REPRESENTATIONS OF CYCLOTOMIC ORIENTED BRAUER CATEGORIES

MENGMENG GAO, HEBING RUI, LINLIANG SONG

Abstract. Let \( A \) be the locally unital algebra associated to a cyclotomic oriented Brauer category over an arbitrary algebraically closed field \( \mathbb{k} \) of characteristic \( p \geq 0 \). The category of locally finite dimensional representations of \( A \) is used to give the tensor product categorification (in the general sense of Losev and Webster) for an integrable lowest weight with an integrable highest weight representation of the same level for the Lie algebra \( \mathfrak{g} \), where \( \mathfrak{g} \) is a direct sum of copies of \( \mathfrak{sl}_\infty \) (resp., \( \hat{\mathfrak{sl}}_p \)) if \( p = 0 \) (resp., \( p > 0 \)). Such a result was expected in \([4]\) when \( \mathbb{k} = \mathbb{C} \) and proved previously in \([2]\) when the level is 1.

1. Introduction

Throughout this paper, \( \mathbb{k} \) is an arbitrary algebraically closed field of characteristic \( p \geq 0 \). Unless otherwise stated, all algebras, categories and functors are assumed to be \( \mathbb{k} \)-linear.

Fix \( \ell \in \mathbb{Z}_{>0} \), \( \mathbf{u}, \mathbf{u}' \in \mathbb{k}^\ell \) and set

\[
\mathbf{u} = (u_1, \ldots, u_\ell), \quad \mathbf{u}' = (u'_1, \ldots, u'_\ell). \tag{1.1}
\]

In \([4]\), Brundan et al. introduced the notion of the cyclotomic oriented Brauer category \( \mathcal{OB}(\mathbf{u}, \mathbf{u}') \). When \( \ell = 1 \), \( \mathcal{OB}(\mathbf{u}, \mathbf{u}') \), called the oriented Brauer category, is monoidally equivalent to the category in \([12]\) Example 1.26. The aim of this paper is to give a tensor product categorification of any integrable lowest weight with any integrable highest weight module of the same level for \( \mathfrak{g} \) by using locally finite dimensional representations of cyclotomic oriented Brauer categories, where \( \mathfrak{g} \) is a direct sum of copies of \( \mathfrak{sl}_\infty \) (resp., \( \hat{\mathfrak{sl}}_p \)) if \( p = 0 \) (resp., \( p > 0 \)).

Let \( A_1 \) be the locally unital algebra associated to \( \mathcal{OB}(\mathbf{u}, \mathbf{u}') \). Reynolds proved that \( A_1 \) admits a triangular decomposition \([20]\). Brundan further showed that the locally unital algebra \( B \) associated to the oriented skein category (which was introduced by Turaev and was called HOMFLY-PT skein category \([24]\)) also admits a triangular decomposition. For any locally unital and locally finite dimensional algebra \( C \), let \( -\text{lfddmod} \) be the category of all locally finite dimensional left \( C \)-modules. The (upper finite) triangular decomposition property is the key in the proof that both \( -\text{lfddmod} \) and \( B\text{-lfddmod} \) are upper finite stratified categories in the sense of Brundan and Stroppel \([10]\). Brundan also constructed certain endofunctors of \( -\text{lfddmod} \) and \( B\text{-lfddmod} \) and its associated graded categories so as to give a tensor product categorification of an integrable lowest weight with an integrable highest weight module of the same level 1 for \( \mathfrak{g} \). However, it is unclear to us whether \( A_1 \) admits an upper finite triangular decomposition in general. So, one could not use arguments on upper finite triangular decompositions to show that \( -\text{lfddmod} \) is an upper finite fully stratified category.

In \([14]\), we introduced and studied the upper finite weakly triangular categories. It is easy to see that any category whose associated locally unital algebra admits an upper finite triangular decomposition is an upper finite weakly triangular category. Moreover, cyclotomic oriented Brauer categories \([23]\) and cyclotomic Kauffman categories \([13]\) are upper finite weakly triangular categories \([14]\). We established in \([14]\) that \( C\text{-lfddmod} \) is an upper finite fully stratified category if \( C \) is the locally unital algebra associated to an upper finite weakly triangular category. In particular, \( A_1\text{-lfddmod} \) is an upper finite fully stratified category.

Set

\[
\mathbb{I} = \mathbb{I}_\mathbf{u} \bigcup \mathbb{I}_\mathbf{u}', \tag{1.2}
\]

where \( \mathbb{I}_\mathbf{u} = \{u_j + n|1 \leq j \leq \ell, n \in \mathbb{Z}\} \) and \( \mathbb{I}_\mathbf{u}' = \{u'_j + n|1 \leq j \leq \ell, n \in \mathbb{Z}\} \), and \( u_1, \ldots, u_\ell \) and \( u'_1, \ldots, u'_\ell \) are given in (1.1). Let \( \mathfrak{g} \) be the complex Kac-Moody Lie algebra associated to Cartan
matrix \((a_{i,j})_{i,j\in \mathbb{N}}\), where

\[
a_{i,j} = \begin{cases} 
2, & \text{if } i = j; \\
-1, & \text{if } i = j \pm 1 \text{ and } p \neq 2; \\
-2, & \text{if } i = j - 1 \text{ and } p = 2; \\
0, & \text{otherwise}.
\end{cases}
\]  

(1.3)

Then \(\mathfrak{g}\) is isomorphic to a direct sum of copies of \(\mathfrak{sl}_n\) (resp., \(\mathfrak{sl}_p\)) if \(p = 0\) (resp., \(p > 0\)) depending on the number of \(\mathbb{Z}\)-orbits of \(\{\omega_1, \ldots, u_1, u_1', \ldots, u_\ell\}\) in the sense that, for any \(x, y \in \{u_1, \ldots, u_\ell, u_1', \ldots, u_\ell'\}\), \(x\) and \(y\) are in the same \(\mathbb{Z}\)-orbit if and only if \(x - y \in \mathbb{Z}1_k\). Let \(V(\omega_u)\) be the integrable highest weight \(\mathfrak{g}\)-module of weight

\[
\omega_u = \sum_{i=1}^{\ell} \omega_{u_i},
\]

(1.4)

where \(\omega_{u_i}\)'s are fundamental weights and \(\ell\) is said to be the level. Similarly, let \(\tilde{V}(-\omega_{u'})\) be the integrable lowest weight \(\mathfrak{g}\)-module of weight \(-\omega_{u'}\). The following is the main result of this paper. It was obtained previously in [2] when \(\ell = 1\).

**Theorem 1.1.** Let \(A\) be the locally unital \(k\)-algebra associated to the cyclotomic oriented Brauer category \(\mathcal{OB}(u, u')\). Then \(A\)-lfd\(\text{mod}\) admits the structure of a tensor product categorification of the \(\mathfrak{g}\)-module \(\tilde{V}(-\omega_{u'}) \otimes V(\omega_u)\) in the general sense of Losev and Webster (e.g., Definition 4.3).

If \(k = \mathbb{C}\) and \(u \cup u'\) consists of a unique orbit, then the category \(A\)-lfd\(\text{mod}\) admits the structure of a tensor product categorification of \(\mathfrak{sl}_{\infty}\)-module \(\tilde{V}(-\omega_{u'}) \otimes V(\omega_u)\). Such a result was expected in [2].

In order to prove Theorem 1.1 we study representations of cyclotomic oriented Brauer categories. Our results include a classification of simple \(A\)-modules, a criterion on the complete reducibility of the category of left \(A\)-modules, certain partial results on blocks, and certain endofunctors of \(A\)-lfd\(\text{mod}\) and its associated graded categories and so on.

After posting the paper on ArXiv we were informed that the category \(\mathcal{OB}(u, u')\) is a generalized cyclotomic quotient (GCQ) of the Heisenberg category (of central charge zero) [3]. Brundan, Savage and Webster proved that GCQs of the Heisenberg category are isomorphic to GCQs of a corresponding Kac-Moody 2-category [7]. Webster showed in [20] that for all GCQs of all Kac-Moody 2-categories, the category of locally finite-dimensional modules is a tensor product categorification, in particular, this category is upper finite fully stratified. So Theorem 1.1 could also be obtained by applying the isomorphism between GCQs of Heisenberg and Kac-Moody 2-category and tensor product categorification for GCQs of Kac-Moody 2-category. We are also told recently by Webster that even if one knows that two results are equivalent under a theorem that translates between two contexts, it can still make more sense to prove them in the other context rather than trying to translate when the translation is sufficiently complicated. Moreover, as pointed out in [10], GCQs of the Heisenberg category of central charge different from zero should possess upper finite weakly triangular decompositions and hence Theorem 1.1 can be generalized to those more general GCQs of Heisenberg category by using the theory of upper finite weakly triangular category in [14]. However, it seems hard to prove the expected basis theorem for a GCQ of Heisenberg category with central charge different from zero.

We organize this paper as follows. In section 2, we recall some elementary results on cyclotomic oriented Brauer categories in [4][14]. We study representations of cyclotomic oriented Brauer categories in section 3 and prove Theorem 1.1 in section 4.

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2. CYCLOTOMIC ORIENTED BRAUER CATEGORIES

In this section, we recall some results on cyclotomic oriented Brauer categories. In the usual string calculus for strict monoidal categories, the composition \(g \circ h\) of two morphisms \(g\) and \(h\) is given by vertical stacking and the tensor product \(g \otimes h\) is given by horizontal concatenation. More explicitly,

\[
g \circ h = \begin{array}{c}
\circ
\\
\h
\end{array}, \quad g \otimes h = \begin{array}{c}
\circ
\\
\h
\end{array}
\quad (2.1)
\]
2.1. The category $\text{OB}$. The oriented Brauer category $\text{OB}$ was introduced in [4]. It is the strict monoidal category generated by two generating objects $\uparrow, \downarrow$ and four elementary morphisms
\[ \begin{align*}
\begin{array}{c}
\uparrow: \emptyset \to \uparrow \otimes \downarrow, \\
\downarrow: \emptyset \otimes \uparrow \to \emptyset, \\
\uparrow \otimes \uparrow \to \uparrow \otimes \uparrow, \\
\uparrow \otimes \downarrow \to \downarrow \otimes \uparrow,
\end{array}
\end{align*} \tag{2.2}
\]
which satisfy the relations [4, (1.4)-(1.8)]:
\[ \begin{align*}
\begin{array}{c}
\circledast = \uparrow, \\
\circledast = \downarrow, \\
\circledast = \uparrow, \\
\circledast = \downarrow, \\
is the two-sided inverse to $\circledast$. \tag{2.7}
\end{array}
\end{align*} \]
Here, $\downarrow$ is the identity morphism from $\downarrow$ to $\downarrow$, and $\uparrow$ is the identity morphism from $\uparrow$ to $\uparrow$. In the following, the inverse of $\circledast$ in (2.7) will be denoted by $\circledast$. For any $\delta_1 \in k$, let $\text{OB}(\delta_1)$ be the category obtained from $\text{OB}$ by adding the additional relation
\[ \circledast = \delta_1. \tag{2.8} \]

2.2. The category $\text{AOB}$. The affine oriented Brauer category $\text{AOB}$ was also introduced in [4]. It is the strict monoidal category generated by two generating objects $\uparrow, \downarrow$ and the morphism $\circledast$ together with four morphisms in (2.2) subject to the relations in (2.3)–(2.7) and one extra relation:
\[ \begin{align*}
\begin{array}{c}
\circledast = \circledast + \uparrow \uparrow, \\
is the two-sided inverse to $\circledast$. \tag{2.7}
\end{array}
\end{align*} \]
Here, $\downarrow$ is the identity morphism from $\downarrow$ to $\downarrow$, and $\uparrow$ is the identity morphism from $\uparrow$ to $\uparrow$. In the following, the inverse of $\circledast$ in (2.7) will be denoted by $\circledast$. For any $\delta_1 \in k$, let $\text{OB}(\delta_1)$ be the category obtained from $\text{OB}$ by adding the additional relation
\[ \circledast = \delta_1. \tag{2.8} \]

\begin{align*}
\begin{array}{c}
\circledast = \downarrow, \\
\circledast = \uparrow, \\
\circledast = \uparrow, \\
\circledast = \downarrow, \\
is the two-sided inverse to $\circledast$. \tag{2.7}
\end{array}
\end{align*} \]
Here, $\downarrow$ is the identity morphism from $\downarrow$ to $\downarrow$, and $\uparrow$ is the identity morphism from $\uparrow$ to $\uparrow$. In the following, the inverse of $\circledast$ in (2.7) will be denoted by $\circledast$. For any $\delta_1 \in k$, let $\text{OB}(\delta_1)$ be the category obtained from $\text{OB}$ by adding the additional relation
\[ \circledast = \delta_1. \tag{2.8} \]

\begin{align*}
\begin{array}{c}
\circledast = \downarrow + \uparrow \uparrow, \\
is the two-sided inverse to $\circledast$. \tag{2.7}
\end{array}
\end{align*} \]
Here, $\downarrow$ is the identity morphism from $\downarrow$ to $\downarrow$, and $\uparrow$ is the identity morphism from $\uparrow$ to $\uparrow$. In the following, the inverse of $\circledast$ in (2.7) will be denoted by $\circledast$. For any $\delta_1 \in k$, let $\text{OB}(\delta_1)$ be the category obtained from $\text{OB}$ by adding the additional relation
\[ \circledast = \delta_1. \tag{2.8} \]

Lemma 2.1. There is a $k$-linear monoidal contravariant functor $\tau : \text{AOB} \to \text{AOB}$, which fixes generating objects and both $\circledast$ and $\uparrow$ and switches $\circledast$ (resp., $\downarrow$, resp., $\circledast$) to $\circledast$ (resp., $\circledast$, resp., $\circledast$). Furthermore, $\tau^2 = \text{Id}.$

Proof. The result follows immediately from the defining relations of $\text{AOB}$ above. \hfill \Box

Lemma 2.2. As morphisms in $\text{AOB}$, we have
\[ \begin{align*}
\begin{array}{c}
\circledast := \circledast, \\
\circledast := \circledast, \\
\downarrow := \downarrow, \\
\circledast := \circledast + \uparrow \uparrow, \\
is the two-sided inverse to $\circledast$. \tag{2.7}
\end{array}
\end{align*} \]
Proof. Easy exercise.\]

2.3. The category \( \mathcal{OB}(u, u') \). Recall \( u, u' \) in (1.11) and define

\[
f(u) = \prod_{i=1}^{\ell} (u - u_i), \quad f'(u) = \prod_{i=1}^{\ell} (u - u'_i).
\]

(2.16)

Thanks to [4] (1.13)], there are scalars \( \delta_i \in k, i \in \mathbb{N} \setminus \{0\} \), such that

\[
1 + \sum_{i \geq 1} \delta_i u_i^{-1} = f'(u)/f(u) \in \mathbb{k}[[u^{-1}]].
\]

(2.17)

In [4], Brundan et. al consider the right tensor ideal \( K \) of \( \mathcal{AOB} \) generated by \( f(\uparrow) \) together with \( \bigcirc i - \delta_i \) for all \( i \in \mathbb{N} \), where \( \uparrow \) is the \( i \)th power of \( \uparrow \). The cyclotomic oriented Brauer category [4] is defined to be the quotient category

\[
\mathcal{OB}(u, u') = \mathcal{AOB}/K,
\]

(2.18)

Remark 2.3. By [4] (1.14)], there are \( \delta_j' \in k, j \in \mathbb{N} \setminus \{0\} \), such that

\[
(1 + \sum_{i \geq 1} \delta_i u_i^{-1})(1 - \sum_{j \geq 1} \delta_j' u_j^{-1}) = 1.
\]

It is proved in [4] Remark 1.6] that the previous \( K \) is also generated by \( f'(\downarrow) \) together with \( \bigcirc i - \delta_i' \) for all \( i \in \mathbb{N} \).

By (2.17), \( \delta_1 = u_1 - u'_1 \) and \( \mathcal{OB}(u, u') = \mathcal{OB}(\delta_1) \) if \( \ell = 1 \). We are going to discuss \( \mathcal{OB}(u, u') \) no matter whether \( \ell = 1 \) or not. In any case, the set of objects in \( \mathcal{OB}(u, u') \) is

\[
J = \langle \uparrow, \downarrow \rangle,
\]

(2.19)

the set of finite sequence of the symbols \( \uparrow, \downarrow \), including the empty word \( \emptyset \). Let

\[
A = \bigoplus_{a,b \in J} \text{Hom}_{\mathcal{OB}(u, u')}(a, b).
\]

(2.20)

Since the contravariant functor \( \tau \) in Lemma 2.1 stabilizes the right tensor ideal \( K \), it induces an anti-involution \( \tau_A \) on \( A \).

For any subspace \( B \subseteq A \) and any \( a, b \in J \), let \( B_{a,b} = 1_a B_{1,b} \), where \( 1_a \) is the identity morphism from \( a \) to \( a \). When \( b = a \), \( B_{a,a} \) is also denoted by \( B_a \). Then \( A_{a,b} = \text{Hom}_{\mathcal{OB}(u, u')}(b, a) \) and

\[
A = \bigoplus_{a,b \in J} A_{a,b}.
\]

(2.21)

So, \( A \) is a \emph{locally unital \( k \)-algebra} and the set \( \{1_a \mid a \in J\} \) serves as the system of mutually orthogonal idempotents of \( A \). In this paper \( C\text{-mod} \) is the category of left \( C \)-modules \( M \) such that \( M = \bigoplus_{a \in H} 1_a M \) for any locally unital algebra \( C = \bigoplus_{a,b \in H} 1_a C_{1,b} \). Let \( C\text{-fdmod} \) be the subcategory of \( C\text{-mod} \) consisting of modules \( M \) such that any \( 1_a M \) is finite dimensional. Let \( C\text{-pmod} \) be the category of finite dimensional (resp., finitely generated projective) left \( C \)-modules.

We are going to recall the notion of \emph{normally ordered oriented Brauer diagrams} in [4]. For any \( a, b \in \mathbb{N} \), an \((a, b)\)-Brauer diagram is a diagram on which \( a + b \) points are placed on two parallel horizontal lines, and \( a \) points on the lower line and \( b \) points on the upper line, and each point joins precisely to one other point. If two points at the upper (resp., lower) line join each other, then this strand is called a cup (resp., cap). Otherwise, it is called a vertical strand.

Any \((a, b)\)-Brauer diagram can also be considered as a partitioning of the set \( \{1, 2, \ldots, a + b\} \) into disjoint union of pairs. If \( a + b \) is odd, then there is no \((a, b)\)-Brauer diagram. Two \((a, b)\)-Brauer
diagrams are said to be equivalent if they give the same partitioning of \( \{1, 2, \ldots, a + b\} \) into disjoint union of pairs.

An oriented Brauer diagram is obtained by adding consistent orientation to each strand in a Brauer diagram as above. Given any oriented Brauer diagram \( d \), let \( a \) (resp., \( b \)) be the element in \( J \) which is indicated from the orientation of the endpoints of the lower line (resp., upper line) and \( d \) is called of type \( a \rightarrow b \). For example, the following diagram is of type \( \uparrow \uparrow \downarrow \downarrow \rightarrow \downarrow \uparrow \):

Two oriented Brauer diagrams of type \( a \rightarrow b \) are said to be equivalent if their underlying Brauer diagrams are equivalent.

A dotted oriented Brauer diagram of type \( a \rightarrow b \) is an oriented Brauer diagram of type \( a \rightarrow b \) such that each segment is decorated with some non-negative number of \( \cdot \)'s (called dots), where a segment means a connected component of the diagram obtained when all crossings are deleted. A normally ordered dotted oriented Brauer diagram of type \( a \rightarrow b \) is a dotted oriented Brauer diagram of type \( a \rightarrow b \) such that:

- whenever a dot appears on a strand, it is on the outward-pointing boundary,
- there are at most \( \ell - 1 \) dots on each strand.

Suppose \( \ell = 4 \). In the following pair of diagrams, the right one is a normally ordered dotted oriented Brauer diagram of type \( \uparrow \uparrow \downarrow \downarrow \rightarrow \downarrow \uparrow \) and the left one is not:

Two normally ordered dotted oriented Brauer diagrams are said to be equivalent if the underlying oriented Brauer diagrams are equivalent and there are the same number of dots on their corresponding strands.

**Theorem 2.4.** [3] Theorem 1.5] Suppose \( a, b \in J \).

1. Two equivalent normally ordered dotted oriented diagrams represent the same morphism in \( \mathcal{O}_B(u, u') \).
2. \( \text{Hom}_{\mathcal{O}_B(u, u')}(a, b) \) has basis given by the set of all equivalence classes of normally ordered dotted oriented Brauer diagrams of type \( a \rightarrow b \).

Thanks to Theorem 2.3, \( A \) is locally finite dimensional in the sense that

\[
\dim \mathbb{A}_a A \mathbb{A}_b < \infty
\]

for all \( a, b \in J \). We are going to explain that \( A \) admits an upper finite weakly triangular decomposition [3, Proposition 2.2].

Suppose \( a = a_1 a_2 \cdots a_h \), where \( a_i \in \{\uparrow, \downarrow\}, 1 \leq i \leq h \). Define

\[
\ell_i(a) = |\{ i : a_i = \downarrow\}|, \quad \ell_1(a) = |\{ i : a_i = \uparrow\}|
\]

where \( |D| \) is the cardinality of a set \( D \). When \( h = 0 \), i.e., \( a \) is the empty word, \( \ell_1(a) = \ell_1(a) = 0 \). For any \( a, b \in J \), write \( a \sim b \) if \( (\ell_1(a), \ell_1(b)) = (\ell_1(b), \ell_1(b)) \). Then \( \sim \) is an equivalence relation on \( J \). Let \( I = J/\sim \). As sets,

\[
I \cong \mathbb{N}^2. \tag{2.22}
\]

**Definition 2.5.** For any \( (r, s), (r_1, s_1) \in I \), define \( (r, s) \preceq (r_1, s_1) \) if \( r = r_1 + k \) and \( s = s_1 + k \) for some \( k \in \mathbb{N} \).

Then \( \preceq \) is a partial order on \( I \). Later on, we also use \( a, b \) etc. to denote elements in \( I \). The partial order \( \preceq \) on \( I \) is upper finite in the sense that \( \{ b \in I \mid a \preceq b \} \) is finite for all \( a \in I \). It induces a partial order on \( J \) such that \( a < b \) if \( a \in a, b \in b \) for \( a, b \in I \) and \( a < b \). If \( a \in I \), define

\[
1_a = \sum_{b \in a} 1_b
\]

and \( B_{a, b} = \bigoplus_{c \in a, d \in b} 1_c B_1 d \) for any \( a, b \in I \) and any subspace \( B \) of \( A \). Then \( B_{a, b} = 1_a B_1 b \).

**Definition 2.6.** For any \( a, b, c \in J \) and \( b \sim c \), define
Lemma 2.7. Suppose $a, b \in J$. Then $X(b, a) \neq \emptyset$ if and only if $Y(a, b) \neq \emptyset$, and $X(b, a) = Y(a, b) = \emptyset$ unless $a \leq b$. Furthermore, 

1. if $a = \uparrow^r \downarrow^s$ for some $r, s$, then $X(b, a) \neq \emptyset$ if and only if $b = \downarrow^m \uparrow^n$ for some $m, n$ and $a \leq b$;
2. if $a = \downarrow^r \uparrow^s$ for some $r, s$, then $X(b, a) \neq \emptyset$ if and only if $b = \uparrow^m \downarrow^n$ for some $m, n$ and $a \leq b$.

Proof. Easy exercise. \hfill \Box

In [14], we define three subspaces $A^+$, $A^-$ and $A^\circ$ (not subalgebras!) of $A$ such that

$$A^\pm = \bigoplus_{b,c,d \in J} A_{b,c,d}^\pm, \quad A^\circ = \bigoplus_{a \in I} \bigoplus_{b,c \in a} A_{b,c}^a,$$

where $A_{b,c}^a$ (resp., $A_{b,c}^+$, resp., $A_{b,c}^\circ$) is the $k$-space with basis $Y(b, c)$ (resp., $X(b, c)$, resp., $H(b, c)$).

Proposition 2.8. [14] Proposition 2.2 The data $(I, A^-, A^\circ, A^+)$ satisfies the following conditions:

1. $(I, \leq)$ is upper finite, where $I$ is given in Definition 2.1.
2. $A_{a,b}^- = 0$ and $A_{a,b}^+ = 0$ unless $a \leq b$. Furthermore, $A_a^- = A_a^+ = \bigoplus_{c \in a} k1_c$.
3. $A^- \otimes_k A^\circ \otimes_k A^+ \cong A$ as $k$-spaces where $k = \oplus_{b \in J} k1_b$. The required isomorphism is given by the multiplication on $A$.

In other words, $(I, A^-, A^\circ, A^+)$ is an upper finite weakly triangular decomposition in the sense of [14] Definition 2.1. When $\ell = 1$, $(I, A^-, A^\circ, A^+)$ is the same as that for the oriented Brauer category $\mathcal{OB}(\delta_1)$ in [20]. In fact, it gives a triangular decomposition in the sense of [10].

2.4. Quotient algebras. For any $a \in I$, define

$$A_{\geq a} = A/A_{\leq a}^\circ, \quad \overline{A}_a = \overline{A}_{\geq a} \overline{A}_{\leq a},$$

where $A_{\circ}$ is the two-sided ideal of $A$ generated by $I_{\circ} = \{1_b \mid b \circ a\}$, where $\circ \in \{\uparrow, \downarrow, \neq\}$. For any $x \in A$, let $\overline{x}$ be its image in any quotient algebra of $A$.

Lemma 2.9. [14] Lemma 2.6, Proposition 2.10 Suppose $e, c \in J$ and $a \in I$.

1. $A_{e,c}$ has basis $\{yhx \mid (y, h, x) \in \bigcup_{d \geq c, b \leq c, b \sim d} Y(e, b) \times H(b, d) \times X(d, c)\}$.
2. $\overline{A}_{e,c} \overline{A}_{a} \overline{A}_{c} \overline{A}_{a} \overline{A}_{e} \overline{A}_{c} \overline{A}_{a}$ has basis $\{\overline{yhx} \mid (y, h, x) \in \bigcup_{b, d \in b, b \leq a} Y(e, b) \times H(b, d) \times X(d, c)\}$.

Suppose $a, b \in J$ and $a \sim b$. Following [20], let $a_{\sigma b}$ be the unique element in $H(a, b)$ on which there are neither crossings among strands of the same orientation (up or down) nor dots on each strand. For example,

$$\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow = \begin{array}{c}
\boxtimes \\
\times \\
\times \\
\boxtimes
\end{array}.$$

The following result follows immediately from the definition of $a_{\sigma b}$.

Lemma 2.10. Suppose $a, b, c \in J$ such that $a \sim b \sim c$. Then $a_{\sigma b} \circ b_{\sigma c} = a_{\sigma c}$ and $a_{\sigma a} = 1_a$.

For any $a, b \in a$, let $\overline{A}_{a,b} = \overline{A}_{a} \overline{A}_{b}$. Then $\overline{A}_{a,b}$, which will be denoted by $\overline{A}_a$, is a subalgebra of $\overline{A}_a$. For any $a, b, c, d \in a$, thanks to Lemma 2.10 there is a $k$-linear isomorphism

$$\overline{A}_{a,b} \sim \overline{A}_{a,d}, \quad g \mapsto (\overline{c}_{\sigma a}) \overline{g} (\overline{d}_{\sigma a}), \quad \forall g \in H(a, b).$$

When $a = b$ and $c = d$, the isomorphism in (2.23) is an algebra isomorphism.

Lemma 2.11. Let $\overline{A} := \bigoplus_{a \in I} \overline{A}_a$.

1. If $a = (r, s) \in I$, then there is an algebra isomorphism $\phi : \text{Mat}_{r \times s}(\overline{A}_{r \times s}) \cong \overline{A}_a$, where the rows and columns of matrices are indexed by $b \in a$. 


Lemma 2.12. \( \mathbb{A}^\circ \)-fdmod \( \cong \bigoplus_{r,s \in \mathbb{N}} \mathcal{A}_{r \vdash s} \)\-fdmod.

Proof. Suppose \( a = (r, s) \). Thanks to (2.23), the required algebra isomorphism \( \phi \) in (1) satisfies

\[
\phi \left( \sum_{a,b \in \mathbb{A}} \tau_{a,b} e_{a,b} \right) = \sum_{a,b \in \mathbb{A}} (a \sigma_{a \rightarrow b}) \tau_{a,b} (b \rightarrow \mathbb{A}) \forall \tau_{a,b} \in \mathcal{A}_{r \vdash s},
\]

where \( e_{a,b} \)'s are the corresponding matrix units. Now, (2) immediately follows from (1). The required functor \( \alpha : \mathcal{A}^\circ \)-fdmod \( \rightarrow \bigoplus_{r,s \in \mathbb{N}} \mathcal{A}_{r \vdash s} \)-fdmod and its inverse \( \beta \) are

\[
\alpha = \bigoplus_{r,s \in \mathbb{N}} \mathbb{T}_{r \vdash s}, \quad \beta = \bigoplus_{r,s \in \mathbb{N}} (\mathcal{A}_{r \vdash s} \otimes \mathcal{A}_{s \vdash r})
\]

(2.24)

For any endofunctor \( \mathcal{F} \) of \( \mathcal{A}^\circ \)-fdmod and any endofunctor \( \mathcal{G} \) of \( \bigoplus_{r,s \in \mathbb{N}} \mathcal{A}_{r \vdash s} \)-fdmod, set

\[
\mathcal{F} \sim \mathcal{G}
\]

(2.25)

if there is a natural isomorphism between \( \mathcal{F} \) and \( \beta \mathcal{G} \alpha \) where \( \alpha \) and \( \beta \) are given (2.24). Obviously, \( \mathcal{F} \) is exact if and only if \( \mathcal{G} \) is exact.

2.5. Degenerate cyclotomic Hecke algebras. Given \( e = (e_1, \ldots, e_\ell) \), the degenerate cyclotomic Hecke algebra \( H_{\ell,n}(e) \) is the associative \( \mathbb{k} \)-algebra generated by \( L_1, S_1, \ldots, S_{n-1} \) subject to the relations:

\[
\begin{cases}
S_i^2 = 1, & 1 \leq i \leq n - 1, \\
S_i S_j = S_j S_i, & 1 \leq i < j - 1 \leq n - 2, \\
S_i S_{i+1} S_i = S_i S_{i+1} S_i, & 1 \leq i \leq n - 2, \\
L_1 S_i = S_i L_1, & 2 \leq i \leq n - 1, \\
(S_1 L_1 S_1 + S_1) L_1 = L_1 (S_1 L_1 S_1 + S_1), \\
(L_1 - e_1) (L_1 - e_2) \cdots (L_1 - e_\ell) = 0.
\end{cases}
\]

Let \( L_i = S_{i-1} L_{i-1} S_{i-1} + S_{i-1} \) for \( 2 \leq i \leq n \). Then \( \{ L_i \mid 1 \leq i \leq n \} \) generates a commutative subalgebra of \( H_{\ell,n}(e) \). Thanks to [1 Lemma 6.6], all generalized eigenvalues of \( L_i, 1 \leq i \leq n \), are of forms \( e_j + k, 1 \leq j \leq \ell \) and \( 1 - n \leq k \leq n - 1 \).

Lemma 2.12. For any \( r, s \in \mathbb{N} \), \( \mathcal{A}_{r \vdash s} \) \( \cong H_{\ell,r}(u) \otimes H_{\ell,s}(-u') \) and \( \mathcal{A}_{s \vdash r} \cong H_{\ell,s}(-u') \otimes H_{\ell,r}(u) \).

Proof. Define the \( \mathbb{k} \)-algebra homomorphism \( \phi : H_{\ell,r}(u) \otimes H_{\ell,s}(-u') \rightarrow \mathcal{A}_{r \vdash s} \) such that

\[
1 \otimes g \mapsto 1 \otimes g^t, \quad h \otimes 1 \mapsto h^t \otimes 1,
\]

for any generators \( 1 \otimes g \) and \( h \otimes 1 \) of \( H_{\ell,r}(u) \otimes H_{\ell,s}(-u') \), where

\[
L_1^r \otimes 1 := \begin{array}{c}
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot
\end{array},
\]

\[
S_1^r \otimes 1 := \begin{array}{c}
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot
\end{array},
\]

\[
1 \otimes S_1^r := \begin{array}{c}
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot
\end{array},
\]

In order to verify that \( \phi \) is well-defined, it suffices to verify that the images of generators of \( H_{\ell,r}(u) \otimes H_{\ell,s}(-u') \) above satisfy the defining relations for \( H_{\ell,r}(u) \otimes H_{\ell,s}(-u') \). This can be verified directly by using (2.9)-(2.13), (2.9) and (2.12)-(2.14) together with the fact that dots can be slide freely in any dotted oriented Brauer diagram if we consider it as element in \( \mathcal{A}_{r \vdash s} \) (see Lemma 2.2). We give an example and leave other details to the reader. By Lemma 2.2 (1)-(2),

\[
1 \otimes L_1^r = \begin{array}{c}
\cdot \end{array}
\]
So, by Remark 2.3, \( f'(1 \otimes L_1^s) = 0 \), where \( f'(u) = f'(-u) \), and \( f'(u) \) is given in (2.16).

Thanks to Lemma 2.9, \( \overline{P}(\{ \lambda, \mu \}) = \{ g \otimes h | (g, h) \in \overline{P}(\lambda) \times \overline{P}(\mu) \} \) is a basis of \( \overline{A}_{\lambda, \mu} \).

Since \( \phi \) sends the well-known basis of \( H_{\ell,r}(u) \otimes H_{\ell,s}(-u') \) to \( \overline{P}(\{ \lambda, \mu \}) \), it is an isomorphism.

The last isomorphism can be proved similarly. In fact, the required \( k \)-algebra isomorphism is \( \phi' : H_{\ell,s}(-u') \otimes H_{\ell,r}(u) \to \overline{A}_{\lambda, \mu} \) such that

\[
1 \otimes g \mapsto 1 \otimes g^s, \quad h \otimes 1 \mapsto h^r \otimes 1,
\]

for any generators \( 1 \otimes g \) and \( h \otimes 1 \) of \( H_{\ell,s}(-u') \otimes H_{\ell,r}(u) \), where

\[
1 \otimes L_1^s := \begin{bmatrix} \ddots & 1 & \\ \vdots & 1 & \\ 1 & 1 & 1 \end{bmatrix}, \quad L_1^s \otimes 1 := \begin{bmatrix} r + s \\ 1 \end{bmatrix}, \quad 1 \otimes S_1^s := \begin{bmatrix} \ddots & 1 & \\ \vdots & 1 & \\ 1 & 1 & 1 \end{bmatrix}, \quad S_1^s \otimes 1 := \begin{bmatrix} r + s \\ 1 \end{bmatrix},
\]

(2.26)

The algebra \( H_{\ell,n}(e) \) is a cellular algebra in the sense of [15] with certain cellular basis given in [1] Theorem 6.3. The corresponding cell modules are denoted by \( S(\lambda) \), \( \lambda \in \Lambda_{\ell,n} \), where \( \Lambda_{\ell,n} \) is the set of all \( \ell \)-partitions \((\lambda^{(1)}, \ldots, \lambda^{(\ell)})\) of \( n \). When all \( e_i \)'s are in the same \( Z \)-orb in the sense that \( e_i - e_j \in ZI_k \) for all \( 1 \leq i < j \leq \ell \), the complete set of pairwise inequivalent irreducible modules are given by

\[
\{ D(\lambda) \mid \lambda \in \Lambda_{\ell,n} \},
\]

where \( \Lambda_{\ell,n} \) is the set of \( e \)-restricted \( \ell \)-partitions in the sense of [17] (3.14). Moreover, \( D(\lambda) \) appears as the simple head of \( S(\lambda) \) for all \( \lambda \in \Lambda_{\ell,n} \) (e.g., [17]). If \( e \) is a disjoint union of certain orbits, then the above result on the classification of simple modules is still available (see [14] Remark 6.2).

2.6. Irreducible \( \overline{A}_2 \)-modules. Suppose \( a = (r, s) \in I \), where \( r, s \in \mathbb{N} \). Recall \( u, u' \) in (1.1). Let \( \Lambda_{\ell,s}^r \) (resp., \( \Lambda_{\ell,s}^l \), resp., \( \Lambda_{\ell,s}^e \), resp., \( \Lambda_{\ell,s}^j \), resp., \( \Lambda_{\ell,s}^r \) be the set of \( \ell \)-partitions of \( s \) (resp., \( \ell \)-partitions of \( r \), resp., \( u \)-restricted \( \ell \)-partitions of \( s \), resp., \( u' \)-restricted \( \ell \)-partitions of \( s \)). Define

\[
\Lambda_a = \Lambda_{\ell,s}^r \times \Lambda_{\ell,s}^l, \quad \overline{A}_a = \overline{A}_{\ell,s}^r \times \overline{A}_{\ell,s}^l.
\]

(2.27)

Thanks to Lemma 2.12, \( \overline{A}_{\lambda, \mu} \) is a cellular algebra with a cellular basis given by those of \( H_{\ell,r}(u) \otimes H_{\ell,s}(u) \). In this case, the cell modules can be considered as \( S(\lambda^\ell) \otimes S(\lambda^l) \)'s, where \( \lambda = (\lambda^\ell, \lambda^l) \in \Lambda_a \). Furthermore, \( \{ D(\lambda^\ell) \otimes D(\lambda^l) \mid \lambda \in \overline{A}_a \} \) gives a complete set of pairwise inequivalent irreducible modules. As proved in [14] § 6.2, each indecomposable projective module of degenerate cyclotomic Hecke algebras is also the injective hull of the same irreducible module. Let \( P(\lambda^\ell) \otimes P(\lambda^l) \) be the projective cover (injective hull) of \( D(\lambda^\ell) \otimes D(\lambda^l) \). For any \( N \in \overline{A}_{\lambda, \mu} \)-fmod, \( \beta(N) \) is an \( \overline{A}_a \)-module where \( \beta \) is the functor given in (2.24).

For any \( \lambda = (\lambda^\ell, \lambda^l) \in \Lambda_a \) and \( \mu = (\mu^\ell, \mu^l) \in \overline{A}_a \), define

\[
S(\lambda) = \beta(S(\lambda^\ell) \otimes S(\lambda^l)), \quad D(\mu) = \beta(D(\mu^\ell) \otimes D(\mu^l)), \quad P(\mu) = \beta(P(\mu^\ell) \otimes P(\mu^l)).
\]

Then \( P(\mu) \) is the projective cover (and injective hull) of the irreducible \( \overline{A}_a \)-module \( D(\mu) \).

Recall the anti-involution \( \tau_a \) in subsection 2.3. Mimicking arguments in [13][20], we see that there is an exact contravariant duality functor \( \otimes \) on \( A \)-fmod (resp., \( \overline{A}_a \)-fmod) such that for any \( V \in A \)-fmod and \( W \in \overline{A}_a \)-fmod,

\[
V^\oplus = \bigoplus_{\alpha \in J} \text{Hom}_k(V, k), \quad W^\oplus = \text{Hom}_k(W, k).
\]

(2.28)

Thanks to Lemmas 2.11, 2.12, it is not difficult to verify that \( \overline{A}_a \) is a cellular algebra with a suitable cellular basis such that \( S(\lambda)^\ell \)'s are the corresponding cell modules. By [18] Chapter 2, Exercise 7,

\[
D(\lambda)^\oplus \cong D(\lambda)
\]

(2.29)
for all $\lambda \in \overline{A}_s$. Since $P(\lambda)$ is the projective cover and injective hull of $D(\lambda)$,

$$P(\lambda)^\circ \simeq P(\lambda),$$

(2.30)

2.7. Induction and restriction functors. Suppose $a = 4^{r+s}_t$, $b = 4^{r+1+s}_t$ and $c = 4^{r+s+1}_t$, where $r, s \in \mathbb{N}$. For $2 \leq i \leq r$ and $2 \leq j \leq s$, define

$$L^i_j \otimes 1 = (S^i_{j-1} \otimes 1)(L^i_{j-1} \otimes 1)(S^i_{j-1} \otimes 1) + S^i_{j-1} \otimes 1,$$

$$1 \otimes L^i_j = (1 \otimes S^i_{j-1})(1 \otimes L^i_{j-1})(1 \otimes S^i_{j-1}) + 1 \otimes S^i_{j-1},$$

where $S^i_{j-1} \otimes 1, L^i_j \otimes 1, 1 \otimes S^i_{j-1}$ and $1 \otimes L^i_j$ are given in (2.29). Then $\{L^i_j \otimes 1, 1 \otimes L^i_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ generates a commutative subalgebra of $\overline{A}_{r+s}$. By Lemma 2.12 and the well-known results on bases of cyclotomic Hecke algebras (e.g., [11, Theorem 6.1]), we have four exact functors:

$$\text{res}_{r,s}^{r+s} := \overline{A}_b \otimes \overline{\pi}_b ??, \quad \text{ind}_{r,s}^{r+s} := \overline{A}_b \otimes \overline{\pi}_b ???,$$

$$\text{res}_{r,s}^{r,s+1} := \overline{A}_c \otimes \overline{\pi}_c ?, \quad \text{ind}_{r,s}^{r,s+1} := \overline{A}_c \otimes \overline{\pi}_c ?.$$  

Suppose $\phi : M \to M$ is a $k$-linear map. For any $i \in k$, the $i$-generalized eigenspace of $\phi$ is

$$M_i = \{m \in M \mid (\phi - i)^n m = 0, \forall n \gg 0\}.$$  

Then

$$\text{res}_{r,s}^{r,s+1} = \bigoplus_{i \in k} i \text{-res}_{r,s}^{r,s+1}, \quad \text{ind}_{r,s}^{r,s+1} = \bigoplus_{i \in k} i \text{-ind}_{r,s}^{r,s+1},$$

$$\text{res}_{r,s}^{r,s+1} = \bigoplus_{i \in k} i \text{-res}_{r,s}^{r,s+1}, \quad \text{ind}_{r,s}^{r,s+1} = \bigoplus_{i \in k} i \text{-ind}_{r,s}^{r,s+1},$$

(2.32)

where

$$i \text{-res}_{r,s}^{r,s+1}(M) \quad \text{and} \quad i \text{-ind}_{r,s}^{r,s+1}(N)$$

are the generalized $i$-eigenspaces of $-L^i_{r+s} \otimes 1$ on $M$ and $N$,

$$i \text{-res}_{r,s}^{r,s+1}(M') \quad \text{and} \quad i \text{-ind}_{r,s}^{r,s+1}(N)$$

are the generalized $i$-eigenspaces of $1 \otimes L^i_{r+s+1}$ on $M'$ and $N$,

for any $(M, M', N) \in \overline{A}_b$-fdmod $\otimes \overline{A}_c$-fdmod $\otimes \overline{A}_a$-fdmod.

2.8. Irreducible $A$-modules. Following [14] (3.1)-(3.2)], there are exact functors

$$\Delta = \bigoplus_{a \in I} j^a, \quad \nabla = \bigoplus_{a \in I} j^a_s,$$

(2.33)

from $\bigoplus_{a \in I} \overline{A}_a$-fdmod to $A$-fdmod where $j^a := A_{s+a} \otimes \overline{\pi}_b ?$ and $j^s_a := \bigoplus_{b \in I} \text{Hom}_{\overline{A}_b}(\overline{A}_b A_{s+a} \overline{A}_b, ?)$. Let

$$\Lambda = \bigcup_{b \in I} A_b, \quad \overline{\Lambda} = \bigcup_{b \in I} \overline{A}_b.$$  

(2.34)

Definition 2.13. For any $\lambda \in \Lambda$ and $\mu \in \overline{\Lambda}$, let $\Delta(\lambda) = \Delta(S(\lambda)), \Delta(\mu) = \Delta(P(\mu)), \overline{\Delta}(\mu) = \Delta(D(\mu)), \nabla(\mu) = \nabla(P(\mu))$ and $\overline{\nabla}(\mu) = \nabla(D(\mu))$.

Following [18], $\Delta(\mu), \overline{\Delta}(\mu), \nabla(\mu)$ and $\overline{\nabla}(\mu)$ are called the standard, proper standard, costandard and proper costandard modules, respectively.

Corollary 2.14. Suppose $a = (r, s)$ and $\lambda \in \overline{A}_a$.

(1) $\Delta(\lambda)$ has a unique irreducible quotient $L(\lambda)$ such that $\overline{A}_a L(\lambda) = D(\lambda)$ as $\overline{A}_a$-modules.

(2) $\{L(\mu) \mid \mu \in \overline{\Lambda}\}$ is a complete set of pairwise inequivalent irreducible $A$-modules.

Proof. This result is a special case of [14] Theorem 3.4(2)-(3)] which is available for any locally unital k-algebra admitting an upper finite weakly triangular decomposition. Now, Proposition 2.8 says that $A$ admits such a decomposition and hence (1)-(2) follow from Lemmas 2.11 and 2.12. □

2.9. Stratified categories. A left $A$-module $V$ has a finite $\Delta$-flag if it has a finite filtration such that its sections are isomorphic to $\Delta(\lambda)$ for various $\lambda \in \overline{\Lambda}$. Let

$$\rho : \overline{\Lambda} \to I$$

(2.35)

such that $\rho(\lambda) = a$ for any $\lambda \in \overline{A}_a$, where $I$ is given in Definition 2.3. Following [10], define

$$\Delta_\varepsilon(\lambda) = \left\{ \begin{array}{ll} \Delta(\lambda), & \text{if } \varepsilon(\rho(\lambda)) = +, \\ \overline{\Delta}(\lambda), & \text{if } \varepsilon(\rho(\lambda)) = -, \end{array} \right.$$  

(2.36)

for any given sign function $\varepsilon : I \to \{\pm\}$. Similarly we have the notion of finite $\Delta_\varepsilon$-flag.
Theorem 2.15. [14] Theorem 3.7. The $A\text{-lfdmod}$ is an upper finite fully stratified category in the sense of [10] Definition 3.36 with respect to the stratification $\rho$ in (2.25).

In other words, for each $\lambda \in \mathfrak{T}$, there exists a projective object $P_{\lambda}$ admitting a finite $\Delta_{\omega}(\lambda)$ at the top and other sections $\Delta_{\omega}(\mu)$ for $\mu \in \mathfrak{T}$ with $\rho(\mu) \geq \rho(\lambda)$. Let $A\text{-mod}^\Delta$ be the category of all left $A$-modules with a finite $\Delta$-flag. Since $\Delta(\lambda) \in A\text{-lfdmod}$ for any $\lambda \in \mathfrak{T}$, $A\text{-mod}^\Delta$ is a subcategory of $A\text{-lfdmod}$. For any $V \in A\text{-mod}^\Delta$, let $(V : \Delta(\lambda))$ be the multiplicity of $\Delta(\lambda)$ in a $\Delta$-flag of $V$.

Corollary 2.16. For any $\lambda \in \mathfrak{T}$, let $P(\lambda)$ be the projective cover of $L(\lambda)$.
(1) $P(\lambda) \in A\text{-mod}^\Delta$, and $(P(\lambda) : \Delta(\mu)) = \frac{1}{L(\lambda)}$, which is non-zero for $\mu \neq \lambda$ only if $\mu \in \bigcup_{a \in A} A_{b}$. In particular, $(P(\lambda) : \Delta(\lambda)) = 1$.
(2) $P(\lambda)$ has a finite $\Delta$-flag. If $\Delta(\mu)$ appears as a section, then $\mu \in \bigcup_{b \in A} A_{b}$. Furthermore, the multiplicity of $\Delta(\mu)$ in this flag is $\Delta(\mu) : L(\lambda)]$.

Proof. Thanks to (2.29) and (2.30) (i.e. [14] Assumption 3.12 holds for $A$), (1) is a special case of [14] Proposition 3.9(2), Lemma 3.13(2) and (2) is a special case of [14] Corollary 4.4(2).

3. Endofunctors and Categorical Actions

Motivated by [14][20], we study certain endofunctors so as to give a categorical action on $A\text{-lfdmod}$.

3.1. Endofunctors. For any $\phi \in \{\uparrow, \downarrow\}$, define $A_{\phi} = \bigoplus_{a,b \in J} 1_{a}A_{b}$ and $\phi A = \bigoplus_{a,b \in J} 1_{a}(\phi A)1_{b}$, where

$$1_{a}(\phi A)1_{b} = (1_{a})\phi A_{1}, \quad 1_{a}A_{b}1_{b} = 1_{a}A_{1}1_{b}. \tag{3.1}$$

Then both $\phi A$ and $A_{\phi}$ are $(A, A)$-bimodules such that the right (resp., left) action of $A$ on $\phi A$ (resp., $A_{\phi}$) is given by the usual multiplication, whereas the left (resp., right) action of $A$ on $\phi A$ (resp., $A_{\phi}$) is given as follows:

$$a \cdot m = \begin{pmatrix} a & m \end{pmatrix}, \quad g \cdot a = \begin{pmatrix} g & a \end{pmatrix}, \quad \text{for all} (m, g, a) \in 1_{b}A \times 1_{c}A_{1} \times 1_{c}A_{1}; \tag{3.2}$$

$$a \cdot m = \begin{pmatrix} a & m \end{pmatrix}, \quad g \cdot a = \begin{pmatrix} g & a \end{pmatrix}, \quad \text{for all} (m, g, a) \in 1_{b}A \times 1_{c}A_{1} \times 1_{c}A_{1}.$$ Similarly, we have $(A', \mathfrak{T}')$-bimodules $A'_{\phi} = \bigoplus_{a,b \in J} 1_{a}A'_{b}$ and $\phi A' = \bigoplus_{a,b \in J} 1_{a}(\phi A')1_{b}$ such that

$$1_{a}(\phi A')1_{b} = (1_{a})\phi A'_{1}, \quad 1_{a}A'_{b}1_{b} = 1_{a}A'. \tag{3.3}$$

where $\mathfrak{T}'$ is given in Lemma 2.11.

Proposition 3.1. As $(A, A)$-bimodules, $A\uparrow \cong \downarrow A$ and $A\downarrow \cong A\downarrow$.

Proof. Thanks to Lemma 2.9(1) and (3.3), there are four $k$-linear maps $\phi : \uparrow A \to A_{\downarrow}$, $\psi : A_{\downarrow} \to \uparrow A$, $\phi' : A_{\downarrow} \to \uparrow A$, and $\psi' : \uparrow A \to A_{\downarrow}$ such that

$$\phi(m) = \begin{pmatrix} m \end{pmatrix}, \quad \psi(g) = \begin{pmatrix} g \end{pmatrix}, \quad \phi'(m') = \begin{pmatrix} m' \end{pmatrix}, \quad \psi'(g') = \begin{pmatrix} g' \end{pmatrix}, \quad \text{for all basis elements} m, g, m', g' \text{ of } \uparrow A, A_{\downarrow}, A_{\downarrow}, \text{ and } A\downarrow \text{ given in Lemma 2.9(1), respectively.}$$

By (2.10)- (2.11) (resp., (2.23)- (2.24)), $\phi^{-1} = \psi$ (resp., $\phi'^{-1} = \psi'$). Finally, it is easy to verify that both $\phi$ and $\phi'$ are $(A, A)$-homomorphisms.

Thanks to Proposition 3.1, $A\downarrow \otimes A \cong A_{\downarrow} \otimes A_{\uparrow}$ and $A\downarrow \otimes A \cong A_{\downarrow} \otimes A_{\uparrow}$ as functors. Define

$$E = \uparrow A \otimes A_{\downarrow}, \quad F = \downarrow A \otimes A_{\uparrow}. \tag{3.4}$$

Definition 3.2. Suppose that $E$ and $F$ are two functors in (3.3). Define four natural transformations

$$\eta : Id_{A\text{-mod}} \to FE, \quad \eta' : Id_{A\text{-mod}} \to EF, \quad \varepsilon : EF \to Id_{A\text{-mod}}, \quad \varepsilon' : FE \to Id_{A\text{-mod}}$$
such that
For any 

Proof.

Lemma 3.3.

For any 

(1) \( \eta \) and \( \eta' \) are induced by the \((A, A)\)-homomorphisms \( \alpha : A \to \downarrow A \otimes_A (\uparrow A) \) and

\[
\alpha(f) = f \downarrow \otimes 1_a \uplus \alpha'(f) = f \uparrow \otimes 1_a \uplus, \quad \forall f \in 1_aA \text{ and } a \in J.
\]

Lemma 3.4.

For any 

Proof. Using (2.10)-(2.11) (resp., (2.3)-(2.4)) \((E \varepsilon \circ \eta F = \Id_E, E \varepsilon' \circ \eta' F = \Id_E \) and \(E \varepsilon' \circ \eta' F = \Id_E\). So, \(E\) and \(F\) are biadjoint to each other.

Lemma 3.5.

For any 

Lemma 3.6.

For any 

Proof. Recall \( \varepsilon, \varepsilon', \eta \) and \( \eta' \) in Definition 3.2. Suppose \( M \) and \( N \) are two left \( A\)-modules. Thanks to the proof of Lemma 3.3, there are \( k\)-linear isomorphisms

\[
\begin{align*}
\Hom_A(EM, N) &\xrightarrow{\alpha_{M,N}} \Hom_A(M, FN), \\
\Hom_A(FM, N) &\xrightarrow{\beta_{M,N}} \Hom_A(M, EN), \\
\Hom_A(EN, M) &\xrightarrow{\beta^{-1}_{M,N}} \Hom_A(FM, N), \\
\Hom_A(FN, M) &\xrightarrow{\alpha^{-1}_{M,N}} \Hom_A(EM, N),
\end{align*}
\]

(3.8)

such that

\[
\begin{align*}
\alpha_{M,N}(h) &= F(h) \circ \eta_M, \\
\alpha^{-1}_{M,N}(h') &= \varepsilon_N \circ E(h'), \\
\beta_{M,N}(h_1) &= E(h_1) \circ \eta'_M, \\
\beta^{-1}_{M,N}(h'_1) &= \varepsilon'_N \circ F(h'_1),
\end{align*}
\]

for all \((h, h', h_1, h'_1) \in \Hom_A(EM, N) \times \Hom_A(M, FN) \times \Hom_A(FM, N) \times \Hom_A(M, EN)\).

For any \((h, f, m) \in \Hom_A(E, M, N) \times 1_aA \times M\), we have (under the isomorphism \(A \otimes_A M \cong M\))

\[
\begin{align*}
\alpha_{M,N}(h)(f \otimes m) &= F(h) \circ \eta_M(f \otimes m) \\
&= F(h)(1_a \downarrow \otimes f \uplus \otimes m) = F(h)(f \downarrow \otimes 1_a \uplus \otimes m).
\end{align*}
\]

(3.9)

Write \( f = \sum_{j \in I} f_j \), where \( f_j \downarrow \in (\downarrow A)_j \). Thanks to Lemma 2.2(7), \( f_j \uplus \in (\uparrow A)_j \). Note that \( h(E, M) = 0 \) if \( i \neq j \). By (3.9),

\[
\alpha_{M,N}(h)(f \otimes m) = f_i \downarrow \otimes h(1_a \uplus \otimes m) \in F_iN.
\]

So, \( \alpha_{M,N}(h) \in \Hom_A(M, F_iN) \).
For any \((h, m) \in \text{Hom}_A(M, F_i \times M)\), there are finite numbers of \(f_k \otimes n_k\)'s \((i_i) \otimes N\) such that
\[
h(m) = \sum_k f_k \otimes n_k. \tag{3.10}
\]
For all \((f, m) \in {}^\uparrow A \times M\), we have (under the isomorphism \(A \otimes_A M \cong M\))
\[
\alpha_{M,N}^{-1}(h)(f \otimes m) = \varepsilon_N E(h)(f \otimes m) = \varepsilon_N(f \otimes \sum_k f_k \otimes n_k) = \sum_k \left(\begin{array}{c}
\sum \[f_k \\
\sum f_k \otimes n_k.
\end{array}\right) \tag{3.11}
\]
We claim \(\alpha_{M,N}^{-1}(h)(f \otimes m) = 0\) if \(f \in ({}^\uparrow A)\), for any \(i \neq j\). If so, \(\alpha_{M,N}^{-1}(h) \in \text{Hom}_A(E_i M, N)\) and the restriction \(\alpha_{M,N}\) in (3.8) to \(\text{Hom}_A(E_i M, N)\) is an isomorphism between \(\text{Hom}_A(E_i M, N)\) and \(\text{Hom}_A(M, F_i N)\). This proves that \((E_i, F_i)\) is an adjoint pair.

In fact,
\[
\langle x^+ - j \rangle t \alpha_{M,N}^{-1}(h)(f \otimes m) = \alpha_{M,N}^{-1}(h)\langle (x^+ - j) f \otimes n \rangle = 0
\]
for some integer \(t \geq 0\) if \(f \in ({}^\uparrow A)\) and \(i \neq j\). Similarly, \(\langle x^+ - i \rangle f_k = 0\) for some integer \(s \geq 0\), where \(f_k\)'s are given (3.10). Since there are only finite number of \(f_k\), we can find an \(s \gg 0\) which is independent of \(f_k\) such that \(\langle x^+ - i \rangle f_k = 0\) for all admissible \(k\). Thanks to (3.11) and Lemma 2.2(8),
\[
\langle x^+ - i \rangle f_k \otimes n_k = 0,
\]
forcing \(\alpha_{M,N}^{-1}(h)(f \otimes m) = 0\). This proves our claim.

Similarly, the restriction of \(\beta_{M,N}\) in (3.8) to \(\text{Hom}_A(F_i M, N)\) induces an isomorphism between \(\text{Hom}_A(F_i M, N)\) and \(\text{Hom}_A(M, E_i N)\). The only difference is that one has to replace Lemma 2.2(7)–2.2(8) by Lemma 2.2(b)–(6). This proves that \((F_i, E_i)\) is an adjoint pair.

Recall \(x^L_i\) and \(x^R_i\) in Lemma 3.3 for any \(\sigma \in \{\uparrow, \downarrow\}\). For any \(i \in k\) let \((\sigma A_i)\) \((\sigma_\circlearrowleft)\) be the generalization \(i\)-eigenspace of \(x^L_i\) \((\text{resp. } x^R_i)\) on \(\sigma A\) \((\text{resp. } \sigma_\circlearrowleft)\). Define
\[
E^\sigma = (\sigma A) \otimes_\sigma ? , \quad F^\sigma = (\sigma_\circlearrowleft) \otimes_\sigma ? . \tag{3.12}
\]

**Lemma 3.7.** \(E^\sigma = \bigoplus_{i \in k} E_i^\sigma\) and \(F^\sigma = \bigoplus_{i \in k} F_i^\sigma\), where \(E_i^\sigma = (\sigma A) \otimes_\sigma ?\) and \(F_i = (\sigma_\circlearrowleft) \otimes_\sigma ?\).

**Proof.** The result follows from the fact that \(x^L_i\) \((\text{resp. } x^R_i)\) preserves the finite dimensional \(k\)-spaces \(\bigcap_{i \in k} (\sigma A)_{i, \uparrow} \bigcap_{i \in k} (\sigma A)_{i, \downarrow}\)'s and \(\bigcap_{i \in k} (\sigma_\circlearrowleft)_{i, \uparrow} \bigcap_{i \in k} (\sigma_\circlearrowleft)_{i, \downarrow}\)'s.

Thanks to Lemma 2.11(2) and (2.26), we have
\[
E^\uparrow \sim \bigoplus_{r,s \in \mathbb{N}} \text{res}_{r,s+1}^r , \quad E^\downarrow \sim \bigoplus_{r,s \in \mathbb{N}} \text{ind}_{r,s+1}^r , \quad F^\uparrow \sim \bigoplus_{r,s \in \mathbb{N}} \text{ind}_{r,s+1}^r , \quad F^\downarrow \sim \bigoplus_{r,s \in \mathbb{N}} \text{res}_{r,s+1}^r . \tag{3.13}
\]

So, both \(E^\sigma\) and \(F^\sigma\) are exact. By Lemma 2.12, 2.26 and (3.32),
\[
E_i^\uparrow \sim \bigoplus_{r,s \in \mathbb{N}} i \text{-res}_{r,s+1}^r , \quad E_i^\downarrow \sim \bigoplus_{r,s \in \mathbb{N}} i \text{-ind}_{r,s+1}^r , \quad F_i^\uparrow \sim \bigoplus_{r,s \in \mathbb{N}} i \text{-ind}_{r,s+1}^r , \quad F_i^\downarrow \sim \bigoplus_{r,s \in \mathbb{N}} i \text{-res}_{r,s+1}^r . \tag{3.14}
\]

**Lemma 3.8.** There are two short exact sequence of functors from \(\sigma A\)-fmod to \(A\)-fmod:
\[
0 \rightarrow \Delta \circ \Delta \rightarrow F \circ \Delta \rightarrow \Delta \circ F \rightarrow 0; \tag{3.15}
\]
\[
0 \rightarrow \Delta \circ E \rightarrow E \circ \Delta \rightarrow \Delta \circ E \rightarrow 0. \tag{3.15}
\]

**Proof.** Suppose \(a = (s, r), \ b = (s - 1, r), \ c = (s, r + 1)\), where \(r, s\) are any admissible non-negative integers. We claim that there are short exact sequences of \((A, \sigma A)\)-bimodules
\[
0 \rightarrow A \otimes_{\sigma A} (\sigma \circlearrowleft) \rightarrow A \otimes \sigma A \rightarrow A \otimes \sigma A \rightarrow 0, \tag{3.16}
\]
\[
0 \rightarrow A \otimes_{\sigma A} (\sigma \circlearrowleft) \rightarrow A \otimes \sigma A \rightarrow A \otimes \sigma A \rightarrow 0,
\]
where the required morphisms \( \phi, \psi, \eta \) and \( \varepsilon \) satisfy:

\[
\phi(f \otimes g) = (f \uparrow) \otimes g, \quad \psi(f_1 \otimes g_1) = \cdots \begin{array}{c} \begin{array}{c} f_1 \end{array} \\ \begin{array}{c} g_1 \end{array} \end{array} \quad \eta(h) = \begin{array}{c} h \end{array} \otimes (k_1 \downarrow),
\]

for any admissible basis diagrams \( f, g, f_1, g_1, h, k, h_1, k_1 \) of \( A \) in Lemma 2.11. If the claim is true, then (3.13) follows immediately.

It is routine to check that \( \phi, \psi, \varepsilon \) and \( \eta \) are well-defined \( (A, \overline{A}) \)-homomorphisms. We are going to prove the exactness of the first sequence in (3.16). One can verify the second one similarly.

Suppose \( d \sim c \in J \) and \( a, b \in \mathfrak{a} \). Thanks to Lemma 2.10 and (2.23), there is a commutative diagram

\[
0 \rightarrow \mathcal{T}_dA_{\langle b \rangle} \otimes \mathcal{T}_a(\overline{A}) \mathcal{T}_a \xrightarrow{\phi} \mathcal{T}_dA_{\langle a \rangle} \otimes \mathcal{T}_a \mathcal{T}_a \rightarrow \mathcal{T}_{d}A_{\langle b \rangle} \otimes \mathcal{T}_a \mathcal{T}_a \rightarrow 0
\]

\[
0 \rightarrow \mathcal{T}_dA_{\langle b \rangle} \otimes \mathcal{T}_a(\overline{A}) \mathcal{T}_b \xrightarrow{\phi, \varepsilon} \mathcal{T}_dA_{\langle c \rangle} \otimes \mathcal{T}_b \mathcal{T}_a \rightarrow \mathcal{T}_dA_{\langle b \rangle} \otimes \mathcal{T}_b \mathcal{T}_a \rightarrow 0
\]

where \( \phi_{a,d} \) and \( \phi_{b,c} \) (resp. \( \psi_{a,d} \) and \( \psi_{b,c} \)) are restrictions of \( \phi \) (resp. \( \psi \)) in (3.10). Here, the \( k \)-linear isomorphisms \( \tau_i \) are defined so that

\[
\tau_2(f \otimes g) := c_{\sigma_d}(f \otimes g)_{\sigma_b}, \quad \tau_i(f \otimes g) := c_{\sigma_d}(f \otimes g)_{\sigma_b}, \quad i \in \{1, 3\},
\]

for all admissible basis diagrams \( f, g \) of \( A \), where \( c_{\sigma_d} \) is given in Lemma 2.10. Therefore, it’s enough to verify the exactness of the first sequence in (3.17) as \( k \)-spaces under the assumption \( a = \uparrow \downarrow^s \) and \( d = \uparrow \downarrow^t \), \( \forall m, n \in \mathbb{N} \). If so, we immediately obtain the first short exact sequence in (3.16).

We define \( B_1 \otimes B_2 = \{ (b_1 \otimes b_2) \mid b_1 \in B_1, b_2 \in B_2 \} \) for any sets \( B_1, B_2 \). For any \( B \subset A \), and any quotient of \( A \), we denote by \( \overline{B} = \{ \overline{t} \mid t \in B \} \) the corresponding subset in the quotient algebra. For convenience, \( H(h, e) \) will be denoted by \( H(e) \) if \( h = e \).

By Lemma 2.14, three \( k \)-spaces in the first sequence of (3.17) are zero if \( \uparrow \downarrow^s \nmid \downarrow^d \) and hence there is nothing to be proved. Suppose \( \uparrow \downarrow^s \nmid \downarrow^d \), and define

\[
H_1(\uparrow \downarrow^s) := \begin{cases} g \in D(\uparrow \downarrow^s) \times H(\uparrow) \times H(\downarrow^s) \end{cases},
\]

where

\[
D(\uparrow \downarrow^s) = \left\{ (g, h_1, h_2) \in D(\uparrow \downarrow^s) \mid 0 \leq j \leq \ell - 1, 1 \leq i \leq r + 1 \right\}
\]

Thanks to Lemma 2.12 and the general result on the basis of degenerate cyclotomic Hecke algebra, it is easy to check that \( H_1(\uparrow \downarrow^s) \) is a basis of \( \overline{A}_{\uparrow \downarrow^s} \). Let

\[
H_1(\uparrow \downarrow^s) := \{ g \in D(\uparrow \downarrow^s) \mid g \in H_1(\uparrow \downarrow^s) \}.
\]

By Lemmas 2.14 and (2.23), we have

\[
\begin{align*}
(a) & \mathcal{T}_dA_{\langle b \rangle} \otimes \mathcal{T}_a(\overline{A}) \mathcal{T}_a \text{ has basis } \mathcal{Y}(\uparrow \downarrow^s) \otimes H(\uparrow \downarrow^s), \\
(b) & \mathcal{T}_dA_{\langle a \rangle} \otimes \mathcal{T}_a \mathcal{T}_a \text{ has basis } \mathcal{Y}(\uparrow \downarrow^s) \otimes H(\uparrow \downarrow^s), \\
(c) & \mathcal{T}_dA_{\langle b \rangle} \otimes \mathcal{T}_a \mathcal{T}_d \mathcal{T}_a \text{ has basis } \mathcal{Y}(\uparrow \downarrow^s) \otimes H(\uparrow \downarrow^s)(\sigma).
\end{align*}
\]

Thanks to \( \mathcal{Y}(\mathcal{Y}(\uparrow \downarrow^s, \uparrow \downarrow^s) = 0 \) and \( \mathcal{Y}(\mathcal{Y}(\uparrow \downarrow^s, \uparrow \downarrow^s) = 0 \) in \( \mathcal{T}_dA_{\langle b \rangle} \) for any \( f \in \mathcal{Y}(\uparrow \downarrow^s) \), we have \( \psi_{a,d}\phi_{a,d} = 0 \) and \( \varepsilon_{a,d}\phi_{a,d} = 0 \). Therefore, \( \psi_{a,d} \) and \( \phi_{a,d} \) are injective. Define

\[
Y_1(\uparrow \downarrow^s) := \begin{cases} (f, g) \in D(\uparrow \downarrow^s) \times H(\uparrow \downarrow^s) \end{cases}.
\]
So \( Y_1((m,n+1),^r\downarrow^s) \otimes \overline{H}((r,\downarrow^s)) \subseteq 1_d (\downarrow A) \otimes A \mathcal{T}_a \mathcal{T}_b \). If \( f \in Y((m,n),^r+1\downarrow^s) \), \( g \in D(\downarrow^r+1) \) and \( h \in H((r,\downarrow^s)) \), then

\[
\psi_{a,d} \begin{pmatrix} g & \cdots & j \otimes h \end{pmatrix} = \psi_{a,d} \begin{pmatrix} g_1 & \cdots & j \otimes h \end{pmatrix} \quad \text{by Lemma 2.2(2),}
\]

\[
= \begin{pmatrix} g_1 & \cdots & j \otimes h \end{pmatrix} \quad \text{by Lemma 2.2(2) and (2.3),}
\]

\[
= \mathcal{T} \otimes \begin{pmatrix} g_1 & \cdots & j \otimes (h_1 \uparrow h_2) \otimes (r+1,\downarrow^s) \end{pmatrix} \in \mathcal{Y}((m,n),^r+1\downarrow^s) \otimes \overline{H}((r,\downarrow^s))(\sigma),
\]

where \( g_1 \) is obtained from \( g \) by removing all dots on \( g \), and \((h_1, h_2) \in H((r,\downarrow^s)) \) such that \( h = h_1 \otimes h_2 \). Thanks to (c), \( \psi_{a,d} \) is surjective. Now, the exactness of the first sequence in \( 3.17 \) follows immediately since

\[
\dim T_d A_{\leq b} \otimes \mathcal{T}_b \mathcal{T}_a = |Y((m,n),^r\downarrow^s)| |H((r,\downarrow^s)|
\]

\[
= (|Y((m,n),^r\downarrow^{r-1})| + |Y_1((m,n),^r\downarrow^s)) |H((r,\downarrow^s))|
\]

\[
= |Y((m,n),^r\downarrow s-1)| |H((r,\downarrow^s))| + |Y((m,n),^r+1\downarrow^s)| |D(\downarrow^r+1)| |H((r,\downarrow^s))|
\]

\[
= \dim T_d (\downarrow A) \otimes A \mathcal{T}_{\leq a} + \dim T_d A_{\leq b} \otimes \mathcal{T}_{b,\downarrow} \mathcal{T}_a.
\]

\( \square \)

**Lemma 3.9.** Suppose \( \varphi, \psi, \eta \) and \( \varepsilon \) are \((A, \overline{\mathcal{T}})\)-homomorphisms in \( 3.10 \). Then

1. \((x^1 \otimes 1) \circ \varphi = \varphi \circ (1 \otimes x_1^1)\),
2. \((1 \otimes x^1_2) \circ \varphi = \psi \circ (x^1 \otimes 1)\),
3. \((x^1 \otimes 1) \circ \eta = \eta \circ (1 \otimes x^1_2)\),
4. \((1 \otimes x^1_2) \circ \varepsilon = \varepsilon \circ (x^1 \otimes 1)\).

**Proof.** Suppose \((\overline{f}, \overline{g}) \in A_{\leq b} \mathcal{T}_a \times \mathcal{T}_b, \mathcal{T}_a\), where \( a = (s, r), b = (s - 1, r) \) and \( s \geq 1 \). Thanks to Lemma 3.1,1,

\[
(x^1 \otimes 1) \circ \varphi(\overline{f} \otimes \overline{g}) = (x^1 \otimes 1)(f \otimes g) = (f \otimes g) \circ (1 \otimes x^1_2),
\]

proving (1). One can check (3) similarly. If \((f, g) \in 1_c \downarrow A \otimes A \mathcal{T}_{\leq a} \mathcal{T}_a \) and \( c \in J \), then

\[
(1 \otimes x^1_2) \circ \psi(f \otimes g) = (1 \otimes x^1_2)(f \otimes g) \quad \text{Lemma 2.2(6)}
\]

\[
\psi(f \otimes g),
\]

proving (2). Replacing Lemma 2.2(6) by Lemma 2.2(8), one can verify (4) by arguments similar to those for (2). \( \square \)

**Theorem 3.10.** For each \( i \in k \), there are two short exact sequences of functors from \( \mathcal{A}^\circ \text{-fmod} \) to \( A^{\text{-fmod}} \):

\[
0 \to \Delta \circ F^i_1 \to F_i \to \Delta \circ F^1_1 \to 0,
\]

\[
0 \to \Delta \circ E^i_1 \to E_i \to \Delta \circ E^1_1 \to 0.
\]

**Proof.** Thanks to Lemma 3.9 the short exact sequences in \( 3.18 \) follow from those in \( 3.15 \) by passing to appropriate generalized eigenspaces. \( \square \)
3.2. Characters. Suppose $a = a_1 \cdots a_{r+s} \in A = (r, s)$. For any $i$, $1 \leq i \leq r + s$, define

$$X_i 1_a = \begin{cases} I_{a_1 \cdots a_{i-1}} & \text{if } a_i = \uparrow, \\ I_{a_1 \cdots a_{i-1}} & \text{if } a_i = \downarrow. \end{cases}$$

Then $\{X_i 1_a \mid 1 \leq i \leq r + s, a \in A\}$ generates a finite-dimensional commutative subalgebra of $A_a$. For any $i = (i_1, i_2, \ldots, i_{r+s}) \in \mathbb{N}^{r+s}$, there is an idempotent $1_{a;i} \in A_a$ which projects any $M \in A_{r+s}$ onto $M_{a;i}$, the simultaneous generalized eigenspace of $X_1 1_a, \ldots, X_{r+s} 1_a$ with respect to $i$. When $r = s = 0$, $A_a \cong k$. In this case, there is a unique idempotent $1_{a;\emptyset}$.

**Definition 3.11.** For any $V \in A_{r+s}$, define

$$
\text{ch} V = \sum_{a \in I, i \in k^{(a)}} (\dim 1_{a;i} V) e^a_i,
$$

(3.19)

where $e^a_i$ are formal symbols, $\ell(a) = \ell_1(a) + \ell_2(a)$, and $1_{a;i} V$ is the simultaneous generalized eigenspace of $X_1 1_a, \ldots, X_{r+s} 1_a$ corresponding to $i$.

Suppose $a = (r, s)$ and $\lambda = (\lambda^1, \lambda^2) \in A_a$, where $A_a$ is given in (2.24). Motivated by Lemma 2.12 and (3.13)-(3.14), We define the content $c(x)$ of a node $x$ with respect to $\lambda$ as follows:

$$c(x) = \begin{cases} c_\lambda(x), & \text{if } x \text{ is in } [\lambda^1], \\ c_\lambda(x), & \text{if } x \text{ is in } [\lambda^2], \end{cases}
$$

(3.20)

where $c_\lambda(x) = u_j + k - l$ (resp., $c_\lambda(x) = u_j' - k + l$) if $x$ is at the $l$th row and $k$th column of the $j$th component of the Young diagram $[\lambda^1]$ (resp., $[\lambda^2]$). Recall the cell modules $S(\lambda)'s$ of $A_a$ in subsection 2.10. The following result follows from Lemma 2.12 (3.14) and the branching rules of the cell modules for degenerate cyclotomic Hecke algebras.

**Lemma 3.12.** Suppose $i \in \mathbb{k}$ and $\lambda = (\lambda^1, \lambda^2) \in A_{(r,s)}$, where $r, s \in \mathbb{N}$. We have

1. $E^i_\lambda S(\lambda)$ (resp., $E^{\uparrow}_\lambda S(\lambda)$) has a multiplicity-free filtration with sections $S(\mu)$, where $\mu \in \Lambda_{(r,s-1)}$ (resp., $\mu \in \Lambda_{(r,s-1)}$) is obtained by removing a box in $[\lambda^1]$ (resp., $[\lambda^2]$) of content $i$.
2. $E^i_\lambda S(\lambda)$ (resp., $E^{\uparrow}_\lambda S(\lambda)$) has a multiplicity-free filtration with sections $S(\mu)$, where $\mu \in \Lambda_{(r+1,s)}$ (resp., $\mu \in \Lambda_{(r+1,s)}$) is obtained by adding a box in $[\lambda^1]$ (resp., $[\lambda^2]$) of content $i$.

Recall $\mathbb{I}$, $\mathbb{I}_u$, and $\mathbb{I}_u'$ in (1.2). By Lemma 3.12, $E^i_\lambda$ and $F^{\uparrow}_\lambda$ (resp., $E^\uparrow_\lambda$ and $F^i_\lambda$) are non-zero only if $i \in \mathbb{I}$ (resp., $i \in \mathbb{I}_u$). Thanks to Theorem 3.10 and Lemma 3.5

$$E = \bigoplus_{i \in \mathbb{I}} E_i, \quad F = \bigoplus_{i \in \mathbb{I}_u} F_i.
$$

Recall that $\Lambda = \bigsqcup_{a \in I} A_a$ in (2.33). Let $\Xi$ be the graph such that the set of vertices is $\Lambda$ and any edge is of form $\lambda - \mu$ whenever $\mu$ is obtained from $\lambda$ by either adding a box or removing a box. A path $\gamma : \lambda \rightsquigarrow \mu$ in $\Xi$ is a finite sequence of vertices $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_m = \mu$ with each $\lambda_j - \lambda_{j+1}$ connected by an edge. We color the edge $\lambda_j - \lambda_{j+1}$ by $c(x)$ if $\lambda_{j+1}$ is obtained from $\lambda_j$ by either adding an addable node $x$ or removing a removable node $x$.

**Definition 3.13.** Suppose $i = (i_1, \ldots, i_m)$. A path $\gamma$ is of type $(a; i)$ if $i_j$ is the color of the $j$th edge of $\gamma$, and if $a = a_1 \cdots a_m \in J$ such that

1. $a_{j+1} = \uparrow$ if $\lambda_j - \lambda_{j+1}$ is obtained from $\lambda_j$ by adding a box to $\lambda_j^1$ or removing a box from $\lambda_j^2$,
2. $a_{j+1} = \downarrow$ if $\lambda_j - \lambda_{j+1}$ is obtained from $\lambda_j$ by adding a box to $\lambda_j^2$ or removing a box from $\lambda_j^1$.

When $\lambda = \mu = (\emptyset, \emptyset)$, we say there is a unique path from $\lambda$ to $\mu$ with type $(\emptyset; \emptyset)$.

**Proposition 3.14.** For any $\lambda \in \Lambda$, $c\tilde{\Delta}(\lambda) = \sum_{\gamma \text{ of type } (a; i)} e^{\text{type}(\gamma)}$ where the summation ranges over all paths $\gamma : (\emptyset, \emptyset) \rightsquigarrow \lambda$ and $\tilde{\Delta}(\lambda)$ is given in Definition 2.13.

**Proof.** Suppose $\lambda \in A_a$ and $a \in I$. Thanks to (2.23), $\tilde{\Delta}(\lambda) = \bigoplus_{a \geq 1} I_a \tilde{\Delta}(\lambda)$ and $I_a \tilde{\Delta}(\lambda) = S(\lambda)$. In order to prove the required formula on $c\Delta(\lambda)$, it suffices to show that

$$\dim 1_{a;i} \tilde{\Delta}(\lambda) = |\{ \gamma : (\emptyset, \emptyset) \rightsquigarrow \lambda \mid \gamma \text{ is of type } (a;i) \}|
$$

(3.22)

for all $i \in \mathbb{N}^n$ and all $a = a_1 \cdots a_m \in J$ and all $m \in \mathbb{N}$. We prove (3.22) by induction on $m \in \mathbb{N}$. 


Suppose that \( m = 0 \). If \( \lambda \neq (0, 0) \), then \( 1_{\phi, \theta} \Delta(\lambda) = 0 \). Otherwise, \( 1_{\phi, \theta} \Delta(\lambda) = k \). So, the result holds for \( m = 0 \). In general, by (3.23),

\[
1_a(EV) = 1_{a\uparrow} A \otimes_A V \cong 1_{a\uparrow} V, \quad 1_a(FV) = 1_{a\downarrow} A \otimes_A V \cong 1_{a\downarrow} V
\]

for any \( V \in A\text{-fmd} \). So,

\[
\dim 1_{a\uparrow} E_i V = \dim 1_{a\uparrow} V, \quad \dim 1_{a\downarrow} F_i V = \dim 1_{a\downarrow} V.
\]

(3.23)

Thanks to (3.19) and Lemma 3.12, \( E_i \Delta(\lambda) \) (resp., \( F_i \Delta(\lambda) \)) has a multiplicity-free \( \Delta \)-filtration such that \( \Delta(\mu) \) appears as a section if and only if \( \mu \) is obtained by either removing a box in \([\lambda^\uparrow]\) (resp., \([\lambda^\downarrow]\)) of content \( i \) or adding a box in \([\lambda^\downarrow]\) (resp., \([\lambda^\uparrow]\)) of content \( i \). Now (3.22) follows from (3.23) and induction on \( m \), immediately.

**Corollary 3.15.** Suppose \((\lambda, \mu) \in \Lambda \times \overset{\circ}{\Lambda}, a \in I \). If \([\Delta(\lambda) : L(\mu)] \neq 0 \), then there are two paths \( \gamma : (0, 0) \rightarrow \lambda \) and \( \delta : (0, 0) \rightarrow \mu \) such that \( \gamma \) and \( \delta \) are of the same type \((a; 1)\) and \( a \in a \).

**Proof.** Mimicking arguments in the proof of [14 Corollary 5.11], one can verify this result by using Corollary 2.14(1) and Proposition 3.14 immediately.

**Theorem 3.16.** Suppose \( \mathcal{I}_a \bigcap \mathcal{I}_{a'} = \emptyset \), where \( \mathcal{I}_a \) and \( \mathcal{I}_{a'} \) are in (1.2). Then

1. \( \Delta : \overset{\circ}{\Lambda} - \text{mod} \rightarrow A\text{-mod} \) is an equivalence of categories.
2. \( \Delta : \overset{\circ}{\Lambda} - \text{mod} \rightarrow A\text{-mod} \) is an equivalence of categories.

**Proof.** Take an arbitrary \( \mu \in \overset{\circ}{\Lambda} \). If \( \mathcal{P}(\mu) \neq \Delta(\mu) \), by Corollary 2.14(1), \( [\Delta(\lambda) : L(\mu)] \neq 0 \) for some \( \lambda \in \overset{\circ}{\Lambda} \) and \( \lambda \neq \mu \). Since \( \Delta \) is exact and \( D(\lambda) \) is the simple head of \( S(\lambda) \), there is an epimorphism from \( \Delta(\lambda) \) to \( S(\lambda) \). So, \([\Delta(\lambda) : L(\mu)] \neq 0 \). By Corollary 3.15 there are two paths \( \gamma : (0, 0) \rightarrow \lambda \) and \( \delta : (0, 0) \rightarrow \mu \) such that \( \gamma \) and \( \delta \) are of the same type \((a; 1)\) and \( a \in a \).

We claim \( a = b \). Otherwise, \( b > a \). By Definition 3.13 there is an edge, say \( \lambda_{j-1} \rightarrow \lambda_j \) such that \( \lambda_j \) is obtained by removing a box \( x \) either in \( \lambda_{j-1} \) with \( c(x) \in \mathcal{I}_a \) or in \( \lambda_{j-1} \) with \( c(x) \in \mathcal{I}_{a'} \). In any case, \( c(x) \in \mathcal{I}_{a} \bigcap \mathcal{I}_{a'} = \emptyset \), a contradiction. By the definition of \( \Delta \) in (2.23), \( 1_{\overset{\circ}{\Lambda}}(\lambda) = D(\lambda) \).

Since \( 1_a L(\mu) = D(\mu) \), it is a composition factor of the simple \( \overset{\circ}{\Lambda}\text{-module} \) \( D(\lambda) \), forcing \( \lambda = \mu \), a contradiction. So, \( \mathcal{P}(\mu) = \Delta(\mu) \) for all \( \mu \in \overset{\circ}{\Lambda} \). Now (2) immediately follows from [6 Corollary 2.5] and (1), since the exact functor \( \Delta \) sends projective \( \overset{\circ}{\Lambda} \text{-modules} \mathcal{P}(\lambda) \)’s to projective \( A\text{-modules} \mathcal{P}(\lambda) \)’s for any \( \lambda \in \overset{\circ}{\Lambda} \).

**Corollary 3.17.** The category \( A\text{-mod} \) is completely reducible if

1. \( u_i - u_j \in \mathbb{Z} I_k \) for all \( 1 \leq i, j \leq \ell \),
2. \( u_i - u_j \notin \mathbb{Z} I_k \), \( u_i' - u_j' \notin \mathbb{Z} I_k \) for all \( 1 \leq i < j \leq \ell \) and \( p = 0 \).

**Proof.** Thanks to (3.19) and Theorem 3.16(2), \( \overset{\circ}{\Lambda} - \text{mod} \) is Morita equivalent to \( A\text{-mod} \). By (2) and [11 Theorem 6.11], both \( H_{\ell, s}(u) \) and \( H_{\ell, s}(-u') \) are semisimple for all \( r, s \in \mathbb{N} \). Now, the result follows immediately from Lemma 2.11(2) and Lemma 2.12.

We expect that Corollary 3.17(1)-(2) are necessary and sufficient conditions for \( A\text{-mod} \) being completely reducible.

3.3. **Categorical actions.** Let \( g \) be the complex Kac-Moody Lie algebra \( g \) associated to Cartan matrix \((a_{ij})_{i,j\in I}\) defined by (1.3). Then \( g \) is the Lie algebra generated by its Cartan subalgebra and Chevalley generators \( \{ e_i, f_i \mid i \in I \} \) subject to the usual Serre relations. Furthermore, \( g \) is isomorphic to a direct sum of certain \( sl_{\infty} \) (resp., \( sl_{p} \)) if \( p = 0 \) (resp., \( p > 0 \)) depending on both \( u \) and \( u' \). Let

\[
\Pi = \{ \alpha_i \mid i \in I \},
\]

(3.24)

the set of simple roots. The weight lattice is

\[
P := \{ \lambda \in h^* \mid \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for all } i \in I \},
\]

(3.25)

where \( h_i := [e_i, f_i] \). Let

\[
P^+ := \{ \lambda \in h^* \mid \langle h_i, \lambda \rangle \in \mathbb{N} \text{ for all } i \in I \},
\]

(3.26)

and \( \{ \omega_i \mid i \in I \} \) be the set of fundamental weights of \( g \). There is a usual dominance order on \( P \) in the sense that \( \lambda \leq \mu \) if \( \mu - \lambda \in \sum_{i\in I} N \alpha_i \). The partial order on \( P \) induces a partial order on \( P \times P \) such that

\[
(\lambda_1, \lambda_2) \geq (\mu_1, \mu_2) \quad \text{if} \quad \lambda_1 + \lambda_2 = \mu_1 + \mu_2 \text{ and } \lambda_1 \leq \mu_1.
\]

(3.27)
Define

\[ wt : \Lambda \to P \times P, \quad \lambda \to (wt_\|_1(\lambda), wt_\|_2(\lambda)) \]  
(3.28)

where \( wt_\|_1(\lambda) = -\omega_{u'} + \sum_{y \in [\lambda]} a_c(y), \) \( wt_\|_2(\lambda) = \omega_u - \sum_{x \in [\lambda]} a_c(x), \) and \( \omega_u, \omega_{u'} \) are given in (1.4).

**Proposition 3.18.** Suppose \( \lambda \in \Lambda \) and \( \mu \in \overline{\Lambda}. \) If \( [\overline{\Delta}(\lambda) : L(\mu)] \neq 0, \) then \( wt(\mu) \leq wt(\lambda). \)

**Proof.** Suppose \( \mu \in \overline{\Lambda}(r,s). \) Thanks to Corollary 3.15, there are two paths \( \gamma : (0,0) \rightsquigarrow \lambda \) and \( \delta : (0,0) \rightsquigarrow \mu \) such that \( \gamma \) and \( \delta \) are of the same type \((a;i), \) where \( a = a_1 \cdots a_{r+s} \) and \( i = i_1, \ldots, i_{r+s}. \)

We are going to prove by induction on \( r+s. \)

If \( r+s = 0, \) then \( \lambda = \mu = (0,0) \) and there is nothing to prove. Otherwise, \( r+s > 0. \) Removing the last edge in both \( \gamma \) and \( \delta \) yields two shorter paths \( \gamma' : (0,0) \rightsquigarrow \lambda' \) and \( \delta' : (0,0) \rightsquigarrow \mu' \) such that \( \gamma' \) and \( \delta' \) are of the same type \((b;j), \) where \( b = a_1 \cdots a_{r+s-1} \) and \( j = (i_1, \ldots, i_{r+s-1}). \) We deal with the case \( a_{r+s} = 1 \) and leave the details on \( a_{r+s} = 0 \) to the reader since the proof is similar.

There are two cases we need to consider. If \( \lambda \) is obtained from \( \lambda' \) by adding a box \( x \) in \( [\lambda'] \), then \( wt_1(\lambda') = wt_1(\lambda) + a_c(x) \) and \( wt_2(\lambda') = wt_2(\lambda). \) Since \( \mu \in \overline{\Lambda}(r,s), \)

\[ wt_1(\mu') = wt_1(\mu) + a_c(x), \quad wt_2(\mu') = wt_2(\mu). \]  
(3.29)

By induction assumption, we have \( wt(\mu') \leq wt(\lambda'), \) forcing \( wt(\mu) \leq wt(\lambda). \) Otherwise, \( \lambda \) is obtained from \( \lambda' \) by removing a box \( x \) in \( [\lambda'] \). So, \( wt_1(\lambda') = wt_1(\lambda), \) \( wt_2(\lambda') = wt_2(\lambda) + a_c(x), \) and (3.29) still holds true. By induction assumption, we have \( wt(\mu) \leq wt(\lambda) \) since \( wt_1(\mu) + wt_2(\mu) = wt_1(\lambda) + wt_2(\lambda), \) \( wt_1(\lambda) \leq wt_1(\mu). \)

\[ \square \]

**Corollary 3.19.** Suppose \( \lambda, \mu \in \overline{\Lambda}. \)

1. If \( L(\lambda) \) and \( L(\mu) \) are in the same block of \( A\text{-mod}, \) then \( wt_1(\lambda) + wt_2(\lambda) = wt_1(\mu) + wt_2(\mu). \)
2. If \( [\overline{\Delta}(\lambda) : L(\mu)] \neq 0 \) and \( \lambda \neq \mu, \) then \( wt(\mu) < wt(\lambda). \)

**Proof.** Mimicking arguments in the proof of [13] Theorem 5.17 and using Proposition 3.18 and Corollary 2.16 (2) yields (1). We leave the details to the reader. (2) follows from Proposition 3.18 and Corollary 2.16 \[ \square \]

Let \( j^\| : A_{[\delta]}\text{-ldmod} \to \overline{A}_{\|}\text{-ldmod} \) be the exact idempotent truncation functor. Then \( j^\| V = T \sum \rho,\sigma \) for any \( V \in A_{[\delta]}\text{-ldmod}. \) Recall \( j^\| \) and \( j^\| \) in (2.33). Then \( (j^\|, j^\|, j^\|) \) with the adjoint triple between \( A\text{-ldmod}_{\rho,\sigma} \) and its Serre quotient \( A\text{-ldmod}_{\rho,\sigma}/A\text{-ldmod}_{\rho,\sigma}. \) In fact, this result is available for any locally unital algebra associated to an upper finite weakly triangular category (see the proof of [13] Theorem 3.5). \[ \square \]

**Theorem 3.20.** The \( A\text{-ldmod} \) is an upper finite fully stratified category in the sense of [10] Definition 3.36 with respect to the stratification \( \overline{X} \to \overline{P} \times \overline{P} \) in (5.28) with the order \( \leq \) on \( \overline{P} \times \overline{P}. \)

**Proof.** It is clear that \( wt \) is a new stratification of \( A\text{-ldmod} \) in the sense of [10] Definition 3.1. Moreover, the image of \( wt \) (denoted by \( \overline{P} \)) is upper finite by its definition in (3.28). By the well-known results on block decomposition of degenerate cyclotomic Hecke algebras, \( \overline{P} \) gives a block decomposition of \( \overline{A} \text{-ldmod}. \)

Suppose that \( \overline{A}_{\|}(\rho,\sigma)\text{-ldmod} \) is the block of \( \overline{A}_{\|} \) indexed by \( (\rho,\sigma) \in \overline{P} \) with \( a = (r,s) \) (i.e., \( \rho \) is obtained from \( \omega_{u'} \) by adding \( r \) simple roots and \( \sigma \) is obtained from \( \omega_u \) by subtracting \( s \) simple roots). Let \( A\text{-ldmod}_{\|}(\rho,\sigma) \) be the Serre subcategory of \( A\text{-ldmod} \) generated by \( \{L(\lambda) | wt(\lambda) \leq (\rho,\sigma)\}. \)

Note that \( \lambda \in \overline{\Lambda}_{[r+k,s+k]} \) for some \( k \in \mathbb{N} \) if \( wt(\lambda) \leq (\rho,\sigma). \) So, \( A\text{-ldmod}_{\|}(\rho,\sigma) \) is actually a Serre subcategory of \( A_{[\delta]}\text{-ldmod}. \) Similarly, we have \( A\text{-ldmod}_{\|}(\rho,\sigma). \)

Let \( j^\|_{\rho,\sigma} \) be the restriction of \( j^\| \) to \( A\text{-ldmod}_{\|}(\rho,\sigma). \) Since \( j^\|_{\rho,\sigma}(M) \in \overline{A}_{\|}(\rho,\sigma)\text{-ldmod} \) for any \( M \in A_{[\delta]}\text{-ldmod}_{\|}(\rho,\sigma), \) \( j^\|_{\rho,\sigma} \) is actually a functor from \( A\text{-ldmod}_{\|}(\rho,\sigma) \) to \( \overline{A}_{\|}(\rho,\sigma)\text{-ldmod}. \) Moreover, \( j^\|_{\rho,\sigma} \) induces an equivalence of categories between \( A\text{-ldmod}_{\|}(\rho,\sigma)/A\text{-ldmod}_{\|}(\rho,\sigma) \) and \( \overline{A}_{\|}(\rho,\sigma)\text{-ldmod}. \)

Let \( j_{\!\uparrow}^{\|_{\rho,\sigma}}(\rho,\sigma) \) be the restriction of \( j^\| \) to \( A_{\|}(\rho,\sigma)\text{-ldmod}. \) Then \( j_{\!\uparrow}^{\|_{\rho,\sigma}}(\rho,\sigma) \) is actually a functor from \( A_{\|}(\rho,\sigma)\text{-ldmod} \) to \( A\text{-ldmod}_{\|}(\rho,\sigma). \) In fact, for any \( M \in \overline{A}_{\|}(\rho,\sigma)\text{-ldmod}, \) \( j_{\!\uparrow}^{\|_{\rho,\sigma}}(\rho,\sigma)(M) \) has a \( \Delta \)-flag, and \( \overline{X}(\mu) \) appears as a section if \( [M : D(\mu)] \neq 0. \) In this case, \( wt(\mu) = (\rho,\sigma). \) By Corollary 3.19 (2), we see that \( j_{\!\uparrow}^{\|_{\rho,\sigma}}(\rho,\sigma)(M) \in A\text{-ldmod}_{\|}(\rho,\sigma). \) Furthermore, since \( (j^\|, j^\|) \) is an adjoint pair, so is \( (j_{\!\uparrow}^{\|_{\rho,\sigma}}, j_{\!\downarrow}^{\|_{\rho,\sigma}}). \) Hence the required standard and proper standard objects coincide with those in Definition 2.13.
Suppose $\lambda \in \overline{\Lambda}$. Thanks to Corollaries 2.16(1) and 3.19(2), $P(\lambda)$ has a finite $\Delta$-flag such that $\Delta(\lambda)$ appears as the top section and other sections $\Delta(\mu)$ with $\text{wt}(\lambda) \leq \text{wt}(\mu)$. So, $A\text{-IIdmod}$ is an upper finite $+\text{-stratified category}$ in the sense of [10] Definition 3.36 with respect to the stratification $\text{wt}$. It is fully stratified since $\Delta(\mu)$ has a finite $\overline{\Delta}$-flag with sections $\overline{\Delta}(v)$ such that $\text{wt}(\nu) = \text{wt}(\mu)$.

Let $K_0(\overline{A}^\sigma\text{-pmod})$ be the Grothendieck group of $\overline{A}^\sigma\text{-pmod}$. Recall $V(\omega_u)$ (resp., $\tilde{V}(-\omega'_u)$) is the integrable highest (resp., lowest) weight $g$-module of weight $\omega_u$ (resp., $-\omega'_u$). Let $g^\updownarrow = \{ y^\uparrow | y \in g \}$ and $g^\downarrow = \{ y^\downarrow | y \in g \}$ be the two copies of $g$, where both $y^\uparrow$ and $y^\downarrow$ are $y$.

**Proposition 3.21.** As $g^\updownarrow \oplus g^\downarrow$-modules,

$$C \otimes \mathbb{Z} K_0(\overline{A}^\sigma\text{-pmod}) \cong \tilde{V}(-\omega'_u) \boxtimes V(\omega_u),$$

where the Chevalley generators $e^\uparrow_i$ and $f^\uparrow_i$ (resp., $e^\downarrow_i$ and $f^\downarrow_i$) act on $C \otimes \mathbb{Z} K_0(\overline{A}^\sigma\text{-pmod})$ via the endomorphisms induced by $E^\uparrow_i$ and $F^\uparrow_i$ (resp., $E^\downarrow_i$ and $F^\downarrow_i$) if $i \in I_u$ (resp., $I'_u$) and 0, otherwise.

**Proof.** This follows from [17] § 5.3 (see also [14] Remark 6.2) and [3.14].

Let $U(t)$ be the universal enveloping algebra of any Lie algebra $t$. There is a Lie algebra homomorphism from $g$ to $g^\updownarrow \oplus g^\downarrow$ sending $y$ to $y^\uparrow$ + $y^\downarrow$. This homomorphism induces the usual comultiplication on $U(g)$ since $U(g) \otimes U(g)$ can be identified with $U(g^\updownarrow \oplus g^\downarrow)$. So, $\tilde{V}(-\omega'_u) \boxtimes V(\omega_u)$ becomes the $g$-module $\tilde{V}(-\omega'_u) \otimes V(\omega_u)$ via the above homomorphism. Let $K_0(A\text{-mod}^\Delta)$ be the Grothendieck group of $A\text{-mod}^\Delta$.

**Theorem 3.22.** As $g$-modules, $C \otimes \mathbb{Z} K_0(A\text{-mod}^\Delta) \cong \tilde{V}(-\omega'_u) \otimes V(\omega_u)$, where the Chevalley generators $e_i, f_i$ act on $C \otimes \mathbb{Z} K_0(A\text{-mod}^\Delta)$ via the endomorphisms induced by the $E^\pm_i$ and $F^\pm_i$ for all $i \in I$.

**Proof.** This follows from Proposition 3.21 and 3.18.

4. Proof of Theorem 1.1

Recall that $g$ is the Kac-Moody Lie algebra in subsection 3.3. The quiver Hecke category $Q\mathcal{H}$ associated to $g$ is a $k$-linear strict monoidal category generated by objects $I$ in $\mathbb{1}_2$ and morphisms

$$\uparrow : i \rightarrow i, \quad \otimes : i \otimes j \rightarrow j \otimes i,$$

subject to certain relations in [5] Definition 3.4, where the parameters $\{t_{ij} \in k^\times | i,j \in I\}$ and $\{s_{ij}^m \in k | 0 \leq m < -a_{ij}, 0 < q < -a_{i,j}\}$ (only appear when $p = 2$) are given as follows:

$$t_{ij} = \begin{cases} -1, & i = j - 1; \\ 1, & \text{otherwise.} \end{cases}, \quad s_{ij}^{11} = i_{ij}^{11} = 2 \text{ for } i = j \pm 1.$$

For any $i = (i_1, i_{d-1}, \ldots, i_1) \in \mathbb{I}^d$, $d > 0$, we identify $i$ with the object $i_d \otimes i_{d-1} \otimes \cdots \otimes i_1 \in \text{ob} \ Q\mathcal{H}$. The locally unital algebra associated to $Q\mathcal{H}$ is $\bigoplus_{d \in \mathbb{N}} QH_d$, where

$$QH_d := \bigoplus_{i \in \mathbb{I}^d} \text{Hom}_{Q\mathcal{H}}(1, i).$$

When $d > 0$, $QH_d$ is known as the quiver Hecke algebra associated to $g$ [16, 21]. It is generated by

$$\{ e(i) | i \in \mathbb{I}^d \} \cup \{ y_1, \ldots, y_d \} \cup \{ \psi_1, \ldots, \psi_{d-1} \}$$

subject to the relations (1.7)–(1.15) in [9] Theorem 1.1, where

$$e(\overline{i}) = \uparrow_{i_1} \cdots \uparrow_{i_d}, \quad y_r e(\overline{i}) = \uparrow_{i_d} \cdots \uparrow_{i_r}, \quad \psi_r e(\overline{i}) = \uparrow_{i_d} \cdots \uparrow_{i_1} \uparrow_{i_{d-1}} \cdots \uparrow_{i_r}.$$

and $\overline{i} = (i_1, i_2, \ldots, i_d)$. For any $\mu \in P^+$ given in (3.26), let $QH_d(\mu)$ be the cyclotomic quotient of $QH_d$ by the two-sided ideal generated by $\{ y_1^{(h_1, \mu)} e(\overline{i}) | i \in \mathbb{I}^d \}$.

Following [11], let $A\mathcal{H}$ be the $k$-linear strict monoidal category generated by the single object $\downarrow$ and two morphisms $\downarrow$ and $\downarrow$, satisfying the relations (2.12)–(2.14). Then $A\mathcal{H}$, known as the degenerate affine Hecke category, can be considered as the subcategory of $AOB$ generated by the single object $\downarrow$ and morphisms $\downarrow$ and $\downarrow$. Let

$$A\mathcal{H}_d := \text{Hom}_{A\mathcal{H}}(\downarrow \otimes d, \downarrow \otimes d),$$
the degenerate affine Hecke algebra $\hat{QH}$. Following [11], define
\[ x_r = \begin{array}{c|c|c|c|c} \downarrow & \cdots & \downarrow & \cdots & \downarrow \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \downarrow & \cdots & \downarrow & \cdots & \downarrow \\ \end{array} \in AH_d. \]

For any $\mu \in P^+$, let $AH_d(\mu)$ be the cyclotomic quotient of $AH_d$ by the two-sided ideal generated by
\[ g(x_1) = \prod_{i \in \mathbb{I}} (x_1 - i)^{\langle h_i, \mu \rangle}. \]

The algebra $AH_d(\mu)$ is isomorphic to some degenerated cyclotomic Hecke algebra in subsection 2.5. At moment, we use $AH_d(\mu)$ so as to compare it with $QH_d(\mu)$. It is known that there is a set of mutually orthogonal idempotents of $AH_d(\mu)$, denoted by $\{1_i \mid i \in \mathbb{I}^d\}$ such that, for any $AH_d(\mu)$-module $M$
\[ 1_i M = \bigcap_{k=1}^d M_{ik}, \]
where $M_{ik}$ is the $i_k$-generalized eigenspace of $M$ of $x_k$. If $\langle h_i, \mu \rangle \leq \langle h_i, \mu' \rangle$ for all $i \in \mathbb{I}$, then there are endofunctors
\[ E, \hat{E} : QH_d(\mu') \rightarrow QH_d(\mu), \quad AH_d(\mu') \rightarrow AH_d(\mu). \]

So, $\{QH_d(\mu) \mid \mu \in P^+\}$ and $\{AH_d(\mu) \mid \mu \in P^+\}$ form two inverse systems of locally unital algebras. Taking inverse limits yields two completions
\[ \hat{AH}_d := \lim_{\leftarrow} AH_d(\mu), \quad \hat{QH}_d := \lim_{\leftarrow} QH_d(\mu). \]

It is proved in [9, Theorem 1.1] that there is an isomorphism of locally unital algebras $QH_d(\mu) \cong AH_d(\mu)$ such that
\[ \begin{array}{c|c|c|c|c} \uparrow & \cdots & \uparrow & \cdots & \uparrow \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \uparrow & \cdots & \uparrow & \cdots & \uparrow \\ \end{array} \rightarrow I_1, \quad \begin{array}{c|c|c|c|c} \uparrow & \cdots & \uparrow & \cdots & \uparrow \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \uparrow & \cdots & \uparrow & \cdots & \uparrow \\ \end{array} \rightarrow (x_r - i_r)I_1. \quad (4.1) \]

These isomorphisms induces an isomorphism of locally unital algebras
\[ \hat{QH}_d \cong \hat{AH}_d. \quad (4.2) \]

See also [21, § 3.2.6] and [25, Theorem 3.10]. Moreover, there is a locally unital embedding
\[ QH_d \hookrightarrow \hat{QH}_d. \quad (4.3) \]

Now, we go on studying the cyclotomic oriented Brauer category $\mathcal{OB}(u, u')$. Recall $A$ is the locally unital $\mathbb{k}$-algebra associated to $\mathcal{OB}(u, u')$ in (2.20). There are endofunctors $E, F$ in (3.4). For any $i \in \mathbb{I}$ there are endofunctors $E_i, F_i$ in (3.21). Thanks to Lemma 3.4(1), there is an $(A, A)$-homomorphism $x_\downarrow : A_\downarrow \rightarrow A_\downarrow$ defined on $A_11_k$ by right multiplication by $1_k\downarrow$. So, $x^\uparrow$ and $x_\downarrow$ are intertwined by the isomorphism $\dagger A \cong A_\downarrow$ in Proposition 3.1 where $x^\uparrow$ is given in Lemma 3.4(2). Moreover, $x_\downarrow$ induces a natural transformation $\downarrow : E \rightarrow E$ such that
\[ \begin{array}{c|c|c|c|c} \downarrow \\ M \end{array} = x_\downarrow \otimes \text{Id} : A_\downarrow \otimes_A M \rightarrow A_\downarrow \otimes_A M. \]

Similarly, there is a natural transformation denoted by $\wedge$ : $E^2 \rightarrow E^2$ such that
\[ \begin{array}{c|c|c|c|c} \wedge \\ M \end{array} : A_{\uparrow \downarrow} \otimes_A M \rightarrow A_{\uparrow \downarrow} \otimes_A M, \quad y \otimes m \mapsto y \circ (1_a \wedge) \otimes m, \]
where $(y, m) \in A_{\uparrow \downarrow} \times M$ and $A_{\uparrow \downarrow} = A_\uparrow \otimes_A A_\downarrow$ in the obvious way.

**Lemma 4.1.** There is a strict monoidal functor $\Psi : AH \rightarrow \mathcal{E}nd(A[/ldmod])$ such that
\[ \Psi(\downarrow) = E, \quad \Psi(\wedge) = \wedge, \quad \Psi(\uparrow) = \uparrow. \]

**Proof.** The result follows directly from (2.12–2.14). \qed
Lemma 4.2. For all $i,j \in I$, there are natural transformations
\[
\begin{align*}
\delta : E_i &\to E_i, \\
\otimes : E_i \otimes E_j &\to E_i \circ E_j,
\end{align*}
\]
which induce a strict monoidal functor from $QH$ to $\mathcal{E}(\text{End}(A,\text{lf}d\text{mod}))$.

Proof. By Lemma 4.1 there is an algebra homomorphism $\Psi_d : AH_d \to \text{End}(E_d)$ for any $d > 0$. Thanks to Lemma 3.3, $E$ and $F$ are biaadjoint to each other. So, $E$ is a sweet endofunctor of $A, \text{lf}d\text{mod}$ in the sense of [5, Definition 4.1]. By [5, Theorem 2.11], $E$ preserves the set of finitely generated projective modules. Therefore, $E^dP$ is finitely generated projective for any $P \in A, \text{pmod}$. In particular, $E^dP \in A, \text{lf}d\text{mod}$ since any finitely generated left $A$-module has to be locally finite dimensional (see [5, Section 2.2]). It is proved in [5, Section 2.2] that $\dim_{\text{Hom}}_d(V,W) < \infty$ for any finitely generated left $A$-module $V$, and any locally finite dimensional left $A$-module $W$. In particular, $\text{End}_A(E^dP)$ is finite dimensional for each $P \in A, \text{pmod}$ and hence $(x_1)\rho$ has a minimal polynomial with roots in $I$ (see [22]). So, the composition
\[
ev_P \circ \Psi_d : AH_d \to \text{End}_A(E^dP)
\]
of $\Psi_d$ with the evaluation at $P$ factors through some cyclotomic quotients $AH_d(\mu)$ and hence factors through all sufficient large cyclotomic quotients $AH_d(\mu')$. Note that any two natural transformations $\eta, \xi : E^d \to E^d$ are equal if $\eta_P = \xi_P$ for each $P \in A, \text{pmod}$. So, $ev_P \circ \Psi_d$ factors through all sufficient large cyclotomic quotients $AH_d(\mu')$ for any $M \in A, \text{lf}d\text{mod}$. This shows that $\Psi_d$ factors through $\widehat{AH}_d$.

Let
\[
\hat{\Psi}_d : \widehat{AH}_d \to \text{End}(E^d).
\]
Comparing $\hat{\Psi}_d$ with the isomorphism in (4.2), and the inclusion in (4.3), yields an algebra homomorphism
\[
\Phi_d : QH_d \to \text{End}(E^d).
\]
Moreover, by the argument for the proof of [6] (6.13) (i.e., by Lemma 4.1 and the explicit formulae of the isomorphism $\overline{QH}_d \cong \overline{AH}_d$ induced by [9, Theorem 1.1] or the non-degenerate analogue in [25, Proposition 3.10]), these maps are compatible with the monoidal structure in the categories $QH$ and $\mathcal{E}(\text{End}(A,\text{lf}d\text{mod}))$ such that there is a monoidal functor $\Phi : QH \to \mathcal{E}(\text{End}(A,\text{lf}d\text{mod}))$ given by $\Phi(i) = E_i$ and $\Phi(h) = \Phi_d(h)$ if $h \in QH_d$. \hfill \Box

Definition 4.3. [18, Definition 3.2, Remark 3.6] Fix $\omega_u$ and $\omega_{u'}$ in (1.4). The data of a tensor product categorification of $\tilde{V}(-\omega_{u'}) \otimes V(\omega_u)$ consists of two parts:

(TPC1) A (locally) Schurian category $\mathcal{C}$ such that the isomorphism classes of irreducible objects labeled by the indexing set for the basis of $\tilde{V}(-\omega_{u'}) \otimes V(\omega_u)$.

(TPC2) A nilpotent categorical action (in the sense of [5, Definition 4.25]) making $\mathcal{C}$ into a 2-representation (e.g., [21, Definition 5.1.1] or [5, Definition 4.1]) of the associated Kac-Moody 2-category $\mathcal{U}(\mathfrak{g})$ (e.g., [21, § 4.1.3] or [5, Definition 3.8]).

These two pieces of data need to satisfy some compatibility conditions as follows:

(TPC3) The category $\mathcal{C}$ is an upper finite fully stratified category in the sense of Brundan and Stroppel [10] with poset $P \times P$. Moreover, the poset structure of $P \times P$ is given by the “inverse dominance order” $\preceq$ on $P \times P$ in (3.27).

(TPC4) There is a categorical $(\mathfrak{g}^+ \oplus \mathfrak{g}^\check{+})$-action on $\text{gr} \mathcal{C} := \bigoplus_{(\lambda,\mu) \in P \times P} C_{(\lambda,\mu)}$ such that
\[
\mathcal{C} \otimes K_0(\text{proj-gr} \mathcal{C}) \cong \tilde{V}(-\omega_{u'}) \otimes V(\omega_u)
\]
as $(\mathfrak{g}^+ \oplus \mathfrak{g}^\check{+})$-modules, where
\[
\begin{align*}
(1) &\quad C_{(\lambda,\mu)} := C_{\leq (\lambda,\mu)} / C_{< (\lambda,\mu)}, \\
(2) &\quad \mathfrak{g}^+ = \{x^+ \mid x \in \mathfrak{g}\} \text{ and } \mathfrak{g}^\check{+} = \{x^\check{+} \mid x \in \mathfrak{g}\} \text{ are two copies of } \mathfrak{g}, \\
(3) &\quad \text{proj-gr} \mathcal{C} \text{ is the subcategory of finite generated projective modules of gr} \mathcal{C}, \\
(4) &\quad C_{(-\omega_{u'}, \omega_u)} \cong \text{Vec}_k \text{ (the category of vector space over } k). 
\end{align*}
\]
The categorification functors for $\mathfrak{g}^\check{+}$ will be denoted by $E^\check{+}_i$ and $F^\check{+}_i$, where $i \in I$ and $\circ \in \{\uparrow, \downarrow\}$. 

(TPC5) For any \( i \in I \), denote by \( E_i \) and \( F_i \) the categorification functors for \( g \). There is a compatibility between the categorical \( g \)-action on \( C \) and the categorical \( (g^\downarrow \oplus g^\uparrow) \)-action on \( gr C \) in the sense that there are short exact sequences

\[
0 \to \Delta \circ E_i^\uparrow \to E_i \circ \Delta \to \Delta \circ E_i^\downarrow \to 0,
\]

\[
0 \to \Delta \circ F_i^\downarrow \to F_i \circ \Delta \to \Delta \circ F_i^\uparrow \to 0,
\]

where \( \Delta = \bigoplus_{(\lambda,\mu) \in P \times P} \Delta(\lambda,\mu) \) and \( \Delta(\lambda,\mu) : C_{\Sigma}(\lambda,\mu) \to C(\lambda,\mu) \) is the corresponding standardization functor of the stratified category \( C \).

**Proof of Theorem 1.1.** Thanks to Definition 3.3, we need to verify that \( A\text{-}lfd\text{mod} \) admits the structures in (TPC1)–(TPC5).

By Corollary 2.14 and Lemma 3.21, we have results on the classification of irreducible objects in \( A\text{-}lfd\text{mod} \). Now, (TPC1) follows from Theorem 3.22 which gives a bijection between labellings. (TPC3) follows from Theorem 3.20 which says that \( A\text{-}lfd\text{mod} \) is an upper finite fully stratified category in the sense of Brundan and Stroppel 11 with respect to the stratification function in 3.28. (TPC4) follows from Lemma 3.21 and the well-known categorical \( g \)-action on the representations of degenerate cyclotomic Hecke algebras [5, Theorem 4.18] [22, Theorem 4.25]. The short exact sequences in (4.5) follow from Lemma 3.21 and the well-known categorical \( g \)-action on the representations of degenerate cyclotomic Hecke algebras [5, Theorem 4.18] [22, Theorem 4.25]. The short exact sequences in (4.5) follow from (3.18) in Theorem 3.10 and hence (TPC5) follows. So, it remains to verify (TPC2).

Thanks to [5, Theorem 4.27], it suffices to check the following conditions in [5, Definition 4.25]:

1. A weight decomposition of the category \( A\text{-}lfd\text{mod}= \bigoplus_{\sigma \in P_{\times} P} A\text{-}lfd\text{mod}_\sigma \).
2. Biadjoint endofunctors \( E = \bigoplus_{i \in I} E_i \) and \( F = \bigoplus_{i \in I} F_i \) such that \( (E_i, F_i) \) are biadjoint functors.
3. For all \( i, j \in I \), there are natural transformations

\[
\begin{array}{c}
\uparrow\downarrow : E_i \to E_i, \\
\downarrow\uparrow : E_i \circ E_j \to E_j \circ E_i,
\end{array}
\]

which induce a strict monoidal functor from \( QH \) to \( End(A\text{-}lfd\text{mod}) \).
4. The endomorphisms \( [E_i] \) and \( [F_i] \) make \( C \otimes_{\mathbb{Z}} K_0(A\text{-}mod) \) into a well-defined \( g \)-module with \( \sigma \)-weight space \( C \otimes_{\mathbb{Z}} K_0(A\text{-}mod)_\sigma \).
5. For any \( i \in I \) and any finitely generated left \( A \)-module \( M \), the endomorphism

\[
(\downarrow) : E_i M \to E_i M
\]

is nilpotent.

In fact, (1) follows from the partial result on the blocks of \( A\text{-}lfd\text{mod} \) in Corollary 3.19. In this case, we set \( A\text{-}lfd\text{mod}_\sigma \) to be the Serre subcategory of \( A\text{-}lfd\text{mod} \) generated by \( L(\lambda) \) such that \( wt(\lambda) + wt(\mu) = \sigma \). The biadjoint functors in (2) are given in Lemma 3.6 and (3.21). (3) follows from Lemma 4.2. Thanks to Corollary 2.14, \( C \otimes_{\mathbb{Z}} K_0(A\text{-}mod) \) can be identified with \( C \otimes_{\mathbb{Z}} K_0(A\text{-}mod) \) and hence (4) follows Theorem 3.22. Finally, by (4.1), under the isomorphism in (4.2) we have

\[
\begin{array}{c}
\uparrow \downarrow = (\downarrow) - iId, \\
M
\end{array} = (\downarrow)M - iId_{E_i M}.
\]

Therefore, it suffices to show that there is bound on the Jordan block sizes of

\[
x_i \otimes \text{Id} = (\downarrow) M : A_i \otimes A M \to A_i \otimes A M
\]

for any finitely generated left \( A \)-module \( M \). Since \( E \) is a sweet functor (e.g., Lemma 3.3) of \( A\text{-}lfd\text{mod} \) in the sense of [5, Definition 2.10], by [5, Theorem 2.11], \( E \) preserve the set of finitely generated modules. Therefore, \( EM = A_i \otimes A M \) is finitely generated. Since \( A \) is locally finite dimensional, \( \text{End}_A(EM) \) is finite dimensional and hence \( (\downarrow)_M \) has a minimal polynomial. So, there is bound on the Jordan block sizes. This completes the proof of (5).

\[\square\]

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22 MENGMENG GAO, HEBING RUI, LINLIANG SONG

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