A DIVIDE-AND-CONQUER METHOD FOR COMPUTING THE BETTI NUMBERS OF FINITE TOPOLOGICAL SPACES

PATRICK ERIK BRADLEY

Abstract. A divide-and-conquer algorithm for computing the Betti numbers of finite \( T_0 \)-spaces is presented. It extensively uses the Mayer-Vietoris sequence for open coverings. In the end, the computational costs for a parallelisation of this method are given.

1. Introduction

The theory of finite topological spaces can be traced back to Pavel Alexandrov who found that finite \( T_0 \)-spaces are in one-to-one correspondence with finite posets [1]. McCord associated with any finite topological space \( X \) a finite simplicial complex \( K^*(X) \) whose space is weakly homotopy equivalent with \( X \), and vice versa [7].

Homology is a coarser invariant than weak homotopy type. This allows to use the simplicial complex associated with a finite topological space \( X \) to compute its homology groups. However, the number of \( n \)-simplices of \( X \) can grow exponentially with \( n \). This makes a direct homology computation of this simplicial complex unsuitable for large finite spaces. In [3], a method for computing homology is introduced which uses a spectral sequence built up from homology groups of certain subspaces in a filtration of \( X \). This might lead to a more efficient computation. However, it requires the homology groups of many different smaller subspaces, and the authors of loc. cit. leave open the question of how to compute these. In [6], the question of an efficient homology computation for finite \( T_0 \)-spaces was raised in the context of evaluating the topological inconsistency of geospatial data.

If instead of homology, the aim is to compute the Betti numbers of a finite topological space, then, as will be seen in this article, the situation is much better, because a divide-and-conquer method relying on the Mayer-Vietoris sequence can be employed. But the natural application of this method to computing homology does not work, because in each step of the algorithm, an extension problem for finitely generated abelian groups has to be solved. The rank of the free part of the solution being uniquely determined leads to the successful computation of the next step in the algorithm for computing Betti numbers. However, the full solutions to this extension problem are in general not equivalent, and this makes our approach useless for computing the homology of arbitrary finite topological spaces.

Date: June 5, 2018.
The aim of this article is to develop a divide-and-conquer algorithm for computing the Betti numbers of finite $T_0$-spaces. In the end, the computational costs for a parallelisation of this method are given.

2. THE DIVIDE-AND-CONQUER ALGORITHM

According to McCord [7, Theorem 4], any finite topological space is homotopy equivalent to its Kolmogorov quotient which is a $T_0$-space. Here, two points are equivalent if their minimal open neighbourhoods coincide. Consequently, it is not a loss of generality, if only finite spaces are considered which are $T_0$-spaces. Finite $T_0$-spaces are in a natural way partially ordered sets, and any finite partially ordered set has a natural $T_0$-topology [1].

Throughout the remainder of this article, $X$ will denote a finite $T_0$-space. Its partial order will be denoted as $\leq$. $K_\bullet(X)$ will denote the order complex of $X$ with coefficients in a field.

An $n$-simplex is a totally ordered subset $\{a_0 \prec \cdots \prec a_n\}$ of $n+1$ elements of $X$. It will also be denoted as $a_0 \ldots a_n$. The module $K_n(X)$ consists of all finite linear combinations of $n$-simplices $a_0 \ldots a_n$. Its elements are called $n$-chains. The $n$-th boundary map is $\partial_{n,X}: K_n(X) \to K_{n-1}(X)$ which is defined on $n$-simplices as

$$a_0 \ldots a_n \mapsto \sum_{i=0}^{n} (-1)^i a_0 \ldots \hat{a}_i \ldots a_n$$

where $\hat{x}$ means omission of $x$. The $n$-th homology group of $K_\bullet(X)$ is defined as

$$H_n(X) = \ker \partial_{n,X} / \text{im} \partial_{n+1,X}$$

The length of a simplex $a_0 \ldots a_n$ is $n$. The dimension of a $T_0$-space is defined as the maximum of the lengths of its simplices. We will denote the dimension of a space $X$ as $\dim(X)$. This is not important, but it should be mentioned that this is an abstraction of the Krull dimension of a finitely generated commutative ring [1], and coincides with the dimension of the poset of its prime ideals (cf. also [2]).

$X_{\min}$ denotes the set of minimal points in $X$.

The Hasse diagram $\mathcal{H}(X)$ of a finite topological space is the graph whose vertices are the points of $X$, and an edge between $x$ and $y$ is drawn if and only if $x > y$ and there is no $z \in X$ such that $x > z > y$. We say that $x$ is a parent of $y$ and $y$ is a child of $x$, if $(x,y)$ is an edge in the Hasse diagram of $X$.

A point $x \in X$ is called a beat point, if its corresponding vertex in the Hasse diagram has precisely one parent or precisely one child. If from $X$ all beat points are removed iteratively, then the remaining space is called the core of $X$. Stong proved that removing beat points does not change the homotopy type of a finite space [8]. In particular, the homology of $X$ equals that of its core.
There is a natural filtration on any subspace $Z \subset X$ given by

$$Z_0 := Z_{\text{min}}$$

$$Z_{k+1} := Z_k \cup (Z \setminus Z_k)_{\text{min}}$$

for $k \geq 0$. And further,

$$B_k(Z) := Z_k \setminus Z_{k-1}$$

for $k \geq 0$ with $Z_{-1} := \emptyset$. $B_k(Z)$ is called the $k$-th layer of $Z$ or of $\mathcal{H}(Z)$.

Let $a \in X$. By $X_a$ we denote the minimal open neighbourhood of $a$ in $X$. Let $I \subset X$. Then

$$X_I := \bigcup_{a \in I} X_a$$

Let $X_{\text{min}}$ denote the subset of $X$ consisting of all minimal elements. Then we have the open cover

$$X = X_{\text{min}} = \bigcup_{a \in X_{\text{min}}} X_a$$

In particular, if $X$ has at least two minimal elements, then let

$$X_{\text{min}} = I \cup J$$

be a partition into two non-empty subsets. In this case, there is the Mayer-Vietoris sequence for the open cover $X = X_I \cup X_J$ [5]:

$$0 \to H_n(X_I) \oplus H_n(X_J) \to H_n(X) \to H_{n-1}(X_I \cap X_J) \to \ldots$$

where $n = \dim(X)$. Observe that $H_n(X_I \cap X_J) = 0$ because $\dim(X_I \cap X_J) < n$ as $I \cap J = \emptyset$.

**Algorithm 2.1** (Divide and Conquer). Step 1: Replace $X$ by its core, and proceed as above.

Step $N$: Let $Z$ run through the set $\{Y_I, Y_J, Y_I \cap Y_J\}$ of subsets of $X$ obtained in the previous step. Replace $Z$ by its core and call this also $Z$. If $Z_{\text{min}}$ has at least two points, then let

$$Z_{\text{min}} = A \cup B$$

be a partition into sets of approximately the same cardinality, and obtain the Mayer-Vietoris sequence for the open cover

$$Z = Z_A \cup Z_B$$

This yields the three subsets of $X$: $Z_A$, $Z_B$, $Z_A \cap Z_B$.

The algorithm terminates when the cores of all obtained sets have precisely one minimal element.

Observe that $T_0$-spaces with precisely one minimal element are contractible [7, Lemma 6]. In particular, they are acyclic.

Consider the core of each set in any step of the algorithm as a node of a rooted tree $T$ where the core of $Z$ is the parent of the cores of $Z_A$, $Z_B$ and $Z_A \cap Z_B$ obtained in the $N$-th step. Each node has at most 3 children.
The Betti numbers of each node of $\mathcal{T}$ are given by the Mayer-Vietoris sequence containing the Betti numbers of the child nodes as follows: The long exact sequence

\[ \cdots \rightarrow H_k(Z_I \cap Z_J) \xrightarrow{\psi_k^Z} H_k(Z_I) \oplus H_k(Z_J) \rightarrow H_k(Z) \rightarrow H_{k-1}(Z_I \cap Z_J) \xrightarrow{\psi_{k-1}^Z} H_{k-1}(Z_I) \oplus H_{k-1}(Z_J) \rightarrow \cdots \]

splits into short exact sequences

\[ 0 \rightarrow \text{coker } \psi_k^Z \rightarrow H_k(Z) \rightarrow \ker \psi_{k-1}^Z \rightarrow 0 \]

which allow to compute Betti number $b_k(Z)$ from the Betti numbers of the child nodes. Recall that the map $\psi_k^Z$ is induced by the map

\[ K_\bullet(Z_I \cap Z_J) \rightarrow K_\bullet(Z_I) \oplus K_\bullet(Z_J), \ a \mapsto (a,a) \]

Finally, the spaces associated with the leaves are all acyclic.

Now, in order to compute $\ker \psi_{k-1}^Z$ and $\text{coker } \psi_k^Z$, observe that $|I|, |J| < |Z_{\text{min}}|$ and $\dim(Z_I \cap Z_J) < \dim(Z)$.

This means that after sufficiently many divide steps, $\psi_k^Z$ for $k > 0$ is the map

\[ 0 \rightarrow H_k(Z_I) \oplus H_k(Z_J) \]

so that in the end only maps

\[ \psi_0^Z : H_0(Z_I \cap Z_J) \rightarrow H_0(Z_I) \oplus H_0(Z_J), \ [a] \mapsto ([a],[a]) \]

need to be computed with $Z_I, Z_J$ acyclic and $Z_I \cap Z_J$ zero-dimensional.

**Lemma 2.2.** The height of $\mathcal{T}$ is at most

\[ h(X) := \sum_{i=0}^{n} \log_2 |B_i(X)| \]

and this bound is attained for some space $X$.

**Proof.** There is a path from root $X$ whose nodes are cores of spaces of the form $X_I$ for some $I \subset X_{\text{min}}$. The length of this path is at most $\log_2 |X_{\text{min}}| = \log_2 |B_0(X)|$. In the worst case, the parent of the leaf of this path has a child node whose core $Z$ is $n-1$-dimensional. This one has a downward path as before of length at most

\[ \log_2 |Z_{\text{min}}| \leq \log_2 |B_1(X)| \]

Continuing the worst-case scenario in this way leads to the asserted upper bound. A space for which the height of $\mathcal{T}$ attains the asserted bound is a space $X$ with $n$ layers $B_k(X)$ such that

\[ \delta_k(B_k(X) \cup B_{k+1}(X)) \]

is a complete bipartite graph for $k = 0, \ldots, n-1$. \qed

**Remark 2.3.** In order to use the Mayer-Vietoris sequence for computing the homology $H_k(X,Z)$ with integer coefficients using the same divide-and-conquer method, one needs to solve the extension problem given by the exact
sequence \([\mathbb{Z}]\). However, the solution is in general finite, but not unique: the number of equivalent solutions is the cardinality of
\[
\text{Ext}_1^\mathbb{Z}(\ker \psi_k^{Z-1}, \text{coker } \psi_k^Z)
\]
which is in general a non-trivial abelian group.

3. Parallelisation

A parallelisation of Algorithm 2.1 for computing Betti numbers is given by distributing the divide and the computation parts on several processors.

**Theorem 3.1.** There is a scheduling algorithm such that the parallel computation cost is at most \([N/p]\) and the communication cost is \(O(p \cdot h(X))\) where \(N\) is the number of nodes of \(T\) and \(p\) is the number of processors.

**Proof.** This follows from \([9]\) and Lemma 2.2. \(\square\)

**Acknowledgement**

This work is partially supported by the Deutsche Forschungsgemeinschaft under grant number BR 3513/12-1. Markus Jahn, Miguel Ottina, Norbert Paul, and Bastian Erndüüß are thanked for fruitful discussions and helpful remarks.

**References**

[1] P.S. Alexandrov. Diskrete Räume. *Matematicheskii Sbornik (N.S.)*, 2:501–518, 1937.
[2] P.E. Bradley and N. Paul. Dimension of Alexandrov topologies. arXiv:1305.1815 [math.GN], 2013.
[3] N. Cianci and M. Ottina. A new spectral sequence for homology of posets. *Topology and its Applications*, 217:1–19, 2017.
[4] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, Berlin, 1993. Sixth corrected printing.
[5] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.
[6] M.W. Jahn, P.E. Bradley, M. Al-Doori, and M. Breunig. Topologically consistent models for efficient big geo-spatio-temporal data distribution. *ISPRS Annals of the Photogrammetry, Remote Sensing and Spatial Information Sciences*, IV-4/W5:65–72, 2017.
[7] M.C. McCord. Singular homology groups and homotopy groups of finite topological spaces. *Duke Math. J.*, 33(3):465–474, 1966.
[8] R.E. Stong. Finite topological spaces. *Trans. Amer. Math. Soc.*, 123:325–340, 1966.
[9] I. Wu and H.T. Kung. Communication complexity for parallel divide-and-conquer. In *Foundations of Computer Science, 1991. Proceedings., 32nd Annual Symposium on*, pages 151–162, 1991.