Precision of Quantization of the Hall Conductivity in a Sample of Finite Size: Power Law

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(Dated: March 23, 2022)

A microscopic calculation of the conductivity in the integer quantum Hall effect (IQHE) regime is carried out. The problem of precision of quantization is analyzed for samples of finite size. It is demonstrated that the precision of quantization shows a power-law dependence on the sample size. A new scaling parameter describing a dependence of this kind is introduced. It is also demonstrated that the precision of quantization linearly depends on the ratio between the amplitude of the chaotic potential and the cyclotron energy. The results obtained are compared with the magnetotransport measurements in mesoscopic samples.

Introduction

Despite the considerable progress in the understanding of the quantum Hall effect (QHE), no consistent microscopic theory of this phenomenon has been developed so far. It will be recalled that the Hall resistance $R_H = h/ve^2$ is quantized in a strong magnetic field directed perpendicularly to the plane of a two-dimensional (2D) semiconductor sample [1]. Here $v$ is an integer, and the precision of quantization at sufficiently low temperatures is limited only by the measurement error, being as good as a millionth of a percent [2]. It is important that the quantization occurs in a certain range of magnetic field strengths or concentrations (on the Hall plateau). Such a behavior of the conductivity of a 2D electron gas in a strong magnetic field contradicts the results of the classical kinetic theory and those obtained by the diagram technique of disorder-averaging (e.g. in self-consistent Born approximation).

To describe IQHE, it is necessary to take into account the strong localization of electrons in the chaotic impurity potential in a magnetic field. The allowance for the chaotic potential leads to a power-law dependence of the localization length $\xi$ on the energy of the electronic state $\xi \propto (E - E_n)^{-\nu}$, $\nu \sim 2.3$ [3]. The existence of the scaling law for the energy dependence of the localization length indicates that the chaotic potential should be taken into account microscopically in order to calculate the conductivity in the IQHE regime, which can be done using numerical methods.

Previously, numerical calculations have been successfully applied to study the dependence of the localization length on the energy of electronic states. Several efficient methods are available, which can be used to calculate the localization length for rather large samples. Calculations of this kind have been employed to confirm the theory of finite-size scaling and obtain values of scaling indices that are in a good agreement with the experiments [3, 4]. Unfortunately, these methods are inapplicable to calculations of the Hall conductivity, because a complete knowledge of the carrier spectrum and wave functions is necessary in this case.

In present paper, we calculate ab initio the Hall conductivity of a 2D electron gas in a strong magnetic field. The results obtained suggest that the precision of quantization of the Hall conductivity on the plateau shows a power-law dependence on the sample size and is directly proportional to the ratio between the amplitude of the chaotic potential and the cyclotron energy.

Below we describe our model of the chaotic potential and the method used to calculate the Hall conductivity. Then we present the results of our numerical calculations and their analysis. Finally, we compare the theoretical results with data obtained in magnetotransport measurements in mesoscopic samples.

Model

We consider a 2D electron gas at zero temperature $T = 0$ and take into account the elastic scattering of carriers only. Such an approximation is justified under the following conditions. First, the sample should not be too "pure", so that Coulomb effects could not lead to transition to the regime of fractional QHE. Second, a finite temperature results in the effective limitation of the sample size by the phase-breaking length $L_\phi \propto T^{-\nu/2}$ [3]. Here, the scaling parameter $\nu$ is determined by the prevalent mechanism of inelastic scattering and, according to experimental data [4], is not a universal parameter. Finally, the broadening of the Fermi step in the distribution function can be disregarded because the typical temperatures at which QHE is observed are low, $T < 1K \simeq 0.1meV \ll \hbar\omega_c$. 

The Hamiltonian of noninteracting carriers in the external magnetic field $\mathbf{B}$ and the chaotic impurity potential $U(\mathbf{r})$ has the form:

$$\hat{H}_M = \frac{\mathbf{p}^2 - e\mathbf{cA}}{2m^*} + U(\mathbf{r}), \quad \text{rot} \ \mathbf{A} = \mathbf{H} = \mathbf{B}. \quad (1)$$

Let the magnetic field perpendicular to the plane of a 2D sample be directed along the $z$-axis and let the chaotic potential $U(\mathbf{r})$ to be independent of $z$. We choose the vector potential of the uniform magnetic field in the form $\mathbf{A} = (-By, 0, 0)$ (Landau gauge). When a single level in a quantum well is considered, the problem is described by the following Hamiltonian:

$$\hat{H} = \left(\hat{p}_x - eBy/c\right)^2 + \hat{p}_y^2 + U(\mathbf{r}), \quad \mathbf{r} = (x, y). \quad (2)$$

The model impurity potential used in this study has the following form:

$$U(\mathbf{r}) = \sum_{n=1}^{N} U_n \exp\left\{-\frac{(\mathbf{r} - \mathbf{r}_n)^2}{R^2}\right\}, \quad (3)$$

where the quantities $U_n$ and $\mathbf{r}_n$ are uniformly distributed in the interval $[U_<, U_>\]$ and over the entire plane $(x,y)$, respectively. It makes possible to vary the amplitude and the correlation properties of the potential with the use of the parameters $N$, $U_<$, $U_>$ and $R$. At the same time, a potential of this kind allows for analytical calculation of the matrix elements in the basis of wave functions containing plane waves $\Psi$:

$$\Psi_{nk} = \frac{\exp(ikz)}{\sqrt{2^n n! \sqrt{\pi a_H L_x}}} \exp\left(-\frac{y^2}{2a_H^2}\right) H_n\left(\frac{y}{a_H}\right), \quad (4)$$

where $n \geq 0$ is a Landau level number, $H_n$ are the Hermite polynomials, $a_H$ is a magnetic length. For a sample of finite dimensions $L_x \times L_y$, the set of basis wave functions was determined by the conditions of periodicity over a length $L_x$ and the point $y_0 = ka_H^2$ falls within an interval of length $L_y$ [4]. The eigenenergies and wave functions needed to calculate the conductivity can be found by numerical diagonalization of the Hamiltonian $\hat{H}$ in the basis of the wave functions $\Psi$.

An expression for the Hall conductivity (Kubo formula) can be derived at $T = 0$ using the first-order in electric field perturbation theory:

$$\sigma_{xy} = \frac{e}{S} \sum_{i \leq \mu} \frac{y_{if} (J_z)_{fi} + \text{c.c.}}{\mathcal{E}_i - \mathcal{E}_f}, \quad (5)$$

where $\mathcal{E}_i, f$ are eigenenergies of $\hat{H}$, $y_{if}$ and $(J_z)_{fi}$ are matrix elements of the coordinate and net current in the basis of eigenfunctions of the Hamiltonian $\hat{H}$, $\mu$ is a chemical potential and $S$ is a sample area.

**Results**

The above-described method for calculation of the Hall conductivity of a 2D electron gas makes it possible to analyze the influence of the finite sample size on the precision of quantization. First of all, we note the importance of a consistent calculation of both the density of states and the wave functions of electronic states. Figure 1 shows the calculated density of states $D(E)$ and the typical electron density distribution in localized and extended states. It can be seen that the wave functions corresponding to the minimum density of states are localized on a microscopic scale of the order of the magnetic length, and those corresponding to the maximum density of states are extended.

Figure 2 presents the results of Hall conductivity calculation for a sample with the dimensions that one Landau level contains 200 electron states. For a magnetic field $B = 10T$ this corresponds to a $0.3 \times 0.3 \mu m^2$ sample. The occurrence of strong conductivity fluctuations between QHE plateaus is a mesoscopic effect intrinsic to the small samples ($\lesssim 1 \mu m$) at low temperatures ($\lesssim 0.1 \ K$). For comparison with the results of our calculations, Fig. 3 shows experimental data obtained in Ref. [2], where magnetotransport measurements were carried out on a silicon MOSFET of dimensions $0.6 \times 0.6 \mu m^2$ at a temperature of 100 mK. As can be seen from Fig. 3, noticeable Hall conductivity fluctuations are observed on the first and second QHE plateaus, in addition to those in the transi-
FIG. 2: Results of calculation of the Hall conductivity as a function of the filling factor $\nu$.

FIG. 3: Hall conductivity of a $0.6 \times 0.6 \mu m^2$ silicon sample vs. the gate voltage (reproduced from [7]).

To analyze the dependence of the precision of quantization on sample size, we averaged deviations of the Hall conductivity of a one filled Landau level from the quantized value $\frac{e^2}{h}$ over the realizations of a random potential. The standard deviation of the Hall conductivity at the center of the first plateau is plotted against the sample size in Fig. 4. The presence in Fig. 4 of equidistant straight lines in the double logarithmic scale means that $\delta\sigma_{xy}$ is a power function of the ratio between the sample size and the magnetic length $a_H$:

$$\frac{\delta\sigma_{xy}}{\sigma_{xy}} \propto \frac{U_*}{\hbar \omega_c} \left( \frac{a_H^2}{S} \right)^b.$$  \hspace{1cm} (6)

Here we introduce a new scaling parameter $b$, which describes sample size dependence of the quantization precision. In all our calculations, the parameter $b$ remains universal $b = 0.7 \pm 5\%$. The proportionality of $\delta\sigma_{xy}$ to the amplitude of the chaotic potential is, in the limit $U_* \ll \hbar \omega_c$, an exact analytical result, which is derived in the Appendix. As the estimate based on Eq. (6) shows, fluctuations of the Hall conductivity on the plateau may be as large as $10^{-2}$ for samples of submicrometer dimensions. This estimation is in a qualitative agreement with the experimental data of [7].

It should be noted that the power-law dependence of the precision of quantization on the sample size, obtained here, is qualitatively confirmed by measurements performed on macroscopic samples. In particular, Hall conductivity fluctuations have been observed in the plateau at a level of $10^{-7}$ (at a measurement error less than $10^{-8}$) for a silicon transistor of dimensions $2 \times 2 \, mm^2$ [8]. At the same time, most of the theoretical studies known to us state that the correction to the Hall conductivity, associated with the finiteness of the sample size, is exponentially small [9], [10]. Conclusions of this kind are based on the exponential behavior of the wave functions of localized states. This, indeed, leads to an exponentially small slope of the IQHE plateau, but does not determine the precision with which the Hall conductivity in the plateau takes a quantized value $\nu e^2/h$. The $\sigma_{xy}$ value on the plateau is determined by all electronic states lying below the Fermi level, both localized and extended.

To conclude, we emphasize that the complete understanding of the high precision quantization, observed in experiments on the quantum Hall effect, requires further analysis of how the precision of quantization depends on various factors. In the first place, high-precision experiments where the dependence of the precision of quantization on sample size, temperature, and carrier mobility are to be performed. It is known from the temperature de-
dependence of the Hall conductivity of small \((L \sim 10 \mu m)\) samples that, at temperatures \(T \lesssim 30 mK\), the inelastic scattering length can exceed the sample length \(L\). This means that such experimental conditions open opportunities for direct observation of the deviation of the Hall conductivity from the quantized value as a function of sample size. It can be expected that measurements of this kind should reveal a power-law, rather than exponential dependence of \(\delta \sigma_{xy}/\sigma_{xy}\) on \(L\).

Acknowledgements

The study was supported by the Russian Foundation for Basic Research (04-07-90148, 04-02-16786, 05-02-16679) and by the Program for Support of Leading Scientific Schools. One of the authors (A.A.G.) is grateful to the Dynasty Foundation and the International Center for Fundamental Physics in Moscow.

Appendix

We now demonstrate that the Hall conductivity takes a quantized value \(\nu e^2/h\) at the center of each plateau in the limit \(U_s/\hbar \omega_c \to 0\). Let the amplitude of the chaotic potential \(U_s\) be so small as compared to the cyclotron energy \(\hbar \omega_c\), that the density of states, \(D(E)\), has an energy gap between Landau levels \(N-1\) and \(N\), \(N \geq 1\). Let us calculate the Hall conductivity for the case when the chemical potential lies within this gap, i.e., in the case of \(N\) completely filled Landau levels.

If the condition \(U_s \ll \hbar \omega_c\) is satisfied, the eigenenergies and eigenfunctions of the Hamiltonian \(\hat{H}\) can be expressed in terms of the perturbation theory for degenerate states \[6\]. In this case, the exact eigenenergies and wave functions can be represented as:

\[
E_{n\alpha(n)} = \hbar \omega_c(n + 1/2 + O(t)),
\]

\[
\Psi_{n\alpha(n)} = \sum_k A_{\alpha(n)}(k) \Psi_{nk} + t \sum_{m \neq n, k} B_{\alpha(n)nk} \Psi_{mk},
\]

\[
\sum_k A^*_{\alpha(n)}(k) A_{\beta(n)k} = \delta_{\alpha(n)} \delta_{\beta(n)} + O(t).
\]

Hereinafter we use Greek letters with superscripts \(n(m)\) to denote energy sublevels of the Landau level with a number \(n(m)\): \(\alpha(n), \beta(m)\); \(t\) is a small parameter \(t = U_s/\hbar \omega_c\). The explicit form of the basis functions \(\Psi_{nk}\) is given by Eq. \[4\]. The set of coefficients \(A\) and \(B\) depends on \(t\), having a finite limit at \(t \to 0\); the Eq. \[4\] is a direct consequence of the orthonormality of the set of eigenfunctions \(\Psi_{n\alpha(n)}\). It is convenient to introduce auxiliary wave functions

\[
\tilde{\Psi}_{n\alpha(n)} = \lim_{t \to 0} \Psi_{n\alpha(n)} = \sum_k C_{\alpha(n)k} \Psi_{nk}
\]

in such a way that they are exactly orthonormal and coincide with the eigenfunctions \(\tilde{\Psi}_{n\alpha(n)}\) to within \(O(t)\). A consequence of the orthonormality of the wave functions \(\tilde{\Psi}_{n\alpha(n)}\) is the following identity derived from Eq \[4\]:

\[
\sum_k C^*_{\alpha(n)k} C_{\beta(n)k} = \delta_{\alpha(n)} \delta_{\beta(n)}.
\]

Let us generalize the functions \[10\] in the following way:

\[
\tilde{\Psi}_{n\alpha(m)} = \sum_k C_{\alpha(m)k} \Psi_{nk}.
\]

Despite that at \(m \neq n\) the wave functions \(\tilde{\Psi}_{n\alpha(m)}\) are not eigenfunctions of the Hamiltonian \(\hat{H}\) in the limit \(t \to 0\), they are of use in further calculations. In particular, \(\tilde{\Psi}_{n\alpha(m)}\) can be expressed in terms of the wave functions \[10\]:

\[
\tilde{\Psi}_{n\alpha(m)} = \sum_{\beta(n)} D_{\alpha(m)\beta(n)} \tilde{\Psi}_{n\beta(n)}.
\]

Using Eqs. \[10\]–\[13\], we can readily obtain one more identity:

\[
\sum_{\gamma(n)} D^*_{\alpha(n)\gamma(n)} D_{\beta(m)\gamma(n)} = \delta_{\alpha(m)} \delta_{\beta(m)}.
\]

To calculate the Hall conductivity by the Kubo formula \[5\], it is necessary, first of all, to derive expressions for the matrix elements of the operators of the coordinate, \(y\), and net current, \(J_x\). In the basis of the wave functions \[4\], we have:

\[
\langle \Psi_{nk}| \hat{J}_x | \Psi_{mq} \rangle = \frac{e \hbar}{m a_H} \langle \Psi_{nk}| \hat{\Lambda} | \Psi_{mq} \rangle,
\]

\[
\langle \Psi_{nk}| y | \Psi_{mq} \rangle = a_H \langle \Psi_{nk}| \hat{\Lambda} | \Psi_{mq} \rangle + k a_H^2 \delta_{kq} \delta_{mn},
\]

\[
\langle \Psi_{nk}| \hat{\Lambda} | \Psi_{mq} \rangle = \sqrt{\frac{\max(m,n)}{2}} \delta_{kq} \delta_{[m-n],1}.
\]

Using the above identities, we obtain the following matrix elements between the neighboring Landau levels:

\[
\langle \tilde{\Psi}_{n-1,\alpha(n)}| \hat{\Lambda} | \tilde{\Psi}_{n\beta(n)} \rangle = \sqrt{n/2} \delta_{\alpha(n)} \delta_{\beta(n)},
\]

\[
\langle \tilde{\Psi}_{n-1,\alpha(n-1)}| \hat{\Lambda} | \tilde{\Psi}_{n\beta(n)} \rangle = \sqrt{n/2} D_{\alpha(n-1)} \delta_{\beta(n)}.
\]

In Eqs. \[18\], \[19\] and below the operator \(\hat{\Lambda}\) is given by any of two following expressions:

\[
\hat{\Lambda} \equiv \{ y/a_H, ma_H \hat{J}_x/(e \hbar) \}.
\]

Thus, if the chemical potential level \(\mu\) lies within the energy gap between two Landau levels, calculation of the Hall conductivity by the Kubo formula gives the following result:
\[ \sigma_{xy}^N = \sum_{m \leq N-1, \alpha^{(m)}}^{\sum_{n \geq N, \beta^{(n)}}} \frac{2e}{S(\mathcal{E}_{\alpha^{(m)}} - \mathcal{E}_{\beta^{(n)}})} \Re \langle \Psi_{\alpha^{(m)}} | \hat{y} | \Psi_{\beta^{(n)}} \rangle \langle \Psi_{\beta^{(n)}} | \hat{J}_x | \Psi_{\alpha^{(m)}} \rangle = \]

\[ = \frac{2e^2 a_H^2}{h} \sum_{\alpha^{(N-1)}}^{\sum_{\beta^{(N)}}} \frac{1}{S} | (\tilde{\Psi}_{N-1,\alpha^{(N-1)}} | \hat{\Delta} | \Psi_{\beta^{(N)}}) |^2 + O(t) = \frac{N e^2 a_H^2}{h} \sum_{\alpha^{(N-1)}}^{\sum_{\beta^{(N)}}} D_{\beta^{(N)}}^{\alpha^{(N-1)}} D_{\alpha^{(N)}}^{\beta^{(N)}} + O(t) = \]

\[ = \frac{N e^2 a_H^2}{h} \sum_{\beta^{(N)}}^{\sum_{\alpha^{(N-1)}}} 1 + O(t) = \frac{N e^2}{h} + O(t) \quad (21) \]

Here the summation over \( \beta^{(N)} \) reduces to calculation of the number of states per Landau level in a sample of area \( S \), and, therefore, \( \sum_{\beta^{(N)}} 1 = \frac{S}{2\pi a_H^2} \).

Thus, we proved that the Hall conductivity at the center of a plateau \( \sigma_{xy} = \nu e^2 / h + O(U_c / \hbar \omega_c) \).

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