Study of the non-linear dynamics of micro-resonators based on a Sn-whisker in vacuum and at mK temperatures

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Abstract The dynamics of micro-resonators (or any mechanical resonators) can be studied by two complementary methods allowing the measurements in two different domains: (i) in the frequency domain - by the frequency sweeps using cw-excitation, and (ii) in the time domain - by the pulse techniques, when the free-decay oscillations are investigated. To broaden the knowledge about the intrinsic mechanical properties of micro-resonators based on a Sn-whisker we used both methods. We show that the dynamics of the Sn-whisker can be described by a phenomenological theory of the Duffing oscillator. Furthermore, we present the results of theoretical analysis based on the Duffing’s model provided in the time and frequency domains, and we show that these results are qualitatively the same with those measured experimentally.

Keywords micro-resonators · tin whiskers · pulse-demodulation technique · Duffing oscillator

1 Introduction

Following the significant technological advancements in a semiconductor device fabrication, as a direct consequence of a very popular trend to downscale integrated electronic circuits, the physical size of mechanical resonators was successfully reduced to micro- and nanometer range [1]. The motion of such resonators is usually measured by a coupled electronic circuit, thus composing a micro-electromechanical (MEMS) or nano-electromechanical systems (NEMS). Due to their high sensitivity and simple implementation, these devices have found many commercial applications (e.g. mass sensing [2], frequency standards [3], analogue frequency filters [4]).
When the resonator’s dimensions are reduced, the energy related to its surface states grows and becomes comparable with the energy stored in its inner volume. Simultaneously, the resonator’s surface breaks several symmetries due to which nonlinear effects are gained and emphasised. These effects change the resonator’s dynamics and affect the intrinsic processes of the energy dissipation and exchange between the resonator and surrounding thermal bath, forming new mechanisms and channels of the energy transfer [5,6]. To investigate these processes, we decided to construct a novel type of micro-resonator based on readily available Sn-whiskers [7]. These metallic fibres with diameter up to 1-2 µm and ~1 mm in length grow usually on the stressed surfaces of tin alloys [8,9,10]. Among the principal advantages of Sn-whisker belong that it is a single crystal, characterised by a relatively smooth surface, [0 0 1] as a preferred growth direction [11], high tensile strength (720 - 880 MPa vs 220 MPa in a bulk tin) and Young’s modulus close to the bulk value [12]. Moreover, it becomes superconducting at temperature $T_C = 3.72 \text{ K}$. As our measurements were performed in vacuum, at temperatures $\sim 10 \text{ mK}$ and magnetic field $B = 20 \text{ mT}$, one can expect that our resonator will have a reasonably high Q-factor.

2 Experimental details

In order to construct the micro-resonator based on Sn-whisker, a relatively simple whisker holder needs to be manufactured [13]. Firstly, two parallel copper wires of 100 µm in diameter are glued $\sim 0.5 \text{ mm}$ apart on a graph paper impregnated by the Stycast 1266 epoxy resin. The same substance is then used to attach the whole structure to a small piece of thin copper sheet with a surface insulated by a cigarette paper to ensure a good thermal and electrically non-conductive contact between both parts. The respective pairs of the copper wires are twisted and in order to protect them, a sleeve made of nylon fishing line is applied. Finally, the Sn-whisker can be positioned carefully on the finished whisker holder and secured on its place by a conductive silver epoxy resin (fig. 1). The whisker holder has a convenient mounting hole, so it can be mounted easily on the silver experimental plate. Placed in the middle...
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Fig. 2 The measurement scheme used for the traditional technique of frequency sweeps with continuous voltage excitation (left) and for the pulse-demodulation technique (right).

The non-linear dynamics of Sn-whisker $\mu$-resonators in vacuum and at mK temp. are studied. The measurement scheme includes a frequency generator, a lock-in amplifier, and a phase-sensitive (Lock-in) amplifier. A precise function generator serves as a current source and measures the response of Sn-whisker based micro-resonator in the form of induced voltage $U_i \sim Blv$.

The experimental set-up allows measurements in magnetic fields up to 8 T and at temperatures down to 10 mK, using a strong superconducting magnet thermally connected to the mixing chamber of the commercially available cryogen free dilution refrigerator.

The physical properties and dynamics of micro- and nano-resonators are usually studied in the frequency domain by the traditional technique of frequency sweeps with continuous voltage/current excitation (fig. 2 left). A precise function generator serves as a current source and the response of Sn-whisker based micro-resonator is measured by a phase-sensitive (Lock-in) amplifier. The reference signal is provided by the same function generator, so both response components can be determined. As we are dealing with the forced oscillations driven by excitation at a given frequency, the dynamics of micro-resonator based on Sn-whisker is studied in the frequency domain.

There is a complementary measurement method represented by a pulse technique when the free damped oscillations are examined in the time domain. If the corresponding FFT spectrum is calculated, the dynamics of micro- and nano-resonators can be transformed to the respective frequency domain and results can be compared to the frequency sweeps measurements. We have utilised a novel type of the pulse-demodulation technique (fig. 2 right) implemented successfully for quartz tuning forks [14]. In contrast to the frequency sweeps technique, there is an additional precise function generator providing the reference signal at frequency $f_{\text{ref}}$ for the Lock-in amplifier. Moreover, a fast DMM is connected to the output of Lock-in amplifier and used for data acquisition and storage. At first, the excitation pulse is applied at frequency $f_{\text{exc}}$ for $N$ periods. At the end of excitation pulse, the fast DMM is triggered and the signal of free damped oscillations (originally at $f_{\text{sig}}$) is measured at differential frequency $|f_{\text{ref}} - f_{\text{sig}}|$.

3 Experimental results and discussion

As it was shown during the previous frequency sweeps measurements [13], the response of Sn-whisker based micro-resonator becomes non-linear even at moderate current excitations. The resonant curve is tilted towards the lower frequencies and as the current excitation increases this tendency becomes more prominent. Our new
experimental results confirm that when a critical excitation is exceeded, the direction of frequency sweep starts to play an important role and as a result, the effect of hysteresis is observed (fig. 3). Clearly, there is a frequency region where the resonator can oscillate either with high or low velocity amplitude depending on the sweep’s history.

To deepen our understanding about the intrinsic mechanical properties of micro-resonator based on Sn-whisker, we studied its response in the time domain by the means of pulse-demodulation technique. The frequency of excitation pulse $f_{\text{exc}} = 16 240$ Hz was chosen to be near the resonant frequency $f_0$. There are two parameters of excitation pulse which can be adjusted in order to obtain a reasonable signal-to-noise ratio: the number of periods $N$ and the pulse amplitude $I_{\text{exc}}$. It is worth to note that due to a low signal-to-noise ratio the amplitude of excitation pulse was significantly larger ($\sim 20 \times$) than during the frequency sweeps measurements. Similarly, the number of periods was set to $N = 16 000$. The resulting signal of free damped oscillations is shown in a fig. 4 left. To improve the corresponding FFT spectrum, the pulse measurement for given parameters $f_{\text{exc}}, I_{\text{exc}}$ and $N$ was repeated 20 times, individual FFT spectra were determined and, finally, the average FFT spectrum was calculated (fig. 4 right). As the signal of free damped oscillations is being measured at differential frequency $|f_{\text{ref}} - f_{\text{sig}}|$ the whole FFT spectrum is transposed to the lower frequency range. Moreover, the FFT spectrum is not symmetrical and higher frequencies are more pronounced.

The response of Sn-whisker based micro-resonator during the frequency sweeps measurements resembles the behaviour of Duffing oscillator, especially at higher current excitations. In general, the motion of forced Duffing oscillator can be described by a following ordinary differential equation

$$\ddot{x} + \delta \dot{x} + \omega_0^2 x + \alpha x^3 = \gamma \cos(\omega t), \quad (1)$$

where $\delta$ represents damping, $\omega_0$ is the resonant angular frequency, $\alpha$ determines the resulting non-linearity of the restoring force and $\gamma$ is the amplitude of a harmonic
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Fig. 4 A signal of free damped oscillations measured for the following excitation pulse parameters $f_{\text{exc}} = 16\,240$ Hz, $I_{\text{exc}}$ and $N = 16\,000$ (left) and corresponding FFT spectrum averaged from 20 pulse measurements (right). The solid line represents the FFT spectrum calculated using the model of Duffing oscillator with the parameters $\alpha$, $\omega_0$, and $\delta$ determined from the frequency sweep measurements.

driving force normalized per unit mass and acting at the angular frequency $\omega$. This equation is not analytically solvable; however, there are many mathematical methods available which allow to find an approximate steady-state solution. Using the homotopy analysis [15] or harmonic balance method [16], it is possible to derive the expressions for displacement amplitude $r$ and phase $\phi$

$$r^2 = \frac{\gamma^2}{(\omega^2 - \omega_0^2 - \frac{3}{4}\alpha r^2)^2 + \omega^2 \delta^2}, \quad (2)$$

$$\sin \phi = -\frac{\omega \delta r}{\gamma}. \quad (3)$$

As a side note, there is a special case $\alpha = 0$, when the frequency response of Duffing oscillator has the same shape of Lorentzian function as for the linear harmonic oscillator. The same is true for the case of small driving forces $\gamma \ll 1$ when $\frac{3}{4}\alpha r^2 \to 0$, i.e. in the linear regime, then the quality factor $Q$ of our micro-resonator can be determined as well. The result $Q = \frac{\omega_0}{\delta} \sim 5500$ is in the excellent agreement with our previous measurements [13].

By analysing the equation (2) for the displacement amplitude $r$, it is possible to determine the angular frequency $\omega_{\text{max}}$ when the response of Duffing oscillator is maximal

$$\omega_{\text{max}}^2 = \omega_0^2 + \frac{3}{4}\alpha r^2. \quad (4)$$

As one can see, the response of Duffing oscillator at the higher amplitudes of displacement $r$ is no longer linear. The resonance curve is not symmetrical any more and depending on the sign of $\alpha$, there is an effect of softening (for $\alpha < 0$) or hardening (for $\alpha > 0$) observed. Moreover, at even larger displacement amplitudes, the response of oscillator will depend on the direction of frequency sweep itself and is characterised by a hysteresis behaviour with distinctive abrupt changes in the amplitude of oscillations.

In principle, the expression (2) is a cubic equation for $r^2(\omega)$ and can be solved numerically for the given parameters $\alpha$, $\omega_0$, $\gamma$ and $\delta$. Once a computational algorithm
for the calculation of $r(\omega)$ is available, the relation (3) can be used to determine $\phi(\omega)$. Moreover, by implementing the well-known Levenberg-Marquardt method [17, 18, 19], it is possible to construct a software (numerical) fitting function for the experimentally measured $v(\omega)$ dependencies. In our case, the coefficients $\alpha$, $\omega_0$, $\gamma$ and $\delta$ serve as the initial guess parameters for our fitting function. In addition, the experimental data from the up and down frequency sweeps are being processed at the same time and, finally, the best fit parameters are determined. The computed results (represented by the solid lines in fig. 3) confirm a definite agreement between the measured experimental data and calculated fit.

Now, we apply the model of Duffing oscillator for the theoretical description of the free damped oscillations of Sn-whisker based micro-resonator, with parameters $\alpha$, $\omega_0$ and $\delta$ determined from the frequency sweeps measurements. In this case, the right-hand side of equation (1) is equal to zero (no driving force is applied). In order to simplify the necessary numerical calculations, the fourth-order Runge-Kutta method was chosen with the initial conditions selected as $x_t=0$ and $v_t=0$. The resulting signal was calculated up to 0.5 s time interval and the corresponding FFT spectrum was determined and then transposed to the lower frequency range by subtracting the reference frequency $f_{ref}$. For initial velocity $v_0 = 15 \text{ mm/s}$, there is a reasonable qualitative agreement between measured and simulated FFT spectra (see fig. 4 right, the solid line represents the simulated FFT spectrum). Knowing the initial induced voltage amplitude $U_i \sim 600 \text{ nV}$ (see fig. 4 left), it is possible to estimate $v_0$ as well. For the whisker of $\sim 1 \text{ mm}$ length placed in the magnetic field of 20 mT, the resulting order of magnitude estimate for the initial velocity is $v_0 \sim 30 \text{ mm/s}$. This agrees quite well with the simulated value of 15 mm/s.

4 Conclusions

To conclude, we studied the dynamics of micro-resonator based on Sn-whisker by measuring its response in both (frequency and time) domains. Besides the traditional measurement method - the frequency sweeps using continuous excitation, the modified pulse-demodulation technique was proposed and implemented. Moreover, the phenomenological theory of Duffing oscillator applied to the micro-resonator’s response in the frequency domain resulted in the excellent qualitative agreement between the experimental data and corresponding fits. Using the known values of fitting parameters $\alpha$, $\omega_0$ and $\delta$, obtained from the implemented Duffing’s theory, the free damped oscillations were calculated and the corresponding FFT spectra were compared to the averaged FFT spectra determined from the resonator’s response in the time domain. As it was shown, there is a qualitative agreement between them. Thus, we have proved a one-to-one qualitative correspondence between measurements in time and frequency domains.

Acknowledgements We would like to thankfully acknowledge the support by grants APVV 14-0605, VEGA 2/0157/15 and European Microkelvin Platform. The financial support provided by the U. S. Steel Košice s.r.o. is also gratefully recognised and highly appreciated.
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