String theory explanation of galactic rotation.

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Abstract

The unique spherically symmetric metric which has vanishing weyl tensor, is asymptotically desitter, and can model constant galactic rotation curves is presented. Two types of field equations are shown to have this metric as an exact solution. The first is palatini varied scalar-tensor theory. The second is the low energy limit of string theory modified by inclusion of a contrived potential.

1 Introduction

Galactic rotation curves often exhibit speeds which are a constant independent of distance from the center of the galaxy. This is less than would be expected from solid body rotation where the rotation speed increases with radial distance and more than would be expected from free orbit rotation where the speed would decrease with radial distance. Currently the majority view is that a large amount of ‘dark matter’ occurs in non-luminous places to produce these rotation curves, as opposed to the minority view which is that constant rotation curves are caused by modified laws of gravity. From a newtonian perspective the modification which works is the replacement of the reciprocal potential by a logarithmic potential, the spherically symmetric relativistic generalization of this [2] has one free function which in the present work is fixed by requiring that the weyl tensor vanishes. This leaves the problem of finding which field equations the einstein tensor obeys and both palatini varied scalar-tensor theory [1] and the low energy limit of string theory with added potential are found to work. The usual method of approaching problems by starting with a lagrangian, deriving field equations,
finding solutions, and then comparing with observations is thus reversed: in the present case it is observation, then metric, then field equations and finally lagrangian.

2 The metric

Consider the line element

\[ ds^2 = -(X + \lambda_t r^2)dt^2 + \frac{(X - v^2)^2}{X(X + \lambda_r r^2)}dr^2 + r^2d\Sigma_2^2, \]  

where

\[ X \equiv (1 + 2v^2 \ln(r)), \quad d\Sigma_2^2 \equiv d\theta^2 + \sin(\theta)^2 d\phi^2, \]

and \( v \) is the constant speed of galactic rotation. \( g_{\theta\theta} \) and \( g_{tt} \) are fixed up to a radial coordinate transformation by requiring constant rotation curves, \( g_{rr} \) is fixed by requiring that the weyl tensor vanishes. When \( \lambda_t(\lambda_t - \lambda_r) = 0 \) the weyl tensor vanishes, from now on only consider the solution \( \lambda = \lambda_t = \lambda_r \).

Another property of the curvature is that the ricci scalar obeys a type of conformal wave equation \( R = 6r \Box (1/r) \), but there appears to be no pattern to the higher order ricci curvature invariants. The curvature of the metric (1) is characterized by the ricci scalar and

\[ Y \equiv R^t_t - R^\theta_\theta = \frac{2v^2}{r^2(X - v^2)}, \quad Z \equiv 3R^t_t - R^\theta_\theta = \frac{4v^6}{r^2(X - v^2)^3}. \]

The metric (1) has vanishing weyl tensor so that it can be expressed in conformally flat form \( d\hat{s}^2 = \Omega^2 ds^2_{\text{minkowski}} \), taking \( \lambda = 0 \) and comparing the \( g_{\theta\theta} \) and \( g_{rr} \) terms respectively

\[ \Omega^2 R^2 = r^2, \quad R = \frac{Ar}{\sqrt{X}}, \]

where \( A \) is a constant of integration, to invert the second of (4) requires lambert W functions. Together with (7), (4) implies

\[ \Omega = \pm \frac{r}{R} = \pm \frac{1}{A} \sqrt{X} = \pm \frac{1}{A} \exp \left( -\frac{\alpha \phi}{2} \right). \]

Thus the field equations (8) can be expressed in terms of the conformal factor \( \Omega \) and can be thought of as the difference in the ricci tensor between \( d\hat{s} \) and \( ds \). This implies that any metric of the form \( d\hat{s}^2 = \Omega^2 ds^2_{\text{ricci flat}} \)
will also be a solution of (8). In particular transforming the schwarzschild solution
\[ ds^2 = -X \left( 1 - \frac{2m\sqrt{X}}{r} \right) dt^2 + \frac{(X - v^2)^2}{X^2 \left( 1 - \frac{2m\sqrt{X}}{r} \right)} dr^2 + r^2 d\Sigma_2^2 \] (6)
and this is still a solution of (8) with scalar field given by (7).

3 Palatini scalar-tensor theory

Choosing the scalar field
\[ \phi = -\frac{1}{\alpha} \ln (X), \] (7)
a tensor which vanishes is
\[ Q_{ab} \equiv R_{ab} - \alpha \phi_{,ab} + \frac{\alpha^2}{2} \phi_{a,\phi_b} - \frac{1}{2} g_{ab} \left\{ \alpha \Box \phi + \alpha^2 (\nabla \phi)^2 - 6\lambda \exp(\alpha \phi) \right\}, \] (8)
where \( \alpha \) is an absorbable constant. The field equations expressed in terms of the ricci tensor of palatini varied scalar-tensor theory are
\[ 8\pi \kappa^2 S_{ab} = A R_{ab} - A' \phi_{,ab} - B A \phi_a \phi_b - \frac{1}{2} g_{ab} \left\{ A' \Box \phi + A'' (\nabla \phi)^2 + V(\phi) \right\}, \] (9)
where
\[ B_A = B + A'' - \frac{3}{2} \frac{A'^2}{A}, \] (10)
\( A \) is called the primary dilaton function, \( B \) is called the secondary dilaton function and \( V \) is called the potential. The field equations (8) are a particular case of the field equations (9) with
\[ A = \exp(\alpha \phi), \quad V = -6\lambda \exp(2\alpha \phi), \quad B = 0, \quad S_{ab} = 0. \] (11)
The field equations (9) can be found by performing both metric and palatini variations of the action
\[ S = \int d^4 x \sqrt{-g} \left\{ A(\phi) R - B(\phi) (\nabla \phi)^2 - V(\phi) \right\}. \] (12)
4 Low energy string theory with a potential

The lagrangian for low energy string theory with a potential is

\[ \mathcal{L} = \exp (-2\phi) \left\{ R + \frac{\alpha}{2} \Box \phi - \beta (\nabla \phi)^2 - \frac{\gamma}{12} H^2 - V(\phi) \right\}, \quad H^2 \equiv H_{abc} H^{abc}, \]

performing metric variation and then expressing the field equations in terms of the ricci tensor

\[ 8\pi \kappa^2 \exp(2\phi) S_{ab} = R_{ab} - \alpha \phi_{;ab} + 2\beta \phi_a \phi_b + \frac{\gamma}{2} H_{acd} H^{cd}_{;b} + g_{ab} \left( -\frac{\gamma}{6} H^2 + V \right), \]

the metric (1) is a solution with scalar field given by (7) and

\[ H^{abc} = \pm \sqrt{\frac{\alpha^2 - 4\beta}{2\gamma}} \epsilon^{abcd} \phi_d, \quad V = \frac{v^2}{r^2(X - v^2)^3} (X(X - 4v^2) + v^4). \]

5 Properties and comments

Seventeen properties and comments follow.

Firstly seven points on the metric:

**M1** (1) has constant rotation curves when \( \lambda r \) is negligible, the easiest way to see this is that the circular vector

\[ P^a = \left( \frac{v}{r} \delta^a_\theta + \delta^a_t \right) f(r), \]

where \( f \) is an arbitrary function of \( r \) has acceleration

\[ \dot{P}_a = \lambda r f^2 \delta_r^a, \]

so that when \( \lambda \) vanishes the vector is acceleration free or geodesic. For (6) with the same vector (16)

\[ \dot{P}_a = m\sqrt{X(3v^2)} \frac{f^2}{r^2} \delta_r^a, \]

so that when \( m \) vanishes the vector (16) is again geodesic.

**M2** (1) is asymptotically desitter as \( \lambda r^2 \) increases much faster than \( \ln(r) \). For \( \lambda = 0 \) the line element is not asymptotically flat as the \( \ln \) term in \( g_{tt} \) diverges, this is a problem with many models of galactic rotation which have no natural long radial distance cut off, having an asymptotically desitter spacetime provides such a cut off. Whether this can be thought of as evidence of a non-vanishing cosmological constant or just an indication of the
effect of distant matter does not have to be chosen.

M3 the short distance cut off for the metric is good, for short distances (6) shows that the galactic metric can approach schwarzschild spacetime.

M4 at \( r = 1, \ln(r) \) changes sign and it is not immediate where this is in meters, however note that the solution is still a solution with \( r \rightarrow r/r_o \) so that the length scale \( r_o \) is arbitrary and has to be fixed by other means.

M5 it is not clear what, if anything, corresponds to the vanishing of the three metric functions in (1), \((X - v^2)\) occurs in the denominator of curvature invariants so as it approaches zero they diverge.

M6 the line elements (1) and (6) were taken to be spherically symmetric rather than axi-symmetric, but rotation of galactic spacetime would be expected, spacetime rotation would need a more elaborate model, one could simply choose \( ds^2 = ds^2_{kerr} \) however kerr rotation is fundamentally short range whereas galactic rotation is long range, the simpler case of the newtonian model uses just the log potential \( v^2 \ln(r) \) so that the newtonian model is spherically symmetric and this suggests that the simplest relativistic models are also spherically symmetric.

M7 why choose a line element with vanishing weyl tensor in the first place: from the perspective of the jordan formulation of einstein’s equations one might expect that at large distances the weyl scalar to be larger than the ricci scalar; however from the perspective of the schwarzschild solution which has vanishing ricci tensor and robertson-walker spacetime which has vanishing weyl tensor one might expect that vanishing weyl tensor characterizes large distances, then the question arises as to at what range the weyl tensor becomes non-negligible, presumably this depends on the distance \( r_o \) introduced above, so far there the metric (6) suggests that it is a universal length rather than a length dependent on the mass of the galaxy under consideration.

Secondly four points on the palatini-scalar-tensor solution:

P1 palatini variations work well for general relativity where they reproduce the christoffel connection, but not so for quadratic action theories where they act back on the lagrangian to produces tensors which appear to be unconnected to anything: for scalar-tensor theories they act back on the primary dilaton function producing a non-christoffel conection and this turns out to be necessary in the present case; purely metric variation of the action (12) is unlikely to recover the field equations (8) as can be seen by subtracting off the \( \theta, \theta \) component from the \( t, t \) component: for the palatini case the non-vanishing of \( Y \) in (3) can be matched to the \( \phi_{,ab} \) term, however for the purely metric case there is no \( \phi_{,ab} \) term.
P2 the secondary dilaton function $B$ is taken to vanish (11), this means that there is no explicit kinetic term in (12), although there is an implicit kinetic term after palatini variation; no explicit kinetic term is similar to some inflationary models where the potential $V$ is related to the hubble constant and the kinetic term is less important.

P3 as $k = 0$ robertson-walker cosmology is conformally flat the field equations (8) are obeyed for any choice of the scale-factor, however the field equations (9) and (11) might have as yet unexamined solutions and properties.

P4 because the scalar-tensor theory involves palatini variations the underlying geometry is no longer riemannian but rather weyl with object of non-metricity related to the primary primary dilaton function $A$, see [1].

Thirdly three points on the properties of the string solution (15):

S1 the values of the constants in (15) lead to a real three-form $H$, they could have led to a complex one; although there are three constants after absorption there is only one free parameter.

S2 the potential in (15) is contrived, perhaps with the inclusion of additional fields it will no longer be necessary.

S3 the lagrangian (13) includes a $\Box \phi$ term, as for palatini-scalar-tensor theory this term is necessary in order for the field equation involving $Y$ in (3) to vanish, however the term can be removed from the lagrangian by integration by parts.

Fourthly three general comments:

G1 the introduction of the absorbable constant $\alpha$ is done because in the better known case of spherically symmetric static minimal scalar fields the value of the constant in the equation analogous to (7) depends on schwarzschild mass, but the value of the constant in (8) does not, so it is anticipated that something similar could happen here.

G2 the equations (8) are assumed to be consistent as they can be lagrangian based, however conservation equations and the initial value problem are not looked at: conservation and euler equations are not immediate in palatini scalar-tensor theory, also the initial value problems is not straightforward when there are $\phi_{,ab}$ terms involved; these consistency problems depend on the variables and connection in which they are formulated.

G3 it is not immediate how the result impacts other areas of physics, for example globular star clusters also exhibit unusual dynamics, but usually have no overall $v$ as in the present case, so far the post-newtonian approximation of (1) has not been studied.
6 Conclusion

A relativistic model of galactic rotation curves was produced which has the properties discussed in the last section. A problem with the metric is how many field equations and lagrangians can it be a solution to. The most important problem with the scalar-tensor theory is the physical origin of the scalar field and the most likely explanation is that the scalar field comes from dimensional reduction. The most important problem with the string theory solution is the contrived potential which hopefully will at some time in the future be replaced by fields.

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References

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