THE PARKER MAGNETOSTATIC THEOREM

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ABSTRACT
We demonstrate the Parker Magnetostatic Theorem in terms of a small neighborhood in solution space containing continuous force-free magnetic fields in small deviations from the uniform field. These fields are embedded in a perfectly conducting fluid bounded by a pair of rigid plates where each field is anchored, taking the plates perpendicular to the uniform field. Those force-free fields obtainable from the uniform field by continuous magnetic footpoint displacements at the plates have field topologies that are shown to be a restricted subset of the field topologies similarly created without imposing the force-free equilibrium condition. The theorem then follows from the deduction that a continuous nonequilibrium field with a topology not in that subset must find a force-free state containing tangential discontinuities.

Key words: magnetic fields – magnetohydrodynamics (MHD) – Sun: corona

1. INTRODUCTION
We give a new demonstration of the Magnetostatic Theorem of Parker (1994) within the model of Parker (1972) where the theorem was first proposed. This theorem shows how high electrical conductivity may result in significant resistive dissipation in a hydromagnetic plasma through the spontaneous formation of electric current sheets (Parker 1979, 1994). As a fundamental explanation of the ubiquitous presence of how high electrical conductivity may result in significant resistive dissipation in a hydromagnetic plasma through the spontaneous formation of electric current sheets (Parker 1979, 1994).

2. STATEMENT OF THE MAGNETOSTATIC THEOREM
The induction equation
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})
\]
(1)
describes the evolution of a magnetic field \(\mathbf{B}\) in a perfectly conducting fluid moving with velocity \(\mathbf{v}\). Magnetic flux is frozen into the fluid so that the magnetic flux surfaces (MFSs) and field lines (FLs) move as fluid surfaces and lines of fluid particles. Taking \(\mathbf{v}\) to be continuous in space implies that these geometric surfaces and lines are continuously deformed. Topology as defined in standard mathematics textbooks deals with the properties of geometric objects that are invariantly meaningful under completely general, continuous transformations, including transformations that do not preserve any geometric metric. The fields at two instants of time are the continuous transformation of one into the other with certain topological properties to relate them. Ultimately, these properties are physically expressible as the conservation of magnetic flux across fluid surfaces (Low 2006a; Low & Janse 2009). Given any field \(\mathbf{B}\) rigidly anchored to the boundary of the domain, there is a set of topological properties that we denote formally as \(T(\mathbf{B})\) that is invariant in time.

Consider the infinite domain \(V : |z| < L_0\), \(L_0\) being a constant, in standard Cartesian coordinates. The boundaries \(z = \pm L_0\) are perfectly conducting rigid plates where \(v_z\) vanishes and the field has a fixed normal component,

\[
B_z|_z = \pm L_0 = F_\pm(x, y).
\]
(2)
The continuity of the tangential electric field at the plate, demanded by Maxwell’s equations, together with \(v_z = 0\), imposes the boundary condition

\[
v|_z = \pm L_0 = 0,
\]
(3)
assuming that \(F_\pm \neq 0\). We limit ourselves to fields that thread across \(V\) from one boundary plate to the other, that is, the case of subsystems of flux closed entirely within \(V\) is excluded. Then, the unchanging \(T(\mathbf{B})\) comprises the mapping of points on one plate to those on the other as pairs of boundary footpoints of FLs, the twist in all definable flux tubes, and the braids among flux tubes—topological features easily stated conceptually but difficult to describe in explicit mathematical terms (JLP).

With a tenuous atmosphere like the solar corona in mind, we adopt the approximation that the non-magnetic forces of the fluid are negligible compared to the Lorentz force the magnetic field is capable of producing. Then, the static equilibrium of the magnetized fluid is described by the force-free equations

\[
(\nabla \times \mathbf{B}) \times \mathbf{B} = 0,
\]
(4)
\[
\nabla \cdot \mathbf{B} = 0.
\]
(5)
Under the frozen-in condition, the field, whether in equilibrium or not, has a distinct identity defined by \( T(B) \). So, the construction of a force-free field in the above domain is not a classical boundary value problem, but a problem subject to both boundary condition (2) and a prescribed \( T(B) \). Any expression of this non-classical mathematical problem involves integral equations since topology is a global property (JLP). The Magnetostatic Theorem states that for most prescriptions of \( (F_1(x, y), T(B)) \), Equations (4) and (5) have no continuous solutions, which in classical analysis simply means no solutions.

The problem as posed is nevertheless physically meaningful, and it only means that its solutions must contain magnetic tangential discontinuities (TDs), the so-called weak solutions of mathematical analysis (Courant & Hilbert 1962; JLP). These are discontinuities of \( B \) across MFSs subject to the continuity of \( B^2 \) everywhere, so that the integral version of Equations (4) and (5) is satisfied. The magnetic TDs contain the current sheets and the continuity of \( B^2 \) ensures that these current-sheet surfaces are in macroscopic equilibrium.

Let us concentrate on the case of a force-free field described by the infinite series

\[
B(x, y, z) = \hat{z} + b(x, y, z) = \hat{z} + \sum_{n=1}^{\infty} \epsilon^n b_n(x, y, z),
\]

where \( \epsilon \) is a constant small parameter, assuming that such a series is analytic with a finite radius of convergence independent of spatial position (Rosner & Knobloch 1982). We may think of some solution space of force-free states in which this series picks out a neighborhood around the uniform field of unit strength. Some of these states are produced from the uniform field under the frozen-in condition in the manner first considered by Parker (1972). Starting with the uniform field, first relax the rigid anchoring of the field at \( z = \pm L_0 \), and then shuffle the magnetic footpoints by some continuous displacement to redistribute the boundary \( B_z \) and introduce magnetic twist into the flux tubes in \( V \). Now, take the boundary plates as rigid again, freezing the new \( B_z \) distribution and the new field topology created. The Magnetostatic Theorem then states that for most continuous footpoint displacements, the force-free equilibrium state of the deformed field must contain TDs. In the context of the infinite series (6), the claim is that there are footpoint displacements producing deformed fields whose force-free states are not found in the \( \epsilon \) neighborhood of the uniform field.

3. FIELD DEFORMATION BY BOUNDARY FOOTPOINT DISPLACEMENTS

We take the footpoint displacements to be continuous. Any discontinuous displacement at the boundary naturally produces a current sheet extending from the boundary into the domain. Let us use the acronym CMFD for continuous magnetic footpoint displacement. We treat the properties of CMFDs and relate them to a property of potential fields studied by Low (2007) and Janse & Low (2009), hereafter L07 and JL. This prepares us for the demonstration in Section 4.

3.1. Field Topology and Footpoint Motions

Take the \( z \) component of the induction Equation (1) for a velocity \( \mathbf{v} = \mathbf{u} = (u_x, u_y, 0) \) on some plane of constant \( z \),

\[
\frac{\partial B_z}{\partial t} + \nabla_\perp \cdot (B_z \mathbf{u}) = 0,
\]

where the subscript \( \perp \) indicates partial differentiations in the constant-\( z \) plane. If \( \mathbf{u} \) is given, Equation (7) poses an initial value problem for \( B_z(x, y, z, t) \) evolving from some initial distribution \( B_z(x, y, z, 0) \) at \( t = 0 \). If we know \( B_z(x, y, z, t) \) instead, Equation (7) poses a static problem at each instant of time to determine the velocity \( \mathbf{u} \).

Express the velocity in the general form

\[
\mathbf{u} = \nabla_\perp \phi + \nabla_\perp \times (\psi \hat{z}) = \nabla_\perp \phi + \nabla_\perp \psi \hat{z},
\]

separating the compressible irrotational and the incompressible rotational parts, described by \( \phi \) and \( \psi \), respectively. If we set \( \psi \equiv 0 \), the fluid merely compresses the magnetic flux threading across the constant-\( z \) plane without twisting the field. Similarly, if we set \( \phi \equiv 0 \), the fluid twists the field threading across that plane without compression. In the latter, \( B_z \) can still change with time because of the frozen-in flux transport in the constant-\( z \) plane. To illustrate, with \( \phi \equiv 0 \), set \( \nabla_\perp \cdot \mathbf{u} = 0 \) to obtain

\[
\frac{\partial B_z}{\partial t} + \nabla_\perp B_z = 0.
\]

Rewrite this equation as

\[
\frac{\partial B_z}{\partial t} + \nabla_\perp \cdot (B_z \nabla_\perp \phi) = 0.
\]

In this form, the equation points out the possibility of twisting the field without changing the \( B_z \) distribution by an incompressible rotational displacement with \( \phi \) being a strict function of the unchanging \( B_z \). In other words, \( \mathbf{u} \) is directed along contours of constant \( B_z \).

Consider a given \( B_z(x, y, z_1, t) \) on some fixed plane \( z = z_1 \), with \( B_z(x, y, z_1, 0) = H_0(x, y) \) at \( t = 0 \) and \( B_z(x, y, z_1, t_1) = H_1(x, y) \) at \( t = t_1 \), at two chosen times. A purely compressive irrotational flow always exists to produce \( B_z(x, y, z_1, t) \). We simply set \( \psi \equiv 0 \) and then solve

\[
\frac{\partial B_z}{\partial t} + \nabla_\perp \cdot (B_z \nabla_\perp \phi) = 0
\]

as an elliptic partial differential equation (PDE) for \( \phi \). The solution is unique under physically reasonable boundary conditions on \( \phi \). In other words, the evolution from \( B_z = H_0 \) at \( t = 0 \) to \( B_z = H_1 \) at \( t = t_1 \) can be achieved through a purely irrotational footprint displacement. No magnetic twist is introduced in this process.

The specific form of \( B_z(x, y, z_1, t) \) determines the precise form of the velocity \( \mathbf{u} = \nabla_\perp \phi \) in the period \( 0 < t < t_1 \). An infinite number of functions \( B_z(x, y, z_1, t) \) share a pair of initial and final states \( (H_0, H_1) \). Therefore, there is an infinite number of compressible, irrotational CMFDs taking \( B_z \) from an initial to the same final distribution. Now, let us distinguish between the Eulerian velocity of the fluid and the Lagrangian velocity of a fluid particle. The CMFDs are the Lagrangian displacements of actual fluid particles on the \( z = z_1 \) plane. Different Eulerian velocities may produce the same final \( B_z \) distribution, and their respective actual CMFDs are quite distinct. In other words, the footpoints on the \( z = z_1 \) plane are displaced continuously in different manners although all the different Eulerian velocities bring about the same final \( B_z \) distribution. It should be emphasized that this is possible only if the footpoints are free to move in the full two dimensionality of the \( z = z_1 \) plane.
Returning to the Parker two-plate domain with an initial uniform field, we draw two important conclusions. First, if the CMFDs at the boundaries \( z = \pm L_0 \) are incompressible, then it is obvious that the distribution \( B_z = 1 \) may remain unchanged at those boundaries. The field merely acquires a twist. The different terms in the series (6) must then satisfy \( b_{n,z} = 0 \) at \( z = \pm L_0 \) for all \( n \), the case treated in the study of Low (2010).

Second, if the CMFDs at the boundaries \( z = \pm L_0 \) are compressible but irrotational, then \( B_z \) at those boundaries ceases to be uniform. Thus, we pose the boundary conditions

\[
b_{n,z} \big|_{z = \pm L_0} = f_{n,\pm}(x, y),
\]

for \( n = 1, 2, 3, \ldots \), where \( f_{n,\pm} \) are prescribed. In other words, we have the expansion

\[
F_{\pm}(x, y) = 1 + \sum_{n=1}^{\infty} e^n f_{n,\pm}(x, y)
\]

to describe the final \( B_z \) at the two boundaries produced by an imposed CMFD. Until now, we had simply treated \( \epsilon \) as a constant parameter without attributing a physical origin to it. For compressible irrotational CMFDs, \( \epsilon \) has the physical meaning as the expansion parameter describing the imposed CMFD and the departure of the boundary distribution of \( B_z \) from uniformity produced by that CMFD.

When we displace the footpoints of an initial uniform field at both plates with a rotational motion, i.e., \( \psi \neq 0 \), it does not mean that we would have twisted the field. It is the relative motion of the pair of footpoints of an FL that determines whether a twist has been built into the field. To avoid this complication, we henceforth consider deforming the uniform field with the imposed CMFDs taken only on \( z = -L_0 \), that is, the magnetic footpoints are not displaced on \( z = L_0 \) where there is no change in the \( B_z \) distribution. There is no loss of generality in the principal physical point we wish to make. A purely rotational \( (\psi \neq 0, \phi = 0) \) CMFD twists the uniform field, whereas a purely compressive, irrotational \( (\psi = 0, \phi \neq 0) \) CMFD introduces no twist on all scales.

What is meant by an untwisted field is unambiguously defined within this context, namely, a field deformed from the uniform field with a purely compressive, irrotational CMFD imposed on one plate. Our analysis shows that an infinite number of compressive, irrotational CMFDs exist to take the uniform field to untwisted fields, all sharing the same \( B_z \) distributions at the two plates but not the same topology. These untwisted fields anchored at the two plates are not topologically equivalent by virtue of their distinct footprint connectivities.

3.2. The Janse–Low Result

Such untwisted fields were encountered in the L07 and JL studies, which treated the domain between two concentric spheres and the upright cylindrical domain, respectively. Let us briefly summarize the JL result.

Consider a potential field \( B_{pot}(L) \) in the form of a unidirectional flux entering an upright cylinder of length \( L \) and radius \( R_0 \), at one cylinder end \( z = -L \) and exiting at the other end \( z = L \). This potential field is tangential at \( R = R_0 \), using the usual cylindrical coordinates. The boundary flux distributions \( B_z \), positive definite in this case, at \( z = \pm L \) define this potential field uniquely. By integrating

\[
\frac{dx}{B_z(x, y, z)} = \frac{dy}{B_z(x, y, z)} = \frac{dz}{B_z(x, y, z)},
\]

with \( B = B_{pot} \), we obtain each of its FLs described by \( x(z) \) and \( y(z) \), two coordinates of a point along the FL in terms of the third \( z \) as the independent variable. These FLs define the map \( M [B_{pot}(L)] : (x_B, y_B, -L) \to (x_T, y_T, L) \):

\[
x_T = x_B + \int_{-L}^{z_L} B_z [x(z), y(z), z] dz,
\]

\[
y_T = y_B + \int_{-L}^{z_L} B_z [x(z), y(z), z] dz,
\]

with \( B = B_{pot} \), relating a footpoint \( (x_B, y_B, -L) \) to the footpoint \( (x_T, y_T, L) \). It was discovered in JL that, with \( B_z \) fixed at \( z = \pm L \), the map \( M [B_{pot}(L)] \) is generally dependent on \( L \), the exceptions being cases containing special symmetries. These symmetries include the obvious case of axisymmetry as well as fields that are symmetric about the \( z = 0 \) plane, for example. This \( L \)-dependence of the footprint map is a 3D effect.

Consider continuously deforming a potential field \( B_{pot}(L_{1}) \), along with its domain \( L = L_{1} \neq L_{0} \), under the frozen-in condition into a non-potential field \( B_{af}(L_{1}; L_{0}) \) to fit into the domain \( L = L_{0} \), treating the cylinder ends as rigid conductors. Throughout our discussion, we assume no change in the cylinder radius as we increase or decrease the length \( 2L \) of the cylindrical domain. In this deformation, the footprint map \( M \) is a topological invariant. By the JL result, \( M [B_{pot}(L_{1})] \) and \( M [B_{pot}(L_{0})] \) are not the same, in general. Under the frozen-in condition with magnetic footpoints fixed at the boundary, there is generally no way for the deformed field \( B_{af}(L_{1}; L_{0}) \) to assume the potential state \( B_{pot}(L_{0}) \) because their footprint maps \( M \) are not the same. With its fixed footprint map \( M [B_{pot}(L_{1})] \), the deformed field must find some force-free state distinct from \( B_{af}(L_{0}) \) and the arguments in JL concluded that this force-free state must contain TDs. These properties carry over to the unbounded domain \( V : |z| < L_{0} \).

For our deductive analysis here, it is possible to be rigorously specific about a concept of an untwisted field without having to describe its topological properties explicitly. We call all potential fields, unique in their respective (simply connected) \( L \) domains, untwisted; see JLP for a discussion of complications found in multiply connected domains. This is based on the fact that the magnetic circulation vanishes for the potential field along any closed curve in the simply connected domain, a well-known property. For the field \( B_{pot}(L) = \tilde{z} + \mathbf{b} \), \( \mathbf{b} \ll 1 \), every flux tube extending from \( z = -L_0 \) to \( z = L_0 \) shows no circulation around the approximately straight tube with an axial flux conserved along it. If this field is deformed and kept within the regime \( |\mathbf{b}| \ll 1 \), magnetic circulations, quantities not invariant under the frozen-in condition, can be created in opposite signs along the flux tube. Conversely, the opposite circulations found at any one time along any flux tube can be removed by a suitable mutual cancellation of all of them along the tube. One thing that is forbidden is to deform any of these potential fields to end up with a positive or negative definite circulation along the tube.

With this definition of untwistedness, we conclude that the \( L = L_{0} \) domain contains an infinite number of untwisted fields.
sharing the same boundary flux distribution, namely, fields that are deformed from the potential fields \( B_{\text{pot}}(L) \) originally occupying domains with \( L \neq L_0 \) but deformed to fit into the \( L = L_0 \) domain. The claim is that these deformed fields must find force-free states containing TDs.

The deformation of the uniform field in the fixed \( L = L_1 \neq L_0 \) domain into a potential field of the form \( B_{\text{pot}}(L_1) = \hat{z} + b \) by CMFD does not involve rotational displacements because the magnetic circulation of the potential field vanishes for all closed curves. As a separate exercise, the same particular irrotational CMFD can also be applied to the boundary curves. As a separate exercise, the same particular irrotational CMFD does not involve rotational displacements because the particular untwisted field \( L \) has the same footpoint map. This construction produces an irrotational CMFD. The particular untwisted field \( L \) can be continuously deformed into the unique potential field \( B_{\text{pot}}(L_0) \) to be found in the \( L = L_0 \) domain. All the other untwisted fields with \( M(L) \neq M(L_0) \) cannot be similarly deformed into \( B_{\text{pot}}(L_0) \).

4. THE \( \epsilon \) NEIGHBORHOOD OF THE UNIFORM FIELD

We first solve the magnetostatic equations for the field given by \( \epsilon \)-series (6) without consideration of field topology, and then examine the implications of such analytic solutions in terms of the field topologies they exhibit.

4.1. Perturbational Analysis

Rewrite the force-free Equation (4) in the form

\[
2 (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla B^2 = 0,
\]

and substitute for the field given by series (6) to first and second orders,

\[
\frac{\partial}{\partial z} b_1 - \nabla b_{1,z} = 0,
\]

\[
2 (\mathbf{b}_1 \cdot \nabla) \mathbf{b}_1 + 2 \frac{\partial}{\partial z} \mathbf{b}_2 - \nabla (b_1^2 + 2b_{2,z}) = 0,
\]

subject to the solenoidal conditions

\[
\nabla \cdot \mathbf{b}_1 = \nabla \cdot \mathbf{b}_2 = 0,
\]

and boundary conditions (12) on \( b_{1,z} \) and \( b_{2,z} \). No other boundary conditions are needed in this consideration.

The first-order Equation (17) subject to the solenoidal condition implies

\[
\nabla^2 b_{1,z} = 0,
\]

which has a unique solution satisfying the prescribed boundary condition \( b_{1,z} = f_{1,z} \) at \( z = \pm L_0 \). Here and elsewhere, we assume that all departures from the uniform field vanish in the far region \( x^2 + y^2 \to \infty \). The solenoidal condition shows that the first-order magnetic field has the form

\[
\mathbf{b}_1 = \nabla \Phi_1(x, y, z) + \mathbf{a}_1(x, y),
\]

where \( \mathbf{a}_1(x, y) = (a_{1,1}, a_{1,1}, 0) \) is an arbitrary solenoidal vector produced by an incompressible rotational CMFD. In contrast, \( \nabla \Phi_1 \) is the unique potential field produced by a compressible irrotational CMFD. The \( z \) component of this potential field accounts for the departure \( b_{1,z} \) from the initially uniform field at the boundaries. If there is no rotational displacement of footpoint, \( \mathbf{a}_1 \equiv 0 \) and \( \mathbf{b}_1 \) is a (untwisted) potential field.

First, note that the \( z \) component of Equation (18) is an identity, posing no condition, whereas the other two components take the forms

\[
\frac{\partial b_{2,x}}{\partial z} - \frac{\partial b_{2,z}}{\partial x} - \omega \left[ \frac{\partial \Phi_1}{\partial y} + a_{1,y} \right] = 0,
\]

\[
\frac{\partial b_{2,y}}{\partial z} - \frac{\partial b_{2,z}}{\partial y} + \omega \left[ \frac{\partial \Phi_1}{\partial x} + a_{1,x} \right] = 0,
\]

where we have expressed \( \mathbf{b}_1 \) using Equation (21) and introduced \( \omega = \frac{\partial a_{1,z}}{\partial x} - \frac{\partial a_{1,x}}{\partial z} \). Eliminating \( b_{2,x} \) and \( b_{2,y} \) by using the solenoidal condition in favor of \( b_{2,z} \) gives the Poisson equation

\[
\nabla^2 b_{2,z} + \frac{\partial}{\partial x} \left[ \omega \left( \frac{\partial \Phi_1}{\partial y} + a_{1,y} \right) \right] - \frac{\partial}{\partial y} \left[ \omega \left( \frac{\partial \Phi_1}{\partial x} + a_{1,x} \right) \right] = 0.
\]

The solution \( b_{2,z} \), subject to its boundary values at \( z = \pm L_0 \), is unique. This solution, substituted into Equations (22), determines the \( z \)-dependent parts of \( b_{2,x} \) and \( b_{2,y} \) uniquely. We are free to linearly superpose this second-order magnetic field, denoted by \( \mathbf{b}_2(x, y, z) \), with an arbitrary solenoidal vector \( \mathbf{a}_2(x, y) = (a_{2,1}, a_{2,2}, 0) \), and still have a valid solution. So, we express the solution as

\[
\mathbf{b}_2 = \mathbf{b}_2^0(x, y, z) + \mathbf{a}_2(x, y).
\]

The free vectors \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) describe the twist to be found in the net equilibrium field (6) up to second order.

For example, if \( \mathbf{a}_1 \neq 0 \), this solenoidal field by itself has FLs that are the same closed curves on all constant-\( z \) planes. This property follows from the boundary condition that the perturbations all vanish at \( x^2 + y^2 \to \infty \). The implied magnetic circulations around the principal flux in the \( z \)-direction of any flux tube cannot mutually cancel. Thus, to first order in \( \epsilon \) the FLs wind with a fixed sense about flux tubes directed in the \( z \)-direction. An untwisted field corresponds, at first order in \( \epsilon \), to \( \mathbf{a}_1 \equiv 0 \), in which case \( \mathbf{b}_1 = -\nabla \Phi_1 \), a pure potential field. For such a field, \( \omega \equiv 0 \) and \( b_{2,z} \) is a potential by Equation (23), so that \( \mathbf{b}_2^0(x, y, z) \) is a potential field. In this case, the free, \( z \)-independent vector \( \mathbf{a}_2 \) imposes a magnetic twist at the second order in \( \epsilon \). Thus, if there is no twist, we must set \( \mathbf{a}_2 \equiv 0 \). The magnetic field \( \mathbf{B} = 1 + \epsilon \mathbf{b}_1 + \epsilon^2 \mathbf{b}_2 \) is thus a potential to second order. By induction, the perturbations to all orders are all potentials and so the net field is potential.

4.2. Footpoint Displacements

Substituting an \( \epsilon \)-series solution into Equation (15) defines the footpoint map \( M \) of the points on \( z = -L_0 \) to the points on \( z = L_0 \). Expanding in \( \epsilon \), we obtain the formal expressions:

\[
x_T = x_B + \sum_{n=1}^{\infty} \epsilon^n \Delta x_n(x_B, y_B),
\]

\[
y_T = y_B + \sum_{n=1}^{\infty} \epsilon^n \Delta y_n(x_B, y_B),
\]

where \( \Delta x_n \) and \( \Delta y_n \) are complicated functions defined through the definite integrals with respect to \( z \), straightforward to work out from Equation (15). The topological magnetostatic problem of Parker (1972, 1994) is posed by giving the footpoint
map \( M \) as a part of the topology specified for a force-free field to be constructed. In the \( \epsilon \)-series formulation, this means prescribing the functional forms of \( (\Delta \varphi(x_R, y_R), \Delta \psi(x_R, y_R)) \) among the constraints to determine the fields \( b_\epsilon \). These forms are complicated definite integrals of the components of \( b_\epsilon \) as the unknowns of the problem. We are thus faced with a generally intractable problem that couples the force-free equations as PDEs with these functional forms as integral equations. Fortunately, we can derive a clear conclusion with the two particular topological problems presented below.

Let us first address how the topology of a magnetic field can be fully specified in an operational manner using the induction Equation (1) (JLP). Given an anchored continuous field \( B_R(x, y, z) \) expressed explicitly as a vector function of space, let us denote its topology by \( T(B_R) \). By giving \( B_R(x, y, z) \), we have prescribed a topology \( T \) not in explicit terms but by its realization in the field \( B_R(x, y, z) \) as a reference field, using the subscript “\( R \)” to denote this topological role. Any property included in \( T \) can be computed from \( B_R(x, y, z) \) according to its definition. In particular, the footpoint map \( M \) as one of the properties that define \( T \) can be computed from \( B_R \) using Equation (15). Once \( T \) is specified, an infinite set \( B(T) \) of all continuous fields possessing the same topology \( T \) can be constructed, each one obtained from the deformation of the given \( B_R \) by a continuous velocity \( \mathbf{v} \), subject to boundary conditions (2) and (3). These fields, including \( B_R \), are not necessarily force free. On the other hand, in anticipation of the Magnetostatic Theorem we are demonstrating, it cannot be assumed that \( B(T) \) contains one or more fields that are actually force free. By definition, \( T(B) \equiv T(B_R) \) for every \( B \in B(T) \). Therefore, any member of the set \( B(T) \) may be used as an alternative to the given reference field \( B_R \) to define \( T \) and generate the same set \( B(T) \). For our purpose below, this construction allows to describe field topology without needing a mathematical language for making that description fully explicit.

For the first problem, consider the set \( B(T) \), where \( T = T(B_{pot}) \) is realized in a potential field in \( V : |z| < L_0 \) of the form \( B_{pot} = z + c b_{pot} \), being a given potential field independent of \( \epsilon \). Take \( c \) to be small such that \( |c b_{pot}| \ll 1 \). What we have is a series solution with \( b_1 = b_{pot} \), and \( b_\epsilon = 0 \), for \( n \geq 2 \). In other words, series (6) only has a finite number of terms—just two terms. The \( z \) component of \( b_{pot} \) defines the boundary values \( F_\epsilon(x, y) \) specified by boundary conditions (2) at \( z = \pm L_0 \). All fields belonging to \( B(T) \) satisfy these boundary conditions. Equations (25), giving the footpoint map \( M \), remain a pair of infinite series. Using \( b_{pot} \), each term of these two infinite series can be computed explicitly by evaluating the integrals defining the FLs. All fields belonging to \( B(T) \) have the same footpoint map \( M \). Turning the problem around, let us pose the topological problem of constructing a force-free field satisfying boundary conditions (2) and possessing topology \( T \). In particular, the field must have the footpoint map \( M \). Then, the potential field \( B = z + c b_{pot} \) is a solution to the problem. This is a case of a deformed field whose prescribed field topology allows it to attain a continuous force-free state, namely, \( b_{pot} \). The topology \( T \) does not have to be directly related to \( b_{pot} \), because any non-force-free member of the set \( B(T) \) may be used as a reference field to define the same \( T \). This point is worth of mention. If we are given an arbitrary reference field \( B_R \) to define \( T \) and the set \( B(T) \), without the knowledge that the set \( B(T) \) contains a continuous force-free field, to answer the question of whether this is the case requires solving the above formidable integro-PDEs.

For the second problem, we make use of the JL result reviewed in Section 3.1. Consider the set \( B(T) \) with \( T = T(B_{df}) \), where \( B_{df} = z + c b_{df} \), not in equilibrium, is given in \( V(L_0) : |z| < L_0 \) obtained by the following construction. Take a given potential field \( B_{pot,L} = z + c b_{pot,L} \) in a different domain \( V(L) : |z| < L \) subject to prescribed boundary conditions

\[
 B|_z = \pm L = F_\epsilon(x, y). \tag{26}
\]

Compute the functions \( \Delta \varphi(x_R, y_R) \) and \( \Delta \psi(x_R, y_R) \) that define the footpoint map \( M(L) \) of this field in \( V(L) \). This field is untwisted and can be deformed continuously from the uniform field \( B = z \) by some suitable, continuous footpoint displacement at \( z = \pm L \), leaving the footpoints fixed at \( z = \pm L_0 \). This deformation treats the two boundary plates as rigid so that boundary conditions (26) transform into boundary conditions (2) applied to \( z = \pm L_0 \). Under the frozen-in condition, the field topology \( T \) is unchanged and the field remains untwisted. By the analysis of Section 3.2, the field \( B_{df} \) in \( V(L_0) \) can also be created from the uniform field \( B = z \) by holding the footpoints fixed at \( z = \pm L_0 \) and applying footpoint displacement CFMD(\( L \)) to \( z = \pm -L_0 \). The important point here is that the invariance of \( T \) in the deformation \( [B_{pot,L}, V(L)] \to [B_{df}, V(L_0)] \) means that the footpoint map \( M(L) \) remains unchanged. In other words, if we use \( B_{df} \) in \( V(L_0) \) to compute its footpoint map, the same map \( M(L) \) is obtained from applying Equations (25). Now, there is a unique potential field \( B_{pot,L} \) in \( V(L_0) \) defined by the boundary conditions (2). This field has a footpoint map \( M(L_0) \) obtained from applying Equations (25) that, in general, is not the same as the footpoint map \( M(L) \) of \( B_{df} \), the JL result. By demanding that the deformed field \( B_{df} \), which is untwisted, be further deformed under the frozen-in condition to be force free, we conclude from our perturbational analysis in Section 4.1 that it cannot find a continuous solution in the \( \epsilon \) neighborhood of the uniform field. If it could, that force-free state has to be the unique potential field \( B_{pot} \), ruled out by \( M(L) \neq M(L_0) \). This is the example of the Parker Magnetostatic Theorem we set out to construct.

5. DISCUSSION

We have returned to the model of Parker (1972) to demonstrate the Magnetostatic Theorem. This theorem is described in generality in Parker (1994). The theory is concerned with the infinite set of field topologies \( T_{fff} \) belonging to continuous force-free fields admissible in a physical system. This set is for most 3D systems a true subset of the set \( T_{fff} \) of field topologies belonging to continuous fields, which are not required to be force free, admissible in the same system, that is, \( T_{fff} \subset T_{fff} \). Therefore, a continuous nonequilibrium field with a topology not found in the subset \( T_{fff} \) must develop TDs as the only way to become force free under the frozen-in condition. Let us briefly recall what is accomplished in Parker (1972) and clarify the debate over that work in order to put our new demonstration in a broader physical context.

The Magnetostatic Theorem is a complete departure from the classical analysis of boundary value problems. To construct a physically specific force-free field \( B \) satisfying Equations (4)
and (5), prescribing the boundary flux distribution $F_{\pm}$ is an essential first step in identifying the field. This prescription tells us where the field enters and leaves the domain. Other boundary conditions may be prescribed to formulate a mathematically complete boundary value problem. Such an approach is useful depending on the physical question being addressed. In the case of the topological magnetostatic problem of Parker (1972, 1994), no other boundary condition is posed and the field is identified with a topology $T(B)$. This identification comes from the physics of the perfectly conducting fluid in which $B$ is embedded. We need to think physically here. Since the $B$ preserves $T(B)$ under the frozen-in condition in its evolution into the force-free state we are constructing, this state as a solution to Equations (4) and (5) is subject to a prescribed $T(B)$. The Magnetostatic Theorem then simply states that for most prescriptions of $(F_{\pm}, T(B))$, the force-free field we wish to construct must contain current sheets or magnetic TDs.

The physics of a perfectly conducting fluid admits magnetic TDs as a natural fluid structure, which is no more unusual than to allow for a TD in the velocity of an inviscid fluid. In an ideal fluid, perfectly conducting and inviscid, two volumes of fluids embedding separate magnetic flux systems can slip discontinuously along an MFS. This process is basic in the hydromagnetic behaviors of the fluid. The Magnetostatic Theorem isolates an aspect of it, stating that a field of a prescribed complex 3D topology generally cannot attain equilibrium if it remains continuous. Physically, the equilibrium must exist. In the context of force-free fields, this equilibrium is a state of minimum, total magnetic energy subject to a fixed $(F_{\pm}, T(B))$. Once it is recognized that magnetic TDs are an integral part of the physics, this theorem tells us that generally the available minimum-energy state is one containing discontinuities.

In mathematical terms, the prescribed pair $(F_{\pm}, T(B))$ is not compatible with the force-free Equations (4) and (5) applied at all points in the domain. The incompatibility can be analytically traced to a restriction on the admissible topology of a field when the free functions defining the field are subject to the equilibrium equations (Parker 1989a, 1989b, 1994). The physics can compromise neither $F_{\pm}$ nor $T(B)$, leaving the (fully admissible) process of TD formation to bring about the minimum-energy state. The topological problem of Parker may be characterized in the following manner. Stripped of all other dynamical effects, except the frozen-in condition, the inevitability for the fluid to form TDs is recovered when the field is brought to its minimum-energy state consistent with the prescribed $(F_{\pm}, T(B))$. This physical point of view leads naturally to a generalization of the magnetostatic theorem to include a steady hydromagnetic flow (Tsinganos 1982).

The time-dependent hydromagnetic equations are commonly used only in their differential forms. When TDs are present, these equations are only a complement to another set of time-dependent equations describing the dynamics of TDs as moving surfaces. In this case, the TDs include hydromagnetic shocks as surfaces of discontinuities moving through the continuous part of the fluid. Similarly for the fluid in a minimum-energy state, the force-free Equations (4) and (5) complement a corresponding set of jump conditions for the TDs present in the fluid. For the force-free field, the latter are the integral versions of these force-free PDEs, ensuring the solenoidal condition and continuity of the magnetic pressure required for macroscopic force balance.

To appreciate the Parker (1972) model, it should be noted from the outset that the equilibrium states in small departures from the uniform field generally carry a variation with $z$, in the direction perpendicular to the two plates. This is pointed out in the Parker (1972) investigation. To arrive at a tractable problem, a distinction is then made between the $z$ variations due to the stresses that built up in a boundary layer at $z = \pm L_0$ and the variations that extend into the interior of the domain. The former extends only a boundary-layer thickness into the interior, as first explicitly demonstrated by Zweibel & Li (1987). To avoid the complexity of the boundary layer, Parker develops an $\epsilon$-expansion for footpoint displacements on a scale that is small compared to the separation $2L_0$ between the two plates, and examines it in the limit of that small scale going to zero, in order to concentrate on the variations in the larger scales. That analysis gives the result that these variations are subject to an invariance in the $z$-direction as a pre-requisite for equilibrium. Such a geometric constraint is too strong to accommodate a braided bundle of flux tubes locked between the plates and created by a sequence of topologically uncorrelated footpoint displacements, applied in succession on one plate. This result led Parker (1972) to propose that the topology of the field deformed in this arbitrary manner is incompatible with continuous equilibrium. The field must thus form TDs as a way of achieving equilibrium.

Over the years, the condition $\frac{\partial}{\partial z} \equiv 0$ that first appeared in Parker (1972) has been re-stated in different versions. It seems that the debate over whether this condition is a valid prerequisite for continuous equilibrium has become confused as to the specific physical context this condition is being applied. It is not our purpose to review this debate here. The reader is referred to Craig & Sneyd (2005) and Low (2010) for an example of that debate. To relate to our new results here, we clarify a particular source of the underlying confusion of the debate.

The condition $\frac{\partial}{\partial z} \equiv 0$ as a pre-requisite for equilibrium is obtained by Parker (1972) in the context of taking the $\epsilon$-series in the limit of $\epsilon \to 0$. The Magnetostatic Theorem is not about an unqualified absence of continuous equilibrium varying with $z$ for the two-plate problem formulated in Parker (1972). That such continuously $z$-varying equilibrium solutions do exist is a trivial point to prove. The simplest equilibrium solutions of this type are the potential fields like the ones presented in this paper. Hence, there is no need for the published complex magnetostatic solutions to make this point such as found in Craig & Sneyd (2005) and Bogoyavlenskij (2000a, 2000b, and reply by Parker 2000). As our calculations have shown, the Magnetostatic Theorem is even inherent in the class of $z$-varying continuous potential fields anchored between the two plates. This is an interesting new result that makes use of the topologically untwisted fields of JL.

The topological problem posed by the untwisted field is simpler than the one posed by twisted fields. In a typical 3D system, the untwisted continuous fields, which are not in equilibrium, define an infinite set $T_{all}^{U}$ of field topologies, with the superscript "U" indicating the restriction to untwisted topologies. These topologies are distinguished by their different footprint maps. This is a property of 3D fields. To emphasize this point, the two-dimensional fields with no neutral points in a cylinder under axisymmetry all share a common footprint map; see JL. Now, there is only one force-free state for an untwisted field in the 3D system, the unique potential field. Hence, the set of untwisted topologies $T_{all}^{U}$ realizable in a continuous force-free state is a set with a single member, and, obviously, $T_{all}^{U} \subset T_{all}$, thus demonstrating the theorem.
A similar demonstration was made in the L07 and JL studies, but those studies depend on arguing around topological ideas that are extremely difficult to pin down mathematically. In the present paper, restricting attention to the force-free fields in the $\epsilon$ neighborhood of the uniform field simplifies the general problem. It is the ordering of the terms by magnitudes in the series expansion, beginning with the dominant uniform field, that allows a clear sorting of the untwisted fields from the twisted ones. Basic to this analysis is the property that different footpoint connectivities can be created by footpoint displacements imposed on a uniform field, without twisting the field but all leading to the same boundary flux distribution.

Counter arguments to the JL result, like those recently presented by Aly & Amari (2010), need to be reexamined in light of this new result. Aly and Amari probe and question the JL results in terms of the concepts of twist and helicity in common use. It seems clear that these concepts are not yet fully understood as these and other recent works have revealed (JLP). There is much work to be done to refine and redefine these concepts, both for their interesting physics and application to the Magnetostatic Theorem. We mention the following open questions that qualify the new result. The fact that we could not find a continuous solution in the $\epsilon$ neighborhood of the uniform field does not preclude the existence of the desired continuous solutions represented by some other small-parameter expansions, such as demonstrated by van Ballegooijen (1985); see also further discussions of this important work in Low (1990, 2010), Parker (1994), and JLP. Another possibility is that, despite the small-parameter nature of the CMFDs expressed by Equations (25), continuous force-free solutions may exist in $V$ that do match the prescribed footpoint displacements but can only be attained via deformations of finite amplitudes inside the domain. These questions should motivate future work, but they should not detract from the central feature of our result: the continuous force-free fields in the $\epsilon$ neighborhood of the uniform field are topologically of a more restricted variety than the non-force-free fields created by continuous footpoint displacements. The Parker Magnetostatic Theorem makes the fundamental point that this hydromagnetic feature is general.

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