Square root of a multivector of Clifford algebras in 3D: 
A game with signs

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Abstract

An algorithm to extract the square root from a multivector (MV) in real 
Clifford algebras \( Cl_{p,q} \), where \( n = p + q \leq 3 \), in radicals is presented. It is 
shown that in \( Cl_{3,0} \), \( Cl_{1,2} \) and \( Cl_{0,3} \) algebras there are up to four isolated 
square roots in a case of the most general (generic) MV. The algebra \( Cl_{2,1} \) is 
an exception and there the MV can have up to 16 isolated roots. In addition, 
a continuum of roots has been found in all Clifford algebras except \( p + q = 1 \). 
Examples which clarify algorithm are provided to illustrate the properties of 
roots in all \( n = 3 \) algebras. The results may be useful in solving nonlinear 
equations, for example Clifford-Riccati equation.

Keywords: Square root of multivector, Clifford algebra, geometric algebra, 
computer-aided theory

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1. Introduction

The square root has a long history. Solution by radicals of the cubic equa-
tion was first published in 1545 by G. Cardano. Simultaneously, a concept of 
square root of a negative number has been developed [4]. In 1872 A. Cayley 
was the first to carry over the square root to matrices [5]. In the recent book
by N. J. Higham [12], where an extensive literature is presented on nonlinear functions of matrices, two sections are devoted to matrix square roots. In the context of Clifford algebra (CA) the main attention up till now was concentrated on the square roots of quaternions [20, 21], or their derivatives such as coquaternions (also called split quaternions), or nectarines [10, 21, 22]. The square root of biquaternion (complex quaternion) was considered in [23]. The quaternions and related objects are isomorphic to one of $n = 2$ algebras $\text{Cl}_{0,2}$, $\text{Cl}_{1,1}$, $\text{Cl}_{2,0}$ and, therefore, the quaternionic square root analysis can be easily rewritten in terms of CA (see Appendix B). In this paper we shall mainly be interested in higher, namely, $n = 3$ Clifford algebras (CAs), where the main object is the 8-component MV.

For CAs of dimension $n \geq 3$ the investigation and understanding of square root properties is still in infancy. The most akin to the present paper are the investigation of conditions for existence of square root of $-1$ [23, 13, 14]. The existence of such roots allows to extend the Fourier transform to MVs, where they are used in formulating Clifford-Fourier transform and CA based wavelet theories [15].

Our preliminary investigation [8] on this subject was concerned with square roots of individual MV grades such as scalar, vector, bivector, pseudoscalar, or their simple combinations. For this purpose we have applied the Gröbner basis algorithm to analyze the system of nonlinear polynomial equations that ensue from the MV equation $A^2 = B$, where $A$ and $B$ are the MVs. The Gröbner basis is accessible in symbolic mathematical packages such as Mathematica and Maple. Specifically, the Mathematica commands such as \texttt{Reduce[ ]}, \texttt{Solve[ ]}, \texttt{Eliminate[ ]} and others also employ the Gröbner basis to solve nonlinear problems. With the help of them we were able to find new properties of roots for $n = 3$ case, namely, that the MVs may have no roots, a single or multiple isolated roots, or even an infinite number (continuum) of roots in 4D parameter spaces or smaller dimensions.

In this paper we continue our [8] investigations of the square root problem in real CAs for $n = 3$ case. In particular, we examine and provide explicit conditions for a MV to have discrete and continuum of roots, and how to express real root coefficients in radicals. For this purpose a symbolic package based on Mathematica system was written [4] that appeared to be invaluable both for detecting specific solutions of the nonlinear CA equation $A^2 = B$ and for numerical checks in general.

In Sec. 2 the notation is introduced. The algorithm to calculate the square root of a generic MV and special cases that follow are given in Secs. 3-5.
for $\text{Cl}_{3,0} \simeq \text{Cl}_{1,2}$, $\text{Cl}_{0,3}$, and $\text{Cl}_{2,1}$ algebras, respectively. The algorithm is illustrated by a number of examples. The conclusions are drawn in Sec. 6. For completeness, in Appendix A and Appendix B the MV square roots are presented for lower dimensional CAs.

2. Notation

For $n = 3$, general MV can be expanded in the orthonormal basis that consists of $2^n = 8$ elements listed in inverse degree lexicographic ordering,

$$\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123} \equiv I\},$$

where $e_i$ are basis vectors and $e_{ij}$ are the bivectors (oriented planes). The last term is the pseudoscalar. The number of subscripts indicates the grade of basis element. The scalar is a grade-0 element, the vectors $e_i$ are the grade-1 elements etc. In the orthonormalized basis the geometric (Clifford) products of basis vectors satisfy the anticommutation relation,

$$e_i e_j + e_j e_i = \pm 2 \delta_{ij}.$$  

For $\text{Cl}_{3,0}$ and $\text{Cl}_{0,3}$ algebras the squares of basis vectors, correspondingly, are $e_i^2 = +1$ and $e_i^2 = -1$, where $i = 1, 2, 3$. For mixed signature algebras such as $\text{Cl}_{2,1}$ and $\text{Cl}_{1,2}$ we have $e_1^2 = e_2^2 = 1$, $e_3^2 = -1$ and $e_1^2 = 1$, $e_2^2 = e_3^2 = -1$, respectively. The sign of squares of higher grade elements is determined by squares of vectors and the property (2). For example, in $\text{Cl}_{3,0}$ we have $e_{12}^2 = e_{12} e_{12} = -e_1 e_2 e_2 e_1 = -e_1 (-1) e_1 = -e_1 e_1 = -1$. However, in $\text{Cl}_{1,2}$ similar computation gives $e_{12}^2 = -e_1 e_2 e_2 e_1 = -e_1 (-1) e_1 = e_1 e_1 = +1$.

When $n = 3$, a MV $A$ in real CA can be expanded in the basis (1),

$$A = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_{12} e_{12} + a_{23} e_{23} + a_{13} e_{13} + a_{123} I \equiv a_0 + a + A + a_{123} I,$$

where $a_i$, $a_{ij}$ and $a_{123}$ are the real coefficients, and $a = a_1 e_1 + a_2 e_2 + a_3 e_3$ and $A = a_{12} e_{12} + a_{23} e_{23} + a_{13} e_{13}$ is, respectively, the vector and the bivector. We will seek for a real MV $A$ (with real coefficients), the square of which satisfies

$$AA \equiv A^2 = B = b_0 + b + B + b_{123} I.$$  

\footnote{Note an increasing order of digits in indices. Therefore we write $e_{13}$ instead of $e_{31} = -e_{13}$. This convention is reflected in opposite signs of some terms in formulas.}
The MV $A$ is called a square root of $B$. In Eq. (1) the square $A^2$ has been expanded in the orthonormal basis where $b_0, b, B$ and $I \equiv I_3$ denote, respectively, a scalar, a vector ($b = b_1 e_1 + b_2 e_2 + b_3 e_3$), a bivector ($B = b_{12} e_{12} + b_{23} e_{23} + b_{13} e_{13}$) and a pseudoscalar. The representation (1) is not convenient for our problem, therefore, for all 3D algebras $Cl_{3,0}$, $Cl_{0,3}$, $Cl_{1,2}$, and $Cl_{2,1}$ a more symmetric representation is introduced,

$$A = s + v + (S + V)I,$$

where now both $s$ and $S$ are the real scalars and both $v = v_1 e_1 + v_2 e_2 + v_3 e_3$ and $V = V_1 e_1 + V_2 e_2 + V_3 e_3$ are the vectors with real coefficients $v_i$ and $V_i$. The MV representation (5) allows to disentangle the coupled nonlinear equations in a regular manner for all listed algebras. To select the scalar $s$ in (5), the grade selector $\langle A \rangle \equiv \langle A \rangle_0 = s$ is used. The pseudoscalar part can be extracted by $\langle A \rangle \equiv \langle -AI \rangle_0 = S$, and similarly for other grades. More about CAs and MV properties can be found, for example, in books [9, 17].

When $n = 1,2$ the MV square root algorithm simplifies substantially. All needed formulas are presented in the Appendix A and Appendix B, respectively.

3. Square roots in $Cl_{3,0}$ and $Cl_{1,2}$ algebras

This section describes the method of substitution of variables in CA which paves a direct way to square root algorithm (also see the Appendix C). The Euclidean $Cl_{3,0}$ algebra is the most simple one among $n = 3$ algebras. The algebra $Cl_{1,2}$ is isomorphic to $Cl_{3,0}$, therefore the algorithm for this algebra follows the same route except that there some notational differences appear.

The goal is to solve nonlinear MV equation $A^2 = B$, where $B = b_0 + b_1 e_2 + b_2 e_2 + b_3 e_3 + b_{12} e_{23} + b_{13} e_{13} + b_{23} e_{23} + b_{123} I$ and $A$ is unknown. The latter may have the general form (5). Expanding $A^2$ in components and equating (real) coefficients at basis elements to respective coefficients in $B$ one obtains a system of eight nonlinear equations:

$$b_0 = s^2 - S^2 + v^2 - V^2,$$
$$b_1 = 2(sv_1 - SV_1),$$
$$b_2 = 2(sv_2 - SV_2),$$
$$b_3 = 2(sv_3 - SV_3),$$
$$b_{123} = 2(sS + v \cdot V),$$
$$b_{12} = 2(sV_3 + Sv_3).$$

(6) (7) (8) (9)
For all algebras the square root algorithm splits into two cases: the *generic* case where either $s^2 + S^2 \neq 0$ (in $Cl_{3,0}$ and $Cl_{1,2}$) or $s^2 - S^2 \neq 0$ (in $Cl_{0,3}$ and $Cl_{2,1}$), and the *special* case where $s^2 + S^2 = 0$ (in $Cl_{3,0}$ and $Cl_{1,2}$) or $s^2 - S^2 = 0$ (in $Cl_{0,3}$ and $Cl_{2,1}$).

### 3.1. The generic case $s^2 + S^2 \neq 0$

The system of six Eqs. (7)-(9) is linear in new variables $v_i$ and $V_i$ in (5). It has very simple solution which is a key to analysis that follows,

$$
v_1 = \frac{b_1 s + b_{23} S}{2(s^2 + S^2)}, \quad v_2 = \frac{b_2 s - b_{13} S}{2(s^2 + S^2)}, \quad v_3 = \frac{b_3 s + b_{12} S}{2(s^2 + S^2)},
$$

(10)

$$
V_1 = \frac{b_{23} s - b_1 S}{2(s^2 + S^2)}, \quad V_2 = -\frac{b_{13} s + b_2 S}{2(s^2 + S^2)}, \quad V_3 = \frac{b_{12} s - b_3 S}{2(s^2 + S^2)}.
$$

(11)

The Eqs. (10)-(11) express the components of vectors $\mathbf{v}$ and $\mathbf{V}$ in terms of scalars $s$ and $S$, which are to be determined from a pair of equations (6). The solution is valid when $s^2 + S^2 \neq 0$, i.e., when either $s \neq 0$ or $S \neq 0$, or both $s$ and $S$ are nonzero scalars. If these conditions are not satisfied we have the subcase $s = S = 0$. After substitution of (10)-(11), i.e., of $(v_1, v_2, v_3)$ and $(V_1, V_2, V_3)$, into (6) we get a system of two coupled algebraic equations for two unknowns $s$ and $S$,

$$
4(b_0 - s^2 + S^2)(s^2 + S^2)^2 = + (b_1 s + b_{23} S)^2 + (b_2 s - b_{13} S)^2 + (b_3 s + b_{12} S)^2
- (b_{23} s - b_1 S)^2 - (b_{13} s + b_2 S)^2 - (b_{12} s - b_3 S)^2,
$$

$$
2(b_{123} - 2s S)(s^2 + S^2)^2 = + (b_1 s + b_{23} S)(b_{23} s - b_1 S) - (b_2 s - b_{13} S)
\times (b_{13} s + b_2 S) + (b_3 s + b_{12} S)(b_{12} s - b_3 S).
$$

(12)

The system (12) has exactly four solutions that can be expressed in radicals. If new variables $t$ and $T$ are introduced and substitution

$$
s S = t, \quad \frac{1}{2}(-s^2 + S^2) = T,
$$

(13)

is used the system (12) reduces to

$$
(b_0 + 4T)(4t - b_{123}) - b_t/2 = 0,
$$

$$
b_s - (b_0 - b_{123} + 4T + 4t)(b_0 + b_{123} + 4T - 4t) = 0.
$$

(14)

---

Note, the symmetry of Eqs. (10) and (11) with respect to pairs $(v_2, V_2)$, $(v_1, V_1)$ and $(v_3, V_3)$ differ as explained in Footnote 4. It can be restored if $b_{13}$ in is replaced by $-b_{31}$.
In (14), coordinate-free abbreviations $b_S$ and $b_I$ have been introduced,

$$b_S = \langle \tilde{B} \tilde{B} \rangle_0 = b_0^2 - b_1^2 - b_2^2 - b_3^2 + b_{12}^2 + b_{13}^2 + b_{23}^2 - b_{123}^2,$$

$$b_I = \langle \tilde{B} \tilde{B} I \rangle_0 = 2b_3b_{12} - 2b_2b_{13} + 2b_1b_{23} - 2b_0b_{123}. \tag{15}$$

In (15) the MV $\tilde{\mathcal{B}}$ denotes the Clifford conjugate of $\mathcal{B}$, where tilde is the grade reversion and cap is the grade inversion. Note, that for remaining algebras $\text{Cl}_{0,3}, \text{Cl}_{2,1},$ and $\text{Cl}_{2,1}$ the signs of individual terms inside $b_S$ and $b_I$ all are different. As we shall see below, the square roots for all $n = 3$ algebras are predetermined by four real coefficients only, namely, $b_0, b_{123}, b_S,$ and $b_I$.

After substitution of (13), the resulting system of equations (14) are of degree $\leq 4$. Thus, we conclude that the initial system (12) can be explicitly solved in radicals. In particular, two real solutions of (13) have the form

$$s_{1,2} = \pm \sqrt{-T + \sqrt{T^2 + t^2}}, \quad S_{1,2} = \pm \frac{t}{\sqrt{-T + \sqrt{T^2 + t^2}}}, \tag{16}$$

where the signs in pairs $(s_i, S_i)$ must be identical, plus or minus. The denominator of $S_{1,2}$ becomes zero if $s = S = 0$. The remaining two solutions of (13), which can be obtained from (16) by the substitution $\sqrt{T^2 + t^2} \rightarrow -\sqrt{T^2 + t^2}$, are complex valued due to the inequality $\sqrt{T^2 + t^2} \geq T$ and therefore must be rejected.

The two real-valued solutions of Eq. (14) are

$$\begin{cases} 
(t_{1,2} = \frac{1}{4}(b_{123} \pm \frac{1}{\sqrt{2}} \sqrt{-b_S + \sqrt{D}}), & T_{1,2} = \frac{1}{4}(\pm \frac{b_I}{\sqrt{2}} \sqrt{-b_S + \sqrt{D}} - b_0), \\
& \text{if } -b_S + \sqrt{D} > 0, \\
(t_{1,2} = b_{123}/4, & T_{1,2} = \frac{1}{4}(\pm \sqrt{b_S} - b_0)), \text{ if } -b_S + \sqrt{D} = 0 \& b_S > 0.
\end{cases} \tag{17}$$

No additional conditions are required for the determinant $D = b_S^2 + b_I^2 \geq 0$ of the MV $\mathcal{B}$, since for $\text{Cl}_{3,0}$ algebra it is always positive definite $D \geq b_S$ (refer to [2, 3] how to compute MV determinant). Again, we should take the same signs for $t_i$ and $T_i$. The two complex valued solutions of (14), which can be obtained from (17) by substitution $\sqrt{D} \rightarrow -\sqrt{D}$, must be rejected. The denominator of $T_{1,2}$ in (17) turns into zero when $b_S = \sqrt{D}$, i.e., when $b_I = 0$.

To summarize, starting from (17) and then going to (16), and finally to formulas (10)-(11), one obtains four explicit real solutions which completely
determine the square root $A = \sqrt{B}$ of generic MV $B$ in terms of radicals $A = s + v + (S + V)I$ of real Clifford algebra $Cl_{3,0}$.

3.2. The special case $s^2 + S^2 = 0$

The only special case in $Cl_{3,0}$ corresponds to $s = S = 0$. In the subcases $s = S \neq 0$ and $s = -S \neq 0$ one can rewrite expressions (16) in a simpler form. In particular, when $s = S \neq 0$ we have

$$s_{1,2} = \begin{cases} 
\pm \frac{1}{2} \sqrt{b_{123} + \frac{b_I}{2b_0}} & \text{if } b_0 \neq 0, \\
\pm \frac{1}{2} \sqrt{b_{123} \pm \sqrt{-b_S}} & \text{if } b_0 = 0,
\end{cases} \quad (18)$$

and when $s = -S \neq 0$

$$s_{1,2} = \begin{cases} 
\pm \frac{1}{2} \sqrt{-b_{123} - \frac{b_I}{2b_0}} & \text{if } b_0 \neq 0, \\
\pm \frac{1}{2} \sqrt{-b_{123} \pm \sqrt{-b_S}} & \text{if } b_0 = 0,
\end{cases} \quad (19)$$

where all expressions inside square roots are assumed to be positive.

The case $s = S = 0$ is special, because the condition implies that the number of square roots of $B$ may be infinite.\footnote{The case of simple MV roots is given in [8].} Indeed, in this case the Eqs. (17)-(19) are compatible only if the vector $(b_1, b_2, b_3)$ and bivector $(b_{12}, b_{13}, b_{23})$ coefficients are zeros. Then, the Eq. (6) reduces to

$$b_0 = v^2 - V^2, \quad b_{123} = 2(v \cdot V), \quad (20)$$

where $v^2 = v_1^2 + v_2^2 + v_3^2$ and $V^2 = V_1^2 + V_2^2 + V_3^2$ for $Cl_{3,0}$. Since, in general, we have $3 + 3 = 6$ unknowns which must satisfy Eqs. (20) we are left with four real arbitrary parameters as will be explicitly demonstrated in Example 1. The solution therefore makes a four dimensional (or smaller) set of real-valued MV coefficients. It is interesting that the both expressions in (20) have very clear geometric interpretation. Indeed, if the ends of vectors $v$ and $V$ represent two concentric spheres then the coefficient $b_0$ controls the lengths of radii $|v|$ and $|V|$, while the pseudoscalar coefficient $b_{123}$ controls the angle between the vectors $v$ and $V$. From this follows that, due to periodicity of the angle, one can introduce principal value for coefficient $b_{123}$. Similar property, i.e., the multiplicity of roots and the existence of principal angle in a complex plane are well-known in case complex numbers [16].

The diagram of the described algorithm is presented in Appendix C.
3.3. $Cl_{1,2} \simeq Cl_{3,0}$ algebra

In paper \cite{19} it is shown that “...for odd $n \geq 3$, there are three classes of isomorphic Clifford algebras what is consistent with Cartan’s classification of real Clifford algebras.” In particular, two algebras, $Cl_{3,0}$ and $Cl_{1,2}$, are represented by $2 \times 2$ complex matrices $\mathbb{C}(2)$. The similarity between square root expressions obtained below also confirms that these two algebras fall into the same isomorphism class. On the other hand, the algebras $Cl_{0,3}$ and $Cl_{2,1}$ are represented by blocked $2 \times 2$ and $1 \times 1$ matrices, respectively $^2\mathbb{R}(2)$ and $^2\mathbb{H}(1)$. Therefore, they belong to different classes. Indeed, as we shall show later, the analysis of roots in $Cl_{2,1}$ is only roughly similar to that in $Cl_{0,3}$. However, between $Cl_{2,1}$ and $Cl_{0,3}$ there are distinctions: they are isomorphic to different, real and quaternionic matrices.

As far as $Cl_{1,2}$ algebra is concerned, its difference from $Cl_{3,0}$ is contained in the explicit expression for $b_s$,

$$b_s = \langle \tilde{B} \tilde{B} \rangle_0 = v_0^2 - b_1^2 + b_2^2 + b_3^2 - b_{12}^2 - b_{13}^2 + b_{23}^2 - b_{123}^2,$$  \hspace{1cm} (21)

$$b_I = \langle \tilde{B} \tilde{I} \rangle_0 = 2b_3b_{12} - 2b_2b_{13} + 2b_1b_{23} - 2b_0b_{123},$$ \hspace{1cm} (22)

$$D = b_s^2 + b_I^2, \quad Cl_{1,2}$$ \hspace{1cm} (23)

and expressions for $v_1$ and $V_1$

$$v_1 = \frac{b_1s + b_{23}s}{2(s^2 + S^2)}, \quad v_2 = \frac{b_2s + b_{13}s}{2(s^2 + S^2)}, \quad v_3 = \frac{b_3s - b_{12}s}{2(s^2 + S^2)},$$ \hspace{1cm} (24)

$$V_1 = \frac{b_{23}s - b_1s}{2(s^2 + S^2)}, \quad V_2 = \frac{b_{13}s - b_2s}{2(s^2 + S^2)}, \quad V_3 = \frac{-b_{12}s + b_3s}{2(s^2 + S^2)}.$$ \hspace{1cm} (25)

The expressions for $b_I$ and $D$ (the determinant of $B$) remain the same. Note that in \cite{20} the scalar product in $Cl_{1,2}$ has both plus/minus signs, in particular $v^2 = v_1^2 - v_2^2 - v_3^2$. Before considering other algebras it is helpful to analyze few examples.

3.4. Examples for $Cl_{3,0}$ and $Cl_{1,2}$

Example 1. The case $s \neq S$.

The square root of $B = e_1 - 2e_{23}$ in $Cl_{3,0}$. The coefficients in this case are $b_1 = 1$ and $b_{23} = -2$, and all remaining ones are equal to zero. Then, from \cite{15} follows that $b_I = -4$ and $b_s = 3$. The expression \cite{17} gives $t_{1,2} = \left(\frac{1}{2}, -\frac{1}{2}\right)$ and $T_{1,2} = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Finally, using \cite{16} we find the real solutions for $s$ and $S$,

$$\left(s_{1,2} = \pm \frac{1}{2}c_1, \ S_{1,2} = \pm \frac{1}{2}c_2\right) \quad \text{and} \quad \left(s_{3,4} = \pm \frac{1}{2}c_2, \ S_{3,4} = \pm \frac{1}{2}c_1\right).$$ \hspace{1cm} (26)
where \( c_1 = \sqrt{-2 + \sqrt{5}} \) and \( c_2 = \sqrt{2 + \sqrt{5}} \). Thus, the MV is regular. Using (10) - (11) then we have the following four sets of non-zero coefficients:

\[
\begin{align*}
(s_1 &= -\frac{1}{2}c_1, \quad S_1 = \frac{1}{2}c_2, \quad v_1 = -\frac{1}{2}c_2, \quad V_1 = -\frac{1}{2}c_1), \\
(s_2 &= \frac{1}{2}c_1, \quad S_2 = -\frac{1}{2}c_2, \quad v_1 = \frac{1}{2}c_2, \quad V_1 = \frac{1}{2}c_1), \\
(s_3 &= \frac{1}{2}c_2, \quad S_3 = -\frac{1}{2}c_2, \quad v_1 = -\frac{1}{2}c_2, \quad V_1 = -\frac{1}{2}c_1), \\
(s_4 &= -\frac{1}{2}c_2, \quad S_4 = -\frac{1}{2}c_1, \quad v_1 = -\frac{1}{2}c_1, \quad V_1 = \frac{1}{2}c_2).
\end{align*}
\]  

(27)

The remaining coefficients are equal to zero, \( v_2 = v_3 = V_2 = V_3 = 0 \). Finally, inserting the coefficients (27) into (5) one can find four different roots,

\[
\begin{align*}
A_{1,2} &= \mp \frac{1}{2}c_2 (-2 + \sqrt{5} + e_1 - (2 + \sqrt{5})e_{23} - e_{123}), \\
A_{3,4} &= \pm \frac{1}{2}c_1 (2 + \sqrt{5} + e_1 - (2 + \sqrt{5})e_{23} + e_{123}),
\end{align*}
\]  

(28)

squares of which give the initial MV \( B = e_1 - 2e_{23} \).

**Example 2. The case \( s = S \).**

The square root of \( B = -1 + e_3 - e_{12} + \frac{1}{2}e_{123} \) in \( Cl_{3,0} \). Now \( b_0 = -1, b_{123} = \frac{1}{2}, b_I = -1, b_S = \frac{3}{4} \). Then, from (16) and (17) follows real solutions for \( s_i \) and \( S_i \),

\[
\begin{align*}
(s_{1,2} &= \pm \frac{1}{2}, \quad S_{1,2} = \pm \frac{1}{2}) \quad \text{and} \quad (s_{3,4} = 0, \quad S_{3,4} = \pm 1).
\end{align*}
\]  

(29)

Then, for case \((s_{1,2}, S_{1,2})\) the equations (10) - (11) yield

\[
\begin{align*}
(s_1 &= -\frac{1}{2}, \quad v_1 = v_2 = v_3 = 0, \quad V_1 = V_2 = 0, \quad V_3 = 1), \\
(s_2 &= \frac{1}{2}, \quad v_1 = v_2 = v_3 = 0, \quad V_1 = V_2 = 0, \quad V_3 = -1).
\end{align*}
\]  

(30)

The case \((s_{3,4}, S_{3,4})\) is treated exactly as in Example 1. The final answer consists of four roots too,

\[
\begin{align*}
A_{1,2} &= \pm \frac{1}{2} (-1 + 2e_{12} - e_{123}), \\
A_{3,4} &= \pm \frac{1}{2} (e_3 + e_{12} - 2e_{123}).
\end{align*}
\]  

(31)

**Example 3. The case \( s = S = 0 \).**

The square root of \( B = -1 + e_{123} \), which is the center of \( Cl_{3,0} \). The coefficients \( b_0 = -1, b_{123} = 1 \) give \( b_I = 2, b_S = 0 \). Then, from expressions (17) and (16) follows

\[
\begin{align*}
(s_{1,2} &= \pm c_1, \quad S_{1,2} = \pm c_2) \quad \text{and} \quad (s_3 = 0, \quad S_3 = 0),
\end{align*}
\]  

(32)
where \( c_1 = \sqrt{-1/2 + 1/\sqrt{2}} \) and \( c_2 = \sqrt{1/2 + 1/\sqrt{2}} \).

The case \((s_{1,2}, S_{1,2})\) in (32) can be computed similarly as in Example 1. The two square roots, which are obtained from case \((s_{1,2} = \pm c_1, S_{1,2} = \pm c_2)\), therefore are

\[
A_{1,2} = \pm (c_1 + c_2 e_{123}).
\] (33)

The set of two roots above should be extended by adding a set of roots provided by the case \((s_3 = 0, S_3 = 0)\) in (32), which is special. Indeed, some of coefficients in this case remain unspecified and therefore may be treated as free parameters that yield an uncountable number (continuum) of roots. The coefficients \((b_1, b_2, b_3)\) and \((b_{12}, b_{13}, b_{23})\) in this case are zeroes, however, the compatibility of (7)-(9) is satisfied and the solution set is not empty. Indeed, as seen from (20) the system can be solved for an arbitrary pair of coefficients \((v_1, v_2, v_3, V_1, V_2, V_3)\), for example with \((v_1, V_1)\). If \((v_1, V_1)\) is inserted into (5) one gets MV with four free parameters,

\[
A = f_1(v_2, v_3, V_2, V_3)e_1 + v_2e_2 + v_3e_3 + f_2(v_2, v_3, V_2, V_3)e_{23} - V_2e_{13} + V_3e_{12},
\] (34)

where \( v_1 = f_1(v_2, v_3, V_2, V_3) \) and \( V_1 = f_2(v_2, v_3, V_2, V_3) \) denote explicit solutions of (20),

\[
v_1 = \mp \frac{c_1}{\sqrt{2}}, \quad V_1 = \pm \frac{1}{c_1} \frac{-b_{123} + 2(v_2 V_2 + v_3 V_3)}{\sqrt{2}}, \quad \text{with}
\]

\[
c_1 = \left( \pm \sqrt{(b_0 - v_2^2 - v_3^2 + V_2^2 + V_3^2)^2 + (b_{123} - 2(v_2 V_2 + v_3 V_3))^2}
\right)^{\frac{1}{2}}.
\] (35)

For example, by setting all free parameters to zero, \( v_2 = V_2 = v_3 = V_3 = 0 \), we select from a continuum two roots, which we denote

\[
A_{3,4} = \pm (c_1 e_1 + c_2 e_{123}).
\] (36)

It is important to realize, however, that the number of roots provided by case \((s_3 = 0, S_3 = 0)\) in general is infinite and the two roots in (36) represent the simplest choice of free parameters. All roots \( A_j \) satisfies \( A_j^2 = B = -1 + e_{123} \).

If instead of \( B = -1 + e_{123} \) we would have tried to find the square root of MV that does not belong to the center, for example have worked with
\[ B = \mathbf{e}_1 + \mathbf{e}_{12}, \] which is directly related to polarized electromagnetic wave in \( Cl_{3,0}^\pm \), we would have ended up with an empty solution set. Indeed, in the latter case \( s_1 = 0, S_1 = 0 \) and \( b_0 = b_{123} = b_I = b_S = 0 \). Then, after substitution of \( s \to s_1 = 0 \) and \( S \to S_1 = 0 \) into Eqs. (7)-(9) one obtains the contradiction, \( 1 = 0 \).

**Example 4. The case of quaternion.**

The quaternions are isomorphic to even subalgebra \( Cl_{3,0}^+ \) with elements \( \{1, \mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{13}\} \), therefore, the provided formulas allow to find quaternionic square root too. Taking into account that quaternion imaginary units are \( i = \mathbf{e}_{12}, j = -\mathbf{e}_{13} \) and \( k = \mathbf{e}_{23} \), let’s compute the square root of \( B = 1 + \mathbf{e}_{12} - \mathbf{e}_{13} + \mathbf{e}_{23} = 1 + i + j + k \). In this example we have \( b_0 = 1, b_{123} = 0 \) and \( b_I = 0, b_S = 4 \). Starting from (17) and then using Eq. (16) it is easy to find that the MV represents a regular case with four different coefficients

\[ (s_{1,2} = 0, \quad S_{1,2} = \pm 1/\sqrt{2}) \quad \text{and} \quad (s_{3,4} = \pm \sqrt{3/2}, \quad S_{3,4} = 0). \] (37)

Using (10)-(11) and (5) we can write the answer:

\[
A_{1,2} = \pm (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_{123})/\sqrt{2}, \\
A_{3,4} = \pm (3 + \mathbf{e}_{12} - \mathbf{e}_{13} + \mathbf{e}_{23})/\sqrt{6} = \pm (3 + i + j + k)/\sqrt{6}. \] (38)

The squares of all roots yield the initial MV. It should be noticed that in \( A_{3,4} \) the quaternion imaginary units have remained in even algebra only. The source of this ‘strange’ difference is related with the algorithm used in Appendix C where the roots are computed by \( Cl_{3,0} \) algebra rather than by \( Cl_{0,2} \), i.e. to algebra of quaternions. However, the program for \( Cl_{0,2} \) (see resp. equation in Appendix B) gives two roots only.

**Example 5. The regular case of \( Cl_{1,2} \) algebra.**

Using the same initial MV, \( B = \mathbf{e}_1 - 2\mathbf{e}_{23} \) as in Example 1, one obtains the same values for \((b_S, b_I)\) and \((s, S)\). After substitution into (24), (25) and then into (5) the square roots are found to be

\[
A_{1,2} = \pm \frac{1}{2}(c_2(-\mathbf{e}_1 + \mathbf{e}_{123}) - c_1(1 + \mathbf{e}_{23})), \\
A_{3,4} = \pm \frac{1}{2}(-c_1(\mathbf{e}_1 + \mathbf{e}_{123}) + c_2(-1 + \mathbf{e}_{23})), \] (39)

where \( c_1 = \sqrt{-2 + \sqrt{5}} \) and \( c_2 = \sqrt{2 + \sqrt{5}} \).
4. Square roots in $\text{Cl}_{0,3}$ algebra

The similar approach to the root problem allows to write down explicit square root formulas for $\text{Cl}_{0,3}$ algebra as well. Using the same notation \([5]\) for $\text{A}$ and $\text{B}$ and equating coefficients at same basis elements in $\text{A}^2 = \text{B}$ now we obtain the following system of equations\(^4\)

\[
\begin{align*}
    b_0 &= s^2 + S^2 + v^2 + V^2, & b_{123} &= 2(sS + v \cdot V), \\
    b_1 &= 2(sv_1 + SV_1), & b_{23} &= -2(sv_1 + Sv_1), \\
    b_2 &= 2(sv_2 + SV_2), & b_{13} &= 2(sv_2 + Sv_2), \\
    b_3 &= 2(sv_3 + SV_3), & b_{12} &= -2(sv_3 + Sv_3),
\end{align*}
\]

where now $v^2 = -v_1^2 - v_2^2 - v_3^2$ and $v \cdot V = -v_1V_1 - v_2V_2 - v_3V_3$.

4.1. The generic case $s^2 - S^2 \neq 0$

The solution of Eqs. \((41)-(43)\) is

\[
\begin{align*}
    v_1 &= \frac{b_1s + b_{23}S}{2(s^2 - S^2)}, & v_2 &= -\frac{b_2s - b_{13}S}{2(s^2 - S^2)}, & v_3 &= \frac{b_3s + b_{12}S}{2(s^2 - S^2)}, \\
    V_1 &= -\frac{b_{23}s + b_1S}{2(s^2 - S^2)}, & V_2 &= \frac{b_{13}s - b_2S}{2(s^2 - S^2)}, & V_3 &= -\frac{b_{12}s + b_3S}{2(s^2 - S^2)},
\end{align*}
\]

which is valid when $s^2 - S^2 \neq 0$, and corresponds to the generic case. After substitution of \((44)\) and \((45)\) into \((40)\) one obtains two coupled nonlinear algebraic equations for two unknowns $s$ and $S$,

\[
\begin{align*}
    b_s &= 4s^2(-6s^2 + b_0) + 8sSb_{123} = 4s^4 + (-2S^2 + b_0)^2 + b_{123}^2, \\
    b_I &= 2(2(s^2 + S^2) - b_0)(4sS - b_{123}),
\end{align*}
\]

where again the coordinate-free notation is introduced,

\[
\begin{align*}
    b_s &= \langle \tilde{\text{BB}} \rangle_0 = b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_{12}^2 + b_{13}^2 + b_{23}^2 + b_{123}^2, \\
    b_I &= \langle \tilde{\text{BB}}I \rangle_0 = -2b_3b_{12} + 2b_2b_{13} - 2b_1b_{23} + 2b_0b_{123}.
\end{align*}
\]

\(^4\)The formulas have the same structure and differ in signs of some constituent terms only. Below, for easier reading and application, all formulas, including those for mixed algebras, are written explicitly without introducing a large number of sign epsilons $\varepsilon_\pm = \pm 1$. In fact, appearance of different signs in structurally similar expressions brings in different conditions for real root existence in distinct algebras.
Note change of signs as compared to \( Cl_{3,0} \) case. The determinant \( D \) in \( Cl_{0,3} \) is expressed as a difference, \( D = b_S^2 - b_I^2 \), which is always positive, \( D > 0 \).

To reduce the degree of the above equations, the substitution

\[
s S = t; \quad \frac{1}{2}(s^2 + S^2) = T,
\]

is used that transforms the system (46) into simpler one

\[
b_S = (4T - b_0)^2 + (4t - b_{123})^2, \quad b_I = 2(4T - b_0)(4t - b_{123}).
\] (49)

The solution of (49) when \( (b_S \pm \sqrt{D}) > 0 \) is

\[
\begin{cases}
  (t_{1,2} = \frac{1}{4} b_{123} \pm \frac{1}{\sqrt{2}} \sqrt{b_S - \sqrt{D}}), & T_{1,2} = \frac{1}{4} \left( b_0 \pm \frac{b_I}{\sqrt{2} \sqrt{b_S - \sqrt{D}}} \right), \\
  (t_{1,2} = \frac{1}{4} b_{123}, \quad T_{1,2} = \frac{1}{4} (\pm \sqrt{b_S} + b_0)), & \text{if } b_S - \sqrt{D} > 0,
\end{cases}
\]

(50)

The \( \pm \) signs in the above formulas are mutually related, thus, there are only two possibilities that correspond to either plus or minus signs inside \( t_i \) and \( T_i \) formulas. The remaining two solutions of (49), which were obtained from (50) after replacement \( \sqrt{D} \to -\sqrt{D} \), yield a complex valued expression for \( T \pm \sqrt{T^2 - t^2} \) (see Eq. (51) below) therefore they were dismissed in advance.

Once the equations in (50) are computed they can be substituted back into solutions of (48),

\[
\begin{cases}
  (s_{1,2,3,4} = \pm \sqrt{T \pm \sqrt{T^2 - t^2}}, \quad S_{1,2,3,4} = \pm \frac{t}{\sqrt{T^2 - t^2}}) \text{ if } T \geq 0, \ t \neq 0, \\
  (s_{1,2} = S_{3,4} = \pm \sqrt{2T}, \quad S_{1,2} = s_{3,4} = 0) \text{ if } T \geq 0, \ t = 0.
\end{cases}
\] (51)

In the obtained equations the same signs must be chosen in the same index positions in \( s_{1,2,3,4} \) and \( S_{1,2,3,4} \) (four possibilities).

Thus, starting from pairs \( (t_1, T_1) \) and \( (t_2, T_2) \) in Eq. (50) and then going to (51), and finally to formulas (12), (15) and (5) one obtains explicit real solutions that completely determine the square root of equation \( B = A^2 \) (with \( A = s + v + (S + V)I \)) of the generic MV \( B \) of real \( Cl_{0,3} \) algebra in radicals.

\[\text{To eliminate } s \text{ and } S, \text{ Mathematica commands } \text{Eliminate}[ ], \text{ GroebnerBasis}[ ] \text{ have been used. They allow to rewrite the initial Eqs (48) in a number of equivalent forms.}\]
It appears that, at most, only four real solutions\(^6\) are possible in this algebra too, since other choices of signs in (50) and (51) yield negative expressions inside square roots.

4.2. The special case \(s^2 - S^2 = 0\)

There are three subcases: 1) \(s = S \neq 0\), 2) \(s = -S \neq 0\) and 3) \(s = S = 0\).

4.2.1. The subcase \(s = S \neq 0\)

Here the system of Eqs. (41)–(43) has a special solution,

\[
v_1 = \frac{b_1}{2s} - V_1, \quad v_2 = \frac{b_2}{2s} - V_2, \quad v_3 = \frac{b_3}{2s} - V_3,
\]

if and only if the MV B coefficients satisfy: 
\(b_1 = -b_23, b_2 = b_{13}, b_3 = -b_{12}\).

In (52), \(v_i\) is expressed in terms of \(V_i\). Appearance of \(s\) in the denominators implies that the case \(s = S = 0\) must be investigated separately. After substituting the solution (52) into (40) and taking into account the mentioned conditions \((b_1 = -b_23, b_2 = b_{13}, b_3 = -b_{12})\) one gets two equations,

\[
\begin{align*}
- \frac{b_1^2 + b_2^2 + b_3^2}{4s^2} + \frac{b_1 V_1 + b_2 V_2 + b_3 V_3}{s} - b_0 + 2s^2 + 2(V \cdot V) &= 0, \\
- \frac{b_1 V_1 + b_2 V_2 + b_3 V_3}{s} - b_{123} + 2s^2 - 2(V \cdot V) &= 0,
\end{align*}
\]

that should be kept mutually compatible. To this end we subtract and add the above equations to get

\[
\begin{align*}
- \frac{4s^2 (b_0 - b_{123} - 4(V \cdot V)) - 8s(b_1 V_1 + b_2 V_2 + b_3 V_3) + b_1^2 + b_2^2 + b_3^2}{4s^2} &= 0, \\
- \frac{4s^2 (b_0 + b_{123} - 4s^2) + b_1^2 + b_2^2 + b_3^2}{4s^2} &= 0.
\end{align*}
\]

Then, making use of expanded form of \(b_S = \langle \hat{\mathbf{B}} \rangle_0 = b_0^2 + 2(b_1^2 + b_2^2 + b_3^2) + b_{123}^2\), where the conditions \(b_1 = -b_{23}, b_2 = b_{13}, b_3 = -b_{12}\) have been taken into account, one can express the sum \(b_1^2 + b_2^2 + b_3^2\) from the second equation in (54), \(b_1^2 + b_2^2 + b_3^2 = \frac{1}{2}(b_S - b_0^2 - b_{123}^2)\), and substitute the latter into the first
of equations. The result is the quadratic equations for \( V_i \)'s. After solving, for example, with respect to \( V_1 \), one can express \( V_1 \) in terms of, now, arbitrary free parameters \( V_2 \) and \( V_3 \),

\[
V_1 = \frac{\sqrt{2}}{8s} \left( \sqrt{2}b_1 \pm \left( -8s^2 \left( b_0 - b_{123} + 4 \left( V_2^2 + V_3^2 \right) \right) \right) + 
16s(b_2V_2 + b_3V_3) + b_0^2 + 2b_1^2 + b_{123}^2 - b_S \right)^{1/2},
\]

(55)

that warrants compatibility of the system (53). Thus, further analysis may be restricted to the simplest single equation, the second equation in (54) that after introduction of shortcut \( b_S \) can be cast to form,

\[
b_S = -8b_0s^2 - 8b_{123}s^2 + b_0^2 + b_{123}^2 + 32s^4.
\]

(56)

Solution of (56) with respect to \( s \) can be expressed in radicals,

\[
s_{1,2} = \pm \frac{1}{2\sqrt{2}} \sqrt{2b_S - (b_0 - b_{123})^2 + b_0 + b_{123}},
\]

(57)

where all expressions inside square roots are assumed to be positive. The expressions (57), (55) and (52) after substitution into (5) yield the final answer for this special case under conditions for MV B coefficients: \( b_1 = -b_{23}, b_2 = b_13, b_3 = -b_{12} \) that in an abridged version can be reduce to \( b_S - (b_0 - b_{123})^2 = b_I \). In conclusion, the solution set contains two free parameters, \( V_2 \) and \( V_3 \), and therefore represents continuum of roots on a two dimensional manifold in the parameter space.

4.2.2. The subcase \( s = -S \neq 0 \)

Performing exactly the same analysis as in subcase 4.2.1 one obtains the conditions for existence of solution: \( b_1 = b_{23}, b_2 = -b_{13} \) and \( b_3 = b_{12} \), or in short \( -b_S + (b_0 + b_{123})^2 = b_I \). Similarly, expressing \( v_i \) in terms of \( V_i \) one gets

\[
v_1 = \frac{b_1}{2s} + V_1, \quad v_2 = \frac{b_2}{2s} + V_2, \quad v_3 = \frac{b_3}{2s} + V_3.
\]

(58)

If \( V_1 \) is expressed in terms of \( V_2 \) and \( V_3 \),

\[
V_1 = \frac{\sqrt{2}}{8s} \left( -\sqrt{2}b_1 \pm \left( -8s^2 \left( b_0 + b_{123} + 4(V_2^2 + V_3^2) \right) \right) \right.
- 16s(b_2V_2 + b_3V_3) + b_0^2 + 2b_1^2 + b_{123}^2 - b_S \right)^{1/2},
\]

(59)
we find two real solutions for $s$,

$$s_{1,2} = \pm \frac{1}{2\sqrt{2}} \sqrt{2bS - (b_0 + b_{123})^2 + b_0 - b_{123}},$$

After substitution into (5) the above expressions again yield the final MV provided the conditions $b_1 = b_{23}, b_2 = -b_{13}, b_3 = b_{12}$ are satisfied.

4.2.3. The subcase $s = S = 0$

The analysis of this special subcase is very similar to that in $Cl_{3,0}$. The Eqs. (41)-(43) satisfy compatibility condition if vector $(b_1, b_2, b_3)$ and bivector $(b_{12}, b_{13}, b_{23})$ coefficients are equated to zero. Then the Eqs. (40) assume the following form

$$b_0 = v^2 + V^2, \quad b_{123} = 2(v \cdot V)$$

from which follows that four parameters remain unspecified. For example, if Eq. (61) is solved with respect to pair $(v_1, V_1)$, one gets

$$v_1 = \pm \frac{c_1}{\sqrt{2}}, \quad V_1 = \pm \frac{1}{c_1} \frac{b_{123} + 2(v_2 V_2 + v_3 V_3)}{\sqrt{2}},$$

where

$$c_1 = \left( \pm \sqrt{(b_0 + v_2^2 + v_3^2 + V_2^2 + V_3^2)^2 - (b_{123} + 2(v_2 V_2 + v_3 V_3))^2} - b_0 - v_2^2 - v_3^2 - V_2^2 - V_3^2 \right)^{\frac{1}{2}}.$$  

The pairs $(v_2, V_2)$ and $(v_3, V_3)$ may be interpreted as free parameters that generate a continuum of roots in a four parameter space. The geometric interpretation of Eqs. (61) is similar to those in (20) for $Cl_{3,0}$.

4.3. Examples for $Cl_{0,3}$

Example 6. The regular case.

As in Example 1, let the initial MV be $B = e_1 - 2e_{23}$, the coefficients of which are $b_1 = 1, b_{12} = -2$. The shortcuts $b_t$ and $b_S$ in (47) have the values $b_t = 4$ and $b_S = 5$. The formulas in (50) give $(T_1, t_1) = (\frac{1}{2}, \frac{1}{4})$ and $(T_2, t_2) = (-\frac{1}{2}, -\frac{1}{4})$. Since $T_1$ is positive, the pair $(T_1, t_1)$ is used in the following. Then, from the system (51) we find four values of $(s_i, S_i)$ and then from (44), (55) the coefficients $v_i$ and $V_i$,

$$(s_1 = V_1 = \frac{1}{2}d_3, \quad S_1 = v_1 = -\frac{1}{2}d_2, \quad v_2 = v_3 = V_2 = V_3 = 0),$$
$$(s_2 = V_1 = \frac{1}{2}d_1, \quad S_2 = v_1 = \frac{1}{2}d_2, \quad v_2 = v_3 = V_2 = V_3 = 0),$$
$$(s_3 = V_1 = -\frac{1}{2}d_2, \quad S_3 = -\frac{1}{2}d_2, \quad v_1 = \frac{1}{4}d_3, \quad v_2 = v_3 = V_2 = V_3 = 0),$$
$$(s_4 = V_1 = \frac{1}{2}d_2, \quad S_4 = \frac{1}{2}d_1, \quad v_1 = \frac{1}{2}d_1, \quad v_2 = v_3 = V_2 = V_3 = 0).$$
where \( d_1 = \sqrt{2 - \sqrt{3}} \), \( d_2 = \sqrt{2 + \sqrt{3}} \) and \( d_3 = \sqrt{2 - \sqrt{6}} \). Finally, in the same way as in Example 1 we find four different square roots,

\[
A_1 = \frac{1}{2}(d_1 + d_2e_1 - d_1e_{23} + d_2e_{123}),
A_2 = \frac{1}{2}(\frac{1}{2}d_3 - d_2e_1 - \frac{1}{2}d_3e_{23} - d_2e_{123}),
A_3 = \frac{1}{2}(d_2 + d_1e_1 - d_2e_{23} + \frac{e_{123}}{d_2}),
A_4 = \frac{1}{2}(-d_2 + \frac{1}{2}d_3e_1 + d_2e_{23} - \frac{e_{123}}{d_2}).
\]

(64)

Noting that \( d_3 = -2d_1 \) and \( d_2^{-1} = d_1 \) the roots may rewritten in the standard form

\[
A_{1,2} = \pm \frac{1}{2}(d_1 + d_2e_1 - d_1e_{23} + d_2e_{123}), \\
A_{3,4} = \pm \frac{1}{2}(d_2 + d_1e_1 - d_2e_{23} + d_1e_{123}).
\]

(65)

**Example 7. The case** \( s = S \neq 0 \).

The square root of \( B = -e_3 + e_{12} + 4e_{123} \). The shortcuts in \( (b_I, b_S) \) have values \( b_I = 2 \) and \( b_S = 18 \), and afterwards the expression \( (64) \) gives \( (T_1, t_1) = (\frac{23}{3}, \frac{1}{3}) \), where \( c_1 = (2 + \sqrt{3}) \). The negative \( T \) solution has been omitted. All this gives \( (s_1, S_1) = (-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}) \) and \( (s_2, S_2) = (\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}) \). The coefficients satisfy the relations \( b_1 = -b_{23} \), \( b_2 = b_{13} \), \( b_3 = -b_{12} \), therefore, a special solution consisting of four MVs exists:

\[
A_1 = -\frac{1}{2}(e_1 + e_{23})\sqrt{-4V_2^2 + 4(c_1 - V_3)V_3 + c_3 - e_{12}V_3 + (e_{13} - e_2)V_2}
+ e_3(c_1 - V_3) - \frac{1}{2}c_2(e_{123} + 1),
A_2 = \frac{1}{2}(e_1 + e_{23})\sqrt{-4V_2^2 + 4(c_1 - V_3)V_3 + c_3 - e_{12}V_3 + (e_{13} - e_2)V_2}
+ e_3(c_1 - V_3) - \frac{1}{2}c_2(e_{123} + 1),
A_3 = \frac{1}{2}(-(e_1 + e_{23})\sqrt{-4V_2^2 - 4V_3(V_3 + c_1) + c_3 - 2e_{12}V_3}
+ 2(e_{13} - e_2)V_2 - 2e_3(V_3 + c_1) + c_2(e_{123} + 1)),
A_4 = \frac{1}{2}((e_1 + e_{23})\sqrt{-4V_2^2 - 4V_3(V_3 + c_1) + c_3 - 2e_{12}V_3}
+ 2(e_{13} - e_2)V_2 - 2e_3(V_3 + c_1) + c_2(e_{123} + 1)),
\]

(66)

17
where $c_1 = \sqrt{\sqrt{5} - 2}$, $c_2 = \sqrt{\sqrt{5} + 2}$, $c_3 = -\sqrt{5} + 6$. Assuming concrete values of parameters $V_2$ and $V_3$ one can check that the root formulas give real MVs.

It should be noted however that symbolical expressions do not guarantee that we will always be able to find real parameters $V_2$ and $V_3$, what would ensure real square roots. For example, if instead of the above MV $B = -e_3 + e_{12} + 4e_{123}$ we would try to find square root of MV $B = -e_3 + e_{12}$ in $Cl_{0,3}$ (the MV was used earlier in the Example 2) we would find $s_1 = \frac{1}{2}$ and $s_2 = -\frac{1}{2}$. The first value then yields $v_1 = -V_1 = \frac{1}{2}\sqrt{-4V_2^2 - (1 + 2V_3)^2}$, $v_2 = -V_2$, $v_3 = -1 - V_3$ and the second one yields $V_1 = -v_1 = \frac{1}{2}\sqrt{-4V_2^2 - (1 - 2V_3)^2}$, $v_2 = -V_2$, and $v_3 = 1 - V_3$. Taking the square of symbolical expressions one can easily check that formally we indeed obtain the MV $B = -e_3 + e_{12}$. It is obvious, however, that in both cases ($s_1 = \frac{1}{2}$ and $s_2 = -\frac{1}{2}$) the expression under square root can be made non-negative (i.e. only zero in this case) for a single choice of parameters. In particular, in the case $s_1 = \frac{1}{2}$, the requirement $-4V_2^2 - (1 + 2V_3)^2 \geq 0$ yields $V_2 = 0$, $V_3 = -1/2$. Alternatively, in the case $s_2 = -\frac{1}{2}$ from equation $-4V_2^2 - (1 - 2V_3) \geq 0$ follows $V_2 = 0$, $V_3 = 1/2$. The both cases yield an isolated root $\pm \frac{1}{2}(1 - e_3 + e_{12} + e_{123})$. Therefore, in this algebra in fact there exist only isolated real square root of $B = -e_3 + e_{12}$.

5. Square roots in $Cl_{2,1}$ algebra

5.1. The generic case $s^2 - S^2 \neq 0$

The system of nonlinear equations is

\[
\begin{align*}
 b_0 &= s^2 + S^2 + v^2 + V^2, & b_{123} &= 2(sS + v \cdot V), \\
 b_1 &= 2(sv_1 + SV_1), & b_{23} &= 2(sV_1 + Sv_1), \\
 b_2 &= 2(sv_2 + SV_2), & b_{13} &= -2(sV_2 + Sv_2), \\
 b_3 &= 2(sv_3 + SV_3), & b_{12} &= -2(sV_3 + Sv_3),
\end{align*}
\]

where now $v^2 = v_1^2 + v_2^2 - v_3^2$ and $v \cdot V = v_1V_1 + v_2V_2 - v_3V_3$. When $s^2 - S^2 \neq 0$ the solutions of system (68)-(70) are

\[
\begin{align*}
 v_1 &= \frac{b_1s - b_{23}S}{2(s^2 - S^2)}, & v_2 &= \frac{b_2s + b_{13}S}{2(s^2 - S^2)}, & v_3 &= \frac{b_3s + b_{12}S}{2(s^2 - S^2)}, \\
 V_1 &= \frac{b_{23}s - b_1S}{2(s^2 - S^2)}, & V_2 &= -\frac{b_{13}s + b_2S}{2(s^2 - S^2)}, & V_3 &= -\frac{b_{12}s + b_3S}{2(s^2 - S^2)}.
\end{align*}
\]
Insertion of $v_i$ and $V_i$ into (67) gives two coupled equations for unknowns $s, S$

\[
\begin{align*}
    b_S &= 4s^2(-6s^2 + b_0) + 8sSb_{123} = 4s^4 + (-2S^2 + b_0)^2 + b_{123}^2, \\
    b_I &= 2(2(s^2 + S^2) - b_0)(4sS - b_{123}),
\end{align*}
\]

where $b_S$ and $b_I$ are functions of coefficients in $B$,

\[
\begin{align*}
    b_S &= \langle \tilde{B}\tilde{B} \rangle_0 = b_0^2 - b_1^2 - b_2^2 + b_3^2 + b_{12}^2 - b_{13}^2 - b_{23}^2 + b_{123}^2, \\
    b_I &= \langle \tilde{B}\tilde{I} \rangle_0 = -2b_3b_{12} + 2b_2b_{13} - 2b_1b_{23} + 2b_0b_{123}.
\end{align*}
\]

Because the Eqs. (73) and (47) have the same shape (the concrete equations for $b_S$ and $b_I$, of course, are different) we can make use of (48) with the purpose to lower the order of the system. However, there arises an important difference: the determinant, $D = b_S^2 - b_I^2$, in $Cl_{2,1}$ is not always positive. It may happen that for some $B$ the MV determinant may become negative, $D < 0$. In such a case the solution set becomes empty. The other particularity is that in the solution (50) instead of single sign ($-\sqrt{D}$) we have to take into account both signs, i.e., $\pm\sqrt{D}$, what doubles the number of possible solutions in the case $D > 0$,

\[
\begin{align*}
    \left\{ 
        \begin{array}{l}
            t_{1,2,3,4} = \frac{1}{4}(b_{123} \pm \frac{1}{2}\sqrt{b_S \pm \sqrt{D}}), \\
            T_{1,2,3,4} = \frac{1}{4}(\pm\sqrt{b_S \pm \sqrt{D}} + b_0),
        \end{array}
    \right. \\
    \text{if } b_S \pm \sqrt{D} > 0,
\end{align*}
\]

\[
\begin{align*}
    \left\{ 
        \begin{array}{l}
            t_{1,2} = \frac{1}{4}b_{123}, \\
            T_{1,2} = \frac{1}{4}(\pm\sqrt{b_S} + b_0),
        \end{array}
    \right. \\
    \text{if } b_S \pm \sqrt{D} = 0 \text{ and } b_S > 0.
\end{align*}
\]

Here again the sign of $t_i$ must be taken in all possible combinations, and the sign of $T$ must follow the same upper-lower sign position as in $t_i$. The condition $b_S \pm \sqrt{D} = 0$ implies that $b_I = 0$. Since we already have four sign combinations in the solution for $s, S$ (as in (51)), we end up with 16 different square roots of MV in a generic case of $Cl_{2,1}$.

5.2. The special case $s^2 - S^2 = 0$

The analysis again closely follows $Cl_{0,3}$ case, except that now different signs appear in expressions.
5.2.1. The subcase \( s = S \neq 0 \)

Now the coefficients satisfy the conditions \( b_1 = b_{23}, b_2 = -b_{13}, b_3 = -b_{12} \) which allow to eliminate the singularity at \( s = S \). As a result the system of Eqs. (68)-(70) has a special solution,

\[
v_1 = \frac{b_1}{2s} - V_1, \quad v_2 = \frac{b_2}{2s} - V_2, \quad v_3 = \frac{b_3}{2s} - V_3,
\]

which coincides with the same solution for \( Cl_{0,3} \) (see Eq. (52)). Thus, after similar calculations one finds that Eq. (55) becomes

\[
V_1 = \frac{\sqrt{2}}{8s} \left( \sqrt{2}b_1 \pm \left( 8s^2 \left( b_0 - b_{123} + 4(-V_2^2 + V_3^2) \right) + 16s(b_2V_2 - b_3V_3) \right)
\right.
\]

\[
- b_0^2 + 2b_1^2 - b_{123}^2 + b_s^{1/2}) \right).
\]

The coefficients \( s_1 \) and \( s_2 \) are similar to (57), except that now we have to take into account all sign combinations in inner square root,

\[
s_{1,2} = \pm \frac{1}{2\sqrt{2}} \sqrt{\pm \sqrt{2b_s - (b_0 - b_{123})^2} b_0 + b_{123}}. \tag{78}
\]

The above listed formulas solve square root problem in the case \( s = S \neq 0 \).

5.2.2. The subcase \( s = -S \neq 0 \)

The only formulas which differ from \( Cl_{0,3} \) algebra are connected with the coefficient compatibility condition: \( b_1 = -b_{23}, b_2 = b_{13}, b_3 = b_{12} \). Now, the coefficients must be replaced by

\[
V_1 = \frac{\sqrt{2}}{8s} \left( -\sqrt{2}b_1 \pm \left( 8s^2 \left( b_0 + b_{123} + 4(-V_2^2 + V_3^2) \right)
\right.
\]

\[
- 16s(b_2V_2 - b_3V_3) - b_0^2 + 2b_1^2 - b_{123}^2 + b_s^{1/2} \right) \right)
\]

\[
s_{1,2} = \pm \frac{1}{2\sqrt{2}} \sqrt{\pm \sqrt{2b_s - (b_0 + b_{123})^2} b_0 - b_{123}} \tag{80}
\]

The remaining formulas which are needed for final answer exactly match the formulas in the corresponding subcase of \( Cl_{0,3} \) algebra.
5.2.3. The subcase $s = S = 0$

The only distinct formulas from $Cl_{0,3}$ are listed below,

\[ v_1 = \pm \frac{c_1}{\sqrt{2}}, \quad V_1 = \pm \frac{1}{c_1} \frac{b_{123} + 2(-v_2 V_2 + v_3 V_3)}{\sqrt{2}}, \quad \text{where} \]

\[ c_1 = \left( \pm \sqrt{\left(b_0 - v_2^2 + v_3^2 - V_2^2 + V_3^2\right)^2 - \left(b_{123} + 2(-v_2 V_2 + v_3 V_3)\right)^2} + b_0 - v_2^2 + v_3^2 - V_2^2 + V_3^2 \right)^{\frac{1}{2}}. \] (81)

This ends the investigation of the square root formulas for all real 3D CAs.

5.3. Examples for $Cl_{2,1}$

Example 8. The regular case.

First, we shall show that $MV \mathbf{B} = \mathbf{e}_1 - 2\mathbf{e}_{23}$ has no real square roots. Indeed, we have $b_S = -5$, $b_I = 4$ and $D = b_S^2 - b_I^2 = (3)^2$. As a result the expression under square root in (75), namely $b_S \pm \sqrt{D} = -5 \pm 3$, is always negative and therefore there are no real-valued solutions.

Next, we shall calculate the roots of $\mathbf{B} = 2 + \mathbf{e}_1 + \mathbf{e}_{13}$. The values of $b_I$ and $b_S$ are 0 and 2, respectively. The determinant of the MV is positive, $D = 4 > 0$. The Eqs. (75) give four real values for pairs: $(T_1, t_1) = (\frac{1}{4}(2 - \sqrt{2}), 0)$, $(T_2, t_2) = (\frac{1}{4}(2 + \sqrt{2}), 0)$, $(T_3, t_3) = (\frac{1}{2}, -\frac{1}{2\sqrt{2}})$ and $(T_4, t_4) = (\frac{1}{2}, \frac{1}{2\sqrt{2}})$. After insertion into (71), 16 pairs of scalars $(s_i, S_i)$ are found:

\[
\begin{align*}
(s_1 &= 0, S_1 = -\frac{\sqrt{2}}{2}), & (s_2 &= 0, S_2 = \frac{\sqrt{2}}{2}), & (s_3 &= -\frac{\sqrt{2}}{2}, S_3 = 0), \\
(s_4 &= \frac{\sqrt{2}}{2}, S_4 = 0), & (s_5 &= 0, S_5 = -\frac{\sqrt{2}}{2}), & (s_6 &= 0, S_6 = \frac{\sqrt{2}}{2}), \\
(s_7 &= -\frac{\sqrt{2}}{2}, S_7 = 0), & (s_8 &= \frac{\sqrt{2}}{2}, S_8 = 0), & (s_9 &= -\frac{\sqrt{2}}{2}, S_9 = \frac{\sqrt{2}}{2}), \\
(s_{10} &= \frac{\sqrt{2}}{2}, S_{10} = -\frac{\sqrt{2}}{2}), & (s_{11} &= -\frac{\sqrt{2}}{2}, S_{11} = \frac{1}{\sqrt{2c_1}}), & (s_{12} &= \frac{\sqrt{2}}{2}, S_{12} = -\frac{1}{\sqrt{2c_1}}), \\
(s_{13} &= -\frac{\sqrt{2}}{2}, S_{13} = \frac{\sqrt{2}}{2}), & (s_{14} &= \frac{\sqrt{2}}{2}, S_{14} = \frac{\sqrt{2}}{2}), & (s_{15} &= -\frac{\sqrt{2}}{2}, S_{15} = -\frac{1}{\sqrt{2c_1}}), \\
(s_{16} &= \frac{\sqrt{2}}{2}, S_{16} = \frac{1}{\sqrt{2c_1}}),
\end{align*}
\]

where $c_1 = \sqrt{2 + \sqrt{2}}$ and $c_2 = \sqrt{2 - \sqrt{2}}$. After substitution of $(s_i, S_i)$ into
Eqs. (71) and then into Eq. (5) we obtain 16 roots $A_{i,j} = \pm \sqrt{2 + e_1 + e_{13}}$:

\[
\begin{align*}
A_{1,2} &= \pm \frac{1}{2} (c_1 e_2 - c_1 e_{23} - \sqrt{2} c_2 e_{123}), \\
A_{3,4} &= \pm \frac{1}{\sqrt{2}} (- c_1^{-1} e_2 + c_1^{-1} e_{23} + c_1 e_{123}), \\
A_{5,6} &= \pm \frac{1}{2} (\sqrt{2} c_2 + c_1 e_1 + c_1 e_{13}), \\
A_{7,8} &= \pm \frac{1}{\sqrt{2}} (c_1 + c_1^{-1} e_1 + c_1^{-1} e_{13}), \\
A_{9,10} &= \pm \frac{1}{2\sqrt{2}} (\sqrt{2} c_2 - c_2 e_1 - c_1 e_2 - c_2 e_{13} + c_1 e_{23} + \sqrt{2} c_1 e_{123}), \\
A_{11,12} &= \pm \frac{1}{2\sqrt{2}} (\sqrt{2} c_1 + c_1 e_1 - c_2 e_2 + c_1 e_{13} + c_2 e_{23} - 2c_1^{-1} e_{123}), \\
A_{13,14} &= \pm \frac{1}{2\sqrt{2}} (\sqrt{2} c_1 + c_1 e_1 + c_2 e_2 + c_1 e_{13} - c_2 e_{23} + 2c_1^{-1} e_{123}), \\
A_{15,16} &= \pm \frac{1}{2\sqrt{2}} (- \sqrt{2} c_2 + c_2 e_1 - c_1 e_2 + c_2 e_{13} + c_1 e_{23} + \sqrt{2} c_1 e_{123}).
\end{align*}
\] (82)

In the end it is worth to note, that the necessary (but not sufficient) condition for a square root of MV $B$ to exist in real Clifford algebras $Cl_{p,q}$ requires the positivity of the multivector determinant $\det(B)$ [2, 3]. Indeed, if the MV $A$ exists and $AA = B$, then the determinant of both sides gives $\det(A) \det(A) = \det(B)$, where we have used the multiplicative property of the determinant [18]. Since the determinant of $A$ in real CAs is a real quantity, the condition can be satisfied if and only if $\det(B) \geq 0$. This is in agreement with explicit formulas for $n \leq 3$.

6. Conclusions

First, we have shown analytically that the square root of general MV in $n = p + q \leq 3$ Clifford algebras (CA) can be expressed in radicals and have provided a detailed analysis and algorithm to accomplish the task. For a general MV the algorithm is rather complicated, where many of conditions are controlled by plus/minus signs. In our first paper [8] only the roots of individual grades have been considered, where explicit formulas in a coordinate-free form are provided.

Second, the paper shows that MV roots may be isolated (up to 16 roots in case of $Cl_{2,1}$) and/or continuous, or conversely there may be no roots at all. Thus, the MV algebras may also accommodate a number free parameters that bring in a continuum of roots on respective parameter hypersurface.

Third, the described algorithm was implemented in Mathematica system [4] and applied in checking up algorithms by purely numerical root
search. For this purpose Mathematica universal root search algorithm was realized and used in the system function \texttt{FindInstance[ ]} to check whether there are cases when isolated root algorithm fails. No such cases were found. The only complication we encountered in the algorithm programming was that Mathematica symbolic zero detection algorithm \texttt{PossibleZeroQ[ ]} in the more complicated cases often switched over to numerical procedure to detect that involved symbolic expression with nested radicals indeed represents zero. This is quite understandable, since it is well known that two expression equivalence problem is, in general, undecidable.

Fourth, we found that for algebras, $Cl_{3,0}$ and $Cl_{1,2}$, the square root solution in general is a union of the following sets: 1) when $s^2 \neq S^2$, the set consists of (up to) four different isolated roots, 2) when $s^2 = S^2 \neq 0$, the set consists of two isolated roots and 3) when $s^2 = S^2 = 0$ there appears a continuum of roots that belong to four or smaller dimensional parameter manifolds. Similar sets with minor modifications exist for remaining algebras, $Cl_{2,1}$ and $Cl_{0,3}$ as well.

The proposed algorithm is a step forward in solving general quadratic equations in CAs (examples are given in \cite{3}), and may find new applications in the control and systems theory \cite{1}, partly because presented solutions uncover totally new properties of square root of MV, for example, the root multiplicity and appearance of free parameters in the roots. Due to intricacies of square root algorithms, it is recommended to do all calculations by prepared in advance numerical/symbolic subroutines.

\textbf{Appendix A. Square roots in $Cl_{1,0}$ and $Cl_{0,1}$ algebras}

In $Cl_{1,0}$ and $Cl_{0,1}$, the square root of general MV $B = b_0 + b_1 e_1$ has the solution $A = \sqrt{B} = s + v_1 e_1$, where the real coefficients $s$ and $v_1$ are

$$v_1 = \frac{b_1}{2s}; \quad s = \begin{cases} \pm \frac{1}{\sqrt{2}} \sqrt{b_0 - \sqrt{D}} & \text{if } b_0 - \sqrt{D} > 0 \text{ and } D \geq 0, \\ \pm \frac{1}{\sqrt{2}} \sqrt{b_0 + \sqrt{D}} & \text{if } b_0 + \sqrt{D} > 0 \text{ and } D \geq 0, \end{cases}$$

where

$$D = \begin{cases} b_0^2 - b_1^2, & \text{for } Cl_{1,0}, \\ b_0^2 + b_1^2, & \text{for } Cl_{0,1}. \end{cases}$$
When \( s = 0 \), (i.e. when \( b_0 \pm \sqrt{D} = 0 \)) and \( b_1 = 0 \) the square roots are

\[
A = \begin{cases} 
\pm \sqrt{b_0}, & \text{if } b_1 = 0, \text{ for } Cl_{1,0}, \\
\pm \sqrt{-b_0}, & \text{if } b_1 = 0, \text{ for } Cl_{0,1}.
\end{cases}
\]

Note, that the \( Cl_{0,1} \) algebra is isomorphic to the algebra of complex numbers, so we know in advance that any MV in this algebra has two roots. The MV determinant for this algebra is positive definite \( D = b_0^2 + b_1^2 \geq 0 \) and represents the square of module of a complex number. We shall always assume that expressions under square roots are non-negative. For example, in this case the square root can only exist when \( D \geq 0 \), and either \((b_0 - \sqrt{D}) \geq 0\) or \((b_0 + \sqrt{D}) \geq 0\). If these conditions cannot be satisfied, then square roots are absent.

**Appendix B. Square roots in \( Cl_{2,0} \), \( Cl_{1,1} \) and \( Cl_{0,2} \) algebras**

Square root \( A \) of general MV \( B = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_{12} \) in all three algebras is \( A = s + v_1 e_1 + v_2 e_2 + S e_{12} \). The coefficients \((s, S)\) are

\[
\begin{cases} 
(s = \pm \frac{1}{\sqrt{2}} \sqrt{b_0 - \sqrt{D}}, S = \pm \frac{1}{\sqrt{2}} \frac{b_3}{\sqrt{b_0 - \sqrt{D}}}), & \text{if } b_0 - \sqrt{D} > 0 \text{ and } D \geq 0, \\
(s = \pm \frac{1}{\sqrt{2}} \sqrt{b_0 + \sqrt{D}}, S = \pm \frac{1}{\sqrt{2}} \frac{b_3}{\sqrt{b_0 + \sqrt{D}}}), & \text{if } b_0 + \sqrt{D} > 0 \text{ and } D \geq 0,
\end{cases}
\]

where the determinant of MV \( B \) is \( D \),

\[
D = \begin{cases} 
b_0^2 - b_1^2 - b_2^2 + b_3^2, & \text{for } Cl_{2,0}, \\
b_0^2 - b_1^2 + b_2^2 - b_3^2, & \text{for } Cl_{1,1}, \\
b_0^2 + b_1^2 + b_2^2 + b_3^2, & \text{for } Cl_{0,2}.
\end{cases}
\]

**The case** \( s \neq 0 \). The coefficients \( v_1, v_2 \in A \) then are given by formulas

\[
v_1 = \frac{b_1}{2s}, \quad v_2 = \frac{b_2}{2s}.
\]

**The case** \( s = 0 \). When \( b_0 - \sqrt{D} = 0 \), or \( b_0 - \sqrt{D} = 0 \) and \( b_1 = b_2 = b_3 = 0 \), the coefficients \( v_1, v_2 \) and \( S \) are connected by single equation \( \pm v_1^2 \pm v_2^2 \pm b_0 \pm S^2 = 0 \). Therefore, one can search the solution with respect to any of coefficients \( v_1, v_2 \) or \( S \), and assume that remaining two coefficients are the
free parameters. For example, if we solve with respect to $S$, then the square root for each of algebras is,

$$A = \begin{cases} 
  v_1 e_1 + v_2 e_2 \pm \sqrt{-b_0 + v_1^2 + v_2^2} e_{12}, & \text{for } Cl_{2,0}, \text{ if } b_1 = b_2 = b_3 = 0, \\
  v_1 e_1 + v_2 e_2 \pm \sqrt{b_0 - v_1^2 + v_2^2} e_{12}, & \text{for } Cl_{1,1}, \text{ if } b_1 = b_2 = b_3 = 0, \\
  v_1 e_1 + v_2 e_2 \pm \sqrt{-b_0 - v_1^2 - v_2^2} e_{12}, & \text{for } Cl_{0,2}, \text{ if } b_1 = b_2 = b_3 = 0.
\end{cases}$$

Since the coefficient $S$ is real, the roots exist only when the expressions under square root are positive. The algebra $Cl_{2,0}$ is isomorphic to $Cl_{1,1}$.

**Example.**

The square root of $B = 6 + 2e_1 + 3e_2 - 4e_{12}$ in various 2D algebras:

$$A = \begin{cases} 
  \pm \frac{1}{\sqrt{2(6+\sqrt{39})}}(6 + \sqrt{39} + 2e_1 + 3e_2 - 4e_{12}) & Cl_{2,0}, \\
  \pm \frac{1}{\sqrt{2}}(1 + 2e_1 + 3e_2 - 4e_{12}) \text{ and } \pm \frac{1}{\sqrt{22}}(11 + 2e_1 + 3e_2 - 4e_{12}) & Cl_{1,1}, \\
  \pm \frac{1}{\sqrt{2(6+\sqrt{65})}}(6 + \sqrt{65} + 2e_1 + 3e_2 - 4e_{12}) & Cl_{0,2}.
\end{cases}$$

Note that in $Cl_{1,1}$ there are four roots.
Appendix C. Algorithm for square root

Algorithm 1: Algorithm for square root for Cl3.0.

Sqrt (B)

Input: B = \( b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 + b_{12} e_{12} + b_{13} e_{13} + b_{23} e_{23} + b_{123} e_{123} \)

Output: A = s + v + (S + V)I with the property A^2 = B, or no solution

/* Initialization */
\( b_S = b_0^2 - b_1^2 - b_2^2 - b_3^2 + b_{12}^2 + b_{13}^2 + b_{23}^2 - b_{123}^2; \)
\( b_I = 2b_3b_{12} - 2b_2b_{13} + 2b_1b_{23} - 2b_0b_{123}; \)
\( D = b_S^2 + b_I^2; \)

/* Compute all \((t_i, T_i)\) pairs */

if \(-b_S + \sqrt{D} > 0\) then

\( t_{1,2} = \frac{1}{2}(b_{123} \pm \frac{1}{\sqrt{2}} \sqrt{-b_S + \sqrt{D}}), T_{1,2} = \frac{1}{4}(\pm\frac{b_I}{\sqrt{2}\sqrt{-b_S + \sqrt{D}}} - b_0); \)

else if \(-b_S + \sqrt{D} = 0\) and \(b_S > 0\) then

\( t_{1,2} = \frac{1}{2}b_{123}, T_{1,2} = \frac{1}{4}(\pm\sqrt{b_S} - b_0); \)

else

return \(\emptyset;\) /* no root */

end

/* for each \((t_i, T_i)\) pair (where \(i = 1, 2 = (+, -)\)) find corresponding \((s_i, S_i)\) pair */

foreach \((t_i, T_i)\) do

\( s_i = \pm\sqrt{-T_i + \sqrt{T_i^2 + t_i^2}}, S_i = \pm\frac{t}{\sqrt{-T_i + \sqrt{T_i^2 + t_i^2}}}; \)

/* every index \(i = 1, 2 = (+, -)\) matches two values */

end

/* For each \((s_i, S_i)\) pair compute corresponding \(v_{i_k}, V_{i_k}\) */

foreach \((s_i, S_i)\) do

if \(s^2 + S^2 \neq 0\) then

\( v_{i_1} = \frac{b_1 + b_{23}}{2(s^2 + S^2)}, v_{i_2} = \frac{b_{23} - b_1S}{2(s^2 + S^2)}, v_{i_3} = \frac{b_3 + b_{12}S}{2(s^2 + S^2)}; \)
\( v_{i_1} = \frac{b_{23} + b_1S}{2(s^2 + S^2)}, v_{i_2} = \frac{b_{13} + b_2S}{2(s^2 + S^2)}, v_{i_3} = \frac{b_{12} + b_3S}{2(s^2 + S^2)}; \)

return \(A \leftarrow (s_i, S_i, v_{i_k}, V_{i_k}); \) /* isolated root */

else

if \(b_1 = b_2 = b_3 = b_{12} = b_{13} = b_{23} = 0; \) /* \(s^2 + S^2 = 0\) case */

then

solve any two of \(v_{i_k}, V_{i_j}\) from \(\begin{cases} b_0 = v^2 - v^2; \\ b_{123} = 2(v \cdot v) \end{cases} \)

return \(A \leftarrow (s_i, S_i, v_{i_k}, V_{i_j}); \) /* continuum of roots */

else

return \(A \leftarrow \emptyset; \) /* no root */

end

end

end
The algorithm in Appendix C was implemented by Mathematica using single piecewise function \texttt{Piecewise[]} to which special case solutions then were added in the final output. Such an approach also allows to compute isolated square roots of MV with symbolic coefficients. Algorithms for remaining $n = 3$ algebras are similar to $Cl_{3,0}$ in Appendix C. We have found the algorithms very helpful in checking over and dealing with real MV roots.

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