A classical explanation of quantization

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Abstract

In the context of our group’s recently developed emergent quantum mechanics, and, in particular, based on an assumed sub-quantum thermodynamics, Max Planck’s assumption of energy quantization is explained by means of a purely classical physics. Moreover, under the same premises, also the energy spectrum of the quantum mechanical harmonic oscillator is derived.

Keywords: harmonic oscillator, Brownian motion, Langevin equation, nonequilibrium thermodynamics, quantum mechanics
1. INTRODUCTION

In references [1, 2], the Schrödinger equation was derived in the context of modelling quantum systems via nonequilibrium thermodynamics, i.e., by the requirement that the dissipation function, or the time–averaged work over the system of interest, vanishes identically. The “system of interest” is a “particle” in terms of a harmonic oscillator embedded in a thermal environment of non–zero average temperature (i.e., of the “vacuum”). In more recent papers, we have illustrated the “particle” more concretely by using the concept of a “bouncer” (or “walker”, respectively) gleaned from beautiful experiments by Couder’s group [3–6]. Thus one assumes that the thermal environment is oscillating itself, with the kinetic energy of these latter oscillations providing the energy necessary for the “particle” to maintain a constant energy, i.e., to remain in a nonequilibrium steady state. Regarding the respective (“zero point”) oscillations of the vacuum, one simply assumes the particle oscillator embedded in an environment comprising a corresponding energy bath.

2. A CLASSICAL OSCILLATOR DRIVEN BY ITS ENVIRONMENT’S ENERGY BATH: THE “BOUNCER”

Let us start with the following Newtonian equation for a classical oscillator

\[
mx'' = -m\omega_0^2x - 2\gamma mx + F_0 \cos \omega t.
\]  
(2.1)

Eq. (2.1) describes a forced oscillation of a mass \(m\) swinging around a center point along \(x(t)\) with amplitude \(A\) and damping factor, or friction, \(\gamma\). If \(m\) could swing freely, its resonant angular frequency would be \(\omega_0\). Due to the damping of the swinging particle there is a need for a locally independent (white noise) driving force \(F_0\).

We are only interested in the stationary solution of Eq. (2.1), i.e., for \(t \gg \gamma^{-1}\), where \(\gamma^{-1}\) plays the role of a relaxation time, using the ansatz

\[
x(t) = A \cos(\omega t + \varphi).
\]  
(2.2)

After a short calculation we find for the phase shift between the forced oscillation and the forcing oscillation that

\[
\tan \varphi = -\frac{2\gamma \omega}{\omega_0^2 - \omega^2},
\]  
(2.3)
and for the amplitude of the forced oscillation
\[ A(\omega) = \frac{F_0/m}{\sqrt{\omega_0^2 - \omega^2}^2 + (2\gamma\omega)^2} . \] (2.4)

To analyse the energetic balance, we multiply Eq. (2.1) with \( \dot{x} \) and obtain
\[ m\ddot{x} + m\omega_0^2x\dot{x} = -2m\gamma\dot{x}^2 + F_0\cos(\omega t)\dot{x} , \] (2.5)
and thus,
\[ \frac{d}{dt} \left( \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega_0^2x^2 \right) = -2m\gamma\dot{x}^2 + F_0\cos(\omega t)\dot{x} = 0 , \] (2.6)
where the first term within the brackets is the kinetic energy and the second one is the potential energy of the oscillator system. Therefore, the time–derivative of the sum must be zero since the sum is the whole energy of the system and has hence to be of constant value.

Due to the friction the oscillator loses energy to the energy bath, represented by \(-2m\gamma\dot{x}^2\), whereas \(F_0\cos(\omega t)\dot{x}\) represents the energy which has to be regained from the energy bath via the force \(F_0\cos\omega t\). The two terms on the right–hand side must also be zero and we can hence write down the net work–energy that is taken up from the bouncer in the form of heat during each period \(\tau\) as
\[ W_{\text{bouncer}} = \int_t^{t+\tau} 2m\gamma\dot{x}^2\,dt = 2m\gamma\omega^2A^2 \int_t^{t+\tau} \sin^2(\omega t + \varphi)\,dt = \gamma m\omega^2A^2\tau . \] (2.7)
Inserting \(A\) from Eq. (2.4), we can see that \(\frac{dW}{d\omega} = 0\) at \(\omega = \omega_0\), in which case the energy reaches its maximum.

To derive the stationary frequency \(\omega\), we use the right–hand side of Eq. (2.6) together with Eq. (2.2) to first obtain
\[ 2m\gamma\dot{x} = -2m\gamma A\omega \sin(\omega t + \varphi) = F_0\cos\omega t . \] (2.8)
As all factors, except for the sinusoidal ones, are time independent, we have the necessary condition for the phase given by
\[ -\sin(\omega t + \varphi) = \cos\omega t \quad \Rightarrow \quad \varphi = -\frac{\pi}{2} + 2n\pi \] (2.9)
for all \(n \in \mathbb{Z}\). Substituting this into Eq. (2.3), we obtain
\[ \tan\left( -\frac{\pi}{2} + 2n\pi \right) = \pm\infty = -\frac{2\gamma\omega}{\omega_0^2 - \omega^2} , \] (2.10)
and therefore

$$\omega = \omega_0 \,. \quad (2.11)$$

Therefore, the system is stationary at the resonance frequency $\omega_0$ of the free undamped oscillator. With the notations

$$\tau = \frac{2\pi}{\omega_0} \,, \quad r := A(\omega_0) = \frac{F_0}{2\gamma m \omega_0} \,, \quad (2.12)$$

we obtain

$$W_{\text{bouncer}} = W_{\text{bouncer}}(\omega_0) = \gamma m \omega_0^2 r^2 \tau = 2\pi \gamma m \omega_0 r^2 \,. \quad (2.13)$$

The Hamiltonian of the system is the term within the brackets of the left–hand side of Eq. (2.6),

$$\mathcal{H} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2 = \text{const.} \,. \quad (2.14)$$

We introduce the angle $\theta(t) := \omega_0 t$ and substitute Eq. (2.2) into Eq. (2.14), thus yielding the two equations

$$\ddot{r} - r \dot{\theta} + \omega_0^2 r = 0 \,, \quad (2.15)$$

and

$$r \ddot{\theta} + 2 \dot{r} \dot{\theta} = 0 \,. \quad (2.16)$$

From Eq. (2.16), an invariant quantity is obtained: it is the angular momentum,

$$L(t) = mr^2 \dot{\theta}(t) \,. \quad (2.17)$$

With $\theta(t) = \omega_0 t$, and thus $\dot{\theta} = \omega_0$, the quantity of Eq. (2.17) becomes a time–invariant expression, which we denote as

$$\hbar := mr^2 \omega_0 \,. \quad (2.18)$$

Thus, we rewrite our result (2.13) as

$$W_{\text{bouncer}} = 2\pi \gamma \hbar \,. \quad (2.19)$$
3. BROWNIAN MOTION OF A PARTICLE: THE “WALKER”

Now we concentrate on the motion of our “bouncer” within a more irregular environment. That is, we now assume that the “particle” is not only driven via harmonic oscillation of a wave–like environment, but that there is also a stochastic element in its movement, as, e.g., due to different fluctuating wave–like configurations in the environment. Therefore, our “particle’s” motion will assume a Brownian–type character. The Brownian motion of a thus characterized particle (which we propose to call a “walker”), is then described by a Langevin’s stochastic differential equation with velocity $u = \dot{x}$ and a force $f(t)$,

$$m\ddot{u} = -m\zeta u + f(t), \quad (3.1)$$

with a friction coefficient $\zeta$. The time–dependent force $f(t)$ is stochastic, i.e., one has as usual for the time–averages

$$\bar{f}(t) = 0, \quad \bar{f}(t)f(t') = \phi(t - t'), \quad (3.2)$$

where $\phi(t)$ differs noticeably from zero only for $t < \zeta^{-1}$. The correlation time $\zeta^{-1}$ denotes the time during which the fluctuations of the stochastic force remains correlated.

The standard textbook solution for Eq. (3.1) in terms of the mean square displacements $\bar{x}^2$ is given (e.g., in [7]) by the Ornstein–Uhlenbeck equation,

$$\bar{x}^2 = \frac{2kT}{\zeta^2 m} \left(\zeta |t| - 1 + e^{-\zeta |t|}\right). \quad (3.3)$$

Note that, on the one hand, for $t \ll \zeta^{-1}$, and by expanding the exponential up to second order, Eq. (3.3) provides that

$$\bar{x}^2 = \frac{kT}{m} t^2 = \frac{mu_0^2}{m} t^2 = u_0^2 t^2. \quad (3.4)$$

On the other hand, for $t \gg \zeta^{-1}$, one obtains the familiar relation for Brownian motion, i.e.,

$$\bar{x}^2 \simeq 2Dt, \quad (3.5)$$

with the “diffusion constant” $D$ given by

$$D = \frac{kT}{\zeta m}. \quad (3.6)$$
To obtain a better understanding of Equations (3.5) and (3.6), we want to detail here how they come about. One usually introduces a coefficient $\lambda$ that measures the strength of the mean square deviation of the stochastic force, such that

$$\phi(t) = \lambda \delta(t).$$

(3.7)

Since friction increases in proportion to the frequency of the stochastic collisions, there must exist a connection between $\lambda$ and $\zeta$, respectively. One then solves the Langevin equation (3.1) in order to find this connection. Solutions of this equation are well known from the Ornstein–Uhlenbeck theory of the Brownian motion [7].

Since the dependence of $f(t)$ is known only statistically, one does not consider the average value of $u(t)$, but instead that of its square,

$$u^2(t) = e^{-2\zeta t} \int_0^t d\tau \int_0^t d\tau' e^{\zeta (\tau + \tau')} \phi(\tau - \tau') \frac{1}{m} + u_0^2 e^{-2\zeta t},$$

(3.8)

with $u_0$ being the initial value of the velocity. For $t \gg \zeta^{-1}$, the contribution of $u_0$ becomes negligible, i.e., $\zeta^{-1}$ then plays the role of a relaxation time. We require that our particle attain thermal equilibrium ([1, 2]) after long times so that due to the equipartition theorem the average value of the kinetic energy becomes, as usual,

$$\frac{1}{2}mu^2(t) = \frac{1}{2}kT.$$

(3.9)

Combining Eqs. (3.8) and (3.9), one obtains the Einstein relation

$$\lambda = 2\zeta mkT.$$

(3.10)

Similarly, one obtains the mean square displacement of $x(t)$ for $t \gg \zeta^{-1}$. (Note that, even if we use the same character $x$ as for the oscillating particle, the meaning is different: $x(t)$ then signified a deterministic harmonic displacement of mass point $m$ in the case of an oscillating particle (“bouncer”), whereas $x(t)$ now means a stochastic random walk variable for the particle that carries out a Brownian motion (“walker”).) Concerning the latter, one integrates twice to obtain the confirmation of our result (3.5), i.e.,

$$x^2(t) = \int_0^t d\tau \int_0^t d\tau' \frac{\lambda}{2\zeta m^2} e^{-\zeta |\tau - \tau'|} \simeq \frac{\lambda}{\zeta^2 m^2} t = 2Dt,$$

(3.11)
with the diffusion constant turning out as identical to Eq. (3.6), i.e.,

$$D = \frac{\lambda}{2\zeta^2m^2} = \frac{kT}{\zeta m}.$$  \hspace{1cm} (3.12)

Now we remind ourselves that we have to do with a steady-state system. Just as with the friction $\zeta$ there exists a flow of (kinetic) energy into the environment, there must therefore also exist a work–energy flow back into our system of interest. For its calculation, we multiply Eq. (3.1) by $u = \dot{x}$ and obtain an energy–balance equation. It yields for the duration of time $n\tau$, with the natural number $n > 0$ chosen so that $n\tau$ is large enough to make all fluctuating contributions negligible, the net work–energy of the walker

$$W_{\text{walker}} = m\zeta \int_{t}^{t+n\tau} \dot{x}^2(t) \, dt = m\zeta \int_{t}^{t+n\tau} u^2(t) \, dt.$$  \hspace{1cm} (3.13)

Inserting (3.9), we obtain

$$W_{\text{walker}} = n\tau m\zeta u^2(t) = n\tau m\zeta \frac{kT}{m} = n\tau \zeta kT ,$$  \hspace{1cm} (3.14)

where $\tau = 2\pi/\omega_0$ equals the period of Eq. (2.7), which is chosen in order to make the result comparable with Eq. (2.19). The work–energy for the particle undergoing Brownian motion can thus be written as

$$W_{\text{walker}} = n \frac{2\pi}{\omega_0} \zeta kT .$$  \hspace{1cm} (3.15)

4. THE “WALKING BOUNCER”: DERIVATION OF $E = \hbar \omega$

Let us summarize what we have achieved so far. We have for both systems, i.e., oscillator and particle in Brownian-type motion (or “bouncer” and “walker”, respectively), obtained a net work–energy flow into each system, respectively, in order to compensate for the respective energy losses due to friction. There is a continuous flow from the bath to the oscillator, and vice versa. Moreover, and most importantly, during that flow, for long enough times $n\tau$, the friction of the bouncer can be assumed to be exactly identical with the friction of the walker. For this reason we directly compare the results of Eqs. (2.19) and (3.15),

$$nW_{\text{bouncer}} = W_{\text{walker}} ,$$  \hspace{1cm} (4.1)
providing
\[ n2\pi\gamma\hbar = n\frac{2\pi}{\omega_0}\zeta kT. \] (4.2)

Now, one generally has that the total energy of a sinusoidal oscillator exactly equals twice its average kinetic energy. Moreover, despite having a nonequilibrium framework of our system, the fact that we deal with a steady state means that our oscillator is in local thermal equilibrium with its environment. As the average kinetic energy of the latter is always given, for each degree of freedom, by \( kT/2 \), one has for the corresponding total energy that \( E_{\text{tot}} = kT \). Now, one can express that energy via Eq. (4.2) in terms of the oscillator’s frequency \( \omega_0 \), and one obtains

\[ E_{\text{tot}} = kT = \frac{\zeta}{\gamma}\hbar\omega_0. \] (4.3)

Assuming the same friction coefficient for both the bouncer and the walker, i.e., \( \gamma = \zeta \), we obtain the energy balance between oscillator and its thermal environment as

\[ kT = \hbar\omega_0, \] (4.4)

with the total energy of our model for a quantum “particle”, i.e., a driven steady–state oscillator system, being now derived as

\[ E_{\text{tot}} = \hbar\omega_0. \] (4.5)

Moreover, if we compare Eq. (4.4) with the Langevin equation (3.1), we find the following confirmation of Eq. (4.5). First, we recall Boltzmann’s relation between the heat applied to an oscillating system and a change in the action function \( \delta S = \delta \int E_{\text{kin}}\,dt \), respectively, providing

\[ \nabla Q = 2\omega_0 \nabla (\delta S). \] (4.6)

\( \delta S \) relates to the momentum fluctuation via

\[ \nabla (\delta S) = \delta p =: m\mathbf{u}, \] (4.7)

where, as usual,

\[ \mathbf{u} = -\frac{\hbar}{2P} \nabla P. \] (4.8)
Thus one obtains with (4.4) that the friction in Eq. (3.1) is given by

$$m\zeta \mathbf{u} = m\gamma \mathbf{u} = -m\gamma \frac{\hbar}{2m} \frac{\nabla P}{P} .$$

(4.9)

Then, as $P = P_0 e^{-\delta Q/kT}$, with Eq. (4.4) for the thermal bath as above, and with Eq. (4.6) one obtains that

$$\zeta = \gamma = 2\omega_0 ,$$

(4.10)

and therefore the friction term becomes

$$m\zeta \mathbf{u} = -\hbar \omega_0 \frac{\nabla P}{P} = \nabla Q .$$

(4.11)

Note that with Eqs. (4.4) and (4.10) one obtains the expression for the diffusion constant

$$D = \frac{kT}{\zeta m} = \frac{\hbar}{2m} ,$$

(4.12)

which is exactly the usual expression for $D$ in the context of quantum mechanics.

With Eq. (4.11) one can also introduce the recently proposed concept of an “entropic force” [8]. That is, with the total energy equaling a total work applied to the system, one can write (with $S_e$ denoting the entropy)

$$E_{\text{tot}} = 2E_{\text{kin}} =: F\Delta x = T\Delta S_e = \int_0^{2\pi} \nabla Q \, dr$$

$$= \Delta Q (\text{circle}) = 2 \left[ \frac{\hbar \omega_0}{4} - \left( -\frac{\hbar \omega_0}{4} \right) \right] = \hbar \omega_0 .$$

(4.13)

Eq. (4.13) provides an “entropic” view of a harmonic oscillator in its thermal bath. First, the total energy of a simple harmonic oscillator is given as $E_{\text{tot}} = mr^2\omega_0^2/2 =: \hbar \omega_0/2$. Now, the average kinetic energy of a harmonic oscillator is given by half of its total energy, i.e., by $E_{\text{kin}} = mr^2\omega_0^2/4 = \hbar \omega_0/4$, which — because of the local equilibrium — is both the average kinetic energy of the bath and that of the “bouncer” particle. As the latter during one oscillation varies between 0 and $\hbar \omega_0/2$, one has the following entropic scenario. When it is minimal, the tendency towards maximal entropy will provide an entropic force equivalent to the absorption of the heat quantity $\Delta Q = \hbar \omega_0/4$. Similarly, when it is maximal, the same tendency will now enforce that the heat $\Delta Q = \hbar \omega_0/4$ is given off again to the “thermostat” of the thermal bath. In sum, then, the total energy throughput $E_{\text{tot}}$ along a full circle will equal, according to Eq. (4.13), $2\overline{E_{\text{kin}}}(\text{circle}) = 2\hbar \omega_0/2 = \hbar \omega_0$. In other words, the formula
$E = \hbar\omega_0$ does not refer to a classical “object” oscillating with frequency $\omega_0$, but rather to a process of a “fleeting constancy”: due to entropic requirements, the energy exchange between bouncer and heat bath will constantly consist of absorbing and emitting heat quantities such that in sum the “total particle energy” emerges as $\hbar\omega_0$.

5. ENERGY SPECTRUM OF THE HARMONIC OSCILLATOR FROM CLASSICAL PHYSICS

A characteristic and natural feature of nonequilibrium steady-state systems is given by the requirement that the time integral of the so-called dissipation function $\Omega_t$ over full periods $\tau$ vanish identically [1]. Assuming that our oscillator has a characteristic frequency $\omega_0 = \frac{2\pi}{\tau}$, one defines the dissipation function w.r.t. the force in Eq. (2.1) over the integral

$$\frac{1}{\tau} \int_0^\tau \Omega_t \, dt := \frac{1}{\tau} \int_0^\tau \frac{dF(t)}{kT} = 0 . \quad (5.1)$$

Here, we assume a generalized driving force $F$ to have a periodic component such that $F(t) \propto e^{i\omega_0 t}$. Then one generally has that

$$\int_0^\tau dF \propto e^{i\omega_0 (t+\tau)} - e^{i\omega_0 t}, \quad (5.2)$$

and so the requirement Eq. (5.1) generally provides that

$$\int_0^\tau \omega_0 \, dt = 2n\pi , \quad \text{for } n = 1, 2, \ldots . \quad (5.3)$$

(Incidentally, this condition resolves the problem discussed by Wallstrom [9] about the single-valuedness of the quantum mechanical wave functions and eliminates possible contradictions arising from Nelson-type approaches to model quantum mechanics on a “particle centered” basis alone.)

So, we are dealing with a situation where a “particle” oscillates with an angular frequency $\omega_0$ driven by the external force due to the surrounding (“zero-point”) fluctuation field, with a period $\tau = \frac{1}{\nu} = \frac{2\pi}{\omega_0}$. For the type of oscillation we have assumed simple harmonic motion, or, equivalently [10], circular motion, and we generally have that the total (“zero-point”) energy is

$$E_0 = \frac{1}{2} mr^2 \omega_0^2 = \frac{\hbar\omega_0}{2} . \quad (5.4)$$
Then, for slow, adiabatic changes during one period of oscillation, the action function over a cycle is an invariant,

\[ S_0 = \frac{1}{2\pi} \oint p \cdot dr = \frac{1}{2\pi} \oint m \omega_0 r \cdot dr , \quad (5.5) \]

with \( r = r \cdot \hat{r} \). This provides, in accordance with the corresponding standard relation for integrable conservative systems [1], i.e.,

\[ dS_0 = \frac{dE_0}{\omega_0} , \quad (5.6) \]

that

\[ S_0 = \frac{1}{2}mr^2\omega_0 . \quad (5.7) \]

However, the external driving frequency and the particle’s frequency \( \omega_0 \), respectively, are not just in simple synchrony, since one has to take into account also the type of energy exchanges of the “particle” with its oscillating environment. Generally, there exists the possibility (within the same boundary condition, i.e., on the circle) of periods \( \tau_n = \frac{n\pi}{\nu} = \frac{2n\pi}{\omega_0} \), with \( n = 1, 2, \ldots \), of adiabatical heat exchanges “disturbing” the simple particle oscillation as given by Eq. (5.4). That is, while we have so far considered, via Eqs. (5.5) and (5.6), a single, slow adiabatic change during an oscillation period, we now also admit the possibility of several (i.e., \( n \)) periodic heat exchanges during the same period, i.e., absorptions and emissions as in (4.13). The action integrals over full periods then more generally become

\[ \oint dS(\tau_n) := -\int_0^{\tau} \dot{S} \, dt = \int_0^{\tau} E_{\text{tot}} \, dt = \hbar \int_0^{\tau} \omega_0 \, dt . \quad (5.8) \]

Thus, one can first recall the expressions (5.4) and (5.7), respectively, to obtain for the case of “no additional periods” \( (n = 0) \) the basic “zero–point” scenario

\[ S_0 = \frac{\hbar}{2} , \quad \text{and} \quad E_0 = U = \frac{1}{2}\hbar\omega_0 . \quad (5.9) \]

Secondly, however, using (5.3), one obtains for \( n = 1, 2, \ldots \) that

\[ \oint dS(\tau_n) = 2n\pi\hbar = nh . \quad (5.10) \]

This provides a spectrum of \( n \) additional possible energy values,

\[ E(n) = nh\omega_0 , \quad (5.11) \]
such that, together with Eq. (5.9), the total energy spectrum of the off–equilibrium steady–state harmonic oscillator becomes

\[-\frac{\partial S}{\partial t} = E(n) + U = E(n) + E_0 = \left(n + \frac{1}{2}\right) \hbar \omega_0, \quad \text{with } n = 0, 1, 2, \ldots. \]  

(5.12)

Note that to derive Eq. (5.12) no Schrödinger or other quantum mechanical equation was used. Rather, it was sufficient to invoke Eq. (5.2), without even specifying the exact expression for \( F \).

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