Wake-mediated interaction between driven particles crossing a perpendicular flow

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Abstract. Diagonal or chevron patterns are known to spontaneously emerge at the intersection of two perpendicular flows of self-propelled particles, e.g. pedestrians. The instability responsible for this pattern formation has been studied in previous work in the context of a mean-field approach. Here, we investigate the microscopic mechanism yielding this pattern. We present a lattice model study of the wake created by a particle crossing a perpendicular flow and show how this wake can localize other particles traveling in the same direction as a result of an effective interaction mediated by the perpendicular flow. The use of a semi-deterministic model allows us to characterize the effective interaction between two particles analytically.

Keywords: driven diffusive systems (theory), patterns, traffic models, self-propelled particles

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1. Introduction

The spontaneous formation of patterns from simple interaction rules between individual agents (molecules, particles, humans, etc) has aroused much interest in the statistical physics community. In particular, it is well-known that perpendicular intersecting flows may give rise to diagonal patterns. This has been observed, for example, at the crossing of two perpendicular corridors, when a unidirectional pedestrian flow is coming from each corridor, both experimentally [1] and in simulations [2, 3]. Another example is the Biham–Middleton–Levine (BML) model [4], which models urban road traffic in
Manhattan-like cities, and exhibits similar patterns in the free flow phase. However, it is only recently that the instability responsible for this diagonal pattern formation has been studied more systematically [5, 6], for a lattice model with two species of particles (or pedestrians), hopping eastward ($E$) or northward ($N$), on a square lattice, and interacting through an exclusion principle. It was discovered that in certain geometries the pattern is not strictly diagonal but has the shape of chevrons. The system was analyzed in terms of mean-field equations believed to correctly represent the microscopic particle system.

In this paper we focus on understanding how the diagonal pattern emerges from the interactions between particles at the microscopic scale. We shall characterize the effective interaction between two $E$ particles crossing a flow of $N$ particles. It is the perturbation of the density field of $N$ particles by the $E$ particles that mediates the interaction. This kind of environment-mediated interaction has been extensively studied in the context of equilibrium soft matter [7] and more recently in out-of-equilibrium systems [8, 9], with a focus on the prediction of effective forces. In our case, the interaction between particles is not modeled in terms of forces (indeed, the notion of force is not appropriate for, e.g., pedestrians) but rather derives from the dynamical rules. A similar question was addressed in [10], for another lattice model. We shall discuss further in the conclusion how our approach differs from the previous ones and complements them.

We had found in [5, 6] that pattern formation is generic for updates with low enough noise on the velocity of the particles. Here we consider two of them, frozen shuffle update and alternating parallel update. These updates are deterministic enough to allow for a quantitative description of the effective interaction, not only in terms of ensemble averaged quantities as in [10], but also at the level of specific realizations of the dynamics.

Our paper is organized as follows. Section 2 gives the definition of the model and summarizes previous results. The ensemble averaged wake of a single $E$ particle in a uniform background of $N$ particles (i.e. the perturbation that it induces in the density of $N$ particles) is computed analytically in section 3, showing good agreement with numerical measurements. Then, in section 4, instead of considering an ensemble average, we shall consider the wake obtained in a given realization of the model history and characterize its structure. This level of description will allow us to obtain the central result of this paper in section 5, namely the existence and the properties of an effective interaction between two $E$ particles, mediated by the perturbation of the density of $N$ particles. Section 3, on the one hand, and sections 4 and 5, on the other hand, can be read independently. Although the calculations of sections 3–5 are carried out in the case of frozen shuffle update, most of the results are expected to hold for a larger class of update schemes. Section 6 shows how some of these results can be readily applied to alternating parallel update. In section 7 the discussion and conclusion are presented.

2. The model

2.1. Geometry

The intersection of two perpendicular flows has been modeled using a cellular automaton model. The basic building block of this model is the totally asymmetric simple exclusion process (TASEP), a cellular automaton which consists of a directed unidimensional sequence of sites each of which is either empty or occupied by one particle (for a review see [11]–[13]). The only possible motion of a particle is a hop toward the site directly
following its current position, occurring with probability $p$. New particles can randomly enter the system on the first site with probability $\alpha$ if it is empty and exit it from the last site with probability $\beta$. Here we will only consider the deterministic version of the bulk dynamics, i.e. a particle will hop with probability $p = 1$ if its target site is empty. This gives a smoother motion which is expected to be closer to pedestrian transport. We want to avoid any effect from the exit boundaries as well. Particles are therefore allowed to leave with probability $\beta = 1$ if they stand on an exit site.

We call an ensemble of several parallel TASEPs a street. We shall focus on the intersection formed by two perpendicular streets directed toward the east and the north, as shown in figure 1. A site of the intersection will be denoted by the coordinates $(x, y)$. Each site can be either empty, or occupied by an eastbound ($E$) or a northbound ($N$) particle. For simplicity we consider only the case where there are $M$ TASEPs directed to the north and $M$ to the east, so that the intersection is a square of side $M$.

We symbolically take the limit $M \to \infty$; that is to say that in our analytical approach we will never consider a system with explicit boundaries. However, the injection algorithm will still determine the form of the correlations between particles in the bulk.

### 2.2. Update scheme

The properties of the model also depend on the order in which the particles are updated [13]. In most of this paper, we shall use the frozen shuffle update [14, 15]. This requires one to determine a random but fixed order in which the particles will be updated (attempt to hop) at each time step. This is carried out by giving each particle a phase $\tau \in [0; 1)$, which does not change from one time step to another. Particles are then updated in the order of increasing phase. This is equivalent to updating a particle of phase $\tau$ at all times $t + \tau$, where $t$ is an integer and time is understood as continuous.
Particles are injected at each entrance site with constant rate $a$, provided the site is empty. The injection procedure also inserts the new particles in the updating sequence [15]. We define the parameter $\alpha$ as the probability that a particle enters the system during a time step on a given entrance site when it is empty, so that we have $\alpha = 1 - e^{-a}$. The entrance rate determines the density in the free flow phase $\rho^{fs} = a/(1 + a)$ [15], where the superscript ‘fs’ refers to frozen shuffle.

Note that, with our deterministic choices $p = 1$ and $\beta = 1$, and for the frozen shuffle update, the $\mathcal{N}$ particles move with velocity 1 if there are no $\mathcal{E}$ particles in the intersection, i.e. no blocking should be seen. This property will be important in the following.

In section 6 we shall consider another update scheme, the alternating parallel update, which we do not detail here. It is, however, worth noting that the phenomena described in section 2.3 are also observed using alternating parallel update.

We also claim that most of the calculations made in the following for a particular update scheme are easily generalizable to other updating schemes, provided the free flow phase is deterministic (moving with velocity 1) and there is exactly one time unit between two updates of a given particle. However, for clarity we shall concentrate from now on, except otherwise stated, on the case of the frozen shuffle update.

2.3. Known results

Direct Monte Carlo simulations show that there is a jamming transition at high density [16], which is outside the scope of this work. We rather focus on the low density ($\alpha \lesssim 0.1$) bulk properties of the system, in particular pattern formation. A detailed report of the numerical results can be found in [6]. Here we only summarize the results necessary to the understanding of this paper.

First, the model has been simulated on a torus. Particles with random phases are then dropped on the intersection square at the initial time. For densities under the jamming threshold, the particles are observed to self-organize into stripes of alternating types directed along the $(1, -1)$ vector. Note that the symmetry with respect to the $(1, 1)$ direction on each site of the system imposes the angle of the stripes to be exactly $45^\circ$ (with respect to the vertical).

The more complex case of open boundaries is now considered, schematized in figure 1. Stripes are again observed, far enough from the entrance boundaries. However, they now have the shape of chevrons. More precisely, the intersection square can roughly be divided into two triangles: above (below) the main diagonal, the stripes are observed to be tilted with respect to $45^\circ$ by a constant amount $\pm \Delta \theta_0$ of the order of a degree and growing with the entrance rate $\alpha$ (see figure 2). This effect can be qualitatively understood as resulting from the asymmetry in the organization of the two types of particles.

In [5, 6], a mean-field approach showed how the diagonal pattern can be explained from a linear stability analysis while the chevron effect was shown to be a nonlinear effect. As a complement to this macroscopic approach, we want now to understand the microscopic mechanism through which the chevron structure can emerge in the stochastic particle model.

In this paper we describe and quantify how an effective interaction between two $\mathcal{E}$ particles can be mediated by the perpendicular flow of $\mathcal{N}$ particles. As a first step, we compute the perturbation of the $\mathcal{N}$ density field created by a single $\mathcal{E}$ particle.
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Figure 2. Observed chevron pattern in a simulated open intersection square with \( M = 400, \alpha = 0.105 \) and frozen shuffle update. The blue \( E \) particles hop toward east and the orange \( N \) particle hop toward north. After a penetration length at the boundaries, the particles are observed to self-organize into alternating stripes. These stripes are, however, slightly tilted with respect to 45° depending on the position in the intersection square. In the lower triangle (delimited by green lines), the angle of the stripes with respect to the vertical is somewhat smaller than 45°, with a deviation of the order of a degree. The tilt can be linked to an asymmetry in the organization of the types of particles visible in the zoom on the left; i.e. in the lower triangle \( E \) particles are compacted in alignments whereas \( N \) particles are more randomly positioned.

3. Ensemble averaged wake of a particle

3.1. Definition of the wake

An isolated \( E \) particle propagating in a flow of randomly incoming \( N \) particles will create a perturbation in the average \( N \) density. The ensemble averaged density pattern seen in the frame moving with the \( E \) particle will be called its wake. In this section we compute the shape of the wake for the frozen shuffle update, although the method is intended to be more general.

The system is taken to be infinite so it is translationally invariant. Throughout this section, the moment at which the \( E \) particle has hopped to a site taken to be \((0,0)\) will be chosen as the time origin \( t = 0 \) without loss of generality. The stationary state was established in negative time. We want to compute the stationary ensemble averaged density of \( N \) particles on any site \((k,l)\) at time 0. We call it \( \rho_{kl}^N \). The dependence of \( \rho_{kl}^N \) on the incoming \( N \) particle density \( \rho \) will be implicit in the following.

Before performing the ensemble average, we need to compute the density field of \( N \) particles for a given realization of the dynamics. This will be the aim of section 3.2.

3.2. Density field for a given realization of the \( E \) particle dynamics

One first remark is that, for a given realization, the past dynamics of the \( E \) particle is fully defined by the number of time steps the particle spent on each of the already
visited columns $p = -1, -2, \ldots$, which we call $\{n_p\}$. Several realizations of the $\mathcal{N}$ particles’ dynamics may yield the same values for the $\{n_p\}$s. We average over all these realizations. We are thus considering in this section a given realization of the $\mathcal{E}$ particle dynamics, averaged over all the compatible realizations of the $\mathcal{N}$ particles’ dynamics.

A second remark is that, when the $\mathcal{E}$ particle leaves a column after perturbing the density in it, the density pattern in that column simply evolves independently of the subsequent hops of the $\mathcal{E}$ particle. Thus it will be convenient to decompose the density field into density profiles in each column that will evolve independently in time.

Let us consider a column that particle $\mathcal{E}$ visited in the past, corresponding to the sites $(k, l)$, with $k < 0$ fixed, and $l$ ranging from $-\infty$ to $+\infty$. The $\mathcal{E}$ particle left column $k$ at time $-\sigma_k \equiv -\sum_{k'=k+1}^{l} n_{k'} \leq 0$. We shall show now that the density profile in column $k$ and at time $-\sigma_k + s$ ($s \geq 0$) depends only on three parameters: $n_k$, $n_{k-1}$, and $s$ (the variable $s$ counts how many time steps have elapsed since particle $\mathcal{E}$ left column $k$). As a consequence, we shall denote this density profile by $R_{l,k}^{n_{k-1}n_k}(s)$. We insist that this quantity is not explicitly dependent on $k$.

Let us first determine $R_{l,k}^{n_{k-1}n_k}(0)$. Particle $\mathcal{E}$ was blocked $n_{k-1} - 1$ time steps before entering column $k$, due to the presence of $\mathcal{N}$ particles on column $k$. As, before the arrival of the $\mathcal{E}$ particle, the $\mathcal{N}$ particles are in free flow (i.e. moving with velocity $1$), this indicates that there were $n_{k-1} - 1$ adjacent $\mathcal{N}$ particles on column $k$. This platoon was not altered by the $\mathcal{E}$ particle and continues to move with velocity $1$ at all subsequent time steps.

Particle $\mathcal{E}$ hopped on column $k$ just behind this platoon, and then blocked the column for $n_k$ time steps. The next incoming $\mathcal{N}$ particles were forced to queue up. This created an empty zone of size $n_k + 1$ for $l \geq 0$, and a denser zone for $l < 0$.

As a summary, the density profile $R_{l,k}^{n_{k-1}n_k}(0)$ just after particle $\mathcal{E}$ left column $k$ reads

$$R_{l,k}^{n_{k-1}n_k}(0) = \begin{cases} \rho & \text{for } l \geq n_{k-1} + n_k \\ 1 & \text{for } n_k < l < n_{k-1} + n_k \\ 0 & \text{for } 0 \leq l \leq n_k \\ \geq \rho & \text{for } -\infty \leq l < 0. \end{cases}$$

In principle, the $\mathcal{E}$ particle can block an arbitrary large number of $\mathcal{N}$ particles. However, in the low density regime, an $\mathcal{N}$ particle is added to the queue with a probability $\sim \rho$. The density profile thus decreases very rapidly to its asymptotic value $\rho$ for $l < 0$. An example of such a density profile is given in figure 3. The explicit calculation of the profile for $l < 0$ is described in appendix B.

Once the $\mathcal{E}$ particle has left column $k$, the particles queuing at $l < 0$, if any, may have undergone a short transient until a new free flow configuration was reached. Here we study low densities and thus we have neglected the possible modification of the density profile in the transient. After this transient the density profile in column $k$ moves upwards unchanged, which enables us to compute the $R_{l,k}^{n_{k-1}n_k}(s)$ for all $s \geq 0$.

### 3.3. Ensemble average of the wake

For $k \leq 0$, one can compute the wake $\rho_{kl}^{\mathcal{N}}$ by averaging $R_{l,k}^{n_{k-1}n_k}(s)$ on the past dynamics of the $\mathcal{E}$ particle. The probability $r_{n_p}$ associated with each possible value of $n_p$ is proved
Figure 3. The function $R_{l}^{23}(0)$ for an incoming $\mathcal{N}$ particle density $\rho = 0.067$. For $l \geq 5$, the density is equal to $\rho$ and has not been perturbed by the $\mathcal{E}$ particle. Site 4 is surely occupied, i.e. $R_{l}^{23}(0) = 1$, because the $\mathcal{E}$ particle was blocked once before entering the column. There is an empty region of size $3+1 = 4$ for $0 \leq l < 4$ corresponding to $\mathcal{N}$ particles being blocked by the $\mathcal{E}$ one. The blocked $\mathcal{N}$ particles then accumulate in the region $l < 0$, where the density is larger than $\rho$. However, the density quickly decreases to its asymptotic value $\rho$ as $l$ goes toward negative values.

in appendix A to be equal to

$$r_{n}^{fs} = \begin{cases} 1 - \rho^{fs} & n = 1 \\ \rho^{fs} e^{-a} \sum_{l=n-1}^{\infty} \frac{a^{l}}{l!} & n = 2, 3, \ldots \end{cases}$$

(2)

For $k > 0$, i.e. in the columns that have not yet been reached by the $\mathcal{E}$ particle, $\rho_{kl}^{\mathcal{N}}$ has to be equal to the $\mathcal{N}$ particle density $\rho$. We thus have

$$\rho_{kl}^{\mathcal{N}} = \begin{cases} 0 & k = 0 \\ \rho \sum_{n_{k-1}=1}^{\infty} \sum_{n_{k}=1}^{\infty} \cdots \sum_{n_{1}=1}^{\infty} r_{n_{k-1}l} r_{n_{k}l} \cdots r_{n_{1}l} R_{l}^{n_{k-1}n_{k}} \left( \sum_{k'=k+1}^{\infty} n_{k'} \right) & k \leq 0 \\ \rho \sum_{n_{k-1}=1}^{\infty} \sum_{n_{k}=1}^{\infty} \cdots \sum_{n_{1}=1}^{\infty} r_{n_{k-1}l} r_{n_{k}l} \cdots r_{n_{1}l} R_{l}^{n_{k-1}n_{k}} \left( \sum_{k'=k+1}^{\infty} n_{k'} \right) & k > 0 \end{cases}$$

(3)

Note that $\rho_{kl}^{\mathcal{N}}$ is computed when the $\mathcal{E}$ particle has just entered the $k = 0$ column, which corresponds to taking $n_{0} = 0$.

A comparison between the theoretical and the numerically measured shape of the wake $\rho_{kl}^{\mathcal{N}}$ is shown in figure 4. They are in excellent agreement, justifying the approximation made in section 3.2 where we neglected the relaxation of the density profile after the passage of the $\mathcal{E}$ particle.

4. Microscopic structure of the wake

Our aim is now to understand how a second $\mathcal{E}$ particle can be localized by the wake of a first $\mathcal{E}$ particle. As the second particle will not see the average wake but a specific realization of it, we shall now describe in more detail its microscopic structure. In particular, we have seen in section 3.2 that the passage of an $\mathcal{E}$ particle creates an empty zone. We shall now
define an algorithm that allows us to track the empty sites for a given realization of the dynamics of the $E$ and $N$ particles. Our algorithm tracks effectively empty sites only for updates such that all the unblocked $N$ particles move one step forward at each time step (this is indeed the case for the two updates considered in this paper).

4.1. Tracking algorithm

We define the following algorithm.

1. Just before the $E$ particle attempts to move, put a white dot on the site it occupies.
2. After the move is performed, put a black dot on the site occupied by the $E$ particle, which will replace the white dot if the $E$ particle did not move.
3. During the next time step, let the $N$ particles hop, and let all the dots move one site upwards simultaneously. Start again at step (1).

We call the set of the sites occupied by a dot the shadow. All sites in the shadow are empty for all time steps. After being created, the shadow is just translated upwards, and its shape is kept invariant.

4.2. Properties of the shadow

For a single $E$ particle having an infinite history, an infinitely long line of dots is constructed. The form of the shadow is depicted in figure 5. Note that according to the algorithm the shadow is well defined at the instant of the hop of the $E$ particle. In the following, we shall always consider it just after the hop of the $E$ particle.

Two subsequent black dots are separated by a vertical line each time the $E$ particle was blocked, and by a diagonal line each time the $E$ particle moved forward. The asymptotic

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Color plot of the density perturbation (wake) created by a single $E$ particle that has just arrived on the $(0,0)$ site (white spot) for $\rho = 0.068$ using frozen shuffle update. (a) The theoretical wake as calculated in equation (3). (b) Numerical measurement of the density field immediately after each update of the $E$ particle, averaged over $5 \times 10^6$ time steps.}
\end{figure}
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Figure 5. The shadow of a single $E$ particle as defined in section 4. The blue right-pointing triangle represents the $E$ particle and the orange up-pointing triangles are the $N$ particles. Dotted sites are denoted by black and white circles. The snapshot is taken directly after the hop of the $E$ particle.

angle $\theta$ between the shadow and the $y$ axis is therefore related to the average velocity $v$ of the $E$ particle, through the relation $\tan \theta = v$. In the limit of small density, the average velocity behaves like $1 - \rho$. If $\Delta \theta$ measures the deviation of the angle $\theta$ from $45^\circ$, we thus have to lowest order in density $|\Delta \theta| = \rho/2$ rad $= \rho \times (90/\pi)^\circ$. This formula is coherent with the one found in [5] in an ideal case.

For section 5, it will be also useful to notice that the shadow can only have a width of 1 or 2 dots in each row. It can be seen as a superposition of two types of rows: from right to left, there is either a black site followed by a white site or an $N$ particle followed by a black site. This $N$ particle is precisely the one that blocked the $E$ particle in the past. We call these two types of rows $D$ and $K$, as defined in figure 6(a).

The relative position of two adjacent rows is not arbitrary: below the black dot of a given row, one finds necessarily the leftmost dot (black or white) of the row below.

5. Wake-mediated interaction with another particle

A second $E$ particle will be said to be in the shadow of the first $E$ particle if it occupies any of the dotted sites. The dynamics of the first particle is entirely encoded in the shape of the shadow; it therefore suffices to study the dynamics of the second particle in the shadow to get information on the correlations between the $E$ particles. In particular, we are interested in knowing how long the second particle will stay in the shadow of the first one.

We have just seen that the shadow is a superposition of two types of rows, one with two dots and the other with one dot and one $N$ particle. The second $E$ particle can occupy any of these dotted sites, and as a result can be in three different states, depicted in figure 6(b).
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Figure 6. (a) Possible types of rows. Type $D$ corresponds to having two dots (a white and a black) in the same row, while type $K$ contains only one black dot, associated with an $N$ particle on its right. (b) Possible states of an $E$ particle in the shadow. If the row is of type $D$, the $E$ particle can be either on a white dot (state $D_W$), or on a black dot adjacent to a white one ($D_B$). If the row is of type $K$, the $E$ particle can only be on the black dot. In the case of the frozen shuffle update, an extra parameter is the phase $\tau$ of the $N$ particle, hence a continuum of states $K_{\tau\in[0;1)}$. The algorithm creating and moving the dots is explained in section 4.

We want now to determine whether the second $E$ particle remains in one of these states or exits the shadow, and how the transitions are between the different states. At each time step, the rows of the shadow are moving upwards with the $N$ particles, while the second $E$ particle remains on the same row, and hops forward if the target site is empty. Thus, in the frame moving upwards with the shadow, the shadow itself is invariant and the $E$ particle moves one diagonal step in the right-bottom direction. One has to determine whether this step in the diagonal direction is possible, and whether the $E$ particle will still be in the shadow at the next time step. All the subsequent discussion will be in this shadow-correlated frame.

A first remark is that once the second $E$ particle is in the shadow, it can never leave the shadow by being left behind. The second $E$ particle can only exit the shadow (if it does) by overtaking it. This is a consequence of the fact that there is always a dot (black or white) below a black dot. Indeed, if particle $E$ is on a black dot (state $D_B$ or $K_{\tau}$), and if it is blocked at the next time step, it will still arrive on a dotted site, and thus will not exit the shadow from behind. If it is on a white site (state $D_W$), it will surely hop forward, and again will not be left behind the shadow. This is one of the most striking effects of the shadow on the dynamics of the second particle.

Our aim is to estimate the probability that the second $E$ particle stays in the shadow of the first one after $t$ time steps, averaged over all possible realizations of the flow of $N$ particles, i.e. of the shadow. In this section, we shall calculate this probability in the case of the frozen shuffle update and denote it as $P_{S}^{fs}(t)$. The phase of the $E$ particle creating the shadow is taken to be 0. The phase of the second $E$ particle is denoted as $\tau_{0}$.

We have listed in figure 6(b) the possible states of a particle in a shadow. As long as the second particle remains in the shadow, it is necessarily in one of these three states. Starting from a given initial state, the probability to remain in the shadow after $t$ time
steps is thus given by the sum

\[ P_S(t) = P_W(t) + P_B(t) + \int_0^1 d\tau \ p_\tau(t) \]  

where \( P_W(t) \), \( P_B(t) \) and \( p_\tau(t) \, d\tau \) stand for the probabilities of the particle being in states \( D_W \), \( D_B \) and \( K_{\tau' \in [\tau, \tau+d\tau]} \) at time \( t \), respectively. We have to write some rate equations for these probabilities based on the microscopic dynamics.

We shall now present an intuitive explanation of the way in which rate equations between the possible states of a particle in the shadow can be written to linear order in \( \rho^{fs} \) (equations (5)). The more general equations valid for all densities will be derived in appendix C.

Suppose we have an \( E \) particle in the state \( D_W \), i.e. occupying a white dotted site. The probability that the row directly under the particle is a \( D \) row is \((1 - \rho^{fs})\). In that case the particle will stay in state \( D_W \) at the next time step. The particle can arrive in state \( K_{\tau > \tau_0} \) if the row under it is a \( K \) row, with \( \tau > \tau_0 \) (probability \( \tau_0 \rho^{fs} \)). Similarly, it will arrive in state \( K_{\tau < \tau_0} \) with probability \((1 - \tau_0)\rho^{fs}\). This completes the list of possible arrival states for a particle leaving state \( D_W \).

If the \( E \) particle departs from state \( D_B \), we have to specify whether the site directly to its right is occupied or not. It is empty with probability \((1 - \rho^{fs})\), in which case the particle can arrive either in state \( D_B \) if the row under it is a \( D \) row (probability \((1 - \rho^{fs})\)), or get blocked by an \( N \) particle if the row under it is a \( K_{\tau < \tau_0} \) row (probability \( \tau_0 \rho^{fs} \)), or exit the shadow if the row under it is a \( K_{\tau > \tau_0} \) row (probability \((1 - \tau_0)\rho^{fs}\)). If, in the initial configuration, there is an \( N \) particle on the site to the right of the \( E \) particle, the next row is necessarily a \( D \) row to linear order in \( \rho^{fs} \). The \( E \) particle will therefore stay in the \( D_B \) state if it is not blocked by the \( N \) particle (probability \( \tau_0 \rho^{fs} \)) or arrive in the \( D_W \) state if it is blocked (probability \((1 - \tau_0)\rho^{fs}\)).

Finally, we notice that an \( E \) particle occupying a state \( K_{\tau < \tau_0} \) will necessarily arrive in state \( D_B \) at the next time step and that a particle occupying a state \( K_{\tau > \tau_0} \) will arrive in \( D_W \). We can therefore write the rate equations to linear order in \( \rho^{fs} \)

\[
\begin{align*}
P^{fs}_{W}(t+1) &= (1 - \rho^{fs})P^{fs}_{W}(t) + \rho^{fs}(1 - \tau_0)P^{fs}_{B}(t) + P^{fs}_{>\tau_0}(t) \\
P^{fs}_{B}(t+1) &= (1 - 2\rho^{fs} + \rho^{fs}\tau_0)P^{fs}_{B}(t) + P^{fs}_{<\tau_0}(t) \\
P^{fs}_{<\tau_0}(t+1) &= \rho^{fs}\tau_0 P^{fs}_{W}(t) + \rho\tau_0 P^{fs}_{B}(t) \\
P^{fs}_{>\tau_0}(t+1) &= \rho^{fs}(1 - \tau_0)P^{fs}_{W}(t)
\end{align*}
\]

where we have defined \( P^{fs}_{<\tau_0}(t) \equiv \int_{\tau=0}^{\tau_0} p_\tau(t) \, d\tau \) and \( P^{fs}_{>\tau_0}(t) \equiv \int_{\tau=\tau_0}^{1} p_\tau(t) \, d\tau \). Therefore, the probability to stay in the shadow evolves as

\[ P^{fs}_S(t+1) = P^{fs}_S(t) - \rho^{fs}(1 - \tau_0)P^{fs}_B(t) \]  

and clearly decreases with time.

It can be seen from these equations that the probability \( P^{fs}_S(t) \) depends on the phase \( \tau_0 \) between the two \( E \) particles. For one given \( \tau_0 \) value, one can solve the linear equations (5) and (6) by diagonalizing the transfer matrix to obtain the time evolution of \( P^{fs}_S(t) \). The
Figure 7. Characteristic escape time of a particle starting from state $D_B$ in the shadow as a function of its phase $\tau_0$ (blue). The escape time diverges for $\tau_0 = 0$ and 1. The escape time in the case of uncorrelated particles has been plotted in red for comparison.

Figure 8. Probability that the second particle stays in the shadow during the first $t$ time steps for frozen shuffle update (blue disks, equations (5) and (6)) and alternating parallel update (red squares, equations (8)). For comparison, the same quantity has been plotted assuming the two particles are uncorrelated (black diamonds). The density $\rho$ of $N$ particles is 0.1.

solution is given by a linear combination of exponentials\(^1\). The longest of the characteristic decay times has been plotted as a function of $\tau_0$ in figure 7.

It is worth underlining that the distribution $P_{S}^{fs}(t)$ as given in figure 8 does not depend on the distance between the two $E$ particles, as distance only acts as a delay in the interaction between the $E$ particles.

\(^1\) While the characteristic escape time is independent of the initial condition, the short-time evolution of $P_{S}^{fs}(t)$ does depend on it. In figure 8, we have averaged the distribution $P_{S}^{fs}(t)$ over all $\tau_0$ values, while assuming that the initial state was a $D_B$ state. If we had considered an initial state $D_W$, the slope of the curve at $t = 0$ would have been horizontal, in accordance with the fact that the shadow cannot be exited directly from state $D_W$. 

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The probability to stay in the shadow is compared to a reference situation where the second $\mathcal{E}$ particle meets an unperturbed flow of $\mathcal{N}$ particles (the shape that the shadow would have if the flow was perturbed by a first $\mathcal{E}$ particle is computed only to define the zone from which the exit time is measured). In this reference situation, the displacements of both $\mathcal{E}$ particles are uncorrelated. We find that for any $\tau_0$ value, $P_{\mathcal{S}}(t)$ decays more slowly if the motion of the two $\mathcal{E}$ particles is correlated through the mediation of the shadow than for decorrelated motion (see figure 7). In particular, with the shadow, the escape time diverges when $\tau_0$ becomes close to 0 or 1. As a result, when averaged over $\tau_0$ (see figure 8), the decay in the uncorrelated case is exponential whereas the probability to stay in the shadow decays like $t^{-1}$. Indeed, in the large-time limit the average over $\tau_0$ is dominated by the diverging escape times.

The fact that the exit time is larger in the presence of a shadow shows the relative stability of the two-particle state. Thus we can consider as a bound state with finite life-time the set of two $\mathcal{E}$ particles, one being in the shadow of the other.

For random initial positions of the $\mathcal{E}$ particles (not necessarily in the shadow), various types of collisions can destroy the structure of the shadow, so that in the case of the crossing of two perpendicular flows as in figure 1, the system will not converge toward the pure mode that we have just described. However, this state was observed to be realized locally in direct simulations [5].

6. Alternating parallel update

We want to illustrate the generality of the calculations and discussions of the previous sections 3–5 by applying them to alternating parallel update. Under alternating parallel update, a parallel update is performed alternatively on $\mathcal{E}$ and $\mathcal{N}$ particles at each half time step. More precisely, we define the time scale such that $\mathcal{E}$ particles move at integer time steps, while $\mathcal{N}$ particles move a half time step later.

The wake of a single $\mathcal{E}$ particle has mostly the same structure as for the frozen shuffle update. The $R_{l+k-n}^{n-k}(s)$ function defined in section 3 still verifies the properties detailed in equation (1), although the exact shape is different for $l<0$. One must also take into account that the mean density in the free flow phase as a function of the entrance probability is now $\rho^{ap} = \alpha/(1 + \alpha)$ [13] (the superscript ‘ap’ standing for alternating parallel). Again, $\alpha$ stands for the probability to inject an $\mathcal{N}$ particle in an entrance site during a time step at which it is empty. Equation (3) is also verified provided one uses the correct expression for the $r_{n}^{ap}$, namely

$$r_{n}^{ap} = \begin{cases} 1 - \rho^{ap} & n = 1 \\ \rho^{ap} & n = 2 \\ 0 & n > 2. \end{cases}$$

The tracking algorithm defined in section 4 is still valid as well. Since all the $\mathcal{N}$ particles move at the same time, the continuum of states $K_\tau$ becomes a single state $K$. The equivalents of equations (5) can then be obtained by setting $\tau_0 = 1$, i.e. all the $\mathcal{E}$ particles move at the same time. As the calculations for alternating parallel update are formally obtained as a special case of the frozen shuffle update, all the remarks made in
section 5 also hold here. We get for the evolution equations

$$
\begin{align*}
P_{ap}^{W}(t+1) &= (1 - \rho_{ap})P_{ap}^{W}(t) \\
P_{ap}^{B}(t+1) &= (1 - \rho_{ap})P_{ap}^{B}(t) + P_{ap}^{K}(t) \\
P_{ap}^{K}(t+1) &= \rho_{ap}P_{ap}^{W}(t) + \rho_{ap}P_{ap}^{B}(t)
\end{align*}
$$

which are in fact true for all densities, as shown in appendix C. One can see that in this case, the second $E$ particle cannot exit the shadow. Indeed, equations (8) sum up to $P_{ap}^{S}(t+1) = P_{ap}^{S}(t)$, and the decay time is infinite.

Another way to phrase this is that it is impossible for the second $E$ particle to leave a black site (state $K$ or $D_B$) for a non-black one (state $D_W$ or exit from the shadow). A consequence is that if the second particle stands on a black site, its shadow coincides with the shadow of the first particle. We can then add more than one particle. In particular, the state in which all black sites are occupied by an $E$ particle is a stable state consisting of an infinite line of $E$ particles. We recover a macroscopic mode that had already been proposed in [5] as an explanation for the chevron effect.

7. Conclusion and discussion

The work presented in this paper was triggered by the observation of an instability at the crossing of two perpendicular flows, leading to the formation of stripes that have the shape of chevrons. While a mean-field approach was proposed in [5] and developed in [6], we are interested here in the mechanisms involved at the microscopic scale.

We demonstrate how interactions between particles of the same type can arise from the mediation of the perpendicular flow. In a first stage, we have studied the wake created by a single particle moving in such a perpendicular flow. The averaged wake was predicted analytically, in good agreement with simulations. The microscopic structure of a given realization of the wake was also provided, and allowed us to show that a second particle could be localized in the wake of a first particle. The localization time depends on the type of update that is used. The angle of the wake is the same as that of the long-lived global mode identified in [5].

The calculations here were made for the frozen shuffle update. We have shown in section 6 how it could be generalized to the alternating parallel update. In fact, calculations can be extended to other update schemes, provided the free flow phase is deterministically shifted forward with velocity 1 at each time step.

We have also assumed that the flow of $N$ particles is homogeneous. In the full problem of crossing flows, the $N$ particles themselves will be organized into stripes. As long as these particles move with unit velocity, our calculation leading to a localization phenomenon of one $E$ particle in the wake of another one can be easily generalized. However, we have not described here how, once it has left the wake, the second particle can alter the wake of the first one and also modify the density of $N$ particles. In the complete setting, where a whole flow of $E$ particles crosses a flow of $N$ particles, multiple collisions will result in a finite length for the wakes. This length depends on the type of update and it would be interesting to estimate it. The angle of the wake may also be altered by these collisions. The resolution of the full problem would require us to be able to estimate this new angle—although the order of magnitude should not be modified compared to what was found here, as already mentioned in [5].

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Of special interest is the alternating parallel update, for which a particle can be localized in the wake of another for an infinite time. As a result, the stripes of the global pattern observed in the complete problem have more contrast than for frozen shuffle update [6].

While the analytical work presented in this paper was first triggered by the observation of patterns in pedestrian crossings, it can also be cast into the more general research field of effective interactions. These effective interactions mediated by the environment were first studied in soft condensed matter physics for systems at equilibrium (see the review in [7]). A classical example is the ‘depletion attraction’ due to entropic effects, which appear between two large colloidal particles placed in a dilute bath of smaller colloids [17]. More recently, similar depletion forces were found and studied in out-of-equilibrium systems. Dzubiella et al [8] considered two fixed large particles (or intruders) in a flowing bath of smaller particles, all of them being modeled as soft spheres. Their theory is based on the approximation that the perturbation of the density field due to the two large particles can be written as the superposition of the perturbation due to each large particle separately. The calculation is extended beyond this hypothesis by [9] for equal size colloids, and under the assumption that interactions between bath particles can be neglected. For simplicity, the calculation is made when the intruders move along their line of centers.

Related models defined on a lattice have been studied, in which intruders undergo a biased random walk, while the bath is made of Brownian particles hopping in all directions with equal probability. The perturbation induced by a single intruder in a bath of Brownian particles and the resulting relation between force and velocity distribution has been extensively studied [18, 19]. The case of two intruders was considered in [10], in which a numerical study showed the existence of an attractive force between the intruders resulting in a statistical pairing.

In our case, the use of a semi-deterministic model makes it possible to characterize analytically the interaction between the two intruders. We do not need to make any mean-field assumption because we are able to consider each particular trajectory instead of working directly with ensemble averaged wakes (though we also predict the latter).

In their conclusion, Mejía-Monasterio and Oshanin [10] conjectured that the attractive interaction that they had found between two intruders could be seen as ‘an elementary act’ leading to pattern formation when many intruders are considered. Here we give an example of such a connection between individual and collective behavior which can even be made explicit in the case of the alternating parallel update.

In [8], an experimental realization was suggested to measure the effective depletion forces between two large colloidal particles fixed by optical tweezers, and placed in a flow of charged particles subject to an electric field. Experimental studies of effective interactions were also made with constant force driving in [20], for particles confined on a circle. A first step toward a physical realization of the system studied in this paper could be to use a similar setup. While the optical tweezers would be fixed in the direction of the flow of the bath, there would be a servo mechanism ensuring that a constant force perpendicular to the flow was applied on the two trapped colloids. In such a way, these two colloids would be driven in the direction perpendicular to the electric field and localization times could be measured a priori.
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Appendix A. The coefficients \( r_{fs}^n \)

In this appendix we compute the \( r_{fs}^n \) defined in section 3 for the frozen shuffle update.

Consider an \( E \) particle with phase 0 (without loss of generality) in a flow of \( N \) particles, trying to hop toward site \((0, 0)\) at time 0. This site is either empty, with probability \( 1 - \rho_{fs} = r_{fs}^1 \), or it is occupied with probability \( \rho_{fs} \) \( dx \) by an \( N \) particle that we call \( N_1 \), with phase between \( 1 - x \) and \((1 - x) + dx\). In the second case, and using continuous time, we define \( T_2 \) as the interval separating the departure of particle \( N_1 \) from site \((0, 0)\) from the arrival of its direct follower \( N_2 \) on the same site. From there we see that the \( E \) particle will attempt to hop toward \((0, 0)\) again at time 1, whereas particle \( N_2 \) will try at time \( 1 - x + T_2 \). The \( E \) particle will therefore hop before \( N_2 \) if \( T_2 > x \). For a fixed \( x \) we get an infinitesimal contribution to \( r_{fs}^2 \):

\[
   dr_{fs}^2(x) = \rho_{fs} \text{Prob}[T_2 > x] \, dx.
\]

If the \( E \) particle is blocked again by particle \( N_2 \), we have to consider a third particle \( N_3 \) coming to site \((0, 0)\) after \( N_2 \) after an interval \( T_3 \). Applying the same argument as in the previous paragraph, we see that for fixed \( x \),

\[
   dr_{fs}^3(x) = \rho_{fs} \text{Prob}[(T_2 + T_3 > x) \text{ and } (T_2 < x)] \, dx.
\]

We can finally generalize to higher numbers of blockings:

\[
   dr_{fs}^n(x) = \rho_{fs} \text{Prob}[(T_2 + \cdots + T_n > x) \text{ and } (T_2 + \cdots + T_{n-1} < x)] \, dx.
\]

We have assumed that the incoming \( N \) particles were moving in free flow with velocity 1. Thus, the distribution of arrival times of the \( N \) particles in a given site is the same as the distribution of injection times. The probability density distribution of the \( T_i \) is given by

\[
   P(T) = ae^{-aT}, \quad T > 0,
\]

where \( a \geq 0 \) is an inverse time which determines the injection rate. The normalization reads \( \int_{T=0}^{\infty} P(T) \, dT = 1 \), expressing that every particle is necessarily followed by another particle. One finally computes, for \( n \geq 1 \),

\[
   dr_{fs}^n(x) = \rho_{fs} \int_{T_2=0}^{x} \cdots \int_{T_{n-1}=0}^{x-S_{n-2}T_i} \int_{T_n=x-S_{n-1}T_i}^{\infty} \prod_{i=2}^{n} P(T_i) \, dT_i
   = \rho_{fs} \frac{(ax)^{n-2}}{(n-2)!} e^{-ax}.
\]

By averaging uniformly over \( x \) we obtain equation (2).

Appendix B. Probability distribution of the length of the queue of \( N \) particles.

We have seen in section 3.2 that, when an \( E \) particle occupies a site in column \( k \) for a certain amount of time, \( N \) particles accumulate below. In this appendix we determine the probability distribution for the length of this queue of \( N \) particles in column \( k \), in the case of the frozen shuffle update.

Let column \( k \) be occupied by a single \( E \) particle during \( n_k \) time steps. We want to calculate \( q_m^{n_k} \), defined as the probability that at least \( m = 1, 2 \ldots \) particles of type \( N \) have
been blocked in column $k$ by the $E$ particle, before the latter leaves the column. We stress that this quantity is independent of the column index $k$.

For an unperturbed flow of $N$ particles, the distribution $P(T)$ defined in appendix A represents the probability of having a time delay $T$ between the departure from one site of a given $N$ particle and the arrival of the next particle. As there is no memory in the injection procedure, it also represents the probability of having a time delay $T$ between any instant where a given site is empty and the arrival on this site of the next particle. Using this we can write

$$q_{n_k}^m = \int_{T_1=0}^{n_k} \int_{T_2=0}^{n_k-T_1} \cdots \int_{T_m=0}^{n_k-\sum_{i=1}^{m-1} T_i} \prod_{i=1}^{m} P(T_i) \, dT_1 \cdots dT_m$$

$$= (-1)^m \sum_{l=0}^{\infty} \frac{(-an_{n_k})^l}{l!}.$$  \hspace{1cm} (B.1)

Due to the relatively low density of $N$ particles considered, we chose to ignore the possible relaxation of the queue occurring after the departure of the $E$ particle from column $k$. We can thus directly use the above values to compute $R_{n_k}^{n_k}(0) = \rho(1 - q_{n_k}^l) + q_{n_k}^l$ for $l<0$. Indeed, site $(k,l)$ with $l<0$ is occupied with probability 1 if the queue extends beyond this site, and with probability $\rho$ if the queue is too short to reach this site.

Appendix C. Rate equations for a particle in a wake

C.1. Localization in the wake: general equations

Consider an $E$ particle in the shadow of another $E$ particle in a perpendicular flow of $N$ particles, as defined in section 5. In the case of the frozen shuffle update, the phase of the first $E$ particle creating the shadow is taken to be 0. The phase of the second $E$ particle is denoted as $\tau_0$. The alternating parallel update can be seen formally as a limiting case of the frozen shuffle update in which all the particles of the same type have the same phase, say 0 for the $E$ particles and $1/2$ for the $N$ particles. General equations can therefore be formulated in terms of some quantities depending on the updating scheme. These equations shall then be applied to the frozen shuffle update in section C.2 and to the alternating parallel update in section C.3.

We want to compute the probability $P_S(t)$ that the second particle remains in the shadow of the first one after $t$ time steps given by equation (4). We have shown that the shadow can be seen as a superposition of rows of two types $D$ and $K$ (figure 6(a)). In the frame of the shadow, the second $E$ particle hops from one row to the one below at each time step.

If, before hopping, the $E$ particle was in state $D_W$, then it will arrive in state $D_W$ if and only if the target row is of type $D$ (probability $1 - \rho$).

If, before hopping, it was in state $D_B$, then it will arrive in state $D_W$ only if it was blocked by an $N$ particle with phase $\tau' > \tau_0$ located just in front and if the target row is of type $D$. Note that the probability of the latter is not $1 - \rho$ anymore, because it is conditioned by the fact that there is an $N$ particle on the departure row, which makes it...
smaller than $1 - \rho$. As a result, the probability for this transition is

$$\rho \int_{\tau_0}^{1} d\tau' Q_{0,\tau'},$$

where $Q_{0,\tau'}$ denotes, for an unperturbed vertical flow of $N$ particles, the probability of having an empty site in $(i, j)$, under the condition that the preceding site $(i, j + 1)$ is occupied by an $N$ particle with a phase $\tau'$. If, before hopping, the $E$ particle was in state $K_{\tau'}$, i.e. if there was an $N$ particle of phase $\tau'$ located just in front, then the $E$ particle will arrive in state $D$ only if $\tau' > \tau_0$ and if the target row is of type $D$. Again, the latter probability is conditioned by the presence of the $N$ particle.

As a result of these different contributions, we get the first rate equation

$$P_W(t + 1) = (1 - \rho)P_W(t) + \rho P_B(t) \int_{\tau_0}^{1} d\tau' Q_{0,\tau'} + \int_{\tau_0}^{1} d\tau' p_{\tau'}(t) Q_{0,\tau'}.$$  \hfill (C.2)

In this equation and the following ones, $t$ is an integer but the phases $\tau, \tau'$ are continuous.

Similar reasoning leads to the equations for $P_B(t)$ and $p_{\tau'}(t + 1)$,

$$P_B(t + 1) = 0 + \left[ (1 - \rho)Q_{0,0} + \rho \int_{0}^{\tau_0} d\tau' Q_{0,\tau'} \right] P_B(t) + \int_{0}^{\tau_0} d\tau' p_{\tau'}(t) Q_{0,\tau'}$$ \hfill (C.3)

$$p_{\tau'}(t + 1) = \rho P_W(t) + P_B(t) [(1 - \rho)Q_{\tau,0}\Theta(\tau_0 - \tau)$$

$$+ \rho \int_{0}^{\tau} d\tau' Q_{\tau,\tau'} (\Theta(\tau' - \tau_0) + \Theta(\tau_0 - \tau))]$$

$$+ \Theta(\tau_0 - \tau) \int_{0}^{\tau} d\tau' Q_{\tau,\tau'} p_{\tau'}(t) + \Theta(\tau - \tau_0) \int_{\tau_0}^{\tau} d\tau' Q_{\tau,\tau'} p_{\tau'}(t),$$ \hfill (C.4)

where $\Theta$ is the Heaviside step function. $Q_{0,0}$, $Q_{\tau,0}$ and $Q_{\tau,\tau'}$ are the probabilities of having a site empty/occupied by a particle with phase $\tau$/occupied by a particle with phase $\tau'$, conditioned by the fact that the site in front is empty/empty/occupied by a particle with phase $\tau'$.

We now want to calculate the probability decay $P_S(t + 1) - P_S(t)$ of staying in the wake. One immediately sees that the coefficient of $P_W$ vanishes. Somewhat lengthy but simple calculations allow us to simplify the remaining terms to

$$P_S(t + 1) - P_S(t) = -P_B(t) \left\{ (1 - \rho) \int_{\tau_0}^{1} d\tau Q_{\tau,0} + \rho \int_{\tau_0}^{1} d\tau' \int_{\tau_0}^{\tau} d\tau' Q_{\tau,\tau'} \right\}$$

$$- \int_{0}^{\tau_0} d\tau' p_{\tau'}(t) \int_{\tau_0}^{1} d\tau' Q_{\tau,\tau'}.$$ \hfill (C.5)

One now has to replace the $Q$ by their explicit expressions in the four coupled equations (C.2)–(C.5) in order to solve them and evaluate the decay. This is carried out in section C.2 for the frozen shuffle update and in section C.3 for the alternating parallel update.

C.2. Frozen shuffle update

Using frozen shuffle update, the $N$ particles are injected such that the time delay $T$ (in continuous time) between the liberation of an entrance site and the introduction of a new particle on this site follows the exponential distribution $P(T)$ defined in equation (A.1).
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Actually, $P(T)$ gives the time delay distribution not only on the entrance site, but also on any site if we are in the free flow phase—which is the case considered in this subsection. As a consequence, the transition rates $Q$ can be expressed as follows:

$$Q_{\emptyset,\emptyset} = 1 - \int_0^1 P(T)\,dT = e^{-a} = 1 - \alpha$$
$$Q_{\emptyset,\tau'} = \int_1^\infty P(T - \tau')\,dT = e^{-a(1 - \tau')}$$
$$Q_{\tau,\emptyset} = P(\tau) = ae^{-\alpha}$$
$$Q_{\tau,\tau'} = P(\tau - \tau') = ae^{-a(\tau - \tau')} \quad \tau > \tau'.$$

(C.6)

With these expressions, the four coupled equations (C.2)–(C.5) define completely the time evolution for the probability $P_{fs}(t)$ of staying in the shadow. However, the integral forms for the $p_\tau$ variables make the resolution of these equations difficult for an arbitrary density. The equations become much simpler in the limit of small densities, for which one has $p_\tau = O(\rho_{fs})$ and $\alpha = \rho_{fs} + O(\rho_{fs})$. Indeed, in this case, it is possible to get rid of integrals by introducing new variables $P_{fs}^{<\tau_0}(t) \equiv \int_{\tau=0}^{\tau_0} p_\tau(t)\,d\tau$ and $P_{fs}^{>\tau_0}(t) \equiv \int_{\tau=\tau_0}^{1} p_\tau(t)\,d\tau$. Then equations (C.2)–(C.5) become equations (5) and (6) after some calculations.

C.3. Alternating parallel update

Here we give the expressions of the $Q$ for alternating parallel update. In contrast with the frozen shuffle update for which several $N$ particles could occupy successive sites, here two $N$ particles are separated by at least one hole in the direction of propagation. The conditional probabilities therefore read

$$Q_{\emptyset,\emptyset} = 1 - \alpha$$
$$Q_{\emptyset,\tau'} = \delta(\tau' - 1/2)$$
$$Q_{\tau,\emptyset} = \alpha\delta(\tau - 1/2)$$
$$Q_{\tau,\tau'} = 0,$$

(C.7)

where $\delta(x)$ is Dirac’s delta distribution. Substituting in equations (C.2)–(C.5) and using $\rho_{ap} = \alpha/(1 + \alpha)$ finally gives equations (8), which are true for all densities.

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