f-BIHARMONIC MAPS AND f-BIHARMONIC SUBMANIFOLDS II

YE-LIN OU∗

Abstract
We continue our study [Ou4] of f-biharmonic maps and f-biharmonic submanifolds by exploring the applications of f-biharmonic maps and the relationships among biharmonicity, f-biharmonicity and conformality of maps between Riemannian manifolds. We are able to characterize harmonic maps and minimal submanifolds by using the concept of f-biharmonic maps and prove that the set of all f-biharmonic maps from 2-dimensional domain is invariant under the conformal change of the metric on the domain. We give an improved equation for f-biharmonic hypersurfaces and use it to prove some rigidity theorems about f-biharmonic hypersurfaces in nonpositively curved manifolds, and to give some classifications of f-biharmonic hypersurfaces in Einstein spaces and in space forms. Finally, we also use the improved f-biharmonic hypersurface equation to obtain an improved equation and some classifications of biharmonic conformal immersions of surfaces into a 3-manifold.

1. Introduction

Biharmonic maps and f-biharmonic maps are maps φ : (M, g) → (N, h) between Riemannian manifolds which are the critical points of

the bienergy : \( E_2(\phi) = \frac{1}{2} \int_{\Omega} |\tau(\phi)|^2 v_g, \) and

the f-bienergy : \( E_{2,f}(\phi) = \frac{1}{2} \int_{\Omega} f |\tau(\phi)|^2 v_g, \)

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respectively, where $\Omega$ is a compact domain of $M$ and $\tau(\phi)$ is the tension field of the map $\phi$. The Euler-Lagrange equations of these functionals give the biharmonic map equation (see [Ji1])

\begin{equation}
\tau_2(\phi) := \text{Trace}_g(\nabla^g \nabla^\phi - \nabla^\phi g_M) \tau(\phi) - \text{Trace}_g R^N(d\phi, \tau(\phi)) d\phi = 0,
\end{equation}

where $R^N$ denotes the curvature operator of $(N, h)$ defined by

$$R^N(X, Y) Z = [\nabla^N_X, \nabla^N_Y] Z - \nabla^N_{[X,Y]} Z,$$

and the $f$-biharmonic map equation (see [Lu] and also [Ou4])

\begin{equation}
\tau_{2,f}(\phi) \equiv f \tau_2(\phi) + (\Delta f) \tau(\phi) + 2 \nabla^\phi_{\text{grad} f} \tau(\phi) = 0,
\end{equation}

where $\tau_2(\phi)$ is the bitension field of $\phi$ defined in (1).

As the images of biharmonic isometric immersions, biharmonic submanifolds have become a popular topic of research with many progresses since the beginning of this century (see a recent survey [Ou5] and the references therein for more details). In a recent paper [Ou4], we introduced and studied $f$-biharmonic submanifolds as those submanifolds whose isometric immersions $\phi : M^m \longrightarrow (N^n, h)$ are $f$-biharmonic maps for some positive function $f$ defined on the submanifolds. $f$-Biharmonic submanifolds are generalizations of biharmonic submanifolds and of minimal submanifolds since (i) $f$-biharmonic submanifolds with $f = \text{constant}$ are precisely biharmonic submanifolds, and (ii) any minimal submanifold is $f$-biharmonic for any positive function defined on the submanifold. As it is seen in [Ou4] and Example 1 of Section 3 of this paper, there are many examples of $f$-biharmonic submanifolds which are neither biharmonic nor minimal.

An interesting link among biharmonicity, $f$-biharmonicity and conformality is the following theorem which, although holds only for two-dimensional domain, has played an important role in the study of biharmonic maps from two-dimensional domains (see [Ou4] and [WOY]) and will be used to study biharmonic conformal immersions in the last section of this paper.

**Theorem A.** ([Ou4]) A map $\phi : (M^2, g) \longrightarrow (N^n, h)$ is an $f$-biharmonic map if and only if $\phi : (M^2, f^{-1}g) \longrightarrow (N^n, h)$ is a biharmonic map. In particular, if a surface (i.e., an isometric immersion) $\phi : (M^2, g = \phi^* h) \longrightarrow (N^n, h)$ is an $f$-biharmonic surface if and only if the conformal immersion $\phi : (M^2, f^{-1}g) \longrightarrow (N^n, h)$ is a biharmonic map.
Regarding the topological and the geometric obstacles to the existence of general $f$-biharmonic maps, it was proved in [On4] that an $f$-biharmonic map from a compact manifold into a non-positively curved manifold with constant $f$-bienergy density is a harmonic map, and that any $f$-biharmonic map from a compact manifold into Euclidean space is a constant map.

In this paper, we continue our work on $f$-biharmonic maps and $f$-biharmonic submanifolds by further exploring the applications of $f$-biharmonic maps and the relationships among harmonicity, biharmonicity, $f$-biharmonicity and conformality of maps between Riemannian manifolds. We prove that a map between Riemannian manifold is a harmonic map if and only if it is an $f$-biharmonic map for any positive function $f$, and that a submanifold of a Riemannian manifold is minimal if and only if it is $f$-biharmonic for any positive function $f$ defined on the submanifold. We show that the set of all $f$-biharmonic maps from 2-dimensional domain is invariant under the conformal change of the metric on the domain. We prove some rigidity theorems about $f$-biharmonic hypersurfaces in nonpositively curved manifolds and give some classifications of $f$-biharmonic hypersurfaces in Einstein spaces and in space forms by using an improved form of $f$-biharmonic hypersurface equation. We also use the improved $f$-biharmonic hypersurface equation to obtain an improved equation and some classifications of biharmonic conformal immersions of surfaces into a 3-manifold.

2. Harmonicity, biharmonicity, $f$-biharmonicity and conformality

It is well known that harmonicity of maps from a 2-dimensional domain is invariant under conformal transformations of the domain. It is also known that biharmonic maps do not have this kind of invariance regardless of domain dimensions. However, as it is shown by our next theorem, the set

$$\mathcal{H}_{2,f}(M^2, N^n) = \{(\phi, f) : \phi : (M^2, g) \to (N^n, h), f : (M^2, g) \to (0, \infty), \phi \text{ is an } f \text{-biharmonic map}\}$$

of all $f$-biharmonic maps is invariant under conformal transformation of the domain.

**Theorem 2.1.** The set $\mathcal{H}_{2,f}(M^2, N^n)$ of all $f$-biharmonic maps from a surface into a manifold is invariant under conformal transformation of the domain, more precisely, if $\phi : (M^2, g) \to (N^n, h)$ is $f$-biharmonic for a function $f = \alpha$, then $\phi : (\bar{g} = \lambda^2 g) \to (N^n, h)$ also an $f$-biharmonic map for $f = \alpha\lambda^2$.

**Proof.** Let $\phi : (M^2, g) \to (N^n, h)$ be an $f$-biharmonic map for a function $f = \alpha$, then by Theorem A, the map $\phi : (M^2, \alpha^{-1}g) \to (N^n, h)$ is a biharmonic map.
Noting that \( (M^2, \alpha^{-1} g) = (M^2, (\alpha^{-1} \lambda^{-2}) \lambda^2 g) = (M^2, (\alpha \lambda^2)^{-1} \bar{g}) \), we conclude that the map \( \phi : (M^2, (\alpha \lambda^2)^{-1} \bar{g}) \rightarrow (N^n, h) \) is a biharmonic map. Using Theorem A again we conclude that the map \( \phi : (M^2, \bar{g} = \lambda^2 g) \rightarrow (N^n, h) \) is an \( f \)-biharmonic map with \( f = \alpha \lambda^2 \). Thus, we obtain the theorem. \( \square \)

It is clear from the definition of \( f \)-biharmonic maps that a harmonic map \( \phi : (M^m, g) \rightarrow (N^n, h) \) is an \( f \)-biharmonic map for any function \( f : (M^m, g) \rightarrow (0, \infty) \). Our next theorem shows that this property can actually be used to characterize harmonic maps between Riemannian manifolds.

**Theorem 2.2.** A map \( \phi : (M^m, g) \rightarrow (N^n, h) \) between Riemannian manifold is a harmonic map if and only if it is an \( f \)-biharmonic map for any function \( f : (M^m, g) \rightarrow (0, \infty) \).

**Proof.** We only need to prove the "if" part of the statement, i.e., if a map \( \phi : (M^m, g) \rightarrow (N^n, h) \) is an \( f \)-biharmonic map for any function \( f : (M^m, g) \rightarrow (0, \infty) \), then it is a harmonic map. To prove this is equivalent to show that if the equation

\[
(3) \quad f \tau_2(\phi) + (\Delta f) \tau(\phi) + 2 \nabla^\phi f \tau(\phi) = 0
\]

holds for all positive functions \( f \) on \( (M^m, g) \), then we have \( \tau(\phi) \equiv 0 \).

In fact, take a positive constant function \( f = C \), then equation \( (3) \) implies that the map \( \phi \) is a biharmonic map, i.e., \( \tau_2(\phi) \equiv 0 \). It follows that if a map \( \phi : (M^m, g) \rightarrow (N^n, h) \) is an \( f \)-biharmonic maps for any function \( f \), then we have

\[
(4) \quad (\Delta f) \tau(\phi) + 2 \nabla^\phi f \tau(\phi) = 0 \quad \forall \ f : (M^m, g) \rightarrow (0, \infty).
\]

If there exists a point \( p \in M \) such that \( \tau(\phi)(p) \neq 0 \), then there is an open neighborhood \( U \) of \( p \) so that \( \tau(\phi)(q) \neq 0 \), \( \forall \ q \in U \). By taking dot product of both sides of \( (4) \) with \( \tau(\phi) \), we have, on \( U \),

\[
(5) \quad (\Delta f)|\tau(\phi)|^2 + (\nabla f)|\tau(\phi)|^2 = 0 \quad \forall \ f : (U, g|_U) \rightarrow (0, \infty).
\]

It is well known that there exists a local harmonic coordinate system in a neighborhood of any point of a Riemannian manifold. By this well known fact and a translation in Euclidean spaces, we can choose a neighborhood \( V \subset U \) of \( p \) with harmonic coordinate \( \{x^i\} \) satisfying \( x^i > 0 \) for all \( i = 1, 2, \cdots, m \). By choosing positive harmonic coordinate functions \( f = x^i \) and substituting it into \( (5) \) we have

\[
(6) \quad g^{ik} \frac{\partial}{\partial x^k} |\tau(\phi)|^2 = 0, \quad \forall \ i = 1, 2, \cdots, m.
\]
It follows that \( |τ(ϕ)|^2 \) = constant, and hence Equation (5) reduces to
\[
(Δf)|τ(ϕ)|^2 = 0 \quad ∀ \ f : (V, V|g) → (0, ∞).
\]
Now let \( f = (x^1)^2 \), where \( x^1 \) is the first coordinate function, and substitute it into Equation (7) to have
\[
(Δf)|τ(ϕ)|^2 = 2g^{11}|τ(ϕ)|^2 = 0 \quad \text{on} \ \ V.
\]
Since the Riemannian metric is positive definite, we conclude that \( |τ(ϕ)|^2 \) = 0 on \( V \), which contradicts the assumption that \( τ(ϕ) \neq 0 \) on \( U \). The contradiction shows that we must have \( τ(ϕ) ≡ 0 \) on \( M \), i.e., the map \( ϕ \) is a harmonic map. This completes the proof of the theorem.

As an immediate consequence, we have the following characterization of minimal submanifolds using the concept of \( f \)-biharmonic submanifolds.

Corollary 2.3. A submanifold \( M^m \hookrightarrow (N^n, h) \) is minimal if and only if it is \( f \)-biharmonic for any positive function defined on the submanifold.

To close this section, we take a look at maps which are \( f \)-biharmonic for two different functions, especially those maps which are both biharmonic (i.e., \( f \)-biharmonic for \( f = \text{constant} \)) and \( f \)-biharmonic for \( f \neq \text{constant} \).

It is shown in [Ou4] that conformal immersions \( ϕ : \mathbb{R}^4 \setminus \{0\} → \mathbb{R}^4 \) given by the inversion of the sphere \( S^3 \), \( ϕ(x) = \frac{x}{|x|^2} \) is a biharmonic map and also an \( f \)-biharmonic map with \( f(x) = |x|^4 \). Our next theorem shows that this situation cannot happen for pseudo-umbilical isometric immersions.

Theorem 2.4. If a pseudo-umbilical proper biharmonic submanifold \( ϕ : M^m → (N^n, h) \) is \( f \)-biharmonic, then \( f \) is constant.

Proof. \( ϕ : M^m → (N^n, h) \) be a pseudo-umbilical submanifold, then, by definition, its shape operator with respect to the mean curvature vector field \( η \) is given by
\[
A_η(X) = \langle η, η \rangle X
\]
for any vector field \( X \) tangent to the submanifold. It is well known that the tension field of the submanifold is given by \( τ(ϕ) = mη \). If the submanifold is both biharmonic and \( f \)-biharmonic, then we have both \( τ_2(ϕ) = 0 \) and \( τ_2f(ϕ) = 0 \) identically. Substituting these into Equation (2) we obtain
\[
0 = (Δf)τ(ϕ) + 2∇^N_{gradf}τ(ϕ)
= m[(Δf)η + 2∇^N_{gradf}η]
= m[(Δf)η + 2∇^⊥_{gradf}η - 2A_η(\text{grad } f)]
= m[(Δf)η + 2∇^⊥_{gradf}η - 2⟨η, η⟩ \text{ grad } f],
\]
where the third equality was obtained by using the Weingarten formula for sub-
manifolds and the last equality was obtained by the assumption that the subman-
ifold is pseudo-umbilical. Comparing the normal and the tangent components of
the both sides of (9) we have grad f = 0 since the submanifold is non-minimal,
and hence f = constant. This completes the proof of the theorem. □

Corollary 2.5. Biharmonic hypersurface φ : \( S^m(\frac{1}{\sqrt{2}}) \hookrightarrow S^{m+1} \), φ(x) = (\frac{1}{\sqrt{2}}, x)
is an f-biharmonic hypersurface if and only if f is constant. In particular, there
is no nonconstant function f on \( S^2(\frac{1}{\sqrt{2}}) \) so that the conformal immersion φ :
(\( S^2(\frac{1}{\sqrt{2}}) \), \( f^{-1}g_0 \)) → (\( S^3, h_0 \)), φ(x) = (\frac{1}{\sqrt{2}}, x)
becomes a biharmonic map, where g_0 and h_0 are the metrics on \( S^2(\frac{1}{\sqrt{2}}) \) and \( S^3 \)
induced from their ambient Euclidean spaces respectively.

Proof. The first statement of the corollary follows from Theorem 2.4 and the fact
that the hypersurface φ : \( S^m(\frac{1}{\sqrt{2}}) \hookrightarrow S^{m+1} \), φ(x) = (\frac{1}{\sqrt{2}}, x) is a totally umbilical
biharmonic hypersurface of \( S^{m+1} \) (See [CMO1]). The second statement of the
corollary follows from the first statement and Theorem A. □

3. The equations of f-biharmonic hypersurfaces and some
classification results

In this section, we will give an improved equation for f-biharmonic hypersurfaces
in a general Riemannian manifold and use it to prove some rigidity theorems and
some classifications of f-biharmonic hypersurfaces in Einstein spaces, including
space forms.

Theorem 3.1. A hypersurface φ : \( M^m \rightarrow (N^{m+1}, h) \) with the mean curvature
vector field \( \eta = H\xi \) is f-biharmonic if and only if:

\[
\begin{align*}
\Delta(f H) - (f H)\|A\|_2^2 &- \text{Ric}^N(\xi, \xi) = 0, \\
A(\text{grad}(f H)) + (f H)[\frac{m}{2}\text{grad} H - (\text{Ric}^N(\xi))]^\top &- 0,
\end{align*}
\]

where \( \text{Ric}^N : T_qN \rightarrow T_qN \) denotes the Ricci operator of the ambient space
defined by \( \langle \text{Ric}^N(Z), W \rangle = \text{Ric}^N(Z, W) \), A is the shape operator of the hypersur-
face with respect to the unit normal vector \( \xi \), and \( \Delta, \text{grad} \) are the Laplace and the
gradient operator of the hypersurface respectively.

Proof. It was proved in [Ou4] that an isometric immersion φ : \( M^m \rightarrow N^{m+1} \)
with mean curvature vector \( \eta = H\xi \) is f-biharmonic if and only if:

\[
\begin{align*}
\Delta H - H|A|^2 + H\text{Ric}^N(\xi, \xi) + H(\Delta f)/f + 2(\text{grad} \ln f)H &- 0, \\
A(\text{grad} H) + \frac{m}{2}H\text{grad} H - H(\text{Ric}^N(\xi))^\top + HA(\text{grad} \ln f) &- 0,
\end{align*}
\]

where the third equality was obtained by using the Weingarten formula for sub-
manifolds and the last equality was obtained by the assumption that the subman-
ifold is pseudo-umbilical. Comparing the normal and the tangent components of
the both sides of (9) we have grad f = 0 since the submanifold is non-minimal,
and hence f = constant. This completes the proof of the theorem. □
where $\xi$, $A$, and $H$ are the unit normal vector field, the shape operator, and the mean curvature function of the hypersurface $\varphi(M) \subset (N^{m+1}, h)$ respectively, and the operators $\Delta$, $\text{grad}$ and $|\cdot|$ are taken with respect to the induced metric $g = \varphi^* h = \lambda^2 \bar{g}$ on the hypersurface.

Multiplying $f$ to both sides of each equation in the system (11) we can rewrite the resulting system as (10).

As a straightforward consequence, we have the following equations for $f$-biharmonic hypersurfaces in Einstein spaces and space forms, which improves the equations given in [PA], and [Ou4] respectively.

**Corollary 3.2.** A hypersurface $\varphi: M^m \to (N^{m+1}, h)$ in an Einstein space with $\text{Ric}^N = \lambda h$ is $f$-biharmonic if and only if its mean curvature function $H$ solves the following PDEs

\[
\begin{cases}
\Delta (f H) - (f H)|A|^2 - \lambda = 0, \\
A(\text{grad}(f H)) + \frac{m}{2} (f H) \text{grad} H = 0.
\end{cases}
\]

In particular, a hypersurface $\phi: M^m \to N^{m+1}(C)$ in a space form of constant sectional curvature $C$ is $f$-biharmonic if and only if its mean curvature function $H$ is a solution of

\[
\begin{cases}
\Delta (f H) - (f H)|A|^2 - mC = 0, \\
A(\text{grad}(f H)) + \frac{m}{2} (f H) \text{grad} H = 0.
\end{cases}
\]

As the first application of the improved $f$-biharmonic hypersurface equation, we will prove the following theorem, which gives a rigidity result of $f$-biharmonic isometric immersions under some constraints of Ricci curvatures of the ambient space. Let us recall the following lemma, which will be used in proving our next theorem.

**Lemma A.** ([NU]) Assume that $(M, g)$ is a complete non-compact Riemannian manifold, and $\alpha$ is a non-negative smooth function on $M$. Then, every smooth $L^2$ function $u$ on $M$ satisfying the Schrödinger type equation

\[
\Delta_g u = \alpha u
\]

on $M$ is a constant.

**Theorem 3.3.** A complete hypersurface $\phi: M^m \to (N^{m+1}, h)$ of a manifold $N$ with $\text{Ric}^N(\xi, \xi) \leq |A|^2$ and the mean curvature vector field $\eta = H \xi$ satisfying
\[
\int_M (fH)^2 dv_g < \infty \text{ is } f\text{-biharmonic if and only if it is minimal, or}
\]

\[
\begin{cases}
\text{Ric}^N(\xi, \xi) = |A|^2, \\
(\text{Ric}^N(\xi))^\top = \frac{m}{2} \text{grad } H, \text{ or } H = 0.
\end{cases}
\]

In particular, a complete \( f\)-biharmonic hypersurface \( \phi : M^m \rightarrow (N^{m+1}, h) \) of a manifold \( N \) of nonpositive Ricci curvature and the mean curvature function satisfying \( \int_M (fH)^2 dv_g < \infty \) is minimal.

**Proof.** A straightforward computation yields

\[
\Delta (fH)^2 = 2fH \Delta (fH) + 2|\nabla (fH)|^2.
\]

If the biharmonic hypersurface \( \phi : M^m \rightarrow (N^{m+1}, h) \) is compact, then, by Theorem 3.1 we have Equation (10). Substituting the first equation of (10) into (16) we have

\[
\Delta (fH)^2 = 2(fH)^2[|A|^2 - \text{Ric}^N(\xi, \xi)] + 2|\nabla (fH)|^2 \geq 0
\]

by the assumption that \( \text{Ric}^N(\xi, \xi) \leq |A|^2 \). It follows that the function \( fH \) is a subharmonic function on a compact manifold, and hence it is constant.

If the biharmonic hypersurface \( \phi : M^m \rightarrow (N^{m+1}, h) \) is non-compact but complete, then the first equation of (14) with \( \alpha = |A|^2 - \text{Ric}^N(\xi, \xi) \) being nonnegative. Using Lemma A we conclude that \( fH \) is constant. So, in either case, the function \( fH = C \), a constant. Substituting this into (10) we obtain the first statement of the theorem.

To prove the second statement, notice that \( fH = \text{constant} \) and \( f \) being positive imply that we either have (i) \( H \equiv 0 \) in which case the hypersurface is minimal, or (ii) \( H \) is never zero. If case (ii) happens, then we use \( fH = \text{constant} \neq 0 \) and the first equation \( f\)-biharmonic hypersurface equation to conclude that \( 0 \leq |A|^2 = \text{Ric}^N(\xi, \xi) \leq 0 \). This implies that \( A = 0 \) and hence the hypersurface is totally geodesic. It follows that \( H \equiv 0 \), which contradicts the assumption that \( H \) is never zero. The contradiction shows that we only have case (i), which gives the second statement. \( \square \)

From the proof of Theorem 3.3 we have the following

**Corollary 3.4.** (i) A compact hypersurface \( \phi : M^m \rightarrow S^{m+1} \) with squared norm of the second fundamental form \( |A|^2 \geq m \) is \( f\)-biharmonic if and only if it is biharmonic with constant mean curvature and \( |A|^2 = m \). (ii) A complete hypersurface \( \phi : M^m \rightarrow S^{m+1} \) of the sphere \( S^{m+1} \) with the mean curvature vector field \( \eta = H\xi \) satisfying \( |A|^2 \geq m \) and \( \int_M (fH)^2 dv_g < \infty \) is \( f\)-biharmonic.
if and only if the hypersurface is a biharmonic hypersurface of constant mean curvature and $|A|^2 = m$.

As the second application of the improved $f$-biharmonic hypersurface equation, we give some classifications of $f$-biharmonic hypersurfaces in Einstein spaces and space forms.

**Theorem 3.5.** A compact hypersurface of an Einstein space with nonpositive scalar curvature is $f$-biharmonic if and only if it is a minimal hypersurface.

**Proof.** First of all, by Corollary 2.3 a minimal hypersurface $\phi : M^m \to (N^{m+1}, h)$ is $f$-biharmonic for any positive function $f$ on $M$. Conversely, let $\phi : M^m \to (N^{m+1}, h)$ be a compact $f$-biharmonic hypersurface with the mean curvature vector field $\eta = H \xi$ on an Einstein manifold. Then, one can easily compute that

$$\Delta((fH)^2) = 2(fH)\Delta(fH) + 2|\text{grad}(fH)|^2.$$  

Using the first equation of (12) with $\lambda = \frac{R_N}{m+1}$ and the assumption on the scalar curvature $R_N$, we have

$$\Delta((fH)^2) = 2(fH)^2[|A|^2 - \frac{R_N}{m+1}] + 2|\text{grad}(fH)|^2 \geq 0.$$  

It follows that $(fH)^2$ is a subharmonic function on a compact manifold and hence it is a constant. This implies that $fH = C$, a constant, from which and the fact that the $f$ is a positive function we conclude that either (i) $H \equiv 0$, in which case the hypersurface in minimal, or (ii) $H(p) \neq 0$ for any $p \in M^m$. Now if case (ii) happens, then we $fH = C \neq 0$ and the second equation of the $f$-biharmonic equation of Einstein space to conclude that $H$ is a nonzero constant. Using this and the first equation of (12) again we have $0 \leq |A|^2 = \frac{R_N}{m+1} \leq 0$ and hence $A = 0$. This means the hypersurface is actually totally geodesic and hence $H \equiv 0$, a contradiction to the assumption that $H(p) \neq 0$ for any $p \in M^m$. The contradiction shows that case (ii) cannot happen. Thus, we obtain the theorem. \(\square\)

**Theorem 3.6.** A totally umbilical hypersurface $\phi : M^m \to (N^{m+1}, h)$ in an Einstein space with $\text{Ric}^N = \lambda h$ is $f$-biharmonic if and only if it is totally geodesic or a biharmonic hypersurface with constant mean curvature $H = \pm \sqrt{\lambda/m}$.

**Proof.** If the hypersurface is totally umbilical, then we have $A(X) = H X$ for any vector tangent to the hypersurface and $|A|^2 = mH^2$. Substituting these into (12) we have

$$\begin{cases} \Delta(fH) - (fH)[mH^2 - \lambda] = 0, \\ H\text{grad}(fH) + \frac{m}{2}(fH)\text{grad}H = 0. \end{cases}$$  


On the other hand, it was proved in [Ko] (see also Lemma 2.1 in [JMS]) that a totally umbilical hypersurface in an Einstein space has constant mean curvature, i.e., $H = \text{constant}$. If $H = 0$, then it is easily seen that the totally umbilical minimal hypersurface is actually totally geodesic. If $H = \text{constant} \neq 0$, then Equation of (20) is solved by $f = \text{constant}$ and
\[ mH^2 - \lambda = 0. \]
This implies that the totally umbilical hypersurface is a biharmonic hypersurface with constant mean curvature $H = \pm \sqrt{\lambda/m}$. Summarizing the results in the two cases discussed above we obtain the theorem. □

As a consequence of Theorem 3.6, we have

**Corollary 3.7.** A totally umbilical $f$-biharmonic hypersurface in Euclidean space $\mathbb{R}^{m+1}$ or hyperbolic space $H^{m+1}$ is a totally geodesic hypersurface, any totally umbilical $f$-biharmonic hypersurface in sphere $S^{m+1}$ is, up to isometries, a part of the great sphere $S^m \subset S^{m+1}$ or a part of the small sphere $S^m(\frac{1}{\sqrt{2}})$.\]

**Proof.** This follows from the well-known facts that space forms $\mathbb{R}^{m+1}$, $H^{m+1}$, $S^{m+1}$ of constant sectional curvature $0$, $-1$, $1$ are Einstein spaces with $\lambda = 0$, $-m$, $m$, and that totally umbilical biharmonic hypersurfaces in Euclidean spaces and hyperbolic spaces are totally geodesic, and the only totally umbilical biharmonic hypersurfaces in $S^{m+1}$ is a part of the great sphere $S^m \subset S^{m+1}$ or a part of the small $S^m(\frac{1}{\sqrt{2}})$ ([CMO1]). □

It is well known that biharmonic hypersurfaces of Euclidean space $\mathbb{R}^{m+1}$ with at most two distinct principal curvature are minimal ones. The following example shows that this is no longer true for $f$-biharmonic hypersurfaces.

**Example 1.** Let $\phi : D = \{ (\theta, x) \in (0, 2\pi) \times \mathbb{R}^{m-1} \}$ and $\phi : D \rightarrow (\mathbb{R}^{m+1}, h_0)$ with $\phi(\theta, x_1, \cdots, x_{m-1}) = (R \cos \frac{\theta}{R}, R \sin \frac{\theta}{R}, x_1, \cdots, x_{m-1})$ be the isometric immersion into Euclidean space. One can check that it is a hypersurface with two distinct principal curvatures, which is also an $f$-biharmonic hypersurface for any positive function $f$ from the family $f = C_1 e^{x_1/R} + C_2 e^{-x_1/R}$, where $C_1, C_2$ are constants.

In fact, we can take
\[ e_1 = d\phi(\frac{\partial}{\partial \theta}) = -\sin \frac{\theta}{R} \frac{\partial}{\partial y^1} + \cos \frac{\theta}{R} \frac{\partial}{\partial y^2}, \]
\[ e_i = d\phi(\frac{\partial}{\partial x^{i-1}}) = \frac{\partial}{\partial y^{1+i}}, \quad i = 2, \cdots, m, \]
\[ \xi = \cos \frac{\theta}{R} \frac{\partial}{\partial y^1} + \sin \frac{\theta}{R} \frac{\partial}{\partial y^2} \]
as an orthonormal frame adapted to the hypersurface. Then a straightforward computation gives
\[
\begin{align*}
\begin{cases}
Ae_1 = -\frac{1}{R} e_1, & Ae_i = 0, \forall i = 2, 3, \cdots, m. \\
H = \frac{1}{m} \sum_{i=1}^{m} \langle Ae_i, e_i \rangle = -\frac{1}{mR} \neq 0 \\
|A|^2 = \sum_{i=1}^{m} |Ae_i|^2 = \frac{1}{R^2},
\end{cases}
\end{align*}
\]

which show indeed that the hypersurface has two distinct principal curvatures and constant mean curvature. Let \( f : (0,2\pi) \times \mathbb{R}^{m-1} \rightarrow (0,\infty) \) be a function that does not depend on \( \theta \), i.e., \( f = f(x_1, \cdots, x_{m-1}) \). Substituting this and (21) into the \( f \)-biharmonic hypersurface equation (13) with \( C = 0 \) we have

\[
\Delta_{\mathbb{R}^{m-1}} f = \frac{1}{R^2} f.
\]

It follows that the cylinder \( S^1 \times \mathbb{R}^{m-1} \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^{m-1} \) is a \( f \)-biharmonic hypersurface for any positive eigenfunction which solves equation (22), in particular, any positive function from the family \( f = C_1 e^{x_1/R} + C_2 e^{-x_1/R} \), where \( C_1, C_2 \) are constants, is a solution of (22).

4. Biharmonic conformal immersions of surfaces into 3-manifolds

It is well known (See, e.g., [Ta]) that a conformal immersion \( \varphi : (M^2, \tilde{g}) \rightarrow (N^n, h) \) with \( \varphi^* h = \lambda^2 \tilde{g} \) is harmonic if and only if the surface \( \varphi(M^2) \hookrightarrow (N^n, h) \) is a minimal surface, i.e., the isometric immersion \( \varphi(M^2) \hookrightarrow (N^n, h) \) is harmonic. The following corollary, which follows from Theorem A, can be viewed as a generalization of this well-known result.

**Corollary 4.1.** A conformal immersion \( \varphi : (M^2, \tilde{g}) \rightarrow (N^n, h) \) with \( \varphi^* h = \lambda^2 \tilde{g} \) is biharmonic if and only if the surface (i.e., the isometric immersion) \( \varphi(M^2) \hookrightarrow (N^n, h) \) is an \( f \)-biharmonic surface with \( f = \lambda^2 \).

It was proved in [On1] that a hypersurface, i.e., an isometric immersion \( M^m \hookrightarrow (N^{m+1}, h) \) with mean curvature \( H \) and the shape operator \( A \) is biharmonic if and only if

\[
\begin{align*}
\Delta H - H|A|^2 + H \text{Ric}^N(\xi, \xi) &= 0, \\
2A(\text{grad} H) + \frac{n}{2} \text{grad} H^2 - 2H (\text{Ric}^N(\xi))^T &= 0,
\end{align*}
\]

where \( \text{Ric}^N : T_q N \rightarrow T_q N \) denotes the Ricci operator of the ambient space defined by \( \langle \text{Ric}^N(Z), W \rangle = \text{Ric}^N(Z, W) \), \( \xi \) is the unit normal vector field, and \( \Delta \) and \( \text{grad} \) denote the Laplace and the gradient operators defined by the induced metric on the hypersurface.
Later in [On3], it was proved that a conformal immersion \( \varphi : (M^2, \bar{g}) \rightarrow (N^3, h) \) with \( \varphi^* h = \lambda^2 \bar{g} \) is biharmonic if and only if

\[
\begin{align*}
\Delta H - H [ A^2 - \text{Ric}^N (\xi, \xi) - (\Delta \lambda^2) / \lambda^2 ] + 4 g (\text{grad} \ln \lambda, \text{grad} H) = 0, \\
A (\text{grad} H) + H [ \text{grad} H - \text{Ric}^N (\xi) ] + 2 A (\text{grad} \ln \lambda) = 0,
\end{align*}
\]

where \( \xi, A, \) and \( H \) are the unit normal vector field, the shape operator, and the mean curvature function of the surface \( \varphi(M) \subset (N^3, h) \) respectively, and the operators \( \Delta, \text{grad} \) and \( |.| \) are taken with respect to the induced metric \( g = \varphi^* h = \lambda^2 \bar{g} \) on the surface.

Our next theorem gives an improved equation for biharmonic conformal immersions of surfaces into a 3-manifold by writing it in a form similar to biharmonic surface equation (23), which turns out to be very useful in proving some classification and/or existence results on biharmonic conformal surfaces.

**Theorem 4.2.** A conformal immersion

\[ \varphi : (M^2, \bar{g}) \rightarrow (N^3, h) \]

into a 3-dimensional manifold with \( \varphi^* h = \lambda^2 \bar{g} \) is biharmonic if and only if the surface \( \varphi(M^2) \rightarrow (N^3, h) \) is an \( f \)-biharmonic surface with \( f = \lambda^2 \), which is characterized by the equation

\[
\begin{align*}
\Delta (\lambda^2 H) - (\lambda^2 H) [ A^2 - \text{Ric}^N (\xi, \xi) ] = 0, \\
A (\text{grad} (\lambda^2 H)) + (\lambda^2 H) [ \text{grad} H - \text{Ric}^N (\xi) ] = 0.
\end{align*}
\]

where \( \xi, A, \) and \( H \) are the unit normal vector field, the shape operator, and the mean curvature function of the surface \( \varphi(M) \subset (N^3, h) \) respectively, and the operators \( \Delta, \text{grad} \) and \( |.| \) are taken with respect to the induced metric \( g = \varphi^* h = \lambda^2 \bar{g} \) on the surface.

**Proof.** One can easily see that a map \( \varphi : (M^2, \bar{g}) \rightarrow (N^3, h) \) is a conformal immersion with conformal factor \( \lambda \), i.e., \( \varphi^* h = \lambda^2 \bar{g} \), if and only if the map

\[ \varphi : (M^2, g = \lambda^2 \bar{g}) \rightarrow (N^3, h) \]

is an isometric immersion. By Corollary [11] the conformal immersion \( \varphi : (M^2, \bar{g}) \rightarrow (N^3, h) \) is biharmonic if and only if the surface \( \varphi : (M^2, \lambda^2 \bar{g}) \rightarrow (N^3, h) \) is an \( f \)-biharmonic surface. Applying Equation (10) with \( m = 2 \) and \( f = \lambda^2 \) we obtain the biharmonic conformal equation (26) and hence the theorem. \( \square \)

As an immediate consequence of Theorem 4.2 we have the following
Corollary 4.3. A conformal immersion \( \varphi : (M^2, \bar{g}) \rightarrow (N^3(C), h_0) \) into 3-dimensional space of constant sectional curvature \( C \) with \( \varphi^* h_0 = \lambda^2 \bar{g} \) is biharmonic if and only if

\[
\begin{align*}
\Delta (\lambda^2 H) &- (\lambda^2 H)(|A|^2 - 2C) = 0, \\
A(\text{grad}(\lambda^2 H)) + (\lambda^2 H) \text{grad} H & = 0.
\end{align*}
\]

where \( A \) and \( H \) are the shape operator and the mean curvature function of the surface respectively, and the operators \( \Delta, \text{grad} \) and \( |\cdot| \) are taken with respect to the induced metric \( g = \varphi^* h = \lambda^2 \bar{g} \) on the surface.

Theorem 4.4. Let \( \varphi : (M^2, \bar{g}) \rightarrow (\mathbb{R}^3, h_0) \) be a biharmonic conformal immersion from a complete surface into 3-dimensional Euclidean space with \( \varphi^* h_0 = \lambda^2 \bar{g} \). If the mean curvature \( H \) of the surface \( \varphi : (M^2, g = \lambda^2 \bar{g}) \rightarrow (\mathbb{R}^3, h_0) \) satisfies \( \int_M \lambda^4 H^2 dv_g < \infty \). Then the biharmonic conformal immersion \( \varphi \) is a minimal immersion.

Proof. Note that \( \varphi : (M^2, \bar{g}) \rightarrow (\mathbb{R}^3, h_0) \) is a biharmonic conformal immersion implies that the surface \( (M^2, g = \lambda^2 \bar{g}) \) can be biharmonically conformally immersed into Euclidean space \( (\mathbb{R}^3, h_0) \). By Corollary 4.3, we have

\[
\begin{align*}
\Delta (\lambda^2 H) & = (\lambda^2 H)|A|^2, \\
A(\text{grad}(\lambda^2 H)) + (\lambda^2 H) \text{grad} H & = 0.
\end{align*}
\]

where \( A \) and \( H \) are the shape operator, and the mean curvature function of the surface \( \varphi(M) \subset (\mathbb{R}^3, h_0) \) respectively, and the operators \( \Delta, \text{grad} \) and \( |\cdot| \) are taken with respect to the induced metric \( g = \varphi^* h = \lambda^2 \bar{g} \) on \( M \).

From the assumption \( \int_M \lambda^4 H^2 dv_g < \infty \) and the first equation of (29) we see that the function \( \lambda^2 H \in L^2(M) \) is a solution of the Schrödinger type equation

\[
\Delta u = \alpha u
\]

for a fixed nonnegative function \( \alpha = |A|^2 \) on a complete manifold \( (M^2, g) \). By Lemma A (Section 3), we conclude that \( \lambda^2 H = C \), a constant. Substituting this into the second equation of (29) we have either \( H = 0 \) which means the conformal immersion is minimal, or \( H = C_1 \) a constant. If \( H = C_1 \neq 0 \), then \( \lambda^2 H = C \) implies that \( \lambda^2 = C/H = C/C_1 \) is a constant. In this case, the conformal immersion is a homothetic immersion. By a well-known result of Chen and Jiang, any biharmonic homothetic immersion of a surface into \( \mathbb{R}^3 \) is minimal. Therefore, in either case, we conclude that the conformal biharmonic immersion is a minimal homothetic immersion. \( \square \)
Motivated by the beautiful theory of minimal surfaces as conformal harmonic immersions and the attempt to understand the relationship among biharmonicity, conformality and $f$-biharmonicity of maps from surfaces, we studied biharmonic conformal immersions of surfaces in [Ou2], and [Ou3]. A Fundamental question is: what are the geometric and/or topological obstacles for a surface to admit a biharmonic conformal immersion into a “nice space” like a space form? Recall (See [Ou3]) that a surface in a Riemannian manifold $(N^3, h)$ defined by an isometric immersion $\varphi : (M^2, g) \rightarrow (N^3, h)$ is said to admit a biharmonic conformal immersion into $(N^3, h)$, if there exists a function $f : M^2 \rightarrow \mathbb{R}^+$ such that the conformal immersion $\varphi : (M^2, \bar{g} = f^{-1}g) \rightarrow (N^3, h)$ with conformal factor $f$ is a biharmonic map. In such a case, we also say that the surface $\varphi : (M^2, g) \rightarrow (N^3, h)$ can be biharmonically conformally immersed into $(N^3, h)$. Later in studying $f$-biharmonic maps and $f$-biharmonic submanifolds in [Ou4], Theorem A was proved. It follows from Theorem A that a surface that $\varphi : (M^2, g) \rightarrow (N^3, h)$ that admits a biharmonic conformal immersion into $(N^3, h)$ if and only if the surface is an $f$-biharmonic surface.

Examples of surfaces that can be biharmonically conformally immersed into a 3-manifold include the following

- For any biharmonic conformal immersion $\varphi : (M^2, \bar{g}) \rightarrow (N^3, h)$ with $\varphi^*h = \lambda^2\bar{g}$, its associated surface $\varphi : (M^2, g = \varphi^*h) \rightarrow (N^3, h)$ admits a biharmonic conformal immersion into $(N^3, h)$ as we see that there exists $f = \lambda^2$ such that $f^{-1}(\varphi^*h) = \bar{g}$. For examples of biharmonic conformal immersions of surfaces, see [Ou2] and [Ou3].

- Any minimal surface (i.e., harmonic isometric immersion) $\varphi : M^2 \rightarrow (N^3, h)$ admits a biharmonic conformal immersion into $(N^3, h)$. This is because, by Corollary 2.3, a minimal surface $\varphi : M^2 \rightarrow (N^3, h)$ is $f$-biharmonic for any positive function $f$ defined on the surface.

Using the improved equation for biharmonic conformal immersions of surfaces we can prove the following

**Theorem 4.5.** If a compact surface $M^2 \hookrightarrow S^3$ with the squared norm of the second fundamental form $|A|^2 \geq 2$ can be biharmonically conformally immersed into $S^3$, then $M^2$ is minimal, or up to a homothetic change of the metric, $M^2 = S^2(\frac{1}{2})$.

**Proof.** Suppose the surface is given by an isometric immersion $\phi : M^2 \rightarrow (S^3, h)$, where $h$ denotes the standard metric on $S^3$ with constant sectional curvature 1. By definition, if the surface admits a biharmonic conformal immersion into $S^3$, then...
then there exists a function \( \lambda : M \to (0, \infty) \) such that the conformal immersion \( \phi : (M^2, \bar{g} = \lambda^2 g) \to (S^3, h) \) is biharmonic. So, by (28), we have

\[
\begin{cases}
\Delta(\lambda^2 H) - (\lambda^2 H)(|A|^2 - 2) = 0, \\
A(\text{grad}(\lambda^2 H)) + (\lambda^2 H)\text{grad}H = 0.
\end{cases}
\]  

(31)

We compute

\[
\Delta((\lambda^2 H)^2) = 2(\lambda^2 H)\Delta(\lambda^2 H) + 2|\text{grad}(\lambda^2 H)|^2
\]

(32)

where in obtaining the second equality we have used the first equation of (31). From Equation (32) and the assumption that \(|A|^2 \geq 2\) we conclude that \(\Delta((\lambda^2 H)^2) \geq 0\), i.e., the function \(\lambda^2 H\) is a subharmonic function on the compact manifold \(M\). It follows that \(\lambda^2 H\) is a constant function. Using this and the second equation of (31) we see that \(H\) must be a constant. If \(H = 0\), then the surface \(\phi : M^2 \to (S^3, h)\) is minimal. Otherwise, if \(H = C \neq 0\), then \(\lambda\) is also a constant and hence, up to a homothety, the compact surface \(\phi : M^2 \to (S^3, h)\) is biharmonic. It follows from a well-known result ([CMO1]) of biharmonic surfaces in \(S^3\) that \(M^2 = S^2(\frac{1}{2})\). Thus, we obtain the theorem. \(\square\)

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Department of Mathematics,
Texas A & M University-Commerce,
Commerce, TX 75429,
USA.
E-mail: yelin.ou@tamuc.edu