Perturbed spherical collapse of matter: exact analytical description

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13 August 2018

ABSTRACT
We investigate generic perturbed scenarios for the spherical collapse of a top-hat matter over-density. For this we enable a fluid description in a Lagrangian-coordinates approach, derive simple all-order recursion relations for the time-Taylor coefficients of the Lagrangian displacement field and prove the convergence of its series representation until the time of collapse (“shell-crossing”). As to the nature of the perturbed initial conditions, its amplitude must be sufficiently small in comparison to the one of the pure spherical collapse, but besides that they are kept fairly general in our description: they could be of geometrical origin (“ellipsoidal collapse”), or due to the presence of another clustering fluid component (e.g., massive neutrinos or clustering dark energy). Then, we derive an exact and simple analytical expression for the time of perturbed matter collapse, which is shown to happen generically earlier than in the spherical case. Although the linear matter density receives an additive correction proportional to the perturbation, we show, due to the decreased collapse time, that the threshold of the linear matter density at collapse is reduced – irrespective of the sign of the perturbation. This could have important consequences for models that predict the abundance of biased tracers of matter, such as halos and galaxies.

Key words: cosmology: theory – dark matter – large-scale structure of Universe

1 INTRODUCTION
The spherical collapse model (SCM) is central to many aspects of cosmology. Although being a simplified model for cosmological matter collapse, its use can be justified by statistical arguments from peaks theory of Bardeen et al. (1986), which predict that high-density peaks in the Universe tend to be more spherically symmetric than low-density peaks.

Within the context of General Relativity, the non-linear solution of the spherical collapse is a spherically symmetric space-time of a collapsed region occupied by homogeneous matter, given by the Friedmann–Lemaître–Robertson–Walker (FLRW) solution with positive curvature. Furthermore, it is well-known that there exists an exact parametric solution for the spherical collapse, at least for simplified cosmologies such as for an Einstein–de Sitter universe (see e.g. Peebles 1967; Tomita 1969; Gunn & Gott 1972; Bertschinger & Jain 1994).

Important applications of the SCM include the analytical prediction of the shape and position of the baryon acoustic oscillation feature, as well as galaxies and their host halos (see e.g. Desjacques et al. 2010; Paranjape et al. 2013). Another application of the SCM concerns the determination of the abundance of primordial black holes (see e.g. Carr et al. 2016). In many of these applications – be it relevant for galaxy, halo or primordial black hole formation, the SCM is employed to predict the threshold of linear density fluctuation (at collapse). This density threshold is then used as the input to determine the abundance, mass or shape of a given object. The SCM is also highly relevant for extracting accurately the cosmological parameters from the probability distribution functions of spherically averaged densities and velocity divergences, as shown by Uhlemann et al. (2017). The standard framework of the SCM can also be extended/generalized to incorporate the effects of massive neutrino clustering, see Ichiki & Takada (2012); LoVerde (2014).

Apart from demonstrative examples as given above, the SCM is not a suitable framework for incorporating a host of other physical effects. In particular, it is inappropriate for realistic matter collapse which is well-known to be not exactly spherical. For this reason, perhaps one of the most natural advancements to the SCM is the ellipsoidal collapse model, with pioneering works, amongst others, by Icke (1973); Bond & Myers (1996). Based on these works, Sheth et al. (2001) obtained a fitting formula for ellipsoidal collapse which can be matched to results from numerical simulations.

For many cosmological scenarios, the collapse can not be modelled by the parametric solution to the dynamics of a closed FLRW universe. Instead one is led to investigate the collapse directly at the level of the equations of motion. This is for example necessary when incorporating the effect of shear or rotation on the matter collapse, as has been done by Reischke et al. (2018). Another interesting example is when investigating the collapse within...
the framework of general modifications of the gravitational theory, such as the class of \( f(R) \) theories (Starobinsky 2007; Hu & Sawicki 2007). There, the appearance of non-local terms in the action of gravity violates the validity of Birkhoff’s theorem – a central requirement to model the matter collapse in terms of a simplistic FLRW model. See e.g. Borisov et al. (2012); Lombriser et al. (2013); Kopp et al. (2013) for related semi-analytic works.

In the present paper we pursue a somewhat similar, yet fully analytic approach, and solve the matter collapse by virtue of the cosmological fluid equations. Specifically, we work in a Lagrangian-coordinates formulation for the Euler–Poisson equations, in a Cartesian coordinate system. For the pure spherical case, we use initial conditions that are, firstly and to the zeroth order, identical with the ones that resemble the classical spherical collapse of matter. This way, one obtains an infinity Taylor series for the Lagrangian displacement field, whose low-order results have been derived for the first time by Munshi et al. (1994). In this paper, we formally go to all orders in the Taylor series, which allows us to determine the radius of convergence of the Taylor series.

Then, by going to the leading order in a small expansion parameter \( \varepsilon > 0 \), we switch on the arbitrary perturbation in the initial conditions, and determine the recursion relations for the Taylor coefficients of the perturbed displacement field. As a consequence, we obtain the full particle trajectory which is shown to be an analytically exact solution for the fluid equations until the instance of collapse. This paper is organized as follows. In the subsequent section we briefly review the cosmological fluid equations, first in Eulerian and then in Lagrangian coordinates. In section 3 we formulate the perturbation problem, the appropriate initial conditions, and provide the solution Ansatz formulated in Lagrangian space. Then, in sections 4 and 5 we solve the problem respectively to zeroth or first order in \( \varepsilon \). The busy reader who is interested in the calculation of the time of perturbed collapse and the linear density contrast (see e.g. Peebles 1980). This second-order differential equation has two power-law solutions for the density, one is decaying as \( \alpha^{-3/2} \) and the other is growing linearly in \( \alpha \). From these observations, it becomes evident that the following boundary conditions select the growing-mode and curl-free solution of the fluid equations (Brenier et al. 2003),

\[
\delta^{\text{(init)}} = 0, \quad q^{\text{(init)}} = -\nabla \phi^{\text{(init)}},
\]

where “\text{(init)}” refers to the evaluation at initial time \( \alpha = 0 \). Thanks to these \textit{slaving conditions}, the solutions of the fluid equations are, for sufficiently early times, time-analytic and thus devoid of any catastrophic behaviour. Real singularities nevertheless appear at the instant of shell-crossing.

### 2.2 Basic equations in Lagrangian coordinates

Let us now turn to the Lagrangian formulation of the fluid equations (1). We denote the Lagrangian coordinates by \( q \), with components \( q_i \) (\( i=1,2,3 \)). A partial derivative with respect to \( q_i \) acting on a given function \( f \) is denoted by \( f_i \). Summation over repeated indices is implied. Let \( q \rightarrow \hat{x}(q, \alpha) \) be the Lagrangian map from the initial \((\alpha = 0)\) position \( q \) to the Eulerian position \( x \) at time \( \alpha \). The map satisfies \( \psi(x(q, \alpha), \alpha) = \hat{x}(q, \alpha) \), where the overdot is the Lagrangian time derivative (sometimes also denoted with \( \dot{\delta} \)). At initial time \((\alpha = 0)\), the velocity is

\[
\psi^{\text{(init)}}(q) = \psi(x(q, 0), 0),
\]

which agrees with the initial Eulerian velocity. Mass conservation is, until the first shell-crossing, given by

\[
\delta = 1/J - 1,
\]

where \( J = \text{det}(\hat{x}_{ij}) \) is the Jacobian, which is the determinant of the Jacobian matrix \( \hat{x}_{ij} \). With these definitions, the fluid equations can be written in Lagrangian coordinates in compact form,

\[
\varepsilon_{i(kl} \hat{x}_{jmn} x_{km} x_{ln} \delta_{\alpha} x_{i,j} = 3 (J - 1),
\]

(5a)

\[
\varepsilon_{ijk} \hat{x}_{i,j} x_{i,k} = 0,
\]

(5b)

where we have defined the operator \( \delta_{\alpha} \equiv \alpha^2 (\delta_{ij})^2 + (3a/2)\delta_{ik} \), and \( \varepsilon_{ijk} \) is the fundamental antisymmetric tensor. Equation (5a) is a scalar equation that is obtained by combining equations (1a) and (1c) in Lagrangian coordinates, as well as taking mass conservation (4) into account. Equations (5b) are of vectorial character and state the conservation of the zero-vorticity (which holds until shell-crossing) written in Lagrangian coordinates. Calculational details about equations (5) are given by Ehlers & Buchert (1997) and Zheglovsky & Frisch (2014). General derivations of the Lagrangian evolution equations are given by Buchert & Goetz (1987), Rampf & Buchert (2012) and references therein.

### 3 PROBLEM AND SOLUTION ANSÄTZ

Matter collapse that can be exactly reduced to a spherical problem is degenerate; given the nature of the initial (Gaussian) random density fluctuations, the probability of finding such objects in the LSS is zero. Furthermore, even just a small random perturbation that is
added to, say, a spherical overdensity is crucial as it decides shape and orientation of the collapsed object. Initial conditions (ICs) that resemble such a problem are introduced in the following section, and an appropriate solution Ansatz is given in section 3.2.

3.1 Initial conditions

In the present paper we analyse three-dimensional matter collapse with initial conditions (ICs) that are close to isotropic, i.e., the ICs amount, to the zeroth order in a perturbation parameter \( \epsilon \), to a spherical problem, and, to first order in \( \epsilon \), is otherwise prescribed by some general perturbation. This perturbation could be of geometric nature, i.e., a spatial deformation from sphericity that arises from some perturbation in the seeds. The perturbation could also originate from external sources, such as from the clustering of massive neutrinos or clustering dark energy. We note however that the coupling of other species to matter generally affects the form of the Friedmann and fluid equations. In the following we will not make any assumption about the form of this perturbation, and thus leave it as a free parameter.

For the given scenario, perturbed initial conditions can be formulated in terms of a superposition of two contributions to the initial gravitational potential, the first being the spherical (‘top-hat’) part and the second one a small perturbation. Specifically we write for the double gradients of the initial gravitational potential

\[
\psi_{ij}^{(\text{init})} = \delta_{ij} \frac{A}{3} + \epsilon \phi_{ij}^{(\text{init})},
\]

where \( A \) is a positive constant function, \( \epsilon \) a small perturbation parameter and \( \phi_{ij}^{(\text{init})} \) an arbitrary function of all three space variables.

Taking into account the slaving conditions (2), we have the following relation between the gradients of the initial velocity and gravitational potential,

\[
v_{i,j}^{(\text{init})} = -\psi_{i,j}^{(\text{init})} = -\delta_{ij} \frac{A}{3} - \epsilon \phi_{i,j}^{(\text{init})}.
\]

3.2 The Lagrangian perturbation Ansatz

We employ a perturbation method in which the solutions to the Lagrangian equations are expanded in powers of \( \epsilon \). Specifically, we impose for the particle trajectories

\[
x(q, a) = q + \xi^{(0)}(q, a) + \epsilon \xi^{(1)}(q, a) + \epsilon^2 \xi^{(2)}(q, a) + \ldots,
\]

where \( \xi^{(n)} \) is the \( n \)th coefficient in the \( \epsilon \)-expansion for the displacement \( x - q \). Evidently, for the zeroth order in \( \epsilon \) expansion, we have the spherical problem; we call this the unperturbed problem. We impose for the gradients of the unperturbed displacement

\[
\xi_{i,j}^{(0)} = \delta_{ij} S,
\]

where \( S \) is a time-dependent unknown. In the present paper we only expand to first order in \( \epsilon \), henceforth we write \( \xi^{(1)}(q, a) = \xi(q, a) \) and neglect all higher-order terms. We thus impose for the Jacobian matrix

\[
x_{i,j} = \delta_{ij}(1 + S) + \epsilon \xi_{i,j} + O(\epsilon^2),
\]

and for its determinant, the Jacobian of the perturbed problem,

\[
J_\epsilon = J^{(0)} + \epsilon (1 + S)^2 \xi_{i,j} + O(\epsilon^2),
\]

where \( J^{(0)} = (1 + S)^3 \) is the unperturbed Jacobian.

As evident from the Lagrangian mass conservation (4), the density blows up when the Jacobian vanishes. Thus, the vanishing of the Jacobian can be used as an indicator that matter has collapsed to high-density objects. In more mathematical terms, the first vanishing of the Jacobian marks the instance of first shell-crossing, i.e., the time where particle trajectories begin to intersect and the single-stream description breaks down.

In the following section we solve for the particle trajectories to zeroth order in \( \epsilon \). Then, in section 5, we include the \( \epsilon \) perturbation in the problem and show that the particle trajectories are time analytic and thus representable by a convergent time-Taylor series until the final stage of the non-linear collapse.

4 THE UNPERTURBED PROBLEM

4.1 Spherical collapse: Equations and solutions to order \( \epsilon^0 \)

Exact solutions to the spherical problem are well-known in the literature, but are usually investigated by considering a Friedmann model. Here we approach the problem differently, namely by solving explicitly the fluid equations. We note that a very similar Lagrangian approach to ours, however restricted to low-order approximations, has been applied by Munshi et al. (1994) and Yoshisato et al. (1998). Here we go, formally, to all orders, which allows us to determine the radius of convergence of the perturbation series.

Plugging the Ansatz to order \( \epsilon^0 \) in equations (5b) gives a trivial identity, whereas for equation (5a) we obtain

\[
(1 + S)^2 \delta_{ij} S = \frac{3}{2} \left[ S + S^2 + \frac{3 S^3}{3} \right].
\]

To solve this equation, we seek a nested Ansatz for the unperturbed displacement in terms of a power series in the cosmic scale factor (our time variable),

\[
S(a) = -\sum_{n=1}^{\infty} \sigma_n (Aa)^n,
\]

where \( \sigma_n \) are numerical coefficients to be determined. Note the minus sign and the powers of \( A \) in our Ansatz, due to convergence.

Using this Ansatz and applying the slaving conditions (2), we obtain directly the first-order solution \( \sigma_1 = 1/3 \). To solve for the higher-order Taylor coefficients, we plug (13) into (12); identifying the powers in \( Aa \) we then get \( (n > 1) \)

\[
\left(n + \frac{3}{2}\right)(n - 1) \sigma_n = \sum_{p+q=n} \left(2q^2 + q - \frac{3}{2}\right) \sigma_p \sigma_q
\]

\[-\sum_{k+l+m=n} \left(m^2 + \frac{m}{2} - \frac{1}{2}\right) \sigma_k \sigma_l \sigma_m.
\]

After symmetrizing the terms on the r.h.s. and division of the coefficients in front of the l.h.s., we obtain the recursion relations for the coefficients of the unperturbed displacement \( (n > 1) \)

\[
\sigma_n = \frac{1}{3} \delta_{n,1} + \sum_{q<n} \frac{q^2 + (n - q)^2 - (3 - n)/2}{(n + 3/2)(n - 1)} \sigma_q \sigma_{n-q}
\]

\[-\sum_{k+l+m=n} \frac{k^2 + l^2 + m^2 - (3 - n)/2}{3(n + 3/2)(n - 1)} \sigma_k \sigma_l \sigma_m.
\]

The first Taylor coefficients are

\[
\sigma_1 = \frac{1}{3}, \quad \sigma_2 = \frac{1}{21}, \quad \sigma_3 = \frac{23}{1701}, \quad \sigma_4 = \frac{1894}{392931}.
\]

The first three coefficients \( \sigma_1 - \sigma_3 \) can be found in Munshi et al. (1994). Sahni & Shandarin (1996) derived the Taylor coefficients up to order \( n = 5 \), however for a void spherical top-hat and thus...
some of their coefficients have different signs. Wagner et al. (2015) determined $\sigma_1 - \sigma_2$, within the separate universe approach (see their eq. B.15); the match of these coefficients could reveal an interesting relationship between the separate universe and Lagrangian-coordinates approaches, and should be investigated further.

To our knowledge, the recursion relation (15) has not been reported in the literature. None the less, as we show briefly now, this result can be set in direct context to standard calculations of the density in the spherical collapse model. Indeed, by plugging our results for the displacement into the definition of the density at the Lagrangian position, $\delta = 1/J^{(0)} - 1$, with $J^{(0)} = (1 + S)^3$, and Taylor expanding, we obtain

\[
\delta = \sum_{n=1}^{\infty} \frac{\nu_n}{n!} (Aa)^n, 
\]

with the first non-vanishing coefficients

\[
\nu_1 = 1, \quad \nu_2 = \frac{34}{21}, \quad \nu_3 = \frac{682}{189}, \quad \nu_4 = \frac{446440}{43659},
\]

These coefficients agree with the ones obtained from Bernardou (1992), however we emphasize that our approach is different to theirs; obtaining these coefficients in the present paper is little more than a check that our methodology can be directly connected to other works in the literature.

### 4.2 Spherical case: Convergence until collapse

After having found recursive solutions for the Taylor series of the unperturbed displacement, it is natural to ask the question: is

\[
S(a) = -\sum_{n=1}^{\infty} \sigma_n (Aa)^n
\]

a convergent series and thus defines an exact solution until shell-crossing?

To address this question, we perform the ratio test which states that the radius of convergence $R$ of the series is given by the relation

\[
\frac{1}{R} = \lim_{n \to \infty} \frac{\sigma_n}{\sigma_{n-1}},
\]

where $\sigma_n$ are the Taylor coefficients of the unperturbed displacement field (13). We determine the radius of convergence of the time-Taylor series by drawing the Domb–Sykes plot (Domb & Sykes 1957), shown in fig. 1 (blue solid line). To obtain this plot we have generated Taylor coefficients for the displacement (and the density; red dotted line) up to order $n = 800$; the output, though quite lengthy at large Taylor orders, can be easily obtained by employing standard computer algebra programs and by the use of our recursion relations. As evident from fig. 1, for sufficiently large Taylor orders ($n > 10$), both ratios of Taylor coefficients settle into a linear behaviour. By linearly extrapolating the ratios, shown as dashed lines, we obtain the value 0.593 at the intersection of $1/n = 0$, from which we conclude that the radius of convergence is, for both the displacement and density, given by $Aa^{(0)}_n = 1/0.593 = 1.686$.

The radius of convergence of a series is determined by the nearest singularity in the complex disc of its argument. The singularity is, generally, at a complex value, and in the present case, at least for the displacement, it is a priori not ruled if the first singularity occurs for real times. To clarify the nature of the singularity at the disc of convergence, we perform numerically a Cauchy convergence test for the series coefficients $S_n = -\sigma_n(Aa)^n$ of the displacement ($S = \sum_n S_n$), at the critical vicinity of $Aa^{(0)} = 1.686$.

Evaluating the Cauchy test to orders as large as $n = 1000$, we find that indeed the series converges absolutely until the real time value of $Aa^{(0)} = 1.686$. As a direct consequence, we can solve for the displacement from initial time $a = 0$ until $a = Aa^{(0)} = 1.686/A$, where convergence is guaranteed. Evaluating the unperturbed Jacobian $J^{(0)} = (1 + S)^3$ at this maximal time, one finds

\[
J^{(0)}(Aa^{(0)}) = 0,
\]

and thus, as expected, $a^{(0)}$ marks the time of first shell-crossing / matter collapse in the unperturbed case. Therefore, in Lagrangian coordinates, we can solve for the particle trajectories all the way to the collapse, and for that only a single-time step is required.

Also shown in fig. 1 is the ratio of Taylor coefficients of the density contrast

\[
\delta = \sum_{n=1}^{\infty} \frac{\nu_n}{n!} (Aa)^n = \sum_{n=1}^{\infty} \delta_n (Aa)^n.
\]

The $\nu_n$ coefficients are determined by the recursion relations (Bernardeau et al. 2002)

\[
\nu_n = \sum_{m=1}^{n-1} \frac{n}{m} \mu_m \frac{(2n+1)! \nu_{n-m} + 2 \mu_{n-m}/3}{(2n+3)(n-1)},
\]

\[
\mu_n = \sum_{m=1}^{n-1} \frac{n}{m} \mu_m \frac{3 \nu_{n-m} + 2 \nu_{n-m}/3}{(2n+3)(n-1)},
\]

themselves being the result of a spherical average of the perturbative Eulerian density and velocity divergence, respectively. Similarly as above, we draw the Domb–Sykes plot for the Taylor series of the density. As evident from the red [dotted] line fig. 1, the radius of convergence for the density is identical with the displacement. This coincidence is because of the vanishing of the convective term in the Euler equation (second term on the l.h.s. in (1a)),

![Figure 1. Domb-Sykes plot for the unperturbed collapse. Shown are ratios of the Taylor coefficients $\delta_n = \nu_n/n!$ for the density contrast (red dotted line) and of the displacement coefficients $\sigma_n$ (blue solid line). Both ratios of subsequent coefficients approach 0.593 for $n \to \infty$ (obtained by a linear fit for $10 \leq n < 800$; dashed lines) and thus mark the radius of convergence of the time series at $|Aa^{(0)}| = 1.686$ in the complex time disc. Formally evaluating the unperturbed Jacobian, $J^{(0)}$, at the real time value of $Aa^{(0)} = 1.686$, it is seen that the radius of convergence is limited by the instance of first shell-crossing, where $J^{(0)}$ vanishes and the density becomes infinite.](image-url)
due to spherical symmetry. For non-isotropical ICs, the convective term does generally not vanish, and as a consequence we expect the radius of convergence of the Eulerian density to be much smaller than the radius of convergence of the displacement (cf. Rampf et al. 2015).

There is an even more striking argument why the Lagrangian-coordinates approach is superior compared to the Eulerian one, even for the simplistic case of perfect sphericity. At the instance of shell-crossing, the density is indeed a real singularity; approaching it by Eulerian means, and in a controlled way is impossible. Even slightly before the time of shell-crossing, when the density is not yet infinity but very large, very high orders in the Taylor series of the density are required to resolve the matter collapse in its final stages. In the Lagrangian approach, by contrast, the displacement is the only dynamical variable, and behaves fairly smoothly at shell-crossing [cf. upper panel of fig. 4, showing $J^{(0)} = (1 + S)^3$].

4.3 Further results on the unperturbed problem

Taking the trace of the unperturbed Jacobian matrix $x_{i,\dot{i}} = \delta_{i\dot{i}} + O(\epsilon)$, and multiplying by the scale factor, we arrive at the divergence of the non-comoving particle trajectory.

\[ \nabla^L \cdot \mathbf{r}_{\ast,0}(\mathbf{q},a) = 3a(1 + S) , \]  

shown in fig. (2) for several levels of accuracies in the time-Taylor expansion. Evidently, low-order approximations for the trajectory perform poorly, none the less even the first-order solution, in Lagrangian space, predicts the existence of a turnaround and a collapse. This should be contrasted to Eulerian perturbation theory which, at first order, does not predict a collapse.

Going to higher orders in the time-Taylor coefficients, the trajectory converges quite quickly to a stable answer. More, in detail, beyond order $n \geq 100$ and for times $0 \leq Aa \leq 1.6$, the corrections to the exact solution of the trajectory are less than 0.07\%\text{w.r.t.}, with the largest deviation at the latest time. Higher orders are only required when evaluating the final stages of the collapse. Indeed, resolving this highly non-linear regime, we find that the time-Taylor series must be truncated up to order $n = 950$ to obtain better than 0.6\% precision for $1.6 \leq Aa \leq 1.685$.

We note that the time of unperturbed shell-crossing could also be obtained by numerically evaluating $J_{\ast,0}^{(0)}(a_{\ast,0}^{(0)}) = 0$ to a given order $n$, and the approximative results for $a_{\ast,0}^{(0)}$ are shown in fig. (3). The accuracy for obtaining $a_{\ast,0}^{(0)}$ numerically gets increasingly better at large Taylor orders, and we find that, at $n = 1000$, the time of unperturbed shell-crossing can be obtained to an accuracy of better than 0.05\% w.r.t. the exact prediction of the spherical collapse model, which is (see e.g. Peebles 1980)

\[ Aa_{\ast,0}^{(0)} = \frac{3}{5} \left( \frac{3\pi}{2} \right)^{2/3} \simeq 1.68647 . \]

In our framework, however, a much better approximation can be obtained by using the extrapolation method that leads to our fig. 1. Indeed, by drawing the Domb–Sykes plot for the Taylor coefficients to order $n = 800$, we obtain an approximation for $Aa_{\ast,0}^{(0)}$ which is two orders of magnitude better than the numerical method as employed in fig. 3.

Numerically evaluating for the time of ’turn around’, by contrast, delivers very accurate predictions already at fairly low Taylor orders. Specifically, the time $Aa_{\ast,0}$ of unperturbed turnaround is achieved when the first time derivative of the non-comoving trajectory (25) vanishes. The standard parametric solution in the spherical collapse model gives (see e.g. Bertschinger & Jain 1994)

\[ Aa_{\ast,0,scm} = \frac{3}{20} (67\pi)^{2/3} \simeq 1.06241 . \]

whereas numerically evaluating for $Aa_{\ast,0}$, with our methods and to fixed time-Taylor orders $n = 10, 20$ and 100, yields a precision of better than $10^{-4}$, $10^{-5}$ and $10^{-15}$, respectively.

Overall the agreement of our method with the standard results of spherical collapse is excellent.
5 THE PERTURBED PROBLEM

5.1 Quasi-spherical case: equations and solutions to order $\epsilon^1$

Collecting all terms $O(\epsilon)$, we obtain from equations (5a) and (5b) respectively

\[
(1 + S)^3 \dot{\rho}_{\xi,\xi} = \frac{3}{2} \xi_{,\xi} \left[ 1 + S + S^2 + \frac{S^3}{3} \right],
\]

\[
(1 + S) \xi_{,\xi} = \dot{S} \xi_{,\xi},
\]

where we remind the reader that the overdot stands for a time derivative w.r.t. the scale factor $a$. The last equation dictates that no transverse displacement is generated during the evolution, and thus, to order $\epsilon$, the perturbed displacement is purely potential. Therefore, the perturbed displacement is entirely described by its divergence part, which we define as

\[
\nabla \cdot \xi = Q.
\]

Furthermore, since the perturbed equation (28a) is autonomous in the space variables, and because the only spatial scale is given by the perturbed initial conditions, it follows that we can write $Q$ in separable form,

\[
Q(q, a) = -\chi(a) A^{-1} \Delta^{(\text{init})}(q).
\]

The latter space-dependent function is fully determined by the perturbed initial conditions (6), supplemented with the initial condition $\chi(a = 0) = A$, it is

\[
\Delta^{(\text{init})}(q) = \nabla^2 q^{(\text{init})}. \tag{31}
\]

Thus, the space dependence of the perturbed solution is already imprinted in the initial conditions of the perturbed problem, and we only need to solve for the time dependence, given by $\chi$ which is subject to the time differential equation

\[
(1 + S)^3 \dot{\rho}_{\xi,\xi} = \frac{3}{2} \chi \left[ 1 + S + S^2 + \frac{S^3}{3} \right]. \tag{32}
\]

This is our evolution equation for the perturbed problem that we solve by imposing the time-Taylor series Ansatz

\[
\chi = \sum_{n=1}^{\infty} \chi_n (aA)^n. \tag{33}
\]

The first-order solution, determined by the slaving condition, is simply $\chi_1 = 1$. To get the solutions for the time-Taylor coefficients for $n > 1$, we plug the Ansatz into the evolution equation (32). Matching the time-Taylor coefficients at fixed order, we get for $n > 1$

\[
\left( \frac{n}{2} \right) \left( n - 1 \right) \chi_n = \sum_{p+q=n} \left( q^2 + \frac{q}{2} - \frac{1}{2} \right) \sigma_p \chi_q - 3 \sum_{p+q+r=n} \left( r^2 + \frac{r}{2} - \frac{1}{2} \right) \sigma_p \sigma_q \chi_r + \sum_{p+q+r+s=n} \left( s^2 + \frac{s}{2} - \frac{1}{2} \right) \sigma_p \sigma_q \sigma_r \chi_s, \tag{34}
\]

which, after symmetrization, yields the recursion relations for $\chi_n$. Here we will not show the explicit recursion relations as derived from (34), mainly because the involved symmetrization of the terms on its r.h.s. becomes fairly cluttered. Instead and much simpler, we find that $\chi_n$ are entirely determined by the following recursion relation ($a > 1$)

\[
\chi_n = 3n \sigma_n, \tag{35}
\]

where the $\sigma_n$’s are given by their own recursion relation (15). We prove the validity of this trivial recursion relation in appendix A. Here we report, for future reference, the first Taylor coefficients for the perturbed displacement,

\[
\chi_1 = 1, \quad \chi_2 = \frac{2}{7}, \quad \chi_3 = \frac{23}{189}, \quad \chi_4 = \frac{7576}{130977}, \tag{36}
\]

which, to our knowledge, have not yet been reported in the literature. Furthermore, because the time-Taylor coefficients of the perturbed displacement are intrinsically related to the coefficients of the unperturbed displacement, it is trivial to determine the radius of convergence of the time-Taylor series $\gamma = \sum n \chi_n (aA)^n$. Indeed, performing the ratio test for its coefficients, we find

\[
\frac{1}{R} = \lim_{n \to \infty} \frac{\chi_n}{\chi_{n-1}} = \lim_{n \to \infty} \frac{3n \sigma_n}{3(n-1) \sigma_{n-1}} = \lim_{n \to \infty} \frac{\sigma_n}{\sigma_{n-1}}, \tag{37}
\]

and thus, the radius of convergence of the series representation of $\chi$ is identical with the one for $\sigma$, namely $R = A \sigma^{(0)}$. Note however, that in the perturbed case, the absolute value of that radius of convergence is not identical with the time or perturbed shell-crossing, the latter denoted with $a_*$. Rather, as we show in the following section, generically we have $a_* \lesssim a^{(0)}_*$ which allows for solving for the perturbed particle trajectory before and at the instance of perturbed shell-crossing.

It is also interesting to plug the r.h.s. of equation (35) into the Ansatz for $\chi$, which reveals a novel relationship between the perturbed and unperturbed displacement,

\[
\chi = \sum_{n=1}^{\infty} \chi_n (aA)^n = \sum_{n=1}^{\infty} n \sigma_n (aA)^n \equiv -3aS, \tag{38}
\]

with $S = \partial^2_{\xi} S = -\delta^2_{\xi} \sum n \sigma_n (aA)^n$. Summing up, from the above results we obtain, to order $\epsilon$, respectively the Jacobian matrix and the Jacobian

\[
\begin{align*}
J_{i,j} &= \delta_{ij} (1 + S) - \epsilon \frac{\chi}{A} \delta^3_{\xi} \partial^i \partial^j S^{(\text{init})}, \tag{39a} \\
J_* &= (1 + S)^3 - \epsilon (1 + S)^2 \frac{\chi}{A} \Delta^{(\text{init})}, \tag{39b}
\end{align*}
\]

with $S = \sum \sigma_n (aA)^n$ and $\chi = -3aS$, where the $\sigma_n$’s are given by eq. (15). We remark again, that $S$ is independent of the chosen initial conditions for $\Delta^{(\text{init})} = \nabla^2 q^{(\text{init})}$, and thus the above results hold for arbitrary perturbed scenarios (see however the related discussion in section 6).

Equations (39a) and (39b) constitute the main technical results of this paper, which we will explore in the following two sections.

5.2 The time of perturbed shell-crossing/matter collapse

In the absence of any perturbations, spherical collapse occurs at $A_{\text{A}}^{(0)} = 1.686$. Since the leading-order correction to the displacement is linear in $\epsilon$, it is expected that the time of matter collapse receives a correction linear in $\epsilon$ as well. Our solution Ansatz is therefore

\[
A_{\alpha} = A_{\alpha}^{(0)} + \epsilon C, \tag{40}
\]

where $C$ is a constant which can only depend on the space coordinates. Perturbed shell-crossing occurs at the time $a_*$ for which the Jacobian vanishes,

\[
J_*(a_*) = \left[ 1 + S(a_*) \right]^2 \left( 1 + S(a_*) + \epsilon \frac{3a_* S(a_*)}{A} \Delta^{(\text{init})} \right) = 0. \tag{41}
\]
Evidently, this Jacobian vanishes also at the time of unperturbed shell-crossing, \(a_{\star}^{(0)}\), for which the square bracketed term vanishes. However, as we claim above, in the perturbed case shell-crossing could be shifted to earlier times. Assuming that \(a_{\star} < a_{\star}^{(0)}\) for the moment, and to leading order in \(\epsilon\), we can ignore the overall factor of \([1 + S(a_{\star})]^{2}\) in the last equation, and thus are left with
\[
1 + S(a_{\star}) + \epsilon \frac{3a_{\star}S(a_{\star})}{A} \Delta^{(\text{init})} = 0. \tag{42}
\]
Now, since \(S(a_{\star}) = S(a_{\star}^{(0)}) + \epsilon CS(a_{\star}^{(0)})/A + O(\epsilon^{2})\), and because of \(1 + S(a_{\star}^{(0)}) = 0\), it is straightforward to find from equation (42) that \(C = -3a_{\star}^{(0)}\Delta^{(\text{init})}\). Thus, the time of perturbed shell-crossing is, to order \(\epsilon\), and for times \(a_{\star} < a_{\star}^{(0)}\),
\[
Aa_{\star} = Aa_{\star}^{(0)}(1 - 3\epsilon\Delta^{(\text{init})}/A), \tag{43}
\]
with \(Aa_{\star}^{(0)} = 1.686\) and \(\Delta^{(\text{init})} = \nabla^{2}\varphi^{(\text{init})}\). Since \(\Delta^{(\text{init})}\) can take generally also positive values, we thus conclude that if \(\Delta^{(\text{init})} > 0\) locally, then indeed perturbed shell-crossing occurs earlier than in the unperturbed case. Stated in another way, an initially overdense region will collapse earlier, if the perturbation \(\Delta^{(\text{init})}\) amplifies the initial overdensity.

What about locations \(q = Q\) for which \(\Delta^{(\text{init})}(Q) < 0\)? Will the time of perturbed shell-crossing be delayed w.r.t. the unperturbed case? The answer to this question is no, since what matters physically is the first vanishing of the Jacobian (41), which is guaranteed to happen, the latest, at the time of unperturbed shell-crossing (for which the square bracketed term in (41) vanishes).

Summing up, to leading order in \(\epsilon\), and for \(\Delta^{(\text{init})} > 0\) perturbed shell-crossing occurs as instructed by equation (43), but for \(\Delta^{(\text{init})} \leq 0\), the time of perturbed shell-crossing is identical with the time of unperturbed shell-crossing.

Since the Jacobian vanishes at the time of unperturbed shell-crossing, \(a_{\star}^{(0)}\), it is straightforward to find from equation (42) that \(C = -3a_{\star}^{(0)}\Delta^{(\text{init})}\). Thus, the time of perturbed shell-crossing is, to order \(\epsilon\), and for times \(a_{\star} < a_{\star}^{(0)}\),
\[
Aa_{\star} = Aa_{\star}^{(0)}(1 - 3\epsilon\Delta^{(\text{init})}/A), \tag{43}
\]
with \(Aa_{\star}^{(0)} = 1.686\) and \(\Delta^{(\text{init})} = \nabla^{2}\varphi^{(\text{init})}\). Since \(\Delta^{(\text{init})}\) can take generally also positive values, we thus conclude that if \(\Delta^{(\text{init})} > 0\) locally, then indeed perturbed shell-crossing occurs earlier than in the unperturbed case. Stated in another way, an initially overdense region will collapse earlier, if the perturbation \(\Delta^{(\text{init})}\) amplifies the initial overdensity.

5.3 Further results on the perturbed problem

In fig. 4 we show the unperturbed Jacobian as well as the difference \(\Delta J = J - J^{0}\), for several values of the perturbation parameter. As before we have set \(A = 1\) and \(\Delta^{(\text{init})} = 1\). Noticeable from that figure is that the effect of the perturbation yields the largest deviation from the unperturbed Jacobian at the time of turnaround, the latter defined by the maximum value of the divergence of the non-comoving particle trajectory. This is most easily seen on the following fig. 5, where we plot the physical particle trajectories, for the same values of \(\epsilon\) as in the last figure.

Let us turn to discuss the consequences for the density in the perturbed case. Formally linearising the expression \(\delta = 1/J - 1\) around its steady state, we obtain to leading order
\[
\delta_{\text{lin}}(a) = Aa \left(1 + \epsilon(\Delta^{(\text{init})})/A\right) + O(a^{2}, \epsilon^{2}). \tag{44}
\]
Evaluating the linear density contrast at the critical vicinity of perturbed shell-crossing \(a_{\star}\), which for \(\Delta^{(\text{init})} > 0\) is given by eq. (43), and otherwise is identical with \(a_{\star}^{(0)}\), we then find
\[
\delta_{\text{lin}}(a_{\star}) = 1.686 \left(1 - c\epsilon(\Delta^{(\text{init})})/A\right) + O(a^{2}, \epsilon^{2}), \tag{45}
\]
where
\[
c = \begin{cases} 1, & \text{for } \Delta^{(\text{init})} < 0, \\ 2, & \text{for } \Delta^{(\text{init})} > 0. \end{cases} \tag{46}
\]
Thus, irrespective of the sign of the perturbation to the spherical collapse...
collapse, the threshold for the linear density contrast at collapse is decreased w.r.t. to the unperturbed problem.

6 SUMMARY AND DISCUSSION

The case of exact spherical collapse is highly degenerated. A more realistic collapse model of matter should allow for departures from exact sphericity. We have shown, by departing perturbatively from the pure spherical problem, that we can solve for the collapse exactly, and by fully analytical means, until the instance of shell-crossing. The latter denotes the first crossing of particle trajectories which results in infinite densities, indicating the formation of density caustics on the one side, and the break-down of the fluid description on the other.

Let us briefly summarize the used methodology, which is heavily inspired by the multi-scale technique of Rampf & Frisch (2017). Firstly, we employ a 3D Lagrangian formulation of the cosmological fluid equations for an EdS universe in a Cartesian coordinate system. We solve the equations for a choice of initial conditions, that resemble, to the zeroth order in an expansion parameter, the spherical top-hat profile. The solution of the spherical collapse can be represented as an infinite time-Taylor series for the Lagrangian displacement field, which was used the time variable is not the cosmic time \( t \) but the cosmic scale factor \( a \sim t^{2/3} \). Then, we include an arbitrary perturbation to the top-hat profile at the level of the initial conditions. This perturbation, controlled by the dimensionless perturbation parameter \( \epsilon \), leads to a perturbed Lagrangian displacement field, which is also represented as an infinite time-Taylor series. By formally going to all orders in the Taylor series, obtained by simple extrapolation techniques (see fig. 1), we find that the series converges absolutely until the instance of shell-crossing.

We obtain the perturbed particle trajectory (39a), subject to the initial density \( \Delta^{(\text{init})} \) of the perturbation to the spherical collapse. Investigating the time of collapse \( a_{\text{c}} \), for which the Jacobian vanishes the first time, it is found that \( A a_{\text{c}} = 1.686(1 - 3\epsilon \Delta^{(\text{init})}/A) \) for \( \Delta^{(\text{init})} > 0 \), and otherwise simply \( A a_{\text{c}} = 1.686 \). Here, \( A > 0 \) is the amplitude for the spherical top-hat. For \( \epsilon = 0 \) we are back at the spherical problem, whereas for \( \epsilon > 0 \) and \( \Delta^{(\text{init})} > 0 \), collapse will occur earlier than in the pure spherical case.

The observation that perturbed collapse occurs earlier than in the spherical case has been already made in the literature for specific perturbation problems, although we are only aware of fairly qualitative statements about \( a_{\text{c}} \), thus no analytic formula; see LoVerde (2014) for the spherical collapse in the presence of massive neutrinos, or Monaco (1997) for the ellipsoidal collapse. We also remark that, including the perturbation in the analysis, the time of collapse as well as the (linear) density become inherently dependent on the mass scales of the collapse problem (set by the ratio \( \Delta^{(\text{init})}/A \)). This observation appears to be in qualitative agreement with the numerical analysis of Sheth et al. (2001).

Our collapse criterion is set by the first vanishing of the Jacobian

\[
J_\epsilon = (1 + S)^2 \left( 1 + S + \epsilon \frac{3aS}{A} \Delta^{(\text{init})} \right) ,
\]

which, in the perturbed problem, is triggered by the Laplacian of the perturbed initial gravitational potential \( \Delta^{(\text{init})} = \nabla^2 \phi^{(\text{init})} \). Thus, it is the total source of the perturbation in all coordinate directions, and not the collapse along a single coordinate axis, that sets our collapse criterion. After the time of first vanishing of the Jacobian, the fluid description breaks down and we must resort to other techniques to resolve the final stages of virialization.

As a consequence of the decreased collapse time in perturbed scenarios, we find that the critical linear density at collapse is reduced [see equation (45)], irrespective of the sign of \( \Delta^{(\text{init})} \). Very similar claims, however based on semi-analytic methods, can be found in the literature for specific perturbation problems, see Padmanabhan (1993); Peacock (1999) for the case of a CDM Universe for which, in our framework, the cosmological constant \( \Lambda \) takes the role of the perturbation. Wintergerst & Pettorino (2010) have also reported a decrease in the critical density for certain coupled dark energy cosmologies.

How general are our findings? Although our description delivers a more accurate collapse modelling than the standard SCM, it is still a simplified model. We have made simplifying assumptions, and for specific perturbation problems one should assess if such assumptions are still sufficiently accurate. In particular we assume an EdS cosmology, and thus have ignored the effects from changes in the background evolution, stemming from other fluid components that could provide the perturbation \( \Delta^{(\text{init})} \) to the spherical collapse. Possibly the most restricting assumption used is that we assume that the effect of the perturbation can be included solely at the initial Lagrangian position, and is otherwise immune to the evolution the perturbation may have.

Our model should deliver more accurate thresholds for the critical density at collapse, which can be used as the input to formalisms that predict the abundance, mass or shape of a given tracer, for example in the Press–Schechter formalism, excursion set, or peaks theory (Press & Schechter 1974; Bond et al. 1991; Desjacques 2008a; Paranjape et al. 2013). When suitably generalized, our results could be also incorporated in calculations for the abundance of primordial black holes (PBHs). In particular, one should take into account that PBHs are usually formed during the radiation dominated era (see e.g. Kühl et al. 2016). See however Kühl & Sandstad (2016) where it is argued, that for PBHs which result from ellipsoidal collapse, standard calculations (from Sheth et al. 2001) could be used with only minor adaptions.

Finally, observe that, to leading order in \( \epsilon \), the perturbed displacement is unaffected by tidal/environmental effects. This is due to the fact that, for small departures from sphericity, such effects are negligible. It would be interesting to go to second order in the perturbation parameter \( \epsilon \), because this would allow the inclusion of such environmental corrections (cf. Desjacques 2008b). We leave such investigations for future work.

ACKNOWLEDGEMENTS

I thank Florian Kühl and Björn Malte Schäfer for useful discussions, and Vincent Desjacques for related discussions and comments on the manuscript. I am supported by the DFG through the SFB-Transregio TRR33 “The Dark Universe”. The Taylor coefficients for the displacement up to order \( n = 1000 \), valid for generic perturbations, are available upon request.

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In section 5 we reported the finding of an exact relation between the Taylor coefficients of the displacement, i.e., that $\chi_n = 3n \sigma_n$. Here we prove the validity of this simple relation by considering the evolution equation for the perturbed displacement. Plugging the Ansatz (10) for the Jacobian matrix into the evolution equation (5a) we arrive at first order in $\epsilon$

\[ (1 + S)^2 \partial_a S + 2a \dot{S}(1 + S) \partial_a S = \frac{3}{2} \chi(1 + S)^2. \]  

(A1)

The relation $\chi_n = 3n \sigma_n$ between the time-Taylor coefficients amounts to the following relation, $\chi = 3a \dot{S}$.  

(A2)

Using this in equation (A1) we find

\[ (1 + S)^2 \partial_a(\alpha S) + 2a \dot{S}(1 + S) \partial_a S = \frac{3}{2} a \dot{S}(1 + S)^2. \]  

(A3)

Now, we rewrite the r.h.s. of the last equation in terms of a Lagrangian time derivative

\[ \text{r.h.s.} = a^2 \dot{\partial}_a \left\{ \frac{3}{2} \left[ S + S^2 + \frac{S^3}{3} \right] \right\} = a^2 \dot{\partial}_a \left\{ (1 + S)^2 \partial_a S \right\}. \]  

(A4)

where in the last step we have used equation (12). Equating the last expression with the l.h.s. of (A3), a few terms are cancelling without further actions, and we are left with

\[ \partial_a(\alpha S) = a^2 \dot{\partial}_a \left( \partial_a S \right). \]  

(A5)

This turns out to be an identity, and thus, we have proven that the perturbed evolution equation (A1) is identical with the unperturbed evolution equation, provided that we make use of the identification (A2).

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