Integrable XYZ Spin Chain with Boundaries

Takeo INAMI and Hitoshi KONNO

Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-01, Japan.

ABSTRACT

We consider a general class of boundary terms of the open XYZ spin-1/2 chain compatible with integrability. We have obtained the general elliptic solution of $K$-matrix obeying the boundary Yang-Baxter equation using the $R$-matrix of the eight vertex model and derived the associated integrable spin-chain Hamiltonian.
1+1 dimensional integrable models with boundaries find interesting applications in particle physics as well as condensed matter systems. In view of this, attempts have been made recently at the integrable extension of conformal field theories\cite{1,2}, both massive and massless integrable quantum field theories\cite{3-6} and solvable lattice models\cite{7-13} to those with boundary terms. In the case of lattice models, relying on the earlier work by Cherednik,\cite{8} Sklyanin has given\cite{9} a general framework which enables us to treat this problem on an algebraic footing. Especially, the general solution of integrable boundary terms has been found in the XXZ and the XXX Heisenberg spin chain system\cite{12} based on his framework. In this letter, we will work out the general solution of boundary interactions in the case of XYZ Heisenberg spin chain system.

The Hamiltonian of the XYZ spin-1/2 chain is given by the transfer matrix of the eight vertex model\cite{14}. The eight vertex model is defined in terms of the Boltzmann weights given by the elliptic solution $R(u)$ of the Yang-Baxter (Y-B) equation

$$R_{12}(u - u') R_{13}(u) R_{23}(u') = R_{23}(u') R_{13}(u) R_{12}(u - u').$$

(1)

Here we regard $R(u)$ as linear operators acting on the tensor product of vector spaces $V \otimes V$ with $V = C v_+ \oplus C v_-$ and $R_{12} \equiv R \otimes 1$, $R_{23} \equiv 1 \otimes R$, etc. as those acting on $V_1 \otimes V_2 \otimes V_3$, where $V_i \cong V$, $i = 1, 2, 3$. Setting $R(u) v_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum_{\varepsilon_1, \varepsilon_2} v_{\varepsilon_1} \otimes v_{\varepsilon_2} R(u)^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1 \varepsilon_2}$ and arranging the elements of $R$ in the order $(\varepsilon_1, \varepsilon_2) = (++), (+-), (-+), (--)$, one can express the eight vertex $R$-matrix as follows.

$$R(u) = \begin{pmatrix}
\text{sn}(u + \eta) & 0 & 0 & k \text{ sn } \eta \text{ sn } u \text{ sn}(u + \eta) \\
0 & \text{sn} u & \text{sn} \eta & 0 \\
0 & \text{sn} \eta & \text{sn} u & 0 \\
k \text{sn} \eta \text{ sn } u \text{ sn}(u + \eta) & 0 & 0 & \text{sn}(u + \eta)
\end{pmatrix},$$

(2)

where $\text{sn } u \equiv \text{sn}(u; k)$ is the Jacobi elliptic function of modulus $0 \leq k \leq 1$. 2
Let $P_{ij}$ be the transposition operator on $V_i \otimes V_j$, i.e. $P(x \otimes y) = y \otimes x$. The $R$-matrix (2) is known to have the following desirable properties.

**Regularity**

$$R(0) = r(\eta)P,$$
$$r(\eta) = \text{sn} \eta,$$  \hspace{1cm} (3)

**$P - $symmetry**

$$P_{12}R_{12}(u)P_{12} = R_{12}(u),$$  \hspace{1cm} (4)

**$T - $symmetry**

$$R^{t_{12}}_{12}(u) = R_{12}(u),$$  \hspace{1cm} (5)

**Unitarity**

$$R_{12}(u)R_{12}(-u) = \rho(u)1, \quad \rho(u) = \text{sn}^2 \eta - \text{sn}^2 u,$$  \hspace{1cm} (6)

**Crossing unitarity**

$$R^{t_{12}}_{12}(u)R^{t_{12}}_{12}(-u - \eta) = \tilde{\rho}(u)1,$$
$$\tilde{\rho}(u) = \text{sn}^2 \eta - \text{sn}^2 (u + \eta).$$  \hspace{1cm} (7)

In the case of periodic boundary condition, it is known that the Y-B equation (1) implies a commuting family of transfer matrix. Hence the model is integrable.

We now consider the eight vertex model with boundary interactions. Aiming at describing integrable systems with boundaries, Sklyanin\cite{9} has introduced a pair of matrices $K_+(u)$ and $K_-(u)$. The effects of presence of boundaries at the left and right ends are solely described by $K_+(u)$ and $K_-(u)$, respectively. $K_\pm(u)$ are defined as the solutions to the relations

$$R_{12}(u - u') \frac{1}{K_-(u)}R_{12}(u + u') \frac{2}{K_-(u')},$$
$$=K_-(u')R_{12}(u + u') \frac{1}{K_-(u)}R_{12}(u - u'),$$  \hspace{1cm} (8)

$$R_{12}(-u + u') \frac{1}{K^t_{12} + (u)}R_{12}(-u - u' + 2\eta) \frac{2}{K^t_{12} + (u')},$$
$$=K^t_{12} (u')R_{12}(-u - u' + 2\eta) \frac{1}{K^t_{12} + (u)}R_{12}(-u + u'),$$  \hspace{1cm} (9)

where $K_{\pm} \equiv K_\pm \otimes \text{id}_{V_2}$ and $K_{\pm} \equiv \text{id}_{V_1} \otimes K_\pm$. The equations (8) and (9) are called boundary Y-B equations and $K_{\pm}(u)$ boundary $K$-matrices.

The boundary Y-B equations imply a commuting family of transfer matrix.\cite{9} The transfer matrix $t(u)$, in this case, is defined using the $K_\pm$ and the monodromy
matrix $T(u)$ as
\[ t(u) = \text{tr} \left[ K_+(u)T(u)K_-(u)T^{-1}(-u) \right], \tag{10} \]
where $T(u)$ is given by
\[ T(u) = R_{N0}(u)R_{N-10}(u) \cdots R_{10}(u). \tag{11} \]
The trace in (10) should be taken over $V_0$. Then, the commuting property of $t(u)$,
\[ [t(u), t(u')] = 0, \tag{12} \]
follows from the properties of $R$ and the boundary Y-B equations (8) and (9).

The problem is now to solve the equations (8) and (9) and find general solutions for $K_-$ and $K_+$, using the eight vertex $R$-matrix given in (2). It suffices to consider the first equation, because of the following fact. Suppose $K_-(u)$ is a solution of the first equation, then the function
\[ K_+(u) = K_-^t(-u - \eta) \tag{13} \]
gives the solution for the second equation.

We now proceed to solving Eq.(8). Write $K_-(u)$ as
\[ K_-(u) = \begin{pmatrix} x(u) & y(u) \\ z(u) & v(u) \end{pmatrix}, \tag{14} \]
we have found that, out of the sixteen equations in the boundary Y-B equation (8), only three are independent:
\[ s_-vv' + s_+vx' = s_+vx' + s_-xx', \tag{15} \]
\[ yz' + k s_-s_+zz' = yz' + k s_-s_+yy', \tag{16} \]
\[ S_- s_+ y x' + k \ s_+^2 S_- s_- z x' + k \ s_\eta S_- S_+(s_- v z' + s_+ x z') \]
\[ = S_+ s_- y x' + k \ s_\eta^2 S_+ s_+ z x' + s_\eta (s_- v y' + s_+ x y'), \quad (17) \]

where we set \( x \equiv x(u), \ x' \equiv x(u') \), etc., and

\[ s_\eta \equiv \text{sn} \ \eta, \quad s_\pm \equiv \text{sn} (u \pm u'), \quad S_\pm \equiv \text{sn} (u \pm u' + \eta). \quad (18) \]

In the following, we also use the notations \( \alpha(u) \equiv v(u)/x(u), \ \beta(u) \equiv z(u)/y(u) \)
and \( \gamma(u) \equiv y(u)/x(u) \).

Dividing (15) by \( xx' \), one obtains

\[ \alpha(u') = \frac{\text{sn}(u + u') \alpha(u) + \text{sn}(u - u')}{\text{sn}(u - u') \alpha(u) + \text{sn}(u + u')}. \quad (19) \]

Taking the limit \( u' \to u \) of \( \frac{\alpha(u') - \alpha(u)}{u' - u} \), one obtains the following differential equation.

\[ \frac{d\alpha(u)}{du} = - \frac{1 - \alpha(u)^2}{\text{sn} \ 2u}. \quad (20) \]

After the change of variable \( t = \text{sn} \ u \), the integration of (20) takes the form

\[ \int \frac{d\alpha}{1 - \alpha^2} = - \frac{1}{2} \int \frac{dt}{t(1 - t^2)(1 - k^2 t^2)}. \quad (21) \]

One can easily get the general solution of Eq.(21),

\[ \frac{v(u)}{x(u)} = C \text{cn} u \ \text{dn} u - \text{sn} u \]
\[ \frac{C \text{cn} u}{\text{dn} u + \text{sn} u}, \quad (22) \]

where \( C \) is an arbitrary constant.
In a similar way, from (16) one gets

\[ \beta(u') = \frac{\beta(u) + k \text{ sn}(u + u') \text{ sn}(u - u')} {k \text{ sn}(u + u') \text{ sn}(u - u') \beta(u) + 1}. \]  

(23)

and the differential equation

\[ \frac{d\beta(u)}{du} = -k \text{ sn} \, 2u \, (1 - \beta(u)^2). \]  

(24)

This implies the general solution

\[ z(u) = \frac{\lambda(1 - k \text{ sn}^2 u) - 1 - k \text{ sn}^2 u}{\lambda(1 - k \text{ sn}^2 u) + 1 + k \text{ sn}^2 u}. \]  

(25)

with \( \lambda \) being another arbitrary constant.

Dividing (17) by \( yy' \), the third equation (17) can be written

\[ \frac{\gamma(u)}{\gamma(u')} = \frac{s_\eta \, (s_- \alpha(u) + s_+)(1 - k \, s_+ s_- \beta(u'))}{S_- s_+ - S_+ s_- + k \, \beta(u) s_\eta^2 (S_- s_- - S_+ s_+)} \].  

(26)

Substituting (23) into (26) and replacing \( \alpha(u) \) and \( \beta(u) \) by the RHS of (22) and (25), one can factorize (26) in the form of the ratio of the same functions, one with the argument \( u \) and the other with \( u' \), respectively. We thus find

\[ \frac{y(u)}{x(u)} = \mu \frac{\lambda(1 - k \text{ sn}^2 u) + 1 + k \text{ sn}^2 u}{C \text{ cn} \, u \, \text{ dn} \, u + \text{ sn} \, u}, \]  

(27)

where \( \mu \) is the third arbitrary constant.

From (25) and (27), one can now get the ratio \( z(u)/x(u) \). It is then not difficult to check that the above solutions satisfy all the remaining equations, if one notes the identity

\[ s_\eta^2 \frac{S_+ s_+ - S_- s_-}{S_- s_+ - S_+ s_-} = \frac{S_+ S_- - s_+ s_-}{1 - k^2 S_+ S_- s_+ s_-} = s_\eta \text{ sn}(2u + \eta). \]  

(28)
In summary, we have obtained the general solution of (8) as \( K_-(u) = K(u; \xi, \lambda, \mu) \)
with

\[
K_-(u; \xi, \lambda, \mu) = \frac{1}{\text{sn} \xi} \left( \text{sn}(\xi + u) \begin{pmatrix} \text{sn}(\xi + u) & \mu \text{sn} 2u \frac{\lambda(1-k \text{sn}^2 u) - 1-k \text{sn}^2 u}{1-k^2 \text{sn}^2 \xi \text{sn}^2 u} \\ \mu \text{sn} 2u \frac{\lambda(1-k \text{sn}^2 u) + 1+k \text{sn}^2 u}{1-k^2 \text{sn}^2 \xi \text{sn}^2 u} & \text{sn}(\xi - u) \end{pmatrix} \right),
\]

(29)

where we set \( C = \text{sn} \xi / \text{cn} \xi \text{dn} \xi \) and replaced \( \mu \text{cn} \xi \text{dn} \xi \) by \( \mu \). We normalize the matrix \( K_-(u) \) as \( K_-(0) = 1 \) for later convenience.\[
\]

In the trigonometric limit \( k \to 0 \), where \( \text{sn} u \to \sin u \), we recover the result in the case of the six vertex model given by de Vega and González Ruiz.\[12\] The rational limit is obtained from the trigonometric \( K \)-matrix by scaling \( u \to \eta u, \xi \to \eta \xi \) and taking the limit \( \eta \to 0 \).

Let us next consider the corresponding XYZ spin chain Hamiltonian. Because of the equation (12), one can regard the transfer matrix \( t(u) \) as the generating function of integrals of motion of the system. Its first logarithmic derivative implies the Hamiltonian.

\[
H = 2r(\eta) t^{-1}(0)(t'(0) - \text{tr} K'_+(0))
= 2r(\eta) \sum_{n=1}^{N-1} H_{n,n+1} + \frac{1}{2} K'_-(0) + \frac{\text{tr}_0 K_+(0) H_{N0}}{\text{tr} K_+(0)},
\]

(30)

where the two-site Hamiltonian is given by

\[
H_{n,n+1} = \frac{1}{r(\eta)} \mathcal{P}_{n+1} R'_{nn+1}(0).
\]

(31)

By a direct calculation with the \( K \)-matrices \( K_-(u) = K_-(u; \xi_-, \lambda_-, \mu_-) \) and \( K_+(u) = \)

* The solution obtained in Ref.[12] for the XYZ model are the special cases of (29) associated with the special solutions \( \alpha(u)^2 = 1, \beta(u)^2 = 1 \) of Eqs.(20) and (24).
\( K'_t (-u - \eta; -\xi_+, -\lambda_+, -\mu_+) \) together with the \( R \)-matrix (2), one gets the following result.

\[
H = \sum_{n=1}^{N-1} ((1 + \Gamma)\sigma^x_n \sigma^x_{n+1} + (1 - \Gamma)\sigma^y_n \sigma^y_{n+1} + \Delta \sigma^z_n \sigma^z_{n+1}) \\
+ \text{sn} \, \eta (A_- \sigma^z_1 + B_- \sigma^z_N + C_- \sigma^z_{N-1} + A_+ \sigma^z_N + B_+ \sigma^z_{N-1} + C_+ \sigma^z_{N-2}),
\]
where

\[
\Gamma = k \, \text{sn}^2 \eta, \quad \Delta = \text{cn} \, \eta \, \text{dn} \eta \\
A_\pm = \frac{\text{cn} \, \xi_\pm \text{dn} \, \xi_\pm}{2 \text{sn} \, \xi_\pm}, \quad B_\pm = \frac{\mu_\pm (\lambda_\pm + 1)}{\text{sn} \, \xi_\pm}, \quad C_\pm = \frac{\mu_\pm (\lambda_\pm - 1)}{\text{sn} \, \xi_\pm}.
\]

In conclusion, we have obtained the general elliptic solution of the boundary Y-B equation for the \( K \)-matrices and derived the Hamiltonian of the associated XYZ spin-1/2 chain with boundary terms.

An immediate question is to find the ground state energy and the excitation spectrum of the XYZ Hamiltonian we have derived. The diagonalization of this Hamiltonian can be achieved by means of the generalized algebraic Bethe ansatz by Sklyanin\cite{[17]} with some modification as in the periodic boundary condition case.\cite{[14],[15]} This subject is now under investigation.

It was shown using the results of Bethe ansatz type analysis that, by tuning the X-Y anisotropy coupling (\( \Gamma \) in Eq.(32)), the XYZ Hamiltonian with periodic boundary condition give rise to the quantum sine-Gordon theory in the continuum limit.\cite{[16]} In the case of open XYZ spin chain, it is of interest to ask whether one can tune the coupling of boundary terms together with the \( \Gamma \) so that one can derive its field theory limit. The resulting theory is expected to be the quantum boundary sine-Gordon theory.\cite{[3],[4]} In the limit \( N \to \infty \), we have three boundary terms proportional to \( \sigma^x_1 \), \( \sigma^y_1 \) and \( \sigma^z_1 \) associated with three free parameters in \( K_\pm \), whereas the boundary term proposed by Sklyanin\cite{[17]} and Ghoshal and Zamolodchikov\cite{[3]} has two parameters. It is necessary to explain this difference in the analysis of the continuum limit.
Furthermore, the higher logarithmic derivatives of commuting transfer matrix give infinite number of conserved quantities. In the closed XYZ spin chain case, it was shown\cite{18} that the conservation laws associated with these quantities yield selection rules in the scattering process of the quantum sine-Gordon theory. It is an interesting question to ask how the parameters appearing in the $K$-matrices affect the scattering process of the boundary sine-Gordon theory.

We will present our result on these problems in future publications.

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