Super slowing down in the bond-diluted Ising model

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(Dated: February 17, 2020)

Abstract

In models in statistical physics, the dynamics often slows down tremendously near the critical point. Usually, the correlation time \( \tau \) at the critical point increases with system size \( L \) in power-law fashion: \( \tau \sim L^z \), which defines the critical dynamical exponent \( z \). We show that this also holds for the 2D bond-diluted Ising model in the regime \( p > p_c \), where \( p \) is the parameter denoting the bond concentration, but with a dynamical critical exponent \( z(p) \) which shows a strong \( p \)-dependence. Moreover, we show numerically that \( z(p) \), as obtained from the autocorrelation of the total magnetisation, diverges when the percolation threshold \( p_c = 1/2 \) is approached: \( z(p) - z(1) \sim (p - p_c)^{-2} \). We refer to this observed extremely fast increase of the correlation time with size as super slowing down. Independent measurement data from the mean-square deviation of the total magnetisation, which exhibits anomalous diffusion at the critical point, supports this result.

PACS numbers: 05.10.Gg, 05.10.Ln, 05.40.-a, 05.50.+q, 05.70.Jk

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I. INTRODUCTION

The Ising model has proven to be a staple model in physics for studying phase transitions and critical phenomena \cite{1, 2}. The model was originally conceived to provide a theoretical understanding of the existence of a Curie temperature for “pure” ferromagnetic materials; purity here refers to the fact that all throughout the material, every lattice site contains a spin, and every spin interacts uniformly with the surrounding ones. From that point of view, it can be argued that in nature pure materials are rare, i.e., impurities are by and large inevitable.

In the Ising model, the impurities have been implemented in terms of randomly placed nonmagnetic spins (site-diluted Ising model) \cite{4–8} or missing interactions (bond-diluted Ising model) \cite{9–13}. The inclusion of any kind of randomness into the system can have significant effects on its critical properties \cite{14}. For instance, a new universality class was found in the three-dimensional bond-diluted Ising model \cite{15, 16}, and complex logarithmic corrections for the equilibrium properties were observed \cite{8, 17}. Moreover, a large number of novel crossover behaviour between pure and percolating Ising systems have been found \cite{18–23}. Despite these advances, dynamical properties of the bond-diluted Ising model (i.e., as a function of bond concentration $p$) remains poorly studied.

In this paper, we take on studying the slowing down of the dynamics of the total magnetisation autocorrelation function at the critical temperature $T_c(p)$ in the square $(L \times L)$ two-dimensional bond-diluted Ising model with Monte Carlo simulations. To this end, using the Binder cumulant, we first measure $T_c(p)$ at several values of $p$. We then turn to the calculation of $z(p)$ for several $p > p_c$ from the total magnetisation autocorrelation function: by collapsing this autocorrelation function to a reference curve, we calculate the relative terminal exponential decay time $\tau[T_c(p)]$ for the correlation function. Thereafter, by fitting this data as $\tau[T_c(p)] \sim L^{z(p)}$, we directly extract $z(p)$. As $p \rightarrow p_c^+$, we empirically find that the dynamical exponent $z(p)$ increases continuously as $z(p) - z(1) \sim (p - p_c)^{-2}$, with $z(1) = 2.1665(12)$ the dynamical critical exponent of the ordinary Ising model \cite{24}.

Further, we also consider the mean-square deviation (MSD) of the total magnetisation $M$ of the model, which for $p = 1$ has been shown to exhibit anomalous diffusion as $\langle \Delta M^2(t) \rangle \sim t^\alpha$ with the anomalous exponent $\alpha = \frac{\gamma}{\nu z(1)}$ \cite{3}, with $\gamma = 7/4$ and $\nu = 1$. Given that the equilibrium critical exponents $\gamma$ and $\nu$ are numerically nearly independent of $p$ for

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$p \geq 0.6$ [25], combined with values for $z(p)$ as obtained through the terminal relaxation time for different $p$, the various MSD-curves of the total magnetisation are collapsed on top of each other with a $p$-dependent shift factor $\mathcal{G}(p)$ via

$$\log \left( \frac{(\Delta M^2)/L^{2+\gamma/\nu}}{\alpha(p)} \right) \sim \log \left( \frac{t/L^{z(p)}}{\mathcal{G}(p)/\alpha(p)} \right),$$

with $\alpha(p) = \gamma(p)/[\nu(p) z(p)]$. The result reveals that the magnetisation indeed experiences anomalous diffusion at the critical point, for a range of dilution $p > p_c$. The collapse of the MSD of the magnetisation confirms that the measured values of $z(p)$ are correct.

The paper is organised as follows. In Sec. II we introduce the 2D bond-diluted Ising model and measure its critical temperature at several values of $p$. In Sec. III we obtain the dynamical exponent $z(p)$ from the total magnetisation autocorrelation function. In Sec. IV, we confirm $z(p)$ values from the exponent of anomalous diffusion of the MSD of the total magnetisation. We conclude the paper in Sec. V.

II. BOND-DILUTED ISING MODEL AND ITS CRITICAL TEMPERATURE

We consider the two-dimensional (2D) bond-diluted Ising model on an $L \times L$ square lattice with periodic boundary conditions. For this model the Hamiltonian, without an external field, is given by

$$\mathcal{H} = -\sum_{(ij)} J_{ij} s_i s_j,$$

(1)

where $s_i = \pm 1$ is the spin residing at site $i$, $(ij)$ denotes the sum running over all nearest neighbour sites, and the coupling constant $J_{ij}$ is given by the distribution function

$$P(J_{ij}) = p \delta(J_{ij} - 1) + (1 - p) \delta(J_{ij}),$$

(2)

with $p$ being the bond concentration ($0 \leq p \leq 1$). The function [2] simply means that the value of $J_{ij}$ is 1 with probability $p$, and 0 otherwise.

| $p$ | 0.9 | 0.85 | 0.8 | 0.75 | 0.7 | 0.65 | 0.6 | 0.58 | 0.55 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $N$ | 2000 | 5000 | 20000 | 20000 | 20000 | 20000 | 200000 | 200000 | 400000 |
| $T_c$ | 1.956(10) | 1.804(10) | 1.650(20) | 1.472(20) | 1.310(25) | 1.141(30) | 0.951(20) | 0.869(25) | 0.727(40) |

TABLE I: Number of samples $N(p)$ used to measure $T_c(p)$, and the simulation results for $T_c(p)$ (including error bars) for different bond concentrations $p$. 
For the pure Ising model \((p = 1)\), there is a second-order phase transition at \(T_c(1) = 2/\ln(1 + \sqrt{2})\) [26]. When \(p\) reaches the percolation threshold \(p_c = 1/2\), its critical temperature decreases to zero: \(T_c(p_c) = 0\) [27]. To determine \(T_c(p)\) for in-between values of \(p\), we use the Binder cumulant. It is defined as [28]

\[
U(T, L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2},
\]

where \(\langle M^4 \rangle\) and \(\langle M^2 \rangle\) are the thermal averages of the fourth and second moments of the total magnetisation \(M = \sum_{i=1}^{L \times L} s_i\). For each value of \(p\), the curves of \(U(T, L)\) plotted vs. \(T\) for various values of \(L\) intersect at a fixed point, which determines the critical temperature. The process is illustrated in Fig. 1(a).

We perform Monte Carlo simulations using the Wolff algorithm [29, 30] to calculate \(T_c(p)\). Running many independent samples provide us with fairly accurate values of these critical temperatures, as noted in Table I. In Fig. 1(b) we show that the values for \(T_c(p)\) obtained this way match very well with those in Refs. [25, 31].

![Image of Fig. 1](image)

**FIG. 1:** (a) Example calculation of \(T_c(p)\) for the 2D bond-diluted Ising model for \(p = 0.8\) using Binder cumulant. The \(x\)-value of the intersection point indicates that \(T_c(p = 0.8) = 1.650 \pm 0.020\). (b) Critical temperatures for different values of \(p\), as noted in Table I. Our results match those of Refs. [25, 31] very well.

### III. DYNAMICAL EXPONENT FOR DIFFERENT VALUES OF \(p\)

Having obtained the critical temperatures for a number of \(p\)-values as per above, in this section, we measure the total magnetisation autocorrelation function \(\langle M(t) \cdot M(0) \rangle\). In
order to do so, we first run $2 \times 10^6$ Wolff Moves to thermalise the system. Subsequently, we evolve the system following Glauber dynamics, i.e. spin flips are proposed at random locations, and accepted with the Metropolis acceptance probability. Time is measured in terms of attempted Monte Carlo moves, since every spin attempts to flip statistically once per unit time. As we continue to do so, we keep taking snapshots of the full system at regular intervals over a total time of $2 \times 10^7$ attempted Monte Carlo moves per lattice site, and correspondingly compute the total magnetisation $M$ at every snapshot. This leads us to $\langle M(t) \cdot M(0) \rangle$. For different values of $p$, we run 500 to 2000 independent simulations to achieve decent accuracy. We vary the system size from 10 to 40.

For a given value of $p$ and the corresponding critical temperature $T_c(p)$, we collapse all the curves for the normalised total magnetisation autocorrelation function $\langle M(t) \cdot M(0) \rangle / \langle M(0)^2 \rangle$ to a reference curve ($L = 10$). This allows us to compute the ratio of the terminal decay times $\tau[T_c(p)]/\tau[T_c(p)]_{L=10}$. Fig. 2 demonstrates this procedure for $p = 0.8$: with a properly chosen value of $\tau[T_c(p)]/\tau_{L=10}[T_c(p)]$, the $\langle M(t) \cdot M(0) \rangle / \langle M(0)^2 \rangle$ data for different system sizes collapse on the curve corresponding to $L = 10$.

![Fig. 2](image)

FIG. 2: The collapse of $\langle M(t) \cdot M(0) \rangle / \langle M(0)^2 \rangle$ as a function of $t \tau[T_c(p)]/\tau[T_c(p)]_{L=10}$ for $p = 0.8$. The system size varies from 10 to 40. Inset: correspondingly, $\tau[T_c(p)]/\tau[T_c(p)]_{L=10}$ as a function of $L$. The dynamical exponent is obtained by fitting these data as $\tau[T_c(p)]/\tau[T_c(p)]_{L=10} \sim L^{z(p)}$. The solid line corresponds the function $y = x^{2.285}$. From this we obtain $z(0.8) \approx 2.285$.

Further, given our argument in Appendix A that $L$ is the characteristic length scale for $L \geq 10$ for the 2D bond-diluted Ising model when $p \geq 0.6$, we have, at the critical
temperature,

\[
\tau(p)/\tau_{L=10}(p) \sim L^{z(p)}. \tag{4}
\]

By plotting the \(\tau(p)/\tau_{L=10}(p)\) data (inset Fig. 2), we extract \(z(p)\). The results from this exercise for several values of \(p\) are shown in Fig. 3. Numerically, therein we find that

\[
\Delta z(p) = z(p) - z(1) \sim (p - p_c)^{-2} \quad \text{for} \quad p_c < p < 1,
\tag{5}
\]

where \(z(1) = 2.1665(12)\) [24] is the dynamical exponent for the pure 2D Ising model.

![Plot of \(\Delta z(p)\) vs. \((p - p_c)\)

FIG. 3: The dynamical exponent difference \(\Delta z(p) = z(p) - z(1)\) as a function of \(p - p_c\), where \(z(1) = 2.1665(12)\) [24] is the dynamical exponent for the pure 2D Ising model. The result implies that \(z(p) \to \infty\) as \(p \to p_c^+\).

Based on concepts of renormalisation, we anticipated that away from the percolation threshold \(p_c\), the correlation time as a function of system size would show a crossover from the behaviour for the bond-diluted Ising model at small system sizes to that of the ordinary Ising model at large system sizes, with a crossover size that diverges if \(p_c\) is approached. Instead, we find a fairly clean power-law behaviour of the correlation time for all system sizes, with a single exponent \(z(p)\) that varies strongly with \(p\). As a function of bond dilution \(p\), the dynamical exponent \(z(p)\) increases monotonically when \(p\) decreases from \(p = 1\) to \(p = p_c\). Moreover, the numerical results suggest that \(z(p)\) will become infinitely large when \(p\) approaches the percolation threshold \(p \to p_c^+\), i.e., the dynamics of the system gets extremely slow as \(p \to p_c^+\), a phenomenon we term as “super slowing down”.

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IV. ANOMALOUS DIFFUSION OF THE TOTAL MAGNETISATION

To confirm the observed behaviour of super slowing down [i.e., the Eq. (5)] for the bond-diluted Ising model by means of independent measurements, we now focus on the mean-square deviation (MSD) of the magnetisation as a function of time $t$ as

$$\langle \Delta M^2 \rangle = \langle [M(t) - M(0)]^2 \rangle. \quad (6)$$

At short times ($t \approx 1$), changes in $M$, occurring due to random thermal fluctuations of individual spins, are uncorrelated; hence $\langle \Delta M^2 \rangle \sim L^2 t$ for 2D Ising model. At long times, $t \gtrsim L^{z(p)}$, we expect $\langle M(t) \cdot M(0) \rangle = 0$, meaning that

$$\langle \Delta M^2 \rangle \sim \frac{2\langle M(t)^2 \rangle}{\sim L^{2+\gamma(p)/\nu(p)}}. \quad (7)$$

If we assume that the MSD is given by a simple power law in the intermediate time regime ($1 \gtrsim t \gtrsim L^{z(p)}$), then we obtain

$$\langle \Delta M^2 \rangle \sim L^{2+\gamma(p)/\nu(p)} \left( \frac{t}{L^{z(p)}} \right)^{\alpha(p)}, \quad (8)$$

where $\alpha(p) = \frac{\gamma(p)}{\nu(p)z(p)}$. For the pure Ising model in two dimensions ($p = 1$), we have shown that

$$\langle \Delta M^2 \rangle / L^{2+\gamma/\nu} = f(t/L^z), \quad (9)$$

where $\gamma = 7/4$ and $\nu = 1$ are two equilibrium critical exponents. Here $f(x)$ is a scaling function such that $\lim_{x \to 0} f(x) = x^{\gamma/(\nu z)} \approx x^{0.81}$, and $f(x)$ saturates for $x \gtrsim 1$. Indeed, given that $\gamma(p)$ and $\nu(p)$ are nearly independent of $p$ when $p \geq 0.6$ [25] (see also in Appendix B), if the scaling relation (9) also continues to hold for values of $p$ other than unity, then we can use it to obtain independent confirmation for the super slowing down (5). We demonstrate this below by focusing on $p \geq 0.6$.

Since in the previous section, we have obtained the values of $z(p)$ for different $p$, here, we describe the MSD of the total magnetisation by modifying Eq. (8) as

$$\langle \Delta M^2 \rangle / L^{2+\gamma/\nu} \sim G(p) \left( \frac{t}{L^{z(p)}} \right)^{\alpha(p)}, \quad (10)$$

where $G(p)$ is a $p$-dependent shift factor. We take logarithm of both sides of Eq. (10) to write

$$\log \left( \langle \Delta M^2 \rangle / L^{2+\gamma/\nu} \right) / \alpha(p) \sim \log \left( \frac{t}{L^{z(p)}} \right) + \log (G(p)) / \alpha(p). \quad (11)$$
FIG. 4: The collapse of the mean-square displacement of the total magnetisation via
\[
\log \left( \frac{\langle \Delta M^2 \rangle}{L^{2+\gamma/\nu}} / \alpha(p) \right) \sim \log \left( \frac{t}{L^{z(p)}} \right) + \log \left( \mathcal{G}(p) \right) / \alpha(p),
\]
where the obtained values of \( z(p) \) from the last section are employed and \( \mathcal{G}(p) \) is a \( p \)-dependent shift factor. The slope of the solid line is unity. It confirms that the MSD of the total magnetisation experiences anomalous diffusion at \( T_c(p) \) and the values of \( z(p) \) is increasing when \( p \to p_c^+ \).

Suppose we choose the MSD of the total magnetisation for the normal Ising model as the reference [means that we set \( \mathcal{G}(1) \equiv 1 \)], if the values of \( z(p) \) obtained from the last section are correct, then with these \( z(p) \) values and the shift factor \( \mathcal{G}(p) \), the MSD of the total magnetisation for different \( p \) can be made to collapse onto the data for \( p = 1 \) via Eq. (11).

In order to obtain the \( \langle \Delta M^2(t) \rangle \) data, once again, we first thermalise the system with \( 2 \times 10^6 \) Wolff moves, then measure \( \langle \Delta M^2 \rangle \) in a further simulation over \( 2 \times 10^7 \) attempted Monte Carlo moves per lattice site. We use three different system sizes: \( L = 20, 40, 60 \) for every value of \( p \).

Figure 4 implies that by using the values of \( z(p) \) obtained from the last section, indeed for different \( p \), the MSD of the magnetisation can be collapsed onto the data for \( p = 1 \) via Eq. (11). It confirms that the MSD of the total magnetisation experience anomalous diffusion at \( T_c(p) \) and \( z(p) \) values obtained from the terminal relaxation time are correct.

In summary, with two different methods, we have shown that \( z(p) \) is diverging when \( p \to p_c^+ \), i.e., the dynamics of the system is getting extremely slow when we reduce the bond
concentration to its percolation threshold. We do not have a quantitative explanation for this behaviour. That said, it might arise from the fact that the fraction of ‘unhappy’ bonds (active bonds between sites with opposing spin values) at the critical temperature decreases to zero if \( p_c \) is approached, thereby removing the energetic contribution of restoring forces; we provide some measurements for this in Appendix C.

V. DISCUSSION

In this paper, we study the critical dynamical exponent \( z(p) \) for the 2D bond-diluted Ising model with bond concentration \( p \). We first measure the critical temperature \( T_c(p) \) for different bond concentrations \( p \) using the Binder cumulant. We then calculate the relative values of the terminal decay time \( \tau \) by collapsing the total magnetisation autocorrelation function to a reference value, from which we obtain \( z(p) \) using the relation \( \tau \sim L^{z(p)} \).

We find that \( z(p) \) increases when \( p \to p_c^+ \) as a power-law \( z(p) - z(1) \sim (p - p_c)^{-2} \), which we refer to as super slowing down. We confirm this result from independent measurements of the MSD of the total magnetisation that exhibits anomalous diffusion.

Our results indicate that \( z(p) \to \infty \) as \( p \to p_c^+ \). This leaves us with the interesting question: what happens to \( z(p) \) when \( p < p_c \)? We plan to explore this in future.

Acknowledgement

W.Z. acknowledges financial support from the China Scholarship Council (CSC).

Appendix A: Relevant length scale for critical phenomena of the bond-diluted Ising model

In the pure 2D Ising model of dimension \( L \times L \), the only relevant length scale for critical phenomena is \( L \). For the bond-diluted Ising model there are other length scales, for instance corresponding to the size of the biggest cluster, \( S(p, L) \). Here, a cluster is defined is by the set of spins such that there is at least one continuous (bond-following) path from every spin in the cluster to every other spin in the same cluster. We define the size of the cluster by the total number of spins belonging to the cluster [thus \( S(p, L) \) is the number of spins in
FIG. 5: (a) A snapshot of the biggest knot cluster for $L = 50$ and $p = 0.6$. Spins that are not belong to this cluster are represented by cavities. (b) Plot of $q(p, L) = \langle S(p, L) \rangle / L^2$ for various values of $p$ and $L$: for $p \geq 0.6$, $q(p, L)$ is independent of $L$ for $L \geq 10$.

the biggest cluster for an $L \times L$ system with bond concentration $p$.

In Fig. 5(b) the quantity considered is $q(p, L) = \langle S(p, L) \rangle / L^2$. If this quantity is independent of $L$ then it means that there is no difference between the two differently defined length scales (apart from a scaling factor). For each result, we have generated 500 samples. We see in Fig. 5(b) that for $p \geq 0.6$, $q(p, L)$ is independent of $L$ for $L \geq 10$. This means that for the range of dilution $p \geq 0.6$ used in this paper, we can use $L$ as the relevant length scale for critical phenomena provided $L \geq 10$.

Appendix B: Equilibrium critical exponents $\nu$ and $\gamma$

In this Appendix, we show that the equilibrium critical exponents $\nu$ and $\gamma$ for the bond-diluted Ising system with $p \geq 0.6$ are numerically indistinguishable from their values in the pure Ising model.

Firstly, if we get the values of $T_c$ for different $p$, the Binder cumulant can be scaled as

$$U(T, L) \sim f(T' L^{1/\nu(p)})$$

(12)

which will provide us the value of $\nu(p)$. Here $T' = (T - T_C)/T_C$ is the reduced temperature.

In Fig. 6 we collapse the data of $U(L, T)$ for $L = 20, 40$ and $60$ with $\nu(p) \approx 1$. It indicates that $\nu(p)$ is numerically indistinguishable from unity for $p \geq 0.6$. 
FIG. 6: The Binder cumulant for $p \geq 0.6$. The data is collapse as $U(T,L) \sim f(T'L^{1/\nu(p)})$ for different $p$ and $L$, where $T' = (T - T_C)/T_C$ is the reduced temperature. Here $L = 20, 40$ and $60$ for each bond concentration, and we set all $\nu(p) = 1.0$ to collapse the data.

Next, we turn to measure the magnetic susceptibility $\chi$. For this simulation, we have used 500 independent samples for each value of $p$. It is a well known result \cite{30} that the susceptibility can be scaled as

$$\chi L^{-\gamma/\nu} = \tilde{\chi}(T'L^{1/\nu}),$$

where $\tilde{\chi}$ is a dimensionless function.

After rescaling the susceptibility using Eq. (13), the data shown in Fig. 7 demonstrate that $\gamma$ is numerically indistinguishable from $7/4$ for $p \geq 0.6$.

In other words, in this Appendix we have shown that $\gamma$ and $\nu$ are numerically indistinguishable respectively from $7/4$ and unity for $p \geq 0.6$, confirming the results from Ref. \cite{25}.

**Appendix C: Number of different types of bonds**

In this Appendix, we connect the super slowing down in the $2D$ bond-diluted Ising model with its equilibrium property, i.e., the ensemble average of the number of ‘unhappy’ bonds, i.e. the number of interacting nearest-neighbour spins with opposite signs at the critical temperature.

In the bond-diluted Ising model, we distinguish inactive bonds (with $J_{ij} = 0$), active bonds connecting sites with aligned spins, and active bonds that connect sites with spins of opposite signs. For the active bonds, we denote the numbers of those aligned and nonaligned
FIG. 7: The scaling of magnetic susceptibility as a function of reduced temperature: $\chi L^{-\gamma/\nu} = \tilde{\chi}(T'L^{1/\nu})$. Here $\tilde{\chi}$ is a dimensionless function, the values of $\gamma(p)$ and $\nu(p)$ are chosen to be their values for the normal Ising model, i.e., $\gamma(p) = 1.75$ and $\nu(p) = 1$. For figures (a)-(d), the bond concentrations are $p = 0.9, 0.8, 0.7$ and $0.6$. The well collapse of all the data indicates that both $\gamma$ and $\nu$ are numerically indistinguishable for $p \geq 0.6$.

spins by $\langle n_{++} + n_{--} \rangle$ and $n_{+-}$, respectively. Energetically, $\langle n_{++} \rangle$ and $\langle n_{--} \rangle$ are the bonds that try to keep the system as it is, and $\langle n_{+-} \rangle$ is driving spins to flip. If $\langle n_{+-} \rangle$ decreases, then most of the proposed spin flips will be rejected and the dynamics of the system will get slower.

In our simulations, we have performed 100 independent samples to obtain the number of ‘unhappy’ bonds. The measured values of $\langle n_{+-} \rangle$ at the critical temperature can be found in Fig. 8 with a log-log plot as an inset. In particular, numerically we find that

$$\langle n_{+-} \rangle / L^2 \sim (p - p_c)^{0.97 \pm 0.03} \quad \text{for} \quad p \geq p_c.$$  \hspace{1cm} (14)

When $p \to p_c^+$, the values of $\langle n_{+-} \rangle$ reduces to zero (or a value close to zero), which means that most of the active bonds are ‘happy’ so that spins are unlikely to flip. This might
FIG. 8: The value of $\langle n_{+-} \rangle$ as a function of $p - p_c$ for $L = 50, 80$ and $100$. The solid line goes as $\sim (p - p_c)^{0.97}$. The inset is a log-log plot of the data. It suggests that when $p \rightarrow p_c^+$, $\langle n_{+-} \rangle \rightarrow 0$, then most of the bonds are activated so that nearly all spins are unlikely to flip, resulting in the super slow dynamics of the system.

explain why the system is getting super slow when $p$ approaches the percolation threshold.

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