Geometric Complexity Theory IV: quantum group for the Kronecker problem

Dedicated to Sri Ramakrishna

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Abstract

A fundamental problem in representation theory is to find an explicit positive rule, akin to the Littlewood-Richardson rule, for decomposing the tensor product of two irreducible representations of the symmetric group (Kronecker problem). In this paper a generalization of the Drinfeld-Jimbo quantum group, with a compact real form, is constructed, and also an associated semisimple algebra that has conjecturally the same relationship with the generalized quantum group that the Hecke algebra has with the Drinfeld-Jimbo quantum group. In the sequel [23] it is observed that an explicit positive decomposition rule for the Kronecker problem exists assuming that the coordinate ring of the generalized quantum group has a basis analogous to the canonical basis for the coordinate ring of the Drinfeld-Jimbo quantum group, as per Kashiwara and Lusztig [12, 17], or in the dual setting—the associated algebra has a basis analogous to the Kazhdan-Lusztig basis for the Hecke algebra [13], as suggested by the experimental and theoretical results therein. In the other sequel [24], similar quantum group and algebra are constructed for the generalized plethysm problem, of which the Kronecker problem studied here is a special case.

These problems play a central role in geometric complexity theory—an approach to the $P$ vs. $NP$ and related problems.

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1 Introduction

This is a continuation of the series of articles [26, 27, 28] on geometric complexity theory, an approach to the P vs. NP and related problems. A basic philosophy of this approach is called the flip, which was proposed in [25], with a detailed exposition to appear in [22]. The flip, in essence, reduces the negative lower bound problems in complexity theory to positive problems in mathematics. One central positive problem that arises in the flip turns out to be the following fundamental problem in representation theory of the symmetric group $S_n$; cf. [5, 20, 33] for its history and significance.

Problem 1.1 (Kronecker problem)

Find an explicit positive decomposition rule for the tensor product of two irreducible representations (Specht modules) of $S_n$ over $\mathbb{C}$. Specifically, given partitions (Young diagrams) $\lambda, \mu, \pi$, find an explicit positive formula for the Kronecker coefficient $k_{\pi \lambda \mu}^\lambda$, which is the multiplicity of the irreducible representation (Specht module) $S_{\pi}$ of the symmetric group $S_n$ in the tensor product $S_{\lambda} \otimes S_{\mu}$.

By an explicit positive formula, we mean a formula involving no alternating signs, as in the Littlewood-Richardson rule.

This problem is a special case of the following slightly more general problem [5]. Given a partition $\lambda$ of height at most $n$, let $V_{\lambda}(G)$ denote the corresponding irreducible Weyl module of $G$. Let $V = \mathbb{C}^n$, $W = \mathbb{C}^m$, $X = V \otimes W$, and consider the natural homomorphism

$$H = GL(V) \times GL(W) \rightarrow G = GL(V \otimes W) = GL(X).$$

Any irreducible representation $V_{\lambda}(G)$ of $G$ decomposes as an $H$-module:

$$V_{\lambda}(G) = \bigoplus_{\alpha, \beta} m_{\alpha, \beta}^\lambda V_{\alpha}(GL(V)) \otimes V_{\beta}(GL(W)),$$

where $\alpha$ and $\beta$ range over Young diagrams of height at most $n$ and $m$, respectively, and $V_{\alpha}(GL(V))$ and $V_{\beta}(GL(W))$ denote the corresponding irreducible representations of $GL(V)$ and $GL(W)$, respectively.

Problem 1.2 (Kronecker problem: second version) Find an explicit positive rule for the decomposition (2). Specifically, given partitions $\alpha, \beta, \lambda$, find an explicit positive formula for the (generalized) Kronecker coefficient $m_{\alpha, \beta}^\lambda$, which is the multiplicity of the $H$-module $V_{\alpha}(GL(V)) \otimes V_{\beta}(GL(W))$ in the $G$-module $V_{\lambda}(G)$.
1.1 The basic idea

Our approach to this problem is motivated by a transparent proof for the positive Littlewood-Richardson rule based on the theory of quantum groups, which works for arbitrary symmetrizable Kac-Moody algebras [10, 11, 16, 18, 17, 29].

To recall the basic idea of this proof, we need some notation. Given \( G = GL(Z) \), let \( G \) be the Lie algebra of \( G \), \( U(G) \) its enveloping algebra, and \( O(G) \) its coordinate algebra. There are two equivalent ways of defining a standard quantum group associated with \( G \): either by defining (1) a quantization of the enveloping algebra \( U(G) \), namely, the Drinfeld-Jimbo quantized enveloping algebra \( U_q(G) \) [4, 7], or (2) a quantization \( O_q(G) \) of the coordinate algebra \( O(G) \), namely, the FRT-algebra [32] that is dual to \( U_q(G) \). Let \( GL_q(Z) \) denote the standard quantum group deformation of \( G = GL(Z) \)–which is only a virtual object–whose coordinate ring is \( O_q(G) \).

Now a proof of the generalized Littlewood-Richardson rule mentioned above roughly goes as follows. Let \( H = GL_n(C) \). Given partitions \( \alpha \) and \( \beta \) of height at most \( n \), the tensor product of the Weyl modules \( V_{\alpha}(H) \) and \( V_{\beta}(H) \) decomposes as:

\[
V_{\alpha}(H) \otimes V_{\beta}(H) = \bigoplus_{\lambda} c_{\alpha,\beta}^{\lambda} V_{\lambda}(H),
\]

where \( \lambda \) ranges over Young diagrams of height at most \( n \) and \( c_{\alpha,\beta}^{\lambda} \) is the Littlewood-Richardson coefficient. The problem is to find an explicit positive formula for this coefficient. Equivalently, letting \( G = H \times H \), the problem is to find an explicit positive decomposition of a given \( G \)-module \( V_{\alpha}(H) \otimes V_{\beta}(H) \), when considered as an \( H \)-module via the diagonal embedding:

\[
H \to G = H \times H.
\]

Let \( \mathcal{H} \) be the Lie algebra of \( H \), and \( U(\mathcal{H}) \) its enveloping algebra. The infinitesimal version the diagonal map above is the map

\[
U(\mathcal{H}) \to U(\mathcal{H}) \otimes U(\mathcal{H}),
\]

where \( \Delta(x) = 1 \otimes x + x \otimes 1 \) for \( x \in \mathcal{H} \). Drinfeld and Jimbo [4, 7] have given a quantum deformation of this map:

\[
U_q(\mathcal{H}) \to U_q(\mathcal{H}) \otimes U_q(\mathcal{H}),
\]

where \( U_q(\mathcal{H}) \) is the Drinfeld-Jimbo algebra, and such a deformation works for arbitrary symmetrizable Kac-Moody algebras. The (generalized) Littlewood-Richardson rule then comes out of the properties of canonical (local/global
crystal) bases of Kashiwara and Lusztig [10, 11, 16, 17, 29] for representations of $U_q(\mathcal{H})$.

This suggests the following analogous strategy to address the Kronecker problem (Problem 1.2).

(1) Find a quantization of the homomorphism (1)

$$H = GL(V) \times GL(W) \rightarrow G = GL(V \otimes W) = GL(X).$$

Instead of working in the setting of Drinfeld-Jimbo quantized algebra, we shall work in the dual setting FRT-coordinate algebras [32]. So what we seek is a quantization of the form:

$$GL_q(V) \times GL_q(W) \rightarrow GL_q(\bar{X}),$$

where the quantum groups $GL_q(V)$ and $GL_q(W)$ are standard, and $GL_q(\bar{X})$ is a newly sought quantum group deformation of $GL(V \otimes W)$. Furthermore, like $GL_q(X)$, the new $GL_q(\bar{X})$ should have a compact real form in the sense of Woronowicz [36]. This is required in the present context for two reasons. First, we want representation theory of the new quantum group to be rich like that of the standard quantum group. In particular, we want quantum analogues of the classical complete reducibility theorem, and Peter-Weyl theorem to hold, for which compactness is crucial. Second, we want representations of the new quantum group to have bases akin to the canonical bases for representations of the standard quantum group. Compactness is important for existence of such bases. Indeed, Kashiwara’s theory of local crystal bases [10], which was later globalized in [12], came out of an analysis of $q$-orthonormal Gelfand-Tsetlin bases for representations of the standard quantum group, whose construction depends on compactness of its real form in the sense of Woronowicz.

(2) Develop a theory of canonical (local/global crystal) bases for representations of $GL_q(\bar{X})$ akin to the canonical bases for representations of the standard quantum group $GL_q(X)$, as per Kashiwara and Lusztig [10, 11, 18].

(3) The required explicit positive decomposition rule should then follow from the properties of canonical bases.

This is the route that we shall adopt.
1.2 Quantization

As expected, the theory of Drinfeld-Jimbo quantum group does not work for the Kronecker problem, because it can be shown (as in [8]) that the homomorphism (1) can not be quantized in the category of Drinfeld-Jimbo quantum groups.

What is needed an analogue of the Drinfeld-Jimbo quantum group in the present setting. This is provided by the following result, which addresses the first step.

**Theorem 1.3** (a) The homomorphism

\[ H = GL(V) \times GL(W) \rightarrow G = GL(V \otimes W) = GL(X). \]  

(4)

can be quantized in the form

\[ GL_q(V) \times GL_q(W) \rightarrow GL_q(\bar{X}), \]  

(5)

where the quantum groups \( GL_q(V) \) and \( GL_q(W) \) are standard, and \( GL_q(\bar{X}) \) is a new quantum group deformation of \( GL(V \otimes W) \) defined in this paper.

(b) The quantum group \( GL_q(\bar{X}) \) has a real form \( U_q(\bar{X}) \), which is a compact quantum group in the sense of Woronowicz [36]. This implies that

1. Every finite dimensional representation of \( GL_q(\bar{X}) \) is completely reducible as a direct sum of irreducible representations.

2. Quantum analogue of the Peter-Weyl theorem holds.

This is proved in Sections 3-8. When \( W \) is trivial, the new quantum group specializes to the Drinfeld-Jimbo quantum group \( GL_q(V) \).

Furthermore, let \( \mathbb{C}[S_r] \) be the group algebra of the symmetric group \( S_r \), and \( \mathbb{C}[S_r] \rightarrow \mathbb{C}[S_r] \times \mathbb{C}[S_r] \) the embedding corresponding to the diagonal homomorphism \( S_r \rightarrow S_r \times S_r \). Then

**Proposition 1.4** The embedding \( \mathbb{C}[S_r] \hookrightarrow \mathbb{C}[S_r] \times \mathbb{C}[S_r] \) can be similarly quantized. That is, there is a quantized semisimple algebra \( B_r = B_r(q) \) with an embedding \( B_r \rightarrow H_r \otimes H_r \), where \( H_r = H_r(q) \) denotes the Hecke algebra associated with the symmetric group \( S_r \). It is a quantization of the group algebra \( \mathbb{C}[S_r] \).

This is proved in Sections 10.1-10.3. The relationship between \( B_r(q) \) and \( GL_q(\bar{X}) \) here is conjecturally similar to that between the Hecke algebra \( H_r(q) \) and the standard quantum group \( GL_q(X) \) (Section 10.2).
1.3 Explicit positive decomposition

The following addresses the second step.

Conjecture 1.5 (cf. [24]) The coordinate ring of $GL_q(\bar{X})$ has a basis that is akin to the canonical basis of the coordinate ring of $GL_q(X)$ as per Kashiwara and Lusztig [12, 19, 17].

The algebra $B_r(q)$ has a basis that is akin to the canonical (Kazhdan-Lusztig) basis of $\mathcal{H}_r(q)$ [13, 14].

The precise meaning of “akin to” will be made clear in [24], along with theoretical and experimental evidence in support of the second statement. See also Section 12 for an example of a canonical basis for $B_3(q)$. The first and the second statements here are closely related, in view of the close relationship between analogous two statements for the standard quantum group and Hecke algebra [6].

Assuming this conjecture, it follows [24] that there is an explicit positive rule for the Kronecker problem. This addresses the third step.

1.4 Comparison

The quantum group $GL_q(\bar{X})$ and the algebra $B_r(q)$ are qualitatively similar, but at the same time fundamentally different, in comparison to the standard quantum group $GL_q(X)$ and the Hecke algebra $\mathcal{H}_r(q)$, respectively. We now compare their properties.

(1) Compactness Compactness is a crucial property that $GL_q(\bar{X})$ shares with the standard quantum group $GL_q(X)$. This means Woronowicz’ theory [36] of compact quantum groups is applicable to $GL_q(\bar{X})$. In particular, compactness implies existence of orthonormal bases for representations of $GL_q(\bar{X})$. We can also expect nice orthonormal bases for representations of $GL_q(\bar{X})$ that yield local crystal bases, akin to the orthonormal Gel’fand-Tsetlin bases for representations of $GL_q(X)$, which, after renormalization, yield local crystal bases [10].

It may be remarked that among the several quantum deformations of $GL(\mathbb{C}^n)$ known by now, besides the one due to Drinfeld and Jimbo—e.g., Reshetikhin, Takhadtzhyan, Faddeev [32, 31], Manin [21], Sudbery [34], Artin, Schelter and Tate [2]—the standard Drinfeld-Jimbo deformation $GL_q(\mathbb{C}^n)$ is the only one that has a compact real form in the sense of Woronowicz [36].
Not surprisingly, it is the only deformation of $GL(\mathbb{C}^n)$ that has been studied in depth in the literature.

(2) **Determinants and minors** The quantum determinant and minors of the standard quantum group have simple formulae, very similar to the classical ones. In contrast, explicit formulae for the quantum determinant and quantum minors of $GL_q(X)$ turn out to be nonelementary; cf. Proposition 6.1. These involve Clebsch-Gordon coefficients and certain $q$-special functions that arise in the theory of the standard quantum group, which have been intensively studied (cf. the surveys [15]), but not yet completely understood. This difference in the complexity of the determinants and minors is the source of main problems that arise in the new setting.

(3) **Representations** The irreducible representations of the standard quantum group $GL_q(X)$ are $q$-deformations of the irreducible representations of $GL(X)$. This is no longer so for $GL_q(\bar{X})$. Because the Poincare series of $GL_q(\bar{X})$ is different from the Poincare series of $GL(X)$ (cf. Proposition 5.4). Thus, unlike in the standard case, the representation theory of the new quantum group does not run parallel to the representation theory of the classical general linear group. But conjecturally each irreducible representation $V_\alpha$ of $G = GL(X)$ has a unique $q$-analogue that is a possibly reducible representation of $GL(\bar{X})$; cf. Section 9 for a basic example which illustrates this phenomenon, and [24] for a detailed treatment. Thus the representations of $GL(X)$ are not lost in a virtual sense in the transition from $GL(X)$ to $GL_q(\bar{X})$.

(4) **Standard bases** The main reason for the one-to-one correspondence between representations of $GL(X)$ and of the standard quantum group $GL_q(X)$ is that the formulae for the determinants and minors of $GL_q(X)$ are very similar to those for $GL(X)$. Hence, the standard monomial basis in terms of minors of $X$ can be quantized [30] to get standard monomial bases for irreducible representations of $GL_q(X)$ [15]. Since, the formulae for the determinants and minors of $GL_q(\bar{X})$ are no longer elementary, construction of the irreducible representations of $GL_q(\bar{X})$ and their standard bases turns out to be a challenge that we have not been able to meet.

(5) **Explicit presentation:** The Hecke algebra $\mathcal{H}_r(q)$ has an explicit presentation in terms of generators and relations. These relations are quantizations of the usual relations among the generators of $\mathbb{C}[S_r]$, the degree three relations being the braid relations. This explicit presentation is necessary for the construction of irreducible representations of $\mathcal{H}_r(q)$ and their special bases as in [13]. Analogous explicit presentation for $B_r(q)$ turns out to be
a challenging problem, as an example for $r = 4$ in Section 13 illustrates. Hence, explicit construction of irreducible representations of $B_r(q)$ and their bases turns out to be challenge that we have not been able to meet. The algebraic structure of $B_r(q)$ and its combinatorial properties, which may shed light on this, will be studied in more detail in [1].

1.5 Plethysm and subgroup restriction problem

The Kronecker problem is a special case of the following generalized plethysm problem with $H = GL(V) \times GL(W)$ and $G = GL(V \otimes W) = GL(X)$.

**Problem 1.6 (The plethysm problem)**

Given partitions $\lambda, \mu, \pi$, give an explicit positive formula for the plethysm constant $a_{\lambda \mu}^\pi$. This is the multiplicity of the irreducible representation $V_\pi(H)$ of $H = GL_n(\mathbb{C})$ in the irreducible representation $V_\lambda(G)$ of $G = GL(V_\mu)$, where $V_\mu = V_\mu(H)$ is an irreducible representation $H$. Here $V_\lambda(G)$ is considered an $H$-module via the representation map $\rho : H \to G = GL(V_\mu)$.

(The generalized plethysm problem)

As above, allowing $H$ to be any connected reductive group, and letting $\mu$ and $\pi$ be dominant weights of $H$.

This problem will be addressed in [23]. Specifically, a quantization of the map $H \to G$ in Problem 1.6 is constructed there, which specializes to the one in Theorem 1.3 when $H = GL(V) \times GL(W)$ and $G = GL(X)$. Analogue of $B_r(q)$ in this context will also be constructed. Assuming analogue of Conjecture 1.5 for this quantization, it then follows that there exists an explicit positive formula as sought in Problem 1.6. Also addressed in [23] is a more general subgroup restriction problem, where one is explicitly given a polynomial homomorphism $H \to G$, $H$ and $G$ being arbitrary connected reductive groups.

1.6 Organization

The rest of this paper is organized as follows. In Section 2 we recall basic results and notions concerning the standard (Drinfeld-Jimbo) quantum group in the setting of FRT-algebras [32]. In Sections 3-8 we prove Theorem 1.3. In Sections 10-10.3 we prove Proposition 1.4. Sections 9, 11, 13 describe concrete examples. Section 12 gives a canonical basis of $B_3$. 8
2 The standard quantum group

In this section, we briefly recall basic notions concerning the standard quantum group; cf. [15, 32, 21]. It can be defined by specifying either its enveloping algebra, as in Drinfeld and Jimbo [4, 7], or its coordinate FRT-algebra, as in [32]. In this paper, we shall adopt the latter view. We shall mostly follow the terminology in [15].

Let $V$ be a vector space of dimension $n$, $R = R_{V,V}$ a linear mapping of $V \otimes V$ to itself, and $\hat{R}_{V,V} = \hat{R} = \tau \circ R$, where $\tau$ is the flip of $V \otimes V$. Let $u$ be an $n \times n$ variable matrix, and $\mathbb{C}\langle u \rangle$ the free algebra over the variable entries of $u$. Then the defining relations of the FRT bialgebra $A(R)$ [32] in the matrix form is

$$\hat{R} u_1 u_2 = u_1 u_2 \hat{R}, \quad (6)$$

i.e.,

$$\hat{R}(u \otimes u) = (u \otimes u) \hat{R}, \quad (7)$$

where $u_1 = u \otimes I$ and $u_2 = I \otimes u$, and $I$ denotes the identity matrix. This is equivalent to:

$$R u_1 u_2 = u_2 u_1 R. \quad (8)$$

Thus,

$$A(R) = \mathbb{C}\langle u \rangle / \langle \hat{R} u_1 u_2 - u_1 u_2 \hat{R} \rangle. \quad (9)$$

This is a bialgebra with comultiplication and counit given by

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j, \quad \epsilon(u^i_j) = \delta_{ij},$$

or in the matrix form

$$\Delta(u) = u \otimes u, \text{ and } \epsilon(u) = I,$$

where $\cdot$ denotes matrix multiplication.

The coordinate algebra $\mathcal{O}(M_q(V))$ of the standard quantum matrix space $M_q(V)$ is the FRT-algebra $A(\hat{R})$ for the specific $R$ given by:

$$R_{mn}^{ij} = q^{\delta_{ij}\delta_{in}} \delta_{jm} + (q - q^{-1}) \delta_{im} \delta_{jn} \theta(j - i),$$

where $\theta(k)$ is 1 if $k > 0$ and 0 otherwise. It is known that the corresponding $\hat{R}$ satisfies the quadratic equation

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0, \quad (10)$$
and has the spectral decomposition

\[ \hat{R} = qP_+ - q^{-1}P, \]  

(11)

where the projections \( P_+ = P^V_+ \) and \( P_- = P^V_- \) are

\[ P_+ = \frac{\hat{R} + q^{-1}I}{q + q^{-1}}, \quad P_- = \frac{-\hat{R} + qI}{q + q^{-1}}, \]  

(12)

so that

\[ I = P_+P_- \]  

(13)

is the spectral decomposition of the identity. These projections are quantum analogues of the symmetrization and antisymmetrization operators on \( \mathbb{C}^l \otimes \mathbb{C}^l \), respectively. Specifically, let the symmetric subspace \( S_q(V \otimes V) \) be the image of \( P_+ \), and the antisymmetric space \( A_q(V \otimes V) \) the image of \( P_- \). Thus \( S_q(V \otimes V) \) is defined by the equation

\[ P_-v_1v_2 = 0, \]  

(14)

where \( v_1 = v \otimes I \) and \( v_2 = I \otimes v \). In terms of the entries \( v_i \)'s of \( v \), this becomes

\[ v_iv_j = qv_jv_i, \quad i < j. \]  

(15)

The antisymmetric space \( A_q(V \otimes V) \) is defined by the equation

\[ P_+v_1v_2 = 0, \]  

(16)

or equivalently,

\[ v_i^2 = 0, \quad \text{and} \quad v_iv_j = -q^{-1}v_jv_i, \quad i < j. \]  

(17)

Let \( \mathbb{C}_q[V] \) be the algebra over the entries \( v_i \)'s of \( v \) subject to the relation (15). It will called the quantum symmetric algebra of \( V \). Let \( \wedge_q[V] \) be the algebra over the entries \( v_i \)'s of \( v \) subject to the relation (17). It will called the quantum exterior algebra of \( V \). Let \( C_q^r[V] \) and \( \wedge_q^r[V] \) be the degree \( r \)-components of \( \mathbb{C}_q[V] \) and \( \wedge_q[V] \), respectively.

Both \( \mathbb{C}_q[V] \) and \( \wedge_q[V] \) are left and right corepresentations of \( \mathcal{O}(M_q(V)) \). By (12), the defining relation (6) of \( \mathcal{O}(M_q(V)) \) is equivalent to

\[ P_+u_1u_2 = u_1u_2P_+, \]  

(18)

or equivalently,

\[ P_-u_1u_2 = u_1u_2P_. \]  

(19)
These can also be rewritten as:

\[ P_+(u \otimes u) = (u \otimes u)P_+, \quad (20) \]

or equivalently,

\[ P_-(u \otimes u) = (u \otimes u)P_. \quad (21) \]

It can be shown [15] that (20) is equivalent to saying that the defining relation (15) of the quantum symmetric algebra \( \mathbb{C}_q[V] \) is preserved by the left and right actions of \( u \) on \( v \) given by \( v \rightarrow uv \) and \( v^t \rightarrow v^t u \). Similarly, (21) is equivalent to saying that the defining relation (17) of the quantum antisymmetric algebra \( \wedge_q[V] \) is also preserved by the left and right actions of \( u \) on \( v \).

We think of \( \mathbb{C}_q[V] \) as the coordinate algebra of a virtual symmetric quantum space \( V_{\text{sym}} \), isomorphic to \( V \) as a vector space, with commuting coordinates (in the quantum sense), and \( \wedge_q[V] \) as the coordinate algebra of a virtual antisymmetric quantum space \( V_{\wedge} \), isomorphic to \( V \) as a vector space, with anti-commuting coordinates (in the quantum sense). Thus \( M_q(V) \) is the set of linear transformations of the symmetric quantum space \( V_{\text{sym}} \) or the antisymmetric quantum space \( V_{\wedge} \), on which each transformation acts from the left and as well as the right. This view of the standard quantum group, emphasized by Manin [21], will be a starting point for the definition of the new quantum group \( GL_q(\mathcal{X}) \).

Let \( \phi_{R,r} \) be the right coaction of \( O(M_q(V)) \) on \( \wedge^r_q[V] \):

\[ \phi_{R,r} : \wedge^r_q[V] \rightarrow \wedge^r_q[V] \otimes O(M_q(V)), \]

and \( \phi_{L,r} \) the left coaction. Let \( \Omega_r \) be the set of subsets of \( \{1, \ldots, n\} \) of size \( r \). For a subset \( I \in \Omega_r \), with \( I = \{i_1, \ldots, i_r\} \), \( i_1 < i_2 < \cdots \), let \( v_I = v_{i_1} \cdots v_{i_r} \).

The left quantum \( r \)-minors of \( O(M_q(V)) \) are defined to be the matrix coefficients of the left corepresentation map \( \phi_{L,r} \). Specifically, for \( I, J \in \Omega_r \), the left quantum \( r \)-minors \( D^L_{I,J}(V) \) are such that

\[ \phi_{L,r}(v_I) = \sum_J D^L_{I,J}(V) \otimes v_J. \]

The right quantum \( r \)-minors \( D^R_{I,J}(V) \) are such that

\[ \phi_{R,r}(v_I) = \sum_J v_J \otimes D^R_{I,J}(V). \]
Then
\[ D^I_J = D^{L,I}_J (V) = D^{R,I}_J (V) \neq 0. \] (22)

The quantum determinant \( D_q = D_q(V) \) of \( u \) is defined to be \( D^{L,I}_J (V) = D^{R,I}_J (V) \), with \( I = J = [1, n] \). Explicitly:
\[
D^I_J = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} u_{j_1}^{i_{\sigma(1)}} \cdots u_{j_n}^{i_{\sigma(n)}},
\] (23)
where \( l(\sigma) \) is the number of inversions in the permutation \( \sigma \).

The coordinate algebra \( \mathcal{O}(GL_q(V)) \) of the quantum group \( GL_q(V) \) is obtained by adjoining the inverse \( D_q(V)^{-1} \) to \( \mathcal{O}(M_q(V)) \). We have a non-degenerate pairing
\[
\wedge_{q}^{n-1}[V] \times \wedge_{q}^{1}[V] \to \wedge_{q}^{n}[V],
\]
where \( \wedge_{q}^{1}[V] = V \) is the fundamental vector representation. Its matrix form lets us define the cofactor matrix \( \tilde{u} \) so that
\[
\tilde{u}u = uu \tilde{u} = D_q(V)I.
\]
Then we can formally define \( u^{-1} = D_q(V)^{-1}u \). This gives the following Hopf structure on \( \mathcal{O}_q(GL_q(V)) \):

1. \( \Delta(u) = u \otimes u \).
2. \( \epsilon(u) = I \).
3. \( S(u^i_j) = \tilde{u}_j^i D_q^{-1} \), \( S(D_q^{-1}) = D_q \), where \( u^i_j \) are the entries of \( u \) and \( \tilde{u}_j^i \) are the entries of \( \tilde{u} \).

The Poincare series of \( \mathcal{O}(M_q(V)) \) coincides with the Poincare series of the commutative algebra \( \mathbb{C}[U] = \mathbb{C}[u_{ij}] \). Because, just as in the classical case, \( \mathcal{O}(M_q(V)) \) has a basis consisting of the standard monomials \( (u_{11})^{k_{11}}(u_{12})^{k_{12}} \cdots (u_{nn})^{k_{nn}}, k_{ij} \) being nonnegative integers. To show this \[30\] \[2\], the monomials are ordered lexicographically, and the defining equations \[20\] of \( \mathcal{O}(M_q(V)) \) are recast in the form of a reduction system:
\[
\begin{align*}
    u_{jk} u_{ik} & \rightarrow q^{-1} u_{ik} u_{jk} & (i < j) \\
    u_{kj} u_{ki} & \rightarrow q^{-1} u_{ki} u_{kj} & (i < j) \\
    u_{jk} u_{il} & \rightarrow u_{il} u_{jk} & (i < j, k < l) \\
    u_{jl} u_{ik} & \rightarrow u_{ik} u_{jl} - (q - q^{-1}) u_{il} u_{jk} & (i < j, k < l).
\end{align*}
\] (24)
Then, by the diamond lemma [15], it suffices to show that all ambiguities in this reduction system are resolvable. This means any term of the form $u_{ij}u_{kl}u_{rs}$, when reduced in any way, leads to the same result. This has to be checked for 24 different types of configurations of the three indices $(i, j), (k, l), (r, s)$; see [2, 15, 30] for details.

### 2.1 Compactness, unitary transformations

What sets the standard quantum group apart from other known deformations [2, 21, 32, 31, 34] of $GL(V)$ is that it has a real form that is compact. To see what this means, we have to recall the notion of compactness due to Woronowicz in the quantum setting; cf. [36] or Chapter 11 in [15] for details.

Let $A$ be the coordinate Hopf algebra of a quantum group $G_q$. Suppose there is an involution $\ast$ on $A$ so that it is a Hopf $\ast$-algebra [15]. We say that $\ast$ defines a real form of the quantum group $G_q$. A finite dimensional corepresentation of $A$ on a vector space $V$ with a Hermitian form is called unitary if the matrix $v = (v_{ij})$ of this corepresentation with respect to an orthonormal basis $\{e_i\}$ of $V$ satisfies $v^*v = vv^* = I$. The algebra $A$ is called a compact matrix group algebra (CMQG) if (1) it is the linear span of all matrix elements of finite-dimensional corepresentations of $A$, (2) it is generated as an algebra by finitely many elements. Then

**Theorem 2.1** (Woronowicz [36]; cf. chapter 11 in [15]) (a) A Hopf $\ast$-algebra $A$ is a CMQG algebra if there is a finite dimensional unitary corepresentation of $A$ on a vector space $V$ with a Hermitian form.

(b) If $A$ is a CMQG algebra then the quantum analogue of the Peter-Weyl theorem holds, and any finite dimensional corepresentation of $A$ is unitarizable, and hence, a direct sum of irreducible corepresentations.

There is a unique involution $\ast$ on the algebra $\mathcal{O}(GL_q(V))$ such that $(u^i_j)^\ast = S(u^i_j)$. Equipped with this involution, $\mathcal{O}(GL_q(V))$ becomes a Hopf $\ast$-algebra, called denoted by $\mathcal{O}(U_q(V))$, and called the coordinate algebra of the quantum unitary group $U_q(V)$—which is, again, a virtual object. Furthermore, $\mathcal{O}(GL_q(V))$ is a CMQG algebra.

Woronowicz [36] has shown that the usual results for real compact groups, such as Harmonic analysis, existence of orthonormal bases, and so on, generalize to CMQG algebras.
2.2 Gelfand-Tsetlin bases and Clebsch-Gordon coefficients

In particular, standard results for the unitary group $U(V)$ have their analogues for $U_q(V)$. In this section, we describe results of this kind that we need; cf. [15, 35] for their detailed description.

Given a partition $\lambda$, let $V_{\lambda}(V)$ denote the corresponding Weyl module of $GL(V)$, and $V_{q,\lambda}(V)$ denote the corresponding irreducible representation (q-Weyl module) of the standard quantum group $GL_q(V)$. Let $\{|M\rangle\}$ denote the orthonormal Gelfand-Tsetlin basis for $V_{q,\lambda}(V)$, where $M$ ranges over Gelfand-Tsetlin tableau of shape $\lambda$.

The tensor product of two irreducible representations of $GL_q(V)$ decomposes as

$$V_{q,\alpha}(V) \otimes V_{q,\beta}(V) = \oplus c_{\alpha,\beta,\gamma} V_{q,\gamma,r}(V),$$

(25)

where $r$ labels different copies of $V_{q,\gamma}(V)$—the number of these copies is the Littlewood-Richardson coefficient $c_{\alpha,\beta,\gamma}$.

The Clebsch-Gordon (Wigner) coefficients (CGCs) of this tensor product are defined by the formula

$$|M\rangle_r = \sum_{N,K} C_{\alpha,\beta,\gamma}^{\alpha,\beta,\gamma} |N\rangle \otimes |K\rangle,$$

(26)

where $N$ and $K$ range over Gelfand-Tsetlin tableau of shapes $\alpha$ and $\beta$, respectively, and $|M\rangle_r$ denotes the Gelfand-Tsetlin basis element of $V_{q,\gamma,r}(V)$ in (25) labelled by the Gelfand-Tsetlin tableau $M$ of shape $\gamma$. We denote $C_{N,K,M,r}^{\alpha,\beta,\gamma}$ by simply $C_{N,K,M,r}$ if the shapes are understood in a context. Furthermore, if the multiplicity of $V_{q,\gamma}(V)$ is one in (25), we denote $C_{N,K,M,r}$ by $C_{N,K,M}$. These coefficients have been intensively studied in the literature; cf. [15, 35] and the references therein. An explicit formula for them is known when either $V_{q,\alpha}(V)$ or $V_{q,\beta}(V)$ is a fundamental vector representation, or more generally, a symmetric representation. In the presence of multiplicities, the Clebsch-Gordon coefficients are not uniquely determined, and do not have explicit formulae in general.

We now briefly recall explicit formulae for the fundamental Clebsch-Gordon coefficients—i.e., when $V_{q,\alpha}(V)$ or $V_{q,\beta}(V)$ is a fundamental representation.

Suppose the multiplicity of each irreducible representation in the tensor product decomposition (25) is one, as in the fundamental case. Then each CGC $C_{N,K,M}$ is a product of so called reduced CGC’s, which depend on the consecutive two rows of $N, K, M$. The s-reduced CGC is denoted by the
symbol

\[
\begin{pmatrix}
N_s & K_s & M_s \\
N_{s-1} & K_{s-1} & M_{s-1}
\end{pmatrix},
\]

where \(N_s\) denote the \(s\)-th row of the Gelfand-Tsetlin tableau \(N\). Let us assume that \(V_{q,\beta}(V)\) is a fundamental representation. Then each (valid) reduced CGC is either of the form:

\[
\begin{pmatrix}
N_s & (1,0) & N_s + e_i \\
N_{s-1} & (0,0) & N_{s-1}
\end{pmatrix},
\]

where \(0 = (0,\ldots,0)\) and \(e_i\) denotes a unit-row with \(i\)-th entry one, or of the form

\[
\begin{pmatrix}
N_s & (1,0) & N_s + e_i \\
N_{s-1} & (1,0) & N_{s-1} + e_j
\end{pmatrix}.
\]

The first CGC is equal to

\[
q^{-1/2(i+1+\sum_j N_{j,s-1} - \sum_j N_{j,s} - i)} \times \left( \frac{\prod_{j=1}^{s-1} [N_{j,s-1} - N_{i,s} - j + i - 1]}{\prod_{j \neq i} [N_{j,s} - N_{i,s} - j + i]} \right)^{1/2},
\]

and the second CGC is equal to

\[
\nu(j-i) q^{- (N_{j,s-1} - N_{i,s} - j + i)/2} \times 
\left( \prod_{k \neq i} \frac{[N_{k,s} - N_{j,s} - k + j]}{[N_{k,s} - N_{i,s} - k + i]} \right)^{1/2},
\]

where \([m]_q = (q^m - q^{-m})/(q - q^{-1})\) is the \(q\)-number, \(N_{i,k}\) are the components of \(N_i\), and \(\nu(j-i) := 1\) if \(j-i \geq 0\) and \(-1\) otherwise.

3 Quantization of \(GL(V) \times GL(W) \hookrightarrow GL(V \otimes W)\)

Now we turn to Theorem 1.3.

Let \(H = GL(V) \times GL(W)\), where \(V\) and \(W\) are vector spaces of dimension \(n\) and \(m\). Let \(X = V \otimes W\) be the fundamental representation of \(H\), and \(\rho : H \hookrightarrow G = GL(X) = GL(V \otimes W)\) the corresponding homomorphism. The goal is to quantize this homomorphism in the form:

\[
GL_q(V) \times GL_q(W) \to GL_q(X),
\]
where the quantum groups $GL_q(V)$ and $GL_q(W)$ are standard, and $GL_q(\bar{X})$ is a newly sought quantum group deformation of $GL(V \otimes W)$ such that (1) it has compact real form and (2) its dimension is the same as that of $GL(X)$. This then can be considered to be a correct quantization of $\rho$. We shall prove (1), and conjecture (2) (cf. Conjecture 5.5).

When we think of the vector space $X$ as a fundamental representation of $GL_q(V)$ or $GL_q(W)$—or as a fundamental representation of $GL_q(\bar{X})$ defined later—we denote it by $\bar{X}$, as in eq.(29).

The construction of $GL_q(\bar{X})$ is motivated by the view of the standard quantum group described in Section 2:

1. First we construct (Section 4) symmetric and antisymmetric quantum algebras of $\bar{X}$, $C_q(\bar{X})$ and $\wedge_q(\bar{X})$, which can be thought of as coordinate algebras of (virtual) quantum spaces $\bar{X}_{\text{sym}}$ and $\bar{X}_{\wedge}$, respectively. When $W = V^*$, $\bar{X}_{\text{sym}}$ is isomorphic to the standard quantum space $M_q(V)$; i.e., the defining equations of $\bar{X}_{\text{sym}}$ are the same as (24).

2. The quantum matrix space $M_q(\bar{X})$ is defined (Section 5) as the space of linear transformations of $\bar{X}_{\text{sym}}$, or equivalently, of $\bar{X}_{\wedge}$; i.e., so that the left and right actions of $M_q(\bar{X})$ on $\bar{X}_{\text{sym}}$ or $\bar{X}_{\wedge}$ preserve their defining equations.

3. It is shown that there is a natural bialgebra homomorphism from $O(M_q(\bar{X}))$ to $O(M_q(V)) \otimes O(M_q(W))$; cf. Section 5.2

4. Let $\wedge_q[\bar{X}]_r$ be the degree $r$ component of $\wedge_q[\bar{X}]$. Then the left and right quantum $r$-minors are defined as the matrix coefficients of the left and right coactions of $O(M_q(\bar{X}))$ on $\wedge_q[\bar{X}]$, and their basic properties are proved; cf. Section 6

5. The cofactor and the inverse of a generic matrix $u \in M_q(\bar{X})$ are defined using determinants and quantum minors, and $GL_q(\bar{X})$ is defined as the subset of nonsingular transformations in $M_q(\bar{X})$. Formally, $GL_q(\bar{X})$ is defined by putting a Hopf structure on the coordinate algebra $O(M_q(\bar{X}))$; cf. Section 7.

6. A natural *-structure is put on $O(GL_q(\bar{X}))$, and using Woronowicz’ results [30], it is shown that $GL_q(\bar{X})$ has a compact real form, which we shall denote by $U_q(\bar{X})$—this is the analogue of the unitary group in this setting; cf. Section 8.
7. Finally, it is shown that Theorem 1.3 follows from these results in conjunction with Woronowicz’s results \[36\].

In the subsequent sections we address these steps one at a time.

4 Quantum symmetric and antisymmetric algebras

Let \( \hat{R}_{\bar{X},\bar{X}} = \hat{R}_{V,V} \otimes \hat{R}_{W,W} \) be the \( \hat{R} \)-matrix associated with \( \bar{X} = V \otimes W \) as a representation of \( GL_q(V) \times GL_q(W) \). This is different from the \( \hat{R} \)-matrix \( \hat{R}_{X,X} \) obtained by thinking of \( X = V \otimes W \) as a vector representation of the standard quantum group \( GL_q(X) \).

Both \( \hat{R}_{V,V} \) and \( \hat{R}_{W,W} \) are diagonalizable with eigenvalues \( q \) and \(-q^{-1}\). Hence, \( \hat{R}_{\bar{X},\bar{X}} \) is diagonalizable with eigenvalues \( q^2, -1, q^{-2} \). In what follows, we assume that \( q \) is positive and transcendental. The quantum symmetric subspace \( S_q(\bar{X} \otimes \bar{X}) \subseteq \bar{X} \otimes \bar{X} \) is defined to be the span of the eigenspaces for the positive eigenvalues \( q^2, q^{-2} \) (cf. Chapter 8,\[15\]). The quantum antisymmetric subspace \( A_q(\bar{X} \otimes \bar{X}) \) is defined to be the eigenspace for the negative eigenvalue \(-1\). Let

\[
I = P_+(\hat{R}_{\bar{X},\bar{X}}) + P_-(\hat{R}_{\bar{X},\bar{X}})
\]

be the spectral decomposition of the identity corresponding to the spectral decomposition of the diagonalizable \( \hat{R}_{\bar{X},\bar{X}} \). In what follows, we let

\[
\begin{align*}
P_+^\bar{X} & = P_-(\hat{R}_{\bar{X},\bar{X}}) \\
P_-^\bar{X} & = P_+(\hat{R}_{\bar{X},\bar{X}}).
\end{align*}
\]

Both \( P_+^\bar{X} \) and \( P_-^\bar{X} \) are polynomials in \( \hat{R}_{\bar{X},\bar{X}} \). The symmetric subspace \( S_q(\bar{X} \otimes \bar{X}) \) is the image of \( P_+^\bar{X} \) and the antisymmetric space \( A_q(\bar{X} \otimes \bar{X}) \) is the image of \( P_-^\bar{X} \). In other words, \( S_q(\bar{X} \otimes \bar{X}) \) is defined by the equation

\[
P_-^\bar{X} x_1 x_2 = 0,
\]

where \( x_1 = x \otimes I \) and \( x_2 = I \otimes x \), and \( A_q(\bar{X} \otimes X) \) is defined by the equation

\[
P_+^\bar{X} x_1 x_2 = 0,
\]

Let the quantum symmetric algebra \( C_q[\bar{X}] \) of \( \bar{X} \) be the algebra over the entries \( x_i \)'s of \( x \) subject to relation (31). It will be called the coordinate ring
of the quantum space $\tilde{X}_{\text{SYN}}$, which is only a virtual object. Let the quantum exterior algebra $\wedge_q[\tilde{X}]$ of $X$ be the algebra over the entries $x_i$'s of $x$ subject to relation (32). It will called the coordinate ring of the virtual quantum space here is different from the one in the literature. Let $\mathbb{C}_q[\tilde{X}]$ and $\wedge_q[\tilde{X}]$ be the degree $r$ components of $\mathbb{C}_q[\tilde{X}]$ and $\wedge_q[\tilde{X}]$, respectively.

Let $A_V$ be an eigenbasis of $S_q(V \otimes V)$ for the eigenvalue $q$ of $\hat{R}_{V,V}$, and $B_V$ an eigenbasis for the eigenvalue $-q^{-1}$; $A_W, B_W$ are defined similarly. Given $c \in V \otimes V$ and $d \in W \otimes W$, we define their restitution $c \ast d \in X \otimes \bar{X}$ as follows. Let $c = \sum_i c_i^1 \otimes c_i^2, c_i^k \in V$, and $d = \sum_j d_j^1 \otimes d_j^2, d_j^k \in W$. Then

$$c \ast d = \sum_{i,j} (c_i^1 \otimes d_j^1) \otimes (c_i^2 \otimes d_j^2). \quad (33)$$

Then the following are eigenbases of $S_q(\bar{X} \otimes \bar{X})$ and $A_q(\bar{X} \otimes \bar{X})$:

$$S_q(\bar{X} \otimes \bar{X}) : \{ a \ast a' \mid a \in A_V, a' \in A_W \} \cup \{ b \ast b' \mid b \in B_V, b' \in B_W \}. \quad (34)$$

An explicit form of the defining relation (31) for the symmetric quantum algebra $\mathbb{C}_q[\tilde{X}]$ is obtained by setting the eigenbasis of $A_q(\bar{X} \otimes \bar{X})$ to zero:

$$a \ast a' = 0 \quad a \in A_V, b' \in B_W, \quad \sum_{i,j} (c_i^1 \otimes d_j^1) \otimes (c_i^2 \otimes d_j^2). \quad (35)$$

An explicit form of the defining relation (32) for the antisymmetric quantum algebra $\wedge_q[\tilde{X}]$ is obtained by setting the eigenbasis of $S_q(\bar{X} \otimes \bar{X})$ to zero:

$$a \ast a' = 0 \quad a \in A_V, a' \in a_W, \quad b \ast b' = 0 \quad b \in B_V, b' \in B_W. \quad (36)$$

Example

Let $\dim(V) = \dim(W) = 2$, $V = \langle v_1, v_2 \rangle$, $W = \langle w_1, w_2 \rangle$. Then $A_V = \{A_{11}, A_{22}, B_{12}\}$ and $B_V = \{B_{12}\}$, where, deleting the $\otimes$ sign and letting $p = 1/q$,

$$A_{11}(V) = v_1 v_1$$
$$A_{22}(V) = v_2 v_2$$
$$A_{12}(V) = v_1 v_2 + p v_2 v_1$$
$$B_{12}(V) = v_1 v_2 - q v_2 v_1 \quad (37)$$
When \( \dim(M) \) for the standard quantum matrix space following reduction system:

### Proposition 4.1

1. A basis of the symmetric space \( S_q(\bar{X} \otimes \bar{X}) \):

\[
\begin{align*}
  a_{11} &:= A_{11}(V) \ast A_{11}(W) = x_{11}x_{11} \\
  a_{22} &:= A_{11}(V) \ast A_{22}(W) = x_{12}x_{12} \\
  a_{12} &:= A_{11}(V) \ast A_{12}(W) = x_{11}x_{12} + px_{12}x_{11} \\
  a_{33} &:= A_{22}(V) \ast A_{11}(W) = x_{21}x_{21} \\
  a_{44} &:= A_{22}(V) \ast A_{22}(W) = x_{22}x_{22} \\
  a_{34} &:= A_{22}(V) \ast A_{12}(W) = x_{21}x_{22} + px_{22}x_{21} \\
  a_{13} &:= A_{12}(V) \ast A_{11}(W) = x_{11}x_{21} + px_{21}x_{11} \\
  a_{24} &:= A_{12}(V) \ast A_{22}(W) = x_{12}x_{22} + px_{22}x_{12} \\
  a_{14} &:= A_{12}(V) \ast A_{12}(W) = x_{11}x_{22} + px_{12}x_{21} + px_{21}x_{12} + p^2x_{22}x_{11} \\
  a_{23} &:= B_{12}(V) \ast B_{12}(W) = x_{11}x_{22} - qx_{12}x_{21} - qx_{21}x_{12} + q^2x_{22}x_{11}
\end{align*}
\] (38)

The following is a basis of the antisymmetric space \( A_q(\bar{X} \otimes \bar{X}) \):

\[
\begin{align*}
  b_{12} &:= A_{11}(V) \ast B_{12}(W) = x_{11}x_{12} - qx_{12}x_{11} \\
  b_{34} &:= A_{22}(V) \ast B_{12}(W) = x_{21}x_{22} - qx_{22}x_{21} \\
  b_{13} &:= B_{12}(V) \ast A_{11}(W) = x_{11}x_{21} - qx_{21}x_{11} \\
  b_{24} &:= B_{12}(V) \ast A_{22}(W) = x_{12}x_{22} - qx_{22}x_{12} \\
  b_{14} &:= A_{12}(V) \ast B_{12}(W) = x_{11}x_{22} - qx_{12}x_{21} + px_{21}x_{12} - x_{22}x_{11} \\
  b_{23} &:= B_{12}(V) \ast A_{12}(W) = x_{11}x_{22} + px_{12}x_{21} - qx_{21}x_{12} - x_{22}x_{11}
\end{align*}
\] (39)

2. A basis of \( \wedge_q[\bar{X}] \) is \( \{ \prod_{i,j} x_{ij}^{k_{ij}} \mid k_{i,j} \in \mathbb{Z}_{\geq 0} \} \).

These bases will be called **standard monomial bases** of \( \mathbb{C}_q[\bar{X}] \) and \( \wedge_q[\bar{X}] \).

**Proof:**

1. The relations \( [35] \) (cf. \( [39] \)) can be reformulated in the form of the following reduction system:

   \[
   \begin{align*}
   x_{jk}x_{ik} &\rightarrow q^{-1}x_{ik}x_{jk} \quad (i < j) \\
   x_{kj}x_{ki} &\rightarrow q^{-1}x_{ki}x_{kj} \quad (i < j) \\
   x_{jk}x_{il} &\rightarrow x_{il}x_{jk} \quad (i < j, k < l) \\
   x_{ji}x_{ik} &\rightarrow x_{ik}x_{ji} - (q - q^{-1})x_{il}x_{jk} \quad (i < j, k < l).
   \end{align*}
   \] (40)

When \( \dim(V) = \dim(W) \), these coincide with the defining relations \( [24] \) for the standard quantum matrix space \( M_q(V) \) after the change of variables
\( x_{ij} \rightarrow u_{ij} \). In other words, \( \bar{X}_{\text{sym}} \cong M_q(V) \). In this case, the ambiguities in this reduction system can be resolved just as in the case of the reduction system for \( \mathcal{O}(M_q(V)) \) \cite{30,2}; cf. Section 2. This is also so when \( \dim(V) \neq \dim(W) \). Hence the result follows from the diamond lemma \cite{15}.

(2) The relations \((36)\) can be reformulated in the form of the following reduction system:

\[
\begin{align*}
x_{ij}^2 & \rightarrow 0 \\
x_{jk}x_{ik} & \rightarrow -qx_{ik}x_{jk} \quad (i < j) \\
x_{kj}x_{ki} & \rightarrow -qx_{ki}x_{kj} \quad (i < j) \\
x_{ij}x_{il} & \rightarrow -x_{ik}x_{jl} \quad (i < j, k < l) \\
x_{jk}x_{il} & \rightarrow -x_{il}x_{jk} + (q^{-1} - q)x_{ik}x_{jl} \quad (i < j, k < l).
\end{align*}
\]

Ambiguities in this reduction system can also be resolved just as in (1); we omit the details. So the result again follows from the diamond lemma \cite{15}. Q.E.D.

For future reference, we note down a corollary of the proof above. Let \( Y = \text{End}(V, V) = V^* \otimes V \). Let \( \hat{R}_{Y,Y} = \hat{R}_{V^*,V} \otimes \hat{R}_{V,V} \) be an \( \hat{R} \) matrix associated with \( Y \). It is diagonalizable. Let

\[
I = P_Y^- + P_Y^+
\]

be the associated spectral decomposition of the identity, where \( P_Y^- \) and \( P_Y^+ \) denote the projections onto the eigenspaces of \( \hat{R}_{Y,Y} \) for the eigenvalues with sign – and +, respectively.

**Proposition 4.2** Eq. \((18)\) or eq. \((19)\) defining \( \mathcal{O}(M_q(V)) \) is equivalent to the relation

\[
P_Y^-(u \otimes u) = 0. \quad (42)
\]

At \( q = 1 \), this relation simply says that the entries of \( u \) commute with each other. So this is a quantized version of commutativity.

**Proof:** See the proof of Proposition 4.1 (1), and the remark therein. Q.E.D.

**Proposition 4.3** (1) As a \( GL_q(V) \times GL_q(W) \)-module

\[
\mathbb{C}_d^d[\bar{X}] = Sym_d^d(V \otimes W) \cong \sum_{\lambda} V_{q,\lambda}(V) \otimes V_{q,\lambda}(W), \quad (43)
\]

where \( \lambda \) ranges over all Young diagrams of size \( d \), with at most \( \dim(V) \) or \( \dim(W) \) rows.
Similarly,
\[ \Lambda_q^d [\tilde{X}] = \Lambda_q^d (V \otimes W) \cong \bigoplus_{\lambda} V_{\lambda,\lambda}(V) \otimes V_{\lambda,\lambda'}(W), \]
(44)

where \( \lambda \) ranges over all Young diagrams of size \( d \), with at most \( \dim(V) \) rows and at most \( \dim(W) \) columns, and \( \lambda' \) denotes the conjugate of \( \lambda \) obtained by interchanging rows and columns.

**Proof:** (1) By Proposition 4.1, the character of \( \mathbb{C}^d[X] \) as a \( GL_q(V) \times GL_q(W) \)-module coincides with the character of \( \mathbb{C}^d[X] \) as a \( GL(V) \times GL(W) \)-module. Since, finite dimensional representations of the \( GL_q(V) \times GL_q(W) \) are completely reducible, the irreducible representations of \( GL_q(V) \times GL_q(W) \) are in one-to-one correspondence with the those of \( GL(V) \times GL(W) \), and the characters of the corresponding representations coincide \[15\], (1) follows from the the classical result
\[ \mathbb{C}^d[X] = \text{Sym}^d (V \otimes W) \cong \bigoplus_{\lambda} V_{\lambda}(V) \otimes V_{\lambda}(W). \]
(45)

Similarly (2) is a \( q \)-analogue of the classical result
\[ \wedge^d (V \otimes W) \cong \bigoplus_{\lambda} V_{\lambda}(V) \otimes V_{\lambda'}(W), \]
(46)

Q.E.D.

We have already noted that \( \mathbb{C}^d_q[\tilde{X}] \) is isomorphic to the coordinate algebra \( \mathcal{O}(M_q(V)) \) of the standard quantum matrix space, when \( W = V^* \). In this case, Proposition 4.3 (1) is the \( q \)-analogue of the Peter-Weyl theorem for the standard quantum group \( GL_q(V) \), and Proposition 4.3 (2) is the \( q \)-analogue of the antisymmetric form of the Peter-Weyl theorem.

### 4.1 Explicit product formulae

We wish to give explicit formulae for products in the symmetric and antisymmetric algebras \( \mathbb{C}_q[\tilde{X}] \) and \( \wedge_q[\tilde{X}] \).

Let
\[ \tilde{B}_X^\pm = \bigcup_{\lambda} \{ |M_\lambda \rangle \otimes |N_\lambda \rangle \} \]
(47)

be the Gelfand-Tsetlin basis for \( \mathbb{C}^d_q[\tilde{X}] \) as per the decomposition in Proposition 4.3 (1) and
\[ \tilde{B}_X^\wedge = \bigcup_{\lambda} \{ |M_\lambda \rangle \otimes |N_{\lambda'} \rangle \} \]
(48)

that for \( \Lambda^d_q[\tilde{X}] \) as per Proposition 4.3 (2).
When $W = V^*$, the basis element $|M\rangle \otimes |N\rangle \in V_{q,\lambda}(V) \otimes V_{q,\lambda}(V^*) \subseteq \mathbb{C}_q[\bar{X}]$ stands for the matrix coefficient $u_{NM}^{\lambda\lambda}$ of the representation $V_{q,\lambda}(V)$ of the standard quantum group $GL_q(V)$.

It is of interest to know explicit transformation matrices connecting the Gelfand-Tsetlin bases of $\mathbb{C}_q[\bar{X}]$ and $\wedge_q(\bar{X})$ with their standard monomial bases in Proposition 4.1. In other words, we want to know the decompositions in Proposition 4.3 (1) and (2) explicitly. When $W = V^*$, this amounts to finding explicit formulae for the matrix coefficients of irreducible representations of $GL_q(V)$. This problem has been studied intensively in the literature. When $\dim(V) = 2$, explicit formulae for matrix coefficients in terms little $q$-Jacobi polynomials are known. In general, the problem is not completely understood at present; see the survey [35] and the references therein.

The advantage of working with the Gelfand-Tsetlin bases of $\mathbb{C}_q[\bar{X}], \wedge_q(\bar{X})$, instead of the standard monomial bases in Proposition 4.1 is that multiplication is simpler in terms of the former, and have explicit formulae in terms of Clebsch-Gordon coefficients. We shall now state these formulae, assuming for the sake of simplicity, the multiplicity free case—which is enough for the purposes of this paper. That is, we shall assume that the multiplicity of each $V_{q,\gamma}(V)$ in the tensor product decomposition (25) for $\alpha$’s and $\beta$’s under consideration is one.

When $W = V^*$, we have the following multiplication formula for matrix coefficients; cf. [15, 35]:

$$u_{NR}^\alpha u_{KS}^\beta = \sum_{\gamma,M,L} C_{NK,M}^{\alpha,\beta,\gamma} C_{RS,L}^{\alpha,\beta,\gamma} u_{ML}^\gamma. \tag{49}$$

where $\alpha, \beta, \gamma$ are as in the decomposition (25), which is assumed to be multiplicity free.

We also have the following identity:

$$u_{ML}^\gamma = \sum_{N,K,R,S} C_{NK,M}^{\alpha,\beta,\gamma} C_{RS,L}^{\alpha,\beta,\gamma} u_{NR}^\alpha u_{KS}^\beta. \tag{50}$$

Multiplication formula for the Gelfand-Tsetlin basis [17] is similar. Let $\alpha, \beta$ be Young diagrams of height less than $\dim(V)$ and $\dim(W)$, and $\gamma$ be a Young diagram in the decomposition (25), which is assumed to be multiplicity free. Then

$$\langle |N_{\alpha}\rangle \otimes |R_{\alpha}\rangle |K_{\beta}\rangle \otimes |S_{\beta}\rangle \rangle = \sum_{\gamma,M,L} C_{NaK\beta M\gamma L\gamma} C_{RaS\beta L\gamma} |M_{\gamma}\rangle \otimes |L_{\gamma}\rangle. \tag{51}$$
Also
\[ |M_\gamma \rangle \otimes |L_\gamma \rangle = \sum_{N_\alpha,K_\beta,R_\alpha,S_\beta} C_{N_\alpha \beta M_\gamma} C_{R_\alpha S_\beta L_\gamma} (|N_\alpha \rangle \otimes |R_\alpha \rangle) (|K_\beta \rangle \otimes |S_\beta \rangle). \] (52)

Multiplication formula for the Gelfand-Tsetlin basis (48) is also similar. Let \( \alpha, \beta, \gamma \) be Young diagrams of height at most \( \dim(V) \) and width at most \( \dim(W) \), and \( \alpha', \beta', \gamma' \) their conjugates. Assume that the multiplicity of \( \gamma \) in the decomposition (25) is one –i.e. the Littlewood-Richardson coefficient \( c_{\alpha,\beta,\gamma} \) is one– and also that \( c_{\alpha',\beta',\gamma'} \) is one.

Then,
\[ (|N_\alpha \rangle \otimes |R_\alpha' \rangle)(|K_\beta \rangle \otimes |S_\beta' \rangle) = \sum_{\gamma,M_\gamma,L_\gamma} C_{N_\alpha K_\beta M_\gamma} C_{R_\alpha' S_\beta' L_\gamma} (|N_\alpha \rangle \otimes |R_\alpha' \rangle) (|K_\beta \rangle \otimes |S_\beta' \rangle). \] (53)

Also
\[ |M_\gamma \rangle \otimes |L_\gamma' \rangle = \sum_{N_\alpha,K_\beta,R_\alpha',S_\beta'} C_{N_\alpha K_\beta M_\gamma} C_{R_\alpha' S_\beta' L_\gamma'} (|N_\alpha \rangle \otimes |R_\alpha' \rangle) (|K_\beta \rangle \otimes |S_\beta' \rangle), \] (54)

5 Quantum matrix space

Let \( u = u^X \) be a variable matrix, specifying a generic transformation from \( \bar{X} \) to \( \bar{X} \). Let \( \mathbb{C}(u) \) denote the free algebra over the variable entries of \( u \).

We define the coordinate algebra \( \mathcal{O}(M_q(\bar{X})) \) of the virtual quantum space \( M_q(\bar{X}) \) to be the quotient of \( \mathbb{C}(u) \) modulo the relations
\[ P_+^X (u \otimes u) = (u \otimes u) P_+^X. \] (55)

Since \( I = P_-^X + P_+^X \), these are equivalent to the relations
\[ P_-^X (u \otimes u) = (u \otimes u) P_-^X. \] (56)

Thus \( \mathcal{O}(M_q(\bar{X})) \) is an FRT-algebra \([32]\), with a singular \( P_-^X \) playing the role of an \( R \)-matrix in eq.(4). But, as we shall see below, it is not coquasitriangular, and hence the main theory of FRT-algebras \([32]\) does not apply to \( \mathcal{O}(M_q(\bar{X})) \).
Proposition 5.1  

1. The relation (55) is equivalent to the relations on the entries of \( u \) which say that the defining equation (31) of the quantum symmetric space \( \bar{X}_{\text{sym}} \) is preserved by the left and right actions of \( u \) on \( x \) given by \( x \rightarrow ux \) and \( x^t \rightarrow x^t u \). This means the quantum symmetric algebra \( \mathbb{C}_q[\bar{X}] \) is a left and right comodule-algebra of \( \mathcal{O}(M_q(\bar{X})) \).

2. The relation (56) is equivalent to the relations on the entries of \( u \) which say that the defining equation (32) of the quantum antisymmetric space \( \bar{X}_{\wedge} \) is preserved by the left and right actions of \( u \) on \( x \). This means the quantum antisymmetric algebra \( \wedge_q[\bar{X}] \) is a left and right comodule-algebra of \( \mathcal{O}(M_q(\bar{X})) \).

Proof: Left to the reader. Q.E.D.

In view of this proposition, the quantum matrix space \( M_q(\bar{X}) \) can be viewed as the space of linear transformations of \( \bar{X}_{\text{sym}} \), or equivalently, of \( \bar{X}_{\wedge} \); i.e., so that the left and right actions of \( M_q(\bar{X}) \) on \( \bar{X}_{\text{sym}} \) or \( \bar{X}_{\wedge} \) preserve their defining equations. This definition is akin to that of the standard quantum matrix space \( M_q(X) \), but with the quantum symmetric space \( \bar{X}_{\text{sym}} \) playing the role of the standard quantum symmetric space \( X_{\text{sym}} \) and the quantum antisymmetric space \( \bar{X}_{\wedge} \) playing the role of the standard quantum antisymmetric space \( X_{\wedge} \).

Remark 1:

The relations (55) and (56) are equivalent to the relations

\[
\hat{S}_{\bar{X},\bar{X}}(a,b)(u \otimes u) = (u \otimes u)\hat{S}_{\bar{X},\bar{X}}(a,b),
\]

where

\[
\hat{S}_{\bar{X},\bar{X}}(a,b) = aP_+ + bP_-,
\]

for any constants \( a \), \( b \), not both zero. Thus \( \mathcal{O}(M_q(\bar{X})) \) is an FRT-algebra with the \( R \)-matrix being \( S_{\bar{X},\bar{X}}(a,b) = \tau \circ \hat{S}_{\bar{X},\bar{X}}(a,b) \), where \( \tau \) denotes the flip operator. Fix \( a, b \), and let \( S = S_{\bar{X},\bar{X}}(a,b) \). Given a tensor product \( \bar{X} \otimes d \), let \( S_i \) denote the transformation which acts like \( S \) on the \( i \)-th and \( (i + 1) \)-st factors, the other factors remaining unaffected. Then it can be shown that \( S_i \)'s do not satisfy the Quantum Yang Baxter Equations (QYBE)–equivalently \( \hat{S}_i \)'s do not satisfy the braid identities; cf. Section 11.

Remark 2:

Let \( R_{\bar{X},\bar{X}} = \tau \circ \hat{R}_{\bar{X},\bar{X}} \). One can also consider an FRT-algebra \( A(R_{\bar{X},\bar{X}}) \) associated with the \( R \)-matrix \( R_{\bar{X},\bar{X}} \). It is not isomorphic to \( \mathcal{O}(M_q(\bar{X})) \),
since, $P_+^X$ is nonlinear in $R_{X,X}$. Furthermore, it can be shown that the dimension of the quantum group associated with $A(R_{X,X})$ is much smaller than the classical dimension of $GL(X)$. Hence, it cannot be considered as a deformation of $GL(X)$.

Let $U = \text{End}(X, X) = X^* \otimes X$. Let $\hat{R}_{U,U} = \hat{R}_{X,X} \otimes \hat{R}_{X^*,X^*}$ be an $\hat{R}$ matrix associated with $U$. It is diagonalizable. Let $I = P_U^U + P_U^U$ be the associated spectral decomposition of the identity, where $P_U^-$ and $P_U^+$ denote the projections onto the eigenspaces of $\hat{R}_{U,U}$ for the eigenvalues with sign $-$ and $+$, respectively.

The following is an analogue of Proposition 4.2.

**Proposition 5.2** Eq. (55) or eq. (56) is equivalent to the relation

$$P_U^- (u \otimes u) = 0.$$  \hspace{1cm} (58)

**Proof:** Left to the reader. Q.E.D.

When $q = 1$ this relation says that $u_{ij}$'s commute. Thus it expresses the $q$-analogue of commutativity in the present context.

Let $A_{X^*}, B_{X}$ be bases of $S_q(X \otimes X)$ and $A_q(X \otimes X)$, respectively; let $A_{X^*}$ and $B_{X^*}$ be defined similarly. Given $c \in X^* \otimes \hat{X}^*$ and $d \in \hat{X} \otimes X$, we define the restitution $c \ast d \in (\hat{X}^* \otimes \hat{X}) \otimes (X^* \otimes X) = U \otimes U$ very much as in (33). Then an eigenbasis of $P_U^-$ in $U \otimes U$ is given by

$$\{a \ast b \mid a \in A_{X^*}, b \in B_X\} \cup \{b \ast a \mid b \in B_{X^*}, a \in A_X\}.$$ 

Hence, the following is an explicit form of the defining relation in Proposition 5.2 for $\mathcal{O}(M_q(\hat{X}))$:

$$a \ast b = 0 \quad a \in A_{X^*}, b \in B_X$$
$$b \ast a = 0 \quad b \in B_{X^*}, a \in A_X.$$ \hspace{1cm} (59)

**Proposition 5.3** $\mathcal{O}(M_q(\hat{X}))$ is a bialgebra such that

$$\Delta(u^X) = u^X \otimes u^X, \text{ and } \epsilon(u^X) = I.$$ 

Here $\otimes$ denotes tensor product and $\cdot$ denotes matrix multiplication.

**Proof:** Follows from Proposition 9.1 in [15] applied to the FRT-algebra $\mathcal{O}(M_q(\hat{X})) = A(S_{X,X}(a,b))$. Q.E.D.
5.1 Example

Let \( \dim(V) = \dim(W) = 2 \). Let

\[
A_X = \{ a_{11}, a_{22}, a_{12}, a_{33}, a_{44}, a_{34}, a_{13}, a_{24}, a_{14}, a_{23} \},
\]

and

\[
B_X = \{ b_{12}, b_{34}, b_{13}, b_{24}, b_{14}, b_{23} \},
\]

where \( a_{ij} \) and \( b_{kl} \) are as in eqs. (38) and (39). We shall identify \( A_X \) and \( B_X \) with \( A_X^\ast \) and \( B_X^\ast \), respectively. We use the notation

\[
z_{i_1i_2,k_1k_2} = x_{i_1i_2} \ast x_{k_1k_2},
\]

where \( x_{ij} \in X \) are as in eqs. (38) and (39). We also drop \( \otimes \) symbol. So that \( x_{i_1i_2}x_{j_1j_2} \) actually means \( x_{i_1i_2} \otimes x_{j_1j_2} \). For example,

\[
z_{12,11}z_{21,22} = (x_{12}x_{21}) \ast (x_{11}x_{22}).
\]

The defining relations (59) of \( O(M_q(X)) \) are now 120 in number. We show one such typical relation below:

\[
0 = a_{14} \ast b_{14} = (x_{11}x_{22} + px_{12}x_{21} + px_{21}x_{12} + p^2x_{22}x_{11})^\ast (x_{11}x_{22} - qx_{12}x_{21} + px_{21}x_{12} - x_{22}x_{11}) = z_{11,11}z_{22,22} + \sum p^2z_{12,12}z_{21,21} - z_{12,12}z_{21,21} - p^2z_{22,11}z_{11,22} + p^2z_{22,11}z_{11,22} + p^2z_{22,22}z_{11,11} - p^2z_{22,22}z_{11,11}
\]

(60)

All these 120 relations together, after taking appropriate linear combinations, can be recast in the form of a reduction system—just as the relations (35) were recast in the form of a reduction system (40)—where each reduction rule is of the form

\[
z_Az_{A'} = \sum_{B,B'} \alpha(A, A', B, B')z_Bz_{B'},
\]

each \( z_Bz_{B'} \) being standard; i.e., \( B > B' \) (say, lexicographically). The resulting reduction system is described in Appendix. It turns out that this system does not satisfy the diamond property. For example, the monomial \( mm = z_{1111}z_{1112}z_{1221} \), when reduced in two different ways, yields the following two distinct standard expressions:
\[ l_{121} = (-1 + q^2) \cdot z_{1111} z_{1121} z_{1112} + \frac{q^3 - q}{1 + q^2} \cdot z_{1211} z_{1122} z_{1111} + \frac{q^2 - 1}{1 + q^2} \cdot z_{1212} z_{1121} z_{1111} + \]
\[ \frac{2q^2}{1 + q^2} \cdot z_{1221} z_{1112} z_{1111} + \frac{(1 - q^2) q}{1 + q^2} \cdot z_{1222} z_{1111} z_{1111}, \]

and

\[ l_{212} = \frac{q^6 + q^4 - 3q^2 + 1}{q^2 (1 + q^2)} \cdot z_{1211} z_{1121} z_{1112} + \frac{2q^3 - 2q}{(1 + q^2)^2} \cdot z_{1211} z_{1122} z_{1111} \]
\[ \frac{4q^2 + q^4 - 1}{(1 + q^2)^3} \cdot z_{1221} z_{1112} z_{1111} + \frac{2q^2 - 2}{(1 + q^2)^2} \cdot [z_{1212} z_{1121} z_{1111} - q \cdot z_{1222} z_{1111} z_{1111}]. \]

This means we have the following nontrivial relation among standard monomials:

\[ l_{121} - l_{212} = 0. \]

See Appendix for the details.

The example above has the following consequence:

**Proposition 5.4** The Poincare series of \( \mathcal{O}(M_q(\tilde{X})) \) does not, in general, coincide with the classical Poincare series of \( \mathcal{O}(M(\tilde{X})) \) (at \( q = 1 \)).

Here by Poincare series of \( \mathcal{O}(M_q(\tilde{X})) \) we mean the series

\[ \sum_d \dim(\mathcal{O}(M_q(\tilde{X})))_d t^d, \]

where \( \mathcal{O}(M_q(\tilde{X})))_d \) denotes the degree \( d \) component of \( \mathcal{O}(M_q(\tilde{X}))) \). As an example, when \( \dim(V) = \dim(W) = 2 \), using computer it was found that \( \dim(\mathcal{O}(M_q(\tilde{X})))_3 = 688 \), whereas the classical \( \dim(\mathcal{O}(M(\tilde{X})))_3 = 816 \).

But we have:

**Conjecture 5.5** The dimension of \( M_q(\tilde{X}) \), as a noncommutative algebraic variety, as determined from the Poincare series of \( \mathcal{O}(M_q(\tilde{X})) \), is the same as \( \dim(X)^2 \), the dimension of the algebraic variety \( M(X) \).
This would imply that dimension of the new quantum group $GL_q(\bar{X})$ is the same as the dimension of the classical $GL(X)$.

Proposition 5.4 has important consequences. In the standard case, the Poincare series of $O(M_q(V))$ coincides with the classical Poincare series of $O(M(V)$ (at $q = 1$). By the Peter-Weyl theorem, this implies that the dimensions of the irreducible representations of $GL_q(V)$ are in one-to-one correspondence with the dimensions of the irreducible representations of $GL(V)$. Intuitively, this is why the irreducible representations of $GL_q(V)$ turn out to be deformations of the irreducible representations of $GL(V)$. Proposition 5.4 implies that this would no longer be so for the new quantum group.

5.2 Homomorphism

Let $O(M_q(V))$ and $O(M_q(W))$ be coordinate algebras of the standard quantum matrix spaces $M_q(V)$ and $M_q(W)$, respectively (Section 2).

Proposition 5.6 There is a natural homomorphism $\psi$ from $O(M_q(\bar{X}))$ to $O(M_q(V)) \otimes O(M_q(W))$.

Proof: In the matrix form the homomorphism $\psi$ is $u^X \rightarrow u^V \otimes u^W$. One has to check that the relations obtained by substituting $u^X = u^V \otimes u^W$ in (55) defining $O(M_q(\bar{X}))$ are implied by the equations defining $O(M_q(V))$ and $O(M_q(W))$.

The defining equation (20) of $O(M_q(V))$ is

$$P^V_+ (u^V \otimes u^V) = (u^V \otimes u^V)P^V_+,$$

which is equivalent to (21); i.e.,

$$P^V_+ (u^V \otimes u^V) = (u^V \otimes u^V)P^V_+.$$  \hfill (62)

Similarly, the defining relation of $O(M_q(W))$ is

$$P^W_+ (u^W \otimes u^W) = (u^W \otimes u^W)P^W_+,$$

or equivalently,

$$P^W_+ (u^W \otimes u^W) = (u^W \otimes u^W)P^W_+.$$  \hfill (64)

Since $P^X_+ = P^V_+ \otimes P^W_+ + P^V_+ \otimes P^W_-$, these relations imply (55) when $u^X = u^V \otimes u^W$.

To show that $\psi$ is a bialgebra homomorphism, one has to verify that
1. $\psi(ab) = \psi(a)\psi(b)$.

2. $\Delta \circ \psi = (\psi \otimes \psi) \circ \Delta$.

3. $\epsilon = \epsilon \circ \psi$.

This is easy. Q.E.D.

6 Quantum determinant and minors

Let $\Omega_r$ be the set of subsets of $\{1, \ldots, N = nm\}$ of size $r$, $n = \dim(V)$, $m = \dim(W)$. We write any $I \in \Omega_k$ as $I = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$. We let $x_I$ be the monomial $x_{i_1} \cdots x_{i_r}$. By Proposition 4.1, $\{x_I\}$ is the basis of $\Lambda^q[X]$. In particular, $\Lambda^q[X]$ is a one-dimensional corepresentation of $\mathcal{O}(M_q(X))$ with basis vector $x_1 \cdots x_N$.

Proposition 5.1(2) says that $\Lambda_q[X]$ is a left and right comodule algebra of $\mathcal{O}(M_q(X))$. Let

$$\phi^L : \Lambda_q[X] \to \mathcal{O}(M_q(X)) \otimes \Lambda_q[X]$$

$$\phi^L_r : \Lambda^r_q[X] \to \mathcal{O}(M_q(X)) \otimes \Lambda^r_q[X]$$

be the maps defining the left corepresentations $\Lambda_q[X]$ and $\Lambda^r_q[X]$. The right corepresentation maps $\phi^R$ and $\phi^R_r$ are similar. We define the left determinant $D_q^L = D_q^L(X)$ to be the matrix coefficient of the left comodule $\Lambda^q[X]$. The right determinant $D_q^R = D_q^R(X)$ is the matrix coefficient of the right comodule $\Lambda^q[X]$. Since $\Lambda^q[X]$ is one dimensional like the classical $\Lambda^q[X]$ (Proposition 4.1), both $D_q^L$ and $D_q^R$ are nonzero.

More generally, the left quantum $r$-minors of $M_q(X)$ in the standard monomial basis are defined to be the matrix coefficients of the left corepresentation map $\phi^L_r$ in the standard monomial basis (cf. Proposition 4.1) of $\Lambda^q[X]$. Specifically, for $I, J \in \Omega_r$, we define the left quantum $r$-minors $D^L_{J,I}(X)$ such that

$$\phi^L_{I,J}(x_I) = \sum_J D^L_{J,I}(X) \otimes x_J.$$  

The right quantum $r$-minors $D^R_{I,J}(X)$ are such that

$$\phi^R_{I,J}(x_I) = \sum_J x_J \otimes D^R_{I,J}(X).$$
We can similarly define left and right quantum minors of $M_q(\bar{X})$ in the Gelfand-Tsetlin basis to be the matrix coefficients of the left and right corepresentation maps $\phi_{L,r}$ and $\phi_{R,r}$ in the Gelfand-Tsetlin basis ([38]) of $\wedge^r_q[\bar{X}]$. Specifically, for $|M_\alpha \otimes N_{\alpha'}\rangle, |M_\beta \otimes N_{\beta'}\rangle \in \tilde{B}_{\bar{X}}$, with $\alpha$ and $\beta$ being young diagrams of size $r$, we define the left quantum $r$-minors $D^{L,M_\alpha,N_{\alpha'}}_{M_\beta,N_{\beta'}}(\bar{X})$ so that

$$\phi_{L,r}(|M_\alpha \otimes N_{\alpha'}\rangle) = \sum_{M_\beta,M_\beta'} D^{L,M_\alpha,N_{\alpha'}}_{M_\beta,M_\beta'}(\bar{X}) \otimes |M_\beta \otimes N_{\beta'}\rangle.$$  

The right quantum $r$-minors $D^{R,M_\alpha,N_{\alpha'}}_{M_\beta,N_{\beta'}}(\bar{X})$ are defined similarly.

The quantum determinants $D^L_q$ and $D^R_q$ are the same whether we use the Gelfand-Tsetlin or the standard monomial basis.

### 6.1 Explicit formulae

We wish to give explicit formulae for quantum determinants $D^L_q$ and $D^R_q$ and, more generally, the left and right quantum minors in the Gelfand-Tsetlin basis. This is possible because of the explicit formulae for multiplication in the Gelfand-Tsetlin basis (Section 4.1). We do not have similar formulae in the standard monomial basis.

#### 6.1.1 Example

Let us first give an explicit formula for $D^L_q$ and $D^R_q$ when $\dim(V) = \dim(W) = 2$. Then $\wedge^2_q[\bar{X}]$ has a basis $\{y_I = y_{i_1} \cdots y_{i_r}\}$, $i_1 < i_2 \cdots$, where

$$y_1 = x_{11}, y_2 = x_{12}, y_3 = x_{21}, y_4 = x_{22},$$

satisfying the relations

$$a_{11}, a_{22}, a_{12}, a_{13}, a_{23}, a_{44}, a_{34}, a_{13}, a_{24}, a_{14}, a_{23} = 0;$$

cf. [38] and Proposition 4.1. Let $I_\Lambda$ denote the ideal in $T(\bar{X}) = \sum_d \bar{X}^d$ generated by these relations. The last two relations imply that $y_4y_1 = -y_1y_4$. Since $y_1$ and $y_4$ quasicommute with all $y_i$’s and $y_i^2 = 0$, for all $i$, it is easy to show that $y_iy_jy_ky_l$ is zero modulo $I_\Lambda$, unless it is of the form $\prod y_{\sigma(i)}$, for some permutation $\sigma$, or is either $y_2y_3y_2y_3$ or $y_3y_2y_3y_2$. Furthermore, we have

$$\prod y_{\sigma(i)} = (-1)^{l(\sigma)} q^{r(\sigma)} y_1y_2y_3y_4,$$
where \( l(\sigma) \) is the number of inversions in \( \sigma \), and \( r(\sigma) \) is the number of inversions in \( \sigma \) not involving \((2, 3)\) or \((1, 4)\). Also

\[
y_{23}y_{23}y_3 = (p - q)q^2 y_{12}y_{23}y_4
\]

\[
y_{32}y_{32}y_2 = (q - p)q^2 y_{12}y_{23}y_4.
\]

The left determinant \( D_{q,L} \) is the the matrix coefficient of \( \wedge^4_q(\bar{X}) \), considered as a left comodule, and the right determinant \( D_{q,R} \) is the the matrix coefficient of \( \wedge^4_q(\bar{X}) \), considered as a right comodule. From the preceding remarks, it easily follows that

\[
D_{q,L} = \left( \sum _\sigma (-1)^l(\sigma) q^r(\sigma) u_{i\sigma(i)} \right) + (p - q)q^2 u_{12}u_{23}u_{32}u_{43} + (q - p)q^2 u_{13}u_{22}u_{33}u_{42}.
\]

The expression for \( D_{q,R} \) is similar. Compare this with the formula (23) for the standard quantum determinant.

More generally, the following result gives an explicit formula in terms of the fundamental Clebch-Gordan (Wigner) coefficients (Section 4.1) for expanding the left or the right quantum minor of \( M_q(\bar{X}) \) (in the Gelfand-Tsetlin basis) by row or column.

**Proposition 6.1**

\[
D_{L,M,\gamma,L',\gamma'}^{E,F,\delta,F'}(\bar{X}) = \sum _{N,K,M} C_{N,M,K,L,L'} C_{R,S,M',L',L} \sum _{A,B,C,D} D_{A,B,C,D}^{L,M,L',L'}(\bar{X})D_{C,D}^{L,M,L',L'}(\bar{X})C_{A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z};
\]

where \( C_{N,M,K,L,L'} \)'s etc. denote fundamental Clebsch-Gordon coefficients; cf. (27) and (28).

A formula for the right minor is similar.

**Proof:** Using (53) and (54) we calculate, dropping the symbol \( \bar{X} \):

**31**
\( \phi_{L,r}(|M_\gamma \rangle \otimes |L_{\gamma'} \rangle) \)
\[ = \phi_{L,r}(\sum_{N_\alpha,K_\beta,R_\alpha',S_\beta'} C_{N_\alpha,K_\beta,M_\gamma} C_{R_\alpha',S_\beta' L_{\gamma'}} (|N_\alpha\rangle \otimes |R_\alpha'\rangle)(|K_\beta\rangle \otimes |S_\beta'\rangle)) \]
\[ = \sum_{N_\alpha,K_\beta,R_\alpha',S_\beta'} C_{N_\alpha,K_\beta,M_\gamma} C_{R_\alpha',S_\beta' L_{\gamma'}} \sum_{A_\mu,B_\mu',C_\lambda,D_\lambda'} D_{A_\mu,B_\mu'}^{L,N_\alpha,R_\alpha'} D_{C_\lambda,D_\lambda'}^{L,K_\beta,S_\beta'} \]
\[ |A_\mu\rangle \otimes |B_\mu'\rangle \rangle (|C_\lambda\rangle \otimes |D_\lambda'\rangle) \]
\[ = \sum_{N_\alpha,K_\beta,R_\alpha',S_\beta'} C_{N_\alpha,K_\beta,M_\gamma} C_{R_\alpha',S_\beta' L_{\gamma'}} \sum_{A_\mu,B_\mu',C_\lambda,D_\lambda'} \]
\[ D_{A_\mu,B_\mu'}^{L,N_\alpha,R_\alpha'} D_{C_\lambda,D_\lambda'}^{L,K_\beta,S_\beta'} \sum_{E_\delta,F_\delta'} C_{A_\mu C_\lambda E_\delta} C_{B_\mu' D_\lambda' F_\delta'} \]
\[ |E_\delta\rangle \otimes |F_\delta'\rangle \]
\[ = \sum_{E_\delta,F_\delta'} \left( \sum_{N_\alpha,K_\beta,R_\alpha',S_\beta'} C_{N_\alpha,K_\beta,M_\gamma} C_{R_\alpha',S_\beta' L_{\gamma'}} \sum_{A_\mu,B_\mu',C_\lambda,D_\lambda'} \right) \]
\[ D_{A_\mu,B_\mu'}^{L,N_\alpha,R_\alpha'} D_{C_\lambda,D_\lambda'}^{L,K_\beta,S_\beta'} C_{A_\mu C_\lambda E_\delta} C_{B_\mu' D_\lambda' F_\delta'} \]
\[ |E_\delta\rangle \otimes |F_\delta'\rangle. \]

Now the result follows from the definition of a left minor. Q.E.D.

We can also give an analogue of a general Laplace expansion in this context. But that would require general Clebsch-Gordon coefficients for which no explicit expressions are known so far.

### 6.2 Symmetry of the determinants and minors

The following property of determinants is needed later for defining the real (unitary) form \( U_q(\bar{X}) \) of the new quantum group \( GL_q(\bar{X}) \) (Cf. Proposition 8.1).

**Proposition 6.2**

\( D_q^L(\bar{X}) = D_q^R(\bar{X}) \). More generally,

\( D_I^{L,J}(\bar{X}) = D_I^{R,J}(\bar{X}) \)

and

\( D_{E_\delta,F_\delta'}^{L,M_{\gamma},L_{\gamma'}}(\bar{X}) = D_{E_\delta,F_\delta'}^{R,M_{\gamma},L_{\gamma'}}(\bar{X}) \).
This allows us to define the quantum determinant \( D_q = D_q(\bar{X}) = D_q^L(\bar{X}) = D_q^R(\bar{X}) \), and the quantum minors

\[
D_I^I(\bar{X}) = D_I^{L,J}(\bar{X}) = D_I^{R,J}(\bar{X})
\]

and

\[
D_{E_{\delta},F_{\delta'}}^{M,\gamma,L_{\gamma'}}(\bar{X}) = D_{E_{\delta},F_{\delta'}}^{L,M,\gamma,L_{\gamma'}}(\bar{X}) D_{E_{\delta},F_{\delta'}}^{R,M,\gamma,L_{\gamma'}}(\bar{X}).
\]

By a standard property of matrix coefficients (cf. Proposition 1.13 in [15]), we also have that

\[
\Delta(D_I^I(\bar{X})) = \sum_K D_K^I(\bar{X}) \otimes D_K^I(\bar{X}), \quad \epsilon(D_I^I(\bar{X})) = \delta_{IJ}, \quad I, J \in \Omega_r,
\]

and a similar property for the minors in the Gelfand-Tsetlin basis. In particular,

\[
\Delta(D_q(\bar{X})) = D_q(\bar{X}) \otimes D_q(\bar{X}), \quad \epsilon(D_q(\bar{X})) = 1. \tag{66}
\]

Thus the quantum determinant is a nonzero group like element of \( O(M_q(\bar{X})) \).

The rest of this section is devoted to the proof of Proposition 6.2. The proof is technical and may be skipped on the first reading. The main problem that needs to be addressed is the following. In the standard case, the right determinant, when rewritten using the reduction system (24), coincides with the left determinant. Analogously, one may wish to prove that the explicit expressions for the left and right determinants (minors) in Proposition 6.1 are equal by reducing them with respect to the reduction system corresponding to the relations (59) defining \( O(M_q(\bar{X})) \). But this reduction system does not satisfy the diamond lemma, and the standard monomials do not form a basis of \( O(M_q(\bar{X})) \); cf. Section 5.1. Hence, the expressions for the left and right minors in Proposition 6.1 need not reduce to the same expressions. So this strategy does not work.

We start with a few general observations. Given a coalgebra \( A \), and a right corepresentation \( \phi : V \to V \otimes A \) given by

\[
\phi(e_j) = \sum_k e_k \otimes v_{kj}, \tag{67}
\]

there is a left corepresentation \( \phi_R : V \to A \otimes V \) given by

\[
\phi_L(e_k) = \sum_j v_{kj} \otimes e_j. \tag{68}
\]

We shall denote this dual left corepresentation to \( V \) by \( V_L \). The two corepresentations \( V \) and \( V_L \) share the same matrix \([v_{kj}]\) of coefficients, which we
shall denote by $M(V) = M(V_L)$. Furthermore, the span $C(\phi) = \langle v_{kj} \rangle$ of the matrix coefficients has a left and right $A$-coaction, and hence, is a $A$-bicomodule.

Similarly, given a left corepresentation $W \to A \otimes W$:

$$\tau(e_k) = \sum_j w_{kj} \otimes e_j,$$

there is a corresponding right corepresentation $W_R$:

$$\tau_R(e_j) = \sum_k e_k \otimes w_{kj},$$

with the same matrix $M(W) = M(W_R) = [w_{kj}]$ of coefficients. Furthermore, $C(\tau) = C(\tau_R) = \langle w_{kj} \rangle$ has the left and right coaction, and hence, is a $A$-bicomodule.

Now we return to the case $A = \mathcal{O}(M_q(\bar{X}))$. Let $\wedge_q^{d,L}(\bar{X})$ denote the left corepresentation of $\mathcal{O}(M_q(\bar{X}))$ on the space $\wedge_q^d[\bar{X}]$. We define $\wedge_q^{d,R}(\bar{X}), \wedge_q^{d,L}(\bar{X})$ and $\wedge_q^{d,R}(\bar{X})$ similarly. Let $M_{\wedge}^{d,L}$ and $M_{\wedge}^{d,R}$ denote the coefficient matrices of $\wedge_q^{d,L}(\bar{X})$ and $\wedge_q^{d,R}(\bar{X})$ in the standard monomial bases (Proposition 4.1), and $\tilde{M}_{\wedge}^{d,L}$ and $\tilde{M}_{\wedge}^{d,R}$ the coefficient matrices in the Gelfand-Tsetlin basis [48]. The coefficient matrices $M_{sym}^{d,L}$ and $M_{sym}^{d,R}$, and $\tilde{M}_{sym}^{d,L}$ and $\tilde{M}_{sym}^{d,R}$ are defined similarly.

Proposition 6.2 follows from the following:

**Lemma 6.3** For every $d$, $M_{\wedge}^{d,L} = M_{\wedge}^{d,R}$, and $\tilde{M}_{\wedge}^{d,L} = \tilde{M}_{\wedge}^{d,R}$. Hence, $\wedge_q^{d,L}(\bar{X})_R = \wedge_q^{d,R}(\bar{X})$, $\wedge_q^{d,L}(\bar{X})_L = \wedge_q^{d,L}(\bar{X})$.

In particular, with $d = \dim(X) = \dim(V) \dim(W)$, this implies that the left determinant $D_q^L(\bar{X})$ is equal to the right determinant $D_q^R(\bar{X})$.

We now turn to the proof of the lemma.

**Proposition 6.4** $M_{\wedge}^{2,L} = M_{\wedge}^{2,R}$ and $M_{sym}^{2,L} = M_{sym}^{2,R}$. Hence,

$$
\begin{align*}
C_q^{2,R}(\bar{X})_L &\equiv C_q^{2,L}(\bar{X}), \\
\wedge_q^{2,L}(\bar{X})_R &\equiv \wedge_q^{2,R}(\bar{X}), \\
\wedge_q^{2,R}(\bar{X})_L &\equiv \wedge_q^{2,L}(\bar{X}), \\
\wedge_q^{2,L}(\bar{X})_R &\equiv \wedge_q^{2,R}(\bar{X}).
\end{align*}
$$
Proof: Assume that \( \dim(V) = \dim(W) = 2 \); the general case is very similar. The result follows by explicit computation of the coefficient matrices and checking that they are equal. We omit the details.

The matrix coefficients of \( C^2_q, R(\overline{X}) \) and \( C^2_q, L(\overline{X}) \) (ignoring constant factors) are

\[
\begin{align*}
a(23) \ast a(23) \\
 a(ij) \ast a(i'j'), && (i, j), (i', j') \neq (2, 3) \\
 b(ij) \ast b(i'j'), && (i, j), (i', j') = (1, 2), (3, 4) \text{ or } (1, 4) \\
 b(ij) \ast b(i'j'), && (i, j), (i', j') = (1, 3), (2, 3) \text{ or } (2, 4),
\end{align*}
\]  

(71)

where, for clarity, we denote \( a_{ij} \) and \( b_{kl} \) in (38) and (39) by \( a(ij) \) and \( b(kl) \) here and in what follows.

Similarly, the matrix coefficients of \( \wedge^2_q, L(\overline{X}) \) and \( \wedge^2_q, R(\overline{X}) \) (ignoring constant factors) are

\[
\begin{align*}
a(ij) \ast a(23), a(23) \ast a(ij) && (i, j) \neq (2, 3) \\
 \{b(12), b(34), b(14)\} \ast \{b(13), b(23), b(24)\} \\
 \{b(13), b(23), b(24)\} \ast \{b(12), b(34), b(14)\}.
\end{align*}
\]

Q.E.D.

Let \( I \subseteq C(U) \cong \oplus_d (\overline{X} \otimes \overline{X}^*)^{\otimes d} \) be the ideal generated by (55) or (56) so that \( O(M_q(\overline{X})) = C(U) / I \). Equivalently, it is generated by

\[
a(ij) \ast b(rs) \text{ and } b(rs) \ast a(ij);
\]

cf. (59). Let \( J \) be the two sided ideal in \( O(M_q(\overline{X})) \) generated by \( M^2_q, L = M^2_q, R = M^2_{sym}, \) i.e., the matrix coefficients of \( C^2_q, L(\overline{X}) \), or equivalently, \( C^2_q, R(\overline{X}) \). It is a \( O(M_q(\overline{X})) \)-bicomodule. Let \( O(M_q(\overline{X}))_d \) and \( J_d \) be the degree \( d \) components of \( O(M_q(\overline{X})) \) and \( J \) respectively. We have chosen \( J \) so that it will turn out later (cf. (77) and (78)) that, as \( O(M_q(\overline{X})) \)-bicomodules,

\[
O(M_q(\overline{X}))_d / J_d \cong \wedge^d_q L(\overline{X}) \otimes \wedge^d_q R(\overline{X}) \cong \wedge^d_q R(\overline{X}) \otimes \wedge^d_q L(\overline{X}).
\]

(72)

(This is obvious at \( q = 1 \).)

We shall find an explicit basis of \( R = O(M_q(\overline{X})) / J \), and then deduce Lemma 6.3 from the form of this basis.

We have

\[
R = O(M_q(\overline{X})) / J \cong \oplus_d (\overline{X} \otimes \overline{X}^*)^{\otimes d} / (I + J).
\]
Hence, by the diamond lemma \[15\], to get an explicit standard monomial basis of $\mathcal{R}$, it suffices to give a reduction system for the ideal $I + J$ in which all ambiguities can be resolved.

Let $L(I + J)$ denote the set of explicit generators for $I + J$; cf. (59), (38), (39), and the proof of Proposition 6.4. For example, when $\dim(V) = \dim(W) = 2$, these are

\[
\begin{align*}
    a(ij) \ast b(i'j') & \text{ and } b(i'j') \ast a(ij) \\
    a(23) \ast a(23) & \\
    a(ij) \ast a(i'j'), & (i, j), (i', j') \neq (2, 3) \\
    b(ij) \ast b(i'j'), & (i, j), (i', j') = (1, 2), (3, 4) \text{ or } (1, 4). \\
    b(ij) \ast b(i'j'), & (i, j), (i', j') = (1, 3), (2, 3) \text{ or } (2, 4).
\end{align*}
\]

The basic lemma is:

**Lemma 6.5** The equations $\{l = 0 | l \in L(I + J)\}$ can be recast in the form of an equivalent reduction system—just as eq.(20) was recast in the form a reduction system \[24\]—whose ambiguities can be resolved. Hence by the diamond lemma \[15\], the standard monomials as per this reduction system form a basis of $\mathcal{R} = \mathcal{O}(M_4(\bar{X})/J \cong \bigoplus_d(\bar{X} \otimes \bar{X}^*)^{\otimes d}/(I + J)$.

**Proof:** We begin with introducing some notation that will be convenient in the proof.

By an abuse of notation, we shall denote the set of basis elements of $V$, $\bar{X}$ and $U = \bar{X}^* \otimes \bar{X}$ by $V$, $\bar{X}$ and $U$ again; what is intended will be clear from the context. Furthermore, we shall identify $\bar{X}^*$ and $\bar{X}$, $V^*$ and $V$, so that we write $U = \bar{X} \otimes \bar{X}$.

In the proof we shall explicitly distinguish between the two tensoring operations that we use. The first one, which we call restitution, and denote by $\ast$, creates new variables from old. The second, which we call tensor, and denote by $\otimes$, raises the degree of the variable, and implements the usual tensor algebra.

Thus for example, when $V = W = \{v_1, v_2\}$, $\bar{X} = V \ast V$ creates the new variables (basis vectors) $x_{ij} = v_i \ast v_j$. On the other hand, $V \otimes V$ creates the set

\[
\{v_i \otimes v_j | 1 \leq i, j \leq 2\}
\]

The element $v_i \otimes v_j$ is also denoted by $v_i v_j$.

In this notation $\bar{X}$ is the set $V \ast W$, whence if $V = W$,

\[
\bar{X} = \{x_{ij} = v_i \ast v_j | 1 \leq i, j \leq 2\}
\]
The set $\bar{X} \otimes \bar{X}$ is thus elements of the type

$$x_{ij} x_{kl} = x_{ij} \otimes x_{kl} = (v_i \star v_j) \otimes (v_k \star v_l) = (v_i \otimes v_k) \star (v_j \otimes v_l) = v_i v_k \star v_j v_l$$

In this sense $\bar{X} \otimes \bar{X} = (V \otimes V) \star (V \otimes V)$.

The set $U$ of the matrix coefficients for the set $\bar{X}$ is given by $\bar{X} \star \bar{X} = V \star V \star V \star V$. Thus

$$u_{ijkl} = v_i \star v_j \star v_k \star v_l$$

We also see that:

$$U \otimes U = (V \otimes V) \star (V \otimes V) \star (V \otimes V) \star (V \otimes V)$$

We define the set of $z$-variables $Z^m$ as the set obtained by the $m$-way restitution of the variables $v_i$. In other words, if $V = \{v_1, v_2\}$, then $\bar{X} = V \star V$ and

$$U = Z^4 = V \star V \star V \star V$$

Note that $\text{dim}(Z^4) = 16$, when $\text{dim}(v) = \text{dim}(w) = 2$. We use the notation $z_{a_1, a_2, \ldots, a_m} = v_{a_1} \star v_{a_2} \cdots \star v_{a_m}$.

Now let $V$ be 2-dimensional as above, and let $W = V$. Recall (cf. [37]) the elements $A_{11}, A_{12}, A_{22}$ and $B_{12}$ of $U \otimes V$, listed below:

- $A_{11} = v_1 v_1$
- $A_{22} = v_2 v_2$
- $A_{12} = v_1 v_2 + pv_2 v_1$
- $B_{12} = v_1 v_2 - qv_2 v_1$

In the space $Z^m \otimes Z^m$, the subspace $P^-$ (corresponding to eigenvectors of $\hat{R}_{Z^m, Z^m}$ with negative eigenvalues) is spanned by the $m$-way restitutions $T_1 \star T_2 \star \ldots \star T_m$, where

(i) each $T_i \in \{A_{11}, A_{12}, A_{22}, B_{12}\}$.

(ii) the term $B_{12}$ appears exactly odd number of times.

We use the following notation:

$$n_1 = \mathbb{C} \cdot B_{12}$$
$$p_1 = \mathbb{C} \cdot A_{11} \oplus \mathbb{C} \cdot A_{22} \oplus \mathbb{C} \cdot A_{12}$$
$$n_i = n_{i-1} \star p_1 \oplus p_{i-1} \star n_1$$
$$p_i = n_{i-1} \star n_1 \oplus p_{i-1} \star p_1$$

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Note that $p_i, n_i \subseteq \mathbb{Z}^i \otimes \mathbb{Z}^i$.

The algebra $O(M_q(\bar{X}))$ is the tensor algebra $T(U)$ modulo the ideal generated by $n_4 \subseteq U \otimes U$. Thus $I$ in the statement of the lemma is $n_4$ in this terminology, since $U = \mathbb{Z}^4$.

First we shall assume that $\dim(V) = \dim(W) = 2$. Then $Z^1 \otimes Z^1 = n_1 \oplus p_1$, with $\dim(n_1) = 1$ and $\dim(p_1) = 3$. $\bar{X} = Z^2$ is 4-dimensional and $\mathbb{C}_q^2(\bar{X}) = (Z^2 \otimes Z^2)/n_2$ is 10-dimensional. A representative decomposition of $\mathbb{C}_q^2(\bar{X})$ is given by

$$\mathbb{C}_q^2(\bar{X}) = n_1 \star n_1 \oplus p_1 \star p_1.$$

The space $M$ spanned by the matrix coefficients of $\mathbb{C}_q^2(\bar{X})$, either as a left or right corepresentation (cf. Proposition 6.1), is thus given by the 100-dimensional

$$\mathbb{C}_q^2(\bar{X}) \star \mathbb{C}_q^2(\bar{X}) = (n_1 \star n_1 \oplus p_1 \star p_1) \star (n_1 \star n_1 \oplus p_1 \star p_1). \quad (74)$$

We now turn to the algebra $R = O(M_q(\bar{X}))/J$, where $J$ is the ideal generated by the subspace $M \subseteq U \otimes U$. Thus $O(M_q(\bar{X})) = T(U)/I$, where $I = n_4$ is the ideal in $T(U)$ generated by $P^- \subseteq U \otimes U$. Thus $R = T(U)/(I + J)$.

We consider the degree 2-component of $R$ above and construct an appropriate reduction system. Remembering that $U = \mathbb{Z}^4$, our task is to construct a reduction system for $(Z^4 \otimes Z^4)/(M \oplus n_4)$.

Note that this space is 36-dimensional, since $\dim(V) = \dim(W) = 2$. We use the fact that $Z^4 = Z^2 \star Z^2$ and given $A = (a_1, \ldots, a_4)$ and $B = (b_1, \ldots, b_4)$, we write $z_A z_B = z_{A_1 B_1} \star z_{A_2 B_2}$, where $A_1 = (a_1, a_2)$ and $A_2 = (a_3, a_4)$, with $B_1$ and $B_2$ similarly defined.

We say that $z_{A_1} z_{A_2} \star z_{B_1} z_{B_2}$ is standard if $A_1 \succ B_1$ and $A_2 \succ B_2$; in other words, it must be that $a_1 > b_1$ or $a_1 = b_1$ and $a_2 > b_2$, and a similar condition on $A_2$ and $B_2$. We represent this as $A \succ B$. We say that $z_{A_1} z_{A_2} \star z_{B_1} z_{B_2}$ is nonstandard if $A \npreceq B$.

There are exactly 6 choices of tuples for $A_1, B_1$ that satisfy $A_1 \succ B_1$. These are

$$\sigma = \{(22, 11), (22, 12), (22, 21), (21, 11), (21, 12), (12, 11)\}$$

If

$$\Sigma = \{(A, B) | A \npreceq B\}$$

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then $|\Sigma| = |\sigma|^2 = 36$. Our reduction system will specify for each nonstandard $z_A z_B$ an element $\beta_{A, B} \in M \oplus n_4$ which contains $z_A z_B$ and all other terms in it are either standard or lower than $z_A z_B$ in a certain order.

We first note that for $z_A z_B \in Z^2 \otimes Z^2$ with $A \not\sim B$, there is an $\alpha_{A, B} \in p_2$, such that

$$\alpha_{A, B} = z_A z_B + \sum_i a_i z_{A_i} z_{B_i}$$

with $A_i \succ B_i$. These expressions are readily computed as follows. Let $Aij$’s and $Bij$’s be as in (37). Let

$$\begin{align*}
\alpha_{11, 11} &= A11 \ast A11 = z_{11} z_{11} \\
\alpha_{22, 22} &= A22 \ast A22 = z_{22} z_{22} \\
\alpha_{12, 12} &= A11 \ast A22 = z_{12} z_{12} \\
\alpha_{21, 21} &= A22 \ast A11 = z_{21} z_{21} \\
\alpha_{11, 12} &= A11 \ast A12 = z_{11} z_{12} + p z_{12} z_{11} \\
\alpha_{11, 21} &= A12 \ast A11 = z_{11} z_{21} + p z_{21} z_{11} \\
\alpha_{21, 22} &= A22 \ast A12 = z_{21} z_{22} + p z_{22} z_{21} \\
\alpha_{12, 22} &= A12 \ast A22 = z_{12} z_{22} + p z_{22} z_{12}
\end{align*}$$

The more complicated ones are:

$$\begin{align*}
\alpha_{11, 22} &= (qA12 \ast A12 + pB12 \ast B12)/(p + q) = z_{11} z_{22} + z_{22} z_{11} \\
\alpha_{12, 21} &= (A12 \ast A12 - B12 \ast B12)/(p + q) = z_{12} z_{21} + z_{21} z_{12} + (p - q) z_{22} z_{11}
\end{align*}$$

Our next observation is that for any $\alpha_{A, B}$ as above, and any $i, j, k, l$ we have $\alpha_{A, B} \ast z_{ij} z_{kl} \in M \oplus n_4$. Indeed, we may wend through the following sequence of relations:

$$p_2 \ast Z^2 = (n_1 \ast n_1 \oplus p_1 \ast p_1) \ast (n_1 \oplus p_1) \ast (n_1 \ast p_1) \ast (n_1 \ast n_1) \ast n_1 \ast n_1 \ast n_1 \ast p_1 \ast n_1 \ast n_1 \ast p_1 \ast n_1 \ast n_1 \ast p_1 \ast p_1 \ast n_1 \ast p_1 \ast p_1 \ast (p_1 \ast p_1) \ast Z^2$$

The first 4 terms are easily seen to belong either to $M$ or to $n_4$. The other unexpanded terms behave similarly.

Now, the promised $\beta_{A, B}$ are available forthwith. Given $A, B$ with the nonstandard expression $z_A z_B = (z_{A1} z_{B1}) \ast (z_{A2} z_{B2})$, if $A_1 \not\prec B_1$, consider

$$\beta_{A, B} = \alpha_{A1, B1} \ast z_{A2} z_{B2} \quad (75)$$

In this, other than the term $z_A z_B$, we will have all terms $z_C z_D$ with $C_1 \succ D_1$. Next, when $A, B$ are such that $A_1 \nless B_1$, we construct

$$\beta_{A, B} = z_{A1} z_{B1} \ast \alpha_{A2, B2} \quad (76)$$

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This defines the reduction system completely.

As an example, we have

\[
\beta_{(1212,2122)} = \alpha_{12,21} \times z_{12} z_{22}
\]
\[
= (z_{12} z_{21} + z_{21} z_{12} + (p - q) z_{22} z_{11}) \times z_{12} z_{22}
\]
\[
= z_{1212} z_{2122} + z_{2112} z_{1222} + (p - q) z_{2212} z_{1122}
\]
\[
\beta_{(2212,1122)} = z_{22} z_{11} \times \alpha_{12,22}
\]
\[
= z_{22} z_{11} \times (z_{12} z_{22} + p z_{22} z_{12})
\]
\[
= z_{2212} z_{1122} + p z_{2222} z_{1112}
\]

Next, for arbitrary \( \dim(V) \) and \( \dim(W) \), modification of the \( \beta_{A,B} \) as above is straightforward. Let \( A = (a_1, \ldots, a_4) \) and \( B = (b_1, \ldots, b_4) \). For each index \( i \), depending on the three conditions \( a_i < b_i \) or \( a_i = b_i \) or \( a_i > b_i \), we replace \( a_i \) and \( b_i \) by either of 1 or 2 which satisfy the same relation. We term this new sequence as \( A' \) and \( B' \). After obtaining \( \beta_{A'B'} \) we replace each occurrence of 1 or 2 in each subscript by its original entry. This is illustrated by the following example:

Let \( A = (1313) \) and \( B = (3223) \). We put \( A' = (1212) \) and \( B' = (2122) \).

and note that \( a_i' < b_i' \) iff \( a_i < b_i \) and similarly for other comparisons. We obtain \( \beta_{(1212,2122)} \) from the above example and make the substitutions to get:

\[
\beta_{(1212,2122)} = z_{1212} z_{2122} + z_{2112} z_{1222} + (p - q) z_{2212} z_{1122}
\]
\[
\beta_{(1313,3222)} = z_{1313} z_{3222} + z_{3213} z_{1323} + (p - q) z_{3313} z_{1223}
\]

Explicit reduction rules which arise from \( \beta_{(2212,1122)} \) and \( \beta_{(1313,3223)} \) are shown below:

\[
z_{2212} z_{1122} \rightarrow -p z_{2222} z_{1112}
\]
\[
z_{1313} z_{3223} \rightarrow -z_{3213} z_{1323} - (p - q) z_{3313} z_{1223}
\]

The reduction system arising from the above elements \( \beta_{A,B} \) does satisfy the diamond lemma. This is an easy consequence of the same fact for \( Z^2/p^2 \), the case that we have exhibited earlier as the antisymmetric algebra \( \wedge \bar{X} \) for \( \bar{X} \) (cf. the proof of Proposition 4.1 (2)). Q.E.D.

**Lemma 6.6**

1. \( \dim(\mathcal{O}(M_q(\bar{X}))/J_d) = \dim(\wedge_q^{d,L}(\bar{X}))^2 = \dim(\wedge_q^{d,R}(\bar{X}))^2 \).

2. The matrix elements of \( M^{d,L}_{\wedge} \) or \( M^{d,R}_{\wedge} \) (modulo \( J_d \)) form a basis of \( \mathcal{O}(M_q(\bar{X}))/J_d \).
Proof: From the diamond lemma [15] and the form of the reduction system above (cf. (75) and (76)), it follows that the standard monomials of the form
\[ z_{A_1}z_{A_2} \cdots z_{A_r}, \]
where \( A_1 \supset A_2 \supset \cdots \), form a basis of \( \mathcal{O}(M_q(\bar{X}))/J \). This immediately implies the first statement.

The second statement is easy to check at \( q = 1 \). Since the dimension of \( \mathcal{O}(M_q(\bar{X}))_d \) is the same at \( q = 1 \) and general transendental real \( q \), the second statement follows. Q.E.D.

Now let us turn to the proof of Lemma 6.3. Since \( \mathcal{O}(M_q(\bar{X})) \) and \( J \) are \( \mathcal{O}(M_q(\bar{X})) \)-bicomodules, so is \( \mathcal{O}(M_q(\bar{X}))/J \). The second statement of Lemma 6.6 implies that, as \( \mathcal{O}(M_q(\bar{X})) \)-bicomodules,
\[ \mathcal{O}(M_q(\bar{X}))/J \cong \wedge_q^{d,L}(\bar{X}) \otimes \wedge_q^{d,L}(\bar{X})_R, \] (77)
and, similarly,
\[ \mathcal{O}(M_q(\bar{X}))/J \cong \wedge_q^{d,R}(\bar{X})_L \otimes \wedge_q^{d,R}(\bar{X}). \] (78)
Hence,
\[ \wedge_q^{d,L}(\bar{X}) \cong \wedge_q^{d,R}(\bar{X})_L \quad \text{and} \quad \wedge_q^{d,R}(\bar{X}) \cong \wedge_q^{d,L}(\bar{X})_R. \]

It follows that coefficient matrices of \( \wedge_q^{d,L}(\bar{X}) \) and \( \wedge_q^{d,R}(\bar{X}) \) are similar; i.e., there exists a nonsingular similarity matrix \( Q \) such that \( M_{\wedge}^{d,L} = Q^{-1}M_{\wedge}^{d,R}Q \). We have to show that \( Q \) is the identity matrix. This follows from Proposition 5.6, Proposition 4.3, and the theory of the standard Drinfeld-Jimbo quantum group, the main point being uniqueness of the orthonormal Gelfand-Tsetlin basis (48) for \( \wedge_q^d[\bar{X}] \); we leave details to the reader. This proves Lemma 6.3 and hence Proposition 6.2.

7 Hopf structure

To define a cofactor matrix of \( u \) we need the following.

Proposition 7.1 There is a nondegenerate pairing
\[ \wedge_q^r[\bar{X}] \otimes \wedge_q^{nm-r}[\bar{X}] \to \wedge_q^{nm}[\bar{X}], \]
given by \((x_I, x_J) \to x_Ix_J;\)
Proof: This follows from Proposition 4.1 and nondegeneracy of the pairing in the classical $q = 1$ case. Q.E.D.

**Proposition 7.2** There exists a cofactor matrix $\tilde{u}$ so that

$$\tilde{u}u = u\tilde{u} = D_q(\bar{X})I.$$ 

Proof: The matrix form of the nondegenerate pairing in Proposition 7.1 yields a $q$-analogue of Laplace expansion for $O(M_q(\bar{X}))$ in the present context. In particular, we have nondegenerate pairings

$$\wedge_{q}^{nm-1}[\bar{X}] \otimes \wedge_{q}^{1}[\bar{X}] \rightarrow \wedge_{q}^{nm}[\bar{X}], \quad (79)$$

and

$$\wedge_{q}^{1}[\bar{X}] \otimes \wedge_{q}^{nm-1}[\bar{X}] \rightarrow \wedge_{q}^{nm}[\bar{X}], \quad (80)$$

where $\wedge_{q}^{1}[\bar{X}] = \bar{X}$ is the fundamental vector representation.

Let $B = \{x_{ij}\}$ be the standard basis of $\bar{X}$, and $B'$ the Gelfand-Tsetlin basis of $\wedge_{q}^{nm-1}[X]$; cf. (48). Then $B$ and $B'$ are dual bases as per the pairings in (79) and (80). This follows from (1) the multiplication formula in the Gelfand-Tsetlin basis (cf. (53), and (2) the symmetry and other detailed properties of Clebsch-Gordon coefficients proved in [3]; e.g. the fundamental CBC's $C_{NK'M}$ and $C_{KNM}$ are obtained from each other by just replacing $q$ by $q^{-1}$. We leave the details to the reader.

Let $\{b_r\}$ denote the elements of $B$ and $\{b'_s\}$ the elements of $B'$ so that

$$b_r b'_s = \delta^s_r, \quad (81)$$

and

$$b'_s b_r = \delta^s_r. \quad (82)$$

Applying the homomorphism $\phi_R$ (cf. Section 6) to the first equation here implies that there exists a cofactor matrix $\tilde{u}_R$ so that

$$u\tilde{u}_R = D_q(\bar{X})I.$$ 

Applying the homomorphism $\phi_L$ to the second equation here implies that there exists a cofactor matrix $\tilde{u}_L$ so that

$$\tilde{u}_L u = D_q(\bar{X})I.$$ 

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It follows from the symmetry result (Proposition 6.2) that \( \tilde{u}_L = \tilde{u}_R \). So we let

\[ \tilde{u} = \tilde{u}_L = \tilde{u}_R. \]

Q.E.D.

This result implies just as in the standard case that \( D_q(\bar{X}) \) belongs to the center of \( \mathcal{O}(M_q(\bar{X})) \). The coordinate algebra \( \mathcal{O}(GL_q(\bar{X})) \) of the sought quantum group \( GL_q(\bar{X}) \) is obtained by adjoining the inverse \( D_q(\bar{X})^{-1} \) to \( \mathcal{O}(M_q(\bar{X})) \). We formally define \( u^{-1} = D_q(\bar{X})^{-1} \tilde{u} \). This allows us to define a Hopf structure on \( \mathcal{O}(GL_q(\bar{X})) \), just as in the standard case (Section 2).

**Proposition 7.3** There is a unique Hopf algebra structure on \( \mathcal{O}(GL_q(\bar{X})) \) so that

1. \( \Delta(u) = u \otimes u \).
2. \( \epsilon(u) = I \).
3. \( S(u_i^j) = \tilde{u}_j^i D_q(\bar{X})^{-1}, S(D_q(\bar{X})^{-1}) = D_q(\bar{X}) \), where \( u_i^j \) are the entries of \( u \) and \( \tilde{u}_j^i \) are the entries of \( \tilde{u} \).

**Proof:** Since \( \mathcal{O}(M_q(\bar{X})) \) is an FRT-algebra, the proof is similar to that of Proposition 9.10 in \([15] \). Q.E.D.

### 8 Compact real form

**Proposition 8.1** The algebra \( \mathcal{O}(GL_q(\bar{X})) \) is a Hopf \(*\)-algebra with the involution \(*\) determined by \( (u_i^j)^* = S(u_i^j) \).

**Proof:** This follows from Proposition 3, Chapter 9 in \([15] \). The crucial fact here is that the left and the right determinants coincide (Proposition 6.2). Q.E.D.

**Proposition 8.2** The Hopf \(*\)-algebra \( \mathcal{O}(GL_q(\bar{X})) \) is a CMQG algebra (cf. Section 2.1).

**Proof:** The fundamental corepresentation \( \bar{X} \) of \( \mathcal{O}(GL_q(\bar{X})) \) is unitary by Proposition 7.2 and \( D_q(\bar{X})^{-1} \) is a unitary element of \( \mathcal{O}(GL_q(\bar{X})) \). Furthermore, \( \mathcal{O}(GL_q(\bar{X})) \) is generated by the matrix elements of the unitary
corepresentation $u \oplus (D_q(X)^{-1})$. Hence, the result follows from Theorem 2.1 (a). Q.E.D.

Finally, Theorem 1.3 can be restated as:

**Theorem 8.3** (a) The quantum group $GL_q(X)$ defined above has a compact real form, which we shall denote by $U_q(X)$.

(b) There is a homomorphism

$$GL_q(V) \times GL_q(W) \to GL_q(X), \quad (83)$$

(c) Every finite dimensional representation of $GL_q(X)$ is unitarizable, and hence, is a direct sum of irreducible representations.

(d) The quantum analogue of the Peter-Weyl theorem holds. That is,

$$\mathcal{O}(GL_q(X)) = \oplus_{S_\lambda} S_\lambda^* \otimes S_\lambda,$$

where $S_\lambda$ ranges over all irreducible representations of $GL_q(X)$.

This follows from Propositions 7.3, 8.1, 8.2 and Theorem 2.1.

9 Example

Let $X = V \otimes W$, $\dim(V) = \dim(W) = 2$. Recall that $V_{q,\alpha}(V)$ denotes the $q$-Weyl module of the standard quantum group $GL_q(V)$ labelled by the partition $\alpha$.

Let $V_\lambda(X)$ denote the Weyl module of $G = GL(X)$ labelled by the partition $\lambda$. There are three partitions of size 3, namely $(3), (2,1), (1,1,1)$. We have $V_{(3)}(X) = \text{Sym}^3(X)$, with dimension 20, $V_{(1,1,1)}(X) = \wedge^3(X)$, with dimension 4, and $V_{(2,1)}(X)$ with dimension 20. Hence, by Peter-Weyl,

$$\dim(\mathcal{O}(M(X))_3) = 20^2 + 4^2 + 20^2 = 816.$$ 

In contrast, by computer it was verified that $\dim(\mathcal{O}(M_q(X))_3) = 688$. Since, 688 $= 20^2 + 4^2 + 16^2 + 4^2$, by the Peter-Weyl theorem for $GL_q(X)$ (Theorem 8.3), we can suspect that $GL_q(X)$ has four irreducible representations of dimensions 20, 4, 16 and 4, respectively.

That is indeed so. The representations $C^3_q[X]$ and $\wedge^3_q[X]$ of $GL_q(X)$ (cf. Proposition 5.1) turn out to be irreducible representations of dimensions 20 and 4 respectively. These are the $q$-deformations of the classical
representations $\mathbb{C}^3(X)$ and $\wedge^3(X)$ in this setting. We shall see below that the classical representation $V_{(2,1)}(X)$ also has a 20-dimensional $q$-analogue $V_{q,(2,1)}(\bar{X})$, which, however, is reducible. It decomposes into two irreducible representations $V_{q,(2,1),1}(\bar{X})$ and $V_{q,(2,1),2}(\bar{X})$ of $GL_q(\bar{X})$ of dimensions 16 and 4, respectively.

The irreducible representations of $GL_q(\bar{X})$ above decompose as $GL_q(V) \times GL_q(W)$-modules, via the homomorphism in Theorem 8.3, as follows:

$$\mathbb{C}^3_q[\bar{X}] = V_{q,(3)}(V) \otimes V_{q,(3)}(W) \oplus V_{q,(2,1)}(V) \otimes V_{q,(2,1)}(W),$$

$$\wedge^3_q(X) = V_{q,(2,1)}(V) \otimes V_{q,(2,1)}(W),$$

$$V_{q,(2,1),1}(X) = V_{q,(2,1)}(V) \otimes V_{q,(3)}(W) \oplus V_{q,(3)}(V) \otimes V_{q,(2,1)}(W),$$

$$V_{q,(2,1),2}(X) = V_{q,(2,1)}(V) \otimes V_{q,(2,1)}(W).$$

The module $V_{q,(2,1)}(\bar{X})$ can be explicitly constructed as follows.

Let $M = \bar{X} \otimes \wedge^2_q[\bar{X}]$. It is a 24-dimensional representation of $GL_q(\bar{X})$, and it can be shown to contain, just as in the classical case, the four-dimensional module $\wedge^3_q(X)$. Since $GL_q(\bar{X})$ has a compact real form $U_q(\bar{X})$, we can define an inner product on $M$. Then $V_{q,(2,1)}(\bar{X})$ is the orthogonal complement of $\wedge^3_q[\bar{X}]$ in $M$. Alternatively, it is the orthogonal complement of $\mathbb{C}^3_q[\bar{X}]$ in $N = \bar{X} \otimes \mathbb{C}^3_q[\bar{X}]$.

The irreducible modules $V_{q,(2,1),1}(\bar{X})$ and $V_{q,(2,1),2}(\bar{X})$ can also be explicitly constructed. In fact, at the end of the the next section we shall see how all irreducible representations of $GL_q(\bar{X})$ can be constructed, in principle, for general $GL_q(\bar{X})$.

### 10 Deformation $B_r$ of $\mathbb{C}[S_r]$

By the Brauer-Schur-Weyl duality in the standard case [15], the left action of $GL_q(V)$ on $V^\otimes r$ commutes with the right action of the Hecke algebra $H_r$ on $V^\otimes r$, and the two actions determine each other.

We now wish to construct a semisimple sub-algebra $B_r \subseteq H_r \otimes H_r$ which will play the role of the Hecke algebra in the present context. The embedding

$$B_r \rightarrow H_r \otimes H_r,$$
here can be considered as a deformation of the embedding
\[ \mathbb{C}[S_r] \to \mathbb{C}[S_r] \otimes \mathbb{C}[S_r], \]
determined by the diagonal embedding \( S_r \to S_r \times S_r \) of the symmetric group \( S_r \). Just as the Poincare series of \( GL_q(\bar{X}) \) does not coincide with the Poincare series of the classical \( GL(X) \), the dimension of \( \mathcal{B}_r \) will turn out to be different (larger) than that of \( \mathbb{C}[S_n] \). In conjunction with the Wederburn structure theorem, this would imply that the irreducible representations of \( \mathcal{B}_r \) are no longer in one-to-one correspondence with those of \( \mathbb{C}[S_n] \). Thus \( \mathcal{B}_r \) would turn out to be qualitatively similar, but at the same time, fundamentally different from \( \mathcal{H}_r \).

We shall define \( \mathcal{B}_r \) by essentially translating the definition of \( GL_q(\bar{X}) \) acting on the left in the dual setting of right action.

### 10.1 Hecke algebra

Before we can state its definition, we need to review some notation and results for standard affine Hecke Algebras. We largely follow the notation of [9]. Let \( \mathbb{K} = \mathbb{C}(q) \) be the field of rational functions in the indeterminate \( q \) with complex coefficients.

The algebra \( \mathcal{H}_r = \mathcal{H}_r(q) \) is the associative \( \mathbb{K} \)-algebra, generated by the symbols \( T_1, \ldots, T_{r-1} \) with the following relations:

\[
\begin{align*}
T_i T_j &= T_j T_i \quad \text{whenever } |i - j| > 1 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad \text{for } i = 1, \ldots, r - 2 \\
T_i^2 &= (q - 1)T_i + q \quad \text{for } i = 1, \ldots, r - 1
\end{align*}
\]

(84)

If \( M \) is a right-\( \mathcal{H}_r \)-module, then eigenvalues of the right-multiplication on \( M \) by each \( T_i \) are \( q \) and \( -1 \). The element \( T_j \) is invertible:

\[ T_j^{-1} = q^{-1}T_j - (1 - q^{-1}) \]

It is well known that \( \mathcal{H}_r \) is finite-dimensional and semi-simple over \( \mathbb{K} \) with dimension \( r! \).

Since the eigenvalues of \( T_i \) must be \( q \) and \( -1 \), the element \( T_i \in \mathcal{H}_r(q^2) \) has eigenvalues \( q^2 \) and \( -1 \). Thus the element \( \tilde{T}_i = q^{-1}T_i \in \mathcal{H}_r(q^2) \) satisfies

- \( (\tilde{T}_i - qI)(\tilde{T}_i + q^{-1}I) = 0 \), and
- \( \tilde{T}_i \tilde{T}_j \tilde{T}_i = \tilde{T}_j \tilde{T}_i \tilde{T}_j \) whenever \( |j - i| > 1 \).
• $\hat{T}_i\hat{T}_j = \hat{T}_j\hat{T}_i$, if $|i - j| > 1$.

These coincide with the relations satisfied by the $\hat{R} = \hat{R}_{V,V}$ operator. Specifically, the first relation corresponds to (10) and the second relation corresponds to the braid relation satisfied by $\hat{R}$ [15]. It follows that $\mathcal{H}_r(q^2)$ has a right action on $V^{\otimes r}$ given by $\hat{T}_i \rightarrow \hat{R}_i^{V,V}$, where $\hat{R}_i^{V,V}$ acts on the $i$-th and $(i + 1)$-st factors of $V^{\otimes r}$ as $\hat{R}_{V,V}$. This action clearly commutes with the left action of $GL_q(V)$, and the two actions determine each other by Brauer-Schur-Weyl duality [15].

**Remark 10.1** We will henceforth ignore the power of 2 and assume that $\mathcal{H}_r(q)$ itself acts on $V^{\otimes r}$.

The algebra $\mathcal{H}_r$ has other natural set of generators that will be crucial to us. Specifically, let

$$p_i = \frac{T_i + 1}{q + 1} \quad \text{and} \quad q_i = 1 - p_i = \frac{q - T_i}{q + 1}. \quad (85)$$

The $p_i$’s generate $\mathcal{H}_r$. The relations defining them are:

$$p_ip_j = pjp_i \quad (86)$$

if $|i - j| > 1$,

$$p_i - (q + 2 + 1/q)p_ip_{i+1}p_i = p_{i+1} - (q + 2 + 1/q)p_{i+1}p_ip_{i+1} \quad (87)$$

for $i = 1, \ldots, r - 2$, and

$$p_i^2 = p_i \quad (88)$$

for $i = 1, \ldots, r - 1$. The second equation here is a reformulation of the braid relation—the second relation in (84)—in terms of $p_i$’s.

If we consider the right action of $\mathcal{H}_r(q^2)$ on $V^{\otimes r}$ as above, then $p_i$ corresponds to the operator $P_i^+$, which acts as $P_i^+$, defined in (12), on the $i$-th and $(i + 1)$-st factors of $V^{\otimes r}$, and $q_i$ corresponds to the the operator $P_i^-$, which acts as $P_i^-$, defined in (12), on the $i$-th and $(i + 1)$-st factors of $V^{\otimes r}$. The following rescaled versions of $p_i$ and $q_i$ are also useful:

$$\tilde{p}_i = (q^{1/2} + q^{-1/2})p_i = q^{-1/2}(T_i + 1)$$

$$\tilde{q}_i = (q^{1/2} + q^{-1/2})q_i = q^{-1/2}(q - T_i). \quad (89)$$

The advantage of this rescaling is that $C_i = -\tilde{q}_i$ is then the basic Kazhdan-Lusztig basis element of degree one [14].
Another useful set of generators of $\mathcal{H}_r$ is given by

$$f_i = \frac{2T_i + 1 - q}{q + 1}.$$ 

There is an important $\mathbb{C}$-involution $\iota$ of $\mathcal{H}_r$ which is crucial in the Kazhdan-Lusztig theory, namely,

$$\iota(q) = \frac{1}{q},$$
$$\iota(T_j) = T_j^{-1} = q^{-1}T_j - (1 - q^{-1}).$$

It is extended to $\mathcal{H}_r$ naturally. We have $\iota(p_i) = p_i$ and $\iota(f_i) = f_i$.

Another involution is the $K$-involution $\theta$ given by:

$$\theta(T_i) = -T_i + q - 1$$

It is easy to check that $\theta(p_i) = q_i$, and $\theta(f_i) = -f_i$.

10.2 The algebra $B_r$

The algebra $\mathcal{H}_r \otimes \mathcal{H}_r$ acts on $\hat{X} = V \otimes W$ in the obvious manner. Of special importance are its elements

$$P_i = p_i \otimes p_i + q_i \otimes q_i,$$
$$Q_i = p_i \otimes q_i + q_i \otimes p_i.$$ (90)

Their actions on $\hat{X} \otimes r$ correspond to the operators $P^X_{i,+}$ and $P^X_{i,-}$, which act as $P^X_+$ and $P^X_-$, defined in (30), on the $i$-th and $(i+1)$-st factors of $\hat{X} \otimes r$.

The algebra $B_r = B_r(q)$ is defined to be the subalgebra of $\mathcal{H}_r \otimes \mathcal{H}_r$ generated by the elements $P_1, \ldots, P_{r-1}$, or equivalently, the elements $Q_i$’s. We have the natural right action $B_r(q)$ on $\hat{X} \otimes r$. By the defining relation [55] of $O(M_q(\hat{X}))$, the right action of $B_r(q)$ commutes with the left action of the quantum group $GL_q(\hat{X})$.

Remark 10.2 The algebra $B_r$ can be defined by letting arbitrary Coxeter group play the role of the symmetric group. The proof of the semisimplicity result below can be extended to any finite Weyl group in place of the symmetric group.
10.3 Basic properties of $B_r$

Let $A_r = H_r \otimes H_r$. The involution $\theta$ lifts to $A_r$ and will also be denoted as $\theta$. Thus, for example,

$$\theta(a \otimes b) = \theta(a) \otimes \theta(b)$$

There is also the involution $\tau$ with

$$\tau(a \otimes b) = b \otimes a$$

Since $\theta(P_i) = \tau(P_i) = P_i$, $B_r \subseteq A_r^{\tau,\theta}$, the invariant subring of $A_r$. We also have the involution $\Theta : A_r \to A_r$ where $\Theta(a \otimes b) = \theta(a) \otimes b$. We see that:

$$\Theta(P_i) = \theta(p_i) \otimes p_i + \theta(1 - p_i) \otimes (1 - p_i)$$

$$= (1 - p_i) \otimes p_i + p_i \otimes (1 - p_i)$$

$$= 1 - P_i$$

Thus $\Theta$ is an involution on $B_r$ as well.

**Proposition 10.3** The algebra $B_r$ is a semi-simple sub-algebra of $A_r$.

**Proof:** We know that there is a right action of $H_r$ on $V^\otimes r$, which is faithful when $\dim(V)$ is large, and where the matrix of each $p_i$ is symmetric. Consequently there is a faithful representation of $A_r$ on $(V \otimes V)^\otimes r$, which is faithful, and where the matrix of $P_i = p_i \otimes p_i + (1 - p_i) \otimes (1 - p_i)$ is symmetric. We may now use the following fact to complete the proof.

**Fact:** Let $A \subseteq M_m(\mathbb{R})$ be a sub-algebra of the real matrix algebra $M_m(\mathbb{R})$. Furthermore, let $A$ be such that if $a \in A$ then the transpose $a^T$ is also in $A$. Then $A$ is semi-simple.

**Proof:** We produce a vector-space basis $C = \{c_1, \ldots, c_k\}$ of $A$ such that the $k \times k$-matrix $G = Tr(c_i c_j)$ is non-singular. The non-singularity of this matrix is equivalent to the semi-simplicity of $A$.

Since transpose is an involution, and $A$ is closed under transposition, we have $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_s\}$ such that

(i) $C = A \cup B$.

(ii) $a_i^T = a_i$ for all $i$, and $b_j^T = -b_j$ for all $j$.

Now consider $Tr(a_i b_j) = Tr(b_j^T a_i^T) = -Tr(b_j a_i)$, whence $Tr(a_i b_j) = 0$.

Thus $G$ has the following format:

$$G = \begin{bmatrix}
G_A & 0 \\
0 & G_B
\end{bmatrix}$$
where $G_A = (Tr(a_i a_j))$ and $G_B = (Tr(b_i b_j))$. But $Gr_A = (Tr(a_i a_j^T))$ is the Gram-matrix for $A$, and similarly $Gr_B$ is that for $B$, whence both $Gr_A$ and $Gr_B$ are non-singular. Since $G_A = Gr_A$ and $G_B = -Gr_B$, $G$ is nonsingular.

\[ \Box \]

10.4 Relationship with $GL_q(\bar{X})$

In analogy with the Brauer-Schur-Weyl duality in the standard case [15], it is a reasonable conjecture that:

1. The commuting actions of $GL_q(\bar{X})$ and $B_r(q)$ determine each other.

2. There is a one-to-one correspondence between the irreducible (polynomial) representations of $GL_q(\bar{X})$ of degree $r$ and the irreducible representations of $B_r$ so that, as a Bimodule,

\[ \bar{X}^{\otimes r} = \oplus_{\alpha,\alpha'} V(\alpha) \otimes W(\alpha'), \] (91)

where $V(\alpha)$ runs over irreducible polynomial representations of $GL_q(\bar{X})$ of degree $r$, and $W(\alpha')$ denotes the irreducible representation of $B_r(q)$ in correspondence with $V(\alpha)$.

**Proposition 10.4** The third statement above holds for $r = 3$.

**Proof:** This is sketched at the end of Section [11] Q.E.D.

Assuming the conjecture above, irreducible representations of $GL_q(\bar{X})$ of degree $r$ can be constructed, in principle, by constructing the idempotents of the algebra $B_r$ and taking their right actions on $\bar{X}^{\otimes r}$, very much as in the standard case [7]. Specifically, if $a \in B_r$ is an idempotent, then $\bar{X}^{\otimes r} \cdot a$ is (conjecturally) an irreducible representation of $GL_q(\bar{X})$, and all irreducible polynomial representations of $GL_q(\bar{X})$ of degree $r$ can be (conjecturally) obtained in this way.

11 The algebra $B_3$

As an example, now we wish to analyze the structure of the algebra $B = B_3$, in particular, its decomposition into irreducible modules. To conform with the convention in the Kazhdan-Lusztig paper [14], instead of the generators $P_i$’s of $B_r$ in [90], we shall consider the generators

\[ P_i = \tilde{p}_i \otimes \tilde{p}_i + \tilde{q}_i \otimes \tilde{q}_i = \mathbf{f} P_i, \]
where $\tilde{p}_i$ and $\tilde{q}_i$ are as in (89), and $f = (q + 2 + 1/q) = (q^{1/2} + q^{-1/2})^2$. We see that

$$P_i^2 = f \cdot P_i. \quad (92)$$

Through experimentation, by letting

$$c_1 = \frac{q^6 + 2q^5 + 3q^4 + 4q^3 + 3q^2 + 2q + 1}{q^3}$$

$$c_2 = \frac{q^4 + q^3 + 4q^2 + q + 1}{q^2}$$

we see that:

$$c_1 P_1 - c_2 P_1 P_2 P_1 + P_1 P_2 P_1 P_2 P_1 = c_1 P_2 - c_2 P_2 P_1 P_2 + P_2 P_1 P_2 P_1 P_2 \quad (93)$$

This identity is quite different from the braid identity (87) for $H_r$. This means the algebra $O(GL_q(\bar{X}))$ is not coquasitriangular in the terminology of [32].

We name $\Sigma$ as either of the two sides of (93). Thus

$$\Sigma = c_1 P_1 - c_2 P_1 P_2 P_1 + P_1 P_2 P_1 P_2 P_1$$

and at once see that $\Sigma P_1 = \Sigma P_2 = f \cdot \Sigma$, and thus $C \cdot \Sigma$ is a trivial one-dimensional $B$-module. At $q = 1$, this specializes to the trivial representation of the symmetric group $S_3$.

Next, let

$$\alpha = \frac{1}{f^4 - c_2 \cdot f^2 + c_1}$$

$$\beta_1 = P_1 - \alpha \Sigma$$

$$\beta_2 = P_2 - \alpha \Sigma$$

$$\beta_{12} = P_1 P_2 - f \cdot \alpha \Sigma$$

$$\beta_{21} = P_2 P_1 - f \cdot \alpha \Sigma$$

$$\beta_{121} = P_1 P_2 P_1 - f^2 \cdot \alpha \Sigma$$

$$\beta_{212} = P_2 P_1 P_2 - f^2 \cdot \alpha \Sigma$$

$$\beta_{1212} = P_1 P_2 P_1 P_2 - f^3 \cdot \alpha \Sigma$$

$$\beta_{2121} = P_2 P_1 P_2 P_1 - f^3 \cdot \alpha \Sigma \quad (94)$$

It can be verified that $\beta$’s span an 8-dimensional $B$-module, with the following multiplication table.
This 8-dimensional module splits as follows:

|     | $P_1$       | $P_2$       |
|-----|-------------|-------------|
| $\beta_1$ | $f\beta_1$ | $\beta_{12}$ |
| $\beta_2$ | $\beta_{21}$ | $f\beta_2$ |
| $\beta_{12}$ | $\beta_{121}$ | $f\beta_{12}$ |
| $\beta_{21}$ | $f\beta_{21}$ | $\beta_{212}$ |
| $\beta_{121}$ | $f\beta_{121}$ | $\beta_{1212}$ |
| $\beta_{212}$ | $\beta_{2121}$ | $f\beta_{212}$ |
| $\beta_{1212}$ | $-c_1\beta_1 + c_2\beta_{121}$ | $f\beta_{1212}$ |
| $\beta_{2121}$ | $f\beta_{2121}$ | $-c_1\beta_2 + c_2\beta_{212}$ |

The representation $\chi_1$ (and similarly $\chi_2$) may be split further as follows. Let

$$
\gamma_1 = \beta_1/a + \beta_{121},
\gamma_{12} = \beta_{12}/a + \beta_{1212},
$$

for an indeterminate $a$. We get the multiplication table:

|     | $P_1$       | $P_2$       |
|-----|-------------|-------------|
| $\gamma_1$ | $f\gamma_1$ | $\gamma_{12}$ |
| $\gamma_{12}$ | $-c_1\beta_1 + (1+ac_2)/a\beta_{121}$ | $f\gamma_{12}$ |

For $\gamma_{12} \cdot P_1$ to be a multiple of $\gamma_1$ we need:

$$
\frac{1 + ac_2}{-ac_1} = a
$$

Or in other words:

$$
c_1a^2 + c_2a + 1 = 0
$$

This yields:

$$
a = \frac{-c_2 \pm \sqrt{c_2^2 - 4c_1}}{2c_1}
$$

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As expected, \(c_2^2 - 4c_1\) is a perfect square and we have:

\[
\begin{align*}
a_1 &= \frac{-q^2}{(q^2+1)^2} \\
a_2 &= \frac{-q}{(q+1)^2}
\end{align*}
\]

By choosing \(a_1\) and \(a_2\), we form the vectors \(\gamma_1, \gamma_{12}\) and \(\gamma_{1}, \gamma_{12}^2\). We thus have:

\[
\begin{align*}
\chi_1 &= \chi_1^1 \oplus \chi_1^2 \\
\chi_2 &= \chi_2^1 \oplus \chi_2^2
\end{align*}
\]

Further, \(\chi_1^1 \cong \chi_1^2\) and \(\chi_2^1 \cong \chi_2^2\).

At \(q = 1\), the nonisomorphic two-dimensional \(B\)-modules \(\chi_1^1\) and \(\chi_2^1\) specialize to the Specht module of \(S_3\) corresponding to the partition \((2, 1)\).

Finally, we have a 1-dimensional (alternating) module \(\mathbb{C}\mu\), which specializes at \(q = 1\) to the alternating representation of \(S_3\). This can be computed as follows.

Let

\[
\mu = 1 + \theta_1 \beta_1 + \theta_2 \beta_2 + \theta_{12} \beta_{12} + \theta_{21} \beta_{21} + \theta_{121} \beta_{121} + \theta_{212} \beta_{212} + \theta_{1212} \beta_{1212} + \theta \Sigma,
\]

with the various constants being as in the following table:

|       | \(\theta_1\)   | \(\theta_{12}\) | \(\theta_{121}\) | \(\theta_{1212}\) | \(\theta\) |
|-------|----------------|-----------------|-----------------|-----------------|---------|
| \(\Sigma\) | \(\frac{f}{t^4-t^2-c_2}\) | \(\frac{f}{t^4-t^2-c_2}\) | \(-\frac{f}{t^4-t^2-c_2}\) | \(-\frac{1}{t^4-t^2-c_2}\) | \(-\alpha\) |
| \(\chi_1^1\) | \(\frac{f}{t^4-t^2-c_2}\) | \(\frac{f}{t^4-t^2-c_2}\) | \(-\frac{f}{t^4-t^2-c_2}\) | \(-\frac{1}{t^4-t^2-c_2}\) | \(-\alpha\) |
| \(\chi_2^1\) | \(\frac{f}{t^4-t^2-c_2}\) | \(\frac{f}{t^4-t^2-c_2}\) | \(-\frac{f}{t^4-t^2-c_2}\) | \(-\frac{1}{t^4-t^2-c_2}\) | \(-\alpha\) |
| \(\mu\) | \(\frac{f}{t^4-t^2-c_2}\) | \(\frac{f}{t^4-t^2-c_2}\) | \(-\frac{f}{t^4-t^2-c_2}\) | \(-\frac{1}{t^4-t^2-c_2}\) | \(-\alpha\) |

Then it can be verified that \(\mu \cdot \mathcal{P}_1 = \mu \cdot \mathcal{P}_2 = 0\).

To summarize, we have 4 non-isomorphic representations of \(B\), namely

|       | dim. | mult. |
|-------|------|-------|
| \(\Sigma\) | 1    | 1     |
| \(\chi_1^1\) | 2    | 2     |
| \(\chi_2^1\) | 2    | 2     |
| \(\mu\) | 1    | 1     |

where, at \(q = 1\), \(\Sigma\) specializes to the trivial one-dimensional representation of \(S_3\), \(\mu\) to the one-dimensional alternating representation, and \(\chi_1^1\) and \(\chi_2^1\) to the 2-dimensional Specht module of \(S_3\) corresponding to the partition \((2, 1)\).

Note that \(1^2 + 2^2 + 2^2 + 1^2 = 10 = \text{dim}(B)\).
The analysis above can be extended to obtain explicit expressions for the idempotents of $B_3$; we skip the details. Using these idempotents, we can obtain all irreducible representations of degree 3 of the quantum group $GL_q(\bar{X})$, $\bar{X} = V \otimes W$, $\dim(V) = \dim(W) = 2$. Indeed, if $a$ is an idempotent of $B_r$ then $\bar{X} \otimes^3 a$ is a representation of $GL_q(\bar{X})$. These turn out to be irreducible when $r = 3$ (as per the general conjecture in Section 10.2). In this way all irreducible representations of $GL_q(\bar{X})$ within $\bar{X} \otimes^3$ can be computed, and Proposition 10.4 can be verified for $r = 3$.

12 Canonical basis

We now ask if $B_r \subseteq \mathcal{H}_r \otimes \mathcal{H}_r$ has a canonical basis $B$ that is akin to the Kazhdan-Lusztig basis of $\mathcal{H}_r$. We shall not specify here precisely what “akin to” means; cf. [24] for the precise meaning. But two properties (among others) that this basis should satisfy are as follows. Let $C$ be the Kazhdan-Lusztig basis of $\mathcal{H}_r \otimes \mathcal{H}_r$; i.e., each element in $C$ is of the form $c_1 \otimes c_2$, where $c_i$'s are Kazhdan-Lusztig basis elements of $\mathcal{H}_r$. Then

1. Each coefficient of any $b \in B$, when expressed in terms of the basis $C$, should be of the form $q^{a/2}f(q)$, for some integer $a$ and a polynomial $f(q)$ with nonnegative integral coefficients, and furthermore, each coefficient should be $\iota$-invariant. This is analogous to the fact the coefficients of the Kazhdan-Lusztig basis of $\mathcal{H}_r$ in terms of the standard basis of $\mathcal{H}_r$ are (Kazhdan-Lusztig) polynomials with nonnegative coefficients (up to a factor of the form $+ or − q^{a/2}$). Thus the role of the standard basis of $\mathcal{H}_r$ is played here by $C$.

2. $B$ should have cellular decompositions into left, right, and two-sided cells, just like the Kazhdan-Lusztig basis.

12.1 Canonical basis of $B_3$

We now construct such a basis $B$ of $B_3$. We follow the notation as in Section 11. Let
\[ \hat{\beta}_1 = P_1 \]
\[ \hat{\beta}_2 = P_2 \]
\[ \hat{\beta}_{12} = P_1 P_2 \]
\[ \hat{\beta}_{21} = P_2 P_1 \]
\[ \hat{\beta}_{121} = P_1 P_2 P_1 \]
\[ \hat{\beta}_{212} = P_2 P_1 P_2 \]
\[ (95) \]

These are modified versions of the elements \( \beta \)'s in eq.(94).

We define the modified forms \( \hat{\gamma}_1, \hat{\gamma}_{12}, \ldots \) of the elements \( \gamma_1, \gamma_{12}, \ldots \) in Section 11 by substituting \( \hat{\beta} \)'s in place of \( \beta \)'s in their definitions. Let \( \hat{\mu} = 1 \) and \( \hat{\Sigma} = \Sigma \). Finally, let

\[ B = U_\sigma \cup U_\mu \cup V_1 \cup V_2 \cup W_1 \cup W_2, \]
where

\[ U_\sigma = \{ \hat{\Sigma} \} \]
\[ V_1 = \{ \hat{\gamma}_1, \hat{\gamma}_{12} \} \]
\[ V_2 = \{ \hat{\gamma}_2, \hat{\gamma}_{21} \} \]
\[ W_1 = \{ \hat{\gamma}_{12}, \hat{\gamma}_{21} \} \]
\[ W_2 = \{ \hat{\gamma}_2, \hat{\gamma}_{21} \} \]
\[ U_\mu = \{ \hat{\mu} \}. \]

The discussion in Section 11 shows that \( B \) has a cellular decomposition into left cells. Indeed, \( U_\sigma, U_\mu, V_i \)'s and \( W_i \)'s are the left cells, with an obvious partial order among these cells, with \( U_\sigma \) at the bottom of the partial order, and \( U_\mu \) at the top. Similarly, it can be verified that \( B \) also has cellular decomposition into right or two-sided cells.

Furthermore, coefficients of each \( b \in B \) in the Kazhdan-Lusztig basis \( \mathcal{C} \) of \( \mathcal{H}_3 \otimes \mathcal{H}_3 \) are indeed polynomials in \( q \) with nonnegative coefficients (up to a factor of the form \( q^{a/2} \)). These coefficients are shown in the following tables; cf. Figures 1-3. The coefficients are shown in the symmetrized Kazhdan-Lusztig basis \( \mathcal{C}' \) of \( \mathcal{H}_3 \otimes \mathcal{H}_3 \) defined as follows. Let \( c_w, w \in S_3 \), denote the Kazhdan-Lusztig basis element corresponding to the permutation \( w \in S_3 \). We order these permutations as per the lexicographic order on the words denoting their reduced decomposition:

\[ \text{id} < s_1 < s_1 s_2 < s_1 s_2 s_1 < s_2 < s_2 s_1, \]
where \( s_i \) denotes a simple transposition. Let \( c_i \) denote the \( i \)-th element of the Kazhdan-Lusztig basis of \( H_3 \) in this order. Let

\[
C' = \{c_i \otimes c_i\} \cup \{c_i \otimes c_j \oplus c_j \otimes c_i \mid i < j\}.
\]

When \( r = 3 \), \( C' \) contains 21 elements. Figure 1 shows the 21-dimensional coefficient vectors of \( \hat{\Sigma} \) and \( \hat{\mu} \), Figure 2 those of the elements of \( V_1 \) and \( V_2 \), and Figure 3 those of the elements of \( W_1 \) and \( W_2 \).

13 The algebra \( B_4 \) and beyond

The algebra \( B_4 \) turns out to be considerably more complicated and is of dimension 114. This was verified on a computer by a simple procedure of generating monomials systematically and of increasing degree while retaining only those which were not linear combinations of earlier monomials. The top degree obtained thus was 9. In other words, every monomial of degree 10 and above is a linear combination of some smaller monomials. However, this linear combination seems fairly complicated. The ideal of all relations among these monomials is not generated by relations of the type \( \langle 93 \rangle \)–letting arbitrary \( i \) and \( i + 1 \) there in place of 1 and 2–and \( \langle 92 \rangle \). We do not know an explicit presentation in terms of generators and relations for the algebra \( B_r \), in general, akin to the explicit presentation \( \langle 83 \rangle \) for the Hecke algebra. This means the construction procedure for a canonical basis for \( B_3 \) in Section 12.1 cannot be generalized to general \( r \). What is needed is a procedure that does not need explicit presentation; this problem is studied in [24].

To give an idea of the difficulties involved, we give below the simplest relation among the generators of \( B_4 \) which cannot be deduced from the relations of the type \( \langle 93 \rangle \) or \( \langle 92 \rangle \). This is done by expressing the degree 7 monomial \( P_{3232123} \in B_4 \) as a linear combination of smaller monomials, in degree or in lexicographical order, where \( P_{i_1,i_2,\ldots,i_r} \) denotes the monomial \( P_{i_1}P_{i_2}\cdots P_{i_r} \).

There are 74 terms in this linear combination which are reported in the table in Figures 4-6. Every coefficient \( c(q) \) is a rational function in \( q \) such that \( c(q) = c(1/q) \). The table contains 74 rows and each row lists the term, the coefficient and finally, the monomial. Thus, for example, the 33-rd row corresponds to the term:

\[
-\frac{9q^0 - 6q^1 - 55q^2 + 12q^3 - 55q^4 - 6q^5 - 9q^6}{2q^1 + 12q^2 + 4q^3 + 12q^4 + 2q^5} P_{13232}.
\]  (96)
The computation of these coefficients was done in MATLAB, but not directly. A large prime $p$ was chosen and these rational functions were calculated on $\mathbb{Z}_p$. The form was verified on another large prime $q$.

It may be observed that, unlike in the relation (93), coefficients of some monomials in the reported relation—e.g. the one above—involve polynomials whose coefficients are of mixed signs. This suggests that there is no natural monomial basis for $B_r$, unlike the standard monomial basis for Hecke algebras. Fortunately, $B_r$ still seems to possess a canonical basis in general [24].

References

[1] B. Adsul, M. Sohoni, K. Subrahmanyam, Geometric complexity theory IX: algebraic and combinatorial aspects of the Kronecker problem, under preparation.

[2] M. Artin, W. Schelter, J. Tate: Quantum deformations of $GL_n$, Commun. Pure. Appl. Math. 44 (1991), 879-895.

[3] E. Date, M. Jimbo, T. Miwa, Representations of $U_q(gl(n, \mathbb{C}))$ at $q = 0$ and the Robinson-Schensted correspondence, in Physics and Mathematics of strings, World Scientific, Singapore, 1990.

[4] V. Drinfeld: Quantum groups, in proceedings of the International Congress of Mathematicians (A. M. Gleason, ed), Amer. Math. Soc., Providence, RI, 1986, pp. 254-258.

[5] W. Fulton, J. Harris: Representation theory, Springer Verlag, 1991.

[6] I. Grojnowski, G. Lusztig, On bases of irreducible representations of quantum $GL_n$, in Kazhdan-Lusztig theory and related topics, Chicago, IL, 1989, Contemp. Math. 139, 167-174.

[7] M. Jimbo: a $q$-analogue of $U(\mathcal{G})$ and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 63-69.

[8] T. Hayashi: Nonexistence of homomorphisms between quantum groups, Tokyo J. Math. 15 (1992), 431-435.

[9] J. Humphreys, Reflection groups and Coxeter groups, Cambridge studies in advanced mathematics, 29 (1990), Cambridge University Press.
[10] M. Kashiwara: Crystallizing the $q$-analogue of universal enveloping algebra, Commun. Math. Phys. 133 (1990) 249-260.

[11] M. Kashiwara: On crystal bases of the $q$-analogue of the universal enveloping algebras, Duke Math. J. 63 (1991), pp. 465–516.

[12] M. Kashiwara, Global crystal bases of quantum groups, Duke Mathematical Journal, vol. 69, no.2, 455-485.

[13] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165-184.

[14] D. Kazhdan, G. Lusztig, Schubert varieties and Poincare duality, Proc. Symp. Pure Math., AMS, 36 (1980), 185-203.

[15] A. Klimyk, K. Schmüdgen, Quantum groups and their representations, Springer, 1997.

[16] P. Littelmann: A Littlewood-Richardson rule for symmetrizable Kac-Moody Lie algebras, Invent. Math. 116 (1994), 329-346.

[17] G. Lusztig: Introduction to quantum groups, Birkhäuser, Boston, 1993.

[18] G. Lusztig: Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990), pp. 447–498.

[19] G. Lusztig, Canonical bases in tensor products, Proc. Nat. Acad. Sci. USA, vo. 89, pp 8177-8179, 1992.

[20] I. Macdonald: Symmetric functions and Hall polynomials, Oxford Science Publications, 1995.

[21] Y. Manin: Quantum groups and noncommutative geometry, CRM, Montreal, 1988.

[22] K. Mulmuley, On $P$ vs. $NP$, geometric complexity theory and the flip, under preparation.

[23] K. Mulmuley, Geometric complexity theory VII: a quantum group for the plethysm problem, under preparation.

[24] K. Mulmuley, Geometric complexity theory VIII: towards canonical bases for the Kronecker problem, under preparation.
[25] K. Mulmuley, M. Sohoni: Geometric complexity theory, P vs. NP and explicit obstructions, in “Advances in Algebra and Geometry”, Edited by C. Musili, the proceedings of the International Conference on Algebra and Geometry, Hyderabad 2001. Available at authors’ websites.

[26] K. Mulmuley, M. Sohoni: Geometric complexity theory: An approach to the P vs. NP and related problems, SIAM J. comput. vol. 31, no. 2, pp 496-526, (2001)

[27] K. Mulmuley, M. Sohoni: Geometric complexity theory II: towards explicit obstructions for embeddings among class varieties, arXiv cs.CC/0612134, December, 2006.

[28] K. Mulmuley, M. Sohoni: Geometric complexity theory III: on deciding positivity of Littlewood-Richardson coefficients, arXiv cs.CC/0501076, January 2005.

[29] T. Nakashima: Crystal base and a generalization of Littlewood-Richardson rule for the classical Lie algebras, Commun. Math. Phys. 154 (1993), 215-243.

[30] M. Noumi, H. Yamada, K. Mimachi: Finite dimensional representations of the quantum group $GL_q(n, \mathbb{C})$ and the zonal spherical functions on $U_q(n)/U_q(n-1)$, Jap. J. Math. 19 (1993), 31-80.

[31] N. Reshetikhin: Multiparameter quantum groups and twisted quasitriangular Hopf algebras, Lett. Math. Phys. 20 (1990), 331-335.

[32] N. Reshetikhin, L. Takhtajan, L. Faddeev: Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1990), 193-225.

[33] R. Stanley: Positivity problems and conjectures in algebraic combinatorics, In Mathematics: frontiers and perspectives, 295-319, Amer. Math. Soc. Providence, RI (2000).

[34] A. Sudbery: Consistent multiparametric quantization of $GL(n)$, J. Phys. A 23 (1990), L697-L704.

[35] N. Vilenkin, A. Klimyk, Representations of Lie groups and special functions, vol. 3, Kluwer Acad. Publ. 1992.

[36] S. Woronowicz: Compact matrix pseudogroups, Commun. Math. Phys. 111 (1987), 613-665.
Appendix: Reduction system for $\mathcal{O}(M_q(\bar{X}))$

In this section we reformulate the relations (59) for $\mathcal{O}(M_q(\bar{X}))$ in the form of a reduction system, which is used in Section 5.1. We follow the same terminology as in the proof of Lemma 6.5.

In that terminology, the algebra $\mathcal{O}(M_q(\bar{X}))$ is the tensor algebra $T(U)$ modulo the ideal generated by $n_4 \subseteq U \otimes U$.

We assume first that $\dim(V) = \dim(W) = 2$. Then the dimension of the degree 2 component of this algebra, viz., $\dim((Z^4 \otimes Z^4)/n_4)$ equals $136 = \binom{17}{2}$. For this, we construct a total order $\succeq$ on the variables $Z = \{z_{a_1,a_2,a_3,a_4} | 1 \leq a_1, a_2, a_3, a_4 \leq 2\}$. This order is specified easily enough: $z_{a_1,a_2,a_3,a_4} \succeq z_{b_1,b_2,b_3,b_4}$ iff there is an $1 \leq i \leq 4$ so that $a_j = b_j$ for all $1 \leq j \leq i$ and $a_{i+1} \geq b_{i+1}$. Thus, for example $z_{2112} \succeq z_{1222}$.

Let $A$ and $B$ be the tuples $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$. We say that a monomial $z_A z_B$ is standard only if $z_A \succeq z_B$, otherwise, $z_A z_B$ is called non-standard. Note that if $\Gamma = \{(A, B) | A \succeq B\}$ then $|\Gamma| = 136$. We shall now see that the degree 2 component above is spanned by the standard monomials.

We say that the monomial $z_A z_B$ is exceptional of order $i$ if $a_{i+1} < b_{i+1}$ and $a_j \geq b_j$ for all $1 \leq j \leq i$. Thus $z_{2212} z_{1222}$ is non-standard of order 2. Clearly, every non-standard monomial $z_A z_B$ is exceptional of order $m$ for some $m < 4$. For every exceptional monomial $z_A z_B$ of order $m$ we exhibit an element $\psi_{A, B}$ of $P^- = n_4$ such that:

$$\psi_{A, B} = z_A z_B + \sum \alpha_i z_{C_i} z_{D_i}$$

where either (i) each monomial $z_{C_i} z_{D_i}$ is either standard, or (ii) $z_{C_i} z_{D_i}$ is exceptional of order exceeding $m$.

We construct $\psi_{A, B}$ via another simpler element $\psi_{A', B'}$ and “exploding” this simpler element by the term $\eta_{A, B}$. The tuples $A'$ and $B'$ are defined as follows. Let $I$ be the set $\{i | a_i \neq b_i\}$ arranged as a tuple. Let $J = (j_1, \ldots, j_r)$ be the complement of $I$. The term $\eta_{A, B}$ is defined as $z_J z_J$ where

$$z_J = x_{j_1} \ast \ldots \ast x_{j_r}$$
Note that
\[ \eta_{A,B} = T_1 \ldots T_r \]
where \( T_i = A_{11} \) if \( a_{ji} = 1 \) and \( T_i = A_{22} \) otherwise. Thus \( \eta_{A,B} \in P^+ \) for all \( A, B \).

We define \( A' = A_I \) and \( B' = B_I \), i.e., the tuples \( A \) and \( B \) restricted to the coordinates \( I \). Thus, for example, if \( A = (1221) \) and \( B = (2212) \) then \( I = (134) \) and \( A' = (121) \) and \( B' = (212) \). The factor \( \eta_{A,B} \) is \( (x_2)^2 \) and thus equals \( A_{22} \). It is now clear that
\[ \psi_{A,B} = \psi_{A',B'} \star \eta_{A,B} \]
where every coordinate of \( \eta_{A,B} \) goes in the right place in every term of \( \psi_{A',B'} \). Furthermore, if \( \psi_{A',B'} \) is an element of \( P^- \) then so is \( \psi_{A,B} \). This reduces the computation for those monomial \( z_A z_B \) which are non-standard and for which \( a_i \neq b_i \). The non-standardness implies that \( a_1 = 1 \). For \( m = 4 \) there are 15 such tuples, the \( A \) values are listed below. \( B \) is precisely the complement of \( A \).

\[
\begin{array}{ccccccc}
1111 & 1112 & 1121 & 1122 & 1211 \\
1212 & 1221 & 1222 & 111 & 112 \\
121 & 122 & 11 & 12 & 1 \\
\end{array}
\]

For each \( A \) above, we inductively construct \( \psi_A \in P^- \) and \( \phi_A \in P^+ \) such that either of them is of the form
\[ z_A z_B + \sum_i \alpha_i z_{A_i} z_{B_i} \]
where the first entry of \( A_i \) is 2. Thus, \( z_A z_B \) is indeed the solitary leading term, and all other terms, if non-standard are exceptional of a higher order.

We illustrate this process by a small example:
\[
\psi_1 = z_1 z_2 - q z_2 z_1 \\
\phi_1 = z_1 z_2 + p z_2 z_1
\]
Thus, \( \psi_1 = B12 \) and \( \phi_1 = A12 \). Next, we construct the terms:
\[
(\phi_1 \star A12 - \psi_1 \star B12)/(p + q) = z_{12} z_{21} + z_{21} z_{12} + (p - q) z_{22} z_{11} \\
(q \cdot \phi_1 \star A12 + p \cdot \psi_1 \star B12)/(p + q) = z_{11} z_{22} + z_{22} z_{11}
\]
Since both these terms are in \( P^+ \), these qualify to be called \( \phi_{11} \) and \( \phi_{12} \). The terms \( \psi_{11} \) and \( \psi_{12} \) are constructed similarly from \( \phi_1 \star B12 \) and \( \psi_1 \star A12 \).
The net results is expressed simply as:

\[
\begin{bmatrix}
\phi_{11} \\
\phi_{12} \\
\psi_{11} \\
\psi_{12}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{q}{p+q} & \frac{p}{p+q} & 0 & 0 \\
0 & 0 & \frac{q}{p+q} & \frac{p}{p+q} \\
0 & 0 & \frac{q}{p+q} & \frac{p}{p+q}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \ast A_{12} \\
\psi_1 \ast B_{12} \\
\phi_1 \ast B_{12} \\
\psi_1 \ast A_{12}
\end{bmatrix}
\]

In general, we note that:

\[
\begin{bmatrix}
\phi_{A1} \\
\phi_{A2} \\
\psi_{A1} \\
\psi_{A2}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{q}{p+q} & \frac{p}{p+q} & 0 & 0 \\
0 & 0 & \frac{q}{p+q} & \frac{p}{p+q} \\
0 & 0 & \frac{q}{p+q} & \frac{p}{p+q}
\end{bmatrix}
\begin{bmatrix}
\phi_A \ast A_{12} \\
\psi_A \ast B_{12} \\
\phi_A \ast B_{12} \\
\psi_A \ast A_{12}
\end{bmatrix}
\]

This gives us the construction of all \( \psi_{A,B} \). As an example, for \( A = [1122] \) and \( B = [1212] \), we have

\[
\psi_{A,B} = A_{11} \ast \psi_{12} \ast A_{22}
\]

\[
= -z_{1122}z_{1212} - z_{1212}z_{1122}
\]

This yields a reduction system for \((Z \otimes Z)/P^-\) wherein we have

\[
z_Az_B = \sum_i \alpha_iz_{A_i}z_{B_i}
\]

where all \( z_{A_i}z_{B_i} \) are standard.

By this reduction rule, there is a method of expanding any monomial \( z_Az_Bz_C \) as

\[
z_Az_Bz_C = \sum_i z_{A_i}z_{B_i}z_{C_i}
\]

wherein \( A_i \geq B_i \geq C_i \). Thus every non-standard monomial may be expanded into a linear combination of standard monomials. Unfortunately, this reduction system does not obey the diamond lemma. In other words, there exist monomials \( z_Az_Bz_C \) wherein two different simplifications using the above reduction rules yield two different expansions into standard monomials.

Consider the monomial \( mm = z_{1111}z_{1112}z_{1221} \). For any monomial \( z_Az_Bz_C \), if \( A \not\geq B \), then we may apply the reduction system above for the first two terms and this is denoted as \((z_Az_B)z_C\). We say that \( R_1 \) applies and display the result. Similar, we say that \( R_2 \) applies if \( z_B \not\geq z_C \) and denote this
application by $z_A(z_B z_C)$. The monomial above has two expansions, viz.,
$R_1 R_2 R_1 R_2$ and $R_2 R_1 R_2$, reading both strings from left to right.

The first expansion yields

\[(m) R_1 = l_1 = (q) \cdot z_{1112} z_{1111} z_{1211}\]

\[(m) R_1 R_2 = l_{12} = (-1 + q^2) \cdot z_{1112} z_{1211} z_{1121} + q \cdot z_{1112} z_{1221} z_{1111}\]

The expression $l_{12}$ has two terms $t_1$ and $t_2$. We have:

\[(t_1) R_1 = z_{1211} z_{1112} z_{1121} = z_{1211} z_{1112} z_{1112};\]

the last equality follows since $z_{1121}$ and $z_{1112}$ commute, as is easy to show.

\[(t_2) R_1 = (-1 + q^2) \cdot z_{1211} z_{1122} z_{1111} + \frac{1 - q^2}{1 + q^2} \cdot z_{1212} z_{1121} z_{1111} + \frac{2q}{1 + q^2} \cdot z_{1221} z_{1112} z_{1111} + \frac{1 - q^2}{1 + q^2} \cdot z_{1122} z_{1111} z_{1111}\]

Combining all this, we have $l_{121} = (m) R_1 R_2 R_1$:

\[l_{121} = (-1 + q^2) \cdot z_{1211} z_{1121} z_{1112} + \frac{q^3 - q}{1 + q^2} \cdot z_{1211} z_{1122} z_{1111} + \frac{q^2 - 1}{1 + q^2} \cdot z_{1221} z_{1112} z_{1111} + \frac{2q^2}{1 + q^2} \cdot z_{1222} z_{1112} z_{1111} + \frac{1 - q^2}{1 + q^2} \cdot z_{1122} z_{1111} z_{1111}\]

$l_{121}$ has 5 terms, viz., $t_1, \ldots, t_5$. Applying $R_2$ to each we get:

\[Term1 = (1) \cdot z_{1211} z_{1121} z_{1112}\]

\[Term2 = (1) \cdot z_{1211} z_{1122} z_{1111}\]

\[Term3 = (1) \cdot z_{1212} z_{1121} z_{1111}\]

\[Term4 = (1) \cdot z_{1221} z_{1112} z_{1111}\]

\[Term5 = (1) \cdot z_{1222} z_{1111} z_{1111}\]

Finally, $(m) R_1 R_2 R_1 R_2 = l_{1212}$ equals:

\[l_{1212} = (q^2 - 1) \cdot z_{1211} z_{1121} z_{1112} + \frac{q^3 - q}{1 + q^2} \cdot z_{1211} z_{1122} z_{1111} + \frac{q^2 - 1}{1 + q^2} \cdot z_{1221} z_{1112} z_{1111} + \frac{2q^2}{1 + q^2} \cdot z_{1222} z_{1112} z_{1111} + \frac{1 - q^2}{1 + q^2} \cdot z_{1122} z_{1111} z_{1111}\]
The second expansion is \((m)R_2R_1R_2\). We have \(l_2 = (m)R_2\) as below:

\[
l_2 = \frac{q^2 - 1}{1 + q^2} z_{1111} z_{1211} z_{1122} + \frac{q^2 - 1}{q(1 + q^2)} z_{1111} z_{1212} z_{1121} + \frac{2q}{1 + q^2} z_{1111} z_{1221} z_{1112} + \frac{1 - q^2}{1 + q^2} z_{1111} z_{1222} z_{1111}
\]

This has 4 terms, and applying \(R_1\) to each term yields:

\[
\begin{align*}
Term 1 &= q \cdot z_{1211} z_{1111} z_{122} \\
Term 2 &= \frac{q^2 - 1}{q} \cdot z_{1211} z_{1112} z_{121} + z_{1212} z_{1111} z_{1121} \\
Term 3 &= \frac{q^2 - 1}{q} \cdot z_{1211} z_{1212} z_{1112} + z_{1221} z_{1111} z_{1112} \\
Term 4 &= \frac{(q^2 - 1) q}{1 + q^2} z_{1211} z_{1212} z_{1111} + \frac{q^2 - 1}{1 + q^2} z_{1212} z_{1211} z_{1112} + \frac{2q^2 - 2}{1 + q^2} z_{1211} z_{1121} z_{1112} + \frac{2q^3}{(1 + q^2)^2} z_{1211} z_{1122} z_{1111} + \frac{1 - 2q^2 + q^4}{1 + q^2} z_{1211} z_{1112} z_{1121} + \frac{2q}{1 + q^2} z_{1221} z_{1112} z_{1111} - \frac{1 - 2q^2 + q^4}{1 + q^2} z_{1212} z_{1112} z_{1111} + \frac{2q}{1 + q^2} z_{1221} z_{1112} z_{1111} - \frac{(2q^2 - 2) q}{(1 + q^2)^2} z_{1222} z_{1111} z_{1111}
\end{align*}
\]

Whence, \(l_{21} = (m)R_2R_1\) equals:

\[
l_{21} = \frac{(q^2 - 1) q}{1 + q^2} z_{1211} z_{1111} z_{122} + \frac{1 - 2q^2 + q^4}{q(1 + q^2)} z_{1211} z_{1112} z_{121} + \frac{2q^2 - 2}{1 + q^2} z_{1211} z_{1121} z_{1112} + \frac{2q^3}{(1 + q^2)^2} z_{1211} z_{1122} z_{1111} + \frac{q^2 - 1}{q(1 + q^2)} z_{1212} z_{1111} z_{121} - \frac{1 - 2q^2 + q^4}{(1 + q^2)^2} z_{1212} z_{1112} z_{1121} - \frac{2q}{1 + q^2} z_{1221} z_{1111} z_{1112} - \frac{1 - 2q^2 + q^4}{(1 + q^2)^2} z_{1221} z_{1112} z_{1111} + \frac{2q}{1 + q^2} z_{1222} z_{1111} z_{1111} - \frac{(2q^2 - 2) q}{(1 + q^2)^2} z_{1222} z_{1111} z_{1111}
\]

This has 9 terms, which on applying \(R_2\) yield:

\[
\begin{align*}
Term 1 &= \frac{q^2 - 1}{q} \cdot z_{1211} z_{1112} z_{1112} + z_{1211} z_{1122} z_{1111} \\
Term 2 &= z_{1211} z_{1121} z_{1112} \\
Term 3 &= z_{1211} z_{1121} z_{1112} \\
Term 4 &= z_{1211} z_{1122} z_{1111} \\
Term 5 &= q \cdot z_{1212} z_{1121} z_{1111}
\end{align*}
\]
Term 6 = \( z_{1212} z_{1121} z_{1111} \)

Term 7 = \( q \cdot z_{1221} z_{1112} z_{1111} \)

Term 8 = \( z_{1221} z_{1112} z_{1111} \)

Term 9 = \( z_{1222} z_{1111} z_{1111} \)

Finally, collating this, we get \( l_{212} = (m)R_2R_1R_2 \) as follows:

\[
l_{212} = \frac{q^6 + q^4 - 3q^2 + 1}{q^2 (1 + q^2)} \cdot z_{1211} z_{1121} z_{1112} + \frac{2q^3 - 2q}{(1 + q^2)^2} \cdot z_{1211} z_{1122} z_{1111} \\
4q^2 + q^4 - 1 \quad \frac{2q^2 - 2}{(1 + q^2)^2} \cdot z_{1221} z_{1112} z_{1111} + \frac{2q^3 - 2q}{(1 + q^2)^2} \cdot [z_{1212} z_{1121} z_{1111} - q \cdot z_{1222} z_{1111} z_{1111}] \\
\]

It is easy to see that \((m)R_1R_2R_1R_2 \neq (m)R_2R_1R_2\).
| Term                                      | Coefficient |
|-------------------------------------------|-------------|
| $4 \left( \frac{q^2+q+1}{q^4} \right)^2$ | 1           |
| $4 \left( \frac{q^2+q+1}{q^{4/2}} \right)^3$ | 0           |
| $4 \left( \frac{q^2+q+1}{q^{4/2}} \right)^4$ | 0           |
| $4 \left( \frac{q^2+q+1}{q^{4/2}} \right)^5$ | 0           |
| $4 \left( \frac{q^2+q+1}{q^{4/2}} \right)^6$ | 0           |
| $6 \frac{(1+q)^6}{q^3}$                   | 0           |
| $6 \frac{(1+q)^5}{q^{3/2}}$               | 0           |
| $6 \frac{(1+q)^4}{q^2}$                   | 0           |
| $2 \frac{(q^2+q+1)(1+q)^4}{q^3}$          | 0           |
| $6 \frac{(1+q)^5}{q^{5/2}}$               | 0           |
| $2 \frac{(q^2+4q+1)(1+q)^4}{q^3}$         | 0           |
| $2 \frac{(q^2+4q+1)(1+q)^3}{q^{5/2}}$     | 0           |
| $6 \frac{(1+q)^5}{q^{5/2}}$               | 0           |
| $6 \frac{(1+q)^4}{q^2}$                   | 0           |
| $4 \frac{(q^2+4q+1)(1+q)^2}{q^4}$         | 0           |
| $6 \frac{(1+q)^4}{q^3}$                   | 0           |
| $2 \frac{(q^2+4q+1)(1+q)^3}{q^{5/2}}$     | 0           |
| $6 \frac{(1+q)^6}{q^4}$                   | 0           |
| $6 \frac{(1+q)^5}{q^{5/2}}$               | 0           |
| $2 \frac{(q^2+4q+1)(1+q)^4}{q^5}$         | 0           |

Figure 1: Coefficient vectors of the elements $\hat{\Sigma}$ and $\hat{\mu}$ in the symmetrized Kazhdan-Lusztig basis.
Figure 2: Coefficient vectors of the elements of $V_1$ and $V_2$ in the symmetrized Kazhdan-Lusztig basis.
\[
\begin{array}{cccc}
\frac{(q^2+q+1)(1+q)^4}{q^4} & \frac{(q^2+q+1)(1+q)^6}{q^4} & \frac{(q^2+q+1)(1+q)^4}{q^4} & \frac{(q^2+q+1)(1+q)^6}{q^4} \\
\frac{(1+q)^5}{q^{7/2}} & \frac{(1+q)^7}{q^{7/2}} & \frac{(1+q)^5}{q^{7/2}} & \frac{(1+q)^7}{q^{7/2}} \\
\frac{(1+q)^4}{q^{3/2}} & \frac{(1+q)^6}{q^{3/2}} & \frac{(1+q)^4}{q^{3/2}} & \frac{(1+q)^6}{q^{3/2}} \\
\frac{(1+q)^3}{q^{7/2}} & \frac{(1+q)^5}{q^{7/2}} & \frac{(1+q)^3}{q^{7/2}} & \frac{(1+q)^5}{q^{7/2}} \\
\frac{(1+q)^2}{q^{7/2}} & \frac{(1+q)^4}{q^{7/2}} & \frac{(1+q)^2}{q^{7/2}} & \frac{(1+q)^4}{q^{7/2}} \\
\frac{2}{q^2} & \frac{2}{q^2} & \frac{2}{q^2} & \frac{2}{q^2} \\
\frac{3 q^3 + 1 + 3 q + 8 q^2 + q^4}{q^2} & \frac{3 q^3 + 1 + 3 q + 8 q^2 + q^4}{q^2} & \frac{3 q^3 + 1 + 3 q + 8 q^2 + q^4}{q^2} & \frac{3 q^3 + 1 + 3 q + 8 q^2 + q^4}{q^2} \\
\frac{(1+q)(2q^2 + 6q + 1)}{q^{7/2}} & \frac{(1+q)(2q^2 + 6q + 1)}{q^{7/2}} & \frac{(1+q)(2q^2 + 6q + 1)}{q^{7/2}} & \frac{(1+q)(2q^2 + 6q + 1)}{q^{7/2}} \\
8 & \frac{(q^2 + 4q + 1)(1+q)^2}{q^7} & \frac{(q^2 + 4q + 1)(1+q)^2}{q^7} & \frac{(q^2 + 4q + 1)(1+q)^2}{q^7} \\
\frac{(1+q)(q^2 + 6q + 1)}{q^{7/2}} & \frac{(1+q)(q^2 + 6q + 1)}{q^{7/2}} & \frac{(1+q)(q^2 + 6q + 1)}{q^{7/2}} & \frac{(1+q)(q^2 + 6q + 1)}{q^{7/2}} \\
2 & \frac{3 q^4 + 6 q^3 + 14 q^2 + 6 q + 3}{q^2} & \frac{3 q^4 + 6 q^3 + 14 q^2 + 6 q + 3}{q^2} & \frac{3 q^4 + 6 q^3 + 14 q^2 + 6 q + 3}{q^2} \\
\frac{4 (1+q)}{q} & \frac{4 (1+q)}{q} & \frac{4 (1+q)}{q} & \frac{4 (1+q)}{q} \\
\frac{4 + q}{\sqrt{q}} & \frac{4 (1+q)^3}{q^{7/2}} & \frac{4 + q}{\sqrt{q}} & \frac{4 (1+q)^3}{q^{7/2}} \\
2 & \frac{(1+q)(3 q^2 + 1 + 3 q + 8 q^2 + q^4)}{q^{7/2}} & \frac{(1+q)(3 q^2 + 1 + 3 q + 8 q^2 + q^4)}{q^{7/2}} & \frac{(1+q)(3 q^2 + 1 + 3 q + 8 q^2 + q^4)}{q^{7/2}} \\
\frac{(1+q)^3}{q^{7/2}} & \frac{(1+q)^2}{q} & \frac{(1+q)^3}{q^{7/2}} & \frac{(1+q)^2}{q} \\
8 & \frac{(q^2 + 6 q + 1)(1+q)^2}{q^2} & \frac{(q^2 + 6 q + 1)(1+q)^2}{q^2} & \frac{(q^2 + 6 q + 1)(1+q)^2}{q^2} \\
\frac{8}{q} & \frac{8}{q} & \frac{8}{q} & \frac{8}{q} \\
2 & \frac{8 (1+q)^2}{q} & \frac{8 (1+q)^2}{q} & \frac{8 (1+q)^2}{q} \\
\frac{(q^2 + 6 q + 1)(1+q)^2}{q^2} & \frac{(q^2 + 6 q + 1)(1+q)^2}{q^2} & \frac{(q^2 + 6 q + 1)(1+q)^2}{q^2} & \frac{(q^2 + 6 q + 1)(1+q)^2}{q^2} \\
\frac{4 (1+q)}{q^{7/2}} & \frac{4 (1+q)}{q^{7/2}} & \frac{4 (1+q)}{q^{7/2}} & \frac{4 (1+q)}{q^{7/2}} \\
2 & \frac{(3 q^3 + 1 + 3 q + 8 q^2 + q^4)(1+q)^2}{q^2} & \frac{(3 q^3 + 1 + 3 q + 8 q^2 + q^4)(1+q)^2}{q^2} & \frac{(3 q^3 + 1 + 3 q + 8 q^2 + q^4)(1+q)^2}{q^2} \\
\frac{2 (1+q)^3}{q^{7/2}} & \frac{(q^2 + 6 q + 1)(1+q)^3}{q^{7/2}} & \frac{(q^2 + 6 q + 1)(1+q)^3}{q^{7/2}} & \frac{(q^2 + 6 q + 1)(1+q)^3}{q^{7/2}} \\
2 & \frac{(1+q)(3 q^3 + 1 + 3 q + 8 q^2 + q^4)}{q^{7/2}} & \frac{(1+q)(3 q^3 + 1 + 3 q + 8 q^2 + q^4)}{q^{7/2}} & \frac{(1+q)(3 q^3 + 1 + 3 q + 8 q^2 + q^4)}{q^{7/2}} \\
\frac{4 (1+q)^4}{q^2} & \frac{4 (1+q)^4}{q^2} & \frac{4 (1+q)^4}{q^2} & \frac{4 (1+q)^4}{q^2} \\
4 & \frac{4 (1+q)^2}{q} & \frac{4 (1+q)^2}{q} & \frac{4 (1+q)^2}{q} \\
\frac{2}{q^2} & \frac{2}{q^2} & \frac{2}{q^2} & \frac{2}{q^2} \\
\frac{3 q^2 + 6 q^3 + 14 q^2 + 6 q + 3}{q^2} & \frac{3 q^2 + 6 q^3 + 14 q^2 + 6 q + 3}{q^2} & \frac{3 q^2 + 6 q^3 + 14 q^2 + 6 q + 3}{q^2} & \frac{3 q^2 + 6 q^3 + 14 q^2 + 6 q + 3}{q^2} \\
\end{array}
\]

Figure 3: Coefficient vectors of the elements of \( W_1 \) and \( W_2 \) in the symmetrized Kazhdan-Lusztig basis.
\begin{figure}
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Number & Coefficient & Monomial \\
\hline
1 & $20q^{11}+104q^{12}+256q^{20}+113q^{21}+49q^{22}+131q^{23}+256q^{24}+104q^{25}+20q^{26}$ & 1 \\
2 & $-16q^{11}-64q^{12}-128q^{20}-192q^{21}-224q^{22}-192q^{23}-128q^{24}-64q^{25}-16q^{26}$ & 2 \\
3 & $4q^{11}-16q^{12}-28q^{20}-32q^{21}-28q^{22}-16q^{23}-4q^{24}$ & 3 \\
4 & $1q^{11}-4q^{12}+6q^{20}-4q^{21}+q^{22}$ & 12 \\
5 & $-1q^{11}-18q^{12}-65q^{20}-128q^{21}-190q^{22}-220q^{23}-190q^{24}-65q^{25}-18q^{26}-1$ & 13 \\
6 & $1q^{11}+q^{12}+3q^{20}+6q^{21}+9q^{22}+6q^{23}+3q^{24}+q^{25}$ & 21 \\
7 & $7q^{11}+26q^{12}+75q^{20}+152q^{21}+174q^{22}+152q^{23}+75q^{24}+26q^{25}+7q^{26}$ & 23 \\
8 & $-1q^{11}-8q^{12}-20q^{20}-24q^{21}-22q^{22}-24q^{23}-20q^{24}-8q^{25}-1q^{26}$ & 32 \\
9 & $20q^{11}+2q^{12}+12q^{20}+14q^{21}+8q^{22}+14q^{23}+12q^{24}+2q^{25}+2q^{26}$ & 121 \\
10 & $-2q^{11}-12q^{20}-40q^{21}-52q^{22}-52q^{23}-40q^{24}-12q^{25}-2q^{26}$ & 212 \\
11 & $1q^{11}+2q^{12}+12q^{20}+14q^{21}+8q^{22}+14q^{23}+12q^{24}+2q^{25}+2q^{26}$ & 213 \\
12 & $1q^{11}+22q^{20}+88q^{21}+170q^{22}+206q^{23}+170q^{24}+88q^{25}+22q^{26}+1$ & 232 \\
13 & $2q^{11}+12q^{20}+2q^{21}$ & 232 \\
14 & $2q^{11}$ & 323 \\
15 & $2q^{11}$ & 1212 \\
16 & $2q^{11}$ & 1213 \\
17 & $2q^{11}$ & 1232 \\
18 & $2q^{11}$ & 1321 \\
19 & $2q^{11}$ & 1323 \\
20 & $2q^{11}$ & 2121 \\
21 & $2q^{11}$ & 2123 \\
22 & $2q^{11}$ & 2321 \\
23 & $2q^{11}$ & 2323 \\
24 & $2q^{11}$ & 3212 \\
25 & $2q^{11}$ & 3213 \\
26 & $2q^{11}$ & 3232 \\
27 & $2q^{11}$ & 12121 \\
28 & $2q^{11}$ & 12123 \\
29 & $2q^{11}$ & 12132 \\
30 & $2q^{11}$ & 12321 \\
\hline
\end{tabular}
\end{center}
\caption{A relation in $B_4$}
\end{figure}
| Number | Coefficient | Monomial |
|--------|-------------|----------|
| 31     | $-4q^2-8q^3-40q^4-8q^5-4q^6$ | $2q^4+12q^5+2q^6$ |
| 32     | $-3q^2-4q^3-2q^4-3q^5$ | $2q^4+2q^5$ |
| 33     | $-9q^6-6q^7-55q^8+12q^9+2q^4$ | $2q^4+12q^5+4q^6+12q^7+2q^8$ |
| 34     | $9q^6-6q^7-55q^8+12q^9+55q^{10}+6q^{11}+9q^{12}$ | $2q^4+12q^5+4q^6+12q^7+2q^8$ |
| 35     | $1q^8-1q^9+3q^{10}-6q^{11}+3q^{12}-1q^{13}+1q^{14}$ | $2q^4+2q^5$ |
| 36     | $-1q^9+2q^{10}-q^{11}$ | $2q^5$ |
| 37     | $2q^{10}+3q^{11}+6q^{12}-3q^{13}-3q^{14}-6q^{15}+3q^{16}+2q^{17}$ | $2q^4+12q^5+4q^6+12q^7+2q^8$ |
| 38     | $3q^4+4q^5+2q^6+4q^7+3q^8$ | $2q^4+2q^5$ |
| 39     | $-16q^9-32q^{10}+16q^{11}$ | $2q^4+12q^5+2q^6$ |
| 40     | $3q^9+4q^{10}+2q^{11}+4q^{12}+3q^{13}$ | $2q^4+2q^5$ |
| 41     | $8q^3/2q^2$ | $2q^2$ |
| 42     | $1q^{11}-2q^{12}+1q^{13}$ | $2q^2$ |
| 43     | $-3q^5-4q^6-2q^7-4q^8-3q^9$ | $2q^4+2q^5$ |
| 44     | $-5q^6-16q^7-8q^8$ | $2q^4+12q^5+2q^6$ |
| 45     | $-1q^9-14q^{10}+15q^{11}-4q^{12}+16q^{13}+14q^{14}+1q^{15}$ | $2q^4+12q^5+4q^6+12q^7+2q^8$ |
| 46     | $-2q^9-4q^{10}-2q^{11}$ | $2q^2$ |
| 47     | $-2q^9+4q^{10}-2q^{11}$ | $2q^4+12q^5+2q^6$ |
| 48     | $8q^5+16q^6+8q^7$ | $2q^4+12q^5+2q^6$ |
| 49     | $-1q^6-2q^7-1q^8$ | $2q^2$ |
| 50     | $2q^6-4q^7+2q^8$ | $2q^4+12q^5+2q^6$ |
| 51     | $2q^9+8q^{10}+12q^{11}+8q^{12}+2q^{13}$ | $2q^4+12q^5+4q^6+2q^7$ |
| 52     | $2q^9+4q^{10}+2q^{11}$ | $2q^2$ |
| 53     | $1q^9+14q^{10}+15q^{11}+4q^{12}+15q^{13}+14q^{14}+1q^{15}$ | $2q^4+12q^5+14q^6+12q^7+2q^8$ |
| 54     | $3q^9+8q^{10}+10q^{11}+8q^{12}+3q^{13}$ | $2q^4+12q^5+2q^6$ |
| 55     | $-2q^9-8q^{10}-12q^{11}-8q^{12}-2q^{13}$ | $2q^4+12q^5+2q^6$ |
| 56     | $-1q^9-2q^{10}-1q^{11}$ | $2q^2$ |
| 57     | $-3q^9-10q^9-8q^{10}-3q^{11}$ | $2q^4+12q^5+2q^6$ |
| 58     | $1q^9+2q^{10}+1q^{11}$ | $2q^2$ |
| 59     | $-2q^9-4q^{10}-2q^{11}$ | $2q^2$ |
| 60     | $2q^9+4q^{10}+2q^{11}$ | $2q^2$ |

Figure 5: A relation in $B_4$ continued.
| Number | Coefficient | Monomial |
|--------|-------------|----------|
| 61     | $\frac{1}{2}q^{3}+2q^{1}+1q^{2}$ | 323212   |
| 62     | $\frac{1}{2}q^{3}-2q^{1}+1q^{2}$ | 1212132  |
| 63     | $2q^{2}$   | 1213213  |
| 64     | $\frac{1}{2}q^{3}-2q^{1}+1q^{2}$ | 1213232  |
| 65     | $2q^{2}$   | 1232121  |
| 66     | $\frac{2q^{3}-1q^{1}+2q^{2}}{2q^{1}+2q^{2}}$ | 1232132  |
| 67     | $\frac{16q^{1}}{2q^{3}+12q^{1}+2q^{2}}$ | 1321323  |
| 68     | $\frac{-2q^{2}}{3q^{2}}$ | 1321323  |
| 69     | $\frac{-1q^{3}+2q^{1}-1q^{2}}{2q^{3}+2q^{2}}$ | 2121323  |
| 70     | $\frac{-4q^{3}-8q^{1}-4q^{2}}{2q^{3}+12q^{1}+2q^{2}}$ | 2123212  |
| 71     | $\frac{-16q^{1}}{3q^{3}+12q^{1}+2q^{2}}$ | 2123213  |
| 72     | $\frac{-1q^{3}+2q^{1}-1q^{2}}{2q^{3}+2q^{2}}$ | 2123232  |
| 73     | $\frac{-2q^{3}+4q^{1}-4q^{2}}{3q^{3}+12q^{1}+2q^{2}}$ | 2132123  |
| 74     | $\frac{4q^{3}+8q^{1}+4q^{2}}{2q^{3}+12q^{1}+2q^{2}}$ | 2321232  |

Figure 6: A relation in $B_4$ continued.