Reynolds Pressure and Relaxation in a Sheared Granular System

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(Dated: May 2, 2014)

We describe experiments that probe the evolution of shear jammed states, occurring for packing fractions \( \phi_S \leq \phi \leq \phi_J \), for frictional granular disks, where above \( \phi_J \) there are no stress-free static states. We use a novel shear apparatus that avoids the formation of inhomogeneities known as shear bands. This fixed \( \phi \) system exhibits coupling between the shear strain, \( \gamma \), and the pressure, \( P \), which we characterize by the ‘Reynolds pressure’, and a ‘Reynolds coefficient’, \( R(\phi) = (\partial^2 P/\partial \phi^2)/2 \). \( R \) depends only on \( \phi \), and diverges as \( R \sim (\phi_c - \phi)^\alpha \), where \( \phi_c \simeq \phi_J \), and \( \alpha \simeq -3.3 \). Under cyclic shear, this system evolves logarithmically slowly towards limit cycle dynamics, which we characterize in terms of pressure relaxation at cycle \( n \): \( \Delta P \simeq -\beta \ln(n/n_0) \). \( \beta \) depends only on the shear cycle amplitude, suggesting an activated process where \( \beta \) plays a temperature-like role.

PACS numbers: 83.80.Fg, 62.20.D-, 83.85.Vb

Keywords: Granular materials, jamming, shear jamming

Much recent work has focused on the mechanical behavior of disordered solids, including granular materials, colloids, foams and molecular glass formers. These systems are well known for their glassy flow behavior and surprising rigidity. Notably, Bi et al. [4, 5] recently showed that in frictional systems, e.g. most common granular materials, shear strain, \( \gamma \), can ‘shear jam’ [4] a loose, low density packing of particles, enabling it to support a shear stress. The nature of these shear jammed states, particularly how they form and evolve, is an unsolved problem with obvious relevance, whose understanding is the goal of the current paper.

To set the context, we note that Bi et al. [5] showed that there is a lowest packing fraction \( \phi_J \), such that below (above) this density, there are (no) zero-stress states. Application of shear to a zero-stress state in \( \phi_S \leq \phi \leq \phi_J \) leads to highly anisotropic contact and force networks, and to non-zero shear stress, \( \tau \), and pressure, \( P \). Here, \( \tau = (\sigma_1 - \sigma_2)/2 \) and \( P = (\sigma_1 + \sigma_2)/2 \), where the \( \sigma_i \) are the principal stresses of the 2D stress tensor, \( \sigma \). Starting from zero stress, the system traverses a fragile regime, and with additional shear strain, the system arrives at a fully jammed state where the force/contact networks percolate in all directions. These shear jammed states may occur naturally in many granular systems, such as geophysical flows, sand and suspensions. Improved understanding of shear jammed states is thus crucial for both a better understanding of the concept of jamming for (frictional) materials, and to shed light on the complex rheology of dense granular media [6].

At the heart of shear jamming are classic studies by Reynolds, who showed that under fixed pressure, granular systems can dilate in response to shear [7]. Despite its relevance, a quantitative understanding of this effect has remained elusive over the last century. This is partly due to a complication in the study of sheared frictional materials: Shear typically induces the formation of dilated localized shear bands, where most of the shear strain is confined. System-wide measures may tend to reflect the band properties rather than the whole system, making it difficult to interpret experiments.

To understand the important physics underlying shear jamming, it is crucial to have an experimental approach that avoids shear banding. In this Letter, we describe such an approach that, for the first time to our knowledge, avoids shear banding. Measurements using this method provide the first characterizations of, and key insights into, the mechanical response and dynamics of shear jammed frictional packings. In these fixed volume experiments, the response to shear is manifested as a nonlinearly growing pressure with shear strain, which is related to Reynolds’ dilatancy. Associated with this pressure effect are structural rearrangements that lead to a surprising Arrhenius-like stress relaxation dynamics in periodically sheared disk packings.

Key Findings In these experiments, we shear a disordered disk packing (2D) at fixed density. In such a system, dilatancy cannot occur, but a related phenomenon occurs: the stresses \( \sigma \) respond to the shear strain. We find that \( P \) increases roughly as \( \gamma^2 \), which we describe by a “Reynolds coefficient”, \( R = (\partial^2 P/\partial \gamma^2)/2 \). We find that \( R \) depends only on \( \phi \), and it provides a simple parametrization of the coupling between \( P \) and \( \gamma \). \( R \) seems to diverge as \( \phi \) approaches \( \phi_c \simeq \phi_J \), thus identifying a special role for \( \phi_J \) for the shear jamming states.

An additional key observation from this work is that for \( \phi_S \leq \phi \leq \phi_J \) the stress response to cyclic shear strain shows slow relaxational dynamics to a limit cycle, that depends on driving. The deviation from a limit cycle, measured by pressure, shows a logarithmic decay over time/cycle number. The data for stress relaxation exhibit a totally unexpected scaling form, as developed below.

Experimental Setup Key to these experiments is a novel
apparatus that provides (simple) shear throughout the system, in contrast to wall-driven shear. The base of the apparatus consists of narrow, parallel, horizontal, and transparent slats. Shear is applied by deforming the slats and boundary uniformly in the ‘y’ direction, keeping the ‘x’ dimension fixed at \( L \), and particle orientation images under UV light (lower). The \( x \)– and \( y \)-displacement of particles vs. their horizontal positions in the system, and (d) the coarse grained \([15, 16]\) density profile after 27% linear shear.

![Image](a) Setup schematics (b) The three close-up images that the camera captures at each step: particle positions (upper), force response under polariscope (middle), and particle orientation images under UV light (lower). (c) The \( x \)– and \( y \)-displacement of particles vs. their horizontal positions in the system, and (d) the coarse grained \([15, 16]\) density profile after 27% linear shear.

FIG. 1: (Color online) (a) Setup schematics (b) The three close-up images that the camera captures at each step: particle positions (upper), force response under polariscope (middle), and particle orientation images under UV light (lower). (c) The \( x \)– and \( y \)-displacement of particles vs. their horizontal positions in the system, and (d) the coarse grained \([15, 16]\) density profile after 27% linear shear.

For the larger \( \phi \)'s considered here, we could not apply the full 54% strain because \( P \) became so large that the layer was unstable to out-of-plane buckling. If buckling occurred, we terminated the forward shear experiment. The forward shear results, Fig. 2a, indicate that the shear-induced ‘Reynolds pressure’ increases roughly as \( \gamma^2 \) with a density dependent prefactor which we characterize by the ‘Reynolds coefficient’:

\[
R = \left( \frac{\partial^2 P}{\partial \gamma^2} \right) / 2.
\]

For linear isotropic elastic materials, no coupling between shear strain and pressure is expected. But, as we apply shear, the system becomes increasingly anisotropic, so a \( P - \gamma \) coupling might be possible, as expressed by, \( \partial P / \partial \gamma \). In our system, this derivative grows roughly as
FIG. 2: (Color online) (a) Reynolds pressure $P(\gamma^2)$ observed in forward shear (see text) tests for $\phi = 0.691 - 0.816$. (b) Reynolds coefficient $R$ extracted from linear fitting, obtained from up to 54% forward shear (red squares), up to 27% forward shear (blue dots), and cyclic shear tests under limit cycle behavior (black triangles). The inset shows the same data on double logarithmic scales with $\phi_c = 0.841 \pm 0.004$. The error bar is smaller than the size of the symbols unless marked. The dashed line shows a fit to a power law. A line corresponding to an exponent $-3.3$ is also shown for reference.

$\phi$, and linear elasticity is not a particularly useful concept. $R$ grows strongly with $\phi$, and shows an apparent, but unexpected divergence at $\phi = \phi_c \approx \phi_f$. Fig. 2b and inset, show a log-log plot of $R$ vs. $\Delta \phi = \phi_c - \phi$. A power-law fit to $R = A(\phi_c - \phi)^\alpha$, yields $\alpha = -3.3 \pm 0.1$ and $\phi_c = 0.841 \pm 0.004$. By contrast, $\phi_c$ lies in the range $0.83 \leq \phi_c \leq 0.84$, so here, $\phi_c$ is not distinguishable from $\phi_f$, which is also comparable to $\phi_f$ for systems of frictionless 2D particles. For $\phi \leq 0.75$, the system is very loose, and it does not form a percolating contact network, even after 54% strain. $R(\phi)$ behavior in this case is affected by small experimental ‘noise’ effects, discussed above, and deviates from the power-law behavior (Fig. 2b(inset)). We identify $\phi_S \simeq 0.75$, the lower limit in this system for shear jamming.

Limit Cycles To characterize the evolution/reproducibility/relaxation of the stresses, we carried out multiple shear cycles. This also allowed us to determine $R$ for $\phi$ closer to $\phi_f$, where shear strains are limited due to buckling; we obtain good statistics by many smaller-amplitude strain cycles. The oscillatory shear experiments were started from initially stress-free states for $\phi$’s in the shear jamming regime, $\phi_S \leq \phi \leq \phi_f$. In a cycle, we sheared by strain steps of 0.45% up to $\gamma_{\text{max}}$ in the ‘forward direction’, followed by a shear strain decrease $(-0.45\% \text{ per step})$ to a smaller strain, $\gamma_{\text{min}}$. For symmetric shear cycles: $\gamma_{\text{min}} = -\gamma_{\text{max}}$, and asymmetric shear cycles: $\gamma_{\text{min}} \neq -\gamma_{\text{max}}$.

For symmetric cycles, $P$ was symmetric about $\gamma = 0$, approximately quadratic in $\gamma$, and virtually reproducible over many cycles, as shown in Fig. 3a. However, details of the network were generally not reproducible from cycle to cycle. The Reynolds coefficient $R(\phi)$ followed the same trend as in the forward shear tests (Fig. 2b), further confirming the Reynolds effect. After transients, the shear stress $\tau$ also followed a reproducible path over cycles, but unlike $P$, $\tau$ was strongly hysteretic, with non-zero values at $\gamma = 0$. There were $\gamma$’s for which $\tau = 0$ but $P \neq 0$, for example, in Fig. 3a and b, at $\gamma \approx 1\%$. However, in such cases, $\tau$ coarse grained at smaller scales than the system size was locally non-zero, even though the global $\tau$ was 0 (e.g. because of spatial variations of the principal stress orientations). Due to length limitations, we consider only the dynamics exhibited by $P$, and we will present the full stress dynamics elsewhere.

The evolution of $P(\gamma)$ for asymmetric shear cycles differed from the symmetric case. Here, $P(\gamma)$ was initially asymmetric, but evolved towards a symmetric shape centered around the mean strain, $\bar{\gamma}$, after many cycles. Thus, the long term $P - \gamma$ dynamics was a limit cycle. The system relaxed quickly (slowly) to the limit cycle if sheared symmetrically (asymmetrically). Fig. 3b shows an example of slow evolution, where a limit cycle was reached after about 28 cycles. In this case $P(\gamma)$ evolved to a symmetric shape, similar to the forward shear experiment, except for a shift; i.e., the system did not reach a completely stress-free state at the mid-point of strain. However, a long term limit cycle was still reached with the same Reynolds coefficient for the given density, $\phi = 0.825$.

Slow Relaxation For asymmetric strain cycles, $\Delta P(n) = P(\gamma_{\text{max}}) - P(\gamma_{\text{min}})$ was initially nonzero, but it decreased and ultimately vanished, within fluctuations, for $n = n_0$. When the limit cycle was reached, $P$ was symmetric about $\bar{\gamma} = (\gamma_{\text{max}} + \gamma_{\text{min}})/2$. The slow relaxation of $\Delta P$ for asymmetric shear shows striking and novel scaling behavior, which we characterize in terms of $\phi$, $\bar{\gamma}$ and the shear amplitude $\gamma_A$. Experiments to characterize this relaxation spanned $\phi$’s from above $\phi_S$ to just below isotropic jamming $\phi_f$: $0.780 \leq \phi \leq 0.828$.
strain amplitudes of $\gamma_A = 6.75, 4.5, 3, 1.5\%$ and a range of starting strains $0 \leq \tilde{\gamma} \leq 21.35\%$. Experiments were 100-500 cycles long; for convenience we measured $G^2$ only at $\gamma_{\text{max}}, \gamma_{\text{min}}$, and then converted $G^2$ to $\Delta P$ using a calibration. Fig. 4a shows $\Delta P$ for a particular $\gamma_A$.

For $\phi$ in the shear jamming region, $\Delta P(n)$ decayed logarithmically slowly towards 0:

$$\Delta P(n) \simeq -\beta \log(n/n_0),$$

implying a natural ‘time scale’ for relaxation, $n_0$, that we obtained through least squares fits of the logarithmic part of the relaxation. All the relaxation data, for a given $\gamma_A$, collapse onto a single curve when expressed in terms of $n/n_0$ (Fig. 4b), regardless of $\phi$ and $\tilde{\gamma}$. The factor $\beta(\gamma_A)$ differs for each $\gamma_A$ (Fig. 4c), but $\Delta P/\beta$ is a universal function of $n/n_0$, as in Fig. 4d, which shows all $\sim 170$ datasets. We emphasize the remarkable role that $\beta(\gamma_A)$ plays, and the fact that it is independent of $\phi$.

We then consider what determines $n_0$. Eq. 2 implies:

$$n_0 = n \cdot \exp(\Delta P(n)/\beta(\gamma_A)).$$

Initially, at $n = 1$, $\Delta P = \Delta P_0$. According to the approximately quadratic relation between $P$ and $\gamma$, $\Delta P_0$ is given by:

$$\Delta P_0 = R(\phi)(\gamma_{\text{max}}^2 - \gamma_{\text{min}}^2)/2 = R(\phi)\gamma_{\gamma A}. $$

Therefore,

$$n_0 = \exp(R(\phi)\gamma_{\gamma A}^A).$$

Eq. 2 also implies an evolution $d\Delta P/dn = -\beta n_0^{-1} \exp(\Delta P/\beta)$ or, with a cutoff, $d\Delta P/dn = -\beta n_0^{-1} \exp(\Delta P/\beta - 1)$, which produces the logarithmic form of Eq. 2 for small $n$, with saturation at $n = n_0$. This suggests an activated process, perhaps involving a generalized ensemble, such as the stress ensemble, as discussed by several authors[17][20].

To summarize: for frictional granular systems in/near the shear jamming regime, $\phi_S \leq \phi \leq \phi_J$, we generated sheared states without shear bands, even with large strains or over many cycles of shear, making it possible to experimentally probe the constitutive relations of granular materials. These experiments show two key and highly novel results: 1) We find a novel Reynolds effect for fixed $\phi$ that is approximately quadratic in $\gamma$ using $R = (\partial^2 P/\partial \gamma^2)/2$. We note that the specific form for $R(\gamma)$ may well depend on the particle interaction force; a more general form might be $P = R\gamma^\delta$, where for our experiments, $\delta \simeq 2$. 2) We find that under cyclic shear, frictional granular systems evolve logarithmically slowly, as one might expect for an activated process, towards a state where the pressure is symmetric, modulo fluctuations, about the mid-point of strain. The pressure at the symmetry point may not be zero. This slow evolution is characterized by highly novel scaling behavior, such that there is good collapse of all data.

These results point towards several interesting directions. First, it is reasonable to search for a description of these states in terms of an ensemble picture, such as the stress ensemble, given the activated process character of the slow relaxation. Such a theory would need to explain some of the striking scaling properties observed here. In addition, we have not considered the properties of the shear stress under cyclic shearing, nor have we considered the particle dynamics of details of the force/contact networks. We will present these results elsewhere.

We thank Jie Zhang for sharing code to perform rotational particle tracking. IGUS generously supplied us with a free linear stage under the Young Engineers Support program. Discussions with Dapeng Bi, Bulbul Chakraborty, Martin van Hecke, Stefan Ludwig and Corey O’Hern are gratefully acknowledged. Work supported by NSF grants DMR-0906908, DMR-1206351, ARO grant W911NF-11-1-0110, and NSF grant DMS0835742.

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