The nature of discrete space-time: The atomic theory of space-time and its effects on Pythagoras’s theorem, time versus duration, inertial anomalies of astronomical bodies, and special relativity at the Planck scale

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Abstract

In this work, resolutions will be given for commonly stated problems associated with a model that assumes that space and time are discretized (i.e., atomized). This model is in contrast to the continuous space-time model that is used in all common physical theories and equations – a model that assumes that spatial coordinates and time are continuous variables. The resolutions to the problems are arrived at, not by proposing any new theories or postulates, but by strictly adhering to: Ernst Mach’s principle of non-absolute space, the tenets of logical positivism, quantum mechanics and general relativity. The problems associated with discrete space-time addressed in this paper include: Lorentz contraction (time dilation) of the ostensibly smallest spatial (temporal) interval, maintaining isotropy, violations of causality, and conservation of energy and momentum. Importantly, this work yields modifications to the standard formulae for time dilation and length contraction, with these modifications preserving the quanta of space and time and allowing for temporary travel at the speed of light. Also given are: a resolution to Weyl’s tile argument, a modification of the 2500 year old Pythagoras’s theorem, a reassessment of Henri Bergson’s theory advocating a distinction between the time durations measured by scientists and an immutable “Time”, and a discussion of whether Einstein’s light-clocks are as ideal as most scientist believe. Also included is a demonstration of how discrete space imposes order upon John Wheeler’s quantum foam such that the foam becomes a gravity crystal permeating all space and producing measurable inertial anomalies of astronomical bodies.

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1. Introduction

“For me, both philosophy and science lose all their attraction when they give up that pursuit [of knowledge and understanding of the world] – when they become specialisms and cease to see, and to wonder at, the riddles of our world. Specialization maybe a great temptation for the scientist. For the philosopher it is the mortal sin.”

- Karl Popper, The World of Parmenides

While the atomic theory of materials, proposed in the 5th century BC by Epicurus, Leucippus and Democritus has been embraced and experimentally verified, the question of the atomization (i.e., discretization) of space-time (S-T) has received far less attention. Even though the number of people who have studied S-T discretization is less, the group includes eminent philosopher-scientists such as René Descartes, Isaac Newton, Gottfried Leibniz, George Berkeley, David Hilbert, Werner Heisenberg, John A. Wheeler and others - as excellently documented by Amit Hagar in [3]. According to its proponents, a discrete S-T would entail space and time being built up from indivisible units, with space arrayed in a grid of unit cells of equal size (or arrayed in an aperiodic tiling, such as Penrose tiling), and time progressing in indivisible equal steps, as shown in Fig. 1.

It is widely believed however, that discrete S-T suffers from the following problems:

1. Lorentz Contraction: The ostensibly smallest possible unit of length (i.e., $\chi$) in one inertial reference frame is Lorentz contracted to yet smaller lengths in moving reference frames.

2. Isotopy: Discrete space will introduce preferred directions in space; the motion of particles would be dependent on the direction of travel, even in matter-free space.

3. Causality: Forces and acceleration on both sides of the incompressible fundamental spatial unit cell are experienced perfectly simultaneously.
(i.e., no time delay). This seemingly violates the postulates of special relativity (SR) and causality.

4. Nonconservation of Energy and Momentum: Particles would be able to gain or lose momentum in units of $\frac{2\hbar\pi}{\chi}$.

5. Weyl Tiles: In 1949, Hermann Weyl claimed that if space is discrete, the length of the side of any size square must be equal to its diagonal. This non-adherence to Pythagoras’s theorem is not observed, hence space must not be discrete [5].

6. “Jerkiness” of motion [6]: Motion of a particle in discrete S-T occurs through discrete jumps from one grid point to the next - something thought to be unphysical [6]. Each single spatial jump (of length $\chi$) occurs over one fundamental duration of time ($\beta$). This jump is followed by several $\beta$s of duration where no jump occurs, and then followed by the next jump. During the jump, the particle is presumably traveling at the speed of light $c = \chi/\beta$.

Besides these problems, a discretized S-T must not predict any behavior that is at odds with what has already been physically measured. All current physical theories are built upon the continuous-space model and have been fantastically successful in describing practically all observed physical phenomena, from the structure of the atom to the motion of planets and stars. However, there are a few physical phenomena that are unexplained by conventional continuous-space-based physics, as well as some lingering philosophical issues:

7. Inertial anomalies (i.e., dark matter and dark energy): Can a correct fusion of quantum mechanics and gravity predict the motion of planets, stars and galaxies without resorting to any modern day “ether”? 

8. Selective quantization of observables: Why are some “measurables” quantized and not others?

9. Constancy of the speed of light: Velocity is determined by measurements of position and temporal duration. Why is it then, that a particular velocity, namely $c (= 3 \times 10^8 \text{ m/s})$, is elevated to the status of a sacrosanct constant of nature and not its foundational quantities of spatial and temporal intervals?
10. Absolute versus nonabsolute space: Does any aspect of space exist independent of matter?

11. Time: Is the time used by scientists (namely, “measured time”) the only time, or does a separate “Time” exist, namely Henri’s Bergson’s “real” or “psychological” time [7]?

A few preliminary notes on Problems 7-11 are warranted at this point. Concerning Problem 7, despite many decades of work, the unification of quantum mechanics (QM) and gravity remains unresolved, as well as explanations of inertial anomalies of some astronomical bodies. All approaches being studied (e.g., string theory, m-brane theory) use continuous S-T and absolute or pseudo-absolute S-T. This is seen by QM’s use of space and time differentials, general relativity’s (GR) assumption of the existence of S-T even in the absence of all particles [8], and the fact that GR allows absolute rotations (of the whole universe) [9]. We do not presume to unify QM with GR in this work, but we do show that there may be straightforward explanations for the inertial and gravitational anomalies of astronomical bodies (see Section 8). Issues 8-11 pertain to space, time, measurement, and the quantization of measurables. Concerning space, time and other measurables, most scientists and philosophers believe that the infinitesimally small and the infinitesimally large do not exist in nature [10], whether it be of energy, momentum, spin or any other measurable quantity. However, beliefs and opinions have no place in science. Thus the discretization of S-T needs to either be demonstrated using existing postulates and theories, or form a new, more fundamental set of postulates from which existing accepted postulates of physics can be derived, or are mere consequences. In this work we do both.

In the seven main sections of this paper, we take Popper’s advice to heart; we do not specialize and use tools from just one narrow philosophical or scientific field to address the nature of discrete S-T, but draw from humanity’s rich tool-chest of knowledge that includes epistemology, quantum mechanics, special and general relativity, solid-state physics and mathematics. Section 2 provides an outline of a deductive proof of S-T discretization, starting from existing and widely accepted scientific and philosophical theories. One reason for only an outline of a proof is that exact values of the space and time quantums are not necessary in addressing the main focuses of this paper: Pythagoras’s theorem, Lorentz transformations and motion in discrete S-T, philosophical implications concerning time and duration, and the inertial
anomalies of massive particles. Goethe provides inspiration for the second reason when he stated “the greatest art in theoretical and practical life consists in changing the problem into a postulate; that way one succeeds” \[11\]. Thus, after developing the proof, we elevate S-T discretization to the status of a postulate. Along with two postulates used in the proof, the complete set of postulates are \[1\]

1. Logical Positivism (LP) is the correct philosophy pertaining to the nature of existence and reality, and physical probes are required to perform a measurement (championed by David Hume, George Berkeley, Albert Einstein and others \[3\]) \[2\].

2. Space and time are non-absolute (the foundation of Mach’s Principle \[12, 13\]).

3. Space and time are quantized, or discretized (first considered by Paramides and Zeno \[3\] and later more explicitly by Maimonides \[2\]).

The proof in Section 2 makes the more conservative assumptions of Postulate 1 and 2 only, and then uses these to demonstrate the validity of Postulate 3. We do so because it is enlightening and useful (for later parts of this paper) to work through this demonstration that describes concepts of measurement, distance, the nature of reality, and the consistency of S-T quantization with, and its naissance from QM, GR and LP. In Section 3, a description is given of how continuous space is an artificial mathematical construct in which (or with which) to more easily obtain solutions to physical problems, and how to map coordinates of continuous space to coordinates of discrete space. Then in Section 4, a new solution is developed to the Weyl-tile argument that historically has been the unassailable argument against discrete space. In Section 5, resolutions are given for Problems 2, 3, 8, 9. Section 6 contains a discussion on how to calculate $\gamma$ assuming discretized S-T, and this model’s impact on time dilation and length contraction. Also in this section

\[1\] Again, these are not new postulates but have been proposed by a number of eminent philosopher-physicists. What we do in this work is to strictly adhere to the postulates and let them lead us where they may – to their ultimate conclusions.

\[2\] If one is concerned that LP involves a logically unjustified negation, then this postulate can be dropped if Postulate 2 and 3 are accepted and this paper’s conclusions remain unchanged.
is a discussion of how discrete S-T allows for particles with nonzero rest mass to temporarily travel at the speed of light. Section 7 contains a discussion of light-clocks with the conclusion that light-clocks with different “tick-tock” rates experience different amounts of time dilation and other irregularities, but in ways that ensure agreement amongst sets of light-clocks. Thus, as Henri Bergson suspected, Einstein’s ideal light-clock is not as ideal as once thought [7]. While Bergson may have been right in that regard, it is ironic that we find that the scientists’ measured time is more in line with what Bergson had in mind for his “psychological” or real “Time”, and that the only candidate for Bergson’s immutable Time is the atom of time itself (a duration on the order of $10^{-44}$ s) that has little or no impact on the daily lives or perceptions of humans. In Section 8 descriptions are given of physical effects that can be measured and used to verify and quantify properties of discrete S-T. These effects include the the possibility of an inertial mass $m_{\text{inertial}}$ of an object (how a particle accelerates due to an applied force) that is different than the gravitational mass $m_g$ (how much gravity a particle produces), with $m_{\text{inertial}}$ even being negative under certain circumstances.

2. Logical Positivism and the Quantums of Space and Time

One thing I am concerned about: you might, as you commence philosophy, decide you see impiety therein, and that the path you enter is the avenue to sin.

- Titus Lucretius Carus, The Nature of Things

“In this way, that [the question of being] which the ancient philosophers found continually disturbing as obscure and hidden has taken on a clarity and self-evidence that if anyone continues to ask about it he is charged with an error of method.”

- Martin Heidegger, Being and Time

Philosophers have debated for thousands of years about the acquisition of knowledge, measurement and being, and the nature and proof of existence. Throughout this time, philosophers have frequently been either ignored or held in contempt. In the quotes above, both Lucretius and Heidegger were lamenting the contempt for philosophy. For Lucretius, the thought-police
were the priests, for more modern-day philosophers such as Heidegger, the authority figures are the physicists. And even though Heidegger was discussing the question of being from an ontological perspective in his statement, as opposed to the epistemological perspective taken in this work, we both agree that contemporary scientists should not eschew philosophy, but embrace it and integrate its concepts within their scientific works and theories. As we shall see in this work, these philosophical issues are not just abstract concepts devoid of any physical consequences, but play a central role in our argument for a discretized S-T, with this discretization leading to measurable effects on the motion of massive particles.

The philosophical school of thought called empiricism, and is offshoot logical positivism (LP), connect measurement to existence ("What cannot be measured, ipso facto, does not exist" [14]). This measurement-being definition can be misinterpreted to mean that whatever cannot be measured with existing technology does not exist. However, the proper interpretation of LP is: whatever can never be measured by any observer, regardless of how advanced one’s technology, does not exist. This is the view taken in this paper. Note that the results of this section are themselves not entirely new; it is generally known that the quantums of space and time are on the order of the Planck length ($l_p = 1.62 \times 10^{-35}$ m) and Planck time ($\tau_p = 5.39 \times 10^{-44}$ s) respectively. However, a clear step-by-step demonstration of this seems to be lacking in the literature.

To calculate the shortest distance of space, we first have to review how one measures distances. Einstein instructed us that two probe-particles and a light-pulse (that serves as a signal-particle) are needed when measuring a distance in space (Fig. 2) [15]. Because any larger distance will be built up from multiples of the fundamental length (i.e., the quantum of spatial distance ($\chi$)), it is adequate to focus on what is the measurable minimum separation of these probe-particles, as shown in Fig. 3. Far into the future, technology may have advanced to such a degree that probe-particles can be made to be as precise and compact as nature allows. Such probes may be elementary particles with the same mass $m_{\text{probe}}$ but perform different functions. One probe ($P_A$) emits and receives a signal-particle ($P_S$) (e.g., a

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3For example, it has been stated that Ernst Mach doubted the existence of atoms because they could not be directly and individually measured at the time [13]. This is nonsense of course.
photons, with $P_S$ also being as spatially compact as possible. A separate and distinct probe ($P_B$) reflects $P_S$. The questions then are: how small can one make the probes (and therefore how precisely can their positions be determined), and what is the minimum separation between their two centers? This minimum separation is $\chi$ - a quantity called a “hodon” by Silberstein \[16\].

To estimate this minimum separation, one requires that the probe-particles remain distinct (i.e., non-overlapping). If both probe-particles have the same mass $m_{\text{probe}}$, then the minimum required separation is given by the Compton wavelength ($\lambda_c = \hbar/m_{\text{probe}}c$) of the particles, as shown in Fig. 3. As for how small the probes can be: they have to be of a size such that they are able to perform their assigned roles. Hence, $P_A$ needs to be able to emit some type of signal-particle, therefore it must have a radius at least as large as the Schwarzschild radius $R_s = 2Gm_{\text{probe}}/c^2$ otherwise the particle would be a black hole from which no particle can escape. This emitted signal-particle must be reflected by $P_B$; this sets the minimum radius of $P_B$ to $R_s$ as well. Equating $\lambda_c$ and $2R_s$ yields the mass of the probes and the minimum separation of the probes:

$$m_{\text{probe}} = \frac{1}{2} \sqrt{\frac{\hbar c}{G}} = \frac{1}{2} m_p = 1.09^{-8} \text{kg} \quad \chi = 2 \sqrt{\frac{\hbar G}{c^3}} = 2l_p = 3.24 \times 10^{-35} \text{m}$$

with $m_p = \sqrt{\hbar c/G} = 2.18 \times 10^{-8} \text{ kg}$ and $l_p = \sqrt{\hbar G/c^3} = 1.62 \times 10^{-35} \text{ m}$ as the Planck mass and Planck length respectively \[19\].

To determine the value of the quantum of time ($\beta$), usually called a chronon, assume that we have an elementary particle (e.g., $P_A$ of Fig. 2 or 3) serving as part of a perfect “clock”, with this particle emitting photons at perfectly regular intervals. These photons are received by another receiver-particle ($P_B$) and analyzed \[5\]. Assume that the position of $P_A$ is known

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as precisely as QM allows, namely within a spherical volume of diameter $\lambda_c$. We use the time intervals between successive photons incident upon $P_B$ to define a unit of time (i.e., $t_{\text{interval}}$). However, due to the uncertainty in where within $P_A$ the photon is emitted (namely anywhere within the clock’s spherical volume), each “tick” at the receiver can arrive either sooner or later than it ideally should (in comparison to the case where $P_A$ is infinitely small), by an amount of time $t_q = \lambda_c/c$, where $c$ is the speed of light. To minimize this uncertainty, one would choose an emitter-particle with as small of $\lambda_c$ as possible while having the photons still able to escape the clock. This leads to $P_A$ having a radius of $\lambda_c = 2l_p$, and the smallest measurable duration in time as:

$$\beta = \frac{2l_p}{c} = 2\tau_p = 1.08 \times 10^{-43} \text{s} \quad (2)$$

One can argue that the values of the quantums of length and time may be somewhat different upon a more careful analysis of the processes of signal-particle emission, reflection and detection. For example, one may say that conservation of momentum would require $P_A$ and $P_B$ to be more massive than $P_S$, such that $P_S$ can catch up to $P_A$ ($P_A$ may experience recoil upon emission of $P_S$) on its return trip. Also, the act of reflection and detection of $P_S$ may not be instantaneous with the arrival of $P_S$, but may each require an additional $2\tau_p$ of duration; this would make the $\beta$ be $6\tau_p$ instead of $2\tau_p$. However, whatever the value of $\beta$ is ultimately determined to be, the ratio $\chi/\beta$ must be $c$. And, as was said before, exact values of the quantums of space and time are not needed for the main focuses of this paper, what is needed is only a demonstration that minimums exist for these entities.

As was said in the introduction, now that a deductive argument has been developed to justify discrete S-T, we make the significant step of elevating $\chi$ and $\beta$ to the status of fundamental constants of nature, rather than derived quantities.

3. Mapping Points from Continuous Space to Discrete Space

For the purposes of this paper, we only need to know how to treat space between pairs of particles, and this is clearly described in Fig. 3. We then
use this information to counter the Weyl-tile argument against discrete space, derive a modified Pythagoras theorem, and discuss all other matters in this paper. However, before moving on to these issues, a short discussion on the issue of mapping points from continuous space to discrete space is interesting, even if it is not entirely necessary for the rest of this work.

After showing that points in space have a nonzero size, one can follow two different paths. One path involves divorcing ourselves completely of the current prevailing view of space and time as the “arena” in which particles interact, “space is not a stage, which might be either empty or full, onto which things come and go” [20]. Instead, one should view S-T as nothing more than a collection of spatial (temporal) separations between particles (events), which themselves are just more quantum mechanical parameters within a system’s wavefunction $\Psi$. And just like other quantum mechanical parameters (e.g., energy, spin, momentum), a system makes spatial transitions by a raising or lowering of this quantum parameter by a hodon. Similarly for time, but presumably a system can only experience a raising (not a lowering) of the time parameter by a chronon. This viewpoint is gaining increased interest by the research community due to its elegance and simplicity [20], however this concept involves an extreme reevaluation of space and time, and many problems with this concept remain unresolved.

Another, more conservative path is to view continuous space as a “dual” space of discrete space, with continuous space being a convenient but artificial space in which to solve problems. Once solutions are obtained in this dual space, one must map these solutions to real space (i.e., discrete space). For example, in the field of crystallography, one typically performs calculations in reciprocal space (the dual space of coordinate space) to determine the allowed directions of x-ray scattering by a crystal [21]. Even though this viewpoint has some attractive aspects, it also has some problems. One of them is that there is not a single unambiguous mapping from continuous space to discrete space. There are different mappings for each pair of particles; there are as many continuous-to-discrete space mappings as there are particle-pairs in the universe. But once again, we do not have to subscribe to either of these viewpoints; only the information in Fig. 3 is necessary.

4. Weyl Tiles and Leopold’s Theorem

In 1949 Hermann Weyl introduced what has come to be viewed as the definitive argument against discrete S-T [5]. Weyl’s argument uses a con-
struction (shown in Fig. 4) in which space is defined a priori – namely, an absolute space laid out in a grid for all particles to reside within, and on which all measurements made. Distances in this model are required to be some integer multiple of $\chi$; no fractional values of $\chi$ are allowed. Weyl maintained that the distance ($AB$) from the center of one tile (Tile A in Fig. 4) to the center of the next tile (Tile B) along the triangle’s side, and the distance ($AC$) from the center of Tile A to the center of Tile C along the triangle’s diagonal are equal: $AB = AC = \chi$. If the concept of absolute space and Weyl’s conception of distances in discrete space are accepted, then the length of the side is equal to the length of the diagonal for a square of any size. Weyl then states that because all measurements to date have yielded a length for the diagonal of a square to be a factor of $\sqrt{2}$ longer than the length of the square’s side, space cannot be discrete. Weyl’s argument is unassailable, provided that his assumption of absolute space is valid. However, such an assumption is wrong, and Mach’s concept of non-absolute (NA) space provides a refutation to Weyl’s argument and a path forwards towards a modified Pythagoras’s theorem.

Our counter-argument starts by rejecting the first step in Weyl’s construction where he assumed absolute space and we instead assume NA-space. In NA-space, a particle can jump in any direction as long as the magnitude of the jump is $\chi$. We then construct a system to measure the distances of a square’s side and diagonal. The system is composed of three particles $P_A$, $P_B$, $P_C$ at positions $A$, $B$, and $C$ respectively, with $P_A$, $P_B$ and $P_C$ able to emit, reflect or receive of a signal-particle $P_S$ (Fig. 5). The particles $P_A$, $P_B$, $P_C$, and $P_S$ all have diameters equal to $\chi$. We first construct the smaller right triangle shown in Fig. 5 such that the time (duration) between emission (at $A$) → reflection (at $B$) → reception (at $A$) of $P_S$ is $2\beta$. This time duration corresponds to a length for the path $AB$ of $\chi$, the smallest length that is possible to measure. (Note that $P_S$ is only shown along the segment $A \rightarrow C$, but signal-particles also traverse the segments $A \rightarrow B$ and $B \rightarrow C$ when the measurement of these segments are performed.) Additionally, the system is constructed such that a similar measurement yields a length of $\chi$ for the path $BC$. Thus the system is an isosceles right triangle with $AB = BC = \chi$. However the length of the diagonal $AC$ is not $\sqrt{2}\chi$. A signal-particle emitted by $P_A$ (centered about $A$) towards $P_C$ (centered about $C$) makes its first

\[^{6}\text{It is assumed that reflection of } P_S \text{ by } P_B \text{ is instantaneous.}\]
discrete jump of $\chi$ and already, the sphere that specifies the position of $P_S$ overlaps with the sphere defining the position $C$. Hence, $P_S$ has arrived at $C$, will be instantaneously reflected by $P_C$, and will propagate back to $P_A$; a process that takes the same duration $2\beta$ as compared with $P_S$ traveling the path $P_A \to P_B \to P_A$. Thus, the length of the hypotenuse is equal to the lengths of the sides, all being $\chi$, and thus Pythagoras’s theorem is violated.

Looking at a larger triangle in Fig. 6, the lengths of the sides are $AB = BC = 3\chi$. When determining the distance $AC$, one can focus on points (from the perspective of continuous space) on $P_S$’s and $P_C$’s boundaries that are the closest to each other – these points are denoted as $\alpha_n$ and $\theta$ in Fig. 6 with $n$ as an integer denoting the jump number. When $\alpha_n$ is within the sphere that defines the point $C$, then $P_S$ has arrived at position $C$ and interacted with $P_C$. For the larger triangle with sides of length $3\chi$, $n = 4$ jumps of $P_S$ are necessary for this to occur. Thus the measured length of the side is $3\chi$ and the length of the diagonal is $4\chi$, with a ratio of $4/3 = 1.33$. This value is closer to $\sqrt{2} = 1.41$ than what was obtained for the smaller triangle, but still is in large disagreement with Pythagoras’s Theorem.

Consider an arbitrarily large isosceles right triangle with the lengths of the two sides as $m\chi$. It is easy to derive an equation for the distances from $A$ to $\alpha_n$ and from $A$ to $\theta$ (these quantities are shown in Fig. 6). Then using these equations, one can determine the lowest number of jumps ($n$) necessary for $P_S$ to arrive at point $C$:

$$n \geq \sqrt{2}m - 1$$

(3)

where again, $m$ is an integer, $m\chi$ is the length of each side, and $n$ is the smallest positive integer that satisfies Eq. (3). Figure 7 shows a plot of the lengths (relative to $\chi$) of hypotenuses versus the lengths of the sides of isosceles right triangles. It is seen that for $m = 1$ and $m = 2$, the length of the hypotenuse is equal to the lengths of the sides. However, as the sides of the triangle become larger, the hypotenuse converges to $\sqrt{2}$ times the length of the side and Pythagoras’s Theorem is restored.

If the sides of the right triangle are not equal, say being $m\chi$ and $p\chi$ with $m$ and $p$ possibly being different integers, then Eq. (3) can be generalized to yield a new, more accurate form of Pythagoras’s Theorem called Leopold’s Theorem:
Leopold’s Theorem

\[
\begin{align*}
nx & \geq \sqrt{(m\chi)^2 + (p\chi)^2} - \chi \\
\end{align*}
\]

with \( n \) being the smallest positive integer that satisfies Eq. (4).

Already, we see that this model contains several attractive properties:

1. It fully embraces the concept of NA-space, and maintains isotropy.

2. Measurements of lengths are performed in ways accepted by science and adhering to the tenets of LP.

3. A single equation applies to all size scales and accommodates both Pythagoras’s Theorem for any practical distance, and the requirement of discretized space (i.e., distances as integer multiples of \( \chi \)).

This result is quite simple, and in hindsight obvious. However, the results were obtained in a logical, step-by-step way that provides a strong foundation on which to base discussions of: motion in discrete space, modification of the Lorentz transformations, time versus duration, and inertial anomalies of massive particles. Note that alternative distance formulas (i.e., metrics) have been proposed in the past. Herman Minkowski proposed and studied several different metrics in the early 20\(^{th}\) century, the most well known of which is the taxicab (or Manhattan) geometry [22]. In fact, Hermann Weyl’s tile argument is an example of the use of the Chebyshev distance formula [23].

It is important to note that Bendegem proposed a solution to the Weyl tile argument that has similarities to the solution described in this paper [24]. In his work, he assumed that points and lines have finite extensions. He then postulated that the lengths of the triangle’s sides and hypotenuse are given by the number of squares contained in the rectangle formed by the length to be measured and the finite “width of the line segment” [24]. He certainly was on a similar track compared to what is done in this paper, but
– and this is not to imply that there is any deficiency in his excellent work – he did not couch the argument in the language of QM, GR and LP.

5. Motion, Isotropy of Space, Constancy of the Speed of Light and Causality

In QM, one can construct a system such that it has one value (from within a set of \(N\) discrete or continuous values) for a particular “measurable” (e.g., energy, angular momentum, linear momentum, and position). Such states are called eigenstates of that measurable, and can be expressed by the wave function \(\Psi_n\) (i.e., the eigenfunction), with \(n = 1, 2, \ldots, N\). When operating on \(\Psi_n\) with the quantum mechanical operator associated with the measurable, the result is the product of a value of the measurable (i.e., the eigenvalue) and \(\Psi_n\). A system can undergo a transition from one eigenstate to another, either spontaneously or in response to external stimuli. For example, Niels Bohr accurately described the spectrum of light emitted from a hydrogen atom by assuming that angular momentum (AM) is quantized in multiples of \(\hbar\), such that an electron in a hydrogen atom can only make transitions from an initial eigenstate with \(AM_{\text{initial}}\) to a final eigenstate with \(AM_{\text{final}}\) with \(AM_{\text{final}} = AM_{\text{initial}} + m\hbar\) with \(m\) being a positive or negative integer. Standard techniques have been developed that provide the probability that such quantum mechanical transitions occur over a certain period of time \([25]\). These concepts can be applied to a particle’s position, as discussed below.

In continuous space, it is assumed that the position operator \(\hat{x}\) is equal to the variable \(x\) and the eigenfunctions are \(\delta(x - x_n)\), with an infinite set of continuous eigenvalues \(x_n\). For discrete space, the eigenvalues form an infinite but discrete set \(x_n = \{\ldots, -2\chi, -\chi, 0, \chi, 2\chi, \ldots\}\)\(^7\) and the eigenfunctions (expressed using continuous-space coordinates and expressing only their \(x\) dependence) are:

\[
\Psi_n = \frac{1}{\sqrt{\chi}} [H(x - x_n + \chi/2) - H(x - x_n - \chi/2)]
\]

where \(H(x)\) is the Heaviside function.

During a single chronon of duration, a particle in a particular position eigenstate \(\Psi_n\) can either remain in that eigenstate (i.e., stay at the position

\(^7\)The eigenvalues really denote the separation between two particles and not an absolute position.
\(x_n\) or make a transition to another position eigenstate \(\Psi_m\) (i.e., move to a position \(x_m = x_n \pm \chi\)). In this model, when a particle is said to be “moving” at a velocity \(v\), one really means that the particle is undergoing \(M\) spatial translations (of magnitude \(\chi\)) during \(N\) temporal durations (of magnitude \(\beta\)) with \(M\) and \(N\) being large integers, such that \(v \approx M\chi/N\beta = (M/N)c\). But when assessing the system at the finest possible temporal resolution (i.e., every \(\beta\) in time), one sees that the instantaneous velocity of a particle is either zero or \(c\) – the particle either makes a \(\chi\) spatial translation or it does not over this temporal duration. However, a common question is whether such staccato movement is physical \[6\]. Is not the particle traveling at a velocity \(c\) over this time duration (\(\beta\)), and therefore does not SR predict that time is maximally dilated, length maximally contracted and mass maximally burgeoned, thereby invalidating this model of motion? In the next section we discuss one of the most important conclusions of this work that shows that this is not the case.

A couple of other important things concerning motion should be noted. First, for each time step of \(\beta\), a particle can only make spatial translations of magnitude \(\chi\), but this jump can be in any direction, as determined either by an external stimulus or due to a spontaneous translation. This rule then provides the important property of isotropy. Second, because any particle can translate only one hodon per chronon duration, a maximum velocity of \(c = \chi/\beta\) is established – a speed limit that cannot be exceeded by any particle. To justify this statement, consider if a particle has transitioned from \(x_n\) to \(x_n + 2\chi\) over a duration of \(\beta\), then one would naturally ask the question: at what duration was the particle at a position \(x_n + \chi\)? Presumably the answer is: at a duration less than \(\beta\), perhaps \(\beta/2\). But since \(\beta\) is the smallest possible duration, this is not possible. Also, there is no “skipping” position states; QM tunneling or any other transition that would allow a particle to skip one or more position states are prohibited \[^8\] \[^8\]. In its strict adherence to the principle of NA-space and its postulate of discretized S-T, this theory clearly predicts a constant speed of light that is independent of the velocity of a nonaccelerating reference frame. This is an important result because it shows that the constantancy of \(c\) is not fundamental, but rather a consequence of the more fundamental principles of S-T discretization and

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\[^8\] This is different than other measurables in quantum mechanics, such as energy or angular momentum, in which a system can skip intermediate states.
Continual consternation concerning causality can cease. Inquiring about displacements, positions, mechanics, kinetics or anything else within any one discrete point (i.e., within a sphere of diameter $\chi$) is meaningless. Moot then, is the debate as to how a force is instantaneously transmitted across one Weyl tile (or across a sphere of diameter $\chi$) such that both sides accelerate identically and synchronously in response to a force. The important point that has been missed in this debate is that either side of the sphere (in fact the entire sphere) is the *same point in real space* (i.e., *discrete space*) — one only encounters apparent causality problems when incorrectly viewing the situation from the artificial perspective of the continuous space.

6. Modified Time Dilation and Travel at the Speed of Light

In SR, the typical calculation of time-dilation and length contraction starts with the consideration of a clock by two observers in different reference frames (RFs), as shown in Fig. 8. One observer (O1) is at rest in the train station, and the other observer (O2) is on the train traveling at a speed $v$. Einstein envisioned the use of a “light-clock” as an ideal clock with which to measure the passage of time from the two observers’ perspectives. This light-clock is on the train and is composed of an emitter/receiver (E/R) of a photon at position $P$ and a mirror placed at a position $M$ that is a distance $h$ vertically (in O2’s perspective) above $P$. Now consider a photon emitted vertically (again in O2’s RF) from E/R towards $P$. After emission, it propagates to the mirror at $M$ where it is reflected, and then propagates back to $P$. The duration of this process is $\Delta t' = 2h/c$. (Note that primed (unprimed) coordinates correspond to O2’s (O1’s) RF.) Changing perspectives to that of O1’s, the photon’s trajectory is not vertical but maps out two back-to-back right triangles (Fig. 8). Since the speed of the photon is $c$ in both RFs, the total duration of the process $P \to M \to P$ is $\Delta t = 2d/c$. Also, the distance traveled by the E/R is $v\Delta t$. So far, this calculation has been setup in the conventional way that can be found in any textbook on relativity, but this is the juncture at which a continuous S-T model and discrete S-T model diverge.

Conventional SR uses Pythagoras’s theorem that gives the hypotenuse $d$ as equal to $\sqrt{x^2 + h^2}$. With $h = c\Delta t'/2$ and $x = v\Delta t/2$, one obtains the well known formula:
\[ \Delta t = \Delta t_{\text{Einstein}} = \frac{\Delta t'}{\sqrt{1-v^2/c^2}} = \gamma(v)_{\text{Einstein}} \Delta t' \]  

This equation predicts a shorter duration measured in a moving RF compared to the corresponding duration measured in a RF at rest. Such time dilations have been experimentally verified numerous times by studying the lifetimes of muons created by cosmic rays bombarding the atmosphere [27] and the time dilation experienced by atomic clocks on airplanes [28]. However the durations involved in these two cases, as well as all other experiments done to date, are much larger than \( \beta \).

Upon analyzing time dilation in discrete space, we will find that \( \gamma \) depends not only on the velocity of the object (as conventional SR predicts), but that \( \gamma \) also depends on the time between two measurements of some property of the object, i.e., the duration of the measurement. We therefore express \( \gamma \) as \( \gamma(v, \Delta t') \) in this paper when the need arises to be explicit about \( \gamma \)'s dependencies. We therefore also construct a system shown in Fig. 8 composed of an array of clocks, all at rest in O2’s RF, but with different values of \( h_n \): Clock 1 has \( h_1 = 1\chi \), Clock 2 has \( h_2 = 2\chi \), and so on until Clock \( N \) with \( h_N = N\chi \).  

The general procedure to calculate \( \gamma \) for these clocks involves solving Eq. (4) as you scan through two sets of parameters in a nested fashion (Fig. 9). (Matlab codes to implement the algorithms described in this paper are available as supplemental material posted online, as well as at the website given in [29].) The outer loop scans through values of \( p \) and provides the dependence of \( \gamma \) on the duration of the measurement. Thus, a particular clock, namely Clock \( p \), is first chosen, corresponding to a height \( h = p\chi = c\Delta t'/2 \) with \( p \in \{1, 2, 3 \ldots \} \). Once \( p \) is set, an inner loop scans through the \( x \) values, starting with \( x = 1\chi \), then \( x = 2\chi \) and so on to \( x = m_{\text{max}}\chi \); in general \( x = m\chi \) with \( m \in \{0, 1, 2, \ldots m_{\text{max}}\} \). With each \( x \) value, one calculates the hypotenuse \( d \) using Eq. (4) and determines how many integer multiples of \( \chi \) are along the hypotenuse, say \( n \). And because it is a light pulse that travels along the hypotenuse, the length of the hypotenuse is \( d = n\chi = c\Delta t'/2 \). Because any particle cannot travel faster than \( c \), \( n \) sets the upper limit of \( m \) of the \( x \) scan, namely \( m_{\text{max}} = n \). One then simply collects the results

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\( ^9 \)It is more accurate to have all the clocks oriented along the width of the train car, namely aligned perpendicular to the direction of motion.
of these calculations for \( n \) and uses the following equations to calculate the velocity \( v \) and \( \gamma \):

\[
v = \frac{m}{n}c \quad m \in \{0, 1, 2, \ldots, n\} \quad \text{(7a)}
\]

\[
\gamma = \frac{n}{p} \quad \text{(7b)}
\]

Equation (7b) contains \( \gamma \)'s dependencies on velocity and duration. Note that in this model, the velocity of the system is one value within a finite set of discrete values, rather than an infinite set of continuous values from 0 to \( c \). Figure 10 shows \( \gamma \) for durations corresponding to the first 15 clocks, namely durations from \( \Delta t' = 2\beta \) to \( \Delta t' = 30\beta \). In Fig. 10, the discrete set of \( \gamma \) values (at the allowed velocities) are indicated with the “○” markers. The solid black lines show \( \gamma_{Einstein} \) that assumes continuous S-T and a continuum of possible velocities from 0 \( \rightarrow \) \( c \) (of course the black curves are the same for each duration).

The dashed line denotes a very interesting and important phenomena, namely a finite value of \( \gamma \) for a speed \( c \). If a particle has this value for \( \gamma \) (denoted as \( \gamma_{\text{critical}} \)), then it will be measured as traveling at a velocity \( c \) over a certain duration \( \Delta t \). This result arises from the fact that in discrete space, the hypotemuse and one side of a right triangle can have equal lengths, something that is not possible in continuous space and in the conventional special theory of relativity. To obtain these important values of \( \gamma_{\text{critical}} \) (which are dependent on \( \Delta t' \)), assume that an object is moving at the speed of light, hence \( m = n \). Using Eq. (4) with \( m = n \) in Fig. 9 one can derive the equation given below:

\[
n = n_{\text{critical}} \geq \frac{1}{2} \left( p^2 - 1 \right) \quad \text{(8)}
\]

where \( n_{\text{critical}} \) is the smallest positive integer that satisfies Eq. (8). For large \( p \), Eq. (8) yields \( p = \sqrt{2n_{\text{critical}}} \); this, in conjunction with \( \gamma_{\text{critical}} = n_{\text{critical}}/p \), \( \Delta t' = 2p\beta \), and \( \Delta t = \gamma_{\text{critical}}\Delta t' \) yields:

\[
\gamma_{\text{critical}} = \frac{1}{2} \sqrt{\frac{\Delta t}{\beta}} \quad \text{(9)}
\]

Since the kinetic energy (\( KE \)) of a particle is related to \( \gamma \) according to \( KE = (\gamma - 1)mc^2 \), the energy needed to be provided to a particle such that
it can be measured as traveling at a speed $c$ over a particular duration $\Delta t$ is:\(^{10}\)

$$KE_{\text{critical}} = (\gamma_{\text{critical}} - 1) mc^2$$  \(\text{(10)}\)

For example, consider an experiment where two probes are separated by a distance $\Delta d = 300$ nm. The probes precisely measure the time at which an electron (of mass $9.11 \times 10^{-31}$ kg) passes them. If the electron is traveling at a speed $c$, then $\Delta t = \Delta d/c = 1 \times 10^{-15} = 1$ fs. Equation \(\text{(10)}\) yields a value of $2.5 \times 10^{19}$ eV, or $25,000,000$ TeV. This value exceeds what is possible with the most powerful existing particle accelerators by a factor of $10^6$, but may not be impossible to reach with future accelerators. Note that values of $\gamma$ larger than $\gamma_{\text{critical}}$ are possible for any measurement duration, but the speed of the particle will be measured as $c$ for any $\gamma \geq \gamma_{\text{critical}}$ (see Fig. \[10\]). Also, the longer the duration under which the measurement of the speed is performed, the greater $\gamma$ needs to be (meaning that more energy needs to be delivered to the system) in order for the measured speed of the object to remain as $c$. Of course, any object cannot travel faster than $c$, and no causality issues arise with this result since $\gamma$ always remain bounded and any particle’s velocity must be equal to, or less than $c$. Also, these results do not conflict with results from any experiments performed to date – any realistic measurement duration has been at least twenty-five orders of magnitude greater than $\beta$.\(^{30}\)

7. Time versus Temporal Duration and Inherent Problems with Ideal Light Clocks

This section discusses the important implications of a duration-dependent $\gamma$ on concepts of time, measured duration, and clocks. It was assumed by scientists that the light-clock was the ideal clock with which to study, nay define, time \[31\]. Using these light-clocks, Einstein derived a $\gamma_{\text{Einstein}}$ factor (describing time dilation and Lorentz contraction) that appeared to be independent of the clock’s tick-tock rates. He then made the significant conceptual leap of stating that the dilation of the physical clocks’ tick rates represented the dilation of the flow of time itself (let us call this “Time”).

\(^{10}\)Note that two position measurements at times $t_1$ and $t_2$ with $\Delta t = t_2 - t_1$ are necessary to determine the speed of a particle.
Thus he said that duration and Time are one in the same, that there is only one time - the time measured by the scientists’ ideal light-clocks. In a famous 1924 debate with the leading thinker of the day on the concept of time, namely the philosopher Henri Bergson - and in a room full of philosophers - Einstein made the provocative statement: “The time of the philosophers does not exist” [31]. Bergson vehemently objected to the conflation of the scientists’ “measured time” and what he called “real time”, “psychological time”, or simply “Time”. Bergson thought that Time was connected to the flow of consciousness, and was an intuitive concept different than scientists’ measured time. He also objected to the perceived ideal nature of the light-clock and the use of clocks at all to measure time [31]. To address this debate and reach some conclusions, let us consider two of the clocks shown in Fig. 8: Clock 10\(^24\) with \(h = 10^{24}\chi\), and Clock 1 with \(h = 1\chi\).

**Einstein’s Clock(s):** Consider any “enormous” clock with \(h\) much larger than \(\chi\), say Clock 10\(^24\), such that \(h(= 10^{24}\chi)\) is approximately the size of the hydrogen atom. In this case, the hypotenuse agrees with Pythagoras’s theorem to within 10\(^{-9}\)%. Thus for all practical purposes, time is dilated in the moving RF by the amount predicted by standard SR, given by Eq. [6], and this clock’s tick rate largely agrees with Einstein’s predictions. This applies to any clock with larger values of \(h\), i.e., slower tick rates.

**Bergson’s Clock:** Next, consider Clock 1 in Fig. 8 - a clock we shall call a Bergson clock (or B-clock) in the hopes that it may be able to measure Bergson’s immutable “Time”. This clock has the E/R and mirror separated only by a single hodon. From Fig. 10 we see that \(\gamma = 1\) for a measurement duration of \(2\beta\), regardless of the velocity of the system. This means that no time dilation occurs over this temporal duration, therefore O1 measures the same duration for one tick of this clock as that measured by O2. Perhaps this clock is the ideal tool with which to probe, to mirror, to represent Bergson’s diaphanous “Time”? …Alas, this is not the case. To see why, we need to consider simultaneous events in the O1’s and O2’s RFs. But prior to this, we have to more clearly describe a couple of key aspect of the clocks on the train.

Let us assume that for each clock (Clocks 1 through \(N\) in Fig. 8), whenever a photon completes a round trip, its detection and the emission of a subsequent photon is immediate, namely there is no temporal dura-
tion between these events. Additionally, we line up the clocks along the width of the train so that they are aligned in a perpendicular direction (i.e. \( \hat{y} \)) relative to direction of motion (i.e., \( \hat{x} \)). This eliminates any factors (e.g., length contraction, differences in positions) that may bring the results of this analysis into question. Also, assume that O2 starts all clocks at the same time \( t = t' = 0 \), such that the first \( N \) photons are simultaneously emitted by the \( N \) clocks at exactly \( t = t' = 0 \). Also, each clock instantly emits a signal-particle every time a photon is received by the E/R (see Fig. 8); this signal-particle propagates to, and is detected by O1 such that the durations between ticks of each clock can be assessed by O1. And finally, we note that because the train is traveling at a velocity \( v \), the tick rate (as measured by the arrival of the signal-particles at O1) experiences the Doppler effect. In this paper, Doppler effects will be factored out of the results so that we can focus on the changes to the clocks’ tick rates due only to relativistic effects and to the effects introduced by the discrete nature of S-T. We now study the simultaneity of photon detection by O2’s B-clock (i.e. Clock 1) and O2’s other clocks \( 2 \to N \), with the ultimate aim of producing a correspondence between the ticks of the B-clock in O2’s RF and the ticks of a second B-clock in O1’s RF.

O2’s B-clock’s first emitted photon returns to the clock’s E/R after a duration \( \Delta t_1' = 2\beta \) and instantly emits a signal-particle to O1. O1 receives the signal-particle and assesses the duration \( \Delta t_1 \) and concludes that \( \Delta t_1 = \gamma(v, 2\beta) \Delta t_1' \), in accordance with \( \gamma(v, \Delta t') \) for the velocity of the clock and duration. (Note that in this section, we will express \( \gamma \) as \( \gamma(v, \Delta t') \) so as to indicate what velocity and duration are used in the calculation; also, the subscripts of \( \Delta t_n \) and \( \Delta t'_n \) represent the \( n^{th} \) tick number of O2’s B-clock.) O2’s B-clock’s second photon is detected by the clock after a duration (relative to \( t' = 0 \)) of \( \Delta t_2' = 2 \times 2\beta = 4\beta \), this is the same duration required for the first photon of O2’s Clock 2 to make a round trip and be detected by Clock 2. Thus, these two events are simultaneous (in both O2’s and O1’s RFs), namely the reception of O2’s B-clock’s second photon and the reception of O2’s B-clock’s first photon.

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11 Even if this assumption needs to be modified, say by there there being at least one chronon between reception of the photon and emission of a subsequent photon, this will not affect the qualitative conclusions of this work but only modify quantitative results in a straightforward fashion.

12 These signal-particles are photons and are as spatially as possible, namely Planck particles.
O2’s Clock 2’s first photon. Subsequently, two signal-particles are emitted simultaneously, one from each clock, and later received (simultaneously) by O1. The question then arises as to what duration (relative to $t' = 0$) O1 measures that corresponds to two ticks of Clock 1 and one tick of Clock 2? Is it $\Delta t_2 = 2 \times \gamma(v, 2\beta) \Delta t'_1$? This would be the case in the standard treatment of SR. The answer is no – in order for simultaneity to be conserved in measurement as it is in fact, the duration measured by O1 must be $\Delta t_2 = \gamma(v, 4\beta) \Delta t'_2$. We continue this process with O1 assessing the duration (relative to $t' = 0$) of each tick of O2’s Clock 1, finding that the $M^{th}$ tick of Clock 1 is received at $\Delta t_M = \gamma(v, M \times 2\beta) \Delta t'_M = M \times \gamma(v, M \times 2\beta) \Delta t'_1$.

In Fig. 11 we plot the correspondence between each tick of O2’s B-clock and a tick of a B-clock in O1’s RF for two cases: Case 1 is where the train is traveling approximately $v = 0.5c$, Case 2 is where $v = c$. In Case 1, the velocity cannot always be exactly $v = 0.5c$ because it is calculated by the ratio of hodons along the hypotenuse and base of the triangle in Fig. 9 as per Eq. (7a). Table 1 provides (for the first 10 clocks only) the number of hodons along: the height, base and hypotenuse of the triangle traced out by the photon; the corresponding speed of the train is also provided. It is seen in Fig. 11 that O1 observes something rather odd with O2’s B-clock, namely that its tick rate is very irregular. As measured by O1, the first six ticks of O2’s B-clock (with $v = 0.5c$) occur at the same time as the first six ticks of O1’s B-clock, even though for ticks 3 and 5, the velocity of the train cannot be exactly $0.5c$. But then things go awry with this “perfect” clock. The 7th tick of O2’s B-clock does not arrive until the 8th tick of O1’s B-clock. Also, both the 7th and 8th tick of O2’s B-clock arrive at the 8th tick of O1’s B-clock. Then, the 9th tick of O2’s B-clock corresponds to the 10th tick of O1’s B-clock. These irregular receptions of signal-particles from O2’s B-clock continue, sometimes skipping and sometimes doubling-up on a tick of O1’s B-clock. As the tick number $N$ becomes large, $\gamma(v, N \times 2\beta) \rightarrow \gamma_{\text{Einstein}}(v)$, thereby agreeing with standard SR.

Also included in Table 1 and Fig. 11 is the correspondence between O1’s and O2’s B-clocks for a train traveling at $c$ (and with $\gamma = \gamma_{\text{critical}}$) – something not possible within the framework of standard SR, but allowed in discrete S-T. Upon commencing the measurements at $t = t' = 0$, it is seen that the arrival of subsequent ticks of O2’s B-clock arrive at O1 after durations that grow rapidly (again, Doppler effects are excluded). In general, the $p^{th}$ tick of O2’s B-clock arrives at the $n^{th}$ tick of O1’s B-clock, with $p$ and $n$ related according to Eq. (8).
Concluding with the discussion of these B-clocks, the important point is that they are not ideal in the sense that time dilation still occurs with these clocks and that the tick rate is irregular. Thus, it cannot represent the immutable “Time” that Bergson envisioned. However, these clocks are the best that nature allows. Thus, there is no clock that can measure Bergson’s immutable “Time” – and if this Time cannot be measured, *ipso facto*, it does not exist. Also, all physical processes occur on time scales much larger than $\beta$ - the fastest measured chemical reactions in the human body occur in the femtosecond time frame, over $10^{30}$ times larger than $\beta$ [30]. Thus, for all practical purposes, chemicals react at rates, objects age, and time can be said to “flow” according to the time described by the standard rules of SR. Thus Bergson was correct in only two regards: there does exist an immutable component of time, namely the chronon; and Einstein’s light clocks are not quite as ideal as most scientists believe. Concerning the former, the chronon is far from the concept of human-lived time Bergson envisioned; concerning the later, the tick rate of light clocks can indeed be irregular and dilated, but in a predictable way as described in this work. Thus, Einstein’s conflation of measured time with Time is valid – and ironically, rather than the “Time” that Bergson championed, Einstein’s measured time is the psychological or lived time.

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13Potentially, one could slightly improve upon these clocks by replacing the mirror above each E/R with a receiver. Thus, temporal resolution could potentially be increased from $2\beta$ to $\beta$ – however this will not eliminate the time dilation that occurs with these clocks.
Table 1: The integer multiples of $\chi$ for the triangles traced out by the photons of O2’s B-clock in O1’s RF (see Figs. 8-9). The integers $p$ and $n$ provide the tick correspondence between O2’s B-clock and O1’s B-clock (after Doppler effects are subtracted out). The height, base and hypotenuse are relative to $\chi$ and the speed is relative to $c$.

| Height ($p$) or O2’s Tick | Base ($m$) or O1’s Tick | Hypotenuse ($n$) or O1’s Tick | Speed | Height ($p$) or O2’s Tick | Base and Hypotenuse or O1’s Tick |
|---------------------------|-------------------------|-------------------------------|-------|---------------------------|---------------------------------|
| 1                         | 0(1)                    | 1(1)                          | 0(1)  | 1                         | 1                               |
| 2                         | 1                       | 2                             | 0.5   | 2                         | 2                               |
| 3                         | 1(2)                    | 3(3)                          | 0.33(0.66) | 3                     | 4                               |
| 4                         | 2                       | 4                             | 0.5   | 4                         | 8                               |
| 5                         | 2(3)                    | 5(5)                          | 0.4(0.6) | 5                     | 12                              |
| 6                         | 3                       | 6                             | 0.5   | 6                         | 18                              |
| 7                         | 4                       | 8                             | 0.5   | 7                         | 24                              |
| 8                         | 4                       | 8                             | 0.5   | 8                         | 32                              |
| 9                         | 5                       | 10                            | 0.5   | 9                         | 40                              |
| 10                        | 5(6)                    | 11(11)                        | 0.45(0.55) | 10                   | 50                              |

Note: If the desired velocity is not possible, results for the two speed neighboring the desired speed are given. For example, for $p = 1$, only the speeds of 0 and 1c are possible, corresponding respectively to $m = 0$ and $m = 1$ for the base and with $n = 1$ in both cases for the hypotenuse – this is recorded in the table under the column labeled “Speed” as 0(1), “Base” as 0(1), and for “Hypotenuse” as 1(1).
Goya’s etching was a critique of pre-Enlightenment practices of Spanish society in his time and the “vulgar prejudices and lies authorized by custom, ignorance or interest, those that he has thought most suitable matter for ridicule”. However, a new generation of philosopher-physicists has strove to awaken Reason from its recent slumber \[33, 13\]. In a caption to the work, Goya makes clear that he is not advocating for society to use reason alone, but rather a synthesis of reason and imagination: “Imagination abandoned by reason produces impossible monsters; united with her, she is the mother of the arts and source of their wonders”. Similarly, Popper laments the fact that no contemporary scientist would dare propose such a bold concept as Anaximander did, devoid of observational evidence, that the Earth is freely poised and stationary in mid-space and of a shape of a drum - a thesis that Popper believes is the naissance of modern science \[32\]. In this section, we use our imagination and propose possible mechanisms to account for some (but not all) of the observed effects attributed to dark matter and dark energy.

In 1957 John A. Wheeler sought to show that all of classical physics, particle physics included, is “purely geometrical and based throughout on the most firmly established principles of electromagnetism and general relativity” \[34\]. Wheeler made use of well developed concepts in quantum
electrodynamics, and showed that the fine structure of space is composed of a random array of quantum “wormholes”, with each wormhole having a pair of charges, \( q = \pm \sqrt{4\pi\varepsilon_o \hbar c} \), and each charge having a mass \( m = m_p = E/e^2 = \sqrt{\hbar c/G} = 2.18 \times 10^{-8} \) kg. He stated that these charges (i.e., Planck particles) have an average spacing of \( l_p = 1.62 \times 10^{-35} \) m, or half of \( \chi \). Otherwise the particles are randomly distributed; hence he called this “quantum foam”.

However, if space is discretized, a random distribution of Planck particles involving fractional distances of \( \chi \) is not allowed. Order must be imposed on the structure, changing the foam into a crystal (Fig. 12). The constituent particles of this crystal all have a very large mass \( (m_p) \) relative to their charge \( (q_p) \) when compared to any other naturally occurring elementary particle. This structure then forms a gravity crystal (GC) that is described in detail in [35].

Since both the gravitational and electromagnetic forces have a \( 1/r^2 \) dependence, one can use techniques within the field of solid-state physics [21] to calculate the behavior of particles traveling within the GC. Let the GC be composed of an array of particles, all with identical mass \( m_c \), and with one particle at each position given by \( \vec{R} = n_x \chi \hat{x} + n_y \chi \hat{y} + n_z \chi \hat{z} \) with \( n_x, n_y, \) and \( n_z \) being integers. The GC creates the following potential energy profile (produced by all the GC’s constituent particles) for a particle (electrically neutral and mass \( m_{\text{particle}} \)) traveling within it:

\[
V(\vec{r}) = -Gm_{\text{particle}} m_c \sum_{\vec{R}} \frac{1}{|\vec{r} - \vec{R}|} \tag{11}
\]

An important quantity to calculate is the particle’s dispersion curve (\( \omega - k \) curve) that plots the energy \( (\mathcal{E} = \hbar \omega) \) of the particle versus its wave vector \( k \), with \( \mathcal{E} \) and \( \omega \) both being functions of \( k \). To calculate this dispersion curve, one can use the tight-binding method (TBM), a nearly free particle method, a empirical pseudo-potential method (EPM), or several other methods [21, 36, 35, 37] to solve the full Schrödinger’s equation for this system:

\[
\frac{\hbar^2}{2m_{\text{particle}}} \nabla^2 \psi + V(\vec{r}) \psi = \mathcal{E} \psi \tag{12}
\]

Matlab code to implement the EPM and TBM is available as supplemental material posted online, as well as at the website given in [29]. Note that in the calculations of this section, we have used the result of discretization of space in units of \( \chi \), but have used the conventional Pythagoras’s theorem. Future
work is needed to implement Leopold’s theorem into EPM, TBM and other band diagram algorithms; it is reasonable to suspect that the use of Leopold’s theorem will significantly reduce the anisotropy of the band diagram shown later in this section.

Many interesting properties can be gleaned from a particle’s dispersion curve, including bandgaps, Brillouin zones (BZs) and their boundaries, and effective inertial mass. Each one of these things provide important information on how particles behave in crystals, sometimes predicting seemingly bizarre behavior. For example, energy bandgaps indicate forbidden energy ranges for particles, but particles can “jump” this gap by acquiring the necessary energy from another particle. BZs provide information about the range of momentum that a particle can have, including an effective maximum momentum. Finally, one can calculate an effective inertial mass \( m_{\text{inertial}} \) of a particle traveling within a crystal. This parameter is usually written as the inverse of \( m_{\text{inertial}} \), and also as a tensor \( (1/m_{\text{inertial}})_{i,j} \), with \( i, j \in \{x, y, z\} \). This then allows an applied external force \( F_j \) in one direction (with \( j \in \{x, y, z\} \)) to produce an acceleration \( a_i \) in the same or different direction (with \( i \in \{x, y, z\} \)).

The effective mass method \[21\] allows one to lump all the effects of the crystal particles into this one parameter \( 1/m_{\text{inertial}} \) and then use this term in a simplified Schrödinger’s equation:

\[
-\frac{\hbar^2}{2m_{\text{inertial}}} \nabla^2 \psi + V_{\text{external}}(\vec{r}) \psi = E \psi
\]  

(13)

where \( V_{\text{external}} \) is the potential energy profile produced only by non-crystal sources, e.g., galaxies, stars, planets, dust. Once \( \mathcal{E}(\vec{k}) \) of Eq. \[12\] has been calculated, \( m_{\text{inertial}} \) can be calculated using the following equation \[21\]:

\[
\left( \frac{1}{m_{\text{inertial}}} \right)_{i,j} = \frac{1}{\hbar^2} \frac{\partial^2 \mathcal{E}(\vec{k})}{\partial k_i \partial k_j}
\]  

(14)

In \[34\], Wheeler showed that if the constituent crystal particles (again, of charge \( q_p = \pm \sqrt{4\pi \epsilon_o \hbar c} \)) are separated from each other by an average distance of \( l_p \), then the positive mass produced by electromagnetic energy (via \( E = mc^2 \)) is totally compensated by negative mass produced by gravitational energy, such that “to the extent this compensation holds locally, nearby wormholes exert no gravitational attraction on remote concentrations of mass-energy”. In \[35\], Crouse considered a GC where this compensation
does not happen, and the particles that compose the crystal all had mass \( m_c = m_p \). It was seen in [35] that a particle traveling within this crystal can exhibit negative and near-zero values for \( m_{\text{inertial}} \). However, no justification was given in [35] as to why no compensation (of the positive electromagnetic mass and negative gravitational mass) occurs. In this work however, we have shown that a spacing of \( l_p \) is not possible, because \( l_p \) is less than the fundamental length \( \chi (= 2l_p) \). If \( \chi \) is the lattice constant of the GC, then it is easy to show (using Wheeler’s methods described in [34]) that there is an uncompensated mass of \( 3m_p/8 \) for each crystal particle; thus \( m_c = 3m_p/8 \). Similar phenomena (e.g., negative effective mass \( \cdots \) ) occur for a crystal with constituent particles all with this mass, compared to the case when \( m_c = m_p \); therefore, we refer the reader to [35] for a more detailed discussion of a GC of this type.

Instead, let us consider an interesting case where the constituent crystal particles are Higgs bosons (\( m_{\text{Higgs}} = 2.25 \times 10^{-25} \) kg), configured in a cubic lattice with lattice constant \( a_o = 2l_p = \chi \). First, let us estimate the mass of elementary particles that would “feel” the effects of the crystal as they travel within it. To do so, one would equate the kinetic energy term and the dominant potential energy term in Eq. (12):

\[
\frac{\hbar^2 k^2}{2m_{\text{particle}}} = \frac{Gm_{\text{particle}}m_c}{a_o}
\]

(15)

The effects of the crystal most often manifest themselves at the BZ boundary at \( k = \pi/a_o \). Using this value of \( k \) in Eq. (15), one arrives at the following approximation for the mass \( (m_{\text{particle}}) \) of a particle that will interact strongly with the GC:

\[
m_{\text{particle}} = \frac{\pi \hbar}{\sqrt{2Gm_c a_o}}
\]

(16)

With \( a_o = \chi \) and \( m_c = m_{\text{Higgs}} \), Eq. (16) yields a value of 10.62 kg. \[14\] The band diagram and \( m_{\text{inertial}} \) as functions of \( k \) are shown in Fig. 13 and 14 respectively. It is seen that \( m_{\text{inertial}} \) can be significantly different than the gravitational mass \( m_{\text{particle}} \), with it even being negative for various ranges of momenta. A negative \( m_{\text{inertial}} \) would predict that a particle would accelerate

\[14\] It is certainly debatable whether such a massive elementary particle (relative to \( m_{\text{Higgs}} \) ) will obliterate the crystal around and along the particle’s trajectory.
in the opposite direction of the external force\textsuperscript{15}. In the case of the universe, the cumulative gravitational force (due to all planets, stars and galaxies) is in a direction towards the “center” of the universe. Particles with a negative value of $m_{\text{inertial}}$ will be observed to be accelerating in the opposite direction, that is, away from the center of the universe – these particles will be “pushed” by the “pull” of gravity. This effect can be easily detected and measured using the latest telescopes.

Instead of Higgs bosons composing the crystal, we can investigate other candidate particles. For example, we could use either of the two commonly stated values for the vacuum energy density, namely $\xi_1 = 10^{-9} \text{ J/m}^3$ \textsuperscript{38} and $\xi_2 = 10^{113} \text{ J/m}^3$ \textsuperscript{39}, and calculate the corresponding mass $m_c$ (via $m = E/c^2$ with $E = \xi \chi^3$) of each constituent crystal particle assuming a lattice constant of $\chi$. For $\xi_1$, we obtain $m_c = 3.78 \times 10^{-130}$ kg, and using this value in Eq. (16) we obtain $m_{\text{particle}} = 2.59 \times 10^{53}$ kg. In this case, $m_{\text{particle}}$ is approximately the mass of the entire universe (as stated in \textsuperscript{40}), thus no particle with a realistic mass would ever feel the effects of this crystal. For $\xi_2 = 10^{113} \text{ J/m}^3$, we obtain $m_c = 3.78 \times 10^{-8}$ kg and $m_{\text{particle}} = 2.59 \times 10^{-8}$ kg; particles of this mass scale are realistic, are described in \textsuperscript{35}, and can produce measurable physical effects.

Another interesting result of these calculations is that it predicts that black holes (BHs) are more complicated than widely believed. Any current textbook on GR or astronomy states that BHs have only three properties: total mass, spin, and electric charge. However, the results in this paper predict that the distribution of the mass within the event horizon is very important in determining the BH’s motion in response to external gravitational forces. Consider two cases: Case 1 with a 10.62 kg BH where all this mass is concentrated into the “singularity” (of volume $V_{\text{BH}} = \chi^3$); and Case 2 where the 10.62 kg mass is composed of a uniform distribution of particles over the BH’s volume of $V_{\text{BH}} = (4\pi/3) R_s^3$, with $R_s = 2Gm/c^2 = 1.57 \times 10^{-26}$ m being nine orders of magnitude greater than $\chi$. For Case 1, $m_{\text{inertial}}$ may vary dramatically, as already described (see Fig. \textsuperscript{14}). For Case 2 however, $m_{\text{inertial}}$ will always be equal to the sum of the constituent particles’ gravitational masses because each particle that composes the BH has too little mass to

\textsuperscript{15}Conservation of energy and momentum are still conserved since the system includes not only the particle, but also the entire crystal. The universe-wide crystal can act as an infinite reservoir for energy and momentum.
“feel” the effects of the GC. Thus, by studying the inertial properties of a BH (i.e., how it responds to an external force), one can glean some knowledge of its internal structure.

9. Conclusion

In this work, the discretization of space and time was studied, along with its affects on mathematical and physical theories. First, using well accepted concepts within quantum mechanics and general relativity, and adhering to the tenets of logical positivism, the quantums of space and time were derived. Next, using Mach’s principle of strict non-absolute space, a method was developed to map points from continuous space to discrete space. Using these results, and again using concepts from quantum mechanics and Mach’s principle, a modified Pythagoras’s theorem (Leopold’s theorem) was deduced. Leopold’s theorem agrees with the Pythagoras’s theorem at large size scales but differs from it at the Planck scale, in such a way that it preserves the concept and requirements of discrete space-time. Particle motion in discrete space-time was then described, and it was shown that the constancy of the speed of light is a consequence of the immutability of the quantums of space and time, and non-absolute space. These results were then used in the theory of special relativity to calculate a modified $\gamma$ function that is a function of both speed and measurement duration. Such results significantly alter time dilation and length contraction, as well as allowing particles (even with nonzero rest mass) to temporarily travel at the speed of light. The issue of the existence and nature of a “real” immutable Time was studied in the context of both logical positivism and the modified laws of special relativity developed in this work. It was seen that the only candidate for an immutable Time was a single quantum of time, the chronon, and a single tick of a Bergson clock. In contrast, scientists’ measured time preserves and contains within it the chronon, but it also is the “lived”, the experienced, the psychological time important for practically all particle interactions and human experiences. Finally, it was shown how the discretization of space imposes order on Wheeler’s quantum foam, turning it into a universe-wide gravity crystal that can affect the inertial properties of dense, high momentum particles such as black holes. By studying the inertial properties of black holes, some aspects of the mass-distribution within their event horizons can be gleaned.
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Figure 1: A schematic of an elementary particle (shaded in gray) moving in discrete space (showing only two of the three spatial dimensions and time). For ease of visualization, time is shown increasing along the horizontal axis, but these are really different “snap-shots” in time. Both the spatial size of the unit cells and the time duration between snap-shots are the smallest that nature allows —the quantum of space (χ) and time (β). Shown in this figure is the common view of discrete space (i.e., the problematic Weyl-tile picture described later in this paper). Motion in this space involves a particle either remaining at a lattice site or moving to an adjacent lattice site during a single β of temporal duration. The particle shown in this figure is making one χ spatial translation for every β temporal duration, thereby moving as fast as nature allows, namely the speed of light (c = χ/β).

Figure 2: To measure distances, the most accurate “ruler” consists of two probe-particles $P_A$ and $P_B$ between which the spatial interval is to be measured. $P_A$ emits and receives the signal particle $P_S$, and $P_B$ instantaneously reflects $P_S$. 
Figure 3: **Top:** A system to measure the smallest spatial separation between two distinct probe-particles (in darker gray). Note that $P_S$ is not shown in this figure. The probe-particles need to remain entirely distinct, as spatially compact as possible, and be able to emit, receive or reflect a signal particle. These conditions set the radii of $P_A$ and $P_B$ to $2l_p$ and their masses to $m_p/2$.

**Bottom:** The set of all continuous space $x$-values (denoted as $\Delta x_n$) within each sphere that is mapped to a single $x$-value in discrete space, namely $\tilde{x}_n$. This mapping is done sequentially, starting with $\Delta x_0 \rightarrow \tilde{x}_0$ and $\Delta x_1 \rightarrow \tilde{x}_1$, and then to the lighter gray circles in the figure that represent the subsequent positions in discrete space, i.e., the mappings $\Delta x_2 \rightarrow \tilde{x}_2$ and $\Delta x_3 \rightarrow \tilde{x}_3$.

Figure 4: The Weyl construction that shows an *a priori* defined lattice. All distances, from the center of one tile to the center of any neighboring tile have to be separated by integer multiples of $\chi$; *no fractional values of $\chi$ are allowed*. Thus the length of the diagonal is equal to the length of the side of the square, regardless of the size of the square.
Figure 5: The modified Weyl construction that does not assume absolute space. Distances are measured along each path according to the rules described in the text. For this particular triangle formed by points $A$, $B$ and $C$, the distance along the diagonal is equal to the lengths of the triangle’s base and height. This is because only one jump of $\chi$ is needed along the diagonal for the sphere defining $P_S$ to partially overlap the sphere centered about $C$, and therefore be at the same position in discrete space.

Figure 6: For a larger triangle with the lengths of its sides equal to $3\chi$, $P_S$ needs 4 jumps of $\chi$ along the diagonal such that its defining sphere overlaps the sphere defining point $C$. Thus the length of the hypotenuse relative to the length of the sides is $4/3 = 1.333$. 

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Figure 7: A plot of the length of the hypotenuse of an isosceles right triangle versus the length of the sides, both for continuous space (solid line) and discrete space (points). Note that the lengths are normalized relative to $\chi$. The length of the hypotenuse is always less than $\sqrt{2}a$ (with $a$ being the length of the side), but never by more than $\chi$.

Figure 8: (a) An array of light clocks on a train traveling at a speed $v$. The clocks have values of $h$ as integer multiples of $\chi$. (b) One of the clocks in O2’s RF on the train, and (c) the same clock in O1’s RF at the station.
Figure 9: The general procedure to calculate $\gamma$ as a function of duration and velocity using light-clocks. The algorithm scans through integer values of $p$ starting at $p = 1$. For each value of $p$, $m$ is scanned from 1 to a maximum value at which the condition $n = m$ is satisfied, with $n$ being obtained from Eq. (4). When this occurs, the maximum velocity has been reached, namely $c$. In this figure, $p = 7$, $m = 4$, and $n = 8$, corresponding to a duration of $\Delta t' = 7 \times 2\beta$, a velocity of $v = (m/n)c = (4/8)c = 0.5c$, and $\gamma = n/p = 8/7 = 1.14$. 
Figure 10: The calculated values for $\gamma$ versus duration and velocity (colored solid lines) with allowable values indicated by the point markers “◦”, and $\gamma_{\text{critical}}$ is shown as the dotted black line. Also shown is $\gamma_{\text{Einstein}}$ versus $v$ (solid black). Speed is given relative to $c$, and duration is relative to $2\beta$. Note that a speed of $c$ is allowed for any duration, even for a particle with nonzero rest mass.
Figure 11: The red squares, black circles and blue diamonds represent the corresponding ticks of O1’s B-clock for each tick of O2’s B-clock. O2’s RF (and B-clock) has a velocity of approximately 0.5c. Red squares indicate times at which the velocity 0.5c is not strictly possible (because of Eq. (7a)), black circles indicate times that allow \( v = 0.5c \), and blue diamonds indicate two sequential ticks of O2’s B-clock that arrive at the same tick of O1’s B-clock. Purple triangles represent the correspondence of the ticks of O2’s and O1’s B-clocks in the case where O2’s RF is traveling at \( v = c \) relative to O1’s RF. This shows that O2’s time is not entirely “frozen” from O1’s perspective, as would be the case with standard SR.
Figure 12: The universe-wide gravity crystal that has a cubic lattice, a lattice constant of $\chi$, and with a basis of one particle of mass $m_c$.

Figure 13: The dispersion curve (calculated using EPM [35, 36]) for a particle with $m_{\text{particle}} = 10.62$ kg traveling within a cubic GC composed of Higgs bosons (one per unit cell) with lattice constant $\chi = 2l_p$. Nonparabolicity of the bands occur, which is indicative of a $m_{\text{inertial}}$ that varies with momentum $k$, as shown in Fig. [14]. **Inset:** One unit cell of reciprocal space showing the crystal directions.
Figure 14: The inertial mass $m_{\text{inertial}}$ (blue line) as a function of momentum $k$ of a particle (with gravitational mass $m_{\text{particle}} = 10.62$ kg) traveling within the GC of Fig. [12] The red dotted line is $m_{\text{particle}}$ and the vertical dotted lines are BZ boundaries. It is seen that away from the Γ-point, $m_{\text{inertial}}$ differs significantly from $m_{\text{particle}}$, with $m_{\text{inertial}}$ being much greater than $m_{\text{particle}}$, near zero, or even negative.