Algorithms for the Nonclassical Method of Symmetry Reductions

by

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Abstract

In this article we present first an algorithm for calculating the determining equations associated with so-called “nonclassical method” of symmetry reductions (à la Bluman and Cole) for systems of partial differential equations. This algorithm requires significantly less computation time than that standardly used, and avoids many of the difficulties commonly encountered. The proof of correctness of the algorithm is a simple application of the theory of Gröbner bases.

In the second part we demonstrate some algorithms which may be used to analyse, and often to solve, the resulting systems of overdetermined nonlinear PDEs. We take as our principal example a generalised Boussinesq equation, which arises in shallow water theory. Although the equation appears to be non-integrable, we obtain an exact “two-soliton” solution from a nonclassical reduction.

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1 Introduction

Nonlinear phenomena have many important applications in several aspects of physics as well as other natural and applied sciences. Essentially all the fundamental equations of physics are nonlinear and, in general, such nonlinear equations are often very difficult to solve explicitly. Consequently perturbation, asymptotic and numerical methods are often used, with much success, to obtain approximate solutions of these equations; however, there is also much current interest in obtaining exact analytical solutions of nonlinear equations. Symmetry group techniques provide one method for obtaining such solutions of partial differential equations (PDES). These have many mathematical and physical applications, and usually are obtained either by seeking a solution in a special form or, more generally, by exploiting symmetries of the equation. This provides a method for obtaining exact and special solutions of a given equation in terms of solutions of lower dimensional equations, in particular, ordinary differential equations (ODES). Furthermore they do not depend upon whether or not the equation is “integrable” (in any sense of the word).

Symmetry groups have several different applications in the context of nonlinear differential equations (for further details see, for example, [8, 31] and the references therein):

- Derive new solutions from old solutions. Applying the symmetry group of a differential equation to a known solution yields a family of new solutions (quite often interesting solutions can be obtained from trivial ones).
- Integration of ODES. Symmetry groups of ODES can be used to reduce the order of the equation (such as to reduce a second order equation to first order).
- Reductions of PDES. Symmetry groups of PDES are used to reduce the total number of dependent and independent variables (for example, reduce a PDE with two independent and one dependent variable to an ODE).
- Classification of equations. Symmetry groups can be used to classify differential equations into equivalence classes.
- Asymptotics of solutions of PDES. Since solutions of PDES asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. Furthermore exact solutions arising from symmetry methods can often be effectively used to study properties such as asymptotics and “blow-up”.
- Numerical methods and testing computer coding. Symmetry groups and exact solutions of physically relevant PDES are used in the design, testing and evaluation of numerical algorithms; these solutions provide an important practical check on the accuracy and reliability of such integrators.

The classical method for finding symmetry reductions of PDES is the Lie group method of infinitesimal transformations and the associated determining equations are an overdetermined, linear system (cf., [8, 31]). Though this method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which can become virtually unmanageable if attempted manually, and so symbolic manipulation programs have been developed to facilitate the calculations. An excellent survey of the different packages presently available and a discussion of their strengths and applications is given by Hereman [Э].
There have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [35] developed the method of partially invariant solutions. Bluman and Cole [7], in their study of symmetry reductions of the linear heat equation, proposed the so-called "nonclassical method of group-invariant solutions" (in the sequel referred to as the nonclassical method), which is also known as the "method of conditional symmetries" [25] and the "method of partial symmetries of the first type" [14]. This method involves considerably more algebra and associated calculations than the classical Lie method. In fact, it has been suggested that for some PDEs, the calculation of these nonclassical reductions might be too difficult to do explicitly, especially if attempted manually since the associated determining equations are now an overdetermined, nonlinear system. Furthermore, it is known that for some equations such as the Korteweg-de Vries equation, the nonclassical method does not yield any additional symmetry reductions to those obtained using the classical Lie method, though there are PDEs which possess symmetry reductions that are not obtained using the classical Lie group method. Olver and Rosenau [33, 34] proposed an extension of the nonclassical method and concluded that "the unifying theme behind finding special solutions of PDEs is not, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of PDEs".

Clarkson and Kruskal [15] developed an algorithmic and direct method for finding symmetry reductions (hereafter referred to as the direct method) and using it obtained previously unknown symmetry reductions of the Boussinesq equation. The novel characteristic about the direct method in comparison to those mentioned above, is that it involves no use of group theory; additionally, for many equations the method appears to be simpler to implement than either the classical or nonclassical methods.

Olver [32] (see also [2, 36]) has recently shown that the direct method is equivalent to the nonclassical method when the infinitesimals for the independent variables are autonomous with respect to the dependent variables, generating a group of "fibre-preserving transformations".

It has been known for several years that there do exist PDEs which possess symmetry reductions that are not obtained using the classical Lie group method (cf., [33, 34]). Recently the direct and nonclassical methods have been used to generate many new symmetry reductions and exact solutions for several physically significant PDEs, which represents important progress (cf., [14, 21] and the references therein).

In §2 we present an algorithm for calculating the determining equations for the nonclassical method for a system of PDE. These are usually calculated by reducing the so-called infinitesimal equations, obtained from the group prolongation of Σ, with respect to both Σ and the invariant surface conditions Ψ. However, in practise, reducing an equation with respect to a system is not well-defined. Indeed, unless one is careful about the choice of term in each equation from which to back-substitute, infinite loops can occur in the reduction process. The theory used to overcome these difficulties is that of Gröbner bases, a powerful computational tool in algebra, geometry and logic. Our essential idea is to first reduce the given system Σ using the invariant surface conditions Ψ to generate a simplified system ΣΨ. Then we apply the classical Lie method to ΣΨ. We use the theory of Gröbner bases, as they apply to algebraic systems, to provide a proof of correctness of an algorithm for finding the determining equations for the nonclassical method which eliminates the problems and moreover proceeds efficiently.

In §2.1 we describe the process of applying the nonclassical method, and the difficulties encountered, in more detail. Then we give an elementary introduction to Gröbner bases adapted to our purposes, and show why they solve the problems. Next we give the algorithm for finding
the determining equations and prove it is correct, followed by some worked examples that are prototypical.

Since “nonclassical symmetries” of $\Sigma$ are actually classical symmetries of the system consisting of $\Sigma$ augmented by the invariant surface conditions, the basic idea of using Gröbner bases to make the reduction of the infinitesimal equations well-defined applies to finding classical symmetries of any system. We stress that the “nonclassical symmetries” obtained by the nonclassical method are not symmetries of $\Sigma$ itself, since they do not necessarily transform all solutions of the system to other solutions. Rather, they are symmetries of the augmented system given by $\Sigma$ together with specified auxiliary conditions.

In §3 we discuss some algorithms and strategies that have proved useful in making solving the determining equations for the nonclassical method, which are overdetermined and nonlinear, tractable. We illustrate this with some examples.

The difficulties of the nonclassical method due to the determining equations being an overdetermined nonlinear system makes the use of symbolic manipulation programs more important. Levi and Winternitz [25] and Clarkson and Winternitz [18] in their applications of the nonclassical method to the Boussinesq and Kadomtsev-Petviashvili equations respectively, interactively used the MACSYMA program SYMMGRP.MAX [4]. Nucci [30] has also developed an interactive program NUSY in REDUCE for the nonclassical method. Here, we use the MAPLE package diffgrob2 [26], which appears to be the only differential algebra package available that can handle equations not solvable for their leading derivative term. In the appendix we give details of how our algorithm to obtain the determining equations may be implemented using SYMMGRP.MAX. The interesting thing here is that we are using the SYMMGRP.MAX program for a purpose for which it was not originally designed, however, it can be adapted to generate the determining equations for the nonclassical method since the latter can be interpreted as the determining equations for the classical method applied to an appropriate system of equations.

2 Determining Equations for Symmetries

2.1 The Classical and Nonclassical Methods

Suppose one is given a system of partial differential equations

$$\Sigma = \{ f_1 = 0, \ldots, f_r = 0 \},$$

where each $f_i$ is some polynomial expression in the independent variables $\{x_1, \ldots, x_n\}$, the dependent variables $\{u_1, \ldots, u_m\}$ and the derivative terms $\{u_{k,\alpha} | \alpha \in \mathbb{N}^n \}$ where

$$u_{k,\alpha} = \frac{\partial^{|\alpha|} u_k}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$  

(2.2)

One also can have transcendental and arbitrary functions of the dependent variables in what follows without affecting the theory.

We recall briefly the method of finding the determining equations for classical Lie point symmetries thereby fixing our notation. Let $u^{(N)}$ denote the list $u_{k,\alpha}$, where $k = 1, \ldots, m$ and $|\alpha| = N$. The index $\alpha + i$ is given by $(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n)$, while $\alpha + \gamma = (\alpha_1 + \gamma_1, \ldots, \alpha_n + \gamma_n)$. 

To find the classical Lie point symmetries of the system $\Sigma$, one takes a group action defined infinitesimally by
\[
\begin{align*}
    x_i^* &= x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), & i = 1, 2, \ldots, n, \\
    u_j^* &= u_j + \epsilon \phi_j(x, u) + O(\epsilon^2), & j = 1, 2, \ldots, m,
\end{align*}
\]
where $x = x_1, \ldots, x_n$ and $u = u_1, \ldots, u_m$. Then one requires that this transformation leaves the set of solutions
\[
S_\Sigma = \{ u(x) | f_1 = 0, f_2 = 0, \ldots, f_r = 0 \},
\]
invariant.

The $N^{th}$ order partial derivatives transform according to (with $|\alpha| = N$)
\[
\frac{\partial^N u_j^*}{\partial x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}} = u_j + \epsilon \phi_j^{[\alpha]} (x, u, u^{(1)}, \ldots, u^{(N)}) + O(\epsilon^2),
\]
where the $N^{th}$ extension, denoted by $\phi_j^{[\alpha]} (x, u, u^{(1)}, \ldots, u^{(N)})$, is given by the recursive formula
\[
\phi_j^{[\alpha+i]} (x, u, u^{(1)}, \ldots, u^{(N)}) = \frac{D\phi_j^{[\alpha]}}{Dx_i} - \sum_{\ell=1}^n \sum_{\lambda=1}^{\ell} D\xi_{\lambda, \alpha+i} u_{j, \alpha+\ell}.
\]

and
\[
\frac{D}{Dx_i} \equiv \frac{\partial}{\partial x_i} + \sum_{\lambda=1}^m \sum_{\alpha} u_{\lambda, \alpha+i} \frac{\partial}{\partial u_{\lambda, \alpha}}.
\]
is the total derivative operator. We make the obvious definition
\[
D\alpha = \frac{D\alpha_1}{Dx_1} \frac{D\alpha_2}{Dx_2} \ldots \frac{D\alpha_n}{Dx_n}.
\]

Now consider the system
\[
\{ f_i (x^*, u^*, u^{(1)} (x^*), \ldots, u^{(N)} (x^*)) = 0 | i = 1, 2, \ldots, r \},
\]
which is (2.1) with $u$ replaced by $u^*$ and $x$ by $x^*$. It is easily seen that
\[
\begin{align*}
    f_i (x^*, u^*, u^{(1)} (x^*), \ldots, u^{(N)} (x^*)) &= f_i (x, u, u^{(1)} (x), \ldots, u^{(N)} (x)) + \epsilon \text{pr}^{(N)} \mathbf{v} (f_i) + O(\epsilon^2),
\end{align*}
\]
where
\[
\text{pr}^{(N)} \mathbf{v} \equiv \sum_{j=1}^n \xi_j \frac{\partial}{\partial x_j} + \sum_{k=1}^m \phi_m \frac{\partial}{\partial u_k} + \sum_{k=1}^m \sum_{|\alpha| \geq 1} \phi_k^{[\alpha]} \frac{\partial}{\partial u_k, \alpha}
\]
is known as the $N^{th}$ prolongation (or $N^{th}$ extension) of the infinitesimal operator
\[
\mathbf{v} \equiv \sum_{j=1}^n \xi_j (x, u) \frac{\partial}{\partial x_j} + \sum_{k=1}^m \phi_k (x, u) \frac{\partial}{\partial u_k}.
\]
Let \( N \) be the order of the system \( \Sigma \). Requiring that (2.1) is invariant under the transformation, i.e.

\[
\left. \text{pr}^{(N)} v(f_i) \right|_{\Sigma=0} = 0, \quad i = 1, \ldots, r
\]  

(2.7)
yields an overdetermined, linear system of equations for the infinitesimals \( \xi(x, u) \) and \( \phi(x, u) \), obtained by setting the coefficients of the different monomials in the derivative terms \( \{u_{k,\alpha} \mid 1 \leq k \leq m, \ |\alpha| \neq 0 \} \) in the \( \text{pr}^{(N)} v(f_i) \big|_{\Sigma=0} = 0 \) to zero. This means that the \( N \)th prolongation of \( f_i \), \( i = 1, 2, \ldots, r \), is zero whenever \( u \) is a solution of the original system (2.1).

The important point to note is that we are considering the invariance under the group action of the system viewed as an \textit{algebraic system} in the indeterminants \( \{x_i, u_j, u_j, \alpha\} \), in the relevant jet bundle (\[2\]. Ch. 2).

The nonclassical method of Bluman and Cole [7] for finding symmetry reductions of a system of pdes involves appending the invariant surface conditions to the system and finding the classical symmetries of the appended system. The invariant surface conditions are given by

\[
\psi_i \equiv \xi_1 u_{i,x_1} + \xi_2 u_{i,x_2} + \ldots + \xi_n u_{i,x_n} - \phi_i = 0, \quad i = 1, 2, \ldots, m.
\]  

(\(\Psi\))

In this method one requires only the subset of \( S_\Sigma \) given by

\[
S_{\Sigma,\Psi} = \{u(x, t)|f_1 = 0, f_2 = 0, \ldots, f_r = 0, \psi_1 = 0, \psi_2 = 0, \ldots, \psi_m = 0\},
\]
is invariant under the infinitesimal transformation. The idea is that since the invariant surface conditions map to themselves under the prolonged group action, they are not a restriction on the infinitesimal equations of the system \( \Sigma \), but rather since \( u_{1,x_1} \) and \( u_{1,x_2} \), for example, are no longer independent, the determining equations will be more general than those for the classical method. Hence imposing these conditions leads to the possibility that there are more solutions, not less.

The usual approach to finding the determining equations for the nonclassical method of the system \( \Sigma \) consists of calculating the infinitesimal equations \( \text{pr}^{(N)} v(f) \) for \( f \in \Sigma \), where \( N \) is the order of the system \( \Sigma \), and reducing them with respect to the prolonged system \( G = \Sigma \cup \Psi \). By reduction with respect to \( G \) is meant elimination (or back-substitution) from \( \text{pr}^{(N)} v(f) \) of derivatives of one pre-determined term from each of the equations in \( G \). One then reads off the coefficients of the different monomials in the derivatives of the \( u_j \); setting these to zero are the determining equations.

In practice there are several difficulties.

**Example 2.1.1.** Consider the equation

\[
\Delta_1 \equiv u_{xt} - f(u) = 0.
\]  

(2.8)
The infinitesimal equation \( \text{pr}^{(2)} v(\Delta_1) \) (with \( x_1 = x, \ x_2 = t, \ \xi = \xi_1, \ \xi_2 \equiv 1 \) and \( \phi = \phi_1 \)), is

\[
\phi[xt] - f'(u)\phi \equiv \phi_{xt} + \phi_{xu}u_t + (\phi_{tu} - \xi_{xt})u_x + (\phi_{uu} - \xi_{xu})u_xu_t - \xi_{tu}u_x^2 - \xi_{uu}u_tu_x^2 + (\phi_u - \xi_x)u_{xt} - \xi_t u_{xx} - 2\xi_u u_x u_{xt} - \xi_u u_t u_{xx} - f'(u)\phi.
\]

When reducing this equation before reading off the determining equations, does one reduce the derivative term \( u_{xt} \) using the original equation, or using the \( x \) derivative of the invariant surface condition

\[
\psi \equiv \xi(x, t, u)u_x + u_t - \phi = 0?
\]  

(2.9)
The difference of the two reductions is proportional to a differential consequence of the system \( \{2.8, 2.9\} \), namely,

\[
\frac{D}{Dx} \psi - \Delta_1 \equiv f(u) + (\xi_x + \xi_u u_x) u_x + \xi u_{xx} - \phi_x - \phi_u u_x = 0.
\]

Using this equation, one can eliminate all \( u_{xx} \) terms, given \( \xi \neq 0 \). But should one? This leads to the next question: “By which differential consequences, if any, do we need to reduce the infinitesimal equations in order to obtain the determining equations for the nonclassical method?”

**Example 2.1.2.** Second, consider the equation,

\[
\Delta_2 \equiv u_{tt} - u_{xx} = 0.
\]  

(2.10)

With the same notation as the previous example, \( pr^{(2)}_v(\Delta_2) \) is:

\[
\phi^{[xx]} - \phi^{[tt]} \equiv \\
\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu} u_x^3 + (\phi_u - 2\xi_x)u_{xx} - 3\xi_{uu} u_x u_{xx} \\
- \{\phi_{tt} + 2\phi_{tu} u_t - \xi_{tt} u_x + \phi_{uu} u_{tt}^2 - 2\xi_{tu} u_t u_x - \xi_{uu} u_{tt}^2 u_x + \phi_u u_{tt} - 2\xi_{tt} u_t - \xi_{uu} u_{tt} u_x - 2\xi_{uu} u_{tt} u_t\}.
\]

Instinctively one would eliminate the \( t \) derivatives using the invariant surface condition \( u_t = \phi - \xi u_x \), and then eliminate the \( u_{xx} \) terms using \( u_{xx} = u_{tt} \). But eliminating \( u_{xx} \) introduces a \( u_{tt} \) term, and eliminating this leads to a \( u_{tt} \) term, and eliminating that leads to a \( u_{xx} \) term again. Clearly care must be taken in the reduction procedure (by which is meant the successive eliminations or back-substitutions) to prevent infinite loops occurring.

These difficulties are all part of the problem of finding the “normal form” of an equation \( f \) with respect to some given system of equations \( G \). By “normal form” is meant some well-defined reduction of \( f \) such that no further eliminations from \( G \) are possible.

For questions concerning “normal forms” to be answered in a well-defined way, the concept of a Gröbner basis is required. The algorithm to calculate a Gröbner basis for a system of polynomials over a field was developed by Buchberger [2, 3] and since then has been extended to a wide variety of algebraic scenarios; the concept has a large number of applications (see [10, 13] and references therein). In the following section, we discuss Gröbner bases as they are used in our particular application.

### 2.2 Gröbner Bases for Differential Polynomials

Consider the set

\[
\mathcal{J} = \{x_i, \xi, \phi_{j,\delta}, u_j, \alpha \mid 1 \leq i \leq n, 1 \leq j \leq m, \delta \in \mathbb{N}^{n+m}, \alpha \in \mathbb{N}^n, \ l, \varepsilon \leq N\},
\]

where \( N \) is some finite number. The system of equations \( \Sigma \), and the infinitesimal equations obtained by the prolongation of the group action on \( \Sigma \), can be considered to be polynomials in the elements of \( \mathcal{J} \) with complex coefficients. In this section, we discuss Gröbner bases for systems of polynomials. We give those definitions required for the sections that follow, and examples, relevant to our application. An excellent introduction to Gröbner bases can be found in [13].

We denote the set of polynomials in the indeterminates \( \{\zeta_1, \ldots, \zeta_s\} \) with coefficients in \( \mathbb{C} \) by \( \mathcal{R} = \mathbb{C}[\zeta_1, \ldots, \zeta_s] \), and use the multi-index notation for multiplication defined for \( \beta = (\beta_1, \ldots, \beta_s) \in \mathbb{N}^s \), where

\[
\alpha \cdot \beta = (\alpha_1 + \beta_1, \ldots, \alpha_s + \beta_s).
\]
\( N^s \) by \( \zeta^\beta = \zeta_1^{\beta_1} \ldots \zeta_s^{\beta_s} \). Take an ordering \( O \) on the indeterminates, say \( \zeta_1 < \zeta_2 < \ldots < \zeta_s \), and define the lexicographic ordering \( \text{lex}(O) \) on the set of monomials \( \{\zeta^\beta \mid \beta \in N^s\} \) to be

\[
\zeta^\beta > \text{lex}(O) \zeta^\delta
\]

if for some \( 0 \leq j \leq s - 1 \),

\[
\beta_s = \delta_s, \ldots, \beta_s - j + 1 = \delta_s - j + 1, \quad \beta_s - j > \delta_s - j.
\]

Following Bayer and Stillman [4], another monomial ordering denoted here by \( \text{BS}(r) \) is given by \( \zeta^\beta > \text{BS}(r) \zeta^\delta \) if \( \beta_r + \beta_{r+1} + \cdots + \beta_s > \delta_r + \delta_{r+1} + \cdots + \delta_s \), else for some \( j, \beta_s = \delta_s, \ldots, \beta_{s-j+1} = \delta_{s-j+1} \), \( \beta_{s-j} < \delta_{s-j} \). There is a wide variety of monomial orderings available [1]. They are required to have the so-called compatibility, or multiplicative, property, that is,

\[
\zeta^\beta > \zeta^\delta \implies \zeta^\gamma \zeta^\beta > \zeta^\gamma \zeta^\delta \quad \text{and} \quad \zeta^\gamma \zeta^\delta > \zeta^\gamma.
\]

Given an ordering on the indeterminates \( \{\zeta_i\} \), we have defined above several orderings on the monomials \( \{\zeta^\delta\} \). We now discuss orderings on the indeterminates \( \zeta \).

We assume that \( u_{j,\alpha} > \phi_{k,\delta} > \xi_{l,\delta} > x_i \) for all \( j, k, l, i, \alpha \) and \( \delta \), and require that the ordering chosen on our indeterminates is compatible with the differential structure, that is, we take the ordering on the indeterminates to be such that

\[
u_{j,\alpha} > u_{k,\beta} \implies u_{j,\alpha + \gamma} > u_{k,\beta + \gamma} \quad \text{and} \quad u_{j,\alpha + \gamma} > u_{j,\alpha}, \quad |\gamma| \neq 0
\]

and similarly for the \( \xi \) and the \( \phi \).

In Example 2.1.2 where the reduction process was infinite, an incompatible ordering had been chosen. In that example we were eliminating \( t \)-derivatives using \( \xi u_x + u_t - \phi = 0 \), which implies \( u_t > u_x \). Then by compatibility (2.14), we must have that \( u_{tt} > u_{xt} \) and \( u_{xt} > u_{xx} \), so that \( u_{tt} > u_{xx} \). Thus using \( u_{xx} - u_{tt} = 0 \) to eliminate occurrences of \( u_{xx} \) is an incompatible choice.

Finally, we require that derivatives of the \( u \) with respect to one particular pre-chosen variable, \( x_k \) (say), are greater in the order than derivatives with respect to other independent variables. This is needed to apply the elimination ideals property of Gröbner bases [13] in the proof of correctness of our algorithm. Thus we want a compatible ordering which is decided first with respect to some ordering on the dependent variables, say \( u_m > u_{m-1} > \ldots > u_1 \), and then the number of derivatives with respect to \( x_k \), and then any compatible choice thereafter. Such an ordering on the indeterminates we will denote by the term \( k \)-order, or \( O(k) \). For a \( k \)-order \( O(k) \) choose \( r = r(k) \) such that for \( l \geq r \), \( \zeta_l \) represents a derivative term \( u_{j,\alpha} \) with \( \alpha_k \neq 0 \), and for \( l < r \), \( \zeta_l \) represents either a derivative term \( u_{j,\alpha} \) with \( \alpha_k = 0 \) or one of the remaining indeterminates. We then denote the monomial ordering \( \text{BS}(r) \) by \( \text{BS}(O(k)) \).

Example 2.2.1. For second order systems of PDE in two independent variables, suppose we want a compatible ordering on the derivative terms of \( u \) such that \( u_t > u_x \). Two possibilities are \( u_{tt} > u_{xt} > u_{xx} > u_t > u_x > u \) and \( u_{tt} > u_{xt} > u_t > u_{xx} > u_x > u \). The first is an ordering where total degree of differentiation is one of the deciding factors in the ordering, the second ordering is a \( t \)-order.

To find the coefficient of a monomial \( M \) in \( p \), denoted \( \text{coef}(p, M) \), one looks for all summands in \( p \) of the form \( r_i M \) where \( r_i \in \mathbb{C} \). Then one defines \( \text{coef}(p, M) = \sum r_i \). If \( \text{coef}(p, M) \neq 0 \), we say
that \( M \) occurs in \( p \). The highest monomial term occurring in a polynomial \( p \) is denoted \( \text{hmt}(p) \) and its coefficient is denoted \( \text{hcoef}(p) \).

**Example 2.2.2.** Consider the polynomial

\[
p = \xi_u \phi_x u t^3 u_{xx} u t + (\xi_{xx} - \xi_u) u_{tt} u_{xx} u_x + x^2 (\phi_u - \xi_x) u_{xt} u_{tt} + u \xi_{xt} u_{xx} u_{tt} u_t^2.
\]

The independent variables are \( x, t \), the dependent variable is \( u \), the level of prolongation is 2, so that \( \mathcal{O}(x) \) is given initially by \( u_{xx} > u_{xt} > u_x > u_{tt} > u_t \), and \( \mathcal{O}(t) \) is given initially by \( u_{tt} > u_{xt} > u_t > u_{xx} > u_x \). The orders \( \mathcal{O}(k) \) are not unique. Then in the ordering \( \text{lex}(\mathcal{O}(x)) \), \( \text{hmt}(p) = u_{tt} u_{xx} u_x \xi_{xx} \), in \( \text{lex}(\mathcal{O}(t)) \), \( \text{hmt}(p) = u_{tt} u_{xx} u_x \xi_{xx} \), while in \( \text{BS}(\mathcal{O}(t)) \), \( \text{hmt}(p) = u_{xx} u_{tt} u_t^2 u_{xt} \).

**Definition 2.2.3.** Suppose for some polynomial \( q \) that the \( \text{hmt}(q) \) divides some monomial \( M \) that occurs in the polynomial \( p \), so that \( \zeta^\delta \text{hmt}(q) = M \), and that \( \text{hcoef}(q) = a \in \mathbb{C} \). Then the reduction of \( p \) at \( M \) with respect to \( q \) is given by

\[
p \rightarrow_q p - \text{coef}(f, M) \zeta^\delta q / a.
\]

Thus reduction depends on the ordering used. The use of a compatible ordering (2.13, 2.14) eliminates the infinite loops observed possible in the Introduction; see [9, 27] where it is proved that with respect to a compatible ordering, reduction is a noetherian relation, that is, it must terminate after a finite number of steps.

The definition of reduction uses no differentiation, since we wish to remain within the algebraic domain. Suppose \( N \) is the highest degree of differentiation occurring in the given system \( \Sigma \). When considering the system \( \Sigma \cup \Psi \), the \( \Psi \) equations need to be prolonged to order \( N \), and the system to which our theory will apply will be

\[
G = \Sigma \cup \{ D_\alpha \psi = 0 \mid \psi \in \Psi, \alpha \in \mathbb{N}^n, |\alpha| \leq N - 1 \}
\]

where \( D_\alpha \) is defined in (2.7). We assume that all equations in \( G \) are of the same order of differentiation; if not, those of lesser order need to be prolonged.

**Definition 2.2.4.** A normal form of a polynomial \( p \) with respect to a system of polynomials \( G \), denoted \( \text{normal}(p, G) \), is achieved when no further reduction of \( p \) with respect to any member of \( G \) is possible.

**Definition 2.2.5.** The ideal generated by a finite system of polynomials \( G \subset \mathbb{C}[\zeta_1, \ldots, \zeta_s] \), is the set \( I(G) = \{ \sum_{g \in G} f_g g \mid f_g \in \mathbb{C}[\zeta_1, \ldots, \zeta_s] \} \).

**Definition 2.2.6.** A \( \text{Gröbner basis} \) of an ideal \( I(G) \) is a finite set \( H \subset \mathbb{C}[\zeta_1, \ldots, \zeta_s] \) such that \( I(H) = I(G) \) and where for all \( p \in I(G) \), one has \( \text{normal}(p, H) = 0 \). Thus a \( \text{Gröbner basis} \) depends upon the ordering used. We denote a \( \text{Gröbner basis} \) for the ideal generated by \( G \) in the monomial ordering \( \text{ORDER} \) to be \( \text{GB}(G, \text{ORDER}) \).

Sufficient conditions to characterise a GB and an algorithm to calculate the GB for an ideal over a field in any compatible ordering were given by Buchberger [3], and it and its generalisations can be found now in textbooks (see for example [19]). This algorithm has since been implemented in various symbolic algebra programs, for example in \textsc{Mathematica} [18], \textsc{Maple} [13], \textsc{Reduce} [29] or the specialist package \textsc{macaulay} [6].

Given a \( \text{Gröbner basis} \) \( G \) of a polynomial ideal \( I(G) \), the normal form of any polynomial with respect to \( G \) is well-defined. Let \( f \) be a polynomial, and let \( h_1 \) and \( h_2 \) be two normal forms of \( f \).
with respect to $G$. Then $h_1 - h_2 \in I(G)$ by the reduction formula. Since $G$ is a Gröbner basis for $I(G)$, $h_1 - h_2$ reduces to zero with respect to $G$. Since neither $h_1$ nor $h_2$ reduces with respect to $G$, neither can their difference (reduction must take place at some monomial which must occur in at least one of $h_1$ or in $h_2$), so that difference must already be zero. Thus different reductions of $f$ with respect to a Gröbner basis are equal.

**Example 2.2.7.** For the standard example

$$u_{xx} = \Delta(x, t, u, u_x, u_t, u_{xt}, u_{tt}),$$

with invariant surface condition

$$\xi u_x + u_t = \phi,$$

one needs to prolong the invariant surface condition to be of the same order as the given equation, namely 2, so one has the system $G$:

$$u_{xx} - \Delta(x, t, u, u_x, u_t, u_{xt}, u_{tt}) = 0,$$

$$(\xi + \xi u_x)u_x + \xi u_{xx} + u_{xt} - \phi_x - \phi u_x = 0,$$

$$(\xi + \xi u_t)u_x + \xi u_{xt} + u_{tt} - \phi_t - \phi u_t = 0,$$

$$\xi u_x + u_t - \phi = 0.$$

Take the ordering given initially by $u_{tt} > u_{xt} > u_{xx} > u_t > u_x$. Finding the Gröbner basis of these equations is equivalent to considering the first three of them as equations in $u_{xx}$, $u_{xt}$ and $u_{tt}$ and calculating the “echelon” form of the system. In the ordering given initially by $u_{tt} > u_{xt} > u_t > u_{xx} > u_x$, we eliminate all the $u_t$ terms using $\xi u_x + u_t - \eta = 0$ from the other conditions, and then calculate the echelon form of the reduced system to obtain conditions for $u_{tt}$, $u_{xt}$ and $u_{xx}$.

To prove the correctness of our algorithm to generate the determining equations for the nonclassical method, we need the elimination ideals property of Gröbner bases [19, 13]:

**Theorem 2.1** Let $\mathbb{C}[\zeta_1, \ldots, \zeta_r]$ be the set of all polynomials over $\mathbb{C}$ in the first $r$ indeterminates. Suppose $G$ is a Gröbner basis of the ideal $I(G)$ in the ordering lex($\mathcal{O}$) where $\mathcal{O}$ is $\zeta_1 < \zeta_2 < \ldots < \zeta_s \ (2.11, 2.12)$. Then for for all $1 \leq r \leq s$, $G \cap \mathbb{C}[\zeta_1, \ldots, \zeta_r]$ generates the elimination ideal $I(G) \cap \mathbb{C}[\zeta_1, \ldots, \zeta_r]$.

In words, this property means that for every $r$, any polynomial in the first $r$ indeterminates that can be found from the generators by addition and by multiplication by elements of $\mathbb{C}[\zeta_1, \ldots, \zeta_r]$, can be “read off” from the Gröbner basis.

For the Bayer and Stillman ordering BS($r$), with $r$ pre-determined, we have that $G \cap \mathbb{C}[\zeta_1, \ldots, \zeta_r]$ generates $I(G) \cap \mathbb{C}[\zeta_1, \ldots, \zeta_r]$, that is, we obtain only the one elimination ideal $\mathcal{I}[19]$. For our application, we only need one particular elimination ideal, that given by using BS($\mathcal{O}(k)$), while the Bayer and Stillman orderings are more efficient than the lexicographic orderings [3].

### 2.3 The Algorithm to calculate the determining equations for the nonclassical method.

Let $\Sigma$ be a system of PDE of “polynomial type”, with $n$ independent variables $\{x_1, \ldots, x_n\}$, and $m$ dependent variables $\{u_1, \ldots, u_m\}$. Terms like $\sin(u_j)$ or $h(u_j)$ where $h$ is undetermined or arbitrary
are simply considered to be additional indeterminates in the algebraic computations and do not affect what follows.

There are $n$ cases to consider, namely where the invariant surface conditions ($\Psi$) have the forms, for $1 \leq j \leq m$,

$$
\begin{align*}
\xi_1 u_{j,x_1} + \xi_2 u_{j,x_2} + \ldots + \xi_{n-1} u_{j,x_{n-1}} + u_{j,x_n} &= \phi_j, \quad \text{Case } n \\
\xi_1 u_{j,x_1} + \xi_2 u_{j,x_2} + \ldots + u_{j,x_{n-1}} &= \phi_j, \quad \text{Case } n - 1 \\
& \vdots \\
u_{j,x_1} &= \phi_j, \quad \text{Case } 1
\end{align*}
$$

That is, we have successively for $1 \leq k \leq n$, that $\xi_k$ has been set equal to 1 and $\xi_{k+1} = \ldots = \xi_n = 0$. We consider each case separately. In the following algorithm, in the $k^{th}$ case, we require $O(k)$ to be a $k$-order, and ORDER to be one of $\text{lex}(O(k))$ or $\text{BS}(O(k))$.

Algorithm: Determining equations for the Nonclassical Method

INPUT: $\Sigma$, a system of pde with highest order of differentiation $N$; 
$k \in \{1, \ldots, n\}$; $O(k)$, a $k$-order, ORDER $\in \{\text{lex}(O(k)), \text{BS}(O(k))\}$. 

OUTPUT: $\text{DetEqs}$, the determining equations for the nonclassical method of the system $\Sigma$ in the $k^{th}$ case

for $j$ from 1 to $m$ let

$$
\begin{align*}
\psi_j := \xi_1 u_{j,x_1} + \ldots + \xi_{k-1} u_{j,x_{k-1}} + u_{j,x_k} - \phi_j \\
\Psi^\ast := \{D_\alpha \psi_j \mid 1 \leq j \leq m, \alpha \in \mathbb{N}^n, |\alpha| \leq N - 1\} \\
\mathcal{K} := \{\text{normal}(f, \Psi^\ast) \mid f \in \Sigma\} \\
\mathcal{I}nf := \{\text{pr}(f) \mid f \in \mathcal{K}\} \\
\mathcal{GB} := \text{gb}(\mathcal{K}, \text{ORDER}) \\
\mathcal{RI}nf := \{\text{normal}(f, \mathcal{GB}) \mid f \in \mathcal{I}nf\} \\
\text{DetEqs} := \{\text{coeff}(f, u_{1,\alpha_1} \ldots u_{m,\alpha_m}) = 0 \mid f \in \mathcal{RI}nf, \alpha_j \in \mathbb{N}^n \setminus \{0\}\}
\end{align*}
$$

end

In words, we reduce the given system $\Sigma$ with respect to the invariant surface conditions, form the infinitesimal equations of the result, reduce the infinitesimal equations with respect to an algebraic Gröbner basis of the reduced system, and then read off the coefficients of the result in the usual manner.

By reducing the equations in $\Sigma$ with respect to $\Psi^\ast$, that is, eliminating all derivatives of the $u_j$ with respect to $x_k$, before finding the infinitesimal equations, the Gröbner basis calculation is greatly diminished. Indeed, for systems consisting of a single equation, the calculation is eliminated altogether, and in that case the algorithm becomes the classical method, but on the reduced equation. Since the algorithm to calculate a Gröbner basis has poor complexity [28], our method is more efficient.

Before discussing the correctness of the algorithm, let us demonstrate it on Example 2.2.8,

Example 2.3.1. Given the general second order equation (2.17), with invariant surface condition (2.18), the algorithm splits into two cases, $\tau \neq 0$ and $\tau \equiv 0$.

Case I. $\tau \neq 0$. In this case we set $\tau = 1$ (without loss of generality), and eliminate $u_t$, $u_{xt}$ and $u_{tt}$
in \((2.17)\) using \((2.18)\), i.e.

\[
\begin{align*}
\dot{u}_t &= \phi - \xi u_x \\
\dot{u}_{xt} &= \phi_x + \phi_u u_x - (\xi_x u_x + \xi_u u_x^2 + \xi u_{xx}) \\
\dot{u}_{tt} &= \phi_t + \phi_u (\phi - \xi u_x) - \xi_t u_x - \xi_u u_x (\phi - \xi u_x) - \xi [\phi_x + \phi_u u_x - (\xi_x u_x + \xi_u u_x^2 + \xi u_{xx})].
\end{align*}
\]

Substituting these into \((2.17)\) yields the ODE (with \(t\) a parameter)

\[
\hat{\Delta}(x, t, u, u_x, u_{xx}; \xi, \xi_x, \xi_t, \xi_u, \phi, \phi_x, \phi_t, \phi_u) = 0.
\]

Now apply the classical Lie algorithm to this equation. Thus we apply the second prolongation \(\text{pr}^{(2)} v\) to \((2.22)\) and require that the resulting expressions vanish for \(u \in \hat{\mathcal{S}} = \{ u : \hat{\Delta} = 0 \} = \mathcal{S}_\psi\), i.e.,

\[
\text{pr}^{(2)} v \left( \hat{\Delta} \right)_{\hat{\Delta}=0} = 0,
\]

where \(\xi\) and \(\phi\) appear both in \((2.22)\) and \(\text{pr}^{(2)} v\). Equating coefficients of powers of \(u_x\) to zero then generates the determining equations.

**Case II.** \(\tau \equiv 0\) In this case we set \(\xi = 1\) and so the invariant surface condition reduces to \(u_x = \phi(x, t, u)\). Hence we obtain the differential consequences

\[
\begin{align*}
\dot{u}_{xt} &= \phi_t + \phi_u u_t, \\
\dot{u}_{xx} &= \phi_x + \phi_u u_x = \phi_x + \phi \phi_x.
\end{align*}
\]

Substituting these into \((2.17)\) yields the ODE (with \(x\) a parameter)

\[
\hat{\Delta}(x, t, u, u_x, u_t; \phi, \phi_x, \phi_t, \phi_u) = 0.
\]

Now apply the classical Lie algorithm to this equation. Thus we apply the second prolongation \(\text{pr}^{(2)} v\) to \((2.23)\) and require that the resulting expressions vanish for \(u \in \hat{\mathcal{S}} = \{ u(x, t) : \hat{\Delta} = 0 \}\), i.e.,

\[
\text{pr}^{(2)} v \left( \hat{\Delta} \right)_{\hat{\Delta}=0} = 0.
\]

Equating coefficients of powers of \(u_t\) to zero then generates the determining equations.

**Proof of Correctness:** We apply the theory described in §2.1. We have that \(\Sigma\) is a system of PDE in the dependent variables \(\{u_1, \ldots, u_m\}\) and the independent variables \(\{x_1, \ldots, x_n\}\) of order \(N\). In the \(k\)th case, we have

\[
\begin{align*}
\Psi &= \{ \xi_1 u_j x_1 + \xi_2 u_j x_2 + \ldots + u_j x_k - \phi_j | 1 \leq j \leq m \}, \\
\Psi^* &= \{ D_\alpha \psi | \psi \in \Psi, \alpha \in \mathbb{N}^n, |\alpha| \leq N - 1 \}, \\
\mathcal{R} &= \mathbb{C}[x_k, u_j, \phi_j, \xi_j, \beta u_j, \alpha] 1 \leq k \leq n, 1 \leq j \leq m, \beta \in \mathbb{N}^{n+m}, \alpha \in \mathbb{N}^n, |\alpha|, |\beta| \leq N, \\
\text{and} \ G &= \Sigma \cup \Psi^* \subset \mathcal{R}.
\end{align*}
\]

By definition, in the notation used in this article, the determining equations for the nonclassical method of the system \(\Sigma\) are

\[
\left\{ \text{coef} \left( \text{normal}(\text{pr}^{(N)} v(f), \text{GB}(\Sigma \cup \Psi^*)), u_1, \alpha_1 \ldots u_m, \alpha_m \right) = 0 | f \in \Sigma, \alpha_i \in \mathbb{N}^n \setminus \{0\} \right\}
\]

The following Lemma is proved in [31] (Proposition 3.33 and Theorem 3.38).
Lemma 2.3.2 The group prolongations of the invariant surface conditions satisfy \( \text{pr}(\nabla(\psi)) \in I(\Psi^*) \) for \( \psi \in I(\Psi^*) \).

Lemma 2.3.3 In the \( k \)-th case of the algorithm, where \( \Psi^* = \{ D_\alpha \psi \mid \psi \in \Psi, \alpha \in \mathbb{N}^n, |\alpha| \leq N - 1 \} \) and \( K = \{ \text{normal}(f, \Psi^*) \mid f \in \Sigma \} \), then \( \text{pr}(\nabla) \varphi \), for \( g \in K \) have all derivative terms of the form \( u_{j, \alpha} \) satisfying \( \alpha_k = 0 \).

Proof: Consider the term \( \phi_j^{[\alpha]} \). By examining the formulae for the prolongation of the group action, (2.24) and (2.20), there are two ways that a derivative of \( u_j \) can occur in the \( \text{pr}(\nabla) \varphi \): if either a derivative of \( \xi_k \) is non-zero, or if \( g \) has a term of the form \( u_{j, \alpha} \) with \( \alpha_k \neq 0 \). Since \( \xi_k \) is constant, and since no equation in \( \Sigma \) after reduction by \( \Psi^* \) contains a derivative of \( u_j \) with respect to \( x_k \) (for any \( j \)), the lemma is proved.

Let \( f \in \Sigma \) and let \( f' \) denote normal\( (f, \Psi^*) \). By Lemma 2.3.2 and the formula for reduction, noting that we have \( \text{hcof}(\psi) = 1 \) for all \( \psi \in \Psi^* \), \( f' = f + \sum_{\psi \in \Psi^*} r_\psi \psi \) implying

\[
\text{pr}(\nabla) (f') = \text{pr}(\nabla) (f) + \kappa \tag{2.25}
\]

where \( r_\psi \in \mathcal{R} \) and \( \kappa \in I(\Psi^*) \). Hence reducing both sides of (2.23) with respect to \( \text{GB}(\Sigma \cup \Psi^*) \) yields the same result. However, in reducing \( f \) by \( \Psi^* \), the choices of the \( r_\psi \) have removed the need to reduce the left hand side of (2.23) by \( \Psi^* \) at all. This is the content of Lemma 2.3.3. Hence there is no need to reduce the elements of \( \text{Inf} \) by the invariant surface conditions.

In \( \Psi^* \) we have a set of equations for the \( u_{j, \alpha+k} \) occurring linearly. Now a \( \text{GB} \) obtained from a reduced set of generators is still a \( \text{GB} \) so that \( \text{GB}(K, \text{ORDER}) \) is equal to the elimination ideal

\[ \text{GB}(G, \text{ORDER}) \cap \mathbb{C}[u_{j, \alpha} \mid |\alpha| \leq N, \alpha_k = 0, 1 \leq j \leq m]. \]

Hence it is sufficient to reduce the elements of \( \text{Inf} \) with respect to \( \text{GB}(K, \text{ORDER}) \). 

Since the system is regarded as an algebraic system on the relevant jet bundle, we see why differential consequences obtainable only by further differentiation, the so-called integrability conditions, are not relevant. Of course, one can always investigate the result of reducing with respect to additional integrability conditions [37]. The following remark by Olver and Rosenau [34] is relevant here: “... the reason why Bluman and Cole find nontrivial conditions on their groups in order to apply their nonclassical method is that they fail to take into account the additional restrictions on the derivatives of \( u \) coming from ... integrability conditions.” According to [34], every group is a “weak symmetry group” if all the integrability conditions are taken into account.

Another idea is find the integrability conditions first and then calculate the determining equations for the enlarged system, discussed by Schwarz [12] for classical symmetries.

2.4 Examples

In this section we give some prototypical examples of what our method looks like in action. The two examples cover those cases mentioned as being problematic in the Introduction. In the Appendix, we give the input files with which the MACSYMA package SYMMGRP.MAX [12] calculates the determining equations for the second example. Provided at some point a program written to calculate classical symmetries recognises that the terms representing infinitesimals in the input equations are the same functions used by it to represent the infinitesimals, the correct equations will be obtained. For this it may be necessary to input the infinitesimals in the internal representation.
used for them by the package. Since we are using the package for a purpose for which is was not originally designed, it is important to know precisely how it can be adapted.

In this section we set \( x_1 = x, \ x_2 = t, \ \xi_1 = \xi \) and \( \xi_2 = \tau \).

**Example 2.4.1.** The Nonlinear Wave Equation \( u_{xt} - f(u) = 0 \).

For the nonlinear wave equation in characteristic coordinates

\[
\frac{u_{xt}}{\alpha} = f(u),
\]

we use the differential consequence of the invariant surface condition with \( \tau \equiv 1 \) to eliminate \( u_{xt} \) in (2.26) and so obtain

\[
\phi_x + \phi_u u_x - (\xi_x u_x + \xi_u u_x^2 + \xi u_{xx}) = f(u).
\]

(2.27)

Now apply the second prolongation \( \text{pr}^{(2)}v \) to this equation and eliminate \( u_{xx} \) using (2.27). Actually, one can only eliminate \( \xi u_{xx} \); if one's program eliminates \( u_{xx} \) so that the equation being reduced is multiplied by \( \xi \), that is equivalent to putting \( \xi \) in the coefficient ring and to assuming that \( \xi \) is non-zero. In some cases it may be necessary to consider the case \( \xi = 0 \) separately. In this case, allowing \( \xi = 0 \) is equivalent to the next case, \( \tau = 0 \). Finally, equating coefficients of powers of \( u_x \) to zero yields the following four determining equations:

\[
\begin{align*}
\xi_\xi u_{xx} - \xi_x^2 &= 0, \\
\xi_\xi u_{xx} - \xi_x^2 &= 0, \\
\phi_x + \phi_u u_x - (\xi_x u_x + \xi_u u_x^2 + \xi u_{xx}) &= f(u).
\end{align*}
\]

In the case \( \tau \equiv 0 \) we set \( \xi = 1 \) and so the invariant surface condition reduces to \( u_x = \phi(x, t, u) \). Then we use the differential consequence of this to eliminate \( u_{xt} \) in (2.26) and so obtain

\[
\frac{u_{tt}}{\beta} = f(u) + \phi_u u_t.
\]

(2.28)

Now apply the first prolongation \( \text{pr}^{(1)}v \) to this equation and then eliminate \( u_t \) using (2.28). This yields one determining equation

\[
\phi_x + \phi_u (\phi - f) + \phi_{xx} (\phi - f) - \phi_u \phi_{xt} - \phi_u \phi_{tu} - \phi_u^2 f_u + \phi_u f_u \frac{df}{du} = 0.
\]

**Example 2.4.2.** A generalised Boussinesq equation.

Here we derive the determining equations for a “generalised” Boussinesq equation

\[
\begin{align*}
u_{tt} + u_{xx} + \alpha u_x u_{xt} + \beta u_t u_{xx} + u_{xxxx} &= 0, \quad (2.29)
\end{align*}
\]

where \( \alpha \) and \( \beta \) are arbitrary nonzero constants. This equation, together with several variants, can be derived from the classical shallow water theory in the so-called Boussinesq approximation [16]. Furthermore the Painlevé PDE test due to Weiss, Tabor and Carnevale [13], suggests that the equation is non-integrable for any non-zero choice of \( \alpha \) and \( \beta \).

In the case when \( \tau \neq 0 \), we set \( \tau = 1 \). Using the invariant surface condition \( \xi u_x + u_t = \phi \), we eliminate \( u_t, \ u_x \) and \( u_{tx} \) in (2.24) to yield

\[
\begin{align*}
\phi_t &= \phi_u (\phi - \xi u_x) - \xi_t u_x - \xi u_{xx} (\phi - \xi u_x) - \xi \left[ \phi_x + \phi_u u_x - (\xi_x u_x + \xi_u u_x^2 + \xi u_{xx}) \right] \\
&\quad + u_{xx} + \alpha u_x \left[ \phi_x + \phi_u u_x - (\xi_x u_x + \xi_u u_x^2 + \xi u_{xx}) \right] + \beta u_{xx} (\phi - \xi u_x) + u_{xxxx} = 0, \quad (2.30)
\end{align*}
\]
Apply the classical Lie algorithm to this equation using the fourth prolongation \( \text{pr}^{(4)} v \) and eliminate \( u_{xxx} \) using (2.30), to yield the following determining equations,

\[
\begin{align*}
\xi_u &= 0, \\
\phi_{uu} &= 0, \\
2\phi_xu - 3\xi_x &= 0, \\
(\beta + \alpha)(\xi_u + 2\xi_x + \xi_t) &= 0, \\
-2\beta\xi_xu - \alpha\xi_xu + \alpha\phi_u^2 + \alpha\xi_x\phi_u + \alpha\phi_{tu} + \beta\xi_xu - 2\alpha\xi_x^2 - \alpha\xi_{xt} &= 0, \\
-\alpha\xi_x + 6\phi_{xxx} + \beta\phi_u + \beta\phi_t + 2\beta\xi_x\phi - 4\xi_x^2 + 4\xi_x^2\xi_x + 2\xi_x + 2\xi_t &= 0, \\
\phi_{xxxx} + \beta\phi_{xx} + \phi_{xx} + \alpha\phi_x^2 - 4\xi_x\phi_x - 2\xi_t\phi_x + 4\xi_x\phi_u + \phi_{tt} + 2\phi_{tu} + 4\xi_x\phi_t &= 0, \\
\beta\xi_{xx} - 2\alpha\phi_u\phi_x - \alpha\xi_x\phi_x - 4\phi_{xxx} - 2\beta\phi_{xx} - \alpha\phi_{xx} - 2\phi_{xx} + 8\xi_x\phi_u \\
+ 2\xi_t\phi_u - \alpha\phi_{xt} + 2\xi_t\phi_u + \beta\xi_{xx}\phi + \xi_{xxx} + \xi_x + 4\xi_x^2 + 2\xi_t\xi_x + \xi_{tt} &= 0.
\end{align*}
\]

In the Appendix we give the MACSYMA input files used to calculate these determining equations using the package SYMGRP.MAX. We discuss the solution of these determining equations and the associated symmetry reductions in Example 3.2.2 below.

In the case when \( \tau \equiv 0 \), we set \( \xi = 1 \). Using the invariant surface condition \( u_x = \phi(x, t, u) \) we eliminate \( u_x, u_{xx}, u_{xt} \) and \( u_{xxx} \) in (2.29) to yield

\[
\begin{align*}
u_{tt} &= \alpha\phi(\phi_t + \phi_u u_t) + (1 + \beta u_t)(\phi_x + \phi_u) \\
+ \phi_{xxx} + \phi_u\phi_{xx} + 3\phi_x\phi_{uu} + 3\phi_x\phi_{xx} + \phi_x\phi_u^2 + \phi_x^2\phi_{uu} \\
+ 3\phi_x^2\phi_{xx} + 4\phi_x^2\phi_{uu} + 3\phi_x\phi_{xx} + 5\phi_x\phi_{xx} + \phi_u^3 & = 0. \\
\end{align*}
\]

Applying the second prolongation \( \text{pr}^{(2)} v \) to this equation yields the following determining equations;

\[
\begin{align*}
\phi_{uu} &= 0, \\
\beta\phi_{xx} + (\alpha + \beta)(\phi_x\phi_u + \phi_u^2) + (\alpha + 2\beta)\phi_{xx} + 2\phi_{tu} &= 0, \\
\phi_{xxxx} + \phi_{xx} + \phi_{tt} + 2\phi_{xx} + 4\phi_x\phi_{xx} + 6\phi_x\phi_{xx} + 4\phi_{xxx} + 4\phi_x\phi_u\phi_{xx} + 6\phi_u\phi_{xx} \\
+ 4\phi_u^2\phi_{xx} + 8\phi_x^2 + (\alpha + \beta)(\phi_x\phi_t + \phi_t\phi_u) + \alpha\phi(\phi_{xt} + \phi_{tu}) &= 0.
\end{align*}
\]

3 Algorithms for solving the systems of determining equations

3.1 Introduction

Finding the determining equations for the nonclassical method is only half the story. Since the systems are overdetermined and nonlinear, it is necessary to have additional algorithms and strategies to aid in their analysis and solution. In this section we demonstrate the use of the Kolchin-Ritt algorithm [11, 16, 27, 38, 39] in conjunction with the DirectSearch [16] and Reid [40] strategies to solve the systems of determining equations.

In §2 of this article, algebraic procedures were used. The process of calculating algebraic Gröbner bases of successive prolongations of a system (see for example [11]) for the purpose of finding integrability or compatibility conditions is impractical, even if one has an \textit{a priori} bound on the level of prolongation necessary to find all such conditions. Instead, we employ a true differential analogue of Buchberger’s algorithm, with cross-multiplication replaced by cross-differentiation, and
algebraic reduction replaced by differential reduction, and so on. For fully nonlinear systems, that is, containing equations that are not linear in their highest derivative terms, it is necessary to use pseudo-reduction instead of reduction else the differential algorithm will not terminate. The resulting algorithm is called the Kolchin-Ritt algorithm and has been implemented in packages in Maple \([25, 40]\).

In the Kolchin-Ritt algorithm, each pair of equations in the system given or obtained en route is cross-differentiated so that their highest derivative terms become equal. One then cross-multiplies by the relevant coefficients so that these terms cancel. The result is then pseudo-reduced by all known conditions, and if non-zero, is called an integrability or compatibility condition of the system. This process continues until no new conditions are found. In pseudo-reduction one may multiply the equation being reduced by nontrivial terms in order to effect the reduction. A simple example will illustrate our meaning. Suppose one wishes to reduce the equation \(f: u_{xyy} - u_{xx}u_y = 0\) by the equation \(g: u_yu_{xy} - u_x^2 = 0\). In \text{lex}(O(x)), we have that the highest derivative term in \(g\) is \(u_{xy}\). Then a (one-step) pseudo-reduction of \(f\) with respect to \(g\) is given by

\[
f \rightarrow_g u_yf - \frac{\partial g}{\partial y}.
\]

In strict reduction, one is allowed to multiply the equation being reduced by at most expressions in the independent variables. The use of pseudo-reduction means that the build up of differential coefficients needs to be taken into account when interpreting the results of the calculation. Precise statements describing the output and the limitations of the Kolchin-Ritt algorithm can be found in \([27]\), with additional algorithms that eliminate some (but not all) of the problems. If the output equations of Kolchin-Ritt are linear in their highest derivative terms, then one will have obtained either the integrability conditions or the differential analogue of the elimination ideals of the system, depending on whether the ordering used is total degree or lexicographic. In other cases, more work may be required to obtain the maximum amount of information possible.

The Reid strategy is designed to overcome as far as possible the problem of the build up of differential coefficients by adjusting the order in which pairs are chosen to be cross-differentiated. In this strategy, the system is divided into four groups of increasing complexity; single term equations, linear equations, equations linear in their highest derivative terms, and fully nonlinear equations. Into the fourth class is also put those equations with excessively many terms or high degree of differentiation. Each class is reduced with respect to those in the lower classes. Beginning with the linear class, the compatibility or integrability conditions are calculated and the result used to (pseudo)-reduce the equations in the higher classes. This can lead to nonlinear equations becoming linear for example. If a condition is found that belongs to a lower class, the strategy is to recalculate the algorithm on the lower class with the new condition added. If a condition is found that belongs to a higher class, the strategy is to put it there and “keep it for later”. In this way the need to multiply by differential coefficients in the reduction process is delayed for as long as possible, and kept to an absolute minimum. This strategy has been implemented in Maple in the StandardForm procedure of the Reid-Wittkopf Differential Algebra Package \([40]\). We give an example to demonstrate this strategy, Example 3.3.2. During the course of the calculations, if any equation is of the form of an expression raised to a power, we throw away the power to keep the complexity of the calculation down. In addition, we throw away factors known to be non-zero, but we do not throw away factors involving arbitrary parameters.

The DirectSearch algorithm is a strategy that searches specifically for conditions less than those in a given set in the ordering in use. This strategy was used effectively in \([16]\) to solve a
classification problem. The first example demonstrates the use of the DirectSearch strategy to make the calculation of the Kolchin-Ritt algorithm tractable.

### 3.2 Examples

We start with a simple example, namely the determining equations for the nonclassical method for Burgers’ Equation. We then study a more complicated example, the generalised Boussinesq equation of Example 2.4.2. All the calculations in this section were performed using the MAPLE package diffgrob2 [28]. By a lexicographic ordering based on \( u_1 < u_2 \ldots < u_n \) and \( x_1 < x_2 < \ldots < x_n \) is meant an ordering on the derivative terms such that \( u_j, \alpha < u_k, \beta \) if \( j < k \) else \( \alpha_n < \beta_n \) else \( \alpha_{n-1} < \beta_{n-1} \) and so on. It is assumed that \( u_j, \alpha > x_k \) (of course). Arbitrary constants of integration in this section are denoted by \( \kappa_i \).

**Example 3.2.1.** The determining equations for the nonclassical method in the \( \tau = 1 \) case for Burgers’ equation

\[
 u_t = u_{xx} + 2uu_x
\]  

are

\[
 \begin{align*}
 \xi_{uu} & = 0, \\
 \phi_{uu} - 2\xi_{ux} + 2\xi_u + 2u\xi_u & = 0, \\
 2\phi\xi_u - \phi - 2\phi_{ux} + \xi_{xx} - 2\xi_x\xi - \xi_xu - \xi_t & = 0, \\
 \phi_{xx} - \phi_t - 2\phi\xi_x + u\phi_x & = 0.
\end{align*}
\]

Calculating the Kolchin-Ritt algorithm on this system leads to expression swell, so we use a little finesse. Take a lexicographic ordering based on \( \phi > \xi \) and \( x > t > u \). In this ordering, the determining equations are listed in ascending order. The DirectSearch on the first three equations consists of the following procedure. We cross-differentiate the second and third equations so that the leading terms cancel, and then reduce with respect to \( \xi_{uu} = 0 \). We iterate this procedure recursively using the result of the previous calculation and the second determining equation until we have eliminated \( \phi \). We then continue the process using the first determining equation until we get zero. This procedure terminates yielding

\[
 \xi_u(\xi_u + 1)(2\xi_u - 1) = 0.
\]

Thus not only is \( \xi_u \) a constant, we know that it can take at most three values. Taking each value of \( \xi_u \) in turn yields three tractable calculations.

**Case 1.** Reducing the determining equations with respect to \( \xi_u = 0 \) and performing the Kolchin-Ritt algorithm yields, after simplification,

\[
 \begin{align*}
 \xi_u & = 0, \\
 \xi_{xx} & = 0, \\
 2\xi_{xt}\xi + \xi_{tt} + 4\xi_x^2\xi + 4\xi_x\xi_t & = 0, \\
 \phi + u\xi_u + 2\xi\xi_x + \xi_t & = 0.
\end{align*}
\]

The equations have been listed in ascending order. The aim of using a lexicographic ordering is to try to reduce the integration of the system to that of a series of ODE by starting with the least
equation and successively integrating up. Setting $\xi = G(t)x + H(t)$ into the third equation and equating coefficients of powers of $x$ to zero yields

$$G'''(t) + 6G(t)G'(t) + 4G^3(t) = 0,$$

$$H'''(t) + 2G'(t)H(t) + 4G^2(t)H(t) + 4G(t)H'(t) = 0.$$ 

The first can be linearized to $\zeta'''(t) = 0$ by setting $G(t) = z'(t)/2z(t)$ while the second is linear in $H$ given $G$ so that

$$G(t) = \frac{2t + \kappa_1}{2(t^2 + \kappa_1 t + \kappa_2)} \quad H(t) = \left\{ \kappa_3 \left[ \frac{\kappa_2^2 - \kappa_2}{8(2t + \kappa_1)} + \frac{1}{4} \right] + \kappa_4 \right\} G(t).$$

Thus we have $\xi$, and given $\xi$ the fourth equation gives $\phi$.

**Case 2.** Reducing the original determining equations with respect to $\xi_u + 1 = 0$ and performing the Kolchin-Ritt algorithm yields $\phi = 0$ and $\xi = -u$, in other words the solution is output!

**Case 3.** Reducing the second of the determining equations by $2\xi_u - 1 = 0$ we obtain $\phi_{uu} + \xi + u = 0$, implying $2\phi_{uu} + 3 = 0$. Thus $\phi$ is cubic in $u$. Substituting $\phi = -\frac{1}{4}a^3 + G(x,t)u^2 + H(x,t)u + K(x,t)$ into the remaining equations, eliminating $\xi$ using $\phi_{uu} + \xi + u = 0$, and setting the coefficients of different powers of $u$ to zero yields the system $\Sigma$ for $G$, $H$ and $K$,

$$-G_t + G_{xx} + 4G_xG_H + H_x = 0,$$

$$K_{xx} - K_t + 4KG_x = 0,$$

$$H_{xx} + K_x - H_t + 4HG_x = 0.$$ (3.2)

In a total degree ordering, the Kolchin-Ritt algorithm on this system yields no new integrability conditions, and the formal general solution space appears to depend on six arbitrary functions. Since the system is highly symmetric in its derivative terms, is nonlinear and is not overdetermined, we need to proceed with caution. Reverting to a lexicographic ordering, with $K > H > G$ and $t > x$, cross-differentiating the second and third equations and reducing yields

$$4KG_{xx} + 2H_{xx} - H_{tt} + 8H_tG_x + 4HG_{xt} - H_{xxxx} - 8H_{xx}G_x - 8H_xG_{xx} - 4HG_{xxx} - 16G^2_x = 0.$$ 

Thus, unless $G_{xx} = 0$, we have solved for $K$. We first solve this singular case by calculating the Kolchin-Ritt algorithm on (3.2) together with $G_{xx} = 0$ to yield,

$$G_{x} = 0,$$

$$-G_{ttt} + 4G_{xtt}G + 12G_{xt}G_t + 12G_xG_{tt} - 48G_xG_{xt}G - 48G^2_xG_t + 64G^3_xG = 0,$$

$$H_x - G_t + 4GX_xG = 0,$$

$$-H_{tt} + 8H_tG_x + 4HG_{xt} - 16HG^2_x + 2G_{xtt} - 24G_xG_{xt} + 32G^3_x = 0,$$

$$K_x - H_t + 4HG_x + G_{xt} - 4G^2_x = 0,$$

$$-K_t + 4KG_x + G_{ttt} - 4G_{xt}G - 8G_xG_t + 16G^2_xG = 0.$$ (3.3)

Setting $G(x,t) = xa(t) + b(t)$ in the second equation of (3.3) and reading off coefficients of $x$ yields,

$$12a_t^2 - a_{ttt} - 96a_t a^2 + 16a_{tt}a + 64a^4 = 0,$$

$$4a_{tt}b - b_{ttt} + 12a_t b_t + 12b_{tt}a - 48a_t a^2 - 48a_t ab + 64a^3 b = 0.$$ 

The first of these can be linearised to $\zeta'''(t) = 0$ by setting $a(t) = -\zeta'(t)/(4\zeta(t))$, and substituting this into the second equation yields $(b(t)\zeta(t))''' = 0$. Using this and the final four equations in the
output \((3.3)\), which are linear in \(H\) and \(K\), yields the singular solution. This solution generalises that in \([36]\).

We return to the general case of \((3.2)\), and use the condition obtained for \(K\) above to eliminate \(K\). Using a Direct Search method we can solve for \(H\) in terms of \(G\), and then obtain two very long expressions for \(G\) whose cross-derivative is zero. None of the singular cases that arise lead to solutions other than that obtained in the \(G_{xx} = 0\) case. This is as far as elementary differential algebra can take one with \((3.2)\), and further analysis will depend on additional input, such as any geometrical information that comes from knowledge of what the \(G, H\) and \(K\) actually mean. This could lead to a transformation in which the analysis is simpler and the geometric content clearer. One possibility is the connection, for Burgers’ equation, between the nonclassical method and the so-called Singular Manifold method of obtaining solutions using a truncated Painlevé expansion discussed in \([20]\). This uses an ansatz in which \(G\) satisfies a Burgers’ type equation, so that the process of finding the nonclassical reductions becomes a Bäcklund transformation for the Burgers’ equation. Another ansatz, used in \([2]\), is \(G_t = 0\). However, the full solution appears to be unknown at present. Further, the meaning of such a large solution space of nonclassical reductions appears to be unknown.

**Example 3.2.2.** In this example we apply the classical and nonclassical methods for symmetry reductions of a “generalised Boussinesq equation” \((2.29)\). The classical symmetry reductions are calculated not only for comparison purposes, but to show just how effective the Kolchin-Ritt algorithm can be for linear systems. We do not consider the case where either \(\alpha\) and \(\beta\) is zero.

### 3.2.2.A Classical Method

Applying the classical Lie point method to this equation yields the system of linear overdetermined equations given below (with \(\xi_1 \equiv \xi\) and \(\xi_2 \equiv \tau\)).

\[
\begin{align*}
\tau_u &= 0, \\
\tau_x &= 0, \\
\xi_u &= 0, \\
\phi_{uu} &= 0, \\
2\phi_{xx} - 3\xi_{xx} &= 0, \\
(\beta + \alpha)\xi &= 0, \\
\alpha\phi_x - 2\xi_t &= 0, \\
4\phi_{xxxx} + 2\phi_{xx} + \alpha\phi_t - \xi_{xxxx} - \xi_{xx} - \xi_{tt} &= 0.
\end{align*}
\]

Before applying the Kolchin-Ritt algorithm to a linear system, the set is simplified to be an “auto-reduced” set. In the diffgrob2 package the procedure used is reduceall, while in the Reid-Wittkopf package the procedure is called Clearderivdep. What these procedures do is reduce every equation with respect to every other equation until no further reductions are possible. With linear systems this results in no loss of information or solutions. The simplification procedure is also applied to the output of Kolchin-Ritt (or StandardForm in the Reid-Wittkopf package). Using a lexicographic ordering based on \(\xi_2 > \phi > \xi_1\) and \(x > t > u\), (recall in these equations, \(u\) is an independent variable), the result of simplification, Kolchin-Ritt and further simplification appears in the left hand column of Table 1. Clearly, it is necessary to run the calculation again having set \(\alpha + \beta = 0\). The result of this second calculation appears in the right hand column of Table 1. It will be necessary to recalculate the determining equations with the special values of the parameters from the beginning if the package used removes factors in the parameters. The output is considerably simpler to solve than the original set of equations, while the solutions sets are the same. The solutions of these determining equations are given in Table 2.

We shall now describe symmetry reductions that are generated by these infinitesimals which are
Table 1: The classical determining equations (3.4) in Standard Form

| $\alpha + \beta \neq 0$  | $\alpha + \beta = 0$ |
|-------------------------|----------------------|
| $\alpha \xi_{xx}(\beta + \alpha) = 0$ | $\beta \xi_{xx} = 0$ |
| $(\beta + \alpha) \xi_t = 0$ | $\beta \xi_t = 0$ |
| $\xi_u = 0$ | $\xi_u = 0$ |
| $\alpha(\beta + \alpha) \phi_x = 0$ | $\beta \phi_x + 2\xi_t = 0$ |
| $\alpha(\beta + \alpha)(\beta \phi_t + 2\xi_x) = 0$ | $\beta(\beta \phi_t + 2\xi_x) = 0$ |
| $\phi_u = 0$ | $\phi_u = 0$ |
| $\tau_x = 0$ | $\tau_x = 0$ |
| $\tau_t - 2\xi_x = 0$ | $\tau_t - 2\xi_x = 0$ |
| $\tau_u = 0$ | $\tau_u = 0$ |

Table 2: Classical Infinitesimals

| $\alpha + \beta \neq 0, \alpha \neq 0, \beta \neq 0$  | $\alpha + \beta = 0$ |
|-------------------------|----------------------|
| $\xi$ | $\kappa_1 x + \kappa_2$ |
| $\tau$ | $2\kappa_1 t + \kappa_3$ |
| $\phi$ | $-2\kappa_1 t/\beta + \kappa_4$ |

obtained by solving solving the invariant surface condition $\xi(x,t,u)u_x + \tau(x,t,u)u_t - \phi(x,t,u) = 0$.

In the following $\mu_i$ are constants of integration.

**Case A.1.** $\alpha + \beta \neq 0$.

(i) If $\kappa_1 \neq 0$, then we set $\kappa_1 = 1$ and $\kappa_2 = \kappa_3 = 0$, and obtain the symmetry reduction

$$u(x,t) = w(z) - t/\beta + \frac{1}{2}\kappa_4 \ln t, \quad z = x/t^{1/2},$$

where $w(z)$ satisfies

$$w''' - \frac{1}{2}(\alpha + \beta)zw'' - \frac{1}{2}\alpha(w')^2 + \left(\frac{1}{4}z^2 + \frac{1}{2}\beta \kappa_4\right)w'' + \frac{3}{4}zw' - \frac{1}{2}\kappa_4 = 0. \quad (3.6)$$

(ii) If $\kappa_1 = 0$, then we set $\kappa_3 = 1$, and obtain the symmetry reduction

$$u(x,t) = w(z) + \kappa_4 t, \quad z = x - \kappa_2 t,$$

where $w(z)$ satisfies

$$w''' = (\alpha + \beta)\kappa_2 w'' - (1 + \kappa_2^2 + \beta \kappa_4)w'. \quad (3.8)$$

Setting $W = w'$, this equation can be integrated twice to yield

$$(W')^2 = \frac{2}{3}(\alpha + \beta)\kappa_2 W^3 - (1 + \kappa_2^2 + \beta \kappa_4)W^2 + \mu_1 W + \mu_0, \quad (3.9)$$

which is solvable in terms of the Weierstrass elliptic function $\wp(\theta; g_2, g_3)$ (cf. [47]).

**Case A.2.** $\alpha + \beta = 0$. 
(i) If $\kappa_1 \neq 0$, then we set $\kappa_1 = 1$ and $\kappa_3 = \kappa_4 = 0$, and obtain the symmetry reduction
\[
u(x, t) = w(z) + \left(\kappa_2^2 - 1\right)t - 2\kappa_2x / \beta + \frac{1}{2} \kappa_5 \ln t, \quad z = (x + \kappa_2 t) / t^{1/2}, \tag{3.10}\]
where $w(z)$ satisfies
\[
 w''' + \frac{1}{2} \beta (w')^2 + \left(\frac{1}{4} z^2 + \frac{1}{2} \beta \kappa_5\right) w'' + \frac{3}{4} zw' - \frac{1}{4} \kappa_5 = 0. \tag{3.11}\]
(ii) If $\kappa_1 = 0$, then set $\kappa_4 = 1$, and we obtain the symmetry reduction
\[
u(x, t) = w(z) - 2\kappa_2 \left[xt - \frac{1}{5} \kappa_2 t^3 - \frac{1}{2} \kappa_3 t^2 + \kappa_5 t\right] / \beta, \quad z = x - \frac{1}{2} \kappa_2 t^2 - \kappa_3 t, \tag{3.12}\]
where $w(z)$ satisfies the linear equation
\[
 w''' = \left[2 \kappa_2 z + (2\kappa_2 \kappa_5 - \kappa_3^2 - 1)\right] w'' - \kappa_2 w' - 2\kappa_2 \kappa_3 / \beta. \tag{3.13}\]

3.2.2.B Nonclassical Method. The determining equations for the nonclassical method are given in Example 2.4.2. As in the classical case, we need to do the cases $\alpha + \beta \neq 0$ and $\alpha + \beta = 0$ separately. We assume $\xi \neq 0$.

Case B.1: $\alpha + \beta \neq 0$. Take a lexicographic ordering based on $\phi > \xi$ and $x > t > u$. The first step of the Reid strategy involves reducing with respect to small linear equations, in this case $2\phi_x - 3\xi_{xx} = 0$, then calculating the compatibility conditions of each pair of determining equations. This leads to several conditions in $\xi$ alone, including
\[
 (2\beta - 3\alpha)(2\beta + \alpha)(3\alpha + 4\beta)(\xi_{tt} \xi - 2\xi_t^2) = 0. \tag{3.14}\]

The precise coefficients in front of this equation depends on the order in which pairs are chosen to be cross-differentiated; this affects which equations appear before others and thus which equations are used to do the reduction of new compatibility conditions. Continuing with the Reid strategy on the equations for $\xi$ yields for generic $\alpha$ and $\beta$ that $\xi_{xx} = \xi_{tt} \xi - 2\xi_t^2 = \xi_{tt} \xi - \xi_x \xi_t = 0$. Choosing all the special values of $\alpha$ for which (3.14) is zero and redoing the calculation leads to the same conclusion so that
\[
 \xi = \frac{\sigma x + \kappa_3}{\kappa_1 t + \kappa_2}, \tag{3.15}\]
for all values of $\alpha$ and $\beta$, where $\sigma$ is either 0 or 1.

Subcase B.1.1: $\xi_x \neq 0$. We set $\sigma = 1$.

Reducing the determining equations by (3.13) yields one small equation $\phi_u = (\kappa_1 - 2) / (\kappa_1 t + \kappa_2)$ and reducing the remaining equations by this still leaves three lengthy equations for $\phi$. In this situation, the DirectSearch strategy is beneficial. Using a DirectSearch strategy with the equation for $\phi_u$ and each of the lengthy equations leads to two consistency conditions $(\kappa_1 - 2)(\beta \kappa_1 - \beta + 2 \alpha) = 0$ and $\alpha(\kappa_1 - 2)(2\kappa_2 \alpha - 4 \alpha + \beta(\kappa_1^2 - 5 \kappa_1 + 2)) = 0$. Needless to say, inserting each subcase in the parameters leads to considerable simplification. Continuing in this manner we obtain the solutions for each case. There are two cases which yield solutions.

(i) For all $\alpha$ and $\beta$, with $\kappa_1 = 2$ we obtain the infinitesimals
\[
 \xi = \frac{x + \kappa_3}{2t + \kappa_2}, \quad \phi = \frac{-2t + \kappa_3}{\beta(2t + \kappa_2)}; \tag{3.16}\]
these correspond to classical symmetries.
(ii) If \( \beta = 2\alpha \) and \( \kappa_1 = 0 \), then we obtain the infinitesimals

\[
\xi = \frac{x + \kappa_4}{\kappa_2}, \quad \phi = -\frac{2\alpha u + t}{\alpha\kappa_2} + 2\left(\frac{x + \kappa}{\kappa_2}\right)^2 + \kappa_3,
\]

which do not correspond to classical symmetries.

Setting \( \kappa_2 = 1/\kappa \) and \( \kappa_4 = 0 \) in (3.17) yields the nonclassical reduction

\[
u(x,t) = w(z)\exp(-2\kappa t) + \left(\kappa x^2 - t\right)/(2\alpha) + \kappa_3, \quad z = x\exp(-\kappa t),
\]

where \( w(z) \) satisfies

\[
w'''' = 3\kappa\alpha zw''' + 4\kappa\alpha w'' + 3\kappa\alpha(w')^2.
\]

**Subcase B.1.2:** \( \xi_x = 0 \).

In this case we set \( \xi = 1/(\kappa_1 t + \kappa_2) \). Reducing the determining equations with the solution for \( \xi \) and calculating the compatibility conditions according to the Reid strategy yields \( \kappa_1 = 0 \). If \( \alpha \neq \beta \) then unless \( \beta = 0 \) we obtain only the trivial solution, \( \phi = \phi_0 \), a constant.

If \( \alpha = \beta \), we can reduce our equations to the simple set

\[
\xi = \kappa, \quad \phi_t = \kappa\phi_x, \quad \phi_{xxxx} + \phi_{xx} \{\kappa^2 + 1 + \beta\phi\} + \beta\phi_x^2 = 0.
\]

Setting \( \phi = \Phi(y), \) where \( y = x + \kappa t \) and \( \phi = -12(\Phi + \kappa^2 + 1)/\beta \), the second of the two equations can be written as an ODE and integrated twice to yield

\[
\frac{d^2\Phi}{dy^2} = 6\Phi^2 + \kappa_1 y + \kappa_2.
\]

If \( \kappa_1 \neq 0 \) then this equation is equivalent the first Painlevé equation PI (cf., [24]), whilst if \( \kappa_1 = 0 \) then it is solvable in terms of the Weierstrass elliptic functions. Hence we obtain the nonclassical reduction

\[
u(x,t) = v(y) + w(z), \quad y = x + \kappa t, \quad z = x - \kappa t,
\]

where \( v(y) \) and \( w(z) \) satisfy

\[
v_{yyyy} + (1 + \kappa^2)v_{yy} + 2\kappa\beta v_y v_{yy} = -\lambda,
\]

and

\[
w_{zzzz} + (1 + \kappa^2)w_{zz} - 2\kappa\beta w_z w_{zz} = \lambda,
\]

respectively, where \( \lambda \) is a “separation” constant. Integrating (3.21,3.22) and setting \( V = v_y \) \( W = w_z \), yields

\[
V_{yy} + (1 + \kappa^2)V + \kappa\beta V^2 = -\lambda y + \mu_1,
\]

and

\[
W_{zz} + (1 + \kappa^2)W - \kappa\beta W^2 = \lambda z + \mu_2,
\]

respectively, where \( \mu_1 \) and \( \mu_2 \) are arbitrary constants. If \( \lambda \neq 0 \) then these equations are equivalent to the first Painlevé equation PI, whilst if \( \lambda = 0 \) then they are solvable in terms of Weierstrass elliptic functions. In particular, if \( \lambda = \mu_1 = \mu_2 = 0 \), then equations (3.23,3.24) possess the solutions

\[
V(y) = -\frac{3(1 + \kappa^2)}{2\kappa\beta} \sec^2 \left[\frac{1}{2} \sqrt{1 + \kappa^2} y\right], \quad W(z) = \frac{3(1 + \kappa^2)}{2\kappa\beta} \sec^2 \left[\frac{1}{2} \sqrt{1 + \kappa^2} z\right].
\]
Thus we obtain the exact solution of (2.29) with $\alpha = \beta$ given by

$$u(x,t) = -\frac{3\sqrt{1 + \kappa^2}}{\kappa\beta} \tan \left[ \frac{1}{2} \sqrt{1 + \kappa^2} (x + \kappa t + \delta_1) \right] + \frac{3\sqrt{1 + \kappa^2}}{\kappa\beta} \tan \left[ \frac{1}{2} \sqrt{1 + \kappa^2} (x - \kappa t + \delta_2) \right].$$

(3.26)

Making the transformation $x \rightarrow ix$, $t \rightarrow it$ and $u \rightarrow iu$ in (2.29) with $\alpha = \beta$ yields

$$u_{tt} + u_{xx} + \alpha(u_x u_{xt} + u_t u_{xx}) - u_{xxxx} = 0.$$

(3.27)

Thus from (3.26) we obtain the exact solution of (3.27)

$$u(x,t) = -\frac{3\sqrt{1 + \kappa^2}}{\kappa\beta} \tanh \left[ \frac{1}{2} \sqrt{1 + \kappa^2} (x + \kappa t + \delta_1) \right] + \frac{3\sqrt{1 + \kappa^2}}{\kappa\beta} \tanh \left[ \frac{1}{2} \sqrt{1 + \kappa^2} (x - \kappa t + \delta_2) \right].$$

(3.28)

A plot of this solution is given in Figure 1 and a plot of its derivative with respect to $x$ in Figure 2. Figure 1 shows that the solution looks like the elastic interaction of a “kink” and an “anti-kink” solution. Figure 2 looks like the elastic interaction of two “soliton” solutions. These are of particular interest since such solutions are normally associated with integrable equations, whereas they arise here for a nonintegrable equation. Furthermore, to our knowledge, this is the first time that a “two-soliton” solution has arisen from a nonclassical reduction. Normally such solutions are associated with so-called Lie-Bäcklund transformations (cf., [1]).

**Case B.2:** $\alpha + \beta = 0$. The analysis of the determining equations in this case is similar but more complicated. We give here only the solution that does not correspond to a classical symmetry,

$$\xi = \kappa_1 t + \kappa_2, \quad \phi = \frac{u}{t + \kappa_0} + \frac{2(\kappa_0 \kappa_1 - \kappa_2)x + 2\kappa_1^2 \kappa_2 t^3 + 2\kappa_1 \kappa_2 t^2 + 2\kappa_2^3 t + \kappa_3}{\alpha(t + \kappa_0)}.$$

Setting $\kappa_0$ we obtain the nonclassical reduction

$$u(x,t) = tw(z) + \left[ 6\kappa_2 x + \kappa_1^2 t^3 - 6\kappa_2^2 t \right] / \alpha + \kappa_3, \quad z = x - \frac{1}{2} \kappa_1 t^2 - \kappa_2 t,$$

(3.29)

where $w(z)$ satisfies

$$w''' + \alpha \left[ (w')^2 - w w'' \right] + (1 + \kappa_2^2)w'' - 3\kappa_1 w' + 2\kappa_1^2 / \alpha = 0.$$

(3.30)

It is interesting to note that the values of the parameters for which nonclassical reductions were found, not corresponding to classical symmetry reductions, were precisely $\beta = -\alpha$, $\alpha$ and $2\alpha$. We will not analyse the $\tau = 0$ case of the determining equations here.

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Appendix 1

We give here the MACSYMA input files used to calculate the determining equations for nonclassical reductions, in the $\tau \equiv 1$ case, of the equation (2.29). They show how to adapt a package designed to calculate classical symmetries.

The batchfile `pbq1.case` containing the MACSYMA commands to implement the program SYMMGRP.MAX is

```
batchload("symmgrp.max");
writefile("pbq.out");
/* PERTURBED BOUSSINESQ EQUATION - NONCLASSICAL (TAU = 1) */
/* u_{tt} + u_{xx} + a u_x u_{xt} + b u_t u_{xx} + u_{xxxx} =0 */
batch("pbq1.dat");
symmetry(1,0,0);
printeqn(lode);
save("lodepbq1.lsp",lode);
closefile();
```

This file in turn batches the file `pbq1.dat` which contains the requisite data about (2.29):

```
p:2$
q:1$
m:1$
parameters:[a,b]$
warnings:true$
sublisteqs:[all]$
subst_deriv_of_vi:true$
info_given:true$
highest_derivatives:all$
deps([eta1,eta2,phi1],[x[1],x[2],u[1]])
eta2:1;
ut:phi1-eta1*u[1,[1,0]];
uxt:diff(phi1,x[1])+diff(phi1,u[1])*u[1,[1,0]] -
diff(eta1,x[1])*u[1,[1,0]]
  - diff(eta1,u[1])*u[1,[1,0]]**2 - eta1*u[1,[2,0]];
utt:diff(phi1,x[2])+diff(phi1,u[1])*ut - diff(eta1,x[2])*u[1,[1,0]]
  - diff(eta1,u[1])*u[1,[1,0]]*ut - eta1*uxt;
e1:utt+u[1,[2,0]]+a*u[1,[1,0]]*uxt+b*ut*u[1,[2,0]]+u[1,[4,0]];
v1:u[1,[4,0]];
```

The important thing to note is that SYMMGRP.MAX recognises that $\eta_1$ and $\phi_1$ represent the infinitesimals $\xi$ and $\phi$ that are to be determined. We refer the reader to the paper by Champagne, Hereman and Winternitz [12] for an explanation of the syntax used and the purpose of the various commands.

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