A Class of Ring-like Objects

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Abstract

We introduce the notions of one-sided dirings, 3-irreducible left modules, 3-primitive left dirings, 3-semi-primitive left dirings, 3-primitive ideals and 3-radicals. The main results consists of two parts. The first part establishes two external characterizations of a 3-semi-primitive left diring. The second part characterizes the 3-radical of a left diring by using 3-primitive ideals.

By forgetting some structures of a 7-tuple introduced in Chapter 4 of [3], we get three roads of generalizing the notion of a ring $R$. The first one is to keep the additive group structure of $R$ and to replace the multiplicative monoid structure of $R$ by a dimonoid with a one-sided bar-unit. The second one is to replace the additive group structure of $R$ by a commutative digroup and to keep the multiplicative monoid structure of $R$. The third one is to replace the additive group structure of $R$ by a commutative digroup and to replace the multiplicative monoid structure of $R$ by a dimonoid with a one-sided bar-unit. Although we do not know how far we can go along the third road now, the first two roads are good enough to develop the counterpart of the basic ring theory. The purpose of this paper is to study the counterpart of the Jacobson radical for rings along the first road.

This paper consists of five sections. In Section 1 we introduce the notion of a one-sided diring and discusses its basic properties. In Section 2 we consider some fundamental concepts and results about a left module over a left diring. In Section 3 we introduce the notion of a 3-irreducible left module and prove that Schur Lemma is still true for 3-irreducible left modules over a left diring. In Section 4 we introduce the notions of 3-primitive left dirings and 3-semi-primitive left dirings, and establish two external characterizations of a 3-semi-primitive left diring. In Section 5 we introduce the notion of the 3-radical of a left diring by using the intersection of the annihilators of all 3-irreducible left $R$-modules, and prove that the 3-radical of a left diring $R$ is equal to the intersection of the 3-primitive ideals of $R$. 

1
The Notion of One-sided Dirings

We begin this section with the definition of a one-sided diring.

Definition 1.1 A nonempty set $R$ is called a left diring (or right diring) if there are three binary operations $+,$ $\leftarrow,$ and $\rightarrow$ on $R$ such that the following three properties hold

(i) $(R, +)$ is an Abelian group with the identity 0.

(ii) $(R, \leftarrow, \rightarrow)$ is a dimonoid with a left bar-unit $e_l$ (or a right bar-unit $e_r$).

(iii) The distributive laws

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad (y + z) \cdot x = y \cdot x + z \cdot x$$

hold for all $x, y, z \in R$ and $\cdot \in \{\leftarrow, \rightarrow\}$.

A left diring or a right diring is called a one-sided diring and is denoted by $(R, +, \leftarrow, \rightarrow)$.

Definition 1.2 A one-sided diring $(R, +, \leftarrow, \rightarrow)$ is called a diring if $(R, \leftarrow, \rightarrow)$ is a dimonoid with a bar-unit.

By forgetting the vector space structure of a dialgebra with a bar-unit([5]), we can regard a dialgebra with a bar-unit as a diring.

If $(R, +, \leftarrow, \rightarrow)$ is a one-sided diring, then the binary operations $+,$ $\leftarrow,$ and $\rightarrow$ are called the addition, the left product and the right product, respectively. The Abelian group $(R, +)$ is called the additive group of $R$, and the identity 0 of the additive group is called the zero element of $R$. If $x$ is an element of $R$, then the group inverse of $x$ in the additive group is denoted by $-x$. A one-sided bar-unit of the dimonoid $(R, \leftarrow, \rightarrow)$ is called a one-sided multiplicative bar-unit of $R$. The left halo of the dimonoid $(R, \leftarrow, \rightarrow)$ is called the left multiplicative halo of $R$ and is denoted by $\bar{h}_\times^\ell(R)$. The right halo of the dimonoid $(R, \leftarrow, \rightarrow)$ is called the right multiplicative halo of $R$ and is denoted by $\bar{h}_\times^r(R)$. The halo of the dimonoid $(R, \leftarrow, \rightarrow)$ is called the multiplicative halo of $R$ and is denoted by $\bar{h}_\times(R)$. Thus, we have

$$\bar{h}_\times^\ell(R) = \{ \alpha \in R \mid \alpha \leftarrow x = x \text{ for all } x \in R \},$$

$$\bar{h}_\times^r(R) = \{ \alpha \in R \mid x = x \rightarrow \alpha \text{ for all } x \in R \}$$

and

$$\bar{h}_\times(R) = \{ \alpha \in R \mid \alpha \leftarrow x = x \rightarrow \alpha \text{ for all } x \in R \}.$$
Example Let $H := \{0, a, b, c\}$ be a set of four distinct elements. We define three binary operations $+$, $\mapsto \cdot$ and $\mapsto \cdot$ on $H$ as follows:

\[
\begin{array}{c|cccc}
+ & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & b & 0 \\
b & 0 & b & b & 0 \\
c & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\mapsto \cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & a & b & c \\
c & 0 & 0 & 0 & 0 \\
\end{array}
\]

One can check that $(H, \mapsto \cdot, \mapsto \cdot)$ is a left diring with $h^x_\ell(H) = \{a, b\}$. Since $h^x_\ell(H) = \emptyset$, $H$ does not have any bar-unit. Hence, $H$ is not a diring.

It is clear that if $(R, +, \mapsto, \mapsto)$ is a left diring with a left bar-unit $e_\ell$, then $(\hat{R}, +, \mapsto, \mapsto)$ is a right diring with a right bar-unit $\hat{e}$, where $\hat{e} := e_\ell$ and the binary operations $\mapsto$ and $\mapsto$ are defined by

\[
\begin{align*}
x \mapsto y : &= y \mapsto x, \\
x \mapsto y : &= y \mapsto x,
\end{align*}
\]

where $x, y \in R$. $\hat{R}$ is called the opposite one-sided diring of an one-sided diring $R$. Using the opposite one-sided diring, a fact about a left diring can be converted to a fact about a right diring, and vice versa. Hence, we will only discuss left dirings.

The following are direct consequences of the distributive laws:

\[x * 0 = 0 = 0 * x,\]

\[(-x) * y = x * (-y) = -(x * y),\]

where $x, y$ are elements of an one-sided diring $(R, +, \mapsto, \mapsto)$ and $* \in \{\mapsto, \mapsto\}$.

Definition 1.3 Let $(R, +, \mapsto, \mapsto)$ be a left diring with a left multiplicative bar-unit $e_\ell$. The set

\[h_\ell^+(R) = \{x \in R \mid e_\ell \mapsto \cdot x = 0\}\]

is called the additive halo of $R$.

It is clear that the definition of the additive halo $h_\ell^+(R)$ does not depend on the choice of the left multiplicative bar-unit of $R$. Since

\[e_\ell \mapsto \cdot x = 0 \Rightarrow x \mapsto \cdot e_\ell = (e_\ell \mapsto \cdot x) \mapsto \cdot e_\ell = (e_\ell \mapsto \cdot x) \mapsto \cdot e_\ell = 0 \mapsto \cdot e_\ell = 0,\]
the additive halo $h^+(R)$ can be also described as follows

$$h^+(R) = \{ x \in R \mid e_\ell \cdot x = 0 = x \cdot e_\ell \}.$$  

The notion of the additive halo is indispensable to rewrite commutative ring theory in the context of dirings. The motivation of introducing the notion comes from the following facts, which were obtained in our attempt to generalize the Lie correspondence between connected linear Lie groups and linear Lie algebras.

Let $(R, +, \cdot)$ be a diring with a multiplicative bar-unit $e$. According to what we did in Section 4.1 of [3], there are three more binary operations $\uplus, \uplus$ and $\cdot$ on $R$. Their definitions are as follows:

\[
\begin{align*}
  x \uplus y & : = x + e \cdot y, \\
  x \uplus y & : = x \cdot e + y, \\
  x \cdot y & : = x \cdot y + x \cdot y - x \cdot e \cdot y, 
\end{align*}
\]

(1)

where $x, y \in R$.

One can check that $(R, \uplus, \uplus, \cdot)$ is a digroup\(^2\) with respect to the bar-unit $0$, and the halo of the digroup is the additive halo $h^+(R)$ of $R$. The binary operation defined by (1) is called the Liu product induced by $+$, $\cdot$, and $e$.

Since the Liu product is associative, a diring $(R, +, \cdot)$ can be regarded as a ring $(R, \cdot)$ with the identity $e$.

Let $A$ and $B$ be two subsets of a one-sided diring $(R, +, \cdot)$. We shall use $A \ast B$ to indicate the following subset of $R$

$$A \ast B := \{ a \ast b \mid a \in A, b \in B \},$$

where $\ast \in \{ +, \cdot, \cdot \}$. We also use $x \equiv y (\text{mod } A)$ to indicate that $x - y \in A$ for $x, y \in R$.

**Proposition 1.1** Let $(R, +, \cdot, \cdot)$ be a left diring with a left multiplicative bar-unit $e_\ell$.

(i) For all $x, y \in R$ and $\ast, \diamond \in \{ \cdot, \cdot \}$, we have

$$x \ast y \equiv x \diamond y (\text{mod } h^+(R)).$$

(ii) $h^+(R) \cdot R = 0 = R \cdot h^+(R)$.

(iii) $h^+(R) \ast R \subseteq h^+(R)$ and $R \ast h^+(R) \subseteq h^+(R)$ for $\ast \in \{ \cdot, \cdot \}$.

(iv) $e_\ell + h^+(R) \subseteq h^+(R)$.\footnote{The notion of a digroup we shall use in this paper was introduced in Definition 1.1 of [4]. In other words, the left inverse of an element $x$ of a digroup may be not equal to the right inverse of $x$.}
If $e \in h^x(R)$, then $e + h^+(R) = h^x(R)$.

Proof

(i) Since

$$e_\ell \cdot (x * y - x \circ y) = e_\ell \cdot x \cdot y - e_\ell \cdot x \cdot y = 0$$

for all $x, y \in R$ and $*, \circ \in \{ \cdot, \cdot \}$, (i) holds.

(ii) This part follows from

$$h^+(R) \cdot R = h^+(R) \cdot (e_\ell \cdot R) = (h^+(R) \cdot e_\ell) \cdot R = 0 \cdot R = 0$$

and

$$R \cdot h^+(R) = R \cdot (e_\ell \cdot h^+(R)) = R \cdot (e_\ell \cdot h^+(R)) = R \cdot 0 = 0.$$

(iii) For $* \in \{ \cdot, \cdot \}$, we have

$$e_\ell \cdot (h^+(R) * R) = (e_\ell \cdot h^+(R)) \cdot R = 0 \cdot R = 0$$

and

$$e_\ell \cdot (R * h^+(R)) = e_\ell \cdot (R \cdot h^+(R)) = e_\ell \cdot 0 = 0,$$

which imply that (iii) holds.

(iv) By (ii), we have

$$(e_\ell + h^+(R)) \cdot x = e_\ell \cdot x + h^+(R) \cdot x = x + 0 = x.$$ 

Hence, $e_\ell + h^+(R) \subseteq h^x(R)$.

(v) By (iv), we have $e + h^+(R) \subseteq h^x(R)$. Conversely, if $\alpha \in h^x(R)$, we have

$$e \cdot (\alpha - e) = e \cdot \alpha - e \cdot e = e - e = 0,$$

which implies that $\alpha - e \in h^+(R)$. Hence, $\alpha = e + (\alpha - e) \in e + h^+(R)$. Thus, $h^x(R) \subseteq e + h^+(R)$. This proves (v).

Let $(R, +, \cdot, \cdot)$ be a one-sided diring. A subgroup $I$ of the additive group $(R, +)$ is called an ideal of $R$ if

$$R * I \subseteq I, \quad I * R \subseteq I$$

for $* \in \{ \cdot, \cdot \}$. It is clear that if $h^+(R) \neq 0$, then every one-sided diring $R$ always has three distinct ideals: $0, h^+(R)$ and $R$ by Proposition (iii).

Definition 1.4 A one-sided diring $R$ is said to be 3-simple if $h^+(R) \neq 0$ and $R$ has no ideals other than 0, $h^+(R)$ and $R$. 

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A one-sided diring \( R \) is said to be \( 2 \)-simple if \( R \) has exactly two distinct ideals. It is clear that if \( R \) is 2-simple, then \( \overline{h^+}(R) = 0 \). Hence, the notion of a 2-simple diring is the same as the notion of a simple ring.

Let \( I \) be an ideal of a left diring \((R, +, \cdot, \circ)\), and let \( e_\ell \) be a left multiplicative bar-unit of \( R \). We define two binary operations \( \cdot \) and \( \circ \) on the quotient group
\[
\frac{R}{I} := \{ x + I \mid x \in I \}
\]
by
\[
(x + I) \cdot (y + I) : = x \cdot y + I,
\]
\[
(x + I) \circ (y + I) : = x \circ y + I,
\]
where \( x, y \in R \). The two binary operations above make the quotient group \( \frac{R}{I} \) into a left diring with a left multiplicative bar-unit \( e_\ell + I \), which is called the \textbf{quotient left diring} of \( R \) with respect to the ideal \( I \).

It is clear that if \( I \) is an ideal of a left diring \( R \) and \( I \supseteq \overline{h^+}(R) \), then the quotient left diring \( \frac{R}{I} \) is a rng with a left identity.

Let \((R, +, \cdot, \circ)\) be a left diring. A subset \( S \) of is called a \textbf{subdiring} of \( R \) if \((S, +)\) is a subgroup of the additive group \((R, +)\), \((S, \cdot, \circ)\) is a dimonoid and \( S \cap \overline{h^x}(R) \neq \emptyset \).

\textbf{Definition 1.5} Let \( R \) and \( \tilde{R} \) be left dirings. A map \( \phi : R \to \tilde{R} \) is called a \textbf{left diring homomorphism} if
\[
\phi(a + b) = \phi(a) + \phi(b),
\]
\[
\phi(a \ast b) = \phi(a) \ast \phi(b),
\]
\[
\phi(\overline{h^x}(R)) \cap \overline{h^x}(\tilde{R}) \neq \emptyset,
\]
where \( a, b \in R \) and \( \ast \in \{\cdot, \circ\} \). A bijective left diring homomorphism is called a \textbf{left diring isomorphism}.

Let \( \phi : R \to \tilde{R} \) be a left diring homomorphism from a left diring \( R \) to a left diring \( \tilde{R} \). The \textbf{kernel} \( \text{Ker}\phi \) and the \textbf{image} \( \text{Im}\phi \) of \( \phi \) are defined by
\[
\text{Ker}\phi := \{ a \mid a \in R \text{ and } \phi(a) = 0 \}
\]
and
\[
\text{Im}\phi := \{ \phi(a) \mid a \in R \}.
\]

It is clear that \( \text{Ker}\phi \) is an ideal of the left diring \( R \), \( \text{Im}\phi \) is a subdiring of the left diring \( \tilde{R} \) and
\[
\tilde{\phi} : a + \text{Ker}\phi \mapsto \phi(a) \quad \text{for } a \in R
\]
is a left diring isomorphism from the quotient diring $\frac{R}{\ker \phi}$ to the subdiring of $\bar{R}$.

2 Modules Over One-sided Dirings

We begin this section with the definition of a left module over a left diring.

**Definition 2.1** Let $(R, +, \cdot, \cdot)$ be a left diring with a left multiplicative bar-unit $e$. A left $R$-module $(M, \cdot, \cdot)$ is an Abelian group $M$ together with two maps $(a, x) \mapsto a \cdot x$ and $(a, x) \mapsto a \cdot x$ from $R \times M$ to $M$ satisfying the following conditions:

\[
\begin{align*}
    a \cdot (x + y) &= a \cdot x + a \cdot y, \\
    (a + b) \cdot x &= a \cdot x + x \cdot b, \\
    (a \cdot b) \cdot x &= a \cdot (b \cdot x), \\
    (a \cdot b) \cdot x &= a \cdot (b \cdot x), \\
    (a \circ b) \cdot x &= a \circ (b \cdot x), \\
    e \cdot x &= x, \\
    \end{align*}
\]

where $a, b \in R$, $x, y \in M$, $\cdot \in \{\cdot, \circ\}$ and $\circ \in \{\cdot, \cdot\}$.

Let $\operatorname{End}(M)$ be the ring of endomorphisms of an Abelian group $M$. If $a$ is an element of a left $R$-module $(M, \circ, \cdot)$ over a left diring $R$, then both $\widetilde{L}_a$ and $\widetilde{L}_a$ are endomorphisms of $M$, where $\widetilde{L}_a$ and $\widetilde{L}_a$ are defined by

\[
\begin{align*}
    \widetilde{L}_a (x) := a \cdot x, \\
    \widetilde{L}_a (x) := a \cdot x \\n    \text{for } x \in M. \\
\end{align*}
\]

$\widetilde{L}_a$ and $\widetilde{L}_a$ are called the **left translations** determined by $a$, which have been used to study digroups in Chapter 2 of [3].

It is easy to check that the two maps $\widetilde{L}: a \mapsto \widetilde{L}_a$ and $\widetilde{L}: a \mapsto \widetilde{L}_a$ are two group homomorphisms from the additive group $(R, +)$ to the additive group $(\operatorname{End}(M), +)$ and the following are true:

\[
\begin{align*}
    \widetilde{L}_a \widetilde{L}_b &= \widetilde{L}_a \widetilde{L}_b = \widetilde{L}_a \cdot \widetilde{b}, \\
    \end{align*}
\]

\[
\begin{align*}
    \widetilde{L}_a \widetilde{L}_b &= \widetilde{L}_a \cdot b, \quad \widetilde{L}_a \widetilde{L}_b &= \widetilde{L}_a \cdot \widetilde{b}, \\
    \end{align*}
\]

\[
\begin{align*}
    \widetilde{L}_a \widetilde{L}_e &= \widetilde{L}_a, \quad \widetilde{L}_e &= 1, \\
\end{align*}
\]
Conversely, if there are two group homomorphisms \( \hat{L}: a \mapsto \hat{L}_a \) and \( \check{L}: a \mapsto \check{L}_a \) from \( (R, +) \) to \( (\text{End}(M), +) \) satisfying (10), (11) and (12), then the Abelian group \( M \) becomes a left \( R \)-module under the module actions defined by (9).

Let \( R \) be a left diring. A subset \( N \) of a left \( R \)-module \( (M, \circ, \bar{\circ}) \) is called a submodule of \( M \) if the following conditions are satisfied:

(i) \( N \) is a subgroup of the Abelian group \( M \),

(ii) For all \( a \in R, x \in N \) and \( * \in \{\circ, \bar{\circ}\} \), \( a * x \in N \).

Let \( M \) be a left \( R \)-module over a left diring \( R \). It is clear that both \( 0 \) and \( M \) are submodules of \( M \). Let \( K \) and \( J \) be two submodules of \( M \). We say that a submodule \( N \) of \( M \) is a proper submodule between \( K \) and \( J \) if \( K \neq N \neq J \) and \( K \subseteq N \subseteq J \).

Let \( e \) be a multiplicative bar-unit of \( R \). The additive halo \( h^+(M) \) of \( M \) is defined by

\[
h^+(M) := \{ x \in M | e \bar{\circ} x = 0 \}. \tag{13}
\]

It is immediate that \( h^+(M) \) is a submodule of \( M \). Hence, every left \( R \)-module \( M \) always has three submodules: \( 0, h^+(M) \) and \( M \).

A left diring \( (R, +, \circ, \bar{\circ}) \) can be regarded as a left \( R \)-module \( (R, \circ, \bar{\circ}) \), where \( \bar{\circ} := \bar{\circ} \) and \( \bar{\circ} := \bar{\circ} \). This module is denoted by \( _R R \) and is called the left regular module over \( R \). A submodule of a left regular module \( _R R \) is called a left ideal of \( R \).

**Proposition 2.1** Let \( (R, +, \circ, \bar{\circ}) \) be a left diring with a left multiplicative bar-unit \( e \). If \( (M, \circ, \bar{\circ}) \) is a left \( R \)-module, then

(i) \( R \circ h^+(M) = 0 \).

(ii) \( h^+(R) \circ M = 0 \) and \( h^+(R) \bar{\circ} M \subseteq h^+(M) \).

(iii) \( M = (e \circ M) \oplus h^+(M) \), where \( \oplus \) denotes the direct sum of groups.

**Proof**

(i) For \( a \in R \) and \( x \in h^+(M) \), we have

\[
a \circ x = a \circ (e \circ x) = a \circ (e \circ x) = a \circ 0 = 0
\]

by (8) and (9). This proves (i).

(ii) For \( a \in h^+(R) \) and \( y \in M \), we have

\[
a \circ y = a \circ (e \circ y) = (a \circ e) \circ y = 0 \circ y = 0
\]
by (8) and (7). Hence, \( \tilde{h}^+(R) \circ M = 0 \).

Using (5), we have
\[
e \circ (a \circ y) = (e \cdot a) \circ y = 0 \circ y = 0,
\]
which proves that \( \tilde{h}^+(R) \circ M \subseteq \tilde{h}^+(M) \).

(iii) For any \( z \in M \), we have
\[
z = e \circ z + (z - e \circ z).
\]

By (8), (8) and (5), we have
\[
e \circ (z - e \circ z) = e \circ z - e \circ (e \circ z)
\]
\[
= e \circ (e \circ z) - e \circ (e \circ z) = 0,
\]
which implies that
\[
z - e \circ z \in \tilde{h}^+(M) \text{ for } z \in M.
\]

By (14) and (15), we get
\[
M = (e \circ M) + \tilde{h}^+(M).
\]

If \( e \circ u \in (e \circ M) \cap \tilde{h}^+(M) \) with \( u \in M \), then
\[
e \circ u = e \circ (e \circ u) = e \circ (e \circ u) \in R \circ \tilde{h}^+(M) = 0
\]
by (i). Hence, we get
\[
(e \circ M) \cap \tilde{h}^+(M) = 0.
\]

It follows from (16) and (17) that (iii) holds.

By Proposition 2.1(iii), every left multiplicative bar-unit \( e \) of a left diring \( R \) induces a decomposition of a left \( R \)-module \( M \):
\[
M = M_0 \oplus M_1,
\]
where
\[
M_0 := e \circ M, \quad M_1 := \tilde{h}^+(M).
\]

By (15), every element \( x \) of a left \( R \)-module \( M \) can be expressed uniquely as
\[
x = x_0 + x_1, \quad x_i \in M_i \text{ for } i = 0, 1.
\]
x_0 and \( x_1 \) are called the even component of \( x \) and the odd component of \( x \) induced by \( e \), respectively.

A useful property of even components is
\[
e \circ x_0 = x_0 \quad \text{for } x_0 \in e \circ M.
\]
Let $M$ and $\bar{M}$ be two left modules over a left diring $R$. A map $\phi : M \to \bar{M}$ is called a $R$-homomorphism (or module homomorphism) if
\[
\phi(x + y) = \phi(x) + \phi(y), \\
\phi(a * x) = a * \phi(x),
\]
for $x, y \in M, a \in R$ and $* \in \{\circ, \overrightarrow{\circ}\}$. A bijective $R$-homomorphism is called a $R$-isomorphism. The kernel $\text{Ker} \phi$ and the image $\text{Im} \phi$ of a $R$-homomorphism $\phi : M \to \bar{M}$ are defined by
\[
\text{Ker} \phi := \{x \mid x \in M \text{ and } \phi(x) = 0\}
\]
and
\[
\text{Im} \phi := \{\phi(x) \mid x \in M\}.
\]

It is easy to check that $\text{Ker} \phi$ is a submodule of $M$, $\text{Im} \phi$ is a submodule of $\bar{M}$ and
\[
\phi(\bar{h}^+(M)) \subseteq \bar{h}^+(\text{Im} \phi) = \bar{h}^+(M) \cap \text{Im} \phi. \tag{20}
\]

Let $N$ be a submodule of a left module $(M, \overrightarrow{\circ}, \overleftarrow{\circ})$ over a left diring $R$. Since $a * N \subseteq N$ for $a \in R$ and $* \in \{\circ, \overrightarrow{\circ}\}$, we know that
\[
a * (x + N) := a * x + N \quad \text{for } x \in M \tag{21}
\]
is a well defined map from $R \times \left(\frac{M}{N}\right)$ to the quotient group $\frac{M}{N}$. One can check that \textbf{21} makes $\frac{M}{N}$ into a left $R$-module, which is called the quotient module of $M$ with respect to the submodule $N$. The additive halo of the quotient module $\frac{M}{N}$ is given by
\[
\bar{h}^+ \left(\frac{M}{N}\right) = \frac{N + \bar{h}^+(M)}{N}. \tag{22}
\]

3 3-Irreducible Modules

We now introduce the notion of a 3-irreducible left module over a left diring.

**Definition 3.1** Let $R$ be a left diring. A left $R$-module $M$ is called a 3-irreducible module if $\bar{h}^+(M)$ is the unique proper submodule between $0$ and $M$.

Let $R$ be a left diring with a left multiplicative bar-unit $e$ and $M$ a left $R$-module. A submodule $N$ of $M$ is said to be 3-maximal if the quotient module
\( \frac{M}{N} \) is 3-irreducible. By (22), a submodule \( N \) of \( M \) is 3-maximal if and only if \( N + h^+(M) \) is the unique proper submodule between \( N \) and \( M \). A 3-maximal submodule of the left regular module \( R R \) is called a 3-maximal left ideal.

The next proposition gives the characterizations of a 3-irreducible left \( R \)-module.

**Proposition 3.1** Let \( R \) be a left diring with a left multiplicative bar-unit \( e \). If \((M, \odot, \oslash)\) is a left \( R \)-module with \( M \neq h^+(M) \) and \( h^+(M) \neq 0 \), then the following are equivalent:

(i) \( M \) is 3-irreducible.

(ii) \( M = R \odot x_0 \) for any nonzero element \( x_0 \) of \( e \odot M \), and \( \overrightarrow{h} + (M) = R \oslash x_1 \) for any nonzero element \( x_1 \) of \( h^+(M) \).

(iii) \( M \cong \frac{R}{I} \) as left \( R \)-modules, where \( I \) is a 3-maximal left ideal of \( R \).

**Proof** This is a direct consequence of Definition 3.1.

Let \( \{ M'_\lambda | \lambda \in \Lambda \} \) be a family of left modules over a left diring \( R \). The (external) direct sum \( \bigoplus_{\lambda \in \Lambda} M'_\lambda \) of the left \( R \)-modules \( M'_\lambda \) is defined by

\[
\bigoplus_{\lambda \in \Lambda} M'_\lambda := \left\{ f : \Lambda \to \bigcup_{\lambda \in \Lambda} M'_\lambda \, | \, f : \Lambda \to \bigcup_{\lambda \in \Lambda} M'_\lambda \text{ is a map} \right. \\
\left. \text{such that } f(\lambda) \in M'_\lambda \text{ for } \lambda \in \Lambda \text{ and } \supp f := \{ \lambda | \lambda \in \Lambda \text{ and } f(\lambda) \neq 0 \} \text{ is a finite set} \right\}.
\]

For \( f, g \in \bigoplus_{\lambda \in \Lambda} M'_\lambda \), \( a \in R \) and \( * \in \{ \odot, \oslash \} \), we define \( f + g \) and \( a * f \) by

\[
(f + g)(\lambda) : = f(\lambda) + g(\lambda), \quad (23)
\]

\[
(a * f)(\lambda) : = a * f(\lambda), \quad (24)
\]

where \( \lambda \in \Lambda \). It is easy to check that \( \bigoplus_{\lambda \in \Lambda} M'_\lambda \) is a left \( R \)-module.

**Definition 3.2** Let \( R \) be a left diring. A left \( R \)-module \( M \) is said to be completely 3-reducible if \( M \) is a direct sum of 3-irreducible left \( R \)-modules.

Let \( M \) and \( N \) be left modules over a left diring \( R \). The set of all \( R \)-homomorphisms from \( M \) to \( N \) is denoted by \( \text{Hom}_R(M, N) \). It is clear that \( \langle \text{Hom}_R(M, N), +, 0 \rangle \) is an Abelian group, where the addition \( + \) is defined by

\[(f + g)(x) := f(x) + g(x) \quad \text{for } x \in M \]
and the $R$-homomorphism $0 \in \text{Hom}_R(M, N)$ is defined by

$$0(x) := 0 \quad \text{for } x \in M.$$ 

The additive inverse $-f$ of an element $f \in \text{Hom}_R(M, N)$ is given by

$$(-f)(x) := -f(x) \quad \text{for } x \in M.$$ 

If $M = N$, the Abelian group

$$\text{End}_R M := \text{Hom}_R(M, M)$$

is a ring with respect to the associative product $fg$, where $fg$ is defined by

$$(fg)(x) := f(g(x)) \quad \text{for } x \in M.$$ 

The next proposition shows that Schur’s Lemma is still true for 3-irreducible left modules over a left diring.

**Proposition 3.2** Let $R$ be a left diring. If $M$ and $N$ are 3-irreducible left $R$-modules, then any $R$-homomorphism from $M$ to $N$ is either 0 or a $R$-isomorphism. In other words, $\text{End}_R M$ is a division ring.

**Proof** Let $f$ be a nonzero $R$-homomorphism from $M$ to $N$. Then $\text{Ker} f = h^+(M)$ or $\text{Ker} f = 0$, and $\text{Im} f = h^+(N)$ or $\text{Im} f = N$. Hence, we have four possible cases.

**Case 1:** $\text{Ker} f = h^+(M)$ and $\text{Im} f = h^+(N)$, in which case, we have

$$f(e \odot x) = e \odot f(x) \in e \odot h^+(N) = 0 \quad \text{for } x \in M.$$ 

Hence, $e \odot M \subseteq \text{Ker} f = h^+(M)$, which is impossible.

**Case 2:** $\text{Ker} f = h^+(M)$ and $\text{Im} f = N$, in which case, we have

$$\frac{M}{h^+(M)} = \frac{M}{\text{Ker} f} \cong N \quad \text{as left } R\text{-modules.}$$

Since $h^+(N)$ is a proper submodule between 0 and $N$, there is a proper submodule between $h^+(M)$ and $M$, which is impossible.

**Case 3:** $\text{Ker} f = 0$ and $\text{Im} f = h^+(N)$, in which case, $0 \neq e \odot M \subseteq \text{Ker} f = 0$, which is impossible.

**Case 4:** $\text{Ker} f = 0$ and $\text{Im} f = h^+(N)$, in which case, $f$ is a $R$-isomorphism. 

\[ \Box \]
4 3-Primitivity and 3-Semi-Primitivity

Let \(( R, +, \circlearrowleft, \circlearrowright )\) be a left diring. The annihilator \( \text{ann}_RM \) of a left \( R \)-module \(( M, \circlearrowleft, \circlearrowright )\) is defined by

\[
\text{ann}_RM := \left\{ a \in R \mid a \ast M = 0 \text{ for } \ast \in \{ \circlearrowleft, \circlearrowright \} \right\}.
\]  

It is clear that \( \text{ann}_RM \) is an ideal of \( R \).

A left \( R \)-module \(( M, \circlearrowleft, \circlearrowright )\) is said to be faithful if \( \text{ann}_RM = 0 \). It is easy to check that \(( M, \circlearrowleft, \circlearrowright )\) is a faithful left \( R \)-module and the module actions are defined by

\[
(b + \text{ann}_RM) \circlearrowleft x : = b \circlearrowleft x,
\]

\[
(b + \text{ann}_RM) \circlearrowright x : = b \circlearrowright x,
\]

where \( b \in R \) and \( x \in M \).

**Definition 4.1** A left diring \( R \) is said to be 3-primitive if there is a faithful 3-irreducible left \( R \)-module. A left diring \( R \) is said to be 3-semi-primitive if for any \( a \neq 0 \) in \( R \) there exists a 3-irreducible left \( R \)-module \( M \) such that \( a \not\in \text{ann}_RM \).

Let \( \{ R_\lambda \mid \lambda \in \Lambda \} \) be a family of left dirings indexed by a set \( \Lambda \). The set

\[
\prod_{\lambda \in \Lambda} R_\lambda := \left\{ f : \Lambda \to \bigcup_{\lambda \in \Lambda} R_\lambda \text{ is a map such that } f(\lambda) \in R_\lambda \text{ for } \lambda \in \Lambda \right\}.
\]  

is called the direct product of the left dirings \( R_\lambda \) with \( \lambda \in \Lambda \). For \( f, g \in \prod_{\lambda \in \Lambda} R_\lambda \), we define \( f + g \), \( f \circlearrowleft g \) and \( f \circlearrowright g \) by

\[
(f + g)(\lambda) : = f(\lambda) + g(\lambda),
\]

\[
(f \circlearrowleft g)(\lambda) : = f(\lambda) \circlearrowleft g(\lambda),
\]

\[
(f \circlearrowright g)(\lambda) : = f(\lambda) \circlearrowright g(\lambda)
\]

for all \( \lambda \in \Lambda \). Let \( 0_\lambda \) and \( e_\lambda \) be the zero element of \( R_\lambda \) and a left multiplicative bar-unit of \( R_\lambda \), respectively. We define \( 0_\Lambda \) and \( e_\Lambda \) by

\[
0_\Lambda(\lambda) := 0_\lambda, \quad e_\Lambda(\lambda) := e_\lambda \quad \text{for all } \lambda \in \Lambda.
\]

Then the direct product \( \left( \prod_{\lambda \in \Lambda} R_\lambda, +, \circlearrowleft, \circlearrowright \right) \) is a left diring, where \( 0_\Lambda \) is the zero element of the direct product, and \( e_\Lambda \) is a left multiplicative bar-unit of the
direct product. The additive halo and the left multiplicative halo of the direct product are given by

\[ h^+ \left( \prod_{\lambda \in \Lambda} R_\lambda \right) = \{ f | f(\lambda) \in h^+ (R_\lambda) \text{ for all } \lambda \in \Lambda \} \]  

(29)

and

\[ h^x_\ell \left( \prod_{\lambda \in \Lambda} R_\lambda \right) = \{ f | f(\lambda) \in h^x_\ell (R_\lambda) \text{ for all } \lambda \in \Lambda \} \].  

(30)

For \( \lambda \in \Lambda \), the map \( \pi_\lambda : \prod_{\alpha \in \Lambda} R_\alpha \rightarrow R_\lambda \) defined by

\[ \pi_\lambda(f) := f(\lambda) \text{ for } f \in \prod_{\alpha \in \Lambda} R_\alpha \]  

(31)

is a surjective left diring homomorphism. \( \pi_\lambda \) is called the projection from \( \prod_{\alpha \in \Lambda} R_\alpha \) onto \( R_\lambda \).

**Definition 4.2** Let \( \{ R_\lambda | \lambda \in \Lambda \} \) be a family of left dirings indexed by a set \( \Lambda \). A left diring \( R \) is called a subdirect product of \( R_\lambda \) with \( \lambda \in \Lambda \) if there is an injective left diring homomorphism \( \phi : R \rightarrow \prod_{\lambda \in \Lambda} R_\lambda \) such that \( \text{Im}(\pi_\lambda \phi) = R_\lambda \) for all \( \lambda \in \Lambda \).

We now establish two external characterizations of 3-semi-primitivity.

**Proposition 4.1** The following conditions on a left diring \( R \) are equivalent:

(i) \( R \) is 3-semi-primitive.

(ii) There exists a faithful completely 3-reducible left \( R \)-module.

(iii) \( R \) is a subdirect product of 3-primitive left dirings.

**Proof** (i) \( \Rightarrow \) (ii): For each \( a \neq 0 \) in \( R \), we have a 3-irreducible left modules \( M_a \) such that \( a \notin \text{ann}_R M_a \). Form \( M = \bigoplus_{a \in R \setminus \{0\}} M_a \), which is the direct sum of the left modules \( M_a \) with \( a \in R \setminus \{0\} \). By (21), we have

\[ \text{ann}_R M = \bigcap_{a \in R \setminus \{0\}} \text{ann}_R M_a = 0. \]

Hence, the direct sum is a faithful completely 3-reducible left \( R \)-module.
(ii) ⇒ (iii): Let $M$ be a faithful completely 3-reducible left $R$-module. Then $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$ is the direct sum of 3-irreducible left $R$-modules $M_{\lambda}$. Hence, we have

$$0 = \text{ann}_R M = \bigcap_{\lambda \in \Lambda} \text{ann}_R M_{\lambda}, \quad (32)$$

Since $\text{ann}_R M_{\lambda}$ is an ideal of $R$ for $\lambda \in \Lambda$, we have a left diring homomorphism $\phi$ from $R$ to the direct product $\prod_{\lambda \in \Lambda} R_{\lambda}$ of the left dirings $R_{\lambda}$, where $R_{\lambda} := \frac{R}{\text{ann}_R M_{\lambda}}$ is the quotient left diring of $R$ with respect to the ideal $\text{ann}_R M_{\lambda}$, and $\phi$ is defined by

$$\phi(a) : \lambda \mapsto a + \text{ann}_R M_{\lambda} \quad \text{for } a \in R \text{ and } \lambda \in \Lambda. \quad (33)$$

It follows from (32) and (33) that

$$\phi(a) = 0 \quad \iff \quad a + \text{ann}_R M_{\lambda} = \text{ann}_R M_{\lambda} \quad \text{for all } \lambda \in \Lambda$$

$$\iff \quad a \in \bigcap_{\lambda \in \Lambda} \text{ann}_R M_{\lambda} = 0,$$

which proves that $\phi$ is injective.

For any $\lambda \in \Lambda$, we have

$$\pi_{\lambda}(\phi(a)) = \phi(a)(\lambda) = a + \text{ann}_R M_{\lambda} \quad \text{for } a \in R,$$

which implies that

$$\text{Im}(\pi_{\lambda} \phi) = \frac{R}{\text{ann}_R M_{\lambda}} = R_{\lambda} \quad \text{for all } \lambda \in \Lambda.$$

This proves that $R$ is a subdirect product of left dirings $R_{\lambda}$.

Since $M_{\lambda}$ is a faithful 3-irreducible left module over $\frac{R}{\text{ann}_R M_{\lambda}} = R_{\lambda}$ under the module actions $\cdot$ and $\cdot$, $R_{\lambda}$ is a 3-primitive left diring. Therefore, (iii) holds.

(iii) ⇒ (i): Let $R$ be a subdirect product of the 3-primitive left dirings $R_{\lambda}$ with $\lambda \in \Lambda$. Hence, there is an injective left diring homomorphism $\phi : R \to \prod_{\lambda \in \Lambda} R_{\lambda}$.

Let $M_{\lambda}$ be a faithful 3-irreducible left $R_{\lambda}$-module, where $\lambda \in \Lambda$. For $x_{\lambda} \in M_{\lambda}$, we define

$$a \ast x_{\lambda} := (\pi_{\lambda} \phi)(a) \ast x_{\lambda} = \phi(a)(\lambda) \ast x_{\lambda}, \quad (34)$$

where $a \in R$, $\ast \in \{\circlearrowleft, \circlearrowright\}$, and $\pi_{\lambda} : \prod_{\alpha \in \Lambda} R_{\alpha} \to R_{\lambda}$ is the projection defined by (31). It is clear that $M_{\lambda}$ becomes a left $R$-module under (34).
Note that $\phi \left( h^x(R) \right) \cap \left( \prod_{\lambda \in \Lambda} R_\lambda \right) \neq \emptyset$. Hence, we have $e \in h^x(R)$ such that $\phi(e) \in \left( \prod_{\lambda \in \Lambda} R_\lambda \right)$. By (30), $\phi(e)(\lambda) \in h^x(R_\lambda)$. For $x_\lambda \in M_\lambda$, we have $e \circ x_\lambda = 0 \iff \phi(e)(\lambda) \circ x_\lambda = 0$ by (34). This proves that the additive halo of the $R$-module $M_\lambda$ is equal to the additive halo of the $R_\lambda$-module $M_\lambda$.

Since $Im(\pi_\lambda \phi) = R_\lambda$ for $\lambda \in \Lambda$, a subgroup $N$ of $(M_\lambda, +)$ is a left $R_\lambda$-submodule of $M_\lambda$ if and only if $N$ is a left $R$-submodule of $M_\lambda$ by (24). This proves that $M_\lambda$ is a 3-irreducible left $R$-module under (34).

We now consider the direct sum $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ of the 3-irreducible left $R$-modules $M_\lambda$. Using (34) and the fact that $M_\lambda$ is a faithful $R_\lambda$-module, we get

$$a \in \text{ann}_R M \iff \pi_\lambda(\phi(a)) \in \text{ann}_{R_\lambda} M_\lambda = 0 \quad \text{for } \lambda \in \Lambda$$

$$\iff \phi(a) \in \bigcap_{\lambda \in \Lambda} \text{Ker} \pi_\lambda = 0$$

$$\iff a = 0.$$

Hence, $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a faithful completely 3-irreducible $R$-module. In other words, $R$ is 3-semi-primitive.

$$\square$$

5 3-Primitive Ideals

Let $I$ be a left ideal of a left diring $(R, +, \circlearrowleft, \circlearrowright)$, We define

$$(I : R) := \{ a \in R \mid a \circlearrowleft R \subseteq I \text{ and } a \circlearrowright R \subseteq I \}.$$  \hspace{1cm} (35)

After regarding $I$ as a submodule of the left regular $R$-module $R_R$, the annihilator of the quotient $R$-module $\frac{R}{T}$ is $(I : R)$. Thus, we know that

$$(I : R) = \text{ann}_R \left( \frac{R}{T} \right)$$  \hspace{1cm} (36)

is an ideal of $R$. If $K$ is an ideal of $R$ contained in $I$, then $K \circ R \subseteq K \subseteq I$ for $\circ \in \{ \circlearrowleft, \circlearrowright \}$. Hence, $K \subseteq (I : R)$ by (36). This proves that

$$(I : R) \text{ is the largest ideal of } R \text{ contained in } I,$$  \hspace{1cm} (37)
where $I$ is a left ideal of a left diring $R$.

Note that

$$R \text{ is a diring } \Rightarrow (I : R) \subseteq I.$$ 

Let $(M, \overrightarrow{\circ}, \overleftarrow{\circ})$ be a left module over a left diring $R$. If $H$ is an ideal of $R$ and $H \subseteq \text{ann}_RM$, then $M$ is a left module over the quotient left diring $\overline{R} := \frac{R}{H}$ under the following module actions:

$$(a + H) \ast x := a \ast x,$$

where $a \in R$, $x \in M$ and $\ast \in \{\overrightarrow{\circ}, \overleftarrow{\circ}\}$. It is clear that

$M$ is a 3-irreducible left $R$-module

$\iff M$ is a 3-irreducible left $\overline{R}$-module

and

$$\text{ann}_\overline{R}M = \frac{\text{ann}_RM}{H}.$$  \hspace{1cm} (40)

**Definition 5.1** An ideal $H$ of a left diring $R$ is called a **3-primitive ideal** if the quotient left diring $\frac{R}{H}$ is a 3-primitive left diring.

**Proposition 5.1** Let $H$ be an ideal of a left diring $R$. Then $H$ is 3-primitive if and only if $H = (I : R)$ for some 3-maximal left ideal $I$ of $R$.

**Proof** If $H$ is a 3-primitive ideal, then there exists a 3-irreducible $\overline{R} := \frac{R}{H}$ module $\overline{R}M$ such that $\text{ann}_{\overline{R}}M = \{H\}$.

It is clear that $M$ becomes a left $R$-module $\underline{R}M$ under the following module actions:

$$a \ast x := (a + H) \ast x,$$

where $a \in R$, $x \in M$ and $\ast \in \{\overrightarrow{\circ}, \overleftarrow{\circ}\}$. Since

$$a \in \text{ann}_R M \iff a + H \in \text{ann}_R M = \{H\} \iff a + H = H \iff a \in H,$$

we have $\text{ann}_R M = H$. By (39), $M$ is also 3-irreducible as a left $R$-module.

Using Proposition 5.1(iii), $M \simeq \frac{R}{I}$ as left $R$-modules, where $I$ is a 3-maximal left ideal of $R$. Thus, we get

$$H = \text{ann}_R M = \text{ann}_R \left(\frac{R}{I}\right) = (I : R).$$

Conversely, if $H = (I : R)$ for a 3-maximal left ideal of $R$, then $M \simeq \frac{R}{I}$ is a 3-irreducible left $R$-module such that

$$\text{ann}_R M = \text{ann}_R \left(\frac{R}{I}\right) = (I : R) = H.$$
Using (38), (39) and (40), $M$ is a faithful 3-irreducible left module over the quotient left diring $\frac{R}{H}$. This prove that $\frac{R}{H}$ is a 3-primitive left diring. Hence, $H$ is a 3-primitive ideal.

Let $R$ be a left diring. The intersection of the annihilators of all 3-irreducible left $R$-modules is called the 3-radical of $R$ and is denoted by $\text{rad}_3 R$. Since

$$\text{rad}_3 R = \bigcap_{M \text{ runs over all 3-irreducible left } R\text{-module}} \text{ann}_R M$$ (42)

and $\text{ann}_R M$ is an ideal of $R$, $\text{rad}_3 R$ is an ideal of $R$.

**Proposition 5.2** If $R$ is a left diring, then $\text{rad}_3 R$ is the intersection of the 3-primitive ideals of $R$.

**Proof** By Proposition 3.1(iii), (36) and (42), we have

$$\text{rad}_3 R = \bigcap_{I \text{ runs over all 3-maximal ideal of } R} (I : R),$$

which can be written as

$$\text{rad}_3 R = \bigcap_{H \text{ runs over all 3-primitive ideal of } R} H$$ (43)

by Proposition 5.1.

**Proposition 5.3** Let $R$ be a nonzero left diring.

(i) $R$ is 3-semi-primitive if and only if $\text{rad}_3 R = 0$.

(ii) $\text{rad}_3 \left( \frac{R}{\text{rad}_3 R} \right) = 0$.

**Proof** (i) If $\text{rad}_3 R = 0$, then

$$\bigcap_{H \text{ runs over all 3-primitive ideal of } R} H = 0$$

by (43). Hence, $R$ is a subdirect product of the 3-primitive left dirings $\frac{R}{H}$, where $H$ runs over the 3-primitive ideals of $R$. It follows from Proposition 4.1 that $R$ is 3-semi-primitive.
Conversely, if \( R \) is 3-semi-primitive, then \( R \) is a subdirect product of the 3-primitive left dirings \( R_\lambda \) with \( \lambda \in \Lambda \). Hence, there is an injective left diring homomorphism \( \phi : R \to \prod_{\lambda \in \Lambda} R_\lambda \) such that \( \text{Im}(\pi_\lambda \phi) = R_\lambda \). Thus, we have

\[
\frac{R}{\text{Ker}(\pi_\lambda \phi)} \simeq R_\lambda
\]
as left dirings. This proves that \( \text{Ker}(\pi_\lambda \phi) \) is a 3-primitive ideal of \( R \). If \( a \in \bigcap_{\lambda \in \Lambda} \text{Ker}(\pi_\lambda \phi) \), then

\[
0 = \pi_\lambda(\phi(a)) = \phi(a)(\lambda)
\]
for \( \lambda \in \Lambda \), which proves that \( \phi(a) \) is the zero element of the diring \( \prod_{\lambda \in \Lambda} R_\lambda \). Since \( \phi \) is injective, \( a = 0 \). It follows from (43) that

\[
0 = \bigcap_{\lambda \in \Lambda} \text{Ker}(\pi_\lambda \phi) \supseteq \bigcap_{H \text{ runs over all 3-primitive ideal of } R} H = \text{rad}_3 R.
\]

Hence, we get \( \text{rad}_3 R = 0 \).

(ii) \( \bar{H} \) is an ideal of \( \bar{R} := \frac{R}{\text{rad}_3 R} \) if and only if \( \bar{H} = \frac{H}{\text{rad}_3 R} \) for some ideal \( H \) of \( R \) containing \( \text{rad}_3 R \). Moreover, we have

\[
\frac{R}{H} \simeq \frac{\frac{R}{\text{rad}_3 R}}{\frac{H}{\text{rad}_3 R}} = \frac{\bar{R}}{\bar{H}}
\]
as left dirings.

By Proposition 5.2 every 3-primitive ideal \( H \) of \( R \) contains \( \text{rad}_3 R \). Hence, we have

\[
\text{rad}_3 \bar{R} = \bigcap_{\bar{H} \text{ runs over all 3-primitive ideals of } \bar{R}} \bar{H} = \bigcap_{H \text{ runs over all 3-primitive ideals of } R} \left( \frac{H}{\text{rad}_3 R} \right).
\]

(44)

If \( a + \text{rad}_3 R \in \text{rad}_3 \bar{R} \), then \( a \in H \) for any 3-primitive ideal \( H \) of \( R \) by (44). Hence, we get

\[
a \in \bigcap_{H \text{ runs over all 3-primitive ideals of } R} H = \text{rad}_3 R.
\]

Thus, \( a + \text{rad}_3 R = \text{rad}_3 R \) is the zero element of \( \bar{R} \). This proves (ii).
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