Quantum Kaluza-Klein Cosmologies (V)

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Abstract

In the No-boundary Universe with $d = 11$ supergravity, under the $S_n \times S_{11-n}$ Kaluza-Klein ansatz, the only seed instanton for the universe creation is a $S_7 \times S_4$ space. It is proven that for the Freund-Rubin, Englert and Awada-Duff-Pope models the macroscopic universe in which we are living must be 4- instead of 7-dimensional without appealing to the anthropic principle.

PACS number(s): 98.80.Hw, 11.30.Pb, 04.60.+n, 04.70.Dy
Key words: quantum cosmology, Kaluza-Klein theory, supergravity, gravitational instanton
In a series of papers [1] the origin of the dimension of the universe was investigated for the first time in quantum cosmology. As far as I am aware, in the No-Boundary Universe [2], the only way to tackle the dimensionality of the universe is through Kaluza-Klein cosmologies. In the Kaluza-Klein model with \( d = 11 \) supergravity, under the \( S_n \times S_{11-n} \) ansatz, it has been shown that the macroscopic universe must be 4- or 7-dimensional. The motivation of this paper is to prove that the universe must be 4-dimensional.

In \( d = 11 \) simple supergravity, in addition to fermion fields, a 3-index antisymmetric tensor \( A_{MNP} \) is introduced into the theory by supersymmetry [3]. In the classical background of the WKB approximation, one sets the fermion fields to vanish. Then the action of the bosonic fields can be written

\[
\tilde{I} = \int \sqrt{-g_{11}} \left( \frac{1}{2} R - \frac{1}{48} F_{MNPQ} F^{MNPQ} + \frac{\sqrt{2}}{6 \cdot (4!)^2} \eta^{M_1 M_2 \cdots M_{11}} F_{M_1 M_2 M_3 M_4} F_{M_5 M_6 M_7 M_8} A_{M_9 M_{10} M_{11}} \right) d^{11}x, \tag{1}
\]

where

\[
F_{MNPQ} \equiv 4! \partial_{[M} A_{NPQ]}, \tag{2}
\]

\[
\eta^{A \cdots N} = \frac{1}{\sqrt{-g_{11}}} \epsilon^{A \cdots N} \tag{3}
\]

and \( R \) is the scalar curvature of the spacetime with metric signature \((-+, +, +, \cdots +)\). The theory is invariant under the Abelian gauge transformation

\[
\delta A_{MNP} = \partial_{[M} \zeta_{NP]}. \tag{4}
\]

It is also noticed that the action is invariant under the combined symmetry of time reversal with \( A_{MNP} \rightarrow -A_{MNP} \).

The field equations are

\[
R_{MN} - \frac{1}{2} R g_{MN} = \frac{1}{48} (8 F_{MPQR} F^P_{NQR} - g_{MN} F_{SPQR} F^{SPQR}), \tag{5}
\]

and

\[
F^{MNPQ} = \left[ \frac{-\sqrt{2}}{2 \cdot (4!)^2} \right] \cdot \eta^{M_1 \cdots M_8 N P Q} F_{M_1 \cdots M_8} F_{M_9 \cdots M_8}. \tag{6}
\]

At the WKB level, it is believed that the Lorentzian evolution of the universe originates from a compact instanton solution, i.e. a stationary action solution of the Euclidean Einstein and other field equations. In order to investigate the origin of the dimension of the universe, we are trying to
find the following minisuperspace instantons: the $d = 11$ spacetime takes a product form $S_n \times S_{11-n}$ with an arbitrary metric signature and all components of the $F$ field with mixed indices in the two factor spaces to be zero. In the factor space $S_n$ ($n = 1, 2, 3$) the $F$ components must be vanish due to the antisymmetry of the indices. Then $F$ must be a harmonic in $S_{11-n}$ since the right hand side of the field equation (6) vanishes. It is known in de Rham cohomology that $H^4(S_4) = 1$ and $H^4(S_m) = 0$ ($m \neq 4$). So there is no nontrivial instanton for $n = 1, 2, 3$. For $n = 5, 6$, both $F$ components in $S_5$ and $S_6$ must be harmonics and so vanish. By the dimensional duality, there does not exit nontrivial instanton either for $n = 10, 9, 8$. The case $S_4 \times S_7$ is the only possibility for the existence of a nontrivial instanton, the $F$ components must be a harmonic in $S_4$, but do not have to in $S_7$. The no-boundary proposal and the ansatz are very strong, otherwise the nonzero $F$ components could live in open or closed $n$-dimensional factor spaces ($4 \leq n \leq 10$) [1].

Four compact instantons are known, their Lorentzian versions are the Freund-Rubin, Englert, Awada-Duff-Pope and Englert-Rooman-Spindel spaces [4][5][6][7]. They are products of a 4-dimensional anti-de Sitter space and a round or squashed 7-sphere. These spaces are distinguished by their symmetries from other infinitely many solutions with the same $F$ field. From now on, Greek letters run from 0 to 3 for the indices in $S_4$ and small Latin letters from 4 to 10 for the indices in $S_7$.

One can analytically continue the $S_7$ or $S_4$ space at the equator to form a 7- or 4-dimensional de Sitter or anti-de Sitter space, which is identified as our macroscopic spacetime, and the $S_4$ or $S_7$ space as the internal space. One may naively think, since in either case the seed instanton is the same, that the creation of a macroscopic 7- or 4-dimensional universe should be equally likely. However, a closer investigation shows that this is not the case, it turns out that the macroscopic universe must be 4-dimensional, regardless whether the universe is habitable.

The Freund-Rubin is of the $N = 8$ supersymmetry [4]. Here the only nonzero $F$ components are in the $S_4$ factor space of the instanton

$$F_{\mu\nu\sigma\delta} = i\kappa \sqrt{g_4} \epsilon_{\mu\nu\sigma\delta},$$

where $g_4$ is the determinant of the $S_4$ metric, the $F$ components are set imaginary in $S^4$ such that their values become real in the anti-de Sitter space, which is an analytic continuation of the $S_4$ space, as shown below. The $F$ field plays the role of an anisotropic effective cosmological constant, which is $\Lambda_7 = \kappa^2/3$ for $S_7$ and $\Lambda_4 = -2\kappa^2/3$ for $S_4$, in the sense that $R_{mn} = \Lambda_7 g_{mn}$ and $R_{\mu\nu} = \Lambda_4 g_{\mu\nu}$, respectively. The $S_4$ space must have radius $r_4 = (3/\Lambda_4)^{1/2}$ and metric signature $(-,-,-,-)$, while
the $S_7$ space is of radius $r_7 = (6/\Lambda_7)^{1/2}$ and metric signature $(+,+,\cdots+)$. Since the metric signature of the factor space $S_4$ is not appropriate, one has to analytically continue the $S_4$ manifold into an anti-de Sitter space with the right metric signature $(-,+,+,+)$. The $S_4$ metric can be written

$$ds_4^2 = -dt^2 - \frac{3}{\Lambda_4} \sin^2 \left( \sqrt{\frac{\Lambda_4}{3}} t \right) (d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)).$$

One can obtain the 4-dimensional anti-de Sitter space by setting $\rho = i\chi$. However, if one looks closely in the quantum creation scenario, this continuation takes two steps. First, one has to continue on a three surface where the metric is stationary. One can choose $\chi = \frac{\pi}{2}$ as the surface, set $\omega = i(\chi - \frac{\pi}{2})$ and obtain the metric with signature $(-,-,-,+)$

$$ds_4^2 = -dt^2 - \frac{3}{\Lambda_4} \sin^2 \left( \sqrt{\frac{\Lambda_4}{3}} t \right) (-d\omega^2 + \cosh^2 \omega (d\theta^2 + \cos^2 \theta d\phi^2)).$$

Then one can analytically continue the metric through the null surface at $t = 0$ by redefining $\rho = \omega + \frac{\pi i}{2}$ and get the anti-de Sitter metric

$$ds_4^2 = -dt^2 + \frac{3}{\Lambda_4} \sin^2 \left( \sqrt{\frac{\Lambda_4}{3}} t \right) (d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)).$$

In the No-Boundary Universe, the relative probability of the creation, at the WKB level, is the exponential to the negative of the Euclidean action of the instanton $S_7 \times S_4$

$$P = \Psi^* \cdot \Psi \approx \exp \{-I\},$$

where $\Psi$ is the wave function of the configuration at the quantum transition. The configuration is the metric and the matter field at the equator. $I$ is the Euclidean action.

If we are living in the section of the 7-dimensional de Sitter universe with the $S_4$ space of metric (8) or the Euclidean version of (10) as the internal space, then the Euclidean action $I$ should take the form

$$I = -\int \sqrt{g_{11}} \left( \frac{1}{2} R - \frac{1}{48} F_{MNPQ} F^{MNPQ} + \frac{\sqrt{2}}{6 \cdot (4!)^2} \eta^{M_1 M_2 \cdots M_{11}} F_{M_1 M_2 M_3 M_4} F_{M_5 M_6 M_7 M_8} A_{M_9 M_{10} M_{11}} \right) d^{11}x.$$ (12)

This is obtained through analytical continuation as in the usual 4-dimensional Euclidean quantum gravity.

However, if we are living in the section of the 4-dimensional anti-de Sitter universe, due to the metric signature, the Euclidean action will gain an extra negative sign in the continuation. This is
also supported by cosmological implications. The $R$ term in the actions can be decomposed into $R_7 - R_4$, where $R_7$ and $R_4$ are the scalar curvatures for the two factor spaces with the positive-definite metric signatures. The negative sign in front of $R_4$ is required so that the perturbation modes of the gravitational field in the $S_4$ background would take the minimum excitation state allowed by the Heisenberg uncertainty principle [8]. The perturbation modes are the origin for the structure of the Lorentzian universe in both the closed and open models. By the same argument, if we consider 7-dimensional factor space as our macroscopic spacetime, then one has to turn the sign around, as the analytic continuation has taken care of automatically.

The Euclidean action $I$ of the $AdS_4 \times S_7$ space can be calculated

$$I = \frac{1}{3} \kappa^2 V_7 V_4,$$

where the volume $V_7$ ($V_4$) of $S_7$ ($S_4$) is $\pi^7 r_7^7 / 3$ ($8 \pi^2 r_4^4 / 3$).

The field equation (6) is derived from the action (1) for the condition that the tensor $A_{MNP}$ is given at the boundary. Therefore, if one uses the action (1) in the evaluation of the wave function and the probability, then the induced metric and tensor $A$ on it must be the configuration of the wave function. The wave function is expressed by a path integral over all histories with the configuration as the only boundary. In deriving Eq. (11), one adjoins the histories in the summation of the wave function to their time reversals at the equator to form a manifold without boundary and discontinuity. If the configuration is given, then one obtains a constrained instanton for the stationary action solution. If one lifts the restriction at the equator, the stationary action solution is a regular instanton.

The induced metric and scalar field (if there is any) at the equator will remain intact under the reversal operation. However, for other fields, one has to be cautious. This occurs to our $A_{MNP}$ field. For convenience, we choose the following gauge potential

$$A = i\kappa \left( \frac{3}{A_4} \right)^2 \left( \sin \left( \sqrt{\frac{A_4}{3}} \tau \right) - \frac{1}{3} \sin^3 \left( \sqrt{\frac{A_4}{3}} \tau \right) + \frac{2}{3} \right) \sin^2 \chi \sin \theta d\chi \wedge d\theta \wedge d\phi,$$

where $\tau = i(t - \frac{\pi}{2})$, the gauge is chosen such that $A$ is regular at the south pole ($\tau = -\pi/2$) of the hemisphere ($0 \geq \tau \geq -\pi/2$). The gauge potential for the north hemisphere will take the same form with a negative sign in front of the constant term $\frac{2}{3}$. The sign change of the potential is consistent with the time reversal, as we mentioned earlier.
One can see that \( A_{MNP} \) is subjected to a discontinuity at the equator. Therefore, \( A_{MNP} \) is not allowed to be the argument for the instanton probability calculation in (11).

In order for the instanton approach to be valid, one has to use the canonical conjugate representation. One can make a Fourier transform of the wave function \( \Psi(h_{ij}, A_{123}) \) to get the wave function \( \Psi(h_{ij}, P^{123}) \),

\[
\Psi((h_{ij}, P^{123}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iA_{123}P^{123}} \Psi(h_{ij}, A_{123}).
\]

(15)

where \( P^{123} \) is the canonical momentum conjugate to \( A_{123} \), the only degree of freedom of the matter content under the minisuperspace ansatz

\[
P^{123} = \int_{\Sigma} \sqrt{-g_{11}} \left( -F^{0123} + \frac{\sqrt{2}}{3(4!)} F^{0123m_5...m_{11}} F_{m_5m_6m_7m_8} A_{m_9m_{10}m_{11}} \right) d^{10} x,
\]

(16)

where \( \Sigma \) denotes the 10-dimensional surface \( t = const. \). The quantum transition should occur at the equator \( \chi = \pi/2 \). However, the calculation at \( \tau = 0 \) or \( t = \pi/2 \) is simpler. Apparently, the result does not depend on the choice of the equator (this has been confirmed), since all equators are congruent for the round \( S_4 \) sphere. Strictly speaking, one cannot use \( A_{123} \) as the argument of the wave function without gauge condition. The only meaning of this argument is its flux at the surface. We shall not use the wave function \( \Psi(h_{ij}, A_{123}) \) anyway.

The discontinuity occurred at the equator instanton is thus avoided using the momentum representation, although it is due to the two distinct patches covering the whole sphere and can be glued through a gauge transformation. At the WKB level, the Fourier transform of the wave function is equivalent to the Legendre transform of the action. The Legendre transform has introduced an extra contribution \(-2A_{123}P^{123}\) to the Euclidean action, where all quantities are in the Euclidean version, and the factor 2 is due to the two sides of the equator in the adjoining. Then the effective action becomes

\[
I_{\text{effect}} = -\frac{2}{3}\kappa^2 V_7 V_4.
\]

(17)

If we consider the quantum transition to occur at the equator of \( S_7 \) instead, using the same argument, then it turns out that the corresponding canonical momentum using the time coordinate in \( S_7 \) vanishes, and the effective action should be the negative of (13), taking account of the sign of the factor \( \sqrt{g_{11}} \) in the action (12).

Since the creation probability is the exponential to the negative of the Euclidean action, the probability of creating a 7-dimensional macroscopic universe is exponentially suppressed relative to
that of the 4-dimensional case.

In the classical framework, the $S_7$ factor space in the Freund-Rubin model can be replaced by $S_2 \times S_5$, $S_2 \times S_2 \times S_3$, $S_4 \times S_3$ or other Einstein spaces. However, all these product spaces have volumes smaller than that of $S_7$. It would lead to an exponential suppression of the creation probability. Therefore, the internal space must be the round $S_7$ space.

Now we consider the Englert model [5]. Then, in addition to the components of the space $S_4$ in (7), the $F_{mnpq}$ components of the $S_7$ space can be non-vanishing and satisfying

$$F^{mnpq}_{;m} = \left[ \frac{\sqrt{2}}{(4!)\sqrt{g_7}} \right] \kappa \epsilon^{npqrst} F_{rstu}. \quad (18)$$

Two nontrivial solutions are

$$F_{mnpq} = \frac{4}{\kappa} \partial_{[m} S^\pm_{npq]}, \quad (19)$$

where $S^\pm_{mnp} = S^\pm_{[mnp]}$ are the two torsion tensors which can flatten the $S_7$ space in the Cartan-Schouten sense [9]

$$R^m_{n pq} \{ \Gamma^r_{st} + S^r_{st} \} = 0, \quad (20)$$

where $+ (-)$ is for the case $\kappa > 0$ ($\kappa < 0$). It is noted that $S_7$ is the only compact manifold to allow this, apart from group manifolds. The potential can be chosen as

$$A_{mnp} = \frac{1}{6\kappa} S^\pm_{mnp}. \quad (21)$$

The anisotropic cosmological constants are $\Lambda_7 = 3\kappa^2/4$ and $\Lambda_4 = -5\kappa^2/4$.

The tensor $A_{mnp}$ satisfies the gauge condition $A^{mnp}_{;p} = 0$. The following properties of the torsion tensor will be used in later calculations

$$S^{tr}_{\ m} S_{trn} = \frac{3}{4} \kappa^2 g_{mn}, \quad (22)$$

$$S^\pm_{mnp} = \mp \frac{2\sqrt{2}}{4!|\kappa|\sqrt{g_7}} \epsilon^{mnpqrst} S^\pm_{[rst,q]} \quad (23)$$

As in the Freund-Rubin model, before we take account of the Legendre term, the Euclidean action of the Englert $AdS_4 \times S_7$ space is

$$I = -\frac{1}{4} \kappa^2 V_7 V_4. \quad (24)$$

After including the Legendre term the effective action becomes

$$I_{\text{effect}} = -\frac{2}{3} \kappa^2 V_7 V_4. \quad (25)$$
It is surprising that after the long calculation, the effective action remains the same as that in the Freund-Rubin case.

If the quantum transition occurred at an equator of the $S_7$ space, one has to include the Legendre terms correspondingly. In contrast to the Freund-Rubin model, the canonical momenta do not vanish. Fortunately, due to the symmetries of the torsion tensor, the sum of the $C_6^3 = 20$ Legendre terms cancel exactly. The action is the negative of that in (24)

$$I_{\text{effect}} = \frac{1}{4} \kappa^2 V_7 V_4.$$  (26)

Again, comparing the results of (25) and (26), one can conclude that the universe we are living is most likely 4-dimensional.

In the Freund-Rubin model, the $S_7$ factor space can be replaced by a general Einstein space with the same cosmological constant $\Lambda_7$. The Awada-Duff-Pope model [6] is most interesting. The round 7-sphere is replaced by a squashed one, so that the $N = 8$ supersymmetry breaks down to $N = 1$. As far as the scenario of the quantum creation is concerned, the argument for the Freund-Rubin model remains intact, the only alternations are that the quantum transition should occur at one of its stationary equators and $V_7$ should be the volume of the squashed 7-sphere.

There is no supersymmetry in the Englert model [5]. Englert, Rooman and Spindel also discussed the model with a squashed $S_7$ factor space [7]. Here the $A$ components in the $S_7$ space are proportional to the torsion which renders the squashed sphere Ricci-flat, instead. It is believed that our conclusion should remain the same.

The right configuration for the wave function has also been chosen in the problem of quantum creation of magnetic and electric black holes [10]. If one considers the quantum creation of a general charged and rotating black holes, this point is even more critical. It is become so acute that unless the right configuration is used, one even cannot find a seed constrained instanton [11].

Many previous studies on dimensionality have essentially been restricted to the classical framework. For $d = 11$ supergravity, there is no way to discriminate the $d = 4$ and $d = 7$ macroscopic universes in the classical framework, as in other similar but more artificial models. This discrimination can be realized only through quantum cosmology.

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