G-TORSORS OVER A DEDEKIND SCHEME

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ABSTRACT. We prove the equivalence of three “points of view” of the notion of a G-torsor when the base scheme is a Dedekind scheme. As an application, we show that the fibered category of G-torsors on a curve over a field k is representable by an Artin stack locally of finite presentation over k.

1. Introduction

Let us first fix some notation. We fix a Dedekind scheme X (the base scheme), that is, a scheme that has a finite affine cover by the spectra of Dedekind domains. Unless stated otherwise, any unadorned product is assumed to be over X, and for two X-schemes Y and T we often write Y × T = Y × T. If Y is a scheme over X, we use the “functor of points notation” and write y ∈ Y to denote a morphism y : T → Y of schemes over X. In the same spirit, if V is a locally free OX-module of finite rank, we denote also by V the functor V : R → V ⊗ R, which is represented by Spec (SymV∗). Here, V∗ = HomO (V, OX) denotes the dual of V. For any Y, if M is an OY-module and N ⊂ M is an OY-submodule, we say N is locally split if N is Zariski locally on Y a direct summand of M.

We fix G a flat algebraic group over X, by which we mean a flat, affine group scheme of finite type over X. Unless specified otherwise, by simply a representation of G, we mean a finite rank, locally free OX-module V with a linear G-action. If V is a representation of G, we denote by V0 the same underlying OX-module with the trivial G-action. If Y is an X-scheme, a GY-torsor is a scheme P faithfully flat and affine over Y, provided with a right G-action such that the following two conditions hold.

(i) The map P → Y is GY-invariant.
(ii) The natural map

P × G → P ×Y P; (p, g) ↦ (p, pg)

is an isomorphism.

A map P → P′ of GY-torsors is a GY-equivariant map of Y-schemes. A trivial GY-torsor is a GY-torsor P → Y that is isomorphic as a GY-torsor to the projection map Y × G → Y. Given this terminology, condition (ii) is equivalent to:

(ii′) The map P → Y is locally trivial in the fppf topology.

Let RepG denote the category of G-representations on locally free OX-modules of finite rank. For a scheme Y over X, denote by BunY the category of vector bundles of finite rank over Y. For any representation V of G, we denote by t(V) some finite iteration of the operations ⊗, ∧i, Symi, ⊕, and (·)∗. We call such an iteration a tensorial construction. If V is a vector bundle on X, and L ⊂ V is a

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locally split line bundle, we denote by $\text{Aut}(V, L)$ the representable functor whose $T$-points are automorphisms $f$ of $V \otimes \mathcal{O}_T$ such that $f(L \otimes \mathcal{O}_T) = L \otimes \mathcal{O}_T$. We now state our main theorem.

**Theorem 1.1.** There is a representation $V$ of $G$, a tensorial construction $t(V)$, and a locally split line bundle $L \subset t(V)$, such that $G = \text{Aut}(V, L)$. Fix such a tensorial construction and pair $(V, L)$. For any $X$-scheme $Y$, there are natural equivalences of the following groupoids that are functorial in $Y$:

(i) the groupoid of $G_Y$-torsors, and 
(ii) the groupoid of pairs $(\mathcal{E}, \mathcal{L})$ consisting of a vector bundle $\mathcal{E}$ on $Y$ equipped with a locally split line bundle $\mathcal{L} \subset t(\mathcal{E})$ that is fppf locally isomorphic as a pair to $(V, L)$.

Furthermore, if $Y$ is faithfully flat over $X$, (i) and (ii) are equivalent to 
(iii) the groupoid of faithful tensor functors $F : \text{Rep} G \to \text{Bun}_Y$ that preserve monomorphisms.

**Proof.** We prove the theorem in separate pieces. The existence of $V$, $t(V)$ and $L \subset t(V)$ is Theorem 1.3. The equivalence (i) $\iff$ (iii) is given by Theorem 1.4 and the equivalence (i) $\iff$ (ii) is given by Theorem 1.6. We remark that composing the equivalences given in those two theorems, the implication (iii) $\implies$ (ii) has a simple description. Namely, given a functor $F : \text{Rep} G \to \text{Bun}_Y$, we get a pair as in (ii) by $F \mapsto (F(V), F(L))$. □

Let us briefly elucidate item (ii). Suppose that we can write $G = \text{Aut}(V, L)$. For an $X$-scheme $Y$, we define a $G_Y$-twist of $(V, L)$ (or simply $G$-twist), to be a pair $(\mathcal{E}, \mathcal{L})$ consisting of a locally free sheaf $\mathcal{E}$ on $Y$ provided with a locally split line bundle $\mathcal{L} \subset t(\mathcal{E})$ that is fppf locally isomorphic as a pair to $(V, L)$. That is, there is an fppf cover $Y' \to Y$ and an isomorphism $f : \mathcal{E}_{Y'} \simto V_{Y'}$ that induces $f(\mathcal{L}_{Y'}) = L_{Y'}$. In particular, such a bundle $\mathcal{E}$ must have $\text{rk} \mathcal{E} = \text{rk} V$. A map of $G$-twists $f : (\mathcal{E}, \mathcal{L}) \to (\mathcal{E}', \mathcal{L}')$ is a map of vector bundles $f : \mathcal{E} \to \mathcal{E}'$ such that $f(\mathcal{L}) = \mathcal{L}'$. Allowing only isomorphisms, we arrive at the following proposition.

**Theorem 1.2.** The groupoid of $G_Y$-torsors is equivalent to the groupoid of $G_Y$-twists of $(V, L)$. This equivalence is functorial in $Y$.

**Proof.** Given a $G_Y$-torsor $P$, we form the associated vector bundle $P \times^G V = (P \times V)/((p, v) \sim (pg^{-1}, gv))$. Forming $P \times^G t(V)$, we get an induced line bundle $L'$ from $L$. To construct a quasi-inverse, suppose we are given a $G_Y$-twist $(\mathcal{E}, \mathcal{L})$ of $(V, L)$. Then, the fppf sheaf $T \mapsto \text{Isom}_{\mathcal{E}_T}((\mathcal{E}_T, \mathcal{L}_T), (V_T, L_T))$ is representable by a $G_Y$-torsor. It is clear that these constructions are functorial in $Y$. □

We remark that the idea of confining oneself to locally free, finite rank representations of $G$ (rather than all quasicoherent sheaves with $G$-action) over Dedekind schemes is already present in Saavedra’s book on Tannakian categories. Nonetheless, the equivalence of (i) and (iii) in Theorem 1.1 is only proven when the base is a field (cf. [10, II.4.2.2]). It is not known to the author whether one can replace “monomorphism” with “exact” in the statement of Theorem 1.1 (iii) (they are of course equivalent over a field).

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2. Application to the moduli of $G$-torsors

Before proceeding with the proof of Theorem 1.1, we give an application to the representability of the stack of $G$-torsors over a curve. For this section only, let $k$ be a field, and assume that $X$ is a connected, regular, proper curve over $k$. In particular, $X$ is a Dedekind scheme. Again for this section only, for a $k$-scheme $T$, we write $X_T = X \times_{\text{Spec} \, k} T$. We assume for this section that $G$ has connected generic fibre. Let $\text{GTor}_X$ denote the fibered category that assigns to a $k$-scheme $T$ the groupoid of $G_X T$-torsors. The goal of this section is to prove the following theorem. We are grateful to Brian Conrad for pointing out this application of Theorem 1.1.

**Theorem 2.1.** The fibered category $\text{GTor}_X$ is an Artin stack, locally of finite presentation over $k$.

We recall the following definition from [9, 3.3.3], a key input into the proof of the theorem, although the reader can take the statements of the subsequent theorem and lemmas as a black box. Let $S$ be a scheme and $T$ a scheme locally of finite presentation over $S$. For a point $s \in S$, denote by $(\tilde{S}, \tilde{s})$ a henselization of the pair $(S, s)$. Let $\tilde{T} = T \times_S \tilde{S}$. We say that $T$ is **pure along** $T \otimes k(\tilde{x})$ if for each element $\tilde{t} \in \bigcup_{x \in S} \text{Ass} (T \otimes k(\tilde{x}))$, the closure of $\tilde{t}$ in $\tilde{T}$ meets $\tilde{T} \otimes k(\tilde{s})$. We say that $T$ is **$S$-pure** if it is pure along each $s \in S$.

The reason why we introduce this notion is that pure maps have “flattening stratifications.” More precisely, we have the following theorem.

**Theorem 2.2.** Suppose that $T \to S$ is pure. Then there is a monomorphism $Z \hookrightarrow S$ that is locally of finite presentation such that for any $S$-scheme $S'$, $T \times_S S' \to S'$ is flat if and only if $S' \to S$ factors through $Z$.

**Proof.** This is Theorem 4.3.1 from Part I of [9].

**Lemma 2.3.** With $G$ and $X$ as above, $G$ is $X$-pure.

**Proof.** Let $\xi \in X$ be the generic point of $X$. By assumption $G_\xi$ is connected, so it is in fact geometrically irreducible by [1, VI, A 2.4]. In particular, $G_\xi$ is irreducible, and so $G$ is irreducible since $G$ is flat over $X$. Furthermore, the only associated prime of $G$ is its generic point. Indeed, if $G$ had an embedded component $Z$, $G_\xi$ would have infinitely many embedded components over the algebraic closure $\bar{k}$ by taking translates of $Z \otimes \bar{k}$. The result then follows from [9, I 3.3.4(iii)].

**Lemma 2.4.** The property that $T$ is $S$-pure is local for the fppf topology. That is, if $T \times_S S' \to S'$ is $S'$-pure if and only if $T$ is $S$-pure.

**Proof.** This is Corollary 3.3.7 of Part I of [9].

**Remark 2.5.** For the proof of Lemma 2.3, we only need to assume that $X$ is a connected Dedekind scheme. By [9 I 3.3.5], it follows that in this situation $\mathcal{O}_X$ is a locally free $\mathcal{O}_X$-module.
Proof of Theorem 2.1. By Theorem 1.1, we can find a representation of $G$ on a rank $n$ vector bundle $V$, a tensorial construction $t(V)$ and a locally split line bundle $L \subset t(V)$ such that $G \xrightarrow{\sim} \text{Aut}(V, L)$. Furthermore, the choice of $(V, L)$ respects base change in the sense that for any $X$-scheme $Y$, this identification pulls back to $G_Y \xrightarrow{\sim} \text{Aut}(V_Y, L_Y)$. We now fix such a pair $(V, L)$. Then $G_{\text{Tot}}$ is isomorphic to the fibered category that assigns to a $k$-scheme $T$ the groupoid of $G_{X_T}$-twists of $(V, L)$. Let $\text{Bun}_X^n$ denote the stack of rank $n$ vector bundles over $X$ (where $n$ is the rank of $V$). That is, to each $k$-scheme $T$, $\text{Bun}_X^n(T)$ is the groupoid of rank $n$ vector bundles over $X_T = X \times_k T$. Then $\text{Bun}_X^n$ is an Artin stack, locally of finite presentation over $k$.

Let $\mathcal{E}^{\text{univ}}$ denote the universal rank $n$ vector bundle on $X \times \text{Bun}_X^n$. Let $\mathcal{D}$ denote the relative quot scheme over $\text{Bun}_X^n$ classifying all rank 1, locally split subbundles of $t(\mathcal{E}^{\text{univ}})$ (where $t$ is the same tensorial construction as that defining $G$). That is, for a scheme $T$ over $\text{Bun}_X^n$, $\mathcal{D}(T)$ is the groupoid of locally split line bundles $\mathcal{L}_{X_T} \subset t(\mathcal{E}^{\text{univ}})_{X_T} = t(\mathcal{E}^{\text{univ}}_{X_T})$ on $X_T$. By [6, 2.2.4], $\mathcal{D} \to \text{Bun}_X^n$ is representable and locally of finite presentation. Let $\mathcal{L}^{\text{univ}} \subset t(\mathcal{E}^{\text{univ}})$ denote the universal line bundle on $X_{\mathcal{D}}$. Finally, over $X_{\mathcal{D}}$, define the fibered category $\mathcal{F}$, where for an $X_{\mathcal{D}}$-scheme $Y$, by $\mathcal{F}(Y) = \text{Isom}((V_Y, L_Y), (\mathcal{E}^{\text{univ}}_Y, \mathcal{L}^{\text{univ}}_Y))$. Then $\mathcal{F}$ is representable, in fact affine, and locally of finite presentation over $X_{\mathcal{D}}$.

For any scheme $T$ and any map $f : T \to \mathcal{D}$, we have an induced map $X_T \to X_{\mathcal{D}}$. We denote the pullback of $\mathcal{F}$ along this latter map by $f^* \mathcal{F}$. The map $X_T \to X_{\mathcal{D}}$ gives rise to a pair $(W, M) = (f^* \mathcal{E}^{\text{univ}}_{X_{\mathcal{D}}}, f^* \mathcal{E}^{\text{univ}}_{X_{\mathcal{D}}})$. We claim that $(W, M)$ is a $G_{X_T}$-twist of $(V, L)$ if and only if the projection $f^* \mathcal{F} \to X_T$ is flat. Note that the map $f^* \mathcal{F} \to \mathcal{F}$ gives an isomorphism $(\mathcal{E}^{\text{univ}}_{X_{\mathcal{D}}}, \mathcal{L}^{\text{univ}}_{X_{\mathcal{D}}}) \cong (V_{X_{\mathcal{D}}}, L_{X_{\mathcal{D}}})$. Thus, if $f^* \mathcal{F} \to X_T$ is flat, then it gives the desired fpqc cover of $X_T$. Conversely, if $(W, M)$ is a $G_{X_T}$-twist of $(V, L)$, then $f^* \mathcal{F}$ is a $G_{X_{\mathcal{D}}}$-torsor, so flat. Furthermore, since $G$ is $X$-pure, it also follows that $f^* \mathcal{F}$ is $X_T$-pure when it is flat.

Let $Q \to \mathcal{D}$ and $I \to \mathcal{F}$ be presentations. It suffices to show that there is an algebraic space $Z$ locally of finite presentation over $Q$ such that $f : T \to Q$ factors through $Z$ if and only if $f^* I \to X_T$ is flat and pure (with notation as above). We first represent the purity condition. By [8, 3.3.8] purity is an open condition. That is, there is an open immersion $U' \hookrightarrow X_Q$ such that $X_T \to X_Q$ factors through $U'$ if and only if $f^* I$ is pure over $X_T$. To get an open subspace of $Q$ representing the purity condition, we take the (closed) image of the closed complement of $U'$ under $X_Q \to Q$ and let $U$ be complement of that image. It then follows that $T \to Q$ factors through $U$ if and only if $f^* I$ is pure over $X_T$.

Thus, replacing $Q$ by $U$, we may assume that $I \to X_Q$ is pure. In this case, by Theorem 2.2 there is a representable monomorphism $Z' \to X_Q$ such that $Y : Y \to X_Q$ factors through $Z'$ if and only if $Y \times_{X_Q} Y \to Y$ is flat. We now want to represent the condition on $Q$-schemes $T$ that $X_T \to X_Q$ factors through $Z'$. These are exactly the $T$-points of the restriction of scalars $\text{Res}_{X_Q}^X(Z')$, which we denote $Z$. By [8, 1.5], since $X_Q \to Q$ is a proper, flat, and locally finitely presented map of algebraic spaces, and $Z' \to X_Q$ is separated, locally of finite presentation with finite diagonal, $Z$ is represented by an algebraic space, locally of finite presentation over $Q$. \qed
3. Algebraic groups over Dedekind schemes

With notation as in the introduction, let $G$ be a flat algebraic group scheme over $X$. Recall that this means that $G$ is a flat affine group scheme of finite type over $X$. If $f : G \to X$ denote the structure morphism, we will abuse notation and denote the $\mathcal{O}_X$-bialgebra $f^*(\mathcal{O}_G)$ simply by $\mathcal{O}_G$. Let $\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$ denote the comultiplication map and $\epsilon : \mathcal{O}_G \to \mathcal{O}_X$ the counit. Throughout, if $V$ is an $\mathcal{O}_G$-comodule, we denote by $V_0$ the underlying $\mathcal{O}_X$-module of $V$ with the trivial $G$-action. Unless noted otherwise, we reserve the term representation for the case where $V$ is a finite rank vector bundle on $X$. As above, if $W \subset V$ is Zariski locally a direct summand as an $\mathcal{O}_X$-module, we will call the inclusion locally split. If $W$ and $V$ are $\mathcal{O}_G$-comodules, that the inclusion $W \subset V$ is locally split does not imply in general that $W \subset V$ is locally a direct summand as an $\mathcal{O}_G$-comodule. Our presentation follows [13, Chap. 3] and [2, Chap. 5], generalizing to our current situation.

**Lemma 3.1.** Let $V$ be a flat quasicoherent $\mathcal{O}_G$-comodule. Then $V$ is the union of $\mathcal{O}_G$-comodules that are locally free $\mathcal{O}_X$-modules of finite rank.

**Proof.** For $X$ affine, this is the Corollary to Proposition 1.2 in [11]. We quickly sketch the proof in the nonaffine case as the details are the same as in *ibid*. Since $X$ is noetherian, by [1] 9.4.9 any quasicoherent sheaf is the limit of its coherent subsheaves. Since a coherent $\mathcal{O}_X$-submodule of $V$ is a vector bundle, it suffices to show that for any coherent submodule $W \subset V$, $W$ is contained in a coherent $\mathcal{O}_G$-subcomodule of $V$. Let $\rho : V \to V \otimes \mathcal{O}_G$ denote the comodule map. Since $\rho(W)$ is coherent, there is a coherent submodule $W' \subset V$ such that $\rho(W) \subset W' \otimes \mathcal{O}_G$. Define an $\mathcal{O}_X$-module $E$ where for any open affine $U \subset X$, $E(U) = \{v \in V(U) \mid \rho(v) \in (W' \otimes G)(U)\}$. Then $E \subset W'$, so it is coherent, and one can show that $E$ is a $\mathcal{O}_G$-comodule. $\square$

**Theorem 3.2.** There is a representation $V$ of $G$ such that the map $G \to GL(V)$ is a closed embedding.

**Proof.** Consider the regular representation $\Delta : \mathcal{O}_G \to (\mathcal{O}_G)_0 \otimes \mathcal{O}_G$. By Lemma 3.1, there is a $\Delta$-stable, finite rank vector bundle, $V \subset \mathcal{O}_G$, that locally contains the algebra generators of $\mathcal{O}_G$. Thus, we have a map of $\mathcal{O}_G$-comodules $\Delta : V \to V_0 \otimes \mathcal{O}_G$. If we tensor this comodule map with $V_0^*$, we get the sequence of comodule maps

$$V \otimes V_0^* \to V_0 \otimes V_0^* \otimes \mathcal{O}_G \to \mathcal{O}_G,$$

where the second map is induced by the natural evaluation map $V_0 \otimes V_0^* \to \mathcal{O}_X$. The composite of these two maps extends to a surjective map of $\mathcal{O}_X$-algebras $\text{Sym}(V \otimes V_0^*) \to \mathcal{O}_G$. Recall that $\mathcal{O}_{GL(V)} = \text{Sym}(V \otimes V_0^*)[1/\det]$. Since $G$ is a group scheme, the above surjection in turn extends to the desired surjection $\mathcal{O}_{GL(V)} \to \mathcal{O}_G$. $\square$

For the remainder of the section, we will fix such a representation $V$ of $G$.

**Lemma 3.3.** Let $W$ be a finite rank vector bundle on $X$, and suppose $U \subset W$ is a locally split, rank $d$ subbundle. Let $L = \bigwedge^d U \subset \bigwedge^d W$. Let $g \in GL(W)$. Then $gL = L \iff gU = U$. 

Proof. The statement is local on $X$, so we suppose that $X = \text{Spec } A$ for a Dedekind domain, $A$, and that $U \subset W$ is a direct summand. The direction $\Leftarrow$ is immediate by functoriality, so we assume now that $gL = L$. First, note that for any $A$-algebra $B$,

$$U \otimes B = \{ \omega \in W \otimes B \mid \omega \wedge (L \otimes B) = 0 \}.$$  

If $g \in GL(W \otimes B)$ and $u \in U \otimes B$, then

$$gu \wedge (L \otimes B) = g(u \wedge g^{-1}(L \otimes B)) = g(u \wedge L \otimes B) = 0.$$  

It follows from the previous remark that $gu \in U \otimes B$, as desired. \hfill \Box

**Lemma 3.4.** Let $W$ be a representation of $G$ and $U' \subset W$ a subrepresentation. Let $\mathcal{K}_X$ denote the fraction field of $\mathcal{O}_X$, and let $U = (U' \otimes \mathcal{K}_X) \cap W \subset W \otimes \mathcal{K}_X$. Then, $U$ is a locally split subrepresentation of $W$.

Proof. That $U \subset W$ is locally split is straightforward, so it suffices to show that it is $G$-stable. Let $\rho : W \to W \otimes \mathcal{O}_G$ denote the comodule map. We wish to show that $\rho(U) \subset U \otimes \mathcal{O}_G$. We can check this on stalks, so we may assume that $X = \text{Spec } A$, where $A$ is a DVR with uniformizer $\pi$. In this case $W$ and $U'$ are both free, say of ranks $n$ and $d$, respectively. By the elementary divisors theorem, we can choose a basis $\{e_1, \ldots, e_n\}$ of $W$ so that $\{\pi^r e_1, \ldots, \pi^r e_d\}$ is a basis for $U'$. Then, $U$ is the $A$-submodule of $W$ with basis $\{e_1, \ldots, e_d\}$. Let $e_i \in \{e_1, \ldots, e_d\}$. Write

$$\rho(e_i) = \sum_{j=1}^n e_j \otimes x_{ij}.$$  

Since $U'$ is $G$-stable, we can also write

$$\rho(\pi^r e_i) = \sum_{j=1}^d e_j \otimes y_{ij}.$$  

Thus, we have that

$$\sum_{j=1}^d e_j \otimes (\pi^r x_{ij} - y_{ij}) + \sum_{j=d+1}^n e_j \otimes \pi^r x_{ij} = 0.$$  

Since $\{e_1 \otimes 1, \ldots, e_n \otimes 1\}$ forms an $\mathcal{O}_G$-basis for $W \otimes \mathcal{O}_G$, we conclude in particular that $\pi^r x_{ij} = 0$ for $d + 1 \leq j \leq n$. Since $\mathcal{O}_G$ is flat, hence torsion-free, this then implies that $x_{ij} = 0$ for $d + 1 \leq j \leq n$. Thus, $\rho(e_i) \in U \otimes \mathcal{O}_G$ for $1 \leq i \leq d$, as desired. \hfill \Box

**Theorem 3.5.** There is a representation of $GL(V)$ on a tensorial construction $t(V)$, and a locally split line bundle $L \subset t(V)$ such that

$$G = \{ g \in GL(V) \mid gL = L \}.$$  

Proof. As above, write $\mathcal{O}_{GL(V)} = \text{Sym} (V \otimes V_0^*)[1/\det]$. Then, we can write

$$\mathcal{O}_{GL(V)} = \lim_{\longrightarrow \atop i} \left( \bigoplus_{m \geq 0} \text{Sym}^m (V \otimes V_0^*) \cdot \det^{-1} \right).$$  

(3.1)  

Identifying $G$ as a closed subgroup of $GL(V)$, $G$ is defined by a coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{GL(V)}$. We first construct a tensorial construction $t(V)$. Choose
a finite open affine cover \( \{ X_i \} \) of \( X \). On each \( X_i \), \( \mathcal{F} | X_i \) is finitely generated in \( \mathcal{O}_{GL(V)} | X_i \) as an \( \mathcal{O}_{X_i} \)-algebra. Hence, by taking integers \( M \) and \( N \) sufficiently large, we can ensure that the algebra generators of \( \mathcal{F} \) on each \( X_i \) are contained in

\[
t'(V) = \bigoplus_{m=0}^M \text{Sym}^m (V \otimes V_0^*) \cdot \det^{-N}.
\]

Let \( U' = I \cap t(V) \). Let \( G' = \{ g \in GL(V) \mid gU' = U' \} \). We claim that \( G = G' \).

First, note that

\[
G = \{ g \in GL(V) \mid g\mathcal{F} = \mathcal{F} \}.
\]

In particular, \( G \subseteq G' \). On the other hand, if \( g \in G'(B) \), then by definition the induced map \((1 \otimes g) \circ \Delta : U' \to \mathcal{O}_{GL(V)} \otimes B \) factors through \( U' \otimes B \). However, since \((1 \otimes g) \circ \Delta \) is an \( \mathcal{O}_X \)-algebra map, it follows that \( \mathcal{F} \to \mathcal{O}_{GL(V)} \otimes B \) factors through \( \mathcal{F} \otimes B \). That is, \( G' \subseteq G \), thus \( G = G' \).

Let \( \mathcal{K}_X \) be the fraction field of \( \mathcal{O}_X \), and let \( U = (U' \otimes \mathcal{K}_X) \cap t'(V) \). By Lemma 3.4, \( U \) is \( G \)-stable and locally split in \( t(V) \). It is clear that we still have that \( G = \{ g \in GL(V) \mid gU = U \} \). Finally, we consider \( L = \bigwedge^d U \subset \bigwedge^d t'(V) = t(V) \). By Lemma 3.3, we have that \( G = \{ g \in GL(V) \mid gL = L \} \), as claimed.

4. FUNCTIONAL VIEWPOINT

As usual, \( G \) denotes a flat algebraic group over a Dedekind scheme \( X \). In this section, we fix a faithfully flat \( X \)-scheme \( Y \) and let \( P \) be a \( G_Y \)-torsor. We write \( \text{Rep} G \) for the category of representations of \( G \) on finite rank, locally free \( \mathcal{O}_X \)-modules. Then, \( \text{Rep} G \) is an \( \mathcal{O}_X \)-linear, rigid tensor category. Here, rigid means that \( \text{Rep} G \) has internal homs. Of course, unless \( \mathcal{O}_X \) is a field, this will not be an abelian category. Let \( \text{Bun}_Y \) denote the category of finite rank vector bundles on \( Y \) (not to be confused with \( \text{Bun}_Y \) in [2]).

**Lemma 4.1.** Denote by \( F_P : \text{Rep} G \to \text{Bun}_Y \) the functor given by \( V \mapsto P \times^G V \). Then \( F_P \) is a faithful tensor functor that preserves monomorphisms.

**Proof.** Each of the properties can be checked fppf locally, so we can assume \( P = G \times Y \) is the trivial \( G_Y \)-torsor. In that case, \( P \times^G V = Y \times V \), and the claimed properties are evident since \( Y \) is assumed to be faithfully flat over \( X \).

Thus, \( F_P \) is a faithful tensor functor that preserves monomorphisms. We now prove the converse is true. Let \( F : \text{Rep} G \to \text{Bun}_Y \) be a faithful tensor functor that preserves monomorphisms. We show that \( F \cong F_P \) for a uniquely defined \( G_Y \)-torsor \( P \). We denote by \( \text{Rep}' G \) the category of representations of \( G \) on flat quasicoherent \( \mathcal{O}_X \)-modules. Denote by \( \text{QCoh}_Y \) the category of quasicoherent \( \mathcal{O}_Y \)-modules. We use Nori’s construction in [2 II], and closely follow the presentation in *ibid*.

**Lemma 4.2.** The functor \( F \) extends uniquely to a faithful tensor functor \( F : \text{Rep}' G \to \text{QCoh}_Y \) that preserves monomorphisms. Furthermore, if \( V \neq 0 \), \( F(V) \) is faithfully flat.

**Proof.** To extend \( F \), let \( V \) be a flat, quasicoherent \( \mathcal{O}_X \)-module, and define

\[
F(V) = \lim F(W),
\]

where the direct limit is over all coherent \( G \)-stable subsheaves \( W \subset V \) (as in the proof of Lemma 3.1). We remark that this is a filtered direct limit. Since filtered
direct limits are exact and commute with tensor, $F(V)$ is flat, and the extended functor is a tensor functor that preserves monomorphisms. So, it remains to show that for $V \neq 0$, $F(V)$ is faithful.

By [5, 2.2.1], $F(V)$ is faithfully flat if and only if it is flat and has the property that $F(V) \otimes M = 0$ implies $M = 0$, for any quasicoherent $\mathcal{O}_X$-module $M$. So, let $M$ be a quasicoherent $\mathcal{O}_X$-module such that $F(V) \otimes M = 0$. Then

$$0 = M \otimes F(V) = M \otimes \lim (F(W)) = \lim (M \otimes F(W)).$$

Since $F$ preserves monomorphisms, the transition maps in the direct limit are injective. This implies that $M = 0$. \qed

If $T$ is a flat affine $X$-scheme with $G$-action, $\mathcal{O}_T$ is a quasicoherent $\mathcal{O}_X$-algebra and $\mathcal{O}_G$-comodule. In this case, since $F$ is a tensor functor, $F(\mathcal{O}_T)$ is a quasicoherent $\mathcal{O}_Y$-algebra. We are thus justified in abusing notation by writing $F(T)$ for $\text{Spec} F(\mathcal{O}_T)$. In particular, we define $\mathcal{P} = F(G) = \text{Spec} F(\mathcal{O}_G)$.

**Theorem 4.3.** Let $P = F(G)$ be defined as in the previous paragraph. Then $P$ is a $G_Y$-torsor.

**Proof.** By Lemma 4.2, $P$ is faithfully flat over $Y$. We must define the right $G$-action on $P$. Let $P_0 = F(G_0)$, where $G_0$ is the underlying $X$-scheme of $G$ with trivial (right) $G$-action. Since $\mathcal{O}_G|_0 \to \mathcal{O}_X$ has a section as an $\mathcal{O}_G$-comodule, $P_0$ is isomorphic to the trivial $G_Y$-torsor $G \times Y \to Y$. Thus, we have a $G$-action on $P$ given by the composition

$$P \times G \xrightarrow{\sim} P \times_Y (G \times Y) \to P,$$

where the second map is induced from the $\mathcal{O}_G$-comodule map $\mathcal{O}_G \to \mathcal{O}_G \otimes (\mathcal{O}_G)_0$.

Finally, since $1 \otimes \Delta : \mathcal{O}_G \otimes \mathcal{O}_G \to \mathcal{O}_G \otimes (\mathcal{O}_G)_0$ is an isomorphism, the corresponding map induced by $F$, $P \times G \to P \times_Y P$ is an isomorphism. That is, $P$ is a $G_Y$-torsor, as claimed. \qed

**Theorem 4.4.** Let $Y$ be faithfully flat scheme over $X$. The functor from the category of $G_Y$-torsors to the category of monomorphism-preserving faithful tensor functors $F : \text{Rep} G \to \text{Bun}_Y$ given by

$$P \mapsto [F_P : V \mapsto P \times^G V]$$

is an equivalence of categories. The quasi-inverse is given by $F \mapsto F(G)$.

**Proof.** We must show that the two functors are quasi-inverses. Given a $G_Y$-torsor $P$, that $F_P(G)$ is naturally isomorphic to $P$ follows directly from the definition of the fibre bundle associated to a $G$-scheme. To wit,

$$F_P(G) = P \times^G G = P \times G/\{(p, x) \sim (pg, g^{-1}x)\} = P.$$  

Let $F : \text{Rep} G \to \text{Bun}_Y$ be given. Let $P = F(G)$. We must show that $F_P$ is naturally equivalent to $F$. Let $V$ be a representation of $G$. Applying $F$ to $G \times V_0 \to V$ induces a map $P \times V \to F(V)$. We wish to show that this factors through a map $P \times^G V \to F(V)$.  

Denote by \( \alpha : G \times G_0 \times V_0 \to G \times V_0 \) the \( G \)-map \((g, h, v) \mapsto (gh, h^{-1}v)\). Then it is immediate that the following diagram commutes.

\[
\begin{array}{ccc}
G \times G_0 \times V_0 & \xrightarrow{\pi_{1,3}} & G \times V_0 \\
\alpha \downarrow & & \downarrow \rho \\
G \times V_0 & \xrightarrow{\rho} & V
\end{array}
\]

From this it follows that we have an induced map \( \phi : P \times^G V \to F(V) \), which it remains to show is an isomorphism.

Since \( P \to X \) is faithfully flat, it suffices to show that \( \phi \) is an isomorphism after pulling back to \( P \). Then, one checks from the definitions that we have the following isomorphisms:

\[
P \times V \xrightarrow{\sim} (P \times G) \times_G V \xrightarrow{\sim} (P \times_X P) \times_G V \xrightarrow{\sim} P \times_X (P \times^G V).
\]

Thus, it remains to show that the induced map \( \psi : P \times V \to P \times_X F(V) \) is an isomorphism. Following the construction, one sees that \( \psi \) arises via \( F \) from the map of \( G \)-schemes \( G \times V_0 \to G \times V \) given by \((g, v) \mapsto (g, gv)\). Since this latter map is an isomorphism, it follows that \( \psi \) is, whence the result follows. \( \square \)

5. \( G \)-TORSORS OVER A PRINCIPAL IDEAL DOMAIN

In this final section, we restrict our attention to the case where \( X = \text{Spec} \, A \) for a principal ideal domain \( A \). Let \( K \) denote the fraction field of \( A \). We assume further that \( K \) has characteristic zero. Recall that as usual, we reserve the term representation for finite free representations of \( G \). Modifying techniques from [3, 1.3.1], we have the following corollary to Theorem 1.1.

**Theorem 5.1.** Suppose that \( G \otimes K \) is reductive. Then, there is a representation \( V \) of \( G \), a tensorial construction \( t(V) \), and an element \( \omega \in t(V) \) such that \( G = \text{Aut}(V, \omega) \). Fixing such a pair \((V, \omega)\), the groupoid (i) in Theorem 1.1 is equivalent to

(ii') The groupoid of pairs \((\mathcal{E}, \nu)\) consisting of a finite rank vector bundle \( \mathcal{E} \) on \( Y \) and an element \( \nu \in t(\mathcal{E}) \) that is locally isomorphic as a pair to \((V, \omega)\).

Furthermore, if we assume that \( Y \) is faithfully flat over \( X \), then both of these groupoids are equivalent to (iii) in Theorem 1.1.

**Proof.** By Lemma 3.3 we know there is \( V \), a tensorial construction \( U = t(V) \), and a locally split line bundle \( L \subset U \) such that \( G = \text{Aut}(V, L) \). Consider \( L \otimes K \subset U \otimes K \). Since \( G \otimes K \) is reductive and \( \text{char} \, K = 0 \), \( L \otimes K \) is a direct summand of \( U \otimes K \) (as a \( G \)-representation). Thus, the dual, \((L \otimes K)^*\) can be realized as a subrepresentation of \((U \otimes K)^*\). Let \( L' = (L \otimes K)^* \cap U^* \).

By Lemma 3.4 we may assume that \( L' \subset U^* \) is a split submodule. Choose bases \( \{e_1, \ldots, e_n\} \) of \( U \) and \( \{f_1, \ldots, f_n\} \) of \( U^* \) such that \( l = e_1 \) is a generator for \( L \) and \( l' = f_1 \) is a generator for \( L' \). Let \( \omega = l \otimes l' \), a generator for \( L \otimes L' \subset W = U \otimes U^* \).

Consider the group

\[ G' = \{ g \in GL(V) \mid g \omega = \omega \}. \]

First we show that \( G' \subseteq G \). Let \( \rho : L \to U \otimes \mathcal{O}_G \) and \( \rho' : L \to U^* \otimes \mathcal{O}_G \) be the restrictions to \( L \) and \( L' \) of the comodule maps for \( U \) and \( U' \). These induce
\[ \sigma = \rho \otimes \rho' : L \otimes L' \to (U \otimes U^*) \otimes \mathcal{O}_G, \] which by assumption is the trivial representation \( \omega \rightarrow \omega \otimes 1. \) If we write \( \rho(l) = \sum e_i \otimes a_i \) and \( \rho(l') = \sum f_j \otimes b_j, \) then
\[ e_1 \otimes f_1 \otimes 1 = \omega \otimes 1 = \sigma(\omega) = \sigma(l \otimes l') = \sum e_i \otimes f_j \otimes a_i b_j. \]
Comparing coefficients, we see that \( a_1 b_1 = 1 \) and \( a_i = b_j = 0 \) for all \( i, j \neq 1. \) That is, \( L \) is a submodule of \( U, \) hence \( G' \subseteq G. \)

Finally, to show the equality \( G = G', \) first note that by [3, 1.3.1], \( G \otimes K = G' \otimes K. \) That is, we have an inclusion \( G' \subseteq G \) that becomes an equality on the generic fiber. Since \( G \) is flat, this then implies that \( G' = G, \) as claimed.

References

[1] Schémas en groupes I-III, Lecture Notes in Mathematics, vol. 151–153, Springer-Verlag, 1970.
[2] Armand Borel, Linear algebraic groups, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, 1991.
[3] Pierre Deligne and J.S. Milne, Tannakian categories, Hodge cycles, motives, and Shimura varieties, 1982.
[4] Alexander Grothendieck, Éléments de géométrie algébrique. I. Le langage des schémas, Inst. Hautes Études Sci. Publ. Math. 4 (1960).
[5] ———, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, Inst. Hautes Études Sci. Publ. Math. 24 (1965).
[6] Max Lieblich, Remarks on the stack of coherent algebras, International Mathematics Research Notices (2006).
[7] Madhav Nori, On the representation of the fundamental group, Compositio Mathematica 33 (1976), no. 1, 29–41.
[8] Martin Olsson, Hom-stacks and restriction of scalars, Duke Mathematical Journal 134 (2006), no. 1.
[9] Michel Raynaud and Laurent Gruson, Critères de platitude et projectivité: Techniques de “platification” d’un module, Inventiones Mathematicae 13 (1971), 1–89.
[10] Neantro Saavedra Rivano, Catégories Tannakiennes, Lecture Notes in Mathematics, vol. 265, Springer-Verlag, 1972.
[11] Serre, Jean-Pierre, Groupes de Grothendieck des schémas en groupes réductifs déployés, Institut des Hautes Études Scientifiques. Publications Mathématiques 34 (1968), 37–52.
[12] Christoph Sorger, Lectures on moduli of principal G-bundles over algebraic curves, School on Algebraic Geometry (Trieste, 1999), 2000, pp. 1–57.
[13] William C. Waterhouse, Introduction to affine group schemes, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, 1979.
[14] Torsten Wedhorn, On Tannakian duality over valuation rings, Journal of Algebra 282 (2004), no. 2, 575–609.

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