GABOR ANALYSIS FOR SCHRODINGER EQUATIONS AND PROPAGATION OF SINGULARITIES

ELENA CORDERO, FABIO NICOLA, AND LUIGI RODINO

Abstract. We consider the Schrödinger equation

$$i \frac{\partial u}{\partial t} + Hu = 0, \quad H = a(x, D),$$

where the Hamiltonian $a(z), z = (x, \eta)$, is assumed real-valued and smooth, with bounded derivatives $|\partial^\alpha a(z)| \leq C_\alpha$, for every $|\alpha| \geq 2, z \in \mathbb{R}^{2d}$. For such equation results are known concerning well-posedness of the Cauchy problem for initial data in $L^2(\mathbb{R}^d)$ and local representation of the propagator $e^{itH}$ by means of Fourier integral operators.

In the present paper we give a global expression for $e^{itH}$ in terms of Gabor analysis and we deduce boundedness in modulation spaces. Moreover, by using time-frequency techniques, we obtain a result of propagation of micro-singularities for $e^{itH}$.

1. Introduction

Time-frequency Analysis, also named Gabor Analysis cf. [22, 26], has found important applications in Signal Processing and related problems in Numerical Analysis, see for example [8, 42] and references therein. More recently, time-frequency methods have been applied to the study of the partial differential equations, in particular constant coefficient wave, Klein-Gordon, parabolic and Schrödinger equations [2, 3, 12, 21, 31, 32, 33, 36, 45, 46], let us also refer to the survey [41] and the monograph [47]. The analysis of variable coefficient Schrödinger type equations was carried out in [1, 9, 10, 13, 14, 15, 16, 19, 43], see also [17, 18] in the analytic category.

In the present paper we address to the Schrödinger equation

$$\begin{cases}
    i \frac{\partial u}{\partial t} + a(x, D)u = 0 \\
    u(0, x) = u_0(x).
\end{cases}$$

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where the symbol \( a(z), z = (x, \xi) \), is real-valued and smooth, satisfying
\[
|\partial^\alpha a(z)| \leq C\alpha, \quad |\alpha| \geq 2, z \in \mathbb{R}^{2d}.
\]
In (1) we understand
\[
a(x, D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i(x, \xi)} a(x, \xi) \hat{f}(\xi) d\xi,
\]
with
\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i(x, \xi)} f(x) dx.
\]
As basic example one can consider the case when \( a(z) \) is a quadratic form in \( z \), including the free particle operator \( a(x, D) = -\Delta \) and the harmonic oscillator
\[
a(x, D) = -\Delta + |x|^2.
\]
Under the assumption (2), it is easy to show by energy methods that the Cauchy problem (1) is well-posed in \( L^2(\mathbb{R}^d) \) and \( S(\mathbb{R}^d) \), see for example [43] and we may consider the propagator
\[
ed^{itH} : S(\mathbb{R}^d) \rightarrow S(\mathbb{R}^d), \quad H = a(x, D),
\]
mapping the initial datum \( u_0 \) to the solution \( u(t, x) \) at time \( t \in \mathbb{R} \).

Going further, one would like to obtain an explicit expression for \( e^{itH} \), from which one may obtain precise estimates for the solutions, and numerical analysis. To this end, by using Fourier methods, one expects for small values of \( t \) a representation as type I Fourier integral operator
\[
ed^{itH} u_0)(t, x) = \int_{\mathbb{R}^d} e^{2\pi i\Phi(t, x, \eta)} b(t, x, \eta) \hat{u}_0(\eta) d\eta.
\]
In fact, under our assumption (2), the phase function \( \Phi \) has quadratic growth with respect to \( x, \eta \) and the amplitude \( b \) is in the Hörmander class \( S^0_{0,0} \), i.e., bounded together with its derivatives. The case of Hamiltonians \( a(x, \eta) \) of polyhomogeneous type was first treated in [7, 28, 29]. For the present general case, where homogeneity is not assumed, see e.g. [1, 5, 6, 11, 24, 25, 34, 35, 38, 44].

Time-frequency analysis enters the picture at this moment, giving a global expression for \( e^{itH} \). Let us recall the notation for the time-frequency shifts:
\[
\pi(z)f(t) = M_y T_x f(t) = e^{2\pi i(t, \eta)} f(t - x), \quad x, \eta, t \in \mathbb{R}^d, \quad z = (x, \eta).
\]
The short-time Fourier transform (STFT) of a function or distribution \( f \) on \( \mathbb{R}^d \) with respect to a Schwartz window function \( g \in S(\mathbb{R}^d) \setminus \{0\} \) is defined by
\[
V_g f(x, \eta) = \langle f, \pi(z)g \rangle = \int_{\mathbb{R}^d} f(v)g(v - x) e^{-2\pi i(v, \eta)} dv, \quad z = (x, \eta) \in \mathbb{R}^{2d}.
\]
The time-frequency representation of a linear continuous operator \( P : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \) is provided by the (continuous) Gabor matrix
\[
k(w, z) := \langle P \pi(w) g, \pi(z) g \rangle, \quad w, z \in \mathbb{R}^{2d}
\]
so that
\[
V_g(P f)(z) = \int_{\mathbb{R}^{2d}} k(w, z) V_g f(w) \, dw.
\]
Following the pattern of [20], we study the Gabor matrix \( k(t, w, z) \) of the propagator \( e^{itH} \). Its structure is linked with the Hamiltonian field of \( a(x, \xi) \). Precisely, consider
\[
\begin{cases}
2\pi \dot{x} = -\nabla_\xi a(x, \xi) \\
2\pi \dot{\xi} = \nabla_x a(x, \xi) \\
x(0) = y, \quad \xi(0) = \eta,
\end{cases}
\]
(the factor \( 2\pi \) depends on the normalization of the STFT). The solution \( \chi_t(y, \eta) = (x(t, y, \eta), \xi(t, y, \eta)) \) exists for all \( t \in \mathbb{R} \) thanks to our hypothesis, and defines a symplectic diffeomorphism \( \chi_t : \mathbb{R}_y^d \to \mathbb{R}_x^d \). The components of \( \chi_t \) are functions with bounded smooth derivatives of any order in \( \mathbb{R} \times \mathbb{R}^d \).

**Theorem 1.1.** Let \( k(t, w, z) \) be the Gabor matrix of the Schrödinger propagator \( e^{itH} \). Then for every \( s > 0 \) there exists \( C = C(t, s) > 0 \) such that
\[
|k(t, w, z)| \leq C(z - \chi_t(w))^{-s}, \quad z = (x, \xi), \quad w = (y, \eta) \in \mathbb{R}^{2d}.
\]

For \( t \) small enough our assumptions yield \( \det \frac{\partial x}{\partial y}(t, y, \eta) \neq 0 \) in the expression of \( \chi_t \), and (10) is then equivalent to (4) with the phase \( \Phi \) linked to \( \chi_t \) as standard and \( b(t, \cdot) \in S_{0,0}^r, \) see the next Section 2. In the classical approach, cf. [1], the occurrence of caustics makes the validity of (4) local in time and for global time \( t \in \mathbb{R} \) multiple compositions of local representations are used, with unbounded number of variables possibly appearing in the expression. Observe instead that \( k(t, w, z) \) keeps meaning for every \( t \in \mathbb{R} \), and the estimates (10) hold for \( \chi_t \) with \( t \in \mathbb{R} \).

A natural functional frame to express boundedness and propagation results for \( e^{itH} \) is given by the modulation spaces, see [23] and the short survey in Section 2. For \( 1 \leq p \leq \infty, \, r \in \mathbb{R}, \) the modulation space \( M^p_r(\mathbb{R}^d) \) is defined by the space of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) for which

\[
\|f\|_{M^p_r(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} |V_g f(z)|^p \langle z \rangle^r dz < \infty
\]
(with obvious changes for \( p = \infty \).)

From Theorem 1.1 it is easy to deduce the following
Theorem 1.2. For every $r \in \mathbb{R}$, $1 \leq p \leq \infty$, $t \in \mathbb{R}$, we have
\begin{equation}
\label{eq:12}
e^{itH} : M^p_r(\mathbb{R}^d) \to M^p_r(\mathbb{R}^d).
\end{equation}

The proofs of Theorems 1.1 and 1.2 are given in [20] when the symbol $a(x, \eta)$ is a polyhomogeneous symbol. The proof for the present non-homogeneous situation follows closely the one in [20]. Our aim being to introduce non-expert readers to the methods of the time-frequency analysis, we shall reproduce in the sequel the main lines of the argument.

Novelty with respect to [20] will be a result of propagation of (micro) singularities for the solutions of (11). In fact, in [20] by taking advantage of the homogeneous structure, the propagation was expressed in terms of the global (Gabor) wave front set, whereas in (11), (2) homogeneity is lost in general. This will require the use of a more refined notion, namely the filter of the Gabor singularities, see below.

Let us first recall that the global wave front set was introduced by Hörmander [30] in 1991, see also [40], where the name of Gabor wave front set was given. The work of Hörmander [30] was addressed to the study of the hyperbolic equations with double characteristics, and also provided propagation of singularities for Schrödinger equations (11) with quadratic Hamiltonian $a(z)$. Such result was generalized to different classes of linear and nonlinear equations, beside [20] see for example [37, 39] and [18], concerning the analytic category. One may find in these papers references to the wide previous literature on the subject.

The renewed and increasing interest for the Gabor wave front set derives from the fact that, under the action of a metaplectic operator, it moves according to the associated linear symplectic transformation. More generally, if $P$ is a Fourier integral operator as in (4) with a phase function $\Phi$ homogeneous of degree 2 in $(x, \eta)$, then the Gabor wave front set is determined by the corresponding map $\chi_t$.

Returning to the present non-homogeneous context, our definition of Gabor (micro) singularity will be as follows. Let $\Gamma$ be a subset of $\mathbb{R}^{2d}$. Given a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, we say that $f$ is $M^p_r$ regular in $\Gamma$, $1 \leq p \leq \infty$, $r \in \mathbb{R}$, if there exists a neighborhood $\Gamma_\delta$ of $\Gamma$ (see precise definitions in Section 4) such that
\begin{equation}
\label{eq:13}
\int_{\Gamma_\delta} |V_\delta f(z)|^p \langle z \rangle^{pr} dz < \infty.
\end{equation}

We shall call filter of the $M^p_r$ singularities of $f$ the collection of subsets of $\mathbb{R}^{2d}$
\begin{equation}
\label{eq:14}
\mathcal{F}^p_r(f) = \{ \Lambda \subset \mathbb{R}^{2d}, \ f \text{ is } M^p_r \text{ regular in } \Gamma = \mathbb{R}^{2d} \setminus \Lambda \}.
\end{equation}

Note that $f$ is regular in any bounded set $\Gamma \subset \mathbb{R}^{2d}$.

We may now state our main result.

Theorem 1.3. For every $r \in \mathbb{R}$, $1 \leq p \leq \infty$, $t \in \mathbb{R}$, $u_0 \in \mathcal{S}'(\mathbb{R}^d)$, we have
\begin{equation}
\label{eq:15}
\chi_t(\mathcal{F}^p_r(u_0)) = \mathcal{F}^p_r(e^{itH} u_0).
\end{equation}
The content of the next sections is the following. Section 2 is devoted to some preliminaries, concerning properties of the short-time Fourier transform, modulation spaces and canonical transformations. Section 3 presents the theory of the Fourier integral operators in terms of Gabor analysis, cf [10]. The proofs of Theorems 1.1 and 1.2 are given as application. Section 4 contains the analysis of the Gabor singularities and the proof of Theorem 1.3.

2. Preliminaries

In what follows there are the basic concepts of time-frequency analysis. For details we refer to [26]. We also discuss the properties of phase functions and canonical transformations.

2.1. The Short-time Fourier Transform. Given a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ (the so-called window), the short-time Fourier transform (STFT) of $f$ with respect to $g$ is defined as in (6). The short-time Fourier transform is well-defined whenever the bracket $\langle \cdot, \cdot \rangle$ makes sense for dual pairs of function or distribution spaces, in particular for $f \in \mathcal{S}'(\mathbb{R}^d)$, $f, g \in L^2(\mathbb{R}^d)$. Consider $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$.

We recall the following pointwise inequality for the short-time Fourier transform [26, Lemma 11.3.3], used to change window functions.

**Lemma 2.1.** If $g_0, g_1, \gamma \in \mathcal{S}(\mathbb{R}^d)$ such that $\langle \gamma, g_1 \rangle \neq 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, then the inequality

$$|V_{g_0} f(x, \xi)| \leq \frac{1}{|\langle \gamma, g_1 \rangle|} (|V_{g_1} f| * |V_{g_0} \gamma|)(x, \xi)$$

holds pointwise for all $(x, \xi) \in \mathbb{R}^{2d}$.

2.2. Modulation spaces. Weighted modulation spaces measure the decay of the STFT on the time-frequency (phase space) plane. They were introduced by Feichtinger in the 80’s [23].

**Weight Functions.** A weight function $v$ is submultiplicative if $v(z_1 + z_2) \leq v(z_1) v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$. We shall work with the weight functions

$$(16) \quad v_s(z) = \langle z \rangle^s = (1 + |z|^2)^{\frac{s}{2}}, \quad s \in \mathbb{R},$$

which are submultiplicative for $s \geq 0$.

For $s \geq 0$, we denote by $\mathcal{M}_{v_s}(\mathbb{R}^{2d})$ the space of $v_s$-moderate weights on $\mathbb{R}^{2d}$. These are measurable positive functions $m$ satisfying

$$m(z + w) \leq C v_s(z)m(w)$$

for every $z, w \in \mathbb{R}^{2d}$. 
Definition 2.2. Given $g \in S(\mathbb{R}^d)$, $s \geq 0$, $m \in M_{v s}(\mathbb{R}^{2d})$, and $1 \leq p, q \leq \infty$, the modulation space $M_{m}^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_{m}^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_{m}^{p,q}(\mathbb{R}^d)$ is

$$\|f\|_{M_{m}^{p,q}} = \|V_g f\|_{L_{m}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p m(x, \xi)^p \, dx \right)^{q/p} \, d\xi \right)^{1/q}$$

(obvious modifications if $p = \infty$ or $q = \infty$).

When $p = q$, we write $M_{m}^{p}(\mathbb{R}^d)$ instead of $M_{m}^{p,p}(\mathbb{R}^d)$ and $M_{s}^{p}(\mathbb{R}^d)$ for $M_{m}^{p,v s}(\mathbb{R}^d)$.

The spaces $M_{m}^{p,q}(\mathbb{R}^d)$ are Banach spaces and every nonzero $g \in M_{m}^{1}(\mathbb{R}^d)$ yields an equivalent norm in (17) and hence $M_{m}^{p,q}(\mathbb{R}^d)$ is independent on the choice of $g \in M_{m}^{1}(\mathbb{R}^d)$.

We recover the Hörmander class

$$S^0_{0,0} = \bigcap_{s \geq 0} M_{1 \otimes v s}^{\infty}(\mathbb{R}^{2d}).$$

For any $1 \leq p, q \leq \infty$,

$$\bigcap_{s \geq 0} M_{v s}^{p,q}(\mathbb{R}^d) = S(\mathbb{R}^d), \quad \bigcup_{s \geq 0} M_{v s}^{p,q}(\mathbb{R}^d) = S'(\mathbb{R}^d).$$

Fix $g \in S(\mathbb{R}^d) \setminus \{0\}$. The adjoint operator of $V_g$, defined by $\langle V_g^* F, h \rangle = \langle F, V_g h \rangle$, can be written as

$$V_g^* F = \int_{\mathbb{R}^{2d}} F(x, \xi) \pi(x, \xi) g dx d\xi,$$

The adjoint $V_g^*$ maps the Banach space $L_{m}^{p,q}(\mathbb{R}^{2d})$ into $M_{m}^{p,q}(\mathbb{R}^d)$, in particular it maps $S(\mathbb{R}^{2d})$ into $S(\mathbb{R}^d)$ and the same for their dual spaces. If $F = V_g f$ we obtain the inversion formula for the STFT

$$\text{Id}_{M_{m}^{p,q}} = \frac{1}{\|g\|_2^2} V_g^* V_g$$

(the same holds when replacing $M_{m}^{p,q}(\mathbb{R}^d)$ by $S(\mathbb{R}^d)$ or $S'(\mathbb{R}^d)$).

2.3. Phase functions and canonical transformations. Let $a$ be as in the Introduction, real-valued and satisfying (2). The related classical evolution, given by the linear Hamilton-Jacobi equations following our normalization can be written as

$$\begin{cases}
2\pi \partial_t x(t, y, \eta) = -\nabla_\xi a(x(t, y, \eta), \xi(t, y, \eta)) \\
2\pi \partial_t \xi(t, y, \eta) = \nabla_x a(x(t, y, \eta), \xi(t, y, \eta)) \\
x(0, y, \eta) = y, \\
\xi(0, y, \eta) = \eta.
\end{cases}$$


The solution \((x(t, y, \eta), \xi(t, y, \eta))\) exists for every \(t \in \mathbb{R}\). Indeed, setting \(u := (x, \xi), F(u) := (-\nabla_x a(u), \nabla_x a(u))\), the initial value problem \((22)\) can be rephrased as
\[
(23) \quad u'(t) = F(u(t)), \quad u(t_0) = u_0,
\]
in the particular case \(t_0 = 0\). Observe that our assumptions on \(a\) imply the boundedness of \(\partial^k \Phi_j\), for every \(|\alpha| > 0, j = 1, \ldots, 2d\), hence in particular \(F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}\) is a Lipschitz continuous mapping. The previous ODE is an autonomous ODE with a mapping \(F \in C^\infty(\mathbb{R}^{2d} \to \mathbb{R}^{2d})\) having at most linear growth, hence \(\|F(u)\| \lesssim 1 + \|u\|\). Hence for each \(u_0 \in \mathbb{R}^{2d}\) and \(t_0 \in \mathbb{R}\) there exists a unique classical global solution \(u : \mathbb{R} \to \mathbb{R}^{2d}\) (in this case \(u \in C^\infty(\mathbb{R} \to \mathbb{R}^{2d})\) since \(F \in C^\infty(\mathbb{R}^{2d} \to \mathbb{R}^{2d})\)) to \((23)\). The solution maps \(S_{t_0}(t) : \mathbb{R}^{2d} \to C^\infty(\mathbb{R} \to \mathbb{R}^{2d})\), defined by \(S_{t_0}(t)u_0 = u(t)\), and \(S_{t_0}(t) = Id\), the identity operator on \(\mathbb{R}^{2d}\), are Lipschitz continuous mappings, obeying \(S_{t_0}(t) = S_0(t - t_0)\) and the group laws
\[
(24) \quad S_0(t)S_0(t') = S_0(t + t'), \quad S_0(0) = Id.
\]
The mapping \(S_0(t)\) is a bi-Lipschitz diffeomorphism with \(S_0^{-1}(t) = S_0(-t)\). Following the notations of \([10, 20]\), we call the bi-Lipschitz diffeomorphism
\[
(25) \quad \chi_t(y, \eta) := S_0(t)(y, \eta), \quad (y, \eta) \in \mathbb{R}^{2d}.
\]

The theory of Hamilton-Jacobi allows to find a \(T > 0\) such that for \(t \in ]-T, T[\) there exists a phase function \(\Phi(t, x, \eta)\), solution of the eiconal equation
\[
(26) \quad \begin{cases} 
2\pi \partial_t \Phi + a(x, \nabla_x \Phi) = 0 \\
\Phi(0, x, \eta) = x\eta
\end{cases}
\]
The phase \(\Phi(t, x, \eta)\) is real-valued since the symbol \(a(x, \xi)\) is real-valued, moreover \(\Phi\) fulfills the condition of non-degeneracy:
\[
(27) \quad |\det \partial_{x, \eta}^2 \Phi(t, x, \eta)| \geq c > 0, \quad (t, x, \eta) \in ]-T, T[ \times \mathbb{R}^{2d},
\]
after possibly shrinking \(T > 0\), and satisfies
\[
(28) \quad |\partial_t^k \partial_{x, \eta}^{\alpha} \Phi(t, x, \eta)| \leq c_{k, \alpha}, \quad |\alpha| \geq 2, k \geq 0, \quad (t, x, \eta) \in ]-T, T[ \times \mathbb{R}^{2d}.
\]
The relation between the phase \(\Phi\) and the canonical transformation \(\chi\) is given by
\[
(29) \quad (x, \nabla_x \Phi(t, x, \eta)) = \chi_t(\nabla_\eta \Phi(t, x, \eta), \eta), \quad t \in ]-T, T[.
\]
In particular,
\[
(30) \quad \begin{cases} 
y(t, x, \eta) = \nabla_\eta \Phi(t, x, \eta) \\
\xi(t, x, \eta) = \nabla_x \Phi(t, x, \eta),
\end{cases}
\]
and there exists \(\delta > 0\) such that
\[
(31) \quad |\det \frac{\partial_x}{\partial y}(t, y, \eta)| \geq \delta, \quad t \in ]-T, T[.
\]
Observe that each component of $\chi_t$ is a function with bounded smooth derivatives of any order in $]-T, T[ \times \mathbb{R}^{2d}$. Using (24) we observe that the same holds in fact for every $t \in \mathbb{R}$.

For $t \in ]-T, T[$, the phase function $\Phi(t, \cdot)$ is a tame phase, and similarly for the canonical transformation $\chi_t$, according to the following definition [10, Definition 2.1]:

A real and smooth phase function $\Phi(x, \eta)$ on $\mathbb{R}^{2d}$ is called tame if:

(i) For $z = (x, \eta)$,

$$|\partial_\alpha^z \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2;$$

(ii) There exists $c > 0$ such that

$$|\det \partial_{x, \eta}^2 \Phi(x, \eta)| \geq c.$$

The mapping defined by $(x, \xi) = \chi(y, \eta)$, which solves the system

$$\begin{cases}
y(x, \eta) = \nabla_\eta \Phi(x, \eta) \\
\xi(x, \eta) = \nabla_x \Phi(x, \eta),
\end{cases}$$

is called tame canonical transformation.

Observe that we have no assumption of homogeneity for large $(x, \eta)$, nevertheless the mapping $\chi$ is well-defined by the global inverse function theorem. The mapping $\chi$ is a smooth bi-Lipschitz canonical transformation (i.e. it preserves the symplectic form) and satisfies, for $(x, \xi) = \chi(y, \eta)$,

$$|\partial_z^\alpha x_i(z)| + |\partial_z^\alpha \xi_i(z)| \leq C_\alpha, \quad |\alpha| \geq 1, \quad z = (y, \eta), \quad i = 1, \ldots, d.$$

The mapping $\chi$ enjoys

$$|\det \partial_x \frac{\partial}{\partial y}(y, \eta)| \geq \delta$$

(that is (31) for the canonical transformations of the Hamilton-Jacobi theory), which allows to uniquely determine, up to a constant, the related tame phase function $\Phi_\chi$ (see [10, Section 2]).

3. Fourier Integral Operators

3.1. The classes $FIO(\chi)$. The class $FIO(\chi)$ was introduced in [10] and its definition can be rephrased as follows.

Definition 3.1. Let $g \in S(\mathbb{R}^d)$ be a non-zero window function. Consider a canonical transformation $\chi$ which is a smooth bi-Lipschitz diffeomorphism and satisfies (35). A continuous linear operator $T : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ is in the class $FIO(\chi)$ if its (continuous) Gabor matrix satisfies for all $s > 0$ the decay condition

$$|\langle T \pi(w)g, \pi(z)g \rangle| \leq C(z - \chi(w))^{-s}, \quad \forall z, w \in \mathbb{R}^{2d},$$
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for a constant \( C > 0 \) depending on \( s \).

Note that we do not require (36) to be valid.

The class \( FIO(\Xi) = \bigcup_{\chi} FIO(\chi) \) is the union of these classes where \( \chi \) runs over the set of all smooth bi-Lipschitz canonical transformations satisfying (35).

The following properties are proved in [10].

(i) **Boundedness of \( T \) on \( M^p(\mathbb{R}^d) \) ([10, Theorem 3.4]):** If \( T \in FIO(\chi) \), then \( T \) can be extended to a bounded operator on \( M^p(\mathbb{R}^d) \) (in particular on \( L^2(\mathbb{R}^d) \)).

(ii) **The algebra property ([10, Theorem 3.6]):** For \( i = 1, 2 \),

\[
T^{(i)} \in FIO(\chi_i) \quad \Rightarrow \quad T^{(1)}T^{(2)} \in FIO(\chi_1 \circ \chi_2).
\]

(iii) **The Wiener property ([10, Theorem 3.7]):** If \( T \in FIO(\chi) \) and \( T \) is invertible on \( L^2(\mathbb{R}^d) \), then \( T^{-1} \in FIO(\chi^{-1}) \).

These properties imply that the union \( FIO(\Xi) \) is a Wiener subalgebra of \( \mathcal{L}(L^2(\mathbb{R}^d)) \), the class of linear bounded operators on \( L^2(\mathbb{R}^d) \). Property (ii) can be refined as follows.

**Lemma 3.2.** For \( T^{(i)} \in FIO(\chi_i), i = 1, 2 \), the continuous Gabor matrix of the composition \( T^{(1)}T^{(2)} \) is controlled for every \( s \geq 0 \) by

\[
|\langle (T^{(1)}T^{(2)}(w)g, \pi(z)g\rangle| \leq C_0C_1C_2(z - \chi_1 \circ \chi_2(w))^{-s}, w, z \in \mathbb{R}^d,
\]

where \( C_i > 0 \) is the constant of \( T^{(i)} \) in (37), \( i = 1, 2 \), whereas \( C_0 > 0 \) depends only on \( s \) and on the Lipschitz constants of the mappings:

\[
\chi_1, \chi_1^{-1}, \chi_1 \circ \chi_2, (\chi_1 \circ \chi_2)^{-1}, \ldots, \chi_1 \circ \chi_2 \circ \cdots \circ \chi_{n-1}, (\chi_1 \circ \chi_2 \circ \cdots \circ \chi_{n-1})^{-1}.
\]

The proof is an easy consequence of Lemma 2.11 in [20].

By induction we immediately obtain

**Proposition 3.3.** For \( n \in \mathbb{N}, n \geq 2, T^{(i)} \in FIO(\chi_i), i = 1, \ldots, n \), we have

\[
|\langle (T^{(1)}T^{(2)} \cdots T^{(n)}(w)g, \pi(z)g\rangle| \leq C_0C_1 \cdots C_n(z - \chi_1 \circ \chi_2 \circ \cdots \circ \chi_n(w))^{-s}.
\]

where \( C_0 \) depends on \( s \) and on the Lipschitz constants of the mappings:

\[
\chi_1, \chi_1^{-1}, \chi_1 \circ \chi_2, (\chi_1 \circ \chi_2)^{-1}, \ldots, \chi_1 \circ \chi_2 \circ \cdots \circ \chi_{n-1}, (\chi_1 \circ \chi_2 \circ \cdots \circ \chi_{n-1})^{-1}.
\]

Observe that, using Schur’s test and the same techniques as in the proof [10, Theorem 3.4], it is straightforward to obtain the following weighted version of [10, Theorem 3.4].

**Theorem 3.4.** Consider \( \mu \in \mathcal{M}_{\mu_{\infty}}. \) Then for every \( 1 \leq p \leq \infty, T \in FIO(\chi) \) extends to a continuous operator from \( M^p_{\mu_{\infty}} \) into \( M^p_\mu. \)
Let us underline that $\mu \circ \chi \in M_{v_r}$, since $v_r \circ \chi \simeq v_r$, due to the bi-Lipschitz property of $\chi$. In particular

$$T : M^p_r(\mathbb{R}^d) \to M^p_r(\mathbb{R}^d), \quad 1 \leq p \leq \infty, \quad r \in \mathbb{R}.$$  

If $\chi = \text{Id}$, then the corresponding Fourier integral operators are simply pseudodifferential operators, as already shown in [27].

The characterization below, written for pseudodifferential operators in the Kohn-Nirenberg form $\sigma(x, D)$, works for any $\tau$-form (in particular Weyl form $\sigma^w(x, D)$) of a pseudodifferential operator.

**Proposition 3.5.** Fix $g \in \mathcal{S}(\mathbb{R}^d)$ and let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$. For $s \in \mathbb{R}$, the symbol $\sigma$ belongs to $S^0_{0,0}(\mathbb{R}^{2d})$ if and only if for every $s \geq 0$, and for suitable constants $C = C_s$ depending on $s$,

$$|\langle \sigma(x, D)\pi(w)g, \pi(z)g \rangle| \leq C_s(z - w)^{-s}, \quad \forall w, z \in \mathbb{R}^{2d}. \quad (42)$$

Similarly, under additional assumptions on the classes $FI\sigma(\chi)$, their operators can be written in the following integral form ($FI\sigma$s of type I):

$$I(\sigma, \Phi)f(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x, \eta)} \sigma(x, \eta)\hat{f}(\eta) d\eta, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (43)$$

where $\sigma \in S^0_{0,0}(\mathbb{R}^{2d})$ and $\Phi$ a tame phase function. More precisely, this particular form is allowed starting from the class $FI\sigma(\chi)$ whenever the mapping $\chi$ enjoys the additional property (36) as explained in the following characterization [10, Theorem 4.3].

**Theorem 3.6.** Consider $g \in \mathcal{S}(\mathbb{R}^d)$. Let $I$ be a continuous linear operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and $\chi$ be a tame canonical transformation satisfying (36). Then the following properties are equivalent.

(i) $I = I(\sigma, \Phi, \chi)$ is a $FI\sigma$ of type I for some $\sigma \in S^0_{0,0}(\mathbb{R}^{2d})$.

(ii) $I \in FI\sigma(\chi)$.

For $\chi = \text{Id}$ we recapture the characterization for pseudodifferential operators of Proposition 3.5.

**Remark 3.7.** We shall apply the preceding results to the mappings $\chi_t(x, \eta)$ coming from the Hamilton-Jacobi system (22), so we need to be more precise on the estimate (37), namely we have to show how the constants $C$ depend on the time variable $t$. To this end, in short: gluing together the results [9, Theorem 3.3] and [10, Theorem 4.3], we obtain that the constants $C = C_s$ in (37), $s \geq 0$, can be estimated in terms of the $S^0_{0,0}$-semi-norms of $\sigma$. Provided the Lipschitz constants of $\chi_t$ are uniformly bounded with respect to $t$, as we have in (29), we may assume $C_s(t) \in C([-T, T])$ in (37) if the semi-norms of $\sigma_t$ can be estimated by functions in $C([-T, T])$. 
4. Proof of Theorems 1.1 and 1.2

Let us consider the Cauchy problem (1) with \(a(z)\) real-valued satisfying (2). From the results recalled in the Introduction we have for \(e^{itH}\) the representation (4). More precisely:

**Proposition 4.1.** There exists a constant \(T > 0\), a symbol \(\sigma(t, x, \eta) \in C^\infty([-T, T], S_{0,0})\) a real-valued phase function \(\Phi(t, x, \eta)\) satisfying (26) and (27) such that the evolution operator can be written as

\[
(e^{itH}u_0)(t, x) = (F_tu_0)(t, x)
\]

where \(F_t\) is the FIO of type I

\[
(F_tu_0)(t, x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(t, x, \eta)} \sigma(t, x, \eta) \hat{u}_0(\eta) d\eta.
\]

**Remark 4.2.** Notice that the function \(\Phi(t, \cdot)\) of Proposition 4.1 and the related canonical transformation \(\chi_t\) in (25) are tame. The Lipschitz constants of \(\chi_t\) and \(\chi_t^{-1}\) can be controlled by a continuous function of \(t\) on the interval \([-T, T]\) and thus can be chosen uniform with respect to \(t\) on \([-T, T]\).

From Theorem 3.6 we infer that the propagator \(e^{itH}\) belongs to \(FIO(\Xi)\) for every fixed \(t \in ]-T, T[\). More precisely:

**Proposition 4.3.** Under the assumptions of Proposition 4.1 we have

\[
e^{itH} \in FIO(\chi_t), \quad t \in ]-T, T[,
\]

where \(\chi_t\) is defined in (25). Moreover for every \(s \geq 0\) there exists \(C(t) = C_s(t) \in C([-T, T])\) such that, for every \(g \in S(\mathbb{R}^d)\) the Gabor matrix satisfies

\[
|\langle e^{itH} \pi(w)g, \pi(z)g \rangle| \leq C(t) \langle z - \chi_t(w) \rangle^{-s}, \quad w, z \in \mathbb{R}^{2d}.
\]

The last part of the statement follows from Remark 4.2 and Remark 3.7.

The previous proposition gives a representation of \(e^{itH}\) for \(|t| < T\). Using the group property of the propagator \(e^{itH}\) we may obtain an expression of \(e^{itH}\) for every \(t \in \mathbb{R}\). Indeed, a classical trick, jointly with the group property of \(e^{itH}\), applies. Namely, we consider \(T_0 < T/2\) and define

\[
I_h = [hT_0, (h + 2)T_0[, \quad h \in \mathbb{Z}.
\]

For \(t \in I_h\), by the group property of \(e^{itH}\):

\[
e^{itH} = e^{i(t-hT_0)H}(e^{i(hT_0)H}|h|)|h|
\]

and using Proposition 4.1 one can write

\[
e^{itH} = F_{t-hT_0}(F_{hT_0}^{|h|})|h|.
\]
In general, $e^{itH}$ or even the composition $F_{t-hT_0}(F_{hT_0}f)[h]$ cannot be represented as a type I FIO in the form (11). We shall prove below that the evolution $e^{itH}$ is in the class $FIO(\chi_t)$ for every $t \in \mathbb{R}$, with $\chi$ defined in (25), so that this class is proven to be the right framework for describing the evolution $e^{itH}$.

**Theorem 4.4.** Given the Cauchy problem (1) with $a(z)$ real valued satisfying (2), consider the mapping $\chi_t$ defined in (25). Then

$$e^{itH} \in FIO(\chi_t), \quad t \in \mathbb{R}$$

and for every $s \geq 0$ there exists $C(t) \in C(\mathbb{R})$ such that

$$|\langle e^{itH} \pi(w)g, \pi(z)g \rangle| \leq C(t)\langle z - \chi_t(w) \rangle^{-s}, \quad w, z \in \mathbb{R}^{2d}, \quad t \in \mathbb{R}.$$  

**Proof.** We fix $T_0 < T/2$ as above. For $t \in \mathbb{R}$, there exists $h \in \mathbb{Z}$ such that $t \in I_h$. Using Proposition 4.3 for $t_1 = t - hT_0 \in [T, T]$ we have that $e^{it_1H} \in FIO(\chi_{t_1})$ and for $t_2 = \frac{h}{|h|}T_0 \in [-T, T]$, $e^{it_2H} \in FIO(\chi_{t_2})$, and for every $s \geq 0$, there exists a continuous function $C(t)$ on $[T, T]$ such that (47) is satisfied for $t = t_1$ and $t = t_2$. Using the algebra property (38), we have

$$e^{it_1H}(e^{it_2H})|^{[h]} \in FIO(\chi_{t_1} \circ (\chi_{t_2})|^{[h]}),$$

and the group law (24) for $\chi_t(y, \eta) = S_0(t)(y, \eta)$ gives

$$\chi_{t_1} \circ (\chi_{t_2})|^{[h]} = \chi_{t_1 + |h|t_2} = \chi_t,$$

as expected. Then, using (40) we obtain that the Gabor matrix of the product $e^{it_1H}(e^{it_2H})|^{[h]}$ is controlled by a continuous function $C_h(t)$ on $I_h$. Finally, from the estimates

$$|\langle e^{itH} \pi(w)g, \pi(z)g \rangle| \leq C_h(t)\langle z - \chi_t(w) \rangle^{-s}, \quad t \in I_h,$$

with $C_h \in C(I_h)$, it is easy to construct a new continuous controlling function $C(t)$ on $\mathbb{R}$ such that (51) is satisfied.

In particular, the estimate (51) gives Theorem 1.1. Using Theorem 3.4 we obtain Theorem 1.2.

### 5. Gabor singularities and proof of Theorem 1.3

We want now to localize in $\mathbb{R}^{2d}$ the Gabor singularities of a distribution and study the action on them of $e^{itH}$.

For $\Gamma \subset \mathbb{R}^{2d}$ we define the $\delta$-neighborhood $\Gamma_\delta$, $0 < \delta < 1$, as

$$\Gamma_\delta = \{ z \in \mathbb{R}^{2d} : |z - z_0| < \delta \langle z_0 \rangle \text{ for some } z_0 \in \Gamma \}.$$  

We begin to list some properties of the $\delta$-neighborhoods, for the proofs we refer to [18, Lemmas 7.1, 7.2].
Lemma 5.1. Given $\delta$, we can find $\delta^*$, $0 < \delta^* < \delta$, such that for every $\Gamma \subset \mathbb{R}^{2d}$

\[(53)\quad (\Gamma_{\delta^*})_{\delta^*} \subset \Gamma_{\delta},\]

\[(54)\quad (\mathbb{R}^{2d} \setminus \Gamma_{\delta^*})_{\delta^*} \subset \mathbb{R}^{2d} \setminus \Gamma_{\delta^*}.
\]

Lemma 5.2. Let $\chi$ be a smooth bi-Lipschitz canonical transformation as in the preceding sections. For every $\delta$ there exists $\delta^*$, $0 < \delta^* < \delta$, such that for every $\Gamma \subset \mathbb{R}^{2d}$

\[(55)\quad \chi(\Gamma_{\delta^*}) \subset \chi(\Gamma)_{\delta},\]

\[(56)\quad \chi(\Gamma)_{\delta^*} \subset \chi(\Gamma_{\delta}).\]

The constant $\delta^*$ depends on $\chi$ and $\delta$ but it is independent of $\Gamma$.

In the following we shall argue on $f \in S'((\mathbb{R}^d))$, and take windows $g \in S(\mathbb{R}^d)$. Since $\cup_{s \geq 0} M_p^r(\mathbb{R}^d) = S'((\mathbb{R}^d))$ we have for some $s_0 \geq 0$

\[(57)\quad \int_{\mathbb{R}^{2d}} |V_g f(z)|^p \langle z \rangle^{-ps_0} dz < \infty.
\]

Definition 5.3. Let $f \in S'((\mathbb{R}^d))$, $g \in S(\mathbb{R}^d) \setminus \{0\}$, $\Gamma \subset \mathbb{R}^{2d}$, $1 \leq p \leq \infty$, $r \in \mathbb{R}$. We say that $f$ is $M_p^r$-regular in $\Gamma$ if there exists $\delta > 0$ such that

\[(58)\quad \int_{\Gamma_{\delta}} |V_g f(z)|^p \langle z \rangle^{pr} dz < \infty.
\]

(obvious changes if $p = \infty$).

Of course, (58) gives us some nontrivial information about $f$ only when $\Gamma$ is unbounded. We shall prove later that Definition 5.3 does not depend on the choice of the window $g \in S(\mathbb{R}^d)$.

Theorem 5.4. Let $e^{itH}$ and $\chi_t$ be defined as in the previous Sections, fix $u_0 \in S'((\mathbb{R}^d))$ and $\Gamma \subset \mathbb{R}^{2d}$. If $u_0$ is $M_p^r$-regular in $\Gamma$, then $e^{itH} u_0$ is $M_p^r$-regular in $\chi_t(\Gamma)$.

Proof. For sufficiently small $\delta > 0$ we have (58) in $\Gamma_\delta$ whereas (57) is valid in $\mathbb{R}^{2d}$ for some $s_0 \geq 0$. Now, from Theorem 1.1 we have

\[(59)\quad V_g (e^{itH} u_0)(z) = \int k(t, w, z)V_g u_0(w) \, dw
\]

with

\[(60)\quad |k(t, w, z)| = \langle e^{itH} \pi(w)g, \pi(z)g \rangle \lesssim \langle z - \chi_t(w) \rangle^{-s},
\]

for every $s \geq 0$. 

We want to show that $e^{itH}u_0$ is $M^p_\nu$-regular in $\chi_\nu(\Gamma)$. To this end, using (56) in Lemma 5.1 we take first $\delta^* < \delta$ such that $\chi_\nu(\Gamma)_{\delta^*} \subset \chi_\nu(\Gamma_\delta)$ and then using (53) in Lemma 5.1 we fix $\delta' < \delta^*$ such that

$$\left(\chi_\nu(\Gamma)_{\delta'}\right)_{\delta'} \subset \chi_\nu(\Gamma)_{\delta'} \subset \chi_\nu(\Gamma_\delta).$$

Note that for $w \not\in \Gamma_\delta$, i.e. $\chi_\nu(w) \not\in \chi_\nu(\Gamma_\delta)$, and $z \in \chi_\nu(\Gamma_{\delta'})$ we have

$$|z - \chi_\nu(w)| \geq \max\{\langle z \rangle, \langle w \rangle\}$$

since $\chi_\nu(w) \not\in \left((\chi_\nu(\Gamma)_{\delta'})_{\delta'}\right)$ in view of (61), and we may use as well (54).

Assuming for simplicity $p < \infty$, we shall prove

$$\int_{\chi_\nu(\Gamma)_{\delta'}} |V_g(e^{itH}u_0)(z)|^p \langle z \rangle^{pr} \, dz < \infty,$$

with $\delta'$ determined as before. Using (59) and (60), we estimate

$$|\langle z \rangle^r V_g(e^{itH}u_0)(z)| \lesssim \int_{\mathbb{R}^d} I(z, w) \, dw,$$

with

$$I(z, w) = \langle z \rangle^r \langle z - \chi_\nu(w) \rangle^{-s} |V_g u_0(w)|.$$

To show that $e^{itH}u_0$ is $M^p_\nu$-regular in $\chi_\nu(\Gamma)$ it will be sufficient to show that

$$\left\| \int_{\mathbb{R}^d} I(\cdot, w) \, dw \right\|_{L^p(\chi_\nu(\Gamma)_{\delta'})} < \infty.$$

First, we estimate $\int_{\mathbb{R}^d} I(z, w) \, dw$ for $z \in \chi_\nu(\Gamma)_{\delta'}$. We split the domain of integration into two domains $\Gamma_\delta$ and $\mathbb{R}^d \setminus \Gamma_\delta$. In $\mathbb{R}^d \setminus \Gamma_\delta$ we use (62) to obtain

$$\int_{\mathbb{R}^d \setminus \Gamma_\delta} I(z, w) \, dw \leq \int_{\mathbb{R}^d \setminus \Gamma_\delta} \langle z \rangle^r \langle w \rangle^{s_0} \langle z - \chi_\nu(w) \rangle^{-s} \frac{|V_g u_0(w)|}{\langle w \rangle^{s_0}} \, dw$$

$$\lesssim \int_{\mathbb{R}^d \setminus \Gamma_\delta} \langle z - \chi_\nu(w) \rangle^{r + s_0 - s} \frac{|V_g u_0(w)|}{\langle w \rangle^{s_0}} \, dw$$

$$\lesssim \left(\frac{\langle \cdot \rangle^{r+s_0-s} |V_g u_0(\cdot)|}{\langle \cdot \rangle^{s_0}} \right)(z).$$

So by (57) and choosing $s$ in (60) so that $r + s_0 - s < -2d$,

$$\left\| \int_{\mathbb{R}^d \setminus \Gamma_\delta} I(\cdot, w) \, dw \right\|_{L^p(\chi_\nu(\Gamma)_{\delta'})} \lesssim \|\langle \cdot \rangle^{r+s_0-s}\|_{L^1(\mathbb{R}^d)} \|V_g u_0(\cdot)^{-s_0}\|_{L^p(\mathbb{R}^d)} < \infty.$$
In the domain $\Gamma_\delta$, we have
\[
\int_{\Gamma_\delta} I(z, w) \, dw \leq \int_{\Gamma_\delta} \langle z \rangle^{-r} \langle z - \chi_t(w) \rangle^{-r} \langle z - \chi_t(w) \rangle^{-s} |V_g u_0(w)\rangle (w)^r \, dw \\
\lesssim \int_{\Gamma_\delta} \langle z - \chi_t(w) \rangle^{-s} |V_g u_0(w)\rangle (w)^r \, dw \\
\lesssim \langle \chi_t^{-1}(\cdot) \rangle^{-s} (\text{Char}_{\Gamma_\delta} \cdot |V_g u_0\rangle (\cdot)^r)(z)
\]
where $\text{Char}_{\Gamma_\delta}$ is the characteristic function of the set $\Gamma_\delta$. The assumption (58) yields to the estimate
\[
\left\| \int_{\Gamma_\delta} I(\cdot, w) \, dw \right\|_{L^p(\chi_t(\Gamma)_{\delta}')} \lesssim \|\langle \chi_t^{-1}(\cdot) \rangle^{-s}\|_{L^1(\mathbb{R}^{2d})} \|V_g u_0\rangle (\cdot)^r\|_{L^p(\Gamma_\delta)} \\
\asymp \|\langle \cdot \rangle^{-s}\|_{L^1(\mathbb{R}^{2d})} \|V_g u_0\rangle (\cdot)^r\|_{L^p(\Gamma_\delta)} < \infty,
\]
because $\chi_t$ is a bi-Lipschitz diffeomorphism and we may take $s$ in (60) so large that $r - s < -2d$. This concludes the proof. \(\Box\)

Remark 5.5. It is now easy to check that Definition 5.3 does not depend on the choice of the window $g \in \mathcal{S}(\mathbb{R}^d)$. In fact, assume that the estimate (58) in Definition 5.3 is satisfied for some $\delta > 0$, and some choice of $g \in \mathcal{S}(\mathbb{R}^d)$. Then (58) is still satisfied, for some new $\delta > 0$, if we replace $g$ with $h \in \mathcal{S}(\mathbb{R}^d)$. To prove this claim, observe that for every $s \geq 0$,
\[
|V_{\delta} g(z)| \lesssim \langle z \rangle^{-s}, \quad z \in \mathbb{R}^{2d}.
\]
Lemma 2.1 then gives
\[
|V_h f(w)| \lesssim \int \langle w - z \rangle^{-s} |V_g f(z)| \, dz.
\]
The claim follows by splitting the domain of integration into $\Gamma_\delta$ and $\mathbb{R}^{2d} \setminus \Gamma_\delta$, and arguing as in the proof of the preceding Theorem 5.4, being now $\chi_t = \text{identity}$.

The following definition allows one to describe the position in phase space of the singularities of a function $f$.

Definition 5.6. Given $f \in \mathcal{S}'(\mathbb{R}^d)$, we shall call filter of the Gabor singularities of $f$ the collection of subsets of $\mathbb{R}^{2d}$:
\[
\mathfrak{F}^p_r(f) = \{ \Lambda \subset \mathbb{R}^{2d} : f \ is \ M^p_r\text{-regular in } \Gamma = \mathbb{R}^{2d} \setminus \Lambda \},
\]
cf. Definition 5.3.

Let us write $\mathfrak{F}(f) = \mathfrak{F}^p_r(f)$ for short. $\mathfrak{F}(f)$ is a filter since if $\Lambda \in \mathfrak{F}(f)$ and $\Lambda \subset \Lambda'$, then also $\Lambda' \in \mathfrak{F}(f)$, and moreover if $\Lambda_1, \ldots, \Lambda_n \in \mathfrak{F}(f)$ then also $\bigcap_{j=1}^n \Lambda_j \in \mathfrak{F}(f)$. Note that any neighborhood of $\infty$ i.e.
the complementary of a bounded set, belongs to $\mathcal{F}(f)$. We have $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if $\emptyset \in \mathcal{F}(f)$, that is equivalent to saying that there exists $\Lambda_1, \ldots, \Lambda_n \in \mathcal{F}(f)$ such that $\cap_{j=1}^n \Lambda_j = \emptyset$.

Theorem 5.4 now follows. Indeed, the inclusion $\chi_t(\mathcal{F}(f)) \subset \mathcal{F}(e^{itH}f)$ is just a restatement of Theorem 5.4. The opposite inclusion is equivalent to $\chi_t^{-1}\mathcal{F}(e^{itH}f) \subset \mathcal{F}(f)$, namely to $\chi_t^{-1}\mathcal{F}(g) \subset \mathcal{F}(e^{-itH}g)$, which is true by reversing the time.

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**Elena Cordero, Fabio Nicola, and Luigi Rodino**

**Università di Torino, Dipartimento di Matematica, via Carlo Alberto 10, 10123 Torino, Italy**

*E-mail address: elena.cordero@unito.it*

**Dipartimento di Scienze Matematiche, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy**

*E-mail address: fabio.nicola@polito.it*

**Università di Torino, Dipartimento di Matematica, via Carlo Alberto 10, 10123 Torino, Italy**

*E-mail address: luigi.rodino@unito.it*