Induced Matchings and the v-Number of Graded Ideals †

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Abstract: We give a formula for the v-number of a graded ideal that can be used to compute this number. Then, we show that for the edge ideal $I(G)$ of a graph $G$, the induced matching number of $G$ is an upper bound for the v-number of $I(G)$ when $G$ is very well-covered, or $G$ has a simplicial partition, or $G$ is well-covered connected and contains neither four, nor five cycles. In all these cases, the v-number of $I(G)$ is a lower bound for the regularity of the edge ring of $G$. We classify when the induced matching number of $G$ is an upper bound for the v-number of $I(G)$ when $G$ is a cycle and classify when all vertices of a graph are shedding vertices to gain insight into the family of $W_2$-graphs.

Keywords: graded ideals; v-number; induced matchings; edge ideals; regularity; very well-covered graphs; $W_2$-graphs; simplicial vertices

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1. Introduction

Let $S = K[t_1, \ldots, t_s] = \bigoplus_{d=0}^\infty S_d$ be a polynomial ring over a field $K$ with the standard grading, and let $I$ be a graded ideal of $S$. A prime ideal $p$ of $S$ is an associated prime of $S/I$ if $I: f = p$ for some $f \in S_d$, where $(I: f)$ is the set of all $g \in S$ such that $gf \in I$. The set of associated primes of $S/I$ is denoted by Ass$(I)$, and the set of maximal elements of Ass$(I)$ with respect to inclusion is denoted by Max$(I)$. The v-number of $I$, denoted $v(I)$, is the following invariant of $I$ that was introduced in [1] to study the asymptotic behavior of the minimum distance of projective Reed–Muller-type codes, Corollary 4.7 in [1]:

$$v(I) := \min\{d \geq 0 \mid \exists f \in S_d \text{ and } p \in \text{Ass}(I) \text{ with } (I: f) = p\}.$$

One can define the v-number of $I$ locally at each associated prime $p$ of $I$:

$$v_p(I) := \min\{d \geq 0 \mid \exists f \in S_d \text{ with } (I: f) = p\}.$$

For a graded module $M \neq 0$, we define $\alpha(M) := \min\{\deg(f) \mid f \in M \setminus \{0\}\}$. By convention, we set $\alpha(0) := 0$. Part (d) of the next result was shown in Proposition 4.2 in [1] for unmixed graded ideals. The next result gives a formula for the v-number of any graded ideal.

Theorem 1. Let $I \subset S$ be a graded ideal, and let $p \in \text{Ass}(I)$. The following hold:

(a) If $\mathcal{G} = \{g_1, \ldots, g_r\}$ is a homogeneous minimal generating set of $(I: p)/I$, then:

$$v_p(I) = \min\{\deg(g_i) \mid 1 \leq i \leq r \text{ and } (I: g_i) = p\};$$

(b) $v(I) = \min\{v_q(I) \mid q \in \text{Ass}(I)\};$
(c) \( v_p(I) \geq n((I: p)/I) \) with equality if \( p \in \text{Max}(I) \);
(d) If \( I \) has no embedded primes, then \( v(I) = \min\{ n((I: q)/I) : q \in \text{Ass}(I) \} \).

The formulas of Parts (a) and (b) give an algorithm to compute the \( v \)-number using Macaulay2 [2] (Example 1, Procedure A1 in Appendix A).

The \( v \)-number of nongraded ideals was used in [3] to compute the regularity index of the minimum distance function of affine Reed–Muller-type codes, Proposition 6.2 in [3]. In this case, one considers the vanishing ideal of a set of affine points over a finite field. For certain classes of graded ideals, \( v(I) \) is a lower bound for \( \text{reg}(S/I) \), the regularity of the quotient ring \( S/I \) (Definition 1); see [1,4,5]. There are examples of ideals where \( v(I) > \text{reg}(S/I) \) [4]. It is an open problem whether \( v(I) \leq \text{reg}(S/I) + 1 \) holds for any squarefree monomial ideal. Upper and lower bounds for the regularity of edge ideals and their powers were given in [6–15]; see Section 2. Using the polarization technique of Fröberg [16], we give an upper bound for the regularity of a monomial ideal \( I \) in terms of the dimension of \( S/I \) and the exponents of the monomials that generate \( I \) (Proposition 2).

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). If \( V(G) = \{ t_1, \ldots, t_s \} \), we can regard each vertex \( t_i \) as a variable of the polynomial ring \( S = K[t_1, \ldots, t_s] \) and think of each edge \( \{ t_i, t_j \} \) of \( G \) as the quadratic monomial \( t_it_j \) of \( S \). The edge ideal of \( G \) is the squarefree monomial ideal of \( S \), defined as:

\[
I(G) := \langle t_it_j \mid \{ t_i, t_j \} \in E(G) \rangle.
\]

This ideal, introduced in [17], has been studied in the literature from different perspectives; see [18–26] and the references therein. We use induced matchings of \( G \) to compare the \( v \)-number of \( I(G) \) with the regularity of \( S/I(G) \) for certain families of graphs.

A subset \( C \) of \( V(G) \) is a vertex cover of \( G \) if every edge of \( G \) is incident with at least one vertex in \( C \). A vertex cover \( C \) of \( G \) is minimal if every proper subset of \( C \) is not a vertex cover of \( G \). A subset \( A \) of \( V(G) \) is called stable if no two points in \( A \) are joined by an edge. Note that a set of vertices \( A \) is a (maximal) stable set of \( G \) if and only if \( V(G) \setminus A \) is a (minimal) vertex cover of \( G \). The stability number of \( G \), denoted by \( \beta_0(G) \), is the cardinality of a maximum stable set of \( G \), and the covering number of \( G \), denoted \( \alpha_0(G) \), is the cardinality of a minimum vertex cover of \( G \). We introduce the following two families of stable sets:

\[
\mathcal{F}_G := \{ A \mid A \text{ is a maximal stable set of } G \};
\]
\[
\mathcal{A}_G := \{ A \mid A \text{ is a stable set of } G, \text{ and } N_G(A) \text{ is a minimal vertex cover of } G \}.
\]

According to Theorem 3.5 in [4], \( \mathcal{F}_G \subset \mathcal{A}_G \) and the \( v \)-number of \( I(G) \) is given by:

\[
v(I(G)) = \min\{ |A| : A \in \mathcal{A}_G \}.
\]

The \( v \)-number of \( I(G) \) is a combinatorial invariant of \( G \) that has been used to characterize the family of \( W_2 \)-graphs (see the discussion below after Corollary 1). We can define the \( v \)-number of a graph \( G \) as \( v(G) := v(I(G)) \) and study \( v(G) \) from the viewpoint of graph theory.

A set \( P \) of pairwise disjoint edges of \( G \) is called a matching. A matching \( P = \{ e_1, \ldots, e_r \} \) is perfect if \( V(G) = \bigcup_{i=1}^{r} e_i \). An induced matching of a graph \( G \) is a matching \( P = \{ e_1, \ldots, e_r \} \) of such that the only edges of \( G \) contained in \( \bigcup_{i=1}^{r} e_i \) are \( e_1, \ldots, e_r \). The matching number of \( G \), denoted \( \beta_1(G) \), is the maximum cardinality of a matching of \( G \), and the induced matching number of \( G \), denoted \( \text{im}(G) \), is the number of edges in the largest induced matching.

The graph \( G \) is well-covered if every maximal stable set of \( G \) is of the same size, and \( G \) is very well-covered if \( G \) is well-covered, has no isolated vertices, and \( |V(G)| = 2\alpha_0(G) \). The class of very well-covered graphs includes the bipartite well-covered graphs without isolated vertices [27,28] and the whisker graphs [24] (p. 392) (Lemma 1). A graph without isolated vertices is very well-covered if and only if \( G \) is well-covered and \( \beta_1(G) = \alpha_0(G) \) (Proposition 1). One of the properties of very well-covered graphs that will be used to show
the following theorem is that they can be classified using combinatorial properties of a perfect matching, as was shown by Favaron, Theorem 1.2 in [29] (Theorem 7, cf. Theorem 6).

We come to one of our main results.

**Theorem 2.** Let \( G \) be a very well-covered graph, and let \( P = \{e_1, \ldots, e_r\} \) be a perfect matching of \( G \). Then, there is an induced submatching \( P' \) of \( P \) and \( D \in \mathcal{A}_G \) such that \( D \subset V(P') \) and \(|e \cap D| = 1 \) for each \( e \in P' \). Furthermore, \( v(I(G)) \leq |P'| = |D| \leq \text{im}(G) \leq \text{reg}(S/I(G)) \).

Let \( G \) be a graph, and let \( W_G \) be its whisker graph (Section 2). As a consequence, we recover a result of [4] showing that the v-number of \( I(W_G) \) is bounded from above by the regularity of the quotient ring \( K[V(W_G)]/I(W_G) \) (Corollary 3). The independent domination number of \( G \), denoted by \( i(G) \), is the minimum size of a maximal stable set, Proposition 2 in [30]:

\[
i(G) := \min\{|A| : A \in \mathcal{F}_G\},
\]

and \( i(G) \) is equal to the v-number of the whisker graph \( W_G \) of \( G \), Theorem 3.19(a) in [4].

A cycle of length \( s \) is denoted by \( C_s \). The inequality \( v(I(G)) \leq \text{reg}(S/I(G)) \) of Theorem 2 is false if we only assume that \( G \) is a well-covered graph, since the cycle \( C_5 \) is a well-covered graph, but one has \( \text{im}(C_5) = 1 < 2 = v(I(C_5)) \). We prove that \( C_5 \) is the only cycle where the inequality \( v(I(C_5)) \leq \text{im}(C_5) \) fails.

**Theorem 3.** Let \( C_s \) be an s-cycle, and let \( I(C_s) \) be its edge ideal. Then, \( v(I(C_s)) \leq \text{im}(C_s) \) if and only if \( s \neq 5 \).

If \( v \in V(G) \), we denote the closed neighborhood of \( v \) by \( N_G[v] \). A vertex \( v \) of \( G \) is called simplicial if the induced subgraph \( H = G[N_G[v]] \) on the vertex set \( N_G[v] \) is a complete graph. A subgraph \( H \) of \( G \) is called a simplex if \( H = G[N_G[v]] \) for some simplicial vertex \( v \). A graph \( G \) is simplicial if every vertex of \( G \) is either simplicial or is adjacent to a simplicial vertex of \( G \).

If \( A \) is a stable set of a graph \( G \), \( H_i \) is a complete subgraph of \( G \) for \( i = 1, \ldots, r \), and \( A \cup \{V(H_i)\}_{i=1}^r \) is a partition of \( V(G) \), then \( \text{reg}(S/I(G)) \leq r \), Theorem 2 in [15]. We consider a special type of partition of \( V(G) \) that allows us to link \( \mathcal{A}_G \) with induced matchings of \( G \). A graph \( G \) has a simplicial partition if \( G \) has simplexes \( H_1, \ldots, H_r \) such that \( \{V(H_i)\}_{i=1}^r \) is a partition of \( V(G) \). Our next result shows that \( v(I(G)) \leq \text{im}(G) \) if \( G \) has a simplicial partition.

**Theorem 4.** Let \( G \) be a graph with simplexes \( H_1, \ldots, H_r \), such that \( \{V(H_i)\}_{i=1}^r \) is a partition of \( V(G) \). If \( G \) has no isolated vertices, then there is \( D = \{y_1, \ldots, y_k\} \in \mathcal{A}_G \), and there are simplicial vertices \( x_1, \ldots, x_k \) of \( G \) and integers \( 1 \leq j_1 < \cdots < j_k \leq r \) such that \( P = \{\{x_i, y_{j_i}\}_{i=1}^k \} \) is an induced matching of \( G \) and \( H_j \) is the induced subgraph \( G[N_G[x_i]] \) on \( N_G[x_i] \) for \( i = 1, \ldots, k \). Furthermore, \( v(I(G)) \leq |D| = |P| \leq \text{im}(G) \leq \text{reg}(S/I(G)) \).

As a consequence, using a result of Finbow, Hartnell, and Nowakowski that classifies the connected well-covered graphs without four and five cycles, Theorem 1.1 in [31] (Theorem 8), we show other families of graphs where the induced matching number of \( G \) is an upper bound for the v-number of \( I(G) \).

**Corollary 1.** Let \( G \) be a well-covered graph, and let \( I(G) \) be its edge ideal. If \( G \) is simplicial or \( G \) is connected and contains neither four, nor five cycles, then:

\[
v(I(G)) \leq \text{im}(G) \leq \text{reg}(S/I(G)) \leq \beta_0(G).
\]

A vertex \( v \) of a graph \( G \) is called a shedding vertex if each stable set of \( G \setminus N_G[v] \) is not a maximal stable set of \( G \setminus v \). We prove that every vertex of \( G \) is a shedding vertex if and only if \( \mathcal{A}_G = \mathcal{F}_G \) (Proposition 4).
A graph $G$ belongs to class $W_2$ if $|V(G)| \geq 2$ and any two disjoint stable sets $A_1, A_2$ are contained in two disjoint maximum stable sets $B_1, B_2$ with $|B_i| = \beta_0(G)$ for $i = 1, 2$. A graph $G$ is in $W_2$ if and only if $G$ is well-covered, $G \setminus v$ is well-covered for all $v \in V(G)$, and $G$ has no isolated vertices, Theorem 2.2 in [32]. A graph $G$ without isolated vertices is in $W_2$ if and only if $v(\beta(G)) = \beta_0(G)$, Theorem 4.5 in [4]. As an application we recover the only if implication of this result (Corollary 5). Using the fact that a graph $G$ without isolated vertices is in $W_2$ if and only if $G$ is well-covered and every $v \in V(G)$ is a shedding vertex, Theorem 3.9 in [32]. For other characterizations of graphs in $W_2$, see [32,33] and the references therein.

In Section 5, we show examples illustrating some of our results. In particular, in Example 3, we compute the combinatorial and algebraic invariants of the well-covered graphs $C_7$ and $T_{10}$ that are depicted in Figure 1. These two graphs occur in the classification of connected well-covered graphs without four and five cycles, Theorem 1.1 in [31] (Theorem 8). A related result is the characterization of well-covered graphs of girth at least five given in [34].

![Figure 1](image_url)  
Figure 1. Two well-covered graphs with no 4 or 5 cycles.

For all unexplained terminology and additional information, we refer to [35,36] for the theory of graphs and [19,21,25] for the theory of edge ideals and monomial ideals.

2. Preliminaries

In this section, we give some definitions and present some well-known results that will be used in the following sections. To avoid repetition, we continue to employ the notations and definitions used in Section 1.

**Definition 1** ([37]). Let $I \subset S$ be a graded ideal, and let $F$ be the minimal graded free resolution of $S/I$ as an $S$-module:

$$F : \quad 0 \rightarrow \bigoplus_{j} S(-j)^{b_{j,i}} \rightarrow \cdots \rightarrow \bigoplus_{j} S(-j)^{b_{j,i}} \rightarrow S \rightarrow S/I \rightarrow 0.$$  

The Castelnuovo–Mumford regularity of $S/I$ (regularity of $S/I$) is defined as:

$$\text{reg}(S/I) := \max\{j - i \mid b_{j,i} \neq 0\}.$$  

The integer $g$, denoted $\text{pd}(S/I)$, is the projective dimension of $S/I$.

Let $G$ be a graph with vertex set $V(G)$. Given $A \subset V(G)$, the induced subgraph on $A$, denoted $G[A]$, is the maximal subgraph of $G$ with vertex set $A$. The edges of $G[A]$ are all the edges of $G$ that are contained in $A$. The induced subgraph $G[V(G) \setminus A]$ of $G$ on the vertex set $V(G) \setminus A$ is denoted by $G \setminus A$. If $v$ is a vertex of $G$, then we denote the neighborhood of $v$ by $N_G(v)$ and the closed neighborhood $N_G(v) \cup \{v\}$ of $v$ by $N_G[v]$. Recall that $N_G(v)$ is the set of all vertices of $G$ that are adjacent to $v$. If $A \subset V(G)$, we set $N_G(A) := \bigcup_{a \in A} N_G(a)$. 


**Theorem 5** ([38]). If a graph $G$ is well-covered and is not complete, then $G_v := G \setminus N_G[v]$ is well-covered for all $v$ in $V(G)$. Moreover, $\beta_0(G_v) = \beta_0(G) - 1$.

If $G$ is a graph, then $\beta_1(G) \leq a_0(G)$. We say that $G$ is a König graph if $\beta_1(G) = a_0(G)$. This notion can be used to classify very well-covered graphs (Proposition 1).

**Theorem 6** ([39], Theorem 5, and [40], Lemma 2.3). Let $G$ be a graph without isolated vertices. If $G$ is a graph without 3, 5, and 7 cycles or $G$ is a König graph, then $G$ is well-covered if and only if $G$ is very well-covered.

**Definition 2.** A perfect matching $P$ of a graph $G$ is said to have Property $(P)$ if for all $\{a, b\}$, $\{a', b'\} \in E(G)$, and $\{b, b'\} \in P$, one has $\{a, a'\} \in E(G)$.

**Remark 1.** Let $P$ be a perfect matching of a graph $G$ with Property $(P)$. Note that if $\{b, b'\} \in P$ and $a \in V(G)$, then $\{a, b\}$ and $\{a, b'\}$ cannot be both in $E(G)$ because $G$ has no loops. In other words, $G$ has no triangle containing an edge in $P$.

**Theorem 7** ([29], Theorem 1.2). The following conditions are equivalent for a graph $G$:

1. $G$ is very well-covered;
2. $G$ has a perfect matching with Property (P);
3. $G$ has a perfect matching, and each perfect matching of $G$ has Property (P).

Let $G$ be a graph with vertex set $V(G) = \{t_1, \ldots, t_k\}$, and let $U = \{u_1, \ldots, u_s\}$ be a new set of vertices. The whisker graph or suspension of $G$, denoted by $W_G$, is the graph obtained from $G$ by attaching to each vertex $t_i$ a new vertex $u_i$ and a new edge $\{t_i, u_i\}$. The edge $\{t_i, u_i\}$ is called a whisker or pendant edge. The graph $W_G$ was introduced in [24] as a device to study the numerical invariants and properties of graphs and edge ideals.

**Lemma 1.** Let $G$ be a graph without isolated vertices. The following hold:

(a) If $G$ is a bipartite well-covered graph, then $G$ is very well-covered;
(b) The whisker graph $W_G$ of $G$ is very well-covered.

**Proof.** (a) A bipartite well-covered graph without isolated vertices has a perfect matching $P$ that satisfies Property (P), Theorem 1.1 in [28]. Thus, by Theorem 7, $G$ is very well-covered; (b) The perfect matching $P = \{\{t_i, u_i\}\}_{i=1}^n$ of the whisker graph $W_G$ satisfies Property (P) and, by Theorem 7, $G$ is very well-covered. 

**Proposition 1** ([41], Lemma 17). Let $G$ be a graph without isolated vertices. Then, $G$ is a very well-covered graph if and only if $G$ is well-covered and $\beta_1(G) = a_0(G)$.

**Proof.** $\Rightarrow$ Assume that $G$ is very well-covered. Then, $|V(G)| = 2a_0(G)$. It suffices to show that $\beta_1(G) = a_0(G)$. In general, $\beta_1(G) \leq a_0(G)$. By Theorem 7, $G$ has a perfect matching $P = \{e_1, \ldots, e_r\}$. Then, $|V(G)| = 2r = 2a_0(G)$ and $r = a_0(G)$. Thus, $a_0(G) = |P| \leq \beta_1(G)$, and one has $a_0(G) = \beta_1(G)$.

$\Leftarrow$ Assume that $G$ is well-covered and $\beta_1(G) = a_0(G)$. Let $P = \{e_1, \ldots, e_r\}$ be a matching of $G$ with $r = \beta_1(G)$. We need only to show that $|V(G)| = 2a_0(G)$. Clearly, $|V(G)|$ is greater than or equal to $2a_0(G)$ because $\bigcup_{i=1}^r e_i \subseteq V(G)$. We argue by contradiction assuming that $\bigcup_{i=1}^r e_i \not\subseteq V(G)$. Pick $v \in V(G) \setminus \bigcup_{i=1}^r e_i$. As $v$ is not an isolated vertex of $G$, there is a minimal vertex cover $C$ of $G$ that contains $v$. As $G$ is well-covered, one has that $|C| = a_0(G) = r$. Since $e_i \cap C \neq \emptyset$ for $i = 1, \ldots, r$ and $v \in C$, we obtain $|C| \geq r + 1$, a contradiction. 

We say that a graph $G$ is in the family $\mathcal{F}$ if there exists $\{x_1, \ldots, x_k\} \subseteq V(G)$ where for each $i$, $x_i$ is simplicial, $|N_G[x_i]| \leq 3$, and $\{N_G[x_i] \mid i = 1, \ldots, k\}$ is a partition of $V(G)$. 


Theorem 8 ([31], Theorem 1.1). Let G be a connected graph that contains neither four, nor five cycles, and let C7 and T10 be the two graphs in Figure 1. Then, G is a well-covered graph if and only if G ∈ {C7, T10} or G ∈ F.

Theorem 9. Let G be a graph. The following hold:
(a) ([7], Theorem 4.5, [42]) 2(n − 1) + \im(G) ≤ \reg(S/I(G)^n) for all n ≥ 1;
(b) ([7], Theorem 4.7, [43]) If G is a forest or G is very well-covered, then:
\begin{equation*}
\reg(S/I(G)^n) = 2(n − 1) + \im(G) \text{ for all } n ≥ 1;
\end{equation*}
(c) ([44], Theorem 1.3) If G is very well-covered, then \( \reg(S/I(G)) = \im(G) \).

The projective dimension of the edge ideal of a graph, the Wiener index, the independence polynomial, the h-vector, and the symbolic powers of cover ideals of graphs have been studied for very well-covered graphs [45–51].

3. The v-Number of a Graded Ideal

Let \( S = K[t_1, \ldots, t_s] = \bigoplus_{d=0}^\infty S_d \) be a polynomial ring over a field \( K \) with the standard grading, and let \( I \) be a graded ideal of \( S \). In this section, we prove a formula for the v-number of \( I \) that can be used to compute this number using Macaulay2 [2]. To avoid repetition, we continue to employ the notations and definitions used in Sections 1 and 2.

Lemma 2. Let \( I \subseteq S \) be a graded ideal. If \( (1 : f) = p \) for some prime ideal \( p \) and some \( f \in S_d \), \( d \geq 0 \), then \( I \subseteq (1 : p) \), and there is a minimal homogeneous generator \( \mathcal{G} := g + I \) of \( (1 : p)/I \) such that \( \deg(f) ≥ \deg(g) \) and \( (1 : g) = p \).

Proof. The strict inclusion \( I \subseteq (1 : p) \) is clear because \( f \in (1 : p) \setminus I \). Let \( \mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_r\} \) be a minimal generating set of \( (1 : p)/I \) such that \( g_i \) is a homogeneous polynomial for all \( i \). As \( (1 : f) = p \), one has \( f \notin \mathcal{G} \) and \( f \in (1 : p) \). Then, we can choose homogeneous polynomials \( h_1, \ldots, h_r \) in \( S, p \) in \( I \), such that \( f = \sum_{i=1}^r h_i g_i + p \) and \( d = \deg(h_i g_i) \) for all \( i \) with \( h_i \neq 0 \).

One has the inclusion \( \bigcap_{i=1}^r (1 : g_i h_i) \subseteq (1 : f) \). Indeed, if we take \( h \) in \( \bigcap_{i=1}^r (1 : g_i h_i) \), then \( h h_i g_i \in I \) for all \( i \) and \( h f = \sum_{i=1}^r h_i g_i + h p \in I \), thus \( h \in (1 : f) \). Therefore, using the fact that all \( g_i \)'s are in \( (1 : p) \), one has the inclusions:

\[ p \subseteq \bigcap_{i=1}^r (1 : g_i) \subseteq \bigcap_{i=1}^r (1 : g_i h_i) \subseteq (1 : f) = p, \]

and consequently, \( p = \bigcap_{i=1}^r (1 : g_i h_i) \). Hence, by [25] (p. 74, 2.1.48), we obtain \( (1 : g_i h_i) = p \) for some \( 1 ≤ i ≤ r \). As \( g_i \in (1 : p) \), we obtain:

\[ p \subseteq (1 : g_i) \subseteq (1 : g_i h_i) = p. \]

Hence, \( p = (1 : g_i) \) and \( d = \deg(f) = \deg(g_i h_i) ≥ \deg(g_i) \).

Theorem 10 (The same as Theorem 1). Let \( I \subseteq S \) be a graded ideal, and let \( p \in \Ass(I) \). The following hold:
(a) If \( \mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_r\} \) is a homogeneous minimal generating set of \( (1 : p)/I \), then:
\[ v_p(I) = \min \{ \deg(g_i) \mid 1 ≤ i ≤ r \text{ and } (1 : g_i) = p \}; \]
(b) \( v(I) = \min \{ v_q(I) \mid q \in \Ass(I) \} \);
(c) \( v_p(I) ≥ a((1 : p)/I) \) with equality if \( p \in \Max(I) \);
(d) If \( I \) has no embedded primes, then \( v(I) = \min \{ a((1 : q)/I) \mid q \in \Ass(I) \} \).
Proof. (a) Take any homogeneous polynomial \( f \) in \( S \) such that \((I: f) = p\). Then, by Lemma 2, there is \( g_j \in \mathcal{G} \) such that \( \deg(f) \geq \deg(g_j) \) and \((I: g_j) = p\). Thus, the set \( \{g_i \mid (I: g_i) = p\} \) is non-empty and the inequality:

\[
\nu_p(I) \leq \min\{\deg(g_i) \mid 1 \leq i \leq r \text{ and } (I: g_i) = p\}
\]

follows by the definition of \( \nu_p(I) \). Now, we can pick a homogeneous polynomial \( f \) in \( S \) such that \( \deg(f) = \nu_p(I) \) and \((I: f) = p\). Then, by Lemma 2, there is \( g_j \in \mathcal{G} \) such that \( \deg(f) \geq \deg(g_j) \) and \((I: g_j) = p\). Thus, \( \deg(f) = \deg(g_j) \) and the inequality “\( \geq \)” holds;

(b) This follows at once from the definitions of \( \nu(I) \) and \( \nu_q(I) \);

(c) Pick a homogeneous polynomial \( g \) in \( S \) such that \( \deg(g) = \nu_p(I) \) and \((I: g) = p\).

Then, \( g \notin I \) and \( gp \subset I \), that is \( g \in (I: p) \setminus I \). Thus, \( \nu_p(I) \geq \deg((I: p)/I) \). Now, assume that \( p \in \text{Max}(I) \). To show the reverse inequality, take any homogeneous polynomial \( f \) in \((I: p) \setminus I \). Then, \( fp \subset I \) and \( p \subset (I: f) \). Since \( \text{Ass}(I : f) \) is contained in \( \text{Ass}(I) \), there is \( q \in \text{Ass}(I) \) such that \( p \subset (I: f) \subset q \). Hence, \( p = q \) and \( p = (I : f) \). Thus, \( \nu_p(I) \leq \deg(f) \) and \( \nu_p(I) \leq \deg((I: p)/I) \);

(d) This follows immediately from (b) and (c). \( \square \)

We give a direct proof of the next result, which in particular relates the \( \nu \)-number of a Cohen–Macaulay monomial ideal \( I \subset S \) to that of \((I,h)\), where \( h \in S_1 \) and \((I : h) = I \).

Corollary 2 ([4], Proposition 4.9). Let \( I \subset S \) be a Cohen–Macaulay nonprime graded ideal whose associated primes are generated by linear forms, and let \( h \in S_1 \) be a regular element on \( S/I \). Then, \( \nu(I, h) \leq \nu(I) \).

Proof. Since the ideal \( I \) has no embedded primes, by Theorem 10d, there are \( p \in \text{Ass}(I) \) and \( f \in (I: p) \setminus I \) such that \( \overline{f} = f + I \) is a minimal generator of \( M_p = (I : p)/I \) and \( \deg(f) = \nu(I) \). The associated primes of \((I : f)\) are contained in \( \text{Ass}(I) \); thus, there is \( q \in \text{Ass}(I) \) such that \( p \subset (I : f) \subset q \). Hence, \( p = q \) because \( I \) has no embedded associated primes, and one has the equality \((I : f) = p\). We claim that \( f \) is not in \((I, h)\). We assume, by contradiction, that \( f \in (I, h) \). Then, we can write \( f = f_1 + hf_2 \), with \( f_i \) a homogeneous polynomial for \( i = 1, 2, f_1 \in I, f_2 \in S \). Hence, one has:

\[
p = (I : f) = (I : hf_2) = (I : f_2).
\]

Therefore, \( f_2 \in (I : p) \setminus I \) and \( \overline{f} = h \overline{f_2} \), a contradiction because \( \overline{f} \) is a minimal generator of \( M_p \). This proves that \( f \notin (I, h) \). Next, we show the equality \((p, h) = ((I, h) : f) \). The inclusion “\( \subset \)” is clear because \((I : f) = p\). Take an associated prime \( p' \) of \(((I, h) : f)\). The height of \( p' \) is equal to \( \text{ht}(I) + 1 \) because \((I, h)\) is Cohen–Macaulay and the associated primes of \(((I, h) : f)\) are contained in \( \text{Ass}(I, h) \). Then:

\[
p = (I : f) \subset ((I, h) : f) \subset p',
\]

and consequently, \((p, h) \subset ((I, h) : f) \subset p' \). Now, \((p, h)\) is prime because \( p \) is generated by linear forms, and \( \text{ht}(p, h) = \text{ht}(p) + 1 = \text{ht}(I) + 1 \) because \( I \) is Cohen–Macaulay and \( h \) is a regular element on \( S/I \). Thus, \((p, h) = p', (p, h) = ((I, h) : f) \), and \( \nu(I, h) \leq \nu(I) \). \( \square \)

Proposition 2. Let \( I \subset S \) be a monomial ideal minimally generated by \( G(I) \), and for each \( t_i \) that occurs in a monomial of \( G(I) \), let \( \gamma_i := \max\{\deg_i(g) \mid g \in G(I)\} \). Then:

\[
\text{reg}(S/I) \leq \dim(S/I) + \sum_i(\gamma_i - 1).
\]

Proof. To show the inequality, we use the polarization technique due to Fröberg (see [52] and [25] (p. 203)). To polarize \( I \) we use the set of new variables:

\[
T_I = \bigcup_{i=1}^{\nu_I} \{t_1, \ldots, t_i, \gamma_i\}.
\]
where \( \{t_{1,2}, \ldots, t_{i,\gamma_i}\} \) is empty if \( \gamma_i = 1 \). Note that \( \lvert T_i \rvert = \sum_i (\gamma_i - 1) \). A power \( t_i^{\rho_i} \) of a variable \( t_i \), \( 1 \leq c_i \leq \gamma_i \), polarizes to \( (t_i^{\rho_i})^{\text{pol}} = t_i \) if \( \gamma_i = 1 \), to \( (t_i^{\rho_i})^{\text{pol}} = t_{i,1} \cdots t_{i,\gamma_i+1} \) if \( c_i < \gamma_i \), and to \( (t_i^{\rho_i})^{\text{pol}} = t_{i,2} \cdots t_{i,\gamma_i+1} \) if \( c_i = \gamma_i \). Setting \( G(I) = \{g_1, \ldots, g_r\} \), the polarization \( I^{\text{pol}} \) of \( I \) is the ideal of \( S[T_i] \) generated by \( g_i^{\text{pol}}, \ldots, g_r^{\text{pol}} \). According to Corollary 1.6.3 in [21], one has:

\[
\text{reg}(S/I) = \text{reg}(S[T_i]/I^{\text{pol}}) \text{ and } \text{ht}(I) = \text{ht}(I^{\text{pol}}).
\]

As \( I^{\text{pol}} \) is squarefree, by Proposition 3.2 in [4], one has \( \text{reg}(S[T_i]/I^{\text{pol}}) \leq \dim(S[T_i]/I^{\text{pol}}) \). Hence, we obtain:

\[
\text{reg}(S/I) = \text{reg}(S[T_i]/I^{\text{pol}}) \leq \dim(S[T_i]/I^{\text{pol}}) = \dim(S[T_i]) - \text{ht}(I).
\]

To complete the proof, notice that \( \dim(S[T_i]) - \text{ht}(I) = \dim(S/I) + |T_i| \). \( \Box \)

Given \( a = (a_1, \ldots, a_s) \in \mathbb{N}^s \), where \( \mathbb{N} = \{0, 1, \ldots\} \), the monomial \( t_i^{a_1} \cdots t_i^{a_s} \) is denoted by \( t^a \). A result of Beintema [53] shows that a zero-dimensional monomial ideal is Gorenstein if and only if it is a complete intersection. (This is also true in dimension one; see Exercise 4.4.19 in [54].) The next result classifies the complete intersection property using regularity.

**Proposition 3.** Let \( I \) be a monomial ideal of \( S \) of dimension zero minimally generated by \( G(I) = \{t_1^{a_1}, \ldots, t_s^{a_s}\} \), where \( a_i \geq 1 \) for \( i = 1, \ldots, s \) and \( d_i \in \mathbb{N} \setminus \{0\} \) for \( i > s \). Then, \( \text{reg}(S/I) \leq \sum_{i=1}^s (d_i - 1) \), with equality if and only if \( I \) is a complete intersection.

**Proof.** The inequality \( \text{reg}(S/I) \leq \sum_{i=1}^s (d_i - 1) \) follows directly from Proposition 2 because \( \dim(S/I) = 0 \). If \( I \) is a complete intersection, then \( I = (t_i^{d_1}, \ldots, t_i^{d_s}) \), and by Lemma 3.5 in [55], we obtain \( \text{reg}(S/I) = \sum_{i=1}^s (d_i - 1) \). Conversely, assume that \( \text{reg}(S/I) \) is equal to \( \sum_{i=1}^s (d_i - 1) \). We argue by contradiction assuming that \( m > s \). Then, the exponents of the monomial \( t_1^{a_1} \cdots t_s^{a_s} \) satisfy \( c_i \leq d_i - 1 \) for \( i = 1, \ldots, s \) because \( t_1^{a_1} \cdots t_s^{a_s} \in G(I) \). The regularity of \( S/I \) is the largest integer \( d \geq 0 \) such that \( (S/I)_d \neq 0 \), Proposition 4.14 in [37]. Pick a monomial \( t^a = t_1^{a_1} \cdots t_s^{a_s} \) such that \( t^a \in S_d \setminus I \) and \( d = \sum_{i=1}^s (d_i - 1) \). Then, \( a_i \leq d_i - 1 \) for \( i = 1, \ldots, s \) because \( t^a \) is not in \( I \), and consequently, \( a_i = d_i - 1 \) for \( i = 1, \ldots, s \). Hence, \( t^a = t_1^{d_1} \cdots t_s^{d_s} \) for some \( \delta \in \mathbb{N}^s \), a contradiction. \( \Box \)

**Remark 2.** Note that Proposition 3 follows also from Corollary 3.17 in [56]. Indeed, assume that \( \text{reg}(S/I) = \text{reg}(S/I_{q_k}) \) is equal to \( \sum_{i=1}^s (d_i - 1) \). Let \( I = \bigcap_{k=1}^s q_k \) be the irreducible decomposition of \( I \), where the \( q_k \)'s are irreducible monomial ideals of \( S \), i.e., ideals generated by powers of variables in \( S \). We argue by contradiction assuming that \( I \) is not a complete intersection. Then, \( I \) is not irreducible and \( \text{reg}(S/I_{q_k}) < \sum_{i=1}^s (d_i - 1) \) for all \( k \) because \( (t_1^{d_1}, \ldots, t_s^{d_s}) \subseteq q_k \) for all \( k \). Therefore, by Corollary 3.17 in [56], it follows that \( \text{reg}(S/I) < \sum_{i=1}^s (d_i - 1) \) because \( I \) is \( m \)-primary, \( m = (t_1, \ldots, t_s) \), and \( \text{reg}(S/I) = \max_k \{\text{reg}(S/I_{q_k})\} \), a contradiction.

4. Induced Matchings and the v-Number

In this section, we show that the induced matching number of a graph \( G \) is an upper bound for the v-number of \( I(G) \) when \( G \) is very well-covered, or \( G \) has a simplicial partition, or \( G \) is well-covered connected and contains neither four, nor five cycles. We classify when the induced matching number of \( G \) is an upper bound for the v-number of \( I(G) \) when \( G \) is a cycle and classify when all vertices of a graph are shedding vertices to gain insight into the family of \( W_2 \)-graphs. To avoid repetition, we continue to employ the notations and definitions used in Sections 1 and 2.

**Theorem 11** ([4], Theorem 3.5). If \( I = I(G) \) is the edge ideal of a graph \( G \), then \( F_G \subset A_G \) and the v-number of \( I \) is:

\[
v(I) = \min \{|A| : A \in A_G\}.
\]

**Lemma 3.** Let \( A \) be a stable set of a graph \( G \). If \( N_G(A) \) is a vertex cover of \( G \), then \( A \in A_G \).
Proof. We take any \( b \in N_G(A) \), then there is \( e \in E(G) \) such that \( e \subseteq A \cup \{ b \} \). Furthermore, \( N_G(A) \cap A = \emptyset \), since \( A \) is a stable set of \( G \). Thus,

\[
e \cap N_G(A) \subseteq (A \cup \{ b \}) \cap N_G(A) \subseteq \{ b \},
\]

and consequently, \( e \cap (N_G(A) \setminus \{ b \}) = \emptyset \). Hence, \( N_G(A) \setminus \{ b \} \) is not a vertex cover of \( G \), since \( e \in E(G) \). Therefore, \( N_G(A) \) is a minimal vertex cover of \( G \) and \( A \in A_G \). □

**Theorem 12** (The same as Theorem 2). Let \( G \) be a very well-covered graph, and let \( P = \{ e_1, \ldots, e_r \} \) be a perfect matching of \( G \). Then, there is an induced submatching \( P' \) of \( P \) and \( D \subset A_G \) such that \( D \subseteq V(P') \) and \( |e \cap D| = 1 \) for each \( e \in P' \). Furthermore, \( v(I(G)) \leq |P'| = |D| \leq \text{im}(G) \leq \text{reg}(G) \).

Proof. To show the first part, we use induction on \( |P| \). If \( r = 1 \), we set \( P' = P = \{ e_1 \} \) and \( D = \{ x_1 \} \), where \( e_1 = \{ x_1, y_1 \} \). Assume \( r > 1 \). We set \( e_r = \{ x, y \}, G_1 := G \setminus \{ x, y \} \) and \( P_1 := P \setminus \{ e_r \} \). By Theorem 7, \( P \) satisfies Property (P). Then, \( P_1 \) satisfies Property (P) as well. Thus, by Theorem 7, \( G_1 \) is very well-covered with a perfect matching \( P_1 \). Hence, by the induction hypothesis, there is an induced submatching \( P_1' \) of \( P_1 \) and \( D_1 \subset A_G \) such that \( D_1 \subseteq V(P_1') \) and \( |e \cap D_1| = 1 \) for each \( e \in P_1' \). Consequently, \( N_{G_1}(D_1) \) is a minimal vertex cover of \( G_1 \). We consider two cases: \( e_r \cap N_{G_1}(D_1) \neq \emptyset \) and \( e_r \cap N_{G_1}(D_1) = \emptyset \):

Case (I). Assume that \( e_r \cap N_{G_1}(D_1) \neq \emptyset \). Thus, we may assume that there is \( \{ x, d \} \in E(G) \) with \( d \in D_1 \). Then, \( N_G(x) \subseteq N_G(d) \subset N_{G_1}(D_1) \), since \( P \) satisfies Property (P). Hence, \( N_{G_1}(D_1) \) is a vertex cover of \( G \), since \( N_{G_1}(D_1) \) is a vertex cover of \( G \) and \( \{ x \} \subset N_G(x) \subset N_{G_1}(D_1) \). Therefore, by Lemma 3, \( D_1 \in A_G \), so this case follows by making \( D = D_1 \) and \( P' = P_1' \).

Case (II). Assume that \( e_r \cap N_{G_1}(D_1) = \emptyset \). We set \( D_2 := V(P_1') \setminus D_1 \), then \( D_2 \) is a stable set of \( G_1 \) and also of \( G \), since \( P_1' \) is an induced matching of \( G_1 \) and also of \( G \). One has the inclusion:

\[
V(P_1') \cap (N_G(x) \cup N_G(x')) \subseteq D_2,
\]

indeed taking \( z \in V(P_1') \cap N_G(x) \) (the case \( z \in V(P_1') \cap N_G(x') \) is similar). If \( z \notin D_2 \), then \( z \in D_2 \setminus N_G(x), \{ z, x \} \in E(G) \), and \( x \in e_r \cap N_{G_1}(D_1) \), a contradiction. We claim that \( |e_r \cap N_{G_1}(D_1)| \leq 1 \). We assume, by contradiction, that \( x, y \in N_{G_1}(D_1) \). Then, there are \( d_1, d_2 \in D_2 \) such that \( \{ x, d_1 \}, \{ x, d_2 \} \in E(G) \). Thus, \( \{ d_1, d_2 \} \in E(G), \) since \( P \) satisfies Property (P), a contradiction, since \( D_2 \) is a stable set of \( G \). Hence, \( |e_r \cap N_{G_1}(D_1)| \leq 1 \), and we may assume:

\[
e_r \cap N_{G_1}(D_1) \subset \{ x \}.
\]

Next we show that \( V(P_1') \cap N_G(x') = \emptyset \). If the intersection is nonempty, by Equation (1), we can pick \( z \) in \( D_2 \cap N_G(x') \), then \( \{ z, x' \} \in E(G) \) and \( x' \in N_G(D_2) \), a contradiction to Equation (2). Therefore, by Equation (1), we obtain the inclusion:

\[
V(P_1') \cap (N_G(x) \cup N_G(x')) \subseteq D_2 \cap N_G(x) =: A_2.
\]

Thus, the edge set \( Q := \{ e \in P_1' \mid e \cap A_2 = \emptyset \} \cup \{ e_r \} \) is an induced matching, since \( P_1' \) is an induced matching. Setting:

\[
D_3 := \{ y \in D_1 \mid \{ y, y' \} \in P_1' \text{ with } y' \notin A_2 \} \cup \{ x \},
\]

i.e., \( D_3 = (D_1 \cap V(Q)) \cup \{ x \} \), we obtain \( \{ e \cap D_3 \} = 1 \) for each \( e \in Q \), since \( |e \cap D_3| = 1 \) for each \( e \in P_1' \). Note that \( D_3 \) is a stable set of \( G \), since \( D_1 \) is a stable set and \( \{ x \} \cap N_G(D_1) = \emptyset \). Now, take \( e \in E(G) \). We prove that \( e \cap N_G(D_3) \neq \emptyset \). Clearly, \( N_G(x) \subset N_G(D_3) \) because \( x \in D_3 \). If \( x' \in e \), then \( x' \in e \cap N_G(x) \subset e \cap N_G(D_3) \). Now, if \( x \in e \), then \( e = \{ x, y \} \) for some \( y \) in \( V(G) \), and \( y \in e \cap N_G(x) \subset e \cap N_G(D_3) \). Therefore, we may assume \( e \cap \{ x, x' \} = \emptyset \), then \( e \in E(G) \). Thus, there is \( z \in e \cap N_G(D_1) \), since \( N_G(D_1) \) is a vertex cover of \( G_1 \). Then, there is \( d \in D_1 \), such that \( z \in N_G(d) \). If \( d \in D_3 \), then \( z \in N_G(D_3) \cap e \). Finally, if \( d \notin D_3 \), then by Equation (3) and the inclusion \( D_1 \subseteq V(P_1') \),
there is $d' \in A_2$ such that $\{d, d'\} \in P'$. Therefore, $\{x, d'\} \in E(G)$, since $d' \in A_2$. This implies, $\{x, z\} \in E(G)$, since $\{d, z\} \in E(G)$, $\{d, d'\} \in E(G)$, $\{d, d'\} \in P$, and $P$ satisfies Property (P). Thus, $z \in c \cap N_G(x) \subset c \cap N_G(D_3)$. Hence, $N_G(D_3)$ is a vertex cover, and by Lemma 3, $D_3 \in A_G$. Therefore, this case follows by making $P' = Q$ and $D = D_3$. This completes the induction process.

Next, we show the equality $|P'| = |D|$. By the first part, we may assume that $P' = \{e_1, \ldots, e_\ell\}$, $1 \leq \ell \leq r$, $e_i = \{x_i, y_i\}$ for $i = 1, \ldots, \ell$, and $x_1, \ldots, x_\ell \in D$. Thus, $\ell = |P'| \leq |D|$, and since $D \subseteq V(P')$, we obtain $2|D| \leq 2|P'|$. Then, $|P'| = |D|$. The inequality $v(I(G)) \leq |D|$ follows by Theorem 11, and $|P'| \leq \text{im}(G)$ is clear by the definition of $\text{im}(G)$. Finally, the inequality $\text{im}(G) \leq \text{reg}(S/I(G))$ follows directly from Theorem 9. □

**Corollary 3** ([4], Theorem 3.19(b)). Let $G$ be a graph, and let $W_G$ be its whisker graph. Then:

$$v(I(W_G)) \leq \text{reg}(K[V(W_G)]/I(W_G)).$$

**Proof.** By Lemma 1, $W_G$ is very well-covered. Thus, by Theorem 12, the $v$-number of $I(W_G)$ is bounded from above by the regularity of $K[V(W_G)]/I(W_G)$. □

**Lemma 4.** Let $\ell \geq 0$ and $s = 4\ell + r$ be integers with $r \in \{0, 1, 2, 3\}$. If $s \geq 3$ and $s \neq 5$, then:

$$\left\lfloor \frac{s}{3} \right\rfloor \geq \ell \text{ if } r = 0 \text{ and } \left\lfloor \frac{s}{3} \right\rfloor \geq \ell + 1 \text{ otherwise.}$$

**Proof.** By the division algorithm, $s \equiv r' \pmod{3}$, where $r' \in \{0, 1, 2\}$. Then:

$$\left\lfloor \frac{s}{3} \right\rfloor = \frac{4\ell + r - r'}{3} = \ell + \frac{r + r' - 2}{3} \in \mathbb{Z}.$$

Thus, $a := \frac{\ell + r - r'}{3} \in \mathbb{Z}$. If $r = 0$, then $a \geq 0$. This follows using the fact that $0 \leq r' \leq 2$ and $\ell \geq 0$. Hence, $\left\lfloor \frac{s}{3} \right\rfloor \geq \ell$. Now, assume $r \in \{1, 2, 3\}$. We claim that $a \geq 1$. We assume, by contradiction, that $a < 1$, then $\ell + r \leq r'$. If $r = 0$, then $s = r = 3$, since $s \geq 3$, a contradiction, since $3 = \ell + r \leq r'$ and $r' \leq 2$. Thus, $\ell \geq 1$, and we have $2 \leq \ell + 1 \leq \ell + r \leq r' \leq 2$. This implies $\ell = 1 = r$ and $r' = 2$. Consequently $s = 5$, a contradiction. Therefore, $a \geq 1$ and $\left\lfloor \frac{s}{3} \right\rfloor \geq \ell + 1$. □

**Theorem 13** (The same as Theorem 3). Let $C_s$ be an $s$-cycle, and let $I(C_s)$ be its edge ideal. Then, $v(I(C_s)) \leq \text{im}(C_s)$ if and only if $s \neq 5$.

**Proof.** $\Rightarrow$ Assume that $v(I(C_s)) \leq \text{im}(C_s)$. If $s = 5$, then $v(I(C_s)) = 2$ and $\text{im}(C_s) = 1$, a contradiction. Thus, $s \neq 5$.

$\Leftarrow$ Assume that $s \neq 5$. We can write $C_s = (t_1, e_1, t_2, \ldots, t_L, e_L, t_{L+1}, \ldots, t_s, e_s, t_1)$. The matching $P = \{e_1, e_4, \ldots, e_{3s-2}\}$, where $q := \left\lfloor \frac{s}{3} \right\rfloor$, is an induced matching of $C_s$ and $|P| = q$.

Now, we choose a stable set $A$ of $C_s$ for each one of the following cases:

- Case $s = 4\ell$. If $A = \{t_2, t_6, \ldots, t_{4\ell+4}\}$, then $N_{C_s}(A) = \{t_1, t_3, t_5, t_7, \ldots, t_{s-3}, t_{s-1}\}$ is a vertex cover of $G$ and $|A| = \ell + 1$;
- Case $s = 4\ell + 1$. If $A = \{t_2, t_6, \ldots, t_{4\ell}\} \cup \{t_{4\ell+2}\}$, then $N_{C_s}(A) = \{t_1, t_3, t_5, t_7, \ldots, t_{s-4}, t_{s-2}\}$ is a vertex cover of $G$ and $|A| = \ell + 1$;
- Case $s = 4\ell + 2$. If $A = \{t_2, t_6, \ldots, t_{4\ell}\} \cup \{t_{4\ell+2}\}$, then $N_{C_s}(A) = \{t_1, t_3, t_5, t_7, \ldots, t_{s-3}, t_{s-1}\}$ is a vertex cover of $G$ and $|A| = \ell + 1$;
- Case $s = 4\ell + 3$. If $A = \{t_2, t_6, \ldots, t_{4\ell+2}\}$, then $N_{C_s}(A) = \{t_1, t_3, t_5, t_7, \ldots, t_{s-3}, t_{s-2}, t_s\}$ is a vertex cover of $G$ and $|A| = \ell + 1$.

In each case, $N_{C_s}(A) = \{t_i\}$ if $l$ is odd and $N_{C_s}(A)$ is a vertex cover of $G$. Therefore, by Lemma 3, $A \in A_{C_s}$. Now, assume $s = 4\ell + r$, with $r \in \{0, 1, 2, 3\}$ and $\ell \geq 0$ an integer. Then, by Lemma 4, $\left\lfloor \frac{s}{3} \right\rfloor \geq \ell$ if $r = 0$ and $\left\lfloor \frac{s}{3} \right\rfloor \geq \ell + 1$ otherwise. Hence, $|P| = \left\lfloor \frac{s}{3} \right\rfloor \geq |A|$. Therefore, $\text{im}(C_s) \leq v(I(C_s))$, since $\text{im}(C_s) \geq |P|$ and $|A| \geq v(I(C_s))$. □
Remark 3. The induced matching number of the cycle $C_s$ is equal to $\lfloor \frac{s}{2} \rfloor$. The regularity of $S/I(C_s)$ is equal to $\lfloor (s + 1)/3 \rfloor,$ Proposition 10 in [15].

Lemma 5. Let $G$ be a graph without isolated vertices, and let $z_1, \ldots, z_m$ be vertices of $G$ such that \{\(N_G(z_i)\)\}_{i=1}^m$ is a partition of $V(G)$. If $G_1 = G \setminus N_G(z_m)$, then:

(i) \(N_G(z)_i = N_G(z_i)\) for \(i < m\);
(ii) \(G_1[N_G(z_i)] = G[N_G(z_i)]\) for \(i < m\).

Proof. (i) Assume that \(1 \leq i \leq m - 1\). Clearly, \(N_{G_1}(z_i) \subseteq N_G(z_i)\) because $G_1$ is a subgraph of $G$. To show the inclusion “\(\subset\)”, take $z \in N_{G_1}(z_i)$. Then, $z = z_i$ or \(\{z, z_i\} \in E(G)\). If $z \in N_G(z_m)$, then $z \in N_G(z_m) \cap N_G(z_i)$, a contradiction. Thus, \(z \notin N_G(z_m)\), and since $G_1$ is an induced subgraph of $G$, we obtain $z = z_i$ or \(\{z, z_i\} \in E(G_1)\). Thus, $z \in N_{G_1}(z_i)$;

(ii) By Part (i), one has \(N_{G_1}(z_i) = N_G(z_i) \subseteq V(G) \setminus N_G(z_m) = V(G_1)\). Then:

\[
E(G[N_G(z_i)]) = \{e \in E(G) \mid e \subseteq N_G(z_i)\} = \{e \in E(G) \mid e \subseteq N_{G_1}(z_i)\}
\]

Thus, $E(G[N_G(z_i)]) = E(G[N_{G_1}(z_i)])$. \(\square\)

Theorem 14 (The same as Theorem 4). Let $G$ be a graph with simplices $H_1, \ldots, H_r$, such that \(\{V(H_i)\}_{i=1}^r\) is a partition of $V(G)$. If $G$ has no isolated vertices, then there is $D = \{y_1, \ldots, y_k\} \in A_G$, and there are simplicial vertices $x_1, \ldots, x_k$ of $G$ and integers $1 \leq j_1 < \cdots < j_k \leq r$ such that $P = \{(x_i, y_{j_i})\}_{i=1}^k$ is an induced matching of $G$ and $H_{j_i}$ is the induced subgraph $G[N_G(x_i)]$ on $N_G[x_i]$ for $i = 1, \ldots, k$. Furthermore, $v(I(G)) \leq |D| = |P| \leq \text{im}(G) \leq \text{reg}(S/I(G))$.

Proof. We proceed by induction on $r$. If $r = 1$, then $V(H_1) = V(G)$, and there is a simplicial vertex $x_1$ of $G$ such that $H_1 = G[N_G[x_1]]$ is a complete graph with at least two vertices. Picking $y_1, y_2 \in N_G[x_1], y_1 \neq y_2$, one has $\{x_1\} \in A_G$ and \(\{x_1, y_1\}\) is an induced matching. Now, assume that $r > 1$. We set $G_1 := G \setminus V(H_r)$. Note that $H_1, \ldots, H_{r-1}$ are simplices of $G_1$ (Lemma 5) and \(\{V(H_i)\}_{i=1}^{r-1}\) is a partition of $V(G_1)$. Then, by the induction hypothesis, there is $D_1 = \{y_1, \ldots, y_{k'}\} \in A_{G_1}$, and there are simplicial vertices $x_1, \ldots, x_{k'}$ of $G_1$ and integers $1 \leq j_1 < \cdots < j_{k'} \leq r - 1$, such that $P_1 = \{\{x_i, y_{j_i}\}_{i=1}^{k'}\}$ is an induced matching of $G_1$ and $H_{j_i} = G_1[N_G[x_i]]$ for $i = 1, \ldots, k'$. By Lemma 5, one has $G_1[N_G[x_i]] = G[N_G[x_i]]$ for $i = 1, \ldots, k'$. We can write $H_r = G[N_G[x]]$ for some simplicial vertex $x$ of $G$.

Case (I). Assume that $V(H_r) \setminus \{x\} \subseteq N_G(D_1)$. Then, $N_G(D_1)$ is a vertex cover of $G$. Indeed, take any edge $e$ of $G$. If $e \not\in V(H_r)$, then $e$ is an edge of $G_1$ and is covered by $N_{G_1}(D_1)$. Assume that $e \cap V(H_r) \neq \emptyset$. If $x \notin e$, then there is $z \in e$ with $z \in V(H_1) \setminus \{x\} \subseteq N_G(D_1)$. Now, if $x \in e$, then $e = \{x, z\}$ with $z \in N_G[x] \setminus \{x\} = V(H_r) \setminus \{x\} \subseteq N_G(D_1)$. This proves that $N_G(D_1)$ is a vertex cover of $G$. Hence, by Lemma 3, $D_1 \in A_G$, and, noticing that $P_1$ is an induced matching of $G$, this case follows by making $D = D_1$ and $P = P_1$;

Case (II) Assume that there is $y \in V(H_r) \setminus \{x\}$ such that $y \notin N_G(D_1)$. Then, $D_2 := D_1 \cup \{y\}$ is a stable set of $G$. Furthermore, $N_G(D_2)$ is a vertex cover of $G$, since $N_G(D_1)$ is a vertex cover of $G_1, H_r$ is a complete subgraph of $G$, and $V(H_r) \subseteq N_G[y]$. Thus, by Lemma 3, $D_2$ is in $A_G$. We set $y_{k'+1} := x, y_{k'+2} := y$, and $H_{j_{k'+1}} := H_{r-1}$. Then, $\{x_{k'+1}, y_{k'+2}\} \subseteq E(H_r)$ and $P_2 := P_1 \cup \{\{x_{k'+1}, y_{k'+2}\}\}$ is an induced matching of $G$, since $P_1$ is an induced matching of $G_1$, $y \in V(H_r) \setminus N_G(D_1)$ and $H_{j_{k'+1}} = G[N_G[x_i]]$, for $i = 1, \ldots, k'+1$. Therefore, this case follows by making $D = D_2$ and $P = P_2$.

The equality $|D| = |P|$ is clear. The inequality $v(I(G)) \leq |D|$ follows from Theorem 11, and $|P| \leq \text{im}(G)$ is clear by the definition of $\text{im}(G)$. Finally, the inequality $\text{im}(G) \leq \text{reg}(S/I(G))$ follows directly from Theorem 9. \(\square\)
Corollary 4 (The same as Corollary 1). Let $G$ be a well-covered graph, and let $I(G)$ be its edge ideal. If $G$ is simplicial or $G$ is connected and contains neither four, nor five cycles, then:

$$v(I(G)) \leq \text{im}(G) \leq \text{reg}(S/I(G)) \leq \beta_0(G).$$

Proof. Assume that $G$ is simplicial. Let $\{z_1, \ldots, z_\ell\}$ be the set of all simplicial vertices of $G$. Then, $V(G) = \bigcup_{i=1}^\ell N_G[z_i]$. As $G$ is well-covered, by Lemma 2.4 in [31], for $1 \leq i < j \leq \ell$, either $N_G[z_i] = N_G[z_j] = N_G[z_i] \cap N_G[z_j] = \emptyset$. Thus, there are simplicial vertices $x_1, \ldots, x_k$ of $G$ such that $\{N_G[x_i]\}_{i=1}^k$ is a partition of $V(G)$. Setting $H_i = G[N_G[x_i]]$ for $i = 1, \ldots, k$ and applying Theorem 14, we obtain that $v(I(G)) \leq \text{im}(G) \leq \text{reg}(S/I(G))$. Noticing that $\dim(S/I(G)) = \beta_0(G)$, the inequality $\text{reg}(S/I(G)) \leq \beta_0(G)$ follows from Proposition 2.

Next, assume that $G$ is connected and contains neither four, nor five cycles. Then, by Theorem 8, $G \in \{C_7, T_{10}\}$ or $G \in F$. The cases $G = C_7$ or $G = T_{10}$ are treated in Example 3 (cf. Theorem 13). If $G \in F$, then there exists $\{x_1, \ldots, x_k\} \subset V(G)$ where for each $i$, $x_i$ is simplicial, $|N_G[x_i]| \leq 3$, and $\{N_G[x_i] | i = 1, \ldots, k\}$ is a partition of $V(G)$. In particular, $G$ is simplicial, and the asserted inequalities follow from the first part of the proof. □

Proposition 4. Let $G$ be a graph. The following conditions are equivalent:

1. Every vertex of $G$ is a shedding vertex;
2. $\mathcal{A}_G = \mathcal{F}_G$.

Proof. $(1) \Rightarrow (2)$ The inclusion $\mathcal{A}_G \supset \mathcal{F}_G$ follows from Theorem 11. To show the inclusion $\mathcal{A}_G \subset \mathcal{F}_G$, we argue by contradiction assuming that there is $D \in \mathcal{A}_G \setminus \mathcal{F}_G$. Then, $D$ is a stable set of $G$ and $N_G(D)$ is a vertex cover of $G$. Thus, $D \cap N_G(D) = \emptyset$. Furthermore, since $D \notin \mathcal{F}_G$, there is $x \in V(G) \setminus D$ such that $D \cup \{x\}$ is a stable set of $G$. Then, $x \notin N_G(D)$. However, $N_G(D)$ is a vertex cover of $G$, then $N_G(x) \subset N_G(D)$ and $A := V(G) \setminus N_G(D)$ is a stable set of $G$. Therefore, $A \subset V(G) \setminus N_G(x)$ and $A' := A \setminus x$ is a stable set of $V(G) \setminus N_G(x)$. Now, we prove that $A'$ is a maximal stable set of $G \setminus x$. We argue by contradiction assuming that there is $a \in V(G \setminus x) \setminus A'$, such that $A' \cup \{a\}$ is a stable set. Then, $a \notin N_G(D)$, since $V(G) = A \cup N_G(D)$. Furthermore, $D \subset A'$, since $D \cap N_G(D) = \emptyset$ and $x \notin D$, a contradiction, since $a \notin N_G(D)$ and $A' \cup \{a\}$ is a stable set. Hence, $A'$ is a maximal stable set of $G \setminus x$. Therefore, $x$ is not a shedding vertex of $G$, a contradiction.

$(2) \Rightarrow (1)$ We assume, by contradiction, that there is $x \in V(G)$ such that $x$ is not a shedding vertex. Thus, there is a maximal stable set $A$ of $G \setminus x$ such that $A \subset V\setminus N_G[x]$. Then, $C := V(G \setminus x) \setminus A$ is a minimal vertex cover of $G \setminus x$ and $A \cup \{x\}$ is a stable set of $G$. Therefore, $A \notin \mathcal{F}_G$. Since $C$ is a minimal vertex cover of $G \setminus x$, we have that for each $z \in C$, there is $z' \in V(G \setminus x) \setminus C = A$ such that $\{z, z'\} \in E(G)$. Consequently, $C \subset N_G(A)$. Furthermore, if $a \in N_G(x)$, then $a \in G \setminus x$ and $a \notin A$. Thus, $a \in N_G(A)$, since $A$ is a maximal stable set of $G \setminus x$. Hence, $N_G(x) \subset N_G(A)$. This implies that $N_G(A)$ is a vertex cover of $G$, since $C \subset N_G(A)$. Therefore, by Lemma 3, $A \in \mathcal{A}_G$, a contradiction since $A \notin \mathcal{F}_G$. □

Lemma 6 ([32], cf. Corollary 3.3). If $G \in W_2$, then every $v \in V(G)$ is a shedding vertex.

Proof. Let $v$ be a vertex of $G$. We may assume that $G$ is not a complete graph. Let $A$ be a stable set of $G_v := G \setminus N_G[v]$. We argue by contradiction assuming that $A$ is a maximal stable set of $G \setminus v$. Then, as $G$ and $G \setminus v$ are well-covered, we obtain:

$$\beta_0(G) = \beta_0(G \setminus v) = |A|.$$

According to [57], Theorem 5, the graph $G_v$ is in $W_2$ and $\beta_0(G_v) = \beta_0(G) - 1$. In particular, $G_v$ is well-covered and $\beta_0(G_v) = \beta_0(G) - 1$ (cf. Theorem 5). However, $A$ is a stable set of $G_v$ and $|A| = \beta_0(G)$, a contradiction. □

Corollary 5 ([4], Theorem 4.5). If $G$ is a $W_2$-graph and $I = I(G)$, then $v(I) = \beta_0(G)$.  

Proof. By Theorem 11, there is $D \in A_G$ such that $v(I) = |D|$. Since $G$ is a $W_2$-graph, by Lemma 6, every vertex of $G$ is a shedding vertex. Thus, by Proposition 4, $D \in F_G$, i.e., $D$ is a maximal stable set of $G$. Furthermore, $G$ is well-covered, since $G$ is a $W_2$-graph. Hence, $|D| = \beta_0(G)$. Therefore, $v(I) = \beta_0(G)$. □

5. Examples

Example 1. Let $S = \mathbb{Q}[t_1, t_2, t_3]$ be a polynomial ring and $I = (t_1^5, t_2^5, t_3^5, t_1^3 t_2^2, t_1^2 t_3, t_2 t_3^2)$. Then, an irredundant primary decomposition of $I$ is given by:

$$I = (t_1^5, t_2^5) \cap (t_3^5, t_1^3 t_2^2, t_1^2 t_3, t_2 t_3^2).$$

The associated primes of $I$ are $p_1 = (t_1, t_2)$ and $p_2 = (t_1, t_2, t_3)$. Setting $g_1 = t_1^2 t_2$, $g_2 = t_1^3 t_2^2 t_3$, and $g_3 = t_1^4 t_2 t_3^3$, and using Procedure A1 in Appendix A, we obtain that $(I : p_1) / I$ and $(I : p_2) / I$ are minimally generated by $\{\overline{g_1}, \overline{g_2}\}$ and $\{\overline{g_3}\}$, respectively. Using Theorem 10 and the equalities:

$$(I : g_1) = (t_1, t_2, t_3^5), \quad (I : g_2) = p_1, \quad (I : g_3) = p_2,$$

we obtain that $v(I) = 11$. The regularity of the quotient ring $S / I$ is equal to 12.

Example 2. Let $S = \mathbb{Q}[t_1, \ldots, t_6]$ be a polynomial ring; let $I$ be the ideal:

$$I = (t_1 t_2, t_2 t_3, t_3 t_4, t_4 t_5, t_5 t_6, t_5 t_7, t_4 t_5, t_1 t_6, t_2 t_6, t_3 t_6, t_4 t_6);$$

let $G$ be the graph defined by the generators of $I$. The associated primes of $I$ are:

$p_1 = (t_1, t_2, t_3, t_4), \quad p_2 = (t_1, t_3, t_5, t_6), \quad p_3 = (t_2, t_4, t_5, t_6).$

Thus, $I(G)$ is unmixed, $G$ is well-covered, and $a_0(G) = 4$. The graph $G$ is not very well-covered because $|V(G)| \neq 2a_0(G)$. The $v$-number of $I$ is one because $N_G(t_6) = \{t_1, t_2, t_3, t_4\}$ is a vertex cover of $G$. Using Macaulay2 [2], we obtain that $\text{reg}(S / I) = 1$. Note that $\text{im}(G) = 1$.

Example 3. Let $C_7$ and $T_{10}$ be the well-covered graphs of Figure 1. Let $R$ and $S$ be polynomial rings over the field $\mathbb{Q}$ in the variables $\{t_1, \ldots, t_7\}$ and $\{t_1, \ldots, t_{10}\}$, respectively. Using Macaulay2 [2] and Procedure A1 in Appendix A, we obtain $\text{ht}(I(C_7)) = a_0(C_7) = 4$, $\text{pd}(R / I(C_7)) = 5$, and:

$$v(I(C_7)) = 2 = \text{im}(C_7) = \text{reg}(R / I(C_7)) \leq \text{dim}(R / I(C_7)) = \beta_0(C_7) = 3.$$ 

The neighbor set of $A = \{t_1, t_4\}$ in $C_7$ is $N_{C_7}(A) = \{t_2, t_3, t_5, t_7\}$, and $N_{C_7}(A)$ is a minimal vertex cover of $C_7$, that is $A \in A_{C_7}$. Using Macaulay2 [2] and Procedure A1 in Appendix A, we obtain $\text{ht}(I(T_{10})) = a_0(G) = 6$, $\text{pd}(S / I(T_{10})) = 7$, and:

$$v(I(T_{10})) = 2 = \text{im}(T_{10}) \leq \text{reg}(S / I(T_{10})) = 3 \leq \text{dim}(S / I(T_{10})) = \beta_0(T_{10}) = 4.$$ 

The neighbor set of $A = \{t_1, t_4\}$ in $T_{10}$ is $N_{T_{10}}(A) = \{t_2, t_3, t_5, t_7, t_8, t_{10}\}$, and $N_{T_{10}}(A)$ is a minimal vertex cover of $T_{10}$, that is $A \in A_{T_{10}}$.

Example 4. Let $G$ be the graph consisting of two disjoint three cycles with vertices $x_1, x_2, x_3$ and $y_1, y_2, y_3$. Take two disjoint independent sets of $G$, say $A_1 = \{x_1\}$ and $A_2 = \{y_1\}$. To verify that $G$ is a graph in $W_2$, note that $B_1 = \{x_1, y_2\}$ and $B_2 = \{y_1, x_2\}$ are maximum independent sets of $G$ containing $A_1$ and $A_2$. And that $|B_i| = \beta_0(G) = 2$ for $i = 1, 2$.

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Appendix A. Procedures

Procedure A1. Computing the v-number and other invariants of a graded ideal I with Macaulay2 [2]. This procedure corresponds to Example 1. One can compute other examples by changing the polynomial ring S and the generators of the ideal I.

S=QQ[t1,t2,t3]
I=ideal(t1^5,t2^5,t2^4*t3^5,t1^4*t3^5)
--This gives the dimension and the height of I
--If I=I(G), G a graph, this gives the stability
--number and the covering number of G
dim(I), codim I
--This gives the associated primes of I
--If I=I(G), this gives the minimal vertex covers of G
L=ass I
--This determines whether or not I has embedded primes
--If I=I(G), this determines whether or not G is well-covered
apply(L,codim)
p=(n)->gens gb ideal(flatten mingens(quotient(I,L#n)/I))
--This computes a minimal generating set for (I:p)/I
MG=(n)->flatten entries p(n)
MG(0), MG(1)
--This gives the list of all minimal generators g of
--(I:p)/I such that (I: g)=p
F=(n)->apply(MG(n),x-> if not quotient(I,x)==L#n then 0 else x)-set{0}
F(0), F(1)
--This computes the v-number of a graded ideal I
vnumber=min flatten degrees ideal(flatten apply(0..#L-1,F))
M=coker gens gb I
regularity M
--This gives the projective dimension of S/I
pdim M

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