On modular double of semisimple quantum groups

Pavel Sultanich

Moscow Center for Continuous Mathematical Education, 119002, Bolshoy Vlasyevsky Pereulok 11, Moscow, Russia

Abstract

In this note we propose a construction of the Hopf algebra of a complex analog of divided powers of the Weyl generators of a semisimple simply-laced quantum group. Here we consider the generators as positive, self-adjoint operators. In particular, we generalize the Lusztig relations [18] on the usual divided powers of generators of a quantum group to the case of complex divided powers of generators. These relations, some of which were known [14], present the complete set of defining relations. As a by-product result, the pure algebraic definition of the Faddeev modular double in the case of semisimple simply-laced quantum groups is formulated. Finally, we introduce an infinite dimension version of the Gelfand-Zetlin finite-dimensional representation of the modular double $M_{q,\tilde{q}}(gl(N))$.

1 Introduction

The notion of modular double of a quantum group was originally introduced by Faddeev [6]. In case $U_q(sl(2,\mathbb{R}))$, $q = e^{\pi i b^2}$, $b^2 \in \mathbb{R} \setminus \mathbb{Q}$ he noticed that the class of representations of modular double, analogous to the principal series representations of $SL_2(R)$ has a remarkable duality, similar to the duality of non-commutative tori, discovered in [23]. More generally, modular double plays an important role in Liouville theory [21], [7], relativistic Toda model [17] and some other problems of mathematical physics. Recently a considerable progress has been made in generalization of modular double to the case of $U_q(g)$, where $g$ is a semisimple Lie algebra of rank $r$ [8], [14], [15]. In these papers certain powers of generators have been used in order to construct the dual quantum group. This idea was first introduced in the case of $U_q(sl(2,\mathbb{R}))$ in [1].

Let $U_q(g)$ be the quantum group corresponding to semisimple simply-laced Lie algebra $g$ of rank $r$ with Cartan matrix $a_{ij} = a_{ji} \in \{0, -1\}$, for $i \neq j$, $1 \leq i, j \leq r$. Let us assume that generators of $U_q(g)$ are positive self-adjoint operators, then their complex powers can

\[E-mail: sultanichp@gmail.com\]
be defined \([24]\). In Theorem 4.1 the Hopf algebra of arbitrary complex analogs of devided powers of generators has been constructed, which is a generalization of the algebra of divided powers \(X^N_{[N]}\) studied by Lusztig \([18]\).

The Hopf algebra of complex devided powers is generated by elements \(K_j^q, E_j^{(s)}, F_j^{(t)}\), \(1 \leq j \leq r\) subjected to certain integral relations. After specification \(s = N, t = M, up = L\), where \(N, M, L\) are positive integers, all the defining integral relations are reduced to the identities for the devided powers of generators of quantum group, proved in \([18]\). Some of these relations were known \([14]\), however the other part of the relations including the generalized Kac identities is novel (see Theorem 3.1).

The Hopf algebra of complex devided powers is constructed using the classical Drinfeld’s double technique \([3]\). This Hopf algebra contains an important subalgebra generated by \(K_j, E_j, F_j\) and \(\tilde{K}_j = K_j^{b^{-2}}, \tilde{E}_j = E_j^{b^{-2}}, \tilde{F}_j = F_j^{b^{-2}}, 1 \leq j \leq r\). The integral relations of the full Hopf algebra of complex devided powers are reduced to the standard relations of quantum groups \(U_q(\mathfrak{g})\), generated by \(K_j, E_j, F_j\), \(1 \leq j \leq r\) and \(U_{\tilde{q}}(\mathfrak{g})\) generated by \(\tilde{K}_j, \tilde{E}_j, \tilde{F}_j\), \(1 \leq j \leq r\) with \(\tilde{q} = e^{\pi lb^{-2}}\), together with non-trivial cross-relations. This subalgebra has been the subject of active research recently \([1, 8, 14, 15]\), and it is usually associated with the modular double construction. However, the phenomenon of non-triviality of cross-relations was studied only for special representations. In the present paper we propose a natural derivation of full set of relations of modular double from the integral relations of the algebra of complex powers (see Theorem 5.1).

Next we study an algebraic definition of modular double. The modular double is a Hopf algebra with the two sets of generators: generators \(K_j, E_j, F_j\), \(1 \leq j \leq r\) subjected to standard relations of \(U_q(\mathfrak{g})\), and additional generators \(\tilde{K}_j, \tilde{E}_j, \tilde{F}_j\), \(1 \leq j \leq r\) subjected to the standard relations of \(U_{\tilde{q}}(\mathfrak{g})\), together with the same cross relations but free of the constraints \(\tilde{K}_j = K_j^{b^{-2}}, \tilde{E}_j = E_j^{b^{-2}}, \tilde{F}_j = F_j^{b^{-2}}, 1 \leq j \leq r\). The algebraic definition is motivated by the fact that these Hopf algebras have two types of realizations of the principal series representations. In Theorem 5.2 a realization of representation of the modular double of \(U_q(\mathfrak{gl}(N))\) of the first type is given. This realization of the principal series representation \([2, 10, 11]\) of the modular double can be viewed as an infinite-dimensional analog of the Gelfand-Zetlin finite-dimensional representation for classical groups \([13]\).

The realizations of second type \([8, 15]\) are the principal series representations in Lusztig parametrization. These representations are \(q\)-deformed versions of representations of universal enveloping algebra \(U(\mathfrak{g})\) by differential operators in Lusztig parametrization introduced in \([12]\) (sections 2.4.1-2.4.4) for classical series Lie algebras. The non-trivial properies of the representations are transcendental relations \(\tilde{K}_j = K_j^{b^{-2}}, \tilde{E}_j = E_j^{b^{-2}}, \tilde{F}_j = F_j^{b^{-2}}, 1 \leq j \leq r\). Using the classical limit of this realization leads to construction of Whittaker function in terms of the stationary phase integral, generalizing Givental’s formula, see \([12]\).

Acknowledgements: The research was supported by RSF (project 16-11-10075). I am grateful to D.R. Lebedev for the statement of the problem and his interest in this work.
2 Preliminaries

We start with the definition of quantum groups following [2], [19]. Let $(a_{ij})_{1 \leq i, j \leq r}$ be Cartan matrix of semisimple Lie algebra $g$ of rank $r$. Let $b_{±} \subset g$ be opposite Borel subalgebras. For simplicity let us restrict ourselves to the simply-laced case $a_{ii} = 2$, $a_{ij} = a_{ji} \{0, -1\}, i \neq j$. Let $U_q(g) (q = e^{πib^2}, b^2 \in \mathbb{R} \setminus \mathbb{Q})$ be the quantum group with generators $E_j, F_j, K_j = q^{H_j}, 1 \leq j \leq r$ and relations

$$K_i K_j = K_j K_i,$$  
(2.1)

$$K_i E_j = q^{a_{ij}} E_j K_i,$$  
(2.2)

$$K_i F_j = q^{-a_{ij}} F_j K_i,$$  
(2.3)

$$E_i F_j - F_j E_i = δ_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$  
(2.4)

For $a_{ij} = 0$ we have

$$E_i E_j = E_j E_i,$$  
(2.5)

$$F_i F_j = F_j F_i.$$  
(2.6)

For $a_{ij} = -1$ we have

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0,$$  
(2.7)

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0,$$  
(2.8)

Coproduct is given by

$$\Delta E_j = E_j \otimes 1 + K_j^{-1} \otimes E_j,$$  
(2.9)

$$\Delta F_j = 1 \otimes F_j + F_j \otimes K_j,$$  
(2.10)

$$\Delta K_j = K_j \otimes K_j.$$  
(2.11)

Non-compact quantum dilogarithm $G_b(z)$ is a special function introduced in [5] (see also [4], [7], [25], [16], [17], [1]). It is defined as follows

$$\log G_b(z) = \log \overline{ζ_b} - \int_{\mathbb{R} + i0} \frac{dt}{t} \frac{e^{zt}}{(1 - e^{bt})(1 - e^{b^{-1}t})},$$  
(2.12)

where $Q = b + b^{-1}$ and $ζ_b = e^{\frac{πi}{4} + \frac{πi(b^2 + b^{-2})}{12}}$. Note, that $G_b(z)$ is closely related to the double sine function $S_2(z|ω_1, ω_2)$, see eq.(A.22) in [17].

Below we outline some properties of $G_b(z)$, for details see appendix.

1. The function $G_b(z)$ has simple poles and zeros at the points

$$z = -n_1 b - n_2 b^{-1},$$  
(2.13)

$$z = Q + n_1 b + n_2 b^{-1},$$  
(2.14)

respectively, where $n_1, n_2$ are nonnegative integer numbers.

2. $G_b(z)$ has the following asymptotic behavior:

$$G_b(z) \sim \begin{cases} \overline{ζ_b}, & Imz \rightarrow +\infty, \\
ζ_b e^{πiz(z-Q)}, & Imz \rightarrow -\infty. \end{cases}$$  
(2.15)
3. Functional equation:
\[ G_b(z + b^{\pm 1}) = (1 - e^{2\pi ib^{\pm 1}z})G_b(z). \] (2.16)

4. Reflection formula:
\[ G_b(z)G_b(Q - z) = e^{\pi iz(z - Q)}. \] (2.17)

For \( 1 \leq j \leq r \) let us introduce the following rescaled generators
\[ E_j = -ı(q - q^{-1})E_j, \] (2.18)
\[ F_j = -ı(q - q^{-1})F_j. \] (2.19)

We will assume the elements \( E_j, F_j, K_j, 1 \leq i \leq r \) to be represented by positive self-adjoint operators. For such operator \( A \) one can define its arbitrary complex powers \([24]\) so that the following properties are satisfied
1. \( A^{is} \) is holomorphic in the half-plane \( Im(s) > -k \) for any \( k \in \mathbb{Z} \).
2. Group property
\[ A^{is_1}A^{is_2} = A^{is_1 + is_2}. \] (2.20)
3. For \( is = k, k \in \mathbb{Z}, A^k \) is an ordinary power of \( A \).

Next, we define the arbitrary divided powers of \( A \) by
\[ A^{(is)} = G_b(-ib)sA^{is}. \] (2.21)

We are going to consider the algebra spanned by the elements \( E_j^{(is)}, F_j^{(it)}, K_j^{up}, 1 \leq j \leq r \).

3 Generalized Kac’s identity

Below we give a generalization of the Kac’s identity \([18]\), eq.(4.1a) to the case of complex powers of generators.

**Theorem 3.1** Let \( q = e^{\pi b^2}, (b^2 \in \mathbb{R} \setminus \mathbb{Q}) \) and let \( K_j = q^{H_j}, E_j = -ı(q - q^{-1})E_j, F_j = -ı(q - q^{-1})F_j, 1 \leq j \leq r \) be positive self-adjoint operators and let \( A^{(is)} \) be defined by (2.21). Then the following generalized Kac’s identity holds:
\[ E_j^{(is)}F_j^{(it)} = \int_C d\tau e^{\pi bQ\tau}F_j^{(it + \tau)}K_j^{-\tau}G_b(-bH_j + ib(s + t + \tau)) \frac{G_b(-ib\tau)G_b(-bH_j + ib(s + t + 2\tau))}{G_b(-bH_j + ib(s + t + 2\tau))}E_j^{(is + \tau)}, \] (3.1)

where the contour \( C \) goes along the real axis above the sequences of poles going down:
\[ \tau = -s - m_1 - m_2b^{-2}, \tau = -t - m_1 - m_2b^{-2}, \tau = -\frac{b^{-1}Q}{2} - \frac{H_j}{2} - \frac{s}{2} - \frac{t}{2} - \frac{1}{2} - \frac{m_1}{2} - \frac{b^{-2}m_2}{2}, \]
and below the sequences of poles going up:
\[ \tau = m_1 + m_2b^{-2}, \tau = -H_j - s - t + m_1 + m_2b^{-2}, \]
where \( n_1, n_2 \) are non-negative integers.
Proof. The proof is based on the Drinfeld’s double construction [3] and will be published elsewhere. □

Let \( N, M \) be positive integers. Define the integer divided powers of generators \( E_j, F_j, 1 \leq j \leq r \) of quantum group \( U_q(\mathfrak{g}) \):

\[
E_j^{(N)} = \frac{\prod_{k=1}^{N} (q - q^{-1})}{\prod_{k=1}^{N} (q^k - q^{-k})} E_j^N, \quad (3.2)
\]

\[
F_j^{(M)} = \frac{\prod_{k=1}^{M} (q - q^{-1})}{\prod_{k=1}^{M} (q^k - q^{-k})} F_j^M. \quad (3.3)
\]

**Corollary 3.1** Let \( is = N, it = M, \) where \( N, M \) are non-negative integers. Then the generalized Kac’s identity reduces to the standard Kac’s identity [13], eq.(4.1a):

\[
E_j^{(N)} F_j^{(M)} = \sum_{n=0}^{\min(N,M)} F_j^{(M-n)} \prod_{k=1}^{n} (q^{-N-M+n+k} K_j - q^{N-M-n-k} K_j^{-1}) \prod_{k=1}^{n} (q^k - q^{-k}) E_j^{(N-n)}, \quad (3.4)
\]

where \( 1 \leq j \leq r \). Substituting into this formula \( N = M = 1 \) one recovers the relation

\[
[E_j, F_j] = \frac{K_j - K_j^{-1}}{q - q^{-1}}, \quad (3.5)
\]

for \( 1 \leq j \leq r \).

Proof. Rewrite the generalized Kac’s identity in the following form:

\[
\mathcal{E}_j^{st} \mathcal{F}_j^{st} = \int d\tau e^{z b Q \tau} \frac{G_b(i b \tau) G_b(-i b \tau) G_b(-i b s - i b \tau) G_b(-i b s)}{G_b(-i b t) G_b(-i b s)} \mathcal{F}_j^{st+\tau} \mathcal{E}_j^{st+\tau} \frac{\prod_{j=1}^{K_j} K_j}{\prod_{j=1}^{K_j} K_j} \frac{G_b(-b H_j + i b (s + t + \tau))}{G_b(-b H_j + i b (s + t + 2 \tau))} E_j^{st+\tau}.
\]

Set \( it = M \) to be some nonnegative integer. Then using delta-distribution formula [14]

\[
\frac{G_b(x) G_b(-N_1 b - N_2 b^{-1} - x)}{G_b(-N_1 b - N_2 b^{-1})} = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \prod_{k_1=1}^{n_1} (1 - q^{-2k_1}) \prod_{k_1=1}^{n_1} (1 - q^{-2k_1})\prod_{k_1=1}^{n_1} (1 - q^{-2k_1})
\]

\[
\times \prod_{k_2=1}^{n_2} (1 - \bar{q}^{-2k_2}) \prod_{k_2=1}^{n_2} (1 - \bar{q}^{-2k_2}) \delta(x + n_1 b + n_2 b^{-1}), \quad (3.6)
\]
we obtain
\[
\mathcal{E}_j^{is} \mathcal{F}_j^M = \sum_{n=0}^{M} \frac{\prod_{k=1}^{M} (1 - q^{-2k})}{\prod_{k=1}^{n} (1 - q^{-2k}) \prod_{k=1}^{M-n} (1 - q^{-2k})} \times 
\]
\[
\int d\tau e^{\pi bQ\tau} \delta(ib\tau + nb) \frac{G_b(-ibs - ib\tau)}{G_b(-ibs)} \mathcal{F}_j^{M+\tau} K_j^{-\tau} \frac{G_b(-bH_j + ibs + Mb + ib\tau)}{G_b(-bH_j + ibs + Mb + 2ib\tau)} \mathcal{E}_j^{is+\tau} = 
\]
\[
\sum_{n=0}^{M} \frac{\prod_{k=1}^{M} (1 - q^{-2k})}{\prod_{k=1}^{n} (1 - q^{-2k}) \prod_{k=1}^{M-n} (1 - q^{-2k})} (-1)^n q^n \times 
\]
\[
\frac{G_b(-ibs + nb)}{G_b(-ibs)} \mathcal{F}_j^{M-n} K_n \frac{G_b(-bH_j + ibs + Mb - 2nb + nb)}{G_b(-bH_j + ibs + Mb - 2nb)} \mathcal{E}_j^{is-n} = 
\]
\[
\sum_{n=0}^{M} \frac{\prod_{k=1}^{M} (1 - q^{-2k})}{\prod_{k=1}^{n} (1 - q^{-2k}) \prod_{k=1}^{M-n} (1 - q^{-2k})} (-1)^n q^n \prod_{k=0}^{n-1} (1 - q^{2k} e^{2\pi b^2 s}) \mathcal{F}_j^{M-n} K_n \times 
\]
\[
\times \prod_{k=0}^{n-1} (1 - q^{2k} e^{2\pi b(-bH_j + ibs + Mb - 2nb)}) \mathcal{E}_j^{is-n}. 
\]
To write the last line we have used the functional equation for quantum dilogarithm:
\[
\frac{G_b(x + n_1 b + n_2 b^{-1})}{G_b(x)} = \prod_{k_1=0}^{n_1-1} (1 - q^{2k_1} e^{2\pi b x}) \prod_{k_2=0}^{n_2-1} (1 - q^{-2k_2} e^{2\pi b^{-1} x}). \quad (3.7) 
\]
We have
\[
\mathcal{E}_j^{is} \mathcal{F}_j^M = \sum_{n=0}^{M} \frac{\prod_{k=1}^{M} (q^k - q^{-k})}{\prod_{k=1}^{n} (q^k - q^{-k}) \prod_{k=1}^{M-n} (q^k - q^{-k})} (-1)^n \prod_{k=0}^{n-1} (q^{-k} e^{-\pi b^2 s} - q^{k} e^{\pi b^2 s}) \times 
\]
\[
\mathcal{F}_j^{M-n} \prod_{k=0}^{n-1} (e^{\pi b^2 s - \pi b^2 M + 2\pi b^2 n - \pi b^2 k} K_j - e^{-\pi b^2 s + \pi b^2 M - 2\pi b^2 n + \pi b^2 k} K_j^{-1}) \mathcal{E}_j^{is-n}. 
\]
If \(M = 1\) then we get the formula from \([\mathbb{I}]\), eq.(3.25)
\[
[\mathcal{E}_j^{is}, \mathcal{F}_j] = -(q^{is} - q^{-is})(q^{H_j-is+1} - q^{-H_j+is-1}) \mathcal{E}_j^{is-1}, \quad (3.8) 
\]
which was derived in representation in \([\mathbb{I}]\).
Now let $M$ again be arbitrary positive integer, set $i\pi = N$ and substitute

$$E_j^N = (-i)^N \prod_{k=1}^{N}(q^k - q^{-k})E_j^{(N)},$$

$$F_j^M = (-i)^M \prod_{k=1}^{M}(q^k - q^{-k})F_j^{(M)},$$

to obtain the Kac formula:

$$E_j^{(N)} F_j^{(M)} = \sum_{n=0}^{\min(N, M)} \prod_{k=1}^{n}(q^{-N-M+n+k}K_j - q^{N+M-n-k}K_j^{-1}) \prod_{k=1}^{n}(q^k - q^{-k})E_j^{(N-n)}.$$

Let in the generalized Kac’s identity $i\pi = 1$ and let $i\tau$ be arbitrary. Then by evaluating the integral as above one can prove the formula [1], eq.(3.25)

$$[E_j, F_j^{i\tau}] = -(q^{i\tau} - q^{-i\tau})F_j^{i\tau - 1}(q^{H_j - i\tau + 1} - q^{-H_j + i\tau - 1}).$$

(3.9)

\[\square\]

Let $\tilde{q} = e^{\pi ib^{-2}}$. Define the dual generators $\tilde{K}_j, \tilde{E}_j, \tilde{F}_j, 1 \leq j \leq r$ by

$$\tilde{K}_j = K_j^{b^{-2}},$$

$$\tilde{E}_j = \frac{t}{\tilde{q} - \tilde{q}^{-1}}E_j^{b^{-2}},$$

$$\tilde{F}_j = \frac{t}{\tilde{q} - \tilde{q}^{-1}}F_j^{b^{-2}},$$

and divided powers of $\tilde{E}_j$ and $\tilde{F}_j, 1 \leq j \leq r$:

$$\tilde{E}_j^{(N)} = \prod_{k=1}^{N}(\tilde{q} - \tilde{q}^{-1}) \tilde{E}_j^N \prod_{k=1}^{N}(\tilde{q}^k - \tilde{q}^{-k}),$$

$$\tilde{F}_j^{(M)} = \prod_{k=1}^{M}(\tilde{q} - \tilde{q}^{-1}) \tilde{F}_j^M \prod_{k=1}^{M}(\tilde{q}^k - \tilde{q}^{-k}).$$

(3.13)

(3.14)

**Corollary 3.2** By setting $i\pi = Nb^{-2}, i\tau = Mb^{-2}$ in the generalized Kac’s identity, where $N, M$ are non-negative integers, one obtains the standard Kac’s identity for the dual quantum group:

$$\tilde{E}_j^{(N)} \tilde{F}_j^{(M)} = \sum_{n=0}^{\min(N, M)} \tilde{E}_j^{(N-n)} \prod_{k=1}^{n}(\tilde{q}^{-N-M+n+k}\tilde{K}_j - \tilde{q}^{N+M-n-k}\tilde{K}_j^{-1}) \prod_{k=1}^{n}(\tilde{q}^k - \tilde{q}^{-k})\tilde{E}_j^{(N-n)},$$

(3.15)
where $1 \leq j \leq r$. Taking $N = M = 1$ we obtain one of the defining relations of the dual quantum group

\[
[E_j, F_j] = \frac{K_j - K_j^{-1}}{\tilde{q} - \tilde{q}^{-1}},
\]

(3.16)

$1 \leq j \leq r$.

**Proof.** Following the same steps as in the proof of previous corollary with the only difference that now $it = Mb^{-2}$ and $is = Nb^{-2}$ and substituting for $1 \leq j \leq r$

\[
E_j^{Nb^{-2}} = (-i)^N \prod_{k=1}^{N} (\tilde{q}^k - \tilde{q}^{-k}) \tilde{E}_j^{(N)},
\]

\[
F_j^{Mb^{-2}} = (-i)^M \prod_{k=1}^{M} (\tilde{q}^k - \tilde{q}^{-k}) \tilde{F}_j^{(M)},
\]

we obtain the Kac’s identity for the modular dual group with the parameter $\tilde{q} = e^{\pi ib^{-2}}$

\[
\tilde{E}_j^{(N)} \tilde{F}_j^{(M)} = \sum_{n=0}^{\min(N,M)} \tilde{F}_j^{(M-n)} \frac{\prod_{k=1}^{n} (\tilde{q}^{-N-M+n+k} \tilde{K}_j - \tilde{q}^{N+M-n-k} \tilde{K}_j^{-1})}{\prod_{k=1}^{n} (\tilde{q}^k - \tilde{q}^{-k})} \tilde{E}_j^{(N-n)}.
\]

□

**Corollary 3.3**

\[
[E_j^{is}, F_j] = -(q^{is} - q^{-is})(q^{H_j-is+1} - q^{-H_j+is-1})E_j^{is-1},
\]

(3.17)

\[
[E_j, F_j^{it}] = -(q^{it} - q^{-it})F_j^{it-1}(q^{H_j-it+1} - q^{-H_j+it-1}),
\]

(3.18)

where $1 \leq j \leq r$. Let $\tilde{E}_j = E_j^{b^{-2}}, \tilde{F}_j = F_j^{b^{-2}}, 1 \leq j \leq r$. Then setting $is = b^{-2}$ and $it = b^{-2}$ one obtains

\[
[\tilde{E}_j, \tilde{F}_j] = [E_j, F_j] = 0.
\]

(3.19)

**Proof.** See the proof of the Corollary 3.1. □

### 4 Hopf algebra of arbitrary complex divided powers

Let $U_q(\mathfrak{g})$ be quantum group corresponding to semisimple simply-laced Lie algebra $\mathfrak{g}$ with Cartan matrix $(a_{ij}), 1 \leq i, j \leq r$. Recall, that we have introduced the rescaled generators $E_j, F_j$ of $U_q(\mathfrak{g})$ by the formulas

\[
E_j = -i(q - q^{-1})E_j,
\]

(4.1)
\[ F_j = -i(q - q^{-1})F_j. \] (4.2)

We have also defined arbitrary divided powers
\[ A^{(is)} = G_b(-ib_2)A^{is}. \] (4.3)

For \( a_{ij} = -1 \) define non-simple root generators by
\[ \mathcal{E}_{ij} = \frac{q^\frac{i}{2} \mathcal{E}_i \mathcal{E}_j - q^{-\frac{i}{2}} \mathcal{E}_i \mathcal{E}_j}{q - q^{-1}}, \] (4.4)
\[ \mathcal{F}_{ij} = \frac{q^\frac{i}{2} \mathcal{F}_i \mathcal{F}_j - q^{-\frac{i}{2}} \mathcal{F}_i \mathcal{F}_j}{q - q^{-1}}. \] (4.5)

The next theorem is a generalization of subalgebra of divided powers \( \frac{X^n}{\langle q \rangle} \), \( X \in U_q(g) \) and relations [13], eq.(4.1 a) - eq.(4.1 j) to the entire Hopf algebra of complex divided powers of generators.

**Theorem 4.1** The elements \( K^{rp}_i, \mathcal{E}_j^{(is)}, \mathcal{F}_j^{(it)}, 1 \leq j \leq r \) generate associative and coassociative Hopf algebra \( A(g) \) with the following commutation relations and coproduct
\[ K^{rp}_i K^{rp}_j = K^{rp}_j K^{rp}_i, \] (4.6)
\[ K^{rp}_i K^{rp+1}_j = K^{rp+1}_j K^{rp}_i, \] (4.7)
\[ \mathcal{E}_j^{(is1)} \mathcal{E}_j^{(is2)} = \frac{G_b(-ib_1)G_b(-ib_2)}{G_b(-ib_1 - ib_2)} \mathcal{E}_j^{(is1+is2)} = \mathcal{E}_j^{(is2)} \mathcal{E}_j^{(is1)}, \] (4.8)
\[ \mathcal{F}_j^{(it1)} \mathcal{F}_j^{(it2)} = \frac{G_b(-ib_1)G_b(-ib_2)}{G_b(-ib_1 - ib_2)} \mathcal{F}_j^{(it1+it2)} = \mathcal{F}_j^{(it2)} \mathcal{F}_j^{(it1)}, \] (4.9)
\[ K^{rp}_i \mathcal{E}_j^{(is)} = e^{-\pi ib^2 a_{ij} p} \mathcal{E}_j^{(is)} K^{rp}_i, \] (4.10)
\[ K^{rp}_i \mathcal{F}_j^{(it)} = e^{\pi ib^2 a_{ij} pt} \mathcal{F}_j^{(it)} K^{rp}_i. \] (4.11)

If \( a_{ij} = 0 \), then
\[ \mathcal{E}_i^{(is)} \mathcal{E}_j^{(it)} = \mathcal{E}_j^{(it)} \mathcal{E}_i^{(is)}, \] (4.12)
\[ \mathcal{F}_i^{(is)} \mathcal{F}_j^{(it)} = \mathcal{F}_j^{(it)} \mathcal{F}_i^{(is)}. \] (4.13)

For \( a_{ij} = -1 \) we have
\[ \mathcal{E}_i^{(is)} \mathcal{E}_j^{(it)} = e^{-\pi ib^2 st} \int_{\Gamma_1} d\tau e^{\frac{\pi ib^2 \pi Q}{2} \tau} \mathcal{E}_j^{(it-\tau)} \mathcal{E}_i^{(is-\tau)}, \] (4.14)
\[ \mathcal{F}_i^{(is)} \mathcal{F}_j^{(it)} = e^{-\pi ib^2 st} \int_{\Gamma_1} d\tau e^{\frac{\pi ib^2 \pi Q}{2} \tau} \mathcal{F}_j^{(it-\tau)} \mathcal{F}_i^{(is-\tau)}, \] (4.15)
where the contour $\Gamma_1$ goes above the pole at $\tau = 0$ and below the poles at $\tau = s, \tau = t$.

For $i \neq j$ we have
\[ \mathcal{E}_i^{(is)} \mathcal{F}_j^{(st)} = \mathcal{F}_j^{(st)} \mathcal{E}_i^{(ss)}. \tag{4.16} \]

\[ \mathcal{E}_j^{(is)} \mathcal{F}_j^{(st)} = \int_C d\tau e^{\pi b \eta \tau} \mathcal{F}_j^{(s+it)} K_j^{(it-ir)} \frac{G_b(ib\tau)G_b(-bH_j + ib(s+t+\tau))}{G_b(-bH_j + ib(s+t+2\tau))} \mathcal{E}_j^{(ss+ir)}, \tag{4.17} \]

where the contour $C$ is defined in Theorem 3.1.

Coproduct consistent with commutation relations has the form
\[ \Delta K_j^{(ip)} = K_j^{(ip)} \otimes K_j^{(ip)}, \tag{4.18} \]
\[ \Delta \mathcal{E}_j^{(is)} = \int_{\Gamma_2} d\tau \mathcal{E}_j^{(is-ir)} K_j^{(ir)} \otimes \mathcal{E}_j^{(ir)}, \tag{4.19} \]

where $\Gamma_2$ goes above the pole at $\tau = 0$ and below the pole at $\tau = s$.

\[ \Delta \mathcal{F}_j^{(st)} = \int_{\Gamma_3} d\tau \mathcal{F}_j^{(ir)} \otimes \mathcal{F}_j^{(s-ir)} K_j^{(ir)}. \tag{4.20} \]

$\Gamma_3$ goes above the pole at $\tau = 0$ and below the pole at $\tau = t$.

**Remark 4.1** The relations (4.14), (4.15) appeared in [14], eq.(6.16). The formula (4.17) is a new one.

**Corollary 4.1** Let $\tilde{K}_i = K_i^{b^{-2}}, 1 \leq i \leq r$. Then any element from both sets $K_i, 1 \leq i \leq r$ and $\tilde{K}_j, 1 \leq j \leq r$ commute with any other element from these sets.

**Proof.** The statement of the corollary follows straightforwardly from formulas (4.6), (4.7) if we take $\eta p_1 = 1$, $\eta p_2 = b^{-2}$. \(\square\)

**Corollary 4.2** Let $\tilde{\mathcal{E}}_j = \mathcal{E}_j^{b^{-2}}$ and $\tilde{\mathcal{F}}_j = \mathcal{F}_j^{b^{-2}}, 1 \leq j \leq r$. Then
\[ \tilde{\mathcal{E}}_j \mathcal{E}_j = \mathcal{E}_j \tilde{\mathcal{E}}_j, \tag{4.21} \]
\[ \tilde{\mathcal{F}}_j \mathcal{F}_j = \mathcal{F}_j \tilde{\mathcal{F}}_j. \tag{4.22} \]

where $1 \leq j \leq r$.

**Proof.** The statement of the corollary follows immediately from formulas (4.8), (4.9) in the case of $\eta s_1 = 1, \eta s_2 = b^{-2}$ and $\eta t_1 = 1, \eta t_2 = b^{-2}$. \(\square\)
Corollary 4.3  The following relations hold

\[ K_i \mathcal{E}_j = q^{a_{ij}} \mathcal{E}_j K_i, \quad (4.23) \]
\[ K_i \mathcal{F}_j = q^{a_{ij}} \mathcal{F}_j K_i, \quad (4.24) \]
\[ \tilde{K}_i \mathcal{E}_j = \tilde{q}^{a_{ij}} \tilde{E}_j \tilde{K}_i, \quad (4.25) \]
\[ \tilde{K}_i \mathcal{F}_j = \tilde{q}^{a_{ij}} \tilde{F}_j \tilde{K}_i, \quad (4.26) \]
\[ K_i \tilde{\mathcal{E}}_j = (-1)^{a_{ij}} \tilde{\mathcal{E}}_j K_i, \quad (4.27) \]
\[ K_i \tilde{\mathcal{F}}_j = (-1)^{a_{ij}} \tilde{\mathcal{F}}_j K_i, \quad (4.28) \]
\[ \tilde{K}_i \mathcal{E}_j = (-1)^{a_{ij}} \mathcal{E}_j \tilde{K}_i, \quad (4.29) \]
\[ \tilde{K}_i \mathcal{F}_j = (-1)^{a_{ij}} \mathcal{F}_j \tilde{K}_i. \quad (4.30) \]

Proof. All these relations follow from (4.10), (4.11) if we take \( \ell p,\, \ell s,\, \ell t \) to be equal to 1 and \( b^{-2} \) in different combinations. \( \square \)

Corollary 4.4  Let \( a_{ij} = 0,\, 1 \leq i, j \leq r. \) Then

\[ [\mathcal{E}_i, \mathcal{E}_j] = [\tilde{\mathcal{E}}_i, \mathcal{E}_j] = [\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_j] = [\mathcal{E}_i, \tilde{\mathcal{E}}_j] = 0, \quad (4.31) \]
\[ [\mathcal{F}_i, \mathcal{F}_j] = [\tilde{\mathcal{F}}_i, \mathcal{F}_j] = [\tilde{\mathcal{F}}_i, \tilde{\mathcal{F}}_j] = [\mathcal{F}_i, \tilde{\mathcal{F}}_j] = 0. \quad (4.32) \]

Proof. These relations are consequences of (4.12), (4.13) if we take \( \ell s,\, \ell t \) to be equal to 1 and \( b^{-2} \) in different combinations. \( \square \)

Corollary 4.5  Let \( i \neq j,\, 1 \leq i, j \leq r. \) Then

\[ [\mathcal{E}_i, \mathcal{F}_j] = [\tilde{\mathcal{E}}_i, \mathcal{F}_j] = [\tilde{\mathcal{E}}_i, \mathcal{F}_j] = [\mathcal{E}_i, \tilde{\mathcal{F}}_j] = 0. \quad (4.33) \]

Proof. These relations follow from (4.16) if we take \( \ell s,\, \ell t \) to be equal to 1 and \( b^{-2} \) in different combinations. \( \square \)

Corollary 4.6  Let \( a_{ij} = -1 \) and let \( N,\, M \) be non-negative integers. Then the following identities hold

\[ \mathcal{E}_i^N \mathcal{E}_j^M = q^{NM} \sum_{n=0}^{\min(N,M)} (-1)^n q^{-\frac{n^2}{2}+n} \prod_{k=1}^{N-n} (1-q^{-2k}) \prod_{k=1}^{M-n} (1-q^{-2k}) \prod_{k=1}^{n} (1-q^{-2k}) \mathcal{E}_j^{-n} \mathcal{E}_i^{M-n} \mathcal{E}_j^n \mathcal{E}_i^{N-n}, \quad (4.34) \]
\[ \mathcal{F}_i^N \mathcal{F}_j^M = q^{NM} \sum_{n=0}^{\min(N,M)} (-1)^n q^{-\frac{n^2}{2}+n} \prod_{k=1}^{N-n} (1-q^{-2k}) \prod_{k=1}^{M-n} (1-q^{-2k}) \prod_{k=1}^{n} (1-q^{-2k}) \mathcal{F}_j^{-n} \mathcal{F}_i^{M-n} \mathcal{F}_j^n \mathcal{F}_i^{N-n}. \quad (4.35) \]
Substituting $N = M = 1$, one recovers the definition of non-simple roots [14], eq.(6.18)

$$\mathcal{E}_{ij} = \frac{q^{\frac{i}{2}}\mathcal{E}_i \mathcal{E}_j - q^{-\frac{i}{2}}\mathcal{E}_j \mathcal{E}_i}{q - q^{-1}},$$  \hspace{1cm} (4.36)

$$\mathcal{F}_{ij} = \frac{q^{\frac{i}{2}}\mathcal{F}_i \mathcal{F}_j - q^{-\frac{i}{2}}\mathcal{F}_j \mathcal{F}_i}{q - q^{-1}}.$$  \hspace{1cm} (4.37)

Substituting $N = 2$, $M = 1$ together with the definition of non-simple roots one recovers $q$-deformed Serre relations

$$\mathcal{E}_i^2 \mathcal{E}_j - (q + q^{-1})\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i + \mathcal{E}_j \mathcal{E}_i^2 = 0,$$  \hspace{1cm} (4.38)

for $a_{ij} = -1$, $1 \leq i, j \leq r$. Analogous relation is true for $\mathcal{F}_i$, $\mathcal{F}_j$.

Proof.

$$\mathcal{E}^{st}_{ij} \mathcal{E}^{st}_{ji} = e^{-\pi b^2 st} \int dz e^{\frac{\pi b^2 z^2}{2} - \pi b Q z} \frac{G_b(-ibz + ibz)G_b(-ibz)}{G_b(-ibt)} \mathcal{E}^{st-iz}_{ij} \mathcal{E}^{st+iz}_{ji}.$$  

Let $zs = N$. Using delta distribution formula

$$\frac{G_b(x)G_b(-Nb - x)}{G_b(-Nb)} = \sum_{n=0}^{N} \prod_{k=1}^{N} (1 - q^{-2k}) \prod_{k=1}^{N-n} (1 - q^{-2k}) \delta(x + nb),$$  

we evaluate the integral

$$\mathcal{E}^{N}_{i} \mathcal{E}^{N}_{j} = e^{-\pi b^2 N t} \int dz e^{\frac{\pi b^2 z^2}{2} - \pi b Q z} \frac{G_b(-Nb + ibz)G_b(-ibz)}{G_b(-ibt)} \mathcal{E}^{st-iz}_{ij} \mathcal{E}^{st+iz}_{ji} \mathcal{E}^{N-iz} =$$

$$e^{-\pi b^2 N t} \sum_{n=0}^{N} \prod_{k=1}^{N} (1 - q^{-2k}) \prod_{k=1}^{N-n} (1 - q^{-2k}) \int dz e^{\frac{\pi b^2 z^2}{2} - \pi b Q z} \delta(-ibz + nb) \mathcal{E}^{st-iz}_{ij} \mathcal{E}^{st+iz}_{ji} \mathcal{E}^{N-iz} =$$

$$e^{-\pi b^2 N t} \sum_{n=0}^{N} (-1)^n q^{\frac{n^2}{2} + n} \prod_{k=1}^{N} (1 - q^{-2k}) \prod_{k=1}^{N-n} (1 - q^{-2k}) \mathcal{E}^{st-n}_{ij} \mathcal{E}^{st+n}_{ji} \mathcal{E}^{N-n}.$$  

Using functional equation for $G_b(z)$

$$\frac{G_b(x + nb)}{G_b(x)} = \prod_{k=0}^{n-1} (1 - q^{2k}e^{2\pi ibx}),$$  \hspace{1cm} (4.39)
we obtain

\[
E_i^n E_j^t = e^{-\pi b^2 N t} \sum_{n=0}^{N} (-1)^n q^{-n^2/2 + n} \frac{\prod_{k=1}^{N} (1 - q^{-2k})^{n-1}}{\prod_{k=1}^{n} (1 - q^{-2k}) \prod_{k=1}^{N-n} (1 - q^{-2k})} E_i^{n-n} E_j^{n} E_i^{N-n}.
\]

By taking \( it = M \) we prove the corollary

\[
E_i^n E_j^M = q^{NM} \sum_{n=0}^{N} (-1)^n q^{-n^2/2 + n} \frac{\prod_{k=1}^{N} (1 - q^{-2k})^{M-n}}{\prod_{k=1}^{n} (1 - q^{-2k}) \prod_{k=1}^{N-n} (1 - q^{-2k})} E_j^{M-n} E_j^{n} E_i^{N-n} = q^{NM} \sum_{n=0}^{\min(N,M)} (-1)^n q^{-n^2/2 + n} \frac{\prod_{k=1}^{N-n} (1 - q^{-2k}) M-n \prod_{k=1}^{n} (1 - q^{-2k})}{\prod_{k=1}^{M-n} (1 - q^{-2k}) \prod_{k=1}^{n} (1 - q^{-2k})} E_j^{M-n} E_j^{n} E_i^{N-n}.
\]

Now, let \( it = b^{-2}, N = 1 \) to prove the second corollary

\[
E_i E_j^{b-2} = -(E_j^{b-2} E_i - q^{-1/2+1} (1 - e^{-2\pi i}) E_j^{b-2} E_i) = -E_j^{b-2} E_i.
\]

Analogously one can prove the same relations for \( F_j \), \( 1 \leq j \leq r \).

\[\square\]

In the proof of the previous corollary we have also obtained the following

**Corollary 4.7** Let \( a_{ij} = -1 \) and let \( \tilde{E}_j = E_j^{b-2}, \tilde{F}_j = F_j^{b-2}, 1 \leq j \leq r \). Then the following identities hold

\[
E_i \tilde{E}_j = -\tilde{E}_j E_i, \quad F_i \tilde{F}_j = -\tilde{F}_j F_i.
\]

**Corollary 4.8** Let \( a_{ij} = -1 \) and let \( \tilde{E}_j = E_j^{b-2}, \tilde{F}_j = F_j^{b-2}, \tilde{E}_{ij} = E_{ij}^{b-2}, \tilde{F}_{ij} = F_{ij}^{b-2}, 1 \leq i, j \leq r \). Then the following identities hold

\[
\tilde{E}_i^n \tilde{E}_j^M = q^{NM} \sum_{n=0}^{\min(N,M)} (-1)^n q^{-n^2/2 + n} \frac{\prod_{k=1}^{N-n} (1 - \tilde{q}^{-2k}) M-n \prod_{k=1}^{n} (1 - \tilde{q}^{-2k})}{\prod_{k=1}^{M-n} (1 - \tilde{q}^{-2k}) \prod_{k=1}^{n} (1 - \tilde{q}^{-2k})} \tilde{E}_j^{M-n} \tilde{E}_j^{n} \tilde{E}_i^{N-n}, \quad (4.42)
\]

\[
\tilde{F}_i^n \tilde{F}_j^M = q^{NM} \sum_{n=0}^{\min(N,M)} (-1)^n q^{-n^2/2 + n} \frac{\prod_{k=1}^{N-n} (1 - \tilde{q}^{-2k}) M-n \prod_{k=1}^{n} (1 - \tilde{q}^{-2k})}{\prod_{k=1}^{M-n} (1 - \tilde{q}^{-2k}) \prod_{k=1}^{n} (1 - \tilde{q}^{-2k})} \tilde{F}_j^{M-n} \tilde{F}_j^{n} \tilde{F}_i^{N-n}. \quad (4.43)
\]
Taking in these formulas $N = M = 1$ one obtains the following expressions for the dual non-simple roots
\[
\tilde{E}_{ij} = \frac{q^{1-\frac{1}{2}} \tilde{E}_i \tilde{E}_j - \tilde{q}^{-1-\frac{1}{2}} \tilde{E}_i \tilde{E}_j}{\tilde{q} - \tilde{q}^{-1}}, \tag{4.44}
\]
\[
\tilde{F}_{ij} = \frac{q^{1-\frac{1}{2}} \tilde{F}_j \tilde{F}_i - \tilde{q}^{-1-\frac{1}{2}} \tilde{F}_i \tilde{F}_j}{\tilde{q} - \tilde{q}^{-1}}. \tag{4.45}
\]
Taking $N = 2$, $M = 1$ and using these expressions one obtains the dual $q$-deformed Serre relations
\[
\tilde{E} \tilde{E} - (\tilde{q} + \tilde{q}^{-1}) \tilde{E} \tilde{E} + \tilde{E} \tilde{E} = 0, \tag{4.46}
\]
for $a_{ij} = -1, 1 \leq i, j \leq r$. Similar relation is valid for $\tilde{F}_i, \tilde{F}_j$.

**Proof.** The proof is similar to the proof of analogous relation for untilded generators. \(\square\)

**Corollary 4.9** The following formulas for coproduct hold
\[
\Delta K_j = K_j \otimes K_j, \tag{4.47}
\]
\[
\Delta E_j = E_j \otimes 1 + K_j^{-1} \otimes E_j, \tag{4.48}
\]
\[
\Delta F_j = 1 \otimes F_j + F_j \otimes K_j, \tag{4.49}
\]
\[
\Delta \tilde{K}_j = \tilde{K}_j \otimes \tilde{K}_j, \tag{4.50}
\]
\[
\Delta \tilde{E}_j = \tilde{E}_j \otimes 1 + \tilde{K}_j^{-1} \otimes \tilde{E}_j, \tag{4.51}
\]
\[
\Delta \tilde{F}_j = 1 \otimes \tilde{F}_j + \tilde{F}_j \otimes \tilde{K}_j. \tag{4.52}
\]

**Proof.** Coproduct for generators $K_j, 1 \leq j \leq r$ are trivial consequences of the formula (4.18) for $ip$ equal to 1 and $b^2$. The rest expressions follow from (4.19), (4.20) for $is, it$ equal to 1 and $b^2$ by evaluating the integral using the delta distribution formula \(6.11\). \(\square\)

## 5 Modular double $M_{q\tilde{q}}(g)$

Let $q = e^{ib^2}, \tilde{q} = e^{ib^{-2}}, b^2 \in \mathbb{R} \setminus \mathbb{Q}$ and let $K_j, E_j, F_j, 1 \leq j \leq r$ be generators of quantum group $U_q(g)$. The rescaled generators are defined by
\[
E_j = -i(q - q^{-1})E_j, \tag{5.1}
\]
\[
F_j = -i(q - q^{-1})F_j. \tag{5.2}
\]
The second set of generators $\tilde{K}_j, \tilde{E}_j, \tilde{F}_j, 1 \leq j \leq r$ is defined as follows
\[
\tilde{K}_j = K_j^{ib^{-2}}; \tag{5.3}
\]
\[
\tilde{E}_j = \frac{i}{\tilde{q} - \tilde{q}^{-1}}\tilde{E}_j^{ib^{-2}}, \tag{5.4}
\]
\[
\tilde{F}_j = \frac{i}{\tilde{q} - \tilde{q}^{-1}}\tilde{F}_j^{ib^{-2}}. \tag{5.5}
\]
**Theorem 5.1** The elements \( K_i, E_j, F_j \) and \( \tilde{K}_i, \tilde{E}_j, \tilde{F}_j \), \( 1 \leq j \leq r \) generate a Hopf subalgebra of \( \mathcal{A}(\mathfrak{g}) \) with the first set of generators satisfying the relations of \( U_q(\mathfrak{g}) \) and the second set of generators satisfying the relations of \( \tilde{U}_q(\mathfrak{g}) \) and both sets of generators satisfying the cross-relations:

\[
\begin{align*}
K_i \tilde{K}_j &= \tilde{K}_j K_i, \\
K_i \tilde{E}_j &= (-1)^{a_{ij}} \tilde{E}_j K_i, \\
\tilde{K}_i E_j &= (-1)^{a_{ij}} E_j \tilde{K}_i, \\
K_i \tilde{F}_j &= (-1)^{a_{ij}} \tilde{F}_j K_i, \\
\tilde{K}_i F_j &= (-1)^{a_{ij}} F_j \tilde{K}_i, \\
E_i \tilde{E}_j &= (-1)^{a_{ij}} \tilde{E}_j E_i, \\
F_i \tilde{F}_j &= (-1)^{a_{ij}} \tilde{F}_j F_i, \\
E_i \tilde{F}_j &= \tilde{F}_j E_i, \\
\tilde{E}_i F_j &= F_j \tilde{E}_i,
\end{align*}
\]

where \( 1 \leq i, j \leq r \).

**Proof.** The results of the Corollaries 3.1-3.3 and Corollaries 4.1-4.9 give the statement of the theorem. \( \square \)

The defining relations of modular double have been observed in representations in [6] for the case of \( U_q(\mathfrak{sl}(2)) \), in [10] for the case of \( U_q(\mathfrak{gl}(N)) \) and in [15] for the case of other Lie algebras \( \mathfrak{g} \).

**Definition 5.1** Modular double \( M_{q\bar{q}}(\mathfrak{g}) \) is a Hopf algebra generated by generators \( K_i, E_j, F_j \), \( 1 \leq j \leq r \) satisfying the relations of \( U_q(\mathfrak{g}) \) and generators \( \tilde{K}_i, \tilde{E}_j, \tilde{F}_j \), \( 1 \leq j \leq r \) satisfying the relations of \( \tilde{U}_q(\mathfrak{g}) \) with the cross-relations (5.6-5.14). The transcendental relations (5.3-5.5) are not imposed.

Note, that in this algebraic definition of modular double we do not require transcendental relations between two sets of generators. The modular double defined in such a way has two types of representations. Below we give an example of the representation of the modular double \( M_{q\bar{q}}(\mathfrak{gl}(N)) \) for which there are no transcendental relations.

Let us introduce more general parametrization \( q = e^{\frac{2\pi i}{\omega_2}} \), \( \bar{q} = e^{\frac{2\pi i}{\omega_1}} \). This notation is reduced to the one we used in this paper if we set \( \omega_1 = b, \omega_2 = b^{-1} \). \( U_q(\mathfrak{gl}(N)) \) is generated by the elements \( K_n, n = 1, ..., N \) and \( E_{n,n+1}, E_{n+1,n}, n = 1, ..., N - 1 \) subjected to the following set of relations

\[
\begin{align*}
E_{n,n+1}E_{m+1,m} - E_{m+1,m}E_{n,n+1} &= \delta_{nm} \frac{K_n K_{n+1}^{-1} - K_{n+1}^{-1} K_n}{q - q^{-1}}, \\
K_n E_{m,m+1} &= q^{\delta_{nm}-\delta_{n,m+1}} E_{m,m+1} K_n, \\
K_n E_{m+1,m} &= q^{\delta_{n,m+1}-\delta_{nm}} E_{m+1,m} K_n,
\end{align*}
\]

15
Analogously, the dual quantum group $U_q^{\#}(\mathfrak{gl}(N))$ is generated by the elements $\tilde{K}_n$, $n = 1, ..., N$ and $\tilde{E}_{n+1,n}$, $n = 1, ..., N - 1$ subject to the relations

\begin{align*}
\tilde{E}_{n,n+1}\tilde{E}_{m+1,n} - \tilde{E}_{m+1,n}\tilde{E}_{n,n+1} &= \delta_{nm} \frac{K_n\tilde{K}_{n+1} - \tilde{K}_n^{-1}K_{n+1}}{q - \tilde{q}^{-1}}, \\
\tilde{K}_n\tilde{E}_{m,n+1} &= q^{\delta_{nm}}\delta_{n,m+1}\tilde{E}_{m+1,n}\tilde{K}_n, \\
\tilde{K}_n\tilde{E}_{m+1,n} &= q^{\delta_{nm}}\delta_{n,m+1}\tilde{E}_{m+1,n}\tilde{K}_n.
\end{align*}

(5.24)

(5.25)

(5.26)

together with quantum analogues of Serre relations

\begin{align*}
E_{n,n+1}^2 - E_{m,m+1}^2 - 2qE_{n,n+1}E_{m,m+1} = qE_{n,n+1}E_{m,m+1} + E_{m,m+1}E_{n,n+1} = 0, \\
E_{n,n+1, n+2} - (q + q^{-1})E_{n,n+1, n+2} + E_{n,n+1, n+2}E_{n,n+1} + E_{n,n+1, n+2}E_{n,n+1}^2 = 0, \\
E_{n,n+1, n+2} - (q + q^{-1})E_{n,n+1, n+2} - E_{n,n+1, n+2}E_{n,n+1} - E_{n,n+1, n+2}E_{n,n+1}^2 = 0,
\end{align*}

(5.27)

(5.28)

(5.29)

together with quantum analogues of Serre relations

\begin{align*}
E_{n,n+1}E_{m,n+1} - E_{m,n+1}E_{n,n+1} &= 0, m \neq n \pm 1, \\
E_{n,n+1, n+2} - (q + q^{-1})E_{n,n+1, n+2} - E_{n,n+1, n+2}E_{n,n+1} - E_{n,n+1, n+2}E_{n,n+1}E_{n,n+1}^2 = 0, \\
E_{n,n+1, n+2} - (q + q^{-1})E_{n,n+1, n+2} - E_{n,n+1, n+2}E_{n,n+1} - E_{n,n+1, n+2}E_{n,n+1}E_{n,n+1}^2 = 0,
\end{align*}

(5.30)

(5.31)

(5.32)

The modular double $M_{q\tilde{q}}(\mathfrak{gl}(N))$ is generated by the generators of $U_q^{\#}(\mathfrak{gl}(N))$ and generators of $U_{\tilde{q}}(\mathfrak{gl}(N))$ with the following cross-relations:

\begin{align*}
E_{n,n+1}\tilde{K}_m &= -1^{\delta_{nm}}\delta_{n,m+1}^{-1}\tilde{K}_mE_{n,n+1}, \\
E_{n+1,n}\tilde{K}_m &= -1^{\delta_{nm}}\delta_{n,m+1}^{-1}\tilde{K}_mE_{n+1,n}, \\
E_{n,n+1}\tilde{E}_{m,m+1} &= -1^{\delta_{nm}}\delta_{n,m+1}^{-1}\tilde{E}_{m,m+1}E_{n,n+1}, \\
E_{n,n+1}\tilde{E}_{m+1,m} &= \tilde{E}_{m+1,m}E_{n,n+1}, \\
E_{n+1,n}\tilde{E}_{m+1,m} &= -1^{\delta_{nm}}\delta_{n,m+1}^{-1}\tilde{E}_{m+1,m}E_{n+1,n}, \\
\tilde{E}_{n,n+1}\tilde{K}_m &= -1^{\delta_{nm}}\delta_{n,m+1}^{-1}\tilde{K}_m\tilde{E}_{n,n+1}, \\
\tilde{E}_{n+1,n}\tilde{K}_m &= -1^{\delta_{nm}}\delta_{n,m+1}^{-1}\tilde{K}_m\tilde{E}_{n+1,n}, \\
\tilde{E}_{n,n+1}\tilde{E}_{m+1,m} &= \tilde{E}_{m+1,m}\tilde{E}_{n,n+1}.
\end{align*}

(5.33)

(5.34)

(5.35)

(5.36)

(5.37)

(5.38)

(5.39)

(5.40)
Theorem 5.2 \[10\] Let \( q = e^{\frac{\pi \omega_1}{2\omega_2}} \) and \( \tilde{q} = e^{\frac{\pi \omega_2}{2\omega_1}} \). The following operators define a representation of modular double \( M_{q\tilde{q}}(\mathfrak{gl}(N)) \)

\[
E_{n,n+1} = 2e^{\frac{\pi(\omega_1 + \omega_2)(n-1)}{2\omega_2}} \sum_{j=1}^{n+1} \frac{\sinh \frac{\pi}{\omega_2}(\gamma_{n_j} - \gamma_{n_{j+1}})}{\prod_{s \neq j} \sinh \frac{\pi}{\omega_2}(\gamma_{n_j} - \gamma_{n_s})} e^{-\omega_1 \partial_{\gamma_{n_j}}}, \tag{5.41}
\]

\[
E_{n+1,n} = 2e^{\frac{\pi(\omega_1 + \omega_2)(n-1)}{2\omega_1}} \sum_{j=1}^{n} \frac{\sinh \frac{\pi}{\omega_1}(\gamma_{n_j} - \gamma_{n_{j+1}})}{\prod_{s \neq j} \sinh \frac{\pi}{\omega_1}(\gamma_{n_j} - \gamma_{n_s})} e^{\omega_2 \partial_{\gamma_{n_j}}}, \tag{5.42}
\]

\[
K_n = e^{\frac{\pi}{\omega_2} \sum_{j=1}^{n} \gamma_{n_j} \sum_{j=1}^{n-1} \gamma_{n_{j+1}}} \tag{5.43}
\]

\[
\tilde{E}_{n,n+1} = 2e^{\frac{\pi(\omega_1 + \omega_2)(n-1)}{2\omega_1}} \sum_{j=1}^{n+1} \frac{\sinh \frac{\pi}{\omega_1}(\gamma_{n_j} - \gamma_{n_{j+1}})}{\prod_{s \neq j} \sinh \frac{\pi}{\omega_1}(\gamma_{n_j} - \gamma_{n_s})} e^{-\omega_2 \partial_{\gamma_{n_j}}}, \tag{5.44}
\]

\[
\tilde{E}_{n+1,n} = 2e^{\frac{\pi(\omega_1 + \omega_2)(n-1)}{2\omega_1}} \sum_{j=1}^{n} \frac{\sinh \frac{\pi}{\omega_1}(\gamma_{n_j} - \gamma_{n_{j+1}})}{\prod_{s \neq j} \sinh \frac{\pi}{\omega_1}(\gamma_{n_j} - \gamma_{n_s})} e^{\omega_2 \partial_{\gamma_{n_j}}}, \tag{5.45}
\]

\[
\tilde{K}_n = e^{\frac{\pi}{\omega_2} \sum_{j=1}^{n} \gamma_{n_j} \sum_{j=1}^{n-1} \gamma_{n_{j+1}}} \tag{5.46}
\]

Proof. This is the same representation as was introduced in [10], (sections 3.2-3.3) if one replaces \( 2\pi \) by \( \pi \). After this replacement it is simple to check that the cross-relations instead of commutativity relations appear. \( \square \)

There is another type of realization of the principal series representations \([8], [15]\) which are \( q \)-analogues of principal series representations of universal enveloping algebra of semisimple Lie algebra \( \mathfrak{g} \) in Lusztig’s parametrization. The classical limit of these representations was introduced earlier in [12] in sections 2.4.1-2.4.4 for classical series of Lie algebras.

6 Appendix

6.1 Quantum dilogarithm and its properties

The basic properties of non-compact quantum dilogarithm/double sine listed below are extracted mainly from [17], [1], [25]. Introduce the following notation \( q = e^{\pi b^2}, \tilde{q} = e^{\pi b^{-2}} \),
\( Q = b + b^{-1}, \ \zeta_b = e^{\frac{\pi i}{2^2b^2-1}}. \)

The integral representation of \( G_b(z) \):

\[
\log G_b(z) = \log \zeta_b - \int_{\mathbb{R}^+0} \frac{dt}{t} \left( 1 - e^{bt} \right) \left( 1 - e^{b^{-1}t} \right). \tag{6.1}
\]

Noncompact analog of \( q \)-exponential \( g_b(z) \):

\[
g_b(z) = \frac{\zeta_b^{\frac{Q}{2}} + \frac{1}{2\pi ib} \log z}{G_b(\frac{Q}{2}) + \frac{1}{2\pi ib} \log z}. \tag{6.2}
\]

Product representation:

\[
G_b(x) = \zeta_b \prod_{n=1}^{\infty} \left( 1 - e^{2\pi ib^{-1}(x-nb^{-1})} \right) \prod_{n=0}^{\infty} \left( 1 - e^{2\pi ib(x+nb)} \right), \tag{6.3}
\]

\[
g_b(x) = \prod_{n=0}^{\infty} \left( 1 + xq^{2n+1} \right) \prod_{n=0}^{\infty} \left( 1 + x^{b^2}q^{-2n-1} \right). \tag{6.4}
\]

Functional equations:

\[
G_b(x + b^{\pm1}) = (1 - e^{2\pi ib^{\pm1}x})G_b(x), \tag{6.5}
\]

or more generally

\[
\frac{G_b(x + n_1b + n_2b^{-1})}{G_b(x)} = \prod_{k_1=0}^{n_1-1} (1 - q^{2k_1}e^{2\pi ibx}) \prod_{k_2=0}^{n_2-1} (1 - \bar{q}^{2k_2}e^{2\pi ib^{-1}x}), \tag{6.6}
\]

\[
g_b(q^{-1}x) = (1 + x)g_b(qx). \tag{6.7}
\]

Reflection formula:

\[
G_b(x)G_b(Q - x) = e^{\pi ix(x-Q)}. \tag{6.8}
\]

Poles and zeros:

\[
\lim_{x \to 0} xG_b(x - n_1b - n_2b^{-1}) = \frac{1}{2\pi} \prod_{k_1=1}^{n_1} (1 - q^{-2k_1})^{-1} \prod_{k_2=1}^{n_2} (1 - \bar{q}^{-2k_2})^{-1}, \tag{6.9}
\]

\[
\lim_{x \to 0} xG_b^{-1}(x + Q + n_1b + n_2b^{-1}) = \frac{1}{2\pi}(-1)^{n_1+n_2+1}q^{-n_1(n_1+1)}\bar{q}^{-n_2(n_2+1)} \prod_{k_1=1}^{n_1} (1 - q^{-2k_1})^{-1} \prod_{k_2=1}^{n_2} (1 - \bar{q}^{-2k_2})^{-1}.
\]
Tau-binomial integral [7, 16, 21]:
\[
\int_{C} d\tau e^{-2\pi b\beta \tau} \frac{G_b(\alpha + ib\tau)}{G_b(Q + ib\tau)} = \frac{G_b(\alpha)G_b(\beta)}{G_b(\alpha + \beta)},
\]
where the contour $C$ goes along the real axis above the sequences of poles going down and below sequences of poles going up.

Delta distributions [14]:
\[
\frac{G_b(x)G_b(-N_1b - N_2b^{-1} - x)}{G_b(-N_1b - N_2b^{-1})} = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \prod_{k_1=1}^{N_1} \left(1 - q^{-2k_1}\right) \prod_{k_2=1}^{N_2} \left(1 - \bar{q}^{-2k_2}\right) \delta(x + n_1b + n_2b^{-1}).
\]

4-5 relation [25]:
\[
\int_{C} d\tau e^{2\pi i(\alpha + \tau)(\beta + \tau)} G_b(\alpha - i\tau)G_b(\beta - i\tau)G_b(\gamma + i\tau)G_b(i\tau) = \frac{G_b(\alpha)G_b(\beta)G_b(\alpha + \gamma)G_b(\beta + \gamma)}{G_b(\alpha + \beta + \gamma)},
\]
where the contour $C$ goes along the real axis above the sequences of poles going down and below sequences of poles going up.

6-9 identity [25]:
\[
\frac{G_b(A)G_b(B)G_b(C)G_b(A + D)G_b(B + D)G_b(C + D)}{G_b(A + B + D)G_b(A + C + D)G_b(B + C + D)} = \int_{C} d\tau e^{2\pi i\tau^2 - 2\pi D\tau} \frac{G_b(A + i\tau)G_b(B + i\tau)G_b(C + i\tau)G_b(D - i\tau)G_b(-i\tau)}{G_b(A + B + C + D + i\tau)},
\]
where the contour $C$ goes along the real axis above the sequences of poles going down and below sequences of poles going up.

$q$-binomial theorem [1]:
Let $u, v$ be positive self-adjoint operators subject to the relations $uv = q^2 vu$. Then:
\[
(u + v)^{\alpha s} = \int_{C} d\tau \frac{G_b(-ib\tau)G_b(-ibs + ib\tau)}{G_b(-ibs)} u^{\alpha - i\tau} v^{i\tau},
\]
where the contour $C$ goes along the real axis above the sequences of poles going down and below sequences of poles going up.
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