Finite size vertex corrections to the three-gluon decay widths of $J/\psi$ and $\Upsilon$ and a redetermination of $\alpha_s(\mu)$ at $\mu = m_c$ and $m_b$

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Abstract

We calculate the corrections to the three-gluon decay widths $\Gamma_{3g}$ of $c\bar{c}$ and $b\bar{b}$ quarkonia due to the finite extension of the $Q\bar{Q} \to 3g$ vertex function. The widths computed with zero range vertex are reduced by a factor $\gamma$ where $\gamma = 0.31$ for the $J/\psi$ and $\gamma = 0.69$ for the $\Upsilon$. These large corrections necessitate a redetermination of the values $\alpha_s(\mu)$ extracted from $\Gamma_{3g}$. We find $\alpha_s(m_c) = 0.28 \pm .01$ and $\alpha_s(m_b) = 0.20 \pm .01$. 

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Since the discoveries of the charmonium and bottomonium states, the physics of heavy quark systems has been of continuous interest [1, 2]. The bound state structure of heavy quarkonia becomes simpler with increasing mass, because the constituents move more and more non-relativistically in a static potential which is dominated for \( r < 0.1 \text{ fm} \) by the Coulomb-like field and for \( r > 0.2 \text{ fm} \) by the confining potential. The Schrödinger equation with this potential reproduces successfully the energies of the bound states \( 1s, 2p, ... \). The leptonic widths \( \Gamma(Q\bar{Q} \to \mu\mu) \) of the \( ^3S_1 \)-states, modified by radiative corrections, can also be obtained by an expression which is proportional to \( |\psi(0)|^2 \), where \( \psi(\vec{r}) \) is the non-relativistic bound state wave function. The value \( |\psi(0)|^2 \) is rather sensitive to the potential [3, 4]. Problems exist with the gluonic width \( \Gamma(Q\bar{Q} \to 3g) \) and with the photo-gluonic width \( \Gamma(Q\bar{Q} \to \gamma gg) \). One usually calculates these quantities in the approximation of a zero-range vertex, i.e. \( \Gamma(Q\bar{Q} \to 3g) \propto |\psi(0)|^2 \) and analogous for \( \Gamma(Q\bar{Q} \to \gamma 2g) \). From a comparison with experiment one extracts a value for the strong interaction coupling constant \( \alpha_s(\mu) \), where \( \mu \) is the scale (\( \mu = m_c \) for \( J/\psi \) and \( \mu = m_b \) for \( \Upsilon \)) [3, 4]. The analysis usually proceeds in three steps. In the first step, expressions for the ratios \( R_1 \) and \( R_2 \)

\[
R_1 = \frac{\Gamma(Q\bar{Q} \to \gamma gg)}{\Gamma(QQ \to ggg)} \\
R_2 = \frac{\Gamma(Q\bar{Q} \to 3g)}{\Gamma(QQ \to \mu\mu)} \tag{1}
\]

are derived in leading order \( \alpha_s \). In this order and using the zero-range approximation for the vertex, the decay rates are obtained from analogous calculations for the three-photon decay of the triplet state of positronium with the photon-electron coupling replaced by the \( SU3_c \) coupling of gluons to quarks. Because of the zero-range approximation, the rate is proportional to \( |\psi(0)|^2 \), the square of the non-relativistic wavefunction, which cancels out in the ratios \( R_1 \) and \( R_2 \). In the second step, the lead-
ing order result for \( R_1 \) and \( R_2 \) is multiplied by the first order \( \alpha_s \) corrections, which represent a factor of the form \((1 + B(\mu)\alpha_s(\mu)/\pi)\). The coefficient \( B(\mu) \) depends both on the scale \( \mu \) and the renormalization scheme. In a third step, one usually takes into account corrections of order \( \langle q^2/m^2 \rangle \) phenomenologically \[3\). These corrections may come from the finite extension of the decay vertex and from relativistic corrections in the wavefunction or from other nonperturbative physics like condensates. In this paper we give the first ab initio calculation of the effect from the finite extension of the decay vertex, which we take into account by a calculated factor \( \gamma \), where \( \gamma = 1 \) represents the zero range situation. The status of theoretical calculations for \( R_1 \) and \( R_2 \) is given in Table 1 for the \( J/\psi \) and \( \Upsilon \) together with the experimental values for the two ratios. Values for \( \alpha_s(\mu) \) extracted from a comparison of the theoretical expressions (\( \gamma = 1 \)) with the data are also shown in this table.

The following observations can be made for the zero-range case: (i) The ratios \( R_1 \) and \( R_2 \) give rather consistent values for \( \alpha_s(\mu) \) for the same decaying system (\( J/\psi \) or \( \Upsilon \)). (ii) Using the values for \( R_2 \) which have small error bars, one obtains precise values for \( \alpha_s(m_c) \) and \( \alpha_s(m_b) \) from \( J/\psi \) and \( \Upsilon \) decays, respectively. The two values are nearly equal. This result is unexpected, because in a simple-minded QCD analysis, the scale dependence of \( \alpha_s(\mu) \) would predict

\[
\alpha_s(m_c) \simeq \alpha_s(m_b) \frac{\ln m_b^2/\Lambda^2}{\ln m_c^2/\Lambda^2} \simeq 0.28 \pm .02
\]

instead of \( \alpha_s(m_c) = 0.19 \pm .01 \) where we used \( \Lambda = 200 \text{ MeV} \). This discrepancy has been observed before \[3\] and a correction factor has been parametrized but never calculated.

We improve on the previous calculations by taking the quark-momentum dependence (or finite size) of the transition operator \( Q\bar{Q} \rightarrow 3g \) into account. This correction cancels in the ratio \( R_1 \), since the rates in the numerator and denominator are affected in the same way. In the ratio \( R_2 \), only the three-gluon state is modi-
fied, whereas the s-channel photon annihilation still samples \( |\psi(0)|^2 = \int d^3q \tilde{\psi}(\vec{q})|^2 \), where \( \tilde{\psi}(q) \) is the Fourier transform of \( \psi(\vec{r}) \). The three-gluon decay amplitude can be written as

\[
T(k_1, k_2, k_3) = \int \frac{d^4q}{(2\pi)^4} \tilde{\psi}_{BS}(p, q) iM(p, q, k_1, k_2, k_3),
\]

where \( k_i, i = 1, 2, 3 \) are the four momenta of the gluons, \( M \) is the expression for the \( Q\bar{Q} \rightarrow 3g \) vertex and \( \tilde{\psi}_{BS}(p, q) \) stands for the Bethe-Salpeter wavefunction of the quarkonium state. In eq. (3) \( p \) denotes the c.m. momentum and \( q \) the relative momentum of the two quarks. In our calculation, we replace \( \tilde{\psi}_{BS} \) by the non-relativistic wavefunction \( \tilde{\psi}(q) \) according to (4)

\[
\tilde{\psi}_{BS}(p, q) = \frac{2\pi}{i} \delta(q^0) u(p/2 + q) \tilde{\psi}(\vec{q}) \bar{v}(p/2 - q).
\]

The reduction of the Bethe-Salpeter wavefunction to the nonrelativistic wavefunction is a necessary approximation to proceed with the calculation. At short distances, i.e. for high momenta \( |\vec{p}| > m_Q \), it is expected that the relativistic wavefunction falls off more slowly than the nonrelativistic wavefunction, since the relativistic transverse one-gluon exchange kernel is less damped. Only a part of this effect is taken into account in the radiative corrections \( \mathcal{R} \propto \alpha_s \) which include transverse gluon exchange between the heavy \( Q \) and \( \bar{Q} \). The amplitude \( M \) factorizes into an expectation value for the colour part of the gluons with fixed colors \( i, j, k \) and the rest

\[
M(p, q, k_1, k_2, k_3) = M_{ijk} \text{Tr} \left( \frac{\lambda_i}{2} \frac{\lambda_j}{2} \frac{\lambda_k}{2} \frac{1}{\sqrt{3}} \right)
= \frac{1}{4\sqrt{3}} (d_{ijk} M^s(p, q, k_1, k_2, k_3) + 2i f_{ijk} M^a(p, q, k_1, k_2, k_3)).
\]

The first part \( M^s \) includes the symmetric sum of the six diagrams arising from the interchange of the three gluons, while the antisymmetric part \( M^a \) sums the
same diagrams with a negative sign for the odd permutations. $d_{ijk}$ and $f_{ijk}$ are the structure constants of the group $SU_3$. For $\vec{q} = 0$, $M^a = 0$, and one finds for the three-gluon rate the three-photon decay rate of positronium with the modification $\alpha^3 \to \frac{5}{18} \alpha_s^3$. For $\vec{q} \neq 0$, the contribution of $M^a$ is numerically found non-zero but negligible in comparison to $M^s$. The vertex $M$ is calculated from the expression

$$M_{123}(p, q, k_1, k_2, k_3, \epsilon_1, \epsilon_2, \epsilon_3) = g^3 \text{Tr} \left\{ \frac{1}{\vec{k}_1 - \vec{p}/2 + \vec{q}/2 + \vec{k}_1 - m_Q} \frac{1}{\vec{k}_2 - \vec{p}/2 + \vec{q}/2 + \vec{k}_3 - m_Q} \vec{k}_3 \cdot P_s(p, q) \right\}. \quad (6)$$

Here, the vectors $\epsilon_i, i = 1, 3$ describe the gluon polarization and $P_s(p, q)$ is the spin projection operator in the $^3S_1$ state. We calculate the decay width $\Gamma(Q\bar{Q} \to 3g)$ including the full $\vec{q}$-dependence in the matrix element $M$

$$\Gamma(Q\bar{Q} \to 3g) = \frac{1}{3!} \sum_{\text{pol}} \int \frac{d^3k_1 d^3k_2 d^3k_3}{8\omega_1 \omega_2 \omega_3(2\pi)^9} (2\pi)^4 \delta^3(\Sigma k_i) \delta(\Sigma \omega_i - 2m_Q)|T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{q}/2 + \vec{k}_1 - \vec{k}_3)|^2, \quad (7)$$

$$T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{q}/2 + \vec{k}_1 - \vec{k}_3) = \int \frac{d^3q}{(2\pi)^3} M(p, \vec{q}, \vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{q}/2 + \vec{k}_1 - \vec{k}_3) \cdot \psi(\vec{q}). \quad (8)$$

The extensive numerical calculation to determine $\Gamma$ is performed in the following way. First we use the symbolic program REDUCE to evaluate the trace in eq. (6). The output of this symbolic program is a rather long function $M$ (50 kbytes) of $\vec{q}, \vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{q}/2 + \vec{k}_1 - \vec{k}_3$. The $T$-matrix (8) is obtained after integrating $M \cdot \psi(\vec{q})$ over $q_x, q_y, q_z$ with an adaptive routine which gives a precision of 5%. We use wavefunctions from a numerical solution of the Schrödinger equation with Richardson’s potential given in ref. [4]. This potential goes over into the Coulomb potential for short distances and becomes a confining linear potential at large distances. Richardson’s potential gives a wavefunction $\psi(q)$ which falls off asymptotically as $\alpha_s(\vec{q}^2)/|\vec{q}|^4$. The matrix element $M(q) \propto |\vec{q}| \ (\text{cf. fig. 1})$ together with the wavefunction leads analytically to a very weakly diverging integral proportional to $\ln \ln |\vec{q}|_{\text{max}}$, which
is in agreement with our numerical calculations. These show no sensitivity to a variation of the cutoff between $|\vec{q}|_{\text{max}} = 3m_Q$ and $|\vec{q}|_{\text{max}} = 7m_Q$.

Finally the square of the $T$-matrix is summed over the gluon polarization vectors and gluon momenta $\vec{k}_1 = (0, 0, \omega_1)$, $\vec{k}_2 = (-\sqrt{1-x^2}\omega_3, 0, -\omega_1 - x\omega_3)$, $\vec{k}_3 = (\sqrt{1-x^2}\omega_3, 0, x\omega_3)$. Energy conservation in eq. (6) fixes $x$:

$$x = \frac{1}{2\omega_1\omega_3}[(2m_Q - \omega_1 - \omega_3)^2 - \omega_1^2 - \omega_3^2].$$

(9)

The two polarization vectors of each gluon are chosen to be $\vec{\epsilon}_i(1) = \vec{e}_y$ and $\vec{\epsilon}_i(2) = \vec{k}_i \times \vec{e}_y/|\vec{k}_i|$. We use a seven point two-dimensional integration routine to do the final phase space integration over the Dalitz triangle given by $0 \leq \omega_1 \leq m$, $0 \leq \omega_2 \leq m$ and $\omega_1 + \omega_2 \geq m$ (cf. eq. (7)). We estimate our total numerical accuracy to be 10%. For consistency we checked our routines by setting $\vec{q} = 0$ in the matrix element $M$, which leads to the zero-range approximation $T^{(0)}$. The corresponding decay width $\Gamma^0(Q\bar{Q} \rightarrow 3g)$ agrees with the well-known result for positronium decay modified by the gluonic couplings (eq. (5)) and the strong interaction wavefunction $\psi(r = 0)$. We denote by

$$\gamma = \Gamma(Q\bar{Q} \rightarrow 3g)/\Gamma^{(0)}(Q\bar{Q} \rightarrow 3g)$$

(10)

the reduction factor for the three-gluon decay due to the finite size effect in the matrix element $M(\vec{q})$. We obtain

$$\gamma = \begin{cases} 0.31 \pm 0.03 & \text{for } J/\psi \\ 0.69 \pm 0.07 & \text{for } \Upsilon \end{cases}$$

(11)

At first sight this correction is surprisingly large, especially for charmonium. In order to demonstrate the origin of the finite-size reduction, we plot in fig. 1 the two factors in the integral eq. (8), the radial Gaussian wavefunction $\tilde{\psi}(\vec{q})$ for $J/\psi$ and $\Upsilon$ as a function of $|\vec{q}|/m_Q$ where $m_Q = m_c$ for the $J/\psi$ and $m_Q = m_b$ for the $\Upsilon$, together with the matrix element $\tilde{M}^s(\vec{q}, k_1, k_2, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ which has been averaged
over the directions of $\vec{q}$, but is still a function of the polarization directions. The matrix element $\tilde{M}^s(q)$ has its maximum at $q = 0$ and falls off to a minimum at $q/m_Q \approx 1$. The exact shape depends on the gluon polarization vectors and the spin projection $s_z$. We have checked the shape of the $q$-dependence of $M^s(q)$ for various combinations of momenta $\vec{k}_i$ and polarizations $\vec{\epsilon}_i$. The shown case is rather typical. It is evident from fig. 1 that $\tilde{M}^s(q)$ cannot be well approximated by a parabolic function

$$\tilde{M}^s(q) = \tilde{M}^s(0)(1 - a q^2/m_Q^2)$$  \hspace{1cm} (12)

over the whole domain of values $q$ of interest. If we approximate $\tilde{\psi}(q)$ by a Gaussian, the form of eq. (12) would lead to a correction factor

$$\gamma_{\text{est}} \approx \left(1 - a \frac{<v^2>}{c^2}\right),$$  \hspace{1cm} (13)

where $<v^2>$ is the expectation value of the velocity of the quarkonium state under consideration \[10\]. The authors of ref. \[5\] assume the correction factor to be of the form eq. (13) and fit the coefficient $a$. They find $a$ between 2.9 and 3.5. For $a = 3$, the correction amounts to $\gamma = .28$ for the $J/\psi$ and $\gamma = .78$ for the $\Upsilon$. These values are not too far from our numerical values eq. (10). This agreement is gratifying, since it suggests that the finite-size correction is probably the most important effect.

The correction factor $\gamma$ calculated in this paper leads to new values of $\alpha_s(m_c)$ and $\alpha_s(m_b)$ if $\gamma$ is inserted into the theoretical expressions of $R_2$ in Table 1. (The ratio $R_1$ is unaffected by this correction.) The QCD corrections to the gluonic width for heavy mesons have been calculated in ref. \[6\] for the $\Upsilon$-meson. Special care is taken in this reference to avoid double counting of the one gluon Coulomb exchange. The contribution to radiative corrections due to the Coulomb exchange between $Q$ and $\bar{Q}$ must be dropped (i.e. class $f$ in fig. 1 of ref. \[6\]), because it is included already in the wavefunction. Only transverse gluons contribute to this class of radiative
corrections. Ref. [5] provides a practical concise summary of all QCD corrections to the decays of quarkonia, also for \(c\bar{c}\)-quarkonia. The values of \(\alpha_s(\mu)\) deduced from the ratio \(R_2\) are shown in Table 1. The new values of \(\alpha_s\) calculated with \(\gamma \neq 1\) have to be compared with those from \(R_1\) and are consistent. Now the values for \(\alpha_s(m_c)\) and \(\alpha_s(m_b)\) are rather different from each other, as is expected for the running coupling constant. We can use the values for \(\alpha_s(\mu)\) to determine the QCD scale parameter \(\Lambda = \Lambda^{(4)}_{\overline{MS}}\), which is related to \(\alpha_s(\mu)\) by

\[
\alpha_s(\mu) = \frac{12\pi}{(33 - 2n_f)\ln \mu^2/\Lambda^2} \left( 1 - \frac{6(153 - 19n_f)\ln(\mu^2/\Lambda^2)}{(33 - 2n_f)^2 \ln(\mu^2/\Lambda^2)} + \ldots \right). \tag{14}
\]

We use \(n_f = 4\) and \(\mu = m_c = 1.5\) GeV for \(J/\psi\) and \(\mu = m_b = 4.9\) GeV for \(\Upsilon\). We take the values of \(\alpha_s(\mu)\) with the finite size corrections from Table 1 (for \(\Upsilon\) we use an average of the values deduced from \(R_1\) and \(R_2\)), insert them into eq. (14) and solve for \(\Lambda\). The values of \(\alpha_s(\mu)\) and the calculated scale parameters \(\Lambda\) are

\[
\begin{align*}
\alpha_s(m_c) &= 0.28 \pm 0.01 \quad \Lambda^{(4)}_{\overline{MS}} = 200 \pm 20 \text{ MeV} \\
\alpha_s(m_b) &= 0.195 \pm 0.007 \quad \Lambda^{(4)}_{\overline{MS}} = 255 \pm 45 \text{ MeV}.
\end{align*} \tag{15}
\]

We have also investigated the dependence on the choice of the scale \(\mu\) discussed in ref. [11] by going from \(\mu = m_Q\) to \(\mu = 2m_Q/3\). The radiative corrections for the strong decay widths calculated from the expressions of ref. [5] change rather drastically for the two cases. The new values of \(\alpha_s(2m_Q/3)\) and the calculated values of the scale parameter \(\Lambda^{(4)}_{\overline{MS}}\) are the following

\[
\begin{align*}
\alpha_s\left(\frac{2}{3}m_c\right) &= 0.35 \pm 0.01 \quad \Lambda^{(4)}_{\overline{MS}} = 210 \pm 20 \\
\alpha_s\left(\frac{2}{3}m_b\right) &= 0.24 \pm 0.01 \quad \Lambda^{(4)}_{\overline{MS}} = 300 \pm 40.
\end{align*} \tag{16}
\]

Eqs. (15,16) summarize the main results of our paper. The values for \(\Lambda^{(4)}_{\overline{MS}}\) agree with the present world average [12] of \(\Lambda^{(4)}_{\overline{MS}} = 238\pm30\pm60\) MeV, where the first error is statistical and systematic and the second reflects the uncertainty in the scale.

In this paper, we have shown that the inclusion of the finite size of the \(Q\bar{Q} \to 3g\) vertex reduces significantly the \(3g\) decay widths of the \(J/\psi\) and the \(\Upsilon\). The finite size
modifications necessitate a redetermination of the values for $\alpha_s(\mu)$. These new $\alpha_s$-values are in agreement with the QCD evolution of the coupling constant $\alpha_s(\mu)$ with the scale $\mu$ and lead to consistent values of $\Lambda$. The corrections due to finite size of the $3g$ vertex can be calculated without free parameter. Because of the large size of the reduction of the 3-gluon width of charmonium, the question arises whether there are not other relativistic corrections. Indeed we believe that the use of a nonrelativistic wavefunction is the most crucial approximation in our calculation. To improve on this approximation, much more conceptual and numerical effort would have to be spent both on the relativistic bound-state problem and the radiative corrections.

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Figure Caption

Fig. 1: The wavefunctions $q^2 \tilde{\psi}(q)$ for the $\Upsilon_0(m_Q = m_b$, full line) and for the $J/\psi(m_Q = m_c$, dashed line) are given in arbitrary units as functions of $q/m_Q$. The matrix elements $M(q)$ (eq. (3)) integrated over the angle of $\hat{q}$ are indicated for different spin directions of the $^3S_1$-quarkonia. Squares give $M(q)$ for $J_z = -1$; crosses for $J_z = 1$ and diamonds for $J_z = 0$. The $z$-axis is defined by the momentum of one fast gluon. The energies and polarizations ($\omega_1 = \omega_2 = 0.97m_Q, \omega_3 = 0.06m_Q$ and $\vec{\epsilon}_1 = \vec{\epsilon}_2 = \vec{\epsilon}_3 = (0, 1, 0)$) of the gluons are chosen to show typical variations of $M(q)$ with $q/m_Q$.

$R_1 = \frac{\Gamma(^3S_1 \rightarrow \gamma gg)}{\Gamma(^3S_1 \rightarrow \gamma gg)}$

|     | Exp          | Theory          | $\alpha_s(\mu)$ |
|-----|--------------|-----------------|-----------------|
| $J/\psi$ | 0.10 ± 0.04 | $\frac{16\alpha_s}{5\alpha_s} (1 - 3.0 \frac{\alpha_s}{\pi})$ | 0.19 ± .09 .05 |
| $\Upsilon$ | 0.0274 ± 0.0016 | $\frac{16\alpha_s}{5\alpha_s} (1 - 2.5 \frac{\alpha_s}{\pi})$ | 0.189 ± 0.011 |
Table 1: Experimental values and theoretical expressions for the ratios \( R_1 \) and \( R_2 \) for the \( J/\psi \) and \( \Upsilon \) states. The experimental values are taken from Kwong et al. [5] for \( R_1(J/\psi) \) and from the review talk by Kobel [7]. We used \( \alpha^{-1}(m_\psi) = 133.7 \) and \( \alpha^{-1}(m_\mu) = 132.0 \) [6]. The coupling constants \( \alpha_s(\mu) \) are computed from the experimental ratios with the help of the theoretical expressions. Finite-range corrections cancel in the ratio \( R_1 \). In the case of \( R_2 \), the finite-range corrections are contained in the factors \( \gamma \). The values for \( \alpha_s(\mu) \) obtained in the zero-range approximation \( \gamma = 1 \) are compared with those which include the finite-range corrections.
This figure "fig1-1.png" is available in "png" format from:

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