Combinatorics of Intervals in the Plane I: Trapezoids

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Abstract
We study arrangements of intervals in $\mathbb{R}^2$ for which many pairs form trapezoids. We show that any set of intervals forming many trapezoids must have underlying algebraic structure, which we characterise. This leads to some unexpected examples of sets of intervals forming many trapezoids, where an important role is played by degree 2 curves.

Keywords Arrangements of intervals · Point-line incidences · Trapezoids in the plane

Mathematics Subject Classification 52C10 · 52C30 · 52C45

1 Introduction

The combinatorics of points in the plane has received much attention, as has the combinatorics of other geometric and algebraic objects such as lines and curves. A typical problem involves estimating the number of possible occurrences of a given combinatorial or geometric relation among all arrangements of the object of study, and beyond this understanding the structure of the extremal arrangements.

Line segments have also been investigated in various areas of discrete and computational geometry, for example in [1, 4, 6, 7]—see also [5] for more examples—with research focusing on properties such as intersections and visibility. We will study the combinatorics of some geometric properties of intervals. This is the first in a series of papers on geometric configurations of intervals; in this first part, we consider arrangements for which there are many pairs forming trapezoids.
An interval in $\mathbb{R}^2$ is a directed line segment. We will denote an interval by a four-tuple $(a, b; c, d) \in \mathbb{R}^4$, where $(a, b), (c, d) \in \mathbb{R}^2$ denotes the coordinates of the initial and terminal point, respectively. We require that $(a, b) \neq (c, d)$, so the interval has positive length. We call the interval $(c, d; a, b)$ the reverse of the interval $(a, b; c, d)$. Consider a pair of distinct intervals, $(a, b; c, d)$ and $(a', b'; c', d')$. We say that this pair forms a trapezoid if the convex hull of the two intervals is a trapezoid or a triangle. Arithmetically, this occurs if

\[
\begin{align*}
(a - a')(d - d') &= (b - b')(c - c'), \quad \text{or} \\
(a - c')(d - b') &= (c - a')(b - d'), \quad \text{or} \\
(d - b')(c' - a') &= (d' - b')(c - a).
\end{align*}
\]

Equations (1) and (2) represent the case when a pair of endpoints from each interval is parallel to the other pair, whereas (3) represents the case when the two intervals are parallel. Note that these equations allow some acceptable degenerate cases, namely where the two intervals lie on the same line, or where they share an endpoint. Note also that both (1) and (2) can be satisfied simultaneously, as in the case when the two intervals form the diagonals of a parallelogram. All of these scenarios are illustrated in Fig. 1.

Given a set of intervals in the plane, a natural combinatorial property of the set is the number of pairs of intervals forming trapezoids. Clearly there are arrangements of intervals for which every pair forms a trapezoid: for example if a set of intervals are all parallel to a single direction or if all the endpoints lie on two fixed parallel lines. These examples are highly structured and one can therefore ask if all arrangements with many pairs forming trapezoids must have some shared underlying structure. We answer this question, showing that if the number of pairs forming trapezoids is above a certain threshold, then many of the endpoints of intervals satisfy particular algebraic relations.

Throughout the paper we use the notation $A \gtrsim B$ to mean there exists an absolute constant $C > 0$ for which $A \geq CB$. The following is our main result and gives a description of sets of intervals forming many trapezoids.

**Theorem 1.1** Let $\mathcal{I}$ be a set of $N$ distinct intervals in $\mathbb{R}^2$. If $\gtrsim N^{3/2} \log N$ pairs of intervals form trapezoids then at least one of the following holds.

0. There are $\gtrsim N^{1/2} \log N$ parallel intervals.
1. There is a centre $S$ and a ratio $\lambda$ such that for $\gtrsim N^{1/2}$ intervals from $\mathcal{I}$, one endpoint is the image of the other under a homothety with centre $S$ and ratio $\lambda$.
2. There are two curves in $\mathbb{R}^2$ of degree at most 2 such that $\gtrsim N^{1/2}$ intervals from $\mathcal{I}$ have an endpoint on each curve.

If a set of intervals are mutually parallel, then any pair will form a trapezoid. Similarly, for a set of intervals created by a single homothety, any pair will also form a trapezoid. For case 0 and case 1 above, only $\gtrsim N$ pairs of intervals are guaranteed to form trapezoids. The reason for this is that there can be $\gtrsim N^{1/2}$ separate instances of that structure, giving to a total of $\gtrsim N^{3/2}$ trapezoids. Case 2 in Theorem 1.1 is more surprising. Based on our proof of this theorem, we construct examples of sets of $N$ intervals forming $\gtrsim N^2$ trapezoids and with underlying conic curve structure. In order for many intervals to form trapezoids via the structure of case 2, additional constraints must be satisfied that govern the relationship between the two curves and the placement of endpoints upon the curves. In Sect. 4 we will see that the precise placement of the endpoints on the curves can be parameterized. For examples of sets forming many trapezoids with homothety or conic curve structure, see Figs. 2 and 3, respectively.

2 Main Lemma

In this section, we will prove the following technical lemma.
Lemma 2.1  Let \( I \) be a set of \( N \) distinct intervals in \( \mathbb{R}^2 \). If \( \geq N^{3/2} \log N \) pairs of intervals form trapezoids then at least one of the following holds.

0. There are \( \geq N^{1/2} \log N \) parallel intervals.
1. There are two parallel lines in \( \mathbb{R}^2 \) such that \( \geq N^{1/2} \) intervals have an endpoint on each line.
2. There are two parallel lines \( \ell_1, \ell_2 \subset \mathbb{R}^2 \) such that \( \geq N^{1/2} \) intervals \((a, b; c, d) \in I \) satisfy \((a, c) \in \ell_1 \) and \((b, d) \in \ell_2 \).
3. There are two subsets \( I_1, I_2 \subset I \) such that for any \( I_1 \in I_1 \) and any \( I_2 \in I_2 \), the intervals \( I_1, I_2 \) form a trapezoid. In addition, \( |I_1| \cdot |I_2| \geq N \).

Note that in property 2, the endpoints \((a, b)\) and \((c, d)\) of the intervals are not themselves contained in two lines; rather, the linear relationship is satisfied separately by the \( x \)-coordinates and the \( y \)-coordinates.

Our first aim of this section is to set up a correspondence between intervals in \( \mathbb{R}^2 \) and lines in \( \mathbb{R}^3 \) such that a pair of intervals forms a trapezoid when the corresponding pair of lines intersect. In (1), (2), and (3) we saw that there are three ‘ways’ for two intervals to form a trapezoid. If many pairs of intervals form a trapezoid via the third ‘way’ (3), then many intervals are parallel simply by pigeonholing. This accounts for conclusion 0 in Theorem 1.1. Therefore we are more interested in determining the structure of a set of intervals when (1) or (2) is satisfied for many pairs. With this in mind, we make the following definitions.

Definition 2.2  Two intervals \( I_1 = (a, b; c, d), I_2 = (a', b'; c', d') \) form a Type 1 trapezoid if they satisfy (1). Similarly \( I_1, I_2 \) form a Type 2 trapezoid if they satisfy (2).

Note that Type 1 and Type 2 trapezoids are not mutually exclusive, see for example the middle pair of intervals in Fig. 1. Moreover, if \( I'_1 \) denotes the reverse of \( I_1 \), then \( I'_1, I_2 \) form a Type 2 trapezoid if and only if \( I_1, I_2 \) form a Type 1 trapezoid. Also note the degenerate case when \( I_1, I'_1 \) form both a Type 1 and a Type 2 trapezoid.

In the following lemma we show that there is a correspondence between intervals in \( \mathbb{R}^2 \) and lines in \( \mathbb{R}^3 \) such that a pair of intervals forms a Type 1 trapezoid if and only if the corresponding pair of lines in \( \mathbb{R}^3 \) intersect.

Lemma 2.3  Let \( I \) be a set of intervals in \( \mathbb{R}^2 \) such that any vertical line contains at most one distinct endpoint from the intervals in \( I \). Then there is a bijection \( \mathcal{L} \) from intervals in \( \mathbb{R}^2 \) to lines in \( \mathbb{R}^3 \) that are not parallel to the \( xy \)-plane, such that a pair of intervals forms a Type 1 trapezoid if and only if their images under \( \mathcal{L} \) intersect.

Proof  To every interval \((a, b; c, d)\) we associate the unique line

\[
\mathcal{L}(a, b; c, d) = \left\{ \begin{pmatrix} b \\ d \\ 0 \\ \end{pmatrix} + t \begin{pmatrix} a \\ c \\ 1 \\ \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathbb{R}^3. \tag{4}
\]

Conversely, every line \( \ell \subset \mathbb{R}^3 \) that is not parallel to the \( xy \)-plane can be normalised to have the same form as the line in (4). Define \( \mathcal{I}(\ell) \) to be this corresponding interval. Clearly \( \mathcal{L} \) is a bijection from intervals in \( \mathbb{R}^2 \) to lines not parallel to the \( xy \)-axis in \( \mathbb{R}^3 \), and \( \mathcal{I} \) is its inverse.
Let \((a, b; c, d)\) and \((a', b'; c', d')\) be intervals such that \((a, b; c, d) \neq (a', b'; c', d')\), though we allow that they are the reverse of each other. Observe that the lines \(L(a, b; c, d)\) and \(L(a', b'; c', d')\) intersect if and only if there is a solution \(t \in \mathbb{R}\) to the system of equations

\[
t(a - a') = b' - b; \quad t(c - c') = d' - d. \tag{5}
\]

We can see that (1) follows from multiplying the two equations in (5). Conversely, note that \(b = b'\) implies \(a = a'\) and similarly \(d = d'\) implies \(c = c'\) since we’ve assumed that two distinct endpoints cannot lie on the same vertical line. Consequently, (1) implies there is a unique solution \(t \in \mathbb{R}\) to (5). \(\Box\)

We remark that our assumption that vertical lines contain only one endpoint is easy to achieve for any finite set of intervals by applying a generic rotation of \(\mathbb{R}^2\). Henceforth we will always assume a set of intervals has this property. Crucially, the intersection of \(L(a, b; c, d)\) and \(L(a', b'; c', d')\) does not imply (2). Instead by the argument of Lemma 2.3 we have that if \((a, b; c, d)\) and \((a', b'; c', d')\) are intervals without two endpoints on a single vertical line then they form a Type 2 trapezoid if and only if \(L(c, d; a, b)\) and \(L(a', b'; c', d')\) intersect. As a result, in order to ‘capture’ all of the pairs of intervals forming trapezoids as intersecting lines, each interval will correspond to two lines in \(\mathbb{R}^3\), one for it and one for its reverse.

**Definition 2.4** Let \(I = \{(a_i, b_i; c_i, d_i) : 1 \leq i \leq N\}\) be a set of \(N\) intervals in the plane. Define the following set of \(2N\) lines in \(\mathbb{R}^3\),

\[
L(I) = \{L(a_i, b_i; c_i, d_i) : 1 \leq i \leq N\} \cup \{L(c_i, d_i; a_i, b_i) : 1 \leq i \leq N\},
\]

where \(L\) is defined as in (4).

As remarked upon earlier, an interval and its reverse form a Type 1 and Type 2 trapezoid. The following lemma shows how the number of pairs of intervals in \(I\) forming trapezoids compares to the number of pairs of intersecting lines in \(L(I)\).

**Lemma 2.5** Let \(I\) be a set of \(N\) distinct intervals in \(\mathbb{R}^2\) such that an interval and its reverse do not both appear in \(I\). Let \(T\) be the number of pairs of intervals in \(I\) that satisfy (1) or (2), where a pair of intervals satisfying both (1) and (2) is counted with multiplicity two. Then

\[
2|T| = \# \{\text{pairs of intersecting lines in } L(I)\} - N.
\]

**Proof** Let \(I = \{(a_i, b_i; c_i, d_i) : 1 \leq i \leq N\}\). For every pair of intervals \((a, b; c, d), (a', b'; c', d') \in I\) satisfying one (resp. two) of equation(s) (1), (2), there are two (resp. four) intersections among the lines

\[
L(a, b; c, d), \quad L(c, d; a, b), \quad L(a', b'; c', d'), \quad L(c', d'; a', b').
\]

Note that \(L(a_i, b_i; c_i, d_i)\) and \(L(c_i, d_i; a_i, b_i)\) intersect exactly once for all \(1 \leq i \leq N\). These intersections correspond to an interval forming a trapezoid with its reverse.
direction. All other pairs of intersecting lines in $L$ correspond to a trapezoid formed by distinct line segments.

So far we roughly have an equivalence between intervals forming trapezoids, and intersecting lines in $\mathbb{R}^3$. The main tool that we will use to prove that our sets of intervals have particular structure is a theorem of Guth and Katz on incidences of lines in $\mathbb{R}^3$. The following is a corollary of [2, Thm. 1.2] via a standard dyadic summation.

**Theorem 2.6** (Guth and Katz [2])  
Let $L$ be a set of $N$ lines in $\mathbb{R}^3$. If the number of pairs of intersecting lines in $L$ is $\gtrsim N^{3/2} \log N$, then at least one of the following holds.

1. There are $N^{1/2}$ concurrent lines in $L$.
2. There exists a plane containing $\gtrsim N^{1/2}$ lines of $L$.
3. There exists a regulus containing a subset of lines $L_R \subset L$ such that the number of pairs of intersecting lines in $L_R$ is $\gtrsim N$.

**Proof** Let $L$ be a set of $N$ lines in $\mathbb{R}^3$ such that none of the cases 1, 2, or 3 hold. For $2 \leq k \leq N^{1/2}$, let $S_k$ denote the number of points in $\mathbb{R}^3$ that lie on exactly $k$ lines of $L$. By [2, Thm. 1.2] we have for all $2 \leq j \leq N^{1/2}$ that $\sum_{k \geq j} S_k \lesssim N^{3/2}/j^2$. Since, by assumption, at most $\sqrt{N}$ lines pass through a single point, the number of pairs of intersecting lines in $L$ is bounded from above by

$$\sum_{k=2}^{\lfloor \sqrt{N} \rfloor} k^2 S_k \leq \sum_{j=1}^{\lfloor \log_2 N^{1/2} \rfloor} \sum_{2^j \leq k \leq 2^{j+1}} k^2 S_k \leq \sum_{j=1}^{\lfloor \log_2 N^{1/2} \rfloor} 2^{j+2} \sum_{k \geq 2^j} S_k \lesssim N^{3/2} \log N.$$ 

Therefore an appropriate choice of implicit constants gives the result.

The reason that reguli and planes appear here is that they are the only doubly ruled surfaces in $\mathbb{R}^3$, planes are in fact infinitely ruled. Recall that a doubly ruled surface is one for which at every point on the surface, there are two distinct lines contained within the surface and containing the point. A regulus contains two families of lines such that there are no intersections within a family but every pair of lines from different families intersect. That is, if we select $M$ lines from one ruling within a regulus and $N$ lines from the other, then there will be precisely $MN$ pairs of intersecting lines.

By combining our preliminary results with Theorem 2.6 we are able to prove Lemma 2.1.

**Proof of Lemma 2.1** Let $I$ be a set of $N$ distinct intervals such that $I$ does not contain both an interval and its reverse. Put $L = L(I)$. For each pair of intervals in $I$ forming a trapezoid, one of (1), (2), and (3) must be satisfied by the corresponding coordinates.

**Case 0: Parallel intervals.** First suppose that at least half of the trapezoids are formed by pairs of intervals satisfying (3). For each such pair, the intervals are parallel so by pigeonholing we find a single direction to which $\gtrsim N^{1/2} \log N$ intervals are parallel.

We can now assume that at least half of the trapezoids are formed by pairs satisfying either (1) or (2). By Lemma 2.5, if there are $\gtrsim N^{3/2} \log N$ such trapezoids, then
\( \geq N^{3/2} \log N \) pairs of lines in \( \mathcal{L} \) intersect, excluding the intersections between an interval and its reverse. By Theorem 2.6, many lines are concurrent, lie in a plane, or are contained in a regulus. We consider these three cases separately, which correspond to 1, 2, and 3 in Lemma 2.1.

**Case 1: Concurrent lines.** Suppose that \( \geq N^{1/2} \) lines of \( \mathcal{L} \) pass through the point \((u, v, w) \in \mathbb{R}^3\). Then \( \geq N^{1/2} \) lines of \( \mathcal{L} \) are of the form \([u - aw, v - cw, 0] + t[a, c, 1]\). These lines correspond to \( \geq N^{1/2} \) intervals with one endpoint on \( y = u - wx \) and the other on \( y = v - wx \). This implies property 1 of Lemma 2.1. Note that if \( u = v \), then many intervals are contained entirely in the line \( y = u - wx \). In this case, an interval and its reverse might be represented in the set of concurrent lines passing through \((u, v, w)\).

**Case 2: Lines in a plane.** Suppose that \( \geq N^{1/2} \) lines lie in the plane \( Ax + By + Cz + D = 0 \). A line in \( \mathcal{L} \) of the form (4) belongs to this plane if \( t(Aa + Bc + C) + Ab + Bd + D = 0 \), for all \( t \in \mathbb{R} \). Thus \((a, c)\) is on the line \( Ax + By + C = 0 \) and \((b, d)\) is on the line \( Ax + By + D = 0 \). Note that \( A, B \) are not both zero, since no line in \( \mathcal{L} \) is parallel to the \( xy \)-plane. We conclude that \( \geq N^{1/2} \) intervals \((a, b; c, d)\) of \( \mathcal{I} \) satisfy \( Aa + Bc + C = Ab + Bd + D = 0 \). That is, property 2 of Lemma 2.1 holds.

**Case 3: Lines in a regulus.** Suppose there exists a subset \( \mathcal{L}_R \subset \mathcal{L} \) of lines on some regulus, such that the number of pairs of intersecting lines in \( \mathcal{L}_R \) is \( \geq N \). As noted earlier, \( \mathcal{L}_R \) can be partitioned into \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), where the lines in \( \mathcal{L}_1 \) belong to one ruling of the regulus and the lines in \( \mathcal{L}_2 \) belong to the other ruling. There are no intersecting pairs of lines within the same ruling, thus \( |\mathcal{L}_1| \cdot |\mathcal{L}_2| \geq N \). Pulling back \( \mathcal{L}_1, \mathcal{L}_2 \) to intervals in \( \mathbb{R}^2 \) gives \( \mathcal{I}_1, \mathcal{I}_2 \) satisfying property 3 of Lemma 2.1.

### 3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by analysing the geometry underlying each of the situations in the conclusion of Lemma 2.1.

**Proof of Theorem 1.1**  If the intervals in \( \mathcal{I} \) form \( \geq N^{3/2} \log N \) trapezoids, then we can apply Lemma 2.1. We will now consider each of the cases implied by this lemma.

**Case 0 (Parallel) or Case 1 (Concurrent).** In both of these cases, there is nothing left to prove as they already give conclusions 0 and 2 of Theorem 1.1, respectively. An example of the Concurrent case, where the intervals have endpoints contained in two parallel lines, is given in Fig. 4.

**Case 2 (Coplanar).** Suppose that many intervals \((a, b; c, d)\) of \( \mathcal{I} \) satisfy \( Aa + Bc + C = Ab + Bd + D = 0 \), for suitable coefficients \( A, B, C, D \), where \( A \) and \( B \) are not both zero. Note that \( A = B = 0 \) does not occur since all lines of \( \mathcal{L}(\mathcal{I}) \) have a vertical component. If \( A + B \neq 0 \), we have
Discrete & Computational Geometry (2023) 69:232–249 239

Fig. 4 Case 1 of Lemma 2.1: Intervals corresponding to a set of concurrent lines in \( \mathcal{L} \). There are two fixed parallel lines and each interval has one endpoint on each line.

\[
\frac{A}{A+B}(a, b) + \frac{B}{A+B}(c, d) = \left(-\frac{C}{A+B}, -\frac{D}{A+B}\right).
\]

In other words, the line containing the points \((a, b)\) and \((c, d)\) always contains \((-\frac{C}{A+B}, -\frac{D}{A+B})\) \(\in\mathbb{R}^2\).

this point will be the centre \(S\) of the homothety. Moreover, since

\[
A\left((a, b) - \left(-\frac{C}{A+B}, -\frac{D}{A+B}\right)\right) = -B\left((c, d) - \left(-\frac{C}{A+B}, -\frac{D}{A+B}\right)\right),
\]

we see that the ratio \(\lambda\) of the distance between \((a, b)\) and \((-C/(A+B), -D/(A+B))\) to the distance between \((c, d)\) and \((-C/(A+B), -D/(A+B))\) is \(-B : A\). Hence the endpoints \((a, b)\) and \((c, d)\) obey a fixed homothety with respect to the centre \((-C/(A+B), -D/(A+B))\). On the other hand, if \(A + B = 0\) then since \(A \neq 0\) we obtain \(a - c = -C/A\) and \(b - d = -D/A\) and so all intervals are translations of each other. For illustrations of this case, see Fig. 5.

Case 3 (Regulus). Let \(\mathcal{J}_1, \mathcal{J}_2, \mathcal{L}_1, \mathcal{L}_2\) be as in the proof of Lemma 2.1. Let \(\ell_j = \{[b_j, d_j, 0] + t[a_j, c_j, 1] : t \in \mathbb{R}\} \) for \(j = 1, 2, 3\), be distinct lines in \(\mathcal{L}_2\). Note that if all intervals in \(\mathcal{J}_2\) have a common endpoint, then all lines in \(\mathcal{L}_2\) lie in a common plane, a case discussed above. We can therefore assume that \(\mathcal{L}_2\) cannot be contained by a plane. Recall that we have previously assumed that no pair of distinct endpoints of intervals lie on the same vertical line. Hence we can choose \(\ell_j, j = 1, 2, 3\), such that \(a_1, a_2, a_3\) are pairwise distinct.

A line \(\ell = \{[b, d, 0] + t[a, c, 1] : t \in \mathbb{R}\} \in \mathcal{L}_1\) intersects each \(\ell_j, j = 1, 2, 3\), and therefore satisfies

\[
(a - a_j)(d - d_j) = (b - b_j)(c - c_j), \quad j = 1, 2, 3.
\]

The above is a system of three degree 2 polynomial equations in \(a, b, c, d\). Subtracting the first equation from the second equation and second equation from the third equation
\[ A = 1, \quad B = 3 \]

\[ A = -1, \quad B = 3 \]

\[ A = 0, \quad B = 1 \]

\[ A = 1, \quad B = -1 \]

**Fig. 5** Intervals coming from lines contained in a plane. Each of the four arrangements of intervals is formed by applying a single homothetic transformation to a set of points; each interval connects a point to its image under the transformation. In the bottom-left arrangement, the centre of the homothety is itself an endpoint so the other endpoints can be anywhere; in the bottom-right arrangement, the centre of the homothety is at infinity, so all the intervals are parallel and the analysis shows that they have the same length.

The system in (6) gives two linear equations in \( a, b, c, d \);

\[
\begin{align*}
(d_1 - d_2)a + (c_2 - c_1)b + (b_2 - b_1)c + (a_1 - a_2)d &= a_1d_1 - b_1c_1 - a_2d_2 + b_2c_2, \\
(d_2 - d_3)a + (c_3 - c_2)b + (b_3 - b_2)c + (a_2 - a_3)d &= a_2d_2 - b_2c_2 - a_3d_3 + b_3c_3.
\end{align*}
\] (7)

The system (7) belongs to one of the following three subcases.

**Subcase** (i). The \( 2 \times 2 \) coefficient matrix induced by the coefficients of \( a \) and \( b \) in (7) is invertible, or the \( 2 \times 2 \) coefficient matrix induced by the coefficients of \( c \) and \( d \) in (7) is invertible. If the \( 2 \times 2 \) matrix induced by the coefficients of \( a \) and \( b \) is invertible, then Gaussian elimination on (7) gives \( a \) and \( b \) linearly in terms of \( c \) and \( d \). Substituting these linear relations into one equation of (6) gives a polynomial of degree at most 2 in \( c \) and \( d \), i.e., the set of all \( (c, d) \) lie on a conic or a line. Thus the set of all \( (a, b) \)
Each arrangement has two fixed ellipses and an interval has one endpoint on each ellipse. Note that in the first figure, the two ellipses (which are in fact, circles) coincide.

Each arrangement has two hyperbolas and an interval has one endpoint on each hyperbola also lies on a conic or a line. An analogous result follows if the $2 \times 2$ matrix induced by the coefficients of $c$ and $d$ in (7) is invertible. See Figs. 6 and 7 for examples.

Subcase (ii). Neither $2 \times 2$ coefficient matrix of (i) is invertible, but (7) has rank 2 as a system of linear equations in $a$, $b$, $c$, $d$. Then all solutions $(a, b)$ lie on one line, and solutions $(c, d)$ on another line, i.e., $b$ can be given linearly in $a$, and $d$ linearly in $c$. Substituting these linear relations into (6) shows the set of all $(a, c)$ lie on a conic and similarly the set of all $(b, d)$ lie on another conic. By the same argument, the pairs $(a, d)$ and $(b, c)$ also lie on two respective conics. See Fig. 9 for an example.

Subcase (iii). (7) has rank 1 as a system of linear equations in $a$, $b$, $c$, $d$. Therefore the $2 \times 2$ submatrix induced by the coefficients of variables $c$, $d$ has determinant 0. Consequently, the points $\{(a_j, b_j)\}_{j=1}^3$ are all collinear. Similarly, the points of $\{(a_j, c_j)\}_{j=1}^3$, and $\{(a_j, d_j)\}_{j=1}^3$ are also all collinear. Since $a_1$, $a_2$, $a_3$ are pairwise distinct, there exist

$$m_1, m_2, m_3, r_1, r_2, r_3 \in \mathbb{R}$$

such that

$$(a_j, b_j, c_j, d_j) = (a_j, m_1a_j + r_1, m_2a_j + r_2, m_3a_j + r_3), \quad j = 1, 2, 3. \quad (8)$$

Now we consider (6) as a single degree 2 equation in $a_j$, $b_j$, $c_j$, $d_j$ and substitute the relation (8) to obtain a quadratic equation in $a_j$ with a $a_j^2$ coefficient of $m_3$ –
Since this quadratic equation has at least three solutions (namely \(a_1, a_2, a_3\)), the leading coefficient is zero, i.e., \(m_3 - m_1 m_2 = 0\). Using the parameterisation \(\ell_j = \{ [b_j, d_j, 0] + t[a_j, c_j, 1] : t \in \mathbb{R} \}\) and setting \(t = -m_1\) we see that \(\ell_1, \ell_2, \ell_3\) all pass through the point \((r_1, -r_2 m_1 + r_3, -m_1)\). This cannot happen, since lines of the same ruling in a regulus do not intersect. We conclude that (iii) never occurs. \(\square\)

Returning for a moment to subcase (i) above, in the case that the solution sets of \((a, b)\) and of \((c, d)\) lie on a line, we see that three variables out of \(a, b, c, d\) can be given linearly in terms of a fourth. Hence the argument of (iii) also shows that in subcase (i), the solution set of \((a, b)\) and of \((c, d)\) lies on a conic, not on a line.

### 4 Examples of Sets Forming Many Trapezoids

In this section, we give some specific constructions of sets forming many trapezoids based on our proof of Theorem 1.1. The conic section case in particular leads to interesting examples that appear difficult to construct without using the correspondence established in Sect. 2.

#### 4.1 Intervals from Coplanar Lines

As observed above, it is easy to see that for any set of intervals all satisfying the homothety conclusion of Theorem 1.1, every pair forms a trapezoid. We will nevertheless highlight some special cases of this case. If \(AB > 0\) then the point \((-C/(A + B), -D/(A + B))\) is on \((a, b; c, d)\), and if \(AB = 0\) it is an endpoint of each interval. A special case of this is when \(A = B\) and pairs of intervals are the diagonals of a parallelogram. On the other hand, if \(A + B = 0\), then \(a + C/A = c\) and \(b + D/A = d\). This corresponds to a set of intervals that are translates of each other. Hence, in this case the intervals are contained on a pencil of lines. See Fig. 5 for examples.

#### 4.2 Intervals from Reguli

As was mentioned previously, the lines in a regulus have a complete bipartite structure with respect to intersections. The conic curve case therefore reveals an interesting class of examples of sets of intervals forming trapezoids in a complete bipartite way, i.e., there are two families of intervals such any pair of intervals, one from each family, form a trapezoid.

From Theorem 1.1, we know that one possibility is that the endpoints of \(I_1\) and \(I_2\) lie on a conic (subcase (i)), or a line (subcase (ii)). We will now showcase examples involving each type of conic, as well as the subcase (ii) situation. To facilitate simpler computation, we examine axis-parallel reguli. Up to a rigid motion (rotation and translation) of \(\mathbb{R}^3\), any regulus takes the form

\[
\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1, \quad \text{or} \quad (9)
\]
\[ z = \frac{x^2}{A^2} - \frac{y^2}{B^2}. \] (10)

(9) describes a hyperboloid of one sheet, and (10) describes a hyperbolic paraboloid. We determine what the corresponding interval set looks like in each case, and then discuss the arrangements that result from rigid transformations of these standard forms of reguli.

4.2.1 Axis-Parallel Hyperboloids

The hyperboloid (9) is ruled by the following two families of lines, parameterised by \( \theta \in [0, 2\pi] \):

\[
\begin{pmatrix}
A \sin \theta \\
-B \cos \theta \\
0
\end{pmatrix} + t \begin{pmatrix}
A C \cos \theta \\
B C \sin \theta \\
1
\end{pmatrix}, \quad \begin{pmatrix}
A \sin \theta \\
B \cos \theta \\
0
\end{pmatrix} + t \begin{pmatrix}
A C \cos \theta \\
-B C \sin \theta \\
1
\end{pmatrix}, \quad t \in \mathbb{R}.
\]

This corresponds to the following two families of intervals parameterised by \( \theta \in [0, 2\pi] \):

\[
\left( \frac{A}{C} \cos \theta, A \sin \theta; \frac{B}{C} \sin \theta, -B \cos \theta \right) \quad \text{and} \quad \left( \frac{A}{C} \cos \theta, A \sin \theta; -\frac{B}{C} \sin \theta, B \cos \theta \right).
\]

Observe that the above intervals have their first endpoint on the ellipse \( C^2 x^2 + y^2 = A^2 \) and their second on \( C^2 x^2 + y^2 = B^2 \). Furthermore, the second endpoint in the first family has a phase shift of \(-\pi/2\) radians compared to the first endpoint. In the second family, the second endpoint has a phase shift of \(+\pi/2\) compared to the first endpoint. If \( A = B \) and \( C = 1 \), then each interval in the first family is the reverse of an interval in the second family. Hence any pair of intervals from the same family or different families form a trapezoid. In all other cases, trapezoids are formed between all and only pairs of intervals from different families. Examples of both of these cases are shown in Fig. 6.

4.2.2 Axis-Parallel Paraboloids

The hyperbolic paraboloid (10) is ruled by the following two families of lines, parameterised by \( \lambda \in \mathbb{R} \setminus \{0\} \).

\[
\begin{pmatrix}
\frac{A}{2\lambda} \\
\frac{B}{2\lambda} \\
0
\end{pmatrix} + t \begin{pmatrix}
\frac{A\lambda}{2} \\
\frac{B\lambda}{2} \\
1
\end{pmatrix}, \quad \begin{pmatrix}
\frac{A}{2\lambda} \\
\frac{B}{2\lambda} \\
0
\end{pmatrix} + t \begin{pmatrix}
\frac{A\lambda}{2} \\
\frac{B\lambda}{2} \\
1
\end{pmatrix}, \quad t \in \mathbb{R}.
\]
This corresponds to the following two families of intervals parameterised by \( \lambda \in \mathbb{R} \setminus \{0\} \):

\[
\left( \frac{A \lambda}{2}, \frac{A}{2}; \frac{B \lambda}{2}, -\frac{B}{2} \right) \quad \text{and} \quad \left( \frac{A \lambda}{2}, \frac{A}{2}; \frac{-B \lambda}{2}, \frac{B}{2} \right).
\]

The left endpoint of all the above intervals lie on the hyperbola \( y = A^2/(4x) \) and the right endpoint lies on \( y = -B^2/(4x) \). The intervals in the first family intersect the \( x \)-axis, and the intervals in the second family intersect the \( y \)-axis. See Fig. 7 for an example of intervals coming from hyperbolic paraboloids.

4.2.3 Reguli Under Affine Transformation

It is easy to understand the effect of translations in \( \mathbb{R}^3 \) on sets of intervals in the plane, and by this we can completely describe the sets of intervals corresponding to axis-parallel reguli in \( \mathbb{R}^3 \). A translation by the vector \( (p, q, r) \in \mathbb{R}^3 \) maps the line \((b, d, 0) + t(a, c, 1)\) to \((b + p, c + q, r) + t(a, c, 1)\). Hence the corresponding transformation on intervals maps \((a, b; c, d)\) to \((a, b + p - ra; c, d + q - rc)\). The effect of the translation by \((p, q, r)\) in \( \mathbb{R}^3 \) on \( I \) can therefore be described by the composition of three basic maps. First, a vertical shift of all left endpoints of intervals in \( I \) by \( p \). Second, a vertical shift of all right endpoints of intervals in \( I \) by \( q \). Third, an affine shear transformation acting on all of \( \mathbb{R}^2 \) by a factor \( r \). See Fig. 3 for an example of a set of intervals resulting from a translation of a hyperboloid, given by translating the hyperboloid of the form (9) with \( A = 2, B = 1, C = 1/2 \) by \((2, 0, 1/2)\).

Evidently, \( \mathbb{R}^3 \) translations do not change the type of conic that the endpoints lie on, so it is now clear that axis-parallel reguli only produce intervals with endpoints lying on either ellipses or hyperbolas. In what follows, we will see that by rotating the axis-parallel reguli it is also possible to produce parabolas and pairs of lines (which is a degenerate conic), thereby showing that each type of conic can be realised in this way.

If an interval is considered as a point in \( \mathbb{R}^4 \), then \( \mathbb{R}^3 \) translations induce an affine transformation on \( \mathbb{R}^4 \). The effect of \( \mathbb{R}^3 \) rotations on \( \mathcal{I} \) is more complicated: the map induced by rotations of \( \mathbb{R}^3 \) is in general not affine. Rotations in \( \mathbb{R}^3 \) are described by well-known matrices. For example, calculation with such a matrix shows that rotation around the \( x \)-axis by an angle \( \alpha \) induces the map

\[
(a, b; c, d) \mapsto \left( a \left( \frac{bc - ad}{c \sin \alpha + \cos \alpha} \right) \frac{b \cos \alpha + \sin \alpha}{c \sin \alpha + \cos \alpha}, d \left( \frac{bc - ad}{c \sin \alpha + \cos \alpha} \right) \frac{b \sin \alpha}{c \sin \alpha + \cos \alpha} \right).
\]

on intervals in \( \mathbb{R}^2 \). Thus, the type of conic containing the endpoints of the intervals is in general not preserved under these transformations, and indeed there are examples where parabolas and pairs of lines arise as a result of rotating the axis-parallel reguli—see Fig. 8.
4.2.4 Endpoints on a Degenerate Conic

All the Case 3 examples that we have seen so far belong to subcase (i). We finish this section with an example from (ii). Recall that we want a family of intervals \([(a_i, b_i; c_i, d_i)]\) such that \((a_i, b_i)\) and \((c_i, d_i)\) lie on lines, and \((a_i, c_i)\) lie on a conic, for all \(i\). By rotating and scaling, we assume that \((a_i, b_i)\) lie on the line \(y = x\) and \((a_i, c_i)\) lie on the hyperbola \(y = 1/x\). These choices determine the following two families of intervals

\[
\left\{(t, t; \frac{1}{t}, \frac{u}{t} + v) : t \in \mathbb{R}\right\} \quad \text{and} \quad \left\{(t, ut; \frac{1}{t}, v + \frac{1}{t}) : t \in \mathbb{R}\right\},
\]

where \(u, v \in \mathbb{R}\), and \(u \neq 1\). These intervals correspond to lines belonging to the hyperboloid

\[
xy = z^2 + z(u + 1) + u + vx. \quad (11)
\]

When \(u = 1\), the two families are identical, and (11) is a cone. See Fig. 9 for a drawing of this case.

5 Orthodiagonal Quadrilaterals

An orthodiagonal quadrilateral is a convex quadrilateral with perpendicular diagonals. Several geometric and arithmetic characterisations of orthodiagonal quadrilaterals are known. For example, a convex quadrilateral is orthodiagonal if and only if the midpoints of the sides are the vertices of a rectangle. Another well known characterisation is that the sum of the lengths of each pair of opposite sides is equal—see for example [3] and the references contained therein.
The hyperboloid \((11)\) with \(u = -1, v = 1\) produces intervals with endpoints on two pairs of lines.

Fig. 9 The hyperboloid \((11)\) with \(u = -1, v = 1\) produces intervals with endpoints on two pairs of lines.

Our proof of Lemma 2.1 can easily be modified to deal with some variants of the problem we have considered. An example is sets of intervals for which there are many pairs forming orthodiagonal quadrilaterals, meaning that the convex hull of the two intervals has perpendicular diagonals. This property is illustrated in the leftmost diagram of Fig. 10. Arithmetically, two intervals \((a, b; c, d), (a', b'; c', d')\) forming an orthodiagonal quadrilateral satisfy

\[
(b - b')(d - d') = -(a - a')(c - c'), \quad \text{or} \quad (12)
\]

\[
(b - d')(d - b') = -(a - c')(c - a'), \quad \text{or} \quad (13)
\]

\[
(d - b)(d' - b') = -(c - a)(c' - a'). \quad (14)
\]

The arithmetic conditions \((12), (13), \) and \((14)\) are not exclusive to orthodiagonal quadrilaterals, i.e., other pairs of intervals can satisfy them and we illustrate such possibilities in Fig. 10.

The similarity of \((12), (13), \) and \((14)\) to \((1), (2), \) and \((3)\) allows a reuse of the previous techniques to create a result on orthodiagonal quadrilaterals, similar to Lemma 2.1. One notable difference is that two intervals coming from two different rulings of reguli may form any of the arrangements in Fig. 10, instead of exclusively forming orthodiagonal quadrilaterals.
Theorem 5.1 Let \( I \) be a set of \( N \) distinct intervals in \( \mathbb{R}^2 \). If \( N \gtrsim N^{3/2} \log N \) pairs of intervals form orthodiagonal quadrilaterals, then one of the following holds.

0. There exist subsets \( I_1, I_2 \subseteq I \) such that all intervals within \( I_i \) are parallel for \( i = 1, 2 \) and all intervals in \( I_1 \) are perpendicular to all intervals in \( I_2 \). Furthermore, \( |I_1| \cdot |I_2| \gtrsim N \log^2 N \).

1. There exist \( u, v, w \in \mathbb{R} \) such that \( \gtrsim N^{1/2} \) intervals \((a, b; c, d)\) satisfy
   \[
   \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} -v \\ u \end{pmatrix}.
   \]

2. There are two curves in \( \mathbb{R}^2 \) of degree at most 2 such that \( \gtrsim N^{1/2} \) intervals from \( I \) have an endpoint on each curve.

Proof First suppose that at least half of the orthodiagonal quadrilaterals are formed by pairs of intervals satisfying (14). Then by a pigeonholing argument we have conclusion 0 of the theorem.

Now assume that at least half of the orthodiagonal quadrilaterals are formed by pairs of intervals satisfying (12) or (13). Note that one obtains (13) from (12) by exchanging interval \((a, b; c, d)\) with its reverse \((c, d; a, b)\). Define
   \[
   \mathcal{L}^\perp(a, b; c, d) = \left\{ \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} + t \begin{pmatrix} c \\ d \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subseteq \mathbb{R}^3
   \]
   and let
   \[
   \mathcal{L}^\perp = \{ \mathcal{L}^\perp(a_i, b_i; c_i, d_i) : 1 \leq i \leq N \} \cup \{ \mathcal{L}^\perp(c_i, d_i; a_i, b_i) : 1 \leq i \leq N \}.
   \]

Under this alternative correspondence, a pair of lines \( \mathcal{L}^\perp(a, b; c, d) \) and \( \mathcal{L}^\perp(a', b'; c', d') \) intersect when (12) is satisfied. Similarly, \( \mathcal{L}^\perp(c, d; a, b) \) and \( \mathcal{L}^\perp(a', b'; c', d') \) intersect when (13) is satisfied. It follows that \( \mathcal{L}^\perp \) is a set of \( 2N \) lines having \( \gtrsim N^{3/2} \log N \) intersections. By Theorem 2.6 we conclude one of three cases.

Case 1. There are \( N^{1/2} \) concurrent lines in \( \mathcal{L}^\perp \). Let \((u, v, w)\) be the point of concurrency. Then from (15) we have conclusion 1.

Case 2. There are \( \gtrsim N^{1/2} \) coplanar lines in \( \mathcal{L}^\perp \). Let \( Ax + By + Cz + D = 0 \) be a plane containing \( \gtrsim N^{1/2} \) lines of \( \mathcal{L}^\perp \). The from (15) we conclude
   \[
   Ac + Bd + C = Ab + B(-a) + D = 0.
   \]

This is an instance of conclusion 2.

Case 3. There exists a regulus containing a subset \( \mathcal{R}^\perp \subseteq \mathcal{L}^\perp \) such that the number of pairs of intersecting lines in \( \mathcal{R}^\perp \) is \( \gtrsim N \). Now we can follow the same analysis in the proof of Theorem 1.1, Case 3, to arrive at conclusion 2 again. \( \square \)
Fig. 11 Two configurations of intervals with many pairs having endpoints on two perpendicular lines. In the first figure, there are two ellipses containing the endpoints; in the second figure, one of the conics is a parabola and the other is a pair of parallel lines.

In this case there are two perpendicular lines such that $\gtrapprox N^{1/2}$ intervals have an endpoint on each line. In Fig. 11 we showcase two instances of sets of intervals resulting from pulling back rulings of reguli by $L_\perp$.

One can also adapt the method to treat a generalisation of the trapezoids problem considered above. Two intervals form a trapezoid if two of the edges of their convex hull are parallel, but our proof did not rely in an important way on this parallel property. Indeed by mapping the interval $(a, b; c, d)$ to the line

$$L^\subset(a, b; c, d) = \left\{ \begin{pmatrix} b \\ d \\ 0 \end{pmatrix} + t \begin{pmatrix} a \\ \rho c \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathbb{R}^3,$$

one obtains a correspondence under which a pair of lines $L^\subset(a, b; c, d)$ and $L^\subset(a', b'; c', d')$ intersect precisely when

$$(a - a')(d - d') = \rho(b - b')(c - c'),$$

i.e., the slopes formed by the endpoints of the intervals have ratio $\rho$. Thus, the arguments from Sect. 2 can now be applied, leading to an analogue of Lemma 2.1 in the case of a fixed ratio $\rho$.

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