Solutions for one class of nonlinear fourth-order partial differential equations

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Abstract

Some solutions for one class of nonlinear fourth-order partial differential equations

\[ u_{tt} = (\kappa u + \gamma u^2)_{xx} + \nu uu_{xxxx} + \mu u_{xtt} + \alpha u_x u_{xxx} + \beta u_x^2 \]

where \( \alpha, \beta, \gamma, \mu, \nu \) and \( \kappa \) are arbitrary constants are presented in the paper. This equation may be thought of as a fourth-order analogue of a generalization of the Camassa-Holm equation, about which there has been considerable recent interest. Furthermore, this equation is a Boussinesq-type equation which arises as a model of vibrations of harmonic mass-spring chain. The idea of travelling wave solutions and linearization criteria for fourth-order ordinary differential equations by point transformations are applied to this problem.

Key words: Linearization problem, point transformation, nonlinear ordinary differential equation, travelling wave solution

1 Introduction

Almost all important governing equations in physics take the form of nonlinear differential equations, and, in general, are very difficult to solve explicitly. While solving problems related to nonlinear ordinary differential equations it is often expedient to simplify equations by a suitable change of variables.

Many methods of solving differential equations use a change of variables that transforms a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there is the problem of transforming a given differential...
equation into a linear equation. This problem, which is called a linearization problem. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows constructing an exact solution of the original equation.

One of the most interested nonlinear problems but also difficultly in solving is the problem of nonlinear fourth-order partial differential equations \[1\]

\[
u_{tt} = \left(\kappa u + \gamma u^2\right)_{xx} + \nu \mu_{xxxx} + \mu u_{xxx} + \alpha u_xu_{xxx} + \beta u^2_{xx} \tag{1}
\]

where \(\alpha, \beta, \gamma, \mu, \nu\) and \(\kappa\) are arbitrary constants. The main difficulty in solving this problem comes from the terms of nonlinear partial differential equations and the large number of order. Because of this difficulty, there are only a few attempts to solve this problem.

In 2008, Suksern, Meleshko and Ibragimov \[2,3\] found the explicit form of the criteria for linearization of fourth-order ordinary differential equations by point transformations. Moreover, the procedure for the construction of the linearizing transformation are presented. By the virtue of \[2,3\] to bring about the idea for solving the problem of nonlinear fourth-order partial differential equations \(1\).

The way of solving the problem is organized as follows. Firstly, reducing the nonlinear fourth-order partial differential equations to the nonlinear fourth-order ordinary differential equations by substituting the form of travelling wave solutions. Secondly, reducing the nonlinear fourth-order ordinary differential equations to the linear fourth-order ordinary differential equations by applying the criteria for linearization in \[2,3\]. Finally, finding the exact solutions of linear equations and then substituting back to the exact solutions of the original problem.

2 Linearization criteria for fourth-order ordinary differential equations by point transformations

The important tools for this research is the linearization criteria for fourth-order ordinary differential equations by point transformations. From \[2,3\] we have the following theorems.

**Theorem 1** Any fourth-order ordinary differential equation

\[ y^{(4)} = f(x, y, y', y'', y''') , \]
can be reduced by a point transformation

\[ t = \varphi(x,y), \quad u = \psi(x,y), \quad (2) \]

to the linear equation

\[ u^{(4)} + \alpha(t) u' + \beta(t) u = 0, \quad (3) \]

where \( t \) and \( u \) are the independent and dependent variables, respectively, if it belongs to the class of equations

\[ y^{(4)} + (A_1 y' + A_0) y''' + B_0 y''^2 + (C_2 y'^2 + C_1 y' + C_0) y'' + D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0 = 0, \quad (4) \]

or

\[ y^{(4)} + \frac{1}{y + r} (-10 y'' + F_2 y'^2 + F_1 y' + F_0) y''' + \frac{1}{(y + r)^2} [15 y'^3 + (H_2 y'^2 + H_1 y' + H_0) y''^2 + (J_4 y'^4 + J_3 y'^3 + J_2 y'^2 + J_1 y' + J_0) y''] + K_7 y'^7 + K_6 y'^6 + K_5 y'^5 + K_4 y'^4 + K_3 y'^3 + K_2 y'^2 + K_1 y' + K_0 = 0, \quad (5) \]

where \( A_j = A_j(x,y), \ B_j = B_j(x,y), \ C_j = C_j(x,y), \ D_j = D_j(x,y), \ r = r(x,y), \ F_j = F_j(x,y), \ H_j = H_j(x,y), \ J_j = J_j(x,y) \) and \( K_j = K_j(x,y) \) are arbitrary functions of \( x, y \).

Since this research deals with the first class, let us emphasize to the first class for other theorems that we need to use.

**Theorem 2** Equation \((4)\) is linearizable if and only if its coefficients obey the
following conditions:

\[ A_{0y} - A_{1x} = 0, \quad (6) \]
\[ 4B_0 - 3A_1 = 0, \quad (7) \]
\[ 12A_{1y} + 3A_1^2 - 8C_2 = 0, \quad (8) \]
\[ 12A_{1x} + 3A_0A_1 - 4C_1 = 0, \quad (9) \]
\[ 32C_{0y} + 12A_{0x}A_1 - 16C_{1x} + 3A_0^2A_1 - 4A_0C_1 = 0, \quad (10) \]
\[ 4C_{2y} + A_1C_2 - 24D_4 = 0, \quad (11) \]
\[ 4C_{1y} + A_1C_1 - 12D_3 = 0, \quad (12) \]
\[ 16C_{1x} - 12A_{0x}A_1 - 3A_0^2A_1 + 4A_0C_1 + 8A_1C_0 - 32D_2 = 0, \quad (13) \]
\[ 192D_{2x} + 36A_{0x}A_0A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 - 288D_{1y} + 9A_0^3A_1 \]
\[ - 12A_0^2C_1 - 36A_0A_1C_0 + 48A_0D_2 + 32C_0C_1 = 0, \quad (14) \]

\[ 384D_{1xy} - \left[ 3(3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1) \right. \]
\[ - 16(A_1C_0 - D_2)A_0A_0 - 32(4(C_1D_1 - 2C_2D_0 + C_0D_2) \]
\[ + (3A_1D_0 - C_0^2A_1) - 96D_{1y}A_0 + 384D_{0y}A_1 + 1536D_{0yy} \]
\[ - 16(3A_0A_1 - 4C_1)C_0x + 12((3A_0A_1 - 4C_1)A_0 \]
\[ - 4(A_1C_0 - 4D_2))A_{0x} \right] = 0. \quad (15) \]

**Theorem 3** Provided that the conditions (6)-(15) are satisfied, the linearizing transformation (2) is defined by a fourth-order ordinary differential equation for the function \( \varphi(x) \), namely by the Riccati equation

\[ 40 \frac{d\chi}{dx} - 20\chi^2 = 8C_0 - 3A_0^2 - 12A_{0x}, \quad (16) \]

for

\[ \chi = \frac{\varphi_{xx}}{\varphi_x}, \quad (17) \]

and by the following integrable system of partial differential equations for the function \( \psi(x, y) \)

\[ 4\psi_{yy} = \psi_yA_1, \quad 4\psi_{xy} = \psi_y(A_0 + 6\chi), \quad (18) \]
and

\[
1600\psi_{xxxx} = 9600\psi_{xxx}\chi + 160\psi_{xx}(-12A_{0x} - 3A_{0}^2 - 90\chi^2 + 8C_{0}) \\
+ 40\psi_{x}(12A_{0x}A_{0} + 72A_{0x}\chi - 16C_{0x} + 3A_{0}^3 + 18A_{0x}\chi - 12A_{0}C_{0} \\
+ 120\chi^3 - 48\chi C_{0} + 24D_{1} - 8\Omega) + \psi(144A_{0x}^2 + 72A_{0x}A_{0}^2 - 352A_{0x}C_{0} \\
- 160C_{0xx} - 80C_{0x}A_{0} - 160D_{0y} + 640D_{1}x - 80\Omega_{x} + 9A_{0}^4 - 88A_{0x}C_{0} \\
+ 160A_{0}D_{1} + 30A_{0}\Omega - 400A_{1}D_{0} + 300\chi\Omega + 144C_{0}^2)) + 1600\psi_{y}D_{0},
\]

(19)

where \(\chi\) is given by equation \((17)\) and \(\Omega\) is the following expression

\[
\Omega = A_{0}^3 - 4A_{0}C_{0} + 8D_{1} - 8C_{0x} + 6A_{0x}A_{0} + 4A_{0xx}.
\]

(20)

Finally, the coefficients \(\alpha\) and \(\beta\) of the resulting linear equation \((3)\) are

\[
\alpha = \frac{\Omega}{8\psi_{x}^3},
\]

(21)

\[
\beta = (1600\psi_{x}^4)^{-1}(-144A_{0x}^2 - 72A_{0x}A_{0}^2 + 352A_{0x}C_{0} + 160C_{0xx} + 80C_{0x}A_{0} \\
+ 1600D_{0y} - 640D_{1}x + 80\Omega_{x} - 9A_{0}^4 + 88A_{0x}C_{0} - 160A_{0}D_{1} - 30A_{0}\Omega \\
+ 400A_{1}D_{0} - 300\chi\Omega - 144C_{0}^2).
\]

(22)

### 3 Method and Result

Let us consider the nonlinear fourth-order partial differential equation (Clarkson and Priestley, 1999)

\[
u u_{tt} = (\kappa u + \gamma u^2)_{xx} + \nu uu_{xxxx} + \mu u_{xxtt} + \alpha u_{x}u_{xxx} + \beta u_{xx}^2,
\]

(23)

where \(\alpha, \beta, \gamma, \mu, \nu\) and \(\kappa\) are arbitrary constants.

Of particular interest among solutions of equation \((23)\) are traveling wave solutions:

\[u(x, t) = H(x - Dt),\]

where \(D\) is a constant phase velocity and the argument \(x - Dt\) is a phase of the wave. Substituting the representation of a solution into equation \((23)\), one finds

\[(\nu H + \mu D^2)H^{(4)} + \alpha H'H'' + \beta H'^2 + (2\gamma H + \kappa - D^2)H'' + 2\gamma H'^2 = 0.
\]

(24)

This is an equation of the form \((14)\) with coefficients

\[A_1 = \frac{\alpha}{\nu H + \mu D^2}, \quad A_0 = 0, \quad B_0 = \frac{\beta}{\nu H + \mu D^2}.
\]
\[ C_2 = C_1 = 0, \quad C_0 = \frac{2\gamma H + \kappa - D^2}{\nu H + \mu D^2}, \]
\[ D_4 = D_3 = 0, \quad D_2 = \frac{2\gamma}{\nu H + \mu D^2}, \quad D_1 = D_0 = 0. \]

Substituting these coefficients into the linearization conditions \((6)-(15)\), one obtains the following results.

3.1 Case 1 : \(\nu = 0\)

If \(\nu = 0\), then equation \((24)\) is linearizable if and only if
\[ \alpha = \beta = \gamma = 0. \]

This means, with these conditions the original equation \((24)\) becomes the linear equation
\[ (\mu D^2)H^{(4)} + (\kappa - D^2)H'' = 0. \]

The solution of this equation is
\[ H(x - Dt) = C_1 \sin \sqrt{\frac{\kappa - D^2}{\mu D^2}} (x - Dt) + C_2 \cos \sqrt{\frac{\kappa - D^2}{\mu D^2}} (x - Dt), \]
where \(C_1\) and \(C_2\) are arbitrary constants. Hence,
\[ u(x, t) = C_1 \sin \sqrt{\frac{\kappa - D^2}{\mu D^2}} (x - Dt) + C_2 \cos \sqrt{\frac{\kappa - D^2}{\mu D^2}} (x - Dt). \]

3.2 Case 2 : \(\nu \neq 0\)

Case 2.1 : \(\gamma = 0\)

- If \(\nu \neq 0\), \(\gamma = 0\) and \(\beta = 0\), then equation \((24)\) is linearizable if and only if
  \[ \alpha = 0, \quad \kappa = D^2. \]

Next, to find the linear form of equation \((24)\) and its solutions.

Note that : For applying the theorems to our problem, here \(x = x - Dt\) and \(y(x) = H(x - Dt)\).

From \((16)\) one has
\[ 2 \frac{d\chi}{dx} - \chi^2 = 0. \]
Let us take its simplest solution $\chi = 0$. Then invoking (17), we let

$$\varphi = x.$$ 

Now the equations (18)-(19) are written as

$$\psi_{yy} = 0, \quad \psi_{xy} = 0,$$

and yield

$$\psi_y = K_0, \quad K_0 = \text{const}.$$ 

Hence,

$$\psi = K_0 y + K_1(x).$$ 

Since one can use any particular solution, we set $K_0 = 1$, $K_1(x) = 0$ and take

$$\psi = y.$$ 

Noting that (20) yields $\Omega = 0$, one can readily verify that the function $\psi = y$ solves equation (19) as well. Hence, one obtains the following transformations

$$\tilde{t} = x, \quad \tilde{u} = y.$$ (25)

Since $\Omega = 0$, equations (21) and (22) give

$$\tilde{\alpha} = 0, \quad \tilde{\beta} = 0.$$ 

Hence, the equation (24) is mapped by the transformations (25) to the linear equation

$$\tilde{u}^{(4)} = 0.$$ 

The solution of this linear equation is

$$\tilde{u} = C_0 + C_1 \tilde{t} + C_2 \tilde{t}^2 + C_3 \tilde{t}^3,$$

where $C_0, C_1, C_2$ and $C_3$ are arbitrary constants. That is

$$H(x - Dt) = C_0 + C_1(x - Dt) + C_2(x - Dt)^2 + C_3(x - Dt)^3.$$ 

Hence,

$$u(x, t) = C_0 + C_1(x - Dt) + C_2(x - Dt)^2 + C_3(x - Dt)^3.$$ 

- If $\nu \neq 0$, $\gamma = 0$ and $\beta = 3\nu$, then equation (24) is linearizable if and only if

$$\alpha = 4\nu, \quad \kappa = D^2.$$ 

Next, to find the linear form of equation (24) and it’s solutions. Considering (16) one has

$$2 \frac{d\chi}{dx} - \chi^2 = 0.$$ 

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Let us take its simplest solution $\chi = 0$. Then invoking (17), we let
\[ \varphi = x. \]

Now the equations (18)-(19) are written as
\[ \psi_{yy} = \frac{\nu \psi_y}{(\nu y + D^2 \mu)}, \quad \psi_{xy} = 0, \]
and yield
\[ \psi_y = K_0(x) \left( y + \frac{D^2 \mu}{\nu} \right). \]
Since $\psi_{xy} = 0$, that means $K_0(x) = K_0 = \text{const.}$ One arrives at
\[ \psi = K_0 \left( \frac{y^2}{2} + \frac{D^2 \mu}{\nu} y \right) + K_1(x). \]
Since one can use any particular solution, we set $K_0 = 1$, $K_1(x) = 0$ and take
\[ \psi = \frac{y^2}{2} + \frac{D^2 \mu}{\nu} y. \]
Observing that (20) yields $\Omega = 0$, one can readily verify that the function $\psi = \frac{y^2}{2} + \frac{D^2 \mu}{\nu} y$ solves equation (19) as well. Hence, one obtains the following transformations
\[ \tilde{t} = x, \quad \tilde{u} = \frac{y^2}{2} + \frac{D^2 \mu}{\nu} y. \quad (26) \]
Since $\Omega = 0$, equations (21) and (22) give
\[ \tilde{\alpha} = 0, \quad \tilde{\beta} = 0. \]
Hence, the equation (24) is mapped by the transformations (26) to the linear equation
\[ \tilde{u}^{(4)} = 0. \]
The solution of this linear equation is
\[ \tilde{u} = C_0 + C_1 \tilde{t} + C_2 \tilde{t}^2 + C_3 \tilde{t}^3, \]
where $C_0, C_1, C_2$ and $C_3$ are arbitrary constants. So that we obtain the implicit solution in the form
\[ \frac{H^2}{2} + \frac{D^2 \mu}{\nu} H = C_0 + C_1(x - Dt) + C_2(x - Dt)^2 + C_3(x - Dt)^3. \]
Hence,
\[ \frac{u^2}{2} + \frac{D^2 \mu}{\nu} u = C_0 + C_1(x - Dt) + C_2(x - Dt)^2 + C_3(x - Dt)^3. \]
Case 2.2: $\gamma \neq 0$

Equation (24) is linearizable if and only if

$$\alpha = 4\nu, \quad \beta = 3\nu, \quad \kappa = \frac{(2\gamma\mu + \nu)D^2}{\nu}. $$

Because of (16) one obtains

$$\frac{d\chi}{dx} - \frac{1}{2} \chi^2 = \frac{2\gamma}{5\nu}. $$

Solving this equation, one gets

$$\chi = 2\sqrt{\frac{\gamma}{5\nu}} \tan \left( \sqrt{\frac{\gamma}{5\nu}} (x + C) \right),$$

where $C$ is an arbitrary constant. Since one can use any particular solution, we set $C = 0$, so that

$$\chi = 2\sqrt{\frac{\gamma}{5\nu}} \tan \left( \sqrt{\frac{\gamma}{5\nu}} x \right).$$

Then invoking (17),

$$\frac{\varphi_{xx}}{\varphi_x} = 2\sqrt{\frac{\gamma}{5\nu}} \tan \left( \sqrt{\frac{\gamma}{5\nu}} x \right).$$

Thus,

$$\varphi_x = K_0 \sec^2 \left( \sqrt{\frac{\gamma}{5\nu}} x \right),$$

where $K_0$ is an arbitrary constant. By solving this equation, one obtains

$$\varphi = \sqrt{\frac{5\nu}{\gamma}} K_0 \tan \left( \sqrt{\frac{\gamma}{5\nu}} x \right) + K_1,$$

where $K_1$ is an arbitrary constant. One can choose $K_0 = \sqrt{\frac{\gamma}{5\nu}}$ and $K_1 = 0$. One arrives at

$$\varphi = \tan \left( \sqrt{\frac{\gamma}{5\nu}} x \right).$$

Now the equations (18)-(19) are written as

$$\psi_{xy} = 3\sqrt{\frac{\gamma}{5\nu}} \tan \left( \sqrt{\frac{\gamma}{5\nu}} x \right) \psi_y, \quad (27)$$

and

$$\psi_{yy} = \frac{\nu \psi_y}{D^2 \mu + \nu y}. \quad (28)$$

Equation (27) and (28) give

$$\psi = \sec^3 \left( \sqrt{\frac{\gamma}{5\nu}} x \right) \left[ \frac{y^2}{2} + \frac{D^2 \mu}{\nu} y \right].$$
One can readily verify that the function $\psi$ solves equation (19) as well. Hence, one obtains the following transformations

$$
\tilde{t} = \tan \left( \sqrt{\frac{\gamma}{5\nu}} x \right), \quad \tilde{u} = \sec^3 \left( \sqrt{\frac{\gamma}{5\nu}} x \right) \left[ \frac{y^2}{2} + \frac{D^2 \mu}{\nu} y \right].
$$

Equations (21) and (22) give

$$
\tilde{\alpha} = 0, \quad \tilde{\beta} = -9 \cos^8 \sqrt{\frac{\gamma}{5\nu}} x.
$$

Hence, the equation (24) is mapped by the transformations (29) to the linear equation

$$
\tilde{u}^{(4)} - 9 \cos^8 \sqrt{\frac{\gamma}{5\nu}} x \tilde{u} = 0.
$$

Since this is an unsolved linear equation, so that the result of this case we obtained only at the linear form of nonlinear equation (24).

4 Discussion and Conclusion

In the present work, we found some following solutions for nonlinear fourth-order partial differential equations (23).

- If $\nu = \alpha = \beta = \gamma = 0$, then the solution of (23) is

$$
u(x,t) = C_1 \sin \sqrt{\frac{\kappa - D^2}{\mu D^2}} (x - Dt) + C_2 \cos \sqrt{\frac{\kappa - D^2}{\mu D^2}} (x - Dt).
$$

- If $\nu \neq 0$, $\alpha = \beta = \gamma = 0$ and $\kappa = D^2$, then the solution of (23) is

$$
u(x,t) = C_0 + C_1 (x - Dt) + C_2 (x - Dt)^2 + C_3 (x - Dt)^3.
$$

- If $\nu \neq 0$, $\alpha = 4\nu$, $\beta = 3\nu$, $\gamma = 0$ and $\kappa = D^2$, then the solution of (23) is

$$
u^2(x,t) - \frac{D^2 \mu}{\nu} \nu(x,t) = C_0 + C_1 (x - Dt) + C_2 (x - Dt)^2 + C_3 (x - Dt)^3.
$$

- If $\gamma \neq 0$, $\alpha = 4\nu$, $\beta = 3\nu$ and $\kappa = \frac{(2\mu + \nu)D^2}{\nu}$, then the linear form of (23) is

An interesting aspect of the results in this paper is that the class of exact solutions of the original nonlinear problems, which can not find by the classical methods.
5 Acknowledgements

This work was financially supported by Faculty of Science, Naresuan University. The author wishes to express thanks to Prof. Dr. Sergey V. Meleshko, Suranaree University of Technology for his guidance during the work.

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