On the theory of symmetric polynomials

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Introduction

If \( g(x, y) \) is a finite or infinite expression depending on \( x \) and \( y \)

\[
g(x, y) = \sum_{i, k=0}^{m} g_{ik} x^i y^k \quad (g_{ik} = \overline{g}_{ik}, \ m \leq \infty),
\]

then by the symbol \([g(x, y)]_n\)

we denote the Hermitian form

\[
\sum_{i, k=0}^{n-1} g_{ik} x_i \overline{x}_k
\]

(where the bar denotes the complex conjugation).

As is known, due to the works by Schur \[15\], Cohn \[6\], and due to additional notice by Fujiwara \[7\], the following theorem was established.

If the Hermitian form

\[
\mathcal{S}[g; x_0, x_1, x_2, \ldots, x_{n-1}] = \left[ \frac{g^*(x)g(y) - g(x)\overline{g}(y)}{1 - xy} \right]_n = \\
= \sum_{\lambda=0}^{n-1} |\overline{g}_n x_\lambda + \overline{g}_{n-1} x_{\lambda+1} + \ldots + \overline{g}_0 x_n|^2 - \sum_{\lambda=0}^{n-1} |a_0 x_\lambda + a_1 x_{\lambda+1} + \ldots + a_{n-\lambda} x_n|^2
\]

constructed from the polynomials

\[
g(x) = a_0 + a_1 x + \ldots + a_n x^n,
\]

\[
g^*(x) = x^n \overline{g}\left(\frac{1}{x}\right) = \overline{a}_n + \overline{a}_{n-1} x + \ldots + \overline{a}_0 x^n,
\]

has \( \pi \) positive and \( \nu \) negative squared terms, and the dimension of its kernel is \( d \) (\( \pi + \nu + d = n \)), then the polynomials \( g(x) \) and \( g^*(x) \) have the greatest common divisor \( D(x) \) of degree \( d \), and the polynomial \( \frac{g(x)}{D(x)} \) has \( \pi \) roots inside the circle \( |x| = 1 \) and \( \nu \) roots outside it.

Therefore, to count exactly the number of roots of \( g(x) \) inside the circle \( |x| = 1 \), we should be able to count the number of roots of \( D(x) \) inside this circle.

At the same time, the polynomial \( D(x) \) with the proper normalization by a constant factor is a symmetric polynomial, that is,

\[
D(x) = D^*(x) = x^n \overline{D}\left(\frac{1}{x}\right).
\]

Meanwhile, for symmetric polynomials A. Cohn established the following theorem.

\[1\] The original paper was published in \( \text{Math. Sb.}, \ vol. 40, \ no. 3, \ 1933, \ pp. 271-283. \) Translated from the Russian original by Mikhail Tyaglov.

\[2\] Here \( \overline{f}(x) \) denotes the expression whose coefficients are complex conjugate to ones of \( f(x) \).
Cohn’s Theorem. The number of roots of a symmetric polynomial inside the circle $|x| = 1$ equals the number of roots of $g'(x)$ outside this circle.

But the derivative $g'(x)$ of a symmetric polynomial is not a symmetric polynomial anymore, so one can apply Schur-Cohn’s theorem to $g'(x)$. Thus, this theorem together with Cohn’s theorem completely solves the posed problem.

A. Cohn proved his theorem with the help of Rouché’s theorem and a number of laborious considerations “by continuity”.

While Schur-Cohn’s theorem can be established by the pure algebraic technique of Liénard and Chipart [13] (see also Fujiwara [7]), we do not know whether the same was done for Cohn’s theorem.

In Section 1 of this note, we construct two quadratic forms which allow us to count the number of roots of a symmetric polynomial inside the circle $|x| = 1$. Comparing one of those forms with the form $H$, we obtain Cohn’s theorem and also its generalization (see Theorem 3) pure algebraically.

In Section 2, we give a criterion for two symmetric polynomial to have interlacing roots on the circle $|x| = 1$. We also prove a theorem analogous to V. Markov’s theorem [14] (see also [8]) and a series of some other statements.

In Section 3, we discuss analogies between symmetric and real polynomials.

Let us denote by $s_k$ $(k = 0, \pm 1, \pm 2, \ldots)$ the sum of $k$th powers of roots of a polynomial $g(x)$.

It is easy to see that a polynomial $g(x)$ is symmetric or differs from symmetric by a constant factor if, and only if, together with a root $\alpha$, $|\alpha| \neq 1$, the polynomial $g(x)$ has the root $\alpha^* = 1/\alpha$ of the same multiplicity as the root $\alpha$.

This implies that $s_{-k} = \overline{s}_k$ $(k = 0, 1, 2, \ldots)$ for every symmetric polynomial. Indeed, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ be all distinct roots of $g(x)$ with absolute value equal to 1. Let their multiplicities equal $\rho_1, \rho_2, \ldots, \rho_p$, respectively. Furthermore, let $\beta_1, \beta_1^*, \beta_2, \beta_2^*, \ldots, \beta_q, \beta_q^*$ be all the distinct pairs of roots symmetric w.r.t. the circle $|x| = 1$ ($\beta_i^* = 1/\beta_i$) with correspondent multiplicities $\sigma_1, \sigma_2, \ldots, \sigma_q$.

So for $s_k$ we have the expression

$$s_k = \sum_{s=1}^p \rho_s \varepsilon_s^k + \sum_{t=1}^q \sigma_t \left( \beta_t^k + \beta_t^* k \right). \quad (1)$$

On the other hand, $\overline{s}_k = \varepsilon_{-k}, \overline{\beta}_t^k = \beta_t^{*-k}$ $(s = 1, 2, \ldots, p; t = 1, 2, \ldots, q)$. Therefore, in fact,

$$s_{-k} = \overline{s}_k.$$

Now we prove the following theorem.

Theorem 1. If the Hermitian form

$$S = \sum_{i,k=0}^{n-1} s_{i-k} x_i \overline{x}_k,$$

constructed for a given symmetric polynomial $g(x)$ has $\pi$ positive squared terms and $\nu$ negative squared terms, then the polynomial $g(x)$ has $\pi - \nu$ distinct roots $\varepsilon_i$ with absolute values equal to 1 and $\nu$ distinct pairs $\beta_i, \beta_i^*$ symmetric w. r. t. the circle $|x| = 1$.

Remark. Thus, Theorem 1 implies that the greatest common divisor $D(x)$ of polynomials $g(x)$ and $g'(x)$ has degree $d$ which is equal to dimension $d$ $(d \geq 0)$ of the kernel of the form $S$ $(\pi + \nu + d = n)$.  

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\textbf{Proof.} According to the formula (1), we obviously have

\[ s_{i-k} = \sum_{s=1}^{p} \rho_s \varepsilon_s^{i-k} + \sum_{t=1}^{q} \sigma_t [\beta_t^{i-k} + \beta^*_t^{i-k}] = \sum_{s=1}^{p} \rho_s \varepsilon_s^{i-k} + \sum_{t=1}^{q} \sigma_t [\beta_t^{i-k} + \beta^*_t^{i-k}] . \]

Hence

\[ \mathcal{G} = \sum_{s=1}^{p} \rho_s |X_s|^2 + \sum_{t=1}^{q} \sigma_t (Y_t Z_t + Z_t Y_t) = \sum_{s=1}^{p} \rho_s |X_s|^2 + \sum_{t=1}^{q} 2|U_t|^2 - \sum_{t=1}^{q} 2|V_t|^2 , \]

where

\[ U_t = \frac{Y_t + Z_t}{2}, \quad V_t = \frac{Y_t - Z_t}{2} . \]

From the last representation of the form \( \mathcal{G} \) as the sum of independent positive and negative squared terms, we deduce that \( \nu = q \) and \( \pi = p + q \), so \( p = \pi - \nu \), as required. \( \square \)

In addition to the form \( \mathcal{G} \), one can also construct another form that allows us to count the number of roots of the symmetric polynomial \( g(x) \) inside the circle \( |x| = 1 \).

In order to do this, let us consider the function

\[ F(x) = \frac{n}{2} - \frac{xg'(x)}{g(x)} = \frac{g_0(x)}{g(x)} , \]

where, thus, in what follows, by \( g_0(x) \) we denote the polynomial

\[ g_0(x) = \frac{n}{2} g(x) - xg'(x) . \]

It is easy to see that

\[ F(x) = \frac{s_0}{2} + \sum_{i=1}^{+\infty} s_{-i} x^i . \]

As well, it is not difficult to check the identity

\[ G(x, y) = \frac{F(x) + F(y)}{1 - xy} = \sum_{i,k=1}^{+\infty} s_{i-k} y^i x^k , \]

from which, by multiplying both sides of this identity by \( g(x)\overline{g}(y) \), we find that

\[ \frac{g(x)\overline{g}_s(y) + g_s(x)\overline{g}(y)}{1 - xy} = g(x)\overline{g}(y)G(x, y) = \]

\[ = \sum_{i,k=1}^{+\infty} s_{i-k} (a_0 x^k + a_1 x^{k+1} + \ldots + a_n x^{n+k})(\overline{a}_0 y^i + \overline{a}_1 y^{i+1} + \ldots + \overline{a}_n y^{n+i}) . \]
But if \( g(x) = a_0 + a_1x + \ldots + a_nx^n \) is a symmetric polynomial \((a_k = \overline{a}_{n-k})\), then the polynomial \( g_\delta(x) \) is skew-symmetric, that is, the following holds
\[
g_\delta'(x) = x^n \overline{g_\delta} \left( \frac{1}{x} \right) = -g_\delta(x).
\]

Due to this property the left part of (2) is a polynomial of \( x \) and \( y \). So, let
\[
g(x)\overline{g}(y) + g_\delta(x)\overline{g}(y) \overline{g}(y) \overline{g}(y) \overline{g}(y) = \sum_{i,k=0}^{n-1} a_{ik}x^iy^k.
\]

(3)

Obviously, the identity (2) remains the same if we change \( x^i \) to \( x_i \) and \( y^k \) to \( \overline{x}_k \). Then according to (3), we obtain
\[
\sum_{i,k=0}^{\infty} s_{i-k} z_k \overline{x}_k = \sum_{i,k=0}^{n-1} a_{ik}x_i \overline{x}_k
\]

(4)

if we set
\[
z_k = a_0x_k + a_1x_{k+1} + \ldots + a_nx_{n+k} \quad (k = 0, 1, 2, \ldots, \infty).
\]

Now putting \( x_n = x_{n+1} = \ldots = 0 \) in (4) we get
\[
\sum_{i,k=0}^{n-1} s_{i-k} z_k = \sum_{i,k=0}^{n-1} a_{ik}x_i \overline{x}_k,
\]

(4a)

where
\[
\begin{align*}
z_0 &= a_0x_0 + a_1x_1 + \ldots + a_nx_n, \\
z_1 &= a_1x_0 + \ldots + a_n-2x_{n-1}, \\
&\quad \vdots \\
z_{n-1} &= a_0x_{n-1}.
\end{align*}
\]

Since for the symmetric polynomial \( g(x) \), we have \( a_0 = \overline{a}_n \neq 0 \), the transformation above is non-singular.

Thus, the identity (4a) shows that the form \( \mathcal{G} \) in Theorem 1 can be changed by the form
\[
\mathcal{R} = \sum a_{ik}x_i \overline{x}_k
\]

So, the following theorem holds.

**Theorem 2.** If the Hermitian form
\[
\mathcal{R}(g; x_0, x_1, \ldots, x_{n-1}) = \left[ \frac{g(x) \overline{g}(y) + g_\delta(x)\overline{g}(y)}{1 - xy} \right],
\]

constructed for a given symmetric polynomial \( g(x) \) has \( \pi \) positive squared terms and \( \nu \) negative squared terms, then the polynomial \( g(x) \) has \( \pi - \nu \) distinct roots \( \varepsilon_i \) with absolute values equal to 1 and \( \nu \) distinct pairs \( \beta_i, \beta^*_i \) symmetric w. r. t. the circle \( |x| = 1 \).

This theorem can also be proved by the method of Liénard and Chipart [13] taking into account the fact that if \( g = g_1g_2 \), then \( g_\delta = g_1g_2 + g_1g_2^* \), but here we will not go into details.

Now we prove the following theorem.

**Theorem 3.** If \( g(x) \) is a symmetric polynomial, then for \( \text{Re} \, z > 0 \), the polynomial
\[
f(x) = g_\delta(x) - zg(x),
\]

has as many roots outside the circle \( |x| = 1 \) as many roots of the polynomial \( g(x) \) lie inside or outside this circle.
Before we prove this theorem, let us note that A. Cohn’s theorem is a consequence of this theorem corresponding to \( z = \frac{n}{2} \), since in this case,

\[
f(x) = -xg'(x).
\]

**Proof.** Let us construct Schur-Cohn’s form

\[
\mathcal{S}[f; x_0, x_1, \ldots, x_{n-1}] = \left[ \frac{f^*(x)f^*(y) - f(x)f(y)}{1 - xy} \right]_n
\]

for the polynomial \( f(x) \). Since

\[
f(x) = g_\delta(x) - zg(x),
\]

\[
f^*(x) = -g_\delta(x) - zg(x),
\]

(5) after a simple calculation we find that

\[
\mathcal{S}[f; x_0, x_1, \ldots, x_{n-1}] = 2\xi k[g; x_0, x_1, \ldots, x_{n-1}],
\]

(6) where \( \xi + i\eta = z \). By (6) the greatest common divisor \( D(x) \) of the polynomials \( f \) and \( f^* \) is the greatest common divisor of the polynomials \( g(x) \) and \( g_\delta(x) \) and, consequently, of the polynomials \( g(x) \) and \( g'(x) \). Now the identity (6) establishes our theorem.

Indeed, by Schur-Cohn’s theorem the number \( \nu_{\mathcal{S}} \) of negative squared terms of the form \( \mathcal{S} \) equals the number of roots of \( f \) outside the unit disk. But according to the identity (6), for \( \xi = \text{Re} z > 0 \), this number equals \( \nu_k \), so by Theorem 2 it equals the number of roots of \( g \) inside (or outside) the circle \( |x| = 1 \).

It can be analogously proved that if \( \xi = \text{Re} z < 0 \), then the polynomial

\[
f(x) = g_\delta(x) - zg(x)
\]

has the same number of roots inside the unit disk as the polynomial \( g(x) \) does. These conclusions have also the following curious interpretation.

If \( g(z) \) is a symmetric polynomial, then the function

\[
z = \frac{g_\delta(x)}{g(x)}
\]

maps the disk \( |x| < 1 \) into a domain consisting of \( k \) sheets with \( \text{Re} z < 0 \) and \( n - k \) sheets with \( \text{Re} z > 0 \), where \( k \) is the number of poles of the function \( z \) in the disk \( |x| < 1 \).

We conclude this Section noticing that it is not difficult to separate a positive squared term from the form \( \mathcal{R} \), that is, it is easy to check that

\[
\mathcal{R}[g; x_0, x_1, \ldots, x_{n-1}] = \frac{1}{n} \mathcal{S}[g'; x_0, x_1, \ldots, x_{n-1}] + \frac{1}{n} [g'(x)g(y)]_n.
\]

This again implies A. Cohn’s theorem.

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Let us correspond to every symmetric polynomial \( g(x) \) the following \( 2\pi \)-periodic function

\[
G(\varphi) = e^{-\frac{ni\varphi}{2}} g(e^{i\varphi}).
\]

Then it is easy to see that

\[
G'(\varphi) = -ie^{-\frac{ni\varphi}{2}} g_\delta(e^{i\varphi}).
\]

(A)
We also agree that two symmetric polynomials $g(x)$ and $h(x)$ have *interlacing* roots if all their roots are simple, lie on the circle $|x| = 1$, and between any two consecutive roots of one polynomial there lies one, and only one, root of the second polynomial.

It is evident that polynomials $g(x)$ and $h(x)$ have interlacing roots if, and only if, $g(x)$ and $h(x)$ are of the same degree, and the function

$$H(\varphi) = e^{\frac{ni \varphi}{2}} h(e^{i \varphi})$$

has different signs at any two consecutive roots of $g(x)$.

If all the roots of the symmetric polynomial $g(x)$ are simple and lie on the circle $|x| = 1$, then by Rolle’s theorem, the formula (A) implies that the roots of $g_0(x)$ interlace the roots of $g(x)$.

Furthermore, this implies that the roots of the polynomial $h(x)$ interlace the roots of such a polynomial $g(x)$ if, and only if, the ratio

$$\frac{H(\varphi)}{G'(\varphi)} = i \frac{h(\varphi)}{g_0(\varphi)} \quad (x = e^{i \varphi})$$

is of the same sign at all roots of the polynomial $g(x)$. This remark allows us to prove the following theorem.

**Theorem 4.** Two symmetric polynomials $g(x)$ and $h(x)$ have interlacing roots if and only if the form

$$\Re[g, h; x_0, x_1, \ldots, x_n] = \left[ i \frac{g(x_0)h(y) - g(y)h(x)}{1 - xy} \right]_n$$

is sign-definite.

**Proof.** We first prove the sufficiency.

If the form $\Re$ is of a definite sign, then the expression

$$\Re[g, h; 1, \alpha, \ldots, \alpha^{n-1}] = \frac{i g(\alpha)h(\alpha) - g(\alpha)h(\alpha)}{1 - \alpha \overline{\alpha}}$$

(7)

for all $\alpha$, $|\alpha| \neq 1$ preserves its sign. Therefore, $g(\alpha) \neq 0$ for $|\alpha| \neq 1$ that follows from the right hand side of (7).

Thus, $g(x)$ has all its roots lying on the circle $|x| = 1$.

On the other hand, if $|\alpha| = 1$, then it is easy to see that

$$\Re[g, h; 1, \alpha, \ldots, \alpha^{n-1}] = -i \frac{g(\alpha)\overline{h'(\alpha)} - \overline{g(\alpha)h'(\alpha)}}{\alpha},$$

(8)

since $\overline{\alpha} = \frac{1}{\alpha}$ in this case, so by L’Hôpital’s rule,

$$\lim_{y \to \frac{x}{x}} \left[ i \frac{g(x_0)h(y) - g(y)h(x)}{1 - xy} \right] = -i \frac{g(x)h\left(\frac{1}{x}\right)}{x} - \frac{g\left(\frac{1}{x}\right)h(x)}{x}.$$ 

The expression (8) can also be transformed as follows

$$\Re[g, h; 1, \alpha, \ldots, \alpha^{n-1}] = -i |g(\alpha)\overline{h'(\alpha)} - h(\alpha)\overline{g'(\alpha)}| =$$

$$= i |g(\alpha)\overline{h'_s(\alpha)} - h(\alpha)\overline{g'_s(\alpha)}|,$$

(9)

since

$$\frac{n}{2} |h(\alpha)\overline{g(\alpha)} - g(\alpha)\overline{h(\alpha)}| = \frac{n}{2} \alpha^n [h(\alpha)g(\alpha) - g(\alpha)h(\alpha)] = 0.$$
Inasmuch as $\mathcal{R}$ does not change its sign, from (5) it is follows that $g(\alpha)$ and $g'(\alpha)$ are non-zero simultaneously, that is, all the roots of $g(x)$ are simple. On the other hand, putting in (9) $\alpha = \alpha_k$ ($k = 1, 2, \ldots, n$), where $\alpha_k$ is a root of $g(x)$, we find that

$$\Re[g, h; 1, \alpha, \ldots, \alpha^{n-1}] = -ih(\alpha_k)g_\alpha(\alpha_k)g_\bar{\alpha}(\alpha_k) = -\frac{h(\alpha_k)}{g_\alpha(\alpha_k)}g_\bar{\alpha}(\alpha_k).$$

(B)

From this formula we infer that the expression $ih(\alpha_k)g_\alpha(\alpha_k)$ has the same sign for all $\alpha_k$. According to the remark before Theorem 4 this implies the sufficiency of the statement of Theorem 4.

We now prove the necessity. Let the roots of $g(x)$ and $h(x)$ be interlacing. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ ($|\alpha_i| = 1$) be all distinct roots of $g(x)$. We denote by

$$\Re[g, h; x_0, x_1, \ldots, x_n]$$

the bilinear form $\sum a_{ik}x_k \bar{y}_k$ corresponding to the Hermitian form

$$\sum_{i,k=0}^{n-1} a_{ik}x_i \bar{y}_k = \Re[g, h; x_0, x_1, \ldots, x_{n-1}].$$

It is easy to see that

$$\Re[g, h; 1, \alpha, \ldots, \alpha^{n-1}] = 0$$

for $k \neq l$. Therefore,

$$\Re[g, h; x_0, x_1, \ldots, x_{n-1}] = \sum_{k=1}^{n} \Re[g, h; x_0, x_1, \ldots, x_{n-1}] \Re[g, h; 1, \alpha, \ldots, \alpha^{n-1}]$$

since the left hand side identically w.r.t. $x$ equals the right hand side for $n$ independent systems of particular values of $y$, namely, for the following systems of values

$$y_0 = 1, y_1 = \alpha_k, \ldots, \alpha_k^{n-1}, \quad (k = 1, 2, \ldots, n).$$

The identity (10) for $y_i = x_i$ ($i = 0, 1, \ldots, n-1$) gives us an expansion of our form into the sum of squared terms with coefficients, which are of the same sign by (B) and by the conditions that $g$ and $h$ satisfy. Theorem is proved.

Let us now establish the following curious statement.

**Theorem 5.** If the roots of symmetric polynomials $g(x)$ and $h(x)$ are interlacing, then the roots of the polynomials $g_\alpha(x)$ and $h_\alpha(x)$ interlace, as well.

**Proof.** In fact, if the roots of the polynomials $g(x)$ and $h(x)$ are interlacing, then as we proved above, the expression

$$i[g(\alpha)h_\alpha(\alpha) - h(\alpha)g_\alpha(\alpha)]$$

is real and does not change its sign on the circle $x = e^{i\varphi}$.

Let us consider the functions

$$H(\varphi) = e^{-\frac{ni\varphi}{2}} h(e^{i\varphi}) \quad \text{and} \quad G(\varphi) = e^{-\frac{ni\varphi}{2}} g(e^{i\varphi}).$$

Since the polynomials $g_\alpha(x)$ and $h_\alpha(x)$ are skew-symmetric, then we have

$$H'(\varphi) = -ie^{-\frac{ni\varphi}{2}} h_\alpha(e^{i\varphi}) = ie^{\frac{ni\varphi}{2}} h_\alpha(e^{-i\varphi}),$$

$$G'(\varphi) = -ie^{-\frac{ni\varphi}{2}} g_\alpha(e^{i\varphi}) = ie^{\frac{ni\varphi}{2}} g_\alpha(e^{-i\varphi}).$$
Consequently, the expression (11) equals the following
\[ G(\varphi)H'(\varphi) - G'(\varphi)H(\varphi). \] (12)

Let now \( \varphi_k \) and \( \varphi_l \) be the arguments of two consecutive roots of \( G'(\varphi) \). Putting into (12) \( \varphi = \varphi_k \) and then \( \varphi = \varphi_l \), we obtain that the expressions
\[ G(\varphi_k)H'(\varphi_k) \quad \text{and} \quad G(\varphi_l)H'(\varphi_l) \]
are of the same sign. But \( G(\varphi_k) \) and \( G(\varphi_l) \) obviously have different signs, consequently, \( H'(\varphi_k) \) and \( H'(\varphi_l) \) have different signs, as well. Therefore, between \( \varphi_k \) and \( \varphi_l \) there lies at least one root of \( H'(\varphi) \). Now the statement of the theorem follows from the fact that \( G'(\varphi) \) and \( H'(\varphi) \) have equal number of roots.

If the polynomials \( g(x) \) and \( h(x) \) are of an even degree, \( n = 2m \), then \( G(\varphi) \) and \( H(\varphi) \) are trigonometric sums of the form
\[ I. \quad a_0 + \sum_{k=1}^{m} (a_k \cos k\varphi + b_k \sin k\varphi) \]

If the polynomials \( g(x) \) and \( h(x) \) are of an odd degree, \( n = 2m - 1 \), then \( G(\varphi) \) and \( H(\varphi) \) are trigonometric sums of the form
\[ II. \quad \sum_{k=1}^{m} \left[ a_k \cos \left(k - \frac{1}{2}\right) \varphi + b_k \sin \left(k - \frac{1}{2}\right) \varphi \right]. \]

Thus, our theorem can also be formulated as follows.

*If roots of two trigonometric sums of type I or II are interlacing, then the roots of their derivatives are interlacing, as well.*

Also it is easy to prove the following theorem.

**Theorem 6.** If symmetric polynomials \( g(x) \) and \( h(x) \) have interlacing roots, then the roots of the symmetric polynomial
\[ f(x) = g(x) + th(x) \quad (-\infty < t < \infty) \]
interlace the roots of the polynomials \( g(x) \) and \( h(x) \). The arguments of roots of \( f(x) \) are monotone functions of parameter \( t \).

*Proof.* The first part of the statement of the theorem follows from the formulæ
\[ \mathfrak{R}[f; h; x_0, x_1, \ldots, x_{n-1}] = \mathfrak{R}[g; h; x_0, x_1, \ldots, x_{n-1}], \]
\[ \mathfrak{R}[f; g; x_0, x_1, \ldots, x_{n-1}] = -t\mathfrak{R}[g; h; x_0, x_1, \ldots, x_{n-1}]. \]

To prove the second part of the statement we consider a root \( \alpha \) of the function \( f \) and differentiate by \( t \) the left hand side of the equation
\[ f(\alpha) = g(\alpha) + th(\alpha) = 0, \]
considering \( \alpha \) a function of \( \varphi \) (\( \alpha = e^{i\varphi} \)), which, in its turn, depends on \( t \). Then we obtain
\[ i\alpha f'(\alpha) \frac{d\varphi}{dt} = -h(\alpha). \]
Taking into account that \( f(\alpha) = 0 \), this implies that
\[ \frac{d\varphi}{dt} = i \frac{h(\alpha)}{f(\alpha)} \]

Since the roots of \( f(x) \) and \( h(x) \) are interlacing, the right hand side of the last equality does not change its sign, as required. \( \square \)
Concluding this Section we note that the interlacing criterion has a function-theoretical interpretation. Namely, the following theorem holds.

**Theorem 7.** The roots of two symmetric polynomials \( g(x) \) and \( h(x) \) are interlacing if and only if the function

\[
z = \frac{h(x)}{g(x)}
\]

maps the disk \(|x| < 1\) into one of two \( n \)-sheet half-planes, \( \text{Im} \, z > 0 \) or \( \text{Im} \, z < 0 \).

**Proof.** Indeed, let \( z = \xi + i\eta \, (\eta \neq 0) \). We consider the function

\[
f(x) = h(x) - zg(x).
\]

Evidently,

\[
f^*(x) = h(x) - zg(x).
\]

Easily, this implies that Schur-Cohn’s form constructed for \( f \) equals the following

\[
\mathcal{R}[f; x_0, x_1, \ldots, x_{n-1}] = -2\eta \mathcal{R}[g, h; x_0, x_1, \ldots, x_{n-1}].
\]

Consequently, for the roots of \( g(x) \) and \( h(x) \) to be interlacing it is necessary and sufficient the form \( \mathcal{R} \) to be sign-definite. In this case, by Schur-Cone’s theorem, all the roots of \( f(x) \) lie either inside the unit circle (if \( \mathcal{R} \) is positive definite) or outside the unit circle (if \( \mathcal{R} \) is negative definite). This fact establishes the statement of the theorem.

The last theorem shows that there is a close connection between the theory of symmetric polynomials and the well-known Carathéodory’s problem on positive harmonic functions [3, 4].

Note also that if we expand \( \frac{ih(x)}{g(x)} \) into the series

\[
\frac{ih(x)}{g(x)} = \sigma_0 + i\tau_0 + \sum_{i=1}^{+\infty} \sigma_i x^i \quad (\sigma_0 \leq 0, \tau_0 \leq 0),
\]

then, analogously to what we do in Section I instead of the form \( \mathcal{R} \) one can consider the form

\[
\mathcal{R}(g, h; x_0, x_1, \ldots, x_{n-1}) = \sum_{i,k=0}^{n-1} \sigma_{i-k} x_i \overline{x_k} \quad (\sigma_{-i} = \overline{\sigma_i}),
\]

which is connected to the form \( \mathcal{R} \) by the identity

\[
\mathcal{R}(g, h; x_0, x_1, \ldots, x_{n-1}) = \sum_{i,k=0}^{n-1} \sigma_{i-k} z_i \overline{z_k},
\]

where

\[
\begin{align*}
z_0 &= a_0 x_0 + a_1 x_1 + \ldots + a_{n-1} x_{n-1}, \\
z_1 &= a_1 x_0 + \ldots + a_{n-2} x_{n-1}, \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
z_{n-1} &= a_0 x_{n-1}, \end{align*}
\]

\[
[g(x) = a_0 + a_1 x + \ldots + a_n x^n].
\]

It is known that in the Carathéodory’s problem, exactly the following Toeplitz forms are considered [10].

\[
\sum \sigma_{i-k} z_i \overline{z_k}
\]

\footnote{In this work, it is pointed out to the possibility of establishing a criterion of root interlacing for two symmetric polynomials}
Since the transformation
\[ f(x) = (x - i)^n g \left( \frac{x + i}{x - i} \right) \]
transforms a symmetric polynomial \( g(x) \) into a real one, there must exist some analogies between the theory of symmetric and real polynomials.

In fact, Theorem 1 corresponds to Borchardt’s theorem [1, 2]:

I. If the real form
\[ \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k \]
constructed for a real polynomial \( f(x) \) has \( \pi \) positive squared terms and \( \nu \) negative squared terms, then the polynomial \( f(x) \) has \( \pi - \nu \) distinct real roots and \( \nu \) distinct pairs of non-real roots.

Using the technique of Section 1, it is not difficult to transform this criterion into the following one.

II. If the form
\[ \left[ \frac{f(x)f'(y) - f'(x)f(y)}{x-y} \right]_n = K[f; x_0, x_1, \ldots, x_{n-1}] \]
has \( \pi \) positive squared terms and \( \nu \) negative squared terms, then the polynomial \( f(x) \) has \( \pi - \nu \) distinct real roots and \( \nu \) distinct pairs of non-real roots.

Hermite [11] suggested, instead of the form \( K \), to consider the form
\[ K_1[f; x_0, x_1, \ldots, x_{n-1}] = \left[ \frac{\hat{f}(x)f'(y) - f'(x)\hat{f}(y)}{x-y} \right]_n, \]
where
\[ \hat{f}(x) = nf(x) - xf'(x). \]

Hermite’s rule (we do not formulate it here) can be easily obtained from the rule II if one takes into account the identity
\[ K[f; x_0, x_1, \ldots, x_{n-1}] = \frac{1}{n} K_1[f; x_0, x_1, \ldots, x_{n-1}] + \frac{1}{n} [f'(x)f'(y)]_n. \]

Theorem 3 corresponds to the following theorem.

III. If \( f(x) \) is a real polynomial, then the polynomial
\[ f'(x) - zf(x), \]
for \( \text{Im} \ z > 0 \), has as many roots in the half-plane \( \text{Im} \ x > 0 \) as many pairs of non-real roots the polynomial \( f(x) \) has.

We agree to say that roots of two real polynomials interlace if all their roots are real and distinct, and between any two consecutive roots of one polynomial there lies one, and only one, root of the other polynomial. Then the analogue of Theorem 3 is the proposition that was established, in another form, by Hurwitz [12]:

IV. Roots of two real polynomials \( f \) and \( F \) interlace if and only if the form
\[ \left[ \frac{F(x)f(y) - F(y)f(x)}{x-y} \right]_n \]
is sign-definite.

This proposition can certainly be proved by the same technique that we used in the proof of Theorem 3 but one can find for it a very beautiful proof based on Sturm’s theorem.

Theorem 5 turns into Markov’s theorem [14]:
V. If roots of polynomials \( f(x) \) and \( F(x) \) interlace, then the roots of their derivatives \( f'(x) \) and \( F'(x) \) interlace, as well.

It is obvious how one should formulate the analogue of Theorem 6.

Theorem 7 turns into the following proposition.

VII. The roots of two real polynomials \( f(x) \) and \( F(x) \) are interlacing if, and only if, the function

\[
z = \frac{f(x)}{F(x)}
\]

maps the half-plane \( \text{Im} \, x > 0 \) into one of two \( n \)-sheet half-planes, \( \text{Im} \, z > 0 \) or \( \text{Im} \, z < 0 \).

For proving Propositions III and VII, it is better to use Hermite's theorem (analogue of Schur-Cohn's theorem).

**Hermite's Theorem.** If the form

\[
\left[ -i \frac{F(x)\overline{F(y)} - \overline{F(x)}F(y)}{x - y} \right]_n
\]

has \( \pi \) positive and \( \nu \) negative squared terms, and if the dimension of its kernel is \( d \), then the polynomials \( F(x) \) and \( \overline{F(x)} \) have the greatest common divisor \( D(x) \) of degree \( d \), and the polynomial \( \frac{F(x)}{D(x)} \) has \( \pi \) roots in the upper half-plane \( \text{Im} \, x > 0 \) and \( \nu \) roots in the lower half-plane \( \text{Im} \, x < 0 \).

**Remarks during proofread.** After this work was submitted to publication, the author got an opportunity to read the paper [9] by G. Herglotz, where two forms constructed in Section I was also introduced, and Theorems I and II were proved. Since the technique of the author is more elementary and completely differs from Herglotz's method which is connected to the theory of characteristics, the author allows himself to leave Section I without any changes.

**References**

[1] C. Borchardt, *Développements sur l’équation à l’aide, de laquelle on détermine les inégalités séculaires du mouvement des planètes*. (French) [On the Equation to the Secular Inequalities in the Movement of Planets], J. Math. pures appl., 12, no. 5, 1847, pp. 50–67.

[2] C. Borchardt, *Bemerkung über die beiden vorstehenden Aufsätze*. (German) [Remark on two papers mentioned above.], J. reine angew. Math., 53, no. 1, 1857, pp. 281–283.

[3] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*. (German) [On the domain of variation of coefficients of a power series that do not achieve a given value.], Math. Annalen, 64, 1907, pp. 95–115.

[4] C. Carathéodory, *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen*. (German) [On the domain of variation of the Fourier constants of positive harmonic functions.], Rendiconti Circ. Mat. di Palermo, 32, 1911, pp. 193–217.

[5] N. Tschebotareff, *Über den Realität von Nullstellen ganzer transzendenten Funktionen*. (German) [On the reality of zeros of entire transcendental functions.], Math. Annalen, 99, 1928, pp. 660–686.

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4 Regarding Propositions III and VII, compare them with Chebotarev’s work [5].

5 See [11], pp. 41–44.
[6] A. Cohn, Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise. (German) [On the number of roots of an algebraic equation inside a circle], Mathem. Zeitschrift, 14, 1922, pp. 110–148.

[7] M. Fujiwara, Über die algebraischen Gleichungen, deren Wurzeln in einem Kreise oder in einer Halbebene liegen. (German) [On algebraic equations whose roots lie in a circle or an a half-plane], Mathem. Zeitschrift, 24, 1925, pp. 161–169.

[8] D. Grave, Элементы высшей алгебры. (Russian) [Elements of higher algebra], Kiev, 1914, p. 698.

[9] G. Herglotz, Über die Wurzelanzahl algebraischer Gleichungen innerhalb und auf dem Einheitskreis. (German) [On the number of roots of algebraic equations inside and outside of the unit circle], Mathem. Zeitschrift, 19, 1924, pp. 26–34.

[10] G. Herglotz, Über Potenzreihen mit positivem reellen Teil im Einheitskreis. (German) [On power series with positive real part in the unit disk], Der. Verh. sächs. Akad. Leipzig, 8, 1911, pp. 501–509.

[11] C. Hermite. Extrait d'une lettre de Mr. Ch. Hermite de Paris à Mr. Borchardt de Berlin sur le nombre des racines d'une équation algébrique comprises entre des limites données. (French) [Extract of a letter from Mr. Ch. Hermite of Paris to Mr. Borchardt of Berlin on the number of roots of an algebraic equation contained between given limits], J. reine angew. Math., 52, 1856, pp. 39–51.

English transl.: On the number of roots of an algebraic equation contained between given limits. Translated from the French original by P. C. Parks. Routh centenary issue. Internat. J. Control, 26, No. 2, 1977, pp. 183–195.

[12] A. Hurwitz. Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativem reellen Teil besitzt. (German), Math. Ann., 46, 1895, pp. 273–284.

[13] A. Liénard et M. Chipart. Sur la signe de la partie réelle des racines d'une équation algébrique. (French) [On the signs of the real parts of the roots of an algebraic equation], J. de Math. Pures et Appl., 10, no. 6, 1914, pp. 291–346.

[14] V. Markov, О функциях, наименее уклоняющихся от нуля в данном промежутке. (Russian) [On functions deviating least from zero in a given interval], Saint-Petersburg, 1892.

[15] I. Schur, Über Potenzreihen, die im innern des Einheitskreises beschränkt sind. (German) [On series bounded in the unit circle], J. reine angew. Math., 148, 1918, pp. 122–135.

Unter Benutzung von elementaren Betrachtungen konstruiert der Verfasser zwei Formen, mit deren Hilfe die Wurzelanzahl des symmetrischen Polynoms

\[ f(x) = x^n \left( \frac{1}{x} \right) \]

innerhalb des Einheitskreises aufgezählt werden kann; zu diesen Formen kommt auch G. Herglotz, indem er von der Charakteristikentheorie ausgeht. Auf Grund dieser Resultate wird rein algebraisch der Satz von A. Cohn abgeleitet, nach dem das symmetrische Polynom dieselbe Anzahl von Wurzeln außerhalb des Einheitskreises besitzt, wie eine Derivierte. Dieser Satz wird etwas verallgemeinert. Zum Schluss werden symmetrische Polynome mit sich trennenden Wurzeln betrachtet, und ein Analogon des W. Markowschen Satzes für symmetrische, also auch für trigonometrische Polynome, aufgestellt.