WELL-POSEDNESS AND ASYMPTOTIC BEHAVIOR OF THE DISSIPATIVE OSTROVSKY EQUATION

HONGWEI WANG
School of Mathematics and Statistics
Anyang Normal University
Anyang 455000, China

AMIN ESFAHANI*
School of Mathematics and Computer Science
Damghan University, Damghan 36715-364, Iran
and
Institute for Research in Fundamental Sciences (IPM)
Tehran 19395-5746, Iran

(Communicated by Remi Carles )

Abstract. In this paper we study the global well-posedness and the large-time behavior of solutions to the initial-value problem for the dissipative Ostrovsky equation. We show that the associated solutions decay faster than the solutions of the dissipative KdV equation.

1. Introduction. The Ostrovsky equation

\[(u_t + u_{xxx} + uu_x)_x + \gamma u = 0, \quad \gamma = \pm 1, \quad (1.1)\]

was originally derived first by Ostrovsky [25] (see also [8, 9]) to model internal waves in the ocean or surface waves in a shallow channel with an uneven bottom and also capillary waves on the surface of the liquid or for oblique magneto-acoustic waves in plasma. The sign of \(\gamma\) is related to the type of dispersion. See also [20] for the first rigorous derivation of the Ostrovsky equation. A model of the propagation of long internal waves in a deep rotating fluid can be found in [6]. The structure of (1.1) is very similar to that of the KdV equation

\[u_t + uu_x + \alpha D^3 u = 0; \quad (1.2)\]

but unlike the KdV equation, the Ostrovsky equation is evidently nonintegrable by the method of inverse scattering transform [9]. When the effects of friction is considered in modeling (1.1), it turns into (see [11, Chapter 1] and [21])

\[
\begin{cases}
(u_t + u_{xxx} + \beta D^3 u + uu_x)_x + \gamma u = 0, & (x, t) \in \mathbb{M} = \mathbb{R} \times (0, +\infty), \\
u(x, 0) = u_0(x),
\end{cases}
\]

2000 Mathematics Subject Classification. Primary: 35Q53, 35B40; Secondary: 35G25, 35K55. Key words and phrases. Dissipative Ostrovsky equation, Bourgain spaces, Cauchy problem, asymptotic behavior.

The second author is partially supported by a grant from IPM (No. 96470043).

* Corresponding author: Amin Esfahani.
where \( \alpha \in (0, 2] \) and \( \beta \in \mathbb{R} \). The term \( \beta D^\alpha u \) represents the dissipation, where \( D^\alpha \) is the Lévy operator defined through its Fourier transform by \( \hat{D}^\alpha u(\xi) = |\xi|^\alpha \hat{u}(\xi) \).

Ignoring the rotation term \( \gamma u \), Equation (1.3) leads to

\[
    u_t + u_{xxx} + \beta D^\alpha u + uu_x = 0.
\]

When \( \alpha = 1 \), (1.4) models the evolution of the free surface for shallow water waves damped by viscosity [26], while \( \alpha = 0 \) corresponds to linear Rayleigh damping. In the case \( \alpha = 2 \), (1.4) is the so-called KdV-Burgers equation and models the propagation of weakly nonlinear dispersive long waves in some contexts when dissipative effects occur [26].

The Cauchy problem for the Ostrovsky equation was first studied by Linares and Milanés in [19]. They proved by the use of certain regularizing effects of the linear part of the equation that (1.1) is locally well-posed in \( H^s(\mathbb{R}) \), with \( s > 3/4 \), such that \( \partial_x^{-1} u_0 = \partial_x^{-1} u(0) \in L^2(\mathbb{R}) \).

**Theorem 1.1** ([19]). Let \( u_0 \in X_s \) with \( s > 3/4 \). Then there exists a \( T > 0 \) such that the Cauchy problem (1.1) has a unique solution \( u \in C([0, T]; X_s) \).

The space \( X_s \) will be defined below. Isaza and Mejía proved in [14], by using Bourgain spaces and the technique of elementary calculus inequalities, introduced by Kenig, Ponce, and Vega in [16], that if \( s > -3/4 \), the Ostrovsky equation (1.1) is locally well-posed in \( H^s(\mathbb{R}) \), and it is not quantitatively well-posed (see [2, Definition 1]) in \( H^s(\mathbb{R}) \), if \( s < -3/4 \). The index \( s_c = -3/4 \) is critical in studying the well-posedness of the KdV equation. One can see [18] to study a well-posedness result of (1.1) at the critical regularity. By working on suitable Bourgain-type spaces, equation (1.4) was proved in [12, 29] to be globally well-posed in \( H^s(\mathbb{R}) \), \( s > s_{\alpha,c} \), where

\[
    s_{\alpha,c} = \begin{cases} 
        -3/4, & 0 < \alpha \leq 1, \\
        -3/(2 - \alpha), & 1 \leq \alpha \leq 2.
    \end{cases}
\]

We note that the index \( s_{\alpha,c} \) shows that the dissipation somehow is weaker than dispersion when \( \alpha \leq 1 \), while in the case \( \alpha \geq 1 \) the well-posedness index \( s_{\alpha,c} \) is improved, compared with the KdV case. This will be expected for equation (1.3). Here we borrow the ideas of [23] and introduce a Bourgain-type space. This space is in fact the intersection of the space introduced in [4] and of a Sobolev space. The advantage of this space is that it contains both the dissipative and dispersive parts of the linear symbol of (1.3). By using Strichartz’s type estimates injected into the framework of Bourgain spaces and some techniques introduced in [16, 27], we derive a bilinear estimate in these spaces which yields the local well-posedness of (1.3) in \( H^s(\mathbb{R}) \) for \( s > s_\alpha \), where

\[
    s_\alpha = \begin{cases} 
        -3/4, & 0 < \alpha \leq 1, \\
        -3/(6 + 5\alpha), & 1 \leq \alpha \leq 2.
    \end{cases}
\]

The main difficulty of proving the associated bilinear estimate arises from the rotation term which contains somehow a singularity.

Concerning the asymptotic behavior of the solution of (1.4) with \( \alpha = 2 \), Amick, Bona and Schonbek used the Hopf-Cole transformation and proved in [1] that the solution of (1.4) satisfies

\[
    \| u(t) \|_{L^2(\mathbb{R})} \lesssim (1 + t)^{-1/4},
\]
if \( u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R}) \). Later, Karch in [15] improved this result by applying the standard scaling argument to show that for \( u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \)

\[
\lim_{t \to +\infty} t^{(1-1/q)/2} \|u(t) - U_M(t)\|_{L^q(\mathbb{R})} = 0,
\]

where \( M = \int_{\mathbb{R}} u_0(x) \, dx \) and \( U_M(x,t) = t^{-1/2} U_M(x t^{-1/2}, 1) \) with

\[
U_M(y,1) = \frac{e^{-y^2/4}}{K + \frac{1}{2} \int_0^{\infty} e^{-\tau^2/4} \, d\tau}.
\]

The constant \( K \) is uniquely determined as a function of \( M \) using the condition \( \int_{\mathbb{R}} U_M(x,1) \, dx = M \). This means that the dispersion is negligible compared to dissipation and nonlinear effects. Vento extended the results of [1] to (1.4) and showed that

\[
\|u(t)\|_{L^2(\mathbb{R})} \lesssim (1 + t)^{-\frac{1}{2q}}.
\]

We will study the effects of the rotation in the asymptotic behavior of the solutions of (1.3). We prove that the solutions of (1.3) satisfies

\[
\|u(t)\|_{L^2(\mathbb{R})} \lesssim (1 + t)^{-\frac{2}{2q}}.
\]

(1.5) shows that the solutions of (1.3) decay faster than the solutions of (1.4). To do that, by using the properties of the generalized heat-type kernel, we give some asymptotic estimates of the free solution. Then we derive decay rates estimate of the solution of the nonlinear problem.

This paper is organized as follows: After stating our main results, we prove the bilinear estimate in Section 2. Section 3 is devoted to obtain some decay estimates of the linear equation of (1.3). Finally the decay estimates of the nonlinear problem is established in Section 4.

1.1. Notations. We introduce some notations which are standard. For \( s \geq 0 \) and \( 1 \leq p \leq \infty \), the Sobolev spaces \( \dot{W}^{s,p} = W^{s,p}(\mathbb{R}) \) and \( \dot{W}^{s,p} = \dot{W}^{s,p}(\mathbb{R}) \) are respectively endowed with the norms \( \|f\|_{\dot{W}^{s,p}} = \|D^s f\|_{L^p(\mathbb{R})} \) and \( \|f\|_{W^{s,p}} = \|f\|_{L^p(\mathbb{R})} + \|D^s f\|_{L^p(\mathbb{R})} \). When \( p = 2 \), we simplify by the notations \( \dot{H}^s = \dot{H}^s(\mathbb{R}) \) and \( H^s = H^s(\mathbb{R}) \). We also denote the Fourier transform of \( f \) by \( \hat{f} \) or \( \mathcal{F}(f) \). Define

\[
X_s = \{ f \in H^s(\mathbb{R}), \partial_x^{-1} f \in H^s(\mathbb{R}) \}
\]

with the norm

\[
\|f\|_{X_s} = \|f\|_{H^s} + \|\partial_x^{-1} f\|_{H^s}.
\]

Remark 1. The space \( X_1 \) is embedded into \( L^q(\mathbb{R}), q \geq 2 \). Indeed, we have

\[
\|u\|_{L^q(\mathbb{R})} \lesssim \|u\|^q_{L^2(\mathbb{R})} \|\partial_x^{-1} u\|^1_{L^2(\mathbb{R})}, \quad \text{if} \quad q \geq 2.
\]

(1.6) More precisely, (1.6) is obtained by combining the classical inequality

\[
\|u\|^q_{L^q(\mathbb{R})} \lesssim \|u_x\|^q_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^{(1-\theta)q}, \quad \theta = \frac{1}{2} - \frac{1}{q},
\]

and the elementary inequality

\[
\|u\|_{L^2(\mathbb{R})} \lesssim \|u_x\|^2_{L^2(\mathbb{R})} \|\partial_x^{-1} u\|^2_{L^2(\mathbb{R})}.
\]

See also [3].
Remark 2. The dissipative part in (1.3), when becoming small enough, has no effect on the low regularity of the equation. In the sequel, we shall prove well-posedness for (1.3) in the case of $\alpha \in [1, 2]$. To completeness, in the appendix, we use the ideas of [22] and sketch the proof of the global well-posedness of (1.3) when $\alpha \in [0, 1]$.

Related to the symbol of the linear equation, we define the Bourgain-type space $X^{s,b} = \left\{ f \in \mathcal{S}'(\mathbb{R}^2); \| f \|_{X^{s,b}}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} |i(\tau - m(\xi)) + |\xi|^\alpha |\tau|^b \hat{f}(\xi, \tau)|^2 d\xi d\tau < \infty \right\}$, where $\langle \cdot \rangle = 1 + |\cdot|$, $s, b \in \mathbb{R}$ and $m(\xi) = \xi^3 + \frac{2}{\xi}$. It is readily seen that

$$\| f \|_{X^{s,b}} \simeq \| U(-t) f \|_{H^{s,b}} + \| f \|_{L^2_t H^{s,b}_x},$$

where $U$ is the unitary group extracted from (1.1) and $H^{s,b}$ with $s, b \in \mathbb{R}$ is the space-time version of $H^s(\mathbb{R})$ defined by the norm

$$\| f \|_{H^{s,b}} = \left( \int_{\mathbb{R}^2} \tau^{2b} \langle \xi \rangle^{2s} |\tau(\xi)|^2 d\tau d\xi \right)^{1/2}.$$  

For $T > 0$, $b > 1/2$ we denote by $X^{s,b}_T$ the space of the restrictions to the interval $[0, T]$ of the elements $g$ in $X^{s,b}$ with norm defined by

$$\| f \|_{X^{s,b}_T} = \inf \left\{ \| g \|_{X^{s,b}}, \ g \in X^{s,b} \text{ and } g(t) = f(t) \text{ on } [0, T] \right\}.$$  

By Duhamel’s principle, the solution of (1.3) can be locally written in the integral form as

$$u(t) = \psi(t) \left( S_\alpha(t) u_0 - \frac{\chi_{\mathbb{R}^+}}{2} \int_0^t S_\alpha(t - \tau) \partial_x \left( \psi_T(t') u^2(t') \right) \, dt' \right),$$  

(1.7)

where $\psi \in C_0^\infty(\mathbb{R})$ is a time cut-off function satisfying $\text{supp}(\psi) \subset [-2, 2]$ and $\psi \equiv 1$ on $[-1, 1]$, $\psi_T(t) = \psi(t/T)$ and the $S_\alpha$ is the semigroup associated with the free evolution of (1.3),

$$S_\alpha(t) f = \int_{\mathbb{R}} e^{ix\xi t + (\text{im}(\xi) - |\xi|^\alpha)} \hat{f}(\xi) d\xi, \quad t \geq 0, \quad f \in \mathcal{S}'(\mathbb{R}),$$  

(1.8)

which can be extended to a linear operator on $\mathbb{R}$ by setting

$$S_\alpha(t) f = \int_{\mathbb{R}} e^{ix\xi t + (\text{im}(\xi) - |\xi|^\alpha)} \hat{f}(\xi) d\xi.$$  

(1.9)

1.2. Main results. We first state our crucial bilinear estimate.

**Theorem 1.2.** Given $s > s_\alpha$, there exists $\nu, \delta > 0$ such that for any $u, v \in X^{s,1/2}$ with compact support in $[-T, T]$,

$$\| \partial_x (uv) \|_{X^{s-1/2+\delta}} \lesssim T^\nu \| u \|_{X^{s,1/2}} \| v \|_{X^{s,1/2}}.$$  

(1.10)

The following local existence is a consequence of Theorem 1.2 together with linear estimates of Section 2 (see the proof of Theorem 1.3 in [29]).

**Theorem 1.3.** Let $\alpha \in (0, 2]$ and $u_0 \in H^s(\mathbb{R})$ with $s > s_\alpha$. Then there are $T = T(\| u_0 \|_{H^s(\mathbb{R})}) > 0$ and a unique solution $u$ of (1.3) in $\mathcal{Z}_T = C([0, T]; H^s(\mathbb{R})) \cap X^{s,1/2}_T$. Moreover, the flow map $u_0 \mapsto u$ is analytic from $H^s(\mathbb{R})$ to $\mathcal{Z}_T$.  


Remark 3. It is worth noting that it was proved in [30] that the pure dissipative equation

\[ u_t + D^\alpha u + uu_x = 0 \]

is well-posed in \( H^s(\mathbb{R}) \) for \( s > s_{d,\alpha} \) with \( \alpha \in [1, 2] \), where \( s_{d,\alpha} = \frac{3}{2} - \alpha \), and that the solution map fails to be smooth when \( s < s_{d,\alpha} \). Hence, Theorem 1.3 can be reduced to \( s_\alpha < s < s_{d,\alpha} \). We also should note by the triangle inequality that \( s \) can be small enough and we can obtain the same result for large \( s \). See [23] for the details.

The following conserved quantities of (1.3) will be very important in proving the global well-posedness of (1.3) and some estimates of the solution. The proof is similar to [23].

Theorem 1.4. Let \( u \) be as in Theorem 1.3, then one has for all \( t \in [0, T) \) that

\[ \|u(t)\|_{L^2(\mathbb{R})}^2 + 2\beta \int_0^t \|D^{\alpha/2} u(\tau)\|_{L^2(\mathbb{R})}^2 d\tau = \|u_0\|_{L^2(\mathbb{R})}^2 \tag{1.11} \]

and

\[ H(u(t)) + \beta \int_0^t \left( \|D^{\alpha/2+1} u(\tau)\|_{L^2(\mathbb{R})}^2 + \gamma \|D^{\alpha/2-1} u(\tau)\|_{L^2(\mathbb{R})}^2 \right) d\tau \]

\[ = \frac{\beta}{2} \int_0^t \int_\mathbb{R} u^2 D^\alpha u \, dx \, d\tau + H(u_0), \tag{1.12} \]

where

\[ H(u) = \frac{1}{2} \int_\mathbb{R} (u_x^2 + \gamma (\partial_x^{-1} u)^2) dx - \frac{1}{6} \int_\mathbb{R} u^3 dx. \]

The global well-posedness will follow from Theorems 1.3 and 1.4 (see the proofs of Theorem 1.1 in [23] and Theorem 1.3 in [29]).

Theorem 1.5. Let \( u_0 \in H^s(\mathbb{R}) \) with \( s > s_\alpha \). The existence time of the solution \( u \) of (1.7) can be extended to infinity and \( u \in C([0, +\infty); H^\infty(\mathbb{R})) \).

Our main result on the decay estimates of the solutions of (1.3) reads as follows. Although the global well-posedness is guaranteed by Theorem 1.5, the results of Theorems 1.1 and 1.4 are sufficient for our purposes.

Theorem 1.6. Let \( \alpha \geq 1, \ell \geq 0, 2 \leq p \leq \infty \) and \( u_0 \in X_s \) with \( s > 3/2 \) such that \( \partial_x^{-1} u_0 \in L^1(\mathbb{R}) \). Then the associated solution \( u \) of (1.3) satisfies

\[ \|u(t)\|_{W^{\ell-1, p}} \lesssim t^{-\frac{1}{2}(\ell+1-\frac{1}{p})}, \quad t \geq 0. \]

2. Linear and bilinear estimate. Before proving Theorem 1.2, we state several linear estimates (for the free and forcing terms) and some well-known Strichartz-type estimates. The linear estimates are principally contained in [5, 23]. Because these proofs are dependent with dissipative term \( ||\xi||^\alpha \), and independent with the dispersive term. So the factor \( 1/\xi \) have no effect in linear estimates. But in the nonlinear estimates, the factor \( 1/\xi \) is important. See also [24], where a factor similar to \( 1/\xi \) appears in the associated semigroup.

Lemma 2.1. For all \( s \in \mathbb{R} \) and \( f \in H^s(\mathbb{R}) \) we have

\[ \|\psi(t) S_\alpha(t) f\|_{X_{s-1/2}} \lesssim \|f\|_{H^s(\mathbb{R})}. \tag{2.1} \]

Proof. Similar to [5, Lemma 1] or [23, Proposition 2.1], the proof is followed by using the effects of the dissipation term in the semigroup \( S_\alpha \).
Lemma 2.2. Let $s \in \mathbb{R}$ and $\delta > 0$.
(a) For all $u \in \mathcal{S}(\mathbb{R}^2)$,
\[
\left\| \chi_{\mathbb{R}^+}(t) \psi(t) \int_0^t S_\alpha(t)(t-t')u(t')dt' \right\|_{X^{s,1/2}} \lesssim \|u\|_{X^{s,-1/2}} + \left( \int_{\mathbb{R}} |\xi|^{2\alpha} \left( \int_{\mathbb{R}} |\hat{u}(\tau + m(\xi),\xi)|^2 d\tau \right) d\xi \right)^{1/2}.
\]
(b) For all $\delta \in (0,1/2)$ and $u \in X^{s,-1/2+\delta}$,
\[
\left\| \chi_{\mathbb{R}^+}(t) \psi(t) \int_0^t S_\alpha(t)(t-t')u(t')dt' \right\|_{X^{s,1/2}} \lesssim \|u\|_{X^{s,-1/2+\delta}}.
\]
(c) For all $u \in X^{s,-1/2+\delta}$,
\[
t \mapsto \int_0^t S_\alpha(t-t')u(t')dt' \in C(\mathbb{R}^+, H^{s+\alpha\delta}(\mathbb{R})).
\]
Moreover, if $\{f_n\}$ is a sequence with $f_n \to 0$ in $X^{s,-1/2+\delta}$, then
\[
\left\| \int_0^t S_\alpha(t-t')f_n(t')dt' \right\|_{L^\infty(\mathbb{R}^+, H^{s+\alpha\delta}(\mathbb{R}))} \to 0.
\]
Proof. The parts (a) and (b) is obtained by a little modifications of Lemma 2 and Lemma 3 in [5] and Propositions 2.2 and 2.3 in [23]. The result (c) will imply a gain of regularity for the nonhomogeneous part of (1.7), and can be proved by an argument similar to the proof of Proposition 2.4 in [23] or Proposition 2.6 in [24].

Lemma 2.3. [10, Lemma 3.1] Let $f \in L^2(\mathbb{R}^2)$ with compact support (in time) in $[-T,T]$. For any $\theta > 0$, there exists $\nu = \nu(\theta) > 0$ such that
\[
\left\| F^{-1} \left( \frac{\hat{f}(\tau,\xi)}{(\tau - m(\xi))^\theta} \right) \right\|_{L^2_{\tau \xi}} \lesssim T^\nu \|f\|_{L^2_{\tau \xi}}.
\]

Lemma 2.4. [13, Lemma 2.5] Let $f \in L^2(\mathbb{R}^2)$ with compact support (in time) in $[-T,T]$. For $0 \leq \theta \leq 1/8$ and $\rho > 3/8$, there exists $\nu > 0$ such that
\[
\left\| F^{-1} \left( \chi_{|\xi| \leq 1} \langle \xi \rangle^{\theta} \frac{\hat{f}(\tau,\xi)}{(\tau - m(\xi))^\rho} \right) \right\|_{L^2_{\tau \xi}} \lesssim T^\nu \|f\|_{L^2_{\tau \xi}}.
\]

2.1. Bilinear estimate for the case $\gamma = 1$. Let $\xi_2 = \xi - \xi_1$, $\tau_2 = \tau - \tau_1$, $\sigma = \tau - m(\xi)$, $\sigma_1 = \tau_1 - m(\xi_1)$, $\sigma_2 = \tau_2 - m(\xi_2)$. By duality, (1.10) is equivalent to
\[
\left| \int_{\mathbb{R}^2} \partial_x(uv)w \right| \lesssim T^\nu \|w\|_{X^{s,1/2-\delta}} \|u\|_{X^{s,1/2}} \|v\|_{X^{s,1/2}}
\]
for all $w \in X^{-s,1/2-\delta}$. Setting
\[
\hat{f}(\tau,\xi) = (i(\tau - m(\xi)) + |\xi|^{\alpha})^{1/2} \hat{u}(\tau,\xi),
\]
\[
\hat{g}(\tau,\xi) = (i(\tau - m(\xi)) + |\xi|^{\alpha})^{1/2} \hat{v}(\tau,\xi),
\]
\[
\hat{h}(\tau,\xi) = (i(\tau - m(\xi)) + |\xi|^{\alpha})^{1/2-\delta} \hat{w}(\tau,\xi),
\]

\[
\left\| \chi_{\mathbb{R}^+}(t) \psi(t) \int_0^t S_\alpha(t)(t-t')u(t')dt' \right\|_{X^{s,1/2}} \lesssim \|u\|_{X^{s,-1/2}} + \left( \int_{\mathbb{R}} |\xi|^{2\alpha} \left( \int_{\mathbb{R}} |\hat{u}(\tau + m(\xi),\xi)|^2 d\tau \right) d\xi \right)^{1/2}.
\]
it is equivalent to show that

\[ I = \int_{\mathbb{R}^4} J(\tau, \tau_1, \xi, \xi_1) \hat{h}(\tau, \xi) \hat{f}(\tau_1, \xi_1) \hat{g}(t - \tau_1, \xi - \xi_1) d\tau d\tau_1 d\xi d\xi_1 \]

\[ \lesssim T^\alpha \|f\|_{L^2_x} \|g\|_{L^2_x} \|h\|_{L^2_x} \]

with

\[ J(\tau, \tau_1, \xi, \xi_1) = \frac{|\xi| \langle \xi \rangle^s}{\langle i\sigma + |\xi|^\alpha \rangle^{1/2 - \delta}} \frac{\langle \xi_1 \rangle^{-s}}{\langle i\sigma_1 + |\xi_1|^\alpha \rangle^{1/2}} \frac{\langle \xi_2 \rangle^{-s}}{\langle i\sigma_2 + |\xi_2|^\alpha \rangle^{1/2}}. \]

By Fubini's theorem, we can always assume \( \hat{f}, \hat{g}, \hat{h} \geq 0 \). By symmetry, one can reduce the integration domain of \( I \) to \( \Omega = \{ (\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^4, |\sigma_1| \geq |\sigma_2| \} \). Split \( \Omega \) into four regions

\[ \Omega_1 = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \leq 1, |\xi_2| \leq 1 \}, \]

\[ \Omega_2 = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \leq 1, |\xi_2| \geq 1 \}, \]

\[ \Omega_3 = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \geq 1, |\xi_2| \leq 1 \}, \]

\[ \Omega_4 = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \geq 1, |\xi_2| \geq 1 \}. \]

Let \( A_j = \{ (\tau, \xi) : \exists (\tau_1, \xi_1) \in \mathbb{R}^2, (\tau_1, \xi_1) \in \Omega_j \} \) and \( \tilde{\Omega}_j = \{ (\tau_1, \xi_1) : (\tau_1, \xi_1, \xi) \in \Omega \} \) for \( j = 1, 2, 3 \). Similar notations for the subdomains of \( \Omega_j \) are defined. In each subdomain \( \Omega_k \) of \( \Omega_4 \), we set \( A_k = \{ (\tau_1, \xi_1) : \exists (\tau, \xi) \in \mathbb{R}^2, (\tau, \xi, \xi_1) \in \Omega_k \} \) and \( \tilde{\Omega}_k = \{ (\tau, \xi) : (\tau, \tau_1, \xi, \xi_1) \in \Omega_k \} \).

**Estimate in \( \Omega_1 \).**

Using the Cauchy-Schwarz inequality and Lemma 2.3, we easily obtain

\[ I_1 \lesssim \sup_{A_1} \left[ \frac{|\xi| \langle \xi \rangle^s}{\langle i\sigma + |\xi|^\alpha \rangle^{1/2 - \delta}} \left( \int_{\tilde{\Omega}_1} \frac{\langle \xi_1 \rangle^{-s}}{\langle i\sigma_1 + |\xi_1|^\alpha \rangle^{1/2}} \frac{\langle \xi_2 \rangle^{-s}}{\langle i\sigma_2 + |\xi_2|^\alpha \rangle^{1/2}} d\tau d\xi_1 \right)^{1/2} \right] \times T^\alpha \|f\|_{L^2_x} \|g\|_{L^2_x} \|h\|_{L^2_x}. \tag{2.2} \]

In \( \Omega_1 \), one has \( |\xi| \leq 2 \) and thus if \( K_1 \) denotes the term between brackets in (2.2), then

\[ K_1 \lesssim \left( \int_{\tilde{\Omega}_1} \frac{d\tau_1 d\xi_1}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle} \right)^{1/2} \lesssim \left( \int_{|\xi_1| \leq 1} \left( \int_{\mathbb{R}} \frac{d\tau_2}{\langle \sigma_2 \rangle^2} \right) d\xi_1 \right)^{1/2} \lesssim 1, \]

without any restriction on \( s \).

**Estimate in \( \Omega_2 \).**

When \( |\xi| \leq 100 \), then \( |\xi_2| \leq |\xi| + |\xi_1| \leq 101 \) and so arguing as in the first case we obtain the required estimate. When \( |\xi| \geq 100 \), then \( |\xi_2| \sim |\xi| \), we see for any \( s \) that \( \langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \leq C \langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-2s} \leq C \). Denote

\[ \lambda = \sigma - \sigma_1 - \sigma_2 = m(\xi_1) + m(\xi_2) - m(\xi) = -3\xi_1 \xi_2 + \frac{1}{\xi_1} + \frac{1}{\xi_2} - \frac{1}{\xi}. \tag{2.3} \]

To estimate the integral in \( \Omega_2 \), we treat \( \frac{1}{\xi_1} \ll |\xi_1| \ll 1 \) and \( |\xi_1| \gtrsim \frac{1}{\xi_1} \) separately. On the case \( \frac{1}{|\xi_1|} \ll |\xi_1| \ll 1 \), (2.3) implies \( \max \{|\sigma|, |\sigma_1|, |\sigma_2| \} \gtrsim |\xi_1| \xi_2| \). Thus, when \( |\sigma| \gtrsim |\sigma_1| \), we have that

\[ K_2 \lesssim |\xi_1| \left( \int_{\tilde{\Omega}_2} \frac{d\tau_1 d\xi_1}{\xi_1 \xi_2 \langle \sigma_1 \rangle \langle \sigma_2 \rangle \langle \xi \rangle^{\alpha(1-\epsilon)}} \right)^{1/2} \]
For the case when $|\sigma_1| \geq |\sigma|$, we can get

\[
K_2 \lesssim \frac{|\xi|}{|\xi|^{\alpha(1/2-\delta)}} \left( \int_{\Omega_2} \frac{d\tau_1 d\xi_1}{\frac{|\xi_1|^{1-\delta}}{\sigma_1}(\sigma_2)^{1-\varepsilon}} \right)^{1/2} \\
\lesssim \frac{|\xi|}{|\xi|^{\alpha(1/2-\delta)}} \left( \int_{|\xi_1| \leq 1} \frac{d\tau_1 d\xi_1}{|\xi_1|^{1-\delta} (\sigma_2)^{1-\varepsilon}} \right)^{1/2} \\
\lesssim \left( \int_{|\xi_1| \leq 1} \frac{d\xi_1}{|\xi_1|^{1-\delta}} \right)^{1/2} \\
\lesssim 1.
\]

For the case $|\xi_1| \lesssim \frac{1}{|\xi_1|}$, we obtain

\[
K_2 \lesssim \frac{|\xi|}{|\xi|^{\alpha(1/2-\delta)}} \left( \int_{\Omega_2} \frac{d\tau_1 d\xi_1}{(\sigma_1)(\sigma_2)^{1-\varepsilon}} \right)^{1/2} \\
\lesssim \frac{|\xi|}{|\xi|^{\alpha(1/2-\delta)}} \left( \int_{|\xi_1| \leq 1} \frac{d\tau_1 d\xi_1}{(\sigma_1)(\sigma_2)^{1-\varepsilon}} \right)^{1/2} \\
\lesssim \frac{|\xi|}{|\xi|^{\alpha(1/2-\delta)}} \left( \int_{|\xi_1| \leq 1} \frac{d\xi_1}{(\sigma_1)(\sigma_2)^{1-\varepsilon}} \right)^{1/2} \\
\lesssim \frac{|\xi|}{|\xi|^{\alpha(1/2-\delta)}} \left( \int_{|\xi_1| \leq 1} \frac{d\xi_1}{(\sigma_1)(\sigma_2)^{1-\varepsilon}} \right)^{1/2} \\
\lesssim \frac{|\xi|}{|\xi|^{\alpha(1/2-\delta)}} \left( \int_{|\xi_1| \leq 1} \frac{d\xi_1}{(\sigma_1)(\sigma_2)^{1-\varepsilon}} \right)^{1/2} \\
\lesssim 1.
\]

**Estimate in $\Omega_3$.**
In this region, by symmetry, the required estimate is obtained in the same way as $\Omega_2$.

**Estimate in $\Omega_4$.**
First we split $\Omega_4$ in $\Omega_4 = \Omega_{41} \cup \Omega_{42}$ where

\[
\Omega_{41} = \left\{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_4 : |\sigma| \geq |\sigma_1| \right\}, \\
\Omega_{42} = \left\{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_4 : |\sigma_1| \geq |\sigma| \right\}.
\]

Nextly, we write $\Omega_{41} = \Omega_{411} \cup \Omega_{412}$ with

\[
\Omega_{411} = \left\{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{41} : |\xi_1| \leq 100|\xi| \right\}, \\
\Omega_{412} = \left\{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{41} : |\xi_1| \geq 100|\xi| \right\}.
\]

**Estimate in $\Omega_{411}$.** In $\Omega_{411}$, $\langle \xi_1 \rangle \lesssim \langle \xi \rangle$, $\langle \xi_2 \rangle \lesssim \langle \xi \rangle$ and $|\lambda| \gtrsim |\xi_1\xi_2|$. Then we get

\[
J_{411} \lesssim \frac{|\xi| |\langle \xi \rangle^s|}{|\xi_1\xi_2|^{2/1-\delta}} \frac{|\xi_1|^{-s}}{|\sigma_1|^{3/8+\varepsilon}} \frac{|\xi_2|^{-s}}{|\sigma_2|^{3/8+\varepsilon}} \\
\lesssim \frac{|\xi_1|^{-s}}{|\sigma_1|^{3/8+\varepsilon}} \frac{|\xi_2|^{-s}}{|\sigma_2|^{3/8+\varepsilon}}.
\]

Consequently, using Plancherel’s theorem, the Hölder inequality and Lemma 2.4, we have

\[ I_{411} \lesssim \int_{\mathbb{R}^2} F^{-1}(\hat{h}) F^{-1} \left( \frac{\langle \xi \rangle^{s/2-1/4-\alpha/8+\alpha\varepsilon+3\delta/2} \hat{f}}{\langle \sigma \rangle^{3/8+\varepsilon}} \right) d\sigma d\xi \]

\[ \times F^{-1} \left( \frac{\langle \xi \rangle^{s/2-1/4-\alpha/8+\alpha\varepsilon+3\delta/2} \hat{g}}{\langle \sigma \rangle^{3/8+\varepsilon}} \right) d\sigma d\xi \]

\[ \lesssim ||h||_{L^2_{\xi}} \left\| F^{-1} \left( \frac{\langle \xi \rangle^{s/2-1/4-\alpha/8+\alpha\varepsilon+3\delta/2} \hat{f}}{\langle \sigma \rangle^{3/8+\varepsilon}} \right) \right\|_{L^4_{\sigma \xi}} \]

\[ \times \left\| F^{-1} \left( \frac{\langle \xi \rangle^{s/2-1/4-\alpha/8+\alpha\varepsilon+3\delta/2} \hat{g}}{\langle \sigma \rangle^{3/8+\varepsilon}} \right) \right\|_{L^4_{\sigma \xi}} \]

\[ \lesssim T^\prime \|f\|_{L^2_{\xi}} \|g\|_{L^2_{\xi}} ||h||_{L^2_{\xi}}, \]

provided that \( s > -3/4 - \alpha/4 \) and \( \delta, \varepsilon > 0 \) small enough. Note in this case that \( s_\alpha \geq -3/4 - \alpha/4 \).

**Estimate in** \( \Omega_{412} \). We split \( \Omega_{412} \) into two sub-domains

\[ \Omega_{4121} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{412} : |\lambda| \leq \frac{1}{2} |\sigma| \}, \]

\[ \Omega_{4122} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{412} : |\lambda| \geq \frac{1}{2} |\sigma| \}. \]

**Estimate in** \( \Omega_{4121} \). First consider the case \( |\xi| \geq 1 \). In this region one has that

\[ |\sigma - \lambda| \geq \frac{1}{2} |\sigma| \]

and since \( \sigma_1 + \sigma_2 = \sigma - \lambda \) it follows that

\[ |\sigma| \sim |\sigma_1| \gtrsim |\xi_1 \xi_2|. \]

Moreover, to get the estimate of \( I_{4121} \), it suffices to bound the following integral

\[ M_{4121} = \frac{\langle \xi_1 \rangle^{-s}}{\langle i \sigma_1 + |\xi_1|^\alpha \rangle^{1/2}} \left( \int_{\Omega_{4121}} \frac{|\xi|^2 \langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-2s}}{(i \sigma + |\xi|^\alpha)^{1-2b} (i \sigma_2 + |\xi_2|^\alpha)} d\sigma d\xi \right)^{1/2}. \]

Therefore, one obtain

\[ M_{4121} \lesssim \frac{\langle \xi_1 \rangle^{-s}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\Omega_{4121}} \frac{|\xi|^2 \langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-2s}}{(\sigma_1)^{1-2b} (\sigma_2)} d\sigma d\xi \right)^{1/2} \]

\[ \lesssim \frac{1}{\langle \sigma_1 \rangle^{1-2b/2}} \left( \int_{\Omega_{4121}} \frac{\langle \sigma_1 \rangle^{-2s} \langle \xi \rangle^{2s+4s}}{(\sigma_2)^{1+s}} d\sigma d\xi \right)^{1/2} \]

\[ \lesssim \langle \sigma_1 \rangle^{-s-1+3\delta/2} \left( \int_{|\xi| \leq |\sigma_1|^{1/3}} \frac{d\xi}{\langle \xi \rangle^{-4s-2}} \right)^{1/2} \]

\[ \lesssim 1. \]
Nextly, we consider the case $|\xi| \leq 1$. When $\frac{1}{|\xi|} \ll |\xi| \leq 1$, then $|\sigma| \sim |\sigma_1| \gtrsim |\xi_1\xi_2|$, we can get

$$J_{4121} \lesssim \frac{\langle \xi \rangle^{-s}}{\sigma^1/2-\delta} \frac{\langle \xi_1 \rangle^{-s}}{\sigma_1^{3/8+\varepsilon}} \frac{\langle \xi_2 \rangle^{-s}}{\sigma_2^{3/8+\varepsilon}} \lesssim \langle \xi_1 \rangle^{-s-5/8-\alpha/16+(1+\alpha/2)\varepsilon+\delta} \langle \xi_2 \rangle^{-s-5/8-\alpha/16+(1+\alpha/2)\varepsilon+\delta} \langle \sigma_1 \rangle^{3/8+\varepsilon} \langle \sigma_2 \rangle^{3/8+\varepsilon}.$$ 

One deduces that for $s > -3/4 - \alpha/16$ and $\delta, \varepsilon > 0$ small enough,

$$I_{4121} \lesssim T^v \|f\|_{L^2_{\tau}} \|g\|_{L^2_{\tau}} \|h\|_{L^2_{\tau}}.$$ 

We should note in this case that $s_\alpha \geq -3/4 - \alpha/16$. When $|\xi| \lesssim \frac{1}{|\xi|}$, we easily check that

$$M_{4121} \lesssim \langle \xi_1 \rangle^{-s-3\alpha/2} \left( \int_{\xi_1 \leq |\xi|} \frac{|\xi|^{2} \langle \xi_2 \rangle^{2s-2s} \langle \sigma_1 \rangle^{1-s/2} \langle \sigma_2 \rangle^{1-3s} \, d\tau d\xi \right)^{1/2} \lesssim \langle \xi_1 \rangle^{-2s-3\alpha/2+3\alpha/2} \left( \int_{|\xi| \leq |\xi_1|} \frac{|\xi|^{2} \langle \sigma_1 \rangle^{1-s} \langle \sigma_2 \rangle^{3/8+\varepsilon} \, d\xi \right)^{1/2} \lesssim 1,$nach 4122. First consider the case $|\xi| \geq 1$. In this region one has that $|\sigma| \sim |\lambda| \gtrsim |\xi_1\xi_2|$. When $|\xi| \sim |\xi_1|$, then the estimate of $\Omega_{4122}$ is the same as $\Omega_{411}$. When $|\xi_1|^{\alpha-1} \lesssim |\xi|$, then

$$J_{4122} \lesssim \frac{|\xi_1|^{s}}{\langle \xi_1 \xi_2 \rangle^{1/2-\delta}} \frac{\langle \xi_1 \rangle^{-s}}{\sigma_1^{3/8+\varepsilon}} \frac{\langle \xi_2 \rangle^{-s}}{\sigma_2^{3/8+\varepsilon}} \lesssim \langle \xi_1 \rangle^{-s-3/2-3/4+\alpha+\varepsilon+\delta} \langle \xi_2 \rangle^{-s-3/2-3/4+\alpha+\varepsilon+\delta} \langle \sigma_1 \rangle^{3/8+\varepsilon} \langle \sigma_2 \rangle^{3/8+\varepsilon},$$

hence we can get the desired estimate for $s > \frac{\alpha-7}{4(3-\alpha)}$, noting $s_\alpha \geq \frac{\alpha-7}{4(3-\alpha)}$. When $|\xi| \lesssim |\xi_1|^{\alpha-1}$, we obtain that

$$M_{4122} \lesssim \langle \xi_1 \rangle^{-s} \left( \int_{\xi_1 \leq |\xi|} \frac{|\xi|^{2} \langle \xi_2 \rangle^{2s} \langle \sigma_1 \rangle^{-2s} \langle \sigma_2 \rangle^{1-3s} \, d\tau d\xi \right)^{1/2} \lesssim \langle \xi_1 \rangle^{-2s-\alpha/2+1+3\delta/2} \left( \int_{\xi_1 \leq |\xi|} \frac{|\xi|^{1+2s+3\delta} \langle \sigma_2 \rangle^{1+3s} \, d\tau d\xi \right)^{1/2} \lesssim 1,$$

which is valid for $s > \frac{\alpha-4}{2(3-\alpha)}$. Note that $s_\alpha \geq \frac{\alpha-4}{2(3-\alpha)}$. 


Nextly, we consider the case $|\xi| \leq 1$. When $\frac{1}{|\xi|} \ll |\xi| \leq 1$, we have $|\sigma| \sim |\lambda| \gtrsim |\xi_1 \xi_2|$, hence
\[
J_{4122} \lesssim \frac{|\xi|}{(\sigma)\frac{1}{2}-\delta (\sigma_1)^{3/8+\varepsilon} (\xi_1)^{\alpha(1/8-\varepsilon)}} \frac{(\xi_2)^{-s}}{(\sigma_2)^{3/8+\varepsilon}},
\]
the desired estimate is valid for $s > s_\alpha = -5/8 - \alpha/8$. When $|\xi| \lesssim \frac{1}{|\xi_1|}$, we have $|\sigma| \sim |\lambda| \gtrsim |\xi_1|$, hence
\[
J_{4122} \lesssim \frac{1}{(\sigma_1)^{3/8+\varepsilon} (\xi_1)^{\alpha(1/8-\varepsilon)}} \frac{(\xi_2)^{-s}}{(\sigma_2)^{3/8+\varepsilon}},
\]
the desired estimate is valid for $s > -7/8 - \alpha/8$.

Estimate in $\Omega_{12}$. We write $\Omega_{42} = \Omega_{421} \cup \Omega_{422}$ with
\[
\Omega_{421} = \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{42} : |\xi_1| \leq 100 |\xi_1|, \Omega_{422} = \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{42} : |\xi_1| \geq 100 |\xi_1|,\n\]

Estimate in $\Omega_{421}$. In $\Omega_{421}$, $(\xi_1) \lesssim (\xi_2) \lesssim (\xi)$ and $|\lambda| \gtrsim |\xi_1 \xi_2|$. Then we get
\[
J_{421} \lesssim \frac{|\xi| (\xi_1)^{s}}{(\sigma_1)^{3/8+\varepsilon} (\xi_1)^{\alpha(1/8-\varepsilon)}} \frac{(\xi_2)^{-s}}{(\sigma_2)^{3/8+\varepsilon}},
\]
Using Plancherel’s theorem, Hölder inequality and Lemma 2.4, we obtain the estimate for $s > -3/4 - \alpha/4$. As before, $s_\alpha \geq -3/4 - \alpha/4$.

Estimate in $\Omega_{422}$. We split $\Omega_{422}$ into two sub-domains
\[
\Omega_{4221} = \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{422} : |\lambda| \leq \frac{1}{2} |\sigma_1|, \Omega_{4222} = \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{422} : |\lambda| \geq \frac{1}{2} |\sigma_1|,\n\]

Estimate in $\Omega_{4221}$. First consider the case $|\xi| \geq 1$. In this region one has that $|\sigma_1| \sim \max(|\sigma_1|, |\sigma_2|) \gtrsim |\xi_1 \xi_2|$. When $|\sigma| \gtrsim |\sigma_2|$, one obtains the estimate of $M_{4221}$ in the same way as $M_{4121}$. When $|\sigma| \lesssim |\sigma_2|$, one obtains
\[
M_{4221} \lesssim \frac{(\xi_1)^{-s}}{(\sigma_1)^{1/2}} \left( \int_{\Omega_{4121}} \frac{|\xi|^2 (\xi_2)^{2s}}{(\sigma_1)^{1-2s}} d\tau d\xi \right)^{1/2},
\]
\[
\lesssim \frac{1}{(\sigma_1)^{1-3\delta/2}} \left( \int_{\Omega_{4121}} \frac{(\xi_1)^{-2s}}{(\sigma_1)^{1-\delta}} d\tau d\xi \right)^{1/2},
\]
Nextly, we consider the case $|\xi| \lesssim 1$. If $|\sigma_2| \lesssim |\sigma|$, then the required estimate is obtained in the same way as previously. Now we assume $|\sigma_2| \gtrsim |\sigma|$. When $\frac{1}{|\xi|} \ll |\xi| \lesssim 1$, we have

$$M_{4221} \lesssim \frac{\langle \xi_1 \rangle^{-s}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{4221}} \frac{|\xi|^2 (\xi_2)^{-2s}}{\langle \sigma \rangle^{1-2\delta} \langle \sigma_2 \rangle} \, d\tau d\xi \right)^{1/2}$$

and

$$\lesssim \frac{\langle \xi_1 \rangle^{-s}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{4221}} \frac{|\xi|^2 (\xi_2)^{-2s}}{\langle \sigma \rangle^{1+\delta} \langle \sigma_2 \rangle^{1-3\delta}} \, d\tau d\xi \right)^{1/2}$$

Nextly, we consider the case $|\xi| \lesssim 1$. If $|\sigma_2| \lesssim |\sigma|$, then the required estimate is obtained in the same way as previously. Now we assume $|\sigma_2| \gtrsim |\sigma|$. When $\frac{1}{|\xi|} \ll |\xi| \lesssim 1$, we have

$$M_{4222} \lesssim \frac{\langle \xi_1 \rangle^{-s-1/2+\delta}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{4222}} \frac{|\xi|^2 (\xi_2)^{-2s+1+2\delta}}{\langle \sigma \rangle^{1-\delta} \langle \sigma_2 \rangle^{1+\delta}} \, d\tau d\xi \right)^{1/2}$$

$$\lesssim \frac{\langle \xi_1 \rangle^{-s-1/2+\delta}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{4222}} \frac{d\xi}{\langle \sigma_1 \rangle^{1+\delta} \langle \sigma_2 \rangle^{1-3\delta}} \right)^{1/2}$$

The change of variables $\mu = \sigma_1 + \lambda$ gives the inequalities

$$M_{4222} \lesssim \frac{\langle \xi_1 \rangle^{-s-3/2+\delta}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{|\mu| \lesssim 2|\sigma_1|} \frac{d\mu}{\langle \mu \rangle^{1-2\delta}} \right)^{1/2}$$

which is bounded on $\mathbb{R}^2$.

Nextly, we consider the case $|\xi| \lesssim 1$. When $\frac{1}{|\xi|} \ll |\xi| \lesssim 1$, we have

$$M_{4222} \lesssim \frac{\langle \xi_1 \rangle^{-s}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{4222}} \frac{|\xi|^2 (\xi_2)^{-2s}}{\langle \sigma \rangle^{1-2\delta} \langle \xi_2 \rangle^{1+\alpha}} \, d\tau d\xi \right)^{1/2}$$

$$\lesssim \frac{\langle \xi_1 \rangle^{-s}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{4222}} \frac{|\xi|^2 (\xi_2)^{-2s}}{\langle \sigma \rangle^{1+\delta} \langle \xi_2 \rangle^{1-3\delta}} \, d\tau d\xi \right)^{1/2}$$

$$\lesssim \frac{\langle \xi_1 \rangle^{-s}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{|\xi| \leq 1} \frac{|\xi|^2}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \, d\xi \right)^{1/2}$$

$$\lesssim 1,$$

for $s > s_\alpha \geq -1/2 - \alpha/4$. When $|\xi| \lesssim \frac{1}{|\xi|}$, we easily check that

$$M_{4222} \lesssim \langle \xi_1 \rangle^{-s} \left( \int_{\tilde{\Omega}_{4222}} \frac{|\xi|^2 (\xi_2)^{-2s}}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \, d\tau d\xi \right)^{1/2}$$
for \( s > -1 - \alpha/4 \).

### 2.2. Bilinear estimate for the case \( \gamma = -1 \)

We divide the integral domain into four subregions as the case \( \gamma = 1 \). The estimate of \( \Omega_1 \) is the same as the case \( \gamma = 1 \). The sets \( A_j \) and \( \bar{\Omega}_j \) are defined the same as the case \( \gamma = 1 \). Let \( u = \frac{\xi}{\bar{\xi}} \), and denote

\[
\lambda' = \sigma - \sigma_1 - \sigma_2 = 3\xi_1\xi_2 + \frac{1}{\xi} - \frac{1}{\xi_1} - \frac{1}{\xi_2} = 3\xi^3 u(1-u) - \frac{1}{\xi} \left[ 1 - \frac{1}{u(1-u)} \right] = \phi(\xi_1) \]

then

\[
\frac{d\lambda}{d\xi_1} = 3\xi^2 (1-2u) \left[ 1 - \frac{1}{3\xi^4 (u-u^2)^2} \right].
\]

#### Estimation in \( \Omega_2 \).

When \( |\xi| \leq 100 \), then \( |\xi_2| \leq |\xi| + |\xi_1| \leq 101 \) and so arguing as in \( \Omega_1 \) we obtain the required estimate.

When \( |\xi| \geq 100 \), then \( |\xi_2| \approx |\xi| \), we see that \( \langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \leq C \langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-2s} \leq C_2 \). Also, we have \(|u| = \left| \frac{\xi_1}{\xi} \right| < \frac{1}{50} \), and therefore \(|\frac{1}{2} - u| > \frac{1}{10} \). Let \( A = \frac{1}{3\xi^4 (u-u^2)^2} \),

if \( I = \{ \xi_1 : 1/4 \leq A \leq 5/4 \} \), then \( |I| \leq \frac{C}{|\xi|^2} \), and for \( \xi_1 \notin I \), we have \(|\frac{d\lambda}{d\xi_1}| \geq C \xi^2 \). On the other hand, the minimum of \( |\lambda| \) is attained when \( A = 1 \), that is, the minimum is of the order of \( \xi \). Therefore, there is \( C > 0 \) such that \(|\lambda'| \geq C|\xi| \). Making the change of variable \( \lambda' = \phi(\xi_1) \), we conclude that

\[
K_2 \lesssim \left( |\xi| \langle \sigma + |\xi|^2 \rangle^{1/2} \right)^{1/2} \left( \int_{\Omega_2} \frac{d\xi_1}{\langle \sigma_1 + |\xi|^2 \rangle^{1/2}} \right)^{1/2}
\]

If \( |\sigma| \geq |\sigma_1| \), then

\[
K_2 \lesssim \left( \langle \sigma \rangle^{1-2\delta-\varepsilon} \frac{|\xi|^2}{\langle \sigma_1 \rangle^{1-2\delta-2\varepsilon}} \int_{\xi \notin I} \frac{d\xi_1}{\langle \sigma_1 \rangle^{1+\varepsilon} \langle \xi_1 \rangle^{1-\varepsilon}} \right)^{1/2}
\]

\[
\lesssim \left( \frac{|\xi|^2}{\langle \sigma \rangle^{1-2\delta-2\varepsilon}} \int_{\xi \notin I} \frac{d\xi_1}{\langle \sigma_1 + |\xi|^2 \rangle^{1+\varepsilon}} \right)^{1/2}
\]

\[
\lesssim \left( \frac{1}{\langle \sigma \rangle^{1-2\delta-2\varepsilon}} \int_{-\infty}^{\infty} \frac{d\lambda'}{\langle \lambda' - \sigma \rangle^{1+\varepsilon}} \right)^{1/2}
\]

\[
\lesssim 1.
\]
If $|\sigma_1| \geq |\sigma|$, then
\[
K_2 \lesssim \left( \frac{|\xi|^2}{(\sigma_1)^{1-\delta}} \right) \left( \int_{\xi_1 \in I} \int_{-\infty}^{+\infty} \frac{d\tau_1}{(\sigma_1)^{1-\varepsilon}(\sigma_2)^{1+\varepsilon}} d\xi_1 \right)^{1/2}
+ \int_{\xi_1 \notin I} \int_{-\infty}^{+\infty} \frac{|\xi|^2 d\tau_1}{(\sigma_1)^{1-2\delta}(\sigma_2)^{1+\varepsilon}} d\xi_1 \right)^{1/2}
\lesssim \left( \frac{|\xi|^2}{|\sigma_1|^\varepsilon(1-\delta)} \right) + \left( \frac{|\xi|^2}{|\sigma_1|^{1-2\delta(1-\varepsilon)}} \right) \left( \int_{\xi_1 \notin I} \frac{d\xi_1}{(\sigma_1)^{1+\varepsilon}(\sigma_2)^{1+\varepsilon}} \right)^{1/2}
\lesssim \left( 1 + \frac{1}{|\xi|^{1-2\delta(1-\varepsilon)}} \int_{-\infty}^{+\infty} \frac{d\lambda'}{(\lambda' + \sigma_1)^{1+\varepsilon}} \right)^{1/2}
\lesssim 1.
\]

**Estimate in $\Omega_3$.**

In this region, by symmetry, the required estimate is obtained in the same way as $\Omega_2$.

**Estimate in $\Omega_4$.**

Firstly, we split $\Omega_4$ into $\Omega_4 = \Omega_{41} \cup \Omega_{42}$, where
\[
\Omega_{41} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_4 : |\sigma| \geq |\sigma_1| \},
\Omega_{42} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_4 : |\sigma_1| \geq |\sigma| \}.
\]
Nextly, we write $\Omega_{41} = \Omega_{411} \cup \Omega_{412}$ with
\[
\Omega_{411} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{41} : |\xi_1| \leq 100|\xi| \},
\Omega_{412} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{41} : |\xi_1| \geq 100|\xi| \}.
\]
The estimate in $\Omega_{411}$ is the same as the case $\gamma = 1$.

**Estimate in $\Omega_{412}$.** We split $\Omega_{412}$ into two sub-domains
\[
\Omega_{4121} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{412} : |\lambda'| \leq \frac{1}{2} |\sigma| \},
\Omega_{4122} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{412} : |\lambda'| \geq \frac{1}{2} |\sigma| \}.
\]

**Estimate in $\Omega_{4121}$.** Firstly, consider the case $|\xi| \geq 1$. In this region one has that $|\sigma| \sim |\sigma_1| \gtrsim |\xi_1|\xi_2|$. Therefore, one obtains
\[
K_{4121} \lesssim \left( \int_{\Omega_{4121}} \frac{|\xi|^2}{(\sigma_1)^{1-\delta}} d\tau d\xi \right)^{1/2}
\lesssim \left( \int_{\Omega_{4121}} \frac{1}{(\sigma_1)^{1-\delta}} d\tau d\xi \right)^{1/2}
\lesssim (\sigma_1)^{-s-1+\delta} \left( \int_{|\xi| \leq |\sigma_1|^{1/3}} \frac{d\xi}{(\xi)^{-4s-2}} \right)^{1/2}
\lesssim 1.
\]
Nextly, we consider the case $|\xi| \leq 1$. When $\frac{1}{\lambda'} \ll |\xi| \leq 1$, we can get
\[
M_{4121} \lesssim \left( \frac{|\xi_1|^{-4s}}{(\sigma_1 + |\xi_1|^d)} \right) \left( \int_{\Omega_{4121}} \frac{1}{(\sigma_1 + |\xi_1|^d)} \right)^{1/2} + \int_{|\xi_1|^{1/2} < |\xi| \leq 1} \frac{d\xi}{(\xi_1)^{4s-2}} \lesssim 1.
\]
for $s > -3/2 + \alpha/4$. Note that $s_{\alpha} \geq -3/2 + \alpha/4$. When $|\xi| \lesssim \frac{1}{|\xi|}$, we easily check that

\[
M_{4121}^2 \lesssim \frac{<\xi_1>^{-2s}}{(i\sigma_1 + |\xi_1|^\alpha)} \int_{|\xi| \leq |\xi|} \frac{|\xi|^2 <\xi_2>^{2s}}{(i\sigma_2 + |\xi|^\alpha)^{-1-2\delta}} \frac{<\xi_2>^{-2s}}{(i\sigma_2 + |\xi|^\alpha)^{1-2\delta}} d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s - \alpha(2-2\delta - \epsilon)} \int_{|\xi| \leq |\xi|} \int \frac{|\xi|^2}{(\sigma)^{1+\epsilon}} d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s - 3\alpha(2-2\delta - \epsilon)}
\]

for $s > -3/4 - \alpha/2$.

Estimate in $\Omega_{4122}$. When $|\xi| \geq 1$, we have

\[
J_{4122} \lesssim \frac{|\xi|^s}{(\sigma)^{1/2-\delta}} \frac{<\xi_1>^{-s}}{(\sigma_1)^{3/8+\epsilon} <\xi_1>^{\alpha(1/8 - \epsilon)}} \frac{<\xi_2>^{-s}}{(\sigma_2)^{3/8+\epsilon} <\xi_2>^{\alpha(1/8 - \epsilon)}}
\]

\[
\lesssim \frac{|\xi|^s}{(\sigma_1)^{3/8+\epsilon}} \frac{<\xi_1>^{-s}}{\langle \xi_1 \rangle^{3/8+\epsilon} <\xi_1>^{\alpha(1/8 - \epsilon)} <\xi_2>^{\alpha(1/8 + \epsilon)}}
\]

\[
\lesssim \frac{<\xi_1\xi_2>^{1/2-\delta}}{(\sigma_1)^{3/8+\epsilon}} \frac{<\xi_2>^{-s}}{(\sigma_2)^{3/8+\epsilon}}
\]

\[
\lesssim \frac{<\xi_1>^{s-1/2-\alpha/8+\alpha\epsilon + \delta}}{(\sigma_1)^{3/8+\epsilon}} \frac{<\xi_2>^{s-1/2-\alpha/8+\alpha\epsilon + \delta}}{(\sigma_2)^{3/8+\epsilon}}
\]

hence the desired estimate is valid for $s > -5/8 - \alpha/8$. When $\frac{1}{|\xi|} \ll |\xi| \leq 1$, we can obtain

\[
M_{4121}^2 \lesssim \frac{|\xi_1|^{-4s}}{(i\sigma_1 + |\xi_1|^\alpha)} \int_{|\xi| \leq |\xi|} \int \frac{|\xi|^2}{(\sigma)^{1-2\delta}} \frac{1}{(i\sigma_2 + |\xi|^\alpha)^{1+\epsilon}} d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s - \alpha(2+2\delta + \epsilon)} \int_{|\xi| \leq |\xi|} \int \frac{|\xi|^{1+2\delta + \epsilon}}{(\sigma_2)^{1+\epsilon}} d\tau d\xi
\]

\[
\lesssim 1,
\]

provided that $s > -1/2 - \alpha/4$. When $|\xi| \lesssim \frac{1}{|\xi|}$, we can get

\[
M_{4122}^2 \lesssim \frac{<\xi_1>^{-4s}}{(|\xi_1|^\alpha)} \int_{|\xi| \leq |\xi|} \frac{|\xi|^2}{(\sigma)^{1-2\delta}} \frac{1}{(\sigma_2)^{1+\epsilon}} d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s - \alpha(1-2\delta - \epsilon)} \int_{|\xi| \leq |\xi|} \int \frac{|\xi|^2}{(\sigma_2)^{1+\epsilon}} d\tau d\xi
\]
\[ |\xi_1|^{-4s - \alpha - (4\delta - \epsilon)} \lesssim 1, \]

for \( s > -1 - \alpha/4 \).

Now we write \( \Omega_{42} = \Omega_{421} \cup \Omega_{422} \) with

\[
\Omega_{421} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{42} : |\xi_1| \leq 100|\xi| \},
\]

\[
\Omega_{422} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{42} : |\xi_1| \geq 100|\xi| \}.
\]

The estimate in \( \Omega_{421} \) is the same as case \( \gamma = 1 \).

**Estimate in** \( \Omega_{422} \). We split \( \Omega_{422} \) into two sub-domains

\[
\Omega_{4221} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{422} : |\lambda| \leq \frac{1}{2}|\sigma_1| \},
\]

\[
\Omega_{4222} = \{ (\tau, \tau_1, \xi, \xi_1) \in \Omega_{422} : |\lambda| \geq \frac{1}{2}|\sigma_1| \}.
\]

**Estimate in** \( \Omega_{4221} \). Firstly consider the case \( |\xi| \geq 1 \). In this region one has that

\[
|\sigma_1 + \lambda| \geq \frac{1}{2}|\sigma_1|\]

and

\[
|\xi| \lesssim |\sigma_1|^{1/3}.
\]

Therefore, one obtains

\[
M_{4221} \lesssim \frac{1}{|\sigma_1|^{1/2}} \left( \int_{\Omega_{4221}} \frac{(\xi_1 \xi_2)^{-2s/\lambda}}{(\sigma_1^{1-\delta} (\sigma_2)^{1-\delta})} d\tau d\xi \right)^{1/2}
\]

\[
\lesssim |\sigma_1|^{-s-1/2} \left( \int_{\Omega_{4221}} \frac{(\xi_1^{2+4s}/(\sigma_1 + \lambda)^{1-\delta}} d\xi \right)^{1/2}
\]

\[
\lesssim |\sigma_1|^{-s-1+\delta/2} \left( \int_{|\xi| \lesssim |\sigma_1|^{1/3}} \frac{d\xi}{(\xi^{4s-2})} \right)^{1/2}
\]

\[
\lesssim 1.
\]

Nextly, we consider the case \( |\xi| \leq 1 \). If \( |\sigma_2| \lesssim |\sigma| \), then \( |\sigma_1| \sim |\sigma| \), the required estimate is obtained in the same way as previously. Now we assume \( |\sigma_2| \geq |\sigma| \). The cases \( |\xi| \lesssim \frac{1}{|\sigma_1|} \) are treated as \( K_{4121} \), we only consider the case \( \frac{1}{|\sigma_1|} \ll |\xi| \leq 1 \). In this way, we can get

\[
M_{4222}^2 \lesssim \frac{|\xi_1|^{-4s}}{(i\sigma_1 + |\xi_1|^{\alpha})} \left( \int_{|\xi| \ll |\xi_1|^{\alpha-2}} + \int_{|\xi| \ll |\xi_1|^{\alpha-2}} \right)
\]

\[
\int \frac{|\xi|^2}{(|\xi|^2 + |\xi_1|^{\alpha})} d\tau d\xi
\]

\[
\lesssim \frac{|\xi_1|^{-4s}}{|\xi_1|^{\alpha} |\xi_2|^{\alpha(1-2\delta - \epsilon)}} \int_{|\xi| \ll |\xi_1|^{\alpha-2}} \int \frac{|\xi|^2}{(|\xi_1|^{\alpha})^{1+\epsilon}} d\tau d\xi
\]

\[
+ \frac{|\xi_1|^{-4s}}{|\sigma_1| |\xi_2|^{1-2\delta - \epsilon}} \int_{|\xi| \ll |\xi_1|} \int \frac{|\xi_1|^{1+2\delta + \epsilon}}{|\sigma_1|^{1+\epsilon}} d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s + \alpha - 6 + \alpha(2\delta + \epsilon)} + |\xi_1|^{-4s - 4 + 2\epsilon + 4\delta} \int_{|\xi| \ll 1} \int \frac{|\xi_2|^{2\delta + \epsilon}}{|\sigma_2|^{1+\epsilon}} d\tau d\xi
\]

\[
\lesssim 1.
\]
provided that \( s > -3/2 + \alpha/4 \), noting \( s_\alpha \geq -3/2 + \alpha/4 \).

**Estimate in \( \Omega_{422} \).** Firstly consider the case \( |\xi| \geq 1 \). In this region one has that \(|\sigma_1| \sim |\lambda| \gtrsim |\xi_1\xi_2| \), then

\[
M_{422} \lesssim \frac{\langle \xi_1 \rangle^{-s-1/2+\delta}}{\langle \sigma_1 \rangle^\delta} \left( \int_{\Omega_{422}} \frac{\langle \xi \rangle^{2s+1+2\delta} \langle \xi_2 \rangle^{-2s-1+2\delta}}{\langle \sigma \rangle^{1-\delta} \langle \sigma_2 \rangle^{1-\delta}} \, d\tau d\xi \right)^{1/2}
\]

\[
\lesssim \frac{\langle \xi_1 \rangle^{-2s+1+2\delta}}{\langle \sigma_1 \rangle^\delta} \left( \int_{\Omega_{422}} \frac{d\xi_1}{\langle \sigma_1 + \lambda \rangle^{1-2\delta}} \right)^{1/2} .
\]

The change of variables \( \mu = \sigma_1 + \lambda \) gives the inequalities

\[
M_{422} \lesssim \frac{\langle \xi_1 \rangle^{-s-3/2+\delta}}{\langle \sigma_1 \rangle^\delta} \left( \int_{|\mu| \leq 2|\sigma_1|} \frac{d\mu}{\langle \mu \rangle^{1-2\delta}} \right)^{1/2}
\]

which is bounded on \( \mathbb{R}^2 \).

Nextly, we consider the case \( |\xi| \leq 1 \). When \( \frac{1}{|\xi_1|} \lesssim |\xi| \leq 1 \), we have

\[
M_{422}^2 \lesssim \frac{|\xi_1|^{-4s}}{(i\sigma_1 + |\xi_1|^\alpha)^{1-2\delta-\epsilon}} \int_{\frac{1}{|\xi_1|} \lesssim |\xi| \leq 1} \frac{|\xi|^2}{\langle \sigma \rangle^{1+\epsilon} (i\sigma_2 + |\xi_2|^\alpha)} \, d\tau d\xi
\]

\[
\lesssim \frac{|\xi_1|^{-4s}}{|\xi_1\xi_2|^{1-2\delta-\epsilon}} \int_{\frac{1}{|\xi_1|} \lesssim |\xi| \leq 1} \frac{|\xi|^2}{\langle \sigma \rangle^{1+\epsilon} (|\xi_2|^\alpha)} \, d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s-a-2+2\delta+\epsilon} \int_{|\xi| \leq 1} \frac{\langle \xi \rangle^{1+2\delta+\epsilon}}{\langle \sigma \rangle^{1+\epsilon}} \, d\tau d\xi
\]

\[
\lesssim 1,
\]

for \( s > -1/2 - \alpha/4 \). When \( |\xi| \lesssim \frac{1}{|\xi_1|} \), we can get

\[
M_{422}^2 \lesssim \frac{\langle \xi_1 \rangle^{-4s}}{(\sigma_1)^{1-2\delta-\epsilon}} \int_{\Omega_{422}} \frac{|\xi|^2}{\langle \sigma \rangle^{1+\epsilon} (i\sigma_2 + |\xi|^\alpha)} \, d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s-a-(1-2\delta-\epsilon)} \int_{|\xi| \leq \frac{1}{|\xi_1|}} \frac{|\xi|^2}{\langle \sigma \rangle^{1+\epsilon}} \, d\tau d\xi
\]

\[
\lesssim |\xi_1|^{-4s-a-(4-2\delta-\epsilon)}
\]

for \( s > -1 - \alpha/4 \). We note in this case for \( \alpha \in [1, 2] \) that \( s_\alpha \geq -1/2 - \alpha/4 \geq -1 - \alpha/4 \).

This completes the proof of Theorem 1.2. \( \square \)

**Remark 4.** The quantitatively well-posedness was defined in [2]: for \( u_0 \in H^s \), and \( T \in (0, 1] \), we say that \( u \in X^{s,b} \) is a solution of (1.3) in \([0, T]\) with initial datum \( u_0 \), if there is an extension \( v \in X^{s,b} \) of \( u \) such that

\[
u(t) = L(u_0) + B(v, v)(t), \quad t \in [0, T],
\]

where \( L(u_0) := \psi(t) S_\alpha(t) u_0 \) and \( B(u, v) = \int_0^t S_\alpha(t-\tau) \partial_x (uv) (\tau) \, d\tau \). Let \( (s, \| \cdot \|_s) \) be a Banach space of space-time functions such that

(i) \( S \cap X \) is dense in \( S \), where \( X = \cap_{s \in \mathbb{R}} \{ u \in \mathscr{S}(\mathbb{R}^2); \| \langle \xi \rangle^s \hat{u} \|_{L_x^2 L_t^\infty} < +\infty \}; \)

(ii) \( S \hookrightarrow C_{bd}(\mathbb{R}^+; H^s) \);

(iii) for \( u_0 \in H^s \), \( L(u_0) \in S \) and \( \| L(u_0) \|_S \leq C \| u_0 \|_{H^s} \).
(iv) for \( u, v \in S \cap X \), \( B(u, v) \in S \) and \( \|B(u, v)\|_S \leq C\|u\|_S\|v\|_S \), and thus \( B \) has a unique continuous extension to \( S \times S \).

If such \( S \) exists, we say that the problem \( u = L(u_0) + B(u, u) \), for given \( u_0 \in H^s \) is quantitatively well posed in \( H^s \). Using the ideas of [2], it was proved in [14] that the Ostrovsky equation (1.1) is not quantitatively well-posed in \( H^s \) if \( s < -3/4 \). But (1.3) contains the dissipation term which does not allow us to use the ideas of [2, 14]. On the other hand, it was proved in [29] that the associated bilinear estimate for the dissipative KdV equation is optimal in \( H^s \) for \( s > -3/4 \). Unfortunately, because of the singular term \( 1/\xi \) in the symbol of (1.3), we are not able to show that our well-posedness result is sharp. However, based on the proof of Theorem 1.2, our conjecture is that the critical index is \( \alpha = \frac{4}{2(3-\alpha)} \).

3. Decay estimates of linear equation. In this section we will obtain some decay estimates of the solution of the linear equation

\[
(ut + u_{xxx} + \beta D^\alpha u)_x + \gamma u = 0, \quad \gamma = \pm 1. \tag{3.1}
\]

**Lemma 3.1.** Let \( u_0 \in X_s \) with \( s \geq 1 \) such that \( \partial_x^{-1}u_0 \in L^1(\mathbb{R}) \). Then the solution \( u \) of (3.1) satisfies

\[
\lim_{t \to +\infty} \frac{1}{\epsilon^2 t} \|D^{\ell-1}u(t)\|^2_{L^2(\mathbb{R})} = C_{\ell, \beta} \|\partial_x^{-1}u_0\|^2_{L^1(\mathbb{R})}, \tag{3.2}
\]

where \( \ell \in [0, s] \).

**Proof.** We consider the case \( \ell = 0 \). The case \( \ell > 0 \) is similar.

First we note that the map \( \xi \mapsto \partial_x^{-1}u_0(\xi) \) is continuous over \( \mathbb{R} \), so for any \( \epsilon > 0 \) there is \( \delta \in (0, 1) \) such that

\[
|\partial_x^{-1}u_0|^{2}(\xi) - |\partial_x^{-1}u_0|^{2}(0)| < \epsilon, \quad \text{for all } \xi \in (-\delta, \delta).
\]

Then, we have from the assumptions of lemma, the properties of \( S_\alpha \) and the Plancherel theorem that

\[
\|\partial_x^{-1}u(t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} e^{-|\xi|\alpha t} |\partial_x^{-1}u_0|^{2}(\xi) d\xi
\]

\[
= \int_{|\xi|<\delta} e^{-|\xi|\alpha t} \left(|\partial_x^{-1}u_0|^{2}(\xi) - |\partial_x^{-1}u_0|^{2}(0)\right) d\xi
\]

\[
+ \int_{|\xi|<\delta} e^{-|\xi|\alpha t} |\partial_x^{-1}u_0|^{2}(0) d\xi + \int_{|\xi|\geq\delta} e^{-|\xi|\alpha t} |\partial_x^{-1}u_0|^{2}(\xi) d\xi
\]

\[
\leq \epsilon \int_{|\xi|<\delta} e^{-|\xi|\alpha t} d\xi + \int_{|\xi|<\delta} e^{-|\xi|\alpha t} |\partial_x^{-1}u_0|^{2}(0) d\xi + \epsilon e^{-\delta\alpha t} \|\partial_x^{-1}u_0\|^2_{L^2(\mathbb{R})}
\]

\[
\leq \epsilon t^{-1/\alpha} + \int_{|\xi|<\delta} e^{-|\xi|\alpha t} |\partial_x^{-1}u_0|^{2}(0) d\xi + e^{-\delta t} \|\partial_x^{-1}u_0\|^2_{L^2(\mathbb{R})}.
\]

Hence,

\[
t^{1/\alpha} \|\partial_x^{-1}u(t)\|^2_{L^2(\mathbb{R})} = O(\epsilon) + t^{1/\alpha} \int_{|\xi|<\delta} e^{-|\xi|\alpha t} |\partial_x^{-1}u_0|^{2}(0) d\xi
\]

\[
= O(\epsilon) + \|\partial_x^{-1}u_0\|^2_{L^1(\mathbb{R})}.
\]

\[\square\]
Corollary 1. Let $2 \leq p \leq \infty$. Under the assumption of Lemma 3.1, the solution $u$ of (3.1) satisfies

$$\|u(t)\|_{W^{s-1,p}} = C t^{-\frac{s}{2} (\ell+1+\frac{1}{2})} \|\partial_x^{-1} u_0\|_{L^1(\mathbb{R})}.$$ 

Proof. We have from the Gagliardo-Nirenberg inequality and Lemma 3.1 that

$$\|\partial_t^{\alpha} u(t)\|_{L^\infty(\mathbb{R})} \lesssim \|\partial_t^{\alpha} u(t)\|^{1/2}_{L^2(\mathbb{R})} \|\partial_t u(t)\|^{1/2}_{L^2(\mathbb{R})}$$

$$\lesssim t^{-\frac{\alpha}{2} (1+2\ell)} \|\partial_x^{-1} u_0\|^{1/2}_{L^1(\mathbb{R})} t^{-\frac{\alpha}{2} (3+2\ell)} \|\partial_x^{-1} u_0\|^{1/2}_{L^1(\mathbb{R})}$$

$$\lesssim t^{-\frac{\alpha}{2} (1+\ell)} \|\partial_x^{-1} u_0\|_{L^1(\mathbb{R})}.$$ 

4. Decay of nonlinear problem. In this section, attention is turned to decay estimates of the nonlinear problem.

Lemma 4.1. If $u_0 \in L^2(\mathbb{R})$ then the corresponding solution $u$ of (1.3) satisfies

$$\sup_t \|u(t)\|_{L^2(\mathbb{R})} \leq C \quad \text{and} \quad \|D^{\alpha/2} u\|_{L^2(\mathcal{M})} \leq C.$$ (4.1)

Proof. This is a direct consequence of the conservation law (1.11).

Lemma 4.2. Let $\gamma = 1$.

(i) If $u_0 \in X_1$ then the corresponding solution $u$ of (1.3) satisfies

$$\sup_t \|u(t)\|_{X_1} \leq C$$

and

$$\frac{\beta}{2} \int_0^t \left( \|D^{\alpha/2-1} u\|_{L^2(\mathbb{R})} + \|D^{\alpha/2+1} u\|_{L^2(\mathbb{R})} \right) \, d\tau \leq C.$$ (4.2)

(ii) If $\alpha \geq 1/2$ and $u_0 \in X_2$ then

$$\sup_t \|u(t)\|_{X_2} \leq C$$

and

$$\|D^{\alpha/2+1} u\|_{L^2(\mathcal{M})} \leq C.$$ (4.3)

Proof. First we note from the Cauchy-Schwarz inequality, inequality (1.6) and Lemma 4.1 that

$$\frac{\beta}{2} \int_\mathbb{R} u^2 D^\alpha u \, dx \leq \|u\|_{L^4(\mathbb{R})}^4 + \frac{\beta}{2} \|D^\alpha u\|_{L^2(\mathbb{R})}^2$$

$$\leq \|u\|_{L^4(\mathbb{R})}^4 + \frac{\beta}{2} \left( \|D^{\alpha/2+1} u\|_{L^2(\mathbb{R})}^2 + \|D^{\alpha/2+1} u\|_{L^2(\mathbb{R})}^2 \right)$$

$$\lesssim \sup_t \left( \|u_x\|_{L^2(\mathbb{R})}^{3/2} \right) \|\partial_x^{-1} u\|_{L^2(\mathbb{R})}^{1/2}$$

Therefore we have from (1.12) that

$$H(u) + \frac{\beta}{2} \int_0^t \left( \|D^{\alpha/2-1} u\|_{L^2(\mathbb{R})}^2 + \|D^{\alpha/2+1} u\|_{L^2(\mathbb{R})}^2 \right) \, d\tau$$

$$\leq H(u_0) + \sup_{t} \|u_x\|_{L^2(\mathbb{R})} \|\partial_x^{-1} u\|_{L^2(\mathbb{R})}.$$ 

Hence we deduce from (1.6) that

$$\sup_t \left( \|u_x\|_{L^2(\mathbb{R})} + \|\partial_x^{-1} u\|_{L^2(\mathbb{R})} + \beta \int_0^t \left( \|D^{\alpha/2-1} u\|_{L^2(\mathbb{R})}^2 + \|D^{\alpha/2+1} u\|_{L^2(\mathbb{R})}^2 \right) \, d\tau \right)$$

$$\lesssim C \sup_{t} \left( 1 + \sup_{t} \|u_x\|_{L^2(\mathbb{R})} + \|\partial_x^{-1} u\|_{L^2(\mathbb{R})} \right) + H(u_0).$$
Lemma 4.3. Let the left hand side of (1.12) is not positive. However the proof is slightly different because the coefficient of the third term of

Proof. Some estimates, similar to ones in Lemma 4.2, hold for the case (ii) If \( u \) satisfies

Therefore we obtain after some integration by parts that

Now by using the Gagliardo-Nirenberg inequality we get after some integration by parts that

Therefore we obtain from (4.4) and (4.5) that

And this proves the case (ii).

Proof. Some estimates, similar to ones in Lemma 4.2, hold for the case \( \gamma = -1 \), however the proof is slightly different because the coefficient of the third term of the left hand side of (1.12) is not positive.

Lemma 4.3. Let \( \gamma = -1 \) and \( \alpha \geq 1 \).

(i) If \( u_0 \in X_1 \) then the corresponding solution \( u \) of (1.3) satisfies

(ii) If \( u_0 \in X_2 \) then

\[ \sup_{t} \| u(t) \|_{X_2} \leq C \quad \text{and} \quad \| D^{\alpha/2+1} u \|_{L^2(\mathbb{R})} \leq C. \]

We multiply (1.3) by \( 2u_x \) and integrate over \( \mathbb{R} \) and then over the time interval \([0, t]\) to obtain

\[ \| u_x \|_{L^2(\mathbb{R})}^2 + 2 \beta \int_0^t \| D^{1+\alpha/2} u \|_{L^2(\mathbb{R})}^2 d\tau - 2 \int_0^t \int_{\mathbb{R}} u u_x u_x dxd\tau = \int_{\mathbb{R}} (\partial_x u_0)^2 dx. \]
We derive by subtracting (4.9) from (1.12) that
\[
\frac{1}{2} \int_{\mathbb{R}} (u_t^2 + (\partial_t^{-1}u)^2) \, dx + \beta \int_0^t \left( \|u\|_{H^{\alpha/2+1}}^2 + \|u\|_{H^{\alpha/2-1}}^2 \right) \, d\tau \\
= \frac{1}{6} \int_{\mathbb{R}} (u_0^3 - u^3) \, dx + \frac{1}{2} \int_{\mathbb{R}} ((\partial_x u_0)^2 + (\partial_x^{-1}u_0)^2) \, dx \\
+ 2 \int_0^t \int_{\mathbb{R}} uu_tu_{xx} \, dx - \frac{\beta}{2} \int_0^t \int_{\mathbb{R}} u^2 \, dxd\tau. 
\] (4.10)

We have from the Plancherel theorem, the fractional Leibniz rule [7] and Lemma 4.1 that
\[
\int_0^t \int_{\mathbb{R}} u^2 \, dxd\tau \leq \int_0^t \int_{\mathbb{R}} |D^{\alpha/2}u||D^{\alpha/2}u|^2 \, dxd\tau \\
\leq C \epsilon \int_0^t \|u\|_{H^{\alpha/2}}^2 + \epsilon \int_0^t \|u^2\|_{H^{\alpha/2}}^2 \, d\tau \\
\approx 1 + \epsilon \int_0^t \|u\|_{H^{\alpha/2}}^2 \|u\|_{L^\infty(R)}^2 \, d\tau \\
\leq 1 + \epsilon \sup_t \|u\|_{H^{\alpha/2}}^2 \leq C. 
\] (4.11)

Moreover we have from Lemma 4.1 that
\[
\int_0^t \|u\|_{L^2(R)}^2 \, dx \lesssim \|u_0\|_{L^2(R)}^2 \sup_t \|u\|_{L^\infty(R)} \lesssim \sup_t \|u\|_{L^\infty(R)}. 
\] (4.12)

To estimate the third term of the right hand side of (4.10), we consider two cases. If $\alpha = 2$, then (4.11) implies that
\[
\int_0^t \int_{\mathbb{R}} uu_tu_{xx} \, dx \, d\tau \lesssim \frac{1}{2} \int_0^t \|u_x\|_{L^2(R)}^2 \, d\tau \left( \sup_t \|u\|_{L^\infty(R)} \right) + \frac{1}{2} \int_0^t \|u_{xx}\|_{L^2(R)}^2 \, d\tau \\
\lesssim \sup_t \|u\|_{L^2(R)} \|u_x\|_{L^2(R)} + \frac{1}{2} \int_0^t \|u_{xx}\|_{L^2(R)}^2 \, d\tau \\
\lesssim \sup_t \|u_x\|_{L^2(R)} + \frac{1}{2} \int_0^t \|u_{xx}\|_{L^2(R)}^2 \, d\tau. 
\] (4.13)

Hence we deduce by inserting (4.11)-(4.13) into (4.10) that
\[
\frac{1}{2} \int_{\mathbb{R}} (u_t^2 + (\partial_t^{-1}u)^2) \, dx + \beta \int_0^t \left( \|u\|_{H^{1+\alpha/2}}^2 + \|u\|_{H^{\alpha/2-1}}^2 \right) \, d\tau \leq C.
\]

Now we consider the case $\alpha < 2$. First we observe that $\sup_t \|u(t)\|_{L^\infty(R)} < C$. Indeed we have from the integral form (1.7), the properties of $S_\alpha$ and Lemma 4.1 that
\[
\|u(t)\|_{L^\infty(R)} \leq \|u_0\|_{L^2(R)} + \int_0^t \|S_\alpha(t - \tau)\partial_x(u^2)\|_{L^\infty(R)} \, d\tau \\
\lesssim \int_0^t (t - \tau)^{-2/\alpha} \|u(\tau)\|_{L^2(R)}^2 \, d\tau \\
\lesssim \int_0^t (t - \tau)^{-2/\alpha} \, d\tau \\
\lesssim t^{1-2/\alpha} \leq C. 
\] (4.14)
Next we get by the Cauchy-Schwarz inequality, the fractional Leibniz rule, the Sobolev interpolation, (4.14) and Lemma 4.1 that
\[
\int_0^t \int_R uu_x u x x dx dr \leq \int_0^t \|u^2\|_{H_{2-\alpha/2}} \|u\|_{H_{1+\alpha/2}} d\tau
\]
\[
\leq \int_0^t \|u\|_{H_{2-\alpha/2}} \|u\|_{L^\infty(R)} \|u\|_{H_{1+\alpha/2}} d\tau
\]
\[
\leq \int_0^t \|u\|_{H_{2-\alpha/2}} \|u\|_{L^\infty(R)} \|u\|_{H_{1+\alpha/2}}^{\alpha-1} d\tau
\]
\[
\leq \varepsilon \int_0^t \|u\|_{H_{1+\alpha/2}}^2 d\tau + C_\varepsilon \sup_t \|u\|_{L^\infty(R)} \int_0^t \|u\|_{H_{\alpha/2}}^2 d\tau
\]
\[
\leq \varepsilon \int_0^t \|u\|_{H_{1+\alpha/2}}^2 d\tau + C_\varepsilon.
\]
By combining Equations (4.11), (4.12), (4.15) with (4.10), we complete the proof of (i) in this case. The proof of (ii) is similar to the case (ii) of Lemma 4.2.

The higher order estimates can be established by an argument similar to Lemma 4.2 and Lemma 4.3.

**Corollary 2.** If \( u_0 \in X_s, \alpha \geq 1 \) and \( s \geq 3 \), then the corresponding solution \( u \) of (1.3) satisfies
\[
\sup_t \|u(t)\|_{X_s} \leq C \quad \text{and} \quad \|D_{\alpha/2}u\|_{L^2(\mathbb{R})} \leq C.
\]

**Lemma 4.4.** Let \( u_0 \in X_s, \alpha \geq 1 \) and \( s > 3/2 \). Then the solution \( u \) of (1.3) satisfies
\[
\|u(t)\|_{H^m(R)} \lesssim t^{-1/2}, \quad m \in \mathbb{N}.
\]

**Proof.** Applying the operator \( \Lambda^s = (I - \partial_x^2)^{s/2} \) to equation (1.3) and multiplying by \( \Lambda^s u \) and integrating over \( R \), we obtain that
\[
\|u\|_{H^s(R)}^2 - \|u_0\|_{H^s(R)}^2 \lesssim C_\beta \int_0^t \|u(\tau)\|_{H^s(R)}^2 d\tau.
\]
By Corollary 2, the integrand in the right hand side of (4.17) is in \( L^1(\mathbb{R}^+) \), so that
\[
\lim_{t \to +\infty} \|u(t)\|_{L^2(R)} = 0.
\]

by an argument same as above and Theorem 1.5, we arrive for any \( m \in \mathbb{N} \) at
\[
\lim_{t \to +\infty} \|\partial_x^m u(t)\|_{L^2(R)} = 0.
\]

Now we multiply equation (1.3) by \( D^{2j-1} u \), \( j \geq 1 \), and integrate over \( R \), we obtain after some integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \|D^j u\|_{L^2(R)}^2 + \beta \|D^{j+\alpha/2} u\|_{L^2(R)}^2 \leq \left| \int_R uu_x D^{2j} u \ dx \right|.
\]
We have from the fractional Leibniz rule and the Sobolev embedding that
\[
\left| \int_R uu_x D^{2j} u \ dx \right| \lesssim \|u\|_{H^{j+\alpha/2}} \|u\|_{H^{j+1-\alpha/2}} \|u\|_{L^\infty(R)}
\]
\[
\lesssim \|u\|_{H^{j+\alpha/2}}^{1+\theta} \|u\|_{H^{j-1+\alpha/2}}^{1-\theta} \|u\|_{L^\infty(R)}
\]
\[
\lesssim \varepsilon \|u\|_{H^{j+\alpha/2}}^2 + C_\varepsilon \|u\|_{H^{j-1+\alpha/2}}^2 \|u\|_{L^\infty(R)}.
\]

\[
(4.19)
\]
where $\theta = 2 - \alpha$ and $\varepsilon > 0$ is sufficiently small. By using the embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, $s > 1/2$, (4.18) and Lemma 4.1, we obtain that

$$\frac{1}{2} \frac{d}{dt} \left( \|u\|_{L^2(\mathbb{R})}^2 + \|du\|_{H^s}^2 \right) + \frac{\beta}{4} \|u\|_{H^{1+s/2}}^2 \leq 0.$$ 

Therefore, we obtain

$$\lim_{t \to +\infty} t\|u\|_{H^1(\mathbb{R})}^2 = 0.$$

Now we are ready to give the proof of Theorem 1.6.

Proof of Theorem 1.6. We first treat the case $p = 2$. Multiplying equation (1.3) by $D^{2\ell-3}u$ and then integrating over $\mathbb{R}$, one obtains

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{2\ell-1}}^2 + \|u(t)\|_{H^{2\ell-1}}^2 \leq \left| \int_{\mathbb{R}} uu_x D^{2(\ell-1)}u \, dx \right|.$$ 

In the case $\ell \geq 1$, similar to (4.19) with $j = \ell - 1$, we can show from the Plancherel theorem, the fractional Leibniz rule and an interpolation that that

$$\left| \int_{\mathbb{R}} uu_x D^{2(\ell-1)}u \, dx \right| \lesssim \|u(t)\|_{H^{2\ell-1}}^2 \|u\|_{L^\infty(\mathbb{R})}^2$$

$$\lesssim \|u(t)\|_{H^{2\ell-1}}^2 \|u\|_{L^\infty(\mathbb{R})}^2 \lesssim \|u(t)\|_{H^{2\ell-1}}^2 \|u\|_{L^\infty(\mathbb{R})}^2 \lesssim \frac{\varepsilon}{2} \|u(t)\|_{H^{2\ell-1}}^2 + C \|u\|_{L^\infty(\mathbb{R})}^2 \|u\|_{H^{2\ell-1}}^2.$$ 

Hence, it follows from the Sobolev embedding and Lemma 4.4 that

$$\frac{d}{dt} \|u(t)\|_{H^{2\ell-1}}^2 + \|u\|_{H^{2\ell-1}}^2 \leq \|u\|_{H^{2\ell-1}}^2 \|u\|_{L^\infty(\mathbb{R})}^2$$

$$\lesssim \|u\|_{H^{2\ell-1}}^2 \|u\|_{L^\infty(\mathbb{R})}^2 \lesssim \|u\|_{H^{2\ell-1}}^2 \|u\|_{L^\infty(\mathbb{R})}^2 \lesssim \|u\|_{H^{2\ell-1}}^2 \|u\|_{L^\infty(\mathbb{R})}^2.$$ 

The case $\ell = 0$ is a little bit more delicate. Indeed, we have from the following Gagliardo-Nirenberg inequality

$$\|D^{1-\frac{2}{\alpha}}u\|_{L^1(\mathbb{R})} \lesssim \|u\|_{L^2(\mathbb{R})} \|D^{3-\alpha}u\|_{L^2(\mathbb{R})},$$

and a Sobolev interpolation that

$$\left| \int_{\mathbb{R}} uu_x D^{-2}u \, dx \right| \lesssim \|D^{-1}u\|_{L^2(\mathbb{R})} \|u\|_{L^4(\mathbb{R})}^2$$

$$\lesssim \|D^{-1}u\|_{L^2(\mathbb{R})} \|u\|_{H^{2-\frac{1}{\alpha}}}^2 \|u\|_{H^{\frac{3-\alpha}{2}}}$$

$$\lesssim \varepsilon \|u\|_{H^{2-\frac{1}{\alpha}}}^2 + C \|u\|_{L^2(\mathbb{R})} \|u\|_{H^{2-\frac{1}{\alpha}}}^2.$$ 

Again Lemma 4.4 implies that

$$\frac{d}{dt} \|u(t)\|_{H^{2-\frac{1}{\alpha}}}^2 + \beta \|u\|_{H^{2-\frac{1}{\alpha}}}^2 \leq \|u\|_{H^{2-\frac{1}{\alpha}}}^2 \|u\|_{H^{\frac{3-\alpha}{2}}}^2$$

$$\lesssim \|u\|_{H^{2-\frac{1}{\alpha}}}^2 \|u\|_{H^{\frac{3-\alpha}{2}}}^2 \lesssim t^{-1} \|u\|_{H^{2-\frac{1}{\alpha}}}^2.$$
Therefore, we get from the above inequalities, in both cases $\ell \geq 1$ and $\ell = 0$, combined with an interpolation that
\[
\frac{d}{dt} (\ell^{s+1} \|u(t)\|_{H^s}^2) \lesssim t^\ell \left( C\|u\|_{H^{s+\delta}}^2 - \beta t \|u\|_{H^s}^2 \right), \quad \ell \geq 0. \tag{4.20}
\]
But, the second term of the right hand side of (4.20) can be written as
\[
C\|u\|_{H^{s+\delta}}^2 - \beta t \|u\|_{H^s}^2 = \int_\mathbb{R} \left( C|\xi|^{2(\ell-1)} - \beta t|\xi|^{2(\ell-1) + \alpha} \right) |\hat{u}(\xi)|^2 d\xi
\]
\[
\lesssim \int_{A_\ell} |\xi|^{2\ell} |\hat{u}(\xi)|^2 d\xi
\]
\[
\lesssim t^{-2\ell + 1},
\]
where $A_\ell = \{|\xi| \leq (\frac{C}{\beta t})^{1/\alpha}\}$. Equation (4.20) implies that
\[
\|u(t)\|_{H^{s+\delta}} \lesssim t^{-\frac{2\ell + 1}{\alpha}}. \tag{4.21}
\]
In the case $p = \infty$, we use the Gagliardo-Nirenberg inequality and (4.21) to obtain
\[
\|u\|_{H^{s+\delta}} \lesssim \|u\|_{H^{s+\delta}}^{1/2} \|u\|_{H^s}^{1/2}
\]
\[
\lesssim t^{-\frac{2\ell + 1}{\alpha} t^{-\frac{1}{2}}} t^{-\frac{1}{2}(2\ell + 3)}
\]
\[
\lesssim t^{-\frac{1}{2}(\ell + 1)}.
\]
The case $2 < p < \infty$ is obtained by an interpolation.

**Acknowledgments.** The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

**Appendix.** Here, we sketch the proof the global well-posedness of (1.3) in the case $\alpha \in [0, 1]$. The proofs proceed along the same lines of the argument provided in [22, 23]. Indeed, the idea is to work in the Bourgain space associated with the Ostrovsky equation, i.e., related only to the dispersive part of the symbol of the dissipative Ostrovsky equation. Let $U(t)$ be the unitary group which defined the free evolution of the Ostrovsky equation (1.1), i.e.,
\[
U(t)f = \int_\mathbb{R} e^{ix\xi - i\tau m(\xi)} \hat{f}(\xi) d\xi, \quad t \in \mathbb{R}, \quad f \in \mathcal{S}'(\mathbb{R}). \tag{4.22}
\]
We denote by $Y^{s,b}$ the Bourgain-type space associated with $H^{s,b}$ for (1.1) by the norm
\[
\|u\|_{Y^{s,b}} = \|U(-t)u\|_{H^{s,b}} = \|\langle \tau - m(\xi)\rangle^{b} \hat{u}(\xi, \tau)\|_{L^2(\mathbb{R}^2)}
\]
For $T \geq 0$, the localized version of $Y^{s,b}$ is defined by the norm
\[
\|u\|_{Y_T^{s,b}} = \inf \left\{ \|g\|_{Y^{s,b}}, \quad g \in Y^{s,b} \text{ and } g(t) = u(t) \text{ on } [0, T] \right\}.
\]
To prove the local well-posedness result, a fixed point argument shall be applied to the integral form (1.7). The following linear estimates for the free and forcing terms are derived by mimicking the argument provided in [22], and we omit the details.

**Lemma 4.5.** Let $s \in \mathbb{R}$, $\delta \in [0, 1/2]$ and $b \in [0, 1]$ with $b + \delta \leq 1$. 


(a) For all $f \in H^s(\mathbb{R})$ we have
\[ \|\psi(t)S_\alpha(t)f\|_{Y^{s,b}} \lesssim \|f\|_{H^{s+\frac{3}{2}(2b-1)}(\mathbb{R})}. \tag{4.23} \]

(b) For all $u \in \mathcal{S}(\mathbb{R}^2)$,
\[ \left\| \chi_+(t)\psi(t) \int_0^t S_\alpha(t)(t-t')u(t')dt' \right\|_{Y^{s,b}} \lesssim \|u\|_{Y^{s,b}} \frac{1}{(2b-1)\alpha} \|\psi\|_{Y^{s,b}} \|\varphi\|_{Y^{s,b}}. \]

(c) For all $u \in \mathcal{S}'(\mathbb{R}^2)$ with $u_x \in Y^{s-\frac{3}{4}(1-2b),-\delta}$, we have
\[ \delta \mapsto \int_0^t S_\alpha(t-t')(f_n)(t')dt' \in C(\mathbb{R}^+, H^s(\mathbb{R})). \]

Moreover, if $\{f_n\}$ is a sequence with $(f_n)_x \to 0$ in $Y^{s-\frac{3}{4}(1-2b),-\delta}$ as $n \to \infty$, then
\[ \left\| \int_0^t S_\alpha(t-t')(f_n)(t')dt' \right\|_{L^\infty(\mathbb{R}^+, H^s(\mathbb{R}))} \to 0. \]

Now we recall the following bilinear estimate proved by Isaza and Mejía [14].

**Proposition 1.** Given $s > -\frac{3}{4}$, $b > \frac{1}{2}$ and $\beta \in (\frac{5}{12}, \frac{1}{2})$ such that $\frac{1}{2} - \beta \leq \min\{\frac{1}{4} - r, 2b - 1\}$ and $b + \beta < 1$, where $r = \max\{-s, 0\}$, then
\[ \|uv\|_{Y^{s,-\beta}} \lesssim \|u\|_{Y^{s,b}}\|v\|_{Y^{s,b}}. \]

for all $u, v \in Y^{s,b}$.

The local existence of a solution is a consequence of the following modified version of Proposition 1.

**Proposition 2.** Given $s > -\frac{3}{4}$, choose $b$ and $\beta$ satisfying Proposition 1. Then
\[ \|uv\|_{Y^{s,-\beta}} \lesssim T^\mu \left( \|u\|_{Y^{s,b}}\|v\|_{Y^{s,-\beta}} + \|v\|_{Y^{s,b}}\|u\|_{Y^{s,-\beta}} \right), \tag{4.24} \]

for some $\mu > 0$.

Estimate (4.24) is obtained thanks to Lemma 2.3 and the triangular inequality and Proposition 1.

**Theorem 4.6.** For $s > -\frac{3}{4}$ and $b$ as in Proposition 2, if $u_0 \in H^s(\mathbb{R})$, then there exists $T = T(\|u_0\|_{H^s(\mathbb{R})})$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R})) \cap Y^s_{T} \mathbb{R}$ of (1.3). Moreover, the flow map is analytic and $u \in C((0, +\infty); H^s(\mathbb{R})).$

**Proof.** Let $u_0 \in H^s(\mathbb{R})$ with $s > -3/4$. Let $F(u)$ be the right hand side of (1.7). We shall prove that for $T \ll 1$, $F$ is contraction in a ball of the Banach space
\[ Z = \left\{ u \in Y^s_{T} \mathbb{R} \cap Y^s_{T} \mathbb{R} \mathbb{R} : \|u\|_Z = \left\| u \right\|_{Y^s_{T} \mathbb{R}} + \nu \|u\|_{Y^s_{T} \mathbb{R}} \right\}, \]

where $\nu$ is defined for all nontrivial $\varphi$ by
\[ \nu = \frac{\|\varphi\|_{H^{-\frac{3}{4}}}}{\|\varphi\|_{H^{s-\frac{3}{4}(2b-1)}}}. \]

Choosing $b$ suitably and combining Lemma 4.5 with (4.24), it is easy to derive that
\[ \|F(u)\|_Z \lesssim \|u_0\|_{H^{-\frac{3}{4}} + \nu} + \nu\|u_0\|_{H^{s}} + T^\mu \|u\|_Z^2 \]
and
\[ \|F(u) - F(v)\|_Z \lesssim T^\mu \|u - v\|_Z \|u + v\|_Z \]

THE DISSIPATIVE OSTROVSKY EQUATION 733
for some $\mu > 0$. Thus, taking $T = T \left(\|u_0\|_{H^{\frac{3}{4}+}}\right)$ small enough, we deduce that $F$ is contractive on the ball of radius $4C(H^{\frac{3}{4}+} + \nu\|u_0\|_{H^s})$ in $Z$. This proves the existence of a solution $u$ to $u = F(u)$ in $Y^s_{T}(\nu(2b-1))$. Following similar arguments of [23], it is not too difficult to see the uniqueness of the solution.

It is straightforward to check that $S_{\alpha}u_0 \in C([R^+; H^s(R)]) \cap C([R^+; H^{s+\alpha(1-b-\beta)}(R)])$. Then it follows from Proposition 1, Lemma 4.5 and the local existence of the solution that $u \in C([0,T],H^s(R)) \cap C([0,T],H^{s+\alpha(1-b-\beta)}(R))$ for some $T = T \left(\|u_0\|_{H^{\frac{3}{4}+}}\right)$. We have by induction that $u \in C((0,T],H^{s+\alpha(1-b-\beta)}(R))$.

Taking the $L^2$-scalar product of (1.3) with $u$, we obtain that $t \mapsto \|u(t)\|_{H^{\frac{3}{4}+}}$ is nonincreasing on $(0,T]$. Since the existence time of the solution depends only on the norm $\|u_0\|_{H^{\frac{3}{4}+}}$, this implies that the solution can be extended globally in time.

REFERENCES

[1] C. J. Amick, J. L. Bona and M. E. Schonbek, Decay of solutions of some nonlinear wave equations, J. Differential Equations, 81 (1989), 1–49.
[2] I. Bejenaru and T. Tao, Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation, J. Funct. Anal., 233 (2006), 228–259.
[3] O. Besov, V. Ilin and S. Nikolski, Integral Representation of Functions and Embedding Theorems. Vol. I., New York: J. Wiley, 1978.
[4] J. Bourgain, On the Cauchy problem for the Kadomtsev-Petviashvili equation, Geom. Funct. Anal., 3 (1993), 315–341.
[5] W. Chen, C. Miao and J. Li, On the well-posedness of the Cauchy problem for dissipative modified Korteweg-de Vries equations, Differential Integral Equations, 20 (2007), 1285–1301.
[6] A. Esfahani and S. Levandosky, Solitary waves of the rotation-generalized Benjamin-Ono equation, Discrete Contin. Dyn. Syst., 33 (2013), 663–700.
[7] K. Fujiwara, V. Georgiev and T. Ozawa, Higher order fractional Leibniz rule, J. Fourier Anal. Appl., 24 (2018), 650–665.
[8] V. N. Galkin and Y. A. Stepanyants, On the existence of stationary solitary waves in a rotating field, J. Appl. Math. Mech., 55 (1991), 939–943.
[9] O. A. Gilman, R. Grimshaw and Y. A. Stepanyants, Approximate and numerical solutions of the stationary Ostrovsky equation, Stud. Appl. Math., 95 (1995), 115–126.
[10] J. Ginibre, Y. Tsutsumi and G. Velo, On the cauchy problem for the Zakharov system, J. Funct. Anal., 151 (1997), 384–436.
[11] R. Grimshaw, Internal solitary waves, in: R. Grimshaw (Ed.), Environmental Stratified Flows, Kluwer, Boston, (2001), pp. 1–27.
[12] Z. Guo and B. Wang, Global well-posedness and inviscid limit for the Korteweg-de Vries-Burgers equation, J. Differential Equations, 246 (2009), 3864–3901.
[13] Z. Huo and Y. Jia, Low-regularity solutions for the Ostrovsky equation, Proc. Edinb. Math. Soc., 49 (2006), 87–100.
[14] P. Isaza and J. Mejía, Local well-posedness and quantitative ill-posedness for the Ostrovsky equation, Nonlinear Anal., 70 (2009), 2306–2316.
[15] G. Karch, Self-similar large time behavior of solutions to Korteweg-de Vries-Burgers equations, Nonlinear Anal., 35 (1999), 199–219.
[16] C. Kenig, G. Ponce and L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc., 9 (1996), 573–603.
[17] S. Levandosky and Y. Liu, Stability of solitary waves of a generalized Ostrovsky equation, SIAM J. Math. Anal., 38 (2006), 985–1011.
[18] Y. Li, J. Huang and W. Yan, The Cauchy problem for the Ostrovsky equation with negative dispersion at the critical regularity, J. Differential Equations, 259 (2015), 1379–1408.
[19] F. Linares and A. Milanes, Local and global well-posedness for the Ostrovsky equation, J. Differential Equations, 222 (2006), 325–340.
[20] B. Melinand, Long wave approximation for water waves under a Coriolis forcing and the Ostrovsky equation, Proc. Roy. Soc. Edinburgh Sect. A, 148 (2018), 1201–1237.
[21] H. Mitsudera and R. Grimshaw, Effects of friction on a localized structure in a baroclinic current, J. Physical Oceanography, 23 (1993), 2265–2292.
[22] L. Molinet and F. Ribaud, The Cauchy problem for dissipative Korteweg-de Vries equations in Sobolev spaces of negative order, Indiana Univ. Math. J., 50 (2001), 1745–1776.
[23] L. Molinet and F. Ribaud, On the low regularity of the Korteweg-de Vries-Burgers equation, Inter. Math. Research Notices, 37 (2002), 1979–2005.
[24] L. Molinet and F. Ribaud, The global Cauchy problem in Bourgain’s-type spaces for a dispersive dissipative semilinear equation, SIAM J. Math. Anal., 33 (2002), 1269–1296.
[25] L. A. Ostrovsky, Nonlinear internal waves in a rotating ocean, Oceanologia, 18 (1978), 181–191.
[26] E. Ott and R. N. Sudan, Damping of solitary waves, Physics of Fluids, 13 (1970), 1432–1434.
[27] T. Tao, Multilinear weighted convolution of $L^2$ functions, and applications to non-linear dispersive equations, Amer. J. Math., 123 (2001), 839–908.
[28] S. Vento, Asymptotic behavior of solutions to dissipative Korteweg-de Vries equations, Asymptot. Anal., 68 (2010), 155–186.
[29] S. Vento, Global well-posedness for dissipative Korteweg-de Vries equations, Funkcial. Ekvac., 54 (2011), 119–138.
[30] S. Vento, Well-posedness and ill-posedness results for dissipative Benjamin-Ono equations, Osaka J. Math., 48 (2011), 933–958.

Received July 2018; revised January 2019.

E-mail address: wanghwxxu@gmail.com
E-mail address: esfahani@du.ac.ir