New qualitative criteria for solutions of Volterra integro-differential equations

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ABSTRACT
In this paper, we consider certain non-linear scalar Volterra integro-differential equations and Volterra integro-differential systems of first order. We investigate the boundedness, stability, uniformly asymptotic stability, integrability and square integrability of solutions to the scalar equations and the system considered. The technique used to prove the results of the paper is based on the second method of Lyapunov. From the obtained results, we extend and improve some related results that can be found in the literature.

1. Introduction
The theory of linear and non-linear Volterra integro-differential equations and systems provides important mathematical models for many real-world phenomena in science and engineering. Therefore, it is very important during investigations which are related to sciences and engineering to have information about the qualitative properties of solutions of linear and non-linear Volterra integro-differential equations and systems without solving them. However, relatively few kinds of Volterra integro-differential equations and systems can be solved explicitly. Therefore, during scientific investigations, researchers need to find analytical methods which allow them to study the qualitative behaviour of solutions of linear and non-linear Volterra integro-differential equations and systems without solving them. The main theories, techniques and methods in the literature used to investigate the qualitative behaviour of paths of solutions of Volterra integro-differential equations and systems, without needing to find their analytical solutions, include the second method of Lyapunov, continuation methods, coincidence degree theory, perturbation theory, fixed point method or theory, iterative techniques and the variation of constants formula.

Furthermore, investigations on stability, boundedness, asymptotic stability, uniform asymptotic stability, integrability and square integrability of solutions, the existence of periodic solutions, etc., of linear and non-linear Volterra integro-differential equations and systems have great importance in many scientific fields such as atomic energy, biology, bi-dimensional gravity chemistry, control theory, differential geometry, economy, engineering techniques, fluid mechanics, information theory, Jacobi fields, medicine, population dynamics, physics and many others (e.g. Corduneanu, 1977; Staffans, 1988; Lakshmikantham and Rama Mohana Rao, 1995; Burton, 2005; Wazwaz, 2011). Through these sources, readers can find a lot of detailed information on the theory and applications of integral and integro-differential equations in some of these scientific fields.

For more information on the theory and applications of linear and non-linear scalar Volterra integro-differential equations and Volterra integro-differential systems, the results obtained in the literature on the various qualitative behaviours of solutions of several kinds of these integro-differential equations and systems, the applications of the second method of Lyapunov, perturbation theory, fixed point methods, iterative techniques, the variation of constants formula and other methods to verify the results that can be found in the literature, we refer readers to the books by Corduneanu (1977), Burton (2005),...
Lakshmikantham and Rama Mohana Rao (1995) and Wazwaz (2011), and the paper by Staffans (1988).

The early results in the relevant literature by the second method of Lyapunov, perturbation theory or the variation of constants formula on the qualitative properties of solutions of equations such as boundedness, stability, asymptotic stability, uniform asymptotic stability, exponential asymptotic stability, integrability, square integrability, exponential decay of solutions of linear and non-linear Volterra integro-differential equations and systems without delay are presented in the papers by Grossman and Miller (1970), Miller (1971), Grimmer and Seifert (1975), Burton and Mahfoud (1983), Mahfoud (1984, 1985, 1987), Rama Mohana Rao and Srînivas (1985), Lakshmikantham and Rama Mohan Rao (1987), Rama Mohana Rao and Raghavendra (1987), Engler (1988), Hara et al. (1989 1990), Murakami (1991), Wang (1993) and Furumochi and Matsouka (1999), and the references therein.

Finally, some recent works can be summarized as follows: uniform asymptotic stability in linear Volterra integro-differential equations (Eloe et al., 2000), stability properties and boundedness of solutions of linear and non-linear Volterra integro-differential equations in Banach space (Martinez, 2002; Islam and Al-Eid, 2004; Hino and Murakami, 2005; Jin and Luo, 2009), uniform asymptotic stability of solutions of linear integro-differential Volterra equations (Funakubo et al., 2006), positive periodic solutions of perturbed linear and non-linear Volterra integro-differential equations (Eloe et al., 2000), stability properties and boundedness of solutions of equations such as boundedness, stability, asymptotic stability, uniform asymptotic stability, exponential asymptotic stability, integrability, square integrability, exponential decay of solutions of linear and non-linear Volterra integro-differential equations (Chang and Wang, 2011), asymptotic behaviour of solutions of integro-differential equations (Becker, 2009), stability of solutions of perturbed n-dimensional Volterra integro-differential equations (Chang and Wang, 2011), inequalities of solutions, exponential stability and instability of solutions in Volterra integro-differential equations (Talpalaru, 2001; Vanualailai, 2002; Vanualailai and Nakagiri, 2003; Raffoul, 2007; Graef and Tunç, 2015; Graef et al., 2016; Tunç, 2016a, 2016b, 2016c, 2017a, 2017b, 2017c; Tunç and Mohammed, 2017a, 2017b, 2017c; Tunç and Tunç, 2018a, 2018b; and many related papers and books in the references of these sources).

For the reasons mentioned above, the qualitative behaviour of Volterra integro-differential equations and systems require further investigation.

We now summarize some works which are will be useful for readers of this paper.

Burton and Mahfoud (1983) considered the following linear and homogeneous scalar Volterra integro-differential equations:

\[ \frac{dx}{dt} = Ax(t) + \int_{0}^{t} C(t-s)x(s)ds \] (1)

and

\[ \frac{dx}{dt} = A(t)x + \int_{0}^{t} C(t,s)x(s)ds \] (2)

respectively.

Burton and Mahfoud (1983) investigated various kinds of stability and some relations between them, and they also proved their results by means of suitable Lyapunov functionals.

Burton and Mahfoud (1983) also proved the following three theorems for Equations (1) and (2), respectively.

**Theorem A** (Burton and Mahfoud, 1983). Suppose \( A<0, C(t)>0 \) and \( A + \int_{0}^{\infty} |C(t)|dt \neq 0 \). The following statements are equivalent:

i. All solutions of Equation (1) tend to zero.

ii. \( A + \int_{0}^{\infty} C(t)dt < 0 \).

iii. Each solution of Equation (1) is in \( L^1[0, \infty) \).

iv. The zero solution of Equation (1) is uniformly asymptotically stable.

v. The zero solution of Equation (1) is uniformly asymptotically stable.

**Theorem B** (Burton and Mahfoud, 1983). If \( A + \int_{0}^{\infty} |C(t)|dt < 0 \), then the zero solution of Equation (1) is uniformly asymptotically stable.

**Theorem C** (Burton and Mahfoud, 1983). Suppose there exists a constant \( \alpha > 0 \) such that

\[ -2|A(t)| + \int_{0}^{t} |C(t,s)|ds + \int_{t}^{\infty} |C(u,t)|du \leq -\alpha. \]

Then Equation (4) is stable if and only if \( A(t)<0 \).

In this paper, motivated by Volterra integro-differential Equation (1), at the first stage, we consider a non-linear Volterra integro-differential equation of the form

\[ x' = - p(t)x + \int_{0}^{t} C(t-s)h(x(s))ds + q(t,x) \] (3)
where $t \in I, I = [0, \infty), x \in \mathbb{R}, \ p : \mathbb{R} \rightarrow \mathbb{R}^+, \mathbb{R}^+ = (0, \infty), h : \mathbb{R} \rightarrow \mathbb{R}$ with $h(0) = 0, q : I \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions for the arguments displayed explicitly, and $C(t-s)$ is a continuous function for $0 \leq s < t < \infty$.

In the next stage, in the same way, motivated by linear Volterra integro-differential Equation (2), we consider the following non-linear Volterra integro-differential equation of the form

$$x' = -p(t)r(x) + \int_0^t C(t,s)\psi(x(s))ds + q(t,x)$$  \hspace{1cm} (4)

where $t \in I, I = [0, \infty), x \in \mathbb{R}, \ p : \mathbb{R} \rightarrow \mathbb{R}^+, \mathbb{R}^+ = (0, \infty), \ r : \mathbb{R} \rightarrow \mathbb{R}$ with $r(0) = 0, \psi : \mathbb{R} \rightarrow \mathbb{R}$ with $\psi(0) = 0, q : I \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions for the arguments shown explicitly, such that the function $r$ is differentiable, and $C$ is a continuous function for $0 \leq s < t < \infty$.

Let us define a function $r_1$ by:

$$r_1(x) = \begin{cases} \frac{r(x)}{x}, & x \neq 0 \\ r'(0), & x = 0 \end{cases}$$

Then, in view of the non-linear Volterra integro-differential Equation (4) and the former expression, we can clearly write that:

$$x' = -p(t)r_1(x)x + \int_0^t C(t,s)\psi(x(s))ds + q(t,x)$$

Burton and Mahfoud (1983) extended their results from the scalar case to the $n$-dimensional linear Volterra integro-differential system:

$$x'(t) = Ax(t) + \int_0^t C(t,s)x(s)ds$$  \hspace{1cm} (5)

Burton and Mahfoud (1983) established necessary and sufficient conditions which guarantee the stability of the zero solution of the Volterra integro-differential system (5).

In this paper, in the third stage, motivated by the Volterra integro-differential system (5), we consider the Volterra integro-differential system of the form:

$$x' = -A(t)x + \int_0^t C(t,s)H(x(s))ds + D(t)$$  \hspace{1cm} (6)

where $t \geq 0, x \in \mathbb{R}^n, A$ is an $n \times n$-continuous matrix function for $0 \leq t < \infty$, $C$ is an $n \times n$-continuous matrix function for $0 \leq s < t < \infty$, $D$ is a continuous $n$-dimensional function for $t \in I$, and $H, H(0) = 0$, is a continuous $n$-dimensional function for $x \in \mathbb{R}^n$.

Finally, we discuss the uniform asymptotic stability of the zero solution and square integrability of solutions of the Volterra integro-differential system (6).

In brief, to summarize the above information, it follows that, motivated by the papers, books and monographs mentioned above, and by the paper of Burton and Mahfoud (1983) in particular, the aim of this paper is to prove new five theorems on the stability, boundedness, uniform asymptotic stability and integrability of solutions of scalar Volterra integro-differential Equations (3) and (4) and uniform asymptotic stability of the zero solution and square integrability of solutions of the Volterra integro-differential system (6).

When we compare the results of Burton and Mahfoud (1983) and the results of this paper, it can be clearly seen that the equations and results of this paper improve and extend the equations and results of Burton and Mahfoud (1983), and they make a contribution to the related literature. In addition, when we look at the literature mentioned above, it can be seen that in the most of the papers, sufficient conditions are obtained for stability, boundedness, uniform asymptotic stability and integrability of solutions, instead of necessary and sufficient conditions. In this paper, for a result to be given, necessary and sufficient conditions on the stability of the solutions are given. Furthermore, the integrability of solutions was not discussed by Burton and Mahfoud (1983). In this paper, the integrability and square integrability of the solutions are also studied. This case makes a contribution to the results of Burton and Mahfoud's (1983) Theorems A, B and C. Finally, the results and assumptions presented here are different from those that can be found in the literature. These are the contributions of this paper to the literature.

For convenience, in what follows, we write $x$ instead of $x(t)$.

2. Qualitative criteria for solutions

Let

$$q(t,x) \equiv 0$$

A. Hypotheses

We assume that the following hypotheses hold.

(A1) There exists a positive constant $\gamma$ such that $h(0) = 0, |h(x)| \leq \gamma|x|$ for $x \in \mathbb{R}, x \neq 0$

(A2) There exists a positive constant $\delta$ such that $p(t) - \gamma \int_0^t |C(u-t)|du > \delta$ for $t \in I$

(A3) $|q(t,x)| \leq r(t)|x|$ for $t \in I$ and $x \in \mathbb{R}$

where $r(t)$ is a non-negative and continuous function for all $t \in I$ and $r \in L^1[0, \infty)$.

Theorem 1. If hypotheses (A1) and (A2) hold, then all solutions of scalar Volterra integro-differential equation (3) are bounded and the zero solution of Volterra integro-differential equation (3) is uniformly asymptotically stable, and $x(t) \in L^1[0, \infty)$, where $L^1[0, \infty)$ is the space of all Lebesgue integrable functions on $[0, \infty)$. 
Proof. We define a Lyapunov function $W_0 = W_0(t, x(t))$ by

$$W_0 = |x| + \int_0^t |C(u - s)| du |h(x(s))| ds$$

Hence, it is clear that $W_0(t, 0) = 0$ and $W_0(t, x) \geq |x|$. Differentiating the Lyapunov function $W_0$ with respect to $t$ and using hypotheses (A1) and (A2), we obtain

$$W'_0 = \frac{x}{|x|} x' + \int_0^t |C(u - t)| du |h(x)|$$

$$- \int_0^t |C(t - s)||h(x(s))| ds$$

$$\leq -p(t)|x| + \int_0^t |C(t - s)||h(x(s))| ds$$

$$+ \int_0^t |C(u - t)| du |h(x)|$$

$$- \int_0^t |C(t - s)||h(x(s))| ds$$

$$= -p(t)|x| + \int_0^t |C(u - t)| du |h(x)|$$

$$\leq -[p(t) - \gamma \int_0^t |C(u - t)| du] |x|$$

$$\leq -\delta |x|$$

This last inequality completes the proof of Theorem 1. That is, the zero solution of Volterra integro-differential Equation (3) is uniformly asymptotic stable.

For the next step, in view of the above information, since $W'_0 \leq -\delta |x|$ we can write

$$W'_0(t, x(t)) \leq 0$$

Integrating the former inequality from zero $t_0$ to $t$, we can find

$$|x(t)| \leq W_0(t, x(t)) \leq W_0(t_0, x(t_0)) = K, K > 0, K \in \mathbb{R}$$

This last estimate implies the boundedness of all solutions of Volterra integro-differential Equation (3).

Finally, integrating the inequality

$$W'_0(t, x(t)) \leq -\delta |x|$$

from $t_0$ to $t$, we have

$$W_0(t, x(t)) - W_0(t_0, x(t_0)) \leq -\int_{t_0}^t |x(s)| ds$$

Hence,

$$\delta \int_{t_0}^t |x(s)| ds \leq W_0(t_0, x(t_0)) - W_0(t, x(t))$$

It is known that the Lyapunov function $W_0(t, x(t))$ is positive definite and a decreasing function. Then, it is obvious that

$$\delta \int_{t_0}^t |x(s)| ds \leq W_0(t_0, x(t_0))$$

Let

$$W_0(t_0, x(t_0)) = L, L > 0, L \in \mathbb{R}$$

Hence, it follows that

$$\int_{t_0}^t |x(s)| ds \leq \delta^{-1} L$$

Therefore, we can conclude that $x(t) \in L^1[0, \infty)$. That is, the solution $x(t)$ is integrable on the interval $[0, \infty)$.

For the next result, let $q(t, x) \neq 0$.

Theorem 2. If hypotheses (A1)–(A3) are satisfied, then all solutions of Volterra integro-differential equation (3) are bounded.

Proof. In the light of hypotheses (A1)–(A3), it is clear that the time derivative of Lyapunov function $W_0(t, x)$ implies

$$W'_0(t, x) \leq |q(t, x)|$$

$$\leq |r(t)||x|$$

$$\leq |r(t)|W_0(t, x)$$

By integrating the former inequality on the interval $[t_0, t]$ and using the Gronwall inequality, we have

$$W_0(t, x(t)) \leq W_0(t_0, x(t_0)) \exp \left[ \int_{t_0}^t |r(s)| ds \right]$$

Then, in view of the definition of the Lyapunov function $W_0(t, x(t))$ and the last inequality, we have

$$|x| \leq W_0(t, x(t)) \leq W_0(t_0, x(t_0)) \exp \left[ \int_{t_0}^t |r(s)| ds \right]$$

Since $r(t) \in L^1[0, \infty)$, then we can say

$$L_0 = W_0(t_0, x(t_0)) \exp \left[ \int_{t_0}^t |r(s)| ds \right], L_0 > 0, L_0 \in \mathbb{R}$$
Hence, we obtain $|x| \leq L_0$. Thus, we can conclude the desired result. That is, all solutions are bounded. This is the end of the proof of Theorem 2.

**B. Hypotheses**

We assume that the following hypotheses hold.

1. There exists a positive constant $\mu$ such that $\psi(0) = 0, |\psi(x)| \leq \mu|x|$ for $x \in \mathbb{R}, x \neq 0$.
2. $\int_0^\infty |C(t,s)| ds < \infty$ and $\int_0^\infty |C(u,t)| du < \infty$.
3. There exist positive constants $\mu$ and $\beta$ such that

$$Zp(t)r_1(x) - \int_0^t |C(t,s)| ds - \mu \int_0^\infty |C(u,t)| du \geq \beta x^2$$

for $t \in I$ and $x \in \mathbb{R}, x \neq 0$.

**Theorem 3.** If hypotheses (C1)–(C3) are satisfied, then all solutions of scalar Volterra integro-differential equation (4) are stable if and only if $p(t)r_1(x) > 0$.

**Proof.** ($\Rightarrow$): First, we suppose that $p(t)r_1(x) > 0$.

We define a Lyapunov function $W_1 = W_1(t,x(t))$ by

$$W_1 = x^2 + \int_0^\infty \int |C(u,s)| du \psi^2(x(s)) ds$$

If the hypotheses of Theorem 3 hold, then it is obvious that

$$W_1(t,0) = 0 \text{ and } W_1(t,x) \geq x^2.$$

Differentiating the Lyapunov function $W_1(t,x(t))$ with respect to $t$, subject to the hypotheses of Theorem 3 and some elementary inequalities, we can obtain

$$W'_1 = -2p(t)r_1(x)x + 2x \int_0^t C(t,s) \psi(x(s)) ds$$

$$+ \int_0^\infty |C(u,t)| du h^2(x) - \int_0^t |C(t,s)| \psi^2(x(s)) ds$$

$$\leq -2p(t)r_1(x)x^2 + \int_0^t |C(t,s)| [x^2(t) + \psi^2(x(s))] ds$$

$$+ \int_0^\infty |C(u,t)| du \psi^2(x) - \int_0^t |C(t,s)| \psi^2(x(s)) ds$$

$$\leq -2p(t)r_1(x) - \int_0^t |C(t,s)| ds - \mu^2 \int_0^\infty |C(u,t)| du x^2$$

($\Leftarrow$): Conversely, we now suppose that $p(t)r_1(x) < 0$.

We define a Lyapunov function $W_2 = W_2(t,x(t))$ by

$$W_2 = x^2 - \int_0^\infty \int |C(u,s)| du h^2(x(s)) ds$$

Differentiating the Lyapunov function $W_2$ with respect to $t$ and taking into consideration hypotheses (C1)–(C3), then an easy calculation implies that

$$W'_2 = -2p(t)r_1(x)x + 2x \int_0^t C(t,s) h(x(s)) ds$$

$$- \int_0^\infty |C(u,t)| du h^2(x) + \int_0^t |C(t,s)| h^2(x(s)) ds$$

$$\geq 2p(t)r_1(x)x^2 - \int_0^t |C(t,s)| [x^2(t) + h^2(x(s))] ds$$

$$- \int_0^\infty |C(u,t)| du h^2(x) + \int_0^t |C(t,s)| h^2(x(s)) ds$$

$$\geq \left[ -2p(t)r_1(x) - \int_0^t |C(t,s)| ds - \mu^2 \int_0^\infty |C(u,t)| du \right] x^2$$

$$\geq \beta x^2$$

That is, we have

$$W'_2(t,x(t)) \geq \beta x^2$$

Integrating the above estimate from zero to $t$, we obtain

$$x^2(t) \geq W_2(t,x(t)) \geq W_2(t_0,x(t_0)) + \beta \int_{t_0}^t x^2(s) ds$$

Given any $t_0 \geq 0$ and any $\delta > 0$, it can be found any continuous function $\phi : [0,t_0] \rightarrow \mathbb{R}$ with $|\phi(t)| < \delta$ and $W_2(t_0, \phi(.)) > 0$ such that if $x(t) = x(t_0, \phi)$ is a solution of Volterra integro-differential Equation (4), then we can obtain

$$x^2(t) \geq W_2(t,x(t)) \geq W_2(t_0, \phi(t_0)) + \beta \int_{t_0}^t x^2(s) ds$$

$$\geq W_2(t_0, \phi(t_0)) + \beta \int_{t_0}^t W_2(t_0, \phi(t_0)) ds$$

$$= W_2(t_0, \phi(t_0)) + \beta W_2(t_0, \phi(t_0))(t-t_0)$$

If $t \rightarrow \infty$, then $|x(t)| \rightarrow \infty$. This is a contradiction. This result completes the proof of Theorem 3.

**C. Hypotheses**

We suppose the following hypotheses hold:

1. $H(0) \neq 0, |H(x)| \leq \delta |x|$, where $\delta > 0$ and $\delta \in \mathbb{R}$.
2. Let $A(t)$ be an $n \times n$-real, symmetric and continuous matrix for all $t \in I$. There exists an $n \times n$-constant symmetric matrix $B$ such that

$$x^T[A(t)B + BA(t)]x \leq -\delta_0 |x|^2$$

for all $t \in I$, $x \in \mathbb{R}^n$, $x \neq 0$ where $\delta_0 > 0$ and $\delta_0 \in \mathbb{R}$.
\[ \int_0^\infty |C(t,s)|ds \text{ is defined and bounded, and} \int_0^\infty |C(u,t)|du < \infty. \]

\[ \delta_0 - |B| \int_0^\infty |C(t,s)|ds - |B| \int_0^\infty |C(u,t)|du \geq M \text{ where } M \in \mathbb{R}, M > 0. \]

\[ (H4) \int_0^\infty |D(s)|ds < \infty \text{ and } |D(t)| \to 0 \text{ as } t \to \infty. \]

\[
D(t) \equiv 0
\]

**Theorem 4.** If hypotheses (H1)–(H3) hold, then the zero solution of Volterra integro-differential system (6) is uniformly asymptotically stable. All solutions of this system are square integrable.

**Proof.** We define a Lyapunov function \( W_3 = W_3(t, x(t)) \) by

\[
W_3 = x^T(t)Bx(t) + |B| \int_0^\infty |C(u, s)|du |H(x(s))|^2 ds
\]

By a similar discussion as in Burton and Mahfoud (1983), the positive definite of the Lyapunov function \( W_3 \) can be easily shown. We omit the details of this discussion.

In the light of hypotheses (H1)–(H3), the time derivative of the Lyapunov function \( W_3 \) with respect to Volterra integro-differential system (6) leads to

\[
W'_3 = -x^TA(t)Bx + \left[ \int_0^\infty H^T(x(s))C^T(t, s)ds \right]Bx
\]

\[
+ x^T \left[ -A(t)x + \int_0^\infty C(t, s)H(x(s))ds \right]
\]

\[
+ |B| \int_0^\infty |C(u, t)|du |H(x)|^2 - |B| \int_0^\infty |C(t, s)| |H(x(s))|^2 ds
\]

\[
= -x^T(A(t)B + BA(t))x + 2x^T \int_0^\infty C(t, s)H(x(s))ds
\]

\[
+ |B| \int_0^\infty |C(u, t)|du |H(x)|^2 - |B| \int_0^\infty |C(t, s)| |H(x(s))|^2 ds
\]

\[
\leq - \delta_0 |x|^2 + 2|x| \int_0^\infty |C(t, s)| |H(x(s))|ds
\]

\[
+ |B| \int_0^\infty |C(u, t)|du |H(x)|^2 - |B| \int_0^\infty |C(t, s)| |H(x(s))|^2 ds
\]

\[
\leq - \delta_0 |x|^2 + |B| \int_0^\infty |C(t, s)| \left[ |x|^2 + |H(x(s))|^2 \right]ds
\]

\[
+ |B| \int_0^\infty |C(u, t)|du |H(x)|^2 - |B| \int_0^\infty |C(t, s)| |H(x(s))|^2 ds
\]

\[
\leq - M|x|^2 \leq 0
\]

When we look of the first and the last terms of the former inequalities, then

\[
W'_3(t, x(t)) \leq - M|x|^2 \leq 0
\]

Thus, in view of the above discussion, we can conclude that the zero solution of Volterra integro-differential system (6) is uniformly asymptotically stable.

In addition, integrating the last inequality, that is, \( W'_3(t, x(t)) \leq - M|x|^2 \) from \( t_1 \) to \( t \), we find

\[
W_3(t, x(t)) - W_3(t_1, x(t_1)) \leq - M \int_{t_1}^t |x|^2 ds
\]

From the above discussion, it can be seen that the function \( W_3(t, x(t)) \) is positive definite and a decreasing function. Therefore, we can choose

\[
W_3(t_1, x(t_1)) = N, N > 0, N \in \mathbb{R}
\]

Hence, it is clear that

\[
M \int_{t_1}^t |x|^2 ds \leq W_3(t_1, x(t_1)) - W_3(t, x(t)) \leq W_3(t_1, x(t_1))
\]

As a result of the above inequalities, it follows that

\[
\int_{t_1}^t |x|^2 ds \leq M^{-1}N
\]

Thus, we can conclude that the solution \( x(t) \) of Volterra integro-differential system (6) is square integrable, that is, \( x(t) \in L^2[0, \infty) \), where \( L^2[0, \infty) \) is the space of all Lebesgue square-integrable functions on \( [0, \infty) \).

This is the end of the proof of Theorem 4.

Finally, let

\[
D(t) \neq 0
\]

**Theorem 5.** Let hypotheses (H1)–(H4) hold. Then, all solutions of Volterra integro-differential equation (6) are bounded.

**Proof.** We benefit from the Lyapunov function \( W_3(\cdot) = W_3(t, x(t)) \). In light of the assumptions of (H1)–(H4), we can find

\[
W'(\cdot) \leq -M|x|^2 + 2|D(t)||B|x|^2
\]
The rest of the proof is similar to the proof of Theorem 2. We omit the rest of the proof.

3. Conclusion

In this paper, we have presented sufficient conditions for stability, boundedness, uniformly asymptotic stability, integrability and square integrability of solutions of a few scalar non-linear Volterra integro-differential equations and a Volterra integro-differential system. The technique of the proof is based on the second method of Lyapunov. The obtained results include and improve upon some results that can be found in the literature (Burton and Mahfoud, 1983, Theorems A, B and C).

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