Stars on trees

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Abstract

For a positive integer \( r \) and a vertex \( v \) of a graph \( G \), let \( \mathcal{I}_G^{(r)}(v) \) denote the set of all independent sets of \( G \) that have exactly \( r \) elements and contain \( v \). Hurlbert and Kamat conjectured that for any \( r \) and any tree \( T \), there exists a leaf \( z \) of \( T \) such that \( |\mathcal{I}_T^{(r)}(v)| \leq |\mathcal{I}_T^{(r)}(z)| \) for each vertex \( v \) of \( T \). They proved the conjecture for \( r \leq 4 \). For any \( k \geq 3 \), we construct a tree \( T_k \) that has a vertex \( x \) such that \( x \) is not a leaf of \( T_k \), \( |\mathcal{I}_{T_k}^{(r)}(z)| < |\mathcal{I}_{T_k}^{(r)}(x)| \) for any leaf \( z \) of \( T_k \) and any \( 5 \leq r \leq 2k + 1 \), and \( 2k + 1 \) is the largest integer \( s \) for which \( \mathcal{I}_{T_k}^{(s)}(x) \) is non-empty. Therefore, the conjecture is not true for \( r \geq 5 \).

1 Introduction

We shall use small letters such as \( x \) to denote non-negative integers or elements of a set, capital letters such as \( X \) to denote sets or graphs, and calligraphic letters such as \( \mathcal{F} \) to denote families (that is, sets whose members are sets themselves). The set \( \{1, 2, \ldots\} \) of positive integers is denoted by \( \mathbb{N} \). For any \( m,n \in \mathbb{N} \), the set \( \{i \in \mathbb{N} : m \leq i \leq n\} \) is denoted by \([m,n]\), and we abbreviate \([1,n]\) to \([n]\). For a set \( X \), the family \( \{A \subseteq X : |A| = r\} \) of all \( r \)-element subsets of \( X \) is denoted by \( \binom{X}{r} \). If \( x \in X \) and \( \mathcal{F} \) is a family of subsets of \( X \), then the family \( \{F \in \mathcal{F} : x \in F\} \) is denoted by \( \mathcal{F}(x) \) and is called a star of \( \mathcal{F} \). All arbitrary sets are assumed to be finite.

A graph \( G \) is a pair \((X,Y)\), where \( X \) is a set, called the vertex set of \( G \), and \( Y \) is a subset of \( \binom{X}{2} \) and is called the edge set of \( G \). The vertex set of \( G \) and the edge set of \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. An element of \( V(G) \) is called a vertex of \( G \), and an element of \( E(G) \) is called an edge of \( G \). We may represent an edge \( \{v,w\} \) by \( vw \). A vertex \( v \) of \( G \) is a leaf of \( G \) if there exists exactly one vertex \( w \) of \( G \) such that \( vw \in E(G) \).

If \( H \) is a graph such that \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \), then we say that \( G \) contains \( H \).
If \( n \geq 2 \) and \( v_1, v_2, \ldots, v_n \) are the distinct vertices of a graph \( G \) with \( E(G) = \{v_i v_{i+1} : i \in [n-1]\} \), then \( G \) is called a \((v_1, v_n)\)-path or simply a path.

A graph \( G \) is a tree if \( |V(G)| \geq 2 \) and \( G \) contains exactly one \((v, w)\)-path for every \( v, w \in V(G) \) with \( v \neq w \).

Let \( G \) be a graph. A subset \( I \) of \( V(G) \) is an independent set of \( G \) if \( uv \notin E(G) \) for every \( v, w \in I \). Let \( I_G^{(r)} \) denote the family of all independent sets of \( G \) of size \( r \). An independent set \( J \) of \( G \) is maximal if \( J \not\subseteq I \) for each independent set \( I \) of \( G \) such that \( I \neq J \). The size of a smallest maximal independent set of \( G \) is denoted by \( \mu(G) \).

Hurlbert and Kamat \(^8\) conjectured that for any \( r \geq 1 \) and any tree \( T \), there exists a leaf \( z \) of \( T \) such that \( I_T^{(r)}(z) \) is a star of \( I_T^{(r)} \) of maximum size.

**Conjecture 1.1** (\(^8\) Conjecture 1.25) For any \( r \geq 1 \) and any tree \( T \), there exists a leaf \( z \) of \( T \) such that \( |I_T^{(r)}(v)| \leq |I_T^{(r)}(z)| \) for each \( v \in V(T) \).

Hurlbert and Kamat \(^8\) also showed that the conjecture is true for \( r \leq 4 \). In the next section, we show that the conjecture is not true for \( r \geq 5 \). For any \( k \geq 3 \), we construct a tree \( T_k \) that has a vertex \( x \) such that \( x \) is not a leaf of \( T_k \), \( |I_{T_k}^{(r)}(z)| < |I_{T_k}^{(r)}(x)| \) for any leaf \( z \) of \( T_k \) and any \( r \in [5, 2k+1] \), and \( 2k+1 \) is the largest integer \( s \) for which \( I_{T_k}^{(r)}(x) \) is non-empty.

Conjecture \(^1\) was motivated by a problem of Holroyd and Talbot \(^3\) \(^7\). A family \( \mathcal{A} \) is intersecting if every two sets in \( \mathcal{A} \) intersect. We say that \( I_G^{(r)} \) has the star property if at least one of the largest intersecting subfamilies of \( I_G^{(r)} \) is a star of \( I_G^{(r)} \). Holroyd and Talbot introduced the problem of determining whether \( I_G^{(r)} \) has the star property for a given graph \( G \) and an integer \( r \geq 1 \). The Holroyd–Talbot (HT) Conjecture \(^7\) claims that \( I_G^{(r)} \) has the star property if \( \mu(G) \geq 2r \). By the classical Erdős–Ko–Rado Theorem \(^4\), the HT Conjecture is true if \( G \) has no edges. The HT Conjecture has been verified for certain graphs \(^2\) \(^3\) \(^6\) \(^7\) \(^8\) \(^9\) \(^10\). It is also verified in \(^1\) for any graph \( G \) with \( \mu(G) \) sufficiently large depending on \( r \); this is the only result known for the case where \( G \) is a tree that is not a path (the problem for paths is solved in \(^3\)), apart from the fact that \( I_G^{(r)} \) may not have the star property for certain values of \( r \) (indeed, if \( G \) is the tree \( ([0] \cup [n], \{0, i : i \in [n]\}) \) and \( 2 \leq n/2 < r < n \), then \( I_G^{(r)} = \binom{n}{2} \) and \( \binom{n}{r} \) is intersecting). One of the difficulties in trying to establish the star property lies in determining a largest star. Our counterexample to Conjecture \(^1\) indicates that the problem for trees is more difficult than is hoped.

### 2 The counterexample

Let \( x_0 = 0, x_1 = 1 \) and \( x_2 = 2 \). For any \( k \in \mathbb{N} \), let \( y_i = 2 + i \) for each \( i \in [2k] \), let \( z_i = 2k + 2 + i \) for each \( i \in [2k] \), and let \( T_k \) be the graph whose vertex set is

\[
\{x_0, x_1, x_2\} \cup \{y_i : i \in [2k]\} \cup \{z_i : i \in [2k]\}
\]

and whose edge set is

\[
\{x_0x_1, x_0x_2\} \cup \{x_1y_i : i \in [k]\} \cup \{x_2y_i : i \in [k+1, 2k]\} \cup \{y_iz_i : i \in [2k]\}.
\]
Theorem 2.1 Let $k \in \mathbb{N}$.
(a) The graph $T_k$ is a tree, and the leaves of $T_k$ are $z_1, \ldots, z_{2k}$.
(b) The largest integer $s$ such that $I^{(s)}_{T_k}(x_0) \neq \emptyset$ is $2k + 1$.
(c) If $k \geq 3$, then $|I^{(r)}_{T_k}(z)| < |I^{(r)}_{T_k}(x_0)|$ for any leaf $z$ of $T_k$ and any $r \in [5, 2k + 1]$.

Proof. (a) is straightforward.
Let $G = T_k$. Let $Y = \{y_i : i \in [2k]\}$ and $Z = \{z_i : i \in [2k]\}$.
We have $\{x_0\} \cup Z \in I_G^{(2k+1)}(x_0)$. Suppose that $S$ is a set in $I_G^{(s)}(x_0)$. Then $S \setminus \{x_0\} \in (Y \cup Z)$ and $|(S \setminus \{x_0\}) \cap \{y_i, z_i\}| \leq 1$ for each $i \in [2k]$. Thus $s - 1 \leq 2k$, and hence $s \leq 2k + 1$. Hence (b).
Suppose $k \geq 3$ and $r \in [5, 2k + 1]$. Let $J = I^{(r)}_G$. Let $E = \{I \in J : x_0, z_1 \in I\}$. Let
$$
\begin{align*}
A_1 &= \{I \in J(x_0) : y_1 \in I\}, \\
A_2 &= \{I \in J(x_0) : y_1, z_1 \notin I\}, \\
B_1 &= \{I \in J(z_1) : x_0 \notin I, x_1 \in I, x_2 \notin I\}, \\
B_2 &= \{I \in J(z_1) : x_0 \notin I, x_1 \notin I, x_2 \in I\}, \\
B_3 &= \{I \in J(z_1) : x_0 \notin I, x_1, x_2 \in I\}, \\
B_4 &= \{I \in J(z_1) : x_0, x_1, x_2 \notin I\}.
\end{align*}
$$
We have $J(x_0) = E \cup A_1 \cup A_2$ and $J(z_1) = E \cup B_1 \cup B_2 \cup B_3 \cup B_4$. Since $y_1, z_1 \in E(G)$, $\{y_1, z_1\} \not\subset I$ for each $I \in J$. Thus $E, A_1$ and $A_2$ are pairwise disjoint, and hence
$$
|J(x_0)| = |E| + |A_1| + |A_2|.
$$
(1)
Since $E, B_1, B_2, B_3$ and $B_4$ are pairwise disjoint,
$$
|J(z_1)| = |E| + |B_1| + |B_2| + |B_3| + |B_4|.
$$
(2)
Let $Y' = Y \setminus \{y_1\}$ and $Z' = Z \setminus \{z_1\}$. Since $x_0x_1, x_0x_2 \in E(G)$, we have $\{x_0, x_1\}$, $\{x_0, x_2\} \not\subset I$ for each $I \in J$. Thus $A \setminus \{x_0\} : A \in A_2 = I^{(r-1)}_G \cap (Y' \cup Z') = \{B \setminus \{z_1\} : B \in B_4\}$, and hence
$$
|A_2| = |B_4|.
$$
(3)
Let $Y_1 = \{y_i : i \in [2, k]\}$ and $Y_2 = \{y_i : i \in [k + 1, 2k]\}$. Let
$$
\begin{align*}
A_1' &= \{A \setminus \{x_0, y_1\} : A \in A_1\}, \\
B_1' &= \{B \setminus \{z_1, x_1\} : B \in B_1\}, \\
B_2' &= \{B \setminus \{z_1, x_2\} : B \in B_2\}, \\
B_3' &= \{B \setminus \{z_1, x_1, x_2\} : B \in B_3\}.
\end{align*}
$$
We have $A_1' = I^{(r-2)}_G \cap (Y' \cup Z')$, $B_1' = I^{(r-2)}_G \cap (Y' \cup Z')$, $B_2' = I^{(r-2)}_G \cap (Y' \cup Z')$ and $B_3' = (Z')$. Let $C = \{I \in I^{(r-2)}_G \cap (Y' \cup Z') : I \cap Y_1 \neq \emptyset \neq I \cap Y_2\}$. Thus $A_1' = B_1' \cup B_2' \cup C$. We have $(B_1' \cup B_2') \cap C = \emptyset$ and $B_1' \cap B_2' = (Z')$. Thus
$$
|A_1'| = |B_1' \cup B_2'| + |C| = |B_1'| + |B_2'| - |B_1' \cap B_2'| + |C| = |B_1'| + |B_2'| - \binom{2k - 1}{r - 2} + |C|.
$$
Let \( a = |A'_1| - (|B'_1| + |B'_2| + |B'_3|) \). Then
\[
a = |C| - \binom{2k-1}{r-2} - \binom{2k-1}{r-3} = \sum_{i=1}^{k-1} \sum_{j=1}^{k} \binom{|Y_1|}{i} \binom{|Y_2|}{j} \binom{|Z'| - i - j}{r - 2 - i - j} - \binom{2k-1}{r-2} - \binom{2k-1}{r-3}.
\]

\[
= \sum_{i=1}^{k-1} \sum_{j=1}^{k} \binom{k - 1}{i} \binom{k}{j} \binom{2k - 1 - i - j}{r - 2 - i - j} - \binom{2k-1}{r-2} - \binom{2k-1}{r-3}.
\]

(4)

We show that \( a > 0 \). If \( r = 2k + 1 \), then
\[
a = \sum_{i=1}^{k-1} \binom{k - 1}{i} \sum_{j=1}^{k} \binom{k}{j} - 2k = (2k-1)(2k-1) - 2k > 0.
\]

Suppose \( r \leq 2k \). We have
\[
a \geq \binom{k - 1}{1} \binom{k}{1} \binom{2k - 3}{r - 4} - \binom{2k-1}{r-2} - \binom{2k-1}{r-3} = \binom{2k - 3}{r - 4} \binom{k}{1} \binom{2k - 1}{r - 3} - \frac{(2k - 1)(2k - 2)}{(r - 2)(r - 3)} - \frac{(2k - 1)(2k - 2)}{(r - 3)(2k + 2 - r)}.
\]

If \( r \geq 6 \), then
\[
a \geq \binom{2k - 3}{r - 4} \binom{k}{1} \binom{2k - 1}{r - 4} - \frac{(2k - 1)(2k - 2)}{(4)(3)} - \frac{(2k - 1)(2k - 2)}{(3)(2)} > 0.
\]

If \( r = 5 \) and \( k \geq 5 \), then
\[
a \geq \binom{2k - 3}{r - 4} \binom{k}{1} \binom{2k - 1}{r - 4} - \frac{(2k - 1)(2k - 2)}{(3)(2)} - \frac{(2k - 1)(2k - 2)}{(2)(7)} > 0.
\]

If \( r = 5 \) and \( 3 \leq k \leq 4 \), then \( a > 0 \) is easily obtained from (4).

Since \( a > 0 \), \( |A'_1| > |B'_1| + |B'_2| + |B'_3| \). Now \( |A'_1| = |A_1|, |B'_1| = |B_1|, |B'_2| = |B_2| \) and \( |B'_3| = |B_3| \). Thus \( |A'_1| > |B_1| + |B_2| + |B_3| \). By (1), (2) and (3), it follows that \( |J(x_0)| > |J(z_1)| \). Clearly, for each \( i \in [2k] \), we have \( |J(z_i)| = |J(z_1)| \), and hence \( |J(z_i)| < |J(x_0)| \). By (a), (c) follows.

If \( I \) is a maximal independent set of \( T_k \), then \( |I \cap \{x_0, x_1, x_2\}| \geq 1 \) and \( |I \cap \{y_i, z_i\}| = 1 \) for each \( i \in [2k] \). Thus \( \mu(T_k) = 2k + 1 \). Therefore, if \( 5 \leq r \leq (2k + 1)/2 \), then the condition \( \mu(T_k) \geq 2r \) of the HT Conjecture is satisfied, but, by Theorem 2.1, no leaf of \( T_k \) yields a star of \( T^{(r)}_{k} \) of maximum size.
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