Finiteness of Multi-Body Neutrino Exchange Potential Energy in Neutron Stars

JIRO ARAFUNE and YUKIHIRO MIMURA

Institute of Cosmic Ray Research, University of Tokyo
Midori-Cho, Tanashi, Tokyo 188-0002, Japan

Abstract

The multi-body neutrino potential energy is analytically estimated for a spherical neutron star with a vector potential model for neutrinos. We show that the self-energy and the neutrino number of the neutron star coincide with the semi-classical values in the large volume limit, and confirm that there is no catastrophe in neutron stars with massless neutrinos.
1. Introduction

The massless neutrino exchange leads to a long-range force \[^{1}\] and in particular to a multi-body potential force whose astrophysical effects were discussed long ago. \[^{2}\] Recently Fischbach \[^{4}\] argued that the multi-body potential gives an unphysically large self-energy to stellar objects like neutron stars and concluded that neutrinos should have non-zero masses to resolve this paradox. Soon after that work several authors \[^{5, 6, 7}\] made claims in contradiction to this conclusion. Smirnov and Vissani \[^{5}\] argued that the mechanism by which the neutrino sea in neutron stars leading to blocking of long-range forces due to the Pauli principle should resolve the paradox. Abada et al. \[^{6}\] argued that such self-energy is small, exactly solving a (1+1) dimensional potential model and a (3+1) dimensional flat boarder model. Kiers and Tytgat \[^{7}\] made numerical analysis of the self-energy and the neutrino number of the ground state of a spherical neutron star, and found they tend to the semi-classical values

\[
W_{cl} = -\frac{\mu^4 R^3}{18\pi}, \quad (1.1)
\]

\[
q_{cl} = \frac{2(\mu R)^3}{9\pi} \quad (1.2)
\]

in the large volume limit and hence that no paradox exists.

In this paper we calculate the self-energy and the neutrino number of a spherical neutron star without recourse to numerical analysis and derive the same conclusion as Kiers et al. based on a large volume approximation. In Section 2, we introduce the Schwinger formula for the ground state energy, \(W\), in the case of a two-component (Weyl) spinor system. This is given by the difference between the ground state neutrino energy of the “vacuum” containing a neutron

\[2\]
star, $|0\rangle$, and that of the true vacuum $|0\rangle$. In the former, the neutrino propagates in the vector potential inside the neutron star due to $Z^0$ exchange,

$$W = \langle \hat{0}|H|\hat{0}\rangle - \langle 0|H_0|0\rangle,$$

(2.1)

where $H_0$ is the free Hamiltonian, and $H$ is the total Hamiltonian for the neutrinos with the neutron star. The latter corresponds to the Lagrangian for the (2-component) neutrino field $\chi$ and its hermitian conjugate $\bar{\chi}$,

$$\mathcal{L} = \bar{\chi}i\sigma^\mu \partial_\mu \chi + \frac{G_W}{\sqrt{2}} \bar{\chi}\sigma^\mu j_\mu,$$

(2.2)

with $\sigma^0 = \sigma_0 = 1$, and $\sigma^i = -\sigma_i$ ($i = 1, 2, 3$) the Pauli matrices. Here $j_\mu$ is the weak current of the neutrons proportional to the neutron density,

$$j_\mu \sim \langle n^\dagger n \rangle g_{\mu 0}.$$  

(2.3)

In the following we consider the model Lagrangian of the neutron stars [6]

$$\mathcal{L} = \bar{\chi}(i\sigma^\mu \partial_\mu + \mu(x))\chi,$$

(2.4)

where

$$\mu(x) = \mu \theta(R - |x|),$$

(2.5)

with $R$ the neutron star radius and $\mu = \frac{G_W}{\sqrt{2}}\langle n^\dagger n \rangle$ a constant, typically on the order of several eV.

Since the Hamiltonian densities in Eq. (2.1) are given by $-i \lim_{x' \to x} \frac{\partial}{\partial x_0} \text{tr}\chi(x)\bar{\chi}(x')$, the Schwinger formula for the energy is given by

$$W = -i \int d^3x \frac{\partial}{\partial x_0} \text{tr}(S(x, x') - S^0(x, x'))_{x' \to x}$$

(2.6)

$$= -\frac{1}{2\pi i} \int dE \text{Tr}(\ln(E - H) - \ln(E - H_0)),$$

(2.7)

where the symbol Tr is the trace over both spinor and configuration space indices, $S(x, x')$ is the Feynman propagator defined as

$$(i\sigma^\mu + \mu(x))S(x, x') = i\delta(x, x'),$$

(2.8)

$$\langle 0|T\chi(x)\bar{\chi}(x')|0\rangle = S(x, x'),$$

(2.9)

and $S^0(x, x')$ is that of the free neutrino. In Eq. (2.7) and thereafter we use the same notation $H$ for the quantum mechanical Hamiltonian, $H = -i\sigma\nabla - \mu(x)$, as
for the field theoretical one in Eq. (2.1), since this should have no confusion. To derive Eq. (2.7) from Eq. (2.6) we have used the formula

$$S(x, x') = -\frac{1}{2\pi i} \lim_{\epsilon \to +0} \int dE(x) \frac{1}{E(1 + i\epsilon) - H} |x'\rangle e^{-iE(x' - x)},$$

(2.10)

and a similar one for $S^0(x, x')$. The prescription with $i\epsilon$ is introduced here to satisfy the boundary condition of the time ordering propagator. The neutrino number, $q$, is similarly given as

$$q = -\int d^3x \text{tr}(S(x, x') - S^0(x, x'))_{x' \to x}$$

(2.11)

$$= \frac{1}{2\pi i} \int dE \text{Tr} \left( \frac{1}{E - H} - \frac{1}{E - H_0} \right).$$

(2.12)

In order to avoid the apparent divergence of the expressions for $W$ and $q$, we differentiate the energy with respect to $\mu$ twice and the neutrino number once to obtain

$$\frac{d^2W}{d\mu^2} = \frac{1}{2\pi i} \int dE \int d^3x \int d^3y \text{ tr} \left( \frac{1}{E - H} \right)_{x,y} \theta(R - |x|) \left( \frac{1}{E - H} \right)_{y,x} \theta(R - |y|),$$

(2.13)

$$\frac{dq}{d\mu} = -\frac{1}{2\pi i} \int dE \int d^3x \int d^3y \left( \frac{1}{E - H} \right)_{x,y} \left( \frac{1}{E - H} \right)_{y,x} \theta(R - |x|).$$

(2.14)

In Eq. (2.13) we see that the neutrino propagates from a point $x$ inside the neutron star to another point $y$ also inside the neutron star. Noting $\mu R \sim 10^{12}$, we see that the neutrino propagator rapidly oscillates in the neutron star, and thus only a short distance with $|x - y| \ll R$ or finite $\mu |x - y|$ contributes to the integration of Eq. (2.13) and (2.14). Thus it is safe to replace the full propagator of the neutrino

$$\left( \frac{1}{E - H_0 + \mu \theta(R - r)} \right)_{x,y}$$

(2.15)

with the approximate one

$$\left( \frac{1}{E - H_0 + \mu} \right)_{x,y}$$

(2.16)

to evaluate Eqs. (2.13) or (2.14), neglecting the small contribution of such neutrinos that virtually propagate out of the neutron star (see Fig. 1).
Figure 1: We neglect the neutrino propagating through path (b). We only take the path (a) into account.

This approximation leads to

\[
\left( \frac{1}{E - H_0 + \mu \theta(R - r)} \right) x \cdot y \approx \left( \frac{1}{E - H_0 + \mu} \right) x \cdot y
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{E + \mu - \sigma p} e^{i p(x - y)}
\]

\[
= -\left( E + \mu + \frac{\sigma r}{r} \left( \frac{i}{r} + p^+ \right) \right) \frac{e^{ip^+ r}}{4\pi r},
\]

(2.17)

where \( r = x - y \) and \( p^+ = \varepsilon(E)(E + \mu) \) (\( \varepsilon(E) = 1 \) \((-1)\) for \( E > 0 \) \((E < 0)\)). The Eq. (2.17) represents nothing but a full propagator in condensed matter of infinite volume.

The Schwinger formula (2.7) has an ultraviolet divergence proportional to \( \mu^2 \), which corresponds to just the familiar vacuum polarization diagram for \( Z^0 \) and should be subtracted. Thus, the right-hand side of Eq. (2.13), a second derivative of \( W \), has a constant divergence which should be subtracted. Equation (2.12) for the neutrino number also has a divergence. This divergence, linear in \( \mu \), corresponds to a one-loop correction to the neutrino density operator, familiar in the correction in the operator product expansion, and is caused by the operator mixing of the neutrino density and neutron density. This is also to be subtracted by renormalization. Thus the right-hand side of Eq. (2.14), the first derivative of \( q \), should be renormalized to be zero at \( \mu = 0 \).

* Here the \( i\varepsilon \) prescription is similar to that used in Eq. (2.10).

† We note that a divergence cannot be subtracted in case of an electron-condensate medium, since the exchange of charged weak boson is involved. In such a case we need a cut-off scale \( \Lambda_c \), which is the weak boson mass or the inverse of the mean distance of the electrons.
3. Evaluation of Energy and Neutrino Number

In this section, we evaluate the self-energy and neutrino number of a neutron star with Eqs. (2.13) and (2.14) using the approximation Eq. (2.17). It is easy to formally obtain

$$\frac{d^2W}{d\mu^2} = \frac{1}{16\pi^3} \int_{|x|\leq R} d^3x \int_{|y|\leq R} d^3y \, r^{-5}(\cos 2\mu r + \mu r \sin 2\mu r), \quad (3.1)$$

with $r = y - x$. We renormalize it by subtracting the divergent term to obtain

$$\frac{d^2W}{d\mu^2} \bigg|_{\text{ren}} \approx \frac{1}{8\pi^3} \int_{|x|\leq R} d^3x \int_{|y|\leq R} d^3y \, r^{-5}(\cos 2\mu r + \mu r \sin 2\mu r - 1). \quad (3.2)$$

Noting the convergence of the integrand of Eq. (3.3) both at $r = 0$ and $r = \infty$, we see that the change of the integration region $\int_{|x|\leq R} d^3x \int_{|y|\leq R} d^3r \rightarrow \int_{|x|\leq R} d^3x \int_{|r|<\infty} d^3r$ is allowed in the large volume limit. Then the integration in Eq. (3.3) is analytically performed to give

$$\frac{d^2W}{d\mu^2} \bigg|_{\text{ren}} \approx \frac{1}{8\pi^3} \int_{|x|\leq R} d^3x \int_{r<\infty} 4\pi r^2 dr \, r^{-5}(\cos 2\mu r + \mu r \sin 2\mu r - 1) \quad (3.4)$$

$$= \frac{1}{8\pi^3} \int_{|x|\leq R} d^3x (-4\pi\mu^2) \quad (3.5)$$

$$= -\frac{2}{3\pi}\mu^2 R^3. \quad (3.6)$$

By integration we obtain the renormalized self-energy

$$W = -\frac{1}{18\pi}\mu^4 R^3, \quad (3.7)$$

where we have used the fact [7] that $W$ is an even function of $\mu$ and $W|_{\mu=0} = \frac{d^2W}{d\mu^2}|_{\mu=0} = 0$. The result Eq. (3.7) coincides with the semi-classical value of Eq. (1.1).

Here we make some remarks regarding some subtleties of the calculation. If we differentiate $W$ with respect to $\mu$ more times, we obtain better convergence at short distance. In this sense it is safer to evaluate $W$ in terms of higher derivatives. In fact we have no divergence in $W^{(n)}(\mu)$ for $n \geq 3$. In the case of $W^{(1)}(\mu)$, apparently
we may have divergence worse than in the case of $W^{(2)}$. When we calculate $W^{(1)}(\mu)$ and use the approximation Eq. (2.17), however, we obtain a finite result,

$$\frac{dW}{d\mu} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE \int_{|x|<R} d^3x \left( 2(E+\mu)e^{ip+r} - 2(E+\mu)e^{i-p+r} \right) \Bigg|_{r \to 0} \tag{3.8}$$

$$\approx \frac{1}{2\pi} \int_{|x|<R} d^3x \left( \frac{\mu r \cos \mu r - \sin \mu r}{r^3} \right) \Bigg|_{r \to 0} \tag{3.9}$$

$$= -\frac{2}{9\pi} \mu^3 R^3, \tag{3.10}$$

which is consistent with the renormalized one, Eq. (3.7). The ultraviolet divergence linear in $\mu$ is accidentally cancelled in this calculation. Since Eq. (3.9) has a superficially divergent dimension, it requires a more careful calculation by taking account of the boarder effect. In order to study the boarder effect, let us take $\frac{d^2W}{d\mu dR}$

$$\frac{d^2W}{d\mu dR} = -\frac{1}{2\pi i} \int dE \text{Tr} \left( \frac{1}{E-H} \delta(R-|x|) \right)$$

$$+ \frac{\mu}{2\pi i} \int dE \text{tr} \int d^3x \int d^3y \left( \frac{1}{E-H} \right)_x \delta(R-|y|) \left( \frac{1}{E-H} \right)_y \theta(R-|x|). \tag{3.11}$$

We note this expression is exact. We again approximate it with the full propagator by Eq. (2.17). The result has an ultraviolet divergence linear in $\mu$.

After renormalizing as explained above, we obtain

$$\frac{d^2W}{d\mu dR} \bigg|_{\text{ren}} = -\frac{2}{3\pi} \mu^3 R^2 + \mu \frac{\cos 4\mu R + 4\mu R \text{Si}(4\mu R)}{8\pi R}, \tag{3.12}$$

where $\text{Si}(x) = \int_0^x dt \frac{\sin t}{t}$. This gives the self-energy of the neutron star as

$$W = -\frac{\mu^4 R^3}{18\pi} \left( 1 + O \left( \frac{1}{R} \right) \right), \tag{3.13}$$

which again coincides with the classical value Eq. (1.1) in the large $R$ limit.

Now let us evaluate the neutrino number. Equation (2.14) for $\frac{dq}{d\mu}$ resembles Eq. (2.13) for $\frac{d^2W}{d\mu^2}$, except for the sign and the range of the $y$ integration. Since the quantity

$$\int_{|x|<R} d^3x \int_{|y|>R} d^3y \text{tr} \left( \frac{1}{E-H} \right)_x \left( \frac{1}{E-H} \right)_y \delta(R-|x|) \tag{3.14}$$

7
should be negligibly small for large $\mu R$, they should coincide (except for the sign) to give
\begin{equation}
\left. \frac{dq}{d\mu} \right|_{\text{ren}} = - \left. \frac{d^2 W}{d\mu^2} \right|_{\text{ren}} = \frac{2}{3\pi} \mu^2 R^3, \tag{3.15}
\end{equation}
with the aid of Eq. (3.6). This gives the neutrino number,
\begin{equation}
q = \frac{2}{9\pi} \mu^3 R^3, \tag{3.16}
\end{equation}
which is equal to the semi-classical value in Eq. (1.2).

4. Discussion

In this paper we estimated the self energy and the neutrino number of neutron stars without recourse to numerical analysis, and confirmed that they coincide with the semi-classical values in the large volume limit. They do not lead to paradox nor to any lower bound for the neutrino masses. While our estimation was made non-perturbatively, it may be instructive to evaluate the self-energy perturbatively by expanding the integrand of Eq. (3.3) with respect to $\mu$ and integrating each term over $x$ and $y$ without using the approximation Eq. (3.4). The integration is easy to perform and leads to an alternating series in powers of $\mu R$. This series is similar (though not exactly equal) to that considered in Ref. [4] and an example where only the sum (but not each term individually) is meaningful in large $R$ limit.

Note Added in Proof

In Ref. [6] Abada et al. estimated the self energy and the neutrino number by independent methods and got results different from ours. Their self energy is linear in the external potential ($\mu$ in our notation), while it should be even to keep CP invariance. We think that the difference is due to their incomplete Hamiltonian violating CP invariance without taking the average of it and its CP conjugation. Our method keeps the invariance, though implicitly, by taking the neutrino propagator with adequate $i\epsilon$ procedure. Kiers et al. [7] also analyzed them independently with the result in good agreement with ours.
Acknowledgement

The authors are grateful to V. Rubakov for valuable comments and to E. Fischbach, M. Koike, and K. Ogure for stimulating discussions.

References

[1] G. Feinberg and J. Sucher, Phys. Rev. 166 (1968) 1638.
[2] J.B. Hartle, Phys. Rev. D1 (1970) 394.
[3] R. Feynman, in Feynman Lectures on Gravitation, Ed. R. Feynman et al. (Addison-Wesley, 1995, 232 p.).
[4] E. Fischbach, Ann. Phys. 247 (1996) 213.
[5] A.Y. Smirnov and F. Vissani, hep-ph/9604443.
[6] A. Abada, M. B. Gavela and O. Pène, Phys. Lett. B387 (1996) 315.
    A. Abada, O. Pène and J. Rodríguez-Quintero, Phys. Lett. B423 (1998) 355; Phys. Rev. D58 (1998) 073001.
[7] K. Kiers and M. H. G. Tytgat, Phys. Rev. D57 (1998) 5970; hep-ph/9807412.
[8] J. Schwinger, Phys. Rev. 94 (1954) 1362.