Scaling Identities for Solitons beyond Derrick’s Theorem

Nicholas S. Manton *

DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

September 17, 2008

Abstract

New integral identities satisfied by topological solitons in a range of classical field theories are presented. They are derived by considering independent length rescalings in orthogonal directions, or equivalently, from the conservation of the stress tensor. These identities are refinements of Derrick’s theorem.

1 Introduction

We will be concerned here with static soliton solutions occurring in a range of field theories in two, three or four dimensional space. Solitons are smooth, localized, finite energy stationary points of some energy function, either minima or saddle points [12]. The energy is the integral of a sum of terms involving the field and its first partial derivatives. Our solitons are generally stabilised by some topological charge that prevents their decay to the vacuum. Whereas the vacuum has zero energy, a soliton has positive energy. Topological aspects of solitons will actually be unimportant here. However, it will be important that the field of a soliton rapidly approaches the vacuum field at spatial infinity, at least up to a gauge transformation, so that the energy density falls off rapidly too.

It is well known that by rescaling lengths one can derive some identities relating different contributions to the total energy of a soliton. The basic point, noted by Derrick [6], is that the energy at the soliton solution is stationary with respect to a uniform spatial rescaling (or dilation). Setting the variation with respect to an infinitesimal dilation to zero gives non-trivial identities. In some field theories, as Derrick showed, the scaling argument can be used to prove that no finite energy solitons can exist. We shall not be interested in this situation here.

In this letter we shall consider the effect of independent length rescalings in different Cartesian directions. In various examples, we obtain simple and novel identities connecting contributions to the total energy. Our results are refinements of the Derrick scaling identities, in that simple combinations of our results reproduce those of Derrick.

* N.S.Manton@damtp.cam.ac.uk
It is unlikely that many of our results are original, and in some sense they are not new at all. From the conservation of the stress tensor (the purely spatial analogue of the energy-momentum tensor), which is a local identity that follows from the field equations, one can derive an infinite number of integral identities, and ours are special cases, albeit interesting ones.

We shall not develop our results in a completely general way. Instead we shall obtain scaling identities for Skyrmions, which are solitons in a purely scalar field theory in three dimensions, and then show how to obtain scaling identities in theories with gauge fields coupled to scalar fields, whose solitons are vortices in two dimensions and monopoles in three dimensions. We shall also consider instantons, which are solitons in a pure gauge theory in four dimensions.

In Section 2 devoted to Skyrmions, we carry out the scaling manipulations directly on the energy function. In Section 3 we show how the same results are derivable from the stress tensor of the field theory. Finally, in Section 4 we introduce gauge fields coupled to scalar fields, and use the gauge invariant stress tensor to obtain scaling identities for vortices and monopoles, and also for instantons and other solutions in pure gauge theories. Some solitons satisfy first order Bogomolny equations, and for these solitons the stress tensor vanishes identically, and our identities become trivial.

2 Skyrmions

The Skyrme model in its original version is defined in three space dimensions [17]. We use Cartesian coordinates $x = (x^1, x^2, x^3)$. The scalar field $U(x)$ takes values in $SU(2)$, so it is convenient to work not directly with the derivatives of $U$ but with the currents $R_i = \partial_i U U^{-1}$, each component of which takes values in the Lie algebra of $SU(2)$. The Skyrme energy function, in simplified units, is

$$E = \int \left\{ -\frac{1}{2} \text{Tr} (R_i R_i) - \frac{1}{16} \text{Tr} ([R_i, R_j][R_i, R_j]) + m^2 \text{Tr} (1 - U) \right\} d^3 x. \quad (2.1)$$

Each term in $E$ is non-negative. What is important here is not so much the Lie algebra structure of the commutators and traces, but which derivatives occur in each term. Note that because of the commutators, no current component occurs raised to the fourth power. $m^2 \text{Tr} (1 - U)$ is the pion mass term and is sometimes omitted.

A Skyrmion is a smooth, finite energy minimum or stationary point of this energy, satisfying the boundary condition $U \to 1$ as $|x| \to \infty$. Skyrmions are distinguished by their topological charge, their baryon number $B$, which is the degree of the map $U: \mathbb{R}^3 \to SU(2)$. Many Skyrmions are known, at least numerically, for a large range of values of $B$ [2].

Suppose now that the field $U(x)$ is a Skyrmion, and consider an independent rescaling of the three coordinates:

$$x^1 \to \lambda_1 x^1, \quad x^2 \to \lambda_2 x^2, \quad x^3 \to \lambda_3 x^3. \quad (2.2)$$

That is, we replace $U(x^1, x^2, x^3)$ by $\tilde{U}(x^1, x^2, x^3) = U(\lambda_1 x^1, \lambda_2 x^2, \lambda_3 x^3)$. Clearly $\partial_i \tilde{U}$ equals $\lambda_i \partial_i U$. By changing variables, we can relate each term in the energy $E$ of the
transformed field $\tilde{U}(x)$ to the corresponding term in $E$. The result is

$$E = \int \left\{ \frac{1}{2} \frac{\lambda_1}{\lambda_2} \text{Tr} (R_1 R_1) - \frac{1}{2} \frac{\lambda_2}{\lambda_3} \text{Tr} (R_2 R_2) - \frac{1}{2} \frac{\lambda_3}{\lambda_1} \text{Tr} (R_3 R_3) \\
- \frac{1}{8} \lambda_1^2 \text{Tr} ([R_1, R_2][R_1, R_2]) - \frac{1}{8} \lambda_2 \lambda_3 \text{Tr} ([R_2, R_3][R_2, R_3]) \\
- \frac{1}{8} \lambda_2 \lambda_3 \text{Tr} ([R_3, R_1][R_3, R_1]) + \frac{1}{\lambda_1 \lambda_2 \lambda_3} m^2 \text{Tr} (1 - U) \right\} d^3 x,$$

(2.3)

where the expressions on the right hand side are evaluated for the field $U(x)$. Because $U(x)$ is a Skyrmion, the function $E$ is stationary with respect to $\lambda_1$, $\lambda_2$ and $\lambda_3$ at $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Rather than work out the general consequences of this, it is simpler to restrict to special rescalings (which together span all possibilities). Particularly worthwhile is to set

$$\lambda_1 = \lambda_2 = \lambda$$

and this is Derrick’s theorem for Skyrmions, which is also obtained by considering the uniform rescaling $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

The difference of identities (2.6) and (2.7) is

$$\int \left\{ -\frac{1}{2} \text{Tr} (R_1 R_1) + \frac{1}{8} \text{Tr} ([R_1, R_2][R_1, R_2]) + \frac{1}{8} \text{Tr} ([R_2, R_3][R_2, R_3]) \\
- \frac{1}{8} \text{Tr} ([R_1, R_3][R_1, R_3]) + \frac{1}{8} \text{Tr} ([R_2, R_3][R_2, R_3]) \right\} d^3 x = 0,$$

(2.9)
and rotating this by $45^\circ$ in the $(x^1, x^2)$ plane gives
\[
\int \left\{ -\frac{1}{2} \text{Tr} (R_1 R_2) - \frac{1}{8} \text{Tr} ([R_1, R_3][R_2, R_3]) \right\} d^3x = 0, \quad (2.10)
\]
which is a further novel identity. Two more identities are obtained by permuting the indices:
\[
\int \left\{ -\frac{1}{2} \text{Tr} (R_2 R_3) - \frac{1}{8} \text{Tr} ([R_2, R_1][R_3, R_1]) \right\} d^3x = 0, \quad (2.11)
\]
\[
\int \left\{ -\frac{1}{2} \text{Tr} (R_3 R_1) - \frac{1}{8} \text{Tr} ([R_3, R_2][R_1, R_2]) \right\} d^3x = 0. \quad (2.12)
\]

These identities could be useful in the following way. In the Skyrme model, and in other field theories with solitons, one is often interested in finding exact numerical solutions, by relaxation of the energy. One often starts with an analytical approximation to a solution, given by simple formulae. For example, in the Skyrme model, the rational map ansatz provides a good starting point \cite{7}. It is helpful to get the size right by imposing the Derrick identity, and if it is not initially satisfied, a simple uniform rescaling will ensure that it is.

With the new identities (2.10) – (2.12) we can also see how to improve a trial solution. For example, the rational map ansatz has no squashing degrees of freedom, but now we can introduce them. Given a field configuration, one should calculate the integrals
\[
I_1 = \int \left\{ -\frac{1}{2} \text{Tr} (R_3 R_3) + m^2 \text{Tr} (1 - U) \right\} d^3x, \\
I_2 = \int \left\{ -\frac{1}{8} \text{Tr} ([R_1, R_2][R_1, R_2]) \right\} d^3x. \quad (2.13)
\]

Then a rescaling in the $(x^1, x^2)$ plane, with no rescaling of $x^3$, by the factor
\[
\lambda = \left( \frac{I_1}{I_2} \right)^{\frac{1}{4}} \quad (2.14)
\]
will optimally reduce the energy. Following this one can rescale in, say, the $(x^2, x^3)$ plane. This procedure can be iterated until the optimal squashing is attained. The process converges, as the energy cannot go below that of the true solution. Moreover, at each step the integrals contributing to the energy change, but in a simple way, so they do not need to be numerically calculated more than once.

This process could be particularly useful for finding good approximations to Skyrmions with $B = 8, 10$ and $12$ (among others). Here it is known that the usual rational map ansatz gives trial solutions that are too spherical, whereas the true solutions are rather prolate (for $B = 8$), or oblate (for $B = 10$ and $B = 12$) \cite{1, 3}. The cyclic symmetries of these solutions are a guide to which rescalings are likely to be helpful.

It is possible that from a given starting field configuration, a single step squashing/shearing transformation can be found, after which all six identities (2.5) – (2.7) and (2.10) – (2.12) are satisfied, but we have not found an algorithm for this.
3 Stress Tensor

For a generic field theory in Euclidean space $\mathbb{R}^n$, with translational invariance, one can derive using Noether’s theorem a conserved stress tensor. This is the static analogue of the energy-momentum tensor of a dynamical field theory. If the energy density is $\mathcal{E}$, and is a function just of some fields $\phi$ and their spatial derivatives $\partial_i \phi$, then the stress tensor is

$$T_{ij} = \frac{\partial \mathcal{E}}{\partial (\partial_i \phi)} \partial_j \phi - \delta_{ij} \mathcal{E}. \tag{3.1}$$

The stress tensor satisfies the conservation law $\partial_i T_{ij} = 0$, which can be verified by using the field equation

$$\partial_l \left( \frac{\partial \mathcal{E}}{\partial (\partial_i \phi)} \right) - \frac{\partial \mathcal{E}}{\partial \phi} = 0. \tag{3.2}$$

Now let $V_j(x)$ be an arbitrary vector field (classical functions, not a dynamical quantity) and define

$$P_i = V_j T_{ij}. \tag{3.3}$$

Then $\partial_i P_i = (\partial_i V_j) T_{ij}$. Integrating this over $\mathbb{R}^n$, and using the divergence theorem, gives

$$\int (\partial_i V_j) T_{ij} \, d^n x = 0, \tag{3.4}$$

provided $P_i$ decays rapidly enough towards spatial infinity. We shall consider field configurations $\phi$ which rapidly approach the vacuum at infinity, so that $\mathcal{E}$ and $T_{ij}$ rapidly approach zero too. $V_i$ may grow towards infinity, but must not do so too fast.

We can rederive Derrick’s theorem and our new scaling identities by choosing $V_j$ to depend linearly on Cartesian coordinates. Let

$$V_j = A_{jk} x^k \tag{3.5}$$

with $A_{jk}$ an arbitrary constant matrix. Then $\partial_i V_j = A_{ji}$, so (3.4) implies that

$$\int T_{ij} \, d^n x = 0 \tag{3.6}$$

for any labels $i, j$.

Let us see how this works for a Skyrmion. From the energy expression (2.1) we obtain, by applying (3.1), the stress tensor

$$T_{ij} = -\text{Tr} (R_i R_j) - \frac{1}{4} \text{Tr} ([R_i, R_k][R_j, R_k]) - \delta_{ij} \mathcal{E}. \tag{3.7}$$

The identity obtained by integrating $T_{11}$ gives a rather complicated expression, but the combination

$$\int (T_{11} + T_{22}) \, d^3 x = 0 \tag{3.8}$$

reproduces the identity (2.5). The identities (2.6) and (2.7) are obtained similarly. The trace combination

$$\int T_{ii} \, d^3 x = 0 \tag{3.9}$$
is Derrick’s theorem (2.8). The vanishing of the integrals of the off-diagonal elements of the stress tensor give the remaining identities. For example, integrating $T_{12}$, one reproduces (2.10).

For Skyrmions, there are no surface terms at infinity to worry about, because $1 - U$ decays as $|x|^{-2}$ when $m = 0$ [11], so $R_i$ decays as $|x|^{-3}$ and $T_{ij}$ as $|x|^{-6}$. The decay is exponentially fast for non-zero $m$.

4 Gauge Fields

In this section we derive scaling identities in a selection of field theories with gauge fields included. Our examples have both abelian and non-abelian gauge fields.

We start by recalling that in pure abelian gauge theory in $n$ space dimensions the energy is

$$E = \frac{1}{4} \int f_{ij} f_{ij} \, d^n x ,$$

where $f_{ij} = \partial_i a_j - \partial_j a_i$. The “improved”, gauge invariant stress tensor is

$$T_{ij} = f_{ik} f_{jk} - \frac{1}{4} \delta_{ij} f_{lm} f_{lm} .$$

The field equations, $\partial_i f_{ij} = 0$, imply that $\partial_i T_{ij} = 0$. However, this theory has no solitons.

Scalar electrodynamics has a complex scalar field $\phi$ coupled to the gauge potential $a_i$. In two space dimensions, the gauge invariant energy is

$$E = \int \left( \frac{1}{4} f_{ij} f_{ij} + \frac{1}{2} D^2 \phi + V \right) \, d^2 x .$$

The only non-vanishing component of $f_{ij}$ is the magnetic field $B = f_{12}$, and $D_i \phi = \partial_i \phi - i a_i \phi$ is the covariant derivative of $\phi$. $V$ is a function of $\bar{\phi} \phi$. We assume that the minimal value of $V$ is zero. If $V$ attains this minimal value at non-zero $\phi$ then there is spontaneous symmetry breaking, and finite energy vortex solutions are possible [12]. A vortex has the property that $\bar{\phi} \phi$ minimizes $V$ around the circle at infinity, and the phase of $\phi$ has non-trivial winding there. The winding number $N$, being an integer, gives the vortex topological stability. $D_i \phi$ and $B$ both vanish exponentially fast towards infinity; however $a_i$ has a non-zero integral around the circle at infinity, related to the winding of $\phi$, and hence the vortex carries a magnetic flux. The flux is $2\pi N$ in our units.

For the energy function (4.3), the stress tensor is

$$T_{ij} = f_{ik} f_{jk} + \frac{1}{2} D_i \phi D_j \phi + \frac{1}{2} D_i \phi D_j \phi - \delta_{ij} \left( \frac{1}{4} f_{lm} f_{lm} + \frac{1}{2} D_i \phi D_l \phi + V \right) .$$

Explicitly, the components are

$$T_{11} = \frac{1}{2} B^2 + \frac{1}{2} D_1 \phi D_1 \phi - \frac{1}{2} D_2 \phi D_2 \phi - V ,$$

$$T_{22} = \frac{1}{2} B^2 - \frac{1}{2} D_1 \phi D_1 \phi + \frac{1}{2} D_2 \phi D_2 \phi - V ,$$

$$T_{12} = \frac{1}{2} D_1 \phi D_2 \phi + \frac{1}{2} D_2 \phi D_1 \phi .$$
For any vortex solution, the integral of each component of \( T_{ij} \) over \( \mathbb{R}^2 \) vanishes. Working with \( T_{11} + T_{22}, T_{11} - T_{22} \) and \( T_{12} \), one obtains the identities

\[
\int (B^2 - 2V) \, d^2 x = 0, \tag{4.8}
\]
\[
\int (\nabla_1 \phi \nabla_1 \phi - \nabla_2 \phi \nabla_2 \phi) \, d^2 x = 0, \tag{4.9}
\]
\[
\frac{1}{2} \int (\nabla_1 \phi \nabla_2 \phi + \nabla_2 \phi \nabla_1 \phi) \, d^2 x = 0. \tag{4.10}
\]

The first of these is Derrick’s theorem, and the remaining two are perhaps less familiar identities associated with squeezing and stretching a vortex in orthogonal directions.

In three dimensions, a non-abelian gauge theory with scalar Higgs fields can have monopoles as soliton solutions. We consider here the standard, simplest example of such a theory. The gauge group is \( SU(2) \) and the gauge potential \( A_i \) is coupled to an adjoint scalar field \( \Phi \), with no Higgs potential term. The energy is

\[
E = \int \left\{ -\frac{1}{8} \text{Tr} \left( F_{ik} F_{jk} \right) - \frac{1}{4} \text{Tr} \left( D_i \Phi D_i \Phi \right) \right\} \, d^3 x \tag{4.11}
\]

where \( F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \) and \( D_i \Phi = \partial_i \Phi + [A_i, \Phi] \). For details about the boundary conditions, field equations, and the interpretation of the soliton solutions as magnetic monopoles, see \[12, 15\].

The gauge invariant stress tensor is

\[
T_{ij} = -\frac{1}{2} \text{Tr} \left( F_{ik} F_{jk} + D_i \Phi D_j \Phi \right) + \delta_{ij} \left\{ \frac{1}{8} \text{Tr} \left( F_{lm} F_{lm} \right) + \frac{1}{4} \text{Tr} \left( D_l \Phi D_l \Phi \right) \right\}, \tag{4.12}
\]

which satisfies \( \partial_i T_{ij} = 0 \) as usual. The scaling identities obtained from the vanishing of the integrals of each component of the stress tensor are simplest if one works with the combinations \( T_{11} + T_{22}, T_{12} \), and related quantities obtained by permuting indices. One finds

\[
-\frac{1}{2} \int \text{Tr} \left( F_{12} F_{12} - D_3 \Phi D_3 \Phi \right) \, d^3 x = 0, \tag{4.13}
\]
\[
-\frac{1}{2} \int \text{Tr} \left( F_{13} F_{23} + D_1 \Phi D_2 \Phi \right) \, d^3 x = 0, \tag{4.14}
\]

plus permutations.

These identities are valid for any solutions of the field equations approaching the vacuum at infinity. Such solutions are not just the stable monopole and multi-monopole solutions, but also the unstable monopole-antimonopole solutions \[18, 13, 8\], and the various, recently found solutions that consist of chains of monopoles, antimonopoles, and Higgs vortex rings \[9\]. Note that for all these solutions, \( F_{ij} \) and \( D_i \Phi \) decay as \( |x|^{-2} \) or better for large \( |x| \), so \( T_{ij} \) decays as \( |x|^{-4} \), fast enough to avoid surface corrections to the identities above.

The identities we have found are actually trivial for the stable monopole solutions of minimal energy, because these satisfy the Bogomolny equations \[5\]

\[
F_{ij} = \epsilon_{ijk} D_k \Phi \tag{4.15}
\]
(or \( F_{ij} = -\epsilon_{ijk}D_k\Phi \) for antimonopoles). As has been noted before [10], the stress tensor vanishes identically if the Bogomolny equations are satisfied. Non-trivial identities are obtained for the unstable solutions mentioned above. Non-trivial identities are also obtained for monopoles in modified field theories where, for example, there is a non-vanishing Higgs potential \( V(\text{Tr } \Phi^2) \) as part of the energy density, and hence no Bogomolny equations.

Let us return for a moment to scalar electrodynamics in two dimensions. Here the vortices satisfy the Bogomolny equations

\[
D_1 \phi + iD_2 \phi = 0 ,
\]

\[
B - \frac{1}{2}(1 - \phi\phi) = 0 ,
\]

if the potential \( V \) has the special form

\[
V(\phi\phi) = \frac{1}{8}(1 - \phi\phi)^2 .
\]

In this case also, the stress tensor vanishes at each point [10], and the scaling identities (4.8) – (4.10), obtained earlier, are all trivial.

Our final example is pure Yang–Mills theory in four space dimensions, with energy (euclidean action)

\[
E = -\frac{1}{8} \int \text{Tr} (F_{\mu\nu}F_{\mu\nu}) \, d^4x ,
\]

and stress tensor

\[
T_{\mu\nu} = -\frac{1}{2} \text{Tr} (F_{\mu\sigma}F_{\nu\sigma}) + \frac{1}{8} \delta_{\mu\nu} \text{Tr} (F_{\sigma\tau}F_{\sigma\tau}) .
\]

The stress tensor is traceless here. Scaling identities satisfied by finite energy solutions of the Yang–Mills field equations \( D_\mu F_{\mu\nu} = 0 \) are obtained by integrating \( T_{11} + T_{22}, T_{12}, \text{ etc.} \). They are

\[
-\frac{1}{2} \int \text{Tr} (F_{12}F_{12} - F_{34}F_{34}) = 0 ,
\]

\[
-\frac{1}{2} \int \text{Tr} (F_{13}F_{23} + F_{14}F_{24}) = 0 .
\]

and the similar identities obtained by permuting indices.

These identities rely, as usual, on there being no contribution from surface terms. Now, since the Yang–Mills equations are conformally invariant, many solutions on \( \mathbb{R}^4 \), including all instantons, arise from smooth solutions on \( S^4 \) through stereographic projection (conformal decompactification). For these, the field tensor in \( \mathbb{R}^4 \) decays as \( |x|^{-4} \), and the energy density and stress tensor decay as \( |x|^{-8} \), so surface terms vanish.

Instantons, which satisfy the (anti-)self-dual equations [4]

\[
F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\sigma\tau}F_{\sigma\tau} ,
\]

in fact satisfy the identities above trivially, because the stress tensor vanishes everywhere. However, there are non-trivial solutions of the Yang–Mills equations which are not instantons [16, 14], and for these, the identities are non-trivial.
5 Conclusion

We have established a set of scaling identities satisfied by static, localized solutions of field theories in various dimensions. They are proved either by considering variations of the energy function, or by considering the conserved stress tensor. A particular combination of these identities is Derrick’s theorem. We have presented our results using a number of examples of field theories known to have soliton solutions, and it would be straightforward to generalize them to other field theories.

One possible application of our results is to the improvement of approximate soliton solutions by suitable squeezing and stretching transformations. We have not found an algorithm that takes a generic field configuration, and in one step transforms it to a field configuration that satisfies all the scaling identities. However, we have suggested an iterative approach that reduces the energy at each step, and could be useful if applied to some known, approximate Skyrmion solutions.

The identities become trivial for solutions satisfying first order Bogomolny equations or the (anti-)self-dual Yang–Mills equations.

Acknowledgements

I am grateful to Mihalis Dafermos and Gary Gibbons for discussions about the stress tensor, and for drawing my attention to some of the relevant literature.

References

[1] R. A. Battye, N. S. Manton and P. M. Sutcliffe, Skyrmions and the α-particle model of nuclei, Proc. Roy. Soc. A463, 261 (2006).

[2] R. A. Battye and P. M. Sutcliffe, Skyrmions, fullerenes and rational maps, Rev. Math. Phys. 14, 29 (2002).

[3] R. A. Battye and P. M. Sutcliffe, Skyrmions with massive pions, Phys. Rev. C73, 055205 (2006).

[4] A. A. Belavin, A. M. Polyakov, A. S. Schwarz and Yu. S. Tyupkin, Pseudoparticle solutions of the Yang–Mills equations, Phys. Lett. B59, 85 (1975).

[5] E. B. Bogomolny, The stability of classical solutions, Sov. J. Nucl. Phys. 24, 449 (1976).

[6] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, J. Math. Phys. 5, 1252 (1964).

[7] C. J. Houghton, N. S. Manton and P. M. Sutcliffe, Rational maps, monopoles and Skyrmions, Nucl. Phys. B510, 507 (1998).

[8] B. Kleihaus and J. Kunz, Monopole-antimonopole solution of the SU(2) Yang–Mills–Higgs model, Phys. Rev. D61, 025003 (2000).
[9] B. Kleihaus, J. Kunz and Ya. Shnir, Monopole-antimonopole chains and vortex rings, Phys. Rev. D70, 065010 (2004).

[10] M. A. Lohe, Two- and three-dimensional instantons, Phys. Lett. B70, 325 (1977).

[11] N. S. Manton, Skyrmions and their pion multipole moments, Acta Phys. Pol. B25, 1757 (1994).

[12] N. Manton and P. Sutcliffe, Topological Solitons, Cambridge University Press, 2004.

[13] B. Rüber, Eine axialsymmetrische magnetische Dipollösung der Yang–Mills–Higgs-Gleichungen, Diplomarbeit, Universität Bonn, 1985.

[14] L. Sadun and J. Segert, Non-self-dual Yang–Mills connections with non-zero Chern number, Bull. Am. Math. Soc. 24, 163 (1991); Stationary points of the Yang–Mills action, Commun. Pure Appl. Math. 45, 461 (1992).

[15] Ya. M. Shnir, Magnetic Monopoles, Springer, Berlin Heidelberg, 2005.

[16] L. M. Sibner, R. J. Sibner and K. Uhlenbeck, Solutions to Yang–Mills equations that are not self-dual, Proc. Natl. Acad. Sci. USA 86, 8610 (1989).

[17] T. H. R. Skyrme, A non-linear field theory, Proc. Roy. Soc. A260, 127 (1961).

[18] C. H. Taubes, The existence of a non-minimal solution to the SU(2) Yang–Mills–Higgs equations on $\mathbb{R}^3$: Parts I and II, Commun. Math. Phys. 86, 257 (1982); ibid 86, 299 (1982).