The Quantum Clock:
a critical discussion on (space-)time

Luciano Burderi (University of Cagliari)
Tiziana di Salvo (University of Palermo)
Operationalism
Percy Williams Bridgman (1882-1961)

The concept is defined on the measurements

SR: “time” is the quantity measured by a light-clock (Einstein)

GR: “mass” is defined by
(a) Newton’s Second Law of Motion (inertial)
(b) Newton’s law of universal gravitation (gravitational)

Equivalence Principle (Einstein):
(a) = (b)
Operational definition of “time”

time ≡ a physical quantity that is measured by an appropriate clock

Light Clock

- time measured with strictly periodic events

\[ \Delta t = D/c \geq \Delta t_{\text{MIN}} = \frac{\hbar}{m_e c^2} \approx 1.3 \times 10^{-21} \text{ s} \]

(see e.g. Garay 1995)

- shortest time interval ever measured: \(2 \times 10^{-17} \text{ s}\) (Schultze et al. 2010)
Distance measurements using the “multi-pulley tackle” principle

\[ \ell = c \tau : \]
\[ \ell_{\text{TOT}} = c \ T_{\text{FLIGHT}} = n \ c \ \tau \]
\[ n = T_{\text{FLIGHT}} / \tau \]
\[ \ell = c \ T_{\text{FLIGHT}} / n \]
\[ \sigma_{\ell} = c \ \sigma_T / n \]
\[ \sigma_{\ell} = c \ \tau \ \sigma_T / T_{\text{FLIGHT}} \propto (T_{\text{FLIGHT}})^{-1} \]

\[ \sigma_{\ell} = 3 \times 10^{-17} \text{ cm}/\sqrt{\text{Hz}} = 7.7 \times 10^{-6} \lambda_C \]
(at 450 Hz laser frequency)
achieved with the \( \ell = 40 \text{ m} \) Caltech interferometer
Operational definition of “time”

time ≡ a physical quantity that is measured by an appropriate clock

Quantum Clock

time measured with totally random events (e.g. Salecker & Wigner, 1958)

decay products (particles or photons)
detectors/counters (quantum efficiency 1)

Counts: 14,392

radioactive matter
The Quantum Clock with radioactive substance

Completely random process: a statistical process whose probability of occurrence is constant (independent of time)

\[ dP = \lambda \, dt \]

Radioactive decay:

\[ dN = -\lambda N \, dt \quad (\text{where } \lambda^{-1} = \tau_{\text{PART}}) \]

Assume: \( \Delta t \ll \tau_{\text{PART}} \)

Number of expected decays in the interval \( \Delta t \):

\[ \Delta N_{\Delta t} = \lambda N \, \Delta t \]

Fluctuations with Poissonian statistics:

\[ \sigma_{\Delta N} = (\lambda N \, \Delta t)^{1/2} \]

Quantum Clock working principle: **compute time by counting the decays**

\[ \Delta t = \Delta N_{\Delta t} / (\lambda N) \]

Relative error in time = relative error in number of decays

\[ \sigma_{\Delta t} / \Delta t = \varepsilon = \sigma_{\Delta N} / \Delta N_{\Delta t} = (\lambda N \, \Delta t)^{-1/2} = 1 / (\Delta N_{\Delta t})^{1/2} \leq 1 \quad \rightarrow \quad \Delta N_{\Delta t} = 1 / \varepsilon^2 \]

Mass of the Quantum Clock: \( M = N \times m_{\text{PART}} \)

Energy of the decaying particle: \( E_{\text{PART}} = m_{\text{PART}} c^2 \)

\[ \Delta t = (m_{\text{PART}} c^2) / (\varepsilon^2 \lambda M c^2) = E_{\text{PART}} / (\varepsilon^2 \lambda M c^2) = (E_{\text{PART}} \times \tau_{\text{PART}}) / (\varepsilon^2 M c^2) \]
The Quantum Clock and Quantum Mechanics

Heisenberg uncertainty relation between the energy and the duration of a phenomenon:

$$\delta E \times \delta t \geq \hbar/2$$

Asssume (for simplicity) that the radioactive substance is destroyed in the decay (e.g. $\pi_0 \rightarrow 2\gamma$).

The whole energy of the particle is involved and therefore: $E_{PART} \geq \delta E$

The decay time must be measurable and therefore:

$$E_{PART} \times \tau_{PART} \geq \hbar/2$$

$$\Delta t = (E_{PART} \times \tau_{PART}) / (\epsilon^2 Mc^2) \geq \hbar / (2\epsilon^2 Mc^2)$$

(compare to Salecker & Wigner 1958, and Ng & van Dam 2003)
The Quantum Clock and General Relativity

To let the decaying particle escape and be detected the size of the Quantum Clock must be larger than its Schwarzschild Radius (Hoop Conjecture, Thorne, 1972):

\[ \Delta r \geq R_{\text{SCH}} = \frac{2GM}{c^2} \]

Therefore:

\[ \frac{1}{M} \geq \frac{2G}{c^2 \Delta r} \]

see Amelino-Camelia (1995) for a lower bound in the uncertainty for the measurement of a distance, in which this condition is included. Therefore, the Quantum Clock equation is:

\[ \Delta t \geq \frac{\hbar}{(2\varepsilon^2 Mc^2)} \geq \frac{G\hbar}{(\varepsilon c^4 \Delta r)} \]

Finally, since at least one decay occurred, \( \varepsilon = 1 / (\Delta N_{\Delta t})^{1/2} \leq 1 \), and therefore we get the new Space-Time Uncertainty Relation:

\[ \Delta r \Delta t \geq G\hbar/c^4 \]
Uncertainty relations proposed in the literature

(see Hossenfelder 2012 review)

1) The Salecker Wigner limit (1958) (see e.g. Camelia 1999):
\[ \Delta r \geq \left[ \hbar T_{\text{OBS}} \left( \frac{1}{M_{\text{BODIES}}} + \frac{1}{M_{\text{DEVICE}}} \right) / 2 \right]^{1/2} \]
uncertainty on the distance of two bodies of total mass \( M_{\text{BODIES}} \) with a device of mass \( M_{\text{DEVICE}} \) operating over a time such that \( r = c T_{\text{OBS}} / 2 \)

2) The Fundamental-Length Hypotheses (Mead 1964, 1966):
\[ \Delta r \geq \left( \frac{G \hbar}{c^3} \right)^{1/2} \]

3) The Generalized Uncertainty Principle (see e.g. Capozziello et al. 1999):
\[ \Delta r \geq \frac{\hbar}{2 \Delta p} + \left( \frac{\alpha}{c^3} \right) G \Delta p \]

4) In String Theory Yoneya (1987, 1989, 1997) proposed:
\[ \Delta X_1 \times c \Delta T \geq \ell_S^2 \]
similar to the uncertainty relation proposed above (see also Doplicher et al. 1995), although:
a) \( \ell_S \) is a free parameter of the theory (sometimes identified with the Planck length).
b) the proposed relation is “speculative and hence rather vague yet” (Yoneya).

5) Space-Time Uncertainty Principle (this work Phys. Rev. D, submitted):
\[ \Delta r \Delta t \geq G \hbar / c^4 \]
“demonstrated” by means of a Gedankenexperiment
The Quantum Clock and Special Relativity

In SR “true” temporal and spatial intervals are defined by a combined measure of space and time:
“true” temporal intervals: TIMELIKE intervals measured at the same place ($\Delta r \approx 0$)
“true” spatial intervals: SPACELIKE intervals measured at the same time ($\Delta t \approx 0$)

Generalized “true” temporal interval: any TIMELIKE interval with $|c\Delta t| \geq |\Delta r|$  
Generalized “true” spatial interval: any SPACELIKE interval with $|\Delta r| \geq |c\Delta t|$  

We represent space and time intervals in a space-time intervals diagram. 
We choose the space and time units in order to have $c = 1$, or $c\Delta t$ as the ordinate.
In this representation the bisector defines the null intervals, separating the TIMELIKE intervals, above the bisector, from the SPACELIKE intervals, below.

The extremal relation $\Delta r \times c\Delta t = G\hbar/c^3$ is an hyperbola in the space-time diagram. 
Asymptotes: $\Delta r$ axis and $c\Delta t$ axis. 
Vertex at: $\Delta r_{\text{VERTEX}} = c\Delta t_{\text{VERTEX}} = (G\hbar/c^3)^{1/2} \equiv \text{Planck Length} \equiv c \times \text{Planck Time}$
The Uncertainty Relation and the space-time diagram for the intervals

\[ \Delta r \times c \Delta t = \frac{\hbar}{c^3} \]

\[ \Delta r_{\text{MIN}} = \left(\frac{\hbar}{c^5}\right)^{1/2} \]

\[ \Delta t_{\text{MIN}} = \left(\frac{\hbar}{c^5}\right)^{1/2} \]
The Quantum Clock and Special Relativity

The following can be deduced:

I) TIMELIKE INTERVALS:  \( \Delta t_{\text{MIN}} = \left( \frac{G\hbar}{c^5} \right)^{1/2} \equiv \text{Planck Time} \)

II) SPACELIKE INTERVALS:  \( \Delta r_{\text{MIN}} = \left( \frac{G\hbar}{c^3} \right)^{1/2} \equiv \text{Planck Length} \)

III) The Uncertainty Relation is invariant under Lorentz Transformation since:
   \[ \Delta r' = \gamma^{-1} \Delta r \quad \text{(Lorentz contraction)} \]
   \[ \Delta t' = \gamma \Delta t \quad \text{(time dilation)} \]
   \[ \gamma = \left( 1 - \left( \frac{v}{c} \right)^2 \right)^{-1/2} \quad \text{(Lorentz factor)} \]
The Quantum Clock with radioactive substance: feasibility for an “advanced civilization”

Decaying substance $^7\text{H}$ (1 proton + 6 neutrons, Gurov et al. 2004))

$m_{\text{PART}} \approx 7 \, m_{\text{PROTON}}$

$\tau_{\text{PART}} \approx 2.3 \times 10^{-23} \, \text{s}$

Asssume:

a) $\Delta t = 0.1 \times \tau_{\text{PART}}$ ($\Delta t << \tau_{\text{PART}}$)

b) $\sigma_{\Delta t} = 0.1 \times t_{\text{PLANCK}}$ (to test below the Planck scale in a SPACELIKE interval)

We found:

$\sigma_{\Delta t} / \Delta t = \sigma_{\Delta N} / \Delta N_{\Delta t} = [\lambda N \Delta t]^{-1/2} = [\tau_{\text{PART}} / (N \Delta t)]^{-1/2}$

Therefore: $(0.1 \times t_{\text{PLANCK}})/(0.1 \times \tau_{\text{PART}}) = [\tau_{\text{PART}} / (N \times 0.1 \times \tau_{\text{PART}})]^{1/2}$

$t_{\text{PLANCK}} / \tau_{\text{PART}} = (10 / N)^{1/2}$

$N = 10 \times (\tau_{\text{PART}} / t_{\text{PLANCK}})^2 = 10 \times (2.3 \times 10^{-23} \, \text{s} / 5.4 \times 10^{-44} \, \text{s})^2 = 1.8 \times 10^{42}$

$M_{\text{CLOCK}} = N \times 7 \, m_{\text{PROTON}} = 2.2 \times 10^{19} \, \text{g} = 3.6 \times 10^{-9} \, M_{\text{EARTH}}$
The Quantum Clock with Blackbody Radiation: the BlackBody Clock

Consider a spherical box of radius $R$ where a small (negligible) amount of matter is in equilibrium with an electromagnetic radiation field at a temperature $T$.

$$L = 4\pi R^2 \sigma_B T^4; \quad \sigma_B = ac/4; \quad a = (8\pi^5 k^4)/(15c^3h^3); \quad \langle h\nu \rangle = 3kT; \quad E_{BB} = M_{BB}c^2 = (4/3)\pi R^3 a T^4$$

$$dN_{PH}/dt = (4\pi R^2 \sigma_B T^4)/(3kT) = [((4/3)\pi R^3 a T^4] \times [c/(4RkT)] = M_{BB}c^2 \times [c/(4RkT)]$$

The number of photons detected in the time $\Delta t$ is $\Delta N_{PHOT\Delta t}$

Poisson statistics holds, therefore:

$$\varepsilon = \sigma_{\Delta t}/\Delta t = \sigma_{\Delta N}/\Delta N_{PHOT\Delta t} = (\Delta N_{PHOT\Delta t})^{-1/2} \quad \text{or} \quad \Delta N_{PHOT\Delta t} = \varepsilon^{-2}$$

$$\Delta N_{PHOT\Delta t} = \Delta t \times dN_{PH}/dt = \varepsilon^{-2}$$

$\Delta t = 1/(\varepsilon^2 M_{BB}c^2) \times (4RkT/c)$ \quad (as before if: $E_{PART} \times \tau_{PART} \rightarrow 4RkT/c$)

But $(4RkT/c) = (4/3)(R/c)(3kT) = (4/3)R\langle h\nu/c \rangle = (4/3)R\langle p_{PHOT} \rangle$

Since $(\langle p_{PHOT} \rangle) \geq (\delta \langle p_{PHOT} \rangle)$ and $R = \delta r$ we have $R \times (\langle p_{PHOT} \rangle) \geq \delta r \times (\delta \langle p_{PHOT} \rangle) \geq h/2$

Therefore: $\Delta t = 1/(\varepsilon^2 M_{BB}c^2) \times (4RkT/c) \geq 1/(\varepsilon^2 M_{BB}c^2) \times (4/3) \times h/2$

$\Delta t \geq (2/3)h/(\varepsilon^2 M_{BB}c^2)$ and inserting the GR constraint, $1/M_{BB} \geq 2G/(c^2\Delta r)$

$\Delta r \Delta t \geq (4/3)Gh/(\varepsilon^2 c^4)$

Dropping $\varepsilon^2 > 1$, we finally get $\Delta r \Delta t \geq (4/3)Gh/c^4$ which is the uncertainty relation again.
The “extreme” Quantum Clock: the Hawking Clock

This is a BlackBody Clock which uses Hawking-Beckenstein radiation emitted from the event Horizon of a Black Hole.

\[ \frac{dN_{PH}}{dt} = \frac{(4\pi R_{BH}^2 \sigma_B T_{BH}^4)}{(3kT_{BH})} = \frac{(4\pi \sigma_B/3k)}{R_{BH}^2 T_{BH}^3} \]

where: \[ R_{BH} = \frac{2GM_{BH}}{c^2} \]; \[ T_{BH} = \frac{\hbar c}{8\pi kGM_{BH}} \]

Consider as before \[ \Delta t = (dN_{PH}/dt \times \epsilon^2)^{-1} \] and \[ \Delta r \geq R_{BH} \]

we get:

\[ \Delta t \times \Delta r \geq R_{BH} / (dN_{PH}/dt \times \epsilon^2) = \epsilon^{-2} \times (3k/4\pi \sigma_B) \times (R_{BH}T_{BH}^3)^{-1} = 2^8 3^2 5 \epsilon^{-2} (GM_{BH})^2/c^5 \]

For the Hawking Clock the minimum occurs for the smallest Black Hole mass.

This Black Hole radiates Hawking-Beckenstein radiation with \[ \langle E_{PART} \rangle = 3kT_{BH} \]

At the end of the evaporation process we must have \[ \langle E_{PART} \rangle = 3kT_{BH} \approx M_{MIN}c^2 \]

This gives: \[ M_{MIN} = (3/8\pi)^{1/2} (\hbar c/G)^{1/2} = (3/8\pi)^{1/2} m_{PLANCK} \]

Inserting the relation above into \[ \Delta t \times \Delta r \geq 2^8 3^2 5 \epsilon^{-2} (GM_{MIN})^2/c^5 \], we get:

\[ \Delta r \Delta t \geq (2^5 3^3 5/\pi) \epsilon^{-2} (G\hbar/c^4) \]

Dropping \( \epsilon^{-2} > 1 \), we finally get \[ \Delta r \Delta t \geq (2^5 3^3 5/\pi) G\hbar/c^4 \] which confirms the uncertainty relation again.
The new Uncertainty Principle and the Minkowski metric

\[ \Delta s^2 = 0 \]

\[ \Delta s^2 = c \Delta t \]

\[ \Delta r \]
Conclusions

- by means of a *Gedankenexperiment* with a Quantum Clock, based on random rather than periodic events, we propose a new uncertainty principle:
  \[ \Delta r \Delta t \geq G\hbar/c^4 \]
- the principle is quite general being a necessary consequence of the very first principles of QM (Heisenberg Uncertainty Relations) and of GR (the formation of an Event Horizon for sufficiently high densities)
- when combined with the constrain imposed by SR, the new uncertainty principle gives:
  \[ \Delta t_{\text{MIN}} = (G\hbar/c^5)^{1/2} \]
  \[ \Delta r_{\text{MIN}} = (G\hbar/c^3)^{1/2} \]
- the principle is invariant in SR (GR, Schwarzschild?)
- the principle makes Space and Time non-commuting quantities (starting point for Quantum Gravity?)
- if, below the Plank scale, space-time has no meaning, Gravity, which is a curvature of space-time, could vanish at those scales (no singularity?)
- we discussed two similar albeit different clocks for which the new uncertainty principle holds
That’s all Folks!