MOTIVIC INVARIANT OF REAL POLYNOMIAL FUNCTIONS
AND NEWTON POLYHEDRON

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Abstract. We propose a computation of real motivic zeta functions for real polynomial functions, using Newton polyhedron. As a consequence we show that the weights are blow-Nash invariants of convenient weighted homogeneous polynomials in three variables.

In Singularity Theory, one aim to classify singular objects with respect to a given equivalence relation. We focus on the singularities of function germs. We are mainly interested in the case of weighted homogeneous polynomial functions, that is polynomial functions that become homogeneous by assigning a particular weight to each of the variables. Concerning such functions, considered as germs at the origin, we tackle the question of the invariance of the weights under a given equivalence relation between germs.

Concerning complex analytic function germs, the first result is this direction is due to K. Saito [17] who proved in 1971 that the weights are local analytic invariants of the pair \((\mathbb{C}^n, f^{-1}(0))\) at the origin, for \(f\) a weighted homogeneous polynomial. Concerning the topological equivalence, E. Yoshinaga and M. Suzuki [20] in 1979 (and later T. Nishimura [15] in 1986) proved the topological invariance of the weights in dimension two, whereas O. Saeki [16] in 1988 treated the three dimensional case.

In this paper, we are concerned with the real counterpart of this question, considering equivalence relation on real analytic function germs. If the topological equivalence is by far too weak in the real setting, the most relevant equivalence relation to consider is the blow-analytic equivalence introduced by T.-C. Kuo (cf. [13], and also [9, 10] for surveys). Real analytic function germs \(f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)\) are said to be blow-analytically equivalent in the sense of [13] if there exist real modifications \(\beta_f : M_f \rightarrow \mathbb{R}^n\) and \(\beta_g : M_g \rightarrow \mathbb{R}^n\) and an analytic isomorphism \(\Phi : (M_f, \beta_f^{-1}(0)) \rightarrow (M_g, \beta_g^{-1}(0))\) which induces a homeomorphism \(\phi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\) such that \(f = g \circ \phi\).

For polynomial functions, or more generally Nash functions (i.e. real analytic functions with semi-algebraic graph), a natural counterpart exists, called blow-Nash equivalence, that takes into account the algebraic nature of Nash functions. This equivalence relation have been proved to have nice properties (cf. [9, 6, 7]).

The question of the invariance of the weights for weighted homogeneous polynomial functions under blow-analytic equivalence already appeared as a conjecture in [8] and as a question in [12]. A positive answer has been given by O. M. Abdelrahmane [11] for the two variables case, using two invariants of the blow-analytic
equivalence: the Fukui Invariants [8] and zeta functions [12] constructed by S. Koike and A. Parusiński using Motivic Integration [4] with Euler characteristic with compact supports as a measure.

We prove as theorem 4.7 that the result holds true in the three variables case for weighted homogeneous polynomial functions that are convenient, under the blow-Nash equivalence. To prove this, we investigate the zeta function introduced in [6] as an invariant of the blow-Nash equivalence, using as a measure the virtual Poincaré polynomial [14]. This polynomial is an additive and multiplicative invariant for real algebraic sets, which degree is equal to the dimension of the variety.

The main tool for the proof of theorem 4.7 is to estimate the degrees of the coefficient of the zeta functions in terms of the Newton polyhedron of a given polynomial function. Zeta functions in Motivic Integration have already been computed in terms of Newton polyhedron [3, 5, 11], and our theorem 2.3 is just an adaptation in order to focus on the question of degrees. The main result in this paper, theorem 3.3, gives a bound for the degree of the coefficient of the zeta function, which leads to the notion of leading exponent in section 3.1. In the case of convenient weighted homogeneous polynomial functions, this leading exponent enable to recover precious informations on the weights. These informations will be sufficient to conclude for theorem 4.7.

1. Motivic measure for arc space

In this section we recall briefly how we can measure arc spaces in the context of real geometry, using the more general theory of motivic integration as developed by Denef & Loeser [4].

The measure takes its value in the Grothendieck ring of real algebraic varieties [14]. It is defined as the free abelian group $K_0$ generated by isomorphism classes $[X]$ of real algebraic varieties modulo the subgroup generated by the relation $[X] = [Y] + [X \setminus Y]$ for $Y \subset X$ a closed subvariety. The ring structure comes from cartesian product of varieties.

1.1. Motivic zeta functions. Let $M$ be a real analytic manifold and $S$ a subset of $M$. Consider the space of formal arcs

$$\mathcal{L}(M, S) := \{ \alpha : (\mathbb{R}, 0) \to (M, S) : \alpha \text{ formal} \}.$$

We set $\mathcal{L}(M, x)$ when $S = \{x\}$ is reduced to one point, and let $L_k(M, x)$ is the set of $k$-jets of elements of $\mathcal{L}(M, x)$. We set $L_k = L_k(\mathbb{R}^n, 0)$.

Let $p_m : \mathcal{L}(\mathbb{R}^n, 0) \to L_m$ denote the map defined by taking $m$-jet. For a so-called constructible subset $A$ of $\mathcal{L}(\mathbb{R}^n, 0)$, we define

$$[A] = \lim_{m \to \infty} \frac{[p_m(A)]}{L^{mn}},$$

where $[p_m(A)]$ is the measure of $p_m(A)$ in $K_0$ and $L$ the measure of $\mathbb{R}$, when the limit exists as an element of $K_0(\text{Var}_{\mathbb{R}})[[L^{-1}]]$. This is for instance the case when $A$ is the preimage under a truncation map $p_m$ of a Zariski constructible subset of $L_k(M, x)$ (cf. [4]). The subsets of the arc space we will consider in this paper will all be constructible.

As an important example, we focus on the measure of the space of arcs with a specified order.
Example 1.1. For \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n, a_i \geq 0 \), we consider the set \( \mathcal{L}_a \) of arcs in \( \mathbb{R}^n \) whose \( i \)-th component vanishes if \( a_i = 0 \) or is of order \( a_i \) otherwise. Namely
\[
\mathcal{L}_a = \{ \alpha \in \mathcal{L}(\mathbb{R}^n, 0) : \text{ord} x_i \circ \alpha = a_i \ (i \in I(a)), \ x_i \circ \alpha \equiv 0 \ (i \notin I(a)) \}
\]
where \( I(a) = \{ i : a_i > 0 \} \). If \( m \) is greater than the maximal value of \( a_i, i = 1, \ldots, n \), then
\[
[p_m(\mathcal{L}_a)] = \mathbb{L}^m |I(a)| - \sum_i a_i (\mathbb{L} - 1)^{|I(a)|} = (\mathbb{L} - 1)^{|I(a)|} \mathbb{L}^m |I(a)| - s(a)
\]
where \( s(a) = \sum_{i=1}^n a_i \). Therefore
\[
[\mathcal{L}_a] = \lim_{m \to \infty} \frac{[p_m(\mathcal{L}_a)]}{\mathbb{L}^m n} = \begin{cases} (\mathbb{L} - 1)^n \mathbb{L}^{-s(a)}, & \text{if } |I(a)| = n, \\ 0, & \text{if } |I(a)| < n. \end{cases}
\]

In other words, arcs with some components equal to zero can be seen, as a finite order, as being the image under truncation of arcs with bigger order. We will use this description in order to compute in section 2.2 the arc spaces associated to a given real analytic function germ as follows.

Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) be a real analytic function germ. For \( k \in \mathbb{N} \), we define the arc space \( \mathcal{A}_k(f) \) by
\[
\mathcal{A}_k(f) = \{ \alpha \in \mathcal{L}(\mathbb{R}^n, 0) : f \circ \alpha(t) = ct^k + \cdots, \ c \neq 0 \}.
\]
Similarly, we defined arc spaces with signs \( \mathcal{A}_k^\pm(f) \) by
\[
\mathcal{A}_k^\pm(f) = \{ \alpha \in \mathcal{L}(\mathbb{R}^n, 0) : f \circ \alpha(t) = ct^k + \cdots, \ c = \pm 1 \}.
\]

Since the \( k \)-jet of \( f \circ \alpha \) determines the \( k \)-jet of \( f \circ \alpha \), we obtain
\[
[A_k(f)] = \frac{[p_m(\mathcal{A}_k(f))]}{\mathbb{L}^m n}, \quad [A_k^\pm(f)] = \frac{[p_m(\mathcal{A}_k^\pm(f))]}{\mathbb{L}^m n}.
\]
for \( m \geq k \). The associated zeta function, and zeta functions with signs, are defined by
\[
Z(f) = \sum_{k=1}^{\infty} [A_k(f)] t^k, \quad Z^\pm(f) = \sum_{k=1}^{\infty} [A_k^\pm(f)] t^k.
\]

Example 1.2. Consider the one variable polynomial function given by \( f(x) = x^d \). Then
\[
[A_k(f)] = \begin{cases} (\mathbb{L} - 1)^{1-a} & (k = ad) \\ 0 & (d \not| k) \end{cases}
\]
so that
\[
Z(f) = \frac{(\mathbb{L} - 1) t^d / \mathbb{L}}{1 - t^d / \mathbb{L}}.
\]

1.2. Virtual Poincaré polynomial. For real algebraic varieties, the best realization known of the Grothendieck ring is given by the virtual Poincaré polynomial [14]. It assigns to a Zariski constructible set a polynomial with integer coefficients in such a way that the coefficients coincide with the Betti numbers with \( \mathbb{Z}_2 \)-coefficients for proper regular real algebraic sets. Denoting by \( u \) the indeterminacy, the virtual Poincaré polynomial of a \( n \)-dimensional sphere is equal to \( 1 + u^n \), and a consequence of the additivity the virtual Poincaré polynomial of an affine \( n \)-dimensional space is \( u^n \). Moreover, the virtual Poincaré polynomial specializes to the Euler characteristic with compact supports when it is evaluated at \( u = -1 \).
A crucial property of the virtual Poincaré polynomial is that its degree is equal to the dimension of the set. In particular, and contrary to the Euler characteristics with compact supports, the virtual Poincaré polynomial cannot be zero for a non-empty set. In particular we will be interested in the degree of \([A_k(f)]\) in section 3.

2. ARC SPACES AND NEWTON POLYHEDRON

In this section we are interested in expressing the mesure of the arc spaces associated to a germ in terms of its Newton polyhedron. Similar results are already used in [3, 5, 11]. Here we focus mainly on a formula that will enable us to estimate efficiently the degrees of the virtual Poincaré polynomial of the arc spaces in terms of the Newton polyhedron.

We begin by introducing some standard notations for Newton polyhedron.

2.1. Newton polyhedron. Let \(f : \mathbb{R}^n \to \mathbb{R}\) denote a polynomial function. Consider its Taylor expansion at 0:

\[
f(x) = \sum_{\nu \in \mathbb{N}^n} c_{\nu} x^\nu, \quad x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n.
\]

Let \(\Gamma(f)\) denote the Newton polyhedron of \(f\), defined as the convex hull of the set

\[
\bigcup \{ \{\nu\} + \mathbb{R}^n_\geq : c_{\nu} \neq 0 \}.
\]

The Newton boundary \(\Gamma(f)\) of \(f\) is the union of the compact faces of \(\Gamma(f)\). We denote by \(\gamma < \Gamma(f)\) the belonging of the face \(\gamma\) to \(\Gamma(f)\).

For \(a = (a_1, \ldots, a_n) \in \mathbb{R}^n\), \(\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n\), we set \(\langle a, \nu \rangle = a_1 \nu_1 + \cdots + a_n \nu_n\), and define the multiplicity \(m_f(a)\) of \(a\) along \(f\) by

\[
m_f(a) = \min \{ \langle a, \nu \rangle : \nu \in \Gamma(f) \}
\]

and the face \(\gamma_f(a)\) of the Newton polyhedron associated to \(a\) by

\[
\gamma_f(a) = \{ \nu \in \Gamma(f) : \langle a, \nu \rangle = m_f(a) \}.
\]

Define also

\[
f_S(x) = \sum_{\nu \in S} c_{\nu} x^\nu
\]

for a subset \(S\) of \(\mathbb{R}^n\).

We define an equivalence relation in \(\mathbb{R}^n_\geq\) by

\[
a \sim b \iff \gamma_f(a) = \gamma_f(b)
\]

for \(a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n_\geq\). We call the quotient space \(\mathbb{R}^n_\geq / \sim\) by the dual Newton polyhedron and denote it by \(\Gamma^*(f)\). This is often identified with a cone subdivision of \(\mathbb{R}^n_\geq\).

Let \(\Gamma^{(1)}(f)\) denote the set of primitive generators of the 1-cones of \(\Gamma^*(f)\). We denote by \(\Gamma^{(1)}_+(f)\) the set of \(a \in \Gamma^{(1)}(f)\) with \(m_f(a) > 0\).
2.2. Motivic invariant of polynomial function. We want to express the measure of the arc spaces associated to a polynomial function in terms of its Newton polyhedron. The set of exponents $k$ for which the arc spaces $A_k(f)$ are not empty have already been studied in the context of blow-analytic equivalence (they are the so-called Fukui Invariants, cf [8]). This set coincides with the set of exponents that do appear in the zeta function with non-zero coefficients since there measure under the virtual Poincaré polynomial cannot be equal to zero (cf. section 1.2). Actually, define subsets $A(f)$ and $A^\pm(f)$ of $\mathbb{N}$ by

$$A(f) = \{ k : A_k(f) \neq \emptyset \}$$

and

$$A^\pm(f) = \{ k : A^\pm_k(f) \neq \emptyset \}.$$ 

Set

$$m_0(f) = \min\{ m_{f(a)} : f_{\gamma(a)} \text{ is not definite} \},$$

and

$$T(f) = \{ m \in \mathbb{N} : m \geq m_0(f) \},$$

$$S(f) = \{ m_{f(a)} : \exists c \in (\mathbb{R}^*)^n, f_{\gamma(a)}(c) \neq 0 \},$$

$$S^\pm(f) = \{ m_{f(a)} : \exists c \in (\mathbb{R}^*)^n, f_{\gamma(a)}(c) = \pm 1 \}.$$ 

Then

$$A(f) = S(f) \cup T(f), \quad A^\pm(f) = S^\pm(f) \cup T(f).$$

We will now be interested in computing the measure of the non-empty arc spaces in terms of the Newton polyhedron of $f$. Define algebraic subsets $X_\gamma$ and $X^\pm_\gamma$ of $(\mathbb{R}^*)^n$ associated to a face $\gamma$ of the Newton polyhedron by

$$X_\gamma = \{ c \in (\mathbb{R}^*)^n : f_\gamma(c) = 0 \},$$

$$X^\pm_\gamma = \{ c \in (\mathbb{R}^*)^n : f_\gamma(c) = \pm 1 \}.$$ 

Remark 2.1. The measure of $X_\gamma$ and $X^\pm_\gamma$ contain $(L - 1)^{n-\dim \gamma}$ as a factor in the sense that there exist varieties $\hat{X}_\gamma$, $\hat{X}^\pm_\gamma$ in $(\mathbb{R}^*)^{\dim \gamma}$ so that

$$X_\gamma = (\mathbb{R}^*)^{n-\dim \gamma} \times \hat{X}_\gamma$$

and

$$X^\pm_\gamma = (\mathbb{R}^*)^{n-\dim \gamma} \times \hat{X}^\pm_\gamma.$$ 

Looking at measure, we therefore obtain

$$[X_\gamma] = (L - 1)^{n-\dim \gamma}[\hat{X}_\gamma]$$

and

$$[X^\pm_\gamma] = (L - 1)^{n-\dim \gamma}[\hat{X}^\pm_\gamma].$$

We say $f$ is non-degenerate if all singular points of $f_\gamma$ are concentrated in the hyperplane coordinates for all are compact faces $\gamma$ of $\Gamma_+(f)$, namely

$$\left( \frac{\partial f_\gamma}{\partial x_1}(c), \ldots, \frac{\partial f_\gamma}{\partial x_n}(c) \right) \neq (0, \ldots, 0)$$ 

for each $c \in (\mathbb{R}^*)^n$ with $f_\gamma(c) = 0$, where $\gamma$ runs along the compact faces of $\Gamma_+(f)$.
If \( f \) is non-degenerate, then \( X_\gamma \) (resp. \( \hat{X}_\gamma \)) is a non-singular submanifold of \( (\mathbb{R}^*)^n \) (resp. \( (\mathbb{R}^*)^{l(\gamma)} \)) of codimension 1, whenever it is not empty.

Next lemma computes the measure of the arc spaces associated to \( f \) for arcs with a specified order \( a \in \mathbb{N}^n \). As motivated in section \([\square]\) it is sufficient to consider strictly positive orders.

**Lemma 2.2.** Take \( a > 0 \) in \( \mathbb{N}^n \). If \( f \) is non-degenerate and \( \gamma = \gamma(a) \), then we have

\[
[L_a \cap A_k(f)] = \begin{cases}
0 & \text{if } m_f(a) > k \\
((L - 1)^n - [X_\gamma]) L^{-s(a)} & \text{if } m_f(a) = k, \\
(L - 1) [X_\gamma] L^{-s(a) - k + m_f(a)} & \text{if } m_f(a) < k.
\end{cases}
\]

In the case with signs

\[
[L_a \cap A^\pm_k(f)] = \begin{cases}
0 & \text{if } m_f(a) > k \\
[X_\gamma^\pm] L^{-s(a)} & \text{if } m_f(a) = k, \\
[X_\gamma] L^{-s(a) - k + m_f(a)} & \text{if } m_f(a) < k.
\end{cases}
\]

**Proof.** Take \( \alpha \in L_a \) and define \( \phi(t) = (\phi_1(t), \ldots, \phi_n(t)) \) by

\[
\alpha(t) = (t^{a_1} \phi_1(t), \ldots, t^{a_n} \phi_n(t)).
\]

We remark that \( \phi_i(0) \neq 0 \) for \( i = 1, \ldots, n \). Then we have

\[
f(\alpha(t)) = \sum_{\nu \in \mathbb{N}^n} c_{\nu} (t^{a_1} \phi_1(t))^{\nu_1} \cdots (t^{a_n} \phi_n(t))^{\nu_n} = \sum_{\nu \in \mathbb{N}^n} c_{\nu} \phi(t)^{\nu} t^{a_{\nu}}.
\]

Setting \( F(t) = t^{-m_f(a)} f(\alpha(t)) \), we obtain therefore

\[
F(t) = f_a(\phi(t)) + R(t)
\]

where \( R(t) \) is a function, depending on \( t \) and the coefficient of \( \phi(t) \), with \( R(t) \to 0 \) (\( t \to 0 \)). This implies the assertion in case \( m_f(a) = k \).

We focus now on the case \( m_f(a) < k \). By differentiating \( (1) \) by \( t \), we obtain

\[
F'(t) = \sum_{i=1}^{n} \frac{\partial f_a}{\partial x_i}(\phi(t))(x_i \phi)'(t) + R'(t).
\]

Differentiating again, the result is

\[
F''(t) = \sum_{i=1}^{n} \frac{\partial^2 f_a}{\partial x_i^2}(\phi(t))(x_i \phi)''(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f_a}{\partial x_i \partial x_j}(\phi(t))(x_i \phi)'(t)(x_j \phi)'(t) + R''(t).
\]

Repeating differentiation \( \ell \) times, we obtain

\[
F^{(\ell)}(t) = \sum_{i=1}^{n} \frac{\partial^{\ell} f_a}{\partial x_i^\ell}(\phi(t))(x_i \phi)^{(\ell)}(t) + \cdots + R^{(\ell)}(t).
\]

We remark that \( (x_i \phi)^{(\ell)}(0) \) concerns only on the first term of the equation when we evaluate \( t \) at \( t = 0 \). We consider the condition

\[
F'(0) = F''(0) = \cdots = F^{(k - m(a) - 1)}(0) = 0.
\]

Since \( f \) is non-degenerate, one of the partial derivatives of \( f_a \) is non-zero on the zero locus of \( f_a \), and \( (2) \) determines the coefficient of \( \phi(t) \), inductively. The equation

\[
F^{(k - m(a))}(0) \neq 0 \quad \text{resp. } F^{(k - m(a))}(0) = \pm 1
\]
imposes that a coefficient of \( \phi(t) \) should be non-zero (resp. determines a coefficient of \( \phi(t) \)). So this implies the assertion in case \( m_f(a) < k \).

As a consequence we obtain a nice description of the measure of the arc spaces associated to \( f \) in terms of the geometry of \( f \) and of the combinatorics of its Newton polyhedron. For a face \( \gamma \) of \( \Gamma_+ (f) \) and \( k \in \mathbb{N} \), set

\[
P_k (\gamma) = \sum_{a > 0, \, \gamma(a) = \gamma, \, m_f(a) = k} \mathbb{L}^{-s(a)}
\]

and

\[
Q_k (\gamma) = \sum_{a > 0, \, \gamma(a) = \gamma, \, m_f(a) < k} \mathbb{L}^{-k + m_f(a) - s(a)}.
\]

Note that, even if the summation may be infinite (we consider it in the completed Grothendieck ring \([4]\)), \( P_k (\gamma) \) and \( Q_k (\gamma) \) are element of the Grothendieck ring of varieties localised at \( \mathbb{L} \) since it measure arcs with a fixed order along \( f \).

**Theorem 2.3.** If \( f \) is a non-degenerate polynomial, the measure of the arc spaces associated to \( f \) can be decomposed as:

\[
[\mathcal{A}_k (f)] = \sum_{\gamma \in \Gamma_+ (f)} ((\mathbb{L} - 1)^n - [X_\gamma]) P_k (\gamma) + (\mathbb{L} - 1) \sum_{\gamma \in \Gamma_+ (f)} [X_\gamma] Q_k (\gamma)
\]

In the case with signs, we obtain similarly:

\[
[\mathcal{A}_k^\pm (f)] = \sum_{\gamma \in \Gamma_+ (f)} [X_\gamma^\pm] P_k (\gamma) + \sum_{\gamma \in \Gamma_+ (f)} [X_\gamma] Q_k (\gamma).
\]

**Remark 2.4.** That decomposition of the measure of the arc spaces into two sums is motivated by the difference between arcs of order \( a \) that directly contribute to the coefficient of \( t^{m_f(a)} \) of the zeta functions and arcs that contribute to coefficient for bigger orders than \( m_f(a) \). To understand both contribution will be the main step in Section 3.3 in order to recover the weights of a weighted homogeneous polynomial.

**Remark 2.5.** We can rewrite \( P_k (\gamma) \) as

\[
P_k (\gamma) = \sum_{a > 0, \, \gamma(a) = \gamma, \, m_f(a) = k} \mathbb{L}^{-k + m_f(a) - s(a)}
\]

since \( m_f(a) = k \). Therefore the difference between \( P_k (\gamma) \) and \( Q_k (\gamma) \) lies in the value of \( m_f \). In particular, we are lead to focus on the levels of the piecewise linear function \( m_f - s \) defined on the dual of the Newton polyhedron of \( f \), and more precisely on the subsets defined by \( m_f = k \) and \( m_f < k \) for a given integer \( k \in \mathbb{N} \). In the sequel, we denote by \( h \) the function \( h = m_f - s \).

**Proof of theorem 2.3** By Lemma 2.2, we obtain

\[
[\mathcal{A}_k (f)] = \sum_{a > 0} [\mathcal{L}_a \cap \mathcal{A}_k (f)]
\]

\[
= \sum_{\gamma \in \Gamma_+ (f)} ((\mathbb{L} - 1)^n - [X_\gamma]) \sum_{a > 0, \, \gamma(a) = \gamma, \, m_f(a) = k} \mathbb{L}^{-s(a)}
\]

\[
+ (\mathbb{L} - 1) \sum_{\gamma \in \Gamma_+ (f)} [X_\gamma] \sum_{a > 0, \, \gamma(a) = \gamma, \, m_f(a) < k} \mathbb{L}^{-k + m_f(a) - s(a)}.
\]
Similarly, we have
\[ [A_k^\pm(f)] = \sum_{\gamma: \sigma} [L_{\sigma} \cap A_k^\pm(f)] \]
\[ = \sum_{\gamma:\Gamma_+} [X_{\gamma}^\pm] \sum_{a>0, \gamma(a)=\gamma, m_f(a)=k} L^{-s(a)} \]
\[ + \sum_{\gamma:\Gamma_+} [X_{\gamma}] \sum_{a>0, \gamma(a)=\gamma, m_f(a)<k} L^{-k+m_f(a)-s(a)}. \]

\[ \square \]

Remark 2.6. Note that it is possible to write down \([A_k(f)]\) in terms of closed algebraic subsets \(\bar{X}_\gamma\). This form will be useful in section 4. To this aim, we remark that there are integers \(\sigma\) are faces of \(\Gamma_+(f)\), so that
\[ \gamma = \sum_{\sigma \subset \gamma} m_{\gamma,\sigma} [\bar{X}_\sigma] \]
This is based on the following equalities: \(\bar{X}_\gamma = \sum_{\tau:\text{face of } \gamma} \bar{X}_\tau\).

For instance, if \(\dim \gamma = 1\), we clearly have \([\bar{X}_\gamma] = [\bar{X}_\gamma]\), whereas if \(\dim \gamma = 2\), we have \(\bar{X}_\gamma = [\bar{X}_\gamma] + \sum_{\tau:\text{1-face of } \gamma} [\bar{X}_\tau]\), and thus
\[ [\bar{X}_\gamma] = \bar{X}_\gamma - \sum_{\tau:\text{1-face of } \gamma} [\bar{X}_\tau]. \]

More generally we obtain a decomposition of the form
\[ [A_k(f)] = (L-1)^n \sum_{\gamma} P_k(\gamma) - \sum_{\sigma} (L-1)^{n-\dim \sigma} [\bar{X}_{\sigma}]P_k(\sigma) + \sum_{\sigma} (L-1)^{n-\dim \sigma+1} [\bar{X}_{\sigma}]Q_k(\sigma) \]
where
\[ P_k(\sigma) = \sum_{a>0, \gamma(a)=\gamma, m_f(a)=k} m_{\gamma(a),\sigma} (L-1)^{\dim \gamma(a)-\dim \sigma} L^{-s(a)} : \gamma(a) \supset \sigma, m_f(a) = k; \]
\[ Q_k(\sigma) = \sum_{a>0, \gamma(a)=\gamma, m_f(a)<k} m_{\gamma(a),\sigma} (L-1)^{\dim \gamma(a)-\dim \sigma} L^{-s(a)-k+m_f(a)} : \gamma(a) \supset \sigma, m_f(a) < k. \]

Remark 2.7. Let \(\Sigma\) be a nonsingular subdivision of \(\Gamma^*(f)\). Let \(\Sigma^{(k)}\) denote the set of \(k\)-cones of \(\Sigma\). We identifies \(\Sigma^{(1)}\) with the set of primitive vectors which generate 1-cones. We set
\[ N_k(m_1, \ldots, m_p) = \{(u_1, \ldots, u_p) \in \Z^p : u_1 m_1 + \cdots + u_p m_p = k, \ u_1 > 0, \ldots, u_p > 0\} \]
\[ N_{<k}(m_1, \ldots, m_p) = \{(u_1, \ldots, u_p) \in \Z^p : u_1 m_1 + \cdots + u_p m_p < k, \ u_1 > 0, \ldots, u_p > 0\} \]
for positive integers \(k, m_1, \ldots, m_p\). Then we have
\[ P_k(\gamma) = \sum_{\sigma \in \Sigma} \sum_{(u_1, \ldots, u_p) \in N_k(m(a^{(1)}), \ldots, m(a^{(p)}))} L^{-u_1 s(a^{(1)}) - \cdots - u_p s(a^{(p)})}, \]
where
\[ \sigma = (a^{(1)}, \ldots, a^{(p)}) \geq a^{(1)} = \cdots = a^{(p)} \in \Sigma^{(1)} \]
\[ \gamma(a^{(l)}) \supset \gamma (1 \leq l \leq p) \]
2.5, we investigate first the level of the piecewise linear function $h_{\text{polynomials}}$. We will use this description in the proof of proposition 2.10 below.

2.3. Two variables case. Theorem 2.3 enables to give a precise formula for the zeta functions (without using resolution of singularities). We will give in this section a complete description of the naïve zeta functions for two variables convenient zeta functions (without using resolution of singularities). We will give in this section a complete description of the naïve zeta functions for two variables convenient polynomials.

We consider a polynomial function $f(x_1, x_2)$ in two variables. As noticed in remark 2.2 we investigate first the level of the piecewise linear function $h = m_f - s$ on the dual of the Newton polyhedron of $f$.

**Lemma 2.8.** For any $a \in \mathbb{N}^2$ we have $h(a) \geq 0$. Moreover,

- $h(a) = 0$ if and only if $(a; m_f(a)) = (1, 1; 2)$.
- $h(a) = 1$ if and only if $(a; m_f(a)) = (1, 1; 3), (1, 2; 4), (2, 1; 4), (2, 3; 6), (3, 2; 6)$.

**Proof.** Let $p$ and $q$ be positive coprime integers. Let $(p_0 + qg, q_0)$ and $(p_0, q_0 + pg)$ be two successive vertices of the Newton polyhedron $\Gamma_+(f)$. The vector $(p, q)$ supports the face connecting these two points. Since $m_f(p, q) = p_0p + q_0q + pqg$, we have

$$m_f(p, q) - s(p, q) = (p_0 - 1)p + (q_0 - 1)q + pqg.$$  

- If $p_0 \geq 1$ and $q_0 \geq 1$, then
  $$m_f(p, q) - s(p, q) = (p_0 - 1)p + (q_0 - 1)q + pqg \geq 1.$$  
  The equality holds if and only if $p_0 = q_0 = p = q = g = 1$.
- If $p_0 \geq 1$ and $q_0 = 0$, then
  $$m_f(p, q) - s(p, q) = (p_0 - 1)p - q + pqg = (p_0 - 1)p + (p - 1)q \geq 0.$$  
  The equality holds if and only if $p_0 = p = g = 1$.
- If $p_0 = q_0 = 0$, then
  $$m_f(p, q) - s(p, q) = pqg - p - q.$$  
  The equality holds if and only if $g \geq 2$, $(p, q) = (1, 1)$, then $m_f(p, q) - s(p, q) = g - 2 \geq 0$.
  - If $g \geq 2$, then
    $$m_f(p, q) - s(p, q) = pqg - p - q = (g - 2)pq + (p - 1)q + g(p - 1) \geq 0.$$  
    The equality holds if and only if $g = 2$ and $p = q = 1$.
  - If $g \geq 2$ and $pq \geq 2$, then
    $$m_f(p, q) - s(p, a) = (g - 2)pq + (p - 1)q + (q - 1)p \geq 1.$$  
    The equality holds if and only if $g = 2$ and $(p, q) = (1, 2)$ or $(2, 1)$.
Proposition 2.10. Newton principal part. With the notations introduced upstairs, we have

\[ \frac{[A_k(f)]}{(L-1)^2} = L^{-d_1'} - L^{-d_2'} + \sum_{a_2 > d_1'} L^{-d_2' - a_2} + \sum_{a_1 > d_2'} L^{-a_1 - d_1'} = L^{-d_1'} - L^{-d_2'} + 2 \sum_{s \geq 1} L^{-d_1' - d_2' - s}. \]

Let \( f(x_1, x_2) \) be a polynomial with non-degenerate Newton principal part. Choose primitive vectors \( a^j \), for \( j = 0, 1, \ldots, q \), so that

\[ \Gamma^{(1)}(f) \subset \{ a^0, a^1, \ldots, a^q \}, \quad \det(a^j a^{j+1}) = 1. \]

We assume that \( f \) is convenient. Then \( a^0 = (1, 0) \) and \( a^q = (0, 1) \), and the vectors \( a^0, a^1, \ldots, a^q \) define a nonsingular subdivision of \( \Gamma^*(f) \).

Example 2.9. Let \( f(x_1, x_2) = x_1^{d_1} + x_2^{d_2} \), where \( d_1 \) and \( d_2 \) are even integers. Set \((d_1, d_2)\) the greatest common divisor of \( d_1 \) and \( d_2 \) and \( d_1' = d_1/(d_1, d_2), \ d_2' = d_2/(d_1, d_2) \). If \( k = d_1' d_2'(d_1, d_2) \), then we have

\[ \frac{[A_k(f)]}{(L-1)^2} = L^{-d_1'} - L^{-d_2'} + \sum_{a_2 > d_1'} L^{-d_2' - a_2} + \sum_{a_1 > d_2'} L^{-a_1 - d_1'} = L^{-d_1'} - L^{-d_2'} + 2 \sum_{s \geq 1} L^{-d_1' - d_2' - s}. \]

Proposition 2.10. Let \( f(x_1, x_2) \) be a convenient polynomial with non-degenerate Newton principal part. With the notations introduced upstairs, we have

\[ \frac{[A_k(f)]}{(L-1)^2} = \sum_{m_f(a^j)k} L^{-\frac{ks(a^j)}{m_f(a^j)}} + \sum_{j=1}^{q-1} \sum_{(u,v) \in N_k(m_f(a^j), m_f(a^{j+1}))} L^{-us(a^{j+1}) - vs(a^j)} \]

\[ + \sum_{m_f(a^q)k} L^{-\frac{ks(a^q-1)}{m_f(a^q)}} + \sum_{j=1}^{q-1} ((L-1) - [\hat{X}_{\gamma(a^j)}]) \sum_{m(a^j)k} L^{-\frac{ks(a^j)}{m(a^j)}} \]

\[ + (L-1) \sum_{j=1}^{q-1} [\hat{X}_{\gamma(a^j)}] \sum_{a: a > 0, \gamma(a)=\gamma, m_f(a)=k} L^{-s(a) - h(a^j)} \left( 1 - L^{-\frac{h(a^j)}{h(a^j)}} \right). \]

Proof. By theorem 2.3, we have the following description of the measure of \( A_k(f) \):

\[ \frac{[A_k(f)]}{(L-1)} = (L-1) \sum_{a: a > 0, m_f(a)=k, \dim \gamma(a)=0} L^{-s(a)} \]

\[ + \sum_{\gamma: \dim \gamma=1} ((L-1) - [\hat{X}_{\gamma}]) \sum_{a: a > 0, \gamma(a)=\gamma, m_f(a)=k} L^{-s(a)} \]

\[ + (L-1) \sum_{\gamma: \dim \gamma=1} [\hat{X}_{\gamma}] \sum_{a: a > 0, \gamma(a)=\gamma, m_f(a)<k} L^{-s(a) - k + m_f(a)} \]

Then

\[ \frac{[A_k(f)]}{(L-1)} = (L-1) \sum_{m_f(a^j)k} L^{-u - \frac{ks(a^j)}{m_f(a^j)}} + \sum_{j=1}^{q-1} \sum_{(u,v) \in N_k(m_f(a^j), m_f(a^{j+1}))} L^{-us(a^{j+1}) - vs(a^j)}. \]
As a consequence, we can recover from the zeta function of a polynomial germs a
notations of section 2.1 concerning the Newton polyhedron of \( f \).

Let \( u, v \in \mathbb{N} \) and we complete the proof.

\[
\left( \mathbb{L} - 1 \right) \sum_{j=1}^{q-1} \left( \mathbb{L} - 1 \right)^{j-1} \mathbb{L}^{k} \frac{a_j}{m_j} : l > k
\]

and we complete the proof.

3. Estimate of degrees

This section is the heart of the paper. We prove a linear bound for the degree of
the virtual Poincaré polynomial of the arc spaces. We show that this bound is sharp. As a consequence, we can recover from the zeta function of a polynomial germs a
leading tangent from a leading exponent, denoted by \( L_\varepsilon \). We show that this leading exponent gives back information about the weights for a weighted homogeneous polynomial with is non-degenerate and convenient.

Recall that \( u \) stands for the Poincaré polynomial of the affine line.

3.1. Leading exponent. Let \( f \) be a non-degenerate polynomial. We keep the notations of section 2.1 concerning the Newton polyhedron of \( f \). We define the
leading exponent of \( f \) to be

\[
L_e(f) = \sup \left\{ 0, 1 - \frac{s(a)}{m_f(a)} : a \in \Gamma^{(1)}_+(f) \right\}.
\]

The sign of the leading exponent will be of major importance in the sequel. The following statements are direct consequences of the definitions:

- If there is \( a \in \Gamma^{(1)}_+(f) \) with \( h(a) > 0 \), then \( L_e(f) > 0 \),
- If \( h(a) \leq 0 \) for any \( a \in \Gamma^{(1)}_+(f) \), then \( L_e(f) = 0 \).

We set \( \Gamma^{(1)}_{\max}(f) = \{ a \in \Gamma^{(1)}(f) : 1 - \frac{s(a)}{m_f(a)} = L_e(f) \} \).

**Proposition 3.1.**

- If \( (1, \ldots, 1) \not\in \Gamma_+(f) \), then \( L_e(f) > 0 \).
- If \( (1, \ldots, 1) \in \Gamma_+(f) \), then \( L_e(f) = 0 \) and \( \Gamma^{(1)}_{\max}(f) \neq \emptyset \).
- If \( (1, \ldots, 1) \in \text{Int} \Gamma_+(f) \), then \( L_e(f) = 0 \) and \( \Gamma^{(1)}_{\max}(f) = \emptyset \).

**Proof.** If \( (1, \ldots, 1) \in \Gamma_+(f) \), then we have \( h(a) \leq 0 \) for any \( a > 0 \). In fact, we have

\[
m_f(a) = \min \{ \langle a, \nu \rangle : \nu \in \Gamma_+(f) \} \leq \langle a, (1, \ldots, 1) \rangle = s(a).
\]

Since \( \frac{h(a)}{m_f(a)} \leq 0 \), we obtain \( L_e(f) = 0 \). In this case \( (1, \ldots, 1) \in \Gamma(f) \) if and only if there exist \( a \in \Gamma^{(1)}(f) \) with \( (1, \ldots, 1) \in \gamma(a) \), which implies \( m_f(a) = s(a) \), \( h(a) = 0 \) and \( \Gamma^{(1)}_{\max}(f) \neq \emptyset \). So \( (1, \ldots, 1) \in \text{Int} \Gamma_+(f) \) is equivalent to the fact that \( h(a) = m_f(a) - s(a) \) is strictly negative for any \( a > 0 \).

If \( (1, \ldots, 1) \not\in \Gamma_+(f) \), then \( h(a) > 0 \) for some \( a > 0 \), and we obtain \( L_e(f) > 0 \). □

Set \( R_k = \{ a = (a_1, \ldots, a_n) \in \mathbb{R}^n_+ : 0 \leq m(a) \leq k \} \).

**Lemma 3.2.** \( L_e(f) = \sup \{ h(a) : a \in R_1 \} \).

**Proof.** We first remark that there exist a finite polyhedral partition \( \{ P_i \} \) of \( R_1 \) so that \( m_f(a) \) is linear on each \( P_i \). Since \( \sup h|_{P_i} = \max \{ \sup h|_{P_i} \} \), it is enough to consider \( \sup h|_{P_i} \). So the supremum of \( h(a) \) on \( P_i \) is attained by a vertex of \( P_i \). We remark that multiple of \( a \in \Gamma^{(1)}(f) \) with \( m_f(a) = 0 \) cannot attain the maximum, since \( h(a) = -s(a) < 0 = h(0) \). So the possible vertices of \( P_i \) are 0, or \( \frac{k}{m_f(a)}a \) for \( a \in \Gamma^{(1)}_+(f) \). The values of \( h \) at these points are 0 or 1 - \( \frac{s(a)}{m_f(a)} \), and we obtain the result. □

The importance of the leading exponent lies in the fact that it gives rise to a bound for the degree of the virtual Poincaré polynomial of the arc spaces.

**Theorem 3.3.** We have the following inequality:

\[
\deg [\mathcal{A}_k(f)] \leq n - k + kL_e(f).
\]

- If \( (1, \ldots, 1) \not\in \text{Int} \Gamma_+(f) \), then there are arbitrary big \( k \) so that the equality in (3) holds.
- If \( (1, \ldots, 1) \in \text{Int} \Gamma_+(f) \), then \( h(a) < 0 \) for all \( a \in \Gamma^{(1)}_{\max}(f) \) and the equality in (3) does not hold. In that case the degree of \( [\mathcal{A}_k(f)] \) is

\[
n - k + \sup \{ h(a) : a \in R_k \cap \mathbb{Z}^n, a > 0 \}.
\]
Proof of Theorem 3.3. First, note that

\[ \sup\{ h(a) : a \in \mathbb{R} \} = k L_e(f). \]

Then

\[ L_e(f) = \sup \left\{ 0, 1 - \frac{s(la)}{m_f(la)} : a \in \Gamma_+^{(1)}(f), \ l > 0 \right\} \]

\[ = \sup \left\{ 0, 1 - s(la) : a \in \Gamma_+^{(1)}(f), \ m_f(la) = 1 \right\} \]

\[ = \sup \left\{ 0, 1 - s(la) : m_f(a) > 0, \ m_f(la) = 1 \right\} \]

by linearity of the function \( \frac{s(a)}{m_f(a)} \) on \( P_i \cap \{ a : m_f(a) = 1 \} \). Finally

\[ L_e(f) = \sup \left\{ 0, 1 - \frac{s(a)}{m_f(a)} : m_f(a) > 0 \right\}. \]

The leading monomial of \( \sum_{\gamma}((u-1)^{n} - [X_{\gamma}])P_k(\gamma) \) is attained by \( a^0 \) with

\[ h(a^0) = \sup \{ h(a) : a_i \geq 0, \ m_f(a) = k \}. \]

Thus we have

\[ \deg \sum_{\gamma}((u-1)^{n} - [X_{\gamma}])P_k(\gamma) = n + \max \{-s(a) : m_f(a) = k\} \]

\[ \leq n + kL_e(f) - k. \]

The leading monomial of \( \sum_{\gamma}[X_{\gamma}]Q_k(\gamma) \) is attained by \( a^0 \) with

\[ h(a^0) = \sup \{ h(a) : a \neq 0, \ a_i \geq 0, \ 0 < m_f(a) < k, \ X_{\gamma(a)} \neq \emptyset \}. \]

Thus we have

\[ \deg \sum_{\gamma}[X_{\gamma}]Q_k(\gamma) = -k + \max \{ \dim X_{\gamma(a)} + m_f(a) - s(a) : 0 < m_f(a) < k \} \]

\[ \leq n + kL_e(f) - k - 1. \]

By Theorem 2.3, we obtain the result. Since

\[ m(ta + (1-t)b) \geq tm_f(a) + (1-t)m(b), \quad s(ta + (1-t)b) = ts(a) + (1-t)s(b) \]

for \( t \in [0,1] \), we obtain

\[ h(ta + (1-t)b) \geq th(a) + (1-t)h(b). \]

If \( a \) and \( b \) attain the maximum of \( h \) on \( R_k \), then \( h(ta + (1-t)b) \) should be the maximum whenever \( ta + (1-t)b \in R_k \) \( (t \in [0,1]) \). \hfill \Box

Remark 3.4. We can give a precise description of those \( k \) that give equality in (3) in case \( (1, \ldots , 1) \notin \text{Int} \Gamma_+(f) \). Actually, it follows from the proof of theorem 2.3 that for \( a^1, \ldots , a^p \in \Gamma_+^{(1)}(f) \) and \( b_j > 0 \) \( (j = 1, \ldots , p) \) so that

\[ \gamma(a^1) \cap \cdots \cap \gamma(a^p) \neq \emptyset \] and \( \sum_{j=1}^{p} b_j a^j \in \mathbb{Z}^n \),

then the equality in (3) holds for \( k = \sum_{j=1}^{p} b_j m_f(a^j) \). Conversely, if the equality in (3) holds, then there exists \( a = \sum_{j=1}^{p} b_j a^j \in R_k \cap \mathbb{Z}^n \) \( (a > 0) \) with above conditions satisfied.
We focus now on how we can compute the leading exponent of \( f \) from its zeta function. Define \( \alpha_0(f) \) by
\[
\alpha_0(f) = \sup \{ \alpha : \lim_{u \to \infty} \frac{Z_f(u^{\alpha}t)}{u^n} = 0 \}.
\]

**Proposition 3.5.**

- We have \( \text{Le}(f) = 1 - \alpha_0(f) \).
- If \( \Gamma_{\max}^{(1)}(f) = \{ a \} \) with \( h(a) > 0 \), then
  \[
  \lim_{u \to \infty} \frac{Z_f(u^{\alpha_0(f)}t)}{u^n} = \frac{t^{m_f(a)}}{1 - t^{m_f(a)}}.
  \]

**Proof.** First we remark that
\[
\frac{Z_f(u^{\alpha}t)}{u^n} = \sum_{k=1}^{\infty} \deg [A_k(f)] u^{\alpha k} t^k.
\]
Since \( \deg [A_k(f)] \leq n - k + k\text{Le}(f) \), we can write
\[
\frac{[A_k(f)]u^{\alpha k}}{u^n} = c_k u^{(\text{Le}(f)-1)k} + \text{(lower order terms)}
\]
where
\[
c_k = \# \left\{ (b_a)_{a \in \Gamma_{\max}^{(1)}(f)} : a = \sum_{a \in \Gamma_{\max}^{(1)}(f)} b_a a \in \mathbb{Z}^n \cap R_k, m_f(a) = k, b_a \geq 0 \right\}.
\]
This implies that
\[
\frac{[A_k(f)]u^{\alpha k}}{u^n} = c_k u^{(\text{Le}(f)-1+\alpha)k} + \text{(lower order terms)},
\]
so that tending \( u \) to \( \infty \) implies
\[
\frac{[A_k(f)]u^{\alpha k}}{u^n} \to \begin{cases} 0 & \text{if } \alpha < 1 - \text{Le}(f), \\ c_k & \text{if } \alpha = 1 - \text{Le}(f), \\ \infty & \text{if } \alpha > 1 - \text{Le}(f). \end{cases}
\]
If \( \Gamma_{\max}^{(1)}(f) = \{ a \} \) and \( h(a) > 0 \), then \( c_k = 1 \) when \( m_f(a) \mid k \). We thus have
\[
\lim_{u \to \infty} \frac{Z_f(u^{\alpha_0(f)}t)}{u^n} = \sum_{k=1}^{\infty} t^{km_f(a)} = \frac{t^{m_f(a)}}{1 - t^{m_f(a)}},
\]
and the proof is achieved. \( \square \)

### 3.2. Contribution of facets

Let \( \gamma \) be an \( (n-1) \)-dimensional face of \( \Gamma_+(f) \). Take \( v \) with \( \gamma(v) = \gamma \).

**Lemma 3.6.** Then we obtain
\[
\deg P_k(\gamma) = \begin{cases} -\infty & \text{if } m_f(v) \not| k, \\ -k \frac{x(v)}{m_f(v)} & \text{if } m_f(v) \mid k \end{cases}
\]
and
by example 1.1, we obtain

We remark that

For

set

which shows the first statement. If

Proof. These are consequences of the following equalities:

\[
\deg P_k(\gamma) = \max\{-s(a) : a = mv \ (m \in \mathbb{Z}_+), \ m_f(a) = k\}
\]

\[
\deg Q_k(\gamma) = -k + \max\{m_f(a) - s(a) : a = mv \ (m \in \mathbb{Z}_+), \ m_f(mv) \leq k - 1\}
\]

which follow from the previous discussion. □

Corollary 3.7. When \(m_f(v)|k\), we have

- If \(h(v) > 0\), then \(\deg P_k(\gamma) > \deg Q_k(\gamma)\).
- If \(h(v) = 0\), then \(\deg P_k(\gamma) = \deg Q_k(\gamma)\).
- If \(h(v) < 0\), then \(\deg P_k(\gamma) < \deg Q_k(\gamma)\).

Proof. If \(m_f(v) - s(v) > 0\) and \(m_f(v)|k\), we obtain

\[
\deg P_k(\gamma) - \deg Q_k(\gamma) = -k \frac{s(v)}{m_f(v)} + k - \left\lfloor \frac{k-1}{m_f(v)} \right\rfloor (m_f(v) - s(v))
\]

\[
= \left(\frac{k}{m_f(v)} - \left\lfloor \frac{k-1}{m_f(v)} \right\rfloor \right) (m_f(v) - s(v))
\]

\[
= (m_f(v) - s(v)) > 0,
\]

which shows the first statement. If \(m_f(v) - s(v) \leq 0\) and \(m_f(v)|k\), we obtain

\[
\deg P_k(\gamma) - \deg Q_k(\gamma) = -k \frac{s(v)}{m_f(v)} + k - (m_f(v) - s(v))
\]

\[
= \left(\frac{k}{m_f(v)} - 1\right) (m_f(v) - s(v)),
\]

and this implies the remaining statement. □

A contribution of facet to \(Z_f(t)\) can be seen as follows.

Lemma 3.8. We have

\[
\lim_{u \to 1} \frac{[A_k(f)]}{u - 1} = -\sum_{a>0, \ m(a)=k, \ \dim \gamma(a)=n-1} [X_{\gamma(a)}]_{u=1}.
\]

We prepare some notations for the proof of lemma 3.8. For \(I \subset \{1, 2, \ldots, n\}\), we set

\[
f_I(x) = \sum_{\nu : \nu_i=0, \ i \notin I} c_\nu x^\nu.
\]

For \(a = (a_1, \ldots, a_n)\), we define

\[
m_I(a) = \min\{\langle a, \nu \rangle : \nu \in \Gamma_+(f_I(a))\},
\]

\[
\gamma_I(a) = \{\nu \in \Gamma_+(f_I(a)) : \langle a, \nu \rangle = m_I(a)\}.
\]

We remark that \(L_a \cap A_k(f) = L_a \cap A_k(f_I(a))\). Since \([L_I(a)] = 0\) for \(I \subset \{1, \ldots, n\}\) by example 1.1, we obtain

\[
[A_k(f)] = \sum_{a : I(a) \neq \{1, \ldots, n\}} [L_a \cap A_k(f)].
\]
Lemma 3.9. consequence of proposition 3.5. even thus we obtain nondegenerate with respect to its Newton polyhedron.

We obtain the result tending $u$ to 1. \hfill \Box

3.3. Weighted homogeneous polynomials. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a weighted homogeneous polynomial, namely there exists weights $w_1, \ldots, w_n \in \mathbb{N}$ relatively prime and a weighted degree $d \in \mathbb{N}$ such that $f(x_1^{w_1}, \ldots, x_n^{w_n})$ is homogeneous of degree $d$.

In that case $\Gamma(f)$ has a unique facet $\gamma_f$, with the associated 1-cone generated by the primitive vector $v = \text{lcm}(w_1, \ldots, w_n)(\frac{1}{w_1}, \ldots, \frac{1}{w_n})$, and then $m_f(v) = \text{lcm}(w_1, \ldots, w_n)$ and $h(v) = \text{lcm}(w_1, \ldots, w_n)(1 - \sum_{i=1}^{n} \frac{1}{w_i})$. Moreover $L_e(f) = \sup\{0, 1 - \sum_{i=1}^{n} \frac{1}{w_i}\}$. In particular, if we are able to compute $h(v)$ and $m_f(v)$ from the zeta function, then we can recover the sum of the inverse of the weights of $f$.

Let us assume that the Newton polyhedron of $f$ is convenient, that is the monomials $x_1^{\frac{d}{w_i}}$ do appear in the expression $f$ with non-zero coefficient, and that $f$ is nondegenerate with respect to its Newton polyhedron.

By proposition 3.5 we recover $L_e(f)$ from the zeta function of $f$, and recover even $m_f(v)$ if $h(v) > 0$. We focus now on recovering $h(v)$. Next lemma is a direct consequence of proposition 3.5.

Lemma 3.9. $L_e(f) > 0$ if and only if $h(v) > 0$, and more precisely $h(v) = m_f(v)L_e(f)$. 
In case $L_e(f) = 0$, the situation is more difficult to handle. Note that generically, the degree of $[A_k(f)]$ is given by $n + \max \{ \deg P_k(\gamma_f), \deg Q_k(\gamma_f) \}$ — it may be different if some $X_\gamma$ are empty. The degree of $P_k(\gamma_f)$ and $P_k(\gamma_f)$ may be express as

$$\deg P_k(\gamma) = \max \{-s(a) : m_f(a) = k\} = \max \{-k + m_f(a) - s(a) : m_f(a) = k\} = -k + \max \{ h(a) : m_f(a) = k\}$$

whereas

$$\deg Q_k(\gamma) = \max \{-k + m_f(a) - s(a) : m_f(a) < k\} = -k + \max \{ h(a) : m_f(a) < k\}.$$

Therefore we are lead to understand the levels of the function $h = m_f - s$ on $\mathbb{N}^n$, and more precisely on the subsets of $\mathbb{N}^n$ defined by $\{ m_f(a) = k \}$ and $\{ m_f(a) < k \}$.

To begin with, let us forget that we are interested in integral points and describe its levels on $\{ a \in \mathbb{R}^n, a_i \geq 0 \}$. The function $h$ is linear on each cone of $\Gamma^v(f)$, therefore its levels are completely described by its value on $v$ and on the canonical basis $\{ e_1, \ldots, e_n \}$ of $\mathbb{R}^n$. Note that $h(e_i) = m_f(e_i) - 1 \geq -1$, with equality in case the Newton polyhedron is convenient.

In particular,

- if $h(v) = 0$ there are only negative levels that are cylinder parallel to the line generated by $v$,
- if $h(v) < 0$ they are (bounded) simplices with a vertex on the line generated by $v$ and other vertices on the positive coordinates axis, whereas
- if $h(v) > 0$, they are unbounded simplices with a vertex on the line generated by $v$.

Coming back to the computation of the degree of $P_k(\gamma)$ and $Q_k(\gamma)$, we need to investigate the integral points on these levels.

**Lemma 3.10.** Assume $L_e(f) = 0$ and $[X_\gamma] \neq 0$.

- $\deg [A_k(f)] \leq n - k$ with equality for $k$ big enough if and only if $h(v) = 0$,
- $\deg [A_k(f)] < n - k$ if and only if $h(v) < 0$. In that case $\deg [A_k(f)] = n - k + \max \{ h(a) : a > 0, \; [X_\gamma(a)] \neq 0 \}$.

In particular in the case $h(v) < 0$, we can only recover the sign of $h(v)$, but not its value.

**Proof.** It suffices to compute the maximum of $h$ on $\{ m_f(a) = k \}$ and on $\{ m_f(a) < k \}$. In the convenient case the levels of $m_f$ are given by translation of the positive part of the hyperplane coordinate along the line generated by $v$. Then if $h(v) = 0$ the maximum on $\{ m_f(a) = k \}$ as a real number is attained on the line generated by $v$. In particular, if $k > m_f(v)$, this maximum is attained in $v$.

In case $h(v) < 0$, the levels of $h$ decreases along the line generated by $v$, therefore for $k > m_f(v)$, this maximum is bigger than $h(v)$ and strictly negative. □

**Remark 3.11.** It may happen that $\max \{ h(a) : a > 0 \}$ is strictly bigger then $h(v)$ in the case $h(v) < 0$. Consider for example $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^p$ with $p$ odd. A direct computation gives that $v = (2, 2, m)$ and $h(v) = -2$ whereas $h(1, 1, 1) = -1$. 
Assume $L_e(f) = 0$ and $[X_{\gamma}] = 0$. Then $[X_\gamma] = 0$ for all face $\gamma$ of $\Gamma(f)$, therefore the degree of $[A_k(f)]$ is only given by $P_k(\gamma)$ and the discussion is simplified.

**Lemma 3.12.** Assume $L_e(f) = 0$ and $[X_{\gamma}] = 0$.
- $\deg[A_k(f)] \leq n - k$ with equality for infinitely many $k$ if and only if $h(v) = 0$,
- $\deg[A_k(f)] < n - k$ if and only if $h(v) < 0$.

**Proof.** As a consequence of the proof of theorem 2.3 we have

$$\deg P_k(\gamma) = \max\{-s(a) : m_f(a) = k\} \leq s\left(\frac{k}{m_f(v)}\right),$$

with equality when $m_f(v)$ divides $k$. Now if $h(v) = 0$ then $s\left(\frac{k}{m_f(v)}\right) = k$ whereas $s\left(\frac{k}{m_f(v)}\right) > k$ if $h(v) < 0$. \hfill \Box

Therefore we are able to recognize the sign of $h(v)$ from the zeta function of $f$, and its value if it is positive or zero.

**Proposition 3.13.** Assume $f$ is a convenient weighted homogeneous polynomial non degenerate with respect to its Newton polyhedron. Denote by $v$ the primitive vector associated to $f$.
- $h(v) > 0$ if and only if $L_e(f) > 0$, and more precisely $h(v) = m_f(v)L_e(f)$.
- $h(v) = 0$ if and only if $\deg[A_k(f)] \leq n - k$, with equality for infinitely many $k$.
- $h(v) < 0$ if and only if $\deg[A_k(f)] < n - k$.

**Remark 3.14.** In [1], it is shown that the weights of two variables non-degenerate weighted homogeneous polynomials are invariants under blow-analytic equivalence, using the zeta function defined with the Euler characteristic with compact support. Because of the properties of the virtual Poincaré polynomials, we can recover easily the same result, in the setting of blow-Nash equivalence. Actually the first exponent of the zeta function combined with the leading exponent $L_e(f)$ give the weights.

## 4. Recovering the weights

We prove that we can recover the weights of a convenient non-degenerate weighted homogeneous polynomials in three variables. Similarly to the two variables case (cf. remark 3.14), it easy to recover the multiplicity whereas the inverse of the sum of the weights is obtained by proposition 3.13. We prove below that in the three variables case, we are able to recover the ultimate weight.

Let $f(x_1, x_2, x_3)$ be a weighted homogeneous polynomial whose Newton polyhedron is convenient. Let $(p_1, 0, 0)$, $(0, p_2, 0)$ and $(0, 0, p_3)$ denote the vertices of $\Gamma_+(f)$. Assume $p_1 \leq p_2 \leq p_3$ without lost of generality.

Let $\gamma$ denote the compact 2-dimensional face of $\Gamma_+(f)$. Set $\gamma_i = \gamma \cap \{\nu_i = 0\}$. Set $p_{ij} = \text{LCM}(p_i, p_j)$ and $p_{123} = \text{LCM}(p_1, p_2, p_3)$. As a consequence of theorem 2.3 and remark 2.6 we can describe completely the zeta function of $f$.

**Proposition 4.1.**

$$\frac{[A_k(f)]}{u - 1} = \frac{P_k}{u^{\frac{1}{p_1}} + \frac{1}{p_2} + \frac{1}{p_3}} + \sum_{1 \leq l < k} \frac{(u - 1)Q_l}{u^{k - l + \frac{1}{p_1}} + \frac{1}{p_2} + \frac{1}{p_3}}.$$
where

\[
P_k = \begin{cases} 
0 & p_i \nmid k \ (i = 1, 2, 3) \\
1 & p_i k, \ p_{ij} \nmid k \ (i \neq j) \\
1 + u - [\hat{X}_{\gamma_1}] & p_{ij} \mid k, \ p_{123} \nmid k, \ \{i, j, s\} = \{1, 2, 3\} \\
1 + u + u^2 - [X_\gamma] & p_{123} \mid k 
\end{cases}
\]

and

\[
Q_l = \begin{cases} 
0 & p_i \nmid l \ (i = 1, 2, 3) \ \text{or} \ p_i \mid l, \ p_{ij} \nmid l \ (i \neq j) \\
[X_{\hat{\gamma}_1}] & p_{ij} \mid l, \ p_{123} \nmid l, \ \{i, j, s\} = \{1, 2, 3\} \\
[X_{\hat{\gamma}_1}] + [X_{\hat{\gamma}_2}] + [X_{\hat{\gamma}_3}] + [\hat{X}_\gamma^u] & p_{123} \mid l 
\end{cases}
\]

Corollary 4.2. the following inequality holds:

\[
\deg[A_k(f)] \leq 3 - \frac{k}{p_1} - \frac{k}{p_2} - \frac{k}{p_3}.
\]

Remark 4.3. Note in particular that

- if \( p_1 < p_2 \leq p_3 \), we have \( [A_{p_1}(f)] = \frac{u-1}{u} \frac{u-1}{(u-1)(1+u-[X_{\gamma_1}])} \).
- if \( p_1 = p_2 < p_3 \), we have \( [A_{p_1}(f)] = \frac{u-1}{u^2} \frac{u-1}{(u-1)(1+u+u^2-[X_\gamma])} \).
- if \( p_1 = p_2 = p_3 \), we have \( [A_{p_1}(f)] = \frac{u-1}{u^3} \frac{u-1}{(u-1)(1+u+u^2-[X_\gamma])} \).

However, in order to recover the weights, it will be enough to concentrate the study on some specific part of the zeta function. Actually, it is enough to recover the integers \( p_1, p_2 \) and \( p_3 \) from the zeta function of \( f \). Note that we already recover the multiplicity of \( f \), that is \( p_1 \), as the order of the zeta function. Moreover we know from proposition 3.13 how to recover the sign of \( h(v) \). In the particular case of \( h(v) < 0 \) the function \( f \) has only simple singularities in the sense of Arnold 2 and we already know how to recover the weights from 7. In the general situation, if \( h(v) \geq 0 \) we obtain \( L_\varepsilon(f) \) by proposition 3.5 which is equal to \( 1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} \). It is therefore sufficient to find \( p_2 \) in order to recover all the weights.

The idea is to recover \( p_2 \) in the zeta function as the first contribution that does not come from the smallest face of the Newton polyhedron \((p_1, 0, 0)\). Denote by \( A(f) \) the set of Fukui invariants \( A(f) = \{k : [A_l] \neq 0\} \).

We treat the cases \( p_1 \) even and \( p_1 \) odd separately. In case \( p_1 \) is odd, note that \( [X_{\gamma_1}] \neq 0 \), and therefore \( A(f) \cap \mathbb{N}_{\geq p_2} = \mathbb{N}_{\geq p_2} \).

Set

\[
\alpha = \min\{l \in \mathbb{N} : A(f) \cap \mathbb{N}_{\geq l} = \mathbb{N}_{\geq l}\},
\]

\[
\beta = \min\{l \in \mathbb{N} : [A_l] \neq 0, \ p_1 \nmid l\}.
\]

As \( p_1 \) is odd, note that \( \alpha \leq p_{12} \).

Lemma 4.4. Assume \( p_1 \) is odd.

- If \( p_1 \nmid \beta - 1 \), then \( p_2 = \beta \).
- If \( p_1 \mid \beta - 1 \) and \( \beta - 1 < \alpha \), then \( p_2 = \beta \).
- If \( p_1 \mid \beta - 1 \) and \( \beta - 1 = \alpha \), then either \( p_2 = \beta - 1 \) or \( p_2 = p_3 = \beta \).
\textbf{Proof.} If \( p_1 \) divides \( p_2 \), then \( \alpha = p_2 \) et \( \beta = \alpha + 1 \), so that \( p_1 \) divides \( \beta - 1 \). As \( \beta = p_2 \) if \( p_1 \) does not divide \( p_2 \), we obtain the first point.

Now, if \( p_1 \) divides \( \beta - 1 \), then either \( p_1 \) divides \( p_2 = \beta - 1 \) and \( \alpha = p_2 \) or \( p_1 \) divides \( p_2 - 1 \) and \( p_2 = \beta \).

In particular, if \( p_1 \mid \beta - 1 \) and \( \beta = \alpha + 1 \), we obtain two possibilities for the value \( p_2 \). We show below that only one of these possibilities gives the correct value for the sum \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \), except in one particular case that we need to treat separately.

\textbf{Lemma 4.5.} Assume \( p_1 \mid \beta - 1 \) and \( \beta = \alpha + 1 \). Assume moreover that the value of \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \) is given. Then we can decide whether \( p_2 = \alpha \) or \( p_2 = \beta \), except in the cases \( (p_1, p_2, p_3) = (3, 4, 4) \) or \( (3, 3, 6) \).

\textbf{Proof.} Assume \((p_2, p_3) = (\beta, \beta)\) and \((p'_2, p'_3) = (\alpha, l)\) with \( l \geq \alpha \) satisfying

\[
\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p'_2} + \frac{1}{p'_3}.
\]

Then \( l = \frac{\alpha(\alpha+1)}{\alpha-1} \) should be integer therefore \( \alpha = 3 \) and \( l = 6 \).

In case \( p_1 \) is even, it may arrive that \( \alpha = \infty \) (if \([X\gamma]\) = 0), and even \( \beta = \infty \) (if \( p_1 \) divides \( p_2 \) and \( p_3 \)). Therefore we need to take care also about the coefficients of the zeta function.

Set \( \delta = \min\{l : -1 \text{ is a root of } [A_l]\} \).

\textbf{Lemma 4.6.} Assume \( p_1 \) is even. Then \( p_2 = \min\{\alpha, \beta, \delta\} \) except if \( \alpha = \beta - 1 \) and \([A_\alpha] = u^{-\frac{\alpha}{\alpha-1}}(u-1)\). In that case \( p_2 = \beta \).

\textbf{Proof.} If \( \alpha = \infty \), then \([A_{p_2}] = \) if \( p_2 < p_3 \) whereas \([A_{p_2}] = \) in case \( p_2 = p_3 \). Therefore \( p_2 = \delta \leq \beta \).

If \( p_1 \) does not divide \( p_2 \) and \( p_2 < p_3 \), then \( p_2 = \beta \leq \delta \). We claim that necessarily \( \beta \leq \alpha \). Actually if \( \alpha < \beta \) then \( \alpha = \beta - 1 \) and so \( p_1 \) divides \( p_2 - 1 \). In order to obtain \( p_2 + 1 \) and \( p_2 + 2 \) in \( A(f) \cap N_{\geq \alpha} \), we then have \( p_1 = 2 \) and \( p_3 = p_2 + 2 \). But in that case \((p_1, p_2, p_3) = (2, 3, 5)\) and \( h(v) < 0 \).

If \( p_1 \) does not divide \( p_2 \) and \( p_2 = p_3 \), then \( p_2 = \beta \leq \min\{\alpha, \delta\} \) unless \( p_1 \) divides \( p_2 - 1 \) and \([X\gamma]\) \( \neq 0 \) (and then \( \beta = \alpha + 1 \leq \delta \)). In that case \([A_\alpha] = u^{-\frac{\alpha}{\alpha-1}}(u-1)\).

If \( p_1 \) divides \( p_2 \), assume first that \( p_2 < p_3 \). Then \( p_2 = \delta \leq \min\{\alpha, \beta\} \) if \([X\gamma]\) = 0 whereas \( p_2 = \alpha \leq \min\{\beta, \delta\} \) if \([X\gamma]\) \( \neq 0 \). In the latter case, note that \([A_\alpha] = u^{-\frac{\alpha}{\alpha-1}}(u^2 - 1 - (u-1)[X\gamma],[X\gamma]) \) with \([X\gamma]\) even (indeed \([X\gamma]\) is the number of real solutions of a real polynomial of even degree not vanishing at zero, with only simple real roots because \( f \) has isolated singularities).

Finally in the case \( p_1 \) divides \( p_2 = p_3 \), then \( p_2 = \alpha \leq \{\beta, \delta\} \) if \( \alpha \neq \infty \).

\textbf{Theorem 4.7.} Convenient weighted homogeneous polynomials which share the same zeta functions have the same weights.

\textbf{Proof.} If \( h(v) < 0 \) we refer to \cite{[7]}. Otherwise, by the preceeding lemmas and by proposition \( 3.3 \) we know how to recover \( p_1, p_2 \) and \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \) except in the particular
case where $p_1 = 3$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{5}{6}$. Therefore it suffices to be able to distinguish the cases $(p_1, p_2, p_3) = (3, 4, 4)$ and $(3, 3, 6)$.

A direct computation shows that $[A_4] = u^{-3}(u - 1)^2$ if $(p_1, p_2, p_3) = (3, 3, 6)$ whereas $[A_4] = u^{-3}(u^2 - 1 - (u - 1)[\hat{X}_{\gamma_1}])$ if $(p_1, p_2, p_3) = (3, 4, 4)$, so that the spaces of arcs of level 4 are different, except when $[\hat{X}_{\gamma_1}] = 2$. However $[A_5] = u^{-4}(u - 1)^2$ if $(p_1, p_2, p_3) = (3, 3, 6)$ whereas $[A_5] = u^{-4}(u - 1)[\hat{X}_{\gamma_1}]$ if $(p_1, p_2, p_3) = (3, 4, 4)$, so at the level 5 the spaces of arcs are different in that case.

\[ \square \]

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