ŁOJASIEWICZ EXPONENTS OF NON-DEGENERATE
HOLOMORPHIC AND MIXED FUNCTIONS

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ABSTRACT. We consider Łojasiewicz inequalities for a non-degenerate
holomorphic function with an isolated singularity at the origin. We
give an explicit estimation of the Łojasiewicz exponent in a slightly
weaker form than the assertion in Fukui [9]. For a weighted homogeneous
polynomial, we give a better estimation in the form which is conjectured
by [4] under some condition (the Łojasiewicz non-degeneracy). We
also introduce Łojasiewicz inequality for strongly non-degenerate mixed
functions and generalize this estimation for mixed functions.

1. HOLOMORPHIC FUNCTIONS AND ŁOJASIEWICZ EXPONENT

Consider an analytic function \( f(z) \) with an isolated singularity at the
origin. We consider the inequality

\[
\|\partial f(z)\| \geq c\|z\|^\theta, \quad \exists c > 0, \forall z \in U
\]

where \( U \) is a sufficiently small neighborhood of the origin and \( \partial f(z) \) is
the gradient vector \((\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})\). The Łojasiewicz exponent \( \ell_0(f) \) of \( f(z) \)
at the origin is the smallest positive number among \( \theta \)'s which satisfy the
inequality (1). It is known that there exists such number \( \ell_0(f) \) and it is
a rational number [16, 26]. We assume that the Newton boundary of \( f \) is
non-degenerate hereafter. The purpose of this paper is to give an explicit
upper bound of the Łojasiewicz exponent in term of the combinatorics of the
Newton boundary. There is a similar estimation proposed by Fukui [9] but
he uses some incorrect equality (2.4), [9] in his proof. Thus the proof has a
gap and the assertion must be proved in a different way, even if it is true.
There is also an estimation by Abderrahmane [2] using Newton number.
In this paper, we give an estimation along Fukui’s way. Our estimation
is apparently a little weaker than that of Fukui but it is enough for our
purpose. In the last section of this paper (§4), we will generalize the notion
of Łojasiewicz exponent for mixed functions.

1.1. Newton boundary and the dual Newton diagram. Let \( f(z) = \sum_\nu c_\nu z^\nu \) be an analytic function with \( f(0) = 0 \). Recall that the Newton
diagram \( \Gamma_+(f) \) is the minimal convex set in the first quadrant of \( \mathbb{R}_+^n \) con-
taining \( \cup_{\nu, c_\nu \neq 0} (\nu + \mathbb{R}_+^n) \). The Newton boundary \( \Gamma(f) \) is the union of compact

\[2000 \text{ Mathematics Subject Classification.} \quad 14P05,32S55.\]

\[\text{Key words and phrases.} \quad \text{Łojasiewicz inequality, vanishing coordinate, non-convenient.}\]
faces of $\Gamma_+(f)$. Let $N_+$ be the space of non-negative weight vectors. Using the canonical basis, it can be identified with the first quadrant of $\mathbb{R}^n_+$. Let $P = (p_1, \ldots, p_n) \in N_+$ be a non-zero weight. It defines a canonical linear mapping on $\Gamma_+(f)$ by $P(\nu) := \sum_{i=1}^n p_i \nu_i$. We denote the minimal value of $P$ on $\Gamma_+(f)$ by $d(P, f)$ and put $\Delta(P, f) = \{ \nu \in \Gamma_+(f) \mid P(\nu) = d(P, f) \}$. This is a face of $\Gamma_+(f)$. We simply write as $d(P)$ or $\Delta(P)$ if no confusion is likely. The dimension of $\Delta(P)$ can be $0, 1, \ldots, n - 1$. Recall that $P, Q \in N_+$ are equivalent if $\Delta(P) = \Delta(Q)$. This equivalence classes gives $N_+$ a rational polyhedral cone subdivision $\Gamma^*(f)$ and we call $\Gamma^*(f)$ the dual Newton diagram. We also define a partial order in $\Gamma^*(f)$ by

$$P \leq Q \iff \Delta(P) \subset \Delta(Q).$$

Denote the set of weights which are equivalent to $Q$ by $[Q]$. Note that the closure $\overline{[P]}$ is equal to the union $\cup_{Q \geq P} [Q]$ and

$$\dim \overline{[P]} = \dim [P] = n - \dim \Delta(P).$$

The generators of the $1$-dimensional cones of $\Gamma^*(f)$ are called vertices. A weight $P \in N_+$ is a vertex if and only if $\dim \Delta(P) = n - 1$. We denote the set of vertices by $\mathcal{V}$, $P = (p_1, \ldots, p_n)$ is called strictly positive if $p_i > 0$ for any $i = 1, \ldots, n$. Note that $\Delta(P)$ is compact if and only if $P$ is strictly positive. A vertex which is not strictly positive is either the canonical basis $e_i = (0, \ldots, 1, \ldots, 1)$, $1 \leq i \leq n$ or corresponds to a vanishing coordinate subspace (see §3.1 for the definition).

1.1.1. Face function. For $\Delta \subset \Gamma(f)$, put $f_\Delta(z) := \sum_{\nu \in \Delta} c_\nu z^\nu$ and we call $f_\Delta$ the face function of $\Delta$. For a weight vector $P \in N_+$, the face function associated with $P$ is defined by $f_P(z) = f_{\Delta(P)}(z)$. The monomial $z^\nu$ and the integer point $\nu \in \Gamma_+(f)$ correspond each other. If $\Delta$ is a compact face, $f_\Delta$ is a weighted homogeneous polynomial.

1.2. Normalized weight vector. Take a weight vector $P = (p_1, \ldots, p_n)$. Let $d = d(P)$ and assume that $d > 0$. The hyperplane $\Pi$ in $\mathbb{R}^n$, defined by $p_1 \nu_1 + \cdots + p_n \nu_n = d$, contains the face $\Delta(P)$ and all other points $\nu \in \Gamma_+(f) \setminus \Delta(P)$ are above $\Pi$. Namely $p_1 \nu_1 + \cdots + p_n \nu_n > d$. We call $\Pi$ the supporting hyperplane of the weight vector $P$. For a weight vector $P$ with $d(P) > 0$, we define the normalized weight vector $\hat{P}$ of $P$ (with respect to $f$) by

$$\hat{P} := (\hat{p}_1, \ldots, \hat{p}_n), \quad \hat{p}_i = p_i / d(P).$$

Hereafter we use this notation $\hat{P}$ throughout this paper. Using the normalized weight vector, $d(\hat{P}) = 1$ and the supporting hyperplane $\Pi$ is written as

$$\Pi : \quad \hat{p}_1 \nu_1 + \cdots + \hat{p}_n \nu_n = 1.$$

Note that the $\nu_j$-coordinate of the intersection of $\Pi$ and $\nu_j$-axis is $1/\hat{p}_j$.

Assume that $\Delta(P) \cap \Delta(Q) \neq \emptyset$ and consider the line segment $\hat{P}_t := tP + (1 - t)Q$, $0 \leq t \leq 1$. Note that $\Delta(P_t) = \Delta(P) \cap \Delta(Q)$ for any $0 < t < 1$
and the normalized weight vector $\hat{P}_t$ is simply given by $\hat{P}_t = t\hat{P} + (1 - t)\hat{Q}$, provided $d(P) > 0$ and $d(Q) > 0$.

The purpose of this paper is to give an upper bound explicitly for the Lojasiewicz exponent using the combinatorial data of the Newton boundary. Then we give an application for the characterization of the monomials which do not change the topology by adding to $f$. For a weighted homogeneous non-degenerate polynomial, we prove the estimation conjectured in [4] under the Lojasiewicz non-degeneracy. In §4, we generalize these results for mixed functions.

2. LOJASIEWICZ EXPONENT FOR CONVENIENT FUNCTIONS

2.1. Preliminary consideration. We first consider the estimation of Lojasiewicz exponent along an analytic curve $C(t)$ which is parametrized as follows. Put $I := \{ i \mid z_i(t) \neq 0 \}$ and $I^c$ be the complement of $I$.

\[
(2) \quad C(t) : \begin{cases} 
    z(t) = (z_1(t), \ldots, z_n(t)), \quad z(0) = 0, \quad z(t) \in \mathbb{C}^*I \\
    z_i(t) = a_{i}t^{p_{i}} + (\text{higher terms}), \quad p_{i} \in \mathbb{N}, \quad i \in I
\end{cases}
\]

Here we use the following notations:

- $\mathbb{C}^I := \{ \mathbf{z} = (z_1, \ldots, z_n) \mid z_j = 0, j \notin I \}$
- $\mathbb{C}^{*I} := \{ \mathbf{z} = (z_1, \ldots, z_n) \mid z_i \neq 0, \iff i \in I \}$
- $N_+^I := \{ P = (p_1, \ldots, p_n) \in N_+ \mid p_j = 0, j \notin I \}$
- $N_+^{*I} := \{ P = (p_1, \ldots, p_n) \in N_+ \mid p_i \neq 0 \iff i \in I \}$
- $f^I := f|_{\mathbb{C}^I}$.

Put $P = (p_i)_{i \in I} \in N_+^I$ and $d = d(P, f^I)$. We define

\[
M(P) := \max\{ p_j \mid j \in I \}, \quad m(P) := \min\{ p_j \mid j \in I \}.
\]

Note that $\text{ord}\ z(t) = m(P)$. Put $q := \text{ord}\ \partial f^I(\mathbf{z}(t))$. Under the non-degeneracy assumption, we have the equalities:

\[
(3) \quad d - M(P) \leq q \leq d - m(P) \quad \text{or} \quad d - M(P) \leq \frac{q}{m(P)} \leq d - \frac{m(P)}{m(P)} = 1 - \frac{1}{m(P)} - 1.
\]

Put $\text{Vari}(P) = \{ z_j \mid \frac{\partial f}{\partial z_j} \neq 0 \}$. Namely $\text{Vari}(P)$ is the set of variables which appear in $f_P$. Then we have the obvious estimations:

\[
(5) \quad \frac{\partial f}{\partial z_j}(\mathbf{z}(t)) = \left( \frac{\partial f}{\partial z_j} \right)_{P} (a)t^{d_j} + (\text{higher terms}), \quad d_j = d(P, \frac{\partial f}{\partial z_j})
\]

\[
(6) \quad \text{ord}\ \frac{\partial f}{\partial z_j}(\mathbf{z}(t)) \geq d_j \geq d - p_j.
\]

If $z_j \in \text{Vari}(P)$, $d_j = d(P, f) - p_j$ and otherwise $d_j > d - p_j$. If $m(P) = p_j$, $d/m(P) = 1/\hat{p}_j$ and this is equal to the $j$-th coordinate of the intersection
of $\Pi(P)$ and $\nu_j$ axis. We define the Lojasiewicz exponent of $f$ along $C(t)$ by

$$\ell_0(C(t)) := \frac{\text{ord } \partial f(z(t))}{\text{ord } z(t)}.$$  

By (3) and by the non-degeneracy assumption, we have

$$(7) \quad \text{ord } \partial f(z(t)) \leq d - m(P)$$

$$(8) \quad \text{ord } f_j(z(t)) \geq d(P, f_j) \geq d - p_j$$

$$(9) \quad \ell_0(C(t)) \leq \frac{d - m(P)}{m(P)}.$$  

For a strictly positive weight vector $P = (p_1, \ldots, p_n)$, we define positive invariants

$$\eta_{i,j}(P) := \frac{d - p_j}{p_i} = 1 - \hat{p}_j, \quad \eta'_{i,j}(P) := \frac{d_j}{p_i} = \frac{d_j}{\hat{p}_i},$$

$$\eta(P) := \frac{d - m(P)}{m(P)} = \frac{1}{m(\hat{P})} - 1.$$  

where $d_j = d(P, f_j)$ and $\hat{d}_j = d_j/d$.

2.2. Lojasiewicz exceptional monomial. We say that $f(z)$ is convenient if for any $1 \leq j \leq n$, Newton boundary $\Gamma(f)$ intersects with $\nu_j$-axis at a point $B_j = (0, \ldots, b_j, \ldots, 0)$. Recall that a non-degenerate function $f(z)$ has an isolated singularity at the origin, if it is convenient ([18]). Assume that $f(z)$ is convenient as above. Define an integer $B := \max\{b_j | j = 1, \ldots, n\}$ and let $\mathcal{L} = \{i | b_i = B\}$. We call $z_i^B, i \in \mathcal{L}$ a Lojasiewicz monomial of $f$. We say that a Lojasiewicz monomial $z_i^B$ is Lojasiewicz exceptional if there exists $j, j \neq i$ and a monomial of the form $z_j z_i^{B'}$, with $B' < B - 1$ which has a non-zero coefficient in $f$.

Consider the curve parametrized as (2) and assume that $I = \{1, \ldots, n\}$. Then we have seen

$$\text{ord } \partial f(z(t)) \leq \frac{d}{m(P)} - 1 \leq B - 1.$$  

This implies the following inequality holds in a small neighbourhood of the origin.

$$(11) \quad \|\partial f(z(t))\| \geq c\|z(t)\|^{B-1}, \quad c \neq 0.$$  

If $z_i^B$ is Lojasiewicz exceptional and let $z_j z_i^{B'}, B' < B - 1$ be as above. Then $\frac{\partial f}{\partial z_j}$ has the monomial $z_i^{B'}$ with non-zero coefficient and ord $\frac{\partial f}{\partial z_j}(z(t))$ can be $p_i B'$ which is smaller than $p_i (B - 1)$. In fact, this is the case for the $i$-axis curve $z(t)$ where $z_i(t) = t$ and $z_j(t) \equiv 0$ for $j \neq i$. We assert
Assertion 1. The inequality $\ell_0(f) \leq B - 1$ holds for any analytic curve $z(t)$.

Proof. As we have shown the assertion for the case $I = \{1, \ldots, n\}$, we need only consider the case where some of $z_i(t)$ is identically zero. In this case, put $I := \{i \mid z_i(t) \neq 0\}$. Then $f^I := f|_{C^I}$ is a non-degenerate convenient function. Thus by the above argument applied for $f^I$, we have

$$\text{ord } \partial f^I(z(t)) \leq (\text{ord } z(t)) \times (B_I - 1).$$

Here $B_I$ is defined similarly for $f^I$. By the obvious inequality $\text{ord } \partial f(z(t)) \leq \text{ord } \partial f^I(z(t))$ and $B_I \leq B$, we get the inequality \( \square \).

For the practical calculation of the Lojasiewicz exponent, we use the following criterion. This can be proved by the Curve Selection Lemma ([17, 10]).

Proposition 2. A positive number $\theta$ satisfies the Lojasiewicz inequality \( \square \) if the inequality

$$\text{ord } \partial f(z(t)) \leq \theta \times \text{ord } z(t)$$

is satisfied along any non-constant analytic curve $C(t)$ parametrized by an analytic path $z(t)$ with $z(0) = 0$. That is $\ell_0(f) = \sup \ell_0(C(t))$ where $C(t)$ moves every possible analytic curves starting from the origin.

Now we have the following result for convenient non-degenerate functions.

Theorem 3. Let $f(z)$ be a non-degenerate convenient analytic function. Then Lojasiewicz exponent $\ell_0(f)$ satisfies the inequality: $\ell_0(f) \leq B - 1$.

Furthermore if $f$ has a Lojasiewicz non-exceptional monomial, $\ell_0(f) = B - 1$.

Proof. We have shown that $\ell_0(f) \leq B - 1$. We only need to show the existence of a curve $C(t)$ which takes the equality $q = B - 1$, assuming that $f$ has a Lojasiewicz non-exceptional monomial. For this purpose, we assume for simplicity $B = b_1$ and $z_1^B$ is non-exceptional. Note that the Newton boundary of $\frac{\partial f}{\partial z_1}$ does not touch the $\nu_1$ axis under $B - 1$ for any $i > 1$ by the assumption. Thus we can take a sufficiently large integer $N$ and put $P = (1, N, \ldots, N)$. Note that $d(P, \frac{\partial f}{\partial z_1}) = B - 1$ and $d(P, \frac{\partial f}{\partial z_1}) \geq B - 1$ for any $i \geq 2$. Consider the curve $C(t)$ defined by $z(t) = (t, t^N, \ldots, t^N)$. Then the above observation tells us that

$$\partial f(z(t)) = (B, *, \ldots, *)t^{B - 1} + (\text{higher terms}).$$

Thus $\|\partial f(z(t))\| \approx \|z(t)\|^B - 1. \square$

In the above proof, if there is a monomial $z_1^{B'}z_j$ with $j \neq 1, B' < B - 1$, we see that $\text{ord } f_j(z(t)) = B'$. Thus we have $\text{ord } f_1(z(t)) > \text{ord } f_j(z(t))$. The importance of Lojasiewicz exceptional monomial is observed by Lemarcik [14]. For plane curves ($n = 2$), we have also observed that it gives a fake effect to computation of the complexity of plane curve singularity but exceptional
monomials can be eliminated without changing the non-degeneracy (Le-Oka [L5]). Suppose that \( c z_1^B + c' z_1^{B'} z_2 \) with \( B' \leq B - 2, c, c' \neq 0 \) is in a face function of \( f \). Then take the coordinate change \((z_1, z_2') := (z_1, z_2 + (c/c') z_1^{B-B'})\) to kill the monomial \( z_1^B \). This operation does not work for mixed polynomials.

3. Lojasiewicz exponents for non-convenient functions

In this section, we consider again a non-degenerate function \( f(z) \) with isolated singularity at the origin without assuming the convenience of the Newton boundary. It turns out that the estimation of Lojasiewicz exponent is much more complicated without the convenience assumption. We assume that \( \Gamma(f) \) has dimension \( n - 1 \) hereafter. If the multiplicity at the origin is greater than 2, this condition is always satisfied.

3.1. Lojasiewicz non-degeneracy along a vanishing coordinate subspace.

Let \( I \) be a subspace of \( \{1, \ldots, n\} \). We say that \( \mathbb{C}^I \) is a vanishing coordinate subspace ([20] [21] [S]) if \( f^I(\mathbb{C}^I) \equiv 0 \). Here \( f^I \) is the restriction of \( f \) to \( \mathbb{C}^I \). We use the notation \( \mathbb{C}^I = \{z | z_j = 0, j \notin I\} \) and \( \mathbb{C}^I = (z_i)_{i \in I} \). If further \( I = \{i\} \) is a vanishing coordinate subspace, we say \( \mathbb{C}^{\{i\}} \) a vanishing axis. We say a face \( \Xi \subseteq \Gamma_+(f) \) is essentially non-compact if there exists a non-strictly positive weight function \( Q = (q_1, \ldots, q_n) \) such that \( d(Q, f) > 0 \) and \( \Delta(Q) = \Xi \). Let \( I(Q) = \{i | q_i = 0\} \). We say also \( I(Q) \) the vanishing direction of \( \Xi \) and write also as \( I(\Xi) = I(Q) \). Then the assumption \( d(Q, f) > 0 \) implies \( \mathbb{C}^I \) is a vanishing coordinate subspace.

Put \( I = I(Q) \) and we assume that \( I = \{1, \ldots, m\} \). Take an \( i \in I \). By the assumption, the gradient vector \( \partial f \) does non vanish in a neighbourhood of the origin of \( i \)-axis except at the origin. This is possible only if there exists a monomial \( z_i^{n_i} z_j \) with a non-zero coefficient for some \( j \neq i \) in the expansion of \( f \). Then we observe that \( j \notin I \), because \( \mathbb{C}^I \) is a vanishing coordinate subspace. Let \( J_i \) be the set of \( j \in I^c \) for which such a monomial \( z_i^{n_i} z_j \) exists with a non-zero coefficient in the expansion of \( f(z) \). Then \( J_i \neq \emptyset \) for any \( i \in I \).

We define an integer \( n_{ij} \) by

\[
n_{ij} := \min \{n_i | z_i^{n_i} z_j \text{ has a non-zero coefficient}\}
\]

for a fixed \( i \in I \) and \( j \in J_i \). For brevity, we put \( n_{ij} = \infty \) if \( j \notin J_i \).

Put \( B_{ij} := (0, \ldots, n_{ij}, \ldots, 1, \ldots, 0) \). Note that \( B_{ij} \in \Gamma(f) \). Put \( J(I) = \cup_{i \in I} J_i \). We say that \( f \) is Lojasiewicz non-degenerate if for any strictly positive weight vector \( P \in \mathbb{N}^I \), the following condition is satisfied. Put \( I' := \{i | p_i = m(P)\} \) and \( J(P) := \cup_{i \in I'} J_i \subseteq J(I) \). Then the variety

\[
\{z \in \mathbb{C}^I | (f_j)^I (z_{I}) = 0, \forall j \in J(P)\}.
\]
Let $V$ the Jacobian dual Newton diagram of a polyhedral cone subdivision of $N$. Proposition 5. Any $i$. We consider $n + 1$ Newton boundaries, we denote by $\Delta(P, f_i)$ the face of $\Gamma(f_i)$ where $P$ takes minimal value, $d(P, f_i)$. We consider the following stronger equivalence relation in the space of non-negative weight vectors. Two weight vectors $P, Q$ are Jacobian equivalent if $\Delta(P, f_i) = \Delta(Q, f_i)$ for any $i = 1, \ldots, n$ and $\Delta(P, f) = \Delta(Q, f)$. We denote it by $P \sim Q$. This gives a polyhedral cone subdivision of $N_+$ and we denote this as $\Gamma^*_j(f)$ and we call it the Jacobian dual Newton diagram of $f$. $\Gamma^*_j(f)$ is a polyhedral cone subdivision of $N_+$ which is finer than $\Gamma^*(f)$. The Jacobian dual Newton diagram can be understood alternatively as follows. Let us consider the function $F(z) = f(z)f_1(z) \cdots f_n(z)$. Then $\Gamma^*_j(f)$ is essentially equivalent to the dual Newton diagram $\Gamma^*(F)$ of $F$. For any weight vector $P$, we have $\Delta(P, F) = \Delta(P, f) + \Delta(P, f_1) + \cdots + \Delta(P, f_n)$ where the sum is Minkowski sum. See [5] for the definition. For a weight vector $P$, the set of equivalent weight vectors in $\Gamma^*(f)$ and $\Gamma^*_j(f)$ is denoted as $[P]$ and $[P]_J$ respectively. We consider the vertices of this subdivision. We denote the set of strictly positive vertices of $\Gamma^*(f)$ and $\Gamma^*_j(f)$ by $V^+, V^*_J$ respectively. Recall that $e_i = (0, \ldots, 1, \ldots, 0)$.

**Proposition 4.**

(1) $P \sim Q$ implies $P \sim Q$ in $\Gamma^*(f)$. Conversely if $P \sim Q$ and $f_P(z)$ contains all $n$-variables, $P_J \sim Q$.

(2) A strictly positive weight vector $P$ is in $V^+$ or $V^*_J$ if and only if $\dim \Delta(P, f) = n - 1$ or $\dim (\Delta(P, f) + \sum_i \Delta(P, f_i)) = n - 1$ respectively where the summation is Minkowski sum.

Let $\mathcal{V}_0$ be the set of vertices of $\Gamma^*(f)$ which is not strictly positive.

**Proposition 5.** Assume that $P \in \mathcal{V}_0$ and $C^{I(P)}$ is a non-vanishing subspace. Then $P$ is one of $e_1, \ldots, e_n$.

**Lemma 6.** Let $P = (p_1, p_2, \ldots, p_n)$ be a non-elementary vanishing vertex of $\Gamma^*(f)$ in $\mathcal{V}_0$ and put $I = I(P)$. Then the following holds.

(1) $f_P$ contains every variable $z_1, \ldots, z_n$. In particular, $\hat{p}_i \leq 1$ for any $i$.

(2) Any monomial $z^a_i z_j$, $i \in I$ must be contained in $f_P(z)$, as $\hat{p}_i = 0$ and $\deg_P z^a_i z_j = \hat{p}_j \leq 1$.

(3) There are no monomials $z^a \in \mathbb{C}[z_I]$ in $f_P(z)$.

*Proof.* Suppose that $f_P$ does not contain the variable $z_i$ for some $i$. Then $\Delta(P) \subset \{v_i = 0\}$. Then $e_i \in [P]$ and thus a contradiction $d(P, f) =$
Take a weight vector \( P \) with \( \deg_{\hat{P}} z^0_i \leq 1 \). Then \( \hat{P}_i \leq 1 \) by the assertion (1) and as \( \deg_{\hat{P}} z^0_i \leq 1 \), this implies \( \hat{P}_j = 1 \) and \( z^0_i \) must be in \( fP \). If there is a monomial \( z^\alpha \) as in the assertion, \( \deg_{\hat{P}} z^\alpha = 0 \) and an obvious contradiction.

Example 7. Consider \( f(z) = (z_1^0 + z_2^3 + z_3^6)z_2 + z_3^7 + z_4^7 \). \( V_f \) has vertices \( e_1, e_2, e_3, e_4 \) and \( R, P, S \) where

\[
R = \left( \frac{1}{12}, \frac{1}{4}, \frac{1}{7}, \frac{1}{7} \right), \quad f_R(z) = z_1^0 z_2 + z_2^3 + z_3^7 + z_4^7
\]

\[
P = \left( \frac{2}{21}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7} \right), \quad f_P(z) = z_3^7 + z_4^7
\]

\[
S = (0, 1, \frac{1}{2}, \frac{1}{7}), \quad f_S(z) = z_1^0 z_2 + z_3^7 + z_4^7.
\]

Note that \( P \in V_f^+ \setminus V^+ \) as \( f_{2P} = z_1^0 + z_2^3 + z_3^6 \) and \( \deg_P f_{2P} + \frac{2}{1} = \frac{8}{1} > 1 \). \( S \in V_0 \) corresponds to the vanishing coordinate subspace \( C^{(1)} \). The vertex \( P \) is in the simplicial cone \( \text{Cone}(R, e_1, e_2) \) as \( P = R + \frac{1}{12} e_1 + \frac{1}{21} e_2 \). Note that \( \text{Cone}(R, e_1, e_2) \) is a regular boundary region. See the definition below.

3.3. Boundary region. We consider equivalence classes \( [P] \) and \( [P]_J \) in \( \Gamma^*(f) \) and \( \Gamma^*_J(f) \) respectively. There exist three different cases.

1. An equivalent class \( [P] \) (respectively \( [P]_J \)) is called an inner region if the closure \( \overline{[P]} \) (resp. \( \overline{[P]}_J \)) does not contain any vertex of \( V_0 \) on the boundary.

2. \( [P] \) (respectively \( [P]_J \)) is called a regular boundary region if the closure \( \overline{[P]} \) (resp. \( \overline{[P]}_J \)) contains some vertex \( e_i \) but contains no vanishing vertex on the boundary.

3. \( [P] \) (resp. \( [P]_J \)) is called a vanishing boundary region, if \( \overline{[P]} \) (resp. \( \overline{[P]}_J \)) contains a vanishing vertex \( Q \in \Gamma^*(f) \) (resp. \( Q \in \Gamma^*_J(f) \)) on the boundary.

3.4. Special admissible paths. Two weight vectors \( P, Q \) are called admissible, (respectively \( J \)-admissible) if \( \Delta(P) \cap \Delta(Q) \neq \emptyset \) (resp. \( \Delta(P) \cap \Delta(Q) \neq \emptyset \) and \( \Delta(P, f_i) \cap \Delta(Q, f_i) \neq \emptyset \) for any \( i \)). Any weight \( R \) in the interior of an admissible line segment \( PQ \) satisfies \( \Delta(R) = \Delta(P) \cap \Delta(Q) \) (resp. \( \Delta(R) = \Delta(P) \cap \Delta(Q) \) and \( \Delta(R, f_i) = \Delta(P, f_i) \cap \Delta(Q, f_i), i = 1, \ldots, n \)). Take a weight vector \( P = (p_1, \ldots, p_n) \) which is not strictly positive. Put \( I = \{ i \mid p_i = 0 \} \). We say \( P \) is a vanishing weight (respectively non-vanishing weight) if \( f^I \equiv 0 \) (resp. \( f^I \neq 0 \)).

Proposition 8. (A path in a regular boundary region) Suppose that two weight \( P, Q \) are admissible, \( Q \) is strictly positive weight and \( P \) is a non-vanishing weight vector with \( I = \{ i \mid p_i = 0 \} \). Then weight vector \( R \in PQ \) on this line segment (except \( P \)) is given in the normalized form as \( \hat{R}_t = \hat{Q} + tP \) with \( 0 \leq t < \infty \). In this expression, \( \hat{R}_t \to P \) when \( t \to \infty \) and there exists a sufficiently large \( \delta > 0 \) so that \( m(\hat{R}_t) \equiv m(\hat{Q}_t) \) and \( \eta(R_t) \equiv \eta(\hat{Q}_t) \) for \( t \geq \delta \) and \( \eta(\hat{Q}_t) \leq \eta(\hat{Q}) \).
Proof. For \( j \notin I \), \( \hat{q}_j + tp_j \to \infty \) and the assertion follows immediately. Here \( m(\hat{Q}_t) = \min\{\hat{q}_j | j \in I\} \). \( \square \)

**Proposition 9.** (A path in a vanishing boundary region) Suppose that \( Q \) is strictly positive and \( P \) is a vanishing weight vector. Put \( I = \{i | p_i = 0\} \). Then \( \hat{Q}_t = (1 - t)\hat{Q} + t\hat{P} \), \( 0 \leq s \leq 1 \) parametrize the weights on the line segment \( \overline{QP} \) and we have the following.

1. Suppose that \( \hat{q}_j \geq \hat{q}_i \) for some \( i \in I \), \( j \notin I \). Then \( (1 - t)\hat{q}_j + t\hat{p}_j \geq (1 - t)\hat{q}_i \) for \( 0 \leq t \leq 1 \).

2. If there is a \( j \notin I \) such that \( \hat{q}_j < \hat{q}_i \) for some \( i \in I \), there exists \( 0 < t_0 < 1 \) which satisfies \( (1 - t_0)\hat{q}_j + t_0\hat{p}_j = (1 - t_0)\hat{q}_i \).

Proof. Third assertion follows from the following property.

\[
(1 - t)\hat{q}_j + t\hat{p}_j \mid_{t \to 1} > 0, \quad (1 - t)\hat{q}_i \mid_{t \to 1} = 0.
\]

\( \square \)

3.5. **Key lemma.** First we prepare an elementary lemma.

**Lemma 10.** Consider a linear fractional function \( \varphi(s) = \frac{as + b}{cs + d} \) where \( a, b, c, d \) are real numbers such that \( (c, d) \neq (0, 0) \) and \( cs + d \neq 0 \) for \( 0 \leq s \leq 1 \). Then if \( \varphi'(s) \neq 0 \), the sign of \( \varphi'(s) \) does not change i.e., \( \varphi'(s) > 0 \) or \( \varphi'(s) < 0 \) for any \( s \), \( 0 \leq s \leq 1 \). Thus \( \varphi(s) \) is a monotone function on \( [0, 1] \).

Proof. Assertion follows from

\[
\varphi'(s) = \frac{ad - bc}{(cs + d)^2}.
\]

\( \square \)

Assume that \( P, Q \) are strictly positive weight vectors. Then the weights on this line segment \( \overline{P\hat{Q}} \) can be parametrized normally as \( \hat{R}_s \), \( 0 < s < 1 \):

\[
\hat{R}_s = s\hat{P} + (1 - s)\hat{Q}.
\]

Putting \( \hat{P} = (\hat{p}_1, \ldots, \hat{p}_n) \) and \( \hat{Q} = (\hat{q}_1, \ldots, \hat{q}_n) \), we can write \( \hat{R}_s = (sp_1 + (1 - s)\hat{q}_1, \ldots, sp_n + (1 - s)\hat{q}_n) \). We consider the quantities defined in (10):

\[
\eta_{ij}(\hat{R}_s) = \frac{1 - \hat{r}_{s,i}}{\hat{r}_{s,j}} = \frac{1 - (sp_j + (1 - s)\hat{q}_j)}{sp_i + (1 - s)\hat{q}_i},
\]

\[
\eta(\hat{R}_s) = \frac{1 - m(\hat{R}_s)}{m(\hat{R}_s)}
\]

\[
\eta'_{ij}(\hat{R}_s) = \frac{\deg(\hat{R}_s, f_j)}{\hat{r}_{s,i}} = \frac{\deg(\hat{R}_s, f_j)}{sp_i + (1 - s)\hat{q}_i}
\]

Applying Lemma [10] we have

**Lemma 11.** Assume that \( P, Q \) are strictly positive weight vectors.
Assume that \( P, Q \) are addmissible. Then we have
\[
\eta_{ij}(\hat{R}_s) \leq \max\{\eta_{ij}(\hat{P}), \eta_{ij}(\hat{Q})\}, \quad 0 < s < 1.
\]
In particular, we have
\[
\eta(\hat{R}_s) \leq \max\{\eta(\hat{P}), \eta(\hat{Q})\}
\]
(2) Assume that \( P, Q \) are \( J \)-addmissible. Then we have
\[
\eta'_{ij}(\hat{R}_s) \leq \max\{\eta'_{ij}(\hat{P}), \eta'_{ij}(\hat{Q})\}, \quad 0 < s < 1.
\]

3.6.1. Invariants to be used for the estimation. Let \( \mathcal{V}^+_j \) be the set of strictly positive vertices of \( \Gamma_j^*(f) \) and consider the subset \( \mathcal{V}^+_j \subset \mathcal{V}^+_j \) which are in a vanishing boundary region \([Q]\) of \( \Gamma^*_j(f) \) for some \( Q \). The numbers of \( \mathcal{V^+}, \mathcal{V^+_j}, \mathcal{V^+_j} \) are finite. We define the following invariants.
\[
\eta_{max}(f) := \max\{\eta(P) \mid P \in \mathcal{V}^+\}
\]
\[
\eta_{I, \text{max}}(f) := \max\{\eta(P) \mid P \in \mathcal{V}^+ \cup \mathcal{V}^+_j\},
\]
\[
\eta'_{I, \text{max}}(f) := \max\{\eta'_{k,i}(P) \mid R \in \mathcal{V}^+_j, k, i = 1, \ldots, n\}
\]
\[
\eta''_{I, \text{max}}(f) := \max\{\eta_{I, \text{max}}(f), \eta'_{I, \text{max}}(f)\}.
\]

Here \( \eta'_{k,i}(R) = d(R, f_i)/r_k \).

3.6. Main theorem. The following estimation is our main result which is a modified weaker version of the assertion in [9]. Recall that we assume that \( \dim \Gamma(f) = n - 1 \).

**Theorem 12.** Let \( f(z) \) be a non-degenerate Lojasiewicz non-degenerate function with an isolated singularity at the origin. Then Lojasiewicz exponent \( \ell_0(f) \) has the estimation
\[
\ell_0(f) \leq \eta''_{I, \text{max}}(f).
\]
If \( \mathcal{V}_j^+ = \emptyset \), the estimation can be replaced by a better one
\[
\ell_0(f) \leq \eta_{\text{max}}(f).
\]

The proof of Theorem 3.6 will be given in 3.7.

3.6.1. Test Curve. We consider an analytic curve \( C(t), -\varepsilon < t \leq \varepsilon \) parametrized as before (2),
\[
\begin{align*}
C(t) : & \quad \{ z(t) = (z_1(t), \ldots, z_n(t)), \quad z(0) = 0, \quad z(t) \in \mathbb{C}^I \} \\
& \quad \begin{aligned}
z_i(t) &= a_i t^{p_i} + \text{(higher terms), } \quad i \in I \\
z_j(t) &\equiv 0, \quad j \notin I
\end{aligned}
\end{align*}
\]
(12)

For simplicity, we assume that \( I = \{1, \ldots, m\} \). Put \( P = (p_1, \ldots, p_m) \in \mathbb{N}_+^I \)
and \( a = (a_1, \ldots, a_m) \in \mathbb{C}^I \). We are interested in a best possible upper bound for the positive quantity \( \ell_0(C(t)) = \text{ord} f(z(t))/\text{ord} z(t) \). Using the notation \( f_j = \partial f/\partial z_j \), we have the expansion
\[
f_j(z(t)) = (f_j)_P(a) t^{d(P, f_j)} + \text{(higher terms)}.
\]

(13)
Note that if \( f_P(z) \) contains the variable \( z_j \),
\[
(f_j)_P(a) = (f_P)_j(a), \quad d(P, f_j) = d(P, f) - p_j. \tag{14}
\]

### 3.6.2. Curves corresponding to a strictly positive weight vector.

In the previous section, we have seen that a test curve \( C(t) \) gives a pair \( (P, a) \in \mathbb{N}_I^* \times \mathbb{C}^*I \). We consider the converse in the case \( I = \{1, \ldots, n\} \). Assume we have a strictly positive integer weight vector \( P = (p_1, \ldots, p_n) \in \mathbb{N}_+^n \cap \mathbb{Z}^n \).

Taking a coefficient vector \( a = (a_1, \ldots, a_n) \in \mathbb{C}^n \), we associate an analytic curve
\[
C_P(t, a) : z(t) := (a_1 t^{p_1}, \ldots, a_n t^{p_n}).
\]

The test curve \( C(t) \) gives the data \((P, a)\) and if \( I = \{1, \ldots, n\} \), \( C_P(t, a) \) and \( C(t) \) differs only higher terms. In this case, we also use the notation as \( \ell_0(P) \) instead of \( \ell_0(C_P(t, a)) \) or \( \ell_0(C(t)) \) by an abuse of notation. Then by the above discussion and by the non-degeneracy assumption, we have
\[
\begin{align*}
\text{ord} \partial f(z(t)) &\leq d(P, f) - m(P) \tag{15} \\
\frac{\text{ord} \partial f(z(t))}{\text{ord} z(t)} &\leq \frac{d(P, f)}{m(P)} - 1 \tag{16} \\
&= \frac{1}{m(\hat{P})} - 1. \tag{17}
\end{align*}
\]

Note that \( \eta(\hat{P}) = \eta(P) \). This estimation does not depend on the choice of representative of the equivalence class of \( P \) and the choice of \( a \). The weakness of the above estimation is that \( m(\hat{P}) \) can be arbitrary small in the vanishing boundary region which makes \( \eta(\hat{P}) = \eta(P) \) unbounded.

### 3.7. Proof of Theorem \ref{thm:main}

Take a test curve as \( \ref{eq:test_curve} \) and we consider the weight vector \( P \). To prove the theorem, it is enough to prove that \( \ell_0(f)(C(t)) \leq \eta''_{J, \max}(f) \) by Proposition \ref{prop:eta_max}

We first consider the case that \( P \) is either strictly positive which is most essential.

#### 3.7.1. Strictly positive case.
We first assume that \( I = \{1, \ldots, n\} \) and \( P \) is strictly positive. We divide the situation into three cases.

- **C-1** \([P] \) is an inner region. That is, \([-P] \) has only strictly positive weight vectors in the boundary.
- **C-2** \([P] \) is a regular boundary region.
- **C-3** \([P] \) is a vanishing boundary region. In this case, we need to consider the subdivision by Jacobian dual Newton diagram. There are three subcases:
  - **C-3-1.** \([P]_J \) is an inner region.
  - **C-3-2.** \([P]_J \) is a regular boundary region.
  - **C-3-3.** \([P]_J \) is also a vanishing boundary region.

We need consider the Jacobian dual diagram only in vanishing boundary regions of \( \Gamma^*(f) \).
3.7.2. Cases C-1 and C-3-1. We start from the inequality \( \ell_0(P) \leq \eta(P) \) and then estimate \( \eta(P) \) by the strictly positive vertices. We use an induction on \( \dim [P] \) or \( \dim [P]_J \) in Case C-3-1 to show that \( \ell_0(P) \leq \eta_{\text{max}}(f) \) (resp. \( \ell_0(P) \leq \eta_{I,\text{max}}(f) \)). The induction starts from the case \( \dim [P] = 1 \) (respectively \( \dim [P]_J = 1 \)). In this case, the assertion is obvious. If \( \dim [P] = r > 1 \), we take a line segment \( RS \) with \( R, S \in \partial [P] \) (resp. in \( [P]_J \)) passing through \( P \) we apply Lemma 13 to get the estimation

\[
\ell_0(C(t)) \leq \eta(P) \leq \max\{\eta(R), \eta(S)\} \leq \eta_{\text{max}}(f), R, S : \text{admissible}
\]

\[
\ell_0(C(t)) \leq \eta(P) \leq \max\{\eta(R), \eta(S)\} \leq \eta_{I,\text{max}}(f), S : J-\text{admissible}.
\]

As \( \dim [R], \dim [S] < r \) (resp. \( \dim [R]_J, \dim [S]_J < r \)), the induction works. For the other cases, we prepare a simple lemma.

**Lemma 13.** Assume that \( \dim \Gamma(f) = n - 1 \). Then for any face \( \Delta \subset \Gamma(f) \), there exists an \( (n - 1) \) dimensional face \( \Xi \) such that \( \Xi \supset \Delta \). Equivalently for any weight \( P \), the closure \( [P] \) contains a strictly positive vertex \( Q \). Respectively \( [P]_J \) contains a strictly positive vertex \( Q \in \mathcal{V}_j^+ \).

The assertion is immediate from the assumption that \( \dim \Gamma(f) = n - 1 \) as \( \Gamma_+(f) \) is a \( n \)-dimensional convex polyhedral region and \( \Gamma(f) \) is the union of compact boundary faces. The assertion for \( \Gamma_+(f) \), as we can use \( F = f f_1 \cdot f_n \) instead of \( f \).

3.7.3. Case C-2 and C-3-2. We start again from the inequality \( \ell_0(P) \leq \eta(P) \). Assume that \( [P] \) (respectively \( [P]_J \)) is a regular boundary region. We prove that \( \eta(P) \leq \eta_{\text{max}}(f) \) (respectively \( \eta(P) \leq \eta_{I,\text{max}}(f) \)) by the induction of \( \dim [P] \) (resp. \( \dim [P]_J \)). The argument is completely same in the case \( [P]_J \). Take a strictly positive vertex \( R \) in \( [P] \cap \mathcal{V}_j^+ \) (resp. in \( R \in [P]_J \cap \mathcal{V}_j^+ \)), using Lemma 13. Take the segment \( RR' \) and extend it further to the right so that it arrives to a boundary point of the region, say \( Q \). Then \( [Q] \) is either an inner region or a regular boundary region.

If \( Q \) is strictly positive and \( [Q] \) is an inner region, we can apply the argument of Case C-1 or C-3-1 and we consider the estimation in \( [Q] \) by the inductive argument.

Similarly if \( Q \) is strictly positive and \( [Q] \) is a regular boundary region, we apply the inductions assumption, as \( \dim [P] > \dim [Q] \).

So we assume that \( Q \) is not strictly positive. Then \( Q \) is a non-vanishing weight vector i.e., \( d(Q) = 0 \). The normalized form of weight vectors on this segment is given as \( \hat{R}_s = \hat{R} + sQ \) with \( 0 \leq s < \infty \) and \( \hat{R} = \hat{R}_{s_0} \) for some \( s_0 > 0 \). Put \( I = I(Q) \) and \( m_I(\hat{R}) = \min \{\hat{r}_i | i \in I\} \), \( I' := \{i | \hat{r}_i = m_I(\hat{R})\} \) and \( I_R := \{j | \hat{r}_j = m(\hat{R})\} \).

-If \( I' \cap I_R \neq \emptyset \), \( m(\hat{R}) = m_I(\hat{R}) \) and \( m(\hat{R}_s) \equiv \hat{r}_{i_0} \) for any \( s \) and \( i_0 \in I' \). Thus

\[
\ell_0(C(t)) \leq \eta(\hat{P}) = \eta(\hat{R}_{s_0}) = \eta(\hat{R}) \leq \eta_{\text{max}}(f) \text{ (resp. } \leq \eta_{I,\text{max}}(f)\text{).}
\]

-If \( I' \cap I_R = \emptyset \), i.e., \( m(\hat{R}) < m_I(\hat{R}) \), take \( j_0 \in I_R \) such that there exists a small positive number \( \varepsilon \) and \( m(\hat{R}_s) = \hat{r}_{j_0} + sq_{j_0} \) for \( s \leq \varepsilon \). As \( m_I(\hat{R}) > \hat{r}_{j_0} \)
but $\hat{r}_{j_0} + sq_{j_0}$ is monotone increasing in $s$, there exists some $s_1$ such that $m_I(\hat{R}) = r_{j_0} + s_1q_{j_0}$ and for $s \geq s_1$, $m(\hat{R}_s) = m_I(\hat{R})$. Thus $\eta(\hat{R}_s)$ is monotone decreasing for $0 \leq s \leq s_1$ and constant for $s \geq s_1$. Thus in any case we get

$$\ell_0(C(t)) \leq \eta(P) = \eta(\hat{R}_0) \leq \eta(\hat{R}) \leq \eta_{\max}(f) \quad (\text{resp. } \leq \eta_{I,\max}(f)).$$

3.7.4. Case C-3-3. This case requires careful choice of the line segment for the estimation. For this purpose, we prepare the following Proposition 14 and Lemma 15. A strictly positive weight vector $P$ is simplicially positive (respectively $J$-simplicially positive) if there exist strictly positive linearly independent vertices $P_1, \ldots, P_s$ of $\Gamma^+(f)$ such that $P_i \in [P]_i, i = 1, \ldots, s$ (resp. vertices $P_1, \ldots, P_s$ of $\Gamma_J^+(f)$ such that $P_i \in [P]_j, i = 1, \ldots, s$) and $P$ is in the interior of the simplex $(P_1, \ldots, P_s)$. Here by a simplex $(P_1, \ldots, P_s)$, we mean the simplicial cone

$$\text{Cone}(P_1, \ldots, P_s) = \left\{ \sum_{i=1}^s \lambda_i P_i | \lambda_i \geq 0 \right\}.$$

Thus a line segment $\overline{PQ}$ is equal to the simplex $(P, Q)$.

**Proposition 14.** Let $P$ be a strictly positive weight vector. Then there are two possibilities.

1. $P$ is simplicially positive (respectively $J$-simplicially positive).
2. There are linearly independent vertices $P_1, \ldots, P_{q-1}, q \leq \dim [P]$ of $[P]$ (resp. linearly independent vertices $P_1, \ldots, P_{q-1}, q \leq \dim [P]_j$ of $[P]_j$) and a weight vector $P_q$ which is not strictly positive so that $P$ is in the interior of the simplex $(P_1, \ldots, P_q)$.

**Proof.** The assertion follows easily from the fact that $[P]$ (resp. $[P]_j$) is a polyhedral convex cone. We use induction on $r = \dim [P]$ (resp. $r = \dim [P]_j$). As the proof is completely parallel, we show the assertion in the case of $\Gamma^+(f)$. Take a strictly positive vertex $P_1 \in [P]$ using Lemma 13 and take the line segment $\overline{P_1P}$ and extending to the right, put $Q_1$ be the weight on the boundary of $[P]$. Thus $P$ is contained in the interior of $\overline{P_1Q_1}$. Consider $[Q_1]$. Then $\dim [Q_1] < r$. If $Q_1$ is not strictly positive, we stop the operation. Then $q = 2$ and this case corresponds to Case (2). If $Q_1$ is still strictly positive but not a vertex, we repeat the argument on $[Q_1]$. Take a strictly positive vertex $P_2 \in [Q_1]$ and so on. Apply an inductive argument. The operation stops if we arrive at a weight vector which is not strictly positive (then case (2)) or a strictly positive vertex $P_q$ (Case (1)).

Using this proposition, we have the following choice of a nice line segment.

**Lemma 15.** Assume that $P$ is a strictly positive weight. If $P$ is not simplicially positive (respectively not $J$-simplicially positive), there is a line segment $\overline{RQ}$ such that $R$ is simplicially positive (resp. $J$-simplicially positive) and $Q$ is not strictly positive.
Proof. We give the proof for $\Gamma^* (f)$ as the proof is completely parallel for $\Gamma^*_j (f)$. Assume that $P$ is not simplicially positive. Using Proposition 12, we suppose that $P$ is in the interior of the simplex $(P_1, \ldots, P_q)$ where $P_1, \ldots, P_{q-1}$ are strictly positive vertices and $P_q$ is not strictly positive. Write $P$ by a barycentric coordinates as $P = \sum_{i=1}^{q} \lambda_i P_i$ with $\lambda_i > 0$ and we may assume $\sum_{i=1}^{q} \lambda_i = 1$. Put $\lambda := \sum_{i=1}^{q} \lambda_i$, $\mu := 1 - \lambda$ and define $R := (\sum_{i=1}^{q-1} \lambda_i P_i)/\lambda$ and $Q := P_q/\mu$. Then $P = \lambda R + \mu Q$ and $R \in (P_1, \ldots, P_{q-1})$. Thus $R$ is simplicially positive. \[\square\]

Remark 16. For the proof of Case 3-3 below, the Jacobian dual Newton diagram is essential. So we use Lemma 15 for $\Gamma^*_j (f)$.

Proof of Case 3-3. Now we are ready to have an estimation for the Lojasiewicz exponent of our test curve $C(t)$. Suppose that $[P]$ and $[P]_J$ are non-banishing boundary region. We apply Lemma 15. If $P$ is $J$-simplicially positive, we have the estimation $\ell_0 (C(t)) \leq \eta_{J, \max} (f)$ by the same argument as in Case 3-1. Thus using Lemma 15, we may assume that $P$ is in the line segment $RQ$ where $R$ is $J$-simplicially positive and $Q$ is not strictly positive. If $Q$ is a non-vanishing weight, we proceed as the case C-3-2 to get the estimation $\ell_0 (C(t)) \leq \eta_{J, \max} (f)$. Thus we assume that $Q$ is a vanishing weight vector and $d(Q, f) > 0$. Assume that $Q = (q_1, \ldots, q_n)$ and put $I := \{ i \mid q_i = 0 \}$. We assume $I = \{ 1, \ldots, m \}$ for simplicity. Note that $C^I$ is a vanishing coordinate subspace. For each $i \in I$, there exists some $j \not\in I$ and a monomial $z_j^{n_{i,j}} z_j$ with a non-zero coefficient, as $f$ has an isolated singularity at the origin. Put $J_i$ be the set of such $j$ for a fixed $i \in I$ and put $J(I) = \bigcup_{i \in I} J_i$. Here $n_{i,j}$ is assumed to be the smallest when $j$ is fixed. Put $\xi_I := \max \{ n_{i,j} \mid i \in I, j \in J_i \}$ and $\xi (f)$ be the maximum of $\xi_I$ where $I$ corresponds to a vanishing coordinate subspace. Put $\eta'_{J, \max} (f) := \max \{ \eta'_{k,i} (R) \mid R \in V^{++}, 1 \leq k, i \leq n \}$ where $\eta'_{k,i} (R) = d(R, f_i)/r_j$. Under the above situation, we will prove that

\[(*) \quad \ell_0 (C(t)) \leq \max \{ \xi (f), \eta_{J, \max} (f), \eta'_{J, \max} \}.
\]

Consider the normalized weight vector $\tilde{R}_s := (1 - s) \tilde{R} + s \tilde{Q}$, $0 \leq s \leq 1$. Note that $\tilde{R}_0 = \tilde{R}$, $\tilde{R}_1 = \tilde{Q}$ and putting $\tilde{R}_s = (\tilde{r}_{s,1}, \ldots, \tilde{r}_{s,n})$,

\[
\tilde{r}_{s,i} = \begin{cases} 
(1 - s) \tilde{r}_i, & 1 \leq i \leq m \\
(1 - s) \tilde{r}_i + s \tilde{q}_i, & m < i \leq n.
\end{cases}
\]

Put $I' = \{ i \in I \mid \tilde{r}_i = m(\tilde{R}_I) \}$ and $J' = \cup_{i \in I'} J_i$. Thus for $i \leq m$, the normalized weight $\tilde{r}_{sj}$ goes to 0, when $s$ approaches to 1. On the other hand, for $j > m$, $\tilde{r}_{sj} \geq \delta$, $0 \leq s \leq 1$ for some $\delta > 0$. Thus there exists an $\varepsilon$, $1 > \varepsilon > 0$ so that for $1 - \varepsilon \leq s \leq 1$, $m(\tilde{R}_s)$ is taken by $i \in I'$. Note that for $j \in J'$, there exists a small enough $\varepsilon_2$, $\varepsilon_2 \leq \varepsilon_1$ so that $(f_j)_{\tilde{R}_s} = ((f_j)^I)_{\tilde{R}_s}$ as $\tilde{r}_{sj} \geq \delta$ for $j > m$ for $1 - \varepsilon_2 \leq s \leq 1$. Here $(f_j)^I$ is the restriction of $f_j$ to $C^I$ and $(\tilde{R}_s)^I$ is the I projection of $\tilde{R}_s$ to $N^I_+$. That is, $(f_j)_{\tilde{R}_s}$ contains only
variable $z_1, \ldots, z_m$. By the Lojasiewicz non-degeneracy, there exists $i_0 \in I'$ and $j_0 \in J_{i_0}$ such that
\[(f_{j_0})_{\hat{R}_s}(a) \neq 0, \quad s \geq 1 - \varepsilon_2.\]

By the definition of Jacobian dual Newton diagram, for any $0 < s < 1$, $(f_{j_0})_{\hat{R}_s}$ does not depend on $s$ and thus $(f_{j_0})_{\hat{R}_s}(a) \neq 0$ for any $0 < s < 1$. Then $d(\hat{R}_s, f_{j_0}) \leq (1 - s)\hat{r}_{i_0} \times n_{i_0,j_0}$ for any $0 < s \leq 1$. The equality takes place if the monomial $z_{i_0,j_0}$ is on the face function $(f_{j_0})_{\hat{R}_s}(z)$. Assume that $\bar{P} = \hat{R}_{s_0}, 0 < s_0 < 1$. Thus we start from the estimation
\[(18) \quad \ell_0(C(t)) \leq d(\hat{R}_{s_0}, f_{j_0})/m(\hat{R}_{s_0}).\]

First for $s \geq 1 - \varepsilon_2$, we see that
\[\ell_0(\hat{R}_s) \leq \frac{(1 - s)\hat{r}_{i_0} n_{i_0,j_0}}{(1 - s)\hat{r}_{i_0}} = n_{i_0,j_0}.\]

$(s_0 = \mu$ in the proof of Lemma $15$). Put $I_R : = \{k \mid \hat{r}_k = m(\hat{R})\}$.

(a) If there exists a $k \in I' \cap I_R$ i.e. $m(\hat{R}) = m(\hat{R}_k)$, we have the estimation $\ell_0(\hat{R}_s) \leq n_{i_0,j_0}$ for any $s$. In particular, $\ell_0(C(t)) \leq n_{i_0,j_0}$.

(b) Assume that $I' \cap I_R = \emptyset$ i.e. $m(\hat{R}) < m(\hat{R}_k)$. Choose $k \in I_R$. Then there exists a small number $\varepsilon > 0$ so that $m(\hat{R}_s) = \hat{r}_{s,k}$ for $0 \leq s \leq \varepsilon$. By the definition of $I'$, this implies that $k \notin I$. There exists $0 < s_1 < 1$ such that $m(\hat{R}_{s_1}) = (1 - s_1)\hat{r}_k + s_1 \hat{q}_k = (1 - s_1)\hat{r}_{i_0}$. We divide this case into two subcases.

(b-1) Assume that $(1 - s)\hat{r}_k + s\hat{q}_k$ is monotone increasing in $s$. Then
\[\ell_0(P) = \ell_0(\hat{R}_{s_0}) \leq \eta(\hat{R}_0) = \eta(\hat{R}) \leq \eta_{I,J,max}(f), \quad \text{if } s_0 \leq s_1.\]

\[\ell_0(P) = \ell_0(\hat{R}_{s_0}) \leq \frac{(1 - s_0)\hat{r}_{i_0} n_{i_0,j_0}}{(1 - s_0)\hat{r}_{i_0}} = n_{i_0,j_0}, \quad \text{if } s_0 > s_1.\]

(b-2) Assume that $(1 - s)\hat{r}_k + s\hat{q}_k$ is monotone decreasing in $s$. Then
\[(19) \quad \ell_0(\hat{R}_{s_0}) \leq \frac{(1 - s_0)\hat{r}_{i_0} n_{i_0,j_0}}{(1 - s_0)\hat{r}_{i_0} + s_0\hat{q}_k} \leq \frac{\hat{r}_{i_0} n_{i_0,j_0}}{\hat{r}_k} \leq \eta_{k,j_0}'(R), \quad \text{if } s_0 \leq s_1\]
\[(20) \quad \ell_0(\hat{R}_{s_0}) \leq \frac{(1 - s_0)\hat{r}_{i_0} n_{i_0,j_0}}{(1 - s_0)\hat{r}_{i_0}} = n_{i_0,j_0}, \quad \text{if } s_0 \geq s_1.\]

where the first estimation (19) for $s_0 \leq s_1$ follows from the fact that
\[s \mapsto \eta_{k,j_0}'(\hat{R}_s) = \frac{(1 - s)\hat{r}_{i_0} n_{i_0,j_0}}{(1 - s)\hat{r}_k + s\hat{q}_k}, \quad 0 \leq s \leq 1\]
is monotone decreasing on $s$. Then we apply Lemma $14$ to get the estimation
\[\ell_0(C(t)) \leq \eta_{k,j_0}'(R) \leq \eta_{I,J,max}(f).\]

Thus we have proved the estimation $(\star)$. Assuming the next lemma for a moment, we can ignore the first term $\xi(f)$ in $(\star)$ and the estimation reduces to
\[\ell_0(C(t)) \leq \eta_{I,J,max}''(f), \quad \text{if } z(t) \in \mathbb{C}^n.\]
Furthermore if $\mathcal{V}_j^+ = \emptyset$, $\eta_{i,j}(P) = \eta_{i,j}(P)$ for $P \in \mathcal{V}^+$, as $f_P(z)$ contains all the variables and therefore $\eta_{i,j}(P) = \eta_{i,j}(P) \leq \eta(f)$ or $\eta_{I,max}(f) \leq \eta_{I,max}(f)$.

**Lemma 17.** The following inequality holds.

$$n_{i,j} \leq \eta_{\text{max}}(f), \forall i \in I, \forall j \in J.$$ 

**Proof.** Let $\Xi$ be a maximal face of $\Gamma(f)$ for which the face function $f_\Xi(z)$ contains the monomial $z_i^{n_{i,j}} z_j$. The corresponding weight vector $P$ (i.e., $\Delta(P) = \Xi$) is in $\mathcal{V}^+ \subset \Gamma^*(f)$ such that $\Xi$ is subset of the hyperplane $p_1\nu_1 + \cdots + p_n\nu_n = d(P)$ and $f_P(z)$ contains the monomial $z_i^{n_{i,j}} z_j$. Thus $n_{i,j}p_i + p_j = d(P)$. Consider an analytic curve $z(t)$ corresponding to the weight $P$. Then we have

$$n_{i,j} = \frac{d(P) - p_j}{p_i} = \eta_{i,j}(P) \leq \eta(P) \leq \eta_{\text{max}}(f).$$

$\square$

To complete the proof, we have to consider the case where the test curve is in a proper coordinate subspace.

### 3.7.5. Test curves in a proper subspace.

We consider the situation of the test curve $z(t)$ defined in (12) for which $I^c \neq \emptyset$. Recall that $I = \{ i \mid z_i(t) \neq 0 \}$ and we assume $I = \{1, \ldots, m\}$, $m < n$ for simplicity.

**Case 1** Assume that $f^I \neq 0$. Recall that $P = (p_1, \ldots, p_m)$ and $a = (a_1, \ldots, a_m) \in \mathbb{C}^m$. Let $\Delta_1 = \Delta(P) \subset \Gamma^*(f^I)$. Consider the weight vector $\tilde{P} = (p_1, \ldots, p_m, K, \ldots, K) \in \mathbb{N}_+$ where $K$ is sufficiently large so that $\Delta(\tilde{P}, f) = \Delta_1$, $d := d(P, f^I) = d(\tilde{P}, f)$ and $m(\tilde{P}) = m(P)$. Put $\tilde{a} = (a, 1, \ldots, 1)$. Consider $\theta_k := \text{ord} \frac{\partial f}{\partial k}(z(t))$ and put $\theta := \min\{\theta_k \mid \theta_k \neq \infty\}$. Here $\theta_k = \infty$ if $\partial f/\partial k(z(t)) \equiv 0$ by definition. Then $\ell_0(C(t)) = \theta/m(P)$. Consider the modified curve

$$\tilde{C}(t) : \tilde{z}(t) = (z_1(t), \ldots, z_m(t), t^N, \ldots, t^N).$$

Taking $N$ sufficiently large, say $N > \max\{\theta_k \mid \theta_k \neq \infty\}$, it is easy to see that ord $\frac{\partial f}{\partial k}(\tilde{z}(t)) = \theta_k$ for any $k$ with $\theta_k \neq \infty$. Thus for such an $N$, $\ell_0(C(t)) = \ell_0(\tilde{C}(t))$. Combining with the previous argument, we get $\ell_0(\tilde{C}(t)) \leq \eta_{I,max}(f)$.

**Case 2** Assume that $f^I \equiv 0$. This implies $C^I$ is a vanishing coordinate subspace. Thus each monomial in the expansion of $f$ must contain one of $\{z_j \mid j \in I^c\}$. Thus $f_i(z(t)) \equiv 0$ for $i \in I$. Put $m(P) := \min\{p_i \mid i \in I\}$ and $I' = \{i \in I \mid p_i = m(P)\}$ and $J_i$ be the set of $j \in I^c$ such that a monomial $z_i^{n_{i,j}} z_j$ exists. Then by the Lojasiewicz non-degeneracy, there exists $i_0 \in I'$ and $j_0 \in J_{i_0}$ such that $f_{j_0}(a) \neq 0$ and thus

$$\frac{\text{ord} \frac{\partial f(z(t))}{\partial z(t)}}{\text{ord} z(t)} \leq \max\{n_{i,j} \mid i \in I', j \in I_i\} \leq \eta_{I,max}(f).$$
This completes the proof of Theorem 12.

**Remark 18.** It is possible to have \( \eta_{I, \text{max}} = \eta_{\text{max}} \) or \( \eta_{I, \text{max}}' = \eta_{I, \text{max}} \) in some cases. For example, see the next section. In Example 7, we have the equality \( \eta_{I, \text{max}} = \eta_{\text{max}} \) as the simplex \((R, e_1, e_2)\) is a regular boundary region. In fact, we have \( \eta(R) = 11, \eta(P) = 9 \).

### 3.8. Weighted homogeneous polynomials

We consider a weighted homogeneous polynomial with isolated singularity at the origin. There are nice results by Abderrhmane [11] and Brzostowski [3]. I thank to Tadeusz Krasniński for informing me these papers. Here we will give a slightly different proof as a special case of Main theorem.

**Theorem 19.** Let \( f(z) \) be a non-degenerate, Łojasiewicz non-degenerate weighted homogeneous polynomial with isolated singularity at the origin and \( \dim \Gamma(f) = n - 1 \). Let \( R \) be the weight vector of \( f \). Then we have the estimation \( \ell_0(f) \leq \eta(R) \).

*Proof.* Let \( \hat{R} = (\hat{r}_1, \ldots, \hat{r}_n) \) be the normalized weight of \( f \). Put \( I_R := \{ i \mid \hat{r}_i = m(\hat{R}) \} \). We work in \( \Gamma^+(f) \). Consider a strictly positive integral weight vector \( P \) and consider the test curve \([12]\) with \( z(t) \in \mathbb{C}^n \). So ignoring the higher terms, we may assume \( \hat{C}_{P, \alpha} : z(t) = (a_1 t^{p_1}, \ldots, a_n t^{p_n}) \) with \( \alpha = (a_1, \ldots, a_n) \in \mathbb{C}^n \) as before. We may assume that \( P \not\ni [\hat{R}] \). There is a line segment the \( SQ \) guaranteed by Lemma [12] so that \( P \in SQ \) where \( S \) is simplicially positive and \( Q \) is not strictly positive. As \( V^+ = \{ R \} \), \( \Gamma(f) \) has only one face, \( S = R \) and \( Q \) is not strictly positive. Put \( I = I(Q) \) and assume that \( I = \{ 1, \ldots, m \} \) as before. If \( f^I \neq 0 \), we know that \( \ell_0(C(t)) \leq \eta(R) \).

Thus, we assume that \( C^I \) is a vanishing coordinate subspace. For each \( i \in I \), there exists a monomial \( z_i^{n_i} \) wit \( j \not\in I \). Put \( J_i \) be the set of such \( j \) for a fixed \( i \in I \) and put \( J = \bigcup_{i \in I} J_i \). Consider the normalized weight vector \( \hat{R}_s := (1 - s)\hat{R} + sQ, 0 \leq s \leq 1 \). Assume that \( \hat{P} = R_{s_0}, 0 < s_0 < 1 \) as before. Note that

\[
\hat{R}_s = (\hat{r}_{s,1}, \ldots, \hat{r}_{s,n}), \hat{R}_0 = \hat{R}, \hat{R}_1 = \hat{Q}
\]

\[
\hat{r}_{s,i} = \begin{cases} (1 - s)\hat{r}_i, & 1 \leq i \leq m \\ (1 - s)\hat{r}_i + s\hat{q}_i, & m < i \leq n. \end{cases}
\]

Thus for \( i \leq m \), the normalized weight \( \hat{r}_{s,i} \) goes to 0, as \( s \to 1 \), while for \( j > m \), \( \hat{r}_{s,j} \geq \delta > 0 \) for some \( \delta > 0 \). Put \( I' := \{ i \in I \mid \hat{r}_i = m(\hat{R}_I) \} \). There exists a positive number \( \varepsilon \) such that \( (f_j)_{\hat{R}_s} = (f_j)_{\hat{R}_0} \) and it contains only variables \( z_1, \ldots, z_m \) for \( 1 \leq s \leq 1 - \varepsilon \). By the Łojasiewicz non-degeneracy, there exists \( i_0 \in I' \) and \( j_0 \in J_0 \) so that \( (f_{j_0})_{\hat{R}_s}(a) = 0 \) for \( 1 > s \geq 1 - \varepsilon \). Note that for any \( j \), \( \Delta(\hat{R}, f_j) = \Gamma(f_j) \), as \( f_j \) is also weighted homogeneous of degree \( d(R) - r_j \) under the weight \( R \). Thus \( \Delta(\hat{R}, f_j) \subset \)
The polynomials appear in the classification of weighted homogeneous polynomials, the Lojasiewicz non-degeneracy for all or almost all such polynomials. We leave the further discussion to the readers. For the following typical case.

\[ f_i(z) = z_1^{a_1} + \cdots + z_{n-1}^{a_{n-1}} + z_n^{a_n} \]

\[ f_{II}(z) = z_1^{a_1}z_2 + \cdots + z_{n-1}^{a_{n-1}}z_n + z_n^{a_n}z_1. \]

These polynomials appear in the classification of weighted homogeneous polynomials by Orlik-Wagreich [24]. The corresponding mixed polynomials are studied in [11] in the problem of isotopy construction to holomorphic links.
3.9. **When the equality** $\ell_0(f) = \eta_{\text{max}}(f)$ **holds?** Suppose that $f$ is a non-degenerate, Lojasiewicz non-degenerate weighted homogenous polynomial with isolated singularity at the origin and let $R = (r_1, \ldots, r_n)$ be the integral weight vector. Assume $r_i \leq r_{i+1}$ for $i = 1, \ldots, n - 1$ for simplicity. Assume also that $r_1 = \cdots = r_k < r_{k+1}$. If there exists an $i_0 \leq k$ such that $a_{i_0} \neq 0$ and there exists a polar curve $z = z(t)$ such that

$$\Gamma : \begin{cases} z_i(t) = a_it^{r_i}, & i = 1, \ldots, n \\ \frac{\partial f}{\partial z_j}(a) = 0, & \forall j \neq i_0. \end{cases}$$

Then $\text{ord} z(t) = p_1 = m(R)$ and $\text{ord} \partial f(z(t)) = d - p_1$. Thus we have $\ell_0(C(t)) = \ell_0(f) = d/p_1 - 1$. For $n = 3$, there is an affirmative result by [12]. The following example shows that in general, we cannot take such a polar curve with $a \in \mathbb{C}^m$.

**Example 21.** 1. Consider $f(z) = z_1^2z_2 + z_2^3z_3 + z_3^4z_1 + z_1^2$. Then $f$ is a weighted homogenous polynomial and $\mathcal{V}^+$ has a single vertex $P = (9/25, 7/25, 4/25, 1/2)$. We see that $\ell_0(f) = \eta_{\text{max}}(f) = 21/4$. This number is taken on the face of $f^I$, $I = \{1, 2, 3\}$ by the family

$$= \sqrt[4]{3}it^9, \quad z_2(t) = \sqrt[12]{1}it^7, \quad z_3(t) = -t^4, \quad z_4(t) \equiv 0.$$ 

and $\text{ord} z(t) = 4$, $\text{ord} \partial f(z(t)) = 21$. As this example shows, the maximal Lojasiewicz number $\ell_0(C(t))$ need not to be taken on a vertex $R \in \mathcal{V}^+$. Consider $g(z') = z_1^2z_2 + z_2^3z_3 + z_3^4z_1$. Then $f$ is a join of two functions $f(z) = g(z') + z_1^2$ and $\ell_0(f) = \ell_0(g)$ by Join Theorem below.

3.10. **Lojasiewicz Join Theorem.** Consider a join type function $f(z, w) = g(z) + h(w)$ where $z \in \mathbb{C}^m$ and $w \in \mathbb{C}^n$. Assume that both $g(z)$ and $h(w)$ have isolated singularities at the respective origin. Then

**Lojasiewicz Join Theorem 22.** ([26], Corollary 2, §2) We have the equality:

$$\ell_0(f) = \max\{\ell_0(g), \ell_0(h)\}.$$ 

**Proof.** Put $u = (z, w) \in \mathbb{C}^{n+m}$. Take any analytic curve $C(t) : u(t) = (z(t), w(t)), 0 \leq t \leq 1$.

Case 1. If $z(t) \equiv 0$ (respectively $w(t) \equiv 0$), $\frac{\text{ord} \partial f(u(t))}{\text{ord} u(t)} \leq \ell_0(h)$ (resp. $\leq \ell_0(g)$).

Case 2. Assume that $z(t) \not\equiv 0$ and $w(t) \not\equiv 0$. If $\text{ord} z(t) \leq \text{ord} w(t)$, we have

$$\frac{\text{ord} \partial f(u(t))}{\text{ord} u(t)} = \min \left\{ \frac{\text{ord} \partial g(z(t))}{\text{ord} z(t)}, \frac{\text{ord} \partial h(w(t))}{\text{ord} z(t)} \right\} \leq \min \left\{ \ell_0(g), \frac{\ell_0(h)\text{ord} w(t)}{\text{ord} z(t)} \right\} \leq \ell_0(g).$$
If \( \text{ord} \, z(t) > \text{ord} \, w(t) \), by the same argument,
\[
\frac{\partial f(u(t))}{\partial u(t)} \leq \ell_0(h).
\]
Thus we have \( \ell_0(f) \leq \max\{\ell_0(g), \ell_0(h)\} \). The equality can be taken by a curve \( u(t) = (z(t),0) \) or \( u(t) = (0,w(t)) \) which takes \( \ell_0(g) \) for \( g(z) \) or \( \ell_0(h) \) for \( h(w) \) respectively.

**Remark 23.** As we see in the proof of Theorem 12, it is not necessary to take the Jacobian dual Newton diagram everywhere. We only need consider \( \Gamma_j(f) \) in the vanishing regions of \( \Gamma^*(f) \). Namely if \( P \) is a vertex of \( \Gamma_j(f) \) which is in an inner or a regular boundary region of \( \Gamma^*(f) \), we have an estimation \( \eta(P) \leq \eta_{\max}(f) \).

3.11. **Application.** We consider the hypersurface \( V = f^{-1}(0) \) and the transversality problem with the sphere \( S_r := \{z \mid \|z\| = r\} \). Certainly transversality has been shown by Milnor [17]. However we want to see this property from a slightly different view point. Recall first that \( S_r \) and \( V \) does not intersect transversely at \( z = a \) if and only if
(i) \( a \) and \( \partial f(a) \) are linear dependent over \( \mathbb{C} \), or
(ii) \( |(a, \partial f(a))|_{\text{norm}} = 1 \).

3.11.1. **Orthogonality at the limits (Whitney \((b)\)-regularity).**

**Lemma 24.** Assume that \( f \) is a non-degenerate and Lojasiewicz non-degenerate holomorphic function with an isolated singularity at the origin. Consider a non-constant analytic curve \( z(t) \) with \( z(0) = 0 \) defined as [2] and assume that \( f_P(a) = 0 \). Then we have

1. \( \lim_{t \to 0} (z(t), \partial f(z(t)))_{\text{norm}} = 0 \). Geometrically this implies the limit direction of \( z(t) \) is contained in the limit of the tangent space \( T_{z(0)}V \) of \( V \).
2. In particular, there exists a positive number \( r_0 \) so that the sphere \( S_r \) intersects \( V \) transversely for any \( r \leq r_0 \).

**Proof.** Let \( J = \{j \mid z_j(t) \equiv 0\} \) and let \( I \) be the complement of \( J \). We assume that \( I = \{1, \ldots, m\} \). Consider the Taylor expansion
\[
z_i(t) = a_i t^{p_i} + \text{(higher terms)}, \quad i \in I.
\]
Put \( P = (p_i)_{i \in I} \in \mathbb{N}^I_0 \) as before.

Case 1. Assume that \( f^I \neq 0 \) and assume for simplicity \( p_1 = \cdots = p_t < p_{t+1} \leq \cdots \leq p_m \) and put \( d = d(P, f^I) \). Note that \( \text{ord} \, z(t) = p_1 \). Put \( q = \text{ord} \, \partial f(z(t)) \). By the assumption, \( \lim_{t \to 0} t^{-p_1} z(t) = a_\infty \) where \( a_\infty := (a_1, \ldots, a_t, 0, \ldots, 0) \). Put \( v_\infty := \lim_{t \to 0} t^{-q} \partial f(z(t)) \). By the non-degeneracy, we have \( q \leq d - p_1 \). If \( q < d - p_1 \), the assertion (1) is immediate,
as $v_\infty \in \mathbb{C}^K$ where $K = \{ j \mid j > \ell \}$. Assume that $q = d - p_1$. This implies
\[
\frac{\partial f^I}{\partial z_j}(a) = 0, \quad j \geq \ell + 1.
\]
By the assumption $f^I_P(a) = 0$ and the Euler equality of $f^I_P(z)$, we get
\[
f^I_P(a) = \sum_{i=1}^{m} p_i a_i \frac{\partial f^I}{\partial z_i}(a) = p_1 \sum_{i=1}^{\ell} a_i \frac{\partial f^I}{\partial z_i}(a) = 0.
\]
In this case, note that $v_\infty = (\frac{\partial f^I}{\partial z_1}(a), \ldots, \frac{\partial f^I}{\partial z_\ell}(a), 0, \ldots, 0)$. This implies also
\[
\lim_{t \to 0} (z(t), \overline{f}(z(t)))_{\text{norm}} = (a_\infty, v_\infty)_{\text{norm}} = c \sum_{i=1}^{\ell} a_i \frac{\partial f^I}{\partial z_i}(a) = 0
\]
where $c$ is a non-zero scalar.

Case 2. Assume that $f^I \equiv 0$. Then $z(t) \in \mathbb{C}^t$ and $\mathbb{C}^t$ is a vanishing coordinate subspace, and thus $\partial f(z(t)) \in \mathbb{C}^t_c$. Thus the assertion is obvious. The assertion (2) follows from (1). Of course, (2) is nothing but the existence of a stable radius which is well known by [17] for a general holomorphic function with an isolated singularity at the origin. □

Remark 25. The assertion of the lemma says that the stratification $\{ V \setminus \{0\}, \{0\} \}$ is a Whitney $b$-regular stratification.

3.11.2. Making $f$ convenient without changing the topology. Let $f$ be a non-degenerate, Lojasiewicz non-degenerate function with isolated singularity at the origin. Choose integer $N_i$ with
\[
N_i > \eta''_{I,\max}(f) + 1, \quad i = 1, \ldots, n
\]
and consider a polynomial
\[
R(z) = c_1 z_1^{N_1} + \cdots + c_n z_n^{N_n}.
\]
Consider the family of functions:
\[
f_s(z) = f(z) + sR(z), \quad 0 \leq s \leq 1.
\]
The coefficients are chosen generically so that $f_s$ is non-degenerate. Note that $f_0 = f$ and $f_1$ is a convenient and non-degenerate.

Theorem 26. Consider the family of hypersurface $V_s := f_s^{-1}(0), 0 \leq s \leq 1$. There exists a positive number $r_0$ such that $V_s \cap B_{r_0}$ has a unique singular point at the origin and for any $r \leq r_0$ the sphere $S_r$ and $V_s$ intersect transversely for any $0 \leq s \leq 1$. In particular, the links of $f$ and $f_1$ are isotopic and their Milnor fibrations are isomorphic.
Proof. Take \( r_0 \) so that there exists a positive number \( c \) and the following inequality is satisfied:

\[
\|\partial f(z)\| \geq c \|z\|^{\ell_0(f)}, \quad 0 < \|z\| \leq r_0.
\]

Assume that the assertion does not hold. Then we can find an analytic curve \((z(t), s(t)), 0 \leq t \leq 1\) and Laurent series \(\lambda(t)\) such that \(z(0) = 0\) and \(s(0) = s_0 \in [0, 1]\) and

\[
(22) \quad \partial f_{s(t)}(z(t)) = \lambda(t)z(t), \quad f_{s(t)}(z(t)) \equiv 0.
\]

Expand \(z(t)\) and \(s(t)\) in Taylor expansions

\[
z_i(t) = a_it^{p_i} + \text{(higher terms)}, \quad i = 1, \ldots, n
\]

\[
s(t) = s_0 + \beta t^b + \text{(higher terms)}, \quad 0 \leq s_0 \leq 1
\]

and let \(I = \{i \mid z_i(t) \not\equiv 0\}\), \(P = (p_i) \in N_+^n\), and \(a = (a_i) \in \mathbb{C}^n\). Then by the definition of \(R(z)\),

\[
(23) \quad \text{ord } s(t)\partial R(z(t)) \geq m(P)(N - 1) > m(P)\eta''_{l,max}
\]

where \(N = \min\{N_1, \ldots, N_n\}\). By Theorem 33,

\[
(24) \quad \text{ord } \partial f(z(t)) \leq m(P)\eta''_{l,max}.
\]

Thus

\[
\text{ord } \partial f_{s_0}(z(t)) = \text{ord } \partial f(z(t))
\]

and

\[
\lim_{t \to 0} (\partial f_{s(t)}(z(t)))_{\text{norm}} = \lim_{t \to 0} (\partial f_{s(0)}(z(t)))_{\text{norm}}.
\]

This already implies the family of hypersurfaces \(V_s\) have isolated singularities at the origin. Note that the equality \(f_P(a) = 0\) follows from the equality \(f_{s(t)}(z(t)) \equiv 0\). The assumption gives us the contradicting equalities

\[
\|(z(t), \partial f_{s(t)}(z(t)))_{\text{norm}}\| \equiv 1, \quad \text{and}
\]

\[
\lim_{t \to 0} (z(t), \partial f_{s(t)}(z(t)))_{\text{norm}} = 0
\]

where the first equality follows from the assumption (22) and the second convergence follows from (24) and Lemma 24. Thus the family \(f_s, 0 \leq s \leq 1\) has a uniform stable radius and the isomorphisms of the Milnor fibrations are easily obtained using tubular Milnor fibre and Ehresman's fibre theorem (27). \qed

Remark 27. By the same argument, it is easy to see that \(f_t(z) = f(z) + t \nu\) with \(\sum_{i=1}^n \nu_i \geq \eta''_{l,max}(f) + 1\) does not change the topology at the origin. A similar result for \(C^0\)-sufficiency is proved in Kuo [13].
4. Lojasiewicz Inequality for Mixed Functions

In this section, we will introduce the notion of Lojasiewicz exponent for a mixed function and we generalize the estimation obtained for non-degenerate holomorphic functions in previous sections for \( f(z, \bar{z}) \) which are strongly non-degenerate.

4.1. Newton Boundaries and Various Gradients. Let \( f \) be a mixed function expanded as

\[
f(z, \bar{z}) = \sum_{\nu, \mu} c_{\nu \mu} z^\nu \bar{z}^\mu.
\]

The Newton diagram \( \Gamma_+(f) \) is defined as the convex hull of

\[
\bigcup_{c_{\nu \mu} \neq 0} ((\nu + \mu) + (\mathbb{R}_+)^n)
\]

and the Newton boundary \( \Gamma(f) \) is defined by the union of compact faces of \( \Gamma_+(f) \) as in the holomorphic case, using the radial weighted degree of the monomial \( z^\nu \bar{z}^\mu \). Here the radial weight degree with respect to the weight vector \( P = (p_1, \ldots, p_n) \), is defined by

\[
r_{\text{deg}}_P z^\nu \bar{z}^\mu := \sum_{i=1}^n p_i (\nu_i + \mu_i).
\]

The notion of strong non-degeneracy is introduced in [20] to study Milnor fibration defined by \( f \). Let us recall the definition. Take an arbitrary face \( \Delta \) of \( \Gamma(f) \) of any dimension. The face function is defined by \( f_\Delta(z, \bar{z}) = \sum_{\nu+\mu \in \Delta} c_{\nu \mu} z^\nu \bar{z}^\mu \). Let \( P \) be the weight vector. Then \( f_P \) is defined as \( f_\Delta(P) \) as in the holomorphic case. \( f_P(z, \bar{z}) \) is a radially weighted homogeneous polynomial with weight \( P \). A mixed function \( f \) is called strongly non-degenerate if \( f_\Delta : \mathbb{C}^n \to \mathbb{C} \) has no critical point for any face function \( f_\Delta \). It is known that such a mixed function admits a Milnor fibration at the origin ([20]).

We assume that \( \Gamma(f) \) has dimension \( n - 1 \). We consider two gradient vectors:

\[
\partial f := \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)
\]

and

\[
\bar{\partial} f := \left( \frac{\partial f}{\partial \bar{z}_1}, \ldots, \frac{\partial f}{\partial \bar{z}_n} \right).
\]

To study the behavior of the tangent spaces, it is more useful to use the real and imaginary part of \( f \). Put \( f = g + ih \) where \( g \) and \( h \) are real and imaginary parts of \( f \). Putting \( z_j = x_j + iy_j \), \( g, h \) are real analytic functions of \( 2n \) variables \( x_1, y_1, \ldots, x_n, y_n \). Substituting \( x_j = (z_j + \bar{z}_j)/2 \) and \( y_j = (z_j - \bar{z}_j)/2i \), we consider \( g, h \) as mixed functions. As they are real valued mixed functions, we have

\[
T_ag^{-1}(0) = \{ v \mid \Re(v, \partial g) = 0 \}, \quad T_a h^{-1}(0) = \{ v \mid \Re(v, \partial h) = 0 \}.
\]
Here \((\mathbf{v}, \mathbf{w})\) is the hermitian inner product. As we have \(\overline{\partial g} = \bar{\partial} g\) and \(\overline{\partial h} = \bar{\partial} h\) (\([21]\)), various gradients are related by

\[
\begin{align*}
\bar{\partial} f & = \bar{\partial} g + i\partial h, \quad \overline{\partial f} = \bar{\partial} g - i\partial h \\
\partial g & = \overline{(\partial f + \overline{\partial f})/2}, \quad \partial h = (\bar{f} - \overline{\partial f})/2i.
\end{align*}
\]

For a weight vector \(P = (p_1, \ldots, p_n)\), the real part and imaginary part \(g_P = \Re f_P, h_P = \Im f_P\) of \(f_P\) are real-valued radially weighted homogeneous polynomials with weight \(P\) and the Euler equality take the form (\([19]\)):

\[
\begin{align*}
g_P(\mathbf{z}, \bar{\mathbf{z}}) &= 2 \sum_{i=1}^{n} p_i z_i \frac{\partial g_P}{\partial z_i}, \quad h_P(\mathbf{z}, \bar{\mathbf{z}}) = 2 \sum_{i=1}^{n} p_i z_i \frac{\partial h_P}{\partial z_i}.
\end{align*}
\]

The strong non-degeneracy implies that \(\{\bar{\partial} g_P(\mathbf{a}, \bar{\mathbf{a}}), \bar{\partial} h_P(\mathbf{a}, \bar{\mathbf{a}})\}\) are linearly independent over \(\mathbb{R}\) for any \(\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n\). We consider the Lojasiewicz inequalities of real-valued mixed functions \(g\) and \(h\). For a non-zero vector \(\mathbf{w} \in \mathbb{C}^n\), we denote the real hyperplane orthogonal to \(\mathbf{w}\) by \(\mathbf{w}^\perp := \{\mathbf{v} \in \mathbb{C}^n | \Re(\mathbf{v}, \mathbf{w}) = 0\}\). We denote the normalized vector \(\mathbf{w}/\|\mathbf{w}\|\) by \(\mathbf{w}_{\text{norm}}\). The problem (which do not happen for holomorphic functions) is that along a given analytic curve \(\mathbf{z}(t), 0 \leq t \leq 1\) with \(\mathbf{z}(0) = \mathbf{0}\), the limit directions \(\lim_{t \to 0} (\bar{\partial} g(\mathbf{z}(t)))_{\text{norm}}\) and \(\lim_{t \to 0} (\bar{\partial} h(\mathbf{z}(t)))_{\text{norm}}\) can be linearly dependent over \(\mathbb{R}\). In this case, we have a proper inclusion:

\[
\lim_{t \to 0} T_{\mathbf{z}(t)} V = \lim_{t \to 0} \left((\bar{\partial} g(\mathbf{z}(t)))^\perp \cap (\bar{\partial} h(\mathbf{z}(t)))^\perp\right) \subseteq \lim_{t \to 0} \left((\bar{\partial} g(\mathbf{z}(t)))_{\text{norm}}^\perp \cap (\bar{\partial} h(\mathbf{z}(t)))_{\text{norm}}^\perp\right)
\]

The following lemma plays a key role to solve this problem. Let \(I = \{1 \leq j \leq n | z_j(t) \neq 0\}\) and and \(I^c\) be the complement of \(I\).

\[
\begin{align*}
C(t) : & \{ \mathbf{z}(t) = (z_1(t), \ldots, z_n(t)), \quad \mathbf{z}(0) = 0, \quad \mathbf{z}(t) \in \mathbb{C}^{*I}, \quad t > 0 \\
& z_i(t) = a_i t^{p_i} + \text{(higher terms)}, \quad a_i \neq 0, \quad i \in I
\end{align*}
\]

Consider the weight vector \(P = (p_i) \in N^{*I}\), \(\mathbf{a} = (a_i)_{i \in I} \in \mathbb{C}^{*I}\) and we assume that \(f|_I \neq 0\). For any real valued analytic function \(b(t)\) defined on an open neighborhood of \(t = 0\), we consider the modified gradient vectors, defined as follows.

\[
\begin{align*}
(\bar{\partial} g(\mathbf{z}(t)))_{b(t)} & := \bar{\partial} g(\mathbf{z}(t)) + b(t) \bar{\partial} h(\mathbf{z}(t)) \\
(\bar{\partial} h(\mathbf{z}(t)))_{b(t)} & := \bar{\partial} h(\mathbf{z}(t)) + b(t) \bar{\partial} g(\mathbf{z}(t)).
\end{align*}
\]

Note that any of the following three pairs generate the same real dimension 2 subspace over \(\mathbb{R}\) at \(T_{\mathbf{z}(t)} \mathbb{C}^n\).

\[
\{\bar{\partial} g(\mathbf{z}(t)), \bar{\partial} h(\mathbf{z}(t))\}, \{\bar{\partial} g(\mathbf{z}(t)), (\bar{\partial} h(\mathbf{z}(t)))_{b(t)}\}, \{(\bar{\partial} g(\mathbf{z}(t)))_{b(t)}, \bar{\partial} h(\mathbf{z}(t))\}
\]

The following is a key lemma to generalize the assertions obtained in previous sections for mixed functions. Consider the family of hypersurface \(V_t := \{\mathbf{z} \in \mathbb{C}^n | f(\mathbf{z}, \bar{\mathbf{z}}) = f(\mathbf{z}(t), \bar{\mathbf{z}}(t))\}\) for \(-\varepsilon \leq t \leq \varepsilon\) where \(V_t\) passes through \(\mathbf{z}(t)\).
Lemma 28. Assume that $\text{ord} \bar{g}(z(t)) \leq \text{ord} \bar{h}(z(t))$ for simplicity. 

(i) There exists a real valued analytic function $b(t)$ so that two analytic curves $\bar{g}(z(t))$, $(\bar{h}(z(t)))_{b(t)}$ have normalized limits 

\[ v_{g,\infty} := \lim_{t \to 0} (\bar{g}(z(t)))_{\text{norm}} \text{ and } v'_{h,\infty} := \lim_{t \to 0} ((\bar{h}(z(t)))_{b(t)})_{\text{norm}} \]

which are linearly independent over $\mathbb{R}$. The limit of the tangent space $T_{z(t)}V_t$ is equal to the intersection of the hyperplanes $v_{g,\infty}^\perp \cap (v'_{h,\infty})^\perp$.

(ii) The orders of the vectors $\bar{g}(z(t))$, $(\bar{h}(z(t)))_{b(t)}$ satisfy the inequality: 

\[ \text{ord} \bar{g}(z(t)), \text{ord} (\bar{h}(z(t)))_{b(t)} \leq d(P, f^I) - m(P). \]

(iii) If further $f_P(a) = 0$, the limit vector $\lim_{t \to 0} z(t)_{\text{norm}}$ is real orthogonal to $v_{g,\infty}$ and $v'_{h,\infty}$ i.e., $\lim_{t \to 0} z(t)_{\text{norm}} \in v_{g,\infty}^\perp \cap (v'_{h,\infty})^\perp$.

(iv) For any analytic functions $b(t), c(t)$, 

\[ \text{ord} \bar{g}(z(t))_{c(t)}, \text{ord} \bar{h}(z(t))_{b(t)} \leq d(P, f^I) - m(P). \]

Here two vectors $v, w \in \mathbb{C}^n$ are called to be real orthogonal if $\Re(v, w) = 0$.

Proof. The proof is essentially the same as the proof of Theorem 3.14, [8]. For the convenience, we repeat the proof briefly. For further related discussion, see [21, 8]. Put $I = \{ i \mid z_i(t) \neq 0 \}$ and $J = I^c$. We may assume that $I = \{1, \ldots, m\}$, $a = (a_1, \ldots, a_m)$ and 

\[ p_1 = p_2 = \cdots = p_k < p_{k+1} \leq \cdots \leq p_m. \]

Put $d = d(P, f^I)$. Under the above assumption, $\text{ord} z(t) = p_1$ and 

\[ \lim_{t \to 0} (z(t))_{\text{norm}} = a(\infty)_{\text{norm}} \quad \text{where} \quad a(\infty) := (a_1, \ldots, a_k, 0, \ldots, 0). \]

By the definition of $P$ and the assumption $f^I \neq 0$, 

\[ \frac{\partial g^I}{\partial z_i}(z(t)) = \frac{\partial g_P}{\partial z_i}(a)t^{d-p_i} + \text{(higher terms)} \]

\[ \frac{\partial h^I}{\partial z_i}(z(t)) = \frac{\partial h_P}{\partial z_i}(a)t^{d-p_i} + \text{(higher terms)} \]

Thus by the strong non-degeneracy assumption, we have the inequality: 

\[ \text{ord} \bar{g}^I(z(t)) \leq d - p_1, \quad \text{ord} \bar{h}^I(z(t)) \leq d - p_1. \]

For an analytic curve $v(t)$ with $v(0) = 0$, we associate scalar vector 

\[ \beta(v(t)) := (\beta_1, \ldots, \beta_m), \quad \text{where} \quad \beta_i = \text{Coeff}(v_i(t), t^{d-p_i}) \]

and integers 

\[ d(v(t)) := \min\{\text{ord } v_i(t) \mid i = 1, \ldots, m\}, \]

\[ \gamma_v := \max\{i \mid \text{ord } v_i(t) = d(v(t))\}. \]

Note that $\gamma_v$ is the largest index for which $\lim_{t \to 0} v(t)_{\text{norm}}$ has non-zero coefficient. We call $\gamma_v$ the leading index of $v(t)$.
We start from two analytic curves \( \partial g^I(z(t)) \) and \( \partial h^I(z(t)) \). Put \( d_g = \text{ord}(\partial g^I(z(t))) \), \( d_h = \text{ord}(\partial h^I(z(t))) \) and \( \gamma_g := \gamma_{\partial g^I(z(t))}, \gamma_h := \gamma_{\partial h^I(z(t))}. \) We assume that \( d_g \leq d_h \). First we associate \( 2 \times m \)-matrix with complex coefficients by

\[
A(\partial g^I, \partial h^I) := \begin{pmatrix}
\beta(\partial g^I(z(t))) \\
\beta(\partial h^I(z(t)))
\end{pmatrix} = \begin{pmatrix}
\frac{\partial g^I}{\partial z_1}(a) & \cdots & \frac{\partial g^I}{\partial z_m}(a) \\
\frac{\partial h^I}{\partial z_1}(a) & \cdots & \frac{\partial h^I}{\partial z_m}(a)
\end{pmatrix}
\]

By the strong non-degeneracy assumption, two raw complex vectors are linearly independent over \( \mathbb{R} \). The normalized limit, \( \lim_{t \to 0}(\partial g^I(z(t)))_{\text{norm}}, \) has non-zero \( j \)-th coefficient if and only if \( \text{ord}(\partial g^I(z(t))) = d_g \). The following three cases are possible.

1. \( \gamma_g \neq \gamma_h \).
2. \( \gamma_g = \gamma_h \) and \( \{\text{Coeff}(\frac{\partial g^I}{\partial z_1}(z(t)), t^{d_g}), \text{Coeff}(\frac{\partial h^I}{\partial z_1}(z(t)), t^{d_h})\} \) are linearly independent over \( \mathbb{R} \).
3. \( \gamma_g = \gamma_h \) and \( \{\text{Coeff}(\frac{\partial g^I}{\partial z_1}(z(t)), t^{d_g}), \text{Coeff}(\frac{\partial h^I}{\partial z_1}(z(t)), t^{d_h})\} \) are linearly dependent over \( \mathbb{R} \).

In the cases of (1), (2), their normalized limits are linearly independent over \( \mathbb{R} \) and there is no operation necessary. In the case of (3), we put \( b_1(t) = \rho_1 t^{d_h - d_g} \) and put \( (\partial h)'(t) := \partial h^I(z(t)) - b_1(t) \partial g^I(z(t)) \). Here \( \rho_1 \) is the unique real number such that

\[
\rho_1 \text{Coeff}(\frac{\partial g^I}{\partial z_1}(z(t)), t^{d_g}) - \text{Coeff}(\frac{\partial h^I}{\partial z_1}(z(t)), t^{d_h}) = 0.
\]

After this operation, we have three possible cases.

1. \( \gamma(\partial h)'(t) \neq \gamma_{\partial g^I(z(t))} \)
2. \( \gamma(\partial h)'(t) = \gamma_{\partial g^I(z(t))} \) but the leading coefficients of \( (\partial h)'(t) \) and \( \partial g^I(z(t)) \) are linearly independent over \( \mathbb{R} \).
3. \( \gamma(\partial h)'(t) = \gamma_{\partial g^I(z(t))} \) and the leading coefficients of \( (\partial h)'(t) \) and \( \partial g^I(z(t)) \) are still linearly dependent over \( \mathbb{R} \).

In the case of (1)’ and (2)’, we stop the operation. Otherwise we have (3)’ and we continue this operation till we get a modified gradient vector

\[
(\partial h)^{(j)}(t) = \partial h^I(z(t)) - \rho(t) \partial g^I(z(t)), \quad \rho(t) := \sum_{i=1}^j b_i(t)
\]

for which either its leading index is different from \( \gamma_g \) (case (1)) or the coefficients of the leading index are linearly independent over \( \mathbb{R} \) (case (2)). Note that in this operation, the order of \( (\partial h)^{(j)}(t) \) is strictly increasing in \( j \) while the matrix \( A(\partial g, (\partial h)^{(j)}(t)) \) is simply changed in the second raw vector by \( \beta((\partial h)^{(j)} - \rho(0)\beta((\partial g)^{(j)})). \) Therefore \( (\partial h)^{(j)}(t) := \partial h^I(z(t)) - \rho(t) \partial g^I(z(t)) \) satisfies

\[
\text{ord}(\partial h)^{(j)}(t) \leq d - p_1
\]
and therefore the operation should stop after finite steps, say $k$. After the operation is finished, the normalized vector $((\partial h)^{(k)}(t))_{\text{norm}}$ has linearly independent limit with that of $\partial g^I(z(t))$.

Suppose further $f^I_p(a) = 0$. This implies $g^I_p(a) = h^I_p(a) = 0$. We will show now $a_\infty = (a_1, \ldots, a_k, 0, \ldots, 0)$ is orthogonal to the limits of the normalized vectors

$$v^g_\infty := \lim_{t \to 0} (\partial g^I(z(t)))_{\text{norm}} \quad \text{and} \quad v^{(\partial h)^{(k)}}_\infty := \lim_{t \to 0} ((\partial h)^{(k)}(t))_{\text{norm}}.$$  

First we consider $v^g_\infty$. If $d_g < d - p_1$, $j$-coefficient of $v^g_\infty$ is zero for $j \leq k$ and $\Re(v^g_\infty, a_\infty) = 0$ is obvious. If $d_g = d - p_1$, we must have

$$\frac{\partial g^I_p(a)}{\partial z_j} = 0, \quad k + 1 \leq j \leq m$$

and the $i$-th coefficient of $v^g_\infty$ is $\frac{\partial g^I_p(a)}{\partial z_i}$ for $1 \leq i \leq k$ up to a scalar multiplication. Thus the assertion follows from the Euler equality

$$g^I_p(a) = 0 = \sum_{i=1}^m p_i a_i \frac{\partial g^I_p(a)}{\partial z_i} = p_1 \sum_{i=1}^k a_i \frac{\partial g^I_p(a)}{\partial z_i}.$$ 

Now we consider $v^{(\partial h)^{(k)}}_\infty$. We start from the equality $h^I_p(a) - \rho(0)g^I_p(a) = 0$.

If $d_{(\partial h)^{(k)}(t)} < d - p_1$, $v^{(\partial h)^{(k)}(t)}$ and $a_\infty$ are orthogonal by the same reason. Suppose that $d_{(\partial h)^{(k)}(t)} = d - p_1$. Then we must have

$$\frac{\partial h^I_p(a)}{\partial z_j} - \rho(0)\frac{\partial g^I_p(a)}{\partial z_j} = 0, \quad k + 1 \leq j \leq m.$$ 

Thus the assertion follows from the Euler equality of the real valued radially weighted homogeneous polynomial $h^I_p(z, \bar{z}) - \rho(0)g^I_p(z, \bar{z})$. The assertion (iv) can be shown in a similar way looking at the matrix $A$ before and after.

\textbf{Definition 29.} Let $z(t)$ be an analytic curve starting at the origin. Assume that $\partial g(z(t)) \leq \text{ord} \partial h(z(t))$ and $\{\partial g(z(t)), \partial h(z(t))_{c(t)}\}$ (respectively $\text{ord} \partial g(z(t)) > \text{ord} \partial h(z(t))$ and $\{\partial g(z(t))_{c(t)}, \partial h(z(t))\}$) is a good modified gradient pair if they have linearly independent normalized limits over $\Re$.

Take an arbitrary analytic curve $C(t) : z = z(t)$ with $z(0) = 0$ and a good modified gradient pair, say $\{\partial g(z(t)), \partial h(z(t))_{c(t)}\}$ assuming $\partial g(z(t)) \leq \text{ord} \partial h(z(t))$ for simplicity and suppose that the following inequality is satisfied for sufficiently small $t$, $0 \leq t \leq \varepsilon$,

\begin{align*}
(32) \quad \text{ord}(\partial g(z(t), \bar{z}(t)))/\text{ord} z(t) & \leq \theta \\
(33) \quad \text{ord}(\partial h(z(t), \bar{z}(t))_{c(t)}/\text{ord} z(t) & \leq \theta. 
\end{align*}

(If $\partial g(z(t)) > \text{ord} \partial h(z(t))$, we exchange $g$ and $h$ in the above inequalities so that $\partial g$ is to be modified.) We define the Lojasiewicz exponent $\varepsilon_0(C(t))$ along an analytic curve $C(t)$ as the infimum of such $\theta$ satisfying the above inequality. The Lojasiewicz exponent of $f$ is defined by the supremum of
\( \ell_0(C(t)) \) for all analytic curves \( C(t) \).

These inequalities are equivalent to the inequality
\[
\| \partial g(z(t), \bar{z}(t)) \|, \| \partial h(z(t), \bar{z}(t)) \|_c(t) \| \geq C \| z(t) \|^\theta, \exists C > 0
\]
for \( 0 \leq t \leq \varepsilon \).

Taking \( c(t) \equiv 0 \), such \( \theta \) satisfies the usual Łojasiewicz inequalities:
\[
\begin{align*}
\| \partial g(z, \bar{z}) \| &\geq C \| z \|^\theta \\
\| \partial h(z, \bar{z}) \| &\geq C \| z \|^\theta, \exists C > 0.
\end{align*}
\]
in a sufficiently small neighbourhood of the origin.

Now we are ready to generalize the results which are obtained in previous sections for holomorphic functions.

4.2. Convenient case. We consider a strongly non-degenerate mixed function \( f(z, \bar{z}) \) with an isolated singularity at the origin. A mixed function \( f(z, \bar{z}) \) is called convenient if for each \( i = 1, \ldots, n \), there is a point \( B_i = (0, \ldots, \bar{b}_i, \ldots, 0) \) on the Newton boundary \( \Gamma(f) \). In the mixed function case, there might exist several corresponding mixed monomial \( z_i^{\nu_i} \bar{z}_i^{\mu_i} \) with \( \nu_i + \mu_i = b_i \) in the expansion of \( f(z, \bar{z}) \). Such a monomial is called an i-axis monomial. Let \( B := \max\{b_i | i = 1, \ldots, n\} \). An i axis monomial \( z_i^{\nu_i} \bar{z}_i^{\mu_i} \) is called Łojasiewicz monomial if \( \nu_i + \mu_i = B \). A Łojasiewicz monomial \( z_i^{\nu_i} \bar{z}_i^{\mu_i} \) is exceptional if there exists a monomial \( z_i^{\nu_i'} \bar{z}_i^{\mu_i'} w_j \) where \( w_j = z_j \) or \( \bar{z}_j \) in the expansion of \( f(z, \bar{z}) \) such that \( \nu_i' + \mu_i' < B - 1 \).

**Theorem 30.** Let \( f(z, \bar{z}) \) be a strongly nondegenerate convenient mixed function. Then Łojasiewicz exponent \( \ell_0(f) \) satisfies the inequality: \( \ell_0(f) \leq B - 1 \). Furthermore if \( f \) has a Łojasiewicz non-exceptional monomial, we have the equality \( \ell_0(f) = B - 1 \).

Using Lemma 28, the proof is completely parallel to that of Theorem 3.

4.3. Non-convenient mixed polynomials. We consider the case of non-degenerate mixed polynomials with an isolated singularity at the origin. One point is how to define "Łojasiewicz non-degeneracy" for mixed functions.

Let \( \mathbb{C}^I \) be a vanishing coordinate subspace and let \( J \) be the complement of \( I \). For each \( i \), there must exist a mixed monomial \( z_i^{n_{ij}} \bar{z}_i^{m_{ij}} w_j \) with \( j \in J \) as we have assumed that the origin is an isolated singularity. Hereafter we use variable \( w_j \) for either \( w_j = z_j \) or \( w_j = \bar{z}_j \) for simplicity. We take \( \ell_{ij} \) to be the minimum of \( \{n_{ij} + m_{ij}\} \) for fixed \( i, j \). The point \( B_{ij} := (0, \ldots, \ell_{ij}, \ldots, 1 \ldots, 0) \) is a point of \( \Gamma(f) \). We call a monomial \( z_i^{n_{ij}} \bar{z}_i^{m_{ij}} w_j \) an almost i-axis monomial if \( n_{ij} + m_{ij} = \ell_{ij} \). Let \( J_i \) be the set of \( j \) for which such almost i-axis monomial exists and put \( J(I) = \cup_{i \in I} J_i \) as in Part I.

Define \( \ell(I) := \max\{\ell_{ij} | i \in I, j \in J_i\} \). For our present purpose, we take the following definition of Łojasiewicz non-degeneracy. Consider a strictly
positive weight vector \( Q \in N^*l \). Consider \( F(j) := z_j(\partial F(j)/\partial z_j)^t + \bar{z}_j(\partial F(j)/\partial \bar{z}_j)^t \).

Note that \( \partial F(j)/\partial z_j \) and \( \partial F(j)/\partial \bar{z}_j \) are polynomials in \( \mathbb{C}[z_1, \bar{z}_1, \ldots, z_m, \bar{z}_m] \). Here \( F(j)_Q := (F(j))_Q \) with the weight of \( z_j \) being 0 by definition. We use the notations \( f_j := \partial f/\partial z_j \) and \( \bar{f}_j := \partial f/\partial \bar{z}_j \) for simplicity. Note that
\[
\frac{\partial F(j)_Q}{\partial z_j} = \begin{cases} (f_j)_Q, & \text{rdeg}_Q f_j \leq \text{rdeg}_Q \bar{f}_j \\ 0, & \text{rdeg}_Q f_j > \text{rdeg}_Q \bar{f}_j \end{cases}
\]
\[
\frac{\partial F(j)_Q}{\partial \bar{z}_j} = \begin{cases} (f_j)_Q, & \text{rdeg}_Q \bar{f}_j \geq \text{rdeg}_Q f_j \\ 0, & \text{rdeg}_Q \bar{f}_j < \text{rdeg}_Q f_j \end{cases}
\]

Put \( I' := \{ i \mid q_i = m(Q) \} \) and \( J(Q) := \cup_{i \in I'} J_i \). \( f(z, \bar{z}) \) is called Lojasiewicz non-degenerate if under any such situation and for any \( a \in \mathbb{C}^*l \), there exists \( i_0 \in I', j_0 \in J(Q) \) such that
\[
\left| \frac{\partial F(j_0)_Q}{\partial z_{j_0}}(a) \right| \neq \left| \frac{\partial F(j_0)_Q}{\partial \bar{z}_{j_0}}(a) \right|
\]
Writing \( F(j) := g(j) + ih(j) \), this is equivalent to

**Proposition 31.** Under the above notations,
\[
\left\{ \frac{\partial g(j)_Q}{\partial z_{j_0}}(a), \frac{\partial h(j)_Q}{\partial z_{j_0}}(a) \right\}
\]
are linearly independent over \( \mathbb{R} \).

The assertion follows from Proposition 1 [19]. Assume that such a \( Q \) is associated with an analytic family \( z(t) \) and \( d = \text{rdeg}_Q F(j) \). Then \( d \leq \ell_{i_0, j_0} q_{i_0} \) and
\[
\frac{\partial g}{\partial z_j}(z(t)) = \frac{\partial g(j)_Q}{\partial z_j}(a) t^d + \text{(higher terms)}, \quad \text{ord} \frac{\partial g(j)_Q}{\partial z_j}(z(t)) = d
\]
\[
\frac{\partial h}{\partial z_j}(z(t)) = \frac{\partial h(j)_Q}{\partial z_j}(a) t^d + \text{(higher terms)}, \quad \text{ord} \frac{\partial h(j)_Q}{\partial z_j}(z(t)) = d.
\]

**Proposition 32.** Using Proposition 31, there exists a good modified gradient pair \( \{ \partial g(z(t)), \partial h(z(t)) \}_{c(t)} \) or \( \{ \partial g(z(t))_{c(t)}, \partial h(z(t)) \} \). Their order in \( t \) has an upper bound \( d = \text{rdeg}_Q F(j_0) \leq \ell_{i_0, j_0} q_{i_0} \).

### 4.4. Jacobian dual Newton diagram

We consider the derivatives \( f_i(z) := \partial f/\partial z_i \) and \( \bar{f}_i(z) := \partial f/\partial \bar{z}_i \), \( i = 1, \ldots, n \). Put \( F_i(z, \bar{z}) = f_i(z, \bar{z}) \bar{f}_i(z, \bar{z}) \). If one of the derivatives vanishes identically, we consider only non-zero derivatives. For example, if \( \bar{f}_i \equiv 0 \), we put \( \bar{F}_i = f_i \). We consider their Newton boundary \( \Gamma(F_i), i = 1, \ldots, n \). Two weight vectors \( P, Q \) are Jacobian equivalent if \( \Delta(P, F_i) = \Delta(Q, F_i) \) for any \( i = 1, \ldots, n \) and \( \Delta(P, f) = \Delta(Q, f) \). We denote it by \( P \sim Q \). This gives a polyhedral cone subdivision of \( N_+ \) and we denote this as \( \Gamma^*_j(f) \) and we call it the Jacobian dual Newton diagram of \( f \). \( \Gamma^*_j(f) \) is a polyhedral cone subdivision of \( N_+ \) which is finer than \( \Gamma^*(f) \).
Alternatively we can consider the function \( F(z) = f(z)F_1(z) \cdots F_n(z) \). Then \( \Gamma^*_j(f) \) is nothing but the dual Newton diagram \( \Gamma^*(F) \) of \( F \). For any weight vector \( P \), we have \( \Delta(P, F) = \Delta(P, f) + \Delta(P, F_1) + \cdots + \Delta(P, F_n) \) where the sum is Minkowski sum. For a weight vector \( P \), the set of equivalent weight vectors in \( \Gamma^*(f) \) and \( \Gamma^*_j(f) \) is denoted as \([P]\) and \([P]_j\) respectively. We consider the vertices of these dual Newton diagrams. We denote the set of strictly positive vertices of \( \Gamma^*(f) \) and \( \Gamma^*_j(f) \) by \( \mathcal{V}^+ \), \( \mathcal{V}^+_j \) as before. Now we can generalize Theorem 14. Let \( \mathcal{V}^+_j \subset \mathcal{V}^+_j \) be the set of the vertices of \( \Gamma^*_j(f) \) which are in a vanishing boundary region of \( \Gamma^*(f) \) as in the holomorphic case. The numbers of \( \mathcal{V}^+, \mathcal{V}^+_j, \mathcal{V}^+_j \) are finite. We define basic invariants, as before

\[
\eta_{I,\text{max}}(f) := \max\{\eta(P) \mid P \in \mathcal{V}^+ \cup \mathcal{V}^+_j \},
\eta_{\text{max}} := \max\{\eta(P) \mid P \in \mathcal{V}^+ \}
\eta'_{I,\text{max}} := \max\{\eta'_{k,i}(R) \mid R \in \mathcal{V}^+_j, k, i = 1, \ldots, n\}
\eta''_{I,\text{max}}(f) := \max\{\eta_{I,\text{max}}(f), \eta'_{I,\text{max}}(f)\},
\eta'_{k,i}(R) := \min\{d(R, f_i), d(R, f_i)\}/m(R).
\]

**Theorem 33.** Let \( f(z) \) be a non-degenerate, Lojasiewicz non-degenerate mixed function with an isolated singularity at the origin. Then Lojasiewicz exponent \( \ell_0(f) \) satisfies the estimation \( \ell_0(f) \leq \eta''_{I,\text{max}}(f) \).

**Proof.** The proof is completely parallel to that of Theorem 12. We consider an analytic curve \( C(t) \) parametrized as \( z(t) = (z_1(t), \ldots, z_n(t)) \) and put \( I := \{i \mid z_i(t) \neq 0\} \). Consider the Taylor expansion of \( z_i(t) \) as before:

\[
(38) \quad \begin{cases} 
 z(t) = (z_1(t), \ldots, z_n(t)), \ z(0) = 0, \ z(t) \in \mathbb{C}^* \I \n z_i(t) = a_i t^p_i + \text{(higher terms)}, \ i \in I 
\end{cases}
\]

We put \( P = (p_i) \in N_+^* \) as before. Assume first \( I = \{1, \ldots, n\} \). We divide the situation into three cases as before.

C-1 \( [P] \) is an inner region. That is, \( \overline{[P]} \) has only strictly positive weight vectors in the boundary.

C-2 \( [P] \) is a regular boundary region.

C-3 \( [P] \) is a vanishing boundary region. In this case, we need to consider the subdivision by \( [P]_j \). There are three subcases.

C-3-1. \( [P]_j \) is an inner region.

C-3-2. \( [P]_j \) is a regular boundary region.

C-3-3. \( [P]_j \) is also a vanishing boundary region.

Then the proof goes exactly as that of Theorem 12 using Lemma 28. For the cases C-1, C-3-1, C-2, C-3-2, we start from a given good modified gradient pair and the estimation of these gradient by Lemma 28. Then the argument is completely the same. We have the estimation \( \ell_0(C(t)) \leq \eta''_{I,\text{max}}(f) \) in these cases.
For the case C-3-3, consider the situation that \( P \) is not a simplicially positive and \( R, Q \) as in Lemma 15 so that \( P \) is on the line segment \( \overline{RQ} \), \( R \) is simplicially positive and \( Q \) is not strictly positive. If \( Q \) is non-vanishing, it reduced to Case 3-2. Thus we assume that \( Q \) is a vanishing weight vector and \( d(Q, f) > 0 \). Assume that \( Q = (q_1, \ldots, q_n) \) and \( I = \{i \mid q_i = 0\} \) and assume \( I = \{1, \ldots, m\} \) for simplicity. Note that \( \mathbb{C}^I \) is a vanishing coordinate subspace. For each \( i \in I \), there exists some \( j \notin I \) and a monomial \( z_i^{n_{i,j}} z_j \) with a non-zero coefficient, as \( f \) has an isolated singularity at the origin. Put \( J_i \) be the set of such \( j \) for a fixed \( i \in I \) and put \( J(I) = \cup_{i \in I} J_i \). Here \( n_{i,j} \) is assumed to be the smallest when \( j \) is fixed. Put \( \xi_I := \max\{n_{i,j} \mid i \in I, j \in J_i\} \) and \( \xi(f) \) be the maximum of \( \xi_I \) where \( I \) moves in the coordinate subspaces corresponding to vanishing coordinate subspaces. Put \( \eta_{I,\text{max}}(f) := \max\{\eta_{k,i}(R) \mid R \in \mathcal{V}^+\} \) where \( \eta_{j,i}(R) = d(R, f_j)/r_j \). Under the above situation, we will prove, as in the holomorphic case, that

\[
\ell_0(C(t)) \leq \max\{\xi(f), \eta_{I,\text{max}}(f), \eta_{I,\text{max}}\}.
\]

Consider the normalized weight vector \( \hat{R}_s := (1-s)\hat{R} + s\hat{Q}, 0 \leq s \leq 1 \). Note that \( \hat{R}_0 = \hat{R}, \hat{R}_1 = \hat{Q} \) and putting \( \hat{R}_s = (\hat{r}_s, 1, \ldots, \hat{r}_s) \),

\[
\hat{r}_{s,i} = \begin{cases} (1-s)\hat{r}_i, & 1 \leq i \leq m \\ (1-s)\hat{r}_i + s\delta_i, & m < i \leq n. \end{cases}
\]

Put \( I' = \{i \in I \mid \hat{r}_i = m(\hat{R}_I)\} \) and \( J' = \cup_{i \in I'} J_i \). Thus for \( i \leq m \), the normalized weight \( \hat{r}_{s,j} \) goes to 0, when \( s \) approaches to 1. On the other hand, for \( j > m \), \( \hat{r}_{s,j} \geq \delta, 0 \leq \forall s \leq 1 \) for some \( \delta > 0 \). Thus there exists an \( \varepsilon, 1 > \varepsilon > 0 \) so that for \( 1 - \varepsilon \leq s \leq 1 \), \( m(\hat{R}_s) \) is taken by \( i \in I' \). Note that for \( j \in J' \), there exists a small enough \( \varepsilon_2, \varepsilon_2 \leq \varepsilon_1 \) so that \( (f_j)_{\hat{R}_s} = \left((f_j)^I\right)_{\hat{R}_s} \) as \( \hat{r}_{s,j} \geq \delta \) for \( j > m \) and \( 1 - \varepsilon_2 \leq s \leq 1 \). Here \( (f_j)^I \) is the restriction of \( f_j \) to \( \mathbb{C}^I \) and \( (\hat{R}_s)^I \) is the I projection of \( \hat{R}_s \) to \( N^+_I \). That is, \( (f_j)_{\hat{R}_s} \) contains only variable \( z_1, \ldots, z_m \). By the Lojasiewicz non-degeneracy, there exists \( i_0 \in I' \) and \( j_0 \in J_{i_0} \) such that

\[
(39) \quad \left\{ \frac{\partial g(j_0)Q}{\partial z^{j_0}}(a), \frac{\partial h(j_0)Q}{\partial z^{j_0}}(a) \right\}
\]

are linearly independent over \( \mathbb{R} \). Here we use the same notation as in [39]. By the definition of Jacobian dual Newton diagram and Proposition 39, there is a good modified gradient pair, say \( \bar{\partial}g(z(t)), \bar{\partial}h(z(t)) \) (we assume \( \text{ord } \bar{\partial}g(z(t)) \leq \text{ord } \bar{\partial}h(z(t)) \) for simplicity) so that their orders are estimated from above by \( \ell_{i_0,j_0} q_{i_0} \).

The rest of the argument is simply the evaluation of the number \( \ell_{i_0,j_0} q_{i_0} / m(\hat{R}_{s_0}) \) and the proof is completed by the exact same argument as in the proof of Theorem 32 Case 3-3-3 using the following. The case \( C(t) \in \mathbb{C}^{*I} \) with \( I^c \neq \emptyset \) is treated also by the exactly same argument. \( \square \)
Lemma 34. (Restatement of Lemma 17) We have the inequality: \( \ell(I) \leq \eta_{\text{max}}(f) \).

Theorem 19 for non-degenerate, Lojasiewicz non-degenerate (radially) weighted homogeneous polynomial and Lojasiewicz Join Theorem 22 also hold in the exactly same way for mixed functions. For example, we can state

Theorem 35. Let \( f(z) \) be a strongly non-degenerate, Lojasiewicz non-degenerate mixed weighted homogeneous polynomial with isolated singularity at the origin and \( \dim \Gamma(f) = n - 1 \). Let \( R \) be the weight vector of \( f \). Then we have the estimation \( \ell_0(f) \leq \eta(R) \).

4.5. Making \( f \) convenient. We also generalize Theorem 21. Take an integer \( N > \eta_{J,\text{max}}(f) + 1 \). Consider mixed polynomial \( R(z, \bar{z}) := \sum_{i=1}^{n} c_i z_1^{m_i} \bar{z}_i^{n_i} \) where \( n_i, m_i \) are any fixed non-negative integers with \( m_i + n_i = N_i \geq N \) and \( n_i \neq m_i \). We choose such \( \{(m_i, n_i) | i = 1, \ldots, n\} \) and fix them. The coefficients \( c_1, \ldots, c_n \) are generic so that \( f_1 := f(z, \bar{z}) + R(z, \bar{z}) \) is strongly non-degenerate. Consider the family \( f_s(z, \bar{z}) = f(z, \bar{z}) + sR(z, \bar{z}) \). Then we have the following.

Theorem 36. There exists a \( r_0 > 0 \) such that for any \( r \leq r_0 \), the sphere \( S_r \) and the family of hypersurface \( V_s := f_s^{-1}(0) \) intersect transversely for any \( 0 \leq s \leq 1 \). In particular, the links of \( f \) and \( f_1 \) are isotopic and their Milnor fibrations are isomorphic.

The proof is similar and we leave it to the reader.

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