The quantum phase transition of high dimensional Yang-Mills theories

N. Irges, G. Koutsoumbas and K. Ntrekis

Department of Physics, National Technical University of Athens
GR-15780 Athens, Greece

Abstract

We determine the critical value of the coupling where the first order quantum phase transition takes place for lattice $SU(2)$ Yang-Mills theories in dimensions higher than four. Within a Mean-Field approach we derive an approximate law valid for any dimension $d$ and in the context of a Monte Carlo approach, in addition to the already known $d = 5$ case, we look at $d = 6, 7, 8$. 
Even though high \((d > 4)\) dimensional Yang-Mills theories are perturbatively non-renormalizable, one can not exclude the possibility that there exists a regime in their phase diagram where a physically useful cut-off effective theory can be constructed. Also from the theoretical point of view, these theories contain the basic ingredients (the gauge fields) of various field theories motivated by string theory.\(^1\) Therefore, their study is potentially useful.

A possible universal property of high dimensional Yang-Mills theories that emerges from previous studies comes from the fact that in five dimensions a "bulk" or "quantum" phase transition appears dividing the confined phase from a Coulomb phase \([2]\). Recall that \(4d\) pure Yang-Mills at zero temperature has only a confined phase where the gauge fields always form flux tubes. Intuition says that as the number of dimensions transverse to the surface of the tube increases, the harder is to sustain a stable flux tube thus requiring a stronger coupling. In phase diagram terminology, we expect the confined phase to persist but also to shrink as \(d\) increases. Here we will perform a first check of this statement. We take \(SU(2)\) as our model and regularize it on a \(d\)-dimensional periodic lattice with linear dimension \(L\). The action is the standard Wilson plaquette action

\[
S_W = \beta \sum_x \sum_{1 \leq M,N \leq d} \left[ 1 - \frac{1}{2} \text{tr} \{U_{MN}(x)\} \right],
\]

where \(U_{MN}(x)\) is the elementary plaquette located at the site \(x\) and \(\beta\) is the dimensionless lattice coupling. We will employ two methods, on one hand the Mean-Field (MF) approximation \([3, 4]\), an analytic method expected to work well near the phase transition and in general increasingly well as \(d\) grows\(^2\) and Monte Carlo (MC) simulations on the other.

For \(SU(2)\) in \(d\) dimensions, the (ungauge-fixed) MF approach to zeroth order determines the confined and Coulomb phases via the solution to the coupled equations for the MF background \(v_0\) \([3]\]

\[
v_0 = \frac{I_2(h_0)}{I_1(h_0)} , \quad h_0 = 2v_0^3(d-1)\beta
\]

with \(I_\nu(h_0)\) the modified Bessel function, by defining the Coulomb phase as the regime of \(\beta\) where there is a solution with \(v_0 \neq 0\) and as the confined phase otherwise. In \([4]\) the equations were solved for \(d = 5\) by a numerical, iterative method. The smallest positive and real non-vanishing value of the background \(v_{0c}\) satisfying eqs.(2) (i.e. where the iteration stabilizes), determines the critical value of the lattice coupling \(\beta_c\) where the phase transition takes place. It is expected to be a quantum phase transition since the MF at this order is volume independent. This of course needs to be checked. In fact,

\(^1\)For Monte Carlo studies of high dimensional (supersymmetric) Yang-Mills theories from the point of view of matrix models, see \([1]\).

\(^2\)Or when \(N\) of \(SU(N)\) grows; this however tends to shrink the Coulomb phase instead \([5]\)!
for \( d = 5 \) it was found by a MC simulation on a 4\(^5\) lattice in 1979 by Creutz to be a quantum, first order phase transition. Subsequently this was confirmed (and extended to anisotropic lattices) by several authors [6]. Apart from the fact that both methods agree on the order of the transition, their estimates for the value of the critical coupling are also quite close: the \( \beta_c^{\text{MF}} \simeq 1.6762017 \) of the MF (corresponding to \( v_{0c} \simeq 0.73333 \)) [4] is to be compared with the \( \beta_c^{\text{MC}} = 1.642 \pm 0.015 \) of the MC [2].

An observation stemming from eqs.(2) is that the quantity \( B = (d-1)\beta_c^{\text{MF}} \) and therefore also \( v_{0c} \) are \( d \)-independent. We can then solve eqs.(2) for general \( d \) by noting that the zero of the function \( F = I_2(h_0)/I_1(h_0) - v_0 \) that signals the phase transition is one where \( F(v_0) \) has an extremum. This, using the identity \( I_\nu(h_0) = I_{\nu-2}(h_0) - \frac{2(\nu-1)}{h_0} I_{\nu-1}(h_0) \), translates into the relation

\[
v_{0c} = \frac{1}{\sqrt{2}} \left[ 1 \pm \sqrt{1 - \frac{20}{3} B} \right].
\]

Substituting the above in the function \( F \) results in an algebraic expression with only parameter \( B \), whose relevant root can be found numerically to be \( B \simeq 6.704840 \), determining \( v_{0c} \simeq 0.7333 \) from the upper sign of eq.(3), as expected. Thus, we find that the equation

\[
(d-1)\beta_c^{\text{MF}} \simeq 6.704840
\]

fixes the \( SU(2) \) critical coupling in any dimension \( d > 4 \).

What makes it possible to go high in \( d \) with Monte Carlo simulations is that we are dealing with a bulk phase transition. This means that as long as the lattice extent is large enough so that finite size effects do not interfere, the phase transition is visible. Most times a \( 4^d \) lattice will suffice to observe the effect, even though larger lattices will be clearly needed to describe it with better precision. The order parameter we use in order to determine the phase transition is the plaquette \( P = \frac{2}{d(d-1)L^2} \sum_x \sum_{1 \leq M < N \leq d} \left[ 1 - \frac{1}{d} \text{tr} \{ U_{MN}(x) \} \right] \).

The Kennedy-Pendleton heat bath algorithm [7] has been used to update the gauge field, while overrelaxation hits have been employed to decorrelate the measurements. The phase diagram is obtained for \( d = 6, 7, 8 \). The lattice sizes used have a linear dimension \( L = 4 \) and after thermalization, hysteresis loops have been performed. The step in \( \beta \) was 0.07 (starting at \( \beta = 0.40 \) and going up to \( \beta = 1.8 \) and back) and 3000 sweeps through the lattice have been done at each \( \beta \) value. There exist well known approximations that we have used as guides. In the strong coupling regime (small values of \( \beta \)) the plaquette is well approximated by \( P \simeq \frac{\beta}{4} \), while in the weak coupling the approximation is \( P \simeq 1 - \frac{3}{\beta^2} \).
Figure 1: Left: Hysteresis loops for 6, 7 and 8 dimensions. The strong and weak coupling predictions are also included. Right: The pseudocritical values for $\beta_c^{MC}$ versus $\frac{1}{d-1}$ and their estimated errors. The Mean-Field prediction eq.(4) is represented by the straight line.

errors. On the other hand, the right part of Figure 1 depicts a comparison between the leading order MF prediction for $\beta_c^{MF}$ based on the analytical expression eq.(4) and the results of the Monte Carlo runs. We also quote for completeness the original $d = 5$ result from [2]. We notice a quite good agreement between the two methods.

Extending the work of [2] we determined the critical value of the coupling where a first order bulk phase transition takes place for high dimensional $SU(2)$ lattice gauge theories. We first derived a law valid in any dimension $d > 4$ based on the Mean-Field method and then performed corresponding Monte Carlo checks for the first time in $d = 6, 7$ and 8 dimensions.

Acknowledgements We would like to thank K. Farakos for illuminating discussions. This research is implemented under the ARISTEIA II action of the operational programme education and long life learning and is co-funded by the European Union (European Social Fund) and National Resources of Greece.

References

[1] K.N. Anagnostopoulos, T. Azuma and J. Nishimura, JHEP 1311 (2013) 009.

[2] M. Creutz, Phys. Rev. Lett. 43 (1979) 553.

[3] J.M. Drouffe, J.B. Zuber, Phys. Rept. 102, (1983) 1.
[4] N. Irges and F. Knechtli, Nucl. Phys. B822 (2009) 1. N. Irges and F. Knechtli, Phys. Lett. B685, (2010) 86.

[5] N. Irges and G. Koutsoumbas, JHEP 1208 (2012) 103.

[6] S. Ejiri, J. Kubo and M. Murata, Phys. Rev. D62 (2000) 105025. P. de Forcrand, A. Kurkela and M. Panero, JHEP 06 (2010) 050. K. Farakos and S. Vrentzos, Nucl. Phys. B862 (2012) 633. F. Knechtli, M. Luz and A. Rago, Nucl. Phys. B856 (2012) 74. L. Del Debbio, R.D. Kenway, E. Lambrou and E. Rinaldi, Phys. Lett. B724, (2013) 133.

[7] A. Kennedy, B. Pendleton, Phys. Lett. B156, (1985) 393.