Excitation of electrostatic standing wave in the superposition of two counter propagating relativistic whistler waves

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Abstract
The problem of standing wave formation by superposing two counter-propagating whistler waves in an overdense plasma, studied recently by Sano \textit{et al} (2019 \textit{Phys. Rev. E} \textbf{100}, 053 205 and 2020 \textit{Phys. Rev. E} \textbf{101}, 013 206), has been revisited in the relativistic limit. A detailed theory along with simulation has been performed to study the standing wave formation in the interaction of two counter propagating relativistically intense whistler waves. The relativistic theory explains such interaction process more precisely and predicts correct field amplitudes of the standing wave for a much wider range of physical parameters of the problem as compared to its non-relativistic counterpart. The analytical results are compared with 1-D Particle-in-Cell (PIC) simulation results, performed using OSIRIS 4.0. The results are of relevance to ion heating and fast ignition scheme of inertial confinement fusion.

1. Introduction

Acceleration and heating of plasma ions by using large amplitude electromagnetic waves or ultra-intense laser pulses is a very useful and efficient method to directly couple the electromagnetic field energy to ions and thermalize them \cite{1–6}. The basic idea of one such ion heating mechanism associated with whistler waves has been recently examined by Sano \textit{et al} \cite{5, 6}. They have investigated the problem of ion heating in the interaction of whistler waves analytically by adopting a simple theoretical model and also numerically by performing Particle-in-cell simulation. Whistler waves have some important and essential characteristic features which make this electromagnetic plasma wave mode a suitable candidate for this purpose \cite{7, 8}. In this ion acceleration and heating mechanism, initially a standing wave is formed by the superposition of two counter propagating whistler waves. The ions are accelerated by the electrostatic field of the standing wave, resulting in the excitation of an unstable ion wave through streaming instabilities \cite{8–10}, which eventually heats the ions through collapse of the ion wave itself. Ion heating by whistler waves has importance in many research field starting from laboratory plasmas to space and astrophysical plasmas, in the diagnostics of the material structures, medicine, security and industrial engineering \cite{11, 12}. Also in plasma-based fusion science, it is necessary to heat the plasma ions for efficient fusion reaction to occur \cite{1–4}.

Whistler wave is basically a low frequency right circularly polarized (RCP) electromagnetic mode having a frequency which is lower than the electron cyclotron frequency associated with the ambient constant magnetic field. Such a mode is excited in a plasma system when an external magnetic field with sufficient amplitude is applied along the propagation direction. The characteristic features and propagation characteristics of whistler waves have been studied extensively in the past \cite{5–7, 8, 11–13}. This mode has a unique attribute of non-existence of cut-off which enables it to propagate through over-dense plasma system \cite{5, 14–17}. The relativistic theory developed by Akheizer and Polovin in 1956 \cite{7} clearly shows how even in over-dense plasma system whistler waves can readily propagate without any cut-off density. In a recent investigation the transmission and propagation properties of whistler waves in over dense plasma system have been studied by Luan \textit{et al} \cite{13}.
Particle-in-cell simulation. In another investigation the interaction of relativistic whistler waves with dense plasma has been performed [18]. The impact of external axial magnetic field on the proton acceleration has been discussed when the target plasma is irradiated by the whistler modes. It has been shown that proton can be accelerated to around 15 MeV energy with an external magnetic field of the order of 20–25 kT with femtosecond laser of intensity around $5 \times 10^{18}$ W cm$^{-2}$.

Our present investigation is primarily motivated by the work of Sano et al [5, 6]. The non-relativistic theory which they have developed for the excitation of standing waves by superposition of whistler waves, in single ion species plasma, reveals beautifully the interesting physical processes associated with it and demonstrates the ion heating process. It has been shown that ion can be heated up to several keV by the collapse of the standing wave, for their chosen system parameters. They have further extended their analysis by choosing a multi-ion species target like deuterium-tritium (DT) ices and solid ammonia borane ($H_4BN$) and reported similar findings [6]. However, a complete and general theory of high intensity laser matter interaction process must take relativistic effects into account. For ultra intense laser pulses, the relativistic effects can significantly change the propagation characteristics of whistler waves and also the energy exchange process with background ions. The relativistic theory enables us to correctly estimate the field amplitude of the standing wave for a much wider range of physical parameters (laser intensity and external magnetic field) as compared to its non-relativistic counterpart.

In this paper we have attempted to establish a relativistic theory of superposition of whistler waves and consequent standing wave formation. In contrast to the non-relativistic theory, the refractive index of the plasma associated with relativistically intense whistler waves is not a constant, but it varies with the relativistic Lorentz factor associated with the electron fluid velocity [7]. This sole feature differentiates this present investigation on standing wave formation from its non-relativistic counterpart and brings out the new physics associated with it. One essential outcome of the relativistic theory is that, it restricts the possible values of the external magnetic field which are needed to excite the standing wave, for a given laser intensity. It turns out that for intensities of the order $\sim 10^{21} W/cm^2$ or more, the required magnetic field becomes several times higher than that allowed by non-relativistic theory. Existence of such strong external magnetic field helps to greatly enhance the conversion efficiency of the laser energy to the plasma ions or electrons [19, 20]. In astrophysical objects like pulsars and magnetars, very high magnetic field of the order of giga gauss to tera gauss can be observed [21–23]. Experimentally, a highest magnetic field of 70 kT has been reported so far in interaction of high intensity laser ($9 \times 10^{19} W/cm^2$) with a plasma of density $10^{19}$–$10^{20}$ cm$^{-3}$ [24]. So, our present investigation is quite timely and it has its relevance to laboratory laser matter interaction as well as in the astrophysical situations.

The paper is organized as follows. In section 2, we present the dispersion relation for relativistic whistler waves by following the theory developed by Akheizer and Polovin [7]. This is presented here for the sake of completeness. In third section, the relativistic theory is developed to estimate the amplitude of the electric field of the standing wave which is excited in the superposition of counter propagating whistler waves. The results of our investigation in the non-relativistic limit is also discussed and compared with the earlier works. In section 4, we present 1-D PIC simulation results obtained using OSIRIS 4.0 and a comparison with our theoretical findings. Finally we summarize our work in section 5.

2. Basic Formulation

The derivation presented in this section has been described earlier by Akheizer and Polovin [7]. As stated in the introduction here we present it for the sake of completeness and to get a clear idea of the problem. The relativistic fluid-Maxwell set of equations describing a cold plasma in the presence of an external magnetic field $B_0$ can be combined to yield the following evolution equations for the fluid momentum, in the stationary wave frame as

\[
\frac{d^2\rho_x}{d\tau^2} + \beta^2\Omega^2 \frac{d}{d\tau}\left(\frac{u_x}{\beta - u_z}\right) + \frac{\beta^2}{\beta^2 - 1}\beta - u_z\omega_p^2 u_x = 0
\]

\[
\frac{d^2\rho_y}{d\tau^2} - \beta^2\Omega^2 \frac{d}{d\tau}\left(\frac{u_y}{\beta - u_z}\right) + \frac{\beta^2}{\beta^2 - 1}\beta - u_z\omega_p^2 u_y = 0
\]

\[
\frac{d^2(\beta \rho_x - \sqrt{1 + \rho^2})}{d\tau^2} + \frac{\beta^2}{\beta - u_z}\omega_p^2 u_z = 0
\]

where, $\tau = (t - z/V)$, $\beta = p/mc$, $u = v/c$, $\Omega = eB_0/mc$, and $\beta = V/c$ being the normalized phase velocity of the plasma wave, $z$ being the direction of the wave vector which coincides with the direction of the external field. Further, the components of electromagnetic fields may be computed in terms of fluid momenta as
\[
\mathbf{\dot{B}} = -\frac{1}{\omega\beta} \left( \mathbf{\hat{z}} \times \frac{d\mathbf{\hat{p}}}{dt} \right) + \left( \frac{\mathbf{\Omega} - \mathbf{u}\mathbf{\Omega}'}{\beta - u_z} \right)
\]

(4)

and

\[
\mathbf{\dot{B}} = \frac{1}{\beta} (\mathbf{\hat{z}} \times \mathbf{\dot{E}}) + \mathbf{\Omega}'
\]

(5)

where, \( \mathbf{\dot{B}} = e\mathbf{B}/m\omega_c, \mathbf{\dot{E}} = e\mathbf{E}/m\omega_c, \) and \( \mathbf{\Omega}' = \mathbf{\Omega}/\omega = e\mathbf{B}_0/m\omega_c \).

For purely transverse oscillation \( u_z = 0, \) and if we require bounded solution then Eq.(3) gives \( \rho^2 = \) constant \([7, 25]\). This implies that \( u_z^2 + u_y^2 = u^2 = \) constant. Substituting \( \mathbf{u} = u(\cos \omega t \mathbf{\hat{x}} + \sin \omega t \mathbf{\hat{y}}) \) in one of the the momentum equations [equation (1) or (2)] we obtain the following relativistic dispersion relation for a purely transverse mode \([7, 8, 17]\)

\[
\omega = \frac{\Omega}{2} \pm \sqrt{\left( \frac{\Omega^2}{4} + \frac{\beta^2\omega_p^2}{\beta^2 - 1} \right)^{1/2}}
\]

(6)

where, \( \Omega = \sqrt{1 - u^2} \) and \( \omega_p^2 = \omega_c^2 = \sqrt{1 - u^2} \) are respectively the relativistically correct cyclotron frequency and plasma frequency. The dielectric constant of the plasma is then given by

\[
\varepsilon = \beta^{-2} = 1 - \frac{\omega_p^2}{\omega^2 - \omega\Omega}
\]

(7)

It is similar to the non-relativistic expression, with plasma frequency and cyclotron frequency replaced by their relativistically correct expressions. For \( \omega < \Omega, \) this is the dispersion relation for the Whistler branch, which may also be written differently as

\[
\varepsilon = 1 - \frac{\omega_p^2}{\gamma - \Omega'}
\]

(8)

where, \( \omega_p' = \omega_p/\Omega, \Omega' = \Omega/\omega. \) From this expression it is evident that when \( \gamma < \Omega' \) there is no cut-off for the whistler waves even in highly dense plasma system. Finally using equations (4) and (5), the magnitude of electric and magnetic field amplitude in terms of fluid velocity (and momentum) may be respectively written as

\[
\mathbf{E} = \rho (\Omega' \sqrt{1 - u^2} - 1)
\]

(9)

\[
\mathbf{B} = \frac{\mathbf{E}}{\beta} = \frac{\rho}{\beta} (\Omega' \sqrt{1 - u^2} - 1)
\]

(10)

These fields are entirely transverse.

### 3. Standing wave formation: superposition of two counter propagating whistler waves

It is quite evident from the analysis in the preceding section, that for a single whistler wave propagating through plasma, formation of electrostatic standing wave is not possible. Whistler waves do not couple to longitudinal perturbations. If however, we launch two counter propagating whistler waves from two sides of a plasma slab, an electrostatic standing wave will form due to the superposition of these counter propagating waves. In this section we shall discuss elaborately the excitation mechanism of such standing wave in the superposition of two counter propagating relativistically intense whistler waves. The right circularly polarized forward (+ve) and backward propagating (−ve) whistler modes in the presence of an external constant magnetic field \( \mathbf{B}_0 \) directed along the positive \( z \)-axis, can be described as

\[
\mathbf{E}_\pm = \mathbf{E}_\pm \exp i(\omega t \mp k\hat{z})(\hat{x} + \hat{y}),
\]

(11)

\[
\mathbf{B}_\pm = \mp \mathbf{B}_\pm \exp i(\omega t \pm k\hat{z})(\hat{x} - i\hat{y}).
\]

(12)

The associated fluid velocities are

\[
\mathbf{u}_\pm = \mathbf{u}_\pm \exp i(\omega t \pm k\hat{z})(\hat{x} - i\hat{y}).
\]

(13)

These two counter propagating whistler waves superpose to form a standing wave pattern in the plasma system producing an electrostatic field \( \mathbf{E}_\pm \) in the longitudinal direction. The longitudinal electric field is a self-consistent electrostatic standing wave solution in the presence of two counter-propagating whistler waves; it arises as a consequence of momentum conservation which may be understood as follows. Each whistler by itself do not give any longitudinal momentum to the electron fluid. In the presence of two counter-propagating whistler waves, the oscillating transverse velocity \( \mathbf{u}_\pm \) due to one whistler couples with the oscillating magnetic field \( \mathbf{B}_\pm \) of the counter-propagating whistler resulting in a force on the electron fluid along the longitudinal direction. Now application of conservation of momentum, results in the appearance of a longitudinal electric field such that there is no net force on the electron fluid along the longitudinal direction; the electric field being
computed from the force free condition on the electron fluid along the longitudinal direction. This longitudinal electric field is a time stationary solution; the electron density rearranges itself to generate this field, so that there is no velocity of electrons in the longitudinal direction. As stated above, the expression for the electric field of this standing wave is calculated by applying the force free condition to the \( z \)-component of the electron momentum equation and is given by

\[
0 = \dot{E}_z + (u \times \dot{B})_z
\]

\[
\Rightarrow \dot{E}_z = -[(u_+ + u_-) \times (\dot{B}_+ + \dot{B}_-)]_z
\]

\[
= -[(u_{+x} \dot{B}^y_{-y} + u_{-x} \dot{B}^y_{+y}) - (u_{-y} \dot{B}^y_{-x} + u_{x} \dot{B}^y_{+y})]
\]

\[
= -(u_+ \dot{B}_{-x} + u_- \dot{B}_{+x}) \sin(2k_0Nz)
\]

(14)

where \( k = \omega / V = (\omega / c)/(V / c) = k_0 / \beta = k_0 N \), \( N \) being the refractive index. Here \( \dot{B}_\perp \) is related to \( u_\perp \) through equation (10), thus the only remaining unknown in the above expression is the constant \( u_\perp \). This unknown constant is determined from the boundary conditions, which through Fresnel equations, relate the transmitted amplitude of the whistler wave (\( \dot{E}_\perp \)) given by equation (9)) to the electric field amplitude of the incident laser pulse. The transmitted fields are given by Fresnel equations [26] as \( \dot{E}_\perp = 2a_0/(N + 1) \) where \( a_0 = eE_0/(m\omega c) \) is the free space laser amplitude. Here \( N \), the refractive index of the plasma, which can be obtained from the relativistic dispersion relation derived in the previous section, is further related to \( u_\perp \) as

\[
N = \beta^{-1} = \left[ 1 - \frac{\omega'^2}{1 - \Omega'^2} \right]^{1/2}
\]

(15)

where \( \omega' = \omega / \sqrt{\Omega} \) and \( \Omega' = \Omega / \omega'. \) This is important, as it differentiates the relativistic theory from its non-relativistic version. Putting all the components together, leads to an algebraic equation for \( u_\perp \) as \( f(x) = 0 \), where \( f(x) \) is given by

\[
f(x) = \frac{1}{2a_0} \left[ 1 + \left( 1 - \frac{\omega'^2 x^2}{1 - \Omega'^2} \right)^{1/2} \right] \times \left[ \sqrt{1 - \frac{x^2}{\Omega' x}} (\Omega' x - 1) \right] = 1
\]

(16)

with \( x = \sqrt{1 - u_\perp^2} \). This equation is numerically solved for \( u_\perp \) and the values of \( \dot{E}_\perp \) and \( \dot{E}_z \) thus obtained are compared with simulation, as discussed in the next section. (Although the above equation yields multiple values of \( u_\perp \), the simulation results, as expected, correspond to the lowest positive value.)

The analysis becomes comparatively simpler in the non-relativistic limit (\( u_\perp \ll 1, x \approx 1 \)), which gives

\[
\ddot{u} \pm = \frac{2a_0}{(N + 1)(\Omega' - 1)} \sin 2k_0Nz
\]

(17)

with \( a_0 = eE_0/m\omega c \) [13].

A comparison has been made between equation (16) and equation (17) for the fluid velocities in the relativistic and non-relativistic limit, in figures 1 and 2. It is observed that in both cases increasing the external magnetic field or the plasma density reduces the fluid velocity. The difference in the fluid velocities results in different standing wave field amplitudes for relativistic and non-relativistic cases. This has been discussed elaborately in the following sections.

In the non-relativistic wave, the refractive index \( N \) becomes constant as evident from equation (15). Thus using \( \dot{E}_\perp = u_\perp (\Omega' - 1) \) and \( \dot{B}_\perp = \dot{E}_\perp / \beta \), equation (14) gives the expression for electrostatic field, under non-relativistic approximation, as

\[
\dot{E}_z = -\frac{1}{\beta} [u_+ \dot{E}_- + u_- \dot{E}_+] \sin 2k_0Nz
\]

\[
= -\frac{2\dot{E}_\perp \dot{E}_z}{\beta (\Omega' - 1)} \sin 2k_0Nz
\]

\[
= -\frac{8N a_0^2}{(N + 1)^2 (\Omega' - 1)} \sin 2k_0Nz
\]

This is exactly the same expression for the static electric field generated by superposition of whistler waves as obtained in the non-relativistic theory developed by Sano et al [5]. In the next section, we present a comparison of the relativistic theory with 1-D Particle-in-cell simulation results.
Here we present the results obtained using fully relativistic 1D PIC simulation which is performed to study the interaction of two oppositely moving right circularly polarized electromagnetic waves in a dense plasma. The code used is fully electromagnetic and massively parallel code 'OSIRIS 4.0' \cite{27, 28}. The geometry of the problem is similar to simulation performed by Sano et al \cite{5} Here a slab of preformed plasma of density $\sim 3.37 \times 10^{22}/\text{c.c.}$ corresponding to the solid hydrogen density has been considered. The system is extended $\sim 69.5 \mu\text{m}$ along the longitudinal direction. Number of particles per cell is $\sim 200$ and the chosen time step is $\sim 0.005 \, \omega_p^{-1}$. The laser profile is considered to be Gaussian with both longitudinal rise and fall to be of $\sim 9 \mu\text{m}$ and it is used throughout all our simulation runs. Number of grids are $\sim 3000$ along the longitudinal direction with the resolution of order $\sim \lambda_0/40$. The open boundary conditions are taken for both the fields and the particles. The EM waves are right circularly polarized with the transverse field expressed as $E_t = \sqrt{E_x^2 + E_y^2}$. Before we discuss the standing wave formation by superposition of counter propagating whistler waves, the transmission characteristics of whistler waves in overdensed plasma medium is investigated first.

### 4.1. Transmission of whistler waves

In the first simulation experiment, a single circularly polarized electromagnetic wave is launched from the left side of the plasma slab. The transmitted field amplitude of the wave inside the plasma is determined from the Fresnel equations and the relativistic theory of whistler wave propagation, as discussed in the previous section.
The 1D PIC simulation shows that for the chosen parameters $a_0 = 2.65, \frac{B_0}{B_c} = 7.47, \frac{n_0}{n_c} = 19.3$ with $B_c = \frac{m\omega_c}{e}$ and $n_c = (\frac{m\omega^2}{4\pi e^2})$, the measured maximum field amplitude of the transmitted wave ($eE_y/m\omega c$) is around $\sim 1.7678$. According to the relativistic theory, the calculated transmitted field amplitude is also around $\sim 1.7676$ corresponding to the lowest root of fluid velocity ($u = 0.20$), thus closely matching the measured value. The measured value of $u = 0.203$ which closely matches the relativistic value (the value calculated using non-relativistic expression equation (17) is $u = 0.27$). The field profiles in the vacuum and in the plasma slab are respectively shown in figures 3 and 4.

The 1D PIC simulation has also been performed for another set of parameters. The measured maximum transmitted field amplitude for the parameters $a_0 = 18.0, \frac{B_0}{B_c} = 50.0, \frac{n_0}{n_c} = 19.3$ is around $\sim 1.65074$ which also closely matches with the relativistic theory ($eE_y/m\omega_c \sim 16.5072$) corresponding to the lowest root ($u = 0.34$). The measured value of $u = 0.34$ which closely matches the relativistic value (the value calculated using non-relativistic expression equation (17) is $u = 0.51$). The results are shown in figures 5 and 6. In the following subsection we discuss the standing wave formation when we launch right circularly polarized EM waves from both sides of the plasma slab.
4.2. Superposition of whistler waves

In the next simulation run we consider that the plasma slab is irradiated by two counter propagating right circularly polarized waves launched from both sides in the longitudinal direction. The circularly polarized waves are propagating in the \( \pm z \) direction after they are launched from the two opposite boundaries. The electric field of the standing wave generated in the longitudinal direction \( \left( E_z \right) \) as a result of the superposition of the two counter propagating EM waves is observed.

The PIC simulation is performed throughout the paper considering the relativistic effect. We compare the results obtained in relativistic and non-relativistic theory with our simulation results. It turns out that relativistic results give a good matching with simulation over a long range of parameter space. In this simulation experiment we consider an external magnetic field of strength \( \sim 100 \text{ kT} \) which is applied along the propagation direction. A laser pulse with relativistic intensity of \( a_0 = 2.65 \) is taken, corresponding to a laser intensity of \( I = 3 \times 10^{19} \text{ W cm}^{-2} \) with 30 fs pulse-width and wavelength of 0.8 \( \mu \text{m} \). The simulation results are shown in figures 7 and 8. It is observed that the normalized maximum electrostatic field amplitude of the excited standing wave is around \( \sim 1.5 \) for \( a_0 = 2.65 \) as evident from figure 7. The slow spatial variation of electrostatic field \( E_z \) seen in simulation is due to the Gaussian envelope of the incoming light pulse. This maximum amplitude matches with the field

**Figure 5.** Incident right circularly polarized EM wave field \( (eE_y / m_\omega c) \) in vacuum corresponding to the parameters: \( a_0 = 18.0, B_0 / B_c = 50.0, n_0 / n_c = 19.3 \).

**Figure 6.** The transmitted and reflected EM wave fields \( (eE_y / m_\omega c) \) in the dense plasma medium corresponding to the parameters: \( a_0 = 18.0, B_0 / B_c = 50.0, n_0 / n_c = 19.3 \).
amplitude as predicted by the relativistic theory [figure 8] corresponding to the lowest root $u_{\pm} = 0.20$ obtained by solving equation (16). This differs from the maximum field amplitude obtained using non-relativistic theory which predicts $eE_z/m\omega c \sim 1.9$. This disparity occurs because in the non-relativistic situation the refractive index $N$ is constant and related to the electric field amplitude via a simple relation $E = 2a_0/N + 1$. Therefore, it should be emphasized that the relativistic theory which is developed without any approximation is more accurate and explains the correct physics of standing wave formation. This assertion is confirmed by our 1D PIC simulation results.

The simulation has also been performed with three other sets of parameters. For the parameter set $(a_0 = 2.65, B_0/B_c = 7.47, n_0/n_e = 19.3)$ with $B_c = m\omega c/e$ and $n_e = (m\omega^2/4\pi e^2)$.

**Figure 7.** Snap shots of the longitudinal electrostatic field $eE_z/m\omega c$ (in blue) and laser electric field $eE_y/m\omega c$ (in red) from simulation at different time steps. The first panel shows the laser profile when the pulses launched from both sides enters into the plasma. The middle and the third panel depicts the starting and formation of the standing wave in the interaction of counter propagating whistler waves. The chosen parameters for this simulation run are $a_0 = 2.65, B_0/B_c = 7.47, n_0/n_e = 19.3$ with $B_c = m\omega c/e$ and $n_e = (m\omega^2/4\pi e^2)$.

**Figure 8.** Comparison of the standing wave field profiles obtained from theory with the PIC simulation. The longitudinal electrostatic field $eE_z/m\omega c$ of the standing wave is represented by blue curve as obtained from simulation at time $960\omega^{-1}$, by red curve obtained from relativistic theory with the lowest root of fluid velocity ($u = 0.20$) and the non-relativistic result is given by green curve. The laser amplitude, magnetic field and plasma density are chosen as $a_0 = 2.65, B_0/B_c = 7.47, n_0/n_e = 19.3$ with $B_c = m\omega c/e$ and $n_e = (m\omega^2/4\pi e^2)$.
The amplitude is $1.1 \times 10^{20}$ W cm$^{-2}$. Similar to the previous case the field amplitude corresponding to this lowest root gives the closest matching with the simulation result [figures 9 and 10]. Same conclusion can be drawn for the other set ($a_0 = 15.0$, $B_0/B_c = 45$, $n_0/n_c = 19.3$) as well, which is evident from the figures 11 and 12. The chosen parameters here correspond to the magnetic field of ~ 602 kT with laser amplitude of $9.7 \times 10^{20}$ W cm$^{-2}$. In addition, the results corresponding to magnetic field of ~ 670 kT and laser amplitude of $1.4 \times 10^{21}$ W cm$^{-2}$ [$a_0 = 18.0$, $B_0/B_c = 50$, $n_0/n_c = 19.3$] is also shown in figures 13 and 14. In all cases, simulation results closely match with the relativistic theory of electrostatic standing wave formation.

5. conclusion

We have presented a general theory of standing wave excitation in the superposition of two counter propagating relativistic whistler waves. This theory is supplemented by fully relativistic electromagnetic 1D PIC simulations performed using the code OSIRIS 4.0. In the limiting cases we recover results established by Sano et al [5] in the non-relativistic limit. However, in contrast to the non-relativistic theory, it is found that the relativistic theory...
predicts a lower value of electrostatic field for a particular set of parameters. It must be emphasized that correct explanation of the interaction physics can be obtained only with the relativistic theory. The occurrence of lower value of electrostatic field may be understood as a consequence of relativistic increase of electron mass making them ‘sluggish’, thus resulting in lower values of electron current. This leads to lower amplitudes of Whistler fields, which in turn reduces the amplitude of the electrostatic fields. The amplitude of the standing wave depends on the three parameters $a_0$, $B_0/B_c$ and $n_0/n_c$. As far as the relativistic theory of electrostatic standing wave formation goes, there is no hard restriction on the parameter space. It is true that when $\frac{B_0}{B_c} \to 1$ (in the non-relativistic limit) or $\frac{n_0}{n_c} \to \gamma$ (in the relativistic limit), strong cyclotron resonance takes place leading to strong electron heating. However, the purpose of the present article is not to study the direct heating of electrons through cyclotron resonance, but to investigate the affect of relativistic physics on electrostatic standing wave formation through superposition of counter-propagating whistler waves which will eventually accelerate the ions resulting in their heating. This latter part on ion heating is not a part of our present study. Therefore in the present study, we have restricted ourselves to the cold plasma limit, thereby choosing parameters in such a way

Figure 11. Longitudinal electrostatic field $eE_z/m_\infty \omega_c$ (in blue) and laser electric field (in red) from simulation. Here we choose $a_0 = 15.0, B_0/B_c = 45.0, n_0/n_c = 19.3$.

Figure 12. Comparison of longitudinal electrostatic field $eE_z/m_\infty \omega_c$ from simulation (in blue) at time $780 \omega_c^{-1}$, from relativistic theory with lowest root $a = 0.32$ (in red) and from non-relativistic theory (in green). The chosen parameters are $a_0 = 15.0$, $B_0/B_c = 45.0$, $n_0/n_c = 19.3$. 
that we are far away from cyclotron resonance. In all our simulation runs, we have ensured that 
\[ \frac{B_0}{B_c} \ll \gamma, \]
where \( \gamma \) is the relativistic factor associated with the amplitude of transverse electron fluid velocity ‘\( u \)’.

The relativistic theory of formation of electrostatic standing wave, presented in section–3, is a time stationary theory with infinitely massive ions (the electrostatic standing wave arises as a result of longitudinal momentum conservation of the electron fluid, in the presence of counter–propagating whistler waves), whereas the 1D PIC simulation, records the time evolution of the electrostatic wave as the counter-propagating whistler waves (excited by the two incident laser pulses) approach each other, superpose, cross and move away from each other. As a result, in simulation, the amplitude of the electrostatic wave grows, reaches a maximum and eventually dissipates. In section–4, maximum amplitude of the electrostatic wave observed in simulation is compared with the theoretical value and is found to agree. Beyond this stage, the theory is not applicable; in the theoretical formulation the whistler waves exist forever, which is not so in simulation.

It is however possible to estimate the time of decay of the electrostatic standing wave, due to ion motion alone by taking into account finite ion mass. Ballistic motion of ions due to the standing wave electric field, leads to a decay time of order 
\[ \omega t \sim \left[ \frac{4\pi}{N} \frac{1}{rE_{ion}/m_e} \frac{M}{m} \right]^{1/2}, \]
where \( N \) is the relativistically correct refractive index.

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**Figure 13.** Longitudinal electrostatic field \( eE_z/m \omega_c \) (in blue) and laser electric field (in red) from PIC simulation. Here we choose \( a_0 = 18.0, B_0/B_c = 50, n_0/n_e = 19.3 \).

**Figure 14.** Comparison of longitudinal electrostatic field \( eE_z/m \omega_c \) from simulation (in blue) at time 772.5 \( \omega^{-1} \), from relativistic theory with lowest root \( u = 0.34 \) (in red) and from non-relativistic theory (in green). Here we choose \( a_0 = 18.0, B_0/B_c = 50.0, n_0/n_e = 19.3 \).
(λ/λ₀ = 1/2N is the standing wave wavelength), \( eE_{\text{stat}}/m \) is the amplitude of the electrostatic field and \( m/M \) is the electron to ion mass ratio. For our set of parameters with \( M/m \sim 1836 \), this \( \omega t \), which is to be measured from the time after the standing wave is fully formed, lies in the range of several tens of laser periods. In our simulation, the standing wave decays much faster due to moving away of whistler waves from the region of superposition. If the whistler waves are maintained for a longer time, then the standing wave will take several tens of laser periods to decay due to ion motion alone.

Our main concern in this paper is to show that the relativistic theory predicts different field amplitudes of the electrostatic standing wave electric field as compared to non-relativistic theory of Sano et al. The field amplitude measured in 1D PIC simulation agrees with relativistic theory. From energy and momentum conservation, partition of the energy contained in the electrostatic standing wave, to electrons (\( \epsilon_e \)) and ions (\( \epsilon_i \)) may be respectively estimated as \( \epsilon_e \sim \frac{1}{1 + \frac{m}{R}} E \) and \( \epsilon_i \sim \frac{R}{1 + \frac{m}{R}} E \), where \( R \) is the electron to ion mass ratio and \( E \sim E_{\text{stat}}/8\pi \) is the density energy in the electrostatic wave, \( E_{\text{stat}} \) being the amplitude of the wave. This estimate however does not take into account the ion motion as the standing wave is forming. Detail work is in progress for the investigation of the time evolution of the standing wave, due to ion motion and the energy partition between electrons and ions (heating of electrons and ions); this will be reported in a future publication.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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