Fibre bundle formulation of relativistic quantum mechanics

I. Time-dependent approach

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Abstract

We propose a new fibre bundle formulation of the mathematical base of relativistic quantum mechanics. At the present stage the bundle form of the theory is equivalent to its conventional one, but it admits new types of generalizations in different directions.

In the present first part of our investigation we consider the time-dependent or Hamiltonian approach to bundle description of relativistic quantum mechanics. In it the wavefunctions are replaced with (state) liftings of paths or sections along paths of a suitably chosen vector bundle over space-time whose (standard) fibre is the space of the wavefunctions. Now the quantum evolution is described as a linear transportation (by means of the evolution transport along paths in the space-time) of the state liftings/sections in the (total) bundle space. The equations of these transportations turn to be the bundle versions of the corresponding relativistic wave equations.
1. Introduction

This investigation is devoted to the fibre bundle (re)formulation of the (mathematical) base of relativistic quantum mechanics. It can be regarded as a direct continuation of [1–7] where a full self-consistent fibre bundle version of non-relativistic quantum mechanics is elaborated. Many ideas of these works can be transferred to the relativistic case but, as we shall see, they are not enough for the above aim. For example, the right mathematical concept reflecting the peculiarities of relativistic quantum evolution is the (linear) transport along the identity map of the space-time, on the contrary to a (linear) transport along paths (in space-time) in the non-relativistic case. A new problem is the fact that some relativistic wave equations are linear partial differential equations of second order; so for them the methods developed for the treatment of Schrödinger equation can not be applied directly.

In the bundle description the relativistic wavefunctions are replaced by (state) liftings of paths or sections along paths of a suitable vector bundle with the space-time as a base and whose (standard, typical) fibre is the space where the corresponding conventional wavefunctions (or some combinations of them and their partial derivatives) ‘live’. The operators acting on this space are replaced with appropriate liftings of paths or morphisms along paths and the evolution of the wavefunctions is now described as a linear transport (along the identity map of the space-time or along paths in it) of the state liftings/sections.

The present first part of our work is devoted to the time-dependent or Hamiltonian bundle description of relativistic wave equations. This approach is a straightforward generalization of the bundle description of non-relativistic quantum mechanics to relativistic one.

The lay-out of the paper is the following.

A review of the bundle approach to non-relativistic quantum mechanics is presented in Sect. 2. It contains certain basic ideas and equations of [2–7] required as a starting point for the present work.

In Sect. 3 is developed a general scheme for fibre bundle treatment of Schrödinger-type partial differential equations. The method is also applicable to linear partial differential equations of higher orders. This is archived by transforming such an equation to a system of first-order linear partial differential equations (with respect to a new function) which, when written in a matrix form, is just the Schrödinger-like presentation of the initial equation. Of course, this procedure is not unique and its concrete realization depends on the physical problem under exploration.

In Sect. 4 is given a bundle description of Dirac equation. Now the corresponding vector bundle over the space-time has as a (standard) fibre the space of 4-spinors and may be called 4-spinor bundle. Since Dirac equation is of first order, it can be rewritten in Schrödinger-type form. By this reason
the formalism outlined in Sect. 3 is applied to it mutatis mutandis. In particular, here the state of a Dirac particle is described by a lifting of paths or section along paths of the 4-spinor bundle and equivalently rewrite Dirac equation as an equation for linear transportation of this lifting with respect to the corresponding Dirac evolution transport in this bundle.

Sect. 5 contains several procedures for transforming Klein-Gordon equation to Schrödinger-like one. After such a presentation is chosen, we can, analogously to Dirac case in Sect. 4, apply to it the methods of Sect. 2 and Sect. 3.

Comments on fibre bundle description of other relativistic wave equations are given in Sect. 6.

Sect. 7 closes the paper with some remarks.

2. Bundle nonrelativistic quantum mechanics (review)

In the series of papers 2–7 we have reformulated nonrelativistic quantum mechanics in terms of fibre bundles. The mathematical base for this was the Schrödinger equation

\[ i\hbar \frac{d\psi(t)}{dt} = \mathcal{H}(t)\psi(t), \tag{2.1} \]

where \( i \in \mathbb{C} \) is the imaginary unit, \( \hbar (= \frac{\hbar}{2\pi}) \) is the Plank constant divided by \( 2\pi \), \( \psi \) is the system’s state vector belonging to an appropriate Hilbert space \( \mathcal{F} \), and \( \mathcal{H} \) is the system’s Hamiltonian. Here \( t \) is the time considered as an independent variable (or parameter). If \( \psi \) is known for some initial moment \( t_0 \), the solution of (2.1) can be written as

\[ \psi(t) = \mathcal{U}(t, t_0)\psi(t_0) \tag{2.2} \]

where \( \mathcal{U} \) is the evolution operator of the system 3, chapter IV, sect. 3.2 (for details see 4).

In the bundle approach the system’s Hilbert space \( \mathcal{F} \) is replace with a Hilbert bundle \( (\mathcal{F}, \pi, M) \) with (total, fibre) bundle space \( \mathcal{F} \), projection \( \pi \), base \( M \), isomorphic fibres \( \mathcal{F}_x := \pi^{-1}(x), x \in M \), and (standard, typical) fibre coinciding with \( \mathcal{F} \). So, there exist isomorphisms \( l_x: \mathcal{F}_x \to \mathcal{F}, x \in M \).

In the present work the base \( M \) will be identified with the Minkowski space-time \( M^4 \) of special relativity.

In the Hilbert bundle description a state vector \( \psi \) and the Hamiltonian \( \mathcal{H} \) are replaced respectively by a state lifting of paths \( \Psi: \gamma \to \Psi_\gamma \) and the

\[ \Psi_\gamma = \pi^{-1}(\gamma), \gamma \in \gamma \]
bundle Hamiltonian (morphism along paths) $H: \gamma \to H_\gamma$, given by \cite{2, 3, 6}:

$$
\Psi_\gamma: t \to \Psi_\gamma(t) = l^{-1}_{\gamma(t)}(\psi(t)), \quad H_\gamma: t \to H_\gamma(t) = l^{-1}_{\gamma(t)} \circ H(t) \circ l_{\gamma(t)}
$$

(2.3)

where $\gamma: J \to M$, $J$ being an $\mathbb{R}$-interval, is the world line (path) of some (point-like) observer, $t \in J$, and $\circ$ denotes composition of maps.

The bundle analogue of the evolution operator $U$ is the evolution transport $U$ along paths, both connected by

$$
U_\gamma(t, s) = l^{-1}_{\gamma(t)} \circ U(t, s) \circ l_{\gamma(s)}: F_\gamma(s) \to F_\gamma(t), \quad s, t \in J,
$$

(2.4)

which governs the evolution of state liftings via (cf. (2.2))

$$
\Psi_\gamma(t) = U_\gamma(t, s)\Psi_\gamma(s), \quad s, t \in J.
$$

(2.5)

The bundle version of (2.1), the so-called bundle Schrödinger equation, is \cite{3, 7}

$$
D\Psi = 0.
$$

(2.6)

Here $D$ is the derivation along paths corresponding to $U$, viz (cf. \cite{3} or \cite{10, definition 4.1}; see also \cite{2, definition 3.4})

$$
D: \text{PLift}^1(E, \pi, B) \to \text{PLift}^0(E, \pi, B)
$$

where $\text{PLift}^k(F, \pi, M)$ is the set of $C^k$ liftings of paths from $M$ to $F$, and its action on a lifting $\lambda \in \text{PLift}^1(F, \pi, M)$ with $\lambda: \gamma \mapsto \lambda_\gamma$ is given via

$$
D^\gamma_s \lambda := \lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} \left[ U_\gamma(s, s + \varepsilon)\lambda_\gamma(s + \varepsilon) - \lambda_\gamma(s) \right] \right\}
$$

(2.7)

where $D^\gamma_s (\lambda) := ((D\lambda)(\gamma))(s) = (D\lambda)_\gamma(s)$.

If $\{e_a(\gamma(s))\}$, $s \in J$ is a basis in $F_\gamma(s)$, the explicit action of $D$ is \cite{10, proposition 4.2} (see also [3])

$$
D^\gamma_s \lambda = \left( \frac{d\lambda^a(s)}{ds} + \Gamma^a_b(s; \gamma)\lambda^b_\gamma(s) \right) e_a(\gamma(s)).
$$

(2.8)

Here the coefficients $\Gamma^b_a(s; \gamma)$ of $U$ are defined by

$$
\Gamma^b_a(s; \gamma) := \left. \frac{\partial (U_\gamma(s, t))^b_a}{\partial t} \right|_{t=s} = -\left. \frac{\partial (U_\gamma(t, s))^b_a}{\partial t} \right|_{t=s}
$$

(2.9)

where $(U_\gamma(s, t))^b_a$ are given via $U(t, s)e_a(\gamma(s)) =: \sum_b (U_\gamma(s, t))^b_a e_b(\gamma(t))$ and are the local components of $U$ in $\{e_a\}$. 

There is a bijective correspondence between $D$ and the (bundle) Hamiltonian expressed by\footnote{We denote the matrix corresponding to some quantity, e.g. vector or operator, in a given field of bases with the same (kernel) symbol but in \textbf{boldface}; e.g. $U_{\gamma}(t, s) := \left[ U_{\gamma}(s, t) \right]_{ab}$.}

\begin{equation}
\Gamma_{\gamma}(t) := \left[ \Gamma_{ab}^b(t; \gamma) \right] = -\frac{1}{i\hbar} H_{\gamma}^m(t) \tag{2.10}
\end{equation}

with\footnote{Note, the constant $i\hbar$ in (2.10) comes from the same constant in (2.1).}

\begin{equation}
H_{\gamma}^m(t) = i\hbar \frac{\partial U_{\gamma}(t, t_0)}{\partial t} U_{\gamma}^{-1}(t, t_0) = \frac{\partial U_{\gamma}(t, t_0)}{\partial t} \ U_{\gamma}(t_0, t).
\end{equation}

being the \textit{matrix-bundle Hamiltonian} (for details see [3]).

In the Hilbert space description of quantum mechanics to a dynamical variable $A$ corresponds an observable $A(t)$ which is a linear Hermitian operator in $\mathcal{F}$. In the Hilbert bundle description to $A$ corresponds a Hermitian lifting $A_{\gamma}$ of paths whose restriction on $\mathcal{F}_{\gamma}(t)$ is

\begin{equation}
A_{\gamma}(t) = l_{\gamma(t)}^{-1} \circ A(t) \circ l_{\gamma(t)} : \mathcal{F}_{\gamma}(t) \to \mathcal{F}_{\gamma(t)}. \tag{2.11}
\end{equation}

The mean value of $A$ at a state characterized by a state vector $\psi$ or, equivalently, by the corresponding to it state lifting $\Psi_{\gamma}$ is

\begin{equation}
\langle A(t) \rangle_{\psi} := \frac{\langle \psi(t) | A(t) | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle} = \langle A_{\gamma}(t) \rangle_{\Psi_{\gamma}} := \frac{\langle \Psi_{\gamma}(t) | A_{\gamma}(t) | \Psi_{\gamma}(t) \rangle_{\gamma(t)}}{\langle \Psi_{\gamma}(t) | \Psi_{\gamma}(t) \rangle_{\gamma(t)}}. \tag{2.12}
\end{equation}

Here

\begin{equation}
\langle \cdot | \cdot \rangle_x = \langle l_x : | l_x \cdot \rangle, \quad x \in M \tag{2.13}
\end{equation}

is the fibre Hermitian scalar product in $(\mathcal{F}, \pi, M)$ induced by the Hermitian scalar product $\langle \cdot | \cdot \rangle : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ in $\mathcal{F}$.

A summary of the above and other details concerning the Hilbert space and Hilbert bundle description of (nonrelativistic) quantum mechanics can be found in [3].

### 3. General case

Consider now the pure mathematical aspects of the scheme described in Sect. 2. On one hand, it is essential to be notice that the Schrödinger equation (2.1) is a first order (with respect to the time\footnote{The dependence of $\psi$ and $\mathcal{H}$ on the spatial coordinates (and momentum operators) is inessential for the present part of our investigation and, respectively, is not written explicitly.}) linear partial differential
equation solved with respect to the time derivative. On the other hand, the considerations of Sect. 2 are true for Hilbert spaces and bundles whose dimensionality is generically infinity, but, as one can easily verify, they hold also for spaces and bundles with finite dimension.

These observations, as we shall prove below, are enough to transfer the bundle nonrelativistic formalism to the relativistic region. We call the result of this procedure time-dependent or Hamiltonian approach as in it the time plays a privileged rôle and the relativistic covariance is implicit.

Taking into account the above, we can make the following conclusion. Given a linear (vector) space $F$ of $C^1$ functions $\psi: J \to \mathbb{C}$ with $J$ being an $\mathbb{R}$-interval. (We do not make any assumptions on the dimensionality of $F$; it can be finite as well as countable or uncountable infinity.) Let $\mathcal{H}(t): F \to F$, $t \in J$ be (possibly depending on $t$) linear operator. Consider the equation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \mathcal{H}(t)\psi(t).$$

For it are valid all of the results of Sect. 2, viz., for instance, there can be introduced the bundle $(F, \pi, M)$, the evolution operator $U$, etc.; the relations between them being the same as in Sect. 2. If the vector space $F$ is endowed with a scalar (inner) product $\langle \cdot | \cdot \rangle: F \times F \to \mathbb{C}$, then (2.13) induces analogous product in the bundle (i.e. in the bundle’s fibres). Consequently, imposing condition (2.12), we get (2.11). Further, step by step, one can derive all of the results of [2–7].

Further we shall need a slight, but very important generalization of the above. Let now $F$ be a vector space of vector-valued $C^1$ functions $\psi: J \to V$ with $V$ being a complex vector space and $\mathcal{H}(t): F \to F$; the case just considered corresponds to $V = \mathbb{C}$, i.e to $\dim_{\mathbb{C}} V = 1$. It is not difficult to verify that all of the above-said, corresponding to $\dim_{\mathbb{C}} V = 1$, is also mutatis mutandis valid for $\dim_{\mathbb{C}} V \geq 1$. Consequently, the fibre bundle reformulation of the solution of (3.1), the operators, and scalar product(s) in $F$ can be carried out in the general case when $\psi: J \to V$ with $\dim_{\mathbb{C}} V \geq 1$.

Suppose now $K^m$ is the vector space of $C^m$, $m \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, vector-valued functions $\varphi: J \to W$ with $W$ being a vector space. Let $\varphi \in K^m$ satisfies the equation

$$f \left( t, \varphi, \frac{\partial \varphi}{\partial t}, \ldots, \frac{\partial^n \varphi}{\partial t^n} \right) = 0$$

where $f: J \times K^m \times K^{m-1} \times \cdots \times K^{m-n} \to K^{m-n}$, $n \in \mathbb{N}$, $n \leq m$ is a map (multi)linear in $\varphi$ and its derivatives. We suppose (3.2) to be solvable with

---

*We introduce the multiplier $i\hbar \neq 0$ from purely physical reasons and to be able to apply the results already obtained directly, without any changes.*
respect to the highest derivative of \( \varphi \), i.e. (3.2) to be equivalent to
\[
\frac{\partial^n \varphi}{\partial t^n} = G \left( t, \varphi, \frac{\partial \varphi}{\partial t}, \ldots, \frac{\partial^{n-1} \varphi}{\partial t^{n-1}} \right)
\]
(3.3)
for some map \( G: J \times K^m \times K^{m-1} \times \cdots \times K^{m-n+1} \rightarrow K^{m-n} \), linear in \( \varphi \) and its derivatives.

The already developed fibre bundle formalism for the (Schrödinger-type) equation (3.1) can be transferred to (3.3). This can be done in a number of different ways. Below we shall realize the most natural way, but one has to keep in mind that for a concrete equation another method may turn to be more useful, especially from the viewpoint of possible physical applications (see below Sect. 5).

Defining \( F := K^m \times K^{m-1} \times \cdots \times K^{m-n+1} \) and putting
\[
\psi := \left( \varphi, \frac{\partial \varphi}{\partial t}, \ldots, \frac{\partial^{n-1} \varphi}{\partial t^{n-1}} \right)^\top
\]
(3.4)
with \( \top \) being the transposition sign, we can transform (3.3) in the form (3.1) with ‘Hamiltonian’
\[
\mathcal{H}(t) = i\hbar \times
\begin{pmatrix}
0 & \text{id}_{K^{m-1}} & 0 & \cdots & 0 \\
0 & 0 & \text{id}_{K^{m-2}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_0(t) \text{id}_{K^m} & f_1(t) \text{id}_{K^{m-1}} & f_2(t) \text{id}_{K^{m-2}} & \cdots & f_{n-1}(t) \text{id}_{K^{m-n+1}}
\end{pmatrix}
: \mathcal{F} \rightarrow \mathcal{F}
\]
(3.5)
which is a linear matrix operator, i.e. a matrix of linear operators. Here \( \text{id}_X \) is the identity map of a set \( X \) and \( f_i: J \rightarrow \mathbb{C} \), \( i = 0, \ldots, n-1 \) define the (multi)linear map \( G: J \times \mathcal{F} \rightarrow K^{m-n} \) by
\[
G \left( t, \varphi, \frac{\partial \varphi}{\partial t}, \ldots, \frac{\partial^{n-1} \varphi}{\partial t^{n-1}} \right) = f_0(t) \varphi + \sum_{i=1}^{n-1} f_i(t) \frac{\partial^i \varphi}{\partial t^i}.
\]
(3.6)

In this way we have proved that (3.2) can equivalently be rewritten as a Schrödinger-type equation (3.1) with ‘Hamiltonian’ given by (3.5). Such a transformation is not unique. For example, one can choose the components

\(6\) Requiring the equivalence of (3.2) and (3.3), we exclude the existence of sets on which the highest derivative of \( \varphi \) entering in (3.2) may be of order \( k < n \). If we admit the existence of such sets, then on them we have to replace \( n \) in (3.3) with \( k \). (Note \( k \) may be different for different such sets.) The below-described procedure can be modified to include this more general situation but, since we do not want to fill the presentation with complicated mathematical details, we are not going to do this here.

\(7\) Operators of this kind will be considered in the next part of the present investigation.
of $\psi$ to be any $n$ linearly independent linear combinations of $\varphi, \partial\varphi/\partial t, \ldots, \partial^{n-1}\varphi/\partial t^{n-1}$; this will result only in another form of the matrix (3.5). In fact, if $A(t)$ is a nondegenerate matrix-valued function, the change $\psi(t) \mapsto \tilde{\psi}(t) = A(t)\psi(t)$ (3.7) leads to (3.1) with Hamiltonian

$$\tilde{H}(t) = A(t)H A^{-1}(t) + \frac{\partial A(t)}{\partial t}A^{-1}(t).$$

Now to the equation (3.1), with $H$ given via (3.5), we can apply the already described procedure for reformulation in terms of bundles.

4. Dirac equation

The relativistic quantum mechanics of spin $\frac{1}{2}$ particle is described by the Dirac equation (see [11, chapter 2], [12, chapters 1–5], and [13, part V, chapter XX]). The wave function $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^\top$ of such a particle is a 4-dimensional (4-component) spinor satisfying this equation whose Schrödinger-type form is (see, e.g. [13, chapter XX, §6, equation (36)] or [12, chapter 1, §3, equation (1.14)])

$$i\hbar \frac{\partial \psi}{\partial t} = D_H \psi$$

(4.1)

where $D_H$ is a Hermitian operator, called Dirac Hamiltonian, in the space $F$ of state vectors (spinors). For a spin $\frac{1}{2}$ particle with (proper) mass $m$ and electric charge $e$ the explicit form of $D_H$ in an (external) electromagnetic field with 4-potential $(\varphi, A)$ is [13, chapter XX, §6, equation (44)]

$$D_H = e\varphi 1_4 + c\alpha \cdot (p - \frac{e}{c}A) + mc^2\beta,$$

(4.2)

where $1_4$ is the $4 \times 4$ unit matrix, $c$ is the velocity of light in vacuum, $\alpha := (\alpha^1, \alpha^2, \alpha^3)$ is a matrix vector of $4 \times 4$ matrices, $p = -i\hbar \nabla$ is the (3-dimensional) momentum operator ($\nabla_i := \partial/\partial x^i, i = 1, 2, 3$), and $\beta$ is a $4 \times 4$ matrix. The explicit (and general) forms of $\alpha^1, \alpha^2, \alpha^3$, and $\beta$ can be found, e.g. in [11, chapter 2, sect. 2.1.2, equation (2.10)].

Since (4.1) is a first-order equation, we can introduce the Dirac evolution operator $D_U$ via $\psi(t) = D_U(t, t_0)\psi(t_0)$ (cf. (2.2)). Generally it is a $4 \times 4$ integral-matrix operator uniquely defined by the initial-value problem

$$i\hbar \frac{\partial}{\partial t} D_U(t, t_0) = D_H(t) \circ D_U(t, t_0), \quad D_U(t_0, t_0) = id_F$$

(4.3)

For the purposes of this section we do not need the widely known relativistic invariant form of Dirac equation [11, [13].
with \( \mathcal{F} \) being the space of 4-spinors. The explicit form of \( DU \) is derived, e.g. in [11, chapter 2, sect. 2.5.1].

Now the bundle formalism developed in [2–7] can be applied to a description of Dirac particles practically without changes. For instance, the spinor lifting of paths have to be introduced via (2.3) and are connected by (2.5) in which \( U \) has to be replaced by the Dirac evolution transport \( DU \) given by

\[
DU_\gamma(t, s) = l_{\gamma(t)}^{-1} \circ DU(t, s) \circ l_{\gamma(s)}, \quad s, t \in J
\]

(cf. (2.4)). The bundle Dirac equation is

\[
^{D}D_{t}^{\gamma} \Psi_{\gamma} = 0 \quad (4.4)
\]

with \( ^{D}D \) being the assigned to \( DU \) by (2.7) derivation along paths. Again, the matrix of the coefficients of the Dirac evolution transport is connected with the matrix-bundle Dirac Hamiltonian via (2.10), etc.

5. Klein-Gordon equation

The wavefunction \( \phi \in K^{m} \) of spinless special-relativistic particle is a scalar function of class \( C^{m}, \ m \geq 2, \) over the spacetime and satisfies the Klein-Gordon equation [12, chapte 9]. For a particle of mass \( m \) and electric charge \( e \) in an external electromagnetic field with 4-potential \( (\phi, A) \) it reads [13, chapter XX, § 5, equation (30)]

\[
\left( \frac{i\hbar}{\partial t} - e\phi \right)^{2} - c^{2} \left( p - \frac{e}{c} A \right)^{2} \phi = m^{2}c^{4} \phi. \quad (5.1)
\]

This is a second-order linear partial differential equation of type (3.2) with respect to \( \phi \). Solving it with respect to \( \partial^{2} \phi / \partial t^{2} \), we can transform it to equation of type (3.3):

\[
\frac{\partial^{2} \phi}{\partial t^{2}} = f_{0} \phi + \frac{2e}{i\hbar} \frac{\partial \phi}{\partial t},
\]

\[
f_{0} := \left[ \frac{c^{2}}{\hbar^{2}} (p - \frac{e}{c} A)^{2} - \frac{m^{2}c^{4}}{\hbar^{2}} + \frac{e^{2}}{\hbar^{2}} \phi^{2} + \frac{2e}{i\hbar} \frac{\partial \phi}{\partial t} \right] \text{id}_{K^{m}}. \quad (5.2)
\]

As pointed above, there are (infinitely many) different ways to put this equation into Schrödinger-type form. Below we realize three of them, each having applications for different purposes.

The ‘canonical’ way to do this is to define \( \psi := (\phi, \partial \phi / \partial t)^{T} \) and \( K^{m} := \{ \phi \colon J \rightarrow \mathbb{C}, \ \phi \text{ is of class } C^{m} \}, \ m \geq 2. \) Then, comparing (5.2) with (3.6), we see that (5.2), and hence (5.1), is equivalent to (3.1) with \( H = K^{-G}cH \), where the ‘canonical’ Klein-Gordon Hamiltonian is

\[
K^{-G}cH := i\hbar \begin{pmatrix} 0 & 2e \text{id}_{K^{m}} \end{pmatrix} \begin{pmatrix} f_{0} \ 2e/i\hbar \phi \text{id}_{K^{m-1}} \end{pmatrix}. \quad (5.3)
\]
Note, for a free particle, \((\varphi, A) = 0\), this is the anti-diagonal matrix operator 
\[
\begin{pmatrix}
0 & c^2 \nabla^2 - m^2 c^4 / \hbar^2 & 1
\end{pmatrix}
id_{K^m}.
\]

Another possibility is to put 
\[
\psi = \left( \phi + \frac{i \hbar}{mc^2} \frac{\partial \phi}{\partial t}, \phi - \frac{i \hbar}{mc^2} \frac{\partial \phi}{\partial t} \right)^\top.
\]
This choice is good for investigation of the non-relativistic limit, when \(i \hbar \frac{\partial \phi}{\partial t} \approx mc^2 \phi\) [13, chapter XX, § 5], so that in it \(\psi \approx (2 \phi, 0)^\top\).

Now, as a simple verification proves, the Schrödinger-type form (3.1) of (5.2) is realized for the Hamiltonian
\[
K - G \mathcal{H} := \frac{1}{2} \times
\begin{pmatrix}
(mc^2 + 2e\varphi)id_{K^{m-1}} - \hbar^2 f_0 / mc^2 & (-mc^2 - 2e\varphi)id_{K^{m-1}} - \hbar^2 f_0 / mc^2 \\
(mc^2 - 2e\varphi)id_{K^{m-1}} + \hbar^2 f_0 / mc^2 & (-mc^2 + 2e\varphi)id_{K^{m-1}} + \hbar^2 f_0 / mc^2
\end{pmatrix}.
\]

If the electromagnetic field vanishes, \((\varphi, A) = 0\), then 
\[
f_0 = (c^2 \nabla^2 - m^2 c^4 / \hbar^2)id_{K^m}.
\]

The third possibility mentioned corresponds to the choice of \(\psi\) as a 5 × 1 matrix:
\[
\psi = \left( mc^2 \phi, \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \frac{\partial \phi}{\partial x^3} \right)^\top = \left( mc^2 \phi, \frac{\partial \phi}{\partial t}, \nabla \phi \right)^\top.
\]
The corresponding Hamiltonian is 5 × 5 matrix which in the absence of electromagnetic field is [14, chapter 2, sect. 2.1.1] \(K_{n.r.}^G \mathcal{H} = mc^2 \beta + \alpha \cdot \nabla\) (cf. Dirac Hamiltonian (4.2) for \((\varphi, A) = 0\)) where \(\beta\) is 5 × 5 matrix and \(\alpha\) is a 3-vector of 5 × 5 matrices. The full realization of this procedure and the explicit form of the corresponding 5 × 5 matrices is given in [14, § 4, sect. 4.4].

Now choosing some representation of Klein-Gordon equation as first-order (Schrödinger-type) equation, we can in an evident way transfer the bundle formalism to the description of spinless particles.

### 6. Other relativistic wave equations

As we saw in sections 4 and 5, the only problem for a bundle reformulation of a wave equation is to rewrite it as a first-order differential equation and to find the corresponding Hamiltonian. Since all relativistic wave equations are of the form of equation (3.2) [12, 14], this procedure can successfully be performed for all of them.
For instance, the wavefunction $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T$ of particles with spin 1 is a 4-vector satisfying Klein-Gordon equation\footnote{Here we do not concern the additional conditions like the Lorentz one. They lead to a modification of the equations defining the evolution transport, but this does not change the main ideas. E.g. the most general equation for vectorial mesons is the Proca equation\textsuperscript{[1]} chapter 3, equation (3.132)) \[ (\partial^\mu \partial_\mu + \frac{m^2 c^4}{\hbar^2}) \eta_{\mu\nu} - \partial_a \partial_\nu \phi^\nu = 0 \] on which one usually imposes the additional condition $\partial_a \phi^\nu = 0$. (For $m \neq 0$ the last condition is a corollary of the Proca equation; to prove this, simply apply $\partial^\nu$ to Proca equation.)}. Hence for each component $\psi_i$, $i = 0, 1, 2, 3$ we can construct the corresponding Hamiltonian $K - G H_i$ using, e.g., one of the methods described in Sect. 5. Then equation (4.1) holds for a Hamiltonian of the form of a 4 × 4 diagonal block matrix operator $K - G H = \text{diag}(K - G H_0, K - G H_1, K - G H_2, K - G H_3)$ and the corresponding new wavefunction which is now a 8 × 1 matrix.

The just said is, of course, valid with respect to electromagnetic field, the 4-potential playing the rôle of wavefunction\textsuperscript{[1]} chapter I, § 5]. It is interesting to be noted that even at a level of classical electrodynamics the Maxwell equations admit Schrödinger-like form. There are different fashions to do this. For a free field, one of them is to put $\psi = (E, H, E, H)^T$ where $E$ and $H$ are respectively the electric and magnetic field strengths, and to define the Hamiltonian, e.g., by $E - M H = i\hbar \begin{pmatrix} 0 & c \text{rot} & 0 & 0 \\ -c \text{rot} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{div} \\ 0 & 0 & -\text{div} & 0 \end{pmatrix}$.

Almost the same as spin 1 particles is the case of particles with spin 2. Their wavefunction is symmetric tensor field of second rank whose components satisfy also the Klein-Gordon equation\textsuperscript{[1]} p. 24].

The wavefunction of particles with spin 3/2 is a (4-)spin-vector whose ‘vector’ components satisfy the Dirac equation (and some additional conditions)\textsuperscript{[1]} pp. 35–36]. Therefore for them can be applied \textit{mutatis mutandis} the presented in Sect. 4 bundle formalism.

7. Conclusion

In this paper we have developed the time-dependent (Hamiltonian) approach for fibre bundle formulation of relativistic quantum mechanics. As we saw, it is a straightforward generalization of the methods worked out for bundle treatment of non-relativistic quantum mechanics. The generic scheme is to transform a relativistic wave equation into a Schrödinger-like form with corresponding Hamiltonian and then to apply \textit{mutatis mutandis} the results already established for the Schrödinger equation. A new moment in the relativistic region is that some of the wave equations are (system(s) of) partial differential equations of order no less than two. The transformation of such (a system of) equations to a first order Schrödinger-like equation (or system of equations) is, of course, non-unique. The choice of such representation is more a physical than a mathematical subject and depends on the concrete problem posed. It is clear that different Schrödinger-like representations
lead to similar (equivalent) but different bundle description of one and the same initial equation.

A ‘bad’ feature of the Hamiltonian approach to bundle description of relativistic quantum mechanics, presented in this paper, is its explicit time-dependence. A consequence of this fact is the implicit covariance of the bundle description obtained in this way. Evidently, such a situation is not satisfactory from the viewpoint of relativistic character of the theory it represents. This naturally leads us to the idea of explicit-covariant bundle description of relativistic quantum mechanics. It turns out that for the solution of this problem are not enough the methods developed for the Schrödinger equation. The physical reason for this is that in the relativistic wave equations is intrinsically incorporated the fact of absence of world lines (trajectories) in a classical sense of the quantum objects they describe. These problems will be investigated in the second part of this work where the covariant bundle description of relativistic quantum mechanics will be developed.

We shall end with some comments on the material of Sect. 3. We saw there that the fibre bundle formalism developed for the solutions of Schrödinger equation can successfully be applied for the solutions of (systems of) linear ordinary differential equations. For this purpose the system of equations, if they are of order greater than one, has to be transformed into a system of first-order equations. It can always be written in a Schrödinger-like form to which the developed in [2–7] bundle approach can be applied mutatis mutandis.

Therefore, in particular, to a system of linear ordinary (with respect to ‘time’) differential equations corresponds a suitable linear transport along paths in an appropriately chosen fibre bundle. As most of the fundamental equations of physics are expressed by such systems of equations, they admit fibre bundle (re)formulation analogous to the one of Schrödinger equation.

An interesting consequence of this discussion is worth mentioning. Suppose a system of the above-described type is an Ouler-Lagrange (system of) equation(s) for some Lagrangian. Applying the outlined ‘bundle’ procedure, we see that to this Lagrangian corresponds some (evolution) linear transport along paths in a suitable fibre bundle. The bundle and the transport are practically (up to isomorphisms) unique if the Ouler-Lagrange equations are of first order with respect to time. Otherwise there are different (but equivalent) such objects corresponding to the given Lagrangian. Hence, some Lagrangians admit description in terms of linear transports along paths. In more details the correspondence between Lagrangians (or Hamiltonians) and linear transports along paths will be explored elsewhere.

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