Recursive structure in the definitions of gauge-invariant variables for any order perturbations

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Abstract
The construction of gauge-invariant variables for any order perturbations is discussed. Explicit constructions of the gauge-invariant variables for perturbations to the forth order are shown. From these explicit constructions, the recursive structure in the definitions of gauge-invariant variables for any order perturbations is found. Through this recursive structure, the correspondence with the fully non-linear exact perturbations is briefly discussed.

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1. Introduction

Higher-order perturbation theory is a popular topic in the recent research on general relativity and has very wide applications: cosmological perturbations [1]; black hole perturbations [2]; and perturbations of stars [3]. However, the ‘gauge issues’ in higher-order perturbations are very delicate despite their wide applicability. Therefore, it is worthwhile to discuss the higher-order perturbation theory in general relativity from a general point of view. Therefore, we have been formulating the higher-order perturbation theory in general relativity from a gauge-invariant perspective [4–6] and have applied our formulation to cosmological perturbations [7]. Our work mainly concerns second-order perturbations except for reference [4]. In this paper, we discuss the gauge issues for any order perturbations.

General relativity is a theory based on general covariance, and that covariance is the reason that the notion of ‘gauge’ has been introduced into the theory. In particular, in general-relativistic perturbations, the second-kind gauge appears in perturbations, as Sachs pointed out [8]. In general-relativistic perturbation theory, we usually treat the one-parameter family
of spacetimes \( \{ (M_\lambda, Q_\lambda) \mid \lambda \in [0, 1] \} \) to discuss differences between background spacetime \( (M_\lambda, Q_\lambda = 0) \) and physical spacetime \( (M_\lambda, Q_\lambda = 1) \). Here, \( \lambda \) is the infinitesimal parameter for perturbations, \( M_\lambda \) is a spacetime manifold for each \( \lambda \), and \( Q_\lambda \) is the collection of the tensor fields on \( M_\lambda \). Since each \( M_\lambda \) is a different manifold, we have to introduce the point-identification map \( \chi_\lambda : M_\lambda \to M_\lambda \) to compare the tensor field on different manifolds. This point-identification is the gauge choice of the second kind. Since we have no guiding principle by which to choose the identification map \( \chi_\lambda \) due to the general covariance, we may choose a different point-identification \( \chi'_\lambda \) from \( \chi_\lambda \). This degree of freedom of choice is the gauge degree of freedom of the second kind. The gauge-transformation of the second kind is a change in this identification map. We note that this second-kind gauge is a different notion of the degree of freedom of coordinate choices on a single manifold, which is called the gauge of the first kind. Henceforth, we concentrate only on the gauge of the second kind, which we will call simply gauge.

Once we introduce the gauge choice \( \chi_\lambda : M_\lambda \to M_\lambda \), we can compare the tensor fields on different manifolds \( \{ M_\lambda \} \), and perturbations of a tensor field \( Q \) are represented by the difference

\[
\chi_\lambda^* Q - Q_\lambda = 0,
\]

where \( \chi_\lambda^* \) is the pull-back induced by the gauge choice \( \chi_\lambda \) and \( Q_\lambda \) is the background value of the variable \( Q \). We note that this representation of perturbations completely depends on the gauge choice \( \chi_\lambda \). If we change the gauge choice from \( \chi_\lambda \) to \( \chi'_\lambda \), the pulled-back variable of \( Q \) is then represented by \( \chi'^* \lambda Q \). These different representations are related to the gauge-transformation rules as

\[
\eta'^* \lambda Q = \Phi^* \lambda \chi'^* \lambda Q \chi_\lambda
\]

where

\[
\Phi := (\chi)^{-1} \circ \chi'
\]

is a diffeomorphism on \( M_\lambda \).

In the perturbative approach, we treat the perturbation \( \chi_\lambda^* Q \) through the Taylor series with respect to the infinitesimal parameter \( \lambda \) as

\[
\chi_\lambda^* Q = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \chi^n \lambda Q + O(\lambda^{k+1}),
\]

where \( \chi^n \lambda Q \) is the representation associated with the gauge choice \( \chi^n \lambda \) of the \( k \)th order perturbation of the variable \( Q \), with its background value \( \chi^n \lambda Q = 0 \). Similarly, we can have the representation of the perturbation of the variable \( Q \) under the gauge choice \( \chi'_\lambda \), which is different from \( \chi_\lambda \) as mentioned above. Since these different representations are related to the gauge-transformation rule (2), the order-by-order gauge-transformation rule between \( n \)th-order perturbations \( \chi^n \lambda Q \) and \( \eta'^* \lambda Q \) is given from the Taylor expansion of the gauge-transformation rule (2).

Since \( \Phi_\lambda \) is constructed by the product of diffeomorphisms, \( \Phi_\lambda \) is not given by an exponential map [4, 7, 9, 10], in general. For this reason, Sonego and Bruni [10] introduced the notion of a knight diffeomorphism. The knight diffeomorphism, which is generated by many generators, includes a wider class of diffeomorphisms than exponential maps, which are
generated by a single vector field. This knight diffeomorphism is suitable for our order-by-order arguments on the gauge issues of general-relativistic higher-order perturbations. Sonego and Bruni also derived the gauge-transformation rules for any order perturbations.

The purpose of this paper is to point out the recursive structure in the definition of the gauge-invariant variables for n-th-order perturbations. We use the gauge-transformation rules for perturbations derived by Sonego and Bruni. We demonstrate the explicit constructions of gauge-invariant variables to the fourth order. From these explicit constructions, we find the recursive structure in the definitions of the gauge-invariant variables for the n-th-order perturbations by using algebraic recursion relations (conjecture 4.1) and the decomposition of the linear metric perturbation into its gauge-invariant and gauge-variant parts (conjecture 3.1).

This paper is organized as follows. In section 2, we review the knight diffeomorphism introduced by Sonego and Bruni [10] and gauge-transformation rules derived from them. In section 3, we examine the construction of gauge-invariant variables to the fourth-order perturbations. These constructions are based on the conjecture which states that we already know how to construct gauge-invariant variables for linear-order metric perturbation (conjecture 3.1). In section 4, we discuss the recursive structure in the definitions of gauge-invariant variables for n-th-order perturbations. Although this discussion is based on the conjecture for algebraic identities (conjecture 4.1), these algebraic identities are confirmed to fourth-order perturbations in section 3. In section 5, we discuss the application of our formulae to cosmological perturbations as an example. The final (section 6) is devoted to the summary and discussions.

2. Gauge-transformation rules of higher-order perturbations

In this section, we briefly review a knight diffeomorphism, which is a representation of diffeomorphism proposed by Sonego and Bruni [10], and the gauge-transformation rules for n-th-order perturbations. In gauge-invariant perturbation theories, we may concentrate on the diffeomorphism on the background spacetime $M_0$. However, in this section, we denote the spacetime manifold by $M$ instead of $M_0$, since our arguments are not restricted to a specific background spacetime $M_0$ as in perturbation theories.

2.1. Knight diffeomorphism

Let $\phi^{(1)}, \ldots, \phi^{(k)}$ be exponential maps on $M$ which are generated by the vector fields $\xi^{(1)}, \ldots, \xi^{(k)}$, respectively. From these exponential maps, we can define a new one-parameter family of diffeomorphisms $\Psi^{(k)}_\lambda$ on $M$, whose action is given by

$$\Psi^{(k)}_\lambda := \phi^{(k)}_{\lambda^{1/4}} \circ \cdots \circ \phi^{(2)}_{\lambda^{1/2}} \circ \phi^{(1)}_\lambda .$$

$\Psi^{(k)}_\lambda$ displaces a point of $M$, a parameter interval $\lambda$ along the integral curve of $\xi^{(1)}$, then an interval $\lambda^{1/2}$ along the integral curve of $\xi^{(2)}$, and so on. For this reason, Sonego and Bruni, using chess-inspired terminology, called $\Psi^{(k)}_\lambda$ a knight diffeomorphism of rank $k$. The vector fields $\xi^{(1)}, \ldots, \xi^{(k)}$ are called the generators of $\Psi^{(k)}_\lambda$. The notion of this knight diffeomorphism is useful in perturbation theories in the theories of gravity with general covariance because any $C^1$ one-parameter family $\Psi_\lambda$ of diffeomorphisms can always be approximated by a family of knight diffeomorphisms of rank $k$. Actually, in [10], Sonego and Bruni presented the following theorem:
Theorem 2.1. Let $D$ be an appropriate open set in $[\lambda] \times M$ which includes $\{0\} \times M$, $\lambda \in \mathbb{R}$ and $\Phi : D \to M$ be a $C^k$ one-parameter family of diffeomorphisms. Then, there exists a set of exponential maps $\{\phi^{(1)}_\lambda, \ldots, \phi^{(k)}_\lambda\}$ on $M$ such that, up to the order $\lambda^{k+1}$, the action of $\Phi_\lambda$ is equivalent to one of the $C^k$ knight diffeomorphisms

$$
\Phi_\lambda = \Psi^{(k)}_\lambda + O(\lambda^{k+1}) = \phi^{(k)}_{j_1j_2} \circ \cdots \circ \phi^{(2)}_{j_2j_3} \circ \phi^{(1)}_{j_3j_4} + O(\lambda^{k+1}).
$$

If $\Phi$ and $\Psi$ are two diffeomorphisms of $M$ such that $\Phi^*f = \Psi^*f$ for every function $f$, it follows that $\Phi \equiv \Psi$. To show that a family of knight $\Psi^{(k)}_\lambda$ approximates any one-parameter family of diffeomorphisms $\Phi_\lambda$ up to the $\lambda^{k+1}$th order, it is sufficient to prove that $\Psi^{(k)}_\lambda f$ and $\Phi^*f$ differ by a function that is $O(\lambda^{k+1})$ for all $f$. We can always generalize the above approximation property of the action of a knight diffeomorphism $\Psi^{(k)}_\lambda$ for an arbitrary tensor function to that of the action for an arbitrary tensor field. For this reason, Sonogu and Bruni concentrated on Taylor-expansion of the pull-back $\Psi^{(k)}_\lambda f = \phi^{(1)}_{j_1j_2} \cdots \phi^{(k)}_{j_kj_{k+1}}f$ of a knight diffeomorphism for an arbitrary smooth function $f$ on $M$. Then they presented the following proposition:

Proposition 2.1. Let $\Phi$ be a one-parameter family of diffeomorphisms, and $T$ a tensor field such that $\Phi^* T$ is of class $C^k$. Then, $\Phi^* T$ can be expanded around $\lambda = 0$ as

$$
\Phi^* T = \sum_{i=0}^k \sum_{\|j\| = i} C_i \left( \{i\} \right) \xi^{j_1} \cdots \xi^{j_k} T + O(\lambda^{k+1}).
$$

Here, $J_n := \left\{ \{j\} \mid \forall i \in \mathbb{N}, j_i \in \mathbb{N}, s.t. \sum_{i=1}^n i_j = n \right\}$ defines the set of indices over which one has to sum in order to obtain the $n$th-order term,

$$
C_i \left( \{i\} \right) := \prod_{i=1}^k \frac{1}{(\lambda i)^{j_i} i_i!},
$$

and $O(\lambda^{k+1})$ is a remainder with $O(\lambda^{k+1})/\lambda^k \to 0$ in the limit $\lambda \to 0$.

Here we note that the expression on the right-hand side of equation (7) is just the form of the Taylor-expansion on the right-hand side of equation (5). From this fact, proposition 2.1, and the fact that $\Phi \equiv \Psi$ if $\Phi$ and $\Psi$ are two diffeomorphisms such that $\Phi^*f = \Psi^*f$ for every function $f$, we reach to the assertion of theorem 2.1. Therefore, we may regard the Taylor-expansion (7) in proposition 2.1 as the most general expression of the pull-back of diffeomorphism on $M$, and it is sufficient at least when we concentrate on perturbation theories. We also note that the properties of the set $J_n$ of integers are discussed in appendix A.

2.2. Gauge-transformation rule for nth-order perturbations

Through the notion of the knight diffeomorphism in the previous section, we derive the gauge-transformation rules for $\lambda^k$ perturbations. As mentioned in section 1, the gauge-transformation rule between the pulled-back variables $\lambda^k Q_\lambda$ and $\lambda^k Q_\lambda$ is given by (2). In perturbation theories, we always use the Taylor-expansion of these variables as in
To derive the order-by-order gauge-transformation rule for the \( n \)th-order perturbation, we have to know the form of the Taylor-expansion of the pull-back \( \Phi^\lambda \) of diffeomorphism. Then we use the general expression (7) of the Taylor expansion of diffeomorphisms in proposition 2.1 by Sonero and Bruni. Substituting equations (7) and (4) into equation (2), we obtain the order-by-order expression of the gauge-transformation rules between the perturbative variables \( ^{\sigma'}Q \) and \( ^{\sigma}Q \) as

\[
^{(n)}Q = \sum_{j=1}^{n} \frac{n!}{(n-j)!} \sum_{\{i\}} C_i \xi^{i_0} \ldots \xi^{i_{(n-j)}}. 
\]  

(9)

The order-by-order gauge-transformation rule (9) gives a complete description of the gauge behavior of perturbations at any order.

3. Definitions of gauge-invariant variables to fourth-order perturbations

Inspecting the gauge-transformation rule (9), we define gauge-invariant variables for metric perturbations and for perturbations of arbitrary tensor fields. Since the definitions of gauge-invariant variables for perturbations of arbitrary tensor fields are trivial if we accomplish the separation of the metric perturbations into their gauge-invariant and gauge-variant parts, we may concentrate on the metric perturbations.

First, we consider the metric \( \tilde{g}_{ab} \) on the physical spacetime \( (\mathcal{M}_{\mathcal{L}}, Q_{\mathcal{L}}) \). We expand the pulled-back metric \( \tilde{g}_{ab} \) to \( Q_0 + Q_1 \) as

\[
X^a \tilde{g}_{ab} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \tilde{g}_{ab} + O (\lambda^{n+1}). 
\]  

(10)

where \( \tilde{g}_{ab} := g_{ab} \) is the metric on the background spacetime \( \mathcal{M}_b \). Of course, the expansion (10) of the metric depends entirely on the gauge choice \( \lambda_i \). Nevertheless, henceforth, we do not explicitly express the index of the gauge choice \( \lambda_i \) if there is no possibility of confusion.

In [4], we proposed a procedure to construct gauge-invariant variables for higher-order perturbations. Our starting point to construct gauge-invariant variables was the following conjecture for the linear-metric perturbation \( \eta_{ab} \):

**Conjecture 3.1.** If there is a symmetric tensor field \( \eta_{ab} \) of the second rank, whose gauge transformation rule is

\[
y_{\xi} \eta_{ab} = \xi_{a} \eta_{ab}, \]  

(11)

then there exists a tensor field \( \mathcal{H}_{ab} \) and a vector field \( X^a \) such that \( \eta_{ab} \) is decomposed as

\[
\eta_{ab} = \mathcal{H}_{ab} + X^a \tilde{g}_{ab}, \]  

(12)

where \( \mathcal{H}_{ab} \) and \( X^a \) are transformed as

\[
y_{\xi} \mathcal{H}_{ab} = 0, \quad y_{\xi} X^a = \sigma^a \]  

(13)

under the gauge transformation (11), respectively.

In this conjecture, \( \mathcal{H}_{ab} \) is gauge-invariant and we refer to \( \mathcal{H}_{ab} \) as the \textit{gauge-invariant part} of the perturbation \( \eta_{ab} \). On the other hand, the vector field \( X^a \) in equation (16) is gauge dependent, and we refer to \( X^a \) as the \textit{gauge-variant part} of the perturbation \( \eta_{ab} \).
In this paper, we assume conjecture 3.1. This conjecture is quite important in our scenario of the higher-order gauge-invariant perturbation theory. In [6], we proposed an outline of a proof of conjecture 3.1. This outline of a proof is almost complete for an arbitrary background metric \( g_{ab} \). However, in this outline, there are missing modes for perturbations, which are called zero modes; we pointed out the physical importance of these zero modes in [6]. Therefore, we have to say that conjecture 3.1 is still a conjecture in our scenario of the higher-order gauge-invariant perturbation theory. If we can take these zero modes into account in the proof of conjecture 3.1, we may regard conjecture 3.1 as a theorem.

Inspecting the order-by-order gauge-transformation rules (9) and based on conjecture 3.1, we consider the recursive construction of gauge-invariant variables for higher-order metric perturbations. The proposal for this recursive construction has already been given in section 5 of reference [4]. In this paper, we try to carry out this proposal through the gauge-transformation rule (9) and show that this proposal is reduced to conjecture 3.1 and recursive relations of gauge-transformation rules for the gauge-variant variables for metric perturbations (conjecture 4.1 below).

According to equation (9), the order-by-order gauge-transformation rule for the \( n \)th-order metric perturbation \( g_{ab}^{n} \) is given by

\[
\sum_{l} (-1)^{l} \sum_{\{j\}} \frac{n!}{(n-l)!} \sum_{|j| \neq h} C_{l} \left( \{ j \} \right) \xi_{1}^{j} \cdots \xi_{l}^{j} \left( n-l \right) g_{ab}^{\left( n \right)}, \tag{14}
\]

To define the gauge-invariant variables from this gauge-transformation rule, we reconsider the recursive procedure to find the gauge-invariant variables proposed in [4].

3.1. First order

Since we assume conjecture 3.1 in this paper and the gauge-transformation rule for the first-order metric perturbation is given by

\[
\left( \frac{(1)}{a} \right) g_{ab} = \sum_{l} \frac{1}{(1-l)!} \sum_{\{j\}} \xi_{1}^{j} \cdots \xi_{l}^{j} g_{ab}^{1}, \tag{15}
\]

the first-order metric perturbation \( \left( \frac{(1)}{a} \right) g_{ab} \) is decomposed as

\[
\left( \frac{(1)}{a} \right) g_{ab} = \left( \frac{(1)}{a} \right) H_{ab} + \varepsilon_{a \gamma} g_{ab}^{\gamma}, \tag{16}
\]

\[
\left( \frac{(1)}{a} \right) H_{ab} - \varepsilon_{a \gamma} \left( \frac{(1)}{a} \right) H_{ab} = 0, \quad \left( \frac{(1)}{a} \right) X_{a} - \varepsilon_{a \gamma} \left( \frac{(1)}{a} \right) X_{a} = \varepsilon_{a \gamma}^{\gamma}. \tag{17}
\]

Through the gauge-variant vector field \( \left( \frac{(1)}{a} \right) X_{a} \), we can define the gauge-invariant variable \( \left( \frac{(1)}{a} \right) Q \) of the first-order perturbation for an arbitrary tensor field other than the metric as

\[
\left( \frac{(1)}{a} \right) Q := \left( \frac{(1)}{a} \right) Q + \sum_{l=1}^{n} \frac{1}{(1-l)!} \sum_{\{j\}} \xi_{1}^{j} \cdots \xi_{l}^{j} Q_{a}^{n-l} \left( \frac{(1-l)}{a} \right) \tag{18}
\]

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\]

3.2. Second order

The gauge-transformation rule for the second-order metric perturbation is given from equation (14) as
To define the gauge-invariant variables for $g_{ab}$, we consider the tensor field defined by

$$\hat{H}_{ab} = g_{ab} + 2\xi^{(1)ab}g_{ab} + \xi^{(2)ab}g_{ab},$$

where the vector field $^{(1)}X^a$ is defined as the gauge-variant part of the first-order metric perturbation $g_{ab}$ in equation (16) and $\xi^{(0)} = \{ (\xi^{(1)}, \xi^{(2)}, \ldots) = (0, 1, 0, 0, \ldots) \}$ is defined in appendix A. From expressions (19) and (21), it is easy to show that the gauge-transformation rule

$$\hat{H}_{ab} = \hat{H}_{ab} - \nabla^a \xi^b,$$

On the other hand, from expression (22), we obtain

$$\hat{H}_{ab} - \nabla^a \xi^b = g_{ab} + \frac{2!}{2(2 - 1)!} \sum_{\{i\} \in \mathcal{A}} C_1(\{i\}) \left( \xi^{(1)ab} - \xi^{(2)ab} \right) g_{ab} + \frac{2!}{2(2 - 2)!} \sum_{\{j\} \in \mathcal{A}_{2-1}} C_2(\{j\}) \left( \xi^{(1)ab} - \xi^{(2)ab} \right) g_{ab} = 2! \sum_{\{i\} \in \mathcal{A}} C_1(\{i\}) \left( \xi^{(1)ab} - \xi^{(2)ab} \right) g_{ab} + \frac{2!}{2(2 - 2)!} \sum_{\{j\} \in \mathcal{A}_{2-1}} C_2(\{j\}) \left( \xi^{(1)ab} - \xi^{(2)ab} \right) g_{ab} + \frac{2!}{2(2 - 1)!} \sum_{\{i\} \in \mathcal{A}} C_1(\{i\}) \left( \xi^{(1)ab} - \xi^{(2)ab} \right) g_{ab} + \frac{2!}{2(2 - 2)!} \sum_{\{j\} \in \mathcal{A}_{2-1}} C_2(\{j\}) \left( \xi^{(1)ab} - \xi^{(2)ab} \right) g_{ab} + \xi^{(1)ab} g_{ab}.$$ (24)
Furthermore, comparing equations (23) and (24), we obtain the identity
\[
2! \sum_{|i| \leq 2|a|} C_i(\{j\})(\xi_{i(1)}^h + \xi_{i(2)}^h - \xi_{i(1)}^h) + 2! \sum_{|i| \leq 2|a|} C_i(\{j\}) \xi_{j(1)}^h \sum_{|k_m| \leq 2|a|} C_i(\{k_m\}) \xi_{k_m(1)}^h
= \xi_{i(1)}^c.
\] (26)

Then we obtain the gauge-transformation rule for the variable \(\hat{H}_{ab}^{(2)}\) as the first equation in (23).

Since the gauge-transformation rule for the variable \(\hat{H}_{ab}^{(2)}\) is given in the first equation in (23), applying conjecture 3.1 to the variable \(\hat{H}_{ab}^{(2)}\), we can decompose \(\hat{H}_{ab}^{(2)}\) as
\[
\hat{H}_{ab}^{(2)} = \hat{H}_{ab} + \epsilon_{ab} \hat{X}^a + \hat{X}^a \hat{X}^a = \Xi_{ab}^0 + \hat{\Theta}_{ab}^{(2)},
\] (27)
where the gauge-transformation rules \(\hat{H}_{ab}^{(2)}\) and \(\hat{X}^a\) are given by
\[
\hat{H}_{ab}^{(2)} = \hat{H}_{ab}, \quad \hat{X}^a = \xi_{a(2)}^a + \hat{\sigma}_{a(2)}^a.
\] (28)

Thus, we have decomposed the second-order metric perturbation \(\hat{g}_{ab}^{(2)}\) into its gauge-invariant and gauge-variant parts as
\[
\hat{g}_{ab}^{(2)} = \hat{g}_{ab} + 2\epsilon_{(1)}^{(2)} \hat{g}_{ab} + (\xi_{(1)}^{(2)} - \xi_{(1)}^{(2)}) \hat{g}_{ab}.
\] (29)

Through substitution of the second equation in (28) into equation (26), we obtain
\[
2! \sum_{|i| \leq 2|a|} C_i(\{j\})(\xi_{i(1)}^h + \xi_{i(2)}^h - \xi_{i(1)}^h) + 2! \sum_{|i| \leq 2|a|} C_i(\{j\}) \xi_{j(1)}^h \sum_{|k_m| \leq 2|a|} C_i(\{k_m\}) \xi_{k_m(1)}^h
= -\xi_{i(1)}^h - \xi_{j(1)}^h + \xi_{j(1)}^h.
\] (30)

It is easy to see that the identity (30) is expressed as
\[
\sum_{|i| \leq 2|a|} C_i(\{j\})(\xi_{i(1)}^h + \xi_{i(2)}^h - \xi_{i(1)}^h) + \sum_{|i| \leq 2|a|} C_i(\{j\}) \xi_{j(1)}^h \sum_{|k_m| \leq 2|a|} C_i(\{k_m\}) \xi_{k_m(1)}^h = 0.
\] (31)

As shown in [4], through the gauge-variant variables \(\hat{X}^a\) and \(\hat{X}^a\), we can always define the gauge-invariant variables \(\hat{Q}^{(2)}\) for the second-order perturbation of an arbitrary tensor field other than the metric as
\[
\hat{Q}^{(2)} := \hat{Q} + \sum_{l=1}^{2} \frac{2!}{(2-l)!} \sum_{|i| \leq 2|a|} C_l(\{j\}) \xi_{j(1)}^l \xi_{j(2)}^l - \xi_{j(1)}^l \xi_{j(2)}^l \hat{Q}^{(2-l)}
= \hat{Q} + 2\epsilon_{(1)}^{(2)} \hat{Q} + (\xi_{(1)}^{(2)} + \xi_{(2)}^{(2)}) \hat{Q}.
\] (32)
3.3. Third order

The gauge-transformation rule for the third-order metric perturbation is given from equation (14) as

\[ g_{ab}^{(3)} - g_{ab}^{(3)} = \sum_{i=1}^{3} \frac{3!}{(3 - i)!} \sum_{\{ j \} \in \mathcal{L}} C_i\left( \{ j \} \right) \mathcal{E}^{(3)}_{i(1)} \cdots \mathcal{E}^{(3)}_{i(1)} g_{ab} \]  

\[ = 3\mathcal{E}_{i(1)}^{(2)} g_{ab}^{(3)} + 3 \left( \mathcal{E}_{i(i)}^{(1)} + \mathcal{E}_{i(1)}^{(2)} \right) g_{ab}^{(3)} \]  

\[ + \left( \mathcal{E}_{i(i)}^{(3)} + 3\mathcal{E}_{i(i)}^{(1)} \mathcal{E}_{i(1)}^{(2)} + \mathcal{E}_{i(1)}^{(3)} \right) g_{ab}^{(3)}. \]  

(33)

To define the gauge-invariant variables for \( g_{ab}^{(3)} \), we consider the tensor field defined by

\[ (3)H_{ab} := (3)g_{ab} + 3\mathcal{E}_{i(1)}^{(2)} g_{ab} + 3 \left( \mathcal{E}_{i(i)}^{(1)} + \mathcal{E}_{i(1)}^{(2)} \right) g_{ab} \]  

\[ + \left( \mathcal{E}_{i(i)}^{(3)} + 3\mathcal{E}_{i(i)}^{(1)} \mathcal{E}_{i(1)}^{(2)} + \mathcal{E}_{i(1)}^{(3)} \right) g_{ab}. \]  

(34)

As shown in [4], directly from expression (35), we have shown that the gauge-transformation rule for the variable \( (3)H_{ab} \) is given as

\[ (3)H_{ab} - (3)\hat{H}_{ab} = \mathcal{E}_{i(1)}^{(3)} g_{ab} \]  

(37)

\[ \sigma_{(3)}^a := \xi_{(3)}^a + \hat{\sigma}_{(3)}^a, \]  

(38)

\[ \hat{\sigma}_{(1)}^a := 3 \left( \xi_{(1)}^{(1)} , \xi_{(2)}^{(1)} \right) + 3 \left( \xi_{(1)}^{(1)} X \right) + 2 \left( \xi_{(1)}^{(1)} , \left[ \xi_{(1)}^{(1)} X \right] \right) \]  

\[ + \left[ \xi_{(1)}^{(1)} X , \left[ \xi_{(1)}^{(1)} X \right] \right]. \]  

(39)

On the other hand, from expression (36), the gauge-transformation rule for the variable \( (3)H_{ab} \) is also given as
\[ 3! \sum_{|j_i| \in A_i} C_1 \left( \{ j_i \} \right) \left( \mathcal{L}^{h}_{\frac{\partial}{\partial x^a}} - \mathcal{L}^{h}_{\frac{\partial}{\partial x^b}} + \mathcal{L}^{h}_{\epsilon_{i1}} \right) \hat{X} \gamma_{ab} \]

\[ + 3! \sum_{|j_i| \in A_i} C_2 \left( \{ j_i \} \right) \left( \mathcal{L}^{h}_{\frac{\partial}{\partial x^a}} \mathcal{L}^{h}_{\frac{\partial}{\partial x^b}} - \mathcal{L}^{h}_{\frac{\partial}{\partial x^c}} \mathcal{L}^{h}_{\frac{\partial}{\partial x^d}} - \mathcal{L}^{h}_{\epsilon_{i1}} \mathcal{L}^{h}_{\epsilon_{i2}} \right) \hat{X} \gamma_{ab} \]

\[ + \sum_{|j_i| \in A_i} C_1 \left( \{ j_i \} \right) \mathcal{L}^{h}_{\frac{\partial}{\partial x^a}} \sum_{|k_m| \in A_i} C_1 \left( \{ k_m \} \right) \mathcal{L}^{h}_{\epsilon_{i1}} \hat{X} \gamma_{ab} \]

\[ + \mathcal{L}^{h}_{\epsilon_{i1}} \hat{X} \gamma_{ab}, \quad (40) \]

To obtain expression (40), we used lower-order gauge-transformation rules (15) and (19) for the metric perturbations. Furthermore, we used identities (25) and (31) to reach expression (41).

We note that gauge-transformation rule (37) with equation (38) for the variable \( \hat{h} \) yields

\[ 3! \sum_{|j_i| \in A_i \setminus A''} C_2 \left( \{ j_i \} \right) \left( \mathcal{L}^{h}_{\frac{\partial}{\partial x^a}} \mathcal{L}^{h}_{\frac{\partial}{\partial x^b}} - \mathcal{L}^{h}_{\frac{\partial}{\partial x^c}} \mathcal{L}^{h}_{\frac{\partial}{\partial x^d}} - \mathcal{L}^{h}_{\epsilon_{i1}} \mathcal{L}^{h}_{\epsilon_{i2}} \right) \hat{X} \gamma_{ab} \]

\[ + 3! \sum_{|j_i| \in A_i} C_1 \left( \{ j_i \} \right) \mathcal{L}^{h}_{\frac{\partial}{\partial x^a}} \sum_{|k_m| \in A_i} C_1 \left( \{ k_m \} \right) \mathcal{L}^{h}_{\epsilon_{i1}} \hat{X} \gamma_{ab} \]

\[ + 3! \sum_{|j_i| \in A_i} C_2 \left( \{ j_i \} \right) \mathcal{L}^{h}_{\frac{\partial}{\partial x^a}} \sum_{|k_m| \in A_i} C_1 \left( \{ k_m \} \right) \mathcal{L}^{h}_{\epsilon_{i1}} \hat{X} \gamma_{ab} \]

\[ = \mathcal{L}^{h}_{\epsilon_{i1}}, \quad (42) \]

since the background metric \( \gamma_{ab} \) is arbitrary.
On the other hand, gauge-transformation rule (37) together with conjecture 3.1 implies that the variable \((3) \tilde{H}_{ab}\) is decomposed as
\[
(3) \tilde{H}_{ab} = (3) \mathcal{H}_{ab} + \xi_{a3} X_{ab}, \tag{43}
\]
where the gauge-transformation rules \((3) \mathcal{H}_{ab}\) and \((3) X^{a}\) are given by
\[
(3) \mathcal{H}_{ab} - (3) \mathcal{X}^{a} = 0, \quad (3) X^{a} = \mathring{\xi}_{a3} + \mathring{\sigma}_{a3}^{b}. \tag{44}
\]
Thus, we have decomposed the third-order metric perturbation \((3) g_{ab}\) into its gauge-invariant and gauge-variant parts as
\[
(3) g_{ab} = (3) \mathcal{H}_{ab} - \sum_{j=1}^{3!} \frac{1}{(3-j)!} \sum_{|\mathcal{I}|=j} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3-j)_{\mathcal{I}}} \mathcal{H}^{(3-j)_{\mathcal{I}}}_{\mathcal{I}}, \tag{45}
\]
\[
= (3) \mathcal{H}_{ab} + 3 \mathcal{L}_{(3)}^a g_{ab} + 3 \left( - \mathcal{L}_{(3)}^a X + \mathcal{L}_{(3)}^a \right) (3) g_{ab}
+ \left( \mathcal{L}_{(3)}^a - 3 \mathcal{L}_{(2)}^a \mathcal{E}_{(3)} + \mathcal{L}_{(2)}^{a3} \right) g_{ab}. \tag{46}
\]
As shown in [4], through the gauge-variant variables \((1) X^a\), \((2) X^a\), and \((3) X^a\), we can always define the gauge-invariant variables \((3) \mathcal{Q}\) for the third-order perturbation of an arbitrary tensor field other than the metric as
\[
(3) \mathcal{Q} = (3) \mathcal{Q} + \sum_{j=1}^{3!} \frac{1}{(3-j)!} \sum_{|\mathcal{I}|=j} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3-j)_{\mathcal{I}}} \mathcal{Q}^{(3-j)_{\mathcal{I}}}. \tag{47}
\]
Substitution of the second equation in (44) into equation (42) leads to the identity
\[
3! \sum_{|\mathcal{I}|=3} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3)_{\mathcal{I}}} \mathcal{Q}^{(3)_{\mathcal{I}}}
+ \sum_{|\mathcal{I}|=2} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3)} \mathcal{Q}^{(3)}
+ \sum_{|\mathcal{I}|=1} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3)} \mathcal{Q}^{(3)}
= - \mathcal{L}_{(3)}^a X + \mathcal{L}_{(3)}^a \mathcal{E}_{(3)} - \mathcal{E}_{(3)}, \tag{48}
\]
which is equivalent to the identity
\[
\sum_{|\mathcal{I}|=3} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3)_{\mathcal{I}}} \mathcal{Q}^{(3)_{\mathcal{I}}}
+ \sum_{|\mathcal{I}|=2} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3)} \mathcal{Q}^{(3)}
+ \sum_{|\mathcal{I}|=1} \mathcal{C}_{\mathcal{I}} \left( \{ i \} \right) \mathcal{E}^{(3)} \mathcal{Q}^{(3)}
= 0. \tag{49}
\]

3.4. Fourth order

The gauge-transformation rule for the fourth-order metric perturbation is given from equation (14) as

\[
\mathcal{L}_{(4)}^a = \mathring{\xi}_{a4} + \mathring{\sigma}_{a4}^{b}. \tag{50}
\]
Inspecting this gauge-transformation rule, we define the gauge-invariant and gauge-variant variables for \( \hat{g}_{ab}^{(4)} \). To do this, as in the case of the second- and third-order perturbations, we consider the tensor field defined by

\[
\hat{\sigma}_{\hat{a}\hat{b}} := \hat{g}_{\hat{a}\hat{b}} + \sum_{i=1}^{4} \frac{4!}{(4 - i)!} \sum_{\{j\} \in \mathcal{H}} C_i(\{j\}) \hat{\xi}_{\hat{a}j} \cdots \hat{\xi}_{\hat{b}j} \hat{g}_{\hat{a}\hat{b}} + 4! \sum_{\{j\} \in \mathcal{H}} C_i(\{j\}) \xi_{\hat{a}j} \cdots \xi_{\hat{b}j} \hat{g}_{\hat{a}\hat{b}}
\]

(51)

where \( \hat{\xi}^a, \xi^a, (1)X^a \) have been defined previously. Through identities (25), (31), and (49), the gauge-transformation rule for the variable \( \hat{\sigma}_{\hat{a}\hat{b}} \) is given by

\[
\hat{\sigma}_{\hat{a}\hat{b}} = \hat{\xi}_{\hat{a}\hat{b}} + \sum_{i=1}^{4} \frac{4!}{(4 - i)!} \sum_{\{j\} \in \mathcal{H}} C_i(\{j\}) \hat{\xi}_{\hat{a}j} \cdots \hat{\xi}_{\hat{b}j} \hat{\xi}_{\hat{a}\hat{b}} + 4! \sum_{\{j\} \in \mathcal{H}} C_i(\{j\}) \xi_{\hat{a}j} \cdots \xi_{\hat{b}j} \hat{\xi}_{\hat{a}\hat{b}}
\]

(55)

Then we may apply conjecture 3.1 to the variable \( \hat{\sigma}_{\hat{a}\hat{b}} \) and decompose \( \hat{\sigma}_{\hat{a}\hat{b}} \) into its gauge-invariant and gauge-variant parts as

\[
\hat{\sigma}_{\hat{a}\hat{b}} = \sigma_{\hat{a}\hat{b}} + \hat{\xi}_{\hat{a}\hat{b}}
\]

(54)

(55)
where the gauge-transformation rules for the variables $^{(4)}\mathcal{H}_{ab}$ and $^{(4)}X^a$ are given by

$^{(4)}\mathcal{H}_{ab} - ^{(4)}X^a \partial_a \mathcal{H}_{ab} = 0, \quad ^{(4)}X^a = \sigma^a_{\hat{a} a} = \xi^a_{\hat{a} a} + \hat{\sigma}^a_{\hat{a} a}$. (57)

Thus, we have decomposed the fourth-order metric perturbation $^{(4)}g_{ab}$ into its gauge-invariant and gauge-variant parts as

$^{(4)}g_{ab} = ^{(4)}\mathcal{H}_{ab} - \sum_{l=1}^{4} \frac{4!}{(4-l)!} \sum_{\{\hat{l}\}} \sigma^l \left( \{ \hat{l} \} \right) \xi^l_{\hat{l} l} \cdots \xi^l_{\hat{l} l} \left( ^{(4-l)}g_{ab} \right)$. (58)

As in the case of the lower-order perturbations, we can always define the gauge-invariant variables $^{(4)}Q$ for the fourth-order perturbation of an arbitrary tensor field other than the metric through the gauge-variant parts $^{(1)}X^a$, $^{(2)}X^a$, $^{(3)}X^a$, and $^{(4)}X^a$ of the metric perturbations:

$^{(4)}Q := ^{(4)}Q + \sum_{l=1}^{4} \frac{4!}{(4-l)!} \sum_{\{\hat{l}\}} \sigma^l \left( \{ \hat{l} \} \right) \xi^l_{\hat{l} l} \cdots \xi^l_{\hat{l} l} \left( ^{(4-l)}Q \right)$. (59)

We also note that the gauge-transformation rules (52) and (53) and the second equation in (57) imply the identity

$4! \sum_{\{\hat{j}\} \in Q \setminus \{\hat{h}\}} C_4 \left( \{ \hat{j} \} \right) \left( \xi^h_{\hat{j} (1)} \cdots \xi^h_{\hat{j} (3)} + \xi^h_{\hat{j} (1)} \cdots \xi^h_{\hat{j} (3)} \right)
= \xi^h_{\hat{j} (3)} \cdots \xi^h_{\hat{j} (3)}
+ \frac{4!}{3!} \sum_{n=1}^{3} \sum_{\{\hat{j}\} \in \hat{b}} C_3 \left( \{ \hat{j} \} \right) \xi^h_{\hat{j} (n)} \cdots \xi^h_{\hat{j} (n)}
\times \sum_{\{\hat{k}\} \in \hat{a}} C_3 \left( \{ \hat{k} \} \right) \xi^h_{\hat{k} (3)} \cdots \xi^h_{\hat{k} (3)}
= \xi_{\hat{j}(3)}$. (60)

Substituting the second equation in (57) into (60), we obtain the identity

$4! \sum_{\{\hat{j}\} \in Q \setminus \{\hat{h}\}} C_4 \left( \{ \hat{j} \} \right) \left( \xi^h_{\hat{j} (1)} \cdots \xi^h_{\hat{j} (3)} + \xi^h_{\hat{j} (1)} \cdots \xi^h_{\hat{j} (3)} \right)
= \xi^h_{\hat{j} (3)} \cdots \xi^h_{\hat{j} (3)}
+ \frac{4!}{3!} \sum_{n=1}^{3} \sum_{\{\hat{j}\} \in \hat{b}} C_3 \left( \{ \hat{j} \} \right) \xi^h_{\hat{j} (n)} \cdots \xi^h_{\hat{j} (n)}
\times \sum_{\{\hat{k}\} \in \hat{a}} C_3 \left( \{ \hat{k} \} \right) \xi^h_{\hat{k} (3)} \cdots \xi^h_{\hat{k} (3)}
= - \xi_{\hat{j}(3)} - \xi_{\hat{j} (3)}$. (61)
This identity is also expressed as
\[
\sum_{[i]} C_i \left( \{ j \} \right) \left( \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ j \}} + \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ j \}} \right) \\
= \mathcal{E}^{h}_{-\frac{1}{2}X} \cdots \mathcal{E}^{h}_{-\frac{1}{2}X} \\
+ \sum_{i=1}^{3} \sum_{[j]} C_i \left( \{ j \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ j \}} \\
\times \sum_{[k]} C_i \left( \{ k \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ k \}} \\
= 0, \tag{62}
\]

or, equivalently,
\[
\sum_{i=1}^{4} \sum_{[j]} C_i \left( \{ j \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ j \}} \sum_{[k]} C_i \left( \{ k \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ k \}} \\
= \sum_{[j]} C_i \left( \{ j \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ j \}}. \tag{63}
\]

4. Recursive structure in the definitions of gauge-invariant variables for \( n \)-th order perturbations

In the last section, we have shown the construction of gauge-invariant variables to the fourth order. From these constructions, we reasonably expect that we can generalize to \( n \)-th order perturbations. In this section, we show that generalization.

As noted in section 3, the gauge-transformation rule for the \( n \)-th order metric perturbation is given by equation (14). Inspecting this gauge-transformation rule, we construct the gauge-invariant variables for \(^{(i)}g_{ab}\). Through the construction of gauge-invariant variables for \(^{(i)}g_{ab}\) \((i = 1, \ldots, n - 1)\), we can also define the vector fields \(^{(i)}X^a\) \((i = 1, \ldots, n - 1)\) through the construction
\[
^{(i)}X^a - \xi^a = \sigma^a_{\{ i \}} \equiv \xi^a_{\{ i \}} + \hat{\sigma}^a_{\{ i \}}. \tag{64}
\]

Furthermore, we can also obtain the \( n - 1 \) identities, which are expressed as
\[
\sum_{\gamma=1}^{i} \sum_{[\langle i \rangle]} C_i \left( \{ j \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ j \}} \sum_{[k]} C_i \left( \{ k \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ k \}} \\
= \sum_{[j]} C_i \left( \{ j \} \right) \mathcal{E}^{h}_{\{ i \}} \cdots \mathcal{E}^{h}_{\{ j \}}. \tag{65}
\]

To construct the gauge-invariant variables for the metric perturbation \(^{(i)}g_{ab}\), as in the cases in the last section, we consider the tensor field defined by
\[ (\alpha) \hat{H}_{ab} := \alpha^{(a)} g_{ab} + \sum_{j=1}^{n-1} \frac{n!}{(n-j)!} \sum_{\left\{ j \right\} \in \mathcal{A}} \mathcal{C}_{n-1} \left( \left\{ j \right\} \right) \mathcal{E}_{j_{(1)}} \cdots \mathcal{E}_{j_{(n-1)}} \mathcal{E}_{j_{(n-1)}}^{-1} g_{ab} \]

\[ + n! \sum_{\left\{ j \right\} \in \mathcal{A} \backslash \mathcal{A}^{(n)}} \mathcal{C}_{n-1} \left( \left\{ j \right\} \right) \mathcal{E}_{j_{(1)}} \cdots \mathcal{E}_{j_{(n-1)}} \mathcal{E}_{j_{(n-1)}}^{-1} g_{ab}. \]  

(66)

Using the order-by-order identities (65), the gauge-transformation rule is given by

\[ \alpha^{(a)} \hat{H}_{ab} = \mathcal{E}_{\alpha} \mathcal{E}_{\alpha}^{-1} \hat{H}_{ab} \]

\[ + n! \sum_{\left\{ j \right\} \in \mathcal{A} \backslash \mathcal{A}^{(n)}} \mathcal{C}_{n-1} \left( \left\{ j \right\} \right) \mathcal{E}_{j_{(1)}} \cdots \mathcal{E}_{j_{(n-1)}} \mathcal{E}_{j_{(n-1)}}^{-1} g_{ab} \]

\[ = \mathcal{E}_{\alpha} \mathcal{E}_{\alpha}^{-1} \hat{H}_{ab} \]

\[ + \sum_{j=1}^{n-1} \sum_{\left\{ j \right\} \in \mathcal{A}} \mathcal{C}_{n-1} \left( \left\{ j \right\} \right) \mathcal{E}_{j_{(1)}} \cdots \mathcal{E}_{j_{(n-1)}} \mathcal{E}_{j_{(n-1)}}^{-1} g_{ab} \]

\[ \times \sum_{\left\{ k \right\} \in \mathcal{A} \backslash \mathcal{A}^{(n)}} \mathcal{C}_{n-1} \left( \left\{ k \right\} \right) \mathcal{E}_{k_{(1)}} \cdots \mathcal{E}_{k_{(n-1)}} \mathcal{E}_{k_{(n-1)}}^{-1} g_{ab}. \]  

(67)

From the analyses in the last section, we can expect that the following conjecture is reasonable.

**Conjecture 4.1.** There exists a vector field \( \hat{\sigma}^{(n)} \) such that

\[ n! \sum_{\left\{ j \right\} \in \mathcal{A} \backslash \mathcal{A}^{(n)}} \mathcal{C}_{n-1} \left( \left\{ j \right\} \right) \mathcal{E}_{j_{(1)}} \cdots \mathcal{E}_{j_{(n-1)}} \mathcal{E}_{j_{(n-1)}}^{-1} g_{ab} \]

\[ = \mathcal{E}_{\hat{\sigma}^{(n)}}. \]  

(68)

To derive the explicit form of \( \hat{\sigma}^{(n)} \), tough algebraic calculations are necessary. Although we do not go into the details of the proof of this conjecture, we expect that this identity should be proved recursively and there is no difficulty except for the difficult calculations. Actually, in the last section, we have confirmed this conjecture to the fourth order, and it is reasonable to assume that conjecture 4.1 holds.

If conjecture 4.1 holds, then the gauge-transformation rule for the variable \( (\alpha) \hat{H}_{ab} \) is given by
This is the same form as for the gauge-transformation rule for the linear-order metric perturbation. Then we may apply conjecture \(3.1\) for the variable \(\hat{H}_{ab}\). This implies that the variable \(\hat{H}_{ab}\) is decomposed as

\[
\hat{H}_{ab} = \mathcal{H}_{ab} + \mathcal{E}_{ab}, \quad (n)\]

Thus, we have gauge-invariant variables \(\mathcal{H}_{ab}\) for the \(n\)th-order metric perturbation. This implies that the original \(n\)th-order metric perturbation \(g_{ab}^{(n)}\)

\[
g_{ab}^{(n)} = \mathcal{H}_{ab} - \mathcal{E}_{ab} + \sum_{l=1}^{n-1} \frac{n!}{(n-l)!} \sum_{\{i\} \in \delta_l} C_i \left( \begin{array}{c} j \end{array} \right) \mathcal{E}^{(l)}_{ab} \mathcal{H}^{(n-l)}_{ab} g_{ab}, \quad (n)\]

This indicates that the \(n\)th-order metric perturbation \(g_{ab}^{(n)}\) is decomposed into its gauge-invariant and gauge-variant parts. Through the gauge-variant variables \(X^a_i\) \((i = 1, \ldots, n)\), we can also define the gauge-invariant variable \(Q^{(n)}\) for the \(n\)th-order perturbation \(Q^{(n)}\) of any tensor field \(Q\) which is also defined as

\[
Q^{(n)} = \mathcal{Q} + \sum_{l=1}^{n-1} \frac{n!}{(n-l)!} \sum_{\{i\} \in \delta_l} C_i \left( \begin{array}{c} j \end{array} \right) \mathcal{E}^{(l)}_{ab} \mathcal{H}^{(n-l)}_{ab} Q. \quad (n)\]

Furthermore, conjecture \(4.1\) establishes the identity which corresponds to \((25), (31), (49), \) and \((63)\). Substituting the second equation in \((71)\) into equation \((68)\), we obtain

\[
\sum_{l=1}^{n} \sum_{\{i\} \in \delta_l} C_i \left( \begin{array}{c} j \end{array} \right) \mathcal{E}^{(l)}_{ab} \mathcal{H}^{(n-l)}_{ab} \mathcal{E}^{(l)}_{ab} \mathcal{H}^{(n-l)}_{ab} Q = \mathcal{E}^{(n)}_{ab} - \mathcal{E}^{(n)}_{ab} + \mathcal{E}^{(n)}_{ab}, \quad (n)\]
This is equivalent to
\[
\sum_{k_1} \cdots \sum_{k_n} C_{k_1} \left( \sum_{j} \left\{ k_j \right\} \right) \xi_{j, \bar{j}} \cdots \xi_{k_n, \bar{k}_n} = \xi_{i, \bar{i}}.
\]
\[\tag{75}\]

This identity corresponds to the \(i = n\) version of identities \((65)\) and is used when we derive the gauge-transformation rules of perturbations higher than the \(n\)th.

5. Example: cosmological perturbations

Here, we consider the application of our formulae derived in the last section to a specific background spacetime as an example. The example discussed here is the cosmological perturbation whose background metric is given by
\[
\eta_{\alpha\beta} = - \eta_\gamma \delta_{\alpha\beta} + \gamma_{\alpha\beta} \eta, \quad (76)
\]
where \(a = a(\eta)\) is the scale factor, \(\gamma_{pq}\) is the metric on the maximally symmetric 3-space with curvature constant \(K\), and the indices \(p, q, r, \ldots\) for the spatial components from 1 to 3. In this section, we concentrate only on the metric perturbations.

We must note that even in the case of this cosmological perturbation, there is the ‘zero-mode problem’ mentioned in section 3. In this section, we ignore these zero-modes and assume conjecture 3.1 for simplicity, because we have not yet resolved the ‘zero-mode problem’ systematically as mentioned in section 3.

On the background spacetime with the metric \((76)\), we consider the metric perturbation \((1)g_{ab}\) and apply the York decomposition \([11]\):
\[
(1)g_{ab} = \xi_{\alpha\beta} (\eta)_{\alpha\beta} + 2 \left( D_p (1)\eta_{(1)\alpha\beta} + (1)\xi_{\alpha\beta} \right) (dx^p)_{(1)\alpha} (dx^p)_{(1)\beta},
\]
\[\tag{77}\]

where \(\Delta := \gamma_{pq}D_q D_p\) and \(D_p\) is the covariant derivative associated with the metric \(\gamma_{pq}\). Here, \((1)\eta_{(1)\alpha\beta}\), \((1)\xi_{(1)\alpha\beta}\), and \((1)\xi_{(1)\alpha\beta}\) satisfy the properties \(D^{\mu} (1)\eta_{(1)\alpha\beta} = D^{\mu} (1)\xi_{(1)\alpha\beta} = 0\), \((1)\eta_{(1)\alpha\beta} = (1)\eta_{(1)\alpha\beta} = (1)\xi_{(1)\alpha\beta} = (1)\xi_{(1)\alpha\beta} = 0\), and \(D^{\mu} (1)\xi_{(1)\alpha\beta} = 0\).

The gauge-transformation rules for the variables \((1)\eta_{\alpha\beta}, (1)\xi_{(1)\alpha\beta}, (1)\xi_{(1)\alpha\beta}, (1)\xi_{(1)\alpha\beta}, (1)\eta_{(1)\alpha\beta}, (1)\xi_{(1)\alpha\beta}, (1)\xi_{(1)\alpha\beta}, (1)\xi_{(1)\alpha\beta}\) are derived from \((15)\). Inspecting these gauge-transformation rules, we define the gauge-variant part \((1)X_a\) in \((16)\):
\[
(1)X_a := \left( (1)\xi_{(1)\alpha\beta} - \frac{1}{2} a^2 q^2 (1)\xi_{(1)\alpha\beta} \right) (dx^a)_{(1)\alpha} + a^2 \left( (1)\xi_{(1)\alpha\beta} + \frac{1}{2} D^p (1)\eta_{(1)\alpha\beta} \right) (dx^a)_{(1)\beta}.
\]
\[\tag{78}\]
We can easily check that this vector field \( (1) \chi \) satisfies (17). Subtracting the gauge-variant part \( E_{\eta \eta} g_{a b} \) from \( (1) g_{a b} \), we have the gauge-invariant part \( (1) \mathcal{H}_{a b} \) in (16):

\[
(1) \mathcal{H}_{a b} = a^2 \left\{ -2(1) \Phi (d \eta)_a (d \eta)_b + 2(1) \nu_{\eta} (d \eta)_a (d x^\nu)_b \\
+ \left( -2(1) \psi_{pq} + (1) \chi_{pq} \right) (d x^\nu)_a (d x^\nu)_b \right\},
\]

(79)

where the properties \( D^{(1)} \nu = \gamma^{\nu} D_p \nu_q = 0 \), \( (1) \chi^\nu = \gamma^{\nu} (1) \chi_{pq} = 0 \), and \( D^{(1)} \chi_{pq} = 0 \) are satisfied.

We must emphasize that, as shown in reference [7], the one-to-one correspondence between the sets of variables \{ \( \theta_{\eta \eta} g_{a b} \), \( \theta_{\eta \eta} g_{a b} \) \} and \{ \( \eta_{\eta \eta} h_{(1)\nu} \), \( \eta_{\nu \nu} h_{(1)\nu} \), \( \eta_{(1)\nu} h_{(1)\nu} \), \( \eta_{(1)\nu} h_{(1)\nu} \) \} is guaranteed by the existence of the Green functions \( \Delta^{-1}, (\Delta + 2 K)^{-1} \), and \( (\Delta + 3 K)^{-1} \). In other words, in the decomposition (77), some perturbative modes of the metric perturbations which belong to the kernel of the operator \( \Delta \), \( (\Delta + 2 K) \), and \( (\Delta + 3 K) \) are excluded from our consideration. For example, homogeneous modes belonging to the kernel of the operator \( \Delta \) are excluded from our consideration. If we have to treat these modes, we must do so separately. This is the 'zero-mode problem' in the cosmological perturbations, which was pointed out in reference [6].

To define gauge-invariant variables for \( n \)th-order metric perturbation, we apply the York decomposition (77) not to the variable \( (n) g_{a b} \) but to the variable \( (n) \mathcal{H}_{a b} \) defined by (66):

\[
(n) \mathcal{H}_{a b} = (n) h_{\nu} (d \eta)_a (d \eta)_b + 2 \left( D^{(n)} h_{(1)\nu} + (n) h_{(1)\nu} \right) (d \eta)_a (d x^\nu)_b \\
+ a^2 \left\{ (n) h_{(1)\nu} \psi_{pq} + \left( D_p D_q - \frac{1}{3} \gamma_{pq} \Delta \right) (n) h_{(1)\nu} \\
+ 2 D^{(n)} h_{(1)\nu} \psi_{pq} + (n) h_{(1)\nu} \right\} (d x^\nu)_a (d x^\nu)_b
\]

(80)

Since the gauge-transformation rule (69) for the variable \( (n) \mathcal{H}_{a b} \) has the same form as the gauge-transformation rule (11), we can define the gauge-variant parts of \( (n) \mathcal{H}_{a b} \) as

\[
(n) \chi_a : = \left( (n) h_{(1)\nu} - \frac{1}{2} a^2 \delta_a (n) h_{(1)\nu} \right) (d \eta)_a \\
+ a^2 \left( (n) h_{(1)\nu} + \frac{1}{2} D^{(n)} h_{(1)\nu} \right) (d x^\nu)_a
\]

(81)

through the same procedure as for the linear case and we can also define the gauge-invariant part \( (n) \mathcal{H}_{a b} \) by

\[
(n) \mathcal{H}_{a b} = a^2 \left\{ -2(1) \Phi (d \eta)_a (d \eta)_b + 2(1) \nu_{\eta} (d \eta)_a (d x^\nu)_b \\
+ \left( -2(1) \psi_{pq} + (1) \chi_{pq} \right) (d x^\nu)_a (d x^\nu)_b \right\},
\]

(82)

where the properties \( D^{(n)} \nu = \gamma^{\nu} D_{p}^{(n)} \nu_q = 0 \), \( (n) \chi^\nu = \gamma^{\nu} (n) \chi_{pq} = 0 \), and \( D^{(n)} \chi_{pq} = 0 \) are satisfied.

As noted in reference [7], the definitions of gauge-invariant variables are not unique. Therefore, we may choose different gauge-invariant variables for each order of metric
perturbations through by choosing a different of \(^{(i)}X_a\). The above choice corresponds to the longitudinal gauge in linear cosmological perturbations.

6. Summary and discussions

In this paper, we discussed the recursive structure in the construction of gauge-invariant variables for any order perturbations. As gauge-transformation rules for the higher-order perturbations, we applied the knight diffeomorphism introduced by Sonego and Bruni [10]. This diffeomorphism is regarded as a general diffeomorphism in the order-by-order treatment of perturbations. On the basis of gauge-transformation rules for higher-order perturbations derived by Sonego and Bruni [10], we proposed a procedure to construct gauge-invariant variables to the third order in [4]. On the basis of this procedure, in this paper, we considered the explicit and systematic construction of gauge-invariant variables for higher-order perturbations. As a result, we found the recursive structure in the construction of gauge-invariant variables.

Although we do not prove conjecture 4.1 within this paper, we have confirmed this conjecture to the fourth order. Therefore, it is reasonable to assume that the algebraic relation (68) holds. Then the gauge-transformation rule for the variable \(^{(a)}\hat{H}_{ab}\) defined by equation (66) is given as equation (69). This indicates that we may apply conjecture 3.1 to the variable \(^{(a)}\hat{H}_{ab}\) and decompose the metric perturbation \(^{(a)}g_{ab}\) of \(n\)th order into its gauge-invariant part \(^{(a)}\hat{H}_{ab}\) and its gauge-variant part \(^{(a)}X_a\). The gauge-transformation rule of the gauge-variant part \(^{(a)}X_a\) establishes the identity (65) with \(i = n\). We used the identities (65) with \(i = 1, \ldots, n\) when we derived the gauge-transformation rule for the variable \(^{(a+1)}\hat{H}_{ab}\), which is given by equation (67) with the replacement \(n \rightarrow n + 1\). Through conjecture 4.1 with the replacement \(n \rightarrow n + 1\), the gauge-transformation rule for the variable \(^{(a+1)}\hat{H}_{ab}\) is also given in the form (69) with the replacement \(n \rightarrow n + 1\). Thus, we can recursively construct gauge-invariant variables for any order perturbations through conjectures 3.1 and 4.1. In this paper, we have confirmed this recursive structure to the fourth order. This recursive structure is the main point of this paper.

We must note that conjecture 3.1 is highly nontrivial, whereas conjecture 4.1 is merely algebraic. In [6], we proposed a scenario for a proof of conjecture 3.1. However, in this scenario there are missing modes of perturbation which we called ‘zero modes’; we also proposed ‘zero-mode problem’. The recursive structure in this paper is entirely based on conjecture 3.1. Therefore, we have to say that the ‘zero-mode problem’ is also essential to the recursive structure in the construction of gauge-invariant variables for any-order perturbations.

Here, we discuss the correspondence with the recent proposal of the fully non-linear and exact perturbations by Hwang and Noh [12]. Since we can decompose the \(n\)th-order metric perturbation as equation (72), the full metric (10), which is pulled back to \(\mathcal{M}_0\) through a gauge \(X\), is given by
\[ X_{k,ab}^{\eta} = g_{ab} + \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \mathcal{H}_{ab} - \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{|i|\leq k} c_i \left( \{ i \} \right) \mathcal{E}_{i,j} \mathcal{E}_{j,k} \cdots \mathcal{E}_{-j} \mathcal{E}_{-i} \mathcal{H}_{ab} + O \left( \lambda^{k+1} \right). \] (83)

In this equation, the term \( \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \mathcal{H}_{ab} \) is the gauge-invariant part and the second line is the gauge-variant part up to \( k + 1 \) order. If the right-hand side of equation (83) converges in the limit \( k \to \infty \), the limit \( \lim_{k \to \infty} \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \mathcal{H}_{ab} \) corresponds to the gauge-invariant variables in the fully non-linear and exact perturbations proposed by Hwang and Noh [12]. The gauge issue of the fully non-linear and exact perturbations will be justified in this way.

In the case of cosmological perturbations discussed in section 5, the components of the gauge-invariant part \( \lim_{k \to \infty} \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \mathcal{H}_{ab} \) for the fully non-linear and exact perturbations are given by

\[ \lim_{k \to \infty} \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \mathcal{H}_{ab} = \alpha^2 \left\{ -2 \hat{\Phi} (d\eta) (d\eta) + 2 \hat{\nu}_p (d\eta) (d\eta) \right\} + \left\{ -2 \hat{\nu}_{pq} + (\text{full}) \right\} (d\eta) (d\eta), \] (84)

where

\[ \hat{\Phi} := \lim_{k \to \infty} \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \Phi, \] (85)

\[ \hat{\nu}_p := \lim_{k \to \infty} \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \nu_p, \] (86)

\[ \hat{\Psi} := \lim_{k \to \infty} \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \Psi, \] (87)

\[ \hat{\chi}_{pq} := \lim_{k \to \infty} \sum_{n=1}^{k} \frac{\lambda^{(n)}}{n!} \chi_{pq}. \] (88)

However, we have to keep in mind the fact that we ignored ‘zero modes’ to define the variables \( \hat{\Phi}, \hat{\nu}_p, \hat{\Psi}, \) and \( \hat{\chi}_{pq}. \)

Finally, we have to emphasize that the arguments presented in this paper are purely kinematical, since the issue of gauge dependence is purely kinematical. Actually, we do not use any information from field equations such as the Einstein equation. Therefore, the arguments of this paper are applicable to any theory of gravity with general covariance.

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Appendix A. Properties of the set \( J_l \)

In [10], Sonego and Bruni introduced the set of integer \( J_l \) associated with the integer \( l \geq 1 \) defined by

\[
J_l := \left\{ \left( j_1, \ldots, j_m \right) \mid j_i \in \mathbb{N}, \quad \sum_{i=1}^{m} i j_i = l \right\}
\]

\[
=: \mathcal{P}_l, \tag{A.1}
\]

where \( \mathbb{N} \) is the set of natural numbers. Here, it is convenient to introduce the set \( J_0 \) so that

\[
J_0 := \left\{ \left( j_1, \ldots, j_m \right) \mid j_n = 0 \quad \forall \ n \in \mathbb{N} \right\} \tag{A.2}
\]

As a result of introducing \( J_0 \), we may consider the \( \mathcal{P}_l \) of \( J_l \) for \( l \geq 0 \).

To classify the elements of \( J_l \), we first introduce the set

\[
J^+_l := \left\{ \left( j_1, j_2, j_3, \ldots \right) \mid \left( j_1, \ldots, j_i \right) \in \mathcal{P}_l \right\}. \tag{A.3}
\]

We note that

\[
J^+_0 = \{ (1, 0, 0, \ldots) \} = J_l. \tag{A.4}
\]

If we replace \( j_i \rightarrow j_i + 1 \) in the condition \( \sum_{i=1}^{m} i j_i = l \) of the \( \mathcal{P}_l \), we obtain

\[
j_i + \sum_{i=2}^{m} i j_i = l - 1. \tag{A.5}
\]

Therefore, \( J^+_l \) is a subset \( J_l \), namely, the elements of \( J^+_l \) are the elements of \( J_l \) with \( j_i \geq 1 \).

All elements of the set \( J_l \setminus J^+_1 \) have the property \( j_1 = 0 \).

Second, we consider the set \( J_l \setminus J^{+1}_2 \). We define \( J^+_2 \) by

\[
J^+_2 := \left\{ \left( j_1, j_2, \ldots \right) \mid \left( j_1, j_2, \ldots \right) \in J_l \setminus J^+_1 \right\}. \tag{A.6}
\]

Since all elements in the set \( J_l \setminus J^+_1 \) have the property \( j_1 = 0 \), all elements in the set \( J^+_2 \) also have the property \( j_i = 0 \). Furthermore, since the elements in \( J_l \setminus J^+_2 \) satisfy the condition \( \sum_{i=2}^{m} i j_i = l \), the elements of the set \( J^+_2 \) satisfy the condition \( \sum_{i=2}^{m} i j_i = l + 2 \). This implies that the set \( J^+_2 \) is the subset of the set \( J_l \setminus J^{+1}_2 \) with the property \( j_2 \geq 1 \). We note that all elements of the set \( J_l \setminus \left( J^{+1}_2 \oplus J^{+2}_2 \right) \) have the property \( j_1 = j_2 = 0 \). We also note that \( J^+_2 \) is an empty set.

Similarly, we consider the set \( J_l \setminus \left( J^{+1}_3 \oplus J^{+2}_3 \right) \). We also define \( J^+_3 \) by

\[
J^+_3 := \left\{ \left( j_1, j_2, \ldots \right) \mid \left( j_1, j_2, \ldots \right) \in J_l \setminus \left( J^{+1}_2 \oplus J^{+2}_2 \right) \right\}. \tag{A.7}
\]

Since all elements in the set \( J_l \setminus \left( J^{+1}_3 \oplus J^{+2}_3 \right) \) have the property \( j_1 = j_2 = 0 \), all elements in the set \( J^+_3 \) also have the property \( j_1 = j_2 = 0 \). Furthermore, since the elements in \( J_l \setminus \left( J^{+1}_3 \oplus J^{+2}_3 \right) \) satisfy the condition \( \sum_{i=3}^{m} i j_i = l \), the elements of the set \( J^+_3 \) satisfy the property \( \sum_{i=3}^{m} i j_i = l + 3 \). This implies that the set \( J^+_3 \) is the subset of the set \( J_l \setminus \left( J^{+1}_3 \oplus J^{+2}_3 \right) \) with the property \( j_3 \geq 1 \). We note that all elements of the set
$\mathcal{J}\setminus \{J_{1}, J_{2}, J_{3}, J_{4}\}$ have the property $j_1 = j_2 = j_3 = 0$ and the sets $\mathcal{J}^+_l$ with $l = 1, 2$ are empty sets.

We can repeat this classification of the elements in $\mathcal{J}_l$ through the recursive definitions of the sets

$$\mathcal{J}_l^+ := \left\{ \left( j_1, \ldots, j_{k-1}, j_k + 1, j_{k+1}, \ldots \right) \mid \left( j_1, \ldots, j_k, \ldots \right) \in \mathcal{J}\right\} \bigcup \mathcal{J}_{l-p}^+,$$

for $0 \geq k \geq l$. This classification of the elements in $\mathcal{J}_l$ terminates when $k = l$ and we obtain the results

$$\mathcal{J}_l = \bigoplus_{k=1}^l \mathcal{J}_{l-k}^+.$$  \hspace{1cm} (A.8)

We note that

$$\mathcal{J}_{l-k}^+ = \emptyset \quad \text{for} \quad k > l - k > 0.$$  \hspace{1cm} (A.9)

and

$$\mathcal{J}_0^+ = \left\{ \left( 0, \ldots, 0, j_k = 1, 0, \ldots \right) \right\}.$$  \hspace{1cm} (A.10)

The explicit elements of $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3,$ and $\mathcal{J}_4$ are given by

$$\mathcal{J}_1 = \{ (1, 0, 0, 0, 0, 0, \ldots) \},$$

$$\mathcal{J}_2 = \{ (2, 0, 0, 0, 0, 0, 0, \ldots) ,$$

$$\{ 0, 1, 0, 0, 0, 0, \ldots \} \},$$

$$\mathcal{J}_3 = \{ (3, 0, 0, 0, 0, 0, 0, \ldots) ,$$

$$\{ 1, 1, 0, 0, 0, 0, \ldots \},$$

$$\{ 0, 0, 1, 0, 0, 0, \ldots \} \},$$

$$\mathcal{J}_4 = \{ (4, 0, 0, 0, 0, 0, 0, \ldots) ,$$

$$\{ 2, 1, 0, 0, 0, 0, \ldots \},$$

$$\{ 1, 0, 1, 0, 0, 0, 0, \ldots \},$$

$$\{ 0, 2, 0, 0, 0, 0, \ldots \},$$

$$\{ 0, 0, 0, 1, 0, 0, 0, \ldots \} \}. $$  \hspace{1cm} (A.11)

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