Vector bundles on flag varieties

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Abstract
We study vector bundles on flag varieties over an algebraically closed field $k$. In the first part, we suppose $G = G_k(d, n)$ $(2 \leq d \leq n - d)$ to be the Grassmannian parameterizing linear subspaces of dimension $d$ in $k^n$, where $k$ is an algebraically closed field of characteristic $p > 0$. Let $E$ be a uniform vector bundle over $G$ of rank $r \leq d$. We show that $E$ is either a direct sum of line bundles or a twist of the pullback of the universal subbundle $H_d$ or its dual $H_d^\vee$ by a series of absolute Frobenius maps. In the second part, splitting properties of vector bundles on general flag varieties $F(d_1, ..., d_s)$ in characteristic zero are considered. We prove a structure theorem for bundles over flag varieties which are uniform with respect to the $i$th component of the manifold of lines in $F(d_1, ..., d_s)$. Furthermore, we generalize the Grauert–Mülich–Barth theorem to flag varieties. As a corollary, we show that any strongly uniform $i$-semistable $(1 \leq i \leq n - 1)$ bundle over the complete flag variety splits as a direct sum of special line bundles.

KEYWORDS
flag variety, Frobenius map, Grassmannian, uniform vector bundle

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1 INTRODUCTION

It is classically known that any vector bundle on a projective line over arbitrary algebraically closed field $k$ splits as a direct sum of line bundles. However, if the dimension of a projective space is bigger than or equal to two, the situation is pretty involved. So the splitting of vector bundles on higher dimensional projective spaces has long been a major concern among the problems on vector bundles in algebraic geometry. For non-splitting vector bundles, there are many classification results on some special classes of vector bundles. One of the classes that has been studied more widely is uniform vector bundles; that is, bundles whose splitting type is independent of the chosen line. The notion of a uniform vector bundle appears first in a paper by Schwarzenberger [24]. In characteristic zero, much work has been done on the classification of uniform vector bundles over projective spaces. In 1972, Van de Ven [26] proved that for $n > 2$, uniform 2-bundles over $P^n_k$ split and uniform 2-bundles over $P^2_k$ are precisely the bundles $\mathcal{O}_{P^2_k}(a) \oplus \mathcal{O}_{P^2_k}(b)$ and $T_{P^2_k}(a)$, $a, b \in Z$. In 1976, Sato [23] proved that for $2 < r < n$, uniform r-bundles over $P^n_k$ split by using a theorem of Tango [25] about holomorphic mappings from projective spaces to Grassmannians. In 1978, Elencwajg [8] extended the investigations of Van de Ven to show that uniform vector bundles of rank 3 over $P^3_k$ are of the form

$$\mathcal{O}_{P^3_k}(a) \oplus \mathcal{O}_{P^3_k}(b) \oplus \mathcal{O}_{P^3_k}(c), \quad T_{P^3_k}(a) \oplus \mathcal{O}_{P^3_k}(b) \quad \text{or} \quad S^2T_{P^3_k}(a),$$

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where \(a, b, c \in \mathbb{Z}\). Sato [23] had previously shown that for \(n\) odd, uniform \(n\)-bundles over \(\mathbb{P}_k^n\) are of the forms

\[
\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}_k^n}(a_i), \ T_{\mathbb{P}_k^n}(a) \text{ or } \Omega^1_{\mathbb{P}_k^n}(b),
\]

where \(a, a_i, b \in \mathbb{Z}\). So, the results of Elencwajg and Sato yield a complete classification of uniform 3-bundles over \(\mathbb{P}_k^n\). In particular, all uniform 3-bundles over \(\mathbb{P}_k^n\) are homogeneous. Later, Elencwajg, Hirschowitz, and Schneider [9] showed that Sato’s result is also true for \(n\) even. Around 1982, Ellia [10] and Ballico [3] independently proved that for \(n \geq 3\), uniform \((n+1)\)-bundles over \(\mathbb{P}_k^n\) are of the form

\[
\bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}_k^n}(a_i), \ T_{\mathbb{P}_k^n}(a) \bigoplus \mathcal{O}_{\mathbb{P}_k^n}(b) \text{ or } \Omega^1_{\mathbb{P}_k^n}(c) \bigoplus \mathcal{O}_{\mathbb{P}_k^n}(d),
\]

where \(a_i, a, b, c, d \in \mathbb{Z}\). One can go over the good reference by Okonek, Schneider, and Spindler [22] for related topics.

Later, similar results have been extensively studied for uniform vector bundles on varieties swept out by lines, such as quadrics [2,15], Grassmannians, [12] and special Fanom manifolds [21].

In positive characteristic, for \(2 \leq r < n\), uniform \(r\)-bundles over \(\mathbb{P}_k^n\) split by Sato’s result [23]. The classification problem of uniform \(n\)-bundles on \(\mathbb{P}_k^n\) has been solved by Lange [18] for \(n = 2\) and Ein [7] for all \(n\). It seems that the classification of uniform vector bundles over other projective manifolds covered by lines in characteristic \(p\) is still open. In the first part of the paper, we consider uniform vector bundles on Grassmannians in positive characteristic and prove the following main theorem.

**Theorem 1.1** (Theorem 3.3 and Theorem 4.1). Let \(G = G_k(d, n) (d \leq n - d)\) be the Grassannian manifold parameterizing linear subspaces of dimension \(d\) in \(k^n\), where \(k\) is an algebraically closed field of characteristic \(p > 0\). Let \(E\) be a uniform vector bundle over \(G\) of rank \(r \leq d\).

- If \(r < d\), then \(E\) is a direct sum of line bundles.
- If \(r = d\), then \(E\) is either a direct sum of line bundles or a twist of the pullback of the universal subbundle \(H_d\) or its dual \(H_d^\vee\) by a series of absolute Frobenius maps.

**Remark 1.2.** The first part of the theorem holds for any algebraically closed field. The result in characteristic zero is due to Guyot [12]. We generalize the method of Elenwajg–Hirschowitz–Schneider [9], by which the authors considered the case over projective spaces, to deal with Grassmannians over any algebraically closed field. Our basic idea follows from the strategy in [9] but the complexity in the case of Grassmannians is far greater than in the case of projective spaces. For example, Proposition 3.1 cannot be generalized from the case for projective spaces directly. We need to strengthen the condition and use different methods. However, for \(r < d\), our method is independent of the characteristic of the field. For the case \(r = d\), we checked that the first part of Guyot’s paper [12], which is independent of the characteristic of the field \(k\), is for studying the Chow ring and axiomatically computing Chern class of vector bundles of flag varieties. However, Guyot’s method only covers the case \(b = -1\) in the notation of the proof of Theorem 4.1, because this is the only case which occurs in characteristic 0. In characteristic \(p\), we need to use Katz’s Lemma. So for the second part of the theorem, we mainly use Ein’s [7] ideas for projective spaces and Katz’s [16] key lemma to study vector bundles in positive characteristic.

The proof of the first part of the above theorem has a surprising consequence.

**Theorem 1.3** (Theorem 3.4). Let \(E\) be a vector bundle on \(G = G(d, n) (2 \leq d \leq n - d)\) over an algebraically closed field. If all the restrictions of \(E\) to arbitrary subspace \(\mathbb{P}_k^d \subseteq G\) split as direct sums of line bundles, then \(E\) splits also globally as a direct sum of line bundles.

In the second part of the paper, we consider vector bundles over flag varieties in characteristic zero. For fixed integer \(n\), let \(F : = F(d_1, \ldots, d_s)\) be the flag manifold parameterizing flags

\[
V_{d_1} \subseteq \cdots \subseteq V_{d_s} \subseteq k^n,
\]
where \( \text{dim}(V_{d_i}) = d_i, 1 \leq i \leq s \). Let \( F(i) := F(d_1, \ldots, d_{i-1}, d_i - 1, d_i + 1, d_{i+1}, \ldots, d_s) \) be the \( i \)th connected component of the manifold of lines in \( F \). (cf. [17] Theorem 4.3 for the general cases on rational homogeneous spaces. From now, we specify that if the two adjacent integers in the expression of flag varieties such as \( F(i) \) are equal, we keep only one of them. Please see Section 5 for the notations.)

We separate our discussion into two cases:

Case I: \( d_i - 1 = d_{i-1} \) and \( d_i + 1 = d_{i+1} \);  
Case II: \( d_i - 1 \neq d_{i-1} \) or \( d_i + 1 \neq d_{i+1} \).

Then we have the standard diagram

\[
F(d_1, \ldots, d_{i-1}, d_i - 1, d_i, d_i + 1, d_{i+1}, \ldots, d_s) \xrightarrow{q_i} F(i) \\
F = F(d_1, \ldots, d_s).
\]

(1.1)

Note that \( q_i \) is the identity in Case I.

**Definition 1.1.** If there exists some integer \( i \) such that the splitting type of a vector bundle \( E \) is uniform for any line \( L \subseteq F \) given by \( l \in F(i) \), then \( E \) is called uniform vector bundle with respect to \( F(i) \). \( E \) is called strongly uniform on \( F \) if the splitting type of \( E \) is uniform for any line \( L \subseteq F \).

Let \( E \) be an algebraic \( r \)-bundle over \( F \). According to a theorem of Grothendieck, for every \( l \in F(i) \), there is an \( r \)-tuple

\[
a^{(i)}_E(l) = (a^{(i)}_1(l), \ldots, a^{(i)}_r(l)) \in \mathbb{Z}^r \quad \text{with} \quad a^{(i)}_1(l) \geq \cdots \geq a^{(i)}_r(l)
\]

such that \( E|L \cong \bigoplus_{j=1}^r \mathcal{O}_L(a^{(i)}_j(l)) \). We give \( \mathbb{Z}^r \) the lexicographical ordering, i.e., \((a_1, \ldots, a_r) \leq (b_1, \ldots, b_r)\) if the first non-zero difference \( b_i - a_i \) is positive. Let

\[
a^{(i)}_E = \inf_{l \in F(i)} a^{(i)}_E(l).
\]

**Definition 1.2.** \( a^{(i)}_E \) is the generic splitting type of \( E \) with respect to \( F(i) \). \( S^{(i)}_E = \{ l \in F(i) | a^{(i)}_E(l) > a^{(i)}_E \} \) is the set of jump lines with respect to \( F(i) \). We define \( U^{(i)}_E := F(i) \setminus S^{(i)}_E \).

**Theorem 1.4** (Theorem 5.3). Fix an integer \( i, 1 \leq i \leq s \). Let \( E \) be an algebraic \( r \)-bundle over \( F \) of generic splitting type \( a^{(i)}_E = (a^{(i)}_1, \ldots, a^{(i)}_r) \), \( a^{(i)}_1 \geq \cdots \geq a^{(i)}_r \) with respect to \( F(i) \). If for some \( t < r \),

\[
a^{(i)}_t - a^{(i)}_{t+1} \geq \begin{cases} 1, & \text{and } F(i) \text{ is in Case I} \\ 2, & \text{and } F(i) \text{ is in Case II}, \end{cases}
\]

then there is a reflexive subsheaf \( K \subseteq E \) of rank \( t \) with the following properties: over the open set \( V^{(i)}_E = q_1(q_2^{-1}(U^{(i)}_E)) \subseteq F \), the sheaf \( K \) is a subbundle of \( E \), which on the line \( L \subseteq F \) given by \( l \in U^{(i)}_E \) has the form

\[
K|L \cong \bigoplus_{j=1}^t \mathcal{O}_L(a^{(i)}_j).
\]

Let \( F \) be a torsion free coherent sheaf of rank \( r \) over \( F \).

Since the singularity set \( S(F) \) of \( F \) has codimension at least 2, there is some integer \( i (1 \leq i \leq s) \) and lines \( L \subseteq F \) given by \( l \in F(i) \) which do not meet \( S(F) \). Let

\[
F|L \cong \mathcal{O}_L(a^{(i)}_1) \oplus \cdots \oplus \mathcal{O}_L(a^{(i)}_r)
\]
and
\[ c_1^{(i)}(F) = a_1^{(i)} + \cdots + a_r^{(i)}, \]
which is independent of the choice of \( L \). (Notice that \( c_1^{(i)}(F) = c_1(F) \cdot L \), where \( L \) is a line corresponding to a point in \( F^{(i)} \).)

We set
\[ \mu(i)(F) = \frac{c_1^{(i)}(F)}{\text{rk}(F)}. \]

**Definition 1.3.** A torsion free coherent sheaf \( \mathcal{E} \) over \( F \) is \( i \)-semistable if for every coherent subsheaf \( \mathcal{F} \subseteq \mathcal{E} \) with \( 0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E}) \), we have
\[ \mu(i)(\mathcal{F}) \leq \mu(i)(\mathcal{E}). \]

Using the above theorem, we generalize the Grauert–Mülich–Barth theorem (cf. [4], Theorem 1) to flag varieties as follows (see [6] for rational homogeneous spaces).

**Corollary 1.5** (Corollary 5.6). Fix an integer \( i \), \( 1 \leq i \leq s \), for an \( i \)-semistable \( r \)-bundle \( E \) over \( F \) of generic splitting type \( \mathbf{a}_E^{(i)} = (a_1^{(i)}, \ldots, a_r^{(i)}) \), \( a_1^{(i)} \geq \cdots \geq a_r^{(i)} \) with respect to \( F^{(i)} \), we have
\[ a_j^{(i)} - a_{j+1}^{(i)} \leq 1 \quad \text{for all } j = 1, \ldots, r - 1. \]

In Case I, we even have that all the \( a_j^{(i)} \)'s must be equal (\( 1 \leq j \leq r \)).

**Remark 1.6.** A generalization of the Grauert–Mülich–Barth theorem to normal projective varieties in characteristic zero was proved in Huybrechts and Lehn’s famous book (see [13] Theorem 3.1.2). The usual way for defining the semistability attributes to Mumford and Takemoto who embed the variety into a projective space and fix an ample divisor to define the degree and the slope of vector bundles. Another way so called Gieseker-semistability is by replacing the degree in the Mumford–Takemoto’s definition with Hilbert polynomial to define the slope (see, e.g., [13] Definition 1.2.3). However, these two ways both depend on the embeddings. We prefer our definition of semistability for the flag varieties (see Definition 1.3) because it is intrinsic. If one embeds the flag variety into a projective space and takes nef divisors whose intersection is a multiple line on the flag variety, then our definition is the same as Mumford–Takemoto’s definition (cf. [5] Proposition 1.4.1 and Proposition 1.4.3).

**Corollary 1.7** (Corollary 5.7). If \( E \) is a strongly uniform \( i \)-semistable \((1 \leq i \leq n - 1)\) \( r \)-bundle over the complete flag \( F \), then \( E \) splits as a direct sum of line bundles. In addition \( E|L \cong O_L(a)^{\oplus r} \) for every line \( L \subseteq F \), where \( a \in \mathbb{Z} \).

## 2 | PRELIMINARIES

Denote by \( G \) the Grassmannian \( G_k(d, n) \) of \( d \)-dimensional linear subspaces in \( V = k^n \), where \( k \) is an algebraically closed field. Of course, we may also consider \( G_k(d, n) \) in its projective guise as \( G_k(d - 1, n - 1) \), the Grassmannian of projective \((d - 1)\)-planes in \( P^{n-1}_k \).

Let \( \mathcal{V} := G \times V \) be the trivial vector bundle of rank \( n \) on \( G \) whose fiber at every point is the vector space \( V \). We write \( H_d \) for the \( d \)-subbundle of \( \mathcal{V} \) whose fiber at a point \([\Lambda] \in G \) is the subspace \( \Lambda \) itself; that is,
\[ (H_d)_{[\Lambda]} = \Lambda \subseteq V = \mathcal{V}_{[\Lambda]}. \]
$H_d$ is called the universal subbundle on $G$; the rank $n - d$ quotient bundle $Q_{n-d} = \mathcal{V}/H_d$ is called the universal quotient bundle, i.e.,

$$0 \to H_d \to \mathcal{V} \to Q_{n-d} \to 0. \tag{2.1}$$

We write line bundle $\mathcal{O}_G(1)$, which is called the Plücker bundle on $G$, to be the pullback of $\mathcal{O}_p(1)$ under the Plücker embedding

$$\iota : G_k(d, n) \to \mathcal{P} = \mathbb{P} \left( \bigwedge^d k^n \right).$$

**Definition 2.1.** Let $F := F(d_1, \ldots, d_s)$ be the flag manifold parameterizing flags

$$V_{d_1} \subseteq \cdots \subseteq V_{d_s} \subseteq k^n$$

where $\text{dim}(V_{d_i}) = d_i$, $1 \leq i \leq s$. In particular, the flag manifold $F(1, \ldots, n-1)$ is called the complete flag manifold.

Given, a flag $V_{d-1} \subseteq V_{d+1} \subseteq k^n$, the set of $d$-dimensional subspaces $W \subseteq k^n$ such that

$$V_{d-1} \subseteq W \subseteq V_{d+1}$$

is the projectivization of the quotient space $V_{d+1}/V_{d-1}$, so the set is isomorphic to $\mathbb{P}^1_{\mathbb{P}^1}$. It follows that the flag manifold $F(d-1, d+1)$ is the manifold of lines in $G$ and the flag manifold $F(d-1, d, d+1)$ can be written as

$$F(d-1, d, d+1) = \{(x, L) \in G \times F(d-1, d+1) | x \in L\}.$$ 

Let $\bar{F} := F(d-1, d, d+1), F_1 := F(d-1, d)$ and $F_2 := F(d, d+1)$. Then, we have the following two diagrams:

$$\begin{array}{ccc}
F_1 & \xrightarrow{pr_1} & \tilde{F} \\
\downarrow q_{11} & & \downarrow q_1 \\
G & & \downarrow q_{12} \\
\uparrow q_1 & & \uparrow q_{11} \\
F_2 & \xrightarrow{pr_2} & \bar{F} \\
\end{array} \tag{2.2}$$

and

$$\begin{array}{ccc}
\tilde{F} & \xrightarrow{q_1} & F(d-1, d+1) \\
\downarrow q_{1} & & \downarrow q_1 \\
G & & \\
\end{array} \tag{2.3}$$

where all morphisms in the above diagrams are projections.

**Definition 2.2.** Let $X$ be a noetherian scheme and $E$ be a locally free coherent sheaf on $X$. We define the associated projective space bundle $\mathbb{P}(E)$ as follows:

$$\mathbb{P}(E) = \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \otimes E).$$

**Remark 2.1** ([12] Section 2.2). The mapping $q_{11}$ (respectively $q_{12}$) identifies $F_1$ (respectively $F_2$) with the projective bundle $\mathbb{P}(H_d)$ (respectively $\mathbb{P}(Q_{n-d}^*)$) of $G$. Let $\mathcal{H}_{H_d}$ (respectively $\mathcal{H}_{Q_{n-d}}$) be the tautological line bundle on $F$ associated to
\(F_1\) (respectively \(F_2\)), i.e.,
\[
\mathcal{H}_{d'}^* = \text{pr}_1^* \mathcal{O}_{F_1}(-1) \quad \text{(respectively \(\mathcal{H}_{\nu-d} = \text{pr}_2^* \mathcal{O}_{F_2}(-1)\)).}
\]
So
\[
\text{pr}_1^* \mathcal{H}_{d'}^* = \mathcal{O}_{F_1}(-1) \quad \text{(respectively \(\text{pr}_2^* \mathcal{H}_{\nu-d} = \mathcal{O}_{F_2}(-1)\)).}
\]

**Definition 2.3.** Let \(X\) be a scheme in characteristic \(p\). We define the absolute Frobenius map of \(X\) to be \(F_X : X \to X\) such that \(F_X = \text{id}_X\) as a map between two topological spaces and on each open set \(U, F_X^* : \mathcal{O}_X(U) \to \mathcal{O}_X(U)\) takes \(f\) to \(f^p\) for any \(f \in \mathcal{O}_X(U)\).

**Definition 2.4.** Let \(S\) be a scheme in characteristic \(p\) and \(X\) be an \(S\)-scheme. Consider the following diagram:

where \(X^{(p)}\) is defined as the fiber product of \(X\) and \(S\) in the diagram. The induced map \(F_{X/S}\) is called the Frobenius morphism of \(X\) relative to \(S\).

**Remark 2.2.** If we replace the morphism \(f\) by \(f_1\) in the above diagram, then we define \(X^{(p^2)}\) as the fibre product of \(X^{(p)}\) and \(S\). The induced map \(F_{X^{(p^2)}/S} : X \to X^{(p^2)}\) is called the 2-fold relative Frobenius morphism. So \(X^{(p^m)}\) and the \(m\)-fold relative Frobenius morphism \(F_{X^{(p^{m-1})}/S}\) are defined by recursion. Let \(S\) be a noetherian scheme in characteristic \(p\), \(E\) be a locally free coherent sheaf on \(S\) and \(X = \mathbb{P}(E)\). Then \(X^{(p^m)} = \mathbb{P}(F_{X/S}^m E)\) ([7] Lemma 1.5).

One of the key tools in studying vector bundles in characteristic \(p\) is the following lemma of Katz.

**Lemma 2.5** ([7] Lemma 1.4 or [16] for details). Let \(X\) and \(Y\) be two varieties smooth over \(S\), a noetherian scheme in characteristic \(p\), and \(f\) be an \(S\)-morphism from \(X\) to \(Y\). If the induced map on differentials, \(df : f^* \Omega_{Y/S} \to \Omega_X/S\) is the zero map, then \(f\) can be factored through the relative Frobenius morphism \(F_{X/S}\).

**Definition 2.6.** Denote by \(H_i\) the universal subbundle on the complete flag manifold \(F(1, \ldots, n-1)\) of rank \(i\) \((1 \leq i \leq n)\) and \(X_i = c_1(H_i/H_{i-1})\).

Although Guyot’s paper [12] is based on the field of characteristic zero, the following conclusions are also true in positive characteristic.

**Lemma 2.7** ([12] Theorem 3.1, 3.2). Suppose \(Z[X_1, \ldots, X_{d-1}; X_d; X_{d+1}]\) to be the ring of polynomials in \(d + 1\) variables with integral coefficients symmetrical in \(X_1, \ldots, X_{d-1}\) and \(A(\bar{F})\) is the Chow ring of \(\bar{F}\). The natural morphism \(Z[X_1, \ldots, X_{d-1}; X_d; X_{d+1}] \to A(\bar{F})\) is surjective and its kernel is the ideal generated by \(\sum_i(X_1, \ldots, X_{d-1}), (n - d - 1) < i \leq n\), where
\[
\sum_i(X_1, \ldots, X_{d+1}) := \sum_{\alpha_1 + \cdots + \alpha_{d+1} = i} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{d+1}^{\alpha_{d+1}},
\]
and \(\alpha_1, \ldots, \alpha_{d+1}\) are nonnegative integers.
Lemma 2.8 ([12] Lemma 5.1). The Picard group of \( \tilde{F} \) is generated by \( q_1^* \Theta_G(1), H_{H_d^e} \) and \( H_{Q_{n-d}} \).

From the exact sequences in [12] Section 2.3, we have the Chern polynomials

\[
c_{H_{H_d^e}}(T) = T + X_d, \quad c_{H_{Q_{n-d}}}(T) = T - X_{d+1}.
\]

Here, \( c_F(T) := T^r - c_1(E)T^{r-1} + \cdots + (-1)^r c_r(E) \) is the Chern polynomial of rank \( r \)-bundle \( E \).

From Proposition 2.3 and Proposition 2.5 in [12], we can get the following lemma easily.

Lemma 2.9. The restriction of the relative cotangent bundle \( \Omega_{F/G} \) to every \( q_2 \)-fibre \( \tilde{L} = q_2^{-1}(L) \subseteq F \) has the following form:

\[
\Omega_{F/G} \mid_{\tilde{L}} = \Omega_L(1)^{\oplus n-2}.
\]

3 \quad UNIFORM VECTOR BUNDLES OF RANK \( R(R < D) \) ON \( G \)

In this section, we suppose \( k \) is an algebraically closed field of arbitrary characteristic.

Proposition 3.1. Let \( E \) be an algebraic vector bundle of rank \( r \) over \( G = G_k(d, n) \) and assume \( E|_L = \Theta_L^{\oplus r} \) for every line \( L \subseteq G \). Then \( E \) is trivial.

Proof. We prove the theorem by induction on \( d \). For \( d = 1 \), the Grassmannian is just \( \mathbb{P}^{n-1}_k \), the result holds (see [22] Theorem 3.2.1). Let us consider the diagram

\[
F(d - 1, d) \xrightarrow{\eta_2} G_k(d - 1, n) \xrightarrow{\eta_1} G.
\]

It is not hard to see that every \( \eta_2 \)-fibre \( \eta_2^{-1}(x) \) is isomorphic to \( \mathbb{P}^{n-d}_k \) and \( \eta_1(\eta_2^{-1}(x)) \cong \mathbb{P}^{n-d} \) is the set of all \( d \)-dimensional linear subspaces containing the \((d-1)\)-dimensional linear subspace corresponding to \( x \). By assumption, the restriction of \( E \) to every line in \( \eta_1(\eta_2^{-1}(x)) \) is trivial, thus \( E|_{\eta_1(\eta_2^{-1}(x))} \) is trivial. Next, let’s consider the coherent sheaf

\[
E' = \eta_2^* E.
\]

Note that \( E' \) is an algebraic vector bundle of rank \( r \) over \( G_k(d - 1, n) \) and \( \eta_1^* E \cong \eta_2^* E' \), because \( \eta_1^* E|_{\eta_2^{-1}(x)} \cong E|_{\eta_1(\eta_2^{-1}(x))} \) is trivial on all \( \eta_2 \)-fibres.

Claim. \( E'|_L = \Theta_L^{\oplus r} \) for every line \( L \) in \( G_k(d - 1, n) \).

In fact, because \( \eta_1^* E|_{\eta_1^{-1}(y)} \) is trivial for every \( y \in G \),

\[
E'|_{\eta_2(\eta_1^{-1}(y))} \cong \eta_2^* E'|_{\eta_1^{-1}(y)} \cong \eta_2^* E|_{\eta_1^{-1}(y)}
\]

is trivial for every \( y \in G \) if we identify \( \eta_2(\eta_1^{-1}(y)) \) with \( \eta_1^{-1}(y) \). Since every line \( L \subseteq G_k(d - 1, n) \) is contained in some set \( \eta_2(\eta_1^{-1}(y)) \), the restriction \( E'|_L = \Theta_L^{\oplus r} \) for every line \( L \) in \( G_k(d - 1, n) \).

By the induction hypothesis, \( E' \) is trivial. Thus \( \eta_1^* E \cong \eta_2^* E' \) is trivial, so is \( E \cong \eta_1^* E \).

Corollary 3.2. If \( E \) be a globally generated vector bundle of rank \( r \) over \( G \) with \( c_1(E) = 0 \), then \( E \) is trivial.

Proof. Since \( E \) is globally generated, we have an exact sequence

\[
0 \rightarrow K \rightarrow \Theta_G^{\oplus m} \rightarrow E \rightarrow 0
\]
for some $m$. Restricting this sequence to a line $L \subseteq G$, we get

$$0 \rightarrow K|L \rightarrow \mathcal{O}^{\Theta m}_L \rightarrow E|L \rightarrow 0.$$ 

Suppose $E|L = \bigoplus_{i=1}^r \mathcal{O}_L(a_i)$, then together with $E|L$ is globally generated, we have $a_i \geq 0$ for $1 \leq i \leq r$. If $c_1(E) = 0$, then we must have $a_i = 0$ for all $i$. Thus, $E$ is trivial on every line and hence trivial.

**Theorem 3.3.** For $r < d$ every uniform $r$-bundle over $G$ splits as a direct sum of line bundles.

**Proof.** We prove this theorem by induction on $r$. For $r = 1$, there is nothing to prove. Suppose the assertion is true for all uniform $r'$-bundles with $1 \leq r' < r < d$. If $E$ is a uniform $r$-bundle, after twisting with an appropriate line bundle, we can assume that $E$ has the splitting type

$$a_E = (a_1, \ldots, a_r),$$

where $a_1 \leq \cdots \leq a_r$ and $a_1 = \cdots = a_t = 0$, $a_{t+1} > 0$. If $t = r$, then $E$ is trivial by Proposition 3.1. Therefore suppose $t < r$, i.e.,

$$a_E = (0, \ldots, 0, a_{t+1}, \ldots, a_r), \quad a_{t+1} > 0, \quad \text{for } i = 1, \ldots, r - t.$$

Let us consider the standard diagram

$$\tilde{F} = F(d - 1, d, d + 1) \xrightarrow{q_1} F(d - 1, d + 1) \xrightarrow{q_2} G.$$ 

For $L \in F(d - 1, d + 1)$, the $q_2$-fibre

$$\tilde{L} = q_2^{-1}(L) = \{(x, L) | x \in L\},$$

is mapped isomorphically under $q_1$ to the line $L$ in $G$ and we have

$$q_1^*E|\tilde{L} \cong E|L.$$ 

For $x \in G$, the $q_1$-fibre over $x$,

$$q_1^{-1}(x) = \{(x, L) | x \in L\},$$

is mapped isomorphically under $q_2$ to the subvariety

$$\text{VMRT}_x = \{L \in F(d - 1, d + 1) | x \in L\} \cong \mathbb{P}^{d-1} \times \mathbb{P}^{n-d-1}$$

(VMRT$_x$ means the variety of minimal rational tangents at $x$. We refer to [14] for a complete account on the VMRT). Because

$$E|L \cong \mathcal{O}^{\Theta t}_L \oplus \bigoplus_{i=1}^{r-t} \mathcal{O}_L(a_{t+i}), \quad a_{t+i} > 0,$$

$$h^0\left( q_2^{-1}(L), q_1^* (E^\vee) | q_2^{-1}(L) \right) = t$$

for all $L \in F(d - 1, d + 1)$. Thus the direct image $q_2, q_1^* (E^\vee)$ is a vector bundle of rank $t$ over $F(d - 1, d + 1)$. The canonical homomorphism of sheaves
makes $\tilde{N}^\vee := q_2^*q_2^*q_1^*(E^\vee)$ to be a subbundle of $q_1^*(E^\vee)$. Because over each $q_2$-fibre $\bar{L}$, the evaluation map

$$\tilde{N}^\vee|\bar{L} = H^0(\bar{L}, q_1^*(E^\vee)|\bar{L}) \otimes_k O_{\bar{L}} \to q_1^*(E^\vee)|\bar{L}$$

identifies $\tilde{N}^\vee|\bar{L}$ with $O_{\bar{L}}^{\oplus t} \oplus \bigoplus_{t=1}^{r-t} O_L(-a_{t+i}) = E^\vee|L$. Over $F$ we thus obtain an exact sequence

$$0 \to \tilde{M} \to q_1^*E \to \tilde{N} \to 0$$

of vector bundles, whose restriction to $q_2$-fibres $\bar{L}$ looks as follows:

$$
\begin{array}{cccc}
0 & \rightarrow & \tilde{M}|\bar{L} & \rightarrow & q_1^*E|\bar{L} & \rightarrow & \tilde{N}|\bar{L} & \rightarrow & 0 \\
0 & \rightarrow & \bigoplus_{t=1}^{r-t} O_L(a_{t+i}) & \rightarrow & \bigoplus_{t=1}^{r-t} O_{\bar{L}}^{\oplus t} & \rightarrow & 0 & \rightarrow & 0.
\end{array}
$$

By Lemma 3.1 below, there are bundles $M = q_1^*\tilde{M}, N = q_1^*\tilde{N}$ over $G$ with

$$\tilde{M} = q_1^*M, \tilde{N} = q_1^*N.$$

$M$ and $N$ are then necessarily uniform and we obtain by projecting the bundle sequence

$$0 \to q_1^*M \to q_1^*E \to q_1^*N \to 0$$

onto $G$ to get the exact sequence

$$0 \to M \to E \to N \to 0.$$ (3.3)

Because $M$ and $N$ are uniform vector bundles of rank smaller than $r$, by the induction hypothesis,

$$M = \bigoplus_{t=1}^{r-t} O_G(a_{t+i}), N = O_{\bar{G}}^{\oplus t}.$$

It follows from the Kempf vanishing theorem that $H^1(G, N^\vee \otimes M) = 0$. Thus the exact sequence (3.3) splits and hence also $E$. □

**Lemma 3.1.** There are bundles $M, N$ over $G$ with $\tilde{M} = q_1^*M, \tilde{N} = q_1^*N$.

**Proof.** To prove the lemma, it suffices to show that $\tilde{M}, \tilde{N}$ are trivial on all $q_1$-fibres (the canonical morphisms $q_1^*q_1, \tilde{M} \to \widetilde{M}, q_1^*q_1, \tilde{N} \to \widetilde{N}$ are then isomorphisms). Because $\tilde{N}^\vee$ is a subbundle of $q_1^*(E^\vee)$ of rank $t$, for every point $x \in G$, it provides a morphism

$$\varphi : \operatorname{VMRT}_x \to G_k(t-1, P_k(E^\vee_x)).$$

We claim that $\psi := \varphi|_{P_k^{d-1}}$ is constant for any $P_k^{d-1} \subseteq \operatorname{VMRT}_x$.

Let us consider $\psi^*H_t$ and $\psi^*Q_{r-t}$ (the pullback of the universal subbundle $H_t$ and the universal quotient bundle $Q_{r-t}$ under $\psi$), which are vector bundles on $P_k^{d-1}$. We have the exact sequence

$$0 \to \psi^*H_t \to \bigoplus_{t=1}^{r-t} O_{\bar{P}_k^{d-1}}^{\oplus t} \to \psi^*Q_{r-t} \to 0.$$

Then

$$c(\psi^*H_t) \cdot (\psi^*Q_{r-t}) = 1,$$

i.e. $\left(1 + c_1(\psi^*H_t) + \cdots + c_t(\psi^*H_t)\right) \cdot \left(1 + c_1(\psi^*Q_{r-t}) + \cdots + c_{r-t}(\psi^*Q_{r-t})\right) = 1$. 

Because $r < d$ and the Chow ring of $\mathbb{P}^{d-1}_k$ is $A(\mathbb{P}^{d-1}_k) = \mathbb{Z}[H]/H^d$, where $H$ is the rational equivalence class of a hyperplane, this must imply

$$c(\psi^*H_t) = 1, \ c(\psi^*Q_{r-t}) = 1.$$ 

(Note that here we use the condition $r < d$, otherwise the proof doesn’t work.) In particular

$$c_1(\psi^*H_t) = 0, \ c_1(\psi^*Q_{r-t}) = 0.$$ 

Let $\mathcal{O}_{\mathbb{G}^r_k(t-1,r-1)}(1)$ be the Plücker bundle of $\mathbb{G}^r_k(t-1,r-1)$, since $\deg \psi^*\mathcal{O}_{\mathbb{G}^r_k(t-1,r-1)}(1) = c_1(\psi^*H_t) = 0$, $\psi$ is constant.

Because $\mathbb{P}^{d-1}_k \cong \mathbb{P}^{d-1}_k \times \mathbb{P}^{n-d-1}_k$ can be covered by a family of $\mathbb{P}^{d-1}_k$ and for any two different $\mathbb{P}^{d-1}_k$ in the family, there exists a $\mathbb{P}^{d-1}_k$ which intersects them both, we obtain that $\varphi$ is constant. Thus $\tilde{N}$ is trivial on all $q_1$-fibres. Moreover for every point $x \in G$, $\tilde{M}^v|q_1^{-1}(x)$ is globally generated and $c_1(\tilde{M}^v|q_1^{-1}(x)) = 0$, so $\tilde{M}^v|q_1^{-1}(x)$ is trivial due to Corollay 3.2. \hfill \Box

**Theorem 3.4.** Let $E$ be a vector bundle on $G = G(d, n)$ ($2 \leq d \leq n-d$) over an algebraically closed field. If all the restrictions of $E$ to arbitrary subspace $\mathbb{P}^d_k \subseteq G$ split as direct sums of line bundles, then $E$ splits also globally as a direct sum of line bundles.

**Proof.** The condition implies that $E$ is uniform. In fact, any line $L$ in $G$ is given by a $(d-1)$-dimensional vector space $V_{d-1}$ and a $(d+1)$-dimensional vector space $V_{d+1}$. Since $G(d, V_{d+1})$ and $\{W \in G(d, 2d) | W \supset V_{d-1}\}$ are two different subsets of $G$ containing $L$ which are both isomorphic to $\mathbb{P}^d$ and $G$ is covered by lines, we can see that $E$ has uniform splitting type.

We use the same notations in Theorem 3.3 to prove the corollary by induction on $r$ (the rank of $E$). If we have the exact sequence of vector bundles

$$0 \to M \to E \to N \to 0 \quad (3.4)$$

on $G$, where the rank of $M$ and $N$ is smaller than $r$, such that

$$M|Z = \bigoplus_{i=1}^{r-t} \mathcal{O}_Z(a_{t+i}), \quad N|Z = \mathcal{O}_Z^{\oplus t},$$

for every projective subspace $Z$ of dimension $d$, then by the induction hypothesis, $M$ and $N$ split. It follows from the Kempf vanishing theorem that $H^1(G, N^\vee \otimes M) = 0$. Thus, the above exact sequence splits and hence also $E$.

Similar to the proof of Theorem 3.3, we can obtain an exact sequence on $\bar{F}$

$$0 \to \tilde{M} \to q_1^*E \to \tilde{N} \to 0$$

and the map

$$\varphi : \text{VMRT}_x \to \mathbb{G}^r_k(t-1, P_k(E^\vee_x)).$$

If we prove that the morphism $\varphi$ is constant for every $x \in G$, then there exist two bundles $M, N$ over $G$ with $\tilde{M} = q_1^*M, \tilde{N} = q_1^*N$. By projecting the bundle sequence

$$0 \to q_1^*M \to q_1^*E \to q_1^*N \to 0$$

onto $G$, we can get the desired exact sequence (3.4). Thus, to prove the existence of the above exact sequence, it suffices to show that the map

$$\varphi : \text{VMRT}_x \to \mathbb{G}^r_k(t-1, P_k(E^\vee_x))$$

is constant for every $x \in G$. Given, a projective subspace $Z$ of dimension $d$ and a line $L \subseteq Z$, we take any point $x \in L$ and denote by $Z'$ the subspace of VMRT$_x$ corresponding to the tangent directions to $Z$ at $x$. By the hypothesis, $E|Z$ is a direct sum of line bundles, then

$$\varphi|Z' : Z' \to \mathbb{G}^r_k(t-1, P_k(E^\vee_x))$$
is constant. Since VMRT\(x \cong \mathbb{P}^{d-1}_k \times \mathbb{P}^{d-1}_k\) and for any two different \(\mathbb{P}^{d-1}_k\)’s in the family, there exists a \(\mathbb{P}^{d-1}_k\) which intersects them both, \(\varphi\) is constant for every \(x \in G\).

4 | UNIFORM VECTOR BUNDLES OF RANK D ON G

In this section, we suppose the characteristic of the field \(k\) is positive integer \(p\). Although Guyot’s paper [12] is based on the field of characteristic zero, many conclusions are also true for positive characteristic, except for the argument of the splitting type of uniform non-splitting \(d\)-bundle over \(G\) being \((0,\ldots,0,-1)\). The main reason is that the first part of Guyot’s paper, which is independent of the characteristic of the field \(k\), is for studying the Chow ring and axiomatically computing Chern classes of vector bundles on flag varieties.

The uniform vector bundle \(E\) can be characterized as follows [12]. Let (3.2) be the standard diagram and \(L\) be a line in \(G\), then \(E\) is uniform with

\[E|L = \Omega_L(u_1)^{\oplus r_1} \oplus \cdots \oplus \Omega_L(u_t)^{\oplus r_t}, \quad u_1 > \cdots > u_t,\]

if and only if there is a filtration

\[0 = HN^0_q(q_1^*E) \subseteq HN^1_q(q_1^*E) \subseteq \cdots \subseteq HN^i_q(q_1^*E) = q_1^*E\]

of \(q_1^*E\) by subbundles \(HN^i_q(q_1^*E)\) such that \(HN^i_q(q_1^*E)/HN^{i-1}_q(q_1^*E) \cong q_2^*(E_i) \otimes O_q(u_i)\), where \(E_i\) is an algebraic vector bundle of rank \(r_i\) over \(F(d-1,d+1)\), \(O_q(1) = \mathbb{H}^\vee_d\) and

\[HN^i_q(q_1^*E) = \text{Im} \left[ q_2^*q_2^* \left( q_1^*E \otimes O_q(-u_i) \right) \otimes O_q(u_i) \rightarrow q_1^*E \right].\]

This filtration is the relative Harder–Narasimhan filtration of \(q_1^*E\). Since rank \(E = d \leq n - d\), by Whitney’s formula and Lemma 2.7, we have

\[c_{q_1^*E}(T) = \prod_{i=1}^{d-1} c_{HN^i_q(q_1^*E)/HN^{i-1}_q(q_1^*E)(T)} + a \sum_{n-d}^{} (X_1, \ldots, X_{d+1}). \tag{4.1}\]

**Theorem 4.1.** If \(E\) is a uniform vector bundle over \(G\) of rank \(d\), then \(E \cong \bigoplus_{i=1}^{d} \Omega_G(a_i)\), \(E \cong F^m \otimes H_d \otimes \Omega_G(v_1)\) or \(E \cong F^{m+}(H^\vee_d) \otimes \Omega_G(v_2)\), \(m \geq 0\) and \(a_1, v_1, v_2 \in \mathbb{Z}\).

**Proof.** If \(a = 0\) in the equality (4.1) and \(E\) cannot split as a direct sum of line bundles, then by the assertion of Guyot ([12] Corollary 4.1.1 b), the expression of \(c_{q_1^*E}(T)\) can only contain \(u_1\) and \(u_2\) and \(r_1 = d - 1\), \(r_2 = 1\). So, after twisting with an appropriate power of \(\Omega_G(1)\) and dualizing if necessary, we can let \(u_1 = 0\) and assume \(E\) is of type \((0,0,\ldots,0,b)\), \(b < 0\). Then we can write

\[c_{q_1^*E}(T) = \prod_{i=1}^{d} (T + bX_i).\]

So, we get

\[c_{HN^i_q(q_1^*E)}(T) = \prod_{i=1}^{d-1} (T + bX_i) \quad \text{and} \quad c_{q_1^*E/HN^{i-1}_q(q_1^*E)}(T) = T + bX_d.\]

By Lemma 2.8, we get \(q_1^*E/HN^1_q(q_1^*E) \cong \left( (H^\vee_d)^\vee \right)^{\otimes (-b)}\). Hence on \(F\), we have the following exact sequence:

\[0 \rightarrow HN^1_q(q_1^*E) \rightarrow q_1^*E \rightarrow \left( (H^\vee_d)^\vee \right)^{\otimes (-b)} \rightarrow 0. \tag{4.2}\]

By the universal property of \(\mathbb{P}(E)\), there is a unique \(G\)-morphism \(\sigma : F \rightarrow \mathbb{P}(E)\) such that

\[\sigma^* \Omega_{\mathbb{P}(E)}(1) = \left( (H^\vee_d)^\vee \right)^{\otimes (-b)}, \quad \sigma^* \Omega_{\mathbb{P}(E)/G} = HN^1_q(q_1^*E) \otimes \left( H^\vee_d \right)^{\otimes (-b)}.\]
Let us consider the following diagram:

\[ \begin{array}{ccc}
F_1 & \xleftarrow{p_1} & \tilde{F} & \xrightarrow{p_2} & F_2 \\
q_1 \downarrow & & q_1 \downarrow & & q_2 \downarrow \\
G & & & & \\
\end{array} \]

(4.3)

**Case 1.** \( b = -1 \): 
Projecting the exact sequence 
\[ 0 \to HN_1^1(q_1^*E) \to q_1^*E \to (H_{H_d})^\vee \to 0 \]
on to \( F_1 \) and by Remark 2.1, we get the exact sequence (\( R^1 pr_1^*HN_1^1(q_1^*E) = 0 \)) 
\[ 0 \to pr_1^*HN_1^1(q_1^*E) \to q_{11}^*E \to \mathcal{O}_{F_1}(1) \to 0. \]
Restricting the above exact sequence to a fiber of \( q_{11}, q_{11}^{-1}(x) \cong \mathbb{P}^{d-1}_k \), we see that 
\[ pr_1^*HN_1^1(q_1^*E)|_{q_{11}^{-1}(x)} \cong \Omega \mathbb{P}^{d-1}_k(1). \]
Hence, we get 
\[ q_{11}^* (pr_1^*HN_1^1(q_1^*E)) = R^1 q_{11}^* (pr_1^*HN_1^1(q_1^*E)) = 0. \]
In particular, 
\[ E \cong q_{11}^* \mathcal{O}_{F_1}(1) \cong H_d. \]

**Case 2.** \( b < -1 \): 
By restricting the induced map \( d\sigma : \sigma^* \Omega_{\mathbb{P}(E)/G} = HN_1^1(q_1^*E) \otimes (H_{H_d})^\otimes(-b) \to \Omega_{\tilde{F}/G} \) to any \( q_2 \)-fiber \( \tilde{L} = q_2^{-1}(L) \subseteq \tilde{F} \), and by Lemma 2.9, we get 
\[ d\sigma|_{\tilde{L}} : \mathcal{O}_{\tilde{L}}(-b)^{\otimes d-1} \to \mathcal{O}_{\tilde{L}}(1)^{\otimes n-2}. \]
Because \( b < -1 \), we have \( d\sigma|_{q_2^{-1}(L)} = 0 \) for all \( L \in F(d-1, d+1) \), i.e. \( d\sigma = 0 \). By Lemma 2.5, \( \sigma \) can be factored through the relative Frobenius morphism \( F^m_{\tilde{F}/G} \) for some positive integer \( m \):

\[ \begin{array}{ccc}
\tilde{F} & \xrightarrow{\sigma} & P(E) \\
q_1 \downarrow & & \pi \downarrow \\
G & & \\
\end{array} \]

Let us consider the following diagram:

\[ \begin{array}{ccc}
\tilde{F}_1^{(p_m)} & \xleftarrow{p_1'} & \tilde{F}(p_m) & \xrightarrow{p_2'} & \tilde{F}_2^{(p_m)} \\
q_{11} \downarrow & & q_1' \downarrow & & q_{12} \downarrow \\
G & & & & \\
\end{array} \]

(4.4)

By Remark 2.2, we have 
\[ \tilde{F}_1^{(p_m)} = P(\tilde{F}_1^m(H_d)), \tilde{F}_2^{(p_m)} = P(\tilde{F}_2^m(Q_{n-d}^\vee)). \]
Because \( \sigma \) can be factored through the relative Frobenius morphism \( F^m_{E/G} \), hence on \( F(p^m) \), we have the exact sequence

\[
0 \to (HN^1_q(q_1^*E))' \to q_1^*E \to (H'_{H_d})^\vee(\mathcal{O}_{\mathbb{P}(E)^\vee}^\beta) \to 0.
\] (4.5)

where \( H'_{H_d} \) is the tautological bundle on \( F(p^m) \) associated to \( F_1(p^m) \), and the pullback of the exact sequence (4.5) under \( F^m_{E/G} \) is the exact sequence (4.2). We also have the reduced map

\[
d\sigma' : \sigma'^* \Omega_{\mathbb{P}(E)/G} \to \Omega_{F(p^m)/G}.
\]

By restricting the map to any \( q_2 \)-fiber, we get

\[
d\sigma'|\widetilde{L} : \mathcal{O}_{\widetilde{L}}(-b)^{\oplus d-1} \to \mathcal{O}_E(p^m)^{\oplus n-2}.
\]

By Lemma 2.5, we may assume \(-b \leq p^m\). On the other hand, we have \( p^m| - b \), thus \(-b = p^m\). On \( \mathbb{P}(E) \), we now have the following exact sequence:

\[
0 \to (HN^1_q(q_1^*E))' \to q_1^*E \to (H'_{H_d})^\vee \to 0.
\]

By projecting the exact sequence to \( F_1(p^m) \), we get an exact sequence

\[
0 \to pr_1^*(HN^1_q(q_1^*E))' \to q_1^*E \to \mathcal{O}_{F_1(p^m)}(1) \to 0.
\]

As in Case 1, we find \( q_1^*(pr_1^*(HN^1_q(q_1^*E)))' = R^1pr_1^*(HN^1_q(q_1^*E))' = 0 \). In particular,

\[
E \cong q_{11}^* \mathcal{O}_{F_1(p^m)}(1) \cong F^{m^*}H_d.
\]

If \( a \neq 0 \), the equation (4.1) implies that \( d = n - d \). Suppose that \( E \) can’t split as a direct sum of line bundles. After twisting with an appropriate power of \( \mathcal{O}_G(1) \) and dualizing if necessary, we can assume \( E \) is of type \((0,0,\ldots,0,\beta) (\beta < 0)\). By Proposition 4.2 below, we get \( E \cong F^{m^*}Q^\vee_{n-d} (m \geq 0) \), where \( Q^\vee_{n-d} \) can be viewed as the universal subbundle of the dual Grassmannian. Hence we get the desired result.

**Proposition 4.2.** Let \( E \) be a uniform vector bundle over \( G \) of rank \( n - d \), and \( a \neq 0 \). Then \( E \cong \mathcal{O}_{\mathcal{G}(v_1)} \otimes \mathcal{O}_{\mathcal{G}(v_2)} \), where \( m \geq 0 \) and \( v_1, v_2 \in \mathbb{Z} \).

**Proof.** By the assertion of Guyot ([12] Lemma 4.2.3), if \( E \) can’t split as a direct sum of line bundles, then there are two cases:

1) \( r_1 = n - d - 1, r_2 = 1 \): after tensoring \( E \) with a line bundle, we may assume \( c_{q_1^*E}(T) = \sum_{n-d} (T, \beta X_1, \ldots, \beta X_d) (\beta < 0) \).

2) \( r_1 = 1, r_2 = n - d - 1 \): after tensoring \( E \) with a line bundle, we may assume \( c_{q_1^*E}(T) = \sum_{n-d} (T, -\beta X_1, \ldots, -\beta X_d) (\beta < 0) \).

In the first case, we can write

\[
c_{q_1^*E}(T) = -\beta^{n-d} \sum_{n-d} (X_1, \ldots, X_{d+1})
= (T - \beta X_{d+1}) \left( \sum_{n-d} (T, \beta X_1, \ldots, \beta X_{d+1}) \right).
\]

So, we get

\[
c_{HN^1_q(q_1^*E)}(T) = \sum_{n-d} (T, \beta X_1, \ldots, \beta X_{d+1}) \quad \text{and} \quad c_{q_1^*E/HN^1_q(q_1^*E)}(T) = T - \beta X_{d+1}.
\]
By Lemma 2.8, we get $q_1^*E/N^1_q (q_1^*E) \cong ((H_{Q_{n-d}}^y)^{\otimes (-2)})$. Hence on $F$, we have the following exact sequence

$$0 \to HN^1_q (q_1^*E) \to q_1^*E \to ((H_{Q_{n-d}}^y)^{\otimes (-2)}) \to 0 \tag{4.6}$$

According to the proof in the above theorem, we get $E \cong F^{m^*}Q^{n-d}_{n-d}$. For the second case, we get $E \cong F^{m^*}Q_{n-d} (m \geq 0)$ for the similar reason.

Therefore, we have proved Theorem 1.1 completely.

**Remark 4.3.** According to [13] Section 3.3, the family of semistable torsion free sheaves with fixed Hilbert polynomial over normal projective varieties in characteristic zero forms a bounded family. Langer ([19] Theorem 4.1 with Theorem 1.1 and Lemma 1.2) generalized this result in arbitrary characteristic.

## 5 VECTOR BUNDLES ON FLAG VARIETIES

Let $F := F(d_1, \ldots, d_s)$ be the flag manifold parameterizing flags

$$V_{d_1} \subseteq \cdots \subseteq V_{d_s} \subseteq k^n,$$

where $dim(V_{d_i}) = d_i, 1 \leq i \leq s$. In this section, we suppose that the characteristic of $k$ is zero.

For every $i$, we may fix a flag $V_{d_1} \subseteq \cdots \subseteq V_{d_{i-1}} \subseteq V_{d_i-1} \subseteq V_{d_{i+1}} \subseteq \cdots \subseteq V_{d_s} \subseteq k^n$. It's not hard to see that the set of flags $V_{d_1} \subseteq \cdots \subseteq V_{d_{i-1}} \subseteq W \subseteq V_{d_{i+1}} \subseteq \cdots \subseteq V_{d_s} \subseteq k^n$ such that

$$V_{d_{i-1}} \subseteq W \subseteq V_{d_{i+1}}, \quad dim(W) = d_i$$

is the projectivization of the quotient space $V_{d_{i+1}}/V_{d_{i-1}}$, so the set of such flags is isomorphic to $\mathbb{P}^i_k$. It follows that any $l \in F(d_1, \ldots, d_{i-1}, d_i - 1, d_i + 1, d_{i+1}, \ldots, d_s)$ determines a line $L \subseteq F$. We denote by $F^{(i)} := F(d_1, \ldots, d_{i-1}, d_i - 1, d_i + 1, d_{i+1}, \ldots, d_s)$ the $i$th connected component of the manifold of lines in $F$. (We re-emphasize that if the two adjacent integers in the expression of flag varieties such as $F^{(i)}$ are equal, we keep only one of them.)

We can consider $F^{(i)}$ in the following two cases:

Case I: $d_i - 1 = d_{i-1}$ and $d_i + 1 = d_{i+1}$;

Case II: $d_i - 1 \neq d_{i-1}$ or $d_i + 1 \neq d_{i+1}$.

Then we have the standard diagram

$$F(d_1, \ldots, d_{i-1}, d_i - 1, d_i + 1, d_{i+1}, \ldots, d_s) \xrightarrow{q_i} F^{(i)}$$

$$F = F(d_1, \ldots, d_s). \tag{5.1}$$

Note that $q_1$ is the identity in Case I.

On the flag manifold $F$, we denote by $H_{d_i}$ the universal subbundle whose fiber at a point $[\Lambda] = [V_{d_1} \subseteq \cdots \subseteq V_{d_i} \subseteq k^n] \in F$ is the subspace $V_{d_i}$; that is,

$$(H_{d_i})_{[\Lambda]} = V_{d_i};$$

Set $H_{d_i} = H_{d_j} / H_{d_j}(j \geq i)$ as the universal quotient bundle.

Let $E$ be an algebraic $r$-bundle over $F$. According to a theorem of Grothendieck, for every $l \in F^{(i)}$, there is an $r$-tuple

$$\alpha^{(i)}_E(l) = (a^{(i)}_1(l), \ldots, a^{(i)}_r(l)) \in \mathbb{Z}^r \text{ with } a^{(i)}_1(l) \geq \cdots \geq a^{(i)}_r(l)$$
such that $E|L \cong \bigoplus_{i=1}^r \mathcal{O}_Y(a^{(i)}(l))$. We give $\mathbb{Z}^r$ the lexicographical ordering, i.e., $(a_1, \ldots, a_r) \leq (b_1, \ldots, b_r)$ if the first non-zero difference $b_i - a_i$ is positive. Let

$$a^{(i)}_E = \inf_{l \in F(i)} a^{(i)}_E(l)$$

be the generic splitting type of $E$ with respect to $F(i)$. $S_E^{(i)} = \{l \in F(i) | a^{(i)}_E(l) > a^{(i)}_E\}$ is the set of jump lines with respect to $F(i)$. We define $U_E^{(i)} := F(i) \setminus S_E^{(i)}$.

Remark 5.1. Fix integer $i, 1 \leq i \leq s$. Let

$$M_t(a_1, \ldots, a_t) = \{l \in F(i) | (a^{(i)}_1(l), \ldots, a^{(i)}_t(l)) > (a_1, \ldots, a_t)\}.$$ 

Because of the semicontinuity theorem, the set $M_1(a_1) = \{l \in F(i) | \eta_0(L, E(-a_1-1)) | L > 0\}$ is Zariski-closed in $F(i)$. By induction on $t$ we see that $S_E^{(i)} = M_r(a^{(i)}_E)$ is Zariski-closed in $F(i)$. Thus $U_E^{(i)}$ is a non-empty Zariski-open subset of $F(i)$ (see [22] Lemma 3.2.2).

The following result holds over any algebraically closed field.

**Proposition 5.2.** Let $E$ be an algebraic vector bundle of rank $r$ over $F := F(d_1, \ldots, d_s)$ and assume $E|L = \mathcal{O}_L^{dr}$ for every line $L$. Then $E$ is trivial.

**Proof.** We prove the theorem by induction on $s$. For $s = 1$, the flag manifold is just $G(d_1, n)$, the result holds by Proposition 3.1. Suppose the assertion is true for all flag manifolds $F(d'_1, \ldots, d'_{s-1})$. Let us consider the natural projection $q : F = F(d_1, \ldots, d_s) \to F(d_2, \ldots, d_s)$. It is not hard to see that every $q$-fiber $q^{-1}(x)$ is isomorphic to the Grassmannian $G(d_1, d_2)$. Since the restriction of $E$ to every line in the $q$-fiber $q^{-1}(x)$ is trivial by assumption, $E$ is trivial on all $q$-fibers by Proposition 3.1. It follows that $E' = q_* E$ is an algebraic vector bundle of rank $r$ over $F(d_2, \ldots, d_s)$ and $E \cong q^* E'$. 

**Claim.** $E'|L = \mathcal{O}_L^{dr}$ for every line $L$ in $F(d_2, \ldots, d_s)$.

In fact, let $L$ be a line in the $i$th ($2 \leq i \leq s$) connected component of the manifold of lines in $F$. Then $q(L)$ is a line in the $(i-1)$th ($2 \leq i \leq s$) connected component of the manifold of lines in $F(d_2, \ldots, d_s)$. When $L$ runs through all lines in the $i$th ($2 \leq i \leq s$) component of the manifold of lines in $F$, $q(L)$ also runs through all lines in the $(i-1)$th ($2 \leq i \leq s$) component of the manifold of lines in $F(d_2, \ldots, d_s)$. The projection $q$ induces an isomorphism 

$$E'|L \cong q^* E'|L \cong E|L.$$ 

Since $E|L$ is trivial for all lines $L$ in the $i$th ($2 \leq i \leq s$) connected component of the manifold of lines in $F$ by assumption, $E'|L$ is trivial for all lines $L$ in the $i$th ($1 \leq i \leq s-1$) connected component of the manifold of lines in $F(d_2, \ldots, d_s)$. It follows that $E'|L = \mathcal{O}_L^{dr}$ for every line $L$ in $F(d_2, \ldots, d_s)$.

By the induction hypothesis, $E'$ is trivial. Thus $E \cong q^* E'$ is trivial. 

**Lemma 5.1** Descent Lemma [11]. Let $X, Y$ be nonsingular varieties over $k$, $f : X \to Y$ be a surjective submersion with connected fibers and $E$ be an algebraic $r$-bundle over $Y$. Let $\bar{K} \subseteq f^* E$ be a subbundle of rank $t$ in $f^* E$ and $\bar{Q} = f^* E / \bar{K}$ be its quotient. If

$$\text{Hom}(T_{X/Y}, \text{Hom}(\bar{K}, \bar{Q})) = 0,$$

then $\bar{K}$ is the form $\bar{K} = f^* K$ for some algebraic subbundle $K \subseteq E$ of rank $t$.

**Lemma 5.2.** Let $\widetilde{F} := F(d_1, \ldots, d_{i-1}, d_i - 1, d_i, d_i + 1, d_{i+1}, \ldots, d_s), \widetilde{L} = q^{n-1}(l) \subseteq \widetilde{F}$ for all $l \in F(i)$. If $F(i)$ is in Case II, then for the relative cotangent bundle $\Omega_{\widetilde{F}/F}$, we have

$$\Omega_{\widetilde{F}/F} | \widetilde{L} = \mathcal{O}_L(1)^{d_{i+1} - d_{i-1} - 2}.$$
Proof. By the definition of vector bundle $H_{d_i,d_j}$ ($j > i$), it's easy to check that on $\widetilde{F}^{(i)}$, we have the following two exact sequences:

$$0 \to H_{d_i-1,d_i}^{\vee} \to q_1^*H_{d_i-1,d_i}^{\vee} \to q_2^*H_{d_i-1,d_i-1}^{\vee} \to 0,$$

(5.2)

$$0 \to H_{d_i,d_i+1} \to q_1^*H_{d_i,d_i+1} \to q_2^*H_{d_i+1,d_i+1} \to 0.$$  

(5.3)

Let $F^{(i)}_1 := F(d_1, \ldots, d_{i-1}, d_i - 1, d_i, d_{i+1}, \ldots, d_s)$, $F^{(i)}_2 := F(d_1, \ldots, d_{i-1}, d_i, d_i + 1, d_{i+1}, \ldots, d_s)$ and consider the following diagram:

$$\begin{array}{c}
\xymatrix{ 
\widetilde{F}^{(i)}_1 
& \widetilde{F}^{(i)}_2 
& F \\
F^{(i)} 
& \ & F^{(i)} \\
& q_1 
& q_2}
\end{array}$$

(5.4)

All morphisms in the above diagram are projections. It is not hard to see that $\widetilde{F}^{(i)}_1 = \mathbb{P}(H_{d_i-1,d_i})$, $\widetilde{F}^{(i)}_2 = \mathbb{P}(H_{d_i,d_i+1})$ and $H_{d_i-1,d_i}^{\vee}$ (respectively $H_{d_i,d_i+1}$) is the tautological line bundle on $\widetilde{F}^{(i)}$ associated to $F^{(i)}_1$ (respectively $F^{(i)}_2$), i.e.,

$$\text{pr}_1^*H_{d_i-1,d_i}^{\vee} = \mathcal{O}_{\widetilde{F}^{(i)}_1}(-1) \quad (\text{respectively } \text{pr}_2^*H_{d_i,d_i+1} = \mathcal{O}_{\widetilde{F}^{(i)}_2}(-1)).$$

Projecting the exact sequence (5.2) onto $\widetilde{F}^{(i)}_1$ and considering the relative Euler sequence, we have the diagram of exact sequences ($\mathbb{R}^1 pr_1^*H_{d_i-1,d_i} = 0$)

$$\begin{array}{c}
\xymatrix{ 
0 
& pr_1^*H_{d_i-1,d_i}^{\vee} 
& q_1^*H_{d_i-1,d_i}^{\vee} 
& pr_1^*q_2^*H_{d_i-1,d_i-1}^{\vee} 
& 0 \\
& \cong 
& \cong 
& \cong \\
0 
& \mathcal{O}_{F^{(i)}_1}(-1) 
& q_1^*H_{d_i-1,d_i}^{\vee} 
& \mathcal{O}_{F^{(i)}_1}(-1) \otimes T_{\widetilde{F}^{(i)}_2}/F 
& 0.}
\end{array}$$

We get $T_{\widetilde{F}^{(i)}_1}/F \cong pr_1^*(H_{d_i-1,d_i} \otimes q_2^*H_{d_i-1,d_i-1}^{\vee})$, since $H_{d_i-1,d_i} = pr_1^*\mathcal{O}_{F^{(i)}_1}^{\vee}(1)$.

Similarly, on $\widetilde{F}^{(i)}_2$, we have the exact sequences ($\mathbb{R}^1 pr_2^*H_{d_i,d_i+1} = 0$)

$$\begin{array}{c}
\xymatrix{ 
0 
& pr_2^*H_{d_i,d_i+1} 
& q_2^*H_{d_i,d_i+1} 
& pr_2^*q_2^*H_{d_i+1,d_i+1} 
& 0 \\
& \cong 
& \cong 
& \cong \\
0 
& \mathcal{O}_{F^{(i)}_2}(-1) 
& q_2^*H_{d_i,d_i+1} 
& \mathcal{O}_{F^{(i)}_2}(-1) \otimes T_{\widetilde{F}^{(i)}_2}/F 
& 0.}
\end{array}$$

We get $T_{\widetilde{F}^{(i)}_2}/F \cong pr_2^*(H_{d_i,d_i+1}^{\vee} \otimes q_2^*H_{d_i+1,d_i+1})$. Since $\widetilde{F}^{(i)}$ as $F$-scheme is the fiber product of two $F$-schemes $\widetilde{F}^{(i)}_1$ and $\widetilde{F}^{(i)}_2$, we have

$$T_{\widetilde{F}^{(i)}_2}/F \cong pr_1^*T_{\widetilde{F}^{(i)}_1}/F \oplus (pr_2^*T_{\widetilde{F}^{(i)}_2}/F).$$

(5.5)

$$\cong (pr_1^*pr_1^*(H_{d_i-1,d_i} \otimes q_2^*H_{d_i-1,d_i-1}^{\vee})) \oplus (pr_2^*pr_2^*(H_{d_i,d_i+1}^{\vee} \otimes q_2^*H_{d_i+1,d_i+1})).$$

(5.6)

The canonical homomorphism $pr_1^*pr_1^*(H_{d_i-1,d_i} \otimes q_2^*H_{d_i-1,d_i-1}^{\vee}) \cong H_{d_i-1,d_i} \otimes q_2^*H_{d_i-1,d_i-1}^{\vee}$, because over each $pr_1$-fiber $pr_1^{-1}(l)$, the evaluation map

$$pr_1^*pr_1^*(H_{d_i-1,d_i} \otimes q_2^*H_{d_i-1,d_i-1}^{\vee})|_{pr_1^{-1}(l)}$$

(5.7)
is an isomorphism. Similarly, $pr_2^*pr_2^*(H^\vee_{d_i,d_{i+1}} \otimes q_2^*H^\vee_{d_{i+1},d_{i+1}}) \cong H^\vee_{d_i,d_{i+1}} \otimes q_2^*H^\vee_{d_{i+1},d_{i+1}}$. Hence,

$$\Omega_{\tilde{F}(i)/F} \cong (H^\vee_{d_{i-1},d_i} \otimes q_2^*H^\vee_{d_i,d_{i+1}}) \oplus (H^\vee_{d_{i+1},d_{i+2}} \otimes q_2^*H^\vee_{d_{i+2},d_{i+2}})$$

Finally, we get

$$\Omega_{\tilde{F}(i)/F}\mid \tilde{L} = \mathcal{O}_\ell(1)^{\oplus d_i-d_{i-1}-2}.$$  

\[ \square \]

**Theorem 5.3.** Fix an integer $i, 1 \leq i \leq s$. Let $E$ be an algebraic $r$-bundle over $F$ of generic splitting type $a_E^{(i)} = (a_1^{(i)}, \ldots, a_r^{(i)})$, $a_1^{(i)} \geq \cdots \geq a_r^{(i)}$ with respect to $F^{(i)}$. If for some $t < r$,

$$a_t^{(i)} - a_{t+1}^{(i)} \geq \begin{cases} 
1, & \text{and } F^{(i)} \text{ is in Case I} \\
2, & \text{and } F^{(i)} \text{ is in Case II},
\end{cases}$$

then there is a reflexive subsheaf $K \subseteq E$ of rank $t$ with the following properties: over the open set $V_E^{(i)} = q_1(q_2^{-1}(U_E^{(i)})) \subseteq F$, the sheaf $K$ is a subbundle of $E$, which on the line $L \subseteq F$ given by $l \in U_E^{(i)}$ has the form

$$K\mid L \cong \oplus_{j=1}^t \mathcal{O}_L(a_j^{(i)}).$$

**Proof.** After tensoring with an appropriate line bundle, we may assume $a_t^{(i)} = 0$, $a_{t+1}^{(i)} < 0$. Consider the standard diagram

$$\begin{CD}
\bar{F}^{(i)} @> q_2 >> F^{(i)} @> q_1 >> F = F(d_1, \ldots, d_i), \\
\downarrow @V q_1 VV @V q_1 VV \\
\bar{F} = F(d_1, \ldots, d_i).
\end{CD}$$

(5.10)

For every point $l \in U_E^{(i)}$, we have

$$q_1^*E|q_2^{-1}(l) \cong E|L \cong \mathcal{O}_L^{\oplus f_1}(a_j^{(i)}).$$

Then $q_2^*q_1^*E$ is a coherent sheaf over $F^{(i)}$ which is locally free over $U_E^{(i)}$. The morphism $\phi : q_2^*q_1^*E \to q_1^*E$ on each $\bar{L} = q_2^{-1}(l) \cong L$ for an $l \in U_E^{(i)}$ is given by the evaluation of the section of $E|L$. Thus the image of $\phi|L$ is the subbundle

$$\oplus_{j=1}^t \mathcal{O}_L(a_j^{(i)}) \subseteq E|L$$

of rank $t$. Hence, over the open set $q_2^{-1}(U_E^{(i)})$, $\phi$ is a morphism of constant rank $t$ and thus its image $Im\phi \subseteq q_1^*E$ over $q_2^{-1}(U_E^{(i)})$ is a subbundle of rank $t$.

Let $Q' = q_1^*E/Im\phi$ and $T(Q')$ be the torsion subsheaf of $Q'$ and

$$\bar{K} = \ker(q_1^*E \to Q'/T(Q')).$$

Because $Q' = Q'/T(Q')$ is a torsion-free sheaf, $\bar{K}$ is a reflexive subsheaf of rank $t$.

In Case I, since $q_1$ is the identity, $\bar{K}$ is a reflexive subsheaf of $E$ exactly. Denote $K := \bar{K}$. Over the open set $V_E^{(i)} = q_1(q_2^{-1}(U_E^{(i)})) \subseteq F$ the sheaf $K$ is a subbundle of $E$, which on the line $L \subseteq F$ given by $l \in U_{E}^{(i)}$ has the form

$$K\mid L \cong \oplus_{j=1}^t \mathcal{O}_L(a_j^{(i)}).$$
In Case II, \( \tilde{K} \) is a reflexive subsheaf of \( q_1^*E \) and outside the singularity set \( S(\tilde{Q}) \) of \( \tilde{Q} \), the sheaf \( \tilde{K} \) is a subbundle of \( q_1^*E \), which on each \( \tilde{L} = q_2^{-1}(l) \cong L \) given by \( l \in U_E^{(i)} \) has the form
\[
\tilde{K}|\tilde{L} \cong \bigoplus_{j=1}^t \mathcal{O}_L(a_j^{(i)}).
\]
Let \( X = F^{(i)} \setminus S(\tilde{Q}) \). \( X \) is open in \( F^{(i)} \) and contains \( q_2^{-1}(U_E^{(i)}) \). We have the following commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\zeta} & F^{(i)} \\
\downarrow f & & \downarrow q_i \\
Y = q_i(X) & \xrightarrow{\eta} & F
\end{array}
\]
with a surjective submersion \( f \) with connected fibers.

In order to apply the Descent Lemma to the subbundle \( \tilde{K}|X \subseteq f^*(E|Y) \), we need to show that
\[
\text{Hom}(T_{X/Y}, \text{Hom}(\tilde{K}|X, \tilde{Q}|X)) = 0.
\]
By the following Claim 5.4, the hypothesis of the Descent Lemma is satisfied. Hence over the open set \( Y \subseteq F \), we get a subbundle \( K' \subseteq E|Y \) with
\[
f^*K' = \tilde{K}|X \subseteq f^*(E|Y).
\]
\( K' \) can be extended to a reflexive subsheaf \( K = q_1^*\tilde{K} \subseteq E \) on \( F \): To prove this, we need to consider the above diagram (5.11) again. From the diagram and Zariski’s Main Theorem, we deduce that
\[
f_*\mathcal{O}_X = \eta^*\eta_*f_*\mathcal{O}_X = \eta^*q_1_*\zeta_*\mathcal{O}_X = \eta^*q_1_*\mathcal{O}_{F^{(i)}} = \eta^*\mathcal{O}_F = \mathcal{O}_Y.
\]
Next, we only need to prove \( K|Y = K' \), i.e., \( \eta_*K' = q_1_*\tilde{K} \).

Because \( S(\tilde{Q}) \) is of codimension at least 2 and \( \tilde{K} \) is a reflexive sheaf, we have \( \zeta_*(\tilde{K}|X) = \tilde{K} \). Thus
\[
\eta_*K' = \eta_*(K' \otimes f_*\mathcal{O}_X) = \eta_*f_*f^*K' = q_1_*(\zeta_*(f^*K')) = q_1_*\zeta_*(\tilde{K}|X) = q_1_*\tilde{K}.
\]
It’s easy to see that over the open set \( U_E^{(i)} = q_1(q_2^{-1}(U_E^{(i)})) \subseteq F \), the sheaf \( K \) is a subbundle of \( E \), which on the line \( L \subseteq F \) given by \( l \in U_E^{(i)} \) has the form
\[
K|L \cong \bigoplus_{j=1}^t \mathcal{O}_L(a_j^{(i)}).
\]
Claim 5.4. If \( a_{i+1}^{(i)} < -1 \), then
\[
\text{Hom}(T_{X/Y}, \text{Hom}(\tilde{K}|X, \tilde{Q}|X)) = 0.
\]
In fact, it is equivalent to prove
\[
H^0(X, \Omega_{\tilde{F}^{(i)}/F} \otimes \tilde{K}^\vee \otimes \tilde{Q}) = 0.
\]
Since the codimension of \( X \setminus q_2^{-1}(U_E^{(i)}) \) in \( X \) is at least 2 and \( \Omega_{\tilde{F}^{(i)}/F} \otimes \tilde{K}^\vee \otimes \tilde{Q} \) is torsion-free, the restriction
\[
H^0(X, \Omega_{\tilde{F}^{(i)}/F} \otimes \tilde{K}^\vee \otimes \tilde{Q}) \rightarrow H^0(q_2^{-1}(U_E^{(i)}), \Omega_{\tilde{F}^{(i)}/F} \otimes \tilde{K}^\vee \otimes \tilde{Q})
\]
is injective, it suffices to show that \( \Omega_{\tilde{F}^{(i)}/F} \otimes \tilde{K}^\vee \otimes \tilde{Q} \) has no sections over \( q_2^{-1}(U_E^{(i)}) \).

Let \( l \in U_E^{(i)} \) and \( \tilde{L} = q_2^{-1}(l) \cong L \). By the previous assertion,
\[
\tilde{K}^\vee|\tilde{L} \cong \bigoplus_{j=1}^t \mathcal{O}_L(-a_j^{(i)}), \quad \tilde{Q}|\tilde{L} \cong \bigoplus_{j=1}^t \mathcal{O}_L(a_j^{(i)}),
\]
and by Lemma 5.2, we have
\[ \Omega_{F^{(i)} / F}^{\sim} \mathcal{L} = \mathcal{O}_L(1)^{\oplus d_{i+1} - d_{i-1} - 2}. \]

Thus
\[ H^0(\mathcal{L}, \Omega_{F^{(i)} / F}^{\sim} \mathcal{L}) = 0, \] if \( a_{i+1}^{(i)} < -1 \).

Then \( \Omega_{F^{(i)} / F} \otimes \mathcal{K} \otimes \mathcal{Q} \) has no sections over \( q^{-1}(U^{(i)}_E) \) and hence over \( X \).

The above theorem has far-reaching consequences. We give first a series of immediate deductions.

**Corollary 5.5.** Fix an integer \( i, 1 \leq i \leq s \). Let \( E \) be a uniform \( r \)-bundle with respect to \( F^{(i)} \) of type
\[ a_E^{(i)} = (a_1^{(i)}, \ldots, a_r^{(i)}), \quad a_1^{(i)} \geq \cdots \geq a_r^{(i)} \]
If for some \( t < r \),
\[ a_t^{(i)} - a_{t+1}^{(i)} \geq 1, \quad \text{and } F^{(i)} \text{ is in Case I} \]
\[ 2, \quad \text{and } F^{(i)} \text{ is in Case II}, \]
then we can write \( E \) as an extension of uniform (with respect to \( F^{(i)} \)) bundles of smaller rank.

**Proof.** By the above Theorem 5.3, there is a uniform bundle \( K \subseteq E \) of type \( a_K^{(i)} = (a_1^{(i)}, \ldots, a_r^{(i)}) \) with respect to \( F^{(i)} \). Then the quotient bundle \( Q = E/K \) is uniform of type \( (a_1^{(i)}, \ldots, a_r^{(i)}) \) with respect to \( F^{(i)} \). We have the following exact sequence
\[ 0 \to K \to E \to Q \to 0. \]

**Corollary 5.6.** Fix an integer \( i, 1 \leq i \leq s \). For an \( i \)-semistable (see Definition 1.3) \( r \)-bundle \( E \) over \( F \) of generic splitting type \( a_E^{(i)} = (a_1^{(i)}, \ldots, a_r^{(i)}), a_1^{(i)} \geq \cdots \geq a_r^{(i)} \) with respect to \( F^{(i)} \), we have
\[ a_j^{(i)} - a_{j+1}^{(i)} \leq 1 \quad \text{for all } j = 1, \ldots, r - 1. \]

In Case I, we even have that all the \( a_j^{(i)} \)'s must be equal \( (1 \leq j \leq r) \).

**Proof.** If for the fixed \( i \), \( E \) is of type \( a_E^{(i)} = (a_1^{(i)}, \ldots, a_r^{(i)}) \) with \( a_t^{(i)} - a_{t+1}^{(i)} \geq 2 \) for some \( t < r \), then by Theorem 5.3 we can find a reflexive sheaf \( K \subseteq E \) which is of the form
\[ K|L \cong \mathcal{O}_L(a_j^{(i)}) \]
over the line \( L \subseteq F \) given by \( l \in U^{(i)}_E \). Then, we have \( \mu^{(i)}(E) < \mu^{(i)}(K) \), hence \( E \) is not \( i \)-semistable.

If the \( i \)-th connected component of the manifold of lines \( F^{(i)} \) is in Case I and there is some \( t < r \) such that \( a_t^{(i)} \neq a_{t+1}^{(i)} \), then we could find a reflexive sheaf \( K \subseteq E \) such that \( \mu^{(i)}(E) < \mu^{(i)}(K) \), hence \( E \) is not \( i \)-semistable.

**Corollary 5.7.** If \( E \) is a strongly uniform \( i \)-semistable \( (1 \leq i \leq n - 1) r \)-bundle over the complete flag variety \( F \), then \( E \) splits as a direct sum of line bundles. In addition \( E|L \cong \mathcal{O}_L(a)^{br} \) for every line \( L \subseteq F \), where \( a \in \mathbb{Z} \).

**Proof.** By Corollary 5.6, we get \( E|L = \mathcal{O}_L(a)^{br} \) for every line \( L \) in \( F \). After tensoring with an appropriate line bundle, we can assume \( E|L = \mathcal{O}_L^a \) for every line \( L \). So \( E \) is trivial by Proposition 5.2.

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