Scaling Limit and Renormalisation Group in General (Quantum) Many Body Theory

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Abstract

Using the machinery of smooth scaling and coarse-graining of observables, developed recently in the context of so-called fluctuation operators (originally developed by Verbeure et al), we extend this approach to a rigorous renormalisation group analysis of the critical regime. The approach is completely general, encompassing classical, quantum, discrete and continuous systems. Our central theme is the analysis of the famous ‘scaling hypothesis’, that is, we make a general investigation under what cluster conditions of the l-point correlation functions a scale invariant (non-trivial) limit theory can be actually attained.
1 Introduction

One of the crucial ideas of the renormalization group analysis of, say, the critical regime, is scale invariance of the system in the scaling limit. This is the famous scaling hypothesis (as to the underlying working philosophy compare any good text book of the subject matter like e.g. [1] and references therein). Central in this approach is the so-called blockspin transformation [2]. That is, observables are averaged and appropriately renormalized over blocks of increasing size. At each intermediate scale a new effective theory is constructed and the art consists of choosing (or rather: calculating) the critical scaling exponents, so that the sequence of effective theories converge to a (scale invariant) limit theory; provided that the start theory lay on the critical submanifold in the (in general infinite dimensional) parameter space of theories or Hamiltonians.

Usually the calculations can only be performed in an approximative way, the main tools being of a perturbative character and are frequently model dependent. Typically, in the more general discussion, spin systems are employed, to motivate and explain the calculational steps. While the general working philosophy, based on the concepts of asymptotic scale invariance, correlation length and the like, is the result of a deep physical analysis of the phenomena, there is, on the other side, no abundance of both rigorous and model independent results.

This applies in particular to the control of the convergence of the scaled l-point correlation functions to their respective limits if we start from a microscopic theory, lying on the critical submanifold. In this case, correlations are typically long-ranged and the usual heuristic arguments about the interplay between poor clustering, on the one side, and formation of block variables of increasing size, on the other side, become rather obscure as one is usually cavalier as to the interchange of various limit procedures. It is known, that this is a very dangerous attitude in this regime. Furthermore, the clustering of the higher correlation functions in the various channels of phase space may be quite complex and non-uniform in general. A concise and selfcontained discussion of the more general aspects and problems, lurking in the background together with a series of note and references, can be found in [3], section 7.

Usually, the crucial scaling relation (the scaling hypothesis)

$$W^T_l(Lx_1, \ldots, Lx_l; \mu^*) = L^{-l \cdot n} \cdot L^{l \cdot \gamma} \cdot W^T_l(x_1, \ldots, x_l; \mu^*)$$

(1)

which is conjectured to hold at the fixed point (denoted by $\mu^*$ in the parameter space), is the starting point (or physical input) of the analysis. Here, $W^T_l$ denote the truncated l-point functions (see below), $L$ is the diameter of the blocks, $Lx_i$ are the respective centers of the blocks, $n$ is the space dimension, $\gamma$ the statistical renormalisation exponent. If it is different from $n/2$, we have an ‘anomalous’ scale dimension.
In the following analysis we want to concentrate on the derivation of such (and similar) scaling relations for the $l$-point functions from the underlying microscopic characteristics of the theory. We will do this in a completely general way, that is, the underlying model theory can be *classical* or *quantum*, *discrete* or *continuous*. We try to adopt a quite general point of view, making as few model assumptions as possible. Our strategy is it, to deal only with the really characteristic (model independent) aspects of the subject matter.

In this sense our approach is perhaps similar to the one, expounded in e.g. [4] in the context of the analysis of the *ultraviolet behavior* in algebraic quantum field theory. A nice review of the unifying aspects of the cluster of ideas, underlying renormalisation with applications to numerous fields is also [8].

At the end of this introduction we want to remark that there exists a superficially different approach, which is more related to the well established concepts of renormalisation theory in quantum field theory (see, for example, [3], [6] or [7]), that is, renormalising propagators, Greens’s functions and path integrals and in which scale invariance is present on a more implicit level.

We do not intend (and actually feel unable) to relate all these different aspects to each other in this paper. We will instead briefly sketch, what we are going to do in the follow sections. We concentrate our analysis entirely on the hierarchy of correlation functions which define the theory. We construct renormalized limit correlation functions from them which happen to be scale invariant, thus defining a new limit theory. These limits are set into relation to the degree of clustering of the original microscopic correlation functions. We do not openly discuss the flow of, say, the renormalized Hamiltonians through parameter space. The characteristics of these intermediate theories are however implicitly given by their hierarchy of correlation functions as was explained in e.g. [9] or [4].

The same holds for the effective Hamiltonians which emerge on the intermediate scales in the ordinary approach. In our approach the effective time evolution is carried over from the microscopic theory as described in [9] or (in a slightly other context) in [4], see also [11]. In case we work in an environment, defined by ordinary Gibbs states, our framework would exactly yield these effective Hamiltonians. In our paper we treated the infrared limit. As our scaling approach depends on a continuous parameter, $R$, which could as well go to zero, a slight extension of our approach would allow to study also the ultraviolet behavior. In this case we have to work, however, in a space-time picture, that is, the time coordinates have also to be scaled. The cluster properties are then expected to be even more intricate as the clustering with respect to both space and time is of a markedly dynamical nature.
2 Concepts and Tools

As to the general framework we refer the reader to [9]. One of our technical tools is a modified (smoothed) version of averaging (such a possibility is also briefly mentioned in the notes in [3]). Instead of averaging over blocks with a sharp cut off, we employ a smoothed averaging with smooth, positive functions of the type

\[ f_R(x) := f(|x|/R) \quad \text{with} \quad f(s) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases} \] (2)

(one can choose even more general classes of functions). This class of scaled functions has a much nicer behavior under Fourier transformation, as, for example, functions with a sharp cut off, the main reason being that the tails are now also scaled. We have

\[ \hat{f}_R(k) = \text{const} \cdot R^n \cdot \hat{f}(R \cdot k) \] (3)

Remark: One might perhaps think that this choice of averaging will lead to a different limit theory. This is however not the case. Furthermore, the mathematical differences between the two approaches, that is, using sharp or smooth and scaled cut off functions, are relatively subtle and not so apparent. We will investigate these aspects in [10].

Another point, worth mentioning, are the implications of translation invariance. We have for the correlation functions

\[ W(x_1, \ldots, x_n) = W(x_1 - x_2, \ldots, x_{n-1} - x_n) \] (4)

The truncated correlation functions are defined inductively as follows (see [3])

\[ W(x_1, \ldots, x_n) = \sum_{\text{part}} \prod_{p_i} W^T(x_{i_1}, \ldots, x_{i_k}) \] (5)

The (distributional) Fourier transform reads

\[ \hat{W}^T(p_1, \ldots, p_l) = \hat{W}^T(p_1, p_1 + p_2, \ldots, p_1 + \cdots p_{l-1}) \cdot \delta(p_1 + \cdots p_l) \] (6)

The dual sets of variables are

\[ y_i := x_i - x_{i+1}, \quad q_i = \sum_{j=1}^{i} p_j \quad i \leq (l - 1) \] (7)
3 The case of Normal Fluctuations

As in \([9]\), we assume that away from the critical point the truncated \(l\)-point functions are integrable, i.e. \(\in L^1(R^{n(l-1)})\), in the difference variables, \(y_i := x_i - x_{i+1}\). As observables we choose the translates

\[
A_R(a_1, \ldots, A_R(a_l), A_R(a) := R^{-n/2} \cdot \int A(x + a) f(x/R) d^nx
\]

(8)

(where, for convenience, the labels \(1 \ldots l\) denote also possibly different observables). We then get (for the calculational details see \([9]\), the hat denotes Fourier transform, translation invariance is assumed throughout, the \(\text{const}\) may change during the calculation but contains only uninteresting numerical factors):

\[
\langle A_R(a_1) \cdots A_R(a_l) \rangle^T = \text{const} \cdot R^{n/2} \cdot \\
\int \hat{f}(Rp_1) \cdots \hat{f}(-R[p_1 + \cdots + p_{l-1}]) \hat{W}^T(p_1, \ldots, p_{l-1}) e^{-i \sum_{i=1}^{l-1} p_i a_i} e^{i a_i \sum_{i=1}^{l-1} p_i} \prod dp_i \\
= \text{const} \cdot R^{n/2} \cdot R^{-(l-1)n} \cdot \\
\int \hat{f}(p'_1) \cdots \hat{f}(-[p'_1 + \cdots + p'_{l-1}]) \hat{W}^T(p'_1/R, \ldots, p'_{l-1}/R) e^{-i \sum_{i=1}^{l-1} (p'_i/R) a_i} e^{i a_i \sum_{i=1}^{l-1} p'_i/R} \prod dp'_i
\]

(9)

We now scale the \(a_i\)’s like

\[
a_i := R \cdot X_i, \ X_i \text{ fixed}
\]

(10)

This yields

\[
\langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = \\
\text{const} \cdot R^{(2-l)n/2} \cdot \int e^{-i \sum_{i=1}^{l-1} p'_i X_i} e^{i X_i \sum_{i=1}^{l-1} p'_i} \cdot \\
\hat{f}(p'_1) \cdots \hat{f}(-[p'_1 + \cdots + p'_{l-1}]) \hat{W}^T(p'_1/R, \ldots, p'_{l-1}/R) \prod dp'_i
\]

(11)

As the \(\hat{f}\) are of strong decrease and \(\hat{W}^T\) continuous and bounded by assumption, we can perform the limit \(R \to \infty\) under the integral and get:

Case 1 \((l \geq 3)\):

\[
\lim_{R \to \infty} \langle A_R^F(R \cdot X_1) \cdots A_R^F(R \cdot X_l) \rangle^T = 0
\]

(12)

Case 2 \((l = 2)\):

\[
\lim_{R \to \infty} \langle A_R^F(R \cdot X_1) A_R^F(R \cdot X_2) \rangle^T = \text{const} \cdot \int \hat{W}^T(0) e^{-ip'_1(x_1-x_2)} \cdot \hat{f}(p'_1) \cdot \hat{f}(-p'_1) dp'_1
\]

(13)
Conclusion 3.1 In the normal regime, away from the critical point, where we assumed $L^1$-clustering, all the truncated correlation functions vanish in the limit $R \to \infty$ apart from the 2-point function. We hence have a quasi free theory in the limit as described in [9] or in the work of Verbeure et al (cf. the references in [7]).

4 The Relation to the Heuristic Scaling Hypothesis

In the following sections we will develop two rigorous, technically slightly different, approaches which implement the physically well-motivated but, nevertheless, to some extent heuristic scaling hypothesis. The analysis will be performed in coordinate space, Fourier space, respectively. In this section we restrict ourselves to the two-point correlation function, for which the asymptotic behavior is considerably simpler and more transparent.

The pedagogical reason for representing two technical methods is the following. It turns out that, as may be expected on physical grounds, critical systems and their characteristics are sitting exactly on points in the parameter space where also the necessary mathematical methods turn out to be of a singular character. This can best be exhibited by contrasting the two methods developed below.

Remark: In the rest of the paper we replace the exponent $n/2$ in the definition of $A_R(a)$ by an exponent $\gamma$, which will usually be fixed during or at the end of a calculation. It plays the role of a critical scaling exponent.

Let us hence study the behavior of

$$\langle A_R(R \cdot X_1) A_R(R \cdot X_2) \rangle^T = R^{-2\gamma} \cdot \int W^T((x_1 - x_2) + R(X_1 - X_2)) \cdot f(x_1/R) f(x_2/R) dx_1 dx_2$$

$$= R^{-2\gamma + 2n} \int W^T(R[(x_1 - x_2) + (X_1 - X_2)]) \cdot f(x_1) f(x_2) dx_1 dx_2 \quad (14)$$

We make the physically well-motivated assumption that—in the critical regime—$W^T$ decays asymptotically like some inverse power, i.e.

$$W^T(x_1 - x_2) \sim F(x_1 - x_2) \cdot |x_1 - x_2|^{-(n-\alpha)} \quad 0 < \alpha < n \quad (15)$$

for $|x_1 - x_2| \to \infty$, $F$ bounded and well-behaved.

From the last line of (14) we see that, as $f$ has compact support, we can replace $W^T$, for $(X_1 - X_2) \neq 0$ and $R \to \infty$ by its asymptotic expression and
absorb the remaining \((x_1, x_2)\)-integration in an appropriate constant. We hence get for \(R\) large:

\[
(A^F_R(R \cdot X_1)A^F_R(R \cdot X_2))^T \approx const \cdot R^{-2\gamma + 2n} \cdot R^{-(n-\alpha)} \cdot |X_1 - X_2|^{-(n-\alpha)}
\]  

(16)

Choosing now

\[
\gamma = (n + \alpha)/2
\]

(17)

we get a limiting behavior (for \(R \to \infty\)) as

\[
\sim |X_1 - X_2|^{-(n-\alpha)}
\]

(18)

Central in the renormalisation group idea is that systems on the critical surface (i.e., critical systems) are driven towards a fixed point, representing a completely scale invariant theory. This idea is usually formulated in an abstract parameter space of, say, Hamiltonians. In our correlation function approach the fixed point shows its existence via the scaling properties of the correlation functions, that is

\[
W^T_2(L \cdot (X - Y); \mu^*) = L^{-2(n-\gamma)}W^T_2(X - Y; \mu^*)
\]

(19)

with \(\mu^*\) describing the fixed point in the (usually) infinite dimensional parameter space. We see from the above that this is exactly implemented by our limiting correlation functions, as we have (with the choice \(\gamma = (n + \alpha)/2\)):

\[
W^T_2(X - Y; \mu^*) \sim |X - Y|^{-(n-\alpha)}
\]

(20)

That is, the above scaling limit leads to a limit (i.e. fixed point) theory, exactly reproducing the asymptotic behavior of the original (microscopic) theory. But there is yet another important point worth to be mentioned.

In the more general situation of \(l\)-point correlation functions we do not necessarily have such a simple decay behavior. Furthermore, it would be desirable to have precise information on thresholds up to which a certain method leads to rigorous and reliable results. We therefore describe and illustrate two methods with the help of the relatively transparent behavior of the 2-point functions, which will then be applied to the more complex and general situation of the higher correlation functions. Using the assumed simple asymptotic behavior of the 2-point functions, we proceed now as follows.

4.1 Method One

We assume the existence of a certain exponent, \(\alpha\), so that \((x^2\) denoting the vector-norm squared)

\[
G(x) := W^T(x) \cdot (1 + x^2)^{(n-\alpha)/2} = const + F(x)
\]

(21)
with a decaying (non-singular) $F$. Fourier transformation then yields:

\[
R^{-2\gamma} \cdot \int W^T((x_1 - x_2) + R(X_1 - X_2)) f(x_1/R) f(x_2/R) dx_1 dx_2
\]

\[
= R^{-2\gamma} \cdot \int G((x_1 - x_2) + R(X_1 - X_2)) \cdot [1 + ((x_1 - x_2) + R(X_1 - X_2))^2]^{-(n-\alpha)/2} \cdot f(x_1/R) f(x_2/R) dx_1 dx_2
\]

\[
= R^{-2\gamma} \cdot R^{2n-(n-\alpha)} \cdot \int dp \hat{G}(p) \cdot e^{-iRp(X_1 - X_2)} \cdot \left[ \int e^{-iRp(x_1-x_2)}(R^{-2} + ((x_1 - x_2) + (X_1 - X_2))^2)^{-(n-\alpha)/2} f(x_1) f(x_2) dx_1 dx_2 \right]
\]

\[(22)\]

where we made the substitution $x \to R \cdot x$.

We now assume the support of $f$ to be contained in a sufficiently small ball around zero (or, alternatively, $(X_1 - X_2)$ sufficiently large so that $(x_1 - x_2) + (X_1 - X_2) \neq 0$ for $x_i$ in the support of $f$). With

\[
\hat{G}(p) = \text{const} \cdot \delta(p) + \hat{F}(p)
\]

\[(23)\]

the leading part in the scaling limit $R \to \infty$ is the $\delta$-term. Asymptotically we hence get for $R \to \infty$ (setting $y := x_1 - x_2$ $Y := X_1 - X_2$):

\[
R^{n+\alpha-2\gamma} \cdot \text{const} \cdot \int |y + Y|^{-(n-\alpha)} \cdot f * f(y) dy
\]

\[(24)\]

with

\[
f * f(y) := \int f(y + x_2) \cdot f(x_2) dx_2
\]

\[(25)\]

The reason why the contribution, coming from $\hat{F}(p)$, can be neglected for $R \to \infty$ is the following: $f$ is assumed to be in $\mathcal{D}$; by assumption the prefactor never vanishes on the support of $f(x_i)$. Hence the whole integrand in the expression in square brackets is again in $\mathcal{D}$ and therefore its Fourier transform is in $\mathcal{S}$, that is, of rapid decrease. We can therefore perform the $R$-limit under the integral and get a rapid vanishing of the corresponding contribution in $R$ for each $p \neq 0$. As the $\delta$-contribution has been extracted from $\hat{G}(p)$, this proves the statement.

As $f * f$ has again a compact support, we have that, choosing

\[
\gamma = (n + \alpha)/2
\]

\[(26)\]

the limit correlation function behaves as $\sim |X_1 - X_2|^{-(n-\alpha)}$ as in the above heuristic analysis.
4.2 Method Two

As in the case of normal clustering or ([9], last section), one can, on the other hand, improve the too weak decay of $W^T(x_1 - x_2)$ and transform it into an integrable (i.e. $L^1$-) function. So, with a similar notation as in the preceding subsection, we choose a suitable exponent $\alpha$ in

$$P_\alpha(x_1 - x_2) := (1 + |x_1 - x_2|^2)^{\alpha/2}$$  \hfill (27)

so that

$$G(y) := W^T(y) \cdot P_\alpha^{-1} \in L^1 \quad (y := x_1 - x_2)$$  \hfill (28)

In contrast to Method One, there is of course a whole range of such possible exponents, $\alpha > \alpha_{inf}$, so that

$$G(y) = \begin{cases} \in L^1 \quad \text{for } \alpha > \alpha_{inf} \\ \notin L^1 \quad \text{for } \alpha < \alpha_{inf} \end{cases}$$  \hfill (29)

Proceding as in Method One, we get

$$R^{-2\gamma} \int W^T(y + R \cdot Y) \cdot f(x_1/R)f(x_2/R)dx_1dx_2 =$$

$$R^{-2\gamma} \cdot R^{n+\alpha} \int dp \hat{G}(p/R) \cdot e^{-ipY} \cdot \left[ \int e^{-ip(y + (y + Y)^2)^{\alpha/2}} \cdot f \ast f(y) dy \right]$$  \hfill (30)

Again the obvious strategy seems to be to choose

$$\gamma = (n + \alpha)/2$$  \hfill (31)

and perform the limit $R \to \infty$. With the same support properties as above, that is, $y + Y \neq 0$ for $x_1, x_2 \in \text{support of } f$, the integrand in square brackets is again infinitely differentiable with respect to $y$. Hence, its Fourier transform is fast decaying in $p$.

Remark: Note that for $\alpha/2$ non-integer and without the above support restriction, there will show up a singularity in sufficiently high orders of differentiation for vanishing $R^{-2}$. One can however control these singularities and show that the analysis still goes through in the case where the support condition does not hold. One gets however some mild constraint on the admissible $\alpha$’s.

Therefore we can again apply Lebesgues’ theorem of dominated convergence and perform the $R$-limit under the integral. This yields the expression

$$\hat{G}(0) \cdot \int dp e^{-ipY} \cdot \left[ \int e^{-ip} \cdot |y + Y|^{\alpha} \cdot f \ast f(y) dy \right] =$$

$$\text{const} \cdot \hat{G}(0) \cdot \int \delta(y + Y) \cdot |y + Y|^{\alpha} \cdot f \ast f(y) dy = \text{const} \cdot \hat{G}(0) \cdot 0$$  \hfill (32)

(as a result of the above support condition).
Conclusion 4.1 With $\alpha$ chosen so that $G(y) \in L^1$ and $\gamma = (n + \alpha)/2$, the limit can be carried out under the integral and yields the result zero. This shows a fortiori that there is no $\alpha_{\text{min}}$ with the property that there is a non-vanishing limit-two-point function. Put differently, we have an $\alpha_{\text{inf}}$ but no $\alpha_{\text{min}}$ (cf. (29)).

So, in contrast to Method One, the relevant exponent, $\alpha_{\text{inf}}$, is of such a peculiar nature that we definitely cannot apply the above method of interchange of taking the limit $R \to \infty$ and integration. But nevertheless, we will show that

$$\gamma := (n + \alpha_{\text{inf}})/2$$

is the correct critical scaling exponent leading to a sensible limit theory and that this $\alpha_{\text{inf}}$ is exactly the $\alpha$, we have determined in Method One.

We have learned above that in order to arrive at a non-zero limit correlation function, we are definitely forbidden to exploit Lebesgue’s theorem of dominated convergence in the above expression. The reason for the vanishing of the respective expression was that with

$$\lim_{R \to \infty} \hat{G}(p/R) = \hat{G}(0)$$

we have to evaluate $\int \hat{g}(p)dp$ with

$$\hat{g}(p) := \int e^{-ip(y+Y)} |y + Y|^\alpha \cdot f * f(y)dy$$

This integral happens to be zero due to the explicit factor, $|y + Y|^\alpha$ and the assumed support properties.

So, we have to investigate what happens for $\alpha = \alpha_{\text{inf}}$. As we learned above that there is no $\alpha_{\text{min}}$, we can conclude

Observation 4.2 For $\alpha = \alpha_{\text{inf}}$, $G_\alpha(y)$ is no longer in $L^1$, with

$$G_\alpha(y) := W^T(y) \cdot (1 + y^2)^{-\alpha/2}$$

We know that for $\alpha < \alpha_{\text{inf}}$ the decay of $G_\alpha(y)$ is so weak that the Fourier transform develops a power law singularity in $p = 0$; that is, we can conclude

Lemma 4.3 For $\alpha_{\text{inf}} - \alpha := \varepsilon$, $\hat{G}_\alpha$ has a singularity of the form $|p|^{-\varepsilon}$ near $p = 0$.

For $\alpha = \alpha_{\text{inf}}$ the singularity is of logarithmic type near $p = 0$.

This statement can again be proved by a scaling argument. Let $G_\alpha$ have a non-integrable tail of the form $|y|^{-(n-\varepsilon)}$. For the (distributional) Fourier transform we then have

$$\hat{G}_\alpha(\lambda \cdot p) = \text{const} \cdot \int e^{i\lambda y} \cdot G_\alpha(y)dy = \text{const} \cdot \lambda^{-n} \cdot \int e^{ipy'} \cdot G_\alpha(y'/\lambda)dy'$$
For $\lambda \to 0$ we can, as above, replace $G_\alpha$ by its asymptotic expression, which goes as $|y|^{-(n-\varepsilon)}$ and conclude that $\hat{G}_\alpha(\lambda p)$ contains a leading singular contribution $\sim \lambda^{-\varepsilon}$ (modulo logarithmic terms). We hence see that

$$\hat{G}_\alpha(p) \sim |p|^{-\varepsilon} \quad (38)$$

near $p = 0$ as a distribution (that is, the above reasoning is to be understood modulo the smearing with appropriate test functions; see e.g. [12]). For $\alpha = \alpha_{inf}$, the singularity must be weaker than any power, that is, must be of logarithmic type.

By *Method One* we get a limit correlation function which clusters as $|Y|^{-(n-\alpha)}$. One may wonder where this behavior is hidden if we use *Method Two*. Taking only the singular term in $\hat{G}(p/R)$ into account, we have (with $\gamma := (\alpha_{inf} + n)/2$)

$$\lim_{R \to \infty} R^{-2\gamma} \langle A_R(RX_1) \cdot B_R(RX_2) \rangle^T \sim \lim_{R \to \infty} \text{const} \cdot \int \ln(|p|/R) \cdot \hat{g}(p) dp \quad (39)$$

and $\hat{g}(p)$ as in equation (35). We can again neglect the term

$$\ln R \cdot \int \hat{g}(p) dp \quad (40)$$

as $\int \hat{g}(p) dp = 0$.

Assuming at the moment that $\alpha$ were an integer (we will get the general result by a scaling argument), the prefactor $|y + Y|^\alpha$ can be transformed into corresponding $p$-differiations of $\hat{f} \ast f(p)$, which, by partial integration, can then be shifted to corresponding differentiations of $\ln(|p|)$. This transformation yields an expression of the type $|p|^{-\alpha}$ times a smooth and decaying function. That means, we essentially end up with an expression like

$$\int dp |p|^{-\alpha} \cdot e^{-ipY} \cdot \left[ \int e^{-ipy} f \ast f(y) dy \right] \quad (41)$$

By the same reasoning as above we conclude that the singularity, $|p|^{-\alpha}$, goes over, via Fourier transform, into a weak decay proportional to $|X_1 - X_2|^{-(n-\alpha)}$, that is, we arrive at the same result as in *Method One*, whereas the reasoning is a little bit more tricky.

For a general non-integer $\alpha$ the argument could be made precise by analysing the distributional character of an expression like $r^\beta$ with $r := |x|$ and its Fourier transform. As the analysis is a little bit tedious, we refer the reader to [12]. On the other hand, one can use a scaling argument as above (with $Y := \lambda \cdot Y_0$, $Y_0$ fixed as $\lambda \to \infty$). This yields an asymptotic behavior of the form

$$\lambda^{-(n-\alpha)} \cdot \int dp \ln(|p|) \cdot \left[ \int e^{-i\lambda p(y+Y_0)} \cdot |y + Y_0|^\alpha \cdot f \ast f(\lambda y) dy \right] \quad (42)$$
The evaluation of the integral for \( \lambda \to \infty \) can be done as follows: As \( f \ast f \) has compact support, the volume of the support of \( f \ast f(\lambda y) \) shrinks proportional to \( \lambda^{-n} \). Therefore the expression in square brackets scales as \( \sim \lambda^{-n} \). On the other hand (due to an ‘uncertainty principle’ argument), its essential \( p \)-support increases proportional to \( \lambda^n \). That is, the two effects compensate each other and we have again a large-\( Y \) behavior \( \sim |Y|^{-(n-\alpha)} \) as before.

We conclude that both methods lead to the same asymptotic scaling behavior of the renormalized two-point function.

5 The General Cluster-Analysis at the Critical Point

We now study the general situation of the presence of some long-range correlations in the \( l \)-point functions. In contrast to the much simpler situation prevailing in the case of two-point functions, the clustering may be quite complicated, in particular, the dependence on the number, \( l \), i.e. the number of observables, occurring in the expressions, may be non-trivial. Therefore, we have to investigate these aspects in more detail.

From general principles (see e.g. \( [13] \)) we know that in a pure phase there is always a certain degree of clustering. We make the slightly stronger assumption that it is in some way of the kind of an inverse power law at infinity (to be specified below). We want to study the scaling limit of

\[
\langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T
\]

with

\[
A_R(a) := R^{-\gamma} \cdot \int A(x + a)f(x/R)d^n x
\]

and an, at the moment, unspecified exponent, \( \gamma \).

The above expression can be written as

\[
\int W^T((x_1 - x_2) + R(X_1 - X_2), \ldots, (x_{l-1} - x_l) + R(X_{l-1} - X_l)) \cdot \prod_{i=1}^l f(x_i/R) \prod_{i=1}^l dx_i
\]

Fourier transformation yields (with \( \hat{W}^T(q_1, \ldots, q_{l-1}) \) considered as a distribution on \( S(\mathbb{R}^{(l-1)n}) \))

\[
const \cdot R^{l(n-\gamma)} \cdot \int \hat{W}^T(q_1, \ldots, q_{l-1}) \cdot e^{-i \sum_{j=1}^{l-1} R_{pj} Y_j} \cdot \left[ \int e^{-i \sum_{j=1}^{l-1} R_{pj} x_j} \cdot e^{i R_{0j-1} x_1} \cdot \prod_{i=1}^l f(x_i) \prod_{i=1}^l dx_i \right] \prod_{j=1}^{l-1} dp_j
\]
with $Y_j := X_j - X_{j+1}$ and the wellknown relation between the $q$-variables and the $p$-variables (see e.g. section 2 or [3]). For calculational or notational convenience we will employ both sets of variables which are linear combinations of each other.

As $f$ is in $\mathcal{D}$ by assumption, the Fourier transform of $\prod f(x_i)$ is in $\mathcal{S}$ and the function in square brackets is a function of $(Rp_1, \ldots, Rp_{l-1})$ or $(Rq_1, \ldots, Rq_{l-1})$, being of rapid decrease in either set of variables. As a consequence, for $R \to \infty$ and at least one $p_j$ being different from zero, the expression approaches zero faster than any inverse power (together with all its derivatives).

From this we see that, as $R \to \infty$, the region of possible singular behavior is located around $(p)_1^{l-1} = 0$ or $(q)_1^{l-1} = 0$, implying also $p_l = -\sum_{i=1}^{l-1} p_j = 0$. We can hence infer that only the singular behavior of $\hat{W}_T$ in $(q) = 0$ will matter in this limit. As a consequence, it will be our strategy to isolate this singular contribution in $\hat{W}_T$ and transform it in a certain explicit scaling behavior in $R$, which can be encoded in some power, $R^{-\alpha}$, in front of the integral.

The singular behavior of $\hat{W}_T(q)$ at $(q) = 0$ is related to the weak decay of $W_T(y)$ at infinity. The limiting behavior of $W_T(y)$ can, however, not expected to be simple or uniform (at least not in the generic case) as $(y_1, \ldots, y_{l-1})$ or $(x_1, \ldots, x_l)$ can move to infinity in many different ways. We may, for example, have that $(x_i)$ together with all $|x_i - x_j|$ go to infinity or, on the other side, the variables move to infinity in certain fixed clusters of finite diameter. The rate of decay of $W_T(y)$ should of course depend in general on these details. Correspondingly, the singular behavior of $\hat{W}_T(q)$ in the infinitesimal neighborhood of $(q) = 0$ should depend on the direction in which $(q) = 0$ is approached, that is, the limit may be direction-dependent.

In the light of this general situation we must at first decide, in which kind of limit we are mainly interested. Inspecting the expression (43), we actually started from, we choose in a first step our fixed vectors, $(X_i)$, so that

$$X_i - X_j \neq 0 \quad \text{for all } i, j \quad (47)$$

As a consequence, all distances, $|RX_i - RX_j|$, go to infinity for $R \to \infty$. As in the preceding section, we can choose the support of $f$ so small that, with $x_i, x_j \in supp(f)$, we have

$$|R(X_i - x_i) - R(X_j - x_j)| \to \infty \quad (48)$$

In this particular case we may expect a relatively uniform limit behavior on physical grounds.

Remark: Similar problems occur in quantum mechanical scattering theory.

Under this proviso the following assumption seems to be reasonable.

**Assumption 5.1** Under the assumption, being made above, we assume the following decomposition of $W^T_1(y)$ to be valid: It exists a function, $(1 + H(y))$, $H(y)$
homogeneous and positive for \( y \neq 0 \) so that
\[
G(y) := (1 + H(y)) \cdot W^T(y) = \text{const} + F(y)
\] (49)
with \( F \) sufficiently decaying at infinity in the channel, indicated above, i.e. \( \{ |y_i| \to \infty \text{ for all } i = 1, \ldots, l - 1 \} \) and
\[
H(Ry) = R^{\alpha_i'} \cdot H(y)
\] (50)

Remark 5.2 A typical example for \( H(y) \) is \( (\sum y_i^2)^{\alpha_i'/2} \).

Fourier transforming \( G(y) \), we get
\[
\hat{G}(q) = \text{const} \cdot \delta(q) + \hat{F}(q)
\] (51)
and expression (46) becomes (compare the related expression in Method One of the preceding section)

\[
\text{const} \cdot R^{(n-\gamma)} \cdot \int \prod_{1}^{l-1} dp_j \hat{G}(q) \cdot e^{-i \sum q_j Y_j} \cdot \left[ \int e^{-i \sum_{1}^{l-1} R_j x_j \cdot \epsilon \cdot i R_{q_{l-1}} \cdot x_l} \cdot (1 + H(R y + R Y))^{-1} \cdot \prod_{1}^{l} f(x_j) \cdot \prod_{1}^{l} dx_j \right]
\] (52)

By assumption, \( H \) is homogeneous of degree \( \alpha_i' \). So we can extract a negative power of \( R, R^{-\alpha_i'} \), from the expression in square brackets. Furthermore, we observed above that for \( R \to \infty \) only the vicinity of \( q = 0 \) matters. Finally, by assumption, the contribution coming from \( \hat{F}(q) \) can be neglected in this limit (compare the corresponding discussion in the subsection [4.1]; as a consequence of the assumed support properties, the expression in square brackets is again strongly decreasing). We hence have

\[
\lim_{R \to \infty} \langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = \lim_{R \to \infty} \text{const} \cdot R^{(n-\alpha_i'-\gamma)} \cdot \int \prod_{1}^{l-1} dq_j \\
\delta(q) \cdot e^{i \sum_{1}^{l-1} R_{q_j} Y_j} \cdot \left[ \int e^{-i \sum_{1}^{l-1} R_j x_j \cdot \epsilon \cdot i R_{q_{l-1}} \cdot x_l} \cdot \left( R^{-\alpha_i'} + H(y + Y) \right)^{-1} \cdot \prod_{1}^{l} f(x_j) \cdot \prod_{1}^{l} dx_j \right]
\] (53)

Remark: We see again the reason for the special choice being made above as to the support properties of the functions \( f(x_i) \), leading to the result \( y_j + Y_j \neq 0 \) on the support of \( f \). Without this assumption, we see for our above example, \( H(y) = (\sum y_i^2)^{\alpha_i'/2} \), that in the limit, where \( R^{-\alpha_i'} \) vanishes, we would get a singular
contribution at points where \(y + Y = 0\) in the integrand in square brackets. These terms would make the following discussion much more tedious.

If we now make the choice

\[
\gamma := \gamma_l = n - \alpha'_l / l
\]

we arrive at a finite limit expression, depending on the coordinates \((X_i)\):

\[
\lim_{R \to \infty} \langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = \text{const} \cdot \int H_l(y + Y)^{-1} \cdot \prod_{1}^{l} f(x_i) \prod_{1}^{l} dx_i
\]

which is a function of the coarse grained difference coordinates

\[
Y_j = X_j - X_{j+1}
\]

For the \(Y_j\) sufficiently large, it is approximately a function

\[
W_{\text{limit}}(Y) \approx \text{const} \cdot H_l(Y)^{-1}
\]

That is, the renormalized limit correlation functions reproduce the asymptotic power law behavior of the original microscopic correlation functions.

For later use we introduce the new scaling exponent, \(\alpha_l\), via

\[
\alpha'_l + \alpha_l = (l - 1)n
\]

This implies

\[
\gamma_l = (n + \alpha_l) / l
\]

The underlying reason for this choice is that an asymptotic decay, \(\sim r^{-(l-1)n}\), is just the threshold for \(W_l^T\) being integrable or non-integrable (with \(r := (\sum y_j^2)^{1/2}\)).

We have arrived at the following result: We are interested in a scaling-limit theory for \(R \to \infty\). In order to get a non-vanishing and finite limit theory, we have to choose the scaling exponent for \(l = 2\) as

\[
\gamma = \gamma_2 = (n + \alpha_2) / 2
\]

Furthermore, we have extracted the asymptotic form from the higher truncated \(l\)-point functions, \(W_l^T(y)\), and have absorbed it in an explicit scaling factor, \(R\) to some power. If the limit theory is to be finite, the corresponding scaling exponents for \(l > 2\) have to be less or equal to zero. This yields unique \(\gamma_l\)'s as threshold values.

A cornerstone of the philosophy of the renormalisation group is that the scaling exponents of the scaled observables remain the same, irrespectively of the
degree of the correlation functions in which they occur. That is, these exponents are fixed by the exponent, $\gamma_2$, and we have

$$\gamma = \gamma_2 \geq \gamma_l$$

(61)

(the latter exponent being derived from equation (59)), in order that the limit correlation functions remain finite.

**Conclusion 5.3** We have the following alternatives for $R \to \infty$:

- $\gamma_2 > \gamma_l \Rightarrow W_{l,R}^T \to 0$ (62)
- $\gamma_2 = \gamma_l \Rightarrow \lim_{R\to\infty} W_{l,R}^T$ is finite and non-trivial (63)
- $\gamma_2 < \gamma_l \Rightarrow W_{l,R}^T \to \infty$ (64)

If $\gamma_2 > \gamma_l$ for all $l \geq 3$, the fixed point is gaussian or trivial. The limit theory is quasi-free. The limit theory is non-trivial if $\gamma_2 = \gamma_l$ for at least some $l \geq 3$. For $\gamma_2 < \gamma_l$ for some $l$, the limit theory does not exist.

**Remark 5.4** The corresponding analysis can also be done by employing Method Two (discussed in the preceding section). One can even omit the support conditions assumed above. The treatment then becomes more involved but the end result is the same. We discuss one particular case below.

To complete the scaling and/or cluster analysis of the truncated correlation functions, we have to analyze the other channels and the respective consequences for scaling exponents and cluster assumptions.

We mentioned several times that without the support condition

$$ (X_i - X_j) + (x_i - x_j) \neq 0 $$

(65)

for $x_{i,j} \in \text{supp}(f)$, the analysis would become more tedious. On the other side, this assumption is violated if the observables move to spatial infinity in certain clusters. The extreme case occurs when all $X_i$ are chosen to be zero, i.e:

$$ \langle A_{R(1)} \cdots A_{R(l)} \rangle_T, R \to \infty $$

(66)

(the indices 1, \ldots, l denote the different observables). This scenario was already briefly discussed in section 7 of [4] in connection with phase transitions and/or spontaneous symmetry breaking, which are also typically related to poor spatial clustering.

With the same notations as above we have

$$ \langle A_{R(1)} \cdots A_{R(l)} \rangle_T = \text{const} \cdot R^{l(n-\gamma)}. $$

\[
\int \tilde{W}_l^T(q_1, \ldots, q_{l-1}) \left[ \int e^{-i \sum_{i=1}^{l-1} R_{pj} x_j} \cdot e^{i R_{pi} x_i} \cdot \prod_{i=1}^{l} f(x_i) \prod_{i=1}^{l} dx_i \right] \prod_{j=1}^{l-1} dp_j \quad (67)
\]
Assuming again the existence of a suitable homogeneous function, $H_l(y)$, in this channel, we get asymptotically two contributions

$$\text{const} \cdot R^{(n-\gamma)-\alpha'_l} \cdot \int H_l(y)^{-1} \cdot \prod_{i=1}^l f(x_i) \prod_{i=1}^l dx_i$$  \hspace{1cm} (68)$$

and

$$\text{const} \cdot R^{(n-\gamma)-\alpha'_l} \cdot \int \prod_{i=1}^{l-1} dq_j \hat{F}(q_1, \ldots, q_{l-1}) \cdot$$

$$\left[ \int e^{-i \sum_{j=1}^{l-1} R_{pj} x_j} e^{i R_{q_{l-1}} x_l} (H_l(y))^{-1} \cdot \prod_{i=1}^{l-1} f(x_i) \prod_{i=1}^l dx_i \right]$$  \hspace{1cm} (69)$$

The first term has almost the same form as above. But now the function in square brackets in the second contribution is no longer of strong decrease as the integrand (considered as a function of $(x)$or $(y)$) is no longer in $D$ as it will have a singularity in $y = 0$. We can however provide the following estimate on the degree of this singularity of $H_l^{-1}$ in $y = 0$. We assumed throughout in this section that $W^T_l$ is not integrable at infinity, that is, the clustering is weak. On the other side, this asymptotic behavior is exactly encoded in $H_l^{-1}$, as we observed above. The threshold where integrability goes over into non-integrability for $H_l^{-1}$ is a behavior

$$\sim r^{-(l-1)n}, \quad r := \left( \sum_{j=1}^{l-1} y_j^2 \right)^{1/2}$$  \hspace{1cm} (70)$$

We can therefore conclude that

$$\alpha'_l \leq (l - 1)n$$  \hspace{1cm} (71)$$
in the above construction if $W^T_l$ is non-integrable at infinity. If $\alpha'_l$ is even strictly smaller than $(l - 1)n$, which is the ordinary case in the critical region, we have

Observation 5.5

$$\alpha'_l < (l - 1)n$$  \hspace{1cm} (72)$$
implies that $H_l^{-1}$ is integrable near $y = 0$. Hence

$$H_l^{-1}(y) \cdot \prod_{i=1}^l f(x_i) \in L^1$$  \hspace{1cm} (73)$$
due to the compact support of $f$. 
From this we infer again that, with
\[ \gamma_l = n - \alpha_l'/l = (n + \alpha_L)/l \tag{74} \]
the contribution (68) is finite in the scaling limit. For the contribution (69) we have by the same reasoning that the function in square brackets is a continuous function of \( Rq \), which goes to zero for \( Rq \to \infty \) (due to the Riemann-Lebesgue lemma).

On the other side, we have no precise apriori information about \( F(y) \) and \( \hat{F}(q) \). \( F(y) \) goes to zero at infinity as the asymptotic behavior is contained in \( H^{-1} \), but its rate of vanishing is not clear.

Conclusion 5.6 If the integrand of contribution (69) is lying in some \( L^p \), so that the limit, \( R \to \infty \), can be performed under the integral, the whole expression vanishes in the scaling limit.

In this situation we are left with again with the first term, which is the limit of
\[ \text{const} \cdot \int H_l(y + Y)^{-1} \cdot \prod_{i=1}^l f(x_i) \prod_{i=1}^l dx_i \tag{75} \]
for \( Y \to 0 \). That is, in this case it holds

Theorem 5.7 If the situation is as in the conclusion, \( W_{l}^{\lim}(X_1, \ldots, X_l) \) is continuous and we have in particular
\[ W_{l}^{\lim}(0, \ldots, 0) = \lim_{X \to 0} W_{l}^{\lim}(X_1, \ldots, X_l) \tag{76} \]

We can hence resume our findings as follows: If the assumptions, made above, are fulfilled and if the functions, \( H_l \), can be chosen consistently in all channels, so that the \( \gamma_l \)'s, resulting from the relation
\[ \gamma_l = (n + \alpha_l)/l \tag{77} \]
are smaller than or identical to \( \gamma_2 \), we arrive at a full limit theory, being well-defined in all channels. In this case the renormalization group program works and yields a non-trivial scaling limit.

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