Dirac operator
on noncommutative principal circle bundles

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Abstract

We study spectral triples over noncommutative principal $U(1)$-bundles of arbitrary dimension and formulate a compatibility condition between the connection and the Dirac operator on the total space and on the base space of the bundle. Examples of low dimensional noncommutative tori are analyzed in more detail and all connections found that are compatible with an admissible Dirac operator. Conversely, a family of new Dirac operators on the noncommutative tori, which arise from the base-space Dirac operator and a suitable connection is exhibited. These examples are extended to the theta-deformed principal $\mathbb{U}(1)$-bundle $S^3_\theta \to S^2$.

1 Introduction

The spectral triples \cite{7,9} and in particular the Dirac operators on noncommutative odd-dimensional principal $U(1)$-bundles have been studied in \cite{14}. Therein an operatorial definition of a connection and a compatibility condition between the connection and the Dirac operator on the total space and on the base space of the bundle were proposed, basing on the classical situation \cite{1,2} and the abstract algebraic approach. Moreover, the case of the noncommutative 3-torus viewed as a $U(1)$-bundle over the noncommutative 2-torus was analysed and all connections compatible with an admissible Dirac operator were found. Conversely, a family of new Dirac operators on the tori was constructed, which arise from the base-space Dirac operator and a suitable connection.
In this paper by making an ampler use of the real structure we slightly modify the scheme of [14] and weaken the assumptions made there. In addition we extend the study to principal $U(1)$-bundles of arbitrary dimension. As examples we discuss low dimensional noncommutative tori and the theta deformation of the principal Hopf $U(1)$-bundle $S^3 \to S^2$, which was studied in [5].

2 Quantum principal $U(1)$-bundles

In the noncommutative realm the notion quantum principal bundles have been recently formulated in the context of principal comodule algebras. This evolved from earlier studies of quantum principal bundles with universal differential calculus, of principal extensions, and of Hopf-Galois extensions, c.f. [4, 13, 18, 17]. For our purpose we shall also need a notion of the principal connection adapted to the case of Dirac operator induced calculi.

Let $H$ be a unital Hopf algebra over $\mathbb{C}$ with a counit $\varepsilon$ and an invertible antipode $S$. Let $\mathcal{A}$ be a right $H$-comodule algebra (which we also assume to be unital). We will use the Sweedler notation for the right coaction of $H$ on $\mathcal{A}$:

$$\Delta_R(a) = a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes H.$$  

(We shall often skip the adjective right if it is clear from the context). Then $\mathcal{A}$ is called principal $H$-comodule algebra [6] iff there exists a $\mathbb{C}$-linear unital map

$$\ell : H \rightarrow \mathcal{A} \otimes \mathcal{A}$$

such that

$$m_\mathcal{A} \circ \ell = \eta \circ \varepsilon,$$  

(2.1a)

$$(\ell \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta_R) \circ \ell,$$  

(2.1b)

$$(S \otimes \ell) \circ \Delta = (\sigma \otimes \text{id}) \circ (\Delta_R \otimes \text{id}) \circ \ell,$$  

(2.1c)

where $m_\mathcal{A} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is the multiplication in $\mathcal{A}$, $\eta : \mathbb{C} \rightarrow \mathcal{A}$ is the unit map and $\sigma : \mathcal{A} \otimes H \rightarrow H \otimes \mathcal{A}$ is the flip.

It follows then that $\mathcal{A}$ is a Hopf-Galois extension of $B := \{b \in \mathcal{A} \mid \Delta_R(a) = b \otimes 1\}$, its subalgebra of coinvariant elements. Namely

$$\mathcal{A} \otimes H \ni a \otimes h \mapsto a \pi_B(\ell(h)) \in \mathcal{A} \otimes_B \mathcal{A},$$

where $\pi_B : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes_B \mathcal{A}$ is the usual projection, is the (left and right) inverse of the canonical map $\chi := (m_\mathcal{A} \otimes \text{id})(\text{id} \otimes \Delta_R)$,

$$\mathcal{A} \otimes_B \mathcal{A} \ni a' \otimes a \mapsto \chi(a' \otimes a) = a' a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes H.$$  

(2.2)

By quantum principal bundle we shall understand the structure defined as above, and we shall often denote it simply by the inclusion map $B \hookrightarrow \mathcal{A}$. Moreover, we shall call the map $\ell$ strong connection lift. It determines a strong connection (for the universal calculus on $\mathcal{A}$) which can be equivalently described in four other terms (cf. [13]). One of them is by
a certain splitting of the multiplication map $B \otimes A \rightarrow A$ (which shows the equivariant left projectivity of $A$ over $B$). Another one, the so called connection form, will be adapted in section 4.1 to the case of the Dirac-operator induced differential calculus, and represented as an operator on a Hilbert space.

In this paper we restrict our discussion only to the case of the structure group given by the classical one-dimensional Lie group $U(1)$. We shall denote by $H$ its coordinate algebra i.e. the complex polynomial algebra in $z$ and $z^{-1}$, with the star structure given by $zz^* = 1$, the coproduct by $\Delta(z) = z \otimes z$, a counit $\varepsilon(z) = 1$ and an invertible antipode $S(z) = z^{-1}$. We can split the $H$-comodule algebra $A$ as a direct sum

$$\mathcal{A} = \bigoplus_{k \in \mathbb{Z}^n} \mathcal{A}^{(k)}, \quad (2.3)$$

where the $\mathcal{A}^{(k)}$ are the set of elements of homogeneous degree $k$; that is,

$$a \in \mathcal{A}^{(k)} \iff \Delta_R(a) = a \otimes z^k.$$ 

Of course, $\mathcal{A}^{(0)} = B$.

3 Dirac operator and spectral triples

According to A. Connes the noncommutative metric and spin geometry is encoded in terms spectral triples [7, 8, 9]. In what follows $\mathcal{A}$ will always denote a unital complex $^*$-algebra.

**Definition 3.1.** A spectral triple for an algebra $\mathcal{A}$ is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ by bounded operators (which we shall simply denote by $\psi \mapsto a\psi$, for any $a \in \mathcal{A}$, $\psi \in \mathcal{H}$) and $D$ is a densely defined, selfadjoint operator on $\mathcal{H}$ with compact resolvent, such that for any $a \in \mathcal{A}$ the commutator $[D, a]$ is a bounded operator.

In the commutative case, when $\mathcal{A}$ is the algebra of smooth functions over a Riemannian spin manifold $M$, $\mathcal{H}$ corresponds to the Hilbert space of $L^2$-sections of the spinor bundle, and $D$ is the Dirac operator, on the spinor bundle, associated to the Levi-Civita connection. So, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we will call $\mathcal{H}$ the space of spinors and $D$ the Dirac operator. The algebra $\mathcal{A}$ will be always certain suitable $^*$-algebra of operators on $\mathcal{H}$, dense in its norm completion (C*-algebra), in particular it can be a (noncommutative) polynomial algebra presented in terms of generators and relations.

An important structure employed in a study of further properties is that of reality.

**Definition 3.2.** A real spectral triple of $KR$-dimension $j$, where $j \in \mathbb{Z}_8$, consists of the data $(\mathcal{A}, \mathcal{H}, D, J)$ when $j$ is odd, whereas of the data $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ when $j$ is even, where $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, $J$ is antiunitary operator and $\gamma$ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$ such that:

(i) for any $a, b \in \mathcal{A}$, $[a, J b^* J^{-1}] = 0$;
(ii) $J$, $D$ and $\gamma$ satisfy the following commutation relations:

$$J^2 = \varepsilon \text{id}, \quad JD = \varepsilon' DJ$$

and, for $j$ even,

$$J\gamma = \varepsilon'' \gamma J, \quad \gamma D = -D\gamma,$$

where $\varepsilon, \varepsilon', \varepsilon''$ depend on the KR-dimension and are given in the following table (c.f. [12])

| $j$ | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 | 1 | 3 | 5 | 7 |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | + | - | - | + | + | - | - | + | - | - | + |
| $\varepsilon'$ | + | + | + | + | - | - | - | - | + | - | + |
| $\varepsilon''$ | - | + | - | + | + | + | - | + | - | - | + |

Table 1: (Connes’ choice is marked by $\bullet$)

The operator $J$ will be usually called the real structure of the spectral triple. We will often treat together the even and the odd dimensional case. So in general we will write $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ for a real spectral triple, keeping in mind that in the odd case $\gamma$ does not appear.

The antiunitary operator $J$ determines a left action of the opposite algebra $\mathcal{A}^\circ$ (or, equivalently, a right action of the algebra $\mathcal{A}$) on the Hilbert space $\mathcal{H}$,

$$\pi^\circ(b)\psi = \psi b = Jb^*J^{-1}\psi, \quad \forall b \in \mathcal{A}, \psi \in \mathcal{H},$$

which due to the condition (i) of definition [3.2] commutes with the representation of $\mathcal{A}$ on $\mathcal{H}$; that is, $J$ maps $\mathcal{A}$ into its commutant on $\mathcal{H}$. This makes the Hilbert space $\mathcal{H}$ a bi-module over $\mathcal{A}$.

Moreover we shall assume the so-called first order condition, that is the requirement that $\mathcal{A}^\circ$ commutes not only with $\mathcal{A}$ but also with $[D, \mathcal{A}]$ so that, for any $a, b \in \mathcal{A}$, we have:

$$[[D, a], Jb^*J^{-1}] = 0.$$  \hfill (3.2)

Notice that conjugating the above identity with $J$ one can show (3.2) to be equivalent to

$$[[D, Jb^*J^{-1}], a] = 0;$$

and so the first order condition is “symmetric” in $\mathcal{A}$ and $\mathcal{A}^\circ$.

4 $U(1)$–equivariant projectable spectral triples

We will assume that the coaction of $H$ on $\mathcal{A}$ (which in the context of $C^*$-algebras is used to formulate the notion of free and proper action) comes from the usual action of $U(1)$ on $\mathcal{A}$. Moreover, we will require that there is an infinitesimal action $\delta$ of the generator of the Lie algebra $u(1) \simeq \mathbb{R}$.  

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To use more precise language, we shall consider real spectral triples over \( \mathcal{A} \), which are \( U(1) \)-equivariant \([20]\), that is, for which the action of \( U(1) \) (and of \( u(1) \) via \( \delta \)) is implemented on \( \mathcal{H} \) as follows.

**Definition 4.1.** A real spectral triple \( (\mathcal{A}, \mathcal{H}, D, J, \gamma) \) is a \( U(1) \)-equivariant real spectral triple iff there exists a selfadjoint operator \( \hat{\delta} \) on \( \mathcal{H} \) such that

\[
\hat{\delta}(\pi(a)\psi) = \pi(\delta(a))\psi + \pi(a)\hat{\delta}(\psi),
\]

and such that

\[
\hat{\delta}J + J\hat{\delta} = 0, \quad [\hat{\delta}, \gamma] = 0, \quad [\hat{\delta}, D] = 0,
\]
on the common core of \( \hat{\delta} \) and \( D \).

Actually, we require also that the spectrum of \( \hat{\delta} \) is \( \mathbb{Z} \) (it could be also \( \mathbb{Z} + \frac{1}{2} \)): this corresponds to the assumption that the \( U(1) \) action on the tangent bundle lifts to an action and not to a projective action on the spinor bundle. Hence, if \( (\mathcal{A}, \mathcal{H}, D, J, \gamma, \hat{\delta}) \) is a \( U(1) \)-equivariant real spectral triple, we can split the Hilbert space \( \mathcal{H} \) accordingly to the spectrum of \( \hat{\delta} \):

\[
\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k,
\]
and this decomposition is preserved by the Dirac operator \( D \). Moreover \( \pi(\mathcal{A}(k))\mathcal{H}_l \subseteq \mathcal{H}_{k+l} \) for any \( k, l \in \mathbb{Z} \); in particular \( \mathcal{H}_0 \) is stable under the action of the invariant subalgebra \( \mathcal{B} = \mathcal{A}^{e_H} = \mathcal{A}^{(0)} \). Throughout the rest of the paper we shall denote \( \hat{\delta} \) for simplicity by \( \delta \).

### 4.1 Operator of strong connection for the Dirac calculus

With a spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) there is associated the (first order differential) Dirac calculus \( \Omega^1_D(\mathcal{A}) \), given by the linear span of all operators of the form \( \delta[D, a], a, a' \in \mathcal{A} \), with the differential of \( a \) given by \( da = [D, a] \). On the Hopf algebra \( \mathcal{H} \) we shall use the calculus \( \Omega^1(\mathcal{H}) \) induced from the usual de Rham calculus on \( U(1) \) (which can be also viewed as Dirac calculus for the operator \( i\frac{\partial}{\partial \phi} \) on L2(\( U(1) \))).

Following \([14]\) we say that \( \Omega^1_D(\mathcal{A}) \) is compatible with \( \Omega^1(\mathcal{H}) \) iff

\[
\sum_i p_i[D, q_i] = 0, \text{ for } p_i, q_i \in \mathcal{A} \Rightarrow \sum_i p_i\delta(q_i) = 0.
\] (4.1)

This agrees with the compatibility of these calculi viewed as quotients of the universal calculi. Accordingly we can give the notion of connection form adapted to the Dirac-induced calculi, represented as operators on certain Hilbert space, as follows:

**Definition 4.2.** We say that \( \omega \in \Omega^1_D(\mathcal{A}) \) is a strong principal connection for the \( U(1) \)-bundle \( \mathcal{B} \hookrightarrow \mathcal{A} \) if the following conditions hold:

\[
[\delta, \omega] = 0, \quad (U(1) \text{ invariante of } \omega),
\]

if \( \omega = \sum_i p_i[D, q_i] \) then \( \sum_i p_i\delta(q_i) = 1 \), \( (\text{vertical field condition}) \),

\( \forall a \in \mathcal{A}, \nabla_\omega(a) := [D, a] - \delta(a)\omega \in \Omega^1_D(\mathcal{B})\mathcal{A} \), \( (\text{strongness}) \).
Note that the second condition is meaningful due to assumption (4.1).

The expression in the third condition defines a $D$-connection $\nabla_\omega$ on the left $\mathcal{B}$-module $\mathcal{A}$ (covariant derivative with respect to the calculus $\Omega^1_D(\mathcal{B})$), in the sense that

$$
\nabla_\omega(ba) = [D, b]a + b\nabla_\omega(a).
$$

(4.2)

It follows by direct computation, see lemma 5.5 in [14], that a $D$-connection $\nabla_\omega$ is hermitian if $\omega$ is selfadjoint as an operator on $\mathcal{H}$.

### 4.2 Projectable spectral triples: odd case

Now we are ready to study the projectability of the spectral triple in the framework of noncommutative geometry. We have to distinguish the odd dimensional case from the even dimensional one. We begin by considering the former.

Let $\mathcal{B} \hookrightarrow \mathcal{A}$ be a principal $H$-comodule algebra, where $H$ is the coordinate algebra of $U(1)$, and consider a $U(1)$-equivariant odd real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \delta)$ over the total space $\mathcal{A}$. Assume that the Dirac calculus $\Omega^1_D(\mathcal{A})$ is compatible with the de Rham calculus on $H$. Then we give the following definition (c.f. [14]).

**Definition 4.3.** An odd $U(1)$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta)$ of KR-dimension $j$ is said to be projectable along the fibers if there exists a $\mathbb{Z}_2$ grading $\Gamma$ on $\mathcal{H}$, which satisfies the following conditions,

$$
\Gamma^2 = 1, \quad \Gamma^* = \Gamma,
$$

$$
[\Gamma, \pi(a)] = 0 \quad \forall a \in \mathcal{A},
$$

$$
[\Gamma, \delta] = 0,
$$

$$
\Gamma J = \begin{cases} 
J\Gamma & \text{if } j \equiv 1 \pmod{4} \\
-J\Gamma & \text{if } j \equiv 3 \pmod{4}.
\end{cases}
$$

If such a $\Gamma$ exists, we define the horizontal Dirac operator $D_h$ by:

$$
D_h \equiv \frac{1}{2}\Gamma[\Gamma, D]
$$

Now we want to consider a special class of projectable spectral triples, which should represent the noncommutative counterpart of smooth $U(1)$ principal bundles which are Riemannian manifolds with fibers of constant length.

**Definition 4.4.** A projectable spectral triple has fibers of constant length if there is a positive real number $l$ such that, if we set

$$
D_v = \frac{1}{l}\Gamma\delta,
$$

the operator

$$
Z = D - D_h - D_v
$$

is a bounded operator which commutes with the representation of $\mathcal{A}$

$$
[Z, \mathcal{A}] = 0.
$$

(4.3)
In such a case, the operator $D_v$ is called the vertical Dirac operator and the number $l$, up to a $2\pi$ factor is the length of each fiber, as in the commutative (smooth) case.

**Remark 4.5.** Actually in [14] a different condition on $Z$ was imposed, namely that $Z$ commutes with the elements $J\pi(a^*)J^{-1}$ from the commutant of $A$. Our present choice is satisfied e.g if $Z \in J\pi(A)J^{-1}$, or if $Z$ is a related 0-order pseudodifferential operator, and is motivated by a recent example of curved noncommutative torus [15]. The condition [14, 15] implies that

$$[D_h, \pi(b)] = [D, \pi(b)], \quad \forall b \in B$$

and hence $D_h$ and $D$ induce the same first order differential calculus over the invariant subalgebra $B$ which was an additional assumption of 'projectability of differential calculus' in [14].

Consider now a projectable triple $(\mathcal{A}, \mathcal{H}, D, J, \delta, \Gamma)$ of KR-dimension $j$, with $j$ odd, and assume that it has fibers of constant length. Since $\Gamma$ and $D$ commute with $\delta$, also $D_h$ does. Therefore $D_h$ preserves each $\mathcal{H}_k$. Instead, the real structure intertwines $\mathcal{H}_k$ and $\mathcal{H}_{-k}$, $J\mathcal{H}_k \subseteq \mathcal{H}_{-k}$. In particular, it preserves $\mathcal{H}_0$. Now, let us denote, for any $k \in \mathbb{Z}$, by $D_k$, and $\gamma_k$ the restrictions to $\mathcal{H}_k$ of $D_h$ and $\Gamma$, respectively. For each $k \in \mathbb{Z}$ we define also an antiunitary operator $j_k : \mathcal{H}_k \to \mathcal{H}_{-k}$ as follows:

$$j_k = \begin{cases} \Gamma J & \text{if } j \equiv 1 \pmod{4} \\ J & \text{if } j \equiv 3 \pmod{4} \end{cases}$$

(4.5)

(where on the r.h.s. appropriate restrictions of $\Gamma$ and $J$ are understood). Now we can state the following

**Proposition 4.6.** $(\mathcal{B}, \mathcal{H}_0, D_0, \gamma_0, j_0)$ and, for $k \neq 0$, $(\mathcal{B}, \mathcal{H}_k \oplus \mathcal{H}_{-k}, D_k \oplus D_{-k}, \gamma_k \oplus \gamma_{-k}, j_k \oplus j_{-k})$, are even real spectral triples of KR-dimension $j - 1$.

**Proof.** We check here only the commutation relations of the operators $D_k, \gamma_k, j_k$ and the first order condition. For rest of the proof see [14], proposition 4.4. In fact we check these relation only on the subspace $\mathcal{H}_0$, since the extension of the computations below to the general case $k \in \mathbb{Z}$ is straightforward.

For $j = 3$ the result is already proved in [14]. Let us consider now the case $j = 1$: $[\Gamma, J] = 0$, $j_0 = \Gamma J$. We need to check that: $j_0^2 = 1$, $j_0D_0 = D_0j_0$, $\gamma_0j_0 = j_0\gamma_0$. We have:

$$j_0^2 = \gamma_0J\gamma_0J = \Gamma J\Gamma J = \Gamma^2 J^2 = 1,$$

$$j_0D_0 = \frac{1}{2} \Gamma J [\Gamma, D] = \frac{1}{2} (\Gamma JD - \Gamma J D \Gamma)$$

$$= \frac{1}{2} (\Gamma JD - JD \Gamma) = \frac{1}{2} (DJ\Gamma - \Gamma DJ) = \frac{1}{2} (D\Gamma J - \Gamma DJ) = D_0j_0,$$

$$\gamma_0j_0 = \Gamma \Gamma J = \Gamma J = j_0\gamma_0.$$

Now the case $j = 5$: $[\Gamma, J] = 0$, $j_0 = \Gamma J$. We need to check that $j_0^2 = -1$, $j_0D_0 = D_0j_0$, $\gamma_0j_0 = j_0\gamma_0$. Since the only difference with the previous case is that now $J^2 = -1$, the proof of the last two relations is the same as before. For the first one:

$$j_0^2 = \gamma_0J\gamma_0J = \Gamma J\Gamma J = \Gamma^2 J^2 = -1.$$
We are left with the proof of the proposition for \( j = 7 \). In this case we have \( j_0 = J \), \( J\Gamma = -\Gamma J \), and we have to check that \( j_0^2 = 1 \), \( j_0D_0 = D_0j_0 \), \( \gamma_0j_0 = -j_0\gamma_0 \). We have:

\[
j_0^2 = J^2 = 1,
\]

\[
j_0D_0 = \frac{1}{2}J\Gamma[\Gamma, D] = \frac{1}{2}(JD - J\Gamma D\Gamma)
\]

\[
= \frac{1}{2}(DJ - \Gamma D\Gamma J) = D_0j_0,
\]

\[
\gamma_0j_0 = \Gamma J = -J\Gamma = -j_0\gamma_0.
\]

The first order condition follows directly from (4.4), and from the fact that \( \Gamma \) commutes with \( D \) and with the elements of \( \mathcal{B} \), and it either commutes or anticommutes with \( J \).

4.3 Projectable spectral triples: even case

Now we extend the notion of projectable spectral triple to the even dimensional case. We give the following definition, which, as we shall see later, is consistent with the results obtained in the commutative (smooth) case [2, 1].

Definition 4.7. An even \( U(1) \)-equivariant real spectral triple \((A, H, D, J, \gamma, \delta)\) is said to be projectable along the fibers if there exists a \( \mathbb{Z}_2 \) grading \( \Gamma \) on \( H \), which satisfies the following conditions,

\[
\Gamma^2 = 1, \quad \Gamma^* = \Gamma,
\]

\[
[\Gamma, \pi(a)] = 0 \quad \forall a \in A,
\]

\[
[\Gamma, \delta] = 0,
\]

\[
\Gamma\gamma = -\gamma\Gamma,
\]

\[
\Gamma J = -J\Gamma.
\]

If such a \( \Gamma \) exists, we define the horizontal Dirac operator \( D_h \) by

\[
D_h \equiv \frac{1}{2}\Gamma[\Gamma, D]
\]

Also in this case we can introduce the property of constant length fibers, similarly as in definition 4.3. Consider now a projectable triple \((A, H, D, J, \gamma, \delta, \Gamma)\) of \( KR \)-dimension \( j \), with \( j \) even, and assume that it has fibers of constant length. Again \( D_h \) preserves each \( H_k \), and the real structure intertwines \( H_k \) and \( H_{-k} \). In particular, it preserves \( H_0 \). Let us denote, for any \( k \in \mathbb{Z} \), by \( D_k \), and \( \gamma_k \) the restrictions to \( H_k \) of \( D_h \) and \( \Gamma \), respectively. Define an operator \( \nu \) by \( \nu = i\Gamma\gamma \). Then \( \nu^* = \nu \) and \( \nu^2 = 1 \), and we can use it to split \( H_0 \). In particular we obtain the following result.

Proposition 4.8. Decompose \( H_0 \) as \( H_0 = H_0^{(+) \oplus H_0^{(-)}} \), where \( H_0^{(\pm)} \) are the \((\pm 1)\)-eigenspaces of \( \nu \). Then the horizontal Dirac operator \( D_h \) preserves both the subspaces \( H_0^{(\pm)} \). Moreover, if we denote respectively by \( D_0^{(\pm)} \) the restrictions of \( D_h \) to \( H_0^{(\pm)} \), \((B, H_0^{(\pm)}, D_0^{(\pm)})\) are spectral triples.
Proof. Clearly $D_h$ preserves $\mathcal{H}_0$, so for the first part of the proposition that $D_0$ preserves $\mathcal{H}_0^{(\pm)}$ we need only to check that $[D_0, \nu] = 0$. Indeed

$$[D_0, \nu] = \frac{1}{2} [\Gamma [\Gamma, D], \Gamma \gamma]$$

$$= \frac{1}{2} (D \Gamma \gamma + \Gamma \gamma D \Gamma - \Gamma D \gamma - \Gamma \gamma D)$$

$$= \frac{1}{2} (\Gamma \gamma D + D \gamma \Gamma - D \gamma \Gamma - \Gamma \gamma D) = 0.$$

Next since $\Gamma$ and $\gamma$ commute with $\mathcal{B}$, and $[D, b]$ is bounded since $\mathcal{B} \subset \mathcal{A}$, we have that $[D_0, b]$ is bounded for any $b \in \mathcal{B}$. Of course, both $\mathcal{H}_0^{(\pm)}$ are preserved by $\mathcal{B}$. So $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)})$, where $D_0^{(\pm)}$ are the restrictions of $D_0$, are spectral triples.

Moreover, it follows from remark 4 of [14] that the Dirac operators $D_0^{(\pm)}$ have compact resolvent.

\[ \square \]

Remark 4.9. Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$ as in the previous proposition. Notice that $\Gamma \nu = -\nu \Gamma$, so that $\Gamma \mathcal{H}_0^{(\pm)} \subset \mathcal{H}_0^{(\mp)}$. Since $\Gamma^2 = 1$, $\Gamma$ determines an isomorphism $\Gamma : \mathcal{H}_0^{(\mp)} \to \mathcal{H}_0^{(\pm)}$. Moreover, one can easily see that $D_h \Gamma = -\Gamma D_h$. So, $D_0^{(+)} = -D_0^{(-)}$ w.r.t. the isomorphism $\mathcal{H}_0^{(+)} \cong \mathcal{H}_0^{(-)}$ determined by $\Gamma$. This is nothing else than the noncommutative counterpart of the fact that, in the smooth case, the two Dirac operators $D^{(\pm)}$ are associated to the same metric, but they differ by a different choice of orientation $[2, 1]$. So we can say that the two triples $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)})$ differ only by the choice of (the sign of) the orientation. \[ \diamond \]

Now we can check if the spectral triples on $\mathcal{B}$ given by the previous proposition are real. We start with the $KR$-dimension 2 case.

Proposition 4.10. Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta, \Gamma)$ be a projectable real spectral triple of $KR$-dimension 2, fulfilling the condition of fibers of constant length. Then the antiunitary operator $\gamma J$ preserves both the subspaces $\mathcal{H}_0^{(\pm)}$. Moreover, if we denote by $j_0^{(\pm)}$ the restrictions of $\gamma J$ to $\mathcal{H}_0^{(\pm)}$ respectively, then $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0^{(\pm)})$ are real spectral triples of $KR$-dimension 1, and they differ just by the change of the sign of the orientation (see previous remark).

Proof. We know that both $J$ and $\gamma$ preserve $\mathcal{H}_0$ and if $j_0$ is the restriction of $\gamma J$ to $\mathcal{H}_0$ we see that $j_0$ preserves $\mathcal{H}_0^{(\pm)}$, since $[j_0, \nu] = 0$. We have also that $j_0$ commutes with $\Gamma$, since the spectral triple is projectable and has $KR$-dimension 2. Moreover, compute

$$D_0 j_0 = \frac{1}{2} (D \gamma J - \Gamma D \Gamma \gamma J) = -j_0 D_0,$$

as it should in $KR$-dimension 1.

Next since $\gamma$ commutes with $\mathcal{A}$, it is immediate that $j_0$ maps $\mathcal{B}$ to its commutant. Since $-J^2 = \gamma^2 = 1$ and $J \gamma = -\gamma J$, we have also $j_0^2 = 1$. Thus $j_0$, and also $j_0^{(\pm)}$, fulfill all the commutation relations required for a real structure of a real spectral triple of $KR$-dimension 1. Moreover, the first order condition follows from (4.4) and from the first order condition for the spectral triple over $\mathcal{A}$.

The last statement of the proposition follows from the fact that $\Gamma$ intertwines the two triples, as shown in remark 4.9. \[ \square \]
In order to extend the result of proposition 4.10 to higher dimensional even spectral triples we give the following definition.

**Definition 4.11.** Let \((A, \mathcal{H}, D, J, \gamma, \delta, \Gamma)\) be an even dimensional projectable real spectral triple of KR-dimension \(j\). Then we define a real structure \(j_0\) on \(\mathcal{H}_0\) by:

\[
\text{KR - dim } j_0 \equiv 0 \quad \begin{array}{c|c|c}
2 & 4 & 6 \\
\gamma J & J & \gamma J
\end{array}
\]  

(4.6)

where the restriction of \(\gamma\) and \(J\) to \(\mathcal{H}_0\) is understood.

With this definition of \(j_0\) we can prove the following result, which is a generalization of proposition 4.10.

**Proposition 4.12.** Let \((A, \mathcal{H}, D, J, \gamma, \delta, \Gamma)\) be a projectable even real spectral triple of KR-dimension \(j\), fulfilling the constant length fibers condition. Let \(j_0 : \mathcal{H}_0 \to \mathcal{H}_0\) be given by (4.6). Then \(j_0\) restricts to \(\mathcal{H}_0^{(\pm)}\), and \((B, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0^{(\pm)})\) are real spectral triples of KR-dimension \((j - 1)\). Moreover, they differ just by the change of the sign of the orientation.

**Proof.** We have already discussed the case \(n = 2\). So we prove the proposition separately in the other three cases. All what we need to check is that \(j_0^2 = \pm 1\) accordingly to KR-dimension \((j - 1)\), that \([j_0, \nu] = 0\), and that \(D_0^{(\pm)}\) and \(j_0^{(\pm)}\) satisfy the correct commutation relations (see table 1); the other properties (like the first order condition) are fulfilled for the same reasons as the previous proposition. The first condition is easily checked.

Let us check that \(j_0\) commutes with \(\nu\). Let \(j = 4\). Then \(j_0 = J\) and

\[
[j_0, \nu] = [J, i\Gamma\gamma] = [J, i\Gamma]\gamma = 0.
\]

Let \(j = 6\). Then \(j_0 = \gamma J\) and

\[
[j_0, \nu] = [\gamma J, i\Gamma\gamma] = \gamma[J, i\Gamma]\gamma = 0.
\]

Finally, let \(j = 0\). Then \(j_0 = J\) and

\[
[j_0, \nu] = [J, i\Gamma\gamma] = [J, i\Gamma]\gamma = 0.
\]

Now we have to check the commutation relation between \(j_0\) and \(D_0\). But we notice that the commutation relations between \(D_0\) and \(j_0\) are the same of those between \(j_0\) and \(D\). So, if \(j = 0\) or \(j = 4\) then \(j_0 = J\) and thus \(j_0D_0 = D_0j_0\), and it is consistent with the requirements, respectively, of KR-dimension 7 and 3. if \(j = 6\) then \(j_0D_0 = -D_0j_0\), as it should be in KR-dimension 5.

We conclude this section pointing out that, as for the odd dimensional case, for \(k \neq 0\) we can define real spectral triples \((B, \mathcal{H}_k \oplus \mathcal{H}_{-k}, D_k \oplus D_{-k}, j_k \oplus j_{-k})\) of KR-dimension \(j - 1\), simply extending the construction discussed above for the \(k = 0\) case.
5 Twisted spectral triples

In this section we shall use a strong connection on the circle bundle to twist the horizontal Dirac operator. We discuss here only the case of even spectral triples, for the reason that the odd dimensional case, which has less properties to check, follows immediately. The latter was discussed also in details in [14].

5.1 Twisted Dirac operators

Let the data $(A, H, J, D, \gamma, \delta, \Gamma)$ be a $U(1)$-equivariant projectable even real spectral triple, with constant length fibers, over a quantum principal $U(1)$-bundle $B \hookrightarrow A$, and assume the Dirac calculus $\Omega_D^1(A)$ to be compatible with the de Rham calculus on $H$ (here $H$ is the coordinate algebra of $U(1)$). Let $\omega \in \Omega_D^1(A)$ be a strong connection form. Let us notice that, for any $k \in \mathbb{Z}$, the set $A^{(k)}$ acting on the right on $H$ via the right action induced by the real structure $J$ (see eq. (3.1)), can be regarded as a set of bounded left $B$-linear maps between $H^{(\pm)}_0$ and $H^{(\pm)}_k$ (where, we recall, the $(\pm)$-decomposition is done accordingly to $\nu^2 = 1$). We introduce a new notation: we write the action of $A^{(k)}$ on the right, that is $m(h) \equiv \omega$. Then the $B$-linearity reads

$$ (bh)a = b(ha). $$

We observe that from the fact that $A$ is a principal comodule algebra, it follows that for any $k \in \mathbb{Z}$, $A^{(k)}$ is a projective $B$-module, which can be easily seen to be finitely generated. This structure is compatible with the right $B$-module structure induced on $H$ by the real structure $J$:

$$ (ba)(h) = a(hb), \quad \forall a \in A^{(k)}, b \in B, h \in H, $$

which in the right-handed notation becomes

$$ h(ba) = (hb)a. $$

Consider the left $H$-comodule $(V, \rho_L)$, where $V = \mathbb{C}$ and $\rho_L(\lambda) = z \otimes \lambda$. For any $k \in \mathbb{Z}$ define the $H$-comodule $(V^k, \rho_L^k)$ by setting $V^k = \mathbb{C}$, $\rho_L^k(\lambda) = z^k \otimes \lambda$. Then it is straightforward to see that $A^{(k)}$ is isomorphic to $A \Box_H V^k$ (where $\Box_H$ denotes the cotensor product over $H$ [3]). It follows that $A^{(k)}$ is a quantum associated bundle (see [3], def. 2.21).

We introduce now two assumptions:

(i) $H_0A^{(k)} \equiv A^{(k)}(H_0)$ is dense in $H_k$, where $A^{(k)}(H_0)$ is the linear span of elements $a(h)$, $a \in A^{(k)}, h \in H_0$;

(ii) the multiplication map from $H_0 \otimes_B A^{(k)}$ to $H_k$ is an isomorphism.

The first order condition (3.2) implies that there is a right action of $\Omega_D^1(B)$ on $H$, given by:

$$ h_\omega = -J\omega^* J^{-1} h \quad \forall \omega \in \Omega_D^1(B), \quad (5.1) $$
where $\omega^* \in \mathfrak{a}$ is the adjoint of $\omega$, s.t. $([D, b])^* = -[D, b^*]$ and
\[
 h[D, b] = D(hb) - (Dh)b.
\]

Such an action is clearly left $\mathcal{B}$-linear. Moreover, it induces a left action of $\Omega^1_\mathcal{D}(\mathcal{B})$ on $\mathcal{A}^{(k)}$, and $\Omega^1_\mathcal{D}(\mathcal{B})\mathcal{A}^{(k)}$ is just the space of all compositions $a \circ \omega$ of left $\mathcal{B}$-linear maps. For further details we refer to [14].

**Remark 5.1.** As we have seen in the previous sections, the real structure which makes the triple over $\mathcal{B}$ a real spectral triple is not always the simple restriction of $J$ to $\mathcal{H}_0$. Nevertheless both $J$ and the collection of $j_k$ induce the same right action of $\mathcal{A}$ on $\mathcal{H}$. Instead in order to define the right action of $\Omega^1_\mathcal{D}(\mathcal{B})$ on $\mathcal{H}_0$ we shall use the real structure $j_0$, to be consistent with $\mathbf{5.1}$. In this case it would not be correct to use $J$, since in general its commutation relation with $D_0$ is different from that of $j_0$, and the representation of a differential form involves the Dirac operator. ∘

Now we come to the construction of twisted spectral triples over $\mathcal{B}$. Using the $D_0^{(\pm)}$-connection $\nabla_\omega$ on the left $\mathcal{B}$-module $\mathcal{A}^{(k)}$, we twist the Dirac operators $D_0^{(\pm)}$ as follows
\[
 D_\omega^{(k, \pm)}(ha) = (D_0^{(\pm)}ha + h\nabla_\omega(a)) \quad (5.2)
\]
Note that as explained in [14], due to (i) and (ii), $D_\omega^{(k, \pm)}$ depends only on the product $ha \in \mathcal{H}_k$.

Let now $D_\omega^{(\pm)}$ be the respective closures of the direct sums of the two families $D_\omega^{(k, \pm)}$. In order to regard them as two twisted Dirac operators on $\mathcal{H}$ we need another assumption: (iii) given $Z$, there exists bounded selfadjoint $Z'$ such that
\[
 (Zh)a = Z'(ha), \quad \forall h \in \mathcal{H}, a \in \mathcal{A}.
\]
Note that it is weaker than assuming that $[Z, J\mathcal{A}J^{-1}] = 0$.

**Proposition 5.2.** The operators $D_\omega^{(\pm)}$ are selfadjoint if $\omega$ is a selfadjoint one form, and they have bounded commutators with all the elements of $\mathcal{A}$.

**Proof.** Take $h \in \mathcal{H}_0^{(\pm)}$ and $p \in \mathcal{A}^{(k)}$ such that $hp$ is in the domain of $D_\omega^{(\pm)}$. Then we have:
\[
 D_\omega^{(\pm)} = (D_0^{(\pm)}h)p + h[D, p] - khp\omega
 = (D_0^{(\pm)}h)p + [D, j_0\omega^*j_0^{-1}]h + j_0\omega^*j_0^{-1}khp
 = D(hp) + ((D_0^{(\pm)} - D)h)p + j_0\omega^*j_0^{-1}\delta(hp)
 = (D + j_0\omega^*j_0^{-1}\delta - Z')(hp), \quad (5.3)
\]
From $(5.3)$ follows, by standard results of functional analysis, the selfadjointness of $D_\omega$. Next, $D$ has bounded commutator with each $a \in \mathcal{A}$; $\omega$ is a one-form and so, due to the first order condition, the commutator of the second term of $(5.3)$ with $a \in \mathcal{A}$ is simply $j_0\omega^*j_0^{-1}\delta(a)$ and hence is bounded. The commutator with the first term is bounded simply because it is the commutator of two bounded operators. We conclude then that $[D_\omega, a]$ is bounded $\forall a \in \mathcal{A}$. $\square$
This shows that \((B, \mathcal{H}^{(\pm)}, D^{(\pm)}_\omega)\) is a (twisted) spectral triple (non necessarily real). Notice that the \(\pm\) label corresponds just to different choice of the orientation, as follows from remark 4.9. Moreover, since \(D^{(\pm)}_0\)-connection \(\nabla_\omega\) on the left \(B\)-module \(A\), restricts to \(D^{(\pm)}_0\)-connections \(\nabla_\omega\) on each \(B\)-module \(A^{(k)}\), it is clear that \((B, \mathcal{H}_k^{(\pm)}, D^{(k,\pm)}_\omega), k \in \mathbb{Z}\), is a family of twisted spectral triples, obtained by suitable restrictions.

**Corollary 5.3.** The “full” Dirac operator \(D_\omega := D^{(+)}_\omega \oplus D^{(-)}_\omega\) is selfadjoint if \(\omega\) is selfadjoint and has bounded commutator with all the elements of the algebra \(A\).

Finally, with \(Z\) as in definition 4.4 and the assumption (iii) above, we define

\[D_\omega := \Gamma \delta + D_\omega.\]

**Proposition 5.4.** \((A, \mathcal{H}, D_\omega)\) is a projectable spectral triple with constant length fibers and the horizontal part of the operator \(D_\omega\) coincides with \(D_\omega\).

**Proof.** See proof of proposition 5.8 in [14]. \(\square\)

Moreover, as in [14] we introduce the following notion of compatibility.

**Definition 5.5.** We say that a strong connection \(\omega\) is compatible with a Dirac operator \(D\) if \(D\) and \(D_h\) coincide on a dense subset of \(\mathcal{H}\).

Thus given a projectable spectral triple with constant length fibers and a compatible strong connection, the operators \(D^{(k)}\) are just \(D_0\) twisted by \(\nabla_\omega\) on \(A^{(k)}\).

## 6 Noncommutative tori

We will show how the canonical flat spectral triples over \(n\)-dimensional noncommutative tori are projectable and we will work out explicit formulae for the twisted Dirac operators for the noncommutative 2-torus as a bundle over the circle \(S^1\) and the noncommutative 4-torus as a bundle over a noncommutative 3-torus. (The case of 3-torus over the 2-torus is discussed in [14]).

### 6.1 Quantum principal \(U(1)\)-bundle \(C^\infty(S^1) \to \mathbb{T}_\theta^2\)

Let \(\mathcal{A} = \mathbb{T}_\theta^2\) be the unital smooth algebra of the noncommutative 2-torus, with two unitary generators \(U, V\) and the commutation relation \(U V = e^{2\pi i \theta} V U\) (\(\theta\) irrational). Let \(\delta_1, \delta_2\) be two derivations of \(\mathcal{A}\) determined by

\[\delta_2(U) = 0, \quad \delta_2(V) = V; \quad \delta_1(U) = U, \quad \delta_1(V) = 0.\]

Consider the \(U(1)\) action on \(\mathcal{A}\) given, at infinitesimal level, by \(\delta_2\). Then the invariant subalgebra \(\mathcal{B}\) is the (commutative) algebra generated by \(U\) and we can identify \(\mathcal{B}\) with \(C^\infty(S^1)\). Now let \(\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2\), where \(\mathcal{H}_\tau\) is the GNS Hilbert space associated to the unique tracial state on \(\mathcal{A}\). The derivations \(\delta_j\) can be implemented (via the commutator)
by selfadjoint operators on \( \mathcal{H} \) with integer spectrum, which for simplicity we denote by the same symbol \( \delta_j \). Let \( \mathcal{H}_k, k \in \mathbb{Z} \), be the \( k \)-eigenspace of \( \delta_2 \) in \( \mathcal{H} \).

Consider the standard “flat” Dirac operator on \( \mathcal{H} \),

\[
D = \sum_{i=1}^{2} \sigma^i \delta_i,
\]

where the \( \sigma^i \) are the Pauli matrices. There is also the orientation \( \mathbb{Z}_2 \)-grading

\[
\gamma = \text{id} \otimes \sigma_3
\]

and the real structure

\[
J = J_0 \otimes (i \sigma^2 \circ \text{c.c.}),
\]

where \( J_0 : \mathcal{H}_\tau \to \mathcal{H}_\tau \) is the Tomita-Takesaki antiunitary involution and \( \text{c.c.} \) denotes the complex conjugation.

**Proposition 6.1.** There exists a unique (up to a sign) operator

\[
\Gamma = \pm \text{id} \otimes \sigma^2 : \mathcal{H} \to \mathcal{H},
\]

such that \((\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta_2, \Gamma)\) is a projectable real spectral triple.

**Proof.** Since \( \Gamma \) has to commute with \( \pi(\mathcal{A}) \), and \( \mathcal{H}_\tau \) is an irreducible representation of \( \mathcal{A} \), we have that the most general form of an admissible \( \Gamma \) is: \( \Gamma = \alpha_0 \cdot \text{id} + \sum_{i=1}^{3} \alpha_i \sigma^i \) with \( \alpha_i \in \mathbb{C} \). And using \( \Gamma \gamma = -\gamma \Gamma \) we see immediately that \( \alpha_0 = 0 \) and \( \alpha_3 = 0 \). Next, from \( \Gamma^2 = -1 \) we obtain:

\[
\alpha_1^2 + \alpha_2^2 = 1.
\]

The last condition to impose is 4.3. In fact, already 4.4 suffices to get, for any \( v \in \mathcal{H} \),

\[
\sigma^1 U \delta_1 (v) = (\alpha_1 \alpha_2 \sigma^2 - \alpha_2^2 \sigma^1) U \delta_1 (v).
\]

This implies that the only solution is \( \alpha_2 = \pm 1 \). Moreover, \( \Gamma = \pm \sigma^2 \) is consistent with the commutation relation \( J \Gamma = -\Gamma J \).

It follows that \( D_h = \sigma^1 \delta_1 \), and that we can identify both \( \mathcal{H}_0^{(\pm)} \) with \( L^2(S^1, d\varphi) \); then the restriction of \( D_h \) to \( \mathcal{H}_0^{(\pm)} \) is given by:

\[
D_0^{(\pm)} = \pm i \frac{d}{d\varphi}.
\]

Also, the real structure \( j_0^{(\pm)} \) on \( \mathcal{H}_0^{(\pm)} \) is simply the complex conjugation.

**Twisted Dirac operators**

If we set \( D_v = \sigma^2 \delta_2 \) we see that the spectral triple discussed above has the constant length fibers property and in addition the operator \( Z \) turns out to be zero. In order to build the twisted Dirac operators we first need a strong connection over \( \mathcal{A} \). We have:
Lemma 6.2. A $U(1)$ selfadjoint strong connection over $\mathcal{A}$ is a one-form

$$\omega = \sigma^2 + \sigma^1 \omega_1,$$

where $\omega_1$ is a selfadjoint element of $\mathcal{B}$.

Proof. Let $\omega \in \Omega^1_D(\mathcal{A})$; then $\omega$ can be written as:

$$\omega = \sum_i a_i [D, c_i]$$

with $a_i, c_i \in \mathcal{A}$. This implies that we can generally write $\omega$ as

$$\omega = \sum_i \sigma^i \omega_i$$

with $\omega_i \in \mathcal{A}$. In order to be a strong connection, $\omega$ has to fulfill the properties of definition 4.2. In particular we need $[\delta, \omega] = 0$, and this implies $\omega_i \in \mathcal{B}$. Also, $\omega_3$ should be equal to zero, since we cannot obtain an operator such as $\sigma^3 \omega^3$ from a commutator $[D, a]$. Thus we are left with a connection of the form:

$$\omega = \sigma^1 \omega_1 + \sigma^2 \omega_2.$$  \hfill \Box

Now notice that, for $j = 1, 2$, we can write the Pauli matrices $\sigma^j$ as: $\sigma^j = U_j^{-1} [D, U_j]$ (where $U_1 = U, U_2 = V$). Then $\omega$ becomes:

$$\omega = \sum_{j=1}^2 \omega_j U_j^{-1} [D, U_j].$$

But now using the second condition of definition 4.2 we obtain: $\omega_2 = 1$. Thus the most general $U(1)$ strong connection on $\mathcal{A}$ is:

$$\omega = \sigma^2 + \sigma^1 \omega_1, \quad \omega_1 \in \mathcal{B}.$$

Proposition 6.3. For any selfadjoint $U(1)$ strong connection $\omega$, the associated Dirac operator $D_\omega$ has the form

$$D_\omega = D_h - \sigma^1 j_0 \omega_1 j_0^{-1} \delta_2.$$  \hfill \Box

Proof. From previous results we know that the projectability of the spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \delta)$ implies that there are two spectral triples over the invariant subalgebra $\mathcal{B}$, which in this case is simply the algebra of smooth functions on $S^1$. They are given by $(\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0)$. In order to fix the conventions, we say that on $\mathcal{H}_0^{(+)1}$ the Dirac operator $D_0$ it is given by $-\delta_1$, while on $\mathcal{H}_0^{(-)}$ it is given by $\delta_1$ (note that $\nu = -\sigma^1$, and thus $\sigma^1$ is diagonal w.r.t.
the decomposition $\mathcal{H}_0 = \mathcal{H}_0^{(+)} \oplus \mathcal{H}_0^{(-)}$. Now we have to compute the twisted operators $D_{\omega,k}^{(+)}$ on $\mathcal{H}_k^{(+)}$. Take $h_0 \in \mathcal{H}_0^{(+)}$. Then, for any $a \in \mathcal{A}^{(k)}$, we have:

$$D_{\omega,k}^{(+)}(h_0a) = (D_0^{(+)}h_0)a - h_0 \nabla_\omega(a) = -\delta_1(h_0a) + kh_0 a \omega_1.$$ 

Thus, if we take $h \in \mathcal{H}_k^{(+)}$ we see that the action of the twisted Dirac operator is given by

$$D_{\omega,k}^{(+)}(h) = -\delta_1(h) + j_0 \omega_1 j_0^{-1} \delta_2(h).$$

In the same way, one obtains that, for $h \in \mathcal{H}_k^{(-)}$,

$$D_{\omega,k}^{(-)}(h) = \delta_1(h) - j_0 \omega_1 j_0^{-1} \delta_2(h).$$

If now we put them together, and we consider the collection of all of them for any $k \in \mathbb{Z}$, we get that the full twisted Dirac operator $D_\omega$ is given, as an operator on $\mathcal{H}$, by

$$D_\omega = \sigma^1 \delta_1 - \sigma^1 j_0 \omega_1 j_0^{-1} \delta_2,$$

which is equal to $D_h = -\sigma^1 j_0 \omega_1 j_0^{-1} \delta_2$. 

**Corollary 6.4.** The only connection compatible with $D$, i.e. with the fully $U(1)^2$-equivariant Dirac operator, on the noncommutative 2-torus is $\omega = \sigma^2$.

**Proof.** It follows from previous lemma and definition 5.5. 

Now we can compute, given any strong connection $\omega$, the general form of a Dirac operator $D^{(\omega)}$ compatible with such a connection.

**Proposition 6.5.** Let $\omega = \sigma^2 + \sigma^1 \omega_1$ be a selfadjoint connection. Then the following Dirac operator,

$$D^{(\omega)} = D - \sigma^1 j_0 \omega_1 j_0^{-1} \delta_2,$$

is compatible with $\omega$.

**Proof.** It follows from definition 5.5 together with the computation of proposition 6.3. 

### 6.2 The principal extension $\mathbb{T}^3_{\theta'} \hookrightarrow \mathbb{T}^4_{\theta}$

Let $\mathcal{A}$ be the unital (smooth) algebra of the noncommutative 4-torus, generated by four unitaries $U_1, U_2, U_3, U_4$ with the commutation relations $U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i$, where $\theta_{ij}$ is an antisymmetric matrix with no rational entries and no rational relation between them. On $\mathcal{A}$ there is the canonical action of $U(1)^4$, whose generators are the derivations $\delta_j$,

$$\delta_j(U_k) = \delta_{kj} U_j.$$ 

As $U(1)$ quantum principal bundle structure we take the one given by the choice $\delta = \delta_4$, and we assume the relative spin structure to be the trivial one. Thus the invariant subalgebra $\mathcal{B}$ is the algebra generated by $U_1, \ldots, U_3$ and is isomorphic to the algebra of a noncommutative 3-torus.

We recall briefly the structure of (one of the) flat $\mathbb{T}^4$-equivariant spectral triples over $\mathbb{T}^4_{\theta}$. The commutation relations in $KR$-dimension 4 are the following ones:

$$J^2 = -1, \quad JD = DJ, \quad J\gamma = \gamma J. \quad (6.1)$$

In order to work out explicitly the operators, it is useful to recall the structure of the Clifford algebra $\mathbb{C}l(4)$ (so that we can fix the notation).
The Clifford algebra $\text{Cl}(4)$

The Clifford algebra $\text{Cl}(4)$ is generated by four elements, $\gamma^1, \ldots, \gamma^4$, with the relations
\begin{align}
\gamma^i \gamma^i &= 1, \\
\gamma^i \gamma^j &= -\gamma^j \gamma^i \quad \text{for } i \neq j, \\
\gamma^i \gamma^* &= \gamma^i.
\end{align}

(6.2)

We can represent the $\gamma^i$’s as $4 \times 4$ matrices, related to the Dirac matrices. In the so-called Dirac representation we can write the matrices $\gamma^i$ as:
\begin{align}
\gamma^4 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma^j &= \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix}.
\end{align}

(6.3)

Moreover, we can define a matrix $\gamma^5 \equiv \gamma^1 \gamma^2 \gamma^3 \gamma^4$ which satisfies $\gamma^5 \gamma^j = -\gamma^j \gamma^5$, $\gamma^5 \gamma^5 = 1$ and $\gamma^5 \gamma^* = \gamma^5$; in Dirac representation:
\begin{align}
\gamma^5 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align}

We recall, also, that using the Dirac matrices we can write down a basis for $M_4(\mathbb{C})$. In particular, if we define $\sigma^{ij} \equiv [\gamma^i, \gamma^j]$ (for $i, j = 1, \ldots, 4$), then the basis is given by:
\begin{align}
\text{id}, & \quad \gamma^5, \\
\gamma^i & \quad i = 1, \ldots, 4, \\
\gamma^5 \gamma^i & \quad i = 1, \ldots, 4, \\
\sigma^{ij} & \quad i < j.
\end{align}

(6.4)

A projectable spectral triple

Now, let $\mathcal{H}_\tau$ be the GNS Hilbert space associated to the canonical trace $\tau$ on $\mathcal{A}$ [16]. Define $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^4$. We consider the usual flat Dirac operator [16]:
\begin{align}
D = \sum_{j=1}^4 \gamma^j \delta_j.
\end{align}

Then we can take the orientation $\mathbb{Z}_2$-grading to be $\gamma = \gamma^5$. To define $J$, we recall that it is related to the charge conjugation operator; so we take
\begin{align}
J = J_0 \otimes (\gamma^4 \gamma^2 \circ \text{c.c.}),
\end{align}

where $J_0 : \mathcal{H}_\tau \rightarrow \mathcal{H}_\tau$ is the Tomita-Takesaki involution and c.c. denotes the complex conjugation. Then one can see that the spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ satisfies the relations (6.1), and it is also a $U(1)$-equivariant spectral triple, which is projectable:

**Proposition 6.6.** The unique operators $\Gamma : \mathcal{H} \rightarrow \mathcal{H}$, such that $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \delta_4, \Gamma)$ is a projectable real spectral triple with projectable differential calculus, are $\Gamma = \pm \text{id} \otimes \gamma^4$. 

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Proof. Since \([\Gamma, \pi(a)] = [\Gamma, \delta] = 0\) for all \(a \in \mathcal{A}\), \(\Gamma\) must be of the form \(\Gamma = \text{id} \otimes A\) for some matrix \(A \in M_4(\mathbb{C})\). Then using the fact that (6.4) give a basis of \(M_4(\mathbb{C})\), we can write \(\Gamma\) as

\[
\Gamma = a + b\gamma^5 + \sum_j c_j\gamma^j + \sum_j d_j\gamma^5\gamma^j + \sum_{i,j} e_{ij}\sigma^{ij}.
\]

From \(\Gamma\gamma = -\gamma\Gamma\) we deduce \(a = b = e_{ij} = 0\). Thus we are left with

\[
\Gamma = \sum_j (\alpha_j\gamma^j + \beta_j\gamma^5\gamma^j), \quad \alpha_j, \beta_j \in \mathbb{C},
\]

where \(\alpha_j \in \mathbb{R}\) and \(\beta_j \in i\mathbb{R}\), as follows from the condition \(\Gamma = \Gamma^*\). This implies that we can write \(\Gamma^2\) as:

\[
\Gamma^2 = \sum_j \alpha_j^2 + \sum_{i \neq j} 2\alpha_i\beta_j\gamma^i\gamma^j - \sum_j \beta_j^2.
\]

Next, using the condition \(\Gamma^2 = -1\), we deduce:

\[
\begin{aligned}
\begin{cases}
\alpha_i\beta_j = \alpha_j\beta_i & \forall i \neq j \\
\sum_j (\alpha_j^2 - \beta_j^2) = 1
\end{cases}
\end{aligned}
\]

(6.5)

We have now to impose that \(Z\) commutes with \(\mathcal{A}\); in particular we shall require that \([D, b] = [D_h, b]\) for all \(b \in \mathcal{B}\). Let us compute, first of all, \(D_h = \frac{1}{2}\Gamma[D, \Gamma]\) (we use the Einstein convention for the sum over repeated indices):

\[
D_h = \frac{1}{2}(\alpha_i\gamma^i + \beta_i\gamma^5\gamma^i)[\gamma^j\delta_j, \alpha_k\gamma^k + \beta_k\gamma^5\gamma^k]
= \frac{1}{2}(\alpha_i\gamma^i + \beta_i\gamma^5\gamma^i)(\alpha_k\delta^{jk}\delta_j - 2\beta_k\gamma^5\delta_k)
= \alpha_j\alpha_k\gamma^k\delta_j - \alpha_j\alpha_j\gamma^k\delta_k + \varepsilon_{ijkl}\alpha_i\alpha_k\gamma^k\gamma^l\delta_j
- \alpha_i\beta_k\gamma^k\gamma^5\delta_k + \varepsilon_{ijkl}\beta_i\alpha_k\gamma^l\delta_j + \beta_i\beta_k\gamma^l\delta_k.
\]

(6.6)

And now, from the condition \([D, b] = [D_h, b]\), using \([\delta_4, \mathcal{B}] = 0\) and the linear independence of the sixteen generators (6.4), we get:

\[
\begin{aligned}
\begin{cases}
\varepsilon_{ijkl}\alpha_i\alpha_k\gamma^5\gamma^l\delta_j - \alpha_i\alpha_k\gamma^5\gamma^l\delta_k = 0 \\
\sum_{j \neq k}(\alpha_j\alpha_k\gamma^k\delta_j + \beta_i\beta_k\gamma^l\delta_k) + \varepsilon_{ijkl}\beta_i\alpha_k\gamma^l\delta_j = 0 \\
\sum_{j \neq k}-\alpha_j^2\gamma^k\delta_k + \sum_{i} \beta_i^2\gamma^i\delta_i = \sum_{j = 1}^3 \gamma^j\delta_i.
\end{cases}
\end{aligned}
\]

(6.7)

The last condition implies:

\[
\begin{aligned}
\begin{cases}
\beta_i^2 = \sum_{j = 1}^3 \alpha_j^2 \\
\text{for } i \neq 4, \sum_{j \neq i} -\alpha_j^2 + \beta_i^2 = 1.
\end{cases}
\end{aligned}
\]

(6.8)
If now we use (6.8) to compute \( \sum_j \beta_j^2 \) we get:

\[
\sum_j \beta_j^2 = \sum_{j=1}^{3} \alpha_j^2 + \sum_{i=1}^{3} \left( 1 + \sum_{j \neq i} \alpha_j^2 \right).
\]  

(6.9)

Comparing (6.9) with the second equation of (6.5) we obtain the following relation:

\[
\alpha_4^2 + \frac{1}{2} \beta_4^2 = 1.
\]  

(6.10)

Now, we know that \( \alpha_j \in \mathbb{R} \) and \( \beta_j \in i\mathbb{R} \) (therefore \( b_j^2 \leq 0 \)). Thus, from (6.10) we obtain \( \alpha_4^2 \geq 1 \), while from the second equation of (6.5) we get \( \alpha_4^2 \leq 1 \). So the only solutions are \( \alpha_4 = \pm 1 \), \( \alpha_j = \beta_j = \beta_4 = 0 \) for \( j = 1, 2, 3 \). It is easy to see that such solutions fulfill all the other conditions of (6.5), (6.7). We conclude that the unique solutions for \( \Gamma \) are \( \Gamma = \pm \gamma^4 \).

Now we take one of the two solutions of the previous proposition, say \( \Gamma = \gamma^4 \). Then:

\[
D_h = \sum_{i=1}^{3} \gamma^i \delta_i, \quad D_v = \gamma^4 \delta_4.
\]

In particular the spectral triple has the “constant length fibers” property and, \( Z = 0 \).

Now we can build the “3-dimensional orientation”:

\[
\nu = i \Gamma \gamma = i \gamma^5 \gamma^4 = i \gamma^1 \gamma^2 \gamma^3.
\]

We have \( \nu^2 = 1 \), \( \nu^* = \nu \) as it should be. In \( 2 \times 2 \) matrix notation \( \nu \) is given by:

\[
\nu = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}.
\]

It is useful to write the down action of \( D_h \) on the two eigenspaces of \( \nu \). Clearly it is enough to know the action of the matrices \( \gamma_j \) \( (j = 1, 2, 3) \). Let us consider the 0-eigenspace \( \mathcal{H}_0 \) of \( \delta_4 \), and decompose it accordingly to \( \gamma \): \( \mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathcal{H}_0^- \). Then any vector can be written as \( v = v_+ \oplus v_- \). Moreover, since \( \Gamma = \gamma_4 \) is an intertwiner between \( \mathcal{H}_0^\pm \), these two spaces are isomorphic. If \( \mathcal{H}_0 = \mathcal{H}_0^{(+)} \oplus \mathcal{H}_0^{(-)} \) accordingly to \( \nu \), then:

\[
v \in \mathcal{H}_0^{(+)} \Rightarrow v \text{ is of the form } v = w \oplus (-i)w.
\]

\[
v \in \mathcal{H}_0^{(-)} \Rightarrow v \text{ is of the form } v = w \oplus iw.
\]

(6.11)

for some \( w \in \mathcal{H}^+ \). Using (6.3), we see that, for \( j = 1, 2, 3 \), \( \gamma^j \) acts as \( \pm \sigma^j \) on \( \mathcal{H}_0^{(+)} \). We summarize these results in the following lemma.

**Lemma 6.7.** Each Hilbert space \( \mathcal{H}_0^{(\pm)} \) is isomorphic to \( \mathcal{H}_\tau \otimes \mathbb{C}^2 \), where \( \mathcal{H}_\tau \) is the GNS Hilbert space associated to the canonical tracial state on \( \mathcal{B} = \mathcal{T}_0 \). Furthermore, the \( \gamma \) matrices, as operators on \( \mathcal{H} \), when restricted to \( \mathcal{H}_0^{(\pm)} \) act as \( \mp \sigma^j \).
Thus both the spectral triples are isomorphic to the “standard” one \[14\] on the non-
commutative 3-torus, with Dirac operators

\[ D_0^{(\pm)} = \mp \sum_{j=1}^{3} \sigma^j \delta_j. \]

Now we have to discuss the real structure. Since \( J \) is antiunitary, we see that \([J, i\Gamma] = 0\). And, since \( J\gamma = \gamma J \) and \( J^2 = -1 \) we can take \( j_0^{(\pm)} = J \) (restricted to \( \mathcal{H}_0^{(\pm)} \)) and obtain that \((\mathcal{B}, \mathcal{H}_0^{(\pm)}, D_0^{(\pm)}, j_0^{(\pm)})\) are real spectral triples of KR-dimension 3.

### Twisted Dirac operators

As we have done for the noncommutative 2-torus, we can construct now the twisted Dirac operators. We present here only the main results; the proofs are omitted, since they are a direct generalization of the proofs of the analogue results in the 2-dimensional case.

**Lemma 6.8.** A U(1) selfadjoint strong connection over \( \mathcal{A} \) is a one-form

\[ \omega = \gamma^4 + \sum_{j=1}^{3} \gamma^j \omega_j, \]

where \( \omega_j \) are selfadjoint elements of \( \mathcal{B} \).

Given a strong connection \( \omega \), the twisted Dirac operator is given by the following

**Proposition 6.9.** For any selfadjoint U(1) strong connection \( \omega \) the associated Dirac operator \( D_\omega \) has the form

\[ D_\omega = D_h - \sum_{j=1}^{3} \gamma^j J \omega_j J^{-1} \delta_4. \]

**Corollary 6.10.** The only connection compatible with \( D \) is \( \omega = \gamma^4 \). Moreover, let \( \omega = \gamma^4 + \sum_{j=1}^{3} \gamma^j \omega_j \) be a selfadjoint strong connection. Then the following Dirac operator,

\[ D_\omega = D - \sum_{j=1}^{3} \gamma^j J \omega_j J^{-1} \delta_4, \]

is compatible with \( \omega \).

### 7 Theta deformations. \( S^3_\theta \) as a U(1)-bundle over \( S^2 \)

Let \( \lambda \) be a complex number of module one, which is not a root of unity. The Dirac operator and isospectral spectral triple on the Matsumoto three-sphere \( S^3_\theta \) (a Rieffel deformation quantization \[19\] of \( C(S^3) \), called also theta deformation of \( S^3 \)) was presented in \[11\]. In \[10\] a deformation of \( S^{N-1} \) and of \( \mathbb{R}^N \) were introduced, with \( N \times N \) skew-symmetric matrix \( \theta \) of parameters. They were shown to have a more ‘functorial’ realization. Namely,
when $M = S^{N-1}$ or $\mathbb{R}^N$, $C^\infty(M_\theta)$ admits an action $\alpha^*$ of $\mathbb{T}^n$ (where $N = 2n$ or $N = 2n+1$) and there is a ‘splitting’ isomorphism

$$\kappa : C^\infty(M_\theta) \approx (C^\infty(M) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta))^{\alpha^* \otimes \beta^{-1}},$$

(7.1)

with the fixed point subalgebra of the action $\alpha^* \otimes \beta^{-1}$ of $\mathbb{T}^n$, given on the generators by

$$\kappa(z_{m,\theta}) = z_m \otimes u_{m,\theta}, \quad \forall m = 1, 2, \ldots, n,$$

(7.2)

where $z_{m,\theta}, z_m, u_{m,\theta}$ denote the $m$-th generator of $C^\infty(M_\theta), C^\infty(M), T^n_\theta$, respectively. Here $\widehat{\otimes}$ denotes a suitable completion of $\otimes$ and $\alpha^*, \beta$ denote the usual action of $\mathbb{T}^n$ on $C^\infty(\mathbb{T}^n_\theta)$, which consist of multiplying the relevant $m$-th generator by the $m$-th generator $u_m$ of $\mathbb{T}^n$.

In [10] the construction of “twisted” manifolds is extended to vector bundles over those manifolds $M$ which admit a direct lift of the action of $\mathbb{T}^n$.

$$\kappa : C^\infty(M_\theta, \Sigma) \approx (C^\infty(M, \Sigma) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta))^{\alpha^* \otimes \beta^{-1}}.$$  

(7.3)

The result is canonically a topological bimodule over $C^\infty(M_\theta)$. Using the usual trace functional $\tau$ on $C^\infty(\mathbb{T}^n_\theta)$ and given a hermitian structure on $\Sigma$ there is also a hermitian structure

$$(\psi \otimes t, \psi' \otimes t') = (\psi, \psi') \tau(t^* t')$$

with respect to which $C^\infty(M_\theta)$ can be completed to a Hilbert space $\mathcal{H}_\theta$. Moreover, any (also densely defined) operator $D$ which commutes with the action $\alpha^*$ defines an operator

$$D_\theta := (D \otimes I) \upharpoonright C^\infty(M_\theta, \Sigma).$$

Now we assume that $M$ is Riemannian spin manifold and that the action $\alpha$ lifts directly to the spinor bundle $\Sigma$ and to its smooth sections called Dirac spinors. Assuming that $\alpha$-invariant $D$ is the Dirac operator on $\Sigma$, we define the spectral triple $(C^\infty(M_\theta), \mathcal{H}_\theta, D_\theta)$, where $\mathcal{H}_\theta := (L^2(M, \Sigma) \widehat{\otimes} L^2(\mathbb{T}^n_\theta))^{\alpha^* \otimes \beta^{-1}}$, and $D_\theta$ is the closure of $D \otimes I$. Similarly the $\mathbb{Z}_2$-grading $\gamma$ on even-dimensional oriented $M$ and the antilinear charge conjugation operator $\tilde{J}$ can be twisted as follows

$$\tilde{\gamma} = \gamma \otimes I, \quad \tilde{J} = J \otimes \ast.$$  

As shown in [10] the spectral triple $(C^\infty(M_\theta), \mathcal{H}_\theta, D_\theta)$ together with the real structure $J_\theta$ and $\gamma$ satisfies all additional seven axioms [8, 9] required for a ‘noncommutative manifold’.

It is also not difficult to verify that the theta deformation (i.e. the twisting construction as above), behaves ‘functorially’ under the maps between manifolds (in particular under bundle projection) and respects the properties of principal $U(1)$-bundles

---

1otherwise c.f. e.g. [L.D.]

2for simplicity we assume that the $U(1)$-action coincides with the action of the last factor of $\mathbb{T}^n$ above as well as our requirements for projectable spectral triples and compatible connections.

Also the requirements (i-iii) in the previous sections can be seen to be satisfied.
this observations by the example of the $\theta$-deformation of the $U(1)$-Hopf fibration $S^3 \to S^2$ calculated explicitly in a concrete Hilbert space basis. In this case there is a $\mathbb{T}^2$-action on $S^3$ given by left and/or right multiplications by $\text{diag}(u, u^*)$ of the elements in $S^3 \equiv SU(2)$. Since $SU(2) \equiv Spin(3)$ is just a subgroup of $Spin(4)$ which in turn is the lift to spinors of the connected isometry group $SO(4)$ of $S^3$, the both the $U(1)$-factors, and so $\mathbb{T}^2$ itself lift directly (not just projectively) to spinors.

7.1 $\theta$-deformation of the $U(1)$-Hopf fibration $S^3 \to S^2$

The general result for the spectral triple construction of Drinfeld-type twists as isospectral deformations and their equivariance was discussed in [20].

The $U(1)$ action on the algebra $\mathcal{A}(S^3_\theta)$, which was studied in [5], provides an interesting example of a noncommutative Hopf fibration with the commutative algebra (identified with the algebra of functions on the sphere $S^2$) as the invariant subalgebra. We shall study here the Dirac operator on $\mathcal{A}(S^3_\theta)$ which arise from the base-space equivariant Dirac operator and the strong connection of the magnetic monopole (called in [5] Matsumoto magnetic monopoles).

We use the description of the algebra of $\mathcal{A}(S^3_\theta)$ as generated by normal operators $a, b$ and their hermitian conjugates, which satisfy the following relations:

$$ab = \lambda ba, \quad ab^* = \bar{\lambda}^* a, \quad aa^* + bb^* = 1. \quad (7.4)$$

We begin with the representation of the algebras acting on the Hilbert space of square integrable function over $S^3$. Using the standard orthonormal basis, we have the explicit formulae:

$$\pi(a)|l, m, n\rangle = \lambda^{\frac{1}{2}(m-n)} \left( \frac{\sqrt{l+1+m\sqrt{l+n}+1}}{\sqrt{2l+1+\sqrt{2l+2}}} |l^+, m^+, n^-\rangle - \frac{\sqrt{l-m\sqrt{l-n}+1}}{\sqrt{2l+1}} |l^-, m^+, n^-\rangle \right), \quad (7.5)$$

$$\pi(b)|l, m, n\rangle = \lambda^{-\frac{1}{2}(m+n)} \left( \frac{\sqrt{l+1+m\sqrt{l-n}+1}}{\sqrt{2l+1+\sqrt{2l+2}}} |l^+, m^-, n^-\rangle + \frac{\sqrt{l-m\sqrt{l+n}+1}}{\sqrt{2l+1}} |l^-, m^-, n^-\rangle \right), \quad (7.6)$$

where $l^\pm, m^\pm, n^\pm$ is a shortcut notation for $l \pm \frac{1}{2}, m \pm \frac{1}{2}, n \pm \frac{1}{2}$. Here $l = 0, \frac{1}{2}, 1, \ldots$ and both $m, n$ are in $-l, l - 1 + 1, \ldots, l - 1, l$. To pass to a spinor representation we need not only to double the Hilbert space but also to modify slightly the diagonal representation. Therefore, the spinor representation of $S^3_\theta$ is diagonal:

$$\pi(x) = \begin{pmatrix} \pi_+(x) & 0 \\ 0 & \pi_-(x) \end{pmatrix}, \quad (7.7)$$

but where $\pi_\pm$ differ from $\pi_0$ through the rescaling of the generators:

$$\pi_\pm(a) = \lambda^{\frac{1}{2}} \pi_0(a), \quad \pi_\pm(b) = \lambda^{-\frac{1}{2}} \pi_0(b).$$
The reality structure $J$, is off-diagonal:

$$J = \begin{pmatrix} 0 & J_0^- \\ J_0^+ & 0 \end{pmatrix},$$

(7.8)

where $J_0^\pm$ is a canonical equivariant antilinear map, which maps the algebra to its commutant:

$$J_0^\pm |l, m, n\rangle = \pm i^{2(m+n)}|l, -m, -n\rangle.$$

(7.9)

Let us verify that $J^2 = -1$:

$$J^2 |l, m, n, \pm\rangle = \pm J_0^\pm \left( i^{2(m+n)}|l, -m, -n, \mp\rangle \right)$$

$$= -i^{-2(m+n)}i^{-2(m+n)}|l, m, n, \pm\rangle = -|l, m, n, \pm\rangle,$$

where we have used that $m + n$ is always integer.

As the Dirac operator we take the following densely defined operator:

$$D|l, m, n, +\rangle = \left( R(m + \frac{1}{2}) + \frac{\alpha}{4} \right) |l, m, n, +\rangle$$

$$+ S\sqrt{l + 1 + m\sqrt{l - m}}|l, m + 1, n, -\rangle,$$

$$D|l, m, n, -\rangle = \left( -R(m - \frac{1}{2}) + \frac{\alpha}{4} \right) |l, m, n, -\rangle$$

$$+ S^*\sqrt{l - m + 1\sqrt{l + m}}|l, m - 1, n, +\rangle,$$

(7.10)

where $S$ is arbitrary complex number and $R$ and $\alpha$ are real. It is the most general Dirac-type operator, which is equivariant under the slightly reduced symmetry of the algebra, which is a Drinfeld twist of SU(2) and U(1). For $R = S = \alpha = 1$ this operator is isospectral to the fully equivariant Dirac operator over $S^3$ (but with the radius of the sphere $r = \frac{1}{2}$).

It is easy to verify that this operator satisfies the usual conditions $D^\dagger = D$ and $JD = DJ$.

**Lemma 7.1.** The following $U(1)$ action on the Hilbert space:

$$|l, m, n, \pm\rangle \rightarrow \phi \cdot |l, m, n, \pm\rangle = e^{i(2m\pm1)\phi}|l, m, n, \pm\rangle,$$

commutes with the Dirac operator $D$ and generates, by conjugation, the following action on the algebra:

$$a \rightarrow e^{i\phi}a, \quad b \rightarrow e^{i\phi}b.$$

The corresponding unbounded selfadjoint operator $\delta$ on the Hilbert space $\mathcal{H}$ is:

$$\delta|l, m, n, \pm\rangle = (2m \pm 1)|l, m, n, \pm\rangle,$$

and the above real spectral triple is $U(1)$ equivariant.
Proof. We compute the commutation of \( D' = D - \frac{\alpha}{4} \text{id} \) with the action of \( U(1) \):

\[
\phi \cdot (D'|l,m,n\pm) = \phi \cdot \left( \pm R(m \pm \frac{1}{2})|l,m,n,\pm\rangle + S^\pm \sqrt{l + 1 \pm m} \sqrt{l \mp m}|l,m\pm 1,n,\mp\rangle \right) \\
= \pm R(m \pm \frac{1}{2})e^{i(2m\pm 1)}|l,m,n,\pm\rangle + S^\pm \sqrt{l + 1 \pm m} \sqrt{l \mp m}|l,m\pm 1,n,\mp\rangle \\
= e^{i(2m\pm 1)} \left( \pm R(m \pm \frac{1}{2})|l,m,n,\pm\rangle + S^\pm \sqrt{l + 1 \pm m} \sqrt{l \mp m}|l,m\pm 1,n,\mp\rangle \right) \\
= D'(\phi \cdot |l,m,n,\pm\rangle).
\]

Since \( D' \) differs from \( D \) only by multiple of the identity operator, it satisfies the same identity.

The invariant subalgebra \( A^{U(1)} \) is generated by \( B = b^*a, B^* = a^*b \) and \( A = aa^* - \frac{1}{2} \), with the relations:

\[
AB = BA, \quad AB^* = B^*A, \quad A^2 + BB^* = \frac{1}{4},
\]

and therefore we shall identify it with the algebra of functions on the two-dimensional sphere, \( A(S^2) \).

Lemma 7.2. There exists an operator \( \Gamma \), \( \Gamma^2 = 1 \) and \( \Gamma J = -J \Gamma \) such that the spectral triple over \( A(S^3_\theta) \) is projectable.

Proof. If we take:

\[
\Gamma |l,m,n,\pm\rangle = \pm |l,m,n,\pm\rangle,
\]

then it is easy to see that all conditions for \( \Gamma \) are satisfied.

The horizontal Dirac operator is:

\[
D_h|l,m,n,\pm\rangle = S\sqrt{l + 1 \pm m} |l,m\pm 1,n,-\rangle \\
D_h|l,m,n,-\rangle = S^*\sqrt{l - m + 1} \sqrt{l + m} |l,m-1,n,\pm\rangle
\]

and we have:

Lemma 7.3. The noncommutative \( U(1) \) principal bundle has fibers of constant length \( \frac{2}{R} \).

Proof. Indeed, a simple computation shows the identity valid on the dense subset of the Hilbert space \( \mathcal{H} \):

\[
D = D_h + \frac{R}{2} \Gamma \delta + Z, \quad (7.11)
\]

where \( Z = \frac{\alpha}{4} \text{id} \).

Next, let us look at the invariant subspace of the Hilbert space \( \mathcal{H} \), by construction it shall provide us with a real spectral over the invariant subalgebra.

Lemma 7.4. The spectral triple obtained by restriction of \( \mathcal{H}, D, J \), and the operators \( A, B \) to the invariant subspace of \( \mathcal{H} \) is unitarily equivalent with the standard equivariant spectral triple over \( S^2 \).
Proof. The invariant subspace of $\mathcal{H}$, $\mathcal{H}_0 \subset \mathcal{H}$ is a closure of the linear span of the following vectors: $|l, -\frac{1}{2}, n, +\rangle$ and $|l, \frac{1}{2}, n, +\rangle$ for $l = \frac{1}{2}, \frac{3}{2}, \ldots$. Identifying these vectors with the standard equivariant basis of the Hilbert space of the spinor bundle over $S^2$, we obtain the desired isometry. Again, a simple check verifies that $\Gamma$ and $J$ are indeed the same as in the case of the standard triple over $S^2$. The restriction of $D$ to this space is:

$$D|l, -\frac{1}{2}, n, +\rangle = S(l + \frac{1}{2})|l, \frac{1}{2}, n, -\rangle, \quad D_h|l, m, n, -\rangle = S^* (l + \frac{1}{2})|l, -\frac{1}{2}, n, +\rangle,$$

so indeed we recover the standard Dirac operator.

Lemma 7.5. The first order differential calculus generated by $D$ is compatible with the usual de Rham differential calculus on $\mathcal{C}^\infty(S^1)$.

Proof. Let us observe that using the split of $D$ (7.11) we have:

$$[D, x] = [D_h, x] + \frac{R}{2} \Gamma \delta(x).$$

Since $x$ commutes with $\Gamma$ and $D_h$ anticommutes with $\Gamma$, then computing the anticommutator of $\sum_i p_i[D, q_i]$, for any $p_i, q_i \in \mathcal{A}(S^3_\theta)$ we obtain:

$$R \sum_i p_i \delta(q_i).$$

Therefore if the first expression vanishes, so must the latter and therefore the calculus generated by $D$ is compatible with the de Rham calculus over the circle.

We pass now to the strong connection.

Theorem 7.6. The following one-form is a strong connection for the spectral triples over the noncommutative $U(1)$-bundle $\mathcal{A}(S^3_\theta) \rightarrow \mathcal{A}(S^2)$:

$$\omega = \frac{R}{2}(a^* [D, a] + b^* [D, b]).$$

Proof. From construction it is clear that $\omega$ satisfies both $U(1)$ invariance:

$$[\delta, \omega] = 0,$$

as well as the vertical field condition (modified for a suitable length of fibers):

$$a^* \delta(a) + b^* \delta(b) = a^* a + b^* b = 1,$$

so the only nontrivial condition to verify is strongness.

To see that it is satisfied first we observe that the bimodule structure of $\Omega^1_D(\mathcal{A}(S^3_\theta))$ resembles the commutation rules of the algebra, that is $a da = da a$, $adb = \lambda db a$ etc. We skip listing of all relations as they are not used at any point later. The consequence of them, however, is that $da, da^*, db, db^*$ are the generators of $\Omega^1_D(\mathcal{A}(S^3_\theta))$ both as a left and as a right module. Moreover, the one forms $dA, dB, dB^*$ are central elements of this bimodule.
To demonstrate strongness, we need to show that $[D, x] - \delta(x)\omega \in \Omega^1(A(S^2))A(S^3_0)$. First, let us take $x$ to be the generators $a$ and $b$. Then, it could be explicitly verified that:

$$
da - \delta(a)\omega = a dA + b dB,$$
$$
 db - \delta(b)\omega = a dB^* + b dA,$$

and by conjugating both identities we obtain similar ones for $da^*$ and $db^*$. Then, observe that the operator:

$$
x \mapsto dx - \delta(x)\omega,$$

satisfies a Leibniz rule for any $x, y \in A(S^3_0)$:

$$
d(xy) - \delta(xy)\omega = (dx - \delta(x)\omega)y + x(dy - \delta(y)\omega),$$

because $\omega$ is a central one-form.

Putting it all together, for any polynomial in the generators $a, a^*, b, b^*$ we could show the strongness, As both the commutator with $D$ as well as $\delta$ extends easily to smooth functions on $S^3_0$, so does the property of strongness.

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