ON THE STABILITY OF WEIGHT SPACES OF ENVELOPING ALGEBRA IN PRIME CHARACTERISTIC

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Abstract. By the result of Dixmier, any weight space of enveloping algebra of Lie algebra $L$ over a field of characteristic 0 is $adL$ stable. In this paper we will show that this result need not be true, if $F$ is replaced by a field of prime characteristic. A condition will be given, so a weight space will be $adL$ stable.

1. Introduction

Let $L$ be a finite dimensional Lie algebra over a field $F$, $U(L)$ its enveloping algebra with center $Z(U(L))$. For each linear form $\lambda : L \rightarrow F$ we denote $U(L)_\lambda = \{ u \in U(L) \mid [x,u] = \lambda(x)u \ \text{for all} \ x \in L \}$. Clearly $U(L)_\lambda$ is a submodule of $U(L)$ and $U(L)_\lambda U(L)_\mu \subseteq U(L)_{\lambda + \mu}$. We call $U(L)_\lambda$ a weight space of $U(L)$ and the sum of all $U(L)_\lambda$ is direct and is denoted by $Sz(U(L))$, the semi-center of $U(L)$. Semi-center was introduced by Dixmier [4, 4.3] and its crucial properties in characteristic 0 is that any two sided ideal of $U(L)$ has non trivial intersection with semi-center ( [4, 4.4.1] ). If $L$ is nilpotent or $[L,L] = L$ then $Z(U(L)) = Sz(U(L))$. In addition if $L$ is nilpotent and $charF = 0$, then by [4] $Sz(U(L))$ is factorial domain and if $charF = p > 0$ then $Sz(U(L))$ is again factorial by [1]. If $charF = 0$ and $L$ is solvable, then $Sz(U(L))$ is factorial due to Moeglin [5]. However, if $charF = p > 0$, then by recent result [2], it is shown that $Sz(U(L))$ need not be factorial.

Consider $L = Fx + H$ a Lie algebra over an algebraically closed field $F$ where $H$ is an ideal in $L$. A weight space $U(H)_\lambda$ is called $adL$ stable if $[L, U(H)_\lambda] \subseteq U(H)_\lambda$. By Dixmier [4, 1.3.11], when $charF = 0$, $U(H)_\lambda$ is always $adL$ stable which implies that $Sz(U(H))$ is also $adL$ stable. The techniques used by Dixmier are characteristic zero dependent and so we cannot adapt them for the case of $charF = p > 0$. In this paper we focus on the problem related stability of $U(H)_\lambda$ in case $L$ is a finite dimensional Lie algebra over a field $F$ of char$F = p > 0$. Unlike in the $charF = 0$ case, it appears that $U(H)_\lambda$ is not necessarily $adL$ stable. We will provide conditions, so that $U(H)_\lambda$ will be $adL$ stable, as well we will show a surprising result that $Sz(U(H))$ is $adL$ stable if and only if $U(H)_\lambda$ is $adL$ stable for every weight $\lambda$. 

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2. Main result

Let $L = Fx + H$ be a finite dimensional Lie algebra over a field $F$ and $H$ an ideal in $L$. As was mentioned, by [2] 1.3.11 if $\text{char} F = 0$ then $U(H)_{\lambda}$ is ad$L$ stable. The following discussion will show that this result need not be true if $\text{char} F = p > 0$. For the simplicity we assume $p = 3$, although the techniques we use, can be adapted for any $p > 0$.

**Result 2.1.** Let $L$ be 5-dimensional Lie algebra over a field $F$ of char $F = 3$, with the following multiplication table

\[
\begin{align*}
[x, e_1] &= 0, \ [x, e_2] = e_2, \ [x, e_3] = 2e_3, \ [y, e_1] = e_3, \ [y, e_2] = e_1, \ [y, e_3] = e_2, \\
[x, y] &= 2y, \text{ the rest of products 0.}
\end{align*}
\]

The subspace $K \equiv \text{span}\{e_1, e_2, e_3\}$ is an ideal of $L$ and consider $L = Fx + H$, $H = Fy + K$. Clearly $H$ is a codimension one ideal in $L$. Then by [2] Section 9

\[S_z(U(L)) = U(K)^{ad_L}[x^p - y^p], \quad Z(U(H)) = U(K)^{ad_L}[y^p - y^p].\]

Therefore $S_z(U(L)) = Z(U(H))[x^p - x]$ and using direct calculations we can show that

\[Z(U(H)) = F[y^p - y^p, e_1, e_2, e_3, e_1e_3 + e_2^2 + e_3e_2].\]

Consider $u \equiv e_1 + e_2 + e_3$ and $v \equiv e_3^2 + e_1e_3 + 2e_2e_3 + 2e_2^2$. Now $[y, u] = u$ and $[y, v] = 2e_1e_2 + e_1^2 + e_1e_2 + 2e_1e_3 + 2e_2^2 + e_2e_3 = 2e_2^2 + 2e_1e_3 + e_2e_3 + e_3^2 = 2v$.

Hence $u \in U(H)_{\lambda}$ (the weight space of $U(H)$ related to eigen value $1$) and $v \in U(H)_{\lambda}$. It can be seen that no more semi-invariant for $U(H)$ exists and $S_z(U(H)) = Z(U(H))[u, v]$. But then $[x, u] = e_2 + 2e_3$ and this implies that $u$ is not semi-invariant of $U(L)$ and consequently $U(H)_{\lambda}$ is not ad$L$ stable. In a similar way, we see that $U(H)_{\lambda}$ is not ad$L$ stable, since $[x, v] \notin U(H)_{\lambda}$.

We would like to notice that although both $u$ and $v$ are not $L$-semi-invariant, $uv = w$ where $w \equiv e_1e_3^2 + e_1^2e_3 + e_2e_3^2 + e_3^3$ and it can be seen that $w \in Z(U(H))$ or by [1], $w \in S_z(U(L))$. We recall the following result of Moeglin [3] Lemma 2.

Let $L$ be solvable Lie algebra over a field of characteristic 0 and let $u, v$ be non zero elements of $U(L)$, then if $uv$ is semi-invariant, then so are $u$ and $v$. In addition if $F$ is algebraically closed, then $uv \in S_z(U(L))$ implies that $u, v \in S_z(U(L))$. This result was also extended in [3] Proposition 1.3 and it seems to be crucial in showing that $S_z(U(L))$ is factorial in characteristic 0. Similar result appears to be wrong if $\text{char} F = p > 0$. Clearly if $L$ is Lie algebra over a field $F$ of char $F = p > 0$ defined by $[x, y] = y, [x, z] = -z, [y, z] = t$ and $[t, L] = 0$, then $yy^{p-1} = y^p \in U(L)_0$ but both $y, y^{p-1} \notin U(L)_0$. However, we are interested in the following condition

**Condition 2.2.** Let $L$ be solvable finite dimensional Lie algebra over an algebraically closed field $F$ of char $F = p > 0$. Let $u, v \in U(L)$ be non zero such that $u$ and $v$ are linearly independent over $U(L)$. If $uv$ is semi-invariant, then so are $u$ and $v$.

Result (2.1) is the evidence where this condition fails and by [2] $S_z(U(L))$ in this case is not factorial. Based on many observations, we have a strong assumption to believe there is linkage between Condition (2.2) and factoriality of $S_z(U(L))$ where $L$ is a finite dimensional Lie algebra over algebraically closed field $F$ of $\text{char} F = p > 0$, however we are not able not supply any general proof here.
We would like to mention that being $adL$ stable, does not imply that $U(H)_\lambda$ is a weight space of $U(L)$, as can be seen in the following example

**Example 2.3.** Let $L$ be 4 dimensional Lie algebra spanned by $\{x, y, u_1, u_2\}$ with a multiplication table $[x, u_2] = u_1$, $[y, u_1] = u_1$, $[y, u_2] = u_2$ and the rest of the products are 0. Then $L = Fx + H$ where $H = span\{y, u_1, u_2\}$. Then $U(H)_\lambda = span\{u_1, u_2\}$. Clearly $\lambda = 1$ and $U(H)_1$ is $adx$ stable. Notice that $adx(u_2) = u_1$ but $U(H)_1 \not\subset Sz(U(L))$.

The following result will establishes a condition on $L$ for $U(H)_\lambda$ to be $adL$ stable.

**Lemma 2.4.** Let $L = Fx + H$ be a finite dimensional Lie algebra over an algebraically closed field $F$, of characteristic $p > 0$ and $H$ an ideal in $L$. If $[L, L]$ is nilpotent, then $U(H)_\lambda$ is $adx$ stable and consequently is $adL$ stable.

**Proof.** Let $0 \neq u \in U(H)_\lambda$. We need to show that for every $y \in H$, $[y, [x, u]] = \lambda(y)[x, u]$. Indeed $[y, [x, u]] = [[y, x], u] + [x, [y, u]] = [[y, x], u] + \lambda(y)[x, u]$. Consider $[[y, x], u]$. Since $y \in H$ then $[y, x] \in H$. Hence $[[y, x], u] = \lambda([y, x])u$ and we need to show that $\lambda([y, x]) = 0$. Now $[y, x] \in [L, L]$ and so $[y, x]$ is nilpotent, therefore $ad[y, x]$ is nilpotent. Let $k$ be its nilpotency index. Therefore $0 = (ad[y, x])^k(u) = \lambda([y, x])^ku$ hence $\lambda([y, x])^k = 0$ which implies that $\lambda([y, x]) = 0$. \hfill \Box

The previous lemma is somewhat misleading. Unlike in the $\text{char}F = 0$, case, $U(H)_\lambda$ is not necessarily an $L$-module and consequently $\lambda([x, y])$, $x, y \in L$, may be non-zero. The assumption $[L, L]$ being nilpotent is therefore essential here. The next result links between Lemma (2.4) and completely solvable Lie algebra. Recall that a Lie algebra $L$ is called a completely solvable if there is a finite family $\{I_i\}$ of ideals in $L$ such that $L = I_1 \supset I_2 \supset \cdots \supset I_k \supset I_{k+1} = (0)$ and $I_i = Fe_i + I_{i+1}$.

**Corollary 2.5.** Let $L$ be a solvable finite Lie algebra over an algebraically closed field $F$, with $\text{char}F = p$. Then $[L, L]$ is nilpotent if and only if $L$ is completely solvable.

**Proof.** The proof is similar to one of Dixmier [4, 1.3.12] where [4, 1.3.11] which is used in its proof is replaced by Lemma (2.4). \hfill \Box

The following result extends Lemma (2.4).

**Proposition 2.6.** Let $L = Fx + H$ be a finite dimensional Lie algebra over an algebraically closed field $F$ of $\text{char}F = p > 0$ and $H$ a codimension one ideal in $L$. Then $\lambda([L, L]) = 0$ if and only if $U(H)_\lambda$ is $adx$ stable.

**Proof.** Assume first that $U(H)_\lambda$ is $adx$ stable for every $\lambda \in H^*$. Hence for every $w \in U(H)_\lambda$, $[x, w] \in U(H)_\lambda$. Therefore for every $y \in H$,

\begin{equation}
[y, [x, w]] = \lambda([y, x])w + \lambda(y)[x, w].
\end{equation}
If \([x, w] = 0\), then \((2)\) implies that
\[
0 = \lambda([y, x])w + 0
\]
and so \(\lambda([y, x]) = 0\). Assume next that \([x, w] \neq 0\). If \([y, [x, w]] = 0\) then since \([x, w] \in U(H)\) we get \(\lambda(y) = 0\). By \((2)\), \(0 = \lambda([y, x])w + 0\) and so \(\lambda([y, x]) = 0\). If \([y, [x, w]] \neq 0\), then the stability of \(U(H)\) implies that \(\lambda([y, x]) = 0\). Finally if \([y, x] = 0\) then \(\lambda([y, x]) = 0\). Therefore \(\lambda([y, x]) = 0\) for every \(y \in H\) and so \(\lambda([L, L]) = 0\). By Lemma \((2.3)\), the reverse direction is trivial.

**Theorem 2.7.** Let \(L = Fx + H\) be a finite dimensional Lie algebra over a field \(F\) and \(H\) an ideal in \(L\). Then \(S\}(U(H))\) is adL stable if and only if \(U(H)\) is adx stable for every \(x\).

**Proof.** Clearly if \(S\}(U(H)) = Z(U(H))\), then \(S\}(U(H))\) is adL stable. So we assume that \(Z(U(H)) \subset S\}(U(H))\). Let \(\Delta = \{\mu | \mu\) is weight on \(U(H)\}\). Now since \(char F = p > 0\), then \(S\}(U(L))\) is finitely generated as a \(Z(U(L))\) module, which implies that \(\Delta\) is finite and we assume \(dim_F \Delta = r\). Assume that \(S\}(U(H))\) is adx stable and assume by negation that there exists some \(1 \leq k \leq r\) such that \(U(H)\) is not adx stable. Let \(0 \neq u \in U(H)\), hence \([x, u] \in \Sigma_{i \leq r} U(H)\), in particular
\[
[x, u] = \Sigma_{i \leq r} u_i, \quad u_i \in U(H)\mu_i.
\]
Then for every \(y \in H\),
\[
[y, [x, u]] = \Sigma_{i \leq r} \Delta u_i.
\]
In addition
\[
[y, [x, u]] = [(y, x), u] + [x, [y, u]] = \mu_k([y, x])u + \mu_k(y)[x, u].
\]
Therefore combining \((3)\), \((4)\) and \((5)\) we get
\[
\mu_k([y, x])u + \mu_k(y) \Sigma_{i \leq r} u_i = \Sigma_{i \leq r} \Delta u_i.
\]
Since \(u_i\) belongs to different weight spaces, they are linearly independent, hence the only solution to \((3)\) is \(\mu_k([x, y]) = 0\) and \(\mu_k(y) = \mu_i(y)\) for every \(i \leq r\), which is impossible by the assumption, that \(U(H)\) is not adx stable. Hence \([x, u] \in U(H)\mu_k\) and we are done. The second direction is trivial.

**Corollary 2.8.** Let \(L\) be a finite dimensional Lie algebra over an algebraically closed field \(F\), of \(char F = p > 0\) and \(H\) any ideal of \(L\) such that \([L, L] \subset H\). Then \(S\}(U(H))\) is adL stable if and only if \(\lambda([L, L]) = 0\) for every \(\lambda \in H^*\).

As was seen in \((2)\) Theorem 4.7, if \([L, L]\) is nilpotent then \(S\}(U(L))\) is factorial. By Lemma \((2.4)\) \([L, L]\) being nilpotent, implies that \(U(H)\) is adL stable and consequently by Lemma \((2.6)\) \(\lambda([L, L]) = 0\). Therefore \(S\}(U(L))\) is factorial if \(S\}(U(H))\) is adL stable. This make certain relation between factoriality of \(S\}(U(L))\) and stability of semi-center of certain subalgebras. Although the reverse direction is wrong, as \(S\}(U(L))\) is factorial does not implies that \(S\}(U(H))\) is adL stable. We hope this relation will help on further research related factoriality of semi-center.
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