Spin force and intrinsic spin Hall effect in spintronics systems

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Abstract – We investigate the spin Hall effect (SHE) in a wide class of spin-orbit coupling systems by using the spin force picture. We derive the general relation equation between spin force and spin current and show that the longitudinal force component can induce a spin Hall current, from which we reproduce the spin Hall conductivity obtained previously using Kubo’s formula. This simple spin force picture gives a clear and intuitive explanation for SHE.

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Introduction. – Spin Hall effect (SHE) refers to the phenomenon in which a transverse pure spin current is induced in response to a longitudinal applied electric field [1,2]. The generation of spin Hall current is associated with spin separation in the transverse direction, which had been explained in many previous studies [3–5]. Sinova et al. derived a momentum-dependent spin polarization [4], in which electrons moving in the opposite transverse (+y) direction in a Rashba system acquire opposite spin polarization, resulting in spin separation and SHE. On the other hand, Murakami et al. studied the effect in p-doped semiconductors [3], where the separation of electron spins is a result of an anomalous spin-dependent velocity. These two mechanisms were then brought under a unified framework by Fujita et al. invoking the gauge field in time space [5]. In another work by Shen [6], a heuristic picture of spin separation is given in terms of the spin force. In this picture, electrons traveling in a 2DEG Rashba system under the influence of a spin-orbit coupling (SOC) effect experience a transverse spin force, which induces separation of spin. However, the spin separation due to transverse spin force was employed to describe the Zitterbewegung (jitter) motion of electrons, but not the SHE. A similar approach was previously developed to show the link between the spin force and spin Hall current in linear Rashba and/or Dresselhaus systems [7–9]. However, the spin Hall conductivity was not derived. In addition, an extension of the Drude model was presented [10,11] to describe SHE, in which the spin Hall conductivity is obtained in the presence of scattering. Experimentally, the spin Hall conductivity or the spin Hall angle (the ratio of the spin Hall current to the injected charge current) can be measured either via non-local electrical spin injection [12,13] or by ferromagnetic resonance [14,15].

Recently, we have also applied the spin force picture to study the SHE in Rashba-Dresselhaus system [16] using non-Abelian gauge field and shown that the longitudinal spin force can induce a transverse spin Hall current, from which we directly recovered the universal spin Hall conductivities. This picture is consistent with others [3–5,17], which assigned the underlying mechanism of the SHE to the spin precession of electrons under acceleration.

In this paper, we generalize this spin force picture of the SHE for a general SOC system, including the cubic-Dresselhaus [18], and heavy hole system based on III-V semiconductor quantum wells [19]. It may also be extended to other systems governed by the same class of Hamiltonian involving the coupling between momentum and a spin-like degree of freedom, such as the graphene systems [20]. We derived the relation between the spin force and spin current, and showed that the longitudinal force component is responsible for the SHC. From this general relation, we recover the spin Hall conductivities obtained previously using Kubo’s formula for various types of spin-orbit couplings. The spin force framework not only presents a unified picture of SHE in a wide class of SOC

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systems, but also gives an intuitive picture of its underlying mechanism, which is not obvious from the linear response or Kubo’s theory.

**Quantum spin force equation.** – We begin with the general SOC Hamiltonian in the presence of applied electric field:

\[
H = \frac{p^2}{2m} + B(p) \cdot \hat{\sigma} + V(r),
\]

where \(m\) is the effective mass, \(B(p)\) is the momentum-dependent effective magnetic field, which arises from the SOC effect, and \(V(r) = eE \cdot r\), where \(E\) is the applied electric field. The above Hamiltonian has eigenenergies

\[
\epsilon_{\pm} = \frac{p^2}{2m} \pm |B(p)|
\]

corresponding to eigenvectors

\[
|\psi_{p,+}\rangle = \left(\cos \frac{\Theta}{2} e^{-i \Phi/2}, \sin \frac{\Theta}{2} e^{i \Phi/2}\right),
\]

\[
|\psi_{p,-}\rangle = \left(-\sin \frac{\Theta}{2} e^{-i \Phi/2}, \cos \frac{\Theta}{2} e^{i \Phi/2}\right),
\]

respectively, where \(\Theta\) and \(\Phi\) are the spherical polar angles of the vector \(B(p)\) in \(k\)-space.

The dynamics of the electron in this system is described by equations of motion in the Heisenberg picture:

\[
v \equiv \dot{r} = \frac{i}{\hbar} [H, r] = \frac{p}{m} + \nabla_p B \cdot \hat{\sigma},
\]

\[
\dot{p} = \frac{i}{\hbar} [H, p] = eE,
\]

where in eq. (5), we have made use of \(r = i\hbar \nabla_p\). Likewise, the spin dynamics can be shown to be governed by the following equation:

\[
\frac{d\hat{\sigma}}{dt} = \frac{i}{\hbar} [H, \hat{\sigma}] = \frac{2}{\hbar} (B \times \hat{\sigma}).
\]

The force acting on the electron can then be derived by taking the time derivative of eq. (5), and using the results of eqs. (6) and (7) for the time derivatives of \(p\) and \(\hat{\sigma}\), respectively. This yields

\[
\langle F_i \rangle = m \frac{d\langle u_i \rangle}{dt} = eE_i + m (U^0_{ij} + U^1_{ij}) \langle \sigma_j \rangle
\]

with

\[
U^0_{ij} = \frac{2}{\hbar} \epsilon_{jkl} (\nabla_p B_k) B_l, \quad U^1_{ij} = eE_k \frac{\partial^2 B_j}{\partial p_l \partial p_k}.
\]

In the above, we assume that repeated indices are summed up, and \(\langle \ldots \rangle\) denotes taking expectation value in spin space.

We note that, in the presence of an applied electric field, the linear response of the spin polarization can be written as

\[
\langle \sigma_i \rangle = \langle \sigma^0_i \rangle + \langle \sigma^1_i \rangle,
\]

where \(\sigma^0_i\) is the solution of eq. (9) in the absence of electric field, and \(\sigma^1_i = A_i E_j\) is the linear correction due to the electric field. It is obvious that the spin will align along the effective SOC field when the electric field is absent, i.e., \(\langle \sigma^0_i \rangle = \pm B_i/|B|\). To fulfill the normalization of the total spin polarization in eq. (10), i.e., \(\langle \sigma_i \rangle^2 = 1\), we must have

\[
\langle \sigma^1_i \rangle \langle \sigma^0_i \rangle = \pm \langle \sigma^1_i \rangle \frac{B_i}{|B|} = 0,
\]

which means that the electric field induces a spin correction that is perpendicular to the effective SOC field.

With these, the force equation (8) can be rewritten as follows:

\[
\langle F_i \rangle = eE_i + mU^0_{ij} \langle \sigma_j \rangle + mU^1_{ij} \langle \sigma^0_j \rangle,
\]

in which the higher order term in the electric field is ignored, and \(U^0_{ij} \langle \sigma^0_j \rangle = \epsilon_{jkl} (\nabla_p B_k) B_l |B|/|B| = 0\) since \(\epsilon_{jkl}\) is asymmetric while \(B_l B_j\) is symmetric in exchanging \((j, l)\).

**Classical spin force equation.** – Although in quantum mechanics the force concept is not well defined as a consequence of the uncertainty principle, we can still establish a connection between the expectation value of the force operator and the well-defined classical force. While the former is derived from Heisenberg’s equation of motion, the latter can be obtained from the energy of a physical system by applying Hamilton’s equations. Indeed, if a physical system has energy \(e(p, r)\), which is a function of position and conjugate momentum, its dynamics can be described by the coupled equations \(\dot{r} = \nabla_p e\) and \(\dot{p} = -\nabla_r e\). Then, the classical force acting on the system is given by \(\mathbf{F}^c = m \dot{r} = m (\dot{p} \nabla_p e)\). We now relate this classical force to the expectation value of the force operator (\(\mathbf{F}\)) of a quantum system, i.e., \(\langle \mathbf{F} \rangle_{\psi_n} = \mathbf{F}^c = m (e\mathbf{E} \nabla_p) \nabla_p e\), with \(n\) denoting the eigenstate index (we assume that the quantum system can exist in \(n\) different eigenstates). Strictly, this is a semi-classical treatment where the classical Hamiltonian is replaced by the eigenenergies of the quantum system. In the following we show the validity of this semi-classical approach. We consider again the force operator \(F_i = \frac{d}{dt}(\partial_{p_i} H)\). The expectation value of the force operator in a given quantum state \(\psi_n\) is given as \(\langle F_i \rangle = \langle \psi_n | \frac{d}{dt}(\partial_{p_i} H) | \psi_n \rangle\). Explicitly,

\[
\langle \psi_n | \frac{d}{dt}(\partial_{p_i} H) | \psi_n \rangle = \frac{d}{dt} \partial_{p_i} \langle \psi_n | H | \psi_n \rangle + \langle \psi_n | H | \partial_{p_i} \psi_n \rangle
\]

\[
= \frac{d}{dt} \langle \partial_{p_i} | \psi_n \rangle + \frac{d}{dt} (\epsilon_n \text{Re} \langle \psi_n | \partial_{p_i} \psi_n \rangle)
\]

\[
= \frac{d}{dt} (\partial_{p_i} \epsilon_n),
\]

(13)
where in the last line we have used the fact that $2\text{Re}(\langle \psi_n | \partial_p \psi_n \rangle) = \partial_p \langle \psi_n | \psi_n \rangle = 0$. Therefore, the classical derivation of force is actually equivalent to taking the expectation value of the force operator.

In our present case, by considering the eigenenergies in eq. (2), the spin force is readily obtained as

$$\langle F^\text{cl}_i \rangle_{\psi_{p,n}} = eE_i + n \frac{\hbar^2}{4|B|} U_{ij}^0 U_{ij}^0 eE_k + nmU_{ij} B_j \frac{|B|}{|B|},$$

with $n = \pm$ being the eigenbranch index. In the above equation, the second term on the right-hand side is odd either in $p_i$ or $p_j$, if $i \neq j$, and it is even when $j = i$; this means that upon averaging the above force equation over the Fermi sphere, only terms with $j = i$ would contribute to the total force. Therefore, if the electric field is applied just along the longitudinal $x$-direction, there exists only a net longitudinal force in the system. Interestingly, we can express the above force as

$$\langle F^\text{cl}_i \rangle_{\psi_{p,n}} = eE_i + nm \frac{\hbar^2}{4|B|} U_{ij}^0 U_{ij}^0 eE_k + nmU_{ij} B_j \frac{|B|}{|B|},$$

which bears a similarity to the spin force in eq. (12).

We note here that in the absence of impurity scattering, the mixture of states with different momenta can be neglected [17,21], and the distribution function for our system is given by $f_n(k) = f_n^0(k) + \langle \partial f_n^0(k)/\partial p_i \rangle eE_i$ [17,21], where $f_n^0$ is the Fermi function corresponding to the $n$-th eigenbranch. The second term in the above equation represents a linear correction due to the applied electric field. In general, the expectation value of an operator is given by $\langle \hat{O} \rangle = \sum_n f_n \langle \psi_n | \hat{O} | \psi_n \rangle$. However, for an operator that is linearly dependent on the applied electric field, e.g., the spin force or the spin polarization in our study, the distribution function in the above expectation value can be replaced by the Fermi function, i.e., $\langle \hat{O} \rangle = \sum_n f_n^0 \langle \psi_n | \hat{O} | \psi_n \rangle$, up to the first order in the electric field. For simplicity, we will consider this case in the rest of the paper.

Equations (12) and (14) relate the spin polarization to the force (electric field) driving the electric current, and thus enables us to quantify other spin-dependent transport effects, e.g., spin Hall effect, or spin separation, in terms of the spin force. In the following part, we will show that in general, the spin Hall current can be regarded as being induced by the spin force. For any general SOC system, we can thus derive the spin Hall current and the associated spin Hall conductivity, once we have obtained the expression of spin force.

**Spin current operator.** By definition, the spin current operator is $J^\sigma_i = \frac{1}{2} \langle \{ \sigma_i, \psi_\sigma \} \rangle$ where $\{ A, B \}$ denotes the anti-commutation relation, $s$ is the spin of carriers (with $s = \frac{1}{2}$ for the electron, and $s = \frac{3}{2}$ for heavy hole in a Luttinger system). With the velocity operator given in eq. (5), the spin current operator then reads as

$$\langle J^\sigma_i \rangle = \frac{s}{m} \langle \sigma_i^1 \rangle + \langle \sigma_i^0 \rangle - s \frac{\partial B_i}{\partial p_j}.$$ 

In the above, the first term depends on spin polarization which is induced by applied electric field, the second term represents the spin current in equilibrium state, i.e., in the absence of $E$ field, while the last term relates to the variation of effective field in $k$-space. In our study, we focus on the spin Hall current contribution which is proportional to the electric field, i.e., the first term only:

$$\langle J^\sigma_i \rangle = \frac{s}{m} \langle \sigma_i^1 \rangle.$$ 

The total spin Hall current $J^\sigma_i$ can be obtained by integrating above expression over the momentum space. In the framework of linear response theory, the spin Hall current in semiconductors with SOC exhibits the general response of [3,4]

$$J^\sigma_i = \sigma_{SH} eE_i E_k,$$

where $\sigma_{SH}$ is the spin Hall conductivity. Thus, if an electric field is applied along one of the axes, e.g., the $x$-direction, there would be two non-zero transverse spin current components $j^\sigma_y$ and $j^\sigma_z$, which would in turn induce spin accumulation $\sigma^x$ and $\sigma^y$, respectively, and that $j^\sigma_y = -j^\sigma_z$. Moreover, there is only longitudinal spin force (along the $x$-direction) acting on the electron as discussed above. From now on, we will just consider this case for simplicity. With eqs. (12), (17) and (18), we can establish the relation between the longitudinal spin force acting on the electron and the resulting transverse spin current.

From eq. (11), we have the following identity: $\langle \sigma_i^1 \rangle = -\langle B_j/\sigma_i^\sigma \rangle + \langle B_z/\sigma_i^\perp \rangle B_z$. With this, the longitudinal component of the spin force (without the $eE_x$ term) in eq. (12) can be expressed as

$$\langle F_x \rangle = \frac{m^2}{s} \left(U_{xy}^0 - U_{xx}^0 \frac{B_y}{B_x} \right) \langle j^\sigma_y \rangle \frac{p_x}{p_z} + \frac{m^2}{s} \left(U_{yx}^0 - U_{xx}^0 \frac{B_x}{B_y} \right) \langle j^\sigma_x \rangle \frac{p_y}{p_z} + mU_{ij}^{0} \langle j^\sigma_i \rangle,$$

where the spin polarizations have been replaced by the corresponding spin currents in eq. (17). From eq. (19), it can clearly be seen that the longitudinal spin force induces transverse Hall currents (fig. 1). By comparing the above spin force relations with the classical analogue (eqs. (14) or (15)), we can thus obtain the explicit expression for the spin Hall current, as well as the spin Hall conductivity. Indeed, we can rewrite the force in eq. (15) as

$$\langle F_{\pm} \rangle_{\psi_{p,n}} = \pm \frac{m^2 eE_x}{4|B|} \left(U_{xy}^0 - U_{xx}^0 \frac{B_y}{B_x} \right) U_{xy}^0$$

$$\pm \frac{m^2 eE_x}{4|B|} \left(U_{yx}^0 - U_{xx}^0 \frac{B_x}{B_y} \right) U_{yx}^0$$

$$\pm mU_{ij}^{0} \frac{B_j}{|B|},$$

in which we have used the relation $U_{ij}^{0} B_j/|B| = 0$. By substituting the above force expression into eq. (19), the spin...
spin Hall current components

The spin Hall effect is similarly described by a set of mutually perpendicular quantities \(\{F_x, j_y^\alpha, \sigma_z\}\), i.e., the correlation between longitudinal spin force, vertical magnetic fields, and spin Hall current. In a 3D system, there are two spin Hall current components \(j_y^\alpha\) and \(j_y^\beta\) corresponding to the spin polarizations in the \(y\)- and \(z\)-direction, respectively.

Hall current components are readily obtained as follows:

\[
\langle j_z^{y,\alpha}\rangle_{p,\pm} = \pm s \frac{h^2 e E_x}{4m|B|^3} \langle \alpha \rangle U_{xy}^0, \quad (21)
\]

\[
\langle j_z^{y,\beta}\rangle_{p,\pm} = \pm s \frac{h^2 e E_x}{4m|B|^3} \langle \beta \rangle U_{xz}^0. \quad (22)
\]

If the electron motion is confined to a 2D plane (\(x\)-\(y\) plane), we have \(B_z(k) = 0\), and \(U_{xy}^0 = 0\) following eq. (9). This means that the transverse spin Hall current in 2D system is given by eq. (22).

Equations (19), (21), and (22) are our main results. By deriving the spin force in Heisenberg picture and its classical counterpart using Hamilton’s equations, we have explicitly obtained the transverse spin Hall current. In the next section, we will apply our analysis to describe the spin Hall effect in a wide class of SOC systems.

**Results and discussions.** — In the context of the classical charge Hall effect, the basic physical quantity underlying the Hall effect is the Lorentz force. In steady state, the transverse component of the Lorentz force vanishes, resulting in a steady transverse Hall current. Therefore, an intuitive picture of the Hall effect can be provided by a set of mutually perpendicular quantities \(\{F_x, j_y^\alpha, B_z\}\) (see fig. 1(a)). In a two-dimensional system where electrons are accelerated \((F_x)\) by an applied electric field in the presence of an out-of-plane magnetic field \((B_z)\), there is a (Hall) current flowing in the transverse direction \(j_y^\alpha\). In the same spirit, we can represent the spin Hall effect by considering a similar set of quantities, i.e., \(\{F_x, j_y^\alpha, \sigma_z\}\) (fig. 1(b)), in which the spin-orbit coupling takes the role of the vertical magnetic field, inducing spin polarization \((\sigma_z)\). When an electron is accelerated by an applied electric field \((E_x\) or \(F_x\) in the presence of spin-orbit coupling (which induces \(\sigma_z\)), a spin Hall current would be generated in the transverse direction \(j_y^\alpha\). In 3D systems, there is an additional spin Hall current component corresponding to spin in the \(y\)-direction, i.e., \(j_y^\beta\). The correlation between the set of quantities in this case is given by eq. (19). Therefore, the spin force picture can provide an intuitive description of the spin Hall effect, analogous to the well-known Hall effect. More significantly, we can also directly obtain the explicit expression of the spin Hall current (and hence spin Hall conductivity) from the spin force equation, which is applicable for a general class of spin-orbit coupling systems (eqs. (21) and (22)). This explicit derivation was absent in previous works [7-9].

We will now illustrate the utility of the spin force picture in evaluating the spin Hall current in exemplary 2D and 3D SOC systems, the corresponding Hamiltonian of which is listed in table 1, together with their respective effective magnetic fields \(B(p)\). For each system, we will consider the spin force equation (eq. (19)), and evaluate the \(U_{ij}\) matrix based on eq. (9). Next, we calculate the classical spin force based on its expectation value when the electron is in the eigenstate \(\psi_{p,n}\) (eq. (14)). By equating the classical and quantum mechanical spin force expression, we obtain the expression for the spin current in the state \(\psi_{p,n}\) corresponding to the \(n\)-th eigenbranch and momentum. Finally, the total spin current of the system is obtained by summing over the momentum space and eigenbranches of the system.

**a) Linear Rashba-Dresselhaus system.**

We consider the linear Rashba-Dresselhaus Hamiltonian, which is applicable to two-dimensional systems lacking inversion symmetry. From eq. (19), the equation relating the spin force to the spin current in this system is readily found to be

\[
\langle F_x \rangle = \frac{4m^2(\alpha^2 - \beta^2)}{\hbar^2} \langle j_y^\alpha \rangle . \quad (23)
\]

Meanwhile, considering eq. (14), the classical force corresponding to eigenstates \(\psi_{p,\pm}\) is

\[
\langle F_x \rangle \psi_{p,\pm} = \pm \frac{meE_x}{p(\alpha^2 + \beta^2 + 2\alpha\beta \sin 2\theta)^{3/2}}. \quad (24)
\]

for the \(\pm\) eigenbranches. It is obvious that the spin force in both eqs. (23) and (24) will vanish if \(\alpha = \pm \beta\). For the case of \(\alpha \neq \pm \beta\), by equating (23) and (24), integrating over \(p\) and summing over the contribution of the two eigenbranches, the spin current is readily shown to be

\[
J_y^\alpha (sH) = \frac{\alpha^2 - \beta^2}{|\alpha^2 - \beta^2|} \frac{eE_x}{8\pi}, \quad (25)
\]

a result which is consistent with previous calculations based on Kubo’s linear response theory [4,22].
Table 1: Various SOC systems and their corresponding Hamiltonian and effective SOC field. Here, \( k_\pm = k_x \pm ik_y \), \( \sigma_\pm = \sigma^x \pm i\sigma^y \), and c.p. denotes cyclic permutation of \( \{x, y, z\} \).

| SOC system                  | Hamiltonian                                                                 | \( B(p) \)                     |
|----------------------------|------------------------------------------------------------------------------|--------------------------------|
| Rashba-Dresselhaus (2D)    | \( p^2/2m + \alpha (p_x\sigma^y - p_y\sigma^x) + \beta (p_y\sigma^y - p_x\sigma^x) \) | \( \pm \alpha p_y - \beta p_x \) |
| Heavy hole in QW (2D)      | \( p^2/2m + i\lambda/2(k^2 \sigma_+ - k^2 \sigma_-) \)                      | \( \lambda \)                   |
| \( k^3 \)-Dresselhaus (3D) | \( p^2/2m + \eta[k_x(k_y^2 - k_z^2)]\sigma^x + \text{c.p.} \)                | \( \eta \)                      |

In which the magnitude of the SOC field is \( |B| = \frac{\lambda e^2}{h} \), which also yields the classical force

\[
\langle F_x \rangle_{\psi_{\pm}} = \pm \frac{3m\lambda eE_x}{2h} (3 + \cos 2\theta).
\]

From these two equations, the spin current reads

\[
\langle j^z \rangle_y = \pm \frac{9eE_xh^3}{4\lambda mp^2} \sin^2 \theta,
\]

which is summed over momentum space and two branches to give total value:

\[
J_z^y(sH) = -\frac{9eE_xh^3}{16\lambda mp} \left( \frac{1}{p_{F-}} - \frac{1}{p_{F+}} \right).
\]

This result is consistent with previous findings obtained via Kubo’s formula [23].

c) Cubic \( k^3 \)-Dresselhaus.

In \( k^3 \)-Dresselhaus system, the Hamiltonian is invariant under all operations of the point symmetry group \( T_d \) [24]. In this case, there are two spin Hall current components \( j_y^z \) and \( j_z^z \) given in eqs. (21) and (22). However, they can be exchanged under one of the \( T_d \) operations. In view of this, we define a chiral spin current about the x-axis, \( j_{\text{chir},x} = (j_y^z - j_z^z)/2 \) [25], which is invariant under the above operation. Explicitly, the chiral spin current reads as

\[
\langle j_{\text{chir},x} \rangle = \pm \frac{eE_x\eta^2 p_x^2 (p_y^2 - p_z^2)^2}{4h^4 m|B|^3}.
\]

The total spin Hall current is then

\[
J_{\text{chir},x} (sH) = \frac{eE_x\eta^2}{4h^4 m} \int_{p_-}^{p_+} \frac{d^3p}{(2\pi)^3} \frac{p_x^2 (p_y^2 - p_z^2)^2}{|B|^3}.
\]

Using the inter-band relation with small spin-split \( (p_+ - p_-) = -2m|B|/p_F \), where \( p_F = (p_+ + p_-)/2 \) is the average Fermi momentum, the above SHC is simplified to

\[
J_{\text{chir},x} (sH) = eE_x \left( \frac{k_F}{12\pi^2} \right)
\]

which recovers previous results [26].
In summary, we have described the spin Hall effect in various semiconductor SOC systems by invoking the spin force picture, both in the quantum mechanical and the classical sense. The former relates the longitudinal force to a transverse spin current carrying a perpendicular spin polarization via Heisenberg’s equation of motion. For the specific case of the linear Rashba-Dresselhaus system, the spin force can be related to the Lorentz-like force arising from a non-Abelian (Yang-Mills) field. The classical spin force equation then enables an explicit evaluation of the transverse spin current and hence the spin Hall conductivity. The calculated spin Hall conductivities are consistent with those obtained via other methods.

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