Two-solvable and two-bipolar knots with large four-genera

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Abstract. For every integer \( g \), we construct a 2-solvable and 2-bipolar knot whose topological 4-genus is greater than \( g \). Note that 2-solvable knots are in particular algebraically slice and have vanishing Casson-Gordon obstructions. Similarly all known smooth 4-genus bounds from gauge theory and Floer homology vanish for 2-bipolar knots. Moreover, our knots bound smoothly embedded height four gropes in \( D^4 \), an a priori stronger condition than being 2-solvable. We use new lower bounds for the 4-genus arising from \( L^{(2)} \)-signature defects associated to meta-metabelian representations of the fundamental group.

1. Introduction

The 4-genus \( g_4(K) \) of a knot \( K \) in \( S^3 \) is the minimal possible genus \( g(\Sigma) \) of a locally flat embedding \( \Sigma \hookrightarrow D^4 \) of an orientable surface \( \Sigma \) in the 4-ball, where \( \Sigma \) has a single boundary component whose image coincides with \( K \subset \partial D^4 \cong S^3 \).

The simplest lower bounds for 4-genus are the Tristram-Levine signature function and Taylor’s bound \[ TAY79 \], the best possible bound for the 4-genus coming from the Seifert form. For algebraically slice knots these lower bounds vanish. Gilmer \[ Gil82 \] showed that there are algebraically slice knots with arbitrarily large 4-genus using Casson-Gordon signatures \[ CG78, CG86 \]. Cha \[ Cha08 \] showed that there exist knots with arbitrarily large 4-genus which are algebraically slice and have vanishing Casson-Gordon signatures, using Cheeger-Gromov Von Neumann \( L^{(2)} \) \( \rho \)-invariants corresponding to metabelian fundamental group representations. Extending this beyond the metabelian level to give further lower bounds for the 4-genus was left open, essentially because of difficulties arising from non-commutative algebra.

In this paper, we present a new method that avoids the non-commutative algebra problem. It enables us to go one step further than Gilmer and Cha, by combining a Casson-Gordon type approach with \( L^{(2)} \)-signatures associated with representations to 3-solvable or meta-metabelian groups which are not metabelian. In Section 6 we give a construction of knots with large 4-genera, with geometric properties that imply the vanishing of existing lower bounds for the 4-genus, as discussed below.

Theorem 1.1. For each \( g \in \mathbb{N} \), there exists a 2-bipolar and 2-solvable knot \( K \) with \( g_4(K) > g \). Moreover, \( K \) bounds an embedded grope of height 4 in \( D^4 \).

A grope is an approximation to a disc constructed from iterated layers of surfaces. We refer to \[ FQ90, COT03 \], or Section 10 for the definition. Gropes were ingredients of fundamental importance for constructing topologically flat embedded discs in the work of Freedman and Quinn \[ Fre84, FQ90 \] on 4-manifolds, and for obstructing knot concordances in the work of Cochran, Orr and Teichner \[ COT03 \].

A knot bounding a grope of height 4 in \( D^4 \) is 2-solvable in the sense of Cochran, Orr, and Teichner \[ COT03 \], and in particular all algebraic and metabelian slice obstructions vanish for such knots. We give the definition of 2-solvability in Section 2. For now, we encourage
readers not familiar with this terminology to simply consider bounding gropes of height 4 or 2-solvability as shorthand for all the 4-genus lower bounds discussed above vanishing.

Our construction in Section 6.2 produces knots that are also 2-bipolar in the sense of Cochran, Harvey and Horn [CHH13]. We give the definition in Section 2, noting for now that the notion of bipolarity combines the idea of Donaldson’s diagonalization theorem with fundamental group information related to the derived series, and that for a 2-bipolar knot the invariants \( \tau, Y, \varepsilon, \nu^+ \) from Heegaard-Floer homology, as well as the \( d \)-invariants of \( p/q \) surgery, all cannot prove that the knot is not smoothly slice, and consequently cannot detect the smooth 4-genus [CHH13]. This also holds for gauge theoretic obstructions such as those arising from Donaldson’s theorem and the 10/8 theorem.

Theorem 1.1 answers the \( h = 2 \) case of the question asked in [Cha08, Remark 5.6].

The smooth slice genus. We remark that concordance obstructions predicated on a smooth embedding cannot be used to draw conclusions about locally flat surfaces, and hence cannot be used to prove our result. On the other hand our knots have arbitrarily large smooth 4-genus, since a smooth embedding of a surface in \( D^4 \) is in particular a locally flat embedding.

If we were interested in the smooth 4-genus version of Theorem 1.1, currently known techniques using Heegaard Floer homology or gauge theory would not apply. It is unknown whether the Rasmussen \( s \)-invariant, which does provide a lower bound on the smooth 4-genus of a knot, must vanish for 2-bipolar knots. Our knots are even the first examples of 1-bipolar knots with large 4-genus, though for that result one could use a simpler Casson-Gordon signature argument analogous to Gilmer’s.

Horn’s results. The fact that \( g_4(K) \) is large implies that the base surface of any embedded grope in \( D^4 \) with boundary \( K \) must have large genus. The main theorem of Horn [Hor11] gives examples, for each \( g \) and each \( n \), of knots bounding height \( n \) gropes such that the base surface of any height \( n \) grope must have genus at least \( g \). However, Horn’s example knots are not known to have large 4-genera: he was only able to provide lower bounds on the genera of surfaces that extend to an embedding of a height \( n \) grope.

Summary of the construction and proof. In order to construct 2-bipolar knots bounding height four gropes, we take connected sums of sufficiently many (in fact \( N := 4g + 1 \)) copies of the seed ribbon knot \( R := 11_{74} \), and perform satellite operations on a collection of judiciously chosen infection curves \( \{ \alpha_i^+, \alpha_i^- \} \), with \( \alpha_i^+ \) lying in the second derived subgroup \( \pi_1(S^3 \setminus R)^{(2)} \) of the knot group of the \( i \)th copy of \( R \). We use knots \( \{ J_i^+, J_i^- \} \) with Arf invariant zero for the companions of the satellite operations, such that the \( \{ J_i^+ \} \) have increasingly large negative Tristram-Levine signature functions and the \( \{ J_i^- \} \) have increasingly large positive signature functions.

Let \( K \) be the result of these satellite operations. In Proposition 2.3, we show that \( K \) is 2-solvable; in Proposition 2.6, we show that \( K \) is 2-bipolar, and in Section 10, we show that \( K \) bounds a grope of height 4 in \( D^4 \). Writing \( K_i \) for the knot resulting from the satellite construction on \( (R, \alpha_i^+, J_i^+) \), we have \( K = \bigcup_{i=1}^N K_i \). Let \( M_{K_i} \) be the zero-surgery manifold of \( K_i \) and write \( Y := \bigcup_{i=1}^N M_{K_i} \). Here is an outline of our proof of Theorem 1.1.

1) If there were a surface \( \Sigma \) of genus \( g \) embedded in \( D^4 \) with boundary \( K \), then there would be an associated 4-manifold \( Z \) with boundary \( \partial Y \) and a quotient \( \Gamma \) of \( \pi_1(Z) \) such that the \( L^{(2)} \) \( p \)-invariant

\[
\rho^{(2)}(Y, \Gamma) := \rho^{(2)}(Y, \phi: \pi_1(Y) \to \pi_1(Z) \to \Gamma)
\]

would be bounded above by \( 2(2g + 4g + 1 - 1) = 12g \).

2) By choosing the infection knots \( \{ J_i \} \) to have suitably large Tristram-Levine signature functions, we may ensure that \( \rho^{(2)}(Y, \Gamma) \) must be so large that it violates the bound

\[
\ldots
\]
from the previous item, showing that $\Sigma$ cannot exist. The key difficulty is to show that for any putative $\Sigma$ of genus less than or equal to $g$, at least one of the infection curves $\alpha_i^{\pm}$ represents an element of $\pi_1(Y)$ mapping nontrivially to $\Gamma$, recalling that $\Gamma$ depends on $\Sigma$. Then $L^{(2)}$ induction implies that the Tristram-Levine signature of the $J_i^{\pm}$ corresponding to $\alpha_i^{\pm}$ contributes to $\rho^{(2)}(Y, \Gamma)$.

As mentioned above, knots bounding height four gropes are 2-solvable (the converse is an open problem). In Example 6.1 we present a slightly simpler construction of a family of 2-solvable knots with arbitrary 4-genera, starting with connected sums $\#^N8_8$ of the ribbon knot $8_8$ and performing a single satellite construction on each copy of $8_8$.

Coefficient systems: comparison with earlier methods. To show the nontriviality of some $\alpha_i^{\pm}$ in $\Gamma$, we use twisted homology over a metabelian representation to define the coefficient system. Although the representation is non-abelian, we use the ideas of Casson and Gordon [CG86] to define finitely generated twisted homology modules over a commutative domain. The commutativity enables us to consider the “size” of the twisted homology modules in terms of the minimal number of generators, generalizing the abelian representation case in e.g. [Cha08]. Supposing that the 4-genus is small compared to the size of the twisted first homology, we show that there is a meta-metabelian quotient $\Gamma$ of $\pi_1(Z)$ in which one of the $\alpha_i^{\pm}$ is nontrivial in order to eventually obtain a contradiction. In previous approaches to slice obstructions using meta-metabelian (or higher) $L^{(2)}$-signature defects, homology modules associated to non-abelian representations were over non-commutative rings, for which it is still unknown how to implement an analogous generating rank argument.

In our method, it is also crucial to use $L^{(2)}$-signatures over amenable groups that are not torsion-free, which were developed in [CO12, Cha14a] and deployed in a similar context in [MP18].

Further questions. One obvious question is whether there exist knots with arbitrarily large 4-genus bounding gropes of arbitrary height, or that are $h$-solvable for every $h$, as already formulated in [Cha08]. We make a bolder conjecture.

**Conjecture 1.2.** Let $K$ be an $h$-solvable knot which is not torsion in $C$. Then $\{\#^nK\}$ is a collection of $h$-solvable knots containing knots with arbitrarily large 4-genera.

A knot which did not satisfy the second sentence would be an example of a non-torsion knot with stable 4-genus zero i.e. $\lim_{n \to \infty} g_4(nK)/n = 0$, and it is unknown whether any such knots exist [Liv10]. Thus a counterexample to this conjecture would also be very interesting.

One might also wonder whether there exist highly bipolar knots with large smooth 4-genus, especially with the additional requirement that they be topologically slice. The following question seems to be unknown even in the case $h = 0$.

**Question 1.3.** Do there exist topologically slice $h$-bipolar knots with large smooth 4-genus?

As above, there are many reasonable candidate knots with which one might hope to answer ‘yes.’ The main result of [CK17] gave many examples of topologically slice, $h$-bipolar knots $K$ which are of infinite order, even modulo the subgroup of $(h+1)$-bipolar knots, and a smooth/bipolar analogue of Conjecture 1.2 suggests we should expect $\#^N K$ to have arbitrarily large smooth 4-genus as $N \to \infty$.

Organization of the paper. The next four sections are concerned with background theory. Section 2 recalls the definitions of the derived series of a group, a useful variation called the local derived series, and what it means for a knot to be $h$-solvable or $h$-bipolar. We also explain here how to construct $h$-solvable and $h$-bipolar knots using the satellite construction. Section 3 introduces some conventions for dealing with disconnected manifolds, in particular as relates to representations of their fundamental groupoids and associated twisted homology
groups. Section 4 recalls the Cheeger-Gromov von Neumann $L(2)$ $\rho$-invariant $\rho^{(2)}(Y, \phi)$ of a closed 3-manifold $Y$ together with a homomorphism of its fundamental group $\pi_1(Y) \to \Gamma$ to a group $\Gamma$, and gives the facts about this invariant that we will need. Section 5 describes homology twisted with metabelian representations. In particular we consider coefficient systems inspired by Casson-Gordon invariants [CG86].

Section 6 begins the proof of Theorem 1.1 by precisely stating the criteria that will imply certain knots have large topological 4-genus, giving a brief outline of the proof, and providing examples meeting those criteria. Section 7 proves some technical lemmas that are vital in arranging that the representation used for our $\rho$-invariant computation is suitably nontrivial. For this, we control the size of the homology groups of certain covering spaces. In Section 8 we review a standard cobordism used in the proof of Theorem 1.1, and carefully investigate the way metabelian representations extend over this cobordism. Section 9 proves the main theorem by bounding the $\rho$-invariant in two different ways as described above. Section 10 proves that our knots bound height four embedded gropes.

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2. The solvable and bipolar filtrations

In this section we recall the definitions of the solvable and bipolar filtrations, and how to construct highly solvable or bipolar knots. We will also need, later in the article, not just the standard derived series of a group but also the local derived series [CH05, CH08, Cha14a].

Definition 2.1. Let $G$ be a group. The $h$th derived subgroup $G^{(h)}$ of $G$ is defined recursively via $G^{(0)} := G$ and $G^{(h)} = [G^{(h-1)}, G^{(h-1)}]$ for $h \geq 1$. Moreover, for any sequence $S = (S_i)_{i \in \mathbb{N}}$ of abelian groups, define the $h$th $S$-local derived subgroup of $G$ recursively by $G^{(0)}_S := 0$ and, for $h \geq 1$,

$$G^{(h)}_S := \ker \left\{ G^{(h-1)}_S \to G^{(h-1)}_S/[G^{(h-1)}_S, G^{(h-1)}_S] \to (G^{(h-1)}_S/[G^{(h-1)}_S, G^{(h-1)}_S]) \otimes \mathbb{Z}, S_h \right\}.$$

Observe that for any sequence $S$ and any $h \in \mathbb{N}$ we have that $G^{(h)} \subseteq G^{(h)}_S$. We will be particularly interested in $S$ with first three terms $(S_1, S_2, S_3) = (\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q})$ for some prime $p$.

Definition 2.2. A knot $K$ is $h$-solvable if there exists a compact spin 4-manifold $W$ such that $\partial W = M_K$, the inclusion induced map $H_1(M_K) \to H_1(W)$ is an isomorphism, and there exist embedded surfaces with trivial normal bundle $D_1, \ldots, D_k$ and $L_1, \ldots, L_k$ in $W$ such that

1. The surfaces are pairwise disjoint except for $D_j$ and $L_j$, which for each $j = 1, \ldots, k$ intersect transversely in a single point.
2. The second homology classes represented by $D_1, \ldots, D_k, L_1, \ldots, L_k$ generate $H_2(W)$.
3. The inclusion induced maps $\pi_1(D_i) \to \pi_1(W)$ and $\pi_1(L_i) \to \pi_1(W)$ have image contained in $\pi_1(W)^{(h)}$.

This gives a filtration of the knot concordance group by subgroups $F_h$ consisting of the concordance classes of $h$-solvable knots, explored in [COT03, COT04, CT07, CHL09], among others. Every 1-solvable knot is algebraically slice and every 2-solvable knot has vanishing Casson-Gordon invariant sliceness obstruction. In particular, as mentioned in the introduction, the traditional 4-genus lower bounds of Tristram-Levine and Casson-Gordon signatures cannot be usefully employed with 2-solvable knots.

The satellite operation interacts particularly nicely with the solvable filtration. We remind the reader that given a knot $R$, infection curves $\alpha_1, \ldots, \alpha_k$ in $S^3 \setminus \nu(R)$ that form an unlink.
in $S^3$, and infection knots $J_1, \ldots, J_k$, the satellite of $R$ by $\{J_i\}$ along $\{\alpha_i\}$ is defined to be the image of $R$ in

$$
(S^3 \setminus \bigcup_{i=1}^{k} \nu(\alpha_i)) \cup \bigcup_{i=1}^{k} E_{J_i} \cong S^3,
$$

where $E_{J_i}$ is the exterior of $J_i$ and the identification is made so that a 0-framed longitude of $\alpha_i$, denoted by $\lambda(\alpha_i)$, is identified with a meridian of $J_i$ and vice versa. We denote this knot by $R_\alpha(J)$. The next proposition comes from [COT04, Proposition 3.1]. We will apply it with $h = 2$ to see that the knots we construct are 2-solvable.

**Proposition 2.3.** Let $R$ be a slice knot and $\{\alpha_i\}_{i=1}^{k}$ be a collection of unknotted, unlinked curves in $S^3 \setminus R$ such that $[\alpha_i] \in \pi_1(M_R)^{(h)}$ for all $i = 1, \ldots, k$. If for each $i = 1, \ldots, k$ the knot $J_i$ has $\text{Arf}(J_i) = 0$, then $R_\alpha(J)$ is $h$-solvable.

While our discussions have been thus far focused on the topological category, there are analogous notions of smooth sliceness, concordance, and 4-genera of knots. There is considerable interest in understanding the structure of $T$, the collection of topologically slice knots modulo smooth concordance. Here the $h$-solvable filtration is of no use, since every topologically slice knot lies in $\bigcap_{h=0}^{\infty} \mathcal{F}_h$. This prompted Cochran-Harvey-Horn to define the bipolar filtration as follows.

**Definition 2.4.** A knot $K$ is $h$-positive (respectively $h$-negative) if there exists a smoothly embedded disc $D$ in a smooth simply connected 4-manifold $V$ such that $\partial(V, D) = (S^3, K)$ and such that there exist disjointly embedded surfaces $S_1, \ldots, S_k$ in $V \setminus \nu(D)$ which form a basis for $H_2(V)$ such that for each $i = 1, \ldots, k$,

1. The surface $S_i$ has $S_i \cdot S_i = +1$ (respectively $S_i \cdot S_i = -1$).
2. The inclusion induced map $\pi_1(S_i) \to \pi_1(V \setminus D)$ has image contained in $\pi_1(V \setminus D)^{(h)}$.

Note that smoothly slice knots are $h$-positive for all $h \in \mathbb{N}$, that the connected sum of two $h$-positive knots is $h$-positive, and that any knot that can be unknotted by changing crossings from positive to negative (negative to positive) is 0-positive (0-negative) [CHH13].

**Definition 2.5.** We say that a knot $K$ is $h$-bipolar if it is both $h$-positive and $h$-negative.

The following proposition, inspired by [CK17, Lemma 2.3], gives us a way to construct $h$-bipolar knots; we will apply it when $h = 2$.

**Proposition 2.6.** Let $R$ be a smoothly slice knot and let $\eta^+$ and $\eta^-$ be curves in the complement of $R$ that form an unlink in $S^3$. Suppose that each $\eta^\pm$ represents a class in $\pi_1(S^3 \setminus R)^{(h)}$, and that for any knot $J$ we have that $R_{\eta^+}(J)$ and $R_{\eta^-}(J)$ are both smoothly slice. Then for any 0-positive knot $J^+$ and 0-negative knot $J^-$, the satellite knot $R_{\eta^+, \eta^-}(J^+, J^-)$ is $h$-bipolar.

**Proof.** Since $R_{\eta^+}(J^+)$ is slice and $J^-$ is 0-negative, the knot

$$
R_{\eta^+, \eta^-}(J^+, J^-) = (R_{\eta^+}(J^+)) \eta^-(J^-)
$$

is $h$-negative by [CHH13, Proposition 3.3]. We see that

$$
R_{\eta^+, \eta^-}(J^+, J^-) = (R_{\eta^-}(J^-)) \eta^+(J^+)
$$

is $h$-positive by a symmetric argument. \qed

3. Disconnected manifolds, fundamental groups and twisted homology

We will need to understand the twisted homology of a connected 4-manifold $X$ with disconnected boundary $Y$. In this section, we consider some technical details necessary to, for example, define inclusion maps from the twisted homology of $Y$ to that of $X$ and show that
there is a long exact sequence of the homology of the pair \((X, Y)\). On a first reading we encourage the reader to skim this section, focusing on the definitions leading up to Definition 3.1 and the statement of Proposition 3.3. A similar discussion can be found in [FK06, Section 2.1].

We note once and for all that manifolds are oriented and either compact or arising as an infinite cover of a compact manifold. For a manifold \(V\), we write \(p: \tilde{V} \to V\) for the universal cover.

Let \(Y = \bigsqcup_{i=1}^{N} Y_i\) be a compact \(\ell\)-dimensional manifold with \(N\) connected components. Let \(y_i \in Y_i\) be a basepoint for each connected component. Let \(S\) be a ring with unity and let \(A\) be a left \(S\)-module. A representation \(\Phi\) of the fundamental groupoid of \(Y\) into \(\text{Aut}(A)\) is equivalent to a homomorphism

\[
\Phi = \prod_{i=1}^{N} \Phi_i: \prod_{i=1}^{N} \pi_1(Y_i, y_i) \to \text{Aut}(A)
\]

from the free product of the fundamental groups of the connected components to \(\text{Aut}(A)\). We will use the following two examples.

(a) Let \(\Gamma\) be a group. Then we will take \(A = S = \mathbb{Z}\Gamma\), with \(\Phi_i: \pi_1(Y_i, y_i) \to \Gamma \subseteq \text{Aut}(A)\), where \(g \in \Gamma\) acts on \(A\) by left multiplication. We will also take \(A = \mathbb{N}\Gamma\), the group Von Neumann algebra of \(\Gamma\), discussed in Section 4.

(b) The ring \(S\) is a commutative PID and \(A = S^r\), together with a homomorphism

\[
\Phi_i: \pi_1(Y_i, y_i) \to GL_r(S) = \text{Aut}(S^r).
\]

For each \(i\) we use the representation \(\Phi_i: \pi_1(Y_i, y_i) \to \text{Aut}(A)\) to give \(A\) a right \(\mathbb{Z}[\pi_1(Y_i, y_i)]\)-module structure. Then we let \(C_*(\tilde{Y}_i)\) be a cellular chain complex of some CW decomposition of \(Y_i\) (or a CW complex homotopy equivalent to \(Y_i\) in the case that \(Y_i\) is a topological 4-manifold), and define the homology of \(Y\) twisted by \(\Phi\) to be

\[
H_*(Y; A) := \bigoplus_{i=1}^{N} H_*(A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i)).
\]

Now suppose that \(Y = \partial X\), where \(X\) is a compact connected \((\ell + 1)\)-dimensional manifold with \(\partial X = Y = \bigsqcup_{i=1}^{N} Y_i\). A schematic of a similar situation is shown in Figure 6. Let \(x \in X\) be a basepoint and let \(\tau_i: [0, 1] \to X\) be a path from \(x\) to \(y_i\). The paths \(\tau_i\) induce homomorphisms \(\iota_i: \pi_1(Y_i, y_i) \to \pi_1(X, x)\), by \(\gamma \mapsto \tau_i \gamma \tau_i^{-1}\).

**Definition 3.1.** We say that \(\Phi: \bigsqcup_{i=1}^{N} \pi_1(Y_i, y_i) \to \text{Aut}(A)\) extends over \(X\) if there is a homomorphism \(\Psi: \pi_1(X, x) \to \text{Aut}(A)\) such that \(\Psi \circ \iota_i = \Phi_i\) for each \(i = 1, \ldots, N\).

Use the inclusion \(j_i: Y_i \to X\) to define the pullback cover of \(Y_i\) in terms of the universal cover of \(X\) via the diagram:

\[
\begin{array}{ccc}
\tilde{Y}_i^X & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow p \\
Y_i & \longrightarrow & X \\
\end{array}
\]

The pullback \(\tilde{Y}_i^X\) is given by pairs \(\{(y, \tilde{x}) \in Y_i \times \tilde{X} \mid j_i(y) = p(\tilde{x})\}\). Apply the action of the group \(\pi_1(X, x)\) on \(\tilde{X}\) to the second factor to obtain an action of \(\pi_1(X, x)\) on \(\tilde{Y}_i^X\). This is defined since the action on \(\tilde{X}\) is equivariant with respect to \(p\). The action of \(\pi_1(X, x)\) on \(\tilde{Y}_i^X\) induces an action of \(\mathbb{Z}[\pi_1(X, x)]\) on the chain complex \(C_*(\tilde{Y}_i^X)\).

**Lemma 3.2.** We have a homeomorphism

\[
\pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \tilde{Y}_i^X \cong \tilde{Y}_i^X,
\]
\[
\pi_1(X, x) \times \tilde{Y}_i/((\gamma, \tilde{y}) \sim (\gamma', \tilde{y}')) \text{ if there is } g \in \pi_1(Y, y_i) \text{ such that } \gamma_i(g) = \gamma' \text{ and } g \cdot \tilde{y} = \tilde{y}'.
\]

**Proof.** Start with the covering space \(\pi_1(Y_i, y_i) \to \tilde{Y}_i \to Y_i\) with fibre \(\pi_1(Y_i, y_i)\), and then apply the ‘product over \(\pi_1(Y_i, y_i)\)’ construction to obtain a covering space \(\pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \pi_1(Y_i, y_i) \to \pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \tilde{Y}_i \to Y_i\). Since \(\pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \pi_1(Y_i, y_i) \cong \pi_1(X, x)\) as discrete spaces and affine sets over \(\pi_1(X, x)\), this fibre bundle is homeomorphic to \(\pi_1(X, x) \to \pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \tilde{Y}_i \to Y_i\).

Since both \(\pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \tilde{Y}_i\) and \(\tilde{Y}^X\) are covering spaces of \(Y_i\) corresponding to the homeomorphism \(i_i: \pi_1(Y_i, y_i) \to \pi_1(X, x)\), they are homeomorphic by the classification of covering spaces. \(\square\)

It follows from Lemma 3.2 that we have an chain isomorphism \(\mathbb{Z}[\pi_1(X, x)] \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) \cong C_*(\tilde{Y}^X_i)\). Consider the sequence of chain maps:

\[
A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) \xrightarrow{\cong} A \otimes_{\mathbb{Z}[\pi_1(X, x)]} \mathbb{Z}[\pi_1(Y_i, y_i)] C_*(\tilde{Y}_i) \\
\text{and } C_*(\tilde{X}).
\]

The first map sends \(a \otimes c \to a \otimes 1 \otimes c\). The second map uses the isomorphism discussed above, and the third map is induced by \(j_i\).

This chain level map induces a map on homology \((j_i)_*: H_*(Y_i; A) \to H_*(X; A)\), which in turn induces

\[
\bigoplus_{i=1}^N (j_i)_*: \bigoplus_{i=1}^N H_*(Y_i; A) \cong H_*(Y; A) \to H_*(X; A).
\]

Let \(\tilde{Y}^X := \bigsqcup_{i=1}^N \tilde{Y}_i^X\). Then by identifying \(\tilde{Y}^X\) with its image in \(\tilde{X}\) we also have relative twisted homology groups

\[
H_*(X, Y; A) := H_*(A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{X}, \tilde{Y}^X)).
\]

The chain maps above fit into a short exact sequence of chain complexes

\[
0 \to \bigoplus_{i=1}^N A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) \to A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{X}) \to A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{X}, \tilde{Y}^X) \to 0.
\]

That this is exact follows from the chain isomorphism

\[
A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) \cong A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{Y}_i^X).
\]

The short exact sequence of chain complexes gives rise to a long exact sequence in homology, which we record in the next proposition.

**Proposition 3.3.** With a fixed choice of paths \(\{\tau_i\}\) and a representation \(\Phi: \bigsqcup_{i=1}^n \pi_1(Y_i, y_i) \to \text{Aut}(A)\) that extends over \(X\), there is a long exact sequence in twisted homology

\[
\cdots \to H_k(Y; A) \to H_k(X; A) \to H_k(X, Y; A) \to H_{k-1}(Y; A) \to \cdots
\]

with \(H_k(Y; A) \to H_k(X; A)\) the inclusion induced map discussed above.

In later sections we work with many different representations of a given fundamental group(oid), and so we emphasize the representation \(\Phi\) rather than the module \(A\) by writing \(H^\Phi_k(Y)\) for \(H_k(Y; A)\).
4. $L^{(2)}$-signature invariants

In this section we introduce the Von Neumann $L^{(2)}$ $\rho$-invariant of a closed (not necessarily connected) 3-manifold equipped with a representation of its fundamental group or groupoid, and we recall the key properties of this invariant required for the proof of Theorem 1.1. In particular, we review the Cheeger-Gromov bound, a satellite formula, and an upper bound in terms of the second Betti number of a bounding 4-manifold.

**Definition 4.1.** Let $Y$ be a closed oriented 3-manifold, let $\Gamma$ be a discrete group, and let $\phi: \pi_1(Y) \to \Gamma$ be a representation. Note that $Y$ might be disconnected, in which case we use the conventions of Section 3. Suppose that $\phi$ extends to $\Phi: \pi_1(W) \to \Gamma$ where $W$ is a compact oriented 4-manifold with $\partial W = Y$. The von Neumann $L^{(2)}$ $\rho$-invariant of $(Y, \phi)$ is the signature defect

$$\rho^{(2)}(Y, \phi) = \sigma^{(2)}_\Gamma(W, \Phi) - \sigma(W),$$

where $\sigma^{(2)}_\Gamma(W, \Phi)$ is the $L^{(2)}$-signature of the intersection form $\lambda_{\Gamma}: H_2(W, \mathbb{N}) \times H_2(W, \mathbb{N}) \to \mathbb{N}$ and $\sigma(W)$ is the ordinary signature of the intersection form on $H_2(W; \mathbb{Q})$. Here the $L^{(2)}$-signature is defined via the completion $\mathbb{Z}\Gamma \to \mathbb{C}\Gamma \to \mathbb{N}$ to the Von Neumann algebra, and the spectral theory of operators on $\mathbb{N}$-modules. We refer to [COT03, Section 5] and [Cha14a, Section 3.1] for more details. In particular, $\rho^{(2)}(Y, \phi)$ only depends on the pair $(Y, \phi)$ since both the $L^{(2)}$ signature and the ordinary signature satisfy Novikov additivity and also $\sigma^{(2)}_\Gamma(V, \Phi) = \sigma(V)$ for a closed 4-manifold $V$. (See [CW03, p. 323] and [COT03, Lemma 5.9].)

This invariant was originally defined by Cheeger and Gromov via Riemannian geometry and $\eta$-invariants, independently of any bounding 4-manifold, so the above definition could be taken as a proposition that the two definitions coincide. For our purposes, as is common in the knot concordance literature, it is simpler to take the above as the definition; for a discussion, see [COT03, Section 5] and [CT07].

**Example 4.2.** Let $M_J$ be the zero-framed surgery manifold of a knot $J \subset S^3$ and let $\phi: \pi_1(M_J) \to \mathbb{Z}$ be the abelianization map. Then

$$\rho_0(J) := \rho^{(2)}(M_J, \mathbb{Z}) = \int_{\omega \in S^1} \sigma_\omega(J) d\omega,$$

where $\sigma_\omega(J)$ is the Tristram-Levine signature of $J$ at $\omega \in S^1$, that is the signature of $(1 - \omega)V + (1 - \omega)V^T$ for $V$ a Seifert matrix of $J$. See [COT03, Lemma 5.4] for the proof.

We will need the following theorem of Cheeger and Gromov, establishing a universal bound for the $\rho$-invariants of a fixed closed 3-manifold $Y$.

**Theorem 4.3** ([CG85]). Let $Y$ be a closed oriented 3-manifold. Then there exists a constant $C$ such that $|\rho^{(2)}(Y, \phi)| \leq C$ for any discrete group $\Gamma$ and any representation $\phi: \pi_1(Y) \to \Gamma$.

We will refer to the infimum of all such constants $C$ as the Cheeger-Gromov constant of $Y$, denoted $C(Y)$. We note that [Cha16] has given a proof of Theorem 4.3 using the signature defect definition of $\rho^{(2)}(Y, \phi)$ given above, and has given explicit bounds for $C(Y)$ in terms of the triangulation complexity of $Y$.

The following proposition comes from [CHL09].

**Proposition 4.4.** Let $K = R_{\alpha}(J)$ be the result of a satellite operation on a knot $R$ by infection knots $\{J_k\}$ along infection curves $\{\alpha_k\}$. Let $\phi: \pi_1(M_K) \to \Gamma$, and suppose that for some $h \in \mathbb{N}$ we have $\alpha_k \in \pi_1(M_R)^{(h)}$ for all $k$ and $\Gamma^{(h+1)} = 1$. Suppose that for all $k$, either $\phi(\alpha_k) = 1$ or $\phi(\alpha_k)$ is infinite order in $\Gamma$. Then the restriction induced maps $\pi_1(M_R \cup \bigcup \nu(\alpha_k)) \to \Gamma$ and $\pi_1(E_{J_k}) \to \Gamma$ extend uniquely to $\phi_0: \pi_1(M_R) \to \Gamma$ and $\phi_k: \pi_1(M_{J_k}) \to \Gamma$ and we have

$$\rho^{(2)}(M_K, \phi) = \rho^{(2)}(M_R, \phi_0) + \sum_k \rho^{(2)}(M_{J_k}, \phi_k).$$
Proof. The proof of [CHL09, Lemma 2.3] applies, with the following modification. The original statement of this proposition assumes the additional hypothesis that $\Gamma$ is a poly-torsion-free-abelian (PTFA) group. However, an inspection of the proof shows that we need only assume that for each $k$ either $\phi(\alpha_k) = 1 \in \Gamma$ or $\phi(\alpha_k)$ is infinite order. In the case that $\phi(\alpha_k) \neq 1$ in $\Gamma$, they need in the proof of [CHL09, Lemma 2.3] that $H_1(\alpha_k; \mathbb{Z}\Gamma) = 0$. But since $\phi(\alpha_k)$ is infinite order, $H_1(\alpha_k; \mathbb{Z}\Gamma)$ is the first homology of $\mathbb{R} \times \Gamma/\langle \alpha_k \rangle$, which vanishes.

We will apply Proposition 4.4 when $\Gamma = G/G^{(3)}_{\langle \mathbb{Z}, \mathbb{Z}\rho, \mathbb{Q} \rangle}$ for some group $G$ and $h = 2$. For such $\Gamma$, any curves $\alpha_k \in \pi_1(M_{R_{(2)}})$ satisfy the hypothesis of the proposition, since then $\phi(\alpha_k) \in \Gamma^{(2)}$ and $\Gamma^{(2)}/\Gamma^{(3)}$ is torsion-free.

Under the assumptions of Proposition 4.4, we have that the map $\phi_k : \pi_1(M_{J_k}) \rightarrow \Gamma$ factors through the abelianization map. To see this, note that each meridian of $J_k$ is identified with a longitude of $\alpha_k$, which lies in $\pi_1(M_{R_{(h)}})$ and hence is sent to $\Gamma^{(h)}$. So the image of $\phi_k : \pi_1(M_{J_k}) \rightarrow \Gamma$ is contained in $\Gamma^{(h)}$, which is an abelian group since $\Gamma^{(h+1)} = 1$. When $\Gamma^{(h)}/\Gamma^{(h+1)}$ is torsion-free, as occurs when $\Gamma = G/G^{(3)}_{\langle \mathbb{Q}, \mathbb{Z}p, \mathbb{Q} \rangle}$ and $h = 2$, we therefore have that $\phi_k$ is either the zero map or maps onto a copy of $\mathbb{Z}$ in $\Gamma$. By the principle of $L^{(2)}$-induction [COT03, Proposition 5.13] and Example 4.2, we have that

$$\rho^{(2)}(M_{J_k}, \phi_k) = \begin{cases} \rho_0(J_k) & \text{if } \phi_k \neq 0 \\ 0 & \text{if } \phi_k = 0. \end{cases}$$

Finally, note that since a meridian of $J_k$ is identified with a longitude $\lambda(\alpha_k)$ of $\alpha_k$ in $M_K$, we have that $\phi_k$ is the zero map if and only if $\phi(\lambda(\alpha_k)) = 0$. We summarize the results of the above discussion for later use.

**Proposition 4.5.** Let $K = R_\alpha(J)$ be the result of a satellite operation on $R$ by infection knots $\{J_k\}$ along infection curves $\{\alpha_k\}$ lying in $\pi_1(M_{R_{(2)}})$. Let $\Gamma = G/G^{(3)}_{\langle \mathbb{Q}, \mathbb{Z}p, \mathbb{Q} \rangle}$ for some group $G$ and prime $p$, and let $\phi : \pi_1(M_{K}) \rightarrow \Gamma$. Then the restriction induced maps $\pi_1(M_{R} \setminus \bigcup \nu(\alpha_k)) \rightarrow \Gamma$ and $\pi_1(E_{J_k}) \rightarrow \Gamma$ extend uniquely to maps $\phi_0 : \pi_1(M_{R}) \rightarrow \Gamma$ and $\phi_k : \pi_1(M_{J_k}) \rightarrow \Gamma$. Moreover,

$$\rho^{(2)}(M_K, \phi) = \rho^{(2)}(M_R, \psi_0) + \sum_k \rho^{(2)}(M_{J_k}, \phi_k) = \rho^{(2)}(M_R, \phi_0) + \sum_j \delta_k(\psi) \rho_0(J_k)$$

where

$$\delta_k(\psi) = \begin{cases} 0 & \text{if } \psi(\lambda(\alpha_k)) = 0, \\ 1 & \text{if } \psi(\lambda(\alpha_k)) \neq 0. \end{cases}$$

The following straightforward consequence of [Cha14a, Theorem 3.11] will provide our key upper bound on $L^{(2)}$-signatures. Strebel’s class of groups $D(\mathbb{Z}_p)$ was defined in [Str74]; we will not recall the definition. We will use the fact that for any group $G$ and any $h \in \mathbb{N}$, we have that $\Lambda = G/G^{(h)}$ is amenable and lies in $D(\mathbb{Z}_p)$ provided $S_i$ is either $\mathbb{Q}$ or $\mathbb{Z}_{p^i}$ for every $i \in \mathbb{N}$ [CO12, Lemma 6.8].

**Theorem 4.6.** Let $Z$ be a 4-manifold with boundary $\partial Z = Y$ and let $\phi : \pi_1(Y) \rightarrow \pi_1(Z) \rightarrow \Lambda$ be a homomorphism, where $\Lambda$ is amenable and in Strebel’s class $D(\mathbb{Z}_p)$. Then $|\rho^{(2)}(Y, \phi)| \leq 2 \dim_{\mathbb{Z}_p} H_2(Z, \mathbb{Z}_p)$.

**Proof of Theorem 4.6.** Let $\tilde{Z}$ be the cover of $Z$ induced by the homomorphism $\pi_1(Z) \rightarrow \Lambda$. Since $Z$ is a compact 4-manifold with boundary, it has the homotopy type of a finite 3-dimensional CW complex. This follows from [KS69, §1(III)] to get a finite CW complex, combined with [Wal66, Corollary 5.1] to restrict the dimension of the CW complex to three.
Let $C_\ast$ be the corresponding chain complex, and let $\widetilde{C}_\ast := C_\ast(\tilde{Z})$ denote the chain complex of $\tilde{Z}$. Since $\Lambda$ is amenable and in Strebel’s [Str74] class $D(\mathbb{Z}_p)$, [Cha14a, Theorem 3.11] tells us that

$$\dim(\mathbb{Z}_p H_2(Z;\mathbb{Z} \Lambda)) = \dim_{\mathbb{Z}_p} H_2(\mathbb{Z}_p \otimes_{\mathbb{Z} \Lambda} \tilde{C}_\ast) = \dim_{\mathbb{Z}_p} H_2(Z;\mathbb{Z}_p).$$

It follows that

$$|\rho(\mathbb{Z}_p H_2(Z;\mathbb{Z} \Lambda))| = |\sigma(\mathbb{Z}_p H_2(Z;\mathbb{Z} \Lambda))| \leq \dim_{\mathbb{Z}_p} H_2(Z;\mathbb{Z}_p) + \dim_{\mathbb{Z}_p} H_2(Z;\mathbb{Q}) \leq 2 \dim_{\mathbb{Z}_p} H_2(Z;\mathbb{Z}_p).$$

We use the universal coefficient theorem to deduce that $\dim_{\mathbb{Q}} H_2(Z;\mathbb{Q}) \leq \dim_{\mathbb{Z}_p} H_2(Z;\mathbb{Z}_p)$ for the final inequality.

\section{5. Metabelian twisted homology}

In this section we review Casson-Gordon type metabelian representations of knot groups, and the resulting twisted homology. The behavior of infection curves in this twisted homology will be key to our proof of Theorem 1.1.

We now let $S$ denote a commutative PID and let $Q$ denote its quotient field. We will often take $S = \mathbb{F}[t^{\pm 1}]$ and $Q = \mathbb{F}(t)$ for some field $\mathbb{F}$, as well as $S = \mathbb{Z}$ and $Q = \mathbb{Q}$.

Let $X$ be a space homotopy equivalent to a finite CW-complex and let $A$ be a left $S$-module given the structure of a right $\mathbb{Z}[\pi_1(X)]$-module by a homomorphism $\phi: \pi_1(X) \to \text{Aut}(A)$. Note that $\pi_1(X)$ naturally acts on $\widetilde{C}_\ast$, the chain complex of the universal cover $\tilde{X}$ of $X$, on the left. Then as in Section 3, the twisted homology $H_\ast(X; A)$ is defined to be

$$H_\ast(X; A) := H_\ast(A \otimes_{\mathbb{Z}[\pi_1(X)]} \widetilde{C}_\ast).$$

We will be particularly interested in the following metabelian representations. Suppose that we have a preferred surjection $\varepsilon: H_1(X) \to \mathbb{Z}$. For every $r \in \mathbb{N}$, let $p_r: \mathbb{Z} \to \mathbb{Z}_r$ be the usual projection map and let $X^r$ denote the $r$-fold cyclic cover of $X$ corresponding to $\ker(p_r \circ \varepsilon)$. Note that covering transformations give $H_1(X^r)$ the structure of a $\mathbb{Z}[\mathbb{Z}_r]$-module. We write $\mathbb{Z} = \langle t \rangle$, and slightly abuse notation by also writing $t$ for $p_r(t)$, so that $\mathbb{Z}_r = \langle t \rangle$. Choosing a preferred element $\gamma_0 \in \pi_1(X)$ with $\varepsilon(\gamma_0) = +1$ then gives us a map

$$\rho_{\gamma_0}: \pi_1(X) \to \mathbb{Z} \ltimes H_1(X^r) \quad \text{by} \quad \gamma \mapsto (t^{\varepsilon(\gamma)}, [\gamma_0^{\varepsilon(\gamma)} \gamma]),$$

where $\gamma_0^{-\varepsilon(\gamma)} \gamma \in \pi_1(X_r) \leq \pi_1(X)$ and $[\gamma_0^{\varepsilon(\gamma)} \gamma]$ denotes the image of $\gamma_0^{-\varepsilon(\gamma)} \gamma$ under the Hurewicz map.

Given any choice of a homomorphism $\chi: H_1(X^r) \to \mathbb{Z}_m$, we let $\xi_m = e^{2\pi i/m}$ and obtain a map $\theta_\chi: \mathbb{Z} \ltimes H_1(X^r) \to \text{GL}_r(\mathbb{Q}(\xi_m)[[t^{\pm 1}]]$ by

$$(t^j, a) \mapsto \begin{bmatrix} 0 & \cdots & 0 & t^j \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_m^{\chi(a)} & \cdots & \cdots & \cdots \\ 0 & \xi_m^{\chi(t \cdot a)} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_m^{\chi(k^{-1} \cdot a)} \end{bmatrix}.$$
where $\Sigma_r(K)$ is the $r$th cyclic branched cover of $S^3$ along $K$. Our map $\chi: H_1(X') \to \mathbb{Z}_m$ will always be chosen to factor through the projection map to $H_1(\Sigma_r(K))$.

In the case $r = 2$ we have that $t$ must act by $-1$ on $H_1(\Sigma_2(K))$ and so we can conveniently decompose $\theta_\chi \circ \rho_\mu$ differently as $\theta \circ f_\chi$, where

$$f_\chi: \pi_1(M_K) \to \mathbb{Z} \times \mathbb{Z}_m$$

$$\gamma \mapsto (t^\varepsilon(\gamma), \chi([\mu^{-\varepsilon(\gamma)}]\gamma))$$

and

$$\theta: \mathbb{Z} \times \mathbb{Z}_m \to \text{GL}_2(\mathbb{Q}(\xi_m)[t^{\pm 1}])$$

$$(t^1, a) \mapsto \begin{bmatrix} 0 & t^1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_m^a & 0 \\ 0 & \xi_m^{-a} \end{bmatrix}.$$

The following proposition is a slight modification of a result of [MP18, Prop. 7.1], and gives the key connection between a certain derived series and metabelian homology, when $m = q^\alpha$ is a prime power.

**Proposition 5.1.** Let $W$ be a 4-manifold with boundary $\partial W = Y$. Let $\Phi: \pi_1(W) \to \text{Aut}(A)$ be a representation that factors through $\mathbb{Z} \times \mathbb{Z}_q$, for some prime $q$, and let $\phi: \pi_1(Y) \to \text{Aut}(A)$ be the composition of $\Phi$ with the inclusion map $\pi_1(Y) \to \pi_1(W)$. Let $\eta \in \pi_1(Y)(2)$ and suppose $\eta$ is sent to the identity in $\pi_1(W)/\pi_1(W)_{(3)}$, $\mathbb{Q}$. Then for any $v \in A$ and any $\tilde{\eta}$, a lift of $\eta$ to the cover of $W$ induced by $\Phi$, we have that the class $[v \otimes \tilde{\eta}]$ in $H_1^\phi(Y)$ maps to 0 in $H_1^\phi(W)$.

**Proof.** The proof of [MP18, Prop. 7.1] (with its first and last sentences deleted) applies verbatim.

Finally, we recall the twisted Blanchfield form. In analogy to the linking form on the torsion part of the ordinary first homology of a closed oriented 3-manifold, if $\phi: \pi_1(M_K) \to \text{GL}_r(\mathbb{Q}(\xi_m)[t^{\pm 1}])$ arises as above then there is a metabelian twisted Blanchfield form [MP18]

$$B\phi^\phi: H_1^\phi(M_K) \times H_1^\phi(M_K) \to \mathbb{Q}(\xi_m)(t)/\mathbb{Q}(\xi_m)[t^{\pm 1}].$$

Note that in the above circumstance $H_1^\phi(M_K)$ is a torsion $\mathbb{Q}(\xi_m)[t^{\pm 1}]$-module, by the corollary to [CG96, Lemma 4]; see also [FP12]. In Section 6.1, we will need to know that this form is *sesquilinear* [Pow16]. That is, letting $\tilde{-}$ denote the involution of $\mathbb{Q}(\xi_m)[t^{\pm 1}]$ induced by sending $t \to t^{-1}$ and $a + bi \mapsto a - bi$, we have

$$B\phi^\phi(px, qy) = pq B\phi^\phi(x, y) \quad \text{for every } p, q \in \mathbb{Q}(\xi_m)[t^{\pm 1}] \text{ and } x, y \in H_1^\phi(M_K).$$

6. Main theorem and examples

Here is our main technical result, that we use to show that certain satellite knots have large 4-genus.

**Theorem 6.1.** Let $R$ be a ribbon knot and let $\eta^1, \ldots, \eta^r$ be curves in $S^3 \setminus \nu(R)$ that form an unlink in $S^3$ such that each $\eta^j$ represents an element of $\pi_1(M_R)(2)$. Suppose that there is a prime $p$ such that for every nontrivial character $\chi: H_1(\Sigma_2(R)) \to \mathbb{Z}_p$ we have

1. The module $H_1^{\phi, \chi}(M_R)$ is nontrivial and generated by the collection $\{(1, 0) \otimes [\eta^j]\}_{j=1}^r$.
2. The order of $H_1^{\phi, f_\chi}(M_R)$ is relatively prime to $\Delta_R(t)$ over $\mathbb{Q}(\xi_{p^a})(t)$ for all $a > 0$. 

Let \( g \geq 0 \). Suppose that \( N \geq \frac{2g+2}{m} \), where \( m \) denotes the generating rank of the \( p \)-primary part of \( H_1(\Sigma_2(R)) \), and suppose that the collection of knots \( \{ J_i^1 \mid 1 \leq i \leq N, 1 \leq j \leq r \} \) satisfy

\[
|\rho_0(J_i^j)| > 2(2g + N - 1) + NC_R + \sum_{k=1}^{i-1} \sum_{\ell=1}^r |\rho_0(J_k^\ell)| + \sum_{\ell=1}^{j-1} |\rho_0(J_i^\ell)|
\]

for each \( 1 \leq i \leq N, 1 \leq j \leq r \). Then the knot \( K = \#_{i=1}^N R_{\eta_1^1, \ldots, \eta_r^j}(J_i^1, \ldots, J_i^r) \) has 4-genus at least \( g + 1 \).

For convenience, we write \( \eta_i^j \) for the curve \( \eta^j \) in the \( i \)th copy of \( R \) in \( \#_{i=1}^N R \). Note that we can also write \( K = \#_{i=1}^N K_i \), where \( K_i := R_{\eta_1^1, \ldots, \eta_r^j}(J_i^1, \ldots, J_i^r) \). We will prove Theorem 6.1 in Section 9 by assuming that \( g_4(K) \leq g \) and obtaining a contradiction according to the following outline.

1. Construct a 4-manifold \( Z := V \cup M_K U \) with \( H_2(Z) = \mathbb{Z}^{2g+N-1} \), \( H_1(M_K) \to H_1(Z) \) an isomorphism, \( \chi(Z) = 2g \), and \( \partial Z = Y := \bigsqcup_{i=1}^N M_K \).
2. Show that we can choose maps \( \chi_i : H_1(M_K_i) \to \mathbb{Z}_p \) such that the corresponding map \( \phi : \prod_{i=1}^N \pi_1(M_K_i) \to \mathbb{Z} \times \mathbb{Z}_p \) extends to a map \( \Phi : \pi_1(Z) \to \mathbb{Z} \times \mathbb{Z}_p \) for some \( a \geq 1 \) and such that at least \( n := 2g + 1 \) of the \( \chi_i \) are nonzero.
3. Compute the twisted homology \( H_1^\phi(Y) \) and show that for some \( 1 \leq i \leq N \) and \( 1 \leq j \leq r \), the element \([1, 0] \otimes [\eta_i^j] \) does not map to 0 in \( H_1^\phi(Z) \). This step crucially relies on our assumption that for every nontrivial \( \chi : H_1(\Sigma_2(R)) \to \mathbb{Z}_p \) we know that the collection \( \{[1, 0] \otimes [\eta_i^j]\} \) generates \( H_1^\phi \chi(R) \) and that the order of \( H_1^\phi \chi(R) \) is relatively prime to \( \Delta_R(t) \).
4. Construct a local coefficient derived series representation \( \pi_1(Y) \to \Lambda \) extending over \( \pi_1(Z) \) and bound the \( L^{(2)} \)-invariant \( \rho^{(2)}(Y, \Lambda) \) in two different ways to get a contradiction. Essentially, \( \chi(Z) = 2g \) and the fact that our representation extends over \( \pi_1(Z) \) will imply that \( |\rho^{(2)}(Y, \Lambda)| \) is small, while our assumptions on \( |\rho_0(J_i^j)| \) together with Step 3 will imply that \( |\rho^{(2)}(Y, \Lambda)| \) is very large.

We will first give some examples of knots satisfying the hypotheses of the theorem and then prove some technical lemmas in the next two sections. In particular we will need to gain some control over the size of certain homology groups, in order to show that some curve \( \eta_i^j \) always survives into a suitable 3-solvable quotient of the fundamental group of the complement of a hypothesized locally flat embedded surface of genus \( g \). Of course Theorem 1.1 follows immediately from Theorem 6.1 together with the examples exhibited in Section 6.2 below. We remark also that Theorem 6.1 can be generalized to consider higher prime power branched covers by appropriately changing the constants; we leave the details of that to the interested reader.

It is relatively easy to find examples of seed ribbon knots satisfying the hypotheses of Theorem 6.1, at least with the help of a computer program to compute twisted metabelian homology as developed in [MP18]. In Section 6.1 we give one such example of a pair \( (R, \alpha) \), and describe the appropriate infection by knots with Arf invariant zero and large signature in order to obtain 2-solvable large 4-genus knots.

It is a little harder to find suitable seed knots \( R \) that also satisfy Proposition 2.6, and therefore produce 2-solvable and 2-bipolar large 4-genus examples for the proof of Theorem 1.1. Nonetheless, we exhibit such a seed knot \( R \) with suitable infection curves in Section 6.2, and describe the appropriate infection by 0-bipolar knots with large signature in order to obtain 2-bipolar, 2-solvable large 4-genus knots.
6.1. Example 1: a 2-solvable knot with large 4-genus

Let $R$ denote the ribbon knot $8_8$, with the unknotted curve $\alpha$ in $S^3 \setminus \nu(R)$ illustrated in Figure 1.

\[ \text{Figure 1. The knot } R = 8_8 \text{ with a grey Seifert surface and a red infection curve } \alpha. \]

This is the same knot-curve pair $(R, \alpha)$ as in [MP18, Example 8.1], with a slight isotopy to make it more obvious that the curve $\alpha$ bounds a surface in the complement of a Seifert surface for $R$, and hence lies in $\pi_1(M_R)^{(2)}$. We will need a few computations from that paper. First, $H_1(\Sigma_2(R)) \cong \mathbb{Z}_{25}$. Note that rescaling a character $\chi : H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_5$ by a nonzero element of $\mathbb{Z}_5$ does not change the underlying $(\mathbb{Z} \times \mathbb{Z}_5)$-covering space of $M_R$, and hence preserves the isomorphism type of the twisted homology. It is therefore not hard to check that given any nontrivial character $\chi : H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_5$, the corresponding twisted homology $H_1^{\theta_0 f_\chi}(M_R)$ is isomorphic to $\mathbb{Q}(\xi_5)[t^{\pm 1}]/(t^2 - 3t + 1)$. Since $\alpha$ is in $\pi_1(M_R)^{(2)}$, it lifts to a curve $\tilde{\alpha}$ in the $f_\chi$ covering space of $M_R$, and hence $x := [1,0] \otimes \tilde{\alpha}$ is an element of $H_1^{\theta_0 f_\chi}(M_R)$. Finally, the metabelian twisted Blanchfield pairing $Bl^{\theta_0 f_\chi}(x,x)$ is non-zero in $\mathbb{Q}(\xi_5)(t)/\mathbb{Q}(\xi_5)[t^{\pm 1}]$, as was computed in [MP18, Example 8.1].

**Lemma 6.2.** The element $x$ generates $H_1^{\theta_0 f_\chi}(M_R)$.

**Proof.** Supposing for a contradiction that $x$ does not generate. Let $y$ be some generator for $H_1^{\theta_0 f_\chi}(M_R)$. Note that $Bl^{\theta_0 f_\chi}(y,y)$ is of the form $\frac{q(t)}{t^2 - 3t + 1}$ for some $q(t) \in \mathbb{Q}(\xi_5)[t^{\pm 1}]$. The polynomial $t^2 - 3t + 1$ factors as $(t-w_+)(t-w_-)$ for $w_{\pm} = \frac{3\pm \sqrt{5}}{2} \in \mathbb{Q}(\xi_5)$. Therefore, if $x$ does not generate then it must be homologous to $c(t - w_s)y$ for some $c \in \mathbb{Q}(\xi_5)[t^{\pm 1}]$ and $* \in \{\pm\}$; without loss of generality, say $* = +$. But then we can obtain a contradiction, since

\[
Bl^{\theta_0 f_\chi}(x,x) = Bl^{\theta_0 f_\chi}(c(t - w_+)y, c(t - w_+)y) \\
= (t - w_+)(t - w_-)c\bar{c}Bl^{\theta_0 f_\chi}(y,y) \\
= (t - w_+)(t^{-1} - w_-)c\bar{c}\frac{q(t)}{t^2 - 3t + 1} \\
= -t^{-1}w_+(t - w_+)(t - w_-)c\bar{c}\frac{q(t)}{t^2 - 3t + 1} \\
= -t^{-1}w_+c\bar{c}q(t) = 0 \in \mathbb{Q}(\xi)(t)/\mathbb{Q}(\xi_5)[t^{\pm 1}]. \quad \square
\]

We also note that $t^2 - 3t + 1$ is relatively prime to $\Delta_R(t) = 2 - 6t^2 + 9t^3 - 6t^5 + 2t^4$.

Now, fix some $g \geq 0$ and let $N = 8g + 2$, noting that the generating rank of the 5-primary part of $H_1(\Sigma_2(R))$ is 1. By [Cha16, Theorem 1.9], $C_R = 10^9$ is an upper bound for Cheeger-Gromov constant $C(M_R)$ of the 0-surgery on $R$. For $k = 1, \ldots, N$, let $J_k$ be a knot with
Arf$(J_k) = 0$ and
\[
\int_{\omega \in S^1} \sigma_\omega(J_k) \, d\omega > 2(2g + N - 1) + C_R N + \sum_{i=1}^{k-1} \left( \int_{\omega \in S^1} \sigma_\omega(J_i) \, d\omega \right). \tag{1}
\]
We can achieve this by taking $J_k$ to be a sufficiently large even connected sum of negative trefoils, since for the negative trefoil
\[
0 \text{ and } (1) \text{ is satisfied if we define } J_k \text{ to be the connected sum of } 10^k \text{ negative trefoils.}
\]

In fact, the numerically minded reader can easily verify that Equation (1) is satisfied if we define $J_k$ to be the connected sum of $10^k$ negative trefoils.

Now let $R$ be the knot depicted on the left of Figure 2. On the right of Figure 2 we see a genus 2 Seifert surface $F$ for $R$ along with two sets of derivative curves for $R$: each of the red and blue collection of curves generates a half-rank summand of $H_1(F)$, forms a slice link (in fact, an unlink) in $S^3$, and is 0-framed by $F$.

![Figure 2. The knot $R = 11_{74} = P(3, -3, 3, -2)$ (left) and a Seifert surface for $R$ along with two derivative links (right).](image)

Now let $\beta_1, \beta_2, \gamma_1, \gamma_2$ be the curves indicated on the left of Figure 3. These curves are disjoint from $F$ and hence lie in $\pi_1(M_R)^{(1)}$. Note that the indicated basepoints should be thought of as living in a plane ‘far below’ the plane of the diagram; in that plane they can be connected, uniquely up to homotopy, to a single preferred basepoint for $\pi_1(M_R)$.

Let $\alpha^- = [\beta_1, \beta_2]$ and $\alpha^+ = [\gamma_2, \gamma_2]$, where $[v, w] = v w v^{-1} w^{-1}$; unknotted representatives for $\alpha^\pm$ are shown on the right of Figure 3. Note that $\alpha^\pm \in \pi_1(M_R)^{(2)}$. Since $\alpha^+$ has no geometric linking with the blue derivative curves of Figure 2, for any knot $J$ the satellite knot $R_{\alpha^+}(J)$ still has a smoothly slice derivative, and hence is itself smoothly slice. Similarly, since $\alpha^-$ has no geometric linking with the red curves, the satellite knot $R_{\alpha^-}(J)$ is slice for every knot $J$. Therefore, by Proposition 2.6, for any 0-positive knot $J^+$ and 0-negative knot $J^-$, the knot $R_{\alpha^+, \alpha^-}(J^+, J^-)$ is 2-bipolar.

We now proceed to verify the conditions of Theorem 6.1, so that for appropriate choices of $N \in \mathbb{N}$ and of $J_i^\pm$, $i = 1, \ldots, N$, the knot $\#_{i=1}^N R_{\alpha^+, \alpha^-}(J_i^+, J_i^-)$ will have large topological 4-genus.
Note that $H_1(\Sigma_2(R)) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and so (up to rescaling by a nonzero constant), there are four nontrivial characters $\chi_i: H_1(\Sigma_2(R)) \to \mathbb{Z}_3$. We compute that for three of these characters, which we call $\chi_1, \chi_2,$ and $\chi_3$, the resulting twisted homology is $H_{f_i}^R(\Sigma_2(M)) \cong \mathbb{Q}(\xi_3)[t^{\pm 1}]/\langle (t - 1)^2 \rangle =: H$. For the fourth character, denoted by $\chi_4$, we compute that the twisted homology is $H_{f_i}^R(\Sigma_2(M)) \cong \mathbb{Q}(\xi_3)[t^{\pm 1}]/\langle t^2 - 14t + 1 \rangle =: H'$. Crucially, for any nontrivial $\chi_i$, the lifts of $\alpha^+$ and $\alpha^-$ to the cover of $M_R$ induced by $f_i$ generate $H_{f_i}^R(\Sigma_2(M))$. (More precisely, $[1, 0] \otimes [\alpha^+]$ and $[1, 0] \otimes [\alpha^-]$ generate.)

As in [MP18], we computed the twisted homology using a Maple program, available for download on the authors' websites. The program obtains a presentation for the twisted homology using the Wirtinger presentation, taking the Fox derivatives, and then applying the representation. It then simplifies the presentation by row and column operations to obtain a diagonal matrix. Keeping track of how the original generators, which can be identified in the knot diagram, are modified under the sequence of row and column operations, we not only compute the twisted homology $H_{f_i}^R(\Sigma_2(M))$ but also can identify which elements the curves $[1, 0] \otimes [\alpha^\pm]$ represent in $H_{f_i}^R(\Sigma_2(M))$. Note also that the orders of $H$ and $H'$ are both relatively prime to $\Delta_R(t) = (t^2 - t + 1)^2$.

Now let $g \in \mathbb{N}$ be given, and let $N = 4g + 1$. We let $C_R$ denote an upper bound for the Cheeger-Gromov constant $C(M)$. For each $i = 1, \ldots, N$, successively pick $m_i$ to be even and large enough that $J_i^+ := \#^{m_i} T_{2,3}$ satisfies
\[ \rho_0(J_i^+) < -2(2g + N - 1) - NC_R + \sum_{j=1}^{i-1} \rho_0(J_j^+) - \sum_{j=1}^{i-1} \rho_0(J_j^-), \]
and then pick $m_i'$ to be even and large enough that $J_i^- := -\#^{m_i'} T_{2,3}$ satisfies
\[ \rho_0(J_i^-) > 2(2g + N - 1) + NC_R - \sum_{j=1}^{i} \rho_0(J_j^+) + \sum_{j=1}^{i-1} \rho_0(J_j^-). \]
Note that in particular $\rho_0(J_i^+) < 0 < \rho_0(J_i^-)$ for all $i$. 

Figure 3. Unknotted curves $\beta_1, \beta_2, \gamma_1, \gamma_2$ in $\pi_1(M_R)^{(1)}$ (left) and $\alpha^\pm$ in $\pi_1(M_R)^{(2)}$ (right).
We now let $K_i = R_{a_i, a_i -} (J_i^+, J_i^-)$ and $K = \#_{i=1}^N K_i$. Observe that $K$ is 2-solvable by Proposition 2.3 and 2-bipolar by Proposition 2.6; $K$ also satisfies the hypotheses of Theorem 6.1, so $g_4(K) \geq g + 1$. In Section 10, we will show that $K$ bounds an embedded grope of height four in the 4-ball. The knot $K$ therefore gives the example claimed in Theorem 1.1.

It now remains to prove Theorem 6.1.

7. Controlling the size of some homology groups

This section contains some technical results needed for the proof of Theorem 6.1, with the theme that we need to control the size of certain homology groups of some covering spaces.

We start this section with an elementary algebraic lemma. This lemma and the one after it are very similar to, and are inspired by, results of Levine in [Lev94], in particular Lemma 4.3 of Part I and Proposition 3.2 of Part II. To avoid citing lemmas that were written for a different situation, and for the edification of the reader, we provide short self-contained proofs.

**Lemma 7.1.** Let $F: M \to M$ be an endomorphism of a finitely generated free $\mathbb{Z}[\mathbb{Z}_2]$-module $M$ such that

$$\text{Id} \otimes F: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M$$

is an isomorphism. Then

$$\text{Id} \otimes F: \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M \to \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M$$

is also an isomorphism.

**Proof.** Let $A + BT \in \mathbb{Z}[\mathbb{Z}_2]$ be the determinant of $F$, where $A, B \in \mathbb{Z}$ and $T \in \mathbb{Z}_2$ denotes the generator. Then $A + B = \pm 1$ by hypothesis. Thus $A^2 - B^2 = (A + B)(A - B) = \pm (A - B) \neq 0$ since $A + B \equiv A - B$ modulo 2. Now

$$(A + BT) \cdot \frac{1}{A^2 - B^2} (A - BT) = 1,$$

so over $\mathbb{Q}[\mathbb{Z}_2]$ the determinant of $F$ is invertible, and hence $\text{Id}_{\mathbb{Q}[\mathbb{Z}_2]} \otimes F$ is an isomorphism as desired. \qed

Next we apply this lemma to obtain some control on the size of the homology of double covering spaces.

**Lemma 7.2.** Let $f: X \to Y$ be a map of finite CW complexes such that

$$f_*: H_i(X; \mathbb{Z}) \to H_i(Y; \mathbb{Z})$$

is an isomorphism for $i = 0$ and a surjection for $i = 1$. Let $\varphi: \pi_1(Y) \to \mathbb{Z}_2$ be a surjective homomorphism and let $X^2, Y^2$ be the induced 2-fold covers. Then

$$f_*: H_i(X^2; \mathbb{Q}) \cong H_i(X; \mathbb{Q}[\mathbb{Z}_2]) \to H_i(Y^2; \mathbb{Q}) \cong H_i(Y; \mathbb{Q}[\mathbb{Z}_2])$$

is also an isomorphism for $i = 0$ and a surjection for $i = 1$.

**Proof.** The zeroth and first relative homology groups vanish, that is $H_i(Y, Y; \mathbb{Z}) = 0$ for $i = 0, 1$. Thus the cellular chain complex $(C_*(Y, X; \mathbb{Z}), \partial_*)$ admits a partial chain contraction: writing $C_*^X$ to abbreviate $C_*(Y, X; \mathbb{Z})$, the partial chain homotopy comprises maps $s_0: C_0 \to C_1$ and $s_1: C_1 \to C_2$ such that $\partial \circ s_0 = \text{Id}: C_0 \to C_0$ and $\partial \circ s_1 + s_0 \circ \partial = \text{Id}: C_1 \to C_1$.

To see this, follow the proof of the fundamental lemma of homological algebra: for each basis element $x_i \in C_0$, choose a lift $y_i \in C_1$ with $\partial y_i = x_i$, and define $s_0(x_i) = y_i$, and then extend linearly. Such a $y_i$ exists since $\partial: C_1 \to C_0$ is surjective. Then for each generator $z_i \in C_1$, consider $\text{Id}(z_i) - s_0 \circ \partial(z_i)$. Since

$$\partial(\text{Id}(z_i) - s_0 \circ \partial(z_i)) = \partial(z_i) - \partial \circ s_0 \circ \partial(z_i) = \partial(z_i) - \text{Id} \circ \partial(z_i) = 0,$$
we have that \( \text{Id}(z_i) - s_0 \circ \partial(z_i) \) is a cycle. Hence there is a \( \nu_i \in C_2 \) such that \( \partial \nu_i = \text{Id}(z_i) - s_0 \circ \partial(z_i) \). Define \( s_1(z_i) := \nu_i \), and extend linearly to define \( s_1 \) on all of \( C_1 \). Then \( \partial \circ s_1(z_i) = \partial \nu_i = \text{Id}(z_i) - s_0 \circ \partial(z_i) \) for every generator \( z_i \) of \( C_1 \). This completes the construction of a partial chain homotopy.

Now consider the chain complex \( D_* := C_*(Y, X; \mathbb{Z}[\mathbb{Z}_2]) \cong C_*(Y^2, X^2) \), the relative chain complex of the 2-fold covering spaces. Since the cellular chain groups are finitely generated free modules, the partial chain contraction \( s_0, s_1 \) lifts to maps \( \tilde{s}_0 : D_0 \rightarrow D_1 \) and \( \tilde{s}_1 : D_1 \rightarrow D_2 \).

We claim that these maps induce a partial chain contraction after tensoring over \( \mathbb{Q}[\mathbb{Z}_2] \). To see the claim, the maps

\[
F := \tilde{s}_0 \circ \partial : D_0 \rightarrow D_0 \quad \text{and} \quad G := \partial \circ \tilde{s}_1 + \tilde{s}_0 \circ \partial : D_1 \rightarrow D_1
\]

are endomorphisms of the free modules \( D_0 \) and \( D_1 \) respectively, that become automorphisms when tensored over \( \mathbb{Z} \). That is,

\[
\text{Id} \otimes F : \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0 \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0,
\]

\[
\text{Id} \otimes G : \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1 \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1
\]

are isomorphisms. By Lemma 7.1,

\[
\text{Id} \otimes F : \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0 \rightarrow \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0,
\]

\[
\text{Id} \otimes G : \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1 \rightarrow \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1
\]

are also isomorphisms. Thus

\[
\text{Id} \otimes \tilde{s}_i : \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_i \rightarrow \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_{i+1}
\]

is a partial chain contraction and

\[
H_i(\mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_* ) = H_i(Y^2, X^2; \mathbb{Q}) \cong H_i(Y, X; \mathbb{Q}[\mathbb{Z}_2]) = 0
\]

for \( i = 0, 1 \). The lemma follows from the long exact sequence of the pair \( (Y, X) \) (Proposition 3.3).

Our next lemma requires some facts about finitely generated modules over commutative PIDs, which we remind the reader of in order to establish notation.

**Definition 7.3.** Let \( S \) be a commutative PID with quotient field \( Q \), and let \( A \) be a finitely generated module over \( S \).

1. \( TA := \{ a \in A \text{ such that } sa = 0 \text{ for some } s \in S \} \), the \( S \)-torsion submodule of \( A \).
2. \( A^\wedge := \text{Hom}_S(A, Q/S) \). If \( A \) is torsion (i.e. \( A = TA \)), \( A \) and \( A^\wedge \) are non-canonically isomorphic.
3. Given a map of \( S \) modules \( f : A \rightarrow B \), we abbreviate \( f|_{TA} : TA \rightarrow TB \) by \( f|_T \). We emphasize that \( \text{coker}(f|_T) \) is therefore isomorphic to \( TB/\text{Im}(f|_TA) \), not \( B/\text{Im}(f|_TA) \).
4. We say that \( A \) has generating rank \( k \) over \( S \) if \( A \) is generated as an \( S \)-module by \( k \) elements but not by \( k - 1 \) elements, and write \( \text{g-rk}_S A = k \). It follows immediately from the definition that if \( A \) surjects onto \( B \) then \( \text{g-rk}_S B \leq \text{g-rk}_S A \). It is also true and easy to check that if \( B \leq A \) then \( \text{g-rk}_S B \leq \text{g-rk}_S A \), though this is less obvious and uses that \( S \) is a commutative PID.
5. By the fundamental theorem of finitely generated modules over PIDs, there exist \( j, k \in \mathbb{N} \) and elements \( s_1, \ldots, s_k \in S \) such that there is a (non-canonical) isomorphism

\[
A \cong S^j \oplus TA \cong S^j \oplus \bigoplus_{i=1}^k S/\langle s_i \rangle.
\]

When \( j > 0 \) we say that the order of \( A \) is \( |A| = 0 \) and when \( j = 0 \) we say that the order of \( A \) is \( |A| = \prod_{i=1}^k s_i \). This is well-defined up to multiplication by units in \( S \). The key property of order we use is that if \( f : A \rightarrow B \) is a map of \( S \)-modules with \( \text{ker}(f) \) torsion, then \( |\text{Im}(f)| = |A|/|\text{ker}(f)| \).
We will use the next lemma twice, once in the proof of Proposition 7.5, and then again in Step 3 of the proof of Theorem 6.1 in Section 9.

**Lemma 7.4.** Let $X$ be a 4-manifold with boundary $\partial X = Y$. Let $S$ be a commutative PID with quotient field $Q$ and let $A := S^n$ for some $n \in \mathbb{N}$. Suppose there is a representation $\Phi$ of the fundamental groupoid of $(Y, X)$ into $\text{Aut}(A)$ that extends over $X$, as in Section 3. Consider the long exact sequence (Proposition 3.3) of $S$-modules of the pair, with coefficients taken in $A$:

$$\cdots \to H_2(X) \xrightarrow{j_2} H_2(X, Y) \xrightarrow{\partial} H_1(Y) \xrightarrow{i_1} H_1(X) \xrightarrow{j_1} H_1(X, Y) \to \cdots.$$ 

Suppose that $H_1(X, Y)$ is torsion. Then $\ker((j_1)_T) \cong \coker((j_2)_T)$.

**Proof.** Unless otherwise specified, all homology groups are taken with coefficients in $A$. In this setting, Poincaré duality, universal coefficients, and the (inverse of the) Bockstein homomorphism together induce natural isomorphisms fitting into a commutative diagram:

$$\begin{array}{ccc}
\text{TH}_2(X) & \xrightarrow{PD} & \text{TH}^2(X, Y) \\
\downarrow^{j_2|_T} & & \downarrow^{(j_1|_T)} \\
\text{TH}_2(X, Y) & \xrightarrow{PD} & \text{TH}^2(X, Y) \\
\end{array}$$

We need to argue that the Bockstein homomorphism $\beta$ is an isomorphism. This Bockstein arises in the long exact sequence of Ext groups [HS97, IV, Prop. 7.5] associated to the short exact sequence $0 \to S \to Q \to Q/S \to 0$, as follows:

$$\text{Ext}^0_S(H_1(X, Y), Q) \to \text{Ext}^0_S(H_1(X, Y), Q/S) \xrightarrow{\beta} \text{Ext}^1_S(H_1(X, Y), S) \to \text{Ext}^1_S(H_1(X, Y), Q).$$

Since $Q$ is an injective $S$-module, $\text{Ext}^1_S(H_1(X, Y), Q)$ vanishes, and $\text{Ext}^0_S(H_1(X, Y), Q) = 0$ because $H_1(X, Y)$ is torsion. Thus $\beta$ is an isomorphism and so can be inverted as claimed.

By the naturality of the above sequence of maps, the following square commutes and so $\coker((j_2)_T) \cong \coker((j_1)_T)$.

$$\begin{array}{ccc}
\text{TH}_2(X) & \xrightarrow{j_2|_T} & \text{TH}_2(X, Y) \\
\downarrow^{(j_1|_T)^\wedge} & & \downarrow^{(j_1|_T)} \\
\text{TH}_1(X, Y)^\wedge & \xrightarrow{(j_1|_T)} & \text{TH}_1(X)^\wedge \\
\end{array}$$

Now let $H := \ker((j_1|_T)) \subseteq \text{TH}_1(X)$ and define $\Phi : \coker((j_1|_T)^\wedge) \to H^\wedge$ by $\Phi([f]) = f|_H$. Observe that $\Phi$ is well-defined, since for any $g \in \text{TH}_1(X, Y)^\wedge$ we have

$$(j_1|_T)^\wedge(g)(x) = g(j_1(x)) = g(0) = 0 \text{ for all } x \in H = \ker((j_1|_T)),$$

Also, $\Phi$ is onto: given any $f \in H^\wedge$ (i.e. a map $f : H \to Q/S$), since $H \subseteq \text{TH}_1(X)$, and using that $Q/S$ is an injective $S$-module, we can extend $f$ to a map $g : \text{TH}_1(X) \to Q/S$, and will have that $\Phi([g]) = f$. Therefore, in order to show that $\Phi$ is an isomorphism it is enough to show that $|\coker((j_1|_T)^\wedge)| = |H^\wedge|$. Note that

$$|\coker((j_1|_T)^\wedge)| = \frac{|\text{TH}_1(X)^\wedge|}{|\text{Im}(j_1|_T)^\wedge|} = \frac{|\text{TH}_1(X)^\wedge|}{|\ker((j_1|_T)^\wedge)|}. \quad (3)$$

Also, $(j_1|_T)^\wedge(f) = 0$ if and only if $f(j_1|_T(x)) = 0$ for all $x \in \text{TH}_1(X)$ if and only if $f$ vanishes on $\text{Im}(j_1|_T)$, so

$$|\ker((j_1|_T)^\wedge)| = \frac{|\text{TH}_1(X, Y)^\wedge|}{|\text{Im}(j_1|_T)|}. $$
Therefore we can rewrite Equation (3) as

$$|\coker((j_1|_T)^\wedge)| = \frac{|TH_1(X)^\wedge|}{|\text{Im}(j_1|_T)|} = \frac{|TH_1(X)|}{|\text{Im}(j_1|_T)|} = |H| = |H^\wedge|.$$

So $\Phi: \coker((j_1|_T)^\wedge) \rightarrow \ker(j_1|_T)^\wedge$ is an isomorphism. Since $\ker(j_1|_T)$ is a torsion $S$-module, there is an (albeit non-canonical) identification $\ker(j_1|_T) \cong (\ker(j_1|_T))^\wedge$, and so we have as desired that

$$\coker(j_2|_T) \cong \coker((j_1|_T)^\wedge) \cong \Phi \ker(j_1|_T)^\wedge \cong \ker(j_1|_T).$$

$\square$

Note that in particular this implies that $\ker(j_1|_T)_p \cong \coker(j_2|_T)_p$ for any prime $p$, where for a $\mathbb{Z}$-module $G$ we write $G_p$ for the $p$-primary part.

Next we apply the control on homology of double covers gained in Lemma 7.2 along with the homological algebra of Lemma 7.4 to manifolds $M^3$ and $V^4$.

**Proposition 7.5.** Let $M$ be a homology $S^1 \times S^2$, and let $V$ be a 4-manifold with boundary $M$ such that the inclusion induced map $H_1(M) \xrightarrow{i} H_1(V)$ is an isomorphism. Suppose that $H_1(M^2) \cong \mathbb{Z} \oplus G$, where $G$ is torsion. Then for any prime $p$ the $p$-primary part of $TH_1(M^2)/\ker(TH_1(M^2) \rightarrow TH_1(V^2))$ has generating rank at least $n := \frac{m-2\chi(V)}{2}$, where $m$ denotes the generating rank of the $p$-primary part of $TH_1(M^2)$.

**Proof.** Observe that $H_i(V, M; \mathbb{Z}) = 0$ for $i = 0, 1$. It follows from Lemma 7.2 that

$$H_i(V^2, M^2; \mathbb{Q}) \cong H_i(V, M; \mathbb{Q}[\mathbb{Z}_2]) = 0 \quad \text{for } i = 0, 1.$$

Therefore $\dim H_1(V^2, \mathbb{Q}) \leq \dim H_1(M^2, \mathbb{Q}) = 1$ and

$$\dim H_3(V^2, \mathbb{Q}) = \dim H^3(V^2; \mathbb{Q}) = \dim H_1(V_2, M^2; \mathbb{Q}) = 0.$$

Also note that

$$2\chi(V) = \chi(V^2) = 1 - \dim H_1(V^2, \mathbb{Q}) + \dim H_2(V^2, \mathbb{Q}).$$

We therefore have that $\dim H_2(V^2, \mathbb{Q}) = 2\chi(V) + \dim H_1(V^2, \mathbb{Q}) - 1$ is at most $2\chi(V)$. Now consider the following long exact sequence:

$$\cdots \rightarrow H_2(V^2) \xrightarrow{j_2^\wedge} H_2(V^2, M^2) \xrightarrow{j_1^\wedge} H_1(M^2) \xrightarrow{i^\wedge} H_1(V^2) \xrightarrow{j_2} H_1(V^2, M^2) \cdots.$$

Note that from above we have $H_1(V^2, M^2; \mathbb{Q}) = 0$, and so by Lemma 7.4 we have that $g\text{-rk}\ker(j_1|_T)_p = g\text{-rk}\coker(j_2|_T)_p$. Note that if $G \cong \mathbb{Z}^a \oplus TG$ then $g\text{-rk}(G \otimes \mathbb{Z}_p) = a + g\text{-rk}(G_p)$. In particular $g\text{-rk}(\ker(j_1|_T) \otimes \mathbb{Z}_p) = g\text{-rk}(\coker(j_2|_T) \otimes \mathbb{Z}_p)$. It also follows from $g\text{-rk}(G \otimes \mathbb{Z}_p) = a + g\text{-rk}(G_p)$ together with $H_1(M^2) \cong \mathbb{Z} \oplus TH_1(M^2)$, that in order to show as desired that the generating rank of the $p$-primary part of $(TH_1(M^2)/\ker(i_1|_T))$ is at least $n$, it suffices to show

$$g\text{-rk}\left(\left(\ker(i_1|_T) \otimes \mathbb{Z}_p\right) \otimes \mathbb{Z}_p\right) \geq n + 1.$$

Note that $H_1(M^2)/\ker(i_1) \cong \text{Im}(i_1) \cong \ker(j_1)$, so if $g\text{-rk}(\ker(j_1) \otimes \mathbb{Z}_p) \geq n + 1$ then we are done. Similarly, since $g\text{-rk}(H_1(M^2) \otimes \mathbb{Z}_p) = m + 1$ and

$$\ker(i_1) = \text{Im}(\partial) \cong H_2(V^2, M^2)/\ker(\partial) = H_2(V^2, M^2)/\text{Im}(j_2) = \coker(j_2),$$

if $g\text{-rk}(\coker(j_2) \otimes \mathbb{Z}_p) \leq m - n$, then we are also done.

So suppose for a contradiction $g\text{-rk}(\coker(j_2) \otimes \mathbb{Z}_p) > m - n$ and $g\text{-rk}(\ker(j_1) \otimes \mathbb{Z}_p) \leq n$. Note that since $H_3(V^2, M^2) \cong H^1(V^2) \cong H^1(M^2) \cong H_2(M^2)$, the ranks of $H_2(V^2, M^2)$ and $H_2(V^2)$ coincide and so $H_2(V^2, M^2)$ splits (albeit non-canonically) as $\mathbb{Z}_b(V^2) \oplus TH_2(V^2, M^2)$. 


Thus we obtain our desired contradiction as follows:

\[ 2\chi(V) + n = m - n < g\text{-rk} (\text{coker}(j_2) \otimes \mathbb{Z}_p) \]
\[ \leq g\text{-rk} \left( \left( H_2(V^2, M^2) / \text{Im}(j_2|_T) \right) \otimes \mathbb{Z}_p \right) \]
\[ = g\text{-rk} \left( \mathbb{Z}^{b_2(V^2)} \otimes \mathbb{Z}_p \right) + g\text{-rk} \left( (TH_2(V^2, M^2) / \text{Im}(j_2|_T)) \otimes \mathbb{Z}_p \right) \]
\[ = b_2(V^2) + g\text{-rk} (\text{coker}(j_2|_T) \otimes \mathbb{Z}_p) \]
\[ = b_2(V^2) + g\text{-rk} (\ker(j_1|_T) \otimes \mathbb{Z}_p) \]
\[ \leq b_2(V^2) + g\text{-rk} (\ker(j_1) \otimes \mathbb{Z}_p) \leq 2\chi(V) + n. \]

\[ \square \]

8. A standard cobordism

In this section we study a standard cobordism \( U \) between the zero-framed surgery manifold of a connected sum of knots \( M_K = M_{\#_{i=1}^{N} K_i} \) and the disjoint union \( Y := \bigsqcup_{i=1}^{N} M_{K_i} \) of the zero-framed surgery manifolds of the summands \( K_i \). In particular we need to understand the behavior of certain representations of the fundamental groups. We will also explicitly choose the basepoints \( \{ x_i \} \) and paths \( \{ \tau_i \} \) necessary to define twisted homology for disconnected manifolds, as discussed in Section 3.

Let \( U' \) be \( M_K \times [0,1] \) with \((N-1)\) 0-framed 2-handles attached along ‘longitudes of \( K_i \).’ A schematic of a relative Kirby diagram for \( U' \) is given by the black and blue curves of Figure 4. Note that we depict each \( K_i \) as the boundary of a Seifert surface \( G_i \), and hence \( K = \#_{i=1}^{N} K_i \) as the boundary of \( \bigsqcup_{i=1}^{N} G_i \). Since repeatedly sliding the black 0-framed curve over the blue curves gives the standard surgery diagram for \( Y' \), we have \( \partial^+ (U') = Y' := \#_{i=1}^{N} M_{K_i} \). Now let \( U'' \) be \( Y' \times [1,2] \) together with \((N-1)\) 3-handles attached along 2-spheres (whose outline is indicated in green in Figure 4) so that \( \partial^+ U'' = Y = \bigsqcup_{i=1}^{N} M_{K_i} \). Let \( U = U' \cup_Y U'' \).

We now consider the points, arcs, and closed curves shown in Figure 5.

Figure 4. A Kirby diagram for \( U \).

Figure 5. Basepoints \( x_i \), arcs \( \kappa_i \), and meridians \( \mu_i \) for \( i = 1, \ldots, N \) and closed curves \( \delta_j \) for \( j = 1, \ldots, m \) in \( M_K \).
Note that the curves $\delta_j$ for $j = 1, \ldots, m$ form a normal generating set for the first commutator subgroup of $\pi_1(M_K, x_N)$, when suitably based using the arcs $\kappa_i$.

The attaching regions for the 2-handles of $U'$ avoid

$$\bigcup_{i=1}^N \mu_{K_i} \cup \bigcup_{i=1}^{N-1} \kappa_i \cup \bigcup_{j=1}^m \delta_j \subset M_K,$$

and so the points $x_i' := x_i \times \{1\}$, arcs $\kappa_i' := \kappa_i \times \{1\}$, and loops $\mu_{K_i}' := \mu_{K_i} \times \{1\}$ and $\delta_j' := \delta_j \times \{1\}$ lie in $\partial_+ U' = Y'$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, m$. Similarly, the attaching regions for the 3-handles of $U''$ avoid

$$\left( \bigcup_{i=1}^N \mu_{K_i}' \cup \bigcup_{j=1}^m \delta_j' \right) \subset Y'',$n

and so the loops $\mu_{K_i}'' := \mu_{K_i}' \times \{2\}$ and $\delta_j'' := \delta_j' \times \{2\}$ lie in $\partial_+ U'' = Y$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, m$. For each $i = 1, \ldots, N$ we have an inclusion-induced map

$$i_i: \pi_1(M_K, x_i' \times \{2\}) \to \pi_1(U'', x_N)$$

by $\beta \mapsto \kappa_i' \cdot (x_i' \times \{1, 2\}) \cdot \beta \cdot (x_i' \times \{1, 2\}) \cdot \kappa_i'$. Let $U = U' \cup_{Y'} U''$, and note that we also have an inclusion-induced map

$$i: \pi_1(U'', x_N') \to \pi_1(U, x_N)$$

by $\gamma \mapsto (x_N \times [0, 1]) \cdot \gamma \cdot (x_N \times [0, 1])$.

In the language of Section 3, $i \circ i_i$ is induced by the path from $x_N$ to $x_i'$ given by

$$\tau_i = (x_N \times [0, 1]) \cdot \kappa_i' \cdot (x_i' \times \{1, 2\}).$$

We return to using the notation from Section 5 in order to state and prove the following.

**Proposition 8.1.** Let $K = \#_{i=1}^N K_i$ and $U$ be the standard cobordism from $M_K$ to $\bigsqcup_{i=1}^N M_{K_i}$ as above. Let $p \in \mathbb{N}$ and choose maps $\chi_i: H_1(\Sigma_2(\kappa_i)) \to \mathbb{Z}_p$ for $i = 1, \ldots, N$, so $(\chi_i)_{i=1}^N: H_1(\Sigma_2(K)) \to \mathbb{Z}_p$. Let $\mu_{K_N}$ be the preferred meridian for $K$ and for $i = 1, \ldots, N$ let $\mu_{K_i}$ be the preferred meridian for $K_i$. Then the map

$$f^K_{(\chi_i)_{i=1}^N} : \pi_1(M_K, x_N) \to \mathbb{Z} \times \mathbb{Z}_p$$

extends uniquely to a map $F: \pi_1(U, x_N) \to \mathbb{Z} \times \mathbb{Z}_p$. Also, the composition

$$f_i : \pi_1(M_K, x_i' \times \{2\}) \xrightarrow{i_i} \pi_1(U'', x_N') \xrightarrow{i} \pi_1(U, x_N) \xrightarrow{F} \mathbb{Z} \times \mathbb{Z}_p$$

satisfies $f_i = f^K_{(\chi_i)_{i=1}^N}$.

**Proof.** Notice that $\pi_1(U) = \pi_1(M_K)/\langle \lambda_{K_1}, \ldots, \lambda_{K_N-1} \rangle$. Therefore, since each $\lambda_{K_i}$ bounds a subarc of $N_{i=1}^N G_i$ and hence lies in $\pi_1(M_K)^{(2)}$, the map $f^K_{(\chi_i)_{i=1}^N}$ extends uniquely as desired.

Observe that for all $i = 1, \ldots, N$ we have

$$i_i(\mu_{K_i}'') = (\kappa_i' \cdot (x_i' \times \{1, 2\}) \cdot \mu_{K_i}' \cdot (x_i' \times \{1, 2\}) \cdot \kappa_i')$$

$$= (x_N \times [0, 1]) \cdot \kappa_i' \cdot \mu_{K_i}' \cdot \kappa_i' \cdot (x_N \times [0, 1]) = \mu_{K_N} \in \pi_1(U, x_N).$$

Therefore

$$f_i(\mu_{K_i}'') = F(\mu_{K_N}) = f^K_{(\chi_i)_{i=1}^N}(\mu_{K_N}) = (t, 0) = f^K_{(\chi_i)_{i=1}^N}(\mu_{K_i}') \in \mathbb{Z} \times \mathbb{Z}_p.$$

For each $i = 1, \ldots, N$, every element $\gamma \in \pi_1(M_K, x_i')$ can be written as $\gamma = (\mu_{K_i}'')^{c(\gamma_i)} a$ for some element $a \in \pi_1(M_K, x_i')^{(1)}$. Moreover, the collection of $\delta_j''$ corresponding to $K_i$ in Figure 5 normally generate $\pi_1(M_K, x_i')^{(1)}$. It therefore suffices to check that $f_i$ agrees with $f^K_{(\chi_i)}$ on $\mu_{K_i}''$ (as done above) and on the collection of $\delta_j''$ corresponding to $K_i$. 


Supposing that $\delta''_i$ corresponds to $K_i$, we have that
\[
\iota(i_1(\delta''_i)) = \iota(\kappa'_i \cdot (x'_i \times [1, 2]) \cdot \delta''_i \cdot (x'_i \times [1, 2]) \cdot \kappa''_i) \\
= \iota(\kappa'_i \cdot \delta''_i \cdot \kappa''_i) \\
= (x_N \times [0, 1]) \cdot \kappa'_i \cdot \delta''_i \cdot \kappa''_i \cdot (x_N \times [0, 1]) \\
= \kappa_i \delta_j \kappa_i \in \pi_1(U, x_N).
\]

Now fix a lift $\tilde{\gamma}_N$ of $x_N$ to $M^K_{\tilde{K}_i}$, the double cover of $M_{K_i}$. Since $x_N$ does not lie in a tubular neighborhood of $K$, we can think of $\tilde{\gamma}_N$ as lying in $E^2_{\tilde{K}_i} \subseteq \Sigma_2(K)$ as well. The inclusion induced maps $M_{K_i} \to U$ and $M_{\tilde{K}_i} \to U$ induce isomorphisms on first homology, and so the double cover $U^2$ is a cobordism from $M^K_{\tilde{K}_i}$ to $\bigsqcup_{i=1}^N M^K_{\tilde{K}_i}$. For each $i = 1, \ldots, N$, lifting the arc
\[
\kappa_i \cdot (x_i \times [0, 1]) \cdot (x'_i \times [1, 2])
\]
to $U^2$ starting at $\tilde{\gamma}_N$ gives a preferred basepoint $\tilde{\gamma'}_N$ in $M^K_{\tilde{K}_i}$. As before, we also think of this basepoint as lying in $E^2_{\tilde{K}_i} \subseteq \Sigma_2(K_i)$. We can therefore speak of the lift $\tilde{\gamma}$ of a curve $\gamma$ based at $x_N$ (respectively, $x'_N$) to $\Sigma_2(K)$ (respectively, $\Sigma_2(K_i)$) by choosing the lift with basepoint $\tilde{\gamma}_N$ (respectively, $\tilde{\gamma'}_N$).

**Remark 8.2.** A choice of basepoint is technically always necessary to define $f^K_i$, though this was suppressed in Section 5 in our discussion of the connected case, where it was less important.

Therefore
\[
f_i(\delta''_i) = F(\kappa_i \delta_j \kappa_i) = f^K_i(\kappa_i \delta_j \kappa_i) = (0, (\chi_k)_{k=1}^N(\kappa_i \delta_j \kappa_i)).
\]

Similarly,
\[
f^K_i(\kappa_i \delta_j \kappa_i) = (0, \chi_i(\tilde{\delta''}_i)).
\]

It therefore only remains to show that
\[
(\chi_k)_{k=1}^N(\kappa_i \delta_j \kappa_i) = \chi_i(\tilde{\delta''}_i).
\]

First, note that $\chi_k(\kappa_i \delta_j \kappa_i) = 0$ unless $k = i$. Also, the homology class of $\kappa_i \delta_j \kappa_i$ in
\[
H_1(\Sigma_2(K_i)) \subset \bigoplus_{i=1}^N H_1(\Sigma_2(K_i)) \cong H_1(\Sigma_2(K))
\]
is exactly the same as that of $\tilde{\delta''}_i$ in $H_1(\Sigma_2(K_i))$, and so we have that
\[
\chi_i(\kappa_i \delta_j \kappa_i) = \chi_i(\tilde{\delta''}_i).
\]

We note for later use that the inclusion induced maps $M_{K'} \to U$ and $M_{K_i} \to U$ give isomorphisms on first homology, and that $H_2(U) \cong \mathbb{Z}^N$ and $H_3(U) \cong \mathbb{Z}^N$.

### 9. Proof of Theorem 6.1

Suppose for the sake of contradiction that there is some locally flat surface $F'$ embedded in $D^4$ with $\partial F' = K$ and $g(F') = g$.

**Step 1:** Construct a 4-manifold $Z := V \cup_k U$ with $H_2(Z) = \mathbb{Z}^{2g+N-1}$ and $\partial Z = Y := \bigsqcup_{i=1}^N M_{K_i}$.
isomorphism, as are each of the maps $H_2$, $H_1(M_K) \to H_1(V)$ an isomorphism, and $H_2(V) \cong \mathbb{Z}^{2g}$, as follows. Let $X = X_0(K)$ denote the 0-trace of $K$, the 4-manifold obtained from $D^4$ by attaching a 0-framed 2-handle along a neighborhood of $K$. Let $F$ be the closed surface in $X$ obtained by taking the union of $F'$ with a core of the 2-handle. Note that since $F$ is locally flat it has a normal bundle by [FQ90, Section 9.3]. Observe that $F \cdot F = 0$, and so $\nu(F) \cong F \times S^1$. Now, let $V = (X \setminus \nu(F)) \cup_{\partial \nu(F)} H \times S^1$, where $H$ is any handlebody with $\partial H = F$. A Mayer-Vietoris argument shows that $H_1(V) \cong \mathbb{Z}$, with generator a meridian to $F$, and that $H_2(V) \cong \mathbb{Z}^{2g}$. Note that by Poincaré duality and universal coefficients, we have

$$H_3(V) \cong H^1(V, M_K) \cong \text{Hom}(H_1(V, M_K), \mathbb{Z}) \cong \text{Hom}(0, \mathbb{Z}) \cong 0.$$ 

So in particular the Euler characteristic of $V$ is $\chi(V) = 1 - 1 + 2g - 0 + 0 = 2g$.

Let $U$ be the standard cobordism between $M_K$ and $\bigsqcup_{i=1}^N M_{K_i}$ (discussed rather extensively in Section 8). As discussed in that section, we have a standard choice of basepoints and paths inducing inclusion maps; for the rest of the proof, these choices will remain fixed though not explicitly discussed. Now let $Z = V \cup_{M_K} U$, as illustrated schematically in Figure 6.

![Figure 6. A schematic diagram of $Z = U \cup_{M_K} V$.](image)

Note that $H_2(Z) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}^{N-1}$, and the inclusion induced map $H_1(M_K) \to H_1(Z)$ is an isomorphism, as are each of the maps $H_i(M_{K_i}) \to H_i(Y) \to H_i(Z)$ for $i = 1, \ldots, N$. Also, $H_3(Z) \cong \mathbb{Z}^{N-1}$ and $H_4(Z) = 0$. So the Euler characteristic of $Z$ is

$$\chi(Z) = 1 - 1 + (2g + N - 1) - (N-1)) = 2g.$$

We will need the Euler characteristic of $Z$ later to estimate the size of the homology of covering spaces. This completes Step 1: we have constructed the 4-manifold $Z$ with the required properties.

We pause to establish notation. For a knot $J$ in $S^3$, we denote its exterior by $E_J$. For a manifold $X$ with $H_1(X) \cong \mathbb{Z}$, we denote its canonical double cover by $X^2$. The choice of a meridian $\mu_J$ determines a splitting $\pi_1(M_J) \cong \mathbb{Z} \times A(J)$, where $A(J)$ denotes the Alexander module of $J$. Note that $H_1(\Sigma_2(J))$ is naturally identified with $A(J)/(t + 1)$, and so a map $\chi: H_1(\Sigma_2(J)) \to \mathbb{Z}^p$ induces a map

$$f_\chi: \pi_1(M_J) \cong \mathbb{Z} \times A(J) \to \mathbb{Z} \times H_1(\Sigma_2(J)) \xrightarrow{1 + \chi} \mathbb{Z} \times \mathbb{Z}_p.$$

Note that in our case, since $H_1(M_K) \to H_1(Z) \cong \mathbb{Z}$ is an isomorphism, $Z$ also has a canonical double cover $Z^2$. It is easy to check that $Z^2 = V^2 \cup_{M_K^2} U^2$ and that $\partial Z^2 = \bigsqcup_{i=1}^N M_{K_i}^2$.

For each $i = 1, \ldots, N$, we have a canonical, linking form–preserving identification of $H_1(\Sigma_2(K_i))$ with $H_1(\Sigma_2(R))$ coming from the degree one maps $E_{\mu_i} \to E_{\text{unknot}}$. Given a map $\chi^R: H_1(\Sigma_2(R)) \to \mathbb{Z}_p$ we will use $\chi^{K_i}$ to denote the corresponding map from $H_1(\Sigma_2(K_i)) \to \mathbb{Z}_p$. Note that $\chi^R$ is a map $H_1(\Sigma_2(R)) \to \mathbb{Z}_p$ that describes the linking form of $\Sigma_2(R)$, and $\chi^{K_i}$ is the corresponding map for $\Sigma_2(K_i)$. The maps $\chi^R$ and $\chi^{K_i}$ are related by the degree one maps $E_{\mu_i} \to E_{\text{unknot}}$.
$\mathbb{Z}_p$, and vice versa. We will also always identify $H_1(\Sigma_2(K))$ with $\bigoplus_{i=1}^{N} H_1(\Sigma_2(K_i))$ in the canonical, linking form-preserving way.

**Step 2:** Build a highly nonzero metabelian map that extends over $Z$.

Recall that $n := 2g + 1$. We wish to show that there exist $\chi_i^R: H_1(\Sigma_2(R)) \to \mathbb{Z}_p$, for $i = 1, \ldots, N$ such that at least $n$ of the $\chi_i^R$ are nonzero and for some $a > 0$, there exists a map $\pi_1(Z) \to \mathbb{Z} \times \mathbb{Z}_p^n$ such that the composition $\pi_1(M_{K_1}) \xrightarrow{i_{\chi_i}} \pi_1(Z) \to \mathbb{Z} \times \mathbb{Z}_p$ is given by $f_{\chi_i}^R$.

We will accomplish this in a somewhat indirect fashion, by focusing on constructing an appropriate map on $\pi_1(M_K)$ which extends over $\pi_1(U)$ and $\pi_1(V)$ separately. By Proposition 8.1, given any choice of $\chi_1^R, \ldots, \chi_N^R: H_1(\Sigma_2(R)) \to \mathbb{Z}_p$, the map $f_{(\chi_i^R)}: \pi_1(M_K) \to \mathbb{Z} \times \mathbb{Z}_p$ extends uniquely to a map $F: \pi_1(U) \to \mathbb{Z} \times \mathbb{Z}_p$ such that when we consider the composition

$$F \circ \iota \circ \iota_4: \pi_1(M_K) \to \mathbb{Z} \times \mathbb{Z}_p,$$

we have $F \circ \iota \circ \iota_4 = f_{\chi_i}^R$. We therefore focus on extension over $V$.

There is a canonical identification $H_1(M_K^2) \cong \mathbb{Z} \oplus H_1(\Sigma_2(K))$, and so given any $(\chi_i^R)_{i=1}^{N}$ we obtain not just a map $\chi: H_1(\Sigma_2(K)) \to \mathbb{Z}_p$ but also a map $\overline{\chi}: H_1(M_K^2) \to \mathbb{Z}_p$ by sending the $\mathbb{Z}$ coordinate to zero. Since the inclusion $H_1(M_K^2) \to H_1(V)$ is an isomorphism, it therefore suffices to show that there are $(\chi_i^R)_{i=1}^{N}: H_1(\Sigma_2(R)) \to \mathbb{Z}_p$, at least $n$ of which are nonzero, such that the map

$$\overline{\chi} := (\chi_i^R)_{i=1}^{N}: H_1(M_K^2) \to \mathbb{Z}_p$$

extends over $H_1(V^2)$, perhaps after expanding its codomain to $\mathbb{Z}_p^n$ for some $a > 0$. Note that $\overline{\chi}$ extends over $H_1(V^2)$ up to enlarging its codomain if and only if $\overline{\chi}$ vanishes on $H := \ker(H_1(M_K^2) \xrightarrow{i} H_1(V^2))$.

The group of characters $TH_1(M_K^2) \to \mathbb{Z}_p$ is isomorphic to $H_1(\Sigma_2(K), \mathbb{Z}_p)$, which is in turn congruent to $(\mathbb{Z}_p^n)^N$, where we recall that $m$ denotes the generating rank of the $p$-primary part of $H_1(\Sigma_2(R))$. The subgroup of characters vanishing on $H$ is in bijective correspondence with $(TH_1(M_K^2)/H) \otimes \mathbb{Z}_p$.

Recall from Step 1 that $\chi(V) = 2g$. Also $M_K$ is a homology $S^1 \times S^2$ with $g \cdot \text{rk}(TH_1(M^2) \otimes \mathbb{Z}_p) = mN \geq 8g + 2$.

Therefore, since

$$\frac{mN - 2\chi(V)}{2} = \frac{mN - 4g}{2} \geq \frac{8g + 2 - 4g}{2} = 2g + 1 = n$$

by Proposition 7.5, the $p$-primary part of $TH_1(M_K^2)/H$ has generating rank at least $n$. Therefore $TH_1(M_K^2)/H$ has a subgroup isomorphic to $\mathbb{Z}_p^n$. Our desired result now follows from a linear algebra argument (see the proof of [KL05, Theorem 6.1]): every subgroup of $\mathbb{Z}_p^n$ isomorphic to $\mathbb{Z}_p^n$ $(0 \leq n \leq M)$ has an element at least $n$ of whose coordinates are nonzero.

Now, let $\chi = (\chi_1^R, \ldots, \chi_N^R)$ be such a map with at least $n$ of the $\chi_i$ nonzero. By the above work, we obtain a map $F: \pi_1(Z) \to \mathbb{Z} \times \mathbb{Z}_p^n$ such that the composition

$$\pi_1(M_{K_i}) \xrightarrow{i_{\chi_i}} \pi_1(Z) \to \mathbb{Z} \times \mathbb{Z}_p^n$$

is given by $f_{\chi_{K_i}}$ (we will implicitly take the usual inclusion of $\mathbb{Z}_p$ in $\mathbb{Z}_p^n$ without further comment). As discussed in Section 5, we have a map $\theta: \mathbb{Z} \times \pi_1(\Sigma_2(K)) \to \text{GL}_2(\mathbb{Q}(\xi_p)[t^{\pm 1}])$. 
By post-composing $F$ and each $f_{\chi_i}$ with this map, we obtain
\[
\Phi = \theta \circ F: \pi_1(Z) \to \text{GL}_2(\mathbb{Q}(\zeta_{p^\ell}))[t^{\pm 1}],
\]
\[
\phi_i = \theta \circ f_{\chi_i}: \pi_1(M_{K_i}) \to \text{GL}_2(\mathbb{Q}(\zeta_{p^\ell}))[t^{\pm 1}].
\]
We let
\[
\phi = \prod_{i=1}^{N} \phi_i: \prod_{i=1}^{N} \pi_1(M_{K_i}) \to \text{GL}_2(\mathbb{Q}(\zeta_{p^\ell}))[t^{\pm 1}].
\]
This completes Step 2: we have constructed a metabelian map such that at least $n$ of the $\phi_i$ do not factor through $\mathbb{Z}$; i.e. such that the associated characters $\chi_i$ are nontrivial.

**Step 3:** Compute the twisted homology $H^\Phi_1(Y)$ and control $\ker \big( H^\Phi_1(Y) \to H^\Phi_0(Z) \big)$.

For convenience, we will often write $\mathbb{F} = \mathbb{Q}(\zeta_{p^\ell})$, $S = \mathbb{F}[t^{\pm 1}]$, and $Q = \mathbb{F}(t)$ and let $S/p$ be shorthand for $S/(p(t))$ for any polynomial $p(t) \in S$. Since our infection curves $\alpha_i$ live in the second derived subgroup of $M_{R_0}$, the degree one maps $f_i: E_{i,1} \to E_{\text{unknot}}$ give us an identification
\[
f_*: H^\Phi_1(Y) = \bigoplus_{i=1}^{N} H^\phi_i(K_i) \to \bigoplus_{i=1}^{N} H^\phi_{\chi_i}(M_R)
\]
where the maps $\chi_i^R: H_1(\Sigma_2(R)) \to \mathbb{Z}/p \to \mathbb{Z}/p$ are as above. Note that if $\chi_i^R = 0$ then
\[
H^\phi_{\chi_i}(M_R) \cong A_{Q(R)} \otimes_{Q[t^{\pm 1}]} \mathbb{F}[t^{\pm 1}].
\]
If $\chi_i^R \neq 0$ then for some $m_i \geq 1$ there exist nontrivial polynomials $s_1(t), \ldots, s_{m_i}(t) \in S$ such that
\[
H^\phi_{\chi_i}(M_R) \cong \bigoplus_{\ell=1}^{M} S/s_{\ell}.
\]
Moreover, by our assumption on the order of $H^\phi_{\chi_i}(M_R)$, we know that each of the polynomials $s_i$ is relatively prime to $\Delta_R$ over $\mathbb{F}$. We therefore have that
\[
H^\Phi_1(Y, S) = (S/\Delta_R)^{N-k} \bigoplus_{\ell=1}^{M} S/s_{\ell},
\]
where $k$ denotes the number of nonzero $\chi_i^R$, each $s_{\ell}$ is a nontrivial polynomial with $(\Delta_R, s_{\ell}) = 1$, and
\[
M := \sum_{\{i | \chi_i^R \neq 0\}} m_i \geq k \geq n.
\]
Observe that since the longitude $\lambda_{(\eta_i^j)}$ of $\eta_i^j$ is in the second derived subgroup of $\pi_1(M_R)$ it must lift to a curve $l_i^j$ in the cover $\widetilde{M}_R$ of $M_R$ determined by $\phi_i$. (In fact, it lifts to $\pi \times \mathbb{Z}_{p^\ell}$ copies – pick one.) We wish to show that for at least one $1 \leq i \leq N$ and $1 \leq j \leq r$, we have that $i_1(\{1,0\} \otimes [l_i^j]) \neq 0$ in $H^\Phi_1(Z)$. Since whenever $\chi_i^R \neq 0$ we have that the collection $\{[1,0] \otimes [l_i^j]\}_{j=1}^{r}$ generates $H^\phi_{\chi_i}(M_R)$, it is enough to show that $\ker(i_1)$ cannot contain $\bigoplus_{\ell=1}^{M} S/s_{\ell}$ when $M \geq n$ and each $s_{\ell}$ is relatively prime to $\Delta_R$.

First, we compute the rank of $H^\phi_2(Z;Q)$. We can immediately see that $H^\phi_0(Z;Q) \cong H^\phi_0(Y;Q) \cong 0$, since $H^\phi_0(Z)$ and $H^\phi_0(Y)$ are annihilated by $t-1$. Note that for each $i = 1, \ldots, N$ the inclusion map $Y_i \to Z$ induces an isomorphism on $H^0(-;\mathbb{Z})$ and $H_1(-;\mathbb{Z})$. By [FP12], this implies that the map $H^\phi_1(Y_i; Q) \to H^\phi_1(Z; Q)$ is onto. We have already observed that $H^\phi_1(Y_i)$ is a torsion $S$-module and so $H^\phi_1(Y_i; Q) = 0$; it follows that $H^\phi_1(Z; Q) = 0$ as
well. By Poincaré-Lefschetz duality, universal coefficients, and the long exact sequence of $(Z, Y)$ with $Q$-coefficients we have that
\[ H^\Phi_3(Z; Q) \cong H^\Phi_4(Z, Y; Q) \cong \text{Hom}(H^\Phi_1(Z, Y; Q), Q) \cong 0. \]
In particular we note for use below that $H^\Phi_4(Z, Y; Q) = 0$. Finally, since $Z$ is a topological 4-manifold and hence homotopy equivalent to a finite CW complex with cells of dimension at most 3 (see the proof of Theorem 4.6 for references for this fact), we have that $H^\Phi_j(Z, F(t)) = 0$ for all $j \geq 4$. Re-computing $\chi(Z)$ with $Q$-coefficients, we obtain
\[ 2g = \chi(Z) = 0 - 0 + \dim_Q H^\Phi_2(Z; Q) - 0 = \dim_Q H^\Phi_2(Z; Q). \]
We now return to working with $S = \mathbb{F}[t^{\pm 1}]$-coefficients and consider the long exact sequence of Proposition 3.3
\[ \cdots \to H^\Phi_2(Y) \to H^\Phi_2(Z) \to H^\Phi_2(Z, Y) \to H^\Phi_1(Y) \to H^\Phi_1(Z) \to H^\Phi_1(Z, Y) \to \cdots. \]
Suppose now for a contradiction that
\[ \bigoplus_{i : \chi_i \neq 0} H^\Phi_1(M_R) \cong \bigoplus_{\ell = 1}^M S/s_\ell \leq \ker(i_1). \]
It follows that
\[ \bigoplus_{\ell = 1}^M S/s_\ell \leq \ker(i_1) = \text{Im}(\hat{c}) \cong H^\Phi_2(Z, Y)/\ker(\hat{c}) = H^\Phi_2(Z, Y)/\text{Im}(j_2) \cong (\mathbb{S}^{2g} \oplus TH^\Phi_2(Z, Y))/\text{Im}(j_2). \]
Intersect this submodule $\bigoplus_{\ell = 1}^M S/s_\ell$ with $TH^\Phi_2(Z, Y)/\text{Im}(j_2)$ to see that there is a submodule
\[ \bigoplus_{k = 1}^{M-2g} S/s_{\ell_k} \leq TH^\Phi_2(Z, Y)/\text{Im}(j_2). \]
But this certainly implies that there exists a submodule
\[ \bigoplus_{k = 1}^{M-2g} S/s_{\ell_k} \leq TH^\Phi_2(Z, Y)/\text{Im}(j_2|_T) = \text{coker}(j_2|_T) \cong \ker(j_1|_T) = \ker(j_1) = \text{Im}(i_1). \]
Here we used Lemma 7.4 for the second time. We checked above that $H^\Phi_1(Z, Y; Q) = 0$ so that $H^\Phi_2(Z, Y)$ is torsion, meaning that Lemma 7.4 indeed applies.

Since $M \geq n = 2g + 1$, we have that $M - 2g > 0$ and so there is a submodule of $\text{Im}(i_1)$ isomorphic to $S/s$ for some nontrivial polynomial $s$ relatively prime to $\Delta_R$. This is our desired contradiction, since
\[ \bigoplus_{i : \chi_i \neq 0} H^\Phi_1(M_R) \not\cong \ker(i_1) \]
also implies that $\text{Im}(i_1)$ is a quotient of $(S/\Delta_R)_N^{N-k}$ and therefore $\text{Im}(i_1)$ cannot contain a submodule isomorphic to $S/s$. This completes Step 3: we have shown that for at least one $i$ and $j$ with $1 \leq i \leq N$ and $1 \leq j \leq r$, we have $i_1([1, 0] \otimes [l_i]) \neq 0$ in $H^\Phi_1(Z)$.

**Step 4:** Construct a new representation $\psi : \pi_1(Y) \to \Lambda$ extending over $\pi_1(Z)$, and bound $\rho^{(2)}(Y, \psi)$ in two different ways to derive a contradiction.

Let $\psi : \pi_1(Y) \to \pi_1(Z) \to \Lambda := \pi_1(Z)/\pi_1(Z)^{[3]}$ be the map induced by inclusion. Since $\Lambda$ is amenable and in $D(Z_p)$ by [Cha14a, Lemma 4.3] and $\psi$ evidently extends over $\pi_1(Z)$, Theorem 4.6 and the fact from Step 1 that $H_2(Z) \cong \mathbb{Z}^{2g + N - 1}$ tells us that
\[ |\rho^{(2)}(Y, \psi)| \leq 2 \dim_{Z_p} H_2(Z, Z_p) = 2(2g + N - 1). \]
However, we are also exactly in the setting of Proposition 4.5 and can therefore conclude that
\[
\rho^{(2)}(Y, \psi) = \sum_{i=1}^{N} \rho^{(2)}(M_{\alpha(J_i)}, \psi|_{M_{\alpha(J_i)}})
\]
\[
= \sum_{i=1}^{N} \left( \rho^{(2)}(M_R, \psi_i^0) + \sum_{j=1}^{r} \delta_i^j(\psi)\rho_0(J_i^j) \right), \quad \text{where } \delta_i^j(\psi) = \begin{cases} 1, & \psi(\lambda(\eta_i^j)) \neq 0 \\ 0, & \psi(\lambda(\eta_i^j)) = 0. \end{cases}
\]

Note that if at least one of the \(\delta_i^j(\psi)\) is nonzero, we can let \((i, j)\) be the maximal tuple (with respect to the lexicographic ordering) for which that is so, and recall that
\[
|\rho_0(J_i^j)| > 2(2g + N - 1) + NC_R + \sum_{k=1}^{i-1} \sum_{\ell=1}^{r} |\rho_0(J_k^{\ell})| + \sum_{\ell=1}^{j-1} |\rho_0(J_i^{\ell})|.
\]
This combines with Equations (4) and (6) to give our desired contradiction.

It therefore suffices to prove that for some \(i\) and \(j\), we have \(\delta_i^j(\psi) \neq 0\), i.e. \(\lambda(\eta_i^j) \notin \pi_1(Z)^{(3)}(\mathbb{Q}, \mathbb{Z}, \rho, \mathbb{Q})\). By Proposition 5.1, this is implied by the fact that for some \(1 \leq i \leq N\) and \(1 \leq j \leq r\), the element \([1, 0] \otimes [\eta_i^j] \in H_1^0(Y)\) does not map to 0 in \(H_1^0(Z)\), as established in Step 3. This completes Step 4 and therefore completes the proof of Theorem 6.1.

10. Height four gropes

We will show the following: the knot \(K\) in Section 6.2 bounds a framed grope of height 4 embedded in \(D^4\). For the reader’s convenience, we begin by recalling the definition of a (capped) grope, a certain type of 2-complex.

Definition 10.1 (Grope of height \(h\) [FQ90, COT03]). A capped surface, or a capped grope of height 1, is an oriented surface of genus \(g \geq 0\) with nonempty connected boundary, together with discs attached along the \(2g\) curves of a standard symplectic basis for the surface. The discs are called caps. If \(G\) is a capped grope of height \(h - 1\), then a 2-complex obtained by replacing each cap of \(G\) with a capped surface is called a capped grope of height \(h\). A grope of height \(h\) is obtained by removing caps from a capped grope of height \(h\). It is also called the body of the capped grope. The initial surface that the inductive construction starts with is called the base surface, and the boundary of a grope, \(\partial G\), is the boundary of its base surface.

A (capped) grope defined above is often called disc-like. An annulus-like (capped) grope is defined in the same way, starting from a base surface with two boundary components.

Remark 10.2. It is not a priori obvious that a 2-complex \(G\) known to be a grope has a well-defined height, but it is true. For the reader’s convenience, we give a quick argument. Let \(\tau \subset G\) be the singular set of the grope union its boundary, i.e. the 1-complex consisting of the points where \(G\) is not locally homeomorphic to an open disc. Then \(G \setminus \tau\) consists of a collection of open surfaces, many of which are planar. Removing the subset of \(G\) corresponding to the non-planar surfaces (the interior of the ‘top stage’ of \(G\)) gives a new grope with a strictly smaller singular set; we can then repeat the above procedure. In this perspective, the height of a grope is exactly the number of such steps needed to reduce the grope to a circle. We leave to the reader the analogous argument that the height of a capped grope is well-defined, as well as the intrinsic definition of the \(i\)th stage of a grope, \(1 \leq i \leq h\).

A (capped) grope admits a standard embedding in the upper half 3-space \(\mathbb{R}^3_+ = \{z \geq 0\}\) which takes the boundary to \(\mathbb{R}^2\). Compose it with \(\mathbb{R}^3_+ \to \mathbb{R}^4_+\), take a regular neighborhood in \(\mathbb{R}^4_+\), and possibly perform finitely many plumblings. An embedding of the result in a 4-manifold is called an immersed framed (capped) grope. If no plumbing is performed, then we
say that it is *embedded*. Often we will regard an immersed/embedded (capped) grope as a 2-complex, but it is always assumed to be framed in this sense. In addition, we assume that each intersection in an immersed capped grope is always between a cap and a surface in the body, following the convention of [CK16]. Note that in a simply connected 4-manifold, an embedded grope without caps can be promoted to an immersed capped grope.

Returning to our case, recall that the knot $K$ in Section 6.2 is the connected sum of satellite knots of the form $R_{\alpha^+,-\alpha^-}(J^+, J^-)$, where $(R, \alpha^+, \alpha^-)$ is shown in the right of Figure 3, and $J^+$ and $J^-$ are knots with vanishing Arf invariants that are 0-positive and 0-negative respectively. So, it suffices to show that $R_{\alpha^+,-\alpha^-}(J^+, J^-)$ bounds an embedded grope of height 4 in $D^4$.

We will use the following terminology and results from [Cha14b, CK16], which also consider link versions.

**Definition 10.3** (Satellite capped grope [Cha14b, Definition 4.2], [CK16, Definition 4.2]). Suppose that there is a satellite capped grope of height $h$ for $(K, \alpha)$ and a capped grope concordance of height $\ell$ between two knots $J$ and $J'$. Then there is a capped grope concordance of height $h + \ell$ between the satellite knots $K_{\alpha}(J)$ and $K_{\alpha}(J')$.

The height $h + \ell$ capped grope concordance in Proposition 10.5 is obtained by a “product” construction described in [CK16, Definition 4.3]. The last ingredient we need is the following result from [COT03].

**Proposition 10.6** ([COT03, Remark 8.14]). A knot in $S^3$ with trivial Arf invariant bounds a capped grope of height two immersed in $D^4$.

Returning to our case, first observe that the curve $\alpha^-$ in Figure 3 bounds a disjoint capped grope of height two embedded in $S^3$, where the body surfaces are disjoint from the knot $R$ but the caps are allowed to intersect $R$. This is a geometric analogue of the commutator relation $\alpha^- = [\beta_1, \beta_2]$ where the curves $\beta_1$ and $\beta_2$ shown in the left of Figure 3 are again commutators in the fundamental group.

Indeed, in the planar diagram in the right of Figure 3, the bounded region enclosed by $\alpha^-$ is the projection of an obviously seen embedded disc which intersects $R$ in four points, and by tubing on this disc, one obtains a genus one surface, shown in red in Figure 7, which is disjoint from $R$. This surface is the base surface of the promised height two grope bounded by $\alpha^-$. The curves $\beta_1$ and $\beta_2$ are parallel to standard basis curves of the base surface, and they bound disjoint genus one surfaces obtained by tubing the obviously seen discs along the knot $R$, as illustrated in Figure 7. Attach them to the base stage surface to obtain a height two grope.

Note that all the surfaces used above are disjoint from the other curve $\alpha^+$, so by performing the satellite construction, we obtain a height two grope in $S^3 \setminus R_{\alpha^+}(J^+)$ bounded by $\alpha^-$. Identify $S^3$ with $S^3 \times 0 \subset S^3 \times I$, push the interior of the grope into the interior of $S^3 \times I$, and add caps using the simple connectedness of $S^3 \times I$ as noted above. Apply general position to make the caps transverse to $R_{\alpha^+}(J^+) \times I$, to obtain a satellite capped grope for $(R_{\alpha^+}(J^+), \alpha^-)$.

Since the knot $J^-$ has trivial Arf invariant, $J^-$ bounds a capped grope of height two in $D^4$, by Proposition 10.6. Remove, from $D^4$, a small open 4-ball which intersects the capped grope
in an unknotted 2-disc lying in the interior of the base surface, to obtain a capped grope concordance of height two between $J^-$ and the trivial knot. By Proposition 10.5 and the above paragraph, the satellite knot $R_{\alpha^+,\alpha^-}(J^+,J^-) = (R_{\alpha^+}(J^+))_{\alpha^-}(J^-)$ is height 4 capped grope concordant to the knot $R_{\alpha^+}(J^+)$. Forget the caps of this capped grope concordance, and attach a slicing disc for the knot $R_{\alpha^+}(J^+)$, to obtain a grope of height 4 bounded by $R_{\alpha^+,\alpha^-}(J^+,J^-)$. This completes the proof of the promised existence of a height 4 grope.

**Remark 10.7.** A similar argument shows the existence of a bounding grope of height 4 for the simpler example in Section 6.1. In this case, the height two surfaces constructed by “tubing along the knot $R$” in the 3-space are not disjoint, but the intersection can be removed by pushing the surfaces into 4-space. We omit the details.

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