Temporal Conjunctive Query Answering in the Extended DL-Lite Family

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Received: date / Accepted: date

Abstract Ontology-based query answering (OBQA) augments classical query answering in databases by domain knowledge encoded in an ontology. Systems for OBQA use the ontological knowledge to infer new information that is not explicitly given in the data. Moreover, they usually employ the open-world assumption, which means that knowledge that is not stated explicitly in the data and that is not inferred is not assumed to be true or false. Classical OBQA however considers only a snapshot of the data, which means that information about the temporal evolution of the data is not used for reasoning and hence lost.

We investigate temporal conjunctive queries (TCQs) that allow to access temporal data through classical ontologies. In particular, we study combined and data complexity of TCQ entailment for ontologies written in description logics from the extended DL-Lite family. Many of these logics allow for efficient reasoning in the atemporal setting and are successfully applied in practice. We show comprehensive complexity results for temporal reasoning with these logics.

Keywords Description logics · Query answering · Temporal queries · DL-Lite

1 Introduction

Ontologies play a central role in various applications: by linking data from heterogeneous sources to high-level concepts and relations, they are used for automated data integration and processing. In particular, queries formulated in the abstract vocabulary of the ontology can then be answered over all the linked datasets. We focus on lightweight description logics as ontology languages, which are known to allow for efficient reasoning in the classical setting and are successfully applied in practice. Well-known

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medical domain ontologies like GALEN\(^1\) may, for example, capture the facts that the varicella zoster virus (VZV) is a virus, that chickenpox is a VZV infection, and that a negative allergy test implies that no allergies are present, by terminological axioms called concept inclusions (CIs):

\[
\text{VZV} \sqsubseteq \text{Virus}, \quad \text{Chickenpox} \sqsubseteq \text{VZVInfection}, \quad \text{NegAllergyTest} \sqsubseteq \neg \exists \text{AllergyTo}.
\]

Here, Virus is a concept name that represents the set of all viruses, and AllergyTo is a role name that represents a binary relation connecting patients to allergens; \(\exists \text{AllergyTo}\) refers to the domain of this relation, i.e., all patients with allergies. A possible data source storing patient data is depicted in Figure 1. The data is linked to the ontology by mappings \(\[68\]\); in our example, the tuple \((1, \text{Chickenpox}, 13.08.2007)\) can be encoded into the facts \(\text{HasFinding}(1, x)\) and \(\text{Chickenpox}(x)\), where \(x\) is a fresh symbol representing the finding and Chickenpox is the type of this finding, which may be contained in a fact base \(A_{13.08.2007}\) (i.e., the individual time points are days).

Ontology-based query answering (OBQA) can then assist in finding appropriate participants for a clinical study, by formulating the eligibility criteria as queries over the mapped patient data. The following are examples of inclusion and exclusion conditions for an existing clinical trial\(^2\):

- The patient should have been previously infected with VZV or previously vaccinated with VZV vaccine.
- The patient should not be allergic to VZV vaccine.

Considering the first condition, OBQA augments standard query answering (e.g., in SQL) in that not only Bob and Chris, but also Ann would be considered as an appropriate candidate. However, in standard OBQA, we can neither express negation (not) nor relate several points in time (previously), both of which are needed to faithfully represent the given criteria. In this article, we study temporal OBQA and allow negation in our query language.

We consider the temporal conjunctive queries (TCQs) proposed by \(\[14, 16\]\), which combine conjunctive queries (CQs) via the operators of propositional linear temporal logic LTL \(\[67\]\). The flow of time is represented by the sequence of natural numbers, i.e., every point in time (also time point or moment) is represented by one number. For example, the above criteria can be specified via the following TCQ, to obtain all eligible patients \(x\):

\[
(\Diamond P (\exists y. \text{HasFinding}(x, y) \land \text{VZVInfection}(y)) \lor \\
\Diamond P (\exists y. \text{VaccinatedWith}(x, y) \land \text{VZVVaccine}(y)) \land \\
\neg (\exists y. \text{AllergyTo}(x, y) \land \text{VZVVaccine}(y))).
\]

We here use the temporal operator \(\Diamond P\) (“at some time in the past”) and consider the symbols AllergyTo and VZVVaccine to be rigid, which means that their interpretation does not change

\(^1\)http://www.co-ode.org/ontologies/galen

\(^2\)https://clinicaltrials.gov/ct2/show/NCT01953900
over time. Hence, we assume someone having an allergy to VZV vaccine to have this allergy for his or her life.

We focus on the problem of evaluating a TCQ w.r.t. a temporal knowledge base (TKB), which contains the domain ontology and a finite sequence of fact bases. Each fact base contains the data associated to a specific point in time—from the past until the current time point \( n \) (“now”). In contrast, the domain knowledge is assumed to hold globally, meaning at every point in time. In this setting, the information within the ontology and the fact bases does not explicitly refer to the temporal dimension, but is written in a classical (atemporal) description logic (DL); only the query is temporal.

1.1 Related Work

There are various ways to represent time in DL modeling; for example, by considering time points as concrete datatypes \([19, 60]\) or formalisms inspired by action logics \([5, 45]\). Good overviews of different such approaches are provided in \([6, 7]\). We focus on temporal description logics that are two-dimensional combinations of standard temporal logics with DLs, which is nowadays the common approach, though there is no formal definition of what a temporal description logic should look like\( ^4 \). Such combinations still offer a wealth of degrees of freedom; for instance w.r.t. the base DL and temporal logic considered. Figure 2 depicts various description logics that are relevant for this article, and their relations in terms of expressivity. Earlier works investigate temporal versions of standard reasoning problems w.r.t. combined complexity and target applications such as terminologies with temporal aspects or temporal conceptual modeling \([63]\). In contrast, most recent investigations focus on temporal OBQA with the goal of accessing temporal data and also consider data complexity \([9]\).

Schild proposed the first combination of a DL and a point-based temporal logic based on a two-dimensional semantics \([72]\). Subsequent studies have focused on classifying different combinations of LTL and (extensions of) \( ALC \) according to expressivity and complexity, with results mostly in the range of \( \text{EXPSPACE} \), if rigid roles are disregarded \([7, 17, 58]\). An important outcome of that research is the observation that rigid roles and other forms of temporal roles usually lead to undecidability. Because decidability represents one major feature of DLs, research since then has been dedicated to the study of decidable temporal DLs. Lower and, in particular, tractable complexities are obtained by restricting the temporal logic or the DL component.

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\(^3\) Temporal description logics represent special kinds of combinations of DLs with modal logics \([58]\).
In contrast to TCQs (which are temporalized queries), many of these logics support temporalizing either concepts, CIs, or facts. Temporalized concepts allow to describe the temporal development of individuals, such as the collection of individuals that were vaccinated and, at the next $\bigcirc_F$ time point, had some allergic reaction:

$$\exists \text{VaccinatedWith} \sqcap \bigcirc_F \exists \text{HasFinding}. \text{AllergicReaction}$$

Temporalized facts can express, for example, that the patient with ID 1 did not have an allergic reaction since ($S$) the last vaccination:

$$\left( (\neg \exists \text{HasFinding}. \text{AllergicReaction})(1) \right) S \left( (\exists \text{VaccinatedWith})(1) \right).$$

Temporalized CIs describe concept inclusions that only hold temporarily (instead of globally, i.e., at all time points), e.g., to describe changing policies using the operators “always in the past” ($\square_P$) and “always in the future” ($\bigcirc_F$):

$$\square_P \left( \exists \text{HasFinding}. \text{Pneumonia} \sqsubseteq \exists \text{Recommend}. \text{Fluoroquinolone} \right) \land$$

$$\bigcirc_F \bigcirc_P \left( \exists \text{HasFinding}. \text{Pneumonia} \sqsubseteq \neg \exists \text{Recommend}. \text{Fluoroquinolone} \right).$$

Without rigid symbols, temporalizing $\mathcal{ALC}$ concepts or CIs does not lead to an increase in complexity from the ExpTime complexity of satisifiability in $\mathcal{ALC}$. However, rigid names lead to ExpSpace-completeness. The logics $\mathcal{ALC}$-LTL [18], $\mathcal{SHOQ}$-LTL [59], $\mathcal{EL}$-LTL [23], and $\mathcal{DL}$-Lite-LTL [10, 75] allow for combining DL axioms (CIs and facts) via LTL operators, but no temporal concepts. The setting where the concept inclusions are required to hold globally represents a simplified variant of the TCQ answering scenario, since assertions can be seen as simple CQs (without variables) and the fact bases are empty. Even more, negated CQs can simulate non-global CIs in $\mathcal{ALC}$, one main feature of $\mathcal{ALC}$-LTL. The satisifiability problem in $\mathcal{ALC}$-LTL has the same complexity as in the atemporal case, but rigid concepts and roles lead to Nexptime and 2-ExpTime-completeness, respectively, and thus to a considerable increase in complexity. This increase can be overcome, at least for rigid concepts, if CIs are only allowed to occur globally [18]. These results have been extended to $\mathcal{SHOQ}$-LTL [59]. Rigid names lead to Nexptime-completeness even in $\mathcal{EL}$-LTL and $\mathcal{DL}$-Lite-LTL [23, 75].

In another line of work, tractable DLs in combination with (subsets of) LTL have been investigated. The KB consistency problem is investigated for temporal extensions of $\mathcal{DL}$-Lite that allow for temporalizing concepts, including rigid roles, with positive results including containment in NLogspace or P, obtained by reduction to fragments of propositional temporal logic, but only for formalisms strongly constrained on the temporal side [11]. In most cases, more variety in that direction leads to NP-completeness and, if more expressive role expressions are allowed (e.g., arbitrary role inclusions), even to undecidability [11]. For both $\mathcal{DL}$-Lite and $\mathcal{EL}$, the integration of temporalized concepts, axioms, and rigid roles yields complexities such as PSpace for $\mathcal{DL}$-Lite$^{\text{atom}}$; ExpSpace for $\mathcal{DL}$-Lite$^{\text{horn}}$; and ExpSpace for $\mathcal{DL}$-Lite$^{\text{bool}}$; and even undecidability for $\mathcal{EL}$ [10]. These results are particularly interesting because, in the atemporal case, complexity results for $\mathcal{DL}$-Lite$^{\text{atom}}$ are usually worse than corresponding ones for $\mathcal{DL}$-Lite$^{\text{horn}}$ [3].

There are also recent works that temporalize concepts and/or axioms with temporal logic operators more expressive than LTL, such as from computation tree logic CTL [40, 42, 43].

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2. $\mathcal{ALC}$ allows to qualify the existential restriction, e.g., to restrict the range of the relation HasFinding. Most logics of the $\mathcal{DL}$-Lite family can only express qualified existential restrictions on the right-hand side of CIs.
and metric temporal logics \cite{13,41,76}. However, this mostly results in quite high complexities and is out of the scope of our work.

Query answering with the goal of retrieving data is the focus of recent research in DLs in general, and this is reflected in the latest explorations on reasoning about temporal knowledge. The different works can be classified w.r.t. the considered ontologies, depending on whether they are also temporal or written in a classical DL. The former approaches offer more expressivity but, on the other hand, tend to lead to higher reasoning complexities. For this reason, they are usually studied w.r.t. lightweight DLs.

Different temporal extensions of DL-Lite have been investigated with the goal of first-order rewritability results for temporal OBQA \cite{8}. The queries are arbitrary combinations of temporalized concept and role atoms using the operators of first-order temporal logic and thus very expressive, which is why epistemic semantics is employed\footnote{Queries with negation are not tractable, even in the atemporal setting \cite{38}.}. Although the setting studied is very restricted (roles are disregarded, only a subset of LTL is applied, the ABoxes contain a single individual name only), the results can, amongst others, be applied to show first-order rewritability of instance query answering in temporal DL-Lite. The rewritings are constructed based on temporal canonical interpretations for TKBs in these logics. Also for TKBs based on $\mathcal{EL}$ and allowing for both $\forall_F$ and $\forall_P$ there are canonical models, which can be used to show that the satisfiability problem is tractable w.r.t. data complexity if rigid roles are disallowed \cite{39}. The latter paper also identifies a certain periodicity of the ontology to ensure decidability and proposes several acyclicity notions for ontologies in temporal $\mathcal{EL}$ that yield tractable combined and data complexity.

Research in the second direction (temporal query answering with classical ontologies) has been first considered in a general way in \cite{44} for expressive operators on both the temporal and the DL side. A particular result shows first-order rewritability for DL-Lite$_{core}$, but this is achieved by considering epistemic semantics for CQs \cite{49,52}. Temporal conjunctive queries have first been studied for ALC \cite{14} and later for expressive extensions such as SHOQ \cite{15,16}, focusing on the complexity of the entailment problem under open-world semantics, resulting in very high combined complexities. Moreover, for many of the considered DLs this problem is $\text{co}-NP$-complete in data complexity, as in the atemporal case, even in the presence of rigid concept names. Subsequent works have investigated TCQ answering and entailment w.r.t. the most prominent lightweight description logics \cite{20,22,23,74,77}. There are also some works on non-standard reasoning problems for TCQs that are different from query answering and entailment, such as ABox abduction \cite{50,51}, but these are less relevant to our work.

There are also recent proposals of interval temporal description logics that allow for tractable OBQA \cite{3,12,55}, but their setting is rather different from the one we consider since there the basic units are intervals instead of time points.

This article can be seen as an extension of the studies on TCQs \cite{14,16,22} to the DLs of the extended DL-Lite family. In particular, it enhances previous results \cite{22,74} by an improved presentation and full and revised versions of all proofs. From another point of view, we extend a previously considered query language over temporal DL-Lite knowledge bases \cite{20} with unrestricted negation in TCQs (and consider infinite instead of finite temporal semantics). Our introductory example clearly shows that, even though most DL-Lite logics can only express a weak form of negation, i.e., disjointness constraints like $\text{NegAllergyTest} \sqsubseteq \neg \exists \text{AllergyTo}$, the negation in the query language can be meaningfully used to query for negative information entailed by the ontology. The impact of negation inside of CQs, i.e., nested inside existential quantification, has been shown to cause a large
increase in complexity even for atemporal \textit{DL-Lite}, even making it undecidable in very restricted cases \cite{38}. In contrast, our extension is much more well-behaved (see Section 1.2), though it clearly does not come for free. We also study the impact of allowing full negation in the ontology language, i.e., the very expressive logic \textit{DL-Lite}_{\text{pool}}. An alternative, more tractable approach is to consider epistemic semantics for negation \cite{44,52}, similar to approaches for closed-world reasoning in DLs \cite{1,62}. This semantics makes quite strong assumptions on the input data, by presupposing that all missing information is indeed false, which is at odds with the open-world nature of standard DL semantics. However, both closed-world and open-world semantics (or a combination of both) may be more or less suitable, depending on the application scenario.

1.2 Contributions

In this article, we study TCQs over the lightweight DLs of the extended \textit{DL-Lite} family, which are depicted in Figure 2 and were tailored for efficient (atemporal) query answering \cite{28,30}. Of particular interest in this setting is the question to what extent ontology-based temporal query answering is \textit{first-order rewritable}, which means that the queries can be rewritten into FO queries (e.g., in SQL) over a database, and then can be executed using standard database systems. This is possible in the atemporal case for the logic \textit{DL-Lite}_{\text{H horn}} \cite{29,31}. Note that \textit{DL-Lite}_{\text{R}}, the DL closest to the OWL 2 QL profile \cite{65}, extends \textit{DL-Lite}_{\text{core}} (a subset of \textit{DL-Lite}_{\text{H horn}}) only by disjointness axioms for roles. Although TCQ entailment over \textit{DL-Lite}_{\text{H horn}} ontologies turns out to be not first-order rewritable, certain parts of this problem can be solved using FO rewritings (see Section 5). We also study related, but more expressive logics such as \textit{DL-Lite}_{\text{bool}}{\text{H}}, where reasoning becomes harder \cite{4}.

We investigate both combined and data complexity of TCQ entailment and, as usual, distinguish three different settings for the rigid symbols:

(i) no symbols are allowed to be rigid,
(ii) only rigid concept names are allowed, and
(iii) both concept names and role names can be rigid.

As in \textit{ALC-LTL} \cite{18}, the fourth case is irrelevant since rigid concepts \(C\) can be simulated by rigid roles \(R_C\) via two CIs \(C \sqsubseteq R_C\) and \(C \sqsupseteq R_C\). Tables 1 and 2 summarize our results and compare them to the baseline complexity of atemporal query answering (the logics are ordered by complexity). On the one hand, for expressive members of the extended \textit{DL-Lite} family, we obtain complexities similar to those for very expressive DLs such as \textit{ALCHI}. In data complexity, there is not even a difference between the lightweight DL \textit{EL} and \textit{ALCHI} if rigid symbols are considered. On the other hand, for the logics below \textit{DL-Lite}_{\text{H horn}}, we get results that are even better than those for \textit{EL}; interestingly, here rigid names do not affect the complexity. The ALogTIME-hardness result for the data complexity of TCQ entailment in \textit{DL-Lite}_{\text{core}} shows however that it is not possible to find a (pure) first-order rewriting of TCQs in this setting. Nevertheless, our analysis gives hope for an efficient implementation of temporal query answering in \textit{DL-Lite}_{\text{H horn}}.

The article is structured as follows. In Section 2 we recall the preliminaries and describe a general approach for solving TCQ entailment \cite{14}; it is based on splitting the problem into separate problems in LTL and in (atemporal) DLs. In Section 3 we propose a characterization of the DL part of the TCQ satisfiability problem that is tailored to \textit{DL-Lite}_{\text{H horn}}. We use this characterization in Section 4 to obtain the \textit{PSPACE} combined complexity result. Based on our characterization, we then also show that parts of the TCQ entailment problem
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Table 1: Combined complexity of TCQ entailment compared to standard CQ entailment. New results are highlighted in gray. The complexity of CQ entailment in DL-Lite $[^{[H]}]$ is still open (marked by ?).

|       | CQ        | (i)     | (ii)   | (iii)   |
|-------|-----------|---------|--------|---------|
| $DL-Lite[^{[H]}]$ | NP       | PSpace  | PSpace | PSpace  |
| $\mathcal{EL}$ | NP       | PSpace  | PSpace | co-NExpTime |
| $\mathcal{ALC}[^{[H]}]$ | ExpTime | ExpTime | co-NExpTime | 2-ExpTime |
| $DL-Lite[^{[krom]}]$ | B| ExpTime | 2-ExpTime | 2-ExpTime | 2-ExpTime |
| $\mathcal{ALC}[^{[H]}][^{[I]}]$ | ExpTime | ExpTime | ExpTime | ExpTime |

Table 2: Data complexity of TCQ entailment compared to standard CQ entailment. New results are highlighted in gray.

|       | CQ        | (i)     | (ii)   | (iii)   |
|-------|-----------|---------|--------|---------|
| $DL-Lite[^{[H]}]$ | AC^0     | ALogTime | ALogTime | ALogTime |
| $\mathcal{EL}$ | P        | P       | co-NP  | co-NP  |
| $DL-Lite[^{[krom]}]$ | co-NP    | co-NP  | co-NP  | co-NP  |
| $\mathcal{ALC}[^{[H]}][^{[I]}]$ | co-NP    | co-NP  | co-NP  | co-NP  |

are first-order rewritable (see Section 5). This, in turn, allows us to develop an algorithm that proves membership in ALogTime in data complexity in Section 6. Section 7 covers DL-Lite logics beyond the Horn fragments.

2 Preliminaries

We first recall description logics of the DL-Lite family, conjunctive queries, linear temporal logic, and their combination into temporal conjunctive queries. Then, we discuss approaches for solving the classical (atemporal) and the temporal query entailment problems.

2.1 DL-Lite

Description logics can be seen as fragments of first-order logic. A DL signature (I,C,P) contains three sorts of non-logical symbols representing constants, unary and binary
Table 3: Semantics of roles, concepts, and axioms for an interpretation $I = (Δ^I, τ^I)$.

| Name                      | Syntax | Semantics                                                                 |
|---------------------------|--------|---------------------------------------------------------------------------|
| inverse role              | $R^-$  | $\{ (y, x) \in Δ^I \times Δ^I \mid (x, y) \in R^I \}$                  |
| existential restriction   | $\exists R$ | $\{ x \in Δ^I \mid \exists y \in Δ^I : (x, y) \in R^I \}$              |
| concept inclusion         | $B_1 \sqcap \cdots \sqcap B_m \sqsubseteq B_{m+1} \sqcup \cdots \sqcup B_{m+n}$ | $B_1^I \sqcap \cdots \sqcap B_m^I \sqsubseteq B_{m+1}^I \sqcup \cdots \sqcup B_{m+n}^I$ |
| role inclusion            | $R \sqsubseteq S$ | $R^I \subseteq S^I$                                                      |
| concept assertion         | $B(a)$ | $a^I \in B^I$                                                             |
| negated concept assertion | $\neg B(a)$ | $a^I \notin B^I$                                                          |
| role assertion            | $R(a, b)$ | $(a^I, b^I) \in R^I$                                                     |
| negated role assertion    | $\neg R(a, b)$ | $(a^I, b^I) \notin R^I$                                                   |

predicates, respectively: individual names $I$, concept names (primitive concepts) $C$, and role names (primitive roles) $P$, which are countably infinite, non-empty, pairwise disjoint sets. In the following, we fix such a signature.

**Definition 2.1 (Syntax of DL-Lite).** Roles and (basic) concepts are defined, respectively, by the following rules, where $A \in C$ and $P \in P$:

$$
R ::= P \mid P^-, \quad B ::= A \mid \exists R.
$$

The sets of all roles and basic concepts are denoted by $R$ and $B$, respectively.

Axioms are the following kinds of expressions: concept inclusions (CI)s of the form

$$
B_1 \sqcap \cdots \sqcap B_m \sqsubseteq B_{m+1} \sqcup \cdots \sqcup B_{m+n}
$$

where $B_1, \ldots, B_{m+n} \in B$; role inclusions (RI)s of the form $S \sqsubseteq R$, where $R, S \in R$; and assertions of the form $B(a), \neg B(a), R(a, b), \text{or } \neg R(a, b)$, where $B \in B, R \in R$, and $a, b \in I$. An ontology is a finite set of concept and role inclusions, and an ABox is a finite set of assertions. Together, an ontology $O$ and an ABox $A$ form a knowledge base (KB) $K := O \cup A$, also written as $K = (O, A)$. The set of all assertions is denoted by $A$.

We distinguish several members of the extended DL-Lite family as presented in [4]. For $c \in \{ \text{core, horn, krom, bool} \}$, we denote by DL-Lite$_c^R$ the logic that restricts the concept inclusions (1) as follows:

- if $c = \text{core}$, then $m + n \leq 2$ and $n \leq 1$;
- if $c = \text{horn}$, then $n \leq 1$;
- if $c = \text{krom}$, then $m + n \leq 2$;
- if $c = \text{bool}$, there are no restrictions.

We also consider the sublogics DL-Lite, that further disallow role inclusions. We use the term DL-Lite for a generic member of this family of logics.

We use the generic notation $X(\mathcal{Y})$ to denote those elements of $X$ that can be built from the names occurring in $\mathcal{Y}$. For example, if $X$ is the set $B$ of basic concepts and $\mathcal{Y}$ is a KB $K$, then $B(K)$ denotes the basic concepts that can be built from the concept and role names occurring in $K$ (either in concept inclusions or assertions). Similarly, $I(A)$ simply denotes the set of individual names occurring in the ABox $A$; we apply this notation more generally also to ontologies and single axioms, and later to queries, temporal queries, and so on.

The semantics is specified in a model-theoretic way, based on interpretations.
Definition 2.2 (Semantics of DL-Lite). An interpretation \( I = (\Delta^T, \cdot^I) \) consists of a non-empty set \( \Delta^T \), the \emph{domain} of \( I \), and an interpretation function \( \cdot^I \), which assigns to every \( A \in \mathcal{C} \) a set \( A^I \subseteq \Delta^T \), to every \( P \in \mathcal{P} \) a binary relation \( P^I \subseteq \Delta^T \times \Delta^T \), and to every \( a \in I \) an element \( a^I \in \Delta^T \) such that, for all \( a, b \in I \) with \( a \neq b \), we have \( a^I \neq b^I \) (unique name assumption; UNA). This function is extended to all roles and concepts as described in the first part of Table 3. An interpretation \( I \) is a model of an axiom \( \alpha \), if the corresponding condition given in Table 3 is satisfied. It is a model of a knowledge base \( \mathcal{K} \), if it is a model of all axioms contained in it.

Following the standard notation for first-order logic, we denote the fact that \( I \) is a model of \( \mathcal{K} \) by \( I \models \mathcal{K} \), and in this case also say that \( I \) satisfies \( \mathcal{K} \). Further, \( \mathcal{K} \) is consistent (or satisfiable) if it has a model, and inconsistent (or unsatisfiable) otherwise. \( \mathcal{K} \) entails an axiom \( \alpha \), written \( \mathcal{K} \models \alpha \), if all models of \( \mathcal{K} \) also satisfy \( \alpha \). Two KBs are equivalent if they have the same models. This terminology and notation for \( \mathcal{K} \) is extended to axioms, ontologies, and ABoxes by viewing each as a (singleton) KB. Moreover, we freely apply these terms to any “model of” relation that we define in the following (for queries, temporal queries, etc.).

Given two domain elements \( d, e \), a role \( R \), and an interpretation \( I \) such that \( (d,e) \in R^I \), we say that \( d \) is an \( R \)-predecessor of \( e \), and \( e \) is an \( R \)-successor of \( d \). We use the terms “(domain) elements” and “individuals” interchangeably for the elements of \( \Delta^T \). We call them “named” if they are used to interpret individual names. In concept inclusions of the form (1) (see Definition 2.1), we denote the empty conjunction by \( \top \) and the empty disjunction by \( \bot \), which are interpreted as \( \Delta^T \) and \( \emptyset \), respectively (cf. Table 3). We use the abbreviation \( \bigcap B \) for the conjunction \( B_1 \cap \cdots \cap B_m \) if \( B = \{B_1, \ldots, B_m\} \), and set \( (P^-)^I := P \) for all \( P \in \mathcal{P} \). We assume every KB to be such that all concept and role names occurring in the ABox also occur in the ontology.

Definition 2.3 (Syntax of CQs). Let \( V \) be a countably infinite set of variables disjoint from \( I, \mathcal{C} \), and \( \mathcal{P} \), and \( T := I \cup V \) be the set of terms. A conjunctive query (CQ) is of the form \( \varphi \in T \) (concept atom) with \( A \in \mathcal{C} \) and \( t \in T \); or

\[ (A(t)) \text{ (concept atom) with } A \in \mathcal{C} \text{ and } t \in T; \text{ or} \]

\[ (P(s,t)) \text{ (role atom) with } P \in \mathcal{P} \text{ and } s,t \in T. \]

A union of conjunctive queries (UCQ) is a disjunction (\( \lor \)) of CQs with the same free variables.

In general, CQs may contain free variables, also called answer variables. However, without loss of generality, and unless stated otherwise, in the following we assume all (U)CQs to be Boolean, i.e., that all variables are existentially quantified in the CQs. We sometimes stress this again, but actually make only one exception (in Section 5.2). For ease of presentation, we sometimes treat a CQ as a set, thereby meaning the set of all of its atoms.

Definition 2.4 (Semantics of CQs). A mapping \( \pi: T(\varphi) \rightarrow \Delta^T \) is a homomorphism of a CQ \( \varphi \) into an interpretation \( I \) if

\[ \pi(a) = a^I \text{ for all } a \in I(\varphi), \]

\[ \pi(t) \in A^I \text{ for all concept atoms } A(t) \text{ in } \varphi, \]

\[ (\pi(s), \pi(t)) \in P^I \text{ for all role atoms } P(s,t) \text{ in } \varphi. \]

An interpretation \( I \) is a model of \( \varphi \) if there is such a homomorphism, and is a model of a UCQ if it satisfies one of its disjuncts.
We also allow basic concept atoms of the form $\exists R(x)$ to occur in CQs, since such an atom can be expressed via a role atom $R(x, y)$ using a fresh, existentially quantified variable $y$. Similarly, inverse role atoms $R^-(x, y)$ can be expressed by $R(y, x)$.

For the interested reader, in Appendix A we introduce additional notions about DL-Lite and the temporal logic LTL that are relevant for our proofs.

2.2 Temporal Conjunctive Queries

Temporal conjunctive queries are a temporal query language introduced in [14]. They are basically formulas of LTL, but the variables are replaced by CQs and the semantics is suitably lifted from sequences of propositional worlds to sequences of DL interpretations. The flow of time is represented by the sequence of natural numbers, i.e., every point in time (also time point or moment) is represented by one number. We additionally assume that a subset of the concept and role names is designated as being rigid. The intuition is that the interpretation of rigid names does not change over time. All individual names are implicitly assumed to be rigid, i.e., to refer to the same domain element at all time points. If a concept (axiom) contains only rigid symbols, then we call it a rigid concept (axiom). We hence extend the signature $\Sigma$ by two sets $C^R \subseteq C$ and $P^R \subseteq P$ of rigid concept names and rigid role names, respectively. The elements of $C^F := C \setminus C^R$ and $P^F := P \setminus P^R$ are called flexible. We denote the sets of rigid roles, basic concepts, and assertions by $R^R$, $B^R$, and $A^R$, respectively, and their complements by $F^R$, $B^F$, and $A^F$, respectively.

In this temporal setting, the knowledge base contains a global ontology that holds at all time points, as well as a series of ABoxes describing a finite sequence of initial time points [14].

Definition 2.5 (Syntax of TKBs). A temporal knowledge base (TKB) $\mathcal{K} = (\mathcal{O}, (\mathcal{A}_i)_{0 \leq i \leq n})$ consists of an ontology $\mathcal{O}$ and a non-empty, finite sequence of ABoxes $\mathcal{A}_i$, $i \in [0, n]$.

As mentioned above, the semantics is given by sequences of DL interpretations.

Definition 2.6 (Semantics of TKBs). An infinite sequence $\mathcal{I} = (\mathcal{I}_i)_{i \geq 0}$ of interpretations $\mathcal{I}_i = (\Delta_i, \cdot \mathcal{I}_i)$ is a DL-LTL structure if it respects rigid names, i.e., we have $X^{\mathcal{I}_i} = X^{\mathcal{I}_j}$ for all $X \in 1 \cup C^R \cup P^R$ and $i, j \geq 0$. Such an interpretation $\mathcal{I}$ is a model of a TKB $\mathcal{K}$ if we have $\mathcal{I}_i \models \mathcal{O}$ for all $i \geq 0$, and $\mathcal{I}_i \models \mathcal{A}_i$ for all $i \in [0, n]$.

Observe that the interpretations in a DL-LTL structure share a single domain (constant domain assumption). Similarly, we say that any finite collection of interpretations $\mathcal{I}_1, \ldots, \mathcal{I}_l$ respects rigid names if they have the same domain and agree on the interpretation of all rigid symbols. As with atemporal KBs, we assume all concept and role names occurring in some ABox of a TKB to also occur in its ontology.

As outlined above, TCQs combine conjunctive queries via LTL operators. We again restrict our focus to Boolean queries.

Definition 2.7 (Syntax of TCQs). The set of temporal conjunctive queries (TCQs) is defined as follows, where $\varphi$ is a CQ:

$$\Phi, \Psi ::= \varphi | \neg \Phi | \Phi \land \Psi | \bigcirc_{F} \Phi | \bigcirc_{P} \Phi | \Phi / \Psi | \Phi \cdot \Psi.$$

A CQ literal is of the form $\varphi$ (positive CQ literal) or $\neg \varphi$ (negative CQ literal), for a CQ $\varphi$. 
Table 4: Definitions of derived temporal operators.

| Operator | Definition                                                                 | Name       |
|----------|---------------------------------------------------------------------------|------------|
| true     | \( \phi \lor \neg \phi \) for some CQ \( \phi \)                      | tautology  |
| false    | \( \neg \neg \phi \)                                                   | contradiction |
| \( \phi \lor \psi \) | \( \neg (\neg \phi \land \neg \psi) \)                                      | disjunction |
| \( \phi \rightarrow \psi \) | \( \neg \phi \lor \psi \)                                                                 | implication |
| \( \phi \leftrightarrow \psi \) | \( (\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \)                  | bi-implication |
| \( \overline{P}_F \Phi \) | \( \circ_F \ldots \circ_F \Phi (i \text{ times}) \)                      | iterated next |
| \( \overline{P}_R \Phi \) | \( \circ_R \ldots \circ_R \Phi (i \text{ times}) \)                      | iterated previous |
| \( \overline{S}_F \Phi \) | true\( \overline{U} \Phi \)                                            | eventually (at some time in the future) |
| \( \overline{S}_R \Phi \) | \( \neg \circ_R \rightarrow \Phi \)                                    | globally (always in the future) |
| \( \overline{S}_P \Phi \) | \( \neg \circ_P \rightarrow \Phi \)                                    | historically (always in the past) |

Table 5: Semantics of TCQs for a DL-LITE structure \( \mathcal{J} = (I_i)_{i \geq 0} \).

| TCQ \( \Phi \) | Condition for \( \mathcal{J}, i, j \models \Phi \) |
|-----------------|-----------------------------------------------------|
| \( C \Phi \)    | \( I_i \models \phi \) \land \( I_j \notmodels \phi \) |
| \( \Phi \lor \psi \) | \( I_i \notmodels \Phi \) \land \( I_j \models \Phi \lor \psi \) |
| \( \Phi \land \psi \) | \( I_i \models \Phi \) \land \( I_j \models \psi \) |
| \( \overline{P}_F \Phi \) | \( I_i \models \Phi \) \land \( I_{i+1} \models \Phi \) |
| \( \overline{P}_R \Phi \) | \( I_i \notmodels \Phi \) \land \( I_{i-1} \models \Phi \) |
| \( \Phi \overline{U} \psi \) | there is a \( k \geq i \) such that \( I_k \models \psi \) and, for all \( j, i \leq j < k \), we have \( I_j \notmodels \Phi \) |
| \( \Phi \overline{S} \psi \) | there is a \( k < i \) such that \( I_k \models \psi \) and, for all \( j, k < j \leq i \), we have \( I_j \models \Phi \) |

The operators \( \overline{P}_F \) and \( \overline{P}_R \) are called “next” and “previous”, respectively. The formula \( \Phi \overline{U} \psi \) stands for “\( \Phi \) until \( \psi \)”, and \( \Phi \overline{S} \psi \) is read “\( \Phi \) since \( \psi \)”. The operators \( \overline{P}_F \) and \( \overline{U} \) are the future operators, and \( \overline{P}_R \) and \( \overline{S} \) are the past operators. Together, they represent the temporal operators. Further, derived operators are defined in Table 4.

**Definition 2.8 (Semantics of TCQs).** For a given DL-LITE structure \( \mathcal{J} = (I_i)_{i \geq 0} \), an \( i \geq 0 \), and a Boolean TCQ \( \Phi \), the satisfaction relation \( \mathcal{J}, i \models \Phi \) is as defined in Table 5.

\( \Phi \) is **satisfiable** w.r.t. a TKB \( \mathcal{K} = \langle \mathcal{O}, \mathcal{A} \rangle_{0 \leq i \leq n} \) if there is a model \( \mathcal{J} \) of \( \mathcal{K} \) such that \( \mathcal{J}, n \models \Phi \), i.e., where \( \Phi \) is satisfied at the current time point \( n \). Similarly, \( \Phi \) is **entailed** by \( \mathcal{K} \) if we have \( \mathcal{J}, n \models \Phi \) for all models \( \mathcal{J} \models \mathcal{K} \).

The satisfiability and entailment problems are mutually reducible. Indeed, \( \Phi \) is **not** entailed by \( \mathcal{K} \) iff the TCQ \( \neg \Phi \) is satisfiable w.r.t. \( \mathcal{K} \), and vice versa. Hence, the complexity of both problems is always complementary. Since the techniques we use apply to satisfiability, we will consider this problem for most of the technical development. However, we formulate our results in terms of entailment since that problem is more interesting from a practical point of view. Hence, in the main part of this article, we investigate the **satisfiability problem of TCQs** w.r.t. TKBS, i.e., the problem of finding a common model of both.

We denote by \( Q_b \) the CQs in the Boolean TCQ \( \Phi \) and assume without loss of generality that these CQs use disjoint sets of variables. We further assume that TCQs contain only individual names that occur in the ABoxes, and only concept and role names that occur in the ontology; this is clearly without loss of generality, especially because of the assumption that all concept and role names occurring in a TKB occur in its ontology. We further assume all CQs \( \phi \) to be connected, i.e., that all elements in \( T(\phi) \) can be reached from the others via a series of role atoms in \( \phi \). This assumption is also without loss of generality, because a
disconnected CQ can be split into a conjunction of several CQs, which is a special kind of TCQ; see [4, 16] for details.

We often consider TCQs $\Phi$ that do not contain temporal operators; for example, UCQs or conjunctions of CQ literals. In this case, the satisfaction of $\Phi$ in a DL-LTL structure $\mathcal{J} = (\mathcal{I}_i)_{i \geq 0}$ at time point $i$ only depends on the interpretation $\mathcal{I}_i$. For simplicity, we then often write $\mathcal{I}_i \models \Phi$ instead of $\mathcal{J}, i \models \Phi$. In this context, it is also sufficient to consider classical knowledge bases $\langle \mathcal{O}, \mathcal{A} \rangle$, which can be viewed as TKBs with a single ABox.

We investigate the influence of the different DL-Lite logics on the combined and data complexity of TCQ entailment. For combined complexity, the size of all the input is taken into consideration (i.e., the size of both the query and the entire KB), while data complexity only refers to the size of the data [78] (i.e., the number and size of the ABoxes).

2.3 Reasoning with TCQs

We recall the general approach to decide TCQ satisfiability from [14], where the problem of finding a common model $(\mathcal{I}_i)_{i \geq 0}$ of a TCQ $\Phi$ and a TKB $\mathcal{K} = \langle \mathcal{O}, \mathcal{A} \rangle_{0 \leq i \leq m}$ is split into an LTL satisfiability problem and several DL satisfiability problems [16, Lemma 4.7].

The former consider the propositional abstraction $\Phi^p$ of $\Phi$, which is obtained from $\Phi$ by replacing the CQs $\varphi_1, \ldots, \varphi_m \in \mathcal{Q}_\Phi$ by propositional variables $p_1, \ldots, p_m$, respectively. The idea is that the world $w_i$ in an LTL model $(w_i)_{i \geq 0}$ of $\Phi^p$ determines which CQs from $\mathcal{Q}_\Phi$ should be satisfied at time point $i$. To obtain an interpretation $\mathcal{I}_i$ from $w_i$, we have to check the satisfiability of the CQ literals that are induced by $w_i$, where $\mathcal{W}_w := \{p_1, \ldots, p_m\} \setminus w_i$ denotes the complement of $w_i$. For this, it is enough to consider the atemporal KB $\langle \mathcal{O}, \mathcal{A}_i \rangle$, for which we assume that $\mathcal{A}_i = \emptyset$ whenever $i > n$. However, the problem is that independent satisfiability tests for each time point are not enough, since the DL interpretations should also respect the rigid names.

Hence, we need to connect the LTL and DL satisfiability problems more closely. For this, we consider a set $W = \{W_1, \ldots, W_k\} \subseteq 2^{\{p_1, \ldots, p_m\}}$ of possible worlds which is given as input to both problems. Intuitively, these are the worlds that are allowed to occur in the LTL model. Moreover, for each of the initial time points $0, \ldots, n$, we fix one of these worlds, via a mapping $\lambda: [0, n] \rightarrow [1, k]$. The idea is to simultaneously look for models of the conjunctions

$$X_{\lambda(i)} := \bigwedge_{p_j \in W_{\lambda(i)}} \varphi_j \land \bigwedge_{p_j \in W_{\lambda(i)}} \neg \varphi_j$$

w.r.t. the atemporal KBs $\langle \mathcal{O}, \mathcal{A}_i \rangle$. For the time points after $n$, we do not have to consider ABoxes; however, for ease of presentation, we set $\mathcal{A}_{n+i} := \emptyset$ and $\lambda(n+i) := i$ for all $i \in [1, k]$, which means that we artificially extend the ABox sequence to cover $n + k + 1$ "time points": $n + 1$ time points for the worlds associated to an input ABox, and $k$ additional time points for all worlds in the model that are not influenced by the input ABoxes.

The LTL part is characterized by temporal satisfiability ($t$-satisfiability), and rigid satisfiability ($r$-satisfiability) summarizes the DL part.

**Definition 2.9** ($t$-satisfiable). The LTL formula $\Phi^p$ is $t$-satisfiable w.r.t. $W$ and $\lambda$ if there is an LTL structure $\mathcal{J} = (w_i)_{i \geq 0}$ such that $w_i \in W$ for all $i \geq 0$, $w_i = W_{\lambda(i)}$ for all $i \in [0, n]$, and $\mathcal{J}, n \models \Phi^p$.

*In the following, we denote propositional worlds both by lower case $w$ and by upper case $W$ (with indices). Usually, the (finitely many) elements of $W$ are denoted by $W_i$, and the (infinitely many) elements of an LTL structure by $w_i$.*
Definition 2.10 (r-satisfiable). The set \( W \) is r-satisfiable w.r.t. \( \lambda \) and \( K \) iff there are interpretations \( J_0, \ldots, J_{n+k} \) as follows:

- the interpretations share the same domain and respect rigid names,
- for all \( i \in [0, n+k], J_i \) is a model of \( O, A_i \), and \( \chi_{\lambda(i)} \).

The satisfiability of \( \Phi \) w.r.t. \( K \) can then be decided by combining the above definitions.

Lemma 2.11 (see [16, Lem. 4.7]). A TCQ \( \Phi \) has a model w.r.t. a TKB \( K \) iff there exist a set \( W = \{ W_1, \ldots, W_k \} \subseteq 2^{\{p_1, \ldots, p_m\}} \) and a mapping \( \lambda : [0, n] \rightarrow [1, k] \) such that \( \Phi^{\text{ps}} \) is t-satisfiable w.r.t. \( W \) and \( \lambda \), and \( W \) is r-satisfiable w.r.t. \( \lambda \) and \( K \).

The original proof in [16] considers the DL\(SHQ\), but it is independent of the description logic under consideration, and hence also applies in our setting. This result shows that TCQ satisfiability can be split into the three subproblems of

(i) obtaining \( W \) and \( \lambda \),
(ii) solving the LTL satisfiability test (t-satisfiability), and
(iii) solving the DL satisfiability test(s) (r-satisfiability).

However, for solving these problems in DL-Lite\(H_{\text{horn}}\), we cannot in general apply the existing methods from [14–16] since we want to show considerably lower complexity bounds, namely ALogTime in data complexity and PSPACE in combined complexity. The linear size of \( \lambda \) is an obstacle for designing algorithms of sublinear complexity, and the exponential size of \( W \) makes it impossible to guess (and store) this set using only a polynomial amount of space. Further, known results only allow to solve Problems (ii) and (iii) in EXPTIME in combined complexity.

Our first step is to provide a new characterization of r-satisfiability that is tailored to DL-Lite\(H_{\text{horn}}\), and can be decided using only polynomial space (see Section 3). It also allows us to show that r-satisfiability is first-order rewritable (see Section 5). In Sections 4 and 6 we then determine the combined and data complexity of satisfiability in DL-Lite\(H_{\text{horn}}\), respectively. There we integrate our characterization of r-satisfiability, which solves Problem (iii), into algorithms that additionally solve Problems (i) and (ii) while satisfying the corresponding resource constraints.

3 Characterizing r-Satisfiability in DL-Lite\(H_{\text{horn}}\)

We consider a DL-Lite\(H_{\text{horn}}\) TKB \( K = (O, (A_i)_{0\leq i < n}), \) a Boolean TCQ \( \Phi \), and a set \( W \) and a mapping \( \lambda \) as in the previous section. We consider these objects fixed, and do not always explicitly include them in the notation we use in the following. The goal is to find interpretations \( J_i \) satisfying \( (O, A_i) \) and \( \chi_{\lambda(i)}, i \in [0, n+k] \), that respect the rigid names.

To solve this problem in PSPACE (and later in ALogTime in data complexity), the idea is to guess a polynomial amount of additional information that allows us to split the above tests into independent satisfiability tests. The additional information enforces a certain connection between these tests, which simulates the following effects of the shared domain:

(F1) The interpretation of rigid names over the named individuals is synchronized.
(F2) The satisfiability of \( \chi_{\lambda(i)} \) at time point \( i \) cannot be contradicted by the interpretation of the rigid names at the other time points.

In this section, we first describe the precise form that this additional information takes, then present the characterization itself, and lastly prove its correctness.
We augment the consistency tests for the KBs \( \langle O, A_i \rangle \) and for the conjunctions \( \chi_{\lambda(i)} \) by additional ABoxes and conditions that encode the information that is shared between the time points. More formally, the additional information is a tuple \((A^R, Q^+, Q^-, S)\), where

- \( A^R \) is a set of rigid assertions over the names occurring in \( \mathcal{K} \), specifying the behaviour of the rigid concept and role names on all named individuals;
- \( Q^+ \subseteq Q_\Phi \) contains those CQs that are satisfied in at least one of the interpretations;
- \( Q^- \subseteq Q_\Phi \) specifies which CQs are not satisfied by at least one of the interpretations; and
- \( S \) is a set of assertions of the form \( \exists S(b) \), where \( S \) is a flexible role name; such an assertion encodes the information that \( b \) has an \( S \)-successor at some point in time, which means that the influence of this successor on the interpretation of the rigid names needs to be taken into account also at the other time points.

Roughly speaking, \( A^R \) simulates the effect \([F1]\) of the common domain, while \( Q^+, Q^- \), and \( S \) express \([F2]\). The additional information thus consists of a number of assertions and queries that is polynomial in the size of \( \Phi \). Each of these four items gives rise to (i) additional ABoxes and/or (ii) external conditions that have to be satisfied by \( O, A_i \), and \( W_{\lambda(i)} \). In the following subsections, we define those precisely.

### 3.1 Rigid ABox Type

The set \( A^R \) is a so-called *rigid ABox type*, which completely fixes the interpretation of the rigid names on the individual names.

**Definition 3.1 (Rigid ABox Type).** A rigid ABox type for \( \mathcal{K} \) is a set \( A^R \subseteq A^R(\mathcal{K}) \) such that, for all \( \neg \alpha \in A^R(\mathcal{K}) \), we have \( \neg \alpha \in A^R \) iff \( \alpha \notin A^R \).

We require that all interpretations \( \mathcal{I}_i \) must satisfy the assertions in \( A^R \).

### 3.2 Rigid Consequences

The second set, \( Q^+ \), contains all CQs \( \varphi_j \in Q_\Phi \) for which \( p_j \) occurs in some \( W_{\lambda(i)} \) with \( i \in [0, n + k] \) (which means that \( \varphi_j \) occurs positively in \( \chi_{\lambda(i)} \), and hence must be satisfied by \( \mathcal{I}_i \)). We keep track of these CQs because their satisfaction implies the presence of certain rigid structures at all time points. To explicitly refer to these structures, we instantiate all variables in CQs by fresh individual names. Formally, given a set \( Q \subseteq Q_\Phi \) (e.g., \( Q^+ \)), the ABox \( A_Q \) is obtained by replacing every variable \( x \) in \( Q \) with a fresh individual name \( a_x \), and viewing the resulting (ground) CQ as a set of assertions.

**Definition 3.2 (Rigid Consequences).** The set \( A^R_Q \) of rigid consequences of a set of CQs \( Q \) (w.r.t. \( O \)) contains exactly those assertions \( \alpha \in A^R(\langle O, A_Q \rangle) \) that are entailed by \( \langle O, A_Q \rangle \).

The set \( Q^+ \) contributes the assertions in \( A^R_Q^+ \) to the individual consistency tests, as well as the additional condition mentioned above; i.e., every CQ that is satisfied at some time point must be included in \( Q^+ \).
3.3 Rigid Witnesses

For the set $Q^-$ of all CQs from $Q_\Phi$ that are not satisfied at some time point, we have to enforce a dual condition to the one above. That is, we have to disallow the presence of any rigid structures that imply the satisfaction of such a CQ. We consider first the case that a CQ is satisfied by the unnamed part of an interpretation $I_i$. The case where the CQ is satisfied (partly) by named individuals is captured by the set $S$.

**Definition 3.3 (Rigid Witness Query).** A CQ $\psi$ is a rigid witness query of a set of CQs $Q$ if there exists $\varphi \in Q$ such that

- $\langle O, A(\psi) \rangle \models \varphi$, i.e., whenever $\psi$ is satisfied, the same must hold for $\varphi$;
- $C(\psi) \cup P(\psi) \subseteq C^R(O) \cup P^R(O)$, i.e., $\psi$ uses only rigid names; and
- $|T(\psi)| \leq |T(\varphi)|$, i.e., the size of $\psi$ is bounded by a polynomial in the sizes of $\varphi$ and $O$.

The condition imposed by $Q^-$ requires that no witness of $Q^-$ is satisfied at any time point, because that would imply that an element of $Q^-$ has to be satisfied at every time point, contradicting the purpose of $Q^-$.

3.4 Flexible Successors of Named Elements

The set $S$ represents the last part of the additional information we have to guess. It contains information about flexible role successors of named individuals, to capture possible effects of RIs involving both rigid and flexible roles, as sketched in the following example.

**Example 3.4.** At $n = 1$, the TCQ

$$\Phi := (\bigcirc_p A(a)) \land \neg (\exists x. B(a) \land R(a, x) \land T(a, x))$$

is not satisfiable w.r.t. the TKB $K = (O, \{\emptyset, B(a)\})$, where $O$ contains CLs $A \sqsubseteq S$, $S \sqsubseteq R$, and $S \sqsubseteq T$, and $R$ and $T$ are the only rigid symbols. This is because every model $I_0 \models I_1 \models \dots$ of $K$ and $\Phi$ must satisfy $I_0 \models O, I_1 \models O, I_0 \models \varphi_1, I_1 \models B(a)$, and $I_1 \models \varphi_2$, where $\varphi_1 = A(a)$ and $\varphi_2 = \exists x. B(a) \land R(a, x) \land T(a, x)$. Thus, there has to be an element $e$ such that the tuple $(a, e)$ is contained in $S^{I_0}, R^{I_0}$, and $T^{I_0}$. Since $I$ respects the rigid names, this means that this tuple is also contained in $S^{I_1}$ and $T^{I_2}$. Hence, $I_1 \models \exists x. B(a) \land R(a, x) \land T(a, x)$, which is a contradiction.

Rigid ABox types, consequences, and witnesses, however, do not help in this case, since $K$ and $\Phi$ imply that a named individual has a flexible role successor $e$ that implies several rigid relations to the same element $e$. To see this, consider $W_1 = \{p_1\}$, $W_2 = \emptyset$, a mapping $\lambda = \{0 \mapsto 1, 1 \mapsto 2\}$, and arbitrary interpretations $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_1$ as in Definition 2.10 except that they do not share one domain. We must then have $Q^+ = \{\varphi_1\}$ and $Q^- = \{\varphi_2\}$. The corresponding rigid ABox type $A^R = \{\exists R(a), \exists T(a)\}$ captures the existence of the rigid relations, but not the fact that they refer to the same domain element. That is, even if we require $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ to all satisfy $A^R$, this only means that $a$ must have one $R$-successor and one $T$-successor (except in $\mathcal{J}_0$ and $\mathcal{J}_2$, where they must be the same element). Likewise, the only rigid consequences of $\varphi_2$ are $\{\exists R(a), \exists T(a)\}$. There are no rigid witnesses for $\varphi_2$ since this query is not entailed by a combination of rigid names. Hence, the additional information in the rigid ABox type, consequences, and witnesses alone cannot detect the contradiction in $I_1$ as described above.
Formally, $S$ contains assertions of the form $\exists S(b)$ with $S \in R^F(O)$ and $b \in I(\mathcal{K}) \cup I^{aux}$, where $I^{aux} \subseteq I$ contains all individual names $a_i$ we introduced in Section 3.2 for the variables $x$ occurring in the CQs of $Q_\varphi$. Because of our assumption that the CQs have no variables in common, each $a_i \in I^{aux}$ can be associated to the unique CQ containing $x$.

The set $S$ captures information about which named elements can have which kinds of flexible role successors, which gives rise to the (rigid) ABox $A_S := \bigcup \exists S(b) \in S \cdot A_{\exists S(b)}$, where $A_{\exists S(b)}$ is constructed as follows.

1. For every domain element $u_{bp}$ of the canonical interpretation $I(O, \exists S(b))$, where the length of $\rho$ is at most $\max \{|\varphi| \mid \varphi \in Q_\varphi\}$, introduce a new individual name $a_{bp}$. These new individual names are collected in the set $I^{free}$.
2. For every $a_{pS} \in I^{free}$, add the following rigid assertions to the set $A_{\exists S(b)}$, which is initially empty:
   - for every $b \in B^R(O)$ with $O \models S \subseteq b$, the concept assertion $B(a_{pS})$;
   - for every $r \in R^R(O)$ with $O \models S \subseteq r$, the role assertion $R(a_{pS})$ if $\rho \not\in I(\mathcal{K})$, otherwise $R(\rho, a_{pS})$.

Observe that these consequences only have to be considered up to a depth which ensures that possible matches of the CQs in $Q_\varphi$ can be fully characterized. The ABox $A_S$ is of exponential size, but this does not affect the complexity results (see Sections 4 and 6).

### 3.5 R-complete Tuples

Based on the tuple $(A^R, Q^+, Q^-, S)$, we hence consider the ABoxes $A^R_i$, $A^{Q^+}_i$, and $A_S$ described above. In addition, at each $i \in \{0, n+k\}$, we check consistency also w.r.t. $A_i$ and $A_{Q^+_i}$, where $Q_{\lambda(i)}$ contains all CQs $\varphi_j$ that occur positively in the conjunction $\chi_{\lambda(i)}$, i.e., for which we have $p_j \in W_{\lambda(i)}$. This gives rise to the knowledge bases

$$\mathcal{K}^\epsilon_{\lambda} := (O, A^R \cup A^{Q^+} \cup A_{Q^+_i} \cup A_S \cup A_i)$$

which we need to check for consistency. Observe that $A^{Q^-}_i$, $A_Q$, and $A_S$ may share individual names from $I(\mathcal{K})$ and $I^{aux}$. Different ABoxes $A_{Q^+_i}$ and $A_Q$ share individual names from $I^{aux}$ for the CQs in $Q_i \cap Q_j$, while individual names from $I^{free}$ only occur in $A_S$.

We finally summarize the conditions outlined above in the following definition.

**Definition 3.5** (r-complete). A tuple $(A^R, Q^+, Q^-, S)$ as above is r-complete (w.r.t. $W$ and $\lambda$) if the following hold for all $i \in \{0, n+k\}$.

(\text{C}1) $\mathcal{K}^\epsilon_{\lambda}$ is consistent.
(\text{C}2) For all $p_j \in W_{\lambda(i)}$, we have $\mathcal{K}^\epsilon_{\lambda} \not\models \varphi_j$.
(\text{C}3) For all $p_j \in W_{\lambda(i)}$, we have $\varphi_j \in Q^-$.
(\text{C}4) For all $p_j \in W_{\lambda(i)}$, we have $\varphi_j \in Q^-$.
(\text{C}5) For all CQs $\varphi \in Q^-$ and rigid witness queries $\psi$ for $\varphi$ w.r.t. $O$, we have $\mathcal{K}^\epsilon_{\lambda} \not\models \psi$.
(\text{C}6) For all $S \in R^F(O)$ and $b \in I(\mathcal{K}) \cup I^{aux}$, we have $\exists S(b) \in S$ if there is an index $j \in \{0, n+k\}$ such that $(O, A^R \cup A^{Q^+}_i \cup A_{Q^+_j} \cup A_j) \models \exists S(b)$.

Note that all conditions except \text{C6} refer only to a single index $i$. Conditions \text{C1} and \text{C2} ensure that we can actually satisfy $(O, A_i)$ and $\chi_{\lambda(i)}$, together with the additional ABoxes. As described in Sections 3.2 and 3.3, Conditions \text{C3} and \text{C4} make sure that only the queries from $Q^+$ (resp., $Q^-$) can occur in some $W \in W$ (resp., $\overline{W}$). Condition \text{C5}
checks that the queries corresponding to some \( W \) are not entailed because of the rigid names, by requiring that the KBs \( K^i_{rc} \) do not entail any of their witnesses. And the last condition ensures that \( S \) is minimal; that is, that it contains only those assertions \( \exists S(b) \) that are required by one of the KBs \( K^i_{rc} \) (excluding \( A_S \)).

**Lemma 3.6.** \( W \) is \( r \)-satisfiable w.r.t. \( \lambda \) and \( K \) iff there is an \( r \)-complete tuple w.r.t. \( W \) and \( \lambda \).

A full proof of this lemma can be found in Appendix [B], we only sketch the main ideas here. For the “only if”-direction, let \( J_0, \ldots, J_{n+k} \) be interpretations over a common domain \( \Delta \), which exist according to the \( r \)-satisfiability of \( W \) (see Definition [2.10]). Finding \((\tilde{A}^R, Q^+, Q^-, S)\) is then straightforward: since the interpretations respect rigid names, \( \tilde{A}^R \) is uniquely defined, \( Q^+, Q^- \) are determined by \( W \) (see Conditions (C3) and (C4)), and \( S \) is then given by Condition (C6). It is rather easy to show that each of the knowledge bases \( K^i_{rc} \) has a model (Condition (C1)) which satisfies neither a CQ from \( Q_{\lambda} \) \( \setminus \) \( Q_{j(i)} \) (Condition (C2)) nor a witness of the queries in \( Q^- \) (Condition (C5)). However, special attention needs to be given to the UNA and Condition (C6). The problem is that the homomorphisms witnessing \( J_i \models X_{j(i)} \) may map several variables to the same domain element, but the new individual names \( a_i \in I^{aux} \bigcap A_{Q_{j(i)}} \) are required to be different by the UNA. Similar considerations apply to the individual names \( I^{tree} \) in \( A_S \). The solution is to introduce enough copies of these domain elements in order to satisfy the UNA.

For the “if”-direction, we assume an \( r \)-complete tuple \((\tilde{A}^R, Q^+, Q^-, S)\) to be given, and need to find interpretations \( J_0, \ldots, J_{n+k} \) that satisfy the requirements of Definition [2.10]. The idea is to start with the canonical interpretations of the KBs given by Condition (C1) and then merge them to obtain a common domain \( \Delta \) while satisfying the rigid names.

We introduce a few auxiliary notions, for \( i \in \{0, n + k\} \):

- \( I_i \) is an abbreviation for the canonical interpretation \( I_{K^i_{rc}} \) of \( K^i_{rc} \) as specified in Definition [A.1].
- We rename every element \( a_i \in I^{aux} \bigcap I_i \) to \( a_i' \), and collect all these elements in the set \( I_i^{aux} \). We similarly define \( I_i^\text{tree} \) and \( I_i^{\text{anon}} \) based on the elements of \( I_i^\text{tree} \bigcap I_i^{\text{anon}} \) and the unnamed domain elements of \( I_i \), respectively.
- \( \text{Witnesses} \) are similar to rigid witness queries, but simpler. A witness is a set of rigid basic concepts that, if satisfied in a model of \( O \), implies the existence of a particular role chain. We use them to detect whether we need to include certain anonymous domain elements from \( I_j \) also at other time points \( j \neq i \).

**Definition 3.7 (Witness).** Let \( I \) be the canonical interpretation for a knowledge base \((O, A)\). A set \( B^R \subseteq B^R(O) \) is a witness of \( u_p \in I^{\text{anon}} \) w.r.t. \((O, A)\) if \( \rho = \sigma R_0 \ldots R_t \sigma \) is such that \( O \models \bigcap B^R \subseteq \exists R_0 \) and either \( u_p \in (\bigcap B^R)^T \) or \( \sigma \in I(A) \bigcap (\bigcap B^R)^T \). The set of all witnesses of \( u_p \) is denoted by \( \text{Wit}(u_p) \). For all \( e \in I^{\Delta^\text{anon}} \), we define \( \text{Wit}(e) := \emptyset \).

Hence, the domain of \( I_i \) is composed of the pairwise disjoint sets \( I(K), I_i^{\text{anon}}, I_i^{\text{tree}}, \) and \( I_i^{\text{anon}} \). Moreover, the domains of different interpretations \( I_i, I_j \) only overlap in \( I(K) \). The common domain is now defined as \( \Delta := I(K) \bigcup \bigcup_{i=0}^{n+k} (I_{i^{\text{anon}}} \bigcup I_{i^{\text{tree}}} \bigcup I_{i^{\text{anon}}}^\text{anon}) \).

Next, we construct the interpretations \( J_0, \ldots, J_{n+k} \) over \( \Delta \) as required by Definition [2.10].

The canonical interpretation \( I_i \) represents the parts specific to \( J_i \) and, for the interpretation of the rigid names in \( J_i \), all \( I_j \) with \( j \in \{0, n + k\} \) are considered. The interpretation of the flexible names then can obviously not be solely based on \( I_i \), but has to be adjusted. Intuitively, we include in \( J_i \) all consequences of the rigid names in \( I_j \). Formally, for all \( i \in \{0, n + k]\), we define \( J_i \) as follows.

\[ \Delta^{J_i} := \Delta. \]
Lemma 3.8. For all $i$, $J_i := a_i$. For all rigid concept names $A$, $A^{R}_{i} := \bigcup_{j=0}^{i+k} A^{R}_{j}$. For all flexible concept names $A$, $A^{F}_{i} := A^{\forall} \cup \bigcup_{j=0}^{i+k} A^{F}_{j}$, where

$$A^{R}_{j} := \{ u_{j,R} \in A^{T}_{j} | O \models \exists R \subseteq A, \ \text{Wit}(u_{j,R}) \neq \emptyset \} \cup$$

$$\bigcup \{ (\bigcap_{i \in I^{R}_{j}} B^{i}) \cap B^{R} \subseteq B^{R}(O), O \models \bigcap_{i \in I^{R}_{j}} B^{i} \subseteq A \}$$

captures the flexible consequences of the rigid names in $I_{j}$. For all rigid role names $R$, $R^{R}_{i} := \bigcup_{j=0}^{i+k} R^{R}_{j}$. For all flexible role names $R$, $R^{F}_{i} := R^{\forall} \cup \bigcup_{j=0}^{i+k} R^{F}_{j}$, where

$$R^{R}_{j} := \{ (d,e) \in R^{T}_{j} | \text{Wit}(d) \neq \emptyset \text{ or } \text{Wit}(e) \neq \emptyset \} \cup$$

$$\bigcup \{ S^{T}_{j} | S \in R^{R}(O), O \models S \subseteq R \}.$$ The interpretations $J_0, \ldots, J_{n+k}$ share the same domain and respect the rigid names. We next point out an important characterization of $B^{J_i}$ for all basic concepts $B$, in terms of the original interpretations $I_0, \ldots, I_{n+k}$.

**Lemma 3.8.** For all $i, j \in [0, n+k]$ and basic concepts $B \in B(O)$, the following hold.

a) For all $e \in I(J_i)$, we have $e \in B^{J_i}$ iff $e \in B^{T}_i$.
b) If $B$ is rigid, then, for every $e \in A^{\forall}_{j} \cup A^{\exists}_{j} \cup A^{\text{slot}}$, we have $e \in B^{T}_j$ iff $e \in B^{T}_j$.
c) If $B$ is flexible, then, for every $e \in A^{\forall}_{j} \cup A^{\exists}_{j} \cup A^{\text{slot}}$, we have $e \in B^{T}_j$ iff

i) $i = j$ and $e \in B^{T}_i$, or

ii) there is a $B^{R} \subseteq B^{R}(O)$ with $e \in (\bigcap_{i \in I^{R}_{j}} B^{i})$ and $O \models \bigcap_{i \in I^{R}_{j}} B^{i} \subseteq B$, or

iii) $e \in B^{T}_i \cap A^{\text{slot}}$ and $\text{Wit}(e) \neq \emptyset$.

Based on Lemma 3.8, we can show that $J_i$ is a model of $(O, A_i)$. The main part of the proof, however, is to show that $J_i$ satisfies the corresponding conjunction $\chi_{A(i)}$ of CQ literals. For the positive literals, this is easy given that the ABox $A_{Q(i)}$ contains an instantiation of all these CQs and is satisfied by $I_i$. For the negative literals $\neg \phi$, we show that, if $J_i$ satisfies $\phi$ via a homomorphism $\pi$, then one of the following cases must apply:

I. $\pi$ maps all terms to unnamed domain elements of a single $I_j$, and a rigid witness query of $\phi$ is satisfied in $I_j$.

II. The image of $\pi$ includes named elements, and either it maps directly into $I_i$ or we can construct such a homomorphism. The latter holds because, if $\pi$ maps some terms to named domain elements from $I_j, j \neq i$, corresponding rigid knowledge on the named elements must be contained in the additional ABoxes and thus also be satisfied in $I_i$.

Thus, the first case contradicts Condition [C5] and the second case is impossible due to Condition [C2]. This concludes the proof of Lemma 3.6.

4 Combined Complexity

Our characterization shows that there is no need to store the exponentially large set $\mathcal{W}$ in order to check $r$-satisfiability. That is, given an $r$-complete tuple $(A^{R}, Q^{+}, Q^{-}, S)$ and a time point $i$ with associated world $W := W_{A(i)}$ and ABox $A := A_i$, the conditions of Definition 3.5 (except for one direction of [C6]) can be checked independently for the KB
Algorithm 4.1: PSPACE procedure for deciding TCQ satisfiability

**Input:** TCQ \( \Phi \), TKB \( K = (\mathcal{O}, (A_i)_{i \in \Sigma}) \)

**Output:** true if \( \Phi \) is satisfiable w.r.t. \( K \), otherwise false

1. Guess \( (A^R, Q^+, Q^-, S) \) and set \( S' := S \)
2. \( i := 0 \)
3. \( s := \) Guess a number between \( 0 \) and \( p^{\Phi_{pa}} \cup [u] \)
4. \( p := \) Guess a number between \( 0 \) and \( q^{\Phi_{pa}} \cup [u] \)
5. \( T_{next} := \emptyset, T_i := \emptyset, T_{curr} := \emptyset \)
6. \( T_{curr} := \) Guess an element of \( Typ(\{\Phi_{pa}\}) \)
7. if \( T_{curr} \) is not initial then return false
8. while \( i < s + p \) do
9. \( T_{next} := \) Guess an element of \( Typ(\{\Phi_{pa}\}) \)
10. if \( (T_{curr}, T_{next}) \) is not t-compatible then return false
11. if \( i = s \) then \( T_s := T_{curr}, T_i := \{\varphi, U \varphi \in T_i\} \)
12. if \( i > s \) then \( T_i := \{\varphi, U \varphi \in T_i \mid \varphi \notin T_{curr}\} \)
13. if \( i = n \) and \( \Phi_{pa} \notin T_{curr} \) then return false
14. \( W := T_{curr} \cap \{p_1, \ldots, p_m\} \)
15. if not RSATISFIABLE(\( \Phi, A, (A^R, Q^+, Q^-, S) \)) then return false
16. \( S' := \{\exists! b \in S' \mid (O, A^R \cup A^R_Q \cup A^R_Q \cup A^R \cup A) \not\models \exists! b\} \)
17. \( i := i + 1 \)
18. \( T_{curr} := T_{next} \)
19. if \( T_i = \emptyset \) and \( S' = \emptyset \) and \( (T_{curr}, T_i) \) is t-compatible then return true
20. return false

\( (O, A^R \cup A^R_Q \cup A^R_Q \cup A^R_W \cup A^R_S \cup A) \) if we define \( Q_W := \{\varphi_j \in Q_{\Phi} \mid p_j \in W\} \). This allows us to show that TCQ satisfiability (and hence also entailment) is in PSPACE w.r.t. combined complexity, which matches the complexity of satisfiability in LTL (cf. Lemma A.6).

We adapt the procedure for LTL [73] as described in Algorithm 4.1. It constructs the propositional types \( T_0, \ldots, T_s, T_{s+1}, \ldots, T_{s+p} \), one after the other, without storing the whole sequence. It keeps in memory two types \( T_{curr} \) and \( T_{next} \) for the current and next time point, respectively, and checks whether these sets are t-compatible (see Appendix A.2 for the definition of t-compatibility). The algorithm additionally guesses the start \( s \) and length \( p \) of the period (Lines 3, 4), stores the type \( T_i \) (Line 11), and checks if the period is valid by comparing \( T_{next} \) to \( T_i \) at time point \( s + p \) (Line 19). It also ensures that all \( U \)-formulas are satisfied within the period (Lines 11, 12, 19).

Our modifications (highlighted in gray) ensure that we consider at least \( n \) time points (Line 3). For t-satisfiability, we check that \( \Phi_{pa} \) is satisfied at \( n \) instead of at 0 (Line 13). Now, r-satisfiability can be tested in a modular fashion (Line 15) in the procedure RSATISFIABLE (see Algorithm A.2), given a tuple \( (A^R, Q^+, Q^-, S) \) guessed in the beginning (Line 1) and the current world \( W \) (Line 14). The additional set \( S' \) checks the global part of Condition (C6) by ensuring that all elements \( \exists R(b) \in S \) are entailed by one of the KBs encountered by the algorithm. We hence integrate the r-satisfiability and t-satisfiability tests from Lemma A.11 such that \( V \) and \( \lambda \) are implicitly represented by the worlds induced by the sequence of guessed types.

**Lemma 4.1.** Algorithm 4.1 decides TCQ satisfiability using only polynomial space.
Algorithm 4.2: The procedure $\mathsf{RSATISFIABLE}$

\textbf{Input:} TCQ, ontology $\mathcal{O}$, ABox $A$, tuple $(A^R, Q^+, Q^-, S)$, world $W$

\textbf{Output:} true if Conditions [(C1)]–[(C6)] hold for time point $i$, otherwise false

1. $K_{rc} := \langle \mathcal{O}, A^R \cup A^R_{Q^+} \cup A^R_{Q^-} \cup A_j \rangle$
2. if $K_{rc}$ is inconsistent then return false
3. for $p_j \in W$ do
   4. \hspace{1em} Guess a set $A_S^j \subseteq A_S$ of size polynomial in $|\varphi_j|
   5. \hspace{1em} if $K_{rc} \cup A_S^j \models \varphi_j$ then return false
   6. \hspace{1em} if $\varphi_j \notin Q^-$ then return false
7. for $p_j \in W$ do
   8. \hspace{1em} if $\varphi_j \notin Q^+$ then return false
9. for all rigid witness queries $\psi$ of $Q^-$ do
   10. \hspace{1em} if $K_{rc} \models \psi$ then return false
11. for all $S \in R^T(\mathcal{O})$ and $b \in I(K) \cup I^{aux}$ do
12. \hspace{1em} if $K_{rc} \models \exists S(b)$ and $\exists S(b) \notin S$ then return false
13. return true

Proof. We consider the conditions in Lemma 2.11. Let the set $\mathcal{W} = \{W_1, \ldots, W_k\}$ be defined as the set of all worlds $W$ encountered during a run of the procedure. The mapping $\lambda : [0,n] \rightarrow [1,k]$ is defined as $\lambda(i) := \ell$ if $W_{\ell}$ is the world encountered at time point $i$. Regarding t-satisfiability (see Definition 2.9), it is easy to see that the above definitions of $\mathcal{W}$ and $\lambda$ fulfill the first two conditions. The last condition follows from the correctness of the original LTL satisfiability algorithm [73, Thm. 4.1, 4.7], which is not affected by our restriction that $s > n$ nor by the other extensions.

It thus remains to show that $\mathcal{W}$ is t-satisfiable iff these extensions do not cause the algorithm to return false. By Lemma 3.6 we can consider Conditions [(C1)]–[(C6)] from Definition 3.5. Conditions [(C3)]–[(C5)] are obviously captured by $\mathsf{RSATISFIABLE}$.

For [(C1)] observe that $\mathsf{RSATISFIABLE}$ only checks the consistency of the knowledge base $K_{rc}$, which does not include $A_S$.

However, this exponentially large ABox can be ignored for this consistency test since, once Condition [(C6)] is verified, we know that for each $A_{33(b)} \subseteq A_S$ there is at least one index $j \in [0, n + k]$ for which the existence of the elements described in $A_{33(b)}$ follows from the KB $\langle \mathcal{O}, A^R \cup A^R_{Q^+} \cup A^R_{Q^-} \cup A_j \rangle$. Hence, the rigid consequences of the assertion $\exists S(b)$ with $b \in I(K) \cup I^{aux}$, which are represented by $A_{33(b)}$, must follow from $A^R$ or $A^R_{Q^+}$ (depending on the kind of $b$). We can thus disregard the assertions from $A_{33(b)}$ including the elements in $I^{aux}$ since any model of the KB we consider must have such domain elements.

For [(C2)] we have to check whether $K_{rc} \cup A_S \models \varphi_j$ holds for each $p_j \in W$. Considering the nondeterministic variant of the algorithm in [23], it is easy to see that, in order to check for a homomorphism from $\varphi_j$, it suffices to consider only a nondeterministically chosen part of $A_S$ of size polynomial in $|\varphi_j|$, the cardinality of $\varphi_j$. Additionally, we have to check if there is a named individual from which we can reach this part, but this can also be done while using only polynomial space.

Finally, we consider [(C6)] The “if”-direction of the equivalence is captured by Lines [11], [12] in Algorithm 4.2. The other direction of Condition [(C6)] is checked globally in Line [16] in Algorithm 4.1. Observe that our global condition corresponds to the extension of $\Phi$ with linearly many additional conjuncts of the form $\diamond_p \diamond_f \exists S(b)$, which may require us to look
for an LTL structure with a longer period. However, the required period is still exponential in the input.

We analyze the complexity. For the original parts of Algorithm 4.1, we refer to \[73\]. The nondeterministic guessing of the polynomially large sets \(A_R^+, Q^+, Q^-, \) and \(S\) can be done using polynomial space only. The set \(A_R^+\) can be computed in polynomial time since it involves only a polynomial number of P subsumption test in DL-Lite\(^H\) \[4\, Thm. 8.2\]. Moreover, \(K_{rc}\) is of polynomial size (recall that we drop \(A_S\)) and hence can be tested for consistency in P \[4\, Thm. 8.2\]. The various UCQ entailment tests can be done in NP by the nondeterministic variant of the algorithm in \[24\] (see the sketch after Theorem 12 in that paper). The guess in Line 4 of Algorithm 4.2 is clearly also possible in polynomial space, and we can enumerate all rigid witness queries in Line 9 of Algorithm 4.2 in polynomial space since their size is bounded by the size of the largest CQ in \(Q\).

\[\square\]

Since the nondeterminism is not relevant for PSPACE complexity according to the well-known result of Savitch \[71\], we obtain the desired complexity result.

**Theorem 4.2.** TCQ entailment in DL-Lite\(^H\) is in PSPACE in combined complexity, even if \(P^H \neq \emptyset\).

### 5 First-Order Rewriting of r-Satisfiability

Towards our goal of obtaining a low data complexity for TCQ entailment in DL-Lite\(^H\), we first reconsider the r-completeness conditions from Definition 3.5 and show that they are partially first-order rewritable. As before, we consider a TCQ \(\Phi\) and a DL-Lite\(^H\) TKB \(\mathcal{K} = \langle \mathcal{O}, \mathcal{A} \rangle\) with \(\mathcal{A} = (A_i)_{0 \leq i \leq n}\). Since we focus on data complexity, we disregard the impact of \(\mathcal{O}\) and \(\Phi\) on the computational resources in the following. In particular, the size of the set \(W = \{W_1, \ldots, W_k\} \subseteq 2^\{p_1, \ldots, p_m\}\) is constant; we consider a fixed such set for now. However, the same does not hold for the mapping \(\lambda\), which depends on the length of \(\mathcal{A}\). In addition to \(W\), we guess a set \(B \subseteq \{B(a) \mid B \in \mathcal{B} (\mathcal{O}), a \in I(\Phi)\}\) of basic concept assertions over individual names occurring in \(\Phi\). The size of this set is also constant in data complexity, and hence we also assume \(B\) to be fixed throughout this section. The set \(B\) captures additional basic concept assertions that are not contained in the ABoxes but, due to their consequences, critical for determining r-satisfiability.

This allows us to show that:

- to verify the r-satisfiability of \(\Phi\) w.r.t. \(\mathcal{K}\), it suffices to check r-completeness of a representative tuple \((A_R^+, Q^+, Q^-, S)\) that depends on \(W\) and \(B\); and
- the r-completeness conditions for this tuple can be encoded into (linearly many) FO formulas that are evaluated over a fixed, finite structure TDB(\(\mathcal{A}\)) (constructed based on \(\mathcal{A}\) in Definition 5.6).

These two steps are described in Sections 5.1 and 5.2, respectively.
5.1 A Tuple for Testing r-Satisfiability

We first describe the tuple \((\mathcal{A}^R, \tilde{Q}^+, \tilde{Q}^-)\) and the corresponding KBs that are relevant for the r-completeness tests. We define the sets such that they are minimal w.r.t. the r-completeness conditions. The definition of \(\tilde{Q}^+\) and \(\tilde{Q}^-\) is straightforward:

\[
\tilde{Q}^+ := \{ p_j \in Q_\Phi \mid W \in W, p_j \in W \}, \\
\tilde{Q}^- := \{ p_j \in Q_\Phi \mid W \in W, p_j \not\in W \}.
\]

To define \(\mathcal{A}^R\), we can use the sets \(B\) and \(\mathcal{A}^R_{\tilde{Q}^-}\) (restricted to \(I(\mathcal{K})\)), but we also have to consider the rigid consequences of the input ABoxes \(A_i\). We give an inductive construction of these consequences that allows us to consider the ABoxes \(A_i\) in isolation. We define sets of (positive) rigid assertions inductively as follows for \(j \geq 0\):

\[
\begin{align*}
\mathcal{A}^R_0 & \colon= \mathcal{A}^R(\mathcal{K}) \cap (B \cup \mathcal{A}^R_{\tilde{Q}^-}), \\
\mathcal{A}^R_{j+1} & \colon= \{ \alpha \in \mathcal{A}^R(\mathcal{K}) \mid \text{there is } i \in [0,n] \text{ with } \langle \mathcal{O}, \mathcal{A}^R_{j} \cup A_i \rangle \models \alpha \}.
\end{align*}
\]

After at most \(r \colon= |B|^{|O|}\) iterations, this computation becomes stable, i.e., we do not add any more assertions, because

- by Definition \([A.1]\) and Lemma \([A.2]\) all role assertions about \(I(\mathcal{K})\) follow in one entailment step from some role inclusions in \(\mathcal{O}\) and a role assertion in \(\mathcal{A}^R_0\) or \(A_i\);
- entailment of basic concept assertions \(B(a)\) does not depend on basic concept assertions on individual names other than \(a\), and so all possible assertions about \(a\) are added after at most \(r\) steps.

\(\mathcal{A}^R\) is now defined as the union of \(\mathcal{A}^R_0\) and the set of all negative assertions \(\neg \alpha \in \mathcal{A}^R(\mathcal{K})\) for which \(\alpha \not\in \mathcal{A}^R_0\). The following is a direct consequence of this definition.

\textbf{Lemma 5.1.} For \(\alpha \in \mathcal{A}^R(\mathcal{K})\), we have \(\alpha \in \mathcal{A}^R\) iff there is \(i \in [0,n]\) with \(\langle \mathcal{O}, \mathcal{A}^R_{j} \cup A_i \rangle \models \alpha\).

It remains to define the last component, \(\tilde{S}\). Recall that this set of flexible assertions of the form \(\exists b\{\}\) can refer to the individual names \(b\) in \(I(\mathcal{K})\) and \(I^{\text{aux}}\); moreover we want to define it as the minimal such set that satisfies the r-completeness conditions (in particular \([C.6]\)). With respect to the elements of \(I^{\text{aux}}\), the set \(\tilde{S}\) thus only depends on the fixed query \(\Phi\); in contrast, the names in \(I(\mathcal{K})\) may occur in the input ABoxes \(A_i\) as well as in \(\Phi\). For the rewriting, it is important to separate these cases since

- the parts about \(I^{\text{aux}}\) are known at the time of the rewriting (depending on \(W\));
- the parts about \(I(\mathcal{K})\) \(\setminus I(\Phi)\) depend on the input ABoxes, so they are not fixed; and
- the parts about \(I(\Phi)\) depend on the input ABoxes as well as on the TCQ \(\Phi\).

Hence, we define \(\tilde{S}\) as the disjoint union of the three sets \(\tilde{S}^{\text{aux}}, \tilde{S}_F, \text{and } \tilde{S}_O\) (for “other”), referring only to names from \(I^{\text{aux}}, I(\Phi), \text{and } I(\Phi) \setminus I(\Phi)\), respectively:

- \(\tilde{S}^{\text{aux}}\) is constant and hence its size is not relevant. In particular, the elements of \(I^{\text{aux}}\) occur neither in the rigid ABox type \(\mathcal{A}^R\), nor in the input ABoxes \(A_i\), and are uniquely associated to one of the CQs in \(\Phi\). For constructing \(\tilde{S}^{\text{aux}}\) in line with Condition \([C.6]\), it is thus sufficient to focus on the consequences of these CQs (see Lemma \([A.3]\)):

\[
\tilde{S}^{\text{aux}} := \{ \exists a_i \{ \} \mid S \in R^{F}(\mathcal{O}), a_i \in I^{\text{aux}}, W \in W, \langle \mathcal{O}, A_{w} \rangle \models \exists S(a_i) \}.
\]
– The set $\tilde{S}_\Phi$ is also of constant size. However, the elements of $I(\Phi)$ may occur in the input ABoxes, and hence their behavior cannot be fully determined by a computation that is independent of the input. To tackle this problem, we assume the set $B$ of assertions to be given first, and postpone the test whether $\tilde{S}_\Phi$ actually satisfies Condition [C6] to a later point (see Lemma 5.5). For now, we simply set

$$\tilde{S}_\Phi := \{ \exists S(a) \in B \mid S \in R^F(\mathcal{O}) \}.$$  

– The set $\tilde{S}_\sigma$ concerns the remaining individual names from $I(\mathcal{K}) \setminus I(\Phi)$. Here, we can refer to the flexible consequences of the ABoxes $\mathcal{A}_i$ together with $\bar{A}^R$. Since the elements under consideration do not occur in $\Phi$, this computation actually does not depend on $\mathcal{W}$ or $B$:

$$\tilde{S}_\sigma := \{ \exists S(a) \mid S \in R^F(\mathcal{O}), a \notin I(\Phi) \text{ there is } i \in [0,n] \text{ with } (\mathcal{O}, \bar{A}^R \cup \mathcal{A}_i) \models \exists S(a) \}.$$  

This finishes the definition of $(\bar{A}^R, \bar{Q}^+, \bar{Q}^-, \bar{S})$. Observe that, apart from $\mathcal{A}_\subseteq$ and $\bar{A}^R$, which depend on the input ABoxes $\mathcal{A}$, all of the ABoxes induced by this tuple (see Section 3) are constant. Moreover, the tuple is indeed as intended.

**Lemma 5.2.** For all $\mathcal{W} = \{W_1, \ldots, W_k\} \subseteq 2^{\{p_1, \ldots, p_m\}}$ and $\lambda : [0,n] \to [1,k]$, there is an r-complete tuple w.r.t. $\mathcal{W}$ and $\lambda$ iff there is a set $B \subseteq \{B(a) \mid B \in B(\mathcal{O}), a \in I(\Phi)\}$ such that $(\bar{A}^R, \bar{Q}^+, \bar{Q}^-, \bar{S})$ is r-complete w.r.t. $\mathcal{W}$ and $\lambda$.

**Proof sketch.** Given an r-complete tuple $(\bar{A}^R, \bar{Q}^+, \bar{Q}^-, \bar{S})$, we define

$$B := \{B(a) \in \bar{A}^R \cup S \mid B \in B(\mathcal{O}), a \in I(\Phi)\}$$

and show that $(\bar{A}^R, \bar{Q}^+, \bar{Q}^-, \bar{S})$ is r-complete as well. The KBs used in the r-completeness tests in Definition 5.5 look as follows, for all $i \in [0,n+k]$:

$$K^i_{rc} := (\mathcal{O}, \bar{A}^R \cup \bar{A}^R_\sigma \cup \bar{A}_{Q^+} \cup \bar{A}_{Q^-} \cup \bar{A}_\subseteq \cup \mathcal{A}_i).$$

Conditions [C3] and [C4] are satisfied by construction. For Conditions [C1] [C2] and [C5], we can find a model of $K^i_{rc}$ that is homomorphically embeddable into the canonical interpretation of $K^i_{rc}$ that is consistent by Condition [C1]. This is because $A^R_\subseteq \subseteq A^R_{Q^+}$, $A_{\subseteq a} \cup A_{\subseteq b} \cup A_{\subseteq c} \subseteq A_S$, and $A^R_\subseteq \subseteq A^R$.

5.2 Rewriting Consistency and Entailment

Now, we can focus on testing the r-completeness of a single, (mostly) fixed tuple in the r-completeness test. Observe that the tests for r-completeness consist of consistency and non-entailment tests for atemporal KBs, which are standard.

Many query answering problems in lightweight DLs can be encoded into first-order logic formulas, called *rewritings*, which are then evaluated over the following structures, in which ABoxes are viewed under the closed-world assumption, i.e., as databases.

**Definition 5.3 (DB(\mathcal{A}))**. For an ABox $\mathcal{A}$, the first-order structure $DB(\mathcal{A}) = (I(\mathcal{A}), DB)$ over the domain $I(\mathcal{A})$ contains the following relations for all $B \in B(\mathcal{A})$ and $R \in P(\mathcal{A})$:

$$B^{DB} := \{ a \mid B(a) \in \mathcal{A} \}, \quad R^{DB} := \{ (a,b) \mid R(a,b) \in \mathcal{A} \}.$$
There are FO rewritings for KB inconsistency and for UCQ entailment in $DL\text{-}Lite^H_{\text{horn}}$, which we here denote by $[\bot]$ and $[\varphi]$, respectively (see, e.g., [24]). These can be easily adapted to our slightly modified setting with assertions about (negated) basic concepts.

**Lemma 5.4.** Let $K = \langle O, A \rangle$ be a $DL\text{-}Lite^H_{\text{horn}}$ knowledge base and $\varphi$ be a Boolean UCQ. Then $K$ is inconsistent iff $DB(A) \models [\bot]$. If $K$ is consistent, then $K \models \varphi$ iff $DB(A) \models [\varphi]$.

The idea is to apply these UCQ rewritings $[\bot]$ and $[\varphi]$, which are evaluated over the FO structure $DB(A)$, where $A$ is a single ABox $A$. However, the conditions for r-completeness involve the sequence of input ABoxes $\Xi$, as well as additional ABoxes such as $\bar{A}\bar{R}$. In this section, we describe how these ABoxes can be incorporated into the rewritings such that the resulting FO formulas can be answered over $\Xi$ alone. Then, the r-completeness check for the tuple $(\bar{A}\bar{R}, \bar{Q}^+, \bar{Q}^-, \bar{S})$ can be reduced to the evaluation of (multiple) FO formulas over $\Xi$ (see Lemma 5.5).

To illustrate the main idea of how to extend the rewritings, consider an atemporal KB $(O, A_i)$ formulated in $DL\text{-}Lite^H_{\text{horn}}$ and a CQ $\varphi$. By [24], there is a rewriting $[\varphi]$ such that

\[
\langle O, A_i \rangle \models \varphi \text{ iff } DB(A_i) \models [\varphi].
\]

Assume that we want to incorporate the additional ABox $\bar{A}\bar{R}$ into the entailment test, without modifying $DB(A_i)$. Specifically, the goal is to extend $[\varphi]$ to an FO formula $[\varphi, \bar{A}\bar{R}]$ (interpreted under the standard first-order semantics) such that

\[
\langle O, \bar{A}\bar{R} \cup A_i \rangle \models \varphi \text{ iff } DB(A_i) \models [\varphi, \bar{A}\bar{R}].
\]

If $\bar{A}\bar{R}$ consists of the single assertion $A(a)$, this can be achieved, for instance, by replacing every atom $A(x)$ in $[\varphi]$ by the disjunction $((x = a) \lor A(x))$.

The various additional ABoxes we consider, such as $A_\Xi$, contain individual names that do not occur in the input sequence $\Xi$ (namely those in $I^{\text{aux}}(\Xi) I^{\text{free}}$). This makes the required adaptations of the rewritings even more complex, as our FO formulas have to quantify over elements that are not in the interpretation domain.

We now present the main lemma that will be shown in this section. It characterizes the r-completeness of $(\bar{A}\bar{R}, \bar{Q}^+, \bar{Q}^-, \bar{S})$ by a series of FO-formulas.

**Lemma 5.5.** For all $W = \{W_1, \ldots, W_k\} \subseteq 2^{\{p_1, \ldots, p_n\}}$, mappings $\lambda: [0,n] \rightarrow [1,k]$, and sets $B \subseteq \{B(a) \mid B \in B(O), a \in I(\Phi)\}$, the tuple $(\bar{A}\bar{R}, \bar{Q}^+, \bar{Q}^-, \bar{S})$ is r-complete w.r.t. $W$ and $\lambda$ iff the following hold:

(a) For all $i \in [0,n]$, we have $TDB(\Xi) \models rSat_{\lambda,i}(i)$.

(b) For all $W \in W$, we have $TDB(\Xi) \models rSat_W(-1)$.

(c) For all $S \in R^O(\bar{O})$ and $a \in I(\Phi)$, we have $\exists S(a) \in B$ iff there is an $i \in [0, n]$ such that $TDB(\Xi) \models [\exists S(a) | A_\Xi \gamma](i)$, where this rewriting is w.r.t. the world $W_{A_\Xi}(i)$.

In the following subsections, we describe how to obtain the temporal database $TDB(\Xi)$ and the rewritings $rSat_W(i)$ and $[\exists S(a) | A_\Xi \gamma](i)$ used in this characterization.

So far, we have restricted our attention to Boolean queries. However, the queries that we rewrite in this section may also be non-Boolean, i.e., they may contain variables that are not existentially quantified, called free variables. This is necessary for the presentation of the intermediate queries we construct in the rewriting process. The rewritings must thus preserve entailment w.r.t. all possible groundings, as defined next. A **grounding** of a UCQ $\varphi$ w.r.t. an atemporal KB $K$ to be a function $\gamma$ that maps the free variables of $\varphi$ to individual names from $I(K)$. Such a grounding $\gamma$ is a certain answer to $\varphi$ over $K$ if $K \models \gamma(\varphi)$, where
\( \gamma(\varphi) \) denotes the Boolean UCQ resulting from \( \varphi \) by replacing all free variables according to \( \gamma \). Similarly, \( \gamma \) is an answer to \( \varphi \) over the first-order structure DB(\( A \)) if DB(\( A \)) \models \gamma(\varphi).

The rewriting \([\varphi] \) can be assumed to be also correct for non-Boolean UCQs \( \varphi \), in the sense that it has the same free variables as \( \varphi \) and that the certain answers to \( \varphi \) over \( \mathcal{K} \) coincide with the answers to \([\varphi] \) over DB(\( A \)); i.e., we have \( \mathcal{K} \models \gamma(\varphi) \) iff DB(\( A \)) \models \gamma([\varphi]) \) (cf. Lemma 5.3) for every possible grounding \( \gamma \) (see, e.g., [24]).

### 5.2.1 From \( A \) to \( A_i \)

Since we have a temporal semantics, as a first step, we need to lift the definition of DB(\( A \)) to the temporal sequence of ABoxes \( \mathfrak{A} \), and to adapt the rewritings \([\varphi] \) and \([\perp] \) accordingly.

In the following, we usually talk only about \([\varphi] \), since the procedures for \([\perp] \) are analogous; however, note that \([\perp] \) is always Boolean. For now, \( \varphi \) is simply an arbitrary CQ, which we later instantiate with the concrete CQs relevant for the r-completeness test, e.g., with the rigid witness queries for the CQs occurring in \( \Phi \).

**Definition 5.6 (TDB(\( \mathfrak{A} \))).** For the ABox sequence \( \mathfrak{A} = (\mathcal{A}_i)_{0 \leq i \leq n} \), the two-sorted first-order structure TDB(\( \mathfrak{A} \)) = (I(\( \mathfrak{A} \)), \([-1, n] \), TDB) over the object domain I(\( \mathfrak{A} \)) and temporal domain \([-1, n] \) contains the following relations, for all \( B \in B(\mathfrak{A}) \) and \( R \in P(\mathfrak{A}) \):

\[
B^{\text{TDB}} := \{(a,i) \mid i \in [0,n], B(a) \in \mathcal{A}_i\}, \quad R^{\text{TDB}} := \{(a,b,i) \mid i \in [0,n], R(a,b) \in \mathcal{A}_i\}.
\]

We use the temporal domain element \(-1\) to describe the prototypical empty ABox \( \mathcal{A}_{-1} := \emptyset \); all formulas of the form \( B(a, -1) \) and \( R(a,b, -1) \) thus evaluate to false. As before, we may use atoms of the form \( R^{-}(b,a,i) \) to refer to \( R(a,b,i) \). The relations \( B^{\text{TDB}}/R^{\text{TDB}} \) for symbols \( B/R \) that do not occur in \( \mathfrak{A} \) are considered to be empty; for simplicity, we do not explicitly consider this case in the following.

We now define a first-order formula \([\varphi](i)\) with an additional argument \( i \) that allows us to explicitly refer to time points. The formula \([\varphi](i)\) adapts \([\varphi] \) to TDB(\( \mathfrak{A} \)); given \( i \in [-1,n] \), it checks whether \( \varphi \) is entailed by \( (\mathcal{O},A_i) \). It is obtained from \([\varphi] \) by simply replacing all atoms \( B(t) \) and \( R(s,t) \) by \( B(t,i) \) and \( R(s,t,i) \), respectively. Given Lemma 5.3 it is easy to see that this is correct in the following sense.

**Lemma 5.7.** For all CQs \( \varphi \), groundings \( \gamma \), and \( i \in [-1,n] \), we have

\[
(\mathcal{O},A_i) \models \gamma(\varphi) \text{ iff } \text{TDB}(\mathfrak{A}) \models \gamma([\varphi](i)).
\]

### 5.2.2 From \( A_i \) to \( \bar{A}^R \cup \mathcal{A}_i \)

In the next step, we incorporate the inductive computation of \( \bar{A}^R \) (see Section 5.1) into \([\varphi](i)\), yielding the FO formulas \([\varphi,\bar{A}^R](i)\) for all \( j \in [0,r] \) (i.e., \( |B^R(\mathcal{O})| + 1 \) formulas):

- \([\varphi,\bar{A}^R](i)\) is obtained from \([\varphi] \) by replacing all rigid basic concept and role atoms \( \alpha(t) \) (where \( t \) is either \( t_1 \) or \( (t_1,t_2) \), depending on the type of \( \alpha \)) by \( A(t,i) \vee \bigwedge_{\alpha(a) \in \bar{A}^R_0} t = a \).

where, if \( \alpha \) is a basic concept \( B \), then \( A \) denotes \( B \), and if \( \alpha \) is a role \( R \), then \( A \) denotes \( R \). The big disjunction over the component-wise equality \( t = a \) encodes that the atom \( \alpha(t) \) is satisfied by \( \bar{A}^R_0 \).
The rewriting $[\varphi|\bar{A}^R](i) := [\varphi|\bar{A}^R](i)$ can be shown to be correct by induction on $j$.

**Lemma 5.8.** For all CQs $\varphi$, groundings $\gamma$, and $i \in [-1, n]$, we have

$$(\mathcal{O}, \bar{A}^R \cup A_i) \models \gamma(\varphi) \text{ iff } \text{TDB}(\mathfrak{A}) \models \gamma([\varphi|\bar{A}^R](i)).$$

5.2.3 From $\bar{A}^R \cup A_i$ to $\bar{A}_{\text{rc}} \cup A_i$

The final rewriting needs to consider an ABox of the form $\bar{A}_{\text{rc}} \cup A_i$, where

$$\bar{A}_{\text{rc}} := A^R \cup A^R_{\text{Q}} \cup A_{QW} \cup A_S$$

is the part that is independent of $i$ (cf. Section 3.5). Note that we use $QW$ instead of $Q_{\text{aux}}$ since we do not explicitly consider a mapping $\lambda$ yet; and, as in Section 4, we assume for now that a world $W$ (e.g., $W_{\lambda(i)}$) is given explicitly. It should be kept in mind that the rewriting depends on $W$ (as well as on $\mathfrak{W}$ and $\mathfrak{B}$), although we do not explicitly specify this in the notation.

The rewritings introduced so far only cover the individual names in $I(\mathcal{K})$. However, $\bar{A}_{\text{rc}}$ also contains auxiliary individual names from $\Gamma^{\text{aux}} \cup \Gamma^{\text{free}}$, which do not occur in $\text{TDB}(\mathfrak{A})$. The set $\Gamma^{\text{aux}}$ and those individuals in $\Gamma^{\text{free}}$ stemming from $A_{\Sigma_0}$ and $A_{\Sigma_{\text{aux}}}$ do not depend on the input ABoxes, and hence are relatively unproblematic. However, $\bar{A}_{\Sigma_0}$ contains the ABoxes $A_{\Sigma_0}$, whose number is not bounded in the size of $\mathcal{O}$ or $\Phi$. Our next goal is thus to separate $A_{\Sigma_0}$ as much as possible from the input data.

We introduce prototypes $[S]$ which are fresh individual names $[S], a_{[S],S}, a_{[S],S}, \ldots$, with the intention that $[S]$ is used to replace the concrete individual names from $I(\mathcal{K})$. We collect all these new names except $[S]$ in the set $\varphi$. The ABoxes $A_{\Sigma_0}$ for all $S \in R^I(\mathcal{O})$ are prototypical versions of $A_{\Sigma_0}$ with $b \in I(\mathcal{K})$ (see Section 3.4), and are obtained from $A_{\Sigma_0}$ by replacing the individual name $b$ everywhere by $[S]$, e.g., $a_{b,0}$ becomes $a_{[S],0}$. In the rewriting, we can then use $A_{\Sigma_0}$ to refer to $A_{\Sigma_0}$ without mentioning $b$ explicitly. In the following, we denote by $\Gamma^{\text{free}}$ the restriction of $\Gamma^{\text{free}}$ to those individual names that occur in $A_{\Sigma_0}$ and $A_{\Sigma_{\text{aux}}}$, and we denote by $\Gamma := \Gamma^{\text{aux}} \cup \Gamma^{\text{free}} \cup \Gamma^{\text{pro}}$ the set of all additional individual names we consider in the following (apart from the original ones in $I(\mathcal{K})$).

We can now continue to extend the rewriting $[\varphi|\bar{A}^R](i)$ to accommodate the remaining parts of $\bar{A}_{\text{rc}}$, i.e., $A^R_{\text{Q}}, A_{QW}$, and $A_S$. As before, our goal is a rewriting $[\varphi|\bar{A}_{\text{rc}}](i)$ that reflects entailment w.r.t. $(\mathcal{O}, \bar{A}_{\text{rc}} \cup A_i)$.

We again start from the original rewriting $[\varphi]$, which is a UCQ. Since $\text{TDB}(\mathfrak{A})$ only contains the individual names from $I(\mathcal{K})$, we adapt the existential quantifiers in the CQs to simulate quantification over the extended set $I(\mathcal{K}) \cup \Gamma$ as follows. We consider each CQ $\varphi = \exists x_0, \ldots, x_{\ell-1}, \psi$ in $[\varphi]$ separately. The idea is to expand $\psi$ into a disjunction of $2^\ell$ variants $v_0, \ldots, v_{2^\ell-1}$ that cover all cases of the variables $x_j$ being mapped either to $I(\mathcal{K})$ or $[S]$.

---

Footnote:

Similar in function to, but not to be confused with, the prototypical elements in canonical interpretations.
to $\Gamma$. More formally, to define the disjunct $v_k, k \in [0, 2^\ell - 1]$, we first represent the number $k$ by the binary vector $(b_0, \ldots, b_{\ell-1}) \in \{0,1\}^\ell$, i.e., such that $k = b_0 \cdot 2^0 + \ldots + b_{\ell-1} \cdot 2^{\ell-1}$. Then, for a variable $x_j$, $j \in [0, \ell - 1]$, we replace the original quantifier $\exists x_j$ from $v$ by the expression $\exists x_j$, which is defined as follows:

- if $b_j = 1$, it remains $\exists x_j$; and
- if $b_j = 0$, it is the disjunction $\bigvee_{s_j \in \Gamma}$. 

Observe that we use the symbol $x_j$ now in two different ways. If $b_j = 0$, then $x_j$ is a variable, as before. However, if $b_j = 1$, then $x_j$ is an element of $\Gamma$; the big disjunction over all these elements simulates the quantification over the additional sets of individual names. We now set

$$v_k := \exists x_0 \ldots \exists x_{\ell-1}. \mathcal{C}_{\text{rep}}(i) \land \mathcal{C}_{\text{filter}},$$

where it remains to define the formula $\mathcal{C}_{\text{rep}}(i) \land \mathcal{C}_{\text{filter}}$. The idea is that $\mathcal{C}_{\text{rep}}(i)$ replaces the atoms of $\mathcal{C}$ in a similar way to the rewritings considered before. Since atoms in $\mathcal{C}$ can refer to prototypes, the additional formula $\mathcal{C}_{\text{filter}}$ is needed to ensure that the structure of the ABoxes $A_{\mathcal{S}_{\mathcal{O}}} \subseteq A_{\mathcal{S}}$ is respected (see Example 5.9 below). This is inspired by a technique described in [53].

The formula $\mathcal{C}_{\text{rep}}(i)$ is constructed by replacing every atom $\alpha$ in $\mathcal{C}$ by $\mathcal{C}_{\text{rep}}(i)$, depending on the form of $\alpha$ as described below. To simplify the notation, here, we do not mention the parameters $\mathcal{W}, W, E$, and $k$, on which the operation $\mathcal{C}_{\text{rep}}(i)$ implicitly depends. First, we define the abbreviation

$$[s \equiv t]_{\text{rep}} := \begin{cases} 
  s = t & \text{if } s, t \in \Gamma, \\
  \text{true} & \text{if } s = t, \\
  \text{false} & \text{otherwise.}
\end{cases}$$

for $s, t \in \mathcal{T}(v)$, which allows us to express equality between two terms.

For all concept and role atoms $\alpha(t)$, we now define

$$[\alpha(t)]_{\text{rep}}(i) := [\alpha(t)]_{1}(i) \lor [\alpha(t)]_{2}(i) \lor [\alpha(t)]_{3}(i).$$

The formulas $[\alpha(t)]_{1}(i)$ and $[\alpha(t)]_{2}(i)$ are the rewritings of $\alpha(t)$ relative to the ABoxes $A_{\mathcal{S}_{\mathcal{O}}} \cup A_{\mathcal{R}}$ and $A_{\mathcal{R}} \cup A_{\mathcal{Q}_{\mathcal{O}}} \cup A_{\mathcal{R}_{\mathcal{O}}} \cup A_{\mathcal{S}_{\mathcal{O}}} \cup A_{\mathcal{S}}$, respectively, and are defined as below:

$$[\alpha(t)]_{1}(i) := \begin{cases} 
  \text{false} & \text{if } t \text{ contains elements of } \Gamma, \\
  [\alpha(t)]_{A_{\mathcal{R}}}(i) & \text{if } \alpha \text{ is rigid,} \\
  A(t,i) & \text{if } \alpha \text{ is flexible,}
\end{cases}$$

$$[\alpha(t)]_{2}(i) := \bigvee_{\alpha(a) \in A_{\mathcal{R}} \cup A_{\mathcal{Q}_{\mathcal{O}}} \cup A_{\mathcal{R}_{\mathcal{O}}} \cup A_{\mathcal{S}_{\mathcal{O}}} \cup A_{\mathcal{S}}} [t = a]_{\text{rep}}.$$ 

It remains to simulate the influence of $A_{\mathcal{S}_{\mathcal{O}}}$ via $[\alpha(t)]_{3}(i)$. We start with the formula

$$[\exists S]_{\text{rep}}(x) := \exists p. [\exists S(x)]_{A_{\mathcal{R}}}(p) \land \bigwedge_{a \in \Gamma(\Phi)} (x \neq a)$$
which expresses that the variable $x$ is bound to an individual name $a$ with $\exists S(a) \in S_0$ (see Section 5.1). We now define

$$[\alpha(t)]_t(i) := \begin{cases} \exists x.[\exists S] \text{rep}(x) & \text{if } t_2 = a_{\text{rep}} \text{ and } \alpha(t) \in A_{\exists S}, \\ [\exists S] \text{rep}(t_1) & \text{if } t_2 = a_{\text{rep}}, \alpha([S], a_{\text{rep}}) \in A_{\exists S}, t_1 \not\in I_r \\ \text{false} & \text{otherwise}. \end{cases}$$

Intuitively, an assertion $\alpha(t)$ involving an individual name from $I_{\text{PO}}$ holds whenever it directly follows from some $A_{\exists S}$ (which is independent of the input ABoxes), or it is a role atom $R(t_1, a_{\text{rep}})$ and $t_1$ is mapped to the root $b$ of some $A_{\exists S(b)} \subseteq A_{\exists S_0}$ such that $R(b, a_{\text{rep}}) \in A_{\exists S(b)}$ (which corresponds to the assertion $R([S], a_{\text{rep}}) \in A_{\exists S}$). In both cases, the formula needs to check that a relevant ABox $A_{\exists S(b)}$ is actually part of $A_{\exists S_0}$.

It remains to define $\psi_{\text{filter}}$, whose purpose is to ensure that the structure of $A_{\exists S_0}$ is preserved, even though its elements cannot be explicitly mentioned in the rewriting.

**Example 5.9.** Consider the CQ $\nu = \exists x, y, z. \psi$, where $\psi = S(y, x) \land S(z, x)$ and $S$ is rigid, and the disjunct

$$v_k = \bigvee_{x \in I_r} \exists y, z. [\nu] \text{rep}(i) \land \psi_{\text{filter}}$$

of the rewriting. In particular, we consider the disjunct of $v_k$ where $x$ is considered to be equal to $u_{\text{rep}}$. It addresses the case where both atoms in $\nu$ are satisfied by an ABox of the form $A_{\exists S(b)}$, by mapping $x$ to $u_{\text{rep}}$ and both $y$ and $z$ to $b$ (note that $y$ and $z$ are still quantified over $I(\Phi)$). Since $x = u_{\text{rep}}$ is a prototype, we must have $b \in I(\Phi) \setminus I(\Phi)$. As $R([S], a_{\text{rep}}) \in A_{\exists S}$, and thus $[S(y, x) \land S(z, x)] \text{rep}(i)$ is equal to $[\exists S] \text{rep}(y) \land [\exists S] \text{rep}(z)$. This formula expresses that both $y$ and $z$ must be mapped to roots of an ABox of the form $A_{\exists S(b)}$, and hence neglects the fact that $y$ and $z$ must actually be mapped to the same individual name $b$. The formula $\psi_{\text{filter}}$ addresses this issue by adding the atom $y = z$ to the rewriting.

Formally, we define

$$\psi_{\text{filter}} := \bigwedge \{ [s = t] \text{rep} | R(x_j, s), S(x_j, t) \in \nu, x_j \in I_{\text{PO}}, s, t \not\in I_{\text{PO}} \}$$

(cf. [54]). Hence, any two terms that are not prototypes and occur together with the same prototype in role atoms of $\nu$ must be mapped to the same individual name in $I(\Phi) \setminus I(\Phi)$. As described above, we construct $[\phi] A_{\text{rep}}(i)$ by replacing each CQ $\nu$ in $[\phi]$ by $v_0 \lor \cdots \lor v_{2^k-1}$.

In the same way, we obtain the formula $[\bot] A_{\text{rep}}(i)$ from $[\bot]$. The next lemma establishes the correctness of this translation.

**Lemma 5.10.** For all Boolean CQs $\phi$ and $i \in [-1, n]$, we have:

- $\langle \mathcal{O}, A_{\text{rep}} \cup A_l \rangle$ is inconsistent iff TDB($\mathfrak{A}$) $\models [\bot] A_{\text{rep}}(i)$.
- $\langle \mathcal{O}, A_{\text{rep}} \cup A_l \rangle \models \phi$ iff TDB($\mathfrak{A}$) $\models [\phi] A_{\text{rep}}(i)$. 

5.3 Rewriting r-Satisfiability

We can now use the above rewritings to capture r-satisfiability via the r-completeness conditions. Given \( \mathcal{W} \subseteq 2^{\{p_1, \ldots, p_n\}} \), \( \mathcal{B} \subseteq \{ B(a) \mid B \in \mathcal{B}(O), a \in I(\Phi) \} \), and a single world \( W \in \mathcal{W} \), define \( rSat_W(i) := fC_1(i) \land fC_2(i) \land fC_5(i) \), where

\[
\begin{align*}
    fC_1(i) & := \neg \exists [\exists A_{\mathcal{R}}] (i), \\
fC_2(i) & := \bigwedge_{p \in \mathcal{P}} \neg [\exists p] A_{\mathcal{R}} (i), \\
    fC_5(i) & := \bigwedge \{ \neg [\exists \psi] A_{\mathcal{C}} (i) \mid \psi \text{ rigid witness query for } \tilde{Q}^{-} \},
\end{align*}
\]

in which all rewritings are w.r.t. \( W \) (which is not mentioned explicitly in the notation).

We can finally prove Lemma 5.5 which we state here again for convenience.

**Lemma 5.5.** For all \( \mathcal{W} = \{ W_1, \ldots, W_k \} \subseteq 2^{\{p_1, \ldots, p_n\}} \), mappings \( \lambda : [0, n] \rightarrow [1, k] \), and sets \( \mathcal{B} \subseteq \{ B(a) \mid B \in \mathcal{B}(O), a \in I(\Phi) \} \), the tuple \( (\tilde{A}^R, \tilde{Q}^+, \tilde{Q}^-, \tilde{S}) \) is r-complete w.r.t. \( \mathcal{W} \) and \( \lambda \) iff the following hold:

(a) For all \( i \in [0, n] \), we have \( TDB(\tilde{A}) \models rSat_{W_{\lambda(i)}}(i) \).

(b) For all \( W \in \mathcal{W} \), we have \( TDB(\tilde{A}) \models rSat_{W}(-1) \).

(c) For all \( S \in \mathcal{R}(O) \) and \( a \in I(\Phi) \), we have \( \exists S(a) \in B \) iff there is an \( i \in [0, n] \) such that \( TDB(\tilde{A}) \models [\exists S(a)] A_{\mathcal{R}} (i) \), where this rewriting is w.r.t. the world \( W_{\lambda(i)} \).

**Proof.** We consider Definition 5.5 Conditions [C3] and [C4] are trivially satisfied. Lemmas 5.10 and 5.4 show that [a] and [b] take care of Conditions [C1] and [C2]. It remains to prove Condition [C5]. For \( S_{aux} \) and \( S_{bottom} \), we show this in the proof of Lemma 5.2 in the appendix, even independent of [c]. For \( S_{aux} \), we show that [c] is equivalent to the corresponding part of Condition [C6]. For this, consider any \( a \in I(\Phi) \).

(\( \Leftarrow \)) If [c] holds, then the definition of \( S_{aux} \) based on \( \mathcal{B} \) and Lemmas 5.10 and 5.4 yield that \( \exists S(a) \in S_{aux} \) iff there is an \( i \in [0, n] \) such that

\[
\langle O, \tilde{A}^R \cup A_{\mathcal{R}}^{Q^{-}}, \cup A_{\mathcal{Q}^{Q^{-}}}, \cup A_{\mathcal{R}} \cup A_{\mathcal{S}_{aux}} \rangle \models \exists S(a),
\]

since \( S_{aux} \) and \( S_{bottom} \) do not contain relevant assertions. However, by Lemma A.3 all parts of \( A_{\mathcal{S}_{aux}} \) are relevant to obtain the conclusion \( \exists S(a) \) are contained in \( \tilde{A}^R \), given the definition of these ABoxes. Since \( A_{\mathcal{S}_{aux}} \) does not contain basic concept assertions over \( I(\Phi) \), we obtain that \( \exists S(a) \in S_{aux} \) iff there is an \( i \in [0, n] \) such that \( \langle O, \tilde{A}^R \cup A_{\mathcal{R}}^{Q^{-}}, \cup A_{\mathcal{Q}^{Q^{-}}}, \cup A_{\mathcal{R}} \rangle \models \exists S(a) \), as required.

(\( \Rightarrow \)) This follows from the definition of \( S_{aux} \) and Lemmas 5.4 and 5.10 \( \square \)

6 Data Complexity

Based on the FO rewritability of r-satisfiability, we now show that the low data complexity of query answering in DL-Lite does not increase dramatically in our temporal setting and prove \( \text{ALG-OPTIME}-\text{completeness} \). Nevertheless, FO rewritability is lost. The lower bound holds already for DL-Lite-core without rigid names, which can be shown by reducing the word problem of deterministic finite automata to TCQ entailment, by translating the construction of [8, Thm. 9] to our setting.
Theorem 6.1. TCQ entailment in DL-Lite\textsubscript{core} is A\textsc{log}\textsc{time}-hard in data complexity, even if \( C^R = \emptyset \) and \( P^R = \emptyset \).

We show A\textsc{log}\textsc{time}-membership by describing an alternating Turing machine that solves our problem in logarithmic time. For t-satisfiability, we need additional notation and auxiliary results, which are described next.

6.1 Separating the LTL Satisfiability Test

As before, we consider a TCQ \( \Phi \) and a DL-Lite\textsubscript{horn} TKB \( \mathcal{K} = (\mathcal{A}, \mathcal{V}) \) with \( \mathcal{A} = (A_i)_{0 \leq i < n} \). Similar to Algorithm 4.1, we do not consider the t-satisfiability test for \( \Phi^{pa} \) as a black box, but rather split it into multiple parts, which are then integrated with the test for r-satisfiability using the rewritings of the previous section. By Lemma A.7, we can assume \( \Phi^{pa} \) to be separated, i.e., that no future operator occurs in the scope of a past operator and vice versa.

A subformula of \( \Phi^{pa} \) is a top-level future formula (top-level past formula) if it is of the form \( \bigcirc \phi \lor \phi H \psi \) (\( \bigcirc \phi \phi \lor \phi S \psi \)) and occurs in \( \Phi^{pa} \) at least once in the scope of no other temporal operator; we denote the set of all such formulas and their negations by \( \mathcal{F} \) (\( \mathcal{P} \)), and assume without loss of generality that all propositional variables from \( \{p_1, \ldots, p_m\} \) occur in both \( \mathcal{F} \) and \( \mathcal{P} \). Since we require \( \Phi^{pa} \) to be satisfied at time point \( n \), the crucial part of this formula thus concerns the past formulas in \( \mathcal{P} \), whose satisfaction depends on the number \( n \).

The goal is to separate this dependency as much as possible.

The Boolean abstraction \( \Phi^{ba} \) of \( \Phi^{pa} \) is obtained by replacing the top-level future and past formulas \( f_1, \ldots, f_o \) of \( \Phi^{pa} \) by propositional variables \( q_1, \ldots, q_o \), respectively. We consider the set \( \mathcal{V} \) of all valuations \( v: \{q_1, \ldots, q_o\} \to \{true, false\} \) of these variables for which \( v(\Phi^{pa}) \models true \). For \( v \in \mathcal{V} \), the set \( \mathcal{F}^v := \{ f_i \in \mathcal{F} \mid v(q_i) = true \} \cup \{ \neg f_i \in \mathcal{F} \mid v(q_i) = false \} \) collects the induced future subformulas of \( \Phi^{pa} \), and \( \mathcal{P}^v \) can be defined similarly. Given a set of worlds \( \mathcal{W} \subseteq 2^{\{p_1, \ldots, p_m\}} \) and \( v \in \mathcal{V} \), the set \( \text{Fut}_v \subseteq \mathcal{W} \) contains the worlds that can serve as the start of an LTL model of \( \mathcal{F}^v \) (restricted to \( \mathcal{W} \)):

\[
\text{Fut}_v := \{ w_0 \mid \text{there is } \mathcal{W} = (w_i)_{i \geq 0} \text{ such that } \mathcal{W}, 0 \models \mathcal{F}^v \text{ and, for all } i \geq 0, w_i \in \mathcal{W} \}.
\]

All of these sets are independent of the data and can hence be considered constant.

Lemma 6.2. Let \( \mathcal{W} = \{w_1, \ldots, w_k\} \subseteq 2^{\{p_1, \ldots, p_m\}} \) and \( w_0, \ldots, w_n \in \mathcal{W} \). The following are equivalent.

(a) There is an LTL structure \( \mathcal{W} \) that only contains worlds from \( \mathcal{W} \), starts with \( w_0, \ldots, w_n \), and satisfies \( \mathcal{W}, n \models \Phi^{pa} \).

(b) There is a valuation \( v \in \mathcal{V} \) such that \( w_n \in \text{Fut}_v \) and \( (w_0, \ldots, w_n, w_n, \ldots) \models \mathcal{P}^v \).

Proof. (\( \Rightarrow \)) Given \( \mathcal{W} \), \( v \) can be obtained by checking which elements of \( \{f_1, \ldots, f_o\} \) are satisfied at \( n \); then, the LTL structure needed to justify \( w_n \in \text{Fut}_v \) is defined as the substructure of \( \mathcal{W} \) that starts at \( n \). Since the satisfaction of the past formulas \( \mathcal{P}^v \) in the structure \( (w_0, \ldots, w_n, \ldots) \) at \( n \) does not depend on any time point after \( n \), the remaining worlds can be chosen arbitrarily.

(\( \Leftarrow \)) \( \mathcal{W} \) can be constructed by joining \( (w_0, \ldots, w_n) \) and the LTL structure obtained from the fact that \( w_n \in \text{Fut}_v \), since the satisfiability of past (future) subformulas at \( n \) is not affected by the worlds after (before) that time point. \( \Box \)
As mentioned above, the critical part is to test whether \((w_0, \ldots, w_n, w_n, \ldots) \models P^\nu\) holds, since that depends on \(n\). However, we can employ Lemma A.6 to separate the time points from each other. Since \(P^\nu\) does not contain any future operators, its satisfaction depends only on the time points before \(n\), and we do not have to be concerned with finding a period or satisfying \(U\)-formulas. Our task is thus to find types \(T_0, \ldots, T_n \in \text{Typ}(P)\) such that

- \(T_0\) is initial and \(P^\nu \subseteq T_n\);
- for all \(i \in [0, n-1]\), the pair \((T_i, T_{i+1})\) is compatible;
- for all \(i \in [0, n]\), the world \(T_i \cap \{p_1, \ldots, p_m\}\) belongs to \(\mathcal{W}\).

We can then use the world \(w_n = T_n \cap \{p_1, \ldots, p_m\}\) induced by the last type to check satisfiability of \(F^\nu\), i.e., whether \(w_n \in \text{Fut}_{\mathcal{W}}\).

### 6.2 An Alternating Logarithmically Time-Bounded Turing Machine

Based on this abstraction, we describe an alternating Turing machine (ATM) \([34]\) that solves the TCQ satisfiability problem in \(DL-Lite_{Horn}\) in logarithmic time, in the size of the ABox sequence \(\mathfrak{A}\). We use a random access model, where the read-only input tape is accessed by writing the address of the symbol to be read (in binary) on a specific address tape. Next to those two tapes, the machine may use a constant number of work tapes. If \(l\) is the size of the input, such machines can add, subtract, compare, and compute the logarithm of numbers with \(O(\log l)\) bits \([64, \text{Lem. 7.1}]\).

As usual, our ATM \(\mathfrak{M}\) deciding satisfiability of \(\Phi\) w.r.t. \(\mathcal{K}\) is based on Lemma 2.11 where for \(t\)-satisfiability of \(\mathcal{W}\) and \(\lambda\) we only need to find types \(T_0, \ldots, T_n\) (Lemmas A.6 and 5.2), and the \(r\)-satisfiability is checked via FO rewritings based on an additional set \(\mathcal{B} \subseteq \{B(a) \mid B \in \text{B}(\mathcal{O}), a \in I(\Phi)\}\) (see Lemmas 3.6 and 5.5). The sets \(\mathcal{W}\) and \(\mathcal{B}\) (of constant size) are guessed in the beginning, and the mapping \(\lambda\) can be obtained from the types \(T_0, \ldots, T_n\) guessed during the computation of \(\mathfrak{M}\).

**Example 6.3.** Figure 3 gives an overview of the computation tree of \(\mathfrak{M}\), given an ABox sequence with \(n = 15\). The nodes represent points at which the alternating machine splits.
at which time point we have \( \Phi \) is satisfiable w.r.t. \( \mathcal{K} \) (i.e., the ATM accepts the input), otherwise false

```pseudo
0 v := Guess an element of \( \mathcal{V} \)
1 \forall w := \text{Guess a subset of } 2^{\{p_1, \ldots, p_n\}}
2 B := \text{Guess a subset of } \{ B(a) \mid B \in \mathcal{B}(\mathcal{O}), a \in \mathcal{I} (\Phi) \}
3 T_i := \text{Guess an initial type in } \text{Typ}(P)
4 T_i := \text{Guess a type in } \text{Typ}(P) \text{ that contains } P^v
5 if \( T_i \cap \{ p_1, \ldots, p_n \} \notin \text{Fut} \), then return false
6 foreach \( w \in \mathcal{V} \) do
7 if \( \text{TDB}(\mathfrak{A}) \neq \text{rSat}_w (-1) \) then return false
8 // start recursion with \( i = 0, \ell = \log(n+1) \), and \( B' = B \)
9 return ATMRECURSION(\( \Phi, \mathcal{K}, v, \mathcal{W}, B, B, T_i, \ell, \log(n+1), 0 \))
```

into two copies. Each node is responsible for constructing a subsequence of \( T_0, \ldots, T_{15} \) for which only the first and last types are given. For the root node, this means that we initially guess \( T_0 \) and \( T_{15} \) in order to start this process. Given the label \( i \) of a node, and its level \( \ell \), it is responsible for the subsequence starting at index \( i \) and ending at \( i+2^\ell-1 \).

The root node is labeled by \( i = 0 \) and has level \( \ell = 4 \), which makes it responsible for the subsequence from \( T_0 \) to \( T_{15} \), i.e., the full sequence. It then delegates this responsibility to its successors at level 3 in the following way: it guesses the types \( T_7 \) and \( T_8 \) in the exact middle of the sequence and verifies their t-compatibility, and then it splits the sequence in half. The machine splits into two copies, each of which is responsible for one half of the remaining computation. The left successor deals with the sequence from \( T_0 \) to \( T_7 \), where we already know the first type and the last type. Correspondingly, the right successor (marked in gray in the figure) is labeled by \( i = 8 \), because its designated subsequence starts at \( T_8 \) and ends at \( T_{15} \). Again, we already know the types \( T_8, T_{15} \) for the start and end points. In turn, this copy of the machine then guesses a t-compatible pair \( (T_{11}, T_{12}) \) (also marked in the figure), and splits the subsequence again into the two shorter sequences \( T_7, \ldots, T_{11} \) and \( T_{12}, \ldots, T_{15} \).

Since all copies of \( \mathfrak{M} \) proceed in this way, those at level \( \ell = 1 \) consider only two types \( T_i, T_{i+1} \) that have already been guessed before. Each copy then verifies the t-compatibility of this pair of types. Finally, the copies at level 0 each know only one type. Throughout the whole computation, each type \( T_i \) is guessed only once, which prevents conflicting guesses for one time point. Moreover, the copies require no knowledge about what happens in other branches of the computation tree.

The copies at level \( \ell = 0 \) are each responsible only for one type \( T_i \), which induces the world \( w_i = T_i \cap \{ p_1, \ldots, p_n \} \) that implicitly corresponds to \( W_{\lambda(i)} \) in Lemma 5.5. We can thus apply this lemma to check the r-completeness conditions. By this lemma, we have to check the satisfaction of FO formulas in TDB(\( \mathfrak{A} \)), which can be done in \( \text{AC}^0 \) [4] Thm. 9.1], a subclass of LogTIME. There are two points that deserve special attention. First, Condition [b] in Lemma 5.5 refers to satisfaction problems w.r.t. the empty ABox (\( i = -1 \)) for all elements of \( \mathcal{W} \). To this end, \( \mathfrak{M} \) splits into \( |\mathcal{W}| \) (constantly many) further copies that then verify the corresponding problems. Second, Condition [c] in Lemma 5.5 imposes a global condition over all time points \( i \in [0, n] \). Therefore, \( \mathfrak{M} \) additionally guesses, for each \( \exists S(a) \in B \), at which time point \( i \) we have \( \text{TDB}(\mathfrak{A}) \models [\exists S(a)]_{\lambda(i)}(i) \), where this rewriting is w.r.t. the
Algorithm 6.2: ATMRECURSION

**Input:** TCQ $\Phi$, TKB $K = (\mathcal{O}, (A_i)_{i \in \mathcal{O}(\mathcal{A})})$, valuation $v$, worlds $W$, sets of basic concept assertions $B$ and $B'$, types $T_i$ and $T_r$, level $\ell$, time point $i$

**Output:** true if all recursively created ATM copies accept, otherwise false

1. $B' := \text{Guess a partition of } B' \text{ into two sets}$
2. if $\ell > 1$ then
   3. $\ell := \ell - 1$
   4. $(T^{(1)}, T^{(2)}) := \text{Guess a } t\text{-compatible pair from Type}(P)$ // Lemma 6.2
   5. // split into two copies and accept iff both accept
   6. return ATMRECURSION($\Phi, K, v, W, B, B'^{(i)}, T, T^{(1)}, \ell, i$) and ATMRECURSION($\Phi, K, v, W, B, B'^{(i)}, T, T^{(2)}, \ell, i, i+2$)
3. else
   4. if $(T_i, T_r)$ are not $t\text{-compatible}$ then return false // Lemma 6.2
   5. // split into two copies one last time
   6. return ATMFINAL($\Phi, K, v, W, B, B'^{(i)}, T, i$) and ATMFINAL($\Phi, K, v, W, B, B'^{(i)}, T, i+1$)

world $W = w_i$. The elements of $B$ for which a copy of $\mathcal{M}$ is responsible are then propagated along the branches of the computation tree, and split accordingly.

The ATM’s behavior is specified in Algorithm 6.1, where the valuation set $V$, propositions $\{p_1, \ldots, p_m\}$, top-level past formulas $P$, and the past formulas $P^*$ are constructed based on $\Phi$ as described in the beginning of Section 6.1. Because of the data complexity assumptions, all constructions depending only on $\Phi$ and $O$ are of constant size and encoded directly into the states of $\mathcal{M}$. In addition to $V$, $B$, and a valuation $v \in V$, which are guessed at the beginning, this includes

- a set $B' \subseteq B$ that contains the elements of $B$ the current copy of the machine is responsible for (see Lemma 5.4(1));
- two types $T_i$ and $T_r$ for the left-most and the right-most types of the current subsequence of $T_0, \ldots, T_n$.

The (read-only) input tape of $\mathcal{M}$ contains only the FO structure $TDB(\mathcal{I})$, which implicitly contains the number $n$. In each configuration, $\mathcal{M}$ stores the index $i$ and level $\ell$ as described in Example 6.3 on its work tapes, which requires only a logarithmic number of bits. The different ATM configurations are described by the recursive Algorithm 6.2(ATMRECURSION).

At the end, there are $n+1$ copies of $\mathcal{M}$, one for each index $i \in [0, n]$, and each of them knows only one type $T_i$ which induces a unique world $w_i = T_i \cap \{p_1, \ldots, p_m\}$. These copies execute the final tests described in Algorithm 6.3(ATMFINAL), in line with Lemmas 5.4 and 6.2. It is easy to show that a successful run of $\mathcal{M}$ indeed reflects the satisfiability of $\Phi$ w.r.t. $K$.

**Theorem 6.4.** TCQ entailment in $DL-Lite^H$ is in $A\log \text{TIME}$ in data complexity, even if $P^\mathbb{R} \neq \emptyset$.

### 7 TCQ Entailment Beyond the Horn Fragment

Having established the good computational behavior of TCQ entailment in Horn fragments of $DL-Lite$, we now consider the more expressive $krom$ and $bool$ fragments. Even without role inclusions, it turns out that TCQ entailment in these logics is as hard as for $\mathcal{ALC}$, which allows to express qualified existential restrictions (but no inverse roles). With role inclusions,
Lemma 7.1. Let $I \models \Phi$ and $\Phi$ be a model of $\Phi$. Then, we have $W_e \models A$.

We show that TCQs, together with CIs of the form

\[ \exists R \text{ supports } \Phi, \quad \Phi = \exists R \wedge \psi \]

We use negated CQs to simulate complex CIs. We use (fresh) symbols $\bar{x}$ to simulate the complements of concept names $A$.

### Table 6: Representing complex CIs in DL-Lite$\text{kom}$ via TCQs.

| CI | TCQ |
|----|-----|
| $\exists R A_1 \sqsubseteq A_2$ | $\exists x, y \exists R(x, y) \wedge A_1(x) \wedge \bar{A}_2(y)$ |
| $A_1 \sqsubseteq \forall R A_2$ | $\exists x, y \exists R(x, y) \wedge A_1(x) \wedge \bar{A}_2(y)$ |
| $A_1 \sqcap \cdots \sqcap A_m \sqsubseteq A_{m+1} \sqcup \cdots \sqcup A_{m+n}$ | $\exists x, y \exists R(x, y) \wedge A_1(x) \wedge \cdots \wedge A_m(x) \wedge \bar{A}_{m+1}(y) \wedge \cdots \wedge \bar{A}_{m+n}(y)$ |

the complexity even increases to the same level as for $\text{ALCI}$ (with inverse roles) (see Table 1). As an auxiliary result, we first show that there is no difference between $\text{DL-Lite}_{\text{kom}}$ and $\text{DL-Lite}_{\text{bool}}$ in our setting, as TCQs can be used to simulate CIs that are usually only expressible in $\text{DL-Lite}_{\text{bool}}$.

### 7.1 Reducing $\text{DL-Lite}_{\text{bool}}$ to $\text{DL-Lite}_{\text{kom}}$

We show that TCQs, together with CIs of the form $\top \sqsubseteq A \sqcup \bar{A}$, can simulate several kinds of CIs that go beyond $\text{DL-Lite}_{\text{kom}}$, covering $\text{DL-Lite}_{\text{bool}}$ and even parts of $\text{ALCI}$. The description logic $\text{ALCT}$ supports qualified existential / value restrictions of the form $\exists R.C \sqcap \forall R.C$, where $R$ is a role and $C$ a concept, with the following semantics:

\[
(\exists R.C)^T := \{ x \in \Delta^T \mid \text{there is } y \in \Delta^T \text{ such that } (x, y) \in R^T \text{ and } y \in C^T \}
\]

\[
(\forall R.C)^T := \{ x \in \Delta^T \mid \text{for all } y \in \Delta^T \text{ such that } (x, y) \in R^T \text{ implies } y \in C^T \}
\]

That is, a qualified existential restriction $\exists R.C$ checks for the existence of an $R$-successor of type $C$, whereas $\forall R.C$ requires that all $R$-successors are of type $C$.

In the following construction, we employ negated CQs to simulate complex CIs. We use (fresh) symbols $\bar{x}$ to simulate the complements of concept names $A$.

Lemma 7.1. Let $(C \subseteq D, \neg \phi)$ be one of the pairs of a CI and a TCQ given in Table 6, and let $\mathcal{I}$ be a model of $\top \subseteq A \sqcup \bar{A}$ and $A_1 \cap \bar{A} \subseteq \bot$ for all concept names $A_i$ occurring in $D$. Then, we have $\mathcal{I} \models C \subseteq D$ iff $\mathcal{I} \models \neg \phi$.

This means that, given a TCQ $\Phi$ and a TKB $K = \langle \mathcal{O}, (A_i)_{0 \leq i < n} \rangle$, we have

\[ \langle \mathcal{O}, (A_i)_{0 \leq i < n} \rangle \models \Phi \iff \langle \mathcal{O}', (A_i)_{0 \leq i < n} \rangle \models ((\Box \neg \neg \phi) \rightarrow \Phi), \quad \text{where} \]

Algorithm 6.3: ATMFIJAL

**Input:** TCQ $\Phi$, TKB $\langle \mathcal{O}, \mathcal{I} \rangle$ with $\mathcal{I} := \langle A_i \rangle_{0 \leq i < n}$, valuation $v$, worlds $W$, sets of basic concept assertions $B$ and $B'$, type $T$, time point $i$.

**Output:** true if all final tests are successful, otherwise false.

1. $w_i := T_i \cap \{ p_1, \ldots, p_m \}$
2. if $w_i \notin W$ then return false // Lemma 6.2
3. if $\text{TDB}(\mathcal{I}) \not\models \text{Sat}_v(i)$ then return false // Lemma 5.5(c)
4. foreach $S \in \mathcal{R}^T(\mathcal{O}, a \in I(\Phi))$ do // Lemma 6.1(c), "if"-direction
5. if $\text{TDB}(\mathcal{I}) \not\models [\exists S(a)|A_{\mathcal{I}}(i)](i)$ w.r.t. $w_i$ and $\exists S(a) \notin B$ then return false
6. foreach $\exists S(a) \in B'$ do // Lemma 5.5(c), "only if"-direction
7. if $\text{TDB}(\mathcal{I}) \not\models [\exists S(a)|A_{\mathcal{I}}(i)](i)$ w.r.t. $w_i$ then return false
8. return true
With the same construction, $\Phi$ is satisfiable w.r.t. $(\mathcal{O}, (A_i)_{0 \leq i < n})$ iff $(\square \mathcal{F} \wedge \Psi) \land \Phi$ is satisfiable w.r.t. $(\mathcal{O}', (A_i)_{0 \leq i < n})$. This means that we can use all CI\s listed in Table 5 also in $DL-Lite_{krom}$ for our purposes. In particular, we have the following corollary.

**Corollary 7.2.** TCQ entailment in $DL-Lite_{krom}$ can be logspace-reduced to TCQ entailment in $DL-Lite_{krom}$.

This means that it suffices to show our complexity upper bounds for $DL-Lite_{krom}$, and the lower bounds for $DL-Lite_{krom}$. But even more than that, we can use CIs with qualified existential restrictions on the left-hand side (or, equivalently, value restrictions on the right-hand side) to prove hardness results for TCQ entailment in $DL-Lite_{krom}$. We can even nest these concept constructors arbitrarily.

**Example 7.3.** The CI $A_1 \sqcup A_2 \sqcup \exists R_1, A_3 \sqsubseteq A_4 \sqcup \forall R_1, (A_1 \sqcap \exists R_2)$ can be expressed by the following CIs, assuming $A'_1, \ldots, A'_7$ to be fresh concept names:

$$A_1 \sqsubseteq A_4 \sqcup A'_5, \quad A_2 \sqsubseteq A_4 \sqcup A'_5, \quad A'_6 \sqsubseteq A_4 \sqcup A'_5,$$

$$\exists R_1, A_3 \sqsubseteq A'_6, \quad A'_5 \sqsubseteq \forall R_1, A'_7, \quad A'_7 \sqsubseteq A_1, \quad A'_1 \sqsubseteq \exists R_2.$$

These CIs can then, in turn, be simulated by negated CQs as described in Lemma 7.1.

As usual, we now consider a Boolean TCQ $\Phi$ and a TKB $\mathcal{K} = (\mathcal{O}, (A_i)_{0 \leq i < n})$ written in a DL between $DL-Lite_{krom}$ and $DL-Lite_{krom}^H$, depending on the context, and investigate the combined and data complexity of TCQ entailment.

### 7.2 Combined Complexity

Given Corollary 7.2 we directly get two rather strong hardness results from atemporal query answering, i.e., even without rigid names, from $\text{EXPTIME}$-hardness of UCQ entailment in $DL-Lite_{krom}$ and $2-\text{EXPTIME}$-hardness of UCQ entailment in $DL-Lite_{krom}^H$ [25].

**Corollary 7.4.** TCQ entailment in $DL-Lite_{krom}$ is $\text{EXPTIME}$-hard in combined complexity, which increases to $2-\text{EXPTIME}$-hardness for $DL-Lite_{krom}^H$, even if $\mathcal{C}^R = \emptyset$ and $\mathcal{P}^R = \emptyset$.

Since $DL-Lite_{krom}^H$ is a sublogic of $\text{AACLHI}$, for which TCQ entailment with rigid concepts and roles is in $2-\text{EXPTIME}$ [15] Thm. 12, the $2-\text{EXPTIME}$ lower bound is already tight. To match the $\text{EXPTIME}$ lower bound for $DL-Lite_{krom}$ without role inclusions, we first establish an auxiliary result for satisfiability of (atemporal) conjunctions of CQ literals in $DL-Lite_{krom}$.

**Lemma 7.5.** Satisfiability of Boolean conjunctions of CQ literals w.r.t. $DL-Lite_{krom}$ KBs is in $\text{EXPTIME}$ in combined complexity.

**Proof.** By grounding the positive literals using fresh individual names, the problem can be reduced to UCQ non-entailment [16]. The complexity bound then follows from the fact that UCQ entailment w.r.t. so-called frontier-one disjunctive inclusion dependencies is in $\text{EXPTIME}$ [26] Thm. 8], and such dependencies can simulate $DL-Lite_{krom}$ CIs. \qed

Following the approach of Lemma 2.11 this allows us to show the following.
Theorem 7.6. TCQ entailment in DL-Lite\textsubscript{bool} is in \textsc{ExpTime} in combined complexity if \(\mathbb{C}_R = \emptyset\) and \(\mathbb{P}_R = \emptyset\).

Proof. By Corollary 7.2, it suffices to describe a decision procedure for DL-Lite\textsubscript{krom}, which can be done by following Lemma 2.11. By [16, Lem. 6.1], we can assume without loss of generality that the TKB is of the form \(\langle \mathcal{O}, \emptyset \rangle\), which means that we do not have to find a mapping \(\lambda\). Since there are no rigid names to enforce dependencies between time points, in order to check the \(r\)-satisfiability of a set \(W = \{W_1, \ldots, W_k\}\), it suffices to check the satisfiability of \(\chi_i\) for all \(i \in [1, k]\) individually (see also [16]). Hence, we can define \(W\) as the set of all those sets \(W_i\) for which \(\chi_i\) is satisfiable w.r.t. \(\mathcal{O}\). According to Lemma 7.5, this can be done in exponential time. Moreover, t-satisfiability of \(\Phi_{\text{pa}}\) w.r.t. \(W\) can also be checked in \textsc{ExpTime} [16], and hence we obtain the claim by Lemma 2.11. \(\square\)

In the presence of rigid names, the complexity of TCQ entailment in DL-Lite\textsubscript{krom} increases.

Theorem 7.7. TCQ entailment in DL-Lite\textsubscript{krom} is \textsc{co-NExpTime-hard} w.r.t. combined complexity if \(\mathbb{C}_R \neq \emptyset\), even if \(\mathbb{P}_R = \emptyset\).

Proof. The proof is by reduction from the satisfiability problem for \(\mathcal{EL}_\bot\)-\textsc{LTL}, which is already \textsc{NExpTime-hard} if no role names are available (neither rigid nor flexible) [21,23]. \(\mathcal{EL}_\bot\) is the DL extending \(\mathcal{EL}\) by the \(\bot\) constructor. The formulas \(\Phi\) of that language are similar to TCQs, but instead of CQs they use assertions and CIs formulated in \(\mathcal{EL}\) (plus \(\bot\)) as atomic formulas. However, if \(\Phi\) contains no role names, we can assume that all assertions in \(\Phi\) are of the form \(A(a)\), where \(A \in \mathbb{C}\) and \(a \in \mathbb{I}\), which can hence be directly treated as CQs. Moreover, all CIs in \(\Phi\) are of the form \(\top \sqsubseteq A\), \(A \sqsubseteq \bot\), or \(A_1 \sqcap \cdots \sqcap A_m \sqsubseteq A_{m+1}\), which can be replaced by equivalent negated CQs according to Lemma 7.1, if we add certain DL-Lite\textsubscript{krom} CIs to the global ontology. We can hence obtain a TCQ \(\Phi'\) and an ontology \(\mathcal{O}\) such that \(\Phi'\) is satisfiable w.r.t. \(\langle \mathcal{O}, \emptyset \rangle\) iff the \(\mathcal{EL}_\bot\)-\textsc{LTL} formula \(\Phi\) is satisfiable. \(\square\)

For proving containment in \textsc{co-NExpTime}, we use a technique from [16,18] that is again based on Lemma 2.11 and additionally guesses an exponential set \(\mathcal{X} \subseteq 2^{\mathbb{C}_R(\mathcal{K})}\) that represents all combinations of rigid concept names that are allowed to occur in a model of the TCQ \(\Phi\) w.r.t. the TKB \(\mathcal{K}\). Using this set, we can separate the satisfiability tests required for r-satisfiability in a similar fashion as in Lemma 3.6.

Theorem 7.8. TCQ entailment in DL-Lite\textsubscript{bool} is in \textsc{co-NExpTime} in combined complexity if \(\mathbb{P}_R = \emptyset\), even if \(\mathbb{C}_R \neq \emptyset\).

Finally, we prove 2-\textsc{ExpTime-hardness} of TCQ satisfiability in DL-Lite\textsubscript{krom} in the presence of rigid role names, by reducing the word problem of exponentially space-bounded alternating Turing machines. Our reduction is based on the 2-\textsc{ExpTime-hardness} proof for \(\mathcal{ALC}\)-\textsc{LTL} in [18], which we adapt to our setting using ideas from [37]. Recall that containment in 2-\textsc{ExpTime} follows from the corresponding result for \(\mathcal{ALC}\) [15, Thm. 12].

Theorem 7.9. TCQ entailment in DL-Lite\textsubscript{krom} is 2-\textsc{ExpTime-hard} in combined complexity if \(\mathbb{P}_R \neq \emptyset\).
7.3 Data Complexity

As the final result of this paper, we show that the data complexity of TCQ entailment in DL-Lite$^H_{\text{kon}}$, is in co-NP, by showing that satisfiability is in NP. We follow an approach similar to the r-complete tuples from Section 3 however, since the goal is NP instead of Alogtime, we do not need to be as careful with our constructions as in Sections 3, 5 and 6. For example, we can simply guess the set $\mathcal{W}$ and mapping $\lambda$ in constant and linear time, respectively. Similarly, in order to separate the satisfiability tests for each $i \in [0, \ldots, n+k]$ as in Section 3 we can guess one flexible ABox type per time point, instead of only one rigid ABox type. Of course, the individual ABox types need to agree on the rigid assertions.

Definition 7.11 (ABox Type). A (flexible) ABox type for $\mathcal{K}$ is a set $\mathcal{A}$ of assertions formulated over $\mathcal{I}(\mathcal{K}) \cup \mathcal{F}^{\text{ax}}, \mathcal{C}(\mathcal{O})$, and $\mathcal{P}(\mathcal{O})$ such that $\neg \alpha \in \mathcal{A}$ iff $\alpha \notin \mathcal{A}$.

The set $\mathcal{F}^{\text{ax}}$ is defined similarly as in Section 3 it contains an individual name $a_i^d$ for each $i \in [0, n+k]$ and variable $x$ occurring in a CQ in $\mathcal{Q}_k(i)$. We again consider the ABoxes $\mathcal{A}\mathcal{Q}_k(i)$ that contain the assertions obtained from the CQs $\phi \in \mathcal{Q}_k(i)$ by replacing each variable $x$ by $a_i^d$. In contrast to Section 3 we also distinguish the time points $i$ here. The reason is that, even if the same CQ is satisfied at two different time points, the elements that satisfy it may behave differently due to the nondeterminism inherent in DL-Lite$^H_{\text{kon}}$.

As in Section 3 we must ensure that the interactions between $\mathcal{J}_0, \ldots, \mathcal{J}_{n+k}$, which are caused by the rigid names, do not lead to the satisfaction of some $\phi_j \in \mathcal{Q}_k$ in the unnamed part of some $\mathcal{J}_i$ although we have $p_j \in \mathcal{W}_k(i)$. However, it is clear that we cannot guess the whole unnamed part of all the interpretations $\mathcal{J}_0, \ldots, \mathcal{J}_{n+k}$. Instead, we consider these tree-shaped parts only up to a constant depth, and moreover abstract from the actual individual names that are the roots of these trees by considering only their general behavior as it is relevant to the TCQ $\Phi$.

To this end, we define types, which sufficiently characterize the interpretations of individual names and their unnamed successors. A type captures the basic concepts satisfied at a named individual $a$, as well as relevant homomorphisms of CQs from $\mathcal{Q}_k$ into the unnamed successors of $a$. However, it does not refer to the actual individual name itself, and is therefore independent of the input ABoxes. A temporal type is a set of types, which describe the possible behaviors of the unnamed parts of the interpretations over time.

Definition 7.10 (ABox Type). A (flexible) ABox type for $\mathcal{K}$ is a set $\mathcal{A}$ of assertions formulated over $\mathcal{I}(\mathcal{K}) \cup \mathcal{F}^{\text{ax}}, \mathcal{C}(\mathcal{O})$, and $\mathcal{P}(\mathcal{O})$ such that $\neg \alpha \in \mathcal{A}$ iff $\alpha \notin \mathcal{A}$.

The set $\mathcal{F}^{\text{ax}}$ is defined similarly as in Section 3 it contains an individual name $a_i^d$ for each $i \in [0, n+k]$ and variable $x$ occurring in a CQ in $\mathcal{Q}_k(i)$. We again consider the ABoxes $\mathcal{A}\mathcal{Q}_k(i)$ that contain the assertions obtained from the CQs $\phi \in \mathcal{Q}_k(i)$ by replacing each variable $x$ by $a_i^d$. In contrast to Section 3 we also distinguish the time points $i$ here. The reason is that, even if the same CQ is satisfied at two different time points, the elements that satisfy it may behave differently due to the nondeterminism inherent in DL-Lite$^H_{\text{kon}}$.

As in Section 3 we must ensure that the interactions between $\mathcal{J}_0, \ldots, \mathcal{J}_{n+k}$, which are caused by the rigid names, do not lead to the satisfaction of some $\phi_j \in \mathcal{Q}_k$ in the unnamed part of some $\mathcal{J}_i$ although we have $p_j \in \mathcal{W}_k(i)$. However, it is clear that we cannot guess the whole unnamed part of all the interpretations $\mathcal{J}_0, \ldots, \mathcal{J}_{n+k}$. Instead, we consider these tree-shaped parts only up to a constant depth, and moreover abstract from the actual individual names that are the roots of these trees by considering only their general behavior as it is relevant to the TCQ $\Phi$.

To this end, we define types, which sufficiently characterize the interpretations of individual names and their unnamed successors. A type captures the basic concepts satisfied at a named individual $a$, as well as relevant homomorphisms of CQs from $\mathcal{Q}_k$ into the unnamed successors of $a$. However, it does not refer to the actual individual name itself, and is therefore independent of the input ABoxes. A temporal type is a set of types, which describe the possible behaviors of the unnamed parts of the interpretations over time.

Definition 7.11 (Type). A basic type $\mathcal{B}$ is a set of basic concepts from $\mathcal{B}(\mathcal{O})$ and their negations, such that $B \in \mathcal{B}$ iff $\neg B \notin \mathcal{B}$ for all $B \in \mathcal{B}(\mathcal{O})$; it induces the set of assertions $\mathcal{A}\mathcal{B}(a) := \{ \neg B(a) \mid \neg B \in \mathcal{B} \}$. A type $\mathcal{T}$ is a triple $\langle \mathcal{B}, \mathcal{Q}, \mathcal{M} \rangle$ containing a basic type $\mathcal{B}$, a set $\mathcal{Q} \subseteq \mathcal{Q}_k$ of CQs, and a set $\mathcal{M} \subseteq \bigcup_{\phi \in \mathcal{Q}_k} 2^{\mathcal{T}(\phi)}$ of term sets. A temporal type $\tau$ is a set of types. We denote the set of all temporal types by $\mathcal{T}$.

The intuition behind a type $\mathcal{T} = \langle \mathcal{B}, \mathcal{Q}, \mathcal{M} \rangle$ is that it describes the (negated) basic concepts $\mathcal{B}$ that are satisfied at some individual name $a$, the CQs $\mathcal{Q} \subseteq \mathcal{Q}_k$ that are satisfied by $a$ and/or its unnamed successors, and the sets of terms $\mathcal{M}$ that can be partially satisfied by $a$ and its unnamed successors, i.e., for which there is a partial homomorphism of these terms and the associated atoms of a CQ $\phi \in \mathcal{Q}_k$ into the unnamed part of an interpretation that is reachable from $a$ via role connections. Observe that the CQs in $\mathcal{Q}$ describe CQs satisfied somewhere in the successor tree of $a$, whereas the sets in $\mathcal{M}$ are used to describe CQs partially satisfied at $a$.

We thus guess, for each $a \in \mathcal{I}(\mathcal{K}) \cup \mathcal{F}^{\text{ax}}$ and $i \in [0, n+k]$, a type $\mathcal{T}_i^a$, which is a polynomial amount of information. In addition, for each of the constantly many temporal types $\tau \in \mathcal{T}$, we guess so-called tree ABoxes $\mathcal{A}_{\tau}^a$ (of constant size) that describe prototypical trees of
unnamed successors for each type \( \exists \in \tau \). For instance, if \( \exists \) specifies a CQ \( \varphi \in Q_\varphi \) to be not satisfied, then \( \varphi \) is not satisfied in \( A_{\exists}^x \). For a fixed \( \tau \), the ABoxes \( A_{\tau}^x \) must contain the same individual names and agree on the rigid assertions, which allows us to test the satisfaction of the negative CQ literals at each of the \( n + k + 1 \) time points individually.

**Definition 7.12** (Tree ABoxes). Given a temporal type \( \tau \in T \), ABoxes \( (A_{\tau}^x)_{\exists \in \tau} \) are called tree ABoxes (for \( \tau \)) if, for all \( \exists = (B, Q, M) \in \tau \),

- (T1) we have \( I(A_{\exists}^x) = I(A_{\exists}^x) \) and \( A_{\exists}^x \cap A^R = A_{\exists}^x \cap A^R \) for all \( \exists' \in \tau' \);
- (T2) the elements of \( I(A_{\exists}^x) \) are of the form \( u_{\alpha_i} \cdot \lambda_i \), where each \( \lambda_i \) is either of the form \( R^{\exists'} \)
  for \( \exists' \in \tau \) and \( R \in R^f(O) \), or simply a rigid role \( R \in R^f(O) \);
- (T3) for all assertions \( \alpha \) of the form \( B(u_{\varphi}) \) or \( S(u_{\varphi}, u_\beta^{[\exists']} \) with \( B \in B(O) \) and \( S \in R(O) \), we have either \( \alpha \in A_{\tau} \) or \( \neg \alpha \in A_{\tau}^x \), and \( A_{\tau}^x \) contains no other assertions;
- (T4) the knowledge base \( \langle O, A_{\tau}^x \rangle \) is consistent;
- (T5) \( A_{G}(u_\tau) \subseteq A_{\tau}^x \);
- (T6) for all \( \varphi \in Q_\varphi \), we have \( \varphi \in Q_\varphi \) iff \( \langle O, A_{\tau}^x \rangle \models \varphi \);
- (T7) for all \( S \in \bigcup_{\varphi \in Q_\varphi} 2^T(\varphi) \), we have \( S \in M \) iff there are a CQ \( \varphi \in Q_\varphi \) and a partial homomorphism \( \pi : T(\varphi) \rightarrow I(A) \) of \( \varphi \) into \( A \) with \( u_\tau \in range(\pi) \) and \( S = domain(\pi) \), where the functions range and domain yield the range and domain of a given mapping, respectively.

Condition (T1) expresses that the tree ABoxes should respect the rigid names. Conditions (T2) and (T5) establish the tree shape of the ABoxes. Conditions (T3) and (T4) further ensure that the ABoxes are consistent w.r.t. \( O \) and that nondeterministic choices forced by \( O \) are realized in them. Finally, Conditions (T5)–(T7) reflect the satisfaction of the types \( \exists \) in each \( A_{\tau}^x \).

However, so far we have said nothing about the size of tree ABoxes. The idea is that they should contain just enough elements to allow us to expand them into proper models of the TKB \( K \). That is, an element \( u_\rho \) without role successors in \( (A_{\exists}^x)_{\exists \in \tau} \) should have an ancestor \( u_\rho \) that satisfies the same basic concepts, which allows us to continue the construction of a model at \( u_\rho \) by copying the successors of \( u_\sigma \). Moreover, since we want to preserve the satisfaction of queries from \( Q_\varphi \) in this process, we require that a whole subtree of depth \( d := \max \{ |I(\varphi)| \mid \varphi \in Q_\varphi \} \) repeats, which is the maximum amount of elements that a match for a connected CQ from \( Q_\varphi \) may require. For tree ABoxes \( (A_{\tau}^x)_{\exists \in \tau} \) and \( u_\rho, u_\sigma \in I(A_{\tau}^x) \), we say that \( u_\rho \) is an ancestor of \( u_\sigma \) (and \( u_\sigma \) is a descendant of \( u_\rho \)) if \( \rho = \sigma \rho' \) for some \( \rho' \). Moreover, the subtree of depth \( d \) below \( u_\rho \) is the tule \( (C_{\tau}^x)_{\exists \in \tau} \) of ABoxes, where each \( C_{\tau}^x \) is obtained from \( A_{\tau}^x \) by restricting this ABox to the individual names from \( \{ u_\sigma \mid |\rho| \leq d \} \).

We say that two such subtrees \( (C_{\tau}^x)_{\exists \in \tau}, (D_{\tau}^x)_{\exists \in \tau} \) are isomorphic if there is a bijection \( f \) between \( I(C_{\tau}^x) \) and \( I(D_{\tau}^x) \) such that each \( D_{\tau}^x \) is equal to \( C_{\tau}^x \) after replacing all individual names according to \( f \).

**Definition 7.13** (Complete Tree ABoxes). Tree ABoxes \( (A_{\tau}^x)_{\exists \in \tau} \) for \( \tau \) are called complete if, for all \( u_\rho \in I(A_{\tau}^x) \), exactly one of the following conditions is satisfied:

- (T8) for all \( \exists = (B, Q, M) \in \tau \) and \( R \in R^f(O) \), \( \exists R(u_\rho) \in A_{\tau}^x \) iff \( u_\rho \in I(A_{\tau}^x) \) and \( R(u_\rho, u_\rho^{[\exists']} \) \( \in A_{\tau}^x \); or
- (T9) \( u_\rho \) has an ancestor \( u_\rho \) such that \( \rho = \rho' \sigma \) with \( |\sigma| = d \), and \( u_\rho \) has an ancestor \( u_\rho' \) such that the subtree of depth \( d \) below \( u_\rho \) is isomorphic to the subtree of depth \( d \) below \( u_\rho' \); in this case, \( u_\rho \) does not have any descendants in \( I(A_{\tau}^x) \).

---

5 The notation \( R^{[\exists']} \) refers to the fact that \( R \) can be either a rigid role without superscript, or a flexible role with some type \( \exists' \) as superscript.
Condition \([T9]\) ensures that complete tree ABoxes contain all role successors required by existential restrictions, but only up to the first time that a subtree of depth \(d\) repeats \([T9]\).
Condition \([T9]\) implies that the size of complete tree ABoxes is constant in data complexity. To see that this holds, let \(t\) be the number of possible types; clearly \(t \leq 2^{B(O) + |\mathcal{Q}_a| + 2^d |\mathcal{Q}_a|}\). Then, the number of possible successors of an element of \(I(A_\tau)\) is \(s \leq t \cdot |R(\mathcal{O})|\), and a subtree of depth \(d\) has at most \(m \leq s^d\) elements. Since each individual name can satisfy different basic concepts and have different role connections to each of its subtree of depth \(d\), there are at most \(2^{|\mathcal{R}(\mathcal{O})| + s |\mathcal{R}(\mathcal{O})|}\) different ABoxes. Since both KB consistency and CQ non-entailment for DL-Lite \(H\) can be guessed in polynomial time, in particular, Definition 7.12 is independent of the input.

To see that this holds, let \(t\), the input. By Lemma 2.11, TCQ satisfiability is hence in NP.

A\(k \leq \) by a \(\in\) A\(\subseteq\mathcal{Q}_a\) and agree on the rigid assertions;
- \(T_\tau^i = (B_i^\tau, Q_i^\tau, A_i^\tau)\) are types such that \(A_i^\tau(a) \subseteq A_i^\tau\); we denote the denoting the residual
type of an individual name \(a \in I(\mathcal{K}) \cup P^{aux}\) by \(t_a^\tau := (T_\tau^0 | i \in [0,n+k])\);
- \((A_\tau^i)_{\tau \in \tau^i}\) are sequences of complete tree ABoxes for \(\tau\) if we have \(\tau = \tau_1^i\) for some \(a \in I(\mathcal{K}) \cup P^{aux}\), and \(A_\tau^i\) are empty otherwise.

Given a complete tree ABox \(A\) and an individual name \(a\), we construct \(A[a]\) by replacing \(u_e\) by \(a\) and all other individual names \(u_p\) by \(u_p\). Given the tuple \(t\) as above, we define the ABoxes \(A_i^\tau\), for each point \(i \in [0,n+k]\), as the union of all ABoxes \(A_i^\tau[a]\) with \(\tau = \tau_1^i\) and \(T = T_\tau^i\), for all individual names \(a \in I(\mathcal{K}) \cup P^{aux}\).

We can now characterize r-satisfiability similarly to Definition 5.5

**Lemma 7.14.** \(W\) is r-satisfiable w.r.t. \(\lambda\) and \(\mathcal{K}\) iff there is a tuple \(t\) as above such that, for all \(i \in [0,n+k]\):

(C1') \(K^i_{rec} := (\mathcal{O}, A_i^\tau \cup A_i^\tau)\) is consistent.
(C2') For all \(p_j \in \mathcal{W}_{\lambda}(i)\), we have \(K^i_{rec} \not\models \psi_j\).

As mentioned above, \(W\) and \(\lambda\) can be guessed in polynomial time in data complexity, and the LTL satisfiability test can be done in P [15] Lem. 4.12]. Similarly, the tuple \(t\) can be guessed in polynomial time; in particular, Definition 7.12 is independent of the input ABoxes. Since both KB consistency and CQ non-entailment for DL-Lite\(\text{H}_{\text{aux}}\) are decidable in NP [4] Thm. 8.2], [C1'] and [C2'] can be decided nondeterministically in polynomial time, since the size of each \(A_i^\tau\) only depends linearly on the number of individual names in the input. By Lemma 2.11 TCQ satisfiability is hence in NP.

This result actually applies even to \(ALCHI\), which extends DL-Lite\(H_{\text{aux}}\) by qualified existential restrictions (value restrictions can be eliminated); Lemma 7.1 shows how to simulate such restrictions on the left-hand side of CIs. However, it is well-known that the presence of role inclusions also allows us to express qualified existential restrictions on the right-hand side of CIs [9].

**Theorem 7.15.** TCQ entailment in \(ALCHI\) is in co-NP in data complexity, even if \(P^R \neq \emptyset\).
8 Conclusions

In this article, we have studied the complexity of TCQ entailment in DL-Lite logics between DL-Lite$_{core}$ and DL-Lite$^H_{horn}$. We have thus focused on a scenario that reflects the needs of the applications of today: the temporal queries are based on LTL, one of the most important temporal logics; the ontologies are written in standard lightweight logics; and the data allows to capture data streams [9, 27, 47, 48]. Since the complexities we have shown for the Horn fragments of DL-Lite are considerably better than known results for other DLs, including the lightweight DL EL, and do not even depend on the rigid symbols considered, we have identified a fragment that is interesting for applications that need efficient reasoning.

In contrast, TCQ entailment in the Krom and Bool fragments of DL-Lite turned out to be as complex as in more expressive DLs, such as SHQ [16]. In particular, we have shown that TCQ entailment in expressive DLs such as ALC$^H_{horn}$ can be reduced to TCQ entailment in DL-Lite$^H_{krom}$. While the combined complexity thus strongly depends on which symbols are considered to be rigid, we have shown the contrary for data complexity. More precisely, we have shown co-NP containment for the case with rigid roles, and thus closed an important gap. Altogether, our results show that the features we have studied can often be considered “for free”. For combined complexity, TCQ entailment w.r.t. a DL-Lite$^H_{horn}$ TKB is in PSPACE, even if rigid symbols are considered; this matches the complexity of satisfiability in LTL, which is much less expressive given the fact that ontologies are not considered at all. Similarly, the co-NP-containment we have shown for many expressive DLs for data complexity matches the complexity of conjunctive query entailment in these DLs.

In future work, we want to study extensions of TCQs with operators from metric temporal logics and investigate extensions of DL-Lite$^H_{horn}$. On the other hand, a restriction of the set of temporal operators could yield even better results, such as first-order rewritability [20]. It would be also interesting to find out if there are efficient parallel implementations of TCQ answering in DL-Lite$^H_{horn}$ for which we have ALOGTIME data complexity.

Acknowledgements We thank Carsten Lutz and the anonymous reviewers for their many helpful comments. This work was supported by the German Research Foundation (DFG) within the Collaborative Research Centre 912 (HAEC), in the joint DFG-ANR project BA 1122/19-1 (GOASQ), and in the grant 389792660 as part of TRR 248 (see https://perspicuous-computing.science).

A Additional Background on DL-Lite and LTL

A.1 Reasoning in Horn Fragments of DL-Lite

The logics below DL-Lite$_{horn}^H$ do not allow to express disjunction on the right-hand side of CIs, which means that the CIs can be represented as first-order Horn clauses. Reasoning in such DLs is easier since it can be done using deterministic algorithms, often based on canonical interpretations. We recall (and slightly adapt) the construction of the canonical interpretation from [24, 30], which is based on the standard chase [35]. This interpretation contains prototypical domain elements of the form $u_\rho (u$ for “unnamed”), where $\rho$ is a path $\rho := aR_1…R_\ell$ with an individual name $a$ and roles $R_1,…,R_\ell$. We assume that the KB does not already contain the symbols $u_\rho$. The expression $|\rho|$ denotes the length $\ell$ of $\rho$.

\footnote{Containment in ALOGTIME is considered as an indicator for the existence of efficient parallel implementations [3] Thm. 6.27].}
Definition A.1 (Canonical Interpretation). Let $K = \langle O, A \rangle$ be a DL-Lite$_{horn}^H$ knowledge base. First, for all $A \in C$ and $P \in P$, define the interpretation $I^0$ as follows:
\[
\begin{align*}
\Delta^{I^0} & := I(K), \\
A^{I^0} & := \{a \mid A(a) \in A\}, \\
P^{I^0} & := \{(a, b) \mid P(a, b) \in A\} \cup \{(a, u_{up}) \mid \exists P(a) \in A\} \cup \{(u_{up}, a) \mid \exists P^-(a) \in A\}.
\end{align*}
\]
Then, for each $i \geq 0$, do the following: for all $X \in C \cup P$, define $X^{T^{i+1}} : = X^{T^i}$, apply one of the following rules, and increment $i$:

- If $\bigcap B^R \subseteq B \in O$ and $e \in (\bigcap B^R)T^i$, then do the following:
  - if $B \in C$, then add $e$ to $B^{T^{i+1}}$;
  - if $B = \exists R$ and $e \in I(A)$, then add $u_{up}$ to $\Delta^{T^{i+1}}$ and $(e, u_{up})$ to $R^{T^{i+1}}$;
  - if $B = \exists R$ and $e = u_{up}$, then add $u_{up}$ to $\Delta^{T^{i+1}}$ and $(e, u_{up})$ to $R^{T^{i+1}}$.

We also apply this rule to the CIs of the form $\exists R \subseteq \exists R$, for all $R \in P(O)$, and $\exists S \subseteq \exists R$ and $\exists S^{-} \subseteq \exists R^{-}$, for all $S \subseteq R \in O$, which we consider to be implicitly present in $O$ (**).

- If $S \subseteq R \in O$ and $(d, R) \in S^{T^i}$, then add $(d, e)$ to $R^{T^{i+1}}$.

The canonical interpretation $I_K$ is obtained as the limit of this (possibly infinite) sequence of rule applications, assuming that each rule that becomes applicable at some point is executed exactly once. We denote by $\Delta^{T^{i+n}}_{\text{anon}} : = \Delta^{T^i} \setminus I(K)$ the set of all unnamed elements $u_{\rho}$ that are introduced in this process.

Our assumption (**) about additional axioms in the ontology ensures that, whenever there is an element $a \in I(K) \cap (\exists R)^T$ for some $i \geq 0$, then $a$ has an R-successor $u_{\rho}$ in the canonical interpretation, and similarly for the new unnamed elements. This assumption simplifies some of our proofs (specifically, those of Lemmas A.3 and A.7). This is also the main difference to the constructions in [24][30], where $u_{\rho}$ would only be created if $a$ does not already have an R-successor, e.g., another named element. Additionally, we are dealing with negated assertions and assertions about basic concepts (i.e., not just concept names) in ABoxes, which are not explicitly considered in [24][30]. However, it is straightforward to adapt the following results, which show that the canonical interpretation $I_K$ can be used for reasoning about $K$.

Lemma A.2 (see [24]). A DL-Lite$_{horn}^H$ knowledge base $K$ is consistent iff $I_K \models K$. Moreover, if $K$ is consistent, then for every Boolean UCQ $\varphi$, we have $K \models \varphi$ iff $I_K \models \varphi$.

We now describe the behavior of the domain elements of $I_K$ in more detail. In particular, the basic concepts satisfied by an element $e \in \Delta^{T^\omega}$ are uniquely determined by the basic concepts satisfied by $e$ at the point where it was first introduced into $\Delta^{T^i}$. For an unnamed element $u_{\rho}$, there is only a single such basic concept, namely $\exists R^{-}$.

Lemma A.3. Let $e \in \Delta^{T^\omega}$, $i \geq 0$ be the minimal index for which $e \in \Delta^{T^i}$, and $B_e$ be the set of all $B \in B(O)$ such that $e \in B^{T^i}$. Then, for all $B \in B(O)$, we have $e \in B^{T^\omega}$ iff $O \models \bigcap B_e \subseteq B$.

In particular, if $e = u_{\rho}$, then $u_{\rho} \in B^{T^\omega}$ iff $O \models \exists R^{-} \subseteq B$.

A.2 Propositional Linear Temporal Logic

Propositional linear temporal logic (LTL) extends propositional logic with modal operators to represent past and future moments. We implicitly fix a finite signature $P = \{p_1, \ldots, p_l\}$ of propositional variables.
Lemma A.4 (Syntax of LTL). The set of LTL formulas is defined by the following rule, where \( p \in \mathcal{P} \):  

\[
\varphi, \psi ::= p \mid \neg \varphi \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi U \psi \mid \varphi S \psi.
\]

Again, more operators can be derived as in Table[3].

Definition A.5 (Semantics of LTL). An LTL structure is an infinite sequence \( \mathcal{M} = (\mathcal{W}, \models) \) of worlds \( w_i \subseteq \mathcal{P} \). Given an LTL formula \( \varphi \) and time point \( i \geq 0 \), the satisfaction relation \( \mathcal{M}, i \models \varphi \) is defined by induction on the structure of \( \varphi \). For propositional variable \( p \in \mathcal{P} \), \( \mathcal{M}, i \models p \) holds if \( p \in w_i \); for complex formulas, the corresponding condition of Table[3] has to be satisfied. If \( \mathcal{M}, 0 \models \varphi \), then \( \mathcal{M} \) is a model of \( \varphi \).

We briefly review some known constructions and results for LTL satisfiability checking. For a set \( \mathcal{F} \) of LTL formulas, \( \text{Clo}(\mathcal{F}) \) denotes the set of all subformulas occurring in \( \mathcal{F} \), together with all their negations. The set \( \text{Typ}(\mathcal{F}) \) contains all types for \( \mathcal{F} \), which are sets \( T \subseteq \text{Clo}(\mathcal{F}) \) such that

- for all \( \neg \varphi \in \text{Clo}(\mathcal{F}) \), we have \( \neg \varphi \in T \iff \varphi \notin T \);
- for all \( \varphi \land \psi \in \text{Clo}(\mathcal{F}) \), we have \( \varphi \land \psi \in T \iff \{ \varphi, \psi \} \subseteq T \).

Each type \( T \) uniquely defines a world \( w = T \cap \mathcal{P} \). A type \( T \) is initial if

- for all \( \circ_p \varphi \in \text{Clo}(\mathcal{F}) \), we have \( \circ_p \varphi \notin T \);
- for all \( \varphi S \psi \in T \), we have \( \psi \in T \).

A pair \( (T_1, T_2) \in \text{Typ}(\mathcal{F}) \times \text{Typ}(\mathcal{F}) \) is t-compatible if

- for all \( \circ_p \varphi \in \text{Clo}(\mathcal{F}) \), we have \( \circ_p \varphi \in T_2 \iff \varphi \in T_1 \);
- for all \( \circ_p \varphi \in \text{Clo}(\mathcal{F}) \), we have \( \circ_p \varphi \in T_1 \iff \varphi \in T_2 \);
- for all \( \varphi S \psi \in \text{Clo}(\mathcal{F}) \), we have \( \varphi S \psi \in T_2 \iff (i) \psi \in T_2 \), or (ii) \( \varphi \in T_2 \) and \( \varphi S \psi \in T_1 \);
- for all \( \varphi U \psi \in \text{Clo}(\mathcal{F}) \), we have \( \varphi U \psi \in T_1 \iff (i) \psi \in T_1 \), or (ii) \( \varphi \in T_1 \) and \( \varphi U \psi \in T_2 \).

These local properties are the basis for the following characterization, which says that every satisfiable LTL formula has a periodic model with a period of at most exponential length.

Lemma A.6 (see [73]). An LTL formula \( \varphi \) is satisfiable iff there is a sequence of types \( T_0, \ldots, T_i, \ldots, T_{i+p} \) for \( \{ \varphi \} \) such that

- \( s \) and \( p \) are bounded by an exponential function in the size of \( \varphi \);
- \( T_0 \) is initial and contains \( \varphi \);
- for all \( i \in [0, s + p - 1] \), the pair \( (T_i, T_{i+1}) \) is t-compatible;
- the pair \( (T_{i+p}, T_i) \) is t-compatible;
- for each \( \varphi U \psi \in T_i \), there is an index \( i \in [s, s + p] \) such that \( \psi \in T_i \).

The last condition says that each \( U \)-subformula has to be satisfied at some point within the period. This result forms the basis of a PSPACE decision procedure for LTL satisfiability [73], which we describe in more detail in Section[3].

To conclude this section, we recall the separation theorem, which was originally shown in [37] using a strict semantics for \( U \) and \( S \), but also holds in our setting since the strict and non-strict variants of these operators are mutually expressible in the presence of \( \circ F \circ F \). An LTL formula is called separated if no future operators occur in the scope of past operators and vice versa.

Lemma A.7 (see [37]). Every LTL formula is equivalent to a separated LTL formula.
B Proofs for Section [3]

Lemma B.1. If \( \mathcal{W} \) is r-satisfiable w.r.t. \( \lambda \) and \( K \), then there is an r-complete tuple w.r.t. \( \mathcal{W} \) and \( \lambda \).

Proof. Let \( \mathcal{J}_0, \ldots, \mathcal{J}_{n+k} \) be the interpretations over a domain \( \Delta \) that exist according to the r-satisfiability of \( \mathcal{W} \) (see Definition 2.10). We assume w.l.o.g. that \( \Delta \) contains \( I(\lambda) \) and that all individual names are interpreted as themselves in all of these interpretations.

We first define the tuple \( (\mathcal{A}^R, \mathcal{Q}^+, \mathcal{Q}^-, \mathcal{S}) \) as follows:

\[
\begin{align*}
\mathcal{A}^R & := \{ \alpha \in \mathcal{A}^R(\mathcal{K}) \mid \mathcal{J}_0 \models \alpha \}; \\
\mathcal{Q}^+ & := \{ \phi_j \in \mathcal{Q}_\phi \mid p_j \in W \text{ for some } W \in \mathcal{W} \}; \\
\mathcal{Q}^- & := \{ \phi_j \in \mathcal{Q}_\phi \mid p_j \not\in W \text{ for some } W \in \mathcal{W} \}; \\
\mathcal{S} & := \{ \exists S(b) \mid S \in \mathcal{R}^F(\mathcal{O}), b \in I(\lambda) \cup \mathcal{I}_{\text{aux}}, i \in [0, n+k] \}; \\
\end{align*}
\]

We prove that the tuple is r-complete by showing that it satisfies all the conditions in Definition 3.5. It is easy to see that \( \mathcal{A}^R \) is a rigid ABox type for \( \mathcal{O} \), that \( \mathcal{Q}^+ \) satisfies Condition (C4) and that \( \mathcal{Q}^- \) complies with Condition (C5), based on the given interpretations; however, special attention needs to be given to the UNA and Condition (C6). The crucial point is that the given interpretations may satisfy CQs in a conjunction \( \mathcal{J}_{i(\lambda)} \) by a homomorphism that can however not be used to satisfy the corresponding ABox \( \mathcal{A}_{\mathcal{I}_{i(\lambda)}} \), because \( \pi \) maps different variables to the same domain element, which are represented as named individuals from \( \mathcal{I}_{\text{aux}} \) in the ABox; the same applies to the ABox \( \mathcal{A}_S \) w.r.t. the elements from \( \mathcal{I}_{\text{aux}} \) and \( \mathcal{I}_{\text{free}} \). For Condition (C6) observe that the “if”-direction does not directly yield that each of the given interpretations satisfies the assertions in \( \mathcal{A}_S \).

The idea is therefore to extend the given interpretations \( \mathcal{J}_i \) in two steps. First, we construct models \( \mathcal{I}_g \) of \( \langle \mathcal{O}, \mathcal{A}^R \cup \mathcal{A}^Q_{\mathcal{I}_g}, \mathcal{A}_{\mathcal{I}_{i(\lambda)}} \cup \mathcal{A}_i \rangle \) that interpret the elements of \( \mathcal{I}_{\text{aux}} \) and \( \mathcal{I}_{\text{free}} \) by using duplicates of elements from \( \Delta \). In order to overcome the issue with Condition (C6), we ensure that these interpretations still share one domain and also interpret the rigid symbols in the same way. This allows us to then adapt these interpretations in a second step for the elements from \( \mathcal{I}_{\text{free}} \) in order to get models of the respective knowledge bases \( \mathcal{K}_{\text{rc}} \), which include \( \mathcal{A}_S \).

For this extension, we consider different canonical interpretations for all CQs \( \phi_j \in \mathcal{Q}^+ \) satisfied in one of the given interpretations: \( \mathcal{I}_g \), the one of \( \mathcal{K}_g := \langle \mathcal{O}, \mathcal{A}_g \rangle \); \( \mathcal{I}_i \), the one of \( \mathcal{K}_i := \langle \mathcal{O}, \mathcal{A}_i \rangle \); and \( \mathcal{I}_F \), which collects the rigid consequences of \( \mathcal{I}_j \) and is inductively defined according to Definition 3.1 with the adaptation that all symbols \( X \in \mathcal{C} \cup \mathcal{P} \) are initially interpreted as follows:

\[
X^0 := \begin{cases} 
X_{\mathcal{I}_j} & \text{if } X \in \mathcal{C}^R \cup \mathcal{P}^R, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Being defined in this way, \( \mathcal{I}_F^R \) behaves exactly as \( \mathcal{I}_j \) w.r.t. the rigid names, but the interpretations of the flexible names only contain those tuples that are implied by the rigid information. Note that the domain of \( \mathcal{I}_F^R \) is a subset of the domain of \( \mathcal{I}_j \) since all elements \( \mathcal{I}_r \) that would
be created by the iteration in Definition A.1 for \( I_j^R \) are also created by the one for \( I_j \) and are hence contained in the initial interpretation above. We consider the interpretations \( I_j^R \) to make sure that all interpretations we construct also satisfy \( A_{Q_{\ell}}^R \), even if they do not satisfy an ABox \( A_{\phi} \).

Observe that \( K_j \) is consistent since \( \phi_j \in Q^+ \) implies that there is an interpretation \( J_i \) that satisfies \( \varphi \) and \( O \) and is thus a model of \( K_j \) (if two variables are mapped by the homomorphism to the same domain element, we obtain a model respecting the UNA by creating a copy of this element that satisfies exactly the same concept names and participates in the same role connections as the original element). Moreover, all the given interpretations satisfy the rigid consequences of \( A_{\phi_j} \), w.r.t. \( O \) (i.e., particularly \( A_{\phi_j}^R \), see Definition A.2), because, by assumption, they share one domain and respect the rigid names. The following properties of all \( \phi_j \in Q^+ \) that are crucial for our construction:

- \( I_j \) can be homomorphically embedded into each \( J_i \) for which we have \( p_j \in W_\lambda(i) \) since there must be a homomorphism of \( \varphi_j \) into \( J_i \). Hence, domain elements that satisfy at least the symbols satisfied by the elements of \( I_j \) must exist.

- \( I_j^R \) can be homomorphically embedded into all \( J_i \), because there must be an index \( \ell \in [0, n + k] \) such that \( J_i \) satisfies \( \varphi_j \) (and \( O \)), and the rigid consequences of \( \varphi_j \), represented by \( I_j^R \), are satisfied in all the given interpretations.

These facts imply that we can, as the first step, extend all given interpretations \( J_i \) to models \( I_j \) of \( A_{\phi_j}^R \), because they already contain elements that behave in the same way—at least w.r.t. the symbols that need to be satisfied to obtain such models:

- The common domain \( \Delta \) is extended by the union of the domains of all \( I_j \) with \( \varphi_j \in Q^+ \) (including the domains of \( I_j^R \)). These domains may overlap in \( I \) and \( I^{aux} \).

- The individual names from \( I^{aux} \) are interpreted as themselves.

- For each \( j \in [1, m] \) and \( p_j \in W_\lambda(i) \), all symbols are, on the domain of \( I_j \), interpreted exactly as in \( I_j \). There are no role connections between the old and the new domains except between \( I(K) \) and the elements of \( I^{aux} \).

- If \( p_j \in W_\lambda(i) \), then all symbols are, on the domain of \( I_j \), interpreted exactly as in \( I_j^R \).

Recall that \( \Delta \) is not yet complete; it is still to be extended by the domain of \( I_S \). However, this definition already meets our requirement that \( I_j^R \models \chi_\lambda(i) \) for all \( i \in [0, n + k] \), which can be seen given the following observations:

- \( I_j^R \) satisfies \( O \) and \( A^R \). This is because, given \( (\dagger) \), the interpretation of symbols on the unnamed elements in the original domain \( \Delta \) does not change (i.e., the interpretation of basic concepts on these elements remains the same); further, the new domain elements do not exhibit new behavior that was not already present in \( J_i \); and the latter also implies that the interpretation of basic concepts on the elements of \( I \) does not change.

- If \( i \leq n \), then \( I_j^R \) satisfies \( A_i \) by the same reasons as in the previous item. Otherwise, \( A_i \) is trivially satisfied.

- \( I_j^R \) is a model of \( A_{\phi_j}^{Q_{\ell}} \), since all \( I_j^R \) for \( \phi_j \in Q^+ \), are part of \( I_j^R \) and \( I_j^R \models A_{\phi_j}^R \).

- Similarly, \( I_j^R \) satisfies \( A_{Q_{\ell}(i)} \) since that ABox consist exactly of the ABoxes \( A_{\phi_j} \) with \( p_j \in W_\lambda(i) \), satisfied by \( I_j \), which are part of \( I_j^R \).

- For each \( p_j \in W_\lambda(i) \), we get \( I_j^R \models \phi_i \) since any homomorphism of \( \phi_j \) into \( I_j^R \) would allow us to also find one into \( J_i \), which contradicts the assumptions that \( J_i \models \chi_\lambda(i) \). Hence, \( I_j^R \models \chi_\lambda(i) \).

We come to the second part. Since the interpretations \( I_j \) are models of the knowledge bases \( \langle O, A^R \cup A_{Q_{\ell}}, A_{Q_{\ell}(i)} \cup A_{Q_{\ell}(n) \cup A_{Q_{\ell}(n+1)}} \rangle \), every assertion \( \exists b \in S \) is satisfied in one of
them by the definition of $S$. That interpretation thus also satisfies the rigid consequences described in $A_S$ since it is a model of $O$. But then this holds for all the interpretations because they interpret the rigid symbols on the named elements in the common domain (i.e., those in $\Delta \cap I(K) \cap I^{\text{aux}}$) in the same way and satisfy $A^R$ and $A^\Omega$, which contain all the relevant rigid information. We can thus extend the domain $\Delta$ by the domain of $I_S$ and all interpretations $I'_i$ as follows. The names from $I^{\text{tree}}$ are interpreted by themselves and behave exactly in the same way as the corresponding elements that already exist in each interpretation since they describe the consequences of assertions in $S$; note that we may again have to copy elements if the UNA would be violated otherwise. Note that this extension does not introduce new role connections between the old and the new domains except between $I(K)$ and the elements of $I^{\text{aux}}$ and $I^{\text{tree}}$, similar to (†). We can therefore argue similarly as above that the final interpretations $I'_i$ are models as required for Condition $\text{[C1]}$

We now use these constructed interpretations to show that $(A^R, Q^+, Q^-, S)$ also satisfies Conditions $\text{[C2]}$ and $\text{[C5]}$ i.e., it is r-complete. For Condition $\text{[C2]}$ we assume that there are an $i \in [0, n + k]$ and a $p_j \in W_0(i)$ such that $K^r_i \models \varphi_j$, which yields $I'_i \models \varphi_j$. This directly contradicts the fact that $I'_i \models \chi_{\emptyset}(\emptyset)$. The proof for Condition $\text{[C5]}$ is also by contradiction. We assume that there are an index $i \in [0, n + k]$ and a rigid witness query $\psi$ for some $\varphi_j \in Q^-$ such that $K^r_i \models \psi$, and thus also $I'_i \models \psi$. By the definition of $Q^+$, there must be a $W^i \in W^R$ with $i \in [1, k]$ such that $p_j \notin W^i$, and thus $I'_{n+i} \not\models \varphi_j$. By Definition 3.3 we know that $I'_{n+i} \not\models \varphi_j$. But this contradicts the facts that $\psi$ contains only rigid names and that $I'_{n+i}$ and $I'_i$ respect the rigid names.

We now prove the “if”-direction of Lemma 3.6. Let $I_i$ and $I'_i$, for $i \in [0, n + k]$, be defined as in the main text. In the following for elements $e \in I(K) \cup I^{\text{aux}} \cup I^{\text{tree}}$, we use $e'$ to denote the corresponding element in $I(K) \cup \Delta^R \cup \Delta^\Omega_{\text{tree}}$; we thus consider $a' := a$ for all $a \in I(K)$.

We state the following fact for future reference, and then establish connections between $J_0, \ldots, J_{n+k}$ and $I_0, \ldots, I_{n+k}$.

**Fact B.2.** The sets $I(K), \Delta^R, \Delta^\Omega_{\text{tree}}, \Delta^\Omega_{\text{aux}}$ and $\Delta^\Omega_{\text{non}}$, for all $i \in [0, n + k]$, are pairwise disjoint.

**Lemma B.3.** For all $i \in [0, n + k]$, $d, e \in \Delta^R$, and $R \in P$, $(d, e) \in R^R$ iff $(d, e) \in R^T$.

**Proof.** $(\Leftarrow)$ This direction follows directly from the definition of $R^T$. $(\Rightarrow)$ We focus on the definition of $R^R$. If $R$ is flexible, we need to consider two cases (i) and (ii); in both, we assume $i \neq j$: (i) $(d, e) \in S^T_j$, for some rigid subrole $S$ of $R$, and (ii) $(d, e) \in R^T_j$ and either $d$ or $e$ has a witness w.r.t. $O$. Given $d, e \in \Delta^R$, Fact B.2, however implies $d, e \in I(K)$ in both cases. Hence, (ii) is impossible since named domain elements cannot have witnesses according to Definition 3.4. In case (i), we get $S(d, e) \in A^R$ since $S \in P^R$, that $A^R$ is a rigid ABox type, and $I_j \models A^R$. Then, $I_j \models A^R$ implies $(d, e) \in R^T_j$ and $I_j \models O$ yields $(d, e) \in R^T_j$ since we assume $O \models S \subseteq R$.

If $R$ is rigid, we consider the case that $(d, e) \in R^T_j$ for some $j \neq i$ and get $d, e \in I(K)$, as above. Since $R \in P^R$, $A^R$ is a rigid ABox type, and $I_j \models A^R$, we must have $R(d, e) \in A^R$. $I_j \models A^R$ then leads to $(d, e) \in R^T_j$.

The following is a direct consequence of the fact that the interpretation of roles in $J_i$ is based on the canonical interpretations, which are models of $O$, and Fact B.2.

**Lemma B.4.** For all $i, j \in [0, n + k]$, $d \in \Delta_j$, $e \in \Delta^\Omega_{\text{non}}$, and $R \in R$, we have that $(d, e) \in R^T_j$ implies $(d, e) \in R^R_j$.

**Lemma B.5.** For all $e \in I(K) \cup I^{\text{aux}} \cup I^{\text{tree}}$ and flexible basic concepts $B$ with $e \in B^R$, either $e \in B^T$ or there is a $\mathbf{B}^R \subseteq \mathbb{B}^R(O)$ and a $j \in [0, n + k]$ with $e \in (\bigcap B^R_j) \setminus B$. 


Proof. For flexible concept names, the claim follows directly from the definition of \( J_i \). It remains to consider \( B \) to be of the form \( \exists R \) with \( R \in R(O) \). If \( e \notin B^R \), then by the definition of \( J_i \), there exists a \( j \in [0,n+k] \) for which one of the following cases applies:

- There is an \( S \in \mathbf{P} \) such that \( O \models S \sqsubseteq R \) and \( e \in \langle \exists S \rangle J_i \). In this case, we can set \( B^R := \langle \exists S \rangle \) since \( O \models \exists S \sqsubseteq \exists R \).
- There exists \( d \in \Delta^J_i \) with \( (e,d) \in R^J_i \) and either \( d \) or \( e \) has a witness w.r.t. \( O \). Since \( e \) is a named domain element, Definition 3.7 yields \( \text{Wit}(e) = \emptyset \), and thus \( \text{Wit}(d) \neq \emptyset \), which implies that \( d \) is of the form \( u_{l/k}^J \) with \( S \in R(O) \) and there exists \( B^R \subseteq B^R(O) \) with \( e \in \langle \exists R \rangle J_i \) and \( O \models \exists B^R \sqsubseteq \exists S \). Since \( (e,u_{l/k}^J) \in R^J_i \), by Definition 3.1 we infer that \( O \models S \sqsubseteq R \), which implies \( O \models \exists B^R \sqsubseteq \exists R \).

\[ \square \]

Lemma 3.8. For all \( i, j \in [0,n+k] \) and basic concepts \( B \in B(O) \), the following hold.

\begin{enumerate}
\item For all \( e \in I(K) \), we have \( e \in B^{J_i} \) iff \( e \in B^{\exists} \).
\item If \( B \) is rigid, then, for every \( e \in \Delta^\exists_{\text{dom}} \cup \Delta^\exists_{\text{tree}} \cup \Delta^\exists_{\text{anon}} \), we have \( e \in B^{J_i} \) iff \( e \in B^{\exists} \).
\item If \( B \) is flexible, then, for every \( e \in \Delta^\exists_{\text{not}} \cup \Delta^\exists_{\text{tree}} \cup \Delta^\exists_{\text{anon}} \), we have \( e \in B^{J_i} \) iff
  \begin{enumerate}
  \item \( i = j \) and \( e \in B^{\exists} \), or
  \item there is a \( B^{R} \subseteq B^{R}(O) \) with \( e \in \langle \bigcap B^{R} \rangle J_i \) and \( O \models \bigcap B^{R} \sqsubseteq B \), or
  \item \( e \in B^{J_i} \cap \Delta^\exists_{\text{anon}} \) and \( \text{Wit}(e) \neq \emptyset \).
  \end{enumerate}
\end{enumerate}

Proof. For (a) \( e \in B^{\exists} \) clearly implies \( e \in B^{J_i} \) since \( X^{J_i} \subseteq X^{\exists} \) for \( X \in \mathbf{C} \cup B \). To prove the converse, we first consider the case that \( B \) is rigid. By the definition of the rigid names in \( J_i \), \( e \in B^{J_i} \) yields that there is a \( j \in [0,n+k] \) such that \( e \in B^{J_i} \). Since \( J_i \) and \( J_j \) are both models of the rigid ABox type \( A^{R} \), we get \( B(e) \in A^{R} \) and \( e \in B^{\exists} \).

For a flexible \( B \) with \( e \in B^{J_i} \), by Lemma 3.5 we either immediately get \( e \in B^{\exists} \), or there is a \( B^{R} \subseteq B^{R}(O) \) with \( O \models \bigcap B^{R} \sqsubseteq B \). For \( j \in [0,n+k] \) such that \( e \in \langle \bigcap B^{R} \rangle J_i \). But this yields \( B^{R}(e) \in A^{R} \) for all \( B^{R} \), which together with \( J_i \models A^{R} \) implies \( e \in B^{\exists} \).

Hence, \( J_i \models O \) leads to \( e \in B^{\exists} \). This concludes the proof of (a).

(b) is a direct consequence of Fact 3.2 and the definition of \( J_i \).

(c) and the case that \( e \in B \), the equivalence with one of (ii) (iii) is covered by the definition of \( J_i \) if, for (iii) Lemma 3.1 is taken into account.

It remains to consider \( B \) to be of the form \( \exists R \) with \( R \in R \). For the case that \( i = j \), the claim can be restricted to Item (ii) since the other two are subsumed by ii; for (iii) this holds because \( J_i \models O \). Then, it is a direct consequence of Fact 3.2 and the definition of \( J_i \), because the interpretation of the elements from \( \Delta^\exists_{\text{not}} \cup \Delta^\exists_{\text{tree}} \cup \Delta^\exists_{\text{anon}} \) in \( J_i \) is not influenced by any \( J_j \) with \( j \neq i \). We consider the case \( i \neq j \).

- For \( e \in \mathbf{P}^{\text{aux}} \cup \mathbf{1}^{\text{tree}} \), we only have to consider (ii):
  \( \Rightarrow \) This follows directly from Lemma 3.5.
  \( \Leftarrow \) Given \( e \in \langle \exists R \rangle J_i \), there is an element of the form \( u_{l/k}^J \in \Delta^\exists_{\text{anon}} \), with \( (e,u_{l/k}^J) \in R^J_i \) by Definition 3.1. Since Definition 3.7 implies that \( B^R \) is a witness of \( u_{l/k}^J \), the definition of \( J_i \) yields \( u_{l/k}^J \in R^J_i \); that is, \( e \in \langle \exists R \rangle J_i \).

- Let \( e \in \Delta^\exists_{\text{anon}} \).
  \( \Rightarrow \) Again, there are two options, by the definition of \( J_i \):
  \begin{enumerate}
  \item there is an \( S \in \mathbf{R}^{R} \) such that \( e \in \langle \exists S \rangle J_i \) and \( O \models S \sqsubseteq R \), or
  \item there is an \( R \)-successor \( d \) of \( e \) in \( J_i \), and either \( e \) or \( d \) has a witness w.r.t. \( O \).
  \end{enumerate}

For (ii), we can set \( B^{R} := \langle \exists S \rangle \), as in Lemma 3.5. For (iii), observe that we have \( e \in \langle \exists R \rangle J_i \). If \( \text{Wit}(d) \) is undefined or empty, then (iii) holds directly. Otherwise, we
Lemma B.6. For all \( i \in \{0, n+k\} \), \( J_i \) is a model of \((\mathcal{O}, \mathcal{A})\).

Proof. For every assertion \( \alpha \in \mathcal{A}_i \), by Lemma A.2 we have \( \mathcal{I}_i \models \alpha \), and thus Lemmas B.3 and A.3 yield \( J_i \models \alpha \).

We consider a CI \( B_1 \cap \cdots \cap B_m \subseteq B \subseteq \mathcal{O} \) and element \( e \) such that \( e \in B_1 \cap \cdots \cap B_m \); note that \( B_1, \ldots, B_m \) are basic concepts and \( B \) is either a basic concept or \( \perp \). For the case that \( e \in I(K) \), Lemma 3.3 yields \( e \in B_1 \cap \cdots \cap B_m \). Since \( I_i \models \mathcal{O} \), this implies that \( e \in B^{\mathcal{O}} \), which is impossible if \( B = \perp \). Otherwise, we get \( e \in B^{\perp}_i \), again by Lemma 3.3.

Let now \( e \in \Delta^T_{\text{free}} \cup \Delta^T_{\text{plan}} \) for some \( j \in \{0, n+k\} \). If \( i = j \), then we get the same conclusion as in the previous case since, given that \( I_j \models \mathcal{O} \), Items (iii) and (iii) collapse to (i) More precisely, we can argue analogously by referring to Lemma 3.3 and (i) instead of Lemma 3.3. In case that \( i \neq j \), Lemma 3.3 implies that, for each \( B_l \) with \( \ell \in \{1, m\} \), we have either that (i) there is a \( B^{R}_l \subseteq B^{\mathcal{O}} \) such that \( e \in (\bigcap B^{R}_l) \cap \mathcal{O} \), (ii) \( e \in (\bigcap B^{\perp}_l) \cap \mathcal{O} \), or that (ii) \( e \in B^{\perp}_l \cap \Delta^T_{\text{plan}} \cap \mathcal{O} \). In this case, \( J_i \models \mathcal{O} \), following from Lemma A.2, thus leads to \( e \in B^{\perp}_l \) for (i) and to \( e \in B^{\perp}_l \) for both cases. If \( B = \perp \), this is again impossible. If \( B \) is rigid, then Lemma 3.3 yields \( e \in B^{\perp}_l \), as required. If \( B \) is flexible and Case (ii) applies to at least one \( B_l \) with \( \ell \in \{1, m\} \), then Item (iii) of Lemma 3.3 yields the claim. Otherwise, it is easy to see that we can define \( B^{R} := \bigcup_{l=1}^{m} B^{R}_l \) and have \( e \in (\bigcap B^{R}) \cap \mathcal{O} \), \( \mathcal{O} \models B^{R} \subseteq B_1 \cap \cdots \cap B_m \), and \( \mathcal{O} \models B^{R} \subseteq B \). Hence, the Item (ii) of Lemma 3.3 applies, and we also get \( e \in B^{\perp}_l \).

It remains to consider role inclusions of the form \( S \preceq R \). We consider a tuple \( (d, e) \in S^{\perp}_l \), i.e., there is a \( j \in \{0, n+k\} \) such that \( (d, e) \in S^{\perp}_j \). Since \( I_j \models \mathcal{O} \), we get \( (d, e) \in S^{\perp}_j \). For the case that \( R \) is rigid, we immediately get \( (d, e) \in R^{\perp}_j \). For the case that \( R \) is flexible, we assume \( (d, e) \in R^{\perp}_j \). Then, we must have that \( i \neq j \), that \( R \) has no rigid subrole \( S' \) with \( (d, e) \in S'^{\perp}_j \), and that neither \( d \) nor \( e \) have a witness w.r.t. \( \mathcal{O} \). By the second observation, \( S' \) cannot be rigid nor have a rigid subrole \( S' \) with \( (d, e) \in (S')^{\perp}_j \), because then \( S' \) would be a rigid subrole of \( R \). But this means \( (d, e) \not\in S'^{\perp}_j \), which contradicts our assumption.

Lemma B.7. For all \( i \in \{0, n+k\} \), \( J_i \) is a model of \( \chi_{\lambda(i)} \).

Proof. We show that \( J_i \) is a model of every CQ literal in \( \chi_{\lambda(i)} \). Let \( \varphi \) first be a positive such literal. Since \( \mathcal{A}_i \) contains an instantiation of \( \varphi \) and \( I_i \models \mathcal{A}_i \), by Lemma A.2 we know

In the remaining parts of the proof, we do not always explicitly refer to Lemma A.2 to justify the argument that \( I_i \models \mathcal{A}_i \) for all \( i \in \{0, n+k\} \).
that there is a homomorphism \( \pi \) of \( \varphi \) into \( I_i \) that maps all variables in \( \varphi \) to elements of \( \Delta^2_{\text{anon}} \); that is, \( \pi \) maps each such variable \( x \) to \( a_i \). By the fact that \( \Delta^2_{\text{anon}} \subseteq \Delta \) and Lemmas 3.3 and 3.8 \( \pi \) is then also a homomorphism of \( \varphi \) into \( J_i \).

Let now \( \neg \varphi \) be a negative literal in \( X_i \). We proceed by contradiction and assume \( \pi \) to be a homomorphism of \( \varphi \) into \( J_i \). First, observe the following.

\begin{enumerate}
  \item[(O1)] Condition \([C4] \) implies \( \varphi \in Q^* \), and hence Condition \([C5] \) and Lemma \( \ref{lem:1} \) yield that none of the rigid witness queries of \( \varphi \) is satisfied in any of the canonical interpretations.
  \item[(O2)] Condition \([C2] \) implies \( \Delta^i_{\text{anon}} \not= \varphi \), and thus Lemma \( \ref{lem:1} \) yields that \( I_i \models \neg \varphi \).
\end{enumerate}

We derive contradictions to these observations by distinguishing the two cases (I) and (II) outlined in Section 3.

\((I)\) Let first \( \pi \) be such that it maps no terms into \( \Delta_i := I(K) \cup \bigcup_{j=0}^{n+k} \Delta^j_{\text{anon}} \cup \Delta^j_{\text{free}} \). Because of the UNA, we thus have \( I(\varphi) \cap I(K) = \emptyset \), which yields that \( I(\varphi) = \emptyset \) since \( \varphi \) does not contain names from \( I^{\text{aux}} \) or \( I^{\text{free}} \). Moreover, we can then assume that there is a single index \( j \in [0,n+k] \) such that \( \pi \) maps all terms of \( \varphi \) to elements of \( \Delta^j_{\text{anon}} \), by the definition of \( \Delta \). To see this, note that \( \varphi \) is connected and that, for a role \( R \), a tuple \((d,e) \in R^{\pi} \) without named individuals exists only if the elements belong to the same domain \( \Delta^j \) by the definition of \( R^{\pi} \) and Fact \( \ref{fact:1} \).

Given \([O2] \) we then directly get a contradiction for the case that \( j = i \) by Lemmas \ref{lem:2} and \ref{lem:3} which imply that \( \pi \) is a homomorphism of \( \varphi \) into \( I_j \).

For the case \( j \neq i \), we show that there is a rigid witness query \( \psi \) for \( \varphi \) such that \( I_j \models \psi \), in contradiction to \([O1] \). Since \( \pi \) is connected, by Lemma \ref{lem:2} and by considering how the elements in \( \Delta^j_{\text{anon}} \) are related by roles within \( I_j \) (see Definition \ref{def:1}), it is easy to see that there is a variable \( x \in V(\varphi) \), for which \( \pi(x) = u_{j,\rho}^i \) is such that the length of \( \rho \) is minimal compared to the paths of all elements of range(\( \pi \)); further, all other \( \pi(y) \) for \( y \in V(\varphi) \) must then be of the form \( u_{j,\rho \sigma}^i \) with \( \sigma \in R^i \). Moreover, we know by the definition of \( I_j \) on the elements of \( \Delta^j_{\text{anon}} \) that \( \pi \) is also a homomorphism of \( \varphi \) into \( I_j \).

We now construct the rigid witness query \( \psi \) for \( \varphi \), distinguishing two cases. We first consider the case that \( \text{Wit}(u_{j,\rho}^i) \neq \emptyset \). By Definition \ref{def:1} \( \rho \) must be of the form \( \rho_0 R_0 \ldots R_l \), and there are a set \( B^R \subseteq B^R(\mathcal{O}) \) such that \( \mathcal{O} \models B^R \subseteq \exists R_0 \ldots \exists R_l \) and an element \( u_{j,\rho}^i \in \Delta^j_{\text{anon}} \) that satisfies \( \mathcal{O} \models B^R \) in \( I_j \). By Definition \ref{def:1} and Lemma \ref{lem:1} \( u_{j,\rho}^i \in \Delta^j_{\text{anon}} \) implies \( \mathcal{O} \models \exists R^i \subseteq \exists R_{i+1} \), for all \( i \in [0,l+1] \). Consider now the ABox \( A_{\text{gen}} := \{ B(a) \mid B \in B^R \} \), where \( a \) is an arbitrary individual name. By Definition \ref{def:1} and the above observations on \( \pi \), the canonical model \( I' \) of \( (\mathcal{O},A_{\text{gen}}) \) is isomorphic to a subtree of \( I_j \) starting at \( u_{j,\rho}^i \). In particular, it contains the elements \( u_{\rho_0 \rho_1}^i, \ldots, u_{\rho_0 R_l \rho_i}^i \) and similar elements corresponding to all other successors of \( u_{j,\rho}^i \) in \( I_j \). Hence, \( I' \) also satisfies \( \varphi \), namely via the homomorphism \( \pi' \) given by \( \pi'(x) = u_{\rho \sigma}^i \) whenever \( \pi(x) = u_{\rho \sigma}^i \), for each variable \( x \in V(\varphi) \). By Lemma \ref{lem:1} \( (\mathcal{O},A_{\text{gen}}) \models \varphi \), which shows that the CQ \( \exists x. B^R(x) \) is a rigid witness query for \( \varphi \).

In the remaining case that \( \text{Wit}(u_{j,\rho}^i) = \emptyset \), then we cannot find a rigid witness query so easily. Instead, we iteratively replace atoms of \( \varphi \) by rigid atoms until no more flexible atoms remain. We do this by induction on the length \( |\sigma| \) of the elements \( \pi(x) = u_{\rho \sigma}^i, x \in V(\varphi) \). That is, we start with those variables \( x \) that are mapped to the “root” \( u_{j,\rho}^i \) and then proceed along the tree-like structure of \( \Delta^j_{\text{anon}} \). Initially, we set \( \psi := \varphi \), and maintain the invariants that \( \pi \) is a homomorphism of \( \psi \) into \( I_j \), and that every homomorphism of \( \psi \) into an interpretation \( I \) can be extended to a homomorphism of \( \varphi \) into \( I \) (in the end, this implies that \( \psi \) is a rigid witness query for \( \varphi \) that is satisfied by \( I_j \)).
In the first step, we consider all concept atoms $A(x)$ in $\varphi$ such that $\pi(x) = u_0^\varphi$. Since $u_0^\varphi \in A^{\Delta}$ and $\text{Wit}(u_0^\varphi) = \emptyset$, by Lemma 3.8, either (i) $A$ is rigid, or (ii) there is a $B^R \subseteq B^R(\mathcal{O})$ such that $u_0^\varphi \in (\bigcap B^R)^J$ and $\mathcal{O} \models \bigcap B^R \subseteq A$. In case (i), we do not have to replace $A(x)$ since it is already rigid, and in case (ii) we replace it by $B^R(x)$, which satisfies our invariants.

Next, we consider the role atoms $R(x,y)$ in $\varphi$ such that either $x$ or $y$ is mapped by $\pi$ to $u_0^\varphi$. Without loss of generality, we assume that $\pi(x) = u_0^\varphi$ and $\pi(y) = u_0^\varphi$; if this is not the case, then we simply consider $R^-$ instead of $R$. If $\text{Wit}(u_0^\varphi) \neq \emptyset$, then, since $u_0^\varphi$ has no witnesses, there must exist some $B^R \subseteq B^R(\mathcal{O})$ such that $u_0^\varphi \in (\bigcap B^R)^J$ and $\mathcal{O} \models \bigcap B^R \subseteq A$. But then the canonical model for $(\mathcal{O}, A_{\text{gen}})$ already satisfies all atoms of $\varphi$ that are mapped by $\pi$ to the elements below $u_0^\varphi$, namely via the homomorphism $\pi'$ defined by $\pi'(y) := u_{k,n}$ whenever $\pi(y) = u_{k,n}$. Moreover, $\pi'$ also satisfies $R(x,y)$ since, by Definition A.1, $\mathcal{O} \models S \subseteq R$. This means that we can replace all these atoms by the conjunction $B^R(x)$ (for all $x$ with $\pi(x) = u_0^\varphi$). It remains to consider the case that neither $u_0^\varphi$ nor $u_0^\varphi$ have any witnesses. Since $(u_0^\varphi, u_0^\varphi) \in R^J$, by the definition of $J$, there is a rigid role $S_1$ such that $\mathcal{O} \models S \subseteq R$ and $(u_0^\varphi, u_0^\varphi) \in S_1$. Hence, we can replace $R(x,y)$ by the rigid atom $S_1(x,y)$, which is satisfied in $I_j$ and implies $R(x,y)$.

We have thus dealt with all atoms involving a variable $x$ with $\pi(x) = u_0^\varphi$. Assume now that we have continued this process up to some $u_{k,n}^\varphi$, and consider all atoms involving variables $x$ mapped to one of its direct successors $u_{k,n}^\varphi$. By our construction above, we can assume that $u_{k,n}^\varphi$ has no witnesses; otherwise, we would already have replaced all atoms involving $x$. Hence, we are in the same position as in the case above, and can continue replacing the atoms in $\varphi$ by rigid atoms in exactly the same way. We can do this until $\varphi$ contains no more flexible atoms, and hence is a rigid witness query for $\varphi$ that is satisfied in $I_j$.

This contradicts (O1) and hence finishes the proof of Case (I) of Lemma B.7.

(II) In the remainder of the proof, let $\pi$ be such that at least one term is mapped into $\Delta_0 = I(\mathcal{K}) \cup \bigcup_{i=0}^{+k} (\Delta^i_{\text{sub}} \cup \Delta^i_{\text{free}})$. We directly define a homomorphism $\pi'$ of $\varphi$ into $I_j$ to contradict (O2), i.e., $I_j \models \neg \varphi$. This is done in three phases, by considering the terms $\pi$ maps to elements from $\Delta_0$, those that are directly connected to the latter, and all others. After phase one, all terms considered are thus mapped to elements from $\bigcup_{i=0}^{+k} \Delta^i_{\text{free}}$ by $\pi$.

Phase 1) For all $t \in T(\varphi)$, where $\pi(t) \in \Delta_0$, and assuming $\pi(t) = e'$, let $\pi'(t) := e'$. We first prove an auxiliary result and subsequently show that, for the terms mapped so far, $\pi'$ is a homomorphism of $\varphi$ into $I_j$.

Claim B.8. For all $B \in \mathcal{B}(\mathcal{O})$, $t \in T(\varphi)$ with $\pi(t) \in \Delta_0$, $\pi(t) \in B^J$ implies $\pi'(t) \in B^J$.

Proof of the claim. By Lemma B.8, $\pi(t) \in B^J$ implies two options: either (i) $\pi(t) \in B^J$, or (ii) $\pi(t) \in \Delta^i_{\text{sub}} \cup \Delta^i_{\text{free}}$ and there is a $B^R \subseteq B^R(\mathcal{O})$ with $\pi(t) \in (\bigcap B^R)^J$ and $\mathcal{O} \models \bigcap B^R \subseteq B$. In Case (i), we have $\pi'(t) = \pi(t)$ by definition, hence the claim holds. In Case (ii), the claim follows if $i = j$, as in Case (i); otherwise, we distinguish the following two cases.

The first case is for $t \in \Delta^i_{\text{free}}$; then $\pi(t)$ is of the form $a_{0,j}^i$. Since $a_{0,j}$ then occurs in some $ABox$ by Definition A.1 only $A_S$ contains assertions on elements of $\Gamma_{\text{free}}$, and $A_S$ is the same w.r.t. all time points, the element $a_{0,j}$ also exists. By Lemma A.3, and the definition of $A_S$, $\pi(t) \in B^J$ implies that $B$ subsumes the conjunction of all rigid basic concepts satisfied by $a_{0,j}$ in some $I_\mathcal{O}(\mathcal{A}_{(\exists S(b))})$ where $\exists S(b) \in S$. Another application of Lemma A.3 then yields that $B$ is also satisfied by $\pi'(t) = a_{0,j}^i$ in $I_j$. 
implies that all variables that occur in role atoms together with a term mapped to an element
∆ here), it follows that (π(t) ∈ (∏BR)′; BR ⊆ BR(∅); and ARQ, and AS contain all rigid assertions on a;, taking O into account (i.e., in particular, all following from ARQ(∅), are also in ARQ,); Definition [A.1] and Lemma [A.3] yield that the elements of BR are implied by the conjunction of

– all basic concepts for which there are assertions on a; in A((O, AR(∅))) and
– all rigid concepts ∃R for which there is an assertion ∃S(a;) ∈ S with O |= S ⊆ R.

But, for all concepts of the latter form, Condition [C6] yields that ∃R(a;) is (already) implied by some KB ⟨O, AR∪AQ∪ARu, u ∪ AR(∅)∪A∅⟩, and must hence be contained in the set of basic concepts from item one, because AR and A∅ do not contain assertions on a; Denoting the set of these basic concepts by B(φ′, a;), we thus obtain O |= ∏B(φ′, a;) ⊆ ∏BR. Given φ′ ∈ Q, the definition of ARQ, yields BR(a;) ∈ ARQ, for all BR ∈ BR. From I; |= ARQ, we obtain a;′ ∈ (∏BR)′ together which with I; |= O leads to a;′ ∈ BR, as required.

We continue with the proof of Lemma [B.7].

As a consequence, all concept atoms A(t) in φ with π(t) ∈ ∆, are satisfied by π′ in I;.

We next show that also holds for the role atoms that only contain such terms. Let hence R(t,t′) ∈ φ be such that π(t), π(t′) ∈ ∆,. If π(t) and π(t′) are both contained in ∆, which especially holds for the elements in I(I); the claim follows immediately from Lemma [B.3] and the fact that π′(t) = π(t) and π′(t′) = π(t′). Otherwise, both π(t) and π(t′) belong to some I(I)∪AL,∪AF for a fixed j ∈ [0, n + k] such that j ≠ i; note that there are no role connections between elements of different sets AL, and AF in I; by definition. We can thus distinguish the following cases: (A) R is rigid, π(t) or π(t′) is contained in AL, and neither is contained in AF; (B) R is rigid and one of π(t) and π(t′) is contained in AF; and (C) R is flexible.

In Case (A), (π(t), π(t′)) ∈ R′ implies (π(t), π(t′)) ∈ R′, By Definition [A.1] considering relations between named elements, and the fact that AR and K do not contain assertions on elements from Faux, there must be an assertion S′(t′, t′) ∈ ARQ, such that O |= S ⊆ R: t = e if π(t) = e. By the definition of ARQ, (see Definition [A.2]), we get R′(t′, t′) ∈ ARQ. Hence, I; |= ARQ implies (π′(t), π′(t′)) ∈ R′.

In Case (B), we argue similarly to the previous case. Again, from (π(t), π(t′)) ∈ R′, we obtain (π(t), π(t′)) ∈ R′. By Definition [A.1] since only AS contains assertions on elements from Faux, and because AS contains all rigid assertions on the elements of Faux that follow by O (see the definition of AS), there must be an assertion R′(t′, t′) ∈ AS. From I; |= AS, we then get (π′(t), π′(t′)) ∈ R′.

In Case (C), given (π(t), π(t′)) ∈ R′ and the fact that witnesses are not defined for elements of ∆, there must be a rigid role S such that (π(t), π(t′)) ∈ RS′ and O |= S ⊆ R by the definition of R′. As in the respective previous case (i.e., based on the kind of π(t) and π(t′) here), it follows that (π(t), π(t′)) ∈ RS′. Since I; |= O, we then obtain (π′(t), π′(t′)) ∈ R′.

It remains to define π′ for the variables of φ that are mapped by π into \\( ∪_{j=0}^{n+k} Δ_{\text{anon}} \), Since the relations in R′ are based on those in the canonical interpretations, Definition [A.1] implies that all variables that occur in role atoms together with a term mapped to an element e ∈ ∆, by π are of the form u′S; and the role atom must be an R-atom such that O |= S0 ⊆ R. Moreover, since φ is connected, if any variable is mapped to an element u′S; ∈ Δanon,
then there is a variable that is mapped to $u_{iS_0}$, one directly connected to it and mapped to $u_{jS_0}$, etc. We hence can proceed as follows.

**Phase 2)** We consider all $y \in V(\phi)$ for which there is an atom $R(t,y) \in \phi$ where $\pi(t) \in \Delta_n$ and $\pi(y) \in \Delta^*$. Assume $\pi(t) = e^i$, $\pi(y) = u^j_{iS_0}$, $S_0 \in R$, and $O \models S_0 \subseteq R$. Recall that $\pi'(t) = e^i$. The goal is to choose an element of $\Delta^*$ as value for $\pi'(y)$ so that $\pi'$ can be (extended to) a homomorphism of $\phi$ into $I_j$. For now, we however only show that our definition of $\pi'(y)$ satisfies $(\pi(t), \pi'(y)) \in R^\phi$ for all these role atoms. The remaining atoms that contain $y$ are then covered in Part 3.

If $i = j$, then we can directly define $\pi'(y) := \pi(y)$ by Lemma [8.3] and [8.8]. Otherwise, we distinguish the following two cases. Note that $(e^i, u^j_{iS_0}) \in R^\phi$ implies $(e^i, u^j_{iS_0}) \in R^\phi$ by Lemma [8.4].

If $\text{Wit}(u^j_{iS_0}) \neq \emptyset$, then $(e^i, u^j_{iS_0}) \in R^\phi$ implies $(e^i, u^j_{iS_0}) \in R^\phi$ by the definition of $J_i$. We then get $e' = \pi'(t) \in (\exists S_0)\beta$ by Claim [8.8]. According to Definition [A.1], the element $u^j_{iS_0}$ then exists and the pair $(e', u^j_{iS_0})$ is related in $I_i$ as $(e^i, u^j_{iS_0})$ is related in $I_j$. And the latter tuple is interpreted in the same way in $I_j$. We can thus set $\pi'(y) := u^j_{iS_0}$.

If $\text{Wit}(u^j_{iS_0}) = \emptyset$, then the definition of $J_i$ yields that, for all atoms $R(t,y)$ as above, there is a rigid role $S$ such that $O \models S_0 \subseteq S$, $O \models S \subseteq R$, and $(e^i, u^j_{iS_0}) \in S^\phi$. Note that $S_0$ must be flexible since otherwise $\exists S_0$ would be a witness for $u^j_{iS_0}$. Furthermore, since $(e^i, u^j_{iS_0}) \in S_0^\phi$, Lemma [8.3] yields that the assertion $\exists S_0(e)$ is a consequence of the basic concepts obtained from the assertions involving $e$ in $K_{\Delta_n}$. We now show that this is still the case if $A_S$ is disregarded, by a case distinction on whether $e$ belongs to $I(K), I^{\text{aux}}$, or $I^{\text{free}}$.

If $e \in I(K)$, then any (rigid) basic concept assertion on $e$ that is a consequence of $A_S$ and $O$ must be contained in $A_R$, since $A_R$ is a rigid ABox type and $I_j$ is a model of both these ABoxes. Since $A_S$ does not contain flexible assertions, $\exists S_0(e)$ is also a consequence of $K_{\Delta_n}$ if $A_S$ is disregarded.

If $e \in I^{\text{aux}}$ is of the form $e = a_x$ and $\phi'$ is the CQ in which $x$ appears, then $\exists S_0(e)$ is a consequence of the assertions in $A_R^0 \cup A_{Q_2(i)} \cup A_S$ (i.e., again, taking $O$ into account). For $A_S$, we thus consider rigid concept assertions $\exists R'(a_x) \in R$, for which there is a flexible assertion $\exists S'(a_j) \in S$ such that $O \models S'_0 \subseteq R'$. By Condition [C6] all those assertions $\exists R'(a_x)$ follow however from some set of assertions $A_R^0 \cup A_{Q_2(i)}$, $j \in [0, n + k]$, and hence from $A_R^0$. By the definition of $A_R^0$, they are thus contained in $A_R^{\Delta_n}$, which means that $A_S$ can be disregarded.

If $e \in I^{\text{free}}$, then $\exists S_0(e)$ must similarly follow from ABox assertions by Lemma [A.3]. Particularly, it follows exclusively from $A_S$ (and $O$) because elements from $I^{\text{free}}$ do not occur in other ABoxes. Since $A_S$ contains only rigid assertions, the corresponding rigid basic concepts constitute a witness for $u^j_{iS_0}$, which contradicts our assumption and yields $e \notin I^{\text{free}}$.

In all three cases, we have shown the entailment required to apply the “only if”-direction of Condition [C6] to infer that $\exists S_0(e) \in A_S$. Since $I_j \models A_S$, $(e', a^j_{S_0}) \in S^\phi$ holds for all rigid roles $S$ as above. Given $I_j \models O$, the above assumptions on $O$, and $\text{Wit}(u^j_{iS_0}) = \emptyset$, $(e', a^j_{S_0})$ satisfies all the role atoms $R(t,y)$ in $I_j$ that are mapped to $(e^i, u^j_{iS_0})$ by $\pi$. We can therefore define $\pi'(y) := a^j_{S_0}$.

It thus remains to consider the satisfaction of the atoms we left out in Phase 2) and, in particular, the other variables of $\phi$ mapped by $\pi$ to elements of $J_{\Delta_n}$. As described above, we can assume them to be related in a tree structure and also to the elements we focused on in 2).
Phase 3) We finish the definition of \( \pi' \) using an induction over the structures of unnamed elements in the image of the homomorphism \( \pi \), starting with the elements we considered in Phase 2). For all variables \( y \) with \( \pi(y) \in \Delta^\text{anon}_{\pi} \), we can obviously set \( \pi'(y) := \pi(y) \). We therefore only consider the case that \( \pi(y) \in \Delta^\text{anon}_{\pi} \) and \( j \neq i \) in the following. Note that this is valid for the induction by Fact (\text{B.2}) and the interpretation of roles in \( J_i \), which show that elements from different sets \( \Delta^\text{anon}_{\pi} \) and \( \Delta^\text{anon}_{\pi} \) cannot be related in \( J_i \).

Given the latter observations, we can also maintain the following invariant while finishing the construction of \( \pi' \) for the remaining variables \( y \) (i.e., we do not have to satisfy the invariant at all for variables mapped by \( \pi \) to elements from \( \Delta^\text{anon}_{\pi} \) since \( \pi' \) is already defined for all variables directly connected to them); let \( m \) denote the number of variables for which \( \pi' \) is defined already at the respective moments in the induction: If \( \pi(y) = u^J_{\rho S_1} \), then either

\[
\begin{align*}
\text{Wit}(u^J_{\rho S_1}) &= \emptyset, \ \pi'(y) = d^J_{\rho S_1}, \text{ and } |\rho| < m, \text{ or} \\
\text{Wit}(u^J_{\rho S_1}) &\neq \emptyset, \ \pi'(y) \text{ is of the form } u^J_{\sigma S_1}, \text{ and } |\sigma| < m.
\end{align*}
\]

As induction hypothesis, we assume that the (partial) definition of \( \pi' \) satisfies all role atoms that only contain variables for which it is already defined and the invariant. It can readily be checked that our definitions from Phase 2), which represent the case base, satisfy both these requirements.

To show that \( \pi' \) is a homomorphism of \( \varphi \) into \( J_i \) for the variables mapped so far, it remains to consider the concept atoms. We assume \( \pi(y) = u^J_{\rho S_1} \in \Delta^\text{anon}_{\pi} \) and \( \pi'(y) \) to be defined already and consider all concept atoms \( A(y) \in \varphi \). Since \( u^J_{\rho S_1} \in \mathcal{A}^{J_i} \), by Lemma (A.8) either

\[
\text{there is a } \mathcal{B}^R \subseteq \mathcal{B}^R(\mathcal{O}) \text{ with } u^J_{\rho S_1} \in ( \bigcap \mathcal{B}^R )^{J_1} \text{ such that } \mathcal{O} \models \bigcap \mathcal{B}^{R} \subseteq A, \text{ or} \]

\[
u^J_{\rho S_1} \in \mathcal{A}^{J_i} \text{ and } \text{Wit}(u^J_{\rho S_1}) \neq \emptyset.
\]

If Case (7) applies, meaning \( \pi'(y) = u^J_{\rho S_1} \), then two applications of Lemma (A.3) yield that \( \mathcal{O} \models \exists S_1 \subseteq A \) and \( \pi'(y) = u^J_{\rho S_1} \in \mathcal{A}^{J_i} \). Otherwise, (8) holds because of \( u^J_{\rho S_1} \in \mathcal{A}^{J_i} \) by the definition of \( J, J_i \models \mathcal{O} \) then implies \( u^J_{\rho S_1} \in \mathcal{A}^{J_i} \), and Lemma (A.3) yields \( \mathcal{O} \models \exists S_1 \subseteq \bigcap \mathcal{B}^{R} \). From \( \pi'(y) = d^J_{\rho S_1} \), we then get \( d^J_{\rho S_1} \in ( \bigcap \mathcal{B}^R )^{J_1} \) by the definition of \( \mathcal{A}^{J_i} \). Given \( J_i \models \mathcal{O} \), we conclude that \( \pi'(y) \in \mathcal{A}^{J_i} \).

To continue the definition of \( \pi' \), we consider an element \( u^J_{\rho S_1 \sigma 2} \in \text{range}(\pi') \) and all role atoms \( R(y, z) \in \Phi \) with \( \pi(y) = u^J_{\rho S_1} \) and \( \pi(z) = u^J_{\rho S_1 \sigma 2} \). We hence can assume that \( \varphi \) contains a variable \( z \) such that \( \pi(z) = u^J_{\rho S_1 \sigma 2} \) for which \( \pi' \) has been defined already. For all these role atoms, we have that \( \mathcal{O} \models S_2 \subseteq R \), by the definition of \( J_i \) and Definition (A.1). Once again, we make a case distinction whether \( \text{Wit}(u^J_{\rho S_1 \sigma 2}) \) is empty or not.

If \( \text{Wit}(u^J_{\rho S_1 \sigma 2}) = \emptyset \), then \( \text{Wit}(u^J_{\rho S_1 \sigma 2}) = \emptyset \) holds, and hence Case (6) of our invariant applies, meaning \( \pi'(y) = d^J_{\rho S_1} \). In addition, for every of the role atoms \( R(y, z) \) under consideration, there is a rigid role \( S \) such that \( \mathcal{O} \models S \subseteq R \) and \( (u^J_{\rho S_1}, u^J_{\rho S_1 \sigma 2}) \in S^{J_1} \) by the definition of \( J_i \). Definition (A.1) then yields \( \mathcal{O} \models S \subseteq S_2 \). Together with \( \mathcal{O} \models \exists S_1 \subseteq \exists S_2 \) and the given bound on the length of \( \rho \), this implies that the element \( d^J_{\rho S_1 \sigma 2} \) exists in \( \mathcal{A} \), in all rigid assertions \( S(a^J_{\rho S_1}, a^J_{\rho S_1 \sigma 2}) \). Thus, \( J_i \models \mathcal{A} \) and \( J_i \models \mathcal{O} \), because of the fact that \( \mathcal{O} \models S \subseteq R \) yield that \( (a^J_{\rho S_1}, a^J_{\rho S_1 \sigma 2}) \) satisfies all relevant role atoms \( R(y, z) \). We set \( \pi'(z) := a^J_{\rho S_1 \sigma 2} \) for all such variables \( z \) and obtain Case (3) of our invariant.

In the remaining case that \( \text{Wit}(u^J_{\rho S_1 \sigma 2}) \neq \emptyset \), we know that \((u^J_{\rho S_1}, u^J_{\rho S_1 \sigma 2}) \in S^{J_1} \) since the pair is contained in \( S^{J_1} \) and also \( \mathcal{O} \models \exists S_1 \subseteq \exists S_2 \) by the definitions of the two interpretations. We make one final case distinction on whether \( \text{Wit}(u^J_{\rho S_1 \sigma 2}) \neq \emptyset \) is empty or not. 
If $\text{Wit}(u_{\rho_5}^{\mathcal{S}}) \neq \emptyset$, then (7) implies that $\pi'(y)$ is of the form $\pi'(y) = u_{\rho_5}^{\mathcal{S}}$. Given $I_i \models \mathcal{O}$, the element $u_{\rho_5}^{\mathcal{S}}$ exists, and $(u_{\rho_5}^{\mathcal{S}}, u_{\rho_5}^{\mathcal{S}})$ satisfies all role atoms $R(y, z)$ of the above form in $I_i$ by Definition 3.5. Hence, the definition $\pi'(z) := u_{\rho_5}^{\mathcal{S}}$ for all such variables $z$ maintains the invariant (Case (7)).

Finally, if $\text{Wit}(u_{\rho_5}^{\mathcal{S}}) = \emptyset$, then there must be a set $B^R \subseteq B^R(\mathcal{O})$ with $\mathcal{O} \models \bigwedge B^R \subseteq \exists S_2$ and $u_{\rho_5}^{\mathcal{S}} \in (\bigwedge B^R)_{\exists S_2}$, which implies $O \models \exists S_1 \subseteq \bigwedge B^R$ by Lemma 3.3. Since (6) yields that $\pi'(y)$ is of the form $a_{\rho_5}^{\mathcal{S}}$, the definition of $A_S$ implies $B^R(a_{\rho_5}^{\mathcal{S}}) \subseteq A_S$, and $I_i \models A_S$ then yields $\pi'(y) \in (\bigwedge B^R)_{\exists S_1}$. Together with $I_i \models \mathcal{O}$, we get $\pi'(y) \in (\exists S_2)^{\exists S_1}$. We can thus argue as in the previous case and set $\pi'(z) := u_{\rho_5}^{\mathcal{S}}, S_2$.

This concludes the construction of $\pi'$ and shows that it is a homomorphism of $\varphi$ into $I_i$, which contradicts (O2).

\[ \square \]

C Proofs for Section 5

**Lemma 5.2.** For all $W = \{W_1, \ldots, W_k\} \subseteq 2^{\{p_1, \ldots, p_n\}}$ and $\lambda : [0, n] \rightarrow [1, k]$, there is an $r$-complete tuple w.r.t. $W$ and $\lambda$ if and only if there is a set $B \subseteq \{B(a) \mid B \in B(\mathcal{O}), a \in I(\Phi)\}$ such that $(A^R, Q^+, Q^-, S)$ is $r$-complete w.r.t. $W$ and $\lambda$.

**Proof.** $(\Leftarrow)$ This direction is trivial. $(\Rightarrow)$ We assume $(A^R, Q^+, Q^-, S)$ to be an $r$-complete tuple, define

$$B := \{B(a) \in A^R \cup S \mid B \in B(\mathcal{O}), a \in I(\Phi)\},$$

and show that the tuple $(A^R, Q^+, Q^-, S)$ is $r$-complete as well. We focus on the conditions in Definition 5.5. Our tuple obviously satisfies Conditions (C3) and (C4) by construction.

For Conditions (C1), (C2) and (C5) we describe a model of $K^r_{\mathcal{I}}$ that can be homomorphically embedded into the canonical interpretation of the consistent KB $K^r_{\mathcal{I}}$ that exists for the given tuple $(A^R, Q^+, Q^-, S)$, since it satisfies Condition (C1). Observe that all positive assertions contained in one of the ABoxes of $K^r_{\mathcal{I}}$ must also be contained in $K^r_{\mathcal{I}}$.

- $A^R_{Q^+} \subseteq A^R_{\overline{Q}^+}$ follows from the facts that both $\overline{Q}^+$ and $Q^+$ satisfy Condition (C3) and $\overline{Q}^+$ is the minimal set satisfying that condition.
- $A_{\exists S_2} \cup A_{\overline{\exists} S_2} \subseteq A_S$ is a consequence of the following observations. By definition, every $\exists S(b) \in \overline{S}_{aux}$ is a consequence of an ABox $\langle O, A_{Q^+} \rangle$, $W_i \in W$. Since the given tuple satisfies Condition (C6) and we have $\lambda(n + j) = j$ in that definition, the assertion is also contained in $S$. $A_{\overline{\exists} S_2} \subseteq A_S$ follows from the construction. Each $\exists S(b) \in \overline{S}_{aux}$ follows, by definition, from $A^R$ together with some $A_j$ (and $\mathcal{O}$). Since this entailment does not depend on $B$ or $A^R_{\overline{Q}^+}$, and the given tuple satisfies Condition (C1), we know that $A^R$ must contain all assertions relevant for this entailment; since Condition (C6) is satisfied, $\exists S(b)$ is thus also contained in $S$.
- All positive assertions in $A^R$ have to be positive in $A^R$, too, by the definition of $A^R$ and the observations in the previous items. More precisely, $A^R(\mathcal{K}) \cap (B \cup A^R_{\overline{Q}^+}) \subseteq A^R$, and all other positive assertions in $A^R$ are implied by these initial assertions together with some $A_i$, $i \in [0, n]$. Since each $K^r_{\mathcal{I}}$ is consistent by assumption, the rigid ABox type $A^R$ also contains the latter rigid consequences.

Hence, any difference between $K^r_{\mathcal{I}}$ and $K^r_{\mathcal{I}}$ (i.e., focusing on the assertions in $K^r_{\mathcal{I}}$ and disregarding additional assertions in $K^r_{\mathcal{I}}$) must be due to negative rigid assertions in $A^R$ that
occur positively in $A^R$ (because $A^R$ is a rigid ABox type) and may cause the inconsistency of $K_{rc}^i$. By providing a model for $K_{rc}^i$, we show that such assertions cannot exist. Since the given tuple satisfies Condition (C1) and $K_{rc}^i$ contains all positive assertions occurring in $K_{rc}^i$, the KB $K_{rc,pos}$, obtained from $K_{rc}^i$ by dropping the negative assertions, is also consistent.

We focus on the canonical interpretation $I$ of that KB and show that it also satisfies $K_{rc}^i$. We consider negative role and basic concept assertions in $K_{rc}^i$.

- Let $\neg R(a, b) \in A^R$. We prove $I \models R(a, b)$ by contradiction, assuming that some of the ABoxes in $K_{rc,pos}^i$ contains a role assertion $S(a, b)$ such that $\mathcal{O} \models S \subseteq R$. We thus consider the positive assertions in $A^R, A^R_{Q^+}, A^R_{Q_{ij}}, A^R_S$, and $A^R_A$.

If $S$ is rigid, then we can disregard $A^R_{Q_{ij}}$ and $A^R_S$, since all rigid assertions in the former ABox are also contained in $A^R_{Q^+}$, and because $A^R_S$ does not contain assertions on two elements of $I$. Hence, by definition, the assertion $R(a, b)$ is contained in $A^R$. Since $A^R$ is a rigid ABox type (i.e., exactly one of $R(a, b)$ and $\neg R(a, b)$ is contained in it), this contradicts the assumption.

If $S$ is flexible, then $S(a, b)$ occurs in $A_i$ or $A^R_{Q_{ij}}$, which implies that $W, a(i) \in W$. By the definition of $A^R$ and $A^R_{Q^+}$, based on $Q^+$ and Definition 3.2, we then get $R(a, b) \in A^R$ or $R(a, b) \in A^R_{Q^+}$, which also implies $R(a, b) \in A^R$ and thus a contradiction to the assumption.

- Let $\neg B(a) \in A^R$. If $a \in I(\Phi)$, then $\neg B(a) \in A^R$ by the definitions of $B$ and $A^R$. Since $A^R$ is a rigid ABox type and the given tuple satisfies Condition (C1), Lemma A.2 yields $I' \models B(a)$, assuming $I'$ to be the canonical interpretation of $K_{rc}^i$. By our above observation on the positive assertions in $K_{rc}^i$, this interpretation must also satisfy $K_{rc,pos}^i$. Hence, $B(a)$ cannot be a consequence of that KB, and Lemma A.2 yields $I \not\models B(a)$.

If $a \not\in I(\Phi)$, then we again proceed by contradiction and assume $I \models B(a)$. Lemma A.3 then yields that there are positive assertions about $a$ in $A^R \cup A^R_S \cup A_i, j \in [0, n]$, that together imply $B$; the other ABoxes do not contain assertions on such names. By that lemma, we can also disregard $A^R_S$, since $A^R$ already contains all relevant assertions, i.e., if $A^R_S$ contains a rigid role assertion $R(a, e)$, then it must follow from some flexible concept assertion $\exists S(a)$ entailed by $(\mathcal{O}, A^R \cup A_i)$ for some $i$, and hence $\exists R(a)$ is already included in $A^R$. Then, Lemma A.1 implies that we can actually focus on $A^R$ alone and obtain $B(a) \in \bar{A}^R$. This contradicts the assumption since $A^R$ is a rigid ABox type.

Since $I$ is a model of $K_{rc,pos}^i$ by Lemma A.2, and we have shown that it satisfies all negative assertions in $A^R$, it is also a model of $K_{rc}^i$. Hence, our tuple satisfies Condition (C1).

If one of Conditions (C2) and (C5) is contradicted, then Lemma A.2 yields that there is a homomorphism of the CQ that causes the contradiction into $I$. Again, the above observation that the positive assertions contained in $K_{rc}^i$ must be contained in $K_{rc}^i$ is important. By Definition A.1 and the semantics, every such homomorphism into $I$ is also a homomorphism into the canonical interpretation of the positive part of $K_{rc}^i$. This contradicts the assumption that $K_{rc}^i$ satisfies Conditions (C2) and (C5) again by Lemma A.2.

It remains to consider Condition (C6) and we make a case distinction between the three parts of $S$. Observe that, w.r.t. the ABoxes considered in that condition, the individual names occurring in $S_{aux}$ can only occur within $A^R_{Q^+} \cup \bigcup_{W \in W} A^R_{QW}$, and those in $S_e$ only in $A^R \cup \bigcup_{0 \leq i < n} A_i$. 

We consider the assertions in $\bar{S}_{\text{aux}}$. ($\implies$) For every $\exists S(a_i) \in \bar{S}_{\text{aux}}$, the definition of $\bar{S}_{\text{aux}}$ directly yields that there is a $W_j \in W$ such that $(\mathcal{O}, A_{Q_j}) \models \exists S(a_i)$. This solves the claim given that Condition [C6] considers $\lambda$ to be such that $j = \lambda(n + j)$.

($\iff$) If there is a world $W \in W$ such that $(\mathcal{O}, A_{\mathcal{R}Q_j} \cup A_{Q_j}) \models \exists S(a_i)$, $a_i \in I_{\text{aux}}$, then Lemma A.3 implies that $\exists S(a_i)$ can only follow from assertions involving $a_i$. But $a_i$ can be associated to a unique query $\phi_j \in Q_\delta$ that contains the variable $x$ and corresponding ABox $A_{\phi_j}$; no other such ABoxes contains assertions on $a_i$. This implies $\phi_j \in Q^+$. By the definition of $\bar{Q}^+$, there is a $W' \in W$ with $p_j \in W'$ and, in particular, $A_{Q_\delta'}$ implies all assertions on $a_i \in A_{\mathcal{R}Q_j}$. This shows that $\exists S(a_i)$ already follows from $A_{Q_\delta'}$ (and $\mathcal{O}$), which yields $\exists S(a_i) \in \bar{S}_{\text{aux}}$.

For $\bar{S}_{\delta}$, its definition directly yields the claim.

We consider the assertions in $\bar{S}_{\delta}$. Since the given tuple satisfies Condition [C6] and, by the definition of $\bar{S}_{\delta}$, $S$ and $\bar{S}_{\delta}$ coincide w.r.t. $I(\Phi)$, we have that there is $i \in [0,n]$ such that $(\mathcal{O}, A_{\mathcal{R}} \cup A_{\mathcal{R}Q_{\delta'}}, \cup \cup A_{\mathcal{Q}_{\delta'}}) \models \exists S(a_i)$ if and only if $\exists S(a_i) \in \bar{S}_{\delta}$ for all $a_i \in I(\Phi)$.

($\iff$) This direction then directly follows from the above observations that all positive assertions in $A_{\mathcal{R}}$ occur in $A_{\mathcal{R}}$ and $A_{\mathcal{R}Q_{\delta'}} \subseteq A_{\mathcal{R}Q_{\delta}}$.

($\implies$) Since $A_{\mathcal{R}}$ is a rigid ABox type, the fact that the given tuple satisfies Condition [C1] yields that all basic concept assertions that can be derived from assertions in $A_{\mathcal{R}}$ or $A_{\mathcal{R}Q_{\delta'}}$ are also contained in $A_{\mathcal{R}}$; note that these ABoxes both contain only rigid assertions. Moreover, by definition, $B$ contains all rigid basic concept assertions on elements of $I(\Phi)$. Hence, Lemma A.3 yields that $\exists S(a) \in \bar{S}_{\delta}$ implies that there is an $i \in [0,n]$ with $(\mathcal{O}, A_{\mathcal{R}} \cup A_{\mathcal{R}Q_{\delta'}}, \cup \cup A_{\mathcal{Q}_{\delta'}}) \models \exists S(a)$.

Thus, Condition [C6] is also satisfied. □

Lemma 5.8. For all CQs $\varphi$, groundings $\gamma$, and $i \in [-1,n]$, we have

$$(\mathcal{O}, A_{\mathcal{R}} \cup A_i) \models \gamma(\varphi) \iff \text{TDB}(\lambda i) \models \gamma(\varphi)[A_{\mathcal{R}}(i)]$$

Proof. By Lemma A.2 we have $(\mathcal{O}, A_{\mathcal{R}} \cup A_i) \models \gamma(\varphi) \iff (\mathcal{O}, A_{\mathcal{R}} \cup A_i) \models \gamma(\varphi)$. Thus, it is sufficient to show the claim for all $A_{\mathcal{R}}$ by induction on $j \in [0,r]$. Moreover, since our rewriting is based on $[\varphi]$ by replacing all atoms individually, by Definition 5.3, Lemma 5.4 and the substitution lemma for first-order logic it suffices to show that

$$\pi(\alpha(t)) \in A_{\mathcal{R}} \cup A_i \iff \text{TDB}(\lambda i) \models \pi(\alpha'(t,i))$$

holds for all $i \in [-1,n]$, all atoms $\alpha(t)$, where $\alpha'(t,i)$ is the corresponding replacement used for the definition of $[\varphi][A_{\mathcal{R}}(i)]$, and all functions $\pi : V(\alpha(t)) \rightarrow I(\mathcal{K})$.

For the base case $j = 0$, this is easy to see by Definition 5.6 and a case analysis on whether $\pi(\alpha(t)) \in A_{\mathcal{R}} = A_{\mathcal{R}}(\mathcal{K}) \cap (B \cup A_{\mathcal{R}Q_{\delta'}}) \cup \pi(\alpha(t)) \in A_i$.

Assume that the claim holds for an arbitrary $j \in [0,r - 1]$. Then the case $\pi(\alpha(t)) \in A_i$ is again captured by $A(t,i)$ and, by induction, the satisfaction of $\pi(\exists \exists[p.[\alpha(t),A_{\mathcal{R}}(p)])$ in TDB(\lambda i)\ equivalent to the existence of a $p \in [0,n]$ such that $(\mathcal{O}, A_{\mathcal{R}} \cup A_{\mathcal{R}}) \models \pi(\alpha(t))$, which is exactly the condition for $\pi(\alpha(t))$ being included in $A_{\mathcal{R}j+1}$. □

Lemma 5.10. For all Boolean CQs $\varphi$ and $i \in [-1,n]$, we have:

- $(\mathcal{O}, A_{\mathcal{R}} \cup A_i)$ is inconsistent iff TDB(\lambda i) $\models [\bot][A_{\mathcal{R}}(i)]$. 

\[ \langle O, \vec{A}_w \cup A_i \rangle \models \varphi \ \text{iff} \ \text{TDB}(\mathfrak{A}) \models [\varphi]\vec{A}_w \langle i \rangle. \]

**Proof.** By Lemma 5.8, \( \langle O, \vec{A}_w \cup A_i \rangle \) is inconsistent iff we have \( \text{DB}(\vec{A}_w \cup A_i) \models [\bot] \), and it entails \( \varphi \) iff \( \text{DB}(A_w \cup A_i) \models [\varphi] \). It is thus sufficient to show that, for any Boolean CQ \( \mathfrak{v} = \exists x_0 \ldots \exists x_{\ell-1}, \psi \), we have \( \text{DB}(\vec{A}_w \cup A_i) \models \mathfrak{v} \) iff \( \text{TDB}(\mathfrak{A}) \models \mathfrak{v} \lor \cdots \lor \mathfrak{v}_{\ell-1} \), where the latter disjunction represents our rewriting of \( \mathfrak{v} \).

(\( \Rightarrow \)) We assume that there is a homomorphism \( \pi \) of \( \mathfrak{v} \) into \( \text{DB}(\vec{A}_w \cup A_i) \), and show that \( \pi \) is also a homomorphism of one of the formulas \( \mathfrak{v}_k \) into \( \text{TDB}(\mathfrak{A}) \). We choose the disjunct

\[ \mathfrak{v}_k := \exists x_0 \ldots \exists x_{\ell-1}. [\psi]_{\rep}(i) \land \psi_{\text{filter}} \]

where

\[ k = b_0 \cdot 2^0 + \ldots + b_j \cdot 2^j + \ldots + b_{\ell-1} \cdot 2^{\ell-1} \]

with \( b_j = 0 \) iff \( \pi(x_j) \in \mathcal{I}^\text{aux} \cup \mathcal{I}^\text{free} \), for all \( j \in [0, \ell - 1] \). Moreover, for each \( j \) with \( b_j = 0 \), we consider the disjunct of \( \mathfrak{v}_k \) in which \( x_j \) is equal to

- the prototype \( a_{\delta, p} \), if \( \pi(x_j) = a_{\delta p} \in \mathcal{I}(A_{\exists S(b)}) \setminus \{ b \} \) with \( b \in \mathcal{I}(K) \setminus \mathcal{I}(\Phi) \), or
- the auxiliary individual name \( \pi(x_j) \in \mathcal{I}^\text{aux} \cup \mathcal{I}^\text{free} \).

We show that, the formula \( \pi([\psi]_{\rep}(i) \land \psi_{\text{filter}}) \) is satisfied in \( \text{TDB}(\mathfrak{A}) \). First, consider a conjunct \([x = t]_{\rep}\) of \( \psi_{\text{filter}} \), for which there must exist two role atoms \( R(x_j, s) \), \( S(x_t, t) \) in \( \mathfrak{v} \) such that \( x_j \) is a prototype element in the disjunct of \( \mathfrak{v}_k \) we consider, and \( x_t \), \( x_p \) are not. By construction, this means that \( \pi(x_j) \in \mathcal{I}(A_{\exists S(b)}) \setminus \{ b \} \) for some \( b \in \mathcal{I}(K) \setminus \mathcal{I}(\Phi) \), and that this does not hold for \( s \) and \( t \). Since \( R(\pi(x_j), \pi(s)) \) and \( S(\pi(x_j), \pi(t)) \) are satisfied in \( \text{DB}(\vec{A}_w \cup A_i) \), the elements of \( \mathcal{I}(A_{\exists S(b)}) \setminus \{ b \} \) only occur in \( \vec{A}_w \), and the only other individual name that occurs in \( A_{\exists S(b)} \) is \( b \), we must have \( \pi(s) = b = \pi(t) \), i.e., the formula \([x = t]_{\rep}\) is satisfied in \( \text{TDB}(\mathfrak{A}) \).

It remains to consider each atom \( \alpha(t) \) in \( \mathfrak{v} \) and show that \( \pi([\alpha(t)]_{\rep}(i)) \) is satisfied in \( \text{TDB}(\mathfrak{A}) \). By assumption, \( \pi(\alpha(t)) \in \vec{A}_w \cup A_i \), and we now consider the specific parts of \( \vec{A}_w \cup A_i \) that \( \alpha(t) \) is mapped into.

- If \( \pi(\alpha(t)) \in \vec{A}^R_w \cup A_i \), then \( t \) cannot contain a variable \( x_j \) with \( b_j = 1 \), since these ABoxes only contain individual names in \( \mathcal{I}(K) \). If \( \alpha \) is rigid, then due to \( \langle O, \vec{A}^R \cup A_i \rangle \models \pi(\alpha(t)) \) the part \([\alpha(t)]_{\rep}(i) = [\alpha(t)]_{\vec{A}^R}(i) \) of the formula \([\alpha(t)]_{\rep}(i) \) is satisfied in \( \text{TDB}(\mathfrak{A}) \) by Lemma 5.8. If \( \alpha \) is flexible, then we must have \( \pi(\alpha(t)) \in A_i \), and thus a corresponding disjunct \( B(t_1, i) \) or \( R(t_1, t_j, i) \) (depending on the shape of \( \alpha \)) is satisfied in \( \text{TDB}(\mathfrak{A}) \) by Definition 5.6.

- If \( \pi(\alpha(t)) \in \vec{A}^Q_w \cup A_{\exists S} \cup A_{\text{aux}} \cup A_{\text{base}} \), then there is a disjunct of \([\alpha(t)]_{\rep}(i) \) that is of the form \([t_1 = \pi(t_1)]_{\rep}\) or \([t_1 = \pi(t_1)]_{\rep} \land \[t_2 = \pi(t_2)]_{\rep} \), which is obviously satisfied under \( \pi \).

- If \( \pi(\alpha(t)) \in \vec{A}_w \), then either (i) \( \pi(t) \) contains only elements from \( \mathcal{I}(A_{\exists S(b)}) \setminus \{ b \} \) with \( b \in \mathcal{I}(K) \setminus \mathcal{I}(\Phi) \), or (ii) it contains both one element from this set and \( b \) itself.

In both cases, we have \( \exists S(b) \in \mathcal{S} \). Moreover, by our construction, in \( \mathfrak{v}_k \) the elements of \( \mathcal{I}(A_{\exists S(b)}) \setminus \{ b \} \) are considered via the corresponding prototypes. In case (i), we hence have \( \alpha(t) \in \vec{A}_{\exists S} \), and thus

\[ [\alpha(t)]_{\rep}(i) = \exists x. [\exists S]_{\rep}(x) = \exists x. \exists p. [\exists S(x)]_{\vec{A}^R}(p) \land \bigwedge_{\alpha \in \mathcal{I}(\Phi)} (x \neq a) \]

can be satisfied by mapping \( x \) to \( b \) and \( p \) to \( j \) such that \( \langle O, \vec{A}^R \cup A_i \rangle \models \exists S(b) \); note that such a time point must exist by the definition of \( \mathcal{S} \). This is correct due to Lemma 5.8.
case (ii), we can similarly show that \([\alpha(t)]_{3}(i) = [\exists S]_{\text{rep}}(t_{1})\) is satisfied since \(\pi(t_{1}) = b\) holds without loss of generality (if \(\pi(t_{2}) = b\), we can consider the equivalent atom \(\alpha^{-}(t_{2}, t_{1})\) instead of \(\alpha(t_{2}, t_{1})\)).

(\(\leftarrow\)) We assume that \(\text{TDB}(\mathcal{A})\) satisfies one of the disjuncts \(\forall t, k \in [0, 2^{l} - 1]\), via a homomorphism \(\pi\), and extend \(\pi\) to the variables \(x_{j}\) with \(b_{j} = 1\) in such a way that \(\pi(V)\) is satisfied in \(\text{DB}(\mathcal{A}_{\mathcal{R}} \cup \mathcal{A}_{t})\). We consider a satisfied disjunct of \(\forall t\) that corresponds to some assignment of the variables \(x_{j}\) with \(b_{j} = 1\) to elements of \(I_{\text{aux}} \cup I_{\text{free}} \cup I_{\text{pro}}\). If \(b_{j} = 1\) and \(x_{j}\) is considered to be an element of \(I_{\text{aux}} \cup I_{\text{free}}\) in this disjunct, then we set \(\pi(x_{j})\) to this element.

If \(b_{j} = 1\) and \(x_{j} \in I_{\text{pro}}\) in this disjunct, then we consider the largest connected subset of atoms in \(V\) that contains \(x_{j}\) and for which at least one variable in each atom is considered to be an element of \(I_{\text{pro}}\). Since, for each of these atoms, the corresponding formula in \([\alpha(t)]_{3}(i)\) must be satisfied, we know that they can all be mapped into an ABox of the form \(A_{\mathcal{S}}\). If this mapping does not involve the root of that ABox, all these rewritings are of the form \(\exists S\cdot [\exists S]_{\text{rep}}(x)\), which by Lemma 5.8 and the definition of \(S_{o}\) implies that there is an element \(b \in I(K)\setminus I(\Phi)\) such that the above atoms can be satisfied in \(A_{\mathcal{S}}\); instead of \(A_{\mathcal{S}}\). In this case, we arbitrarily choose one such \(b\) and let \(\pi\) map each such variable \(x_{j}\) to the corresponding element in \(I(\mathcal{A}_{\mathcal{S}}(b))\). Otherwise, for at least one of the atoms, a formula of the form \([\exists S]_{\text{rep}}(t_{1})\) must be satisfied in \(\text{TDB}(\mathcal{A})\), which means that \(\pi(t_{1})\) is an element of \(I(K)\setminus I(\Phi)\). Similar arguments as above yield that the ABox \(A_{\mathcal{S}}(\pi(t_{1}))\) is part of \(A_{\mathcal{S}}\). Moreover, by \(\mathcal{S}_{\text{iter}}\), any terms \(t_{1}\) occurring in such a way in \(V\) must be mapped by \(\pi\) to the same element \(b\) of \(I(K)\setminus I(\Phi)\). Hence, we can extend \(\pi\) to the variables above by considering the corresponding elements in \(A_{\mathcal{S}}\).

To see that this definition of \(\pi\) is correct, consider an arbitrary atom \(\alpha(t)\) in \(V\) and its rewriting \([\alpha(t)]_{3}(i)\) in the disjunct of \(\forall t\) that we considered above, i.e., it is satisfied in \(\text{TDB}(\mathcal{A})\) under \(\pi\). If this is the case because \([\alpha(t)]_{3}(i)\) is satisfied in \(\text{TDB}(\mathcal{A})\), we have argued in the previous paragraph that \(\pi(\alpha(t))\) is satisfied in \(\text{DB}(\mathcal{A}_{\mathcal{R}} \cup \mathcal{A}_{t})\). If \([\alpha(t)]_{3}(i)\) is satisfied, then we directly obtain \(\pi(\alpha(t)) \in A_{\mathcal{S}}\). Since all of these ABoxes are part of \(\mathcal{A}_{\mathcal{R}} \cup \mathcal{A}_{t}\), the claim follows.

Finally, consider the case that \([\alpha(t)]_{3}(i)\) is satisfied in \(\text{TDB}(\mathcal{A})\). This can only be the case if no element of \(t\) is considered to be in \(I_{\text{aux}} \cup I_{\text{free}} \cup I_{\text{pro}}\), i.e., \(t\) contains only variables or individual names from \(I(\Phi)\). If \(\alpha\) is flexible, the definition of the rewriting and Definition 5.6 yield that \(\pi(\alpha(t)) \in A_{t}\). Similarly, if \(\alpha\) is rigid, we obtain \(\pi(\alpha(t)) \in A_{\mathcal{R}}\) from Lemmas 5.8 and 5.1. \(\square\)

D Proofs for Section 6

Theorem 6.1. TCQ entailment in DL-Lite-core is ALOGTIME-hard in data complexity, even if \(C^{R} = \emptyset\) and \(P^{R} = \emptyset\).

Proof. It is well-known that every finite monoid \(M\) (i.e., a finite, closed set having an associative binary operation and an identity element) can be directly translated (in logarithmic time) to a deterministic finite automaton (DFA) that decides the word problem for that monoid, by treating the elements of \(M\) as states and considering transitions according to the associative operation. Moreover, for some such monoids (e.g., the group \(S_{5}\)), this problem is known to be ALOGTIME-hard in data complexity, even if \(C^{R} = \emptyset\) and \(P^{R} = \emptyset\).

\(\square\)

We refer the reader to [1] for details about monoids, groups, and the word problem in that context.
lem is complete for \textsc{LogTime}-uniform \textsc{NC}^1 under \textsc{LogTime}-uniform \textsc{AC}^0 reductions \cite[Cor. 10.2]{[a]}; and \textsc{LogTime}-uniform \textsc{NC}^1 equals \textsc{ALogTime} \cite[Lem. 7.2]{[a]}.

We hence can establish \textsc{ALogTime}-hardness by considering an arbitrary DFA \( \mathfrak{M} \) and reducing its word problem to TCQ entailment in logarithmic time. For that, we adapt a construction of \cite[Thm. 9]{[a]}.

Let \( \mathfrak{M} \) be a tuple of the form \( (Q, \Sigma, \Delta, q_0, F) \), specifying the set of states \( Q \), the alphabet \( \Sigma \), the transition relation \( \Delta \), the initial state \( q_0 \), and the set of final states \( F \). Because we consider data complexity, the task is to specify a TCQ \( \Phi_{\mathfrak{M}} \) based on \( \mathfrak{M} \) and an ABox sequence \( \mathfrak{A}_n \) based on an arbitrary input word \( w \in \Sigma^* \) such that \( \mathfrak{M} \) accepts \( w \) iff \( (\emptyset, \mathfrak{A}_n) \models \Phi_{\mathfrak{M}} \). We consider concept names \( A_\sigma \) and \( Q_i \) for all characters \( \sigma \) of the input alphabet \( \Sigma \) and states \( q \in Q \), respectively, and define the following TCQ:

\[
\Phi_{\mathfrak{M}} := \Box \left( \bigwedge_{q \rightarrow q' \in \Delta} \left( (Q_i(a) \wedge A_\sigma(a)) \rightarrow \bigcirc_F Q_{q'}(a) \right) \right) \to \bigvee_{q_i \in F} Q_{q_i}(a).
\]

For a given input word \( w = \sigma_0 \ldots \sigma_{n-1} \), we then define the sequence \( \mathfrak{A}_n = (A_i)_{0 \leq i < n} \) of ABoxes as follows: \( A_0 := \{ Q_0(a) \} \) and, for all \( i \in [0, n] \), \( A_i := \{ A_{\sigma_i}(a) \} \). It is easy to see that this reduction can be computed in logarithmic time.

Given that the semantics of TCQ entailment focus on time point \( n \), it can readily be checked that the model of \( (\emptyset, \mathfrak{A}_n) \) that satisfies the premise of \( \Phi_{\mathfrak{M}} \) at \( n \) represents the run of \( \mathfrak{M} \) on \( w \). Observe that there is only one such model relevant for entailment since \( \mathfrak{M} \) is deterministic. Hence, \( \mathfrak{M} \) accepts \( w \) iff all models of \( (\emptyset, \mathfrak{A}_n) \) that satisfy the premise also satisfy the disjunction \( \bigvee_{q_i \in F} Q_{q_i}(a) \) at \( n \). This is equivalent to the entailment \( (\emptyset, \mathfrak{A}_n) \models \Phi_{\mathfrak{M}} \).

\( \square \)

\textbf{Theorem 6.4.} TCQ entailment in DL-Lite\(_h^H\) is in ALogTime in data complexity, even if \( P^R \neq \emptyset \).

\textbf{Proof.} The ATM \( \mathfrak{M} \) accepts the input \( n \) and \( \text{TDB}(\mathfrak{A}) \) (in logarithmic time) iff there are sets \( \mathcal{W} = \{ W_1, \ldots, W_k \} \subseteq 2^{\{ p_1, \ldots, p_n \}} \) and \( \mathcal{B} \subseteq \{ B(a) \mid B \in \mathfrak{B}(O), a \in \mathfrak{I}(\Phi) \} \), a valuation \( \nu \in \mathcal{V} \), and types \( T_0, \ldots, T_n \) as follows, where \( w_i := T_i \cap \{ p_1, \ldots, p_n \} \):

\begin{itemize}
  \item \( T_0 \) is initial and \( T^n \subseteq T_i \);
  \item for every \( i \in [0, n] \), the pair \( (T_i, T_{i+1}) \) is t-compatible;
  \item \( w_n \in \text{Fut}_i \);
  \item for every \( i \in [0, n] \), we have \( w_i \in \mathcal{W} \);
  \item for every \( i \in [0, n] \), we have \( \text{TDB}(\mathfrak{A}) \models r\text{Sat}_w(i) \);
  \item for every \( i \in [0, n] \), we have \( \exists s(a) \in \text{B} \) iff there is an \( i \in [0, n] \) such that \( \text{TDB}(\mathfrak{A}) \models e\text{Sat}_{w_i}(i) \) (w.r.t. \( w_i \));
  \item for every \( W \in \mathcal{W} \), we have \( \text{TDB}(\mathfrak{A}) \models r\text{Sat}_{w}(-1) \);
\end{itemize}

By Lemmas 6.6 and 6.2, the first four points are equivalent to the existence of a set \( \mathcal{W} \) and worlds \( w_i \) as above and an \text{LTL}-structure \( \mathfrak{M} \) as follows:

\begin{itemize}
  \item \( \mathfrak{M} \) only contains worlds from \( \mathcal{W} \);
  \item \( \mathfrak{M} \) starts with \( w_0, \ldots, w_n \);
  \item \( \mathfrak{M}, w_n \models \Phi^a \).
\end{itemize}

Moreover, because of the condition that each \( w_i \) is an element of \( \mathcal{W} \), the sequence \( w_0, \ldots, w_n \) can equivalently be expressed by a mapping \( \lambda : [0, n] \rightarrow [1, k] \) such that \( w_i = W_{\lambda(i)} \) for all \( i \in [0, n] \). Finally, by \textbf{Definition 6.9} and Lemmas 6.11, 6.6, 6.2, and 6.3, the above items characterize the satisfiability of \( \Phi \) w.r.t. \( \mathcal{K} \). The claim now follows from the fact that the class \text{ALogTime} is closed under complement (see \cite[Thm. 2.5]{[a]}). \( \square \)
E Proofs for Section 7

Lemma 7.1. Let \( (C \subseteq D, \neg \varphi) \) be one of the pairs of a CI and a TCQ given in Table \( \mathcal{T} \) and let \( \mathcal{I} \) be a model of \( \top \subseteq A \cup A_i, A_i \cap \overline{A_i} \subseteq \bot \) for all concept names \( A_i \) occurring in \( D \). Then, we have \( \mathcal{I} \models C \subseteq D \text{ iff } \mathcal{I} \models \neg \varphi \).

Proof. (\( \Rightarrow \)) We assume \( \mathcal{I} \not\models \neg \varphi \), which yields \( \mathcal{I} \models \varphi \), and hence that there is a corresponding homomorphism by Definition 2.3. Observe that the atoms in the CQ \( \varphi \) always refer to the concepts and roles of the corresponding CI \( C \subseteq D \) in the same way, so that \( C \) and \( \neg D \) are modeled in the CQ. Thus, the shape of \( \varphi \) together with our assumption that \( \mathcal{I} \) satisfies the CIs w.r.t. \( \mathcal{T} \) and the semantics of the constructor \( \forall \) yield that there is an element \( e \) in the domain of \( \mathcal{I} \) such that \( e \in C^\mathcal{I} \) and \( e \not\in D^\mathcal{I} \). This directly yields \( C^\mathcal{I} \not\subseteq D^\mathcal{I} \), and thus \( \mathcal{I} \not\models C \subseteq D \).

(\( \Leftarrow \)) The proof for this direction is by dual arguments. \( \square \)

Theorem 7.8. TCQ entailment in DL-Lite\textsubscript{bool} is in co-NExpTime in combined complexity if \( P^{\text{NP}} = \emptyset \), even if \( C^{\text{NP}} \neq \emptyset \).

Proof. We consider DL-Lite\textsubscript{bool} (see Corollary 7.7) and use Lemma 7.1 for checking satisfiability of \( \Phi \) w.r.t. \( \mathcal{K} \). As in the proof of Theorem 7.6, we can assume \( \mathcal{K} \) to be of the form \( (\mathbb{Q}, \emptyset) \), since integrating the ABoxes into the TCQ does not influence combined complexity. This means that \( \mathbb{A} \) is irrelevant. We can then guess a set \( \mathcal{W} \subseteq 2^{(p_1, \ldots, p_m)} \) in exponential time and check t-satisfiability of \( \Phi^{\text{ta}} \) w.r.t. this set in ExpTime \([16]\).

For r-satisfiability, we adapt a technique from \([16],[18]\). We guess a set \( \mathcal{F} \subseteq 2^{C^R(\mathcal{O})} \), which specifies the combinations of rigid concept names that are allowed to be satisfied by domain elements in the models of the conjunctions \( \chi_i \), and a mapping \( \tau : I(\Phi) \rightarrow \mathcal{F} \) that fixes the rigid concepts for each individual occurring in \( \mathcal{K} \)—similar to the rigid ABox types we considered in previous sections. Based on \( \tau \), we define a polynomial-sized ontology \( \mathcal{O}_\tau \) and CQ \( \psi_\tau \), as follows:

\[
\mathcal{O}_\tau := \{ A_{\tau(i)} \equiv C_{\tau(i)} \mid a \in I(\Phi) \} \cup \{ \top \subseteq A \cup A_i, A \cap \overline{A} \subseteq \bot \mid A \in C^R(\mathcal{O}) \},
\]

\[
\psi_\tau := \bigwedge_{a \in I(\Phi)} A_{\tau(i)}(a)
\]

where \( \equiv \) is an abbreviation for both \( \subseteq \) and \( \sqsubseteq \), and \( C_F \) with \( F \subseteq C^R(\mathcal{O}) \) is defined as \( C_F := \bigcap_{a \in F} A \cap \bigcap_{a \in C^R(\mathcal{O}) \setminus F} \overline{A} \). We further say that an interpretation \( \mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J}) \) respects \( \mathcal{F} \) if

\[
\mathcal{F} = \{ F \subseteq C^R(\mathcal{O}) \mid (C_F)^\mathcal{J} \neq \emptyset \}.
\]

In \([16] \text{ Lem. } 6.2\) , it is shown that \( \mathcal{W} \) is r-satisfiable w.r.t. \( \mathcal{K} \) iff there are a set \( \mathcal{F} \) and mapping \( \tau \) as above such that each conjunction \( \chi_i \wedge \psi_\tau \) with \( i \in [1,k] \) has a model w.r.t. \( \mathcal{O} \cup \mathcal{O}_\tau \) that respects \( \mathcal{F} \). The proof considers the DL \( \mathcal{K} \), but similarly holds for DL-Lite\textsubscript{bool}.

Although it seems that the NExpTime result now directly follows from Lemma 7.5 stating that satisfiability of conjunctions of CQ literals can be decided in exponential time, this is not the case. The restriction that \( \mathcal{F} \) should be respected causes an exponential blowup. For that reason, we consider the proof of \([26] \text{ Thm. } 8\) , which addresses UCQ entailment, in more detail. In that paper, an exponentially large looping tree automaton is constructed that recognizes exactly those (forest-shaped) canonical models of the KB—in a wider sense—that do not satisfy the given UCQ. We integrate the check that the interpretations respect \( \mathcal{F} \) into the automaton. To this end, we restrict the state set to consider only models where every domain element satisfies some \( C_F \) with \( F \in \mathcal{F} \). To ensure that each \( F \in \mathcal{F} \) is represented...
somewhere in the model, we check $|F|$ variants of this automaton for emptiness, each of which considers an ABox of the form $\{A(a) \mid A \in F\} \cup \{\overline{A}(a) \mid A \in C^R(O) \setminus F\}$, where $a$ is a fresh individual name. The disjoint union of all resulting interpretations is still a model of the original KB that does not satisfy the UCQ. It can readily be checked that this modified procedure for deciding UCQ non-entailment is sound and complete given the result of \cite{26} Thm. 8]. Satisfiability of the conjunctions $\chi \land \psi_r$ can thus be decided in exponential time, because the constructed automata are of exponential size and emptiness of looping tree automata can be decided in polynomial time \cite{29} Thm. 2.2].

Theorem 7.9. TCQ entailment in DL-Lite$_{krom}$ is $2$-ExpTime-hard in combined complexity if $P^R \neq \emptyset$.

Proof Sketch. We adap a reduction proposed in \cite{18} (see the proof of Theorem 4.1), where the word problem for exponentially space-bounded ATMs is reduced to the satisfiability problem in $\mathit{ACC}$-LTL with global CIs and rigid names. While the assertions in the proposed $\mathit{ACC}$-LTL formula can be directly viewed as conjuncts of a TCQ, the global CIs cannot all be transferred into a DL-Lite$_{krom}$ ontology since \mathit{ACC} is much more expressive than DL-Lite$_{krom}$. However, we show how some of the critical CIs can be adapted to comply with the shapes given in Table \ref{tab:table} and how the remaining ones—with qualified existential restrictions on the right-hand-side—can be replaced by equivalent new constructions. The latter are inspired by \cite{37} (see Section 6.2 in that paper).

As in the original proof, we assume w.l.o.g. that the ATM never moves to the left when it is on the left-most tape cell; that it has an accepting state $q_a$ and a rejecting state $q_r$, designating accepting and rejecting configurations, respectively; that any configuration where the state is neither $q_a$ nor $q_r$ has at least one successor configuration; and that all computations of the ATM are finite (see \cite{34} Thm. 2.6]). We disregard transitions that do not move the head ($N$). Further, we assume for simplicity that the length of every computation on a word $w \in \Sigma^*$ is bounded by $2^k$, and that every configuration in such a computation can be represented using $\leq 2^k$ symbols, plus one to represent the state.

According to \cite{34} Cor. 3.5], there is an exponentially space-bounded alternating TM $\mathbb{M} = (O, \Sigma, \Gamma, q_0, \Delta)$ whose word problem is 2-ExpTime-hard; $O$ represents the set of states, which is partitioned into the sets $O^e$ of existential states and $O^u$ of universal states; $\Sigma$ is the input alphabet; $\Gamma$ is the work alphabet of the TM, containing the blank symbol $B$ and all symbols from $\Sigma$; $q_0$ is the initial state; and $\Delta$ denotes the transition relation. We show that this problem can be reduced to TCQ satisfiability in DL-Lite$_{krom}$ with rigid role names.

To this end, let $w = \sigma_0 ... \sigma_{k-1} \in \Sigma^*$ be an arbitrary input word given to $\mathbb{M}$. We next construct a TCQ $\phi_{\mathbb{M}, w}$ and a TKB $\langle O_{\mathbb{M}, w}, (A_0) \rangle$ in DL-Lite$_{krom}$ such that $\mathbb{M}$ accepts $w$ iff $\phi_{\mathbb{M}, w}$ is satisfiable w.r.t. $\langle O_{\mathbb{M}, w}, (A_0) \rangle$. We use two counters modulo $2^k$, $A$ and $A'$. We can consider a tree describing the computations of the ATM in that one path describes one computation. The individual configurations are represented explicitly, one after the other, and each as a chain, such that every tree node represents one of the $2^k$ tape cells of a configuration; these cells are numbered by the rigid counter $A$. Each tree node or cell is represented by an individual in the reduction and, since these individuals are related by rigid roles, the computation tree “exists” at all time points; the time points are numbered by the $A'$ counter. Branching models the universal transitions. Different from usual computation trees, the tree however splits at the node representing the cell under the head of the machine; the remaining parts of the configuration where the splitting occurs are replicated in each of the subtrees. Before specifying the TCQ and ontology, we introduce all symbols we use below:

\footnote{\label{footnote}Strictly speaking, the space bound is $2^{\kappa^{(k)}}$ for a polynomial $p$, but omitting $p$ does not affect the proof.}
– A single named individual \( a \) identifies the root of the tree.
– Rigid role names \( R_{q,\rho,M} \) where \( q \in Q, \rho \in \Gamma, \) and \( M \in \{L, R\} \) represent the edges of the tree. We collect all these role names in the set \( R \).

Note that these roles represent the major difference to the reduction of [18], where a single rigid role fulfills this purpose, but is used within qualified existential restrictions on the right-hand side of CIs.

– Rigid concept names \( A_0, \ldots, A_{k-1} \) are used to model the bits of a binary counter numbering the tape cells in the configurations.
– Rigid concept names \( I \) and \( H \) point out special cells. In particular, \( I \) is satisfied by the nodes representing the initial configuration, and \( H \) is satisfied by all nodes representing a tape cell that is located (anywhere) to the right of the head in the current configuration.
– A rigid concept name, for each element in \( Q \cup \Gamma \), represents the tape content, the current state, and the head position in each configuration in the tree: if \( 2R \) is in a state \( q \) and the head is on the \( i \)-th tape cell, then the individual (tree node) representing this cell satisfies the concept name \( q \); we correspondingly represent the symbols in \( \Gamma \).
– The rigid concept names \( T_{q,\rho,M} \), for all \( q \in Q, \rho \in \Gamma, \) and \( M \in \{L, R\} \), are satisfied by an individual, representing a cell, if the head is on the left neighboring cell and the ATM executes the transition \( (q, \sigma, M) \) in the described configuration.

We use the temporal dimension to synchronize successor configurations in accordance with the chosen transition in order to model the change in the tape contents, the head position, and the state from one configuration to the next:

– Flexible concept names \( A'_0, \ldots, A'_{k-1} \) are used to model a counter in the temporal dimension. Its value is incremented (modulo \( 2^k \)) similar to the counter \( A_0, \ldots, A_{k-1} \) but along the temporal dimension and, at every time point, all individuals of the domain share the value of this counter. It is used for the synchronization of successor configurations: if the \( A' \)-counter has value \( i \), then the symbol in the \( i \)-th tape cell of any configuration (where \( i \) is not the head position) is propagated to the \( i \)-th tape cell of its successor configuration. Similarly, the state is propagated from the cells \( c \) directly right of the head position, each pointing out a specific transition (via the symbols \( T_{q,\rho,M} \)), to the corresponding cells of the successor configurations (i.e., these cells have the same position on the tape as \( c \) for right-moves and otherwise lie two to the left).
– We further use a flexible concept name, for each element in \( Q \cup \Gamma \), which as above is distinguished from the rigid version by a prime. Considering a fixed time point, these names are used for the propagation of the state \( q \) or cell content \( \sigma \) of a cell \( c \) to the corresponding cell in the successor configuration(s). This propagation happens via the right neighboring cells of that configuration, which then satisfy \( q' \) and \( \sigma' \), respectively, at the time point whose \( A' \)-counter corresponds to the \( A \)-counter at \( c \).

We may further use concept names of the form \( \overline{A} \) for given concept names \( A \) as detailed in Lemma 7.1.

In the remainder of the proof, we define the TCQ \( \Phi_{20, w} \) and the TKB \( \langle O_{20, w}, (A_0) \rangle \) by describing the conjuncts of \( \Phi_{20, w} \) and listing the CIs contained in \( O_{20, w} \). To enhance readability, we may use CIs that are not in \( DL-Lite_{ext} \), but can be transformed as described in the beginning of this section (see Table 6 and Example 7.3). We first express the tree structure in general.
We enforce all elements to have some successor except if they satisfy \( q_a \) or \( q_r \). Since the only elements satisfying a symbol from \( Q \) are the ones representing the position of the head, the tree generation thus is only stopped if we meet a halting configuration:

\[
\overline{q_a} \cap \overline{q_r} \subseteq \bigcup_{R \in R} \exists R_\delta.
\]

Using a big disjunction over all possible roles, we can correctly represent the nondeterminism of the machine.

The \( A \)-counter is incremented alongside the tree modulo \( 2^k \) and modeled using the following CIs for all \( i \in [0, k - 1] \):

\[
\bigcap_{0 \leq j < i} A_j \subseteq \bigcap_{R \in R} \forall R_\delta A_i,
\]

\[
\bigcap_{0 \leq j < i} A_j \cap \overline{A}_i \subseteq \bigcap_{R \in R} \forall R_\delta A_i,
\]

\[
\left( \bigcup_{0 \leq j < i} \overline{A}_j \right) \cap A_i \subseteq \bigcap_{R \in R} \forall R_\delta A_i,
\]

\[
\left( \bigcup_{0 \leq j < i} \overline{A}_j \right) \cap \overline{A}_i \subseteq \bigcap_{R \in R} \forall R_\delta A_i.
\]

For example, if the bits \( A_0, \ldots, A_i \) are all true in the current tape cell, then in the successor cell these bits are all false.

We thus have described a sequence of configurations where we can address single tape cells in all the configurations using the \( A \)-counter. The latter restarts every time it has reached \( 2^k - 1 \), and thus with each new configuration.

The counter is initialized with value 0 at \( a \). Hence, all elements representing the first tape cell in some configuration in the tree satisfy the auxiliary concept name \([A = 0]\), defined as follows:

\[
[A = 0] \equiv \overline{A}_0 \cap \ldots \cap \overline{A}_{k-1}.
\]

Below, we use additional concept names of the form \([A = i]\), for (polynomially many) different values \( i \), which we assume to be defined similarly. Moreover, we assume that \([A \neq i]\) is defined as the negation of \([A = i]\) as described in Lemma 7.1.

We further add the assertion

\[
[A = 0](a)
\]

to \( A_0 \). Since the names \( \overline{A}_0, \ldots, \overline{A}_{k-1} \) are rigid, this assertion must be satisfied at every time point.

We now enforce basic conditions which help to ensure that the tree actually represents a successful computation of \( M \) on \( w \). To formulate these conditions, we use the rigid concept name \( H \) to identify the tape cells that are to the right of the head:

\[
\left( H \cup \bigcup_{q \in Q} \right) \cap [A \neq 2^k - 1] \subseteq \bigcap_{R \in R} \forall R_\delta H.
\]

Thus, the propagation stops at tree levels whose elements represent the last cell in a configuration, since these elements satisfy \([A = 2^k - 1]\).

There is only one head position per configuration:

\[
H \subseteq \bigcap_{q \in Q} \overline{q}.
\]
Note that we do not have to consider the elements representing the cells left to the head since, if such a cell satisfies a concept name from $Q$, then all its successors in the tree are enforced to satisfy $H$.

Each tape cell is associated with at most one state (which, at the same time, represents the position of the head):

$$\top \subseteq \bigcap_{q_1, q_2 \in Q, q_1 \neq q_2} \mathcal{P}_1 \cup \mathcal{P}_2.$$  

Each tape cell contains exactly one symbol:

$$\top \subseteq \bigcup_{\sigma \in \Gamma} \left( \sigma \cap \bigcap_{\sigma' \in \Gamma \setminus \{\sigma\}} \neg \sigma' \right).$$

Before specifying the remaining, more intricate conditions for the synchronization of the configurations, we describe the first configuration in the tree (starting at $a$) as the initial configuration.

In particular, we mark the corresponding elements by adding the assertion $I(a)$ to $A_0$ and by propagating the concept alongside the first configuration as follows:

$$I \cap [A \neq 2^k - 1] \subseteq \bigcap_{R_\delta \in R} \forall R_\delta.$$

The first configuration is modeled by adding the assertion $q_0(a)$ to $A_0$ and by considering the following CIs for all $i \in [0, k - 1]$:

$$I \cap [A = i] \subseteq \sigma_i,$$

$$I \cap [A = k] \subseteq B,$$

$$I \cap B \cap [A \neq 2^k - 1] \subseteq \bigcap_{R_\delta \in R} \forall R_\delta.$$

where $w = \sigma_0 \ldots \sigma_{k-1}$ is the input word.

We finally come to the most involved part, the synchronization of the configurations, which includes the modeling of the transitions.

We first introduce the $A'$-counter, which is incremented along the temporal dimension. For every possible value of this counter, there is a time point where $a$ belongs to the concepts from the corresponding subset of $\{A'_0, \ldots, A'_{k-1}\}$. This is expressed using the following conjunct of $\Phi_M$:

$$\Box_F \bigwedge_{0 \leq i < k} \left( \bigwedge_{0 \leq j < i} A'_j(a) \leftrightarrow (A'_i(a) \leftrightarrow \Box_F \neg A'_i(a)) \right).$$

This formula expresses that the $i$-th bit of the $A'$-counter is flipped from one world to the next iff all preceding bits are true. Thus, the value of the $A'$-counter at the next world is equal to the value at the current world incremented by one.

Note that it is not necessary to initialize this counter to 0 in $A_0$; we only need to know that all possible counter values are represented at some time point.

The value of the $A'$-counter is always shared by all individuals:

$$\Box_F \left( \bigwedge_{0 \leq i < k} \exists x A'_i(x) \rightarrow \neg \exists x \overline{A}_i(x) \right).$$
For the application of the \(A'\)-counter, we introduce the abbreviation \([A = A']\) describing the equality of the two counters:

\[
[A_i = A' + i] \equiv (A_i \cap A'_i) \sqcup (\overline{A_i} \cap \overline{A'_i}),
\]
\[
[A = A'] \equiv \bigcap_{0 \leq i < k} [A_i = A_i'].
\]

Furthermore, we define similar abbreviations as follows (note that we consider addition modulo \(2^k\)):

\[
[A = A' + 1] \equiv \bigcap_{0 \leq j < k} (A_j' \cap \overline{A_j}) \sqcup \\
\bigcup_{0 \leq j < k} \left( \bigcap_{0 \leq j < i} (A_j' \cap \overline{A_j}) \cap A_i \cap \bigcap_{i+1 \leq j < k} [A_j = A'_j] \right),
\]
\[
[A = A' + 2] \equiv [A_0 = A'_0] \sqcap \left( \bigcap_{1 \leq j < i} (A_j' \cap \overline{A_j}) \sqcup \\
\bigcup_{1 \leq j < k} \left( \bigcap_{1 \leq j < i} (A_j' \cap \overline{A_j}) \cap A_i \cap \bigcap_{i+1 \leq j < k} [A_j = A'_j] \right) \right).
\]

We now can use the temporal dimension to propagate information from one level of the tree to the next one as outlined above, and hence specify the transitions.

Symbols not under the head are copied:

\[
\sigma \sqcap \bigcap_{q \in Q} [A = A'] \sqsubseteq \bigcap_{R \in \mathcal{R}} \forall R. \sigma',
\]
\[
\sigma' \sqcap [A \neq A'] \sqsubseteq \bigcap_{R \in \mathcal{R}} \forall R. \sigma',
\]
\[
\sigma' \sqcap [A = A'] \sqsubseteq \sigma.
\]

To describe the transitions, we explicitly store chosen transitions with the help of the rigid concepts \(T_{p, q, M}\), by enforcing them to be satisfied by the elements representing the cells directly right-neighbored to the head position. Recall that there may be several such cells; we are now at the point where we specify the branching of the tree. Hence, we model the transitions for all \(q \in Q\) and \(\sigma \in \Gamma\) using the following CIs:

\[
q \sqcap \sigma \sqsubseteq \bigcup_{\delta \in \Delta(q, \sigma)} \exists R_\delta, \text{ if } q \in Q_3,
\]
\[
q \sqcap \sigma \sqsubseteq \bigcap_{\delta \in \Delta(q, \sigma)} \exists R_\delta, \text{ if } q \in Q_4,
\]
\[
q \sqcap \sigma \sqsubseteq \bigcap_{\delta \in \Delta(q, \sigma)} \forall R_\delta, \forall T_\delta.
\]

Observe that our main adaptation of the proof in \([13]\) is that we, instead of considering a single role \(R\), deal with all those in \(\mathcal{R}\) and, instead of considering one \(R\)-successor per
successor configuration $\delta$, consider an $R_\delta$-successor. This enables us to simulate the qualified existential restriction of the form $\prod_{\delta \in \Delta} \exists R_\delta$ on the right-hand side of a CI in the original proof, via the last two of the above CIs.

The (possible) replacement of the symbols under the head is described with the help of the transition concepts $T_{q, \sigma, M}$ for all $q \in Q$, $\sigma \in \Gamma$, and $M \in \{L, R\}$:

$$T_{q, \sigma, M} \cap [A = A'] \subseteq \bigcap_{R \in \mathcal{R}} \forall R \Delta \sigma'. $$

Recall that the transition concepts are only enforced to hold at the cell to the right of the current head position (hence the $+1$).

The state information is similarly propagated for all $q \in Q$ and $\sigma \in \Gamma$ as follows:

$$T_{q, \sigma, L} \cap [A = A'] \subseteq \bigcap_{R \in \mathcal{R}} \forall R \Delta \sigma', $$

$$T_{q, \sigma, R} \cap [A = A' + 2] \subseteq \bigcap_{R \in \mathcal{R}} \forall R \Delta \sigma', $$

$$q' \cap [A \neq A'] \subseteq \bigcap_{R \in \mathcal{R}} \forall R \Delta \sigma', $$

$$q' \cap [A = A'] \subseteq q. $$

We finally enforce the computation to be an accepting one by disallowing the state $q_r$ entirely using the CI $q_r \subseteq \bot$. Note that this is correct since we assume all the computations of $\mathcal{M}$ to be terminating. This finishes the definition of the Boolean TCQ $\Phi_{2R, w}$ and the global ontology $O_{2R, w}$, which consist of the conjuncts and CIs specified above. We further collect all assertions in the ABox $A_0$. Given our descriptions above, it is easy to see that the size of $\Phi_{2R, w}, O_{2R, w}$, and $A_0$ is polynomial in $k$. Moreover, it can readily be checked that our constructions are equivalent to those in the proof of [13, Thm. 4.1]. Hence, $\Phi_{2R, w}$ is satisfiable w.r.t. $(O_{2R, w}, (A_0))$ if $\mathcal{M}$ accepts $w$.

**Lemma 7.14.** $W$ is r-satisfiable w.r.t. $\lambda$ and $K$ iff there is a tuple $t$ as above such that, for all $i \in [0, n + k]$:  

(C1') $K_{tc}^i := \langle O, A^+_i \cup A^i \rangle$ is consistent.

(C2') For all $p_j \in W_{\lambda(i)}$, we have $K_{tc}^i \not\models \varphi_j$.

**Proof.** ($\Rightarrow$) Let $J_0, \ldots, J_{n+k}$ be the interpretations over the common domain $\Delta$ that exist by the r-satisfiability of $W$ w.r.t. $\lambda$. As in the proof of Lemma 3.6 we can assume that they interpret the individual names in $I^{\mu*}$ in such a way that each $J_i$ satisfies $A_{\mu*}(\Delta)$. We can thus already fix the ABox types $A^+_i := \langle \alpha | J_i \models \alpha \rangle$ for all $i \in [0, n + k]$, where $\alpha$ ranges over all assertions formulated over $I(\lambda) \cup I^{\mu*}$, $C(\lambda)$, and $P(\lambda)$ (see Definition 7.10). To find the types $T_{\lambda}^i$, we first unravel the interpretations $J_i$ into tree-shaped models $J_i^\ell$. However, in contrast to classical (atemporal) unraveling techniques, we need to construct a common domain for all time points, and hence need to unravel the interpretations $J_0, \ldots, J_{n+k}$ simultaneously.

For this purpose, we iteratively extend the interpretations in a sequence of interpretations $J_{0}^\ell, \ldots, J_{n+k}^\ell$ over a common domain $\Delta^\ell$, for increasing $j \geq 0$. At each step $j$ of this construction, we also maintain a function $g: \Delta^\ell \to \Delta$ that maps the domain elements of the tree-shaped interpretations to the “original” domain elements from $\Delta$, such that, for all $d \in \Delta^\ell$ and $i \in [0, n + k]$, $d$ satisfies the same basic concepts in $J_i^\ell$ as $g(d)$ does in the
original interpretation $\mathcal{J}_i$. We start with the domain $\Delta^0 := \{u_a \mid a \in \mathbf{I}(\mathcal{K}) \cup \mathbf{I}^{\text{aux}}\}$, the function $\mathcal{g}$ that maps each $u_a$ to $a^{\mathcal{g}},$ and the interpretations $\mathcal{J}_i^0$, $i \in [0, n+k]$, defined such that $a^{\mathcal{g}^0} = u_a$ for all $a \in \mathbf{I}(\mathcal{K}) \cup \mathbf{I}^{\text{aux}}$, and otherwise uniquely determined by $\mathcal{A}_j^+$. However, the elements $u_a$ may not yet satisfy all existential restrictions in $\mathcal{A}_j^+$. At each index $j > 0$ and for each domain element $u_a \in \Delta^j$ added in the previous step, we therefore introduce a number of fresh (i) flexible role successors $u_{\rho R}$ for all $i \in [0, n+k]$ and $R \in \mathbf{R}^\mathcal{K} (\mathcal{O})$, and (ii) rigid role successors $u_{\rho R}$ for $R \in \mathbf{R}^\mathcal{K} (\mathcal{O})$ if $\mathcal{g}(u_{\rho R}) \in (\exists R)\mathcal{J}_i^0$. Then, there must also be a $d \in \Delta$ such that $(\mathcal{g}(u_{\rho R}), d) \in R^{\mathcal{J}_i^0}$, and we can add $u_{\rho R}$ to $\Delta^j+1$ and set $\mathcal{g}(u_{\rho R}) := d$. We also add the pair $(u_{\rho R}, u_{\rho R})$ to all $S^{\mathcal{J}_j^+}$ for which we have $(\mathcal{g}(u_{\rho R}), d) \in S^{\mathcal{J}_j^0}$, for all $d' \in [0, n+k]$; hence, $u_{\rho R} \in (\exists R)^{\mathcal{J}_j^+}$, meaning that the existential restriction at $u_{\rho R}$ is satisfied. We further interpret the basic concepts on $u_{\rho R}$ in $\mathcal{J}_j^+1$ as on $d$ in $\mathcal{J}_j$ (p ending the introduction of role successors for $u_{\rho R}$). For all rigid roles $R \in \mathbf{R}^\mathcal{K} (\mathcal{O})$ (case (ii)), we proceed in the same way, but only choose one successor $u_{\rho R}$ for all time points. The unraveled interpretations $\mathcal{J}_j^*$, $i \in [0, n+k]$, over the domain $\Delta^*$ are obtained as the limit of this construction. It is easy to show that these interpretations still satisfy all the properties required for Definition 2.10 in particular, none of the negative CQ literals in the conjunctions $\chi_{\lambda(i)}$ can become violated by unraveling. Moreover, by construction, each $\mathcal{J}_j^*$ still satisfies $\mathcal{A}_j^*$. We denote by $\mathcal{A}_j^*$ the subtree of $\Delta^*$ starting at $u_a$, i.e., the set of all domain elements of the form $u_{\rho R}$, and by $\mathcal{J}_j^*|_a$ the interpretation $\mathcal{J}_j^*$ restricted to this subtree.

We now extract the types $\Sigma_i^* := \Sigma(\mathcal{J}_j^*|_a)$ of these subtrees, which are triples of the form $(\mathbf{B}_i^*, \mathbf{Q}_i^*, \mathbf{M}_i^*)$, where

- $\mathbf{B}_i^*$ contains exactly those (negated) basic concepts $(-)B$ for which $\mathcal{J}_j^*|_a = (-)B(a)$,
- $\mathbf{Q}_i^*$ contains all CQs $\phi \in \mathcal{Q}_a$ that are satisfied in $\mathcal{J}_j^*|_a$, and
- $\mathbf{M}_i^*$ contains all $S \subseteq \mathcal{T}(\phi)$ for which there is a partial homomorphism $\pi$ of $\phi$ into $\mathcal{J}_j^*|_a$ with $u_a \in \text{range}(\pi)$ and $S = \text{domain}(\pi)$.

We thus immediately obtain the temporal types $\tau_i^* = \{\Sigma_i^* \mid i \in [0, n+k]\}$ for all individual names $a \in \mathbf{I}(\mathcal{K}) \cup \mathbf{I}^{\text{aux}}$.

The next step is to make the structure of the models independent of the time points $i \in [0, n+k]$, by grouping the time points according to the types $\Sigma_i^*$, of which there are constantly many (for a fixed $a$). Hence, we define a mapping $f_a : \tau_i^* \to [0, n+k]$ that arbitrarily chooses, for each $\Sigma \in \tau_i^*$, one representative time point $f_a(\Sigma)$ such that $\Sigma = \Sigma_{f_a(\Sigma)}$. We now replace each subtree $\mathcal{J}_j^*|_a$ in $\mathcal{J}_j^*$ by the subtree $\mathcal{J}_{f_a(\Sigma)}^*|_a$ that is representative for the type $\Sigma_i^*$; that is, we interpret all concept (role) names on (pairs of) elements from $\Delta^*$ in the same way as at time point $f_a(\Sigma)$. These changes are “local” to each individual name $a$, i.e., the other individual names $b$ do not affect the subtrees below $a$. We denote the resulting interpretations by $\mathcal{J}_j^\Sigma_i^*$, $i \in [0, n+k]$, which have the same domain $\Delta^\Sigma = \Delta^*$ as before. It is easy to show that the types of the time points remain the same as in $\mathcal{J}_j^*$, i.e., we have $\Sigma(\mathcal{J}_j^\Sigma_i^*) = \Sigma(\mathcal{J}_j^*|_a)$. Moreover, $\mathcal{O}$ and $\mathcal{A}_j^*$ clearly remain satisfied at each time point $i \in [0, n+k]$, which implies that the positive CQ literals in $\chi_{\lambda(i)}$ are also satisfied. For a negative CQ literal $-\phi$ in $\chi_{\lambda(i)}$, assume that it becomes satisfied when replacing $\mathcal{J}_j^*|_a$ with $\mathcal{J}_j^\Sigma_i^*$ with $\mathcal{J}_{f_a(\Sigma)}^*|_a$. Since $i$ and $f_a(\Sigma_i^*)$ have the same type $\Sigma_i^*$, and in particular agree on the second component $\mathbf{Q}_i^*$ of that type, the homomorphism that maps $\phi$ into $\mathcal{J}_j^\Sigma_i^*$ must map some terms to elements outside of $\Delta^\Sigma|_a$. However, this also means that the set of all terms of $\phi$ that are mapped by $\pi$ into $\Delta^\Sigma|_a$ is included in the third component of the type, $\mathbf{M}_i^*$. Thus, there is a similar partial homomorphism $\pi'$ also into $\mathcal{J}_j^\Sigma_i^*$, which can be merged with the remainder
of $\pi$ to obtain a homomorphism of $\varphi$ into $J_i^\ast$, contradicting the fact that this interpretation satisfies $\mathcal{L}_i(\iota)$.

We can now remove the domain elements $u_{ap}\mathbf{R}$ (and all their successors) that refer to any time point $i$ that is not contained in $\text{range}(f_a)$, since the domain elements are introduced only to satisfy existential restrictions at $i$, which are now replaced by those of time point $f_a(T_{i}^{\tau})$.

We further assume that all time points $i$ occurring in the names of the remaining domain elements from $\Delta^\tau$ are replaced by the corresponding types $T_i^\tau$, e.g., $u_{ap}\mathbf{R^\tau}$ is replaced by $u_{ap}\mathbf{R^i}$.

Hence, our domain already has the form required for the tree ABoxes in Definition 7.12. The only remaining obstacle to obtain models of $K_{\mathcal{T}}$ that we can use for $[C^1\mathcal{T}]$ and $[C^2\mathcal{T}]$ is the fact that different named elements $a$ and $b$ having the same temporal type $T_a^\tau = T_b^\tau$ may still have non-isomorphic subtrees $J_i^{|a}$ and $J_i^{|b}$ of unnamed successors.

We follow a similar approach as above, and take a partial function $h: T \rightarrow I(\mathcal{K}) \cup \mathbf{P}^{aux}$ that maps each temporal type $\tau \in T$ to an arbitrary individual name $a$ with $T_a^\tau = \tau$, if such an individual exists, and that is undefined on all other temporal types (i.e., those that are not realized by any individual name in our interpretations). We now replace all subtrees $\Delta_i^{|a}$ by $\Delta_i^{|h(T_a^\tau)|a}$, which denotes the set obtained from $\Delta_i^{|h(T_a^\tau)|}$ by replacing the name $h(T_a^\tau)$ with $a$, i.e., each $u_{ap}(T_a^\tau)|a$ becomes $u_{ap}$.

We denote the resulting domain by $\Delta^h$. Correspondingly, we interpret the concept and role names on $\Delta^i|a = \Delta^i{|h(T_a^\tau)|a}$ in corresponding interpretations $J_i^{|a}$ as on $\Delta_i^{|h(T_a^\tau)|}$ in $J_i^{|h(T_a^\tau)|}$, where $j = f_a(T_a^\tau)(T_j^\tau)$ is the time point corresponding to $T_j^\tau$ in the subtree belonging to the individual name $h(T_a^\tau)$; this is well-defined since $a$ and $h(T_a^\tau)$ have the same temporal type. For example, we have $(u_{aR}\mathbf{R})(x) \in S^j_i$ if $(u_{h(T_a^\tau)}R\mathbf{R}(x)) \in S^j_i$ etc. We first show that the types remain the same, i.e., we have $\mathcal{I}(J_i^{|a}) = \mathcal{I}(J_i^{|b})$ for all $a \in I(\mathcal{K}) \cup \mathbf{P}^{aux}$ and $i \in [0, n + k]$. Since $J_i^{|a} = J_i^{|h(T_a^\tau)|a}$, where $j$ is defined as above, and $\mathcal{I}(J_i^{|h(T_a^\tau)|}) = \mathcal{I}(J_i^{|a})$, it is clear that the basic type remains the same. Moreover, any (partial) homomorphism $\pi$ of some $\varphi \in Q_\mathcal{F}$ into $J_i^{|a}$ that does not include the individual name $h(T_a^\tau)$ in its domain is clearly valid also in $J_i^{|a}$ (after renaming), and vice versa. But if $\pi$ does refer to $h(T_a^\tau)$ (or $a$), then it follows from $\mathcal{I}(J_i^{|h(T_a^\tau)|}) = \mathcal{I}(J_i^{|a})$ that $a = h(T_a^\tau)$, since the individual name must be mentioned explicitly in either the second or third component of that type. In this case, it follows that $J_i^{|a} = J_i^{|a}$, and the claim trivially holds. It is easy to show that the interpretations $J_i^{|a}$ still satisfy $O$ and $A_i^\tau$. The fact that $J_i^{|a} \models \mathcal{L}_i(\iota)$ follows as for $J_i^{|a}$ from what we have shown above, namely that the types remain the same.

Our interpretations now have the following properties. For each temporal type $\tau \in T$ that is realized in $J_i^{|0}, \ldots, J_i^{|n+k}$, there is a representative individual name $a = h(\tau)$ such that $T_a^\tau = \tau$ and all other individual names $b$ with the same temporal types have subtrees $\Delta_i^{|b}$ that behave in the same way as $\Delta_i^{|a}$ (after reshuffling the time points to match the types, using $f_a$). Similarly, for each type $\xi \in \tau$, there is a representative time point $i = f_a(\xi)$ such that $T_i^\tau = \xi$ and all other time points $j$ with the same type have subtrees $J_j^{|a}$ that are isomorphic to $J_j^{|a}$. Hence, we can define the complete tree ABoxes $(A_i^\tau)_{\tau \in \mathcal{T}}$ by introducing a new individual name $u_\mathcal{F}$ for each $u_{ap}$ that appears in $J_i^{|a}$, and collecting all assertions about these individual names that hold in $J_i^{|a}$—until the termination condition (19) of Definition 7.13 is satisfied. It is easy to show that $(A_i^\tau)_{\tau \in \mathcal{T}}$ are actually tree ABoxes for $\tau$ according to Definition 7.12. Moreover, we have constructed $J_i^{|a}$, $i \in [0, n + k]$, in such a way that they satisfy $O$, $A_i^\tau$, $A_i^0$, and $\mathcal{L}_i(\iota)$, which means that $[C^1\mathcal{T}]$ and $[C^2\mathcal{T}]$ are satisfied.

($\Rightarrow$) Let $t$ be a tuple for which $[C^1\mathcal{T}]$ and $[C^2\mathcal{T}]$ are satisfied. We iteratively construct tree-shaped interpretations $J_0, \ldots, J_{n+k}$ to satisfy Definition 7.10. To start, we define these
interpretations up to the point where they are uniquely determined by the ABox types $A^+_i$ and the renamed complete tree ABoxes in $A^+_i$. It remains to extend these interpretations to actual (possibly infinite) models of $O$, $A_i$, and $X(A(i))$. The only thing we have to be careful about in this process is to not accidentally satisfy some CQ that occurs negatively in $X(A(i))$. Hence, consider a path from a root $a$ to a leaf $u_{ag}$, in these tree ABoxes, for which $u_{ag}$ is lacking some necessary role successors. By $(T9)$, we know that there exist an ancestor $u_{ag'}$ of $u_{ag}$ such that $\rho = \rho'\sigma$ with $|\sigma| = d$, and an ancestor $u_{ag''}$ of $u_{ag'}$ such that the subtree of depth $d = \max\{|\mathcal{T}(\varphi)|, \varphi \in Q_p\}$ below $u_{ag''}$ is isomorphic to the subtree of depth $d$ below $u_{ag'}$ (in case that $\rho'' = e$, the element $u_{ag''}$ is actually equal to $a$). We now extend the subtree below $u_{ag'}$ by a copy of the subtree below $u_{ag''}$, which in particular introduces the required role successors of $u_{ag}$. Note that no existing assertions are replaced by this operation, due to the requirements that the subtrees are isomorphic up to depth $d$, and the fact that $u_{ag}$ does not contain any successors beyond depth $d$, by $(T9)$. We can continue this process indefinitely to obtain the desired models. Moreover, a CQ $\varphi$ that occurs negatively in $X(A(i))$ can never become satisfied by the copied domain elements, because it could only be mapped into a subtree of maximal depth $v$, which, by our construction, must have an isomorphic image in the original tree ABoxes in $A^+_i$, which is a contradiction to $(C2)$.

References

1. Ahmetaj, S., Ortiz, M., Simkus, M.: Polynomial datalog rewritings for expressive description logics with closed predicates. In: S. Kamath (ed.) Proc. of the 25th Int. Joint Conf. on Artificial Intelligence (IJCAI’16), pp. 878–885. AAAI Press (2016). URL https://www.ijcai.org/Abstract/16/129
2. Arora, S., Barak, B.: Computational Complexity - A Modern Approach. Cambridge University Press (2009)
3. Artale, A., Bresolin, D., Montani, A., Sciacicco, G., Ryzhikov, V.: DL-Lite and interval temporal logics: a marriage proposal. In: T. Schaub (ed.) Proc. of the 21st Eur. Conf. on Artificial Intelligence (ECAI’14), Frontiers in Artificial Intelligence and Applications, vol. 263, pp. 957–958. IOS Press (2014).
4. Artale, A., Calvanese, D., Kontchakov, R., Zakharyaschev, M.: The DL-Lite family and relations. Journal of Artificial Intelligence Research 36, 1–69 (2009). doi:10.1613/jair.2820
5. Artale, A., Franconi, E.: A temporal description logic for reasoning about actions and plans. Journal of Artificial Intelligence Research 9, 463–506 (1998). doi:10.1613/jair.516
6. Artale, A., Franconi, E.: A survey of temporal extensions of description logics. Annals of Mathematics and Artificial Intelligence 30(1-4), 171–210 (2000). doi:10.1023/A:1016636131405
7. Artale, A., Franconi, E.: Temporal description logics. In: M. Fisher, D.M. Gabbay, L. Vila (eds.) Handbook of Temporal Reasoning in Artificial Intelligence, pp. 375–388. Elsevier Science Inc. (2005)
8. Artale, A., Kontchakov, R., Kovunova, A., Ryzhikov, V., Wolter, F., Zakharyaschev, M.: First-order rewritability of ontology-mediated temporal queries. In: Q. Yang, M. Wooldridge (eds.) Proc. of the 24th Int. Joint Conf. on Artificial Intelligence (IJCAI’15), pp. 2706–2712. AAAI Press (2015). URL http://ijcai.org/Abstract/15/383
9. Artale, A., Kontchakov, R., Kovunova, A., Ryzhikov, V., Wolter, F., Zakharyaschev, M.: Ontology-mediated query answering over temporal data: A survey. In: S. Schewe, T. Schneider, J. Wijsen (eds.) Proc. of the 24th Int. Symp. on Temporal Representation and Reasoning (TIME’17), Leibniz International Proceedings in Informatics, vol. 90, pp. 1:1–1:37. Schloss Dagstuhl–Leibniz-Zentrum für Informatik (2017). doi:10.4230/LIPIcs.TIME.2017.1
10. Artale, A., Kontchakov, R., Litj, C., Wolter, F., Zakharyaschev, M.: Temporalising tractable description logics. In: V. Goranko, X.S. Wang (eds.) Proc. of the 14th Int. Symp. on Temporal Representation and Reasoning (TIME’07), pp. 11–22. IEEE Press (2007). doi:10.1109/TIME.2007.62
11. Artale, A., Kontchakov, R., Ryzhikov, V., Zakharyaschev, M.: A cookbook for temporal conceptual data modelling with description logics. ACM Transactions on Computational Logic 15(3), 25 (2014). doi:10.1145/2592956
12. Artale, A., Kontchakov, R., Ryzhikov, V., Zakharyaschev, M.: Tractable interval temporal propositional and description logics. In: B. Bonet, S. Koenig (eds.) Proc. of the 29th AAAI Conf. on Artificial Intelligence (AAAI’15), pp. 1417–1423. AAAI Press (2015). URL https://www.aaai.org/ocs/index.php/AAAI/AAAI15/paper/view/9630
Temporal Conjunctive Query Answering in the Extended DL-Lite Family

Intelligence, and Reasoning (LPAR'13), Lecture Notes in Computer Science, vol. 8312, pp. 536–551. Springer-Verlag (2013). doi:10.1007/978-3-642-45221-5_36

Klarman, S., Meyer, T.: Complexity of temporal query abduction in DL-Lite. In: M. Bienvenu, M. Ortiz, R. Rosati, M. Simkus (eds.) Proc. of the 27th Int. Workshop on Description Logics (DL’14), CEUR Workshop Proceedings, vol. 1193, pp. 233–244 (2014). URL http://www.ceur-ws.org/Vol-1193/paper_45.pdf

Klarman, S., Meyer, T.: Querying temporal databases via OWL 2 QL. In: R. Kontchakov, M. Mugnier (eds.) Proc. of the 8th Int. Conf. on Web Reasoning and Rule Systems (RR’14), Lecture Notes in Computer Science, vol. 8741, pp. 92–107. Springer-Verlag (2014). doi:10.1007/978-3-319-11113-1_7

Kontchakov, R., Lutz, C., Toman, D., Wolter, F., Zakharyaschev, M.: The combined approach to query answering in DL-Lite. In: F. Lin, U. Sattler, M. Truszczynski (eds.) Proc. of the 12th Int. Conf. of Knowledge Representation and Reasoning (KR’10), pp. 247–257. AAAI Press (2010). URL http://aaai.org/ocs/index.php/KR/KR2010/paper/view/1282

Kontchakov, R., Toman, D., Wolter, F., Zakharyaschev, M.: The combined approach to ontology-based data access. In: T. Walsh (ed.) Proc. of the 22nd Int. Joint Conf. on Artificial Intelligence (IJCAI’11), pp. 2656–2661. AAAI Press (2011). doi:10.5591/978-1-57735-516-8/IJCAI11-442

Kontchakov, R., Pandofo, L., Pulina, L., Ryzhikov, V., Zakharyaschev, M.: Temporal and spatial OBDA with many-dimensional Halpern-Shoham logic. In: S. Kambhampati (ed.) Proc. of the 23rd Int. Joint Conf. on Artificial Intelligence (IJCAI’16), pp. 1160–1166. AAAI Press (2016). URL http://ijcai.org/Abstract/16/168

Kroetzsch, M., Rudolph, S., Hitzler, P.: Complexities of Horn description logics. ACM Transactions on Computational Logic 14(1), 2:1–2:36 (2013). doi:10.1145/2422085.2422087

Kurucz, A., Wolter, F., Zakharyaschev, M., Gabbay, D.M.: Many-Dimensional Modal Logics: Theory and Applications, vol. 148. Gulf Professional Publishing (2003)

Lippmann, M.: Temporalised description logics for monitoring partially observable events. PhD thesis, Technische Universität Dresden (2014). URL http://nbn-resolving.de/urn:nbn:de:bsz:14-qucosa-147977

Lutz, C.: Interval-based temporal reasoning with general TBoxes. In: B. Nebel (ed.) Proc. of the 17th Int. Joint Conf. on Artificial Intelligence (IJCAI’01), pp. 89–96. Morgan Kaufmann (2001). URL https://www.ijcai.org/proceedings/2001-1

Lutz, C.: The complexity of conjunctive query answering in expressive description logics. In: A. Armando, P. Baumgartner, G. Dowek (eds.) Proc. of the 4th Int. Joint Conf. on Automated Reasoning (IJCAR’08), Lecture Notes in Artificial Intelligence, vol. 5195, pp. 179–193. Springer-Verlag (2008). doi:10.1007/978-3-540-70760-7_16

Lutz, C., Seylan, I., Wolter, F.: Ontology-based data access with closed predicates is inherently intractable (sometimes). In: F. Rossi (ed.) Proc. of the 23rd Int. Joint Conf. on Artificial Intelligence (IJCAI’13), pp. 1024–1030. AAAI Press (2013). URL https://www.ijcai.org/Abstract/13/156

Lutz, C., Wolter, F., Zakharyaschev, M.: Temporal description logics: A survey. In: S. Demri, C.S. Jensen (eds.) Proc. of the 15th Int. Symp. on Temporal Representation and Reasoning (TIME’08), pp. 3–14. IEEE Press (2008). doi:10.1109/TIME.2008.14

Mix Barrington, D.A., Immerman, N., Straubing, H.: On uniformity within NC1. Journal of Computer and System Sciences 43(3), 274–306 (1990). doi:10.1016/0022-0000(90)90022-D

Motik, B., Cuenca Grau, B., Horrocks, I., Wu, Z., Fokoue, A., Lutz, C. (eds.): OWL 2 Web Ontology Language: Profiles (Second Edition). W3C Recommendation 11 December 2012 (2012). Available at http://www.w3.org/TR/owl2-profiles/

Ortiz, M., Calvanese, D., Eiter, T.: Data complexity of query answering in expressive description logics via tableaux. Journal of Automated Reasoning 41(1), 61–98 (2008). doi:10.1007/s10817-008-9102-9

Pnueli, A.: The temporal logic of programs. In: Proc. of the 18th Annual Symp. on Foundations of Computer Science (FOCS’77), pp. 46–57. IEEE Press (1977). doi:10.1109/SFCS.1977.32

Poggi, A., Calvanese, D., De Giacomo, G., Lembo, D., Lenzerini, M., Rosati, R.: Linking data to ontologies. Journal on Data Semantics 10, 133–173 (2008). doi:10.1007/978-3-540-77688-8_5

Rosati, R.: On the limits of querying ontologies. In: T. Schewnitck, D. Suciu (eds.) Proc. of the 11th Int. Conf. on Database Theory (ICDT’07), Lecture Notes in Computer Science, vol. 4353, pp. 164–178. Springer-Verlag (2007). doi:10.1007/11965893_12

Rosati, R.: On conjunctive query answering in EL. In: D. Calvanese, E. Franconi, V. Haarslev, D. Lembo, B. Motik, A.Y. Turhan, S. Tessaris (eds.) Proc. of the 2007 Int. Workshop on Description Logics (DL’07), CEUR Workshop Proceedings, vol. 250, pp. 451–458 (2007). URL http://ceur-ws.org/Vol-250/paper_83.pdf
71. Savitch, W.J.: Relationships between nondeterministic and deterministic tape complexities. Journal of Computer and System Sciences 4(2), 177–192 (1970). doi:10.1016/S0022-0000(70)80006-X
72. Schild, K.: Combining terminological logics with tense logic. In: Proc. of the 6th Portuguese Conf. on Artificial Intelligence: Progress in Artificial Intelligence (EPIA’93), pp. 105–120. Springer-Verlag (1993). doi:10.1007/3-540-57287-2_41
73. Sistla, A.P., Clarke, E.M.: The complexity of propositional linear temporal logics. Journal of the ACM 32(3), 733–749 (1985). doi:10.1145/3828.3837
74. Thost, V.: News on temporal conjunctive queries. In: D. Dell’Aglio, D. Anicic, P. Barnaghi, E. Della Valle, D.L. McGuinness, L. Bozzato, T. Eiter, M. Horváth, D. Porrello (eds.) Joint Proc. of the Web Stream Processing workshop (WSP’17) and the 2nd Int. Workshop on Ontology Modularity, Contextuality, and Evolution (WOMoCoE’17). CEUR Workshop Proceedings, vol. 1936, pp. 1–16 (2017). URL http://ceur-ws.org/Vol-1936/paper-01.pdf
75. Thost, V.: Using ontology-based data access to enable context recognition in the presence of incomplete information. Ph.D. thesis, TU Dresden (2017). URL http://nbn-resolving.de/urn:nbn:de:bsz:14-qucosa-227633
76. Thost, V.: Metric temporal extensions of DL-Lite and interval-rigid names. In: F. Wolter, M. Thielscher, F. Toni (eds.) Proc. of the 16th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR’18), pp. 665–666. AAAI Press (2018). URL https://aaai.org/ocs/index.php/KR/KR18/paper/view/18035 Short paper.
77. Thost, V.; Holste, J.; Özçep, Ö.L.: On implementing temporal query answering in DL-Lite (extended abstract). In: D. Calvanese, B. Konev (eds.) Proc. of the 28th Int. Workshop on Description Logics (DL’15). CEUR Workshop Proceedings, vol. 1350, pp. 552–555 (2015). URL http://ceur-ws.org/Vol-1350/paper-63.pdf
78. Vardi, M.Y.: The complexity of relational query languages (extended abstract). In: H.R. Lewis, B.B. Simons, W.A. Burkhard, L.H. Landweber (eds.) Proc. of the 14th Annual ACM Symposium on Theory of Computing, pp. 137–146, ACM (1982). doi:10.1145/800070.802186
79. Vardi, M.Y., Wolper, P.: Automata-theoretic techniques for modal logics of programs. Journal of Computer and System Sciences 32(2), 183–221 (1986). doi:10.1016/0022-0000(86)90026-7