LAWS OF THE ITERATED LOGARITHM FOR OCCUPATION TIMES OF MARKOV PROCESSES

SOOBIN CHO, PANKI KIM AND JAEHUN LEE

Abstract. In this paper, we discuss the laws of the iterated logarithm (LIL) for occupation times of Markov processes $Y$ in general metric measure space both near zero and near infinity under some minimal assumptions. We first establish LILs of (truncated) occupation times on balls $B(x, r)$ of radii $r$ up to an function $\Phi(r)$, which is an iterated logarithm of mean exit time of $Y$, by showing that the function $\Phi$ is optimal. Our first result on LILs of occupation times covers both near zero and near infinity regardless of transience and recurrence of the process. Our assumptions are truly local in particular at zero and the function $\Phi$ in our truncated occupation times $r \mapsto \int_0^{\Phi(r)} 1_{B(x, r)}(Y_s) ds$ depends on space variable $x$ too. We also prove that a similar LIL for total occupation times $r \mapsto \int_0^{\infty} 1_{B(x, r)}(Y_s) ds$ holds when the process is transient. Then we establish LIL concerning large time behaviors of occupation times $t \mapsto \int_0^t 1_A(Y_s) ds$ under an additional condition that guarantees the recurrence of the process. Our results cover a large class of Feller (Levy-like) processes, random conductance models with long range jumps, jump processes with mixed polynomial local growths and jump processes with singular jumping kernels.

Keywords: Jump processes; Feller process; Occupation times; Law of the iterated logarithm;

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1. Introduction

Law of the iterated logarithm (LIL), together with the the central limit theorem and law of large numbers, is considered as the fundamental limit theorem in Probability theory and a lot of beautiful results have been established for LIL for the sample path of Markov processes. See [5] and the references therein. Very recently, in [10][11], the authors have obtained limsup and liminf laws for sample paths of a large class of standard Markov processes both near zero and near infinity under some minimal localized assumptions.

On the other hand, the literature on LIL for occupation times for stochastic processes is quite scarce and, to the best of our knowledge, LIL for occupation times has been studied only for Brownian motion and stable processes partially. See (1.1)–(1.3) below.

The purpose of this paper is to establish various forms of limsup LILs for occupation times of Markov processes in general metric measure space both near zero and infinity. This paper is a continuation of our journey on investigating minimal assumptions in the study of sample path properties for Markov processes via localizing approach.

We first recall known results on the limsup LIL for occupation times. Let $(Y_t)_{t \geq 0}$ be a Brownian motion in $\mathbb{R}^d$, $d \geq 3$ and $B(0, r) \subset \mathbb{R}^d$ be the ball of radius $r$ centered at the origin. Denote by $U(B(0, r), t) := \int_0^t 1_{B(0, r)}(Y_s) ds$ the occupation time on $B(0, r)$ up to time $t$. Ciesielski and Taylor [12] proved that

$$\limsup_{t \to 0} \frac{U(B(0, r), \infty)}{r^2 \log \log r} = \frac{2}{p_d}, \quad \text{a.s.,} \quad (1.1)$$

where $p_d$ is the first positive zero of Bessel function $J_{d/2-2}(z)$ of the first kind. Using the theory of large deviations, Donsker and Varadhan [13] showed a large scale counterpart of (1.1), namely,
there exists a constant $c_1 > 0$ such that
\[
\limsup_{r \to \infty} \frac{\mathcal{U}(B(0, r), \infty)}{r^2 \log \log r} = c_1, \quad \text{a.s.} \tag{1.2}
\]
Motivated by (1.2), Shieh [23] showed that for every strictly stable process $(Y_t)_{t \geq 0}$ in $\mathbb{R}^d$, $d \geq 1$ with index $\alpha \in (0, 2]$ (that is, $Y_t$ and $s^{-1/\alpha}Y_{st}$ have the same distribution for all $s, t > 0$) which has an everywhere strictly positive density, there exists a constant $c_2 > 0$ such that
\[
\limsup_{r \to \infty} \frac{\mathcal{U}(B(0, r), r^2 \log \log r)}{r^2 \log \log r} = c_2, \quad \text{a.s.} \tag{1.3}
\]
Note that only truncated occupation times considered in [23] because $(Y_t)_{t \geq 0}$ can be recurrent in the setting of (1.3) so that $\mathcal{U}(B(0, r), \infty) = \infty$ almost surely.

In the first part of this paper, we study (1.3)-type LIL and its small scale counterpart LIL for general standard (possibly recurrent) Markov processes that have mere weak scaling structures. See Theorems 2.4 and 2.6 below. Then we get (1.2)-type LIL under additional assumptions that guarantee the transience of the process. See Theorem 2.9 below. Our assumptions on a scaling structure of processes are quite weak so that our result even covers some random conductance model.

In the second part of the paper, we study large time behaviors of occupation times $t \mapsto \int_0^t 1_A(Y_s)ds$ and some additive functionals when the process is recurrent. See Theorem 2.10 below.

Our assumption is general enough to cover a large class of Levy-like processes, jump processes with mixed polynomial local growths, jump processes with singular jumping kernels. In particular, similar to [10][11], our assumptions at infinity allow irregular behaviors of the process when it is on regions far away from the origin, which are controlled by a parameter $v \in (0, 1)$. See Definition 2.1(ii) and Assumption B below. Thanks to such weak assumptions, our results on LIL at infinity cover some random conductance model including ones with long range jumps. In Section 3, we give two important examples in detail: (1) Feller processes on $\mathbb{R}^d$ and (2) Random conductance model.

**Notations:** Without loss of generality, throughout this paper, we use same fixed positive constants $d_1$, $d_2$, and $C_i$, $i = 0, 1, 2, \ldots$ on conditions and statements both at zero and at infinity except in Section 6. Lower case letters $a_i$ and $c_i$, $i = 0, 1, 2, \ldots$, which denote positive real constants, are fixed in each statement and proof and the labeling of these constants starts anew in each proof unless they are specified to denote particular values. We use the symbol “:=” to denote a definition, which is read as “is defined to be.” We write $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$ and $[a] := \sup\{n \in \mathbb{Z} : n \leq a\}$. $\overline{A}$ denotes the closure of $A$. We extend a function $f$ defined on $M$ to $M_0$ by setting $f(\partial) = 0$. The notation $f(x) \sim g(x)$ means that there exist constants $c_2 \geq c_1 > 0$ such that $c_1g(x) \leq f(x) \leq c_2g(x)$ for a specified range of the variable $x$.

2. Settings and Main results

2.1. Settings. Throughout this paper, we assume that $(M, d)$ is a locally compact separable metric space with a base point $o \in M$, and $\mu$ is a positive Radon measure on $M$ with full support. Denote by $B(x, r)$ the open ball centered at $x$ with radius $r$, and $\mathcal{B}(M)$ the family of all Borel sets on $M$.

For $D \in \mathcal{B}(M)$, we write
\[
\delta_D(x) = \inf\{d(x, y) : y \in M \setminus D\}.
\]
Define
\[
d(x) = d(x, o) + 1. \tag{2.1}
\]
Note that when $(M, o) = (\mathbb{R}^d, 0)$, $d(x)$ is equal to $|x| + 1$. Since $d(x) \geq 1$, the map $v \to d(x)^v$ is non-decreasing on $(0, \infty)$ for all $x \in M$. The function $\delta_D$ will be used to describe assumptions and statements for LILs near zero, and $d$ will be used for LILs near infinity.

Write $V(x, r) = \mu(B(x, r))$. We recall interior and local and weak versions of the *volume doubling and reverse doubling property* from the authors’ previous paper [10].
Definition 2.1. (i) For an open set $U \subset M$ and $R_0 \in (0, \infty]$, we say that an interior volume doubling and reverse doubling property VRD$_{R_0}(U)$ holds if there exist constants $C_0 \in (0, 1)$ and $d_1, d_2, c_\mu, C_\mu > 0$ such that for all $x \in U$ and $0 < s \leq r < R_0 \wedge (C_0 \delta_U(x))$,

$$c_\mu \left( \frac{r}{s} \right)^{d_1} \leq V(x, r) \leq C_\mu \left( \frac{r}{s} \right)^{d_2}.$$  \hfill (2.2)

(ii) For $R_\infty \geq 1$ and $v \in (0, 1)$, we say that a weak volume doubling and reverse doubling property at infinity VRD$^{R_\infty}(v)$ holds if there exist constants $d_1, d_2, c_\mu, C_\mu > 0$ such that (2.2) holds for all $x \in M$ and $r \geq s > R_\infty d(x)^v$.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, t \geq 0; \mathbb{P}^x, x \in M_\partial)$ be a Borel standard Markov process on $M_\partial := M \cup \{\partial\}$ where $\partial$ is a cemetery point added to $M$. We sometimes denote $X(t)$ for $X_t$. According to [4, Theorem 1.1], every Borel standard Markov process on $M_\partial$ has a Lévy system. In this paper, we assume that $X$ has a Lévy system of the form $(J(x, \cdot), ds)$, that is, there exists a family of Borel measures $(J(x, dy) : x \in M)$ on $M_\partial$ such that for any $z \in M$, $t > 0$ and any non-negative Borel function $F$ on $M \times M_\partial$ vanishing on the diagonal,

$$\mathbb{E}^z \left[ \sum_{s \leq t} F(X_{s-}, X_s) \right] = \mathbb{E}^z \left[ \int_0^t \int_{M_\partial} F(X_s, y)J(X_s, dy)ds \right].$$

The measure $J$ is called the Lévy measure of $X$. Note that we count the killing term $J(x, \partial)$ as a part of the Lévy measure.

For $D \in \mathcal{B}(M)$, we denote the first exit time of $X$ from $D$ by

$$\tau_D = \tau[D] := \inf\{t > 0 : X_t \in M_\partial \setminus D\}.$$ 

It is known that the part process $X^D$ of $X$ in $D$, which is defined by $X^D_t := X_t 1_{\{\tau_D > t\}} + \partial 1_{\{\tau_D \leq t\}}$, is a Borel standard process having a Lévy system induced from that of $X$. See [9, Section 3.3] for details. We call a Borel measurable function $p^D : (0, \infty) \times D \times D \to [0, \infty]$ is the heat kernel of $X^D$ if it satisfies the following properties:

1. $\mathbb{E}^x[f(X^D_t)] = \int_D p^D(t, x, y)f(y)\mu(dy)$ for all $f \in L^\infty(D; \mu)$, $t > 0$ and $x \in D$;
2. $p^D(t+s, x, y) = \int_D p^D(t, x, z)p^D(s, z, y)\mu(dz)$ for all $t, s > 0$ and $x, y \in D$.

We write $p(t, x, y)$ for $p^M(t, x, y)$.

Now, we introduce our assumptions; cf. [11]. Fix an open subset $U$ of $M$.

Assumption A. There exist constants $R_0 > 0$, $C_0 \in (0, 1)$, $C_1 > 1$, $C_i > 0$, $2 \leq i \leq 7$ and a function $\phi : U \times (0, R_0) \to (0, \infty)$ such that $\phi(z, \cdot)$ is increasing and continuous for every $z \in U$ and the following properties are satisfied for every $x \in U$ and $0 < r < R_0 \wedge (C_0 \delta_U(x))$:

- $C_1^{-1}\phi(y, r) \leq \phi(x, r) \leq C_1\phi(y, r)$ for all $y \in B(x, r)$;  \hfill (CE$_{R_0}(\phi, U)$)
- $\lim_{r \to 0} \phi(x, r) = 0$, $\phi(x, r) \leq C_2\phi(x, r/2)$;  \hfill (US$_{R_0}(\phi, U)$)
- $J(x, M_\partial \setminus B(x, r)) \leq \frac{C_3}{\phi(x, r)}$;  \hfill (Tail$_{R_0}(\phi, U, \leq)$)
- $C_4e^{-C_5n} \leq \mathbb{P}^x(\tau_{B(x, r)} \geq n\phi(x, r)) \geq C_5e^{-C_7n}$ for all $n \geq 1$.  \hfill (SP$_{R_0}(\phi, U)$)

The most delicate part of Assumption A is SP$_{R_0}(\phi, U)$. We will see in Proposition 3.3 that under VRD$_{R_0}(U)$ and SS$_{R_0}(\phi, U)$ the following condition implies SP$_{R_0}(\phi, U)$. There exist constants $c_1 > 0$ and $\eta \in (0, 1)$ such that for every $x \in U$ and $0 < r < R_0 \wedge (C_0 \delta_U(x))$, the heat kernel $p^{B(x,r)}(t, x, y)$ of $X^{B(x,r)}$ exists and satisfies that

$$p^{B(x,r)}(\phi(x, \eta r), y, z) \geq \frac{c_1}{V(x, r)} \quad \text{for all } y, z \in B(x, \eta^2 r).$$  \hfill (NDL$_{R_0}(\phi, U)$)
Next, we give assumptions for LILs at infinity.

**Assumption B.** There exist constants \( R_\infty \geq 1, v, \eta \in (0, 1), c_1 > 0, \ell > 1, C_i > 0, 2 \leq i \leq 3 \) and an increasing continuous function \( \Phi : (R_\infty, \infty) \to (0, \infty) \) such that the following properties are satisfied for every \( x \in M \) and \( r > R_\infty d(x)^v \):

\[
2\Phi(s/\ell) \leq \Phi(s) \leq C_2 \Phi(s/2) \quad \text{for all } s > R_\infty, \quad \text{(ULS}^{R_\infty}(\Phi))
\]

\[
J(x, M_\delta \setminus B(x, r)) \leq \frac{C_3}{\Phi(r)}, \quad \text{(Tail}^{R_\infty}(\Phi, v, \leq))
\]

the heat kernel \( p^{B(x,r)}(t, x, y) \) of \( X^{B(x,r)} \) exists and satisfies that

\[
p^{B(x,r)}(\Phi(\eta r), y, z) \geq \frac{c_l}{\Phi(x, r)} \quad \text{for all } y, z \in B(x, \eta^2 r). \quad \text{(NDL}^{R_\infty}(\Phi, v))
\]

**Remark 2.2.** (i) The upper scaling condition at zero \( \text{US}_{R_0}(\phi, U) \) is equivalent with that there exist constants \( \beta_2 > 0 \) and \( C_U \geq 1 \) such that

\[
\frac{\phi(x, r)}{\phi(x, s)} \leq C_U \left( \frac{r}{s} \right)^{\beta_2} \quad \text{for all } x \in U, 0 < s \leq r < R_0 \wedge (C_0 \delta_U(x)), \quad (2.3)
\]

and the upper and lower scaling condition at infinity \( \text{ULS}^{R_\infty}(\Phi) \) is equivalent with that there exist constants \( \beta_1, \beta_2 > 0 \) and \( C_U \geq 1 \geq C_L > 0 \) such that

\[
C_L \left( \frac{r}{s} \right)^{\beta_1} \leq \frac{\Phi(r)}{\Phi(s)} \leq C_U \left( \frac{r}{s} \right)^{\beta_2} \quad \text{for all } r \geq s > R_\infty. \quad (2.4)
\]

In particular, \( \text{ULS}^{R_\infty}(\Phi) \) implies \( \lim_{r \to \infty} \Phi(r) = \infty \). Note that we do not impose lower scaling condition at zero.  

(ii) One can check that under \( \text{SP}_{R_0}(\phi, U) \), there exists a constant \( \Lambda \geq 1 \) such that

\[
\Lambda^{-1} \mathbb{E}^x[\tau_{B(x,r)}] \leq \phi(x, r) \leq \Lambda \mathbb{E}^x[\tau_{B(x,r)}] \quad (2.5)
\]

for every \( x \in U \) and \( 0 < r < R_0 \wedge (C_0 \delta_U(x)) \). See [10] (4.12)–(4.13). We will see that \( \text{VRD}^{R_\infty}(v), \text{ULS}^{R_\infty}(\Phi) \) and \( \text{NDL}^{R_\infty}(\Phi, v) \) imply a near infinity counterpart of \( \text{SP}_{R_0}(\phi, U) \). Hence, under these assumptions, there exist constants \( R'_\infty, \Lambda \geq 1 \) such that (2.5) holds with \( \Phi(r) \) instead of \( \phi(x, r) \) for every \( x \in M \) and \( r > R'_\infty d(x)^v \). See Proposition 3.3(ii).

We recall the following remark from the previous paper [11].

**Remark 2.3.** (i) We impose conditions at infinity only for \( r > R_\infty d(x)^v \). By considering such weak assumptions at infinity, our results cover some random conductance models. See Subsection 6.2.

(ii) We will use a proper zero-one law for tail events (Proposition 3.5) to get a deterministic LILs at infinity. To use this zero-one law, we need \( \text{NDL}^{R_\infty}(\Phi, v) \) which is stronger than a near infinity counterpart of \( \text{SP}_{R_0}(\phi, U) \) for LILs at infinity.

2.2. **Main results.** For a Borel measurable function \( F \) on \( M \) and \( t \in [0, \infty) \), we denote \( U(F, t) \) for the (truncated) occupation time of \( X \) for \( F \) until \( t \), namely,

\[
U(F, t)(\omega) := \int_0^t F(X_s(\omega)) ds.
\]

We write \( U(D, t) := U(1_D, t) \) for \( D \in \mathcal{B}(M) \). The first set of LILs concerns occupation times in balls.
Theorem 2.4. Suppose that $\text{CE}_{R_0}(\phi, U)$, $\text{US}_{R_0}(\phi, U)$, $\text{Tail}_{R_0}(\phi, U, \leq)$ and $\text{SP}_{R_0}(\phi, U)$ hold for an open set $U \subset M$. Let $f_0 : U \times (0, \infty) \to [0, 1]$ be a deterministic function such that for every $x \in U$ and $\kappa > 0$,

$$
\lim_{r \to 0} \frac{U(B(x, r), \kappa \Phi(x, r) \log |\log \Phi(x, r)|)}{\kappa \Phi(x, r) \log |\log \Phi(x, r)|} = f_0(x, \kappa), \quad \mathbb{P}^x\text{-a.s.},
$$

(2.6)

which is well-defined by the Blumenthal’s zero-one law. Then $f_0$ satisfies the following properties:

(P1) For each fixed $x \in U$, the map $\kappa \mapsto f_0(x, \kappa)$ is strictly positive and non-decreasing.

(P2) There exists a constant $\kappa_0 > 0$ such that $f_0(x, \kappa) = 1$ for all $x \in U$ and $\kappa \leq \kappa_0$.

(P3) $\lim_{\kappa \to \infty} f_0(x, \kappa) = 0$ uniformly on $U$.

Properties (P1)-(P3) of $f_0$ in the above theorem and (2.11) below show that the results in Theorems 2.4 and 2.6 are optimal. See Remark 2.7 below. The proofs of (P3) in Theorems 2.4 and 2.6 are the most delicate parts of the paper. See Lemmas 4.3-4.5 below.

By Proposition 3.3(i) below, Theorem 2.4 implies

Corollary 2.5. Suppose that $\text{VRD}_{R_0}(U)$, $\text{CE}_{R_0}(\phi, U)$, $\text{US}_{R_0}(\phi, U)$, $\text{Tail}_{R_0}(\phi, U, \leq)$ and $\text{NDL}_{R_0}(\phi, U)$ hold for an open set $U \subset M$. Then the function $f_0 : U \times (0, \infty) \to [0, 1]$ defined by (2.6) satisfies (P1)-(P3) in Theorem 2.4.

To obtain the (uniform) limit of occupation time at infinity, we introduce a regularized function

$$
\varphi(r) := \int_{1}^{r} s^{-1} \Phi(s) ds, \quad r > 1.
$$

(2.7)

See (4.28) below for a reason of this regularizing. Under $\text{ULS}^{R_{\infty}}(\Phi)$, there is a constant $C > 1$ such that

$$
C^{-1} \Phi(r) \leq \varphi(r) \leq C \Phi(r) \quad \text{for all } r \geq 2.
$$

(2.8)

Indeed, by the monotone property of $\Phi$ and (2.4), it holds that for any $r \geq 2$,

$$
2^{-\beta_2} C_U^{-1} \log 2 \leq \frac{\Phi(r/2)}{\Phi(r)} \int_{r/2}^{r} s^{-1} ds \leq \frac{\varphi(r)}{\Phi(r)} \leq C_L^{-1} r^{-\beta_1} \int_{1}^{r} s^{-1 + \beta_1} ds \leq \beta_1^{-1} C_L^{-1}.
$$

By (2.4) and (2.8), there exist constants $C_U' \geq 1 \geq C_L' > 0$ such that

$$
C_L' \left( \frac{r}{s} \right)^{\beta_1} \leq \frac{\varphi(r)}{\varphi(s)} \leq C_U' \left( \frac{r}{s} \right)^{\beta_2} \quad \text{for all } r \geq s \geq 2 R_{\infty}.
$$

(2.9)

Theorem 2.6. Suppose that $\text{VRD}^{R_{\infty}}(\psi)$, $\text{ULS}^{R_{\infty}}(\Phi)$, $\text{Tail}^{R_{\infty}}(\phi, U, \leq)$ and $\text{NDL}^{R_{\infty}}(\phi, U)$ hold. Then there exists a deterministic non-increasing strictly positive function $f_{\infty} : (0, \infty) \to [0, 1]$ such that for every $x \in M$ and $\kappa > 0$,

$$
\lim_{r \to \infty} \frac{U(B(x, r), \kappa \varphi(r) \log \log \varphi(r))}{\kappa \varphi(r) \log \log \varphi(r)} = f_{\infty}(\kappa), \quad \mathbb{P}^x\text{-a.s., } \forall x \in M,
$$

(2.10)

where $\varphi$ is defined by (2.7). Moreover, there exists a constant $\kappa_{\infty} > 0$ such that

$$
f_{\infty}(\kappa) = 1 \quad \text{for } \kappa \leq \kappa_{\infty} \quad \text{and} \quad \lim_{\kappa \to \infty} f_{\infty}(\kappa) = 0.
$$

(2.11)

Remark 2.7. Let $\psi : U \times (0, \infty) \to (0, \infty)$ be an increasing function such that for some $x_0 \in U$,

$$
\lim_{r \to 0} \psi(x_0, r) \log \log \psi(x_0, r) = 0 \quad (\text{resp. } = \infty).
$$

Then under the setting of Theorem 2.4 by (P1)-(P3) and (2.6), it holds that for any $\kappa > 0$,

$$
\limsup_{r \to 0} \frac{U(B(x_0, r), \kappa \psi(x_0, r))}{\kappa \psi(x_0, r)} = \limsup_{r \to 0} \frac{U(B(x_0, r), \kappa \varphi(x_0, r) \log \log \varphi(x_0, r)) \psi(x_0, r) \log \log \varphi(x_0, r)}{(\kappa \varphi(x_0, r) \log \log \varphi(x_0, r)) \psi(x_0, r) \log \log \varphi(x_0, r)}.
$$
We simply denote $\Theta(x,y) = \psi(x_0,r)$, then \( \text{NDU} \) (2.6) and (2.10) are optimal and unique.

Thus, $\psi$ can not satisfy both (P2) and (P3) simultaneously. Analogous result holds concerning $\varphi(r)$ instead of $\phi(x_0,r)$. In this sense, the rate functions $\phi(x,r) \log |\log \phi(x,r)|$ and $\varphi(r) \log \log \varphi(r)$ in (2.6) and (2.10) are optimal and unique.

Next, we consider limsup behaviors of total occupation times $U(B(x,r), \infty)$ when the process $X$ is transient. To state this result, we need the following near diagonal upper heat kernel estimates.

The heat kernel $p(t,x,y)$ of $X$ exists and there exists a constant $c_u > 0$ such that
\[
p(t,x,y) \leq \frac{c_u}{V(x,\Phi^{-1}(t)) \wedge V(y,\Phi^{-1}(t))} \quad \text{for all } x,y \in M, t \geq \Phi(R_\infty(d(x) \vee d(y))^\gamma).
\]

\text{NDU}^R_\infty(\Phi,v)

Remark 2.8. Suppose that $X$ is $\mu$-symmetric and has the heat kernel $p(t,x,y)$. If there exist $R_\infty \geq 1$, $v \in (0,1)$ and $c_1 > 0$ such that
\[
p(t,x,x) \leq \frac{c_1}{V(x,\Phi^{-1}(t))} \quad \text{for all } x \in M, t > \Phi(R_\infty d(x)^v),
\]
then \text{NDU}^R_\infty(\Phi,v) is satisfied. Indeed, by the semigroup property, $\mu$-symmetry and Cauchy–Schwarz inequality, (2.12) implies that for all $x,y \in M$ and $t \geq \Phi(R_\infty(d(x) \vee d(y))^\gamma),$
\[
p(t,x,y) = \int_M p(t/2,x,z)p(t/2,z,y)\mu(dz) \leq \left( \int_M p(t/2,x,z)^2 \mu(dz) \right)^{1/2} \left( \int_M p(t/2,y,z)^2 \mu(dz) \right)^{1/2}
\leq \frac{c_1}{V(x,\Phi^{-1}(t)) \wedge V(y,\Phi^{-1}(t))}.
\]

Theorem 2.9. Suppose that VRD$^R_\infty(v)$, ULS$^R_\infty(\Phi)$, Tail$^R_\infty(\Phi,v,\leq)$, NDL$^R_\infty(\Phi,v)$ and NDU$^R_\infty(\Phi,v)$ hold. Suppose also that the upper inequality in (2.4) holds with $\beta_2 < d_1$, where $d_1$ is the constant from VRD$^R_\infty(v)$. Let $f_\infty$ be the function given in Theorem 2.6. Then the limit $\kappa_1 := \lim_{x \to \infty} \kappa f_\infty(x)$ exists in $(0, \infty)$ and for every $x \in M$,
\[
\limsup_{r \to \infty} \frac{U(B(x,r), \infty)}{\varphi(r) \log \log \varphi(r)} = \kappa_1, \quad \mathbb{P}^x\text{-a.s., } \forall z \in M.
\]

As we mentioned in the introduction, using the theory of large deviations, Donsker and Varadhan [13] showed (2.13) for transient Brownian motions, and Shieh [23] showed (2.10) for every strictly stable process $X_t$ in $\mathbb{R}^d$ ($d \geq 1$) that has an everywhere strictly positive density. But their proofs use the strict scaling property of the process which is not available in the present paper. In this paper, we take a different approach relying on our Theorem 2.6 and a Chung-type liminf LIL.

Lastly, we concern large time behaviors of $t \mapsto U(F,t)$ when $X$ is recurrent (see Remark 2.12). We define a function $\Theta$ on $M \times [0, \infty) \times [0, \infty]$ as
\[
\Theta(x,r,t) = \Theta(t,r,t; \varphi) := \int_0^t \frac{ds}{\varphi^{-1}(s)}.
\]

We simply denote $\Theta(t)$ for $\Theta(\alpha,1,t)$. See Lemma 3.1 for basic properties of $\Theta$.

Denote by $\mathcal{B}_{b,+}(M)$ the family of all bounded non-negative Borel measurable functions on $M$. Recall the definition of $d$ from (2.1). For $\gamma \in (0, \infty]$, we set
\[
\mathcal{B}_{b,+}^\gamma(M) := \begin{cases} 
\{ F \in \mathcal{B}_{b,+}(M) : d(x)^\gamma F(x) \in \mathcal{B}_{b,+}(M) \} & \text{if } \gamma < \infty; \\
\{ F \in \mathcal{B}_{b,+}(M) : F \text{ is compactly supported} \} & \text{if } \gamma = \infty.
\end{cases}
\]
Under VRD$_{R}^{R_{\infty}}(v)$, we have $\mathcal{B}^\gamma_{b,\gamma}(M) \subset L^1(M,\mu) \cap L^\infty(M,\mu)$ if $\gamma \in (d_2,\infty]$ where $d_2$ is the constant from (2.2). Indeed, for every $F \in \mathcal{B}^\gamma_{b,\gamma}(M)$ with $\gamma \in (d_2,\infty]$, we have

$$
\int_M F(y)\mu(dy) \leq \sum_{n=0}^{\infty} \int_{\{y \in M : 2^n \leq d(y) < 2^{n+1}\}} 2^{-n\gamma}d(y)^\gamma F(y)\mu(dy)
\leq c_1 \|d^\gamma F\|_{L^\infty(M,\mu)} \sum_{n=0}^{\infty} 2^{-n\gamma}V(o,2^{n+1}) \leq c_2 \|d^\gamma F\|_{L^\infty(M,\mu)} V(o,1) \sum_{n=1}^{\infty} 2^{-n(\gamma-d_2)} < \infty.
$$

We suppose that one of the following two conditions holds true:

1. $F \in \mathcal{B}^\infty_{b,\gamma}(M)$ or
2. $\beta_1 > d_2$ and $F \in \mathcal{B}^\gamma_{b,\gamma}(M)$ for some $\gamma \in (\frac{\beta_1 d_2}{\beta_1 - d_2},\infty)$.  \hspace{1cm} (2.16)

Note that each of (2.16) implies $\|F\|_{L^1(M,\mu)} < \infty$.

**Theorem 2.10.** Suppose that VRD$_{R}^{R_{\infty}}(v)$, [ULS$_{R_{\infty}}(\Phi)$ Tail$_{R_{\infty}}(\Phi,v,\leq)$], NDL$_{R_{\infty}}(\Phi,v)$ and NDU$_{R_{\infty}}(\Phi,v)$ hold. Suppose also that

$$
\Theta(x_0,1,\infty) = \infty \quad \text{for some} \quad x_0 \in M. \hspace{1cm} (2.17)
$$

Then there are constants $0 < a_1 \leq a_2 < \infty$ such that for every $F \in \mathcal{B}^\gamma_{b,\gamma}(M)$ satisfying (2.16) and $\|F\|_{L^1(M,\mu)} \neq 0$, there exists a constant $a_F \in [a_1,a_2]$ satisfying

$$
\limsup_{t \to \infty} \frac{U(F,t)/\|F\|_{L^1(M,\mu)}}{\Theta(t/\log \log \Theta(t)) \log \log \Theta(t)} = a_F, \quad \mathbb{P}_z^a \text{-a.s., } \forall z \in M. \hspace{1cm} (2.18)
$$

When the index in the lower scaling condition on $\Phi$ at infinity is bigger than the upper dimension in VRD$_{R}^{R_{\infty}}(v)$, the function $\Theta$ can be written explicitly without integral form so that we get the following corollary.

**Corollary 2.11.** Suppose that VRD$_{R}^{R_{\infty}}(v)$, [ULS$_{R_{\infty}}(\Phi)$ Tail$_{R_{\infty}}(\Phi,v,\leq)$], NDL$_{R_{\infty}}(\Phi,v)$ and NDU$_{R_{\infty}}(\Phi,v)$ hold. Suppose also that the lower inequality in (2.4) holds with $\beta_1 > d_2$, where $d_2$ is the constant from VRD$_{R}^{R_{\infty}}(v)$. Then (2.17) is satisfied and there are constants $0 < \tilde{a}_1 \leq \tilde{a}_2 < \infty$ such that for every $F \in \mathcal{B}^{\gamma}_{b,\gamma}(M)$ satisfying (2.16) and $\|F\|_{L^1(M,\mu)} \neq 0$, there exists a constant $\tilde{a}_F \in [\tilde{a}_1,\tilde{a}_2]$ satisfying

$$
\limsup_{t \to \infty} \frac{U(F,t)/\|F\|_{L^1(M,\mu)}}{t/V(o,\phi^{-1}(t/\log t))} = \tilde{a}_F, \quad \mathbb{P}_z^a \text{-a.s., } \forall z \in M. \hspace{1cm} (2.19)
$$

**Remark 2.12.** Suppose that VRD$_{R}^{R_{\infty}}(v)$, [ULS$_{R_{\infty}}(\Phi)$ NDL$_{R_{\infty}}(\Phi,v)$ and NDU$_{R_{\infty}}(\Phi,v)$] hold. Then (2.17) is satisfied if and only if $X$ is Harris recurrent, namely:

For all $z \in M$ and $A \in \mathcal{B}(M)$ with $\mu(A) > 0$, it holds that $\mathbb{P}_z^a(U(A,\infty) = \infty) = 1$.  \hspace{1cm} (2.20)

In particular, (2.17) holds if and only if $\Theta(x,1,\infty) = \infty$ for all $x \in M$.

**Proof of Remark 2.12.** Suppose that (2.20) holds. Let $x_0 \in M$. By Proposition 2.4, (2.22) yields that $\int_0^\infty p(t,x_0,y)dt = \infty$ for $\mu$-a.e. $y \in M$. For any $y \in B(x_0, d(x_0))$, it holds that $R_{\infty}(d(x_0) \vee d(y)) \leq 2R_{\infty}(d(x_0))$ by the triangle inequality. Hence, using NDU$_{R}^{R_{\infty}}(\Phi,v)$ VRD$_{R}^{R_{\infty}}(v)$ and (2.8), we get that for all $t > \phi(2R_{\infty}(d(x_0)))$ and $y \in B(x_0,d(x_0))$,

$$
p(t,x_0,y) \leq \frac{c_u}{V(x_0,\phi^{-1}(t))} \leq \frac{c_u}{V(x_0,\phi^{-1}(t/2))} \leq \frac{c_1}{V(x_0,\phi^{-1}(t/2))}.
$$

In the second inequality above, we used the fact that $B(x_0,\phi^{-1}(t/2)) \subset B(y,d(x,y) + \phi^{-1}(t/2)) \subset B(y,\phi^{-1}(t))$ for all $t > \phi(2R_{\infty}(d(x_0)))$ and $y \in B(x_0,d(x_0))$. Therefore, by (2.8), there is $c_2 > 1$ and $y \in B(x_0,d(x_0))$ such that

$$
\int_{c_2\phi(2R_{\infty}(d(x_0)))}^{\infty} p(t,x_0,y)dt \leq \int_{\phi(2R_{\infty}(d(x_0)))}^{\phi(2R_{\infty}(d(x_0)))} \frac{c_1}{V(x_0,\phi^{-1}(t))}dt \leq c_1 \Theta(x_0,1,\infty).
$$
Conversely, suppose that (2.17) holds for some \( x_0 \in M \). Let \( y, z \in M \). By NDL \( R_\infty (\Phi, \nu) \) \( 2.4 \) and \( 2.8 \), there exist \( T, c_1, c_2 > 0 \) such that \( p(t, y, z) \geq p^{B(x_0, c_1 \phi^{-1}(t))(t), y, z} \geq c_2 / V(x_0, \phi^{-1}(t)) \) for all \( t > T \). Hence, \( \int_0^\infty p(t, y, z) dt = \infty \) by \( 2.8 \) and \( 2.17 \). Now we deduce from \[ 15 \] Proposition 2.4] and \[ 18 \] Theorem 1] that (2.20) holds true. \( \square \)

3. Preliminary

Recall that \( \Theta \) is defined in \( 2.14 \) and that \( \varphi \) is defined in \( 2.7 \). We begin this section with some basic properties of \( \Theta \).

**Lemma 3.1.** (i) For every \( x \in M \), it holds that
\[
\Theta(x, r, t; \varphi) \geq \frac{t}{2V(x, \varphi^{-1}(t))} \quad \text{for all } t \geq 2\varphi(r).
\]
(ii) For every \( x \in M \), it holds that
\[
\frac{\Theta(x, r, t)}{\Theta(x, r, u)} \leq 3 \left( \frac{t}{u} \right)^{\log 3 / \log 2} \quad \text{for all } t > u \geq 2\varphi(r).
\]
(iii) Suppose that VRD \( R_\infty (\nu) \) holds. If \( 2.4 \) holds with \( \beta_1 > d_2 \), where \( d_2 \) is the constant from \( 2.2 \), then there exists \( c_1 > 0 \) such that for all \( x \in M \), \( r > R_\infty d(x)^\nu \) and \( t \geq 2\varphi(r) \),
\[
\frac{t}{2V(x, \varphi^{-1}(t))} \leq \Theta(x, r, t) \leq \frac{c_2 t}{V(x, \varphi^{-1}(t))}.
\]

**Proof.** (i) Using the monotone property of \( \varphi \), we get that for all \( t \geq 2\varphi(r) \),
\[
\Theta(x, r, t) \geq \int_{t/2}^t \frac{ds}{V(x, \varphi^{-1}(s))} \geq \frac{t}{2V(x, \varphi^{-1}(t))}.
\]
(ii) Fix \( t > u \geq 2\varphi(r) \) and let \( n_0 \geq 1 \) be the smallest integer such that \( 2^{n_0} u \geq t \). Using the monotone property of \( \varphi \) and (i), we get that
\[
\Theta(x, r, 2u) - \Theta(x, r, u) = \int_u^{2u} \frac{ds}{V(x, \varphi^{-1}(s))} \leq \frac{u}{V(x, \varphi^{-1}(u))} \leq 2 \Theta(x, r, u).
\]
By iterating the above inequalities, we deduce that
\[
\Theta(x, r, t) \leq \Theta(x, r, 2^{n_0} u) \leq \cdots \leq t^{n_0} \Theta(x, r, u) \leq 3(t/u)^{\log 3 / \log 2} \Theta(x, r, u).
\]
(iii) Using VRD \( R_\infty (\nu) \) and \( 2.9 \), we obtain that for all \( x \in M \), \( r > R_\infty d(x)^\nu \) and \( t \geq 2\varphi(r) \),
\[
\Theta(x, r, t) \leq \frac{c_1}{V(x, \varphi^{-1}(t))} \int_{\varphi(r)}^t \left( \frac{\varphi^{-1}(t)}{\varphi^{-1}(s)} \right)^{d_2} ds \leq \frac{c_2}{V(x, \varphi^{-1}(t))} \int_{\varphi(r)}^t \left( \frac{t}{s} \right)^{d_2/\beta_1} ds
\]
\[
\leq \frac{c_2}{V(x, \varphi^{-1}(t))} \int_0^t \left( \frac{t}{s} \right)^{d_2/\beta_1} ds = \frac{c_2 \beta_1}{(\beta_1 - d_2) V(x, \varphi^{-1}(t))}.
\]

For \( A \in \mathcal{B}(M) \), denote by \( \sigma_A = \inf \{ t > 0 : X_t \in A \} \) the first hitting time of \( A \).

**Lemma 3.2.** Let \( F \) be a non-negative Borel measurable function on \( M \) and \( t \in [0, \infty] \). Suppose that \( \text{supp}[F] \subset A \) for some close set \( A \). Then it holds that
\[
\sup_{x \in A} \mathbb{E}^x \left[ U(F, t) \right] = \sup_{x \in M} \mathbb{E}^x \left[ U(F, t) \right].
\]
**Proof.** For all \( w \notin A \), using the strong Markov property and the fact that \( F(X_s) = 0 \) for all \( s < \sigma_A \), we get that
\[
\mathbb{E}^w[U(F, t)] = \mathbb{E}^w[U(F, t); \sigma_A \leq t] = \mathbb{E}\left[ \mathbb{E}^{X_{\sigma_A}} \int_0^{t-\sigma_A} F(X_s) ds; \sigma_A \leq t \right] \leq \sup_{x \in A} \mathbb{E}^x[U(F, t)].
\]
This proves the lemma. \( \square \)

In the following, we give some consequences of our assumptions given in Section 2. The following is a near infinity counterpart of \([\text{SP}_{R_0}(\phi, U)]\):

There exist constants \( R_\infty \geq 1, v \in (0, 1), C_i > 0, 4 \leq i \leq 7 \) such that for every \( x \in M \) and \( r > R_\infty d(x)^v \),
\[
C_4 e^{-C_7 n} \leq \mathbb{P}^x(\tau_{B(x, r)} \leq n \phi(r)) \leq C_6 e^{-C_7 n} \quad \text{for all } n \geq 1.
\]

By following the arguments in the proof of [10, Proposition 4.3] line-by-line, we obtain the next result. We skip the proof since the proof is almost identical.

**Proposition 3.3.** (i) Suppose that \( \text{VRD}_{R_0}(U), \text{USR}_{R_0}(\phi, U) \) and \( \text{NDL}_{R_0}(\phi, U) \) hold for an open set \( U \subset M \). Then \( [\text{SP}_{R_0}(\phi, U)] \) holds true with \( R_0 > 0 \) and \( C_0 \in (0, 1) \). Moreover, with the redefined \( R_0, (2.5) \) holds for every \( x \in U \) and \( 0 < r < R_0 \land (C_0 \delta_U(x)) \).

(ii) Suppose that \( \text{VRD}^{R_\infty}(v), \text{ULS}^{R_\infty}(\phi) \) and \( \text{NDL}^{R_\infty}(\Phi, v) \) hold. Then \( [\text{SP}^{R_\infty}(\Phi, v)] \) holds with a redefined \( R_\infty \geq 1 \). Moreover, with the redefined \( R_\infty, (2.5) \) holds for every \( x \in M \) and \( r > R_\infty d(x)^v \).

We recall two propositions from [11, Section 5] which will be used in this paper several times.

**Proposition 3.4** ([11, Proposition 5.1]). (i) Suppose that \( \text{CE}_{R_0}(\phi, U), \text{USR}_{R_0}(\phi, U), \text{Tail}_{R_0}(\phi, U, \leq) \) and \( [\text{SP}_{R_0}(\phi, U)] \) hold for an open set \( U \subset M \). Then there exist constants \( q \in (0, 1] \) and \( c_1 > 0 \) such that for all \( x \in U \), \( 0 < r < 3^{-1}(R_0 \land (C_0 \delta_U(x))) \) and \( t > 0 \),
\[
\mathbb{P}^x(\tau_{B(x, r)} \leq t) \leq c_1 \left( \frac{t}{\phi(x, r)} \right)^q.
\]

(ii) Suppose that \( \text{VRD}^{R_\infty}(v), \text{ULS}^{R_\infty}(\phi), \text{Tail}^{R_\infty}(\Phi, v, \leq) \) and \( \text{NDL}^{R_\infty}(\Phi, v) \) hold. Then for every \( v_1 \in (v, 1) \), there exist constants \( c_1 > 0 \) and \( R_1 \geq R_\infty \) such that for all \( x \in M \), \( r > R_1 d(x)^v_{v_1} \) and \( t \geq \varphi(2r^{1/v_1}) \),
\[
\mathbb{P}^x(\tau_{B(x, r)} \leq t) \leq c_1 \left( \frac{t}{\varphi(r)} \right).
\]

Moreover, \( X \) is conservative, that is, \( \mathbb{P}^x(\zeta = \infty) = 1 \) for all \( x \in M \).

Let \( (\theta_t)_{t \geq 0} \) denote the shift operator which is defined by \( (X_s \circ \theta_t)(\omega) = X_{s+t}(\omega) \) for all \( s, t \geq 0 \).

An event \( G \) is called shift-invariant (with respect to \( X \)) if \( G \) is a tail event, namely, \( \cap_{t \geq 0} \sigma(X_s: s > t) \)-measurable, and \( \mathbb{P}^y(G) = \mathbb{P}^y(G \circ \theta_t) \) for all \( y \in M \) and \( t > 0 \).

**Proposition 3.5** ([11, Proposition 5.4]). Suppose that \( \text{VRD}^{R_\infty}(v), \text{ULS}^{R_\infty}(\phi), \text{Tail}^{R_\infty}(\Phi, v, \leq) \) and \( \text{NDL}^{R_\infty}(\Phi, v) \) hold. Then for every shift-invariant event \( G \), it holds either \( \mathbb{P}^y(G) = 0 \) for all \( z \in M \) or else \( \mathbb{P}^y(G) = 1 \) for all \( z \in M \).

We present limsup LILs for \( \tau_{B(x, r)} \) both at zero and at infinity. Note that the following result is closely related to Chung-type liminf LILs [11, Theorem 1.2 and Corollary 1.7].

**Proposition 3.6.** (i) Suppose that \( \text{CE}_{R_0}(\phi, U), \text{USR}_{R_0}(\phi, U), \text{Tail}_{R_0}(\phi, U, \leq) \) and \( [\text{SP}_{R_0}(\phi, U)] \) hold for an open set \( U \subset M \). Then, there exist constants \( c_2 \geq c_1 > 0 \) such that for every \( x \in U \), there exists a constant \( c_x \in [c_1, c_2] \) satisfying
\[
\limsup_{r \to 0} \frac{\tau_{B(x, r)}}{\phi(x, r) \log \log \phi(x, r)} = c_x, \quad \mathbb{P}^x\text{-a.s.}
\]
(ii) Suppose that $\text{VRD}^R_{\sim}(v)$, $\text{ULS}^R_{\sim}(\Phi)$, $\text{Tail}^R_{\sim}(\Phi,v,\leq)$ and $\text{NDLR}^R_{\sim}(\Phi,v)$ hold. Then, there exists a constant $c_3 \in (0,\infty)$ such that for every $x \in M$,
\[
\limsup_{r \to \infty} \frac{\tau_B(x,r)}{\varphi(r) \log \varphi(r)} = c_3, \quad \mathbb{P}^x\text{-a.s., } \forall z \in M.
\] (3.2)

**Proof.** (i) By the Blumenthal’s zero-one law, we get the result from [11 (5.11)].
(ii) Similarly, using Proposition 3.5, we deduce the result from [11 (5.19)].

**Remark 3.7.** In Proposition 3.6(ii), we assumed $\text{VRD}^R_{\sim}(v)$ and $\text{NDLR}^R_{\sim}(\Phi,v)$ which are stronger than $\text{SP}^R_{\sim}(\Phi,v)$ by Proposition 3.3(ii), to use Proposition 3.5 and get a deterministic limit in (3.2).

The following is one of key estimates in the proof of Theorems 2.4 and 2.6.

**Lemma 3.8.** (i) Suppose that $C\mathbb{E}_{R_0}(\phi,\mathbb{U})$, $\mathbb{U}_{R_0}(\phi,\mathbb{U})$, $\text{Tail}_{R_0}(\phi,\mathbb{U},\leq)$ and $\text{SP}_{R_0}(\phi,\mathbb{U})$ hold for an open set $U \subset M$. Then there exist constants $C_8, C_9, C_{10} > 0$ such that for all $x \in U$, $0 < r < 3^{-1}(R_0 \wedge (C_0\delta_U(x)))$ and $y \in B(x,r)$,
\[
\mathbb{E}^y \left[ \exp \left( \frac{C_8\tau_{B(y,r)}}{\phi(y,r)} \right) \right] \leq e^{C_6}.
\] (3.3)

and
\[
\mathbb{E}^y \left[ \exp \left( - \frac{C_6\tau_{B(y,r/2)}}{\phi(y,r)} \right) \right] \leq 2e^{-4}.
\] (3.4)

(ii) Suppose that $\text{ULS}^R_{\sim}(\Phi)$, $\text{Tail}^R_{\sim}(\Phi,v,\leq)$ and $\text{SP}^R_{\sim}(\Phi,v)$ hold. Define a function $\varphi$ by (2.7). There exist constants $R_2 \geq R_{\infty}$ and $C_8, C_9, C_{10} > 0$ such that (3.3) and (3.4) hold for all $x \in M$, $r \geq R_2d(x)^{2/3}$ and $y \in B(x,r^{v-1/3})$, with $\varphi(r)$ in denominators instead of $\phi(x,r)$.

**Proof.** (i) Choose any $x \in U$ and let $r_1 := R_0 \wedge (C_0\delta_U(x))$. By the triangle inequality, we see that for all $0 < r < r_1/3$ and $y \in B(x,r)$,
\[
R_0 \wedge (C_0\delta_U(y)) \geq R_0 \wedge (C_0\delta_U(x) - C_0r) \geq r_1/2.
\]
Hence, we get from $C\mathbb{E}_{R_0}(\phi,\mathbb{U})$ and $\text{SP}_{R_0}(\phi,\mathbb{U})$ that for all $0 < r < r_1/3$ and $y \in B(x,r)$,
\[
\mathbb{E}^y \left[ \exp \left( \frac{C_7\tau_{B(y,r)}}{2C_1\phi(y,r)} \right) \right] \leq \mathbb{E}^y \left[ \exp \left( \frac{C_7\tau_{B(y,r)}}{2\phi(y,r)} \right) \right] \leq \sum_{n=1}^{\infty} e^{Cn/2} \mathbb{P}^y((n-1)\phi(y,r) \leq \tau_{B(y,r)} < n\phi(y,r))
\]
\[
\leq \sum_{n=1}^{\infty} e^{Cn/2} \mathbb{P}^y(\tau_{B(y,r)} \geq (n-1)\phi(y,r)) \leq C_6e^{C_7} \sum_{n=1}^{\infty} e^{-Cn/2} < \infty.
\] (3.5)
Moreover, by $C\mathbb{E}_{R_0}(\phi,\mathbb{U})$, $\mathbb{U}_{R_0}(\phi,\mathbb{U})$ and Proposition 3.4(i), there exists a large constant $c_1 > 0$ independent of $x$ such that for all $0 < r < 3^{-1}(R_0 \wedge (C_0\delta_U(x)))$ and $y \in B(x,r)$,
\[
\mathbb{P}^y(\tau_{B(y,r/2)} \leq 4c_1^{-1}\phi(x,r)) \leq \mathbb{P}^y(\tau_{B(y,r/2)} \leq 4c_1^{-1}C_2\phi(y,r/2)) \leq e^{-4}.
\] (3.6)
Then using Markov inequality, we obtain that for all $0 < r < 3^{-1}(R_0 \wedge (C_0\delta_U(x)))$ and $y \in B(x,r)$,
\[
\mathbb{E}^y \left[ \exp \left( - \frac{C_1\tau_{B(y,r/2)}}{\phi(x,r)} \right) \right] \leq \mathbb{P}^y(\tau_{B(y,r/2)} \leq 4c_1^{-1}\phi(x,r)) + \mathbb{E}^y \left[ \exp \left( - \frac{C_1\tau_{B(y,r/2)}}{\phi(x,r)} \right) : \frac{C_1\tau_{B(y,r/2)}}{\phi(x,r)} > 4 \right] \leq 2e^{-4}.
\] (3.7)

(ii) Choose any $x \in M$ and set $v_1 := v^{2/3} \in (v,1)$. Let $R_1 \geq R_{\infty}$ be the constant from Proposition 3.4(ii). Note that for all $r \geq (2R_1)^{1/(1-v^{1/3})}d(x)^{v_1}$, we have $r > 2R_1d(x)^{v_1}$ and $r^{1-v^{1/3}} \geq 2R_1$ since $R_1 \geq 1$ and $d(x) \geq 1$. Therefore, for all $r \geq (2R_1)^{1/(1-v^{1/3})}d(x)^{v_1}$ and $y \in B(x,r^{v-1/3})$,
\[
R_1d(y)^{v_1} \leq R_1(d(x) + d(x,y))^{v_1} \leq R_1d(x)^{v_1} + R_1r^{v_1v-1/3} = R_1d(x)^{v_1} + R_1r^{v-1/3} \leq \frac{r}{2} + \frac{r}{2} = r.
\]
Hence, using \((2.8)\), \(\text{ULS}^{R_\infty}(\Phi)\), \(\text{SP}^{R_\infty}(\Phi, \nu)\) and Proposition 3.4(ii), by similar arguments as that for (i), we can deduce that \((3.5)\) and \((3.7)\) hold for all \(r \geq (2R_1)^{1/(1-\alpha/3)}d(x)^{\alpha_1}\) and \(y \in B(x, r^{\alpha_1/v})\) with \(\varphi(r)\) in denominators instead of \(\phi(x, r)\). We omit the details. \(\square\)

4. Proofs of Theorems 2.4, 2.6 and 2.9

In this section, we give proofs for the first type of LILs: Theorems 2.4, 2.6 and 2.9. The following lemma shows the monotone properties of the functions \(f_0(x, \cdot)\) and \(f_\infty(\cdot)\) in Theorems 2.4 and 2.6.

**Lemma 4.1.** Let \(f\) and \(g\) be increasing positive continuous functions defined on a subinterval of \((0, \infty)\) such that \(f \leq g\).

(i) If \(\lim_{r \to 0} g(r) = 0\), then for every \(x \in M\),
\[
\limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), f(r))}{f(r)} \geq \limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), g(r))}{g(r)}.
\]
In particular, for each fixed \(x \in U\), the function \(f_0(x, \cdot)\) defined as \((2.6)\) is non-increasing.

(ii) If \(\lim_{r \to \infty} f(r) = \infty\), then for every \(x \in M\),
\[
\limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), f(r))}{f(r)} \geq \limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), g(r))}{g(r)}.
\]
In particular, if the function \(f_\infty\) in \((2.10)\) is well-defined, then it is non-increasing.

**Proof.** (i) Note that \((f^{-1} \circ g)(r) \geq (f^{-1} \circ f)(r) = r\) for all \(r > 0\). Hence by the monotonicity of occupation times, we get
\[
\limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), f(r))}{f(r)} = \limsup_{r \to 0} \frac{\mathcal{U}(B(x, (f^{-1} \circ g)(r)), g(r))}{g(r)} \geq \limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), g(r))}{g(r)}.
\]
(ii) It can be proved by the same way. \(\square\)

Next, we prove the property (P2) in Theorem 2.4 and the first equality in \((2.11)\).

**Lemma 4.2.** (i) Under the setting of Theorem 2.4, there exists a constant \(\kappa_0 > 0\) such that for all \(x \in U\) and \(\kappa \leq \kappa_0\),
\[
\limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), \kappa \phi(x, r) \log |\log \phi(x, r)|)}{\kappa \phi(x, r) \log |\log \phi(x, r)|} = 1, \quad \mathbb{P}^x\text{-a.s.} \tag{4.1}
\]
(ii) Under the setting of Theorem 2.6, there exists a constant \(\kappa_\infty > 0\) such that for all \(x \in M\) and \(\kappa \leq \kappa_\infty\),
\[
\limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), \kappa \varphi(r) \log \log \varphi(r))}{\kappa \varphi(r) \log \log \varphi(r)} = 1, \quad \mathbb{P}^x\text{-a.s., } \forall z \in M. \tag{4.2}
\]

**Proof.** (i) By Proposition 3.6(i), there exists a constant \(\kappa_0 > 0\) independent of \(x \in U\) such that for \(\mathbb{P}^x\text{-a.s } \omega\), there is a decreasing sequence \((r_n)_{n \geq 1} = (r_n(\omega))_{n \geq 1}\) which converges to zero and
\[
\tau_{B(x, r_n)} \geq \kappa_0 \phi(x, r_n) \log |\log \phi(x, r_n)| \quad \text{for all } n \geq 1.
\]
Then for all \(\kappa \leq \kappa_0\) and \(n \geq 1\), it holds that
\[
\mathcal{U}(B(x, r_n), \kappa \phi(x, r_n) \log |\log \phi(x, r_n)|)(\omega) \geq \tau_{B(x, r_n)}(\omega) \wedge (\kappa \phi(x, r_n) \log |\log \phi(x, r_n)|)
\]
\[
= \kappa \phi(x, r_n) \log |\log \phi(x, r_n)|.
\]
This proves \((4.1)\).

(ii) Analogously, one can deduce \((4.2)\) from Proposition 3.6(ii). \(\square\)

In the following, we show that the function \(f_0\) defined as \((2.6)\) satisfies that \(\lim_{\kappa \to \infty} f_0(x, \kappa) = 0\) uniformly on \(U\). In order to prove this, we give a number of definitions first.
Let $x \in U$. For $\kappa > 0$ and $\delta \in (0, 1/4)$, let $(u_{n}^{\kappa,\delta})_{n \geq N} \equiv (u_{n}^{\kappa,\delta}(x))_{n \geq N}$ be a decreasing sequence such that
\[
\kappa \phi(x, u_{n}^{\kappa,\delta}) \log |\log \phi(x, u_{n}^{\kappa,\delta})| = (1 - \delta)^{n} \quad \text{for all } n \geq N. \tag{4.3}
\]
By taking logarithm in (4.3) twice and assuming that $N$ is large enough, we get that
\[
2^{-1} \log n \leq \log |\log \phi(x, u_{n}^{\kappa,\delta})| \leq 2 \log n \quad \text{for all } n \geq N. \tag{4.4}
\]
For $n, m \geq 1$, $\kappa > 0$ and $\delta \in (0, 1/4)$, define an event $O_{n}(\kappa, \delta, m; x)$ as
\[
O_{n}(\kappa, \delta, m; x) = \left\{ \omega \in \Omega ; \mathcal{U}(B(x, u_{n}^{\kappa,\delta}), (1 - \delta)^{n})((\omega) \geq (1 - \delta)^{n+1}(1 - m \delta) \right\}. \tag{4.5}
\]
Lastly, we define a constant $L$ which is independent of $x \in U$ as
\[
L = 4^{1+2\beta_{2}}C_{U}(2C_{9} + 1)C_{10}/C_{8}, \tag{4.6}
\]
where $\beta_{2}, C_{U}, C_{8}, C_{9}, C_{10}$ are the positive constants from (2.3), (3.3) and (3.4).

Now we claim that there exists a constant $\delta_{0} \in (0, 1/4)$ such that for every $x \in U$, $\delta \in (0, \delta_{0}]$ and $m \in \mathbb{N}$ satisfying $(m + L)\delta \leq 1$, there exists a constant $\kappa_{\delta, m} > 0$ independent of $x$ such that
\[
\sum_{n=1}^{\infty} \mathbb{P}^{x}(O_{n}(\kappa_{\delta, m}, \delta, m; x)) < \infty. \tag{4.7}
\]
Before giving the proof of (4.7), we note that (4.7) yields the desired result.

**Lemma 4.3.** If (4.7) is true, then $\lim_{\kappa \to \infty} f_{0}(x, \kappa) = 0$ uniformly on $U$.

**Proof.** Choose any $x \in U$ and $\delta \in (0, \delta_{0}/(L + 1))$. Let $m := \lfloor \delta^{-1} - L \rfloor$ and $\lambda := \kappa_{\delta, m}$. By the Borel-Cantelli lemma, it holds $\mathbb{P}^{x}$-a.s. that
\[
\limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), \lambda \phi(x, r) \log |\log \phi(x, r)|)}{\lambda \phi(x, r) \log |\log \phi(x, r)|} \leq \limsup_{n \to \infty} \sup_{r < u_{n}^{\lambda,\delta}} \left\{ \frac{\mathcal{U}(B(x, r), \lambda \phi(x, r) \log |\log \phi(x, r)|)}{\lambda \phi(x, r) \log |\log \phi(x, r)|} : \lambda u_{n+1}^{\lambda,\delta} \leq r < u_{n}^{\lambda,\delta} \right\} \tag{4.8}
\]
Hence, $f_{0}(x, \kappa) \leq (L + 1)\delta$ for all $\kappa \geq \lambda$ by Lemma 4.1. Recall that the constant $\lambda = \kappa_{\delta, m}$ is independent of $x$. Since $\delta$ can be arbitrarily small, the proof is finished. \hfill \Box

In the following two lemmas, we prove (4.7) by induction. Recall that $C_{i}, i = 8, 9, 10$ are the constants from (3.3) and (3.4).

**Lemma 4.4.** Under the setting of Theorem 2.4, there exist constants $\delta_{0} \in (0, 1/4)$ and $\lambda_{1} > 0$ such that for any $x \in U$ and $\delta \in (0, \delta_{0}]$, (4.7) holds when $m = 1$ and $\kappa_{\delta, 1} = \lambda_{1}$.

**Proof.** Let $\delta_{0} \in (0, 1/4)$ and $\lambda_{1} > 0$ be constants chosen later. Fix $x \in U$ and $\delta \in (0, \delta_{0}]$. We simply write $u_{n} = u_{n}^{\lambda_{1},\delta}(x)$ and $O_{n} = O_{n}(\lambda_{1}, \delta, 1; x)$. For each $n \geq N$, we define sequences of random times $(\sigma_{n}^{j})_{j \geq 1}$ and $(\tau_{n}^{j})_{j \geq 1}$ as
\[
\sigma_{1}^{n} = 0, \quad \tau_{1}^{n} = \inf \{ s > \sigma_{1}^{n} : X_{s} \notin B(x, 2u_{n}) \} \quad \text{and} \quad \sigma_{j+1}^{n} = \inf \{ s > \tau_{j}^{n} : X_{s} \in B(x, u_{n}) \}. \tag{4.9}
\]
Recall that $(\theta_{t})_{t \geq 0}$ denotes the shift operator. By Markov inequality, we have that
\[
\mathbb{P}^{x}(O_{n} : \tau_{\lfloor \log n \rfloor}^{n} \geq (1 - \delta)^{n}) \leq \mathbb{P}^{x}(O_{n} : \sum_{j=1}^{\lfloor \log n \rfloor} \tau_{B(x, 2u_{n})} \circ \theta_{\sigma_{j}^{n}} \geq \mathcal{U}(B(x, u_{n}), (1 - \delta)^{n})). \tag{4.10}
\]
Using the strong Markov property and the fact that $B(x, 2u_n) \subset B(y, 4u_n)$ for all $y \in B(x, 2u_n)$ in the first inequality below and $(3.3)$ in the last inequality, we get that for all $n$ large enough,

$$
\mathbb{E}^x \left[ \exp \left( \frac{C_8(1 - \delta)^{n+2}}{\phi(x, 4u_n)} \sum_{j=1}^{\lfloor \log n \rfloor} \tau_{B(x, 2u_n) \circ \theta_{\sigma_j^n}} \right) \right] 
\leq \mathbb{E}^x \left[ \exp \left( \frac{C_8}{\phi(x, 4u_n)} \sum_{j=1}^{\lfloor \log n \rfloor} \tau_{B(x, 2u_n) \circ \theta_{\sigma_j^n}} \right) \right].
$$

(4.10)

By (4.10), (4.11) and the definition (4.3), using (4.4), (2.3) and the fact that $\delta < 1/4$, we get that for all $n$ large enough,

$$
\mathbb{P}^x \left( \sigma_{\lfloor \log n \rfloor} \geq (1 - \delta)^{n} \right) \leq \exp \left( C_9 \log n - \lambda_1 \frac{C_8(1 - \delta)^2 \phi(x, u_n) \log \log \phi(x, u_n)}{\phi(x, 4u_n)} \right) 
\leq \exp \left( (C_9 - c_1 \lambda_1) \log n \right).
$$

(4.12)

where $c_1 := 4^{-1-\beta_2}C_8/C_U$ is independent of $(x, \kappa_1, \delta)$.

Set $r_0 := 3^{-1}(R_0 \wedge (C_0 \delta_U(x)))$. Using Markov inequality and the fact that $\mathcal{U}(D_1, t) \leq t - \mathcal{U}(D_2, t)$ for all $t \geq 0$ when $D_1, D_2 \in \mathcal{B}(M)$ are disjoint, we see that for all $n \geq N$,

$$
\mathbb{P}^x \left( \sigma_{\lfloor \log n \rfloor} < (1 - \delta)^{n} < \mathcal{U}(x, r_0) \right) 
\leq \mathbb{P}^x \left( \sigma_{\lfloor \log n \rfloor - 1} \sum_{j=1}^{\lfloor \log n \rfloor - 1} \mathcal{U}(x_0, d(x_0) - u_n) \circ \theta_{\tau_j^n} \leq (1 - \delta)^{n} - \mathcal{U}(x, u_n), (1 - \delta)^{n}, \mathcal{U}(x, r_0) \right),
$$

$$
\mathbb{P}^x \left( \sum_{j=1}^{\lfloor \log n \rfloor - 1} \tau_{B(x_0, d(x_0) - u_n) \circ \theta_{\tau_j^n}} \leq 2\delta(1 - \delta)^{n}, X(\tau_j^n) \in B(x, r_0), 1 \leq j \leq \lfloor \log n \rfloor - 1 \right)
\leq \exp \left( 2C_{10}\delta(1 - \delta)^{n} \right) \mathbb{E}^x \left[ \exp \left( - \frac{C_{10}}{\phi(x, 2u_n)} \sum_{j=1}^{\lfloor \log n \rfloor - 1} \tau_{B(x_0, d(x_0) - u_n) \circ \theta_{\tau_j^n}} \right) \right].
$$

(4.13)

For every $y \in B(x, r_0) \setminus B(x, 2u_n)$, we get from (3.4) that

$$
\mathbb{E}^x \left[ \exp \left( - \frac{C_{10}}{\phi(x, 2u_n)} \tau_{B(y,d(x,y) - u_n)} \right) \right] \leq \mathbb{E}^x \left[ \exp \left( - \frac{C_{10}}{\phi(x, d(x,y))} \tau_{B(y,d(x,y)/2)} \right) \right] \leq e^{-3}.
$$

(4.14)
Since $X(\tau^n) \notin B(x, 2u_n)$, by the strong Markov property, it follows that for all $n$ large enough,

$$
E^x \left[ \exp \left( -\frac{C_{10}}{\phi(x, 2u_n)} \sum_{j=1}^{[\log n] - 1} \tau_{B(X_0, d(x, x_0) - u_n)} \circ \theta_{\tau^n} \right) : X(\tau^n) \in B(x, r_0), 1 \leq j \leq [\log n] - 1 \right]
$$

$$
= E^x \left[ \exp \left( -\frac{C_{10}}{\phi(x, 2u_n)} \sum_{j=1}^{[\log n] - 1} \tau_{B(X_0, d(x, x_0) - u_n)} \circ \theta_{\tau^n} \right) : X(\tau^n) \in B(x, r_0), 1 \leq j \leq [\log n] - 1 \mid \mathcal{F}_{[\log n] - 1} \right]
$$

$$
\leq \left( \sup_{y \in B(x, r_0) \setminus B(x, 2u_n)} E^y \left[ \exp \left( -\frac{C_{10} \tau_{B(y, d(x, y) - u_n)}}{\phi(x, 2u_n)} \right) \right] \right)^{[\log n] - 1} \leq e^{6 - 3 \log n}.
$$

Thus, by the definition (4.3), the monotone property of $\phi(x, \cdot)$ and (4.4), we obtain that for all $n$ large enough,

$$
P^x \left( O_n : \tau^n_{[\log n]} < (1 - \delta)^n < \tau_{B(x, r_0)} \right) \leq e^{6} \exp \left( -3 \log n + \delta \lambda_1 \frac{2C_{10} \phi(x, u_n) \log \log \phi(x, u_n)}{\phi(x, 2u_n)} \right)
$$

$$
\leq e^{6} \exp \left( -(3 + 4C_{10} \delta \lambda_1) \log n \right).
$$

Take $\lambda_1 = (2 + C_9)/c_1$ and $\delta_0 = 1/(4C_{10} \lambda_1)$. Then by (4.12), (4.16) and Proposition 3.4(i), it holds that for all $n$ large enough,

$$
P^x \left( O_n(\lambda_1, \delta, 1; x) \right)
$$

$$
\leq P^x \left( O_n : \tau^n_{[\log n]} \geq (1 - \delta)^n \right) + P^x \left( O_n : \tau^n_{[\log n]} < (1 - \delta)^n < \tau_{B(x, r_0)} \right) + P^x \left( \tau_{B(x, r_0)} \leq (1 - \delta)^n \right)
$$

$$
\leq \exp \left( (C_9 - c_1 \lambda_1) \log n \right) + e^{6} \exp \left( -(3 + 4C_{10} \delta_0 \lambda_1) \log n \right) + c_2 \phi(x, r_0)^{-q(1 - \delta)^n q}
$$

$$
\leq n^{-2} + e^{6} n^{-2} + c_2 \phi(x, r_0)^{-q(1 - \delta)^n q}.
$$

The proof is complete. 

\textbf{Lemma 4.5.} Under the setting of Theorem 2.4, (4.7) is true.

\textbf{Proof.} Choose any $x \in U$ and $\delta \in (0, \delta_0]$. When $m = 1$, (4.7) is satisfied by Lemma 4.4. For the next step, let $m \in \mathbb{N}$ be such that $(m + 1 + L) \delta \leq 1$ and suppose that (4.7) is satisfied with $m$.

Let $\kappa_{\delta, m+1} \geq 1 + \beta_2 C_1 \kappa_{\delta, m}$ be a constant chosen later. For simplicity, we write as

\begin{align*}
\bar{u}_n &= u_n^{\kappa_{\delta, m}, \delta}(x), \quad \bar{u}_n = u_n^{\kappa_{\delta, m+1}, \delta}(x), \quad O_n = O_n(\kappa_{\delta, m}, \delta, m; x) \quad \text{and} \quad \bar{O}_n = O_n(\kappa_{\delta, m+1}, \delta, m + 1; x),
\end{align*}

where $u_n^{\kappa_{\delta, m}}(x)$ and $O_n(\kappa, \delta, m; x)$ are defined by (4.3) and (4.5) respectively. Using (2.3) and (4.4), we get that for all $n$ large enough,

$$
4 \kappa_{\delta, m} \phi(x, 4 \bar{u}_n) \leq 4^{1 + \beta_2} C_1 \kappa_{\delta, m} \phi(x, \bar{u}_n) \leq \kappa_{\delta, m+1} \phi(x, \bar{u}_n) = \frac{(1 - \delta)^n}{\log \log \phi(x, u_n)}
$$

$$
\leq \frac{2(1 - \delta)^n}{\log n} \leq \frac{4(1 - \delta)^n}{\log \log \phi(x, u_n)} = 4 \kappa_{\delta, m} \phi(x, u_n).
$$

(4.17)
Thus, $4\tilde{u}_n \leq u_n$ for all $n$ large enough.

Our task is to show $\sum_{n=1}^{\infty} \mathbb{P}^x(\tau_n) < \infty$. Since $\sum_{n=1}^{\infty} \mathbb{P}^x(\bar{O}_n) < \infty$ by the induction hypothesis, it suffices to show $\sum_{n=1}^{\infty} \mathbb{P}^x(\bar{O}_n \setminus O_n) < \infty$. Define $(\bar{\sigma}^n_j)_{j \geq 1}$ and $(\bar{\tau}^n_j)_{j \geq 1}$ as (cf. (4.9))

$\bar{\sigma}^n_1 := 0$, $\bar{\tau}^n_j := \inf\{s > \bar{\sigma}^n_j : X_s \notin B(x, 2\tilde{u}_n)\}$ and $\bar{\sigma}^n_{j+1} := \inf\{s > \bar{\tau}^n_j : X_s \in B(x, \tilde{u}_n)\}$.

Following the calculations in (4.10), we see that for any constant $K > 0$,

$$\mathbb{P}^x\left(\bar{O}_n \setminus O_n : \bar{\tau}^n_{2[K \log n]} \geq (1 - \delta)^n\right)$$

$$\leq \mathbb{P}^x\left(\bar{O}_n \setminus O_n : \sum_{j=1}^{2[K \log n]} \tau_{B(x, 2\tilde{u}_n)} \circ \theta_{\bar{\sigma}^n_j} \geq \mathcal{U}(B(x, \tilde{u}_n), (1 - \delta)^n)\right)$$

$$\leq \mathbb{P}^x\left(\sum_{j=1}^{2[K \log n]} \tau_{B(x, 2\tilde{u}_n)} \circ \theta_{\bar{\sigma}^n_j} \geq (1 - \delta)^{n+1}(1 - \delta - m\delta)\right)$$

$$\leq \exp\left(- \frac{C_8(1 - \delta)^{n+1}(1 - \delta - m\delta)}{\phi(x, 4\tilde{u}_n)}\right) \mathbb{E}^x\left[\exp\left(\frac{C_8}{\phi(x, 4\tilde{u}_n)} \sum_{j=1}^{2[K \log n]} \tau_{B(x, 2\tilde{u}_n)} \circ \theta_{\bar{\sigma}^n_j}\right)\right].$$

Also, as in the calculations showing (4.11), we get from the strong Markov property and (3.3) that for all $n$ large enough,

$$\mathbb{E}^x\left[\exp\left(\frac{C_8}{\phi(x, 4\tilde{u}_n)} \sum_{j=1}^{2[K \log n]} \tau_{B(x, 2\tilde{u}_n)} \circ \theta_{\bar{\sigma}^n_j}\right)\right]$$

$$\leq \mathbb{E}^x\left[\exp\left(- \frac{C_8}{\phi(x, 4\tilde{u}_n)} \sum_{j=1}^{2[K \log n]-1} \tau_{B(x, 2\tilde{u}_n)} \circ \theta_{\bar{\sigma}^n_j}\right)\right] \sup_{y \in B(x, 2\tilde{u}_n)} \mathbb{E}^y\left[\exp\left(\frac{C_8 \tau_{B(y, 4\tilde{u}_n)}}{\phi(x, 4\tilde{u}_n)}\right)\right]$$

$$\leq \ldots \leq \left(\sup_{y \in B(x, 2\tilde{u}_n)} \mathbb{E}^y\left[\exp\left(\frac{C_8 \tau_{B(y, 4\tilde{u}_n)}}{\phi(x, 4\tilde{u}_n)}\right)\right]\right)^{2[K \log n]} \leq e^{2C_8 K \log n}.$$

Thus, since $\delta \leq \delta_0 < 1/4$ and $1 - \delta - m\delta \geq L\delta$, by (4.4) and (2.3), it holds that for all $n$ large enough,

$$\mathbb{P}^x\left(\bar{O}_n \setminus O_n : \bar{\tau}^n_{2[K \log n]} \geq (1 - \delta)^n\right)$$

$$\leq \exp\left(2C_8 K \log n - \kappa_{\delta, m+1} C_8 (1 - \delta)(1 - \delta - m\delta)\phi(x, \tilde{u}_n) \log \log \phi(x, \tilde{u}_n)\right)$$

$$\leq \exp\left((2C_9 K - c_1 L\delta\kappa_{\delta, m+1}) \log n\right),$$

where $c_1 := 4^{-1-\beta_2} C_8/C_U$.

Next, we set $r_0 := 3^{-1}(R_0 \wedge (C_9 \delta U(x)))$ as in the proof of Lemma 4.4 and define two random sequences of natural numbers $(a_\ell)_{\ell \geq 0}$ and $(b_\ell)_{\ell \geq 0}$ as

$$a_0 = 0, \quad a_{\ell+1} := \inf\{j > a_\ell : X(\bar{\tau}^n_j) \in B(x, u_n - \tilde{u}_n) \setminus B(x, 2\tilde{u}_n)\}$$

and

$$b_0 = 0, \quad b_{\ell+1} := \inf\{j > b_\ell : X(\bar{\tau}^n_j) \notin B(x, u_n - \tilde{u}_n)\}.$$

Let $\bar{\tau}^{n,1}_\ell := \bar{\tau}^{n}_{a_\ell}$ and $\bar{\tau}^{n,2}_\ell := \bar{\tau}^{n}_{b_\ell}$. Since $X(\bar{\tau}^n_j) \notin B(x, 2\tilde{u}_n)$ by the definition of $\bar{\tau}^n_j$, we have

$$\mathbb{P}^x\left(\bar{O}_n \setminus O_n : \bar{\tau}^n_{2[K \log n]} < (1 - \delta)^n \geq \tau_{B(x, r_0)}\right)$$

$$\leq \mathbb{P}^x\left(\bar{O}_n \setminus O_n : \bar{\tau}^{n,1}_{[K \log n]} < (1 - \delta)^n \geq \tau_{B(x, r_0)}\right)$$
Thus, by following the calculations in (4.13), (4.14), (4.15) and (4.16) in turn, we get that for all $n$ large enough,

$$
\mathbb{P}^x \left( O_n \setminus O_n : \tilde{\tau}_{[K \log n]}^{n,2} < (1 - \delta)^n < \tau_B(x,r_0) \right)
$$

$$
\leq \mathbb{P}^x \left( \sum_{j=1}^{[K \log n] - 1} \tau_{B(\bar{x},d(x,x_0) - \bar{u}_n)} \circ \theta_{\tau_{\hat{j}}} \leq \delta(1 - \delta)^{n+1}, \ X(\tilde{\tau}_{\hat{j}}^{n,1}) \in B(x,r_0), \ 1 \leq j \leq \lfloor K \log n \rfloor - 1 \right)
$$

$$
\leq \left( \sup_{y \in B(x,r_0) \setminus B(2 \bar{u}_n)} \mathbb{E}^y \left[ \exp \left( - \frac{C_{10} \tau_{B(y,d(x,y) - \bar{u}_n)}}{\phi(\bar{x},2 \bar{u}_n)} \right) \right] \right)^{[K \log n] - 1} \exp \left( \frac{C_{10} \delta(1 - \delta)^{n+1}}{\phi(\bar{x},2 \bar{u}_n)} \right)
$$

$$
\leq \delta^6 \exp \left( - 3K \log n + 3C_{10} \delta \kappa \delta_{m+1} \log \phi(x,u) \right)
$$

$$
\leq \delta^6 \exp \left( - 3K + 2C_{10} \delta \kappa \phi(x,u) \log \phi(x,u) \right).
$$

Besides, for all $n$ large enough, using the strong Markov property, (3.4), the definition (4.3), (4.4), (2.3) and the fact that $u_n \geq 4 \bar{u}_n$ follows from (4.17), we get that

$$
\mathbb{P}^x \left( O_n \setminus O_n : \tilde{\tau}_{[K \log n]}^{n,2} < (1 - \delta)^n < \tau_B(x,r_0) \right) \leq \mathbb{P}^x \left( \tilde{\tau}_{[K \log n]}^{n,2} < (1 - \delta)^n \right)
$$

$$
\leq \mathbb{P}^x \left( \sum_{\ell=1}^{[K \log n] - 1} \tau_{B(x,u_n - 2 \bar{u}_n)} \circ \theta_{\tilde{\delta}_{\ell+1}} \leq \tilde{\tau}_{[K \log n]}^{n,2} < (1 - \delta)^n \right)
$$

$$
\leq \mathbb{P}^x \left( \sum_{\ell=1}^{[K \log n] - 1} \tau_{B(x,u_n - 2 \bar{u}_n)} \circ \theta_{\tilde{\delta}_{\ell+1}} < (1 - \delta)^n \right)
$$

$$
\leq \exp \left( \frac{C_{10} \tau_{[K \log n]}^{n,2}}{\phi(x,2(u_n - 2 \bar{u}_n))} \right)^{[K \log n] - 1} \mathbb{E}^x \left[ \exp \left( - \frac{C_{10} \tau_{B(x,u_n - 2 \bar{u}_n)}}{\phi(x,2(u_n - 2 \bar{u}_n))} \right) \sum_{j=1}^{[K \log n] - 1} \tau_{B(x,u_n - 2 \bar{u}_n)} \circ \theta_{\tilde{\delta}_{\ell+1}} \right]
$$

$$
\leq \left( \sup_{y \in B(x,u_n)} \mathbb{E}^y \left[ \exp \left( - \frac{C_{10} \tau_{B(x,u_n - 2 \bar{u}_n)}}{\phi(x,2(u_n - 2 \bar{u}_n))} \right) \right] \right)^{[K \log n] - 1} \exp \left( \frac{C_{10} \tau_{[K \log n]}^{n,2}}{\phi(x,2(u_n - 2 \bar{u}_n))} \right)
$$

$$
\leq \delta^6 \exp \left( - 3K \log n + \kappa \delta \phi(x,u) \log \phi(x,u) \right) \leq \delta^6 \exp \left( - 3K + 2C_{10} \kappa \delta \log n \right).
$$
Finally, using (4.18), (4.21), (4.23), (4.24) and Proposition 3.4(i), we deduce that for all \( n \) large enough,

\[
\begin{align*}
\mathbb{P}^x(O_n \setminus O_n) &\leq \mathbb{P}^x(O_n \setminus O_n : \tau_{2[K \log n]} \geq (1 - \delta)^n) + \mathbb{P}^x(O_n \setminus O_n : \tau_{2[K \log n]} < (1 - \delta)^n < \tau_{B(x, r_0)}) \\
&+ \mathbb{P}^x(\tau_{B(x, r_0)} \leq (1 - \delta)^n) \\
&\leq \exp\left((2C_9 K - c_1 L\delta \kappa_{\delta, m+1}) \log n\right) + e^6 \exp\left((-3K + 2C_{10}\delta \kappa_{\delta, m+1}) \log n\right) \\
&\quad + e^6 \exp\left((-3K + 2C_{10}\delta \kappa_{\delta, m}) \log n\right) + c_2 \phi(x, r_0)^{-\eta}(1 - \delta)^{n\eta}.
\end{align*}
\]

(4.25)

Note that the constant \( L \) defined as (4.6) is equal to \((2C_9 + 1)C_{10}/c_1\). Take

\[
\kappa_{\delta, m+1} = 2\delta^{-1}C_{10}^{-1} \vee \delta^{-1}\kappa_{\delta, m} \vee 4^{1 + \beta_2} C_U \kappa_{\delta, m} \quad \text{and} \quad K = C_{10}\delta \kappa_{\delta, m+1}.
\]

Then one can check that

\[
\begin{align*}
2C_9 K - c_1 L\delta \kappa_{\delta, m+1} &= (2C_9 C_{10} - c_1 L)\delta \kappa_{\delta, m+1} = -C_{10}\delta \kappa_{\delta, m+1} \leq -2, \\
-3K + 2C_{10}\delta \kappa_{\delta, m+1} &= -C_{10}\delta \kappa_{\delta, m+1} \leq -2, \\
-3K + 2C_{10}\delta \kappa_{\delta, m} &\leq -3K + 2C_{10}\delta \kappa_{\delta, m} \leq -2.
\end{align*}
\]

(4.26)

It now follows from (4.25) that \( \mathbb{P}^x(O_n \setminus O_n) \) is summable. The proof is complete. \( \square \)

**Proof of Theorem 2.4** The result follows from Lemmas 4.1(i), 4.2(i), 4.3 and 4.5 \( \square \)

Theorem 2.6 can be proved by a similar way to that of Theorem 2.4 with some modifications.

Recall that \( \phi \) is the function defined in (2.7). Before giving the proof of Theorem 2.6, we first give a lemma about regularity of the function \( \phi \).

**Lemma 4.6.** If [ULS] holds, then for any \( a > 0 \), \( \lim_{r \to \infty} \varphi(r + a)/\varphi(r) = 1 \).

**Proof.** Let \( a > 0 \). Obviously, \( \lim_{r \to \infty} \varphi(r + a)/\varphi(r) \geq 1 \). On the other hand, by the mean value theorem, (2.8) and (2.4), we obtain

\[
\begin{align*}
\limsup_{r \to \infty} \frac{\varphi(r + a) - \varphi(r)}{\varphi(r)} &\leq c_1 \limsup_{r \to \infty} \frac{\sup_{s \in (r, r + a)} a s^{-1} \phi(s)}{\phi(r)} \\
&\leq c_2 \limsup_{r \to \infty} a \left( \frac{r + a}{r} \right)^{\beta_2} = 0.
\end{align*}
\]

Hence, we get \( \limsup_{r \to \infty} \varphi(r + a)/\varphi(r) \leq 1 \) and finish the proof. \( \square \)

**Proof of Theorem 2.6** For \( x \in M, \kappa > 0 \) and \( a \in (0, 1] \), define

\[
E_{x, \kappa}(a) = \left\{ \omega \in \Omega : \limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), \kappa \varphi(r) \log \log \varphi(r))(\omega)}{\kappa \varphi(r) \log \log \varphi(r)} < a \right\}
\]

and \( g(x, \kappa) = \inf \{ a \in (0, 1] : \mathbb{P}^x(E_{x, \kappa}(a)) = 1 \} \). Since \( \lim_{r \to \infty} \varphi(r) = \infty \) by (2.8) and Remark 2.2(i), one can see that \( E_{x, \kappa}(a) \) is shift-invariant for all \( a \in (0, 1] \). Thus, by Proposition 3.3, for each fixed \( a \in (0, 1] \), it holds that either \( \mathbb{P}^x(E_{x, \kappa}(a)) = 1 \) for all \( z \in M \), or \( \mathbb{P}^x(E_{x, \kappa}(a)) = 0 \) for all \( z \in M \). Therefore, for every \( x \in M \) and \( \kappa > 0 \),

\[
\limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), \kappa \varphi(r) \log \log \varphi(r))}{\kappa \varphi(r) \log \log \varphi(r)} = g(x, \kappa), \quad \mathbb{P}^x\text{-a.s., } \forall z \in M.
\]

(4.27)

By the monotone property of occupation times and Lemma 4.6, it holds that for any \( x, y \in M \),

\[
g(x, \kappa) \leq \limsup_{r \to \infty} \frac{\mathcal{U}(B(y, r + d(x, y)), \kappa \varphi(r + d(x, y)) \log \log \varphi(r + d(x, y)))}{\kappa \varphi(r) \log \log \varphi(r)} \\
\leq g(y, \kappa) \limsup_{r \to \infty} \frac{\varphi(r + d(x, y)) \log \log \varphi(r + d(x, y))}{\varphi(r) \log \log \varphi(r)} = g(y, \kappa).
\]

(4.28)
Hence, \( g(x, \kappa) \) is equal to \( g(y, \kappa) \) by symmetry so that is independent of \( x \). Define \( f_\infty(\kappa) = g(o, \kappa) \). By \((4.27)\), \( f_\infty \) satisfies \((2.10)\). Moreover, by Lemmas 4.1(ii) and 4.2(ii), \( f_\infty \) is non-increasing and there exists \( \kappa_\infty > 0 \) such that \( f_\infty(\kappa) = 1 \) for all \( \kappa \leq \kappa_\infty \). Thus it remains to show \( \lim_{\kappa \to \infty} f_\infty(\kappa) = 0 \).

To do this, we follow the proofs of Lemmas 4.3, 4.4 and 4.5.

For \( \kappa > 0 \) and \( \delta \in (0, 1/8) \), let \((w_n^{\kappa, \delta})_{n \geq N} \) be an increasing sequence such that \((4.3)\)

\[
\kappa \varphi(w_n^{\kappa, \delta}) \log \log \varphi(w_n^{\kappa, \delta}) = (1 + \delta)^n \quad \text{for all } n \geq N.
\]

(4.29)

By assuming that \( N \) is large enough, similar to \((4.4)\), we get from \((4.29)\) that

\[
2^{-1} \log n \leq \log \varphi(w_n^{\kappa, \delta}) \leq 2 \log n \quad \text{for all } n \geq N.
\]

(4.30)

For \( \kappa > 0 \), \( \delta \in (0, 1/8) \) and \( m \in \mathbb{N} \), we set

\[
\mathcal{O}_n(\kappa, \delta, m) := \left\{ U(B(o, w_n^{\kappa, \delta}), (1 + \delta)^n) \geq (1 + \delta)^n(1 - m\delta) \right\}, \quad n \geq N.
\]

Let

\[
L' := 4^{1 + \beta_2} C'_U(2C_9 + 1)C_{10}/C_8,
\]

where \( \beta_2, C'_U \) are the constants from \((2.9)\) and \( C_8, C_9, C_{10} \) from Lemma 3.8(ii). Cf. \((4.6)\). We will prove in Lemma 4.7 below that there exists a constant \( \delta_0 \in (0, 1/8) \) such that for any \( \delta \in (0, \delta_0] \) and \( m \in \mathbb{N} \) satisfying \((m + L')\delta \leq 1 \), there exists a constant \( \kappa_{\delta, m} > 0 \) such that

\[
\sum_{n=1}^{\infty} \mathbb{P}^\omega(\mathcal{O}_n(\kappa_{\delta, m}, \delta, m)) < \infty.
\]

(4.31)

Assume \((4.31)\) for the moment. Choose any \( \delta \in (0, \delta_0/(L' + 1)) \), and set \( m := \lfloor \delta^{-1} - L' \rfloor \) and \( \lambda := \kappa_{\delta, m} \). Then, similar to \((4.8)\), using the Borel-Cantelli lemma, we get from \((4.31)\) that \( \mathbb{P}^\omega \)-a.s.,

\[
\limsup_{r \to \infty} \frac{U(B(o, r), \lambda \varphi(r) \log \log \varphi(r))}{\lambda \varphi(r) \log \varphi(r)} \leq \limsup_{n \to \infty} \frac{U(B(o, w_n^{\lambda, 0}), \lambda \varphi(w_n^{\lambda, 0}) \log \varphi(w_n^{\lambda, 0}))}{\lambda \varphi(w_n^{\lambda, 0}) \log \varphi(w_n^{\lambda, 0})} = \limsup_{n \to \infty} \frac{U(B(o, w_n^{\lambda, 0}), (1 + \delta)^n)}{(1 + \delta)^n-1} \leq 1 - m\delta \leq (L' + 1)\delta.
\]

Thus, \( f_\infty(\lambda) \leq (L' + 1)\delta \). By the monotone property of \( f_\infty \), since \( \delta \) can be chosen arbitrarily small, we finish the proof. \( \square \)

**Lemma 4.7.** Under the setting of Theorem \((2.6)\) \((4.31)\) is true.

**Proof.** Let \( \delta_0 \in (0, 1/8) \) and \( \lambda_1 > 0 \) be constants chosen later. Choose any \( \delta \in (0, \delta_0] \), and write \( w_n = w_n^{\lambda_1, \delta} \) and \( \mathcal{O}_n = \mathcal{O}_n(\lambda_1, \delta, 1) \). Then we define

\[
\sigma_n := 0, \quad \tau_n := \inf\{s > \sigma_n : X_s \notin B(o, 2w_n)\} \quad \text{and} \quad \tau_{n+1} := \inf\{s > \tau_n : X_s \in B(o, w_n)\}.
\]

By following the calculations in \((4.10)\) and \((4.11)\), we get from Lemma 3.8(ii) that for all \( n \) large enough,

\[
\mathbb{P}^\omega \left( \mathcal{O}_n : \tau_{[\log n]} \geq (1 + \delta)^n \right) \leq \mathbb{P}^\omega \left( \sum_{j=1}^{[\log n]} \tau_{B(o, 2w_n)} \circ \theta_{\tau_j} \geq (1 + \delta)^n(1 - \delta) \right)
\]

\[
\leq \exp \left( - \frac{C_8(1 + \delta)^{-2}(1 - \delta)}{\varphi(4w_n)} \right) \mathbb{P}^\omega \left[ \exp \left( \frac{C_8}{\varphi(4w_n)} \sum_{j=1}^{[\log n]} \tau_{B(o, 2w_n)} \circ \theta_{\tau_j} \right) \right]
\]

\[
\leq \exp \left( - \frac{C_8(1 + \delta)^{-2}(1 - \delta)}{\varphi(4w_n)} \right) \left( \sup_{y \in B(o, 2w_n)} \mathbb{P}^y \left[ \exp \left( \frac{C_8\tau_{B(y, 4w_n)}}{\varphi(4w_n)} \right) \right] \right)^{\log n}
\]
\[ \leq \exp \left( C_9 \log n - \frac{C_8 (1 + \delta)^{n-2}(1 - \delta)}{\varphi(4w_n)} \right). \]

Note that \((1 + \delta)^{-2}(1 - \delta) > 1/2\) since \(\delta < 1/8\). By the definition \((4.29)\) and \((2.9)\), it follows that for all \(n\) large enough,

\[ P^o \left( \mathcal{O}_n : \tau_n^{\log n} \geq (1 + \delta)^n \right) \leq \exp \left( C_9 \log n - \lambda_1 C_8 \varphi(w_n) \log \log \varphi(w_n) \right) \]

\[ \leq \exp \left( (C_9 - c_1 \lambda_1) \log n \right), \quad (4.32) \]

where \(c_1 := 4^{-1-\beta_2}C_8/C'_{U} \). Next, by similar calculations to that for obtaining \((4.13)-(4.16)\), we get that for \(p := v^{-1/3} > 1\) and all \(n\) large enough,

\[ P^o \left( \mathcal{O}_n : \tau_n^{\log n} < (1 + \delta)^n < \tau_{B(o,w_n^p)} \right) \]

\[ \leq P^x \left( \sum_{j=1}^{[\log n] - 1} \tau_B \left( X_{o,w_n^p} \right) \circ \theta_{\tau_j^n} \leq 2\delta(1 + \delta)^{n - 1}, \quad X(\tau_j^n) \in B(o,w_n^p), \quad 1 \leq j \leq [\log n] - 1 \right) \]

\[ \leq \exp \left( \frac{2C_10\delta(1 + \delta)^{n - 1}}{\varphi(2w_n)} \right) P^o \left[ \exp \left( - \frac{C_10}{\varphi(2w_n)} \sum_{j=1}^{[\log n] - 1} \tau_B \left( X_{o,w_n^p} \right) \circ \theta_{\tau_j^n} \right) : X(\tau_j^n) \in B(o,w_n^p), \quad 1 \leq j \leq [\log n] - 1 \right] \]

\[ \leq \exp \left( \frac{2C_10\delta(1 + \delta)^{n - 1}}{\varphi(2w_n)} \right) \left( \sup_{y \in B(o,w_n^p)} P^y \left[ \exp \left( - \frac{C_10}{\varphi(2w_n)} \tau_B(y,w_n^p) \right) \right] \right)^{[\log n] - 1} \]

\[ \leq e^6 \exp \left( - 3 \log n + \delta \lambda_1 \frac{2C_10\varphi(w_n^p) \log \varphi(w_n)}{(1 + \delta) \varphi(2w_n)} \right) \leq e^6 \exp \left( - 3 \frac{4C_10 \delta \lambda_1}{\log n} \right). \quad (4.33) \]

Lastly, using Proposition 3.4(ii) (with \(v_1 = v_p\)), \((4.30)\) and \((2.9)\), we get that for all \(n\) large enough,

\[ P^o(\tau_B(o,w_n^p) \leq (1 + \delta)^n) = P^o(\tau_B(o,w_n^p) \leq \lambda_1 \varphi(w_n^p) \log \log \varphi(w_n)) \]

\[ \leq c_2 \lambda_1 \frac{\varphi(w_n^p) \log \log \varphi(w_n)}{\varphi(w_n^p)} \leq 2c_2 \lambda_1 C'_{L \cdot w_n^{-\beta_1(p-1)}} \log n. \quad (4.34) \]

Using \((4.30)\) and \((2.9)\), we see from the definition \((4.29)\) that for all \(n\) large enough,

\[ w_n \geq c_3 \left( \frac{\varphi(w_n^p)}{\varphi(2R_{\infty})} \right)^{1/\beta_2} \geq c_3 \left( \frac{(1 + \delta)^n}{2\lambda_1 \varphi(2R_{\infty}) \log n} \right)^{1/\beta_2}, \quad (4.35) \]

which yields that \(\sum_{n=1}^{\infty} w_n^{-\beta_1(p-1)} \log n < \infty\). Take \(\lambda_1 = (2 + C_8)/c_1\) and \(\delta_0 = 1/(4C_10 \lambda_1)\). Then we get from \((4.32),(4.33)\) and \((4.34)\) that for all \(n\) large enough,

\[ P^o(\mathcal{O}_n) \leq (1 + e^6)n^{-2} + 2c_2 \lambda_1 C'_{L \cdot w_n^{-\beta_1(p-1)}} \log n \]

and hence \(\sum_{n=1}^{\infty} P^o(\mathcal{O}_n) < \infty\). This proves the base step.

For the next step, let \(m \geq 1\) be such that \((m + 1 + L)\delta \leq 1\) and suppose that \((4.31)\) holds with \(m\). Let \(\kappa_{\delta,m+1} \geq 4^{1+\beta_2}C_U \cdot \kappa_{\delta,m}\) be a constant chosen later. Denote by

\[ v_n = w_n^{\kappa_{\delta,m}}, \quad \tilde{v}_n = w_n^{\kappa_{\delta,m+1}}, \quad U_n = \mathcal{O}_n(\kappa_{\delta,m}, \delta, m) \quad \text{and} \quad \tilde{U}_n = \mathcal{O}_n(\kappa_{\delta,m+1}, \delta, m + 1). \]

As in the proof of Lemma 4.5, our task is to show \(\sum_{n=1}^{\infty} P^o(\tilde{U}_n \setminus U_n) < \infty\). By similar calculations to \((4.17)\), we see that \(4\tilde{v}_n \leq v_n\) for all \(n\) large enough. Let

\[ \tilde{\sigma}_1^n := 0, \quad \tilde{\tau}_j^n := \inf \{ s > \tilde{\sigma}_j^n : X_s \notin B(o,2\tilde{v}_n) \} \quad \text{and} \quad \tilde{\sigma}_{j+1} := \inf \{ s > \tilde{\tau}_j^n : X_s \in B(o,\tilde{v}_n) \}. \]
By analogous arguments as the ones for obtaining (4.18), using (4.29), (2.9) and the assumption that \((m + 1 + L')\delta \leq 1\), we get that for any \(K > 0\) and all \(n\) large enough,

\[
\mathbb{P}^o \left( \tilde{U}_n \setminus U_n : \tilde{z}_n^{n} \geq (1 + \delta)^n \right)
\]

\[
\leq \mathbb{P}^o \left( \sum_{j=1}^{2\lfloor K \log n \rfloor} \tau_{B(o, 2\tilde{v}_n)} \circ \theta_{\tilde{\sigma}_j} \geq (1 + \delta)^n - 1 (1 - \delta - m\delta) \right)
\]

\[
\leq \exp \left( - \frac{C_8(1 + \delta)^{n-1} (1 - \delta - m\delta)}{\varphi(4\tilde{v}_n)} \right) \cdot \exp \left( \frac{C_8 \sum_{j=1}^{2\lfloor K \log n \rfloor} \tau_{B(o, 2\tilde{v}_n)} \circ \theta_{\tilde{\sigma}_j}}{\varphi(4\tilde{v}_n)} \right)
\]

\[
\leq \cdots \leq \exp \left( - \frac{C_8(1 + \delta)^{n-1} (1 - \delta - m\delta)}{\varphi(4\tilde{v}_n)} \right) \sup_{y \in B(o, 2\tilde{v}_n)} \mathbb{E}^y \left[ \exp \left( - \frac{C_8 \tau_{B(x, 2\tilde{v}_n)} \circ \theta_{\tilde{\sigma}_j}}{\varphi(4\tilde{v}_n)} \right) \right]
\]

\[
\leq \exp \left( 2C_9 \log n - \kappa_\delta, m + 1 \frac{C_8(1 + \delta)^{-1} (1 - \delta - m\delta) \varphi(\tilde{v}_n)}{\varphi(4\tilde{v}_n)} \right)
\]

\[
\leq \exp \left( 2C_9 K - c_1 L' \delta \kappa_\delta, m + 1 \log n \right),
\]

where \(c_1 > 0\) is the constant from (4.32).

Next, with a slight abuse of notation, we define \((a_t)_{t \geq 0}\) and \((b_t)_{t \geq 0}\) as (4.19) and (4.20) with \(v_n\) and \(\tilde{v}_n\) instead of \(u_n\) and \(\bar{u}_n\) therein respectively. Then we write \(\tau_{n, 1}^s = \tau_{a_s}^n\) and \(\tau_{n, 2}^s = \tau_{b_s}^n\).

Similar to (4.22), one can see that on the event \(\tilde{U}_n \setminus U_n, U(B(y, \tilde{v}_n), (1 + \delta)^n) \leq \delta(1 + \delta)^{-1} \text{ for all } n \in B(o, v_n - \tilde{v}_n) \setminus B(o, 2\tilde{v}_n)\). Thus, using the strong Markov property and Lemma 3.8(ii), we get that for any \(K > 0\) and all \(n\) large enough,

\[
\mathbb{P}^o \left( \tilde{U}_n \setminus U_n : \tilde{z}_n^{n, 1} < (1 + \delta)^n < \tau_{B(o, \tilde{v}_n)} \right)
\]

\[
\leq \mathbb{P}^o \left( \sum_{j=1}^{\lfloor K \log n \rfloor - 1} \tau_{B(x, \tilde{v}_n)} \circ \theta_{\tilde{\sigma}_j}^{n} \leq (1 + \delta)^n - 1, \ X_{\tilde{\sigma}_j, 1} \in B(o, \tilde{v}_n) \right)
\]

\[
\leq \exp \left( \frac{C_{10}(1 + \delta)^{n-1}}{\varphi(2\tilde{v}_n)} \right) \left( \sup_{y \in B(o, \tilde{v}_n)} \mathbb{E}^y \left[ \exp \left( - \frac{C_{10} \tau_{B(y, \tilde{v}_n)} \circ \theta_{\tilde{\sigma}_j}}{\varphi(2\tilde{v}_n)} \right) \right] \right)^{\lfloor K \log n \rfloor - 1}
\]

\[
\leq e^6 \exp \left( - 3K \log n + \delta \kappa_{\delta, m + 1} \frac{C_{10} \varphi(\tilde{v}_n) \log \varphi(\tilde{v}_n)}{(1 + \delta) \varphi(2\tilde{v}_n)} \right) \leq e^6 \exp \left( - 3K + 2C_{10} \delta \kappa_{\delta, m} \log n \right).
\]

Moreover, by a similar argument to the one for obtaining (4.24), we see that for all \(n\) large enough,

\[
\mathbb{P}^o \left( \tilde{U}_n \setminus U_n : \tilde{z}_n^{n, 2} < (1 + \delta)^n < \tau_{B(o, \tilde{v}_n)} \right)
\]

\[
\leq \mathbb{P}^o \left( \sum_{t=1}^{\lfloor K \log n \rfloor - 1} \tau_{B(X_o, v_n - 2\tilde{v}_n)} \circ \theta_{\tilde{\sigma}_t}^{n} < (1 + \delta)^n \right)
\]

\[
\leq \exp \left( \frac{C_{10}(1 + \delta)^n}{\varphi(2(v_n - 2\tilde{v}_n))} \right) \mathbb{E}^o \left[ \exp \left( - \frac{C_{10}}{\varphi(2(v_n - 2\tilde{v}_n))} \sum_{j=1}^{\lfloor K \log n \rfloor - 1} \tau_{B(X_o, v_n - 2\tilde{v}_n)} \circ \theta_{\tilde{\sigma}_j}^{n} \right) \right]
\]

\[
\leq \left( \sup_{y \in B(o, \tilde{v}_n)} \mathbb{E}^y \left[ \exp \left( - \frac{C_{10} \tau_{B(y, v_n - 2\tilde{v}_n)} \circ \theta_{\tilde{\sigma}_t}^{n}}{\varphi(2(v_n - 2\tilde{v}_n))} \right) \right] \right)^{\lfloor K \log n \rfloor - 1}
\]

\[
\leq e^6 \exp \left( - 3K \log n + \kappa_{\delta, m} \frac{C_{10} \varphi(v_n) \log \varphi(v_n)}{\varphi(v_n)} \right) \leq e^6 \exp \left( - 3K + 2C_{10} \kappa_{\delta, m} \log n \right).
\]
Lastly, by similar calculations to the ones for obtaining (4.34) and (4.35), we get that for all \( n \) large enough,
\[
\mathbb{P}^\alpha(\tau_{B(o,v_n^R)} \leq (1 + \delta)^n) \leq c_4 \kappa_{\delta,m+1} \frac{\varphi(v_n^R) \log \log \varphi(v_n^R)}{\varphi(v_n^R)} \leq 2c_4 \kappa_{\delta,m+1} C_L^{-1} \nu_n^{-\beta_1(p-1) \log n} \leq c_5 (1 + \delta)^{-n \beta_1(p-1)/\beta_2} \log(n)^{1-1/\beta_2}
\]
and hence \( \sum_{n=N}^{\infty} \mathbb{P}^\alpha(\tau_{B(o,v_n^R)} \leq (1 + \delta)^n) < \infty. \)

Notice that \( L' = (2C_9 + 1)C_{10}/c_1. \) Take \( \kappa_{\delta,m+1} := 2\delta^{-1}C_{10}^{-1} \vee \delta^{-1}\kappa_{\delta,m} \vee 4^{1+\beta_2}c_4 C_L^{-1} \kappa_{\delta,m} \) and \( K = C_{10}\delta \kappa_{\delta,m+1}. \) Then by the above four displays and (4.26), we deduce that \( \sum_{n=N}^{\infty} \mathbb{P}^\alpha(U_n \setminus U_n) < \infty. \)

The proof is complete. \( \Box \)

We end this section with the proof of Theorem 2.9.

**Proof of Theorem 2.9.** For \( x \in M \) and \( a > 0, \) we define
\[
E'_x(a) := \left\{ \limsup_{r \to \infty} \frac{U(B(x,r),\infty)}{\varphi(r) \log \log \varphi(r)} < a \right\}.
\]

Since \( X \) is conservative by Proposition 3.4(ii), \( E'_x(a) \) is shift-invariant. Moreover, using Lemma 4.6 similar to (4.28), one can check that \( E'_x(a) \) is independent of \( x. \) Thus, by Proposition 3.5, it suffices to show that there are constants \( p_2 \geq p_1 > 0 \) such that
\[
\limsup_{r \to \infty} \frac{U(B(o,r),\infty)}{\varphi(r) \log \log \varphi(r)} \in [p_1,p_2], \quad \mathbb{P}^\alpha\text{-a.s.} \quad (4.36)
\]
The lower bound in (4.36) comes from (2.10) and the monotonicity of occupation times.

Write \( B(r) := \{ x \in M : d(o,x) \leq r \}. \) Observe that for all \( r \geq (2R_{\infty})^{1/(1-\nu)}, \) \( x \in B(r), \)
\[
R_{\infty} d(x)^{\nu} \leq R_{\infty}(r+1)^{\nu} < 2R_{\infty} r^{\nu} \leq r.
\]
Hence, by \( \text{NDU}_{\infty}^\Phi,\nu \) and \( \text{VRD}_{\infty}^\nu, \) we see that for all \( r \geq (2R_{\infty})^{1/(1-\nu)}, x,y \in B(r) \) and \( t \geq \varphi(r), \)
\[
p(t,x,y) \leq \frac{c_1}{V(x,\varphi^{-1}(t)) \wedge V(y,\varphi^{-1}(t))} \leq \frac{c_2}{V(x,2\varphi^{-1}(t)) \wedge V(y,2\varphi^{-1}(t))} \leq \frac{c_2}{V(o,\varphi^{-1}(t))}. \quad (4.37)
\]
Using (4.37), \( \text{VRD}_{\infty}^\nu \) and (2.9) twice, since \( \beta_2 < d_1, \) we get that for all \( r \geq (2eR_{\infty})^{1/(1-\nu)}, x \in B(r), \)
\[
\mathbb{E}^\nu\left[ U(B(r),\infty) \right] = \int_0^{\varphi(2r)} \int_{B(r)} p(t,x,y) \mu(dy) dt + \int_0^{\varphi(2r)} \int_{B(r)} p(t,x,y) \mu(dy) dt
\leq \varphi(2r) + c_2 \int_{\varphi(2r)}^{\varphi(2r)} \frac{V(o,2r)}{V(o,\varphi^{-1}(t))} \, dt \leq \varphi(2r) + c_3 \int_{\varphi(2r)}^{\infty} \left( \frac{2r}{\varphi^{-1}(t)} \right)^{d_1/\beta_2} \, dt
\leq \varphi(2r) + c_4 \int_{\varphi(2r)}^{\infty} \left( \frac{\varphi(2r)}{t} \right)^{d_1/\beta_2} \, dt = c_5 \varphi(2r) \leq c_6 \varphi(r/e).
\]
Thus, by Lemma 3.2, we deduce that for all \( r \geq (2eR_{\infty})^{1/(1-\nu)}, \)
\[
\sup_{x \in M} \mathbb{E}^x\left[ U(B(r),\infty) \right] = \sup_{x \in B(r)} \mathbb{E}^x\left[ U(B(r),\infty) \right] \leq c_6 \varphi(r/e).
\]

By Khasminskii’s lemma (see [21], Lemma 3) or [24], it follows that
\[
\sup_{x \in M} \mathbb{E}^x \left[ \exp \left( \frac{U(B(e^{n+1}),\infty)}{2c_6 \varphi(e^n)} \right) \right] \leq 2 \quad \text{for all } n \text{ large enough.}
\]
The lower inequality in (2.9) implies that \( \log \log \varphi(e^n) \geq 2^{-1} \log n \) for all \( n \) large enough. Hence, by Markov inequality and the monotonicity of occupation times, we get that for all \( n \) large enough,

\[
\mathbb{P}^o \left( \mathcal{U}(B(o, e^{n+1}), \infty) > 8c_6 \varphi(e^n) \log \varphi(e^n) \right) \leq \mathbb{P}^o \left( \frac{\mathcal{U}(B(o, e^{n+1}), \infty)}{4c_6 \varphi(e^n)} \geq \log n \right) \leq n^{-2} \mathbb{P}^o \left[ \exp \left( \frac{\mathcal{U}(B(e^{n+1}), \infty)}{2c_6 \varphi(e^n)} \right) \right] \leq 2n^{-2}.
\]

Finally, using the Borel-Cantelli lemma, we arrive at

\[
\limsup_{r \to \infty} \mathcal{U}(B(o, r), \infty) = \limsup_{n \to \infty} \sup_{e^n \leq t \leq e^{n+1}} \varphi(r) \log \varphi(r) \leq \limsup_{n \to \infty} \mathcal{U}(B(o, e^{n+1}), \infty) \leq 8c_6, \quad \mathbb{P}^o\text{-a.s.}
\]

which completes the proof.

5. Proof of Theorem 2.10

In this section except in the proof of Corollary 2.11, we always assume that (2.17) holds for some \( x_0 \in M \), which implies that (2.17) holds with \( x_0 = o \) by Remark 2.12. For simplicity, let us denote \( \|F\|_p \) for \( \|F\|_{L^p(M, \mu)} \), \( p \in \{1, \infty\} \).

**Lemma 5.1.** Let \( a > 0 \). For every \( F \in L^1(M, \mu) \) with \( \|F\|_1 \neq 0 \), the event

\[
E_F(a) := \left\{ \omega \in \Omega; \limsup_{t \to \infty} \frac{\mathcal{U}(F, t)(\omega)/\|F\|_1}{\Theta(t/\log \log \Theta(t)) \log \log \Theta(t)} < a \right\}
\]

is shift-invariant.

**Proof.** Define \( g(t) = \Theta(t/\log \log \Theta(t)) \log \log \Theta(t) \). Then \( \lim_{t \to \infty} g(t) = \infty \) by (2.17). Hence, \( E_F(a) \) is a tail event. Moreover, we have that, for all \( s > 0 \) and \( \omega \in \Omega \),

\[
\limsup_{t \to \infty} \frac{\mathcal{U}(F, t)(\omega)}{g(t)} = \limsup_{t \to \infty} \frac{\mathcal{U}(F, t + s) - \mathcal{U}(F, s)}{g(t)}(\omega) \leq \limsup_{t \to \infty} \frac{\mathcal{U}(F, t)(\omega) + \mathcal{U}(F, s)(\omega)}{g(t)} \leq \limsup_{t \to \infty} \frac{\mathcal{U}(F, t)(\omega) + s\|F\|_\infty}{g(t)} = \limsup_{t \to \infty} \frac{\mathcal{U}(F, t)(\omega)}{g(t)}
\]

and

\[
\limsup_{t \to \infty} \frac{\mathcal{U}(F, t + s) - \mathcal{U}(F, s)}{g(t)}(\omega) \geq \limsup_{t \to \infty} \frac{\mathcal{U}(F, t) - \mathcal{U}(F, s)}{g(t)}(\omega) \geq \limsup_{t \to \infty} \frac{\mathcal{U}(F, t)(\omega) - s\|F\|_\infty}{g(t)} = \limsup_{t \to \infty} \frac{\mathcal{U}(F, t)(\omega)}{g(t)}.
\]

Therefore, \( \limsup_{t \to \infty} \mathcal{U}(F, t)(\omega)/g(t) = \limsup_{t \to \infty} \mathcal{U}(F, t)(\omega)/g(t) \). \( \square \)

**Proof of Theorem 2.10.** In Lemmas 5.3 and 5.4 below, we will prove that there are constants \( a_2 \geq a_1 > 0 \) such that for every \( F \in \mathfrak{L}_{b_+}(M) \) satisfying (2.16) and \( \|F\|_1 \neq 0 \),

\[
\limsup_{t \to \infty} \frac{\mathcal{U}(F, t)/\|F\|_1}{\Theta(t/\log \log \Theta(t)) \log \log \Theta(t)} \in [a_1, a_2], \quad \mathbb{P}^o\text{-a.s.} \tag{5.1}
\]

Then using Proposition 3.5 and Lemma 5.1, we deduce (2.18) from (5.1). \( \square \)

The following lemma will be used in the proof for the upper bound in (5.1).
Lemma 5.2. Suppose that VRD$^{R_{\infty}}(v)$, ULS$^{R_{\infty}}(\Phi)$ and NDU$^{R_{\infty}}(\Phi, v)$ hold. Then there exists a constant $K_0 > 0$ such that for every $F \in \mathcal{B}_{w+}(M)$ satisfying (2.16) and $\|F\|_1 \neq 0$,

$$\sup_{w \in M} \mathbb{E}^w[U(F, t)] \leq K_0 \|F\|_1 \Theta(t) \quad \text{for all } t > T_0$$

(5.2)

with some constant $T_0 > 0$.

Proof. When $F \in \mathcal{B}_{w+}^{\infty}(M)$, let $r_1 > R_{\infty}^{-1/\theta_1}$ be a constant satisfying supp$[F] \subset B(o, r_1)$. When $\beta_1 > d_2$ and $F \in \mathcal{B}_{\beta_1+}^{\gamma}(M)$ for $\gamma \in (\beta_1 d_2/(\beta_1 - d_2), \infty)$, let $\delta_1 \in (0, 1)$ be a constant such that

$$d_2 \left(\frac{1}{\beta_1} + \frac{1}{\gamma}\right) + \beta_2 \delta_1 < 1.$$  

(5.3)

Recall that $\mathcal{B}(r) := \{x \in M : d(o, x) \leq r\}$. We define for $t > 0$,

$$r(t) = \begin{cases} t^\delta_1 \varphi^{-1}(t)d_2/\gamma, & \text{if } \gamma < \infty, \\ r_1, & \text{if } \gamma = \infty, \end{cases}$$

where $F_{1,t} = F \cdot 1_{\mathcal{B}(r(t))}$ and $F_{2,t} = F - F_{1,t}$.

Observe that for each $t > 0$,

$$\sup_{w \in M} \mathbb{E}^w[U(F, t)] \leq \sup_{w \in M} \mathbb{E}^w[U(F_{1,t}, t)] + \sup_{w \in M} \mathbb{E}^w[U(F_{2,t}, t)].$$

(5.4)

By VRD$^{R_{\infty}}(v)$, for all large $t$ such that $r(t)^{-\theta_1} > R_{\infty}$, any $y \in \mathcal{B}(r(t))$ and $s > \varphi(r(t))$, we have

$$V(o, \varphi^{-1}(s)) \leq V(y, 2\varphi^{-1}(s)) \leq c_1 V(y, \varphi^{-1}(s)).$$

Hence, by NDU$^{R_{\infty}}(\Phi, v)$, it holds that for all $t$ large enough and all $w \in \mathcal{B}(r(t))$,

$$\mathbb{E}^w[U(F_{1,t}, t)] = \int_0^{\varphi(r(t))} \int_{\mathcal{B}(r(t))} p(s, w, y)F(y)\mu(dy)ds + \int_{\varphi(r(t))}^t \int_{\mathcal{B}(r(t))} p(s, w, y)F(y)\mu(dy)ds$$

$$\leq \varphi(r(t))\|F\|_\infty + c_2 \int_{\varphi(r(t))}^t \int_{\mathcal{B}(r(t))} \frac{1}{V(w, \varphi^{-1}(s))} \sqrt{\frac{1}{V(y, \varphi^{-1}(s))}} F(y)\mu(dy)ds$$

$$\leq \varphi(r(t))\|F\|_\infty + c_1 c_2 \|F\|_1 \left(\int_{\varphi(r(t))}^t \frac{ds}{V(o, \varphi^{-1}(s))}\right) \leq \varphi(r(t))\|F\|_\infty + c_1 c_2 \|F\|_1 \Theta(t).$$

Thus, by Lemma 3.2, it holds that for all $t$ large enough,

$$\sup_{w \in M} \mathbb{E}^w[U(F_{1,t}, t)] = \sup_{w \in \mathcal{B}(r(t))} \mathbb{E}^w[U(F_{1,t}, t)] \leq \varphi(r(t))\|F\|_\infty + c_1 c_2 \|F\|_1 \Theta(t).$$

(5.5)

On the other hand, we have that, if $\gamma < \infty$, then

$$\sup_{w \in M} \mathbb{E}^w[U(F_{2,t}, t)] = \sup_{w \in \mathcal{B}(r(t))} \int_0^t \int_{\mathcal{B}(r(t))} p(s, w, y)F(y)\mu(dy)ds$$

$$\leq (1 + r(t))^{-\gamma} \|d(y)^\gamma F(y)\|_\infty \int_0^t ds \sup_{w \in M} \int_{\mathcal{B}(r(t))} p(s, w, y)\mu(dy)$$

$$\leq t(1 + r(t))^{-\gamma} \|d(y)^\gamma F(y)\|_\infty$$

and if $\gamma = \infty$, then $F_{2,t} = 0$ so that $\sup_{w \in M} \mathbb{E}^w[U(F_{2,t}, t)] = 0$. Combining with (5.5), we get from (5.4) that for all $t$ large enough,

$$\sup_{w \in M} \mathbb{E}^w[U(F, t)] \leq c_1 c_2 \|F\|_1 \Theta(t) + \varphi(r(t))\|F\|_\infty + 1_{\{\gamma < \infty\}}t(1 + r(t))^{-\gamma} \|d(y)^\gamma F(y)\|_\infty.$$
If $\gamma = \infty$, then (5.6) immediately follows since $\lim_{t \to \infty} \Theta(t) = \infty$. Now suppose that $\gamma < \infty$ and $\beta_1 > d_2$. By Lemma 3.1(i) and VRD$^{R_\infty}(v)$, we have
\[
\lim_{t \to \infty} \frac{t(1 + r(t))^{-\gamma}}{\Theta(t)} \leq 2V(o, 2R_\infty) \lim_{t \to \infty} \left( t^{-\beta_1}(r^{-1}(t) - d_2) \frac{V(o, \varphi^{-1}(t))}{V(o, 2R_\infty)} \right) \leq c_3 R_\infty^{-d_2} V(o, 2R_\infty) \lim_{t \to \infty} t^{-\beta_1} = 0.
\]
Further, using Lemma 3.1(i), VRD$^{R_\infty}(v)$ and (2.9) several times, we get from (5.3) that
\[
\lim_{t \to \infty} \frac{\varphi(t)}{\Theta(t)} \leq 2 \lim_{t \to \infty} \frac{\varphi(t^{\beta_1}(r^{-1}(t))^{d_2/\gamma})}{t^{d_2/\gamma}} V(o, \varphi^{-1}(t)) = 2 \lim_{u \to \infty} \frac{\varphi(u)}{\varphi(u^{d_2/\gamma})} V(o, u) = 2V(o, 2R_\infty) \lim_{u \to \infty} \left( \frac{\varphi(2R_\infty)^{\beta_1} \varphi(u) u^{d_2/\gamma}}{\varphi(2R_\infty)^{\beta_1}} \right) \frac{V(o, u)}{V(o, 2R_\infty)} \leq c_4 R_\infty^{-d_2} \lim_{u \to \infty} \left( \frac{\varphi(c_5 u^{\beta_1} u^{d_2/\gamma})}{\varphi(u)} u^{d_2} \right) \leq c_5 \lim_{u \to \infty} u^{d_2 - \beta_1(1 - \beta_2 d_1 - d_2/\gamma)} = 0.
\]
Thus, (5.6) holds and we finish the proof. □

**Lemma 5.3.** Under the setting of Theorem 2.10, the upper bound in (5.1) holds.

**Proof.** By Markov inequality and Lemma 5.2, it holds that for all $t \geq T_0$,
\[
\sup_{w \in M} P^w \left( \mathcal{U}(F, t) / \|F\|_1 \geq 2K_0 \Theta(t) \right) \leq \frac{1}{2K_0 \|F\|_1 \Theta(t)} \sup_{w \in M} E[\mathcal{U}(F, t)] \leq \frac{1}{2}.
\]
Hence, using (2.17) (with $x_0 = o$) and Lemma 3.1(ii), we see that with $F(t) = \mathcal{U}(F, t) / \|F\|_1$ and $g(t) = \Theta(t)$, conditions (1)–(3) of Lemma 7.1 are satisfied. Now the upper bound in (5.1) follows from Lemma 7.1. □

**Lemma 5.4.** Under the setting of Theorem 2.10, the lower bound in (5.1) holds.

**Proof.** By (2.17), there exist $N \geq 10$ and an increasing sequence $(t_n)_{n \geq N}$ such that
\[
\Theta(t_n / \log n) = \exp(n^2), \quad n \geq N. \tag{5.7}
\]
Set $u_n := \log \log \Theta(t_n)$. Then $u_n \geq \log \log \exp(n^2) = 2 \log n$ for $n \geq N$. Further, by Lemma 3.1(ii), we see that $u_n \leq \log \log (3(\log n)^{\log 3/\log 2} \exp(n^2)) \leq \log \log \exp(n^2) = 3 \log n$ for $n \geq N$. Thus,
\[
2 \log n \leq u_n \leq 3 \log n \quad \text{for all} \quad n \geq N. \tag{5.8}
\]
Using Lemma 3.1(ii) twice, we get from (5.7) and (5.8) that for all $n \geq N \geq 10$,
\[
\Theta(n^2 t_n) \leq 3(n^2 \log n)^2 \Theta(t_n / \log n) = 3(n^2 \log n)^2 e^{-2n-1} \Theta(t_{n+1} / \log(n+1)) \leq 250(n^2 \log n)^2 e^{-2n-1} \Theta(t_{n+1}/2u_{n+1}) \leq \frac{1}{2} \Theta(t_{n+1}/2u_{n+1}). \tag{5.9}
\]
In particular, since $\Theta$ is increasing, we see that $t_{n+1} \geq 100 t_n$ for all $n \geq N$.

Let $r_F > 0$ be a constant such that $2 \|F \cdot 1_{B(r_F)}\|_1 \geq \|F\|_1$ and set $F_0 := F \cdot 1_{B(r_F)}$. By NDL$^{R_\infty}(\varphi, v)$, there exist constants $c_1 > 1$, $c_2 > 0$ such that for all $n$ large enough, all $w \in B(\varphi^{-1}(nt_n))$ and $s > \varphi(c_1 \varphi^{-1}(nt_n))$,
\[
\int_{B(r_F)} p_{B(o, c_1 \varphi^{-1}(s))}(s, w, y) F_0(y) \mu(dy) \geq \frac{c_2 \|F_0\|_1}{V(o, \varphi^{-1}(s))}. \tag{5.10}
\]
Note that by (2.4), it holds that for all $n$ large enough,
\[
\varphi(c_1 \varphi^{-1}(nt_n)) \leq c_2 n t_n \leq n^2 t_n.
\]
Thus, using (5.10), (5.9), Lemma B.1.1(ii) and the fact that $2\|F_0\|_1 \geq \|F\|_1$, we get that for all $n$ large enough and any $w \in B(\varphi^{-1}(nt_n))$,

$$\mathbb{E}^w \left[ \mathcal{U}(F_0, \frac{t_{n+1} - t_n}{u_{n+1}}) \right] \geq \int^{t_{n+1} / (2u_{n+1})}_{t_n / u_{n+1}} \int_{\mathcal{B}(r_F)} p^{B(o,c_1\varphi^{-1}(s))}(s,w,y)F_0(y)\mu(dy)ds$$

$$\geq c_2\|F_0\|_1 \int^{t_{n+1} / (2u_{n+1})}_{t_n / u_{n+1}} ds = c_2\|F_0\|_1 \left( \Theta(t_{n+1} / 2u_{n+1}) - \Theta(n^2t_n) \right)$$

$$\geq \frac{c_2}{2} \|F_0\|_1 \Theta(t_{n+1} / 2u_{n+1}) \geq \frac{c_2}{18} \|F_0\|_1 \Theta(t_{n+1} / u_{n+1}) \geq \frac{c_2}{36} \|F\|_1 \Theta(t_{n+1} / u_{n+1}).$$

(5.11)

On the other hand, by Lemma 5.2 and Khasmin’skii’s lemma (see [21] Lemma 3) or [24] Lemma B.1.2), it holds that for all $n$ large enough,

$$\sup_{w \in M} \mathbb{E}^w \left[ \mathcal{U}(F_0, \frac{t_{n+1} - t_n}{u_{n+1}})^2 \right] \leq \sup_{w \in M} \mathbb{E}^w \left[ \mathcal{U}(F, \frac{t_{n+1} - t_n}{u_{n+1}})^2 \right] \leq (K_0 \|F\|_1 \Theta(t_{n+1} / u_{n+1}))^2.$$

Therefore, by combining the above with (5.11) and using the Paley-Zygmund inequality, we deduce that there are constants $c_3 > 0$ and $p \in (0, e^{-1})$ independent of $F$ such that for all $n$ large enough,

$$\inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w \left( \mathcal{U}(F_0, \frac{t_{n+1} - t_n}{u_{n+1}}) \geq 2c_3 \|F\|_1 \Theta(t_{n+1} / u_{n+1}) \right) \geq p.$$

(5.12)

Set $\delta := (3 \log(1/p))^{-1}$ and define for $n \geq N$ and $i \geq 1$,

$$E_n := \left\{ \frac{\mathcal{U}(F,t_n)}{u_n \Theta(t_n/u_n)} \geq \delta c_3 \|F\|_1 \right\}, \quad G_n := \{ X_{t_n} \in B(\varphi^{-1}(nt_n)) \},$$

$$H_{n,i} := \{ \exists s_1, ... , s_i \in (0,(t_{n+1} - t_n)/u_{n+1}) \text{ such that for all } 1 \leq j \leq i, \quad X_{s_j} \in B(r_F) \text{ and } \mathcal{U}(F_0, \sum_{m=1}^{j} s_m) - \mathcal{U}(F_0, \sum_{m=1}^{j-1} s_m) \geq c_3 \|F\|_1 \Theta(t_{n+1} / u_{n+1}) \}.$$

Then by the strong Markov property, we get that for all $n \geq N$,

$$\mathbb{P}^\omega(E_{n+1} \mid F_{t_n}) \geq \mathbb{P}^\omega(E_{n+1} \cap G_n \mid F_{t_n})$$

$$\geq \inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w \left( \mathcal{U}(F,t_{n+1} - t_n) \geq \delta c_3 u_{n+1} \|F\|_1 \Theta(t_{n+1} / u_{n+1}) \right) \cdot 1_{G_n}$$

$$\geq \inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w \left( \mathcal{U}(F_0, t_{n+1} - t_n) \geq \delta c_3 u_{n+1} \|F\|_1 \Theta(t_{n+1} / u_{n+1}) \right) \cdot 1_{G_n}$$

$$\geq \inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w(H_{n,\{\delta u_{n+1}\}+1}) \cdot 1_{G_n}. \quad (5.13)$$

Using the strong Markov property, (5.8) and (5.12), we get that for all $n$ large enough,

$$\inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w(H_{n,\{\delta u_{n+1}\}+1}) = \inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{E} \left[ \mathbb{P}^w \left( H_{n,\{\delta u_{n+1}\}+1} \mid \mathcal{F}_{s_j=1} s_j \right) \right]$$

$$\geq \inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w \left( H_{n,\{\delta u_{n+1}\}+1} \right) \cdot 1_{G_n}$$

$$\geq \inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w \left( H_{n,\{\delta u_{n+1}\}+1} \right) \cdot 1_{G_n}$$

$$\geq \cdots \geq \left( \inf_{w \in B(\varphi^{-1}(nt_n))} \mathbb{P}^w(H_{n,1}) \right)^{\delta u_{n+1}+1} \cdot 1_{G_n}. \quad (5.14)$$
Observe that for any \(0 < u < t\), if \(U(F_0, t) - U(F_0, u) > 0\), then there exists \(s \in [u, t]\) such that \(X_s \in \text{supp}[F_0] \subset B(r_F)\). Thus, \(\inf_{w \in \mathcal{B}(\varphi^{-1}(nt_n))} \mathbb{P}^{\omega}(H_n, 1) \geq p\) by \((5.12)\). It follows from \((5.14)\) and \((5.8)\) that for all \(n\) large enough,
\[
\inf_{w \in \mathcal{B}(\varphi^{-1}(nt_n))} \mathbb{P}^{\omega}(H_n, [\delta u_{n+1}] + 1) \geq p^{[\delta u_{n+1}] + 1} \geq p^{3d \log(n+1) + 1} = p(n+1)^{-1}.
\]
Hence, \(\sum_{n=N}^\infty \inf_{w \in \mathcal{B}(\varphi^{-1}(nt_n))} \mathbb{P}^{\omega}(H_n, [\delta u_{n+1}] + 1) = \infty\).

Further, recall that \(t_{n+1} \geq 100t_n\) for all \(n \geq N\). This implies that \(t_n \geq 99^n\) for all \(n\) large enough. Then by \((2.4)\), it holds that for all \(n\) large enough,
\[
2\varphi^{-1}(nt_n) v^{1/2} \leq 2(2R_\infty) v^{1/2} - 1 \left( \frac{\varphi^{-1}(nt_n)}{\varphi^{-1}(t_n)} \right) v^{1/2}  = 2R_\infty \left( \frac{\varphi^{-1}(nt_n)}{\varphi^{-1}(t_n)} \right)^{1-v^{1/2}} 
\leq c_4 t^{1/2} t_n^{-1} (v^{1/2} - 1/\beta_2) \leq c_4 t^{1/2} 99^{-n(1-v^{1/2})/\beta_2} < 1.
\]
Therefore, we can apply Proposition \((3.4)\) with \(v_1 = v^{1/2}\), \(r = \varphi^{-1}(nt_n)\) and \(t = t_n\) for all \(n\) large enough and get that
\[
\lim_{n \to \infty} \mathbb{P}^o(G_n^o) \leq \lim_{n \to \infty} \mathbb{P}^o(\tau_B(o, \varphi^{-1}(nt_n)) \leq t_n) \leq \lim_{n \to \infty} c_3 n^{-1} = 0.
\]
Finally, in view of \((5.13)\), we deduce from Lemma \((7.2)\) that \(\mathbb{P}^s(\limsup E_n) = 1\) and hence the lower bound in \((5.1)\) holds with \(a_1 = \delta c_3\). The proof is complete.

**Proof of Corollary \((2.11)\)**: By VRD\((R_\infty)(v)\) and \((2.9)\), we see that for all \(t > \varphi(2R_\infty)\),
\[
\frac{t}{V(o, \varphi^{-1}(t))} = \frac{t}{V(o, 2R_\infty)} \frac{V(o, 2R_\infty)}{V(o, \varphi^{-1}(t))} \leq c_1 t \left( \frac{2R_\infty}{\varphi^{-1}(t)} \right)^{d_1} \leq c_2 t \left( \frac{\varphi(2R_\infty)}{t} \right)^{d_1/\beta_2}
\]
and
\[
\frac{t}{V(o, \varphi^{-1}(t))} \geq c_3 t \left( \frac{2R_\infty}{\varphi^{-1}(t)} \right)^{d_2} \geq c_4 t \left( \frac{\varphi(2R_\infty)}{t} \right)^{d_2/\beta_1}.
\]
Hence, since \(\beta_1 > d_2\), by Lemma \((3.1)(i)\), we see that \((2.17)\) holds with \(x_0 = o\) and
\[
\Theta(t) \simeq \Theta(o, 2R_\infty, t) \simeq \frac{t}{V(o, \varphi^{-1}(t))} \quad \text{and} \quad \log \log \Theta(t) \simeq \log \log t \quad \text{for} \quad t > 2\varphi(2R_\infty).
\]
Then using VRD\((R_\infty)(v)\) and \((2.4)\), we deduce from \((5.1)\) that there are constants \(a_4 \geq a_3 > 0\) independent of \(F\) such that
\[
\lim_{t \to \infty} \sup \frac{U(F, t)}{\|F\|_1} / \frac{t}{V(o, \varphi^{-1}(t/\log \log t))} \in [a_3, a_4], \quad \mathbb{P}^o-\text{a.s.}
\]
For each \(a > 0\), one can see that \(\left\{ \limsup_{t \to \infty} \frac{U(F, t)}{\|F\|_1} / \frac{t}{V(o, \varphi^{-1}(t/\log \log t))} < a \right\}\) is shift-invariant by a similar proof to that of Lemma \((5.1)\). Applying Proposition \((3.5)\) again, we conclude \((2.19)\).

6. Examples

The results of this paper cover a large class of subordinate diffusions and symmetric jump processes considered in the authors’ previous paper \([10]\). Precisely, one can apply the small time LIL result Theorem \((2.4)\) to \([10]\) Examples 2.4, 2.5, 2.6(i), 2.8, 2.9), and large time LIL results Theorem \((2.6)\) and one of Theorem \((2.9)\) or \((2.10)\) to \([10]\) Examples 2.6(ii)]. In this section, we give two important examples: (1) Feller processes on \(\mathbb{R}^d\), and (2) Random conductance model. We begin with a lemma about stability of scale functions \(\phi(x, r)\) and \(\varphi(r)\) in \([2.6]\) and \([2.10]\).
Lemma 6.1. Let $g$ and $h$ be increasing positive continuous functions on a subinterval of $(0, \infty)$, and $\varepsilon \in (0, 1)$ be a constant.

(i) If $\lim_{r \to 0} g(r) = 0$, then there exists a constant $a_1 \in (0, 1]$ such that

$$\limsup_{r \to 0} \mathcal{U}(B(x, r), g(r))/g(r) = a_1, \quad \mathbb{P}^x\text{-a.s.},$$

then there exists a constant $a_2 \in (0, 1]$ such that

$$\limsup_{r \to 0} \mathcal{U}(B(x, r), h(r))/h(r) = a_2, \quad \mathbb{P}^x\text{-a.s.},$$

(ii) Suppose that zero-one law for shift-invariant event holds (see the paragraph above Proposition 3.5 for the definition of this) and $\lim_{r \to \infty} h(r + s)/h(r) = 1$ for every $s > 0$. If $\lim_{r \to \infty} g(r) = \infty$, then there exists a constant $a_3 \in (0, 1]$ such that

$$\limsup_{r \to \infty} \mathcal{U}(B(x, r), g(r))/g(r) = a_3, \quad \mathbb{P}^x\text{-a.s.}, \quad (6.1)$$

then there exists a constant $a_4 \in (0, 1]$ such that

$$\limsup_{r \to \infty} \mathcal{U}(B(x, r), h(r))/h(r) = a_4, \quad \mathbb{P}^x\text{-a.s., \forall} z \in M.$$

**Proof.** Since the proofs are similar, we only give the proof for (ii).

As in the proof of Theorem 2.6, since zero-one law for shift-invariant event holds and $\lim_{r \to \infty} h(r + s)/h(r) = 1$ for every $s > 0$, there exists a constant $a \in (0, 1]$ such that

$$\limsup_{r \to \infty} \mathcal{U}(B(x, r), h(r))/h(r) = a, \quad \mathbb{P}^x\text{-a.s., \forall} z \in M.$$

To conclude the result, it remains to show that $a > 0$. Since $g(r) \approx h(r)$ for $r > 1/\varepsilon$, there exists $c_1 \in (0, 1]$ such that $c_1 g(r) \leq h(r) \leq c_1^{-1} g(r)$ for $r > 1/\varepsilon$. Let $f(r) := g^{-1}(c_1 g(r))$ which is well-defined for all $r$ large enough. Then $f(r) \leq r$ since $g$ is increasing. Using (6.1) and the monotone property of occupation times, we get that $\mathbb{P}^x$-a.s.,

$$a = \limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), h(r))}{h(r)} \geq \limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), c_1 g(r))}{c_1^{-1} g(r)} = c_1^2 \limsup_{r \to \infty} \frac{\mathcal{U}(B(x, r), g(f(r))}{g(f(r))}$$

$$\geq c_1^2 \limsup_{r \to \infty} \frac{\mathcal{U}(B(x, f(r)), g(f(r))))}{g(f(r))} = c_1^2 a_3 > 0.$$

This completes the proof. \qed

### 6.1. LILs for Feller processes on $\mathbb{R}^d$

Let $X$ be a (rich) Feller process on $\mathbb{R}^d$ with the generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ such that $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L})$. It is well-known that $\mathcal{L}$ restricted to $C_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator, which has the following representation: For every $u \in C_c^\infty(\mathbb{R}^d)$,

$$\mathcal{L}u(x) = -q(x, D)u(x) := -(2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} q(x, \xi) \int_{\mathbb{R}^d} e^{-i(y \cdot \xi)} u(y) dy d\xi$$

with

$$q(x, \xi) = k(x) - i \langle b(x), \xi \rangle + \langle \xi, a(x) \xi \rangle + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{i\langle z, \xi \rangle} + i\langle z, \xi \rangle 1_{\{|z| \leq 1\}}) \nu(x, dz)$$

where $k : \mathbb{R}^d \to \mathbb{R}$ is a non-negative measurable function, $b : \mathbb{R}^d \to \mathbb{R}^d$ a measurable function, $a : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ a non-negative definite matrix-valued function, and $\nu(x, dz)$ a non-negative $\sigma$-finite kernel on $\mathbb{R}^d \times B(\mathbb{R}^d \setminus \{0\})$ satisfying $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) \nu(x, dz) < \infty$ for all $x \in \mathbb{R}^d$. The function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is called the symbol of $X$ and the quadruplet $(k(x), b(x), a(x), \nu(x, dz))_{x \in \mathbb{R}^d}$ is called the Lévy characteristics of $X$. We refer to [6]. In this subsection, we always assume that $k(x)$ is identically zero so that $X$ has no killing inside.
Define for \( x \in \mathbb{R}^d \) and \( r > 0 \),
\[
\Phi(x, r) = \left( \sup_{|\xi| \leq 1/r} \Re q(x, \xi) \right)^{-1}.
\]
We consider the following conditions on \( X \). Fix an open set \( U \subset \mathbb{R}^d \).

**Assumption C.** Suppose that there exist constants \( R_0, a_0, a_1, a_2, a_3, \varepsilon_0 \in (0, 1) \) such that for any \( x \in U \) and any \( \xi \in \mathbb{R}^d \) with \( |\xi| > 1/(R_0 \wedge (a_0 \delta_U(x))) \),
\[
\lim_{r \to 0} \Phi(x, r) = 0 \quad \text{(or, equivalently, either} \ a(x) \neq 0 \text{ or} \ \nu(x, \mathbb{R}^d) = \infty) \tag{C1}
\]
\[
\sup_{|\xi| \leq 1} \Re q(x, \xi) \geq a_1 |\Im q(x, \xi)|, \quad \tag{C2}
\]
\[
\inf_{x, \xi : |x, \xi| \geq 1} \Re q(y, \xi) \geq a_2 \sup_{|x-y| \leq 1/|\xi|} \Re q(y, \xi), \quad \tag{C3}
\]
\[
\inf_{\nu \in \mathbb{R}^d, |\xi| \leq 1} \mathbb{P}^x (2|X_t - x, z| - |X_t - x|) \geq a_3 \quad \text{for all} \ 0 < t < \varepsilon_0 \Phi(x, R_0 \wedge (a_0 \delta_U(x))). \tag{C4}
\]

**Remark 6.2.** When \( d = 1 \) and \( X \) is symmetric, \( (\mathbf{C4}) \) holds true with \( a_3 = 1/2 \).

From Theorem 2.4, we obtain the following limsup LIL for occupation times of \( X \).

**Proposition 6.3.** Let \( X \) be a Feller process on \( \mathbb{R}^d \) with symbol \( q \). Suppose that Assumption C holds for an open set \( U \subset \mathbb{R}^d \). Let \( f_0 : U \times (0, \infty) \to [0, 1] \) be a deterministic function such that for every \( x \in U \) and \( \kappa > 0 \),
\[
\limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), \kappa \Phi(x, r) \log |\log \Phi(x, r)|)}{\kappa \Phi(x, r) \log |\log \Phi(x, r)|} = f_0(x, \kappa), \quad \mathbb{P}^x\text{-a.s.} \tag{6.2}
\]
Then \( f_0 \) satisfies all properties (P1)-(P3) in Theorem 2.4.

**Proof.** Following the arguments in the proof of [11, Theorem 2.4], we deduce that if Assumption C holds true for \( U \) then so \( \mathbf{CE}_{R_0}(\phi, U) \) \( \mathbf{US}_{R_0}(\phi, U) \) \( \mathbf{Tail}_{R_0}(\phi, U, \leq) \) and \( \mathbf{SP}_{R_0}(\phi, U) \) do for \( U \). Now the result follows from Theorem 2.4.

A Lévy process on \( \mathbb{R}^d \) is a Feller process whose characteristics is independent of \( x \in \mathbb{R}^d \). Hence, \( q(x, \xi) = \psi(\xi) \) for a negative definite function \( \psi \) when \( X \) is a Lévy process. The function \( \psi \) is called the Lévy exponent. For a given Lévy exponent \( \psi \), we write
\[
\Phi_\psi(r) := \left( \sup_{|\xi| \leq 1/r} \Re \psi(\xi) \right)^{-1}. \tag{6.3}
\]

**Example 6.4. (Isotropic Lévy processes)** Let \( X \) be a Lévy process on \( \mathbb{R}^d \) with exponent \( \psi \). Suppose that \( \psi \) is a radial function, namely, \( \psi(\xi) = \psi^*(|\xi|) \) for some \( \psi^* \) and \( \lim_{r \to \infty} \psi^*(r) = \infty \). Here we do not assume the absolute continuity of the Lévy measure \( \nu \).

Note that \( \overline{\psi(\xi)} = \psi(-\xi) = \psi(\xi) = \Re \psi(\xi) \) for all \( \xi \in \mathbb{R}^d \). Hence, \( (\mathbf{C1}) \) and \( (\mathbf{C2}) \) are satisfied for \( U = \mathbb{R}^d \). Clearly, \( (\mathbf{C3}) \) holds true for \( U = \mathbb{R}^d \). Lastly, we get \( (\mathbf{C4}) \) for \( U = \mathbb{R}^d \) from [17] Proposition 5.2] and [11, Lemma 2.2]. Therefore, we can apply Proposition 6.3 and deduce that there exist constants \( c_1, \kappa_1 > 0 \) and a non-increasing function function \( f_0 \) on \( (0, \infty) \) satisfying \( f_0(\kappa) = 1 \) for \( \kappa \leq \kappa_1 \) and \( \lim_{\kappa \to \infty} f_0(\kappa) = 0 \) such that for every \( x \in \mathbb{R}^d \) and \( \kappa > 0 \),
\[
\limsup_{r \to 0} \frac{\mathcal{U}(B(x, r), \kappa \Phi_\psi(r) \log |\log \Phi_\psi(r)|)}{\kappa \Phi_\psi(r) \log |\log \Phi_\psi(r)|} = f_0(\kappa), \quad \mathbb{P}^x\text{-a.s.}, \tag{6.4}
\]
where the function \( \Phi_\psi \) is defined as \( (6.3) \).

Note that the LIL \( (6.4) \) covers Lévy processes whose exponent is slowly varying at infinity. \( \square \)
In [11], the authors give a sufficient condition for Assumption C in terms of the symbol \( q(\cdot, \xi) \) only by using the symmetrization argument from [22]. Fix an open set \( U \subset \mathbb{R}^d \) as before.

**Assumption S.** \( C_{c}^{\infty}(\mathbb{R}^d) \) is an operator core for \((\mathcal{L}, \mathcal{D}(\mathcal{L}))\), i.e. \( \mathcal{L}|_{C_{c}^{\infty}(\mathbb{R}^d)} = \mathcal{L} \), and there exist constants \( R_0, a_0 \in (0, 1) \), \( c_L, c_U > 0 \) and \( K \geq 1 \) such that the following properties are satisfied for every \( x \in U \):

(S1) There exists an increasing function \( g(x, \cdot) \) and constants \( 0 < \alpha(x) \leq \beta(x) \) such that

\[
\frac{1}{\alpha(x)} - \frac{1}{\beta(x)} < \frac{1}{d^2 + d},
\]

\[
c_L \left( \frac{r}{s} \right)^{\alpha(x)} \leq \frac{g(x, r)}{g(x, s)} \leq c_U \left( \frac{r}{s} \right)^{\beta(x)} \quad \text{for all } r > s > 1/(R_0 \wedge (a_0 \delta_U(x)))
\]

and

\[
K^{-1} g(x, |\xi|) \leq \text{Re} \, q(x, \xi) \leq Kg(x, |\xi|) \quad \text{for all } \xi \in \mathbb{R}^d, \ |\xi| > 1/(R_0 \wedge (C_0 \delta_U(x))).
\]

(S2) For every \( 0 < R < R_0 \wedge (C_0 \delta_U(x)) \), there exists a Feller process \( Y = Y^{x,R} \) with symbol \( q_Y(\cdot, \xi) \) such that

(i) \( q(y, \xi) = 2 \text{Re} \, q_Y(y, \xi/2) \) for all \( y \in \overline{B}(x, R) \) and \( \xi \in \mathbb{R}^d \),

(ii) \( K^{-1} \inf_{|z-x| \leq r} \text{Re} \, q_Y(z, \xi) \leq \text{Re} \, q_Y(y, \xi) \leq K \sup_{|z-x| \leq r} \text{Re} \, q_Y(z, \xi) \)

for all \( y \in \mathbb{R}^d \setminus \overline{B}(x, R) \) and \( \xi \in \mathbb{R}^d, \ |\xi| > 1/(R_0 \wedge (C_0 \delta_U(x))).

We refer to [11] Examples 2.13 and 2.14 for concrete examples of Feller processes satisfying Assumption S.

**Remark 6.5.** \( C_{c}^{\infty}(\mathbb{R}^d) \) is an operator core for \((\mathcal{L}, \mathcal{D}(\mathcal{L}))\) if and only if the martingale problem for \((-q(\cdot, D), C_{c}^{\infty}(\mathbb{R}^d))\) is well-posed. See [22] Proposition 4.6.

From Proposition 6.3 and [11] Proposition 2.12], we conclude

**Corollary 6.6.** Let \( X \) be a Feller process on \( \mathbb{R}^d \) with symbol \( q \). Suppose that (C3) and Assumption S hold true for an open set \( U \subset \mathbb{R}^d \). Then the deterministic function \( f_0 : U \times (0, \infty) \to [0, 1] \) defined by (6.2) satisfies all properties (P1)-(P3) in Theorem 2.4.

### 6.2. Random conductance model

In this section, we give LILs for occupations times at infinity for some random conductance models. We repeat the setting of the random conductance models in [10] Section 3 here for the readers’ convenience.

Let \( G = (\mathbb{L}, E_{\mathbb{L}}) \) be a locally finite connected infinite undirected graph, where \( \mathbb{L} \) is the set of vertices and \( E_{\mathbb{L}} \) the set of edges. We denote \( d(x, y) \) for the graph distance between \( x, y \in \mathbb{L} \), and \( \mu_c \) for the counting measure on \( \mathbb{L} \). A random conductance \( \eta = (\eta_{xy} : x, y \in \mathbb{L}) \) on \( \mathbb{L} \) is a family of nonnegative random variables defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \( \eta_{xx} = 0 \) and \( \eta_{xy} = \eta_{yx} \) for all \( x, y \in \mathbb{L} \). Associated with a random conductance \( \eta \), for each \( \omega \in \Omega \), the variable speed random walk (VSRW) \( X^\omega = (X^\omega_t, t \geq 0; \mathbb{P}^\omega_x, x \in \mathbb{L}) \) is defined by the symmetric Markov process on \( \mathbb{L} \) with \( L^2(\mathbb{L}, \mu_c) \)-generator

\[
\mathcal{L}^\omega_{\mathbb{L}} f(x) = \sum_{y \in \mathbb{L}} \eta_{xy}(\omega)(f(y) - f(x)),
\]

and the constant speed random walk (CSRW) \( Y^\omega = (Y^\omega_t, t \geq 0; \mathbb{P}^\omega_x, x \in \mathbb{L}) \) is the symmetric Markov process on \( \mathbb{L} \) with \( L^2(\mathbb{L}, \nu) \)-generator

\[
\mathcal{L}^\omega_{\mathbb{L}} f(x) = \nu_x^{-1}(\omega) \sum_{y \in \mathbb{L}} \eta_{xy}(\omega)(f(y) - f(x)),
\]
where $\nu_x := \sum_{y \in U} \eta_{xy}$.

In this section, we denote by $U_X(F, t)$ and $U_Y(F, t)$ the occupation times of VSRW $X$ and CSRW $Y$ for $F$ respectively.

6.2.1. Random walks on supercritical bond percolation clusters.

Let $L = \mathbb{Z}^d$, $d \geq 2$ and $p \in (0, 1]$. Let $(\eta_{xy})_{x,y \in \mathbb{Z}^d}$ be independent random variables such that

$$\eta_{xy} = \begin{cases} 
\text{Bernoulli distribution with mean } p, & \text{if } |x - y| = 1, \\
0, & \text{otherwise},
\end{cases} \quad (6.9)$$

An edge $\{x, y\}$ in $\mathbb{Z}^d$ is called open if $\eta_{xy} = 1$ and a set $C \subset \mathbb{Z}^d$ is called an open cluster if every $x, y \in C$ are connected by an open path. It is known that there exists a critical value $p_c = p_c(d) \in (0, 1)$ such that if $p > p_c$, then $P$-a.s., there exists a unique infinite open cluster in each configuration, which we denote $C_\infty = C_\infty(\omega)$. It is obtained in [19] that $p_c(2) = 1/2$, but the exact values of $p_c(d)$ for $d \geq 3$ are still open. For details and a comprehensive bibliography, we refer to [16,20].

Recall the definition of $B_{b,+}^Y(M)$ from (2.15). Below, we give LILs for occupation times of both VSRW $X$ and CSRW $Y$ on $C_\infty$ associated with the random conductance $\eta$ defined by (6.9).

**Proposition 6.7.** Let $p \in (p_c, 1]$. There exist $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ and constants $\kappa_3 \geq \kappa_2 \geq \kappa_1 > 0$, $a_2 \geq a_1 > 0$ which depend only on the parameter $p$ such that the following statements hold true for every $\omega \in \Omega_0$.

(i) There exists an non-increasing function $f_\infty(\cdot, \omega) : (0, \infty) \to (0, 1]$ such that $f_\infty(\kappa, \omega) = 1$ if $\kappa \leq \kappa_1$, $\lim_{\kappa \to \infty} f_\infty(\kappa, \omega) = 0$ uniformly on $\omega \in \Omega_0$ and that for every $\kappa > 0$,

$$\limsup_{r \to \infty} \frac{U_X(B(x, r), \kappa r^2 \log \log r)}{\kappa r^2 \log \log r} = f_\infty(\kappa, \omega), \quad P_\omega^z \text{-a.s., } \forall z \in C_\infty(\omega).$$

(ii) (a) If $d \geq 3$, then there exists a constant $\kappa(\omega) \in [\kappa_2, \kappa_3]$ such that

$$\lim_{r \to \infty} \frac{U_X(B(x, r), \infty)}{r^2 \log \log r} = \kappa(\omega), \quad P_\omega^z \text{-a.s., } \forall z \in C_\infty(\omega).$$

(b) If $d = 2$, then for every $F \in B_{b,+}^Y(C_\infty(\omega); \mu_c)$ with $\|F\|_{L^1(C_\infty(\omega); \mu_c)} \neq 0$, there exists a constant $a_F(\omega) \in [a_1, a_2]$ such that

$$\limsup_{t \to \infty} \frac{U_X(F, t)}{\log t \log \log t} = a_F, \quad P_\omega^z \text{-a.s., } \forall z \in C_\infty(\omega). \quad (6.10)$$

(iii) The assertions in (i-ii) hold true for $U_Y$ instead of $U_X$ with different constants $\kappa_1, \kappa_2, \kappa_3, a_1, a_2$.

**Proof.** According to [1] Theorem 2.18(c), Lemma 2.19, Theorem 5.7 and Lemma 5.8 and the Borel-Cantelli lemma, for $\omega$, there is a constant $R_\infty(\omega) \in [1, \infty)$ such that $\text{VRD}^{R_\infty(\omega)}(\Phi, 4/5)$ with respect to $\mu_c$ holds, and NDL$^{R_\infty(\Phi, 4/5)}$ and NDU$^{R_\infty(\Phi, 4/5)}$ are satisfied with $\Phi(r) = r^2$. Tail$^{R_\infty(\Phi, 4/5, \leq)}$ trivially holds because there are no long range jumps. Therefore, the assertions in (i) and (ii) are consequences of Theorems 2.6, 2.9 and 2.10 Since $\mu_c \leq \nu \leq \mu_c \leq 4 \mu_c$, we also obtain (iii).

6.2.2. Random conductance model with unbounded conductances.

Let $L = \mathbb{Z}^d$, $d \geq 2$ and $\xi$ be any random variable such that $\xi \geq 1$, $\mathbf{P}$-a.s. Let $(\eta_{xy})_{x,y \in \mathbb{Z}^d}$ be independent random variables such that

$$\eta_{xy} = \begin{cases} 
an \text{independent copy of } \xi, & \text{if } |x - y| = 1, \\
0, & \text{otherwise}.
\end{cases} \quad (6.11)$$

We establish LILs for occupation times of VSRW $X$ associated with (6.11).
Proposition 6.8. Assertions (i) and (ii) of Proposition 6.7 hold true for $\mathcal{U}_X(\cdot, \cdot)$ with $\mathbb{Z}^d$ instead of $C_\infty(\omega)$, where $X$ is the VSRW associated with the random conductance defined by (6.11).

Proof. Clearly VRD$^2(4/5)$ holds for $(\mathbb{Z}^d, \mu_c)$. Using [2] Lemma 3.5 and Theorems 4.3 and 4.6 and the Borel-Cantelli lemma, we deduce that there exists for $\mathbb{P}$-a.s. $\omega$, there is a constant $R_\infty(\omega) \in (1, \infty)$ such that NDL$^R_{R\infty}(\Phi, 4/5)$ and NDU$^R_{R\infty}(\Phi, 4/5)$ hold with $\Phi(r) = r^2$. Since Tail$^R_{R\infty}(\Phi, 4/5, \leq)$ clearly holds, using Theorems 2.6, 2.9 and 2.10 we get the result.

6.2.3. Random conductance model with stable-like jumps.

In [10,11], the authors have obtained limsup and liminf LILs at infinity for random conductance model with long range jumps using results from [7,8]. Here we give LILs for occupation times at infinity for this model.

Suppose that there is a constant $d > 0$ such that

$$
\mu_c(B(x, r)) \approx r^d \quad \text{for all } x \in \mathbb{L}, \ r > 10
$$

(6.12)

Let $\alpha \in (0 \vee (2 - \frac{d}{2}), 2)$ and $\eta$ be a random conductance on $\mathbb{L}$ such that $w_{xy} := \eta_{xy}|x-y|^{d+\alpha}$ satisfies the following properties:

$$
\sup_{x, y \in \mathbb{L}, x \neq y} \mathbb{P}(w_{xy} = 0) < 1/2 \quad \text{and} \quad \sup_{x, y \in \mathbb{L}, x \neq y} \mathbb{E}[w_{xy}^p + w_{xy}^{-q}1_{w_{xy} > 0}] < \infty,
$$

(6.13)

where the constants $p$ and $q$ satisfy

$$
p > \frac{d + 2}{d} \vee \frac{d + 1}{4 - 2\alpha}, \quad q > \frac{d + 2}{d}.
$$

When we consider the CSRW $Y^\omega$, we additionally assume that there exist constants $m_2 \geq m_1 > 0$ such that for $\mathbb{P}$-a.s. $\omega$,

$$
\eta_{xy}(\omega) > 0 \quad \text{for all } x, y \in \mathbb{L}, \ x \neq y \quad \text{and} \quad m_1 \leq \sum_{y \in \mathbb{L}} \eta_{xy}(\omega) \leq m_2 \quad \text{for all } x \in \mathbb{L}.
$$

(6.14)

Proposition 6.9. Suppose that (6.12) and (6.13) are satisfied. There exist $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and constants $\kappa_3 \geq \kappa_2 \geq \kappa_1 > 0$, $a_2 \geq a_1 > 0$ such that the following statements hold true for every $\omega \in \Omega_0$.

(i) There exists an non-increasing function $f_{\infty}(\cdot, \omega) : (0, \infty) \to (0, 1]$ such that $f_{\infty}(\kappa, \omega) = 1$ if $\kappa \leq \kappa_1$, $\lim_{\kappa \to \infty} f_{\infty}(\kappa, \omega) = 0$ uniformly on $\omega \in \Omega_0$ and that for every $\kappa > 0$,

$$
\lim_{r \to \infty} \frac{\mathcal{U}_X(B(x, r), \kappa r^\alpha \log \log r)}{\kappa r^\alpha \log \log r} = f_{\infty}(\kappa, \omega), \quad \mathbb{P}^z_{\omega}-\text{a.s.}, \ \forall z \in \mathbb{L}.
$$

(ii) (a) If $d < \alpha$, then for every $\gamma \in (\frac{ad}{\alpha-d}, \infty]$ and $F \in \mathcal{B}_{b+}^\gamma(\mathbb{L}; \mu_c)$ with $\|F\|_{L^1(\mathbb{L}; \mu_c)} \neq 0$, there exists a constant $a_F(\omega) \in [a_1, a_2]$ such that

$$
\limsup_{t \to \infty} \frac{\mathcal{U}_X(F, t)/\|F\|_{L^1(\mathbb{L}; \mu_c)}}{t^{-1/\alpha} (\log \log t)^{1/\alpha}} = a_F, \quad \mathbb{P}^z_{\omega}-\text{a.s.}, \ \forall z \in \mathbb{L}.
$$

(b) If $d > \alpha$, there then exists a constant $a_F(\omega) \in (\kappa_2, \kappa_3]$ such that

$$
\limsup_{r \to \infty} \frac{\mathcal{U}_X(B(x, r), \infty)}{\kappa(\omega) \ r^\alpha \log \log r} = 1, \quad \mathbb{P}^z_{\omega}-\text{a.s.}, \ \forall z \in \mathbb{L}.
$$

(iii) If (6.14) are also satisfied, then the assertions in (i-ii) hold true for $\mathcal{U}_Y$ instead of $\mathcal{U}_X$ with different constants $\kappa_1, \kappa_2, \kappa_3, a_1, a_2$. 
Then, there exists a constant \( a \) such that be a continuous non-decreasing adapted functionals of \( \nu \). By the proof of \cite[Theorem 3.1]{10} and \cite[Proposition 2.3]{7}, there exist \( v \in (0, 1) \) such that for some \( p > \frac{d+2}{d} \vee \frac{d+1}{2-2a} \) and \( q > \frac{d+2}{d} \), by the proof for the second statement of \cite[Theorem 3.1]{10}. We arrive at the desired result. \( \square \)

**Remark 6.10.** The assertions of Proposition \cite[6.9]{3} hold true for \( \alpha \in (0 \vee (1 - \frac{d}{2}), 1) \), if \( (6.13) \) is satisfied with some \( p > \frac{d+2}{d} \vee \frac{d+1}{2-2a} \) and \( q > \frac{d+2}{d} \), by the proof for the second statement of \cite[Theorem 3.1]{10}.

### Appendix

**Lemma 7.1.** Let \( X \) be a topological space and \( Y = (Y_t, t \geq 0; \mathbb{P}^y, y \in X_0) \) a strong Markov process on \( X \) with a cemetery point \( \partial \). Denote by \( \theta^Y_t \) the shift operator with respect to \( Y \). Let \( (F_t)_{t \geq 0} \) be a continuous non-decreasing adapted functionals of \( Y \) and \( g : (0, \infty) \to (0, \infty) \) be an increasing continuous function. Suppose that there exists \( T > 0 \) such that the following conditions are satisfied:

1. \( F(t) - F(s) \leq F(t-s) \circ \theta^Y_s \) for all \( T < s \leq t \).
2. \( \lim_{t \to \infty} g(t) = \infty \) and there exists a constant \( a_1 > 1 \) such that \( g(2t) \leq a_1 g(t) \) for all \( t > T \).
3. There exist constants \( a_2, \varepsilon_1 > 0 \) such that
   \[
   \sup_{x \in X} \mathbb{P}^x (F(t) \geq a_2 g(t)) \leq e^{-\varepsilon_1} \quad \text{for all } t > T. \tag{7.2}
   \]

Then, there exists a constant \( a_3 \in (0, \infty) \) which depends only on \( a_1, a_2 \) and \( \varepsilon_1 \) such that
\[
\limsup_{t \to \infty} \frac{F(t)}{g(t/\log \log g(t)) \log \log g(t)} \leq a_3, \quad \mathbb{P}^x \text{-a.s., } \forall \, z \in X. \tag{7.5}
\]

**Proof.** Let \( K := \log a_1 \). Since \( \lim_{t \to \infty} g(t) = \infty \), there exist \( N > 3 \) and a sequence \( (t_n)_{n \geq N} \) such that
\[
F(t_n/\log(nK)) = e^{nK} \quad \text{for all } n \geq N. \tag{7.3}
\]
By (7.1), it holds that
\[
g(2t_n/\log(nK)) \leq a_1 e^{nK} \leq e^{(n+1)K} = g(t_{n+1}/\log(nK+K)).
\]
Since \( g \) is increasing, this yields that
\[
t_{n+1} \geq 2t_n \quad \text{for all } n \geq N. \tag{7.4}
\]
Define \( u_n = \log \log g(t_n) \) for \( n \geq N \). By (7.3) and (7.1), we have
\[
e^{nK} \leq g(t_n) \leq c_1 (\log(nK))^{c_2} g(t_n/\log(nK)) = c_1 (\log(nK))^{c_2} e^{nK}.
\]
Since \( t_n \) grows at least exponentially by (7.4), we deduce that there exists \( N' \geq N \) such that
\[
\log n \leq u_n \leq 2 \log n < t_n/T \quad \text{for all } n \geq N'. \tag{7.5}
\]
Applying the property (1) yields that for every \( n \geq N' \),
\[
F(t_n) \leq \sum_{j=0}^{[u_n]} F(t_{n+1}/u_n) \circ \theta^Y_{t_{n+1}/u_n}. \tag{7.6}
\]
We claim that for all \( n \geq N' \) and \( m \geq 1 \),
\[
\sup_{z \in X} \mathbb{P}^z (F(t_n/u_n) \geq a_2 mg(t_n/u_n)) \leq e^{-m\varepsilon_1}. \tag{7.7}
\]
By (7.2), (7.7) holds true when \( m = 1 \). Assume that (7.7) holds for \( m \leq k \in \mathbb{N} \). Set

\[
T_1 := \inf \{ s > 0 : F(s) \geq a_2 g(t_n/u_n) \}.
\]

Using the strong Markov property, (7.2) and the induction hypothesis, we get that for all \( z \in \mathcal{X} \),

\[
\mathbb{P}^z(F(t_n/u_n) \geq a_2(k + 1)g(t_n/u_n)) = \mathbb{P}^z(F(t_n/u_n) \geq a_2(k + 1)g(t_n/u_n), T_1 \leq t_n/u_n)
\leq \mathbb{E}^z\mathbb{P}^Z_{T_1}(F(t_n/u_n) \geq a_2kg(t_n/u_n))\mathbb{P}^z(T_1 \leq t_n/u_n) \leq e^{-(k+1)\varepsilon_1}.
\]

Hence, we conclude that (7.7) is true by induction.

Using Markov inequality and (7.7), we arrive at

\[
sup_{z \in \mathcal{X}} \mathbb{E}^z \left[ \exp \left( \frac{\varepsilon_1}{2a_2} \frac{F(t_n/u_n)}{g(t_n/u_n)} \right) \right]
\leq \sum_{m=0}^\infty e^{\varepsilon_1(m+1)/2} \sup_{z \in \mathcal{X}} \mathbb{P}^z \left( \frac{F(t_n/u_n)}{g(t_n/u_n)} \in \left[ a_2m, a_2(m + 1) \right) \right)
\leq e^{\varepsilon_1/2} \sum_{m=0}^\infty e^{-m\varepsilon_1/2} = c_3. \tag{7.8}
\]

By Markov inequality, (7.5) and (7.6), (7.8) yields that for all \( z \in \mathcal{X} \) and \( n \geq N' \),

\[
\mathbb{P}^z(F(t_n) > 2a_2\varepsilon_1^{-1}(3 + \log c_3)g(t_n/u_n)u_n) \leq e^{-(3+\log c_3)u_n} \mathbb{E}^z \left[ \exp \left( \frac{\varepsilon_1}{2a_3} \frac{F(t_n)}{g(t_n/u_n)} \right) \right]
\leq e^{-(3+\log c_3)u_n} \prod_{j=0}^{|u_n|} \sup_{w \in \mathcal{X}} \mathbb{E}^w \left[ \exp \left( \frac{\varepsilon_1}{2a_3} \frac{F(t_n/u_n)}{g(t_n/u_n)} \right) \right]
\leq c_3 e^{-3u_n} \leq c_3 n^{-3/2}.
\]

Finally, using the Borel-Cantelli lemma, (7.1), (7.3) and (7.5), it holds that for all \( z \in \mathcal{X} \) and \( \mathbb{P}^z \)-a.s.,

\[
\limsup_{t \to \infty} \frac{F(t)}{g(t/\log \log g(t)) \log \log g(t)}
\leq \limsup_{n \to \infty} \frac{F(t_n)}{g(t/\log \log g(t)) \log \log g(t)} \leq \limsup_{n \to \infty} \frac{F(t_n)}{g(t_n/u_n-1)u_n-1}
\leq c_4 \limsup_{n \to \infty} \frac{F(t_n)}{u_n-1 e^{(n-1)K}} \leq c_5 e^K \limsup_{n \to \infty} \frac{F(t_n)}{g(t_n/u_n)u_n} \leq 2a_2\varepsilon_1^{-1}(3 + \log c_3)c_5 e^K.
\]

The proof is complete. \( \square \)

The second lemma is a version of conditional Borel-Cantelli lemma which was proved in [10 Proposition 5.1].

**Lemma 7.2.** Let \((\tilde{\Omega}, \tilde{\mathbb{P}}, \mathcal{G}, (G_t)_{t \geq 0})\) be a filtered probability space and \((s_n)_{n \geq 1}\) an increasing sequence with \( \lim_{n \to \infty} s_n = \infty \). Assume that families of events \((E_n)_{n \geq 1}\) and \((G_n)_{n \geq 1}\) satisfy the following properties:

1. \( E_n, G_n \in \mathcal{G}_{s_n} \) for all \( n \geq 1 \).
2. There exist a constant \( p > 0 \) and a sequence \((b_n)_{n \geq 1}\) with \( \sum_{n=1}^{\infty} b_n = \infty \) such that

\[
\tilde{\mathbb{P}}(G_n^c) \leq p \quad \text{and} \quad \tilde{\mathbb{P}}(E_{n+1} | G_{s_n}) \geq b_n 1_{G_n} \quad \text{for all} \ n \geq 1.
\]

Then, \( \tilde{\mathbb{P}}(\limsup E_n) \geq 1 - p \). In particular, if \( \lim_{n \to \infty} \tilde{\mathbb{P}}(G_n^c) = 0 \), then \( \tilde{\mathbb{P}}(\limsup E_n) = 1 \).
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