On the Cohomology of Twisting Sheaves on Toric Varieties

Mahmoud Nikbakht–Tehrani
Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstraße 8–10, A-1040 Wien, AUSTRIA

Abstract

Using the homogeneous coordinate ring construction of a toric variety $\mathbb{P}_\Sigma$ defined by a complete simplicial fan $\Sigma$ and the methods of local cohomology theory we develop a framework for the calculation of cohomology groups $H^\bullet(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(p))$ of twisting sheaves $\mathcal{O}_{\mathbb{P}_\Sigma}(p)$ on $\mathbb{P}_\Sigma$.

February 1998
1 Introduction

The study of $(0, 2)$ heterotic string compactifications leads us to consider holomorphic stable vector bundles [1] (or more generally stable torsion-free or reflexive sheaves [2]) $\mathcal{E}$ on Calabi-Yau varieties $X$. The physical properties of the resulting string vacua such as their stability and massless particle spectrum are closely related to the topological properties of these bundles or sheaves.

In a large class of $(0, 2)$ heterotic compactifications which has been constructed in the framework of gauged linear sigma models [3, 4] the Calabi-Yau varieties $X$ are realized as complete intersections of hypersurfaces in toric varieties $\mathbb{P}_S$ and the relevant sheaves $\mathcal{E}$ are defined by short exact sequences or by the cohomology of monads in terms of the restriction of twisting sheaves $\mathcal{O}_{\mathbb{P}_S}(p)$ to $X$. The knowledge of the cohomology groups of such sheaves in these models enables us to get useful information about the various topological invariants associated to $\mathcal{E}$ such as its cohomology groups and the like. The development of an efficient method for the calculation of such cohomology groups is the aim of the present work.

The basic ideas used here are those of the homogeneous coordinate ring construction of toric varieties defined by complete simplicial fans [5] and local cohomology theory [6]. Combining these ideas with the methods of computational commutative algebra we present an algorithmic way for approaching such cohomology calculations.

The outline of the paper is as follows. The next section is devoted to a short survey of relevant concepts from the homogeneous coordinate ring approach as they apply to our work. In section 3 we first review the basic definitions and results on the sheaf cohomology and then go on to discuss its relation to local cohomology theory. Afterwards we explain how these ideas apply to our situation. In the last part of this section we describe some algorithms from commutative algebra which will be used in the calculation of the local cohomology. In section 4, performing a few examples, we demonstrate explicitly the application of the developed methods. An appendix on the definition of the inductive limit and of the Ext functor concludes this work.

2 Homogeneous coordinate ring approach

In this section we briefly discuss the relevant concepts from the homogeneous coordinate ring approach as they apply to our work. The original motivation for its development was the desire to have a construction of toric varieties and related objects similar to
those of \( \mathbb{P}^n \) in classical algebraic geometry. For details on the constructions that follow and for proofs we refer to [3, 7].

To begin with we first introduce some notation. Let \( N \) and \( M = \text{Hom}(N, \mathbb{Z}) \) denote a dual pair of lattices of rank \( d \) and \( \langle \cdot, \cdot \rangle \) be the canonical pairing on \( M \times N \). Further, let \( N_R = N \otimes \mathbb{Z} R \) and \( M_R = M \otimes \mathbb{Z} R \) be the \( R \)-scalar extensions of \( N \) and \( M \), respectively. \( T = N \otimes \mathbb{Z} C^* = \text{Hom}(C^*, \mathbb{Z}) \) is the \( d \)-dimensional algebraic torus which acts on the toric variety \( \mathbb{P}_\Sigma \) defined by the (complete simplicial) fan \( \Sigma \) in \( N_R \). For a cone \( \sigma \in \Sigma \) the dual cone, \( \sigma^\vee \), is defined as usual by \( \sigma^\vee = \{ m \in M_R \mid \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma \} \) and \( \text{cosp}\sigma^\vee \) is the greatest subspace of \( M_R \) contained in \( \sigma^\vee \). The open affine variety in \( \mathbb{P}_\Sigma \) associated to \( \sigma \) is denoted by \( X_{\sigma^\vee} \). Let \( \Sigma^{(k)} \) be the set of \( k \) dimensional cones in \( \Sigma \). By \( e_i \) we denote the primitive lattice vectors on the one dimensional cones in \( \Sigma^{(1)} = \{ \rho_1, \ldots, \rho_n \} \). This set will play an important role in what follows.

Each one dimensional cone \( \rho_i \) defines a \( T \)-invariant Weil divisor, denoted by \( D_i \), which is the closed subvariety \( X_{\text{cosp}\rho_i^\vee} \) in \( X_{\rho_i^\vee} \). This is indeed the closed \( T \)-orbit associated to \( \rho_i \). The finitely generated free abelian group \( \bigoplus_{i=1}^n \mathbb{Z} \cdot D_i \) is the group of \( T \)-invariant Weil divisors in \( \mathbb{P}_\Sigma \). Each \( m \in M \) gives a character \( \chi_m : T \to C^* \), and hence \( \chi^m \) is a rational function on \( \mathbb{P}_\Sigma \). It defines the \( T \)-invariant Cartier divisor \( \text{div}(\chi^m) = \sum_{i=1}^n \langle m, e_i \rangle D_i \).

In this way we obtain the map \( \alpha \)

\[
\alpha : M \to \bigoplus_{i=1}^n \mathbb{Z} \cdot D_i , \quad m \mapsto \sum_{i=1}^n \langle m, e_i \rangle D_i .
\]  

(1)

It follows from the completeness of the fan \( \Sigma \) that the map \( \alpha \) is injective. The cokernel of this map defines the Chow group \( A_{d-1}(\mathbb{P}_\Sigma) \) which is a finitely generated abelian group of rank \( n - d \). Therefore we have the following exact sequence

\[
0 \to M \xrightarrow{\alpha} \bigoplus_{i=1}^n \mathbb{Z} \cdot D_i \xrightarrow{\text{deg}} A_{d-1}(\mathbb{P}_\Sigma) \to 0 ,
\]  

(2)

where \( \text{deg} \) denotes the canonical projection. Now consider \( G = \text{Hom}_\mathbb{Z}(A_{d-1}(\mathbb{P}_\Sigma), C^*) \) which is in general isomorphic to a product of \( (C^*)^{n-d} \) and a finite group. By applying \( \text{Hom}_\mathbb{Z}(\cdot, C^*) \) to \( (2) \) we get

\[
1 \to G \to (C^*)^n \to T \to 1 ,
\]

which defines the action of \( G \) on \( C^n \):

\[
g \cdot (x_1, \ldots, x_n) = (g(\text{deg } D_1) \ x_1, \ldots, g(\text{deg } D_n) \ x_n)
\]

for \( g \in G \) and \( (x_1, \ldots, x_n) \in C^n \).
Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbb{C}$ in the variables $x_1, \ldots, x_n$, where the $x_i$ correspond to the one-dimensional cones $\rho_i$ in $\Sigma$. Each monomial $x_1^{a_1} \ldots x_n^{a_n}$ in $S$ determines a divisor $\sum_{i=1}^n a_i D_i$. This ring is graded in a natural way by $\deg(x_i) := \deg D_i$.

$$S = \bigoplus_{q \in \mathbb{A}^{d-1}(\mathbb{P}_\Sigma)} S_q,$$

where $S_q$ is generated by all monomials $x_1^{a_1} \ldots x_n^{a_n}$ such that $\deg(\sum_{i=1}^n a_i D_i) = q$. Let $I$ denote the monomial ideal in $S$ generated by $x_\sigma = \prod_{\rho_i \not\prec \sigma} x_i$ for all $\sigma \in \Sigma$. Note that the set of monomials $\{x^\sigma \mid \sigma \in \Sigma(d)\}$ is the unique minimal basis of this ideal. The ring $S$ defines the $n$-dimensional affine space $A^n = \text{Spec } S$. The ideal $I$ gives the variety $Z_\Sigma = V(I)$ which is denoted as the exceptional set. Removing the exceptional set $Z_\Sigma$ we obtain the Zariski open set $U_\Sigma = A^n \setminus Z_\Sigma$ which is invariant under the action of $G$. For the case of a complete simplicial fan the geometric quotient of $U_\Sigma$ by $G$ exists and gives rise to $\mathbb{P}_\Sigma[5]$.

Using this construction of a toric variety $\mathbb{P}_\Sigma$, the sheaves of $\mathcal{O}_{\mathbb{P}_\Sigma}$-modules can be studied in a way similar to that of $\mathbb{P}^n$. For example, the twisting sheaves $\mathcal{O}_{\mathbb{P}_\Sigma}(p)$ are sheaves which are associated to the graded $S$-module $S(p)$, i.e. $\mathcal{O}_{\mathbb{P}_\Sigma}(p) = S(p)^\sim$, where $S(p)_q = S_{p+q}$ for all $q$. It can be shown that $\mathcal{O}_{\mathbb{P}_\Sigma}(p) \simeq \mathcal{O}_{\mathbb{P}_\Sigma}(D)$, where $D$ is a $T$-invariant divisor with $\deg D = p$. In particular, $\mathcal{O}_{\mathbb{P}_\Sigma} = S^\sim$. Furthermore, we have $S \simeq \bigoplus_p H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(p))$. We conclude this section with the following theorem [5].

**Theorem A**: Let $\Sigma$ be a complete simplicial fan. Then every coherent sheaf $F$ on $\mathbb{P}_\Sigma$ is of the form $F = M^\sim$, where $M$ is a finitely generated graded $S$-module.

### 3 Cohomology of twisting sheaves

We begin this section with recalling some definitions and facts from the cohomology of sheaves. For more details on the following concepts and the proofs we refer to the standard works [5, 8, 11].

Let $F$ be an Abelian sheaf on a topological space $X$. Consider the sheaf $C^0(F)$ defined by

$$C^0(F)(U) = \prod_{x \in U} F_x.$$

The sheaf $F$ is canonically embedded in $C^0(F)$ by associating to $s \in F(U)$ the family $(s(x)) \in \prod_{x \in U} F_x$. The sheaf $C^0(F)$ is always flabby. Now consider the family
\{C^n(F)\}_{n \geq 1} of sheaves that is recursively defined by
\[ C^1(F) := C^0(C^0(F) / F), \]
\[ C^{n+1}(F) := C^0(C^n(F) / d^{n-1}C^{n-1}(F)), \]
where \( d^n : C^n(F) \rightarrow C^{n+1}(F) \) is defined as a composition
\[ C^n(F) \rightarrow (C^n(F) / d^{n-1}C^{n-1}(F)) \rightarrow C^0\left( C^n(F) / d^{n-1}C^{n-1}(F) \right). \]

In this way we obtain the Godement canonical flabby resolution \( 0 \rightarrow F \rightarrow C^\bullet(F) \).

The cohomology of the complex \( C^\bullet(F)(X) \) of Abelian groups is said to be the cohomology of the sheaf \( F \) and is denoted by \( H^\bullet(X,F) \). Clearly \( \Gamma(X,F) = H^0(X,F) \) and \( H^p(X,F) = 0 \) for \( p < 0 \). If \( F \) is flabby, then \( H^p(X,F) = 0 \) for \( p > 0 \). The definition of the cohomology group given here is not very enlightening. For this reason we now discuss some other methods which provide us with more useful tools for the cohomology computations.

First, we recall the definition of the Čech cohomology group. Let \( U = \{U_i\}_{i \in I} \) be an open covering of \( X \), where the index set is well-ordered. For an Abelian sheaf \( F \) on \( X \) we set
\[ C^p(U,F) := \bigoplus_{i_0 < \ldots < i_p} \Gamma(U_{i_0} \cap \ldots \cap U_{i_p}, F) \]
and define \( d^p : C^p(U,F) \rightarrow C^{p+1}(U,F) \) by
\[ (d^p \alpha)_{i_0 \ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \ldots \hat{i}_k \ldots i_{p+1}}|_{U_{i_0} \cap \ldots \cap U_{i_{p+1}}}, \]
where the \( \hat{\cdot} \) over \( i_k \) means that this index is to be omitted. \( C^\bullet(U,F) \) is called the Čech complex for the open covering \( U \) with values in \( F \). The cohomology groups \( \check{H}^\bullet(U,F) \) of this complex are called the Čech cohomology groups of \( F \) for the open covering \( U \).

We are primarily concerned with the coherent sheaves (of \( O_X \)-modules) on an algebraic variety \( (X,O_X) \). The following theorem of Serre plays a fundamental role in the cohomology theory of such sheaves.

**Theorem (Serre):** Let \( F \) be a quasi-coherent sheaf on an affine variety \( X \). Then \( H^p(X,F) = 0 \) for \( p > 0 \).

If the variety \( X \) is separated, then the following theorem gives us a useful tool for the computation of cohomology groups.

**Theorem B:** Let \( F \) be a quasi-coherent sheaf on a separated variety \( X \) and \( U = \{U_i\}_{i=0}^n \) be an open covering of \( X \) by open affine subvarieties. Then \( \check{H}^p(U,F) \simeq H^p(X,F) \).
The key point in the proof of this theorem is to show that $H^q(U_{i_0} \cap \ldots \cap U_{i_p}, \mathcal{F}) = 0$ for all $(p + 1)$-tuples $i_0 < \ldots < i_p$ and $q > 0$. This follows already from Serre’s theorem because the separability of $X$ implies that $U_{i_0} \cap \ldots \cap U_{i_p}$ are affine.

Having reduced the calculation of cohomology to that of Čech cohomology for a finite covering, we now come to the question how these cohomology groups can be effectively calculated. The Koszul complex method provides us with the necessary tools. Before going further, we mention the following vanishing theorem of Grothendieck.

**Theorem (Grothendieck):** Let $X$ be an $n$-dimensional Noetherian topological space. Then, $H^p(X, \mathcal{F}) = 0$ for all $p > n$ and all Abelian sheaves $\mathcal{F}$ on $X$.

Now, let $R$ be a commutative ring with unit. Consider the free $R$-module $R^d$ with the canonical basis $\{e_i\}_{i=1}^d$. Further, let $d_1 : R^d \to R$, $e_i \mapsto a_i := d_1(e_i)$ be a homomorphism of $R$-modules. We are going to construct a complex of $R$-modules. Let

$$K_p(a) = K_p(a_1, \ldots, a_d) = \bigwedge^p R^d = \bigoplus_{i_1 < \ldots < i_p} R e_{i_1} \wedge \ldots \wedge e_{i_p}.$$ 

Define $d_p : K_p(a) \to K_{p-1}(a)$ by

$$d_p(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{k=1}^p (-1)^k d_1(e_{i_k}) e_{i_1} \wedge \ldots \wedge \hat{e}_{i_k} \wedge \ldots \wedge e_{i_p}.$$ 

It can be easily seen that $d_{p-1} \circ d_p = 0$. Therefore, we get the following complex

$$0 \to \bigwedge^d R^d \to \bigwedge^{d-1} R^d \to \ldots \to \bigwedge^1 R^d \xrightarrow{d_1} R \to 0,$$

which is called a (homological) Koszul complex. Let $M$ be a finitely generated $R$-module. Then $K_\bullet(a, M) := K_\bullet(a) \otimes_R M$ (resp. $K^\bullet(a, M) := \text{Hom}_R(K_\bullet(a), M)$) is called the homological (resp. cohomological) Koszul complex of $M$ with respect to $(a_1, \ldots, a_d)$. Its corresponding homology (cohomology) groups are denoted by $H_\bullet(a, M)$ ($H^\bullet(a, M)$). Note that $K^\bullet(a, M) = \bigoplus_{i_1 < \ldots < i_p} M$. Therefore, $\alpha \in K^\bullet(a, M)$ can be considered as a sequence $\alpha = (m_{i_1 \ldots i_p})_{i_1 < \ldots < i_p}$ of elements of $M$. The coboundary operator $d^p : K^p(a, M) \to K^{p+1}(a, M)$ is given by

$$(d^p \alpha)_{i_1 \ldots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^k d_1(e_{i_k}) m_{i_1 \ldots \hat{i}_k \ldots \hat{i}_k \ldots i_{p+1}}.$$ 

It is not hard to see that $H_0(a, M) = M / \langle a_1, \ldots, a_d \rangle M$ and $H_d(a, M) = \{ m \in M \mid \langle a_1, \ldots, a_d \rangle m = 0 \}$.

**Theorem C:** If $\langle a_1, \ldots, a_d \rangle = R$, then $K_\bullet(a, M)$ is acyclic.
A sequence \((b_1, \ldots, b_r)\) in \(R\) is said to be \(M\)-regular if \(\langle b_1, \ldots, b_r \rangle \neq R\) and the image of \(b_i\) in \(M / \langle b_1, \ldots, b_{i-1} \rangle M\) is no zerodivisor for \(i = 1, \ldots, r\).

**Theorem D:** If \(H_p(a, M) = 0\) for \(p > r\) while \(H_r(a, M) \neq 0\), then every maximal \(M\)-sequence in \(\langle a_1, \ldots, a_d \rangle\) has length \(r\). In particular, if \((a_1, \ldots, a_d)\) is an \(M\)-regular sequence in \(R\), then \(H_0(a, M) = M / \langle a_1, \ldots, a_d \rangle M\) and \(H_p(a, M) = 0\) for \(p = 1, \ldots, d\).

**Theorem E:** \(H_p(a, M) \cong H^{d-p}(a, M)\).

After these preliminaries, we now turn our attention to the relation between Koszul and Čech complexes. We will show that the Čech complex for a given covering is related to the limit of an inductive system of Koszul complexes. Let \(U_i\) be the principal open subsets in \(\text{Spec} \ R\) associated to \(a_i\) and \(\mathcal{U} = \{U_i\}_{i=1}^d\). Let \(U = \bigcup_{i=1}^d U_i\) and \(F = M^\sim\). Then \(\Gamma(U_{i_1} \cap \ldots \cap U_{i_p}, F)\) is equal to the localization of \(M\) at \(a_{i_1} \ldots a_{i_p}\), i.e. \(M_{a_{i_1} \ldots a_{i_p}}\).

Now, consider the family \(\{K^\bullet(a^m, M)\}_{m \in \mathbb{N}}\) of Koszul complexes of \(M\) with respect to \(a^m := (a_1^m, \ldots, a_d^m)\). Note that \(d^p_m : K^p(a^m, M) \to K^{p+1}(a^m, M)\) is given by

\[
(d^p_m \alpha)_{i_1 \ldots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^k a^m_{i_k} \cdot \alpha_{i_1 \ldots \hat{i}_k \ldots i_{p+1}}.
\]

Let \(f^p_m : K^p(a^m, M) \to K^p(a^n, M)\) be defined by

\[
f^p_m : (m_{i_1 \ldots i_p})_{i_1 < \ldots < i_p} \mapsto (a_{i_1} \ldots a_{i_p})^{n-m} \cdot (m_{i_1 \ldots i_p})_{i_1 < \ldots < i_p}
\]

for \(n \geq m\). It can be easily checked that \(f^p_{mn} = \text{id}\) and \(f^p_m \circ f^p_{lm} = f^p_{ml}\). Using the definitions of coboundary operators and \(f^p_{mn}\) one can show that the following diagram commutes

\[
\begin{array}{ccc}
K^p(a^m, M) & \longrightarrow & K^{p+1}(a^m, M) \\
\downarrow & & \downarrow \\
K^p(a^n, M) & \longrightarrow & K^{p+1}(a^n, M)
\end{array}
\]

Therefore, the family \(\{K^\bullet(a^m, M)\}_{m \in \mathbb{N}}\) together with \(\{f^\bullet_m\}_{m \leq n}\) build an inductive system of Koszul complexes. Let \(K^\bullet(a^\infty, M) = \lim_{\longrightarrow} K^\bullet(a^m, M)\).

We now define the map \(\varphi^p_m : K^p(a^m, M) \to C^{p-1}(\mathcal{U}, F)\) by

\[
(m_{i_1 \ldots i_p}) \mapsto \left(\frac{(m_{i_1 \ldots i_p})}{(a_{i_1} \ldots a_{i_p})^m}\right).
\]

Obviously, the following diagram
commutes for all \( m \leq n \). Therefore, \( C^{p-1}(\mathbb{U}, \mathcal{F}) \) is a target object for the inductive system \( \{K^p(\alpha^m, M)\}_{m \in \mathbb{N}} \). Because of the universal property of the inductive limit there exists a map \( \psi^p : K^p(\alpha^\infty, M) \to C^{p-1}(\mathbb{U}, \mathcal{F}) \) such that the diagram

\[
\begin{array}{ccc}
K^p(\alpha^m, M) & \xrightarrow{\varphi_m^p} & \psi^p \\
\downarrow f_{mn} & & \downarrow \psi^p \\
K^p(\alpha^n, M) & \xrightarrow{\varphi_n^p} & C^{p-1}(\mathbb{U}, \mathcal{F}) \\
\end{array}
\]

commutes. We now prove that \( \psi^p \) is actually an isomorphism. Let \( \beta \in C^{p-1}(\mathbb{U}, \mathcal{F}) \).

Then, for \( i_1 < \ldots < i_p \), \( \beta_{i_1 \ldots i_p} = (a_{i_1} \ldots a_{i_p})^{-m} \cdot m_{i_1 \ldots i_p} \), where \( m_{i_1 \ldots i_p} \in M \). Therefore, \( \alpha = (m_{i_1 \ldots i_p})_{i_1 < \ldots < i_p} \in K^p(\alpha^m, M) \) and \( \beta = \psi^p(f_m^p(\alpha)) \). This shows the surjectivity of \( \psi^p \).

Injectivity of \( \psi^p \): Let \( f_m^p(\alpha) \in \text{Ker} \psi^p \), i.e. \( \psi^p(f_m^p(\alpha)) = 0 \). It follows from the commutativity of the above diagram that \( \psi^p(f_m^p(\alpha)) = (\psi^p \circ f_m^p)(\alpha) = \varphi_m^p(\alpha) = 0 \), i.e. \( (a_{i_1} \ldots a_{i_p})^{-m} \cdot m_{i_1 \ldots i_p} = 0 \) for each \( i_1 < \ldots < i_p \). Consequently, there exists some \( k \) such that \( (a_{i_1} \ldots a_{i_p})^k \cdot m_{i_1 \ldots i_p} = 0 \). But \( (a_{i_1} \ldots a_{i_p})^k \cdot m_{i_1 \ldots i_p} = (f_{m+k}^p(\alpha))_{i_1 \ldots i_p} \). That is \( f_{m, m+k}^p(\alpha) = 0 \).

On the other hand, \( f_m^p(\alpha) = (f_{m+k}^p \circ f_{m, m+k}^p)(\alpha) = f_{m+k}^p(f_{m, m+k}^p(\alpha)) = 0 \). Hence \( \text{Ker} \psi^p = \{0\} \).

Summarizing the above discussion, we can write down the following exact sequence of complexes

\[
\begin{array}{cccccc}
0 & \longrightarrow & 0 & \longrightarrow & C^0(\mathbb{U}, \mathcal{F}) & \longrightarrow & C^1(\mathbb{U}, \mathcal{F}) & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\tilde{C}^\bullet(\mathbb{U}, \mathcal{F}) & : & 0 & \longrightarrow & 0 & \longrightarrow & C^0(\mathbb{U}, \mathcal{F}) & \longrightarrow & C^1(\mathbb{U}, \mathcal{F}) & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^\bullet(\alpha^\infty, M) & : & 0 & \longrightarrow & K^0(\alpha^\infty, M) & \longrightarrow & K^1(\alpha^\infty, M) & \longrightarrow & K^2(\alpha^\infty, M) & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M[0] & : & 0 & \longrightarrow & K^0(\alpha^\infty, M) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & \cdots
\end{array}
\]
which yields
\[ 0 \to H^0(\mathfrak{a}^\infty, M) \to M \to \tilde{H}^0(\mathcal{U}, \mathcal{F}_|\mathcal{U}) \to H^1(\mathfrak{a}^\infty, M) \to 0 \] (3)
\[ \tilde{H}^{p-1}(\mathcal{U}, \mathcal{F}_|\mathcal{U}) \simeq H^p(\mathfrak{a}^\infty, M) \quad \text{for } p > 1 . \]

It is noteworthy that there is a close relation between \( H^p(\mathfrak{a}^\infty, M) \) and \( \text{Ext}^p_R \) providing us with yet another useful computational tool:
\[ H^p(\mathfrak{a}^\infty, M) \simeq \lim_{\to} \text{Ext}^p(R / I^m, M) , \]
where \( I^m = \langle a_1^m, \ldots, a_d^m \rangle \). For the details of the proof we refer to [6] §2.

Now, the calculation of the cohomology of twisting sheaves on \( \mathbb{P}_\Sigma \) can be approached in the following way. As mentioned before, in the homogeneous coordinate ring construction the toric variety \( \mathbb{P}_\Sigma \) is realized as the geometric quotient \( \mathcal{U}_\Sigma / G \) provided that \( \Sigma \) is a complete simplicial fan. An open covering of \( \mathcal{U}_\Sigma \) is given by the principal open subsets \( \mathcal{U}_i := D(\mathbf{x}^{\sigma_i}) \), where the \( \mathbf{x}^{\sigma_i} \) belong to the minimal basis of the ideal \( I : \mathcal{U}_\Sigma \). We have the following lemma.

**Lemma:** Let \( f : X \to Y \) be an affine morphism of separated varieties \( X \) and \( Y \). Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. Then \( H^p(X, \mathcal{F}) \simeq H^p(Y, f_*\mathcal{F}) \).

**Proof:** Let \( \mathcal{U} = \{U_i\}_{i=1}^m \) be a covering of \( Y \) by open affine subsets and \( \check{\mathcal{U}} = \{f^{-1}(U_i)\}_{i=1}^m \). It follows from the definition of \( f_*\mathcal{F} \) that \( C^0(\check{\mathcal{U}}, \mathcal{F}) = C^0(\mathcal{U}, f_*\mathcal{F}) \). It is also obvious that \( C^p(\check{\mathcal{U}}, \mathcal{F}) = C^p(\mathcal{U}, f_*\mathcal{F}) \) for \( p > 0 \). Therefore, \( \tilde{H}^p(\check{\mathcal{U}}, \mathcal{F}) = \tilde{H}^p(\mathcal{U}, f_*\mathcal{F}) \). Since the Čech cohomology groups with respect to an affine covering are isomorphic to the cohomology groups, we have \( H^p(X, \mathcal{F}) \simeq H^p(Y, f_*\mathcal{F}) \).

It follows from our discussion in the section on the homogeneous coordinate ring approach that \( \pi_*\mathcal{O}_{\mathcal{U}_\Sigma} = \bigoplus_p \mathcal{O}_{\mathbb{P}_\Sigma}(p) \). Consequently, if we calculate \( H^p(\mathcal{U}_\Sigma, \mathcal{O}_{\mathcal{U}_\Sigma}) \) using (3), then the above lemma yields the desired result!

We now go on to consider the question how an object like \( \text{Ext}^i_R(M, N) \) can be algorithmically calculated. It should be clear that the basic algorithms being discussed here can be equally applied to the calculations involving Koszul complexes. In what follows we assume that \( R = \mathbb{C}[x_1, \ldots, x_n] \) and all \( R \)-modules are finitely generated. Let \( M \) be an \( R \)-module and
\[ 0 \to K \overset{i}{\to} R^p \overset{\pi}{\to} M \to 0 \] (4)

a presentation thereof. The $R$-module $K$ is said to be a (first) syzygy module of $M$. The algorithmic calculation of syzygy modules, which we will sketch below, is the heart of the computational approach to many questions of commutative algebra (cf. [9] for more details).

Let $M = \langle f_1, \ldots, f_p \rangle$ as a submodule of $R^m$ and $\{g_1, \ldots, g_q\}$ be a Gröbner basis for $M$ such that the leading coefficient of each $g_i$ is one. Let $x_i := \text{lm}(g_i)$ be the leading monomial of $g_i$ and $x_{ij} := \text{lcm}(x_i, x_j)$ be the least common multiple of $x_i$ and $x_j$. Further, let

$$S(g_i, g_j) := \frac{x_{ij}}{x_i} g_i - \frac{x_{ij}}{x_j} g_j .$$

It can be shown that $S(g_i, g_j)$ belongs to $M$ and therefore can be written as

$$S(g_i, g_j) = \sum_{k=1}^q h_{ij}^k g_k.$$  

For $i \neq j$ we define

$$s_{ij} := \frac{x_{ij}}{x_i} e_i - \frac{x_{ij}}{x_j} e_j - (h_{ij}^1, \ldots, h_{ij}^q),$$

where $\{e_i\}_{i=1}^q$ is the canonical basis of $R^q$. The two generating systems $F = (f_1, \ldots, f_p)$ and $G = (g_1, \ldots, g_q)$ of $M$ are related through the relations $F = G \cdot A$ and $G = F \cdot B$, where $A$ and $B$ are $q \times p$ and $p \times q$ matrices over $R$, respectively. The matrix $A$ can be obtained using the division algorithm. While, the matrix $B$ is obtained during the calculation of the Gröbner basis using the Buchberger’s algorithm. Let $r_i$, $i = 1, \ldots, p$, denote the columns of the matrix $1 - B \cdot A$. Then we have

**Theorem F:** $\{r_1, \ldots, r_p\} \cup \{B s_{ij} \mid 1 \leq i < j \leq q \}$ generates the syzygy module $K = \text{Syz}(f_1, \ldots, f_p)$.

Using the above theorem we can easily compute the kernel of a surjective homomorphism

$$\phi : R^p \to M/N ,$$

where $M = \langle f_1, \ldots, f_p \rangle$ and $N = \langle f_{p+1}, \ldots, f_{p+q} \rangle$ are submodules of $R^m$ and $\phi(e_i) = f_i + N$ for $i = 1, \ldots, p$. For that purpose we only need to calculate $\text{Syz}(f_1, \ldots, f_{p+q})$. Let $\text{Syz}(f_1, \ldots, f_{p+q}) = \langle p_1, \ldots, p_\ell \rangle$ and $h_i$ denote the vector of the first $p$ components of $p_i$. Then Ker $\phi = \langle h_1, \ldots, h_\ell \rangle$.

Having the necessary computational tools at our disposal, we are now faced with the computation of $\text{Hom}(M, N)$ as a first step in the calculation of the Ext functor (cf. appendix). We begin with a presentation of $M$ and $N$

$$0 \to K_0 \xrightarrow{\text{id}} R^p \xrightarrow{\pi_0} M \to 0 , \quad 0 \to L_0 \xrightarrow{\text{id}} R^k \xrightarrow{\pi'_0} N \to 0 .$$
Similarly, we can write down presentations of the first syzygy modules $K_0$ and $L_0$

$$0 \to K_1 \xrightarrow{i_1} R^q \xrightarrow{\pi_1} K_0 \to 0 \quad , \quad 0 \to L_1 \xrightarrow{i'_1} R^\ell \xrightarrow{\pi'_1} L_0 \to 0 .$$  

(7)

Combining (6) and (7) we obtain

$$R^q \xrightarrow{\alpha} R^p \xrightarrow{\pi_0} M \to 0 \quad , \quad R^\ell \xrightarrow{\beta} R^k \xrightarrow{\pi'_0} N \to 0 ,$$  

(8)

where $\alpha = i_0 \circ \pi_1$ and $\beta = i'_0 \circ \pi'_1$. By considering $K_0$ (resp. $L_0$) as a submodule of $R^p$ (resp. $R^k$) $\alpha$ (resp. $\beta$) can be identified with a $p \times q$ (resp. $k \times \ell$) matrix whose columns are the generators of $K_0$ (resp. $L_0$). (Note that theorem F provides us with such a generating system.) Applying the (left exact contravariant functor) $\text{Hom}(\cdot, N)$ to the first sequence in (8) we get

$$0 \to \text{Hom}(M, N) \to \text{Hom}(R^p, N) \xrightarrow{\alpha^*} \text{Hom}(R^q, N) .$$  

(9)

Therefore, the calculation of $\text{Hom}(M, N)$ has been reduced to that of the kernel of the map $\alpha^* : \text{Hom}(R^p, N) \to \text{Hom}(R^q, N)$, $\psi \mapsto \alpha^*(\psi) := \psi \circ \alpha$. For the maps $\xi_0$ we know how it can be calculated. Hence, we are led to find presentations of modules $\text{Hom}(R^p, N)$ and $\text{Hom}(R^q, N)$. For that purpose, we apply the exact functors $\text{Hom}(R^p, \cdot)$ and $\text{Hom}(R^q, \cdot)$ to the second sequence in (8) to get

$$\text{Hom}(R^p, R^\ell) \xrightarrow{\beta_*} \text{Hom}(R^p, R^k) \to \text{Hom}(R^p, N) \to 0 ,$$  

(10)

$$\text{Hom}(R^q, R^\ell) \xrightarrow{\tilde{\beta}_*} \text{Hom}(R^q, R^k) \to \text{Hom}(R^q, N) \to 0 .$$  

(11)

From this we see that the cokernels of the maps $\beta_*$ and $\tilde{\beta}_*$ yield the desired presentations. Now by calculating the kernel of the map

$$\tilde{\alpha}^* : \text{Hom}(R^p, R^k) \longrightarrow \text{Hom}(R^q, R^k)/\text{Im} \tilde{\beta}_* ,$$

where $\tilde{\alpha}^*$ is the composition of $\alpha^*$ with the canonical projection $\text{Hom}(R^p, R^k) \to \text{Hom}(R^q, R^k)/\text{Im} \beta_*$ and taking its quotient by $\text{Im} \beta_*$ we arrive at $\text{Hom}(M, N)$, i.e.

$$\text{Hom}(M, N) = \text{Ker} \tilde{\alpha}^*/\text{Im} \beta_* .$$

We now turn to the calculation of $\text{Ext}^i_R(M, N)$. According to the definition of $\text{Ext}^i_R(M, N)$ we first compute a free resolution of $M$, which amounts to the repeated application of the ‘Theorem F’ for the calculation of syzygy modules. Note that this procedure terminates in at most $n$ steps due to Hilbert’s syzygy theorem.
Next we apply the functor $\text{Hom}(\cdot, N)$ to (12) to obtain

$$0 \to \text{Hom}(M, N) \xrightarrow{d^0} \text{Hom}(F_0, N) \xrightarrow{d^1} \text{Hom}(F_1, N) \to \ldots$$

(13)

Following the steps discussed above for the calculation of the Hom functor we can find a presentation for each term of the sequence (13) and then, by calculating the kernel of $d^i$ (cf. (5)) and the image of $d^i-1$, we get the desired result!

4 Examples

In the following examples we will calculate the cohomology groups $H^\bullet(U_\Sigma, O_{U_\Sigma})$ from which $H^\bullet(P_\Sigma, O_{P_\Sigma}(p))$ can be obtained through the projection map $\pi$ as described above.

**Example 1** (the weighted projective space $P(w_0, \ldots, w_d)$): Let $w_0, \ldots, w_d$ be relatively prime positive integers. Further, let $\{v_0, \ldots, v_d\}$ be a spanning set of a $d$-dimensional real vector space $V$ satisfying the linear relation $w_0 v_0 + \ldots + w_d v_d = 0$. Let the integer span of $v_0, \ldots, v_d$ define the lattice $N$ whose $\mathbb{R}$-extension is obviously $V$. The fan $\Sigma$ consists of all simplicial cones generated by proper subsets of $\{v_0, \ldots, v_d\}$. The corresponding toric variety will be the weighted projective space $P(w_0, \ldots, w_d)$. It follows from these data that $S = \mathbb{C}[x_0, \ldots, x_d]$ with $\deg x_i = w_i$ and $I = \langle x_0, \ldots, x_d \rangle$. Since the sequence $(x_0, \ldots, x_d)$ is regular in $S = \mathbb{C}[x_0, \ldots, x_d]$ we find according to theorem D and (3) that the only nontrivial cohomology groups are $H^0(U_\Sigma, O_{U_\Sigma})$ and $H^d(U_\Sigma, O_{U_\Sigma})$:

$$H^p(U_\Sigma, O_{U_\Sigma}) = \begin{cases} 
S & \text{for } p = 0 \\
0 & \text{for } 0 < p < d \\
\bigoplus_{n_i > 0} \mathbb{C} \cdot x_0^{-n_0} \cdots x_d^{-n_d} & \text{for } p = d
\end{cases}$$

where $U_\Sigma = \mathbb{A}^{d+1} \setminus \{0\} = \bigcup_{i=0}^d U_i$. We have also made use of the fact that the inductive limit commutes with the cohomology (cf. the next example for the details on the calculation of inductive limit).

**Example 2**: Let $P_\Sigma$ denote the Fano toric variety defined by the reflexive polytope $\Delta$ in $N$ with the vertices

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| (1, 0, 0, 0, 0) | (0, 1, 0, 0, 0) | (0, 0, 1, 0, 0) | (0, 0, 0, 1, 0) | (0, 0, 0, 0, 1) | (0, 0, 0, 0, -1) | (-1, -1, -3, -3, -6) |
which is a blowup of the weighted projective space $\mathbb{P}(1, 1, 1, 3, 3, 6)$. (The one-dimensional cone $\langle e_6 \rangle$ corresponds to the resulting exceptional divisor in $\mathbb{P}_5.$) The big cones of its defining simplicial fan $\Sigma$ are given by

$$
\sigma_1 = \langle e_1 e_2 e_3 e_4 e_5 \rangle, \quad \sigma_2 = \langle e_1 e_2 e_3 e_4 e_6 \rangle, \quad \sigma_3 = \langle e_1 e_2 e_3 e_6 e_7 \rangle, \quad \sigma_4 = \langle e_1 e_2 e_4 e_6 e_7 \rangle, \\
\sigma_5 = \langle e_1 e_3 e_4 e_6 e_7 \rangle, \quad \sigma_6 = \langle e_2 e_3 e_4 e_6 e_7 \rangle, \quad \sigma_7 = \langle e_1 e_2 e_3 e_5 e_7 \rangle, \quad \sigma_8 = \langle e_1 e_2 e_4 e_5 e_7 \rangle, \\
\sigma_9 = \langle e_1 e_3 e_4 e_5 e_7 \rangle, \quad \sigma_{10} = \langle e_2 e_3 e_4 e_5 e_7 \rangle.
$$

It follows from the above data that $S = \mathbb{C}[x_1, \ldots, x_7]$ with $\deg x_1 = (1, 0)$, $\deg x_2 = (1, 0)$, $\deg x_3 = (3, 0)$, $\deg x_4 = (3, 0)$, $\deg x_5 = (6, 1)$, $\deg x_6 = (0, 1)$, $\deg x_7 = (1, 0)$, and $I = \langle x_6 x_7, x_4 x_6, x_3 x_6, x_2 x_6, x_1 x_6, x_5 x_7, x_4 x_5, x_3 x_5, x_2 x_5, x_1 x_5 \rangle$. Using the algorithm described above we get a free resolution of $S/I^m$: $0 \to S \xrightarrow{d_1} S^7 \xrightarrow{d_2} S^{20} \xrightarrow{d_3} S^{30} \xrightarrow{d_4} S^{25} \xrightarrow{d_5} S^{10} \xrightarrow{d_6} S \to S/I^m \to 0$ with

$$
\begin{align*}
&d_1 = (x_6 x_7 x_4 x_6 x_3 x_6 x_2 x_6 x_1 x_6 x_5 x_7 x_4 x_5 x_3 x_5 x_2 x_5 x_1 x_5) \\
&d_2 = \\
&d_3 = (x_6 x_7 x_4 x_6 x_3 x_6 x_2 x_6 x_1 x_6 x_5 x_7 x_4 x_5 x_3 x_5 x_2 x_5 x_1 x_5)
\end{align*}
$$
Applying $\text{Hom}(\cdot, S)$ to $0 \to S \xrightarrow{d_6} S^7 \xrightarrow{d_5} S^{20} \xrightarrow{d_4} S^{30} \xrightarrow{d_3} S^{25} \xrightarrow{d_2} S^{10} \xrightarrow{d_1} S \to 0$ and calculating the cohomology of the resulting complex as described above yields

\[
\begin{align*}
\text{Ext}_S^2(S/I^m, S) &= S/\langle x_5^m, x_6^m \rangle, \\
\text{Ext}_S^3(S/I^m, S) &= 0, \\
\text{Ext}_S^4(S/I^m, S) &= 0, \\
\text{Ext}_S^5(S/I^m, S) &= S/\langle x_1^m, x_2^m, x_3^m, x_4^m, x_7^m \rangle, \\
\text{Ext}_S^6(S/I^m, S) &= S/\langle x_1^m, x_2^m, x_3^m, x_4^m, x_5^m, x_6^m, x_7^m \rangle.
\end{align*}
\]

Now we turn to the calculation of the inductive limits. We begin with $\{\text{Ext}_S^2(S/I^m, S)\}$, $\{\tilde{f}_{mn}\}$. The maps $\tilde{f}_{mn}$ are induced by $f_{mn} : S \to S$, $1 \mapsto (x_5 x_6)^{n-m} \cdot 1$ which are in turn induced by the canonical maps $S/I^m \to S/I^n$. Let $\mathcal{I}_m$ denote the $S$-module $\langle x_5^m, x_6^m \rangle$ and $\tilde{f}_{mn} := f_{mn}^* \mathcal{I}_m$. Then it can be easily seen that $\{\mathcal{I}_m\}_m$ together with the maps $\tilde{f}_{mn}$ is an inductive system. Since $\lim S/I^m = S/\lim I_m$ we only need to consider the latter two limits. The first one is well-known: $\lim S = S_{x_5 x_6}$, where $S_{x_5 x_6}$ is the localization of $S$ at $x_5 x_6$ (cf. [1], EGA, 4, p.19). We now show that $\lim S = S_{x_5} / S \oplus S_{x_6} / S$. It is not difficult to verify that $(S_{x_5} / S \oplus S_{x_6} / S, \{\varphi_m\})$ with $\varphi_m(x_5^m) = 0 \oplus \frac{1}{x_6}$ and $\varphi_m(x_6^m) = \frac{1}{x_5} \oplus 0$ (modulo $S!$) is a target object for
(\{J_m\}, \{\tilde{f}_{mn}\}). According to the universal property of the inductive limit there exists a unique homomorphism \(\psi\) such that the diagram

\[
\begin{array}{ccc}
J_m & \xrightarrow{f_m} & S_{x_5}/S \oplus S_{x_6}/S \\
\downarrow \varphi_m & & \downarrow \psi \\
\lim J_m & \rightarrow & S_{x_5}/S \oplus S_{x_6}/S
\end{array}
\]

commutes for all \(m\). We now prove that \(\psi\) is actually an isomorphism. Let \(\beta\) be an arbitrary element of \(S_{x_5}/S \oplus S_{x_6}/S\). Then \(\beta\) can be written as \(\frac{g}{x_5^5} \oplus \frac{h}{x_6^6}\). Therefore, it follows from \(\beta = \varphi_m(h x_5^m + g x_6^m) = \psi \circ f_m(h x_5^m + g x_6^m) = \psi(f_m(h x_5^m + g x_6^m))\) that \(\psi\) is surjective. Now, let \(\alpha = h x_5^m + g x_6^m\) such that \(f_m(\alpha)\) lies in the kernel of \(\psi\), i.e. \(\psi(f_m(\alpha)) = 0\). Since \(\psi \circ f_m = \varphi_m\) then \(\varphi_m(\alpha) = 0\), which means \(\frac{g}{x_5^5} \oplus \frac{h}{x_6^6} = 0 \iff (x_5^k g = 0 \text{ and } x_6^6 h = 0)\). Because of the commutativity of

\[
\begin{array}{ccc}
J_m & \xrightarrow{f_m} & \lim J_m \\
\downarrow \tilde{f}_{m+k} & & \downarrow \lim \tilde{f}_{m+k} \\
J_{m+k} & \xrightarrow{f_{m+k}} & \lim J_m
\end{array}
\]

we have \(f_m(\alpha) = f_{m+k}(\tilde{f}_{m,m+k}(\alpha))\). Since \(\tilde{f}_{m,m+k}(h x_5^m + g x_6^m) = x_5^k h x_5^{m+k} + x_5^k g x_6^{m+k} = 0\) we obtain \(f_m(\alpha) = 0\), i.e. \(\ker \psi = \{0\}\). Summarizing the above discussion we find out that

\[
H^1(U_\Sigma, \mathcal{O}_{U_\Sigma}) = \bigoplus_{k,l,m,n,r \geq 0} \mathbb{C} \cdot x_1^k x_2^l x_3^m x_4^n x_5^p x_6^q x_7^r \oplus \mathbb{C}[x_1, x_2, x_3, x_4, x_7]
\]

as a \(\mathbb{C}\)-vector space. The maps \(\tilde{f}_{mn}\) in the next nontrivial inductive system \((\{\text{Ext}^5_5(S/I^m, S)\}, \{\tilde{f}_{mn}\})\) are induced by \(f_{mn} : S \rightarrow S, 1 \mapsto (x_1 x_2 x_3 x_4 x_7)^{n-m} 1\). As before the family \(\{J_m\}_m\), where \(J_m\) now denotes the \(S\)-module \(\langle x_1^1, x_2^2, x_3^3, x_4^4, x_7^7 \rangle\), together with the maps \(\tilde{f}_{mn} := f_{mn} \mid J_m\) form an inductive system. Following the same lines of argumentation as presented above one can easily verify that \(\lim J_m = S_{x_{13233447}/S} \oplus S_{x_{13233447}/S} \oplus S_{x_{13233447}/S} \oplus S_{x_{13233447}/S} \oplus S_{x_{13233447}/S} \oplus S_{x_{13233447}/S} \oplus S_{x_{13233447}/S} \oplus S_{x_{13233447}/S}\). Taking \(\lim S = S_{x_{13233447}}\) into account we therefore obtain

\[
H^4(U_\Sigma, \mathcal{O}_{U_\Sigma}) = \bigoplus_{k,l,m,n,r \geq 0} \mathbb{C} \cdot x_1^{-k} x_2^{-l} x_3^{-m} x_4^{-n} x_5^p x_6^q x_7^r \oplus \mathbb{C}[x_5, x_6]
\]

as a \(\mathbb{C}\)-vector space. Similar considerations in the case of \((\{\text{Ext}^6_6(S/I^m, S)\}, \{\tilde{f}_{mn}\})\), where \(\tilde{f}_{mn}\) is now induced by \(f_{mn} : S \rightarrow S, 1 \mapsto (x_1 x_2 x_3 x_4 x_5 x_6 x_7)^{n-m} 1\), lead us to the following result

\[
H^5(U_\Sigma, \mathcal{O}_{U_\Sigma}) = \bigoplus_{k,l,m,n,p,q,r \geq 0} \mathbb{C} \cdot x_1^{-k} x_2^{-l} x_3^{-m} x_4^{-n} x_5^p x_6^q x_7^r.
\]
Example 3: Let $\Delta$ be a reflexive polytope in $\mathbb{N}$ whose vertices are given by

$$
e_1 = (1, 0, 0, 0, 0) \quad e_3 = (0, 0, 1, 0, 0) \quad e_5 = (0, 0, 0, 1) \quad e_7 = (0, -1, -1, -1, -3) \\
e_2 = (0, 1, 0, 0, 0) \quad e_4 = (0, 0, 0, 1, 0) \quad e_6 = (0, 0, 0, 0, -1) \quad e_8 = (-1, -2, -2, -3, -6).
$$

We take a maximal triangulation of the reflexive polytope $\Delta$, which leads to a simplicial fan $\Sigma$ whose big cones are defined by

$$
\sigma_1 = \langle e_1 e_2 e_3 e_4 e_5 \rangle \quad \sigma_2 = \langle e_1 e_2 e_3 e_4 e_6 \rangle \quad \sigma_3 = \langle e_1 e_2 e_3 e_5 e_8 \rangle \quad \sigma_4 = \langle e_1 e_2 e_3 e_6 e_8 \rangle \\
\sigma_5 = \langle e_1 e_2 e_4 e_5 e_7 \rangle \quad \sigma_6 = \langle e_1 e_2 e_4 e_6 e_7 \rangle \quad \sigma_7 = \langle e_1 e_2 e_5 e_7 e_8 \rangle \quad \sigma_8 = \langle e_1 e_2 e_6 e_7 e_8 \rangle \\
\sigma_9 = \langle e_1 e_3 e_4 e_5 e_7 \rangle \quad \sigma_{10} = \langle e_1 e_3 e_4 e_6 e_7 \rangle \quad \sigma_{11} = \langle e_1 e_3 e_5 e_7 e_8 \rangle \quad \sigma_{12} = \langle e_1 e_3 e_6 e_7 e_8 \rangle \\
\sigma_{13} = \langle e_2 e_3 e_4 e_5 e_8 \rangle \quad \sigma_{14} = \langle e_2 e_3 e_4 e_6 e_8 \rangle \quad \sigma_{15} = \langle e_2 e_4 e_5 e_7 e_8 \rangle \quad \sigma_{16} = \langle e_2 e_3 e_6 e_7 e_8 \rangle \\
\sigma_{17} = \langle e_2 e_4 e_5 e_7 e_8 \rangle \quad \sigma_{18} = \langle e_3 e_4 e_6 e_7 e_8 \rangle.
$$

The Fano toric variety $\mathbb{P}_\Sigma$ constructed from $\Sigma$ is a blowup of the weighted projective space $\mathbb{P}(1,1,2,3,6)$. (The one-dimensional cones $\langle e_6 \rangle$ and $\langle e_7 \rangle$ correspond to the resulting exceptional divisors in $\mathbb{P}_\Sigma$.) It follows from the above data that $S = \mathbb{C}[x_1, \ldots, x_8]$ with $\deg x_1 = (1, 0, 0), \deg x_2 = (2, 1, 0), \deg x_3 = (2, 1, 0), \deg x_4 = (3, 1, 0), \deg x_5 = (6, 3, 1), \deg x_6 = (0, 0, 1), \deg x_7 = (0, 1, 0), \deg x_8 = (1, 0, 0)$ and $I = \langle x_6 x_7 x_8, x_5 x_7 x_8, x_4 x_6 x_7, x_4 x_5 x_7, x_3 x_6 x_8, x_3 x_5 x_8, x_3 x_4 x_6, x_3 x_4 x_5, x_2 x_6 x_8, x_2 x_5 x_8, x_2 x_4 x_6, x_2 x_4 x_5, x_1 x_6 x_7, x_1 x_5 x_7, x_1 x_3 x_6, x_1 x_3 x_5, x_1 x_2 x_6, x_1 x_2 x_5 \rangle$. Using the algorithm described above we get a free resolution of $S/I^m$: $0 \to S \xrightarrow{d_6} S^8 \xrightarrow{d_5} S^{27} \xrightarrow{d_{14}} S^{48} \xrightarrow{d_9} S^{15} \xrightarrow{d_2} S^{18} \xrightarrow{d_1} S \to S/I^m \to 0$. Proceeding as in the last example we get the following results

$$
\text{Ext}_2^2(S/I^m, S) = S/\langle x_5^m, x_6^m \rangle, \\
\text{Ext}_3^2(S/I^m, S) = S/\langle x_2^m, x_3^m, x_7^m \rangle \oplus S/\langle x_1^m, x_4^m, x_8^m \rangle, \\
\text{Ext}_4^2(S/I^m, S) = S/\langle x_2^m, x_3^m, x_5^m, x_6^m, x_7^m \rangle \oplus S/\langle x_1^m, x_4^m, x_5^m, x_6^m, x_8^m \rangle, \\
\text{Ext}_5^2(S/I^m, S) = S/\langle x_1^m, x_2^m, x_3^m, x_4^m, x_7^m, x_8^m \rangle, \\
\text{Ext}_6^2(S/I^m, S) = S/\langle x_1^m, x_2^m, x_3^m, x_4^m, x_5^m, x_7^m, x_8^m \rangle.
$$

By taking the inductive limit and using the results of the foregoing example we arrive at

$$
H^1(U_\Sigma, \mathcal{O}_{U_\Sigma}) = \bigoplus_{k,l,m,n,r,s \geq 0 \atop p,q \geq 0} \mathbb{C} \cdot x_1^k x_2^l x_3^m x_4^p x_5^p x_6^q x_7^r x_8^s \oplus \mathbb{C}[x_1, x_2, x_3, x_4, x_7, x_8] \\
H^2(U_\Sigma, \mathcal{O}_{U_\Sigma}) = \bigoplus_{k,n,p,q,r \geq 0 \atop l,m,s \geq 0} \mathbb{C} \cdot x_1^{-k} x_2^l x_3^m x_4^n x_5^p x_6^q x_7^r x_8^s \oplus \mathbb{C}[x_2, x_3, x_5, x_6, x_7].
$$
\[
H^3(U_{\Sigma}, \mathcal{O}_{U_{\Sigma}}) = \bigoplus_{k,n,s \geq 0} \mathbb{C} \cdot x_1^k x_2^{-l} x_3^{-m} x_4^n x_5^p x_6^{-q} x_7^{-r} x_8^s \oplus \mathbb{C}[x_1, x_4, x_8] \oplus \\
\bigoplus_{l,m,r \geq 0} \mathbb{C} \cdot x_1^{-k} x_2^l x_3^m x_4^{-n} x_5^{-p} x_6^q x_7^{-r} x_8^{-s} \oplus \mathbb{C}[x_2, x_3, x_7] \\
H^4(U_{\Sigma}, \mathcal{O}_{U_{\Sigma}}) = \bigoplus_{k,l,m,n,r,s \geq 0} \mathbb{C} \cdot x_1^{-k} x_2^l x_3^{-m} x_4^{-n} x_5^p x_6^q x_7^{-r} x_8^{-s} \oplus \mathbb{C}[x_5, x_6] \\
H^5(U_{\Sigma}, \mathcal{O}_{U_{\Sigma}}) = \bigoplus_{k,l,m,n,p,q,r,s \geq 0} \mathbb{C} \cdot x_1^{-k} x_2^l x_3^{-m} x_4^{-n} x_5^p x_6^{-q} x_7^{-r} x_8^{-s}.
\]

Acknowledgement

I would like to thank M. Kreuzer and P. Michor for helpful discussions and comments. I would also like to thank SINGULAR research group at the mathematics department of the university of Kaiserslautern. This work has been supported by the Austrian Research Fund (FWF) under grant Nr. P10641-PHY and ÖNB under grant Nr. 6632.
Appendix

Inductive Limit

Let \((I, \prec)\) be a partially ordered set. \((I, \prec)\) is said to be directed if \(\forall i, j \in I \exists k \in I\) such that \(i \prec k\) and \(j \prec k\). An inductive system of sets \([\{M_i\}_{i \in I}\) consists of a family of sets \(\{M_i\}_{i \in I}\) together with a family of maps \(\{f_{ij} : M_i \to M_j\}_{i \prec j}\) such that 1) \(f_{ii} = \text{id}_{M_i}\) for all \(i \in I\), 2) \(f_{jk} \circ f_{ij} = f_{ik}\) for all \(i \prec j \prec k\).

A target object for the inductive system \(S\) is a set \(M\) together with a family of maps \(\{\varphi_i : M_i \to M\}_{i \in I}\) such that \(\varphi_i \circ f_{ij} = \varphi_j\) commutes for all \(i \prec j\). ‘The inductive limit of \(S\)’ is a target object \((\text{lim}^{-\to} M_i, \{f_i\}_{i \in I})\) satisfying the following ‘universal property’: for any target object \((M, \{\varphi_i\}_{i \in I})\) there is a unique map \(f : \text{lim}^{-\to} M_i \to M\) such that the diagram

\[
\begin{array}{ccc}
M_i & \xrightarrow{\varphi_i} & M \\
\downarrow{f_i} & & \downarrow{f} \\
\text{lim}^{-\to} M_i & \xrightarrow{f} & M
\end{array}
\]

commutes for all \(i \in I\).

The Ext Functor

We first recall some elementary concept of homological algebra. Let \(R\) be a ring. A chain complex of \(R\)-modules is a sequence of \(R\)-modules and homomorphisms

\[(M_\bullet, d_\bullet) : \ldots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \ldots\]

with \(d_i \circ d_{i+1} = 0\) for all \(i \in \mathbb{Z}\). A cochain complex of \(R\)-modules is a sequence of \(R\)-modules and homomorphisms

\[(M^\bullet, d^\bullet) : \ldots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \ldots\]

with \(d^i \circ d^{i-1} = 0\) for all \(i \in \mathbb{Z}\). The homology of a chain complex \((M_\bullet, d_\bullet)\) is given by \(H_i(M_\bullet) := \text{Ker} \ d_i / \text{Im} \ d_{i+1}\). The cohomology of a cochain complex \((M^\bullet, d^\bullet)\) is given by \(H^i(M^\bullet) := \text{Ker} \ d^i / \text{Im} \ d^{i-1}\).

*Analogously one can define the same concepts in the category of groups, rings and modules.
Let \((M_\bullet, d_\bullet)\) and \((N_\bullet, d'_\bullet)\) be two chain complexes. A homomorphism of chain complexes \(f_\bullet : M_\bullet \to N_\bullet\) is a family of homomorphisms \(f_i : M_i \to N_i\) such that \(f_{i-1} \circ d_i = d'_i \circ f_i\) for all \(i \in \mathbb{Z}\). Clearly \(f_\bullet : M_\bullet \to N_\bullet\) induces a well-defined map \(H_i(f_\bullet) : H_i(M_\bullet) \to H_i(N_\bullet)\) for all \(i \in \mathbb{Z}\). Two homomorphisms \(f_\bullet, g_\bullet : M_\bullet \to N_\bullet\) are said to be homotopic, written \(f_\bullet \simeq g_\bullet\), if there are maps \(h_i : M_i \to N_{i+1}, i \in \mathbb{Z}\), such that \(f_i - g_i = d'_{i+1} \circ h_i + h_{i-1} \circ d_i\). The chain complexes \((M_\bullet, d_\bullet)\) and \((N_\bullet, d'_\bullet)\) are called homotopy equivalent, written \(M_\bullet \simeq N_\bullet\), if there are homomorphisms \(f_\bullet : M_\bullet \to N_\bullet\) and \(g_\bullet : N_\bullet \to M_\bullet\) such that \(f_\bullet \circ g_\bullet \simeq \text{id}_{N_\bullet}\) and \(g_\bullet \circ f_\bullet \simeq \text{id}_{M_\bullet}\). Homotopy equivalent chain complexes have the same homology.

An \(R\)-module \(P\) is called projective if for any surjective homomorphism \(f : M \to N\) of \(R\)-modules \(M\) and \(N\) and any homomorphism \(g : P \to N\) there exists a homomorphism \(h : P \to M\) such that \(g = f \circ h\). It follows from the definition that a free module is projective. By a projective (resp. free) resolution of an \(R\)-module \(M\) we mean an exact sequence \(\ldots \to P_i \xrightarrow{d_i} P_{i-1} \to \ldots \to P_0 \xrightarrow{d_0} M \to 0\), where \(P_i\) are projective (resp. free) for \(i = 0, 1, \ldots\). It can be shown that any two projective resolutions of \(M\) are homotopy equivalent.

Let \(M\) and \(N\) be two \(R\)-modules and \(P_\bullet \to M \to 0\) a projective resolution of \(M\). Then

\[
\text{Ext}^i_R(M, N) := H^i(\text{Hom}(P_\bullet, N)).
\]

References

[1] E. Witten, New issues in manifolds of SU(3) holonomy, Nucl. Phys. B268 (1986) 79
   J. Distler, B. Greene, Aspects of (2,0) string compactifications, Nucl. Phys. B304 (1988) 1

[2] J. Distler, B.R. Green, D.R. Morrison, Resolving singularities in (0,2) models, Nucl. Phys. B481 (1996) 289, [hep-th/9605222]

[3] E. Witten, Phases of N=2 theories in two dimensions, Nucl. Phys. B403 (1993) 159, [hep-th/9301042]

[4] J. Distler, S. Kachru, (0,2) Landau–Ginzburg theory, Nucl. Phys. B413 (1994) 213, [hep-th/9309110]
   J. Distler, Notes on (0,2) superconformal field theories, Proceedings of the 1994 Trieste Summer School, [hep-th/9502012]

[5] D.A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebr. Geom. 4 No.1 (1995) 17-50, [alg-geom/9210008]
   D.A. Cox, Recent developments in toric geometry, [alg-geom/9606016]
[6] A. Grothendieck, *Local cohomology*, (Notes by R. Hartshorne), Lecture Notes in Math. 41 (1967)

[7] V.I. Danilov, *The geometry of toric varieties*, Russian Math. Survey 33, n.2 (1978) 97
T. Oda, *Convex bodies and algebraic geometry*, Springer Verlag, (1988)
W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, Princeton, (1993)
G. Ewald, *Combinatorial convexity and algebraic geometry*, GTM 168, Springer Verlag, (1996)

[8] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. 61 (1955) 197
R. Godement, *Topologie algébrique et théorie des faisceaux*, Herman, Paris (1964)
EGA: A. Grothendieck, J. Dieudonné, *Eléments de géométrie algébrique*, Publ. Math. IHES 4 (1960), 11 (1961), 17 (1963)
R. Hartshorne, *Algebraic geometry*, GTM 52, Springer Verlag, (1977)

[9] R. Brüiske, F. Ischebeck, F. Vogel, *Kommutative Algebra*, BI-Wissenschaftsverlag, (1989)
D. Cox, J. Little, D. O'Shea, *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra*, UTM, Springer Verlag, (1992)
W. W. Adams, P. Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Math., Vol. 3, AMS, (1994)
D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, GTM 150, Springer Verlag, (1995)

[10] I. Dolgachev, *Weighted projective varieties*, Lecture Notes in Math. 956 (1981) 34

[11] G.-M. Greuel, G. Pfister, H. Schoenemann, *Singular Reference Manual*, In: Reports On Computer Algebra, number 12, Centre for Computer Algebra, University of Kaiserslautern, May 1997, [http://www.mathematik.uni-kl.de/~zca/Singular](http://www.mathematik.uni-kl.de/~zca/Singular)