On some products taken over the prime numbers

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Abstract
This paper is devoted to study some expressions of the type

\[ \prod_p p^{\lfloor x f(p) \rfloor} \]

where \( x \) is a nonnegative real number, \( f \) is an arithmetic function satisfying some conditions, and the product is over the primes \( p \). We begin by proving that such expressions can be expressed by using the lcm function, without any reference to prime numbers; we illustrate this result with several examples. The rest of the paper is devoted to study the two particular cases related to \( f(m) = m \) and \( f(m) = m - 1 \). In both cases, we found arithmetic properties and analytic estimates for the underlying expressions. We also put forward an important conjecture for the case \( f(m) = m - 1 \), which depends on the counting of the prime numbers of a special form.

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1 Introduction and Notation

Throughout this paper, we let \( \mathbb{N}^* \) denote the set of positive integers and \( \mathcal{P} \) the set of prime numbers. We let \( \text{Card} \mathcal{A} \) denote the cardinal of a given finite set \( \mathcal{A} \). For a given prime number \( p \), we let \( \vartheta_p \) denote the usual \( p \)-adic valuation. For \( x \in \mathbb{R} \), we let \( \lfloor x \rfloor \) denote the integer-part of \( x \). For \( N, b \in \mathbb{N} \), with \( b \geq 2 \), the expansion of \( N \) in base \( b \) is denoted by \( N = a_0 + b a_1 + b^2 a_2 + \cdots + b^k a_k \) (with \( k \in \mathbb{N} \), \( a_0, a_1, \ldots, a_k \in \{0, 1, \ldots, b - 1\} \) and

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In such a context, we let $S_b(N)$ denote the sum of base-$b$ digits of $N$, that is $S_b(N) := a_0 + a_1 + \cdots + a_k$. Further, we let $\pi$ and $\theta$ respectively denote the prime-counting function and the Chebyshev theta function, defined by:

$$
\pi(x) := \sum_{p \text{ prime}} 1 \quad \text{and} \quad \theta(x) := \sum_{p \text{ prime}} \log p \quad (\forall x \in \mathbb{R}^+).
$$

The prime number theorem states that $\pi(x) \sim_{+\infty} \frac{x}{\log x}$. Other equivalent statements are: $\theta(x) \sim_{+\infty} x$ and $\log \text{lcm}(1, 2, \ldots, n) \sim_{+\infty} n$ (see e.g., [3, Chapter 4]). The weaker estimates $\pi(x) = O\left(\frac{x}{\log x}\right)$, $\theta(x) = O(x)$ and $\log \text{lcm}(1, 2, \ldots, n) = O(n)$ are called Chebyshev’s estimates. In section 4, we use extensively Landau’s big $O$ notation which we sometimes specify as follows: if $f$ and $g$ are two real functions, with $g > 0$, defined on some interval $I$ of $\mathbb{R}$, and depending on a parameter $t$, then we write $f = O_{\perp t}(g)$ if there exists a positive constant $M$, not depending on $t$, such that $|f(x)| \leq Mg(x)$ $(\forall x \in I)$.

In number theory, it is common that a prime factorisation of some special numbers $N$ makes appear, as exponents of each prime $p$, expressions of the form $\left\lfloor \frac{n}{p^a} \right\rfloor$ or a sum of such expressions. The most famous example is perhaps the Legendre formula stating that for any natural number $n$, we have

$$
n! = \prod_{p \text{ prime}} p^\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots, \quad (1.1)
$$

which may be also reformulate in terms of base expansions as follows:

$$
n! = \prod_{p \text{ prime}} p^{n-S_b(n)}. \quad (1.2)
$$

(See e.g., [4, pages 76-77]). Another famous example is the formula of the least common multiple of the first consecutive positive integers:

$$
\text{lcm}(1, 2, \ldots, n) = \prod_{p \text{ prime}} p^{\left\lfloor \frac{\log n}{\log p} \right\rfloor} \quad (\forall n \in \mathbb{N}). \quad (1.3)
$$

Among other examples which are less known, we can cite the following

$$
\text{lcm}\left\{i_1i_2\cdots i_k ; k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, i_1+i_2+\cdots+i_k \leq n\right\} = \prod_{p \text{ prime}} p^{\left\lfloor \frac{x}{p} \right\rfloor}, \quad (1.4)
$$

which is pointed out in the book of Cahen and Chabert [1, page 246] and also by Farhi [2] in the context of the integer-valued polynomials. Basing on the remark that in both Formulas (1.1), (1.3) and (1.4), the right-hand side (which is a product taken over the primes) is interpreted without any reference to prime numbers, we may naturally ask if an expression of a general type $\prod_{p \text{ prime}} p^{\left\lfloor \frac{x}{f_i(p)} \right\rfloor + \left\lfloor \frac{x}{f_2(p)} \right\rfloor + \cdots}$ (where $x \in \mathbb{R}^+$ and $(f_i)_i$ is a sequence of positive functions, satisfying some regularity
conditions) possess the same property; that is, it has an interpretation without reference to the primes. In this paper, we study only the case of the products

$$\pi_f(x) := \prod_{p \text{ prime}} p^{\left\lfloor \frac{x}{f(p)} \right\rfloor},$$

for which we answer affirmatively to the previous question (under some hypothesis on $f$). After giving several applications of our result, we focus our study on the two particular cases $f(p) = p$ and $f(p) = p - 1$. Because in both cases, there is no loss of generality to take $x$ an integer, we are led to define for any $n \in \mathbb{N}$:

$$\rho_n := \prod_{p \text{ prime}} p^{\left\lfloor \frac{n}{p} \right\rfloor} \quad \text{and} \quad \sigma_n := \prod_{p \text{ prime}} p^{\left\lfloor \frac{n}{p-1} \right\rfloor}.$$

We begin with the arithmetic study of $\rho_n$ and $\sigma_n$ by establishing several arithmetic properties concerning them; especially, we obtain for $\sigma_n$ a nontrivial divisor and a nontrivial multiple. Moreover, we determine the $p$-adic valuations of the integers $\frac{n}{k}$ when the prime $p$ is large enough compared to $\sqrt{n}$; we discover that the prime numbers of the form $\left\lfloor \frac{n}{k} + 1 \right\rfloor$ ($k \in \mathbb{N}^*$, $k < \sqrt{n+1} + 1$) play a vital role in the arithmetic nature of the $\sigma_n$’s (this phenomenon will pop up again when studying analytically the $\sigma_n$’s). In another direction, we find asymptotic estimates for $\log \rho_n$ and $\log \sigma_n$. However, due to the difficulties encountered in counting the prime numbers of the form $\left\lfloor \frac{n}{k} + 1 \right\rfloor$ (1 \leq k \leq \sqrt{n}$), the optimal estimate of $\log \sigma_n$ is only given conjecturally by leaning on heuristic reasoning. We finally conclude the paper by pointing out the connection between our arithmetic and analytic studies concerning the numbers $\sigma_n$.

2 An expression of $\pi_f$ using the lcm’s

Our stronger result of expressing $\pi_f$ in terms of the lcm’s (without any reference to prime numbers) is the following:

**Theorem 2.1.** Let $f : \mathbb{N}^* \to \mathbb{R}_+$ be an arithmetic function such that $f(\mathbb{N}^* \setminus \{1\}) \subset \mathbb{R}_+$ (i.e., $f$ does not vanish except at 1 eventually). Consider the set $\mathbb{N}^* \setminus \{1\}$ equipped with the partial order relation “divide” and the set $\mathbb{R}_+$ equipped with the usual total order relation “$\leq$”, and suppose that the map:

$$\tilde{f} : \mathbb{N}^* \setminus \{1\} \to \mathbb{R}_+$$

$$n \mapsto \frac{f(n)}{\log n}$$

is nondecreasing with respect to these two orders. Then, we have for any $x \in \mathbb{R}^+$:

$$\prod_{p \text{ prime}} p^{\left\lfloor \frac{x}{\sigma_k} \right\rfloor} = \text{lcm}\left\{i_1i_2\cdots i_k : k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, f(i_1) + f(i_2) + \cdots + f(i_k) \leq x\right\}.$$
In order to present a clean proof of Theorem 2.1, we go through the following lemma:

**Lemma 2.2.** Let \( f : \mathbb{N}^* \to \mathbb{R}_+ \) as in Theorem 2.1. Then, for any prime number \( p \) and any positive integer \( a \), we have

\[ \vartheta_p(a) \leq \frac{f(a)}{f(p)}. \]

**Proof.** Let \( p \) be a prime number and \( a \) be a positive integer. Since the inequality of the lemma is trivial when \( \vartheta_p(a) = 0 \), we may suppose that \( \vartheta_p(a) \geq 1 \); that is \( p \mid a \). Setting \( \alpha = \vartheta_p(a) \), we can write \( a = bp^\alpha \) for some \( b \in \mathbb{N}^* \) with \( p \nmid b \). Thus

\[ \alpha \leq \frac{\log b}{\log p} = \frac{\log(bp^\alpha)}{\log p} = \frac{\log a}{\log p}. \]

(2.1)

Next, the fact that \( p \mid a \) implies (according to our assumptions on \( f \)) that:

\[ \frac{f(p)}{\log p} \leq \frac{f(a)}{\log a}; \]

that is

\[ \frac{\log a}{\log p} \leq \frac{f(a)}{f(p)} \]

(2.2)

Combining (2.1) and (2.2), we get

\[ \alpha \leq \frac{f(a)}{f(p)}, \]

as required. This completes the proof of the lemma.

**Proof of Theorem 2.1.** Let \( x \in \mathbb{R}_+ \) be fixed. For a given prime number \( p \), the \( p \)-adic valuation of the left-hand side of the identity of the theorem is equal to \( \lfloor \frac{x}{f(p)} \rfloor \), while the \( p \)-adic valuation of the right-hand side of the same identity is equal to

\[ \ell_p := \max\{\vartheta_p(i_1i_2\ldots i_k); k \in \mathbb{N}, i_1, \ldots, i_k \in \mathbb{N}^*, f(i_1) + \cdots + f(i_k) \leq x\}. \]

So, we have to show that \( \ell_p = \lfloor \frac{x}{f(p)} \rfloor \) (for any prime number \( p \)). To do so, we are going to prove the two inequalities \( \ell_p \geq \lfloor \frac{x}{f(p)} \rfloor \) and \( \ell_p \leq \lfloor \frac{x}{f(p)} \rfloor \) (where \( p \) is a given prime number).

First, for a given prime number \( p \), let us show that \( \ell_p \geq \lfloor \frac{x}{f(p)} \rfloor \). By considering the particular natural number:

\[ k = \left\lfloor \frac{x}{f(p)} \right\rfloor \]

and the particular positive integers:

\[ i_1 = i_2 = \cdots = i_k = p, \]

we get

\[ f(i_1) + f(i_2) + \cdots + f(i_k) = kf(p) = \left\lfloor \frac{x}{f(p)} \right\rfloor f(p) \leq x. \]
Thus (according to the definition of $\ell_p$):

$$\ell_p \geq \vartheta_p(i_1 i_2 \cdots i_k) = \vartheta_p(p^k) = k = \left\lfloor \frac{x}{f(p)} \right\rfloor,$$

as required.

Now, for a given prime number $p$, let us show that $\ell_p \leq \left\lfloor \frac{x}{f(p)} \right\rfloor$. For any $k \in \mathbb{N}$ and any $i_1, i_2, \ldots, i_k \in \mathbb{N}^*$, with $f(i_1) + f(i_2) + \cdots + f(i_k) \leq x$, we have

$$\vartheta_p(i_1 i_2 \cdots i_k) = \vartheta_p(i_1) + \vartheta_p(i_2) + \cdots + \vartheta_p(i_k) \leq \frac{f(i_1)}{f(p)} + \frac{f(i_2)}{f(p)} + \cdots + \frac{f(i_k)}{f(p)} \leq \frac{x}{f(p)},$$

but since $\vartheta_p(i_1 i_2 \cdots i_k) \in \mathbb{N}$, it follows that:

$$\vartheta_p(i_1 i_2 \cdots i_k) \leq \left\lfloor \frac{x}{f(p)} \right\rfloor.$$

The definition of $\ell_p$ concludes that

$$\ell_p \leq \left\lfloor \frac{x}{f(p)} \right\rfloor,$$

as required. This completes the proof.

Remarks 2.3. Let us put ourselves in the situation of Theorem 2.1.

1. If the map $\tilde{f}$ is nondecreasing in the usual sense (i.e., with respect to the usual orders of the two sets $\mathbb{N}^* \setminus \{1\}$ and $\mathbb{R}^*_+$) then it remains nondecreasing in the sense imposed by Theorem 2.1 (this immediately follows from the implication: $a \mid b \Rightarrow a \leq b$, $\forall a, b \in \mathbb{N}^*$).

2. More generally than the previous item, if the restriction of the map $\tilde{f}$ on $\mathbb{N}^* \setminus \{1, 2\}$ is nondecreasing in the usual sense and $\tilde{f}(2) \leq \tilde{f}(4)$ then $\tilde{f}$ is nondecreasing in the sense imposed by Theorem 2.1.

Now, from Theorem 2.1 we derive the following corollary in which the condition imposed on $f$ is made simpler.

Corollary 2.4. Let $f : \mathbb{N}^* \to \mathbb{R}^*_+$ be an arithmetic function satisfying $f(\mathbb{N}^* \setminus \{1\}) \subset \mathbb{R}^*_+$. Suppose that the map

$$\begin{align*}
\mathbb{N}^* \setminus \{1\} & \to \mathbb{R}^*_+ \\
n & \mapsto \frac{f(n)}{n}
\end{align*}$$

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Section 2. An expression of $\pi_f$ using the lcm’s

is nondecreasing in the usual sense (i.e., with respect to the usual order of $\mathbb{R}$).

Then we have for any $x \in \mathbb{R}_+$:

$$\prod_{p \text{ prime}} p^{\left\lfloor \frac{x}{\log p} \right\rfloor} = \operatorname{lcm}\left\{ i_1 i_2 \cdots i_k ; k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, \quad f(i_1) + f(i_2) + \cdots + f(i_k) \leq x \right\}.$$

Proof. We use Theorem 2.1 together with Item 2 of Remarks 2.3. We remark that $\tilde{f}$ (defined as in Theorem 2.1) is the product of the two functions: $n \mapsto \frac{f(n)}{n}$ (supposed nondecreasing in the usual sense on $\mathbb{N}^* \setminus \{1\}$) and $n \mapsto \frac{x}{\log n}$ (which is nondecreasing on $\mathbb{N}^* \setminus \{1, 2\} = \{3, 4, 5, \ldots\}$, as a simple study of function shows).

So, $\tilde{f}$ is nondecreasing on $\mathbb{N}^* \setminus \{1, 2\}$ in the usual sense. In addition, we have

$$\tilde{f}(2) = \frac{f(2)}{\log 2} = \frac{\frac{2}{\log 2}}{2} \cdot \frac{\frac{4}{\log 4}}{4} \leq \frac{f(4)}{4} \cdot \frac{4}{\log 4}$$

(since $n \mapsto \frac{f(n)}{n}$ is supposed nondecreasing in the usual sense on $\mathbb{N}^* \setminus \{1\}$). That is

$$\tilde{f}(2) \leq \frac{f(4)}{\log 4} = \tilde{f}(4).$$

The conclusion follows from Item 2 of Remarks 2.3 and Theorem 2.1.

Some applications:

1. By applying Theorem 2.1 for $f(m) = \log m$ (which clearly satisfies the required conditions), we obtain that for any $x \in \mathbb{R}_+$, we have

$$\prod_{p \text{ prime}} p^{\left\lfloor \frac{x}{\log p} \right\rfloor} = \operatorname{lcm}\left\{ i_1 i_2 \cdots i_k ; k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, \quad \log i_1 + \log i_2 + \cdots + \log i_k \leq x \right\}$$

$$= \operatorname{lcm}\left\{ i_1 i_2 \cdots i_k ; k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, i_1 i_2 \cdots i_k \leq e^x \right\}$$

$$= \operatorname{lcm}\left\{ 1, 2, \ldots, \lfloor e^x \rfloor \right\}.$$

By taking in particular $x = \log n$ ($n \in \mathbb{N}^*$), we obtain the well-known formula:

$$\prod_{p \text{ prime}} p^{\left\lfloor \frac{\log n}{\log p} \right\rfloor} = \operatorname{lcm}\{1, 2, \ldots, n\} \quad (\forall n \in \mathbb{N}^*).$$

2. By applying Corollary 2.4 for the function $f(m) = m$ (which clearly satisfies the imposed conditions), we obtain in particular that for all $n \in \mathbb{N}$, we have

$$\prod_{p \text{ prime}} p^{\left\lfloor \frac{n}{\log p} \right\rfloor} = \operatorname{lcm}\left\{ i_1 i_2 \cdots i_k ; k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, \quad i_1 + i_2 + \cdots + i_k \leq n \right\}, \quad (2.3)$$

which is already pointed out by Cahen and Chabert [1] and by Farhi [2].
3. (Generalization of (2.3)). Let $\alpha \geq 1$. By applying Corollary 2.4 for the function $f(m) = m^\alpha$ (which clearly satisfies the imposed conditions), we obtain in particular that for all $n \in \mathbb{N}$, we have

$$\prod_{p \text{ prime}} p^{\left\lfloor \frac{n}{p^\alpha} \right\rfloor} = \text{lcm}\left\{i_1i_2\cdots i_k : k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, i_1^\alpha + i_2^\alpha + \cdots + i_k^\alpha \leq n\right\}. \quad (2.4)$$

4. For all $n, k \in \mathbb{N}$, with $n \geq k$, let us define (as in [2]):

$$q_{n,k} := \text{lcm}\left\{i_1i_2\cdots i_k : i_1, i_2, \ldots, i_k \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_k \leq n\right\}. \quad (2.5)$$

Note that these numbers have been already encountered and studied by Farhi [2] in a context relating to the integer-valued polynomials. By applying Corollary 2.4 for the function $f(m) = m - 1$ (which clearly satisfies the imposed conditions), we obtain that for all $n \in \mathbb{N}$, we have

$$\prod_{p \text{ prime}} p^{\left\lfloor \frac{n}{p^{n-1}} \right\rfloor} = \text{lcm}\left\{i_1i_2\cdots i_k : k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, (i_1 - 1) + (i_2 - 1) + \cdots + (i_k - 1) \leq n\right\}$$

$$= \text{lcm}\left\{i_1i_2\cdots i_k : k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_k \leq n + k\right\}$$

$$= \text{lcm}\left\{q_{n+k,k} : k \in \mathbb{N}\right\}, \quad (2.4)$$

which remarkably represents the least common multiple of the $n^{th}$ diagonal of the arithmetic triangle of the $q_{i,j}$’s, beginning as follows (see [2]):

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 6 \quad 2 \quad 1$$
$$1 \quad 12 \quad 12 \quad 2 \quad 1$$
$$1 \quad 60 \quad 12 \quad 12 \quad 2 \quad 1$$
$$1 \quad 60 \quad 360 \quad 24 \quad 12 \quad 2 \quad 1$$
$$1 \quad 420 \quad 360 \quad 360 \quad 24 \quad 12 \quad 2 \quad 1$$

Table 1: The triangle of the $q_{n,k}$’s for $0 \leq k \leq n \leq 7$

For a given $n \in \mathbb{N}$, let $D_n = (d_{n,k})_{k \in \mathbb{N}}$ denote the sequence of the $n^{th}$ diagonal of the above triangle, that is

$$d_{n,k} := q_{n+k,k}$$

$$= \text{lcm}\left\{i_1i_2\cdots i_k : i_1, i_2, \ldots, i_k \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_k \leq n + k\right\} \quad (2.5)$$

$$(\forall k \in \mathbb{N})$$. In order to simplify Formula (2.4), we are going to show that the sequences $D_n$ ($n \in \mathbb{N}$) are all nondecreasing in the divisibility sense and eventually constant. Precisely, we have the following proposition:
Proposition 2.5. For all \( n, k \in \mathbb{N} \), we have
\[
d_{n,k} \text{ divides } d_{n,k+1}.
\]
If in addition \( k \geq n \), then we have
\[
d_{n,k} = d_{n,n}.
\]

Proof. Let \( n, k \in \mathbb{N} \) be fixed and let \( i_1, i_2, \ldots, i_k \in \mathbb{N}^* \) such that \( i_1 + i_2 + \cdots + i_k \leq n + k \). By setting \( i_{k+1} = 1 \), we have \( i_1 + i_2 + \cdots + i_k + i_{k+1} \leq n + k + 1 \); thus (by \( 2.3 \)) \( d_{n,k+1} \) is a multiple of \( i_1i_2 \cdots i_{k+1} = i_1i_2 \cdots i_k \). Since this holds for any \( i_1, i_2, \ldots, i_k \in \mathbb{N}^* \) such that \( i_1 + i_2 + \cdots + i_k \leq n + k \), we derive that \( d_{n,k+1} \) is a multiple of \( d_{n,k} \), as required.

Now, let us prove the second part of the proposition. So, suppose that \( k \geq n \) and let us prove that \( d_{n,k} = d_{n,n} \). It follows from an immediate induction leaning on the result of the first part of the proposition (proved above) that \( d_{n,n} \mid d_{n,k} \). So, it remains to prove that \( d_{n,k} \mid d_{n,n} \). Let \( i_1, i_2, \ldots, i_k \in \mathbb{N}^* \) such that \( i_1 + i_2 + \cdots + i_k \leq n + k \). Let \( \ell \in \mathbb{N} \) denote the number of indices \( i_r \) (\( 1 \leq r \leq k \)) which are equal to 1; so we have exactly \( (k - \ell) \) indices \( i_r \) which are \( \geq 2 \). Thus we have
\[
i_1 + i_2 + \cdots + i_k \geq \ell + 2(k - \ell) = 2k - \ell.
\]
But since \( i_1 + i_2 + \cdots + i_k \leq n + k \), we derive that \( 2k - \ell \leq n + k \), which gives \( \ell \geq k - n \). This proves that we have at least \( (k - n) \) indices \( i_r \) which are equal to 1. By assuming (without loss of generality) that those indices are \( i_{n+1}, i_{n+2}, \ldots, i_k \) (i.e., \( i_{n+1} = i_{n+2} = \cdots = i_k = 1 \)), we get
\[
i_1i_2\cdots i_n = i_1i_2\cdots i_k
\]
and
\[
i_1 + i_2 + \cdots + i_n = (i_1 + i_2 + \cdots + i_k) - (k - n)
\leq (n + k) - (k - n)
= 2n.
\]
This shows that each product \( i_1i_2\cdots i_k \) occurring in the definition of \( d_{n,k} \) reduces (by permuting the \( i_r \)'s and eliminate those of them which are equal to 1) to a product \( j_1j_2\cdots j_n \) which occurs in the definition of \( d_{n,n} \). Consequently \( d_{n,k} \mid d_{n,n} \), as required. This completes the proof of the proposition.

Using Proposition 2.5 we have for any \( n \in \mathbb{N} \):
\[
\text{lcm}\{q_{n+k,k} ; k \in \mathbb{N}\} = \text{lcm}\{d_{n,k} ; k \in \mathbb{N}\}
= d_{n,n}
= \text{lcm}\{i_1i_2\cdots i_n ; i_1, i_2, \ldots, i_n \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_n \leq 2n\}.
\]
This concludes to the following interesting corollary, simplifying Formula (2.4):
Corollary 2.6. For any $n \in \mathbb{N}$, we have
\[
\prod_{p \text{ prime}} p^{\text{ord}_p(n)} = \text{lcm}\{i_1i_2 \cdots i_n ; i_1, i_2, \ldots, i_n \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_n \leq 2n\}. \quad \square
\]

3 Arithmetic results on the numbers $\rho_n$ and $\sigma_n$

A certain number of arithmetic properties concerning the numbers $\rho_n$ and $\sigma_n$ are either immediate or quite easy to prove. We have gathered them in the following proposition:

Proposition 3.1. For any natural number $n$, we have

(i) $\rho_n \mid \rho_{n+1}$, $\sigma_n \mid \sigma_{n+1}$, and $\rho_n \mid \sigma_n$;

(ii) $\rho_n \mid n!$;

(iii) $n! \mid \sigma_n$ and $\sigma_n \mid (2n)!$;

(iv) $\sigma_{2n+1} = 2\sigma_{2n}$.

Proof. Let $n \in \mathbb{N}$ be fixed. The properties of Item (i) are trivial. The property of Item (ii) immediately follows from the Legendre formula providing the decomposition of $n!$ into a product of prime factors. For Item (iii), the fact that $n! \mid \sigma_n$ follows from the inequality:

\[
\frac{n}{p-1} = \frac{n}{p} + \frac{n}{p^2} + \cdots \geq \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots
\]

together with the Legendre formula. Next, to prove that $\sigma_n \mid (2n)!$, we use Corollary 2.6. For any $i_1, i_2, \ldots, i_n \in \mathbb{N}^*$ satisfying $i_1 + i_2 + \cdots + i_n \leq 2n$, we have that $i_1i_2 \cdots i_n \mid i_1!i_2! \cdots i_n! \mid (i_1 + i_2 + \cdots + i_n)! \mid (2n)!$. Thus $\text{lcm}\{i_1i_2 \cdots i_n ; i_1, i_2, \ldots, i_n \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_n \leq 2n\} \mid (2n)!$; that is (according to Corollary 2.6): $\sigma_n \mid (2n)!$.

Let us finally prove Item (iv). First, we have $\vartheta_2(\sigma_{2n+1}) = \left\lfloor \frac{2n+1}{2} \right\rfloor = 2n + 1$ and $\vartheta_2(2\sigma_{2n}) = 1 + \vartheta_2(\sigma_{2n}) = 1 + \left\lfloor \frac{2n}{2-1} \right\rfloor = 2n + 1$; hence $\vartheta_2(\sigma_{2n+1}) = \vartheta_2(2\sigma_{2n})$. Next, for any odd prime $p$, since the odd number $(2n + 1)$ cannot be a multiple of the even number $(p - 1)$ then we have

\[
\left\lfloor \frac{2n + 1}{p-1} \right\rfloor = \left\lfloor \frac{2n}{p-1} \right\rfloor;
\]

that is

\[
\vartheta_p(\sigma_{2n+1}) = \vartheta_p(\sigma_{2n}) = \vartheta_p(2\sigma_{2n}).
\]

Consequently, we have $\vartheta_q(\sigma_{2n+1}) = \vartheta_q(2\sigma_{2n})$ (for any prime number $q$), concluding that $\sigma_{2n+1} = 2\sigma_{2n}$, as required. This completes the proof of the proposition. \quad \square
In the following proposition, we shall improve Item (iii) of Proposition 3.1. It appears that this improvement is optimal (understanding that we use uniquely simple expressions).

**Proposition 3.2.** For any natural number \( n \), we have

\[
(n + 1)! \mid \sigma_n \text{ and } \sigma_n \mid n! \text{lcm}(1, 2, \ldots, n, n+1).
\]

**Proof.** Let \( n \in \mathbb{N} \) be fixed. We have to show that for any prime \( p \), we have

\[
\vartheta_p ((n + 1)!) \leq \vartheta_p (\sigma_n) \leq \vartheta_p (n! \text{lcm}(1, 2, \ldots, n, n+1)). \tag{3.1}
\]

Let \( p \) be a fixed prime number and let us prove (3.1). By setting \( e \) the largest prime number satisfying \( p^e \leq n + 1 \), we have that \( \vartheta_p (n!) = \sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \) (according to the Legendre formula), \( \vartheta_p (\sigma_n) = \left\lfloor \frac{n}{p-1} \right\rfloor \) (by definition of \( \sigma_n \)), and \( \vartheta_p (\text{lcm}(1, 2, \ldots, n, n+1)) = e \). So (3.1) reduces to

\[
\sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \leq \left\lfloor \frac{n}{p-1} \right\rfloor \leq \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor + e. \tag{3.2}
\]

On the one hand, we have

\[
\sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \leq \sum_{i=1}^{e} \frac{n+1}{p^i} = \frac{n+1}{p-1} \left(1 - \frac{1}{p^e}\right) \leq \frac{n}{p-1}
\]

(since \( p^e \leq n + 1 \)). But since \( \sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \) is an integer, we derive that

\[
\sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \leq \left\lfloor \frac{n}{p-1} \right\rfloor,
\]

confirming the left inequality in (3.2). On the other hand, by leaning on the refined inequality \( \left\lfloor \frac{n}{b} \right\rfloor \geq \frac{n}{b} - 1 \), which holds for any positive integers \( a, b \), we have

\[
\left\lfloor \frac{n}{p-1} \right\rfloor - \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor \leq \frac{n}{p-1} - \sum_{i=1}^{e} \left( \frac{n+1}{p^i} - 1 \right) = \frac{n}{p-1} - \frac{n+1}{p-1} \left(1 - \frac{1}{p^e}\right) + e = \frac{1}{p-1} \left( \frac{n+1}{p^e} - 1 \right) + e.
\]

But from the definition of \( e \), we have \( p^{e+1} > n + 1 \), that is \( \frac{n+1}{p^e} < p \). By reporting this into the last estimate, we get

\[
\left\lfloor \frac{n}{p-1} \right\rfloor - \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor < e + 1.
\]

Next, since \( \left\lfloor \frac{n}{p-1} \right\rfloor - \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor \in \mathbb{Z} \), we conclude to

\[
\left\lfloor \frac{n}{p-1} \right\rfloor - \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor \leq e,
\]

confirming the right inequality of (3.2). This completes this proof. \( \square \)
From Proposition\ref{thm:asymptotic_estimate} we derive an asymptotic estimate for the number $\log \sigma_n$ when $n$ tends to infinity. We have the following

**Corollary 3.3.** We have

$$\log \sigma_n \sim_{+\infty} n \log n.$$  

**Proof.** According to Proposition\ref{thm:asymptotic_estimate} we have for any $n \in \mathbb{N}^*$:

$$\log (n+1)! \leq \log \sigma_n \leq \log (n!) + \log \text{lcm}(1, 2, \ldots, n, n+1).$$

Then the asymptotic estimate of the corollary follows from the facts: $\log (n+1)! \sim_{+\infty} \log(n!) \sim_{+\infty} n \log n$ (according to Stirling’s formula) and $\log \text{lcm}(1, 2, \ldots, n, n+1) \sim_{+\infty} n$ (according to the prime number theorem). \hfill \Box

Note that the asymptotic estimate of the above corollary will be specified in §4.

We now turn to establish a result evaluating the $p$-adic valuations of the positive integers $\frac{\sigma_n}{n!}$ ($n \in \mathbb{N}^*$) for sufficiently large prime numbers. We discover as a remarkable phenomenon that primes of a special type play a vital role. We find again this phenomenon in §4 when estimating asymptotically $\log \sigma_n$. We have the following theorem:

**Theorem 3.4.** Let $n$ be a positive integer and $p$ be a prime number such that:

$$\sqrt{n+1} < p \leq n+1.$$  

Then, we have

$$\vartheta_p \left( \frac{\sigma_n}{n!} \right) \in \{0, 1\}.$$  

Besides, the equality $\vartheta_p \left( \frac{\sigma_n}{n!} \right) = 1$ holds if and only if $p$ has the form

$$p = \left\lfloor \frac{n}{k} + 1 \right\rfloor,$$

with $k \in \mathbb{N}^*$ and $k < \sqrt{n+1} + 1$.

**Proof.** By the definition of $\sigma_n$ and the Legendre formula\ref{eq:legendre}, we have that

$$\vartheta_p \left( \frac{\sigma_n}{n!} \right) = \vartheta_p (\sigma_n) - \vartheta_p (n!)$$

$$= \left\lfloor \frac{n}{p-1} \right\rfloor - \frac{n - S_p(n)}{p-1}$$

$$= \left\lfloor \frac{n}{p-1} - \frac{n - S_p(n)}{p-1} \right\rfloor$$

(since $\frac{n - S_p(n)}{p-1} = \vartheta_p(n!) \in \mathbb{Z}$)

$$= \left\lfloor \frac{S_p(n)}{p-1} \right\rfloor.$$  

(3.3)
The first part of the theorem is then equivalent to the fact $\frac{S_p(n)}{p-1} \in \{0, 1\}$. So, let us prove this last fact. The hypothesis on $p$ insures that $n < p^2 - 1$, which implies that the representation of the positive integer $n$ in base $p$ has the form $n = a_1a_0(p)$, with $a_0, a_1 \in \{0, 1, \ldots, p-1\}$ and $(a_0, a_1) \neq (p-1, p-1)$. Consequently, we have $S_p(n) = a_0 + a_1 < 2(p-1)$, implying that $\frac{S_p(n)}{p-1} < 2$; hence $\frac{S_p(n)}{p-1} \in \{0, 1\}$, as required. This achieves the proof of the first part of the theorem. Now, let us prove the second part of the theorem.

- Suppose that $\vartheta_p \left( \frac{a_n}{n!} \right) = 1$ and let us show the existence of $k \in \mathbb{N}^*$, with $k < \sqrt{n+1} + 1$ such that $p = \left\lfloor \frac{n}{k} + 1 \right\rfloor$. As seen above, the representation of $n$ in base $p$ has the form $n = a_2a_1a_0(p) = a_0 + pa_1$, where $a_0, a_1 \in \{0, 1, \ldots, p-1\}$ and $(a_0, a_1) \neq (p-1, p-1)$. We will show that $k = a_1 + 1$ is suitable. According to (3.3), we have $\vartheta_p \left( \frac{a_n}{n!} \right) = \left\lfloor \frac{S_p(n)}{p-1} \right\rfloor = \left\lfloor \frac{a_0 + a_1}{p-1} \right\rfloor$. So the supposition $\vartheta_p \left( \frac{a_n}{n!} \right) = 1$ implies that $\frac{a_0 + a_1}{p-1} \geq 1$, that is $a_0 + a_1 \geq p - 1$. This last inequality together with $a_0 < p$ imply that

$$p - 1 \leq \frac{a_0 + a_1p}{a_1 + 1} < p,$$

which is equivalent to

$$\left\lfloor \frac{n}{a_1 + 1} \right\rfloor = p - 1.$$

Thus

$$p = \left\lfloor \frac{n}{a_1 + 1} + 1 \right\rfloor.$$

Besides, we have $a_1 = \left\lfloor \frac{n}{p} \right\rfloor \leq \frac{n}{p} < \sqrt{n+1}$ (since $p > \sqrt{n+1} > \frac{n}{\sqrt{n+1}}$). Thus $k = a_1 + 1$ satisfy the required properties (i.e., $p = \left\lfloor \frac{n}{k} + 1 \right\rfloor$ and $k < \sqrt{n+1} + 1$).

- Conversely, suppose that there exists $k \in \mathbb{N}^*$, with $k < \sqrt{n+1} + 1$, such that $p = \left\lfloor \frac{n}{k} + 1 \right\rfloor$, and let us show that $\vartheta_p \left( \frac{a_n}{n!} \right) = 1$. Setting $a_0 := n - (k-1)p$ and $a_1 := k - 1$, we first show that the representation of $n$ in base $p$ is $n = a_1a_0(p)$. Since it is immediate that $n = a_0 + pa_1$, it just remains to prove that $a_0, a_1 \in \{0, 1, \ldots, p-1\}$. Since $k < \sqrt{n+1} + 1 < p + 1$ then $k - 1 < p$; that is $a_1 \in \{0, 1, \ldots, p-1\}$. Next, since $p = \left\lfloor \frac{n}{k} + 1 \right\rfloor$ then

$$p \leq \frac{n}{k} + 1 < p + 1,$$

implying that

$$p - k \leq n - (k-1)p < p,$$

that is

$$p - k \leq a_0 < p.$$

But $p - k = (p-1) - a_1 \geq 0$; thus $a_0 \in \{0, 1, \ldots, p-1\}$. We have confirmed that the representation of $n$ in base $p$ is $n = a_1a_0(p)$. Consequently, we have (according to (3.3)):

$$\vartheta_p \left( \frac{a_n}{n!} \right) = \left\lfloor \frac{S_p(n)}{p-1} \right\rfloor = \left\lfloor \frac{a_0 + a_1}{p-1} \right\rfloor = \left\lfloor \frac{n - (k-1)(p-1)}{p-1} \right\rfloor = \left\lfloor \frac{n}{p-1} \right\rfloor - k + 1.$$
Then, since $\frac{n}{p-1} \geq k$ (because $\frac{n}{k} + 1 \geq \lceil \frac{n}{k} + 1 \rceil = p$), it follows that $\vartheta_p \left( \frac{a_p}{m^2} \right) \geq 1$.

But since $\vartheta_p \left( \frac{a_p}{m^2} \right) \in \{0, 1\}$ (according to the first part, already proved, of the theorem), we conclude that $\vartheta_p \left( \frac{a_p}{m^2} \right) = 1$, as required. This completes the proof of the theorem. $\square$

4 Analytic estimates of the numbers $\log \rho_n$ and $\log \sigma_n$

Throughout this section, we let $c$ denote the absolute positive constant given by:

$$c := \sum_{p \text{ prime}} \frac{\log p}{p(p-1)} = 0.755 \ldots$$

Our goal is to find asymptotic estimates for $\log \rho_n$ and $\log \sigma_n$ as $n$ tends to infinity. The obtained main results are the following:

**Theorem 4.1.** We have

$$\log \rho_n = n \log n - (c + 1)n + O \left( \sqrt{n} \right).$$

**Theorem 4.2.** We have

$$\log \sigma_n = n \log n - n + O \left( \sqrt{n \log n} \right).$$

**Conjecture 4.3** (improving Theorem 4.2). We have

$$\log \sigma_n = n \log n - n + O \left( \sqrt{n} \right).$$

Note that an explanation for the validity of Conjecture 4.3 is given latter; actually, it depends on a conjecture on counting the prime numbers of a certain form, which is heuristically plausible. To establish the above results, we need the following auxiliary results:

**Lemma 4.4.** For any $x \geq 1$, we have

$$\sum_{p \text{ prime}, p > x} \log p \frac{p}{p(p-1)} = O \left( \frac{1}{x} \right).$$

**Proof.** Since $\frac{\log p}{p(p-1)} \leq 2 \frac{\log p}{p}$ (for any prime number $p$), then it suffices to show that $\sum_{p \text{ prime}, p > x} \frac{\log p}{p^2} = O \left( \frac{1}{x} \right)$. According to the Abel summation formula (see e.g., [3, Proposition 1.4]), we have for any positive real numbers $x, y$, with $x < y$:

$$\sum_{p \text{ prime}, \frac{1}{x} < p \leq \frac{1}{y}} \frac{\log p}{p^2} = \left( \sum_{p \text{ prime}, \frac{1}{x} < p \leq \frac{1}{y}} \frac{\log p}{p} \right) \frac{1}{y^2} - \int_{x}^{y} \left( \sum_{p \text{ prime}, \frac{1}{x} < p \leq t} \frac{\log p}{p^2} \right) \frac{1}{t^2} \, dt$$

$$= \frac{\theta(y) - \theta(x)}{y^2} + 2 \int_{x}^{y} \frac{\theta(t) - \theta(x)}{t^3} \, dt.$$
Then, by setting $y$ to infinity, it follows (since $\theta(y) = O(y)$) that:

$$\sum_{p \text{ prime}} \frac{\log p}{p^2} = 2 \int_x^{+\infty} \frac{\theta(t) - \theta(x)}{t^3} dt = 2 \int_x^{+\infty} \frac{\theta(t) - \theta(x)}{t^3} dt - \frac{\theta(x)}{x^2}. $$

Using finally $\theta(t) = O(t)$, we get

$$\sum_{p \text{ prime}} \frac{\log p}{p^2} = O\left(\int_x^{+\infty} \frac{dt}{t^2}\right) + O\left(\frac{1}{x}\right) = O\left(\frac{1}{x}\right),$$

as required. The proof is complete. \hfill $\Box$

Lemma 4.4 above is used in the proof of the following proposition:

**Proposition 4.5.** For any positive integer $n$, we have

$$\sum_{p \text{ prime}} \left(\left\lfloor \frac{n}{p^2}\right\rfloor + \left\lfloor \frac{n}{p^3}\right\rfloor + \ldots\right) \log p = c \cdot n + O\left(\sqrt{n}\right).$$

**Proof.** Let $n$ be a fixed positive integer. For any prime number $p$, let $e_p$ denote the greatest natural number satisfying $p^{e_p} \leq n$; explicitly $e_p = \left\lfloor \frac{\log n}{\log p}\right\rfloor$. So we have $p^{e_p+1} > n$. On the one hand, we have

$$\sum_{p \text{ prime}} \left(\frac{n}{p^2} + \frac{n}{p^3} + \ldots\right) \log p \leq \sum_{p \text{ prime}} \left(\frac{n}{p^2} + \frac{n}{p^3} + \ldots\right) \log p$$

$$= \sum_{p \text{ prime}} \frac{n}{p(p-1)} \log p;$$

that is

$$\sum_{p \text{ prime}} \left(\frac{n}{p^2} + \frac{n}{p^3} + \ldots\right) \log p \leq c \cdot n. \quad (4.1)$$

On the other hand, we have (according to the definition of the $e_p$’s):

$$\sum_{p \text{ prime}} \left(\frac{n}{p^2} + \frac{n}{p^3} + \ldots\right) \log p = \sum_{p \text{ prime}} \left(\frac{n}{p^2} + \frac{n}{p^3} + \ldots\right) \log p$$

$$\geq \sum_{p \text{ prime}, p \leq \sqrt{n}} \left(\frac{n}{p^2} - 1\right) + \left(\frac{n}{p^3} - 1\right) + \ldots + \left(\frac{n}{p^{e_p}} - 1\right) \log p$$

$$= n \sum_{p \text{ prime}, p \leq \sqrt{n}} \left(\frac{1}{p^2} + \frac{1}{p^3} + \ldots + \frac{1}{p^{e_p}}\right) \log p - \sum_{p \text{ prime}, p \leq \sqrt{n}} (e_p - 1) \log p$$

$$= n \sum_{p \text{ prime}, p \leq \sqrt{n}} \left(\frac{1}{p(p-1)} - \frac{1}{p^{e_p}(p-1)}\right) \log p - \sum_{p \text{ prime}, p \leq \sqrt{n}} (e_p - 1) \log p$$

$$= n \sum_{p \text{ prime}, p \leq \sqrt{n}} \frac{\log p}{p(p-1)} - n \sum_{p \text{ prime}, p \leq \sqrt{n}} \frac{\log p}{p^{e_p}(p-1)} - \sum_{p \text{ prime}, p \leq \sqrt{n}} (e_p - 1) \log p$$

$$= n \left(c - \sum_{p \text{ prime}, p > \sqrt{n}} \frac{\log p}{p(p-1)}\right) - n \sum_{p \text{ prime}, p \leq \sqrt{n}} \frac{\log p}{p^{e_p}(p-1)} - \sum_{p \text{ prime}, p \leq \sqrt{n}} (e_p - 1) \log p;$$
that is
\[
\sum_{p \text{ prime}} \left( \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots \right) \log p \geq c \cdot n - n \sum_{p \text{ prime} \atop p > \sqrt{n}} \frac{\log p}{p(p-1)} - n \sum_{p \text{ prime} \atop p \leq \sqrt{n}} \frac{\log p}{p^{p^2}(p-1)} - \sum_{p \text{ prime} \atop p \leq \sqrt{n}} (e_p - 1) \log p. \quad (4.2)
\]

But, by using Lemma 4.3, we have
\[
\sum_{p \text{ prime} \atop p > \sqrt{n}} \frac{\log p}{p(p-1)} = O \left( \frac{1}{\sqrt{n}} \right).
\quad (4.3)
\]

Next, by using the fact \( p^{p^2} > \frac{n}{p} \) (for any prime \( p \)), we have
\[
\sum_{p \text{ prime} \atop p \leq \sqrt{n}} \frac{\log p}{p^{p^2}(p-1)} < \frac{1}{n} \sum_{p \text{ prime} \atop p \leq \sqrt{n}} \frac{p}{p-1} \log p \leq \frac{2}{n} \sum_{p \text{ prime} \atop p \leq \sqrt{n}} \log p = \frac{2}{n} \theta(\sqrt{n}) = O \left( \frac{1}{\sqrt{n}} \right),
\quad (4.4)
\]

and by using the fact \( e_p - 1 < e_p := \left\lfloor \frac{\log n}{\log p} \right\rfloor \leq \frac{\log n}{\log p} \), we have
\[
\sum_{p \text{ prime} \atop p \leq \sqrt{n}} (e_p - 1) \log p < \sum_{p \text{ prime} \atop p \leq \sqrt{n}} \log n = (\log n)\pi(\sqrt{n}) = O \left( \sqrt{n} \right).
\quad (4.5)
\]

Then, by inserting \((4.3), (4.4)\) and \((4.5)\) into \((4.2)\), we get
\[
\sum_{p \text{ prime}} \left( \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots \right) \log p \geq c \cdot n + O \left( \sqrt{n} \right).
\quad (4.6)
\]

Finally, \((4.1)\) and \((4.6)\) conclude to
\[
\sum_{p \text{ prime}} \left( \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots \right) \log p = c \cdot n + O \left( \sqrt{n} \right),
\]

as required.

We are now able to prove Theorem 4.1.

**Proof of Theorem 4.1.** For any sufficiently large integer \( n \), we have according to Legendre’s formula:
\[
\log \rho_n = \sum_{p \text{ prime}} \left\lfloor \frac{n}{p} \right\rfloor \log p = \sum_{p \text{ prime}} \left( \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \ldots \right) \log p
\]
\[
- \sum_{p \text{ prime}} \left( \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots \right) \log p
\]
\[
= \log(n!) - \sum_{p \text{ prime}} \left( \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots \right) \log p.
\]
Then, the weaker form of Stirling’s approximation formula \(\log(n!) = n \log n - n + O(\log n)\) and Proposition 4.5 conclude to:

\[
\log \rho_n = n \log n - (c + 1)n + O(\sqrt{n}),
\]
as required.

We now turn to estimate \(\log \sigma_n\) by leaning on the estimate of \(\log \rho_n\) (given by Theorem 4.1 proved above). To do so, we shall first establish an important formula relating \(\log \rho_n\) and \(\log \sigma_n\). This is done with the following proposition:

**Proposition 4.6.** For any positive integer \(n\), we have

\[
\log \rho_n = \sum_{k=1}^{n} \theta \left( \frac{n}{k} \right), \tag{4.7}
\]
\[
\log \sigma_n = \sum_{k=1}^{n} \theta \left( \frac{n}{k} + 1 \right), \tag{4.8}
\]
\[
\log \sigma_n - \log \rho_n = \sum_{1 \leq k \leq n, \left( \frac{n}{k} + 1 \right) \text{ is prime}} \log \left( \frac{n}{k} + 1 \right). \tag{4.9}
\]

**Proof.** Let \(n\) be a fixed positive integer. We have

\[
\log \rho_n = \sum_{p \text{ prime}} \left\lfloor \frac{n}{p} \right\rfloor \log p = \sum_{p \text{ prime}} \left( \sum_{1 \leq k \leq \frac{n}{p}} 1 \right) \log p = \sum_{1 \leq k \leq n} \left( \sum_{p \text{ prime} \atop p \leq \frac{n}{k}} \log p \right) = \sum_{1 \leq k \leq n} \theta \left( \frac{n}{k} \right),
\]
proving (4.7). Similarly, we have

\[
\log \sigma_n = \sum_{p \text{ prime}} \left\lfloor \frac{n}{p-1} \right\rfloor \log p = \sum_{p \text{ prime}} \left( \sum_{1 \leq k \leq \frac{n}{p-1}} 1 \right) \log p = \sum_{1 \leq k \leq n} \left( \sum_{p \text{ prime} \atop p \leq \frac{n}{k} + 1} \log p \right) = \sum_{1 \leq k \leq n} \theta \left( \frac{n}{k} + 1 \right),
\]
proving (4.8). Finally, using (4.7) and (4.8), let us prove (4.9). We have

\[
\log \sigma_n - \log \rho_n = \sum_{1 \leq k \leq n} \theta \left( \frac{n}{k} + 1 \right) - \sum_{1 \leq k \leq n} \theta \left( \frac{n}{k} \right)
\]
\[
= \sum_{1 \leq k \leq n} \left( \theta \left( \frac{n}{k} + 1 \right) - \theta \left( \frac{n}{k} \right) \right)
\]
\[
= \sum_{1 \leq k \leq n} \left( \sum_{p \text{ prime} \atop \frac{n}{k} < p \leq \frac{n}{k} + 1} \log p \right).
\]
But since for any $1 \leq k \leq n$, the interval $\left(\frac{n}{k}, \frac{n}{k} + 1 \right]$ contains a unique integer which is $\lfloor \frac{n}{k} + 1 \rfloor$, it follows that:

$$\log \sigma_n - \log \rho_n = \sum_{1 \leq k \leq n} \log \left\lfloor \frac{n}{k} + 1 \right\rfloor,$$

proving (4.9). The proof is complete. $\square$

To deduce an asymptotic estimate for $\log \sigma_n$ from that of $\log \rho_n$ by means of Proposition 4.6, we must estimate (asymptotically) the sum

$$S(n) := \sum_{1 \leq k \leq n} \log \left\lfloor \frac{n}{k} + 1 \right\rfloor.$$

To do so, we split $S$ into two sums:

$$S_1(n) := \sum_{1 \leq k \leq \sqrt{n}} \log \left\lfloor \frac{n}{k} + 1 \right\rfloor \quad \text{and} \quad S_2(n) := \sum_{\sqrt{n} < k \leq n} \log \left\lfloor \frac{n}{k} + 1 \right\rfloor.$$

The estimate of $S_2(n)$ is within our reach; it is given by the following proposition:

**Proposition 4.7.** For any positive integer $n$, we have

$$S_2(n) = c_n + O \left(\sqrt{n}\right).$$

**Proof.** For a given $n \in \mathbb{N}^*$, we have

$$S_2(n) := \sum_{\sqrt{n} < k \leq n \atop \lfloor \frac{n}{k} + 1 \rfloor \text{ is prime}} \log \left\lfloor \frac{n}{k} + 1 \right\rfloor = \sum_{p \text{ prime}} \sum_{\sqrt{n} < k \leq n \atop \lfloor \frac{n}{k} + 1 \rfloor = p} \log p.$$

But since for any prime number $p$ satisfying $p < \sqrt{n} + 1$ and any integer $k$ satisfying $\sqrt{n} < k \leq n$, we have

$$\left\lfloor \frac{n}{k} + 1 \right\rfloor = p \iff p \leq \frac{n}{k} + 1 < p + 1 \iff \frac{n}{p} < k \leq \frac{n}{p - 1},$$

it follows that:

$$S_2(n) = \sum_{p \text{ prime}} \sum_{\sqrt{n} < k \leq n \atop \lfloor \frac{n}{k} + 1 \rfloor = p} \log p = \sum_{p \text{ prime}} \left( \sum_{\max \left(\sqrt{n}, \frac{n}{p}\right) < k \leq \frac{n}{p - 1}} 1 \right) \log p. \quad (4.10)$$

Next, we remark that for any prime number $p$ satisfying $p < \sqrt{n} + 1$, we have

$$\frac{n}{p} > \frac{n}{\sqrt{n} + 1} > \sqrt{n} - 1,$$

implying that $\frac{n}{p} \leq \max \left(\sqrt{n}, \frac{n}{p}\right) < \frac{n}{p + 1}$. Consequently, the interval $\left(\max \left(\sqrt{n}, \frac{n}{p}\right), \frac{n}{p - 1}\right]$ contains at least

$$\frac{n}{p - 1} - \max \left(\sqrt{n}, \frac{n}{p}\right) - 1 > \frac{n}{p - 1} - \frac{n}{p} - 2 = \frac{n}{p(p-1)} - 2$$

integers and at most
\[ \frac{n}{p-1} - \max \left( \sqrt{n}, \frac{n}{p} \right) + 1 \leq \frac{n}{p-1} - \frac{n}{p} + 1 = \frac{n}{p(p-1)} + 1 \text{ integers. So, for any prime number } p \text{ satisfying } p < \sqrt{n} + 1, \text{ we have} \]

\[
\sum_{\max(\sqrt{n}, \frac{n}{p}) < k \leq \frac{n}{p-1}} 1 = \frac{n}{p(p-1)} + O_{\perp p}(1). 
\]

By reporting this into (4.10), we get

\[ S_2(n) = \sum_{p \text{ prime}} \sum_{p < \sqrt{n} + 1} \frac{n}{p(p-1)} + O_{\perp p}(1) \log p 
\]

\[ = \left( \sum_{p \text{ prime}} \frac{\log p}{p(p-1)} \right) n + O \left( \theta(\sqrt{n} + 1) \right) 
\]

\[ = \left( c - \sum_{p \text{ prime}} \frac{\log p}{p(p-1)} \right) n + O \left( \theta(\sqrt{n} + 1) \right) . 
\]

But since

\[
\sum_{p \text{ prime}} \frac{\log p}{p(p-1)} \leq 2 \sum_{p \text{ prime}} \frac{\log p}{p^2} = O \left( \frac{1}{\sqrt{n}} \right) \quad \text{(according to Lemma 4.4)}
\]

and

\[
\theta(\sqrt{n} + 1) = O\left( \sqrt{n} \right) \quad \text{(according to the Chebyshev estimates)},
\]

we conclude to

\[ S_2(n) = c n + O \left( \sqrt{n} \right), \]

as required. \qed

We now turn to estimate the crucial sum \( S_1(n) \). To facilitate the task, we begin by estimating \( S_1(n) \) in terms of the cardinality of a specific set of prime numbers. For any \( n \in \mathbb{N}^* \), we set

\[ \mathcal{A}_n := \left\{ \left\lfloor \frac{n}{k} + 1 \right\rfloor ; k \in \mathbb{N}^*, k \leq \sqrt{n} \right\} \cap \mathcal{P} \]

(in other words, \( \mathcal{A}_n \) is the set of prime numbers having the form \( \left\lfloor \frac{n}{k} + 1 \right\rfloor \), where \( k \leq \sqrt{n} \) is a positive integer). Above all, it is important to note that for any \( n \in \mathbb{N}^* \), the positive integers \( \left\lfloor \frac{n}{k} + 1 \right\rfloor \) \( (k \in \mathbb{N}^*, k \leq \sqrt{n}) \) (appearing in the definition of \( \mathcal{A}_n \)) are pairwise distinct. Indeed, for all \( k, \ell \in \mathbb{N}^* \), with \( k \leq \sqrt{n}, \ell \leq \sqrt{n} \), and \( k \neq \ell \), we have

\[
\left| \left\lfloor \frac{n}{k} + 1 \right\rfloor - \left\lfloor \frac{n}{\ell} + 1 \right\rfloor \right| = \frac{n|\ell - k|}{k\ell} \geq \frac{n}{k\ell} \geq 1,
\]

18
implying that:

\[ \left\lfloor \frac{n}{k} + 1 \right\rfloor \neq \left\lfloor \frac{n}{\ell} + 1 \right\rfloor. \]

It follows from this fact that \( \mathcal{A}_n \) (\( n \in \mathbb{N}^* \)) has the same cardinality with the set of positive integers \( k \) satisfying \( k \leq \sqrt{n} \) and for which \( \left\lfloor \frac{n}{k} + 1 \right\rfloor \) is prime. That is

\[ \text{Card} \mathcal{A}_n = \sum_{1 \leq k \leq \sqrt{n}} 1. \] (4.11)

The estimate of \( S_1(n) \) in terms of \( \text{Card} \mathcal{A}_n \) (\( n \in \mathbb{N}^* \)) is given by the following proposition:

**Proposition 4.8.** We have

\[ S_1(n) = (\text{Card} \mathcal{A}_n) \cdot O(\log n). \]

**Proof.** Let \( n \in \mathbb{N}^* \) be fixed. From the obvious double inequality

\[ \sqrt{n} < \left\lfloor \frac{n}{k} + 1 \right\rfloor \leq n + 1 \quad (\forall k \in \mathbb{N}^*, \text{ with } k \leq \sqrt{n}), \]

we have that

\[ \frac{1}{2} \log(n) \cdot \sum_{1 \leq k \leq \sqrt{n}} 1 \leq S_1(n) \leq \log(n + 1) \cdot \sum_{1 \leq k \leq \sqrt{n}} 1; \]

\[ \left\lfloor \frac{n}{k} + 1 \right\rfloor \text{ is prime} \]

\[ \left\lfloor \frac{n}{\ell} + 1 \right\rfloor \text{ is prime} \]

which immediately implies (taking into account (4.11)):

\[ S_1(n) = (\text{Card} \mathcal{A}_n) \cdot O(\log n), \]

as required. \( \square \)

By combining Formula (4.9) of Proposition 4.6, Theorem 4.1, Proposition 4.7, and Proposition 4.8, we immediately derive the following proposition:

**Proposition 4.9.** We have

\[ \log \sigma_n = n \log n - n + O \left( \sqrt{n} \right) + (\text{Card} \mathcal{A}_n) \cdot O(\log n). \] \( \square \)

At this point, the whole problem is now to estimate \( \text{Card} \mathcal{A}_n \) (\( n \in \mathbb{N}^* \)). First, let us do it heuristically. For \( n \in \mathbb{N}^* \), since the numbers constituting the set \( \left\{ \left\lfloor \frac{n}{k} + 1 \right\rfloor ; k \in \mathbb{N}^*, k \leq \sqrt{n} \right\} \) (of cardinality \( \lfloor \sqrt{n} \rfloor \)) do not satisfy (apparently) any particular congruence, we may conjecture with considerable confidence that the quantity prime numbers in it (i.e., \( \text{Card} \mathcal{A}_n \)) is \( O \left( \sqrt{n} \log \sqrt{n} \right) = O \left( \frac{\sqrt{n}}{\log n} \right) \). We precisely make the following
Conjecture 4.10. There exist two positive absolute constants \( \alpha \) and \( \beta \) (with \( \alpha < \beta \)) such that for any sufficiently large positive integer \( n \), we have
\[
\alpha \left( \frac{\sqrt{n}}{\log n} \right) \leq \text{Card} \mathcal{A}_n \leq \beta \left( \frac{\sqrt{n}}{\log n} \right).
\]

The proof of Conjecture 4.3 through Conjecture 4.10 is then immediate:

Proof of Conjecture 4.3 through Conjecture 4.10. It suffices to insert the estimate \( \text{Card} \mathcal{A}_n = O \left( \frac{\sqrt{n}}{\log n} \right) \) (provided by Conjecture 4.10) into the estimate of Proposition 4.9.

Unfortunately, we were unable to confirm Conjecture 4.10, so the best result we have achieved is Theorem 4.2. Before setting out the proof of that theorem, it is important to note that the trivial estimate \( \text{Card} \mathcal{A}_n = O \left( \frac{\sqrt{n}}{\log n} \right) \) leads us (through Proposition 4.9) to the estimate:
\[
\log \sigma_n = n \log n - n + O \left( \sqrt{n} \log n \right).
\]

We shall improve the later by counting more intelligently the elements of \( \mathcal{A}_n \). We have the following

Proposition 4.11. We have
\[
\text{Card} \mathcal{A}_n = O \left( \frac{\sqrt{n}}{\log n} \right).
\]

Proof. Let \( n \) be a fixed sufficiently large positive integer and let \( t \) be a real parameter with \( 1 \leq t \leq \sqrt{n} \) (we will choose \( t \) later in terms of \( n \) in order to optimize the result). We have (according to (4.11))
\[
\text{Card} \mathcal{A}_n = \sum_{1 \leq k \leq \sqrt{n}} 1
\]
\[
= \sum_{1 \leq k \leq \sqrt{n}, \left\lceil \frac{n}{k} \right\rceil + 1} 1 + \sum_{\sqrt{n} < k \leq \sqrt{n}, \left\lceil \frac{n}{k} \right\rceil + 1} 1. \quad (4.12)
\]

Next, we have on the one hand:
\[
\sum_{1 \leq k \leq \sqrt{n}, \left\lceil \frac{n}{k} \right\rceil + 1} 1 \leq \frac{\sqrt{n}}{t}, \quad (4.13)
\]
and on the other hand:
\[
\sum_{\sqrt{n} < k \leq \sqrt{n}, \left\lceil \frac{n}{k} \right\rceil + 1} 1 = \text{Card}\{ p \text{ prime} ; p \leq \sqrt{n}t + 1 \} \quad \text{(by putting } p = \left\lceil \frac{n}{k} + 1 \right\rceil \}
\]
\[
= \pi \left( \sqrt{n}t + 1 \right)
\]
\[
= O \left( \frac{\sqrt{n}t}{\log(\sqrt{n}t)} \right) \quad \text{(according to the Chebyshev estimate)},
\]

\[\]
implying (since $1 \leq t \leq \sqrt{n}$) that:

$$
\sum_{\sqrt{n} < k \leq \sqrt{n}} 1 = O \left( \frac{\sqrt{n} t}{\log n} \right).
$$  \hfill (4.14)

By inserting (4.13) and (4.14) into (4.12), we get

$$
\text{Card } \mathcal{A}_n = O \left( \sqrt{\frac{n}{t}} \right) + O \left( \sqrt{\frac{n}{\log n}} \right).
$$

To obtain an optimal result, we must take $t = O \left( \sqrt{\log n} \right)$, providing

$$
\text{Card } \mathcal{A}_n = O \left( \sqrt{\frac{n}{\log n}} \right),
$$

as required.

We are finally ready to prove Theorem 4.2.

\textbf{Proof of Theorem 4.2.} It suffices to insert the estimate of Proposition 4.11 into that of Proposition 4.9. \hfill \Box

\section{Concluding remarks about the connection between the arithmetic and the analytic studies}

In this section, we briefly explain how we can derive our asymptotic estimates concerning $\log \sigma_n$ (i.e., Theorem 4.2 and Conjecture 4.3) rather from our arithmetic study. For the sequel, we let $c_1, c_2, c_3, \text{ etc.}$ denote suitable absolute constants greater than 1. For any $n \in \mathbb{N}^*$, we can write:

$$
\frac{\sigma_n}{n!} = \prod_{\text{prime } p \leq \sqrt{n} + 1} p^{\vartheta_p \left( \frac{\sigma_n}{n!} \right)} \cdot \prod_{\text{prime } p \leq n+1} p^{\vartheta_p \left( \frac{\sigma_n}{n!} \right)}.
$$  \hfill (5.1)

For the primes $p \leq \sqrt{n + 1}$, we estimate $\vartheta_p \left( \frac{\sigma_n}{n!} \right)$ as follows: let $n = a_r a_{r-1} \ldots a_0(p)$ be the representation of $n$ in base $p$ ($r \in \mathbb{N}, a_0, a_1, \ldots, a_r \in \{0, 1, \ldots, p-1\}$, and $a_r \neq 0$). Then we have (according to the Legendre formula (1.2)):

$$
\vartheta_p \left( \frac{\sigma_n}{n!} \right) = \vartheta_p (\sigma_n) - \vartheta_p (n!) = \left[ \frac{n}{p-1} \right] - \frac{n - S_p(n)}{p-1} \leq \frac{S_p(n)}{p-1} \leq \frac{(p-1)(r+1)}{p-1} = r + 1.
$$

Thus

$$
p^{\vartheta_p \left( \frac{\sigma_n}{n!} \right)} \leq p^{r+1} \leq pn.
$$
Consequently, we have
\[
\prod_{\substack{p \text{ prime} \leq \sqrt{n+1} \leq \sqrt{n+1}}} p^{\vartheta_p \left( \frac{\sigma_n}{n!} \right)} \leq \left( \prod_{p \text{ prime} \leq \sqrt{n+1}} p \right) \cdot n^{\pi(\sqrt{n+1})} \leq c_1 \sqrt{n} \tag{5.2}
\]
(according to the Chebyshev estimates).

However, for the primes \( p > \sqrt{n+1} \), we estimate \( \vartheta_p \left( \frac{\sigma_n}{n!} \right) \) by using Theorem 3.4 saying us that for any such prime \( p \), we have \( \vartheta_p \left( \frac{\sigma_n}{n!} \right) \in \{0, 1\} \) and \( \vartheta_p \left( \frac{\sigma_n}{n!} \right) = 1 \) if and only if \( p \in \mathcal{A}_n' \), where
\[
\mathcal{A}_n' := \left\{ \left\lfloor \frac{n}{k} + 1 \right\rfloor ; k \in \mathbb{N}^*, k < \sqrt{n+1} + 1 \right\} \cap \mathcal{P}.
\]
(Note the resemblance between \( \mathcal{A}_n' \) and \( \mathcal{A}_n \)). So we have
\[
\prod_{\substack{p \text{ prime} \leq \sqrt{n+1} \leq \sqrt{n+1}}} p^{\vartheta_p \left( \frac{\sigma_n}{n!} \right)} = \prod_{p \in \mathcal{A}_n'} p.
\]

By using the results of \( \S 4 \) concerning \( \text{Card} \mathcal{A}_n \) (almost the same with \( \text{Card} \mathcal{A}_n' \)), we easily derive that the quantity \( \prod p^{\vartheta_p \left( \frac{\sigma_n}{n!} \right)} \), where the product is over the primes \( p \) satisfying \( \sqrt{n+1} < p \leq n+1 \), is conjecturally bounded below by \( c_2 \sqrt{n} \) and bounded above by \( c_3 \sqrt{n \log n} \) (or conjecturally by \( c_3 \sqrt{n} \)). By inserting this together with (5.2) into (5.1), we conclude that \( \frac{\sigma_n}{n!} \) is conjecturally bounded below by \( c_2 \sqrt{n} \) and bounded above by \( c_4 \sqrt{n \log n} \) (or conjecturally by \( c_4 \sqrt{n} \)). Notice that this is exactly what obtained in \( \S 4 \). By this approach, the constant \( c \) (present in the analytic approach when estimating \( \log \rho_n \) and then eliminated when estimating \( \log \sigma_n \)) does remarkably not appear!

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