The Geometry of the solution space of first order
Hamiltonian field theories I: from particle dynamics to free
Electrodynamics

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Abstract

We start the program of defining a Poisson bracket structure on the space of solutions of the equations of motion of first order Hamiltonian field theories. The cases of Hamiltonian mechanical systems and field theories without gauge symmetries are addressed by showing the existence of a symplectic (and, thus, a Poisson) structure on the solution space. Also the easiest case of gauge theory, namely Free Electrodynamics, is considered. Here the structure on the solution space is a pre-symplectic one and the Poisson structure is defined by the aid of a flat connection on a particular bundle associated to the theory.

Introduction

The need for a systematic study of the geometry of the space of solutions of differential equations describing physical systems was first pointed out by Souriau around the 70s of the last century [Sou97]. Using his own words:

Analytical mechanics is not an outdated theory, but it appears that the categories which one classically attributes to it such as configuration space, phase space, Lagrangian formalism, Hamiltonian formalism, are, simply because they do not have the required covariance; in other words, because these categories are in contradiction with Galilean relativity. A fortiori, they are inadequate for the formulation of relativistic mechanics in the sense of Einstein.

Indeed, with these arguments in mind, he suggested to abandon the phase space usually adopted within the description of (Hamiltonian) dynamical systems as main object to study and to pass to what he called space of motions and we will refer to as solution space throughout the work. For him it was the space of solutions of the equations of the motion, representing the space of physical trajectories of the system. In the present work, it will be actually the space of extrema of an action functional in terms of which the dynamical content of the theory will be described.

In particular, having in mind the Quantum/Classical Physics transition, the geometrical structure one hopes to bring out for the solution space is that of a Poisson manifold. Indeed, a first formulation of the so called correspondence principle is due to Niels Bohr (see [RN76] for a collection of
papers on the correspondence principle) who stated that he expected that the predictions of the 'new theory' he was developing (i.e. the Quantum Theory) should agree, in the limit of high energies, with the predictions of the classical theory of radiations. Such a principle was made more mathematically rigorous by P. M. Dirac in [Dir25] where he stated that to the Lie algebra structure on quantum observables (i.e. linear operators on a Hilbert space) should correspond, in the classical limit, a Poisson bracket structure on their classical counterparts.

In this work, we will carry on Souriau’s point of view aiming to exhibit a Poisson manifold structure for the solution space of a large class of field theories.

This is not a new task. Indeed, on the one hand, regarding the necessity of better studying the geometry of the space of solutions of the equations of motion, a few contributions by C. Crnkovic and E. Witten appeared along the ’80s of the last century. In particular, in [CW86] and [Crn88] a first attempt to define a symplectic structure on the solution space was made, even if at a formal level from the differential geometric point of view. Being interested in examples such as Yang-Mills theories, General Relativity and String Theory, because of the gauge invariance of those theories the authors actually ended up to work with a pre-symplectic structure rather than with a symplectic one.

On the other hand, the search for a Poisson bracket between functions of the solutions within field theories motivated by the wish of better understanding their quantum counterpart dates back to R. E. Peierls who introduced his covariant commutation rules in relativistic field theories in 1952 [Pei52]. Essentially, he gave an algorithm to construct a covariant bracket between functions of the solutions of the equations of motion via the Green’s function of the linearized equations of the motion. As testified by B. DeWitt’s words in [DeW03]:

There exists an anomaly today in the pedagogy of physics. When expounding the fundamentals of quantum field theory physicists almost universally fail to apply the lessons that relativity theory taught them early in the twentieth century. Although they usually carry out their calculations in a covariant way, in deriving their calculational rules they seem unable to wean themselves from canonical methods and Hamiltonians, which are holdovers from the nineteenth century and are tied to the cumbersome $(3 + 1)$-dimensional baggage of conjugate momenta, bigger-than-physical Hilbert spaces, and constraints. There seems to be a feeling that only canonical methods are 'safe'; only they guarantee unitarity. This is a pity because such a belief is wrong, and it makes the foundations of field theory unnecessarily cluttered. One of the unfortunate results of this belief is that physicists, over the years, have almost totally neglected the beautiful covariant replacement for the canonical Poisson bracket that Peierls invented in 1952.

Peierls’ construction was neglected by a great part of physical community for many years with only a few exceptions. Indeed, B. DeWitt himself was the one who carried on Peierls’s point of view in the successive decades. Indeed, in [DeW60] he applied Peierls’ procedure to the Gravitational field and in [DeW65, DeW03] used the Peierls’ bracket as the fundamental ingredient over which one should construct a coherent covariant description of field theories.
Even if they represented a breakthrough in the formulations of field theories from a relativistic point of view, the above mentioned contributions lack of a deep analysis of the very geometrical structure involved in the construction of the covariant brackets. Indeed, the mathematical framework that, in our opinion, best fit the geometrical description of covariant field theories, is the multisymplectic geometry which started to be developed along the same years and, at that time, was not yet mature enough. The seminal contributions in this sense are due to the Polish’s school of W. M. Tulczyjew [Kij73,KS76,TK79] motivated by the success of symplectic geometry within the description of Classical Mechanics and by the aim of constructing an analogous symplectic framework for field theories. Along the same years, also the contributions of P. L. García and A. Pérez-Rendón [GPR69,GPR71] should be mentioned. In the successive decades a lot of contributions to the development of multisymplectic geometry appeared. We cite the contributions of D. Krupka and his collaborators [KKS10,KKS10,KKS12,Krn97] (and references therein) focused on the geometrical formulation of Lagrangian variational principles and on the generalizations of the definition of the Poincarè-Cartan form within higher order field theories and the contributions of M. Francaviglia and his collaborators (see [FF03] and references therein) aimed to a geometrical formulation of General Relativity as a field theory. Other relevant contributions to the development of multisymplectic geometry are in the following, far from exhaustive, list [GS73,Zuc87,Got91a,Got91b,GMS97,CCI91,IS17,BSF88,EEMLRR00,EEMLRR96,RR09,BMMRL21,MBV21].

After the development of multisymplectic geometry, the construction of a covariant Poisson bracket for field theories has received a renewed interest even if with a slightly different perspective, that of studying the very differential geometrical structures involved in the construction of the brackets introduced in the physical literature cited so far (see [MMMT86,FR05,FS15,Kha14,ACD+17,ACDI17,CDI+20b,CDI+20a,Gie21]).

The aim of this work, which consists of two parts of which the present paper is the first one, is exactly that of putting together the previous considerations and offer a unified account of the description of covariant brackets together with a consistent analysis of the geometric and global-analytic structures carried by the constructed bracket.

With respect to the existing literature, we make a step further in two directions. First, we extend the construction given in [FR05] for the scalar field theory, to the more general setting of gauge theories. Such an attempt was already made by Khavkine in [Kha14]. However, his construction makes use of the possibility of performing a gauge fixing and, thus, if one aims to construct a bracket given by a Poisson bivector which is globally defined on the solution space, it can not be applied for those gauge theories for which the gauge fixing can not be global in the space of fields, i.e. for theories presenting the so called Gribov’s ambiguities (note that, for instance, all non-Abelian gauge theories lie in this case). Therefore, for such a class of gauge theories the construction in [Kha14] still remain valid but only locally in the space of fields. In this paper, we encode the condition to construct the Poisson bracket on the solution space into a precise geometrical condition, that is the flatness of a connection on a bundle associated with the gauge theory. Even if such a condition exclude the same class of theories (those presenting Gribov’s ambiguities), our different point of view has the great advantage to be ready to be extended to the realm of non-Abelian gauge theories by a consistent
use of the coisotropic embedding theorem (following an idea introduced by the authors in [CDI+22]).
We perform such an extension in the second part of this series of papers [CDI+] while, in this first part
we actually focus on the Abelian example given by free Electrodynamics in order to introduce
our geometrical construction within the easiest case of gauge theory. In particular, in the present
paper we show that the solution space of a gauge theory is a pre-symplectic manifold and we avoid
the usual approach of quotienting with respect to the kernel of the pre-symplectic form to obtain a
symplectic structure because, in our opinion, the quotient manifold is less manageable both from the
mathematical point of view and from the point of view of the physical interpretation. Indeed, we will
construct a Poisson bracket directly on the pre-symplectic manifold by the aid of a flat connection.
Second, we avoid the widespread approach of dealing with the space of smooth fields as a formal
differential manifold on which a formal differential calculus à la Cartan is defined and we actually pay
attention in all the examples considered to deal with well defined Hilbert manifolds on which all the
geometrical objects and the differential calculus are rigorously defined.

Let us briefly introduce the main ideas of the work. An important fact we will use throughout
the whole paper is that in all the situations considered there exists a bijection between the solution
space $\mathcal{E}\mathcal{L}$ and the space of Cauchy data $\mathcal{E}\mathcal{L}\Sigma$ by virtue of existence and uniqueness theorem
for the (at least weak) solutions of the equations of motion arising from the variational principle
(see [CDI+20b, CDI+20a] where this idea is already introduced by some of the authors). We will
denote it by $\Psi$ ($\Psi^{-1}$ denoting its inverse):
$$\Psi : \mathcal{E}\mathcal{L}\Sigma \rightarrow \mathcal{E}\mathcal{L}. \quad (1)$$
In the situations considered we will be able also to prove that both spaces are at least Banach
manifolds with a well defined notion of differential calculus and that, with respect to these differential
structures, the bijection is also a diffeomorphism.
We will see that in the multysimplectic formulation of first order Hamiltonian field theories a
canonical 2-form on the solution space naturally emerge from the variational principle and that it has
the form $\Omega = \Psi^{-1}\tilde{\Omega}$ where $\tilde{\Omega}$ is a 2-form on $\mathcal{E}\mathcal{L}\Sigma$. Now, depending on the properties of $\mathcal{E}\mathcal{L}\Sigma$ and
$\tilde{\Omega}$ we discuss three difficulty increasing cases in which we will prove how to use the structure $\Psi^{-1}\tilde{\Omega}$
to define a Poisson bracket on $\mathcal{E}\mathcal{L}$:

- In mechanical systems without constraints we will see that $\mathcal{E}\mathcal{L}\Sigma$ is a finite-dimensional manifold
  and that $\tilde{\Omega}$ is a symplectic form. Therefore, since $\Psi^{-1}$ is a diffeomorphism, the structure $\Psi^{-1}\tilde{\Omega}$
is symplectic as well and gives rise to a Poisson bracket with the usual construction of symplectic
  manifolds;

- In field theories $\mathcal{E}\mathcal{L}\Sigma$ is infinite-dimensional and $\tilde{\Omega}$ is proved to be the structure emerging
  from the pre-symplectic constraint algorithm used in Field Theory to deal with constraints.

\footnote{This notation is justified by the fact that we will usually denote by $\Sigma$ the hypersurface of the space-time on which we will take Cauchy data.}
Thus, if the theory only exhibits first class constraints, it is symplectic. Consequently, $Ψ^{-1}Ω$ is symplectic and gives rise to a Poisson bracket as in the previous case.

- The last case analyzed is the case of gauge theories where the pre-symplectic constraint algorithm ends up with a degenerate 2-form and, thus, $Ω$ and $Ψ^{-1}Ω$ are pre-symplectic. We will see that in this case a Poisson bracket can be defined as well by the aid of a flat connection on principal bundle associated with the gauge theory.

The work is organized as follows. Section 1 is devoted to recall the main ingredients of the multisymplectic formalism used to describe first order Hamiltonian field theories.

Section 2 deals with the formulation of the Schwinger-Weiss action principle within the multisymplectic framework and with the construction of the solution space of the theory.

In Section 3 the existence of the canonical 2-form on the solution space is exhibited while in Section 4 we argue how such a 2-form is a symplectic one in theories presenting only first class constraints while it is pre-symplectic in gauge theories.

To conclude, in section 5 we show how to construct a Poisson bracket on the solution space related with the canonical 2-form both in the symplectic case and within gauge theories. The construction of the bracket in the three relevant examples cited above is presented in 5.1, 5.2 and 5.3.

## 1 Multisymplectic formulation of first order Hamiltonian field theories

The basic ingredients for developing a covariant Hamiltonian description of a geometric classical field theory are a smooth manifold $\mathcal{M}$ and a bundle $π: \mathcal{E} → \mathcal{M}$ whose sections $φ$ model the fields of the theory. The base manifold $\mathcal{M}$ is considered to be a smooth manifold, possibly with non-empty boundary $\partial \mathcal{M}$, where a chart is denoted by $(U_#, ψ_{U_#})$, $ψ_{U_#}(m) = (x^0, ..., x^d)$ ($m$ being a point in $U_# \subset \mathcal{M}$), hence $\mathcal{M}$ has dimension $n = 1 + d$. In the case of particle dynamics, seen as a trivial example of field theory, it is an interval of the real line. The only global aspect required on $\mathcal{M}$ is that it is orientable and, consequently, a reference volume $(d + 1)$-form $vol_#$ is chosen. In all the example considered, $\mathcal{M}$ will be a space-time, that is, it will carry a Lorentzian metric. Unless stated otherwise, the chart $(U_#, ψ_{U_#})$ is chosen such that $vol_# = dx^0 ∧ ... ∧ dx^d$ (in the case of particle dynamics, it is $vol_# = dt$). Regarding the boundary, it is assumed to be not only a smooth manifold (not necessarily connected) of dimension $d$, but that it is properly embedded in $\mathcal{M}$, that is, that there exists a collar around it, $C_ε = [0, ε) × \partial \mathcal{M}$ and an embedding $i_{C_ε}: C_ε → \mathcal{M}$, such that $i\{0\} × \partial \mathcal{M} = \partial \mathcal{M}$, and $i_{C_ε}^* vol_# = dx^0 ∧ vol_{\partial \mathcal{M}}$, where $vol_{\partial \mathcal{M}}$ is a volume form on $\partial \mathcal{M}$.

As we said above, the fields of the theory are represented by sections of $π$, that is, they are local maps from open sets in $\mathcal{M}$ to some space $\mathcal{E}$, playing the role of the configuration manifold $Q$ of Mechanics. Such space of local configurations of the field is assumed to be a smooth manifold again to be specified on each situation. It may be (and this is the case analysed within the present paper) a
linear space (real or complex). More generally it may be a Lie group, a Poisson or symplectic manifold, etc. We consider an adapted fibered chart on \( E \) given by \((U_E, \psi_{U_E})\), \( \psi_{U_E}(e) = (x^0, ..., x^d, u^1, ..., u^r) \), \( r \) being the dimension of \( E \) and \( e \) being a point in \( U_E \subset E \). Local sections of \( \pi \), are denoted as: \( \phi = (\phi^1(x), ..., \phi^r(x)) \).

Out of these ingredients it is possible to construct a space which is a natural extension of the phase space of Mechanics and that provide a natural framework to develop the theory. Such space is called the covariant phase space associated to \( \pi : E \rightarrow \mathcal{M} \) and is denoted as \( \mathcal{P}(E) \). We recall its construction and we refer to the literature cited in the introduction for a more extensive account.

First, it is well known that the natural setting to describe a first order Lagrangian field theory is the first jet bundle \( J^1\pi \) (see for instance, [Sni70, BSF88, EEMLRR96, GMS97, Kru97, FF03, Kru15]).

\[ J^1\pi = \pi_1 \]
\[ \psi_{J^1 \pi}(\bar{p}) = (x^0, ..., x^d, u^1, ..., u^r, \rho_0^0, ..., \rho_0^d, \rho_0), \bar{p} \text{ being a point on } U_{J^1 \pi} \subset J^1 \pi. \] The variables \( \rho_a^\mu \) are identified with the \( \text{(generalized)} \) momenta of the theory, while the inhomogeneous term \( \rho_0 \) is identified with the Hamiltonian of the theory and is defined up to the addition of a constant. Then we consider the space obtained by quotienting the extended dual space with respect to the addition of constants and one gets the covariant phase space \( \mathcal{P}(\mathbb{E}) \) of the theory, which turns out to be again a fibre bundle both over \( \mathbb{E} \) and over \( \mathcal{M} \). As adapted fibered chart on \( \mathcal{P}(\mathbb{E}) \) we take \( (U_{\mathcal{P}(\mathbb{E})}, \psi_{U_{\mathcal{P}(\mathbb{E})}}) \), \( \psi_{U_{\mathcal{P}(\mathbb{E})}}(p) = (x^0, ..., x^d, u^1, ..., u^r, \rho_0^0, ..., \rho_0^d) \), \( p \) being a point in \( U_{\mathcal{P}(\mathbb{E})} \subset \mathcal{P}(\mathbb{E}) \). The fibre bundles constructed so far fit in the diagram

\[ \begin{array}{ccc}
J^1 \pi & \xrightarrow{\kappa} & \mathcal{E} \\
\tau_1 \downarrow & & \downarrow \pi \\
\mathcal{M} & \xrightarrow{\delta_1} & \mathcal{P}(\mathbb{E})
\end{array} \]

\( \kappa \) being the projection associated to the quotient procedure described above.

An interesting interpretation of such dual bundles in terms of spaces of differential forms on \( \mathbb{E} \) exists. Indeed, the extended dual bundle \( J^1 \pi \) is isomorphic to the set of \( 1 \)-semi-basic \( d+1 \)-forms on \( \mathbb{E} \), namely \( \Lambda^{d+1}_1(\mathbb{E}) \), this isomorphism requiring the choice of a reference volume form on \( \mathcal{M} \), say \( \text{vol}_{\mathcal{M}} \). The elements in \( \Lambda^{d+1}_1(\mathbb{E}) \) can be expressed as:

\[ w = \rho_a^\mu du^a \land dx^\mu + \rho_0 d^{d+1}x, \]

where \( d^{d+1}x = i \_ \_ \_ \_ \_ \_ \_ vol_{\mathcal{M}} \) and \( d^{d+1}x = vol_{\mathcal{M}} \). Then, it is straightforward to prove that \( \mathcal{P}(\mathbb{E}) \) can be identified with the quotient \( \Lambda^{d+1}_1(\mathbb{E})/\Lambda^{d+1}_0(\mathbb{E}) \), where \( \Lambda^{d+1}_0(\mathbb{E}) \) is the set of basic \( (d+1) \)-forms on \( \mathbb{E} \) whose elements are written as:

\[ w_0 = \rho_0 d^{d+1}x. \]

The extended dual space \( J^1 \pi \) carries a canonical \( d+1 \)-form, also called \textsc{tautological form}, defined as \( \Theta_w(X_1, ..., X_{d+1}) = w(\nu_1 X_1, ..., \nu_r X_{d+1}) \), \( \nu \) denoting the projection \( \nu : \Lambda^{d+1}_1(\mathbb{E}) \to \mathbb{E} \), and which has exactly the expression (6).

Now, we introduce the second fundamental ingredient to describe the theory, i.e., the Hamiltonian function of the theory as a function defined on the covariant phase space \( H = H(p) \). By this choice we are actually selecting a hypersurface on the extended dual space which is transverse to the projection \( \kappa \) (in other words a section of the bundle \( \tau : J^1 \pi \to \mathcal{P}(\mathbb{E}) \)), by means of:

\[ \rho_0 = H(p). \]

\( ^2 \)They are differential forms on \( \mathbb{E} \) vanishing when contracted along 2 \( \pi \)-vertical vectors.

\( ^3 \)Namely, the set of differential forms on \( \mathbb{E} \) vanishing when contracted along a \( \pi \)-vertical vector.
The pullback of the tautological \((d + 1)\)-form on \(\mathcal{J}^{1}\pi\) to the covariant phase space \(\mathcal{P}(E)\) via (minus)\(^4\) the Hamiltonian section determines a \((d + 1)\)-form on \(\mathcal{P}(E)\):

\[
\Theta_{H} = \rho_{a}^{\mu} du^{a} \wedge d^{4}x_{\mu} - H d^{d+1}x .
\] (8)

Note that in the case of particle dynamics, where \(\mathcal{M}\) is an interval of the real line, the differential form above is the well known Poincarè-Cartan form:

\[
\Theta = p_{j}dq^{j} - Hdt .
\] (9)

The \(n + 1\)-form \(d\Theta_{H}\) is a closed and non-degenerate form, i.e., what is defined to be a multisymplectic (or \(n\)-plectic) form. Thus, the covariant phase space \(\mathcal{P}(E)\) equipped with the \(n + 1\)-form \(d\Theta_{H}\) is a multisymplectic manifold (hence the name of multisymplectic formalism given to this framework) whose properties have been studied by many authors (see references cited in the introduction).

2 Schwinger-Weiss variational principle, the first fundamental formula and the solution space

The dynamical content of the theory is encoded into an Action functional which is defined as follows.

**Definition 2.1 (Action functional).** The action functional of the theory is the function from \(\Gamma(\delta_{1})\) to \(\mathbb{R}\):

\[
\mathcal{S}_{\chi} := \int_{\mathcal{M}} \chi^{\ast} \Theta_{H} ,
\] (10)

where \(\chi\) denotes an element of \(\Gamma(\delta_{1})\) which is the space of smooth sections of \(\delta_{1}\) that factorize in the following way:

\[
\begin{array}{c}
\mathcal{M} \\
\bigcirc \\
\chi \\
\phi \bigcirc \delta_{1} \\
P \bigcirc \delta_{0}
\end{array} \rightarrow \mathcal{P}(E)
\] (11)

The sections \(\chi\) are couples of the type \(\chi = (\phi, P)\) that are locally expressed as \(\phi^{a}(x^{\mu})\) and \(P_{a}^{\mu}(x^{\mu}, \phi^{a}(x^{\mu}))\), respectively. The space of factorizable sections will be denoted by \(\Gamma^{\text{split}}(\delta_{1})\). The coordinate expression of \(\mathcal{S}\) is:

\[
\mathcal{S}_{\chi} = \int_{\mathcal{M}} \left[ P_{a}^{\mu}(x, \phi(x))\partial_{\mu}\phi^{a}(x) - H(x, \phi(x), P(x, \phi(x))) \right] \text{vol}_{\mathcal{M}} .
\] (12)

\(^{4}\)This sign is a matter of convention.
The problem with this definition is that $\Gamma_{\text{split}}(\delta)$ is not a Banach manifold. The structure of a Banach manifold for the space of fields $\chi$ would be desirable in order to have the usual Cartan differential calculus on it at our disposal\(^5\). For this reason, we assume that $\Gamma_{\text{split}}(\delta)$ can be completed with respect to a Banach norm in which the action functional is continuous so that it can be extended by continuity to the completion. Such a completion will be what we will call THE SPACE OF DYNAMICAL FIELDS and we will denote by $\mathcal{F}_{\text{P}(E)}$. With a slight abuse of notation, elements of $\mathcal{F}_{\text{P}(E)}$ will be called again $\chi = (\phi, P)$ and the extension of $\mathcal{I}$ to $\mathcal{F}_{\text{P}(E)}$ will be called again $\mathcal{I}$. We call the space of $\phi$'s the SPACE OF CONFIGURATION FIELDS and we denote by $\mathcal{F}_{\text{E}}$. With a slight abuse of notation, elements of $\mathcal{F}_{\text{E}}$ will be called again $\chi = (\phi, P)$ and the extension of $S$ to $\mathcal{F}_{\text{P}(E)}$ will be called again $S$.

We call the space of $\phi$'s the SPACE OF DYNAMICAL FIELDS and we denote by $\mathcal{F}_{\text{P}(E)}$. With a slight abuse of notation, elements of $\mathcal{F}_{\text{P}(E)}$ will be called again $\chi = (\phi, P)$ and the extension of $S$ to $\mathcal{F}_{\text{P}(E)}$ will be called again $S$. We proceed in developing the general theory by taking the above assumption and, thus, from now on, we consider $\mathcal{I}$ to be directly defined on $\mathcal{F}_{\text{P}(E)}$. Then, in the explicit examples considered throughout the work we will take care of performing the completion described above with respect to a suitable Banach norm.

Being $\mathcal{F}_{\text{P}(E)}$ a Banach manifold, it has a well defined notion of tangent space to each point $\chi$, say $T_{\chi}\mathcal{F}_{\text{P}(E)}$, whose elements will be denoted by $\mathcal{X}_{\chi}$ and can be identified with $\delta_1$-vertical vector fields on $\mathcal{P}(E)$ along the image of $\chi$.

The “trajectories” of the physical system under investigation are defined to be the extrema of $\mathcal{I}$. In particular they can be obtained as solutions of a set of PDEs by imposing a Schwinger-Weiss variational principle. It states that the first variation of $S$ along any direction only could depend on boundary terms.

**Definition 2.2 (First variation of the action functional).** Given the action functional $\mathcal{I}$ and a tangent vector $\mathcal{X}_{\chi} \in T_{\chi}\mathcal{F}_{\text{P}(E)}$, the first variation of $\mathcal{I}$ along the direction $\mathcal{X}_{\chi}$ is defined to be the Gateaux derivative of $S$ in the direction $\mathcal{X}_{\chi}$.

It reads\(^6\):

$$
\delta_{\mathcal{X}_{\chi}} \mathcal{I}_{\chi} = \frac{d}{ds} \mathcal{I}_{\chi_s} \bigg|_{s=0} = \frac{d}{ds} \int_{\mathcal{M}} \chi_s^* \Theta_H \bigg|_{s=0} = \frac{d}{ds} \int_{\partial \mathcal{M}} \left( F^X_s \circ \chi \right)^* \Theta_H \bigg|_{s=0} = \frac{d}{ds} \int_{\partial \mathcal{M}} \chi^* \left[ L_X \Theta_H \right],
$$

where $X$ is any vector field defined in a neighborhood of the image of $\chi$ which, restricted to $\chi$, coincides with $\mathcal{X}_{\chi}$ and where $F^X_s$ represents its flow. In the previous chain of equalities we used the fact that $(f \circ g)^* = g^* \circ f^*$ and the definition of the Lie derivative along the vector field $X$. The previous expression, by means of the Cartan identity for the Lie derivative, gives rise to two terms:

$$
\delta_{\mathcal{X}_{\chi}} \mathcal{I}_{\chi} = \int_{\mathcal{M}} \chi^* \left[ i_X d\Theta_H \right] + \int_{\partial \mathcal{M}} \delta \left[ i_X \Theta_H \right],
$$

\(^5\) Actually, that of being a Banach manifold is not the minimal requirement in order for a differential calculus to be well defined. However, we will restrict to the Banach case in order to avoid technicalities related with Frechet manifolds and locally convex spaces for which we refer to [MK97].

\(^6\) $\chi_s$ represents the one parameter family of sections in terms of which the Gateaux derivative is defined.
with \( i_{\partial M} \) being the embedding of the boundary \( \partial M \) in \( M \), and \( \chi_{0,M} = \chi \circ i_{\partial M} \), the restriction of the field \( \chi \) to the boundary.\(^7\) It is worth noting that the pullback map \( \chi^* \), restricts the bracketed expression along the image of the field \( \chi \) (or its restriction to the boundary). Therefore, such a variation only depends on \( \mathcal{X}_\chi \) and not on the chosen extension. Then, a Schwinger-Weiss-like principle imposes the first term on the r. h. s. of the previous expression to vanish for all \( X \in \mathcal{X}(U^{(x)}) \), \( U^{(x)} \) being an open neighborhood of the image of \( \chi \):

\[
\chi^* [ i_X d\Theta_H ] = 0 \quad \forall \mathcal{X}(U^{(x)}).
\]

An interpretation of the objects appearing in (14) in terms of differential forms on the infinite dimensional manifold \( F_{\mathcal{P}(\mathcal{E})} \) exists. Having in mind that \( \mathcal{S} \) is a function on \( F_{\mathcal{P}(\mathcal{E})} \), the quantity on the l. h. s. of (14) is nothing but the “component along \( \mathcal{X}_\chi \)” of the differential of \( \mathcal{S} \), that is, \( i_{\mathcal{X}_\chi} d\mathcal{S} \) at \( \chi \), where \( d \) denotes the differential on \( F_{\mathcal{P}(\mathcal{E})} \). On the other hand, the first term in the r. h. s. of (14) can be read as the evaluation at the point \( \chi \) on the tangent vector \( \mathcal{X}_\chi \) of a differential one-form on \( F_{\mathcal{P}(\mathcal{E})} \), that we call the Euler-Lagrange form and we denote by \( \Pi \).

**Definition 2.3 (Euler-Lagrange Form).** Given an action functional \( \mathcal{S} \), the first term appearing in the first fundamental formula (14) is the evaluation of the following Euler-Lagrange form on the tangent vector \( \mathcal{X}_\chi \) representing the direction of the variation:

\[
\Pi \mathcal{X}_\chi = \int_{\mathcal{M}} \mathcal{X}_\chi^* [ i_X d\Theta_H ],
\]

where \( X \) denotes a vector field on \( \mathcal{P}(\mathcal{E}) \) defined over an open neighborhood of the image of \( \chi \) which is an extension of \( \mathcal{X}_\chi \).

Regarding the second term, an analogous interpretation in terms of a differential form on \( F_{\mathcal{P}(\mathcal{E})} \) exists but requires more comments. Recall that the chart considered on \( M \) is such that in a neighborhood of \( \partial M \), \( x^0 \) is the coordinate transversal to the boundary, that is, \( x^0 \) together with a system of coordinates on \( \partial M \) provides a system of coordinates for the full \( M \) in a neighborhood of \( \partial M \). Consequently, in that neighborhood we recover the components of the momenta transversal to \( \partial M \), namely, \( P_a^0 \), and split the fields restricted to the boundary in the following way:

\[
\Pi \mathcal{X}_{\partial M} | F_{\mathcal{P}(\mathcal{E})} \rightarrow F_{\mathcal{P}(\mathcal{E})}^{\partial \mathcal{M}} : (\phi, P) \mapsto (\phi |_{\partial M}, P_a |_{\partial M}) = (\phi |_{\partial M}, P_a^0 |_{\partial M}, P_a^1 |_{\partial M}).
\]

Thus, the space of fields restricted to the boundary, \( F_{\mathcal{P}(\mathcal{E})}^{\partial \mathcal{M}} \), is split as \( F_{\mathcal{P}(\mathcal{E})}^{\partial \mathcal{M}} = F_{\mathcal{P}(\mathcal{E}),0}^{\partial \mathcal{M}} \times \mathcal{B}^{\partial \mathcal{M}}, \)

where \( (\phi |_{\partial M}, P_a^0 |_{\partial M}) =: (\phi^a, p_a) \in F_{\mathcal{P}(\mathcal{E}),0}^{\partial \mathcal{M}} \) and \( P_a^1 |_{\partial M} =: \beta_a^j \in \mathcal{B}^{\partial \mathcal{M}} \). We depict schematically the projections onto \( F_{\mathcal{P}(\mathcal{E})} \) and \( F_{\mathcal{P}(\mathcal{E}),0}^{\partial \mathcal{M}} \) in the following way:

\[
F_{\mathcal{P}(\mathcal{E})} \xrightarrow{\Pi \mathcal{X}_{\partial M}} F_{\mathcal{P}(\mathcal{E})}^{\partial \mathcal{M}} \xrightarrow{\Pi \mathcal{X}_{\partial M}} F_{\mathcal{P}(\mathcal{E}),0}^{\partial \mathcal{M}} \xrightarrow{\mathcal{P}_{\mathcal{E}}^{\partial \mathcal{M}}} F_{\mathcal{P}(\mathcal{E}),0}^{\partial \mathcal{M}}.
\]

---

\(^7\)Actually these are the sections of the pull-back bundle of \( \mathcal{P}(\mathcal{E}) \) via \( i_{\partial M} \).
It turns out that $\mathcal{F}_P^{0,\mathcal{M}}$ is isomorphic to the cotangent bundle of $\mathcal{F}_E^{0,\mathcal{M}}$, say $T^*\mathcal{F}_E^{0,\mathcal{M}}$ where we denote a point by $(\varphi,\varrho)$ and we choose a fibered chart $(U_{T^*\mathcal{F}_E^{0,\mathcal{M}}},\psi_{U_{T^*\mathcal{F}_E^{0,\mathcal{M}}}})$, $\psi_{U_{T^*\mathcal{F}_E^{0,\mathcal{M}}}}[(\varphi, \varrho)] = (\varphi_1, \ldots, \varphi_r, \varrho_1, \ldots, \varrho_r)$. The isomorphism\footnote{It is also continuous since it is just an identification map and, thus, the pull-back $\lambda^*$ is well defined.} is denoted by:

$$\lambda : \mathcal{F}_P^{0,\mathcal{M}} \rightarrow T^*\mathcal{F}_E^{0,\mathcal{M}} : (\varphi, p) \mapsto (\varphi, q).$$

$T^*\mathcal{F}_E^{0,\mathcal{M}}$ has a canonical non-degenerate 2-form, say $\tilde{\omega}^0_{\partial \mathcal{M}}$:

$$\tilde{\omega}^0_{(\varphi, \varrho)}(\mathcal{X}_{(\varphi, \varrho)}, \mathcal{Y}_{(\varphi, \varrho)}) = \int_{\partial \mathcal{M}} \left( X^a v_a - \gamma^a v_a \right) \text{vol}_{\partial \mathcal{M}},$$

where $\mathcal{X}_{(\varphi, \varrho)}$ and $\mathcal{Y}_{(\varphi, \varrho)}$ are elements of $\mathcal{T}_{(\varphi, \varrho)}T^*\mathcal{F}_E^{0,\mathcal{M}}$ and $X^a, \gamma^a$ (resp. $\gamma^a, \gamma_a$) are their components with respect to the adopted chart. The two-form $\tilde{\omega}^0_{\partial \mathcal{M}}$ is (minus) the differential of the following one-form:

$$\tilde{\chi}^0_{(\varphi, \varrho)}(\mathcal{X}_{(\varphi, \varrho)}) = \int_{\partial \mathcal{M}} \left( p_a X^a \right) \text{vol}_{\partial \mathcal{M}}.$$

On the other hand, a direct computation shows that the second term on the r. h. s. of (14) reads:

$$\int_{\partial \mathcal{M}} \chi_{\partial \mathcal{M}}^* [i_X \Theta_H] = \int_{\partial \mathcal{M}} \left( \rho^0 \chi^a \right) \chi_{\partial \mathcal{M}} \text{vol}_{\partial \mathcal{M}} = \int_{\partial \mathcal{M}} \left( p_a \chi^a \right) \text{vol}_{\partial \mathcal{M}},$$

where $X$ is, again, a vector field on $\mathcal{P}(\mathbb{E})$ defined over an open neighborhood of the image of $\chi$ which is an extension of $\mathcal{X}$ and where $\chi^a$ is the restriction to $\partial \mathcal{M}$ of the $a^\mu$-component of $\chi$. Thus, if we want to interpret the term $\int_{\partial \mathcal{M}} \chi_{\partial \mathcal{M}}^* [i_X \Theta_H]$ appearing in the first fundamental formula as a differential form on $\mathcal{F}_P(\mathbb{E})$, it is actually the pull-back, via $\lambda \circ \rho_{\partial \mathcal{M}} \circ \Pi_{\partial \mathcal{M}}$, of the canonical form $\tilde{\omega}^0_{\partial \mathcal{M}}$ of $T^*\mathcal{F}_E^{0,\mathcal{M}}$ acting on a tangent vector:

$$\int_{\partial \mathcal{M}} \chi_{\partial \mathcal{M}}^* [i_X \Theta_H] = \left( \lambda \circ \rho_{\partial \mathcal{M}} \circ \Pi_{\partial \mathcal{M}} \right)^* \tilde{\omega}^0_{\chi_{\partial \mathcal{M}}} = \Pi_{\partial \mathcal{M}}^* \rho^* \lambda^* \tilde{\omega}^0_{\chi_{\partial \mathcal{M}}} =: \Pi_{\partial \mathcal{M}}^* \rho^* \chi_{\partial \mathcal{M}}^* \tilde{\omega}^0_{\chi_{\partial \mathcal{M}}} =: \Pi_{\partial \mathcal{M}}^* \alpha^0_{\chi_{\partial \mathcal{M}}}.$$ \hspace{1cm} (23)

The variational formula (14) can then be written in terms of differential forms on $\mathcal{F}_P(\mathbb{E})$ and reads:

$$d \mathcal{F}_\chi = \mathbb{E} \chi + \Pi_{\partial \mathcal{M}}^* \alpha^0_{\chi_{\partial \mathcal{M}}},$$

Thus, solutions of the equations of the motion can be seen as the zeroes of the Euler-Lagrange form:

$$\mathbb{E} \chi = 0,$$ \hspace{1cm} (25)

which is equivalent to:

$$\chi^* [i_X \Omega_H] = 0 \quad \forall \chi \in \chi(U(\chi)),$$ \hspace{1cm} (26)

\footnote{$\mathcal{F}_E^{0,\mathcal{M}}$ is the space of restrictions of elements of $\mathcal{F}_E$ to $\partial \mathcal{M}$.}
and whose local expression is:

\[
\frac{\partial \phi^a}{\partial x^\mu} - \frac{\partial H}{\partial \rho^a_\mu} = 0, \quad \frac{\partial \rho^a_\mu}{\partial x^\mu} + \frac{\partial H}{\partial u^a} = 0, \tag{27}
\]

that are the COVARIANT HAMILTON EQUATIONS of the theory, also called DE DONDER-WEYL
EQUATIONS. We will equivalently refer to them as EQUATIONS OF THE MOTION, or as EULER-
LAGRANGE EQUATIONS. Thus, we give the following definition.

**Definition 2.4 (Solution Space).** Given an action functional \( \mathcal{S} \) we define SOLUTION SPACE the following subset of \( \mathcal{F}_{\mathcal{P}(\mathcal{E})} \):

\[
\mathcal{E}_{\mathcal{L},\mathcal{E}} = \{ \chi = (\phi, P) \mid \mathbb{E}_\chi = 0 \} = \{ (\phi, P) \mid \chi^*(i_X d\Theta_H) = 0 \forall X \in \mathfrak{X}(U(\chi)) \}. \tag{28}
\]

From now on, we will assume it to be a smooth immersed submanifold of \( \mathcal{F}_{\mathcal{P}(\mathcal{E})} \), \( i_{\mathcal{E}_{\mathcal{L},\mathcal{E}}} \) denoting the immersion map.

Before moving on to the next section we provide some explicit examples in order to fix the ideas about the machinery we have introduced so far.

**Example 2.5. Free Point Particle on the Line** The first example we consider is that of a mechanical system seen as a field theory over a one dimensional space-time. In particular we consider a free particle moving on a line. In this case the “space-time” is just an interval of the real line \( I \subset \mathbb{R} \) where a point has coordinate \( t \). The bundle \( \mathcal{E} \to I \) has typical fibre \( \mathcal{E} = \mathbb{R} \) where we denote by \( q \) a (global) coordinate. Therefore, configuration fields are real valued functions \( q : I \to \mathbb{R} \), i.e. they are curves with support on the real line. The Covariant Phase Space is \( \mathcal{P}(\mathcal{E}) = I \times \mathbb{R}^2 \) where a set of (global) coordinates is denoted by \( (t, q, p) \). The space of smooth sections \( \chi \) of the bundle \( \pi^1 : \mathcal{P}(\mathcal{E}) \to I \) equipped with the sup-norm is evidently a Banach space which is our space of dynamical fields \( \mathcal{F}_{\mathcal{P}(\mathcal{E})} \). Therefore, in this example there is no need for an actual completion of the space of sections of \( \delta_1 \) to the space of dynamical fields. The Hamiltonian of the theory reads

\[
H : \mathcal{P}(\mathcal{E}) \to J^1, \pi : (t, q, p) \mapsto H(t, q, p) = \frac{p^2}{2m}, \tag{29}
\]

and, thus, the canonical form \( \Theta_H \) is

\[
\Theta_H = pdq - \frac{p^2}{2m} dt, \tag{30}
\]

giving the multisymplectic form

\[
d\Theta_H = dp \wedge dq - \frac{p}{m} dp \wedge dt. \tag{31}
\]
The associated multisymplectic form is:

\[
\omega = X_\mu dq - X_q dp - \frac{p}{m} X_\mu dt.
\]

Its pull-back via \(\chi\) reads:

\[
\chi^* (i_\chi \omega) = \left[ X_p \left( \dot{q} - \frac{p}{m} \right) - X_q \dot{p} \right] dt.
\]

The Euler-Lagrange form is:

\[
\mathcal{E} \chi (\chi, \xi) = \int \left[ X_p \left( \dot{q} - \frac{p}{m} \right) - X_q \dot{p} \right] dt,
\]

whose zeroes satisfies the following set of Euler-Lagrange equations:

\[
\dot{q} = \frac{p}{m}, \quad \dot{p} = 0.
\]

**Example 2.6. Free vector meson field theory**  A further example we consider is the PDE describing the evolution of the free (real) vector meson field theory. In this case, the space-time \(\mathcal{M}\) is the Minkowski space-time, i.e., \(\mathbb{M}^{1,3}\) equipped with the Minkowski metric, \(\eta = dx^0 \otimes dx^0 - \sum_{j=1}^3 dx^j \otimes dx^j\), where a (global) chart is given by \((U_{\mathfrak{V}_{M}^1, \mathfrak{V}_{M}^3}, \psi_{U_{\mathfrak{V}_{M}^1, \mathfrak{V}_{M}^3}}(m) = (x^0, ..., x^3) = x, m\) being a point in \(U_{\mathfrak{V}_{M}^1, \mathfrak{V}_{M}^3} \subset \mathbb{M}^{1,3}\). The bundle \(\mathcal{E} \rightarrow \mathbb{M}^{1,3}\) has typical fiber \(\mathcal{E} \cong \mathbb{R}^r\), thus, configuration fields are vector valued functions \(\phi \colon \mathbb{M}^{1,3} \rightarrow \mathbb{R}^r\). Here, the covariant phase space is \(\mathcal{P}(\mathbb{E}) = \mathbb{M}^{1,3} \times (\mathbb{R} \times \mathbb{R}^4)^r\), where we chose the adapted fibered chart \((U_{\mathcal{P}(\mathbb{E})}, \psi_{U_{\mathcal{P}(\mathbb{E})}}, \psi_{U_{\mathcal{P}(\mathbb{E})}}(p) = (x^0, ..., x^3, u^1, ..., u^r, \rho_1^1, ..., \rho_3^3) = (x, u, \rho)\) with \(p\) being a point in \(U_{\mathcal{P}(\mathbb{E})} \subset \mathcal{P}(\mathbb{E})\).\(^{10}\) The Hamiltonian of the theory is given by\(^{11}\):

\[
H : U_{\mathcal{P}(\mathbb{E})} \rightarrow U_{\mathcal{P}(\mathbb{E})} : \psi_{U_{\mathcal{P}(\mathbb{E})}}^{-1} (x, u, \rho) =
\]

\[
\psi_{U_{\mathcal{P}(\mathbb{E})}}^{-1} (x, u, \rho, \rho_0 = \frac{1}{2} (\eta_{\mu\nu} \delta^{ab} \rho^\mu_\alpha \rho^\nu_\beta + m^2 \delta_{ab} u^a u^b)),
\]

and, thus, the canonical one-form on the covariant phase space takes the form:

\[
\Theta_H = \rho^\mu_\alpha du^\alpha \wedge i_\mu d^4x - \frac{1}{2} (\eta_{\mu\nu} \delta^{ab} \rho^\mu_\alpha \rho^\nu_\beta + m^2 \delta_{ab} u^a u^b) d^4x.
\]

The associated multisymplectic form is:

\[
d\Theta_H = d\rho^\mu_\alpha \wedge du^\alpha \wedge i_\mu d^4x - \eta_{\mu\nu} \delta^{ab} \rho^\mu_\alpha \rho^\nu_\beta \wedge d^4x - m^2 \delta_{ab} u^a d^4x \wedge d^4x.
\]

\(^{10}\)Here the base manifold has a global chart and the bundle considered is a globally trivial vector one, i.e., this chart is also global.

\(^{11}\)Actually, this section is also global since the bundle is a trivial one.
The action functional (10) reads:

$$\mathcal{S}_c = \int_{\mathcal{M}} \chi^* \Theta_H = \int_{\mathcal{M}} \left[ P^a_\mu \partial_\mu \phi^a - \frac{1}{2} \left( P^a_\mu P^a_\mu + m^2 \phi^a \phi^a \right) \right] d^4 x. \tag{39}$$

In order for it to be well defined, we should restrict to sections for which $\phi^a$, $P^a_\mu$ and $\partial_\mu \phi^a$ are square integrable functions with respect to $d^4 x \forall a, \mu$. This means that $\mathcal{S}$ is well defined on the subset of $\Gamma^{\text{split}}(\delta_1)$ given by sections in $\prod_a \mathcal{H}^1(\mathbb{M}^{1,3}, \text{vol}_{\mathbb{M}^{1,3}})^a \times \prod_{\mu, a} L^2(\mathbb{M}^{1,3}, \text{vol}_{\mathbb{M}^{1,3}})^a$. For technical reasons that will be clear in Sect. 5.2, we will restrict ourselves to the subset of $\Gamma^{\text{split}}(\delta_1)$ given by sections in $\prod_a \mathcal{H}^2(\mathbb{M}^{1,3}, \text{vol}_{\mathbb{M}^{1,3}})^a \times \prod_{\mu, a} \mathcal{H}^1(\mathbb{M}^{1,3}, \text{vol}_{\mathbb{M}^{1,3}})^a =: \mathcal{C}$. Such subset is a linear subspace which is indeed a Hilbert space with respect to the Hilbert norm:

$$\| \chi \|_\mathcal{C} = \sum_a \| \phi^a \|_{\mathcal{H}^2} + \sum_{\mu, a} \| P^a_\mu \|_{\mathcal{H}^1}. \tag{40}$$

The functional $\mathcal{S}$ is continuous in $\| \cdot \|_\mathcal{C}$. Indeed the following inequality holds:

$$| \mathcal{S}_c - \tilde{\mathcal{S}} | \leq \frac{1}{2} \int_{\mathcal{M}} | (P^a_\mu - \tilde{P}^a_\mu)(\partial_\mu \phi^a - \partial_\mu \tilde{\phi}^a) | vol_{\mathcal{M}} + \frac{1}{2} \sum_{\mu, a} \| P^a_\mu - \tilde{P}^a_\mu \|_{\mathcal{H}^2} \| \phi^a - \tilde{\phi}^a \|_{\mathcal{H}^2} \leq \frac{1}{2} \sum_{\mu, a} \| P^a_\mu - \tilde{P}^a_\mu \|_{\mathcal{H}^1} + \frac{1}{2} m^2 \int_{\mathcal{M}} | (\phi^a - \tilde{\phi}^a)(\phi^a - \tilde{\phi}^a) | vol_{\mathcal{M}}, \tag{41}$$

where $\chi = (\phi, P)$ and $\tilde{\chi} = (\tilde{\phi}, \tilde{P})$. It is easy to see that the right hand side is zero when $\chi$ approaches $\tilde{\chi}$ with respect to the norm $\| \cdot \|_\mathcal{C}$ and then, the left hand side vanishes as well. This proves the continuity of $\mathcal{S}$ which can then be extended by continuity to the completion of $\mathcal{C}$ with respect to $\| \cdot \|_\mathcal{C}$, say $\mathcal{F}_{\mathcal{C}(\mathcal{E})}$.

In order to compute the Euler-Lagrange form, let us compute the contraction of $d\Theta_H$ along a $\tau_1$-vertical vector field $X$:

$$i_X d\Theta_H = X^\mu_\rho d\rho^\mu \wedge i_\rho d^4 x - X^a d\rho^a \wedge i_\rho d^4 x - \eta_{\mu \nu} \delta^{ab} \rho^a_\mu X^b \rho^\nu_x d^4 x - m^2 \delta_{ab} \mu^a X^b_a d^4 x. \tag{42}$$

Its pull-back via $\chi$ is:

$$\chi^* [ i_X d\Theta_H ] = \left[ X^\mu_\rho^a \chi \partial_\rho \phi^a - X^a_\rho^\mu \chi \partial_\mu \phi^a - \eta_{\mu \nu} \delta^{ab} \rho^a_\mu X^b \rho^\nu_x \chi \right] d^4 x =: \left[ \mathcal{X}^\mu_\rho^a \partial_\rho \phi^a - \mathcal{X}^a_\rho^\mu \partial_\mu \phi^a - \eta_{\mu \nu} \delta^{ab} \rho^a_\mu \mathcal{X}^b \rho^\nu_x - m^2 \delta_{ab} \phi^a \mathcal{X}^b_a \chi \right] d^4 x. \tag{43}$$

Thus the Euler-Lagrange form is:

$$\mathcal{E} \mathcal{L}_\chi(\mathcal{X}) = \int_{\mathbb{R}^4} \left[ \mathcal{X}^\mu_\rho^a \partial_\rho \phi^a - \mathcal{X}^a_\rho^\mu \partial_\mu \phi^a - \eta_{\mu \nu} \delta^{ab} \rho^a_\mu \mathcal{X}^b \rho^\nu_x - m^2 \delta_{ab} \phi^a \mathcal{X}^b_a \right] d^4 x, \tag{44}$$
whose zeroes satisfies the following set of Euler-Lagrange equations:

\[ \partial_{\mu} \phi^{a} = \eta_{\mu\nu} \delta^{ab} P^{b}_{a}, \quad \partial_{\mu} P^{a}_{b} = -\delta_{ab} m^{2} \phi^{b}. \]  

(45)

Notice that the two equations together show that the configuration fields satisfy the Klein-Gordon equation.

**Example 2.7. Free electrodynamics** Another example we consider is the PDEs describing the evolution of a sourceless electromagnetic field. Such equations come as Yang-Mills equations for a $U(1)$ gauge theory. For the sake of simplicity we are going to consider electrodynamics in Minkowski space time $(\mathbb{M}^{1,3}, \eta)$, as in the previous example. In a gauge field theory fields are represented by connection one forms on a principal bundle. In the electromagnetic case the structure group is the Abelian group $G = U(1)$, whereas the principal bundle is $P = \mathbb{M}^{1,3} \times G$, which is a trivial bundle. Under these assumptions electromagnetic fields are represented by Lie-Algebra valued differential forms on $\mathbb{M}^{1,3}$, the Lie-Algebra being $i\mathbb{R}$, and they are sections of the fibre bundle $\pi : T^{*}\mathbb{M}^{1,3} = \mathbb{E} \rightarrow \mathbb{M}^{1,3}$. Therefore, its first jet bundle is $\pi_{1} : J^{1}\mathbb{E} = J^{1}T^{*}\mathbb{M}^{1,3} \rightarrow \mathbb{M}^{1,3}$ which is a trivial bundle over $\mathbb{M}^{1,3}$ whose typical fibre is $T^{*}_{m}\mathbb{M}^{1,3} \times (\bigotimes^{2} T^{*}_{m}\mathbb{M}^{1,3})$. Consequently, the covariant phase space is a trivial bundle over $\mathbb{M}^{1,3}$, $\delta_{1} : \mathcal{P}(\mathbb{E}) \rightarrow \mathbb{M}^{1,3}$, whose typical fibre is $T^{*}_{m}\mathbb{M}^{1,3} \times (\bigotimes^{2} T^{*}_{m}\mathbb{M}^{1,3})$. Thus, sections of $\delta_{1}$ are contravariant tensors of rank 2. In particular, in order to correctly describe the properties of gauge fields within the multisymplectic framework, one has to consider the space of antisymmetric tensors as fibre of the covariant phase space. This choice is made to correctly reproduce Maxwell equations as it will be clear in a moment. With an abuse of notation we still denote this bundle by $\mathcal{P}(\mathbb{E})$. An adapted fibered chart is written as $(U_{\mathcal{P}(\mathbb{E})}, \psi_{U_{\mathcal{P}(\mathbb{E})}}, \psi_{U_{\mathcal{P}(\mathbb{E})}}(p) = (x^{1}, ..., x^{3}, u_{1}, ..., u_{3}, \rho^{00}, ..., \rho^{33}) = (x, u, \rho), p$ being a point on $U_{\mathcal{P}(\mathbb{E})} \subset \mathcal{P}(\mathbb{E})$ \(^{12}\). We denote sections of $\mathcal{P}(\mathbb{E}) \rightarrow \mathbb{M}^{1,3}$ by $\chi = (A_{\mu}, \rho^{\mu\nu})$. The Hamiltonian of the theory is:

\[ H : U_{\mathcal{P}(\mathbb{E})} \rightarrow U_{J^{1}\pi} \quad : \quad \psi_{U_{\mathcal{P}(\mathbb{E})}}^{-1}(x, u, \rho) \mapsto \psi_{U_{\mathcal{P}(\mathbb{E})}}^{-1}(x, u, \rho, \rho_{0} = \frac{1}{2} \rho^{\mu\nu} \rho^{\alpha\beta} \eta_{\mu\alpha} \eta_{\nu\beta}) \],

(46)

and, thus, the canonical 4-form on the Covariant Phase Space takes the form:

\[ \Theta_{H} = \rho^{\mu\nu} du_{\nu} \wedge i_{a} d^{4}x - \frac{1}{2} \rho^{\mu\nu} \rho^{\alpha\beta} \eta_{\mu\alpha} \eta_{\nu\beta} d^{4}x. \]

(47)

The associated multisymplectic form is:

\[ d\Theta_{H} = d\rho^{\mu\nu} \wedge du_{\nu} \wedge i_{a} d^{4}x - \rho^{\mu\nu} \eta_{\mu\alpha} \eta_{\nu\beta} d\rho^{\alpha\beta} \wedge d^{4}x. \]

(48)

With the same motivations of the previous example, the action functional $\mathcal{S}$ can be defined via (10) on the space of smooth sections of the covariant phase space and extended by continuity to the space

\[ \mathcal{F}(\mathcal{P}(\mathbb{E})) = \prod_{\mu=0,\ldots,3} \mathcal{H}^{2}(\mathbb{M}^{1,3}, vol_{\mathbb{M}^{1,3}})_{\mu} \times \prod_{\mu,\nu=0,\ldots,3} \mathcal{H}^{1}(\mathbb{M}^{1,3}, vol_{\mathbb{M}^{1,3}})^{\mu\nu}. \]

(49)

\(^{12}\)Again, this chart is global.
Now, following the same calculations of the previous two example one gets the contraction of the Euler-Lagrange form on a tangent vector \( X \chi \) to be:

\[
\mathcal{E}_\chi(X \chi) = \int_{\Omega, 3} \left[ \chi_{\chi, \rho}^{\mu\nu} F_{\mu\nu} - \chi_{\chi,\nu} \partial_\nu P_{\mu\nu} + \frac{1}{2} P_{\mu\nu} \chi_{\chi, \rho}^{\mu\nu} \right] \, d^4 x ,
\]

where \( F_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \). From the latter expression one derives the following set of equations of motion:

\[
F_{\mu\nu} + \frac{1}{2} P_{\mu\nu} = 0 , \quad \partial_\nu P_{\mu\nu} = 0 .
\]

Here it is clear that the choice of \( P_{\mu\nu} \) to be antisymmetric is necessary to obtain the correct equations of motion. Indeed, let us notice that these equations coincide with Maxwell equations for the electromagnetic field in absence of external sources.

3 The canonical structure on the solution space

We devote this section to comment further on the boundary term emerging in the variational principle since it gives rise to a canonical 2-form on the solution space \( \mathcal{E} \mathcal{L} \mathcal{M} \) which in turn will allow for the definition of a canonical bracket on the solution space.

First, let us recall that \( \mathcal{M} \) is a differential manifold with boundary. What is more, it is an orientable differential manifold on which we could fix the orientation to be the positive (outer) one, for instance. We assume the boundary \( \partial \mathcal{M} \) to have a finite number of connected components, that we denote by \( \partial \mathcal{M}^j \), for \( j = 1, \ldots, k \). \( \partial \mathcal{M} \) is in turn a topological manifold embedded into \( \mathcal{M} \), actually an orientable one with the orientation inherited from \( \mathcal{M} \) by means of its embedding into \( \mathcal{M} \). We denote by \( \sigma^j \) (\( j \) going from 1 to \( k \)) the sign of the orientation inherited from the connected component \( \partial \mathcal{M}^j \) of the boundary in its immersion inside \( \mathcal{M} \). This means that explicitly the boundary term emerging in the variational principle reads:

\[
\Pi_{\partial \mathcal{M}^j} \alpha_{\partial \mathcal{M}^j} (\chi) = \sum_{j=1}^{k} \sigma^j \int_{\partial \mathcal{M}^j} \chi_{\partial \mathcal{M}^j} [i_X \Theta_H] = \sum_{j=1}^{k} \sigma^j \Pi_{\partial \mathcal{M}^j} \alpha_{\partial \mathcal{M}^j} (\chi) ,
\]

where \( \alpha_{\partial \mathcal{M}^j} \) are differential forms on the space of dynamical fields restricted to the component \( \partial \mathcal{M}^j \) of the boundary, i. e., \( \mathcal{F}_{\partial \mathcal{M}^j} \). Its differential is

\[
- \frac{d \left( \Pi_{\partial \mathcal{M}^j} \alpha_{\partial \mathcal{M}^j} \right)}{d \Pi_{\partial \mathcal{M}^j} \alpha_{\partial \mathcal{M}^j}} = - \Pi_{\partial \mathcal{M}^j} \left( d \alpha_{\partial \mathcal{M}^j} \right) = \Pi_{\partial \mathcal{M}^j} \Omega_{\partial \mathcal{M}^j} = - \sum_{j=1}^{k} \sigma^j \Pi_{\partial \mathcal{M}^j} \Omega_{\partial \mathcal{M}^j} = - \sum_{j=1}^{k} \sigma^j \Pi_{\partial \mathcal{M}^j} \Omega_{\partial \mathcal{M}^j} ,
\]

where \( \sigma^j \) is the sign of the orientation inherited from the connected component \( \partial \mathcal{M}^j \) of the boundary in its immersion inside \( \mathcal{M} \).
where, again $\Omega^{\partial,\bar{\partial}}$ are differential forms on the space of sections restricted to $\partial \mathcal{M}$. Let us look at its explicit expression in terms of differential forms on the finite-dimensional manifold $\mathcal{P}(E)$. Recall that by definition of exterior differential, the differential of a 1-form $\alpha$ on a manifold reads:
\[
d\alpha(X, Y) = i_X d[\alpha(Y)] - i_Y d[\alpha(X)] - i_{[X,Y]} \alpha, \tag{54}
\]
where $X$ and $Y$ are vector fields defined on an open neighborhood of the point of our manifold in which we want to compute the differential. In our case we want to compute such a differential at the point $\chi$ of the infinite-dimensional manifold $\mathcal{F}_{\mathcal{P}(E)}$, thus we need a couple of vector fields on $\mathcal{F}_{\mathcal{P}(E)}$ defined on a neighborhood of $\chi$. This can be actually done since by construction $\mathcal{F}_{\mathcal{P}(E)}$ has a Banach manifold structure and, thus, vector fields on it are well defined. We denote by $\mathcal{X}$ and $\mathcal{Y}$ a couple of vector fields on $\mathcal{F}_{\mathcal{P}(E)}$. Now:
\[
d\left( \Pi^*_{\partial,\bar{\partial}} \alpha^{\partial,\bar{\partial}} \right)(\mathcal{X}, \mathcal{Y}) = i_{\mathcal{X}} d \left[ \Pi^*_{\partial,\bar{\partial}} \alpha^{\partial,\bar{\partial}}(\mathcal{Y}) \right] - i_{\mathcal{Y}} d \left[ \Pi^*_{\partial,\bar{\partial}} \alpha^{\partial,\bar{\partial}}(\mathcal{X}) \right] - i_{[\mathcal{X},\mathcal{Y}]} \Pi^*_{\partial,\bar{\partial}} \alpha^{\partial,\bar{\partial}}. \tag{55}
\]
Recalling that $\Pi^*_{\partial,\bar{\partial}} \alpha^{\partial,\bar{\partial}}(\mathcal{X}) = \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} [i_X \Theta_H]$ we get that:
\[
i_{\mathcal{X}} \mathcal{X} \mathcal{X} d \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} [i_Y \Theta_H] = i_{\mathcal{X}} d \int_{\partial,\bar{\partial}} \chi^* \left[ d i_Y \Theta_H \right] = \int_{\partial,\bar{\partial}} \chi^* \left[ \mathcal{X} d i_Y \Theta_H \right] = \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} [i_Y d i_Y \Theta_H], \tag{56}
\]
where, in the second equality, we used Eq. (13). Consequently:
\[
\Pi^*_{\partial,\bar{\partial}} \Omega^{\partial,\bar{\partial}}(\mathcal{X}, \mathcal{Y}) = -d \Pi^*_{\partial,\bar{\partial}} \alpha^{\partial,\bar{\partial}}(\mathcal{X}, \mathcal{Y}) = -i_{\mathcal{X}} d \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} [i_Y \Theta_H] + i_{\mathcal{Y}} d \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} [i_X \Theta_H] + i_{[\mathcal{X},\mathcal{Y}]} \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} \Theta_H = \tag{57}
\]
\[
= - \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} \left[ i_X d i_Y \Theta_H - i_Y d i_Y \Theta_H - i_{[\mathcal{X},\mathcal{Y}]} \Theta_H \right],
\]
where $X$ and $Y$ are extensions of $\mathcal{X}$ and $\mathcal{Y}$ to an open neighborhood of the image of $\mathcal{X}$. A straightforward application of the identity $i_{[\mathcal{X},\mathcal{Y}]} = [\mathcal{X}, i_{\mathcal{Y}}]$ leads to:
\[
\Pi^*_{\partial,\bar{\partial}} \Omega^{\partial,\bar{\partial}}(\mathcal{X}, \mathcal{Y}) = - \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} [i_Y i_X d \Theta_H] = \int_{\partial,\bar{\partial}} \chi^*_{\partial,\bar{\partial}} [i_X i_Y d \Theta_H]. \tag{58}
\]
Now, with an analogous definition, a differential 2-form on $\mathcal{F}_{\mathcal{P}(E)}$ can be defined for any slice $\Sigma$ of $\mathcal{M}$, that is, for any codimension-1 hypersurface $\Sigma \subset \mathcal{M}$ such that $\mathcal{M} \setminus \Sigma$ is the disjoint union of two regions $\mathcal{M}_+$ and $\mathcal{M}_-$:
\[
\Pi^*_{\Sigma} \Omega^{\Sigma}(\mathcal{X}, \mathcal{Y}) = \int_{\Sigma} \chi_{\Sigma}^* [i_X i_Y d \Theta_H]. \tag{59}
\]
The aim of this section is to prove that the structure $\Pi^*_{\Sigma} \Omega^{\Sigma}$ does not depend on the considered slice $\Sigma$ if evaluated at $\chi \in \mathcal{E}_{\mathcal{L},\bar{\partial}}$, i.e. that $\Pi^*_{\Sigma} \Omega^{\Sigma}$ is a canonical structure on the solution space of the considered variational problem.

Let us start by proving the following claims.
**Proposition 3.1.** $\mathcal{E} \mathcal{L}_\mathcal{M}$ is an isotropic\(^{13}\) manifold for the differential 2-form $\Pi^{\star}_{\partial,\mathcal{M}} \Omega_{\mathcal{M}}^{0,\mathcal{H}}$.

**Proof.** The action of the differential operator upon both sides of (24) gives:

$$\frac{d}{d\tau} \mathcal{L}_\chi = d\mathcal{L}_\chi + d\Pi^{\star}_{\partial,\mathcal{M}} \Omega_{\mathcal{M}}^{0,\mathcal{H}} = d\mathcal{L}_\chi - \Pi^{\star}_{\partial,\mathcal{M}} \Omega_{\mathcal{M}}^{0,\mathcal{H}},$$

i.e.:

$$d\mathcal{L}_\chi - \Pi^{\star}_{\partial,\mathcal{M}} \Omega_{\mathcal{M}}^{0,\mathcal{H}} = 0.$$

Consider the pull-back of such a differential form to $\mathcal{E} \mathcal{L}_\mathcal{M}$ via $i_{\mathcal{E} \mathcal{L}_\mathcal{M}}$. The pull-back acts naturally with respect to $d$ and $\mathcal{E} \mathcal{L}_\mathcal{M}$ is the space of zeroes of $\mathcal{E} \mathcal{L}$, thus, $i_{\mathcal{E} \mathcal{L}_\mathcal{M}} ^\star d\mathcal{L} = di_{\mathcal{E} \mathcal{L}_\mathcal{M}} \mathcal{L} = 0$. Therefore:

$$i_{\mathcal{E} \mathcal{L}_\mathcal{M}} ^\star \Pi^{\star}_{\partial,\mathcal{M}} \Omega_{\mathcal{M}}^{0,\mathcal{H}} = 0.$$

In particular, by looking at (53), the previous proposition gives

$$\sum_{j=1,...,k} o^{j} \Pi^{\star}_{\partial,\mathcal{M}} \Omega_{\mathcal{M}}^{0,\mathcal{H}^1} \chi = 0$$

if $\chi \in \mathcal{E} \mathcal{L}_\mathcal{M}$.

**Proposition 3.2.** Consider a submanifold $\tilde{\mathcal{M}}$ open into $\mathcal{M}$ and having the same dimension of $\mathcal{M}$. Denote by $\mathcal{E} \mathcal{L}_{\tilde{\mathcal{M}}}$ the solution space related to the variational principle formulated on $\tilde{\mathcal{M}}$ instead of $\mathcal{M}$. Then the restriction to $\tilde{\mathcal{M}}$ of elements of $\mathcal{E} \mathcal{L}_\mathcal{M}$ are elements of $\mathcal{E} \mathcal{L}_{\tilde{\mathcal{M}}}$.

**Proof.** Let us denote by $\tilde{i}$ the immersion of $\tilde{\mathcal{M}}$ into $\mathcal{M}$. Given $i$, the space $\tilde{\mathcal{E}} = \tilde{\mathcal{M}} \times \mathcal{E}$ is immersed into $\mathcal{E}$ via the map $\tilde{i}_\mathcal{E} := \tilde{i} \times 1_\mathcal{E}$. Analogously $\mathcal{P}(\tilde{\mathcal{E}})$ is immersed into $\mathcal{P}(\mathcal{E})$ via $\tilde{i}_\mathcal{P} := \tilde{i} \times 1_{\delta_1}$, $\delta_1$ being the fibre of $\delta_1$. We denote by $\tilde{\chi}$ and by $\chi$ sections of $\mathcal{P}(\tilde{\mathcal{E}})$ and of $\mathcal{P}(\mathcal{E})$ respectively. For any $\chi$ there exists a $\tilde{\chi}$ (its restriction $\chi|_{\tilde{\mathcal{M}}}$) such that the following diagram commutes:

$$\begin{array}{c}
\mathcal{P}(\tilde{\mathcal{E}}) \\
\tilde{i}_\mathcal{P} \downarrow \quad \quad \downarrow \tilde{i}_\mathcal{P} \\
\mathcal{P}(\mathcal{E}) \\
\tilde{\chi} \\
\tilde{i} \downarrow \quad \quad \downarrow \chi \\
\tilde{\mathcal{M}} \\
\mathcal{M}
\end{array}$$

It represents, indeed, a section of the pull-back bundle of $\delta_1$ via $\tilde{i}$. Viceversa, for all $\tilde{\chi}$ there exists at least one $\chi$ such that the previous diagram commutes. The variational principle formulated on $\mathcal{P}(\tilde{\mathcal{E}})$ gives the following equations of motion:

$$\tilde{\chi} ^\star \left( i_{\tilde{\chi}} \tilde{i}_\mathcal{P} d\Theta_{\mathcal{H}} \right) = 0 \quad \forall \, \tilde{X} \in \mathcal{X}^{\delta_1}(U^{(\tilde{\chi})}) \, .$$

\(^{13}\)By isotropic submanifold we mean a submanifold upon which the restriction of the differential 2-form vanishes.
The left hand side of the previous equation can be rewritten as:

\[ \tilde{\chi}^* \left( i_{\tilde{X}} \tilde{\iota}_P d\Theta_H \right) = \tilde{\chi}^* \left( (\tilde{\iota}_P \circ \tilde{\chi})^* \left( i_X d\Theta_H \right) \right) = \tilde{\chi}^* \left[ \chi^* \left( i_X d\Theta_H \right) \right], \]

for any \( X \) which is \( \tilde{i}_P \)-related with \( \tilde{X} \) and where \( \chi \) is one of the sections of \( \delta_1 \) that restrict to \( \tilde{\chi} \). The previous chain of equalities clearly shows that if \( \chi \) is a solution for the variational problem on \( \mathcal{M} \), i.e., if \( \chi^* \left( i_X \Omega_H \right) = 0 \), then its restriction \( \tilde{\chi} \) is a solution of the variational principle on \( \mathcal{\tilde{M}} \).

Now, using the results of the present section the following can be proved.

**Proposition 3.3.** The structure \( \Pi^*_\Sigma \Omega^\Sigma \) does not depend on the slice \( \Sigma \) if evaluated on solutions of the equations of the motion.

**Proof.** Consider a region of the space-time, say \( \mathcal{M}_{12} \) being a submanifold with an open interior of the type of Prop. 3.2 and whose boundary is made of two slices, say \( \Sigma_1 \) and \( \Sigma_2 \), both carrying the orientation pointing outside the region. Then, a straightforward application of (63) and propositions 3.1 and 3.2 gives:

\[ \Pi^*_\Sigma_1 \Omega^\Sigma_1 = \Pi^*_\Sigma_2 \Omega^\Sigma_2, \]

if \( \chi \in \mathcal{E}_L_{12} \). Now, any slice \( \Sigma \) define a family made by all those slices such that the union with \( \Sigma \) is the boundary of a region of \( \mathcal{M} \) of the type of Prop. 3.2. Thus, Eq. (67) says that \( \Pi^*_\Sigma \Omega^\Sigma \) is the same on any slice \( \Sigma \) belonging to the same family.

Now, the thesis follows from the fact that for any couple of \( \Sigma \)'s belonging to different families, there always exists a third one belonging to both families.

\[ \square \]

### 4 The symplectic case vs gauge theories

This section is devoted to prove that the canonical structure \( \Pi^*_\Sigma \Omega^\Sigma \) introduced in the previous section, which is closed by construction, may be non-degenerate or just pre-symplectic depending on the theory we are considering. We refer to the first case as **symplectic case** whereas to the second one as **gauge theories**.

We will proceed as follows. First, we will prove that the structure \( \Omega^\Sigma \) on \( \mathcal{F}^\Sigma_{\mathcal{P}(E)} \) which \( \Pi^*_\Sigma \Omega^\Sigma \) comes from is pre-symplectic. Then, we will prove that, at least locally, the solution space is isomorphic to the space of solutions of a pre-symplectic Hamiltonian system associated with \( \Omega^\Sigma \) that can be solved via the so called **pre-symplectic constraint algorithm** (PCA) (see Appendix A). Finally, we will see that the structure \( \Pi^*_\Sigma \Omega^\Sigma \) is associated via a diffeomorphism to the pull-back of \( \Omega^\Sigma \), say \( \Omega^\Sigma_\infty \), to the final stable manifold resulting from the PCA. As a matter of fact, the latter may be symplectic or not and, consequently, \( \Pi^*_\Sigma \Omega^\Sigma \) may be symplectic or not.

As we saw in the previous sections, the one-form \( \alpha^{0,\#} \) is the pull-back, under the action of \( \lambda \circ \rho_{0,\#} \) of a one-form which is the potential of a non-degenerate 2-form. The same holds for the structure \( \alpha^\Sigma \)
for any $\Sigma$:

$$\alpha^\Sigma = \rho_\Sigma^* \lambda \bar{\vartheta}^\Sigma = \rho_\Sigma^* \vartheta^\Sigma,$$

where:

$$-d\vartheta^\Sigma =: \omega^\Sigma,$$

and, thus:

$$-d\Pi^\Sigma_\Sigma \alpha^\Sigma = \Pi^\Sigma_\Sigma \Omega^\Sigma = \Pi^\Sigma_\Sigma \rho_\Sigma^* \lambda \bar{\omega}^\Sigma = \Pi^\Sigma_\Sigma \rho_\Sigma^* \omega^\Sigma,$$

with $\rho_\Sigma$, $\bar{\vartheta}^\Sigma$, $\bar{\omega}^\Sigma$, $\vartheta^\Sigma$, $\omega^\Sigma$ being the obvious analogous of $\rho_{0.\#}$, $\bar{\vartheta}_{0.\#}$, $\bar{\omega}_{0.\#}$, $\vartheta_{0.\#}$, $\omega_{0.\#}$ and where $\bar{\omega}^\Sigma$ is non-degenerate for the same reasons as $\bar{\omega}_{0.\#}$. Being $\lambda$ an isomorphism, $\omega^\Sigma = \lambda^* \bar{\omega}$ is non-degenerate as well, while, being $\rho_\Sigma$ a surjection, $-d\alpha^\Sigma = \Omega^\Sigma = \rho_\Sigma^* \omega^\Sigma$ is only pre-symplectic. Therefore, the following holds.

**Proposition 4.1.** The two-form $\omega^\Sigma$ is closed and non-degenerate, i.e. symplectic, on $F_{\Sigma(\mathcal{E})}^\Sigma$, while $\Omega^\Sigma$ is closed and degenerate, i.e. pre-symplectic, on $F_{\Sigma(\mathcal{E})}^\Sigma$.

Consider a hypersurface $\Sigma$ and a collar $C^\Sigma_\epsilon = [x^0_\Sigma, x^0_\Sigma + \epsilon] \times \Sigma =: \mathbb{I}^\Sigma_\epsilon \times \Sigma$. As usual, we choose a chart on $C^\Sigma_\epsilon$ of the type $(U, \psi_U)$, $\psi_U(m) = (x^0, x^j)_{j=1,...,d}$ such that $x^0$ is actually the coordinate transversal to $\Sigma$ and such that the volume reads $vol_{C^\Sigma_\epsilon} = dx^0 \wedge vol_\Sigma$. We denote by $(U_\Sigma, \psi_{U_\Sigma})$, $\psi_\Sigma(r) = (x^k)_{k=1,...,d}$ the corresponding adapted chart on $\Sigma$, where $r \in \Sigma$ and by $(U_{\Sigma^c}, \psi_{U_{\Sigma^c}})$, $\psi_{U_{\Sigma^c}}(r) = s$ the corresponding adapted chart on $\mathbb{I}^\Sigma_\epsilon$, where $s \in U_{\Sigma^c} \subset \mathbb{I}^\Sigma_\epsilon$. Consider the space of dynamical fields restricted to the collar $C^\Sigma_\epsilon$, say $F_{\Sigma(\mathcal{E})}^C$. We denote solutions restricted to $C^\Sigma_\epsilon$ by $\mathcal{E}L^{\epsilon}_{\mathcal{E}}$. Denote by $i_{\Sigma^c}$ the immersion of $C^\Sigma_\epsilon$ into $\mathcal{M}$ and by $\mathcal{E}^\epsilon := i_{\Sigma^c}^{*} \mathcal{E}$. The space $F_{\Sigma(\mathcal{E})}^C$ is isomorphic to the space of curves on $F_{\Sigma(\mathcal{E})}^C$, whose elements are denoted by $\gamma_s = (\varphi_s, p_s, \beta_s)$. The isomorphism, say $\varpi$, reads:

$$\varphi_s^a (\psi_{U_\Sigma}^{-1}(x^k)) = \phi^a (\psi_{U_\Sigma}^{-1}(s), \psi_{U_\Sigma}^{-1}(x^k)),$$

$$p_{as} (\psi_{U_\Sigma}^{-1}(x^k)) = P^0_a (\psi_{U_\Sigma}^{-1}(s), \psi_{U_\Sigma}^{-1}(x^k)),$$

$$\beta^s_{as} (\psi_{U_\Sigma}^{-1}(x^k)) = P^1_a (\psi_{U_\Sigma}^{-1}(s), \psi_{U_\Sigma}^{-1}(x^k)),$$

with $s \in [x^0_\Sigma, x^0_\Sigma + \epsilon]$. In other words, $\varpi$ is the identification of the coordinate transversal to the collar with the evolution parameter describing the curve on $F_{\Sigma(\mathcal{E})}^C$. Therefore, being the identity map up to the identification of the coordinate $s$ on one space with the coordinate $x^0$ on the other space, it is also continuous and differentiable together with its inverse and, thus, it is a diffeomorphism. Let $\Gamma \left( F_{\Sigma(\mathcal{E})}^C \right) = \varpi \left( F_{\Sigma(\mathcal{E})}^C \right)$ denote the space of curves $\gamma(\cdot)$. Then, the pull-back of the action functional

---

14Here "transversal coordinate" has the same meaning as after Eq. (16).
to \( \Gamma \left( \mathcal{F}_p^{\Sigma} \right) \) reads:

\[
\hat{\mathcal{J}}^\epsilon_\gamma = (\varpi^{-1} \mathcal{J}^\epsilon)_\gamma = \int_{s_0}^{s_0+\epsilon} ds \int_{\Sigma} \left( p_a \dot{\varphi}_s^a + \beta_a^k \partial_k \varphi_s^a - H(\gamma_s) \right) \text{vol}_\Sigma =: \int_{s_0}^{s_0+\epsilon} \mathcal{L}(\gamma_s, \dot{\gamma}_s) ds, \tag{72}
\]

where the dot denotes the derivative with respect to \( s \) and where we are interpreting the integrand as a Lagrangian function \( \mathcal{L} : T(\mathcal{F}_p^{\Sigma}) \to \mathbb{R} \) which, evaluated along the curve \( t \gamma_s = (\gamma_s, \dot{\gamma}_s) \) being the tangent lift of \( \gamma_s \) reads:

\[
\mathcal{L} (\varphi, v_\varphi, p, v_p, \beta, v_\beta) \big|_{t \gamma_s} = \int_{\Sigma} \left( p_a \dot{\varphi}_s^a + \beta_a^k \partial_k \varphi_s^a - H(\gamma_s) \right) \text{vol}_\Sigma = \langle p, \dot{\varphi} \rangle + \langle \beta, d_\Sigma \varphi \rangle - \int_{\Sigma} H(\varphi_s, p_s, \beta_s) \text{vol}_\Sigma =: \langle p, \dot{\varphi} \rangle - \mathcal{H}(\gamma_s), \tag{73}
\]

where \((U_{\mathcal{F}_p^{\Sigma}}, \psi_{\mathcal{F}_p^{\Sigma}}), \psi_{\mathcal{F}_p^{\Sigma}}(\chi_\Sigma) = (\varphi^1, ..., \varphi^r, p_1, ..., p_r, \beta_1^1, ..., \beta_r^d) = (\varphi, p, \beta)\) is a chart on \( \mathcal{F}_p^{\Sigma}\),

with \( \chi_\Sigma \) denoting a point in \( U_{\mathcal{F}_p^{\Sigma}} \subset \mathcal{F}_p^{\Sigma}\) and \((U_{T(\mathcal{F}_p^{\Sigma})}, \psi_{T(\mathcal{F}_p^{\Sigma})}), \psi_{T(\mathcal{F}_p^{\Sigma})}(\chi_\Sigma, \dot{\chi}_\Sigma) = (\varphi, v_\varphi, p, v_p, \beta, v_\beta)\) is a chart on \( T(\mathcal{F}_p^{\Sigma}) \) with \((\chi_\Sigma, \dot{\chi}_\Sigma) \in U_{T(\mathcal{F}_p^{\Sigma})} \subset T(\mathcal{F}_p^{\Sigma})\).

In the previous formula \( d_\Sigma \text{denotes the differential over the smooth 3-manifold } \Sigma, \text{the symbol } \langle \cdot, \cdot \rangle \text{ denotes both the integration over } \Sigma \text{ and the contraction over the internal degrees of freedom of the fields, whereas the functional } \mathcal{H} \text{ is:}

\[
\mathcal{H}(\gamma_s) = \langle \beta, d_\Sigma \varphi \rangle - \int_{\Sigma} H(\gamma_s) \text{vol}_\Sigma. \tag{74}
\]

It is a direct consequence of proposition 3.2 that extrema of \( \mathcal{J}^\epsilon \) are in one-to-one correspondence, via \( i_{\mathcal{C}_\epsilon}^{\Sigma} \), with extrema of \( \mathcal{J} \) for \( x^0 \) restricted to \([x_0^0, x_0^0 + \epsilon]\). What is more, the above discussion shows that extrema of \( \mathcal{J}^\epsilon \) are in one-to-one correspondence (via \( \varpi \)) with extrema of \( \varpi^{-1} \mathcal{J}^\epsilon =: \hat{\mathcal{J}}^\epsilon \). It is possible to prove ( [IS17] theorem 3.1) the following.

**Proposition 4.2.** Extrema of the functional \( \hat{\mathcal{J}}^\epsilon \), say \( \mathcal{E}_a^\epsilon \), are the solutions of the pre-symplectic systems \( (\mathcal{F}_p^{\Sigma}, \Omega^\Sigma, \mathcal{H}) \).

**Proof.** The proof is actually a direct computation that we report here for the sake of completeness.

Solutions of the pre-symplectic system above are the integral curves of the vector field \( \Gamma \in \mathcal{X}(\mathcal{F}_p^{\Sigma}) \) satisfying:

\[
\Omega^\Sigma(\Gamma, \mathcal{X}) = d\mathcal{H}(\mathcal{X}) \quad \forall \mathcal{X} \in \mathcal{X}(\mathcal{F}_p^{\Sigma}). \tag{75}
\]

By definition of \( \Omega^\Sigma \) and \( \mathcal{H} \), the latter equation gives:

\[
\int_{\Sigma} \left( \Gamma^a_{\varphi} \mathcal{X}_p - \Gamma_p \mathcal{X}^a_{\varphi} \right) \text{vol}_\Sigma = \int_{\Sigma} \left( \frac{\delta \mathcal{H}}{\delta \varphi^a} \mathcal{X}_p^a + \frac{\delta \mathcal{H}}{\delta p_a} \mathcal{X}_p^a + \frac{\delta \mathcal{H}}{\delta \beta_a^i} \mathcal{X}^a_{\beta} \right) \text{vol}_\Sigma \tag{76}
\]

\[\forall \mathcal{X}^a_{\varphi}, \mathcal{X}_p^a, \mathcal{X}^a_{\beta} \]

\[\text{Restricted to its range.}\]
where $\Gamma^a_\varphi$, $\Gamma^a_p$, $\chi^a_p$ and $\chi^a_{\beta^j}$ are the components of the vector fields $\Gamma$ and $\chi$. The latter equation reads:

$$
\Gamma^a_\varphi = \frac{d\varphi^a_s}{ds} = \frac{\delta H}{\partial \rho^a_p}(\varphi_s, p_s, \beta_s),
$$

$$
\Gamma^a_p = \frac{dp^a_s}{ds} = \frac{\delta H}{\partial \varphi^a}(\varphi_s, p_s, \beta_s),
$$

$$
0 = \frac{\delta H}{\delta \beta^j_a} = \frac{\partial \varphi^a_s}{\partial x^j} - \frac{\partial H}{\partial \rho^a_p}(\varphi_s, p_s, \beta_s).
$$

These equations formally coincide with the covariant Hamilton equations by identifying the coordinate $x^0$ with the parameter $s$ and, thus, solutions of the latter system of equations are actually the extrema of $\mathcal{S}^\epsilon$.

Thus, we obtained that extrema of $\mathcal{S}^\epsilon$, i.e., the elements of the solution space restricted to the collar $C^\Sigma_\epsilon$, are in one-to-one correspondence, via $\varpi$, with solutions of the pre-symplectic system $(\mathcal{F}^\Sigma_{\mathcal{P}(E)}, \Omega^\Sigma, H)$.

**Remark 4.3.** It should be noted that all the previous arguments are local in the sense that they are all valid for the parameter $\epsilon$ small enough. The possibility of considering them as global ones strongly depends on the particular space-time considered. In particular in all the previous results the space $E^L_{\Sigma}C^\epsilon$ can be substituted with the whole $E^L_{\Sigma}M$ in case $M$ is homeomorphic with $\Sigma \times \mathbb{R}$ for some codimension-one hypersurface $\Sigma$. This is actually the definition of global hyperbolic space-time. They are a large class of physically interesting space-times and in the examples considered in this paper only space-times of this kind will appear.

It should be stressed that equation (76) can not be interpreted as an equation to determine only the vector field $\Gamma$ because, since $\Omega^\Sigma$ has a non-trivial kernel, it can not determine $\Gamma$ uniquely. Indeed, it should be thought of as an equation determining both the components of $\Gamma$ outside the kernel of $\Omega^\Sigma$ and the points (determined by the third set of equations in (77)) of $\mathcal{F}^\Sigma_{\mathcal{P}(E)}$ where the equation makes sense.

How to rigorously (from the geometrical point of view) deal with this situation is the goal of the so called PRE-SYMPLECTIC CONSTRAINT ALGORITHM (PCA) (see [GNH78] where the algorithm was introduced and see Appendix A where we briefly recall how it works). Essentially the goal of the algorithm is to construct (when possible), starting from a pre-symplectic system of the type (76), a suitable constraint manifold (usually denoted by $\mathcal{M}_\infty$) where the equation (76) is well defined and then to try to determine its solution.

The fact that solutions of the equations of the motion for any collar close to a hypersurface $\Sigma$ are in one-to-one correspondence with solutions of a pre-symplectic system on $\mathcal{F}^\Sigma_{\mathcal{P}(E)}$, clearly shows that the space of Cauchy data (and, thus, the solution space) does not coincide with the whole space of dynamical fields restricted to $\Sigma$. Eventually, it coincides with the constraint submanifold $\mathcal{M}_\infty$.
because, as it is recalled in Appendix A, points of $\mathcal{M}_\infty$ are those being in one-to-one correspondence with solutions of the pre-symplectic system. That $\mathcal{M}_\infty$ is a smooth immersed submanifold of the original pre-symplectic manifold will be assumed in developing the general theory and must be verified case by case. We will denote by $i^\infty$ the immersion map which, in the examples considered will be actually an embedding.

As a matter of fact, two instances may occur at this stage. The first is the case where $\Omega^\Sigma_\infty := i^\infty_* \Omega^\Sigma$ is non-degenerate. We refer to this case as the SYMPLECTIC CASE. The second one is the case where $\Omega^\Sigma_\infty$ is again a pre-symplectic manifold. We refer to this case as GAUGE THEORIES.

In case $\Omega^\Sigma_\infty$ is non-degenerate, $\mathcal{M}_\infty$ is said to be weakly symplectic. It is actually strongly symplectic if $\mathbf{T}_{m_\infty} \mathcal{M}_\infty \simeq \mathbf{T}^*_m \mathcal{M}_\infty \ \forall \ m_\infty \in \mathcal{M}_\infty$. From now on, within the symplectic case, we assume that $\mathcal{M}_\infty$ is strongly symplectic so that for any function on $\mathcal{M}_\infty$ its Hamiltonian vector field with respect to $\Omega^\Sigma_\infty$ is uniquely defined. We are motivated in making this assumption by the fact that in the examples we are going to consider $\mathcal{M}_\infty$ is actually a Hilbert manifold for which the isomorphism above is canonically given by the Hermitian structure of the model Hilbert space. However, we stress that in general, if $\mathcal{M}_\infty$ is only weakly symplectic, Hamiltonian vector fields may not be globally defined, or they may be defined only on some subset of $\mathcal{M}_\infty$. In this sense, a case by case analysis must be performed to take care of this possibility.

Now, keeping the assumption that $\mathcal{M}_\infty$ is strongly symplectic, solutions of the original pre-symplectic system can be obtained from the integral curves of the vector field $\Gamma^\infty$ over $\mathcal{M}_\infty$ satisfying:

\[ \Omega^\Sigma_\infty(\mathcal{F}, \mathcal{X}) = dH^\infty(\mathcal{X}) \ \forall \ \mathcal{X}_\infty \in \mathcal{X}(\mathcal{M}_\infty), \]

(78)

(\(\Omega^\Sigma_\infty = i^\infty_* \Omega^\Sigma, \ H^\infty = i^\infty_* H\)) by immersing them into $\mathcal{F}_{P(E)}$ via $i^\infty$. Being $\Omega^\Sigma_\infty$ strongly symplectic, for each point (Cauchy datum) $m_\infty \in \mathcal{M}_\infty$ there exists a unique such integral curve and consequently a unique solution in $\mathcal{E}(\mathcal{L}_E^\epsilon)$. In this sense, the space of Cauchy data is actually the constraint submanifold $\mathcal{M}_\infty$. The unique solution associated to the Cauchy datum $m_\infty$ reads $\chi = i^\infty \circ F^{\Gamma^\infty}(m_\infty) =: \Psi(m_\infty)$ where $F^{\Gamma^\infty}$ is the flow of $\Gamma^\infty$. The map $\Psi$ is bijective and differentiable. It is bijective by virtue of the existence and uniqueness theorem for the solutions of (78) which holds since $\Omega^\Sigma_\infty$ is non-degenerate. Its inverse reads $\Psi^{-1} = F^{\Gamma^\infty \Gamma^{-1}} \circ i^{-1}$ where $i^{-1}$ is the inverse of $i^\infty$ which exists on $\mathcal{E}(\mathcal{L}_E^\epsilon)$ since it is, by definition, the image of $i^\infty$. It is differentiable because it is the composition of $F^{\Gamma^\infty}$ which is a smooth (resp. $C^k$) map of the Cauchy datum $m_\infty$ since it is a flow of a smooth (resp. $C^k$) vector field, and $i^\infty$ which is a smooth map by assumption. Consequently, for each point in $\mathcal{M}_\infty$ there exists a unique solution in $\mathcal{E}(\mathcal{L}_E^\epsilon)$, i.e., the image via $\varpi$ of the latter solution. Eventually, if regularity conditions mentioned in remark 4.3 are met, these solutions would also be global. The situation is
depicted in the following diagram:

\[
\begin{array}{c}
\mathcal{E}\mathcal{L}^\epsilon_{\#} \xrightarrow{\pi_\Sigma} \mathcal{E}\mathcal{L}^\epsilon_{\#} \\
\uparrow \psi \downarrow \pi_\Sigma \\
\mathcal{F}^\Sigma_{\mathcal{P}(E)} \xrightarrow{\rho_\Sigma} \mathcal{F}^\Sigma_{\mathcal{P}(E),0} \xrightarrow{\iota_\infty} \mathcal{M}_\infty \\
\end{array}
\]

(79)

Now, note that \(\Pi_\Sigma = i_\infty \circ \Psi^{-1} \circ \varpi\). Thus the canonical structure on the solution space \(\Pi^* \Omega^\Sigma\) reads:

\[
\Pi^* \Omega^\Sigma = \left( i_\infty \circ \Psi^{-1} \circ \varpi \right)^* \Omega^\Sigma = \left( \Psi^{-1} \circ \varpi \right)^* \Omega^\Sigma
\]

that is, \(\Pi^* \Omega^\Sigma\) is the pull-back via \(\Psi^{-1} \circ \varpi\) of the strongly symplectic structure \(\Omega^\Sigma\). But, as we argued above, \(\Psi\) and \(\varpi\) are diffeomorphisms and, thus, \(\Pi_\Sigma\) is a diffeomorphism as well. Therefore, the canonical structure \(\Pi^* \Omega^\Sigma\) on the solution space is a strongly symplectic one being the image via a diffeomorphism of the strongly symplectic structure \(\Omega^\Sigma\).

The situation where \(\mathcal{M}_\infty\) is pre-symplectic is different. Indeed, \(\mathcal{M}_\infty\) and \(\mathcal{E}\mathcal{L}^\epsilon_{\#}\) are not diffeomorphic anymore. This is because, being \(\Omega^\Sigma\) pre-symplectic, the vector field \(\Gamma^\Sigma\) satisfying (78) can not be uniquely determined and, consequently, so is for the map \(\Psi\). The first observation is that in this case, if the structure \(\Omega^\Sigma\) has constant rank, then the quotient manifold \(\mathcal{R}_\infty = \mathcal{M}_\infty / \ker(\Omega^\Sigma)\) is a symplectic manifold where a symplectic structure \(\tilde{\Omega}^\Sigma\) and an Hamiltonian \(\tilde{H}^\Sigma\) exist such that \(\Omega^\Sigma = \pi_\infty^* \tilde{\Omega}^\Sigma\) and \(H^\Sigma = \pi_\infty^* \tilde{H}^\Sigma\) (\(\pi_\infty\) denoting the projection associated with the quotient). The relevant spaces of fields in this case are the following:

\[
\begin{array}{c}
\mathcal{E}\mathcal{L}^\epsilon_{\#} \xrightarrow{\pi_\Sigma} \mathcal{E}\mathcal{L}^\epsilon_{\#} \\
\uparrow \psi \downarrow \pi_\Sigma \\
\mathcal{F}^\Sigma_{\mathcal{P}(E)} \xrightarrow{\rho_\Sigma} \mathcal{F}^\Sigma_{\mathcal{P}(E),0} \xrightarrow{\iota_\infty} \mathcal{M}_\infty \xrightarrow{\pi_\infty} \mathcal{R}_\infty \\
\end{array}
\]

(81)

However, elements of the quotient manifold are equivalence classes of fields which are not easy to manage from the computational point of view. In particular it could be very involved to explicitly write the geometric objects on \(\mathcal{R}_\infty\) that pulls-back to \(\mathcal{M}_\infty\) and in terms of which we would construct the analogue of the diffeomorphism \(\Psi\).
However, in order to keep working on the pre-symplectic manifold $\mathcal{M}_\infty$, we could follow an idea from [DGMS93]. Indeed, as we said above, the problem in constructing the analogue of the diffeomorphism $\Psi$ in this case is that being $\Omega^\Sigma_\infty$ only presymplectic, Hamiltonian vector fields are not uniquely determined. However, it is possible to use a connection in order to associate uniquely Hamiltonian vector fields with functions in $\mathfrak{F}(\mathcal{M}_\infty)$. Indeed, a connection on $\pi_\infty$ can be defined via an idempotent operator $\mathcal{P} : \mathfrak{X}(\mathcal{M}_\infty) \to \mathfrak{X}(\mathcal{M}_\infty)$, which acts on the module of vector fields on the manifold $\mathcal{M}_\infty$ and whose kernel are the so called horizontal vector fields associated with the connection. In this case, the image of the operator $\mathcal{P}$ is chosen to be the distribution $\text{Ker} (\Omega^\Sigma_\infty)$. Then, among the vector fields $\Gamma_\infty$ satisfying:

$$\Omega^\Sigma_\infty(\Gamma_\infty, \cdot ) = d\mathcal{H}(\cdot),$$

(82)
a connection allows to uniquely fix that $\Gamma_\infty$ satisfying

$$\mathcal{P}(\Gamma_\infty) = 0,$$

(83)

namely, the one which is horizontal with respect to the chosen connection. Now, if the connection is flat, it is integrable in the sense that the distribution of horizontal vector fields it generates is a completely integrable one. The integral manifolds of such a distribution give rise to a foliation of $\mathcal{M}_\infty$ whose leaves are transversal to the fibers of $\pi_\infty$. Thus, each of these leaves amounts to a section of $\pi_\infty$, say $\sigma_\mathcal{P}$. Therefore, the discussion above amounts to say that the restriction of $\Omega^\Sigma_\infty$ to the range of $\sigma_\mathcal{P}$, is strongly symplectic and the Hamiltonian vector fields are uniquely defined.

**Remark 4.4.** It is worth stressing that in this work we will deal with an example of gauge theory for which the flatness condition is fulfilled, i.e. Electrodynamics. Thus, here we are excluding the same class of examples for which the construction given in [Kha14] can not be performed globally on the space of fields. However, the advantage of our point of view is that the construction of the Poisson bracket we will make can be immediately generalized to the realm of non-Abelian gauge theories for which the flatness condition is not fulfilled. Indeed, we will do it in the second part of this series $[\text{CDI}^+]$ by using a construction related with the coisotropic embedding theorem already used by the authors to construct a momentum map in the pre-symplectic setting $[\text{CDI}^+ 22]$.

The analogue of the diffeomorphism $\Psi$ of the previous section exists on each $\sigma_\mathcal{P}(\mathcal{R}_\infty) \subset \mathcal{M}_\infty$ and
is the one appearing in the following diagram:

\[
\begin{array}{c}
\text{\(E_L^{\epsilon}(\sigma_p)\)} \\
\xymatrix{
\mathcal{F}_{\mathcal{E}_{\mathcal{L}_{\mathcal{A}}}}^\Sigma(E) & \mathcal{F}_{\mathcal{E}_{\mathcal{L}_{\mathcal{A}}}}^\Sigma(0) \\
\mathcal{F}_{\mathcal{E}_{\mathcal{L}_{\mathcal{A}}}}^\Sigma(R) & \mathcal{F}_{\mathcal{E}_{\mathcal{L}_{\mathcal{A}}}}^\Sigma(M) \\
\sigma_p(R) & \sigma_p(M) \\
\sigma_p(R) & \sigma_p(M) \\
\end{array}
\]

The map \(\Psi\) is the flow of the unique vector field, say \(\Gamma \in \mathfrak{X}(\sigma_p(R))\) satisfying the restriction of (78) to \(\sigma_p(R)\), composed with \(i_\infty\). It is worth stressing that since we are considering initial conditions on the image of a particular \(\sigma_p\) rather than on the whole \(M_\infty\) we are not guaranteed that the image of \(\Psi\) is actually the whole \(E_L^{\epsilon}(M)\). Rather, it is a subset of it that we call \(E_L^{\epsilon}(\sigma_p)\) and which is diffeomorphic (via \(\varpi\)) with a subset of \(E_L^{\epsilon}(M)\) that we call \(E_L^{\epsilon}(\sigma_p)\) and which represents the solution space of the variational problem associated with the chosen connection (and, physically speaking, to a gauge choice).

### 5 Poisson brackets on the solution space

This section is devoted to the main aim of the paper, namely, to write a Poisson bracket on the solution space both in the symplectic case and within Abelian gauge theories.

In the previous sections we constructed a strongly symplectic structure on the solution space of the theory in both situations. Within the symplectic case the strongly symplectic structure was defined on the whole solution space, whereas within gauge theories it was defined on a subset of it associated with a particular gauge fixing, geometrically encoded in the choice of a flat connection. With the above mentioned strongly symplectic structures at hand, the construction of the Poisson bracket is straightforward, being the standard construction usually done in symplectic geometry. In particular, we may use either the structure \(\Pi_\Sigma^*\Omega^\Sigma\) on \(\mathcal{E}_{\mathcal{L}_{\mathcal{A}}}^\epsilon\) or the structure \(\Omega^\Sigma_\infty\) on \(M_\infty\) in order to construct a Poisson bracket. Note that, from the computational point of view, the second way is more convenient since \(\Omega^\Sigma_\infty\) is a 2-form over the manifold \(M_\infty\), while \(\Pi_\Sigma^*\Omega^\Sigma\) is a 2-form on the restriction of \(\mathcal{F}_{\mathcal{E}_{\mathcal{L}_{\mathcal{A}}}}^{\epsilon(0)}\) to a space of solutions of a system of PDE’s. However, being the latters manifolds diffeomorphic, we obtain two equivalent brackets in the two cases as we will show in a moment.

Let us focus for a moment on the symplectic case for simplicity. Consider a function \(F\) on \(E_L^{\epsilon}\). Its Hamiltonian vector field with respect to the strongly symplectic structure \(\Pi_\Sigma^*\Omega^\Sigma\) is the one
The relation between these two brackets is as follows. Consider \( \Pi_\Sigma^* \Omega^\Sigma \) and recall that the Hamiltonian vector field associated to \( \phi \) and, in terms of it a Poisson bracket can be written between \( \mathcal{X}_\phi \) and another function \( g \) on \( \mathcal{M}_\infty \):

\[
\{ g, \phi \} = \Pi_\Sigma^* \Omega^\Sigma (\mathcal{X}_\phi, \mathcal{X}_g) = \mathcal{X}_f (g).
\]

On the other hand, consider the pull-back of \( \phi \) to the space of Cauchy data, i.e., \( \phi := (\varpi^{-1} \circ \Psi)^* \phi \). The Hamiltonian vector field associated to \( \phi \) with respect to \( \Omega^\Sigma_\infty \) is the one satisfying:

\[
\Omega^\Sigma_\infty (\mathcal{X}_\phi, \cdot) = df(\cdot),
\]

and, in terms of it a Poisson bracket can be written between \( \phi \) and another function \( g \) on \( \mathcal{M}_\infty \):

\[
\{ g, \phi \} = \Omega^\Sigma_\infty (\mathcal{X}_\phi, \mathcal{X}_g) = \mathcal{X}_f (g).
\]

The relation between these two brackets is as follows. Consider (85) and recall that \( \Pi_\Sigma^* \Omega^\Sigma = (\Psi^{-1} \circ \varpi)^* \Omega^\Sigma_\infty \) and \( \phi = (\Psi^{-1} \circ \varpi)^* \phi \). Then, equation (85) reads:

\[
\Omega^\Sigma_\infty \left( (\Psi^{-1} \circ \varpi) \mathcal{X}_\phi, (\Psi^{-1} \circ \varpi) \mathcal{X}_g \right) = df \left( (\Psi^{-1} \circ \varpi) \mathcal{X}_g \right) \forall \mathcal{X} \in \chi (\mathcal{E} \mathcal{L}_s^\Sigma).
\]

Now, comparing (87) and (89) we get the following relation between \( \mathcal{X}_\phi \) and \( \mathcal{X}_f \):

\[
\mathcal{X}_\phi = (\varpi^{-1} \circ \Psi)^* \mathcal{X}_f = (\varpi^{-1} \circ i_\infty \circ F^{T\infty} ) \mathcal{X}_f = \varpi^{-1} \circ i_\infty \circ F^{T\infty} \mathcal{X}_f.
\]

Now, by definition of tangent lift of a vector field, \( F^{T\infty} \) is the flow of the tangent lift of \( \Gamma_\infty \), i.e., \( F^{T\infty} = F^{TT} \). Again, following the definition of tangent lift of a vector field, the flow of the tangent lift of some vector field \( \mathcal{X} \), say \( TX \), takes a tangent vector at some point of the manifold and gives the solution of the linearization of the ODEs associated with \( \mathcal{X} \). This means that \( \mathcal{X}_\phi \) is reconstructed from \( \mathcal{X}_f \) in the following way:

- solve the linearization of the ODEs associated with \( \Gamma \) for the initial condition \( \mathcal{X}_f \);
- act on it with \( i_\infty \), which, as we will see in the examples means imposing the tangent lift of the constraint equations emerging from the PCA;
- act with \( \varpi^{-1} \) which, again, is the identification of the parameter \( s \) of the ODEs above with the coordinate transversal to the hypersurface \( \Sigma \).

The bracket associated with \( \Pi_\Sigma^* \Omega^\Sigma \) between any two functions is the pull-back, via \( \Pi_\Sigma \) of the bracket associated with \( \Omega^\Sigma_\infty \) between their restrictions to \( \mathcal{M}_\infty \) as it is seen from the following computation:

\[
\{ G, F \} = \left( \Pi_\Sigma^* \Omega^\Sigma \right)_\chi (\mathcal{X}_G, \mathcal{X}_F) = \left( (\Psi^{-1} \circ \varpi)^* \Omega^\Sigma_\infty \right)_\chi (\mathcal{X}_G, \mathcal{X}_F) = \Omega^\Sigma_\infty (\mathcal{X}_G, \mathcal{X}_F) = \{ g, f \} \Pi_\Sigma (\chi).
\]
Before concluding the section with examples, we want to point out that the whole construction can be reproduced in the case of a gauge theory (as we will see in the example in section 5.3) by recalling that anytime one should compute an Hamiltonian vector field with respect to $\Omega^\Sigma_\infty$ one must impose the additional condition of the vector field being horizontal with respect to some flat connection, i. e., $\mathcal{P}(\chi) = 0$.

### 5.1 Mechanical systems: the free particle on the line

Here we give the explicit construction of the Poisson bracket on the solution space of the equations of the motion of the free particle moving on the line introduced in example 2.5.

In particular, we write down the action of the bracket upon the following functions of the solutions:

$$F = q(t_1), \quad G = q(t_2), \quad (92)$$

giving the position of the particle at two particular instants, say $t_1$ and $t_2$.

In this case the Cauchy hypersurface $\Sigma$ is just a point of the interval $\bar{\mathbb{I}} \subset \mathbb{R}$, say $\bar{t}$, and the space $\mathcal{F}_P^{\Sigma(E)}$ is made by trajectories $\chi(t) = (q(t), p(t))$ restricted to $\bar{t}$, say $\bar{\chi} = \chi|_{\bar{t}}$. The canonical structure $\Pi^\Sigma_\Sigma \Omega^\Sigma$ on the solution space is:

$$\Pi^\Sigma_\Sigma \Omega^\Sigma(\chi_1, \chi_2) = \int_\Sigma \left( X_q^\chi Y_p - X_p^\chi Y_q \right) \omega = (X_q^\chi Y_p - X_p^\chi Y_q)|_{t=\bar{t}}, \quad (93)$$

and, consequently, the structure $\Omega^\Sigma$ reads:

$$\Omega^\Sigma(\chi_1, \chi_2) = (X_q^{\chi_1} Y_p^{\chi_1} - X_p^{\chi_1} Y_q^{\chi_1}), \quad (94)$$

which is indeed the symplectic structure of the cotangent bundle of $\mathbb{R}$. Thus, in this example $\mathcal{F}_P^{\Sigma(E)}$ is already a (finite-dimensional) symplectic manifold and, thus, there is no need to apply the PCA.

The solutions of the equations of the motion written in example 2.5 for the Cauchy datum $(\bar{q}, \bar{p})$ at $\bar{t}$ are:

$$q(t) = \bar{q} + \frac{\bar{p}}{m} (t - \bar{t}), \quad p(t) = \bar{p}, \quad (95)$$

and, consequently the pull-back of $F$ and $G$ to $\mathcal{F}_P^{\Sigma(E)}$, say $f$ and $g$, explicitly read:

$$f = \bar{q} + \frac{\bar{p}}{m} (t_1 - \bar{t}), \quad g = \bar{q} + \frac{\bar{p}}{m} (t_2 - \bar{t}), \quad (96)$$

and they are seen as functions of $\bar{q}$ and $\bar{p}$. The Poisson bracket between $f$ and $g$ is readily computed using the structure $\Omega^\Sigma$ written above. It reads:

$$\{g, f\}_{(\bar{q}, \bar{p})} = \Omega^\Sigma(\chi_1, \chi_2) = \chi_f(g) \quad (97)$$
where:
\[ \Omega^\Sigma_{(\bar{q}, \bar{p})}(X_f, \cdot) = df(\cdot) \]  
which gives:
\[ X_f = \frac{t_1 - \bar{t}}{m} \frac{\partial}{\partial \bar{q}} - \frac{\partial}{\partial \bar{p}}. \]  
Consequently, the bracket is the following function of \((\bar{q}, \bar{p})\) depending on the parameters \(t_1\) and \(t_2\):
\[ \{g, f\}_{(\bar{q}, \bar{p})} = \frac{t_1 - t_2}{m}. \]
In order to compute the bracket between the original functions \(F\) and \(G\) we need to write the solutions of the linearized equations of motion with \(X_f\) as Cauchy datum. Since the equations of motion are linear, the lineatization has no effect, and, thus, solutions of the linearized equations of motion read:
\[ X_{Fq} = X_{f\bar{q}} + \frac{X_{f\bar{p}}}{m} (t - \bar{t}), \quad X_{Fp} = X_{f\bar{p}}. \]
Then, the bracket between \(F\) and \(G\) is computed as follows:
\[ \{G, F\}_\chi = \Pi^*_\Sigma \Omega^\Sigma(X_{Fq}, X_{Gq}) = X_{Fq}(G), \]
which reads:
\[ \{G, F\}_\chi = \frac{t_1 - t_2}{m} \]
now seen as a function of \(\chi\). Note that the bracket between \(F\) and \(G\) is the pull-back of the bracket between \(f\) and \(g\) via the restriction map to \(\bar{t}\) as it comes from (91).

5.2 The symplectic case: massive vector boson field

Here we give the explicit construction of the bracket between two specific functions on the solution space of equations (45). In particular we consider the following functions:
\[ F = \phi^{a_1} \left( \psi^{-1}_{\bar{u}_{\bar{q}1,3}}(x^{0}_{1}, \bar{z}_1) \right) \quad G = P^{0}_{a_2} \left( \psi^{-1}_{\bar{u}_{\bar{q}1,3}}(x^{0}_{2}, \bar{z}_2) \right), \]
giving, respectively, the value of the field and the (0-component of the) momentum, solutions of the equations of motion, at two given points in \(M_{1,3}\), say \(\psi^{-1}_{\bar{u}_{\bar{q}1,3}}(x^{0}_{1}, \bar{z}_1)\) and \(\psi^{-1}_{\bar{u}_{\bar{q}1,3}}(x^{0}_{2}, \bar{z}_2)\). Consider the setting of example 2.6. What is more, we fix the hypersurface \(\Sigma\) to be \(\{x^{0}_{\Sigma}\} \times \mathbb{R}^3 \subset M_{1,3}\). Consequently, the collar \(C^\Sigma_{\epsilon}\) is of the type \([x^{0}_{\Sigma}, x^{0}_{\Sigma} + \epsilon] \times \mathbb{R}^3\) where \(\epsilon\) can be \(+\infty\) and the space of curves \(\gamma(\cdot)\) on \(F^\Sigma_{P(E)}\) is defined for the parameter \(s\) belonging to \([x^{0}_{\Sigma}, +\infty)\). Before computing the bracket let us stress
that with the regularity chosen for the elements in $\mathcal{F}_{\Sigma}(\Sigma)$, $\mathcal{F}_{\Sigma}^\Sigma(\Sigma)$ turns out to be a Hilbert space again. Indeed:

$$\mathcal{F}_{\Sigma}(\Sigma) = \prod_a \mathcal{H}^2(\mathbb{M}^{1,3}, \text{vol}_{\mathbb{M}^{1,3}})^a \times \prod_{\mu,a} \mathcal{H}^1(\mathbb{M}^{1,3}, \text{vol}_{\mathbb{M}^{1,3}})^a,$$  

and by using the trace theorem\(^{17}\) (see [DL90]) for restricting elements of $\mathcal{F}_{\Sigma}(\Sigma)$ to $\Sigma$ we get that

$$\mathcal{F}_{\Sigma}^\Sigma(\Sigma) = \prod_a \mathcal{H}^2(\Sigma, \text{vol}_\Sigma)^a \times \prod_{\mu,a} \mathcal{H}^1(\Sigma, \text{vol}_\Sigma)^a,$$  

which is indeed a Hilbert space whose elements will be denoted by:

$$\chi_{\Sigma} = (\varphi, p, \beta).$$  

Following the procedure outlined in the previous sections, we first need to pull-back $F$ to $\mathcal{M}_\infty$ via $\varpi^{-1} \circ \Psi$, where $\Psi$ is the flow of the dynamical vector field on $\mathcal{M}_\infty$ composed with the immersion of $\mathcal{M}_\infty$ into $\mathcal{F}_{\Sigma}(\Sigma)$. As we will prove in a moment, it is $\mathcal{M}_\infty \cong \mathcal{F}_{\Sigma}^\Sigma(\Sigma)$. In this case, the structure $\Omega^\Sigma = \rho_\Sigma^\star \psi^\ast \omega^\Sigma$ is:

$$\Omega^\Sigma(X, Y) = \int_{\Sigma} \left( X^a_p Y^a_p - X^a_p Y^a_{\varphi} \right) \text{vol}_\Sigma$$  

where $X$, $Y$ are vector fields on $\mathcal{F}_{\Sigma}(\Sigma)$. Clearly, this structure is degenerate with kernel spanned at each point by tangent vectors with only $X_{\beta}^a$ components. We denote them by $X_{\varphi}$. Taking into account the expression (36), the Hamiltonian functional at $\Sigma$, $\mathcal{H}$ evaluated on the curve $\gamma_s$, reads:

$$\mathcal{H}(\gamma_s) = \int_{\Sigma} \left[ \beta^i_{as} \partial_k \varphi^a_s - \frac{1}{2} \left( \eta_{jk} \delta^{ab} \beta^i_{as} \beta^k_{bs} + \delta^{ab} p_{as} p_{bs} + m^2 \delta_{ab} \varphi^a_s \varphi^b_s \right) \right] \text{vol}_\Sigma.$$  

The manifold obtained from the first step of the PCA is the following:

$$i_1(\mathcal{M}_1) = \left\{ \chi_{\Sigma} \in \mathcal{F}_{\Sigma}^\Sigma(\Sigma) : d \mathcal{H}(\chi_{\beta}) = 0 \ \forall \chi_{\beta} \in \ker(\Omega^\Sigma) \right\}.$$  

A direct computation shows that:

$$i_1(\mathcal{M}_1) = \left\{ \chi_{\Sigma} \in \mathcal{F}_{\Sigma}^\Sigma(\Sigma) : \beta^i_a = \delta_{ab} \varphi^a \partial_k \varphi^b \right\}.$$  

Thus, $\mathcal{M}_1$ coincides with $\mathcal{F}_{\Sigma}^\Sigma(\Sigma)$, i.e. the space of $\varphi$s and $p$s, whose immersion into $\mathcal{F}_{\Sigma}(\Sigma)$ is given by the latter equation. The pull-back of $\Omega^\Sigma$ and $\mathcal{H}$ are:

$$i_1^\ast \Omega^\Sigma(X, Y) = \Omega_1^\Sigma(X, Y) = \int_{\Sigma} \left( X^a_p Y^a_p - X^a_p Y^a_{\varphi} \right) \text{vol}_\Sigma,$$

$$i_1^\ast \mathcal{H} = \mathcal{H}_1 = \frac{1}{2} \int_{\Sigma} \left( \delta_{ab} \eta^{jk} \partial_j \varphi^a \partial_k \varphi^b - \delta^{ab} p_{a} p_{b} - m^2 \delta_{ab} \varphi^a \varphi^b \right) \text{vol}_\Sigma.$$  

\(^{17}\)It is exactly the aim of using this theorem to find the restrictions of our fields to $\Sigma$ that lead us to further restrict the fields in 2.6.
The two-form $\Omega_1^\Sigma$ is now symplectic and then the PCA stabilizes. Thus, in this case $\mathcal{M}_\infty = \mathcal{M}_1 = \mathcal{F}_{\mathcal{P}(E),0}$ and $i_\infty = i_1$ which is differentiable with respect to the differential structure of $\mathcal{M}_\infty$ inherited from $\mathcal{F}_{\mathcal{P}(E)}$ and represents the section of the projection $\rho_\Sigma$ given by (111), say $\sigma$:

\[
\mathcal{M}_\infty \equiv \mathcal{F}_{\mathcal{P}(E),0}
\]

The pre-symplectic system reduced to $\mathcal{M}_\infty$ gives rise to the following equations:

\[
\Omega_\infty^\Sigma (\Gamma_{(\varphi,p)}, \Psi_{(\varphi,p)}) = d\mathcal{H}_\infty (\Psi_{(\varphi,p)}),
\]

that reads:

\[
\begin{cases}
\frac{d\varphi^a}{ds} = -\delta^{ab} p_a \\
\frac{dp_a}{ds} = \delta_{ab} \eta^{jk} \partial_j \varphi^b + m^2 \delta_{ab} \varphi^b
\end{cases}
\]

(115)

whose solutions are the integral curves of $\Gamma$. A direct computation shows that solutions of the previous equations are:

\[
\begin{align*}
\varphi^a_s(x) &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \varphi^a_0(x') \cos \left[ \omega_k \left( s - x_0^0 \right) \right] - \delta^{ab} p_b \varphi^a_0(x') \sin \left[ \omega_k \left( s - x_0^0 \right) \right] \right) \frac{s}{\omega_k} e^{ik \cdot (x-x')} d^3k d^3x', \\
p_{as}(x) &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \omega_k \varphi^a_0(x') \sin \left[ \omega_k \left( s - x_0^0 \right) \right] + \delta^{ab} p_b \varphi^a_0(x') \cos \left[ \omega_k \left( s - x_0^0 \right) \right] \right) \frac{s}{\omega_k} e^{ik \cdot (x-x')} d^3k d^3x',
\end{align*}
\]

(116)

where $\varphi^a_0$ and $p_{as}$ are the Cauchy data of the equations on the hypersurface $\Sigma$, $\omega_k = \sqrt{|k|^2 + m^2}$ and $x$, $x'$ and $k$ are points in three (different) $\mathbb{R}^3$. One has that:

\[
f = (\varpi^{-1} \circ \Psi)^* F = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \varphi^a_0(x') \cos \left[ \omega_k \left( x_1^0 - x_0^0 \right) \right] - \delta^{ab} p_b \varphi^a_0(x') \sin \left[ \omega_k \left( x_1^0 - x_0^0 \right) \right] \right) \frac{s}{\omega_k} e^{ik \cdot (x-x')} d^3k d^3x',
\]

(117)

whose Hamiltonian vector field, with respect to the symplectic structure $\Omega_\infty^\Sigma$ has the following components:

\[
\begin{align*}
\mathbb{X}_{f_\varphi} (x) &= -\frac{\delta f}{\delta \varphi^a} (x) = \int_{\mathbb{R}^3} \delta_{a1} \sin \left[ \omega_k \left( x_1^0 - x_0^0 \right) \right] \frac{s}{\omega_k} e^{ik \cdot (x_1-x)} d^3k, \\
\mathbb{X}_{f_{p_a}} (x) &= \frac{\delta f}{\delta \varphi^a} (x) = \int_{\mathbb{R}^3} \delta_{a1} \cos \left[ \omega_k \left( x_1^0 - x_0^0 \right) \right] e^{ik \cdot (x_1-x)} d^3k.
\end{align*}
\]

(118)
Then, the bracket between \( f \) and \( g \) with respect to \( \Omega^\Sigma \) is the (constant) function of \( \varphi \) and \( p^\Sigma \):

\[
\{ g, f \} = \mathcal{X}_f(g) = \int_{\mathbb{R}^3} \cos[\omega_k(x_1^0 - x_2^0)] e^{ik \cdot (\xi - \zeta)} d^3 k .
\] (119)

The Hamiltonian vector field \( \mathcal{X}_F \) with respect to \( \Pi^\Sigma \Omega^\Sigma \) is the image under \( (\varpi \circ \Psi)_\ast \) upon \( \mathcal{X}_f \). Integral curves of \( T\Gamma \) are easily seen to be elements of the type \((\varphi^a, p_a, \mathcal{X}_\varphi^a, \mathcal{X}_{p_a})\) where \((\varphi^a, p_a)\) are solutions of the previous equations and \((\mathcal{X}_\varphi^a, \mathcal{X}_{p_a})\) are solutions of the linearization:

\[
\begin{cases}
\frac{d\varphi^a}{ds} = -\delta^{ab} p^b_a \\
\frac{dp^a}{ds} = \delta_{ab} \eta \partial_j \partial_k \varphi^b + m^2 \delta_{ab} \varphi^b
\end{cases}
\] (120)

which is identical to the original equations. The solutions obtained are:

\[
\begin{align*}
\mathcal{X}_\varphi^a(x) &= \int_{\mathbb{R}^3} \delta_0 \sin \left[ \frac{\omega_k(x^1 - s)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k , \\
\mathcal{X}_{p_a}(x) &= \int_{\mathbb{R}^3} \delta_0 \cos \left[ \frac{\omega_k(x^1 - s)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k .
\end{align*}
\] (121)

Their immersion into \( T\mathcal{F}^\Sigma P(x) \) gives the curve:

\[
\begin{align*}
\mathcal{X}_{\varphi^a}(x) &= \int_{\mathbb{R}^3} \delta_0 \sin \left[ \frac{\omega_k(x^0)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k , \\
\mathcal{X}_{p_a}(x) &= \int_{\mathbb{R}^3} \delta_0 \cos \left[ \frac{\omega_k(x^0)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k , \\
\mathcal{X}_{\varphi^b}(x) &= \int_{\mathbb{R}^3} \eta^i \xi_k \delta_0 \sin \left[ \frac{\omega_k(x^0)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k .
\end{align*}
\] (122)

The action of \( \varpi_\ast \) is given by the identification of the parameter \( s \) with the coordinate \( x^0 \) in the (global) chart chosen on \( \mathbb{M}^{1,3} \). This gives the vector field \( \mathcal{X}_F \) whose components read:

\[
\begin{align*}
\mathcal{X}_{F \varphi^a}(x, \zeta) &= \int_{\mathbb{R}^3} \delta_0 \sin \left[ \frac{\omega_k(x^0 - x^0)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k , \\
\mathcal{X}_{F p_a}(x, \zeta) &= \int_{\mathbb{R}^3} \delta_0 \cos \left[ \frac{\omega_k(x^0 - x^0)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k , \\
\mathcal{X}_{F \varphi^b}(x, \zeta) &= \int_{\mathbb{R}^3} \eta^i \xi_k \delta_0 \sin \left[ \frac{\omega_k(x^0 - x^0)}{\omega_k} \right] e^{ik \cdot (\xi - \zeta)} d^3 k .
\end{align*}
\] (123)

Now, recalling that \( G = P_a^0(x^0_2, x^i_2) \), then the bracket between \( F \) and \( G \) is computed to be:

\[
\{ G, F \}_\chi = \mathcal{X}_F(G) = \int_{\mathbb{M}^{1,3}} \mathcal{X}_{F p_a}(x^0, \zeta) \frac{\delta G}{\delta P_a^0} d^4 x = \int_{\mathbb{R}^3} \cos \omega_k(x^1 - x^0) e^{ik \cdot (\xi - \zeta)} d^3 k .
\] (124)

Note that, as we already argued in general, the bracket just obtained is the pull-back of the function (119) to \( \mathcal{E} \mathcal{L} \) via \( \Pi \); and, since (119) was a constant function on the space of Cauchy data, they have the same expression.
5.3 Gauge theories: Electrodynamics

Consider the example 2.7 where Maxwell equations have been considered within a multisymplectic Hamiltonian formalism. \( \mathcal{M}^{4,3, \eta} \) denotes the Minkowski spacetime and let \( \Sigma \) be the surface \( x^0 = x^0_\Sigma \), so that \( \mathcal{M}^{4,3} \simeq \Sigma \times \mathbb{R} \). A (global) chart on \( \Sigma \) is denoted by \( (U_\Sigma, \psi_\Sigma) \), \( \psi_\Sigma(x) = x \), with \( x \in U_\Sigma \subset \Sigma \), whereas a chart on \( \mathbb{R} \) is denoted by \( (U_\mathbb{R}, \psi_\mathbb{R}) \), \( \psi_\mathbb{R}(r) = x^0 \), with \( r \in U_\mathbb{R} \subset \mathbb{R} \). The space \( \xi_{\mathcal{L}^p G} \) is made by curves \( \gamma : \mathbb{R} \rightarrow \mathcal{F}_\Sigma^G \). The components of these curves are denoted as follows:

\[
\begin{align*}
a_{0,s}(\psi^{-1}_\Sigma(x)) &= A_0(\psi^{-1}_\Sigma(x^0)) = s, \\
a_{j,s}(\psi^{-1}_\Sigma(x)) &= A_j(\psi^{-1}_\Sigma(x^0)) = s, \\
p^k_s(\psi^{-1}_\Sigma(x)) &= P^{kl}(\psi^{-1}_\Sigma(x^0)) = s, \quad \beta^{jk}_s(\psi^{-1}_\Sigma(x)) = P^{jk}(\psi^{-1}_\Sigma(x^0)) = s,
\end{align*}
\]  

(125)  

with the labels \( j, k = 1, 2, 3 \). In analogy to the previous analysis, the space \( \mathcal{F}^G_\Sigma^G \) reads:

\[
\mathcal{F}^G_\Sigma(\mathcal{L}) = \prod_\mu H^2(\Sigma, \text{vol}_\Sigma) \times \prod_{\mu, \nu} H^4(\Sigma, \text{vol}_\Sigma)^{\mu\nu}.
\]  

(126)  

The Hamiltonian introduced in example 2.7 gives rise to the following Lagrangian on \( \mathcal{T}(\mathcal{F}^G_\Sigma^{G}) \) which, evaluated along (the tangent lift of) a curve \( \gamma_s \) (in the next formulas we avoid explicitly denoting the dependence on the parameter \( s \) for the sake of notational simplicity) reads:

\[
\mathcal{L} = \int_{\mathcal{E}_\Sigma} \left[ p^k (\dot{a}_k - \partial_ka_0) - \frac{1}{2} \beta^{kj} (\partial_ka_j - \partial_ja_k) - \frac{1}{4} \delta_{ij} \delta_{km} \beta^{jk} \beta^{lm} - \frac{1}{2} \delta_{jk} p^k p^j \right] \text{vol}_\Sigma,
\]  

(127)  

from which one gets the following Hamiltonian functional:

\[
\mathcal{H}(\gamma) = \int_{\mathcal{E}_\Sigma} \left[ p^k \partial_ka_0 + \frac{1}{2} \beta^{kj} (\partial_ka_j - \partial_ja_k) + \frac{1}{4} \delta_{ij} \delta_{km} \beta^{jk} \beta^{lm} + \frac{1}{2} \delta_{jk} p^k p^j \right] \text{vol}_\Sigma.
\]  

(128)  

On the other hand, the structure \( \Omega^G_\Sigma \) in this case reads:

\[
\Omega^G(\xi, \gamma) = \int_{\Sigma} (\xi_{ak} \gamma^k_p - \xi^k_p \gamma_{ak}) \text{vol}_\Sigma.
\]  

(129)  

The structure \( \Omega^G_\Sigma \) is clearly pre-symplectic, its kernel is spanned, at each point \( \chi_\Sigma \), by tangent vectors having only components \( \xi_{a_0} \) or \( \xi^{jk} \). The first step of the PCA gives the following manifold:

\[
i_1(\mathcal{M}_1) = \left\{ \chi_\Sigma \in \mathcal{F}_\Sigma^G : i_{\xi_{\chi_\Sigma}} \text{d}\mathcal{H} = 0 \right\},
\]  

(130)  

where \( T_{\chi_\Sigma} \mathcal{M}_1 = 0 \) is nothing but the kernel of \( \Omega^G_\Sigma \) at \( \chi_\Sigma \). The latter condition gives the following constraints:

\[
\partial_ka_k = 0, \quad \beta^{jk} = -2 \delta^{jlm} F_{lm}.
\]  

(131)  

The second constraint can be eliminated by inserting the expression on the right hand side in terms of the \( a_k \)s in the \( \beta \)s (which represent the passage from the description in terms of magnetic field to
the description in terms of vector potential), whereas the first constraint, representing the Gauss’
constraint, is a non-holonomic one that cannot be eliminated. Therefore, the manifold \( M_1 \) reads:

\[
M_1 = \left\{ (a_k, p^k) \in F^\Sigma_{\cal P}(E) : \partial_j p^j = 0 \right\}
\]  

(132)

whose immersion into \( F^\Sigma_{\cal P}(E) \) is given by the second of (131) and by fixing any arbitrary \( a_0 \). The
manifold \( M_1 \) is seen to be also the final manifold of the PCA, so that, \( M_\infty = M_1 \). The pull-back of \( \Omega^\Sigma \) and \( \mathcal{H} \) to the final manifold via \( i_1 = i_\infty \) reads:

\[
i_1^* \Omega^\Sigma(\mathcal{X}, \mathcal{Y}) = \Omega^\Sigma_1(\mathcal{X}, \mathcal{Y}) = \int_\Sigma \left( \mathcal{X}_{ak} \mathcal{Y}_p^k - \mathcal{Y}_{ak} \mathcal{X}_p^k \right) \text{vol}_\Sigma,
\]

\[
i_1^* \mathcal{H}(\gamma_1) = \mathcal{H}_1(\gamma_1) = \int_\Sigma \left[ p^k \partial_k a_{0,s} + \frac{1}{4} \delta^{jl} \delta^{km} F_{jks} F_{lms} + \frac{1}{2} \delta_{jk} p_s^k p_s^m \right] \text{vol}_\Sigma = \int_\Sigma \left[ \frac{1}{4} \delta^{jl} \delta^{km} F_{jks} F_{lms} + \frac{1}{2} \delta_{jk} p_s^k p_s^m \right] \text{vol}_\Sigma.
\]

(133)

where \( \gamma_1 \) is a curve on \( M_1 \) and the latter equality is the result of an integration by parts. The structure
\( \Omega^\Sigma_1 \) is again pre-symplectic. Indeed, tangent vectors with \( \mathcal{X}_{ak} \) component of the type \( \mathcal{X}_{ak} = \partial_k \zeta \) (for some function \( \zeta \)) lie in the kernel of \( \Omega^\Sigma_1 \) (at each point \( (a, p) \)) because of the Gauss’ constraint. As
in the previous examples, the immersion map is easily seen to be differentiable with respect to the
differential structure of \( M_\infty \). On \( M_\infty \) the canonical equation reads:

\[
p^k = \delta^{kj} \frac{da_j}{ds}, \quad \frac{dp^k}{ds} = \delta^{km} \delta^{jl} \partial_j \partial_m - \delta^{kl} \delta^{jm} \partial_j \partial_m.
\]

(134)

Since we took \( F_{\cal P}(E) = \prod_{\mu=0,\ldots,3} \mathcal{H}^2(\mathcal{D}^{1,3}, \text{vol}_{\mathcal{D}^{1,3}})_\mu \times \prod_{\mu,\nu=0,\ldots,3} \mathcal{H}^1(\mathcal{D}^{1,3}, \text{vol}_{\mathcal{D}^{1,3}})^{\mu\nu} \), then \( M_\infty \) turns to be:

\[
M_\infty = \prod_k \mathcal{H}^1_{\frac{1}{2}}(\Sigma, \text{vol}_\Sigma)_k \times \prod_k \mathcal{H}^1_{\frac{1}{2}}(\text{div}0; \Sigma, \text{vol}_\Sigma)_k
\]

(135)

where \( \mathcal{H}^1_{\frac{1}{2}}(\text{div}0; \Sigma, \text{vol}_\Sigma) \) denotes the space of divergenceless \( \mathcal{H}^1_{\frac{1}{2}} \)-functions, div denoting the adjoint of the
grad operator acting from \( \mathcal{H}^2_{\frac{1}{2}}(\Sigma, \text{vol}_\Sigma) \) to \( \mathcal{H}^1_{\frac{1}{2}}(\Sigma, \text{vol}_\Sigma) \). The latter space is a closed subspace
of the space of square integrable functions as the following proposition proves, and, thus, it is an Hilbert
space itself.

**Proposition 5.1 (Closedness of grad from \( \mathcal{H}^2_{\frac{1}{2}} \) to \( \mathcal{H}^1_{\frac{1}{2}} \)).** The gradient operator is a closed
operator between \( \mathcal{H}^2_{\frac{1}{2}}(\Sigma, \text{vol}_\Sigma) \) and \( \mathcal{H}^1_{\frac{1}{2}}(\Sigma, \text{vol}_\Sigma) \).

**Proof.** Consider a sequence of functions in \( \mathcal{H}^2_{\frac{1}{2}}(\Sigma, \text{vol}_\Sigma) \), say \( \{ f_n \}_{n \in \mathbb{N}} \), converging to some \( f \) in
\( \mathcal{H}^2_{\frac{1}{2}}(\Sigma, \text{vol}_\Sigma) \). Then, the following inequalities hold:

\[
\sum_j \| \partial_j (f_n - f) \|_{H^2_{\frac{1}{2}}} = \sum_{j,k} \int_{\mathbb{R}^3} |\tilde{f}_n - \tilde{f}|^2 j_k k |k| d^3 k \leq 9 \int_{\mathbb{R}^3} |\tilde{f}_n - \tilde{f}|^3 |k|^3 d^3 k = 9 \| f_n - f \|_{H^2_{\frac{1}{2}}} ,
\]

(136)

which proves that grad maps closed sets into closed sets. \( \square \)
Therefore, by virtue of the closed range theorem, the kernel of the adjoint of grad, i.e. \( \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma}) \), is a closed subspace of \( \mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) \) whose complement is the image of grad into \( \mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) \), i.e. the following decomposition into closed (and, thus, Hilbert subspaces) exists:

\[
\prod_{k} \mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma})^{k} = \prod_{k} \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma})^{k} \oplus \text{grad}\mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}).
\] (137)

A similar proof shows that:

\[
\mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) = \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma}) \oplus \text{grad}\mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}),
\] (138)

where the first term is the space of \( \mathcal{H}^{2} \) functions with zero divergence and the second term is the image via the gradient operator of the space of \( \mathcal{H}^{2} \) functions. According to this splitting, \( \mathcal{M}_{\infty} \) splits as:

\[
\mathcal{M}_{\infty} = \prod_{k} \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma})^{k} \times \text{grad}\mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) \times \prod_{k} \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma})^{k},
\] (139)

where all the terms are closed (and, thus, Hilbert) subspaces of \( \mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) \) and \( \mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) \) respectively. Let us denote by \( m_{\infty} = (\tilde{a}_{k}, \partial \phi^{k}, p^{k}) \) a point in \( \mathcal{M}_{\infty} \) and notice that the tangent space to \( \mathcal{M}_{\infty} \) splits as follows:

\[
T_{m_{\infty}}\mathcal{M}_{\infty} = T\tilde{a}_{0} \prod_{k} \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma})^{k} \oplus T\partial \phi \text{grad}\mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) \oplus T_{p} \prod_{k} \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma})^{k} = \prod_{k} \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma})^{k} \oplus \text{grad}\mathcal{H}^{2}(\Sigma, \text{vol}_{\Sigma}) \oplus \prod_{k} \mathcal{H}^{2}(\text{div}0, \Sigma, \text{vol}_{\Sigma})^{k},
\] (140)

since all the space considered are Hilbert spaces and, thus, isomorphic with their tangent spaces. This decomposition of \( \mathcal{M}_{\infty} \) and of its tangent space is particularly suitable for our purposes because the second component of \( T_{m_{\infty}}\mathcal{M}_{\infty} \) coincide with the kernel of \( \Omega_{\Sigma}^{\infty}(T\partial \phi \text{grad}\mathcal{H}^{2} = \ker \Omega_{\Sigma_{m_{\infty}}}^{\infty}) \), i.e., with the generators of the so called gauge transformations. What is more, to write down such a decomposition amounts to fix a complement of \( \ker \Omega_{\Sigma}^{\infty} \) into \( T_{m_{\infty}}\mathcal{M}_{\infty} \). In particular, the choice made so far amounts to fix a particular connection on the bundle \( \mathcal{M}_{\infty} \rightarrow \mathcal{M}_{\infty}/\ker \Omega_{\Sigma}^{\infty} \), i.e. the so called COULOMB CONNECTION [CI85], given by the following idempotent operator:

\[
P : \mathcal{X}(\mathcal{M}_{\infty}) \rightarrow \mathcal{X}^{H}(\mathcal{M}_{\infty}) : P(\mathcal{X}) = \tilde{\mathcal{X}},
\] (141)

where \( \tilde{\mathcal{X}} \) has components:

\[
\tilde{\mathcal{X}}_{ak} = 0,
\]

\[
\tilde{\mathcal{X}}_{\partial \phi k} = \partial_{k} \int_{\Sigma} G_{\Delta}(y, x) \delta^{ij} \partial_{j} \delta \phi (y) \text{vol}_{\Sigma}^{y},
\]

\[
\tilde{\mathcal{X}}_{p} = 0,
\] (142)
with $G_\Delta(y, x)$ the Green function of the Laplacian. The latter connection is exactly the one we will use to compute Hamiltonian vector fields in this pre-symplectic case, following the general theory presented in the previous sections.

Consider a function $f$ on $\mathcal{M}_\infty$. Its Hamiltonian vector field with respect to $\Omega^\Sigma_\infty$ is the one satisfying:

$$\begin{cases} 
\Omega^\Sigma_\infty(\mathcal{X}_f, \cdot) = df(\cdot) \\
\mathcal{P}(\mathcal{X}_f) = 0
\end{cases}$$

(143)

From the first condition we see that the vector field $\mathcal{X}_f$ has the following components:

$$\tilde{\mathcal{X}}_{f_\alpha k} + \mathcal{X}_f\partial \phi_k = \frac{\delta f}{\delta p^k}, \quad \mathcal{X}_f^k_p = -\frac{\delta f}{\delta a_k}.$$  

(144)

On the other hand, it is readily proven that the second equation actually fixes $\mathcal{X}_f\partial \phi_k$ to vanish. This fixes the bracket between two functions on $\mathcal{M}_\infty$ to be:

$$\{ g, f \} = \Omega^\Sigma_\infty(\mathcal{X}_g, \mathcal{X}_f) = \int_\Sigma \left( \frac{\delta f}{\delta p^k} \frac{\delta g}{\delta a_k} - \frac{\delta f}{\delta a_k} \frac{\delta g}{\delta p^k} \right) \text{vol}_\Sigma.$$  

(145)

Let us consider two functions on $\mathcal{E}_{\mathcal{P}_{1,3}}$, say $F$ and $G$. Following the procedure described in this section, the Hamiltonian vector field associated with $F$, say $\mathcal{X}_F$ with respect to the canonical structure $\Pi_\mathcal{P}\Omega^\Sigma$ is obtained as the solution of the linearized Hamiltonian equations associated with $(\Omega^\Sigma, \mathcal{H})$ (with the gauge condition given by the chosen connection) with Cauchy datum given by the Hamiltonian vector field $\mathcal{X}_f$ associated with $f = (\varpi^{-1} \circ \Psi)^* F$ with respect to $\Omega^\Sigma_\infty$ and the connection $\mathcal{P}$. Then the bracket between $G$ and $F$ is given by the action of the vector field $\mathcal{X}_F$ on $G$.

As a specific example, we consider the following two functions:

$$F_{k_1} = A_{k_1}(x^0_1, x_1) = \int_{\mathcal{P}_{1,3}} A_{k_1}(x^0, x) \delta(x^0 - x_1) \delta(x - x_1) \text{vol}_{\mathcal{P}_{1,3}},$$  

$$G_{k_2} = A_{k_2}(x^0_2, x_2) = \int_{\mathcal{P}_{1,3}} A_{k_2}(x^0, x) \delta(x^0 - x_2) \delta(x - x_2) \text{vol}_{\mathcal{P}_{1,3}},$$

(146)

namely, those giving the value of (the $k_1$-component of) the configuration field, solution of the equations of the motion, at the point $\psi_{V_{1,3},(x^0_1, x_1)(x^0_2, x_2)}^{-1}$. Having in mind the procedure described to find the bracket, we first need to pull-back $F$ to a function $\tilde{F}$ on the space of Cauchy data via the diffeomorphism $\varpi^{-1} \circ \Psi$. The space of Cauchy data is one leaf of the foliation induced by $\mathcal{P}$ on the manifold $\mathcal{M}_\infty$ obtained from the PCA, since any point in $\mathcal{M}_\infty$ belongs to one single leaf $\sigma_\mathcal{P}$ of $\mathcal{P}$ and for any point in $\sigma_\mathcal{P}$ there exists a unique solution of the equations of the motion:

$$\Omega^\Sigma_\infty(\Gamma, \cdot) = d\mathcal{H}_\infty(\cdot),$$

(147)

provided with the condition:

$$\mathcal{P}(\Gamma) = 0,$$  

(148)
which ensures the flow of \( \Gamma \) to entirely lie on the same leaf \( \sigma_P \). In order to compute \( f \) we need to write \( A_{k_1} \) as the image under \( \varpi \) of a solution of the equations of the motion for \( a_{k_1} \) and explicitly show the dependence on the initial conditions. By looking at (134), \( a_{k,s} \) is the solution of:

\[
\frac{d^2 a_{k,s}}{ds^2} = \delta^{ij} \partial_j a_{k,s} - \delta^{ij} \partial_j a_{l,s},
\]

which is unique (for any fixed initial condition \( a_\Sigma^\Sigma(y), p_\Sigma^k(y) \)) after imposing the condition \( P(\Gamma) = 0 \). Indeed, the latter condition imposes \( \delta^{jk} \partial_j a_k = 0 \), that is, the so called COULOMB'S GAUGE, since it is a projector onto the subspace of fields \( \prod_k H^2_3(\text{div}0, \Sigma, \text{vol}_\Sigma) \times \prod_k H^2_2(\text{div}0, \Sigma, \text{vol}_\Sigma) \). Thus, we are lead to the following equation:

\[
\frac{d^2 a_{k,s}}{ds^2} - \delta^{ij} \partial_j a_{k,s} = 0,
\]

namely, the homogeneous wave equation for \( a_{k,s} \) for which an existence and uniqueness theorem for the solutions exists within the considered space of fields. The latter equation is solved by:

\[
a_{k,s}(x) = \frac{1}{4\pi} \left[ \int_\Sigma \left( a_{k}^\Sigma(y) + |s - x_\Sigma^0| \delta_{kj} p_{\Sigma}^k(y) \right) \tilde{G}_{\Sigma,|s-x_\Sigma^0|}(y) \text{vol}_\Sigma^y + \int_\Sigma |s - x_\Sigma^0| a_{k}^\Sigma(y) \frac{\partial}{\partial s} \tilde{G}_{\Sigma,(s-x_\Sigma^0)}(y)(\Theta(s - x_\Sigma^0) - \Theta(x_\Sigma^0 - s)) \text{vol}_\Sigma^y \right],
\]

with \( \tilde{G}_{\Sigma,a}(y) \) being the characteristic function of the surface of the sphere with center \( x \) and radius \( a \) and \( \Theta \) being the Heaviside function. With the expression (151) in mind, the pull-back of \( F \) to the space of Cauchy data is readily computed to be:

\[
f = \frac{1}{4\pi} \left[ \int_\Sigma \left( a_{k}^\Sigma(y) + |s_1 - x_\Sigma^0| \delta_{kj} p_{\Sigma}^k(y) \right) \tilde{G}_{\Sigma,|s_1-x_\Sigma^0|}(y) \text{vol}_\Sigma^y + \int_\Sigma |s_1 - x_\Sigma^0| a_{k}^\Sigma(y) \frac{\partial}{\partial s} \tilde{G}_{\Sigma,(s-x_\Sigma^0)}(y) \bigg|_{s=s_1} (\Theta(s_1 - x_\Sigma^0) - \Theta(x_\Sigma^0 - s_1)) \text{vol}_\Sigma^y \right].
\]

Taking into account this expression for \( f \) and the analogous for \( g \), a direct computation gives the
bracket between the functions $g$ and $f$ on the space of Cauchy data:

$$\{g, f\}_{(\alpha, \beta)} = \int_{\Sigma} \left( \frac{\delta f}{\delta \alpha_k} \frac{\delta g}{\delta \alpha_k} - \frac{\delta f}{\delta \alpha_k} \frac{\delta g}{\delta \beta_k} \right)$$

$$= \frac{\delta_{k_1 k_2}}{16\pi^2} \int_{\Sigma} \left\{ \left( |x_1^0 - x_1^0| - |x_1^0 - x_1^0| \right) \tilde{G}_{\Sigma_1, |x_1^0 - x_1^0|} \left( y \right) \tilde{G}_{\Sigma_1, |x_1^0 - x_1^0|} \left( y \right) + \left| x_1^0 - x_1^0 \right| |x_2^0 - x_2^0| \left[ \frac{\partial}{\partial s} \tilde{G}_{\Sigma_1, (s-x_2^0)} \left( y \right) \right]_{s=x_1^0} \left( \Theta(x_1^0 - x_1^0) - \Theta(x_1^0 - x_1^0) \right) \tilde{G}_{\Sigma_2, |x_2^0 - x_2^0|} \left( y \right) + \left[ \frac{\partial}{\partial x^0} \tilde{G}_{\Sigma_1, (x^0-x_2^0)} \left( y \right) \right]_{x^0=x_1^0} \left( \Theta(x_1^0 - x_1^0) - \Theta(x_1^0 - x_1^0) \right) \tilde{G}_{\Sigma_2, |x_2^0 - x_2^0|} \left( y \right) \right\} v_{\Sigma_1}^2 \cdot (153)

In order to evaluate the action of the bracket upon $F$ and $G$ we need to compute the Hamiltonian vector field associated with $f$ with respect to $\Omega_{\Sigma_0}^2$ and $\mathcal{P}$, say $\mathcal{X}_f$, and then we need to reconstruct the Hamiltonian vector field $\mathcal{X}_F$ via solving the linearized equations of the motion with $\mathcal{X}_F$ as Cauchy datum. In this case the linearization of (150) is equation (150) itself since it is linear. With this in mind, $\mathcal{X}_F$ is the vector field with components:

$$\mathcal{X}_{F, \alpha k}(x, x^0) = \frac{\delta_{k_1 k_2}}{16\pi^2} \int_{\Sigma} \left\{ \left( |x_1^0 - x_1^0| - |x_1^0 - x_1^0| \right) \tilde{G}_{\Sigma_1, |x_1^0 - x_1^0|} \left( y \right) \tilde{G}_{\Sigma_1, |x_1^0 - x_1^0|} \left( y \right) + \left| x_1^0 - x_1^0 \right| |x_2^0 - x_2^0| \left[ \frac{\partial}{\partial s} \tilde{G}_{\Sigma_1, (s-x_2^0)} \left( y \right) \right]_{s=x_1^0} \left( \Theta(x_1^0 - x_1^0) - \Theta(x_1^0 - x_1^0) \right) \tilde{G}_{\Sigma_2, |x_2^0 - x_2^0|} \left( y \right) + \left[ \frac{\partial}{\partial x^0} \tilde{G}_{\Sigma_1, (x^0-x_2^0)} \left( y \right) \right]_{x^0=x_1^0} \left( \Theta(x_1^0 - x_1^0) - \Theta(x_1^0 - x_1^0) \right) \tilde{G}_{\Sigma_2, |x_2^0 - x_2^0|} \left( y \right) \right\} v_{\Sigma_1}^2 \cdot (154)

$$

$$\mathcal{X}_{F, \beta j k} = \delta_{i j} \frac{\partial}{\partial x^0} X_{F, \beta i} \cdot (155)$$

and

$$\mathcal{X}_{F, \beta k} = \delta_{k l} \delta_{j m} \frac{\partial}{\partial x^0} \left( \partial_x \mathcal{X}_{F, a m} - \partial_{x_m} \mathcal{X}_{F, a l} \right) \cdot (156)$$

Such analysis gives

$$\{G, F\}_\chi = \int_{\mathbb{R}^3} \left[ \mathcal{X}_{F, \alpha k} \frac{\delta G}{\delta A_k} (x, x^0) + \mathcal{X}_{F, \beta p} \frac{\delta G}{\delta p_k} \right] v_{\mathbb{R}^3} \cdot (157)$$

which is coherent with (153).
Conclusions

To conclude, we showed how the solution space of a class of first order Hamiltonian field theories can be equipped with a Poisson manifold structure. In particular we saw that such a Poisson manifold structure comes from a pre-symplectic structure that naturally emerges from the Schwinger-Weiss variational principle formulated within the multisymplectic formulation of first order Hamiltonian field theories.

The structure emerging from the variational principle was of the type $\Pi^{\Sigma} \Omega^{\Sigma}$ where $\Pi^{\Sigma}$ is the restriction of the fields to some hypersurface $\Sigma$ in the space-time where the fields are defined. The structure $\Omega^{\Sigma}$ on $F_{\mathcal{P}(E)}$ from which it comes from is related with a structure $\Omega_{\infty}^{\Sigma}$ on the final manifold of the pre-symplectic constraint algorithm, say $\mathcal{M}_{\infty}$, used to deal with the constraints of the theory. In particular $\Omega_{\infty}^{\Sigma} = i_{\infty}^{\ast} \Omega^{\Sigma}$ where $i_{\infty}$ is the immersion of the final manifold into $F_{\mathcal{P}(E)}$. We proved that, in theories without gauge symmetries, $\Omega_{\infty}^{\Sigma}$ is symplectic and a diffeomorphism, say $\Psi$, exists associating to each Cauchy datum in $\mathcal{M}_{\infty}$ a unique solution of the equations of the motion. In particular $\Psi$ is such that $i_{\infty} \circ \Psi^{-1} = \Pi^{\Sigma}$. This allowed to prove that the structure on $\mathcal{E} \mathcal{L}_{\#,}^{\Sigma} \Pi^{\ast}_{\Sigma} \Omega^{\Sigma}$ is equal to $\Psi^{-1} \Omega_{\infty}^{\Sigma}$. Thus, being $\Omega_{\infty}^{\Sigma}$ is symplectic and $\Psi$ a diffeomorphism, $\Pi^{\ast}_{\Sigma} \Omega^{\Sigma}$ turns out to be symplectic as well and determines a Poisson bracket.

Within Abelian gauge theories we slightly modified the construction. Indeed, the structure $\Omega^{\Sigma}_{\infty}$ on the final submanifold, $\mathcal{M}_{\infty}$, of the pre-symplectic constraint algorithm was not symplectic. However, we were able to prove that it is symplectic when restricted to the leaves of a flat (and, thus, integrable) connection on the bundle $\mathcal{M}_{\infty}/\ker \Omega^{\Sigma}_{\infty}$ that is canonically defined for any gauge theory\(^{18}\). From the physical point of view the choice of such a connection amounts to a gauge fixing which, in the case of the connection we chose in Electrodynamics, is the Coulomb gauge. We saw that the structure $\Omega^{\Sigma}_{\infty}$ restricted to the leaves of such a connection is symplectic and, thus, with the same construction used above, so is the structure $\Pi^{\ast}_{\Sigma} \Omega^{\Sigma}|_{\sigma_{\mathcal{P}}}$ ($\sigma_{\mathcal{P}}$ representing a leaf of the foliation). The case where a flat connection can not be chosen because of intrinsic obstructions of the theory is exemplified by non-Abelian gauge theories. We will deal with them in the second part of this series [CDI+ where we will construct a Poisson bracket by working within the framework developed in the present paper and by using the coisotropic embedding theorem. Such a construction will show that the emergence of additional degrees of freedom, interpreted as ghost fields, seems to be necessary in order for properly defining a Poisson structure.

A The pre-symplectic constraint algorithm

For the sake of completeness, we report in this appendix how the pre-symplectic constraint algorithm used in Sect. 4 and 5 works. We refer to [GNH78] for a more extensive discussion.

\(^{18}\)Actually it is flat only for those gauge theories which do not exhibit the so-called Gribov anomaly.
Let us consider a pre-symplectic Hamiltonian system, i.e., a triple \((\mathcal{M}, \omega, \mathcal{H})\) where \(\mathcal{M}\) is a Banach manifold, \(\omega\) a closed 2-form on it and \(\mathcal{H}\) a real-valued function on \(\mathcal{M}\). Denoting by \(\flat\) the map between \(T\mathcal{M}\) and \(T^*\mathcal{M}\) induced by \(\omega\):

\[
\flat : T\mathcal{M} \to T^*\mathcal{M} : (m, X_m) \mapsto (m, \omega_m(X_m, \cdot)) ,
\]

we say that \(\mathcal{M}\) is strongly symplectic if \(\flat\) is an isomorphism, that it is weakly symplectic if \(\flat\) is injective but not surjective and that it is pre-symplectic if \(\flat\) is neither injective nor surjective.

The problem one wants to address is to understand whether and in which sense one is able to find a solution, for a pre-symplectic \(\omega\), of the equation:

\[
i_\Gamma \omega = d\mathcal{H}
\]

where \(\Gamma\) is a vector field on \(\mathcal{M}\). If \(\omega\) has a non-trivial kernel \(K\), such an equation does not make sense unless \(d\mathcal{H}\big|_K = 0\) or, equivalently, \(d\mathcal{H}_m \in T_m\mathcal{M}^\perp\) where \(T_m\mathcal{M}^\perp\) is the image of \(T_m\mathcal{M}\) via \(\flat\). Defining 

\[
T_m\mathcal{M}^\perp = \{ X_m \in T_m\mathcal{M} : i_{X_m} \omega = 0 \} \subset T_m\mathcal{M},
\]

then the points of \(\mathcal{M}\) for which \(d\mathcal{H}_m \in T_m\mathcal{M}^\perp\) give the space \(^{19}\mathcal{M}_1\) defined by

\[
\mathcal{M}_1 = \{ m_1 \in \mathcal{M} : i_{X_{m_1}} d\mathcal{H}_{m_1} = 0 \quad \forall X_{m_1} \in T_{m_1}\mathcal{M}^\perp \} .
\]

**Remark A.1.** Note that \(T_m\mathcal{M}^\perp\) is the kernel of \(\omega\) at \(m\), i.e. \(K_m\). However we used such a notation because it is better suited to formulate the first step of the algorithmic procedure we are going to describe in the rest of the section.

Indeed, a definition of ortosymplectic complement can be given for any immersed submanifold \(\mathcal{N}\) of \(\mathcal{M}\), say \(^{20}\)

\[
T_n\mathcal{N}^\perp = \{ X_n \in T_n\mathcal{M}|_\mathcal{N} : i^*(i_{X_n}\omega) = 0 \}
\]

and, thus, the definition of \(T_m\mathcal{M}^\perp\) given above can be thought of as an example of position (161) in the case where the submanifold \(\mathcal{N}\) is the manifold \(\mathcal{M}\) itself.

Equation (159) makes sense if restricted to \(\mathcal{M}_1\) but nothing is said about the fact that the solution \(\Gamma\) is tangent to \(\mathcal{M}_1\). However, this is exactly what we want to happen, because, from the physical point of view, the fact that we are restricting to the points of the submanifold \(\mathcal{M}_1\) means that we are constraining our "physical variables" via some constraint relations that we want to be preserved along the dynamics, that is, along the flow of \(\Gamma\). For this reason we want \(\Gamma\) to be tangent to \(\mathcal{M}_1\) or, more precisely, we want \(\Gamma\) to be a vector field on \(\mathcal{M}\) being \(i_1\)-related to some \(\Gamma_1 \in \mathfrak{X}(\mathcal{M}_1)\), where \(i_1\) is the immersion of \(\mathcal{M}_1\) into \(\mathcal{M}\). By imposing this, we are further restricting to a submanifold \(^{21}\) of

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\(^{19}\)The set defined by (160) is not necessarily a submanifold embedded into \(\mathcal{M}\). This relies on assumptions depending on specific cases.

\(^{20}\)\(i\) denotes the immersion of \(\mathcal{N}\) into \(\mathcal{M}\).

\(^{21}\)Again, that this is a submanifold relies on specific assumptions.
that we denote by $\mathcal{M}_2$ and that can be characterized in the following way. Assume that an $\mathcal{M}_2$ exists for which the solution $\Gamma$ of

\[
(i_\Gamma \omega - dH)_{\mathcal{M}_2} = 0
\]  

(162)

exists and is $i_2$-related\(^{22}\) to a vector field $\Gamma_2 \in \mathfrak{X}(\mathcal{M}_2)$. Then, a straightforward computation shows that if $X_m \in T_m \mathcal{M}$ is such that $i_2^*(i_{X_m} \omega) = 0$, i.e., if $X_m \in T_m \mathcal{M}_2^\perp$, then $i_2^*(i_{X_m} dH) = 0$. This last condition is equivalent to $i_{X_m} dH = 0 \ \forall \ X_m \in T_m \mathcal{M}_2^\perp$ since, being $X_m \in T_m \mathcal{M}_2^\perp$, $i_{X_m} dH$ is a function defined on $\mathcal{M}_2$ and, thus, its pull-back via $i_2$ coincides with the function itself. Therefore, $i_{X_m} dH = 0 \ \forall \ X_m \in T_m \mathcal{M}_2^\perp$ is a necessary condition for a solution $\Gamma$ of (162) which is tangent to $\mathcal{M}_2$ to exist. By imposing such a necessary condition we restrict to a submanifold\(^{23}\) of $\mathcal{M}_2$, say $\mathcal{M}_3$ where, again, we must check that the canonical equations:

\[
(i_\Gamma \omega - dH)_{\mathcal{M}_3} = 0
\]  

(163)

are well posed, i.e., that $i_{X_m} dH = 0 \ \forall \ X_m \in T_m \mathcal{M}_3^\perp$.

Therefore, we turned into an algorithmic procedure which, at the $n$-th step, defines the submanifold $\mathcal{M}_n$ of the original manifold $\mathcal{M}$ given by:

\[
\mathcal{M}_n = \left\{ m \in \mathcal{M}_{n-1} : i_{X_m} dH = 0 \ \forall \ X_m \in T_m \mathcal{M}_{n-1}^\perp \right\}
\]  

(164)

where

\[
T_m \mathcal{M}_{n-1}^\perp = \left\{ X_m \in T_m \mathcal{M}_{n-1} : i_{X_m}^* (i_{X_m} \omega) = 0 \right\},
\]  

(165)

and where $\mathcal{M}_0 := \mathcal{M}$ and $T_m \mathcal{M}_0^\perp := T_m \mathcal{M}^\perp = K_m$.

Such an algorithm is called the\(^{24}\) pre-symplectic constraint algorithm (PCA) and is said to converge (or to stabilize) when there exists an $n$ such that $\mathcal{M}_n = \mathcal{M}_{n-1} =: \mathcal{M}_\infty$, which is called the final manifold of the algorithm. Now, on the final manifold $\mathcal{M}_\infty$, the canonical equation:

\[
(i_\Gamma \omega - dH)_{\mathcal{M}_\infty} = 0
\]  

(166)

makes sense and has solution $\Gamma$ tangent to $\mathcal{M}_\infty$, i.e., $\Gamma$ is $i_\infty$-related with a $\Gamma_\infty \in \mathfrak{X}(\mathcal{M}_\infty)$. Such a $\Gamma_\infty$ satisfies the equation:

\[
i_{\Gamma_\infty} \omega_\infty = dH_\infty,
\]  

(167)

where $\omega_\infty = i_\infty^* \omega$ and $H_\infty = i_\infty^* H$. The solution $\Gamma_\infty$ of the last equation is also unique if $\omega_\infty$ is non-degenerate and, in this case, gives rise to a unique solution of the original problem (159) which is given by the integral curves of $\Gamma_\infty$ immersed into $\mathcal{M}$ via $i_\infty$.

\(^{22}\)Here $i_2$ is the immersion map of $\mathcal{M}_2$ into $\mathcal{M}$.

\(^{23}\)Again, that this is actually a submanifold is an assumption.
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