Local Plücker formulas for orthogonal groups

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Abstract
Local Plücker formulas relate the vector of metrics on a holomorphic curve in the projective space, induced by the Fubini–Study metrics on projective spaces via Plücker embeddings of associated curves, to the corresponding vector of curvatures. Givental noted that the vector of curvatures is expressed in terms of the vector of metrics via Cartan matrix of type $A_n$, conjectured a version of Plücker formulas for any simple Lie group and proved it in type $C_n$. In this paper we show how local Plücker formulas for special orthogonal groups, i.e. for Cartan matrices of type $B_n$ and $D_n$, can be obtained by reduction to the classical $A_n$ case.

Keywords Plücker formulas · Spin representation · Fubini–Study metric · Cartan matrix

Mathematics Subject Classification 17B10 · 32M05 · 15A66

1 Introduction
Consider a holomorphic curve $\Sigma$ and a nondegenerate map $f = f_1 : \Sigma \rightarrow \mathbb{P}^n$. Define a local lift of $f_1$ in a neighborhood $U$ of any point $p \in U \subset \Sigma$ by a holomorphic vector-valued function $v(z) = (v_0(z), \ldots, v_n(z)) : U \rightarrow \mathbb{C}^{n+1}$ and define the $k$-th, $1 \leq k \leq n$, associated curve as

$$f_k : U \rightarrow \text{Gr}(k, n + 1) \subset \mathbb{P}(\wedge^k \mathbb{C}^{n+1})$$

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by 

\[ f_k(z) = [v(z) \wedge v'(z) \wedge \cdots \wedge v^{(k-1)}(z)]. \]

Fubini–Study metric (FS) on projective space \( \mathbb{P}(\wedge^k \mathbb{C}^{n+1}) \) induces Hermitian metric on the \( k \)-th associated curve and the corresponding curvature forms via the Plücker embedding of \( \text{Gr}(k, n+1) \). In both cases they are \((1, 1)\)-forms on the associated curve. We will denote by \( \varphi_k \) the associated \((1, 1)\)-form of the metric on \( \Sigma \), and by \( \theta_k \) the curvature form corresponding to the metric \( \varphi_k \). The classical local Plücker formulas express the vector of curvatures \( \vartheta \) in terms of the vector of metrics \( \varphi \) using the Cartan matrix \( C \) of the \( A_n \) series in the following form:

\[ \vartheta = C \varphi. \]

One can also construct another vector of associated \((1, 1)\)-forms of the metrics induced by pullback of the Fubini–Study metrics and the corresponding curvatures via the following morphisms of the associated curves to projective spaces. We will follow the treatment from [2, 5].

Let \( f: \Sigma \to F \) be a holomorphic curve on the flag variety \( F := G/B \) where \( G \) is a complex semisimple Lie group of rank \( n \), and \( B \) is a Borel subgroup. We denote the tangent space to the point \( B \in F \) by \( T_B F \). Note that \( T_B F \) is canonically isomorphic to \( g/b \) where \( g, b \) are Lie algebras of the groups \( G \) and \( B \) respectively. Consider the \( n \)-dimensional distribution \( \mathcal{N} \) on the flag variety \( F \):

\[ \mathcal{N}(B) = \{ x \in g \mid [x, [b, b]] \subset b \} / b \subset T_B F. \]

Let us now begin by considering the collection of line bundles \( L_1, \ldots, L_n \) on \( F \) such that each \( L_i \) corresponds to the fundamental characters \( \chi_i \) of the group \( B \). The space \( \Gamma(F, L_i)^\vee \) is the fundamental representation \( V_i \) of the group \( G \). With the line bundle \( L_i \) we can associate the projective morphism \( \pi_i \):

\[ \pi_i: G/B \to \mathbb{P}(V_i) \]

\[ [g] \mapsto [g \cdot v_i], \]

where \( v_i \) is a highest weight vector of the representation \( V_i \) and \( g \in G \).

The representation \( V_i \) is irreducible, and therefore it has a Hermitian inner product invariant with respect to the action of the compact form \( K \) of the group \( G \). Note that this Hermitian inner product is uniquely defined up to multiplication by a constant scalar factor. The Hermitian form determines the Fubini–Study metric \( \text{FS} \) on the space \( \mathbb{P}(V_i) \). The induced metric on the original curve \( \Sigma \) is denoted by \( \varphi_i = f^* \pi_i^*(\text{FS}) \), and we write \( \theta_i \) for the corresponding curvature form. Givental conjectured in [2]:

**Conjecture 1.1** If the holomorphic curve \( f: \Sigma \to F \) is an integral curve of the distribution \( \mathcal{N} \), then the vector \( \theta \) of curvature forms is expressed in terms of the vector of metrics \( \varphi \) using the Cartan matrix \( C \) of the Lie algebra \( g \):

\[ \theta = C \varphi. \]
Consider the case when $G = \text{SL}(n+1, \mathbb{C})$. The condition that the curve $f : \Sigma \to F$ is an integral curve of the distribution $\mathcal{N}$ means that the preimage under $\pi_1 : F \to \mathbb{P}(V)$ of the point $x \in \pi_1(\Sigma)$ is the flag of osculating $i$-planes of the curve $\pi_1(\Sigma)$ at the point $x$, i.e. the curve $\pi_i(\Sigma)$ is the $i$-th associated of $\pi_1(\Sigma)$.

In this case the exterior power $\land^i V$ of the standard representation of $G$ is the fundamental representation $V_i$. The projective morphism $\pi_i$ associated with the line bundle $\mathcal{L}_i$ provides the Plücker embedding of the $i$-th associated curve on $\text{Gr}(i, n+1)$ in $\mathbb{P}(V_i)$.

However Plücker embeddings of associated curves on Grassmannians do not always coincide with embeddings into the projectivization of fundamental representations. In particular this can be seen in the example of the special orthogonal groups.

Positselski proved Conjecture 1.1 in [5] using the general technique for line bundles over flag varieties. At the same time in [2], in addition to the general formulation of the conjecture, Givental proposed the proof of the local Plücker formulas for symplectic group $\text{Sp}(2n, \mathbb{C})$ based on the reduction of the problem to the classical formulas for the group $\text{SU}(2n)$.

However, the question of the geometric interpretation in the case of a special orthogonal group has remained unanswered. In this paper we prove Conjecture 1.1 for orthogonal groups, i.e. for Cartan matrices of the series $B_n$ and $D_n$. Our proof is based on the reduction to the classical case.

2 Local Plücker formulas for the group $\text{SO}(2n, \mathbb{C})$

Let us begin by considering the case when the group $G = \text{SO}(2n, \mathbb{C})$ with the Lie algebra $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Given a $2n$-dimensional vector space $V$ over $\mathbb{C}$ equipped with a nondegenerate symmetric bilinear form $Q$ which is preserved by $G$, we will choose a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ for $V$ in terms of which the quadratic form is given by

$$Q(e_i, f_i) = Q(f_i, e_i) = 1,$$
$$Q(e_i, e_j) = Q(f_i, f_j) = 0,$$

and

$$Q(e_s, f_k) = Q(f_k, e_s) = 0, \quad \text{if} \quad k \neq s.$$

The action of $G$ on the set of isotropic flags in $V$ with a given set of numerical invariants (i.e., the sequence of dimensions of its successive terms) is transitive except the case when maximal isotropic subspace in the flag has dimension $n$. In the latter case there are exactly two $G$-orbits with the given numerical invariants.

We will therefore consider two connected components of the variety of maximal isotropic flags in $V$. We denote these components by $F_+$ and $F_-$ and assume that the Borel subgroup $B$ stabilizes the standard maximal isotropic flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle$ from $F_+$.

In the case of even orthogonal groups the exterior powers of the tautological representation $V_i := \land^i V$ of $G$ are irreducible only for $i = 1, \ldots, n-1$. Moreover, the
exterior powers provide us with the fundamental representations of the group $G$ only for $i = 1, \ldots, n - 2$.

Let us denote by $C(Q)^{\text{even}}$ the subalgebra of the Clifford algebra $C(Q)$ generated by elements of even degree with respect to the natural $\mathbb{Z}/2\mathbb{Z}$ grading. The remaining two fundamental representations $V_i$ for $i = n - 1, n$ are realized as $C(Q)^{\text{even}}$-modules.

For future reference we prove the following lemma. We briefly recall the relevant constructions here. The decomposition $V = W \oplus W'$, where $W = \langle e_1, e_2, \ldots, e_n \rangle$ and $W' = \langle f_1, f_2, \ldots, f_n \rangle$, determines an isomorphism of algebras

$$C(Q) \cong \text{End}(\wedge^* W),$$

and $C(Q)^{\text{even}}$ respects the splitting $\wedge^* W = \wedge^{\text{even}} W \oplus \wedge^{\text{odd}} W$. It then follows that

$$C(Q)^{\text{even}} \cong \text{End}(\wedge^{\text{even}} W) \oplus \text{End}(\wedge^{\text{odd}} W).$$

Moreover we have an embedding of Lie algebras

$$\mathfrak{so}(2n, \mathbb{C}) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\wedge^{\text{even}} W) \oplus \mathfrak{gl}(\wedge^{\text{odd}} W).$$

These two representations $\wedge^{\text{even}} W$ and $\wedge^{\text{odd}} W$ are usually called half-spin representations (see [1]).

Let us denote by $E_{i, i}$ the diagonal matrix that takes $e_i$ to itself and kills $e_j$ for $j \neq i$. With the above-mentioned choice (1), we may take as a Cartan subalgebra $\mathfrak{h}$ the subalgebra of diagonal matrices generated by the $2n \times 2n$ matrices $H_i := E_{i, i} - E_{i + n, i + n}$. We will correspondingly take as basis for the dual vector space $\mathfrak{h}^*$ the dual basis $L_i$ where $L_i(H_j) = \delta_{i,j}$. For concreteness assume that the representation $S_-$ corresponds to the fundamental weight $\omega_{n-1} = \omega_- := \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$ and the representation $S_+$ corresponds to the fundamental weight $\omega_n = \omega_+ := \frac{1}{2}(L_1 + \cdots + L_{n-1} + L_n)$.

We also denote the morphisms $\pi_{n-1}$ and $\pi_n$ associated with the line bundles $\mathcal{L}_{n-1}$ and $\mathcal{L}_n$ by $\pi_-$ and $\pi_+$ respectively, i.e. suppose that $\pi_{\pm} : G/B \to \mathbb{P}(S_{\pm})$, $\pi_{\pm} : g \mapsto [gv_{\pm}]$, where $v_+$ and $v_-$ are highest weight vectors of the representations $S_+$ and $S_-$. Let $i_+ : G/B \to \text{Fl}(V)$ be the embedding that corresponds to the orbit of the standard maximal isotropic flag, i.e. $i_+ : [b] \mapsto \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_{n-1}, e_n \rangle$. Define the analogous embedding that corresponds to the other orbit $i_- : G/B \to \text{Fl}(V), [b] \mapsto \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_{n-1}, f_n \rangle$. Denote the composition of $i_{\pm}$, projection to $\text{Gr}(n - 1, 2n)$ and the Plücker embedding by $p_{n-1}$:

$$p_{n-1} : G/B \to \text{Fl}(V) \to \text{Gr}(n - 1, 2n) \to \mathbb{P}(\wedge^{n-1} V).$$

For future reference we prove the following lemma.
Lemma 2.1 Let $\sigma: \mathbb{P}(S_+) \times \mathbb{P}(S_-) \to \mathbb{P}(S_+ \otimes S_-)$ be the Segre embedding. Then there is a natural linear embedding $e: \mathbb{P}(\wedge^{n-1}V) \to \mathbb{P}(S_+ \otimes S_-)$ such that the following diagram is commutative:

\[ \begin{array}{ccc}
\mathbb{P}(S_+) \times \mathbb{P}(S_-) & \xrightarrow{\sigma} & \mathbb{P}(S_+ \otimes S_-) \\
\downarrow^{(\pi_+ \times \pi_-)} & & \downarrow^{e} \\
G/B & & \mathbb{P}(\wedge^{n-1}V)
\end{array} \]

Proof Note that $\wedge^{n-1}V$ is the irreducible representation of $\mathfrak{g}$ with the highest weight $\omega_+ + \omega_-$ (see [1]). The complete decomposition into irreducible representations of $S_+ \otimes S_-$ looks like (see [4])

\[ S_+ \otimes S_- = \wedge^{n-1}V \bigoplus \bigoplus_{j>0} V_{n-2j-1}, \]  

where $V_{n-2j-1}$ is the irreducible representation of $\mathfrak{g}$ with the highest weight $\omega_{n-2j-1} = L_1 + L_2 + \cdots + L_{n-2j-1}$, i.e. $V_{n-2j-1} = \wedge^{n-2j-1}V$. Note that the vector $v_+ \otimes v_-$ has weight $\omega_n + \omega_{n-1}$ and therefore it is the highest weight vector of the first summand in (3). \qed

The projective space $\mathbb{P}(S_+ \otimes S_-)$ admits a $K$-invariant Fubini–Study metric induced by the $K$-invariant Hermitian inner products on the spaces $S_+$ and $S_-$. We denote the corresponding $(1, 1)$-form of the metric on the space $\mathbb{P}(S_+ \otimes S_-)$ by $FS$. The following standard properties of $(1, 1)$-forms of the metrics will be needed later (see [3]).

Lemma 2.2 Let $N, M$ be complex manifolds, $f: N \to M$ a holomorphic map such that the induced map $f^*: T'_z(N) \to T'_{f(z)}(M)$ is injective for all $z \in N$, then the associated $(1, 1)$-form of the induced metric on $N$ is the pullback of the associated $(1, 1)$-form of the metric on $M$.

Lemma 2.3 Let $\mathbb{P}^n$ and $\mathbb{P}^m$ be projective spaces with homogeneous coordinates $[z_0, \ldots, z_n]$ and $[w_0, \ldots, w_m]$ respectively, equipped with the standard Fubini–Study metrics, $\sigma$ is the Segre embedding $\sigma: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$

\[ \sigma([z_0, \ldots, z_n], [w_0, \ldots, w_m]) = [(z_i w_j)_{0 \leq i \leq n, 0 \leq j \leq m}]. \]

Denote by $\alpha: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $\beta: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ the projections onto the factors, then $\sigma$ is a holomorphic isometry, that is

\[ \sigma^*FS = \alpha^*FS + \beta^*FS. \]

We shall need the following fact.
Lemma 2.4 Let $\varphi_+$ and $\varphi_-$ denote the pullback of the FS metrics on the spaces $\mathbb{P}(S_+)$ and $\mathbb{P}(S_-)$ via morphisms $\pi_+ \circ f$ and $\pi_- \circ f$ respectively and let $\phi_{n-1}$ be the metric on the curve $\Sigma$ induced by pullback of the metric FS on the space $\mathbb{P}(\Lambda^n V)$ via $p_{n-1} \circ f$. Then the following equality holds

$$\varphi_+ + \varphi_- = \phi_{n-1}. \tag{4}$$

Proof The commutative diagram (2) shows that on the one hand

$$(e \circ p_{n-1} \circ f)^* \text{FS} = (p_{n-1} \circ f)^* \text{FS} = \phi_{n-1}. \tag{5}$$

On the other hand, it follows from the above lemmas that the following equalities hold:

$$(\sigma \circ (\pi_+, \pi_-) \circ f)^* \text{FS} = ((\pi_+, \pi_-) \circ f)^*(\alpha^* \text{FS} + \beta^* \text{FS})$$

$$= (\pi_+ \circ f)^* \text{FS} + (\pi_- \circ f)^* \text{FS} = \varphi_+ + \varphi_-.$$

(6)

To prove the lemma it is sufficient to note that the left-hand sides of (5) and (6) coincide.

Consider the following morphisms $p_{\pm}$:

$$p_{\pm} : G/B \to \text{Fl}(V) \to \text{Gr}(n, 2n) \subset \mathbb{P}(\Lambda^n \mathbb{C}^{2n})$$

which are compositions of the embeddings $i_{\pm}$, projections to $\text{Gr}(n, 2n)$ and the Plücker embedding. We shall need the following properties of the morphisms $p_{\pm}$.

Lemma 2.5 Let $v_{\pm} : \mathbb{P}(S_{\pm}) \to \mathbb{P}(\text{Sym}^2 S_{\pm})$ be the quadratic Veronese embeddings, then there exist natural linear embeddings $e_{\pm} : \mathbb{P}(\Lambda^n V_{\pm}) \to \mathbb{P}(\text{Sym}^2 S_{\pm})$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{P}(S_{\pm}) & \overset{\pi_{\pm}}{\longrightarrow} & \mathbb{P}(\text{Sym}^2 S_{\pm}) \\
\pi_{\pm} \downarrow & & \downarrow e_{\pm} \\
G/B & \overset{p_{\pm}}{\longrightarrow} & \mathbb{P}(\Lambda^n V_{\pm})
\end{array} \tag{7}
$$

Proof The representation $\Lambda^n V$ is no longer an irreducible representation of $\mathfrak{g}$ as in the classical case. It decomposes into a direct sum of the irreducible representations $\Lambda^n V_-$ and $\Lambda^n V_+$ with highest weights $2\omega_-$ and $2\omega_+$ respectively (see [1, 4]).

Note also that there is the following direct sum decomposition of $\text{Sym}^2 S_{\pm}$ into irreducible representations of $\mathfrak{g}$ (see [4]):

$$\text{Sym}^2 S_{\pm} = \Lambda^n V_{\pm} \oplus \bigoplus_{j > 0} \Lambda^{n-4j} V.$$

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Since the vectors $v_+ \otimes v_+$ and $v_- \otimes v_-$ are vectors of highest weight $2\omega_+$ and $2\omega_-$ respectively, we get the natural linear embeddings $e_\pm$. 

**Remark 2.6** The morphisms $\pi_+$ and $\pi_-$ decompose into a composition as follows:

$$\pi_\pm : G/B \rightarrow G/P_\pm \xrightarrow{\text{spin}_\pm} P(S_\pm).$$

Note that Pic$(G/P_\pm) \cong \mathbb{Z}$ due to the fact that $P_\pm$ is a maximal parabolic subgroup, and the positive generator of the group corresponds to the very ample bundle $\mathcal{O}_{S_\pm}(1)$ where $S_+$ and $S_+$ are spinor varieties. The embeddings spin$_\pm$ are associated to these generators.

**Lemma 2.7** Let $\phi_+$ and $\phi_-$ denote the pullback of FS metrics from the spaces $P(\wedge^n V_+)$ and $P(\wedge^n V_-)$ via morphisms $p_+ \circ f$ and $p_- \circ f$ respectively and let $\phi_{r-1}$ be the metric on the curve $\Sigma$ induced by pullback of metrics FS from the spaces $P(\wedge^{r-1} V_+)$ and $P(\wedge^{r-1} V_-)$ via morphisms $p_+ \circ f$ and $p_- \circ f$. Then the following equalities hold: $\phi_+ = 2\phi_+$ and $\phi_- = 2\phi_-$. 

**Proof** In order to prove the above statement we need to return to the commutative diagram (7). The spaces $P(\text{Sym}^2 S_+)$ and $P(\text{Sym}^2 S_-)$ have the Fubini–Study metric induced by the Hermitian scalar product on the spaces $S_+ \otimes S_+$ and $S_- \otimes S_-$ respectively.

Note that, the Segre embedding $\sigma_+ : P(S_+) \rightarrow P(S_+ \otimes S_+)$ factors through $P(S_+) \times P(S_+)$:

$$v_+ : P(S_+) \xrightarrow{\Delta} P(S_+) \times P(S_+) \xrightarrow{\sigma} P(S_+ \otimes S_+),$$

where $\Delta$ is the diagonal morphism, $\sigma$ is the Segre embedding. We already know by Lemma 2.3 that $v_+^*(FS) = 2FS$. A similar relation holds for the morphism $v_-$. Thus we get

$$2\phi_\pm = (v_+ \circ \pi_\pm \circ f)^*FS = (e_\pm \circ p_\pm \circ f)^*FS = \phi_\pm. \quad \Box$$

Let $\theta(\mathcal{L})$ denote the curvature of the $K$-invariant metric on an arbitrary line bundle $\mathcal{L}$ with the action of $G$ on $G/B$. Let $l$ be a local section of $\mathcal{L}$, then the following formula holds:

$$\theta(\mathcal{L}) = -\partial \bar{\partial} \ln ||l||.$$ 

We slightly abuse notation and take certainly a standard candidate for Cartan subalgebra of $\mathfrak{so}(2n, \mathbb{C})$, namely the subalgebra of diagonal matrices with zero trace and write $L_i$ for the element of the dual basis such that $L_i(E_j, j) = \delta_{i,j}$. The following lemma shows the relation of curvature forms under morphisms $i_\pm$. These relations can be derived from direct calculations.
Lemma 2.8 Suppose $N_{\pm}$ are line bundles on the projective variety $G/B$ corresponding to the simple roots $\lambda_{\pm} = L_{n-1} \pm L_n$ of $\mathfrak{so}(2n, \mathbb{C})$, and $N_{n-1}$ is the line bundle on the full flag variety $\text{Fl}(V)$ corresponding to the simple root $\lambda_{n-1} = L_{n-1} - L_n$ of $\mathfrak{sl}(2n, \mathbb{C})$. Then there are following identities:

\[ i_{\pm}^* (\theta(N_{n-1})) = \theta(N_{\mp}). \]

Classical Plücker formulas for the group $\text{SL}(2n, \mathbb{C})$ give the following relation on the curve $\Sigma$:

\[ \vartheta = C \phi, \quad (8) \]

where $C$ is a Cartan matrix of type $A_{2n-1}$. We notice that relation (8) descends to the analogous relation on $\text{Fl}(V)$ (see [5]):

\[ \hat{\vartheta} = C \hat{\phi}, \quad (9) \]

where $\hat{\vartheta}$ is the vector of $(1, 1)$-forms $\hat{\vartheta}_i := \theta(N_i)$ on $\text{Fl}(V)$, $N_i$ is the line bundle on $\text{Fl}(V)$ corresponding to the simple root $\lambda_i = L_i - L_{i+1}$ of $\mathfrak{sl}(2n, \mathbb{C})$, $\hat{\phi}$ is the vector of $(1, 1)$-forms with $\hat{\phi}_i := \hat{p}_i^* (\mathbb{P}(\Lambda^1 V))$ and $\hat{p}_i : \text{Fl}(V) \to \mathbb{P}(\Lambda^1 V)$ is the standard projection.

Theorem 2.9 The local Plücker formulas for the group $\text{SO}(2n, \mathbb{C})$ are obtained by taking pullback of the classical Plücker formulas on $\text{Fl}(V)$ via the maps $i_{\pm} \circ f$ of the first $n - 1$ components of the vector $\hat{\vartheta}$.

Proof We need only to deal with the first $n - 1$ components of the vector $\hat{\vartheta}$. The first $n - 3$ components of the vector $\hat{\vartheta}$ remain unchanged as the pullback is taken with respect to morphisms $i_{\pm}$. Note that the result does not depend on the choice of embedding $i_+$ or $i_-$. Consider the $(n - 2)$-th row of the matrix relation (9):

\[ \hat{\vartheta}_{n-2} = - \hat{\phi}_{n-3} + 2\hat{\phi}_{n-2} - \hat{\phi}_{n-1}. \]

Taking into account formulas (4) and applying pullback to the right- and left-hand sides we have

\[ \vartheta_{n-2} = (i_+ \circ f)^* \hat{\vartheta}_{n-2} = - (i_+ \circ f)^* \hat{\phi}_{n-3} + (i_+ \circ f)^* 2\hat{\phi}_{n-2} - (i_+ \circ f)^* \hat{\phi}_{n-1} = - \varphi_{n-3} + 2\varphi_{n-2} - \varphi_+ - \varphi_. \]

Next we consider the $(n - 1)$-th component of the curvature vector $\hat{\vartheta}$ and the corresponding row of the matrix equality (9):

\[ \hat{\vartheta}_{n-1} = - \hat{\phi}_{n-2} + 2\hat{\phi}_{n-1} - \hat{\phi}_n. \quad (10) \]

In view of Lemmas 2.4, 2.7 and 2.8 we observe that pullback of the right- and left-hand sides of equality (10) via the morphism $i_+ \circ f$ yields the desiring equality for the one
of the last two rows of Plücker relations for $SO(2n, \mathbb{C})$:
\[
\theta_- = (i_+ \circ f)^*(\hat{\vartheta}_{n-1}) = -(i_+ \circ f)^*(\hat{\varphi}_{n-2}) + (i_+ \circ f)^*(2\hat{\varphi}_{n-1}) - (i_+ \circ f)^*(\hat{\varphi}_n) = -\varphi_{n-2} + 2\varphi_+ + 2\varphi_- - 2\varphi_+ = -\varphi_{n-2} + 2\varphi_-.
\]

Then the same reasoning as before shows that the remaining row of Plücker formulas for $SO(2n, \mathbb{C})$ is given by:
\[
\theta_+ = (i_- \circ f)^*(\hat{\vartheta}_{n-1}) = -(i_- \circ f)^*(\hat{\varphi}_{n-2}) + (i_- \circ f)^*(2\hat{\varphi}_{n-1}) - (i_- \circ f)^*(\hat{\varphi}_n) = -\omega_{n-2} + 2\omega_+ + 2\omega_- - 2\omega_+ = -\omega_{n-2} + 2\omega_+.
\]

With matrix notation we get the full collection of Plücker formulas on the curve $\Sigma$ for $SO(2n, \mathbb{C})$:
\[
\theta = A\varphi,
\]
where $A$ is a Cartan matrix of type $D_n$:
\[
\begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & \ldots & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & -1 & 2
\end{pmatrix}.
\]

3 Local Plücker formulas for the group $SO(2n + 1, \mathbb{C})$

Consider the case $G = SO(2n + 1, \mathbb{C})$. Let $V$ be a complex vector space of dimension $2n+1$ with a nondegenerate symmetric bilinear form $Q$ which is preserved by $G$. In this case the exterior power $V_i := \wedge^i V$ of the standard representation $V$ of $so(2n + 1, \mathbb{C})$ is the irreducible representation with highest weight $L_1 + \cdots + L_i$ for $i = 1, \ldots, n$. Exterior powers of the standard representation provide us with the fundamental representations of the group $G$ only for $i = 1, \ldots, n - 1$. The remaining fundamental representation $V_n$ with highest weight $\omega_n := (L_1 + \cdots + L_n)/2$ is called the spin representation and is realized on $S := \wedge^\bullet W$ where $W \subset V$ is the $n$-dimensional isotropic subspace. As in the case of even orthogonal Lie algebras, we can consider the projective morphism associated with line bundle $L_i$ on $G/B$ corresponding to the fundamental weight $\omega_i$.

We immediately get the first $(n - 2)$-th rows of Plücker formulas for $SO(2n + 1, \mathbb{C})$. We briefly show that two remaining formulas for $i = n - 1, n$ can be reconstructed from classical formulas.

Note that for the odd case a lemma similar to Lemma 2.5 is true.
Lemma 3.1 Let $v: \mathbb{P}(S) \to \mathbb{P}(S \otimes S)$ be the quadratic Veronese embedding and let $p_n: G/B \to \mathbb{P}(\Lambda^n V)$ be the the composition of the standard projection to Grassmannian with Plücker embedding, then there exists a natural linear embedding $e: \mathbb{P}(\Lambda^n V) \to \mathbb{P}(S \otimes S)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{P}(S) & \xrightarrow{v} & \mathbb{P}(\Lambda^n V) \\
\pi_n & & \downarrow e \\
G/B & \xrightarrow{p_n} & \mathbb{P}(\text{Sym}^2 S)
\end{array}
\]

Proof The representation $S \otimes S$ of $\text{so}(2n + 1, \mathbb{C})$ decomposes into a direct sum (see [1])

\[S \otimes S = \Lambda^n V \oplus \Lambda^{n-1} V \oplus \cdots \oplus \Lambda^0 V.\]

Let $v_n$ be the highest weight vector with weight $\omega_n$ of the representation $S$. Next observe that $v_n \otimes v_n \in \text{Sym}^2 S$ is the vector with weight $L_1 + \cdots + L_n$, i.e. $v_n \otimes v_n$ is the highest weight vector of the irreducible representation $\Lambda^n V \subset \text{Sym}^2 S$ (see [1]). The lemma is an immediate consequence of this fact. \qed

The technical statements in the odd dimensional case can be worked out similarly to even orthogonal groups.

Lemma 3.2 Let $\varphi_n$ denote the pullback of the FS metric on the space $\mathbb{P}(S)$ via morphisms $\pi_n \circ f$ and let $\phi_{n-1}$ be the metric on the curve $\Sigma$ induced by pullback of the metric FS on the space $\mathbb{P}(\Lambda^n V)$ via $p_n \circ f$. Then the following equality holds:

\[2 \varphi_n = \phi_n.\]

Combining the previous two lemmas, we can now formulate the orthogonal version of Plücker formulas for odd orthogonal groups.

Theorem 3.3 The local Plücker formulas for the group $SO(2n + 1, \mathbb{C})$ are obtained by taking pullback of the classical Plücker formulas on $\text{Fl}(V)$ via the map $i \circ f$ of the first $n$ components of the vector $\mathbf{\hat{\theta}}$.

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References

1. Fulton, W., Harris, J.: Representation Theory. Graduate Texts in Mathematics, vol. 129. Springer, New York (2004)
2. Givental, A.B.: Plücker formulas and Cartan matrices. Russian Math. Surveys 44(3), 193–194 (1989)
3. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley Classics Library. Wiley, New York (1994)
4. Manivel, L.: On spinor varieties and their secants. SIGMA Symmetry Integrability Geom. Methods Appl. 5, Art. No. 078 (2009)
5. Positselski, L.E.: Plücker formulas for a semisimple Lie group. Funct. Anal. Appl. 25(4), 291–292 (1991)

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