What is the Schwarzschild radius of a quantum mechanical particle?

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Abstract A localised particle in Quantum Mechanics is described by a wave packet in position space, regardless of its energy. However, from the point of view of General Relativity, if the particle’s energy density exceeds a certain threshold, it should be a black hole. In order to combine these two pictures, we introduce a horizon wave-function determined by the position wave-function, which yields the probability that the particle is a black hole. The existence of a (fuzzy) minimum mass for black holes naturally follows, and we also show that our construction entails an effective Generalised Uncertainty Principle simply obtained by adding the uncertainties coming from the two wave-functions.

1 The Schwarzschild link

In natural units, with $c = 1$ (and $\hbar = \ell_p m_p$), the Newton constant is given by

$$G_N = \ell_p / m_p ,$$

where $\ell_p$ and $m_p$ are the Planck length and mass, respectively, and converts mass (or energy) into length. This naive observation stands behind Thorne’s hoop conjecture [1]: A black hole forms when the impact parameter $b$ of two colliding objects is shorter than the Schwarzschild gravitational radius of the system, that is for

$$R_H \equiv 2 \ell_p \frac{E}{m_p} \gtrsim b ,$$

where $E$ is total energy in the centre-of-mass frame. The emergence of the Schwarzschild radius is indeed easy to understand in a spherically symmetric space-time,
where the metric $g_{\mu\nu}$ can be written as
\[
ds^2 = g_{ij} \, dx^i \, dx^j + r^2 (x') \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\] (3)
with $x' = (x^1, x^2)$ coordinates on surfaces of constant angles $\theta$ and $\phi$. The location of a trapping horizon, a sphere where the escape velocity equals the speed of light, is then determined by
\[
0 = g^{ij} \nabla_i r \nabla_j r = 1 - \frac{2M}{r},
\] (4)
where $\nabla_i r$ is the covector perpendicular to surfaces of constant area $A = 4\pi r^2$. The active gravitational (or Misner-Sharp) mass $M$ represents the total energy enclosed within a sphere of radius $r$, and, if we set $x^1 = t$ and $x^2 = r$, is explicitly given by
\[
M(t, r) = \frac{4\pi \ell_p}{3m_p} \int_0^r \rho(t, \bar{r}) \bar{r}^2 \, d\bar{r},
\] (5)
where $\rho = \rho(x')$ is the matter density. It is usually very difficult to follow the dynamics of a given matter distribution and find surfaces satisfying Eq. (4), but an horizon exists if there are values of $r$ such that $R_H = 2M(t, r) > r$, which is a mathematical reformulation of the hoop conjecture (2).

2 Horizon wave-function

The hoop conjecture was formulated having in mind black holes of astrophysical size [2], for which a classical metric and horizon structure are reasonably safe concepts. However, for elementary particles quantum effects may not be neglected [3]. Consider a spin-less point-like source of mass $m$, whose Schwarzschild radius is given by $R_H$ in Eq. (2) with $E = m$. The Heisenberg principle introduces an uncertainty in its spatial localisation, of the order of the Compton-de Broglie length, $\lambda_m \simeq \ell_p m_p / m$. Assuming quantum physics is a more refined description of reality implies that $R_H$ only makes sense if it is larger than $\lambda_m$,
\[
R_H \gtrsim \lambda_m \Rightarrow m \gtrsim m_p \quad (\text{or } M \gtrsim \ell_p).
\] (6)
Note that this argument employs the flat space Compton length, and it is likely that the particle’s self-gravity will affect it. However, we can still assume the condition (6) holds as an order of magnitude estimate, hence black holes can only exist with mass (much) larger than the Planck scale.

We are thus facing a deeply conceptual challenge: how can we describe systems containing both quantum mechanical particles and classical horizons? For this purpose, we shall define a horizon wave-function that can be associated with any
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localized quantum mechanical particle [4], and that will put on quantitative grounds
the condition (6) that distinguishes black holes from regular particles.

The quantum mechanical state representing and object, which is both located in space and at rest in the chosen reference frame, must be described by a wavefunction \( \psi_S \in L^2(\mathbb{R}^3) \), which can be decomposed into energy eigenstates,

\[
| \psi_S \rangle = \sum_E C(E) | \psi_E \rangle ,
\]

(7)

where the sum represents the spectral decomposition in Hamiltonian eigenmodes,

\[
\hat{H} | \psi_E \rangle = E | \psi_E \rangle ,
\]

(8)

and \( H \) can be specified depending on the model we wish to consider. If we also assume the state is spherically symmetric, we can introduce a Schwarzschild radius \( R_H = R_H(E) \) associated to each component \( \psi_E \) of energy \( E \), by inverting Eq. (2), and define the (unnormalised) horizon wave-function as

\[
\psi_H(R_H) = C \left( E = m_p \frac{R_H}{2 \ell_p} \right) .
\]

(9)

The normalisation is finally fixed by employing the inner product

\[
\langle \psi_H | \phi_H \rangle = 4 \pi \int_0^\infty \psi_H^* (R_H) \phi_H (R_H) R_H^2 dR_H .
\]

(10)

We interpret the normalised wave-function \( \psi_H \) as yielding the probability that we would detect a horizon of areal radius \( r = R_H \) associated with the particle in the quantum state \( \psi_S \). Such a horizon is necessarily “fuzzy”, like the particle’s position, unless the width of \( \psi_H \) is negligibly small. Moreover, the probability density that the particle lies inside its own horizon of radius \( r = R_H \) will be given by

\[
P_<(r < R_H) = P_S(r < R_H) P_H(R_H) ,
\]

(11)

where \( P_S(r < R_H) = 4 \pi \int_0^{R_H} |\psi_S(r)|^2 r^2 dr \) is the probability that the particle is inside the sphere of radius \( r = R_H \), and \( P_H(R_H) = 4 \pi R_H^2 |\psi_H(R_H)|^2 \) is the probability that the horizon is located on the sphere of radius \( r = R_H \). Finally, by integrating (11) over all possible values of the radius,

\[
P_{BH} = \int_0^\infty P_<(r < R_H) dR_H ,
\]

(12)

the probability that the particle is a black hole will be obtained.
2.1 Gaussian particle

The above construction can be straightforwardly applied to a particle described by the Gaussian wave-function

$$\psi_S(r) = \frac{e^{-r^2/\ell^2}}{\ell^{3/2} \pi^{3/4}},$$  \hspace{1cm} (13)

where the width $\ell \sim \lambda_m$. This wave-function in position space corresponds to the momentum space wave-function

$$\psi_S(p) = \frac{e^{-p^2/2\Delta^2}}{\Delta^{3/2} \pi^{3/4}},$$  \hspace{1cm} (14)

where $p^2 = p \cdot p$ and $\Delta = \hbar/\ell = m_p \ell_p/\ell$. For the energy of the particle, we simply assume the relativistic mass-shell relation in flat space, $E^2 = p^2 + m^2$, and we easily obtain the normalised horizon wave-function

$$\psi_H(R_H) = \frac{\ell^{3/2} e^{-R_H^2/4\ell^2}}{2\sqrt{\pi \ell^6} \ell_p^4}. $$  \hspace{1cm} (15)

Note that, since $\langle \hat{p}^2 \rangle \simeq \ell^2$ and $\langle \hat{R}_H^2 \rangle \simeq \ell^4/\ell^2$, we expect the particle will be inside its own horizon if $\langle \hat{p}^2 \rangle \ll \langle \hat{R}_H^2 \rangle$, which precisely yields the condition (6) if $\ell \simeq \lambda_m$.

In fact, the probability density (11) can now be explicitly computed,

$$P_< (r < R_H) = \frac{\ell^3 R_H^2}{2 \sqrt{\pi} \ell^6 \ell_p^4} \left[ \text{Erf} \left( \frac{R_H}{\ell} \right) - 2 \frac{R_H}{\sqrt{\pi} \ell \ell_p} e^{-R_H^2/4\ell^2} \right],$$  \hspace{1cm} (16)

from which we derive the probability (12) for the particle to be a black hole,

$$P_{BH}(\ell) = \frac{2}{\pi} \left[ \arctan \left( \frac{2 \ell^2}{\ell_f^2} \right) + 2 \frac{\ell^2 (4 - \ell^4/\ell_f^4)}{\ell_p^4 (4 + \ell^4/\ell_f^4)} \right]. $$  \hspace{1cm} (17)

In Fig. 1, we show the probability (17) that the particle is a black hole as a function of the Gaussian width $\ell$ (in units of $\ell_p$). From the plot of $P_{BH}$, it appears that the particle is most likely a black hole, $P_{BH} \simeq 1$, if $\ell \lesssim \ell_p$. Assuming $\ell = \lambda_m = \ell_p m_0/m$, we have thus derived a result in qualitative agreement with the condition (6), but from a totally quantum mechanical picture. Strictly speaking, there is no black hole minimum mass in our treatment, but a vanishing probability for a particle of “small” mass (say $m \lesssim m_p/4$, that is $\ell \gtrsim 4 \ell_p$), to be a black hole.
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2.2 Generalised uncertainty principle

For the Gaussian packet described above, the Heisenberg uncertainty in radial position is given by

\[
\langle \Delta r^2 \rangle = 4\pi \int_0^\infty |\psi_S(r)|^2 r^4 dr - \left( 4\pi \int_0^\infty |\psi_S(r)|^2 r^3 dr \right)^2 = \frac{3\pi - 8}{2\pi} \ell^2 , \quad (18)
\]

and, analogously, the uncertainty in the horizon radius will be given by

\[
\langle \Delta R_{H}^2 \rangle = 4 \frac{3\pi - 8}{2\pi} \ell_p^4 \ell . \quad (19)
\]

Since \( \langle \Delta p^2 \rangle = (\frac{3\pi - 8}{2\pi}) m_p^2 \ell_p^2 \ell \equiv \Delta p^2 \), we can also write

\[
\ell^2 = \frac{3\pi - 8}{2\pi} \ell_p^2 \frac{m_p^2}{\Delta p^2} . \quad (20)
\]

Finally, by combining the uncertainty (18) with (19) linearly, we find

\[
\Delta r \equiv \sqrt{\langle \Delta r^2 \rangle} + \gamma \sqrt{\langle \Delta R_{H}^2 \rangle} = \frac{3\pi - 8}{2\pi} \ell_p \frac{m_p}{\Delta p} + 2\gamma \ell_p \frac{\Delta p}{m_p} , \quad (21)
\]

where \( \gamma \) is a coefficient of order one, and the result is plotted in Fig. 2 (for \( \gamma = 1 \)). This is precisely the kind of result one obtains from the generalised uncertainty principles considered in Refs. [5], leading to a minimum measurable length

\[
\Delta r \geq 2 \sqrt{\frac{3\pi - 8}{\pi}} \ell_p \simeq 1.3 \sqrt{\gamma} \ell_p . \quad (22)
\]

Of course, one might consider different ways of combining the two uncertainties (18) and (19), or even avoid this step and just make a direct use of the horizon wave-function. In this respect, the present approach appears more flexible and does not require modified commutators for the canonical variables \( r \) and \( p \).
3 Final remarks

So far, the idea of the horizon wave-function was just applied to the very simple case of a spinless massive particle, and expected results (existence of a minimum black hole mass and generalised uncertainty relation) were recovered and refined [4]. Next, it should be applied to more realistic systems. For example, one could investigate dispersion relations derived from quantum field theory in curved space-time, and a better definition of what a localised state in the latter context should probably be employed as well [6]. Regardless of such improvements, the conceptual usefulness of our construction should already be clear, in that it allows us to deal with very quantum mechanical sources, and to do so in a quantitative fashion. For example, one could review the issue of quantum black holes [7] in light of the above formalism, as well as finally tackle the description of black hole formation and dynamical horizons in the gravitational collapse of truly quantum matter [3, 8].

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