A Note on the Weak Dirac Conjecture

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Abstract

We show that every set \( P \) of \( n \) non-collinear points in the plane contains a point incident to at least \( \lceil \frac{n}{3} \rceil + 1 \) of the lines determined by \( P \).

Keywords: Configurations of points; Incident-line-numbers; Weak Dirac Conjecture, Hirzebruch-type inequalities

In this note we denote by \( P \) a set of non-collinear points in the plane, and by \( \mathcal{L}(P) \) the set of lines determined by \( P \), where a line that passes through at least two points of \( P \) is said to be determined by \( P \). For a point \( P \in P \), we denote by \( d(P) \) the number of lines of \( \mathcal{L}(P) \) that are incident to \( P \), called the incident-line-number or multiplicity of \( P \); see [4] and [14]. Finally, we denote by \( l_r \) the number of lines that pass through precisely \( r \) points of \( P \).

Dirac’s conjecture is a well-known problem in combinatorial geometry. In 1951, Dirac [5] showed that:

**Theorem 1.** Every set \( P \) of \( n \) non-collinear points in the plane contains a point incident to at least \( \lceil \sqrt{n} + 1 \rceil \) lines of \( \mathcal{L}(P) \).

Dirac [5] made (and verified for \( n \leq 14 \)) the following conjecture.

**Conjecture 2 (Dirac Conjecture).** Every set \( P \) of \( n \) non-collinear points in the plane contains a point incident to at least \( \lceil \frac{n}{2} \rceil \) lines of \( \mathcal{L}(P) \).

The conjectured bound is tight, for instance, Dirac [5] constructed a set \( P \) of \( n \) non-collinear points with \( (l_2, l_3, l_4) = \left( \frac{n^2}{4} - \frac{3n}{2} + 3, \frac{n}{2} - 1, 2 \right) \) for every even-integer \( n \geq 6 \). In 2011, Akiyama, Ito, Kobayashi, and Nakamura [1] proved there exists a set \( P \) of \( n \) non-collinear points for every integer \( n \geq 8 \) except \( n = 12k + 11(k \geq 4) \), satisfying \( d(P) \leq \lceil \frac{n}{2} \rceil \) for every point \( P \in P \). However, Dirac’s conjecture is false, some counter-examples were found in [1, 7–11].

The following natural conjecture arises [4].
Conjecture 3 (Strong Dirac Conjecture). Every set $P$ of $n$ non-collinear points in the plane contains a point incident to at least $\left\lfloor \frac{n}{2} \right\rfloor - c_0$ lines of $L(P)$ with $c_0 > 0$.

In 1961, Erdős [6] proposed the following weakened conjecture.

Conjecture 4 (Weak Dirac Conjecture). Every set $P$ of $n$ non-collinear points in the plane contains a point incident to at least $\left\lceil \frac{n}{c_1} \right\rceil$ lines of $L(P)$ with $c_1 > 0$.

In 1983, the Weak Dirac Conjecture was proved independently by Beck [2] and Szemerédi and Trotter [20] with $c_1$ unspecified or very large.

In 2012, based on Crossing Lemma, Szemerédi-Trotter Theorem, and Hirzebruch’s inequality, Payne and Wood [17] proved the following theorem,

Theorem 5. Every set $P$ of $n$ non-collinear points in the plane contains a point incident to at least $\left\lceil \frac{n^3}{37} \right\rceil$ lines of $L(P)$.

In 2016, Pham and Phi [18] refined the result of Payne and Wood to give:

Theorem 6. Every set $P$ of $n$ non-collinear points in the plane contains a point incident to at least $\left\lceil \frac{n}{26} \right\rceil + 2$ lines of $L(P)$.

There are some results in algebraic geometry providing constraints on line arrangements in the projective plane. In [12, 13], Hirzebruch studied algebraic surfaces constructed as abelian covers of the projective plane branched along line arrangements in the context of the so-called ball-quotients. It turned out that he obtained, as a by-product, the following result which is known as Hirzebruch’s inequality.

Theorem 7 (Hirzebruch’s Inequality). Let $P$ be a set of $n$ points in the plane with at most $n - 3$ collinear. Then

$$l_2 + \frac{3}{4}l_4 \geq n + \sum_{r \geq 5} (2r - 9)l_r.$$

In 2003, Langer [15] provided a variation on the classical Bogomolov-Miyaoka-Yau inequality [16] using the so-called orbifold Euler numbers.

Theorem 8 (Orbifold Langer-Miyaoka-Yau Inequality). Let $(X, D)$ be a normal projective surface with a $\mathbb{Q}$-divisor $D = \sum_i a_i D_i$ with $0 \leq a_i \leq 1$. Assume that the pair $(X, D)$ is log canonical and $K_X + D$ is $\mathbb{Q}$-effective. Then

$$(K_X + D)^2 \leq 3e_{orb}(X, D),$$

where $e_{orb}(X, D)$ denotes the global orbifold number for $(X, \sum a_i D_i)$. Moreover, if equality holds, then $K_X + D$ is nef.

Bojanowski in [3] provided the following Hirzebruch-type inequality for line arrangements in the projective plane, which is also a special case of a much stronger result from the same thesis [3, Theorem 2.3]. It is worth pointing out that following Langer’s ideas, Pokora [19] provided some Hirzebruch-type inequalities for curve configurations in the projective plane with transversal intersection points where Bojanowski’s result is a special case.
Theorem 9 (Bojanowski-Pokora Inequality). Let $\mathcal{P}$ be a set of $n$ points in the plane with at most $\left\lfloor \frac{2n}{3} \right\rfloor$ collinear. Then
\[
l_2 + \frac{3}{4}l_3 \geq n + \frac{1}{4} \sum_{r \geq 5} r(r - 4)l_r.
\]

Based on the Bojanowski-Pokora inequality, we show the following result.

Theorem 10. Every set $\mathcal{P}$ of $n$ non-collinear points in the plane contains a point incident to at least $\left\lceil \frac{n}{3} \right\rceil + 1$ lines of $\mathcal{L}(\mathcal{P})$.

Proof. Suppose some line $L$ passes through $\left\lceil \frac{n}{3} \right\rceil + 1$ or more points of $\mathcal{P}$. Since $\mathcal{P}$ is non-collinear, there exists a point $P \in \mathcal{P}$ such that $P \not\in L$. Consider the (distinct) lines determined by $P$ and $\mathcal{P} \cap L$. Then $P$ is incident to at least $\left\lceil \frac{n}{3} \right\rceil + 1$ lines of $\mathcal{L}(\mathcal{P})$, and the theorem holds. Now assume that $\mathcal{P}$ does not contain $\left\lceil \frac{n}{3} \right\rceil + 1$ collinear points.

According to Theorem 9,
\[
l_2 + \frac{3}{4}l_3 \geq n + \frac{1}{4} \sum_{r \geq 5} r(r - 4)l_r = n + \frac{1}{2} \sum_{r \geq 5} \binom{r}{2}l_r - \frac{3}{4} \sum_{r \geq 5} rl_r.
\]

Since $\sum_{r \geq 2} \binom{r}{2}l_r = \binom{n}{2}$,
\[
l_2 + \frac{3}{4}l_3 \geq n + \frac{1}{2} \left( \binom{n}{2} - \frac{4}{2} \binom{r}{2}l_r \right) - \frac{3}{4} \sum_{r \geq 5} rl_r.
\]

That is,
\[
\sum_{r \geq 2} rl_r \geq \frac{n(n + 3)}{3}.
\]

Since $\sum_{P \in \mathcal{P}} d(P) = \sum_{r \geq 2} rl_r$,
\[
\sum_{P \in \mathcal{P}} d(P) \geq \frac{n(n + 3)}{3}.
\]

By the pigeonhole principle, $\mathcal{P}$ contains a point incident to at least $\left\lceil \frac{n}{3} \right\rceil + 1$ lines of $\mathcal{L}(\mathcal{P})$.

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