ON THE LOGARITHMIC
KELLER-SEGEL-FISHER/KPP SYSTEM

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Abstract. We consider the Cauchy problem of a Keller-Segel type chemotaxis model with logarithmic sensitivity and logistic growth. We study the global well-posedness, long-time behavior, vanishing coefficient limit and decay rate of solutions in $\mathbb{R}$. By utilizing energy methods, we show that for any given classical initial datum which is a perturbation around a constant equilibrium state with finite energy (not small), there exists a unique global-in-time solution to the Cauchy problem, and the solution converges to the constant equilibrium state, as time goes to infinity. Under the same initial condition, it is shown that the solution with positive chemical diffusion coefficient converges to the solution with zero chemical diffusion coefficient, as the coefficient goes to zero. Furthermore, for a slightly smaller class of initial data, we identify the algebraic decay rates of the solution to the constant equilibrium state by employing time-weighted energy estimates.

1. Introduction. We consider the following system of partial differential equations:
\[
\begin{align*}
\tag{1.1}
&u_t + (uv)_x = u_{xx} + ru(1 - u), \quad x \in \mathbb{R}, \ t > 0, \\
&v_t + (u - \varepsilon_2 v^2)_x = \varepsilon_1 v_{xx},
\end{align*}
\]
where $u(x, t)$ and $v(x, t)$ are unknown functions, and $r > 0, \varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ are constants. The purpose of this paper is to study the qualitative behavior (global well-posedness, long time behavior, vanishing coefficient limit, decay rate) of large data solutions to the Cauchy problem of the model under generally prepared initial conditions.

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Key words and phrases. Chemotaxis, logarithmic sensitivity, logistic growth, Cauchy problem, classical solution, global well-posedness, long-time behavior, vanishing coefficient limit, decay rate.

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1.1. Background. System (1.1) is derived from the following Keller-Segel type chemotaxis model with logarithmic sensitivity and logistic growth:

\[
\begin{align*}
    u_t &= Du_{xx} - \chi (u \log(c))_x + au \left(1 - \frac{u}{K}\right), \\
    c_t &= \varepsilon c_{xx} - \mu u c - \sigma c,
\end{align*}
\]

where the unknown functions and system parameters are interpreted as follows:

- \( u(x,t) \): density of cellular population at space location \( x \) and time \( t \),
- \( c(x,t) \): concentration of chemical signal at space location \( x \) and time \( t \),
- \( D > 0 \): diffusion coefficient of cellular population,
- \( \chi \neq 0 \): coefficient of chemotactic sensitivity,
- \( a > 0 \): relative growth rate of cellular population,
- \( K > 0 \): relative carrying capacity of cellular population,
- \( \varepsilon \geq 0 \): diffusion coefficient of chemical signal,
- \( \mu \neq 0 \): coefficient of density-dependent production/degradation rate of chemical signal,
- \( \sigma \geq 0 \): natural degradation rate of chemical signal.

Since the classic Fisher/KPP equation is included in (1.2), by following standard terminologies, we shall term the model as Keller-Segel-Fisher/KPP (KSF) system.

The KSF system, (1.2), belongs to a class of nonlinear reaction-diffusion models with chemotactic effects. Biologically, such a model describes the movement of certain biological organism in response to the chemical signal that it releases or consumes in the local environment while both entities are naturally diffusing and reacting (growing, dying, degrading, \textit{et al}), cf. \cite{2, 10, 11, 14}. One of the characteristic features of (1.2) is the logarithmic (singular) sensitivity function in the first equation, which entails that the chemotactic response of the cellular population to the chemical signal follows the Fechner’s law which states that subjective sensation is proportional to the logarithm of the stimulus intensity and has prominent applications in biological modelings (cf. \cite{3, 16}). The singular sensitivity function is also the major source where the technical (analytical, numerical) difficulties for studying the qualitative behavior of the model come from.

Biologically, the sign of \( \chi \) dictates whether the chemotactic movement is attractive (\( \chi > 0 \)) or repulsive (\( \chi < 0 \)). From the mathematical point of view, the sign of the product of \( \chi \) and \( \mu \) plays an indispensable role in the qualitative analysis of the model. In this paper, we shall consider the case when \( \chi \mu > 0 \), since otherwise the foundation for analytical study will be lost (explained later).

Formally, when \( \chi > 0 \) and \( \mu > 0 \), the first equation in (1.2) indicates that while naturally diffusing and growing/dying, the cellular population is additionally driven by the concentration gradient of the chemical signal in the opposite direction of diffusion (due to \( \chi > 0 \)), which suggests that the cellular population may aggregate at certain spatial locations as time evolves. On the other hand, the (exponentially) rapid degradation in the second equation of (1.2) illustrates that the force driving the cellular population to aggregate is diminishing as time goes on. Hence, one may expect that the system will enter an equilibrium state in the long time run, due to the balance between cellular aggregation and chemical degradation. Similarly, when \( \chi < 0 \) and \( \mu < 0 \), because of the interaction between chemotactic repulsion and chemical production, the system is also expected to reach a steady state as time goes on. Collectively, when \( \chi \mu > 0 \), finite time singularities are not anticipated to develop in the system (1.2), and the synergy of diffusion, chemotactic attraction/repulsion,
logistic growth and chemical degradation/production make the dynamics of the model an intriguing problem to pursue.

1.2. Motivations and literature review. Now we would like to point out the facts that motivate this work, and briefly survey the literature in connection with the current work to put things into perspective.

1.2.1. Connection with Fisher/KPP equation. We mentioned that the coupled system (1.2) contains the classic Fisher/KPP equation [5, 17]:

\[ u_t = Du_{xx} + au \left( 1 - \frac{u}{K} \right) , \tag{1.3} \]

which has the dimensionless form:

\[ u_t = u_{xx} + ru(1 - u). \tag{1.4} \]

Indeed, when \( \chi = 0 \), (1.2) is decoupled and the first equation becomes (1.3). Here the positive constants \( a \) and \( K \) in (1.3) are the relative growth rate and carrying capacity per unit length, respectively. Next we would like to point out one of the major qualitative differences between (1.3) and (1.2). For this purpose, let us consider the Cauchy problem of (1.4) with

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \]

where \( u_0 \) is a perturbation of 1. Using the maximum principle, one can show that if \( 0 \leq u_0(x) \leq 1 \), then \( 0 \leq u(x, t) \leq 1 \). Thus in the original equation (1.3), the population density \( u \) stays bounded above by the capacity \( K \). Such a conclusion, however, is not true for the chemotaxis model (1.2) due to the interaction of the cellular population and the chemical signal. In general, we may expect \( u(x, t) \geq 0 \) but we may not have \( u(x, t) \leq K \).

To illustrate the difference we consider the equation for \( u \) in (1.1):

\[ u_t = u_{xx} - u_x v - uv_x + ru(1 - u), \tag{1.5} \]

where \( v = c_x/c \) is a working variable for \( c_x \), assuming \( c > 0 \). The value of \( v \) can be any real number depending on the increasing/decreasing of \( c \). In our discussion later, \( v \) is a perturbation of zero, thus \( v_x \) changes sign in \( \mathbb{R} \). Since \( v_x \) inevitably takes negative values on some intervals, we are not able to apply the maximum principle to conclude \( u(x, t) \leq 1 \).

As an example we consider the Cauchy problem of (1.2) with initial data \( (u_0, c_0) \), or equivalently of (1.1) with \( (u_0, v_0) \). Let

\[ c_0(x) = \tilde{c} + \delta e^{-x^2}, \]

where \( \tilde{c} \) and \( \delta \) are positive constants, and \( \delta \) is small. By direct calculation we have

\[ v_0(x) = -\frac{2\delta x e^{-x^2}}{\tilde{c} + \delta e^{-x^2}}, \]

\[ v'_0(x) = -\frac{2\delta e^{-x^2}(\tilde{c} + \delta e^{-x^2} - 2\tilde{c}x^2)}{(\tilde{c} + \delta e^{-x^2})^2}. \tag{1.6} \]
It is clear that $v_0, v'_0 \in L^1(\mathbb{R})$, and $v'_0(x) < 0$ for $|x| \leq 1/\sqrt{2}$. Now we take $u_0 \in C^2(\mathbb{R})$ satisfying
\begin{equation}
 u_0(x) = \begin{cases} 
 1 - e^{4/(2x^2-1)} & \text{if } |x| < \frac{1}{\sqrt{2}}, \\
 1 & \text{if } |x| \geq \frac{1}{\sqrt{2}}.
 \end{cases}
 \tag{1.7}
 \end{equation}
Then it is straightforward to check that
\begin{align*}
 0 \leq u_0 \leq 1, \quad u_0 - 1 \in H^s(\mathbb{R}), \quad \forall s \geq 0, \quad u'_0(1/\sqrt{2}) = 0 = u''_0(1/\sqrt{2}).
\end{align*}
Hence (1.5) and (1.7) imply
\begin{align*}
 u_t(\frac{1}{\sqrt{2}}, 0) = -v_x(\frac{1}{\sqrt{2}}, 0) = -v'_0(\frac{1}{\sqrt{2}}) > 0,
\end{align*}
which further implies $u(1/\sqrt{2}, 0^+) > u_0(1/\sqrt{2}) = 1$.

To further illustrate the difference between (1.4) and (1.1), we present a numerical simulation to support the above discussion. In the simulation we set $\varepsilon_1 = 0.05$, $\varepsilon_2 = 0.1$ and $r = 1$ in (1.1). For the initial data we take
\begin{align*}
 u_0(x) &= \begin{cases} 
 1 - 0.1 \cos(\pi x/\sqrt{2}), & |x| \leq 1/\sqrt{2}, \\
 1, & |x| > 1/\sqrt{2},
 \end{cases} \\
 v_0(x) &= -\frac{0.2 x e^{-x^2}}{0.5 + 0.1 e^{-x^2}}.
\end{align*}
Here we note that $u_0$ has two corners, which cause computational difficulties. Utilizing a graph functionality we smooth the corners and replace $u_0$ by the blue curve in Figure 1. The computation domain is $[-25, 25]$, with Dirichlet boundary condition (1, 0) on both sides. Figure 1 displays the simulation results of $u(x, t)$ for $t = 0, 1, 2, 3$, over the interval $[-10, 10]$ to highlight the non-trivial parts of the solution. Clearly, $u(x, t)$ is not bounded above by one when $t > 0$. 

**Figure 1.**
Intuitively, due to the interaction of the cellular population and the chemical signal, we do not expect a uniform carrying capacity for everywhere and all time. Hence the parameter $K$ in (1.2) is understood as a typical scale of capacity, and therefore $u$ in the scaled equation (1.1) is about one and not necessarily bounded above by one.

From the analytical point of view, the maximum principle enjoyed by the Fish/KPP equation substantially simplifies the analysis of the qualitative behavior of the model, while the absence of such a property in (1.2) makes the corresponding analysis considerably difficult. Indeed, the question of qualitative behavior of large data solutions to (1.2), such as global well-posedness, long time dynamics, et al, remains largely open. This is the first fact that motivates the current work.

1.2.2. Connection with balance laws. It was mentioned before that the logarithmic sensitivity function in (1.2) is the major source, from which the technical difficulties for studying the qualitative behavior of the model come. To overcome the possible singularity caused by the logarithmic function, we first pre-process the model by carrying out the following actions including non-dimensionalization, change of variable, rescaling and transformation:

- By changing the variable $u \rightarrow u/K$, (1.2) becomes
  \[ \begin{cases} u_t = D u_{xx} - \chi \left( \frac{c}{c_x} \right)_x + a u (1 - u), \\ c_t = \varepsilon c_{xx} - \mu K u - \sigma c. \end{cases} \] (1.8)

- By changing the variable $c \rightarrow e^{\sigma t} c$, (1.8) becomes
  \[ \begin{cases} u_t = D u_{xx} - \chi \left( \frac{c}{c_x} \right)_x + a u (1 - u), \\ c_t = \varepsilon c_{xx} - \mu K u. \end{cases} \] (1.9)

- By applying the transformation $v = c_x/c$, (1.9) becomes a system of balance laws
  \[ \begin{cases} u_t + \chi (uv)_x = D u_{xx} + a u (1 - u), \\ v_t + (\mu K u - \varepsilon v^2)_x = \varepsilon v_{xx}. \end{cases} \] (1.10)

We remark that the characteristics associated with the flux on the left hand side of (1.10) are
  \[ \lambda_{\pm} = \frac{(\chi - 2\varepsilon)v \pm \sqrt{(\chi + 2\varepsilon)v^2 + 4\chi \mu K u}}{2}, \]
from which we see that the principle part of the system (1.10) is hyperbolic in biologically relevant regimes (where $u > 0$) when $\chi \mu > 0$, while it may change type when $\chi \mu < 0$. This is the reason for which we focus on the case when $\chi \mu > 0$ throughout this paper.

- By further applying the re-scalings:
  \[ t \rightarrow \frac{\chi \mu K}{D} t, \quad x \rightarrow \frac{\sqrt{\chi \mu K}}{D} x, \quad v \rightarrow \text{sign}(\chi) \sqrt{\frac{\chi}{\mu K}} v, \]
to the system (1.10), we obtain a clean version of the model:
  \[ \begin{cases} u_t + (uv)_x = u_{xx} + ru (1 - u), \\ v_t + (u - \varepsilon_2 v^2)_x = \varepsilon_1 v_{xx}, \end{cases} \] (1.11)
where
\[ r = \frac{aD}{\chi \mu K}, \quad \varepsilon_1 = \frac{\varepsilon}{D}, \quad \varepsilon_2 = \frac{\varepsilon}{\chi}. \] (1.12)

From (1.11) we see that the possible singularity caused by the logarithmic sensitivity function is removed by the nonlinear (Cole-Hopf) transformation. On the other hand, the pre-processing introduces a quadratic (convection-like) nonlinearity into the second equation of the model. Hence, properly balancing diffusion and convection, especially in the regime of large data solutions, brings a significant challenge to the analysis of the transformed model.

In this paper, we consider the Cauchy problem of (1.11), with
\[ (u, v)(x, 0) = (u_0, v_0)(x), \quad x \in \mathbb{R}. \] (1.13)
In particular, we are interested in \((u_0, v_0) \to (\bar{u}, \bar{v})\) as \(x \to \pm \infty\), where \((\bar{u}, \bar{v})\) is a constant equilibrium state, i.e. \(\bar{u} = 0\) or \(1\). For stability, it is necessary to take \(\bar{u} = 1\). For physically interesting scenario we set \(\bar{v} = 0\). This is to be seen as follows. Suppose \(v_0 - \bar{v} \in L^1(\mathbb{R})\). If \(\bar{v} \geq 0\), we have
\[ \int_0^\infty v_0(y) \, dy = \pm \infty, \quad \int_{-\infty}^0 v_0(y) \, dy = \pm \infty. \]
Thus from the inverse transform
\[ c(x, 0) = c(0, 0)e^{\int_0^x v_0(y) \, dy}, \quad c(0, 0) > 0, \]
we have \(c(x, 0) \to \infty\) either as \(x \to \infty\) or as \(x \to -\infty\), depending on \(\bar{v} > 0\) or \(\bar{v} < 0\). Similar discussion on \(v\) for traveling waves can be found in [28].

If we take \(\bar{v} = 0\) and assume \(v_0 \in L^1(\mathbb{R})\), then \(c_0(x) \equiv c(x, 0)\) has finite limits as \(x \to \pm \infty\):
\[ \lim_{x \to \pm \infty} c_0(x) = c(0, 0)e^{\int_0^\infty v_0(y) \, dy} \equiv \bar{c}_+ > 0. \]
In particular, if we further assume that \(v_0\) has zero mass, see (2.2) below, then
\[ \int_0^\infty v_0(y) \, dy = -\int_{-\infty}^0 v_0(y) \, dy = \int_{-\infty}^\infty v_0(y) \, dy. \]
This implies \(\bar{c}_+ = \bar{c}_- = \bar{c}\). Except for the study of time decay rates, all other results in this paper apply to both \(\bar{c}_+ = \bar{c}_-\) and \(\bar{c}_+ \neq \bar{c}_-\).

System (1.11) is a special case of the more general system of hyperbolic-parabolic balance laws:
\[ w_t + \sum_{j=1}^m f_j(w)x_j = \sum_{j,k=1}^m [B_{jk}(w)w_{x_k}]x_j + r(w), \quad m \geq 1, \] (1.14)
where \(w, f_j, r \in \mathbb{R}^n\) and \(B_{jk} \in \mathbb{R}^{n \times n}\). A set of structural conditions have been proposed in a recent paper [47]. Under those conditions and under the assumption on the smallness of initial perturbations, existence of global-in-time solutions to the Cauchy problem has been established in [47]. \(L^p (p \geq 2)\) decay rates have been obtained under the same hypotheses for \(m \geq 2\) in [48], and for \(m = 1\) in [50]. Asymptotic behavior of solutions has been studied for \(m \geq 2\) in [49].

System (1.11) fits in the framework of (1.14) with \(n = 2\) and \(m = 1\). It is easy to check that (1.11) satisfies the structural conditions proposed in [47], for both \(\varepsilon = 0\) and \(\varepsilon > 0\). As a consequence, if \((u_0 - 1, v_0)\) is small in \(H^2(\mathbb{R})\), the Cauchy problem (1.11), (1.13) has a unique global-in-time solution. In addition, if \((u_0 - 1, v_0)\) is small in \(H^4(\mathbb{R}) \cap L^1(\mathbb{R})\), then the \(L^2\) decay rates of \(u - 1\) and \(v\) are obtained as \((t + 1)^{-3/4}\) and \((t + 1)^{-1/4}\), respectively.
However, we note that all the aforementioned results are obtained under the assumption that the initial perturbations are sufficiently small. Indeed, to the authors’ knowledge, the story for large data solutions is completely unknown. This is the second fact that motivates the current work.

1.2.3. Connection with Keller-Segel model. Except the classic Fisher/KPP equation, the coupled system (1.2) is also closely connected with contemporary models in mathematical biosciences. Indeed, when \( a = 0 \), (1.2) becomes the following Keller-Segel type chemotaxis model:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= Du_{xx} - \chi (u \log(c))_x, \\
\frac{\partial c}{\partial t} &= \varepsilon c_{xx} - \mu uc - \sigma c,
\end{align*}
\]

(1.15)

which was proposed in [32] for describing the movement of chemotactic populations that deposit little- or non-diffusive chemical signals that modify the local environment for succeeding passages, and later found applications in cancer research [20]. By applying the similar pre-processing actions as in the preceding section, one gets the following hyperbolic-parabolic system of conservation laws:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (uv)_x &= u_{xx}, \\
\frac{\partial v}{\partial t} + (u - \varepsilon_2 v^2)_x &= \varepsilon_1 v_{xx},
\end{align*}
\]

(1.16)

Since the model was initiated in the late 1990s, the qualitative behavior of (1.15) has been analyzed to a large extent. The pioneering works of [19, 32], where the authors constructed explicit and numerical solutions to (1.15) when \( \varepsilon = 0 \) exhibiting chemotactic aggregation or collapsing (time asymptotical uniform distribution of cellular population), is followed by a series of recent products [4, 6, 8, 12, 13, 21, 22, 23, 24, 25, 26, 27, 28, 29, 33, 34, 40, 45], in which the global well-posedness and long time behavior of large data classical solutions, local stability of traveling waves, and zero diffusion limit/boundary layer formation are studied for various types of initial and/or boundary value problems of the transformed model (1.16).

We mentioned that the object of this paper is the Cauchy (initial value) problem of (1.2). One of our major interests is the global stability of constant ground states (equilibrium solutions) associated with the model. For (1.15), the results in [22, 29] show that any positive constant associated with \( u \) is globally asymptotically stable, provided the initial function is perturbed around the prescribed constant state. However, it is not difficult to see that such a result can not be inherited by (1.2), due to the presence of the logistic growth term. Indeed, the logistic growth term indicates that, instead of an arbitrarily prescribed positive ground state, the unique possibly stable constant equilibrium associated with \( u \) in (1.2) should be the relative carrying capacity, i.e., \( K \). This is one of the major differences between (1.2) and (1.15). The current work is partially motivated by the question that whether the self-selected constant equilibrium state is globally asymptotically stable or not.

In addition, we observe that in the perturbed system, cf. (3.11), the logistic growth introduces a weak damping mechanism, which may have further influences on the dynamics of the solution. Indeed, by applying the arguments in [47] one can show that, when the initial data are slightly perturbed around the constant equilibrium state, the perturbation associated with (1.2) enjoys faster temporal decay rates than that of (1.16), due to the dissipation induced by the linear damping. Nevertheless, such a phenomenon is not yet clear in the regime of large data solutions, which is the third fact that motivates this work.
1.2.4. Related KSF models. Lastly, we would like to point out that other Keller-Segel-Fisher type models have been studied in the literature. A search in the database shows that those models are studied mostly in bounded domains, in particular, with no-flux boundary condition. They are also studied mostly with constant rate production and degradation of the chemical signal, i.e., the second equation in the model takes the form
\[ c_t = \varepsilon c_{xx} + \mu u - \sigma c. \quad (1.17) \]
This is to compare with our model with a density-dependent rate. With regular sensitivity and (1.17), global existence of large data classical solutions has been established on bounded domains in all space dimensions under suitable conditions, see [9, 30, 35, 43, 42] and references therein. The result of [30] is extended in [1] to apply to systems with singular sensitivity functions such as the logarithmic function. Moreover, Cauchy problem is also considered in \( \mathbb{R}^2 \), with regular sensitivity and (1.17), in [31]. There are also recent works dealing with the existence of large data weak solutions to Keller-Segel type models with logistic growth, and we refer the reader to [18, 36, 37, 38, 39, 41, 44, 46, 51] and the references therein for more information in this direction. However, we notice that with the logarithmic sensitivity function and density-dependent production rate, no result concerning the qualitative behavior of large data solutions is available in the knowledge base. This is the fourth fact that motivates the current work.

1.3. Goals and outcomes. Motivated by the aforementioned facts, we devote this paper to the study of qualitative behavior of classical solutions to the Cauchy problem (1.11)-(1.13). In particular, we aim to remove the restriction of small initial data and obtain similar results as those in [47]. The major outcomes of this paper are listed as follows:

- **(Global Dynamics)** Under the weak assumption of \((u_0 - 1, v_0)\) in \(H^2(\mathbb{R}) \times H^2(\mathbb{R})\) and \(u_0 \geq 0\) (as to be physically meaningful), we establish the well-posedness of global-in-time solutions of (1.11)-(1.13), for both \(\varepsilon = 0\) and \(\varepsilon > 0\). Time asymptotic behavior (in particular, global asymptotic stability of the constant equilibrium state \((1, 0)\)) is further studied. See Theorems 2.1-2.2.

- **(Zero Diffusion Limit)** We mentioned that (1.11) is closely connected with (1.16) which was designed to model the movement of chemotactic populations that deposit \textit{little-} or \textit{non-diffusive} chemical signals that modify the local environment for succeeding passages. Hence, it is natural to ask whether the chemically non-diffusive model (i.e., \(\varepsilon = 0\)) is a good approximation of the diffusive one (i.e., \(\varepsilon > 0\)) in the process of vanishing diffusion coefficient as \(\varepsilon \to 0\). Under the same hypotheses as in the preceding point, we also study the zero diffusion limit (as \(\varepsilon \to 0\)) of the solution and the corresponding convergence rate in terms of \(\varepsilon\). See Theorem 2.3.

- **(Decay Rates)** To further study the qualitative behavior of (1.11)-(1.13), we consider the explicit time-decay rates of the solution. For this purpose, we make an extra assumption that \(v_0\) has zero mass. As discussed above, this is equivalent to \(c_0\) has the same positive end-state \(\bar{c}\) as \(x \to \pm \infty\). (Here \(c_0 - \bar{c}\) may have nonzero mass though.) The hypotheses, which are without smallness assumption on the norms of \((u_0 - 1, v_0)\), allow us to obtain \(L^2\) rates when \(\varepsilon = 0\). The rates are \((t + 1)^{-1}\) for \(u - 1\) and \((t + 1)^{-3/4}\) for \(v\). The better rates here, when comparing with the framework (1.14), are due to the zero mass assumption on \(v_0\). In the case \(\varepsilon > 0\), we do need \(u_0 - 1\),
Throughout this paper, notation, statement of results and ideas of proof.

2. Statement of results and ideas of proof.

Notation. Throughout this paper, \( \| \cdot \|, \| \cdot \|_{\infty} \) and \( \| \cdot \|_{H^s} \) denote the norms of the usual Lebesgue measurable function spaces \( L^2 \), \( L^\infty \) and Hilbert space \( H^s \), respectively. We use \( \| (f_1, f_2, \ldots, f_N) \|_X^2 \) to denote the sum \( \sum_{i=1}^N \| f_i \|_X^2 \). Unless otherwise specified, \( c_j, C \) and \( C_i \) denote generic constants which are independent of the unknown functions. The values of the constants may vary line by line according to the context.

2.1. Results and comments. The first two theorems are concerned with the global-in-time well-posedness and long time behavior of classical solutions to the Cauchy problem of (1.1) for initial data having potentially large energy.

Theorem 2.1 (Global Well-posedness). Consider the Cauchy problem

\[
\begin{align*}
  u_t &= u_{xx} - (uv)_x + ru(1-u), \\
  v_t &= \varepsilon \frac{\partial}{\partial x} v_{xx} + \varepsilon (v^2)_x - u_x, \\
  (u,v)(x,0) &= (u_0,v_0)(x),
\end{align*}
\]

(2.1)

where \( r, D > 0 \), \( \chi \neq 0 \) and \( \varepsilon \geq 0 \) are fixed constants. Suppose that the initial data satisfy \( u_0 > 0 \) and \( (u_0 - 1, v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}) \). Then there exists a unique solution to (2.1) for all \( t > 0 \), such that \( u(x,t)>0 \) for \( x \in \mathbb{R}, t > 0 \), and

\[
\begin{align*}
  \| u(t) - 1 \|_{H^2}^2 + \| v(t) \|_{H^2}^2 + \int_0^t \left( \| u_x(\tau) \|_{H^2}^2 + \varepsilon \| v_x(\tau) \|_{H^2}^2 \right) d\tau &\leq c_1, \\
  \text{and} \quad \int_0^t \| v_x(\tau) \|_{H^1}^2 d\tau &\leq c_2(1 + \varepsilon),
\end{align*}
\]

where the constants \( c_1 \) and \( c_2 \) are independent of \( t \) and \( \varepsilon \), and depend on \( r, D, \chi \) and the initial data.

Theorem 2.2 (Long Time Behavior). Let the conditions of Theorem 2.1 hold. Then the unique global-in-time solution to (2.1) enjoys the following long time behavior:

\[
\lim_{t \to \infty} (\| u(t) - 1 \|_{H^2} + \| v_x(t) \|_{H_1} + \| u(t) - 1 \|_{C^1(\mathbb{R})} + \| v(t) \|_{C^1(\mathbb{R})}) = 0,
\]

for any \( \varepsilon \geq 0 \).

The third theorem addresses the relationship between the chemically diffusive (\( \varepsilon > 0 \)) and non-diffusive (\( \varepsilon = 0 \)) solutions, and characterizes the difference between the solutions in terms of \( \varepsilon \).

Theorem 2.3 (Zero Diffusion Limit and Convergence Rates). Let the conditions of Theorem 2.1 hold. Let \( (u^\varepsilon, v^\varepsilon) \) and \( (u^0, v^0) \) be the solutions to (2.1) with \( \varepsilon > 0 \) and \( \varepsilon = 0 \), respectively, and with the same initial data. Then for any fixed \( t > 0 \) we have

\[
\begin{align*}
  \|(u^\varepsilon - u^0, v^\varepsilon - v^0)(t)\|^2 &\leq e^{ct \varepsilon} c_4 \varepsilon^2 (\varepsilon + 1), \\
  \|(u_x^\varepsilon - u^0_x, v_x^\varepsilon - v^0_x)(t)\|^2 &\leq e^{ct \varepsilon} c_6 \varepsilon (\varepsilon^2 + \varepsilon + 1),
\end{align*}
\]

The rest of the paper is organized as follows. In Section 2, we state and comment on the main results of this paper, and briefly explain the ideas used to prove the results. Sections 3-7 are devoted to the proofs of the main results. We then finish the paper with concluding remarks in Section 8.
where the constants $c_3, ..., c_6$ are independent of $t$ and $\varepsilon$.

**Remark 1.** Theorem 2.3 shows that for fixed $t > 0$, the zeroth and first frequencies of the difference between the diffusive and non-diffusive solutions decay to zero, as $\varepsilon \to 0$, at different rates in terms of $\varepsilon$.

Next, we identify the explicit decay rates of the solution. For this purpose, we further assume
\[ \int_{-\infty}^{\infty} v_0(x)dx = 0, \quad (2.2) \]
which, combined with the second equation of (2.1), allows one to define the antiderivative:
\[ \psi(x, t) = \int_{-\infty}^{x} v(y, t)dy, \quad t \geq 0. \]

Then we have the following.

**Theorem 2.4** (Decay Rates). Let the conditions of Theorem 2.1 hold, and assume that $\psi_0 \in L^2(\mathbb{R})$.

- When $\varepsilon > 0$, let $N > 0$ be an arbitrarily fixed constant. Then there exists a constant $\delta > 0$, such that if $\|u_0\|^2 + \|v_0\|^2 \leq N$ and
  \[ \|\psi_0\|^2 + \|u_0 - 1\|^2 + \|v_0\|^2 \leq \delta, \]
  the unique global-in-time solution to (2.1) satisfies
  \[ (1 + t)^2 \|u(t) - 1\|^2 + (1 + t)\|v(t)\|^2 + \int_0^t (1 + \tau)^2 ((u - 1, v_x))^2(\tau)d\tau \leq c_7, \]
  \[ (1 + t)^2 \|(u_x, v_x)(t)\|^2 + \int_0^t (1 + \tau)^2 ((u_x, v_{xx})\|^2(\tau)d\tau \leq c_8, \]
  \[ (1 + t)^3 \|(u_{xx}, v_{xx})(t)\|^2 + \int_0^t (1 + \tau)^3 (\|u_{xx}\|_{H^1}^2 + \varepsilon\|v_{xxx}\|^2(\tau)d\tau \leq c_9, \]
  where the constants $c_7, c_8, c_9$ are independent of $t$.

- When $\varepsilon = 0$, there exists a finite $T_0 > 0$, such that the global-in-time solution to (2.1) enjoys the same decay rates as in (2.3) for $t > T_0$, and in this case the temporal integrals in (2.3) are taken from $T_0$ to any $t > T_0$.

**Remark 2.** Theorem 2.4 shows that for the chemically diffusive model ($\varepsilon > 0$) one needs the smallness of the low frequency part of the solution to identify the explicit decay rate, while such a requirement is completely unnecessary when $\varepsilon = 0$. This is so partially because when $\varepsilon > 0$, in the equation of the antiderivative (cf. (6.1)), the quadratic nonlinear term $(\psi_x)^2$ somehow can not be effectively controlled by the diffusion $\psi_{xx}$, which is a different scenario from the solution with $\varepsilon = 0$.

2.2. **Ideas.** We prove the results by using $L^p$-based energy methods. For the global well-posedness of large data solutions, we first construct an entropy-type estimate which involves the anti-logarithmic function of $u$ and provides the uniform-in-time estimates of $\|v(t)\|^2$ and $\int_0^t \|\sqrt{u}(\tau)\|^2d\tau$. The $L^2$ estimate of the solution is then carefully crafted by exploring the logistic growth and the uniform estimates derived from the entropy estimate. The $L^2$ estimates of the spatial derivatives of the solution are then obtained in a standard fashion by using the estimates of the low frequency part of the solution. Collectively, these provide the first estimate in the statement of Theorem 2.1.
As a consequence of the preceding uniform-in-time energy estimate, the second estimate in the statement of Theorem 2.1 is obtained by analyzing the lower order dissipation induced by the logistic growth. Such an estimate plays a crucial role in deriving the zero diffusion limit and convergence rates, leading to the result recorded in Theorem 2.3. In addition, the time asymptotic behavior (Theorem 2.2) is proved by applying the uniform-in-time estimate and using the fact that \( f(t) \in W^{1,1}(0, \infty) \implies \lim_{t \to \infty} f(t) = 0 \).

To identify the explicit decay rates of the perturbation, we resort to an equation of the antiderivative of \( v \), from which the uniform temporal integral of \( \|v(t)\|^2 \) is obtained under the conditions of Theorem 2.4 by utilizing the energy estimates obtained in Theorem 2.1. Then by carrying out time-weighted energy estimates, exploring some fine structure of the logistic growth, and using an iteration scheme, we succeed in obtaining the explicit decay rates recorded in Theorem 2.4.

Lastly, we would like to remark that regarding the explicit decay rate of the Keller-Segel type model (1.16), all of the existing results require the smallness of initial perturbation no matter \( \varepsilon > 0 \) or \( \varepsilon = 0 \), see e.g. [22, 29]. On the other hand, for (1.1) when \( \varepsilon = 0 \), one does not need any smallness assumption on the initial perturbation. This is one of the exclusive features that distinguish (1.1) from (1.16).

3. Global well-posedness. In this section we prove Theorem 2.1. For the reader's convenience, we restate the Cauchy problem:

\[
\begin{aligned}
    u_t &= u_{xx} - (uv)_x + ru(1 - u), & x \in \mathbb{R}, \ t > 0, \\
    v_t &= \varepsilon_1 v_{xx} + \varepsilon_2 (v^2)_x - u_x, \quad x \in \mathbb{R}, \\
    (u, v)(x, 0) &= (u_0, v_0)(x), \\
    u_0 > 0, \quad (u_0 - 1, v_0) &\in H^2(\mathbb{R}) \times H^2(\mathbb{R}),
\end{aligned}
\]

where \( r = aD/\chi \mu K, \varepsilon_1 = \varepsilon/D, \varepsilon_2 = \varepsilon/\chi \), see (1.12).

3.1. Local existence. Equation (3.1) can be written in vector form:

\[
w_t + f(w)_x = Bw_{xx} + g(w), \\
w(x, 0) = w_0(x),
\]

where

\[
w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad f(w) = \begin{pmatrix} uv \\ u - \varepsilon_2 v^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon_1 \end{pmatrix}, \\
g(w) = \begin{pmatrix} ru(u - 1) \\ 0 \end{pmatrix}, \quad w_0(x) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}(x).
\]

Local existence of solution to (3.1), or equivalently to (3.2), is an application of Kawashima’s theory [15]. Here we cite Theorem 2.2 from [47]. To simplify our presentation, we restrict the statement of the theorem, which applies to all space dimensions, to one space dimension only:

\textbf{Theorem 3.1 ([47])}. Consider the Cauchy problem (3.2), where \( w, w_0, f, g \in \mathbb{R}^n \), and \( B \in \mathbb{R}^{n \times n} \). Let \( \bar{w} \in \mathbb{R}^n \) be a constant equilibrium state. Assume that there is a strictly convex function \( \eta \) of \( w \) (called entropy function), defined in an open convex set \( \mathcal{O} \) containing \( \bar{w} \), such that \( \eta' f' \) is symmetric, and \( \eta'' B \) is symmetric, semi-positive definite. Here \( \eta'' \) is the Hessian of \( \eta \), and \( f' \) is the Jacobian matrix.
of $f$. Also, assume that there is a diffeomorphism $\varphi \to w$ from an open set of $\mathbb{R}^n$ to $\mathcal{O}$ and a constant orthogonal matrix $P \in \mathbb{R}^{n \times n}$, such that
\[
P^t Bw_\varphi P = \begin{pmatrix} 0_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & B^* \end{pmatrix}.
\] (3.4)

Here $n_1, n_2 \geq 0$ are two constant integers such that $n = n_1 + n_2$, and $B^* \in \mathbb{R}^{n_2 \times n_2}$ is non-singular if $n_2 > 0$. Let
\[
\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} \equiv P^t \varphi(w),
\]
where $\tilde{w}_1 \in \mathbb{R}^{n_1}$ and $\tilde{w}_2 \in \mathbb{R}^{n_2}$. For $s \geq 2$, if $w_0 - \tilde{w} \in H^s(\mathbb{R})$ and $w_0(x)$ takes values in a compact subset of $\mathcal{O}$ for $x \in \mathbb{R}$, then there exists a positive constant $T$, such that the Cauchy problem (3.2) has a unique solution $w$, satisfying $w - \tilde{w} \in C([0,T]; H^s(\mathbb{R}))$ and $\tilde{w}_2x \in L^2([0,T]; H^s(\mathbb{R}))$.

With the specific form of our (3.2), which is supplemented by (3.3), we take $\eta = \frac{1}{2} v^2 + u \ln u - u$ [22]. As in [50], by direct calculation we find
\[
\eta'' = \begin{pmatrix} \frac{1}{2}v^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad f' = \begin{pmatrix} v & u \\ 1 & -2\varepsilon_2 v \end{pmatrix}.
\] (3.5)

Note that the constant equilibrium state for Theorem 2.1 is $\tilde{w} = (1,0)^t \in \mathbb{R}^2$. Thus we take $\mathcal{O} = \{(u,v) | u > 0\}$, which is an open convex set containing $\tilde{w}$. It is clear that in $\mathcal{O}$, $\eta$ is strictly convex, $\eta'' f'$ is symmetric, and $\eta'' B$ is symmetric, semi-positive definite.

Also, we take $\varphi$ as the identity map, and $P$ as the permutation:
\[
P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Then
\[
P^t Bw_\varphi P = \begin{pmatrix} 0 & 0 \\ \varepsilon_1 & 1 \end{pmatrix}.
\]

Therefore, (3.4) is satisfied, with $n_1 = 1$ if $\varepsilon_1 = 0$, or $n_1 = 0$ if $\varepsilon_1 > 0$. Note that in either case, $B^* \in \mathbb{R}^{n_2 \times n_2}$, $n_2 = 2 - n_1$, is non-singular.

Under the assumptions of Theorem 2.1, we apply Theorem 3.1 to (3.2), (3.3), with $s = 2$. In particular, as a consequence of $u_0 > 0$ and $(u_0 - 1, v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$, via Sobolev imbedding theorem we conclude that $w_0(x)$ takes values in a compact set of $\mathcal{O}$ for all $x \in \mathbb{R}$. This gives us the existence and uniqueness of solution $(u, u)$ to (3.1) for $0 \leq t < T$, with some $T > 0$.

The positivity of $u$, obtained via Theorem 3.1, can be justified by two different approaches. First, it is inherited from the invariant set under iterations in Kawashima’s local existence theory, see Proposition 2.8 and Theorem 2.9 in [15]. The invariant set is a bounded, open, convex set whose closure is in $\mathcal{O}$, hence $u(x, t) > 0$ for all $x \in \mathbb{R}$, $t \in [0, T]$.

Also, the positivity of $u$ can be justified separately by a straightforward application of the maximum principle for Cauchy problems as follows, at the cost of the strict inequality. For this, we use $(u, v)$ obtained from Theorem 3.1 to construct a parabolic operator $\mathcal{L}$:
\[
\mathcal{L}h \equiv h_{xx} - v(x,t)h_x + [-v_x(x,t) + r - ru(x,t)]h - h_t.
\] (3.6)

Since $(u - 1, v) \in C([0,T]; H^2(\mathbb{R}))$, by Sobolev imbedding, the coefficients of $\mathcal{L}$ are continuous and bounded in $\mathbb{R} \times (0, T)$. Applying Theorem 9 on page 43 of [7], we
conclude that if \( Lh \leq 0 \) in \( \mathbb{R} \times (0, T] \) and \( h(x, t) \geq -Be^{\beta x^2} \) on \( \mathbb{R} \times [0, T] \) for some positive constants \( B \) and \( \beta \), then \( h(x, t) \geq 0 \) on \( \mathbb{R} \times [0, T] \) provided \( h(x, 0) \geq 0 \) on \( \mathbb{R} \).

From the first equation of (3.1), it is clear that \( Lu = 0 \) in \( \mathbb{R} \times (0, T] \). Also \( |u(x, t)| \leq B e^{\beta x^2} \) on \( \mathbb{R} \times [0, T] \) for some positive constant \( B \) and any positive constant \( \beta \) by Sobolev imbedding. Therefore, \( u(x, t) \geq 0 \) on \( \mathbb{R} \times [0, T] \) provided \( u_0(x) \geq 0 \) on \( \mathbb{R} \).

In what follows, we will concentrate on deriving the \textit{a priori} estimates of the local solution, in order to extend it to a global one.

### 3.2. Entropy estimate.

By testing the first equation in (3.1) by \( \ln u \), we get

\[
\frac{d}{dt} \int_{\mathbb{R}} (u \ln u - u + 1) dx - \int_{\mathbb{R}} u_x v dx + \int_{\mathbb{R}} \frac{|u_x|^2}{u} dx + r \int_{\mathbb{R}} u(u - 1)(\ln u - \ln 1) dx = 0. \tag{3.7}
\]

By testing the second equation in (3.1) by \( v \), we get

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \int_{\mathbb{R}} u_x v dx + \varepsilon_1 \|v_x\|^2 = 0. \tag{3.8}
\]

By adding (3.8) to (3.7), we get

\[
\frac{d}{dt} \left( \int_{\mathbb{R}} (u \ln u - u + 1) dx + \frac{1}{2} \|v\|^2 \right) + \int_{\mathbb{R}} \frac{|u_x|^2}{u} dx + \varepsilon_1 \|v_x\|^2 + r \int_{\mathbb{R}} u(u - 1)(\ln u - \ln 1) dx = 0. \tag{3.9}
\]

Note that

\[
u(u - 1)(\ln u - \ln 1) = u(u - 1) \frac{1}{u} (u - 1) = \frac{u}{u} (u - 1)^2,
\]

where \( \hat{u} \) is between \( u \) and 1. Since \( u \geq 0 \), we see that \( u(u - 1)(\ln u - \ln 1) \geq 0 \).

Hence, by integrating (3.9), we obtain

\[
\frac{1}{2} \|v(t)\|^2 + \int_0^t \left( \int_{\mathbb{R}} \frac{|u_x|^2}{u} dx + \varepsilon_1 \|v_x\|^2 \right) d\tau \leq \int_{\mathbb{R}} (u_0 \ln u_0 - u_0 + 1) dx + \frac{1}{2} \|v_0\|^2 \equiv C_1, \tag{3.10}
\]

where the constant \( C_1 \) is independent of \( t \) and \( \varepsilon \), and we dropped two non-negative terms from the left hand side without affecting the inequality.

### 3.3. \( L^2_x \)-estimate of zeroth frequency.

By letting \( u = \hat{u} + 1 \) in (3.1), we obtain

\[
\begin{cases}
\hat{u}_t = \hat{u}_{xx} - (\hat{uv})_x - v_x - r \hat{u} (\hat{u} + 1), & x \in \mathbb{R}, \ t > 0, \\
v_t = \varepsilon_1 v_{xx} + \varepsilon_2 v_x - \hat{u}_x, \\
(\hat{u}_0, v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}).
\end{cases} \tag{3.11}
\]

By testing the first equation in (3.11) with \( \hat{u} \) and the second with \( v \), then adding the results, we deduce by using integration by parts and the elementary inequality,
By feeding (3.13) into (3.12), we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \| \tilde{u} \|^2 + \frac{1}{2} \|v\|^2 \right) + \| \tilde{u}_x \|^2 + \varepsilon_1 \|v_x\|^2 + r \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \\
= - \int_\mathbb{R} (\tilde{u} v)_x \tilde{u} dx \\
= \int_\mathbb{R} \tilde{u} v \tilde{u}_x dx
\]
(3.12)

which implies
\[
\| \tilde{u}(t) \|^2 \leq \left( \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \right)^{\frac{1}{2}} \left( \int_\mathbb{R} |\tilde{u}_x|^2 dx \right)^{\frac{1}{2}},
\]
(3.13)

where we applied (3.10) in deriving the last inequality. Next, we derive an energy estimate for \( \| \tilde{u} \|^2 \). Note that since \( \lim_{x \to \pm \infty} \tilde{u}(x,t) = 0 \), we deduce by using the Hölder inequality that
\[
\tilde{u}^2(x,t) = 2 \int_{-\infty}^x \tilde{u} \tilde{u}_y dy \leq 2 \int_\mathbb{R} |\tilde{u}| |\tilde{u}_x| dx \\
= 2 \int_\mathbb{R} |\tilde{u}| (\tilde{u} + 1)^{\frac{1}{2}} \frac{|\tilde{u}_x|}{(\tilde{u} + 1)^{\frac{1}{2}}} dx \\
\leq 2 \left( \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \right)^{\frac{1}{2}} \left( \int_\mathbb{R} |\tilde{u}_x|^2 dx \right)^{\frac{1}{2}},
\]
which implies
\[
\| \tilde{u}(t) \|^2 \leq 2 \left( \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \right)^{\frac{1}{2}} \left( \int_\mathbb{R} |\tilde{u}_x|^2 dx \right)^{\frac{1}{2}}.
\]
(3.13)

By feeding (3.13) into (3.12), we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \| \tilde{u} \|^2 + \frac{1}{2} \|v\|^2 \right) + \frac{1}{2} \| \tilde{u}_x \|^2 + \varepsilon_1 \|v_x\|^2 + r \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \\
\leq 2C_1 \left( \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \right)^{\frac{1}{2}} \left( \int_\mathbb{R} |\tilde{u}_x|^2 dx \right)^{\frac{1}{2}}
\]
(3.14)
\[
\leq \frac{r}{2} \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx + \frac{2C_1^2}{r} \int_\mathbb{R} |\tilde{u}_x|^2 dx,
\]

which implies
\[
\frac{d}{dt} \left( \frac{1}{2} \| \tilde{u} \|^2 + \frac{1}{2} \|v\|^2 \right) + \frac{1}{2} \| \tilde{u}_x \|^2 + \varepsilon_1 \|v_x\|^2 + \frac{r}{2} \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \\
\leq \frac{2C_1^2}{r} \int_\mathbb{R} |\tilde{u}_x|^2 dx.
\]
(3.15)

By integrating (3.15) with respect to \( t \), we obtain
\[
\frac{1}{2} \| \tilde{u}(t) \|^2 + \frac{1}{2} \|v(t)\|^2 + \int_0^t \left( \frac{1}{2} \| \tilde{u}_x \|^2 + \varepsilon_1 \|v_x\|^2 + \frac{r}{2} \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) dx \right) d\tau \\
\leq \frac{1}{2} \| \tilde{u}_0 \|^2 + \frac{1}{2} \|v_0\|^2 + \frac{2C_1^2}{r} \int_\mathbb{R} |\tilde{u}_x|^2 dx.
\]
Next, we estimate in (3.11) with \(-\epsilon_1 \|v_x\|^2 \lesssim C_1\), where the constant \(C_2\) is independent of \(t\) and \(\epsilon_1\).

As a direct consequence of (3.13), (3.10) and (3.16), we deduce that

\[
\int_0^t \|\tilde{u}(\tau)\|_\infty^2 \, d\tau \leq 2 \int_0^t \left( \int_\mathbb{R} |\tilde{u}|^2 (\tilde{u} + 1) \, dx \right)^{\frac{1}{2}} \left( \int_\mathbb{R} \frac{|\tilde{u}_x|^2}{(\tilde{u} + 1)} \, dx \right)^{\frac{1}{2}} \, d\tau
\]

\[
\leq C_2 \int_0^t \|\tilde{u}\|_\infty \|\tilde{u}_x\| \, dx \leq 2C_2 \,
\]

where the constant \(C_3\) is independent of \(t\) and \(\epsilon_1\).

3.4. \(L^2_x\)-estimate of 1st order spatial derivatives. By testing the first equation in (3.11) with \(-\tilde{u}_x\) and the second with \(-v_x\), then adding the results, we deduce

\[
\frac{d}{dt} \left( \frac{1}{2} \|\tilde{u}_x\|^2 + \frac{1}{2} \|v_x\|^2 \right) + \|\tilde{u}_{xx}\|^2 + \epsilon_1 \|v_{xx}\|^2
\]

\[
= r \int_\mathbb{R} \tilde{u}(\tilde{u} + 1) \tilde{u}_{xx} \, dx + \int_\mathbb{R} (\tilde{u}v)_x \tilde{u}_{xx} \, dx - \epsilon_2 \int_\mathbb{R} (v^2)_x v_{xx} \, dx
\]

\[
= -r \int_\mathbb{R} \|\tilde{u}_x\|^2 \, dx - 2r \int_\mathbb{R} \tilde{u} \|\tilde{u}_x\|^2 \, dx + \int_\mathbb{R} \tilde{u}_x v \, \tilde{u}_{xx} \, dx
\]

\[
+ \int_\mathbb{R} \tilde{u} v_x \, \tilde{u}_{xx} \, dx - 2\epsilon_2 \int_\mathbb{R} v_x v_{xx} \, dx,
\]

which implies

\[
\frac{d}{dt} \left( \frac{1}{2} \|\tilde{u}_x\|^2 + \frac{1}{2} \|v_x\|^2 \right) + \|\tilde{u}_{xx}\|^2 + \epsilon_1 \|v_{xx}\|^2 + r \|\tilde{u}_x\|^2
\]

\[
\leq 2r \int_\mathbb{R} \|\tilde{u}\| \|\tilde{u}_x\|^2 \, dx + \int_\mathbb{R} \|\tilde{u}_x\| \|v_x\| \, dx + \int_\mathbb{R} \|\tilde{u}\| \|v_x\| \, dx
\]

\[
+ 2\epsilon_2 \int_\mathbb{R} \|v_x\| \|v_{xx}\| \, dx
\]

\[
\equiv I_1 + I_2 + I_3 + I_4.
\]

Next, we estimate \(I_k\) on the right hand side of (3.19). First of all, we have

\[
I_1 \leq \frac{r}{2} \|\tilde{u}_x\|^2 + 2r \|\tilde{u}\|\|\tilde{u}_x\|^2.
\]

Second, by using the Sobolev-type inequality:

\[
\|f\|_\infty \leq \sup_x \int_{-\infty}^{x} \frac{d}{dy} [f(y)]^2 \, dy \leq 2 \|f\| \|f_x\|, \quad \forall f \in H^1(\mathbb{R}),
\]

and the Young inequality, we estimate \(I_2\) as

\[
I_2 \leq \|\tilde{u}_x\| \|v\| \|\tilde{u}_{xx}\|
\]

\[
\leq \sqrt{2} \|\tilde{u}_x\|^{\frac{1}{2}} \|\tilde{u}_{xx}\|^{\frac{1}{2}} \sqrt{2} C_1 \|\tilde{u}_{xx}\|
\]

\[
= 2\sqrt{C_1} \|\tilde{u}_x\|^{\frac{1}{2}} \|\tilde{u}_{xx}\|^{\frac{1}{2}}
\]

\[
\leq \frac{1}{4} \|\tilde{u}_{xx}\|^2 + 108 C_1^2 \|\tilde{u}_x\|^2,
\]

where the constant \(C_2\) is independent of \(t\) and \(\epsilon_1\).
where we applied the estimate (3.10) for $\|v(t)\|$. For $I_3$, we deduce that

$$I_3 \leq \frac{1}{4} \|\tilde{u}_{xx}\|^2 + \|\tilde{u}\|_\infty^2 \|v_x\|^2.$$

Similar to (3.22), we can show that

$$I_4 \leq 2|\varepsilon_2| \|v\| \|v_x\| \|v_{xx}\|$$

$$\leq 2|\varepsilon_2| \sqrt{2C_1} \sqrt{2} \|v_x\|^2 \|v_{xx}\|^2$$

$$= 4|\varepsilon_2| \sqrt{C_1} \|v_x\|^2 \|v_{xx}\|^2$$

$$\leq \frac{\varepsilon_1}{2} \|v_{xx}\|^2 + \frac{216D^4C_1^2}{\chi^4} \varepsilon_1 \|v_x\|^2.$$  

By feeding the estimates for $I_k$ into (3.19), we have

$$\frac{d}{dt} \left( \frac{1}{2} \|\tilde{u}_x(t)\|^2 + \frac{1}{2} \|v_x(t)\|^2 \right) + \frac{1}{2} \|\tilde{u}_{xx}\|^2 + \frac{\varepsilon_1}{2} \|v_{xx}\|^2 + \frac{r}{2} \|\tilde{u}_x\|^2$$

$$\leq 2r \|\tilde{u}\|_\infty^2 \|\tilde{u}_x\|^2 + 108C_1^2 \|\tilde{u}_x\|^2 + \|\tilde{u}\|_\infty^2 \|v_x\|^2 + \frac{216D^4C_1^2}{\chi^4} \varepsilon_1 \|v_x\|^2$$

$$\leq 2 \max\{2r, 1\} \|\tilde{u}\|_\infty^2 \left( \frac{1}{2} \|\tilde{u}_x\|^2 + \frac{1}{2} \|v_x\|^2 \right) + 216C_1^2 \left( \frac{1}{2} \|\tilde{u}_x\|^2 \right)$$

$$+ \frac{216D^4C_1^2}{\chi^4} (\varepsilon_1 \|v_x\|^2).$$

By applying the Gronwall inequality to (3.24), we deduce

$$\frac{1}{2} \|\tilde{u}_x(t)\|^2 + \frac{1}{2} \|v_x(t)\|^2$$

$$\leq \exp \left\{ 2 \max\{2r, 1\} \int_0^t \|\tilde{u}\|_\infty^2 d\tau \right\}$$

$$\times \left[ 216C_1^2 \int_0^t \frac{1}{2} \|\tilde{u}_x\|^2 d\tau + \frac{216D^4C_1^2}{\chi^4} \int_0^t \varepsilon_1 \|v_x\|^2 d\tau \right.$$

$$+ \frac{1}{2} \|\tilde{u}_x(0)\|^2 + \frac{1}{2} \|v_x(0)\|^2 \right]$$

$$\leq \exp \left\{ 2 \max\{2r, 1\} C_3 \right\} \left[ 216C_1^2C_2 + \frac{216D^4C_1^3}{\chi^4} \right.$$

$$+ \frac{1}{2} \|\tilde{u}_x(0)\|^2 + \frac{1}{2} \|v_x(0)\|^2 \right\} = C_4,$$

where we applied (3.17), (3.16) and (3.10), and the constant $C_4$ is independent of $t$ and $\varepsilon$. By feeding (3.25) into (3.24), we obtain

$$\frac{d}{dt} \left( \frac{1}{2} \|\tilde{u}_x\|^2 + \frac{1}{2} \|v_x\|^2 \right) + \frac{1}{2} \|\tilde{u}_{xx}\|^2 + \frac{\varepsilon_1}{2} \|v_{xx}\|^2 + \frac{r}{2} \|\tilde{u}_x\|^2$$

$$\leq 2C_4 \max\{2r, 1\} \|\tilde{u}\|_\infty^2 + 216C_1^2 \left( \frac{1}{2} \|\tilde{u}_x\|^2 \right) + \frac{216D^4C_1^3}{\chi^4} (\varepsilon_1 \|v_x\|^2).$$
After integrating (3.26) with respect to $t$, we conclude
\[
\int_0^t \left( \frac{1}{2} \|	ilde{u}_{xx}\|^2 + \frac{\varepsilon_1}{2} \|v_x\|^2 + \frac{r}{2} \|	ilde{u}_x\|^2 \right) dt
\leq 2C_4 \max\{2r, 1\} C_3 + 216 C_1^2 C_2 + \frac{216D^4C_4^3}{\chi^4}
\]
(3.27)\]
where the constant $C_5$ is independent of $t$ and $\varepsilon$.

3.5. $L^2$-estimate of 2nd order spatial derivatives. By taking $\partial_x$ to the equations in (3.11), we have
\[
\begin{cases}
\tilde{u}_{xt} = \tilde{u}_{xxx} - (\tilde{u}v)_{xx} - v_{xx} - r(2\tilde{u} + 1) \tilde{u}_x, & x \in \mathbb{R}, \ t > 0, \\
v_{xt} = \varepsilon_1 v_{xxx} + \varepsilon_2 (v^2)_{xx} - \tilde{u}_{xx}.
\end{cases}
\]
(3.28)\]
By testing the first equation in (3.28) with $-\tilde{u}_{xxx}$ and the second with $-v_{xxx}$, then adding the results, we deduce
\[
\frac{d}{dt} \left( \frac{1}{2} \|	ilde{u}_{xx}\|^2 + \frac{1}{2} \|v_x\|^2 \right) + \|	ilde{u}_{xxx}\|^2 + \varepsilon_1 \|v_{xxx}\|^2
= r \int_{\mathbb{R}} (2\tilde{u} + 1) \tilde{u}_x \tilde{u}_{xxx} \, dx + \int_{\mathbb{R}} (\tilde{u}v)_{xx} \tilde{u}_{xxx} \, dx - \varepsilon_2 \int_{\mathbb{R}} (v^2)_{xx} v_{xxx} \, dx
= -2r \int_{\mathbb{R}} \tilde{u} \|	ilde{u}_{xx}\|^2 \, dx - r \int_{\mathbb{R}} \|	ilde{u}_{xx}\|^2 \, dx - 2r \int_{\mathbb{R}} |	ilde{u}_x|^2 \tilde{u}_{xx} \, dx
+ \int_{\mathbb{R}} \tilde{u}_{xx} v \tilde{u}_{xxx} \, dx + 2 \int_{\mathbb{R}} \tilde{u}_x v_x \tilde{u}_{xxx} \, dx + \int_{\mathbb{R}} \tilde{u} v_x \tilde{u}_{xxx} \, dx
- 2\varepsilon_2 \int_{\mathbb{R}} v_{xx} v_{xxx} \, dx - 2\varepsilon_2 \int_{\mathbb{R}} |v_x|^2 v_{xxx} \, dx.
\]
(3.29)\]
After rearranging terms, we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|	ilde{u}_{xx}\|^2 + \frac{1}{2} \|v_x\|^2 \right) + \|	ilde{u}_{xxx}\|^2 + \varepsilon_1 \|v_{xxx}\|^2 + r \|	ilde{u}_x\|^2
= -2r \int_{\mathbb{R}} \tilde{u} \|	ilde{u}_{xx}\|^2 \, dx - 2r \int_{\mathbb{R}} |	ilde{u}_x|^2 \tilde{u}_{xx} \, dx
+ \int_{\mathbb{R}} \tilde{u}_{xx} v \tilde{u}_{xxx} \, dx + 2 \int_{\mathbb{R}} \tilde{u}_x v_x \tilde{u}_{xxx} \, dx + \int_{\mathbb{R}} \tilde{u} v_x \tilde{u}_{xxx} \, dx
- 2\varepsilon_2 \int_{\mathbb{R}} v_{xx} v_{xxx} \, dx - 2\varepsilon_2 \int_{\mathbb{R}} |v_x|^2 v_{xxx} \, dx \equiv \sum_{k=1}^7 J_k.
\]
(3.30)\]
Next, we are devoted to estimate the $J_k$’s on the right hand side of (3.30). First, by using Cauchy’s inequality, we can show that
\[
|J_1| \leq 2r \int_{\mathbb{R}} |\tilde{u}| \|	ilde{u}_{xx}\|^2 \, dx \leq \frac{r}{4} \|	ilde{u}_{xx}\|^2 + 4r \|	ilde{u}_x\| \|	ilde{u}_{xx}\|^2
\leq \frac{r}{4} \|	ilde{u}_{xx}\|^2 + 8r \|	ilde{u}\| \|	ilde{u}_x\| \|	ilde{u}_{xx}\|^2
\leq \frac{r}{4} \|	ilde{u}_{xx}\|^2 + 8r \sqrt{2C_2} \cdot \sqrt{2C_4} \|	ilde{u}_{xx}\|^2
= \frac{r}{4} \|	ilde{u}_{xx}\|^2 + 16r \sqrt{2C_2} \cdot \sqrt{2C_4} \|	ilde{u}_{xx}\|^2
\]
(3.31)\]
where we applied (3.21), (3.16) and (3.25). In a similar fashion, we can show that

\[ |J_2| \leq 2r \int_{\mathbb{R}} |\tilde{u}_x|^2 |\tilde{u}_{xx}| \, dx \leq \frac{r}{4} \|\tilde{u}_{xx}\|^2 + 4r \|\tilde{u}_x\| \|\tilde{u}_x\|^2 \]

\[ \leq \frac{r}{4} \|\tilde{u}_{xx}\|^2 + 8r \|\tilde{u}_x\| \|\tilde{u}_x\|^2 \]

\[ \leq \frac{r}{4} \|\tilde{u}_{xx}\|^2 + 8r \|\tilde{u}_x\| \|\tilde{u}_x\|^2 \]

\[ 2C_4 \]

\[ = \frac{r}{4} \|\tilde{u}_{xx}\|^2 + 8r C_4 \left(\|\tilde{u}_x\|^2 + \|\tilde{u}_{xx}\|^2\right). \]

For the next three terms involving \( \tilde{u}_{xxx} \), we can show that

\[ |J_3| \leq \frac{1}{6} \|\tilde{u}_{xxx}\|^2 + \frac{3}{2} \|v\|_\infty^2 \|\tilde{u}_{xx}\|^2 \leq \frac{1}{6} \|\tilde{u}_{xxx}\|^2 + 3 \sqrt{C_1 C_4} \|\tilde{u}_{xx}\|^2, \]

\[ |J_4| \leq \frac{1}{6} \|\tilde{u}_{xxx}\|^2 + 6 \|\tilde{u}_x\|_\infty \|v_x\|^2 \leq \frac{1}{6} \|\tilde{u}_{xxx}\|^2 + 12 C_4 \left(\|\tilde{u}_x\|^2 + \|\tilde{u}_{xx}\|^2\right), \]

\[ |J_5| \leq \frac{1}{6} \|\tilde{u}_{xxx}\|^2 + \frac{3}{2} \|\tilde{u}_x\|_\infty^2 \|v_x\|^2. \]

For the last two terms, recalling (1.12) we can show that

\[ |J_6| \leq \frac{\varepsilon_1}{4} \|v_{xxx}\|^2 + \frac{4D^2}{\chi^2} \varepsilon_1 \|v\|_\infty^2 \|v_x\|^2 \]

\[ \leq \frac{\varepsilon_1}{4} \|v_{xxx}\|^2 + \frac{16D^2 \sqrt{C_1 C_4}}{\chi^2} \varepsilon_1 \|v_x\|^2, \]

\[ (3.32) \]

\[ |J_7| \leq \frac{\varepsilon_1}{4} \|v_{xxx}\|^2 + \frac{4D^2}{\chi^2} \varepsilon_1 \|v_x\|_\infty \|v_x\|^2 \]

\[ \leq \frac{\varepsilon_1}{4} \|v_{xxx}\|^2 + \frac{8D^2 C_4}{\chi^2} \varepsilon_1 \left(\|v_x\|^2 + \|v_{xx}\|^2\right). \]

By feeding the estimates (3.31)–(3.32) into (3.30), we obtain

\[ \frac{d}{dt} \left(\frac{1}{2} \|\tilde{u}_x\|^2 + \frac{1}{2} \|v_x\|^2\right) + \frac{1}{2} \|\tilde{u}_{xxx}\|^2 + \frac{3}{2} \|\tilde{u}_{xx}\|^2 + \frac{\varepsilon_1}{2} \|v_{xxx}\|^2 + \frac{r}{2} \|\tilde{u}_x\|^2 \]

\[ \leq \frac{3}{2} \|\tilde{u}_x\|_\infty \left(\|\tilde{u}_x\|^2 + \|\tilde{u}_{xx}\|^2\right) + (8r + 12) C_4 \left(\|\tilde{u}_x\|^2 + \|\tilde{u}_{xx}\|^2\right) \]

\[ + (16r + 3) \sqrt{C_2 C_4} \|\tilde{u}_{xx}\|^2 + \frac{8D^2 C_4}{\chi^2} \varepsilon_1 \left(\|v_x\|^2 + \|v_{xx}\|^2\right) \]

\[ + \frac{16D^2 \sqrt{C_1 C_4}}{\chi^2} \varepsilon_1 \|v_{xx}\|^2. \]

By applying the Gronwall inequality to (3.33), we get

\[ \frac{1}{2} \|\tilde{u}_x(t)\|^2 + \frac{1}{2} \|v_x(t)\|^2 \]

\[ \leq \exp \left\{3 \int_0^t \|\tilde{u}_x\|_\infty^2 \, d\tau\right\} \cdot \left((8r + 12) C_4 \int_0^t \left(\|\tilde{u}_x\|^2 + \|\tilde{u}_{xx}\|^2\right) \, d\tau \right. \]

\[ + (16r + 3) \sqrt{C_2 C_4} \int_0^t \|\tilde{u}_{xx}\|^2 \, d\tau + \frac{8D^2 C_4}{\chi^2} \int_0^t \varepsilon_1 \left(\|v_x\|^2 + \|v_{xx}\|^2\right) \, d\tau \]

\[ + \frac{16D^2 \sqrt{C_1 C_4}}{\chi^2} \int_0^t \varepsilon_1 \|v_{xx}\|^2 \, d\tau + \frac{1}{2} \|\tilde{u}_{xx}(0)\|^2 + \frac{1}{2} \|v_{xx}(0)\|^2 \right) \]
By integrating (3.35) with respect to time, we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} ||\tilde{u}_{xx}||^2 + \frac{1}{2} ||v_{xx}||^2 \right) + \frac{1}{2} ||\tilde{u}_{xxx}||^2 + \frac{\epsilon_1}{2} ||v_{xxx}||^2 + \frac{r}{2} ||\tilde{u}_{xx}||^2 
\leq 3C_6 ||\tilde{u}||^2_{L^\infty} + (8r + 12)C_4 (||\tilde{u}_x||^2 + ||\tilde{u}_{xx}||^2)
\leq (16r + 3)\sqrt{C_2C_4} ||\tilde{u}_{xx}||^2 + \frac{8D^2C_4}{\chi^2} \frac{\epsilon_1}{2} (||v_x||^2 + ||v_{xx}||^2)
+ \frac{16D^2\sqrt{C_1C_4}}{\chi^2} \frac{\epsilon_1}{2} ||v_{xx}||^2.
\]
By integrating (3.35) with respect to time, we obtain
\[
\int_0^t \left( \frac{1}{2} ||\tilde{u}_{xx}||^2 + \frac{1}{2} ||v_{xx}||^2 + \frac{r}{2} ||\tilde{u}_{xx}||^2 \right) d\tau 
\leq 3C_6C_3 + 2(8r + 12)C_4(C_2 + C_5) + 2(16r + 3)\sqrt{C_2C_4}C_5
+ \frac{8D^2C_4}{\chi^2} (C_1 + 2C_5) + \frac{32D^2\sqrt{C_1C_4}C_5}{\chi^2} + \frac{1}{2} ||\tilde{u}_{xx}(0)||^2 + \frac{1}{2} ||v_{xx}(0)||^2
\equiv C_7,
\]
where the constant $C_7$ is independent of $t$ and $\epsilon$.

3.6. **Improved $L^2_tH^1_x$-estimate of $v_x$.** So far we have established the following energy estimate for the solution to (3.1):
\[
||\tilde{u}(t)||^2_{H^2} + ||v(t)||^2_{H^2} + \int_0^t (||\tilde{u}_x||^2_{H^2} + \epsilon ||v_x||^2_{H^2}) d\tau \leq C,
\]
where the constant $C$ is independent of $t$ and $\epsilon$. This indicates that the $L^2_tH^2_x$ norm of $v_x$ is inversely proportional to $\epsilon$. In this section, we shall show that the $L^2_tH^1_x$ norm of $v_x$ is bounded by some generic constant which is independent of $t$ and is not inversely proportional to $\epsilon$. In particular, the generic constant is finite when $\epsilon = 0$. To see this, let us test the second equation of (3.11) with $v$ to get
\[
\frac{d}{dt} \left( \frac{1}{2} ||v||^2 \right) + \epsilon_1 ||v_x||^2 = - \int_R \tilde{u}_x v d\tau = \int_R \tilde{u}_x v d\tau.
\]
Note that, from the first equation of (3.11),
\[
\tilde{u} = \frac{1}{r} \left( \tilde{u}_{xx} - \tilde{u}_x v - \tilde{u}_{xx} - v_x - r\tilde{u}^2 - \tilde{u}_t \right).
\]
By substituting (3.38) into the integral on the right hand side of (3.37), we have
\[
\int_R v_x d\tau = \frac{1}{r} \int_R (\tilde{u}_{xx} - \tilde{u}_x v - \tilde{u}_{xx} - v_x - r\tilde{u}^2 - \tilde{u}_t) v_x d\tau
= -\frac{1}{r} ||v_x||^2 + \frac{1}{r} \int_R (\tilde{u}_{xx} - \tilde{u}_x v - \tilde{u}_{xx} - r\tilde{u}^2) v_x d\tau - \frac{1}{r} \int_R \tilde{u}_x v_x d\tau.
\]
For the last term on the right hand side of (3.39), we can show that

\[
-\frac{1}{r} \int \tilde{u}_x v_x \, dx = \frac{d}{dt} \left( \frac{1}{r} \int \tilde{u}_x v_x \, dx \right) + \frac{1}{r} \int \tilde{u}_x v_x \, dx \\
= \frac{d}{dt} \left( \frac{1}{r} \int \tilde{u}_x v_x \, dx \right) - \frac{1}{r} \int \tilde{u}_x v_t \, dx \\
= \frac{d}{dt} \left( \frac{1}{r} \int \tilde{u}_x v_x \, dx \right) - \frac{1}{r} \int \tilde{u}_x (\varepsilon_1 v_{xx} + 2\varepsilon_2 v v_x - \tilde{u}_x) \, dx,
\]

where again we used the second equation of (3.11). By substituting (3.40) into (3.39), we obtain

\[
\int \tilde{u}_x v_x \, dx = -\frac{1}{r} \|v_x\|^2 + \frac{1}{r} \int \left( \tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r\tilde{u}^2 \right) v_x \, dx \\
+ \frac{d}{dt} \left( \frac{1}{r} \int \tilde{u}_x v_x \, dx \right) - \frac{1}{r} \int \tilde{u}_x (\varepsilon_1 v_{xx} + 2\varepsilon_2 v v_x) \, dx + \frac{1}{r} \|\tilde{u}_x\|^2.
\]

By substituting (3.41) into (3.37), we have

\[
\frac{d}{dt} \left( \frac{1}{2} \|v\|^2 - \frac{1}{r} \int \tilde{u}_x v_x \, dx \right) + \varepsilon_1 \|v_x\|^2 + \frac{1}{r} \|v_x\|^2 \\
= \frac{1}{r} \int \left( \tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r\tilde{u}^2 \right) v_x \, dx \\
- \frac{1}{r} \int \tilde{u}_x (\varepsilon_1 v_{xx} + 2\varepsilon_2 v v_x) \, dx + \frac{1}{r} \|\tilde{u}_x\|^2.
\]

Next, we are devoted to estimate the terms on the right hand side of (3.42). First, we have

\[
\left| \frac{1}{r} \int \left( \tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r\tilde{u}^2 \right) v_x \, dx \right| \\
\leq \frac{1}{2r} \|v_x\|^2 + \frac{1}{2r} \|\tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r\tilde{u}^2\|^2,
\]

where the second term on the right hand side can be estimated as

\[
\frac{1}{2r} \|\tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r\tilde{u}^2\|^2 \\
\leq \frac{2}{r} \left( \|\tilde{u}_{xx}\|^2 + \|\tilde{u}_x\|^2 \|v\|_{L\infty}^2 + \|\tilde{u}\|_{L\infty}^2 \|v_x\|^2 + r^2 \|\tilde{u}\|_{L\infty}^2 \|\tilde{u}\|^2 \right) \\
\leq \frac{2}{r} \left( \|\tilde{u}_{xx}\|^2 + 2\|\tilde{u}_x\|^2 \|v\|_{L\infty} + 2C_4 \|\tilde{u}\|_{L\infty}^2 + 2r^2 C_2 \|\tilde{u}\|^2 \right) \\
\leq \frac{2}{r} \left( \|\tilde{u}_{xx}\|^2 + 4\sqrt{C_1 C_4} \|\tilde{u}_x\|^2 + 2C_4 \|\tilde{u}\|_{L\infty}^2 + 2r^2 C_2 \|\tilde{u}\|^2 \right),
\]

where we applied (3.21), (3.10), (3.16) and (3.25). By substituting (3.44) into (3.43), we have

\[
\left| \frac{1}{r} \int \left( \tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r\tilde{u}^2 \right) v_x \, dx \right| \\
\leq \frac{1}{2r} \|v_x\|^2 + \frac{2}{r} \left( \|\tilde{u}_{xx}\|^2 + 4\sqrt{C_1 C_4} \|\tilde{u}_x\|^2 + (2C_4 + 2r^2 C_2) \|\tilde{u}\|_{L\infty}^2 \right).
\]
Next, in a similar fashion we can show that
\[
\left| -\frac{1}{r} \int_{\mathbb{R}} \bar{u}_x (\varepsilon_1 v_{xx} + 2\varepsilon_2 v v_x) \, dx \right|
\]
\[
\leq \frac{\varepsilon_1}{2r} \|\bar{u}_x\|^2 + \frac{\varepsilon_1}{2r} \|v_{xx}\|^2 + \frac{\varepsilon_2}{r} \|v\|_{\infty} (\|\bar{u}_x\|^2 + \|v_x\|^2)
\]
\[
\leq \frac{\varepsilon_1}{2r} \|\bar{u}_x\|^2 + \frac{\varepsilon_1}{2r} \|v_{xx}\|^2 + \frac{\varepsilon_2}{r} \sqrt{2} \|v\| \frac{1}{2} \|v_x\| \frac{1}{2} (\|\bar{u}_x\|^2 + \|v_x\|^2)
\]
\[
\leq \frac{\varepsilon_1}{2r} \|\bar{u}_x\|^2 + \frac{\varepsilon_1}{2r} \|v_{xx}\|^2 + \frac{2\varepsilon_2}{r} (C_1 C_4)^\frac{1}{4} (\|\bar{u}_x\|^2 + \|v_x\|^2)
\]
\[
= \frac{\varepsilon_1}{2r} \|\bar{u}_x\|^2 + \frac{\varepsilon_1}{2r} \|v_{xx}\|^2 + \frac{2\varepsilon_1 D}{r |x|} (C_1 C_4)^\frac{1}{4} (\|\bar{u}_x\|^2 + \|v_x\|^2).
\]
By feeding (3.45) and (3.46) into (3.42), we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \|v\|^2 - \frac{1}{r} \int_{\mathbb{R}} \bar{u}_x v \, dx \right) + \varepsilon_1 \|v_x\|^2 + \frac{1}{2r} \|v_{x}\|^2
\]
\[
\leq \frac{2}{r} \left( \frac{2C_5}{2} + 2(C_4 + 2r^2 C_2)\|\bar{u}_x\|^2 \right) + \frac{\varepsilon_1}{r} C_2 + \frac{\varepsilon_1}{r} C_5
\]
\[
+ \frac{2D}{r |x|} (C_1 C_4)^\frac{1}{4} (2\varepsilon_1 C_2 + C_1) + \frac{2}{r} C_2 + \frac{1}{2} \|v(0)\|^2 - \frac{1}{r} \int_{\mathbb{R}} \bar{u}_x (x, 0) v(x, 0) \, dx,
\]
which implies that
\[
\frac{1}{2} \|v(t)\|^2 + \int_0^t \left( \varepsilon_1 \|v_x\|^2 + \frac{1}{2r} \|v_{x}\|^2 \right) \, d\tau
\]
\[
\leq \frac{2}{r} \left( 2C_5 + 2(C_4 + 2r^2 C_2)C_3 + C_2 \right) + \frac{\varepsilon_1}{r} C_2 + \frac{\varepsilon_1}{r} C_5
\]
\[
+ \frac{2D}{r |x|} (C_1 C_4)^\frac{1}{4} (2\varepsilon_1 C_2 + C_1) + \frac{1}{2} \|v(0)\|^2
\]
\[
+ \frac{1}{2r} (\|\bar{u}_x(0)\|^2 + \|v(0)\|^2) + \frac{1}{r} \int_{\mathbb{R}} \bar{u}_x v \, dx
\]
(3.48)
In particular, (3.48) yields
\[ \int_0^t \| v_x \|^2 \, d\tau \leq 4 \left( 2C_5 + 8\sqrt{C_1C_4}C_2 + (2C_4 + 2r^2C_2)C_3 + C_2 \right) + 2\varepsilon C_2 + 2\varepsilon C_5 + \frac{4D}{|\chi|} \left( C_1C_4 + \frac{1}{2} \right) (2\varepsilon C_2 + C_1) + r\| v(0) \|^2 + \left( \| \tilde{u}_x(0) \|^2 + \| v(0) \|^2 \right) + 2(C_4 + C_1) \equiv C_8. \] (3.49)

We note that the constant on the right hand side of (3.49) is independent of $t$ and is not inversely proportional to $\varepsilon$. In particular, the constant is finite when $\varepsilon = 0$. Furthermore, by using the same idea and previous estimates, we can establish a similar result for $v_{xx}$. Since the proof is in the same spirit, we shall omit the technical details here. This completes the proof of Theorem 2.1.

4. **Long time behavior of large data solutions.** In this section we prove Theorem 2.2. From (3.18) we see that
\[ \frac{d}{dt} \left( \frac{1}{2} \| \tilde{u}_x \|^2 + \frac{1}{2} \| v_x \|^2 \right) = -\| \tilde{u}_{xxx} \|^2 - \varepsilon_1 \| v_{xxx} \|^2 - r \int_{\mathbb{R}} |\tilde{u}_x|^2 \, dx - 2r \int_{\mathbb{R}} \tilde{u}_x |\tilde{u}_x|^2 \, dx + \int_{\mathbb{R}} \tilde{u}_x v \tilde{u}_{xx} \, dx + \int_{\mathbb{R}} \tilde{u}_x v_x \tilde{u}_{xx} \, dx - 2\varepsilon_2 \int_{\mathbb{R}} v \cdot v_x \tilde{u}_{xx} \, dx. \]

By applying the arguments in Section 3.3 (cf. (3.20), (3.22)-(3.23)), we can show that
\[ \left| \frac{d}{dt} \left( \frac{1}{2} \| \tilde{u}_x \|^2 + \frac{1}{2} \| v_x \|^2 \right) \right| \leq \frac{3}{2} \| \tilde{u}_{xxx} \|^2 + \frac{3\varepsilon_1}{2} \| v_{xxx} \|^2 + \left( \frac{3r}{2} + 108C_1 \right) \| \tilde{u}_x \|^2 + 2C_4(2r + 1)\| \tilde{u}_x \|^2 + \frac{216D^4C_1^3}{\chi^4} \varepsilon_1 \| v_x \|^2. \] (4.1)

Upon integrating (4.1) with respect to $t$, we obtain
\[ \int_0^t \left| \frac{d}{dt} \left( \frac{1}{2} \| \tilde{u}_x \|^2 + \frac{1}{2} \| v_x \|^2 \right) \right| \, d\tau \leq 3C_5 + (3r + 216C_1^2)C_2 + 2C_4(2r + 1)C_3 + \frac{216D^4C_1^3}{\chi^4}, \]
where the constant on the right hand side is independent of $t$. The preceding estimate implies
\[ \frac{d}{dt} (\| \tilde{u}_x(t) \|^2 + \| v_x(t) \|^2) \in L^1((0, \infty)), \]
which, together with (3.16) and (3.49), yields
\[ (\| \tilde{u}_x(t) \|^2 + \| v_x(t) \|^2) \in W^{1,1}((0, \infty)). \]

Hence, it holds that
\[ \lim_{t \to \infty} (\| \tilde{u}_x(t) \|^2 + \| v_x(t) \|^2) = 0. \] (4.2)
In a similar fashion, by using (3.29) and the arguments in Section 3.4, we can show that
\[
\lim_{t \to \infty} \left( \|\tilde{u}_{xx}(t)\|^2 + \|v_{xx}(t)\|^2 \right) = 0,
\] (4.3)
and we omit the details to simplify the presentation.

Since \(\|\tilde{u}(t)\|_{H^2} \) and \(\|v(t)\|_{H^2}\) are uniformly bounded with respect to \(t\), according to Morrey’s inequality we know that \((\tilde{u}, v) \in C^1(\mathbb{R}) \times C^1(\mathbb{R})\) and \(\|\tilde{u}(t)\|_{C^1(\mathbb{R})}\) and \(\|v(t)\|_{C^1(\mathbb{R})}\) are uniformly bounded with respect to \(t\). By using (3.21) we can easily show that
\[
\|\tilde{u}(t)\|_{C^1(\mathbb{R})} + \|v(t)\|_{C^1(\mathbb{R})} \\
\leq 2 (\|\tilde{u}(t)\|_{C^0(\mathbb{R})} + \|\tilde{u}_x(t)\|_{C^0(\mathbb{R})} + \|\tilde{u}_{xx}(t)\|_{C^0(\mathbb{R})} + \|v(t)\|_{C^0(\mathbb{R})} + \|v_x(t)\|_{C^0(\mathbb{R})} + \|v_{xx}(t)\|_{C^0(\mathbb{R})}).
\]
It then follows from the uniform boundedness of \(\|\tilde{u}(t)\|\) and \(\|v(t)\|\) and (4.2)-(4.3) that
\[
\lim_{t \to \infty} \left( \|\tilde{u}(t)\|^2_{C^1(\mathbb{R})} + \|v(t)\|^2_{C^1(\mathbb{R})} \right) = 0.
\] (4.4)
This implies that there is a finite time \(T > 0\), such that \(|\tilde{u}(x, t)| \leq \frac{1}{2} \) for all \(x \in \mathbb{R}\) and \(t \geq T\). This in turn shows that \(1 + \tilde{u}(x, t) \geq \frac{1}{2} \) for all \(x \in \mathbb{R}\) and \(t \geq T\). By utilizing such a piece of information in (3.15), we have
\[
\frac{d}{dt} \left( \frac{1}{2} |\tilde{u}|^2 + \frac{1}{2} |v|^2 \right) + \frac{1}{2} |\tilde{u}_x|^2 + \epsilon_1 |v_x|^2 + \frac{r}{4} \|\tilde{u}\|^2 \\
\leq 2C_1^2 \int_R \frac{|\tilde{u}_x|^2}{\tilde{u} + 1} \, dx, \quad \forall \, t \geq T.
\] (4.5)
For any \(t > T\), by integrating (4.5) from \(T\) to \(t\), we have in particular,
\[
\frac{r}{4} \int_T^t \|\tilde{u}(\tau)\|^2 \, d\tau \\
\leq 2C_1^2 \int_T^t \int_R \frac{|\tilde{u}_x|^2}{\tilde{u} + 1} \, dx \, d\tau + \frac{1}{2} \|\tilde{u}(T)\|^2 + \frac{1}{2} \|v(T)\|^2 \\
\leq 2C_1^2 + C_2 + C_1 \equiv C_9,
\] (4.6)
where we applied (3.10) and (3.16). By testing the first equation in (3.11) with \(\tilde{u}\), we have
\[
\frac{d}{dt} \left( \frac{1}{2} |\tilde{u}|^2 \right) = -|\tilde{u}_x|^2 + \int_R \tilde{u} \, v \, \tilde{u}_x \, dx - \int_R v_x \, \tilde{u} \, dx - r \int_R |\tilde{u}|^2 (\tilde{u} + 1) \, dx,
\]
which implies
\[
\left| \frac{d}{dt} \left( \frac{1}{2} |\tilde{u}|^2 \right) \right| \leq |\tilde{u}_x|^2 + \frac{1}{2} \|v\|_{\infty} \left( |\tilde{u}|^2 + |\tilde{u}_x|^2 \right) \\
+ \frac{1}{2} \left( |v_x|^2 + |\tilde{u}|^2 \right) + \frac{3r}{2} |\tilde{u}|^2 \\
\leq |\tilde{u}_x|^2 + (C_1C_4)^{\frac{1}{2}} \left( |\tilde{u}|^2 + |\tilde{u}_x|^2 \right) \\
+ \frac{1}{2} \left( |v_x|^2 + |\tilde{u}|^2 \right) + \frac{3r}{2} |\tilde{u}|^2, \quad \forall \, t \geq T.
\] (4.7)
For any \(t > T\), upon integrating (4.7) from \(T\) to \(t\), we get
\[
\int_T^t \left| \frac{d}{dt} \left( \frac{1}{2} |\tilde{u}(\tau)|^2 \right) \right| \, d\tau \leq 2C_2 + (C_1C_4)^{\frac{1}{2}} \left( \frac{4C_9}{r} + 2C_2 \right) \\
+ \frac{1}{2} \left( C_9 + \frac{4C_9}{r} \right) + 6C_9,
\]
where the constant on the right hand side is independent of $t$. The preceding estimate implies
\[
\frac{d}{dt} (\|\tilde{u}(t)\|^2) \in L^1((T, \infty)),
\]
which, together with (4.6), yields
\[
\|\tilde{u}(t)\|^2 \in W^{1,1}((T, \infty)).
\]
Therefore, it holds that
\[
\lim_{t \to \infty} \|\tilde{u}(t)\|^2 = 0.
\]
This completes the proof of Theorem 2.2.

5. Diffusion limit and convergence rate of large data solutions. This section is devoted to the proof of Theorem 2.3. Let $(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)$ and $(\tilde{u}^0, \tilde{v}^0)$ be the solutions to (3.11) and with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, and with the same initial data. Let $(u, v) = (\tilde{u}^\varepsilon - \tilde{u}^0, \tilde{v}^\varepsilon - \tilde{v}^0)$. Then $(u, v)$ satisfies
\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - (u \tilde{v}^\varepsilon + \tilde{u}^0 v)_x - vu - ru(\tilde{u}^\varepsilon + \tilde{u}^0), \\
v_t = \varepsilon \frac{\partial^2 v}{\partial x^2} + 2\varepsilon \tilde{v}^\varepsilon \frac{\partial}{\partial x} - u_x, \\
(u, v)(x, 0) = (0, 0), & x \in \mathbb{R}.
\end{cases}
\]
By testing the first equation of (5.1) with $u$ and the second equation with $v$, we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v\|^2 \right) + \|u_x\|^2 + r \|u\|^2 = \int_{\mathbb{R}} (u \tilde{v}^\varepsilon + \tilde{u}^0 v) u_x dx - r \int_{\mathbb{R}} (\tilde{u}^\varepsilon + \tilde{u}^0) u^2 dx + \epsilon_1 \int_{\mathbb{R}} \frac{\partial}{\partial x} \tilde{v}^\varepsilon v dx + 2\epsilon_2 \int_{\mathbb{R}} \tilde{v}^\varepsilon \frac{\partial}{\partial x} v dx. 
\]
We estimate the right hand side of (5.2) as follows. First, by Cauchy’s inequality, we have
\[
\left| \int_{\mathbb{R}} (u \tilde{v}^\varepsilon + \tilde{u}^0 v) u_x dx \right| \leq \frac{1}{2} \|u_x\|^2 + \|\tilde{v}^\varepsilon\|_\infty^2 \|u\|^2 + \|\tilde{u}^0\|_\infty^2 \|v\|^2 \leq \frac{1}{2} \|u_x\|^2 + 2 \|\tilde{v}^\varepsilon\|_\infty^2 \|u\|^2 + 2 \|\tilde{u}^0\| \|\tilde{u}^0\| \|v\|^2. 
\]
Second, we can show that
\[
\left| -r \int_{\mathbb{R}} (\tilde{u}^\varepsilon + \tilde{u}^0) u^2 dx \right| \leq \frac{r}{2} \|u\|^2 + r \left( \|\tilde{u}^\varepsilon\|_\infty^2 + \|\tilde{u}^0\|_\infty^2 \right) \|u\|^2 \leq \frac{r}{2} \|u\|^2 + 2r \left( \|\tilde{u}^\varepsilon\| \|\tilde{u}^0\| + \|\tilde{u}^0\| \|\tilde{u}^0\| \right) \|u\|^2.
\]
The third term on the right hand side of (5.2) is estimated as
\[
\left| \epsilon_1 \int_{\mathbb{R}} \frac{\partial}{\partial x} \tilde{v}^\varepsilon v dx \right| \leq \frac{1}{2} \|v\|^2 + \left( \frac{\epsilon_1}{2} \right)^2 \|\tilde{v}^\varepsilon\|_\infty^2.
\]
For the last term, we have
\[
\left| 2\epsilon_2 \int_{\mathbb{R}} \tilde{v}^\varepsilon \frac{\partial}{\partial x} v dx \right| \leq \|\tilde{v}^\varepsilon\|_\infty^2 \|v\|^2 + \left( \epsilon_2 \right)^2 \|\tilde{v}^\varepsilon\|_\infty^2 \|\tilde{v}^\varepsilon\| \|\tilde{v}^\varepsilon\| \leq 2 \|\tilde{v}^\varepsilon\| \|\tilde{v}^\varepsilon\| \|v\|^2 + \left( \epsilon_2 \right)^2 \|\tilde{v}^\varepsilon\|_\infty^2. 
\]
By substituting (5.3)-(5.4) into (5.2), we obtain
\[
\frac{d}{dt} \left( \|u\|^2 + \|v\|^2 \right) + \|u_x\|^2 + r \|u\|^2 \\
\leq (4r \|\tilde{\eta}\|^2 + 4r \|\tilde{\omega}\|^2 + 4 \|\tilde{\eta}\|^2 \|\tilde{\omega}\|^2) \|u\|^2 \\
+ (1 + 4r \|\tilde{\omega}\|^2 \|\tilde{\eta}\|^2 + 4 \|\tilde{\eta}\|^2 \|\tilde{\omega}\|^2) \|v\|^2 + (\varepsilon_1)^2 \|\tilde{\omega}_{xx}\|^2 + 2(\varepsilon_2)^2 \|\tilde{\eta}_{xx}\|^2 \\
\leq (1 + 4r \|\tilde{\omega}\|^2 \|\tilde{\eta}\|^2 + 4(r + 1) \|\tilde{\omega}\|^2 \|\tilde{\eta}\|^2 + 4 \|\tilde{\eta}\|^2 \|\tilde{\omega}\|^2) (\|u\|^2 + \|v\|^2) \\
+ (\varepsilon_1)^2 \|\tilde{\omega}_{xx}\|^2 + 2(\varepsilon_2)^2 \|\tilde{\eta}_{xx}\|^2. 
\]
By applying the Gronwall inequality to (5.5), we have
\[
\|u(t)\|^2 + \|v(t)\|^2 \\
\leq \exp \left\{ \int_0^t \left( 1 + 4r \|\tilde{\omega}\|^2 \|\tilde{\eta}\|^2 + 4(r + 1) \|\tilde{\omega}\|^2 \|\tilde{\eta}\|^2 + 4 \|\tilde{\eta}\|^2 \|\tilde{\omega}\|^2 \right) d\tau \right\} \\
\times \left( (\varepsilon_1)^2 \int_0^t \|\tilde{\omega}_{xx}(\tau)\|^2 d\tau + 2(\varepsilon_2)^2 \int_0^t \|\tilde{\eta}_{xx}(\tau)\|^2 d\tau \right). 
\]
From (3.36) we see that the quantity involving exponential on the right hand side of (5.6) is bounded by $e^{Ct}$ for some constant $C$ which is independent of $t$ and $\varepsilon$. Moreover, from the improved estimates in Section 3.5 we know that the remaining integrals on the right hand side of (5.6) are bounded by $\varepsilon^2 C(1 + \varepsilon)$ for some constant which is also independent of $t$ and $\varepsilon$. Hence, (5.6) shows that for any fixed $t > 0$, the $L^2$ norm of the difference between the diffusive and non-diffusive solutions converges to zero, as $\varepsilon \to 0$, at the rate of $O(\varepsilon)$. Next, we investigate the convergence rate of the first order spatial derivatives of the solution.

By testing the first equation in (5.1) with $-u_{xx}$ and the second with $-v_{xx}$, we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|u_x\|^2 + \frac{1}{2} \|v_x\|^2 \right) + \|u_{xx}\|^2 + r \|u_x\|^2 \\
= \int_{\mathbb{R}} (u\tilde{\omega} + \tilde{u}\omega)_x u_{xx} dx + r \int_{\mathbb{R}} u(\tilde{\omega} + \tilde{u}\omega)_x u_{xx} dx - \varepsilon_1 \int_{\mathbb{R}} \tilde{\omega}_{xx} v_{xx} dx \\
- 2\varepsilon_2 \int_{\mathbb{R}} \tilde{\omega}_x v_{xx} dx \\
= \int_{\mathbb{R}} (u\tilde{\omega} + \tilde{u}\omega)_x u_{xx} dx - r \int_{\mathbb{R}} [u(\tilde{\omega} + \tilde{u}\omega)_x u_{xx} + \varepsilon_1 \int_{\mathbb{R}} \tilde{\omega}_{xx} v_{xx} dx + 2\varepsilon_2 \int_{\mathbb{R}} (\tilde{\omega}_x v_{xx}) dx.
\]
The first term on the right hand side of the above equation can be estimated as
\[
\left| \int_{\mathbb{R}} (u\tilde{\omega} + \tilde{u}\omega)_x u_{xx} dx \right| \\
\leq \frac{1}{2} \|u_{xx}\|^2 + 2 \left( \|\tilde{\omega}_x\|^2 \|u_{xx}\|^2 + \|\tilde{\omega}\|^2 \|u_x\|^2 + \|\tilde{u}\|^2 \|\omega\|^2 + \|\tilde{u}_x\|^2 \|\omega_x\|^2 \right),
\]
and the second term can be estimated as
\[
\left| -r \int_{\mathbb{R}} [u(\tilde{\omega} + \tilde{u}\omega)_x u_{xx} + \varepsilon_1 \int_{\mathbb{R}} \tilde{\omega}_{xx} v_{xx} dx + 2\varepsilon_2 \int_{\mathbb{R}} (\tilde{\omega}_x v_{xx}) dx \right| \leq r \|u_{xx}\|^2 + 2r \left( \|\tilde{\omega}_x\|^2 \|u_{xx}\|^2 + \|\tilde{\omega}\|^2 \|u_x\|^2 + \|\tilde{u}\|^2 \|\omega\|^2 + \|\tilde{u}_x\|^2 \|\omega_x\|^2 \right). 
\]
The third term can be estimated as
\[ \| \varepsilon_1 \int_{\mathbb{R}} \hat{v}_{xxx}^\varepsilon v_x dx \| \leq \frac{1}{2} \| v_x \|^2 + \frac{(\varepsilon_1)^2}{2} \| \hat{v}_{xxx}^\varepsilon \|^2. \]

The last term is estimated as
\[ 2\varepsilon_2 \int_{\mathbb{R}} (\hat{v}^\varepsilon \hat{v}_x^\varepsilon) x v_x dx \leq (\| \hat{v}_{x}^\varepsilon \|^2 + \| \hat{v}_{xx}^\varepsilon \|^2) \| v_x \|^2 + (\varepsilon_2)^2 (\| \hat{v}_{x}^\varepsilon \|^2 + \| \hat{v}_{xx}^\varepsilon \|^2). \]

By applying the estimates for \((\hat{u}^\varepsilon, \hat{v}^\varepsilon)\) and \((\hat{u}^0, \hat{v}^0)\), we can show that
\[ \frac{d}{dt} \left( (\| u \|^2 + \| v \|^2) + \| u_x \|^2 + \| u_{xx} \|^2 \right) \leq C (\| u \|^2_{H^1} + \| v \|^2_{H^1}) + 2(\varepsilon_2)^2 (\| \hat{v}_{x}^\varepsilon \|^2 + \| \hat{v}_{xx}^\varepsilon \|^2) + (\varepsilon_1)^2 \| \hat{v}_{xxx}^\varepsilon \|^2. \]

By combining (5.5) and (5.7), we obtain
\[ \frac{d}{dt} \left( (\| u \|^2_{H^1} + \| v \|^2_{H^1}) + \| u_x \|^2_{H^1} + \| u_{xx} \|^2_{H^1} \right) \leq C (\| u \|^2_{H^1} + \| v \|^2_{H^1}) + C \varepsilon^2 (\| \hat{v}_{x}^\varepsilon \|^2_{H^1} + \| \hat{v}_{xx}^\varepsilon \|^2), \]
for some constant \(C\) which is independent of \(t\) and \(\varepsilon\). By applying the Gronwall inequality to (5.8), we have
\[ \| u(t) \|^2_{H^1} + \| v(t) \|^2_{H^1} \leq e^{Ct} \left( C \varepsilon^2 \int_0^t \| \hat{v}_{x}^\varepsilon \|^2_{H^1} d\tau + C \varepsilon \int_0^t \| \hat{v}_{xx}^\varepsilon \|^2 d\tau \right) \]
\[ \leq e^{Ct} C \varepsilon (\varepsilon^2 + \varepsilon + 1), \]
where we used the uniform estimates of the temporal integral of \(\| \hat{v}_{x}^\varepsilon \|^2_{H^1}\) and \(\varepsilon \| \hat{v}_{xx}^\varepsilon \|^2\) (cf. Theorem 2.1), and the constant on the right hand side is independent of \(t\) and \(\varepsilon\). In particular, (5.9) shows that the \(L^2\) norm of the first order spatial derivative of the difference between the diffusive and non-diffusive solutions decays to zero, as \(\varepsilon \rightarrow 0\), at the rate of \(O(\sqrt{\varepsilon})\). This completes the proof of Theorem 2.3. \(\square\)

6. Explicit decay rate of small data solutions. In this section we prove the first part of Theorem 2.4. By integrating the second equation of (3.11) with respect to \(x\) from \(-\infty\) to \(\infty\), we have
\[ \frac{d}{dt} \left( \int_{-\infty}^{\infty} v(x, t) \, dx \right) = 0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} v(x, t) \, dx = \int_{-\infty}^{\infty} v(x, 0) \, dx. \]
Hence, by assuming \(\int_{-\infty}^{\infty} v(x, 0) \, dx = 0\), we can define the anti-derivative:
\[ \psi(x, t) = \int_{-\infty}^{x} v(y, t) \, dy, \quad x \in \mathbb{R}, \ t \geq 0. \]
Then \(\psi\) satisfies
\[ \psi_t = \varepsilon_1 v_x + \varepsilon_2 v^2 - \tilde{u}. \quad (6.1) \]
By testing (6.1) with \(\psi\), we have
\[ \frac{d}{dt} \left( \frac{1}{2} \| \psi \|^2 \right) + \varepsilon_1 \| v \|^2 = \varepsilon_2 \int_{\mathbb{R}} v^2 \psi \, dx - \int_{\mathbb{R}} \tilde{u} \psi \, dx. \quad (6.2) \]
From the first equation of (3.11) we have
\[ \tilde{u} = \frac{1}{r} \left( \tilde{u}_{xx} - (\tilde{u} v)_x - v_x - r\tilde{u}^2 - \tilde{u}_t \right), \]
by using which we can show that
\[- \int_R \bar{u} \psi \, dx = - \frac{1}{r} \int_R (\bar{u}_x - (\bar{u}v)_x - v_x - r\bar{u}^2 - \bar{u}_t) \psi \, dx\]
\[= \frac{1}{r} \int_R (\bar{u}_x - \bar{u} v - v) \psi \, dx + \int_R \bar{u}^2 \psi \, dx + \frac{d}{dt} \left( \frac{1}{r} \int_R \bar{u} \psi \, dx \right) - \frac{1}{r} \int_R \bar{u}_t \psi \, dx.\] 

(6.3)

By substituting (6.3) into (6.2), we can show that
\[
d\left( \frac{1}{2} \|\psi\|^2 - \frac{1}{r} \int_R \bar{u} \psi \, dx \right) + \varepsilon_1 \|v\|^2 + \frac{1}{r} \|\bar{u}\|^2
= \varepsilon_2 \int_R \bar{v}^2 \psi \, dx + \frac{1}{r} \int_R \bar{u}_x \, dx - \frac{1}{r} \int_R \bar{u} \bar{v} \, dx + \int_R \bar{u}^2 \psi \, dx + \frac{1}{r} \|\bar{u}\|^2
\leq \varepsilon_1 \left( \frac{D}{|\chi|} \|v\|_\infty \|\bar{v}\|^2 + \frac{1}{r^2} \|\bar{u}\|^2 \right) + \frac{1}{r} \|\bar{u}\|^2.
\]

(6.4)

where in the first equality we applied the equation (6.1). From the proof of the global well-posedness result in Section 3 (cf. (3.21), (3.16), (3.25)) we know that
\[
\|\bar{u}(t)\|_\infty \leq 2(C_2 C_4)^\frac{1}{4}, \quad \forall \, t > 0,
\]

(6.5)

where the constants \(C_2\) and \(C_4\) depend only on the parameters \(D, r, \chi\) and initial data. Hence, for fixed values of the system parameters, we can choose the initial data such that
\[
2(C_2 C_4)^\frac{1}{4} \leq \min \left\{ \frac{|\chi|}{8D}, \frac{1}{4} \right\},
\]

which, combined with (6.5), implies
\[
\|\bar{u}(t)\|_\infty \leq \min \left\{ \frac{|\chi|}{8D}, \frac{1}{4} \right\}, \quad \forall \, t > 0.
\]

(6.6)

By applying (6.6) to (6.4), we obtain
\[
d\left( \frac{1}{2} \|\psi\|^2 - \frac{1}{r} \int_R \bar{u} \psi \, dx \right) + \varepsilon_1 \|v\|^2 + \frac{1}{2r} \|\bar{u}\|^2
\leq \varepsilon_1 \left( \frac{D}{|\chi|} \|v\|_\infty + \frac{3}{8} \|\bar{v}\|^2 \right) + \frac{1}{r} \|\bar{u}_x\|^2 + \left( \frac{1}{4} + \frac{\varepsilon_1}{r^2} \right) \|\bar{v}\|^2.
\]

Now we make the a priori assumption that for some finite \(t_0 > 0\),
\[
\|\psi(t)\|_\infty \leq \min \left\{ \frac{|\chi|}{8D}, \frac{1}{r} \right\}, \quad \forall \, 0 < t \leq t_0.
\]

(6.7)
Then under the \textit{a priori} assumption, it holds that for $\forall \ 0 < t \leq t_0$,

$$
\frac{d}{dt} \left( \frac{1}{2} \| \psi \|^2 - \frac{1}{r} \int \tilde{u} \psi \, dx \right) + \frac{\varepsilon_1}{2} \| v \|^2 + \frac{1}{r} \| v \|^2 
\leq \left( \frac{1}{r} + \frac{\varepsilon_1}{r^2} \right) \| \tilde{u}_x \|^2 + \frac{2}{r} \| \tilde{u} \|^2.
$$

(6.8)

Now let us revisit the entropy estimate (3.9), which in terms of $\tilde{u}$ reads

$$
\frac{d}{dt} \left( \int \left[ (\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u} \right] \, dx + \frac{1}{2} \| v \|^2 \right) + \int \frac{1}{\tilde{u} + 1} |\tilde{u}_x|^2 \, dx + \varepsilon_1 \| v_x \|^2 
\leq \int \left( \tilde{u} + 1 \right) \tilde{u} \left[ \ln(\tilde{u} + 1) - \ln 1 \right] \, dx = 0.
$$

(6.9)

Note that by Taylor's theorem,

$$
\int \left[ (\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u} \right] \, dx = \int \left\{ \left[ (\tilde{u} + 1) \ln(\tilde{u} + 1) - (\tilde{u} + 1) \right] - 1 \cdot \ln(1) - 1 \right\} \, dx
\leq \int \frac{1}{\rho} |\tilde{u}|^2 \, dx,
$$

(6.10)

where $\rho$ is between $\tilde{u} + 1$ and 1, and

$$
\int (\tilde{u} + 1) \tilde{u} \left[ \ln(\tilde{u} + 1) - \ln 1 \right] \, dx = \int (\tilde{u} + 1) \frac{1}{\sigma} |\tilde{u}|^2 \, dx,
$$

where $\sigma$ is between $\tilde{u} + 1$ and 1. Hence, under the condition (6.6), we know that

$$
\int [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx \geq \frac{4}{5} \| \tilde{u} \|^2,
$$

$$
\int (\tilde{u} + 1) \tilde{u} \left[ \ln(\tilde{u} + 1) - \ln 1 \right] \, dx \geq \frac{3}{5} \| \tilde{u} \|^2.
$$

(6.11)

Moreover, under the same condition, it holds that

$$
\int \frac{|\tilde{u}_x|^2}{\tilde{u} + 1} \, dx \geq \frac{4}{5} \| \tilde{u}_x \|^2.
$$

(6.12)

By applying (6.11) and (6.12) to (6.9), we get

$$
\frac{d}{dt} \left( \int \left[ (\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u} \right] \, dx + \frac{1}{2} \| v \|^2 \right) 
+ \frac{4}{5} \| \tilde{u}_x \|^2 + \varepsilon_1 \| v_x \|^2 + \frac{3r}{5} \| \tilde{u} \|^2 \leq 0.
$$

(6.13)

Let

$$
M \equiv \max \left\{ \frac{5}{4} \left( \frac{1}{r^2} + 1 \right), \frac{5}{4} \left( \frac{1}{r} + \frac{\varepsilon_1}{r^2} + 1 \right), \frac{5}{3r} \left( \frac{2}{r} + 1 \right) \right\}.
$$

Multiplying (6.13) by $M$, then adding the result to (6.8), we obtain

$$
\frac{d}{dt} [G(t)] + K(t) \leq 0,
$$

(6.14)
where by definition,
\[
G(t) = \frac{1}{2} \| \psi \|^2 - \frac{1}{r} \int_{\mathbb{R}} \tilde{u} \psi \, dx + M \int_{\mathbb{R}} [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx + \frac{M}{2} \| v \|^2
\]

\[
\geq \frac{1}{4} \| \psi \|^2 + \frac{1}{4} \left\| \psi - \frac{2}{r} \tilde{u} \right\|^2 + \| \tilde{u} \|^2 + \frac{M}{2} \| v \|^2, \tag{6.15}
\]

\[
K(t) \geq \frac{\varepsilon_1}{2} \| v \|^2 + \frac{1}{2r} \| v \|^2 + \| \tilde{u}_x \|^2 + \varepsilon_1 M \| v_x \|^2 + \| \tilde{u} \|^2.
\]

After integrating (6.14) with respect to \( t \) and using (6.15), we have in particular,
\[
\frac{1}{4} \| \psi(t) \|^2 + \frac{M}{2} \| v(t) \|^2 + \frac{1}{2r} \int_0^t \| v(\tau) \|^2 \, d\tau \leq G(0). \tag{6.16}
\]

Hence, on one hand, in view of (3.21) and (6.16), we see that the \textit{a priori} assumption (6.7) can be verified by properly choosing the initial data. On the other hand, (6.16) shows that the temporal accumulation of \( \| v(t) \|^2 \) is uniformly bounded with respect to time, which is the key ingredient for cooking up the subsequent time-weighted energy estimates leading to the algebraic decay rate of the solution.

### 6.1. Decay rate of zeroth order frequency.

First, by integrating (6.13) with respect to time, we get in particular (cf. (3.10))
\[
\frac{3r}{5} \int_0^t \| \tilde{u}(\tau) \|^2 \, d\tau \leq \int_{\mathbb{R}} [(\tilde{u}_0 + 1) \ln(\tilde{u}_0 + 1) - \tilde{u}_0] \, dx + \frac{1}{2} \| v_0 \|^2 = C_1. \tag{6.17}
\]

By multiplying (6.13) by \((1 + t)\), we have
\[
\frac{d}{dt} \left[ (1 + t) \left( \int_{\mathbb{R}} [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx + \frac{1}{2} \| v \|^2 \right) \right] + \frac{4}{5} (1 + t) \| \tilde{u}_x \|^2
\]
\[
+ \varepsilon_1 (1 + t) \| v_x \|^2 + \frac{3r}{5} (1 + t) \| \tilde{u} \|^2 \leq \int_{\mathbb{R}} [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx + \frac{1}{2} \| v \|^2.
\]  \tag{6.18}

Since \((\tilde{u} + 1)(x, t) \geq \frac{3}{4}\) for all \( x \in \mathbb{R} \) and \( t > 0 \), due to (6.6), we have (cf. (6.10))
\[
\int_{\mathbb{R}} [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx \leq \frac{4}{3} \| \tilde{u} \|^2.
\]

Then we update (6.18) as
\[
\frac{d}{dt} \left[ (1 + t) \left( \int_{\mathbb{R}} [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx + \frac{1}{2} \| v \|^2 \right) \right] + \frac{4}{5} (1 + t) \| \tilde{u}_x \|^2
\]
\[
+ \varepsilon_1 (1 + t) \| v_x \|^2 + \frac{3r}{5} (1 + t) \| \tilde{u} \|^2 \leq \frac{4}{3} \| \tilde{u} \|^2 + \frac{1}{2} \| v \|^2. \tag{6.19}
\]

By integrating (6.19) with respect to time, we obtain
\[
(1 + t) \left( \int_{\mathbb{R}} [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx + \frac{1}{2} \| v \|^2 \right)
\]
\[
+ \frac{4}{5} \int_0^t (1 + \tau) \| \tilde{u}_x(\tau) \|^2 \, d\tau + \varepsilon_1 \int_0^t (1 + \tau) \| v_x(\tau) \|^2 \, d\tau
\]
where the first term on the right hand side can be estimated as

\[
\frac{3r}{5} \int_0^t (1 + \tau) \|\tilde{u}(\tau)\|^2 d\tau
\]

\[
\leq \frac{4}{3} \int_0^t \|\tilde{u}(\tau)\|^2 d\tau + \frac{1}{2} \int_0^t \|v(\tau)\|^2 d\tau \leq \frac{20}{9r} C_1 + r G(0),
\]

(6.20)

where we applied (6.16) and (6.17). By applying the first inequality in (6.11) to (6.20), we have

\[
(1 + t) \left( \frac{4}{5} \|\tilde{u}\|^{2} + \frac{1}{2} \|v\|^{2} \right) + \frac{4}{5} \int_0^t (1 + \tau) \|\tilde{u}_x(\tau)\|^2 d\tau
\]

\[
+ \varepsilon_1 \int_0^t (1 + \tau) \|v_x(\tau)\|^2 d\tau + \frac{3r}{5} \int_0^t (1 + \tau) \|\tilde{u}(\tau)\|^2 d\tau \leq \frac{20}{9r} C_1 + r G(0) \equiv C_{10},
\]

(6.21)

which establishes the decay rate of the zeroth order frequency of the solution.

6.2. Decay rate of spatial derivatives. We recall the estimate (3.24):

\[
\frac{d}{dt} (\|\tilde{u}_x\|^2 + \|v_x\|^2) + \|\tilde{u}_xx\|^2 + \varepsilon_1 \|v_xx\|^2 + r \|\tilde{u}_x\|^2 \\
\leq 2 \max\{2, 1\} \|\tilde{u}\|_\infty^2 (\|\tilde{u}_x\|^2 + \|v_x\|^2) + 216 C_1^2 \|\tilde{u}_x\|^2 \\
+ \frac{432 D^4 C_1^2}{\lambda^4} (\varepsilon_1 \|v_x\|^2),
\]

(6.22)

where the first term on the right hand side can be estimated as

\[
2 \max\{2, 1\} \|\tilde{u}\|_\infty^2 (\|\tilde{u}_x\|^2 + \|v_x\|^2)
\]

\[
= 2 \max\{2, 1\} \|\tilde{u}\|_\infty^2 \|\tilde{u}_x\|^2 + 2 \max\{2, 1\} \|\tilde{u}\|_\infty^2 \|v_x\|^2
\]

\[
\leq 8 \max\{2, 1\} \sqrt{C_2 C_4} \|\tilde{u}_x\|^2 + 4 \max\{2, 1\} C_4 (\|\tilde{u}_x\|^2 + \|\tilde{v}\|^2),
\]

(6.23)

where we applied (3.21), (3.16) and (3.25). By feeding (6.23) into (6.22), we obtain

\[
\frac{d}{dt} (\|\tilde{u}_x\|^2 + \|v_x\|^2) + \|\tilde{u}_xx\|^2 + \varepsilon_1 \|v_xx\|^2 + r \|\tilde{u}_x\|^2 \\
\leq (8 \max\{2, 1\} \sqrt{C_2 C_4} + 4 \max\{2, 1\} C_4 + 216 C_1^2) \|\tilde{u}_x\|^2 \\
+ 4 \max\{2, 1\} C_4 \|\tilde{u}\|^2 + \frac{432 D^4 C_1^2}{\lambda^4} (\varepsilon_1 \|v_x\|^2).
\]

(6.24)

By multiplying (6.24) by \(1 + t\), we have

\[
\frac{d}{dt} \left[ (1 + t) (\|\tilde{u}_x\|^2 + \|v_x\|^2) \right] + (1 + t) \left( \|\tilde{u}_xx\|^2 + \varepsilon_1 \|v_xx\|^2 + r \|\tilde{u}_x\|^2 \right)
\]

\[
\leq \left( 8 \max\{2, 1\} \sqrt{C_2 C_4} + 4 \max\{2, 1\} C_4 + 216 C_1^2 \right) (1 + t) \|\tilde{u}_x\|^2 + \\
4 \max\{2, 1\} C_4 (1 + t) \|\tilde{u}\|^2 + \frac{432 D^4 C_1^2}{\lambda^4} (1 + t) \left( \varepsilon_1 \|v_x\|^2 \right) + \|\tilde{u}_x\|^2 + \|v_x\|^2.
\]

By integrating the result with respect to time and applying (6.21), (3.16) and (3.49), we have

\[
(1 + t) (\|\tilde{u}_x\|^2 + \|v_x\|^2) + \int_0^t (1 + \tau) \left( \|\tilde{u}_xx\|^2 + \varepsilon_1 \|v_xx\|^2 + r \|\tilde{u}_x\|^2 \right) d\tau
\]
which, together with (6.21) and (6.25), implies

\[ \begin{align*}
&\leq \left( 8 \max\{2r, 1\} \sqrt{C_2 C_4} + 4 \max\{2r, 1\} C_4 + 216 C_1^2 \left( \frac{5}{4} C_{10} \right) \right) \\
&\quad + 4 \max\{2r, 1\} C_4 \left( \frac{5}{3r} C_{10} \right) + \frac{432 D^4 C_1^2}{\chi^4} (C_{10}) + 2C_2 + C_8 \equiv C_{11}.
\end{align*} \] (6.25)

In addition, by applying the arguments in Section 3.5 and using (6.21) and (6.25), we can show that for any \( t > 0 \),

\[ \int_0^t (1 + \tau) \| v_x(\tau) \|^2 d\tau \leq C_{12}, \] (6.26)

where the constant \( C_{12} \) is independent of \( t \).

Next, let us revisit the inequality (3.19):

\[ \frac{d}{dt} \left( \frac{1}{2} \| \tilde{u}_x \|^2 + \frac{1}{2} \| v_x \|^2 \right) + \| \tilde{u}_{xx} \|^2 + \epsilon_1 \| v_{xx} \|^2 + r \| \tilde{u}_x \|^2 \]
\[ \leq 2r \int_{\mathbb{R}} |\tilde{u}| |\tilde{u}_x|^2 dx + \int_{\mathbb{R}} |\tilde{a}_x| |v| |\tilde{u}_{xx}| dx + \int_{\mathbb{R}} |\tilde{u}| |v_x| |\tilde{u}_x| dx \]
\[ + 2\epsilon_1 \int_{\mathbb{R}} |v| |v_x| |v_{xx}| dx \]
\[ \leq \frac{r}{2} \| \tilde{u}_x \|^2 + 2r \| \tilde{u}_x \| \| v_x \| \| v_x \| + \frac{1}{2} \| \tilde{u}_{xx} \|^2 + \| v \| \| v \| \| \tilde{u}_x \| + \| \tilde{u} \| \| v_x \| \]
\[ + \frac{\epsilon_1}{2} \| v_{xx} \|^2 + \frac{2\epsilon_1 D^2}{\chi^2} \| v_x \| \| v_x \| \| v_x \|^2, \]

which, together with (6.21) and (6.25), implies

\[ \frac{d}{dt} \left( \| \tilde{u}_x \|^2 + \| v_x \|^2 \right) + \| \tilde{u}_{xx} \|^2 + \epsilon_1 \| v_{xx} \|^2 + r \| \tilde{u}_x \|^2 \]
\[ \leq 4r \| \tilde{u} \| \| \tilde{u}_x \| + 2\| v \| \| \tilde{u}_x \| + \frac{4\epsilon_1 D^2}{\chi^2} \| v_x \| \| v_x \| \]
\[ \leq 4(2r + 1) \left( \| \tilde{u} \| \| \tilde{u}_x \| + \| v \| \| v_x \| \right) \| \tilde{u}_x \| \| v_x \| \]
\[ + \frac{8\epsilon_1 D^2}{\chi^2} \| v_x \| \| v_x \| \| v_x \|^2 \]
\[ \leq 2(2r + 1) \left( \sqrt{5 + \sqrt{8}} \right) \sqrt{C_{10} C_{11}} (1 + t)^{-1} \| \tilde{u}_x \|^2 \]
\[ + 2\sqrt{5 C_{10} C_{11}} (1 + t)^{-1} \| v_x \|^2 + \frac{8\epsilon_1 D^2}{\chi^2} \sqrt{2C_{10} C_{11}} (1 + t)^{-1} \| v_x \|^2. \]

By multiplying (6.27) by \((1 + t)^2\), we obtain

\[ \frac{d}{dt} \left[ (1 + t)^2 \left( \| \tilde{u}_x \|^2 + \| v_x \|^2 \right) \right] + (1 + t)^2 \left( \| \tilde{u}_{xx} \|^2 + \epsilon_1 \| v_{xx} \|^2 + r \| \tilde{u}_x \|^2 \right) \]
\[ \leq 2(2r + 1) \left( \sqrt{5 + \sqrt{8}} \right) \sqrt{C_{10} C_{11}} (1 + t) \| \tilde{u}_x \|^2 + 2\sqrt{5 C_{10} C_{11}} (1 + t) \| v_x \|^2 \] (6.28)
\[ + \frac{8\epsilon_1 D^2}{\chi^2} \sqrt{2C_{10} C_{11}} (1 + t) \| v_x \|^2 + 2(1 + t) \left( \| \tilde{u}_x \|^2 + \| v_x \|^2 \right). \]

By integrating (6.28) with respect to time from 0 to \( t \) and using (6.21) and (6.26), we get

\[ (1 + t)^2 \left( \| \tilde{u}_x \|^2 + \| v_x \|^2 \right) \]
\[ + \int_0^t (1 + \tau)^2 \left( \| \tilde{u}_{xx} \|^2 + \epsilon_1 \| v_{xx} \|^2 + r \| \tilde{u}_x \|^2 \right) d\tau \leq C_{13}, \] (6.29)
where
\[ C_{13} = \left( \frac{5}{2}(2r + 1) \left( \sqrt{5} + \sqrt{8} \sqrt{C_{10}C_{11}} + \frac{5}{2} \right) \right) C_{10} \]
\[ + \left( 2\sqrt{5C_{10}C_{11}} + 8\varepsilon_1D^2 + \sqrt{2C_{10}C_{11} + 2} \right) C_{12}, \]
which is independent of \( t \). Moreover, in a completely similar fashion, one can show that
\[ \int_0^t (1 + \tau)^2 \| v_{xx}(\tau) \|^2 d\tau \leq C_{14}, \]
and
\[ (1 + t)^3 \left( \| \tilde{u}_{xx} \|^2 + \| v_{xx} \|^2 \right) \]
\[ + \int_0^t (1 + \tau)^3 \left( \| \tilde{u}_{xxx} \|^2 + \varepsilon_1 \| v_{xxx} \|^2 + r \| \tilde{u}_{xx} \|^2 \right) d\tau \leq C_{15}, \quad (6.30) \]
where both the constants \( C_{14} \) and \( C_{15} \) are independent of \( t \). Since the idea is exactly the same as in deriving (6.29), we omit the (tedious) technical details to simplify the presentation.

### 6.3. Improved decay rate of zeroth order frequency

In this subsection, we improve the decay rate of \( \| \tilde{u}(t) \|^2 \). From the first equation of (3.11) we have
\[ \tilde{u}_t + r \tilde{v} = \tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r \tilde{u}^2 - v_x. \quad (6.31) \]

Squaring both sides of the above equation, then integrating over \( \mathbb{R} \), we have
\[ \int_{\mathbb{R}} (\tilde{u}_t + r \tilde{u})^2 dx = \int_{\mathbb{R}} (\tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r \tilde{u}^2 - v_x)^2 dx, \quad (6.32) \]
from which we can show that
\[ \frac{d}{dt} \left( r \| \tilde{u} \|^2 \right) + r^2 \| \tilde{u} \|^2 + \| \tilde{u}_t \|^2 \]
\[ = \int_{\mathbb{R}} (\tilde{u}_{xx} - \tilde{u}_x v - \tilde{u} v_x - r \tilde{u}^2 - v_x)^2 dx \]
\[ \leq 5\| \tilde{u}_{xx} \|^2 + 5\| v \|^2_\infty \| \tilde{u}_x \|^2 + 5\| \tilde{u} \|^2_\infty \| v_x \|^2 + 5r^2 \| \tilde{u} \|^2_\infty \| \tilde{u} \|^2 + 5\| v_x \|^2 \]
\[ \leq 5\| \tilde{u}_{xx} \|^2 + 20\sqrt{C_1C_4} \| \tilde{u}_x \|^2 + 20\sqrt{C_2C_4 + 5} \| v_x \|^2 + 20r^2 \sqrt{C_2C_4} \| \tilde{u} \|^2. \quad (6.33) \]

By multiplying (6.33) by \( (1 + t) \), we have
\[ \frac{d}{dt} \left[ (1 + t) \left( r \| \tilde{u} \|^2 \right) \right] + (1 + t) \left( r^2 \| \tilde{u} \|^2 + \| \tilde{u}_t \|^2 \right) \]
\[ \leq (1 + t) \left( 5\| \tilde{u}_{xx} \|^2 + 20\sqrt{C_1C_4} \| \tilde{u}_x \|^2 + 20\sqrt{C_2C_4 + 5} \| v_x \|^2 \right) \]
\[ + 20r^2 \sqrt{C_2C_4} \| \tilde{u} \|^2 + r \| \tilde{u} \|^2. \quad (6.34) \]

By integrating (6.35) with respect to time and using the previous estimates, we can show that
\[ \int_0^t (1 + \tau) \| \tilde{u}_\tau \|^2 \leq C_{16}, \quad (6.35) \]
where the constant \( C_{16} \) is independent of \( t \). Moreover, from the second equation of (3.11) we can show that
\[ \| v_t \|^2 \leq 3\| \tilde{u}_x \|^2 + \varepsilon_1^2 \| v_{xx} \|^2 + 12\varepsilon_2^2 \| v \|^2_\infty \| v_x \|^2 \]
\[ \leq 3\| \tilde{u}_x \|^2 + \varepsilon_2^2 \| v_{xx} \|^2 + 48\varepsilon_2^2 \sqrt{C_1C_4} \| v_x \|^2, \quad (6.36) \]
which, together with the previous estimates, implies that
\[ \int_0^t (1 + \tau) \| v_\tau \|^2 d\tau \leq C_{17} + \varepsilon C_{18}, \]  
for some constants \( C_{17} \) and \( C_{18} \), which are independent of \( t \).

Next, we differentiate the equations in (3.11), then testing the resulting derivatives with the temporal derivatives of the solution to get
\[ \frac{d}{dt} \left( \frac{1}{2} \| \tilde{u}_t \|^2 + \frac{1}{2} \| v_t \|^2 \right) + \| \tilde{u}_{xt} \|^2 + \varepsilon_1 \| v_{xt} \|^2 + r \int_R (2\tilde{u} + 1)|\tilde{u}_t|^2 dx \]
\[ = \int_R \tilde{u}_t \tilde{v} \tilde{u}_{xt} dx + \int_R \tilde{u} v_t \tilde{u}_{xt} dx - 2\varepsilon_2 \int_R v v_t v_{xt} dx \]
\[ \leq \frac{1}{2} \| \tilde{u}_{xt} \|^2 + \| v_t \|^2 \| \tilde{u}_t \|^2 + \| \tilde{u} \|_\infty \| v_t \|^2 + \frac{\varepsilon_1}{2} \| v_{xt} \|^2 + \frac{2\varepsilon_1 D^2}{\lambda^2} \| v \|_\infty^2 \| v_t \|^2, \]  
which implies
\[ \frac{d}{dt} \left( \| \tilde{u}_t \|^2 + \| v_t \|^2 \right) + \| \tilde{u}_{xt} \|^2 + \varepsilon_1 \| v_{xt} \|^2 + r \| \tilde{u}_t \|^2 \]
\[ \leq 2 \| \tilde{u} \|_\infty^2 \| \tilde{u}_t \|^2 + 2 \tilde{u} \| \tilde{u}_t \|_\infty \| v_t \|^2 + \frac{4\varepsilon_1 D^2}{\lambda^2} \| v \|_\infty^2 \| v_t \|^2 \]
\[ \lesssim (1 + t)^{-\frac{3}{2}} (\| \tilde{u}_t \|^2 + \| v_t \|^2), \]
where we applied (6.6), (6.21) and (6.29). By multiplying (6.39) by \((t + 1)^2\), then integrating the result with respect to time and using (6.35) and (6.37), we can show that
\[ (1 + t)^2 \left( \| \tilde{u}_t \|^2 + \| v_t \|^2 \right) + \int_0^t (1 + \tau)^2 \left( \| \tilde{u}_{xt} \|^2 + \varepsilon_1 \| v_{xt} \|^2 + r \| \tilde{u}_t \|^2 \right) d\tau \]
\[ \leq C_{19} + \varepsilon C_{20}, \]
for some constants \( C_{19} \) and \( C_{20} \), which are independent of \( t \). Finally, based on (6.31), we have
\[ \| \tilde{u}(t) \|^2 \lesssim \| \tilde{u}_{xx} \|^2 + \| v(t) \|_\infty^2 \| \tilde{u}_x(t) \|^2 + \| \tilde{u}(t) \|_\infty^2 \| v_x(t) \|^2 + \| \tilde{u}(t) \|_\infty^2 \| \tilde{u}_t(t) \|^2 \]
\[ \lesssim \| \tilde{u}_{xx} \|^2 + \| v(t) \|_\infty \| v_x(t) \|_\infty \| \tilde{u}_x(t) \|^2 + \| \tilde{u}(t) \|_\infty \| \tilde{u}_x(t) \| \| v_x(t) \|^2 + \| \tilde{u}(t) \|_\infty \| \tilde{u}_x(t) \| + \| \tilde{u}_t(t) \|^2 \]
\[ \lesssim (1 + t)^{-3} + (1 + t)^{-\frac{5}{2}} + (1 + t)^{-2} \]
\[ \lesssim (1 + t)^{-2}, \]
where we applied (6.30), (6.29), (6.21) and (6.40). In addition, one can show that the quantity \[ \int_0^t (1 + \tau)^2 \| \tilde{u}(\tau) \|^2 d\tau \] is uniformly bounded with respect to \( t \).

7. Explicit decay rate of large data solutions. In this last section, we establish the explicit decay rate of the global-in-time solution to (3.11) when \( \varepsilon = 0 \), and finish the proof of Theorem 2.4. Interestingly, in this case we do not need any smallness assumption on the initial data. However, we will show that the large data solution
begins to decay with explicit decay rate after some positive finite time, instead of decaying from the beginning. To see this, we revisit (6.4), which reads:

\[
\frac{d}{dt} \left( \frac{1}{2} \| \psi \|^2 - \frac{1}{r} \int_R \tilde{u} \, \psi \, dx \right) + \frac{1}{r} \| v \|^2 = \frac{1}{r} \int_R \tilde{u}_x \, v \, dx - \frac{1}{r} \int_R \tilde{u} \, v^2 \, dx + \int_R \tilde{u}^2 \psi \, dx + \frac{1}{r} \| \tilde{u} \|^2
\]

\[
\leq \frac{1}{r} \| \tilde{u}_x \|^2 + \frac{1}{4r} \| v \|^2 + \frac{1}{r} \| \tilde{u} \|^2 \| v \|^2 + \frac{1}{2} \| \tilde{u} \|^2 + \frac{1}{2} \| \tilde{u} \|^2 \| \psi \|^2 + \frac{1}{r} \| \tilde{u} \|^2
\]

\[
\leq \frac{1}{2r} \| v \|^2 + \frac{1}{2} \| \tilde{u} \|^2 \| \psi \|^2 + \left( \frac{1}{2} + \frac{1}{r} \right) \| \tilde{u} \|^2 + \frac{1}{r} \| \tilde{u}_x \|^2 + \frac{2C_1}{r} \| \tilde{u} \|^2,
\]

which, together with (3.21), implies

\[
\frac{d}{dt} \left( \frac{1}{2} \| \psi \|^2 - \frac{1}{r} \int_R \tilde{u} \, \psi \, dx \right) + \frac{1}{2r} \| v \|^2 \leq \frac{1}{2} \| \tilde{u} \|^2 \| \psi \|^2 + \left( \frac{1}{2} + \frac{1}{r} \right) \| \tilde{u} \|^2 + \frac{1}{2} \| \tilde{u}_x \|^2
\]

\[
\leq \frac{1}{2} \| \tilde{u} \|^2 \| \psi \|^2 + \left( \frac{1}{2} + \frac{1}{r} + \frac{2C_1}{r} \right) \| \tilde{u} \|^2 + \left( \frac{1}{r} + \frac{2C_1}{r} \right) \| \tilde{u}_x \|^2.
\]

Based on Taylor’s theorem and the statement between (4.4) and (4.5) we know that for the same \( T > 0 \) there hold that

\[
\frac{2}{3} \| \tilde{u} \|^2 \leq \int_R [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx \leq 2 \| \tilde{u} \|^2,
\]

\[
\frac{1}{3} \| \tilde{u} \|^2 \leq \int_R (\tilde{u} + 1) \tilde{u} \ln(\tilde{u} + 1) - \ln 1 \, dx \leq 3 \| \tilde{u} \|^2, \quad \forall t \geq T.
\]

\[
\frac{2}{3} \| \tilde{u}_x \|^2 \leq \int_R \frac{\| \tilde{u}_x \|^2}{\tilde{u} + 1} \, dx \leq 2 \| \tilde{u}_x \|^2,
\]

For any \( t \geq T \), by applying (7.2) to (6.9), we have

\[
\frac{d}{dt} \left( \int_R [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx + \frac{1}{2} \| v \|^2 \right) + \frac{2}{3} \| \tilde{u}_x \|^2 + \frac{r}{3} \| \tilde{u} \|^2 \leq 0,
\]

which implies that \( \| \tilde{u}(\tau) \|^2 \) is uniformly integrable with respect to \( \tau \) from \( T \) to any \( t > T \). Next, we show the same property for \( \| v(\tau) \|^2 \). For this purpose, we let

\[
M_1 = \max \left\{ \frac{3}{2} \left( \frac{1}{r^2} + 1 \right), \frac{3}{2} \left( \frac{1}{r^2} + \frac{2C_1}{r} \right), \frac{3}{2} \left( \frac{1}{r} + \frac{1}{r} + \frac{2C_1}{r} \right) \right\}.
\]

By multiplying (7.3) by \( M_1 \), then adding the result to (7.1), we obtain

\[
\frac{d}{dt} [G_1(t)] + K_1(t) \leq \frac{1}{2} \| \tilde{u} \|^2 \| \psi \|^2,
\]

(7.4)
where by definition,
\[
G_1(t) = \frac{1}{2} \| \psi \|^2 - \frac{1}{r} \int_{\mathbb{R}} \tilde{u} \psi \, dx + M_1 \int_{\mathbb{R}} [(\tilde{u} + 1) \ln(\tilde{u} + 1) - \tilde{u}] \, dx + \frac{M_1}{2} \| v \|^2 \\
\geq \frac{1}{4} \| \psi \|^2 + \frac{1}{4} \left\| \psi - \frac{2}{r} \tilde{u} \right\|^2 + \| \tilde{u} \|^2 + \frac{M_1}{2} \| v \|^2,
\]
(7.5)
\[
K_1(t) \geq \frac{1}{2r} \| v \|^2 + \| \tilde{u} \|^2 + \| \tilde{u} \|^2.
\]

The combination of (7.4) and (7.5) then implies
\[
\frac{d}{dt} [G_1(t)] + K_1(t) \leq 2 G_1(t) \| \tilde{u} \|^2_{\infty}, \quad \forall \ t > T.
\]
(7.6)

By applying the Gronwall inequality and (3.17) to (7.6), we get
\[
G_1(t) \leq \exp \left\{ 2 \int_T^t \| \tilde{u}(\tau) \|^2_{\infty} \, d\tau \right\} \cdot G_1(T) \leq \exp \{2C_3\} \cdot G_1(T) = D_1(T), \quad \forall \ t > T.
\]
(7.7)

By feeding (7.7) into (7.6), we have
\[
\frac{d}{dt} [G_1(t)] + K_1(t) \leq 2 D_1(T) \| \tilde{u} \|^2_{\infty}, \quad \forall \ t > T.
\]
(7.8)

For any \( t > T \), by integrating (7.8) with respect to time from \( T \) to \( t \), we obtain
\[
G_1(t) + \int_T^t K_1(\tau) \, d\tau \leq 2 D_1(T) C_3 + G_1(T), \quad \forall \ t > T,
\]
which, together with (7.5), implies
\[
\frac{1}{2r} \int_T^t \| v(\tau) \|^2 \, d\tau \leq 2 D_1(T) C_3 + G_1(T), \quad \forall \ t > T.
\]

This shows that \( \| v(\tau) \|^2 \) is uniformly integrable with respect to \( \tau \) from \( T \) to any \( t > T \). By repeating the arguments in Sections 6.1-6.3, we can establish the same decay rates for the solution. This completes the proof of Theorem 2.4. □

8. Conclusion. We have studied the qualitative behavior of large data classical solutions to the Keller-Segel type chemotaxis model with logarithmic sensitivity and logistic growth, i.e., (1.2), which can be transformed into a system of hyperbolic-parabolic balance laws (1.11) by the Cole-Hopf type transformation. We considered the Cauchy problem of (1.11) on \( \mathbb{R} \) with the initial data being perturbed around the constant equilibrium state \( (u_{\infty}, v_{\infty}) = (1, 0) \). By utilizing energy methods, we showed that for any general initial perturbation (not necessarily small) in \( H^2(\mathbb{R}) \), there exists a unique global-in-time solution to the Cauchy problem, and the solution is shown to converge to the equilibrium state as time goes to infinity. This indicates that the constant ground state is globally asymptotically stable. Under the same hypotheses, we showed that the solution with \( \varepsilon > 0 \) converges to the one with \( \varepsilon = 0 \) as \( \varepsilon \to 0 \) with certain convergence rate depending on the frequency of the difference between the solutions. This indicates that the chemically diffusive and non-diffusive models are consistent when the spatial domain has no physical boundary. In other words, the vanishing diffusion coefficient limit of the diffusive solution as \( \varepsilon \to 0 \) is not a singular limit, and boundary layer does not emerge.
Furthermore, by restricting the initial data to a smaller class of functions and using time-weighted energy methods, we identified the explicit time decay rate of the perturbation towards the constant ground state.

On the other hand, we would like to emphasize that the results obtained in this paper are still at the primary stage of research on (1.2), and a family of related phenomena are still waiting to be explored, such as existence and stability of nonlinear waves (traveling wave, diffusion wave, et al.), global dynamics of large data solutions in higher space dimensions, and qualitative behavior of large data solutions to the model with more general structures on the reaction term, just to mention a few. These are beyond the scope of the current paper, and we leave the investigation for the future.

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