An adaptive framework for quantum-secure device-independent randomness expansion

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Abstract

A device-independent randomness expansion protocol aims to take an initial random seed and generate a longer one without relying on details of how the devices work for security. A large amount of work to date has focussed on a particular protocol based on spot-checking devices using the CHSH inequality. Here we show how to derive randomness expansion rates for a wide range of protocols, with security against a quantum adversary. Our technique uses semi-definite programming and a recent improvement of the entropy accumulation theorem. To support the work and facilitate its use, we provide code that can generate lower bounds on the amount of randomness that can be output based on the measured quantities in the protocol. As an application, we give a protocol that robustly generates up to two bits of randomness per entangled qubit pair, which is twice that established in existing analyses of the spot-checking CHSH protocol in the low noise regime.

1 Introduction

Random numbers are an essential resource in the information processing era, finding applications in gaming, simulations and cryptography. Cryptographic protocols are frequently built upon an assumption of access to a private random seed. Relaxing the privacy of this seed can be fatal to the security of the protocol (see, e.g., [1]). Thus, in order to adhere to these standard protocol assumptions, it is imperative that we are able to certify the privacy of our random numbers.

The intrinsic randomness of quantum theory provides a natural mechanism with which we can generate random numbers: a simple source of perfectly random bits could be a device that prepares a $\sigma_x$ eigenstate and then measures $\sigma_z$. However, the use of such a source comes with a significant caveat: the internal mechanisms of the preparation and measurement devices must be well-characterized and kept stable throughout their use. Any mismatch between the characterization and how the device operates in practice may be an exploitable weakness in the hands of a smart enough adversary; such mismatches have been used to compromise commercially available quantum key distribution (QKD) systems [2].

While weaknesses caused by mismatches may be mitigated by increasingly detailed descriptions of the quantum devices, generating such descriptions rapidly becomes unwieldy and remaining vulnerabilities can be difficult to detect. This is reminiscent of the situation in modern software engineering where security flaws are frequently discovered and patched. Fixing hardware vulnerabilities, such as those exploited in the aforementioned QKD attacks, can be more difficult logistically and economically.

Fortunately, quantum theory provides a means to address this problem. Going back to [3] and using an important insight of [4], device-independent quantum cryptography establishes security independently of the devices involved within a protocol, relying only on the validity of quantum theory and the imposition of certain no-signalling constraints between device-components. Security is subsequently verified through the...
observation of non-local output statistics, which in turn act as witnesses to the inner workings of the devices. Limiting the number of initial assumptions greatly reduces the threat of side-channel attacks.

Randomness expansion in a device-independent setting was proposed in [5,6] with further development and experimental testing following shortly after [7]. Subsequent work provided security proofs against classical adversaries [8,9]. Security against quantum adversaries—who may share entanglement with the internal state of the device—came later [10–12], progressively increasing in noise-tolerance and generality, with the recently introduced entropy accumulation theorem (EAT) [13], on which our work is based, providing asymptotically optimal rates [14,15]. A new proof technique which is also asymptotically optimal has recently appeared [16].

The majority of device-independent security proofs rely on a single Bell-expression for testing non-locality. The technique we introduce here goes beyond this, allowing us to prove security with an arbitrary number of devices, subject to an arbitrary number of Bell-expressions on arbitrary (finite) alphabets. The flexibility of this construction grants a user the freedom to tune the randomness expansion protocol depending on the devices they have to hand. Furthermore, the computational nature of the security proof allows for easily calculable expansion rates, facilitating the exploration of alternate protocols.

The core of our framework is built upon the fusion of two powerful tools from device-independent quantum information, the semidefinite hierarchy [18,19], which is used to bound the device-independent guessing probability (DIGP) [20–22], and the EAT [13, 23]. In [14,15] the EAT was applied to the task of randomness expansion and a general entropy accumulation procedure was detailed. The security of the resulting randomness expansion protocol relies on the construction of a randomness bounding function (known as a min-tradeoff function) that characterizes the average entropy gain during the protocol. However, the analysis in [14] applies only to protocols based on the CHSH inequality, and relies on some analytic steps that don’t directly generalize to arbitrary protocols. Using the DIGP in conjunction with the semidefinite hierarchy [18,19] we obtain computational constructions of the required min-tradeoff functions in more general scenarios, which in turn lead to provably secure randomness expansion protocols. Moreover, as this computational method takes the form of a semidefinite program these constructions are both computationally efficient and reliable, although at the cost of producing potentially suboptimal bounds. To accompany this work, we provide a code package (available at [24]) for the construction and analysis of these randomness expansion protocols.

Our framework is presented in the form of a template protocol, Protocol QRE, from which a user can develop their randomness expansion protocol. This template protocol imposes some mild structural constraints which allow one to use the EAT. Given certain parameters chosen by a user, e.g., time constraints, choice of non-locality tests and security tolerances, the protocol allows the projected randomness expansion rates to be calculated. If these rates are unsatisfactory, then modifications to the protocol’s design can be made and the projections repeated. Thus, we can iteratively optimize the randomness expansion rates for the user’s experimental arrangement. Once a choice of experimental design has been made, the resulting randomness expansion procedure can be performed. Subject to the protocol not aborting, this gives a certifiably private random bit-string.

We apply our technique to several example protocols. In particular, we look at randomness expansion using the complete empirical distribution as well as a simple extension of the CHSH protocol, showing noise-tolerant rates of up to two bits per entangled qubit pair, secure against quantum adversaries. This shows the potential benefits of tuning the protocol. Additionally, we include in the appendices a full non-asymptotic account of input randomness necessary for running the protocols. For an in-depth discussion of the practicality of randomness expansion protocols in reasonable experimental regimes we refer the reader to an accompanying work [25].

The paper is structured as follows: in Sec. 2 we introduce the material relevant for our construction. In Sec. 3 we detail the various components of our framework and present the template protocol with full security statements and proofs. We provide examples of several randomness expansion protocols built within our framework in Sec. 4 before concluding with some open problems in Sec. 5.

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1 An exception to this is [17] where security was established against a classical adversary.

2 In particular, simplifications that arise due to the two party, two input, two output scenario being reducible to qubits.
2 Preliminaries

2.1 General notation

For notational ease, we consider only the bipartite case in this work; the generalization to more parties is straightforward.

Throughout this work, the calligraphic symbols $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$ and $\mathcal{Y}$ denote finite alphabets and we use the notational shorthand $\mathcal{AB}$ to denote the Cartesian product alphabet $\mathcal{A} \times \mathcal{B}$. We refer to a behaviour (or strategy) on these alphabets as some conditional probability distribution, $(p(a,b|x,y))_{ab|xy}$ with $abxy \in \mathcal{AB}|\mathcal{XY}$. That is, by denoting the set of canonical bases vectors of $\mathbb{R}^{[\mathcal{AB}|\mathcal{XY}]}$ by $\{e_{ab|x,y}\}_{abxy}$, we write $p = \sum_{abxy} p(a,b|x,y) e_{ab|x,y}$. We make the distinction between the vector and its elements through the use of boldface, i.e., $p(a,b|x,y) = p \cdot e_{ab|x,y}$. Throughout this work we assume that all conditional distributions obey the no-signalling constraints that $\sum_{a \in \mathcal{A}} p(a,b|x,y) = \text{independent of } x$ and hence can be written $p(b|y)$ and similarly $\sum_{b \in \mathcal{B}} p(a,b|x,y) = p(a|x)$. We denote the set of all such behaviours by $\mathcal{P}_{\mathcal{AB}|\mathcal{XY}} \subset \mathbb{R}^{[\mathcal{AB}|\mathcal{XY}]}$. Given an alphabet $\mathcal{C}$ and a sequence $\mathbf{C} = (c_i)_{i=1}^n$, with $c_i \in \mathcal{C}$ for each $i = 1, \ldots, n$, we denote the frequency distribution induced by $\mathbf{C}$ by

$$\text{freq}_{\mathbf{C}}(x) = \frac{\sum_{i=1}^n \delta_{x,c_i}}{n}, \quad (1)$$

where $\delta_{ab}$ is the Kronecker delta on the set $\mathcal{C}$.

We use the symbol $\mathcal{H}$ to denote a Hilbert space, subscripting with system labels when helpful. For a system $E$, we denote the set of positive semidefinite operators with unit trace acting on $\mathcal{H}_E$ by $\mathcal{S}(E)$ and its subnormalized extension by $\tilde{\mathcal{S}}(E)$. We refer to a state $\rho_{XE} \in \mathcal{S}(XE)$ as a classical-quantum state (cq-state) on the joint system $XE$ if it can be written in the form $\rho_{XE} = \sum_x p(x) |x\rangle \langle x| \otimes \rho_E^x$ where $\{ |x\rangle \}$ is a set of orthonormal vectors in $\mathcal{H}_X$. Letting $\Omega \subseteq \mathcal{X}$ be an event on the alphabet $\mathcal{X}$, we define the conditional state (conditioned on the event $\Omega$) by

$$\rho_{XE|\Omega} = \frac{1}{\text{Pr}[\Omega]} \sum_{x \in \Omega} p(x) |x\rangle \langle x| \otimes \rho_E^x. \quad (2)$$

We denote the identity operator of a system $E$ by $\mathbb{1}_E$. We write the natural logarithm as $\ln(\cdot)$ and the logarithm base 2 as $\log(\cdot)$. The function $\text{sgn} : \mathbb{R} \rightarrow \{-1,0,1\}$ is the sign function, mapping all positive numbers to 1, negative numbers to $-1$ and 0 to 0.

We say that a behaviour $p \in \mathcal{P}_{\mathcal{AB}|\mathcal{XY}}$ is quantum if its elements can be written in the form $p(a,b|x,y) = \text{Tr} [\rho_{AB}|N_{a|x} \otimes M_{b|y}]$ where $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\{|N_{a|x}\}_{a \in \mathcal{A}}|_{x \in \mathcal{X}}$, $\{|M_{b|y}\}_{b \in \mathcal{B}}|_{y \in \mathcal{Y}}$ are sets of POVMs; we denote the set of all quantum behaviours by $\mathcal{Q}$. Additionally, we use $\tilde{\mathcal{Q}}$ to denote the subnormalized extension of this set (i.e., the set that arises when the normalization constraint $\text{Tr}[\rho] = 1$ is relaxed to $0 \leq \text{Tr}[\rho] \leq 1$) and we extend the use of tildes to other sets to denote their subnormalized extensions.

Note that randomness expansion is a single-party protocol; there is one user who wishes to expand an initial private random string. However, that user may work with a bipartite setup in which they use two devices that are prevented from signalling to one another; in such a case we sometimes refer to Alice and Bob as the users of each device. Note though that, unlike in QKD, Alice and Bob are agents of the same party and are within the same laboratory. There may also be a dishonest party, Eve, trying to gain information about the random outputs.

2.2 Entropies and SDPs

Let $\rho \in \mathcal{S}(A)$ be a quantum state, the von Neumann entropy of $\rho$ is

$$H(A)_\rho := -\text{Tr}[\rho \log(\rho)]. \quad (3)$$

For a bipartite state $\rho_{AE} \in \mathcal{S}(AE)$, we use the notation $\rho_E$ for $\text{Tr}_A[\rho_{AE}]$ and define the conditional von Neumann entropy of system $A$ given system $E$ when the joint system is in state $\rho_{AE}$ by

$$H(A|E)_\rho := H(AE)_\rho - H(E)_\rho. \quad (4)$$
In addition, for a tripartite system $\rho_{ABE} \in S(ABE)$, the conditional mutual information between $A$ and $B$ given $E$ is defined by

$$I(A : B|E)_\rho = H(A|BE)_\rho - H(A|E)_\rho.$$  

We drop the state subscript whenever the state is clear from the context.

In this work it is useful to consider the conditional min-entropy [26] in its operational formulation [22]. Given a cq-state $\rho_{XE} = \sum_x p(x) |x\rangle \langle x| \otimes \rho_E^x$, the maximum probability with which an agent holding system $E$ can guess the outcome of a measurement on $X$ is

$$p_{\text{guess}}(X|E) := \max_{\{M_x\}_x} \sum_x p(x) \text{Tr} [M_x \rho_E^x], \quad (5)$$

where the maximum is taken over all POVMs $\{M_x\}_x$ on system $E$. Using this we can define the min-entropy of a classical system given quantum side information as

$$H_{\text{min}}(X|E) := -\log (p_{\text{guess}}(X|E)). \quad (6)$$

The final entropic quantity we consider is the $\epsilon$-smooth min-entropy [27]. Given some $\epsilon \geq 0$ and $\rho_{XE} \in S(XE)$, the $\epsilon$-smooth min-entropy $H_{\text{min}}^\epsilon$ is defined as the supremum of the min-entropy over all states $\epsilon$-close to $\rho_{XE}$,

$$H_{\text{min}}^\epsilon(X|E)_\rho := \sup_{\rho' \in B_\epsilon(\rho)} H_{\text{min}}(X|E)_{\rho'}, \quad (7)$$

where $B_\epsilon(\rho)$ is the $\epsilon$-ball centred at $\rho$ with respect to the purified trace distance [28]. For a thorough overview of smooth entropies and their properties we refer the reader to [29].

In the device-independent scenario we do not know the quantum states or measurements being performed. Instead, our entire knowledge about these must be inferred from the observed input-output behaviour of the devices used. In particular, observing correlations that violate a Bell inequality provides a coarse-grained characterization of the underlying system. In a device-independent protocol, the idea is to use only this to infer bounds on particular system quantities, e.g., the randomness present in the outputs. As formulated above, the guessing probability [5] is not a device-independent quantity because its computation requires knowing $\rho_E^x$. However, the guessing probability can be reformulated in a device-independent way [17,20,21,30] as we now explain.

Consider a tripartite system $\rho_{ABE}$ shared between two devices in the user’s lab and Eve. Because we are assuming an adversary limited by quantum theory, we can suppose that, upon receiving some inputs $(x,y) \in \mathcal{X}\mathcal{Y}$, the devices work by performing measurements $\{E_{a|x}I_a\}$ and $\{F_{b|y}I_b\}$ respectively, which give rise to some probability distribution $p \in \mathcal{Q}_{AB|xy}$, and overall state

$$\sigma_{ABE}^{x,y} = \sum_{ab} |a\rangle \langle a| \otimes |b\rangle \langle b| \otimes \rho_{a}^{abxy},$$

where $\text{Tr}_{AB} [(E_{a|x} \otimes F_{b|y} \otimes I_E) \rho_{ABE}] = \rho_E^{abxy}$, and $p(a,b|x,y) = \text{Tr} [\rho_{E}^{abxy}]$. Note that the user of the protocol is not aware of what the devices are doing.

Consider the best strategy for Eve to guess the value of $AB$ using her system $E$. She can perform a measurement on her system to try to distinguish $\{|E_{c|ab}\rangle\}_c$ (occurring with probability $p(a,b|x,y)$). Denoting Eve’s POVM $\{M_c\}_c$ with outcomes in one-to-one correspondence with the values $AB$ can take (say $c_{ab}$ being the value corresponding to a best guess of $AB = (a,b)$), then given some values of $a,b,x$ and $y$, Eve’s outcomes are distributed as $p(c_{ab}|a,b,x,y) = \text{Tr} [M_{c_{ab}} \rho_E^{abxy}]$, and her probability of guessing correctly is $p(c_{ab}|a,b,x,y) = \text{Tr} [M_{c_{ab}} \rho_E^{abxy}]$. Hence, the overall probability of guessing $AB$ correctly given $E$ and

\footnote{Without loss of generality we can assume Eve’s measurement has as many outcomes as what she is trying to guess.}
The problem can be expressed as matrix equation
\[ W p_c = \omega \]
where \( c \) corresponds to the chosen \( a, b \) value of \( C \).

With this rewriting it is evident that we can think about Eve’s strategy as follows: Eve randomly chooses a \( C \) realisable distributions forms a convex cone). By writing \( \Pr \) the guessing probability depends on the inputs \( x, y \). In the protocols we consider later, there will only be one pair of inputs for which Eve is interested in guessing the outputs. We denote these inputs by \( \tilde{x} \) and \( \tilde{y} \).

In the device-independent scenario, Eve can also optimize over all quantum states and measurements that could be used by the devices. However, she wants to do so while restricting the devices to obey certain relations which depend on the protocol (for example, the CHSH violation that could be observed by the user). For the moment, without specifying these relations precisely, call the set of quantum states and measurements that could be used by the devices. However, she wants to do so while restricting the devices to obey certain constraints ensuring that it then holds for all \( (x, y) \in X Y \).

Optimizing over the set of quantum correlations is a difficult problem, in part because the dimension of the quantum system achieving the optimum could be arbitrarily large. Because of this, we consider a

\[ \sum_c p(c)p_c = \omega \]

This rewriting makes sense provided no information leaks to Eve during the protocol, which is reasonable for randomness expansion since it takes place in one secure lab.
computationally tractable relaxation of the problem, by instead optimizing over distributions within some level of the semidefinite hierarchy \[15,19\]. We denote the \(k\)th level by \(\tilde{Q}(k)\). This relaxation of the problem takes the form of a semidefinite program which can be solved in an efficient manner, at the expense of possibly not obtaining the same optimum value. The corresponding relaxed program is

\[
p_{\text{guess}}(\omega) := \sup_{\tilde{p}_c} \sum_{ab} \tilde{p}_{c}(a, b|\tilde{x}, \tilde{y})
\]

subj. to

\[
\sum_{abc} \tilde{p}_{c}(a, b|\tilde{x}, \tilde{y}) = 1
\]

\[
\sum_{c} W \tilde{p}_{c} = \omega
\]

\[
\tilde{p}_{c} \in \tilde{Q}(k) \quad \forall \ c.
\]  \hspace{1cm} (9)

This program has a dual. In Appendix \[E\] we show that there is an alternative program with the same properties as the standard dual. To specify this, we define the set \(V(k)\) of valid constraint vectors at level \(k\) by the set of vectors \(\nu\) for which there exists \(p \in Q(k)\) such that \(Wp = \nu\).

The alternative dual then takes the form

\[
q_{\text{guess}}(\omega) := \inf_{\alpha, \lambda} \alpha + \lambda \cdot \omega
\]

subj. to

\[
p_{\text{guess}}(\nu) \leq \alpha + \lambda \cdot \nu, \quad \forall \nu \in V(k),
\]  \hspace{1cm} (10)

with \(\alpha \in \mathbb{R}\) and \(\lambda \in \mathbb{R}^{||\omega||_0}\). The constant \(\alpha\) in the objective function comes from the first (normalization) constraint in the primal. Since the NPA hierarchy forms a sequence of outer approximations to the set of quantum correlations, \(Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q\), the relaxed guessing probability provides an upper bound on the true guessing probability, i.e., \(p_{\text{guess}}(\omega) \leq \tilde{p}_{\text{guess}}(\omega)\). Combined with \[6\], one can use the relaxed programs to compute valid device-independent lower bounds on \(H_{\min}\).

Programs \[6\] and \[10\] are parameterized by a vector \(\omega\). We denote a feasible point of the dual program parameterized by \(\omega\) by \((\alpha_\omega, \lambda_\omega)\). Note that for our later analysis we only need \((\alpha_\omega, \lambda_\omega)\) to be a feasible point of the dual program, we do not require it to be optimal.\[8\]

### 2.3 Devices and nonlocal games

In a device-independent protocol, tasks are completed through a series of interactions with some untrusted devices. A device \(D\) refers to some physical system that receives classical inputs and produces classical outputs. Furthermore, we say that \(D\) is untrusted if the mechanism by which \(D\) produces the outputs from the inputs need not be characterized. During the protocol, the user interacts with their untrusted devices within the following scenario.\[7\]

1. The protocol is performed within a secure lab from which information can be prevented from leaking.
2. This lab can be partitioned into disconnected sites (one controlled by Alice and one by Bob).
3. The user can send information freely between these sites without being overheard, while at the same time, they can prevent unwanted information transfer between the sites.\[8\]
4. The user has two devices to which they can provide inputs (taken from alphabets \(X\) and \(Y\)) and receive outputs (from alphabets \(A\) and \(B\)).
5. These devices operate according to quantum theory, i.e., $p_{AB|XY} \in Q_{AB|XY}$. Any eavesdropper is also limited by quantum theory. We use $D_{ABE}$ to denote the collection of devices (including any held by an eavesdropper) and refer to this as an untrusted device network.

6. The user has an initial source of private random numbers and a trusted device for classical information processing.

One of the key advantages of a device-independent protocol is that because no assumptions are made on the inner workings of the devices used, the protocol checks that the devices are working sufficiently well on-the-fly. The protocols hence remain impervious to many side-channel attacks, malfunctioning devices or prior tampering. The idea behind their security is that by testing that the devices exhibit ‘nonlocal’ correlations, their internal workings are sufficiently restricted to enable the task at hand.

In this work we think about the testing of the devices via orthogonal vector games (OVGs), which are related to nonlocal games, which we first introduce. A (two-player) nonlocal game $G = (\mu, V)$ (on $AB\mathcal{X}\mathcal{Y}$) consists of a set of question pairs $(x,y) \in \mathcal{X}\mathcal{Y}$ chosen according to some probability distribution $\mu : \mathcal{X}\mathcal{Y} \rightarrow [0,1]$, a set of answer pairs $(a,b) \in AB$ and a scoring function $V : AB\mathcal{X}\mathcal{Y} \rightarrow \{0,1\}$\footnote{These may be interpreted as ‘win’ and ‘lose’. In general the scoring predicate of a nonlocal game can take non-binary values, however, the restriction to binary scores simplifies some of the later analysis. This restriction does not impact on the generality of our protocol as one can constructively map an arbitrary collection of Bell-expressions to an equivalent collection of nonlocal games with binary scoring rules. We discuss this mapping in detail in Appendix [7].} The game is initiated by a referee who sends the two players their own question chosen according to $\mu$. The players then respond with their answers chosen from $A$ and $B$ respectively. Using the predefined scoring rule $V$, the referee then announces whether or not they won the game. The game is referred to as nonlocal because prior to receiving their questions, the players are separated and unable to communicate until they have given their answers. The question sets, answer sets, distribution $\mu$ and the scoring rule $V$ are all public knowledge. Moreover, the players are allowed to confer prior to the start of the game.

A strategy for a nonlocal game is a conditional distribution $p \in P_{AB|XY}$ defined on the question and answer sets. The probability of winning the nonlocal game $G$ whilst playing according to a strategy $p$ can be written as
\[
\omega(p) = \sum_{abxy} \mu(x,y)p(a,b|x,y)V(a,b,x,y).
\] (11)

We denote the set of all possible expected winning probabilities for a game $G$ when using a quantum strategy by $Q_G$, and for a strategy within the $k^{th}$ level of the NPA hierarchy by $\tilde{Q}_G^{(k)}$.

In order for the inclusion of multiple linear constraints on $D$ to remain compatible with the later analysis, we require a notion of orthogonality between scoring functions. To this end, let $V_{\text{win}} := \{(a,b,x,y) \in AB\mathcal{X}\mathcal{Y} : V(a,b,x,y) = 1\}$ then we say two scoring functions $V$ and $V'$ are orthogonal if $V_{\text{win}} \cap V'_{\text{win}} = \emptyset$. We denote orthogonality between two scoring functions in the canonical manner $V \perp V'$\footnote{In parts of this paper we allow the eavesdropper limited additional power—the bounds will then still apply if the eavesdropper is limited by quantum theory.}. Note that (by padding with unused symbols as necessary) we can always assume that the question and answers sets for all games are identical.

**Definition 2.1** (Orthogonal vector game): An orthogonal vector game (on $AB\mathcal{X}\mathcal{Y}$) is defined by a distribution $\mu : \mathcal{X}\mathcal{Y} \rightarrow [0,1]$ and a tuple of $r$ mutually orthogonal scoring functions $(V_i)_{i=1}^r$. Using $V : AB\mathcal{X}\mathcal{Y} \rightarrow \{0,1\}^r$ which takes a question-answer tuple $(a,b,x,y) \in AB\mathcal{X}\mathcal{Y}$ and outputs the vector of scores $V(a,b,x,y) = \sum_{i=1}^r V_i(a,b,x,y)e_i$, we write the orthogonal vector game $G = (\mu, V)$.

For a strategy $p \in P_{AB|XY}$ we calculate its expected score as
\[
\omega(p) = \sum_i \sum_{abxy} \mu(x,y)p(a,b|x,y)V_i(a,b,x,y)e_i,
\] (12)

and we denote the set of all expected score vectors achievable using quantum strategies by $\tilde{Q}_G$ and those achievable using strategies within the $k^{th}$ level of the NPA hierarchy by $\tilde{Q}_G^{(k)}$. We use the notational shorthand $|G|$ to denote the number of different scores in the game, i.e., the number of entries in vector $V$; in the above

\[11\] If the entries of $V$ and $V'$ are written out in a vector, then this condition is equivalent to orthogonality of the vectors.
definition $|\mathcal{G}| = r$. Note that the orthogonality condition enforces that all expected score vectors satisfy the sub-normalization constraint

$$\|\omega(p)\|_1 \leq 1.$$ (13)

Moreover, the components of $\omega(p)$ are just the probabilities of winning the individual games (11) given the distribution $\mu$. The probability that we do not win any game is hence $p_{\text{lose}} = 1 - \|\omega(p)\|_1$. In addition, both $\tilde{Q}_G$ and $\tilde{Q}^{(k)}_G$ are convex sets for each $k \in \mathbb{N}$. This is a consequence of the linearity of (12) together with the convexity of $\tilde{Q}$ and $\tilde{Q}^{(k)}$, for each $k \in \mathbb{N}$.

**Example 2.1** (Extended CHSH game ($G_{\text{CHSH}}$)): The extended CHSH game has appeared already in the device-independent literature [14, 31, 32]. It extends the standard CHSH game [33] to include a correlation check between one of Alice’s CHSH inputs and an additional input from Bob. In the language of OVGs it is defined by the question-answer sets $X = \{0, 1\}$, $Y = \{0, 1, 2\}$ and $A = B = \{0, 1\}$ along with the mutually orthogonal pair of scoring functions

$$V_{\text{CHSH}}(a, b, x, y) := \begin{cases} 1 & \text{if } x \cdot y = a \oplus b \text{ and } y \neq 2 \\ 0 & \text{otherwise,} \end{cases}$$ (14)

and

$$V_{\text{align}}(a, b, x, y) := \begin{cases} 1 & \text{if } (x, y) = (0, 2) \text{ and } a \oplus b = 0 \\ 0 & \text{otherwise.} \end{cases}$$ (15)

The input distribution we use is defined by $\mu_{\text{CHSH}}(x, y) = \frac{1}{8}$ for $(x, y) \in \{0, 1\}^2$, $\mu_{\text{CHSH}}(02) = \frac{1}{2}$ and $\mu_{\text{CHSH}}(x, y) = 0$ otherwise. This is equivalent to choosing to play either $G_{\text{CHSH}}$ or $G_{\text{align}}$ uniformly at random and then proceeding with the chosen game. The collection

$$\mathcal{G}_{\text{CHSH}} = \left(\mu_{\text{CHSH}}, \left(V_{\text{CHSH}} \atop V_{\text{align}}\right)\right)$$ (16)

forms an OVG. The motivation behind this game can be understood by considering a schematic of the ideal implementation of $\mathcal{G}_{\text{CHSH}}$ on a bipartite qubit system as given in Fig. [1].

### 2.4 Device-independent randomness expansion protocols and their security

A device-independent randomness expansion protocol is a procedure by which one attempts to use a uniform, trusted seed, $D$, to produce a longer uniform bit-string, $Z$, through repeated interactions with some untrusted devices. We consider so-called *spot-checking* protocols, which involve two round types: test-rounds, during which one attempts to produce certificates of nonlocality, and generation rounds in which a fixed input is given to the devices and the outputs are recorded. By choosing the rounds randomly according to a distribution heavily favouring generation rounds, we are able to reduce the size of the seed whilst sufficiently constraining the device’s behaviour, guaranteeing the presence of randomness within outcomes (except with some small probability).

Using the setup described in Sec. [23] our template randomness expansion protocol consists of two main steps.

1. **Accumulation**: In this phase the user repeatedly interacts with the separated devices. Each interaction is randomly chosen to be a generation round or a test round in a coordinated way using the initial random seed. On generation rounds the devices are provided with some fixed inputs $(\tilde{x}, \tilde{y}) \in \mathcal{X}$, while in test rounds, the testing procedure specific to the protocol is followed. After many interactions, the recorded outputs are concatenated to give $AB$. Using the statistics collected during test rounds, a decision is made about whether to abort or not based on how close the observations are to some pre-defined expected device behaviour.

2. **Extraction**: Subject to the protocol not aborting in the accumulation step, a quantum-proof randomness extractor is applied to $AB$. This maps the partially random $AB$ to a shorter string $Z$ that is the output of the protocol.
We define security of a randomness expansion protocol according to a composable definition \cite{35,39}. Using composable security ensures that the output randomness can be used in any other application with only an arbitrarily small probability of it being distinguishable from perfect randomness. To make this more precise, consider a hypothetical device that outputs a string $Z$ that is uniform and uncorrelated with any information held by an eavesdropper. In other words, it outputs $\tau_m \otimes \rho_E$, where $\tau_m$ is the maximally mixed state on $m$ qubits. The ideal protocol is defined as the protocol that involves first doing the real protocol, then, in the case of no abort, replacing the output with a string of the same length taken from the hypothetical device. The protocol is then said to be $\varepsilon_{\text{sound}}$-secure ($\varepsilon_{\text{sound}}$ is called the soundness error) if, when the user either implements the real or ideal protocol with probability $1/2$, the maximum probability that a distinguisher can guess which is being implemented is at most $1 + \varepsilon_{\text{sound}}^2$. If $\varepsilon_{\text{sound}}$ is small, then the real and ideal protocols are virtually indistinguishable. Defining the ideal as above ensures that the real and ideal protocols can never be distinguished by whether or not they abort. We refer to \cite{39} for further discussion of composability in a related context (that of QKD).

There is a second important parameter of any protocol, its completeness error, which is the probability that an ideal implementation of the protocol leads to an abort. It is important for a protocol to have a low completeness error in addition to a low soundness error since a protocol that always aborts is vacuously secure.

**Definition 2.2:** Consider a randomness expansion protocol whose output is denoted by $Z$. Let $\Omega$ be the event that the protocol does not abort. The protocol is an $(\varepsilon_{\text{sound}}, \varepsilon_{\text{comp}})$-randomness expansion protocol if it satisfies the following two conditions.

1. **Soundness:**
   \[
   \frac{1}{2} \Pr[\Omega] \cdot \|\rho_{ZE} - \tau_m \otimes \rho_E\|_1 \leq \varepsilon_{\text{sound}},
   \]  

Figure 1: A measurement schematic for a qubit implementation of the CHSH game. Measurements are depicted in the $x$-$z$ plane of the Bloch sphere and $\sigma_{\varphi} = \cos(\varphi)\sigma_z + \sin(\varphi)\sigma_x$ for $\varphi \in (-\pi, \pi]$. Using the maximally entangled state $|\psi\rangle_{AB} = (|00\rangle + |11\rangle)/\sqrt{2}$ with the measurements depicted, one could achieve an expected score vector $\omega = 1/2 \cdot \left(1 + \frac{\sqrt{2}}{2}, 1\right)$, implying a perfect correlation between the $X = 0$ and $Y = 2$ inputs. In addition, self-testing results \cite{34} give a converse result: these scores completely characterize thedevice up to local isometries. This implies that the state used by the devices is uncorrelated with an adversary and that the measurement pair $(X, Y) = (1, 2)$ yields uniformly random results, certifying the presence of 2 bits of private randomness in the outputs.
where $E$ is an arbitrary quantum register (which could have been entangled with the devices used at the start of the protocol), $m$ is the length of the output string $Z$ and $\tau_m$ is the maximally mixed state on a system of dimension $2^m$.

2. Completeness: There exists a set of quantum states and measurements such that if the protocol is implemented using those

$$\Pr[\mathcal{O}] \geq 1 - \varepsilon_{\text{comp}}. \quad (18)$$

Although we use a composable security definition to ensure that any randomness output by the protocol can be used in any scenario, importantly, this may not apply if the devices used in the protocol are subsequently reused \[40\]. Thus, after the protocol the devices should be kept shielded and not reused until such time as the randomness generated no longer needs to be kept secure. How best to resolve this remains an open problem: the Supplemental Material of \[40\] presents candidate protocol modifications (and modifications to the notion of composability) that may circumvent such problems.

2.5 Entropy accumulation

In order to bound the smooth min entropy $H^c_{\min}(\mathbf{AB} | \mathbf{X} \mathbf{Y} \mathbf{E})$ accumulated during the protocol we employ the EAT \[13\][23]. Roughly speaking, the EAT says that this min-entropy is proportional to the number of rounds, up to square root correction factors. The proportionality constant is the single-round conditional von Neumann entropy optimized over all states that can give rise to the observed scores. In its full form, the EAT is an extension of the asymptotic equipartition property \[41\] to a particular non-i.i.d. regime. For the purposes of randomness expansion we only require a special case of the EAT, which we detail later in this section.

With the goal of maximising our entropic yield, we use the recently improved statement of the entropy accumulation theorem \[23\][17]. For completeness we present the relevant statements including the accumulation procedure (see also \[15\]).

2.5.1 The entropy accumulation procedure

The entropy accumulation procedure prescribes how the user interacts with their untrusted devices and collects data from them. Before beginning this procedure an OVG $\mathcal{G} = (\mu, \mathbf{V})$ that is compatible with the alphabets of the devices is selected.

A round within the entropy accumulation procedure consists of the user giving an input to each of their devices and recording the outputs. We use subscripts on random variables to indicate the round that they are associated with, i.e., $X_iY_i$ are the random variables describing the joint device inputs for the $i$th round. In addition, boldface will be used to indicate that a random variable represents the product random variable over all $n$ rounds of the protocol, $X = X_1X_2 \ldots X_n$.

On the $i$th round, random variable $T_i$ is selected such that $T_i = 0$ with probability $1 - \gamma$ and $T_i = 1$ with probability $\gamma$ for some fixed $\gamma \in (0, 1)$ (we discuss later how this is done). If $T_i = 0$ then the round is a generation round and the devices are given the fixed generation inputs $(\tilde{x}, \tilde{y}) \in \mathcal{X}\mathcal{Y}$ and the outputs recorded as $A_iB_i$. Otherwise, if $T_i = 1$, the round is a test round. On a test round, the inputs to the devices are chosen according to the OVG distribution $\mu$ and the score vector is calculated using the scoring rule $\mathbf{V}$. At the end of each round the outputs, inputs and corresponding score are recorded in some tuple $(A_i, B_i, X_i, Y_i, C_i)$. The score that they record, $C_i$, is some element of $\{0, 1\}^{\mathcal{G}^i+1}$. The first $|\mathcal{G}|$ elements of which (denoted $C_i(\mathcal{G})$) are reserved for recording the OVG-score and the final element $C_i(\bot)$ tracks whether or not a test occurred. Concretely, if the round was a test round then the user sets $C_i = \mathbf{V}(A_i, B_i, X_i, Y_i) + 0 \cdot \mathbf{e}_\bot$, otherwise they set $C_i = \mathbf{e}_\bot$. For notational convenience we often refer to the scores by abstract labels from some set $\mathcal{C}$, making an implicit identification between each $c \in \mathcal{C}$ and a unique basis vector $\mathbf{e}_c$ of $\mathbb{R}^{\mathcal{G}^i+1}$. That is, when we say the honest parties scored $c \in \mathcal{C}$ on round $i$, we mean that they recorded $C_i = \mathbf{e}_c$. Similarly, for some vector $\mathbf{v} \in \mathbb{R}^{\mathcal{G}^i+1}$ we may write $v(c)$ instead of $\mathbf{v} \cdot \mathbf{e}_c$, dropping the boldface to signify that the quantity is no longer a vector.

\[12\] We discuss this EAT statement and compare it to alternatives in Appendix \[C\].
After \( n \) rounds of device interaction, the user performs parameter estimation on the test statistics they have collected, \( C = (C_1, C_2, \ldots, C_n) \). They calculate the total EAT-score

\[
C_{\text{total}} = \sum_{i=1}^{n} C_i.
\]

(19)

If the devices behave in an i.i.d. fashion with an expected OVG score \( \omega \in \mathcal{Q}_G \) then the expected total EAT-score is

\[
E[C_{\text{total}}] = \left( \gamma n \omega \right) (1 - \gamma)^n.
\]

(20)

If the total OVG score calculated is sufficiently close to this expected score then the user completes the randomness expansion protocol by applying an appropriate randomness extractor to the concatenated outputs \( AB \). Otherwise, if the elements of \( C_{\text{total}}(G) \) fall outside of their permitted ranges, the protocol is aborted. The event that the protocol does not abort during the parameter estimation phase is

\[
\Omega = \{ C \mid \gamma n(\omega - \delta) < C_{\text{total}}(G) < \gamma n(\omega + \delta) \},
\]

(21)

where \( \delta \) is a vector of confidence interval widths satisfying \( 0 \leq \delta \leq \omega \) with all vector inequalities being interpreted as element-wise constraints.

**Remark 2.1:** The number of losses that occurred during the protocol, i.e., the number of test rounds with a score of \( 0 \), can be calculated through normalization, \( n_{\text{loss}} = n - \| C_{\text{total}} \|_1 \). Referring to a score of \( 0 \) as a loss is a continuation of the convenient narrative introduced Sec. 2.3. However, it does not necessarily carry with it any negative connotations: any game in which the players want to score as high as possible can be trivially transformed, by swapping the scoring labels, into one in which the players want to lose as often as possible.

**Remark 2.2:** The success event, \( \Omega \), does not constrain the value of \( C_{\text{total}}(\perp) \). This is because the sampling of the test rounds is an entirely trusted procedure and therefore, except with exponentially small probability, \( C_{\text{total}}(\perp) \) will take values within some small interval about its expected value – which is fixed by the protocol to be \( (1 - \gamma)n \). The effect of infrequent testing on the total OVG score is accounted for in (21) and any deviation\(^\text{13}\) in the number of test rounds occurring will only lead to a decrease in \( \Pr[\Omega] \).

### 2.5.2 The entropy accumulation theorem

To complete the protocol, uniform randomness needs to be extracted from the partially random outputs. Doing so requires the user to assume a lower bound on the smooth min-entropy (conditioned on any side information held by an adversary) contained in the devices’ outputs when the protocol does not abort. If \( \varepsilon_{\text{sound}} \) is very small, then the assumption must be correct with near certainty. The EAT provides a method by which one can compute such a lower bound. Loosely, the EAT states that if the interaction between the honest parties occurs in a sequential manner (as described in Sec. 2.5.1), then with high probability the uncertainty an adversary has about the outputs is close to their total average uncertainty. As a mathematical statement it is a particular example of the more general phenomenon of concentration of measure (see [42] for a general overview). In order to state the EAT precisely, we first require a few definitions.

**Definition 2.3** (EAT channels): A set of EAT channels \( \{N_i\}_{i=1}^{n} \) is a collection of trace-preserving and completely-positive maps \( N_i : \mathcal{S}(R_{i-1}) \rightarrow \mathcal{S}(A_iB_iX_iY_iC_iR_i) \) such that for every \( i \in [n] \):

1. \( A_i, B_i, X_i, Y_i \) and \( C_i \) are finite dimensional classical systems, \( R_i \) is an arbitrary quantum system and \( C_i \) is the output of a deterministic function of the classical registers \( A_i, B_i, X_i \) and \( Y_i \).

2. For any initial state \( \rho_{R_iE} \), the final state \( \rho_{ABXYCE} = \text{Tr}_{R_i}[((N_n \circ \cdots \circ N_1) \otimes I_E)\rho_{R_nE}] \) fulfills the Markov chain condition

\[
I(A^{i-1}B^{i-1};X_iY_i|X^{i-1}Y^{i-1}E) = 0 \text{ for every } i \in [n].
\]

\(^\text{13}\) Deviations are unavoidable if \( \gamma n \notin \mathbb{N} \). The relationship between \( \gamma, n, \delta \) and the probability of aborting are made explicit when we look at the completeness error (Lemma 3.4).
The EAT channels formalise the notion of interaction within the protocol. The first condition in Def. 2.3 specifies the nature of the information present within the protocol and, in particular, it restricts the honest parties' inputs to their devices to be classical in nature. The arbitrary quantum register \( R_i \) represents the quantum state stored by the separate devices after the \( i \)th round. The second condition specifies the sequential nature of the protocol. The channels \( N_i \) describe the joint action of both devices and include the generation of the randomness needed to choose the settings. The Markov chain condition implies that the inputs to the devices presented by the honest parties are conditionally independent of the previous outputs they have received. Note that by using a trusted private seed to choose the inputs and supplying the inputs sequentially (as is done in Sec. 2.5.1), this condition will be satisfied. Finally, the adversary is permitted to hold a purification, \( E \) of the initial state shared by the devices, and the state evolves with the sequential interaction through the application of the sequence of EAT channels.

As explained above, the EAT allows the elevation of i.i.d. analyses to the non-i.i.d. setting. To do so requires a so-called \( \text{min-tradeoff function} \) which, roughly speaking, gives a lower bound on the single-round von Neumann entropy produced by any devices that are capable of producing the observed statistics in the i.i.d. limit. In the case of the EAT these distributions are \( \{ \text{freq}_C \}_{C \in \Omega} \), i.e., all frequency distributions induced from EAT-scores that do not lead to an aborted protocol. The EAT asserts that, under sequential interaction, an adversary's uncertainty about the outputs of the non-i.i.d. device will (with high probability) be concentrated within some interval about the uncertainty produced by these i.i.d. devices. In particular, a lower bound on this uncertainty can be found by considering the worst-case i.i.d. device.

**Definition 2.4 (Min-tradeoff functions):** Let \( \{ N_i \}_{i=1}^n \) be a collection of EAT channels and let \( \mathcal{C} \) denote the common alphabet of the systems \( C_1, \ldots, C_n \). An affine function \( f_{\min} : \mathcal{P}_\mathcal{C} \to \mathbb{R} \) is a \( \text{min-tradeoff function} \) for the EAT channels \( \{ N_i \}_{i=1}^n \) if for each \( i \in [n] \) it satisfies

\[
f_{\min}(p) \leq \inf_{\sigma_{R_{i-1}R} : N_i(\sigma)C_i = \tau_p} H(A_iB_i|X_iY_iR'_i)_{N_i(\sigma)},
\]

where \( \tau_p := \sum_{c \in \mathcal{C}} p(c) |c\rangle \langle c| \), \( R'_i \) is a register isomorphic to \( R_{i-1} \) and the infimum over the empty set is taken to be +\( \infty \).

**Remark 2.3:** The min-tradeoff functions are only defined on normalized probability distributions. However, the frequency distributions that were considered in Sec. 2.5.1 may be subnormalized – this follows from the subnormalization of expected OVG scores. To remedy this discrepancy we can include an explicit lose symbol \( (L) \). In such a scenario, any scores \( C_i = 0 \) would be replaced with \( C_i = e_L \) and then all distributions that we could expect our devices to generate within the context of the protocol would be of the form

\[
p = \begin{pmatrix}
\gamma q \\
(1 - \gamma)
\end{pmatrix}
\]

for some \( q \in \mathcal{Q}_G \), where \( p(L) \), \( p(\perp) \) are the penultimate and final elements of \( p \) respectively. This is a consequence of the protocol have a fixed probability of testing and therefore any distribution that results in a finite infimum in (22) necessarily takes this form. We shall refer to any distribution of the form \( (23) \) as a \( \text{protocol-respecting} \) distribution, denoting the set of all such distributions by \( \Gamma \).

Particular properties of the min-tradeoff function appear within the error terms of the EAT:

- The maximum value attainable on \( \mathcal{P}_\mathcal{C} \),

\[
\text{Max}[f] := \max_{p \in \mathcal{P}_\mathcal{C}} f(p).
\]

- The minimum value over protocol-respecting distributions,

\[
\text{Min}[f|\Gamma] := \min_{p \in \Gamma} f(p).
\]

- The maximum variance over all protocol-respecting distributions,

\[
\text{Var}[f|\Gamma] := \max_{p \in \Gamma} \sum_{c \in \mathcal{C}} p(c) (f(e_c) - f(p))^2,
\]

where \( \mathcal{C} = \mathcal{G} \cup \{ L, \perp \} \) is the normalizing alphabet.
be some event that occurs with probability \( p \) and for some 

\[
\text{Definition 2.5 (Quantum-proof strong extractor)}
\]

the standard definition for a quantum-proof randomness extractor \([44]\) gives the following definition.

A randomness extractor can be used. This is a function seeded, quantum-proof to ‘compress’ this randomness into a shorter but almost uniformly random string \( a \) and an adversary with quantum side-information

Subject to the protocol not aborting, the entropy accumulation sub-procedure detailed in Sec. 2.5.1 will result in the production of some bit string \( \text{AB} \in \{0,1\}^{2n} \) with \( H^*_{\text{min}}(\text{AB}|\text{XYE}) > k \) for some \( k \in \mathbb{R} \). In order to ‘compress’ this randomness into a shorter but almost uniformly random string a seeded, quantum-proof randomness extractor can be used. This is a function \( R_{\text{ext}} : \text{AB} \times \text{D} \to \text{Z} \), such that if \( \text{D} \) is a uniformly distributed bit-string, the resultant bit-string \( \text{Z} \) is \( \epsilon \)-close to uniformly distributed, even from the perspective of an adversary with quantum side-information \( \text{E} \) about \( \text{AB} \). More formally, combining \([43, \text{Lemma 3.5}]\) with the standard definition for a quantum-proof randomness extractor \([44]\) gives the following definition.

**Definition 2.5** (Quantum-proof strong extractor): We say that a function \( R_{\text{ext}} : \{0,1\}^{|\text{AB}|} \times \{0,1\}^{|	ext{D}|} \to \{0,1\}^{|	ext{Z}|} \) is a quantum-proof \((k, \epsilon_{\text{ext}} + 2\epsilon_s)\)-strong extractor, if for all eq-states \( \rho_{\text{ABE}} \) with \( H^*_{\text{min}}(\text{AB}|\text{E}) \geq k \) and for some \( \epsilon_s > 0 \) it maps \( \rho_{\text{ABE}} \otimes \tau_{\text{D}} \) to \( \rho'_{R_{\text{ext}}(\text{AB}, \text{D})}\text{DE} \) where

\[
\frac{1}{2} \|\rho'_{R_{\text{ext}}(\text{AB}, \text{D})}\text{DE} - \tau \otimes \tau_{\text{D}} \otimes \rho_{\text{E}}\|_1 \leq \epsilon_{\text{ext}} + 2\epsilon_s,
\]

where \( \tau \) is the maximally mixed state on a system of dimension \( 2^{|	ext{Z}|} \).

Although in general the amount of randomness extracted will depend on the extractor, \( H^*_{\text{min}}(\text{AB}|\text{E}) \) provides an upper bound on the total number of \( \epsilon_s \)-close to uniform bits that can be extracted from \( \text{AB} \) and a well-chosen extractor will result in a final output bit-string with \( |\text{Z}| \approx H^*_{\text{min}}(\text{AB}|\text{E}) \). We denote any loss of entropy incurred by the extractor by \( \ell_{\text{ext}} = k - |\text{Z}| \). Entropy loss will differ between extractors but in general it will be some function of the extractor error, the seed length and the initial quantity of entropy. The extractor literature is rich with explicit constructions, with many following Trevisan’s framework \([45]\). For an in-depth overview of randomness extraction, we refer the reader to \([46]\) and references therein.

**Remark 2.5:** By using a strong quantum-proof extractor, the output of the extractor will remain uncorrelated with the string used to seed it. Since the seed acts like a catalyst, we need not be overly concerned with the amount required. Furthermore, if available, it could just be acquired from a trusted public source immediately prior to extraction without compromising security. However, it is important that the source of public randomness used is not available to Eve too early in the protocol as this could allow Eve to create correlations between the outputs of the devices and the extractor seed.
Remark 2.6: Related to the previous remark is the question of whether the quantity we are interested in is $H_{\text{min}}(AB|XYE)$, rather than $H_{\text{min}}^*(AB|E)$ or $H_{\text{min}}^*(ABXY|E)$. In common QKD protocols (such as BB84), the first of these is the only reasonable choice because the information $XY$ is communicated between the two parties over an insecure channel and hence could become known by Eve. For randomness expansion, this is no longer the case: this communication can all be kept secure within one lab. Whether the alternative quantities can be used then depends on where the seed randomness comes from. If a trusted beacon is used then the first case is needed. If the seed randomness can be kept secure until such time that the random numbers need no longer be kept random then the second quantity could be used. If it is also desirable to extract as much randomness as possible, then the third quantity could be used instead. However, in many protocols the amount of seed required for the entropy accumulation procedure is small enough that its reuse will not be of practical significance (see, e.g., our discussion in Appendix B).

3 An adaptive randomness expansion protocol

The primary purpose of this work is to provide a method whereby one can construct tailored randomness expansion protocols, with a guarantee of security and easily calculable generation rates. We achieve this by providing a template protocol (Protocol QRE), for which we have explicit security statements in terms of the protocol parameters as well as the outputs of some SDPs. Our framework may be divided into three sub-procedures: preparation, accumulation and extraction. The preparation procedure consists of assigning values to the various parameters of the protocol, this includes choosing an OVG to act as the nonlocality test. At the end of the preparation one would have completed the protocol template, constructed a min-tradeoff function and be able to calculate the relevant security quantities. The final two parts of the framework form the process described in Protocol QRE. The accumulation step follows the entropy accumulation procedure detailed in Sec. 2.5.1 wherein the user interacts with their devices using the chosen protocol parameters. After the device interaction phase has finished, the user implicitly evaluates the quality of their devices by testing whether the observed inputs and outputs satisfy the condition (21). Subject to the protocol not aborting, a reliable lower bound on the min-entropy of the total output string is calculated through the EAT (27). With this bound, the protocol can be completed by applying an appropriate quantum-proof randomness extractor to the devices’ raw output strings.

The next three subsections are dedicated to explaining these three sub-procedures in detail. In particular, Sec. 3.1 outlines the min-tradeoff function construction. A bound on the total entropy accumulated in terms of the various protocol parameters is then provided in Sec. 3.2 and finally, in Sec. 3.3 the security statements for the template protocol are presented.

3.1 Preparation

Before interacting with their devices the user must select appropriate protocol parameters (see Fig. 2 for a full list of parameters). In particular, they must choose an OVG to use during the test rounds and construct a corresponding min-tradeoff function.

The parameter values chosen will largely be dictated by situational constraints; e.g., runtime, seed length and the expected performance of the untrusted devices. The user’s choice of parameters, in particular the choice of OVG, will affect the form of their min-tradeoff function derived and in turn their projected total accumulated entropy. Before moving to the accumulation step of the protocol the user can test the entropy rates of many different protocol parameter choices. This enables them to adapt the protocol to the projected performance of their devices.

We now present a constructible family of min-tradeoff functions for a general instance of Protocol QRE. This construction is based on the following idea. As noted in Sec. 2.2 one can numerically calculate a lower bound on the min-entropy of a system based on its observed statistics. Pairing this with the relation, this is a reasonable requirement, because there are other strings that have to be kept secure in the same way, e.g., the raw string $A$.

At first this may seem to conflict with the ethos of device-independence. The point is that although the user of the protocol relies on an expected behaviour to set-up their devices, they do not rely on this expected behaviour being an accurate reflection of the devices for security. This also means that the expected behaviour could be that claimed by the device manufacturer. Using inaccurate estimation of the devices behaviour will not compromise security, but may lead to a different abort probability.
Protocol QRE

**Parameters and notation:**

- $\mathcal{D}_{AB}$ – a collection of two untrusted devices taking inputs from $\mathcal{X}, \mathcal{Y}$ and giving outputs from $\mathcal{A}, \mathcal{B}$
- $\mathcal{G} = (\mu, V)$ – an orthogonal vector game compatible with $\mathcal{D}_{AB}$
- $\omega \in \mathbb{Q}^G$ – the expected score vector for $\mathcal{G}$
- $\delta$ – vector of confidence interval widths (satisfies $0 \leq \delta_k \leq \omega_k$ for all $k \in |G|$)
- $n \in \mathbb{N}$ – number of rounds
- $\gamma \in (0, 1)$ – probability of a test round
- $(\tilde{x}, \tilde{y})$ – distinguished inputs for generation rounds
- $f_{\min}$ – min-tradeoff function
- $\epsilon_{\text{ext}} > 0$ – extractor error
- $\epsilon_{\text{s}} \in (0, 1)$ – smoothing parameter
- $\epsilon_{EAT} \in (0, 1)$ – entropy accumulation error
- $R_{\text{ext}}$ – quantum-proof $(k, \epsilon_{\text{ext}} + 2\epsilon_{\text{s}})$-strong extractor
- $\ell_{\text{ext}}$ – entropy loss induced by $R_{\text{ext}}$

**Procedure:**

1. Set $i = 1$.
2. While $i \leq n$:
   - Choose $T_i = 0$ with probability $1 - \gamma$ and otherwise $T_i = 1$.
   - If $T_i = 0$:
     - Gen: Input $(\tilde{x}, \tilde{y})$ into the respective devices, recording the inputs $X_i, Y_i$ and outputs $A_i, B_i$.
     - Set $C_i = e_\perp$ and $i = i + 1$.
   - Else:
     - Test: Play a single round of $\mathcal{G}$ on $\mathcal{D}_{AB}$ using inputs sampled from $\mu$, recording the inputs $X_i, Y_i$ and outputs $A_i, B_i$. Set $C_i = V(A_{i,j}B_{i,j}X_{i,j}Y_{i,j})$ and $i = i + 1$.
3. Compute the total EAT-vector $C_{\text{total}} = \sum_{i=1}^{n} C_i$.
   - If $\gamma n(\omega - \delta) < C_{\text{total}}(\mathcal{G}) < \gamma n(\omega + \delta)$:
     - Ext: Apply a strong quantum-proof randomness extractor $R_{\text{ext}}$ to the raw output string $AB$ producing $(1 - \gamma)(A_\nu - B_\nu \lambda_\nu \cdot (\omega - \delta_\perp)) - \epsilon_V - \epsilon_K)n - \epsilon_\Omega - \ell_{\text{ext}}$ bits ($\epsilon_{\text{ext}} + 2\epsilon_{\text{s}}$)-close to uniformly distributed.
   - Else:
     - Abort: Abort the protocol.

Figure 2: The template quantum-secure device-independent randomness expansion protocol.

$H_{\min}(X|E) \leq H(X|E)$, we have access to numerical bounds on the von Neumann entropy. In particular, we can use the affine function $g(q) = \alpha + \lambda \cdot q$, where $\alpha$ and $\lambda$ are a feasible point of the dual program [10], in order to build a min-tradeoff function for the protocol [16]. In order for $g$ to meet the requirements of a min-tradeoff function, its domain must be extended to include the symbol $\perp$. To perform this extension we use the method presented in [23, Section 5.1]. As the rounds are split into testing and generation rounds, we may decompose the EAT-channel for the $i^{th}$ round as $\mathcal{N}_i = \gamma \mathcal{N}_{i^{\text{test}}} + (1 - \gamma)\mathcal{N}_{i^{\text{gen}}}$, where $\mathcal{N}_{i^{\text{test}}}$ is the channel that would be applied if the round were a test round and $\mathcal{N}_{i^{\text{gen}}}$ if the round were a generation round. Importantly, this splitting separates $\perp$ from the OVG scores. That is, if $\mathcal{N}_{i^{\text{test}}}$ is the channel applied then $Pr[C_i = e_\perp] = 0$ whereas if $\mathcal{N}_{i^{\text{gen}}}$ is applied then $Pr[C_i = e_\perp] = 1$.

[16] In fact, by relaxing the dual program to the NPA hierarchy, the single round bound is valid against super-quantum adversaries. However, the full protocol is not necessarily secure more widely: to show that we would need to generalise the EAT and the extractor.
Lemma 3.1 (Min-tradeoff extension [23, Lemma 5.5]): Let $g : \mathcal{P}_{\mathcal{G}\cup\{L\}} \to \mathbb{R}$ be an affine function satisfying
\begin{equation}
    g(p) \leq \inf_{\sigma_i : 1 \leq i \leq \mathcal{N}^{\text{min}}(\sigma) \in \mathcal{P}_{\mathcal{G}\cup\{L\}}} H(A_iB_i|X_iY_iR')_{\mathcal{N}^{\text{min}}(\sigma)}
\end{equation}
for all $p \in \mathcal{Q}_{\mathcal{G}\cup\{L\}}$. Then, the function $f : \mathcal{P}_{\mathcal{G}\cup\{L\}} \to \mathbb{R}$, defined by its action on the basis vectors
\begin{align*}
    f(e_c) &= \max [g] + \frac{g(e_c) - \max [g]}{\gamma}, \\
    f(e_L) &= \max [g],
\end{align*}
is a min-tradeoff function for the EAT-channels $\{\mathcal{N}_i\}$. Furthermore, $f$ satisfies the following properties:
\begin{align*}
    \max [f] &= \max [g], \\
    \min [f|_R] &\geq \min [g], \\
    \text{Var}[f|_R] &\leq \frac{(\max [g] - \min [g])^2}{\gamma}.
\end{align*}

Lemma 3.2 (Min-tradeoff construction): Let $\mathcal{G}$ be an OVG and $k \in \mathbb{N}$. For each $\nu \in \tilde{Q}_G^{(k)}$, let $(\alpha_\nu, \lambda_\nu)$ denote a feasible point of Prog. [10] when parameterized by $\nu$. Furthermore, let $\lambda_{\max} = \max \{0, \lambda_1, \ldots, \lambda_{|\mathcal{G}|}\}$ and $\lambda_{\min} = \min \{0, \lambda_1, \ldots, \lambda_{|\mathcal{G}|}\}$, where $\{\lambda_i\}$ are the components of $\lambda_\nu$. Then, for any set of EAT channels $\{\mathcal{N}_i^{\nu} \}_{i=1}^\nu$ implementing an instance of Protocol QRE with the OVG $\mathcal{G}$, the set of functionals $\mathcal{F}_{\min}(\mathcal{G}) = \{f_\nu(\cdot) | \nu \in \tilde{Q}_G^{(k)}\}$ forms a family of min-tradeoff functions, where $f_\nu : \mathcal{P}_C \to \mathbb{R}$ are defined by their actions on the basis vectors
\begin{align*}
    f_\nu(e_c) &= (1 - \gamma) \left( A_\nu - B_\nu \frac{\lambda_\nu \cdot e_c - (1 - \gamma) \lambda_{\min}}{\gamma} \right), \\
    f_\nu(e_L) &= (1 - \gamma) \left( A_\nu + B_\nu \frac{(1 - \gamma) \lambda_{\min}}{\gamma} \right),
\end{align*}
and
\begin{equation}
    f_\nu(e_L) = (1 - \gamma) (A_\nu - B_\nu \lambda_{\min}),
\end{equation}
where $A_\nu = \frac{\lambda_\nu}{(\alpha_\nu + \lambda_\nu \cdot \nu) \ln 2} - \log(\alpha_\nu + \lambda_\nu \cdot \nu)$ and $B_\nu = \frac{1}{(\alpha_\nu + \lambda_\nu \cdot \nu) \ln 2}$ are constants defined by the solution to the dual program.

Moreover, these min-tradeoff functions satisfy the following relations.
\begin{itemize}
    \item **Maximum:**
    \begin{equation}
        \max [f_\nu] = (1 - \gamma)(A_\nu - B_\nu \lambda_{\min})
    \end{equation}
    \item **G-Minimum:**
    \begin{equation}
        \min [f_\nu|_R] \geq (1 - \gamma)(A_\nu - B_\nu \lambda_{\max})
    \end{equation}
    \item **G-Variance:**
    \begin{equation}
        \text{Var}[f_\nu|_R] \leq \frac{(1 - \gamma)^2 B_\nu^2 (\lambda_{\max} - \lambda_{\min})^2}{\gamma}
    \end{equation}
\end{itemize}

Proof. Consider the entropy bounding property [32], but with the score alphabet $\mathcal{C}$ restricted to the set of OVG scores, i.e., we have an affine function $g_\nu : \mathcal{P}_{\mathcal{G}} \to \mathbb{R}$ such that
\begin{equation}
    g_\nu(q) \leq \inf_{\sigma_i : 1 \leq i \leq \mathcal{N}^{\text{min}}(\sigma) \in \mathcal{P}_{\mathcal{G}}} H(A_iB_i|X_iY_iR')_{\mathcal{N}^{\text{min}}(\sigma)},
\end{equation}
for all $q \in \tilde{Q}_G$.  

As conditioning on classical side information will not increase the von Neumann entropy, we may condition on whether or not the round was a test round,
\begin{align*}
H(A_iB_i|X_iY_iR'_i)_{\mathcal{N}_i(\sigma)} & \geq H(A_iB_i|X_iYiT_iR'_i)_{\mathcal{N}_i(\sigma)} \\
& = \gamma H(A_iB_i|X_iY_i, T_i = 1, R'_i)_{\mathcal{N}_i(\sigma)} + (1 - \gamma) H(A_iB_i|X_iY_i, T_i = 0, R'_i)_{\mathcal{N}_i(\sigma)} \\
& > (1 - \gamma) H(A_iB_i|X_i = \tilde{x}, Y_i = \tilde{y}, T_i = 0, R'_i)_{\mathcal{N}_i(\sigma)}
\end{align*}

where in the final line we have used the fact that the inputs are fixed for generation rounds. As the min-entropy lower bounds the von Neumann entropy, we arrive at the bound

\[ H(A_iB_i|X_iY_iR'_i)_{\mathcal{N}_i(\sigma)} > (1 - \gamma) H_{\min}(A_iB_i|X_i = \tilde{x}, Y_i = \tilde{y}, T_i = 0, R'_i)_{\mathcal{N}_i(\sigma)}. \]

Using programs \[ \text{(9)} \] and \[ \text{(10)} \], we can lower bound the right-hand side in terms of the relaxed guessing probability. Specifically, for a single generation round

\[ H_{\min}(AB|X = \tilde{x}, Y = \tilde{y}, T = 0, R') = -\log(p_{\text{guess}}(q)) \geq -\log(\alpha(\nu) + \lambda(\nu) \cdot q), \]

holds for all \( k \in \mathbb{N} \), any \( \nu \in \tilde{Q}^{(k)} \) and any quantum system realising the expected statistics \( q \in \tilde{Q}_\sigma \). In the final line we used the monotonicity of the logarithm together with the fact that a solution to the relaxed dual program, for any parameterization \( \nu \in \tilde{Q}^{(k)} \), provides an affine function \( \alpha(\nu) + \lambda(\nu) \cdot (\cdot) \) that is everywhere on \( \tilde{Q}^{(k)} \) greater than \( p_{\text{guess}} \). Note that this bound is also device-independent and is therefore automatically a bound on the infimum. Dropping the \( k \) for notational ease, we may recover the desired affine property by taking a first order expansion about the point \( \nu \). This results in the function

\[ g(\nu)(q) := (1 - \gamma)(A_\nu - B_\nu \lambda \cdot q), \]

which satisfies

\[ g(\nu)(q) \leq \inf_{\sigma_{R_i-1}R'_i N^{\text{test}}(\sigma) C_i = c_q} H(A_iB_i|X_iY_iR'_i)_{\mathcal{N}_i(\sigma)}, \]

for all \( q \in \tilde{Q}_\sigma \), with \( A_\nu \) and \( B_\nu \) as defined above.

As discussed in Remark 2.3, we can normalize the OVG scores by mapping them to a vector space of one higher dimension. That is, we define a 'lose' vector \( e_L \) and apply the embedding \( P_\sigma \hookrightarrow P_{\sigma \cup \{l\}} \) defined by the mapping \( \omega \mapsto (\omega, 1 - \|\omega\|_1) \) for each \( \omega \in P_\sigma \). The new scores, post embedding, are normalized probability distributions. We then trivially extend the domain of \( g_\nu \) to this new space by mapping \( \lambda(\nu) \mapsto \lambda(\nu) + 0 \cdot e_L \). This preserves the affine and entropy bounding properties of \( g_\nu \) and we now have \( \text{Max}[g(\nu)] = \alpha_\nu + \text{Max}\{0, \lambda_1, \ldots, \lambda_{|\sigma|}\} \) and \( \text{Min}[q] = \alpha_\nu + \text{Min}\{0, \lambda_1, \ldots, \lambda_{|\sigma|}\} \).

As the scores are now normalised, we may apply Lemma 3.1 to finalise the min-tradeoff function construction. Writing \( \lambda_{\text{max}} = \max\{0, \lambda_1, \ldots, \lambda_{|\sigma|}\} \) and \( \lambda_{\text{min}} = \min\{0, \lambda_1, \ldots, \lambda_{|\sigma|}\} \), the result follows.

**Example 3.1:** Taking the OVG \( \mathcal{G}_{\text{CHSH}} \) from Sec. 2.3 we can construct a min-tradeoff function for a particular instance of Protocol QRE. Fixing the probability of testing, \( \gamma = 5 \times 10^{-3} \), we consider a device which behaves (during a test round) according to the expected OVG-score \( \omega = (\omega_{\text{align}}, \omega_{\text{CHSH}}) \). Below, in Fig. 3 we plot the certifiable min-entropy of a single generation round for a range of \( \omega \). We see that as the scores approach the maximum quantum-achievable value of \( \omega = \frac{1}{2} \left( 1, \frac{2 + \sqrt{2}}{4} \right) \), we are able to certify almost \( 17 \) two bits of randomness using \( \mathcal{G}_{\text{CHSH}} \).

### 3.2 Accumulation and extraction

After fixing the parameters of the protocol and constructing a min-tradeoff function \( f_{\min} \), the user proceeds with the remaining steps of Protocol QRE: accumulation and extraction. The accumulation step consists of the device interaction and evaluation sub-procedures that were detailed in Sec. 2.5.1. If the protocol does not abort, then with high probability the generated string \( AB \) contains at least some given quantity of smooth min-entropy. The following lemma applies the EAT to deduce a lower bound on the amount of entropy that they will have accumulated.

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17Due to the infrequent testing we are actually only able to certify a maximum of \( 2 \cdot (1 - \gamma) \) bits per interaction.
Figure 3: A plot of a lower bound on the certifiable min-entropy produced during a single round. This lower bound was calculated using Prog. 9 relaxed to the second level of the NPA hierarchy. In addition, we plot a min-tradeoff function $f_\nu$ evaluated for protocol respecting distributions parameterized by the OVG score $\omega$.

Since $f_\nu$ is the tangent plane to the surface at the point $\nu$ it forms an affine lower bound on the min-entropy of any distribution in $\Gamma$.

**Lemma 3.3** (Accumulated entropy): Let the randomness expansion procedure and all of its parameters be as defined in Fig. 2. Furthermore, let $\Omega$ be the event the protocol does not abort (cf. (21)) and let $\rho_\Omega$ be the final state of the system conditioned on this. Then, for any $\beta, \epsilon_s, \epsilon_{\text{EAT}} \in (0, 1)$ and any choice of min-tradeoff function $f_\nu \in F_{\text{min}}$, either Protocol QRE aborts with probability greater than $1 - \epsilon_{\text{EAT}}$ or

$$H_{\text{min}}(AB|XYE)_{\rho_\Omega} > (1 - \gamma)n(A_\nu - B_\nu, \lambda_\nu \cdot (\omega - \delta_\perp)) - n(\epsilon_V + \epsilon_K) - \epsilon_{\Omega},$$  

where

$$\epsilon_V := \frac{\beta \ln 2}{2} \left( \log(2|AB|^2 + 1) + \sqrt{(1 - \gamma)^2 B_\nu^2(\lambda_{\text{max}} - \lambda_{\text{min}})^2 + 2} \right)^2,$$

$$\epsilon_K := \frac{\beta^2}{6(1 - \beta)^3} \ln 2 2^{\beta \log|AB|/(1 - \gamma)B_\nu(\lambda_{\text{max}} - \lambda_{\text{min}})} \ln 3 \left( 2^{\beta \log|AB|/(1 - \gamma)B_\nu(\lambda_{\text{max}} - \lambda_{\text{min}})} + \epsilon \right)^2,$$

$$\epsilon_{\Omega} := \frac{1}{\beta} (1 - 2 \log(\epsilon_{\text{EAT}} \epsilon_s))$$

and $\delta_\perp$ is the vector with components $\delta_i \cdot \text{sgn}(-\lambda_i)$.

**Proof.** Let $\{N_i\}_{i \in [n]}$ be the set of channels implementing the entropy accumulation sub-procedure of Protocol QRE. As discussed in Sec. 2.5.2 each of these channels necessarily satisfies the definition of an EAT-channel and by Lemma 3.2 $f_\nu$ is a min-tradeoff function for these channels. We therefore satisfy all of the prerequisites required to use the EAT.
Consider now the pass probability of the protocol, \( p_\Omega \). Either \( p_\Omega < \epsilon_{\text{EAT}} \), in which case the protocol will abort with probability greater than \( 1 - \epsilon_{\text{EAT}} \), or \( \epsilon_{\text{EAT}} \leq p_\Omega \). In the latter case we can replace the unknown \( p_\Omega \) in (30) with \( \epsilon_{\text{EAT}} \) as this results in an increase in the error term \( \epsilon_\Omega \). The EAT then asserts that

\[
H^c_{\text{min}}(AB|IE)_{p_\Omega} > n \inf_{C \in \Omega} f_\nu(freq_C) - n(\epsilon_V + \epsilon_K) - \epsilon_\Omega.
\]

for any choice of min-tradeoff function \( f_\nu \in F_{\text{min}} \).

As the min-tradeoff functions are affine, we can lower bound the infimum for the region of possible scores specified by the success event,

\[
\Omega = \{ C \mid \gamma n(\omega - \delta) < C_{\text{total}}(G) < \gamma n(\omega + \delta) \}.
\]

Taking \( p = (\gamma(\omega - \delta_\pm), p_L, (1 - \gamma)) \), where \( p_L \in \mathbb{R} \) is a normalization constant set to make \( \sum_i p_i = 1 \), we have \( f(p) \leq \inf_{C \in \Omega} f_\nu(freq_C) \)\(^{18}\). Note that \( p \) may not correspond to a frequency distribution that could have resulted from a successful run of the protocol – it may not even be a probability distribution. However, it is sufficient for our purposes as an explicit lower bound on the infimum. Further, noting that \( f_\nu(p) = g_\nu(\omega - \delta_\pm) \), we can straightforwardly compute this lower bound as

\[
f_\nu(p) = (1 - \gamma)(A_\nu - B_\nu \lambda_\nu \cdot (\omega - \delta_\pm)).
\]

Inserting the min-tradeoff function properties: (30), (37) and (38): into the the EAT’s error terms we get the explicit form of the quantities \( \epsilon_V, \epsilon_K \) and \( \epsilon_\Omega \) as seen above.

If the protocol does not abort during the accumulation procedure, the user may proceed by applying a quantum-proof strong extractor to the concatenated output string \( AB \) resulting in a close to uniform bit-string of length approximately \( (1 - \gamma)n(A_\nu - B_\nu \lambda_\nu \cdot (\omega - \delta_\pm)) - n(\epsilon_V + \epsilon_K) - \epsilon_\Omega \).

**Example 3.2:** Continuing from Ex. 3.1 we look at the bound on the accumulated entropy specified by (39) for a range of choices of \( f_\nu \in F_{\text{min}} \). Again, we are considering a quantum implementation with an expected score vector \( \omega = (0.49, 0.4225) \). In Fig. 1 we see that our choice of min-tradeoff function can have a large impact on the quantity of entropy we are able to certify. The rough appearance of the EAT-rate surface is an artefact of obtaining local optima when we optimize the choice of \( \beta \). However, the plot gives some reassuring numerical evidence that, for the OVG \( \mathcal{G}_{\text{CHSH}} \), the certifiable randomness is continuous and concave in the family parameter \( \nu \).

The min-tradeoff function indexed by our assumed score vector, \( f_\omega \), is able to certify just under 0.939-bits per interaction. By applying a gradient-ascent algorithm we were able to improve this to 0.946-bits per interaction. In an attempt to avoid getting stuck within local optima we applied the algorithm several times, starting subsequent iterations at randomly chosen points close to the current optimum. The optimization led to an improved min-tradeoff function choice \( f_\nu^* \), where \( \nu^* = (0.491, 0.421) \).

### 3.3 Protocol QRE

Protocol QRE is the concatenation of the accumulation and extraction sub-procedures. It remains to provide the formal security statements for a general instance of Protocol QRE. We refer to an untrusted device network \( \mathcal{D}_{AB} \) as *honest* if during each interaction, the underlying quantum state shared amongst the devices and the measurements performed in response to inputs remain the same (i.e., the devices behave as the user expects). Furthermore, each interaction is performed independently of all others. The following lemma provides a bound on the probability that an honest implementation of Protocol QRE aborts.

**Lemma 3.4** (Completeness of Protocol QRE): Let Protocol QRE and all of its parameters be as defined in Fig. 2. Then, the probability that an honest implementation of Protocol QRE aborts is no greater than \( \epsilon_{\text{comp}} \) where

\[
\epsilon_{\text{comp}} = 2 \sum_{k=1}^{\lfloor \gamma\omega \rfloor} e^{-\frac{\gamma^2}{2\pi k} n}.
\]

\(^{18}\)As was noted in Remark 2.1 it may be the case that reducing a score leads to an increase in entropy and therefore the components of \( \lambda_\nu \) may take positive or negative values. To account for this, we adapt the sign of the elements within the statistical confidence vector allowing us to correctly bound \( \inf_{C \in \Omega} f_\nu(freq_C) \).
Proof. During the parameter estimation step of Protocol QRE, the protocol aborts if the total EAT-score $C_{\text{total}}$ fails to satisfy
\[ \gamma n (\omega - \delta) < C_{\text{total}}(G) < \gamma n (\omega + \delta). \]
Writing $C_{\text{total}}(G) = (c_k)_{k=1}^{|G|}$, $\omega = (\omega_k)_{k=1}^{|G|}$ and $\delta = (\delta_k)_{k=1}^{|G|}$, the probability that an honest implementation of the protocol aborts can be written as
\[
P_{\text{abort}} = \Pr \left[ \bigcup_{k=1}^{|G|} \{ |c_k - \gamma \omega_k n| \geq \gamma \delta_k n \} \right] \leq \sum_{k=1}^{|G|} \Pr \left[ |c_k - \gamma \omega_k n| \geq \gamma \delta_k n \right].
\]
Restricting ourselves to a single element $c_k$ of $C_{\text{total}}(G)$, we can model its final value as the binomially distributed random variable $c_k \sim \text{Bin} (n, \gamma \omega_k)$. As a consequence of the Chernoff bound (cf. Corollary B.1), and that $\delta_k < \omega_k$, we have
\[
\Pr \left[ |c_k - \gamma \omega_k n| \geq \gamma n \delta_k \right] \leq 2e^{-\frac{\gamma^2 \delta_k^2 n}{3\omega_k^2}}.
\]
Applying this bound to each element of the sum individually, we arrive at the desired result.

Remark 3.1: The completeness error in the above lemma only considers the possibility of the protocol aborting during the parameter estimation stage. However, if the initial random seed is a particularly limited resource then it is possible that the protocol aborts due to seed exhaustion. In Lemma B.4 we analyse a sampling algorithm required to select the inputs during device interaction. If required, the probability of failure for that algorithm could be incorporated into the completeness error.
With a secure bound on the quantity of accumulated entropy established by Lemma 3.3 we can apply a $(k, \epsilon_{\text{ext}} + 2\delta_s)$-strong extractor to $AB$ to complete the security analysis. Combined with the input randomness discussed in Appendix B we arrive at the following theorem.

**Theorem 3.1** (Security of Protocol QRE): Let Protocol QRE be implemented with some initial random seed $D$ of length $d$. Furthermore let all other protocol parameters be chosen within their permitted ranges, as detailed in Fig. 2. Then the soundness error of Protocol QRE is

$$\epsilon_{\text{sound}} = \max(\epsilon_{\text{ext}} + 2\delta_s, \epsilon_{\text{EAT}}).$$

**Proof.** Recall from [17] that the soundness error is an upper bound on $\frac{1}{2}\Pr[\Omega] \cdot \|\rho_{ZE} - \tau_m \otimes \rho_E\|_1$. In the case $\Pr[\Omega] \leq \epsilon_{\text{EAT}}$, we have $\frac{1}{2}\Pr[\Omega] \cdot \|\rho_{ZE} - \tau_m \otimes \rho_E\|_1 \leq \epsilon_{\text{EAT}}$.

In the case $\Pr[\Omega] > 1 - \epsilon_{\text{EAT}}$, Lemma 3.3 gives a bound on the accumulated entropy. Combining with the definition of a quantum-proof strong extractor Def. 2.5 and noting that the norm is non-increasing under partial trace we obtain $\frac{1}{2}\Pr[\Omega] \cdot \|\rho_{ZE} - \tau_m \otimes \rho_E\|_1 \leq \epsilon_{\text{ext}} + 2\epsilon_s$, from which the claim follows.

**Remark 3.2:** By choosing parameters such that $\epsilon_{\text{EAT}} \leq \epsilon_{\text{ext}} + 2\epsilon_s$ we can take the soundness error to be $\epsilon_{\text{ext}} + 2\epsilon_s$.

Combining all of the previous results we arrive at the full security statement concerning Protocol QRE.

**Theorem 3.1** (Security of Protocol QRE): Protocol QRE is an $(\epsilon_{\text{comp}}, \epsilon_{\text{sound}})$-secure

$$[d] \to [(1 - \gamma) (A_\nu - B_\nu \lambda_\nu \cdot (\omega - \delta_\omega)) - \epsilon_V - \epsilon_K] n - \epsilon_\Omega - \ell_{\text{ext}}$$

randomness expansion protocol, where $\epsilon_{\text{comp}}, \epsilon_{\text{sound}}$ are given by Lemma 3.4 (cf. Remark 3.1) and Lemma 3.3.

**Remark 3.3:** The expected seed length required to execute Protocol QRE is $d \approx (\gamma H(\mu) + h(\gamma)) n$ (cf. Lemma B.4).

**Example 3.3:** In Ex. 3.1 and Ex. 3.2 we used the following choice of protocol parameters: $n = 10^{10}$, $\gamma = 5 \times 10^{-3}$, $\delta_1 = \cdots = \delta_{|\mathcal{G}|} = 10^{-3}$ and $\epsilon_s = \epsilon_{\text{EAT}} = 10^{-8}$. The resulting implementation of Protocol QRE, using the OVG $\mathcal{G}_{\text{CHSH}}$ with an expected score vector $\omega = (0.49, 0.4225)$, exhibits the following statistics.

| Quantity | Value |
|----------|-------|
| Total accumulated entropy before extraction (no abort) | $9.46 \times 10^8$ |
| Expected length of required seed before extraction | $5.54 \times 10^8$ |
| Expected net-gain in entropy (no abort) | $8.91 \times 10^8 - \ell_{\text{ext}}$ |
| Completeness error ($\epsilon_{\text{comp}}$) | $8.77 \times 10^{-8}$ |

4 Examples

In this section we demonstrate the use of our framework through the construction and analysis of several protocols based on different tests of nonlocality. To this end, we begin by introducing two families of OVGs which we consider alongside $\mathcal{G}_{\text{CHSH}}$.

**Empirical behaviour game ($\mathcal{G}_{\text{EB}}$).** The empirical behaviour game ($\mathcal{G}_{\text{EB}}$) is the OVG that estimates the underlying behaviour of $\mathcal{O}_{AB}$, i.e., each individual probability $p(a, b|x, y)$. The scoring vector for this game is $V_{\text{EB}}$ which is defined by the rule $(a, b, x, y) \mapsto \epsilon_{a b x y}$. Then, for any input distribution $\mu_{\text{EB}}$ with full support on the alphabets $\mathcal{X}\mathcal{Y}$, the collection $\mathcal{G}_{\text{EB}} = (\mu_{\text{EB}}, V_{\text{EB}})$ forms an orthogonal vector game with an expected score corresponding to the behaviour of the device. As $\mathcal{G}_{\text{EB}}$ can be defined for any collection of input-output alphabets, we indicate the size of these alphabets as superscripts, i.e., $\mathcal{G}_{\text{EB}}^{\mathcal{X}\mathcal{Y}[A][B]}$.\footnote{As all output alphabets we consider are binary, we will not include their sizes in the superscript labelling. I.e., we will write $\mathcal{G}_{\text{EB}}^{\mathcal{X}\mathcal{Y}[A][B]}$ instead of $\mathcal{G}_{\text{EB}}^{\mathcal{X}\mathcal{Y}[A][B]}$.}
Remark 4.1: The score vector for $G_{EB}$, as defined above, has several redundant components. These redundancies come from normalization and the no-signalling constraints. After considering these constraints, there are only $|\mathcal{A}| - 1|\mathcal{A}| + 1)(|\mathcal{B}| - 1)|\mathcal{B}| + 1| - 1$ free parameters remaining [47]. With retention of orthogonality in mind, one can define an equivalent but quantitatively smaller set of scoring rules for $G_{EB}$. One such set is $(V_{a|x}, V_{b|y}, V_{ab|xy})_{abxy}$ for $(x, y) \in \mathcal{X}\mathcal{Y}$ and $(a, b) \in \mathcal{A}^i\mathcal{B}^i$, where $\mathcal{A}^i$ and $\mathcal{B}^i$ indicate the sets $\{\tilde{a}\} \in A$ and $\{\tilde{b}\} \in B$. The scoring functions are defined by

$$V_{a'|x'}(a, b, x, y) = \delta_{aa'}\delta_{xx'}\delta_{bb'},$$

and

$$V_{b'|y'}(a, b, x, y) = \delta_{bb'}\delta_{yy'}\delta_{aa'},$$

This is equivalent to the minimal parameterization of no-signalling behaviours, $\{p(a|x), p(b|y), p(a, b|x, y)\}$ [43]. However, in our parameterization we replace the marginal terms with quantities of the form $p(ab|xy) = p(a|x) - \sum_{b \in B} p(a, b|x, y)$ which preserve orthogonality.

Joint correlators game ($G_{(AB)}$). Because we can only collect finite statistics, we are subject to a tradeoff between precision and confidence. That is, in order to use OVGs with larger numbers of scores we may be required to collect substantially more test data. The joint correlators game, which we now introduce, offers an intermediate step between a single score game and $G_{EB}$. Specifically, for every pair of inputs $(x, y) \in \mathcal{X}\mathcal{Y}$, we define a scoring function

$$V_{x'y'}(a, b, x, y) = \delta_{aa'}\delta_{bb'}\delta_{xx'}\delta_{yy'}.$$  

That is for a pair of inputs $(x, y)$ the score is $e_{xy}$ whenever their outcomes agree and 0 otherwise. As each scoring function $V_{xy}$ can only produce a non-zero value for its indexed inputs, the condition of orthogonality is satisfied. Taking, for example, $\mu_{(AB)}$ to be the uniform distribution on $\mathcal{X}\mathcal{Y}$, we define the full alignment game to be $G_{(AB)} = (\mu_{(AB)}, (V_{xy})_{xy})$.

4.1 Noise robustness

We now compare the accumulation rates of protocols built using the OVGs described above. We retain the protocol parameter choices from the previous examples: $n = 10^{10}$, $\gamma = 5 \times 10^{-3}$ and $\epsilon_s = \epsilon_{EAT} = 10^{-8}$, except we now set the confidence interval width parameter to

$$\delta_k = \sqrt{\frac{3\omega_k \log(2|G|/\varepsilon_{comp})}{\gamma n}}, \quad (44)$$

in order to keep a constant completeness error $\varepsilon_{comp} \approx 10^{-12}$ across the different protocols [20].

To discuss robustness to noise, we suppose that the devices operate by using a pure, non-maximally entangled state of the form

$$|\psi(\theta)\rangle_{AB} = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle, \quad (45)$$

for $\theta \in [0, \pi/2]$. We denote the corresponding density operator by $\rho_\theta = |\psi(\theta)\rangle\langle\psi(\theta)|$. For simplicity we restrict to projective measurements within the $x$-$z$ plane of the Bloch-sphere, i.e., measurements $\{\Pi(\varphi), 1 - \Pi(\varphi)\}$, with the projectors defined by

$$\Pi(\varphi) = \begin{pmatrix} \cos^2(\varphi/2) & \cos(\varphi/2)\sin(\varphi/2) \\ \cos(\varphi/2)\sin(\varphi/2) & \sin^2(\varphi/2) \end{pmatrix}, \quad (46)$$

for $\varphi \in [0, 2\pi]$. We denote the projectors associated with the $j^{th}$ outcome of the $i^{th}$ measurement by $A_{ji}$ and $B_{ji}$. The elements of the devices’ behaviour can then be written as

$$p(a, b|x, y) = \text{Tr} \left[ \rho_\theta (A_{a|x} \otimes B_{b|y}) \right], \quad (47)$$

[20] In practice one would fix the soundness error of the protocol. However, because the soundness error is also dependent on the extraction phase we instead assume independence of rounds and fix the completeness error.
Our analysis is focused on how the accumulation rates differ when the devices operate with inefficient detectors. Heralding can be used to account for losses incurred during state transmission and has been used to develop novel device-independent protocols [48]. However, losses that occur within a user’s laboratory cannot be ignored without opening a detection loophole [49]. Inefficient detectors are a major contributor to the total experimental noise, so robustness to inefficient detectors is a necessary property for any practical randomness expansion protocol. We characterize detection efficiency by a single parameter \( \eta \in [0, 1] \), representing the (independent) probability with which a measurement device successfully measures a received state and outputs the result. To deal with failed measurements we assign outcome 0 when this occurs. Combining this with (47), we may write the behaviour is then

\[
p(a, b|x, y) = \eta^2 \text{Tr} \left[ \rho_0 (A_a|x \otimes B_{b|y}) \right] + (1 - \eta)^2 \delta_{0a} \delta_{0b} \\
+ \eta(1 - \eta) \left( \delta_{0a} \text{Tr} \left[ \rho_0 (1 \otimes B_{b|y}) \right] + \delta_{0b} \text{Tr} \left[ \rho_0 (A_{a|x} \otimes 1) \right] \right).
\]

(48)

For each protocol we consider lower bounds on two quantities: the pre-EAT gain in min-entropy from a

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21 For simplicity, we make the additional assumption that the detection efficiencies are constant amongst all measurement devices used within the protocol.
Figure 6: Comparison illustrating the EAT-rates (cf. (39)) converging to the i.i.d. rates for protocols based on different OVGs. The rates were derived by assuming a qubit implementation of the protocols with a detection efficiency $\eta = 0.9$, optimizing the state and measurement angles in order to maximise the i.i.d. rate. Then, for each value of $n$ we optimized the min-tradeoff function choice and $\beta$ parameter and noted the resulting bound on $H_{\min}^{e}$. To ensure that we approach the i.i.d. bound as $n$ increased we set $\gamma = \delta_1 = \cdots = \delta_{|G|} = n^{-1/3}$, resulting in a constant completeness error across all values of $n$.

With inefficient detectors, partially entangled states can exhibit larger Bell-inequality violations than maximally entangled states [50]. To account for this we optimize both the state and measurement angles at each data point using the iterative optimization procedure detailed in [51]. All programs were relaxed to the second level of the NPA hierarchy using [52] and the resulting SDPs were computed using the SDPA solver [53]. The results of these numerics are displayed in Fig. 5.

In Fig. 5a and Fig. 5b we see that in both families of protocols considered, an increase in the number of inputs leads to higher rates. This increase is significant when one moves from the $(2,2)$-scenario to the $(2,3)$-scenario. However, continuing this analysis for higher numbers of inputs we find that any further increases appear to have negligible impact on the overall robustness of the protocol [23]. Whilst all of the protocols achieve i.i.d. rates of 2 bits per round when $\eta = 1$, their respective EAT-rates at this point differ substantially. In Fig. 5c we see a direct comparison between protocols from the different families. The plot shows that, as expected, entropy loss is greater when using the nonlocality test $G_{EB}^{23}$ as opposed to the other protocols. In particular, for high values of $\eta$ we find that we would be able to certify a larger quantity of entropy by considering fewer scores. However, it is still worth noting that this entropy loss could be reduced by choosing a more generous set of protocol parameters, e.g., increasing $n$ and decreasing $\delta$.

Increasing $n$ can be difficult in practice due to restrictions on the overall runtime of the protocol. Not only does it take longer to collect the statistics within the device-interaction phase, but it may also increase the runtime of the extraction phase [54]. In Fig. 6 we observe how quickly the various protocols converge on single interaction, $H_{\min}^{e}(AB|XE)$, and the EAT-rate, $H_{\min}^{e}/n$. The former quantity, which we refer to as the i.i.d. rate, represents the maximum accumulation rate achievable with our numerical technique. It is a lower bound on $H_{\min}^{e}/n$, specified by (39), as $n \to \infty$ and $\gamma, \delta \to 0$. Comparing these two quantities gives a clear picture of the amount of entropy that we lose due to the effect of finite statistics.

22 We would really like to plot $H(AB|XE)$ and the corresponding EAT-rate derived from it. However, in general we don’t have suitable techniques to access these quantities in a device-independent manner.

23 This could also be an artefact of the assumed restriction to qubit systems.
their respective i.i.d. rates as we increase $n$. Again we find that, due to the finite-size effect, entropy loss when using $G_{EB}^{23}$ is greater than that observed in the other protocols. In particular, we see that for protocols with fewer than $10^{10}$ rounds, it is advantageous to use $G_{CHSH}^{23}$. From the perspective of practical implementation, Figs. 5c and 6 highlight the benefits of a flexible protocol framework wherein a user can design protocols tailored to the scenario under consideration.

5 Conclusion

We have shown how to combine device-independent bounds on the guessing probability with the EAT, to create a versatile method for analysing quantum-secure randomness expansion protocols. The construction was presented as a template protocol from which an exact protocol can be specified by the user. The relevant security statements and quantity of output randomness of the derived protocol can then be evaluated numerically. A Python package [24] accompanies this work to help facilitate implementation of the framework. In Sec. 4 we illustrated the framework, applying it to several example protocols. We then compared the robustness of these protocols when implemented on qubit systems with inefficient detectors. Our analyses show that, within a broadly similar experimental setup, different protocols can have significantly different rates, and hence that it is worth considering small modifications to a protocol during their design.

Although the framework produces secure and robust protocols, there remains scope for further improvements. For example, our work relies on the relation $H(AB|XYE) \geq H_{\min}(AB|XYE)$ which is far from tight. The resulting loss can be seen when one compares the i.i.d. rate of $G_{CHSH}$ in Fig. 5c with those presented in [14] (see Fig. 8). Several alternative approaches could be taken in order to reduce this loss. Firstly, the above relation is part of a more general ordering of the conditional Rényi entropies [24] If one were able to develop efficient computational techniques for computing device-independent lower bounds on one of these alternative quantities we would expect an immediate improvement. Furthermore, dimension-dependent bounds may be applicable in certain situations. For example, it is known that for the special case of $n$-party, 2-input, 2-output scenarios it is sufficient to restrict to qubit systems [31]. Approaching the problem from the other direction, one could attempt to make the EAT more accommodating to entropy measures that can be more easily bounded.

Optimising the choice of min-tradeoff function over $\mathcal{F}_{\min}$ is a non-convex and not necessarily continuous problem [55]. Our analysis in Sec. 4 used a simple probabilistic gradient ascent algorithm to approach this problem. A more sophisticated approach to this optimization could yield higher EAT-rates, particularly for protocols with a higher number of scores, e.g., $G_{EB}$.

As the framework permits protocols that rely on any linear test of nonlocality, it is natural to search for tests that provide high EAT-rates from within this set. Investigations into the randomness certification properties of nonlocality tests with larger output alphabets or additional parties could be of interest. However, increasing either of these parameters is likely to increase the influence of finite-size effects. Alternatively, without increasing the complexity of the scenario, one could leverage the fact that we are able to track multiple scores to search for complementary tests of randomness, i.e., collections of Bell-expressions which together form robust witnesses of randomness. Previous work [56] has shown that the relationship between randomness and nonlocality is not straightforward. Identifying and isolating components of Bell-expressions that are crucial for the task of randomness certification would be of great practical and theoretical interest.

Finally, our computational approach to the EAT considered only the task of randomness expansion. Our work could be extended to produce adaptable security proofs for other device-independent tasks. Given that the EAT has already been successfully applied to a wide range of problems [32, 57–60], developing good heuristics for robust min-tradeoff function constructions represents an important step towards practical device-independent security.

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24 The Rényi entropies are one of many different entropic families that include the von Neumann entropy as a limiting case. Any such family could be used if they satisfy an equivalent relation.
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Appendices

A Table of parameters and notation

| Notation | Description | Initial reference |
|----------|-------------|-------------------|
| D        | A collection of untrusted devices. | Sec. 2.3 |
| G        | An orthogonal vector game. | Def. 2.1 |
| Q_G      | Set of expected score vectors for G using quantum strategies. | Sec. 2.3 |
| Q_G(k)   | Set of expected score vectors for G using strategies from k^{th} level of NPA. | Sec. 2.3 |
| ν, ω     | Expected score vectors for an OVG. | Eqn. 12 |
| p(k)_guess, d(k)_guess | Solutions to the k-relaxed primal and dual guessing probability programs. | Prog. 9 and Prog. 10 |
| (α_ν, λ_ν) | Feasible point of the dual guessing probability program with parameter ν. | Sec. 2.3 |
| δ        | Vector of statistical confidence interval widths. | Eqn. 21 |
| δ_i      | δ with elements signed in accordance with a given λ. | Lemma 3.3 |
| A, B     | Devices’ output alphabets. | Sec. 2.1 |
| X, Y     | Devices’ input alphabets. | Sec. 2.1 |
| n ∈ N   | Number of rounds in the device-interaction phase. | Sec. 2.5.1 |
| γ ∈ (0, 1) | Probability that any given round is a test round. | Sec. 2.5.1 |
| A_i, B_i | Devices’ outputs for the i^{th} round. | Sec. 2.5.1 |
| X_i, Y_i | Devices’ inputs for the i^{th} round. | Sec. 2.5.1 |
| C_i      | EAT-score for the i^{th} round. | Sec. 2.5.1 |
| C_{total} | Total EAT-score for the protocol. | Eqn. 19 |
| Ω        | Event that the protocol does not abort. | Eqn. 21 |
| R_{ext}  | Strong quantum-secure randomness extractor. | Def. 2.5 |
| ε_{comp} | Completeness error of Protocol QRE. | Lemma 3.3 |
| ε_{sound} | Soundness error of Protocol QRE. | Lemma 3.3 |
| ε_s      | Smoothing parameter for H_{min}. | Eqn. 7 |
| ε_{EAT} | Tolerance of unlikely success events. | Lemma 3.3 |
| ε_V      | EAT error term (Variance). | Thm. 2.1 |
| ε_K      | EAT error term (Remainder). | Thm. 2.1 |
| ε_Ω      | EAT error term (Pass probability). | Thm. 2.1 |
| ε_{extract} | Extractor error. | Def. 2.5 |
| ℓ_{ext} | Entropy lost during extraction. | Sec. 2.6 |

B Input Randomness

Here we quantify the length of the initial random seed required to execute an instance of Protocol QRE. This supply of random bits is necessary for selecting the devices’ inputs and seeding the extractor. In the forthcoming analysis we ignore the latter as this quantity depends on the choice of extractor. Instead, we look at the process of converting a uniform private seed into device inputs required for running Protocol QRE. We follow a similar procedure to that used in [14], modifying the algorithm slightly in order to extract explicit bounds.

B.1 Statistical bounds

We begin by stating some standard statistical bounds. The first is commonly known as the Chernoff bound [61], although we take our formulation from [62]. This provides a convenient bound on the deviation of the sum of random variables from the expected value.

**Lemma B.1 (Chernoff bound):** Let \( X_i \) be independent binary random variables for \( i = 1, \ldots, n \), \( S = \sum_i X_i \), then for any \( a > 0 \)
and $\mu = \mathbb{E}[S]$. Then for $0 \leq t \leq 1$

$$
\begin{align*}
\Pr[S \geq (1 + t)\mu] &\leq e^{-t^2\mu/3} \\
\Pr[S \leq (1 - t)\mu] &\leq e^{-t^2\mu/2}.
\end{align*}
$$

**Corollary B.1:** For $r \leq \mu$ we have $\Pr[|S - \mu| \geq r] \leq 2e^{-r^2/(3\mu)}$.

In addition to this, we also make use of Hoeffding’s inequality \[63\].

**Lemma B.2** (Hoeffding’s inequality): Let $X_i$ be independent random variables, such that $a_i \leq X_i \leq b_i$ with $a_i, b_i \in \mathbb{R}$ for $i = 1, \ldots, n$. In addition, let $S = \sum_i X_i$ and $\mu = \mathbb{E}[S]$. Then for $t > 0$

$$
\Pr[|S - \mu| \geq t] \leq 2e^{-\sum_i (b_i - a_i)^2/4t^2}.
$$

### B.2 Rounded interval algorithm

The interval algorithm provides an efficient method for simulating the sampling of some target random variable $T$ using another random variable $S$. To aid understanding of our modification to this algorithm and any subsequent results we shall briefly explain how this simulation works. For simplicity we restrict ourselves to the scenario where $S$ is a sequence of uniformly distributed bits, we denote the uniform distribution on an alphabet of size $2^k$ by $U_{2^k}$, for $k \in \mathbb{N}$.

The distribution of the target random variable $T$ forms a partition of the unit interval, one subinterval for each outcome $t$ of $T$. In exactly the same way, the probability distribution for $U_{2^k}$ partitions the unit interval into $2^k$ subintervals. Thus, we can associate a bit-string with its corresponding subinterval, defined by this partitioning. The interval algorithm works by using an increasing sequence of random bits and the corresponding subintervals that the sequence defines. Once the subinterval generated by the sequence of bits is contained inside one of the subintervals defined by the target random variable $T$, say $t$, then we say that we have simulated the sampling of $t$ from $T$ and the algorithm terminates. Denoting by $N$ the length of seed required for the interval algorithm to terminate, then by \[64\] Theorem 3 we have

$$
\mathbb{E}[N] \leq H(T) + 3. \tag{49}
$$

As the algorithm stands, the maximum value that $N$ can take is unbounded (although the probability that the algorithm does not terminate decreases exponentially in $N$). In order to produce large deviation bounds on the number of bits required to execute our protocol, we place an upper limit on the maximum seed length. We thus propose an adapted sampling procedure, the rounded interval algorithm (RIA), which forcefully terminates if the seed length reaches the upper bound of $k_{\text{max}}$ bits.

Should the RIA fail to terminate after $k_{\text{max}}$ steps, then the output sequence generated will correspond to some subinterval $I(r) = \left[\frac{r}{2^k_{\text{max}}}, \frac{r+1}{2^k_{\text{max}}}\right)$, for some $r \in \{0, 1, \ldots, 2^{k_{\text{max}}} - 1\}$, that is not entirely contained within one of the subintervals induced by $T$. If this occurs, we round down: selecting the interval $I_t$ for which $\frac{r}{2^k_{\text{max}}} \in I_t$.

**Remark B.1:** Note that the above procedure depends on the ordering of the intervals generated by $T$ (which should be fixed before sampling). One could imagine a rather pathological scenario where an ordering places extremely unlikely outcomes over rounding points, greatly increasing their simulated outcome probabilities. However, as will be shown in Lemma \[33\], the distance between the simulated random variable and the target random variable decreases exponentially in $k_{\text{max}}$.

**Remark B.2:** The rounding procedure truncates the maximum seed length, $N \leq k_{\text{max}}$, and as such, it is clear that the inequality \[19\] also holds for the RIA.

**Definition B.1** (Statistical distance): Given two random variables $X$ and $X'$, taking values in some common alphabet $\mathcal{X}$. The statistical distance between $X$ and $X'$, is defined by

$$
\Delta(X, X') := \frac{1}{2} \sum_{x \in \mathcal{X}} |p_X(x) - p_{X'}(x)|. \tag{50}
$$
Lemma B.3: Let $T$ be a random variable taking values in some alphabet $T$. Let $T'$ be the distribution sampled using the RIA with target distribution $T$. Then

$$\Delta(T, T') \leq \lvert T \rvert 2^{-(k_{\text{max}}+1)},$$

where $k_{\text{max}}$ is the maximum number of input bits that can be used by the RIA.

Proof. Consider the partitions of the unit interval \( \{I(t)\}_{t \in \mathcal{T}} \) and \( \{I'(t)\}_{t \in \mathcal{T}} \) corresponding to the distributions $p_T$ and $q_{T'}$ of $T$ and $T'$ respectively. The intervals of $T'$ take the form

$$I'(t) = \bigcup_r \left[ \frac{r}{2^{k_{\text{max}}}}, \frac{r+1}{2^{k_{\text{max}}}} \right)$$

where the (potentially empty) union is taken over all $r \in \mathbb{N}_0$ such that $r2^{-k_{\text{max}}} \in I(t)$. The intervals within the union are either contained fully within the corresponding outcome interval of $T$, i.e., $\left[ \frac{r}{2^{k_{\text{max}}}}, \frac{r+1}{2^{k_{\text{max}}}} \right) \subseteq I(t)$, or they are included as a result of rounding. Thus we may write

$$\lvert I'(t) \rvert = \lvert \{r \mid r \cdot 2^{-k_{\text{max}}} \in I(t), r \in \mathbb{N}_0 \} \rvert 2^{-k_{\text{max}}}.$$ 

By a straightforward counting argument, there are at least \( \lceil I(t) \rceil 2^{k_{\text{max}}} \) such values of $r$, and at most \( \lceil I(t) \rceil 2^{k_{\text{max}}} \). We hence have

$$\lvert I(t) \rceil 2^{k_{\text{max}}} - 1 \leq \lvert I'(t) \rceil 2^{k_{\text{max}}} \leq \lvert I(t) \rceil 2^{k_{\text{max}}} + 1,$$

and therefore

$$\lvert p_T(t) - p_{T'}(t) \rvert \leq 2^{-k_{\text{max}}}.$$ 

holds for all $t \in \mathcal{T}$. Applying this bound to each term within the $\Delta(T, T')$ sum completes the proof. \qed

B.3 Input randomness for Protocol QRE

Following the structure of Protocol QRE, we look to use the RIA to sample the devices’ inputs for each round. In adherence with the Markov-chain condition (Def. 2.3), the natural procedure would be to sample at the beginning of each round. However, in practice this requires a much larger seed: because of (49) and the property $H(T^n) = nH(T)$, by sampling the joint distribution the expected saving is about $3n$ bits compared to repeating a single sample $n$ times. Fortunately, this joint sampling can be implemented while maintaining the Markov-chain condition. Within the assumptions of Protocol QRE we allow the honest parties access to a trusted classical computer, which would also contain some trusted data storage—we assume that the parties can record their outcome strings without leakage. Thus, the honest parties could select the devices’ inputs prior to the device interaction phase, store them securely within the classical computer and at the beginning of each round feed the corresponding inputs to the devices. In such a scenario we retain the Markov-chain condition. Due to potential computational constraints and to permit large deviation bounds on the number of bits required we will not assume that all $n$ rounds are sampled at once. Instead, we split the $n$ rounds into at most \( \lceil n/m \rceil \) blocks of size $m$ and apply the RIA to sample the inputs of each block separately. For simplicity, we assume that $n/m \in \mathbb{N}$ and henceforth remove the ceiling function from the analysis.

Recall that for the $i^{\text{th}}$ round, the user first uses $T_i$ to decide whether the round is a test round, and, if so, they choose inputs according to the OVG input distribution $\mu$. Otherwise, if $T_i = 0$, they supply their devices with the fixed inputs $\hat{x}$ and $\hat{y}$. The probability mass function of joint random variables $X_i, Y_i, T_i$, representing the $i^{\text{th}}$ round’s inputs, is therefore

$$\Pr[(X_i, Y_i, T_i) = (x_i, y_i, t_i)] = \begin{cases} 
\gamma \mu(x, y) & \text{for } (x_i, y_i, t_i) = (x, y, 1), \\
(1 - \gamma) & \text{for } (x_i, y_i, t_i) = (\hat{x}, \hat{y}, 0), \\
0 & \text{otherwise}.
\end{cases} \quad (51)$$

Following (49), if $M$ is the seed length required to sample one of the $m$ blocks of rounds, then we have

$$\mathbb{E}[M] \leq \frac{(\gamma H(\mu) + h(\gamma)) n}{m} + 3 \quad (52)$$

32
where $H(\mu)$ is the Shannon entropy of the distribution $\mu$ and $h(\cdot)$ is the binary entropy.

The following lemma gives a probabilistic bound on the total length of the random seed required to sample the inputs for the devices.\(^{25}\)

**Lemma B.4:** Let the parameters of Protocol QRE be as defined in Fig. 3 and let $k_{\text{max}} \in \mathbb{N}$ be the maximum permitted seed length for an instance of the RIA. Then, with probability greater than $(1 - \epsilon_{\text{RIA}})$, we can use $m$ instances of the RIA to simulate the sampling of every device input required to execute Protocol QRE with a uniform seed of length no greater than $N_{\text{max}}$, where

$$N_{\text{max}} = 2\kappa$$

$$\epsilon_{\text{RIA}} = e^{-2\kappa^2/mk_{\text{max}}^2}$$

and $\kappa = (\gamma H(\mu) + h(\gamma)) n + 3m$. Moreover, the sampled distribution lies within a statistical distance of

$$\epsilon_{\text{dist}} = m 2^{n \log(\text{supp}(\mu)+1)/m - (k_{\text{max}}+1)},$$

from the target distribution, where $\text{supp}(\mu) := |\{(x, y) \in \mathcal{X}\mathcal{Y} \mid \mu(x, y) > 0\}|$.

**Proof.** Consider the sequence $(M_i)_{i=1}^m$ of i.i.d. random variables representing the number of random bits required to choose the inputs for the $i^\text{th}$ block and the corresponding random sum $N = \sum_{i=1}^m M_i$. By (52), the expected number of bits required to select all of the inputs for the protocol can be bounded above by $\kappa = (\gamma H(\mu) + h(\gamma)) n + 3m$. Using Hoeffding’s inequality, we can bound the probability that $N$ greatly exceeds this value,

$$\Pr[N \geq \kappa + t] \leq e^{-2t^2/mk_{\text{max}}^2},$$

for some $t > 0$. Setting $t = \kappa$ this becomes

$$\Pr[N \geq 2\kappa] \leq e^{-2\kappa^2/mk_{\text{max}}^2}.$$

Although $\kappa$ is not exactly the expected value of $N$, which is the quantity appearing in Hoeffding’s bound, the bound holds because $\kappa \geq \mathbb{E}[N]$.

It remains to bound the statistical distance between the sampled random variable $I' = (X', Y', T')$ and the target random variable $I = (X, Y, T)$. For each block of rounds, the corresponding random variable $I_i$ can take one of a possible $(\text{supp}(\mu) + 1)^{n/m}$ different values. Therefore, by Lemma B.3, we have for the $i^\text{th}$ block of rounds, the corresponding random variable $I_i$ can take one of a possible $(\text{supp}(\mu) + 1)^{n/m}$ different values. Therefore, by Lemma B.3, we have for the $i^\text{th}$ block of rounds

$$\Delta(I_i, I_i') \leq (\text{supp}(\mu) + 1)^{n/m} 2^{-(k_{\text{max}}+1)} = 2^{n \log(\text{supp}(\mu)+1)/m - (k_{\text{max}}+1)}$$

Since $\Delta(W, V)$ is a metric and hence satisfies the triangle inequality \(^{65}\), the statistical distance between independently repeated samples can grow no faster than linearly, i.e., $\Delta(F^m, F^m) \leq m \Delta(I, I')$. This completes the proof. \(\square\)

### C Incorporating the blocking procedure of [14]

The original statement of the entropy accumulation theorem \(^{13}\) was released alongside an accompanying paper, \(^{14}\), which detailed its application to security proofs of device-independent protocols. Within the appendix of \(^{14}\), it was shown that one could increase the quantity of entropy certified by the original EAT through a modification to the structure of the protocol. In the sections that follow, we show how the family of min-tradeoff functions $\mathcal{F}_{\text{min}}$ can be adapted to this structural change. Furthermore, we compare the accumulation rates achievable with the different structures and EAT statements. In particular, we show that this structural change provides no clear benefits when using the improved EAT statement. To clearly distinguish the different statements of the EAT, we shall indicate with the subscript $\text{DFR16}$, quantities associated with the original EAT \(^{13}\) and similarly we shall indicate with the subscript $\text{DF18}$, quantities associated with the EAT with improved second-order \(^{23}\).

\(^{25}\)We don’t include the extractor’s seed here as its size will depend on the choice of extractor.
C.1 Construction

Let us briefly review the structural modification that was introduced in [14]. Instead of distinguishing the statistics from each interaction separately, rounds are grouped together to form **blocks**. The number of rounds within a block can vary: a new block begins when either a test-round occurs or when the maximum number of rounds permitted within a block, \( s_{\text{max}} \), is reached. On expectation there are \( \bar{s} = \frac{1}{1 - \gamma} s_{\text{max}} \) rounds within a block. The device-interaction phase of the protocol concludes after some specified number of blocks \( m \in \mathbb{N} \) have terminated. We shall use the superscripts \( R \) and \( B \) to indicate whether a quantity is concerned with the round-by-round or block structured protocols respectively.

The collected information is now defined at the level of blocks and not rounds. In particular, at the end of the \( i \)-th block the user records some tuple \((A_i, B_i, X_i, Y_i, C_i)\), where \((A_i, B_i, X_i, Y_i) \in \mathcal{A}_{\text{max}} \mathcal{B}_{\text{max}} \mathcal{X}_{\text{max}} \mathcal{Y}_{\text{max}}\) and the score’s alphabet remains the same, \( C_i \in \mathcal{G} \cup \{L, \perp\} \). The EAT-channels are also defined for each block and the entropy bounding property (cf. (22)) that the min-tradeoff functions must satisfy becomes

\[
f^{B}_{\min}(p) \leq \inf_{\sigma \in \mathcal{N}_{C_i}^{\text{test}}(\sigma) = \tau_p} H(A_i, B_i | X_i, Y_i, R_i)_{\mathcal{N}_i(\sigma)},
\]

for each \( i \in [m] \). The set of distributions compatible with the protocol structure (cf. (23)) now take the form

\[
p^{B} = \begin{pmatrix}
\gamma \bar{s} q \\
(1 - \gamma) s_{\text{max}}
\end{pmatrix}
\]

for \( q \in \mathcal{Q}_\mathcal{G} \).

**Lemma C.1** (Blocked variant of Lemma 3.1): Let \( g : \mathcal{P}_{\mathcal{G} \cup \{L\}} \to \mathbb{R} \) be an affine function satisfying

\[
g(q) \leq \inf_{\sigma \in \mathcal{N}_{C_i}^{\text{test}}(\sigma) = \tau_p} H(A_i, B_i | X_i, Y_i, R_i)_{\mathcal{N}_i(\sigma)}
\]

for all \( q \in \mathcal{Q}_{\mathcal{G} \cup \{L\}} \). Then the function \( f : \mathcal{P}_{\mathcal{G} \cup \{L, \perp\}} \to \mathbb{R} \), defined by its action on the basis vectors

\[
f(e_c) = \text{Max}[g] + \frac{g(e_c) - \text{Max}[g]}{\gamma \bar{s}}, \quad \forall c \in \mathcal{G} \cup \{L\},
\]

\[
f(e_{\perp}) = \text{Max}[g],
\]

is a min-tradeoff function for any EAT-channels implementing Protocol QRE\(^B\). Furthermore, \( f \) satisfies the following properties:

\[
\begin{align*}
\text{Max}[f] &= \text{Max}[g], \\
\text{Min}[f|_{\perp}] &\geq \text{Min}[g], \\
\text{Var}[f|_{\perp}] &\leq \frac{(\text{Max}[g] - \text{Min}[g])^2}{\gamma \bar{s}}.
\end{align*}
\]

**Proof.** This follows from replicating the original proof (23) with the block channels decomposed into the testing and generation channels, \( \mathcal{N}_i = \gamma \bar{s} \mathcal{N}_{i}^{\text{test}} + (1 - \gamma) \mathcal{N}_{i}^{\text{gen}} \).

**Lemma C.2** (Min-tradeoff construction): Let \( \mathcal{G} \) be an OVG and \( k \in \mathbb{N} \). For each \( \nu \in \mathcal{Q}_{\mathcal{G}}^{(k)} \), let \((\nu, \lambda_{\nu})\) be some feasible point of Prog. (10). Furthermore, let \( \lambda_{\text{max}} = \max\{0, \lambda_1, \ldots, \lambda_{|\mathcal{G}|}\} \) and \( \lambda_{\text{min}} = \min\{0, \lambda_1, \ldots, \lambda_{|\mathcal{G}|}\} \), where \( \{\lambda_i\} \) are the components of \( \lambda_{\nu} \). Then, for any set of EAT channels \( \{\mathcal{N}_i\}_{i=1}^{m} \) implementing an instance of Protocol QRE\(^B\) with the OVG \( \mathcal{G} \), the set of functionals \( F^{B}_{\min}(\mathcal{G}) = \{f_{\nu}(\cdot) \mid \nu \in \mathcal{Q}_{\mathcal{G}}^{(k)}\} \) forms a family of min-tradeoff functions, where \( f_{\nu} : \mathcal{P}_{\mathcal{C}} \to \mathbb{R} \) are defined by their actions on the basis vectors

\[
f_{\nu}(e_c) := (1 - \gamma) \bar{s} \left( A_{\nu} - B_{\nu} \lambda_{\nu} \cdot e_c - (1 - \gamma \bar{s}) \lambda_{\text{min}} \right) \quad \text{for } c \in \mathcal{G},
\]

\[
f_{\nu}(e_{\perp}) := (1 - \gamma) \bar{s} \left( A_{\nu} + B_{\nu} \frac{(1 - \gamma \bar{s}) \lambda_{\text{min}}}{\gamma \bar{s}} \right)
\]

for \( c \in \mathcal{G} \), (59) and (60).
and applying the extension Lemma C.1, analogous to the technique of Lemma 3.2.

Moreover, these min-tradeoff functions satisfy the following identities.

- **Maximum:**
  
  $\text{Max}[f_\nu] = (1 - \gamma)\bar{s}(A_\nu - B_\nu \lambda_{\min})$  

- **Γ-Minimum:**
  
  $\text{Min}[f_\nu|_\Gamma] \geq (1 - \gamma)\bar{s}(A_\nu - B_\nu \lambda_{\max})$  

- **Γ-Variance:**
  
  $\text{Var}[f_\nu|_\Gamma] \leq \frac{(1 - \gamma)^2\bar{s}B_\nu^2(\lambda_{\max} - \lambda_{\min})^2}{\gamma}$

**Proof.** The proof follows the same structure as the proof of Lemma 3.2. The only significant difference is the construction of the function $g : \mathcal{P}_\nu \rightarrow \mathbb{R}$ satisfying (58) so we shall explain this part here. Following Appendix B of [14], by repeated application of the chain rule we may decompose a block’s entropy as

$$H(A, B|x, y, T_i R_i)_{\mathcal{N}_i(\sigma)} = \sum_{j=1}^{s_{\max}} (1 - \gamma)^{j-1} H(A_{i,j} B_{i,j}|X_{i,j} Y_{i,j}, T_i^{j-1} = 0, T_{ij}^{s_{\max}} A_{i,i}^{j-1} B_{i_i}^{j-1} R_i),$$

where $T_{ij}$ is the random variable indicating whether a test occurred on the $j$th round of the $i$th block. Considering the individual terms within the sum, we can absorb the majority of the side information into some arbitrary quantum register $E$ leaving us with terms of the form

$$(1 - \gamma)^{j-1} H(A_{i,j} B_{i,j}|X_{i,j} Y_{i,j} T_{i,j} E).$$

As before, we can use the inequality $H(A|B) \geq H_{\min}(A|B)$ and conditioning on $T_{ij}$ to lower bound each term in the sum by the outputs of the semidefinite program,

$$(1 - \gamma)^{j-1} H(A_{i,j} B_{i,j}|X_{i,j} Y_{i,j} T_{i,j} E) = (1 - \gamma)^{j-1} \Pr[T_{ij} = 0] H(A_{i,j} B_{i,j}|X_{i,j} = \tilde{x}, Y_{i,j} = \tilde{y}, T_{i,j} = 0, E) + (1 - \gamma)^{j-1} \Pr[T_{ij} = 1] H(A_{i,j} B_{i,j}|X_{i,j} Y_{i,j} T_{i,j} = 1, E) \geq (1 - \gamma)^{j} H(A_{i,j} B_{i,j}|\tilde{x}, \tilde{y}, E) \geq (1 - \gamma)^{j} H_{\min}(A_{i,j} B_{i,j} | \tilde{x}, \tilde{y}, E) \geq -(1 - \gamma)^{j} \log(\alpha_\nu + \lambda_\nu \cdot \omega_{i,j}),$$

where $\omega_{i,j} \in Q_j$ is the devices’ expected score vector for round $j$ of block $i$. Noting that $-\log(\cdot)$ of an affine function is convex, we can establish a bound on the entire block $i$ through an application of Jensen’s inequality

$$(\gamma - 1) \sum_{j=1}^{s_{\max}} (1 - \gamma)^{j-1} \log(\alpha_\nu + \lambda_\nu \cdot \omega_{i,j}) \geq \bar{s}(\gamma - 1) \log \left( \alpha_\nu + \lambda_\nu \cdot \frac{\sum_{j=1}^{s_{\max}} (1 - \gamma)^{j-1} \omega_{i,j}}{\bar{s}} \right) = \bar{s}(\gamma - 1) \log \left( \alpha_\nu + \lambda_\nu \cdot \frac{\omega_{i,j}}{\bar{s}} \right),$$

we have used the fact that $\sum_{j \in [s_{\max}]} (1 - \gamma)^{j-1} = \bar{s}$ and the notation $\omega_i = \sum_{j \in [s_{\max}]} \gamma(1 - \gamma)^{j-1} \omega_{i,j}$ for the expected OVG score vector for the $i$th block. Taking a first-order expansion of the last line, we get the function $g(\cdot) = (1 - \gamma)\bar{s} (A_\nu - B_\nu \lambda_\nu \cdot (\cdot))$. The proof is then completed by normalizing the OVG scores and applying the extension Lemma C.1 analogous to the technique of Lemma 3.2. \qed
C.2 Blocking with the improved second order

The error term in the original EAT bound is

\[ \epsilon^{R}_{DFR16} := 2 (\log(1 + 2|A||B|) + \| \nabla f_{\min} \|_{\infty}) \sqrt{1 - 2 \log(\epsilon_s \epsilon_{EAT})}. \]  

(65)

The disadvantage of using this bound as-is is that the gradient of \( f_{\min} \) scales like \( 1/\gamma \) and so the total error scales as \( O(\sqrt{n}/\gamma) \). Collating the statistics into \( m \in \mathbb{N} \) blocks, allows some of the \( \gamma \) dependence from the gradient term to be transferred to the \( \log(1 + 2|A||B|) \) term. Moving to the blocked structure and setting \( s_{\max} = [1/\gamma] \) (as was done in [14]), the output alphabets grow exponentially with the size of the block and the logarithmic term acquires a \( 1/\gamma \) scaling. In contrast, the scaling of the derivative of the min-tradeoff function is found to be independent of the block size. Fortunately, as our error is defined for an entire block, we reduce the multiplicative factor on the total error from \( \sqrt{n}/\gamma \) to \( \sqrt{\bar{m}} \approx \sqrt{n/s} \). As \( \bar{s} \in O(1/\gamma) \), we find that the total error term now scales as \( \sqrt{n/\gamma} \). By increasing the size of the blocks we have effectively redistributed the testing probability dependence evenly amongst the components of \( \epsilon^{R}_{DFR16} \).

In light of this block-induced improvement, it is natural to investigate whether similar advantages can be obtained by applying this technique to the improved EAT statement [23]. Recall the error terms

\[ \epsilon^{R}_{V} := \frac{\beta \ln 2}{2} \left( 2|AB|^2 + 1 + \sqrt{\text{Var}[f_{\bar{r}}]} + 2 \right)^2, \]  

(66)

\[ \epsilon^{R}_{K} := \frac{\beta^2}{6(1 - \beta)^3 \ln 2} 2^{3\beta(\log |AB| + \text{Max}[f] - \text{Min}[f_{\bar{r}}])} \ln^3 \left( 2^{2\log |AB| + \text{Max}[f] - \text{Min}[f_{\bar{r}}] + \epsilon^2} \right), \]  

(67)

and

\[ \epsilon^{R}_{\Omega} := \frac{1}{\beta} (1 - 2 \log(p_{\Omega} \epsilon_s)). \]  

(68)

Using the explicit form of the blocked min-tradeoff functions Lemma C.2, we can calculate the asymptotic growth of the error terms as \( s_{\max} \to \infty, \gamma \to 0 \) and \( m \approx n^R/s \). In particular, we find

\[ m \cdot \epsilon^{B}_{V} \leq \frac{\beta m \ln 2}{2} \left( 2|AB|^{2s_{\max}} + 1 + \sqrt{(1 - 2\beta n^2 B_\sigma^2 (\lambda_{\max} - \lambda_{\min})^2)} + 2 \right)^2, \]  

(69)

\[ = O(\beta n s_{\max}) + O(\beta n/\gamma), \]

\[ m \cdot \epsilon^{B}_{K} \leq \frac{m \beta^2}{6(1 - \beta)^3 \ln 2} 2^{3\beta(\log |AB|^{s_{\max}} + (1 - \gamma)\bar{s} B_\nu (\lambda_{\max} - \lambda_{\min}))} \ln^3 \left( 2^{2\log |AB|^{s_{\max}} + (1 - \gamma)\bar{s} B_\nu (\lambda_{\max} - \lambda_{\min}) + \epsilon^2} \right), \]  

(70)

\[ = \beta^2 2^{O(\beta s_{\max})} O(n^2 s_{\max}^2), \]  

(71)

and therefore the total error scales as

\[ \epsilon^{B}_{DF18} = O \left( \beta n s_{\max} + \frac{\beta n}{\gamma} + \beta^2 n^2 s_{\max}^2 2^{O(\beta s_{\max})} + 1 \right). \]  

(72)

In order for \( \epsilon^{B}_{K} \) to have any sensible scaling, we need the exponent to grow no faster than \( O(1) \). Combining this with the inverse dependence of \( \beta \) in \( \epsilon^{B}_{\Omega} \), we would like \( \beta \approx \frac{\gamma}{\sqrt{s_{\max}}} \). Such a choice results in \( \epsilon^{B}_{DF18} \in O \left( s_{\max} \sqrt{n/\gamma} \right) \) which suggests that the blocking procedure is not in general advantageous when used in conjunction with the improved second order statement.

A comparison between the expansion rates obtained when using the improved second order statement [23] and the blocked variant of the original EAT are presented in Fig. 7. The faster convergence to the i.i.d. rate is indicative of the new EAT statement’s strength. Additionally, in Fig. 8 we also plot a comparison between the rates achieved by the protocols presented in the main text with those achieved by the one-sided CHSH based protocol (Protocol ARV) [14], when implemented on qubit systems with inefficient detectors (cf. Fig. 5).
Figure 7: Comparison of the certifiable accumulation rates using the two different statements of the EAT: DFR16$^B$ [14] and DF18$^R$ (39). The rates were derived using the following procedure. We assumed a qubit implementation of the protocols with a detection efficiency $\eta = 0.9$, optimizing the state and measurement angles in order to maximise the i.i.d. bound. Then, for each value of $n$ an optimization of the min-tradeoff function choice was performed – for the rates calculated using (39) we also optimized the $\beta$ parameter at each value of $n$. To ensure that we approached the i.i.d. bound as $n$ increased we set $\gamma = \delta_1 = \cdots = \delta_{|G|} = n^{-1/3}$ as such a choice provides a constant completeness error across all values of $n$.

Figure 8: Comparison between the certifiable accumulation rates of QRNE protocols based on $G_{\text{CHSH}}$, $G_{\text{EB}}^{23}$ and Protocol ARV from [14] on qubit systems with inefficient detectors (cf. Fig. 5).
D From Bell-expressions to OVGs

The template protocol used in the main text relies on the concept of an OVG. However, sometimes the quantities we naturally measure may not be directly expressed as such. In this section we introduce a method to turn a set of linear functions on the observed probabilities into an OVG.

Recall that an OVG on $A\times \mathcal{Y}$ comprises a distribution $\mu$ over the input set $A\times \mathcal{Y}$ and a tuple of orthogonal scoring functions $V_i$, each a map from $A\times \mathcal{Y}$ to $\{0,1\}$. A general Bell expression, on the other hand, is a linear combination of the probabilities. It can be expressed in terms of a vector $s \in \mathbb{R}^{\dim(P)}$ via $s \cdot p$. We wish to use a set, $S = \{s_i\}_{i=1}^r$ of $r$ Bell expressions to construct an OVG in such a way that the expected values of the Bell expressions in $S$ can be inferred from the expected scores in the OVG for all strategies $p \in P$. In other words, we would like there to exist a collection of functions $f_i : \mathbb{R}^{|\mathcal{G}|} \to \mathbb{R}$ such that for each $i$

$$s_i \cdot p = f_i(\omega_\mathcal{G}(p)), \quad (73)$$

for all $p \in P$.

The orthogonal vector game comprising the empirical behaviour of the devices trivially satisfies these conditions (cf. $G_{EB}$). However, this has a large number of scores, which can be sub-optimal from the point of view of statistics (cf. the discussion in Sec. [4.1]). To avoid this issue we instead look for a solution with a low number of scores. Our proposed method is outlined in Fig. [5].

Our algorithm works as follows. We first construct a matrix $M$ that represents the non-signalling and normalization constraints that are necessarily satisfied by any valid distribution in the sense that $p$ is no signalling and normalized if and only if $Mp = b$, where $b$ is a vector of constants. The procedure then proceeds to take each Bell expression and perform on it first a splitting and then a redundancy check.

We start with the first Bell expression, $s_1$. If $s_1$ has $k$ unique non-zero entries, we form $k$ binary vectors $t_1, \ldots, t_k$ indicating the positions of the matching components within $s_1$. For example, if we have

$$s_1 = (a,b,0,b,c,a) \quad (74)$$

with $a, b, c \in \mathbb{R}$ and distinct then the splitting would result in

$$t_1 = (1,0,0,0,0,1),$$
$$t_2 = (0,1,0,1,0,0),$$
$$t_3 = (0,0,0,0,1,0). \quad (75)$$

These are pairwise orthogonal binary vectors, so the collection of them satisfy the constraints required for an OVG’s scores. Given the normalization and no-signalling constraints these may not be linearly independent. By definition of the rank we have that $t_1$ is redundant (depends linearly on previous constraints) iff

$$\text{rank} \begin{pmatrix} M \\ t_1 \end{pmatrix} = \text{rank} (M), \quad (76)$$

where $\begin{pmatrix} M \\ t_1 \end{pmatrix}$ denotes the matrix $M$ with the vector $t_1$ appended as an extra row. If appending $t_1$ leads to an increase in the rank of $M$ then we update $M$ to include $t_1$ and repeat for the rest of the $t_i$.

For the remaining Bell-expressions the procedure is similar, except that before adding a non-redundant vector to $M$, we first check that it is orthogonal to all of the previously retained vectors. If it is orthogonal, we add it to $M$. If not, it requires adjustment before being added. This is done in the following way.

Given a non-redundant vector $t$ that is not orthogonal to a previously retained vector $v$, we redefine $M$ by removing the row $v$ from it and then create three new vectors

$$x' = v \odot t$$
$$v' = v - v \odot t$$
$$t' = t - v \odot t \quad (77)$$

where $\odot$ denotes element-wise multiplication. By construction, these new vectors are pairwise orthogonal and $v'$ and $x'$ are orthogonal to the rows of $M$ (though $t'$ may not be). We perform the redundancy check
Algorithm: Conversion

Inputs:
- \( S \) – collection of linearly independent Bell-expressions
- \( M \) – matrix of normalization and no-signalling constraints

Outputs:
- \( V \) – collection of OVG scoring vectors

Key functions:
- \( \text{triv} (\cdot) \) – Function indicating whether a supplied vector can be written as a linear combination of the vectors comprising \( M \) and \( V \). For \( t \in \mathbb{R}^{\text{dim}(p)} \) it may be written explicitly as
  \[
  \text{triv} (t) := 1 - \left( \text{rank} \left( \begin{bmatrix} M \\ V \\ t \end{bmatrix} \right) - \text{rank} \left( M V \right) \right).
  \]

Procedure:
1: Initialise \( V \) to be a \( 1 \times \text{dim}(p) \) matrix with all components set to zero.
2: For each \( s \in S \):
   2a: Decompose \( s \) into a collection of binary orthogonal vectors \( T = \{ t_1, \ldots, t_k \} \) that indicate the positions of non-zero components of \( s \) with equal values.
   2b: While \( T \neq \emptyset \):
      2b-i: Pick \( t \in T \) and set \( T \mapsto T \setminus \{ t \} \).
      If \( \text{triv} (t) = 0 \) and \( V t = 0 \):
         Add: Set \( V \mapsto \begin{bmatrix} V \\ t \end{bmatrix} \).
      Elseif \( \text{triv} (t) = 0 \) and \( V t \neq 0 \):
         Split: Remove the first row \( v \) from \( V \) that satisfies \( v \cdot t \neq 0 \). Then set
         \( T \mapsto \{ v \odot t, v - v \odot t, t - v \odot t \} \cup T \).
      Else:
         Pass: Do nothing.
3: Remove \( 0 \) from \( V \) and return.

Figure 9: An algorithm for converting Bell-expressions to OVGs

of (76) on \( v' \) and \( x' \), adding them to \( M \) if they pass. If \( t' \) also passes the redundancy check, we check if it is orthogonal to the rows of \( M \) and add it to \( M \) if so. If not, we repeat this extra splitting procedure using this \( t' \). This continues until the resulting vector becomes redundant or orthogonal.

After this procedure has been performed for each of the Bell-expressions, the result is a collection \( \{ v_i \} \) of orthogonal binary vectors that form the score vectors for the OVG. Importantly, as each of the Bell-expressions can be formed from linear combinations of the \( \{ v_i \} \) together with the constraints of \( M \), we are able to recover the expected Bell-values from the expected score vector of the OVG.

Note that generically, a Bell expression will not have any repeated components, and in such a case we will be left with an OVG with many scores. However, commonly-used Bell expressions, or those based on extremal Bell inequalities, have many repeated entries, in which case this procedure will lead to an OVG with relatively few scores.

Note also that using the same Bell expressions in a different order within this algorithm can result in a different OVG.
E Conic program duality

In this section we outline the duality statements for conic programs, introduce the alternative form of dual that we use in this paper and show that it has the required properties to be considered a dual.

**Definition E.1:** A cone is a set $\mathcal{K} \subseteq \mathbb{R}^n$ with the property that if $x \in \mathcal{K}$ then $\lambda x \in \mathcal{K}$ for all $\lambda \geq 0$. A cone is pointed if $\mathcal{K} \cap (-\mathcal{K}) = \emptyset$.

**Definition E.2:** Given a cone $\mathcal{K}$, its dual cone is the set $\mathcal{K}^* \subseteq \mathbb{R}^n$ defined by the property that $y \in \mathcal{K}^*$ if and only if $\langle y, x \rangle \geq 0$ for all $x \in \mathcal{K}$, i.e., $\mathcal{K}^* = \{ y : \langle y, x \rangle \geq 0 \ \forall x \in \mathcal{K} \}$.

**Definition E.3:** A proper cone is a cone that is closed, convex, pointed and non-empty.

**Definition E.4** (Dual for conic programs): Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a proper cone with dual $\mathcal{K}^*$, $M \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and consider the conic program

$$\min_{x \in \mathbb{R}^n} \langle c, x \rangle \quad \text{subj. to} \quad Mx = b, \ x \in \mathcal{K}.$$ 

The optimization

$$\max_{y \in \mathbb{R}^m, z \in \mathbb{R}^n} \langle b, y \rangle \quad \text{subj. to} \quad c = z + MTy, \ z \in \mathcal{K}^*$$

is the dual program.

Note that this is a conic program over $\mathcal{K}^*$, the dual cone to $\mathcal{K}$.

The following two Lemmas are standard results (see, for example [66])

**Lemma E.1** (Weak duality): Let $P$ be a conic program with optimum value $p^*$. If the program $D$, dual to $P$ has optimum $d^*$ then $p^* \geq d^*$.

**Lemma E.2** (Strong duality): Let $P$ be a conic program with optimum value $p^*$ and dual $D$. If $P$ is strictly feasible then $p^* = d^*$.

Consider a family of conic programs parameterized by $b$, denoted $\hat{P}(b)$, with optimum $p^*(b)$. Say that $b$ is valid if there exists some $x \in \mathcal{K}$ such that $Mx = b$, and denote by $\mathcal{B}$ the set of valid $b$. Consider now the program $\hat{D}(b)$ defined by

$$\max_{y \in \mathbb{R}^m} \langle b, y \rangle \quad \text{subj. to} \quad p^*(b') \geq \langle y, b' \rangle \ \forall b' \in \mathcal{B}$$

**Lemma E.3:** If $P(b)$ has optimum $p^*(b)$ and $\hat{D}(b)$ has optimum $\hat{d}^*(b)$, then $\hat{d}^*(b) \leq p^*(b)$. Furthermore, if $P(b)$ is strictly feasible, then $\hat{d}^*(b) = p^*(b) = d^*(b)$.

**Proof.** For the first part, note that the set of constraints in $\hat{D}$ include $p^*(b) \geq \langle y, b \rangle$, so $\hat{d}^*(b) \leq p^*(b)$.

For the second part, consider the dual problem $D(b)$ and write the constraint as $c - MTy \in \mathcal{K}^*$. Take the inner product of $c - MTy$ with $x^*(b')$ (the optimal argument for the primal with parameter $b' \in \mathcal{B}$) to give $\langle c, x^*(b') \rangle - \langle MTy, x^* \rangle = p^*(b') - \langle y, Mx^* \rangle = p^*(b') - \langle y, b' \rangle$. Since $c - MTy \in \mathcal{K}^*$, from $x^*(b') \in \mathcal{K}$ we have that $p^*(b') - \langle y, b' \rangle \geq 0$. Thus, for any $b' \in \mathcal{B}$ we have $\langle y^*(b), b' \rangle \leq p^*(b')$. The constraints in $D$ thus imply those in $\hat{D}$ and so $\hat{d}^*(b) \geq d^*(b)$. If $P(b)$ is strictly feasible then by strong duality, $p^*(b) = d^*(b)$, so, combining with the first part, $\hat{d}^*(b) = p^*(b) = d^*(b)$.

**Remark E.1:** The previous lemma implies that we can think of $\hat{D}$ as an alternative dual to $P$. 

