Second order surfaces in architecture and construction

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Abstract The modern means of computer graphics allow designing a wide variety of architectural forms. An architect's fantasy is not limited by the difficulties of the technical implementation of the project, nor by the materials, nor by the complexity of the design calculations. Of particular interest are the technical problems associated with second-order surfaces (quadrics). Two problems are considered in the paper: the problem of reconstructing a quadric given by nine points, and the problem of joining two quadrics by means of a flat welded seam. To solve these problems, modern computer graphics is used in combination with classical methods of descriptive and projective geometry. The reconstruction task was solved on the basis of the well-known algorithm proposed by German mathematicians Rohn, Papperitz (1896). The algorithm can be practically realized with the help of 3D computer graphics. The problem of a planar compound of quadrics is solved on the basis of Theorem 1 (parabolic cylinders with parallel planes of symmetry intersect along a plane curve) and Theorem 2 (quadrics of rotation with the coincident pair of focal points are in imaginary double contact). The results obtained can be used in construction and architectural design.

1. Introduction

The modern computer graphics tools allow to design a wide variety of surfaces and architectural forms, based on a flat or spatial foundation [1-3]. To implement the projects, high-strength materials are used - steel, ceramics, concrete, glass. Computer graphics tools allow to get a visual image of the projected object of any complexity [4-6]. The fantasy of the architect is practically unlimited - no difficulties in the technical implementation of the project, no materials, or the complexity of design calculations. Computer technologies and specialized CAD systems (computer-aided design systems) allow solving the most complex engineering problems [7,8]. In large world capitals, buildings and complexes of high-tech style buildings are rapidly being built. New buildings do not always fit into the natural landscape or the urban environment. For example, the "fiery towers" in the center of Baku and the traditional urban architecture (Figure 1) are completely unrelated. The towers were built in 2012, but two of them are still empty.

The methods of constructing surfaces must combine the possibilities of the modern computer graphics and known projective methods of surface theory [9-11]. Classical design methods involve the use of algebraic surfaces [6]. The simplest algebraic surfaces are a plane and a surface of the second order.
2. Reconstruction of the second order surface

Consider one of the important problems in the theory of second-order surfaces (quadrics): construction of a quadric passing through nine pre-specified points of space. Let the points 1, 2, ..., 9 be indicated in the three-dimensional Euclidean space. Nine points of the general position define a unique quadric \( \Theta \) passing through these points. It is required to compile a graphical algorithm for constructing the set of points of a quadric given by nine points [12,13].

This problem can be supplemented by the requirement to construct the principal diameters of the required quadric \( \Theta \). It should be noted that the task of constructing a quadric with respect to nine points, due to its fundamental importance, has been repeatedly considered by mathematicians of the 19th century. Consider the algorithm of German mathematicians Rohn, Papperitz [14].

Mark the planes \( \alpha(123), \beta(456), \gamma(789) \). The straight lines \( m=\alpha \cap \beta, n=\beta \cap \gamma, l=\alpha \cap \gamma \) intersect at the point \( P \). The solution reduces to the construction of conic sections \( a^2(1, 2, 3), b^2(4, 5, 6), g^2(7, 8, 9) \), pair wise intersecting on lines \( m, n, l \). These conic sections completely determine the required surface \( \Theta \) (Figure 2). Please follow these instructions as carefully as possible so all articles within a conference have the same style to the title page. This paragraph follows a section title so it should not be indented.

![Figure 1](image1.jpg) Architecture of Baku.

![Figure 2](image2.jpg) Conical sections \( a^2, b^2, g^2 \) belong to the surface \( \Theta(1, 2, ..., 9) \).

![Figure 3](image3.jpg) Conjugate curves \( a^2 \sim b^2 \) and conjugate polars \( p_a \sim p_b \).
Consider the planes $\alpha$ and $\beta$ together. On the line $m=\alpha \cap \beta$ indicate arbitrary points $M, N$. Through points 1, 2, 3, $M, N$ in the plane $\alpha$ draw the conic section $a^2$. Through points 4, 5, 6, $M, N$ in the plane $\beta$ draw the conic section $b^2$ (Figure 3).

Definition. The conic sections $a^2, b^2$, lying in the planes $\alpha, \beta$, will be called "conjugate" if they intersect at points lying on the line $m=\alpha \cap \beta$.

Draw the polars $p_a, p_b$ of the point $P$ relative to the conic sections $a^2, b^2$. The polars of conjugate conic sections are also called "conjugate". The conjugate polars intersect at points lying on the line $m$.

The set of conjugate conic sections $a^2 \sim b^2$ corresponds to the set of conjugate polars $p_a \sim p_b$ [15, 16].

Arbitrarily determining the points $M, N$ on the line $m$, draw in the planes $\alpha, \beta$ three pairs of conjugate conic sections $a^2 \sim b^2, a^2 \sim b^2, a^{n2} \sim b^{n2}$. Three pairs of conjugate conic sections correspond to three pairs of conjugate polars $p_a \sim p_b, p'_a \sim p'_b, p''_a \sim p''_b$. The conjugate polars intersect in pairs on the line $m$, forming a spatial configuration of Desargues with the center $O_{\alpha \beta}$ and the axis $m$. The center $O_{\alpha \beta}$ is at the intersection of lines connecting the corresponding points ($p_a \cap p'_a \leftrightarrow (p_b \cap p'_b), (p'_a \cap p''_a \leftrightarrow (p'_b \cap p''_b)$).

Similarly, perspective relationships are established between pairs of the planes $\alpha \leftrightarrow \gamma$ (with the center $O_{\alpha \gamma}$ and the axis $l=\alpha \cap \gamma$) and $\beta \leftrightarrow \gamma$ (with the center $O_{\beta \gamma}$ and the axis $n=\beta \cap \gamma$). Note an important result.

Let through one of the centers, for example, through the center $O_{\alpha \beta}$, an arbitrary plane $\delta$ be drawn. The plane $\delta$ intersects the planes $\alpha, \beta$ along the straight lines $p_\alpha=\delta \cap \alpha, p_\beta=\delta \cap \beta$. These lines can be considered as conjugate polars (with respect to the pole $P$) of two conjugate conic sections $d_\alpha^2 (1, 2, 3) \sim d_\beta^2 (4, 5, 6)$. The conic section $d_\alpha$, passing through points 1, 2, 3, is completely determined by the pole $P$ and polar $p_\alpha$. The conic section $d_\beta$, passing through points 4, 5, 6, is completely determined by the pole $P$ and polar $p_\beta$ [17].

The arbitrary plane $\delta$ incident to the center $O_{\alpha \beta}$, generates in the planes $\alpha, \beta$ a pair of conjugate conic sections passing through the given points (1, 2, 3) and (4, 5, 6). Consequently, the plane $\Delta(O_{\alpha \beta}O_{\alpha \gamma}O_{\beta \gamma})$ generates in the planes $\alpha, \beta, \gamma$ three pairs of conjugate conic sections $a^2 \sim b^2, a^2 \sim g^2, b^2 \sim g^2$, passing through the given points (1, 2, 3), (4, 5, 6), (7, 8, 9). These conic sections define the required surface $\Theta$. The task of reconstruction is completely solved. The algorithm is practically realized with the help of the modern computer graphics.

3. Connection of the second order surfaces

Connection of parabolic cylinders. The parabolic cylinder sections are used in architecture and construction when constructing arches of sports facilities. When constructing composite arches, it is required to ensure their connection along flat curves.

![Figure 4. Parabolic cylinders: a - orthogonal drawing; b - axonometry.](image)

3.1. Theorem 1

Parabolic cylinders with parallel planes of symmetry intersect along a plane curve (Figure 4).

Proof. In the extended Euclidean space $E^3$ a parabolic cylinder touches an improper plane along its improper generatrix. The improper generator of a parabolic cylinder coincides with the improper line
of its symmetry plane. The planes of symmetry $\Sigma$, $\Gamma$ of parabolic cylinder data are parallel, so the improper lines $s, g$ of the planes $\Sigma, \Gamma$ coincide. Consequently, both cylinders have a common generator $s=g$ along which they touch both the improper plane and between themselves. The contact of the cylinders along the common generatrix is regarded as an intersection along two coincident generators. In other words, a plane curve appears in the intersection of the cylinders (a curve of the second order degenerating into two coincident straight lines $s=g$). As is known from the course of descriptive geometry and from the basic theorem of algebra, in this case second-order surfaces intersect one more plane curve. The theorem is proved.

A parabolic cylinder in extended Euclidean space is regarded as a second-order conic surface that is in contact with an improper plane. An arbitrary projective transformation transforms parabolic cylinders with parallel planes of symmetry into second-order conic surfaces with a common tangent plane and coinciding lines of tangency [18,19]. Consequently, the theorem just proved is a special case of a more general assertion: second-order conic surfaces that touch a rectilinear generator intersect along a plane curve. This assertion, in turn, follows directly from the well-known theorem of descriptive geometry: if two surfaces of the second order are not intersected by a plane curve, then they intersect one more plane curve [19,20].

Connection of rotation surfaces. Consider the engineering problem of joining second-order focal surfaces by means of a flat welded seam. From a technical point of view, flat welds are the most technologically advanced. Among flat welds, preference should be given to the simplest flat curves - straight lines, circles and curves of the second order. The solution of the problem is based on the following theorem.

3.2. Theorem 2
Focal surfaces of the second order with the coincident pair of foci are in imaginary double contact (focal points are present in an elongated ellipsoid of revolution, a two-sheeted hyperboloid of revolution, and a paraboloid of revolution).

The method of joining surfaces of the second order, based on the above theorem, makes it possible to produce and connect special containers and install flanges using flat welds described by an algebraic second-order equation. Simplification of the shape of the welded joint ensures the quality of welding of the elements to be joined and the required penetration depth.

For example, consider the installation of flat round flange 3 on fuel tank 1 with a conical bottom, when the fuel bay configuration does not allow any variation in the spatial position of the flange (Figure 5). In this case, special transition compartment 2 is used. The choice of the shape of the transition compartment as a section of the circular pipe leads to the need to use a spatial weld in the form of a fourth-order curve that connects tank 1 to cylinder 2. The shortcomings of the cylindrical transition compartment should also be to attribute a low margin of stability to torsion, which leads to the need to increase the thickness of the walls of the transition compartment.

Figure 5. Transition compartment in the form of an ellipsoid of rotation: 1 - conical compartment; 2 - transition compartment; 3 - flat flange.

To eliminate these drawbacks, a transition compartment is used in the form of an ellipsoid of rotation. On the basis of Theorem 2, choose the shape of transition compartment 2, which intersects
conical bottom 1 along a planar curve. In the normal section of compartment 2 obtain a circle, which allows to weld flange 3 to compartment 2 by means of a circular weld.

4. Conclusion
It is shown that computer graphics tools, in combination with the known methods of descriptive and projective geometry, allow performing accurate reconstruction of the second-order surface specified by nine points. A technique for joining second-order surfaces by means of flat welds is proposed. The results obtained can be used in construction and architectural design.

References
[1] Voloshinov D V 2010 Constructive geometric modeling. Theory, practice, automation: monograph (Saarbrucken: Lambert Academic Publishing) p 355
[2] Korotkiy V A, Khmarova L I and Usmanova E A 2016 Computer simulation of kinematic surfaces Geometry and Graphics 3 (4) pp 19–26
[3] Korotkiy V A, Usmanova E A and Khmarova L I 2017 Geometric Modeling of Construction Communications with Specified Dynamic Properties IOP Conf. Series: Materials Science and Engineering (ICCATS 2017) 262 012110
[4] Ivanov G S 1987 Designing technical surfaces: monograph (Moscow: Mashinostroenie (Mechanical Engineering)) p 192
[5] Korotkiy V A 2018 Computer visualization of a curve of the second order passing through imaginary points and touching imaginary lines Scientific Visualization 10 (1) pp 56–68
[6] Korotkiy V A and Usmanova E A 2014 Curves of the second order in problems of shaping of architectural shells Proc. of high schools. Series "Construction" 9-10 (669-670) pp 101–107
[7] Korotkiy V A, Usmanova E A and Khmarova L I 2015 The Int. Conf. on Industrial Engineering (ICIE-2015) (Chelyabinsk, Rossiya) pp 775–80
[8] Korotkiy V A and Khmarova L I 2017 Kinematic Methods of Designing Free Form Shells IOP Conf. Series: Materials Science and Engineering (ICCATS 2017) 262 012109
[9] Glagolev N A 1963 Projective geometry: manual (Moscow: Higher School) p 344
[10] Klein F 2004 Higher Geometry: monograph (Moscow: URSS) p 399
[11] Coxter H S M 1959 Real projective plane (Moscow: Fizmatlit) p 280
[12] Korotkiy V A 2017 Fundamental and applied problems of science Materials of the 12th Int. symposium (Moscow: RAS) pp 189–199
[13] Girsh A G and Korotkiy V A 2016 Graphic algorithms for reconstructing a second-order curve given by imaginary elements Geometry and Graphics 4 (4) pp 19–30
[14] Rohn K and Papperitz E 1986 Lehrbuch der Darstellenden Geometrie V II (Leipzig) p 196
[15] Savelov AA 2009 Flat curves (Moscow: The book house "Librocom") p 296
[16] Walker R 2009 Algebraic curves (Moscow: The book house "Librocom") p 240
[17] Korotkiy V A, Usmanova E A and Khmarova L I 2016 Dinamic connection of second-order curves 2nd Int. Conf. on Industrial Engineering, Application and Manufacturing (ICIEAM) (IEEE Conf. Publications) pp 1–4
[18] Korotkiy V A and Khmarova L I 2016 Universal computer conicograph Proc. of the 26th GraphiCon2016 Int. Scientific Conf. (September 19-23, 2016) (Nizhny Novgorod, Rossiya: Nizhny Novgorod State University of Architecture and Building) pp 347–51
[19] Korotkiy V A 2014 Construction of flat conjugation of focal quadrics Privolzhsky scientific journal 1 pp 19–26
[20] Korotkiy V A 2013 On a special case of intersection of quadrics Geometry and Graphics 1 (2) pp 17–21

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