A short note on the concept of free choice

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We argue that the concepts of “freedom of choice” and of “causal order” are intrinsically linked: a choice is considered “free” if it is correlated only to variables in its causal future. We discuss the implications of this to Bell-type scenarios, where two separate measurements are carried out, neither of which lies in the causal future of the other, and where one typically assumes that the measurement settings are chosen freely. Furthermore, we refute a recent criticism made by Ghirardi and Romano in our previous works, [Nat. Commun. 2, 411 (2011)] and [Phys. Rev. Lett. 108, 150402 (2012)].

Consider a simple experiment in which a system is prepared in state $Z$, then a measurement $A$ is chosen and applied to the system, and finally the outcome $X$ is recorded. How should we express the requirement that $A$ is a free choice?

We may think of $Z$, $A$, and $X$ as random variables with joint probability distribution $P_{ZAX}$. For $A$ to be free it is natural to demand that it can be chosen independently of the state $Z$, i.e., $P_{A|Z} = P_A$. However, it would be too restrictive to also require independence from $X$, i.e., $P_{A|ZX} = P_A$, as we expect the outcome of an experiment to depend on how we measure. The notion of free choice is hence intrinsically connected to a causal order: we don’t require that the free choice $A$ is uncorrelated with $X$ since $X$ lies in the causal future of $A$.

This can be easily extended to scenarios involving more than one measurement. As above, a system’s state, measurement choices, and measurement outcomes may be modeled as random variables, the collection of which we denote by $\Gamma$. A causal order is then simply a preorder relation $\preceq$ on $\Gamma$ (see Fig. 1 for examples). $A \rightarrow X$ should be interpreted as “$X$ is in the causal future of $A$” [9]. While the causal order may, in principle, be defined arbitrarily, it is reasonable to demand that it be compatible with time-ordering, defined as follows. For two random variables, $A$ and $X$, the order $A \rightarrow X$ is taken to hold if and only if $A$ occurs at an earlier time than the generation of $X$ (with respect to all relativistic frames [10]) [11].

Given a set $\Gamma$ with an (arbitrary) causal order, we can define the concept of a free choice as follows [12]:

A choice $A \in \Gamma$ is free if $A$ is uncorrelated with the set of all $W \in \Gamma$ that satisfy $A \not\rightarrow W$.

Said another way, $A$ is free if the only variables it is correlated with are those it could have caused. Note that the condition $A \not\rightarrow W$ cannot be replaced by $W \rightarrow A$ [13].

To demonstrate the use of this definition, we consider a Bell-type setup, where two particles are generated in state $Z$ and subsequently measured at two distant locations. Let $A$ and $B$ be the choices of the measurement settings at the two locations, and let $X$ and $Y$ be the corresponding measurement outcomes. The two measurements should be arranged such that neither lies in the causal future of the other, as depicted in Fig. 1(b). We note that, physically, this causal order can be obtained by carrying out the two measurements in two spacelike separated regions, and demanding that the causal order be compatible with time-ordering. Now, assuming that $A$ is free means that $P_{A|BYZ} = P_A$, for example.

![FIG. 1: Two examples of a causal order. In (a), $F$ being free implies $P_{F|EG} = P_F$, while in (b), if $A$ is free then $P_{A|BYZ} = P_A$, for example.](image-url)

In recent work, we have used the assumption of free choice to show that there cannot exist any extension of quantum theory with improved predictive power [1], and that the quantum wave function is in one-to-one correspondence with its elements of reality [2] (see also [3]). The assumption is also used in experimental work that provides a fundamental bound on the maximum probability by which the outcomes of measurements in a Bell-type setup can be predicted correctly [4]. Unhappy with these consequences, Ghirardi and Romano have, in a sequence of two papers [5] and [6], criticized the use of the freedom of choice assumption in these works, calling it “unphysical” [14]. However, the concept of free choice used in [14] is precisely the one explained in this note for a causal order compatible with time-ordering (as defined above), and hence has a clear physical motivation.

We conclude by remarking that this notion of free choice matches what Bell said about “free variables” [7]:

For me this means that the values of such variables have implications only in their future light cones.

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[1] Colbeck, R. & Renner, R. No extension of quantum theory can have improved predictive power. *Nature Communications* **2**, 411 (2011).

[2] Colbeck, R. & Renner, R. Is a system’s wave function in one-to-one correspondence with its elements of reality? *Physical Review Letters* **108**, 150402 (2012).

[3] Colbeck, R. & Renner, R. The completeness of quantum theory for predicting measurement outcomes. e-print *arXiv:1208.4123* (2012).

[4] Stuart, T. E., Slater, J. A., Colbeck, R., Renner, R. & Tittel, W. Experimental bound on the maximum predictive power of physical theories. *Physical Review Letters* **109**, 020402 (2012).

[5] Ghirardi, G. & Romano, R. About possible extensions of quantum theory. e-print *arXiv:1301.5040v1* (2013).

[6] Ghirardi, G. & Romano, R. Comment on “Is a system’s wave function in one-to-one correspondence with its elements of reality?” e-print *arXiv:1302.1635v1* (2013).

[7] Bell, J. S. Free variables and local causality. In *Speakable and unspeakable in quantum mechanics*, chap. 12 (Cambridge University Press, 1987).

[8] A preorder relation is a binary relation that is reflexive (\(A \rightarrow A\)) and transitive (if \(A \rightarrow B\) and \(B \rightarrow C\) then \(A \rightarrow C\)).

[9] That \(X\) is in the causal future of \(A\) does not mean that \(A\) is the cause of \(X\), but only that the causal order does not preclude this.

[10] Note that relativity theory is not assumed here. Relativistic spacetime structure merely offers one possible way to physically motivate a causal order.

[11] In other words, \(X\) is *not* in the causal future of \(A\) if \(X\) occurs at a time earlier than \(A\) (in some reference frame).

[12] See Definition 4 of [3].

[13] If the condition \(A \not\rightarrow W\) was replaced by \(W \rightarrow A\), the requirement for a choice \(A\) to be free would be that \(A\) is uncorrelated only with the set of variables in its causal past. Such an alternative characterization would however contradict our intuitive understanding of “freedom of choice”, as can be seen by a simple example. Consider the causal order defined by Fig. 1(b), where \(Z\) is the state in which a system has been prepared and where \(A\) and \(B\) are choices made by two different experimenters. According to the alternative characterization, \(A\) and \(B\) would both be considered free provided \(A\) and \(B\) are independent of \(Z\) (i.e., \(P_A = P_A|Z\) and \(P_B = P_B|Z\)). However, this does not prevent \(A\) and \(B\) from being perfectly correlated, a situation in which the two experimenters would likely reject any claim that both choices were free.

[14] In addition, in [6], it is claimed that the result of [2] is based on an extra independence assumption of the form \(P_{CZ|ABXY} = P_{CZ}\), this relation being the authors’ interpretation of our remark in [1] that certain information, \((C, Z)\), is considered static. However, this independence is never assumed in our work. The implication of our remark is only that information that is used to predict the outcome of a measurement should not lie in the causal future of the measurement.