - \pi/2) \) and \( e (\varphi + \pi/2) \) on \( \partial D \) where \( \lambda = \tan \omega \), \( \omega \) is the angular measure of \( F (\lambda, \varphi) \). For any \( \varphi \) the arcs \( F (\lambda, \varphi) \) and \( F (\lambda', \varphi) \) are equidistant with distance \( 2 |\lambda - \lambda'| \). This family has generating function

\[ \Phi (x; \lambda, \varphi) = \lambda + \psi, \quad \psi (x, \varphi) = -\frac{2 \langle x, e (\varphi) \rangle}{1 - |x|^2} \]

Check that this function fulfils the conditions of Theorem 3.1. A proof of regularity is a routine. Further we have \( \psi (x, \varphi) - \psi (y, \varphi) = \langle y' - x', e (\varphi) \rangle \) where \( x' = (1 - |x|^2)^{-1} x, \ y' = (1 - |y|^2)^{-1} y \). This is a first order trigonometric polynomial with two real roots which implies \( N (x, y) = 0 \) as \( x \neq y \). We have

\[ |\nabla \psi (x, \varphi)|^2 = \frac{4}{(1 - |x|^2)^2} \left( 1 + z^2 \cos^2 \varphi \right) \]

where \( z = 2 |x| (1 - |x|^2)^{-1} \) and

\[ D (x) = \frac{1}{8\pi} \int_0^{2\pi} \frac{d\varphi}{1 + z^2 \cos^2 \varphi} = \frac{1}{4 \pi^2} \frac{(1 - |x|^2)^3}{1 + |x|^2} \]

**Corollary 7.1** For any function \( f \) with compact support in the unit disc a reconstruction from \( M_\Phi f \) is given by

\[ f (x) = -\frac{1}{\pi^2} (1 + |x|^2) \int_0^{2\pi} \int_{-1}^1 \frac{M_\Phi f (\lambda, \varphi)}{\left( (1 - |x|^2) \lambda - 2 \langle x, e (\varphi) \rangle \right)^2} d\lambda d\varphi \]
Because of \(|\nabla \Phi(x; \lambda, \varphi)| = 2 \left(1 - |x|^2\right)^{-1} \cos^{-1} \omega\) we can express the Funk-Radon operator \(M\) in terms of Euclidean arc integrals

\[
Mf(\lambda, \varphi) = \frac{\cos \omega}{2} Rf^*(\lambda, \varphi), \lambda = \tan \omega, \quad f^*(x) = (1 - |x|^2) f(x)
\]

This yields

\[
f(x) = -\frac{1 + |x|^2}{2\pi^2} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \frac{Rf(\omega, \varphi) \cos \omega d\omega d\varphi}{\left((1 - |x|^2) \sin \omega - 2 \langle x, e(\varphi) \rangle \cos \omega\right)^2}
\]

This formula was stated in \cite{11} by a different method.

\section{Photoacoustic geometries}

\textbf{Elliptical source curve.} D. Finch and Rakesh \cite{8} gave a formula of type \(2\) for reconstruction for the family of spheres centered on a sphere. Another reconstruction formula was proposed by L. Kunyansky \cite{9}; it looks different however can be reduced to a form close to \(2\) after one-fold integration. We show that a reconstruction can be done for ellipse and more general source curves by means of Theorem \(3.1\). F. Natterer applied quite different method for reconstruction from the family of spheres centered in an ellipsoid \cite{12}.

Take first a generating function \(\Phi = \lambda + \psi\) where

\[
\psi(x, \varphi) = |x - e(\varphi)|^2, e(\varphi) = (\cos \varphi e_1, \sin \varphi e_2),
\]

where \(e_1, e_2\) are arbitrary positive numbers. The source curve \(s = e(\varphi), \varphi \in S^1\) is an ellipse with half-axes \(e_1\) and \(e_2\).

\textbf{Proposition 8.1} For any function \(f\) with support in the ellipse \(E = \{x; (x_1/e_1)^2 + (x_2/e_2)^2 \leq 1\}\) a reconstruction is given by

\[
f(x) = -\frac{|e|^2 - |x|^2}{\pi^2} \int \int \frac{Mf(\lambda, \varphi)}{\left(\lambda - |x - e(\varphi)|^2\right)^2} d\lambda d\varphi
\]

\[
= -\frac{|e|^2 - |x|^2}{\pi^2} \int \int \frac{Rf(r, \varphi)}{\left(r^2 - |x - e(\varphi)|^2\right)^2} dr d\varphi
\]

where \(e = (e_1, e_2)\) and \(\lambda = r^2\) since \(Mf(\lambda, \varphi) = Rf(r, \varphi)/2r, \quad Rf(r, \varphi) = \int_{F(r, \varphi)} f ds\).

\textbf{Proof.} We have

\[
\psi(x, \varphi) - \psi(y, \varphi) = |x - e(\varphi)|^2 - |y - e(\varphi)|^2 = 2 \langle y - x, e(\varphi) \rangle + |x|^2 - |y|^2
\]

\[
= 2 \|y - x\|_e \cos (\varphi - \theta) + |x|^2 - |y|^2
\]

\[\tag{12}\]

\[\]
where \( \theta = \arg(y - x) \) and \( \|z\|_e = ((z_1 e_1)^2 + (z_2 e_2)^2)^{1/2} \). This norm and the dual norm \( \|z\|_e^* = ((z_1/e_1)^2 + (z_2/e_2)^2)^{1/2} \) fulfil the inequality
\[
||x|^2 - |y|^2| \leq |y_1 - x_1| |y_1 + x_1| + |y_2 - x_2| |y_2 + x_2|
\leq \|y - x\|_e \|y + x\|_e^* \leq 2 \|y - x\|_e
\]

since \( \|y + x\|_e^* < 2 \) for \( x, y \) belonging to the support of \( f \). This implies that the trigonometric polynomial (12) has only simple real zeros. By Lemma 4.1 \( N(x, y) = 0 \) for arbitrary \( x \neq y \in E \). We have
\[
|\nabla \psi| = 2 |x - e(\varphi)|
\]
and our statement follows. ▶

A similar reconstruction from spherical means in \( \mathbb{R}^n \) can be done by means of Theorem 11.1.

9 Isofocal hyperbolas and parabolas

**Hyperbolas.** The equation \( \lambda = |x| - \varepsilon x_1, \varepsilon > 1 \) defines a fold of the hyperbola
\[
\left( \alpha x_1 + \frac{\lambda}{\alpha} \right)^2 - x_2^2 = \frac{\lambda^2}{\alpha^2}, \quad \alpha = \sqrt{\varepsilon^2 - 1}
\]

with a focus at the origin. The function \( \Phi(x; \lambda, \varphi) = \lambda + \psi(x, \varphi), \psi(x, \varphi) = \varepsilon \langle x, e(\varphi) \rangle - |x| \) generates the family of all one-fold hyperbolae with focuses in the origin. The first order trigonometric polynomial
\[
\psi(x, \varphi) - \psi(y, \varphi) = \varepsilon \langle x - y, e(\varphi) \rangle - |x| + |y|
\]

has two real zeros if \( x \neq y \). since \( |x| - |y| < \varepsilon |x - y| \). By Lemma 4.1, the nucleus \( N(x, y) \) vanishes for \( y \neq x \). We have \( |\nabla \psi|^2 = \frac{1 + \varepsilon^2 - 2\varepsilon |x|^{-1} \langle x, e \rangle}{4 |e|^2 - |x|^2} \) and
\[
D(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{1 + \varepsilon^2 - 2\varepsilon |x|^{-1} \langle x, e(\varphi) \rangle} = \frac{1}{\varepsilon^2 - 1}
\]

(Lemma 4.2).

**Corollary 9.1** For any smooth function \( f \) with compact support in \( \mathbb{R}^2 \) the equation holds
\[
f(x) = -\frac{\varepsilon^2 - 1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{Mf(\lambda, \varphi) \, d\lambda \, d\varphi}{(|x| - \varepsilon \langle x, e(\varphi) \rangle)^2}
\]
Parabolas. A parabola with a focus at the origin can be given by $2px_1 + x_2^2 - p^2 = 0$ where $p$ is a positive parameter. Set $\lambda = p^{1/2}$ and write this equation in the form $(x_1 - \lambda^2)^2 = |x|^2$ which is equivalent to $|x| + x_1 = \lambda^2$ since $x_1 \leq \lambda^2$. Take a function $\Phi (x; \lambda, \varphi) = \lambda + \psi (x, \varphi)$ in $X = \mathbb{R}^2 \setminus \{0\}$ where

$$\psi (x, \varphi) = -\sqrt{|x|} + \langle x, e (\varphi) \rangle = - (2 |x|)^{1/2} \cos (\varphi - \theta) / 2, \ \theta = \arg x$$

This function generates all parabolas with focus at the origin. For any $x \neq y \in \mathbb{R}^2$ the trigonometric polynomial $\psi (x, \varphi) - \psi (y, \varphi)$ is of order 1 and zero constant term. Therefore it has two real roots and by Lemma 4.1 the function $N (x, y)$ vanishes if $y \neq x$. We have

$$|\nabla \psi|^2 = \frac{1}{2 |x|}, \ M f (\lambda, \varphi) = \int_{F(\lambda, \varphi)} (2 |x|)^{1/2} f (x) \, ds, \ D (x) = 2 |x|$$

Corollary 9.2 For any $L_2$-function $f$ with compact support in $\mathbb{R}^2 \setminus \{0\}$ a reconstruction is given by

$$f (x) = -\frac{1}{8\pi^2 |x|} \int_0^{2\pi} \int_0^\infty \frac{M f (\lambda, \varphi) \, d\lambda d\varphi}{\left( \lambda - \sqrt{|x|} + \langle x, e (\varphi) \rangle \right)^2}$$

$$= -\frac{1}{4\pi^2 (2 |x|)^{1/2}} \int_0^{2\pi} \int_0^\infty \frac{R f (\lambda, \varphi) \, d\lambda d\varphi}{\left( \lambda - \sqrt{|x|} + \langle x, e (\varphi) \rangle \right)^2}$$

In [5] Fourier coefficients of $f$ are reconstructed from $M f$ by Cormack’s method.

10 Cormack’s curves

For $\alpha > 0$ an $\alpha$-curve is given in a plane by an equation

$$\lambda = r^\alpha \cos (\alpha (\theta - \varphi)), \ |\theta - \varphi| \leq \pi / 2\alpha$$

where $x = re (\theta)$ and $\lambda, \varphi$ are parameters [2]. Suppose that $\alpha = k$ is natural and take the family of curves generated by a function $\Phi = \lambda + \psi (x, \varphi), \ \psi (x, \varphi) = -r^\alpha \cos (\alpha \theta - \varphi)$. Note that in the case $k = 2$ the curve $\psi (x, \varphi) = -\lambda$ is a hyperbola. The function $\psi$ always admits a symmetry group $\mathbb{Z}_k$ acting by rotation of the plane by the angle $2\pi / k$ about the origin. Consider a quotient manifold $X = \mathbb{R}^2 \setminus \{0\} / \mathbb{Z}_k$; lifting of $\Phi$ to $X$ is a regular generating function. We have $\psi (x, \varphi) = \text{Re} \left( \exp (-i\varphi) (x_1 + ix_2)^k \right)$ and

$$\psi (x, \varphi) - \psi (y, \varphi) = \text{Re} \exp -i\varphi \left( (x_1 + ix_2)^k - (y_1 + iy_2)^k \right)$$
is a first order trigonometric polynomial with two real zeros for arbitrary \( x \neq y \). This implies \( N(x, y) = 0 \). Further we have

\[
|\nabla \psi(x, \varphi)|^2 = k^2 |x|^{2k-2}, \quad D(x) = \frac{1}{k^2} |x|^{2k-2}, \quad Mf(\lambda, \varphi) = \frac{1}{k} \int_{\lambda = \psi(x, \varphi)} f(x) ds |x|^{k-1}
\]

and the reconstruction formula reads: for any \( \mathbb{Z}_k \)-invariant function \( f \) with compact support in \( \mathbb{R}^2 \setminus \{0\} \):

\[
f(x) = -\frac{k |x|^{k-1}}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{f(\lambda, \varphi) d\lambda d\varphi}{\left(\lambda - \text{Re} \exp(-i\varphi)(x_1 + ix_2)^k\right)^2}
\]

The family of \( \beta \)-curves \( \lambda = r^{-\beta} \cos(\beta \theta - \varphi) \) with natural \( \beta \) is studied in the same way.

### 11 Higher dimensions

Let \( \Phi \) be a generating function defined in \( X \times \Sigma \) where \( X \subset \mathbb{R}^n \) is a smooth manifold of dimension \( n \) with a Riemannian metric \( g \), \( \Sigma = \mathbb{R} \times S^{n-1} \) and \( \Phi(x; \lambda, \omega) = \lambda + \psi(x, \omega), \lambda \in \mathbb{R}, \omega \in S^{n-1} \). We say that a generating function \( \Phi \) is regular if a \( n+1 \times n+1 \) determinant like \( (6) \) does not vanish in \( F = \{ \Phi = 0 \} \) and conjugated points are absent, see [10] for details. The Funk-Radon transform is defined by

\[
M_\Phi f(\lambda, \varphi) = \int_{F(\lambda, \varphi)} f dV
\]

where \( dV \) means the Riemannian volume form and \( F(\lambda, \varphi) = \{ x : \Phi(x; \lambda, \varphi) = 0 \} \). The result of Sec. 3 is generalized as follows. Define a function

\[
N(x, y) = \text{Re} \int_{S^{n-1}} \frac{d\omega}{(\psi(x, \omega) - \psi(y, \omega) \pm i0)^n}, x \neq y
\]

where \( d\omega \) the Euclidean volume form in \( S^{n-1} \) and

\[
D(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \frac{d\omega}{|\nabla \psi(x, \omega)|^n}
\]

**Theorem 11.1** If a regular generating function \( \Phi \) is real analytic in \( X \) and \( N(x, y) = 0 \) for any \( x \neq y \in X \) then a reconstruction of type (2) exists for an arbitrary function \( f \in L_2(X) \) with compact support. For even \( n \) it reads

\[
f(x) = \frac{1}{(2\pi i)^n D(x)} \text{Re} \int_{\Sigma} M_\Phi f(\lambda, \omega) d\lambda d\omega
\]

For odd \( n \) it is

\[
f(x) = \frac{1}{2 (2\pi i)^{n-1} D(x)} \int_{\Sigma} \delta^{(n-1)}(\Phi(x; \lambda, \omega)) M_\Phi f(\lambda, \omega) d\lambda d\omega
\]

The integrals converge in \( L_2(X)_{\text{loc}} \).

A proof will be given elsewhere.
References

[1] P. Funk, Über eine geometrische Anwendung der Abelschen Integralgleichung. *Math. Ann.* **77** (1916), 129-135.

[2] A. Cormack, The Radon transform on a family of curves in the plane. *Proc. Amer. Math. Soc.* **83** (1981), 325–330.

[3] G. Beylkin, The inversion problem and applications of the generalized Radon transform. *Commun. Pure Appl. Math.* **37** (1984), 579–599.

[4] L. Hörmander, *The analysis of linear partial differential operators IV. Fourier integral operators*. Springer 1985.

[5] K. Denecker, J. Van Overloop and F. Sommen, The general quadratic Radon transform. *Inverse Problems* **14** (1998), 615-633.

[6] S. Helgason, *The Radon transform*. Second edition Birkhäuser (Boston, 1999).

[7] I. Gelfand, S. Gindikin and M. Graev, *Selected topics in integral geometry*. AMS Providence (Rhode Island, 2003).

[8] D. Finch and Rakesh, The spherical mean value operator with centers on a sphere. *Inverse Problems* **23** (2007), S37-S49.

[9] L. Kunyansky, Explicit inversion formulae for the spherical mean Radon transform. *Inverse Problems* **23** (2007), 373–383.

[10] V. Palamodov, Remarks on the general Funk transform and thermoacoustic tomography *Inverse Probl. Imaging* **4** (2010), 693–702.

[11] V. Palamodov, An analytic reconstruction for Compton scattering tomography in Lobachevski plane, (to be published in Inverse Problems).

[12] F. Natterer, Photo-acoustic inversion in convex domains, Personal communication, 2011.