THE STRONG SYMMETRIC GENUS OF THE FINITE COXETER GROUPS

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Abstract. The strong symmetric genus of a finite group \( G \) is the smallest genus of a closed orientable topological surface on which \( G \) acts faithfully as a group of orientation preserving automorphisms. In this paper we complete the calculation of the strong symmetric genus for each finite Coxeter group excluding the group \( E_8 \).

Keywords: Coxeter groups, strong symmetric genus.

1. Introduction

It is known that the automorphism group of a compact Riemann surface of genus \( g \geq 2 \) is finite; furthermore, the size of this automorphism group must be no larger than \( 84(g-1) \) when \( g \geq 2 \) \([1]\). When a finite group \( G \) can be realized as an automorphism group of a genus \( g \) compact Riemann surface such that \( |G| = 84(g-1) \), we say that \( G \) is a Hurwitz group. If a group \( G \) is not the automorphism group of any Riemann surface of genus 0 or 1, then for any Riemann surface on which the group acts as an automorphism group the genus must be at least \( 1 + \frac{|G|}{84} \). Burnside \([2]\) began the investigation of a related problem: find the least genus \( g \) of a Riemann surface on which a given finite group acts faithfully as a group of automorphisms. Equivalently, given a finite group \( G \), one may want to find the least genus \( g \) of a closed orientable topological surface on which \( G \) acts as a group of orientation preserving symmetries. The latter parameter is denoted \( \sigma^0(G) \) and has become known as the strong symmetric genus of \( G \) (see \([10, \text{Chapter 6}]\) ).

Many results are known concerning the strong symmetric genus of finite groups. All finite groups with strong symmetric genus less than 4 are known \([1, 13]\). It has been shown as well that for each positive integer \( n \), there is a finite group \( G \) with \( \sigma^0(G) = n \) \([14]\) . The strong symmetric genus of several infinite families of finite groups have been found: the alternating and symmetric groups \([3, 4, 5]\), the hyperoctahedral groups \([12]\), the groups \( \text{PSL}_2(q) \) \([8, 9]\), and the groups \( \text{SL}_2(q) \) \([19]\). In addition, the strong symmetric genus has been found for the sporadic finite simple groups \([6, 20, 21, 22]\).

The symmetric groups and the hyperoctahedral groups are two infinite families of finite Coxeter groups. They are often referred to as the \( A \)-type and \( B \)-type Coxeter groups, respectively. As stated above, the strong symmetric genus is known for each group in these families. Another family of finite Coxeter groups is the dihedral groups,
which are automorphism groups of the sphere and, thus, have strong symmetric genus 0. This leaves one infinite family of finite Coxeter groups whose strong symmetric genus has not been found previously: the $D\text{-type}$ groups. In this paper we calculate the strong symmetric genus of the $D\text{-type}$ finite Coxeter groups and of the sporadic finite Coxeter groups, excluding $E_8$. These new results will be given in Theorem 1 below; the previously known results concerning the other finite Coxeter groups will also be stated in Theorem 1.

The strong symmetric genus of the finite Coxeter groups will be shown by demonstrating the existence of certain pairs of generators. If a finite group $G$ has generators $x$ and $y$ of orders $p$ and $q$, respectively, with $xy$ having order $r$, then we say that $(x, y)$ is a $(p, q, r)$ generating pair of $G$. By the obvious symmetries concerning generators, we will use the convention that $p \leq q \leq r$. Following the convention of Marston Conder [5], we say that a $(p, q, r)$ generating pair of $G$ is a minimal generating pair if there does not exist a $(k, l, m)$ generating pair for $G$ with $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$.

Before stating Theorem 1 we define the groups involved. For an $n \geq 3$, $B_n$ will be the group of symmetries of the $n$ dimensional cube, and $D_n$ will be the group of orientation preserving symmetries of the $n$ dimensional cube. For the sporadic finite Coxeter groups, we list the results in Table 1.

| group $G$ | description |
|-----------|-------------|
| $G_2$     | isomorphic to the dihedral group of order 12 |
| $I_3$     | symmetry group of the regular icosahedron |
| $I_4$     | symmetry group of the 4-dimensional, regular 120-cell |
| $F_4$     | symmetry group of the 4-dimensional, regular 24-cell |
| $E_6$     | symmetry group of the $E_6$ polytope |
| $E_7$     | symmetry group of the $E_7$ polytope |
| $E_8$     | symmetry group of the $E_8$ polytope |

Table 1. Sporadic Coxeter group descriptions

**Theorem 1.** Let $G$ be a finite Coxeter group. If $G$ is the dihedral group of size $2n$, then $G$ has a $(2, 2, n)$ minimal generating pair and $\sigma^0(G) = 0$. If $G = \Sigma_n$ for $n > 29$, then $G$ has a $(2, 3, 8)$ minimal generating pair and $\sigma^0(\Sigma_n) = \frac{n!}{48}$. If $G = B_n$ for $n > 8$, then $G$ has a $(2, 4, 6)$ minimal generating pair and $\sigma^0(B_n) = \frac{n!2^n}{24}$. If $G = D_n$ for $n > 29$, then $G$ has a $(2, 3, 8)$ minimal generating pair and $\sigma^0(D_n) = \frac{n!2^{2n-1}}{48}$. The results for $G$ being one of the sporadic cases are given in Table 2. The remaining cases are listed in Table 3.

2. Generating Pairs and Strong Symmetric Genus

Recall that the groups of small strong symmetric genus are well known (see [1, 13]). The only finite Coxeter groups $G$ with $\sigma^0(G) = 0$ are the dihedral groups, $G_2$, $\Sigma_3$, ...
\[
\begin{array}{|c|c|c|c|}
\hline
\text{group } G & \text{size} & \min. \text{ gen. pair} & \sigma^0(G) \\
\hline
G_2 & 12 & (2, 2, 6) & 0 \\
I_3 & 120 & (2, 3, 10) & 5 \\
I_4 & 14400 & (2, 4, 6) & 601 \\
F_4 & 1152 & (2, 6, 6) & 97 \\
E_6 & 51,840 & (2, 4, 8) & 3241 \\
E_7 & 2,903,040 & (2, 4, 7) & 155,521 \\
\hline
\end{array}
\]

\textbf{Table 2.} Sporadic Coxeter group results

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
n & \Sigma_n & \sigma^0(\Sigma_n) & B_n & \sigma^0(B_n) & D_n & \sigma^0(D_n) \\
\hline
n = 3 & \text{Dihedral of order 6} & (2, 4, 6) & \frac{32^2}{24} + 1 = 3 & D_3 = \Sigma_4 \\
\hline
n = 4 & (2, 3, 4) & 0 & (2, 4, 6) & \frac{42^2}{24} + 1 = 17 & (3, 4, 4) & \frac{42^2}{12} + 1 = 17 \\
\hline
n = 5 & (2, 4, 5) & \frac{54}{40} + 1 = 4 & (2, 4, 10) & \frac{52^2}{40} + 1 & (2, 4, 5) & \frac{52^2}{40} + 1 \\
\hline
n = 6 & (2, 5, 6) & \frac{64}{40} + 1 & (2, 6, 6) & \frac{62^2}{24} + 1 & (2, 5, 6) & \frac{62^2}{24} + 1 \\
\hline
n = 7 & (2, 3, 10) & \frac{72}{30} + 1 & (2, 4, 6) & \frac{72^2}{24} + 1 & (2, 4, 6) & \frac{72^2}{24} + 1 \\
\hline
n = 8 & (2, 4, 7) & \frac{8(3)^2}{24} + 1 & (2, 4, 8) & \frac{82^2}{12} + 1 & (2, 4, 7) & \frac{82^2}{12} + 1 \\
\hline
n = 9 & (2, 4, 6) & \frac{92}{24} + 1 & (2, 4, 6) & \frac{92^2}{24} + 1 & (2, 4, 6) & \frac{92^2}{24} + 1 \\
\hline
n = 10 & (2, 3, 10) & \frac{102}{24} + 1 & (2, 4, 6) & \frac{102^2}{24} + 1 & (2, 3, 10) & \frac{102^2}{24} + 1 \\
\hline
n = 11 & (2, 4, 5) & \frac{112}{40} + 1 & (2, 4, 6) & \frac{112^2}{24} + 1 & (2, 4, 5) & \frac{112^2}{24} + 1 \\
\hline
n = 12 & (2, 3, 12) & \frac{122}{24} + 1 & (2, 4, 6) & \frac{122^2}{24} + 1 & (2, 3, 12) & \frac{122^2}{24} + 1 \\
\hline
n = 13 & (2, 3, 12) & \frac{132}{24} + 1 & (2, 4, 6) & \frac{132^2}{24} + 1 & (2, 3, 12) & \frac{132^2}{24} + 1 \\
\hline
n = 14 & (2, 4, 6) & \frac{142}{24} + 1 & (2, 4, 6) & \frac{142^2}{24} + 1 & (2, 3, 14) & \frac{142^2}{24} + 1 \\
\hline
n = 15 & (2, 4, 5) & \frac{152}{40} + 1 & (2, 4, 6) & \frac{152^2}{24} + 1 & (2, 4, 5) & \frac{152^2}{24} + 1 \\
\hline
n = 16 & (2, 4, 5) & \frac{162}{40} + 1 & (2, 4, 6) & \frac{162^2}{24} + 1 & (2, 4, 5) & \frac{162^2}{24} + 1 \\
\hline
n = 17 & (2, 4, 6) & \frac{172}{24} + 1 & (2, 4, 6) & \frac{172^2}{24} + 1 & (2, 4, 6) & \frac{172^2}{24} + 1 \\
\hline
n = 20 & (2, 3, 8) & \frac{20}{24} + 1 & (2, 4, 6) & \frac{20^2}{24} + 1 & (2, 4, 5) & \frac{20^2}{24} + 1 \\
\hline
n = 22 & (2, 3, 10) & \frac{22}{30} + 1 & (2, 4, 6) & \frac{22^2}{24} + 1 & (2, 3, 10) & \frac{22^2}{24} + 1 \\
\hline
n = 23 & (2, 3, 10) & \frac{23}{24} + 1 & (2, 4, 6) & \frac{23^2}{24} + 1 & (2, 3, 12) & \frac{23^2}{24} + 1 \\
\hline
n = 26 & (2, 4, 5) & \frac{26}{24} + 1 & (2, 4, 6) & \frac{26^2}{24} + 1 & (2, 4, 5) & \frac{26^2}{24} + 1 \\
\hline
n = 29 & (2, 3, 12) & \frac{29}{24} + 1 & (2, 4, 6) & \frac{29^2}{24} + 1 & (2, 3, 12) & \frac{29^2}{24} + 1 \\
\hline
\end{array}
\]

\textbf{Table 3.} Exceptional case results

\[\Sigma_4, \text{ and } D_3. \text{ Also there are no finite Coxeter groups } G \text{ with } \sigma^0(G) = 1. \text{ In this paper, we will assume that } \sigma^0(G) > 1 \text{ for each group } G \text{ that we are discussing. It is known that for groups with } \sigma^0(G) > 1, \text{ any generating pair will be a } (p, q, r) \text{ generating pair}\]
with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \). Using the Riemann-Hurwitz equation, we see that given any generating pair of \( G \), we get an upper bound on the strong symmetric genus of \( G \) \cite{17}. If \( G \) has a \((p, q, r)\) generating pair, then \( \sigma^0(G) \leq 1 + \frac{1}{2}|G| \cdot (1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}) \). The following lemma, which is a result of Singerman \cite{16} (see also \cite{13, 18}), shows that the strong symmetric genus for many groups is computed directly from a minimal generating pair.

**Lemma 2** (Singerman \cite{16}). Let \( G \) be a finite group such that \( \sigma^0(G) > 1 \). If \( |G| > 12(\sigma^0(G) - 1) \), then \( G \) has a \((p, q, r)\) generating pair with

\[
\sigma^0(G) = 1 + \frac{1}{2}|G| \cdot (1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}).
\]

In addition, if \( p \neq 2 \), \( p = q = 3 \), and \( r \) is 4 or 5.

We also include a lemma that allows for control of minimal generating pairs when passing to quotient groups.

**Lemma 3.** Let \( G \) be a finite group such that \( G \) has a \((p, q, r)\) generating pair. For any normal subgroup \( N \trianglelefteq G \), any minimal generating pair of \( G/N \) must have a \((p', q', r')\) generating pair such that \( \frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'} \leq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \). In other words, \( \sigma^0(G/N) - 1 \leq \frac{\sigma^0(G) - 1}{|N|} \).

Proof: Suppose that \((x, y)\) is a \((p, q, r)\) generating pair of \( G \). In addition, let \( \bar{x} \) and \( \bar{y} \) be the images under the quotient map \( \pi : G \to G/N \) of \( x \) and \( y \) respectively. Let \( p' \), \( q' \), and \( r' \) be the orders of \( \bar{x} \), \( \bar{y} \), and \( \bar{x}\bar{y} \). Clearly, \( p' \leq p \), \( q' \leq q \), and \( r' \leq r \); therefore, \( \frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'} \geq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \). It is also clear that \( \bar{x} \) and \( \bar{y} \) generate \( G/N \). \( \square \)

In order to use this quotient group result, we recall some cases where certain Coxeter groups are quotients of other Coxeter groups: For a fixed \( n \), \( \Sigma_n \cong B_n/(\mathbb{Z}_n)^n \), and \( \Sigma_n \cong D_n/(\mathbb{Z}_n)^{n-1} \). In addition if \( n \) is odd, \( D_n \cong B_n/Z(B_n) \).

For an example of using quotient groups, we look at \( D_{17} \). We notice that both \( B_{17} \) and \( \Sigma_{17} \) have minimal \((2, 4, 6)\) generating pairs. Now \( D_{17} \) has a minimal generating pair; suppose \( D_{17} \) has a \((p, q, r)\) minimal generating pair. Since \( D_{17} \cong B_{17}/Z(B_{17}) \),

\[
\frac{11}{12} \leq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \quad \text{with} \quad p \leq 2, \quad q \leq 4, \quad \text{and} \quad r \leq 6.
\]

On the other hand, we have \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{11}{12} \), because \( \Sigma_{17} \cong D_{17}/(\mathbb{Z}_2)^{16} \). So we see that \( D_{17} \) must have a \((2, 4, 6)\) minimal generating pair.

### 3. Generators of \( D_n \)

For notation purposes, we recall that \( D_n \) is an index two subgroup of \( B_n \) and that \( B_n \) is the wreath product \( \mathbb{Z}_2 \wr \Sigma_n \). We will use notation that was adopted by V. S. Sikora \cite{15} for the elements of \( B_n \) and thus \( D_n \) (see also \cite{12} ). For an element of \( B_n \), we will write a tuple \([\sigma, b]\) where \( \sigma \) is an element of \( \Sigma_n \) and where \( b \) is a list of \( n \) binary digits representing the element of \( (\mathbb{Z}_2)^n \). The multiplication then becomes \([\sigma, b] \cdot [\tau, c] = [\sigma \cdot \tau, \tau^{-1}(b) + c]\) where addition in the binary digits is a parity computation in each entry. We will use the convention of calling \( b \) even or odd
according to the number of ones appearing as binary digits of \( b \). Notice that if \( b \) and \( c \) have the same parity, then \( b + c \) is even; and if they differ in parity, then \( b + c \) is odd. Using this notation, an element \([\sigma, b] \in B_n\) is contained in \( D_n\) if and only if \( b \) is even.

Notice that if \([\sigma, b] \) and \([\tau, c] \) generate \( D_n\), then \( \sigma \) and \( \tau \) must generate \( \Sigma_n\). Since we are looking to find generators of \( D_n\), we need to first find generators of \( \Sigma_n\). Next we construct generators of \( D_n \) from the generators of \( \Sigma_n\). Also during this construction we would like to have control of the orders of the generators of \( D_n \) as well as to have control of the order of their product as described in Section 2.

Recall that the following sequence is a split exact sequence of groups:

\[
(Z_2)^{n-1} \stackrel{i}{\rightarrow} D_n \stackrel{\pi}{\rightarrow} \Sigma_n
\]

where \( i(b) = [1, b] \) and \( \pi([\sigma, b]) = \sigma \). In addition, we have the following commutative diagram

\[
\begin{array}{ccc}
(Z_2)^{n-1} & \stackrel{i}{\rightarrow} & D_n \\
\downarrow & & \downarrow \\
(Z_2)^n & \stackrel{i}{\rightarrow} & B_n \\
\end{array}
\]

where both horizontal sequences are split exact. Using the bottom split exact sequence, the author has proven the following proposition:

**Proposition 4** (Jackson [12]). For \( n \geq 5 \), let \( G \) be a subgroup of \( B_n\) with \( \pi(G) = \Sigma_n\); then \( G \) is a split extension of \( \Sigma_n\) by one of the following: 1, \( Z_2\), \((Z_2)^{n-1}\), or \((Z_2)^n\). In the first two cases, \( G \) is isomorphic to \( \Sigma_n \) and \( \mathbb{Z}_2 \times \Sigma_n\), respectively; in the third case either

\[
G = \{[\sigma, b] \in B_n | b \text{ is even}\} = D_n \text{ or } G = \{[\sigma, b] \in B_n | b \text{ is odd if } \sigma \in \Sigma_n \setminus A_n\};
\]

in the fourth case \( G = B_n\).

Proposition 4 leads to Corollary 5.

**Corollary 5.** For \( n \geq 5 \), let \( H \) be a subgroup of \( D_n\) with \( \pi(H) = \Sigma_n\); then \( H \) is a split extension of \( \Sigma_n\) by one of the following: 1, \( Z_2\), or \((Z_2)^{n-1}\). In the first case, \( H \cong \Sigma_n\). In the second case, which can only occur when \( n \) is even, \( H \cong \Sigma_n \times Z(D_n)\). In the third case, \( H = D_n\).

Using Corollary 5, we can prove Proposition 6, which we will use to construct generators of \( D_n \) from those of \( \Sigma_n\).

**Proposition 6.** Suppose \( \sigma \) and \( \tau \) generate \( \Sigma_n\) as \( \text{Symm}(\Gamma)\) where \( \Gamma = \{1, 2, \ldots, n\}\) with both \( \sigma \) and \( \sigma \cdot \tau \) having even order. Assume, furthermore, that \( \sigma \) fixes two elements \( i \) and \( j \) of \( \Gamma\), which are in the same cycle of the element \( \sigma \cdot \tau\); assume as well that \( \sigma \) fixes a third element of \( \Gamma\) if \( n \) is even. Let \( b = (0, 0, \ldots, 0, 0)\), and
Now therefore, may assume that $n$ and $\sigma$ since $\sigma$ is the length of the cycle in $k$ listed below, $n$ the elements $\sigma, a$ respectively. Let $\sigma, a$ such that $\sigma, a$ are not equal to $\sigma, a$. Recall that any section $\sigma, a$ for some $\{1, d\} = \langle \alpha, (0, \ldots, 0) \rangle$. Also notice that the last case only occurs if $n$ is even.

We will show first that $[\sigma, a]$ cannot be in the image of any such section via contradiction. Suppose $[\sigma, a]$ is in the image of some section homorphism $s : \Sigma_n \to D_n$. Now $[\sigma, a]$ cannot be $[1, d] \cdot [\sigma, (0, \ldots, 0)] \cdot [1, d]^{-1} = [\sigma, \sigma^{-1}(d) + d]$ for any $[1, d] \in B_n$ since $\sigma$ fixes $i$ and $j$ so that $\sigma^{-1}(d) + d$ has a 0 in both the $i^{th}$ and $j^{th}$ positions. We, therefore, may assume that $n$ is even and

$$[\sigma, a] = [1, d] \cdot [\sigma, (1, \ldots, 1)] \cdot [1, d]^{-1} = [\sigma, \sigma^{-1}(d) + d + (1, \ldots, 1)]$$

for some $[1, d] \in B_n$. $\sigma^{-1}(d) + d + (1, \ldots, 1) \neq a$ since $\sigma$ fixes $k$ and $a$ has a zero in the $k^{th}$ position.

$[\sigma, a]$ is not in the image of any section; thus we only need to show that $[\sigma, a]$ is not equal to $s(\sigma) \cdot [1, (1, \ldots, 1)]$ for any section $s$. Notice that if $[\sigma, a]$ were such an element, then $a$ is either $\sigma^{-1}(d) + d + (1, \ldots, 1)$ or $\sigma^{-1}(d) + d$, which are both ruled out in the previous paragraph. \(

4. Results

From Section 3, we see that the strong symmetric genus of $D_n$ may be shown by demonstrating a particular generating pair of $\Sigma_n$. First we notice that if $\Sigma_n$ has a $(p, q, r)$ generating pair, then at most one of $p$, $q$, or $r$ is odd. This result also holds for generators of $D_n$. So we see that the best possible generating pair for any $D_n$ with $n > 3$ is a $(2, 3, 8)$ generating pair. We notice that Marston Conder has demonstrated $(2, 3, 8)$ generating pairs for each $\Sigma_n$ with $n \geq 168$. If we call this generating pair $(\sigma, \tau)$, we notice that for each $n \geq 168$, excluding those for values of $n$ listed below, $\sigma$ fixes three elements, two of which are in the same cycle of $\sigma \cdot \tau$. In
these cases we apply Proposition 6 to see that $D_n$ has a (2, 3, 8) minimal generating pair. The exceptional values of $n$ are 171, 173, 174, 181, 185, 188, 194, 201, 202, 206, 209, 214, 230, 250, 257, 265, and 286.

In the remaining cases, the results were computed using GAP [7]. For $n \geq 30$, including the exceptional values of $n$ listed above, $D_n$ was shown to have a (2, 3, 8) minimal generating pair. So for each $n \geq 30$, $\sigma^0(D_n) = \frac{n! 2^{n-1}}{48}$. For each $D_n$ with $n < 30$ as well as for each sporadic group listed in Table I, an exhaustive search was performed using GAP [7] to find a minimal generating pair. In this manner the new results found in Table I and Table II, as well as in Theorem 11 were obtained.

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