A REPRESENTATION THEORY APPROACH TO INTEGRAL MOMENTS OF $L$-FUNCTIONS OVER FUNCTION FIELDS

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ABSTRACT. We propose a new heuristic approach to integral moments of $L$-functions over function fields, which we demonstrate in the case of Dirichlet characters ramified at one place (the function field analogue of the moments of the Riemann zeta function, where we think of the character $n^{th}$ as ramified at the infinite place). We represent the moment as a sum of traces of Frobenius on cohomology groups associated to irreducible representations. Conditional on a hypothesis on the vanishing of some of these cohomology groups, we calculate the moments of the $L$-function and they match the predictions of the CFKRS recipe [4].

In this case, the decomposition into irreducible representations seems to separate the main term and error term, which are mixed together in the long sums obtained from the approximate functional equation, even when it is dyadically decomposed. This makes our heuristic statement relatively simple, once the geometric background is set up. We hope that this will clarify the situation in more difficult cases like the $L$-functions of quadratic Dirichlet characters to squarefree modulus. There is also some hope for a geometric proof of this cohomological hypothesis, which would resolve the moment problem for these $L$-functions in the large degree limit over function fields.

1. INTRODUCTION

The Conrey-Farmer-Keating-Rubinstein-Snaith heuristics give precise conjectures for the distribution of special values of $L$-functions in certain families [4]. They were extended to function fields in [2]. Certain constants appearing in these predictions can be related to statistics of random matrices.

While these are conjectures in general, they are known for many families up to an error term of $O(1/\sqrt{q})$ in the function field setting (e.g. [21], [18], [19]). This error term hides everything but the random matrix term. However, the random matrix term appears in a particularly natural way. In the function field setting, the $L$-functions are equal to characteristic polynomials of the matrices giving the action of Frobenius elements on a certain Galois representation, and these matrices are random in a precise technical sense [8, Theorem 3.5.3].

We are not able today to remedy this and prove the full conjecture of [4] over function fields for any family of $L$-functions. However, we propose a middle ground. Using the machinery of étale cohomology, and in particular the interpretation of $L$-functions via representations of monodromy groups, we will describe a new heuristic which matches the predictions of [4]. However, while the heuristics of [4] require multiple manipulations, that do not make sense on their own, we will make a single assumption on vanishing of

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cohomology groups, which could well be true. This assumption also makes predictions for other problems, such as the variance of the divisor function in short intervals.

In this paper, we describe this heuristic, and verify its relationship to \[4\], only for the “short interval” family of characters:

**Definition 1.1.** For \(n\) a natural number and \(\mathbb{F}_q\) a finite field, let \(S_{n,q}\) be the set of all primitive even Dirichlet characters \(\mathbb{F}_q[x]/x^{n+1} \to \mathbb{C}^\times\), which has cardinality \(q^n - q^{n-1}\). View elements of \(S_{n,q}\) as characters of monic polynomials in \(\mathbb{F}_q[T]\) by sending a monic \(f\) of degree \(d\) to \(\chi(f(x^{-1}))x^{\deg f}\), i.e. as characters depending on the \(n + 1\) leading terms. Form the associated \(L\)-functions

\[
L(s, \chi) = \sum_{\chi(f) \in S_{n,q}} \chi(f)|f|^{-s}
\]

where \(|f| = q^{\deg f}\).

**Hypothesis 1.2.** Let \(n, r, s, c\) be natural numbers with \(0 \leq c \leq n\). Let \(\text{Prim}_n \cong \mathbb{A}^n - \mathbb{A}^{n-1}\) be the moduli space of primitive Dirichlet characters defined by Katz \[18, \S 4\], and let \(L_{\text{univ}}\) be the lisse sheaf of rank \(n - 1\) on \(\text{Prim}_n\) defined by Katz \[18, \S 4\].

Let \(\mathcal{F}\) be an irreducible lisse \(\mathbb{Q}_l\)-sheaf on \(\text{Prim}_{n,q}\) that appears as a summand of

\[
\det(L_{\text{univ}})^{-s} \otimes \bigotimes_{i=1}^{r+s} \wedge^d_i(L_{\text{univ}})
\]

for some \(0 \leq d_1, \ldots, d_{r+s} \leq n - 1\), but which does not appear as a summand of \(L_{\text{univ}}^\otimes a \otimes L_{\text{univ}}^\otimes b\) for \(0 \leq a, b \leq n - 1\).

Then

\[
H^j_c(\text{Prim}_n, \mathcal{F}) = 0
\]

for \(j > n + c\).

**Theorem 1.3.** Let \(n, r, s, c\) be natural numbers and \(\mathbb{F}_q\) a finite field. Assume that Hypothesis \[L\] is satisfied for \(n, r, s, c\). Assume also that \(n > 2 \max(r, s) + 1\) and if \(n = 4\) or \(5\) that the characteristic of \(\mathbb{F}_q\) is not \(2\). Let \(C_{r, s} = (2 + \max(r, s))^{\max(r, s) + 1}\). Let \(\alpha_1, \ldots, \alpha_{r+s}\) be imaginary numbers. Let \(\epsilon_\chi\) be the \(\epsilon\)-factor of \(L (\chi)\). Then

\[
\frac{1}{(q^n - q^{n-1})} \sum_{\chi \in S_{n,q}} \epsilon_\chi^{-s} \prod_{i=1}^{r+s} L(1/2 - \alpha_i, \chi)
\]

\[
= \sum_{S \subseteq \{1, \ldots, r+s\}} \prod_{i \notin S} q^{\alpha_i(n-1)} \prod_{i \in S} \prod_{f_i \in \mathbb{F}_q[T]} |f_i|^{-\frac{1}{2} + \alpha_i} \prod_{i \notin S} |f_i|^{-\frac{1}{2} - \alpha_i} + O \left( \frac{q^{\epsilon n}}{2} C_{r, s}^{n} n^{r+s} \right)
\]

Here the implicit constant depends only on \(r, s\) and not on \(n, q, c\), and the term for each \(S\) on the right-hand side is interpreted as a meromorphic function analytically continued from its zone of absolute convergence.

The right side in Theorem \[1.3\] is indeed the prediction of the CFKRS recipe \[4\] for this family.
 Remark 1.4. We can write the error term in Theorem 1.3 as
\[ O\left(q^n - q^{n-1} \sum_{\chi \in S_{n,q}} \epsilon_\chi^{-1} \prod_{i=1}^{r+1} L(1/2 - \alpha_i, \chi) \right). \]

The error term predicted by [4] is always the size of the family raised to the power \(-1/2 + \epsilon \). Our exponent approaches the predicted square-root cancellation as long as \(c = n + 1 - \frac{p-2r}{pr} n \). This gives the following unconditional estimate:

**Corollary 1.5.** Let \(n, r \) be natural numbers and \(\mathbb{F}_q \) a finite field of characteristic \(p\). Assume also that \(n > 2r + 1\) and if \(n = 4\) or \(5\) that the characteristic of \(\mathbb{F}_q \) is not \(2\). Let \(C_{r,1} = (2 + r)^{r+1}\). Let \(\alpha_1, \ldots, \alpha_{r+1}\) be imaginary numbers. Let \(\epsilon_\chi \) be the \(\epsilon\)-factor of \(L(\chi)\). Then

\[ \frac{1}{(q^n - q^{n-1})} \sum_{\chi \in S_{n,q}} \epsilon_\chi^{-1} \prod_{i=1}^{r+1} L(1/2 - \alpha_i, \chi) \]

\[ = \sum_{j=1}^{r+1} q^{(r+1)(n-1)} \sum_{f_1, \ldots, f_{r+1} \in \mathbb{F}_q[T]} \prod_{i \neq j} \frac{f_i}{m_{f_i}} \frac{\prod_{i \neq j} |f_i|^{-1/2 + \alpha_i}|f_j|^{-1/2 - \alpha_i}}{\prod_{i \neq j, f_i/f_j \in \mathbb{T}^2}} + O\left(\sqrt{q} \left(q^{\frac{2r-2n}{pr}} C_{r,1} \right)^n n^{r+1}\right). \]

As \(p\) goes to \(\infty\) with fixed \(r\), this bound converges to a power savings of \(1/2r\).

The key idea of the proof is that if we multiply the \(L\)-function moment on the left side by the Vandermonde determinant that is needed to regularize the individual terms in the estimate on the right side, then the coefficients of monomials in \(q^{\alpha_1}, \ldots, q^{\alpha_{r+s}}\) will be averages over \(\chi\) of Schur functions in the zeroes of \(L(s, \chi)\) corresponding to irreducible representations of \(GL_{n-1}\) (Lemma 2.4). Those irreducible representations which appear as a summand of \(L_{\text{univ}}^a \otimes L_{\text{univ}}^b\) for \(0 \leq a, b \leq n-1\) are exactly those that appear in the region of the space of possible monomials (i.e. the Fourier dual space to the space of \(\alpha_1, \ldots, \alpha_{r+s}\)) where the off-diagonal terms cancel (Lemma 3.3). These coefficients will match the coefficients of one of the main terms on the right side. Because we have multiplied by the Vandermonde, the other terms will be small (Lemma 3.4). Hence to make the identity valid it is sufficient to show that the average over \(\chi\) of Schur functions in the zeroes of \(L(s, \chi)\) corresponding to irreducible representations of \(GL_{n-1}\) that do not appear as a summand of \(L_{\text{univ}}^a \otimes L_{\text{univ}}^b\) for \(0 \leq a, b \leq n-1\) is small, which is exactly what is provided by the Grothendieck-Lefschetz fixed point formula, our assumption on vanishing of cohomology, and some Betti number estimates (Lemma 2.9).

**Remark 1.6.** We present some remarks on the hypothesis.

1. As part of our proof, we will implicitly calculate the trace of Frobenius on the cohomology of sheaves \(\mathcal{F}\) which do appear as a summand of \(L_{\text{univ}}^a \otimes L_{\text{univ}}^b\). So our hypothesis is a version of the usual heuristic that what we cannot calculate
should cancel. Of course, such a heuristic may be overly optimistic. Instead, what is interesting here is that it is a very straightforward and geometrically natural heuristic.

(2) The calculations of the traces for the sheaves which do appear as a summand are closely related to Katz’s calculations in \[19\] §5. (In the case \(N = n < p\), the sheaf \(\mathcal{F}\) defined in \[19\] §4 is the restriction of \(L_{\text{univ}}\) to a hyperplane section, and essentially the same calculations as in \[19\] §5 can be done in this setting.) So the failure of square-root cancellation he observes does not cause a problem for us, as it occurs exactly in the cases where we do not assume square-root cancellation. In fact, we show that the non-square-root terms that he observes correspond exactly to the secondary terms predicted by \[4\].

(3) Hypothesis \([2]\) is known for \(c = n - 1\) by Poincaré duality in étale cohomology. This gives a bound in the \(q\) aspect whose error term is \(O(q^{n-\frac{1}{2}})\). This implies that the main term of Theorem \([3]\) must match, to within \(O(q^{n-\frac{1}{2}})\), the main term obtained by applying Katz’s equidistribution result \([18\) Theorem 1.2] and performing a matrix integral. Our method in this case is simply a (more complicated) variant of the proof of Deligne’s equidistribution theorem \([8\) Theorem 3.5.3], which Katz uses in his proof, combined with the calculation of the matrix integral.

(4) It is possible that some very strong form of Hypothesis \([2]\). For instance, we do not know whether it is true for \(c = 1\) for all \(n, r, s\). However, it is likely to be easier to prove weaker special cases first, which is why we have stated it flexibly using multiple parameters.

**Remark 1.7.** We present some remarks on possible generalizations. We first discuss families that are harmonic in the sense of \([23]\), and then geometric families.

(1) We expect that these results can be generalized to at least some families with orthogonal and symplectic symmetry type. The simplest cases for our method are probably the families of Dirichlet characters studied by Katz in \([20]\), where both orthogonal and symplectic examples are given. One simply replaces the Vandermonde determinant with, for the \(r\)th moment in the orthogonal case,

\[
\prod_{1 \leq i_1 < i_2 \leq r} (q^{\alpha_{i_1} - q^{\alpha_{i_2}}})(q^{\alpha_{i_1}^2} - 1)
\]

or, for the \(r\)th moment in the symplectic case,

\[
\prod_{1 \leq i \leq r} (q^{2\alpha_i} - 1) \prod_{1 \leq i_1 < i_2 \leq r} (q^{\alpha_{i_1}^2} - q^{\alpha_{i_2}^2})(q^{\alpha_{i_1}} - q^{\alpha_{i_2}})(q^{\alpha_{i_1}^2} - 1).
\]

The hypothesis needed then has to do with the cohomology of sheaves generated from the universal sheaves constructed by Katz in that paper.

(2) Similar results can be proven for moments of an \(L\)-function of a fixed Galois representation twisted by a varying Dirichlet character, again conditional on a cohomological hypothesis. However, the dependency on \(n\) in the error term may be worse or even ineffective, as Betti number bounds are more difficult in this case. If the Galois representation is an Artin representation splitting over the function
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field of a curve of bounded degree and genus, it should be possible to make the dependence on \( n \) an effective exponential.

(3) For other harmonic families of Dirichlet characters, such as those of squarefree modulus, stating properly an analogous hypothesis seems to require the use of higher-dimensional sheaf convolution Tannakian categories, which have not yet been connected to equidistribution. If that geometric setup is handled, there should not be any major new difficulties. (In the prime modulus case, the fourth absolute moment was studied in [25]. Our method does not immediately imply anything about this moment, though possibly it could with more geometric work.)

(4) For families of automorphic forms on higher-rank groups, the \( q \to \infty \) equidistribution theory is not yet available, which is a precondition for our method.

(5) New difficulties present themselves in the family of all quadratic Dirichlet characters with squarefree moduli of a given degree. This family has attracted the most attention in the function field setting, beginning with [15] and [1] on the first moment. Recently, improved bounds for the first four moments were obtained in [11, 12, 13]. Improved bounds on the third moment were obtained in [9], demonstrating the existence of a secondary term and thereby verifying a prediction from [10].

The difficulties in applying our method to this case start with the fact that there is no range of short sums where the off-diagonal terms cancel completely. Thus, there is no set of irreducible representations close to the trivial representation in highest weight space whose contributions can be exactly computed. Furthermore, the existence of a secondary term in the cubic case suggests that even for representations very far from the trivial representation in highest weight space, the contribution does not necessarily exhibit square-root cancellation and the term does not vanish above the middle degree. However, neither of these difficulties seems insurmountable, and it is possible that the representation-theoretical and cohomological approach can separate the main term from the secondary terms and shed light, if only conjecturally, on each.

(6) For general geometric families, the situation is likely similar to, but more complicated than, the situation for quadratic Dirichlet characters.

\textit{Remark 1.8.} To obtain predictions for moments, instead of Hypothesis 1.2, we could make a purely analytic conjecture of square-root cancellation in the trace of the cohomology (equivalently, the sum of the Schur polynomial associated to this representation, evaluated at the roots of the \( L \)-function, over all primitive Dirichlet characters) for representations outside this special set.

Such a hypothesis is essentially equivalent to a uniform version of the conjecture of [4] for shifted moments, as we can extract these individual coefficients by a Fourier series after multiplying by the Vandermonde determinant. However, Hypothesis 1.2 would not follow directly from this unless the cohomology groups were proven to be pure.

If made uniform in \( r, s \), such a hypothesis would imply conjectures for ratios and tuple correlations - presumably matching the predictions of [5], and therefore [6]. On the other hand, while Hypothesis 1.2 can be stated uniformly in \( r, s \), it would not imply a good
estimate on the error term in the degree aspect unless stronger Betti number bounds were proven.

Despite these difficulties, we have stated Hypothesis \[\text{Hypothesis 1.2}\] in a geometric way to motivate it as a natural statement (we would not have come up with it if it weren’t for geometry) and to suggest the potential of a geometric proof.

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2. Representation theory and algebraic geometry

For any \(d \geq 0\), define

\[
\lambda_d(\chi) = q^{-d/2} \sum_{f \text{ monic degree } d} \chi(f),
\]

so that \(L(s, \chi) = \sum_{d=0}^{n-1} \lambda_d(\chi)q^{d(1/2-s)}\). Let \(\epsilon_\chi = \lambda_{n-1}\) be the \(\epsilon\)-factor of \(L(s, \chi)\), so that \(\lambda_{n-1-d}(\chi) = \epsilon_\chi \lambda_d(\chi)\). By the Riemann hypothesis or more directly from the explicit formula for \(\epsilon_\chi\) in terms of Gauss sums, we have \(|\epsilon_\chi| = 1\) for all \(\chi\).

Let \(m\) be the order of geometric monodromy group of the determinant of \(L_{\text{univ}}\).

Let \(\mu\) be the (unique) eigenvalue of \(\text{Frob}_q\) on the \(m\)th power of the determinant of \(L_{\text{univ}}(1/2)\).

Let \(R(\text{GL}_{n-1})\) be the representation ring of \(\text{GL}_{n-1}\) over \(\mathbb{Z}\).

We fix an embedding \(\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}\). Let \(F\) be the unique additive group homomorphism: \(R(\text{GL}_{n-1}) \rightarrow \mathbb{C}\) whose value on a representation \(V\) is

\[
F(V) = \sum_{j \in \mathbb{Z}} (-1)^j \left( \text{tr}(\text{Frob}_q, H^j_c(\text{Prim}_{n, \mathbb{Q}}, V(L_{\text{univ}}(1/2)))) \right).
\]

Lemma 2.1. For any \(r, s, d = (d_1, \ldots, d_{r+s})\),

\[
\sum_{\chi} \epsilon_\chi^{-s} \prod_{i=1}^{r+s} \lambda_{d_i}(\chi) = (-1)^{\sum_{i=1}^{r+s} d_i} \left( \det^{-s} \bigotimes_{i=1}^{r+s} \bigwedge^{d_i} \right).
\]

Furthermore, \(\epsilon_\chi^m = (-1)^m(n-1)^\mu\) for all primitive \(\chi\).

Proof. First we observe that, by the Grothendieck-Lefschetz fixed point formula [7 Sommes trig. (1.1.1)]

\[
F(V) = \sum_{x \in \text{Prim}_n(\mathbb{F}_q)} \text{tr}(\text{Frob}_q, x, V(L_{\text{univ}}(1/2))).
\]

By construction, \(\text{Prim}_n(\mathbb{F}_q)\) is in bijection with the set of primitive Dirichlet characters \(\chi\). It is therefore sufficient to prove that

\[
\text{tr}(\text{Frob}_q, x, \wedge^d(V(L_{\text{univ}}(1/2)))) = (-1)^d \lambda_d.
\]

This follows from the relationship between the characteristic polynomial of Frobenius at \(x\) and \(L(s, \chi)\), proved by Katz [18 Lemma 4.1].
A special case is that \( \epsilon_\chi = \lambda_{n-1}(\chi) = (-1)^n \text{tr}(\text{Frob}_{q,x}, \det(V(L_{\text{univ}}(1/2)))) \). The determinant is a lisse sheaf of rank one, with a single Frobenius eigenvalue. By assumption, its \( m \)th power is a constant sheaf with Frobenius eigenvalue \( \mu \). Hence its Frobenius eigenvalue must be an \( m \)th root of \( \mu \), proving the last claim.

For \( 0 \leq d_1 \leq \cdots \leq d_{r+s} \leq n-1 \), let \( V_{d_1,\ldots,d_r,d_{r+1},\ldots,d_{r+s}} \) be the representation of \( GL_n \) whose highest weight character is \( \lambda_1 \cdots \lambda_{d_1} \lambda_{d_1+1}^{-1} \cdots \lambda_{d_r} \lambda_{d_r+1}^{-1} \cdots \lambda_{d_{r+s}}^{-1} \).

**Lemma 2.2.** In the ring \( R(GL_{d-1})[q^{\alpha_1}, \ldots, q^{\alpha_{r+s}}] \) with formal variables \( q^{\alpha_1}, \ldots, q^{\alpha_{r+s}} \),
\[
\left( \sum_{\sigma \in S_{r+s}} \text{sgn}(\sigma) \prod_{i=1}^{r+s} q^{(\sigma(i)-1)\alpha_i} \right) \sum_{0 \leq d_1,\ldots,d_{r+s} \leq n-1} q^{\sum_{i=1}^{r+s} d_i \alpha_i} (-1)^{\sum_{i=1}^{r+s} d_i + \left(\frac{r+s}{2}\right)} \text{det}^{-s} \bigotimes_{i=1}^{r+s} \bigwedge^{d_i}
\]
\[
= \sum_{\sigma \in S_{r+s}} \sum_{0 \leq d_1,\ldots,d_{r+s} \leq n-1} \text{sgn}(\sigma) q^{\sum_{i=1}^{r+s} (d_i + \frac{r+s}{2}) \alpha_{\sigma(i)}} (-1)^{\sum_{i=1}^{r+s} d_i V_{d_1,\ldots,d_r,d_{r+1},\ldots,d_{r+s}}}. 
\]

**Proof.** To check this, observe that both sides are antisymmetric in \( \alpha_1, \ldots, \alpha_{r+s} \). Hence it is sufficient to check that the coefficients of \( q^{\sum_{i=1}^{r+s} (d_i + \frac{r+s}{2}) \alpha_{\sigma(i)}} \) on both sides agree for \( d_1 \leq d_2 \leq \ldots \leq d_{r+s} \). On the right side, only the trivial permutation contributes, so this coefficient is \( (-1)^{\sum_{i=1}^{r+s} d_i + \left(\frac{r+s}{2}\right)} V_{d_1,\ldots,d_r,d_{r+1},\ldots,d_{r+s}} \).

On the left side, every permutation \( \sigma \) contributes the amount
\[
\text{sgn}(\sigma) (-1)^{\sum_{i=1}^{r+s} d_i + \left(\frac{r+s}{2}\right)} \text{det}^{-s} \bigotimes_{i=1}^{r+s} \bigwedge^{d_i+\frac{r+s}{2}} \bigwedge^{d_i+\frac{r+s}{2}}
\]
so it suffices to check that
\[
V_{d_1,\ldots,d_r,d_{r+1},\ldots,d_{r+s}} = \left( \sum_{\sigma} \text{sgn}(\sigma) \text{det}^{-s} \bigotimes_{i=1}^{r+s} \bigwedge^{d_i+\frac{r+s}{2}} \bigwedge^{d_i+\frac{r+s}{2}} \right).
\]

Here we interpret wedge powers as vanishing if the power does not lie between 0 and \( n-1 \).

As \( V_{d_1,\ldots,d_r,d_{r+1},\ldots,d_{r+s}} = \text{det}^{-s} \otimes V_{d_1,\ldots,d_{r+s}} \), it suffices to check that
\[
V_{d_1,\ldots,d_r,d_{r+1},\ldots,d_{r+s}} = \sum_{\sigma} \text{sgn}(\sigma) \text{det}^{-s} \bigotimes_{i=1}^{r+s} \bigwedge^{d_i+\frac{r+s}{2}} \bigwedge^{d_i+\frac{r+s}{2}}.
\]

Observe that
\[
\sum_{\sigma} \text{sgn}(\sigma) \bigotimes_{i=1}^{r+s} \bigwedge^{d_i+\frac{r+s}{2}} \bigwedge^{d_i+\frac{r+s}{2}}
\]
is the determinant of an \( (r+s) \times (r+s) \) matrix whose \( i,j \) entry is \( \bigwedge^{d_i+i-j} \). By the second Jacobi-Trudi identity for Schur functions [16, Formula A6], this determinant is equal to \( V_{d_1,\ldots,d_{r+s}} \) in the representation ring of \( GL_{n-1} \), so the product with \( \text{det}^{-s} \) is equal to \( V_{d_1,\ldots,d_r,d_{r+1},\ldots,d_{r+s}} \). \( \square \)
Lemma 2.3. For $0 \leq d_1 \leq \ldots \leq d_{r+s} \leq n-1$, $V_{d_1,\ldots,d_k|d_{k+1},\ldots,d_{r+s}}$ appears as a summand of the $n$th tensor power of the standard representation of $GL_{n-1}$ with the $b$th tensor power of its dual with $0 \leq a, b \leq n-1$ if and only if $\sum_{i=1}^{k} d_i \leq n-1$ and $\sum_{i=k+1}^{r+s} n-1-d_i \leq n-1$

Proof. For the only if direction, consider the element of $GL_{n-1}$ depending on a parameter $\lambda$ whose eigenvalues are $\lambda$ with multiplicity $d_k$ and 1 with multiplicity $n-1-d_k$. Its eigenvalue on the highest weight vector of $V_{d_1,\ldots,d_k|d_{k+1},\ldots,d_{r+s}}$ is $\lambda \sum_{i=1}^{k} d_i$. On the other hand, its eigenvalues on $\text{std}^{\otimes a} \otimes \text{std}^{\otimes b}$ are at most $\lambda^a$. Similarly, the element whose eigenvalues are 1 with multiplicity $n-1-d_{k+1}$ and $\lambda^{-1}$ with multiplicity $d_k$ acts on the highest weight vector of $V_{d_1,\ldots,d_k|d_{k+1},\ldots,d_{r+s}}$ with eigenvalue $\lambda \sum_{i=k+1}^{r+s} n-1-d_i$, but its eigenvalues on its eigenvalues on $\text{std}^{\otimes a} \otimes \text{std}^{\otimes b}$ are at most $\lambda^b$.

For the if direction, we observe that the highest weight vector of 

$\left( \bigotimes_{i=1}^{k} \lambda^{d_i} \text{std} \right) \otimes \left( \bigotimes_{i=k+1}^{r+s} \lambda^{n-1-d_i} \text{std}^\vee \right)$

matches the highest weight of $V_{d_1,\ldots,d_k|d_{k+1},\ldots,d_{r+s}}$ and that 

$\left( \bigotimes_{i=1}^{k} \lambda^{d_i} \text{std} \right) \otimes \left( \bigotimes_{i=k+1}^{r+s} \lambda^{n-1-d_i} \text{std}^\vee \right)$

is a summand of $\text{std}^{\otimes a} \otimes \text{std}^{\otimes b}$. \hfill \Box

Lemma 2.4. Assume $n \geq 3$, and, if $n = 3$, that the characteristic of $\mathbb{F}_q$ is not 2 or 5.

For $0 \leq d_1 \leq \ldots \leq d_{r+s} \leq n-1$, the geometric monodromy of the sheaf $V_{d_1,\ldots,d_r|d_{r+1},\ldots,d_{r+s}}(L_{\text{univ}}(1/2))$ appears as a summand of $L^{\otimes a}_{\text{univ}} \otimes L^{\otimes b}_{\text{univ}}$ for $0 \leq a, b \leq n-1$ if and only if there is some $k$ such that $\sum_{i=1}^{k} d_i \leq n-1$, $\sum_{i=k+1}^{r+s} (n-1-d_i) \leq n-1$, and $k \equiv r \mod m$.

Proof. By [18, Theorem 7.1], under these assumptions on $n$, the monodromy group of $L_{\text{univ}}$ contains $SL_{n-1}$. Thus two irreducible representations of $GL_{n-1}$ give isomorphic sheaves when composed with $L_{\text{univ}}$ if and only if one is equal to the other twisted by an integer power of the determinant, where the integer is a multiple of $m$. The claim then follows from Lemma 2.3. \hfill \Box

Lemma 2.5. We have the identity

$\prod_{1 \leq i < j \leq r+s} (q^{\alpha_{i1}} - q^{\alpha_{i2}}) \sum_{\chi} \epsilon_{\chi}^{-s} \prod_{i=1}^{r+s} L(1/2 - \alpha_i, \chi)$

$= \sum_{\sigma \in S_{r+s}} \sum_{0 \leq d_1 \leq \ldots \leq d_{r+s} \leq n-1} \text{sgn}(\sigma) q^{\sum_{i=1}^{r+s} (d_i + i - 1)\alpha_{\sigma(i)} (1) \sum_{i=1}^{r+s} d_i + \frac{r+s}{2}} F (V_{d_1,\ldots,d_r|d_{r+1},\ldots,d_{r+s}})$

Proof. We have

$\sum_{\chi} \epsilon_{\chi}^{-s} \prod_{i=1}^{r+s} L(1/2 - \alpha_i, \chi) = \sum_{\chi} \sum_{0 \leq d_1,\ldots,d_{r+s} \leq n-1} q^{d_i \alpha_i (1) \sum_{i=1}^{r+s} d_i + \frac{r+s}{2}} \epsilon_{\chi}^{-s} \prod_{i=1}^{r+s} \lambda_{d_i}(\chi)$.

Applying Lemma 2.1 this is
Now multiply by the Vandermonde factor $\prod_{1 \leq i_1 < i_2 \leq r+s}(q^{a_{i_1}}-q^{a_{i_2}}) = \sum_{\sigma \in S_{2k}} \prod_{i=1}^{r+s} q^{a_i(\sigma(i)-1)}$ and apply Lemma 2.2.

**Lemma 2.6.** The natural number $m$ is divisible by the largest power of the characteristic of $\mathbb{F}_q$ that is greater than or equal to $\frac{n-1}{2}$.

**Proof.** Because $\mu = \epsilon^m_\chi$ for all $\chi$ and $|\epsilon_\chi| = 1$, we have $|\mu| = 1$.

Let $\chi$ be a primitive character of $\mathbb{F}_q[x]/x^{n+1}$ trivial on $\mathbb{F}_q^x$ and nontrivial on $1 + \mathbb{F}_q x^n$. There is a unique nontrivial character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ with $\chi(1 + ax^n) = \psi(-a)$ for all $a \in \mathbb{F}_q$. For this $\psi$,

$$\epsilon_\chi = q^{-\frac{n-1}{2}} \sum_{a_1, \ldots, a_{n-1} \in \mathbb{F}_q} \chi(1 + \sum_i a_i x^i) = q^{-\frac{n-1}{2}} \sum_{a_1, \ldots, a_{n} \in \mathbb{F}_q} \chi(1 + \sum_i a_i x^i) \psi(a_n).$$

For all other $\psi$, this sum vanishes. There are $q^{n-1}$ primitive characters matching each nontrivial $\psi$, so

$$\sum_{\chi} \left( q^{-\frac{n-1}{2}} \sum_{a_1, \ldots, a_{n} \in \mathbb{F}_q} \chi(1 + \sum_i a_i x^i) \psi(a_n) \right)^m = \sum_{\chi, \chi(1+ax^n)=-\psi(a)} \epsilon^m_\chi = \sum_{\chi, \chi(1+ax^n)=-\psi(a)} \mu = q^{n-1} \mu$$

which has absolute value $q^{n-1}$ as $|\mu| = 1$. Here we can sum over all $\chi$ and not just primitive ones as $\sum_{a_n \in \mathbb{F}_q} \chi(1 + \sum_i a_i x^i) \psi(a_n) = 0$ for imprimitive $\chi$. This means we have

$$\sum_{a_{i,j} \in \mathbb{F}_q, i=1, \ldots, n, j=1, \ldots, m} \psi \left( \sum_{j=1}^m a_{m,j} \right) = q^{\frac{m(n+1)}{2}-1}.$$

The left side is a Kloosterman-type sum. By standard stationary phase analysis, inductively for $i$ from 1 to $\left\lceil \frac{n-1}{2} \right\rceil$, the sum over $a_{n-i,j}$ vanishes unless $a_{i,j_1} = a_{i,j_2}$ for all $j_1, j_2$.

If $n$ is odd, after restricting to this subset the number of terms remaining on the left side is $q^{(m-1)\left\lceil \frac{n+1}{2} \right\rceil}$ times the number of $a_1, \ldots, a_{\frac{n-1}{2}}$ satisfying $(1 + \sum_{i=1}^{\frac{n+1}{2}} a_i x^i)^m \equiv 1 \mod x^{\frac{n+1}{2}+1}$. The only way the total size is $q^{m(\frac{n-1}{4})-1}$ is if the number of such $a_1, \ldots, a_{\frac{n-1}{2}}$ is $q^{\frac{n-1}{2}}$, which only happens if the largest power of the characteristic dividing $m$ is at least $\frac{n-1}{2}$.

For $n$ even, we observe that when $a_{1,j_1}, \ldots, a_{\frac{n}{2}-1,j_2}$ for all $i \leq \frac{n}{2} - 1$ and all $j_1, j_2$, then the sum over $a_{\frac{n}{2}+1}, \ldots, a_{\frac{n}{2},m}$ is a quadratic Gauss sum in $m - 1$ variables, which is nondegenerate unless $p|m$, in which case it has one-dimensional degeneracy locus. This means the left side is at most $q^{(m-1)\frac{n+1}{2}+\frac{1}{2}}$ times the number of $a_1, \ldots, a_{\frac{n-1}{2}}$ satisfying $(1 + \sum_{i=1}^{\frac{n}{2}-1} a_i x^i)^m \equiv 1 \mod x^{\frac{n}{2}}$. The only way this can be is at most $q^{m(\frac{n+1}{2})-1}$ if the
number of such $a_1, \ldots, a_{n-1}$ is at least $q^{\frac{m-1}{2}}$, which only happens if the largest power of the characteristic dividing $m$ is at least $\frac{m-1}{2}$.

Let the scheme $\text{Witt}_n$ be $\prod_{m \geq 1 \text{ prime to } p, m \leq n} W_{1}(m,n)$ in the notation of [13, §4]. This is a product of commutative unipotent group schemes and hence is a commutative unipotent group scheme itself, isomorphic to $\mathbb{A}^n$. Each $\mathbb{F}_q$-point corresponds to an even Dirichlet character $\mathbb{F}_q[x]/x^{n+1} \to \mathbb{C}^\times$, and the subscheme $\text{Witt}_{n-1}$ corresponds to the set of imprimitive characters, so $\text{Prim}_n = \text{Witt}_n \setminus \text{Witt}_{n-1}$.

Katz constructs $\mathcal{L}_{\text{univ}}$ as $R^1pr_{2!}\mathcal{L}_{\text{univ}}$ for a certain lisse rank one sheaf $\mathcal{L}_{\text{univ}}$ on $\mathbb{A}^1 \times \text{Prim}_n$, with $pr_2$ the projection onto $\text{Prim}_n$ [13, §4]. However, the same definition constructs a lisse rank one sheaf on $\mathbb{A}^1 \times \text{Witt}_n$. We will also refer to this sheaf as $\mathcal{L}_{\text{univ}}$ and the projection map $\mathbb{A}^1 \times \text{Witt}_n \to \text{Witt}_n$ as $pr_2$.

For natural numbers $m_1, m_2$, let $Z_{n,m_1,m_2}$ be the subspace of $\mathbb{A}^{m_1} \times \mathbb{A}^{m_2}$ consisting of points $(a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})$ such that $\prod_{i=1}^{m_1}(1-a_i x) \equiv \prod_{i=1}^{m_2}(1-b_i x) \bmod x^{n+1}$.

**Lemma 2.7.** There is an $S_{m_1} \times S_{m_2}$-equivariant isomorphism

$$H^j_c \left( \text{Witt}_n, \mathbb{F}_q \right) (Rpr_{2!}\mathcal{L}_{\text{univ}})^{\otimes m_1} \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^{\otimes m_2} = H^j_c(Z_{n,m_1,m_2}, \mathbb{F}_q, \mathbb{Q}_\ell(-n)).$$

**Proof.** By applying the Künneth formula [3, Exposé XVII, Thm. 5.4.3],

$$H^j_c \left( \text{Witt}_n, \mathbb{F}_q \right) (Rpr_{2!}\mathcal{L}_{\text{univ}})^{\otimes m_1} \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^{\otimes m_2} = H^j_c(\mathbb{A}^{m_1} \times \mathbb{A}^{m_2} \times \text{Witt}_n, \mathbb{F}_q, \mathbb{Q}_\ell(-n))$$

where $(a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})$ are coordinates on $\mathbb{A}^{m_1} \times \mathbb{A}^{m_2}$ and $\omega$ is a coordinate on $\text{Witt}_n$.

Let $pr_1$ be the projection $\mathbb{A}^{m_1} \times \mathbb{A}^{m_2} \times \text{Witt}_n \to \mathbb{A}^{m_1} \times \mathbb{A}^{m_2}$ and let $i : Z_{n,m_1,m_2} \to \mathbb{A}^{m_1} \times \mathbb{A}^{m_2}$ be the closed immersion. By applying the projection formula [3, Exposé XVII, Prop. 5.2.9] to $pr_1$ on the left side and $i$ on the right, it suffices to find an isomorphism

$$Rpr_{1!} \left( \bigotimes_{i=1}^{m_1} \mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega) \right) \cong i_! \mathbb{Q}_\ell[-2n](-n).$$

To do this, we will first check that the stalk of $Rpr_{1!} \left( \bigotimes_{i=1}^{m_1} \mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega) \right)$ vanishes outside the image of $i$. To do this, by proper base change [3, Exposé XVII, Prop. 5.2.8], it suffices to check that the compactly supported cohomology of the fiber vanishes. It even suffices to check this for finite field-valued points, as the support is constructible. Let $(a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}) \in \mathbb{A}^{m_1} \times \mathbb{A}^{m_2}(\mathbb{F}_q)$ be a point over a possibly larger finite field extension $\mathbb{F}_q$, and let $\mathcal{L}'$ be the fiber of $\mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega)$ over $(a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})$, a sheaf lisse of rank one on $\text{Witt}_n$. Over any finite field extension of $\mathbb{F}_q$, the trace function of $\mathcal{L}_{\text{univ}}(a_i, \omega)$ is a Frobenius-invariant character of $\text{Witt}_n(\mathbb{F}_q)$ evaluated at $\omega$. Hence the trace function of $\mathcal{L}'$ is also a Frobenius-invariant character. Thus the pullback of the trace function of $\mathcal{L}'$ under the Lang isogeny $\text{Witt}_n \to \text{Witt}_n, g \to \text{Frob}_q(g)g^{-1}$ is trivial, so by Chebotarev the pullback of $\mathcal{L}'$ under the Lang isogeny is trivial. So $\mathcal{L}'$
is a summand of the pushforward of the constant sheaf by the Lang isogeny of \( \text{Witt}_n \), and thus its cohomology is a summand of the cohomology of \( \text{Witt}_n \), which is \( \mathbb{Q}_\ell(-n) \) in degree 2n because \( \text{Witt}_n \cong \mathbb{A}^n \). Thus if \( H_c^*(\text{Witt}_n, \mathcal{L}^\vee) \) is nontrivial, it is equal to \( \mathbb{Q}_\ell(-n) \), which implies that the sum of the trace function is \( q^n \), so the character of \( \text{Witt}_n(\mathbb{F}_q) \) induced by \( (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}) \) is trivial, which contradicts the claim that \( (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}) \notin \mathbb{Z}_{n,m_1,m_2} \).

So in fact \( Rpr_{1!}(( \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega))) \) is supported on the image of \( i \). Restricting to the inverse image under \( pr_1 \) of the image of \( i \), the trace function of \( \mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega) \) is the constant function 1, so in fact \( \mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega) \cong \mathbb{Q}_\ell \), giving an isomorphism

\[
i^* Rpr_{1!} \left( \mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega) \cong \mathbb{Q}_\ell \right) \cong i^* Rpr_{1!} \mathbb{Q}_\ell \cong i^* \mathbb{Q}_\ell[-2n](n)
\]

and thus by the support condition an isomorphism

\[
Rpr_{1!} \left( \mathcal{L}_{\text{univ}}(a_i, \omega) \otimes \bigotimes_{i=1}^{m_2} \mathcal{L}_{\text{univ}}^\vee(b_i, \omega) \cong \mathbb{Q}_\ell \right) \cong i_! \mathbb{Q}_\ell[-2n](n),
\]

as desired.

\[\square\]

**Lemma 2.8.** For \( 0 \leq d_1, \ldots, d_{r+s} \leq n-1 \), there is a long exact sequence of complexes of vector spaces

\[
H_c^* \left( \text{Prim}_{n, \mathbb{F}_q^1}^{r} \bigotimes_{i=1}^{r} \wedge^{d_i}(\mathcal{L}_{\text{univ}})[-d_i] \otimes \bigotimes_{i=r+1}^{r+s} \wedge^{d_i}(\mathcal{L}_{\text{univ}}^\vee)[-d_i] \right)
\]

\[\rightarrow H_c^* \left( \text{Witt}_{n, \mathbb{F}_q^1}(Rpr_{2!}\mathcal{L}_{\text{univ}}) \otimes \sum_{i=1}^{r} d_i \otimes \bigotimes_{i=r+1}^{r+s} d_i \right) S_{d_1} \times \cdots \times S_{d_{r+s}}
\]

\[\rightarrow H_c^* \left( \text{Witt}_{n-1, \mathbb{F}_q^1}(Rpr_{2!}\mathcal{L}_{\text{univ}}) \otimes \sum_{i=1}^{r} d_i \otimes \bigotimes_{i=r+1}^{r+s} d_i \right) S_{d_1} \times \cdots \times S_{d_{r+s}}
\]

where \( S_{d_1} \times \cdots \times S_{d_{r+s}} \subseteq S_{\sum_{i=1}^{r} d_i} \times S_{\sum_{i=r+1}^{r+s} d_i} \), in the obvious way.

**Proof.** In view of the excision long exact sequence [3 Exposé XVII, Eq (5.1.16.2)], it suffices to find an isomorphism

\[
\bigotimes_{i=1}^{r} \wedge^{d_i}(\mathcal{L}_{\text{univ}})[-d_i] \otimes \bigotimes_{i=r+1}^{r+s} \wedge^{d_i}(\mathcal{L}_{\text{univ}}^\vee)[-d_i] \cong \left( (Rpr_{2!}\mathcal{L}_{\text{univ}}) \otimes \sum_{i=1}^{r} d_i \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee) \otimes \sum_{i=r+1}^{r+s} d_i \right) S_{d_1} \times \cdots \times S_{d_{r+s}}.
\]

To do this, observe that by definition

\[
\mathcal{L}_{\text{univ}} = Rpr_{2!}\mathcal{L}_{\text{univ}}
\]

and that

\[
\mathcal{L}_{\text{univ}}^\vee = R^1 pr_{2!}\mathcal{L}_{\text{univ}}^\vee
\]
because they are lisse, irreducible, and have the same trace functions up to scaling. Note too that \( \mathcal{L}_{\text{univ}} \) and its dual have no higher and lower cohomology in the fibers of \( pr_2 \) over \( \text{Prim}_n \), so that

\[
L_{\text{univ}}[-1] = Rpr_{1*} \mathcal{L}_{\text{univ}}, \quad L_{\text{univ}}^\vee[-1] = Rpr_{2*} \mathcal{L}_{\text{univ}}^\vee.
\]

Because the tensor product of complexes is anticommutative in odd degrees, we have

\[
(\wedge^d L_{\text{univ}})[d] = \text{Sym}^d (L_{\text{univ}}[1]) = \left( (L_{\text{univ}}[1]) \otimes d \right)^{S_d} = \left( (Rpr_{2*} \mathcal{L}_{\text{univ}}) \otimes d \right)^{S_d}
\]

and similarly for \( \mathcal{L}_{\text{univ}}^\vee \).

Tensoring these equalities for \( d_i \) from 1 to \( r \), and the dual equalities for \( d_i \) from \( r + 1 \) to \( r + s \), we have the desired isomorphism.

**Lemma 2.7.** For \( 0 \leq d_1, \ldots, d_{r+s} \leq n - 1 \), we have the Betti number bound

\[
\sum_j \dim H^j_c \left( \text{Prim}_n, \mathfrak{r}_q, \bigotimes_{i=1}^r \wedge^{d_i}(L) \otimes \bigotimes_{i=r+1}^{r+s} \wedge^{d_i}(L^\vee) \right) \leq 4(2 + \max(r, s))^{n + \sum_{i=1}^{r+s} d_i}.
\]

**Proof:** We apply the exact sequence of Lemma 2.8 and then evaluate each term using Lemma 2.9. Because of this, it suffices to bound

\[
\sum_j \dim \left( H^j_c \left( Z_{n, \sum_{i=1}^r d_i, \sum_{i=r+1}^{r+s} d_i, \mathfrak{r}_q}, \mathbb{Q}_\ell \right) \right)^{S_{d_1} \times \cdots \times S_{d_{r+s}}}
\]

and

\[
\sum_j \dim \left( H^j_c \left( Z_{n-1, \sum_{i=1}^r d_i, \sum_{i=r+1}^{r+s} d_i, \mathfrak{r}_q}, \mathbb{Q}_\ell \right) \right)^{S_{d_1} \times \cdots \times S_{d_{r+s}}}
\]

separately. We have

\[
\left( H^j_c \left( Z_{n, \sum_{i=1}^r d_i, \sum_{i=r+1}^{r+s} d_i, \mathfrak{r}_q}, \mathbb{Q}_\ell \right) \right)^{S_{d_1} \times \cdots \times S_{d_{r+s}}} = H^j_c \left( Z_{n, \sum_{i=1}^r d_i, \sum_{i=r+1}^{r+s} d_i, \mathfrak{r}_q}, \mathbb{Q}_\ell \right).
\]

Because we can take the coordinates of \( (\mathbb{A}^d)^{S_d} \) to be the coefficients of the polynomial

\[
\prod_{i=1}^T (T - a_i)
\]

for \( d_1, \ldots, a_d \) the coordinates of \( \mathbb{A}^d \), we can view \( Z_{n, \sum_{i=1}^r d_i, \sum_{i=r+1}^{r+s} d_i, \mathfrak{r}_q} \), as the moduli space of tuples of monic polynomials \( f_1, \ldots, f_{r+s} \), with \( f_i \) of degree \( d_i \), such that the leading \( n + 1 \) coefficients of \( \prod_{i=1}^r f_i \) and \( \prod_{i=r+1}^{r+s} f_i \) agree. The equality of the leading coefficient is trivial, while the equality of the remaining \( n \) coefficients is a system of \( n \) polynomial equations of degrees 1, \ldots, \( n \). So this is the solution set of a system of \( n \) equations, of degree at most \( n \), in \( \sum_{i=1}^{r+s} d_i \) variables. By [17, Theorem 12],

\[
\sum_j \dim H^j_c \left( Z_{n, \sum_{i=1}^r d_i, \sum_{i=r+1}^{r+s} d_i, \mathfrak{r}_q}, \mathbb{Q}_\ell \right) \leq 3(2 + \max(r, s))^{n + \sum_{i=1}^{r+s} d_i}.
\]

For \( Z_{n-1} \), the same argument gives a Betti number bound of

\[
3(2 + \max(r, s))^{n - 1 + \sum_{i=1}^{r+s} d_i} \leq (2 + \max(r, s))^{n + \sum_{i=1}^{r+s} d_i}
\]

as \( 2 + \max(r, s) \geq 3 \).

Summing the bounds for \( Z_n \) and \( Z_{n-1} \), we get the stated bound. \( \square \)
For $S \subseteq \{1, \ldots, r + s\}$, let
\[
M_S(\alpha_1, \ldots, \alpha_{r+s}) = \prod_{1 \leq i_1 < \alpha_2 \leq r + s} (q^{\alpha_{i_1} - \alpha_{i_2}}) \sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[T]} \prod_{i \in S} q^{(-1/2 + \alpha_i)} \deg f_i \prod_{i \notin S} q^{(-1/2 - \alpha_i)} \deg f_i.
\]
Note that this is independent of $n$.

**Lemma 3.1.** Let $d_1, \ldots, d_{r+s}$ be integers. The coefficient of $\prod_i q^{\alpha_i d_i}$ in
\[
\sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[T]} \prod_{i \in S} q^{(-1/2 + \alpha_i)} \deg f_i \prod_{i \notin S} q^{(-1/2 - \alpha_i)} \deg f_i
\]
vanishes unless $d_i \geq 0$ for $i \in S$ and $d_i \leq 0$ for $i \notin S$. Furthermore, it is symmetric in the variables $d_i$ for $i \in S$ and also in the $d_i$ for $i \notin S$.

**Proof.** The vanishing is because, in each term of the sum, $q^{\alpha_i}$ appears only in nonnegative powers if $i \in S$ and in nonpositive powers in $i \in S$. The symmetry is because the definition is symmetric in the $\alpha_i$ for $i \in S$ and symmetric in the $\alpha_i$ for $i \notin S$, by permuting the corresponding $f_i$.

**Lemma 3.2.** $M_S$ is antisymmetric in the $\alpha_i$ variables for $i \in S$, and also in the $\alpha_i$ for $i \notin S$. Expressed as a power series, the coefficient of $q^{\sum_i \alpha_i d_i}$ is nonzero only if the multiset $\{d_i| i \in S\}$ consists of one element $\geq j$ for each $j$ from 0 to $|S| - 1$ and the multiset $\{d_i| i \notin S\}$ consists of one element $\leq j$ for each $j$ from $|S|$ to $r + s - 1$.

**Proof.** These follow from Lemma 3.1 once we adjust for the Vandermonde factor. The first claim follows because the Vandermonde is asymmetric, and it is multiplied by a symmetric term, making the product symmetric. The support conditions follow from the support statement in Lemma 3.1 combined with the fact that the Vandermonde determinant is a sum of terms $q^{\sum_i \alpha_i d_i}$ where $d_1, \ldots, d_{r+s}$, is a permutation of $0, \ldots, r + s - 1$.

**Lemma 3.3.** Let $d_1, \ldots, d_{r+s}$ be integers with $\sum_{i \in S} d_i - \sum_{i \notin S} d_i \geq 0$. The coefficient of $\prod_i q^{\alpha_i d_i}$ in
\[
\left(\prod_{i_1 \in S} \prod_{i_2 \notin S} (1 - q^{|\alpha_{i_1} - \alpha_{i_2}|})\right) \sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[T]} \prod_{i \in S} q^{(-1/2 + \alpha_i)} \deg f_i \prod_{i \notin S} q^{(-1/2 - \alpha_i)} \deg f_i
\]
is
\[
O\left(1 + \sum_{i \in S} d_i - \sum_{i \notin S} d_i\right)^{(1)} \min\left(q^{-\max_{i \in S} d_i + \min_{i \notin S} d_i}, q^{-\sum_{i \in S} d_i}, q^{-\sum_{i \notin S} d_i}, q^{-\frac{\sum_{i \in S} d_i + \sum_{i \notin S} d_i}{2}}\right).
\]
Note that, by Lemma 3.1, this coefficient vanishes unless $\sum_{i \in S} d_i - \sum_{i \notin S} d_i \geq 0$. 

3. Analysis of the Main Term

**Analysis of the Main Term**
Proof. We view the upper bound as a conjunction of four upper bounds and prove each separately, by similar methods. In each case, by a contour integration, it suffices to prove that

\[
\left( \prod_{a_1 \in S} \prod_{a_2 \notin S} (1 - q^{a_1 - a_2}) \right) \sum_{f_1, \ldots, f_{r+\ell} \in \mathbb{F}_q[T]} \prod_{i \in S} q^{(-1/2 + \alpha_i) \deg f_i} \prod_{i \notin S} q^{(-1/2 - \alpha_i) \deg f_i} = \left( \frac{1}{1 - q^\epsilon} \right)^{O(1)}
\]

if \( \epsilon > 0, \alpha_1, \ldots, \alpha_{r+\ell} \) satisfy one of the following

1. For some \( i_1 \in S, i_2 \not\in S \), \( \Re \alpha_{i_1} \leq \frac{1}{2} - \epsilon \), \( \Re \alpha_i \leq -\epsilon \) for all other \( i \in S \), \( \Re \alpha_{i_2} \leq \frac{1}{2} + \epsilon \), \( \Re \alpha_i \leq +\epsilon \) for all other \( i \not\in S \).
2. \( \Re \alpha_i \leq \frac{1}{2} - \epsilon \) for \( i \in S \), \( \Re \alpha_i \geq \epsilon \) for \( i \not\in S \).
3. \( \Re \alpha_i \leq -\epsilon \) for \( i \in S \), \( \Re \alpha_i \geq \frac{1}{2} + \epsilon \) for \( i \not\in S \).
4. \( \Re \alpha_i \leq \pm \frac{1}{2} - \epsilon \) for all \( i \in S \), \( \Re \alpha_i \geq \pm \frac{1}{2} + \epsilon \) for \( i \not\in S \).

Indeed, we take \( \epsilon = \frac{1}{1+\sum_{i \in S} d_i + \sum_{i \notin S} d_i \log q} \) and then integrate over all \( \alpha_1, \ldots, \alpha_{r+\ell} \) exactly attaining the inequality.

For all cases, we will use the Euler products

\[
= \left( \prod_{i \in S} (1 - q^{-1/2 + \alpha_i})^{-1} \prod_{i \notin S} (1 - q^{-1/2 - \alpha_i})^{-1} \right) \prod_{\pi \in \mathbb{F}_q[T]} \prod_{\pi \neq \alpha_i} \sum_{d_1, \ldots, d_{r+\ell} \in \mathbb{N}} \pi^{-\sum_{i \in S} d_i + \sum_{i \notin S} \alpha_i d_i - \sum_{i \notin S} \alpha_i d_i}
\]

where the first term is the Euler factor at \( T \), and

\[
(1 - q^{a_1 - a_2}) = \prod_{\pi \in \mathbb{F}_q[T]} (1 - |\pi|^{a_1 - a_2 - 1}).
\]

In all cases, the Euler factors at \( T \) in both products are manifestly \( O \left( (1 - q^{-\epsilon})^{-O(1)} \right) \) so we focus on the other Euler factors, where it suffices to prove that for \( \alpha_1, \ldots, \alpha_{r+\ell} \) in these ranges

\[
\left( \prod_{i_1 \in S} \prod_{i_2 \notin S} (1 - |\pi|^{a_1 - a_2 - 1}) \right) \sum_{d_1, \ldots, d_{r+\ell} \in \mathbb{N}} \pi^{-\sum_{i \in S} d_i + \sum_{i \notin S} \alpha_i d_i - \sum_{i \notin S} \alpha_i d_i} \leq (1 - |\pi|^{-1-\epsilon})^{-O(1)}.
\]

We have

\[
\left( \prod_{i_1 \in S} \prod_{i_2 \notin S} (1 - |\pi|^{a_1 - a_2 - 1}) \right) \sum_{d_1, \ldots, d_{r+\ell} \in \mathbb{N}} \pi^{-\sum_{i \in S} d_i + \sum_{i \notin S} \alpha_i d_i - \sum_{i \notin S} \alpha_i d_i}
\]
so is straightforward. Suppose max_{i \in S} d_i vanishes unless max_{i \in S} d_i does not. In particular, the sum over J vanishes for all but finitely many d_1, \ldots, d_{r+e}. So in each case it suffices to show that each term in the sum is \( O(\pi^{-1-\epsilon}) \).

The first case is the most difficult. However, using the inequality we have verified, it is straightforward. Suppose max_{i \in S} d_i < |\{i \notin S|d_i > 0\}| or max_{i \notin S} d_i < |\{i \in S|d_i > 0\}|. Indeed, if neither of these is satisfied, letting \((i_1, i_2)\) be the maximizer, we see that adding \((i_1, i_2)\) to \( J \) or removing it from \( J \) preserves the conditions \(|J \cap p_{S_r}^{-1}(i)| \leq d_i, i \in S, |J \cap p_{S_r}^{-1}(i)| \leq d_i, i \notin S|\), so defines a sign-reversing involution of \( J \), and thus the sum vanishes.

In particular, the sum over \( J \) vanishes for all but finitely many \( d_1, \ldots, d_{r+e} \). So in each case it suffices to show that each term in the sum is \( O(\pi^{-1-\epsilon}) \).

For the second range and third ranges, we have

\[
|\pi|^{-\sum_{i \in S} d_i + \sum_{i \notin S} \alpha_i d_i - \sum_{i \notin S} \alpha_i d_i} = |\pi|^{-(\frac{1}{2} \epsilon)} \sum_{i \in S} d_i,
\]

so the terms are \( O(\pi^{-1-4\epsilon}) \).

For the last range, we have

\[
|\pi|^{-\sum_{i \in S} d_i + \sum_{i \notin S} \alpha_i d_i - \sum_{i \notin S} \alpha_i d_i} = |\pi|^{-(1+2\epsilon)} \sum_{i \in S} d_i,
\]

so all the terms are \( O(\pi^{-2-4\epsilon}) \).

\[\square\]

**Lemma 3.4.** Let \( d_1, \ldots, d_{r+e} \) be integers satisfying the inequalities of Lemma 3.2. Then the coefficient of \( \prod q^{\alpha d_i} \) in \( M_S \) is bounded by

\[
O \left( \left( O(1) + \sum_{i \in S} d_i - \sum_{i \notin S} d_i \right)^{O(1)} \right)
\]
Proof. Observe that

\[
\min \left( \frac{q^{-\max_{i \in S} d_i + \min_{i \in S} d_i - 1}}{q^{\sum_{i \in S} d_i + \frac{|S|}{2}}}, q^{\sum_{i \in S} d_i + \frac{|S|}{2} - \left( \frac{r + s}{2} \right)} \right)
\]

and so \(M_S\) is equal to the power series bounded by Lemma 3.3 times

\[
\pm \prod_{1 \leq i_1 < i_2 \leq r + s} (q^{\alpha_{i_1} - \alpha_{i_2}}) \prod_{i \in S} q^{\alpha_{i_1} / \alpha_{i}}.
\]

This additional factor has bounded coefficients and is supported on those terms \(q^{\sum_{i \in S} \alpha_i d_i}\) where \(\{d_i | i \in S\} = \{0, \ldots, |S| - 1\}\) and \(\{d_i | i \notin S\} = \{|S|, \ldots, r + s - 1\}\).

Hence we can obtain bounds for \(M_S\) by subtracting from the exponents in Lemma 3.3 the minimal possible contribution of an element in the support of this additional factor to the exponent, which are as stated. In fact, in all cases but the first, we are minimizing a constant function.

\[
\square
\]

Lemma 3.5. Assume \(|S| - r\) is a multiple of \(m\). The coefficients of \(q^{\sum_{i} \alpha_i d_i}\) in the power series

\[
\prod_{1 \leq i_1 < i_2 \leq r + s} (q^{\alpha_{i_1} - \alpha_{i_2}}) \sum_{\chi} \epsilon_{\chi}^{-s} \prod_{i=1}^{r+s} L(1/2 - \alpha_i, \chi)
\]

and

\[
(q^n - q^{n-1})(-1)^{(n-1)(r-|S|)} \mu^{\frac{r-|S|}{m}} \prod_{i \in S} q^{\alpha_{i}(n-1)} M_S
\]

agree as long as

\[
\sum_{i \in S} d_i - \left( \frac{|S|}{2} \right), \sum_{i \notin S} (n - 1 - d_i) + \left( \frac{r + s}{2} \right) - \left( \frac{|S|}{2} \right) \leq n - 1.
\]

Proof. We have

\[
\epsilon_{\chi}^{-s} \prod_{i=1}^{r+s} L(1/2 - \alpha_i, \chi) = \epsilon_{\chi}^{r+s-|S|-s} \prod_{i \in S} L(1/2 - \alpha_i, \chi) \prod_{i \notin S} q^{(n-1)\alpha_i} L(1/2 + \alpha_i, \chi).
\]

Because \(r - |S|\) is divisible by \(m\), \(\epsilon_{\chi}^{r-|S|} = \mu^{\frac{r-|S|}{m}}\). Thus

\[
\sum_{\chi} \epsilon_{\chi}^{-s} \prod_{i=1}^{r+s} L(1/2 - \alpha_i, \chi)
\]
\[
Hence for \alpha_n \prod \deg \text{Lemma 3.2.} \prod \text{unless} \alpha_n \ables \mu = \text{The left side and the main term of the right side are antisymmetric in the variables } \alpha_1, \ldots, \alpha_{r+s}. \text{ For the left side this is clear and for the right side, this follows from Lemma 3.2.}
\]

Proof. The left side and the main term of the right side are antisymmetric in the variables \( \alpha_1, \ldots, \alpha_{r+s} \). For the left side this is clear and for the right side, this follows from Lemma 3.2.
We view both the left side and the main term on the right side as a sum of monomials $q^{\sum_{i=1}^{r+s} \alpha_i d_i}$. Our estimate for their difference will be proved monomial-by-monomial, without proving cancellation among the different monomials, except that we use cancellation among the permuted copies of the same monomial - i.e. we use the estimate

$$\left| \sum_{\sigma \in S_{r+s}} \text{sgn} (\sigma) q^{\sum_{i=1}^{r+s} \alpha_i d_{\sigma(i)}} \right| \leq \prod_{1 \leq i_1 < i_2 \leq r+s} |q^{\alpha_{i_1}} - q^{\alpha_{i_2}}| \frac{\prod_{i=1}^{r+s} (d_{i_1} - d_{i_2})}{\prod_{i=1}^{r+s} (i-1)!}$$

arising from the Weyl character formula and the Weyl dimension formula [16, Theorem 24.2 and Corollary 24.6].

We will cancel certain terms on the left and (the main term on the) right side using Lemma 3.5. We will then bound the remaining terms on the left and right sides separately. Consider first the set of tuples $d_1, \ldots, d_{r+s}$ where, for some $S$,

$$0 \leq \sum_{i \in S} d_i - \left( \frac{|S|}{2} \right), \quad \sum_{i \notin S} (n-1 - d_i) + \binom{r+s}{2} - \left( \frac{|S|}{2} \right) \leq n - 1.$$

Let $k_{d_1, \ldots, d_{r+s}} = \sum_{i \in S} d_i + \sum_{i \notin S} (n-1 - d_i) - 2 \binom{|S|}{2} + (r+s)$. For every element in that range, choose some $S$ satisfying the inequality. We make this choice in such a way that if $S$ is chosen for $d_1, \ldots, d_{r+s}$ then for all $\sigma \in S_{r+s}$, $\sigma(S)$ is chosen for the tuple $d_{\sigma(1)}, \ldots, d_{\sigma(r+s)}$. By Lemma 3.5, the coefficient of the $q^{\sum_{i=1}^{r+s} \alpha_i d_i}$ appearing on the left side equals the coefficient of $q^{\sum_{i=1}^{r+s} \alpha_i d_i}$ in $(q^n - q^{n-1})^{r-s} \mu_{r-s} \prod_{i \notin S} q^{\alpha_i(n-1)} M_S$, so these terms cancel.

The remaining terms on the right side have two forms. There are those satisfying the inequalities

$$0 \leq \sum_{i \in S} d_i - \left( \frac{|S|}{2} \right), \quad \sum_{i \notin S} (n-1 - d_i) + \binom{r+s}{2} - \left( \frac{|S|}{2} \right) \leq n - 1,$$

and those that do not.

For the first type of terms, those inequalities must in fact be satisfied for some other subset $S'$. Applying Lemma 3.5, we see that the coefficients of $q^{\sum_{i=1}^{r+s} \alpha_i d_i}$ in $M_S$ and $M_{S'}$ must equal each other, so it suffices to bound one of them. We again split into two cases - either $|S| = |S'|$ or not.

In the first case, we choose $i_1$ in $S$ but not $S'$ and $i_2$ in $S'$ but not $S$. We have

$$(d_{i_1} + (n-1 - d_{i_2})) + ((n-1 - d_{i_1}) + d_{i_2}) \geq 2(n-1),$$

so one is at least $n-1$. Applying the first part of Lemma 3.4 for $S$ if the first one is smaller and $S'$ if the second one is smaller, we see that the coefficient is $O \left( n^{O(1)} q^{n-n/2} \right)$. There are $n^{O(1)}$ of these and each satisfies

$$\left| \sum_{\sigma \in S_{r+s}} \text{sgn} (\sigma) q^{\sum_{i=1}^{r+s} \alpha_i d_{\sigma(i)}} \right| \leq n^{O(1)} \prod_{1 \leq i_1 < i_2 \leq r+s} |q^{\alpha_{i_1}} - q^{\alpha_{i_2}}|$$

so the total contribution is bounded by the stated error term, using $n^{O(1)} = O(C_{r,s}^{-1})$. 


In the second case, assume without loss of generality that $|S| < |S'|$. Then by Lemma 2.6 $|S'| - |S| \geq m \geq 3$. Hence
\[
\left( \sum_{i \in S} d_i - \left( \frac{r + s}{2} \right) - (r + s - |S|)(n - 1) \right) - \left( \sum_{i \in S} d_i - \left( \frac{r + s}{2} \right) - (r + s - |S'|)(n - 1) \right) \geq 3(n - 1) > 2n - 2
\]
so one of these two terms must have absolute value at least $n$. Without loss of generality, it is the term associated by $S$. Then by the fourth part of Lemma 3.4, the coefficient is $O(n^{O(1)}q^{n-n/2})$, so the total contribution of these terms is $O(n^{O(1)}q^{n-n/2})$ and is bounded by the stated error term.

For the second type of terms, by Lemma 3.2 we know that
\[
0 \leq \sum_{i \in S} d_i - \left( \frac{|S|}{2} \right), \quad 0 \leq \sum_{i \notin S} (n - 1 - d_i) + \left( \frac{r + s}{2} \right) - \left( \frac{|S|}{2} \right),
\]
so as the inequalities are violated we must have either
\[
\sum_{i \in S} d_i - \left( \frac{|S|}{2} \right) \geq n
\]
or
\[
\sum_{i \notin S} d_i - \left( \frac{|S|}{2} \right) \geq n.
\]
Let $k = \max(\sum_{i \in S} d_i - \left( \frac{|S|}{2} \right), \sum_{i \notin S} d_i - \left( \frac{|S|}{2} \right))$. Then there are $k^{O(1)}$ terms with a given $k$, each of which has coefficients $O(k^{O(1)}q^{n-k/2})$ by the second and third parts of Lemma 3.4, so the total contribution is $\sum_{k \geq n} k^{O(1)}q^{n-k/2} = O(n^{O(1)}q^{n/2})$ which is within our stated error term.

But using Lemma 2.2
\[
\frac{\prod_{1 \leq i_1 < i_2 \leq r+s} |d_{i_1} - d_{i_2}|}{\prod_{i=1}^{r+s} (i - 1)!} = \lim_{\alpha_1, \ldots, \alpha_r \to 1} \sum_{\sigma \in S_n} \text{sgn}(\sigma) q^{d_{i_1} - d_{i_2}}
\]
is simply the multiplicity that $V_{d_1, \ldots, d_r}$ appears in $\det^{-s} \otimes \bigotimes_{i=1}^{r+s} \Lambda^{e_i}$. Hence
\[
\sum_{0 \leq d_1 < d_2 < \cdots < d_{r+s} \leq n+r+s-1} \frac{\prod_{1 \leq i_1 < i_2 \leq r+s} |d_{i_1} - d_{i_2}|}{\prod_{i=1}^{r+s} (i - 1)!} \sum_j \dim H^j_c(\text{Prim}_{n, \mathbb{F}_q}, V_{d_1, \ldots, d_r} | d_{r+1}', \ldots, d_{r+s}' (L_{\text{univ}})) \leq \sum_{0 \leq e_1, \ldots, e_{r+s} \leq n-1} \dim H^j_c(\text{Prim}_{n, \mathbb{F}_q}, \det(L_{\text{univ}})^{-s} \otimes \bigotimes_{i=1}^{r+s} \Lambda^{e_i}(L_{\text{univ}})).
\]

The total number of terms here is $n^{r+s}$. For each term, we apply Lemma 2.9 and use the fact that
\[
\det(L_{\text{univ}})^{-s} \otimes \bigotimes_{i=1}^{r+s} \Lambda^{e_i}(L_{\text{univ}}) = \bigotimes_{i=1}^{r} \Lambda^{e_i}(L_{\text{univ}}) \otimes \bigotimes_{i=r+1}^{r+s} \Lambda^{n-1-e_i}(L_{\text{univ}}).
\]
We can assume in each term that \( e_1 \leq e_2 \leq \cdots \leq e_n \), so that \( \sum_{i=1}^r e_i + \sum_{i=r+1}^{r+s}(n-1-e_i) \leq re_r + s(n-1-e_{r+1}) \leq re_r + s(n-1-e_r) \leq (n-1)\max(r,s) \). This gives exactly the stated bound.

Remark 4.2. It is unsurprising, that, in this proof, dimensions of irreducible representations of \( GL_r \) appear as multiplicities of irreducible representations of \( GL_{n-1} \) in

\[
\sum_{0 \leq e_1, \ldots, e_{r+s} \leq n-1} \det^{-s} \otimes_i \wedge e_i,
\]

because \( \otimes_i \wedge e_i \) admits a natural action of \( GL_{n-1} \times GL_r \) as it is the exterior algebra of the tensor product of the standard representations of \( GL_{n-1} \) and \( GL_r \), and this action is preserved after tensoring with the \(-s\) power of the determinant of \( GL_{n-1} \).

Corollary 4.3. Assume \( n \geq 3 \), if \( n = 3 \) that the characteristic of \( \mathbb{F}_q \) is not 2 or 5, and if \( n = 4 \) or 5 that the characteristic of \( \mathbb{F}_q \) is not 2.

Assume Hypothesis 1.2 with constant \( c \). Let \( \alpha_1, \ldots, \alpha_{r+s} \) be imaginary. Let \( C_{r,s} = (\max(r,s)+2)^{\max(r,s)+1} \)

\[
\frac{1}{(q^n-q^{n-1})} \sum_{\chi} \epsilon_{\chi}^{-s} \prod_{i=1}^{r+s} \L(1/2-\alpha_i, \chi)
\]

\[
= \sum_{S \subseteq M} \mu_{r-|S| \in \mathbb{Z}} \prod_{m|r-|S|} q^{\alpha_m(n-1)} \sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[T]} \prod_{i \in S} \left| f_i \right|^{-\frac{1}{2}+\alpha_i} \prod_{i \notin S} \left| f_i \right|^{-\frac{1}{2}-\alpha_i}
\]

\[
\prod_{i \in S} \frac{\left| f_i \right|^{1/2}}{\prod_{j \notin S} \left| f_j \right|^{1/2}}
\]

\[
+ O \left( q^{c-n} C_{r,s}^{n} \right).
\]

If \( n > 2 \max(r,s) + 1 \) then we need only the terms where \( r = |S| \).

Proof. The first claim follows from Theorem 4.1 after dividing both sides by \( (q^n-q^{n-1}) \prod_{1 \leq i_1 < i_2 \leq r+s} (q^{\alpha_{i_1}} - q^{\alpha_{i_2}}) \) and using the definition of \( M \).

The second claim follows from Lemma 2.6 because then \( m > \max(r,s) \).

In particular, the second claim is Theorem 1.3.

5. Verification of the hypothesis in special cases

Lemma 5.1. Let \( \mathcal{F} \) be an irreducible lisse \( \mathbb{Q}_\ell \)-sheaf on \( \text{Prim}_{n,\mathbb{Q}} \) that appears as a summand of

\[
L^a_{\text{univ}} \otimes L^{\vee \otimes b}_{\text{univ}}
\]

for some \( a \geq n > b \). Then

\[
H^j_{\ell}(\text{Prim}_n, \mathcal{F}) = 0
\]

for \( j > n+b + \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor + 1 \).
Lemma 5.2. \( j > a \) is trivial for \( j > a \) whenever \( j > a \). Hence by Lemma 5.1, the action of \( S_a \) on the cohomological dimension of \( S_a \) is trivial in degrees greater than \( a + 2b - \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor + 1 \). Applying this to \( n - 1 \), we can see that the action of \( S_a \) on

\[
H^j_c \left( \text{Witt}_{n,\mathbb{F}_q}, (Rpr_{2L} \otimes \mathcal{L}_{\text{univ}})^\otimes \otimes (Rpr_{2L} \otimes \mathcal{L}_{\text{univ}})^\vee \right)
\]

is trivial whenever \( j > a + 2b - \left\lfloor \frac{n - 1}{p} \right\rfloor + 1 \). Hence by shifting, the action of \( S_a \) on

\[
H^{j-1}_c \left( \text{Witt}_{n-1,\mathbb{F}_q}, (Rpr_{2L} \otimes \mathcal{L}_{\text{univ}})^\otimes \otimes (Rpr_{2L} \otimes \mathcal{L}_{\text{univ}})^\vee \right)
\]

is trivial whenever

\[
j - 1 > a + 2b + n - 1 + \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{n - 1}{p} \right\rfloor + 1 \geq \left( a + 2b + n + \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor + 1 \right) - 1
\]

By the long exact sequence of Lemma 2.8, the action of \( S_a \) on

\[
H^j_c \left( \text{Prim}_{n,\mathbb{F}_q}, \bigotimes_{i=1}^{a} L_{\text{univ}}^{[-1]} \otimes \bigotimes_{i=1}^{b} L_{\text{univ}}^{[-1]} \right)
\]

is trivial for \( j > a + 2b + n + \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor + 1 \). Hence by shifting, the action of \( S_a \) on

\[
H^j_c \left( \text{Prim}_{n,\mathbb{F}_q}, \bigotimes_{i=1}^{a} L_{\text{univ}} \otimes \bigotimes_{i=1}^{b} L_{\text{univ}}^{\vee} \right)
\]

factors through the sign character if \( j > n + b + \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor + 1 \). But the sign-equivariant part of \( \bigotimes_{i=1}^{a} L_{\text{univ}} \) is \( \wedge^a L_{\text{univ}} \), which vanishes, so the sign-equivariant part of the cohomology vanishes as well. Hence the same is true for any summand of \( \bigotimes_{i=1}^{a} L_{\text{univ}} \otimes \bigotimes_{i=1}^{b} L_{\text{univ}}^{\vee} \), such as \( \mathcal{F} \).

Lemma 5.2. Hypothesis 1.2 is satisfied for any \( n, r \), with \( s = 0 \) and \( c = \left\lfloor \frac{(n-1)r}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor + 1 \).

Proof. Any sheaf \( \mathcal{F} \) that is a summand of \( \bigotimes_{i=1}^{r} L_{\text{univ}}^{d_i} \) for some \( 0 \leq d_1, \ldots, d_r \leq n - 1 \) is a summand of \( L_{\text{univ}}^{\otimes \sum_{i=1}^{r} d_i} \). If \( \sum_{i=1}^{r} d_i < n \) then the condition of Hypothesis 1.2 is not satisfied, so it is vacuously true. Otherwise, we apply Lemma 5.1 with \( a = \sum_{i=1}^{r} d_i \leq r(n-1) \) and \( b = 0 \).

However, combining this with Theorem 1.1 would simply recover [24, Theorems 1.2 and 1.3].

Lemma 5.3. Hypothesis 1.2 is satisfied for any \( n, r \), with \( s = 1 \) and \( c = n + 1 - \frac{p-2r}{pr} n \).
Proof. We may assume $p > 2r$ as this follows immediately from cohomological dimension estimates otherwise.

Let $F$ be a summand of $\det^{-1}(L_{\text{univ}}) \otimes \wedge^{d_1} (L_{\text{univ}})$ for some $0 \leq d_1, \ldots, d_r \leq n-1$. Without loss of generality, $0 \leq d_1 \leq \ldots d_r \leq n-1$. Then because $\det^{-1}(L_{\text{univ}}) \otimes \wedge^{d_{r+1}} (L_{\text{univ}}) = \wedge^{n-1-r}(L_{\text{univ}})$, $F$ is a summand of $L_{\text{univ}} \otimes \wedge^{d_1} (L_{\text{univ}}) \otimes \wedge^{d_2} (L_{\text{univ}}) \otimes \cdots \otimes \wedge^{d_r} (L_{\text{univ}})$ for some $0 \leq d_1, \ldots, d_r \leq n-1$. Without loss of generality, $0 \leq d_1 \leq \ldots d_r \leq n-1$. Then because $\det^{-1}(L_{\text{univ}}) \otimes \wedge^{d_{r+1}} (L_{\text{univ}}) = \wedge^{n-1-r}(L_{\text{univ}})$, $F$ is a summand of $L_{\text{univ}} \otimes \wedge^{d_1} (L_{\text{univ}}) \otimes \wedge^{d_2} (L_{\text{univ}}) \otimes \cdots \otimes \wedge^{d_r} (L_{\text{univ}})$. If $\sum_{i=1}^{r} d_i < n$ then Hypothesis 1.2 is vacuously true. Otherwise, $d_{r+1} \geq d_r \geq \frac{n}{r}$ and we apply Lemma 5.1 to see that the cohomology groups vanish for

$$j > n + (n - 1 - d_{r+1}) + \left\lfloor \frac{\sum_{i=1}^{r} d_i}{p} \right\rfloor + 1,$$

and we have

$$(n - 1 - d_{r+1}) + \left\lfloor \frac{\sum_{i=1}^{r} d_i}{p} \right\rfloor + 1 \leq n - 1 - d_{r+1} + \frac{\sum_{i=1}^{r} d_i}{p} - \frac{n}{p} + 2$$

$$\leq n - 1 - \frac{p-r}{p} d_{r+1} - \frac{n}{p} + 2 \leq n - 1 - \frac{p-r}{pr} n - \frac{n}{p} + 2 = n + 1 - \frac{p-2r}{pr} n$$

so we may take $c = n + 1 - \frac{p-2r}{pr} n$.

We could apply the same techniques with $r, s \geq 2$ but we would not obtain a nontrivial bound this way.

6. Spaces defined by Hast and Matei

The results in this section are not directly related to the main results of this paper, but use similar techniques. In it, we recover by a more direct geometric argument calculations by Hast and Matei of certain cohomology groups of certain spaces. We also sketch how more cohomology groups might be computed conditionally on Hypothesis 1.2.

Using Katz’s equidistribution results for the $L$-functions of Dirichlet characters [18], and performing a Fourier transform over the group of Dirichlet characters, Rodgers calculated the variance of certain arithmetic functions in the $q \rightarrow \infty$ limit [22]. Hast and Matei defined schemes in a natural way so that their cohomology would control these variances, and then used Rodgers’ calculations to control the top nonvanishing cohomology groups [14]. This has the effect of starting with geometric information (the monodromy calculations of Katz), proceeding to numerical information, and deriving geometric information again. Using the Witt vector Fourier transform calculation in Lemma 2.7, it is possible to avoid the numerical step and reason entirely geometrically. This answers a question asked in [14].

Lemma 6.1. Let $n$ and $m$ be natural numbers with $m > n$. Assume that $n \geq 3$, and, if $n = 3$, then $p \neq 2, 5$.

Then the cohomology group

$$H^j_c \left( \text{Witt}_{\mathbb{F}_q}, (Rpr_2 L_{\text{univ}})^{\otimes m} \otimes (Rpr_2 L^\vee_{\text{univ}})^{\otimes m} \right)$$

for $j \geq 2n + 2m$ are described as follows:

• If $j > 2n + 2m$ and $j \neq 4m$, then this cohomology group vanishes.
• If $j = 4m$, this cohomology group is $\mathbb{Q}_\ell(-2m)$ with the trivial $S_m \times S_m$ action.
• If $j = 2n + 2m$, this cohomology group is $\text{Hom}_{GL_{n-1}}(V^\otimes m, V^\otimes m)(-n-m)$ where $V$ is the $n-1$-dimensional standard representation of $GL_{n-1}$ over $\mathbb{Q}_\ell$, with $S_m \times S_m$ acting by permuting the factors.

Proof. We stratify $\text{Witt}_n$ into, first, the open subset $\text{Prim}_n = \text{Witt}_n \setminus \{0\}$, second, $\text{Witt}_{n-1} \setminus \{0\}$, and third, the point $\{0\}$. We will calculate the cohomology independently on each of the three sets, then combine the information.

On $\text{Witt}_n \setminus \{0\}$, $Rpr_{2!}\mathcal{L}_{\text{univ}}$ and $Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee$ are supported in degree one, so $(Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m$ is supported in degree $2m$. Thus the cohomology of $\text{Witt}_{n-1}$ with that complex is supported in degree $\leq 2(n-1) + 2m$. Hence by excision, the natural map

$$H^j_c\left(\text{Prim}_n, (Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m\right)$$

$$\rightarrow H^j_c\left(\text{Witt}_n, (Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m\right)$$

is an isomorphism in degrees $2 > (n-1) + 2m + 1$.

Again applying the cohomological dimension bound, $H^j_c\left(\text{Prim}_n, (Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m\right)$ is supported in degrees $\leq 2n + 2m$. Furthermore

$$H^{2m+2n}_c\left(\text{Prim}_n, (Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m\right) = H^{2n}_c\left(\text{Prim}_n, L^\otimes m \otimes L^\vee\otimes m(-m)\right)$$

with the Tate twist because $R^1pr_{2!}\mathcal{L}_{\text{univ}}^\vee = L^\vee\otimes (-1)$. Applying Poincare duality, we have

$$H^{2n}_c\left(\text{Prim}_n, L^\otimes m \otimes L^\vee\otimes m(-m)\right) = H^0\left(\text{Prim}_n, L^\vee\otimes m \otimes L^\otimes m(-m)\right)^\vee(-n-m)$$

$$= \text{Hom}(L^\otimes m, L^\vee\otimes m(-n-m)) = \text{Hom}(L^\vee\otimes m, L^\otimes m(-n-m)).$$

By [15 Theorem 5.1], the geometric monodromy group of $L_{\text{univ}}$ is contained between $SL_{n-1}$ and $GL_{n-1}$. Letting $V$ be the standard representation of $GL_{n-1}$, we have

$$\text{Hom}_{GL_{n-1}}(V^\otimes m, V^\otimes m) \subseteq \text{Hom}(L^\otimes m, L^\vee\otimes m) \subseteq \text{Hom}_{SL_{n-1}}(V^\otimes m, V^\otimes m).$$

Because the center of $GL_{n-1}$ acts by scalars on $V^\otimes m$, every $SL_{n-1}$-equivariant endomorphism is also $GL_{n-1}$-equivariant, so all three of these vector spaces are equal, and

$$H^{2m+2n}_c\left(\text{Witt}_n, (Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m\right) = \text{Hom}_{GL_{n-1}}(V^\otimes m, V^\otimes m)(-n-m).$$

Observe that at the point $0$, $Rpr_{2!}\mathcal{L}_{\text{univ}}$ and $Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee$ are one-dimensional vector spaces in degree 2, so

$$(Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m = \mathbb{Q}_\ell[4m](-2m).$$

Thus

$$H^j_c\left(\{0\}, (Rpr_{2!}\mathcal{L}_{\text{univ}}^\otimes m) \otimes (Rpr_{2!}\mathcal{L}_{\text{univ}}^\vee)^\otimes m\right)$$

is $\mathbb{Q}_\ell(-2m)$ if $j = 4m$ and 0 otherwise.

We now apply the excision exact sequence to calculate the cohomology of $\text{Witt}_n$. In degrees at least $2n + 2m$, the only nonvanishing terms are in degrees $2n + 2m$ and $4m$. As they are both in even degrees the connecting homomorphism between them vanishes,
and thus the cohomology of the total space is simply the sum of the contributions from \(\{0\}\) and \(\text{Witt}_n - \{0\}\), as stated. (In general, it could be an extension, but since \(m > n\) they are in different degrees.)

**Corollary 6.2.** Let \(n\) and \(m\) be natural numbers with \(m > n\). Assume that \(n \geq 3\), and, if \(n = 3\), that \(p \neq 2, 5\). Then we have

- If \(j > 2m\) and \(j \neq 4m\), then \(H^j_c(\mathbb{Z}, \mathbb{Q}_\ell) = 0\).
- If \(j = 4m - 2n\), then \(H^j_c(\mathbb{Z}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-2m)\) with the trivial \(S_m \times S_m\) action.
- If \(j = 2m\), then \(H^j_c(\mathbb{Z}, \mathbb{Q}_\ell) = \text{Hom}_{GL_{n-1}}(V^{\otimes m}, V^{\otimes m})(-n - m)\) where \(V\) is the \(n - 1\)-dimensional standard representation of \(GL_{n-1}\) over \(\mathbb{Q}_\ell\), with \(S_m \times S_m\) acting by permuting the factors.

**Proof.** This follows immediately from Lemmas 2.7 and 6.1. \(\Box\)

**Proposition 6.3.** Let \(n\) and \(m\) be natural numbers with \(m > n\).

Let \(X_{2,n,m}\) be the subset of \(\mathbb{P}^{2m-1}\) with projective coordinates \((a_1, \ldots, a_m, b_1, \ldots, b_m)\) such that \(\prod_{i=1}^m (1-a_i,1-b_i) \equiv 1 \mod x^{n+1}\) (matching the definition of [14, §2]). Assume that \(n \geq 3\), and, if \(n = 3\), that \(p \neq 2, 5\). Then we have

- If \(j > 4m - 2n\) or \(j\) is odd and \(j > 2m - 2\), then \(H^j_c(\mathbb{Z}, \mathbb{Q}_\ell) = 0\).
- If \(j > 2m - 2\) is even, then \(H^j_c(\mathbb{Z}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\frac{1}{2})\) with the trivial \(S_m \times S_m\) action.
- If \(j = 2m - 2\), then \(H^j_c(\mathbb{Z}, \mathbb{Q}_\ell)\) is an extension of \(\mathbb{Q}_\ell(1-m)\) by \(\text{Hom}_{GL_{n-1}}(V^{\otimes m}, V^{\otimes m})(1-n - m)\) where \(V\) is the \(n - 1\)-dimensional standard representation of \(GL_{n-1}\) over \(\mathbb{Q}_\ell\), with \(S_m \times S_m\) acting by permuting the factors.

This matches the description of [14, Theorem A] by a calculation in Schur-Weyl duality.

**Proof.** By comparing definitions, we see that \(Z_{n,m,m}\) is the affine cone on \(X_{2,n,m}\) (which is called \(Y_{2,n,m}\) in [14, §2]). By excision, our calculations of the cohomology of \(Z_{n,m,m}\) hold without modification for \(Z_{n,m,m} \neq 0\).

Applying the Leray spectral sequence to the projection \(Y_{2,n,m} - \{0\} \to X_{2,n,m}\), we obtain a long exact sequence (dropping subscripts for compactness)

\[
H^{j+1}_c(X, \mathbb{Q}_\ell) \to H^{j+2}_c(Z - \{0\}, \mathbb{Q}_\ell) \to H^{j+2}_c(X, \mathbb{Q}_\ell(-1)) \to H^{j+2}_c(X, \mathbb{Q}_\ell) \to H^{j+3}_c(Z - \{0\}, \mathbb{Q}_\ell).
\]

We verify our description of \(H^j_c(X_{2,n,m}, \mathbb{Q}_\ell)\) by descending induction on \(j\), starting from \(4m - 2n - 2\), which is twice the dimension of \(X_{2,n,m}\) and beyond which we know its cohomology vanishes. In particular, we may assume that the cohomologies of \(X\) and \(Z\) vanish in all odd degrees greater than \(j\). Because of this, when \(j\) is odd, the exact sequence reduces to

\[
0 \to H^j_c(X_{2,n,m}, \mathbb{Q}_\ell(-1)) \to 0,
\]

verifying the induction step, and when \(j\) is even, it reduces to

\[
0 \to H^{j+2}_c(Z_{n,m,m} - \{0\}, \mathbb{Q}_\ell)(1) \to H^j_c(X_{2,n,m}, \mathbb{Q}_\ell) \to H^{j+2}_c(X_{2,n,m}, \mathbb{Q}_\ell)(1) \to 0,
\]

again quickly verifying the induction step. \(\Box\)
We sketch in addition how Hypothesis 1.2 for $c = 0$ and $r, s, n$ arbitrary could potentially be used to calculate the cohomology of $Z_{n,m,m}$ (and hence $X_{2,n,m}$) in degrees greater than $2m - n$. One uses Lemma 2.7 and expresses the right side by iterated excision as arising by a spectral sequence from the cohomologies of $\text{Prim}_d$, $0 \leq d \leq n$, with cohomology in $L^\otimes m \otimes L_{\text{univ}}^\vee \otimes m$. One then decomposes into irreducible representations of $GL_{d-1}$ and discards those that do not satisfy the criterion of Hypothesis 1.2. For the remainder, one can calculate the cohomology using Lemma 2.2 to reduce the calculation to the cohomology of tensor products of wedge powers. Using Lemma 2.7 again, we obtain the cohomology of a moduli space of tuples of polynomials of fixed degrees whose product is equal to another tuple of polynomials of fixed degrees. If the cohomology of these spaces can be calculated explicitly, then the cohomology of the Hast-Matei varieties can be calculated, conditionally on the hypothesis.

References

[1] J. C. Andrade and J. P. Keating, The mean value of $L(1/2, \chi)$ in the hyperelliptic ensemble. Journal of Number Theory, 132 (2012), 2793-2816.
[2] J. C. Andrade and J. P. Keating, Conjectures for the integral moments and ratios of $L$-functions over function fields, Journal of Number Theory 142 (2014), 102-148.
[3] M. Artin, A. Grothendieck, J.-L. Verdier, eds. (1972). Séminaire de Géométrie Algébrique du Bois Marie - 1963-64 - Théorie des topos et cohomologie étale des schémas - (SGA 4) - vol. 3 Lecture Notes in Mathematics 305.
[4] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, Integral moments of $L$-functions, Proceedings of the London Mathematical Society 91 (2005), 33-104.
[5] J. B. Conrey, D. W. Farmer, and M. R. Zirnbauer, Autocorrelation of ratios of $L$-functions, Communications in Number Theory and Physics 2 (2008), 593-636.
[6] J. B. Conrey and N. C. Snaith, Applications of the $L$-functions ratios conjectures, Proceedings of the London Mathematical Society 94 (2007), 594-646.
[7] P. Deligne, with the collaboration of J. F. Boutot, A. Grothendieck, L. Illusie, and J. L. Verdier, Séminaire de Géométrie Algébrique du Bous Marie - 1963-64 - Théorie des topos et cohomologie étale des schémas - (SGA 4) - vol. 3 Lecture Notes in Mathematics 569.
[8] P. Deligne, La conjecture de Weil: II, Publications mathématiques de l’I.H.É.S. 52 (1980), 137-252.
[9] A. Diaconu, On the third moment of $L (1/2, \chi_d)$ I: the rational function field case, arXiv preprint: 1801.00486 (2018).
[10] A. Diaconu, D. Goldfeld, and J. Hoffstein, Multiple Dirichlet series and moments of zeta and $L$-functions. Compositio Math., 139 (2003), 297-360.
[11] Improving the error term in the mean value of $L(1/2, \chi)$ in the hyperelliptic ensemble, IRMN (2017), 6119-6148.
[12] The second and third moment of $L(1/2, \chi)$ in the hyperelliptic ensemble, Forum Math. 29 (2017), 873-892.
[13] A. Florea, The fourth moment of quadratic Dirichlet $L$-functions over function fields, Geometric and Functional Analysis 27 (2017), 541-595.
[14] D. R. Hast and V. Matei, Higher Moments of Arithmetic Functions in Short Intervals: A Geometric Perspective, IMRN (2018), rnx310.
[15] J. Hoffstein and M. Rosen, Average values of $L$-series in function fields, J. Reine Angew. Math. 426 (1992) 117-150.
[16] W. Fulton and J. Harris Representation theory: A first course. Graduate Texts in Mathematics, Readings in Mathematics, 129 (1991).
[17] N. M. Katz, Sums of Betti numbers in arbitrary characteristic, *Finite Fields and their Applications*, 7 (2001), 29-44.

[18] N. M. Katz, Witt Vectors and Question of Keating and Rudnick, *IMRN* (2013), 3613-3638.

[19] N. M. Katz, On a question of Rudnick: Do we have square root cancellation for error terms in moment calculations?, *Philosophical Transactions of the Royal Society A* 373 (2014).

[20] N. M. Katz, Witt Vectors and a Question of Rudnick and Waxman, *IMRN* (2016), 3337-3412.

[21] N. M. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues, and Monodromy*, Colloquium Publications 45, 1999.

[22] B. Rodgers Arithmetic functions in short intervals and the symmetric group, to appear in *Algebra and Number Theory*, arXiv:1609.02967.

[23] P. Sarnak and S. W. Shin and N. Templier, Families of L-Functions and Their Symmetry, in W. Müller and S. W. Shin and N. Templier (eds), *Families of Automorphic Forms and the Trace Formula*, SISY, 2016, Springer.

[24] W. Sawin Square-root cancellation for sums of factorization functions over short intervals in function fields arXiv preprint: 1809.015137 (2018).

[25] N. Tamam, The fourth moment of Dirichlet L-Functions for the rational function field *International Journal of Number Theory* 10 (2014), 183-218.