LIMITS OF SATURATED IDEALS

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ABSTRACT. We investigate the question of when a given homogeneous ideal is a limit of saturated ones. We provide cohomological necessary criteria for this to hold and apply them to a range of examples. In small cases we characterise the limits. We also supply a number of auxiliary results on the classical and multigraded Hilbert schemes, for example we prove a very general result on openness of the saturated locus and give scheme structure to “constant Hilbert function” loci. Our results have applications to theory of tensors.

1. Introduction

It is famously untrue that a deformation of a zero-dimensional projective scheme $\Gamma \subseteq \mathbb{P}^n$ induces a deformation of its homogeneous coordinate ring $S/I(\Gamma)$, where $I(\Gamma) \subseteq S = \mathbb{k}[\alpha_0, \ldots, \alpha_n]$ is the saturated ideal. The standard example is given by the varying three points

$$\Gamma_t = \{[1 : 0 : 0], [1 : t : 1], [0 : 0 : 1]\} \subseteq \mathbb{P}^2,$$

which form a flat family over the affine line with parameter $t$. The ideal $I(\Gamma_t|_{t=\lambda})$ contains a linear form exactly when $\lambda = 0$, so if the family $S[t]/I(\Gamma_t)$ existed, it had to be lower semi-continuous in degree one; an absurd. What happens instead is that as $\Gamma_t$ converges towards $\Gamma_0$, the family of ideals $I(\Gamma_t|_{t=\lambda})$ converges degreewise to a nonsaturated ideal

$$I(\Gamma_0)' = (\alpha_0\alpha_2(\alpha_0 - \alpha_2)) + \alpha_1 \cdot (\alpha_0, \alpha_1, \alpha_2).$$

For a homogeneous ideal $I$ we say that it is a limit of saturated ideals or saturable if there is a family of saturated ideals that converges to $I$. Hence $I(\Gamma_0)'$ above is saturable, yet not saturated. In this article we tackle the following.

Problem 1.1. Given a homogeneous ideal, decide whether it is saturable.

There are numerical obstructions to being saturable: an ideal has to have an admissible Hilbert function. In the case of a standard graded polynomial ring such functions are classified by Macaulay [BH93, §4.2]. However, once the Hilbert function is admissible, there are no general criteria for verifying whether an ideal is saturable. We give such criteria below, using deformation theory.

1.1. Results. Our results are of three kinds. First, we prove several results for classical and multigraded Hilbert schemes: see Theorem 1.5, Theorem 2.3 and Theorem A.2. Second, we employ them and our idea of “stickyness” to give explicit necessary criteria for being saturable. These criteria are checkable by computer on specific examples, so they are useful also for researchers outside algebraic
geometry. Third, as an illustration, we give examples and applications to specific low degrees and to wild polynomials (see §5).

Below, we consider (multi)homogeneous ideals but we do not impose any further restrictions, for example, we do not restrict to the zero-dimensional case. Actually, we take care to prove results under minimal assumptions: we allow any multigrading, any dimension, any base ring, see §2.1 for details.

Before we present our results, we informally present the observation which underlies our criteria. Let $S$ be a Cox ring of a smooth projective toric variety. In algebraic terms, this implies that $S$ is a polynomial ring graded by a finitely generated abelian group $\mathcal{A}$ and admitting an irrelevant ideal $\text{Irr}$ with respect to which we consider the saturation (see Example 2.5 for one concrete instance of this setup). We say that an ideal $J$ with $I \subseteq J \subseteq I_{\text{sat}}$ sticks with $I$ (or is a sticky ideal for $I$) if every deformation of $I$ induces a deformation of $J \supseteq I$. If $I$ admits a sticky ideal $J$, then it is not a limit of saturable ideals. Indeed, suppose $I_t \to I$ with $I_{t|t=\lambda}$ saturated for $\lambda \neq 0$. Then we have a corresponding family $J_t \supseteq I_t$. Due to equal Hilbert polynomials, the ideals $J_{t|t=\lambda}$ and $I_{t|t=\lambda}$ agree in “positive enough” degrees, so $(J_{t|t=\lambda})_{\text{sat}} = (I_{t|t=\lambda})_{\text{sat}} = I_{t|t=\lambda}$, in particular $J_{t|t=\lambda} \subseteq I_{t|t=\lambda}$, which is a contradiction.

In this paper, we obtain a toolbox of cohomological criteria for proving that a given ideal $J$ sticks with $I$. The fiber obstruction group for the pair $(I, J)$ is

$$\text{Ob}_{\text{fiber}}(I, J) := \text{Ext}^1_S \left( \frac{J}{I}, \frac{S}{J} \right)_0,$$

where $0 \in \mathcal{A}$ is the zero element. The most succinct version of our criterion is as follows.

**Theorem 1.2** (Theorem 3.1). If $\text{Ob}_{\text{fiber}}(I, J) = 0$, then $J$ sticks with $I$, hence $I$ is not saturable.

This is effective already for the ideal $J := I_{\text{sat}}$, see §4.3. A slightly changed version is the following, which employs the multigraded Hilbert scheme, see §2.1.

**Theorem 1.3** (Theorem 3.2). If the natural map $\text{Hom}_S(J, S/J)_0 \to \text{Hom}_S(I, S/J)_0$ is onto and $[J] \in \text{Hilb}^{HS/J}$ is smooth, then $J$ sticks with $I$, hence $I$ is not saturable.

We say that $I$ is entirely nonsaturable if it is nonsaturated and there is an open neighbourhood of $[I] \in \text{Hilb}^{HS/I}$ such that any ideal $I'$ from this neighbourhood has $S/(I')_{\text{sat}}$ with the same Hilbert function as $S/I_{\text{sat}}$, so “every near enough ideal is as nonsaturated as $I$’. If the conditions of Theorem 1.2 or Theorem 1.3 are satisfied for $J := I_{\text{sat}}$, then $I$ is entirely nonsaturable (see Theorems 3.3-3.4).

The necessary conditions for both theorems above are easily computable with computer systems such as Macaulay2 [GS]. However, the presentation of fiber obstruction (1.1) is opaque. There is a more transparent version when $\mathcal{A} = \mathbb{N}$ and $J = I_{\text{sat}}$ defines a one-dimensional subscheme. Namely, the vector space $\text{Ob}_{\text{fiber}}$ is dual to $(I_{\text{sat}}/I \otimes_S \omega_{S/I_{\text{sat}}})_0$, where $\omega_{S/I_{\text{sat}}}$ is the canonical module, see Proposition 3.12. This reinterpretation is especially useful when we assume additionally that $S/I_{\text{sat}}$ is Gorenstein (which holds for example if $I_{\text{sat}}$ is a complete intersection). In this case, we have

$$\text{Ob}_{\text{fiber}}(I, I_{\text{sat}}) \simeq \left( \frac{I_{\text{sat}}}{I + (I_{\text{sat}})^2} \right)^{\vee}_a,$$

where $a$ is the largest degree for which $H_{S/I_{\text{sat}}}(a) < H_{S/I_{\text{sat}}}(a+1)$ see Corollary 3.14. The obstruction group can be more generally computed for non-Gorenstein $J = I_{\text{sat}}$ if additional conditions on the Hilbert functions are satisfied, see Example 4.2.

Above we discussed necessary conditions for being saturable. It is natural to ask what happens if those are not satisfied. In this case, interestingly, we do get information about possible families
showing saturability of $I$. For example, in the context of Theorem 1.3 for $J = I^{\text{sat}}$, if $\text{Hom}_S(J, S/J)_0 \to \text{Hom}_S(I, S/J)_0$ is not surjective and yet $\text{Hom}_S(I, S/I)_0 \to \text{Hom}_S(I, S/J)_0$ is surjective, then we obtain a tangent vector (see §2.3) at $[I]$, which does not come from a tangent vector at $[I \subseteq I^{\text{sat}}]$. If we are able to integrate along it (for example, if $[I]$ is smooth), then we obtain a one-dimensional family whose general member $I'$ has $H_{S/(I')^{\text{sat}}} > H_{S/I^{\text{sat}}}$, where the inequality means argument-wise inequality with at least one of them strict. Repeating this process we arrive at the case where $H_{S/(I')^{\text{sat}}} = H_{S/I} = H_{S/I'}$, hence $I'$ is saturated. This procedure seems effective on examples even when $[I]$ is not smooth.

1.2. Examples. To illustrate the effectiveness of our ideas, we decide the saturability for all ideals with Hilbert function $(1, d, d, d, \ldots)$ for $d \leq 5$; in this case $A = \mathbb{N}$, see §4. For $d = 4$ and all but one cases for $d = 5$ a nonsaturated ideal $I$ is saturable if and only if the obstruction space (1.2) is nonzero. The unique case for $d = 5$ is when $H_{S/I^{\text{sat}}} = (1, 2, 3, 4, 5, 6, \ldots)$. In this case the nonvanishing of the obstruction boils down to $(I^{\text{sat}})^2 \subseteq I$. It is still necessary, yet not sufficient: there is a second necessary condition of the form $(I^{\text{sat}} \cdot J)_2 \subseteq I$ for a certain ideal $I \subseteq J \subseteq I^{\text{sat}}$ determined by $I$. The same argument would apply to other Hilbert functions eventually equal to $d \leq 5$; we have restricted ourselves to $(1, d, d, \ldots)$ in order to better exhibit the main ideas.

1.3. Application to the classical Hilbert scheme. While our focus is on nonsaturated ideals, we obtain two results about saturated ideals. They are of independent interest.

The first results concerns the openness of saturation.

**Theorem 1.4** (openness of saturation, Theorem 2.3). For any $A$-grading on $S$, any Hilbert function $H: A \to \mathbb{N}$, and any fixed homogeneous ideal $\text{Irr}$, the subset of $[I] \in \text{Hilb}^H$ which are saturated (with respect to $\text{Irr}$) is open.

The above seems not present in the literature even in the case of standard $\mathbb{N}$-grading on the polynomial ring $S$ and $\text{Irr} = S_{\geq 0}$. It is important in itself, but also for applications to tensors, where it allows to define the Slip component(s), see Definition 3.10.

Our second result gives a functorial description of the set of all zero-dimensional ideals with given Hilbert function.

**Theorem 1.5** (Theorem 3.9). Let $S$ be the Cox ring of a smooth projective toric variety $X$ and $A \simeq \text{Pic}(X)$ be the induced grading group. For any function $H: A \to \mathbb{N}$ with multigraded Hilbert polynomial equal to $d \in \mathbb{N}$, the locus of $[\Gamma] \in \text{Hilb}^d(X)$ with $H_{S/I(\Gamma)} = H$ is locally closed in $\text{Hilb}^d(X)$ and admits a scheme structure with which it is isomorphic to $\text{Hilb}^d_{S/I^{\text{sat}}}$, the saturated locus of $\text{Hilb}^d$.

Theorem 1.5 allows in particular for an effective calculation of the tangent space to the locus.

In the case of projective space, we can say more about the smoothness.

**Theorem 1.6** (Theorems 3.18-3.19). In the setting of Theorem 1.5, suppose additionally that $X = \mathbb{P}^{n-1}$. Then, a point $[\Gamma]$ in the saturated locus is smooth whenever the Artin reduction of $S/I(\Gamma)$ is unobstructed. In particular, $[\Gamma]$ is smooth when $n \leq 3$ or when $n \leq 4$ and $S/I(\Gamma)$ Gorenstein.

In the case $n = 3$ and for a field of characteristic zero Gotzmann proved that the above locus is smooth [Got88]. Kleppe proved smoothness in the Gorenstein $n = 4$ case [Kle98]. The interpretation as a functor seems new even in these special cases.
1.4. **Application to nonexistence of wild polynomials.** We apply the above results also to the theory of border rank in the case of $\mathbb{P}^2$. It is an open question whether wild forms (see §5) in three variables exist. By [Mań22, Proposition 3.4] degree $d$ forms of border rank at most $d + 2$ are not wild. Here we improve the bound by one.

**Proposition 1.7** (Proposition 5.12). *Let $F$ be a degree $d$ form in three variables. If the border rank of $F$ is at most $d + 3$, then the smoothable rank of $F$ is equal to its border rank, so $F$ is not wild.*

There are lower bounds for the smoothable rank such as [RS11]. Coupled with the above they yield constraints on the possible forms of low border rank. To give a concrete example, a septic with Hilbert function $(1, 3, 6, 9, 9, 6, 3, 1)$ and border rank 9 has, by the above, smoothable rank 9 and so, by [RS11, Proposition 1], its apolar ideal cannot be generated in degrees at most 4. This also follows from border apolarity [BB21a, Theorem 3.15].

1.5. **Motivation from theory of VSP.** Investigating Problem 1.1 is very natural from the commutative algebra perspective. But, for us, the main motivation comes from the theory of tensors. A central problem there is to determine the border rank of a tensor (see Section 5 for a brief introduction). This problem is solved in very very few cases and is open even for tensors of small size [BCS97, §16], [CHL22].

A recent successful technique is border apolarity [BB21a] (see also [RS00, IR01, RS13, RV17, JRS24]). To a tensor $T$ and integer $r$ it assigns a projective scheme, the (border) variety of sums of powers whose points are certain multigraded ideals $I$ contained in $\text{Ann}(T)$. The tensor $T$ has border rank at most $r$ if and only if the scheme contains a saturable ideal with multigraded Hilbert polynomial $r$ which is a limit of reduced ones. Frequently, even if such an ideal exists, the scheme contains no saturated ideals at all. Producing ideals in the scheme is in practice easy, but deciding whether they are saturable is hard.

1.6. **Previous work.** To our best knowledge, there are almost no papers concerned with Problem 1.1. This problem is mentioned in [BB21a, CHL22]. One may also ask which ideals are limits of saturated and radical ideals. The latter problem was investigated in [Mań22] in the $\mathbb{N}$-graded case. It gives two necessary conditions: a semicontinuity of tangent spaces type of statement [Mań22, Theorem 1.1] and a necessary condition in the case that $V(I)$ is contained in the line [Mań22, Theorem 2.7]. The latter statement is an immediate corollary of Theorem 1.2 above.

Theorem 3.3 is inspired by the works of Cid-Ruiz and Ramkumar [CRR21, CRR22] which in turn were inspired by [CV20]. The main difference is that Cid-Ruiz and Ramkumar employ local cohomology and impose stronger conditions that guarantee that also the higher local cohomology groups $H^i$ deform with the ideal, while we only investigate whether $H^0_{\text{irr}}(S/I) = I_{\text{sat}}/I$ deforms with $I$. This is especially important in the multigraded case, where $S/I_{\text{sat}}$ has high dimension and need not even be Cohen-Macaulay, so intermediate cohomology may well be nonzero and not deform. Finally, our setting does not require standard $\mathbb{Z}$-grading and furthermore, by allowing intermediate ideals $J$ lying between $K$ and $K_{\text{sat}}$ we obtain more flexibility. This is illustrated by Examples 3.5-3.6.

Buczyńska-Buczyński in an unpublished note [BB21b] gave a proof of a slightly weaker version of Theorem 1.4 based on the ideas of the first-named author.

**Acknowledgements.** We thank Alessandra Bernardi and Joseph Landsberg for helpful comments and conversations. We thank an anonymous referee for constructive suggestions regarding the presentation. We especially thank Jarosław Buczyński for his very helpful comments.
2. General results on multigraded ideals and Hilbert schemes

Throughout the article, \( k \) is a Noetherian ring of finite dimension. (On the first reading, any willing reader can assume that \( k := \kappa \) is their favourite field.) The article makes frequent use of deformation theory. One source well aligned with our results is [FGI+05, Chapter 5]. This reference assumes that everything takes place over a field, but the proofs do generalize in a straightforward way. For convenience of reader, we phrase the main theorem of obstruction calculus explicitly as Lemma A.1.

2.1. Conventions and notation. We fix a polynomial ring \( S \) over \( k \). For a \( k \)-algebra \( A \) or a \( k \)-scheme \( U \) we write \( S_A, S_U \) instead of \( S \otimes_k A, S \otimes_k O_U \), respectively.

We assume that \( A \) is a finitely generated abelian group and that \( S \) is \( A \)-graded via a homomorphism \( \mathbb{N}^{\# \text{vars}(S)} \to A \). A homogeneous ideal is an ideal homogeneous with respect to the \( A \)-grading. We remark that even when \( k = \kappa \) is a field it may well happen that for some \( a \in A \) the dimension of the \( \kappa \)-vector space \( S_a \) is infinite (for example, take \( S = \kappa[x,y], A = \mathbb{Z}, \deg(x) = 1, \deg(y) = -1, a = 0 \)). This forces us to take care when defining familiar notions such as Hilbert functions.

For any \( A \)-graded \( S \)-algebra \( S' \) and any point \( \text{Spec}(L) \to \text{Spec}(k) \), where \( L \) is a field, the fiber of \( S' \) over \( L \) is \( S' \otimes_k L \). If for every \( a \in A \) the vector space \( (S' \otimes_k L)_a \) has finite dimension, then the Hilbert function of the fibre \( H_{S' \otimes_k L} : A \to \mathbb{N} \) is given by

\[
H_{S' \otimes_k L}(a) = \dim_L (S' \otimes_k L)_a.
\]

If \( k \) is a field, this definition agrees with the usual definition.

We recall the terminology from [HS04]. Fix a function \( H : A \to \mathbb{N} \). For a \( k \)-scheme \( U \) and a \( A \)-graded sheaf of ideals \( I \subseteq S_U \), we say that \( I \) is admissible with (finite) Hilbert function \( H \) if for every \( e \in A \) the \( O_U \)-module \( ((S_U)/I)_e \) is locally free of rank \( H(e) \). We call a closed subscheme \( Z \subseteq \text{Spec}(S) \times U \) admissible with (finite) Hilbert function \( H \) if its ideal sheaf is admissible with Hilbert function \( H \). The multigraded Hilbert scheme \( \text{Hilb}^H \) is the scheme representing admissible ideals \( I \).

Throughout the article, the symbol \( \kappa \) will denote any field with a homomorphism \( k \to \kappa \). The ideals of interest will lie in some \( S_\kappa = S \otimes_k \kappa \). Below we will use \( S_\kappa \) without explicitly repeating that \( \kappa \) is a field as above. For an admissible ideal \( J \subseteq S_\kappa \) we define the shorthand notation \( \text{Hilb}^H_J \) for the \( k \)-scheme \( \text{Hilb}^{H_{S_\kappa}/J} \).

Likely, many readers are interested in the case where \( k \) itself is a field. They may safely take \( \kappa = k \), so that \( S_\kappa = S \). The more general case is useful for example when considering deformations (with \( k \) Artinian) and arithmetic questions (with \( k = \mathbb{Z} \)).

2.2. Saturations. We fix a homogeneous ideal \( \text{Irr} \subseteq S \) which we will call the irrelevant ideal. For a homogeneous ideal \( I \) in an \( A \)-graded Noetherian \( S \)-algebra \( S' \), its saturation is the ideal

\[
I^{\text{sat}} := \bigcup_{i \geq 0} (I : \text{Irr}^i).
\]

Since \( S' \) is Noetherian, we have \( I^{\text{sat}} = (I : \text{Irr}^e) \) for all \( e \gg 0 \). The ideal \( I \) is saturated if \( I = I^{\text{sat}} \). The saturation of any ideal is saturated. Let \( \text{Irr}' := \text{Irr} \cdot S' \). By prime avoidance [sta23, 00LD, 00DS], a homogeneous ideal \( I \subseteq S' \) is saturated if and only if there exists a homogeneous \( \ell \in \text{Irr}' \) such that the multiplication by \( \ell \) is injective on \( S'/I \). For a homogeneous ideal \( I \subseteq S' \) a transverse element is any homogeneous \( \ell \in \text{Irr}' \) which is a nonzerodivisor on \( S'/I^{\text{sat}} \). For every such \( \ell \), the ideal \( I^{\text{sat}}/I \) is the kernel of the localisation map \( S'/I \to (S'/I)_\ell \). If \( e \) is any integer such that \( I^{\text{sat}} = (I : \text{Irr}^e) \), then
$\ell^e$ is another transverse element and this element annihilates $I^{\text{sat}}/I$. (In this article, the only $S'$ we consider are of the form $S_A$ for $A$ a $k$-algebra, trivially $A$-graded.)

Proposition 2.2 is the technical part of the proof that saturation is an open property, see Theorem 2.3 below for motivation.

Lemma 2.1. Let $U$ be a Noetherian $k$-scheme of finite dimension and $Z \subset \text{Spec}(S) \times U$ be a closed subscheme given by a homogeneous ideal sheaf. Denote by $p: Z \to U$ the projection. For every $u \in U$ consider the “Hilbert function” $H_u: A \to \mathbb{N} \cup \{\infty\}$ defined by

$$H_u(a) := \dim_{k(u)} H^0(O_{p^{-1}(u)})a.$$ 

Then the set $\{H_u \mid u \in U\}$ is finite.

Proof. We do induction on the dimension of $U$. For a given $U$, we replace it with $U_{\text{red}}$ and then by the disjoint union of the irreducible components of its affine open subschemes. For every such component $\text{Spec}(A)$, let $\text{Spec}(B) = p^{-1}(\text{Spec}(A)) \subseteq Z$.

By Generic Flatness ([sta23, Tag 051R] or [Eis95, Theorem 14.4]) there is a nonzero element $a \in A$ such that $B_a$ is a free $A_a$-module.

Choose a degree $e \in A$. The $A_a$-module $(B_a)_e$ is a direct summand of $B_a$, hence is a projective $A_a$-module. Choose a point $p \in \text{Spec}(A_a)$. The module $((B_a)_e)_p$ is a projective $(A_a)_p$-module, hence is free by [sta23, Tag 0593]. Recall that $\text{Spec}(A)$ corresponds to an irreducible scheme, so it has a generic point $\eta$. The points $p, \eta$ lie in $\text{Spec}(A_a)$, so $((B_a)_e)_p \subseteq \text{Spec}(A)$. The freeness of $((B_a)_e)_p$ implies that the Hilbert functions of $p^{-1}(p)$ and $p^{-1}(\eta)$ agree at position $e$. The point $p$ and the position $e \in A$ are arbitrary, so this means that the Hilbert function of $p^{-1}(u)$ is independent of the choice of $u \in (a \neq 0) \subseteq \text{Spec}(A)$.

Repeating this argument for every component of $U$, we obtain an open dense subset $U' \subseteq U$ with only finitely many possible Hilbert functions, at most as many as the number of components. By induction, the set of Hilbert functions for the fibers over $U \setminus U'$ is finite. This concludes the proof. □

Proposition 2.2. Let $f \in S$ be a homogeneous element. Let $U$ be a locally Noetherian $k$-scheme. Assume that $U$ has an open cover by schemes of finite dimension. Let $Z \subset \text{Spec}(S) \times U$ be an admissible closed subscheme with Hilbert function $H$. Let $p: Z \to U$ be the projection. Then the subset

(2.1) $$U' := \{u \in U \mid f \text{ is a nonzerodivisor on } H^0(O_{p^{-1}(u)})\} \subseteq U$$

is open. The multiplication by $f$ on $O_{p^{-1}(U')}$ is injective and $V(f) \cap p^{-1}(U')$ is flat over $U'$.

Proof. Everything is local on $U$, so we assume $U = \text{Spec}(A)$ is Noetherian of finite dimension.

We have $Z = \text{Spec}(B)$ for some $A$-algebra $B$. Pick a point $u_0 \in U$, let $p \subset A$ be the prime ideal corresponding to $u_0$ and $\kappa_0 = A_p/pA_p$. Suppose that $f$ is a nonzerodivisor on $B \otimes_A \kappa_0 = H^0(O_{p^{-1}(u_0)})$. Let $T = A \setminus p$ be the multiplicatively closed subset, so that $T^{-1}A = A_p$. Let $d = \text{deg}(f)$ and let $\mu_f: B \to B$ be the multiplication by $f$. Fix any element $e \in A$. Consider the map $\mu_f: B_e \to B_{d+e}$. By assumption, the map

$$(\mu_f)_e \otimes_A \kappa_0: B_e \otimes_A \kappa_0 \to B_{d+e} \otimes_A \kappa_0$$

is injective. By [Ser06, Lemma A.7] the map $T^{-1}(\mu_f)_e: T^{-1}B_e \to T^{-1}B_{d+e}$ of $A_p$-modules is split injective; in particular its cokernel $T^{-1}(B_{d+e})/fT^{-1}B_e$ is a free $A_p$-module because it is a direct summand of the free $A_p$-module $T^{-1}B_{d+e}$. Since localization is exact, the module $T^{-1}(B_{d+e})/fT^{-1}B_e$ is isomorphic to $(T^{-1}(B/fB))_{d+e}$, so the latter is a free $A_p$-module as well. This holds for any $e \in A$, so, by reindexing, we conclude that $(T^{-1}(B/fB))_e$ is a free $A_p$-module for every $e \in A$. 

Consider the degree decomposition
\[ T^{-1}(B/fB) = \bigoplus_{e \in A} (T^{-1}(B/fB))_e. \]
The summands are finitely generated free \( A_p \)-modules, so \( T^{-1}(B/fB) \) is a free \( A_p \)-module (not necessarily finitely generated!). We now prepare to “smear out” the freeness of \( T^{-1}(B/fB) \) to the freeness of some \( (B/fB)_s \), \( s \in T \). Each individual factor can be “smeread out” to a free \( A_s \)-module for some \( s \in T \), see [sta23, Tag 00NX]. The issue is that the direct sum has possibly an infinite number of nonzero summands, so we need to take care to find a common \( s \).

Let \( H' \) be the Hilbert function of \( (B/fB) \otimes_A \kappa_0 \), so that for every \( e \in A \) we have
\[ H'(e) = H(e) - H(e - d). \]

By Lemma 2.1, the set \( \mathcal{H} \) of possible Hilbert functions for the fibers of \( B/fB \) is finite. Fix a finite subset \( \mathcal{I} \subseteq A \) such that every Hilbert function \( H'' \in \mathcal{H} \setminus \{ H' \} \) differs from \( H' \) on some position \( e \in \mathcal{I} \). Fix \( s \in T \) such that \( (B_e/fB_{e-d})_s \) is a free \( A_s \)-module for every \( e \in \mathcal{I} \).

For every \( u \in \text{Spec}(A_s) \) with residue field \( \kappa \), the Hilbert function \( H' \) of \( (B/fB) \otimes_A \kappa \) is an element of \( \mathcal{H} \) and agrees with \( H' \) for all arguments \( e \in \mathcal{I} \), hence is equal to \( H' \). For every \( e \in A \), in the right-exact sequence
\[ B_{e-d} \otimes_A \kappa \xrightarrow{\mu_f \otimes_A \kappa} B_e \otimes_A \kappa \rightarrow (B/fB)_e \otimes_A \kappa \rightarrow 0 \]
the dimensions of the \( \kappa \)-vector spaces are \( H(e - d), H(e), H'(e) \). By (2.2), we have \( H'(e) = H(e) - H(e - d) \) so this sequence is also left-exact. This proves that \( u \) belongs to the \( U' \) from the statement. Since \( u \) was chosen arbitrarily, we have \( \text{Spec}(A_s) \subseteq U' \) which proves that \( U' \) is open. The injectivity of \( (\mu_f)_s \) and flatness of \( (B/fB)_s \) can be checked after localizing to every maximal ideal of a point \( u \in \text{Spec}(A_s) \) and then follow from the argument given above for \( u_0 \).

\[ \square \]

**Theorem 2.3.** For any Hilbert function \( H : A \to \mathbb{N} \), the set of points of \( \text{Hilb}^H \) which correspond to saturated ideals is open; we denote the resulting open subscheme by \( \text{Hilb}^{H,\text{sat}} \). Moreover, for every morphism \( \varphi : T \to \text{Hilb}^H \) the following are equivalent

1. the (set-theoretic) image of \( \varphi \) is contained in \( \text{Hilb}^{H,\text{sat}} \),
2. the fibers of the universal ideal sheaf pulled back to \( T \) are saturated ideals.

If these conditions holds, then the universal ideal sheaf pulled back to \( T \) is saturated.

**Proof.** By [HS04, Theorem 1.1], the multigraded Hilbert scheme \( \text{Hilb}^H \) is quasi-projective over \( k \), in particular it is locally of finite type over \( k \). Since \( k \) is Noetherian and of finite dimension, the scheme \( \text{Hilb}^H \) has an open cover by affine Noetherian schemes of finite dimension [AM69, Ex 6-7, p. 126]. The universal family is, by definition, admissible with Hilbert function \( H \), so Proposition 2.2 applies and yields openness.

Once we proved openness, the claim about \( \varphi : T \to \text{Hilb}^H \) reduces immediately to the case of \( T = \text{Spec}(L) \), where \( L \) is a field. Let \( \kappa \) be the residue field of the point of \( \text{Hilb}^H \) given by \( T \to \text{Hilb}^H \), so that \( \kappa \subseteq L \) is a field extension. Then the claim is that for a homogeneous ideal \( I \subseteq S_\kappa \) and the ideal \( I' = I \otimes_\kappa L \subseteq S_L \), obtained by a field extension, \( I \) is saturated if and only if so is \( I' \).

Observe that \( S_L/I' \) contains \( S_\kappa/I \) and it is a free \( S_\kappa/I \)-module, since \( L \) is a free \( \kappa \)-module. If \( I' \) is saturated, then so is \( I \) because of the containment. If \( I \) is saturated, then a transverse element \( \ell \in \text{Irr}_\kappa \) is a nonzerodivisor on \( S_\kappa/I \), hence also on the free module \( S_L/I' \), so \( I' \) is saturated.
Finally, in order to show that the pulled back ideal sheaf is saturated, one may replace \( T \) by an open subset \( U' \subseteq T \) as in Proposition 2.2 and apply the final part of that Proposition. \( \square \)

**Definition 2.4.** We define the saturable locus \( \overline{\text{Sat}}^H \) of \( \text{Hilb}^H \) as the closure of \( \text{Hilb}^{H, \text{sat}} \). It is a union of irreducible components of \( \text{Hilb}^H \).

As discussed in the introduction, it is interesting to understand the locus where a deformation of \( I \) induces a deformation of \( I^{\text{sat}} \). To address this, we need to consider deformations of pairs \( I \subseteq I^{\text{sat}} \). The following subsection sets the stage for this (while §3.1 contains the results).

### 2.3. Main diagram

For (any field \( \kappa \) and) ideals \( K \subseteq J \subseteq S_\kappa \) the two long exact sequences coming from \( \text{Hom}_{S_\kappa}(K, -) \) and \( \text{Hom}_{S_\kappa}(-, S_\kappa) \) give the following **main diagram**

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}_{S_\kappa}(K, J_0) & \rightarrow & \text{Hom}_{S_\kappa}(K, S_\kappa)_0 & \rightarrow & \text{Ext}^1_{S_\kappa}(K, J_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Tg}_{\text{flag}} & \rightarrow & \text{Hom}_{S_\kappa}(K, S_\kappa)_0 & \rightarrow & \text{Ext}^1_{S_\kappa}(K, S_\kappa)_0 & \rightarrow & \text{Ob}_{\text{flag}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_{S_\kappa}(J_0, S_\kappa)_0 & \rightarrow & \text{Hom}_{S_\kappa}(K, S_\kappa)_0 & \rightarrow & \text{Ext}^1_{S_\kappa}(J, S_\kappa)_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Ext}^1_{S_\kappa}(K, J_0) & \rightarrow & \text{Ext}^1_{S_\kappa}(J, S_\kappa)_0 & \rightarrow & \text{Ext}^1_{S_\kappa}(K, S_\kappa)_0.
\end{array}
\]

We define the spaces \( \text{Tg}_{\text{flag}} \) and \( \text{Ob}_{\text{flag}} \) as pullbacks of vector spaces: their elements are pairs of elements which map to the same element. Their names are justified by the following theorem. Recall that the flag multigraded Hilbert scheme \( \text{Hilb}_{K \subseteq J} \rightarrow \text{Spec}(\kappa) \) parameterizes pairs of ideals \( K' \subseteq J' \subseteq S_\kappa \) with Hilbert functions \( H_{S_\kappa/K'} = H_{S_\kappa/K} \) and \( H_{S_\kappa/J'} = H_{S_\kappa/J} \), respectively, for every \( \kappa \rightarrow \kappa \). The fibre of \( \text{Hilb}_{K \subseteq J} \rightarrow \text{Spec}(\kappa) \) over the point \( \text{Spec}(\kappa) \rightarrow \text{Spec}(\kappa) \) is the Hilbert scheme parameterizing \( K \subseteq J \subseteq S_\kappa \), where \( S_\kappa \) is fixed. (If \( \kappa \) was a field and \( \kappa = \kappa \), then the fibre is equal to \( \text{Hilb}_{K \subseteq J} \).) The tangent space to this fiber at \( [K \subseteq J] \) is \( \text{Tg}_{\text{flag}} \), see, for example [Jel19, Theorem 4.10]. The flag Hilbert scheme is a closed subscheme of \( \text{Hilb}_K \times \text{Hilb}_J \), so we obtain projections

\[
(2.3) \quad \text{Hilb}_K \xleftarrow{\text{pr}_K} \text{Hilb}_{K \subseteq J} \xrightarrow{\text{pr}_J} \text{Hilb}_J.
\]

Let

\[
\psi: \text{Hom}_{S_\kappa}(K, S_\kappa/K) \oplus \text{Hom}_{S_\kappa}(J, S_\kappa/J) \rightarrow \text{Hom}_{S_\kappa}(K, S_\kappa/J)
\]

be the sum of the natural maps \( \text{Hom}_{S_\kappa}(K, S_\kappa/K), \text{Hom}_{S_\kappa}(J, S_\kappa/J) \rightarrow \text{Hom}_{S_\kappa}(K, S_\kappa/J) \). We prove that if \( \psi_0 \) is surjective, then \( \text{Hilb}_{K \subseteq J} \rightarrow \text{Spec}(\kappa) \) admits an obstruction theory at \( [K \subseteq J] \) with obstruction space \( \text{Ob}_{\text{flag}} \), see Theorem A.2. Since the proof is technical, we delegate it to an appendix. For some smoothness results, we will need the full tangent space to \( [K \subseteq J] \in \text{Hilb}_{K \subseteq J} \), that is, also its image in \( \text{Spec}(\kappa) \). We can control it thanks to Lemma A.4.
Going back to the main diagram, we see that the “linking” group \( \text{Ext}^1_{\mathcal{K}}(K, J/K)_0 \) is the obstruction group for equivariantly deforming \( K \) inside \( J \), in other words, it is the obstruction group of the fiber \( (\text{pr}_J^{-1}([J]), [K]) \), while \( \text{Ext}^1_{\mathcal{S}_\kappa}(J/K, S_\kappa/J)_0 \) is the obstruction group for deforming \( J/K \) in \( S_\kappa/K \), hence the obstruction group of \( (\text{pr}_K^{-1}([K]), [J]) \). We will be principally interested in the map \( \text{pr}_K \), hence we define the fiber obstruction group

\[
(\text{ObFib}) \quad \text{Ob}_\text{fiber}(K, J) := \text{Ext}^1_{\mathcal{S}_\kappa} \left( \frac{J}{K}, \frac{S_\kappa}{J} \right)_0.
\]

The Main Theorem of obstruction calculus (see Lemma A.1) asserts that a map of pointed schemes which induces a surjection of tangent spaces and an injection of obstruction spaces (at the base point) is smooth (at the base point). It follows from the main diagram (and Lemma A.4) that the vanishing of \( \text{Ob}_\text{fiber} \) implies both these conditions for \( \text{pr}_K \) (when \( \psi_0 \) is onto). We will exploit this observation in §3.1.

2.4. Toric varieties setup. In this subsection we restrict the general setup from §2.1 to a more geometric one, namely that of toric varieties. Our general reference for toric varieties are [CLS11, Ful93]. Also [MS04, MS05] are very helpful; in this subsection we do little more than applying them. Finally, the reader uninterested in the general toric case may go straight into Example 2.5.

Let us formally introduce our toric setup. Fix a smooth complete fan \( \Sigma \) corresponding to a projective toric variety. We take \( S \) to be a polynomial ring over \( \mathbb{k} \) with variables given by the rays of \( \Sigma \). The irrelevant ideal \( \text{Irr} = \text{Irr}(\Sigma) \) is the monomial ideal generated by elements \( \prod \{x_i \mid i \notin \sigma \} \) where \( \sigma \) ranges over the maximal cones of \( \Sigma \), see [CLS11, p.207]. The grading is given by the torsion-free abelian group \( \mathcal{A} := \text{Pic}(\Sigma) \), where \( \text{Pic}(\Sigma) \) is defined as in [CLS11, Theorem 4.2.1]. To further distinguish this setup from the general one, we will speak about \( \text{Pic}(\Sigma) \)-homogeneous ideals, rather than abbreviate it to homogeneous ideals.

Arguing as in [CLS11], but relatively over \( \mathbb{k} \), we obtain a smooth projective morphism \( X \to \text{Spec}(\mathbb{k}) \) with toric fibers. All the fibers have the same Nef cone in \( \mathcal{A} \), to which we refer simply as \( \text{Nef}(X) \). Similarly, we refer to \( \text{Pic}(X) \) and \( \text{Irr}(X) \), rather than \( \text{Pic}(\Sigma) \), \( \text{Irr}(\Sigma) \).

Example 2.5. Take \( \Sigma \) to be the fan of the product of projective spaces \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \). In this case, the polynomial ring \( S \) is

\[
S = \mathbb{k}[\alpha_0, \ldots, \alpha_{n_1}, \beta_0, \ldots, \beta_{n_2}, \gamma_0, \ldots, \gamma_{n_3}, \ldots],
\]

the grading is given by \( \deg(\alpha_i) = (1, 0, 0, \ldots, 0) \), \( \deg(\beta_i) = (0, 1, 0, \ldots) \) and so on and the irrelevant ideal is

\[
\text{Irr}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}) = (\alpha_0, \ldots, \alpha_{n_1}) \cdot (\beta_0, \ldots, \beta_{n_2}) \cdot \ldots.
\]

This example is the most important in terms of applications to border apolarity. Even the case \( s = 1 \) is useful.

We now ask how far is a \( \text{Pic}(X) \)-homogeneous ideal \( I \subseteq S_\kappa \) from its saturation. In general, this is a subtle question, see [MS04, MS05]. For any ideal \( I \), we have the following result.

Lemma 2.6. If \( I \) is a \( \text{Pic}(X) \)-homogeneous ideal of \( S_\kappa \), then there exists \( n \in \text{Nef}(X) \) such that \( I_\alpha = I_\alpha^\text{sat} \) for every \( \alpha \in n + \text{Nef}(X) \).

Proof. Let \( M = I^\text{sat}/I \). By [MS04, Corollary 3.8], there exists \( n \in \text{Nef}(X) \) such that \( H^i_B(M)_{n+u} = 0 \) for all \( u \in \text{Nef}(X) \). In particular, \( H^i_B(M)_{n+u} = 0 \) which by the definitions of zeroth local cohomology and saturation implies that \( M_{n+u} = 0 \). \( \square \)
Proposition 2.7. Let $I$ be a homogeneous ideal of $S_{\kappa}$ and $I^{\text{sat}}$ be its saturation. Let $p := \text{pr}_I : \text{Hilb}_{I^{\text{sat}}} \to \text{Hilb}_I$, then $p$ is a locally closed immersion near $[I \subseteq I^{\text{sat}}]$. More precisely, let $U$ be the open neighborhood of $[I \subseteq I^{\text{sat}}] \in \text{Hilb}_{I^{\text{sat}}}$ that consists of pairs $[K \subseteq J]$ with $J$ saturated and let $Z = \text{Hilb}_{I^{\text{sat}}} \setminus U$. Then $p|_U : U \to \text{Hilb}_{I^{\text{sat}}} \setminus p(Z)$ is a closed immersion.

Proof. First, the subset $U$ is equal to $\text{pr}^{-1}_I(\text{Hilb}_{I^{\text{sat}}})$, hence indeed open by Theorem 2.3. Observe that $p^{-1}(p(Z)) = Z$ as closed subsets. Suppose not, then there are points $[I' \subseteq J'] \in U$ and $[I' \subseteq J] \in Z$, that is $J'$ is saturated and $J$ is not. By Lemma 2.6 the Hilbert functions of $I$ and $I^{\text{sat}}$ are equal in all multidegrees far enough into the interior of the Nef cone. Therefore, so are the Hilbert functions of $I'$, $J$ and $J'$. It follows that these three ideals define the same scheme in $X$. As a result, we have $(I')^{\text{sat}} = J^{\text{sat}} = (J')^{\text{sat}} = J'$. The Hilbert functions of $J$ and $J' = J^{\text{sat}}$ are equal, so the inclusion $J \subseteq J^{\text{sat}} = J'$ is an equality, hence $J = J'$. The proof of $p^{-1}(p(Z)) = Z$ is completed.

The morphism $p$ is projective [HS04, Corollary 1.2] and so restricts to a projective morphism $p|_U : U = \text{Hilb}_{I^{\text{sat}}} \setminus p^{-1}(p(Z)) \to \text{Hilb}_I \setminus p(Z)$.

We claim that this morphism is a closed immersion. By [sta23, Tag 04 XV, Tag 05 VH] it is enough to prove that for every point $[I'] \in \text{Hilb}_I \setminus p(Z)$ the fiber $p^{-1}([I'])$ is either empty or maps isomorphically to the point $[I']$. Let $\kappa_0$ be the residue field at $[I']$ and $\kappa$ be its algebraic closure. It is enough to prove the above claim about the fiber after field extension $\kappa_0 \to \kappa$ [sta23, Tag 02 L4]. In particular, we may assume that the base field is algebraically closed and that $[I']$ is closed. Consider the fiber $p^{-1}([I'])$. Arguing as above, we get that as a set the fiber is

- a singleton $[I' \subseteq (I')^{\text{sat}}]$ if $H_{S_{\kappa}/(I')^{\text{sat}}} = H_{S_{\kappa}/I^{\text{sat}}}$,
- empty if $H_{S_{\kappa}/(I')^{\text{sat}}} \neq H_{S_{\kappa}/I^{\text{sat}}}$.

Assume that the fiber is nonempty. Since $\text{Hom}_{S_{\kappa}}((I')^{\text{sat}}/I', S_{\kappa}/(I')^{\text{sat}}) = 0$, by the main diagram 2.3 we get that the tangent map to $p$ at $[I' \subseteq (I')^{\text{sat}}]$ is injective. Hence the fiber has zero tangent space, so as a scheme it is a point. □

2.4.1. Relation to Maclagan-Smith’s and Grothendieck’s Hilbert schemes. We now relate the above to a more geometric setup related to $X$.

Let $P$ be a polynomial in rank Pic($X$) variables. Maclagan and Smith constructed the Hilbert scheme $\text{Hilb}_X^P$ that parameterizes subschemes of $X$ with multigraded Hilbert polynomial equal to $P$, see [MS05, Thm. 6.2]. When $X$ is a projective space, this agrees with Grothendieck’s Hilbert scheme.

We recall the construction, following [MS05, Theorem 6.2].

Fix a Pic($X$)-graded polynomial $P$. By [MS05, Theorem 4.11] there exists an element $n \in \text{Nef}(X)$ such that for every $a \in n + \text{Nef}(X)$ and for every subscheme $Z$ in a fiber $X_{\kappa}$ with multigraded Hilbert polynomial equal to $P$, we have

$$\dim_{\kappa}(S_{\kappa}/I_Z)_a = P(a),$$

where $I_Z$ is the saturated ideal of $Z$.

Fix such an $n$ and consider a function $h : \text{Pic}(X) \to \mathbb{N}$ such that $h(a) = P(a)$ for all $a \in n + \text{Nef}(X)$ and $h(a) = \text{rank}_{\kappa} S_{\kappa}$ otherwise. Consider Haiman-Sturmfels multigraded Hilbert scheme for function $h$. Maclagan-Smith [MS05, Theorem 6.2] prove that for any $n$ as above, the resulting scheme represents $\text{Hilb}_X^P$. (There is a subtlety in this construction, related to the fact that there exist indices $b$ such that
$S_b$ is nonzero and yet $b + \text{Nef}(X) \not\subset \text{Nef}(X)$. It will not be important for us, so we omit discussing it and refer to [MS05] for details.)

For every Hilbert function $H: \text{Pic}(X) \to \mathbb{N}$ such that there exists an associated multigraded Hilbert polynomial $P$, we have a morphism $\text{Hilb}^H \to \text{Hilb}^P(X)$, which is given by forgetting the degrees outside $n + \text{Nef}(X)$.

**Corollary 2.8.** For every Hilbert function $H: \text{Pic}(X) \to \mathbb{N}$ such that there exists an associated multigraded Hilbert polynomial $P$, the morphism

$$p: \text{Hilb}^{H,\text{sat}} \to \text{Hilb}^P(X)$$

is a locally closed immersion.

**Proof.** Fix an $n$ as above. Take a point $[I'] \in \text{Hilb}^{H,\text{sat}}$. Let $I := I'|_{n+\text{Nef}(X)}$. The ideals $I'$ and $I$ define the same subscheme of $X$ and $I'$ is saturated, so $I' = I_{\text{sat}}$. Macalagan-Smith’s result implies that $\text{Hilb}^P(X)$ is isomorphic to $\text{Hilb}_I$. The scheme $\text{Hilb}^{H,\text{sat}}$ is isomorphic to $U$ from Proposition 2.7. Thus the result follows from this Proposition. \hfill \square

### 2.5. Macaulay’s inverse systems.

Throughout the paper, we will frequently consider surjections such as $S_\kappa/I \to S_\kappa/I_{\text{sat}}$ for a field $\kappa$ with $k \to \kappa$. We will also analyse the possible choices of the subideal $I$ for fixed $I_{\text{sat}}$. It seems cleaner to do this dually: to consider superobjects $I^\perp \supseteq (I_{\text{sat}})_{\perp}$ instead. Hence we employ Macaulay’s inverse systems. They are a standard tool in the Artinian case and introduced in the higher-dimensional case at least in [ER17]. Here we loosely summarize exactly as much of the theory as we need.

The idea is quite simple yet requires some notation. Given $S_\kappa = \kappa[\alpha_0, \ldots, \alpha_n]$ we define the dual graded vector space $S_\kappa^* = \kappa_{\text{dp}}[x_0, \ldots, x_n]$ and make it into a graded $S_\kappa$-module by the *contraction* action $S_\kappa \cdot S_\kappa^* \to S_\kappa^*$ defined on a monomial $m \in S_\kappa^*$ by

$$\alpha_i \cdot m = \begin{cases} 0 & \text{if } x_i \text{ does not divide } m, \\ m/x_i & \text{otherwise.} \end{cases}$$

For an ideal $I \subseteq S_\kappa$ we define $I^\perp \subseteq S_\kappa^*$ as $I^\perp = \{ f \in S_\kappa^* | I \cdot f = 0 \}$. The subspace $I^\perp$ is an $S_\kappa$-submodule of $S_\kappa^*$. Similarly, for a submodule $M \subseteq S_\kappa^*$ we define an ideal $M^\perp \subseteq S_\kappa$ as the annihilator of $M$, namely $M^\perp = \{ \sigma \in S_\kappa | \sigma \cdot M = 0 \}$. We have $(-)^{\perp \perp} = \text{id}$, hence $(-)^{\perp}$ gives a bijection between graded ideals of $S_\kappa$ and graded $S_\kappa$-submodules of $S_\kappa^*$. This bijection reverses inclusions and is equivariant with respect to the usual $\text{GL}_{n+1}$-actions. For an element $F \in S_\kappa^*$ we also write $F^\perp$ as a shorthand for $(S_\kappa \cdot F)^\perp$.

**Remark 2.9.** For experts: the subscript in $\kappa_{\text{dp}}$ underlies the fact that $S_\kappa^*$ can be made into a divided powers ring on which $(S_\kappa)_1$ acts as differential operators (we will not use this structure).

In characteristic zero (or large enough) a slightly different, equivalent, action by *partial differentiation* is usually employed. In this setup, we assume that $\kappa$ has characteristic zero, take the polynomial ring $\kappa[x_0, \ldots, x_n]$ (note the lack of subscript $\text{dp}$) and make it into an $S_\kappa$-module where $\alpha_i$ acts as $\frac{\partial}{\partial x_i}$. We denote this action by $\circ$. There is an isomorphism $\kappa[x_0, \ldots, x_n] \to S_\kappa^*$ of $S_\kappa$-modules that sends a monomial $x^a$ to $a! x^a$, and so all the duality claims for $S_\kappa, S_\kappa^*$ hold — still in characteristic zero — also for the pair $S_\kappa, \kappa[x_0, \ldots, x_n]$. See [IK99, Appendix A] or [Jel17] for more details.

We use the following example in §4.4 where we assume that $\kappa$ has characteristic zero. Therefore, we present it under the same assumption and we consider the action $\circ$ of $S_\kappa$ on $\kappa[x_0, \ldots, x_n]$. 
Example 2.10. Assume that $\kappa$ has characteristic zero. Let $L \in \langle x_0, \ldots, x_n \rangle$ be a nonzero linear form and let $I(\{L\})$ be the homogeneous ideal of the point $[L] \in \text{Proj} \kappa[\alpha_0, \ldots, \alpha_n] \simeq \mathbb{P}^n$. For every non-negative integer $i$ we have

$$I(\{L\})_i = \langle L^i \rangle.$$ 

Similarly, if $L_1, \ldots, L_r$ are pairwise different linear forms, then the equality $I(\{L_1, \ldots, L_r\})_i^\perp = \langle L_j \mid 1 \leq j \leq r \rangle$ holds for every non-negative integer $i$.

For further use, we note the following result:

**Proposition 2.11.** Let $F_1, \ldots, F_s$ be homogeneous degree $e$ elements of $S_\kappa[t]$, where $\operatorname{deg}(t) = 0$. Assume that there are homogeneous degree $e$ elements $G_1, \ldots, G_s \in S_\kappa[t]$ such that

1. for every $i$ there exists $d_i \geq 0$ with $t^{d_i}G_i \in \langle F_1, \ldots, F_s \rangle S_\kappa[t]$,
2. $G_1|_{t=0}, \ldots, G_s|_{t=0}$ are linearly independent. (we call them limit forms).

Then $F_1|_{t=\lambda}, \ldots, F_s|_{t=\lambda}$ are linearly independent for general $\lambda$ and the limit of the subspaces $\langle F_1, \ldots, F_s \rangle$ at $t \to 0$ is $\langle G_1|_{t=0}, \ldots, G_s|_{t=0} \rangle$.

**Proof.** Let $N \subseteq (S_\kappa)_\kappa$ be a finite-dimensional $\kappa$-subspace spanned by all monomials in $F_1, \ldots, F_s, G_1, \ldots, G_s$. Let $M := \kappa[t]^{\oplus s}$ and let $\iota: M \to N[t]$ be the $\kappa[t]$-linear map that sends the $i$-th generator to $G_i$. Since $t$ is a nonzerodivisor on $N[t]$, it follows from (2) that $\iota$ is injective and that $t$ is a nonzerodivisor in the $\kappa[t]$-module $N[t]/\text{im } \iota$. By the classification of finitely generated $\kappa[t]$-modules there exists an $f \in \kappa[t] \setminus (t)$ such that after inverting $f$ the homomorphism $\iota$ is injective and $(N[t]/\text{im } \iota)_f$ is a free $\kappa[t]/f$ module. By this freeness it follows that $\iota|_{t=\lambda}$ is injective for every $\lambda$ with $f(\lambda) \neq 0$. This shows that for every nonzero such $\lambda$, the forms $(G_j|_{t=\lambda})_{j=1}^s$ are linearly independent. They are linear combinations of $(F_j|_{t=\lambda})_{j=1}^s$, which are thus linearly independent as well. The last statement is purely formal. $\square$

2.6. **Miscellaneous results.** In this subsection we gather two rather technical yet general results, which form important parts of the arguments in §4 and §3.2, respectively.

**Lemma 2.12** (small tangent space implies large intersection, [Mań22, Lemma 2.6]). Let $X$ be a scheme locally of finite type over a field $\kappa$ and $Z_1, Z_2 \subseteq X$ be irreducible closed subschemes. Let $x \in Z_1 \cap Z_2$ be a $\kappa$-point. Then every irreducible component of $Z_1 \cap Z_2$ has dimension at least

$$\dim Z_1 + \dim Z_2 - \dim T_x X.$$ 

We will also need the following result about derivations. Let $X$ be a $k$-scheme and $\varphi: \text{Spec}(A) \to X$ be a morphism of schemes. The $A$-module

$$\text{Hom}_A(H^0(\varphi^*\Omega_{X/k}), A)$$

is called the module of (pulled back) vector fields on $\text{Spec}(A)$. We have the following description.

**Lemma 2.13** (vector fields and cotangent module). There is a bijection functorial in $A$ between the elements of the module of vector fields on $\text{Spec}(A)$ and the set of morphisms of $k$-schemes $\varphi': \text{Spec}(A[e]/e^2) \to X$ which are equal to $\varphi$ when restricted to $\text{Spec}(A)$.

The proof is standard, yet we could not find a ready reference. Likely the most natural way of arguing is gluing on affine parts, but this seems messy to write down. As a result, the proof is quite hermetic, the reader might like to follow it for affine $X$ first.
Proof. Let $\mathcal{A} := \varphi_*\mathcal{O}_{\text{Spec}(\mathcal{A})}$. To give a morphism $\varphi'$ as in the statement is to give a sheaf-of-$\mathbb{k}$-algebras homomorphism $(\varphi')^\# : \mathcal{O}_X \to \mathcal{A}[\varepsilon]/\varepsilon^2$ that restricts to $\varphi^\#$ when composed with $\mathcal{A}[\varepsilon]/\varepsilon^2 \to \mathcal{A}$. Write $(\varphi')^\# = \varphi^\# + \varepsilon \delta$ for $\delta : \mathcal{O}_X \to \mathcal{A}$. The fact that $(\varphi')^\#$ is a homomorphism is equivalent to the fact that $\delta$ is a $\mathbb{k}$-linear derivation of the $\mathcal{O}_X$-module $\mathcal{A}$. By the universal property of $\Omega_{X/\mathbb{k}}$, such derivations are in bijection with $\text{Hom}_{\mathcal{O}_X}(\Omega_{X/\mathbb{k}}, \mathcal{A})$, which by adjunction is isomorphic $\text{Hom}_{\mathcal{O}_{\text{Spec}(\mathcal{A})}}(\varphi^*\Omega_{X/\mathbb{k}}, \mathcal{O}_A)$, which is in bijection with the module of vector fields on $\text{Spec}(\mathcal{A})$.

3. Main results

We work in the toric setup as in Subsection 2.4. In particular, $X$ is a smooth projective toric variety over a Noetherian ring $\mathbb{k}$.

We start by giving cohomological criteria for the existence of sticky ideals (see introduction). We then study the locus of saturated ideals with a given Hilbert function inside the usual Hilbert scheme of points of $X$.

After that we restrict the setup to the projective space where we are able to do two more things: first, express the obstruction groups from Subsection 3.1 in a more explicit way. Second, in Subsection 3.4 we provide classes of saturated ideals that give smooth points of the corresponding multigraded Hilbert scheme.

3.1. Smoothness of projection maps. Recall that $S$ is the Cox ring of a smooth projective toric variety $X$ over a Noetherian ring $\mathbb{k}$ and $I, J, K$ are Pic($X$)-homogeneous ideals of $S_\mathbb{k}$. If $K \subseteq J$ then recall that $\text{Ob}_{\text{fiber}}(K, J) = \text{Ext}_{S_\mathbb{k}}^1(J/K, S_\mathbb{k}/J)$.

Theorem 3.1. Suppose that we have $\text{Ob}_{\text{fiber}}(K, J) = 0$. Then $\text{pr}_K : \text{Hilb}\{K \subseteq J\} \to \text{Hilb}_K$ is smooth at $[K \subseteq J]$. In particular, if $K \subset J \subseteq K^{\text{sat}}$ then $K$ is nonsaturable.

Proof. Since $\text{Ob}_{\text{fiber}} = 0$, the assumptions of Theorem A.2 are satisfied. By the main diagram §2.3 the obstruction map is injective and the tangent map to $\text{pr}_K|_{\mathbb{k}} : \text{Hilb}_{K \subseteq J}|_{\mathbb{k}} \to \text{Hilb}_K|_{\mathbb{k}}$ is surjective. By Lemma A.4 and the five lemma this implies that the tangent map to $\text{pr}_K : \text{Hilb}_{K \subseteq J} \to \text{Hilb}_K$ is also surjective. This proves that $\text{pr}_K$ is smooth at $[K \subseteq J]$. In particular its image contains an open neighborhood of $[K]$. Suppose that $K$ is saturable and let $K_t$ be a family with special fiber $K$ and general fiber $K_\eta$ such that $K_\eta$ is saturated. Then, after perhaps shrinking the base of the family, we would also get a family $K_t \subseteq J_t$ with special fiber $J$. By Lemma 2.6 there exists $n \in \text{Nef}(X)$ such that $K_a = K_a^{\text{sat}}$ for every $a \in n + \text{Nef}(X)$. As a result, $(J/K)_a = 0$ for such $a$ and the same holds for $(J_\eta/K_\eta)_a$. It follows that $J_\eta^{\text{sat}} = K_\eta^{\text{sat}}$. From this we conclude that $K_\eta \subseteq J_\eta \subseteq J_\eta^{\text{sat}} = K_\eta^{\text{sat}}$ which contradicts the assumption that $K_\eta$ is saturated. □

It may happen that the obstruction group is nonzero, yet the map $\text{pr}_K$ is smooth, so we provide the following refined criterion.

Theorem 3.2. Suppose that $(K, J)$ are such that the map $\text{Hom}_{S_\mathbb{k}}(J, S_\mathbb{k}/J)_0 \to \text{Hom}_{S_\mathbb{k}}(K, S_\mathbb{k}/J)_0$ is surjective and that $[J] \in \text{Hilb}_J$ is a smooth point. Then $\text{pr}_K$ is smooth at $[K \subseteq J]$. In particular, if $K \subset J \subseteq K^{\text{sat}}$ then $K$ is nonsaturable.

Proof. By assumption, the map $\psi_0$ is surjective, so we can apply Theorem A.2. By pullback, the map $d\text{pr}_K|_{\mathbb{k}} : T_\text{flag} \to \text{Hom}_{S_\mathbb{k}}(K, S_\mathbb{k}/J)_0$ is surjective. The map $\text{Ob}_\text{flag} \to \text{Ext}_{S_\mathbb{k}}^1(J, S_\mathbb{k}/J)_0$ is a map of obstruction theories, so the image of any obstruction is the obstruction to deforming $[J]$, which is zero by assumption. Thus the obstructions actually live in a smaller space $O := \ker(\text{Ob}_\text{flag} \to$
we obtain surjectivity on tangent spaces and then

2.2

we see that

and the main diagram, the tangent map to

\[ T \eta 01S4 \]

3.2

for an

4.5

\[ T \eta 025G \]

cannot be used. Yet, for

A.4

3.3

that it is étale on

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\[ (pr,\eta 01S4) \]

\[ (\eta 025G,\eta 01S4) \]

\[ \eta 01S4,\eta 01S4 \]

Theorem 3.4. Let \( I \) be a homogeneous ideal and suppose that \( Ob_{\text{fiber}}(I, I^\text{sat}) = 0 \). Then \( pr_I \) is an open immersion near \( [I \subseteq I^\text{sat}] \). In particular, if \( I \) is non saturated, it is entirely nonsaturable.

Proof. Since \( Ob_{\text{fiber}} = 0 \), the assumptions of Theorem A.2 are satisfied. Let \( \ell \) be a transverse element as in Section 2.2. The multiplication by \( \ell \) is an injective map on \( S_\ell/I^\text{sat} \) and a nilpotent one on \( I^\text{sat}/I \), hence \( \text{Hom}_{S_\ell}(I^\text{sat}/I, S_\ell/I^\text{sat}) = 0 \). By Lemma A.4 and the main diagram, the tangent map to \( pr_I : \text{Hilb}_{I^\text{sat}} \to \text{Hilb}_I \) is bijective, while the obstruction map is injective. This proves that \( pr_I \) is étale at \( [I \subseteq I^\text{sat}] \). In particular its image contains an open neighborhood of \( [I] \), whence if \( I^\text{sat} \neq I \), then \( I \) is entirely nonsaturable. Consider \( U := pr_{I^\text{sat}}^{-1}(\text{Hilb}_{I^\text{sat}}) \) which is an open (Theorem 2.3) neighbourhood of \( [I \subseteq I^\text{sat}] \). For every point \( [I' \subseteq I''] \in U \) we have \( I'' = (I')^\text{sat} \). Thus the map \( (pr_I)|_U : U \to \text{Hilb}_I \) is universally injective [sta23, Tag 01S4]. By shrinking \( U \) to a neighborhood of \( [I \subseteq I^\text{sat}] \), we assume that it is étale on \( U \) as well. By [sta23, Tag 025G] the morphism \( (pr_I)|_U \) is an open immersion.

Again, the vanishing of \( Ob_{\text{fiber}}(I, I^\text{sat}) \) is not necessary for \( I \) to be nonsaturable, see §4.5 for an explicit example. We state a slightly more general version of the theorem.

Theorem 3.4. Suppose that the map \( \text{Hom}_{S_\ell}(I^\text{sat}/I, S_\ell/I^\text{sat})_0 \to \text{Hom}_{S_\ell}(I, S_\ell/I^\text{sat})_0 \) is surjective and that \([I^\text{sat}] \in \text{Hilb}_{I^\text{sat}} \) is a smooth point. Then \( pr_I \) is open immersion near \([I \subseteq I^\text{sat}] \). In particular, if \( I \) is not saturated, it is entirely nonsaturable.

Proof. By Theorem 3.2 the map is smooth and arguing as in the proof of Theorem 3.3 we see that \( pr_I \) is an open immersion near \([I \subseteq I^\text{sat}] \).

Example 3.5. Let \( \kappa \) be a field of characteristic zero and \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) with Cox ring \( S_\kappa = \kappa[\alpha_0, \alpha_1, \beta_0, \beta_1] \) where \( \deg(\alpha_i) = (1, 0) \) and \( \deg(\beta_i) = (0, 1) \). The irrelevant ideal is \( \text{Irr} = (\alpha_0, \alpha_1) \cdot (\beta_0, \beta_1) \). If \( K = (\alpha_0 \alpha_1, \alpha_0 \beta_0, \alpha_0 \beta_1, \beta_0 \beta_1) \) and \( J = K^\text{sat} = (\alpha_0, \beta_0 \beta_1) \), then \( \text{Ext}^1_{S_\kappa}(J/K, S_\kappa/J)_0 = 0 \). Therefore, \( K \) is entirely nonsaturable by Theorem 3.3.

The second example illustrates that choosing \( J \) different than \( \kappa^\text{sat} \) may be useful.

Example 3.6. Let \( \kappa \) be a field of characteristic zero, \( X = \mathbb{P}^2 \) with Cox ring \( S_\kappa = \kappa[\alpha_0, \alpha_1, \alpha_2] \) and consider the ideal \( K = (\alpha_0^2 \alpha_2, \alpha_0 \alpha_2, \alpha_0^3, \alpha_0^2 \alpha_1, \alpha_0^3 \alpha_1, \alpha_0^5, \alpha_1^6) \). We show that \( K \) is nonsaturable. We have \( \text{Ext}^1_{S_\kappa}(K^\text{sat}/K, S_\kappa/K^\text{sat})_0 \neq 0 \) so Theorem 3.3 cannot be used. Yet, for \( J = K^\text{sat} \geq 3 \) we get \( \text{Ext}^1_{S_\kappa}(J/K, S_\kappa/J)_0 = 0 \). Therefore, \( K \) is nonsaturable by Theorem 3.1.

For applications for border rank lower bounds the following variation of the above ideas may be useful. We present two versions: first for an arbitrary smooth projective toric variety and then for the product of projective spaces, since the latter is a main case of interest for applications. The general statement is quite heavy. The informal idea is that \( K \) is a truncation of \( I \), so by replacing \( I \) with \( K \) we reduce the amount of data.

Proposition 3.7. Assume that the ideal \( \text{Irr} \) is minimally generated in degrees \( v_1, \ldots, v_l \). Let \( I \subseteq S_\kappa \) be a Pic(X)-homogeneous ideal and \( A \subseteq \text{Pic}(X) \) be a subset such that \( A + \text{Eff}(X) \subseteq A \). Let \( L \) denote the ideal \( \bigoplus_{a \in A}(S_\kappa)_a \) and \( K = I + L \). If there exist a Pic(X)-graded ideal \( I \subseteq S_\kappa \), a degree \( u \in \text{Eff}(X) \), and positive integers \( k_1, \ldots, k_l \) such that
(1) \( K \subseteq J \),
(2) \( \text{Ob}_{\text{fiber}}(K, J) = 0 \),
(3) \( K_u \subseteq J_u \),
(4) \( K_{u+k_i,v_i} = J_{u+k_i,v_i} \subseteq (S_k)_{u+k_i,v_i} \) for all \( 1 \leq i \leq l \),
then \( I \) is not saturable.

Proof. By part (1) we may consider the flag multigraded Hilbert scheme \( \text{Hilb}_{K \subseteq J} \). As in the proof of Theorem 3.1, it follows from part (2) that there is an open neighborhood \( U \) of \( K \) in \( \text{Hilb}_K \) such that for every \( K' \in U \) there is \( J' \) such that \( K' \subseteq J' \) is a point of \( \text{Hilb}_{K \subseteq J} \). Let \( \pi : \text{Hilb}_J \to \text{Hilb}_K \) be the morphism given on closed points by \( I' \mapsto I' + L \). Let \( V = \pi^{-1}(U) \) and pick \( I' \in V \). Let \( K' = I' + L \) and \( J' \) be such that \( K' \subseteq J' \) is a point in \( \text{Hilb}_{K \subseteq J} \). Let \( \kappa' \) be the residue field at \([K']\). Since \( K'_{u+k_i,v_i} \neq (S_k)_{u+k_i,v_i} \) for all \( i \) we conclude that \( I'_{u} = K'_u \subseteq J'_u \) and \( I'_{u+k_i,v_i} = K'_{u+k_i,v_i} = J'_{u+k_i,v_i} \) for all \( i \). By part (3) there exists \( f \in J'_u \setminus I'_u \) and by part (4) it belongs to \((I')^{\text{sat}}\). Therefore, \( V \) consists of nonsaturated ideals. Thus, \( I \) is nonsaturable. \( \square \)

When \( X \) is the product of projective spaces we obtain the following simplified statement.

**Corollary 3.8.** Let \( X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \) and \( S \) be its Cox ring (see Example 2.5). Let \( I \subseteq S_\kappa \) be a Pic\((X)\)-homogeneous ideal and \( A \subseteq \text{Pic}(X) \) be a subset such that \( A + \text{Eff}(X) \subseteq A \). Let \( L \) denote the ideal \( \bigoplus_{a \in A} (S_\kappa)_a \) and \( K = I + L \). If there exist a Pic\((X)\)-graded ideal \( J \subseteq S_\kappa \) a degree \( u \in \mathbb{Z}_{\geq 0}^s \) and a positive integer \( k \) such that

(1) \( K \subseteq J \),
(2) \( \text{Ob}_{\text{fiber}}(K, J) = 0 \),
(3) \( K_u \subseteq J_u \),
(4) \( K_{u+k_1,1,...,1} = J_{u+k_1,1,...,1} \subseteq (S_\kappa)_{u+k_1,1,...,1} \),

then \( I \) is not saturable.

### 3.2. Saturated locus in the multigraded Hilbert schemes

In this subsection we study the saturated locus of the multigraded Hilbert scheme.

As above, throughout this subsection, \( X \) is a smooth projective toric variety over a Noetherian ring \( k \) with Cox ring \( S \). The symbol \( H \) denotes a Hilbert function admissible for the saturated ideal of a length \( d \) zero-dimensional subscheme of \( X \). We use notation introduced in Subsection 2.4.1.

Let \( P \) be the multigraded Hilbert polynomial of \( H \). There is a natural map

\[
\text{Hilb}^P(X) \to \text{Hilb}^d(X),
\]

that sends an ideal \( I \subseteq S_\kappa \) to the subscheme \( V(I) \subseteq X_\kappa \), see [CLS11, §5.2] for the construction of \( V(I) \). Furthermore, as discussed in Subsection 2.4 there is a natural restriction morphism \( \text{Hilb}^H \to \text{Hilb}^P(X) \) and therefore, we obtain a morphism

\[
\text{Hilb}^H \to \text{Hilb}^d(X).
\]

This allows us to identify the saturated locus in \( \text{Hilb}^H \) with a locally closed subscheme of the usual Hilbert scheme as follows.

**Proposition 3.9.** Let \( P \) be a multigraded polynomial equal to a constant \( d \). Then, the map \( \text{Hilb}^P(X) \to \text{Hilb}^d(X) \) is an isomorphism. In particular, for every Hilbert function \( H \) with multigraded Hilbert polynomial \( P \), the natural map \( \text{Hilb}^{H,\text{sat}} \to \text{Hilb}^d(X) \) is a locally closed immersion with image consisting of subschemes that have Hilbert function \( H \).
Proof. It is enough to construct an inverse. Take a $k$-algebra $R$, an $R$-point of $\text{Hilb}^d(X)$, the associated flat family $Z \subseteq X \times_k \text{Spec}(R)$, and its ideal sheaf $\mathcal{I}_Z$. Let $I_Z \subseteq S_R$ be the homogeneous ideal of $Z$; it is obtained as $I_Z = \bigoplus_{a \in \Phi(X)} H^0(X_R, \mathcal{I}_Z(a))$. We have an exact sequence

\begin{align}
0 \to (I_Z|_{\text{Nef}(X)}) \to S_R|_{\text{Nef}(X)} \to \bigoplus_{a \in \text{Nef}(X)} H^0(X_R, \mathcal{O}_Z(a)) \to \\
\to \bigoplus_{a \in \text{Nef}(X)} H^1(X_R, \mathcal{I}_Z(a)) \to \bigoplus_{a \in \text{Nef}(X)} H^1(X_R, \mathcal{O}_Z(a)).
\end{align}

(3.1)

Fix any $a \in n + \text{Nef}(X)$, where $n$ was defined above (2.5). If $R$ is a field, then (2.5) implies that $(S_R/I_Z)_a$ has rank $d$, the same as $H^0(X_R, \mathcal{O}_Z(a))$. Moreover, in this case $H^1(X_R, \mathcal{O}_Z(a))$ is zero, see [CLS11, Theorem 9.2.3] for $R = \mathbb{C}$ and [ABKW20, Theorem 3.6] in general. This implies that $H^1(X_R, \mathcal{I}_Z(a))$ is zero in this case. For general $R$, by Cohomology and Base Change [Con00, Lemma 5.1.1] applied to the sheaf $\mathcal{I}_Z(a)$, we obtain that $H^j(X_R, \mathcal{I}_Z(a)) = 0$ for every $j \geq 1$. Given that, the sequence (3.1) becomes short exact and $R \rightarrow (I_Z)|_{n+\text{Nef}(X)}$ yields the required inverse map $\text{Hilb}^d(X) \to \text{Hilb}^P(X)$ and thus the isomorphism. The final claims follow from Corollary 2.8. \qed

Recall that $\text{Hilb}^d(X)$ has a distinguished component $\text{Hilb}^{d,\text{sm}}(X)$, called the smoothable component. The general point $[\Gamma]$ of this component corresponds to a reduced $\Gamma$, that is, to a tuple of $d$ points of $X$. Even more, the set of points $[\Gamma] \in \text{Hilb}^d(X)$ such that $\Gamma$ is smooth, is open. For the purposes of this discussion, we call it the locus of smooth subschemes of $\text{Hilb}^d(X)$.

**Definition 3.10.** The Scheme of Limits of Ideals of Points $\text{Slip}^H \subseteq \text{Hilb}^H$ is the closure of the open subscheme: the intersection of $\text{Hilb}^{H,\text{sat}}$ and the preimage of the locus of smooth subschemes under the natural morphism $\text{Hilb}^H \to \text{Hilb}^d(X)$ constructed in Subsection 2.4. For $[I] \in \text{Hilb}^H$, we also denote $\text{Slip}^H$ by $\text{Slip}_I$.

By definition, a general point of $\text{Slip}^H$ corresponds to a saturated ideal $I^{\text{sat}}$ such that $V(I^{\text{sat}})$ is a tuple of points with Hilbert function $H$. The map $\text{Slip}^H \to \text{Hilb}^{d,\text{sm}}(X)$ can be described a bit more precisely when $H$ is the Hilbert function of a general $d$-tuple of points in $X$, that is, $H(a) = \min(d, \text{rank}_a S_a)$ for every $i$. For a zero-dimensional subscheme $\Gamma$ in a smooth quasi-projective $\kappa$-scheme $Y$, we say that $\Gamma$ is unobstructed if $[\Gamma] \in \text{Hilb}^d(Y)$ is a smooth point, where $d = \dim_{\kappa} H^0(\mathcal{O}_Y)$ and $\text{Hilb}^d(Y)$ is a Grothendieck Hilbert scheme. Unobstructedness is an intrinsic property of $\Gamma$, it does not depend on the embedding.

**Corollary 3.11.** Let $H$ be the Hilbert function of a general $d$-tuple of points in $X$. Let $[I] \in \text{Slip}^H$ be such that $V(I^{\text{sat}}) \subseteq X_\kappa$ is unobstructed and the fiber of the map $\text{Slip}^H|_\kappa \to \text{Hilb}^d(X|_\kappa)$ over $V(I^{\text{sat}})$ is not a point. Then there is a divisor $E \subseteq \text{Slip}^H|_\kappa$ with $[I] \in E$ such that the image of $E$ in $\text{Hilb}^d(X)|_\kappa$ has codimension at least two.

**Proof.** The locally closed immersion from Proposition 3.9 defines a birational morphism $\text{Slip}^H|_\kappa \to \text{Hilb}^{d,\text{sm}}(X|_\kappa)$. Indeed, its restriction to the inverse image of the locus $\mathcal{V}$ of all smooth subschemes with Hilbert function $H$ is a surjective locally closed immersion onto the smooth scheme $\mathcal{V}$ and thus an isomorphism. The claim follows from applying [Sha94, Theorem 2, p.120] to this map. \qed

3.3. **Fiber obstruction group for saturation.** In this subsection we analyse the obstruction group further. We keep the toric setup as in the previous subsections, however in Subsection 3.3.1 we restrict to the projective space where more interesting results are obtained.
Let \( I \) be a homogeneous ideal in \( S_\kappa \). For the pair \((K, J) := (I, I^{\text{sat}})\), the fiber obstruction group from (ObFib) can be “underived” as follows. Choose a transverse element \( \ell \) as in Subsection 2.2 and let \( C(I^{\text{sat}}) \) be defined by the exact sequence

\[
0 \to \frac{S_\kappa}{I^{\text{sat}}} \to \left(\frac{S_\kappa}{I^{\text{sat}}}\right)_\ell \to C(I^{\text{sat}}) \to 0.
\]

Applying \( \text{Hom}(I^{\text{sat}}/I, -) \) we obtain a long exact sequence. The multiplication by a large enough power of \( \ell \) annihilates \( I^{\text{sat}}/I \) and is an isomorphism on \((S_\kappa/I^{\text{sat}})_\ell\), hence \( \text{Ext}^1_{S_\kappa}(I^{\text{sat}}/I, (S_\kappa/I^{\text{sat}})_\ell) = 0 \) and the long exact sequence induces a homogeneous isomorphism

\[
\text{Ext}^1_{S_\kappa}(\frac{I^{\text{sat}}}{I}, \frac{S_\kappa}{I^{\text{sat}}}) \simeq \text{Hom}\left(\frac{I^{\text{sat}}}{I}, C(I^{\text{sat}})\right).
\]

We can refine the above taking into account degrees. For \( m, e \in A = \text{Pic}(X) \) we say that \( m \leq e \) if \( e - m \) lies in the effective cone of \( X \). Fix degrees \( m \leq e \). Applying \( \text{Hom}(I^{\text{sat}}_{\geq m}/I_{\geq e}, -) \) to \( 0 \to (S_\kappa/I^{\text{sat}})_{\geq m} \to ((S_\kappa/I^{\text{sat}})_\ell)_{\geq m} \to C(I^{\text{sat}})_{\geq m} \to 0 \) we arrive at the homogeneous isomorphism

\[
\text{Ext}^1_{S_\kappa}\left(\frac{I^{\text{sat}}_{\geq m}}{I_{\geq e}}, \frac{S_\kappa}{I^{\text{sat}}_{\geq m}}\right) \simeq \text{Hom}\left(\frac{I^{\text{sat}}_{\geq m}}{I_{\geq e}}, C(I^{\text{sat}})_{\geq m}\right).
\]

By construction, the module \( M := I^{\text{sat}}_{\geq m}/I_{\geq e} \) is generated in degrees \( \geq m \). The same is true for modules in the minimal resolution of \( M \) since the ring \( S_\kappa \) is positively graded. The \( \text{Ext} \) groups are computed from this resolution, so if we restrict to homogeneous maps, we can drop the subscripts in (3.4) to obtain

\[
\text{Ext}^1_{S_\kappa}\left(\frac{I^{\text{sat}}_{\geq m}}{I_{\geq e}}, \frac{S_\kappa}{I^{\text{sat}}_{\geq m}}\right)_0 \simeq \text{Hom}\left(\frac{I^{\text{sat}}_{\geq m}}{I_{\geq e}}, C(I^{\text{sat}})_{\geq e}\right)_0.
\]

3.3.1. Case of the projective space. In this subsection we assume that \( X \) is a projective space, so that \( S \) becomes standard \( \mathbb{N} \)-graded polynomial ring. As always, take \( I \subseteq S_\kappa \). In this setup the \( S_\kappa/I^{\text{sat}} \)-module \( C(I^{\text{sat}}) \) is pleasantly explicit. If \( \dim S_\kappa/I^{\text{sat}} = 1 \) it admits connections to the canonical module \( \omega_{S_\kappa/I^{\text{sat}}} \) [ILL+07, §13.1], which are useful since the latter is sometimes even easier to describe. We discuss this connection now.

Recall the graded dual operator: for a graded \( \kappa \)-vector space \( M = \bigoplus e M_e \) with every \( M_e \) finite-dimensional we define \( ^*\text{Hom}(M, \kappa) := \bigoplus \text{Hom}(M_{-e}, \kappa) \subseteq \text{Hom}(M, \kappa) \). The inclusion is an equality if \( M \) has finite dimension but not in general. If \( M \) is an \( S_\kappa \)-module, then \( ^*\text{Hom}(M, \kappa) \) is an \( S_\kappa \)-module as well by \( (s \cdot \varphi)(m) := \varphi(sm) \) and the canonical map \( M \to ^*\text{Hom}(^*\text{Hom}(M, \kappa), \kappa) \) is an isomorphism of \( S_\kappa \)-modules.

**Proposition 3.12.** Suppose that \( S_\kappa/I^{\text{sat}} = 1 \). Then the module \( C(I^{\text{sat}}) \) is isomorphic to \( ^*\text{Hom}(\omega_{S_\kappa/I^{\text{sat}}}, \kappa) \) and to the top local cohomology of \( S_\kappa/I^{\text{sat}} \).

**Proof.** Let \( \ell_1 \) be a transversal element for \( I \). Since \( S_\kappa/I^{\text{sat}} \) is one-dimensional, we may pick \( \dim S_\kappa - 1 \) elements \( \ell_2, \ldots, \ell_{\dim S_\kappa} \) of \((S_\kappa)_+\) that annihilate \( S_\kappa/I^{\text{sat}} \) and such that the radical of \((\ell_1, \ell_2, \ldots, \ell_{\dim S_\kappa})\) is \((S_\kappa)_+\). Using properties of local cohomology [ILL+07, Proposition 7.3(b), Theorem 7.13], we conclude that \( C(I^{\text{sat}}) \simeq H^1_{(S_\kappa)_+}(S_\kappa/I^{\text{sat}}) \). Since \( S_\kappa/I^{\text{sat}} \) is one-dimensional, by [ILL+07, Theorem 13.5] we obtain \( ^*\text{Hom}(C(I^{\text{sat}}), \kappa) \simeq \omega_{S_\kappa/I^{\text{sat}}} \). Applying \( ^*\text{Hom}(-, \kappa) \) to this isomorphism we get the claim. \( \square \)
The above Proposition 3.12 becomes easier in the Gorenstein case. Recall that \( S_\kappa/I^{\text{sat}} \) is Gorenstein if for some (or every) transverse element \( \ell \) the socle \( S_\kappa/(I^{\text{sat}} + (\ell)) \) is one-dimensional [Eis95, Proposition 21.5].

**Corollary 3.13** (Gorenstein case). Suppose that \( S_\kappa/I^{\text{sat}} \) is one-dimensional and Gorenstein. Then the module \( C(I^{\text{sat}}) \) is isomorphic to \( \ast\text{Hom}(S_\kappa/I^{\text{sat}}, \kappa)(-a) \) where \( a \) is the largest index such that \( H_{S_\kappa/I^{\text{sat}}}(a) \neq H_{S_\kappa/I^{\text{sat}}}(a+1) \).

**Proof.** It follows from Grothendieck’s local duality [ILL+07, Theorem 18.7] and from Proposition 3.12 that \( C(I^{\text{sat}}) \simeq \ast\text{Hom}(S_\kappa/I^{\text{sat}}, \kappa)(-a) \) for a certain \( a \). It follows from this isomorphism that \( a \) is the largest degree in which \( C(I^{\text{sat}}) \) is nonzero. From (3.2) it follows that this agrees with the \( a \) in the statement. \( \square \)

**Corollary 3.14.** Suppose that \( I^{\text{sat}} \) is one-dimensional and Gorenstein. Then

\[
\text{Ob}_{\text{fiber}}(I, I^{\text{sat}}) \simeq \text{Hom}_\kappa \left( \left( \frac{I^{\text{sat}}}{I + (I^{\text{sat}})^2} \right)_a, \kappa \right).
\]

**Proof.** Let \( R = S_\kappa/I^{\text{sat}} \). Using Corollary 3.13 and (3.3) we obtain that

\[
\text{Ob}_{\text{fiber}}(I, I^{\text{sat}}) \simeq \text{Hom}_R \left( \frac{I^{\text{sat}}}{I + (I^{\text{sat}})^2}, C(I^{\text{sat}}) \right)_0 \simeq \text{Hom}_R \left( \frac{I^{\text{sat}}}{I + (I^{\text{sat}})^2}, \ast\text{Hom}(R, \kappa)(-a) \right)_0 \\
\simeq \text{Hom}_R \left( \frac{I^{\text{sat}}}{I + (I^{\text{sat}})^2}, \ast\text{Hom}(R, \kappa) \right)_{-a} \simeq \text{Hom}_\kappa \left( \frac{I^{\text{sat}}}{I + (I^{\text{sat}})^2}, \kappa \right)_{-a} \simeq \text{Hom}_\kappa \left( \left( \frac{I^{\text{sat}}}{I + (I^{\text{sat}})^2} \right)_a, \kappa \right),
\]

where (*) follows from the graded version of the tensor-Hom adjunction. \( \square \)

### 3.4. Smoothness of the saturated locus in the standard graded case

In this subsection we consider the toric setup but restrict to \( X = \mathbb{P}^n \), so that \( S \) is standard \( \mathbb{N} \)-graded.

In this setup we provide classes of saturated ideals which yield smooth points of the multigraded Hilbert schemes. In addition to being interesting on its own, understanding these cases increases the applicability of Theorem 3.4.

For a saturated homogeneous one-dimensional ideal \( I^{\text{sat}} \subseteq S_\kappa \) with a transverse element \( \ell \) which is a linear form we fix a decomposition \( S_\kappa = S'[\ell] \), where \( S' \) is a polynomial subring. We define the one-parameter family for \( (I^{\text{sat}}, \ell, S') \) as

\[
\text{Spec}(S_\kappa/I^{\text{sat}}) \rightarrow \text{Spec}(S_\kappa) \simeq \text{Spec}(S') \times \mathbb{A}^1 \\
\pi \downarrow \text{pr}_2 \\
\mathbb{A}^1 = \text{Spec}(\kappa[\ell])
\]

where \( \pi \) is induced by the inclusion \( \kappa[\ell] \rightarrow S_\kappa/I^{\text{sat}} \). In the projective space \( \text{Proj}(S_\kappa) \), the quotient \( \text{Proj}(S_\kappa/I^{\text{sat}}) \) corresponds to a zero-dimensional subscheme disjoint from the hyperplane \( (\ell = 0) \). It follows that for every nonzero \( \lambda \in \kappa \) the scheme \( \pi^{-1}(\lambda) = \text{Spec}(S_\kappa/(I^{\text{sat}} + (\ell - \lambda))) \) is isomorphic to \( \text{Proj}(S_\kappa/I^{\text{sat}}) \).

**Lemma 3.15.** For every \( (I^{\text{sat}}, \ell, S') \) as above, the map \( \pi \) is finite and flat.

**Proof.** The \( \kappa \)-algebra \( S_\kappa/(I^{\text{sat}} + (\ell)) \) is finitely generated and zero-dimensional, hence a finite-dimensional \( \kappa \)-vector space. Fix its basis \( \mathcal{B} \). Since \( S_\kappa/I^{\text{sat}} \) is \( \mathbb{N} \)-graded, induction by degree proves that \( S_\kappa/I^{\text{sat}} \) is spanned by \( \mathcal{B} \) as a \( \kappa[\ell] \)-module, so \( \pi \) is finite. Suppose that there is a nonzero element of \( S_\kappa/I^{\text{sat}} \) annihilated by some nonzero polynomial in \( \kappa[\ell] \). Taking leading forms, we find a nonzero element of
Suppose that \( \ell \) is a linear form transverse to a saturated one-dimensional ideal \( I_{\text{sat}} \) and that \( \text{Spec}(S_\kappa/(I_{\text{sat}} + (\ell))) \subseteq \text{Spec}(S_\kappa) \) is unobstructed. Then the natural map

\[
\frac{\text{Hom}_{S_\kappa}(I_{\text{sat}}, S_\kappa)}{\ell \cdot \text{Hom}_{S_\kappa}(I_{\text{sat}}, S_{\text{sat}})} \to \text{Hom}_{S_\kappa}\left(I_{\text{sat}}, \frac{S_\kappa}{I_{\text{sat}} + (\ell)}\right)
\]

is an isomorphism.

The theorem does not hold under the slightly weaker assumption that \( \text{Proj}(S_\kappa/I_{\text{sat}}) \simeq \text{Spec}(S_\kappa/(I_{\text{sat}} + (\ell - 1))) \subseteq \text{Proj}(S_\kappa) \) is unobstructed; one counterexample is a general tuple of 8 points on \( \mathbb{P}^3 \).

**Proof.** By left-exactness of \( \text{Hom} \), it is enough to prove that \( \text{Hom}_{S_\kappa}(I_{\text{sat}}, S_\kappa/I_{\text{sat}}) \to \text{Hom}_{S_\kappa}(I_{\text{sat}}, S_\kappa/(I_{\text{sat}} + (\ell))) \) is surjective. The family \( \pi \) above provides a map \( \tilde{\pi} : \mathbb{A}^1 \to \text{Hilb}^d(\text{Spec}(S')) \), where \( d = \dim \kappa \frac{S_\kappa}{I_{\text{sat}} + (\ell)} \).

By assumption, the point \( \tilde{\pi}(0) \in \text{Hilb}^d(\text{Spec}(S')) \) is smooth, hence a general point of the image is smooth as well. But all \( \kappa \)-points of \( \tilde{\pi}^{-1}(0) \) correspond to different embeddings of the same abstract scheme \( \text{Proj}(S_\kappa/I_{\text{sat}}) \), hence we conclude that every point in the image of \( \tilde{\pi} \) is smooth. In particular, the cotangent sheaf of \( \text{Hilb}^d(\text{Spec}(S')) \) pulled back to \( \mathbb{A}^1 \) is a free sheaf of \( \mathbb{A}^1 \). As a result, we obtain surjectivity of the natural restriction map from the module of pulled back vector fields on \( \text{Spec}(\kappa[\ell]) \) to such a module on \( \text{Spec}(\kappa[\ell]/(\ell)) \).

By Lemma 2.13 and a generalization of [Str96, Theorem 10.1], we conclude that the \( \kappa[\ell] \)-module of (pulled back) vector fields for \( \tilde{\pi} \) is \( \text{Hom}_{S_\kappa}(I_{\text{sat}}, S_\kappa/I_{\text{sat}}) \) and that this is a free \( \kappa[\ell] \)-module. Applying the same Lemma and reference to \( \text{Spec}(\kappa) = (\ell = 0) \to \text{Hilb}^d(\text{Spec}(S')) \) we obtain \( \text{Hom}_{S'}(I_{\text{sat}}/(I_{\text{sat}} \cap (\ell)), S_\kappa/(I_{\text{sat}} + (\ell))) \). As mentioned above, the restriction map of modules of vector fields

\[
\text{Hom}_{S_\kappa}\left(I_{\text{sat}}, \frac{S_\kappa}{I_{\text{sat}}+\ell}\right) \to \text{Hom}_{S'}\left(I_{\text{sat}}/(I_{\text{sat}} \cap (\ell)), 
\frac{S_\kappa}{I_{\text{sat}}+\ell}\right)
\]

is surjective. Since \( \ell \) is a nonzerodivisor in \( S_\kappa/I_{\text{sat}} \), we have \( I_{\text{sat}} \cap (\ell) = I_{\text{sat}} \cdot (\ell) \), so the natural map

\[
\text{Hom}_{S_\kappa}\left(I_{\text{sat}}, \frac{S_\kappa}{I_{\text{sat}}+\ell}\right) \to \text{Hom}_{S'}\left(I_{\text{sat}}/(I_{\text{sat}} \cap (\ell)), \frac{S_\kappa}{I_{\text{sat}}+\ell}\right)
\]

is bijective. It follows that the restriction \( \text{Hom}_{S_\kappa}(I_{\text{sat}}, S_\kappa/I_{\text{sat}}) \to \text{Hom}_{S_\kappa}(I_{\text{sat}}, S_\kappa/(I_{\text{sat}} + (\ell))) \) is surjective as well.

**Corollary 3.17.** Under the assumptions of Proposition 3.16, the multiplication by \( \ell \) on \( \text{Ext}^1_{S_\kappa}(I_{\text{sat}}, S_\kappa/I_{\text{sat}}) \) is injective. Consequently, for every subideal \( K \subseteq I_{\text{sat}} \) such that \( I_{\text{sat}} \cdot (\ell^{\geq 0}) \subseteq K \) the natural map

\[
\text{Ext}^1_{S_\kappa}(I_{\text{sat}}/K, S_\kappa/I_{\text{sat}}) \to \text{Ext}^1_{S_\kappa}(I_{\text{sat}}, S_\kappa/I_{\text{sat}})
\]

is zero.

**Proof.** Applying \( \text{Hom}_{S_\kappa}(I_{\text{sat}}, -) \) to the short exact sequence

\[
0 \to \frac{S_\kappa}{I_{\text{sat}}+1} \to S_\kappa/I_{\text{sat}} \to \frac{S_\kappa}{I_{\text{sat}}+\ell} \to 0
\]

and using Proposition 3.16 we obtain that the multiplication by \( \ell \) on \( \text{Ext}^1_{S_\kappa}(I_{\text{sat}}, S_\kappa/I_{\text{sat}}) \) is injective. The multiplication by \( \ell \) on \( I_{\text{sat}}/K \) is nilpotent, so it is nilpotent on \( \text{Ext}^1_{S_\kappa}(I_{\text{sat}}/K, S_\kappa/I_{\text{sat}}) \) as well, whence the claim.

**Theorem 3.18** (unobstructedness of Artinian reduction implies smoothness in multigraded setting). In this theorem we assume that \( \kappa = \kappa \) is a field (but we keep using \( \kappa \) for notational consistency). Let \( S_\kappa \)}
be a standard graded polynomial ring and $I^\text{sat} \subseteq S_\kappa$ be a saturated one-dimensional homogeneous ideal with transverse element $\ell$ which is a linear form. If $\text{Spec}(S_\kappa/(I^\text{sat} + (\ell))) \subseteq \text{Spec}(S_\kappa)$ is unobstructed, then $[I^\text{sat}] \in \text{Hilb}_{I^\text{sat}}$ is a smooth point.

Proof. Let $d = \dim_\kappa S_\kappa/(I^\text{sat} + (\ell))$ and fix $e \gg 0$. Consider the canonical map defined in §3.2:

$$\xymatrix{ \text{Hilb}_{I^\text{sat}} \ar[r]^-{\sim} & \text{Hilb}_{I^\text{sat} \subseteq I^\text{sat}} \ar[r]^-{\text{pr}_{I^\text{sat}}} & \text{Hilb}_{I^\text{sat} \subseteq I^\text{sat}} \ar[r]^-{\sim} & \text{Hilb}^d/(\mathbb{P}^{\dim S_\kappa-1}). }$$

The natural map $\text{Hom}_{S_\kappa}(I^\text{sat}_{\geq e}, S_\kappa/I^\text{sat}_{\geq e})_0 \to \text{Hom}_{S_\kappa}(I^\text{sat}_{\geq e}, S_\kappa/I^\text{sat})_0$ is surjective, hence Theorem A.2 applies and shows that $[I^\text{sat}_{\geq e}] \in \text{Hilb}_{I^\text{sat}}$ has an obstruction group $\text{Ob}_{\text{flag}}$ given by

$$\xymatrix{ \text{Ob}_{\text{flag}} \ar[r] \ar@{-->}[d] & \text{Ext}_S^1(I^\text{sat}, S_{I^\text{sat}})_0 \ar[d] \ar[r] & \text{Ext}_S^1(I^\text{sat}_{\geq e}, S_{I^\text{sat}})_0. }$$

Using the one-parameter family $\pi$ and the fact that unobstructedness is open, we conclude that $\text{Spec}(S_\kappa/(I^\text{sat} + (\ell - 1))) \subseteq \text{Spec}(S_\kappa)$ is unobstructed, hence also $\text{Proj}(S_\kappa/I^\text{sat}) \subseteq \text{Proj}(S_\kappa)$ is unobstructed, so $[I^\text{sat}] \in \text{Hilb}_{I^\text{sat}}$ is a smooth point. This means that for any obstruction at $[I^\text{sat}_{\geq e}] \subseteq \text{Hilb}_{I^\text{sat}}$ arising from any small extension, its image under the left vertical map is zero. By Corollary 3.17, the right vertical map is injective. But then it follows that every such obstruction is zero, hence $[I^\text{sat}] \in \text{Hilb}_{I^\text{sat}}$ is smooth.

Corollary 3.19. Assume that $\kappa = \kappa$ is a field. Suppose that $I^\text{sat} \subseteq S_\kappa$ is a homogeneous saturated ideal and that one of the following holds

1. $S_\kappa$ is at most three-dimensional (so that $\text{Proj}(S_\kappa) = \mathbb{P}^2$ or $\mathbb{P}^1$),
2. $S_\kappa$ is four-dimensional (so that $\text{Proj}(S_\kappa) \simeq \mathbb{P}^3$) and $S_\kappa/I^\text{sat}$ is Gorenstein,
3. $I^\text{sat}$ is a complete intersection.

Then $I^\text{sat} \in \text{Hilb}_{I^\text{sat}}$ is unobstructed.

This corollary is classical and proven, for example, in the references given in the proof.

Proof. To prove smoothness, we may enlarge the base field $\kappa$, so we assume it is infinite. Then there exists a linear form $\ell$ transverse to $I^\text{sat}$. The ideal $I^\text{sat}/(I^\text{sat} \cap (\ell)) \subseteq S_\kappa/(\ell)$ is respectively

1. an ideal in $S_\kappa/(\ell) \simeq \kappa[\alpha_1, \alpha_2]$ or in $\kappa[\alpha_1]$,
2. a Gorenstein ideal in $\kappa[\alpha_1, \alpha_2, \alpha_3]$,
3. a complete intersection ideal.

In these cases unobstructedness of $S_\kappa/(I^\text{sat} + (\ell))$ follows from classical results: Hilbert-Burch Theorem [Sch77, Theorem 1, Corollary 1], Buchsbaum-Eisenbud theorem [Har10, Theorem 9.7] and finally [Har10, Theorem 9.2], so the results follow from Theorem 3.18.

4. Examples

In this section we assume that $\kappa = \kappa$ is a field.
4.1. **The fiber obstruction group for cases with** \( r = 1 \). Let \( S_{\kappa} \) be standard \( \mathbb{N} \)-graded, with \( \kappa \) infinite. Let \( I \subseteq S_{\kappa} \) be homogeneous and let \( a \) be the largest degree such that \( \dim_{\kappa}(S_{\kappa}/I^{\text{sat}})_{a} \neq \text{deg}(S_{\kappa}/I^{\text{sat}}) \). As proven in Corollary 3.14 if \( S_{\kappa}/I^{\text{sat}} \) is Gorenstein, then \( \text{Ob}_{\text{fiber}} := \text{Ob}_{\text{fiber}}(I, I^{\text{sat}}) \) is dual to the degree \( a \) part of \( I^{\text{sat}}/(I + (I^{\text{sat}})^2) \).

**Example 4.1** (Line). In the special case when \( I^{\text{sat}} \) is a degree \( d \) subscheme of a line, we have \( a = d - 2 \). In this case \( H_{S_{\kappa}/I^{\text{sat}}}(d - 2) = d - 1 \), \( H_{S_{\kappa}/I^{\text{sat}}}(d - 1) = d \) and \( \text{Ob}_{\text{fiber}} = 0 \) if and only if the ideals \( I^{\text{sat}} \) and \( I + (I^{\text{sat}})^2 \) agree in degree \( d - 2 \). Assuming additionally that \( H_{S_{\kappa}/I} \) is nondecreasing, we get \( \dim_{\kappa}(I^{\text{sat}}/I)_{d - 2} \leq 1 \), hence \( \text{Ob}_{\text{fiber}} \neq 0 \) only if \( (I^{\text{sat}})^2_{d - 2} \subseteq I_{d - 2} \subseteq (I^{\text{sat}})_{d - 2} \).

**Example 4.2** (Line and some general points). Generalizing Example 4.1, consider the case where \( S_{\kappa}/I^{\text{sat}} \) has Hilbert function \((1, e + 2, e + 3, e + 4, \ldots)\) and becomes a degree \( d \) subscheme of a line, and there exists an \( J \supseteq I \) with \( S_{\kappa}/J^{\text{sat}} \) having Hilbert function \((1, 2, 3, 4, \ldots)\). A choice of an element \( \ell \) transversal both for \( I \) and \( J \) yields a surjective map \( C(I^{\text{sat}}) \to C(J^{\text{sat}}) \). The spaces \( C(I^{\text{sat}}) \), \( C(J^{\text{sat}}) \) have the same Hilbert functions in positive degrees, so this map is bijective in positive degrees. The quotient \( S_{\kappa}/J^{\text{sat}} \) is Gorenstein, so we have that

\[
\text{Ob}_{\text{fiber}}(I) \simeq \text{Hom}_{S_{\kappa}} \left( \frac{I^{\text{sat}}}{I}, C(I^{\text{sat}})_{\geq 1} \right) \simeq \text{Hom}_{S_{\kappa}} \left( \frac{I^{\text{sat}}}{I + J^{\text{sat}}}, C(J^{\text{sat}})_{\geq 1} \right) \simeq \text{Hom}_{S_{\kappa}} \left( \frac{I^{\text{sat}}}{I + J^{\text{sat}}}, J^{\text{sat}} \right)_{d - 2}, \kappa \right),
\]

which is nonzero if and only if \( I^{\text{sat}} \) and \( I + J^{\text{sat}} \) differ in degree \( d - 2 \).

4.2. **Three points.** Consider the example \( I := I(\Gamma_{0})^{r} \) from the introduction. In this example \( \text{Ob}_{\text{fiber}} \) is one-dimensional.

We claim that all points from \( \text{Hilb}_{I} \) are saturable. The restriction of natural map \( \text{Hilb}_{I} \to \text{Hilb}^{\ell}(\mathbb{P}^2) \) to the saturable locus is dominant and projective, hence onto \( \text{Hilb}^{\ell}(\mathbb{P}^2) \). If \( [I'] \in \text{Hilb}_{I} \) is a nonsaturated ideal then \( S_{\kappa}/(I')^{\text{sat}} \) has Hilbert function \((1, 2, 3, 3, \ldots)\). Therefore, \( [I'] \) is the unique point of \( \text{Hilb}_{I} \) whose saturation is \((I')^{\text{sat}} \). It follows that \( I' \) is saturable. This claim follows also from [Man22, Thm. 1.3].

4.3. **Four points.** Suppose that \( I \) is a nonsaturated homogeneous ideal in a standard graded polynomial ring \( S_{\kappa} = \kappa[\alpha_{0}, \ldots, \alpha_{3}] \) such that the Hilbert function of \( S_{\kappa}/I \) is \((1, 4, 4, 4, \ldots)\), where \( \kappa \) has characteristic zero. The Hilbert function of \( S_{\kappa}/I^{\text{sat}} \) is \((1, 3, 4, 4, \ldots) \) or \( H = (1, 2, 3, 4, 4, \ldots) \). In the first case, \( [I] \) is the unique point in \( \text{Hilb}_{I} \) whose saturation is equal to \( I^{\text{sat}} \). Therefore, we get that \( I \) is saturable arguing as in §4.2. We assume that \( S_{\kappa}/I^{\text{sat}} \) has Hilbert function \( H \). By Theorem 3.3 and Example 4.1 a necessary condition for \( I \) to be saturable is that \((I^{\text{sat}})^2 \subseteq I \). We claim that it is also a sufficient condition.

Let \( Z \) in \( \text{Hilb}^{4}(\mathbb{P}^3) \) consist of subschemes with Hilbert function \( H \). This is a closed, irreducible and 8-dimensional locus parameterized by a line in \( \mathbb{P}^3 \) and a quartic on it. Let \( Y \in \text{Hilb}_{I} \) be the preimage of \( Z \). The fiber of \( Y \to Z \) over any \([R]\) is \( \text{Gr}(6, I(R)_{2}) \cong \mathbb{P}^6 \), so \( Y \) is irreducible and 14-dimensional. Let \( Y' \subseteq Y \) be the locus consisting of \([I']\) such that \((I')^{\text{sat}})^2 \subseteq I' \). The fiber of \( Y' \to Z \) over any \([R]\) is isomorphic with \( \text{Gr}(3, I(R)_{2}/I(R)_{2}^{2}) \cong \mathbb{P}^3 \), so \( Y' \) is irreducible and 11-dimensional.

Let \( X = \text{Slip}^{(1,4,4,\ldots)} \) be the locus of saturable ideals in \( \text{Hilb}_{I} \). It is irreducible and 12-dimensional. By Example 4.1 and Theorem 3.3, all saturable ideals from \( Y \) lie in \( Y' \), so \( X \cap Y \subseteq Y' \). We now prove that this containment is an equality. Take any line and four points \( \Gamma \) on it. Converging four general points to \( \Gamma \) yields an ideal \([I'] \in X \cap Y \) with \( V(I^{\text{sat}}) = \Gamma \); in this way we obtain more than one \( I' \) for given \( \Gamma \). The scheme \( V(I^{\text{sat}}) \) is a complete intersection, hence unobstructed. Applying Corollary 3.11 we obtain a divisor \([I'] \in E \subseteq X \), so \( \dim E = 11 \). Since \( E \) gets contracted in \( \text{Hilb}^{4}(\mathbb{P}^3) \),
the saturation of a point of $E$ cannot have Hilbert function $(1,3,4,\ldots)$, thus is has Hilbert function $H$, so $E \subseteq X \cap Y \subseteq Y'$. But $Y'$ is irreducible and $\dim Y' = 11 = \dim E$, so $Y' = E \subseteq X$.

We have thus proven the following.

**Proposition 4.3.** Let $I \subseteq \kappa[\alpha_0,\ldots,\alpha_3]$ be a nonsaturated ideal such that $H_{S_1/I} = (1,4,4,4,\ldots)$. Then $I$ is a limit of saturated ideals if and only if $\Ob_{\fiber}(I, I^\sat) \neq 0$ if and only if $(I^\sat)_2 \subseteq I$.

4.4. **Five points.** Suppose that $I$ is a (not necessarily saturated) homogeneous ideal in a standard graded polynomial ring $S_\kappa = \kappa[\alpha_0,\ldots,\alpha_4]$ such that the Hilbert function of $S_\kappa/I$ is $(1,5,5,5,\ldots)$. We assume that $\kappa$ has characteristic zero. There are a few possible Hilbert functions of $S_\kappa/I^\sat$. We divide them according to the number of quadrics.

4.4.1. Case $H_{S_\kappa/I^\sat}(2) = 5$. In this case $[I]$ is the unique point of the fiber of the natural map $\Hilb_I \to \Hilb^5(\mathbb{P}^4)$ defined in §2.1, hence is saturable.

4.4.2. Case $H_{S_\kappa/I^\sat}(2) = 4$. Using Macaulay’s Growth Theorem as in [CJN15, Lemma 2.9] we conclude that $H_{S_\kappa/I^\sat} = (1,3,4,5,5,\ldots)$. Geometrically, the scheme $V(I^\sat) \subseteq \mathbb{P}^4$ is a line and a (possibly embedded) point. Let $J \supseteq I^\sat \cap \mathbb{P}^4$ define this line. The locus $\mathcal{L} \subseteq \Hilb^5(\mathbb{P}^4)$ of possible $V(I^\sat)$ is parameterized by the choice of $\mathcal{L}$ in $\Hilb_I \subseteq I^\sat$. For every $\mathcal{L}$ we obtain a closed subvariety $\mathcal{V} \subseteq \mathcal{U}'$ of $\mathcal{U}'$ whose fiber over $[J]$ is a general 5-tuple of points in $\mathcal{V}$. The fiber is isomorphic to $\Gr(5, I^\sat_1/I^\sat_2 \cdot J_1)$, hence is $(5,5,5,5,5,\ldots)$-dimensional. By [Vak17, 11.4.C, 25.2.E] the variety $\mathcal{V}$ is smooth, irreducible and 19-dimensional, hence also $\pr_I(\mathcal{V})$ is irreducible and 19-dimensional.

The locus $\Delta$ contains a point $[I' \subseteq (I^\sat)^\Delta]$ with $I'$ saturable: when a general 5-tuple of points $\Gamma_5$ deforms to $\Gamma$ a 4-tuple on a line and fifth point is “static” outside it, the limit $I'$ of $I(\Gamma_5)$ is one such ideal; for given $\Gamma$ one obtains more than $I'$ (compare §4.3). Let $X = \text{Slip}^{(1,5,5,\ldots)}$. By Corollary 3.11 there is a 19-dimensional family $[I'] \in E \subseteq X$, which gets contracted in $\Hilb^5(\mathbb{P}^4)$. The saturation of a general element $[I'] \in E$ satisfies $H_{S_\kappa/I'}(2) \geq 4$ by semicontinuity and $H_{S_\kappa/I'}(2) \neq 5$ since $E$ gets contracted. Therefore, $E \subseteq \pr_I(\mathcal{U}') \cap X \subseteq \pr_I(\mathcal{V})$. Since $\pr_I(\mathcal{V})$ is irreducible and 19-dimensional, we conclude that $E = \pr_I(\mathcal{V})$.

4.4.3. Case $H_{S_\kappa/I^\sat}(2) = 3$. Using Macaulay’s Growth Theorem as in [CJN15, Lemma 2.9] we conclude that $H_{S_\kappa/I^\sat} = (1,2,3,4,5,\ldots)$, when $(I^\sat)_3 \not\subseteq I$, we obtain by Example 4.1 that $\Ob_{\fiber} = 0$, hence $I$ is entirely nonsaturable.

The space of possible $I^\sat$ has dimension $\dim \Gr(3, (S_\kappa)_1) + \dim \mathbb{P}(\kappa[\alpha_3, \alpha_4]_5) = 11$. Constructing $I$ from given $I^\sat$ is a bit more tricky in the current subcase. Each such ideal requires a choice of a 10-dimensional space of quadrics inside a 12-dimensional space $\Gr(I^\sat)_2$, but a general choice $Q$ of such quadrics will yield $H_{S_\kappa/Q}(3) < 5$, so does not give rise to $I$. Instead of dealing with ideals directly, we employ Macaulay’s inverse systems §2.5 in their partial differentiation flavour.
Up to coordinate change, we have $I_1^{\text{sat}} = (\alpha_0, \alpha_1, \alpha_2)$, so in terms of inverse systems, for every $e \leq 4$ the space $(I_1^{\text{sat}})_e^+$ is $\kappa[x_3, x_4]_e$. To construct $I$, we need to add a one-dimensional space of cubics $(c)$ and a 2-dimensional space of quadrics $\langle q_1, q_2 \rangle$, taking into account that the derivatives of the cubic lie in the 5-dimensional space $\langle x_3^2, x_3 x_4, x_4^2, q_1, q_2 \rangle$.

We construct two families. First, the cubic only family is obtained by taking a cubic $c$ whose space of partial derivatives $(S_\kappa)_1 \circ c$ is two-dimensional modulo $\kappa[x_3, x_4]_2$. The condition $(I^{\text{sat}})_3^2 \subseteq I$ is equivalent to $(\alpha_0, \alpha_1, \alpha_2)^2 \circ c = 0$, which in turn implies that

$$c = x_0 q_0 + x_1 q_1 + x_2 q_2 + c_0,$$

where $q_0, q_1, q_2 \in \kappa[x_3, x_4]_2$ and $c_0 \in \kappa[x_3, x_4]_3$. The choice of $c_0$ does not affect the ideal $I$, hence we have a $(3 \cdot 3 - 1)$-dimensional choice of $c$. Altogether, we obtain a 19-dimensional irreducible family. We now prove that it lies in $\text{Slip}_I$. It is enough to prove this for a general point of this family. Passing to the algebraic closure if necessary, we assume that $V(I^{\text{sat}}) = \{[\mu_1], \ldots, [\mu_5]\} \subseteq \mathbb{P} \langle x_3, x_4 \rangle$, where $[\mu_i] \neq [x_3]$ for $i = 1, 2, \ldots, 5$. Since $H_{S_\kappa/I^{\text{sat}}}(2) = H_{[x_3, x_4]}(2)$, we have $\langle \mu_1^2, \ldots, \mu_5^2 \rangle = \kappa[x_3, x_4]_2$. Rescaling $\mu_i$ if necessary, we assume $\sum_{j=1}^5 \mu_j^2 = 0$. Fix $c_1, c_2, c_3 \in \kappa^5$ such that $\sum_{j=1}^5 (c_j)[j]^2 = q_i$ for $i = 0, 1, 2$. For $j = 1, \ldots, 5$ let $p_j = \mu_j + t \sum_{i=0}^2 (c_i)j x_i$ and consider $\Gamma_1 = \{p_j \mid j = 1, \ldots, 5\} \subseteq \mathbb{P}^4$. By Example 2.10, for every $e > 0$ and nonzero $t$ we have $I(\Gamma_1)_e^3 = \langle p_j^3 \mid j = 1, 2, \ldots, 5 \rangle$. We now use Proposition 2.11 to show that for every $e > 0$ and general $\lambda$, the vector space $I(\Gamma_1)_e^4$ is five-dimensional and that the limit of those spaces is $I_e^4$. In the notation of that Proposition, for every $e > 0$ and $i \in \{1, \ldots, 5\}$ we take $F_i = p_i^e$. If $e > 3$, then $\mu_1^e, \ldots, \mu_5^e$ are linearly independent so we may take $G_i = F_i$ for all $i$ and conclude using that Proposition. Let

$$H = \frac{1}{3t} \sum_{j=1}^5 \left( \mu_j + t \sum_{i=0}^2 (c_i)j x_i \right)^3.$$

Observe that for every homogeneous partial derivative operator $D$ of order $k \leq 3$ we have

$$DH \in \frac{1}{t} \left( (p_1)^{3-k}, \ldots, (p_5)^{3-k} \right) \kappa[t]$$

and

$$\lim_{t \to 0} (DH) = D(x_0 q_0 + x_1 q_1 + x_2 q_2) \equiv Dc \mod \kappa[x_3, x_4]_{3-k},$$

if $D(x_0 q_0 + x_1 q_1 + x_2 q_2) \neq 0$. Assume that $e \in \{1, 2, 3\}$. Let $G_1, \ldots, G_{4-e}$ be $D_1 H_1, \ldots, D_{4-e} H$, where $D_i$ are homogeneous partial derivative operators in $x_3$ and $x_4$ of order $3 - e$ such that $D_1(x_0 q_0 + x_1 q_1 + x_2 q_2), \ldots, D_{4-e}(x_0 q_0 + x_1 q_1 + x_2 q_2)$ are linearly independent. Let $G_{5-e}, \ldots, G_5$ be some of $F_1, \ldots, F_5$ such that the corresponding $\mu_j^3$ are linearly independent. The above claims about $I(\Gamma_1)_e^4$ follow from Proposition 2.11, (4.1) and (4.2). We proved that for general $\lambda$ the Hilbert function of $S_\kappa/I(\Gamma_\lambda)$ is $(1, 5, 5, \ldots)$ and the limit of $[I(\Gamma_1)] \in \text{Hilb}(1, 5, 5, \ldots)$ with $t \to 0$ is the ideal $I$, so $[I] \in \text{Slip}_{I_1^4}$.

Next, the cubic and quadric family is obtained by taking the cubic $c$ to be a cube of a linear form and all possible limits of $c$. The choice of $c$ is 4-dimensional. The only nontrivial possible limit is $\ell_1 \cdot (\ell_2)^2$, where $\ell_1$ is a linear form and $\ell_2 \in \langle x_3, x_4 \rangle$. We call such cubics impure. As for the choice of quadrics, we fix the quadric $q_1$ as any basis element of the one-dimensional space $(S_\kappa)_1 \circ c$. The second quadric can be chosen arbitrarily modulo $\langle x_3^2, x_3 x_4, x_4^2, q_1 \rangle$, hence yields a 10-dimensional choice. It follows that the whole family has dimension $11 + 10 + 4 = 25$. In particular, its general element does not lie in $\text{Slip}_I$. Moreover, from the condition $(I^{\text{sat}})_3^2 \subseteq I$ we see that the only elements that are possibly in $\text{Slip}_I$ have impure $c$. Hence we assume that $c = x_0 x_3^2$. 
We consider the subcase when \(q_2(x_0, x_1, x_2, 0, 0) = \nu x_0^2\) for some \(\nu \in \kappa\), hence \(q_2 = \nu x_0^2 + x_3\ell_1 + x_4\ell_2\) for some \(\ell_1, \ell_2 \in \langle x_0, \ldots, x_4 \rangle\). In this case we claim that the ideal \(I\) is in \(\text{Slip}_I\) for every choice of \(\nu\) and \(q_2\) and the quintic in \(I^{\text{sat}}\). Since \(\text{Slip}_I\) is closed, it is enough to prove this for a general choice and over an algebraically closed field \(\kappa\), thus we assume that the quintic is completely decomposable and \(V(I^{\text{sat}}) = \{[\mu_1], \ldots, [\mu_5]\} \subseteq \mathbb{P} \langle x_3, x_4 \rangle\), where \([\mu_i] \neq [x_3]\) for \(i = 1, 2, \ldots, 5\). The five cubes \(\mu_1^3, \ldots, \mu_5^3\) are linearly dependent as they lie in a four-dimensional space \(\kappa[x_3, x_4]_3\). Rescaling the forms \(\mu_i\), we assume \(\sum_{i=1}^5 \mu_i^3 = 0\). Moreover, since \(H_{S_\kappa/I^{\text{sat}}}(2) = 3\), the squares \(\mu_1^2, \ldots, \mu_5^2\) span \(\kappa[x_3, x_4]_2\). Take \(c_1, c_2, c_3, c_4 \in \kappa^5\) such that
\[
\sum_{i=1}^5 c_i \mu_i^2 = x_3^2, \quad \sum_{i=1}^5 \left(\frac{\partial \mu_i}{\partial x_4}\right) c_i^2 \neq 0
\]
and
\[
\sum_{i=1}^5 \left(\frac{\partial \mu_i}{\partial x_4}\right) (c_2, x_0 + c_3, x_1 + c_4, x_2) \equiv \sum_{i=1}^5 \left(\frac{\partial \mu_i}{\partial x_4}\right) c_i^2 (\ell_1 x_3 + \ell_2 x_4) \mod \kappa[x_3, x_4]_2.
\]
To prove the existence of \(c_1\), it is enough to prove it on a single example, for example with \([\mu_\bullet] = ([x_4 - x_3], [x_4 + x_3], [x_4 - 2x_3], [x_4 + 2x_3], [x_4])\) and \(c_1 = \frac{1}{8}(0, 0, 1, 1, -2)\). For \(i = 1, \ldots, 5\) let \(p_i = \mu_i + t c_1 x_0 + t^2 (c_2 x_0 + c_3 x_1 + c_4 x_2)\) and consider the family \(\Gamma_t = \{p_i \mid i = 1, 2, \ldots, 5\} \subseteq \mathbb{P}^4\). We show that \(S_\kappa/I(\Gamma_t)\) has Hilbert function \((1, 5, 5, \ldots)\) for general \(\lambda\) and the limit of ideals \(I(\Gamma_t)\) taken degree-by-degree (hence, in the multigraded Hilbert scheme) is \(I\). As above, we use Proposition 2.11 and for each degree \(e > 0\) we take \(F_i = p_i^e\) for each \(i = 1, \ldots, 5\). For \(e > 3\) we may take \(G_i = F_i\) for every \(i\) and conclude using that Proposition. Let
\[
H = \frac{1}{3l} \sum_{i=1}^5 \left(\mu_i + t c_1 x_0 + t^2 (c_2 x_0 + c_3 x_1 + c_4 x_2)\right)^3.
\]
For \(e \in \{1, 2, 3\}\) we take \(G_1, \ldots, G_{e+1}\) to be any \(e + 1\) of \(F_1, \ldots, F_5\) such that the corresponding \(\mu_i^e\) are linearly independent. For \(e = 3\) we take \(G_5 = H\) and conclude using Proposition 2.11 since
\[
\lim_{t \to 0} H = \sum_{i=1}^5 \mu_i^2 c_i x_0 = x_0^2 x_3^2.
\]
Before proceeding with cases \(e = 1, 2\) observe that
\[
\lim_{t \to 0} \frac{1}{2l} \frac{\partial H}{\partial x_4} = \left(\frac{1}{2} \sum_{i=1}^5 \frac{\partial \mu_i}{\partial x_4} c_i^2\right) \cdot \left(x_0^2 + \frac{\ell_1 x_3}{\nu} + \frac{\ell_2 x_4}{\nu}\right) \mod \kappa[x_3, x_4]_2.
\]
By assumption, the first factor is nonzero. Replacing \(H\) by \(H\) divided by that scalar and multiplied by \(\nu\) we get
\[
\lim_{t \to 0} \frac{1}{2l} \frac{\partial H}{\partial x_4} = q_2 \mod \kappa[x_3, x_4]_2.
\]
Therefore, for \(e = 2\) we may take \(G_4 = \frac{\partial H}{\partial x_4}\) and \(G_5 = \frac{1}{2l} \frac{\partial H}{\partial x_4}\). Finally, for \(e = 1\) we take \(G_3 = \frac{1}{2l} \frac{\partial^2 H}{\partial x_4^2}\), \(G_4 = \frac{1}{2l} \frac{\partial^2 H}{\partial x_4 \partial x_3}\) and \(G_5 = \frac{1}{2l} \frac{\partial^2 H}{\partial x_4 \partial x_0}\). This finishes the proof of our claims about \(I(\Gamma_t)\) and implies that \([I] \in \text{Slip}_{I^{(S_\kappa)_2}}\).

Finally, we consider the subcase when \(q_2(x_0, x_1, x_2, 0, 0)\) is not a multiple of \(x_0^2\). Consider the natural map \(\pi: \text{Hilb}_I \to \text{Hilb}_{I^{(S_\kappa)_2}}\). The images of \(\text{Slip}_I\) and the cubic and quadric family are both 20-dimensional. The tangent space at the image of any ideal corresponding to \((q_2, c) = (x_0 x_1, x_0 x_2^2)\) is 20-dimensional, so no such ideal is in \(\text{Slip}_I\). If the quadric \(q_2(x_0, x_1, x_2, 0, 0)\) has rank at least two,
then $q_2$ degenerates to $x_0 x_1$ so the corresponding ideal is not in Slip$_I$. We are left with the rank one case, so $q_2(x_0, x_1, x_2, 0, 0) = x_1^2$. It is enough to show that no ideal with $(q_2, c) = (x_1^2, x_0 x_3^2)$ is in Slip$_I$. Let $K$ be the image under $\pi$ of any such ideal. A computation using the VersalDeformations package [Il12] verifies that $[K]$ lies on the intersection of two irreducible components: 20-dimensional and 19-dimensional. It follows that $[K] \notin \pi(\text{Slip}_I)$.

Summing up the five points case, we have the following:

**Proposition 4.4.** Let $I \subseteq \kappa[\alpha_0, \ldots, \alpha_4]$ be an ideal such that $H_{S_n/I} = (1, 5, 5, 5, \ldots)$.

1. Suppose $H_{S_n/I^{\text{sat}}}(2) = 5$. Then $I$ is a limit of saturated ideals.
2. Suppose $H_{S_n/I^{\text{sat}}}(2) = 4$. Then $I$ is a limit of saturated ideals if and only if $\text{Ob}_{\text{fiber}}(I, I^{\text{sat}}) \neq 0$ if and only if $(I^{\text{sat}})_1 \cdot J_1 \subseteq I$, where $J$ was defined in 4.4.2.
3. Suppose $H_{S_n/I^{\text{sat}}}(2) = 3$. Assume that $\kappa$ has characteristic zero. Then $I$ is a limit of saturated ideals if and only if both $(I^{\text{sat}})_2 \subseteq I$ and $(I : (S_n)^2_2) \cdot I^{\text{sat}}_1 \subseteq I$.

**4.5. A partially saturable ideal with nonzero fiber obstruction group.** Let $S_\kappa = \kappa[\alpha_0, \alpha_1, \alpha_2]$ with standard grading and consider the degree 9 ideal

$$I = (\alpha_0^2 \alpha_2, \alpha_0 \alpha_1^3, \alpha_0^2 \alpha_1^2, \alpha_0^3 \alpha_1, \alpha_5^2, \alpha_1^6).$$

Its Hilbert function is $(1, 3, 6, 9, \ldots)$ and it has $I^{\text{sat}} = (\alpha_0^2, \alpha_0 \alpha_1^3, \alpha_1^6)$, whose Hilbert function is $(1, 3, 5, 7, 8, 9, \ldots)$. In this example we check that the map $\text{Hom}_{S_\kappa}(I^{\text{sat}}_n, S_n/I^{\text{sat}}_0) \rightarrow \text{Hom}_{S_\kappa}(I, S/I)_0$ has one-dimensional cokernel and the same holds for $T_{\text{flag}} \rightarrow \text{Hom}_{S_\kappa}(I, S/I)_0$. Choosing a complementary tangent vector we discover the deformation

$$I_I = (t \alpha_0 \alpha_1^2 + \alpha_0^2 \alpha_2, \alpha_0 \alpha_1^3, \alpha_0^2 \alpha_1^2, \alpha_0^3 \alpha_1, \alpha_5^2, \alpha_1^6).$$

The saturation of the ideal $I_I$ is $(\alpha_0 \alpha_1^2 + \alpha_0^2 \alpha_2, \alpha_0 \alpha_1^3, \alpha_0^2 \alpha_1^2, \alpha_0^3 \alpha_1, \alpha_5^2, \alpha_1^6)$ which has Hilbert series $(1, 3, 6, 7, 8, 9, \ldots)$. This shows that $I = I_0$ is not entirely nonsaturable. In fact, with the help of VersalDeformations package [Il12] we check that $[I_1] \in \text{Hilb}_{I_1}$ is smooth and has $\text{Ob}_{\text{fiber}} = 0$, so it is entirely nonsaturable. We check directly that the tangent space dimensions at $I_0$ and $I_1$ are equal, so it also follows that $I$ is a smooth point on the component of $I_1$, so $I$ is nonsaturable as well.

**4.6. An entirely nonsaturable ideal with nonzero fiber obstruction group using Białynicki-Birula decomposition.** Let $S_\kappa = \kappa[\alpha_0, \alpha_1, \alpha_2]$ with standard grading and consider the ideal

$$I = (\alpha_0^3, \alpha_0^2 \alpha_1^2, \alpha_0^2 \alpha_2, \alpha_0 \alpha_1 \alpha_2, \alpha_0 \alpha_2^2, \alpha_2^6).$$

In this case $\text{Ob}_{\text{fiber}}$ is one-dimensional, so Theorem 3.3 does not apply. We will nevertheless prove that $I$ is entirely nonsaturable. Refine the standard grading on $S_\kappa$ to an $\mathbb{N}^2$-grading, where $\text{deg}(\alpha_0) = \text{deg}(\alpha_2) = (1, 0)$ and $\text{deg}(\alpha_1) = (1, 1)$. With the help of Macaulay2 we check that for the pair $(I, I^{\text{sat}})$ we have $\text{Ob}_{\text{fiber}_0,0} = \text{Ob}_{\text{fiber}_0,-1}$, so in particular $\text{Ob}_{\text{fiber}_{0,0},0} = 0$. The second coordinate of the grading gives a $\kappa^*$-action on $\text{Hilb}_I$ and $\text{Hilb}_{I^{\text{sat}}_I}$. We now use the theory of Białynicki-Birula decompositions. Introducing this theory would take too much place, so we only sketch the main results. Repeating the proofs of Theorem A.2 and Theorem 3.1 for degrees $(0, \geq 0)$, we conclude that the map $\text{Hilb}_{I^{\text{sat}}_I} \rightarrow \text{Hilb}_I^+$ on Białynicki-Birula decompositions is smooth, hence also the map

$$\text{id}_{\mathbb{A}^2} \times \text{pr}_I : \mathbb{A}^2 \times \text{Hilb}_{I^{\text{sat}}_I}^+ \rightarrow \mathbb{A}^2 \times \text{Hilb}_I^+.$$
is smooth. Consider the embedding

\[ \mathbb{A}^2 \ni (v_1, v_2) \to \begin{pmatrix} 1 & v_1 & 0 \\ 0 & 1 & 0 \\ 0 & v_2 & 1 \end{pmatrix} \in \text{GL}_3 \]

The scheme \( \mathbb{A}^2 \times \text{Hilb}_I \) maps to \( \text{Hilb}_I \) by the forgetful map \( \text{Hilb}_I^+ \to \text{Hilb}_I \) followed by the \( \mathbb{A}^2 \)-action via \( \mathbb{A}^2 \subseteq \text{GL}_3 \). Under this map, the tangent space to \( \mathbb{A}^2 \) maps to a 2-dimensional subspace of \( \text{Hom}_{\mathbb{A}}(I, S/\mathbb{A})_{0,-1} \). A final \textit{Macaulay2} computation shows that this subspace is the degree \((0, < 0)\) part of the tangent space, hence the above tangent map is surjective in degrees \((0, *)\). Arguing as in [Jel19, Theorem 4.5], we conclude that \( \mathbb{A}^2 \times \text{Hilb}_I \to \text{Hilb}_I \) is an open immersion at \((0, [I])\), hence we conclude that the image of the natural morphism

\[ \mathbb{A}^2 \times \text{Hilb}_I^{\subseteq \text{I}_{\text{sat}}} \to \text{Hilb}_I \]

contains an open neighbourhood of \([I]\). Then the same is true for the morphism

\[ \mathbb{A}^2 \times \text{Hilb}_I^{\subseteq \text{I}_{\text{sat}}} \to \text{Hilb}_I \]

which proves that \([I]\) is entirely nonsaturable. This proof is interesting also because while the \( \text{Tg}_{\text{flag}} \to \text{Hom}_{\mathbb{A}}(I, S/\mathbb{A})_{0} \) is surjective, the map \( \psi_0 \) is not; only the map \( \psi_{0, \geq 0} \) is surjective.

We provide the \textit{Macaulay2} code below.

```plaintext
S = QQ[a_0 .. a_2, Degrees=>{{1,0}, {1,1}, {1,0}}];
I = ideal mingens ideal(a_0^3, a_0*a_1^2, a_0^2*a_2, a_0*a_1*a_2, a_0*a_2^4, a_2^6);
Isat = ideal mingens saturate I;
obfib = Ext^1(Isat/I, S^1/Isat);
hs = hilbertSeries(obfib, Order=>1);
T1 := (gens ring hs)_1;
assert(select(terms hs, el -> first degree el == 0) == {T1^(-1)});
assert(sum select(terms hilbertSeries(Hom(I, S^1/I), Order=>1),
el -> first degree el == 0) == T1^6 + T1^5 + T1^4 + T1^3 + T1^2 + 4*T1 + 3 + 2*T1^(-1));
```

Since \( H_{S_2, I^2}^3 = 17 \) the fact that \( I \) is not saturable could also be deduced from [Mań22, Theorem 1.1]. Here, by a completely different argument, we obtain a stronger result, that \( I \) is entirely nonsaturable.

5. Wild polynomials

For positive integers \( r \) and \( n \) we denote by \( H_{r, \mathbb{P}^n} \) the Hilbert function of \( r \) points in general position on \( \mathbb{P}^n \), that is, we have \( H_{r, \mathbb{P}^n}(i) = \min\{r, \binom{n+i}{i} \} \) for every \( i \). By \( \text{Hilb}_{r, \mathbb{P}^n} \) we denote the multigraded Hilbert scheme \( \text{Hilb}_{r, \mathbb{P}^n} \) and by \( \text{Sat}_{r, \mathbb{P}^n} \) we denote \( \text{Sat}_{r, \mathbb{P}^n} \subseteq \text{Hilb}_{r, \mathbb{P}^n} \).

**Remark 5.1.** If \( r \) and \( n \) are such that \( \text{Hilb}_{r, \mathbb{P}^n} \) is irreducible, then \( \text{Sat}_{r, \mathbb{P}^n} \) coincides with the closure of the locus of saturated and radical ideals. That, is we have \( \text{Sat}_{r, \mathbb{P}^n} = \text{Slip}_{r, \mathbb{P}^n} \) where \( \text{Slip}_{r, \mathbb{P}^n} \) is as defined in [BB21a].

In this section \( k = \kappa \) is an algebraically closed field and \( S_\kappa = \kappa[\alpha_0, \alpha_1, \alpha_2] \) is a polynomial ring with graded dual ring \( S_\kappa^* = \kappa_{dp}[x_0, x_1, x_2] \). Given a positive integer \( d \) by \( \nu_d : \mathbb{P}((S_\kappa^*)_1) \to \mathbb{P}((S_\kappa^*)_d) \) we denote the \( d \)-uple embedding \( \ell \mapsto \ell^{[d]} \). We now define the relevant ranks.
**Definition 5.2.** Let $F \in (S^*_\kappa)_d$ be a nonzero form of positive degree. Given a subscheme $\Gamma$ of $\mathbb{P}(S^*_\kappa)_d$, by $(\Gamma')$ we denote the projective linear span of $\Gamma$. If $[F] \in (\Gamma')$, we say that $\Gamma$ is apolar to $F$. The cactus rank $\text{cr}(F)$ of $F$ is the smallest $r$ such that there exists $\Gamma \subseteq \mathbb{P}((S^*_\kappa)_1) = \text{Proj}(S_\kappa) \simeq \mathbb{P}^2$ with $\nu_d(\Gamma')$ apolar to $F$ and of degree $r$. The smoothable rank $\text{sr}(F)$ of $F$ is defined in the same way, but only considering smoothable $\Gamma$, see §3.2. Finally, the rank $r(F)$ of $F$ is defined by only considering smooth $\Gamma$; for $\Gamma = \{[\ell_1], \ldots, [\ell_r]\}$ we have $\langle \Gamma \rangle = \langle \ell_1^d, \ldots, \ell_r^d \rangle$. The border rank $\text{br}(F)$ of $F$ is the smallest $r$ such that $F$ is a limit of rank $r$ forms. We have $\text{cr}(F) \leq \text{sr}(F) \leq r(F)$ and $\text{br}(F) \leq \text{sr}(F)$. These inequalities can be strict [BB15].

We assume that the characteristic of $\kappa$ is zero or larger than the degree of any form whose rank we want to compute. Therefore, the reader may think of $S^*_\kappa$ as an ordinary polynomial ring, compare §2.5.

The main result of this section is Proposition 5.12, in which we show that if a divided power polynomial $F \in (S^*_\kappa)_d$ satisfies $\text{br}(F) \leq d + 3$ then we have $\text{sr}(F) = \text{br}(F)$.

5.1. Apolarity in three variables. This subsection gathers some technical lemmas necessary for the proof of Proposition 5.12. Throughout we assume that $F \in (S^*_\kappa)_d$ is a nonzero degree $d$ form, for $d > 0$.

**Lemma 5.3.** Assume that $\text{br}(F) \leq r < \text{cr}(F)$ for some positive integer $r$. Then there exists an ideal $I \subseteq F^\perp$ such that $[I] \in \text{Sat}_{r, \mathbb{P}^2}$ and $I^\text{sat}_d \neq I_d$.

**Proof.** By the border apolarity lemma [BB21a, Theorem 3.15], there is an ideal $[I] \in \text{Sat}_{r, \mathbb{P}^2}$ such that $I \subseteq F^\perp$. If $I^\text{sat}_d = I_d$, then $I^\text{sat} \subseteq F^\perp$. Thus, $\text{cr}(F) \leq r$ follows from the cactus apolarity lemma [Tei14, Theorem 4.7].

**Lemma 5.4.** Let $e = \lceil \frac{d + 1}{2} \rceil$. For every linear form $\ell \in (S_\kappa)_1$, there is an element of $(F^\perp)_\kappa$ that is not divisible by $\ell$.

**Proof.** Let $G = \ell \perp F$. Suppose that $(F^\perp)_\kappa \subseteq (\ell)$, then also $(F^\perp)_{\leq e} \subseteq (\ell)$, so the multiplication by $\ell$ is a bijection from $(G^\perp)_a$ to $(F^\perp)_a + 1$ for every $a \leq e - 1$. Let $f$, $g$ be the Hilbert functions of $S_\kappa/F^\perp$, $S_\kappa/G^\perp$, respectively. For $a \leq e - 1$ we have

\begin{equation}
(5.1) \quad f(a + 1) = \dim_\kappa(S_\kappa)_a + 1 - \dim_\kappa(F^\perp)_a = a + 2 + \dim_\kappa(S_\kappa)_a - \dim_\kappa(G^\perp)_a = a + 2 + g(a).
\end{equation}

Assume that $d$ is even. Then $d = 2e - 2$ and we have

\[
\begin{align*}
f(e - 2) &= f((2e - 2) - (e - 2)) = f(e) \overset{(5.1)}{=} g(e - 1) + e + 1
\end{align*}
\]

\[
= g((2e - 3) - (e - 1)) + e + 1 = g(e - 2) + e + 1 \overset{(5.1)}{=} f(e - 1) + 1.
\]

This contradicts the unimodality of the Hilbert function of $S_\kappa/F^\perp$, see [Sta78, Theorem 4.2], [IK99, Theorem 5.25].

Assume that $d$ is odd. Then $d = 2e - 1$ and we have

\[
\begin{align*}
g(e) &= g((2e - 2) - e) = g(e - 2) \overset{(5.1)}{=} f(e - 1) - e
\end{align*}
\]

\[
= f((2e - 1) - (e - 1)) - e = f(e) - e \overset{(5.1)}{=} g(e - 1) + 1.
\]

This contradicts the unimodality of the Hilbert function of $S_\kappa/G^\perp$. \hfill $\square$

**Lemma 5.5.** If there is a linear form $\ell$ in $(S_\kappa)_1$ such that $(\ell^2) \subseteq F^\perp$, then $\text{cr}(F) \leq 2 \cdot \lceil \frac{d + 1}{2} \rceil$.

**Proof.** Let $e = \lceil \frac{d + 1}{2} \rceil$. By Lemma 5.4 there exists a $\sigma \in (F^\perp)_\kappa$ not divisible by $\ell$. The ideal $(\ell^2, \sigma)$ is a complete intersection, hence is saturated and proves that $\text{cr}(F) \leq 2e$. \hfill $\square$
Lemma 5.6. Let \( k \geq 2 \) be an integer. Assume that there exists \( [I] \in \text{Sat}_{d+k,\mathbb{P}^2} \) such that \( I_1^{\text{sat}} \neq 0 \) and \( I \subseteq F^{\perp} \). Then

1. Up to a linear change of variables, we have
   \[
   I_{d+k-2}^\perp = \langle x_0 G, x_1^{d+k-2}, x_1^{d+k-3} x_2, \ldots, x_2^{d+k-2} \rangle
   \]
   for some \( G \in \kappa_{dp}[x_1, x_2]_{d+k-3} \).

2. If \( H_{\kappa[\alpha_1, \alpha_2]/G^{\perp}}(k-2) = k-1 \), then \( \alpha_0^2 \in F^{\perp} \).

Proof. The ideal \( I^{\text{sat}} \) defines a degree \( d + k \) subscheme on a line, hence \( (I^{\text{sat}})^{\perp_{d+k-2}} \subseteq (\ell) \) where \( \ell \in (S_\kappa) \) defines the line. We change coordinates so that \( \ell = \alpha_0 \). Comparing Hilbert functions, we obtain that \( \dim_\kappa(I^{\text{sat}}/I)_{d+k-2} = 1 \), so \( I_{d+k-2}^\perp = \langle x_0 G, x_1^{d+k-2}, x_1^{d+k-3} x_2, \ldots, x_2^{d+k-2} \rangle \) for some \( G \in \kappa_{dp}[x_0, x_1, x_2]_{d+k-3} \). Since \( [I] \in \text{Sat}_{d+k,\mathbb{P}^2} \) we get from Theorem 3.3 that \( \text{Ob}_\text{fiber} = \text{Ext}^1_{S_\kappa} \left( I^{\text{sat}}/I, \frac{S_\kappa}{I^{\text{sat}}} \right)_0 \neq 0 \). It follows from Example 4.1 that \( (\alpha_0^2)^{d+k-2} \subseteq I_{d+k-2} \). As a result, \( I_{d+k-2} \) is as claimed in part (1). If \( H_{\kappa[\alpha_1, \alpha_2]/G^{\perp}}(k-2) = k-1 \), then the space \( \mathcal{L} = x_0 \cdot (\kappa[\alpha_1, \alpha_2]_{k-2} \otimes G) + \langle x_1^{d}, x_1^{d-1} x_2, \ldots, x_2^d \rangle \) is \((d + k)\)-dimensional. It is contained in \( I_d^\perp \), so comparing dimensions we get \( I_d^\perp = \mathcal{L} \), so \( (\alpha_0^2)^d \subseteq F^{\perp} \) and part (2) follows. \( \square \)

For \( r \in \mathbb{Z}_{>0} \) let \( f_r \) be the Hilbert function \((1, 3, 4, 5, \ldots, r-1, r, r, \ldots)\).

Lemma 5.7. Assume that \( d \geq 3 \) and that there exists \( [I] \in \text{Sat}_{d+3,\mathbb{P}^2} \) such that \( S_\kappa/I^{\text{sat}} \) has Hilbert function \( f_{d+3} \) and \( I \subseteq F^{\perp} \). Up to a linear change of variables we have \((\alpha_0^2 \alpha_1, \alpha_0^2 \alpha_2) \subseteq F^{\perp} \) or \((\alpha_0^3, \alpha_0^2 \alpha_1) \subseteq F^{\perp} \).

Proof. We may and do assume that \( I_2^{\text{sat}} = (\alpha_0 \alpha_1, \alpha_0 \alpha_2)^2 \) or \( I_2^{\text{sat}} = (\alpha_0^2, \alpha_0 \alpha_1)^2 \). Let \( J = (\alpha_0, \theta_{d+2}) \), where \( \theta_{d+2} \in S_{d+2} \) is a minimal generator of \( I^{\text{sat}} \). It follows from Theorem 3.3 and Example 4.2 that we have \((\alpha_0 \cdot I^{\text{sat}})^d \subseteq I_d \subseteq (F^{\perp})_d \). From this we deduce that \( \alpha_0 \cdot I_2^{\text{sat}} \subseteq F^{\perp} \). \( \square \)

Lemma 5.8. Assume that \( d \geq 3 \) and that there exists \( [I] \in \text{Sat}_{d+3,\mathbb{P}^2} \) such that \( S_\kappa/I^{\text{sat}} \) has Hilbert function \( H_{d+3,F^1} \) and \( I \subseteq F^{\perp} \). Up to a linear change of variables we have \((\alpha_0^2 \ell, \alpha_0 \alpha_1 \ell) \subseteq F^{\perp} \) for a linear form \( \ell \in (S_\kappa) \).

Proof. By Lemma 5.6(1) applied with \( k = 3 \), we obtain \( I_{d+1}^\perp = \langle x_0 G_d, x_1^{d+1}, x_1^{d} x_2, \ldots, x_2^{d+1} \rangle \) for some nonzero \( G_d \in \kappa_{dp}[x_1, x_2]_d \). Consider \((\alpha_1 \otimes G_d, \alpha_2 \otimes G_d) \). If this space is two-dimensional, then applying Lemma 5.6(2), we get \( \alpha_0^2 \in F^{\perp} \). We thus assume that \( \dim_\kappa(\alpha_1 \otimes G_d, \alpha_2 \otimes G_d) = 1 \). We may and do assume that \( G_d = x_2^2 \). As a result, we have

\[
(5.2) \quad I_{d-1}^\perp \supseteq \langle x_0 x_2^{d-2} \rangle.
\]

If \( H_{S_\kappa/(\alpha_0 \otimes F)\perp}(1) \leq 2 \), then there is a linear form \( \ell \in (S_\kappa) \) such that \( \alpha_0 \ell \in F^{\perp} \) and the proof is complete. Thus, we assume that \( H_{S_\kappa/(\alpha_0 \otimes F)\perp}(1) = 3 \). We have

\[
H_{S_\kappa/(F^{\perp} \cap (\alpha_0))(d-1)} = \dim_\kappa(S_\kappa)_{d-1} - \left( \dim_\kappa(S_\kappa)_{d-2} - H_{S_\kappa/(\alpha_0 \otimes F)\perp}(d-2) \right) = d + H_{S_\kappa/(\alpha_0 \otimes F)\perp}((d-1) - (d-2)) = d + 3.
\]

It follows that \((F^{\perp} \cap (\alpha_0))_{d-1} = I_{d-1} \). Thus, from Equation (5.2) we get

\[
(\kappa \cdot (\alpha_0^2 \cdot x_2^{d-2} \otimes F)) \cap ((\alpha_0^2, \alpha_0 \alpha_1)_{d-1} \otimes F) = 0.
\]

In particular, \( \dim_\kappa((\alpha_0^2, \alpha_0 \alpha_1)_{d-1} \otimes F) \leq 2 \), so there exists a linear form \( \ell \in (S_\kappa) \) such that

\[
((\alpha_0^2, \alpha_0 \alpha_1)_{d-1} \otimes F) \subseteq (\ell)^\perp.
\]
We get \((\alpha_0^2 \ell, \alpha_0 \alpha_1 \ell)_d \subseteq F^\perp\) and we conclude that \(\alpha_0^2 \ell, \alpha_0 \alpha_1 \ell\) are in \(F^\perp\).

### 5.2. Nonexistence of wild polynomials of small border rank.

A form \(F \in (S^*_\kappa)_d\) is called wild if its border rank is smaller than its smoothable rank.

**Lemma 5.9** ([Mañ22, Proposition 3.4]). If \(\br(F) \leq d + 2\) for some \(F \in (S^*_\kappa)_d\), then \(\cr(F) = \sr(F) = \br(F)\). In particular, \(F\) is not wild.

**Proof.** By [BB14, Proposition 2.5] we may and do assume that \(\br(F) = d + 2\). Suppose that \(\cr(F) > d + 2\). By Lemma 5.3 there exists \([I] \in \text{Sat}_{d+2,2,d}\) such that \(I \subseteq F^\perp\) and \(I^\text{sat}_{d+2,2,d} \neq I_d\). Then necessarily, \(S_\kappa/I^\text{sat}\) has Hilbert function \(H_{d+2,2,d}\) and it follows from Lemma 5.6(2) that the square of a linear form annihilates \(F\). From Lemma 5.5 we get \(\cr(F) \leq 2[rac{d+2}{2}] \leq d + 2\).

**Lemma 5.10.** Let \(d \geq 4\) and \(F_d \in (S^*_\kappa)_d\) be of the form \(F_d = ax_0 x_1 x_2^{d-2} + G_d(x_0, x_2) + H_d(x_1, x_2)\) for some \(a \in \kappa\) and homogeneous, degree \(d\) divided power polynomials \(G_d\) and \(H_d\). The cactus rank of \(F_d\) is at most \(d + 3\).

**Proof.** We have \(\cr(G_d) \leq \lceil \frac{d+1}{2} \rceil\) so we may and do assume that \(a = 1\). Assume first that \(d\) is even and let \(d = 2k\). We consider two cases:

1. \(((\alpha_0^2 \cup G_d)^{\perp})_{k-1} \neq 0\) and \(((\alpha_1^2 \cup H_d)^{\perp})_{k-1} \neq 0\);
2. \(((\alpha_0^2 \cup G_d)^{\perp})_{k-1} = 0\) or \(((\alpha_1^2 \cup H_d)^{\perp})_{k-1} = 0\).

In the first case, there are \(\theta_{k-1}(\alpha_0, \alpha_2)\) and \(\eta_{k-1}(\alpha_1, \alpha_2)\) such that \(I = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \theta_{k-1}, \alpha_1^2 \eta_{k-1}) \subseteq F^\perp\). Its initial ideal with respect to any term order is of the form \(J = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \theta_{k-1}', \alpha_1^2 \eta_{k-1}')\) for some monomials \(\theta' \in \kappa[\alpha_0, \alpha_1]_{k-1}\) and \(\eta' \in \kappa[\alpha_1, \alpha_2]_{k-1}\). The ideal \(J\) is saturated and the Hilbert polynomial of \(S_\kappa/J\) is \(2k + 2 = d + 2\). Therefore, \(\cr(F) \leq d + 2\) follows.

If (2) holds, we may and do assume that \(((\alpha_0^2 \cup G_d)^{\perp})_{k-1} = 0\). The map

\[
\kappa[\alpha_0, \alpha_2]_{k-1} \rightarrow \kappa[d][x_0, x_2]_{d-k-1} = \kappa[d][x_0, x_2]_{d-k-1}
\]

is injective and thus bijective. It follows that for some \(\theta_{k-1} \in \kappa[\alpha_0, \alpha_2]_{k-1}\), \(\eta_{k-1} \in \kappa[\alpha_1, \alpha_2]_{k-1}\) and \(c \in \{0, 1\}\), the ideal

\[
I = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \theta_{k-1} - \alpha_0 \alpha_1 \alpha_2^{k-1}, \alpha_1^2 \eta_{k-1} - \alpha_0 \alpha_1 \alpha_2^{k-1})
\]

is contained in \(F^\perp\). If \(c = 0\) or \(\theta_{k-1} \notin \kappa \cdot \alpha^{k-1}\) or \(\eta_{k-1} \notin \kappa \cdot \alpha^{k-1}\), then these generators form a Gröbner basis with respect to the grevlex order with \(\alpha_0 > \alpha_1 > \alpha_2\) or \(\alpha_1 > \alpha_0 > \alpha_2\) and the proof is as in case (1). Therefore, we are left with the case that \(I = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \theta_{k-1} - \alpha_0 \alpha_1 \alpha_2^{k-1}, \alpha_1^2 \eta_{k-1} - \alpha_0 \alpha_1 \alpha_2^{k-1}) \subseteq F^\perp\) for some nonzero \(c'\). This ideal contains \(I' = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \alpha_2^{k-1} - c' \alpha_1^2 \alpha_2^{k-1} = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \alpha_2^{k-1} - c' \alpha_1^2 \alpha_2^{k-1}, \alpha_1^2 \alpha_2^{k-1})\). By taking an initial ideal, one may verify that \(I'\) is saturated and \(S_\kappa/I'\) has Hilbert polynomial \(2k + 3 = d + 3\).

Now assume that \(d\) is odd and let \(d = 2k + 1\). Then for dimension reasons, there are \(\theta_k \in \kappa[\alpha_0, \alpha_2]_k\) and \(\eta_k \in \kappa[\alpha_1, \alpha_2]_k\) such that \(I = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \theta_k, \alpha_1^2 \eta_k) \subseteq F^\perp\). By taking an initial ideal, one may verify that \(I\) is saturated and \(S_\kappa/I\) has Hilbert polynomial \(2k + 4 = d + 3\).

**Lemma 5.11.** Let \(d\) be a positive integer and \(F_d \in (S^*_\kappa)_d\) be of the form \(F_d = ax_0 x_2^{d-1} + G_d(x_0, x_1) + H_d(x_1, x_2)\) for some \(a \in \kappa\) and homogeneous, degree \(d\) divided power polynomials \(G_d\) and \(H_d\). The cactus rank of \(F_d\) is at most \(d + 3\).

**Proof.** Let \(e = \lceil \frac{d+1}{2} \rceil\). The cactus rank of \(G_d\) is at most \(e\), so it suffices to show that the cactus rank of \(x_0 x_2^{d-1} + H_d(x_1, x_2)\) is at most \(e + 1\). For some \(\theta \in \kappa[\alpha_1, \alpha_2]_{e-1}\) and \(a \in \{0, 1\}\), we have
\[
I = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1, 1 - a_0 \alpha_0 \epsilon_{a_0}^{-1}) \subseteq (x_0 x_2^{d-1} + H_d(x_1, x_2))^\perp. \]

The ideal \( I \) is saturated and \( S_\kappa/I \) has Hilbert polynomial \( e + 1 \). \( \square \)

**Proposition 5.12.** Assume that \( \text{br}(F) \leq d + 3 \) for some \( F \in (S_\kappa^e)_d \). If the characteristic of \( \kappa \) is zero or larger than \( d \), then \( \text{cr}(F) = \text{sr}(F) = \text{br}(F) \). In particular, \( F \) is not wild.

**Proof.** By Lemma 5.9 we may and do assume that \( \text{br}(F) = d + 3 \). By the Alexander-Hirschowitz theorem, see [AI95] or [IK99, Theorem 1.61], for \( d \leq 5 \) such a high border rank does not occur, hence we obtain \( d \geq 6 \). Suppose that \( \text{cr}(F) > d + 3 \). By Lemma 5.3 there is an ideal \( I \subseteq F^\perp \) such that \( I_\text{sat}^d \neq I_d \) and \([I] \in \text{Ssat}_{d+3,F^2} \). There are two possibilities for the Hilbert function of \( S_\kappa/I_\text{sat} \). It is either \( H_{d+3,F^2} \) or \( f_{d+3} \). It follows from Lemmas 5.7 and 5.8 that we may assume that one of the following holds

1. \((\alpha_0^2 \alpha_1, \alpha_0 \alpha_2) \subseteq F^\perp \);
2. \((\alpha_0^2, \alpha_0^2 \alpha_1) \subseteq F^\perp \);
3. \((\alpha_0^2 \alpha_1, \alpha_0^2) \subseteq F^\perp \);
4. \((\alpha_0^2 \alpha_2, \alpha_0 \alpha_1 \alpha_2) \subseteq F^\perp \).

Let \( e = \left\lfloor \frac{d+1}{2} \right\rfloor \). Let \( \langle \rangle \) be the grevlex monomial order with \( 0 < \alpha_1 < \alpha_2 \).

Assume that (1) holds. Let \( \eta_e \in (F^\perp)_e \) be not divisible by \( \alpha_0 \) (see Lemma 5.4). We choose one of the form \( \eta_e = \zeta_e + a \alpha_0^2 + a_0 \alpha_1 \zeta_{e-2} + a_0 \alpha_2 \zeta_{e-2}^\prime \), where \( a \in \kappa, \zeta_e \in \kappa[\alpha_1, \alpha_2, e] \setminus \{0\} \) and \( \zeta_{e-2}, \zeta_{e-2}^\prime \in \kappa[\alpha_1, \alpha_2]_{e-2} \). Furthermore, if \( a \neq 0 \), then \( \alpha_0^2 \alpha_1 \zeta_{e-2}^\prime \in F^\perp \). It follows that \( \alpha_0^2 \alpha_1 \zeta_{e-2}^\prime \in F^\perp \), so by Lemma 5.5 the cactus rank of \( F \) is at most \( d + 2 \). Let \( J = \text{in}_e(\alpha_0^2 \alpha_1, \alpha_0^2 \alpha_2, \eta_e) \). Then \( J = (\alpha_0^2 \alpha_1, \alpha_0^2 \alpha_2, M) \) where \( M \) is a monomial in \( \kappa[\alpha_1, \alpha_2]_e \). It follows that \( S_\kappa/J \) has Hilbert polynomial \( 2e + 1 \leq d + 3 \) and \( J \) is saturated.

Assume that (2) holds. Let \( \eta_e \in (F^\perp)_e \) be not divisible by \( \alpha_0 \) (see Lemma 5.4). We choose one of the form \( \eta_e = a \alpha_0^2 + a_1 \zeta_{e-1} + b \alpha_0^2 \zeta_{e-1}^\prime + a_0 \alpha_2 \zeta_{e-1} \), where \( a, b \in \kappa, \zeta_{e-1}, \zeta_{e-1}^\prime \in \kappa[\alpha_1, \alpha_2]_{e-1} \) and \( a_0 \alpha_2 + a_1 \zeta_{e-1} \neq 0 \). If \( a \neq 0 \), then \( \alpha_0^2 \alpha_1 \zeta_{e-1}^\prime \in F^\perp \) and it follows that \( \alpha_0^2 \alpha_1 \zeta_{e-1}^\prime \in F^\perp \). Then by Lemma 5.5 we get \( \text{cr}(F) \leq d + 2 \). Therefore, \( J = \text{in}_e(\alpha_0^2 \alpha_1, \alpha_0 \alpha_1, \eta_e) = (a \alpha_0^2 \alpha_1, \alpha_1 M) \) for a monomial \( M \in \kappa[\alpha_1, \alpha_2]_{e-1} \).

It follows that \( J \) is saturated and \( S_\kappa/J \) has Hilbert polynomial \( 2e + 1 \leq d + 3 \).

If (3) or (4) hold, then \( \text{cr}(F) \leq d + 3 \) by Lemma 5.10 or 5.11, respectively. \( \square \)

**Appendix A. Obstruction theory of flag Hilbert schemes**

Throughout this appendix, we keep the assumptions from §2.1, in particular we fix \( k \to \kappa \), where \( \kappa \) is a field, and take homogeneous ideals \( K \subseteq J \) in the \( A \)-graded algebra \( S_\kappa \). Arguing as in for example [FGI+05, Theorem 6.4.5] or [Jel10, Proposition 4.1], we get that the scheme \( \text{Hilb}_K \) has obstruction theory with obstruction group \( \text{Ext}_{S_\kappa}^1(K, S_\kappa^e)_{l_0} \).

In the case \( k = \kappa \), the basic results of deformation theory are phrased in our language in Chapter 5 of [FGI+05] or [FM98]. In the general case, it is harder to find the reference for the following well-known result, so we sketch a proof.

**Lemma A.1** (Main Theorem of Obstruction Calculus). Let \( Y \) be a Noetherian scheme. Suppose that \( f: (X, x) \to (Y, y) \) is a finitely-presented morphism of pointed schemes that yields an isomorphism on residue fields \( \kappa(y) \simeq \kappa(x) \). Assume that \( (X, x) \) and \( (Y, y) \) admit obstruction theories, that \( f \) is a map of obstruction theories and that

1. \( df|_x: T_x X \to T_y Y \) is surjective,
2. the map of obstruction groups induced by \( f \) is injective.
Then $f$ is smooth at $x$.

**Proof.** The surjectivity of $df|_x$ and injectivity of map on obstruction groups imply that for any small extension $B \to A \to 0$ of Artinian local rings and any diagram

$$
\begin{array}{ccc}
(Spec(A), \ast) & \longrightarrow & (X, x) \\
\downarrow c & & \downarrow f \\
(Spec(B), \ast) & \longrightarrow & (Y, y)
\end{array}
$$

of pointed schemes, there is a lift $(Spec(B), \ast) \to (X, x)$. Having this, the claim follows from [Art73, Proposition 1.1, Remark 1.3(3)]. For an alternative, one can argue using [Gro67, Proposition (17.5.3)]. □

We pass to constructing the obstruction space for the flag Hilbert scheme. As in §2.3, we have natural maps $\operatorname{Hom}_{S_\kappa}(K, S_\kappa/K), \operatorname{Hom}_{S_\kappa}(J, S_\kappa/J) \to \operatorname{Hom}_{S_\kappa}(K, S_\kappa/J)$. Denote their sum as

$$
\psi: \operatorname{Hom}_{S_\kappa}(K, S_\kappa/K) \oplus \operatorname{Hom}_{S_\kappa}(J, S_\kappa/J) \to \operatorname{Hom}_{S_\kappa}(K, S_\kappa/J).
$$

Recall that $\text{Hilb}_{K \subseteq J} \to \text{Spec}(k)$ is the flag Hilbert scheme and that $K \subseteq J \subseteq S_\kappa$ yields its Spec($\kappa$)-point. Below we restrict to $k$-schemes. Therefore, $k$ becomes a base scheme and does not appear in the argument, in particular it is not important that it may not be a field.

**Theorem A.2.** Suppose that $\psi_0$ is surjective. Then the pointed $k$-scheme $\text{Hilb}_{K \subseteq J}$ has an obstruction theory with obstruction group $\text{Ob}_{\text{flag}}$ and the natural projections $\text{Ob}_{\text{flag}} \to \operatorname{Ext}_{S_\kappa}^1(K, S_\kappa/K)_0, \text{Ob}_{\text{flag}} \to \operatorname{Ext}_{S_\kappa}^1(J, S_\kappa/J)_0$ are maps of obstruction theories for the maps $\text{pr}_K$ and $\text{pr}_J$, respectively.

This theorem partially generalizes [Jel19, Theorem 4.10], which has stronger assumptions and is formulated only in the case $k = \kappa$, however considers the case “$\geq 0$” rather than degree zero.

**Proof.** A large part of the proof follows as in Theorem [Jel19, Theorem 4.10]. We only sketch this part, referring the reader to the above mentioned proof for details. The obstruction theories of $\text{Hilb}_K$ and $\text{Hilb}_J$ yield an obstruction theory for $\text{Hilb}_K \times \text{Hilb}_J$ and by direct check, the obstruction at a point of $\text{Hilb}_{K \subseteq J}$ lies in the subgroup $\text{Ob}_{\text{flag}}$, compare [Jel19, Theorem 4.10]. None of this uses the assumption that $\psi_0$ is surjective.

What is left to prove is that if the obstruction vanishes, an extension exists. Let us explain this more precisely: we consider a small extension $\text{Spec}(A) \hookrightarrow \text{Spec}(B)$ of local Artinian $k$-algebras with residue field $\kappa$ and a map $\text{Spec}(A) \to \text{Hilb}_{K \subseteq J}$. If the obstruction vanishes, then we want to prove that there exists a map from $\text{Spec}(B)$ to $\text{Hilb}_{K \subseteq J}$ extending a given one from $\text{Spec}(A)$. A small extension can be composed of the ones when $\ker(B \to A)$ has length one (so it is annihilated by the maximal ideal of $B$), so we assume that our extension is

$$
0 \to \kappa \to B \to A \to 0.
$$

We denote by $\varepsilon \in B$ the image of $1$ in $B$. By vanishing of the obstruction for $\text{Hilb}_K \times \text{Hilb}_J$, there exists a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(A) & \longrightarrow & \text{Hilb}_{K \subseteq J} \\
\downarrow c_l & & \downarrow c_l \\
\text{Spec}(B) & \longrightarrow & \text{Hilb}_K \times \text{Hilb}_J.
\end{array}
$$


We are to prove that there exists such a diagram, with $\text{Spec}(B) \to \text{Hilb}_K \times \text{Hilb}_J$ actually factoring through $\text{Hilb}_K \subseteq J$. We stress the word exists: in the course of the proof we will modify the map $\text{Spec}(B) \to \text{Hilb}_K \times \text{Hilb}_J$ in the diagram above.

![Diagram](attachment:diagram.png)

**Figure 1.** Deformations of $K$ and $J$, but not yet a deformation of a pair $K \subseteq J$.

Let $I_K, I_J \subseteq S_B$ be the ideals of the deformations corresponding to $\text{Spec}(B) \to \text{Hilb}_K \times \text{Hilb}_J$. Let $* \in \{K, J\}$ and let $\bar{\mathcal{T}}_* := \mathcal{T}_* \otimes_B A$ be the restrictions to $A$. By assumption we have $\bar{\mathcal{T}}_K \subseteq \bar{\mathcal{T}}_J$. Let $Q_* := S_B/I_*$ be the quotients and $\bar{Q}_* = S_A/\bar{I}_* \simeq Q_* \otimes_B A$. We have the commutative diagram as in Figure 1.

From its middle horizontal plane we obtain a homomorphism $f: \mathcal{I}_K \to S_B \to Q_J$. The composition of $f$ with $Q_J \to \bar{Q}_J$ is zero by a diagram chase, since $\bar{\mathcal{T}}_K \subseteq \bar{\mathcal{T}}_J$. Thus $f$ lifts to $\mathcal{I}_K \to \varepsilon(S_k/J)$ and hence gives a homomorphism $\bar{f} \in \text{Hom}_{S_0}(K, S_k/K)$. By assumption on $\psi_0$, we can write $\bar{f} = -\bar{\varphi}_k + \bar{\varphi}_J$, where $\varphi_K \in \text{Hom}_{S_0}(K, S_k/K)$ and $\varphi_J \in \text{Hom}_{S_0}(J, S_k/J)$.

Now, recall that $\text{Spec}(B) \to \text{Hilb}_K \times \text{Hilb}_J$ was only an extension of $\text{Spec}(A)$. The other possible extensions admit an action of the vector space $\text{Hom}_{S_0}(K, S_k/K) \oplus \text{Hom}_{S_0}(J, S_k/J)$ by the argument of [FGI+05, Theorem 6.4.5]. We claim that by acting on the above extension with $(\varphi_K, \varphi_J)$ we obtain an extension that satisfies $\bar{f} = 0$. To prove this, we need to unpack the abstract statements in the previous paragraph. For $s \in K$ let $s_A \in \bar{\mathcal{T}}_K \subseteq \bar{\mathcal{T}}_J$ be its lift and $s_K \in I_K$ and $s_J \in I_J$ be lifts of $s_A$.

Let $\delta_s \in S_K$ be such that $s_K - s_J = \delta_s \varepsilon$. Then the map $\bar{f}$ lifts an element $s$ to $s_K$, maps it to $S_B/I_J$ and then lifts the resulting class to $S_K/J$. One such lift is the class of $\delta_s$ in $S_K/J$. So $\bar{f}(s) = \delta_s \mod J$ and we obtain that $\delta_s \mod J = -\varphi_K(s) + \varphi_J(s)$.

Let $\varphi_K'$ be the composition

$$I_K \to I_K \otimes_B K \simeq K \to S_K/K \simeq \varepsilon \cdot S_K/K \hookrightarrow S_B/\varepsilon K,$$

where the middle map is $\varphi_K$. Let $I_K'$ be the preimage in $S_B$ of

$$\text{im}(\text{id}_{I_K} + \varphi_K') = (i + \varphi_K'(i) \mid i \in I_K) \subseteq S_B/\varepsilon K.$$

Unravelling the proof of [FGI+05, Theorem 6.4.5(b)] or by a direct check with [FGI+05, Lemma 6.4.3] we see that $S_B/I_K'$ is flat over $B$. Similarly, from $\varphi_J$ we obtain $\varphi_J'$, $I_J'$ and $Q_J' := S_B/I_J'$. From $(I_K', I_J')$ we obtain a map $\bar{f}' : K \to S_K/J$ by the same procedure as for $\bar{f}$ above. Let us analyse it. Let
us keep the notation \( s, s_A, s_K, s_J \) from above. For an element of \( S_\kappa / K \) by \( \tilde{\varphi} \) we denote any its lift to \( S_\kappa \). The elements \( s'_K = s_K + \varepsilon \varphi_K(s) \in I'_K \) and \( s'_J = s_J + \varepsilon \varphi_J(s) \in I'_J \) are lifts of \( s_A \). We have
\[
s'_K - s'_J = s_K + \varepsilon \varphi_K(s) - s_J - \varepsilon \varphi_J(s) = \varepsilon (\delta_s + \varphi_K(s) - \varphi_J(s)),
\]
so we can take \( \delta'_s = \delta_s + \varphi_K(s) - \varphi_J(s) \). But then \( \delta'_s \equiv 0 \mod J \), which proves that \( \tilde{j}' = 0 \). This implies that \( I'_K \subseteq I'_J \) as claimed. The map \( \text{Spec}(B) \to \text{Hilb}_{K\subseteq J} \) corresponding to \( (I'_K, I'_J) \) lifts the given map \( \text{Spec}(A) \). This concludes the proof.

Lemma A.3. Let \( f: R_1 \to R_2 \) be a surjective homomorphism of local Noetherian \( k \)-algebras with residue fields \( \kappa \). Suppose that for every small extension \( B \to A \) of Artinian local \( k \)-algebras with residue field \( \kappa \) and every commutative diagram of affine \( k \)-schemes

\[
\begin{array}{ccc}
\text{Spec}(R_2) & \xrightarrow{f^\#} & \text{Spec}(R_1) \\
p & & p \\
\text{Spec}(\kappa) & \xleftarrow{\text{Spec}(A)} & \text{Spec}(B)
\end{array}
\]

there exists \( p' : \text{Spec}(B) \to \text{Spec}(R_2) \) such that \( p \) is the restriction of \( p' \) (but we do not assume any compatibility with \( f^\# \)). Then \( \ker(f) \) is generated by \( d \) elements whose classes are independent in \( m_{R_1}/m_{R_1}^2 \), where \( d = \dim_\kappa m_{R_1}/m_{R_1}^2 - \dim_\kappa m_{R_2}/m_{R_2}^2 \).

Proof. Let \( d = \dim_\kappa (m_{R_1}^2 + k')/m_{R_1}^2 \) and let \( K \) be generated by any \( d \) elements of \( \ker f \) whose classes span this space. Let \( \overline{R}_1 := R_1/K \). It follows that the homomorphism \( \pi : \overline{R}_1 \to R_2 \) also satisfies the assumptions of the statement and moreover induces an isomorphism on cotangent spaces. Let \( n = m_{\overline{R}_1} \). Suppose that \( \pi \) is not an isomorphism and let \( s \geq 3 \) be the minimal element such that \( \ker \pi \subseteq n^s \). Let \( A = \overline{R}_1/n^{s-1} \) and \( B = \overline{R}_1/n^s \). By assumption, the canonical surjective homomorphism \( R_2 \to A \) lifts to a homomorphism \( g : R_2 \to B \). We obtain a diagram of homomorphisms of \( k \)-algebras

\[
\begin{array}{ccc}
R_2 & \xrightarrow{\overline{g}} & \overline{R}_1/n^{s-1} \\
\overline{R}_1/n^s & \xleftarrow{\text{can}} & \overline{R}_1/n^s
\end{array}
\]

The canonical maps induce isomorphisms of cotangent spaces, hence so does \( g \). But then \( g \) is surjective. Yet the map \( g \) factors through \( \overline{R}_1/(\ker \pi + n^s) \to \overline{R}_1/n^s \) which cannot be surjective, as \( \dim_\kappa \overline{R}_1/(\ker \pi + n^s) < \dim_\kappa \overline{R}_1/n^s \). The contradiction shows that \( \ker \pi = 0 \) and the proof is completed.

To apply Lemma A.3 to the Hilbert schemes, we need to understand the full tangent spaces, not only the tangent spaces to the fiber.

Lemma A.4 (image in the tangent to the base). Let \( H \to \text{Spec}(k) \) be one of the Hilbert schemes considered above, let \( \text{Ob} \) be its obstruction group and let \( x : \text{Spec}(\kappa) \to H \) be a point. Then the obstruction map yields a \( \kappa \)-linear map \( T_x \text{Spec}(k) \to \text{Ob} \) and the kernel of this map is the image of \( T_x H \in T_\kappa \text{Spec}(k) \). If \( f : H \to H' \) is a morphism of such Hilbert schemes which induces an injection on obstruction groups, then the images of \( T_x H \) and \( T_{f(x)} H' \) in \( T_\kappa \text{Spec}(k) \) coincide.

Proof. This follows as in [Ser06, Proposition 4.4.4]. The key point is that a tangent vector in \( T_x \text{Spec}(k) \) yields a map \( \text{Spec}(\kappa[z]/z^2) \to \text{Spec}(k) \) and this map lifts to a map to \( H \) if and only if the associated obstruction in \( \text{Ob} \) vanishes.

□
Corollary A.5. Under the assumptions of Theorem A.2 the subscheme $\text{Hilb}_{K \subseteq J} \subseteq \text{Hilb}_K \times \text{Hilb}_J$ is cut out near $[K \subseteq J]$ by $\dim_k \text{Hom}(K, S_\kappa/J)_0$ equations which yield linearly independent classes in the cotangent space at $[K \subseteq J]$.

We stress that without additional assumptions on $\text{Hilb}_K \times \text{Hilb}_J$ this does not imply that $\text{Hilb}_{K \subseteq J}$ has codimension $\dim_k \text{Hom}(K, S_\kappa/J)_0$.

Proof. Theorem A.2 shows that closed immersion induces an injective map on obstruction spaces. This by definition of obstruction implies that the assumptions of Lemma A.3 are satisfied for

$$R_1 = \mathcal{O}_{\text{Hilb}_K \times \text{Hilb}_J,([K],[J])} \rightarrow R_2 = \mathcal{O}_{\text{Hilb}_{K \subseteq J},[K \subseteq J]}.$$ 

It remains to understand the difference of dimensions of tangent maps. The images of both tangent spaces in $T_\kappa \text{Spec}(k)$ coincide by Lemma A.4. So we can restrict to tangent spaces to the fibers. Finally, since $\psi_0$ is surjective, we see from the main diagram §2.3 that

$$\dim_k T_{([K],[J])} \text{Hilb}_K \times \text{Hilb}_J - \dim_k T_{[K \subseteq J]} \text{Hilb}_{K \subseteq J} = \dim_k \text{Hom}_{S_\kappa}(K, S_\kappa/J)_0.$$

\[\square\]

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