On the convergence of quasilinear viscous approximations with BV initial data

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Abstract

We show that the almost everywhere limit of quasilinear viscous approximations is the unique entropy solution (in the sense of Bardos-Leroux-Nedelec) of the corresponding scalar conservation laws on a bounded domain in $\mathbb{R}^d$ whenever the initial data is essentially bounded and a function of bounded variation.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial \Omega$. For $T > 0$, denote $\Omega_T := \Omega \times (0, T)$. We write the initial boundary value problem (IBVP) for scalar conservation laws given by

\begin{align}
    u_t + \nabla \cdot f(u) &= 0 \quad \text{in } \Omega_T, \tag{1.1a} \\
    u(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, T), \tag{1.1b} \\
    u(x, 0) &= u_0(x) \quad x \in \Omega, \tag{1.1c}
\end{align}

where $f = (f_1, f_2, \cdots, f_d)$ is the flux function and $u_0$ is the initial condition.

Denote by $f_\varepsilon$, the regularizations of the flux function $f = (f_1, f_2, \cdots, f_d)$ by the sequence of mollifiers $\rho_\varepsilon$ defined on $\mathbb{R}^d$. It is given by $f_\varepsilon := (f_{1_\varepsilon}, f_{2_\varepsilon}, \cdots, f_{d_\varepsilon})$, where

$$f_{j_\varepsilon} := f_j * \tilde{\rho}_\varepsilon \ (j = 1, 2, \cdots, d)$$

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Consider the IBVP for regularized generalized viscosity problem

\[ u_\varepsilon^t + \nabla \cdot f_\varepsilon(u_\varepsilon) = \varepsilon \nabla \cdot (B(u_\varepsilon) \nabla u_\varepsilon) \quad \text{in } \Omega_T, \quad \text{(1.2a)} \]

\[ u_\varepsilon(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T), \quad \text{(1.2b)} \]

\[ u_\varepsilon(x, 0) = u_{0_\varepsilon}(x) \quad x \in \Omega, \quad \text{(1.2c)} \]

indexed by \( \varepsilon > 0 \). Let us now give the hypothesis on \( f, B, u_0 \) and \( u_{0_\varepsilon} \).

**Hypothesis E:**

1. Let \( f \in C^1(\mathbb{R}), f' \in L^\infty(\mathbb{R}), \) and denote

\[ \|f'\|_{L^\infty(\mathbb{R})} := \sup_{y \in \mathbb{R}} |f'(y)|. \]

2. Let \( B \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R}), \) and there exists an \( r > 0 \) such that \( B \geq r \).

3. We denote all those elements of \( L^\infty(\Omega) \) whose essential support are compact subsets of \( \Omega \) by \( L^\infty_c(\Omega) \) and let \( BV(\Omega) \) be the space of functions of bounded variations in \( \Omega \). Let \( u_0 \) be in \( BV(\Omega) \cap L^\infty_c(\Omega) \) and denote by \( u_{0_\varepsilon} \), the regularizations of the initial data \( u_0 \) by the sequence of mollifiers \( \tilde{\rho}_\varepsilon \) defined on \( \mathbb{R} \), i.e., \( u_{0_\varepsilon} := u_0 * \tilde{\rho}_\varepsilon \). Denote

\[ I := [-\|u_0\|_\infty, \|u_0\|_\infty]. \]

**Hypothesis F**

1. Let \( f \in (C^1(\mathbb{R}))^d, f' \in (L^\infty(\mathbb{R}))^d, \) and denote

\[ \|f'\|_{(L^\infty(\mathbb{R}))^d} := \max_{1 \leq j \leq d} \sup_{y \in \mathbb{R}} |f_j'(y)|. \]

2. Let \( B \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R}), \) and there exists an \( r > 0 \) such that \( B \geq r \).

3. We denote the set of all infinitely differentiable functions with compact support in \( \Omega \) by \( D(\Omega) \) and denote \( W^{1,1}(\Omega) := \overline{D(\Omega)} \) in \( W^{1,1}(\Omega) \). Let \( u_0 \in W^{1,1}(\Omega) \cap C(\Omega) \). Let \( u_{0_\varepsilon} \) be in \( D(\Omega) \) such that for all \( \varepsilon > 0 \), there exists a constant \( A > 0 \) such that \( \|u_{0_\varepsilon}\|_{L^\infty(\Omega)} \leq A \) and \( u_{0_\varepsilon} \to u_0 \) in \( W^{1,1}(\Omega) \) as \( \varepsilon \to 0 \). Denote

\[ I := [-A, A]. \]

The aim of this article is to prove that the a.e. limit of sequence of solutions \( (u_\varepsilon) \) to (1.2) (called quasilinear viscous approximations) is the unique entropy solution for IBVP (1.1). In this context, we have two main results (Theorem 1.1 and Theorem 1.2).
depending on the regularity of the initial data. In [3], we have proved that the a.e. limit of quasilinear viscous approximations is the unique entropy solution in the sense of Bardos et.al whenever the initial data lies in $W^1_\infty(\Omega)$ and in [4], we showed that the a.e. limit of quasilinear viscous approximations is the unique entropy solution in the sense of Bardos et.al whenever the initial data lies in $H^1(\Omega) \cap L^\infty_c(\Omega)$ using compensated compactness. In this article, we are able to show that the a.e. limit of quasilinear viscous approximations is the unique entropy solution in the sense of Bardos et.al whenever the initial data lies in $BV(\Omega) \cap L^\infty_c(\Omega)$ (see Hypothesis E) and we state our first result.

**Theorem 1.1** Let $f$, $B$, $u_0$ and $u_{0\varepsilon}$ satisfy Hypothesis E. Then the a.e. limit of the quasilinear viscous approximations $(u^\varepsilon)$ satisfying (1.2) is the unique entropy solution of IBVP (1.1) in the sense of Bardos et.al [1].

For the case artificial viscosity problem, i.e., $B(\cdot) \equiv 1$, it is enough to assume initial data in $BV(\Omega) \cap L^\infty_c(\Omega)$ for establishing the BV estimates of quasilinear viscous approximations $(u^\varepsilon)$. But in the case of regularized viscosity problem, we are unable to achieve $L^1$-estimate of the time derivative of $(u^\varepsilon)$ using usual technique [1], [2] because of the presence of non-constant $B$. In [3], we establish BV estimate with initial data in $W^1_\infty(\Omega)$ and in [4], we establish $L^1$-estimate of time derivative with initial data in $H^1(\Omega) \cap L^\infty_c(\Omega)$. But in the present article, we are able to give an alternate proof of the $L^1$-estimate of time derivative of quasilinear viscous approximations $(u^\varepsilon)$ with initial data in $BV(\Omega) \cap L^\infty_c(\Omega)$.

In [3], we removed compact essential support of initial data by showing the a.e. limit of quasilinear viscous approximations is the unique entropy solution in the sense of Bardos et.al whenever the initial data lies in $H^1_0(\Omega) \cap C(\Omega)$. In this article we show that the a.e. limit of quasilinear viscous approximations is the unique entropy solution in the sense of Bardos et.al whenever the initial data lies in $W^{1,1}_0(\Omega) \cap C(\Omega)$ (see Hypothesis F). We now state our second result.

**Theorem 1.2** Let $f$, $B$, $u_0$ and $u_{0\varepsilon}$ satisfy Hypothesis F. Then the a.e. limit of the quasilinear viscous approximations $(u^\varepsilon)$ satisfying (1.2) is the unique entropy solution of IBVP (1.1) in the sense of Bardos et.al [1].

In this article, the main difficulty is to establish the BV estimate of quasilinear viscous approximations with initial data as given in Hypothesis E and Hypothesis F. Then the proof of Theorem 1.1 and Theorem 1.2 follow from [3].
The plan of the paper is the following. In Section 2, we prove a few properties of quasilinear viscous approximations and in Section 3, we establish the BV estimate of quasilinear viscous approximations \( (u^\varepsilon) \) and prove Theorem 1.1 and Theorem 1.2.

### 2 Properties of quasilinear viscous approximations

We prove the following result for the existence and uniqueness, maximum principle of quasilinear viscous approximations \( (u^\varepsilon) \).

**Lemma 2.1** Let \( f, B, u_0 \) and \( u_0\varepsilon \) satisfy Hypothesis E. Then there exists a unique solution of (1.2) in \( C^{4+\beta, \frac{4+\beta}{2}}(\Omega_T) \), for every \( 0 < \beta < 1 \) and the following estimate

\[
\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \text{ a.e. } t \in (0, T)
\]

holds.

In order to prove Lemma 2.1 we use higher regularity and maximum principle of solutions to generalized viscosity problem from [3]. For generalized viscosity problem and for Hypotheses on \( f, B, u_0 \), we refer the reader to [3, p.1] and [3, p.2] respectively. We now state higher regularity result of generalized viscosity problem from [3, p.18]. This higher regularity result will be used to prove Lemma 2.1.

**Theorem 2.1 (higher regularity)** Let \( f, B, u_0 \) satisfy Hypothesis A of [3, p.2]. Then the solutions of the IBVP for generalized viscosity problem belong to the space \( C^{4+\beta, \frac{4+\beta}{2}}(\Omega_T) \). Further, \( u^\varepsilon_t \in C(\Omega_T) \).

We now state the following maximum principle for solutions of generalized viscosity problem from [3, p.12].

**Theorem 2.2 (Maximum principle)** Let \( f, B \) and \( u_0 \) satisfy Hypothesis A of [3, p.2]. Then any solution \( u \) of generalized viscosity problem in \( C^{4+\beta, \frac{4+\beta}{2}}(\Omega_T) \) satisfies the bound

\[
\|u^\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \text{ a.e. } t \in (0, T).
\]

**Proof of Lemma 2.1**: Applying higher regularity result of solutions to generalized viscosity problem, *i.e.*, Theorem 2.1 to (1.2), we conclude the existence and uniqueness of
quasilinear viscous approximations \((u^\varepsilon)\) to the regularized viscosity problem \((1.2)\).

An application of Maximum principle, \textit{i.e.,} Theorem 2.2 to quasilinear viscous approximations \((u^\varepsilon)\) and using \(\|u_0^\varepsilon\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}\), we obtain \((2.3)\). This completes the proof of Lemma 2.1.

Following exactly the same argument as in the proof of Lemma 2.1, we conclude the following result.

**Lemma 2.2** Let \(f, B, u_0\) and \(u_0^\varepsilon\) satisfy Hypothesis F. Then there exists a unique solution of \((1.2)\) in \(C^{4+\beta, 4+\beta/2}(\Omega_T)\), for every \(0 < \beta < 1\) and the following estimate

\[ \|u^\varepsilon\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \text{ a.e. } t \in (0, T) \] \hspace{1cm} (2.5)

holds.

Applying Theorem 4.2 from [3, p.30] to quasilinear viscous approximations \((u^\varepsilon)\) as asserted in Lemma 2.1, we obtain

**Theorem 2.3** Let \(f, B, u_0\) and \(u_0^\varepsilon\) satisfy Hypothesis E. Let \(u^\varepsilon\) be the unique solution to regularized viscosity problem \((1.2)\). Then

\[ \sum_{j=1}^{d} \left( \varepsilon \left\| \frac{\partial u^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega_T)} \right)^2 \leq \frac{1}{2r} \|u_0^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{1}{2r} \|u_0\|_{L^\infty(\Omega)}^2 \text{ Vol}(\Omega). \] \hspace{1cm} (2.6)

Again applying Theorem 4.2 from [3, p.30] to quasilinear viscous approximations \((u^\varepsilon)\) as asserted in Lemma 2.2, we obtain

**Theorem 2.4** Let \(f, B, u_0\) and \(u_0^\varepsilon\) satisfy Hypothesis F. Let \(u^\varepsilon\) be the unique solution to regularized viscosity problem \((1.2)\). Then

\[ \sum_{j=1}^{d} \left( \varepsilon \left\| \frac{\partial u^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega_T)} \right)^2 \leq \frac{1}{2r} \|u_0^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{1}{2r} \text{ Vol}(\Omega) A^2. \] \hspace{1cm} (2.7)

### 3 BV estimates

In this section we establish uniform \(L^1(\Omega_T)\) estimates of the first order derivatives of quasilinear viscous approximations \((u^\varepsilon)\) with respect to both time and space. We now state the BV estimates result.
3.1 BV estimate with $u_0$ in $BV(\Omega) \cap L_1^\infty(\Omega)$

**Theorem 3.1** Let $f, B, u_0$ and $u_0^\varepsilon$ be as in Hypothesis E and $(u^\varepsilon)$ be as asserted in Lemma 2.1. Then

1. for all $\varepsilon > 0$, the following inequality
   \[ \|\nabla u^\varepsilon\|_{(L^1(\Omega_T))^d} \leq TV_\Omega(u_0) \]  
   (3.8)
   holds.

2. for all $\varepsilon > 0$, the following inequality
   \[ \|\frac{\partial u^\varepsilon}{\partial t}\|_{L^1(\Omega_T)} \leq 2\|B\|_{L^\infty(I)} \frac{Vol(\Omega)}{2r} \|u_0\|_{L^\infty(\Omega)}^2 + 2 \max_{1 \leq j \leq d} \left( \sup_{y \in I} |f_j'(y)| \right) TV_\Omega(u_0) \]
   + \[ 2 \|u_0\|_{L^\infty(\Omega)} Vol(\Omega) \]  
   (3.9)
   holds.

Further there exists a subsequence $(u^\varepsilon_k)$ of $(u^\varepsilon)$, and a function $u \in L^1(\Omega_T)$ such that

\[ u^\varepsilon_k \rightharpoonup u \text{ in } L^1(\Omega_T), \]  
(3.10)

\[ u^\varepsilon_k \to u \text{ a.e. } (x, t) \in \Omega_T \]  
(3.11)

as $k \to \infty$.

We now introduce signum function which will be used in the proof of Theorem 3.1. For $n \in \mathbb{N}$, let $sg_n : \mathbb{R} \to \mathbb{R}$ be the sequence of functions given by

\[ sg_n(s) = \begin{cases} 
1 & \text{if } s > \frac{1}{n}, \\
ns & \text{if } |s| \leq \frac{1}{n}, \\
-1 & \text{if } s < -\frac{1}{n},
\end{cases} \]

which converges pointwise to the signum function $sg : \mathbb{R} \to \mathbb{R}$ defined by

\[ sg(s) = \begin{cases} 
1 & \text{if } s > 0, \\
0 & \text{if } s = 0, \\
-1 & \text{if } s < 0.
\end{cases} \]

The next result follows from [2, p.67] and is useful in proving Theorem 3.1.
Lemma 3.1 Let $u_0 \in BV(\Omega) \cap L^\infty_c(\Omega)$ and $u_{0\varepsilon}$ be as in Hypothesis E. Then $u_{0\varepsilon}$ satisfies the following bounds

$$
\|u_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \quad (3.12)
$$

$$
\|\nabla u_{0\varepsilon}\|_{(L^1(\Omega))^d} \leq TV_\Omega(u_0) \quad (3.13)
$$

There exists a constant $C > 0$ such that for all $\varepsilon > 0$, $u_{0\varepsilon}$ satisfies

$$
\|\Delta u_{0\varepsilon}\|_{L^1(\Omega)} \leq \frac{C}{\varepsilon} TV_\Omega(u_0). \quad (3.14)
$$

Proof of Theorem 3.1: We prove Theorem 3.1 in three steps. In Step 1, we show (3.8), in Step 2, we show (3.9) and in Step 3, we show (3.10) and (3.11).

Step 1: Applying Step 2 in the proof of BV estimate Theorem 4.1 from [3, p.22] with $f = f_\varepsilon$ and $u_0 = u_{0\varepsilon}$, we arrive at

$$
\|\nabla u_\varepsilon\|_{(L^1(\Omega))^d} = \|\nabla u_{0\varepsilon}\|_{(L^1(\Omega))^d} \quad (3.15)
$$

Then applying Lemma 3.1, we conclude (3.8).

Step 2: From equation (1.2a) of the regularized viscosity problem, we get

$$
u_\varepsilon = \varepsilon B(u_\varepsilon) \Delta u_\varepsilon + \varepsilon \sum_{j=1}^d B'(u_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_j} \right)^2 - \sum_{j=1}^d f_j'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_j} \text{ on } \overline{\Omega_T} \quad (3.16)
$$

From equation (3.16), we get

$$
\varepsilon \int_{\Omega_T} B(u_\varepsilon) |\Delta u_\varepsilon| \, dx \, dt \leq \varepsilon \sum_{j=1}^d \int_{\Omega_T} B'(u_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_j} \right)^2 \, dx \, dt + \sum_{j=1}^d \int_{\Omega_T} f_j'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_j} \, dx \, dt 
$$

$$
+ \int_{\Omega_T} \left| \frac{\partial u_\varepsilon}{\partial t} \right| \, dx \, dt 
$$

$$
\leq \|B'\|_{L^\infty(I)} \varepsilon \sum_{j=1}^d \int_{\Omega_T} \left( \frac{\partial u_\varepsilon}{\partial x_j} \right)^2 \, dx \, dt 
$$

$$
+ \max_{1 \leq j \leq d} \left( \sup_{y \in \mathbb{R}} |f_j'(y)| \right) \sum_{j=1}^d \int_{\Omega_T} \left| \frac{\partial u_\varepsilon}{\partial x_j} \right| \, dx \, dt + \int_{\Omega_T} \left| \frac{\partial u_\varepsilon}{\partial t} \right| \, dx \, dt \quad (3.17)
$$

We now compute

$$
\int_{\Omega_T} \left| \frac{\partial u_\varepsilon}{\partial t} \right| \, dx \, dt = \lim_{n \to \infty} \int_{\Omega_T} s g_n \left( \frac{\partial u_\varepsilon}{\partial t} \right) \frac{\partial u_\varepsilon}{\partial t} \, dx \, dt \quad (3.18)
$$
Using integration by parts, we get
\[
\int_{\Omega_T} \frac{\partial u^\varepsilon}{\partial t} \, dx \, dt = - \lim_{n \to \infty} \int_{\Omega_T} s g_n' \left( \frac{\partial u^\varepsilon}{\partial t} \right) \frac{\partial^2 u^\varepsilon}{\partial t^2} u^\varepsilon \, dx \, dt \\
+ \lim_{n \to \infty} \int_{\Omega} \left( s g_n \left( \frac{\partial u^\varepsilon(x, T)}{\partial t} \right) u^\varepsilon(x, T) - s g_n \left( \frac{\partial u^\varepsilon(x, 0)}{\partial t} \right) u_0^\varepsilon(x) \right) \, dx
\]
(3.19)

We want to show that
\[
\lim_{n \to \infty} \int_{\Omega_T} s g_n' \left( \frac{\partial u^\varepsilon}{\partial t} \right) \frac{\partial^2 u^\varepsilon}{\partial t^2} u^\varepsilon \, dx \, dt = 0
\]
(3.20)
The technique that we use to show (3.20), was used by us in Step 2 in the proof of BV estimate Theorem 4.1 in [3, p.25]. We adapt here the argument.

Denote
\[
A^\varepsilon := \left\{ (x, t) \in \Omega_T : \frac{\partial u^\varepsilon}{\partial t} = 0 \right\}.
\]
Since \( \frac{\partial u^\varepsilon}{\partial t} \in C^1(\overline{\Omega_T}) \), for a.e. \((x, t) \in A^\varepsilon\), using Stampacchia’s theorem (see [6]), we conclude \( \nabla_{x,t} \left( \frac{\partial u^\varepsilon}{\partial t} \right) = 0 \). In particular, \( \frac{\partial^2 u^\varepsilon}{\partial t^2} = 0 \) a.e. \((x, t) \in A^\varepsilon\) and we have (3.20).

If \( \Omega_T \setminus A^\varepsilon = \emptyset \), then (3.20) follows trivially. Assume that \( \Omega_T \setminus A^\varepsilon \neq \emptyset \). For each \((x, t) \in \Omega_T \setminus A^\varepsilon\), we have
\[
sg_n' \left( \frac{\partial u^\varepsilon}{\partial t} \right) \frac{\partial^2 u^\varepsilon}{\partial t^2} u^\varepsilon \to 0 \text{ as } n \to \infty
\]
(3.21)

Note that on \( \Omega_T \setminus A^\varepsilon \), we have
\[
\left| sg_n' \left( \frac{\partial u^\varepsilon}{\partial t} \right) \frac{\partial^2 u^\varepsilon}{\partial t^2} u^\varepsilon \right| \leq sg_n' \left( \frac{\partial u^\varepsilon}{\partial t} \right) \left| \frac{\partial^2 u^\varepsilon}{\partial t^2} u^\varepsilon \right|
\]
(3.22)

Let \((x_0^\varepsilon, t_0^\varepsilon) \in A^\varepsilon\) and \(R' > 0\) be a real number. We denote the open ball with center at \((x_0^\varepsilon, t_0^\varepsilon)\) and having radius \(R'\) by \(B((x_0^\varepsilon, t_0^\varepsilon), R')\).

For each \(n \in \mathbb{N}\), denote
\[
C_n^\varepsilon := \left\{ (x, t) \in \Omega_T \setminus A^\varepsilon : 0 < \left| \frac{\partial u^\varepsilon}{\partial t}(x, t) \right| \leq \frac{1}{n} \right\}.
\]
(3.23)

Observe that for each \(n \in \mathbb{N}\), we have \(C_{n+1}^\varepsilon \subseteq C_n^\varepsilon\). Since \(\Omega_T\) is bounded, there exists \(n_0 \in \mathbb{N}\) be such that for all \(n \geq n_0\), the following inclusion holds:
\[
C_n^\varepsilon \subseteq B \left( (x_0^\varepsilon, t_0^\varepsilon), \frac{n}{2} \right)
\]
Define a function $\rho \in C_0^\infty(\mathbb{R}^{d+1})$ by
\[
\rho(x, t) := \begin{cases} 
k \varepsilon \exp\left(- \frac{1}{1 - |(x,t)-(x^\varepsilon_0,t^\varepsilon_0)|^2}\right), & \text{if } x \in B((x^\varepsilon_0, t^\varepsilon_0), 1) \\
0, & \text{if } (x, t) \notin B((x^\varepsilon_0, t^\varepsilon_0), 1),
\end{cases}
\] (3.24)
where the constant $k \varepsilon$ is chosen so that
\[
\int_{\mathbb{R}^{d+1}} \rho(x, t) \, dx = 1.
\] (3.25)

Denote the sequence of mollifiers $\rho_n : \mathbb{R}^{d+1} \to \mathbb{R}$ by
\[
\rho_n(x, t) := \begin{cases} 
k \varepsilon n^{d+1} \exp\left(- \frac{n^2}{n^2 - |(x,t)-(x^\varepsilon_0,t^\varepsilon_0)|^2}\right), & \text{if } (x, t) \in B((x^\varepsilon_0, t^\varepsilon_0), n) \\
0, & \text{if } (x, t) \notin B((x^\varepsilon_0, t^\varepsilon_0), n)
\end{cases}
\] (3.26)
Since for each $n \geq n_0$, we have
\[
sg_n' \left(\frac{\partial u^\varepsilon}{\partial t}\right) = \begin{cases} 
n & \text{if } 0 \leq \left|\frac{\partial u^\varepsilon}{\partial t}\right| \leq \frac{1}{n}, \\
0 & \text{if } \left|\frac{\partial u^\varepsilon}{\partial t}\right| > \frac{1}{n},
\end{cases}
\] (3.27)
therefore we compute
\[
\int_{\Omega_T \setminus A^\varepsilon} sg_n' \left(\frac{\partial u^\varepsilon}{\partial t}\right) \left| \frac{\partial^2 u^\varepsilon}{\partial t^2}\right| u^\varepsilon \, dx dt = \int_{B((x^\varepsilon_0, t^\varepsilon_0), \frac{n}{2})} \chi_{\Omega_T \setminus A^\varepsilon} sg_n' \left(\frac{\partial u^\varepsilon}{\partial t}\right) \left| \frac{\partial^2 u^\varepsilon}{\partial t^2}\right| u^\varepsilon \, dx
\]
\[
= \int_{B((x^\varepsilon_0, t^\varepsilon_0), \frac{n}{2})} \chi_{\Omega_T \setminus A^\varepsilon(x)} sg_n' \left(\frac{\partial u^\varepsilon}{\partial t}\right) \left| \frac{\partial^2 u^\varepsilon}{\partial t^2}\right| u^\varepsilon \, dx
\]
\[
= \int_{B((x^\varepsilon_0, t^\varepsilon_0), \frac{n}{2})} \chi_{\Omega_T \setminus A^\varepsilon(x)} sg_n' \left(\frac{\partial u^\varepsilon}{\partial t}\right) \left| \frac{\partial^2 u^\varepsilon}{\partial t^2}\right| u^\varepsilon \, dx dt
\] (3.28)

For all $n \geq n_0$ and for $(x, t) \in B((x^\varepsilon_0, t^\varepsilon_0), \frac{n}{2})$, we have
\[
\frac{sg_n' \left(\frac{\partial u^\varepsilon}{\partial t}\right)}{\rho_n(x, t)} := \begin{cases} 
\frac{1}{k \varepsilon n^d} e^{\frac{n^2}{n^2 - |(x,t)-(x^\varepsilon_0,t^\varepsilon_0)|^2}} & \text{if } (x, t) \in B((x^\varepsilon_0, t^\varepsilon_0), \frac{n}{2}) \cap C^\varepsilon_{n}, \\
0 & \text{if } (x, t) \notin B((x^\varepsilon_0, t^\varepsilon_0), \frac{n}{2}) \cap C^\varepsilon_{n}.
\end{cases}
\] (3.29)
For $(x, t) \in B((x^\varepsilon_0, t^\varepsilon_0), \frac{n}{2})$, we obtain
\[
\left| \frac{sg_n' \left(\frac{\partial u^\varepsilon}{\partial t}\right)}{\rho_n(x, t)} \right| \leq \frac{1}{k \varepsilon n^d} e^{\frac{4}{n^2}}
\]
Since $n \in \mathbb{N}$, the integrand on the last line of (3.28) is dominated by
\[ \frac{1}{k_\varepsilon} e^{\frac{4}{\varepsilon}} \rho_n(x, t) \left| \frac{\partial^2 u^\varepsilon}{\partial t^2} \right|, \]
which is integrable on $\Omega_T$ as $u^\varepsilon \in C^{4+\beta, 4+\beta, 2} (\Omega_T)$ and
\[ \int_{\mathbb{R}^{d+1}} \rho_n(x, t) \, dx = 1. \]
Therefore an application of dominated convergence theorem gives (3.20). We now pass to the limit in the second term on RHS of (3.19). Note that
\[ sg_n \left( \frac{\partial}{\partial t} u^\varepsilon(x, T) \right) u^\varepsilon(x, T) - sg_n \left( \frac{\partial}{\partial t} u^\varepsilon(x, 0) \right) u_{0\varepsilon}(x) \to sg \left( \frac{\partial}{\partial t} u^\varepsilon(x, T) \right) u^\varepsilon(x, T) \]
\[ - sg \left( \frac{\partial}{\partial t} u^\varepsilon(x, 0) \right) u_{0\varepsilon}(x) \text{ as } n \to \infty \]
(3.30)
and
\[ \left| sg_n \left( \frac{\partial}{\partial t} u^\varepsilon(x, T) \right) u^\varepsilon(x, T) - sg_n \left( \frac{\partial}{\partial t} u^\varepsilon(x, 0) \right) u_{0\varepsilon}(x) \right| \leq 2 \| u_0 \|_{L^\infty(\Omega)}, \]
which is integrable as $\text{Vol}(\Omega) < \infty$. Therefore an application of dominated convergence theorem gives
\[ \int_{\Omega} \left( sg_n \left( \frac{\partial}{\partial t} u^\varepsilon(x, T) \right) u^\varepsilon(x, T) - sg_n \left( \frac{\partial}{\partial t} u^\varepsilon(x, 0) \right) u_{0\varepsilon}(x) \right) \, dx \, dt \]
\[ \to \int_{\Omega} \left( sg \left( \frac{\partial}{\partial t} u^\varepsilon(x, T) \right) u^\varepsilon(x, T) - sg \left( \frac{\partial}{\partial t} u^\varepsilon(x, 0) \right) u_{0\varepsilon}(x) \right) \, dx \, dt, \text{ as } n \to \infty \]
(3.31)
Using equations (3.20), (3.31) in (3.19), we have
\[ \int_{\Omega_T} \left| \frac{\partial u^\varepsilon}{\partial t} \right| \, dx \, dt = \int_{\Omega} \left( sg \left( \frac{\partial}{\partial t} u^\varepsilon(x, T) \right) u^\varepsilon(x, T) - sg \left( \frac{\partial}{\partial t} u^\varepsilon(x, 0) \right) u_{0\varepsilon}(x) \right) \, dx \]
(3.32)
Applying Theorem 2.3 and using equations (3.8) and (3.32) in (3.17), we get
\[ \varepsilon \int_{\Omega_T} B(u^\varepsilon) |\Delta u^\varepsilon| \, dx \, dt \leq \| B' \|_{L^\infty(\mathbb{R}^d)} \frac{\text{Vol}(\Omega)}{2r} \| u_0 \|_{L^\infty(\Omega)}^2 + \max_{1 \leq j \leq d} \left( \sup_{y \in I} \left| f_j'(y) \right| \right) TV_\varepsilon(u_0) \]
\[ + 2 \| u_0 \|_{L^\infty(\Omega)} \text{Vol}(\Omega). \]
(3.33)
Taking absolute value on both sides of (3.16) and integrating over $\Omega_T$, we have
\[ \int_{\Omega_T} \left| \frac{\partial u^\varepsilon}{\partial t} \right| \, dx \, dt \leq 2 \| B' \|_{L^\infty(\mathbb{R}^d)} \frac{\text{Vol}(\Omega)}{2r} \| u_0 \|_{L^\infty(\Omega)}^2 + 2 \max_{1 \leq j \leq d} \left( \sup_{y \in I} \left| f_j'(y) \right| \right) TV_\varepsilon(u_0) \]
\[ + 2 \| u_0 \|_{L^\infty(\Omega)} \text{Vol}(\Omega). \]
(3.34)
Step 3: Denote the total variation of \( u^\varepsilon \) by \( TV_{\Omega_T}(u^\varepsilon) \). Since for each \( \varepsilon > 0 \), \( u^\varepsilon \in H^1(\Omega_T) \), the total variation of \( u^\varepsilon \) is given by

\[
TV_{\Omega_T}(u^\varepsilon) = \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^1(\Omega_T)} + \left\| \nabla u^\varepsilon \right\|_{(L^1(\Omega_T))^d}.
\] (3.35)

Using equations (3.3) and (3.9), we get that \( (TV_{\Omega_T}(u^\varepsilon))_{\varepsilon \geq 0} \) is bounded. Applying the fact that \( BV(\Omega_T) \cap L^1(\Omega_T) \) is compactly imbedded in \( L^1(\Omega_T) \) \cite{2}, we get the existence of a subsequence \( (u^\varepsilon_k) \) and a function \( u \in L^1(\Omega_T) \) such that \( u^\varepsilon_k \rightarrow u \) in \( L^1(\Omega_T) \) as \( k \rightarrow \infty \). We still denote the subsequence by \( (u^\varepsilon) \). Since \( u^\varepsilon \rightarrow u \) in \( L^1(\Omega_T) \) as \( \varepsilon \rightarrow 0 \), there exists a further subsequence \( (u^\varepsilon_k) \) of \( (u^\varepsilon) \) such that we have (3.10) and (3.11).

### 3.2 BV estimate with initial data \( u_0 \) in \( W^{1,1}_0(\Omega) \cap C(\overline{\Omega}) \)

Following the proof of a result (Lemma 7.1) \cite[p.47]{3}, we conclude the following result.

**Lemma 3.2** Let \( u_0 \in W^{1,1}_0(\Omega) \cap C(\overline{\Omega}) \). Then there exists a sequence \( (u_0^\varepsilon) \) in \( D(\Omega) \) such that the following properties hold.

1. As \( \varepsilon \rightarrow 0 \), we have

\[
u_0^\varepsilon \rightarrow u_0 \text{ in } W^{1,1}(\Omega) \quad (3.36)
\]

2. For all \( \varepsilon > 0 \), there exists a constant \( A > 0 \) such that

\[
\left\| u_0^\varepsilon \right\|_{L^\infty(\Omega)} \leq A 
\] (3.37)

Following exactly the same proof of Theorem 3.1 using Hypothesis F and Theorem 2.4, we conclude that

**Theorem 3.2** Let \( f, B, u_0 \) and \( u_0^\varepsilon \) satisfy Hypothesis F and \( (u^\varepsilon) \) be as in Lemma 2.2. Then

1. for all \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that

\[
\left\| \nabla u^\varepsilon \right\|_{(L^1(\Omega_T))^d} \leq C 
\] (3.38)
2. for all \( \varepsilon > 0 \), the following inequality

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^1(\Omega_T)} \leq 2 \|B'\|_{L^\infty(I)} \frac{\text{Vol}(\Omega)}{2r} A^2 + 2 \max_{1 \leq j \leq d} \left( \sup_{y \in I} |f'_j(y)| \right) C + 2 A \text{Vol}(\Omega)
\]

(3.39)

holds.

Further there exists a subsequence \((u^\varepsilon_k)\) of \((u^\varepsilon)\), and a function \(u \in L^1(\Omega_T)\) such that

\[
u^\varepsilon_k \to u \text{ in } L^1(\Omega_T),
\]

(3.40)

\[
u^\varepsilon_k \to u \text{ a.e. } (x,t) \in \Omega_T
\]

(3.41)

as \(k \to \infty\).

**Proof of Theorem 1.1**: The proof of Theorem 1.1 follows from the proof of Theorem 1.2 from [3, p.40].

**Proof of Theorem 1.2**: The proof of Theorem 1.2 follows from the proof of Theorem 1.3 from [3, p.47].
References

[1] C. Bardos, A. Y. le Roux, and J.-C. Nédélec, First order quasilinear equations with boundary conditions Comm. Partial Differential Equations, 4(9):10171034, 1979.

[2] E. Godlewski and P.-A. Raviart, Hyperbolic systems of conservation laws, Mathématiques and Applications, Ellipses (Paris), 1991.

[3] R. Mondal, S. Sivaji Ganesh and S. Baskar, Quasilinear viscous approximations to scalar conservation laws, Preprint, arXiv:1608.07415v3.

[4] R. Mondal and S. Sivaji Ganesh, On the convergence of quasilinear viscous approximations using compensated compactness, Preprint, arXiv:1708.08847.

[5] O.A. Ladyženskaja, V. A. Solonnikov and N.N Ural’ceva, Linear and Quasi-linear Equations of Parabolic Type, American Mathematical Society, 1988.

[6] S. Kesavan, Topics in functional analysis and applications, Wiley, 1989.