Cluster automorphism groups and automorphism groups of exchange graphs

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Abstract

For a coefficient free cluster algebra $\mathcal{A}$, we study the cluster automorphism group $\text{Aut}(\mathcal{A})$ and the automorphism group $\text{Aut}(E_{\mathcal{A}})$ of its exchange graph $E_{\mathcal{A}}$. We show that these two groups are isomorphic with each other, if $\mathcal{A}$ is of finite type, excepting types of rank two and type $F_4$, or $\mathcal{A}$ is of skew-symmetric finite mutation type.

Key words. Cluster algebras; Exchange graphs; Cluster automorphism groups.

Mathematics Subject Classification. 16S99; 16S70; 18E30

1 Introduction

Cluster algebras are introduced by Sergey Fomin and Andrei Zelevinsky in [9]. In this paper we consider a special class so called coefficients free cluster algebras of geometric type, which is a commutative $\mathbb{Z}$-algebra defined through a skew-symmetrizable square matrix. Starting from a seed, which is a pair consisting of a set (cluster) of $n$ indeterminate elements and a skew-symmetrizable square matrix of rank $n$, we get a new seed by a mutation, that makes a birational transform on the cluster and performs an operation upon the matrix. Then as a $\mathbb{Z}$-subalgebra of the rational function field over the initial cluster, the cluster algebra is generated by the elements (cluster variables) in the clusters obtained by recursively mutations. A cluster algebra has nice combinatorical structures which are given by mutations, and these structures are captured by an exchange graph, which is a graph with seeds as vertices and with mutations as edges.

If there are finite clusters in a cluster algebra, then we say that it is of finite type. These cluster algebras are classified in [10], which corresponds to the Killing-Cartan classification of complex semisimple Lie algebras, equivalently, corresponds to the classification of root systems in Euclidean space. If there are finite matrix classes in the seeds of a cluster algebra, then we call it a cluster algebra of finite mutation type, where two matrices are in a same class if one of them can be obtained from the other matrix by simultaneous relabeling of the rows and columns. The cluster algebras of finite mutation type defined by skew-symmetric matrices are classified in [8], a large class of them arises from oriented marked Riemann surfaces [6], and there are 11 exceptional ones. The classification of skew-symmetric cluster algebras of finite mutation type is given in [7] via operations so called unfoldings upon the skew-symmetric cluster algebras of finite mutation type.

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1Supported by the NSF of China (Grants 11131001)
A cluster automorphism of a cluster algebra is a permutation of the cluster variable set, which commutates with mutations. Cluster automorphisms naturally consist a group: cluster automorphism group. This group reveals the symmetries of the cluster algebra, especially the symmetries of combinatorical structures and algebra structures. This group is introduced in [1] for a coefficient free cluster algebra, and in [4] for a cluster algebra with coefficients. An automorphism of the exchange graph of a cluster algebra is firstly studied in [4], it is an automorphism of a graph. All of the automorphisms consist a group, which describes symmetries of the exchange graph, in other words, describes symmetries of the combinatorical structures of the cluster algebra.

In this paper, for a coefficient free cluster algebra \( \mathcal{A} \) with exchange graph \( E_{\mathcal{A}} \), we consider the relations between the cluster automorphism group \( \operatorname{Aut}(\mathcal{A}) \) and the automorphism group \( \operatorname{Aut}(E_{\mathcal{A}}) \). Generally, \( \operatorname{Aut}(\mathcal{A}) \) is a subgroup of \( \operatorname{Aut}(E_{\mathcal{A}}) \), and may be a proper subgroup, for example see Example 2, and Example 4. The main result of this paper is that these two groups are isomorphic with each other, if \( \mathcal{A} \) is of finite type, excepting types of rank two and type \( F_4 \) (Theorem 3.7), or \( \mathcal{A} \) is of skew-symmetric finite mutation type (Theorem 3.8). Therefore in some degree, for these cluster algebras, the algebra structures are also captured by the exchange graphs.

To prove these results, we describe \( E_{\mathcal{A}} \) more precisely. In subsection 3.1, we define layers of geodesic loops of \( E_{\mathcal{A}} \) by using the distance of a vertex to a fixed vertex on \( E_{\mathcal{A}} \). An easy observation is that an isomorphism of exchange graphs should maintain the combinatorial numbers of the layers of geodesic loops based on the corresponding vertices (Remark 3.2(4)). Then by this observation, we directly show in Examples 5, 6, 8, 9 that for a cluster algebra of type \( A_3, B_3, C_3, \tilde{A}_2 \) or \( T_3 \) (the cluster algebra from an once punctured torus), we have \( \operatorname{Aut}(\mathcal{A}) \cong \operatorname{Aut}(E_{\mathcal{A}}) \). For the general cases we reduce them to above five cases (Theorems 3.7, 3.8).

The paper is organized as follows: we recall preliminaries on cluster algebras, cluster algebras of finite mutation type and cluster automorphisms in section 2, then we prove the main Theorems in section 3.

### 2 Preliminaries

#### 2.1 Cluster algebras

We recall basic definitions and properties on cluster algebras in this subsection.

**Definition 2.1.** ([9](Labeled seeds). A labeled seed is a pair \( \Sigma = (\mathbf{x}, B) \), where

- \( \mathbf{x} = \{x_1, x_2, \ldots, x_n\} \) is a set with \( n \) elements;
- \( B = (b_{x_jx_i})_{n \times n} \in \mathcal{M}_{n \times n}(\mathbb{Z}) \) is a matrix labeled by \( \mathbf{x} \times \mathbf{x} \), and it is skew-symmetrizable, that is, there exists a diagonal matrix \( D \) with positive integer entries such that \( DB \) is skew-symmetric.

The set \( \mathbf{x} \) is the cluster of \( \Sigma \), \( B \) is the exchange matrix of \( \Sigma \). We also write \( b_{ji} \) to an element \( b_{x_jx_i} \) in \( B \) for brevity. The elements in \( \mathbf{x} \) are the cluster variables of \( \Sigma \). We always assume through the paper that \( B \) is indecomposable, that is, for any \( 1 \leq i, j \leq n \), there is a sequence \( i_0 = i, i_1, \ldots, i_m, i_{m+1} = j \), such that \( b_{i_k,i_{k+1}} \neq 0 \) for any \( 0 \leq k \leq m \). Given a cluster variable \( x_k \), we produce a new labeled seed by a mutation. We also assume that \( n > 1 \) for convenience.


Definition 2.2. ([9](Seed mutations). The labeled seed \( \mu_k(\Sigma) = (\mu_k(x), \mu_k(B)) \) obtained by the mutation of \( \Sigma \) in the direction \( k \) is given by:

- \( \mu_k(x) = (x \setminus \{x_k\}) \cup \{x'_k\} \) where
  \[
  x_k x'_k = \prod_{1 \leq j \leq n \colon b_{jk} > 0} x_j^{b_{jk}} \prod_{1 \leq j \leq n \colon b_{jk} < 0} x_j^{-b_{jk}}.
  \]

- \( \mu_k(B) = (b'_{ij})_{n \times n} \in M_{n \times n}(\mathbb{Z}) \) is given by
  \[
  b'_{ij} = \begin{cases} 
  -b_{ji} & \text{if } i = k \text{ or } j = k; \\
  b_{ji} + \frac{1}{2}(b_{ij} b_{ik} + b_{ji} b_{jk}) & \text{otherwise}.
  \end{cases}
  \]

It is easy to check that a mutation is an involution, that is a mutation in direction \( k \). The elements of seeds assigned to the endpoints of any edge labeled by \( k \) are obtained from each other by the seed pattern

\[\text{Note that } \Sigma \text{ is of finite type } [10](\text{see in section 2.2 for cluster algebras of finite type}) \text{ or of skew-symmetric type } [12], \text{ then a cluster determines the quiver, and we denote the quiver of a cluster } x \text{ by } Q(x).\]
**Example 1.** Let \( B \) be the following matrix, it is a skew-symmetrizable matrix with diagonal matrix \( D = \text{diag}(2, 2, 1, 1) \). The quiver corresponding to \( B \) is \( Q \), where we always delete the trivial pairs of values \((1, 1)\), and replace a arrow assigning pair \((m, m)\) by \( m \) arrows.

\[
B = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 \\
0 & 2 & 0 & 2 \\
0 & 0 & -2 & 0
\end{pmatrix}
\]

\( Q : 1 \quad \longrightarrow \quad 2 \quad \overset{(2,1)}{\longrightarrow} \quad 3 \quad \longrightarrow \quad 4 \)

**Definition 2.5.** \([11]\) (Seeds) Given two labeled seeds \( \Sigma = (\mathbf{x}, B) \) and \( \Sigma' = (\mathbf{x}', B') \), we say that they define the same seed if \( \Sigma' \) is obtained from \( \Sigma \) by simultaneous relabeling of the sets \( \mathbf{x} \) and the corresponding relabeling of the rows and columns of \( B \).

We denote by \( [\Sigma] \) the seed represented by a labeled seed \( \Sigma \). The cluster \( \mathbf{x} \) of a seed \( [\Sigma] \) is an unordered \( n \)-element set. For any \( x \in \mathbf{x} \), there is a well-defined mutation \( \mu_x([\Sigma]) = [\mu_{x_k}(\Sigma)] \) of \( [\Sigma] \) at direction \( x \), where \( x = x_k \). For two same rank skew-symmetrizable matrices \( B \) and \( B' \), we say \( B \equiv B' \), if \( B' \) is obtained from \( B \) by simultaneous relabeling of the rows and columns of \( B \). Then the exchange matrices in any two labeled seeds representing a same seed are isomorphic. The isomorphism of two exchange matrices induces an isomorphism of corresponding quivers. For convenience, in the rest of the paper, we also denote by \( \Sigma \) the seed \( [\Sigma] \) represented by \( \Sigma \).

**Definition 2.6.** \([11]\) (Exchange graphs) The exchange graph of a cluster algebra is the \( n \)-regular graph whose vertices are the seeds of the cluster algebra and whose edges connect the seeds related by a single mutation. We denote by \( E_{A} \) the exchange graph of a cluster algebra \( A \).

Clearly, the exchange graph of a cluster algebra is a quotient graph of the exchange pattern, its vertices are equivalent classes of labeled seeds. Note that the edges in an exchange graph lost the ‘color’ of labels. The exchange graph not necessary be a finite graph, if it is finite, then we say the corresponding cluster algebra (and its exchange pattern) is of finite type. The exchange graph is related to a cluster complex \( \Delta \) of \( A \), which is a simplicial complex on the ground set \( \mathcal{X} \) with the clusters as the maximal simplices. Then \( \Delta \) is a \( n \)-dimensional complex. If \( A \) is of finite type or skew-symmetric, then the vertices of \( E_{A} \) are clusters, thus the dual graph of \( \Delta \) is \( E_{A} \).

### 2.2 Finite types and finite mutation types

By the classification of cluster algebras of finite type \([9]\), a cluster algebra is of finite type if and only if there is a seed whose quiver is one of quivers depicted in Figure 1. For a quiver mutation equivalent to a quiver in Figure 1, we call it a quiver of corresponding type. Note that the underlying graphs of quivers in Figure 1 are trees, thus any two quivers with the same underlying graph are mutation equivalent, and they have the same type.

Now we recall the classification of skew-symmetric cluster algebras of finite mutation type. Let us start with definition of block-decomposable quivers.
Figure 1: Quivers of finite type

\[
\begin{align*}
A_n : & \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n \\
B_n : & \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \xrightarrow{(2,1)} n \\
C_n : & \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \xrightarrow{(1,2)} n \\
D_n : & \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-2 \xleftarrow{n-1} n \\
E_6 : & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \\
E_7 : & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \xrightarrow{8} 6 \\
E_8 : & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \xrightarrow{(2,1)} 5 \rightarrow 6 \rightarrow 7 \\
F_4 : & \quad 1 \xrightarrow{(2,1)} 2 \rightarrow 3 \rightarrow 4 \\
G_3 : & \quad 1 \xrightarrow{(3,1)} 2 \\
\end{align*}
\]
Definition 2.7. [6] [8] A block is a quiver isomorphic to one of the quivers with black/white colored vertices shown on Figure 2. Vertices marked in white are called outlets. A connected quiver $Q$ is called block-decomposable (decomposable for brevity) if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching (matching of outlets of the same block is not allowed), where two arrows with same endpoints and opposite directions cancel out. If $Q$ is not block-decomposable then we call $Q$ non-decomposable.

Then it is proved in [6] (Theorem 13.3) that a quiver is decomposable if and only if it is a quiver of a triangulation of an oriented marked Riemann surface, and thus a quiver mutation equivalent to a decomposable quiver is also decomposable. Note that all arrow multiplicities of a decomposable quiver are 1 or 2. Therefore decomposable quivers are mutation finite. It is clearly that a quiver of rank two, that is, a quiver with two vertices, is mutation finite. Besides these two kinds of quivers, there are exactly 11 exceptional skew-symmetric quivers of finite mutation type, see Theorem 6.1 in [8]. We list the exceptional quivers in Figure 3.

2.3 Automorphism groups

In this subsection, we recall the cluster automorphism group[1] of a cluster algebra, and the automorphism group of the corresponding exchange graph[4].

Definition 2.8. [1] (Cluster automorphisms) For a cluster algebra $\mathcal{A}$ and a $\mathbb{Z}$-algebra automorphism $f : \mathcal{A} \to \mathcal{A}$, we call $f$ a cluster automorphism, if there exists a labeled seed $(x, B)$ of $\mathcal{A}$ such that the following conditions are satisfied:

1. $f(x)$ is a cluster;

2. $f$ is compatible with mutations, that is, for every $x \in x$ and $y \in x$, we have

$$f(\mu_{s,x}(y)) = \mu_{f(s),f(x)}(f(y)).$$

Then a cluster automorphism maps a labeled seed $\Sigma = (x, B)$ to a labeled seed $\Sigma' = (x', B')$. Note that in a labeled seed, the cluster is a ordered set, then the second item in above definition yields that $B' = B$ or $B' = -B$. In fact, under our assumption that $B$ is indecomposable, we have the following

Lemma 2.9. [1] A $\mathbb{Z}$-algebra automorphism $f : \mathcal{A} \to \mathcal{A}$ is a cluster automorphism if and only if one of the following conditions is satisfied:

1. there exists a labeled seed $\Sigma = (x, B)$ of $\mathcal{A}$, such that $f(x)$ is the cluster in a labeled seed $\Sigma' = (x', B')$ of $\mathcal{A}$ with $B' = B$ or $B' = -B$;
Figure 3: Non-decomposable quivers of finite mutation type
2. for every labeled seed $\Sigma = (x, B)$ of $\mathcal{A}$, $f(x)$ is the cluster in a labeled seed $\Sigma' = (x', B')$ with $B' = B$ or $B' = -B$.

We call those cluster automorphism such that $B = B'$ ($B = -B'$ respectively) the direct cluster automorphism (inverse cluster automorphism respectively). Clearly, all the cluster automorphism of a cluster algebra $\mathcal{A}$ consist a group with homomorphism compositions as multiplications. We call this group the cluster automorphism group of $\mathcal{A}$, and denote it by $\text{Aut}(\mathcal{A})$. We call the group $\text{Aut}^*(\mathcal{A})$ consisting of the direct cluster automorphisms of $\mathcal{A}$ the direct cluster automorphism group of $\mathcal{A}$, which is a subgroup of $\text{Aut}(\mathcal{A})$ with index at most two $[1]$. 

**Definition 2.10.** (Automorphism of exchange graphs) $[4]$ An automorphism of the exchange graph $E_\mathcal{A}$ of a cluster algebra $\mathcal{A}$ is an automorphism of $E_\mathcal{A}$ as a graph, that is, a permutation $\sigma$ of the vertex set, such that the pair of vertices $(u, v)$ forms an edge if and only if the pair $(\sigma(u), \sigma(v))$ also forms an edge.

Clearly, the natural composition of two automorphisms of $E_\mathcal{A}$ is again an automorphism of $E_\mathcal{A}$. We define an automorphism group $\text{Aut}(E_\mathcal{A})$ of $E_\mathcal{A}$ as a group consisting of automorphisms of $E_\mathcal{A}$ with compositions of automorphisms as multiplications.

It is clearly that an cluster automorphism induces a unique automorphism of the exchange graph. Thus $\text{Aut}(E_\mathcal{A})$ is a subgroup of $\text{Aut}(\mathcal{A})$ $[4]$. By the definition, an automorphism $\sigma$ of an exchange graph maps clusters to clusters, and induces an automorphism of its dual graph: cluster complex $\Delta$, we denote this automorphism by $\sigma_\Delta$. Then $\sigma_\Delta$ is a permutation of cluster variables of $\mathcal{A}$, but it may not be compatible with algebra relations of cluster variables in $\mathcal{A}$, thus it is not a cluster automorphism, and in this case $\text{Aut}(E_\mathcal{A})$ is a proper subgroup of $\text{Aut}(\mathcal{A})$, for example see Example $[2]$ and Example $[4]$. However, we have the following:

**Lemma 2.11.** Let $\sigma : E_\mathcal{A} \to E_\mathcal{A}$ be an automorphism which maps a seed $\Sigma = (x, B)$ to a seed $\Sigma' = (x', B')$. If $B \cong B'$ or $B \cong -B'$ under the map $\sigma_\Delta : x \to x'$, then $\sigma_\Delta : x \to x'$ induces a cluster automorphism $\delta$ of $\mathcal{A}$ and the induced automorphism $\delta_\mathcal{E} : E_\mathcal{A} \to E_\mathcal{A}$ is the same as $\sigma$.

**Proof.** Since $B \cong B'$ or $B \cong -B'$ under the map $\sigma_\Delta : x \to x'$, $\sigma : x \to x'$ induces a cluster automorphism $\delta$ of $\mathcal{A}$. To prove $\delta_\mathcal{E} = \sigma : E_\mathcal{A} \to E_\mathcal{A}$, we only need to show that $\delta = \sigma_\Delta$ on the cluster complex $\Delta$, or equivalently, on the cluster variables of $\mathcal{A}$. This is true for the cluster $x$, and the general cases can be proved by inductions on mutations. $\square$

### 3 Automorphism groups of exchange graphs

In this section we consider relations between two groups $\text{Aut}(\mathcal{A})$ and $\text{Aut}(E_\mathcal{A})$ for a cluster algebra $\mathcal{A}$ of finite type or of skew-symmetric finite mutation type. For this, it is needed to describe $E_\mathcal{A}$ more precisely. Firstly we will recall the basic structures of $E_\mathcal{A}$ from $[9]$, and then we introduce layers of geodesic loops on $E_\mathcal{A}$.

#### 3.1 Layers of geodesic loops

Let $\Sigma = (x, B)$ be a labeled seed on the exchange pattern $\mathcal{P}_n$ of $\mathcal{A}$. Let $x'$ be a proper subset of $x$, then $x'$ is a non-maximal simplex in the cluster complex $\Delta$. We denote by $\Delta_{x'}$ the link of $x'$.
which is the simplicial complex on the ground set \( \mathcal{A}_\mathcal{X} = \{ \alpha \in \mathcal{A} : D \cup \{ \alpha \} \in \Delta \} \), such that \( \mathbf{x}'' \) is a simplex in \( \Delta \mathcal{X} \) if and only if \( \mathbf{x}' \cup \mathbf{x}'' \) is a simplex in \( \Delta \). Let \( \Gamma_{\mathcal{X}} \) be the dual graph of \( \Delta \mathcal{X} \). We view \( \Gamma_{\mathcal{X}} \) as a subgraph of \( E_{\mathcal{A}} \) whose vertices are the simplices in \( \Delta \) that contains \( \mathbf{x}' \).

In fact \( \Gamma_{\mathcal{X}} \) is the exchange graph of a cluster algebra \( \mathcal{A}_f \) defined by a seed \( \Sigma_f = (\mathbf{x} \setminus \mathbf{x}', \mathbf{x}', B_f) \), which is the frozenization of \( \Sigma \) at \( \mathbf{x}' \) (see Definition 2.25 [3]), where \( B_f \) is obtained from \( B \) by deleting the columns labeled by variables in \( \mathbf{x}' \). Then elements in \( \mathbf{x}' \) are coefficients of \( \mathcal{A}_f \) (we refer to [9, 11] for a cluster algebra with coefficients).

For a \( \Sigma \) in \( \mathcal{A} \), introduce the following concept. Let \( \mathcal{A}' \) be cluster algebra defined by a seed \( \Sigma' = (\mathbf{x} \setminus \mathbf{x}', B') \), where \( B' \) is obtained from \( B \) by deleting rows and columns labeled by variables in \( \mathbf{x}' \). In our settings, that is, cluster algebras are of finite type or of skew-symmetric finite type, the exchange graph of a cluster algebra (with coefficients) only depends on the principal part of the exchange matrix [10, 2] which is the submatrix labeled by \( \mathbf{x} \setminus \mathbf{x}' \times \mathbf{x} \setminus \mathbf{x}' \), thus the graph \( \Gamma_{\mathcal{X}} \) is the same as the exchange graph \( E_{\mathcal{A}} \).

For a \( n - 2 \)-dimensional subcomplex \( \mathbf{x}' \) of \( \Delta \), we call the dual graph \( \Gamma_{\mathcal{X}} \) a geodesic loop of \( E_{\mathcal{A}} \). If \( \mathcal{A} \) is of finite type, then \( E_{\mathcal{A}} \) is a finite graph, and \( \Gamma_{\mathcal{X}} \) is a polygon. Notice that in the seed \( \Sigma' = (\mathbf{x} \setminus \mathbf{x}', B') \), \( B' \) is of Dynkin type, that is, one of types \( A_2, B_2, C_2 \) or \( G_2 \). Therefore \( \Gamma_{\mathcal{X}} \) is a \( h + 2 \)-polygon, where \( h \) is the Coxeter number of the corresponding Dynkin type [10]. If \( \mathcal{A} \) is of finite mutation type, then \( \Gamma_{\mathcal{X}} \) may be a line.

In fact, the fundamental group of \( E_{\mathcal{A}} \) is generated by geodesic loops pinned down to a fixed basepoint, that is, the generators of \( E_{\mathcal{A}} \) is of the form \( P L P \), where \( P \) is a path originating at the basepoint, and \( L \) is a geodesic loop. We fix a basepoint \( \Sigma = (\mathbf{x}, B) \) and introduce the following concept.

**Definition 3.1.**

1. Let \( \Sigma' \) be a point of \( E_{\mathcal{A}} \), the distance \( \ell(\Sigma, \Sigma') \) between \( \Sigma \) and \( \Sigma' \) is the minimal length of paths between \( \Sigma \) and \( \Sigma' \);
2. Let \( L \) be a geodesic loop of \( E_{\mathcal{A}} \), the distance \( \ell(L) \) between \( \Sigma \) and \( L \) is equal to the minimal length \( \min(\ell(\Sigma, \Sigma'), \Sigma' \in L) \);
3. Let \( m \in \mathbb{Z}_{\geq 0} \) be a non-negative integer, denote by \( \ell^m(\Sigma) \) the set of geodesic loop whose distance to \( \Sigma \) is \( m \). We call it the \( m \)-layer of geodesic loops of \( E_{\mathcal{A}} \) originating at \( \Sigma \);
4. For a set \( \ell^m(\Sigma) \), the element in \( N(\ell^m(\Sigma)) \) is the number of edges of geodesic loops in \( \ell^m(\Sigma) \).

**Remark 3.2.** The following observations are directly derived from the definitions:

1. The elements in \( \ell^0(\Sigma) \) are those geodesic loops \( \Gamma_{\mathcal{X}} \) for the \( n - 2 \)-dimensional subcomplex \( \mathbf{x}' \) of \( \Delta \);
2. For \( m_1 \neq m_2 \), \( \ell^{m_1}(\Sigma) \cap \ell^{m_2}(\Sigma) = \emptyset \);
3. The disjoint union \( \sqcup_{m \geq 0} \ell^m(\Sigma) \) is the set of all the geodesic loops of \( E_{\mathcal{A}} \);
4. If \( \sigma : E_{\mathcal{A}} \to E_{\mathcal{A}} \) is an isomorphism of graphs, such that the image of \( \Sigma \) is \( \Sigma' \), then for every \( m \in \mathbb{Z}_{\geq 0} \), \( N(\ell^m(\Sigma)) = N(\ell^m(\Sigma')) \).
3.2 Cases of rank two and rank three

In this subsection, we consider the relations between $\text{Aut}(\mathcal{A})$ and $\text{Aut}(E_{\mathcal{A}})$ for a cluster algebra $\mathcal{A}$ of rank two or rank three.

**Example 2.** For a finite type cluster algebra $\mathcal{A}$ of rank 2, that is, one of types $A_2, B_2, C_2$ or $G_2$, its exchange graph $E_{\mathcal{A}}$ is a $h+2$-polygon, thus $\text{Aut}(E_{\mathcal{A}}) \cong D_{h+2}$. If $\mathcal{A}$ is of type $A_2$, then $\text{Aut}(\mathcal{A}) \cong D_3$ [1], thus $\text{Aut}(\mathcal{A}) \equiv \text{Aut}(E_{\mathcal{A}})$. If $\mathcal{A}$ is of type $B_2, C_2$ or $G_2$, Theorem 3.5 in [5] shows that $\text{Aut}(\mathcal{A}) \equiv D_{h+2/2}$, thus $\text{Aut}(\mathcal{A}) \subseteq \text{Aut}(E_{\mathcal{A}})$.

**Example 3.** For an infinite type skew-symmetric cluster algebra $\mathcal{A}$ of rank 2, its exchange graph $E_{\mathcal{A}}$ is a line, thus $\text{Aut}(E_{\mathcal{A}}) = \langle s \rangle \times \mathbb{Z} \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, where $s$ is a reflection of $E_{\mathcal{A}}$ which maps a cluster to the left adjacent cluster and $r$ is a reflection through a fixed cluster. Then $s$ corresponds to a direct cluster automorphism of $\mathcal{A}$ and $r$ corresponds to an inverse cluster automorphism of $\mathcal{A}$, thus by Lemma 2.11 $\text{Aut}(E_{\mathcal{A}}) \subseteq \text{Aut}(\mathcal{A})$. Therefore $\text{Aut}(E_{\mathcal{A}}) \equiv \text{Aut}(\mathcal{A}) \equiv \mathbb{Z} \rtimes \mathbb{Z}_2$.

**Example 4.** For an infinite type non-skew-symmetric cluster algebra $\mathcal{A}$ of rank 2, its exchange graph $E_{\mathcal{A}}$ is also a line, thus as showed in Example 2 $\text{Aut}(E_{\mathcal{A}}) = \langle s \rangle \times \mathbb{Z} \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, where $s$ corresponds to a direct cluster automorphism of $\mathcal{A}$, while $r$ does not correspond to any cluster automorphism of $\mathcal{A}$, since there is no non-trivial symmetry of the quiver in any seed of $\mathcal{A}$. Thus $\text{Aut}(\mathcal{A}) \equiv \mathbb{Z} \rtimes \text{Aut}(E_{\mathcal{A}})$.

**Example 5.** We consider the cluster algebra $\mathcal{A}$ of type $A_3$ with an initial labeled seed $\Sigma = (\{x_1, x_2, x_3\}, Q)$, where $Q$ is $1 \rightarrow 2 \rightarrow 3$. Then its exchange graph $E_{\mathcal{A}}$ is depicted in Figure 4. Note that there are three quadrilaterals and six pentagons in $E_{\mathcal{A}}$. Then as showed in [5] (Example 2), $\text{Aut}(\mathcal{A}) = \langle f_-, f_0 \rangle \cong D_6$, where $f_-$ is defined by:

$$f_- : \begin{cases} x_1 &\mapsto x_1 \\ x_2 &\mapsto \mu_2(x_2) \\ x_3 &\mapsto x_3 \end{cases}$$

(3)

It maps $\Sigma$ to $\Sigma_1$, and induces a reflection with respect to the horizontal central axis of $E_{\mathcal{A}}$, see in Figure 4. The cluster automorphism $f_0$ is defined by:

$$f_0 : \begin{cases} x_1 &\mapsto x_3 \\ x_2 &\mapsto x_2 \\ x_3 &\mapsto x_1 \end{cases}$$

(4)

It induces a reflection with respect to the vertical central axis of $E_{\mathcal{A}}$. In fact as showed in [5], an cluster automorphism of $\mathcal{A}$ induces a permutation of seeds in $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$, thus $\text{Aut}(\mathcal{A}) \cong D_6$ can be viewed as the symmetry group of the belt consisting of $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$. We prove that these permutations induce all the automorphisms of $E_{\mathcal{A}}$, that is, $\text{Aut}(\mathcal{A}) \equiv \text{Aut}(E_{\mathcal{A}})$. For this purpose, by Lemma 2.11 we only need to show that there exists no automorphism of $E_{\mathcal{A}}$ which maps $\Sigma$ to a vertex excepting for a vertex in $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$. Let $\sigma$ be an automorphism of $E_{\mathcal{A}}$, then due to symmetries of $E_{\mathcal{A}}$, we only show that $\sigma(\Sigma) \neq O_i$, $i = 1, 2, 3$. The layers
Figure 4: The exchange graph of a cluster algebra of type $A_3$

of geodesic loops originating at these vertices are as follows:

\[
\begin{align*}
N(\ell_{\Sigma}^0) &= \{4, 5, 5\}, \quad N(\ell_{\Sigma}^1) = \{4, 5, 5\}, \quad N(\ell_{\Sigma}^2) = \{5, 5, 5\}, \quad N(\ell_{\Sigma}^3) = \{4\}; \\
N(\ell_{O_1}^0) &= \{4, 5, 5\}, \quad N(\ell_{O_1}^1) = \{5, 5, 5\}, \quad N(\ell_{O_1}^2) = \{4, 4\}, \quad N(\ell_{O_1}^3) = \{5\}; \\
N(\ell_{O_2}^0) &= \{5, 5, 5\}, \quad N(\ell_{O_2}^1) = \{4, 4, 4\}, \quad N(\ell_{O_2}^2) = \{5, 5, 5\}; \\
N(\ell_{O_3}^0) &= \{4, 5, 5\}, \quad N(\ell_{O_3}^1) = \{5, 5, 5\}, \quad N(\ell_{O_3}^2) = \{4, 4\}, \quad N(\ell_{O_3}^3) = \{5\};
\end{align*}
\]

Then by Remark 3.2(4), $\sigma(\Sigma) \neq O_i$, $i = 1, 2, 3$. Thus $\text{Aut}(E_A) \cong \text{Aut}(A) \cong D_6$.

**Example 6.** It is well known that the cluster algebras of type $B_n$ and type $C_n$ have the same combinatorial structure. Originating at a seed $\Sigma$, the exchange graph of a cluster algebra $A$ of type $B_3$ or type $C_3$ is depicted in Figure 5. For the cluster algebra of type $B_3$, the quiver of the initial seed $\Sigma$ is

\[
1 \rightarrow (2, 1) 2 \leftarrow 3.
\]

For the cluster algebra of type $C_3$, the quiver of the initial seed $\Sigma$ is

\[
1 \rightarrow (1, 2) 2 \leftarrow 3.
\]
Figure 5: The exchange graph of a cluster algebra of type $B_3$ or type $C_3$

Let $\sigma$ be an automorphism of $E_{A\mathcal{R}}$. Since $N(\ell^0_{\Sigma}) = \{4, 5, 6\}$, there are no rotation symmetries of $E_{A\mathcal{R}}$ at $\Sigma$, thus $\sigma(\Sigma) \neq \Sigma$. As showed by Example 3 in [5], $\text{Aut}(\mathcal{R}) \cong D_4$ and $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7\}$ are all the seeds which quivers are isomorphic to $Q$, then by Lemma 2.11 to prove that $\text{Aut}(E_{A\mathcal{R}}) \cong \text{Aut}(\mathcal{R})$, we only need to show that $\sigma(\Sigma)$ must be a seed in $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7\}$. By the symmetries of $E_{A\mathcal{R}}$, we only prove that $\sigma(\Sigma) \neq O_i (i = 1, 2, 3, 4)$, and this can be obtained by the fact that these seeds have different combinatorial numbers of layers of geodesic loops:

- $N(\ell^0_{\Sigma}) = \{4, 5, 6\}, \ N(\ell^1_{\Sigma}) = \{4, 5, 6\}$;
- $N(\ell^0_{O_1}) = \{5, 6, 6\};$
- $N(\ell^0_{O_2}) = \{4, 5, 6\}, \ N(\ell^1_{O_2}) = \{5, 6, 6\};$
- $N(\ell^0_{O_3}) = \{4, 5, 6\}, \ N(\ell^1_{O_3}) = \{5, 6, 6\};$
- $N(\ell^0_{O_4}) = \{5, 6, 6\}.$
Example 7. For cluster algebra of type $F_4$, let the quiver $Q$ of a seed $\Sigma$ is

```
1 ----> 2 <----(2,1)----> 3 ----> 4.
```

Then $\text{Aut}(\mathcal{A}) \cong D_7 \mathbb{Z}_2$. The variables $x_1, x_2, x_3$ and the corresponding full subquiver of $Q$ form a seed $\Sigma_1$ of type $B_3$, while $x_2, x_3, x_4$ and the corresponding full subquiver of $Q$ form a seed $\Sigma_2$ of type $C_3$. By pinning down $\Sigma$, rotating the graph $E_{\mathcal{A}}$ induces an automorphism $\sigma$ of $E_{\mathcal{A}}$, which exchanges the graph $E_{\mathcal{A}\Sigma_1}$ and the graph $E_{\mathcal{A}\Sigma_2}$. However $\sigma$ does not induces a cluster automorphism of $\mathcal{A}$, and $\text{Aut}(\mathcal{A}) \cong D_7 \mathbb{Z}_2 \cong \text{Aut}(E_{\mathcal{A}})$.

Proposition 3.3. Let $Q$ be a connected quiver with three vertices, which is of finite type. Let $\Sigma = (x, Q)$ and $\Sigma' = (x', Q')$ be two seeds. If there is an isomorphism $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ such that $\sigma(\Sigma) = \Sigma'$, then $\Sigma'$ is a finite type seed with $Q'$ connected, and

1. if $\Sigma$ is of type $A_3$, then $Q' \cong Q$ (or $Q^{op}$);
2. if $\Sigma$ is of type $B_3$ and $\Sigma'$ is not of type $C_3$, then $Q' \cong Q$ (or $Q^{op}$);
3. if $\Sigma$ is of type $C_3$ and $\Sigma'$ is not of type $B_3$, then $Q' \cong Q$ (or $Q^{op}$).

Proof. Clearly, since $E_{\mathcal{A}} \cong E_{\mathcal{A}'}$ is of finite, $Q'$ is a Dynkin type quiver with three vertices. If $Q$ is of type $A_3$, then by Example 5

$N(\ell_0^0) = \{4, 5, 5\}$ or $\{5, 5, 5\}$.

If $Q$ is of type $B_3$ (or $C_3$), then from Example 6

$N(\ell_0^0) = \{4, 5, 6\}$ or $\{5, 6, 6\}$.

If $Q'$ is a union of a quiver of type $A_2$ and a point, then from Example 2

$N(\ell_0^0) = \{4, 4, 5\}.$

If $Q'$ is a union of a quiver of type $B_2$ (or $C_2$) and a point, then from Example 2

$N(\ell_0^0) = \{4, 4, 6\}.$

If $Q'$ is a union of a quiver of type $G_2$ and a point, then from Example 2

$N(\ell_0^0) = \{4, 4, 8\}.$

Thus we get the proof by Remark 5.2. □

Example 8. Let $Q$ be the quiver in Figure 6, we call it of type $\tilde{A}_2$. Then it is not hard to see that if a quiver in the mutation class of $Q$ is not isomorphic to $Q$, then it must be isomorphic to the quiver $Q'$ in Figure 6. Let $\mathcal{A}$ be a cluster algebra with an initial seed $\Sigma = (\{x_1, x_2, x_3\}, Q)$, similar to above examples, to show that $\text{Aut}(\mathcal{A}) \cong \text{Aut}(E_{\mathcal{A}})$, it is only need to notice that:

$N(\ell_0^0) = \{5, 5, \infty\},$

$N(\ell_0^0) = \{5, 5, 5\}$.

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where $\Sigma'$ is a seed of $\mathcal{A}$ with quiver isomorphic to $Q'$. In fact, from section 3.3 in [1], $\text{Aut}(\mathcal{A}) = \langle r_1, r_2 | r_1r_2 = r_2r_1, r_1^2 = r_2 > \triangleright < \sigma^2 = 1 \rangle \cong H_{2,1} \rtimes \mathbb{Z}_2$, where

$$r_1 : \begin{cases} x_1 \mapsto x_3 \\ x_2 \mapsto \mu_1(x_1) \\ x_3 \mapsto x_2 \end{cases}$$

(5)

$$r_2 : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto \mu_3\mu_1(x_3) \\ x_3 \mapsto \mu_1(x_1) \end{cases}$$

(6)

$$\sigma : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto x_1 \\ x_3 \mapsto x_3 \end{cases}$$

(7)

Thus $\text{Aut}(E_{\mathcal{A}}) \cong H_{2,1} \rtimes \mathbb{Z}_2$.

**Example 9.** Let $\mathcal{A}$ be a cluster algebra from an once punctured torus, we call it a cluster algebra of type $T_3$, then it is of finite mutation type with quiver always isomorphic to the quiver in Figure 7. Then by Lemma 2.11 we have $\text{Aut}(\mathcal{A}) \cong \text{Aut}(E_{\mathcal{A}})$.

**Lemma 3.4.** Let $Q$ be a connected skew-symmetric quiver of finite mutation type.

1. If there are 3 vertices in $Q$, then $Q$ is one of the following types:

   (1) $A_3$ type;
   (2) $\tilde{A}_2$ type;
   (3) $T_3$ type.
2. If there are at least 4 vertices in \( Q \), then any full subquiver of \( Q \) with three vertices is of type \( A_3 \) or of type \( \tilde{A}_3 \).

Proof. 1. From the classification of cluster algebras of finite mutation type, \( Q \) must be block-decomposable, then the proof is a straightforward check by gluing the blocks in \( \Sigma \).

2. It is only need to notice that there is no way to glue a quiver of type \( T_3 \) with other blocks. \( \square \)

It is clearly that if for any quiver in the mutation equivalent class of \( Q \), the number of arrows between any two vertices is at most 2, then \( Q \) is of finite mutation type. The above lemma shows that the inverse statement is also true for the cases of vertices at least 3, that is, we have the following:

**Corollary 3.5.** A connected quiver \( Q \) with at least 3 vertices is of finite mutation type if and only if for any quiver in its mutation equivalent class, the number of arrows between any two vertices is at most 2.

**Proposition 3.6.** Let \( Q \) be a connected skew-symmetric quiver with three vertices, which is of finite mutation type. Let \( \Sigma = (x, Q) \) and \( \Sigma' = (x', Q') \) be two seeds. If there is an isomorphism \( \sigma : E_{A} \rightarrow E_{A'} \) such that \( \sigma(\Sigma) = \Sigma' \), then \( Q \cong Q' \) or \( Q' \cong Q^{op} \).

Proof. Similar to Proposition 3.3 this follows from Lemma 3.4 Example 9 Example 5 and Example 8. \( \square \)

**3.3 General cases**

**Theorem 3.7.** Let \( \mathcal{A} \) be a cluster algebra of finite type. Assume that it is not of type \( F_4 \), let \( \Sigma = (x, Q) \) be a labeled seed of \( \mathcal{A} \), where \( Q \) is a connected quiver with at least three vertices. Then we have \( \text{Aut}(\mathcal{A}) \leq \text{Aut}(E_{A}) \).

Proof. We need to show that \( \text{Aut}(E_{A}) \subseteq \text{Aut}(\mathcal{A}) \). Let \( x' \in x \) be a 3-dimensional complex, such that the full subquiver \( Q' \) of \( Q \) corresponding to the variables in \( x' \) are connected. Define a seed \( \Sigma' = (x', Q') \). Denote by \( \mathcal{A}' \) the cluster algebra defined by \( \Sigma' \). Let \( \sigma' \) be an automorphism of \( E_{A} \). Then \( \sigma' \) induces an automorphism \( \sigma \) of \( \Delta \), which maps \( \Delta_{x\setminus x'} \rightarrow \Delta_{x\setminus x'} \), clearly, \( \sigma : \Delta_{x\setminus x'} \rightarrow \Delta_{x\setminus x'} \) is an isomorphism, and induces an isomorphism from \( \Gamma_{x\setminus x'} \). Let \( \Sigma'' = (x'' Q'') \) be a seed, where \( x'' = \sigma(x') \) and \( Q'' \) is a full subquiver of \( Q(x) \) whose vertices are those labeled by elements in \( x'' \). Let \( \mathcal{A}'' \) be the cluster algebra of \( \Sigma'' \). Then as showed in the beginning of subsection 3.3, \( \Gamma_{x\setminus x'} \cong E_{A} \) and \( \Gamma_{x\setminus x'} \cong E_{A'} \). Thus \( E_{A} \cong E_{A'} \). Since \( \mathcal{A} \) is not of type \( F_4 \), if \( Q' \) is of type \( B_3 \) (type \( C_3 \) respectively), then \( Q'' \) is not of type \( C_3 \) (type \( B_3 \) respectively). Thus by Proposition 3.3, \( Q' \cong Q'' \) or \( Q' \cong Q^{op} \). Finally, due to the arbitrariness of the choose of \( x' \) and the connectivity of \( Q \), \( Q(\sigma(x)) \cong Q \) or \( Q(\sigma(x)) \cong Q^{op} \). Therefore \( \sigma : \Delta \rightarrow \Delta \) induces an cluster automorphism of \( \mathcal{A} \). Thus \( \text{Aut}(E_{A}) \subseteq \text{Aut}(\mathcal{A}) \) and \( \text{Aut}(E_{A}) \equiv \text{Aut}(\mathcal{A}) \).

Then combine with above theorem, Table 1 in 11 and Theorem 3.5 in 5, we have the table of automorphism groups of the exchange graphs of cluster algebras of finite type. The cases of rank two and type \( F_4 \) is computed in Example 2 and Example 7 respectively.

**Theorem 3.8.** Let \( \mathcal{A} \) be a connected skew-symmetric cluster algebra of finite mutation type, then \( \text{Aut}(\mathcal{A}) \equiv \text{Aut}(E_{A}) \).
Dynkin type | Automorphism group \( \text{Aut}(E_{\mathcal{A}}) \)  
|----------------|----------------------------------| 
| \( A_n(n \geq 2) \) | \( D_{n+3} \)  
| \( B_2 \) | \( D_6 \)  
| \( B_n(n \geq 3) \) | \( D_{n+1} \)  
| \( C_2 \) | \( D_6 \)  
| \( C_n(n \geq 3) \) | \( D_{n+1} \)  
| \( D_4 \) | \( D_4 \times S_3 \)  
| \( D_n(n \geq 5) \) | \( \mathbb{Z}_2 \)  
| \( E_6 \) | \( D_{14} \)  
| \( E_7 \) | \( D_{10} \)  
| \( E_8 \) | \( D_{16} \)  
| \( F_4 \) | \( D_7 \rtimes \mathbb{Z}_2 \)  
| \( G_2 \) | \( D_8 \)  

Table 1: Automorphism groups of exchange graphs of cluster algebras of finite type

**Proof.** If \( \mathcal{A} \) is of finite type of rank 2, that is, of type \( A_2 \), then the result follows from Example 2. If \( \mathcal{A} \) is of infinite type of rank 2, then the result follows from Example 3. For the case of type \( A_3 \), type \( \tilde{A}_2 \) or type \( T_3 \), we derive the result from Example 5, Example 8 or Example 9 respectively. Otherwise, by Lemma 3.4(2), the connected full subquiver of a quiver of \( \mathcal{A} \) with three vertices is of type \( A_3 \) or of type \( \tilde{A}_2 \). Therefore by Example 8 and Example 5, the proof is similar to the proof of Theorem 3.7. \( \square \)

**Corollary 3.9.** Let \( \mathcal{A} \) be a connected cluster algebra of finite type or of skew-symmetric finite mutation type, then an automorphism of \( E_{\mathcal{A}} \) is determined by the image of any fixed seed and the images of seeds which are adjacent to the fixed seed, more precisely, let \( \Sigma = (x, B) \) be a seed on \( E_{\mathcal{A}} \), then an automorphism \( \sigma : E_{\mathcal{A}} \rightarrow E_{\mathcal{A}} \) is determined by a pair \((\Sigma', \sigma')\), where \( \Sigma' = (x', B') \) is a seed on \( E_{\mathcal{A}} \) and \( \sigma' : x \rightarrow x' \) is a bijection such that \( \sigma(\Sigma) = \Sigma' \) and \( \sigma(\mu_\lambda(x)) = \mu_{\sigma'(\lambda)}(x') \) for any \( x \in x \).

**Proof.** If \( \mathcal{A} \) is of finite type of rank 2 and type \( F_4 \), then the conclusion is clearly. Otherwise, note that a cluster automorphism is determined by such a pair \((\Sigma', \sigma')\), thus the proof follows from Theorem 3.7 and Theorem 3.8. \( \square \)

**Conjecture 3.10.** Let \( \mathcal{A} \) and \( \mathcal{A}' \) be two connected cluster algebras of finite type, or of skew-symmetric finite mutation type. Let \( \Sigma = (x, B) \) and \( \Sigma' = (x', B') \) be two seeds of \( \mathcal{A} \) and \( \mathcal{A}' \) respectively.

1. If \( N(t_k^\Sigma) = N(t_k^{\Sigma'}) \) for any \( k \in \mathbb{Z}^* \), then there exists an isomorphism \( \sigma : E_{\mathcal{A}} \rightarrow E_{\mathcal{A}'} \) such that \( \sigma(\Sigma) = \Sigma' \).
2. Assume that \( \mathcal{A} \) or \( \mathcal{A}' \) is not types of rank 2 and type \( F_4 \). If \( N(t_k^\Sigma) = N(t_k^{\Sigma'}) \) for any \( k \in \mathbb{Z}^* \), then \( B \cong B' \) or \( B \cong -B' \).

It follows from Example 8, Example 6, Example 8 and Example 9 that the conjecture is true for the cases of rank 2 and rank 3. Clearly, the second part of the conjecture can be derived from the first one, Theorem 3.7 and Theorem 3.8.
References

[1] Assem I, Schiffler R, Shramchenko V. Cluster automorphisms. Proceedings of the London Mathematical Society, 2012, 104(6):1271-1302.

[2] Irelli C G, Keller B, Labardini-Fragoso D, Plamondon P G. Linear independence of cluster monomials for skew-symmetric cluster algebras. Compositio Mathematica, 2013, 149(10):1753-1764.

[3] Wen Chang, Bin Zhu. On rooted cluster morphisms and cluster structures in 2-Calabi-Yau triangulated categories, arXiv:1410.5702 (2014).

[4] Wen Chang, Bin Zhu. Cluster automorphism groups of cluster algebras with coefficients.

[5] Wen Chang, Bin Zhu. Cluster automorphism groups of cluster algebras of finite type.

[6] Fomin S, Shapiro M, Thurston D. Cluster algebras and triangulated surfaces. Part I: Cluster complexes. Acta Mathematica, 2008, 201:83-146.

[7] Felikson A, Shapiro M, Tumarkin P. Cluster algebras of finite mutation type via unfoldings. International Mathematics Research Notices, 2011: rnr072.

[8] Felikson A, Shapiro M, Tumarkin P. Skew-symmetric cluster algebras of finite mutation type. Journal of the European Mathematical Society, 2012, 14(4): 1135-1180.

[9] Fomin S, Zelevinsky A. Cluster algebras. I. Foundations. Journal of the American Mathematical Society, 2002, 15(2), 497-529.

[10] Fomin S, Zelevinsky A. Cluster algebras. II. Finite type classification. Inventiones Mathematicae, 2003, 154(1):63-121.

[11] Fomin S, Zelevinsky A. Cluster algebras IV: Coefficients. Compositio Mathematica, 2007, 143:112-164.

[12] M. Gekhtman, M. Shapiro and A. Vainshtein. On the properties of the exchange graph of a cluster algebra, Math. Res. Lett, 15(2), (2008), 321–330.

[13] Keller B. Cluster algebras and derived categories. arXiv:1202.4161. 60 pages.