WAVE ASYMPTOTICS FOR MANIFOLDS WITH INFINITE CYLINDRICAL ENDS

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Abstract. We describe wave decay rates associated to embedded resonances and spectral thresholds for manifolds with infinite cylindrical ends. We show that if the cut-off resolvent is polynomially bounded at high energies, as is the case in certain favorable geometries, then there is an associated asymptotic expansion, up to a $O(t^{-k_0})$ remainder, of solutions of the wave equation on compact sets as $t \to \infty$. In the most general such case we have $k_0 = 1$, and under an additional assumption on the ends of the manifold we have $k_0 = \infty$. If we localize the solutions to the wave equation in frequency as well as in space, our results hold for quite general manifolds with infinite cylindrical ends.

1. Introduction

Wave decay rates on a manifold of infinite volume can be related to the geometry of the manifold via the behavior of the resolvent $(-\Delta - z)^{-1}$ in the vicinity of the spectrum. A particularly important and long-studied class of problems is that of compactly supported perturbations of Euclidean space, an example of which is the classical obstacle scattering problem. In that case the dominant contributions to wave decay rates come from the resolvent behavior near the only threshold in the spectrum, $z = 0$, and as $\text{Re} z \to +\infty$. We think of the former as being related to the geometric infinity—here the spatial dimension is especially important. The latter typically reflects the dynamics of the compact, and possibly empty, set of trapped geodesics. In particular, in this setting we can separate the contributions of the geometric infinity and the trapped set. Similar results hold in many situations where infinity is “large” in a suitable sense, such as on asymptotically Euclidean, conic, and hyperbolic manifolds.

In this paper we consider manifolds which are isometric to a cylinder $(0, \infty) \times Y$ (with $Y$ compact) outside of a compact set. Thus we cannot separate the geometric infinity and the trapped geodesics, since the latter occur outside of arbitrarily large compact sets. Also in contrast with the Euclidean case is the relatively complicated nature of the spectrum of the Laplacian. The continuous spectrum has infinitely many thresholds, given by the eigenvalues of the Laplacian on $Y$, where the multiplicity increases. In addition, there may be up to infinitely many embedded resonances and eigenvalues.

A motivation for the study of such manifolds comes from waveguides and quantum dots connected to leads. The spectral geometry of these is closely related to that of asymptotically cylindrical manifolds, and they appear in certain models of electron motion in semiconductors and of propagation of electromagnetic and sound waves. We give just a few pointers to the physics and applied math literature here [LCM99, Rai00, RBBH12, EK15, BGW].
Our main results concern manifolds with infinite cylindrical ends for which the resolvent is well-behaved at high energies, which is the case in some favorable geometric situations discussed below. In this case we can compute asymptotics of the wave equation in terms of the features of the spectrum discussed above. Roughly speaking, any eigenvalues and zero resonances contribute non-decaying terms; any other embedded resonances, which can occur only at thresholds, contribute terms decaying like $t^{-1/2-k}$, with $k \in \mathbb{N}_0$; and non-resonant thresholds contribute terms decaying like $t^{-3/2-k}$.

More specifically, in this paper we study asymptotic expansions as $t \to \infty$ of solutions to the wave equation

$$\left(\partial_t^2 - \Delta\right) u(t) = 0, \quad u(0) = f_1, \quad \partial_t u(0) = f_2,$$

where $\Delta \leq 0$ is the Laplacian on a suitable Riemannian manifold $(X, g)$ with infinite cylindrical ends, and $f_1$ and $f_2$ are suitable initial conditions.

Our main results allow us to replace $-\Delta$ with a more general self-adjoint operator $H$ but require an assumption on the high energy behavior of the cut-off resolvent of $-\Delta$ or $H$. The companion paper [CD] gives a technique of constructing manifolds with infinite cylindrical ends so that such estimates hold for the resolvent of the Laplacian, or in fact for the resolvent of many Schrödinger operators; see Sections 1.2.1 and 1.2.2 of this paper for examples. However, if we apply a spectral cut off to our solution $u$ of (1.1), the hypothesis on the high energy resolvent is unnecessary—see Proposition 4.2.

Our starting point is the following elementary result for the wave equation on a Riemannian product. We will see below that many aspects of this result carry over to a range of more complicated geometries.

1.1. **The wave equation on a half cylinder.** Let $(X, g) = ((0, \infty)_r \times Y_y, dr^2 + g_Y)$, where $(Y, g_Y)$ is a compact Riemannian manifold without boundary. Let $\Delta_Y \leq 0$ be the Laplacian on $(Y, g_Y)$, and let $\{\phi_j\}_{j=0}^\infty$ be a complete orthonormal set of eigenfunctions of $\Delta_Y$, with $-\Delta_Y \phi_j = \sigma_j^2 \phi_j$, $0 = \sigma_0 \leq \sigma_1 \leq \cdots$.

Let $f_1, f_2 \in C_c^\infty(X)$. We shall consider solutions $u_D$ and $u_N$, satisfying Dirichlet and Neumann boundary conditions respectively, to (1.1) on $X$. Then we can solve (1.1) by separating variables, writing

$$f_\ell = f_\ell(r, y) = \sum_{j=0}^\infty f_{\ell, j}(r) \phi_j(y), \quad u_B(t) = u_B(t, r, y) = \sum_{j=0}^\infty u_{j, B}(t, r) \phi_j(y),$$

(1.2)

where $\ell \in \{1, 2\}$ and $B$ denotes the boundary condition “D” or “N.” We can perhaps most easily solve these initial value problems by extending $f_1, f_2$ to be odd (Dirichlet) or even (Neumann) functions on $\mathbb{R} \times Y$, and solving the wave equation on the full cylinder. For $r$ in a compact set,

$$u_{j, N}(t, r) = \int_0^\infty f_{2, j}(r) dr, \quad u_{j, D}(t) = 0 \text{ if } \sigma_j = 0, \text{ for } t \text{ sufficiently large}$$

(1.3)
by d’Alembert’s formula. For any nonnegative integer \( k_0 \), we have for \( t \) sufficiently large

\[
u_{j,B}(t,r) = \sum_{k=0}^{k_0} \left[ p_{j,k,B}(r) \cos(\sigma_j t + \frac{\pi}{4}) + q_{j,k,B}(r) \sin(\sigma_j t + \frac{\pi}{4}) \right] t^{-k-\frac{3}{2}} + O\left(t^{-k_0-\frac{3}{2}}\right), \quad \text{if } \sigma_j > 0,
\]

by the method of stationary phase (see also [Hör87] for asymptotics as \( r \to \infty \)). Here for each \( j \) \( p_{j,k,B} \) and \( q_{j,k,B} \) are polynomials in \( r \) of degree at most \( 2k \), and the remainders are uniform in \( j \) and as \( r \) varies in a compact set. Moreover, for the Neumann boundary condition

\[
p_{j,0,N} = 2\sqrt{\frac{\sigma_j}{2\pi}} \int_0^\infty f_{1,j}(r')dr', \quad q_{j,0,N} = \frac{2}{\sqrt{2\pi \sigma_j}} \int_0^\infty f_{2,j}(r')dr',
\]

(1.5)

With the Dirichlet boundary condition, \( p_{j,0,D} = 0 = q_{j,0,D} \), and

\[
p_{j,1,D}(r) = 2r \sqrt{\frac{\sigma_j}{2\pi}} \int_0^\infty r' f_{2,j}(r')dr', \quad q_{j,1,D}(r) = 2\sigma_j r' \sqrt{\frac{\sigma_j}{2\pi}} \int_0^\infty f_{1,j}(r')dr'.
\]

(1.6)

To interpret the above result in terms of the spectrum, we can similarly write the resolvent of \(-\Delta_B\) as a direct sum of shifted resolvents of \(-\partial_B^2\):

\[(-\Delta_B - z)^{-1} \cong \bigoplus_{j=0}^\infty (-\partial_B^2 + \sigma_j^2 - z)^{-1}, \quad z \in \mathbb{C} \setminus [0, \infty),\]

from which we see that the spectrum of \(-\Delta_B\) is \([0, \infty)\), is purely absolutely continuous, and has thresholds (points at which multiplicity jumps) at the eigenvalues of \(-\Delta_Y\). For the Neumann Laplacian on the half cylinder, at each threshold the spectrum contains an embedded resonance of multiplicity equal to the multiplicity of the corresponding eigenvalue of \(-\Delta_Y\) (and there are no other resonances embedded in the spectrum). The Dirichlet Laplacian on the half-cylinder has no embedded resonances.

We now see that the coefficient of the constant term in the expansion (1.3) is given by a projection onto the resonant states at zero, with the number of states equal to the number of connected components of \(Y\). The terms of order \( t^{-1/2} \) in (1.4) have coefficients given in (1.5) by projections onto the resonant states at nonzero eigenvalues of \(-\Delta_Y\). From the example of the Dirichlet half-cylinder, we see that in the absence of eigenvalues or resonances embedded in the continuous spectrum we may have a wave decay like \(O(t^{-3/2})\), but we cannot in general expect faster decay.

In the remainder of the paper we adapt the asymptotics above, in a somewhat weaker form, to Schrödinger operators on more general manifolds with cylindrical ends. One difficulty is that such operators can have much nastier behavior of the resolvent near the continuous spectrum, including the possible presence of infinitely many embedded eigenvalues, [CZ95, Par95]. Below we mostly restrict our attention to some particular cases in which the resolvent is better behaved.

1.2. Two term asymptotics for mildly trapping manifolds with cylindrical ends. Our first extension of the results of Section 1.1 is to manifolds with infinite cylindrical ends for which we have polynomial bounds on the cut-off resolvent. Rather than state the theorem in full generality here, for now we let \((X, g)\) be one of the examples in Sections 1.2.1 and 1.2.2.
Let \( \{\lambda_\ell\} \) denote the eigenvalues of \(-\Delta\), repeated with multiplicity, with corresponding orthonormal \( L^2 \) eigenfunctions \( \{\eta_\ell\} \): \(-\Delta \eta_\ell = \lambda_\ell \eta_\ell \). The generalized eigenfunctions \( \{\Phi_j\} \) are defined in (2.2); if \( \Phi_j(\sigma_j) \neq 0 \) then \( \Phi_j(\sigma_j) \) is a resonant state; \( \Phi_j(\sigma_j) = \Phi_j(\sigma_j, \bullet) \in C^\infty(X) \).

Our first result is a two term expansion, an example of which is the following theorem.

**Theorem 1.1.** Let \((X, g)\) be as in the examples in Sections 1.2.1 or 1.2.2, \( f_1, f_2 \in C^\infty_c(X) \) be given, and let \( u(t) \) solve (1.1). Then we can write

\[
  u(t) = u_e(t) + u_{thr}(t) + u_r(t),
\]

where

\[
  u_e(t) = \sum_{\lambda_\ell \in \text{spec}_p(-\Delta) \atop \lambda_\ell \neq 0} \eta_\ell \left( \cos((\lambda_\ell)^{1/2}t) \langle f_1, \eta_\ell \rangle + \frac{\sin((\lambda_\ell)^{1/2}t)}{(\lambda_\ell)^{1/2}} \langle f_2, \eta_\ell \rangle \right) + \sum_{\lambda_\ell \in \text{spec}_p(-\Delta) \atop \lambda_\ell = 0} \eta_\ell \langle f_1, \eta_\ell \rangle + t \langle f_2, \eta_\ell \rangle
\]

and

\[
  u_{thr}(t) = \frac{1}{4} \sum_{\sigma_j = 0} \Phi_j(0) \langle f_2, \Phi_j(0) \rangle + \frac{1}{2\sqrt{t}} \sum_{\sigma_j > 0} \sqrt{\frac{\sigma_j}{2\pi}} \cos(\sigma_j t + \pi/4) \Phi_j(\sigma_j) \langle f_1, \Phi_j(\sigma_j) \rangle + \frac{1}{2\sqrt{t}} \sum_{\sigma_j > 0} \frac{1}{\sqrt{2\pi \sigma_j}} \sin(\sigma_j t + \pi/4) \Phi_j(\sigma_j) \langle f_2, \Phi_j(\sigma_j) \rangle.
\]

Moreover, for any \( \chi \in C^\infty_c(X) \), there is a constant \( C \) so that the remainder, \( u_r(t) \), satisfies

\[
  \|\chi u_r(t)\|_{L^2(X)} \leq C t^{-1}, \text{ for } t \text{ sufficiently large.}
\]

For the manifolds considered here, each of these sums in (1.8) and (1.9) is in fact a finite sum. Since we have assumed the initial data \( f_1, f_2 \in C^\infty_c(X) \), we could replace the bound (1.10) by \( \|\chi u_r(t)\|_{H^m(X)} \leq C t^{-1} \) for any \( m \in \mathbb{N} \), with a new constant \( C \) depending on \( m \).

We compare the expansion of \( u \) in Theorem 1.1 with that of the solution to the wave equation on the Neumann half cylinder given by (1.2-1.5). From the expression for \( u_{thr} \) in (1.9), if \( \sigma_j = 0 \), then \( \langle \Phi_j(0), f_2 \rangle \) corresponds to \( 2 \int_0^\infty f_2, f_{2,j}(r) \, dr \) from (1.3) and (1.2). If \( \sigma_j > 0 \), then \( \Phi_j(\sigma_j) \langle f_1, \Phi_j(\sigma_j) \rangle \) corresponds to \( 2 \sqrt{\frac{2\pi}{\sigma_j}} p_{j,0,N}(r) \phi_j \) from (1.4). In contrast, for the Dirichlet half-cylinder there are no embedded resonances. Hence, for the Dirichlet half-cylinder \( \Phi_j(\sigma_j) = 0 \) for each \( j \).

Theorem 3.2 is a more general version of Theorem 1.1. In Theorem 3.2 we can allow any manifold with infinite cylindrical ends for which we have a polynomial bound on the cut-off resolvent of the Laplacian at high energies. That this condition holds for the manifolds in Sections 1.2.1 and 1.2.2 is shown in [CD], where such estimates are shown for the resolvent of \(-\Delta + V\), for a large class of potentials \( V \).

In fact, our wave expansion, Theorem 3.2, holds for more general compactly supported perturbations of the Laplacian on a manifold with infinite cylindrical ends, see Section 2. Our techniques also would work to generalize this further, for example, to solutions to the wave equation on a
planar waveguide with appropriate boundary conditions, assuming that the condition of the polynomial bound on the cut-off resolvent were satisfied, see the statement of Theorem 3.2. In the interest of clarity we do not pursue this here.

1.2.1. **Examples with minimal trapping.** Let $r$ be the radial coordinate in $\mathbb{R}^d$ for some $d \geq 2$, and let

$$X = \mathbb{R}^d, \quad g_0 = dr^2 + F(r)dS,$$

where $dS$ is the usual metric on the unit sphere, $F(r) = r^2$ near $r = 0$, and $F'$ is compactly supported on some interval $[0, R]$ and positive on $(0, R)$; see Figure 1.

![Figure 1. A cigar-shaped warped product.](image)

Then all $g_0$-geodesics obey, for $r(t) \neq 0$,

$$\ddot{r}(t) := \frac{d^2}{dt^2}r(t) = 2|\eta|^2F'(r(t))F(r(t))^{-2} \geq 0,$$

where $r(t)$ is the $r$ coordinate of the geodesic at time $t$ and $\eta$ is the angular momentum. Consequently, the only trapped geodesics (that is, the only maximally extended geodesics with $\sup_{t \in \mathbb{R}} r(t) < +\infty$) are the ones with $\dot{r}(t) \equiv F'(r(t)) \equiv 0$, that is the ones in the cylindrical end that have no radial momentum. This is the smallest amount of trapping a manifold with a cylindrical end can have.

Let $g$ be any metric such that $g - g_0$ is supported in $\{(r, y) \mid r < R\}$, and such that $g$ and $g_0$ have the same trapped geodesics. For example we may take $g = g_0 + cg_1$, where $g_1$ is any symmetric two-tensor with support in $\{(r, y) \mid r < R\}$, and $c \in \mathbb{R}$ is chosen sufficiently small depending on $g_1$. Alternatively, we may take $g = dr^2 + g_S(r)$, where $g_S(r)$ is a smooth family of metrics on the sphere such that $g_S(r) = r^2dS$ near $r = 0$ and $g_S(r) = F(r)dS$ near $r \geq R$, and such that $\partial_r g_S(r) > 0$ on $(0, R)$. This way we can construct examples where $g - g_0$ is not small.

1.2.2. **Examples based on convex cocompact manifolds.** Let $(X, g_H)$ be a convex cocompact hyperbolic surface, such as the symmetric hyperbolic ‘pair of pants’ surface with three funnels depicted in Figure 2.

In particular, there is a compact set $N \subset X$ (the convex core of $X$) such that

$$X \setminus N = (0, \infty)_r \times Y_y, \quad g_H|_{X \setminus N} = dr^2 + \cosh^2 r \, dy^2,$$

where $Y$ is a disjoint union of $k \geq 1$ geodesic circles, not necessarily all of the same length.

We construct a metric on $X$ which gives it the structure of a manifold with infinite cylindrical ends by modifying the metric on the funnel ends. Take $g$ such that

$$g|_N = g_H|_N, \quad g|_{X \setminus N} = dr^2 + F(r)dy^2,$$
where $F(r) = \cosh^2 r$ near $r = 0$, and $F'$ is compactly supported and positive on the interior of the convex hull of its support.

To construct examples with dimension $d \geq 3$, we can take $(X, g_H)$ to be a conformally compact manifold of constant negative curvature, provided the dimension of the limit set is less than $(d - 1)/2$. In this case the modification of the metric in the ends is a bit more complicated—see [CD, Section 2.2].

1.3. Complete expansions under an additional spacing condition on the thresholds.

In this subsection we suppose that $(X, g)$ is as in Section 1.2 but with an additional assumption on the eigenvalues of $-\Delta_Y$. This assumption always holds in the examples in Section 1.2.1, but holds only sometimes in the other examples.

To state it, let $\{\nu_l\}_{l=0}^{\infty}$ be the sequence of square roots of distinct eigenvalues of $-\Delta_Y$ in increasing order, so that $0 = \nu_0 < \nu_1 < \cdots$. The assumption is that there are positive constants $c_Y$ and $N_Y$, such that

$$\nu_{l+1} - \nu_l \geq c_Y \nu_l^{-N_Y},$$

for all $l \in \mathbb{N}$ with $\nu_l \geq 1$. Note that this assumption allows the eigenvalues of $-\Delta_Y$ to have high multiplicities, but forbids distinct eigenvalues from clustering too closely together. With this assumption and a bound on the cut-off resolvent at high energy we can bound derivatives of the cut-off resolvent at high energy, see Lemmas 4.3 and 4.4. This allows us to refine our expansions.

**Theorem 1.2.** Let $(X, g)$ be as in the examples in Sections 1.2.1 or 1.2.2 and satisfy (1.11), $f_1$, $f_2 \in C^\infty_c(X)$ be given, and let $u(t)$ solve (1.1). Then for each $k_0 \in \mathbb{N}$ we can write

$$u(t) = u_e(t) + u_{thr,k_0}(t) + u_{r,k_0}(t),$$

where $u_e$ is still given by (1.8) and

$$u_{thr,k_0}(t) = \frac{1}{4} \sum_{\sigma_j = 0} \Phi_j(0) \langle f_2, \Phi_j(0) \rangle + \sum_{k=0}^{k_0-1} t^{-1/2-k} \sum_{l=1}^\infty (e^{it\nu_l b_{l,k,+}} + e^{-it\nu_l b_{l,k,-}})$$
for some $b_{l,k,\pm} \in \langle r \rangle^{1/2+2k_0+\epsilon} L^2(X)$. For any $\chi \in C^\infty_c(X)$ there is a constant $C$ so that

$$\sum_{l=1}^\infty \| \chi b_{l,k,\pm} \|_{L^2(X)} < C, \ k = 0, 1, 2, \ldots, k_0$$

and

$$\| \chi u_{r,k_0}(t) \|_{L^2(X)} \leq Ct^{-k_0} \text{ for } t \text{ sufficiently large.}$$

Moreover, the $b_{l,k,\pm}$ are determined by the value $\nu_l$, the initial data $f_1, f_2$, and suitable derivatives of elements of the set $\{ \Phi_j(\lambda) \} \nu_l \leq \sigma_j \leq \nu_l$ evaluated at $\pm \nu_l$.

For further details about how the $b_{l,k,\pm}$ are determined, see Theorem 4.1 and Lemma 4.7 and its proof.

As in Theorem 1.1, because of the smoothness of the initial data we can instead bound $\| \chi u_{r,k_0}(t) \|_{H_m(X)} \leq Ct^{-k_0}$ with the constant depending on $m$ as well as $\chi$ and the initial data, and the series $\sum_{l=1}^\infty \| \chi b_{l,k,\pm} \|_{H_m(X)}$ converges for each value of $m$.

Theorems 1.1 and 1.2, and their more general versions, Theorems 3.2 and 4.1, require high energy bounds on the norm of the cut-off resolvent. The bounds are generally proved under some conditions on the trapping, as we have discussed in the introduction. However, for a general manifold with infinite cylindrical ends, without a bound on the high energy behavior of the resolvent or any restrictions on the trapping, we can find an asymptotic expansion of $\chi \psi_{sp}(-\Delta)u(t)$, provided that $\psi_{sp} \in C^\infty_c(\mathbb{R})$, see Proposition 4.2. Here, as before, $u(t)$ is the solution of (1.1), and we must assume the initial data have support in a fixed compact set.

The literature of the study of local energy decay under the assumptions of no trapping or mild trapping is quite large, and we mention only a few papers. The study of local energy decay for nontrapping perturbations of the Laplacian on Euclidean space was initiated by Morawetz in [Mor61] and continued in, for example, [LMP62, MRS77, Vai89]. The question of wave expansions or wave decay on noncompact manifolds with various kinds of ends and different trapping assumptions is a very active area of research; see [Zwo17] and references therein for some more recent results. The most closely related results of which we are aware are for solutions to the wave equation on a planar waveguide without forcing [Lyf76] and with forcing [HW06], where the expansion is found to order $o(1)$. For energy decay of solutions to a dissipative wave equation on a waveguide, see [MR].

Our results build on studies of the spectral theory of manifolds with cylindrical ends, in particular we mention [Gol73, Lyf76, Gui89, Mel93, Chr95, Par95]. More recent papers include [Chr02, IKL10, MS10, RTdA13] and references therein. We give more precise references as they are used.

1.4. Notation. In this section we collect, for reference, some notation introduced either in the introduction or later in the paper.

- $f_1, f_2$ are initial data, see (1.1) and (3.1).
- $(Y, g_Y)$ is a smooth compact Riemannian manifold without boundary, not necessarily connected, and is the “cross section” of the cylindrical end.
• \( \{ \sigma_j^2 \}_{j=0}^\infty \) are the eigenvalues of \(-\Delta_Y\) on \( Y \), repeated with multiplicity, with \( 0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \).

• \( \{ \phi_j \} \) are a complete orthonormal set of eigenfunctions of \(-\Delta_Y\) with \(-\Delta_Y \phi_j = \sigma_j^2 \phi_j\), and are chosen to be real-valued for simplicity.

• \( \{ \nu_l^2 \}_{l=0}^\infty \) are the distinct eigenvalues of \(-\Delta_Y\) on \( Y \), with \( 0 = \nu_0 < \nu_1 < \nu_2 < \ldots \).

• \( H \) is a “black box” perturbation of the Laplacian on a manifold with infinite cylindrical end, acting on elements of the Hilbert space \( \mathcal{H} \); see Section 2.

• \( \Phi_j(\lambda) \) are generalized eigenfunctions of \( H \): \( (H - \lambda^2) \Phi_j(\lambda) = 0 \), and the \( \Phi_j \)satisfy further conditions; see (2.2).

• \( \{ \eta_l \} \) are orthonormal eigenfunctions of \( H \) with eigenvalue \( \lambda_l \): \( H \eta_l = \lambda_l \eta_l \); see Section 3.1.

Throughout the paper, \( C \) denotes a positive constant whose value may change from line to line.

2. Black box perturbations of the Laplacian on a manifold with infinite cylindrical ends and their spectrum

The results we shall prove about local wave expansions are valid for a large class of “black box” perturbations of the Laplacian on a manifold with an infinite cylindrical end, provided that there are appropriate high-energy bounds on the cut-off resolvent.

We adapt the idea of [SZ91] of a compactly supported black box perturbation of the Laplacian on Euclidean space to give a definition of a black box compactly supported perturbation of the Laplacian on a manifold with an infinite cylindrical end. Moreover, we recall some results of [Gui89, Mel93, Par95, Chr95] to describe the spectrum and spectral measure of such operators.

Let \(( Y, g_Y )\) be a smooth compact Riemannian manifold without boundary. We do not require that \( Y \) be connected. Let \( X_\infty = (0, \infty) \times Y \), with product metric \((dr)^2 + g_Y\). With \( \Delta_Y \leq 0 \) the Laplacian on \( Y \), denote the differential operator \( \partial^2_r + \Delta_Y \) by \( \Delta_{X_\infty} \). Note that we are thinking of this just as a differential operator for now, and not imposing any boundary conditions.

Let \( H \) be a complex Hilbert space with orthogonal decomposition
\[ \mathcal{H} = \mathcal{H}_0 \oplus L^2( X_\infty). \]
Let \( \mathbb{I}_{X_\infty}, \mathbb{I}_{\mathcal{H}_0} \) denote the orthogonal projections \( \mathbb{I}_{X_\infty}: \mathcal{H} \to L^2( X_\infty) \) and \( \mathbb{I}_{\mathcal{H}_0}: \mathcal{H} \to \mathcal{H}_0 \). We shall denote the inner product on \( \mathcal{H} \) by \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \).

Let \( H : \mathcal{H} \to \mathcal{H} \) be a linear self-adjoint operator with domain \( D \subset \mathcal{H} \). We assume \( \mathbb{I}_{X_\infty} D = H^2( X_\infty) \) and \( \mathbb{I}_{X_\infty} H = -\Delta_{X_\infty} |_{X_\infty} \). Moreover, assume \( H \) is lower semi-bounded and \( \mathbb{I}_{\mathcal{H}_0}(H+i)^{-1} \) is compact. We add an assumption which does not have a parallel in [SZ91], but which is used elsewhere (see e.g. [DZ17, Section 4.4]) and which is convenient for us. We suppose that there is an involution of \( \mathcal{H} \), \( f \mapsto \overline{f} \), so that, for any \( f, g \in \mathcal{H} \), \( z \in \mathbb{C} \),
\[ \overline{zf} = z\overline{f}, \quad (\overline{f})|_{X_\infty} = \overline{f|_{X_\infty}}, \quad \langle f, g \rangle_{\mathcal{H}} = \langle g, f \rangle_{\mathcal{H}}. \]
Explicitly, the second condition means that the involution agrees with complex conjugation on \( L^2( X_\infty) \). We suppose that \( H \) commutes with this involution on \( \mathcal{H} \).

We say a Riemannian manifold \(( X, g )\) is a manifold with an infinite cylindrical end if there is an \( X_\infty \subset X \) so that \( X \setminus X_\infty \) is compact, and \( X_\infty \) is as above. That is, there is a compact
Riemannian manifold \((Y, g_Y)\) so that \((X_\infty, g_{|X_\infty})\) is isometric to \((0, \infty) \times Y\) with metric \(dr^2 + g_Y\).

If \(X\) is a Riemannian manifold with infinite cylindrical ends, then the Laplacian \(-\Delta_X\) and the Schrödinger operator \(-\Delta_X + V, V \in L^2_c(X, \mathbb{R})\), are examples of black box type operators, with \(\mathcal{H} = L^2(X)\). It is possible to allow \(X\) to have a boundary which does not meet the end \(X_\infty\), in which case one must include boundary conditions that make the operator \(H\) self-adjoint.

We set the following conventions: If \(\chi \in C^\infty_c(X_\infty)\) is 1 for \(r \leq c_0\), for any \(c_0 > 0\), and \(f \in \mathcal{H}\), then by \(\chi f\) we mean

\[
\chi f = 1_{\mathcal{H}_0} f + \chi 1_{X_\infty} f.
\]

Moreover, by \(f \in (r)^p \mathcal{H}\), we mean \(1_{\mathcal{H}_0} f \in \mathcal{H}_0\) and \((1 + r^2)^{-p/2} 1_{X_\infty} f \in L^2(X_\infty)\).

2.1. **Resolvent and generalized eigenfunctions.** Now let \(H\) be an operator as described above, and for \(\text{Im} \lambda > 0\) define the resolvent

\[
R(\lambda) = (H - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{H}.
\]

The assumptions we have made on \(H\) imply that \(R(\lambda)\) has at most finitely many poles in \(\text{Im} \lambda > 0\), and these poles correspond to the square roots of negative eigenvalues. Moreover, it is straightforward to see with techniques combining those of [SZ91] and [Gui89] that if \(\chi \in C^\infty_c(X_\infty)\), with \(\chi(r) = 1\) for \(r \leq 1\), then \(\chi R(\lambda) \chi\) has a meromorphic continuation to the minimal Riemann surface on which \((\lambda^2 - \sigma_j^2)^{1/2}\) is an analytic function for each \(\sigma_j^2 \in \text{spec}(-\Delta_Y)\). This is done for \(H = -\Delta_X\) on a manifold with cylindrical ends \(X\) in [Gui89], for example. We denote this Riemann surface by \(\tilde{Z}\). By the physical space we mean the copy of the upper half plane \(\{\text{Im} \lambda > 0\}\) in \(\tilde{Z}\) on which the resolvent \(R(\lambda)\) is bounded on \(\mathcal{H}\). We often identify the physical space with the upper half plane without further comment. Thus, by \(\lambda \in \mathbb{R}\) we mean \(\lambda\) lies on the boundary of the physical space.

We recall some results from [Mel93, sections 6.7-6.10], [Chr95, Section 2], and [Par95] on the behavior of \(R(\lambda)\) along the real axis and the consequences for the spectral measure. The results of [Mel93] which we recall are given for a somewhat different class of operators than the operators \(H\) which we consider here. However, it is not hard to check that the results we cite follow from the proofs of the analogous results in [Mel93] with only minor modifications. This is because the properties of \(-\Delta\) essential to the proofs in [Mel93] which we use are shared by our operator \(H\): self-adjointness, the fact that the operator is the Laplacian on the end \(X_\infty\), and the meromorphic continuation of the resolvent to \(\tilde{Z}\).

If the operator \(H\) has any eigenvalues, they are a discrete set with infinity being the only possible accumulation point. The continuous spectrum of \(H\) is \([0, \infty)\). However, the multiplicity of the continuous spectrum changes at points \(\sigma_j^2 \in \text{spec}(-\Delta_Y)\). We call the points \(\{\sigma_j^2\}\) thresholds. When using \(\lambda^2\) as a spectral parameter, we shall abuse terminology a bit and refer to the points \(\{\pm \sigma_j\}\) as thresholds as well. Let \(\mathcal{P}\) denote projection onto the span of the eigenfunctions of \(H\). Then for any \(\chi \in C^\infty_c(X_\infty), \chi = 1\) for \(r \leq 1\), the operator-valued function \(\chi R(\lambda)(I - \mathcal{P})\chi\) is continuous for \(\lambda\) on the boundary of the physical space, except, perhaps, at \(\{\pm \sigma_j, j \in \mathbb{N}_0\}\).

We set, for \(\text{Im} \lambda > 0, j \in \mathbb{N}_0\),

\[
\tau_j(\lambda) = (\lambda^2 - \sigma_j^2)^{1/2},
\]
where we take the square root to have positive imaginary part. When \( \lambda \in \mathbb{R} \), we take \( \tau_j(\lambda) = \tau_j(\lambda + i0) \). Note that for each \( j \), \( \tau_j \) has an analytic continuation to a single-valued function on \( \hat{Z} \), with \( \tau_j(\lambda) = \tau_j(\lambda) \) if \( \sigma_j = \sigma_j' \). We shall use the same notation for this continuation. The physical space can be characterized as the subset of \( \hat{Z} \) on which \( \text{Im} \, \tau_j(\lambda) > 0 \) for all \( j \in \mathbb{N}_0 \).

We define a family of generalized eigenfunctions \( \Phi_j \) of \( H \) depending on the parameter \( \lambda \in \hat{Z} \), though we shall be most interested in them for \( \lambda \in \mathbb{R} \), that is, \( \lambda \) on the boundary of the physical space. Let \( \{\phi_j\} \) be a complete orthonormal set of real-valued eigenfunctions of \(-\Delta_Y\) on \( Y \), with \(-\Delta_Y \phi_j = \sigma_j^2 \phi_j \). Let \( \chi \in C^\infty_c(X_\infty) \) be 1 for \( r \leq 1 \), and set

\[
\Phi_j = \Phi_j(\lambda) = (1 - \chi)e^{-ir\tau_j(\lambda)r}\phi_j(y) - R(\lambda)[\Delta_{X_\infty}, \chi]e^{-ir\tau_j(\lambda)r}\phi_j(y), \quad \text{for} \ \text{Im} \ \lambda > 0. \tag{2.2}
\]

Then \( \chi \Phi_j \in \mathcal{H} \) and \((H - \lambda^2)\Phi_j(\lambda) = 0\). It is easy to see that when \( \text{Im} \ \lambda > 0 \) and \( \lambda^2 \) is not an eigenvalue of \( H \) then \( \Phi_j(\lambda) \) is independent of the choice of \( \chi \) satisfying these conditions: If \( \Phi_j, \tilde{\Phi}_j \) are defined as in (2.2) with different functions \( \chi, \tilde{\chi} \) satisfying the conditions on \( \chi \), then the difference is an element of \( \mathcal{H} \) which is in the null space of \( H - \lambda^2 \), and hence is 0 by necessity. The functions \( \Phi_j \) have a meromorphic continuation to \( \hat{Z} \), which we continue to denote in the same way.

From (2.2), away from poles of the resolvent we have

\[
\mathbb{1}_{X_\infty} \Phi_j(\lambda) = \mathbb{1}_{X_\infty} \left( e^{-ir\tau_j(\lambda)r}\phi_j(y) + \sum_{k=0}^\infty S_{kj}(\lambda)e^{ir\tau_k(\lambda)r}\phi_k(y) \right) \tag{2.3}
\]

for some functions \( S_{kj}(\lambda) \) which determine the scattering matrix. For each \( \lambda \) away from poles of the resolvent the series in (2.3) converges absolutely on compact sets in \( X_\infty \), as do its derivatives with respect to \( r \) or \( y \in Y \). For \( \lambda \in \mathbb{R} \) (i.e., on the boundary of the physical space) with \( |\lambda| > \sigma_j \) and away from the thresholds and the poles of the resolvent, one can equivalently define \( \Phi_j(\lambda) \) to be the element of \((r)^{1/2+\epsilon}\mathcal{H}\) which satisfies \((H - \lambda^2)\Phi_j = 0\) and which has an expansion of the form (2.3) for some \( S_{kj} \).

Both the generalized eigenfunctions \( \Phi_j \) and the functions \( S_{kj} \) are meromorphic functions on \( \hat{Z} \). We can say a bit more. Here we consider \( \lambda \in \mathbb{R} \), that is, on the boundary of the physical space in \( \hat{Z} \). From [Par95, (3.4)], or [Chr95, Lemma 1.2],

\[
\sum_{0 \leq \sigma_m \leq \lambda} \tau_m(\lambda)S_{mj}(\lambda)\mathcal{S}_{mk}(\lambda) = \tau_j(\lambda)\delta_{jk}, \quad \text{if} \ 0 \leq \sigma_j, \sigma_k \leq \lambda.
\]

In particular, this implies

\[
\sum_{0 \leq \sigma_m \leq \lambda} \tau_m(\lambda)|S_{mj}(\lambda)|^2 = \tau_j(\lambda), \quad \text{if} \ 0 \leq \sigma_j \leq \lambda.
\]

Thus \((\tau_m(\lambda))^{1/2}S_{mj}(\lambda)\) is bounded for \( \lambda \geq \sigma_m, \sigma_j \), and since \( S_{mj} \) is meromorphic on \( \hat{Z} \), it is actually continuous in this region. This, along with the fact that \( \Phi_j(\lambda) \) is orthogonal to the eigenfunctions of \( H \) when \( \lambda > \sigma_j \), means that \( \Phi_j(\lambda) \) is continuous for \( \lambda \geq \sigma_j \). Analogous arguments imply \( \Phi_j(\lambda) \) is continuous for \( \lambda \leq -\sigma_j \). This then implies that if \( \sigma_k \geq \sigma_j, \delta_0 > 0, \) and \( \lambda \in (\pm \sigma_k - \delta_0, \pm \sigma_k + \delta_0) \) and if this interval is sufficiently small then \( \Phi_j(\lambda) \) is a smooth function of \( \tau_k(\lambda) \) on this interval.
2.2. The spectral measure. The spectral measure for \((I - \mathcal{P})H\) can be written in terms of the generalized eigenfunctions \(\{\Phi_j\}\). When \(H\) is the Laplacian on a manifold with infinite cylindrical ends (or in fact on a more general \(b\)-manifold), Lemma 2.2 below follows from the results of [Mel93, Section 6.9] and [Chr95, Section 2]. (See particularly [Chr95, (2.2)] and the end of the proof of Lemma 2.5. See also [Lyf75, Section 5].) We show below that it follows rather directly from the meromorphic continuation of the resolvent and an identity due to Vodev [Vod14, (5.4)]. In [Vod14] the identity is stated only for Schrödinger operators on on \(\mathbb{R}^d\). However, it in fact holds in far greater generality for operators which are, in an appropriate sense, compactly supported perturbations of each other. Here we state a version adapted to our circumstance, and give a proof for the convenience of the reader.

In the lemma below, for \(\lambda\) in the physical space (initially identified with the upper half plane) \(R_0(\lambda) = (-\Delta_{X,\infty,D} - \lambda^2)^{-1}\) is the resolvent for the Dirichlet Laplacian on \(X_\infty\), and \(R_0(\lambda)\) denotes the analytic continuation otherwise. Since we shall want to allow \(\lambda \in \hat{Z}\), we define the projection \(p : \hat{Z} \to \mathbb{C}\). When \(\lambda\) lies in the physical space, identified with the upper half plane, we have \(p(\lambda) = \lambda\), and for general \(\lambda \in \hat{Z}\), \(p(\lambda)\) is the analytic continuation of this operator.

**Lemma 2.1.** ([Vod14, (5.4)]) Let \(H\) be a black box perturbation of the Laplacian on a manifold with infinite cylindrical ends as described above. Let \(\chi_1 \in C_c^\infty(X_\infty)\) be 1 for \(r \leq 1\). Choose \(\chi \in C_c^\infty(X_\infty)\) so that \(\chi \chi_1 = \chi_1\) and \(0 \leq \chi \leq 1\). Then for \(\lambda, \lambda_0 \in \hat{Z}\),

\[
\chi R(\lambda) - \chi R(\lambda_0) = (p^2(\lambda) - p^2(\lambda_0))\chi R(\lambda)\chi_1(2 - \chi_1)\chi R(\lambda_0)\chi
+ (1 - \chi_1 - \chi R(\lambda)\chi[\Delta_{X,\infty}, \chi_1]) (\chi R(\lambda)\chi - \chi R(\lambda_0)\chi) (1 - \chi_1 + [\Delta_{X,\infty}, \chi_1]\chi R(\lambda_0)\chi). \tag{2.4}
\]

It is important to note in the identity above that \(\chi R_0\chi\) only appears where it is multiplied both on the left and right by an operator (either \(1 - \chi_1\) or \([\Delta_{X,\infty}, \chi_1]\)) supported on the end \(X_\infty\).

**Proof.** We first assume \(\lambda\) and \(\lambda_0\) are in the physical region, which we identify as usual with \(\{z \in \mathbb{C} : \text{Im } z > 0\}\). Then by the resolvent identity

\[
R(\lambda) - R(\lambda_0) = (\lambda^2 - \lambda_0^2)R(\lambda)R(\lambda_0)
= (\lambda^2 - \lambda_0^2)(R(\lambda)(1 - \chi_1)R(\lambda_0) + R(\lambda)(1 - \chi_1)^2R(\lambda_0)). \tag{2.5}
\]

Now consider

\[
R(\lambda)(1 - \chi_1) : L^2(X_\infty) \to \mathcal{H}.
\]

Then

\[
R(\lambda)(1 - \chi_1) = R(\lambda)(1 - \chi_1)(-\Delta_{X,\infty} - \lambda^2)R_0(\lambda).
\]

But since \(H\) agrees with \(-\Delta_{X,\infty}\) on the support of \(1 - \chi_1\), we may write

\[
R(\lambda)(1 - \chi_1) = R(\lambda)\{(H - \lambda^2)(1 - \chi_1) + [\chi_1, \Delta_{X,\infty}]\}R_0(\lambda)
= \{(1 - \chi_1) - R(\lambda)[\Delta_{X,\infty}, \chi_1]\}R_0(\lambda). \tag{2.6}
\]

Likewise

\[
(1 - \chi_1)R(\lambda_0) = R_0(\lambda_0)\{(1 - \chi_1) + [\Delta_{X,\infty}, \chi_1]R(\lambda_0)\}. \tag{2.7}
\]
Using (2.6) and (2.7) in (2.5), along with \((\lambda^2 - \lambda_0^2)R_0(\lambda)R_0(\lambda_0) = R_0(\lambda) - R_0(\lambda_0)\) proves that when \(\lambda\) and \(\lambda_0\) are in the physical space

\[
R(\lambda) - R(\lambda_0) = (\lambda^2 - \lambda_0^2)R(\lambda)\chi_1(2 - \chi_1)R(\lambda_0)
\]

\[
+ \{(1 - \chi_1) - R(\lambda)[\Delta X_\infty, \chi_1]\}(R_0(\lambda) - R_0(\lambda_0))\{(1 - \chi_1) + [\Delta X_\infty, \chi_1]R(\lambda_0)\}. \quad (2.8)
\]

Multiplying on the left and the right by \(\chi\) and using that \(\chi\chi_1 = \chi_1\) proves the result when \(\lambda, \lambda_0\) are in the physical space. The result holds for general \(\lambda, \lambda_0 \in \hat{Z}\) by analytic continuation. \(\square\)

Below we use the notation

\[
(g \otimes h) f = g(f, h)_{\mathcal{H}}.
\]

When \(\lambda \in \mathbb{R}, R(\lambda) = R(\lambda + i0)\).

**Lemma 2.2.** Let \(\chi \in C_c^\infty(X_\infty)\) be one for \(r \leq 1\). Then for \(\lambda \in \mathbb{R}, \lambda \neq \pm \sigma_k, k \in \mathbb{N}_0\), we have

\[
\frac{1}{i} \chi[R(\lambda) - R(-\lambda)](I - P)\chi = \frac{1}{i} \sum_{0 \leq \sigma_j \leq \lambda^2} \frac{1}{\tau_j(\lambda)} \chi \Phi_j^{0}(\lambda) \otimes \Phi_j^{0}(\lambda). \quad (2.9)
\]

**Proof.** Without loss of generality we may assume that \(\chi = 1\) for \(r \leq 2\), and choose \(\chi_1 \in C_c^\infty(X_\infty)\) so that \(\chi_1\chi = \chi_1\) and \(\chi_1 = 1\) for \(r \leq 1\). We shall use Lemma 2.1. We identify points on the open upper half plane (with \(\lambda\) as the parameter) with the physical space of \(\hat{Z}\). Thus \(\lambda > 0\) corresponds to approaching the spectral parameter \(\lambda^2\) from the upper half plane, and \(\lambda < 0\) corresponds to approaching the spectral parameter \(\lambda^2\) from the lower half plane.

Recall \(R_0(\lambda)\) denotes the resolvent for \(-\Delta X_\infty\) with Dirichlet boundary conditions on \([0, \infty) \times Y\). By explicit computation,

\[
R_0(\lambda) - R_0(-\lambda) = \frac{i}{2} \sum_{0 \leq \sigma_j \leq \lambda} \frac{1}{\tau_j(\lambda)} \Phi_j^{0}(\lambda) \otimes \Phi_j^{0}(\lambda) \quad (2.10)
\]

where

\[
\Phi_j^{0}(\lambda) = \Phi_j^{0}(\lambda, r, y) = (e^{-ir_j(\lambda)r} - e^{ir_j(\lambda)r})\phi_j(y).
\]

From Lemma 2.1 and (2.10), if \(\lambda^2\) is not an eigenvalue of \(H\),

\[
\chi R(\lambda)\chi - \chi R(-\lambda)\chi
\]

\[
= (1 - \chi_1 - \chi R(\lambda) [\Delta X_\infty, \chi_1])(\chi R_0(\lambda)\chi - \chi R_0(-\lambda)\chi)(1 - \chi_1 + [\Delta X_\infty, \chi_1]\chi R(-\lambda)\chi)
\]

\[
= \frac{i}{2} (1 - \chi_1 - \chi R(\lambda) [\Delta X_\infty, \chi_1]) \left( \sum_{0 \leq \sigma_j \leq \lambda} \frac{1}{\tau_j(\lambda)} \chi \Phi_j^{0}(\lambda) \otimes \Phi_j^{0}(\lambda)\chi \right) (1 - \chi_1 + [\Delta X_\infty, \chi_1]\chi R(-\lambda)\chi).
\]

From (2.2) and (2.3),

\[
\Phi_j(\lambda) = (1 - \chi_1)\Phi_j^{0}(\lambda) - R(\lambda)[\Delta X_\infty, \chi_1]\Phi_j^{0}(\lambda).
\]

This finishes the proof if \(\lambda^2\) is not an eigenvalue of \(H\). If \(\lambda^2\) is an eigenvalue, the result follows from the fact that both sides of (2.9) are continuous functions of \(\lambda\) away from the thresholds. \(\square\)

We note that a related proof of an analogous result for the Schrödinger operator on \(\mathbb{R}\) can be found in, for example, [RS79, Appendix to XI.6].
2.3. **Threshold behavior.** We now discuss in more detail the behavior of the resolvent at thresholds. Lemma 2.2 can now be combined with a result of [Mel93] to obtain the following corollary. We note that we are giving a somewhat different formulation, and a rather different proof, of part of [Mel93, Proposition 6.28].

**Corollary 2.3.** Near $\pm \sigma_j$,

$$\chi \left( R(\lambda)(I - \mathcal{P}) - \frac{i}{4\tau_j(\lambda)} \sum_{l: \sigma_l = \sigma_j} \Phi_l(\sigma_j) \otimes \Phi_l(\sigma_j) \right) \chi$$

is bounded if $\chi \in C_c^\infty(X_\infty)$, $\chi = 1$ for $r \leq 1$. Moreover,

$$\sum_{l: \sigma_l = \sigma_j} \Phi_l(\sigma_j) \otimes \Phi_l(\sigma_j) = \sum_{l: \sigma_l = \sigma_j} \Phi_l(-\sigma_j) \otimes \Phi_l(-\sigma_j). \quad (2.11)$$

**Proof.** Using the self-adjointness of $H$ and the meromorphy of $R$ on $\hat{Z}$ as in [Mel93, Proof of Proposition 6.28], there is an operator $A_0$ so that

$$\chi \left( R(\lambda)(I - \mathcal{P}) - \frac{A_0}{\tau_j(\lambda)} \right) \chi$$

is bounded near $\lambda = \sigma_j$. Likewise, there is a $B_0$ so that

$$\chi \left( R(\lambda)(I - \mathcal{P}) - \frac{B_0}{\tau_j(\lambda)} \right) \chi$$

is bounded near $\lambda = -\sigma_j$.

If $\sigma_j = 0$, then trivially $A_0 = B_0$. So suppose temporarily $\sigma_j > 0$. If $\lambda \in \mathbb{R}$ with $0 < \lambda < \sigma_j$, then $\tau_j(-\lambda) = \tau_j(\lambda)$. Thus we find

$$\lim_{\lambda \uparrow \sigma_j} \tau_j(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi = \chi(A_0 - B_0)\chi.$$  

However, since $\Phi_k(\lambda)$ is continuous at $\lambda = \sigma_j$ if $\sigma_k \leq \sigma_j$, we find from (2.9) that

$$\lim_{\lambda \downarrow \sigma_j} \tau_j(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi = 0.$$

Thus $A_0 = B_0$.

Now let $\sigma_j \geq 0$. If $\lambda \in \mathbb{R}$, $\lambda > \sigma_j$, then $\tau_j(-\lambda) = -\tau_j(\lambda)$, so that

$$\lim_{\lambda \downarrow \sigma_j} \tau_j(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi = \chi(A_0 + B_0)\chi = 2\chi A_0 \chi.$$  

Comparing (2.9) this means

$$A_0 = \frac{i}{4} \sum_{l: \sigma_l = \sigma_j} \Phi_l(\sigma_j) \otimes \Phi_l(\sigma_j).$$
To show (2.11), note that using (2.9) with $\lambda$ and then again with $\lambda$ replaced by $-\lambda$, we have, for $\lambda \in \mathbb{R}$ and not a threshold,

$$\frac{i}{2} \sum_{0 \leq \sigma_j^2 \leq \lambda^2} \frac{1}{\tau_j(\lambda)} \chi \Phi_j(\lambda) \otimes \Phi_j(\lambda) \chi = -\frac{i}{2} \sum_{0 \leq \sigma_j^2 \leq \lambda^2} \frac{1}{\tau_j(-\lambda)} \chi \Phi_j(-\lambda) \otimes \Phi_j(-\lambda) \chi = \frac{i}{2} \sum_{0 \leq \sigma_j^2 \leq \lambda^2} \frac{1}{\tau_j(\lambda)} \chi \Phi_j(-\lambda) \otimes \Phi_j(-\lambda) \chi.$$ 

Comparing the singularities at $\lambda = \sigma_j$ as above shows (2.11). \qed

Given $j_0 \in \mathbb{N}_0$, consider the set

$$\{\Phi_j(\sigma_j) : \sigma_j = \sigma_{j_0}\}.$$ (2.12)

If this set contains at least one nonzero element, we may say $\pm \sigma_{j_0}$ is a threshold resonance. The set (2.12) contains a nonzero element if and only if $R(\lambda)(I - P)$ has a pole at $\sigma_{j_0}$ on the boundary of the physical space. Indeed, we can see from Corollary 2.3 that if the set (2.12) contains only 0, then $R(\lambda)(I - P)$ is continuous at $\lambda = \sigma_{j_0}$. If $\sigma_{j_0}^2$ is a simple eigenvalue of $-\Delta_Y$, then the other direction is immediate. Otherwise, if $\sigma_{j_0}^2$ is not a simple eigenvalue of $-\Delta_Y$, see [Mel93, Proposition 6.28] to see that the singularity of $R(\lambda)(I - P)$ is nontrivial at $\sigma_{j_0}$.

The threshold resonances are analogous to the familiar half-bound states of one-dimensional scattering theory.

3. Two term wave expansions

Let $H$ be an operator as in Section 2 and let $u(t)$ be the solution to the wave equation

$$\left(\partial^2_t + H\right)u = 0, \quad u(0) = f_1, \quad u_t(0) = f_2$$ (3.1)

where $f_1, f_2 \in \mathcal{H}$. Later we shall impose more stringent conditions on $f_1, f_2$.

We begin by recalling the contribution of the eigenvalues to $u$. In Section 3.2 we state the main theorem, the two term asymptotics result. In the remainder of Section 3 we give the proof of Theorem 3.2.

3.1. Projection onto the eigenfunctions. We begin by recalling the contribution of the eigenvalues to the behavior of $u$. This requires only that $H$ is self-adjoint.

Let $\{\lambda_\ell\}$ denote the eigenvalues of $H$, repeated with multiplicity, with corresponding orthonormal eigenfunctions $\{\eta_\ell\}$: $H\eta_\ell = \lambda_\ell \eta_\ell$. For a general black box operator $H$ the set $\{\lambda_\ell\}$ could be empty, nonempty but finite, or infinite. However, the assumptions we make on $H$ in Theorem 3.2 imply that $H$ does not have infinitely many eigenvalues. On the other hand, [CD] contains examples of Schrödinger operators on a manifold with infinite cylindrical ends which have embedded eigenvalues but which still have the type of high-energy resolvent estimate which we need for Theorem 3.2.
Lemma 3.1. Let \( u(t) \) be the solution of (3.1). Then, with \( u_e(t) = \mathcal{P}u(t) \),

\[
u_e(t) = \sum_{\lambda_\ell \in \text{spec}_p(H) \lambda_\ell \neq 0} \eta_\ell \left( \cos((\lambda_\ell)^{1/2}t)\langle f_1, \eta_\ell \rangle_H + \frac{\sin((\lambda_\ell)^{1/2}t)}{(\lambda_\ell)^{1/2}}\langle f_2, \eta_\ell \rangle_H \right) + \sum_{\lambda_\ell \in \text{spec}_p(H) \lambda_\ell = 0} \eta_\ell (\langle f_1, \eta_\ell \rangle_H + t(\langle f_2, \eta_\ell \rangle_H)) \tag{3.2}
\]

\( \mathcal{P}u(t) = u_e(t) \), where \( u_e \) satisfies

\[
(\partial_t^2 + H)u_e(t) = 0 \\
u_e(0) = \mathcal{P}f_1, \\
(\partial_t u_e)(0) = \mathcal{P}f_2.
\]

Then a straightforward computation shows that the explicit expression in (3.2) solves this initial value problem. \( \square \)

3.2. Statement of Theorem 3.2. Set

\[
\mathbb{C}_{\text{slit}} := \mathbb{C} \setminus \left( \bigcup_{j} \{ \sigma_j > 0 \} \cup \{ \pm \sigma_j - is, s \geq 0 \} \right). \tag{3.3}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{The set \( \mathbb{C}_{\text{slit}} \) is the complex plane with downward half-lines removed at the square roots of the nonzero eigenvalues of \( -\Delta_Y \).}
\end{figure}

The operator \( \chi R(\lambda) \chi \) continues meromorphically from \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda > 0 \} \) to \( \mathbb{C}_{\text{slit}} \). In fact, we can identify \( \mathbb{C}_{\text{slit}} \) with a subset of the Riemann surface \( \tilde{Z} \). We use the same notation for the continuation of \( R(\lambda) \) to \( \mathbb{C}_{\text{slit}} \), so that when \( \lambda \in \mathbb{R} \), \( R(\lambda) = R(\lambda + i0) \).

In the following theorem, \( u_e \) encodes the contribution of the eigenvalues of \( H \) as in Lemma 3.1 and \( u_{\text{thr}} \) is the leading order contribution from the threshold resonances (if any). The expansion for \( u_e(t) \) is given in Lemma 3.1. Recall that we have assumed that \( H \) is lower semibounded. Here we choose \( M_0 \in \mathbb{R} \) so that \( H + M_0 > 0 \).
Theorem 3.2. Let $H$ be a black box perturbation of $-\Delta$ on $X_\infty$, and suppose that for some $N_1, N_2 \in [0, \infty)$, $\lambda_0 > 0$, and any $\tilde{\chi} \in C^\infty_c(X_\infty)$ with $\tilde{\chi}(r) = 1$ for $r \leq 1$ there are $C_0, C_1$ so that $\tilde{\chi}R(\lambda)\tilde{\chi}$ is analytic on the set
\[
\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda > \lambda_0 - 1 \text{ and } \operatorname{Im} \lambda > -C_0(\operatorname{Re} \lambda)^{-N_1} \}
\] (3.4)
in $\hat{Z}$, and that in this region
\[
\| \tilde{\chi}R(\lambda)\tilde{\chi} \| \leq C_1(1 + |\lambda|)^{N_2}. \tag{3.5}
\]
Fix $M_1 > 0$. Suppose $f_l \in (H + M_0)^{-m_l} \mathcal{H}$ for $l = 1, 2$, and $f_1, f_2$ restricted to $X_\infty$ vanish for $r > M_1 > 0$. Let $u(t)$ be the solution of (3.1). Then
\[
u
\]
where $u_e(t) = \mathcal{P}u(t)$ has an expansion as given in Lemma 3.1,
\[
u
\]
Moreover, if $\chi \in C^\infty(X_\infty)$ is 1 for $r < 1$, then
\[
u
\]
if $t$ is sufficiently large and $m_l \in \mathbb{N}$ satisfies $m_l > (N_2 + 2 - l + d)/2$ and $m_l \geq (N_1 + 3 - l)/2$ for $l = 1, 2$.

We note that our assumptions on $H$ in this theorem ensure that the sums in $u_e$ and $u_{thr}$, see (3.2) and (3.6), are finite. The identity $R(\lambda) = R(\lambda^*)^*$, which is a consequence of the self-adjointness of $H$, and the consequent symmetry of the resonances mean that $\tilde{\chi}R(\lambda)\tilde{\chi}$ is analytic in the region $\{ \lambda \in \mathbb{C} : \lambda < -(\lambda_0 - 1) \text{ and } \operatorname{Im} \lambda > -C_0(-\operatorname{Re} \lambda)^{-N_1} \}$, and satisfies $\| \tilde{\chi}R(\lambda)\tilde{\chi} \| \leq C_1(1 + |\lambda|)^{N_2}$ there.

The assumption that there is a resonance-free region of the form (3.4) follows from the seemingly weaker assumption that the bound (3.5) on the cut-off resolvent holds for $\lambda \in \mathbb{R}$, $|\lambda| > \lambda_0 - 1$. By [CD, Theorem 5.6] this implies the existence of a resonance-free region of the form (3.4), with a corresponding estimate on the cut-off resolvent there. Note that by [CD, Theorem 3.1] and [CD, Sections 3.2, 3.3] this bound on the resolvent for $-\Delta_X$ (or $-\Delta_X + V$, for a large class of $V \in C^\infty_c(\bar{X}; \mathbb{R})$) holds for the examples of manifolds $X$ in Sections 1.2.1 and 1.2.2. Moreover, [CD, Theorem 3.1] gives a more general method of constructing manifolds and Schrödinger operators for which such an estimate holds.

For special choices of the cross-sectional manifold $Y$ the result of Theorem 3.2 holds for smaller values of $m_l$. For example, let $\beta > 0$ be a fixed real number, $\nu_0 \in \mathbb{N}$, and let $(Y, g_Y^\beta) = \bigcup_{\nu=1}^{\nu_0} (S^{d-1}, \beta g_{S^{d-1}})$ where $S^{d-1}$ is the $d - 1$-dimensional unit sphere, and $g_{S^{d-1}}$ is the usual metric on it. Then using Lemma 3.11 in place of Proposition 3.10 in the proof, one can show that in
this special case it suffices to require $m_l \in \mathbb{N}$, $m_l > (N_2 + 4 - l)/2$ and $m_l \geq (N_1 + 3 - l)/2$ for $l = 1, 2$.

Note that without loss of generality we may assume that $\chi|_{X_\infty}(r) = 1$ for $r < M_1$. We do so in the remainder of this section. In particular, this implies that $\chi_{f_l} = f_l$, $l = 1, 2$.

3.3. Reduction to Propositions 3.4 and 3.5. In this section we prove Theorem 3.2 modulo the proofs of two propositions. We have already found the contribution of the discrete spectrum to $u(t)$ in Lemma 3.1. We use the spectral theorem to write $(I - \mathcal{P})u(t)$ as an integral. This, in turn, we write as the sum of three integrals depending on the size of the spectral parameter. Each of these three will be evaluated or bounded using a different technique.

Lemma 3.3. Let $u(t)$ be the solution of (3.1). Then

$$
(I - \mathcal{P})u(t) \partial_t (I - \mathcal{P})u(t) = \text{PV} \frac{1}{2\pi i} \int_{-\infty}^\infty e^{it\lambda} A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P})d\lambda \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right)
$$

where

$$
A(\lambda) = \left( \begin{array}{cc} \frac{\lambda}{\lambda^2 + i^2} \\ i \end{array} \right)
$$

and PV is the principal value.

Proof. We have

$$
u(t) = \cos(t\sqrt{H})f_1 + \sin(t\sqrt{H})\frac{\sqrt{H}}{\sqrt{H}}f_2.
$$

By the functional calculus and Stone's formula,

$$
\cos(t\sqrt{H})(I - \mathcal{P}) = \frac{1}{2\pi i} \int_0^\infty \cos(t\tau)((H - \tau - i0)^{-1} - (H - \tau + i0)^{-1})(I - \mathcal{P})d\tau
$$

$$
= \frac{1}{2\pi i} \int_0^\infty [e^{it\lambda} + e^{-it\lambda}][R(\lambda) - R(-\lambda)](I - \mathcal{P})\lambda d\lambda
$$

$$
= \frac{1}{2\pi i} \int_{-\infty}^\infty e^{it\lambda}[R(\lambda) - R(-\lambda)](I - \mathcal{P})\lambda d\lambda.
$$

Similarly

$$
\sin(t\sqrt{H})\sqrt{H}(I - \mathcal{P}) = -\frac{1}{2\pi} \int_0^\infty [e^{it\lambda} - e^{-it\lambda}][R(\lambda) - R(-\lambda)](I - \mathcal{P})d\lambda
$$

$$
= -\text{PV} \frac{1}{2\pi} \int_{-\infty}^\infty e^{it\lambda}[R(\lambda) - R(-\lambda)](I - \mathcal{P})d\lambda.
$$

Here we do need the principal value if $H$ has 0 as a threshold resonance, since in that case $(R(\lambda) - R(-\lambda))(I - \mathcal{P})$ has a pole of order 1 at 0.

We use the integral representation from Lemma 3.3 to write $(I - \mathcal{P})u(t)$ as the sum of three terms:

$$
(I - \mathcal{P}) \left( \begin{array}{c} u(t) \\ \partial_t u(t) \end{array} \right) = (I_s(t) + I_m(t) + I_t(t)) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right)
$$

(3.9)
Proposition 3.4. Let $f_1$, $f_2$, $\chi$, and $u_{\text{thr}}$ be as given in Theorem 3.2. Then there is a constant $C$ (depending on $\chi$) so that

$$
\left\| \chi \left( \begin{array}{cc} (H + M_0)^{1/2} & 0 \\ 0 & I \end{array} \right) I_s(t) \left( \begin{array}{cc} f_1 \\ f_2 \end{array} \right) - \left( \begin{array}{c} u_{\text{thr}}(t) \\ \partial_t u_{\text{thr}}(t) \end{array} \right) \right\|_{\mathcal{H} \oplus \mathcal{H}} \leq Ct^{-1} \left\| \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) \right\|_{\mathcal{H} \oplus \mathcal{H}}
$$

when $t$ is sufficiently large.
We remark that the error $O(t^{-1})$ in this estimate is sharp and is due to the discontinuous nature of the cutoff at $\lambda = \pm \lambda_0$ in the definition of $I_s(t)$. The error could be improved by instead using a smooth cut-off function in the $\lambda$ variable to define $I_s(t)$ and $I_m(t)$; we do something similar in Section 4. However, since our methods for estimating the contribution of $I_m(t)$ result in an error of size $O(t^{-1})$ even with this change, we would not gain by taking this alternate approach here.

For $I_m(t)$, the corresponding result is

**Proposition 3.5.** Let $0 < \epsilon < 1/N_1$, let $\lambda_0$ be as in the statement of Theorem 3.2 and be chosen so that $\lambda_0 = \sigma_{j_0} > 0$, and, for $t$ sufficiently large, let $\alpha(t)$ be chosen so that $1 \leq \alpha(t) \leq 2$ and $\alpha(t)t^\epsilon = \sigma_{J(t)}$, with $j_0$, $J(t) \in \mathbb{N}$. Choose $m_l \in \mathbb{N}$ so that $m_l > (N_2 + 2 - l + d)/2$ for $l = 1, 2$. Then for $t$ sufficiently large

$$\left\| \chi \left( \frac{(H + M_0)^{1/2}}{0} \right) I_m(t) \left( \begin{array}{cc} 0 & 0 \\ 0 & (H + M_0)^{-m_2} \end{array} \right) \chi \right\|_{\mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}} \leq Ct^{-1}. \quad (3.13)$$

We prove Proposition 3.4 in Section 3.4 and Proposition 3.5 in Section 3.5.

Assuming these two proposition, we may now prove Theorem 3.2.

**Proof of Theorem 3.2.** Writing $u(t) = \mathcal{P}u(t) + (I - \mathcal{P})u(t)$, Lemma 3.1 gives an explicit expression for $u_e(t) = \mathcal{P}u(t)$.

Recall that $f_1$, $f_2$ vanish on $X_\infty$ for $r$ sufficiently large, and without loss of generality we have chosen $\chi$ so that $\chi f_l = f_l$; in particular, $(H + M_0)^m \chi f_l = (H + M_0)^m f_l = \chi f_l$. From (3.9) we see that to understand $(I - \mathcal{P})u(t)$ it suffices to understand the contributions of $I_s(t)$, $I_m(t)$, and $I_l(t)$. Here we choose $\epsilon = 1/(N_1 + 1)$, and $\lambda_0 > 0$ so that it both satisfies the conditions of the Theorem and so that $\lambda_0 = \sigma_{j_0}$ for some $j_0 \in \mathbb{N}$. In addition, for $t$ sufficiently large, we choose $\alpha(t)$ so that $1 \leq \alpha(t) \leq 2$, and $\alpha(t)t^\epsilon = \sigma_{J(t)}$ for some $J(t) \in \mathbb{N}$.

Then, with $m_l \geq (N_1 + 3 - l)/2$, $l = 1, 2$, we have the bound from (3.11) on $I_l(t)$, $Ct^{-\epsilon \min(2m_1 - 1, 2m_2)}$, is less than or equal to $Ct^{-1}$. The results of Propositions 3.4 and 3.5 complete the proof. \qed

### 3.4. The contribution of $I_s(t)$

The goal of this section is to prove Proposition 3.4. We remark that to prove this proposition, we do not need to assume bounds on the resolvent of $H$ at high energy. Moreover, note that the bound is given in terms of $\|f_1\|_\mathcal{H}$, $\|f_2\|_\mathcal{H}$.

In order to prove the proposition, we shall write the integral defining $I_s$ as the sum of three types of terms: an integral over a small neighborhood of 0, an integral over a small neighborhood of $\sigma_j$ or $-\sigma_j$, where $\sigma_j > 0$, and an integral of a function with support disjoint from all $\pm \sigma_j$ with $0 \leq \sigma_j < \lambda_0$.

Let $\psi \in C^\infty_c(\mathbb{R})$ be a function which is 1 in a neighborhood of 0, and set

$$I_0(t) = \text{PV} \frac{1}{2\pi i} \int \psi(\lambda) e^{it\lambda} A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P})d\lambda.$$

We emphasize that the following lemma does not require high energy resolvent estimates for $H$, and is valid for any $f_1$, $f_2 \in \mathcal{H}$ which have $\mathbb{1}_{X_\infty} f_1$, $\mathbb{1}_{X_\infty} f_2$ both supported in a fixed compact subset of $X_\infty$. 
Lemma 3.6. Let $H$ be any operator satisfying the hypotheses of Section 2, and let $f_1, f_2 \in \mathcal{H}$ have $\mathbf{1}_{X \infty} f_1, \mathbf{1}_{X \infty} f_2$ both supported in $r \leq M_1$. Then with the support of $\psi$ chosen sufficiently small, for any $q \in \mathbb{N}_0$, $k \in \mathbb{N}$ there is a constant $C$ depending on $q$, $k$ and the support of $f_1, f_2$ so that

$$
\left\| \chi (H + M_0)^{q/2} I_0(t) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) - \frac{1}{4} \left( \chi \sum_{\sigma_j = 0} M_0^{q/2} \Phi_j(0) \langle f_2, \Phi_j(0) \rangle \right) \right\|_{\mathcal{H} \oplus \mathcal{H}} \leq Ct^{-k} (\|f_1\|_\mathcal{H} + \|f_2\|_\mathcal{H})
$$

when $t$ is sufficiently large.

Proof. Recall that $\chi (R(\lambda) - R(-\lambda)) (I - \mathcal{P}) \chi$ has a singularity at worst like $1/\lambda$ at $\lambda = 0$. Thus, if $p \in \mathbb{N}$, then $\lambda^p (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi (R(\lambda) - R(-\lambda)) (I - \mathcal{P}) \chi$ is a smooth function of $\lambda \in \mathbb{R}$ if the support of $\psi$ is chosen sufficiently small that it contains no $\pm \sigma_j$ with $\sigma_j \neq 0$. Hence for $p \in \mathbb{N}$, by integrating by parts $k$ times, we find for $t > 0$

$$
\left\| \int_{-\infty}^{\infty} e^{it\lambda} \lambda^p (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi (R(\lambda) - R(-\lambda)) (I - \mathcal{P}) \chi d\lambda \right\|_{\mathcal{H} \rightarrow \mathcal{H}} \\
\leq t^{-k} \int_{-\infty}^{\infty} \left\| \frac{d^k}{d\lambda^k} \left( \lambda^p (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi (R(\lambda) - R(-\lambda)) (I - \mathcal{P}) \chi \right) \right\|_{\mathcal{H} \rightarrow \mathcal{H}} d\lambda = O(t^{-k}), \ p \in \mathbb{N}
$$

(3.14)

for any $k \in \mathbb{N}$.

Using (3.14) and considering the expression (3.7) for $A$, this means we need only consider more carefully the entry corresponding to the upper right-hand corner of $A$. We also will use, see Lemma 2.2,

$$(R(\lambda) - R(-\lambda)) (I - \mathcal{P}) = \frac{i}{2\lambda} \sum_{\sigma_j = 0} \Phi_j(0) \otimes \Phi_j(0) + B(\lambda)$$

where $\chi B(\lambda) \chi$ is analytic in a neighborhood of $\lambda = 0$. Hence, using another integration by parts argument,

$$
\text{PV} \int_{-\infty}^{\infty} e^{it\lambda} (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi (R(\lambda) - R(-\lambda)) (I - \mathcal{P}) \chi d\lambda = \\
\frac{i M_0^{q/2}}{2} \sum_{\sigma_j = 0} \chi \Phi_j(0) \otimes \chi \Phi_j(0) \text{PV} \int_{-\infty}^{\infty} e^{it\lambda} \frac{\psi(\lambda)}{\lambda} d\lambda + O(t^{-k}) \quad (3.15)
$$

when the support of $\psi$ is sufficiently small.

Now we use

$$
\text{PV} \int_{-\infty}^{\infty} e^{it\lambda} \frac{1}{\lambda} d\lambda = i\pi, \ t > 0
$$
and the fact that $\psi$ is 1 in a small neighborhood of the origin to find that
\[
\text{PV} \int_{-\infty}^{\infty} e^{it\lambda}(\lambda^2 + M_0)^{q/2}\psi(\lambda)\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi d\lambda = -\frac{\pi M_0^{q/2}}{2} \sum_{\sigma_j = 0} \chi \Phi_j(0) \otimes \chi \Phi_j(0) + O(t^{-k})
\]
for $t$ sufficiently large. \[\square\]

The next lemma follows directly from the more general Lemma A.1.

**Lemma 3.7.** Let $X$ be a Banach space, $\sigma_j > 0$, and set $B_0 = \{z \in \mathbb{C} : |z| < \min(\sigma_j, 1)/2\}$. If $F \in C_c^\infty(B_0; \mathcal{X})$ then there is a $C \in \mathbb{R}$ so that
\[
\left\| \int_0^\infty e^{-i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - (\sigma_j t)^{-1/2}e^{-i\pi/4\sqrt{2\pi}F(0)}e^{-i\sigma_j t} \right\|_X \leq Ct^{-1}, \quad t > 0.
\]
Moreover,
\[
\left\| \int_0^\infty e^{i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda \right\|_X \leq Ct^{-1}, \quad t > 0.
\]

**Proof of Proposition 3.4.** Let $\psi \in C_c^\infty(\mathbb{R}; [0, 1])$ be equal to 1 in a small neighborhood of the origin. Set $L = \max\{l : \nu_l < \lambda_0\}$, and, for $l = 1, \ldots, L$, set $\psi_l(\lambda) = \psi(|\lambda^2 - \nu_l^2|)$. Note that $\psi_l$ is smooth since $\psi$ is 1 in a neighborhood of the origin. Choose the support of $\psi$ sufficiently small that
\[
0 < l, l' \leq L, \quad l \neq l' \Rightarrow \text{supp } \psi_l \cap \text{supp } \psi_{l'} = \emptyset \text{ and } \text{supp } \psi \cap \text{supp } \psi_l = \emptyset.
\]
Then set $\psi_s(\lambda) = \psi(\lambda) + \sum_{l=1}^L \psi_l(\lambda)$. By shrinking the support of $\psi$ if necessary, we can assume that $\psi_s$ is 0 in a neighborhood of $\pm \lambda_0$. Note that $(R(\lambda) - R(-\lambda))(I - \mathcal{P})H$ is smooth on $(-\lambda_0, \lambda_0)$, and continuous on $[-\lambda_0, \lambda_0]$. Moreover, although $\left\| \frac{d}{d\lambda} (\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi(1 - \psi_s(\lambda))) \right\|$ may not be continuous at $\lambda = \pm \lambda_0$ if $\lambda_0 = \nu_{L+1}$, the singularity is at worst like $C|\lambda^2 - \lambda_0^2|^{-1/2} = C|\lambda^2 - \nu_{L+1}^2|^{-1/2}$ at the endpoints, and hence integrable. We note that although here we could easily avoid choosing $\lambda_0$ to be a threshold, later it will in fact be convenient for us to choose it to be a threshold.

To simplify notation, for $q \in \mathbb{N}_0$, set
\[
A_q(\lambda) = (\lambda^2 + M_0)^{q/2}A(\lambda).
\]
Hence, integrating by parts,
\[
\left\| \int_{-\lambda_0}^{\lambda_0} e^{it\lambda}A_q(\lambda)(1 - \psi_s(\lambda))\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi d\lambda \right\|_{\mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}}
= \left\| \frac{-i}{t}[e^{it\lambda_0}A_q(\lambda_0) + e^{-it\lambda_0}A_q(-\lambda_0)]\chi(R(\lambda_0) - R(-\lambda_0))(I - \mathcal{P})\chi + \frac{i}{t} \int_{-\lambda_0}^{\lambda_0} e^{it\lambda} \frac{d}{d\lambda} (A_q(\lambda)(1 - \psi_s(\lambda))\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi) d\lambda \right\|_{\mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}}
= O(t^{-1})
\]
(3.16)
locally well-defined inverse of $\hat{\cdot}$, with $\nu$ be such that $\nu$ smooth function of $\|\cdot\|$. Now consider $\int_{-\lambda_0}^{\lambda_0} e^{it\lambda} A_q(\lambda) \psi_1(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P}) d\lambda$ with $0 < l \leq L$. Let $j = j(l) \in \mathbb{N}$ be such that $\nu_j = \sigma_j$. By our choice of the support properties of $\psi$, $\tau_j(\lambda) \psi_1(\lambda) R(\lambda)(I - \mathcal{P})$ is a smooth function of $\tau_j(\lambda)$ for $\lambda \in \mathbb{R}$, but $\tau_j(\lambda) \psi_1(\lambda) R(-\lambda)(I - \mathcal{P})$ is not. Hence we do a change of variable:

$$\int_{-\lambda_0}^{\lambda_0} e^{it\lambda} A_q(\lambda) \psi_1(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P}) d\lambda = \int_{-\lambda_0}^{0} (e^{it\lambda} A_q(\lambda) - e^{-it\lambda} A_q(-\lambda)) \psi_1(\lambda) R(\lambda)(I - \mathcal{P}) d\lambda + \int_{0}^{\lambda_0} (e^{it\lambda} A_q(\lambda) - e^{-it\lambda} A_q(-\lambda)) \psi_1(\lambda) R(\lambda)(I - \mathcal{P}) d\lambda. \quad (3.17)$$

By shrinking the support of $\psi$ is necessary, we may apply Lemma 3.7 to the second integral on the right-hand side, with $F(\tau) = \chi \tau A_q((\tau^2 + \sigma_j^2)^{1/2}) \psi(|\tau|^2) R(\lambda)(\tau)(I - \mathcal{P})\chi$, where $\lambda(\tau)$ is the locally well-defined inverse of $\tilde{Z} \ni \lambda \mapsto \tau_j(\lambda) \in \mathbb{C}$. Thus, using the meromorphic continuation of $\chi R \chi$ to $\tilde{Z}$ and Corollary 2.3, $F$ is a smooth function supported in a complex neighborhood of the origin.

From Lemma 3.7, for $t > 0$

$$\int_{0}^{\lambda_0} e^{-it\lambda} A_q(-\lambda) \psi_1(\lambda) \chi R(\lambda)(I - \mathcal{P}) \chi d\lambda = \int_{0}^{\lambda_0} e^{-it\lambda} A_q(-\lambda) \psi(|\tau_j(\lambda)|^2) \chi R(\lambda)(I - \mathcal{P}) \chi d\lambda = \sqrt{2\pi} e^{-i(\sigma_j + \pi/4)} A_q(-\sigma_j) [\chi R(\lambda)(I - \mathcal{P}) \chi \tau_j(\lambda)] |_{\lambda = \sigma_j} (\sigma_j t)^{-1/2} + B_{t,1}(t) \quad (3.18)$$

where $\|B_{t,1}(t)\|_{\mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}} = O(t^{-1})$. By a result parallel to Lemma 3.7, for $t > 0$,

$$\int_{-\lambda_0}^{0} e^{-it\lambda} A_q(-\lambda) \psi_1(\lambda) \chi R(\lambda)(I - \mathcal{P}) \chi d\lambda = \int_{-\lambda_0}^{0} e^{-it\lambda} A_q(-\lambda) \psi(|\tau_j(\lambda)|^2) \chi R(\lambda)(I - \mathcal{P}) \chi d\lambda = -\sqrt{2\pi} e^{i(\sigma_j + \pi/4)} A_q(\sigma_j) [\chi R(\lambda)(I - \mathcal{P}) \chi \tau_j(\lambda)] |_{\lambda = -\sigma_j} (\sigma_j t)^{-1/2} + B_{t,2}(t) \quad (3.19)$$

where $\|B_{t,2}(t)\| = O(t^{-1})$. From Lemma 3.7 (and the analogous result for the integral over $\lambda < 0$)

$$\left\| \chi \int_{-\lambda_0}^{\lambda_0} e^{it\lambda} A_q(\lambda) \psi_1(\lambda) R(\lambda)(I - \mathcal{P}) \chi d\lambda \right\|_{\mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}} = O(t^{-1}), \ t \to \infty. \quad (3.20)$$

It follows from Corollary 2.3 that

$$(\tau_j(\lambda) \chi R(\lambda)(I - \mathcal{P}) \chi) |_{\lambda = \pm \sigma_j} = \frac{i}{4} \sum_{j', j'' = \sigma_j = \nu_l} \chi \Phi_j(\sigma_j) \otimes \chi \Phi_j(\sigma_j). \quad (3.21)$$
Hence from (3.18-3.21), we have
\[
\int_{-\lambda_0}^{\lambda_0} e^{it\lambda} A_q(\lambda) \psi_l(\lambda) \chi (R(\lambda) - R(-\lambda))(I - \mathcal{P}) \chi d\lambda
= (\sigma_j t)^{-1/2} \frac{i}{2} \sqrt{\pi/2} \sum_{j: \sigma_j = i\eta} \left[ e^{i(\sigma_j t + \pi/4)} A_q(\sigma_j) - e^{-i(\sigma_j t + \pi/4)} A_q(-\sigma_j) \right] \chi \Phi_j(\sigma_j) \otimes \chi \Phi_j(\sigma_j) + B_{l,\chi}
\]
where \( \|B_{l,\chi}\| = O(t^{-1}). \)

Using (3.16), (3.22) and Lemma 3.6 proves Proposition 3.4. □

3.5. The contribution of \( I_m(t) \). The main result of this section is Proposition 3.5, which provides the needed bound on \( I_m(t) \). This is the portion of the proof of Theorem 3.2 for which we use the resonance-free region and the high energy resolvent estimate. We have some freedom in our choice of \( \lambda_0 \) (we can always choose a larger value) and in our choice of \( \alpha(t) \). The choices of \( \lambda_0 \) and \( \alpha(t) \) we make in Proposition 3.5 are made only so that the results of Proposition 3.9 are sufficient themselves to prove Proposition 3.10 without need of further contour deformation or integration by parts arguments.

In order to prove the bound, we shall perform a contour deformation into \( C_{\text{slit}} \). We recall that
\[
I_m(t) = \frac{1}{2\pi i} \int_{\lambda_0 < |\lambda| < \alpha(t)t^\epsilon} e^{it\lambda} A(\lambda)[R(\lambda) - R(-\lambda)] d\lambda.
\]

There is no need to compose on the right with \((I - \mathcal{P})\) here, since \( \lambda_0^2 \) exceeds the largest eigenvalue of \( H \).

In order to simplify notation, set
\[
G(\lambda) = \begin{pmatrix} (\lambda^2 + M_0)^{1/2} & 0 \\ 0 & 1 \end{pmatrix} A(\lambda) \begin{pmatrix} (\lambda^2 + M_0)^{-m_1} & 0 \\ 0 & (\lambda^2 + M_0)^{-m_2} \end{pmatrix}
\]
and
\[
R_\chi(\lambda) = \chi R(\lambda) \chi.
\]

Then
\[
\left\| \chi \begin{pmatrix} (H + M_0)^{1/2} & 0 \\ 0 & I \end{pmatrix} I_m(t) \begin{pmatrix} (H + M_0)^{-m_1} & 0 \\ 0 & (H + M_0)^{-m_2} \end{pmatrix} \chi \right\|_{\mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}}
= \frac{1}{2\pi i} \int_{\lambda_0 < |\lambda| < \alpha(t)t^\epsilon} e^{it\lambda} G(\lambda)[R_\chi(\lambda) - R_\chi(-\lambda)] d\lambda.
\]

We treat the contributions from the terms \( R_\chi(\lambda) \) and \( R_\chi(-\lambda) \) separately; the second is substantially more difficult than the first.

To bound the term in (3.23) with \( e^{it\lambda} R_\chi(\lambda) \) we shall use the following lemma.
Lemma 3.8. Let $\lambda_0 > 0$ be as in the statement of Theorem 3.2 and let $\alpha(t)$ satisfy $1 \leq \alpha(t) \leq 2$. Then, if $\epsilon, t > 0$ then there is a constant $C$ so that

$$\left\| \int_{\lambda_0 \leq |\lambda| < \alpha(t)|t^\epsilon} e^{it\lambda} G(\lambda) R_\chi(\lambda) d\lambda \right\|_{\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}} \leq Ct^{-1} \quad \text{if } N_2 + \max(2 - 2m_1, 1 - 2m_2) \leq 0 \text{ and } t \text{ is sufficiently large.}$$

We postpone the proof of this lemma to Section 3.6.

To bound an integral of the form

$$\int_{-\lambda_0 - \alpha(t)|t^\epsilon} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda$$

we will use contour deformation. We wish to deform to the region with $\text{Im } \lambda > 0$ to take advantage of the decay provided by $e^{it\lambda}$ in this region. However, to do so means that $R_\chi(-\lambda)$ must be evaluated at a point with $\text{Im } (-\lambda) < 0$. Recalling the continuation of $R_\chi$ is to $\mathbb{C}_{\text{slit}}$, we see that this is complicated. Each distinct value of $\sigma_j$ gives ramification points at $\pm \sigma_j$ in $\hat{\mathbb{Z}}$; this corresponds to the omitted rays in the lower half plane in $\mathbb{C}_{\text{slit}}$. Then to do a contour deformation argument for (some) integrals over $[\sigma_j, \sigma_{j+1}]$, with $J > j$, we are forced to write $[\sigma_j, \sigma_{j+1}] = \bigcup_{j \leq j+1} \left[ \sigma_j, \sigma_{j+1} \right]$ and, when $\sigma_{j+1} > \sigma_j$, do a contour deformation to bound the integral over $[\sigma_j, \sigma_{j+1}]$. We make this precise in Section 3.6, where we prove Proposition 3.9, a substantial step in our proof.

Proposition 3.9. Let $0 < \epsilon < 1/N_1$. Then there are constants $T, C > 0$ so that if $\lambda_0 \leq \sigma_j < \sigma_{j+1} < 2t^\epsilon$ and $t > T$ then

$$\left\| \int_{-\lambda_0 - \alpha(t)|t^\epsilon} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}} \leq Ct^{-1} (1 + \sigma_{j+1} - \sigma_j)(1 + \sigma_j)^{N_2 + \max(2 - 2m_1, 1 - 2m_2)} \quad (3.26)$$

if $N_2 + \max(2 - 2m_1, 1 - 2m_2) \leq 0$. Moreover, under these same assumptions

$$\left\| \int_{\sigma_j}^{\sigma_{j+1}} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}} \leq Ct^{-1} (1 + \sigma_{j+1} - \sigma_j)(1 + \sigma_j)^{N_2 + \max(2 - 2m_1, 1 - 2m_2)}. \quad (3.27)$$

Here $N_1, N_2$ are as in the statement of Theorem 3.2.

The first inequality in this proposition follows from combining the results of Lemmas 3.12, 3.13, and 3.14. We postpone the proof to Section 3.6, and instead turn to the consequences of the proposition.

Proposition 3.10. Let $\lambda_0, \alpha(t)$ be chosen as in the statement of Proposition 3.5. Then, if $0 < \epsilon < 1/N_1$ and $t$ is sufficiently large, there is a constant $C$ so that

$$\left\| \int_{-\lambda_0 - \alpha(t)|t^\epsilon} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}} \leq Ct^{-1} \quad (3.28)$$
if \( m_l > (N_2 + 2 - l + d)/2 \) for \( l = 1, 2 \). Under the same hypotheses

\[
\left\| \int_{\lambda_0}^{\alpha(t)t^{\epsilon}} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}} \leq Ct^{-1}. \quad (3.29)
\]

**Proof.** We give the proof of the first inequality, as the proof of the second is essentially identical. Using our choice of \( \lambda_0 = \sigma_j \) and \( \alpha(t)t^\epsilon = \sigma_{J(t)} \), we write

\[
\int_{-\lambda_0}^{-\alpha(t)t^{\epsilon}} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda = \sum_{j=j_0}^{J(t)-1} \int_{-\sigma_j}^{-\sigma_{j+1}} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda. \quad (3.30)
\]

It may happen that \( \sigma_j = \sigma_{j+1} \), in which case the value of the corresponding integral is 0. Using (3.30) and Proposition 3.9 we find that

\[
\left\| \int_{-\alpha(t)t^{\epsilon}}^{-\lambda_0} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}} \leq \sum_{j=j_0}^{J(t)-1} Ct^{-1}(1 + \sigma_{j+1} - \sigma_j)(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)}.
\]

Here we may be substantially over-counting, particularly if the eigenvalues of the Laplacian on the cross section \( Y \) have high multiplicity; compare Lemma 3.11. Nonetheless, this gives us an upper bound. By the Weyl asymptotics for the Laplacian on the cross section \( Y \), we know that we can bound \( \sigma_{j+1} - \sigma_j \) from above by a constant which is independent of \( j \). Moreover, there is a constant \( c(Y) > 0 \) so that for \( j \gg 0 \), we have \( \sigma_j \sim c(Y)j^{1/(d-1)} \). Hence we can bound

\[
\sum_{j=j_0}^{J(t)-1} Ct^{-1}(1 + \sigma_{j+1} - \sigma_j)(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)}
\]

\[
\leq Ct^{-1} + Ct^{-1} \sum_{j=1}^{J(t)-1} j^{(N_2 + \max(2-2m_1,1-2m_2))/(d-1)}. \quad (3.31)
\]

If we choose \( m_l > (N_2 + 2 - l + d)/2 \), then the sum on the right-hand side is bounded independent of \( t \). \( \square \)

Proposition 3.5 follows directly from (3.25), Proposition 3.10, and Lemma 3.8.

It is the bound in Proposition 3.10 that requires the largest values of \( m_1, m_2 \) in our proof of Theorem 3.2. We include the following lemma, which allows smaller values of \( m_1, m_2 \) as a point of comparison. We note that the cross-sectional manifold \( Y \) in Lemma 3.11 certainly satisfies the assumptions on the cross-section \( Y \) made in Theorem 4.1. However, in Theorem 4.1 we have not made any effort to track the value of \( m \) (the analog of \( m_1, m_2 \)) required.

**Lemma 3.11.** Let \( \beta > 0 \) be a fixed positive number, \( \mu_0 \in \mathbb{N} \), and suppose

\[
(Y, g_Y) = \bigsqcup_{\nu=1}^{\mu_0} (\mathbb{S}^{d-1}, \beta g_{\mathbb{S}^{d-1}})
\]

where \( \mathbb{S}^{d-1} \) is the \( d-1 \)-dimensional unit sphere, and \( g_{\mathbb{S}^{d-1}} \) is the usual metric on it. Let \( \lambda_0 \) be as in the statement of Theorem 3.2 be chosen so that for some \( j_0 \in \mathbb{N} \), \( \lambda_0 = \sigma_{j_0} \), and, for \( t \)
sufficiently large, \( \alpha(t) \) be chosen so that \( 1 \leq \alpha(t) \leq 2 \) and \( \alpha(t)t' = \sigma_J(t) \) for some \( J(t) \in \mathbb{N} \). Then, if \( 0 < \epsilon < 1/N_1 \) and \( t \) is sufficiently large, there is a constant \( C \) so that (3.28) and (3.29) hold if \( m_l > (N_2 + 4 - l)/2, \ l = 1, 2 \).

Proof. As in the proof of the previous proposition, we find

\[
\left\| \int_{-\lambda_0}^{\lambda_0} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}} \leq \sum_{\sigma_{j_0} \leq \sigma_j < \sigma_{j+1} \leq \sigma_{J(t)-1}} Ct^{-1}(1 + \sigma_{j+1} - \sigma_j)(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)}. \tag{3.32}
\]

Note that the sum is over \( \sigma_j \) with \( \sigma_j < \sigma_{j+1} \). Recall that the \( k \)th distinct eigenvalue of \(-\Delta_Y\) is \( k(d + k - 2)/\beta \). If we assume that \( m_l > (N_2 + 4 - l)/2 \), we can bound the sum in (3.32) from above by

\[
Ct^{-1} \left( 1 + \sum_{k=1}^{\infty} k^{N_2 + \max(2-2m_1,1-2m_2)} \right)
\]

and the sum converges by our assumptions on \( m_l \).

Essentially the same argument bounds the integral in (3.29). \( \square \)

### 3.6. Proofs of Proposition 3.9 and Lemma 3.8

It remains to prove Proposition 3.9 and Lemma 3.8. The proofs are similar, involving contour deformations off the real axis to take advantage of the exponential decay of \( e^{\pm it\lambda} \) in the appropriate half-plane. As the proof of Proposition 3.9 is more complicated, we focus on it. In particular, we prove (3.26) carefully, as the proof of (3.27) is completely analogous. A first step in our proof of Proposition 3.9 is

**Lemma 3.12.** Let \( \epsilon < 1/N_1 \). Then there is a \( T > 0 \) so that if \( \lambda_0 \leq \sigma_j < \sigma_{j+1} < 2t^\epsilon \), then for \( t > T \)

\[
\int_{-\lambda_0}^{-\sigma_j} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda = I_{j\downarrow} + I_{j\rightarrow} + I_{j\uparrow}
\]

where

\[
I_{j\downarrow} = \lim_{\delta \downarrow 0} \int_{\gamma_{j\downarrow} + \delta} e^{-it\lambda} G(-\lambda) R_\chi(\lambda) d\lambda
\]

\[
I_{j\rightarrow} = \int_{\gamma_{j\rightarrow}} e^{-it\lambda} G(-\lambda) R_\chi(\lambda) d\lambda
\]

\[
I_{j\uparrow} = \lim_{\delta \downarrow 0} \int_{\gamma_{j\uparrow} - \delta} e^{-it\lambda} G(-\lambda) R_\chi(\lambda) d\lambda
\]

and the paths are given by

\[
(\gamma_{j\downarrow} + \delta)(s) = \sigma_j + \delta - is \log t/t, \quad (\gamma_{j\rightarrow})(s) = s(\sigma_{j+1} - \sigma_j) - i \log t/t,
\]

and \( (\gamma_{j\uparrow} - \delta)(s) = \sigma_{j+1} - \delta - i(1-s) \log t/t, \ 0 \leq s \leq 1 \). \tag{3.34}
Proof. By a change of variable,
\[
\int_{-\sigma_j}^{-\sigma_{j+1}} e^{i\lambda t} G(\lambda) R_\chi(\lambda) d\lambda = \int_{\sigma_j}^{\sigma_{j+1}} e^{-i\lambda t} G(-\lambda) R_\chi(\lambda) d\lambda. \tag{3.35}
\]
Of course, \(e^{-i\lambda t}\) is an analytic function of \(\lambda\), as is \(G(-\lambda)\) for \(\lambda\) near the real axis. Moreover, \(R_\chi(\lambda)\) has an analytic continuation to \(\{ \lambda \in \mathbb{C}_{\text{slit}} : \text{Re } \lambda > \lambda_0 \text{ and Im } \lambda > -C_0(\text{Re } \lambda)^{-N_1} \}\). There is a \(T > 0\) (independent of \(j\) satisfying conditions of the lemma) so that when \(t > T\) and \(\delta > 0\) is sufficiently small, the closed rectangle \(\mathfrak{R}_{jt}(\delta)\) with vertices
\[
\sigma_j + \delta, \quad \sigma_j + \delta - i(\log t)/t, \quad \sigma_{j+1} - \delta, \quad \sigma_{j+1} - \delta - i(\log t)/t
\]
lies in \(\mathbb{C}_{\text{slit}}\) and \(R_\chi(\lambda)\) is analytic in a neighborhood of \(\mathfrak{R}_{jt}(\delta)\). Hence by Cauchy’s theorem
\[
\int_{-\partial \mathfrak{R}_{jt}(\delta)} e^{-i\lambda t} G(-\lambda) R_\chi(\lambda) d\lambda = 0.
\]
In the limit as \(\delta \downarrow 0\) the integral over the top side of the rectangle is the integral in (3.35). Hence taking the limit as \(\delta \downarrow 0\) gives an equality equivalent to the claim of the lemma. \(\square\)

The next lemma bounds the integrals over the vertical sides of the rectangle.

**Lemma 3.13.** With the notation and assumptions as in Lemma 3.12, there is a constant \(C > 0\) independent of \(j\) and \(t\) so that for \(t > T\)
\[
\|I_{j\downarrow}\|_{H \oplus H \rightarrow H \oplus H} \leq Ct^{-1}(1 + \sigma_j)^{N_2 + \max(2-2m_1, 1-2m_2)}
\]
and
\[
\|I_{j\uparrow}\|_{H \oplus H \rightarrow H \oplus H} \leq Ct^{-1}(1 + \sigma_{j+1})^{N_2 + \max(2-2m_1, 1-2m_2)}
\]
where \(N_2\) is as in the statement of Theorem 3.2.

**Proof.** The proofs of the two inequalities are essentially the same, so we prove only the first one. We use that \(|e^{-i\lambda t}| = e^{t \text{Im } \lambda}\). Recall from the definition of \(G\) in (3.24) that for \(\lambda \gg 1\),
\[
G(-\lambda) \sim \begin{pmatrix} -\lambda^{2-2m_1} & -i\lambda^{1-2m_2} \\ i\lambda^{2-2m_1} & -\lambda^{1-2m_2} \end{pmatrix}.
\]
Moreover, for \(\lambda\) lying on the image of \(\gamma_{j\downarrow} + \delta, |\lambda|\) is quite close to \(\sigma_j\), so that on \(\gamma_{j\downarrow} + \delta, \|G(-\lambda)\| \leq C(1 + \sigma_j)^{\max(2-2m_1, 1-2m_2)}\) and \(\|R_\chi(\lambda)\|_{H \rightarrow H} \leq C(1 + \sigma_j)^{N_2}\) by our assumptions on \(R_\chi\) in the
Proof. Arguing as in the proof of the previous lemma, proving the lemma. □

Now we use that since $R_X$ has a meromorphic continuation to $\hat{Z}$ and

$$\|G(-(\sigma_j + is + \delta))R_X(\sigma_j + is + \delta)\|_{H^\pm H \to H^\pm H} \leq C(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)}$$

for all $-\log t/t \leq s \leq 0$, and all $\delta > 0$ sufficiently small, the same estimate holds in the limit as $\delta \downarrow 0$. Thus

$$\|I_{j}\|_{H^\pm H \to H^\pm H} \leq C\int_{-(\log t)/t}^{0} e^{ts}(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)} ds \leq C(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)} t^{-1}(1 - t^{-1}) \leq C(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)} t^{-1}.$$  

□

Next we bound the integral over the bottom side of the rectangle.

**Lemma 3.14.** With the notation and assumptions of Lemma 3.12, if $N_2 + \max(2-2m_1,1-2m_2) \leq 0$ there is a constant $C > 0$ independent of $j$ so that for $t > T$

$$\|I_{j}\|_{H^\pm H \to H^\pm H} \leq Ct^{-1}(\sigma_{j+1} - \sigma_j)(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)}$$

with $N_2$ as in the statement of Theorem 3.2.

Before proving the lemma, we note that if $N_2 + \max(2-2m_1,1-2m_2) > 0$ then a similar estimate holds, with $(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)}$ replaced by $(1 + \sigma_{j+1})^{N_2 + \max(2-2m_1,1-2m_2)}$.

Proof. Arguing as in the proof of the previous lemma,

$$\|I_{j}\|_{H^\pm H \to H^\pm H} = \left\| \int_{\sigma_j}^{\sigma_{j+1}} e^{-i(s-i(\log t)/t)t} G(-s + i(\log t)/t) R_X(s - i(\log t)/t) ds \right\|_{H^\pm H \to H^\pm H} \leq C\int_{\sigma_j}^{\sigma_{j+1}} e^{-\log t(1 + s)}^{N_2 + \max(2-2m_1,1-2m_2)} ds.$$ 

If $N_2 + \max(2-2m_1,1-2m_2) \leq 0$, then on the domain of integration we can bound $(1 + s)^{N_2 + \max(2-2m_1,1-2m_2)}$ from above by $(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)}$, and then

$$\int_{\sigma_j}^{\sigma_{j+1}} (1 + s)^{N_2 + \max(2-2m_1,1-2m_2)} ds \leq (\sigma_{j+1} - \sigma_j)(1 + \sigma_j)^{N_2 + \max(2-2m_1,1-2m_2)},$$

proving the lemma. □
Proof of Proposition 3.9. The proof of (3.26) follows by combining the results of Lemmas 3.12, 3.13, and 3.14. The proof of (3.27) follows in a completely analogous way, using the consequences of the self-adjointness of $H$ for the (continued) resolvent $R(\lambda)$. \hfill \Box

Proof of Lemma 3.8. We can use Cauchy’s theorem to write the integral
\[
\int_{\lambda_0 < \lambda < \alpha(t)t^i} e^{it\lambda} G(\lambda) \chi(\lambda) d\lambda
\]  
(3.36)
as the sum of integrals over the three line segments $\lambda_0 + i[0, 3t^{-1} \log t]$, $[\lambda_0, \alpha(t)t^i] + i3t^{-1} \log t$, and $\alpha(t)t^i + i[0, 3t^{-1} \log t]$, where we reverse the orientation on the last interval. We are deforming into the upper half plane, the physical region, where $R(\lambda)$ is a bounded operator on $\mathcal{H}$ when $\lambda^2$ is not an eigenvalue of $H$—hence the assumption $\lambda_0 > 0$. Note that $|e^{it\lambda}| = e^{-t \Im \lambda}$. Now we can bound the integrals over the vertical segments as in Lemma 3.13 and the integral over the top as in Lemma 3.14. To bound the integral over the sides, it suffices to have $N_2 + \max(2 - 2m_1, 1 - 2m_2) \leq 0$. To bound the integral over the top, where we can use $\|R(\lambda)\|_{\mathcal{H} \to \mathcal{H}} \leq 1/|\Im \lambda^2|$, it would suffice to take $m_1 = m_2 = 1$.

The bound for the portion of the integral over $-\alpha(t)t^i < \lambda < -\lambda_0$ is proved in a similar way. \hfill \Box

4. A WAVE EXPANSION UNDER A HYPOTHESIS ON THE DISTINCT EIGENVALUES OF $-\Delta_Y$

Under an assumption on the distinct eigenvalues of $-\Delta_Y$, we can find an asymptotic expansion of $u(t)$ to order $t^{-k_0}$ for any $k_0 \in \mathbb{N}$. This expansion involves an infinite sum, see (4.3). If multiplied by the cut-off function $\chi$, the infinite sum over $l$ converges absolutely, see (4.16). The main result of this section is Theorem 4.1.

In order to state the theorem, we introduce the notion of a distance on $\hat{Z}$. For two points $\lambda, \lambda' \in \hat{Z}$ we define $d_2(\lambda, \lambda') = \sup_j |\tau_j(\lambda) - \tau_j(\lambda')|$. That this is well-defined is shown in [CD, Lemma and Definition 5.1]. In the statement of Theorem 4.1 below, by $\lambda' \in \mathbb{R}$ we mean that $\lambda'$ lies on the boundary of the physical space. We also recall that $\nu_l^2$ denote the distinct eigenvalues of $-\Delta_Y$, with $0 = \nu_0 < \nu_1 < \nu_2$...

**Theorem 4.1.** Let $H$ be a black box perturbation of $-\Delta$ on $X_\infty$, and suppose that for some $N_1, N_2 \in [0, \infty)$, $\lambda_0 > 0$, and any $\chi \in C^\infty_c(X_\infty)$ with $\chi(r) = 1$ for $r \leq 1$ there are $C_0, C_1$ so that $\bar{\chi} R(\lambda) \bar{\chi}$ is analytic on the set
\[
\{ \lambda \in \hat{Z} : d_2(\lambda, \lambda') < C_0(1 + \lambda')^{-N_1} \text{ for some } \lambda' \in \mathbb{R}, \lambda' > \lambda_0 \}
\]  
(4.1)
and that in this region
\[
\|\bar{\chi} R(\lambda) \bar{\chi}\| \leq C_1(1 + |\lambda|)^{N_2}.
\]  
(4.2)
In addition, suppose that there are $c_Y > 0, N_Y \geq 0$ so that $\nu_{l+1} - \nu_l > c_Y \nu_l^{-N_Y}$ when $\nu_l > 1$. Let $k_0 \in \mathbb{N}$ be given, and $\chi \in C^\infty_c(X_\infty)$ be one for $r \leq 1$. Let $u(t)$ be the solution of (3.1), with $f_1, f_2 \in (H + M_0)^{-m}\mathcal{H}$ for any $m \in \mathbb{N}_0$ and $1_{X_\infty}f_1, 1_{X_\infty}f_2$ supported in $r \leq M_1 < \infty$. Then there are $b_{l,k,\pm} \in (r)^{1/2+2k+\epsilon}\mathcal{H}$, depending on $f_1, f_2$ so that if we set
\[
u_{thr,k_0}(t) = \frac{1}{4} \sum_{j=0}^{k_0 - 1} \Phi_j(0) \langle f_2, \Phi_j(0) \rangle + \sum_{k=0}^{k_0 - 1} t^{-1/2-k} \sum_{l=1}^{\infty} (e^{it\nu_l} b_{l,k,+} + e^{-it\nu_l} b_{l,k,-})
\]  
(4.3)
then there are \( m \in \mathbb{N}, C > 0 \) so that
\[
\| \chi(u(t) - u_c(t) - u_{thr,k_0}(t)) \|_{\mathcal{H}} \leq Ct^{-k_0} (\| (H + M_0)^m f_1 \|_{\mathcal{H}} + \| (H + M_0)^m f_2 \|_{\mathcal{H}})
\]
if \( t \) is sufficiently large. Moreover,
\[
\sum_{l=1}^{\infty} (\| \chi b_{l,k,+} \|_{\mathcal{H}} + \| \chi b_{l,k,-} \|_{\mathcal{H}}) \leq C (\| (H + M_0)^m f_1 \|_{\mathcal{H}} + \| (H + M_0)^m f_2 \|_{\mathcal{H}}).
\]
The value of \( m \) needed depends polynomially on \( k_0 \), and also depends on \( N_1, N_2, \) and \( N_Y \). The \( b_{l,k,\pm} \) are determined by the value of \( \nu_l, f_1, f_2, \) and the derivatives with respect to \( \tau_j \) of order at most \( 2k \) of elements of the set \( \{ \Phi_j \}_{0 \leq \sigma_j \leq \nu_l} \) evaluated at \( \pm \nu_l \), where \( \sigma_j = \nu_l \). Recall \( u_c \) is given in (3.2).

The paper [CD] includes examples of manifolds \( X \) and large classes of potentials \( V \in C_c^\infty(X; \mathbb{R}) \) so that the hypotheses of this theorem hold for \( H = -\Delta + V \) on \( X \); see [CD, Theorems 3.1 and 5.6, and Section 3.2]. These manifolds include the manifolds in Section 1.2.1 and some of the manifolds in Section 1.2.2.

Our Theorems 3.2 and 4.1 require a polynomial bound on the cut-off resolvent at high energies in order to handle the large energy contribution to solutions of the wave equation. If instead we consider only the solution localized in a finite energy regime, neither the bound on the cut-off resolvent nor the assumption made in Theorem 4.1 on the distinct eigenvalues of \( -\Delta_Y \) is necessary. We prove the following Proposition naturally in the course of the proof of Theorem 4.1.

**Proposition 4.2.** Let \( H \) be any operator satisfying all the conditions on the black box perturbation outlined in Section 2. Let \( \psi_{sp} \in C_c^\infty(\mathbb{R}; \mathbb{R}) \), \( M_1 > 0, f_1, f_2 \in \mathcal{H} \) satisfy \( f_1 \big|_{X_\infty}, f_2 \big|_{X_\infty} \) vanish for \( r > M_1 \). Let \( \psi_{sp,1} \in C_c^\infty(\mathbb{R}; \mathbb{R}) \) satisfy \( \psi_{sp,1} \psi_{sp} = \psi_{sp} \). Then if \( k_0 \in \mathbb{N} \), for each \( l \in \mathbb{N} \) with \( \nu_l \in \text{supp} \psi_{sp} \) there are \( b_{l,k,\pm} = b_{l,k,\pm}(f_1, f_2, \psi_{sp}) \in \mathbb{R}^{2k+1/2+\epsilon} \mathcal{H} \), \( k = 0, ..., k_0 - 1 \), so that
\[
\chi \psi_{sp}(H)u(t) = \chi \psi_{sp}(H)u_c(t) + \frac{1}{4} \psi_{sp}(0) \sum_{\sigma_j = 0}^{k_0 - 1} \chi \Phi_j(0) \langle f_2, \Phi_j(0) \rangle
\]
\[
+ \sum_{l=1}^{\infty} \psi_{sp,1}(r_j^2) \sum_{k=0}^{k_0 - 1} \chi (b_{l,k,+,e^{it\nu_l}} + b_{l,k,-,e^{-it\nu_l}}) t^{-1/2-k} + \chi u_{r,k_0,\psi_{sp}}(t) \quad (4.4)
\]
with \( \| \chi u_{r,k_0,\psi_{sp}}(t) \|_{\mathcal{H}} \leq Ct^{-k_0} \) for sufficiently large \( t \). Here \( u_c(t) \) is as given in (3.2).

Note that the assumption that \( \psi_{sp,1} \) has compact support means that the sum in (4.4) is finite. Related results for spectrally cut-off solutions of the wave equation (though with quite different geometry) can be found in [GHS13, VW13]. Again, we note that we need neither the assumption of high-energy bounds on the (cut-off) resolvent nor an assumption on the spacing of the \( \nu_l \) in the hypotheses of Proposition 4.2.

We comment that Proposition 4.2 holds for a planar waveguide under reasonable assumptions. For example, suppose \( X \subseteq \mathbb{R}^2 \) is a connected open set with smooth boundary. Moreover, suppose \( X \) can be decomposed as \( X = X_c \sqcup \bigcup_{i=1}^{n} X_i \), where \( X_c \) is compact and each \( X_i \) can be mapped to \((0, a_i) \times [0, \infty)\) for some \( a_i > 0 \) by a rigid motion. Then if \( H \) is the Laplacian with Neumann boundary conditions on \( X \), and \( \mathcal{H} = L^2(X) \), Proposition 4.2 holds. It holds with Dirichlet
boundary conditions as well, as long as we understand that the sum over $\sigma_j = 0$ is a sum of no elements.

Similarly, an analog of Theorem 4.1 holds for planar waveguides as well, provided the hypotheses on the eigenvalues on the cross section $Y$ and the high energy resolvent estimates are valid.

In the interest of brevity we do not pursue this further here.

4.1. Bounds on the derivatives of the cut-off resolvent. Our proof of Theorem 4.1 will require bounds of the derivatives of the cut-off resolvent along the real axis. It is here that we will use our assumption on the spacing of the distinct eigenvalues of $-\Delta_Y$. Our first lemma, however, does not need these hypotheses as we bound the derivatives away from the thresholds.

Lemma 4.3. Suppose the hypotheses of Theorem 3.2 hold. Let $N \geq N_1$ be fixed and let $\beta \geq 1$. If $\lambda' \in \mathbb{R}$, $|\lambda'| > \max(1, \lambda_0)$ and $\inf_{l, \pm} |\lambda' \pm \nu_l| > |\lambda'|^{-N}/\beta$, then there is a $C > 0$ so that

$$\left\| \frac{\partial^k}{\partial \lambda^k} \chi R(\lambda) \chi \big|_{\lambda = \lambda'} \right\|_{\mathcal{H} \to \mathcal{H}} \leq C k!(1 + |\lambda'|)^{N_2 + kNk^2}, \quad k \in \mathbb{N}. \quad (4.5)$$

Proof. There is a ball in $\mathbb{C}_{\text{slit}}$ centered at $\lambda'$ of radius proportional to $|\lambda'|^{-N}/\beta$ on which $\chi R(\lambda) \chi$ is analytic, with norm bounded by $C|\lambda|^{N_2}$. Hence the estimate (4.5) follows immediately from the Cauchy estimates. \hfill \Box

Away from the thresholds we can use $\lambda$ as a coordinate, as we did in Lemma 4.3. Near a threshold we need to introduce a different local coordinate. In particular, near the threshold $\sigma_j$ (and $-\sigma_j$) we shall use $\tau_j$ as a local coordinate.

For the setting of the next lemma, we think of $\{\nu_l\}$ as denoting not just the square roots of the distinct eigenvalues of $-\Delta_Y$, but also corresponding to a point on the boundary of the physical space in $\hat{Z}$. Given $l \in \mathbb{N}$, choose $\epsilon = \epsilon(l) > 0$ so that $|\nu_l - \nu_{l \pm 1}| \nu_{l - 1} > \epsilon^2$ and let $j = j(l) \in \mathbb{N}$ be such that $\nu_l = \sigma_j$. Then we may, in a natural way, identify $B(0; \epsilon) = \{z \in \mathbb{C} : |z| < \epsilon\}$ with a (particular) neighborhood $U_{\nu_l}(\epsilon)$ of $\nu_l \in \hat{Z}$ by using

$$U_{\nu_l}(\epsilon) \ni \lambda \mapsto \tau_j(\lambda) \in B(0; \epsilon); \quad (4.6)$$

$U_{\nu_l}(\epsilon)$ is defined to the the connected component of $\tau_j^{-1}(B(0; \epsilon))$ containing $\nu_l$, a point on the boundary of the physical space. A completely analogous identification can be done near $-\nu_l$, also using $\tau_j$, where $j = j(l)$.

The assumption on the spacing of the distinct eigenvalues of $-\Delta_Y$ allows us to bound the derivatives of $\chi R \chi$ in a neighborhood of each threshold.

Lemma 4.4. Suppose the hypotheses of Theorem 4.1 hold, and continue to use the notation $j = j(l)$, $U_{\nu_l}(\epsilon)$ introduced above. Set $N_M = \max((N_Y - 1)/2, N_1)$. There are $\alpha > 0, C \in \mathbb{R}$ so that if $\nu_l = \sigma_j \geq \lambda_0 + 1$, then

$$\left\| \left( \frac{\partial^k}{\partial \nu_l^j} \chi R(\lambda) \chi \right) \big|_{\lambda = \lambda'} \right\|_{\mathcal{H} \to \mathcal{H}} \leq C k!|\nu_l|^{N_2 + kN_M}, \quad k \in \mathbb{N} \quad (4.7)$$

for all $\lambda' \in U_{\nu_l}(\alpha \nu_l^{-N_M}) \subset \hat{Z}$. 
Proof. For simplicity, we give the proof only for $U_{\nu_i}(\alpha \nu_i^{-\beta N_i})$.

The assumptions on the spacing of the distinct eigenvalues of $-\Delta_Y$ ensure that there is a $\beta > 0$ so that $|\nu_l - \nu_{l+1}| > \nu_1^{-1} N_Y / \beta$ for all $l$ with $\nu_l > 0$. Moreover, increasing $\beta > 0$ if necessary, our definition of $N_M$ and the hypotheses of Theorem 4.1 ensure that $\chi R(\lambda) \chi$ is analytic on $U_{\nu_i}(1/(\beta \nu_i^{N_M}))$, again for all $l$ with $\nu_l > \lambda_0 + 1$. Moreover, $\|\chi R(\lambda) \chi\| \leq C(1 + \nu) N_2$ in this set, with constant $C$ independent of $l$. Identify $U_{\nu_i}(1/(\beta \nu_i^{N_M}))$ with $B(0; 1/(\beta \nu_i^{N_M}))$. Each point $z$ in $B(0; 1/(\beta \nu_i^{N_M}))$ has the property that the ball with center $z$ and radius $1/(2 \beta \nu_i^{N_M})$ lies in $B(0; 1/(\beta \nu_i^{N_M}))$. Hence, we may prove the lemma by taking $\alpha = 1/(2 \beta)$ and by applying the Cauchy estimates on such a ball, recalling that the coordinate is $\tau_j$.

4.2. The proof of Theorem 4.1. We turn more directly to the proof of Theorem 4.1. As in the proof of Theorem 3.2, we shall write $(I - \mathcal{P})u(t)$ as the sum of several integrals. In order to define these, let $\psi \in C^\infty_c(\mathbb{R}; [0, 1])$ have its support in a small neighborhood of the origin, and be one in a smaller neighborhood of the origin. For convenience later, choose $\psi$ to be even. Set

$$\tilde{N} = \max(N_Y - 1, 2N_1)$$

and

$$\psi_l(\lambda) = \psi \left( \nu_l^{-\beta} (\lambda^2 - \nu_l^2) \right).$$

If we wish to prove Proposition 4.2 (rather than Theorem 4.1), the choice of of $\tilde{N}$ does not matter much—we can take $\tilde{N} = 0$. We choose the support of $\psi$ to be small enough that

$$\text{supp } \psi \cap \text{supp } \psi_l = \emptyset, \text{ for } l \in \mathbb{N}. $$

To prove Theorem 4.1, by shrinking the support of $\psi$ if necessary, we choose $\psi$ to satisfy

$$\text{supp } \psi_l \cap \text{supp } \psi_{l'} = \emptyset \text{ if } l \neq l', \text{ } l, l' \in \mathbb{N}. \tag{4.9}$$

The assumption on the spacing of the distinct eigenvalues of $-\Delta_Y$ and our choice of $\tilde{N} \geq N_Y - 1$ ensure that (4.9) is possible. To prove Proposition 4.2 instead we replace (4.9) by

$$\psi_{s\psi}(\lambda^2) \psi_l(\lambda) \psi_{l'}(\lambda) \equiv 0 \text{ if } l \neq l', \text{ } l, l' \in \mathbb{N}. \tag{4.10}$$

Similarly to (3.9), using the integral representation of Lemma 3.3 we can write

$$(I - \mathcal{P}) \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = (I_0(t) + I_{\text{thr}}(t) + I_r(t)) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{4.11}$$

where

$$I_0(t) = \text{PV} \frac{1}{2\pi i} \int e^{it\lambda} \psi(\lambda) A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P}) d\lambda$$

$$I_{\text{thr}}(t) = \frac{1}{2\pi i} \int e^{it\lambda} \sum_{l=1}^{\infty} \psi_l(\lambda) A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P}) d\lambda$$

$$I_r(t) = \frac{1}{2\pi i} \int e^{it\lambda} \left( 1 - \psi(\lambda) - \sum_{l=1}^{\infty} \psi_l(\lambda) \right) A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P}) d\lambda.$$

Here $A(\lambda)$ is as in (3.7).
We recall that we have already studied $I_0(t)$ in Lemma 3.6. Lemma 4.5 shows that $I_r(t)$ does not contribute to the asymptotic expansion of $(I - P)u(t)$. In Lemma 4.7 we evaluate the contribution from any nonzero threshold. Finally we put these all together to prove the theorem.

**Lemma 4.5.** Under the hypotheses of Theorem 4.1, if the support of $\psi$ is chosen sufficiently small, then for any $k \in \mathbb{N}$, there is an $m \in \mathbb{N}$ depending polynomially on $k$ so that for $t > 0$

$$
\left\| \chi I_r(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_{\mathcal{H} \otimes \mathcal{H}} \leq Ct^{-k} \left( \| (H + M_0)^m f_1 \|_{\mathcal{H}} + \| (H + M_0)^m f_2 \|_{\mathcal{H}} \right).
$$

**Proof.** Set

$$
\psi_{\text{tot}}(\lambda) = \psi(\lambda) + \sum_{l=1}^{\infty} \psi_l(\lambda). \quad (4.12)
$$

Note that by our assumptions on $\psi$, $1 - \psi_{\text{tot}}$ vanishes in a neighborhood of $\lambda = 0$ and in a neighborhood of $\lambda = \pm \sigma_j$ for each $\sigma_j$. Hence

$$
(1 - \psi_{\text{tot}}(\lambda))(\chi R(\lambda)(I - P)\chi - \chi R(-\lambda)(I - P)\chi)
$$

is a smooth function of $\lambda$. Using Lemma 4.3, by choosing $m \in \mathbb{N}$ sufficiently large we can ensure that

$$
\left\| \frac{d^{k'}}{d\lambda^{k'}} \left((\lambda^2 + M_0)^{-m}(1 - \psi_{\text{tot}}(\lambda))\chi(R(\lambda) - R(-\lambda))(I - P)\chi\right) \right\|_{\mathcal{H} \otimes \mathcal{H}} \leq C(1 + |\lambda|)^{-2} \quad (4.13)
$$

for all $k' \in \mathbb{N}_0$, $k' \leq k$. The choice of exponent $-2$ on the right-hand side is somewhat arbitrary, but is made to ensure that the function is integrable. We could replace $-2$ by $-p$, some other $p > 1$, and such a change may change the value of $m$ which is needed on the left hand side. Now we use (3.10) and integrate by parts $k$ times to prove the lemma. □

By way of comparison, we include following lemma.

**Lemma 4.6.** Under the hypotheses of Proposition 4.2, if the support of $\psi$ is chosen sufficiently small, then for any $k \in \mathbb{N}$, there is a $C > 0$ so that

$$
\left\| \chi \psi_{\text{sp}}(H)I_r(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\| \leq Ct^{-k} \left( \| f_1 \|_{\mathcal{H}} + \| f_2 \|_{\mathcal{H}} \right) \text{ for } t > 0.
$$

**Proof.** We use $\psi_{\text{tot}}$ from (4.12). Using the compact support of $\psi_{\text{sp}}$ there is a $C > 0$ so that

$$
\left\| \frac{d^{k'}}{d\lambda^{k'}} \left(\psi_{\text{sp}}(\lambda^2)(1 - \psi_{\text{tot}}(\lambda))\chi(R(\lambda) - R(-\lambda))(I - P)\chi\right) \right\|_{\mathcal{H} \otimes \mathcal{H}} \leq C(1 + |\lambda|)^{-2}
$$

for all $k' \in \mathbb{N}_0$, $k' \leq k$. Then integrating by parts $k$ times proves the lemma. □

**Lemma 4.7.** Let $H$, $f_1$, and $f_2$ satisfy the hypotheses of Theorem 4.1. Let $\psi_l$ be as defined in (4.8) and let $k_0 \in \mathbb{N}$. Then with the support of the function $\psi$ in (4.8) chosen sufficiently small,
for \( l \in \mathbb{N} \) there are \( b_{l,k,\pm}, b_{l,k,\pm}^{(l)} \in \langle r \rangle^{1/2+2k+\epsilon} \mathcal{H} \), \( k = 0, 1, \ldots, k_0 - 1 \) so that
\[
\int_{-\infty}^{\infty} e^{it\lambda} \psi_{l}(\lambda) A(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P}) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda \\
= \sum_{k=0}^{k_0-1} t^{-k-1/2} \left( \chi b_{l,k,\pm} e^{it\nu_k} + \chi b_{l,k,\pm}^{(l)} e^{-it\nu_k} \right) + \chi B_{l,k_0}(t). \tag{4.14}
\]
There is an \( m \in \mathbb{N} \) depending polynomially on \( k_0 \) as well as on \( N_1, N_2, N_Y \) and a constant \( C \) independent of \( l \) so that for \( t > 0 \)
\[
\| \chi B_{l,k_0}(t) \|_{\mathcal{H} \oplus \mathcal{H}} \leq C t^{-k_0}(\| (H + M_0)^m f_1 \| + \| (H + M_0)^m f_2 \|) \tag{4.15}
\]
and
\[
\| \chi b_{l,k,\pm} \| + \| \chi b_{l,k,\pm}^{(l)} \| \leq C t^{-2}(\| (H + M_0)^m f_1 \| + \| (H + M_0)^m f_2 \|), \quad k = 0, \ldots, k_0 - 1. \tag{4.16}
\]
The \( b_{l,k,\pm}, b_{l,k,\pm}^{(l)} \) are determined by the initial data \( f_1, f_2, \) the value of \( \nu_l \), and the derivatives with respect to \( \tau_j \) of order at most \( 2k \) of elements of the set \( \{ \Phi_{j,l} \}_{0 \leq \nu_j \leq \nu_l} \) evaluated at \( \pm \nu_j \), where \( \sigma_j = \nu_j \).

Before proving the lemma, we note that as in (4.13) the choice of exponent \(-2\) for \( l \) in (4.15) and (4.16) is again somewhat arbitrary. We choose it because we wish the sum over \( l \) to converge. As in (4.13), \( l^{-2} \) may be replaced by \( l^{-p}, \) \( p > 1, \) if \( m \) is chosen sufficiently large, depending on \( p \).

**Proof.** Let \( j \in \mathbb{N} \) be such that \( \nu_l = \sigma_j \). We wish to apply Lemma A.1.

By our choice of \( \psi_{l} \), the support of \( \psi_{l} \) contains no thresholds other than \( \pm \nu_l \). Then with \( \sigma_j = \nu_l, \tau_j(\lambda) = R(\lambda)(I - \mathcal{P})(\lambda) \chi \) is a smooth function of \( \tau_j(\lambda) \) on the support of \( \psi_{l}(\lambda), \lambda \in \mathbb{R} \), and \( \chi \tau_j(\lambda)R(-\lambda)(I - \mathcal{P})(\lambda) \chi \) is not in general. Hence we shall rewrite the integral to avoid the use of \( R(-\lambda) \). At the same time, for \( m \in \mathbb{N}_0 \) we write \( f_1 = (H + M_0)^{-m}(H + M_0)^m f_1 \), and similarly for \( f_2 \). Hence we rewrite
\[
\int_{-\infty}^{\infty} e^{it\lambda} \psi_{l}(\lambda) A(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda \\
= \int_{-\infty}^{\infty} (e^{it\lambda} A(\lambda) - e^{-it\lambda} A(-\lambda)) \psi(\lambda) \chi R(\lambda)(I - \mathcal{P}) (\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda. \tag{4.17}
\]
This integral has an asymptotic expansion, with contributions from the neighborhoods of \( \lambda = \sigma_j = \nu_l \) and \( \lambda = -\sigma_j = -\nu_l \); each will contribute both to the \( b's \) and to \( B_{l,k_0} \).

Now we recall from the proof of Lemma 4.4 that there are neighborhoods of \( \nu_l = \sigma_j \) and \( -\nu_l = -\sigma_j \) in \( \tilde{Z} \) on which we may use \( \tau_j \) as a coordinate and on which \( \chi \tau_j(\lambda)R(\lambda)(I - \mathcal{P})(\lambda) \chi \) is a smooth function of \( \tau_j \). Under the hypotheses of Theorem 4.1, the radii of the balls about \( \pm \nu_l \) can be taken proportional to \( \min(\sigma_j^{(1-N_{Y})/2}, \sigma_j^{-N_{Y}}) \) in the \( \tau_j \) coordinate. Hence by our choice of \( \tilde{N} \) we can choose our original function \( \psi \), with \( \psi_{l}(\lambda) = \psi(\nu_l^{\tilde{N}}(\lambda^2 - \nu_l^2)) \), so that we can extend \( \psi_{l}(\lambda) A(\pm \lambda) \chi \tau_j(\lambda) R(\lambda)(I - \mathcal{P})(\lambda) \chi \) to be a smooth, compactly supported function of \( \tau_j \) in this complex ball. In fact, with \( \psi \) chosen to be even, \( \psi(\nu_l^{\tilde{N}}(\lambda^2 - \nu_l^2)) A(\pm \lambda) \chi \tau_j(\lambda) R(\lambda)(I - \mathcal{P})(\lambda) \chi \)
provides such an extension (where we understand \( \lambda \) to be the locally well-defined function of \( \tau_j \)). With the support of \( \psi \) chosen sufficiently small, this will hold for all \( l \in \mathbb{N} \).

Thus we may apply Lemma A.1 in order to find an expansion for the portion of the integral in (4.17) over \((0, \infty)\). From the second part of Lemma A.1 we obtain immediately that

\[
\begin{align*}
\int_0^\infty & (e^{it\lambda} A(\lambda) - e^{-it\lambda} A(-\lambda)) \psi_l(\lambda) R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda \\
= & -\int_0^\infty e^{-it\lambda} A(-\lambda) \psi_l(\lambda) R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda + O(e^{-\nu_0}).
\end{align*}
\]

(4.18)

Now by applying the first part of Lemma A.1 and (A.1), we have an expansion of (4.18) of the form in (4.14) with exponential \( e^{-it\nu} \), and the coefficients in the expansion; that is, the \( b_{l,k,-} \) and \( b_{l,k,-}' \) are determined by \( \sigma_j \) and derivatives with respect to \( \tau_j \) of

\[
\psi_l(\lambda) R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m} \left( \begin{array}{c} (H + M_0)^m f_1 \\ (H + M_0)^m f_2 \end{array} \right)
\]

of order at most \( 2k \) evaluated at \( \lambda = \sigma_j = \nu_l \).

In order to prove the uniformity in \( l \), as in (4.15) and (4.16), we use the consequences of our assumptions in Theorem 4.1. By Lemma 4.4 the derivatives of fixed order of \( R(\lambda)(I - \mathcal{P}) \chi \) with respect to \( \tau_j \) near \( \pm \sigma_j \) grow at worst polynomially in \( j = j(l) \), as do the derivatives of \( \psi_l(\lambda) \) by definition. Thus by taking \( m \) sufficiently large, depending polynomially on \( k_0 \), we can guarantee that the analog of (4.16) holds. Similarly, using the remainder estimate of Lemma A.1, we find that the analog of (4.15) holds.

Thus far we have proved that the coefficients \( b_{l,k,-} \) (and \( b_{l,k,-}' \)) are determined by the value of \( \nu_l = \sigma_j \) and the derivatives with respect to \( \tau_j \) of (4.19) evaluated at \( \lambda = \nu_l \). We can say a bit more. Here we concentrate on describing the origin of the \( b_{l,k,-} \) and do not worry about bounding them uniformly in \( l \), as we have already done so. We write

\[
\begin{align*}
\int_0^\infty & e^{-it\lambda} A(-\lambda) \psi_l(\lambda) R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda \\
= & \int_0^\infty e^{-it\lambda} A(-\lambda) \psi_l(\lambda) [R(-\lambda) + R(\lambda) - R(-\lambda)](I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda \\
= & \int_{-\infty}^0 e^{it\lambda} A(\lambda) \psi_l(\lambda) R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda \\
+ & \int_{-\infty}^\infty e^{-it\lambda} A(-\lambda) \psi_l(\lambda) [R(\lambda) - R(-\lambda)](I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda
\end{align*}
\]

(4.20)

The analog of Lemma A.1 for integration over \((-\infty, 0)\) shows that

\[
\left\| \int_{-\infty}^0 e^{it\lambda} A(\lambda) \psi_l(\lambda) R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) d\lambda \right\| = O(e^{-k_0})
\]

using that \( R(\lambda)(I - \mathcal{P}) \tau_j(\lambda) \) is analytic function of \( \tau_j \) in a neighborhood of \(-\sigma_j\).
Using Lemma 2.2,

\[
\int_0^\infty e^{-it\lambda}A(-\lambda)\psi_1(\lambda)\chi[R(\lambda) - R(-\lambda)](I - P)(\lambda^2 + M_0)^{-m}(H + M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda
\]

\[
= \frac{i}{2} \int_0^{\sigma_j} e^{-it\lambda}A(-\lambda)\psi_1(\lambda)\chi \sum_{0 \leq \sigma_j' < \sigma_j} \frac{\Phi_{j'}(\lambda) \otimes \Phi_{j'}(\lambda)}{\tau_{j'}(\lambda)}(\lambda^2 + M_0)^{-m}(H + M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda
\]

\[
+ \frac{i}{2} \int_{\sigma_j}^{\infty} e^{-it\lambda}A(-\lambda)\psi_1(\lambda)\chi \sum_{0 \leq \sigma_j' \leq \sigma_j} \frac{\Phi_{j'}(\lambda) \otimes \Phi_{j'}(\lambda)}{\tau_{j'}(\lambda)}(\lambda^2 + M_0)^{-m}(H + M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda.
\]

(4.21)

We can apply Lemma A.2 to each of these two integrals. We know from our previous discussion that the sum will have an asymptotic expansion in powers of \( t^{-k-1/2} \), but we learn some more specific information about the coefficients this way. This shows us that the \( b_{l,k,-} \) and \( b_{l,k,-}^{(i)} \) are actually determined by the value \( \nu_l = \sigma_j \), the initial conditions \( f_1 \) and \( f_2 \), and elements of the set

\[
\left\{ (\partial^{k'}_{\tau_j} \Phi_{j'})_{|\lambda = \sigma_j} : 0 \leq \sigma_{j'} \leq \sigma_j, \ 0 \leq k' \leq 2k \right\}.
\]

This also provides a natural way to see that \( b_{l,k,-} \in \langle r \rangle^{1/2+2k+\epsilon} \mathcal{H} \), since \( \partial^{k'}_{\tau_j} \Phi_{j'} \mid_{\lambda = \sigma_j} \in \langle r \rangle^{1/2+k'+\epsilon} \mathcal{H} \) if \( \sigma_{j'} \leq \sigma_j \).

The integral over \((-\infty, 0)\) can be handled in an analogous manner, and gives us the \( b_{l,k,+} \) and \( b_{l,k,+}^{(i)} \). \( \square \)

The next lemma is almost parallel to Lemma 4.7. It differs in that it assumes only the hypotheses of Proposition 4.2, and only achieves uniformity in \( l \) because of the multiplication by the compactly supported function \( \psi_{sp}(\lambda^2) \). We omit the proof because it is so similar.

**Lemma 4.8.** Let \( H, f_1, \) and \( f_2 \) satisfy the hypotheses of Proposition 4.2. Let \( \psi_1 \) be as defined in (4.8) and let \( k_0 \in \mathbb{N} \). Then with the support of the function \( \psi \) in (4.8) chosen sufficiently small, for \( l \in \mathbb{N} \) there are \( b_{l,k,\pm}, b_{l,k,\pm}^{(i)} \in \langle r \rangle^{1/2+2k+\epsilon} \mathcal{H}, \ k = 0, 1, ..., k_0 - 1 \) so that

\[
\int_{-\infty}^{\infty} e^{it\lambda} \psi_1(\lambda)\psi_{sp}(\lambda^2)A(\lambda)\chi(R(\lambda) - R(-\lambda))(I - P) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda
\]

\[
= \sum_{k = 0}^{k_0-1} t^{-k-1/2} \left( \frac{\chi b_{l,k,+} e^{it\nu_k} + \chi b_{l,k,-} e^{-it\nu_k}}{\chi b_{l,k,+}^{(i)} e^{it\nu_k} + \chi b_{l,k,-}^{(i)} e^{-it\nu_k}} \right) + \chi B_{l,k_0}(t) \tag{4.22}
\]

with

\[
\| \chi B_{l,k_0}(t) \|_{\mathcal{H}} \leq C_{l,k} t^{-k}(\| f_1 \|_{\mathcal{H}} + \| f_2 \|_{\mathcal{H}}).
\]

The \( b_{l,k,\pm}, b_{l,k,\pm}^{(i)} \) are determined by the initial data \( f_1, f_2, \) the value of \( \nu_l, \) and the derivatives with respect to \( \tau_j \) of order at most \( 2k \) of elements of the set \( \{ \Phi_{j'}(\lambda), \psi_{sp}(\lambda^2)\Phi_{j'}(\lambda) \}_{0 \leq \sigma_{j'} \leq \nu_l} \) evaluated at \( \pm \nu_l \), where \( \sigma_j = \nu_l \).

Of course, the \( b_{l,k,\pm}, b_{l,k,\pm}^{(i)} \) in Lemma 4.8 are 0 if \( \pm \nu_l \) are not in the support of \( \psi_{sp}(\lambda^2) \).
Proof of Theorem 4.1. We write \( u(t) = Pu(t) + (I - P)u(t) \). The expansion of \( u_c(t) = Pu(t) \) is given in Lemma 3.1. From equation (4.11) we see that \((I - P)u(t)\) is given by a sum of contributions from \( I_0 \), \( I_{thr} \), and \( I_r \). By Lemma 4.5, the contribution of \( I_r \) is of order \( t^{-k} \) for any \( k \). Note that using Lemma 4.7 and summing over \( l \in \mathbb{N} \) evaluates the contribution of \( I_{thr} \); the estimates (4.15) and (4.16) ensure the convergence of the sums over \( l \) to bound the remainder and the absolute convergence of the sum over \( l \) in (4.3), respectively. Lemma 3.6 gives the contribution of \( I_0 \). Summing the contributions of the terms from \( I_0 \) and \( I_{thr} \) gives \( u_{thr,k_0} \). □

Proof of Proposition 4.2. We use (4.11), multiplying on the left hand side by \( \chi_{sp}(H) = \chi_{sp}(H)\psi_{sp,1}(H) \). By Lemma 4.6, \( \|\chi_{sp}(H)I_r(t)\chi\|_{H \otimes H \rightarrow H \otimes H} \leq Ct^{-k} \) for any \( k \). Then by Lemma 3.6 and Lemma 4.8, summing over the finite number of \( l \) with \( \nu^2_l \in \text{supp} \psi_{sp} \), we see that the sum of the contributions of \( \chi_{sp}(H)I_0(t) \) and \( \chi_{sp}(H)I_{thr}(t) \) gives an expansion as claimed. □

Appendix A. Asymptotic expansions of some integrals

In this section we prove two lemmas which are used in evaluating the contribution of the thresholds to the asymptotics of the solutions of the wave equation. The proofs of these lemmas use a change of variables and stationary phase to find asymptotic expansions of two types of integrals.

Lemma A.1. Fix \( c_0 \in (0, 1) \) and set \( B_0 = \{z \in \mathbb{C} : |z| < c_0/2\} \). Let \( \mathcal{X} \) be a Banach space. For each \( k_0 \in \mathbb{N} \) there is a \( C > 0 \) so that if \( \sigma_j > c_0 \), \( F \in C^\infty_0(B_0; \mathcal{X}) \) then there are \( b_k = b_k(F, \sigma_j) \in \mathcal{X} \) so that for \( t > 0 \)

\[
\left\| \int_0^\infty e^{-i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - (\sigma_j t)^{-1/2} e^{-i\sigma_j t} \sum_{k=0}^{k_0-1} b_k(t\sigma_j)^{-k} \right\|_{\mathcal{X}} \\
\leq C \left( (\sigma_j t)^{-k_0} \sum_{k \leq 2k_0+1} \sup_{\tau} \|\sigma_j^k F^{(k)}(\tau)\|_{\mathcal{X}} + t^{-k_0} (1 + \sigma_j)^{k_0} \sum_{k \leq 2k_0+1} \sup_{\tau} \|F^{(k)}(\tau)\|_{\mathcal{X}} \right). \quad (A.1)
\]

Here \( b_0 = e^{-i\pi/4} F(0) \sqrt{2\pi} \), and the coefficients \( b_k \) are determined by the derivatives with respect to \( \tau \) of \( F(\sigma_j \tau)/\sqrt{\tau^2 + 1} \) of order at most \( 2k \), evaluated at \( \tau = 0 \). Moreover, under the same assumptions for \( t > 0 \)

\[
\left\| \int_0^\infty e^{i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda \right\|_{\mathcal{X}} \\
\leq C \left( (\sigma_j t)^{-k_0} \sum_{k \leq 2k_0+1} \sup_{\tau} \|\sigma_j^k F^{(k)}(\tau)\|_{\mathcal{X}} + t^{-k_0} (1 + \sigma_j)^{k_0} \sum_{k \leq 2k_0+1} \sup_{\tau} \|F^{(k)}(\tau)\|_{\mathcal{X}} \right). \quad (A.2)
\]

Proof. In order to simplify notation, we give the proof for \( \mathcal{X} = \mathbb{C} \), with the notation \( |\alpha| = \|\alpha\|_{\mathbb{C}} \). The proof for a general Banach space \( \mathcal{X} \) is essentially identical, though notationally more complicated.
We may write \( F(\tau_j(\lambda)) = F_e(\tau_j(\lambda)) + F_o(\tau_j(\lambda)) \), where
\[
F_e(\tau_j(\lambda)) = \frac{1}{2} \left( F(\tau_j(\lambda)) + F(-\tau_j(\lambda)) \right)
\]
\[
F_o(\tau_j(\lambda)) = \frac{1}{2} \left( F(\tau_j(\lambda)) - F(-\tau_j(\lambda)) \right).
\]
Now \( F_o(\tau_j(\lambda))/\tau_j(\lambda) \) is in fact a smooth function of \( \tau_j^2 = \lambda^2 - \sigma_j^2 \), and hence a smooth function of \( \lambda \). Then, integrating by parts,
\[
\left| \int_0^\infty e^{\pm i\lambda t} \frac{F_o(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda \right| \leq t^{-k_0} \left\| \frac{d^{k_0}}{d\lambda^{k_0}} F_o(\tau_j(\lambda)) \right\|_{L^1} \leq C t^{-k_0} (1 + \sigma_j)^{k_0} \sum_{k \leq 2k_0 + 1} \sup_{\tau} |D^k \tau F(\tau)|.
\]
To evaluate the integral \( \int_0^\infty e^{\pm it\lambda} \frac{F_e(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda \), we make a change of variables. For \( \lambda \in [\sigma_j, \infty) \), \( \tau_j(\lambda) \in [0, \infty) \) and we use the variable \( \tau' = \tau_j \); for \( \lambda \in [0, \sigma_j] \), \( \tau_j(\lambda) \in i[0, \infty) \) and we use the variable \( \tau' = -i\tau_j \). Hence
\[
\int_0^\infty e^{\pm it\lambda} \frac{F_e(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda = \int_0^\infty e^{\pm it\sqrt{\tau'^2 + \sigma_j^2}} \frac{F_e(\tau')}{\sqrt{\tau'^2 + \sigma_j^2}} d\tau' - i \int_0^\infty e^{\pm it\sqrt{\tau'^2 + \sigma_j^2}} \frac{F_e(i\tau')}{\sqrt{\tau'^2 + \sigma_j^2}} d\tau'. \tag{A.3}
\]
For the first integral on the right-hand side of (A.3) we perform a change of variable in order to be able to track dependence on \( \sigma_j \). Using \( \tau' = \sigma_j \tau \), we have
\[
\int_0^\infty e^{\pm it\sqrt{\tau'^2 + \sigma_j^2}} \frac{F_e(\tau')}{\sqrt{\tau'^2 + \sigma_j^2}} d\tau' = \int_0^\infty e^{\pm it\sigma_j \sqrt{\tau'^2 + 1}} \frac{F_e(\tau \sigma_j)}{\sqrt{\tau'^2 + 1}} d\tau = \frac{1}{2} \int_{-\infty}^\infty e^{\pm it\sigma_j \sqrt{\tau'^2 + 1}} F_e(\tau \sigma_j) d\tau \tag{A.4}
\]
where for the second equality we have used that the integrand is even in \( \tau \). For this integral, we may use the method of stationary phase. Note that the only stationary point is at \( \tau = 0 \). By [Hör90, Theorem 7.7.5], we have that there are constants \( \tilde{b}_{k \pm} \), depending on \( F_e \) and \( \sigma_j \), so that
\[
\left| \int_0^\infty e^{\pm it\sigma_j \sqrt{\tau'^2 + 1}} \frac{F_e(\tau \sigma_j)}{\sqrt{\tau'^2 + 1}} d\tau - (\sigma_j t)^{-1/2} e^{\pm i\sigma_j t} \sum_{k=0}^{k_0-1} \tilde{b}_{k \pm}(\sigma_j t)^{-k} \right| \leq C(\sigma_j t)^{-k_0} \sum_{|\alpha| \leq 2k_0} \sup_{\tau} \left| D^\alpha \left( \frac{F_e(\tau \sigma_j)}{\sqrt{\tau'^2 + 1}} \right) \right|. \tag{A.5}
\]
Moreover, the \( \tilde{b}_{k \pm} \) are determined by derivatives with respect to \( \tau \) of \( F_e(\sigma_j \tau)/\sqrt{\tau'^2 + 1} \) of order less than or equal to \( 2k \), evaluated at \( \tau = 0 \). The coefficient \( \tilde{b}_{0 \pm} = F(0) \sqrt{\pi/2} e^{\pm in/4} \). By allowing
the constant to depend on \( k_0 \), we may bound

\[
\sum_{|\alpha| \leq 2k_0} \sup_{\tau} \left| D_\tau^\alpha \left( \frac{F(e(\sigma_0 \tau))}{\sqrt{\tau^2 + 1}} \right) \right| \leq C k_0 \sum_{k \leq 2k_0} \sup_{\tau} \left| \sigma_j^k F^{(k)}(\tau) \right|.
\]

A similar computation gives a similar expansion for the second integral on the right-hand side of (A.3). We note that \( k = 0 \) coefficient for the expansion of the second term on the right-hand side of (A.3) (including the factor of \(-i\) in front) is \(-i\sqrt{\pi/2} F(0) e^{\pm i\pi/4}\). This finishes the proof of (A.1), and shows that the integral on the left in (A.2) has a similar expansion.

To complete the proof of (A.2) it suffices to show that the coefficients in the expansion are 0. We shall give two proofs of this. For the first, we return to (A.3), but with the “+” sign, writing

\[
\int_0^\infty e^{it\lambda} F(e(\tau j(\lambda))) d\lambda = \int_0^\infty e^{it\sqrt{\tau^2 + \sigma_j^2}} \frac{F(e(\tau'))}{\sqrt{\tau^2 + \sigma_j^2}} d\tau' - i \int_0^\infty e^{it\sqrt{\tau^2 + \sigma_j^2}} - \frac{F(e(it\tau'))}{\sqrt{\sigma_j^2 - (\tau')^2}} d\tau'.
\]

As previously for the second equality we have used that the integrands are even in \( \tau' \). Setting \( g(\tau') = ((\tau')^2 + \sigma_j^2)^{1/2} - (\sigma_j + (\tau')^2 / 2) \) we find from using the explicit expression for the stationary phase coefficients (see, for example, [Hör90, Theorem 7.7.5]) and summing the contributions from the two integrals that the coefficients in the asymptotic expansion of (A.6) are linear combinations of

\[
e^{i\pi/4} D^{2\nu}_{\tau'} \left( \frac{g^\mu(\tau') F(e(\tau'))}{\sqrt{(\tau')^2 + \sigma_j^2}} \right) \bigg|_{\tau' = 0} - i e^{-i\pi/4} (-1)^\nu D^{2\nu}_{\tau'} \left( \frac{g^\mu(i\tau') F(e(i\tau'))}{\sqrt{(i\tau')^2 + \sigma_j^2}} \right) \bigg|_{\tau' = 0}
\]

for \( \nu, \mu \in \mathbb{N}_0 \). Since if \( h \) is a smooth function in a complex neighborhood of the origin, \( D^{2\nu}_{\tau'} h(it\tau') \big|_{\tau' = 0} = (i)^{2\nu} D^{2\nu}_{\tau'} h(\tau') \big|_{\tau' = 0} \), we see that the quantity in (A.7) is 0, and the sum of the terms coming from the stationary phase expansions in (A.6) is 0.

\[\text{Figure 5. The contours of integration } \gamma_1 \text{ and } \gamma_2.\]
Now we outline an alternate, perhaps more intuitive, proof that the sum of the stationary phase coefficients in arising from the right-hand side of (A.6) is 0. The middle expression in (A.6) may be written

$$\int_{\gamma_1} e^{it\sqrt{z^2 + \sigma_j^2}} \frac{F_e(z)}{\sqrt{z^2 + \sigma_j^2}} \, dz \quad \text{(A.8)}$$

where $\gamma_1$ is as in Figure A: the path that goes down the positive imaginary axis to the origin, and then to infinity along the positive real axis. We understand the square root to be analytic in the first quadrant and to be positive on the positive real axis; this ensures $\text{Im} \sqrt{z^2 + \sigma_j^2} > 0$ in the first quadrant. If $F_e$ were analytic in a neighborhood of the origin, then by Cauchy’s Theorem we could write

$$\int_{\gamma_2} e^{it\sqrt{z^2 + \sigma_j^2}} \frac{F_e(z)}{\sqrt{z^2 + \sigma_j^2}} \, dz = \int_{\gamma_1} e^{it\sqrt{z^2 + \sigma_j^2}} \frac{F_e(z)}{\sqrt{z^2 + \sigma_j^2}} \, dz$$

where $\gamma_2$ is smooth, differs from $\gamma_1$ only in a suitably small neighborhood of the origin, and is contained in the closure of the first quadrant; see Figure A. Since for $t > 0$, $|e^{it\sqrt{z^2 + \sigma_j^2}}| \leq 1$ on $\gamma_2$ and (with suitably parameterized $\gamma_2$) the phase has no stationary points on $\gamma_2$, by repeated integration by parts we can see that the integral in (A.6) is $O(t^{-k})$, any $k \in \mathbb{N}$, as $t \to \infty$.

If $F_e$ is only smooth, not complex analytic, in a neighborhood of the origin, write $F_e = \tilde{\psi} T_{2k} + (F_e - \tilde{\psi} T_{2k})$ where $T_{2k}$ is the $2k$th Taylor polynomial of $F_e$ at 0 and $\tilde{\psi} \in C^\infty_c(\mathbb{C})$ is 1 in a neighborhood of the origin. Then the argument outlined above may be applied to the integral with $\tilde{\psi} T_{2k}$ if $\gamma_2$ differs from $\gamma_1$ only on the set where $\tilde{\psi}$ is 1. Since $(F_e - \tilde{\psi} T_{2k})$ vanishes to order $2k + 1$ at the origin, we may integrate by parts $k$ times to see that

$$\int_{\gamma_1} e^{it\sqrt{z^2 + \sigma_j^2}} \frac{F_e(z) - \tilde{\psi}(z) T_{2k}(z)}{\sqrt{z^2 + \sigma_j^2}} \, dz = O(t^{-k}).$$

The second argument for (A.2) also gives an intuitive reason for the difference between (A.1) and (A.2). In place of (A.8) we have instead for (A.1) the integral $\int_{\gamma_1} e^{-it\sqrt{z^2 + \sigma_j^2}} F_e(z)(z^2 + \sigma_j^2)^{-1/2} \, dz$. If $F_e$ is analytic near the origin, we can, as in the argument above, use a contour deformation argument to deform the contour of integration to $\gamma_2$. But for $z$ in the open first quadrant, $e^{-it\sqrt{z^2 + \sigma_j^2}}$ is exponentially increasing as $t \to \infty$. If we instead deform $\gamma_1$ to avoid the origin and the first quadrant, the deformed path must have portions in each of quadrants 2, 3, and 4. But $e^{-it\sqrt{z^2 + \sigma_j^2}}$ is exponentially increasing as $t \to \infty$ if $z$ is in the open third quadrant.

We state another lemma, with results similar to those of Lemma A.1. Note that this differs from Lemma A.1 in the domain of integration, the assumptions on where $F$ and $G$ are smooth, and the less explicit bound on the error.

**Lemma A.2.** Let $\mathcal{X}$ be a Banach space and let $\sigma_j > 0$. Let $F \in C^\infty_c([0, \sigma_j/2); \mathcal{X})$ and $G \in C^\infty_c(i[0, \sigma_j/2); \mathcal{X})$. Then given $k_0 \in \mathbb{N}$ there are $\alpha_{k, \pm}, \beta_{k, \pm} \in \mathcal{X}$, $k \in 0, 1, \ldots, 2k_0 - 2$, $C = \ldots$
\[ C(F, G, \sigma_j, k_0) > 0 \text{ so that} \]
\[
\left\| \int_{\sigma_j} e^{\pm i \lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - t^{-1/2} e^{\pm i \sigma_j} \sum_{k=0}^{2k_0-2} \alpha_{k, \pm} t^{-k/2} \right\| \leq C t^{-k_0}, \ t > 0 \tag{A.9}
\]
and
\[
\left\| \int_{0}^{\sigma_j} e^{\pm i \lambda t} \frac{G(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - t^{-1/2} e^{\pm i \sigma_j} \sum_{k=0}^{2k_0-2} \beta_{k, \pm} t^{-k/2} \right\| \leq C t^{-k_0}, \ t > 0. \tag{A.10}
\]

Here the \(\alpha_{k, \pm}\) (respectively \(\beta_{k, \pm}\)) are determined by \(\sigma_j\) and the derivatives of \(F\) (respectively \(G\)) of order at most \(k\), evaluated at 0.

**Proof.** We prove only (A.9), as the proof of (A.10) is almost identical. By introducing \(\tau = \tau_j(\lambda)\) as the variable of integration,
\[
\int_{\sigma_j}^{\infty} e^{\pm i \lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda = \int_{0}^{\infty} e^{\pm i t \sqrt{\tau^2 + \sigma_j^2}} \frac{F(\tau)}{\sqrt{\tau^2 + \sigma_j^2}} d\tau. \tag{A.11}
\]
Then an application of [Erd56, Section 2.9] proves (A.9), with coefficients \(\alpha_{k, \pm}\) determined by \(\sigma_j\) and derivatives of \(F\), evaluated at 0, of order at most \(k\). \(\square\)

**Acknowledgments.** The authors are grateful to the Simons Foundation for its support through the Collaboration Grants for Mathematicians program. It is a pleasure to thank Maciej Zworski for helpful conversations.

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