Research Article

Unrestricted Cesàro summability of \(d\)-dimensional Fourier series and Lebesgue points

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ABSTRACT. We generalize the classical Lebesgue’s theorem to multi-dimensional functions. We prove that the Cesàro means of the Fourier series of the multi-dimensional function \(f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)\) \((1 < p \leq \infty)\) converge to \(f\) at each strong Lebesgue point.

Keywords: Cesàro summability, strong Hardy-Littlewood maximal function, strong Lebesgue points.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

In 1904, Fejér [3] investigated the arithmetic means of the partial sums of the trigonometric Fourier series of a one-dimensional function \(f\), the so called Fejér means and proved that if the left and right limits \(f(x - 0)\) and \(f(x + 0)\) exist at a point \(x\), then the Fejér means

\[
\sigma_n f(x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) \hat{f}(k)e^{ikx}
\]

converge to \((f(x - 0) + f(x + 0))/2\). Here, \(\hat{f}(k)\) denotes the \(k\)-th Fourier coefficient. One year later, Lebesgue [11] extended this theorem and obtained that every one-dimensional integrable function is Fejér summable at each Lebesgue point, thus almost everywhere. Some years later, M. Riesz [15] generalized this theorem for the Cesàro means of one-dimensional integrable functions (the definition can be found later).

The Cesàro summability is investigated in a great number of papers (see e.g. Gát [4, 5, 6], Goginava [7, 8, 9], Simon [17, 18], Nagy, Persson, Tephnadze and Wall [13, 14] and Weisz [19, 20]). In this short note, we generalize the result of Lebesgue and Riesz to this summability of multi-dimensional functions. We generalize the Lebesgue points and introduce the so called strong Lebesgue points. It is known that almost every point is a strong Lebesgue point of \(f \in L_1(\log L)^{d-1}(\mathbb{T}^d)\). We introduce the strong Hardy-Littlewood maximal function \(M_s f\) and show that the Cesàro means of \(f \in L_1(\log L)^{d-1}(\mathbb{T}^d)\) can be estimated by \(M_s f\) pointwise. Our main result is the following. If \(M_s f(x)\) is finite and \(x\) is a strong Lebesgue point of \(f \in L_1(\log L)^{d-1}(\mathbb{T}^d)\), then

\[
\lim_{n \to \infty} \sigma_n^\alpha f(x) = f(x),
\]

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where $\sigma_n^f$ denotes the $n$-th Cesàro means of the Fourier series of $f$. This implies the convergence of the Cesàro means almost everywhere as well as covers the one-dimensional results mentioned above. Note that $L_1((\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ with $1 < p \leq \infty$. The results are not true for $L_1(\mathbb{T}^d)$ if $d > 1$. Similar theorems are known for the $\theta$-means generated by a single function $\theta$ (see Feichtinger and Weisz [2] and the references therein). However, those results and proofs do not contain the results for Cesàro means. For the multi-dimensional Cesàro means, we need new ideas.

2. STRONG MAXIMAL FUNCTION AND STRONG LEBESGUE POINTS

Let us fix $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself $d$ times. We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm

$$\|f\|_p := \left( \int_{\mathbb{T}^d} |f|^p \, d\lambda \right)^{1/p} \quad (1 \leq p < \infty),$$

with the usual modification for $p = \infty$, where $\lambda$ is the Lebesgue measure. We identify the torus $\mathbb{T}$ with $[-\pi, \pi]$. Set $\log^+ u = \max(0, \log u)$. For $k \in \mathbb{N}$ and $1 \leq p < \infty$, a measurable function $f$ is in the set $L_p(\log L)^k(\mathbb{T}^d)$ if

$$\|f\|_{L_p(\log L)^k} := \left( \int_{\mathbb{T}^d} |f|^p (\log^+ |f|)^k \, d\lambda \right)^{1/p} < \infty.$$

For $k = 0$, we get back the $L_p(\mathbb{T}^d)$ spaces. We have for all $k \in \mathbb{N}$ and $1 \leq p < r \leq \infty$

$$L_p(\mathbb{T}^d) \supset L_p(\log L)^{k-1}(\mathbb{T}^d) \supset L_p(\log L)^k(\mathbb{T}^d) \supset L_r(\mathbb{T}^d).$$

For $f \in L_1(\mathbb{T}^d)$, the strong Hardy-Littlewood maximal function is defined by

$$M_s f(x) := \sup_{h \in \mathbb{R}_+^d} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)| \, dt.$$

For $d > 1$, it is known that there is a function $f \in L_1(\mathbb{T}^d)$ such that $M_s f = \infty$ almost everywhere (see Jessen, Marcinkiewicz and Zygmund [10] and Saks [16]). Thus, in contrary to the one-dimensional case, $M_s$ cannot be of weak type $(1,1)$ if $d > 1$. However, we know the following weak type inequality. If $f \in L((\log L)^{d-1}(\mathbb{T}^d)$, then

$$(2.1) \quad \sup_{\rho>0} \rho \lambda(M_s f > \rho) \leq C + C \left\| f \left( \log^+ |f| \right)^{d-1} \right\|_1.$$

Moreover, for $1 < p \leq \infty$, we have

$$(2.2) \quad \|M_s f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbb{T}^d)).$$

In this paper, the constants $C$ and $C_p$ may vary from line to line. If $f \in L_1((\log L)^{d-1}(\mathbb{T}^d)$, then

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-t) \, dt = f(x)$$

for almost every $x \in \mathbb{T}^d$. Here $h \to 0$ means that $h_j \to 0$ for all $j = 1, \ldots, d$. Note that this result does not hold for all $f \in L_1(\mathbb{T}^d)$ if $d > 1$ (see Jessen, Marcinkiewicz and Zygmund [10] and Saks [16]).

Motivated by this convergence result, a point $x \in \mathbb{T}^d$ is called a strong Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)| \, dt = 0.$$
Theorem 2.1. Almost every point \( x \in \mathbb{T}^d \) is a strong Lebesgue point of \( f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \).

This is not true for \( f \in L_1(\mathbb{T}^d) \) if \( d > 1 \). Note that \( L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d) \) for all \( 1 < p \leq \infty \). For the results of this section, see Chang and Fefferman [1], Zygmund [21] or Weisz [19, 20].

3. RECTANGULAR CESÁRO SUMMABILITY

For \( \alpha \neq -1, -2, \ldots \) and \( n \in \mathbb{N} \), let

\[
A_n^\alpha := \left( \frac{n + \alpha}{n} \right) = \frac{(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)}{n!}.
\]

Then \( A_0^\alpha = 1 \), \( A_n^0 = 1 \) and \( A_n^1 = n + 1 \) \((n \in \mathbb{N})\). The \( k \)-th Fourier coefficient of a \( d \)-dimensional integrable function \( f \in L_1(\mathbb{T}^d) \) is defined by

\[
\hat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ik \cdot x} \, dx \quad (k \in \mathbb{Z}^d),
\]

where \( u \cdot x := \sum_{k=1}^d u_k x_k \) for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( u = (u_1, \ldots, u_d) \in \mathbb{R}^d \). Since the Fourier series of \( f \) has bad convergence properties (see e.g. Weisz [20]), we consider the Cesàro summability.

Let \( f \in L_1(\mathbb{T}^d) \), \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \) and \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_+^d \). The \( n \)-th rectangular Cesàro means \( \sigma_n^\alpha f \) of the Fourier series of \( f \) and the Cesàro kernel \( K_n^\alpha \) are introduced by

\[
\sigma_n^\alpha f(x) := \frac{1}{\prod_{i=1}^d A_{n_i}^{\alpha_i}} \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha \hat{f}(k) e^{ik \cdot x}
\]

and

\[
K_n^\alpha(t) := \frac{1}{\prod_{i=1}^d A_{n_i}^{\alpha_i}} \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha e^{ik \cdot t},
\]

respectively. It is easy to see that

\[
\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x - t) K_n^\alpha(t) \, dt
\]

and

\[
K_n^\alpha = K_{n_1}^{\alpha_1} \otimes \cdots \otimes K_{n_d}^{\alpha_d},
\]

where the functions \( K_{n_i}^{\alpha_i} \) are the one-dimensional Cesàro kernels. The Cesàro means are also called \((C, \alpha)\)-means. If all \( \alpha_i = 1 \), then we get back the rectangular Fejér means. For the one-dimensional Cesàro kernels, it is known (see Zygmund [21]) that

\[
(3.3) \quad K_n^\alpha(t) \leq C \min \left( n, \frac{1}{n^\alpha |t|^{|\alpha|+1}} \right)
\]

and

\[
\sup_{n \in \mathbb{N}} \int_{\mathbb{T}} |K_n^\alpha| \, d\lambda \leq C,
\]

where \( n \in \mathbb{N} \), \( 0 < \alpha \leq 1 \) and \( t \in (-\pi, \pi) \).
4. UNRESTRICTED CONVERGENCE AT LEBESGUE POINTS

Before proving the main results of this paper, we introduce the Herz space $E_\infty(\mathbb{R}^d)$ with the norm
\[
\|f\|_{E_\infty} := \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} 2^{k_1+\cdots+k_d} \|f 1_{P_k}\|_\infty < \infty,
\]
where
\[P_k := P_{k_1} \times \cdots \times P_{k_d} \quad (k \in \mathbb{Z}^d)\]
and
\[P_i = \{ x \in \mathbb{R} : 2^{i-1}\pi \leq |x| < 2^i\pi \} \quad (i \in \mathbb{Z}).\]
Obviously, $L_1(\mathbb{R}^d) \supset E_\infty(\mathbb{R}^d)$. First, we will estimate pointwise the maximal operator
\[
\sigma_n^\alpha f := \sup_{n \in \mathbb{N}^d} |\sigma_n^\alpha f|
\]
by the strong Hardy-Littlewood maximal function. To this end, we introduce the functions
\[h^\alpha_j(t) := \min\{1, |t|^{-\alpha_j-1}\} \quad (t \in \mathbb{R})\]
and
\[h^\alpha := h^\alpha_1 \otimes \cdots \otimes h^\alpha_d.\]
We get from (3.3) that
\[
(4.4) \quad \frac{1}{n_j} \left| \left(1_{(-\pi,\pi)} K_{n_j}^\alpha \right) \left(\frac{t}{n_j}\right) \right| \leq C \min \left\{ n_j, \frac{n_j}{|t|^{\alpha_j+1}} \right\} = C h^\alpha_j(t) \quad (t \in \mathbb{R}).
\]
It is easy to see that
\[
(4.5) \quad \|h^\alpha\|_{E_\infty(\mathbb{R}^d)} = \prod_{j=1}^{d} \|h^\alpha_j\|_{E_\infty(\mathbb{R})} \leq C_\alpha.
\]

Theorem 4.2. Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$, then
\[
\sigma_n^\alpha f(x) \leq CM_\alpha f(x).
\]
Proof. Observe that
\[
|\sigma_n^\alpha f(x)| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} f(x-t) \left(1_{(-\pi,\pi)} K_{n_j}^\alpha \right) (t) \, dt \right|
= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)| \left| \left(1_{(-\pi,\pi)} K_{n_j}^\alpha \right) (t) \right| \, dt,
\]
where
\[P_{k_j}(n_j) := \{ x \in \mathbb{R} : 2^{k_j-1}\pi/n_j \leq |x| < 2^{k_j}\pi/n_j \} \quad (j = 1, \ldots, d).\]
Then,
\[ |\sigma_n^\alpha f(x)| \leq \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^\infty \cdots \sum_{k_d=-\infty}^\infty \int_{P_k} \cdots \int_{P_k} |f(x-t)| dt \\
\times \sup_{t \in P_k} \left| \left(1 - \pi, \pi\right)^d K_n^\alpha(t) \right| \\
= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^\infty \cdots \sum_{k_d=-\infty}^\infty \int_{P_k} \cdots \int_{P_k} |f(x-t)| dt \\
\times \sup_{t \in P_k} \left| \left(1 - \pi, \pi\right)^d K_n^\alpha \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right) \right|. \tag{4.6} \]

Consequently, by (4.4),
\[ |\sigma_n^\alpha f(x)| \leq \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^\infty \cdots \sum_{k_d=-\infty}^\infty 2^{k_1+\ldots+k_d} M_s f(x) \sup_{t \in P_k} |h^\alpha(t)| \\
= C \|h^\alpha\|_{E_{\infty}(\mathbb{R}^d)} M_s f(x). \]

Inequality (4.5) finishes the proof. \(\square\)

Inequalities (2.1) and (2.2) imply:

**Corollary 4.1.** Suppose that \(0 < \alpha_j \leq 1\) for all \(j = 1, \ldots, d\). If \(f \in L_1(\log L)^{d-1}(\mathbb{T}^d)\), then
\[ \sup_{\rho>0} \rho \lambda(\sigma_n^\alpha f > \rho) \leq C + C \|f\|_{L_1(\log L)^{d-1}}. \]

If \(1 < p \leq \infty\) and \(f \in L_p(\mathbb{T}^d)\), then
\[ \|\sigma_n^\alpha f\|_p \leq C_p \|f\|_p. \]

The usual density argument due to Marcinkiewicz and Zygmund [12] implies:

**Corollary 4.2.** Suppose that \(0 < \alpha_j \leq 1\) for all \(j = 1, \ldots, d\). If \(f \in L_1(\log L)^{d-1}(\mathbb{T}^d)\), then
\[ \lim_{n \to \infty} \sigma_n^\alpha f = f \quad \text{a.e.} \]

In this paper, \(n \to \infty\) means that \(n_j \to \infty\) for all \(j = 1, \ldots, d\). Now, we prove that the convergence in Corollary 4.2 holds at each strong Lebesgue point, whenever the corresponding strong Hardy-Littlewood maximal function is finite.

**Theorem 4.3.** Suppose that \(0 < \alpha_j \leq 1\) for all \(j = 1, \ldots, d\). If \(M_s f(x)\) is finite and \(x\) is a strong Lebesgue point of \(f \in L_1(\log L)^{d-1}(\mathbb{T}^d)\), then
\[ \lim_{n \to \infty} \sigma_n^\alpha f(x) = f(x). \]

**Proof.** Let
\[ G(u) := \int_{-u_1}^{u_1} \cdots \int_{-u_d}^{u_d} |f(x-t) - f(x)| dt \quad (u \in \mathbb{R}^d_+). \]

Since \(x\) is a strong Lebesgue point of \(f\), for all \(\epsilon > 0\), we can find an integer \(m \leq 0\) such that
\[ \frac{G(u)}{\prod_{j=1}^d (2u_j)} \leq \epsilon \quad \text{if} \quad 0 < u_j \leq 2^m \pi, j = 1, \ldots, d. \tag{4.7} \]

Since
\[ \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n^\alpha(t) dt = 1, \]
we have
\[ |\sigma_n f(x) - f(x)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(x - t) - f(x)| \left| \left( 1_{(-\pi,\pi)^d} K_n^\alpha \right)(t) \right| \ dt := A_1(x) + A_2(x), \]
where
\[
A_1(x) := \frac{1}{(2\pi)^d} \sum_{k_1=\ldots=\infty}^{m+\lfloor \log_2 n_1 \rfloor} \ldots \sum_{k_d=\ldots=\infty}^{m+\lfloor \log_2 n_d \rfloor} \times \int_{P_{k_1}(n_1)} \ldots \int_{P_{k_d}(n_d)} \left| f(x - t) - f(x) \right| \left| \left( 1_{(-\pi,\pi)^d} K_n^\alpha \right)(t) \right| \ dt
\]
and
\[
A_2(x) := \frac{1}{(2\pi)^d} \sum_{\pi_1, \ldots, \pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor + 1}^{m+\lfloor \log_2 n_{\pi_1} \rfloor} \ldots \sum_{k_{\pi_d}=m+\lfloor \log_2 n_{\pi_d} \rfloor + 1}^{m+\lfloor \log_2 n_{\pi_d} \rfloor} \sum_{k_{\pi_j+1}=\ldots=\infty}^{m+\lfloor \log_2 n_{\pi_j+1} \rfloor} \sum_{k_{\pi_d}=\ldots=\infty}^{m+\lfloor \log_2 n_{\pi_d} \rfloor} \times \int_{P_{k_1}(n_1)} \ldots \int_{P_{k_d}(n_d)} \left| f(x - t) - f(x) \right| \left| \left( 1_{(-\pi,\pi)^d} K_n^\alpha \right)(t) \right| \ dt.
\]
Here \( \{\pi_1, \ldots, \pi_d\} \) is a permutation of \( \{1, \ldots, d\} \) and \( 1 \leq j \leq d \). As in (4.6),
\[
A_1(x) \leq C \sum_{k_1=\ldots=\infty}^{m+\lfloor \log_2 n_1 \rfloor} \ldots \sum_{k_d=\ldots=\infty}^{m+\lfloor \log_2 n_d \rfloor} \int_{P_{k_1}(n_1)} \ldots \int_{P_{k_d}(n_d)} \left| f(x - t) - f(x) \right| \ dt
\]
\[
\times \sup_{t \in P_{k_1} \times \ldots \times P_{k_d}} \left| \left( 1_{(-\pi,\pi)^d} K_n^\alpha \right)(t) \right|
\]
\[
\leq C \sum_{k_1=\ldots=\infty}^{m+\lfloor \log_2 n_1 \rfloor} \ldots \sum_{k_d=\ldots=\infty}^{m+\lfloor \log_2 n_d \rfloor} G \left( \frac{2^{k_1} \pi}{n_1}, \ldots, \frac{2^{k_d} \pi}{n_d} \right) \left( \prod_{j=1}^d n_j \right) \sup_{t \in P_k} |h^\alpha(t)|.
\]
Inequalities (4.7), (4.5) and \( 2^{k_j}/n_j \leq 2^m \) imply
\[
A_1(x) \leq C \epsilon \sum_{k_1=\ldots=\infty}^{m+\lfloor \log_2 n_1 \rfloor} \ldots \sum_{k_d=\ldots=\infty}^{m+\lfloor \log_2 n_d \rfloor} 2^{k_1+\ldots+k_d} \sup_{t \in P_k} |h^\alpha(t)| \leq C \epsilon \|h^\alpha\|_{E_\infty(\mathbb{R}^d)} \leq C_\alpha \epsilon.
\]
Similarly,
\[
A_2(x) \leq C \sum_{\pi_1, \ldots, \pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor + 1}^{m+\lfloor \log_2 n_{\pi_1} \rfloor} \ldots \sum_{k_{\pi_d}=m+\lfloor \log_2 n_{\pi_d} \rfloor + 1}^{m+\lfloor \log_2 n_{\pi_d} \rfloor} \sum_{k_{\pi_j+1}=\ldots=\infty}^{m+\lfloor \log_2 n_{\pi_j+1} \rfloor} \sum_{k_{\pi_d}=\ldots=\infty}^{m+\lfloor \log_2 n_{\pi_d} \rfloor} \times \int_{P_{k_1}(n_1)} \ldots \int_{P_{k_d}(n_d)} \left| f(x - t) - f(x) \right| \ dt \sup_{t \in P_k} |h^\alpha(t)|
\]
\[
\leq C_p \sum_{\pi_1, \ldots, \pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor + 1}^{m+\lfloor \log_2 n_{\pi_1} \rfloor} \ldots \sum_{k_{\pi_d}=m+\lfloor \log_2 n_{\pi_d} \rfloor + 1}^{m+\lfloor \log_2 n_{\pi_d} \rfloor} \sum_{k_{\pi_j+1}=\ldots=\infty}^{m+\lfloor \log_2 n_{\pi_j+1} \rfloor} \sum_{k_{\pi_d}=\ldots=\infty}^{m+\lfloor \log_2 n_{\pi_d} \rfloor} \times 2^{k_1+\ldots+k_d} \sup_{t \in P_k} |h^\alpha(t)| \left( M_s f(x) + |f(x)| \right).
\]
Since \( M_s f(x) \) and \( f(x) \) are finite, the fact \( \lfloor \log_2 n_{\pi_j} \rfloor \to \infty \) as \( T \to \infty \) imply that \( A_2(x) \to 0 \) as \( n \to \infty \).
In the one-dimensional case, if \( x \) is a strong Lebesgue point, then \( M_s f(x) \) is finite and \( L_1(\log L)^{d-1}(\mathbb{T}^d) = L_1(\mathbb{T}^d) \), hence we get back the results due to Lebesgue [11] and Riesz [15] mentioned in the introduction. Recall that \( L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d) \) for \( 1 < p \leq \infty \) and \( d > 1 \). Since by Theorem 2.1 and (2.1) almost every point is a strong Lebesgue point and the strong maximal operator \( M_s f \) is almost everywhere finite for \( f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \), Theorem 4.3 implies Corollary 4.2. If \( f \) is continuous at a point \( x \), then \( x \) is also a strong Lebesgue point. So we obtain:

**Corollary 4.3.** Suppose that \( 0 < \alpha_j \leq 1 \) for all \( j = 1, \ldots, d \). If \( M_s f(x) \) is finite and \( f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \) is continuous at a point \( x \), then

\[
\lim_{n \to \infty} \sigma_n^\alpha f(x) = f(x).
\]

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