Spherically symmetrical configurations of self-dual Yang-Mills and Einstein-Plebanski equations

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Abstract

Spherically symmetrical reductions of self-dual Yang-Mills and Einstein-Plebanski equations are constructed at the same manner. As in the first case we come back to known before solutions (under such kind of reduction but in some different form) if in the second one we obtain unknown before equation describing the spherically symmetrical configurations of Plebanski equation and some number of its particular solutions.
1 Introduction

Outstanding similarity the form of self-dual Yang-Mills and Plebanski \[1\] heavenly equations give to many investigators possibility to assume that these two systems (really system in the first case and single equation in the second one) possesses the solutions of the similar kind (of course in some sense).

In the present paper we would like to investigate this problem on the example of spherically symmetrical solutions of such systems. In the case of self-dual Yang-Mills system (for arbitrary semisimple gauge algebra and arbitrary \(A_1\) subalgebra embedding in it) such kind of solution were obtained 20 years ago in \[2\]. Corresponding result with respect to Plebanski equation to the best of our knowledge is absent up to now.

We will use the following forms (in each case one of many other possible ones) for Y-M and P equations

\[
M_{y,\bar{y}} + M_{z,\bar{z}} = [M_y, M_z], \quad P_{y,\bar{y}} + P_{z,\bar{z}} = P_{y,y}P_{z,z} - P_{y,z}^2 \equiv \{P_y, P_z\}_{y,z}\tag{1}
\]

where function \(M\) takes values in arbitrary semisimple algebra, \(P\) is scalar function.

We will use the same reducing procedure to spherically symmetrical case to both equations \([1]\), which allow as to compare and emphasize essential differences in either cases.

2 Yang-Mills case

Let us use for solution of \([1]\) the following ansatz

\[
M = (\bar{y})^{-1}m(r,t), \quad r^2 = y\bar{y} + \left(\frac{z + \bar{z}}{2}\right)^2, \quad t = \frac{z - \bar{z}}{2i}\tag{2}
\]

For derivatives involved in \([1]\) we obtain

\[
M_y = \frac{1}{2}\frac{m_r}{r}, \quad M_{y,\bar{y}} = \frac{y}{4} \left(\frac{m_r}{r}\right)_r
\]

\[
M_z = (\bar{y})^{-1}\left(\frac{z + \bar{z}}{4}\right)\left(\frac{m_r}{r}\right)_r + \frac{1}{2i}m_t, \quad M_{zz} = (\bar{y})^{-1}\left(\left(\frac{z + \bar{z}}{4}\right)^2\left(\frac{m_r}{r}\right)_r + \frac{1}{4}m_{tt}\right)
\]

Substituting these expressions into first equation \([1]\) we obtain

\[
m_{rr} + m_{tt} = \frac{1}{2ir}[m_t, m_r]\tag{3}
\]
After introduction of two complex conjugative variables \( \xi = t + ix, \bar{\xi} = t - ix \) we are able to rewrite (3) in the form of the system of equations of the main chiral field with the moving poles

\[
(\xi - \bar{\xi})m_{\xi, \bar{\xi}} = [m_{\xi}, m_{\bar{\xi}}]
\]  

(4)

The technique of integration of (4) reader can find in [3].

3 Plebanski equation

In this case the ansatz (5) is slightly modifies \[1\]

\[
P = (\bar{y})^{-2}p(x, t), \quad x \equiv r^2 = y\bar{y} + \left(\frac{z + \bar{z}}{2}\right)^2, \quad t = \frac{z - \bar{z}}{2i}
\]

(5)

For the derivatives involved in Plebanski equation we have in a consequent

\[
P_y = (\bar{y})^{-1}p_x, \quad P_{yy} = p_{xx}, \quad P_{yz} = (\bar{y})^{-1}\left(p_{xx}\frac{z + \bar{z}}{2} + p_{xt}\frac{1}{2i}\right),
\]

\[
P_z = (\bar{y})^{-2}(p_x\frac{z + \bar{z}}{2} + p_t\frac{1}{2i}), \quad P_{zz} = (\bar{y})^{-2}(p_{xx}\left(\frac{z + \bar{z}}{2}\right)^2 + p_{xt}\frac{z + \bar{z}}{2i} + \frac{1}{2}p_x - \frac{1}{4}p_t),
\]

\[
P_{y,\bar{y}} = (\bar{y})^{-2}(y\bar{y}p_{xx} - p_x), \quad P_{z,\bar{z}} = (\bar{y})^{-2}(p_{xx}\left(\frac{z + \bar{z}}{2}\right)^2 + p_t\frac{1}{4} + \frac{1}{2}p_x)
\]

Substituting these expressions into Plebanski equation (1) we obtain

\[
p_{xx}p_{tt} - p_{xt}^2 = 2p_xp_{xx} + 2p_x - p_{tt} - 4xp_{xx}
\]

and after exchanging of unknown function \( p \rightarrow p - \frac{x^2}{2} \) we rewrite the previous equation in the form

\[
p_{xx}p_{tt} - p_{xt}^2 = 2p_xp_{xx} - 6xp_{xx} + 4x = (p_x^2 - 6xp_x - p) + 2x^2 \equiv F_x
\]

(6)

The last is nonhomogeneous Monge-Amphere equation. The regular methods for its integration are unknown.

Let us formally rewrite (3) in form of Poisson breackets

\[
(p_x)_F(p_t)_t - (p_x)_t(p_t)_F = 1
\]

\[\uparrow\]The different degrees on \( \bar{y} \) before the invariant function reflects the facts of different spins 1 and 2 in both cases
and resolve the last equation with the help of generating function of canonical transformation $W(p_x, t)$

$$p_t = W_{p_x}(p_x, t), \quad F = p_x^2 - 6xp_x - 2x^2 = W_t(p_x, t) \quad (7)$$

But under such kind of the procedure generating function $W$ is not arbitrary one but satisfy some equation obtaining of which is our nearest goal.

Let us differentiate two equations (7) with respect to $x$ and $t$. Result of differentiation of the first equation is the following

$$p_{tx} = W_{p_x,p_x}p_{xx}, \quad p_{tt} = W_{p_x,p_x}p_{tx} + W_{p_x,t} \quad (8)$$

and the same with respect to second equation (7)

$$2p_xp_{xx} - 6xp_{xx} + 4x = W_{p_x,t}p_{xx}, \quad 2p_xp_{xt} - 6xp_{tx} + 6p_t = W_{p_x,t}p_{xt} + W_{tt} \quad (9)$$

Multiplying the second equation (8) on $p_{xx}$ and keeping in mind two first equations (8) and (16) we conclude that equation (6) is satisfied. Resolving (16) with respect to $p_{xx}, p_{xt}$ we obtain ($p_t = W_{p_x}$)

$$p_{xx} = \frac{4x}{W_{t,p_x} + 6x - 2p_x}, \quad p_{xt} = \frac{6W_{p_x} - W_{tt}}{W_{t,p_x} + 6x - 2p_x} \quad (10)$$

Combining (10) with the first equation (8) we able to express $x$ in terms of derivatives of the generating function

$$4x = \frac{6W_{p_x} - W_{tt}}{W_{p_x,p_x}} \quad (11)$$

Result of differentiation of the last equation with respect to $x, t$ arguments lead to

$$4 = \left(\frac{6W_{p_x} - W_{tt}}{W_{t,p_x}}\right)_{p_x}p_{xx}, \quad 0 = \left(\frac{6W_{p_x} - W_{tt}}{W_{t,p_x}}\right)_{p_xt} + \left(\frac{6W_{p_x} - W_{tt}}{W_{t,p_x}}\right)_t \quad (12)$$

Keeping in mind that $p_{xx}, p_{xt}$ as a corollary of (10) and (11) are expressed in terms of the derivatives of generating function $W$, we conclude that (12) really is a system of equations which this function must satisfy. Introducing the notation $\alpha = \frac{6W_{p_x} - W_{tt}}{W_{p_x,p_x}}$ we rewrite (12) as

$$\alpha_t + 4W_{p_x,p_x} = 0, \quad \alpha p_{xx} = 4W_{p_x,t} + 6\alpha - 8p_x$$
Let us substitute $\alpha = \beta_{p_x}$. The last substitution allows to integrate once the previous system with the result

$$\beta_t = -4W_{p_x} + A(t), \quad \frac{\beta^2_{p_x}}{2} = 4W_t + 6\beta - 4p_x^2 + B(t) \quad (13)$$

From definition of $\alpha$ function

$$\alpha = \beta_{p_x} = \frac{6W_{p_x} - W_{tt}}{W_{p_x,p_x}}$$

and known expressions for derivatives of $W$ function via $\beta$ function (13) it follows immediately that

$$B_t(t) + A(t) = 0$$

And at last after excluding $W$ from system (13) we come to a single equation which function $\beta$ satisfy

$$\beta_{tt} + \left(\frac{\beta^2_{p_x}}{2}\right)_{p_x} = -8p_x + 6\beta_{p_x} + A_t \quad (14)$$

After differentiation of the last equation once with respect to $p_x \equiv y$ we come to equation, which $\alpha$ function satisfy

$$\alpha_{tt} + \left(\frac{\alpha^2}{2}\right)_{yy} = -8 + 6\alpha_y \quad (15)$$

4 Some particular solutions

Here we would like to present some number of particular solutions, arising as a result of the corresponding reductions. We like to emphasize here that all equations (and systems) below allow the general solution.

4.1 Anzats $\alpha = x^2A(t) + xB(t) + C(t)$

Substituting the anzats of the title of this subsection into (15) and equating to zero coefficient on 0, 1, 2 degrees of $x$ variable we obtain

$$x^2\ddot{A}(t) + x\ddot{B}(t) + \ddot{C}(t) = 2A(x^2A(t) + xB(t) + C(t)) + (2Ax + B)^2 = -8 + 6(2Ax + B)$$

$$\ddot{A} + 6A^2 = 0, \quad \ddot{B} + 6AB = -12A, \quad \ddot{C} + 2AC = -(B - 2)(B - 4) \quad (16)$$
Equation for $A$ function can be resolved in terms of particular case of Weirstrasse function $[4]$

$$\dot{A} = \sqrt{C - 4A^3}. $$

Equation for $B$ is integrable in quadratures with the result

$$B = -2 + c_1 A + c_2 \int \frac{dt}{B^2}$$

In terms of the Weirstrasse like function may be resolved in quadratures equation for $C$ function (see $[4]$).

### 4.2 The travelling wave $\alpha \equiv \alpha(x + vt)$

Under such substitution the ordinary differential equation takes the form

$$(\alpha + v^2)\alpha'' + (\alpha')^2 = -8 + 6\alpha'$$

where $\alpha' = \alpha_\xi, \xi = x + vt$. After introducing $\tilde{\alpha} = \alpha + v^2$ the last equation allows integration ones with the result

$$\tilde{\alpha}\tilde{\alpha}' = -8(\xi + c_1) + 6\tilde{\alpha}$$

Consequent exchanging $\xi \to \xi + c_1, \tilde{\alpha} = \xi\nu$ leads the last equation to a form

$$\xi\nu' + \nu = -\frac{8}{\nu} + 6$$

The way of integration of the last equation is obvious with the result

$$\frac{\nu d\nu}{(\nu - 2)(\nu - 4)} = -d\ln\xi, \quad \frac{(\nu - 4)^2}{\nu - 2} \xi = c_2$$

Returning to initial variables and functions we obtain the following general solution of (17) (but only particular solution of (15))

$$\alpha = -v^2 + \frac{(8(x + vt) + \tilde{c}_1) \pm \sqrt{c_2(8(x + vt) + \tilde{c}_1)}}{2}, \quad \tilde{c}_1 = c_2 + 8c_1$$
4.3 Automodel solution

Equation (13) is obviously invariant with respect to transformation

\[ \alpha(t, x) = p^{-1} \alpha(\sqrt{pt}, px) \]

As a corollary it arised automodel solution of this equation

\[ \alpha = x f \left( \frac{x}{t^2} \right) \]

Using anzats above we obtain for corresponding derivatives (\( \xi \equiv \frac{\xi}{t^2} \))

\[ \alpha_x = (\xi f)_\xi, \quad \alpha_{xx} = (\xi f)(\xi f)_\xi \xi \]

\[ \alpha_t = -2t \xi^2 f_\xi, \quad \alpha_{tt} = -2t \xi^2 f_\xi + 4 \xi \xi (f_\xi)_\xi \]

Introducing the new unknown function \( \nu \equiv \xi f \) we rewrite (15) (\( \nu_\xi \equiv \nu' \))

\[ \nu = -\xi^2 + 2 \xi + c, \quad \nu = c \xi - \frac{(c - 2)(c - 4)}{2}, \quad \nu = 4 \xi + c \xi^2 \quad (21) \]

We present below three its different particular solutions, which can be checked by direct not cumbersome calculations

\[ \nu = -\xi^2 + 2 \xi + c, \quad \nu = c \xi - \frac{(c - 2)(c - 4)}{2}, \quad \nu = 4 \xi + c \xi^2 \quad (21) \]

All these solutions are depending on only one arbitrary parameter \( c \), while the general solution of (20) must contain two arbitrary parameters.

To obtain general solution it is necessary to consider not obvious and simple but very important symmetry of (20).

With this goal let us perform two consequent transformations

\[ \xi \rightarrow \xi + q, \quad \nu \rightarrow \tilde{\nu} - 8q \xi - 4q^2 \]

As a result of such transformations equation (20) takes the form

\[ -2(\xi \ddot{\nu} - \dot{\nu}) + 4 \xi^2 \dddot{\nu} + \ddot{\nu} \nu'' = -(\ddot{\nu})^2 + (6 + 18q) \ddot{\nu}' - (2 + 8q)(4 + 8q) - 8q^2 \quad (22) \]

Choosing \( q = -\frac{1}{3} \) we obtain finally

\[ -2(\xi \ddot{\nu} - \dot{\nu}) + 4 \xi^2 \dddot{\nu} + \ddot{\nu} \nu'' = -(\ddot{\nu})^2 \]

(23)
The last equation is obviously invariant with respect to transformation
\[ \tilde{\nu}(\xi) \rightarrow p^{-2}\tilde{\nu}(p\xi) \]
Gathering all results together we obtain in consequent. If \( \nu(\xi) \) is solution of \( (20) \) than both \( \tilde{\nu} \) are solutions of the equation \( (23) \).
\[ \tilde{\nu} = \nu(\xi - \frac{1}{3}) - \frac{8\xi}{3} + \frac{4}{9}, \quad \tilde{\nu} = p^{-2}(\nu(\xi - \frac{1}{3}) - \frac{8p\xi}{3} + \frac{4}{9}) \]
Performing the inverse transformation we return to a new solution of the initial system in the form
\[ \nu = p^{-2}[\nu(p(\xi + \frac{1}{3}) - \frac{1}{3}) - \frac{8p(\xi + \frac{1}{3})}{3} + \frac{4}{9}] + \frac{8(\xi + \frac{1}{3})}{3} - \frac{4}{9}, \quad (24) \]
\( (24) \) is the mentioned above not simple and obvious invariant transformation of the equation \( (20) \).
Two first solutions of \( (21) \) are covariant with respect to such transformation – solution preserve their forms with changed value of the constant of integration. In the first case \( c \rightarrow p^{-2}(c - \frac{p^2}{3} - \frac{1}{3}) + \frac{4}{9} \) in the second one \( c \rightarrow p^{-1}c - p^{-1}\frac{8}{3} + \frac{8}{3} \).
But third solutions after such transformation does not preserve its form and passes to a general solution of the system \( (20) \) dependent on two arbitrary constants \( p \) and \( c \)
\[ \nu = p^{-2}[4(p(\xi + \frac{1}{3}) - \frac{1}{3}) + c\sqrt{p(\xi + \frac{1}{3}) - \frac{1}{3}} - \frac{8p(\xi + \frac{1}{3})}{3} + \frac{4}{9}] + \frac{8(\xi + \frac{1}{3})}{3} - \frac{4}{9}, \quad (25) \]
That \( (23) \) is really (general) solution of \( (20) \) one can convinced by direct computation. Going back to \( \alpha \) we obtain the particular solution of the initial equation \( (15) \)
\[ \alpha = (\frac{4}{3}p^{-1} + \frac{8}{9})x + \frac{4}{9}p^{-2}(p - 1)(p + 2)t^2 + p^{-2}ct\sqrt{px + \frac{p - 1}{3}t^2}, \quad (26) \]

5 Outlook

The main result of the present paper consists in new equation \( (15) \) describing the spherically symmetrical configurations of Plebanski equation and some
number of its particular solutions. At this moment we are unable to say anything definite is this equation integrable or not. At least to the list of known up to now integrable equations it doesn’t belongs. We hope in future publications find the answer on this intrigued question.

References

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