On the spectra of strong power graphs of finite groups

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Abstract

We give the characteristic polynomial of the distance or adjacency matrix of the strong power graph of a finite group, and compute its distance and adjacency spectrum.

Key words: strong power graph; cyclic group; characteristic polynomial; spectrum.

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1 Introduction

Given a connected graph $\Gamma$, denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set, respectively. Let $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. The distance between the vertices $v_i$ and $v_j$, denoted by $d_\Gamma(v_i, v_j)$, is the length of the shortest path between them. The diameter of $\Gamma$, denoted by $\text{diam}(\Gamma)$, is the maximum distance between any pair of vertices of $\Gamma$. The set of neighbours of a vertex $v_i$ in $\Gamma$ is denoted by $N_\Gamma(v_i)$, that is, $N_\Gamma(v_i) = \{v_j \in V(\Gamma) : \{v_i, v_j\} \in E(\Gamma)\}$.

The distance matrix $D(\Gamma)$ of $\Gamma$ is the $n \times n$ matrix, indexed by $V(\Gamma)$, such that $D(\Gamma)_{v_i, v_j} = d_\Gamma(v_i, v_j)$. The characteristic polynomial $\Theta(\Gamma, x) = |xI - D(\Gamma)|$ is the distance characteristic polynomial of $\Gamma$. Note that $D(\Gamma)$ is symmetric. The distance characteristic polynomial has real roots $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t$ are the distinct roots of $\Theta(\Gamma, x)$, then the $D$-spectrum of $\Gamma$ can be written as

$$\text{spec}_D(\Gamma) = \left(\begin{array}{cccc}
\mu_1 & \mu_2 & \cdots & \mu_t \\
m_1 & m_2 & \cdots & m_t
\end{array}\right),$$

where $m_j$ is the algebraic multiplicity of $\mu_j$. Clearly $\sum_{j=1}^t m_j = n$. The adjacency matrix $A(\Gamma)$ of $\Gamma$ is a $n \times n$ matrix and indexed by $V(\Gamma)$. The $ij$-th entry of $A(\Gamma)$ is 1 if the vertices $v_i$ and $v_j$ are adjacent, otherwise it is 0. Denote by $\Phi(\Gamma, x)$ the characteristic polynomial of $A(\Gamma)$. Similarly, we define the adjacency spectrum $\text{spec}(\Gamma)$ of $\Gamma$. The largest root of $\Theta(\Gamma, x)$ (resp. $\Phi(\Gamma, x)$) is called the distance spectral radius (resp. adjacency spectral radius) of $\Gamma$.

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Let $G$ denote a finite group of order $n$. The power graph of $G$ was introduced by Chakrabarty et al. [2], which has the vertex set $G$ and two distinct elements are adjacent if one is a power of the other. Motivated by this, Singh and Manilal [3] defined the strong power graphs as a generalization of the power graphs. The strong power graph $P_s(G)$ of $G$ is a graph whose vertex set consists of the elements of $G$ and two distinct vertices $x$ and $y$ are adjacent if $x^{n_1} = y^{n_2}$ for some positive integers $n_1, n_2 < n$. Very recently, Bhuniya and Berathe [1] investigated the Laplacian of strong power graphs.

In this paper we study the characteristic polynomial of the distance or adjacency matrix of the strong power graph of a finite group, and compute its distance and adjacency spectrum.

2 The results

Throughout this section $G$ denotes a finite group, and $Z_n$ stands for the cyclic group of order $n$. We always assume $Z_n = \{0, 1, \ldots, n-1\}$. For strong power graphs we have the following proposition.

**Proposition 2.1** (1) If $G$ is not cyclic, then $P_s(G)$ is complete.

(2) $P_s(Z_n)$ is not connected if and only if $n$ is a prime number.

(3) $N_{P_s(Z_n)}(0) = \{k \in Z_n : m \neq 0, (m, n) \neq 1\}$, and the subgraph of $P_s(Z_n)$ induced by $Z_n \setminus \{0\}$ is complete. In particular, $\text{diam}(P_s(Z_n)) = 2$ if $n$ is not a prime number.

Now we find the characteristic polynomial of the distance matrix associated with the strong power graph $P_s(Z_n)$ for any composite number $n$.

**Theorem 2.2** For any composite number $n$,

$$\Theta(P_s(Z_n), x) = (x+1)^{n-3}\left(x^3+(3-n)x^2+(3-2n-3\phi(n)x-\phi(n)^2-\phi(n)(4-n)-n+1\right),$$

where $\phi(n)$ is Euler’s totient function.

**Proof.** Write $\phi(n) = t$ and $k = n - \phi(n) - 1$. By Proposition 2.1, the distance matrix $D(P_s(Z_n))$ is the $n \times n$ matrix given below, where the rows and columns are indexed in order by the vertices in $N_{P_s(Z_n)}(0)$ then all generator elements of $Z_n$, and 0 is in last position.

$$D(P_s(Z_n)) = \begin{pmatrix}
0 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 0 & 1 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 0 & \ldots & 1 & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 1 & 1 & \ldots & 0 & 2 \\
1 & 1 & \ldots & 1 & 2 & \ldots & 2 & 0
\end{pmatrix}.$$
The characteristic polynomial of $D(P_s(Z_n))$ is

$$\Theta(P_s(Z_n), x) = \begin{vmatrix}
  x & -1 & \ldots & -1 & -1 & \ldots & -1 & -1 \\
  -1 & x & \ldots & -1 & -1 & \ldots & -1 & -1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & -1 & \ldots & x & -1 & \ldots & -1 & -1 \\
  -1 & -1 & \ldots & -1 & x & \ldots & -1 & -2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & -1 & \ldots & -1 & -1 & \ldots & -2 & -2 \\
  -1 & -1 & \ldots & -1 & -2 & \ldots & -2 & x \\
\end{vmatrix} \tag{1}$$

Subtract the first column from the columns 2, 3, $\ldots$, $n$ of (1) to obtain (2):

$$(x + 1)^{k-1} \begin{vmatrix}
  x & -1 & \ldots & -1 & -1-x & \ldots & -1-x & -1-x \\
  -1 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
  -1 & 0 & \ldots & 0 & x+1 & \ldots & 0 & -1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & 0 & \ldots & 0 & 0 & \ldots & x+1 & -1 \\
  -1 & 0 & \ldots & 0 & -1 & \ldots & -1 & x+1 \\
\end{vmatrix} \tag{2}$$

Adding the rows 2, 3, $\ldots$, $n-1$ to the first row of (2). Then adding columns 2, 3, $\ldots$, $k$ to the first column, we arrive at the determinant (3):

$$(x+1)^{k-1} \begin{vmatrix}
  x-n+2 & 0 & \ldots & 0 & 0 & \ldots & 0 & -1-x-t \\
  0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
  -1 & 0 & \ldots & 0 & x+1 & \ldots & 0 & -1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & 0 & \ldots & 0 & 0 & \ldots & x+1 & -1 \\
  -1 & 0 & \ldots & 0 & -1 & \ldots & -1 & x+1 \\
\end{vmatrix} \tag{3}$$

Subtract the first column from the last column of (3). Then subtract the row $k+1$ from the rows $k+2, k+3, \ldots, n$ to obtain (4):

$$(x+1)^{n-3} \begin{vmatrix}
  x-n+2 & 0 & \ldots & 0 & 0 & \ldots & 0 & -2x+k-2 \\
  0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
  -1 & 0 & \ldots & 0 & x+1 & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & 0 & -1 & \ldots & -1 & 0 \\
  0 & 0 & \ldots & 0 & -2-x & \ldots & -1 & x+2 \\
\end{vmatrix} \tag{4}$$
Adding the rows \( k + 2, k + 3, \ldots, n - 1 \) to the last row of (4), we get (5):

\[
(x + 1)^{n-3} \begin{vmatrix}
 x - n + 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & -2x + k - 2 \\
 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\
 -1 & 0 & \cdots & 0 & x + 1 & 0 & \cdots & 0 \\
 0 & 0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 1 \\
 0 & 0 & \cdots & 0 & -x - t - 1 & 0 & \cdots & 0 \\
\end{vmatrix} \cdot (5)
\]

Expand it along the first row to obtain (6):

\[
(x + 1)^{n-3} \left( (x - n + 2)(x + 1)(x + 2) + (-1)^{1+n}(-2x + k - 2)|A| \right), \tag{6}
\]

where

\[
|A| = \begin{vmatrix}
 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\
 -1 & 0 & \cdots & 0 & x + 1 & 0 & \cdots & 0 \\
 0 & 0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 1 \\
 0 & 0 & \cdots & 0 & -x - t - 1 & 0 & \cdots & 0 \\
\end{vmatrix} \cdot (7)
\]

In (7) the determinant has order \( n - 1 \). Interchange the row \( k - 1 \) and the last row of (7). Then expand it along the first column to obtain \(|A|\) as follows:

\[
|A| = (-1)^n(-x - t - 1).
\]

It follows that

\[
\Theta(P_s(\mathbb{Z}_n), x) = (x + 1)^{n-3} \left( x^3 + (3 - n)x^2 + (3 - 2n - 3t)x - n + nt - t^2 - 4t + 1 \right).
\]

This completes our proof. \( \square \)

By Proposition 2.1, one has that \( P_s(G) \) is connected if and only if \( G \) is not cyclic, or \( G \cong \mathbb{Z}_n \) for some composite number \( n \). Now we may compute the \( D \)-spectrum of any connected strong power graph.

**Theorem 2.3** If \( G \) is not cyclic, then

\[
\text{spec}_D(P_s(G)) = \begin{pmatrix}
 n - 1 & -1 \\
 1 & n - 1
\end{pmatrix}.
\]

If \( G \cong \mathbb{Z}_n \) for some composite number \( n \), then \( \text{spec}_D(P_s(\mathbb{Z}_n)) \) is

\[
\begin{pmatrix}
 -1 & n - 3 + 2 \cos \frac{\theta}{3} \sqrt{n^2 + 9\phi(n)} \\
 n - 3 & 1
\end{pmatrix},
\]

where \( 0 < \theta < \frac{\pi}{2} \) and \( \theta = \arccos \frac{2n^3 + 27\phi(n)^2 + 27\phi(n)}{2\sqrt{(n^2 + 9\phi(n))^3}} \).
Proof. Note that if $G$ is not cyclic then $\mathcal{P}_s(G)$ is complete. Thus, it suffices to compute the distance spectrum of $\mathcal{P}_s(\mathbb{Z}_n)$ for some composite number $n$. Let

$$f(x) = x^3 + (3 - n)x^2 + (3 - 2n - 3\phi(n))x - \phi(n)^2 - \phi(n)(4 - n) - n + 1.$$ 

Suppose that $f(-1) = 0$. Then $\phi(n)(n - \phi(n) - 1) = 0$. It follows that $\phi(n) = n - 1$. Namely $n$ is a prime number, a contradiction. This implies that $D$-spectrum of $\mathcal{P}_s(\mathbb{Z}_n)$ has $-1$ with multiplicity $n - 3$ by Theorem 2.2. Note that the canonical solutions of any quadratic and cubic equation. We conclude that $f(x)$ has three pairwise distinct roots, as presented above. \hfill \blacksquare

**Corollary 2.4** For any composite number $n$, the distance spectral radius of $\mathcal{P}_s(\mathbb{Z}_n)$ is

$$\frac{n - 3 + 2 \cos \frac{\theta}{3} \sqrt{n^2 + 9\phi(n)}}{3},$$

where $0 < \theta < \frac{\pi}{2}$ and $\theta = \arccos \frac{2n^3 + 27\phi(n)^2 + 27\phi(n)}{2\sqrt{(n^2 + 9\phi(n))^3}}$.

For any positive integer $n$, the adjacency matrix $A(\mathcal{P}_s(\mathbb{Z}_n))$ is given below, where the rows and columns are indexed in order by the vertices in $N_{\mathcal{P}_s(\mathbb{Z}_n)}(0)$ then all generator elements of $\mathbb{Z}_n$, and 0 is in last position. Note that $N_{\mathcal{P}_s(\mathbb{Z}_n)}(0)$ may be the empty set.

$$A(\mathcal{P}_s(\mathbb{Z}_n)) = \begin{pmatrix} 0 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

An argument similar to the one used in the computing of $\Theta(\mathcal{P}_s(\mathbb{Z}_n), x)$, we get $\Phi(\mathcal{P}_s(\mathbb{Z}_n), x)$.

**Theorem 2.5** For any positive integer $n$,

$$\Phi(\mathcal{P}_s(\mathbb{Z}_n), x) = (x+1)^{n-3} \left(x^3 + (3-n)x^2 + (3-2n+\phi(n))x + (n-\phi(n)-1)(\phi(n)-1)\right).$$

**Corollary 2.6** For a prime number $p \geq 2$,

$$\Phi(\mathcal{P}_s(\mathbb{Z}_p), x) = x(x+1)^{p-2}(x + 2 - p).$$

As an application of Theorem 2.5, we may obtain the adjacency spectrum of the strong power graph of a finite group.
Theorem 2.7 If $G$ is not cyclic, then
\[ \text{spec}(\mathcal{P}_s(G)) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}. \]

If $G \cong \mathbb{Z}_p$ for some prime number $p$, then
\[ \text{spec}(\mathcal{P}_s(\mathbb{Z}_n)) = \begin{pmatrix} 0 & -1 & p-2 \\ -p & 1 & 1 \end{pmatrix}. \]

If $G \cong \mathbb{Z}_n$ for some composite number $n$, then
\[ \text{spec}(\mathcal{P}_s(\mathbb{Z}_n)) \text{ is } \begin{pmatrix} -1 & n-3+2\cos \frac{\theta}{3} \sqrt{n^2-3\phi(n)} & n-3+2\cos \frac{\theta+2\pi}{3} \sqrt{n^2-3\phi(n)} \\ n-3 & 1 & n-3+2\cos \frac{\theta-2\pi}{3} \sqrt{n^2-3\phi(n)} \end{pmatrix}, \]
where $0 < \theta < \frac{\pi}{2}$ and $\theta = \arccos \frac{2n^3+27\phi(n)^2+27\phi(n)-36n\phi(n)}{2\sqrt{(n^2-3\phi(n))^3}}$.

Corollary 2.8 For any composite number $n$, the adjacency spectral radius of $\mathcal{P}_s(\mathbb{Z}_n)$ is
\[ n - 3 + 2\cos \frac{\theta}{3} \sqrt{n^2-3\phi(n)}, \]
where $0 < \theta < \frac{\pi}{2}$ and $\theta = \arccos \frac{2n^3+27\phi(n)^2+27\phi(n)-36n\phi(n)}{2\sqrt{(n^2-3\phi(n))^3}}$.

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