Abstract: Fluctuations of the number of particles for the dilute interacting gas with Bose-Einstein condensate are considered. It is shown that in the Bogolubov theory these fluctuations are normal. The fluctuations of condensed as well as noncondensed particles are also normal both in canonical and grand canonical ensembles.

Number-of-particle fluctuations in systems with Bose-Einstein condensate

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1. Introduction

Fluctuations in systems with Bose-Einstein condensate have long been a topic of special attention, as can be inferred from reviews [1–3]. This problem plays an important role in the general context of quantum statistical mechanics. Interest in this problem has quickened in the past years due to intensive experimental and theoretical investigations of Bose-Einstein condensed atomic gases (see reviews [4–8]).

The number-of-particle fluctuations in a uniform ideal Bose gas with Bose-Einstein condensate are known [1,2] to be anomalous, having the dispersion \( \Delta^2(\hat{N}) \sim N^2 \), with the power of \( N \) larger than one. This anomalous behaviour comes from the fluctuations of the condensate, the corresponding dispersion being \( \Delta^2(\hat{N}_0) \sim N^2 \). The fluctuations of the noncondensed particles are also anomalous, with the dispersion \( \Delta^2(\hat{N}_1) \sim N^{4/3} \). Such anomalous fluctuations manifest the instability of the Bose-condensed ideal gas [3].

The problem of fluctuations in interacting Bose gases has been the arena of numerous discussions, with controversial conclusions (see review [3]). In the present paper, this problem is analysed in the frame of the Bogolubov theory [9–11] and it is shown that no anomalous fluctuations arise, but all fluctuations are normal.

2. Number-of-particle fluctuations

The measure of fluctuations for the number-of-particle operator \( \hat{N} \) is provided by the dispersion

\[
\Delta^2(\hat{N}) \equiv \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2 ,
\]

where the angle brackets mean, as usual, statistical averaging. Dispersion (1) can be considered as an observable quantity, since it is directly connected with the isothermal compressibility, sound velocity, and structural factor, which are measurable quantities. In order to stress that the linkage between dispersion (1) and the observable quantities is general and exact, it is worth demonstrating how these relations can be rigorously derived.

For an equilibrium system of \( N \) particles in volume \( V \), with temperature \( T \), it is easy to check that

\[
\Delta^2(\hat{N}) = k_B T \left( \frac{\partial N}{\partial \mu} \right)_{TV} ,
\]

where \( k_B \) is the Boltzmann constant, \( N \equiv \langle \hat{N} \rangle \), and \( \mu \) is chemical potential. The Gibbs potential \( G = G(T, P, N) \) is a function of temperature \( T \), pressure \( P \), and the number of particles \( N \). From the differential

\[
dG = -SdT + VdP + \mu dN ,
\]

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The definition of the isothermal compressibility

\[ \kappa_T \equiv -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial P} \right)_T, \]

owing to the above relations, can be rewritten as

\[ \kappa_T = \frac{1}{N} \left( \frac{\partial N}{\partial P} \right)_T = \frac{1}{N \rho} \left( \frac{\partial N}{\partial \mu} \right)_T. \]

Comparing Eqs. (4) with (2) yields

\[ \kappa_T = \frac{\Delta^2(\hat{N})}{N \rho k_B T}. \]

In turn, the isothermal sound velocity \( s \) can be expressed through the compressibility as

\[ s^2 \equiv \frac{1}{m} \left( \frac{\partial P}{\partial \rho} \right)_T = \frac{1}{m \rho \kappa_T}, \]

where \( m \) is particle mass.

On the other hand, representing the operator \( \hat{N} \equiv \int \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r} \) through the field operators \( \psi^\dagger(\mathbf{r}) \) and \( \psi(\mathbf{r}) \), from definition (1) it follows

\[ \Delta^2(\hat{N}) = N + \int \rho(\mathbf{r}) \rho(\mathbf{r}' ) [\hat{g}(\mathbf{r}, \mathbf{r}')] - 1 d\mathbf{r} d\mathbf{r}', \]

which should be compared with the central structure factor

\[ S(0) = 1 + \frac{1}{N} \int \rho(\mathbf{r}) \rho(\mathbf{r}' ) [\hat{g}(\mathbf{r}, \mathbf{r}')] - 1 d\mathbf{r} d\mathbf{r}', \]

where \( \rho(\mathbf{r}) \equiv \langle \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \rangle \), and the pair correlation function is

\[ g(\mathbf{r}, \mathbf{r}') \equiv \frac{\langle \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) \rangle}{\rho(\mathbf{r}) \rho(\mathbf{r}')}. \]

In this way, there exist exact relations

\[ \Delta^2(\hat{N}) = N k_B T / m \hbar^2 = N S(0), \]

valid for any equilibrium system, whether it is uniform or not. A stable equilibrium system requires that its compressibility be positive and finite, \( 0 < \kappa_T < \infty \). An infinite compressibility would mean that the system immediately collapses or explodes. In a stable system, according to relations (10), the sound velocity and the central structure factor are also finite and positive. Thus, relations (10) tell us that in a stable equilibrium system the dispersion \( \Delta^2(\hat{N}) \) must satisfy the stability condition

\[ 0 < \frac{\Delta^2(\hat{N})}{N} < \infty \quad (0 < \kappa_T < \infty) \]

for any \( N > 0 \), including the thermodynamic limit \( N \to \infty \). Fluctuations satisfying the stability condition (11) are termed normal, while if condition (11) is not valid, fluctuations are called anomalous. Clearly, anomalous fluctuations imply instability. For instance, the ideal uniform Bose gas is unstable, since it has anomalous fluctuations with \( \Delta^2(\hat{N}) \sim N^2 \) and a divergent compressibility \( \kappa_T \sim N \). Hydrodynamic equations for this gas with condensate are plagued by the appearance of unbound solutions [12].

Now let us pass to systems experiencing Bose-Einstein condensation. At the present time, a variety of trapped atomic gases is known to demonstrate this phenomenon (see recent review [3]). Several types of molecules, produced by means of Feshbach resonances, have been condensed. Bose-Einstein condensation might also appear in a system of semiconductor biexcitons, consisting of two electrons plus two holes [13,14].

In the presence of condensate, the operator of the total number of particles is written as a sum

\[ \hat{N} = \hat{N}_0 + \hat{N}_1, \]

whose terms correspond to condensed \( (\hat{N}_0) \) and noncondensed \( (\hat{N}_1) \) particles. Then dispersion (1) takes the form

\[ \Delta^2(\hat{N}) = \Delta^2(\hat{N}_0) + \Delta^2(\hat{N}_1) + 2 \text{cov}(\hat{N}_0, \hat{N}_1), \]
in which the last term contains the covariance
\[
\text{cov}(\hat{N}_0, \hat{N}_1) \equiv \frac{1}{2} \left\langle \hat{N}_0 \hat{N}_1 + \hat{N}_1 \hat{N}_0 \right\rangle - \left\langle \hat{N}_0 \right\rangle \left\langle \hat{N}_1 \right\rangle .
\]  
(14)

Since \( \hat{N}_0 \) and \( \hat{N}_1 \) are usually defined as commuting operators, covariance (14) reduces to
\[
\text{cov}(\hat{N}_0, \hat{N}_1) = \langle \hat{N}_0 \hat{N}_1 \rangle - \langle \hat{N}_0 \rangle \langle \hat{N}_1 \rangle .
\]  
(15)

By the stability condition (11), we know that \( \Delta^2(\hat{N}) \sim N \). Then the fundamental question is: Could the fluctuations of either condensed or noncondensed particles, or both, be anomalous, at the same time preserving the validity of the exact Eq. (13)? Recently there have appeared a number of papers stating that such fluctuations could be anomalous (see discussion in review [3]). In the following section we consider an interacting homogeneous system with Bose-Einstein condensate at low temperature and density, when the Bogolubov theory is applicable, and show that all fluctuations are normal.

3. Dilute gas

Current experiments with Bose-condensed atomic gases are typified by a rather low density, such that \( \rho_0 a_s^3 \ll 1 \), where \( a_s \) is a scattering length. Atomic gases can be cooled down to very low temperatures, when practically all atoms are condensed, so that \( |\hat{N}_0 - N| \ll N \). Under these conditions, the Bogolubov theory [9–11] becomes applicable.

Let us start with the standard Hamiltonian for a uniform system of spinless atoms,
\[
H = \int \hat{\psi}(\mathbf{r}) \left( \frac{\hbar^2}{2m} \nabla^2 - \mu \right) \hat{\psi}(\mathbf{r}) d\mathbf{r} + \frac{1}{2} \int \hat{\psi}(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) (\Phi(\mathbf{r}) - \Phi^*(\mathbf{r})) \hat{\psi}(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) d\mathbf{r} d\mathbf{r} ,
\]  
(16)

with an interaction potential \( \Phi(\mathbf{r}) = \Phi(-\mathbf{r}) \). For dilute gas, the latter is usually modelled by the contact interaction \( \Phi(\mathbf{r}) = (4\pi\hbar^2 a_s/\mu) \delta(\mathbf{r}) \). However, for the sake of generality, we keep the form \( \Phi(\mathbf{r}) \). The chemical potential \( \mu \) is included here in order to compare the results, corresponding to the grand canonical ensemble, with the original Bogolubov consideration [9–11] made in the frame of the canonical ensemble.

Following the Bogolubov prescription let us separate the condensate by the shift in the field operator,
\[
\hat{\psi}(\mathbf{r}) = \psi_0(\mathbf{r}) + \psi_1(\mathbf{r}),
\]  
(17)

where the condensate operator \( \psi_0 \) does not depend on \( \mathbf{r} \) owing to the system uniformity. The field operators of noncondensed particles, \( \psi_1(\mathbf{r}) \), are assumed to possess the same Bose commutation relations as \( \psi(\mathbf{r}) \). From it follows that the operator of condensed particles, \( \psi_0 \), commutes with all operators in the thermodynamic limit, when \( N \to \infty \). The operators \( \psi_0 \) and \( \psi_1 \) are assumed to be orthogonal,
\[
\int \psi_0^\dagger(\mathbf{r}) \psi_1(\mathbf{r}) d\mathbf{r} = 0 .
\]

Hence, \( \hat{N} = \hat{N}_0 + \hat{N}_1 \) as in Eq. (12), with
\[
\hat{N}_0 \equiv \int \psi_0^\dagger(\mathbf{r}) \psi_0(\mathbf{r}) d\mathbf{r} , \quad \hat{N}_1 \equiv \int \psi_1^\dagger(\mathbf{r}) \psi_1(\mathbf{r}) d\mathbf{r} .
\]

For a uniform system, the field operators can be expanded in Fourier series
\[
\psi_0 = \frac{a_0}{\sqrt{V}} , \quad \psi_1(\mathbf{r}) = \sum_{k \neq 0} a_k \varphi_k(\mathbf{r})
\]
over the plane waves \( \varphi_k(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V} \). Assuming that the interaction potential can also be Fourier expanded, we have
\[
\Phi(\mathbf{r}) = \frac{1}{V} \sum_k \Phi_k e^{i\mathbf{k}\cdot\mathbf{r}} , \quad \Phi_k = \int \Phi(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} .
\]

For the number-of-particle operators, we get
\[
\hat{N}_0 = a_0^\dagger a_0 , \quad \hat{N}_1 = \sum_{k \neq 0} a_k^\dagger a_k .
\]

Hamiltonian (16) is gauge invariant, hence the field operators are defined up to a global phase factor. Then one may chose a representation, where \( a_0 \) is self-adjoint, such that \( a_0^\dagger a_0 = \hat{N}_0 \) and \( a_0^\dagger a_0 = \hat{N}_0 \). Actually, this choice is absolutely not principal and is accepted with the sole aim to simplify the following formulas.

After the Fourier transformation, Hamiltonian (16) acquires the form
\[
H = \sum_{n=0}^{4} H^{(n)} ,
\]  
(18)
in which the terms are grouped according to the number of noncondensed-particle operators. In the zeroth order,
\[
H^{(0)} = \frac{\hat{N}_0^2}{2V} \Phi_0 - \mu \hat{N}_0 ,
\]
where \( \Phi_0 \equiv \int \Phi(\mathbf{r}) d\mathbf{r} \). The first-order term vanishes, \( H^{(1)} = 0 \). In the second order,
\[
H^{(2)} = \sum_{k \neq 0} \left( \frac{\hbar^2 k^2}{2m} + \frac{\hat{N}_0}{V} \Phi_0 - \mu \right) a_k^\dagger a_k + \frac{\hat{N}_0}{2V} \sum_k \Phi_0 \left( 2a_k^\dagger a_k + a_k^\dagger a_k^\dagger + a_{-k}^\dagger a_{-k} + a_{-k} a_k \right) ,
\]
where it is taken into account that \( \Phi_{-k} = \Phi_k \) because of the symmetry of \( \Phi(\mathbf{r}) \). The third-order term is
\[
H^{(3)} = \frac{1}{V} \sum_{k \neq q \neq 0} \Phi_q \left( a_k^\dagger a_q a_k^\dagger a_q^\dagger + a_q^\dagger a_{-q}^\dagger a_{-q} a_k \right) .
\]
Finally, the fourth-order term is written as
\[ H^{(4)} = \frac{1}{2V} \sum_{k \neq q \neq 0} \Phi_q a_k \dagger a_p a_{p+q} a_{k-q} . \]

Remembering that we consider an almost condensed system, where \( N_0 \approx N \), the terms \( H^{(3)} \) and \( H^{(4)} \) can be treated as small perturbations to the terms up to the second order. Limiting ourselves by the latter terms, we have
\[ H = \frac{\hat{N}_0^2}{2V} \Phi_0 - \mu \hat{N}_0 + \left( \frac{\hbar^2 k^2}{2m} + \frac{\hat{N}_0}{V} \Phi_0 - \mu \right) a_k \dagger a_k + \frac{\hat{N}_0}{2V} \sum_{k \neq 0} \Phi_k \left( 2a_k \dagger a_k + a_k \dagger a_{-k} + a_{-k} a_k \right) . \]

The contraction of the total fourth-order Hamiltonian (18) to its second-order part (19) is the first Bogolubov approximation.

The next approximation is the replacement in Eq. (19) of the operator \( \hat{N}_0 \) by its average value \( N_0 \), which gives
\[ H = \frac{N_0^2}{2V} \Phi_0 - \mu N_0 + \sum_{k \neq 0} \omega_k a_k \dagger a_k + \frac{N_0}{2V} \sum_{k \neq 0} \Phi_k \left( 2a_k \dagger a_k + a_k \dagger a_{-k} + a_{-k} a_k \right) . \]

where the notation
\[ \omega_k \equiv \frac{\hbar^2 k^2}{2m} + \frac{N_0}{V} \Phi_0 - \mu \]

is used.

The Heisenberg equation of motion for \( \psi_0 \) is equivalent to the equation
\[ \frac{\delta H}{\delta N_0} = 0 , \]
which yields
\[ \mu = \frac{N_0 + N_1}{V} \Phi_0 + \frac{1}{2V} \sum_{k \neq 0} \Phi_k \left( 2a_k \dagger a_k + a_k \dagger a_{-k} + a_{-k} a_k \right) . \]

To remain in the frame of the second-order approximation in Hamiltonian (20), one has to use in Eq. (21) the zero-order part of the chemical potential (22), setting there
\[ \mu = \frac{N_0}{V} \Phi_0 . \]

Another approximation is again based on the fact that \( N_0 \approx N \). Therefore one can replace in Eq. (20) \( N_0 \) by \( N \). This gives
\[ H = \frac{N^2}{2V} \Phi_0 + \sum_{k \neq 0} \omega_k a_k \dagger a_k - \mu N + \frac{N}{2V} \sum_{k \neq 0} \Phi_k \left( 2a_k \dagger a_k + a_k \dagger a_{-k} + a_{-k} a_k \right) . \]

Hamiltonian (24) is diagonalized by means of the Bogolubov canonical transformation
\[ a_k = u_k b_k + v_{-k} b_{-k} , \]
resulting in the diagonal form
\[ H = E_0 + \sum_{k \neq 0} \varepsilon_k b_k \dagger b_k - \mu N , \]
in which the first term is the ground-state energy
\[ E_0 = \frac{1}{2} N \rho \Phi_0 + \frac{1}{2} \sum_{k \neq 0} \left( \varepsilon_k - \omega_k - \rho \Phi_k \right) , \]
and the Bogolubov spectrum is
\[ \varepsilon_k = \sqrt{2 \rho \Phi_k \omega_k + \omega_k^2} . \]
The latter, with the notation for the sound velocity
\[ c_k \equiv \sqrt{\left( \rho / m \right) \Phi_k} , \]
can be rewritten as
\[ \varepsilon_k = \sqrt{ \frac{2m c_k^2}{\omega_k} + \omega_k^2} . \]

Note that, due to Eqs. (21) and (23), the spectrum \( c_k \) is gapless. The coefficient functions of the Bogolubov transformation are given by the equations
\[ a_k = \frac{\sqrt{\varepsilon_k^2 + m^2 c_k^2 + \varepsilon_k}}{2c_k} , \quad \varepsilon_k = \frac{\sqrt{\varepsilon_k^2 + m^2 c_k^2}}{2c_k} - \varepsilon_k . \]

In the case of the contact interaction, one has \( \Phi_k = 4 \pi \hbar^2 a_k / m \) and \( c_k \equiv c = (\hbar / m) \sqrt{4 \pi \rho a_s} . \)

The diagonal form (25) has been derived here starting with the Hamiltonian (16) corresponding to the grand canonical ensemble. The same Hamiltonian (25), up to the term \( -\mu N \), was derived by Bogolubov [9–11] in the frame of the canonical ensemble. Therefore all following calculations are actually identical for the grand canonical as well as canonical ensembles.

A principal point must be stressed related to the transition from Eq. (19) to Eq. (20), when the operator \( \hat{N}_0 \) is replaced by its average value \( N_0 \). The equivalence of Hamiltonians (19) and (20) can be understood in two senses. In the strong sense, the equality of operators (19) and (20) requires that \( \hat{N}_0 = N_0 \). This, however, is not compulsory.
And the equivalence of operators (19) and (20) can be understood in the weak sense, implying the equality of all their matrix elements. The latter can be reformulated as the equality on the average, so that the average values \( \langle H \rangle \) for both forms (19) and (20) be coinciding. As is evident from expressions (19) and (20), their average values coincide then and only then, when the operator \( \hat{N}_0 \) is not correlated with the operators \( a_k^\dagger a_k \) and \( a_{-k} a_k \). This means, in particular, the validity of the decoupling

\[
\langle \hat{N}_0 \hat{N}_1 \rangle = \langle \hat{N}_0 \rangle \langle \hat{N}_1 \rangle.
\]

(26)

Taking account of Eq. (26), we notice that covariance (15) vanishes, \( \text{cov}(\hat{N}_0, \hat{N}_1) = 0 \). As a result, dispersion (13) becomes

\[
\Delta^2(\hat{N}) = \Delta^2(\hat{N}_0) + \Delta^2(\hat{N}_1).
\]

(27)

All terms here are non-negative. Hence, if at least one of the dispersions, either \( \Delta^2(\hat{N}_0) \) or \( \Delta^2(\hat{N}_1) \), is anomalous, then the left-hand side, \( \Delta^2(\hat{N}) \), is also anomalous. This, however, would contradict the stability condition (11). Thus we come to an indispensable conclusion: In a stable equilibrium system, the fluctuations of condensed as well as of noncondensed atoms must be normal.

If all fluctuations have to be normal, then how could one explain the appearance of anomalous fluctuations in a number of recent calculations (see review [3]) accomplished on the basis of the same Bogolubov approximations? To answer this question, let us recollect how such anomalous fluctuations arise. The standard origin of their appearance in calculations is as follows. One considers the dispersion \( \Delta^2(\hat{N}_1) \), containing the four-operator terms (\( a_k^\dagger a_k a_{-k}^\dagger a_{-k} \)) or (\( b_k^\dagger b_k b_{-k}^\dagger b_{-k} \)). These are decoupled as (\( b_k^\dagger b_k b_{-k}^\dagger b_{-k} \)) or (\( b_{-k}^\dagger b_{-k} b_k^\dagger b_k \)). Calculating \( \Delta^2(\hat{N}_1) \), one meets the integral \( \Delta^2(\hat{N}_2) \sim N \int dk/k^2 \). To avoid this the infrared divergence, one can replace the integral by a sum over the discretized spectrum of collective excitations [1,15]. This yields \( \Delta^2(\hat{N}_1) \sim N^{4/3} \). Another way could be by limiting the integration from below by \( k_{\text{min}} \approx 1/L \), with \( L \sim N^{1/3} \). Then again \( \Delta^2(\hat{N}_2) \sim N^{4/3} \) in either canonical or grand canonical ensemble. This means, according to equality (27), that the dispersion \( \Delta^2(\hat{N}) \sim N^{4/3} \) is anomalous, hence the system is unstable.

In order to stress that the same type of the anomalous dispersion \( \Delta^2(\hat{N}_1) \) arises in the grand canonical as well as in the canonical ensembles, let us consider \( \Delta^2(\hat{N}_1) \) in the grand canonical ensemble, when it can be represented as

\[
\Delta^2(\hat{N}_1) = k_B T \frac{\partial}{\partial \mu} \langle \hat{N}_1 \rangle - \text{cov}(\hat{N}_0, \hat{N}_1).
\]

Since in the Bogolubov approximation \( \text{cov}(\hat{N}_0, \hat{N}_1) = 0 \), one has

\[
\Delta^2(\hat{N}_1) = k_B T \frac{\partial}{\partial \mu} \langle \hat{N}_1 \rangle.
\]

Keeping in mind that \( V_{-k} = V_k \), the operator \( \hat{N}_1 \) can be written as

\[
\hat{N}_1 = \sum_{k \neq 0} \left( \left[ (u_k^2 + v_k^2) b_k^\dagger b_k + v_k^2 + u_k v_k \left( b_k^\dagger b_{-k} + b_{-k} b_k \right) \right] \right).
\]

From here one gets

\[
\frac{\langle \hat{N}_1 \rangle}{V} = \frac{\sqrt{2}}{2} \int \left[ \sqrt{\frac{\varepsilon_k^2 + m^2 c^4}{\varepsilon_k^2}} \coth \left( \frac{\varepsilon_k}{2 k_B T} \right) - 1 \right] \frac{dk}{(2\pi)^3}.
\]

For the case of the contact interaction, when \( c_k = c \), and involving the relation

\[
\omega_k = \sqrt{\frac{\varepsilon_k^2 + m^2 c^4}{\varepsilon_k^2}} - mc^2,
\]

we come to the form

\[
\frac{\langle \hat{N}_1 \rangle}{V} = \frac{\sqrt{2}}{2} \left( \frac{mc}{\hbar} \right)^3 \int_{x_0}^{\infty} \left[ \sqrt{1 + x^2} - 1 - \delta \right]^{1/2} \times \coth \left( \frac{mc}{2k_B T} x \right) - \frac{x}{\sqrt{1 + x^2}} dx,
\]

in which

\[
x_0 \equiv \sqrt{\delta(2 + \delta)}, \quad \delta \equiv \rho_0 \theta_0 - \mu \frac{mc}{\hbar}.
\]

At zero temperature, one has the known result

\[
\frac{\langle \hat{N}_1 \rangle}{N} = \frac{8}{3\sqrt{\pi}} \sqrt{\rho \alpha^2} \quad (T = 0).
\]

For any nonzero temperatures, taking the derivative \( \partial \langle \hat{N}_1 \rangle / \partial \mu \) and setting \( \mu = \rho_0 \theta_0 \), one recovers the same anomalous behaviour \( \Delta^2(\hat{N}_1) \sim N \int dx/x^2 \sim N^{4/3} \).

However this anomaly is spurious and arises owing to a not self-consistent calculational procedure. Really, the Bogolubov approach is based on the Hamiltonian (19) of the second-order with respect to noncondensed-particle operators \( a_k \). All higher-order terms have been omitted from the initial Hamiltonian (18). Hence, such higher-order terms should also be omitted from the calculations of any other physical quantities. Calculating the fourth-order expression \( \langle \hat{N}_2^2 \rangle \) with the second-order approximation is not self-consistent, hence, is not correct.

The total dispersion \( \Delta^2(\hat{N}) \) can be found from Eq. (7). In order to be self-consistent in defining the pair correlation function \( g(r - r') = g(r, r') \), one has to omit in the latter all terms of the order higher than two. Then one finds

\[
g(r) = 1 + \frac{2}{\rho} \int \left( \langle a_k^\dagger a_k \rangle + \langle a_k a_{-k} \rangle \right) e^{ik \cdot r} \frac{dk}{(2\pi)^3}.
\]

(28)

From here,

\[
\int [g(r) - 1] dr = \frac{2}{\rho} \lim_{k \to 0} \left( \langle a_k^\dagger a_k \rangle + \langle a_k a_{-k} \rangle \right).
\]
This gives for the structural factor (8)

\[ S(0) = \frac{k_B T}{mc^2} \left( \frac{\rho_0}{m} \right). \]  

(29)

And we find the dispersion (10) as

\[ \Delta^2(N) = N\frac{k_B T}{mc^2}, \]  

(30)

which is, of course, normal. If in the pair correlation function (28) we would retain the terms of the orders higher than the second with respect to \(a_k\), we would again get an anomalous dispersion. This, however, would not involve any sensible physics, but would simply mean the inconsistency of the calculations.

The dispersion \(\Delta^2(N_1)\), being a fourth-order expression with respect to \(a_k\), is not a well-defined quantity in the frame of the second-order Bogolubov theory. For its definition, it requires to invoke additional assumptions. Thus, we may follow the Bogolubov approximation, treating \(N_0 = N_0\) as a classical quantity, because of which \(\Delta^2(N_0) = 0\). Then in the grand canonical ensemble,

\[ \Delta^2(N_1) = \Delta^2(\hat{N}) = N\frac{k_B T}{mc^2} \quad (\text{grand}). \]

In the canonical ensemble, when \(\hat{N} = N\) is fixed, we get

\[ \Delta^2(N_0) = \Delta^2(\hat{N}) = N\frac{k_B T}{mc^2} \quad (\text{canonical}), \]

which results from the fact that \(\Delta^2(N_1)\) is the same in both ensembles.

Passing to the ideal gas, one should consider the limit \(\Phi(r) \to 0\), that is, \(c \to 0\). Then \(\Delta^2(N)\) diverges, as it should be for the ideal gas, which is unstable. The character of this divergence can be estimated as follows. The vanishing of \(\Phi(r)\) can be accepted as being analogous to its diminishing at large distance, where it has the standard Lennard-Jones behaviour \(\Phi(r) \sim 1/r^6\). In other words, \(\Phi(r) \sim 1/N^2\). Consequently, \(\Phi_0 \sim 1/N\); hence \(c \sim 1/\sqrt{N}\). Then dispersion (30) diverges as \(\Delta^2(N) \sim N^2\), which is typical of the ideal uniform Bose gas [1,2]. Contrary to the unstable Bose-condensed ideal gas, the interacting gas is stable, possessing always only normal fluctuations.

The conclusion on the absence of anomalous fluctuations can be generalized for any stable equilibrium systems by rigorously proving the following theorem. If an observable quantity is represented as a sum \(\hat{A} + \hat{B}\) of two linearly independent terms, then the total dispersion

\[ \Delta^2(\hat{A} + \hat{B}) = \Delta^2(\hat{A}) + \Delta^2(\hat{B}) + 2\text{cov}(\hat{A}, \hat{B}), \]

with the covariance

\[ \text{cov}(\hat{A}, \hat{B}) \equiv \frac{1}{2} \langle \hat{A}\hat{B} + \hat{B}\hat{A} - \langle \hat{A} \rangle \langle \hat{B} \rangle \rangle, \]

is normal then and only then, when both partial dispersions \(\Delta^2(\hat{A})\) and \(\Delta^2(\hat{B})\) are normal. The proof will be presented in a separate publication.

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