A Probabilistic Characterization of g-Harmonic Functions

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Abstract
This paper gives a definition of g-harmonic functions and shows the relation between the g-harmonic functions and g-martingales. It’s direct to construct such relation under smooth case, but for continuous case we need the theory of viscosity solution. The results show that under the nonlinear expectation mechanism, we also can get the similar relation between harmonic functions and martingales. Finally, we will give a result about the strict converse problem of mean value property of g-harmonic functions.

Key words or phrases: BSDE, g-martingale, g-harmonic function, nonlinear Feynman-Kac formula, viscosity solution.

1 Introduction and Preliminary
Harmonic function ($\Delta u = 0$) has a probabilistic interpretation as that if $\Delta u = 0$ on $\mathbb{R}^n$, then $u(B^x_t)$ is a martingale for any $x \in \mathbb{R}^n$ (see for example [6]). This relation between martingale and harmonic function connects probability with potential analysis. It helps us to give probabilistic characterizations for harmonic function and more generalized X-harmonic function(see [6]). In 1997, Peng([9]) introduced the notions of g-expectation and conditional g-expectation via a backward stochastic differential equation(BSDE) with a generator function $g$. Further, Peng([10]) introduced the notion of g-martingale which provided an heuristic tool to characterize some kind of harmonic function described by elliptic operator with a nonlinear term $g$ by which we will define g-harmonic function. In this paper, with the help of nonlinear Feynman-Kac formula established from BSDE(see for example [8]), we will give a probabilistic characterization of the g-harmonic function.

Now we state our problem in detail. Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by an n-dimensional Brow-
ian motion \(\{B_t^n\}_{t \geq 0}\). i.e.

\[ F_t = \sigma\{B_s : s \leq t\}. \]

Then we can define a g-martingale by an \(F_t\)-adapted process \(\{y_t\}_{t \geq 0}\) which satisfies the following BSDE for any \(0 \leq s \leq t\):

\[ y_s = y_t + \int_s^t g(y_r, z_r)dr - \int_s^t z_r dB_r. \quad (1) \]

Here \(g : R \times R^n \rightarrow R\), satisfies the condition:

[H1]. \(g(y, 0) \equiv 0\) and the Lipschitz condition: \(\exists C > 0\), for any \((y_1, z_1), (y_2, z_2) \in R \times R^n\) we have

\[ |g(y_1, z_1) - g(y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|). \]

And the equality (1) also can be formulated simply as [11]:

\[ \mathcal{E}^g_{s,t}(y_t) := y_s. \]

Then we can also get the definition of g-super(sub)martingale when

\[ \mathcal{E}^g_{s,t}(y_t) \leq (\geq) y_s. \]

This definition derives from the definition of g-expectation in the beginning paper Peng[9]. When \(g(y, z) \equiv 0\) the g-expectation is actually the classical expectation. Except that g-expectation is nonlinear in general, it holds many other important properties as its classical counterpart [2][4][10][12].

Given an n-dimensional Itô diffusion process \(\{X_t^x\}_{t \geq 0}\):

\[ dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dB_t, \quad (2) \]

\[ X_0^x = x \in R^n, \]

where \(b(x) : R^n \rightarrow R^n\), \(\sigma(x) : R^n \rightarrow R^{n \times n}\) satisfy the Lipschitz condition: \(\exists C > 0\) s.t.

\[ |b(x_1) - b(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leq C|x_1 - x_2| \quad \forall x_1, x_2 \in R^n, \]

our problem is that: what kind of function \(u(x) : R^n \rightarrow R\) satisfies that \(u(X_t^x)\) is a g-martingale for any \(x \in R^n\)?

This problem also has its classical counterpart:

First if \(\{X_t^x\}\) is just the Brownian motion \(\{B_t^x\}\), then we have the result that when \(u(x)\) is harmonic on \(R^n\) i.e.

\[ \Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{for any } x \in R^n, \]

the process \(u(B_t^x)\) is a martingale for any \(x\). And conversely if \(u(x)\) satisfies that \(u(B_t^x)\) is a martingale for any \(x\), then \(u(x)\) must be harmonic on \(R^n\). The proof
may have many editions, here we can give a sketch of one which may induce the extension to g-martingale case.

If \( u(x) \) is harmonic on \( \mathbb{R}^n \), then we use Itô formula to \( u(B^x_t) \) and get

\[
du(B^x_t) = \sum_i \frac{\partial u}{\partial x_i}(B^x_t) dB^x_{i,t} + \frac{1}{2} \sum_i \frac{\partial^2 u}{\partial x^2_i}(B^x_t) dt
\]

Then we get \( u(B^x_t) \) is a martingale for any \( x \in \mathbb{R}^n \). Conversely if \( u(x) \) is continuous on \( \mathbb{R}^n \) and for any \( x \in \mathbb{R}^n \), \( u(B^x_t) \) is a martingale, then we have

\[
E[u(B^x_\tau)] = u(x) \quad \text{for any stopping time } \tau.
\]

Particularly for any sphere \( S(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \} \), we have

\[
u(x) = E\left[u(B^x_{\tau_{S(x,r)}})\right] = \int_{\partial S(x,r)} u(y) d\sigma_y
\]

where \( \tau_{S(x,r)} \) is the exit time of \( \{B^x_t\} \) from the sphere \( S(x, r) \), i.e.

\[
\tau_{S(x,r)} = \inf\{ t > 0 : |B^x_t - x| \geq r \},
\]

and \( \sigma_y \) is the harmonic measure on the \( \partial S(x, r) \). Then from the familiar converse of the mean value property for harmonic function, we can get \( u(x) \) must be harmonic function.

Further we can extend the Brownian motion \( \{B^x_t\} \) to the general diffusion process \( \{X^x_t\} \):

If \( u(x) \in C^2_0(\mathbb{R}^n) \) and satisfies

\[
\sum_i b_i \frac{\partial u}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j} \sigma_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0, \tag{3}
\]

then we have \( u(X^x_t) \) is a martingale for any \( x \). The proof also uses the Itô formula. But conversely if \( u(X^x_t) \) is a martingale for any \( x \), we can’t conclude that \( u(x) \) is smooth. Then with additional assumption \( u(x) \in C^2_0(\mathbb{R}^n) \) we can get that \( u(x) \) satisfies the PDE (3) (see [6]).

Then naturally we will ask that what happens when we substitute the expectation mechanism by the g-expectation mechanism. First we will define the infinitesimal generator:

**Definition 1.** Let

\[
\mathcal{A}_g^X f(x) := \lim_{t \downarrow 0} \frac{\mathcal{E}^g_{0,t}[f(X^x_t)] - f(x)}{t}, \tag{4}
\]

then we call \( \mathcal{A}_g^X \) the infinitesimal generator of a diffusion process \( \{X^x_t\} \) under g-expectations.
Thanks to the celebrating nonlinear Feynman-Kac formula \cite{8}, we can get the explicit form of $A_X^g f$ when $f \in C^2_0(\mathbb{R}^n)$ by considering the following type of quasilinear parabolic PDE:

\[
\begin{aligned}
\partial_t u - \mathcal{L} u(t, x) - g(u(t, x), u_x(t, x) \sigma(x)) &= 0, \\
u(0, x) &= f(x).
\end{aligned}
\] (5)

Where

\[
\mathcal{L} u(t, x) = \sum_i b_i \frac{\partial u}{\partial x_i}(t, x) + \frac{1}{2} \sum_{i,j} (\sigma \sigma^\tau)_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(t, x).
\] (6)

When $f \in C^2_0(\mathbb{R}^n)$, we assert that

\[
u(t, x) = \mathcal{E}^g_{0,t}[f(X^x_t)]
\] (7)

is the solution of PDE\cite{8}. Then under the case $t = 0$, we get

\[
A_X^g f(x) = \mathcal{L} f(x) + g(f(x), f_x(x) \sigma(x)).
\] (8)

Then we finish the preliminary and we can introduce our main results. In section 2, we give a characterization of $g$-harmonic function under smooth case. In section 3, we characterize it under continuous case, where the differential operator is interpreted as viscosity solution. In section 4, we will investigate the strict converse problem of mean value property of $g$-harmonic function evoked by its classical counterpart \cite{7}.

## 2 Smooth Case

The equality \cite{8} has the implication about the relation between the $g$-martingales and the $g$-harmonic functions when $f \in C^2_0(\mathbb{R}^n)$. In fact, the left side of \cite{8} is related to a $g$-martingale and the right side is related to a harmonic PDE. At first we will give the definition of $g$-harmonic functions:

**Definition 2.** Let $f \in C^2_0(\mathbb{R}^n)$. We call it a $g$-(super)harmonic function w.r.t. \{${X^x_t}$\} if it satisfies

\[
A_g^X f(x)(\leq) = 0 \quad \text{for any} \quad x \in \mathbb{R}^n.
\] (9)

Then we suffice to construct the relation between the $g$-supermartingales and the $g$-superharmonic functions.

**Theorem 1.** If $f(x) \in C^2_0(\mathbb{R}^n)$, then the following assertions are equivalent:

1. $f(x)$ is a $g$-superharmonic function.
2. \{${f(X^x_t)}$\} is a $g$-supermartingale for any $x \in \mathbb{R}^n$. 

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Proof. (i) (1) ⇒ (2): 
If $f \in C^2(R^n)$, by Itô formula, we can get $f(X_t^x)$ is still an Itô diffusion process:

$$f(X_t^x) = f(X_s^x) + \int_s^t \mathcal{L}f(X_r^x)dr + \int_s^t f_x(X_r^x)\sigma(X_r^x)dB_r, \quad 0 \leq s \leq t.$$ 

and then we insert the term $g(f(X_r^x), f_x\sigma(X_r^x))$ and get:

$$f(X_t^x) = f(X_s^x) - \int_s^t \mathcal{L}f(X_r^x)dr - \int_s^t f_x\sigma(X_r^x)dB_r$$

$$= f(X_t^x) + \int_s^t g(f(X_r^x), f_x\sigma(X_r^x))dr - \int_s^t f_x\sigma(X_r^x)dB_r$$

$$- \int_s^t \{\mathcal{L}f(X_r^x) + g(f(X_r^x), f_x\sigma(X_r^x))\}dr$$

$f(x)$ is a g-superharmonic function, so:

$$\mathcal{L}f(X_t^x) + g(f(X_t^x), f_x\sigma(X_t^x)) = \mathcal{A}^X_g f(X_t^x) \leq 0.$$ 

And then according to the comparison theory of BSDE(cf.[10]), we can get \{f(X_t^x)\} is a g-supermartingale.

(ii) (2) ⇒ (1):
By the definition of the $\mathcal{A}^X_g$: 

$$\mathcal{A}^X_g f(x) = \lim_{t \downarrow 0} \frac{\mathcal{E}_{0,t}^g [f(X_t^x)] - f(x)}{t}.$$ 

\{f(X_t^x)\} is a g-supermartingale, so:

$$\mathcal{E}_{0,t}^g [f(X_t^x)] - f(x) \leq 0,$$

then

$$\mathcal{A}^X_g f(x) \leq 0.$$ 

So we get $f(x)$ is a g-superharmonic function. 

3 Continuous Case

If we generalize the requirement of function $f(x)$ to be only continuous on $R^n$, how we get a function $f$ which satisfies that $f(X_t^x)$ is a g-martingale for any $x \in R^n$? With the help of viscosity solution(cf. [3]) we can also refer to the quasi-linear second order PDEs. Here we need a lemma due to Peng[8].

Lemma 1. Let $0 \leq t \leq T$ and

$$u(t, x) = \mathcal{E}_{0,T-t}^g [f(X_{T-t}^x)].$$
Then $u(t, x)$ is the viscosity solution of the following PDE on $(0, T) \times \mathbb{R}^n$:

$$
\begin{cases}
\frac{\partial u}{\partial t} + Lu(t, x) + g(u(t, x), u_x(t, x)\sigma(x)) = 0 \\
u(T, x) = f(x)
\end{cases}
$$

(10)

Here $g(y, z)$ and $f(x)$ satisfy:

(H2). Let $F(u, p) = g(u, p\sigma(x))$, then $\exists C > 0$ s.t.

$$
|F(u, p)| \leq C(1 + |u| + |p|);
$$

$$
|D_uF(u, p)|, |D_pF(u, p)| \leq C,
$$

and (H3). $f(x)$ is a continuous function with a polynomial growth at infinity.

**Definition 3.** Let $u(t, x) \in C(\mathbb{R} \times \mathbb{R}^n)$, $u(t, x)$ is said to be a viscosity super-solution (resp. sub-solution) of the following PDE (11):

$$
\frac{\partial u}{\partial t} + Lu(t, x) + g(u(t, x), u_x(t, x)\sigma(x)) = 0,
$$

(11)

if for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and $\varphi \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ such that $\varphi(t, x) = u(t, x)$ and $(t, x)$ is a maximum (resp. minimum) point of $\varphi - u$,

$$
\frac{\partial \varphi}{\partial t}(t, x) + L\varphi(t, x) + g(\varphi(t, x), \varphi_x(t, x)\sigma(x)) \leq 0.
$$

(resp. $\frac{\partial \varphi}{\partial t}(t, x) + L\varphi(t, x) + g(\varphi(t, x), \varphi_x(t, x)\sigma(x)) \geq 0$.)

$u(t, x)$ is said to be a viscosity solution of PDE (11) if it is both a viscosity super- and sub-solution of (11).

We also consider the viscosity solution of the following type of quasilinear elliptic PDE (12):

$$
Lu(x) + g(u(x), u_x(x)\sigma(x)) = 0.
$$

(12)

We can directly get an relation between the two solutions of (11) and (12):

**Lemma 2.** Let $\bar{u}(t, x) = u(x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, then we have:

$\bar{u}(t, x)$ is the viscosity super-(sub-)solution of PDE (11) $\Leftrightarrow$ $u(x)$ is the viscosity super-(sub-)solution of PDE (12).

Proof. We only prove the case of viscosity super-solution. The "sub-" case is an immediate conclusion of the "super-" case.

(i). "$\Rightarrow"$

For any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, and a function $\varphi(x) \in C^2(\mathbb{R}^n)$ which satisfies $\varphi(x) \leq u(x), \varphi(x_0) = u(x_0)$, we define $\tilde{\varphi}(t, x) = \varphi(x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Then

$$
\frac{\partial \tilde{\varphi}}{\partial t} = 0, \quad \tilde{\varphi}(t_0, x_0) = \bar{u}(t_0, x_0), \quad \tilde{\varphi}(t, x) \leq \bar{u}(t, x),
$$

and

\begin{align*}
\frac{\partial \tilde{\varphi}}{\partial t} + L\tilde{\varphi}(t, x) + g(\tilde{\varphi}(t, x), \tilde{\varphi}_x(t, x)\sigma(x)) &= 0 \\
\tilde{\varphi}(T, x) &= f(x)
\end{align*}

(11)
and due to the assumption that $\tilde{u}(t, x)$ is the viscosity super-solution of PDE (11), we have
\[
\frac{\partial \tilde{\varphi}}{\partial t}(t_0, x_0) + L\tilde{\varphi}(t_0, x_0) + g(\tilde{\varphi}(t_0, x_0), \varphi_x(t_0, x_0)\sigma(x_0)) \leq 0,
\]
i.e.
\[
L\varphi(x_0) + g(\varphi(x_0), \varphi_x(x_0)\sigma(x_0)) \leq 0.
\]
So $u(x)$ is the viscosity super-solution of PDE (12).

(ii). " $\Rightarrow$ ":

For any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, and a function $\varphi(t, x) \in C^2(\mathbb{R} \times \mathbb{R}^n)$ which satisfies $\varphi(t, x) \leq \tilde{u}(t, x)$, $\varphi(t_0, x_0) = \tilde{u}(t_0, x_0)$, then
\[
\frac{\partial \varphi}{\partial t}(t_0, x_0) = 0,
\]
and due to the assumption that $u(x)$ is the viscosity super-solution of PDE (12), we have
\[
L\varphi(t_0, x_0) + g(\varphi(t_0, x_0), \varphi_x(t_0, x_0)\sigma(x_0)) \leq 0.
\]
Combined with (13), we get
\[
\frac{\partial \varphi}{\partial t}(t_0, x_0) + L\varphi(t_0, x_0) + g(\varphi(t_0, x_0), \varphi_x(t_0, x_0)\sigma(x_0)) \leq 0.
\]
So $\tilde{u}(t, x)$ is the viscosity super-solution of PDE (11).

Then we can introduce our main result of this section:

**Theorem 2.** We have the following two consequences:

(i). For any $f(x) \in C(\mathbb{R}^n)$, and $g(y, z)$ satisfying (H1), if $\forall x \in \mathbb{R}^n$, $f(X^x_t)$ is a $g$-supermartingale, then $f(x)$ is a viscosity super-solution of PDE (12).

(ii). For any $f(x)$ satisfying (H3), and $g(y, z)$ satisfying (H1) (H2), let $f(x)$ is a viscosity super-solution of PDE (12), then $\{f(X^x_t)\}$ is a $g$-supermartingale for all $x \in \mathbb{R}^n$.

Actually, the consequence (ii) is the answer of our main problem and the consequence (i) is the converse of it. But (i) is easier to be proved, so we are going to prove (i) at first:

**Proof.** (i).

For any $x \in \mathbb{R}^n$, let $\varphi \in C^2(\mathbb{R}^n)$, $\varphi(x) = f(x)$ where $x$ is a maximum point of $\varphi - f$. It means $\forall \hat{x} \in \mathbb{R}^n$, we have $\varphi(\hat{x}) \leq f(\hat{x})$. Then from (8), we get
\[
L\varphi(x) + g(\varphi(x), \varphi_x(x)\sigma(x)) = A^X_\varphi(x) = \lim_{t \downarrow 0} \frac{E^\mathbb{P}_t[\varphi(X^x_t)] - \varphi(x)}{t} = \lim_{t \downarrow 0} \frac{E^\mathbb{P}_t[\varphi(X^x_t)] - f(x)}{t}.
\]
According to the comparison theory of BSDE, we get
\[ \mathcal{E}_t^g[\varphi(X_t^x)] \leq \mathcal{E}_t^g[f(X_t^x)], \]
and with the assumption \( \{f(X_t^x)\} \) is a g-supermartingale, we can get:
\[ \mathcal{E}_t^g[\varphi(X_t^x)] - f(x) \leq \mathcal{E}_t^g[f(X_t^x)] - f(x) \leq 0. \]
Then
\[ \mathcal{A}_t^X \varphi(x) = \lim_{\epsilon \downarrow 0} \frac{\mathcal{E}_t^g[\varphi(X_t^x)] - f(x)}{\epsilon} \leq 0. \]
i.e.
\[ \mathcal{L}\varphi(x) + g(\varphi(x), \varphi_x(x)\sigma(x)) \leq 0. \]
By definition, it means \( f(x) \) is a viscosity super-solution of PDE (12).

(ii). We want to prove \( \{f(X_t^x)\} \) is a g-supermartingale for any \( x \in \mathbb{R}^n \). It means that we need to prove \( \forall x \in \mathbb{R}^n \) and \( \forall 0 \leq s \leq t \), we have
\[ \mathcal{E}_s^g[f(X_t^x)] \leq f(X_s^x). \]
Under the assumption, in fact \( b(x), \sigma(x) \) and \( g(y, z) \) are all independent of time \( t \), so we can get the Markovian property of \( \mathcal{E}_{s,t}^g \), i.e.
\[ \mathcal{E}_{s,t}^g[f(X_t^x)] = \mathcal{E}_{t-s}^g[f(X_{t-s}^y)|y=x^t]. \]
Then we get an equivalence relation:
\[ \{f(X_t^x)\} \text{ is a g-(super)martingale for any } x \in \mathbb{R}^n \iff \mathcal{E}_t^g[f(X_t^x)] = (\leq) f(x) \text{ for any } t \geq 0 \text{ and } x \in \mathbb{R}^n. \quad (14) \]
So we suffice to prove the latter assertion.

For any \( T \geq 0 \), the assumption \( f(x) \) is a viscosity super-solution of PDE (12) implies that \( \check{f}(t, x) := f(x) \) is a viscosity super-solution to the following PDE:
\[ \begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u(t, x) + g(u(t, x), u_x(t, x)\sigma(x)) = 0, \\ u(T, x) = f(x), \end{cases} \quad (15) \]
according to lemma 2. And with the help of lemma 11,\[ u(t, x) = \mathcal{E}_{0,T-t}^g[f(X_{T-t}^x)] \]
is actually the viscosity solution of PDE (15). Moreover by the maximum principle of the viscosity solution (see [11]), we can get:
\[ u(t, x) \leq \check{f}(t, x) \quad \text{for any } 0 \leq t \leq T. \]
Especially, we have
\[ u(0, x) \leq \check{f}(0, x), \]
i.e.
\[ \mathcal{E}_0^g[f(X_0^x)] \leq f(x). \]
Corollary 1. (i). For any \( f(x) \in C(\mathbb{R}^n) \), and \( g(y, z) \) satisfying (H1), if \( \forall x \in \mathbb{R}^n \), \( f(X^x_t) \) is a g-martingale, then \( f(x) \) is a viscosity solution of PDE(12).

(ii). For any \( f(x) \) satisfying (H3), and \( g(y, z) \) satisfying (H1) (H2), let \( f(x) \) is a viscosity solution of PDE(12), then \( \{f(X^x_t)\} \) is a g-martingale for all \( x \in \mathbb{R}^n \).

It is an immediate consequence from the theorem 2.

4 Strict Converse of Mean Value Property

For classical harmonic function, many generalized results of the converse problem of mean value property have been investigated (cf. [5][7]). In [7], Øksendal and Stroock give a technique to solve a strict converse of the mean value property for harmonic functions. Now we will generalize it to the case of g-harmonic function. Here the strictness means that for each \( x \in \mathbb{R}^n \) we don’t need justify that for any stopping time \( \tau \) whether \( \mathbb{E}^g_{0,\tau}(f(X^x_\tau)) \) equals \( f(x) \). We only need to justify one appropriate stopping time of each \( x \).

In the sequel we put \( \Delta(x, r) = \{y \in \mathbb{R}^n; |y - x| < r\} \) for any \( x \in \mathbb{R}^n \) and \( r > 0 \). Let \( \tau_U = \inf\{t > 0; X^x_t \in U^c\} \) for any open set \( U \). And we suppose the operator (6) is elliptic on \( \mathbb{R}^n \).

Theorem 3. \( f(x) \) is a local bounded continuous function on \( \mathbb{R}^n \). If for any \( x \in \mathbb{R}^n \), there exists a radius \( r(x) \), the mean value property holds:

\[
\mathbb{E}^g_{0,\tau_x}[f(X^x_{\tau_x})] = f(x) \quad \text{here} \quad \tau_x = \tau_{\Delta(x, r(x))},
\]

(16)

And \( r(x) \) is a measurable function of \( x \) and satisfies that for each \( x \), there exists a bounded open set \( U_x \), \( x \in U_x \) and moreover \( r(y), \ y \in U_x \) should satisfy the following two conditions:

\[
0 \leq r(y) \leq \text{dist}(y, \partial U_x),
\]

(17)

and

\[
\inf \{r(y); y \in K\} > 0
\]

(18)

for all closed subsets \( K \) of \( U_x \) with \( \text{dist}(K, \partial U_x) > 0 \). Then we can get:

(i). For each \( y \in U_x \) the mean value property holds on the boundary:

\[
\mathbb{E}^g_{0,\tau_y}[f(X^y_{\tau_y})] = f(y) \quad \text{here} \quad \tau_y = \inf\{t > 0; X^y_t \in U^c_x\}.
\]

And furthermore:

(ii). \( f(x) \) is the viscosity solution of PDE(12).

Proof. (i) \( \Rightarrow \) (ii) is also based on the nonlinear Feynman-Kac formula for elliptic PDE(cf. [5]). So we sufficiently prove the first conclusion.

For each \( y \in U_x \), we define a sequence of stopping times \( \tau_k \) for \( \{X^y_t\} \) by induction as follows:

\[
\tau_0 \equiv 0
\]

\[
\tau_k = \inf\{t \geq \tau_{k-1}; |X^y_t - X^y_{\tau_{k-1}}| \geq r(X^y_{\tau_{k-1}})\}; \ k \geq 1.
\]
By the mean property, and the strong markovian property we can get
\[
\mathcal{E}_{0,\tau_k}^g[f(Y_{\tau_k})] = \mathcal{E}_{0,\tau_k-1}^g[\mathcal{E}_{\tau_k-1,\tau_k}^g[f(X_{\tau_k}^y)]]
\]
\[
= \mathcal{E}_{0,\tau_k-1}^g[\mathcal{E}_{0,\tau_k-\tau_k-1}^g[f(X_{\tau_k-\tau_k-1}^y)]]
\]
\[
= \mathcal{E}_{0,\tau_k-1}^g[f(X_{\tau_k-1}^y)],
\]
then by induction we get
\[
\mathcal{E}_{0,\tau_k}^g[f(Y_{\tau_k})] = f(y).
\]
In the following we will prove \( \tau_k \to \tau_y \) a.e. when \( k \to \infty \). Obviously
\[
\tau_k \geq \tau_{k-1},
\]
so there exists a stopping time \( \tau \) s.t. \( \tau_k \uparrow \tau \). If \( \tau \neq \tau_y \), then there exists \( \epsilon > 0 \) s.t.
\[
\text{dist}(X_{\tau_k}^y, \partial U_x) \geq \epsilon \quad \text{for any } k.
\]
Let \( \tau_k = r(X_{\tau_k}^y) \), according to the condition, we get there exists \( r > 0 \),
\[
\tau_k \geq r \quad \text{for any } k.
\]
It means
\[
\text{dist}(X_{\tau_k}^y, X_{\tau_k-1}^y) \geq r.
\]
And since \( X_t^y \) is continuous, then \( \tau_k \to \infty \), which implies \( \tau_y = \infty \). So
\[
P(\tau_k \text{ don’t converge to } \tau_y) \leq P(\tau_y = \infty).
\]
But for \( f \) is elliptic and \( U_x \) is bounded, we have \( P(\tau_y < \infty) = 1 \). So
\[
P(\tau_k \text{ converge to } \tau_y) = 1.
\]
Then we get
\[
f(y) = \mathcal{E}_{0,\tau_k}^g[f(Y_{\tau_k})]
\]
\[
= \lim_{k \to \infty} \mathcal{E}_{0,\tau_k}^g[f(X_{\tau_k}^y)]
\]
\[
= \mathcal{E}_{0,\tau_k}^g[f(Y_{\tau_k}^y)]
\]
So we have finished the proof.

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