Singular inverse square potential in coordinate space
with a minimal length

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Abstract

The problem of a particle of mass $m$ in the field of the inverse square potential $\alpha/r^2$ is studied in quantum mechanics with a generalized uncertainty principle, characterized by the existence of a minimal length. Using the coordinate representation, for a specific form of the generalized uncertainty relation, we solve the deformed Schrödinger equation analytically in terms of confluent Heun functions. We explicitly show the regularizing effect of the minimal length on the singularity of the potential. We discuss the problem of bound states in detail and we derive an expression for the energy spectrum in a natural way from the square integrability condition; the results are in complete agreement with the literature.

Keywords: generalized uncertainty principle, minimal length, inverse square potential, singular potential.
I. INTRODUCTION

In spite of its singular features, the inverse square potential $\alpha/r^2$ remains one of the most important interactions in quantum mechanics. This potential function appears in the study of many important problems in different fields of physics, such as Efimov effect \[1\], dipole-bound anions in polar molecules \[2, 3\], atoms interacting with a charged wire \[4\], the analysis of the near-horizon structure of black holes \[5\] and the interaction of a dipole in a cosmic string background \[6\].

The singularity of this potential manifests in the solutions of the Schrödinger equation where the square integrability condition does not lead to an orthogonal set of eigenfunctions with their corresponding eigenenergies \[7\]. This is because the Hamiltonian operator for a singular $\alpha/r^2$ potential is not essentially self-adjoint \[8\], and therefore, to restore a discrete bound states spectrum, one must apply the method of self-adjoint extensions or equivalently require orthogonality of the wave functions \[7\]. Besides, alternative approaches have been used to deal with this potential, such as the standard regularization methods \[3, 9, 10\] or the renormalization techniques \[11, 12\]. Furthermore, it has been shown in Refs. \[6, 13\] that this potential becomes regular in the framework of quantum mechanics with a generalized uncertainty principle (GUP) \[14–18\] due to the presence of a minimal length, which plays the role of a regularizing cutoff at short distances. In this context, the deformed Schrödinger equation has been solved in momentum space and the energy spectrum was computed in a natural way \[13\].

It is noteworthy that the generalization of the Heisenberg uncertainty principle in order to include an elementary length is a common prediction of various studies in quantum gravity \[19–21\] and string theory \[22–24\]. The implications of such modifications on the mathematics of quantum mechanics have first been addressed by Kempf and his collaborators in several papers \[15–18\]. Since then, many studies have been directed to investigate the theoretical and the physical consequences of the GUP, see, for instance, Refs. \[25–28\]. Particularly, special attention was given to the fundamental non-relativistic quantum systems, such as the harmonic oscillator \[15, 29, 30\], the hydrogen atom in one \[31, 32\] and three \[29, 33–35\] dimensions, the singular inverse square potential \[6, 13\] and the gravitational quantum well \[36, 37\]. In most of these works, the Schrödinger equation has been solved in momentum space as the coordinate representation leads to a fourth order differential equation, whose
resolution is generally not possible. This explains the use of the perturbation techniques in the study of certain problems in coordinate space [27, 29, 34, 36, 37]. However, there are some disagreements between the results of the coordinate and momentum representations, as that observed in the case of the hydrogen atom [35]. Let us note that the perturbative approach cannot be applied for the $\alpha/r^2$ potential to investigate the minimal length corrections in coordinate space. Therefore this potential has been studied only in momentum space. On another side, it has been recently proposed, in Ref. [38], an ad hoc transformation to reduce the order of the Schrödinger differential equation in coordinate space. Then, this approach has been applied to the spherical square well potential to investigate the consequence of the GUP on the resonant scattering.

In this paper, we use the transformation of Ref. [38] to study the inverse square potential $\alpha/r^2$ in coordinate space in the presence of a minimal length. We show that the deformed Schrödinger equation can be solved analytically in terms of confluent Heun functions which are also found as solutions in ordinary quantum mechanical systems [40–43]. Then we illustrate the regularizing effect of the minimal length and we investigate the problem of bound states; the energy spectrum will be computed by simply requiring a physical boundary condition. Before doing so, let us mention that this study allows us to check whether the momentum and coordinate representations lead to the same results in the case of the $\alpha/r^2$ potential.

In Sec. II, we review the basic mathematics of quantum mechanics with a GUP. Sec. III is devoted to the study of the deformed Schrödinger equation for the $\alpha/r^2$ potential in coordinate space where the effect of the minimal length on the singularity in this problem will be examined in detail. In Sec. IV, we investigate the problem of bound states and derive an expression of the energy spectrum. In the last section, we summarize our results and conclusions.

II. SCHröDINGER EQUATION IN COORDINATE SPACE WITH A GUP

Diverse forms of the generalized uncertainty principle (GUP) have been proposed in the literature: there is a GUP with a minimal length [15], a GUP which incorporates a minimal length and a minimal momentum [17], a GUP with a Lorentz-covariant algebra [25], and a GUP including a minimal length and a maximal momentum [26]. In this work, we are
interested in the first form, where the GUP can be expressed in $N$-dimensions as \[13]:
\[(\Delta X_i)(\Delta P_i) \geq \frac{\hbar}{2} \left( 1 + \beta \sum_{j=1}^{N} [(\Delta P_j)^2 + \langle \hat{P}_j \rangle^2] + \beta' [(\Delta P_i)^2 + \langle \hat{P}_i \rangle^2] \right), \tag{1}\]
where $\beta$ and $\beta'$ are small positive parameters. This GUP can be obtained from the following modified Heisenberg algebra \[13, 15, 16, 30]:

\[
[\hat{X}_i, \hat{P}_j] = i\hbar[(1 + \beta \hat{P}^2)\delta_{ij} + \beta' \hat{P}_i \hat{P}_j], \quad [\hat{P}_i, \hat{P}_j] = 0,
\]
\[
[\hat{X}_i, \hat{X}_j] = i\hbar\frac{2\beta - \beta' + \beta(2\beta + \beta')\hat{P}^2}{1 + \beta \hat{P}^2}(\hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i). \tag{2}\]

The GUP (1) implies the existence of a minimal length, given by \[16\]
\[(\Delta X_i)_{\text{min}} = \hbar \sqrt{N\beta + \beta'}, \quad \forall i. \tag{3}\]

One of the most used representations of the position and momentum operators satisfying the commutation relations (2) are \[16, 30\]

\[
\hat{X}_i = i\hbar[(1 + \beta p^2) \frac{\partial}{\partial p_i} + \beta' p_i p_j \frac{\partial}{\partial p_j} + \gamma p_i], \quad \hat{P}_i = p_i, \tag{4}\]

where $\gamma$ is a small positive parameter related with $\beta$ and $\beta'$.

As mentioned in Sec. I, the inverse square potential was studied in Refs. \[6, 13\] by the help of representation (4).

In coordinate space, up to the first order in $\beta$, the operators $\hat{X}_i$ and $\hat{P}_i$ can be represented by \[34\]

\[
\hat{X}_i = \hat{x}_i + \frac{2\beta - \beta'}{4}(\hat{p}^2 \hat{x}_i + \hat{x}_i \hat{p}^2), \quad \hat{P}_i = \hat{p}_i(1 + \frac{\beta'}{2}\hat{p}^2), \tag{5}\]

where $\hat{x}_i$ and $\hat{p}_i$ satisfy the standard commutation relations of ordinary quantum mechanics.

In the case $\beta' = 2\beta$, representation (5) reduces to

\[
\hat{X}_i = \hat{x}_i, \quad \hat{P}_i = \hat{p}_i(1 + \beta \hat{p}^2), \tag{6}\]

which was firstly used in Ref. \[29\] to study the hydrogen atom. In this special case, the deformed algebra (2) takes the form

\[
[\hat{X}_i, \hat{P}_j] = i\hbar[(1 + \beta \hat{P}^2)\delta_{ij} + 2\beta \hat{P}_i \hat{P}_j], \quad [\hat{P}_i, \hat{P}_j] = 0, \quad [\hat{X}_i, \hat{X}_j] = 0,
\]

and the minimal length reads in 3-dimensions as

\[(\Delta X_i)_{\text{min}} = \hbar \sqrt{5\beta}, \quad \forall i. \tag{7}\]
Let us now write the Schrödinger equation for a particle of mass $m$ in the external potential $V(r)$ in coordinate space by using the representation (6) as follows

$$\left[\frac{\hat{p}^2}{2m} + V(r) + \frac{\beta}{m} \hat{p}^4\right]\psi(\vec{r}) = E\psi(\vec{r}),$$

(8)

where the terms of order $\beta^2$ have been neglected.

Equation (8) is a fourth order differential equation, and thereby its resolution is not obvious in general. However, it has been recently shown in Ref. [38] that the order of Eq. (8) can be reduced by performing the following transformation:

$$\psi(\vec{r}) = (1 - 2\beta \hat{p}^2)\phi(\vec{r}).$$

(9)

The function $\phi(\vec{r})$ then satisfies the equation

$$\left[(1 + 4m\beta[E - V(r)]) \frac{\hat{p}^2}{2m} + V(r) - E\right] \phi(\vec{r}) = 0.$$  

(10)

By using the factorization $\phi(\vec{r}) = R(r)Y^m(\theta, \varphi)$, the radial function $R(r)$ satisfies the second order differential equation

$$\left[(1 + 4m\beta[E - V(r)]) \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\ell(\ell + 1)}{r^2}\right) + \frac{2m}{\hbar^2}[E - V(r)]\right] R(r) = 0.$$  

(11)

In the following, we will use this equation to study the $\alpha/r^2$ potential in coordinate space with a minimal length depending on the deformation parameter $\beta$.

III. INVERSE SQUARE POTENTIAL IN COORDINATE SPACE WITH A GUP

The deformed Schrödinger equation (11) for a particle of mass $m$ in the field of the attractive inverse square potential $V(r) = -\frac{\alpha}{r^2}$, $(\alpha > 0)$, reads

$$\left[\left(1 - \Omega - \frac{\alpha \Omega}{E r^2}\right) \frac{d^2}{dr^2} + \frac{2}{r} \left(1 - \Omega - \frac{\alpha \Omega}{E r^2}\right) \frac{d}{dr} - \left(1 - \Omega - \frac{\alpha \Omega}{E r^2}\right) \frac{L}{r^2} + \frac{2mE}{\hbar^2} \left(\frac{\alpha}{E r^2} + 1\right)\right] R(r) = 0,$$

(12)

with definitions $L = \ell(\ell + 1)$ and $\Omega = -4m\beta E$.

Let us examine the effect of the deformation parameter $\beta$ on the asymptotic behaviors of the two linearly independent solutions of Eq. (12) in the regions $r \approx 0$ and $r \to \infty$.

In the region $r \approx 0$, we write Eq. (12) by keeping only the dominant terms as

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{L}{r^2}\right] R(r) = 0.$$  

(13)
By putting $R(r) = r^s$ in Eq. (13), we get two values of $s$ ($s = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4L}$), corresponding to two solutions

$$R_1(r \ll 1) = r^{-1-\ell}, \quad R_2(r \ll 1) = r^\ell. \quad (14)$$

When $\ell \neq 0$, the function $R_1$ is not square integrable in the origin. In the case where $\ell = 0$, the two solutions are square integrable and the situation is analogous to that discussed by Landau and Lifshitz [39], where it was concluded that one must always choose the solution which is less divergent at the origin. Therefore, the physical solution has to be chosen as $R_2$.

Note that these behaviors are completely different from that of ordinary quantum mechanics, where there is a difference between two ranges of the coupling $\alpha$ of the potential: the weak coupling regime $2m\alpha/h^2 < 1/4 + L$ and the strong coupling regime $2m\alpha/h^2 \geq 1/4 + L$. In the second regime the two solutions behave similarly at the origin and, both of them are quadratically integrable. Now, in the deformed case the second solution $R_2$ is the physical one regardless the value of the coupling constant $\alpha$. This might be viewed as an indication of the regularization of the singular inverse square potential in this formalism of the GUP.

In the region $r \to \infty$, we proceed in the same manner. We write Eq. (12) by keeping only the dominant terms as

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{2mE}{\hbar^2 (1 - \Omega)} \right] R(r) = 0. \quad (15)$$

We use the transformation $R(r) = \frac{1}{r} u(r)$, then $u(r)$ satisfies the equation

$$\left[ \frac{d^2}{dr^2} + \frac{2mE}{\hbar^2 (1 - \Omega)} \right] u(r) = 0. \quad (16)$$

For bound states ($E < 0$), the solutions of this equation are clearly given by

$$u(r) = \exp \left( \pm \sqrt{-\frac{2mE}{\hbar^2 (1 - \Omega)}} r \right). \quad (17)$$

These asymptotic behaviors are analogous to that of the solutions of ordinary quantum mechanics ($\beta = 0$): one of the solutions behaves at infinity as $u_1(r) \sim \exp(-\eta r)$, with $\eta^2 = -\frac{2mE}{\hbar^2}$, and it is square integrable, and the other solution has the behavior $u_2(r) \sim \exp(\eta r)$, which is not physically acceptable. This result is expected since the minimal length modifies the characteristics of the potential only at short distances, and it has no effect at large distances.
We return now to Eq. (12). By introducing the dimensionless variable $x = -\frac{E_r^2}{\alpha}$, and with the definitions $\kappa = \frac{m^2}{2\hbar^2}$, $\varepsilon = 1 - \Omega$, equation (12) takes the form

$$\left[ (\varepsilon x + \Omega) \frac{d^2}{dx^2} + \frac{3}{2} (\varepsilon x + \Omega) \frac{d}{dx} - \frac{L (\varepsilon x + \Omega)}{4 x^2} + \frac{\kappa (1 - x)}{x} \right] R(x) = 0. \quad (18)$$

We perform the transformation

$$R(x) = x^{-\frac{1}{4}} \sqrt{4L+1} - \frac{1}{4} f(x) = x^{-\frac{1}{4} \ell - \frac{1}{4}} f(x), \quad (19)$$

and the equation (18) becomes

$$\left[ \frac{d^2}{dx^2} + \frac{1}{2} - \ell \frac{d}{dx} + \frac{\kappa (1 - x)}{x (\varepsilon x + \Omega)} \right] f(x) = 0. \quad (20)$$

To move the singular point $-\frac{\Omega}{\varepsilon}$ to the value 1, we simply change our variable as

$$y = \frac{\varepsilon}{\Omega} x, \quad (21)$$

which leads to the equation

$$\left( \frac{d^2}{dy^2} + \frac{1}{2} - \ell \frac{d}{dy} + \frac{\kappa (\Omega y + 1)}{y (y - 1)} \right) f(y) = 0. \quad (22)$$

We then apply another transformation

$$f(y) = (y - 1) g(y), \quad (23)$$

and the equation for $g(y)$ is

$$\left( \frac{d^2}{dy^2} + \frac{1}{2} - \ell \frac{d}{dy} + \frac{\kappa (\Omega y + 1) + 1/2 - \ell}{y (y - 1)} \right) g(y) = 0. \quad (24)$$

Equation (24) is in the form of the following confluent Heun equation [40, 41]

$$\left[ \frac{d^2}{dy^2} + \left( a + \frac{b + 1}{y} + \frac{c + 1}{(y - 1)} \right) \frac{d}{dy} + \frac{\ell}{2} a (b + c + 2) + d \right] g(y) + e + \frac{b + 1}{2} (c - a) (b + 1) \frac{y + e}{y (y - 1)} = 0, \quad (25)$$

with the parameters

$$a = 0, \quad b = -1/2 - \ell, \quad c = 1, \quad d = \frac{\kappa \Omega}{\varepsilon^2}, \quad e = \frac{\kappa}{\varepsilon} + 1/2. \quad (26)$$
Equation (25) is a second order linear differential equation with two regular singularities at \( y = 0 \) and 1, and an irregular singularity of rank 1 at \( y = \infty \). In the vicinity of \( y = 0 \), the two linearly independent solutions of Eq. (25) are

\[
g_1(y) = Hc(a, b, c, d, e; y), \quad g_2(y) = y^{-b}Hc(a, -b, c, d, e; y),
\]

where \( Hc \) is the confluent Heun function. According to the transformations (19) and (23), one has

\[
R(y) = y^{-\frac{\ell}{2}-\frac{1}{2}}(y - 1)g(y), \tag{28}
\]

and consequently the deformed Schrödinger equation (12) admits two solutions as

\[
R_1(y) = y^{-\frac{\ell}{2}-\frac{1}{2}}(y - 1)Hc(a, b, c, d, e; y), \tag{29}
\]

\[
R_2(y) = y^{\frac{\ell}{2}}(y - 1)Hc(a, -b, c, d, e; y), \tag{30}
\]

where

\[
y = -\frac{1 + 4m\beta E}{4m\beta\alpha}r^2 \tag{31}
\]

In the vicinity of \( r = 0 \) \((y = 0)\), the confluent Heun function behaves as \( Hc(a, b, c, d, e; 0) = 1 \). So the asymptotic behaviors of the two solutions in this region are

\[
R_1(y) \approx y^{-\frac{\ell}{2}-\frac{1}{2}}(y - 1) \approx r^{-\ell - 1}, \quad R_2(y) \approx y^{\frac{\ell}{2}}(y - 1) \approx r^{\ell}, \tag{32}
\]

which is exactly what we have already obtained in Eq. (14). It was indicated that \( R_1(y) \) is not square integrable, so that the physical solution is \( R_2(y) \). Then the solution of the deformed Schrödinger equation (12) reads

\[
R(y) = y^{\frac{\ell}{2}}(y - 1)Hc(a, -b, c, d, e; y), \tag{33}
\]

or in terms of the old variable \( r \) as

\[
R(r) = Ar^{\ell}\left(1 + \frac{1 + 4m\beta E}{4m\beta\alpha}r^2\right)Hc\left(a, -b, c, d, e; -\frac{1 + 4m\beta E}{4m\beta\alpha}r^2\right), \tag{34}
\]

where \( A \) is a normalization constant, and the parameters are given by

\[
a = 0, \quad b = -1/2 - \ell, \quad c = 1, \quad d = \frac{\kappa\Omega}{\varepsilon^2} = -\frac{4m^2\alpha\beta E}{2\hbar^2(1 + 4m\beta E)^2},
\]

\[
e = \frac{\kappa}{\varepsilon} + 1/2 = \frac{m\alpha}{2\hbar^2(1 + 4m\beta E)} + 1/2,
\]

in their explicit form.

As illustrated above, the potential is now regularized by the introduction of the minimal length parameter \( \beta \). Accordingly, the quantized energy spectrum should be the manifestation of a physical boundary condition, as it will be shown in the following section.
IV. ENERGY OF BOUND STATES

To obtain the energy spectrum we should guarantee the square integrability condition on the whole interval of \( r \) for all values of eigenenergies \( E \). We must, however, take into account that the solution (34) is well-defined only for \( |y| < 1 \) as it should be for a local Frobenius solution [41]. Then the asymptotic form of the wave-function in the region \( (r \to \infty) \) cannot be examined from the solution (34) as \( \infty \) is an irregular singularity of the confluent Heun equation. Around the irregular singular point at \( \infty \), one can define Thomé-type (asymptotic series) solutions, which have the form given in Eq. (17). To overcome this difficulty, one needs to define a range of the radial distance \( r \) compatible with the physical constraints for studying the asymptotic form of the solution within this range.

To this end, we can consider, as in Refs. [12], very low energy levels such as \( \sqrt{-\frac{2mE}{\hbar^2}} R \ll 1 \) (here the cutoff \( R \) is the minimal length \( (\Delta x)_{\text{min}} = \hbar \sqrt{5\beta} \)), which implies that \( \Omega \ll 1 \).

Let us now rewrite the function (34) as

\[
R(r) = A r^\ell \left( 1 - \frac{(\Omega - 1)r^2}{4m\beta\alpha} \right) H^c \left( a, -b, c, d, e; \frac{(\Omega - 1)r^2}{4m\beta\alpha} \right),
\]

(35)

where \( \Omega = -4m\beta E \) with \( \Omega > 0 \) for bound states. It can be observed that for infinite values of \( r \), the confluent Heun series diverges because the argument of the function, in Eq. (35) grows up, and therefore, a physical condition should be associated with the solution (35).

To deal with this problem, let us focus on large values of \( r \) such as \( \frac{r^2}{4m\beta\alpha} \to \frac{1}{\Omega} \gg 1 \) (or \( r \to \sqrt{-\frac{\alpha}{E}} \)), and require the constraint \( R(r = \sqrt{-\frac{\alpha}{E}}) = 0 \), which yields the following spectral condition,

\[
H^c \left( a, -b, c, d, e; \frac{\Omega - 1}{\Omega} \right) = 0,
\]

(36)

which gives the bound state energy levels of the \( \alpha/r^2 \) potential in the presence of a minimal length.

It would be important to emphasize that a mathematical reasoning lead to the condition (36). Recalling that the wave-function (35) corresponds to a series solution of the confluent Heun equation around \( y = 0 \), where the radius of convergence is found to be \( |y| = 1 \) [40]. Therefore, in the limit \( r \to c \sqrt{-\frac{\alpha}{E}} \) (\( c \) being a positive constant), one has \( y \to c \frac{\Omega - 1}{\Omega} \), and so if \( \Omega < \frac{c}{c+1} \) then \( |y| > 1 \), which implies that the confluent Heun series diverges. In our considerations, \( c = 1 \) and \( \Omega \ll 1 \), consequently, the series always diverges; hence one needs to impose the condition (36).
Before examining Eq. (36), let us show that in this limit (i.e., $\Omega \ll 1$), the confluent Heun function can be reduced to a hypergeometric function by doing the following approximations:

$$d = \frac{\kappa \Omega}{(1 - \Omega)^2} \simeq 0, \quad e = \frac{\kappa}{(1 - \Omega)} + 1/2 \simeq \kappa + 1/2,$$

(37)

In this case the confluent Heun equation (25) reduces to

$$\left[ \frac{d^2}{dy^2} + \left( \frac{b + 1}{y} + \frac{c + 1}{y - 1} \right) \frac{d}{dy} + \frac{e}{y(y - 1)} \right] g(y) = 0,$$

(38)

which is a hypergeometric equation of the form

$$\left[ y (1 - y) \frac{d^2}{dy^2} + \left( \delta - (\alpha + \gamma + 1)y \right) \frac{d}{dy} - \alpha \gamma \right] g(y) = 0,$$

(39)

where the parameters $\alpha, \gamma$ and $\delta$ are

$$\alpha = \frac{3}{4} - \frac{\ell}{2} - \frac{i \nu}{2}, \quad \gamma = \frac{3}{4} - \frac{\ell}{2} + \frac{i \nu}{2}, \quad \delta = \frac{1}{2} - \ell,$$

(40)

with $\nu = \sqrt{4\kappa - (\ell + \frac{1}{2})^2}$ and $\kappa = \frac{m\omega}{2\hbar^2}$. By using the transformation (28), the two solutions now take the forms

$$R_1(y) = y^{-\frac{\delta}{2}}(y - 1)F(\alpha, \gamma, \delta; y),$$

(41)

$$R_2(y) = y^{-\frac{\delta}{2}}(y - 1)y^{1-\delta}F(\alpha', \gamma', \delta'; y),$$

(42)

where

$$\alpha' = \alpha - \delta = \frac{1}{4} + \frac{\ell}{2} - \frac{i \nu}{2}, \quad \gamma' = \gamma - \delta = \frac{1}{4} + \frac{\ell}{2} + \frac{i \nu}{2}, \quad \delta' = 2 - \delta = \frac{3}{2} + \ell.$$

It follows that the regular solution at the origin is $R_2(y)$, which can be written as follows

$$R(y)\big|_{\Omega \ll 1} = A_0 y^{\frac{\delta}{2}}(y - 1)F\left(\alpha', \gamma', \delta'; y\right),$$

(43)

where $A_0$ is a normalization constant.

Now, by using the expression (43), the associated boundary condition that substitutes for Eq. (36) is then

$$F\left(\alpha', \gamma', \delta'; -\frac{1}{\Omega}\right)_{\Omega \ll 1} = 0,$$

(44)
By using the following transformation

\[ F(\alpha, \gamma, \delta; z) = \frac{\Gamma(\delta)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)\Gamma(\delta - \alpha)}(-z)^{-\alpha}F\left(\alpha, 1 - \delta + \alpha, 1 - \gamma + \alpha; \frac{1}{z}\right) \]

\[ + \frac{\Gamma(\delta)\Gamma(\alpha - \gamma)}{\Gamma(\alpha)\Gamma(\delta - \gamma)}(-z)^{-\gamma}F\left(\gamma, 1 - \delta + \gamma, 1 - \alpha + \gamma; \frac{1}{z}\right), \]

and by taking \( F(a, b, c; y) \approx y \ll 1 \), the equation (44) can be rewritten in the form

\[ |B| \left(\frac{1}{\Omega}\right)^{-\frac{3}{2} - \frac{\ell}{2}} \Gamma\left(\frac{3}{2} + \ell\right) \left[ \exp i \left(\left\{\frac{\nu}{2} \ln\left(\frac{1}{\Omega}\right) + \arg(B)\right\}\right) + \exp \left(-i \left\{\frac{\nu}{2} \ln\left(\frac{1}{\Omega}\right) + \arg(B)\right\}\right)\right] = 0, \]

where

\[ B \equiv \frac{\Gamma(iv)}{\Gamma\left(\frac{1}{4} + \frac{\ell}{2} + \frac{i\nu}{2}\right)\Gamma\left(\frac{5}{4} + \frac{\ell}{2} + \frac{i\nu}{2}\right)} = |B| \exp[i \arg(B)]. \]

Then, Eq. (46) leads to the condition

\[ \cos \left[\arg(B) + \frac{\nu}{2} \ln\left(\frac{1}{\Omega}\right)\right] = 0, \]

which gives the following energy spectrum:

\[ E_n = -\frac{1}{4m\beta} \exp\frac{2}{\nu} \left[\arg(B) - (n + \frac{1}{2})\pi\right], \ n = 0, 1, 2, \ldots, \]

which can be written in terms of the minimal length \((\Delta x)_{\text{min}} = \hbar\sqrt{5/\beta}\) as

\[ E_n = -\frac{5\hbar^2}{4m(\Delta x)_{\text{min}}^2} \exp\frac{2}{\nu} \left[\arg(B) - (n + \frac{1}{2})\pi\right], \ n = 0, 1, 2, \ldots. \]

The expression (49) is identical to the one obtained in momentum space in Ref. [13] by using the representation (4). This result is also reached in standard quantum mechanics by regularization techniques, where the potential is cut off at a short-distance radius \(R\), and the potential is replaced in the region \(r < R\) by another interaction, see, for instance, Refs. [3, 9, 11]. It follows that the minimal length plays the same role as the ultraviolet cutoff \(R\). In the standard regularization methods, the limit \(R \to 0\) does not make sense because the energy goes to \(-\infty\) and this explains the need to perform renormalization [9, 11]. However, in the minimal length formalism, the limit \(\beta \to 0\) is not mandatory as \(\beta\) is a physical parameter of this formalism, and hence it should appear in expressions of the observable quantities.

As it has been pointed out in Ref. [13], the condition \(|E_n| \ll 1/4m\beta\) systematically excludes the undesirable values of the number \(n\) in the formula (49), so that there is now a
ground state with finite energy. Furthermore, if the potential is weakly attractive ($4\kappa < 1/4$) there exists no bound state solution for Eq. (44).

To confirm these results in coordinate space, we will now examine the exact spectral equation (36). We have plotted the confluent Heun function in Eq. (36) as a function of $\omega = \Omega/2 = -2m\beta E$ for different values of the parameter $\kappa = m\alpha/\hbar^2$, by taking $\ell = 0$ as in Ref. [13]. The zeros of the function represent the eigenenergies of the potential. For the sake of comparison with the momentum space results, we choose the same values of $\kappa$ as in Ref. [13], namely, $\kappa = 1/20, 3/4$ and 2. One can claim that the energy of the ground state is finite and, as in ordinary quantum mechanics, there are many, almost identical, excited states with $\Omega \approx 0$ (accumulation point). Indeed, in Fig. 1, corresponding to $\kappa = 3/4$, the energy of the ground state is proportional to $\omega_1 \approx 0.0491$ (in Ref. [13], one has $\omega_1 \approx 0.0694$); and for $\kappa = 2$, Fig. 2 shows that $\omega_1 \approx 0.2486$ (we have $\omega_1 \approx 0.3704$ in Ref. [13]).

This difference between the results is expected as the two spectral conditions, namely the Eq. (36) and that of Ref. [13], correspond to the two special cases $\beta' = 2\beta$ and $\beta' = \beta$ respectively; and in the general case ($\beta' \neq \beta$), the spectrum inversely depends on the sum $\beta' + \beta$ [13], giving rise to the factor $3/2$ between the two results. On another side, the momentum representation used in Ref. [13] is exact while the one used here satisfies the GUP algebra in the first order of the deformation parameter.

For $\kappa = 1/20$, Fig. 3 shows that there is no bound state; the value of the critical coupling constant, below which bound states do not exist, is also identical to what was obtained in Ref. [13] ($\kappa^* = 1/16$ for $\ell = 0$), which is the same as in ordinary quantum mechanics.

To complete this analysis, it is important to present some curves of the radial wavefunction in coordinate space to show in particular that the boundary condition (36) leads to a decaying behavior for large values of $r$ such as $r \to \sqrt{-\frac{\omega}{\kappa}}$. We plotted the radial wavefunction in Eq. (35) as a function of the dimensionless variable $\xi = \frac{r}{(\Delta x)_{\min}}$ with $\kappa = 2$, for different values of the parameter $\omega = -2m\beta E$ (solutions and not solutions of the boundary condition (36) by taking $\ell = 0$). Thus, for the two eigenvalues $\omega = 0.0167$ and 0.000167, Figs. 4 and 5 show, respectively, the decaying behavior of $R(\xi)$ when $\xi$ grows up. In contrast, Fig. 6 with the value $\omega = 0.004$, which is not a solution of the quantization condition (36), shows that $R(\xi)$ does not decay for large values of $\xi$ in the aforementioned range of the radial variable.

This study shows that the coordinate space representation of Ref. [29], used firstly to
study the hydrogen atom, gives the same results as that obtained in Ref. [13] in the case of the inverse square potential by using the momentum representation. This result is intriguing because these two representations led to conflicting results in some problems [35], such as the hydrogen atom case.
FIG. 3: $H_C \equiv H_C\left(a, -b, c, d, e; \frac{2\omega - 1}{2\omega}\right)$ for $\kappa = 1/20; \omega = -2m\beta E$.

FIG. 4: $R_\xi$ for $\omega = 0.0167$ (a solution of the energy condition).

V. SUMMARY AND CONCLUSION

We studied the singular inverse square potential in the framework of quantum mechanics with a GUP, which implies the existence of a minimal length. The corresponding deformed Schrödinger equation was established in coordinate space by using the representation of Ref.
By following Ref. [38], we transformed this equation into a second order differential equation. We explicitly illustrated the regularizing effect that the fundamental length plays on the singularity of the problem. Then, we solved this equation analytically in terms of the confluent Heun function. The problem of bound states has been discussed in detail and the energy spectrum was derived in a natural way from the condition of square integrability.
of the wave function. The results obtained here are identical to that of Ref. [13], where the
momentum representation has been used. This situation contradicts the conclusion drawn
in the study of the hydrogen atom problem [35], where the two representations have led
to different results. However, it is consistent with what was concluded in the study of the
harmonic oscillator [30].

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