Elliptic stochastic quantization

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Abstract

We prove an explicit formula for the law in zero of the solution of a class of elliptic
SPDE in \( \mathbb{R}^2 \). This formula is the simplest instance of dimensional reduction, discovered in
the physics literature by Parisi and Sourlas (1979), which links the law of an elliptic SPDE
in \( d+2 \) dimension with a Gibbs measure in \( d \) dimensions. This phenomenon is similar to
the relation between an \( \mathbb{R}^{d+1} \) dimensional parabolic SPDE and its \( \mathbb{R}^d \) dimensional invariant
measure. As such, dimensional reduction of elliptic SPDEs can be considered a sort of elliptic
stochastic quantisation procedure in the sense of Nelson (1966) and Parisi and Wu (1981).

Our proof uses in a fundamental way the representation of the law of the SPDE as a su-
persymmetric quantum field theory. Dimensional reduction for the supersymmetric theory
was already established by Klein et al. (1984). We fix a subtle gap in their proof and also
complete the dimensional reduction picture by providing the link between the elliptic SPDE
and the supersymmetric model. Even in our \( d = 0 \) context the arguments are non-trivial
and a non-supersymmetric, elementary proof seems only to be available in the Gaussian case.

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sional reduction, Wiener space, supersymmetry, Euclidean quantum field theory

1 Introduction

Stochastic quantisation \([16,17]\) broadly refers to the idea of sampling a given probability distribution
by solving a stochastic differential equation (SDE). This idea is both appealing practically
and theoretically since simulating or solving an SDE is sometimes simpler than sampling or
studying a given distribution. If, in finite dimensions, this boils down mostly to the idea of the
Monte Carlo Markov chain method (which was actually invented before stochastic quantisation),
it is in infinite dimensions that the method starts to have a real theoretical appeal.

It was Nelson \([37,38,39]\) and subsequently Parisi and Wu \([43]\) who advocated the constructive
use of stochastic partial differential equations (SPDEs) to realize a given Gibbs measure for
the use of Euclidean quantum field theory (QFT). Indeed the original (parabolic) stochastic
quantisation procedure of [43] can be understood as the equivalence

$$E[F(\phi(t))] \propto \int F(\phi)e^{-S(\phi)}D\phi. \quad (1)$$

Here $F$ belongs to a suitable space of real-valued test functions, $D\phi$ is an heuristic “Lebesgue measure” on $\mathcal{S}'(\mathbb{R}^d)$, while on the left hand side the random field $\phi$ depends on $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and is a stationary solution to the parabolic SPDE

$$\partial_t \phi(t, x) + (m^2 - \Delta)\phi(t, x) + V'(\phi) = \xi, \quad (2)$$

where $\xi$ is a Gaussian white noise in $\mathbb{R}^{d+1}$, $V : \mathbb{R} \to \mathbb{R}$ a generic local potential bounded from below, $m^2$ a positive parameter, and $\phi(t)$ is the fixed time marginal of $\phi$ which has a law independent of $t$ by stationarity and on the right hand side we have the formal expression for a measure on functions on $\mathbb{R}^d$ with weight factor given by

$$S(\phi) := \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 + m^2|\phi(x)|^2 + V(\phi(x))dx. \quad (3)$$

Eq. (1) can be made mathematically precise and rigorous with standard tools from the theory of Markov processes [15, 30, 18], SDE/SPDEs [30, 11, 48] and Dirichlet forms [11], for example when $d = 0$, or when the equation is regularized appropriately and, in certain cases, for suitable renormalized versions of the SPDE [5, 3, 9, 11, 23, 24, 25, 20, 35, 2, 28] when $d = 1, 2, 3$. Let us note for example that in the full space it is easier to make sense of eq. (2) than of the formal Gibbs measure on the right hand side of (1), see [23].

In a slightly different context, and inspired by previous perturbative computations of Imry, Ma [27] and Young [51], Parisi and Sourlas [41, 42] considered the solutions of the elliptic SPDEs

$$(m^2 - \Delta)\phi + V'(\phi) = \xi \quad (4)$$

in $\mathbb{R}^{d+2}$ where $\xi$ is a Gaussian white noise on $\mathbb{R}^{d+2}$ and they discovered that its stationary solutions are similarly related to the same $d$ dimensional Gibbs measure. If we take $x \in \mathbb{R}^d$ then, they claimed that, for “nice” test functions $F$ (e.g. correlation functions) we have

$$E[F(\phi(x, y))] \propto \int F(\phi)e^{-4\pi S(\phi)}D\phi. \quad (5)$$

More precisely the law of the random field $(\phi(0, x, y))_{x, y \in \mathbb{R}^d}$, obtained by looking at the trace of $\phi$ on the hyperplane $\{x = (x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_1 = x_2 = 0\} \subset \mathbb{R}^{d+2}$, should be equivalent to that of the Gibbs measure formally appearing on the right hand side of (5) and corresponding to the action functional (3). Therefore one can interpret eq. (5) as an elliptic stochastic quantisation prescription in the same spirit of eq. (4).

When $V = 0$ one can directly check that the formula (5) is correct. Indeed in this case the unique stationary solution $\phi$ to the elliptic SPDE (4) is given by a Gaussian process with covariance

$$E[\phi(x)\phi(x')] = \int_{\mathbb{R}^{d+2}} \frac{e^{ik(x-x')}}{m^2 + |k|^2} \frac{dk}{(2\pi)^{d+2}}, \quad x, x' \in \mathbb{R}^{d+2}.$$  

Therefore for all $y, y' \in \mathbb{R}^d$ we have

$$E[\phi(0, y)\phi(0, y')] = \int_{\mathbb{R}^d} e^{ik(y-y')} \int_{\mathbb{R}^{2}} \frac{dq}{|q|^2 + m^2 + |k|^2} \frac{dk}{(2\pi)^{d+2}}.$$
\[
\int_{\mathbb{R}^2} \frac{dq}{(|q|^2 + 1)^2} \int_{\mathbb{R}^2} \frac{e^{ik(q-y')}}{m^2 + |k|^2 (2\pi)^{d+2}} \, dk = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{e^{ik(q-y')}}{m^2 + |k|^2 (2\pi)^{d}} \, dk
\]

where we performed a rescaling of the \(q\) integral in order to decouple the two integrations. The reader can easily check that the expression we obtained describes the covariance of the Gaussian random field formally corresponding to the right hand side of (5) for \(V = 0\).

While this last argument is almost trivial, a more general justification outside the Gaussian setting is not so obvious. The equivalence (5) was derived in [41, 42] at the theoretical physics level of rigor going through a representation of the left hand side via a supersymmetric quantum field theory (QFT) involving a pair of scalar fermion fields. This is one of the instances of the dimensional reduction phenomenon which is conjectured in certain random systems where the randomness effectively decreases the dimensionality of the space where fluctuations take place.

A crucial assumption is that the equation (4) has a unique solution, which is already a non-trivial problem for general \(V\). Parisi and Sourlas [42] observed that non-uniqueness can lead to a breaking of the supersymmetry, in which case the relation (5) could fail. So, part of the task of clarifying the situation is to determine under which conditions some relations in the spirit of (5) could anyway be true.

The dimensional reduction (5) of the elliptic SPDEs (4) seems less amenable to standard probabilistic arguments than its parabolic counterpart (4). Let us remark that from the point of view of theoretical physics it is possible [17, 42] to justify also dimensional reduction in the parabolic case (2) using a supersymmetric argument much like in the elliptic setting, the difference in dimensions is only due to the different number of fermions fields needed to represent the law of the SPDE as a quantum field theory.

The only attempt we are aware of to a mathematically rigorous understanding of the relation (5) is the work of Klein, Landau and Perez [31, 32, 33] (see also the related work on the density of states of electronic systems with random potentials [34]) which however do not fully prove eq. (5) but only the equivalence between the intermediate supersymmetric theory in \(d+2\) dimensions and the Gibbs measure in \(d\) dimensions. The reason for this limitation is that the problem of uniqueness of the elliptic SPDE seems to restrict unnecessarily the class of potentials for which (5) can be established and Klein et al. decided to bypass a detailed analysis of the situation by starting directly with the supersymmetric formulation. Their rigorous argument requires a cut-off, both on large momenta in \(d\) “orthogonal” dimensions and on the space variable in \(d+2\) dimensions in order to obtain a well defined, finite volume problem. This regularization breaks the supersymmetry which has to be recovered by adding a suitable correction term, spoiling the final result (see below). A subtle gap in their published proof is pointed out, and closed, in Section 4.

Let us remark that, in a different context, dimensional reduction has been proven and exploited in the remarkable work of Brydges and Imbrie on branched polymers [13, 12] and more recently by Helmut [26].

In the present work we try to bridge the gap and provide a proof of elliptic stochastic quantisation for the SPDE (4) in the \(d = 0\) case.

Fix \(d = 0\) and consider the two dimensional elliptic multidimensional SPDE

\[
(m^2 - \Delta)\phi(x) + f(x)\partial V(\phi(x)) = \xi(x) \quad x \in \mathbb{R}^2
\]

(6)
where $\phi = (\phi^1, \ldots, \phi^n)$ takes values in $\mathbb{R}^n$, $(\xi^1, \ldots, \xi^n)$ are $n$ independent Gaussian white noises, $V : \mathbb{R}^n \to \mathbb{R}$ a smooth potential function, $f(x) := \tilde{f}(|x|^2)$ with $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ a decreasing cut-off function, such that the derivative $f'$ of the function $r \mapsto f(r)$ is defined, tending to 0 at infinity, and $\partial V = (\partial_i V)_{i=1,\ldots,n}$ denotes the gradient of $V$. We will denote $f'(x) := \tilde{f}'(|x|^2)$.

Eq. (6) is the elliptic counterpart of the equilibrium Langevin reversible dynamics for finite dimensional Gibbs measures. Let us note that the elliptic dynamics is already described by an SPDE in two dimensions while in the parabolic setting one would consider a much simpler Markovian SDE [28, 2]. The question of uniqueness of solutions is however quite similar in difficulty, indeed it is non-trivial to establish uniqueness of stationary solutions to the SDE and much work in the theory of long time behavior of Markov processes is devoted precisely to this. In the elliptic context of (6) there is no (easy) Markov property helping and the question of uniqueness of weak stationary solutions seems is more open, even in the presence of the cut-off $f$.

What makes this $d = 0$ problem very interesting, is above all, the fact that while the statements we would like to prove are quite easy to describe (see below), to our surprise their rigorous justification is already quite involved and not quite yet complete in full generality. However we are now in a position to confirm that dimensional reduction is indeed at work for the two dimensional SPDE (6) under a subset of the following assumptions on $V$ and on the finite volume cut-off $f$:

**Hypothesis C. (convexity)** The potential $V : \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function such that

$$y \in \mathbb{R}^2 \mapsto V(y) + m^2 |y|^2$$

is strictly convex and it and its first and second partial derivatives grow at most exponentially at infinity.

**Hypothesis QC. (quasi convexity)** The potential $V : \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function, such that it and its first and second partial derivatives grow at most exponentially at infinity and such that there exists a function $H : \mathbb{R}^n \to \mathbb{R}$ with exponential growth at infinity such that we have

$$-\langle \hat{n}, \partial V(y + r\hat{n}) \rangle \leq H(y), \quad \hat{n} \in \mathbb{S}^{n-1}, y \in \mathbb{R}^n \text{ and } r \in \mathbb{R}_+,$$

with $\mathbb{S}$ is the $n-1$ dimensional sphere.

**Hypothesis CO. (cut-off)** The function $f$ is a real valued, has at least $C^2$ smoothness and in addition satisfies $f' \leq 0$, it decays exponentially at infinity and fulfills $\Delta(f) \leq b^2 f$ for $b^2 \ll m^2$ (some examples of such functions are given in [32]).

**Remark 1** The following families of functions satisfy Hypothesis QC:

- Smooth convex functions (since they satisfy the stronger Hypothesis C),
- Smooth bounded functions,
- Smooth functions having the second derivative semidefinite positive outside a compact set,
- Any positive linear combinations of the previous functions.
By weak solution to equation (6) we understand a probability measure \( \nu \) on fields \( \phi \) under which \( (m^2 - \Delta)\phi + \partial V(\phi) \) is distributed like Gaussian white noise on \( \mathbb{R}^2 \). A strong solution \( \phi \) to equation (6) is a measurable map \( \xi \mapsto \phi = \phi(\xi) \) satisfying the equation for almost all realizations of \( \xi \).

Define the probability measure \( \kappa \) on \( \mathbb{R}^n \) by

\[
\frac{d\kappa}{dy} := Z_\kappa^{-1} \exp \left[-4\pi \left( \frac{m^2}{2} |y|^2 + V(y) \right) \right],
\]

where \( Z_\kappa := \int_{\mathbb{R}^n} \exp \left[-4\pi \left( \frac{m^2}{2} |y|^2 + V(y) \right) \right] dy. \)

The main result of this paper is the following theorem which states that on very general conditions on \( V \) there is always a weak solution which satisfies (an approximate) elliptic stochastic quantisation relation.

**Theorem 1** Under the Hypotheses QC and CO there exists (at least) one weak solution \( \nu \) to equation (6) such that for all measurable bounded functions \( h : \mathbb{R}^n \to \mathbb{R} \) we have

\[
\int_W h(\phi(0))\Upsilon_f(\phi)\nu(d\phi) = Z_f \int_{\mathbb{R}^n} h(y)d\kappa(y)
\]

where \( \Upsilon_f(\phi) := e^{4\int_{\mathbb{R}^2} f'(x)V(\phi(x))dx} \) and \( Z_f := \int_W \Upsilon_f(\phi)\nu(d\phi) \). \( W \) is a suitable Banach space of functions from \( \mathbb{R}^2 \) to \( \mathbb{R}^n \) where \( \nu \) is defined (see Section 2).

The drawback of this result is the lack of constructive determination of the weak solution \( \nu \) for which the dimensional reduction described by eq. (8) is realized. This is of course linked with the possible non-uniqueness of the strong solution to (6). The fact that non-uniqueness is related to a possible breaking of the supersymmetry associated with (4) was already noted by Parisi and Sourlas [42]. If we are willing to assume that the potential is convex we can be more precise, as the following theorem shows.

**Theorem 2** Under Hypothesis C and CO there exists an unique strong solution \( \phi = \phi(\xi) \) of eq. (6) and for all measurable bounded functions \( h : \mathbb{R}^n \to \mathbb{R} \) we have

\[
\mathbb{E}[h(\phi(0))\Upsilon_f(\phi)] = Z_f \int_{\mathbb{R}^n} h(y)d\kappa(y)
\]

where \( Z_f := \mathbb{E}[\Upsilon_f(\phi)] \) and where \( \mathbb{E} \) denotes expectation with respect to the law of \( \xi \).

Both theorems require the presence of a suitable cut-off \( f \neq 1 \) which is responsible for the weighting factor \( \Upsilon_f(\phi) \) on the left hand side of the dimensional reduction statements (8) and (9). If we would be allowed to take \( f = 1 \) then we would have proven the \( d = 0 \) version of eq. (5). However, presently we are not able to do this for all QC potentials but only for those satisfying Hypothesis C (see Sect 4 for the proof). This is the first rigorous result on elliptic stochastic quantisation without any cut-off. In fact in this case the results of Klein, Landau and Perez [32], using only an integral representation of the solution to equation (6), it does not hold.

**Theorem 3** Suppose that \( V \) satisfies Hypothesis C and let \( \phi \) be the unique strong solution in \( C_0^{\exp,\beta}(\mathbb{R}^2;\mathbb{R}^n) \) (see Section 5 for the definition of this space) of equation

\[
(m^2 - \Delta)\phi + \partial V(\phi) = \xi.
\]
Remark 2 It is easy to generalize Theorems 1, 2 and 3 to equations of the form
\[ (m^2 - \Delta)\phi(x) + \sum_{r=1}^{n} \gamma^r_i \phi(x) \partial_{\phi^r} V(\phi(x)) = \gamma_i \xi(x), \]
where \( f \) is as before, \( \Gamma = (\gamma^r_i)_{i,j=1,\ldots,n} \) is an \( n \times n \) invertible matrix and the Hypothesis C and QC generalized accordingly.

Plan. The paper is organized as follows. In Section 2 we study the strong and weak solutions of eq. (6) and also the representation of weak solutions via the theory of transformation of measures on Wiener space developed by Üstünel and Zakai in [49] whose setting and main facts needed here are summarized in Appendix A. Section 3 is devoted to the proof of our results about elliptic stochastic quantisation. The supersymmetric approach to dimensional reduction is detailed in [40]. Finally, Section 4 discusses the proof of Theorem 3 on the cut-off removal.

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2 The elliptic SPDE

In order to study equation (5) we have to recall some definitions, notations and conventions. Fix an abstract Wiener space \((W, \mathcal{H}, \mu)\) where the noise \( \xi \) is defined (for the concept of abstract Wiener space we refer e.g. to [22, 40, 41]). The Cameron-Martin space \( \mathcal{H} \) is the space
\[ \mathcal{H} := L^2(\mathbb{R}^2; \mathbb{R}^n), \]
with its natural scalar product and natural norm given by \( \langle h, g \rangle = \int_{\mathbb{R}^2} h(x)g(x)dx \). Let \( \mathcal{W} \) (in which \( \mathcal{H} \) is densely embedded) be the space
\[ \mathcal{W} = \mathcal{W}_{p,\eta} := W^{p-1-2\epsilon}_{\eta}(\mathbb{R}^2; \mathbb{R}^n) \cap (1-\Delta)(C^0_{\eta}(\mathbb{R}^2; \mathbb{R}^n)), \]
where \( p \geq 1, \eta > 0 \) and \( W^{p-1-2\epsilon}_{\eta}(\mathbb{R}^2; \mathbb{R}^n) \) is a fractional Sobolev space with norm
\[ \|g\|_{W^{p-1-2\epsilon}_{\eta}} := \left( \int_{\mathbb{R}^2} (1 + |x|^{-\eta}) \left( (1-\Delta)^{-\frac{\epsilon}{2}}(g) \right)^p dx \right)^{\frac{1}{p}}, \]
for some \( \epsilon > 0 \) small enough and \((1-\Delta)(C^0_{\eta}(\mathbb{R}^2; \mathbb{R}^n))\) is the space of the second order distributional derivatives of continuous functions on \( \mathbb{R}^n \) growing at infinity at most as \( |x|^\eta \) with norm
\[ \|g\|_{(1-\Delta)^{-1}(C^0_{\eta})} := \| (1 + |x|^{-\eta})((1-\Delta)^{-1}g)(x) \|_{L^\infty}. \]
Thus \( \mathcal{W}_{p,\eta} \) is a Banach space with norm given by the sum of the norms of \( W^{p-1-2\epsilon}_{\eta}(\mathbb{R}^2; \mathbb{R}^n) \) and of \((1-\Delta)^{-1}(C^0_{\eta}(\mathbb{R}^2; \mathbb{R}^n))\). In the following we usually do not specify the indices \( \eta \) and \( p \) in the definition of \( \mathcal{W}_{p,\eta} \) and we write only \( \mathcal{W} \). We also introduce the notation \( \mathcal{V} := (1-\Delta)^{-1}(\mathcal{W}) \).
The Gaussian measure $\mu$ on $\mathcal{W}$ is the standard Gaussian measure with Fourier transform given by $e^{-\frac{1}{2}\|\cdot\|_2^2}$. The white noise $\xi$ is then naturally realized on $(\mathcal{H}, \mathcal{W}, \mu)$, in the sense that $\xi$ is the random variable $\xi : \mathcal{W} \to S'(\mathbb{R}^2; \mathbb{R}^n)$ (where $S'(\mathbb{R}^2; \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued Schwartz distributions on $\mathbb{R}^2$) defined as $\xi(w) = \text{id}_{\mathcal{W}'}(w) = w$. In this way the law of $\xi$ is simply $\mu$ (or, better, it is equal to the pushforward of $\mu$ on $S' := S'(\mathbb{R}^2, \mathbb{R}^n)$ with respect the natural inclusion map of $\mathcal{W}$ in $S'$).

Sometimes it is also useful to consider the space $C^n_{\alpha}$, i.e. the space of $\alpha$-Hölder continuous functions such that they and their derivatives (or Hölder norms) grow at infinity at most like $|x|^\alpha$ for the real number $\tau$ (this notation is used also if $\tau$ is negative in that case the functions decrease at least like $\frac{1}{|x|^\alpha}$). It is important to note that $C^n_{\alpha}$ can be identified with the Besov space $B^{\alpha}_{\infty, \infty}(\mathbb{R}^2)$ (see [5] Chapter 2 Section 2.7). It is also important to realize that $(1-\Delta)^{-1}(\mathcal{W}) \subset C^n_{\alpha}$ if we choose $p$ big enough and $\alpha > 0$ small enough.

We introduce now two notions of solutions for equation (15). For later convenience it is better to discuss the equation in term of the variable $\eta := (m^2 - \Delta)\phi$ for which it reads

$$\eta + f \partial V(\mathcal{I}\eta) = \eta + U(\eta) = \xi,$$

(13)

where

$$\mathcal{I} := (m^2 - \Delta)^{-1}$$

and where we introduced the map $U : \mathcal{W} \to \mathcal{H}$ given by

$$U(w) := f \partial V(\mathcal{I}w), \quad w \in \mathcal{W}. \quad (14)$$

It is simple to prove that under the Hypothesis C or QC, we have indeed $U(w) \in \mathcal{H}$. Furthermore we introduce the map $T : \mathcal{W} \to \mathcal{W}$ as $T(w) := w + U(w)$. It is clear that a map $S : \mathcal{W} \to \mathcal{W}$ satisfies the equation (13), i.e. $T(S(w)) = \xi(w) = w$, for ($\mu$-)almost all $w \in \mathcal{W}$, if and only if $T\xi(w)$ satisfies equation (15). The law $\nu$ on $\mathcal{W}$ associated to a solution of equation (13) must satisfy the relation $T\nu = \mu$. For these reasons we introduce the following definition.

**Definition 1** A measurable map $S : \mathcal{W} \to \mathcal{W}$ is a strong solution to equation (13) if $T \circ S = \text{Id}_{\mathcal{W}}$ $\mu$-almost surely. A probability measure $\nu \in \mathcal{P}(\mathcal{W})$ (where $\mathcal{P}(\mathcal{W})$ is the space of probability measures on $\mathcal{W}$) on the space $\mathcal{W}$ is a weak solution to equation (13) if $T\nu = \mu$.

### 2.1 Strong solutions

In order to study the existence of strong solutions to equation (15) we introduce an equivalent version of the same equation that is simpler to study. Indeed if we write

$$\tilde{\phi} = \phi - \mathcal{I}\xi,$$

and we suppose that $\phi$ satisfies equation (15), then the function $\tilde{\phi}$ satisfies the following (random) PDE

$$(m^2 - \Delta)\tilde{\phi} + f \partial V(\tilde{\phi} - \mathcal{I}\xi) = 0. \quad (15)$$

Equation (15) can be studied pathwise for any realization of the random field $\mathcal{I}\xi$. Hereafter the symbol $\lesssim$ stands for inequality with some positive constant standing on the right hand side.

**Lemma 1** Suppose that $V$ satisfies Hypothesis QC, and let $\tilde{\phi}$ be a classical $C^2$ solution to the equation (15), such that $\lim_{x \to \infty} \tilde{\phi}(x) = 0$, then for any $0 < \tau < 2$ and $\eta > 0$ we have

$$\|\tilde{\phi}\|_{\infty} \lesssim 1 + \|fe^{\alpha_1 \mathcal{I}\xi}\|_{\infty} \quad (16)$$

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\[
\|\hat{\phi}\|_{C^{2-\tau}_{-}} \lesssim 1 + e^{\alpha_1 \|\hat{\phi}\|_{\infty} \|(|x| + 1)^{\eta} (t e^{\alpha_1 |\mathcal{I}|})\|_{\infty}}, \tag{17}
\]
for some positive constant \(\alpha_1\) and where \(\hat{\phi}\) and the constants involved in the symbol \(\lesssim\) depend only on the function \(H\) in Hypothesis QC.

**Proof** Putting \(r_\phi(x) = \sqrt{\sum_i (\phi_i(x))^2} = |\phi(x)|\), \(x \in \mathbb{R}^2\), since the \(C^2\) function \(\phi\) converge to zero at infinity, the function \(r_\phi\) must have a global maximum at some point \(\bar{x} \in \mathbb{R}^2\). This means that 
\[-\Delta (r_\phi^2)(\bar{x}) \geq 0.\]
On the other hand since \(\phi\) solves equation (15) we have
\[
m^2 r_\phi^2(\bar{x}) \leq -\frac{1}{2} \Delta (r_\phi^2)(\bar{x}) + m^2 r_\phi^2(\bar{x}) \leq (-\phi \cdot \Delta \phi - |\nabla \phi|^2 + m^2 |\phi|^2) \leq -f(\bar{x}) r_\phi(\bar{x}) (\hat{n}_\phi(\bar{x}) \cdot \partial V(I\xi(\bar{x}) + \hat{n}_\phi(\bar{x}) r_\phi(\bar{x})))
\]
where \(\hat{n}_\phi = \frac{x}{|x|} \in \mathbb{R}^k\) when \(\phi \neq 0\), and 0 elsewhere. Using Hypothesis QC we obtain
\[
\|r_\phi\|_{\infty} \leq \frac{f(\bar{x}) H(I\xi(\bar{x}))}{m^2} \lesssim 1 + \|f e^{\alpha_1 |\mathcal{I}|}\|_{\infty},
\]
since \(H\) grows at most exponentially at infinity. This result implies inequality (16). The bound (17) can be obtained directly using the inclusion properties of Besov spaces. \(\square\)

**Remark 3** It is simple to prove that the inequalities (16) and (17) can be chosen to be uniform with respect to some rescaling of the potential of the form \(AV\), or satisfying Hypothesis \(V_\lambda\) below where \(\lambda \in [0, 1]\).

In the following we denote by \(\mathcal{F} : \mathcal{W} \to \mathcal{P}(C^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n))\) the set valued function which associates to a given \(w \in \mathcal{W}\) the (possible empty) set of solutions to equation (15) in \(C^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n)\), where \(\tau > 0\), when \(I\xi\) is evaluated in \(w\).

**Theorem 4** For any \(w \in \mathcal{W}\) the set \(\mathcal{F}(w)\) is non-empty and closed. Furthermore \(\mathcal{F}(w) \subset C^2(\mathbb{R}^2; \mathbb{R}^n)\) and if \(B \subset \mathcal{W}\) is a bounded set then \(\mathcal{F}(B) = \bigcup_{w \in B} \mathcal{F}(w)\) is compact in \(C^{2-\tau}_{\eta}(\mathbb{R}^2; \mathbb{R}^n)\) for any \(\tau > 0\) and \(\eta > 0\).

**Proof** We introduce the map \(\mathcal{C}(\mathbb{R}^2; \mathbb{R}^n) \times \mathcal{W} \ni (\phi, w) \mapsto K(\phi, w) \in C^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n)\), given by
\[
K(\phi, w) := -\mathcal{I}(f \partial V(\phi + I\xi(w))).
\]
The map \(K\) is continuous with respect to its first argument, indeed if \(\phi, \phi_1 \in \mathcal{C}(\mathbb{R}^2; \mathbb{R}^n)\),
\[
\|K(\phi, w) - K(\phi_1, w)\|_{C^{2-\tau}_{-}} \lesssim \|(|x| + 1)^{\eta} f(\partial V(\phi, I\xi(w)) - \partial V(\phi_1, I\xi(w)))\|_{\infty}
\]
\[
\lesssim \left\| \int_0^1 \left(\|(|x| + 1)^{\eta} f(\partial^2 V(\hat{n} - t(\hat{n} - \hat{n}_1) + \mathcal{I}(w)) \cdot (\hat{n} - \hat{n}_1)dt\right)_{\infty}
\lesssim \|\hat{n} - \hat{n}_1\|_{\infty} \|(|x| + 1)^{\eta} \sqrt{\mathcal{I}}\|_{\infty} \left(\|\partial^2 V\|_{\infty} + e^{\alpha_1 \|\hat{n} - \hat{n}_1\|_{\infty} \|\sqrt{\mathcal{I}}\|_{\infty}}\right),
\]
where the positive constant \(\alpha\) depends on the exponential growth of \(\partial^2 V\) at infinity. By a similar reasoning we can prove that \(K\) sends bounded sets of \(C^{2-\tau}_{\eta}\) into bounded sets of \(C^{2-\tau}_{\eta'}\), where \(\tau' < \tau\) and \(\eta' > \eta\). Since the immersion \(C^{2-\tau}_{\eta'} \hookrightarrow C^{2-\tau}_{\eta}\) is compact we have that \(K\) is a compact map.
Since $I\xi \in C_0^1$, it is simple to prove, using a bootstrap argument, that if $\tilde{\phi} = K(\phi, w)$ then $\tilde{\phi} \in C^2(\mathbb{R})$. From this fact it follows that, using inequalities (13) and (17) of Lemma 1 and Remark 3, the solutions to the equation $\tilde{\phi} = K(\phi, w)$ are uniformly bounded for $\lambda \in [0,1]$. Thanks to these properties of the map $K$ we can use Schaefer’s fixed-point theorem (see Theorem 4 Section 9.2 Chapter 9) to prove the existence of at least one solution to the equation (13).

Finally using again Lemma 1 we have that $F(B)$ is compact for any bounded set $B \subset W$. □

Theorem 5 There exists a strong solution to equation (13) (or equivalently of (15)).

Proof For proving the existence of a strong solution to the equation (13) (in the sense of Definition 1) it is sufficient to prove that we can choose the solutions to equation (15), whose existence for any $w \in W$ is guaranteed by Theorem 3 in a measurable way with respect $w \in W$.

More precisely we have to prove that there exists a measurable selection for the function set map $F$, i.e. there exists a map $\tilde{S} : W \to C_{-2}\mathbb{R}$ such that $\tilde{S}(w) \in F(w)$.

Fix a sequence of balls $B_1, \ldots, B_n, \ldots \subset W$ of increasing radius and such that $\lim_{n \to +\infty} B_n = W$, then, by Theorem 4 the map $F|_{B_n \setminus B_{n-1}}$ takes values in a compact set. Since $K$ (introduced in the proof of Theorem 1) is a Carathéodory map (since, as it is proven in the proof of Theorem 4 $K$ is continuous in $\phi$ and measurable in $w$) by Filippov’s implicit function theorem (see Theorem 18.17 in [6]), there exists a (Borel) measurable function $S_n$ defined on $B_n \setminus B_{n-1}$ such that $S_n(w) \in F(w)$. The map $S$ defined on $B_n \setminus B_{n-1}$ by $S|_{B_n \setminus B_{n-1}} = S_n$ is the measurable selection that we need (since $B_n \setminus B_{n-1}$ is measurable).

A strong solution $S$ to equation (13) is then given by $S(w) := w + (m^2 - \Delta)S(w), w \in W$. □

Corollary 1 Under the Hypothesis C there exists only one strong solution to equation (13).

Proof Suppose that $S_1, S_2$ are two strong solutions to equation (13) then, putting $\phi_j(x, w) = I(S_j(w)(x)), j = 1, 2$, writing $\delta \phi(x, w) = \phi_1(x, w) - \phi_2(x, w)$ and $\delta r(x, w) = \sqrt{\sum_{i=1}^{2m}(\delta \phi_i(x, w))^2}$, we obtain

$$(m^2 - \Delta)(\delta r^2) + 2 \sum_i (|\nabla \delta \phi_i|^2 + r \delta r [\hat{n}_{\delta \phi} \cdot (\partial V(\phi_1) - \nabla V(\phi_2))]) = 0.$$ 

By Lagrange’s theorem there exists a function $g(x), x \in \mathbb{R}^2$, taking values in the segment $[\phi_1(x), \phi_2(x)] \subset \mathbb{R}$ such that $\hat{n}_{\delta \phi} \cdot (\partial V(\phi_1) - \nabla V(\phi_2)) = \delta r \partial^2 V(g(x))(\hat{n}_{\delta \phi}, \hat{n}_{\delta \phi})$. From this fact we obtain

$$(m^2 - \Delta)(\delta r^2) + (\partial^2 V(g(x))(\hat{n}_{\delta \phi}, \hat{n}_{\delta \phi}))\delta r^2 \leq 0.$$ 

Since $m^2 + \partial^2 V(g(x))(\hat{n}_{\delta \phi}, \hat{n}_{\delta \phi})$ is positive $y \mapsto V(y) + m^2|y|^2$ being convex by our Hypothesis C, and $\delta r(x)$ is positive and goes to zero as $x \to +\infty$, we have $\phi_1 = \phi_2$ and so $S_1(w) = S_2(w)$. □

2.2 Weak solutions

First of all we prove that the map $U$, given by (14), is a $H - C^1$ function (in the sense of [19], see Appendix 4) for the abstract Wiener space $(W, \mathcal{H}, \mu)$.

Proposition 1 If $V$ and its derivatives grow at most exponentially at infinity the map $U$ is a $H - C^1$ function, on the abstract Wiener $(W, \mathcal{H}, \mu)$ and we have

$$\nabla U^i(w)[h] = f(x)\partial_{\phi^i}V(x) \cdot I(h^i).$$
Furthermore $U$ is $C^2$ Fréchet differentiable as a map from $W$ into $H$.

**Proof** The proof is essentially based on the fundamental theorem of calculus and the use of the Fourier transform. In order to give an idea of the proof we prove only the most difficult part, namely that $\nabla U$ is continuous with respect to translations by elements of $H$, where continuity is understood with respect to the Hilbert-Schmidt norm for operators acting on $H$.

For fixed $w \in W$, $h, h' \in H$ we have

$$\nabla U^j(w + h')[h] - \nabla U^j(w)[h] = f(x) \int_0^1 \partial^3_{\phi,\phi,\phi} V((m^2 - \Delta)^{-1}(w + th')) \cdot \mathcal{I}(h') \cdot \mathcal{I}(h'') dt,$$

where the sum over $j, r = 1, \ldots, n$ is implied. We recall that the Hilbert-Schmidt norm of an integral kernel is the integral of the square of the kernel. In our case the Fourier transform of the integral kernel representing the difference (18) is given by

$$\hat{K}_j^i(k, k') = \sum_{r = 1}^n \int_{\mathbb{R}^4} \hat{V}^i_{\phi,\phi,\phi}(k - k_1) \cdot \hat{h}^r(k_1 - k_2) \cdot \frac{dk_1 dk_2}{(\|k_2 - k'|^2 + m^2)(2\pi)^4},$$

where $\hat{V}^i_{\phi,\phi,\phi}(k, l)$ is the Fourier transform of $\partial_{\phi,\phi,\phi} V(I_w + th')$, $t \in [0, 1]$. It is simple to prove that

$$\|\nabla U(w + h')[\cdot] - \nabla U(w)[\cdot]\|_2^2 \lesssim \int_{\mathbb{R}^4} \hat{K}_j^i(k, k') \hat{K}_j^i(k', k) dk dk' \lesssim \|\sqrt{f} e^{\alpha|I_w| + \alpha|\hat{h}'|}\|_2 \|\sqrt{\mathcal{I}}\|_2 \|h'\|_H^2,$$

where $\alpha$ depends on the exponential growth of $\partial^3 V$. Since $\|\sqrt{f} e^{\alpha|I_w| + \alpha|\hat{h}'|}\|_\infty$ is always finite in $W$ (for $\eta$ positive and small enough) we have proved the continuity of the map $h' \mapsto \nabla U(w + h')$ with respect to the Hilbert-Schmidt norm.

Define the measurable map $N : W \to \mathbb{N} \cup \{+\infty\}$

$$N(w) := \text{number of solutions } y \in W \text{ to the equation } T(y) = w,$$

moreover let $M \subset W$ be the set of zeros of $\det(I_H + \nabla U(w))$.

**Theorem 6** The function $N$ is greater or equal to 1 and it is $\mu$-almost surely finite. Furthermore the map $T$ is proper and $N$ is ($\mu$-)almost surely finite.

**Proof** We define $T(\phi, w) = \phi + U(\phi + w)$. Obviously we have that $z$ is a solution to the equation $T(z) = w$ if and only if $\phi = z - w$ is a solution to the equation $T(\phi, w) = 0$. On the other hand $\phi$ is solution to the equation $T(\phi, w) = 0$ if and only if $\tilde{\phi} = T(\phi)$ is a solution to equation (15). By Theorem 4, equation (15) has at least one solution for any $w \in W$ and so $N(w) \geq 1$ for any $w \in W$.

Let $K$ be a compact set in $W$ we have that $T^{-1}(K) \subset K + (m^2 - \Delta)(F(K))$ (where $F$ is the set valued map introduced in Theorem 4). Since $K$ is compact, by Theorem 4 $F(K)$ is compact in $C^2_{\eta}$, which implies that $(m^2 - \Delta)(F(K))$ is compact in $C^0_{\eta}$. Since the immersion $C^0_{\eta} \hookrightarrow W$ is compact and the sum of two compact sets is compact, we obtain that $T$ is a proper map.

Since by Proposition 5 $\mu(T(M)) = 0$, for proving the theorem it is sufficient to prove that $N(w) < +\infty$ for $w \in T(M)$. If $w \in T(M)$ then $\text{id}_W + \nabla U(w)|_H$ is a linear invertible operator on $H$ and so $\text{id}_W + \nabla U(w)$ is a linear invertible operator on $W$. By the Implicit Function Theorem, we have that $T$ is a $C^4$ diffeomorphism between a neighborhood $B_w$ of $w$ onto $T(B_w)$. This
implies that the set \( T^{-1}(w) \) consists of isolated points. Since the map \( T \) is proper, this means that \( T^{-1}(w) \) is a compact set made only by isolated points which implies that \( T^{-1}(w) \) is a finite set.

**Theorem 7** A probability measure \( \nu \) is a weak solution to equation (13) if and only if it is absolutely continuous with respect to \( \mu \) and there exists a non-negative function \( A \in L^\infty(\mu) \) such that \( \sum_{y \in T^{-1}(w)} A(y) = 1 \) for \( \mu \)-almost all \( w \in \mathcal{W} \) and \( \frac{\partial}{\partial \mu} = A|\Lambda_U| \) with

\[
\Lambda_U(w) := \det_2(I + \nabla U(w)) \exp \left( -\delta(U)(w) - \frac{1}{2} \|U(w)\|_H^2 \right),
\]

where \( \delta(U) \) the Skorokhod integral of the map \( U \) and where \( \det_2 \) denotes the regularized Fredholm determinant (see [17] Chapter 9).

**Proof** Recall that, by Proposition 5 \( \mu(T(M)) = 0 \). This implies that for any weak solution \( \nu \) we have \( \nu(T^{-1}(T(M))) = 0 \). Letting \( \mathcal{W}_n := T^{-1}(N = n) \cap T^{-1}(T(M)) \) we deduce that \( \nu(\cup_n \mathcal{W}_n) = \sum_n \nu(\mathcal{W}_n) = 1 \) and therefore if we prove that \( \nu \) is absolutely continuous with respect to \( \mu \) on each \( \mathcal{W}_n \) we have proved that \( \nu \) is absolutely continuous with respect to \( \mu \).

Using \( n \) times iteratively the Kuratowski-Ryll-Nardzewski selection theorem (Theorem 18.13 in [6]) due to the fact that \( T^{-1}(x) \cap \mathcal{W}_n \) is composed by zero or \( n \) elements, we can decompose the set \( \mathcal{W}_n \) into \( n \) measurable subsets \( \mathcal{W}_1, \ldots, \mathcal{W}_n \) where the map \( T|_{\mathcal{W}_i} \) is invertible. This means that if \( \Omega \subset \mathcal{W}_n \) we have \( \nu(\Omega \cap \mathcal{W}_i) \leq \mu(T(\Omega)) \). On the other hand we have that \( \mu(T(\Omega)) = \int_{\Omega \cap \mathcal{W}_i} |\Lambda_U|d\mu \). This implies that if \( \mu(\Omega) = 0 \) then \( \nu(\Omega \cap \mathcal{W}_i) \leq \mu(T(\Omega)) = \int_{\Omega \cap \mathcal{W}_i} |\Lambda_U|d\mu = 0 \). As a consequence \( \nu(\Omega) = \sum_1 \nu(\Omega \cap \mathcal{W}_i) = 0 \) and \( \nu \) is absolutely continuous with respect to \( \mu \).

**Theorem 14** below implies that for any measurable positive functions \( f, A \) we have

\[
\int f \circ T(w) A(w)|\Lambda_U(w)|d\mu = \int f(w) \left( \sum_{y \in T^{-1}(w)} A(y) \right) d\mu.
\]

Taking \( f = \mathbb{1}_{T(M)} \) and \( A = 1 \) we deduce that \( \int_{T^{-1}(T(M))} |\Lambda_U|d\mu = \mu(T(M)) = 0 \). Therefore we can suppose that there exists a specific non-negative function \( A \) such that \( d\nu = A|\Lambda_U|d\mu \) and since \( T_1(\nu) = \mu \) we must have

\[
\int f(w)d\mu = \int f \circ T(w)d\nu = \int f \circ T(w)A(w)|\Lambda_U(w)|d\mu,
\]

for any bounded measurable function \( f \). From a comparison of this formula with (19) we deduce that \( \sum_{y \in T^{-1}(w)} A(y) = 1 \) for (\( \mu \))-almost all \( w \in \mathcal{W} \).

On the other hand, using again Theorem 14 it is simple to prove that if \( d\nu = A|\Lambda_U|d\mu \) and \( \left( \sum_{y \in T^{-1}(w)} A(y) \right) = 1 \) then \( \nu \) is a weak solution to equation (13).

**Remark 4** If \( S \) is any strong solution to equation (13) then \( \nu = S_*\mu \) is a weak solution. Furthermore it is simple to prove that the weak solutions of the form \( S_*\mu \), where \( S \) is some strong solution to (9), are the extremes of the convex set \( \mathcal{W} := \{ \nu \text{ satisfying } T_*\nu = \mu \} \). Using a lemma (precisely Lemma 2) that we shall prove below, it follows from this that \( \mathcal{W} \) is weakly compact and thus, by Krein–Milman theorem (see Theorem 3.21 in [15]), any measure \( \nu \in \mathcal{W} \) can be written as convex combination of measures induced by strong solutions.
Corollary 2 If $V$ satisfies Hypothesis C there exists only one weak solution $\nu$ to equation (13) and we have that $\frac{d\nu}{dm} = |\Lambda_U|$ and $\nu = S_\ast \mu$ (where $S$ is the only strong solution to equation (13) and $\Lambda_U$ is as in Theorem 7).

Proof If $V$ satisfies Hypothesis C, by Corollary 1, $T$ is invertible and by Theorem 7 we have that $\nu$ is unique and $\frac{d\nu}{dm} = |\Lambda_U|$. By Remark 4 we have that $S_\ast \mu$, where $S$ is the unique strong solution of (13), is the unique weak solution to the same equation. ♦

3 Elliptic stochastic quantisation

In this section we want to prove the dimensional reduction of equation (6), namely that the law in 0 of at least a (weak) solution to equation (13), has an explicit expression in terms of the potential $V$.

The original idea of Parisi and Sourlas [41] for proving this relations was to transform expectations involving the solution $\phi$ to equation (6) (taken at the origin) into an integral of the form

$$E[h(\phi(0))] = \int h(Iw(0)) \det(I + \nabla U(Iw)) e^{-\langle U(Iw), Iw \rangle - \frac{1}{2} \|U(Iw)\|^2} d\mu(w),$$

where $U$ is precisely defined in equation (14). Then one can express the determinant on the right hand side of (20) as the exponential $e^{\int V(\Phi) dx d\theta d\bar{\theta}}$ involving the superfield

$$\Phi(x, \theta, \bar{\theta}) = \varphi(x) + \psi(x) \theta + \bar{\psi}(x) \bar{\theta} + \omega(w),$$

(see Section 4.1 for a more precise description) constructed from the real Gaussian free field $\varphi$ over $\mathbb{R}^2$, two additional fermionic (i.e. anticommuting) fields $\psi, \bar{\psi}$ and the complex field $\omega$. Introducing these new anticommuting fields it can be easily proven that the integral (20) admits an invariance property with respect to supersymmetric transformations. This implies the dimensional reduction, i.e.

$$\int h(Iw(0)) \det(U(Iw)) d\nu(w) = \int h(Iw(0)) \det(U(Iw)) d\mu(w) = Z_f \int_{\mathbb{R}^n} h(y) d\kappa(y).$$

Unfortunately this reasoning is only heuristic since the integral on the right hand side of (20) is not well defined without a spatial cut-off, given that both the determinant and the exponential are infinite.

For polynomial potentials $V$, a rigorous version of this reasoning was proposed by Klein et al. [32]. More precisely Klein et al. give a rigorous proof of the relationship (20) introducing a suitable modification due to the presence of the spatial cut-off $f$, but they do not discuss the relationship between equation (6) and the reduction (20).

The goal of this section is to prove the following theorem.

Theorem 8 Under the Hypotheses CO and QC there exists (at least) one weak solution $\nu$ to equation (6), such that for any measurable bounded function $h$ defined on $\mathbb{R}^n$ we have

$$\int_{\mathcal{W}} h(Iw(0)) \Upsilon_f(Iw) d\nu(w) = \int_{\mathcal{W}} h(Iw(0)) \Upsilon_f(Iw) \Lambda_U(w) d\mu(w) = Z_f \int_{\mathbb{R}^n} h(y) d\kappa(y)$$

where $Z_f = \int_{\mathcal{W}} \Upsilon_f(Iw) d\nu(w) > 0$. 

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Let us first discuss the consequences of this result. The relation (22) can be expressed in the following more probabilistic way. Suppose that on a given probability space \((\Omega^\nu, \mathbb{P}^\nu)\), the map \(\phi : \mathbb{R}^2 \times \Omega^\nu \rightarrow \mathbb{R}^n\) gives the weak solution \(\nu\) of Theorem 8, namely that the law of the \(W\)-random variable \((m^2 - \Delta)\phi(\cdot, \omega)\) is the measure \(\nu\). Then we have that, for any real measurable bounded function defined on \(\mathbb{R}^n\),

\[
E_{\mathbb{P}^\nu} \left[ h(\phi(0)) \frac{\mathbf{I}_f(\phi)}{Z_f} \right] = \int_{\mathbb{R}^n} h(y) d\kappa(y),
\]

namely we have proven Theorem 1. If we assume Hypothesis C then by Corollary 1, Corollary 2 and Theorem 8 there exists a unique strong solution satisfying (22) and we have proven as a consequence Theorem 2.

The proof of Theorem 8 will be given in several step of wider degree of generality with respect to the hypothesis on the potential \(V\). For technical reasons we need to first introduce an additional class of potentials.

**Hypothesis \(V_\lambda\).** We have the decomposition

\[
V = V_B + \lambda V_U, \quad V_U(y) = \sum_{i=1}^{n} (y_i)^4, \quad y = (y^1, \ldots, y^n) \in \mathbb{R}^n,
\]

with \(\lambda > 0\) and \(V_B\) a bounded function with all bounded derivatives on \(\mathbb{R}^n\).

In Section 4 below we will exploit a supersymmetric argument for the family of potentials \(V\) satisfying the more restrictive Hypothesis \(V_\lambda\) to prove that in this case a cut-off version of equation (21) holds:

**Theorem 9** Under the Hypotheses \(CO\) and \(V_\lambda\) if \(h\) is any real measurable bounded function defined on \(\mathbb{R}^n\) then we have

\[
\int_{\mathbb{R}^n} h(Iw(0)) \Lambda_U(w) \mathbf{I}_f(Iw) d\mu(w) = Z_f \int_{\mathbb{R}^n} h(y) d\kappa(y),
\]

where \(Z_f = \int_{\mathbb{R}^n} \Lambda_U(w) \mathbf{I}_f(Iw) d\mu(w) > 0\).

**Proof** The proof is given in Section 4 below. \(\square\)

Next we prove that Theorem 9 implies the existence of a weak solution satisfying equation (22) under Hypothesis \(V_\lambda\).

**Theorem 10** Under the Hypothesis \(V_\lambda\) we have that

\[
\int_{\mathbb{W}} g \circ T(w) \Lambda_U(w) d\mu(w) = \int_{\mathbb{W}} g(w) d\mu(w).
\]

where \(g\) is any bounded measurable function defined on \(\mathbb{W}\).

**Proof** Using the methods of Section 2 we can prove that the map \(T\) satisfies Hypotheses DEG1, DEG2, DEG3 of Appendix A. The claim then follows from Theorem 15 and Theorem 16 below, where we can choose the function \(g\) to be any bounded continuous function since \(\Lambda_U \in L^1(\mu)\) under Hypothesis \(V_\lambda\). \(\square\)
Proposition 2  Under the Hypotheses CO and $V_\lambda$ there exists at least a weak solution $\nu$ to equation \eqref{13} satisfying \eqref{22}.

Proof  Let $V \subset L^1(\|\Lambda_U\|d\mu)$ be the span of the two linear spaces $V_1, V_2 \subset L^1(\|\Lambda_U\|d\mu)$ where $V_1$ is composed by the functions of the form $g \circ T$, where $g$ is a measurable function defined on $W$ such that $g \circ T \in L^1(\|\Lambda_U\|d\mu)$, and $V_2$ is formed by the functions of the form $h(Iw(0))Y_f(Iw)$, where $h$ is a measurable function defined on $\mathbb{R}^n$ such that $h(Iw(0))Y_f(Iw) \in L^1(\|\Lambda_U\|d\mu)$. Note that $V_1$ and $V_2$, and so $V = \text{span}\{V_1, V_2\}$, are non-void since, under the Hypotheses $V_1$ and CO (see Lemma \ref{15} below), $\Lambda_U \in L^p(\mu)$ and so $g \circ T, h(Iw(0))Y_f(Iw) \in L^1(\mu)$ whenever $g, h$ are bounded. Define a positive functional $\tilde{L} : V \rightarrow \mathbb{R}$ by extending via linearity the relations

\[
\tilde{L}(h(Iw(0))Y_f(Iw)) := \int h(Iw(0))Y_f(Iw)\Lambda_U(w)d\mu(w) \quad (24)
\]

\[
\tilde{L}(g \circ T) := \int g(w)d\mu(w). \quad (25)
\]

This implies that $\tilde{L}$ is well defined on $V_1 \cap V_2$ and so on $V$. Obviously $\tilde{L}$ is positive on $V_2$, and, by Theorem \ref{9} we have

\[
\int_W \int g \circ T \Lambda_U |w| \mu(w) = \int_W h(Iw(0))Y_f(Iw)\Lambda_U d\mu. \quad (26)
\]

whenever $h$, and so $h(Iw(0))Y_f(Iw)$, is positive. This means that $\tilde{L}$ is positive.

For any $f = g \circ T \in V_1$, by Theorem \ref{13} and Theorem \ref{8} we have

\[
|\tilde{L}(f)| = \left|\int_W g(w)d\mu(w)\right| \leq \int_W |g(w)|N(w)d\mu(w) = \int_W |g \circ T \Lambda_U(w)|d\mu(w) = ||f\Lambda_U||_1.
\]

On the other hand, if $f \in V_2$, by relation \eqref{21}, $\tilde{L}(f) \leq ||f\Lambda_U||_1$. These two inequalities and the positivity of $\tilde{L}$ imply, by Theorem 8.31 of \cite{6} on the extension of positive functionals on Riesz spaces, that there exists at least one positive continuous linear functional $L$ on $L^1(\|\Lambda_U\|d\mu)$, such that $L(f) = \tilde{L}(f)$ for any $f \in V$. The functional $L$ defines the weak solution to equation \eqref{13} we are looking for. Indeed, since $L$ is a continuous positive functional on $L^1(\|\Lambda_U\|d\mu)$ there exists a measurable positive function $B \in L^\infty(\|\Lambda_U\|d\mu) \subset L^\infty(\mu)$ such that $L(f) = \int_W f(w)B(w)|\Lambda_U|d\mu(w)$. Since $\Lambda_U \in L^p$ by Lemma \ref{15} below, we have $1 \in V_1$ and so $L(1) = \int_W 1d\mu(w) = 1$. This implies, since the function $B$ is positive, that the $\sigma$-finite measure $d\nu = B|\Lambda_U|d\mu$ is a probability measure. Furthermore, since $V_1$ contains all the functions $g \circ T$, where $g$ is measurable and bounded, equality \eqref{25} implies that $T_\nu(\nu) = \mu$. This means that $\nu$ is a weak solution to equation \eqref{13}. Finally since $V_2$ contains all the functions of the form $h(Iw(0))Y_f(Iw)$ where $h$ is measurable and bounded on $\mathbb{R}^n$ the measure $\nu$ satisfies the thesis of the theorem.

Unfortunately we cannot repeat this reasoning for general potentials satisfying the weaker Hypothesis QC since both Theorem \ref{9} and Proposition \ref{2} exploit an $L^p$ bound on $\Lambda_U$ (see Lemma \ref{15} below) that cannot be obtained for more general potentials. Thus the idea is to generalize equation \eqref{22} without passing from equation \eqref{21}. Indeed it is possible to approximate any potential
V satisfying Hypothesis QC by a sequence of potentials \( (V_i) \) satisfying Hypothesis VA in such a way that the sequence of weak solutions \( \nu_i \) associated with \( V_i \) converges (weakly) to a weak solution associated with the potential \( V \) (see Lemma 2, Lemma 3 and Lemma 4 below). Since equation (22) involves only integrals with respect to a weak solution to equation (20), we are able to prove that equation (22) holds for any potential \( V \) approximating its weak solution \( \nu \) by the sequence \( \nu_i \) satisfying equation (22).

Let us now set up the approximation argument, starting with a series of lemmas about convergence of weak solutions.

**Lemma 2** Let \( \{T_i\}_{i \in \mathbb{N}} \) be a sequence of continuous maps on \( W \) such that for any compact \( K \subset W \) we have that \( \bigcup_{i \in \mathbb{N}} T_i^{-1}(K) \) is pre-compact and there exists a continuous map \( T \) such that \( T_i \to T \) uniformly on the compact subsets of \( W \). Let \( M_i \) be a set of probability measures on \( W \) defined as follows

\[
 M_i := \{ \nu \text{ probability measure on } W \text{ such that } T_{j,*}(\nu) = \mu \text{ for some } j \geq i \}.
\]

Then \( M := \cap_{i \in \mathbb{N}} M_i \), where the closure is taken with respect to the weak topology on the set of probability measures on \( W \), is non-void and

\[
 M \subset \{ \nu \text{ probability measure on } W \text{ such that } T_*(\nu) = \mu \}.
\]

**Proof** First of all we prove that \( M_i \) is pre-compact for any \( i \in \mathbb{N} \). This is equivalent to prove that the measures in \( M_i \) are tight. Let \( K \) be a compact set such that \( \mu(K) \geq 1 - \epsilon \) for a fixed \( 0 < \epsilon < 1 \), then \( K := \bigcup_{i \in \mathbb{N}} T_i^{-1}(K) \) is a compact set in \( W \). Consider \( \nu \in M_j \) then there exists \( T_k \) such that \( T_{k,*}\nu = \mu \). This implies

\[
 \nu(K) \geq \nu(\bigcup_{i \in \mathbb{N}} T_i^{-1}(K)) \geq \nu(T_{k,1}^{-1}(K)) \geq \mu(K) \geq 1 - \epsilon,
\]

for any \( k \in \mathbb{N} \). Since \( M_i \) are pre-compact, \( M_i \) are compact and \( M_i \subset M_j \) if \( i \geq j \). This implies that \( M \) is non-void. If we consider a \( \nu \in M \) there exists a sequence \( \nu_k \) weakly converging to \( \nu \), for \( k \to +\infty \), such that \( T_{i_k,1}\nu_k = \mu \) and \( i_k \to +\infty \). Proving that \( T_*(\nu) = \mu \) is equivalent to prove that for any \( C^1 \) bounded function \( g \) with bounded derivatives defined on \( W \) taking values in \( \mathbb{R} \) we have \( \int_W g \circ T \, d\nu = \int_W g \, d\mu \). Let \( K \) the compact set defined before, then there exists a \( k \in \mathbb{N} \) such that \( \sup_{w \in K} \frac{||T_{i_k}(w) - T(w)||}{||T(w)||} \leq \epsilon \) and that \( \frac{\int_W g \circ T \, d\nu - \int_W g \circ T_{i_k} \, d\mu}{\int_W g \circ T_{i_k} \, d\nu} \leq \epsilon \), for the arbitrary \( 0 < \epsilon < 1 \). This implies that

\[
 \left| \int_W g \circ T \, d\nu - \int_W g \, d\mu \right| \leq \left| \int_W g \circ T \, d\nu - \int_W g \circ T_{i_k} \, d\nu \right| + \left| \int_K (g \circ T - g \circ T_{i_k}) \, d\nu \right|
\]

\[
 + \frac{||g||}{\epsilon} \frac{\int_W g \circ T \, d\nu}{\int_W g \, d\mu}
\]

\[
 \leq \epsilon + \frac{||g||}{\epsilon} + \frac{||g||}{\epsilon}.
\]

Since \( \epsilon \) is arbitrary, from this it follows that \( \int_W g \circ T \, d\nu = \int_W g \, d\mu. \)

\[\square\]

**Remark 5** The proof of Lemma 2 proves also that given any sequence of \( \nu_i \in M_i \) there exists a subsequence converging weakly to \( \nu \in M \).

**Lemma 3** Let \( \{V_i\}_{i \in \mathbb{N}} \) be a sequence of potentials satisfying the Hypothesis QC converging to the potential \( V \) and such that \( \partial V_i \) converges uniformly to \( \partial V \) on compact subsets of \( \mathbb{R}^n \), \( V_i, V, \)

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\( \partial V_1 \) and \( \partial V \) are uniformly exponentially bounded and there exists a common function \( H \) entering Hypothesis QC for \( \{V_i\}_{i \in \mathbb{N}} \) and \( V \). Let \( T_i \), \( T \) the maps on \( W \) associated to \( V_i \) and \( V \) respectively. Then the sequence \( \{T_i\}_{i \in \mathbb{N}} \) satisfies the hypothesis of Lemma 3.

**Proof** Note that the a priori estimates (16) and (17) in Lemma 1 are uniform in \( i \in \mathbb{N} \) since they depend only on \( H \) and the exponential growth of \( V_1, V, \partial V_1, \partial V \). From this we can deduce the pre-compactness of the set \( K = \bigcup_{i \in \mathbb{N}} T_i^{-1}(\tilde{K}) \) for any compact set \( \tilde{K} \subset W \) using a reasoning similar to the one proposed in Theorem 4 and Theorem 6.

Proving that \( T_i \) converges to \( T \) uniformly on the compact sets is equivalent to prove that the map \( U_i(w)(x) = f(x)\partial V_i(\mathcal{I}w(x)) \) converges to \( U(w)(x) = f(x)\partial V(\mathcal{I}w(x)) \) in \( L^2 \) uniformly on the compact subsets of \( W \). Let \( K \) be a compact set of \( W \), then there exists an \( M > 0 \) such that \( |\mathcal{I}w(x)| \leq M(1 + |x|^\eta) \) (where we suppose without loss of generality that \( \eta < 1 \)). By hypotheses we have that there exist two constants \( \alpha, \beta > 0 \) such that \( |\partial V_i(y)|, |\partial V(y)| \leq e^{\alpha |y| + \beta} \), thus there exists a compact subset \( \mathcal{R} \) of \( \mathbb{R}^2 \) such that \( \int_{\mathcal{R}} (f(x))^2 \exp(2\alpha M(1 + |x|^\eta) + 2\beta) dx \leq \epsilon \), for some \( \epsilon \in (0, 1) \). Denote by \( B_\epsilon \) the ball of radius \( \sup_{y \in \mathcal{R}} M(1 + |x|^\eta) \) then we have

\[
\sup_{w \in K} \|U_i(w) - U(w)\|_{\mathcal{R}}^2 \leq 2 \left| \int_{\mathcal{R}} (f(x))^2 e^{2\alpha M(1 + |x|^\eta) + 2\beta} dx \right| + \sup_{w \in K} \left| \int_{\mathcal{R}} (f(x))^2 |\partial V(\mathcal{I}w) - \partial V_i(\mathcal{I}w)|^2 dx \right| \\
\leq 2\epsilon + \left( \sup_{y \in B_\epsilon} |\partial V(y)| \right)^2 \int_{\mathcal{R}} (f(x))^2 dx \\
\rightarrow 2\epsilon,
\]

as \( i \to +\infty \). This means that \( \lim_{i \to +\infty} (\sup_{w \in K} \|U_i(w) - U(w)\|_{\mathcal{R}}^2) \leq \epsilon \), and since \( \epsilon \) is arbitrary in \( (0, 1) \) the theorem is proved. \( \Box \)

**Lemma 4** Let \( V \) be a potential satisfying Hypothesis QC, then there exists a sequence \( \{V_i\}_{i \in \mathbb{N}} \) of bounded smooth potentials converging to \( V \) and satisfying the hypothesis of Lemma 3.

**Proof** Let \( V \) be a potential satisfying the Hypothesis QC and let \( \tilde{H} \) the function whose existence is guaranteed by Hypothesis QC. Let, for any \( N \in \mathbb{N}, v_N := \sup_{y \in B(0,N)} |V(y)| \) and let \( V^N := G_{v_N} \circ V \) where

\[
G_{v_N}(z) := \begin{cases} 
  z & \text{if } |z| \leq k, \\
  k & \text{if } |z| > k.
\end{cases}
\]

Let \( \rho \) be a smooth compactly supported mollifier and denote by \( \rho_c \), the function \( \rho_c(y) := e^{-\eta} \rho \left( \frac{y}{\eta} \right) \). We want to prove that \( V^N = \tilde{V}^N \ast \rho_{c_N} \), for a suitable sequence \( \epsilon_N \in \mathbb{R}_+ \), is the approximation requested by the lemma. Without loss of generality we can suppose that \( \tilde{H} \) is a positive function depending only on the radius \( |y| \) and increasing as \( |y| \to +\infty \). Under these conditions the request of Hypothesis QC is equivalent to say that for any unit vector \( \hat{n} \in \mathbb{S}^n \) we have that for any \( y \in \mathbb{R}^n \)

\[
\max(-\hat{n} \cdot \partial V(y + r\hat{n}), 0) \leq \tilde{H}(y).
\]

We want to prove that \( \tilde{H}(|y|) = \tilde{H}(|y| + \sup_N \epsilon_N) \) is the function requested by the lemma.

Since for any unit vector \( \hat{n} \in \mathbb{S}^n \) we have \( |\hat{n} \cdot \partial V^N| \leq |\hat{n} \cdot \partial V| \) and since \( V^N \) is absolutely continuous we obtain

\[
-\hat{n} \cdot \partial V^N(y + r\hat{n}) = (\hat{n} \cdot \partial V^N \ast \rho_{c_N})(y + r\hat{n})
\]
$$\leq (\max(-\hat{n} \cdot \partial V(\cdot + r\hat{n}), 0) \ast \rho_N)(y) \leq H \ast \rho_N(y).$$

Furthermore, we have that \(\hat{V}^N = V\) on \(B(0, N - 1)\) and so there exists a sequence \(\{\epsilon_N\}_N\) such that \(\epsilon_N \to 0\) and \(\sup_{x \in B(0, N - 1)} |\partial V^N(x) - \partial V(x)| \leq \frac{1}{N}\). Since \(V^N\) is smooth and bounded and

$$H \ast \rho_N(y) \leq H(|y| + \sup_N(\epsilon_N)) = H(y),$$

we conclude the claim. \(\square\)

Finally, we are able to prove (22) for all QC potentials, which will conclude this section.

**Proof of Theorem 8** By Proposition 2 the equality (22) holds when \(V\) satisfies the Hypothesis \(V_\lambda\) for some \(\lambda > 0\), i.e. if \(V(y) = V_{\lambda, V_n}(y) = V_B(y) + \lambda \sum_{i=1}^{n}(y^i)^4\) for some bounded potential \(V_B\). It is clear that if \(\lambda_i \to 0\) the potentials \(V_{\lambda_i, V_n}\) converge to the potential \(V_B\) and the hypothesis of Lemma 3 hold. This means that if \(\hat{\nu}_i\) is a sequence of probability measures such that \(\hat{\nu}_i\) is a weak solution to the equation associated with \(V_{\lambda_i, V_n}\) satisfying the thesis of Proposition 2 by Remark 5 and Lemma 2 there exists a probability measure \(\hat{\nu}\), that is a weak solution to the equation associated with \(V_B\), such that \(\hat{\nu}_i \to \nu\) in the weak sense, as \(i \to \infty\) and \(\lambda_i \to 0\).

We want to prove that \(\hat{\nu}\) is a weak solution to the equation associated with \(V_B\) satisfying equation (22). The previous claim is equivalent to prove that

$$\int_W g(\mathcal{I}w(0))e^4 \int f'(x)V_{\lambda_i, V_n}(\mathcal{I}w(x))dx d\nu_i(w) \to \int_W g(\mathcal{I}w(0))e^4 \int f'(x)V_B(\mathcal{I}w(x))dx d\nu(w),$$

as \(\lambda \to 0\), for any continuous bounded function \(g\), and that \(\kappa_{\lambda_i} \to \kappa_B\) weakly, where \(d\kappa_{\lambda_i} = \exp(-4\pi V_{\lambda_i, B})dx/Z_{\lambda_i}\) and \(d\kappa_B = d\kappa = \exp(-4\pi V_B)dx/Z_B\).

Proving relation (28) is equivalent to prove that

$$\int f'(x)V_{\lambda_i, B}(\mathcal{I}w(x))dx \to \int f'(x)V_B(\mathcal{I}w(x))dx$$

uniformly on compact sets of \(W\). This assertion can be easily proved using the methods of Lemma 3. The weak convergence of \(\kappa_{\lambda_i}\) to \(\kappa_B\) easily follows from Lebesgue’s dominate convergence theorem.

The previous reasoning proves the theorem for any bounded potential \(V_B\). Using Lemma 3 we can approximate any potential \(V\) satisfying Hypothesis QC by a sequence of bounded potentials \(V_{\lambda_i, V_n}\). Using Lemma 3, Remark 5, Lemma 2 and a reasoning similar to the one exploited in the first part of the proof we obtain the thesis of the theorem for a general potential satisfying Hypothesis QC. \(\square\)

## 4 Dimensional reduction

Let

$$\Xi(h) := \int_W h(\mathcal{I}w(0))\Lambda_U(w)\frac{\mathcal{Y}_f(\mathcal{I}w)}{Z_f}dw,$$

with the notations as in Section 2 (Theorem 7) and Section 3 (Theorem 8). In this section we prove Theorem 9 i.e. the identity

$$\Xi(h) = \int_{R^n} h(y)d\kappa(y),$$

using the supersymmetric representation of the integral. It is important to note that \(\Lambda_U\) appears without the modulus in (29).
4.1 Supersymmetric representation

Let us illustrate how to derive the supersymmetric formulation of $\Xi(h)$. Our goal is to provide a blueprint for the reader to understand the role of supersymmetry in our argument. For an introduction to the mathematical formalism of supersymmetry see e.g. [20, 19]. The details of the rigorous implementation of the ideas exposed here is the main goal of the paper of Klein et al. [32] and of the modifications we implement in the following subsections in order to overcome a gap in their proof.

Let us start by unfolding the definition of $\Lambda_U$ and $\Upsilon_f(Uw)$ in (30) to get the expression

$$Z_f \Xi(h) = \int h(Uw(0)) \det_2 (I_U + \nabla U) \exp \left( -\delta(U) - \frac{1}{2} \|U\|_H^2 + 4 \int V(Uw(x)) f'(x) dx \right) d\mu(w).$$

In order to manipulate the regularized Fredholm determinant we approximate the right hand side by

$$Z_f^\chi \Xi(h) := \int h(\mathcal{J}_\chi w(0)) \det_2 (I_U + \nabla U_\chi) \times$$

$$\times \exp \left( -\delta(U_\chi) - \frac{1}{2} \|U_\chi\|_H^2 + 4 \int V(\mathcal{J}_\chi w(x)) f'(x) dx \right) d\mu(w).$$

where $\chi > 0$ is a regularization parameter, $\mathcal{J}_\chi := \mathcal{I} + \chi = (m^2 - \Delta)^{-1-\chi}$, $Z_f^\chi$ is the unique positive constant such that $\Xi(h) = 1$ and

$$U_\chi(w) := \frac{1}{1 + 2\chi} \mathcal{I}^\chi \partial V(\mathcal{J}_\chi w). \quad (31)$$

We will prove below that $\lim_{\chi \to 0} \Xi(h) = \Xi(h)$. When $\chi > 0$, $\nabla U_\chi(w) = \frac{1}{1 + 2\chi} \mathcal{I}^\chi \partial V(\mathcal{J}_\chi w) \mathcal{J}_\chi$ is almost surely a trace class operator and $U_\chi \in \mathcal{W}^\circ$. This means that we can rewrite the regularized Fredholm determinant $\det_2$ in term of the unregularized one (denoted $\det$) (see equation (60) and the discussion in Appendix A) obtaining

$$Z_f^\chi \Xi(h) = \int h(\mathcal{J}_\chi w(0)) \det(I_U + \nabla U_\chi) \times$$

$$\times \exp \left( -(U_\chi, w) - \frac{1}{2} \|U_\chi\|_H^2 + 4 \int V(\mathcal{J}_\chi w(x)) f'(x) dx \right) d\mu(w). \quad (32)$$

From the expression of $\nabla U_\chi$ and the invariance of the determinant by conjugation we deduce

$$\det(I_U + \nabla U_\chi) = \det(I_U + \omega \mathcal{I}^\chi \partial^2 V(\mathcal{J}_\chi w) \mathcal{J}_\chi) = \det(I_U + \omega \mathcal{I}^{1/2+\chi} \partial^2 V(\mathcal{J}_\chi w) \mathcal{I}^{1/2+\chi}),$$

where $\omega = \frac{1}{1 + 2\chi}$, and featuring the nicer symmetric operator $\omega \mathcal{I}^{1/2+\chi} \partial^2 V(\mathcal{J}_\chi w) \mathcal{I}^{1/2+\chi}$. Let $\gamma$ be the Gaussian measure given by the law of $\varphi = \mathcal{J}_\chi w$ under $\mu$. The expression (32) is then equivalent to

$$\int h(\varphi(0)) \det(I_U + \omega \mathcal{I}^{1/2+\chi} \partial^2 V(\varphi) \mathcal{I}^{1/2+\chi}) \exp(-\langle \omega \partial^2 V(\varphi), (m^2 - \Delta) \varphi \rangle) \times$$

$$\times \exp \left( -\frac{\omega^2}{2} \|\mathcal{I}^\chi \partial^2 V(\varphi)\|_H^2 + 4 \int V(\varphi(x)) f'(x) dx \right) \gamma(d\varphi).$$

At this point we introduce an auxiliary Gaussian field $\eta$ distributed as the Gaussian white noise $\mu$ to write

$$\exp \left( -\frac{\omega^2}{2} \|\mathcal{I}^\chi \partial^2 V(\varphi)\|_H^2 \right) = \int \exp(-i \omega (\partial^2 V(\varphi), \mathcal{I}^\chi \eta)) \mu(d\eta).$$

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We also introduce two free fermion fields \( \psi, \bar{\psi} \) realized as bounded operators on a suitable Hilbert space \( H_{\psi, \bar{\psi}} \) with a state \( \langle \cdot \rangle_{\psi, \bar{\psi}} \) for which
\[ \langle \bar{\psi}(x) \bar{\psi}(x') \rangle_{\psi, \bar{\psi}} = \langle \psi(x) \psi(x') \rangle_{\psi, \bar{\psi}} = 0, \quad \langle \psi(x) \bar{\psi}(x') \rangle_{\psi, \bar{\psi}} = \omega G_{1+2}(x - x'), \]
where \( G_{\alpha} \) is the kernel of the operator \( \mathcal{I}^\alpha \). These additional fields allow to represent the determinant as
\[
\det(I_H + \omega \mathcal{I}^{1/2+x} f \partial^2 V(\varphi) \mathcal{I}^{1/2-x}) = \exp \left( \int \psi^i(x) f(x) \partial^2 \bar{\varphi} \varphi(x) \bar{\psi}^j(x) dx \right)_{\psi, \bar{\psi}}.
\]
By tensorizing the fermionic Hilbert space \( H_{\psi, \bar{\psi}} \) with the \( L^2 \) space of the product measure \( \gamma \otimes \mu \) one can realize the fermionic free fields \( \psi, \bar{\psi} \) and the Gaussian fields \( \varphi, \eta \) as operators on the same Hilbert space \( H_{\Phi} \) with a state which we denote by \( \langle \cdot \rangle_{\Phi} \) or \( \langle \cdot \rangle_{\chi} \) when this does not cause ambiguity. As a consequence, we have

\[
Z_{\chi}(h) = \langle h(\varphi(0)) \exp(S(\varphi, \eta, \psi, \bar{\psi})) \rangle_{\chi},
\]
with
\[
S(\varphi, \eta, \psi, \bar{\psi}) := \int \psi(x) f(x) \partial^2 V(\varphi(x)) \bar{\psi}^i(x) dx +
- \omega(f \partial V(\varphi), (m^2 - \Delta) \bar{\varphi} + i \mathcal{I}^x \eta) + 4 \int \varphi(x) f'(x) dx.
\]

A “more symmetric” form for this expression involving bosonic and fermionic fields can be obtained by introducing the superspace \( \mathcal{S} \) and the superfield \( \Phi \).

**The superspace**
Formally the superspace \( \mathcal{S} \) can be thought as the set of points \( (x, \theta, \bar{\theta}) \) where \( x \in \mathbb{R}^2 \) and \( \theta, \bar{\theta} \) are two additional anticommuting coordinates. A more concrete construction is to understand \( \mathcal{S} \) via the algebra of smooth functions on it.

Let \( \mathcal{S}(\theta_1, \ldots, \theta_n) \) be the Grassmann algebra generated by symbols \( \theta_1, \ldots, \theta_n \), i.e.
\[
\mathcal{S}(\theta_1, \ldots, \theta_n) = \text{span}(1, \theta_1, \theta_1 \theta_2, \theta_1 \theta_2 \theta_3, \ldots, \theta_1 \cdots \theta_n)
\]
with the relations \( \theta_i \theta_j = -\theta_j \theta_i \).

A \( C^\infty \) function \( F : \mathbb{R}^2 \to \mathcal{S}(\theta, \bar{\theta}) \) is just a quadruplet \( (f_\theta, f_\bar{\theta}, f_\theta \bar{\theta}, f_\theta \bar{\theta}) \in (C^\infty(\mathbb{R}^2))^4 \) the identification
\[
F(x) = f_\theta(x) + f_\bar{\theta}(x) \theta + f_\theta(x) \bar{\theta} + f_\theta \bar{\theta}(x) \theta \bar{\theta}.
\]
The function \( F \) can be considered as a function \( F : \mathcal{S} \to \mathbb{R} \) by formally writing
\[
F(x, \theta, \bar{\theta}) = F(x).
\]
In particular we identify \( C^\infty(\mathcal{S}) \) with \( C^\infty(\mathbb{R}^2; \mathcal{S}(\theta, \bar{\theta})) \). \( C^\infty(\mathcal{S}) \) is a non-commutative algebra on which we can introduce a linear functional defined as
\[
F \mapsto \int F(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} := - \int f_\theta \bar{\theta}(x) dx,
\]
induced by the standard Berezin integral on \( \mathcal{S} \) satisfying
\[
\int d\theta d\bar{\theta} = \int \theta d\theta d\bar{\theta} = \int \bar{\theta} d\theta d\bar{\theta} = 0, \quad \int \theta \bar{\theta} d\theta d\bar{\theta} = -1.
\]
Remark 6 A norm on $C^\infty(\mathcal{S})$ can be defined by
\[
\|F\|_{C(\mathcal{S})} = \sup_{x \in \mathbb{R}^2} (|f_\omega(x)| + |f_\theta(x)| + |f_\bar{\theta}(x)|),
\]
and an involution by
\[
F(x, \theta, \bar{\theta}) = \overline{\int_0^1 r(f_\theta(x)) + r'(f_\theta(x))f_\theta(x)\theta + r'(f_\theta(x))f_\bar{\theta}(x)\bar{\theta},
\]
where the bar on the right hand side denotes complex conjugation.

Given $r : C^1(\mathbb{R}; \mathbb{R})$ we define the composition $r \circ F : \mathcal{S} \rightarrow \mathbb{R}$ by
\[
r(F(x, \theta, \bar{\theta})) := r(f_\theta(x)) + r'(f_\theta(x))f_\theta(x)\theta + r'(f_\theta(x))f_\bar{\theta}(x)\bar{\theta},
\]
in accordance with the same expression one would get if $r$ were a monomial. Moreover we can define similarly the space of Schwartz superfunctions $S(\mathcal{S})$ and the Schwartz superdistributions $S'(\mathcal{S}) \approx S'(\mathbb{R}^2; \mathcal{S}(\theta, \bar{\theta}))$ where $T \in S'(\mathcal{S})$ can be written $T = T_\theta + T_\bar{\theta} + T_\theta\bar{\theta}$ with $T_\theta, T_\bar{\theta}, T_\theta\bar{\theta} \in S'(\mathbb{R}^2)$ and duality pairing
\[
T(f) = -T_\theta(f_\theta) + T_\bar{\theta}(f_\bar{\theta}) - T_\theta\bar{\theta}(f_\theta f_\bar{\theta}), \quad f \theta, f_\bar{\theta}, f_\theta f_\bar{\theta} \in S(\mathbb{R}^2).
\]

The superfield We take the generators $\theta, \bar{\theta}$ to anticommute with the the fermion fields $\psi, \bar{\psi}$, and introduce the complex Gaussian field
\[
\omega := -\varpi((m^2 - \Delta)\varphi + i\mathcal{I}\eta)
\]
and put all our fields together in a single object defining the superfield
\[
\Phi(x, \theta, \bar{\theta}) := \varphi(x) + \bar{\psi}(x)\theta + \psi(x)\bar{\theta} + \omega(x)\theta\bar{\theta}.
\]
We define also
\[
V(\Phi(x, \theta, \bar{\theta})) = V(\varphi(x)) + \partial V(\varphi(x))(\bar{\psi}(x)\theta + \psi(x)\bar{\theta}) + [\partial^2 V(\varphi(x))\omega(x) + \partial^2 V(\varphi(x))\psi(x)\bar{\psi}(x)]\theta\bar{\theta}
\]
and since
\[
\tilde{f}(|x|^2 + 4\theta\bar{\theta}) = \tilde{f}(|x|^2) + 4\tilde{f}'(|x|^2)\theta\bar{\theta}
\]
we observe that
\[
- \int V(\Phi(x, \theta, \bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta})dx + \int f_\theta \partial V(\varphi(x))\omega(x)dx + \int [f(\varphi(x))\partial V(\varphi(x))\psi(x)\bar{\psi}(x) + 4V(\varphi(x))f'(x)]dx = S(\varphi, \eta, \psi, \bar{\psi}).
\]
By introducing the superspace distribution $\theta\bar{\theta}\delta_0(dx)$ we have also, by similar computations.
\[
h(\varphi(0)) = - \int h(\Phi(x, \theta, \bar{\theta}))\theta\bar{\theta}\delta_0(dx)dx + dx + \delta_0(dx)dx + \delta_0(dx)
\]
As a consequence we can rewrite $\Xi(h)$ as an average over the superfield $\Phi$:
\[
Z^\chi \Xi_\chi(h) = \left\langle \left(- \int h(\Phi(x, \theta, \bar{\theta}))\theta\bar{\theta}\delta_0(dx)dx + \delta_0(dx)dx + \delta_0(dx)dx \right) \right\rangle.
\]

(33)
While all these rewriting are essentially algebraic (modulo the issue of justifying rigorously each step), the supersymmetric formulation \([33]\) makes appear a symmetry of the expression for \(X(h)\) which was not clear from the original formulation. In some sense the reader can think to the superspace \((x, \theta, \bar{\theta})\) and to the superfield \(\Phi(x, \theta, \bar{\theta})\) as a convenient bookkeeping procedure for a series of relations between the quantities one is manipulating.

A crucial observation is that the superfield \(\Phi\) is a free field with mean zero, namely all its correlation functions can be expressed in terms of the two-point function \(\langle \Phi(x, \theta, \bar{\theta})\Phi(x', \theta', \bar{\theta}')\rangle_x\) via Wick’s theorem. A direct computation of this two point function gives:

\[
\langle \Phi(x, \theta, \bar{\theta})\Phi(x', \theta', \bar{\theta}')\rangle_x = \langle \varphi(x)\varphi(x')\rangle_x - \langle \bar{\psi}(x)\psi(x')\rangle_x \theta \theta' - \langle \psi(x)\bar{\psi}(x')\rangle_x \bar{\theta} \bar{\theta}' + \langle \varphi(x)\omega(x')\rangle_x \theta' \bar{\theta}'
\]

\[
+ (\omega(x)\varphi(x'))_x \theta \bar{\theta} + (\omega(x)\omega(x'))_x \theta \bar{\theta} \theta' \theta'
\]

\[
= G_{2+2\chi}(x - x') + \omega G_{1+2\chi}(x - x')(\theta \theta' - \bar{\theta} \bar{\theta}') - \omega (m^2 - \Delta) G_{2+2\chi}(x - x')(\theta' \bar{\theta} + \theta \bar{\theta}')
\]

\[
+ ((m^2 - \Delta)^2 G_{2+2\chi}(x - x') - G_{2\chi}(x - x'))\theta \theta' \bar{\theta} \bar{\theta}'.
\]

Upon observing that \((m^2 - \Delta)^2 G_{2+2\chi} = G_{1+2\chi}, \ (m^2 - \Delta)^2 G_{2+2\chi} = G_{2\chi}\) and that \(-\theta \bar{\theta}' + \bar{\theta} \theta' + \theta' \theta + \bar{\theta} \bar{\theta} = (\theta - \theta')(\bar{\theta} - \bar{\theta}')\) we conclude

\[
\langle \Phi(x, \theta, \bar{\theta})\Phi(x', \theta', \bar{\theta}')\rangle = C_{\Phi}(x - x', \theta - \theta', \bar{\theta} - \bar{\theta}'), \tag{34}
\]

where

\[
C_{\Phi}(x, \theta, \bar{\theta}) = G_{2+2\chi}(x) - \omega G_{1+2\chi}(x)\theta \bar{\theta}.
\]

**Remark 7** Note that when \(\chi = 0\), the superfield \(\Phi\) corresponds to the formal functional integral

\[
e^{-\frac{1}{2} \int \Phi(m^2 - \Delta_S)\Phi d x d \theta d \bar{\theta} D\Phi}
\]

where \(D\Phi = D\psi D\bar{\psi} D\varphi D\eta\) and where \(\Delta_S = \Delta + \partial \theta \partial \bar{\theta}\) is the superlaplacian. Then

\[
\frac{1}{2} \int \Phi(m^2 - \Delta_S)\Phi d x d \theta d \bar{\theta} = \frac{1}{2} \int [-2\bar{\psi}(m^2 - \Delta)\psi - \omega \omega + 2\omega (m^2 - \Delta)\varphi] d x
\]

\[
= \frac{1}{2} \int [-2\bar{\psi}(m^2 - \Delta)\psi + ((m^2 - \Delta)\varphi)^2 + \eta^2] d x
\]

and this indeed corresponds to the action functional appearing in the formal functional integral for \((\psi, \bar{\psi}, \varphi, \eta)\). This is in agreement to the fact that the two point function satisfies the equation

\[
(m^2 - \Delta_S)C_{\Phi}(x, \theta, \bar{\theta}) = \delta_{\Phi}(x)\delta(\theta)\delta(\bar{\theta}),
\]

where \(\delta(x)\delta(\theta)\delta(\bar{\theta})\) is the distribution such that

\[
\int F(x, \theta, \bar{\theta})\delta_{\Phi}(x)\delta(\theta)\delta(\bar{\theta}) d x d \theta d \bar{\theta} = f_{\theta}(0),
\]

namely, \(C_{\Phi}\) is the Green’s function for \((m^2 - \Delta_S)\).
The supersymmetry  On \( C^\infty(\mathfrak{S}) \) one can introduce the (graded) derivations

\[
Q := 2\theta\nabla + x\partial_\theta, \quad \bar{Q} := 2\bar{\theta}\nabla - x\partial_{\bar{\theta}},
\]

which are such that

\[
Q(|x|^2 + 4\theta\bar{\theta}) = \bar{Q}(|x|^2 + 4\theta\bar{\theta}) = 0,
\]

namely they annihilate the quadratic form \(|x|^2 + 4\theta\bar{\theta})\). Moreover if \( QF = \bar{Q}F = 0 \), for \( F \) as above, then we must have

\[
0 = QF(x, \theta, \bar{\theta}) = 2\theta\nabla f_\theta(x) + x f_\theta(x) + 2\nabla f_\theta(x)\theta - x f_{\theta\bar{\theta}}(x)\theta
\]

\[
0 = \bar{Q}F(x, \theta, \bar{\theta}) = 2\bar{\theta}\nabla f_{\bar{\theta}}(x) + x f_{\bar{\theta}}(x) - 2\nabla f_{\bar{\theta}}(x)\bar{\theta} + x f_{\theta\bar{\theta}}(x)\bar{\theta}
\]

and therefore

\[
\nabla f_{\theta}(x) = \frac{x}{2} f_{\theta\bar{\theta}}(x) \quad \text{and} \quad f_{\theta}(x) = f_{\bar{\theta}}(x) = 0.
\]

If we also request that \( F \) is invariant with respect to \( \mathbb{R}^2 \) rotations, then there exists an \( f \) such that \( f(|x|^2) = f_0(x) \) from which we deduce that \( 2xf'(|x|^2) = \nabla f(|x|^2) = \nabla f_0(x) = \frac{x}{2} f_{\theta\bar{\theta}}(x) \) which implies

\[
f(|x|^2 + 4\theta\bar{\theta}) = f(|x|^2) + 4f'(|x|^2)\theta\bar{\theta} = f_0(x) + f_{\theta\bar{\theta}}(x)\theta\bar{\theta} = F(x, \theta, \bar{\theta}).
\]

Namely any function satisfying these two equations can be written in the form

\[
F(x, \theta, \bar{\theta}) = f(|x|^2 + 4\theta\bar{\theta}).
\]

Observe that if we introduce the linear transformations

\[
\tau(b, \bar{b}) \begin{pmatrix} x \\ \theta \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} x + 2b\theta\rho + 2\bar{b}\bar{\theta}\rho \\ \theta - (x \cdot b)\rho \\ \bar{\theta} + (x \cdot \bar{b})\rho \end{pmatrix} \in \mathfrak{S}(\theta, \bar{\theta}, \rho)
\]

for \( b, \bar{b} \in \mathbb{R}^2 \) and where \( \rho \) is a new odd variable anticommuting with \( \theta, \bar{\theta} \) and itself, then we have

\[
\left. \frac{d}{dt} \right|_{t=0} \tau(tb, \bar{t}\bar{b})F(x, \theta, \bar{\theta}) = \frac{d}{dt} \left|_{t=0} F(\tau(tb, \bar{t}\bar{b})(x, \theta, \bar{\theta})) = (b \cdot Q + \bar{b} \cdot Q)F(x, \theta, \bar{\theta})
\]

so \( \tau(b, \bar{b}) = \exp(b \cdot Q + \bar{b} \cdot Q) \) and \( \tau(tb, \bar{t}\bar{b})\tau(sb, \bar{s}\bar{b}) = \tau((t+s)b, (t+s)\bar{b}) \). In particular \( F \in C^\infty(\mathfrak{S}) \) is supersymmetric if and only if \( F \) is invariant with respect to rotations and for any \( b, \bar{b} \in \mathbb{R}^2 \) we have \( \tau(b, \bar{b})(F) = F \).

By duality the operators \( Q, \bar{Q} \) and \( \tau(b, \bar{b}) \) act also on the space \( S'(\mathfrak{S}) \) and we say that the distribution \( T \in S'(\mathfrak{S}) \) is supersymmetric if it is invariant with respect to rotations and \( Q(T) = \bar{Q}(T) = 0 \). For supersymmetric functions and distribution the following fundamental theorem holds.

**Theorem 11** Let \( F \in S(\mathfrak{S}) \) and \( T \in S'(\mathfrak{S}) \) such that \( T_0 \) is a continuous function. If both \( F \) and \( T \) are supersymmetric, then we have the reduction formula

\[
\int T(x, \theta, \bar{\theta}) \cdot F(x, \theta, \bar{\theta}) \, dx \, d\theta \, d\bar{\theta} = 4\pi T_0(0)F_0(0).
\]
By the supersymmetry of $C$ and $\bar{C}$, the correlation function $\langle \Phi(x, \theta, \bar{\theta}) \rangle$ of $\Phi$ with its complex conjugate $\bar{\Phi}$. The law of $\Phi$ would have been invariant with respect super transformations if and if only $\Phi$ is supersymmetric. Unfortunately the function $\langle \Phi(x, \theta, \bar{\theta}) \rangle$ is not invariant with respect to super transformations.

As a consequence, the superfield $\Phi' := \tau(b, \bar{b})\Phi$ is a Gaussian free field and has the same correlation function $C_{\Phi'}$ as $\Phi$ given by equation (34). However it is important to stress that this does not imply that $\Phi'$ is supersymmetric. Indeed the correlation function given in equations (34) involves only the product $\langle \Phi(x, \theta, \bar{\theta}) \rangle$ of the complex superfield $\Phi$ and not also the product $\langle \Phi(x, \theta, \bar{\theta}) \rangle$ of its complex conjugate $\Phi$. The law of $\Phi$ would have been invariant with respect super transformations if and if only $\langle \Phi(x, \theta, \bar{\theta}) \rangle$ and $\langle \Phi(x, \theta, \bar{\theta}) \rangle$ had been both supersymmetric. Unfortunately the function $\langle \Phi(x, \theta, \bar{\theta}) \rangle$ is not invariant with respect to super transformations.

Expectation of supersymmetric polynomials As explained in Remark 8 the law of $\Phi$ is not supersymmetric. Nevertheless we can deduce important consequences from the supersymmetry of the correlation function $C_{\Phi}$. More precisely, since $\Phi$ is a free field Wick’s theorem hold and

$$\left\langle \prod_{i=1}^{2n} \Phi(x_i, \theta_i, \bar{\theta}_i) \right\rangle_{\chi} = \sum_{\{i_k, j_k\}_k} \prod_{k=1}^{n} C_{\Phi}(x_{i_k} - x_{j_k}, \theta_{i_k} - \theta_{j_k}, \bar{\theta}_{i_k} - \bar{\theta}_{j_k}),$$  \quad (37)

$$\left\langle \prod_{i=1}^{2n+1} \Phi(x_i, \theta_i, \bar{\theta}_i) \right\rangle_{\chi} = 0.$$  \quad (38)

By the supersymmetry of $C_{\Phi}(x - x', \theta - \bar{\theta}, \theta - \bar{\theta})$ and of its products, we obtain that

$$\left\langle \prod_{i=1}^{2n} \tau(b, \bar{b})\Phi(x_i, \theta_i, \bar{\theta}_i) \right\rangle_{\chi} = \left\langle \prod_{i=1}^{2n} \Phi(x_i, \theta_i, \bar{\theta}_i) \right\rangle_{\chi}.$$
The previous equality implies that
\[
\left\langle \prod_{i=1}^{n} P_i(\Phi) \cdot \tau(b, \bar{b})(F^i)dx d\theta d\bar{\theta} \right\rangle_{\chi} = \left\langle \prod_{i=1}^{n} \tau(b, \bar{b})(P_i(\Phi)) \cdot F^i dx d\theta d\bar{\theta} \right\rangle_{\chi} = \left\langle \prod_{i=1}^{n} P_i(\tau(b, \bar{b})(\Phi)) \cdot F^i dx d\theta d\bar{\theta} \right\rangle_{\chi} = \left\langle \prod_{i=1}^{n} P_i(\Phi) \cdot F^i dx d\theta d\bar{\theta} \right\rangle_{\chi},
\]
(39)
where \(P_1, \ldots, P_n\) are arbitrary polynomials and \(F^1, \ldots, F^n\) arbitrary functions on superspace.

**Lemma 5** Let \(F^1, \ldots, F^n \in \mathcal{S}(\mathcal{G})\) be supersymmetric smooth functions and \(P_1, \ldots, P_n\) be \(n\) polynomials then
\[
\left\langle \prod_{i=1}^{n} P_i(\Phi)(x, \theta, \bar{\theta}) \cdot F^i(x, \theta, \bar{\theta})dx d\theta d\bar{\theta} \right\rangle_{\chi} = (4\pi)^n \left\langle \prod_{i=1}^{n} f_i^0(0)P_i(\phi(0)) \right\rangle_{\chi}.
\]

**Proof** We define the distribution \(\mathcal{H}^1 \in \mathcal{S}'(\mathcal{G})\) in the following way:
\[
\mathcal{H}^1(G) := \left\langle \int P_i(\Phi)(x, \theta, \bar{\theta}) \cdot G(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} \prod_{i=2}^{n} \int P_i(\Phi)(x, \theta, \bar{\theta}) \cdot F^i(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} \right\rangle_{\chi}
\]
for any \(G \in \mathcal{S}(\mathcal{G})\). Using the fact that \(F^2, \ldots, F^n\) are supersymmetric and relation (39) we have that
\[
\mathcal{H}^1(\tau(b, \bar{b})(G)) = \left\langle \int P_i(\Phi) \cdot \tau(b, \bar{b})(G) dx d\theta d\bar{\theta} \prod_{i=2}^{n} \int P_i(\Phi) \cdot F^i dx d\theta d\bar{\theta} \right\rangle_{\chi} = \left\langle \int P_i(\Phi) \cdot \tau(b, \bar{b})(F^i) dx d\theta d\bar{\theta} \prod_{i=2}^{n} \int P_i(\Phi) \cdot \tau(b, \bar{b})(F^i) dx d\theta d\bar{\theta} \right\rangle_{\chi} = \mathcal{H}^1(G).
\]
This means that \(\mathcal{H}^1\) is supersymmetric and since \(F^1\) is also supersymmetric, by Theorem \(\square\) we conclude
\[
\mathcal{H}^1(F^1) = f_1^0(0)\mathcal{H}^0(0) = (4\pi)^n \left\langle f_1^0(0)P_1(\phi(0)) \prod_{i=2}^{n} \int F^i \cdot P_i(\Phi) dx d\theta d\bar{\theta} \right\rangle_{\chi} = \mathcal{H}^1(V)
\]
where \(V := (4\pi)\delta_0(dx)\theta\bar{\theta}\). Setting
\[
\mathcal{H}^2(G) := \left\langle \left( \int P_i(\Phi) V dx d\theta d\bar{\theta} \right) \left( \int P_i(\Phi) G dx d\theta d\bar{\theta} \right) \prod_{i=3}^{n} \int P_i(\Phi) F^i dx d\theta d\bar{\theta} \right\rangle_{\chi}
\]
and reasoning similarly we also conclude that \(\mathcal{H}^2(F^2) = \mathcal{H}^2(V)\). Proceeding by transforming each subsequent factor, we can deduce that
\[
\left\langle \prod_{i=1}^{n} P_i(\Phi) F^i dx d\theta d\bar{\theta} \right\rangle_{\chi} = \left\langle \prod_{i=1}^{n} P_i(\Phi) V dx d\theta d\bar{\theta} \right\rangle_{\chi} = (4\pi)^n \left\langle \prod_{i=1}^{n} f_i^0(0)P_i(\phi(0)) \right\rangle_{\chi}.
\]
\(\Box\)
Theorem 12 Let \( V_1, V_2, \ldots, V_n \) be smooth functions such that they and their derivatives grow at most polynomially then
\[
\left\langle \prod_{i=1}^{n} \int V_i(\Phi(x, \theta, \bar{\theta}) \cdot F^i(x, \theta, \bar{\theta})dxd\theta \right\rangle = (4\pi)^n \left\langle \prod_{i=1}^{n} f_i^0(0)V_i(\phi(0)) \right\rangle.
\]

Proof By Lemma 5 the theorem holds for \( V_i = P_i \) polynomials. Then the thesis follows by the density of polynomials in the space of two-times differentiable functions with respect to the Malliavin derivative (see [40] Corollary 1.5.1). \( \square \)

Remark 9 Going back to eq. (33) the idea now would be to expand the exponential getting
\[
Z_f\Xi_\chi(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left( \int h(\Phi(x, \theta, \bar{\theta}))\theta\bar{\theta}\delta_0(dx)d\theta d\bar{\theta} \right)^n \right\rangle \chi \times \left( -\int V(\Phi(x, \theta, \bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta})dx d\theta d\bar{\theta} \right)^n \chi.
\]
and then to use Theorem 12 to prove that each average in the right hand side is equal to
\[
\langle h(\varphi(0))(-4\pi V(\varphi(0)))^n \rangle \chi.
\]
Since
\[
\langle h(\varphi(0))(-4\pi V(\varphi(0)))^n \rangle \chi = Z_f^0 \int_{R^n} g(y)d\nu(y)
\]
equality (30) would be proved taking the limit \( \chi \to 0 \). Unfortunately equation (40) is not easy to prove since the series on the right hand side of (40) does not converge absolutely for a general \( V \). For this reason we present below an indirect proof of (30).

4.2 Outline of the proof of Theorem 9

A proof of Theorem 9 requires a justification of the formal steps in the previous subsection. Much of the work is already present in [32] and indeed our result is analogous, under different hypotheses, to Theorem II in [32]. The proofs of Lemma 11, Lemma 13 and Lemma 14 below follows the same ideas of Lemma 3.1, Lemma 3.2 and Lemma 3.3 in [32]. We decided to propose a detailed proof of Theorem 9 mainly for two reasons:

1. The first reason is that the hypotheses on the potential \( V \) of Theorem 9 and of Theorem II in [32] are quite different. Indeed in [32] only polynomial potentials are considered while Hypothesis \( V_\lambda \) permits to consider polynomial of at most fourth degree perturbed by any bounded function. In order to prove the boundedness of \( \Lambda_U \) in \( L^p(\mu) \) under these different hypotheses we need to prove Lemma 12 which is a trivial consequence of hypercontractivity when the potential \( V \) is polynomial but is based on the non-trivial inequality (47) (proven in [49]) for general potentials \( V \).

2. The second main reason is the difference in the use of supersymmetry and of the supersymmetric representation of the integral [29]. Indeed, in our opinion there is a little gap in the proof of Theorem III of [32] that cannot be fixed without developing a longer proof. More precisely putting
\[
G(F) := \left\langle g(\varphi(0)) \exp \left( -\int V(\Phi) Fd\theta d\bar{\theta} dx \right) \right\rangle \chi.
\]

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in the proof of Theorem III of [32] it is tacitly assumed that the expression \( G(F) \) is supersymmetric with respect to the function \( F \), i.e. if \( F \) is a smooth function in \( S(\mathcal{S}) \) and \( \tau(b, \bar{b}) \) is a supersymmetric transformation, then we have that \( G(\tau(b, \bar{b})(F)) = G(F) \). In our opinion this fact is non-trivial since the law of \( \Phi \) is not supersymmetric (see Remark [5]). What can be easily proven is only that the expressions

\[
G^n(F) := \left\langle g(\varphi(0)) \left( \int V(\Phi) F \theta \theta d\theta dx \right)^n \right\rangle, 
\]

are supersymmetric in \( F \) (see Theorem [12] of Section [3,1]). This fact alone does not easily imply that \( G(F) \) is supersymmetric. Indeed for what we say in the discussion of Remark [9] we cannot guarantee that the series \( [10] \), which is equivalent to \( G(F) = \sum_{n \geq 0} \frac{n!}{2^n} G^n(F) \), converges absolutely when \( V \) growth at infinity at least as a polynomial of fourth degree (and we do not know under which conditions on \( V \) and \( F \) it converges relatively). In order to overcome this problem we propose a proof of Theorem [9] which exploits only indirectly the supersymmetric representation of the integral \( [20] \) in a way which permits to use only the supersymmetry of the expressions \( G^n(F) \) and avoiding the proof of the supersymmetry of the expression \( G(F) \) (see Lemma [8]).

The proof is based on two steps:

1. Prove Theorem [9] under the Hypothesis \( V_\lambda \), \( C \) and \( \text{CO} \).
2. Prove Theorem [9] under the Hypothesis \( V_\lambda \) with \( \lambda \) small enough and \( \text{CO} \).

We introduce some preliminary objects, notations and considerations. First of all, if \( V \) is a potential satisfying the Hypothesis \( V_\lambda \) we write

\[
V_{t,\lambda} = tv_B + \lambda v_U, 
\]

for any \( t \in \mathbb{R} \), denote by \( U_{t,\lambda} \) the corresponding map from \( W \) into \( \mathcal{H} \) and use a similar notation \( U_{t,\lambda}^\chi \) for the corresponding map analogous to \( U_{t,\lambda} \) when \( U \) is replaced by its regularized version \( [31] \). Let \( L : \mathbb{R} \to \mathbb{R} \) be a continuous bounded function. We write

\[
G^{x,L}(t) := \int_W L(J_{\chi}w(0)) \det_2(J_{\mathcal{H}} + \nabla U_{t,\lambda}^\chi) \times \\
\times \exp \left( -\delta(U_{t,\lambda}^\chi) - \frac{1}{2} \|U_{t,\lambda}^\chi\|^2_\mathcal{H} + 4 \int_{\mathbb{R}^2} V_{t,\lambda}(J_{\chi}w(x)) f'(x) dx \right) d\mu 
\]

and

\[
H^{x,L}(t) := Z_{\chi} \int_{\mathbb{R}^n} L(y) \exp \left( -4\pi \left( (1 + 2\chi) m^2 + \frac{|y|^2}{2} + tv_B(y) + \lambda v_U(y) \right) \right) dy. 
\]

It is evident that the thesis of Theorem [9] is equivalent to prove that

\[
G^x(1) := G^{0,L}(1) = H^x(1) := H_0^{x,L},
\]

for any bounded potential \( v_B \), any \( \lambda \) small enough and any \( L \) continuous and bounded. Since the set of bounded potentials is invariant with respect to rescaling, proving the previous equality for any bounded potential is equivalent to proving that \( G^x(t) = H^x(t) \) for any \( t \). Furthermore since \( H^x(t) \) is real analytic in \( t \) proving the previous equality is equivalent to proving that \( \partial^k_t G^x(0) = \partial^k_t H^x(0) \) for any \( k \in \mathbb{N} \) (or equivalently that \( G^x(t) = H^x(t) \) in a neighborhood of 0) and \( G^x(t) \) is real analytic for any \( t \in \mathbb{R} \).

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4.3 Proof under the Hypothesis C

Let $V_B$ be a bounded smooth function with first and second derivatives which are both bounded.

**Lemma 6** Let $V_B$ be a bounded potential satisfying the Hypothesis C, then $\exp(-t\delta(U^\chi)) \in L^1(\mu)$ for any $|t| \in \left[0, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right]$ and $\chi \in [0,1]$. Furthermore the integral $\int \exp(-t\delta(U^\chi))d\mu$ is uniformly bounded for $\chi \in [0,1]$ and $t$ in the compact subsets of $\left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$.

**Proof** Under the Hypothesis of the lemma we have that

$$\|\nabla U_{t,0}^\chi\| \leq |t| \frac{\|\partial^2 V_B\|_\infty}{m^2(1+\chi)}$$

where $\cdot \cdot$ is the usual operator norm on $\mathcal{L}(H)$. Proposition B.8.1 of [49] states that

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}K\right)\right] \leq (\mathbb{E}[\exp(\|K\|^2)])^{\frac{1}{2}} \left(\mathbb{E}\left[\exp\left(\frac{\|\nabla K\|^2}{(1-\|\nabla K\|)}\right)\right]\right)^{\frac{1}{2}}$$

whenever $K$ is a $H-C^1$ map such that $\|\nabla K\| < 1$. Taking $K = 2tU^\chi$ in the previous inequality we obtain the thesis. $\Box$

**Lemma 7** The function $G_0^{\chi,L}(t)$ is real analytic in $\left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$.

**Proof** First of all we have that for any $t \in \mathbb{R}$ the map $r \rightarrow \det_2(I + \nabla U^\chi_{t,0}) =: D_t(r)$ is holomorphic in $r$ (see [47] Theorem 9.3). By Cauchy theorem this means that

$$|\partial^\mu_t(\det_2(I + \nabla U^\chi_{t,0}))| \leq \frac{n! \sup_{t \in \mathbb{R}} |D_t(Re^{i\theta})|}{R^n}.$$ 

On the other hand we have for any $\chi \in [0,1]$ 

$$|D_t(r)| \leq \exp\left(\frac{1}{2}\|\nabla U^\chi_{t,0} + \nabla U^\chi_{t,0}\|_2^2\right) \leq \exp(C(t^2 + |r|^2)\|\partial^2 V_B\|^2_\infty),$$

where $C \in \mathbb{R}_+$ is some positive constant depending on $f$ but not on $V_B$. Thus we obtain

$$|\partial^\mu_t(\det_2(I + \nabla U^\chi_{t,0}))| \leq \frac{n! \exp(C(t^2 + |R|^2)\|\partial^2 V_B\|^2_\infty)}{R^n}.$$ 

With a similar reasoning we obtain a uniform bound of this kind for $\partial^n_t \exp\left(-\frac{1}{2}(U^\chi_{t,0})^2\right)$.

Finally we note that

$$\mathbb{E}[\exp(-\delta(U^\chi_{t+r,0}))] = \sum_{n=0}^{\infty} \frac{(-1)^n r^n}{n!} \mathbb{E}[\exp(-\delta(U^\chi_{t,0}))] \mathbb{E}[\delta(U^\chi_{t,0})]^n].$$

By Lemma 6 we note that

$$\mathbb{E}[\partial^n_t \exp(-\delta(U^\chi_{t+r,0}))] = \mathbb{E}[\exp(-\delta(U^\chi_{t+r,0}))\delta(U^\chi_{t,0})]^n]$$

$$\leq \frac{1}{\epsilon^n} \mathbb{E}[\exp(-\delta(U^\chi_{t+r})) + \exp(-\delta(U^\chi_{t-r}))] < +\infty$$

for any $t \in \left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$ and $0 < \epsilon < \frac{m^2}{2\|\partial^2 V_B\|_\infty} - |t|$. Using the previous inequality it follows that $G_0^{\chi,L}(t)$ is real analytic in the required interval. $\Box$
Lemma 8 For $\chi \in (0, 1)$ we have that $G_0^{\chi,L}(t) = H_0^{\chi,L}(t)$ for $t \in \left(-\frac{m^2}{2\|\sigma^2 V_n\|\|\cdot\|_H}, \frac{m^2}{2\|\sigma^2 V_n\|\|\cdot\|_H}\right)$.

Proof By Lemma 7 the function $G_0^{\chi,L}(t)$ is real analytic in $\left(-\frac{m^2}{2\|\sigma^2 V_n\|\|\cdot\|_H}, \frac{m^2}{2\|\sigma^2 V_n\|\|\cdot\|_H}\right)$. It is enough then to prove $\partial_t^n G_0^{\chi,L}(0) = \partial_t^n H_0^{\chi,L}(0)$ for any $n \in \mathbb{N}$. By the discussion in Section 4.3 we note that

$$G_0^{\chi,L}(t) = \left\langle L(\mathcal{J}_\chi w(0)) \det_2(I_H + \nabla U_{t,0}^{\chi}) \exp \left(-\frac{1}{2}\|U_{t,0}^{\chi}\|_{\mathcal{H}}^2\right) \times \exp \left(4 \int V_{t,0}(\mathcal{J}_\chi w(x)) f'(x) dx \right) d\mu \right\rangle \times$$

$$\times \exp \left(4 \int L(\mathcal{J}_\chi w(x)) f'(x) dx \right) \times \exp \left(4t \int V(\varphi(x)) f'(x) dx \right) \times \exp \left(4t \int V(\varphi(x)) \omega(x) dx \right) \times \exp \left(4t \int V(\varphi(x)) \omega(x) dx + 4t \int V(\varphi(x)) f'(x) dx \right) \times \exp \left(-t \int V(\Phi(x,\theta,\bar{\theta})) \bar{f}(\omega^2 + 4\omega \bar{\theta}) d\omega d\bar{\theta} \right) \times \exp \left(4t \int V(\Phi(x,\theta,\bar{\theta})) \bar{f}(\omega^2 + 4\omega \bar{\theta}) d\omega d\bar{\theta} \right) \times$$

From the previous equality we get

$$\partial_t^n G_0^{\chi,L}(0) = \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \delta \delta_0(dx) d\theta d\bar{\theta}) \times \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \Delta \delta_0(dx) d\omega d\bar{\omega}) \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \Delta \delta_0(dx) d\omega d\bar{\omega}) \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \delta \delta_0(dx) d\theta d\bar{\theta}) \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \delta \delta_0(dx) d\theta d\bar{\theta}) \right\rangle \right\rangle \right\rangle \right\rangle \times$$

Since the distribution $f(\omega^2 + \omega \bar{\theta})$ and $\theta \delta \delta_0(dx)$ are supersymmetric, by Theorem 12 we have

$$\left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \delta \delta_0(dx) d\theta d\bar{\theta}) \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \delta \delta_0(dx) d\theta d\bar{\theta}) \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \delta \delta_0(dx) d\theta d\bar{\theta}) \left\langle (-\int L(\Phi(x,\theta,\bar{\theta})) \theta \delta \delta_0(dx) d\theta d\bar{\theta}) \right\rangle \right\rangle \right\rangle \times$$

And since

$$\partial_t^n H_0^{\chi,L}(0) = (-4\pi)^n \langle L(\varphi(0)) \rangle \times$$

the thesis follows. \qed

Proposition 3 We have that $G_0^L(t) = H_0^L(t)$ in $\left(-\frac{m^2}{2\|\sigma^2 V_n\|\|\cdot\|_H}, \frac{m^2}{2\|\sigma^2 V_n\|\|\cdot\|_H}\right)$.

Proof We need only to prove that $G_0^{\chi,L}(t) \rightarrow G_0^L(t)$ as $\chi \rightarrow 0$. Since $\det_2$, $\delta$, $\|\cdot\|_H$ are continuous with respect to the natural norm of $H$ and the Hilbert-Schmidt norm on $H \otimes H$ (see [17] Theorem 9.2 for the continuity of $\det_2$ and [10] Proposition 1.5.4 for the continuity of $\delta$), and since $\exp(-\delta(U_{t,0}^\chi))$ is bounded uniformly in $L^p$ (for $p$ small enough) we only have to prove that, for $\chi \rightarrow 0$, $\frac{\nabla U_{t,0}^\chi}{\nabla U_{t,0}(w)} \rightarrow \nabla U_{t,0}(w)$ in $H$ and $\nabla U_{t,0}^\chi \rightarrow \nabla U_{t,0}(w)$ in $H \otimes H$ for almost every $w \in W$. We present only the proof of the second convergence, the proof of the first one being simpler and similar.

We have that

$$\nabla U_{t,0}^\chi[h] = (f \partial^2 V_B(\mathcal{J}_\chi w) \cdot \mathcal{J}_\chi h),$$

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thus proving the convergence of $\nabla U_{\phi}(w)$ in $\mathcal{H} \otimes \mathcal{H}$ is equivalent to proving the convergence of $(m^2 - \Delta)^{-1} x$ to $(m^2 - \Delta)^{-1} x$ in $\mathcal{H} \otimes \mathcal{H}$ and the convergence of $f \partial^2 V_B(J_t w)$ to $f \partial^2 V_B(Iw)$ in $C^0(\mathbb{R}^2)$. The first convergence follows from direct computation using the Fourier transform of this operators. The second convergence follows from the fact that $V_B$ is smooth with bounded derivatives, $f$ decays exponentially at infinity and $J_t w$ converges to $Iw$ pointwise and uniformly on compact sets since $(m^2 - \Delta)^{-1} \to \text{id}_{L^2}$, weakly as bounded operator on $L^2(\mathbb{R}^2)$ and $(m^2 - \Delta)^{-1}$ is a compact operator from $L^2(\mathbb{R}^2)$ into $C^0_{\text{loc}}(\mathbb{R}^2)$.

Introduce the following equation for $\phi_t = \tilde{\phi}_t + \mathcal{I} \xi$:

$$(m^2 - \Delta) \tilde{\phi}_t + t f \partial V_B(\tilde{\phi}_t + \mathcal{I} \xi) = 0. \quad (41)$$

Denote by $\lambda_-$ the infimum on $y \in \mathbb{R}^n$ over the eigenvalues of the $y$ dependent matrix $(\partial^2 V_B(y))$, and with $\lambda_+$ the supremum on $y \in \mathbb{R}^n$ over the eigenvalues of the same matrix.

For $t \in \left(-\frac{m^2}{\lambda_+ + 0}, \frac{m^2}{\lambda_- + 0}\right)$, we have that equation (41) has an unique solution that, by the Implicit Function Theorem, is infinitely differentiable with respect to $t$ when $V_B \in C^\infty(\mathbb{R}^n)$. Define the formal series

$$S_t(r) := \sum_{n \geq 1} \sup_{x \in B^2} |\partial_t^n \tilde{\phi}_t(x)| r^n. \quad (42)$$

**Lemma 9** Suppose that $V_B$ is a bounded function with all derivatives bounded such that

$$\|\partial^n V_B\|_\infty \leq C^n n!,$$

where the norm is the one induced by the identification of $\partial^n V_B$ as a multilinear operator and for some $C \in \mathbb{R}_+$, then the $r$ power series $S_t(r)$ is holomorphic for any $t \in \left(-\frac{m^2}{\lambda_+ + 0}, \frac{m^2}{\lambda_- + 0}\right)$.

Furthermore the radius of convergence of $S_t(r)$ can be chosen uniformly for $t$ in compact subsets of $\left(-\frac{m^2}{\lambda_+ + 0}, \frac{m^2}{\lambda_- + 0}\right)$.

**Proof** We define the following functions

$$V^1(r) := \sum_{n \geq 0} \frac{\|\partial^n+1 V\|_\infty}{n!} r^n, \quad V^2(r) := \sum_{n \geq 0} \frac{\|\partial^n+2 V\|_\infty}{n!} r^n.$$

We have that the partial derivative $\partial^n_t \tilde{\phi}_t$ solves the following equation

$$(m^2 - \Delta) \partial^n_t \tilde{\phi}_t + t \partial^2 V_B(\tilde{\phi}_t) \cdot \partial^n_t \tilde{\phi}_t = -\partial^{n-1}_t (\partial V_B(\tilde{\phi}_t) + t \partial^2 V_B(\tilde{\phi}_t) \cdot \partial_t \tilde{\phi}_t) + t \partial^2 V_B(\tilde{\phi}_t) \cdot \partial^n_t \tilde{\phi}_t$$

Using a reasoning similar to the one of Lemma 11, it is easy to prove that

$$\|\partial^n_t \tilde{\phi}_t\|_\infty \leq \frac{\|\partial^n-1(\partial V_B(\tilde{\phi}_t) + t \partial^2 V_B(\tilde{\phi}_t) \cdot \partial_t \tilde{\phi}_t) + t \partial^2 V_B(\tilde{\phi}_t) \cdot \partial^n_t \tilde{\phi}_t\|_\infty}{m^2 - |t|(|\lambda_{\text{sign}(t)} + 0)},$$

where it is important to note that the right hand side of the previous inequality depends only on the derivatives of order at most $n - 1$. The previous inequality and the method of majorants (see 50) of holomorphic functions permit to get the following differential inequality for $S_t(r)$

$$(m^2 - |t|(|\lambda_{\text{sign}(t)} + 0) - r V^2(S_t(r))) \partial_r(S_t)(r) \leq V^1(S_t(r)). \quad (43)$$
From the previous inequality we obtain that $S_i(t)$ is majorized by the holomorphic function $F_i(t)$ that is the solution of the differential equation \( \frac{\partial F_i}{\partial t} = -L_i F_i(t) \) (where the symbol \( \leq \) is replaced by \( = \)) depending parametrically on \( t \) with initial condition $F_i(0) = 0$. Since $F_i(t)$ is majorized by $F_k(t)$ or by $F_{-k}(t)$ if $|t| \leq k$ the thesis follows.

\[ \square \]

**Remark 10** An example of potential satisfying the hypotheses of Lemma 9 is given by the family of trigonometric polynomials in $\mathbb{R}^n$.

**Lemma 10** Under the hypotheses of Lemma 9 and supposing that $L$ is an entire function we have that $G^L_0(t) = H^L_0(t)$ for any $t \in \left( -\frac{n^2}{|\lambda|+\lambda_0^0}, \frac{n^2}{|\lambda|+\lambda_0^0} \right)$. In other words the thesis of Theorem 6 holds if $\lambda = 0$, $V_B$ satisfies Hypothesis C as well as the hypotheses of Lemma 9.

**Proof** By Proposition 4 we need only to prove that $G^L_0$ is real analytic in the required set. By Corollary 2 we have that

\[
G^L_0(t) = \mathbb{E} \left[ L(T\xi(0) + \hat{\phi}(0))e^{\int tv_B(T\xi(x) + \hat{\phi}(x))f(x)dx} \right].
\]

Then the thesis follows from Lemma 9 and the analyticity of $L$ and of the exponential. \[ \square \]

**Proposition 4** Under Hypothesis $V_\lambda$ we have that $G^L_\lambda(t) = H^L_\lambda(t)$ for any $t \in \left( -\frac{n^2}{|\lambda|+\lambda_0^0}, \frac{n^2}{|\lambda|+\lambda_0^0} \right)$. In other words the thesis of Theorem 6 holds if $V$ satisfies Hypothesis C.

**Proof** By Lemma 10 we know that Theorem 6 holds for any $\lambda = 0$ and for any bounded potential satisfying Hypothesis C and the hypothesis of Lemma 9. Thus if we are able to approximate any potential $V$ satisfying Hypothesis $V_\lambda$ and Hypothesis C by potentials of the form requested by Lemma 10 the thesis is proved.

We can use the methods of the proof of Lemma 3 for approximating a potential $V$ satisfying Hypothesis $V_\lambda$ by a sequence of potentials $V_{B,N}$ satisfying the hypothesis of Lemma 9. More in detail, using the notations of Lemma 3, we have that the sequence of functions $V^K$ is composed by smooth, bounded functions and, if $V$ satisfies Hypothesis $V_\lambda$, they are identically equal to $N$ outside a growing sequence of squares $Q_N \subset \mathbb{R}^2$. This means that $V^{N,p}$, which is the periodic extension of $V^K$ outside the square $Q_N$, is a smooth function for any $N \in \mathbb{N}$. Since $V^{N,p}$ is periodic it can be approximated with any precision we want by a trigonometric polynomial $P_N$. Furthermore since $V$ satisfies Hypothesis C, also $V^{N,p}$ satisfies Hypothesis C and we can choose the trigonometric polynomial $P_N$ satisfying Hypothesis C too. In this way we construct a sequence of potentials $V_{B,N} = P_N$ satisfying the hypotheses of Lemma 9 and converging to $V$ uniformly on compact sets. Thus the thesis follows from Lemma 2, Lemma 3, Corollary 2 and the fact that the functions of the form $L(T\xi(0) + \hat{\phi}(0))$, where $L$ is an entire function, are dense in the set of measurable functions in $T\xi(0) + \hat{\phi}(0)$ with respect to the $L^p(\mu)$ norm. \[ \square \]

### 4.4 Extension

**Lemma 11** Under the Hypothesis $V_\lambda$ we have $\det(I + \nabla U(w)) \in L^\infty(\mu)$.

**Proof** We follow the same reasoning proposed in [22] for polynomials. First of all, by the invariance property of the determinant with respect to conjugation, we have that

\[
\det(I + \nabla U(w)) = \det(I + O(w))
\]

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where $O(w)$ is the selfadjoint operator given by
\begin{equation}
O_2^1(w)[h] = (m^2 - \Delta)^{-\frac{1}{4}}(f \partial_{\psi}^2 \cdot V(\mathcal{I}w) \cdot (m^2 - \Delta)^{-\frac{1}{4}}h).
\end{equation}

Since $V$ satisfies the Hypothesis QC the eigenvalues of the symmetric matrix $\partial^2 V(y)$ (where $y \in \mathbb{R}^n$) are bounded from below. Furthermore we can write the matrix $\partial^2 V(y)$ as the difference of two commuting matrices $\partial^2 V(y) = V_+(y) - V_-(y)$ where $V_+(y), V_-(y)$ are symmetric, they have only eigenvalues greater or equal to zero and $\ker V_+(y) \cap \ker V_-(y) = \{0\}$. We denote by $O^+, O^-$ the two operators defined as in equation (44) replacing $\partial^2 V$ by $V_+$ and $V_-$ respectively. Obviously $O^+$ and $O^-$ are positive definite and $O = O^+ - O^-$. By Lemma 3.3 \[32\] we have that
\[
|\det_2(I + O(w))| \leq \exp(2\|O^-(w)\|_2^2).
\]

Using a reasoning similar to the one of Proposition\[1\] and the fact that, under the Hypothesis $V_\lambda$, the minimum eigenvalue $\lambda(y)$ of $\partial^2 V(y)$ has a finite infimum $\lambda_\lambda$ that is the same as the one for $V_-$ we obtain
\[
|\det_2(I + \nabla U(w))| = |\det_2(I + O(w))| \leq \exp(C\lambda_0 \|f\|_2^2)
\]
for some positive constant $C$. In particular we have $\det_2(I + \nabla U(w)) \in L^\infty$. \hfill \Box

In order to prove that $\exp(-\delta(U)) \in L^p$ we split $U$ into two pieces. First of all if $\lambda(y)$ is the minimum eigenvalue of $\partial^2 V(y)$ we recall that $\lambda_- = \inf_{y \in \mathbb{R}^n} \lambda(y)$. Moreover we shall set
\[
\hat{U} := U - (\lambda_- \wedge 0) f \mathcal{I}w,
\]
and $\hat{U} := U - \hat{U}$. We also set $W := V + \frac{1}{\lambda_-} \|\psi\|^2$. We introduce a useful approximation of $\hat{U}(w)$ for proving Theorem \[14\] Let $P_n$ the projection of an $L^2(\mathbb{R}^2)$ function on the momenta less then $n$, i.e.
\[
P_n(h) = \int_{|k| < n} e^{ik \cdot \hat{h}} \hat{h}(k) dk,
\]
where $\hat{h}$ is the Fourier transform of $h$ defined on $\mathbb{R}^2$. We can uniquely extend the operator $P_n$ to all tempered distributions. In this way we define $U_n(w)$ as
\begin{equation}
U_n(w) := P_n[f \partial V(\mathcal{I}P_n w)]
\end{equation}
We shall denote by $U_n$ the expression corresponding to \[45\] where $V$ is replaced by $W$.

**Lemma 12** Under the Hypothesis $V_\lambda$ there exist two positive constants $C, \alpha$ independent on $p \geq 2$ and $n \in \mathbb{N}$ such that
\begin{equation}
\mathbb{E}[|\delta(U_n - U)|^p] \leq C(p - 1)^{2p} n^{-\alpha}.
\end{equation}
Furthermore a similar bound holds also for $\mathbb{E}[\|\nabla U_n\|_2^p - \|\nabla U\|_2^p]$ and $\mathbb{E}[\|\mathcal{I}w\|_2^p - \|P_n(\mathcal{I}w)\|_2^p]$.

**Proof** First of all we write $\hat{U} = U_B + \hat{U}_B$ where $U_B = f \partial V_B(\mathcal{I}w)$, and we consider the corresponding decomposition for $U_n$. If we prove that an inequality analogous to \[40\] holds for $U_B - U_{B,n}$ and $\hat{U}_B - \hat{U}_{B,n}$ separately then the inequality \[46\] holds.

In order to prove the lemma we use the following inequality (proven in \[49\] Proposition B.8.1)
\begin{equation}
\mathbb{E} \left[ \cosh \left( \frac{\mathcal{V}}{2\sqrt{2}} \delta(K) \right) \right] \leq \left( \mathbb{E}[\exp(p\|K\|_2^2)] \right)^{\frac{1}{p}} \cdot \left( \mathbb{E} \left[ \exp \left( \frac{p}{1 - \rho^2} \|\nabla K\|_2^2 \right) \right] \right)^{\frac{1}{p}}.
\end{equation}
that holds when \( \|\nabla K\|_2^2 \in L^\infty, \|\nabla K\| \leq c < 1 \) and \( 0 \leq \rho < \frac{1}{4c} \). Putting \( K = \tilde{c}(U_B - U_{B,n}) \) for \( \tilde{c} \) small enough, since \( \|\nabla(U_{B,n} - U_B)\|_2, \|\nabla(U_{B,n} - U_B)\| \in L^\infty \) with a bound uniform in \( n \), we have that

\[
E[\cosh(\epsilon\delta(U_B - U_{B,n}))] \leq \left( E[\exp(\epsilon\|U_B - U_{B,n}\|_H^2)] \right)^{\frac{1}{2}} : \left( E[\exp(\epsilon\|\nabla(U_B - U_{B,n})\|_H^2)] \right),
\]  

for suitable \( \epsilon, \epsilon' > 0 \) and for all \( n \in \mathbb{N} \). First of all we want to give a bound for the right hand side of (48) providing a precise convergence rate to the constant 1 of the upper bound for the right hand side as \( n \to +\infty \). We first note that

\[
E[\exp(\epsilon\|U_B - U_{B,n}\|_H^2)] = \sum_{k=1}^{\infty} \frac{e^{k\epsilon}}{k!} E[\|U_B - U_{B,n}\|_H^{2k}],
\]

Using a reasoning like the one in the proof of Proposition II we have that

\[
\|U_B - U_{B,n}\|_H^2 \lesssim \|\nabla V_B\|_\infty^2 \|Q_n(f)\|_H^2 + \|\nabla^2 V_B\|_\infty^2 \int \mathbb{R}^2 (f(x)Q_n(Iw)(x))^2 dx,
\]

where \( Q_n = I - P_n \). From the previous inequality and the hypercontractivity of Gaussian random fields we obtain that

\[
E[\|U_B - U_{B,n}\|_H^2] \lesssim k \left( \|Q_n(f)\|_H^2 + \int \mathbb{R}^2 (f(x)^k E[(Q_n(Iw)(x))^2] dx \right) \lesssim k\|Q_n(f)\|_H^2 + k(2k - 1)^k \|f\|_1^2 E[(Q_n(Iw)(x))^2],
\]

where the constants implied by the symbol \( \lesssim \) do not depend on \( k \). The right hand side converges then for \( n \to +\infty \) to 1 as we have announced. Using the Fourier transform, the fact that \( f \) is a Schwartz function, and the fact that \( Iw \) is equivalent to a white noise transformed by the operator \( (m^2 - \Delta) \) it is simple to obtain that \( \|Q_n(f)\|^2, \|E[(Q_n(Iw)(x))^2] \| \lesssim \frac{1}{n^2} \). Then using the fact that \( (2k - 1)^k \lesssim C_k^k k! \) and inserting the previous inequality in equation (49), we obtain

\[
E[\exp(\epsilon\|U_B - U_{B,n}\|_H^2)] \leq 1 + C_3 \frac{\epsilon'}{n^2},
\]

that holds when \( \epsilon' > 0 \) is small enough and for two positive constants \( C_2, C_3 \). Using similar methods it is possible to prove a similar estimate for \( E[\exp(\epsilon'\|\nabla(U_B - U_{B,n})\|_H^2)] \). Inserting these estimates in the inequality (48), we obtain

\[
E[\cosh(\epsilon\delta(U_B - U_{B,n}))] - 1 \lesssim \frac{\epsilon'}{n^2},
\]

where the constants implied by the symbol \( \lesssim \) do not depend on \( n \) and on \( \epsilon' \), when \( \epsilon' \) is smaller than a suitable \( \epsilon_0' > 0 \). Using the inequality (50) we obtain that

\[
\sum_{k,n=1}^{+\infty} n^{1/2} \frac{e^{2k}}{(2k)!} E[\delta(U_B - U_{B,n})^{2k}] = \sum_{n=1}^{+\infty} n^{\frac{1}{2}} (E[\cosh(\epsilon\delta(U_B - U_{B,n}))] - 1) \lesssim \sum_{n=1}^{+\infty} \frac{\epsilon'}{n^2} < +\infty.
\]

Since the terms of an absolutely convergent series are bounded we obtain

\[
E[(\delta(U_B - U_{B,n}))^{2k}] \lesssim \frac{(2k)!}{e^{2kn^2}} \lesssim (2k - 1)^{4k} n^{1/2}.
\]
Using Young inequality we obtain that the inequality (46) holds for any $p \geq 2$. The estimate for $\delta(U_n) - R \int_{\mathbb{R}^2} f(P_n I w)^2 dx - \|\nabla U_n\|_2^2 \geq -C_1 - C_2 c_n^2$ for any $R \in \mathbb{R}_+$. 

**Proof** If $\text{Tr}(\|K\|) < +\infty$ and $K \in \mathcal{W}$ we have that $\delta(K) = (K, w)_H - \text{Tr}(\nabla K)$. Using this relation we obtain that

$$\delta(U_n) = \sum_{i=1}^n \left( \int_{\mathbb{R}^2} P_k(f \partial_{\phi_i} W(P_k I w))(x) w_i(x) dx - \text{Tr}_{L^2}(P_k(f \partial_{\phi_i}^2 W(P_k I w) \cdot P_k(m^2 - \Delta))) \right).$$

From this we obtain the lower bound

$$\int_{\mathbb{R}^2} P_k(f \partial_{\phi_i} W(P_k I w)) w_i dx = \int_{\mathbb{R}^2} f \partial_{\phi_i} W(P_k I w)(m^2 - \Delta)(P_k I w') dx$$

$$= \int_{\mathbb{R}^2} f \partial_{\phi_i} W(I w_k)(m^2 - \Delta)(I w_k^i) dx$$

$$= \int_{\mathbb{R}^2} f \partial_{\phi_i} W(I w_k) \nabla(I w_k^i) \cdot \nabla w_k dx +$$

$$- \int_{\mathbb{R}^2} (\Delta f) W(I w_k) dx + m^2 \int_{\mathbb{R}^2} f w_k^i \partial_{\phi_i} W(I w_k) dx$$

$$\geq \int_{\mathbb{R}^2} f(m^2 I w_k^i \partial_{\phi_i} W(I w_k) - b^2 W(I w_k)) dx$$

On the other hand we have

$$\text{Tr}_{L^2}(P_k(f \partial_{\phi_i}^2 W(I w_k) \cdot P_k(m^2 - \Delta))) = c_n \int_{\mathbb{R}^2} \partial_{\phi_i}^2 W(I w_k) f dx$$

$$\leq \frac{c_n^2}{p} + \frac{1}{q} \int_{\mathbb{R}^2} (\partial_{\phi_i}^2 W(I w_k)(I w_k))^q f dx,$$

where $\frac{1}{q} + \frac{1}{p} = 1$ and $q < 2$. Furthermore we have that

$$\|\nabla U_k\|^2 \leq \int_{\mathbb{R}^2} \frac{1}{|x|^2 + m^2} dx \int_{\mathbb{R}^2} (\partial_{\phi_i}^2 V(I w_k))^2 f dx = \ell \int_{\mathbb{R}^2} (\partial_{\phi_i}^2 V(I w_k))^2 f dx,$$
where $\ell = \int_{\mathbb{R}^2} \frac{1}{(|x|^2 + m^2)^{1/2}} \, dx$. Using the previous inequality we obtain that

$$
\delta(U_n) - R \int_{\mathbb{R}^2} f|Iw_k|^2 \, dx - \|\nabla U_n\|^2_2 \\
\geq - \frac{cp}{p} + \int_{\mathbb{R}^2} f(m^2Iw_k \partial_{\phi} W(Iw_k) - b^2 W(Iw_k)) \, dx + \\
- \int_{\mathbb{R}^2} f \left( \frac{(\partial_{\phi}^2 W(Iw_k))^q}{q} + \ell(\partial_{\phi}^2 V(Iw_k))^2 + R|Iw_k|^2 \right) \, dx
$$

It is simple to see that there exists a $\lambda_0 > 0$ (depending only on $b^2$ and $m^2$) such that for any potential $V$ satisfying the Hypothesis $V_{\lambda}$ with $\lambda < \lambda_0$ the expression

$$
m^2 y_k^i \partial_{\phi} W(y) - b^2 W(y) - \frac{(\partial_{\phi}^2 W(y))^q}{q} - \ell(\partial_{\phi}^2 V(Iw_k))^2 - R|y|^2
$$

is bounded from below and thus the thesis of the lemma holds.

\textbf{Lemma 14} Given a $p \in [1, +\infty)$ there is a $R > 0$ big enough such that

$$
\exp \left( -\delta(U) - R \int_{\mathbb{R}^2} f(x)|Iw(x)|^2 \, dx \right) \in L^p(\mu).
$$

\textbf{Proof} This lemma is proven in [32] Lemma 3.2.

\textbf{Lemma 15} Suppose that $f$ satisfies the Hypotheses CO, then there exists $\lambda_0 > 0$ depending only on $f$ and $m^2$ such that for any $\lambda < \lambda_0$ and any $V$ satisfying the Hypothesis $V_{\lambda}$ we have that

$$
\exp(-\delta(U) + (1 + \|\nabla U\|^2_2)) \in L^p(\mu)
$$

for any $p \in [1, +\infty)$.

\textbf{Proof} The thesis follows from Lemma 11, Lemma 12, Lemma 13 and Lemma 14 using a standard reasoning due to Nelson (see Lemma V.5 of [46]) due to the fact that from the previous results it follows that there exist two constants $\alpha, \beta > 0$ independent on $N$ such that

$$
\mu\{w \in W | \delta(U_N)(w) \geq \beta(\log(N)) \} \leq e^{-N^\alpha}.
$$

\textbf{Proof of Theorem 9} By Proposition 4 in order to prove the theorem it remains only to prove that $G(t)$ is real analytic for any $t \in \mathbb{R}$. The proof of this fact easily follows from Lemma 15 exploiting a reasoning similar to the one used in Lemma 7.

\section{5 Removal of the spatial cut-off}

In this section we prove Theorem 3 on the removal of the spatial cut-off in the setting of Hypothesis C. It is important to note that, differently from Theorem 8 we explicitly require that the potential $V$ satisfies Hypothesis C and not only Hypothesis QC. This is not due to problems in proving the existence of solutions to equation (10) or in proving the convergence of the
cut-offed solution to the non-cut-offed one without the Hypothesis C (see Lemma 16). The main difficulty is instead to prove the convergence of \( \Upsilon_f(\phi)/Z_f \) to 1. Indeed the previous factor does not actually converge and what we can reliably expect is that
\[
\lim_{f \to 1} Z_f^{-1} E[\Upsilon_f(\phi_f)|\mathcal{F}(\phi_f(0))] \to 1,
\]
(51)
where hereafter \( \phi_f \) denotes the solution to the equation (4) with cut-off \( f \), i.e. \( \Upsilon_f(\phi_f)/Z_f \) becomes independent with respect to the \( \sigma \)-algebra generated by \( \phi_f(0) \).

To prove (51) directly is quite difficult due to the non-linearity of the equation or equivalently to the presence of the regularized Fredholm determinant in the expressions (23) and (22) (which is a strongly non-local operator). For this reason we want to exploit a reasoning similar to the one used in Section 4. With this aim we introduce the equation
\[
(m^2 - \Delta)\phi_{f,t} + tf \partial V(\phi_{f,t}) = \xi
\]
and the functions
\[
F^L_f(t) := Z_f^{-1} E \left[ L(\phi_{f,t}(0)) e^{4t f_2 \int_x \partial_x^2 V(\phi_{f,t}(x))dx} \right],
\]
where \( t \) is taken such that \( tf^2 V(0) + m^2 \), and \( F^L_f(t) = E[L(\phi_t(0))] \) (where \( \phi_t \) is the solution to (22) with \( f \equiv 1 \)). By Lemma 2 (whose proof does not use in any point the cut-off \( f \)) \( F^L_f(t) \) is real analytic whenever \( V \) is a trigonometric polynomial, \( tf^2 V(0) + m^2 \) is definite positive for any \( y \in \mathbb{R}^n \) and \( L \) is an entire bounded function. Furthermore, by Theorem 8 \( F^L_f(t) = H^L(t) \) (where \( H^L(t) = \int L(y)dx \)), see Section 4 which is real analytic. Thus if we are able to prove that \( \lim_{f \to 1} \partial^0_t F^L_f(0) = \partial^0_t F^L(0) \) we have that \( H^L(t) = F^L(t) \) whenever \( \partial^2 V + m^2 \) is definite positive proving that Theorem 8 when \( V \) is a trigonometric polynomial satisfying Hypothesis C. The idea, then, is to apply a generalization of Lemma 2, Lemma 3, Lemma 4 and the reasoning in the proof of Proposition 4 and in the proof of Theorem 8 in order to obtain Theorem 8.

**Remark 11** Hypothesis C is required in an essential way in the proof of the holomorphy of \( F^L(t) \), in particular in Lemma 7. The fact that the cutoff is removed does not allow to reason by approximation as we did in Theorem 8.

Since the proof is composed by many steps which are a straightforward generalization of the results of the previous sections of the paper, we write here only some details of the parts of the proof of Theorem 8 which largely differ from what has been obtained before.

Hereafter we denote by \( \omega_\beta(x) \) the function
\[
\omega_\beta(x) := \exp(-\beta \sqrt{1 + |x|^2})
\]
and introduce the space \( W_\beta \) where \( \beta > 0 \) in the following way
\[
W_\beta := (-\Delta + 1)C^0_{\exp,\beta}(\mathbb{R}^2; \mathbb{R}^n),
\]
where \( C^0_{\exp,\beta} \) is the space of continuous function with respect to the weighted \( L^\infty \) norm
\[
\|g\|_{\infty,\exp,\beta} := \sup_{x \in \mathbb{R}^2} |\omega_\beta(x)g(x)|.
\]
The triple \( (W_\beta, \mathcal{H}, \mu) \) is an abstract Wiener space. We introduce the obvious generalization of equation (13)
\[
(m^2 - \Delta)\bar{\phi} + \partial V(\bar{\phi} + \mathcal{L}\xi) = 0,
\]
(53)
where \( \tilde{\phi} = \phi - I\xi \).

Now we want to prove a result that can replace Lemma 1. Indeed Lemma 1 plays a central role in the previous sections of the paper, allowing to prove the existence of strong solutions to equation (43), the characterization of weak solutions in Theorem 6 and Theorem 7 and finally allowing to show the convergence of weak solutions using the convergence of potentials in Lemma 2 and Lemma 3.

**Lemma 16** Suppose that \( V \) satisfies the Hypothesis QC and suppose that \( \tilde{\phi} \) is a classical solution to equation (53), then there exists a \( \beta_0 \) depending only on \( m^2 \) such that, for any \( \beta < \beta_0 \)

\[
\| \tilde{\phi} \|_{\infty, \exp \beta} \lesssim \| \exp(\alpha[I\xi]) \|_{\infty, \exp \beta},
\]

where \( \| \exp(\alpha[I\xi]) \|_{\infty, \exp \beta} \) is almost surely finite and the constants implied by the symbol \( \lesssim \) depend only on \( H \) and \( m^2 \). Furthermore for any \( U \) open and bounded we have

\[
\| \tilde{\phi} \|_{C^{2,\gamma}(U)} \lesssim \| \exp(\alpha[I\xi]) \|_{U, \infty} \exp(\alpha\|\tilde{\phi}\|_{\infty, \exp \beta} \|\omega_2^{-1}\|_U, \infty)}
\]

where \( U_\epsilon := \{ x \in \mathbb{R}^2 \mid \exists y \in U, |y - x| \leq \epsilon \} \) and \( \epsilon > 0 \).

**Proof** The proof is very similar to the proof of Lemma 1. We report here only the passages having the main differences. For any \( \epsilon > 0 \) there is a \( \beta_0 > 0 \) and for any \( \beta < \beta_0 \) we have

\[
\frac{\Delta(\omega_{2\beta}(x))}{\omega_{2\beta}(x)} - \frac{|
abla \omega_{2\beta}(x)|^2}{\omega_{4\beta}(x)} < \epsilon, \quad x \in \mathbb{R}^2.
\]

Without loss of generality (using the result of Lemma 1) we have that \( \lim_{x \to \infty} |\tilde{\phi}(x)|^2 \omega_{2\beta}(x) = 0 \) and so \( x \mapsto |\tilde{\phi}(x)|^2 \omega_{2\beta}(x) \) has a positive maximum at \( \bar{x} \in \mathbb{R}^2 \). This means that \( -\Delta |\tilde{\phi}|^2 \omega_{2\beta}(\bar{x}) \geq 0 \) and \( \nabla \tilde{\phi} = -\frac{\tilde{\phi}}{2\omega_{2\beta}} \nabla \omega_{2\beta} \) we have that

\[
(m^2 - \epsilon)|\tilde{\phi}(\bar{x})|^2 \omega_{2\beta}(\bar{x}) \leq -\Delta |\tilde{\phi}|^2 \omega_{2\beta}(\bar{x}) + m^2 |\tilde{\phi}(\bar{x})|^2 \omega_{2\beta}(\bar{x})
\]

\[
\leq -\omega_{2\beta}(\bar{x}) (\tilde{\phi}(\bar{x}) \cdot \partial V[I\xi(\bar{x}) + \tilde{\phi}(\bar{x})]).
\]

Using a reasoning similar to the one of Lemma 1 the thesis follows. \( \square \)

Since the bounds (54) and (55) in \( C^{0, \exp \beta} \) and \( C^{2, \gamma}_{\text{loc}} \) imply the compactness in \( C^{0, \exp \beta} \), when \( \beta' < \beta \), Lemma 16 permits to prove the existence of strong solutions to equation (43), their uniqueness when \( V \) satisfies Hypothesis C and the generalization of Lemma 2, Lemma 3, Lemma 4, Proposition 4 and Theorem 8 needed in order to prove Theorem 3.

At this point the proof of Theorem 13 requires only the following additional statement.

**Theorem 13** Let \( V \) be a trigonometric polynomial, let \( L \) be a polynomial and let \( f_r \) be a sequence of cut-offs satisfying Hypothesis CO, such that \( f_r \equiv 1 \) on the ball of radius \( r \in \mathbb{N} \) and such that \( f_r'(x) \leq C_1 \exp(-C_2 |x| - r) \) for some positive constants \( C_1, C_2 \in \mathbb{R}_+ \) independent on \( r \), then

\[
\partial^k H^L(0) = \lim_{r \to +\infty} \partial^k F_{f_r}^L(0) = \partial^k F^L(0).
\]

To make the proof easy to follow we restrict ourselves to the scalar case, i.e. the case where \( n = 1 \). The general case is a straightforward generalization. We will also need certain results about iterated Gaussian integrals. So let us introduce first some notations.
We denote by $\mathcal{T}$ the set of rooted trees with at least one external vertex which is not the root. We denote by $\tau_0$ the tree with only one vertex other than the root. In this set we introduce two operations: if $\tau \in \mathcal{T}$ we denote by $[\tau]$ the tree obtained from $\tau$ by adding a new vertex at the root which becomes the new root, and if $\tau' \in \mathcal{T}$ we denote by $\tau \cdot \tau'$ the tree obtained by identifying the root of $\tau$ and $\tau'$. It is possible to obtain any element of $\mathcal{T}$ by applying iteratively a finite number of times the previous operations to $\tau_0$. Furthermore we define $I^\tau_f(x) \in C^0(\mathbb{R}^2)$ by induction in the following way

$$I^\tau_{\tau_0}(x) := I^\xi, \quad I^\tau_{[\tau]}(x) := \int_{\mathbb{R}^2} G(x-y) f(y) I^\tau_f(y) dy,$$

where $G(x)$ is the Green function of the operator $I = (m^2 - \Delta)^{-1}$. We need also to introduce the following notation. Suppose that $\tau, \tau' \in \mathcal{T}$ and let $\Pi_{\tau,\tau'}$ be the set of all possible pairing between the external vertices (excepted their roots) of the forest $\tau \cup \tau'$ and let $\Pi_{\tau,\tau'}^{\text{int}} \subset \Pi_{\tau,\tau'}$ the set of all possible pairing involving separately the vertices of $\tau$ and $\tau'$. If $\pi \in \Pi$ we write

$$\Pi_{\tau,\tau'}^\pi(x,y) = E[I_{\tau,x}^\pi, f(y)],$$

where $\Pi_{\tau,\tau'}^\pi(x,y)$ are the expression $I^\tau_f(x)$ where $\xi$ is replaced by some copies of Gaussian white noises $\xi_V$ one for each vertex $V$ of $\tau$ and $\tau'$ which have correlation 0 if $(V,v') \notin \pi$ and are identically correlated otherwise.

**Lemma 17** With the notations and the hypotheses of Theorem 13 we have that for any $\tau, \tau' \in \mathcal{T}$

$$\lim_{r \to +\infty} \left( E \left[ I^\tau_f(0) \cdot \prod_{i=1}^p \int f_r(x) I^\tau_f(x) dx \right] - E[I^\tau_f(0)] \cdot E \left[ \prod_{i=1}^p \int f_r(x) I^\tau_f(x) dx \right] \right) = 0.$$

**Proof** We present the proof only for the case $p = 1$, since the general case is a straightforward generalization. Since $I^\tau_f$ are Gaussian random variables depending polynomially with respect to the white noise $\xi$, using the notation previously introduced we have

$$E \left[ I^\tau_f(0) \cdot \int f_r(x) I^\tau_f(x) dx \right] = E[I^\tau_f(0)] \cdot E \left[ \int f_r(x) I^\tau_f(x) dx \right] =$$

$$= \sum_{\pi \in \Pi_{\tau,\tau'}^{\text{int}}} \int_{\mathbb{R}^2} \Pi_{\tau,\tau'}^\pi(0,x)f_r(x) dx.$$

Let us consider the simplest case when $\tau = \tau_k := \ldots \tau_0 \ldots k$ times and $\tau' = \tau_{k'} = \ldots \tau_0 \ldots k'$ times. In this case we have

$$\Pi_{\tau,\tau'}^\pi(0,x) = \int G(0-y_1) f_r(y_1) \ldots G(y_k - x_1) G(x_1 - x_2) \times$$

$$\times f_r(x_2) \ldots f_r(x_{k'}) G(x_{k'} - x) dy_1 \ldots dy_k dx_1 \ldots dx_k.$$

In particular, since $G(x) = G \ast G$, which is the Green function of $I^2 = (m^2 - \Delta)^{-1}$, is bounded and positive, and since $G$ is positive we obtain that

$$|\Pi_{\tau,\tau'}^\pi(0,x)| \leq G^{\ldots \ast G}_{k+k'}(0-x) = \int_{\mathbb{R}^2} \frac{e^{-u \cdot x}}{(|l|^2 + m^2)^{k+k'}} dl.$$

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Thus we get
\[
|\mathcal{T}_{r,\tau'}^{f_\tau}(0, x)| \cdot (|x|^2 + 1) \leq \left|\int_{\mathbb{R}^2} (-\Delta + 1) \frac{e^{-il \cdot x}}{(|l|^2 + m^2)^{k+1}} \, dl \right| \leq C_3,
\]
where \( C_3 \in \mathbb{R}_+ \). Thus
\[
\int_{\mathbb{R}^2} \mathcal{T}_{r,\tau'}^{f_\tau}(0, x)f_r'(x) \, dx \leq \int_{\mathbb{R}^2} \frac{C_3}{(|x|^2 + 1)} C_1 \exp(-C_2(|x| - r)) \, dx \lesssim_{C_1,C_2,C_3} \frac{1}{r^2 + 1} \to 0.
\]

For the general case let us note that \( \mathcal{T}_{r,\tau'}^{f_\tau}(0, x) \) is built by taking the product and the convolution with the functions \( \mathcal{G}, f_\tau \) and \( \mathcal{C} = \mathcal{G} \ast \mathcal{G} \). We note that \( \mathcal{C} \) appears one time for every pair of vertices \((V_1, V_2) \in \pi \). Then, since \( \pi \notin \mathcal{P}^{\text{int}}_{r,\tau'} \) there is at least a couple \((V, V') \in \pi \) such that \( V \) is a vertex of \( \tau \) and \( V' \) is a vertex of \( \tau' \). Now we can bound the function \( \mathcal{C} \) with a constant \( C_4 \) for all pairs of vertices \((V_1, V_2) \neq (V', V')\) and \( f_r \) by 1 obtaining, for any \( x \in \mathbb{R}^2 \), that
\[
\mathcal{T}_{r,\tau'}^{f_\tau}(0, x) \lesssim C_4 \mathcal{T}_{r_2,r_3}^{f_\tau}(0, x)
\]
for some \( k_1, k_2, k_3 \in \mathbb{N} \). The thesis follows from the previous inequality and the bounds obtained on \( \mathcal{T}_{r_2,r_3}^{f_\tau}(0, x) \).

**Proof of Theorem 13** We write
\[
\mathcal{L}_{f_r}(t) := L(\phi_{f_r,t}(0)) \quad \mathcal{E}_{f_r}(t) := \exp \left( 4t \int_{\mathbb{R}^2} f_r'(x)V(\phi_{f_r,t}(x)) \, dx \right).
\]

We have
\[
\partial_t^k \mathcal{L}_{f_r}(t) = \sum_{0 \leq l \leq k} \binom{k}{l} \mathbb{E} \left[ \chi^{(k-l)}(0) \partial_t^l \left( \frac{\mathcal{E}_{f_r}(t)}{\mathbb{E}[\mathcal{E}_{f_r}(t)]} \right) \bigg|_{t=0} \right] = \mathbb{E}[\mathcal{L}_{f_r}(0)] + \sum_{1 \leq l \leq k} \sum_{0 \leq p \leq (k-l)-1} \binom{k}{l} \binom{l}{p} \mathbb{E}[\mathcal{L}_{f_r}^{(k-l)}(0) \cdot \mathcal{E}_{f_r}^{(l-p)}(0)] + \mathbb{E}[\mathcal{L}_{f_r}^{(k-l)}(0)\mathcal{E}_{f_r}^{(l-p)}(0))] \cdot \partial_t^p \left( \frac{1}{\mathbb{E}[\mathcal{E}_{f_r}(t)]} \right) \bigg|_{t=0},
\]
where we used the Leibniz rule for the derivative of the product and the relation
\[
\partial_t^l \left( \frac{1}{\mathbb{E}[\mathcal{E}_{f_r}(t)]} \right) \bigg|_{t=0} = - \sum_{0 \leq p \leq (k-l)-1} \binom{l}{p} \mathbb{E}[\mathcal{E}_{f_r}^{(l-p)}(0)] \cdot \partial_t^p \left( \frac{1}{\mathbb{E}[\mathcal{E}_{f_r}(t)]} \right) \bigg|_{t=0}.
\]

Since \( \partial_t^p \left( \frac{1}{\mathbb{E}[\mathcal{E}_{f_r}(t)]} \right) \bigg|_{t=0} \) is bounded from above and below when \( r \to +\infty \) if we are able to prove that \( \mathbb{E}[\mathcal{L}_{f_r}^{(k-l)}(0) \to \partial_t^k \mathcal{L}_{f_r}(t) \) and \( \mathbb{E}[\mathcal{L}_{f_r}^{(k-l)}(0) \cdot \mathcal{E}_{f_r}^{(l-p)}(0)] - \mathbb{E}[\mathcal{L}_{f_r}^{(k-l)}(0)\mathcal{E}_{f_r}^{(l-p)}(0)] \to 0 \) the theorem is proven.

First of all we note that
\[
(m^2 - \Delta) \partial_t^k \phi_{f_r,t}(t = 0) = k f_r \partial_t^{k-1}(V(\phi_{f_r,t})) \bigg|_{t=0} = (m^2 - \Delta) \partial_t^k \phi_{f_r,t}(t = 0) = k f_r \partial_t^{k-1}(V(\phi_{f_r,t})) \bigg|_{t=0} = \tag{56}
\]
for \( k > 0 \) and \( \phi_{f_r,0} = \mathcal{L} \xi \) for \( k = 0 \). This means that \( \mathcal{L}_{f_r}^{(k-l)}(0), \mathcal{E}_{f_r}^{(l-p)}(0) \) are given by a finite combination of convolutions and products between the function \( \mathcal{G} \) (i.e. the Green function of \( \mathcal{I} \)), the functions of the form \( V^{(l)}(\phi_{f_r,0}) \) (where \( V^{(l)} \) is the \( l \)-th derivative of \( V \)), the cut-off \( f_r \) and
Since $V$ is a trigonometric polynomial, by developing $V$ and its derivative by Taylor series, we obtain the following formal expressions

$$
\mathcal{L}_{f_{r}}^{(k)}(0) = \sum_{\tau \in \mathcal{T}} A_{\tau}^{k} \mathcal{I}_{f_{r}}^{(k)}(0), \quad \mathcal{E}_{f_{r}}^{(k)}(0) = \sum_{l} \sum_{\tau_{1}, \ldots, \tau_{l} \in \mathcal{T}} B_{\tau_{1}, \ldots, \tau_{l}}^{k,l}(x) \prod_{i=1}^{l} \int_{\mathbb{R}^{2}} f_{r}(x) \mathcal{I}_{r}^{(k,l)}(x) dx. \quad (57)
$$

The previous series are not only formal but they are actually absolutely convergent series. Furthermore we can change the integral, the expectation and the limit with the series.

In order to prove this we now note that there exist two positive constants $C, \alpha > 0$ such that the function $V$ is majorized (in the meaning of the majorants method) by $C \exp(\alpha x)$ and let $\tilde{L}$ be the polynomial which majorizes the polynomial $L$. We now consider $\tilde{\mathcal{L}}_{f_{r}}(t) = \tilde{L}(\phi_{f_{r}, t}(0))$ and $\tilde{\mathcal{E}}_{f_{r}}(t) = (tC \int_{\mathbb{R}^{2}} f_{r}(x) \exp(\alpha \phi_{f_{r}, t}(x)) dx)$. For what we said, $\tilde{\mathcal{L}}_{f_{r}}^{(k)}(0)$ and $\tilde{\mathcal{E}}_{f_{r}}^{(k)}(0)$ are a finite combination of convolutions and products between $G$, the functions of the form $V^{(l)}(\phi_{f_{0}, 0})$ (where $V^{(l)}$ is the $l$-th derivative of $V$), the cut-off $f_{r}$ and $f_{r}'$. Let $\hat{\mathcal{L}}_{f_{r}}^{(k)}$ and $\hat{\mathcal{E}}_{f_{r}}^{(k)}$ be some random variables having the same expression of $\tilde{\mathcal{L}}_{f_{r}}^{(k)}(0)$ and $\tilde{\mathcal{E}}_{f_{r}}^{(k)}(0)$ where we replace every appearance of $V(\phi_{f_{r}, 0}(x))$ by $C \exp(\alpha \phi_{f_{r}, 0}(x))$, every appearance of $V'(\phi_{f_{r}, 0}(x))$ with $C_{0} \exp(\alpha \phi_{f_{r}, 0}(x))$ and so on.

We introduce the following functions dependent on $\tau \in \mathcal{T}$ and defined recursively as follows

$$
\mathcal{J}_{\tau}^{f_{r}}(x) := |\mathcal{I}_{\tau}^{f_{r}}(x)|, \quad \mathcal{J}_{\tau}^{f_{r}}(x) := \int_{\mathbb{R}^{2}} \mathcal{G}(x - y) f_{r}(y) \mathcal{J}_{\tau}^{f_{r}}(y) dy, \quad \mathcal{J}_{\tau, \tau'}^{f_{r}}(x) := \mathcal{J}_{\tau}^{f_{r}}(x) \cdot \mathcal{J}_{\tau'}^{f_{r}}(x).
$$

We, then, obtain that

$$
\hat{\mathcal{L}}_{f_{r}}^{(k)} = \sum_{\tau \in \mathcal{T}} \hat{A}_{\tau}^{k} \mathcal{J}_{\tau}^{f_{r}}(0), \quad \hat{\mathcal{E}}_{f_{r}}^{(k)} = \sum_{l} \sum_{\tau_{1}, \ldots, \tau_{l} \in \mathcal{T}} \hat{B}_{\tau_{1}, \ldots, \tau_{l}}^{k,l}(x) \prod_{i=1}^{l} \int_{\mathbb{R}^{2}} f_{r}(x) \mathcal{J}_{\tau_{i}}^{f_{r}}(x) dx.
$$

By our construction we have that $\hat{\mathcal{L}}_{f_{r}}^{(k)}, \hat{\mathcal{B}}_{\tau_{1}, \ldots, \tau_{l}}^{k,l}$ are all greater or equal than zero and also the following inequalities hold $|\hat{A}_{\tau}^{k}| \leq \hat{A}_{\tau}^{k}, |\hat{B}_{\tau_{1}, \ldots, \tau_{l}}^{k,l}| \leq \hat{B}_{\tau_{1}, \ldots, \tau_{l}}^{k,l}$. Furthermore we have $|\mathcal{J}_{\tau}^{f_{r}}(x)| \leq \mathcal{J}_{\tau}^{f_{r}}(x)$.

Finally $\mathbb{E}[|\hat{\mathcal{L}}_{f_{r}}^{(k)}|^{p}, \mathbb{E}[|\hat{\mathcal{E}}_{f_{r}}^{(k)}|^{p}]$ are finite for any $p$, since the $x_{1}, \ldots, x_{1}$ function

$$
\mathbb{E} \left[ \exp \left( \beta \alpha \sum_{i=1}^{l} |\phi_{f_{r}, 0}(x_{i})| \right) \right] \leq +\infty,
$$

for any $\beta > 0$. Since $\mathcal{G}$ is positive the bounds on $\mathbb{E}[|\hat{\mathcal{L}}_{f_{r}}^{(k)}|^{p}, \mathbb{E}[|\hat{\mathcal{E}}_{f_{r}}^{(k)}|^{p}]$ can be chosen uniformly on $r$. This implies that the series (57) are absolutely convergent and by Lebesgue’s dominate convergence theorem we can exchange the series with the summation and the limit. This means that

$$
\lim_{r \rightarrow +\infty} \mathbb{E}[\mathcal{L}_{f_{r}}^{(k)}(0) \cdot \mathcal{E}_{f_{r}}^{(k)}(0)] = \mathbb{E}[\mathcal{L}_{f_{r}}^{(k)}(0)] \mathbb{E}[\mathcal{E}_{f_{r}}^{(k)}(0)] = \mathbb{E}[\hat{\mathcal{L}}_{f_{r}}^{(k)}(0)] \mathbb{E}[\hat{\mathcal{E}}_{f_{r}}^{(k)}(0)] = 0.
$$
where in the last line we used Lemma 17. In a similar way it is simple to prove that
\[ \mathbb{E}[\Sigma_{f_r}^{(k)}(0)] \to \partial_f^k F^L(0), \]
and this concludes the proof. \qed

A Transformations in abstract Wiener spaces

This appendix summarizes the results of [49] which are used in the paper and establish the related notations. Hereafter we consider an abstract Wiener space \((W, H, \mu)\) where \(W\) is a separable Banach space, \(H\) is an Hilbert space densely and continuously embedded in \(W\) (with inclusion map denoted by \(i : H \to W\)) called Cameron-Martin space and \(\mu\) is the Gaussian measure on \(W\) associated with the Cameron-Martin space, i.e. \(\mu\) is the centered Gaussian measure on \(W\) such that for any \(w^* \in W^*\) we have \(\mu(w^*) = \int_W \exp(i\langle w^*, w \rangle) d\mu(w) = \exp\left(-\frac{\|i^*(w^*)\|^2_H}{2}\right)\) where \(i^* : W^* \to H\) is the dual operator of \(i\).

If \(u : W \to \mathbb{R}\) is a measurable non-linear functional we denote by \(\nabla u : W \to H\) the following non-linear operator
\[ \nabla u(w)[h] = \langle \nabla u(w), h \rangle_H := \lim_{\epsilon \to 0} \frac{u(w + \epsilon h) - u(w)}{\epsilon}. \]
The operator \(\nabla\) is called Malliavin derivative and it is possible to prove that it is closable (with unique closure) on the set of measurable \(L^p(\mu)\) functions. We denote the domain of \(\nabla\) in \(L^p(\mu)\) by \(\mathbb{D}_{p,1}\). The previous operation can be extended for functional \(u : W \to H^{\otimes k}\) where \(\nabla u : W \to H^{\otimes k+1}\) with its natural topology. Also this extension of the operator \(\nabla\) is closable.

If the measurable non-linear operator \(F : W \to H\), where \(|F|_H \in L^p(\mu)\), is such that \(\mathbb{E}[\langle F, \nabla u \rangle_H] = \mathbb{E}[\tilde{F} u]\) for some \(\tilde{F} \in L^p(\mu)\), we say that \(F\) is in the domain of the operator \(\delta\) and we denote by \(\delta(F) = \tilde{F} \in L^p(\mu)\) the Skorokhod integral of the measurable operator \(F\). The following expression for \(\delta(F)\) used in the following holds: suppose that \(F(w) \in i^*(W)\) and that \(\nabla F(w)\) is a trace class operator on \(H\) for \(\mu\) almost every \(w \in W\) then
\[ \delta(F)(w) = \langle i^* \delta^{-1}(F(w)), w \rangle - \text{Tr}(\nabla F(w)). \] (58)

We introduce a definition for studying the random transformations defined on abstract Wiener spaces.

**Definition 2** Let \(U : W \to H\) be a measurable map. We say that \(U\) is a \(H - C^1\) map if for \(\mu\) almost every \(w \in W\) the map \(U_w : H \to H\), defined as \(h \mapsto U_w(h) := U(w + h)\), is a Fréchet differentiable function in \(H\) and if \(\nabla U_w : H \to H^{\otimes 2}\), defined as \(h \mapsto \nabla U_w(h) := \nabla U(w + h)\) where \(\nabla\) is the Malliavin derivative, is continuous for almost every \(w \in W\) and with respect to the natural (Hilbert-Schmidt) topology of \(H^{\otimes 2}\).

We introduce the shift \(T : W \to W\) associated with \(U\), i.e. the map defined as \(T(w) = w + U(w)\), and the non-linear functional \(\Lambda_U : W \to \mathbb{R}\) as follows
\[ \Lambda_U(w) = \text{det}_2(I_H + \nabla U(w)) \exp \left(-\delta(U)(w) - \frac{1}{2} \|U(w)\|_H^2\right), \] (59)
where \(\text{det}_2(I_H + \cdot)\) is the regularized Fredholm determinant (see [47] Chapter 9) at it is well defined for any Hilbert-Schmidt operator \(K\). In particular if \(K\) is self adjoint we have
\[ \text{det}_2(I + K) = \prod_{i \in \mathbb{N}} (1 + \lambda_i) e^{-\lambda_i}, \]
where \(\lambda_i\) are the eigenvalues of \(K\).
where $\lambda_i$ are the eigenvalues of the operator $K$.

Suppose that $U(w) \in i^s(W)$ and that $\nabla U(w)$ is a trace class operator for almost any $w \in W$, then using the expression (58) and the properties of $\det_2$ we obtain
\[
\Lambda_U(w) = \det(I_H + \nabla U(w)) \exp \left( -\langle i^{*,-1}(U(w)), w \rangle_W - \frac{1}{2} |U(w)|_H^2 \right),
\]
where $\det(I_H + \cdot)$ is the standard Fredholm determinant. The functional $\Lambda_U$ is closely related to the transformation of the measure $\mu$ with respect to the transformation $T$. Indeed suppose that $W$ is finite dimensional then we have
\[
d\mu = \exp \left( -\frac{1}{2} \langle w, w \rangle_H \right) \frac{dx}{Z} = \exp \left( -\frac{1}{2} \langle i^{*,-1}(w), w \rangle_W \right) \frac{dx}{Z},
\]
where $Z \in \mathbb{R}_+$ is a suitable renormalization constant and $dx$ is the Lebesgue measure on $W$.

Thus, if $T$ is a diffeomorphism on $W$, we evidently have, thanks to equation (60),
\[
T_\ast(\mu) = \left| \det(I + \nabla U(w)) \exp \left( -\langle i^{*,-1}(U(w)), w \rangle_W - \frac{1}{2} \langle i^{*,-1}(w), U(w) \rangle_W \right) \right| = |\Lambda_U(w)|.
\]
The previous relation can be extended to the case where $W$ and $H$ are infinite dimensional and the transformation $T$ is not a diffeomorphism but it is only a $H - C^1$ map.

First of all we need the following generalization to the abstract Wiener space context of the finite dimensional Sard Lemma.

**Proposition 5** Let $T$ be a $H - C^1$ map and let $M \subset W$ be the set of the zeros of $\det_2(I + \nabla U(w))$ then the $\mu$ measure of the set $T(M)$ is zero, i.e. $\mu(T(M)) = 0$.

**Proof** See Theorem 4.4.1 [49].

The following is the change of variable theorem for (generally not invertible) $H - C^1$ maps.

**Theorem 14** Let $T$ be an $H - C^1$ map and let be $f, g$ two positive measurable functions then
\[
\int_W f \circ T(w) g(w) |\Lambda_U| d\mu(w) = \int_W f(w) \left( \sum_{y \in T^{-1}(w)} g(y) \right) d\mu(w).
\]
In particular if $K \subset W$ is a measurable subset where $T|_K$ is invertible we
\[
\int_K f \circ T(w) |\Lambda_U| d\mu(w) = \int_{T(K)} f(w) d\mu(w).
\]

**Proof** See Theorem 4.4.1 [49].

In order to prove Theorem 14 and so the relationship between the weak solutions to equation (6) and the integrals with respect to the signed measure $\Lambda_U d\mu$, it is not enough to consider Theorem 14 but we need a relationship analogous to (61) with $|\Lambda_U|$ replaced by $\Lambda_U$. In order to achieve this result we need some more hypotheses on the map $U$:

**Hypothesis DEG1** The map $U : W \rightarrow H \rightarrow W$ is a Fréchet differentiable map from $W$ into itself and furthermore it is $C^1$ with respect to the usual topology of $W$;

**Hypothesis DEG2** The map $T$ is proper (i.e. inverse images of compact subsets are compact) and the equation $T^{-1}(y) = w$ has a finite number of solution $y$ for $\mu$ almost every $w \in W$. 

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Under the Hypothesis DEG1 and DEG2 we can define the following number

$$\text{DEG}(w, T) := \sum_{y \in T^{-1}(w)} \text{sign}(\det_2(I_W + \nabla U(y))).$$

This index is a suitable modification of the Leray-Schauder degree of a Fredholm non-linear operator described, for example, in [10] Section 5.3C, where the following definition is given: if $B$ is a bounded set of $W$ such that $T^{-1}(w) \cap \partial B = \emptyset$ and $\nabla T(y) \neq 0$ for $y \in T^{-1}(w) \cap B$ we have

$$\text{DEG}_B(w, T) = \sum_{y \in T^{-1}(w)} (-1)^{\text{number of negative eigenvalues of } \nabla T(y)}.$$ 

It is evident that under the Hypothesis DEG2 and, as a consequence of Proposition 5, we have

$$\lim_{B \to W} \text{DEG}_B(w, T) = \text{DEG}(w, T)$$

for almost all $w \in W$.

**Theorem 15** Under the Hypotheses DEG1 and DEG2 we have that $\text{DEG}(w, T)$ is $\mu$ almost surely equal to the constant $\text{DEG}(T) \in \mathbb{Z}$ independent on $w$ and for any $f$ bounded function such that $f \circ T \cdot \Lambda_U \in L^1(\mu)$ we have

$$\int_W f \circ T(w)\Lambda_U(w) d\mu(w) = \text{DEG}(T) \cdot \int_W f(w) d\mu(w).$$

**Proof** The proof can be found in [10] Theorem 9.4.1 and Theorem 9.4.6.

In general is not simple to compute $\text{DEG}(T)$ but this computation simplified under the following Hypothesis:

**Hypothesis DEG3** The map $T_\epsilon(w) = w + \epsilon U(w)$ has bounded level set uniformly in $\epsilon \in [0, 1]$, i.e. if $B \subset W$ is bounded $\cup_{\epsilon \in [0, 1]} T_\epsilon^{-1}(B)$ is a bounded set in $W$.

**Theorem 16** Under the Hypotheses DEG1, DEG2 and DEG3 we have that, for any $\epsilon \in [0, 1]$

$$\text{DEG}(T) = \text{DEG}(w, T) = \text{DEG}(w, T_\epsilon) = 1.$$ 

**Proof** The proposition follows from the invariance of $\text{DEG}_B$ under homotopies of the operator $T$. In other words for any $B$ such that $T^{-1}(w) \cap \partial B = \emptyset$ we have $\text{DEG}_B(w, T_\epsilon) = \text{DEG}_B(w, T)$. Under the Hypothesis DEG3 we can choose $B$ big enough such that $\text{DEG}_B(w, T_\epsilon) = \text{DEG}(w, T_\epsilon)$ for any $\epsilon \in [0, 1]$. Since $\text{DEG}(w, T_0) = \text{DEG}(w, \text{id}_W) = 1$ the thesis follows.

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