Extensions of algebraic groups with finite quotient and non-abelian 2-cohomology

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Abstract

For a finite smooth algebraic group $F$ over a field $k$ and a smooth algebraic group $\bar{G}$ over the separable closure of $k$, we define the notion of $F$-kernel in $\bar{G}$ and we associate to it a set of non-abelian 2-cohomology. We use this to study extensions of $F$ by an arbitrary smooth $k$-group $G$. We show in particular that any such extension comes from an extension of finite $k$-groups when $k$ is perfect and we give explicit bounds on the order of these finite groups when $G$ is linear. We prove moreover some finiteness results on these sets.

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1 Introduction

It is a well known fact in abstract group theory that group extensions

$$1 \to G \to E \to \Gamma \to 1,$$

where $G$ is an abelian group and $\Gamma$ is an arbitrary group, are classified by what are called factor systems. In the modern language of cohomology theory, these factor systems turn out to be 2-cocycles for the group action of $\Gamma$ on $G$ given by conjugation in $E$. Extensions like the one above are thus classified by the group cohomology set $H^2(\Gamma, G)$ (cf. for example [Mac95, IV.4]). In a non-abelian context (that is, when $G$ is also arbitrary), one loses the action of $\Gamma$ over $G$ given by conjugation in $E$ but can still define the notion of $\Gamma$-kernel (which we redefine below, cf. also [Mac95, IV.8]) which allows for a description of the set of extensions.

Such a description can be generalized to the case of topological groups if one is careful enough to impose the good conditions on the 2-cocycles one works with, a natural
condition being for example continuity of all considered maps. This was already done for example by Springer in [Spr66] (albeit in a rather sloppy way, see the discussion in section 2.4), where he considers the action of the absolute Galois group of a perfect field \( k \) on the \( k_s \)-points of an algebraic group over \( k \), where \( k_s \) denotes a separable closure of \( k \), obtaining thus the notion of \( k \)-kernel (which we also redefine below). These ideas were later resumed (and treated with much more precision and care) by Borovoi in the case of linear groups over a field of characteristic 0 (cf. [Bor93]) and by Flicker, Scheiderer and Sujatha in the more general case of smooth \( k \)-groups for arbitrary \( k \) (cf. [FSS98]). With such tools one can prove interesting results, as for example Springer’s result stating the existence, for an arbitrary homogeneous space under an algebraic group over a perfect field of cohomological dimension \( \leq 1 \), of a principal homogeneous space surjecting onto it; or Borovoi’s abelianization of the non-abelian Galois cohomology and its known consequences in the arithmetic study of homogeneous spaces.

However, if one wishes to extend such theories to algebraic groups acting on other algebraic groups (having thus in mind studying their extensions), serious problems appear. For starters, an algebraic group, when considered as a group scheme over the base field \( k \), brings with it not a single group, but infinitely many (one for each scheme over \( k \)), making it difficult to define the notion of kernel (let alone that of 2-cocycle) in this context in a “naive” way. These difficulties can be overcome in the case where the group that acts is a finite smooth \( k \)-group. This is done by a slight generalization of the non-abelian Galois cohomology cited above.

The interest for doing so is not only based on the simple question of understanding algebraic group extensions, but also on the fact that one can use these tools in order to exhibit finite subgroups of arbitrary algebraic groups which are defined over the base field and intersect every connected component of the group. This has already been used by the author in the study of Brauer groups of homogeneous spaces with arbitrary stabilizer (cf. [LA14]), but also for example by Gille and Reichstein in the study of essential dimension for linear algebraic groups (cf. [GR09]). In this last work, an important issue is to control the order of the finite group obtained. We thus try to control this order in the main result stated here below (Theorem 1.1).

The existence of such subgroups had already been stated by Borel and Serre for a perfect field \( k \), although they only gave the proof for linear \( G \) and \( k = k_s \) of characteristic zero (cf. [BS64, Lem. 5.11 and footnote on p. 152]). This result was extended shortly after by Platonov to the case of a perfect field, but still for linear groups (cf. [Pla66, Lem. 4.14]). Finally, the same assertion has also been recently proved by Brion in an even more general setting (cf. [Bri15, Thm. 1]), although with no explicit bounds on the order of the finite groups thus obtained.

The structure of this article is then as follows:

Section 2 is devoted to preliminaries on \( \Gamma \)-kernels and \( k \)-kernels which summarize the above cited results of Springer, Borovoi, Flicker, Scheiderer and Sujatha. This takes more than a third of the whole text, but it is necessary in order to avoid citing previous
work all the time and also to repair or precise some of Springer’s statements and proofs in [Spr66].

In section 3 we define the notion of an $F$-kernel in $\bar{G}$ for a finite smooth $k$-group $F$ and a smooth $k_s$-group $\bar{G}$ and we define a non-abelian 2-cohomology associated to such a kernel. We compare later this cohomology set with the set of extensions of $F$ by $G$ in the case where $\bar{G}$ comes from a $k$-group $G$ (cf. Propositions 3.5 and 3.6). We also treat the case of a possibly non smooth commutative group $G$, since it is useful in order to describe the extensions by $G$ via its center (cf. in particular Proposition 3.12).

In section 4, we prove the following theorem on the “reduction” of extensions (Theorem 4.2):

**Theorem 1.1.** Let $k$ be a perfect field of characteristic $p \geq 0$. Let $F$ be a smooth finite $k$-group of order $n$, and $G$ an arbitrary smooth $k$-group. Then, for any extension

$$1 \to G \to H \to F \to 1,$$

there exists a finite smooth $k$-subgroup $S$ of $G$ and a commutative diagram with exact rows

$$
\begin{array}{cccccc}
1 & \to & S & \to & H' & \to & F & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & G & \to & H & \to & F & \to & 1.
\end{array}
$$

Moreover, if $G$ is linear, let $T$ be a maximal torus in $G$, $W$ be the Weyl group of $G$ (that is, the finite group of connected components of the normalizer of $T$) and $K/k$ be a separable algebraic extension splitting $T$. Denote by $r$ the rank of $T$, $w$ the order of $W$ and $d$ the degree of $K/k$. Assume that either $nw$ is prime to $p$ or that $G^0$ is reductive. Then one can take $S$ to be contained in an extension of $W$ by the $ndw$-torsion subgroup of $T$, hence of order dividing $nrdw^2$.

Finally, in section 5, we study the finiteness of the set of extensions of $F$ by $G$. It turns out that this set is actually always finite when $k$ is a finite field and, when $G$ is linear, this is quite often the case also for other fields (cf. Theorem 5.1).

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2 Preliminaries on $\Gamma$-kernels and $k$-kernels

2.1 $\Gamma$-kernels

Let $\Gamma$ be a Hausdorff topological group and $G$ be an abstract group. We denote $\text{Aut}(G)$ (resp. $\text{Int}(G)$) the group of automorphisms (resp. of inner automorphisms) of $G$. It
is well known that \( \text{Int}(G) \) is normal in \( \text{Aut}(G) \). We denote by \( \text{Out}(G) \) the respective quotient and \( \pi : \text{Aut}(G) \to \text{Out}(G) \) the natural projection. We equip \( \text{Aut}(G) \) (and thus \( \text{Int}(G) \) by restriction) with the weak topology with respect to the evaluation maps \( \varphi \mapsto \varphi(g) \) for \( g \in G \), the group \( G \) being considered as a discrete topological group, and \( \text{Out}(G) \) with the quotient topology.

A \( \Gamma \)-kernel in \( G \) is a generalization of the notion of a (discrete) \( \Gamma \)-group, which corresponds to a continuous morphism \( \Gamma \to \text{Aut}(G) \). This generalization allows the construction of a non-abelian 2-cohomology set which will be useful for us below (see \cite{Spr66}, \cite{Bor93} or \cite{FSS98}).

**Definition 2.1.** A \( \Gamma \)-kernel in \( G \) is a morphism of topological groups \( \kappa : \Gamma \to \text{Out}(G) \) such that there exists a continuous map \( \tilde{\kappa} : \Gamma \to \text{Aut}(G) \) lifting \( \kappa \), i.e. such that \( \kappa = \pi \circ \tilde{\kappa} \).

A \( \Gamma \)-kernel \( \kappa \) is said to be **locally trivial** if there exists such an \( \tilde{\kappa} \) which moreover respects the group law on an open subset \( \Delta \) of \( \Gamma \) containing the identity element. It is said to be **trivial** if moreover \( \Delta \) can be taken to be the whole \( \Gamma \). In other words, \( G \) is trivial if it becomes a \( \Gamma \)-group via \( \tilde{\kappa} \).

**Remarks.**

1. The definition of locally trivial kernels will only be of interest to us in the case in which \( \Gamma \) has the following property:

   \( \ast \) There exists a basis of open neighbourhoods of the identity element which are subgroups.

   This is always the case for example if \( \Gamma \) is locally compact and totally discontinuous, so in particular if it is profinite. Note that in this case, one can always take the subset \( \Delta \) to be a subgroup of \( \Gamma \).

2. Note that for a locally trivial \( \Gamma \)-kernel, the continuity on \( \tilde{\kappa} \), when restricted to \( \Delta \), amounts to ask that the stabilizer of any \( g \in G \) is an open subset of \( \Delta \) closed under the group law. This fits accordingly with the abelian case, in which discrete \( \Gamma \)-modules are supposed to have this property (note that a \( \Gamma \)-kernel in a commutative group is precisely a discrete \( \Gamma \)-module).

Let us now talk about extensions. An extension \( E \) of \( \Gamma \) by \( G \) is an exact sequence of topological groups

\[
1 \to G \xrightarrow{\iota} E \xrightarrow{p} \Gamma \to 1,
\]

i.e. \( \iota \) is a homeomorphism with its image, which is a closed subgroup of \( E \), and \( p \) is an open map. A morphism of extensions of \( \Gamma \) by \( G \) is a commutative diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{\iota} & G & \xrightarrow{p} & E & \xrightarrow{\nu} & \Gamma & \xrightarrow{\iota'} & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \xrightarrow{\iota'} & G & \xrightarrow{p'} & E' & \xrightarrow{\nu'} & \Gamma & \xrightarrow{\iota''} & 1.
\end{array}
\]

Remark that every morphism is an isomorphism. An extension is said to be **locally split** if there exists an open subset \( \Delta \) of \( \Gamma \) containing 1 and a continuous section \( s : \Gamma \to E \)
(i.e. a map such that $p \circ s = \text{id}_\Gamma$) such that its restriction to $\Delta$ respects the group law. It is said to be split if moreover $\Delta$ can be taken to be the whole $\Gamma$. Note that in the case where $\Gamma$ has property (*), $E$ is locally split if and only if the preimage of some open subgroup $\Delta$ in $E$ is split as an extension of $\Delta$ by $G$.

Remark.
We will assume hereafter that every extension admits a continuous section $s : \Gamma \to E$. This is always the case for example if $\Gamma$ is profinite and for locally split extensions. We will show later that in fact this condition is equivalent to local splitness.

To every extension $E$ of $\Gamma$ by $G$ one can associate a $\Gamma$-kernel in $G$ in the following way: choose a continuous section $\hat{s} : \Gamma \to E, \sigma \mapsto \hat{\sigma}$ and define $\tilde{f} : \Gamma \to \text{Aut}(G), \sigma \mapsto \text{int}(\hat{\sigma})$, where int denotes conjugation in $E$. This map is well defined since $G$ is normal in $E$ and one can verify that it is continuous since the action of $E$ on itself by conjugation is continuous. Define then $\kappa = \pi \circ \tilde{f}$. The definition is independent of the choice of the section $\hat{s}$. Indeed, if $\tilde{s} : \sigma \mapsto \tilde{\sigma}$ is another section, then $\tilde{\sigma} = g_\sigma \hat{\sigma}$ for some $g_\sigma \in G$, which gives a map $\tilde{f}$ and a morphism $\tilde{\kappa}$ verifying

$$\tilde{f}_\sigma = \text{int}(g_\sigma \hat{\sigma}) = \text{int}(g_\sigma) \circ \text{int}(\hat{\sigma}),$$

therefore $\tilde{f} \equiv f$ mod $\text{Int}(G)$, i.e. $\tilde{\kappa} = \kappa$. Moreover, when $E$ is locally split we get a locally trivial $\Gamma$-kernel. We say that $\kappa$ is the $\Gamma$-kernel associated to $E$ or that $E$ extends the kernel $\kappa$.

Remark.
An ordinary $\Gamma$-kernel $\kappa$ is not necessarily associated to an extension, even if locally trivial. Such a kernel $\kappa$ is said to be extendable. There exists a criterion for extendability using $H^3(\Gamma, Z)$, where $Z$ is the center of $G$ (c.f. [Mac95, IV.8]).

In the sequel, all our kernels will be extendable.

### 2.2 Non abelian 2-cohomology

Throughout this section, we fix $G$ a discrete group, $\Gamma$ a Hausdorff topological group and $\kappa$ a $\Gamma$-kernel in $G$. We give here the notion of (non-abelian) 2-cocycles and their relation to group extensions.

Definition 2.2. A 2-cocycle is a pair $(f, g)$ where $f : \Gamma \to \text{Aut}(G)$ and $g : \Gamma \times \Gamma \to G$ are continuous maps such that, for $\sigma, \tau, \upsilon \in \Gamma$,

$$f_\sigma \text{ mod } \text{Int}(G) = \kappa_\sigma, \quad (1)$$
$$f_{\sigma\tau} = \text{int}(g_{\sigma, \tau}) \circ f_\sigma \circ f_\tau, \quad (2)$$
$$g_{\sigma, \tau\upsilon} f_\sigma(g_{\tau, \upsilon}) = g_{\sigma, \tau} g_{\sigma, \tau}. \quad (3)$$

The set of 2-cocycles is denoted $Z^2(\Gamma, G, \kappa)$. A cocycle of the form $(f, 1)$ is said to be neutral.
Remark.
Note that the mere existence of such a cocycle forces, by continuity of \( g \), the kernel \( \kappa \) to be locally trivial. Indeed, by continuity, there exists an open subset \( \Delta \) of \( \Gamma \) containing \( 1 \) such that \( g_{\sigma, \tau} = g_{1,1} \) for all \( \sigma, \tau \in \Delta \). Then equation (2) gives \( f_1 = \text{int}(g_{1,1}^{-1}) \) and one can verify then (using repeatedly equations (2) and (3)) that translating \( f \) by \( \text{int}(g_{1,1}) \) gives a map compatible with the group law on \( \Delta \). The natural choice of continuous maps for defining 2-cocycles tells us then that locally trivial \( \Gamma \)-kernels are the natural objects to look at.

The group \( C^1(\Gamma, G) \) of continuous maps \( c : \Gamma \to G \) acts on \( Z^2(\Gamma, G, \kappa) \) by means of the formula \( c \cdot (f, g) = (c \cdot f, c \cdot g) \), where

\[
(c \cdot f)_\sigma = \text{int}(c_\sigma) \circ f_\sigma, \\
(c \cdot g)_{\sigma, \tau} = c_{\sigma \tau} g_{\sigma, \tau} f_\sigma(c_\tau)^{-1} c_\sigma^{-1}.
\]

A direct computation shows that this defines indeed an action.

Definition 2.3. We denote \( H^2(\Gamma, G, \kappa) \) the quotient \( Z^2(\Gamma, G, \kappa)/C^1(\Gamma, G) \). It is called the 2-cohomology set of \( \kappa \). A class in \( H^2(\Gamma, G, \kappa) \) will be called neutral if it contains a neutral cocycle. The set of neutral classes will be denoted \( N^2(\Gamma, G, \kappa) \).

We denote by \( \text{Ext}(\Gamma, G, \kappa) \) the set of isomorphism classes of extensions \( E \) of \( \Gamma \) by \( G \) extending \( \kappa \) (and hence admitting a continuous section).

Proposition 2.4. There is a canonical bijection \( \lambda : \text{Ext}(\Gamma, G, \kappa) \to H^2(\Gamma, G, \kappa) \).

Remarks.
1. The set \( H^2(\Gamma, G, \kappa) \) can be empty, even if \( \kappa \) is locally trivial. More specifically, and given this last proposition, it is empty if and only if \( \kappa \) is not extendable.
2. When \( G \) is a \( \Gamma \)-group, we have a morphism \( f : \Gamma \to \text{Aut}(G) \) inducing the trivial \( \Gamma \)-kernel \( \kappa = \pi \circ f \). In this case the set \( \text{Ext}(\Gamma, G, \kappa) = H^2(\Gamma, G, \kappa) \) becomes a pointed set with the isomorphism class of the semi-direct product induced by \( f \) as base point. This extension corresponds to the element \( [f, 1] \in H^2(\Gamma, G, \kappa) \), where \( [f, g] \) stands for the class of \((f, g)\).
3. When \( A \) is a commutative group, there is a canonical bijection between \( H^2(\Gamma, A, \kappa) \) and the classical 2-cohomology group \( H^2(\Gamma, A) \), where the action of \( \Gamma \) on \( A \) is given by \( \kappa \) (remark that for commutative \( A \) we have \( \text{Aut}(A) = \text{Out}(A) \)).

Proof. We only give here the definition of \( \lambda \) and its inverse, leaving verifications to the reader.

Let \( E \) be an extension representing a class in \( \text{Ext}(\Gamma, G, \kappa) \). Choose a continuous section \( \hat{s} : \sigma \mapsto \hat{\sigma} \) and set, as before, \( f : \Gamma \to \text{Aut}(G), \sigma \mapsto \text{int}(\hat{\sigma}) \). This map has already seen to be continuous and, since \( E \in \text{Ext}(\Gamma, G, \kappa) \), one sees that \( f_\sigma \mod \text{Int}(G) = \kappa_\sigma \). On the other hand, for \( e \in E \) there is a unique way of writing \( e = g\hat{\sigma} \) with \( g \in G, \sigma \in \Gamma \). Note moreover that we have \( \hat{\sigma}g = \hat{\sigma}g\hat{\sigma}^{-1}\hat{\sigma} = f_\sigma(g)\hat{\sigma} \).
In order to define a cocycle, note that \( \hat{\sigma}\hat{\tau} \) and \( \hat{\sigma}\hat{\tau} \) have the same image in \( \Gamma \). Thus, there exists a unique \( g_{\sigma,\tau} \in G \) such that \( \hat{\sigma}\hat{\tau} = g_{\sigma,\tau}\hat{\sigma}\hat{\tau} \). We argue that \((f, g)\) is a cocycle. Indeed, it already verifies equation (11), equation (2) follows readily from the definition of \( f \) and, in order to obtain (3), it suffices to explicitly develop \( \hat{\sigma}^{-1}\hat{\tau}^{-1} \) in two different ways. Finally, we already know that \( f \) is continuous and this is also the case for \( g \). Indeed, we have \( g_{\sigma,\tau} = \hat{\sigma}\hat{\tau}^{-1}\hat{\sigma}^{-1} \) and we know that multiplication and inverse in \( \Gamma \) and \( E \) are continuous functions, as well as \( \hat{s} : \Gamma \rightarrow \hat{E} \). Thus \( g : \Gamma \times \Gamma \rightarrow \hat{E} \) is continuous and, since it takes values in \( G \) which has the induced topology as a subset of \( E \), then so is \( g : \Gamma \times \Gamma \rightarrow G \). Thus \((f, g) \in Z^2(\Gamma, G, \kappa) \) and we set \( \lambda(E) = [f, g] \).

One can verify that \( \lambda \) is well defined. For example, if \( \hat{s} : \sigma \mapsto \hat{\sigma} \) is another section, we have \( \hat{\sigma} = c_{\sigma}\hat{\sigma} \) for some continuous map \( c \in C^1(\Gamma, G) \). A direct computation then gives that \((\hat{f}, \hat{g}) = c \cdot (f, g)\), hence \([\hat{f}, \hat{g}] = [f, g] \). On the other hand, if \( E' \) is another extension of \( \Gamma \) by \( G \) isomorphic to \( E \), then pushing \( \hat{s} \) to \( E' \) gives a section \( \hat{s}' \) of \( \Gamma \) in \( E' \). Once again a direct computation gives that the cocycle \((f', g')\) induced by this section is nothing but the cocycle \((f, g)\), whence \([f, g] = [f', g'] \).

Now let \((f, g)\) be a 2-cocycle. We may assume that \( f_1 = \text{id}_G \) and \( g_{1,1} = g_{\sigma,1} = g_{1,\sigma} = 1 \). Indeed, from the definition of cocycles we can easily obtain the following equalities:
\[
\hat{f}_1 = \text{int}(g_{1,1}^{-1}), \quad g_{\sigma,1} = f_\sigma(g_{1,1}), \quad f_1(g_{1,\sigma}) = g_{1,1},
\]

from which we also get \( g_{1,\sigma} = g_{1,1} \). Setting \( c_\sigma := g_{1,1} \) for every \( \sigma \in \Gamma \) and \((f', g') := c \cdot (f, g)\), one sees that \((f', g')\) has the desired properties.

Consider then the set \( E = G \times \Gamma \) with the natural product topology and define a group law via the formula
\[
(g, \sigma)(h, \tau) = (g_f(h)g_{\sigma^{-1}}, \sigma\tau).
\]

Direct computations then show that this gives a continuous group law in which the trivial element is \((1, 1)\) and the inverse of \((g, \sigma)\) is given by \((f_\sigma^{-1}(g^{-1}g_{\sigma,\sigma^{-1}})), \sigma^{-1})\). One can see \( G \) as a subgroup of \( E \) via \( g \mapsto (g, 1) \) and verify that it is closed and normal in \( E \). Indeed,
\[
(h, \sigma)(g, 1)(h, \sigma)^{-1} = (h, \sigma)(g, 1)(\sigma, \sigma^{-1}) = (\sigma, \sigma^{-1}) = (\sigma, 1) \in G.
\]

Moreover, it is clear that the quotient \( E/G \) gives \( \Gamma \) and that the projection is an open map. We have thus constructed an extension of \( \Gamma \) by \( G \) and we must show that it extends \( \kappa \). Now, one can easily verify that using the section \( \hat{s} : \sigma \mapsto \hat{\sigma} = (1, \sigma) \) this extension induces the cocycle \((f, g)\) and hence extends \( \kappa = f \mod \text{Int}(G) \).

**Remarks.**

1. The inverse of \( \lambda \) given above sends neutral classes in \( H^2(\Gamma, G, \kappa) \) into split extensions. Thus \( N^2(\Gamma, G, \kappa) \) can be also characterized as the set of split extensions of \( \Gamma \) by \( G \).
2. The proof above has as a corollary the fact that *every extension admitting a section*
is locally split. Indeed, such an extension gives a continuous cocycle \((f, g)\) and, following the remark after Definition 2.2, the continuity of \(g\) tells us that there exists an open subset \(\Delta\) of \(\Gamma\) containing 1 such that \(g_{\sigma, \tau} = g_{1, 1}\) for all \(\sigma, \tau \in \Delta\). Then, up to modifying the cocycle as in the second part of the proof above, one easily recognizes that the section \(\hat{s} : \sigma \mapsto \hat{\sigma} = (1, \sigma)\) respects the group law on \(\Delta\), proving that \(E\) is locally split.

We now give a nice description of the set \(H^2(\Gamma, G, \kappa)\). Let \(Z\) be the center of \(G\). It is a characteristic subgroup and hence invariant by automorphisms. This gives thus a morphism \(\text{Aut}(G) \rightarrow \text{Aut}(Z) = \text{Out}(Z)\) which is trivial on inner automorphisms. By composition with \(\kappa : \Gamma \rightarrow \text{Out}(G)\) we get a map \(\Gamma \rightarrow \text{Out}(Z)\) which we abusively still denote \(\kappa\). Since \(Z\) is commutative, \(\kappa\) defines a trivial \(\Gamma\)-kernel over \(Z\).

Let \((f, g)\) be a 2-cocycle in \(Z\). Now, since \(\text{Aut}(Z) = \text{Out}(Z)\), we find that \(f = \kappa\) and thus \(g\) is a 2-cocycle in the classic (abelian) sense. This shows that \(H^2(\Gamma, Z, \kappa)\) can be identified with the abelian group \(H^2(\Gamma, Z)\) via \([f, g] \mapsto [g]\). This group acts on \(H^2(\Gamma, G, \kappa)\) in the following way: for \(z \in Z^2(\Gamma, Z)\), \((f, g) \in Z^2(\Gamma, G, \kappa)\), set

\[ [z] \cdot [f, g] = [f, zg]. \]

Using the fact that the \(z_\sigma\) are central in \(G\) for \(\sigma \in \Gamma\), one can easily show by direct computations that the action at the level of the \(H^2\)'s is well defined.

**Proposition 2.5.** If \(H^2(\Gamma, G, \kappa)\) is non empty (i.e. if \(\kappa\) is extendable), then the action of \(H^2(\Gamma, Z)\) defined above is free.

Assume moreover that \(\Gamma\) has property (*) and that, for any two 2-cocycles \((f, g), (f', g') \in Z^2(\Gamma, G, \kappa)\) there exists an open subgroup \(\Delta \subset \Gamma\) and an element \(g \in G\) such that \(f_\sigma = \text{int}(g) \circ f'_\sigma\) for all \(\sigma \in \Delta\). Then the action is also transitive.

**Remarks.**

1. The additional hypotheses added for the second result are naturally verified in the context of Galois cohomology and \(k\)-kernels as we will see in section 2.5 below.
2. The author ignores if there is any natural hypothesis on the abstract group \(G\) that would imply those giving the transitivity of the action. However, this is trivial for example if \(\Gamma\) has the discrete topology and the proof given here below can then be found for example in [Mac95, IV.8.8].

**Proof.** Let us prove the first assertion. Let \(z \in Z^2(\Gamma, Z)\) and \((f, g) \in Z^2(\Gamma, G, \kappa)\) and suppose that we have \([z] \cdot [f, g] = [f, g]\). There exists then a continuous map \(c : \Gamma \rightarrow G\) such that

\[ f_\sigma = \text{int}(c_\sigma) \circ f_\sigma, \]

\[ z_{\sigma, \tau} g_{\sigma, \tau} = c_{\sigma \tau} g_{\sigma, \tau} f_\sigma(c_\tau)^{-1} c_\sigma^{-1}. \]

We deduce from the first equation that \(c_\sigma \in Z\) and hence the second equation becomes

\[ z_{\sigma, \tau} = c_{\sigma \tau} f_\sigma(c_\tau)^{-1} c_\sigma^{-1}, \]

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which tells us that $\mathfrak{z}$ is a coboundary and hence its class is trivial.

Let us prove now the transitivity of the action under the additional assumptions. Let $\eta = [f, g]$, $\eta' = [f', g']$ be two elements in $H^2(\Gamma, G, \kappa)$. Since both $f$ and $f'$ lift $\kappa$, for every $\sigma \in \Gamma$ we have $f_{\sigma} \equiv f'_{\sigma} \mod \text{Int}(G)$ and thus $c_{\sigma} := f_{\sigma}f_{\sigma}^{-1} \in \text{Int}(G) = G/Z$. Moreover, by our assumptions, $c$ is constant over an open subgroup $\Delta \subset \Gamma$. Using equation (2) and the continuity of $g$ and $g'$ one may finally deduce that $c$ may be taken to be continuous for the discrete topology over $G/Z$. We may then lift $c$ into a continuous map $c : \Gamma \to G$ and one easily sees that $c \cdot f' = f$. Up to modifying $(f', g')$ by $c$, we may then assume that $f = f'$ in what follows.

Define $\mathfrak{z}_{\sigma, \tau} := g_{\sigma, \tau}g^{-1}_{\sigma, \tau}$. Then the equalities

$$f_{\sigma, \tau} = \text{int}(g_{\sigma, \tau}) \circ f_{\sigma} \circ f_{\tau},$$

$$f_{\sigma, \tau} = \text{int}(g'_{\sigma, \tau}) \circ f_{\sigma} \circ f_{\tau},$$

tell us that $\text{int}(\mathfrak{z}_{\sigma, \tau}) = \text{int}(g_{\sigma, \tau}) \circ \text{int}(g'^{-1}_{\sigma, \tau}) = \text{id}_G$, hence $\mathfrak{z}_{\sigma, \tau} \in Z$. We argue that $\mathfrak{z}$ is in fact a cocycle. Indeed, we have

$$\mathfrak{z}_{\sigma, \tau \nu}^\sigma \mathfrak{z}_{\tau, \nu}^\sigma = g_{\sigma, \tau \nu}^\sigma g'^{-1}_{\sigma, \tau \nu} g_{\tau, \nu}^\sigma g'^{-1}_{\tau, \nu} = g_{\sigma, \tau \nu}^\sigma g_{\tau, \nu}^\sigma g'^{-1}_{\sigma, \tau \nu} g'^{-1}_{\tau, \nu},$$

since $\mathfrak{z}$ has its values in $Z$ and $Z$ is $\Gamma$-invariant. Using equation (3) one gets then

$$\mathfrak{z}_{\sigma, \tau \nu}^\sigma \mathfrak{z}_{\tau, \nu}^\sigma = g_{\sigma, \tau \nu}g_{\sigma, \tau \nu}^{-1} g'^{-1}_{\sigma, \tau \nu} = g_{\sigma, \tau \nu}g_{\sigma, \tau \nu}^{-1} g'^{-1}_{\sigma, \tau \nu} g_{\sigma, \tau \nu} g_{\sigma, \tau \nu}^{-1} g_{\sigma, \tau \nu} = \mathfrak{z}_{\sigma, \tau \nu} g_{\sigma, \tau \nu} g_{\sigma, \tau \nu}^{-1} g_{\sigma, \tau \nu}.$$

We get thus that $[f, g'] = [\mathfrak{z}] : [f, g]$, which proves transitivity.

\section*{2.3 Morphisms}

Let us look now at morphisms of $\Gamma$-kernels and at their consequences on the sets of 2-cohomology. Recall that we assume that all our kernels are extendable and thus locally trivial (otherwise, there is nothing to be defined in this section).

Let $\lambda : \Gamma' \to \Gamma$ be a morphism of topological groups and let $\kappa$ (resp. $\kappa'$) be a $\Gamma$-kernel (resp. $\Gamma'$-kernel) in $G$ (resp. in $G'$). A group morphism $\mu : G \to G'$ is said to be a morphism from $\kappa$ to $\kappa'$ compatible with $\lambda$ if there exist maps $f, f'$ lifting $\kappa$ and $\kappa'$ verifying, for $g \in G$ and $\sigma' \in \Gamma'$,

$$f'_{\sigma'}(\mu(g)) = \mu(f_{\lambda(\sigma')}(g)).$$

When $\Gamma' = \Gamma$, $\lambda = \text{id}_\Gamma$, $G' = G$ and $\mu \in \text{Aut}(G)$, this amounts to $\bar{\mu} \kappa_\sigma = \kappa'_\sigma \bar{\mu}$ for all $\sigma \in \Gamma$, where $\bar{\mu}$ is the class of $\mu$ in $\text{Out}(G)$.

Given such a morphism of kernels, we define a relation (or “multi-valued map”) $(\lambda, \mu)_*^2 : H^2(\Gamma, G, \kappa) \to H^2(\Gamma', G', \kappa')$ by the means of cocycles: $\eta \in H^2(\Gamma, G, \kappa)$ is related to $\eta' \in H^2(\Gamma', G', \kappa')$ if and only if there exist cocycles $(f, g)$, $(f', g')$ representing $\eta$ and $\eta'$ such that, for $g \in G$ and $\sigma', \tau' \in \Gamma'$,

$$f'_{\sigma'}(\mu(g)) = \mu(f_{\lambda(\sigma')}(g)), \quad g'_{\sigma', \tau'} = \mu(g_{\lambda(\sigma'), \lambda(\tau')}).$$

\numberwithin{equation}{section}
Using the constructions in the proof of Proposition 2.4, one can verify that this definition is equivalent to the existence of a commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & G \\
\mu \downarrow & & \downarrow \\
1 & \longrightarrow & \text{Im}(\mu) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \text{Im}(\mu) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \text{Im}(\mu) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G' \\
\downarrow & & \downarrow \\
1 & \longrightarrow & E' \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \Gamma' \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
\]

where \(E\) represents \(\eta\) and \(E'\) represents \(\eta'\).

**Remark.**
Note that in the particular case where \(\Gamma' = \Gamma\) and \(\lambda = \text{id}_\Gamma\), the diagram simplifies to:

\[
\begin{array}{ccc}
1 & \longrightarrow & G \\
\mu \downarrow & & \downarrow \\
1 & \longrightarrow & \text{Im}(\mu) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \text{Im}(\mu) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
\]

One can even show that in some particular cases, the relation \((\lambda, \mu)_2^\ast\) is in fact a (single-valued) map, namely:

**Proposition 2.6.** Let \(G, G', \Gamma, \Gamma', \lambda, \mu\) as above. Assume either that \(G'\) is abelian or that \(\mu\) is surjective. Then the relation \((\lambda, \mu)_2^\ast\) is in fact a map

\[
H^2(\Gamma, G, \kappa) \to H^2(\Gamma', G', \kappa').
\]

The proof of this proposition is an easy exercise left to the reader. In this case of surjective \(\mu\) however, this is also easily seen with the diagram above: all vertical arrows in the middle are in fact easily seen to be uniquely defined in this case.

Following [Spr66, 1.18-19], let us have a look at a particular case. Let \(\Gamma' = \Gamma\), \(\lambda = \text{id}_\Gamma\) and \(\mu \in \text{Aut}(G)\). Then \(\mu\) is clearly a morphism from \(\kappa\) to \(\kappa'\) compatible with \(\text{id}_\Gamma\), where \(\kappa'_\sigma = \bar{\mu}\kappa_\sigma\bar{\mu}^{-1}\). Proposition 2.6 tells us that \(\mu_2^\ast := (\text{id}_\Gamma, \mu)_2^\ast\) is a map in this case. Furthermore, we have the following:

**Proposition 2.7.** If \(\mu \in \text{Int}(G)\), we have \(\kappa' = \kappa\) and \(\mu_2^\ast\) is the identity map on \(H^2(\Gamma, G, \kappa)\).
Proof. It is clear that $\kappa' = \kappa$. Given that $\lambda = \text{id}_\Gamma$ and $\mu(g) = hgh^{-1}$ for some fixed $h \in G$, the formulas in (2.6) tell us that $\mu_\ast^T(f, g) = (f', g')$ with

$$f'_\sigma(g) = \mu(f\sigma(\mu^{-1}(g))) = hf\sigma(h^{-1})f\sigma(g)f\sigma(h)h^{-1},$$

$$g'_{\sigma, \tau} = \mu(g_{\sigma, \tau}) = hgh^{-1}.$$ 

We deduce that $f' = c \cdot f$ with $c_\sigma = hgh^{-1}$. On the other hand, using equation (2), one easily finds that $g'_{\sigma, \tau} = (c \cdot g)_{\sigma, \tau}$, whence $(f', g') = c \cdot (f, g)$, i.e. $[f', g'] = [f, g]$. \qed

This result can be further generalized. Fix $(f, g) \in Z^2(\Gamma, G, \kappa)$ and set, for $\tau \in \Gamma$,

$$\lambda_\tau : \Gamma \to \Gamma, \sigma \mapsto \tau^{-1}\sigma \tau \quad \text{and} \quad \mu_\tau : G \to G, g \mapsto f_\tau(g).$$

One can see that $\mu_\tau$ is a morphism from $\kappa$ to $\kappa$, compatible with $\lambda_\tau$ which, by Proposition 2.6 gives us a map $(\lambda_\tau, \mu_\tau)^2_\ast$ from $H^2(\Gamma, G, \kappa)$ to itself. Note moreover that if we had started with a cocycle $(f', g')$, then we would have $f'_\tau = \text{int}(g_\tau) \circ f_\tau$ for some $g_\tau \in G$ and we would have thus got the map

$$(\lambda_\tau, \text{int}(g_\tau) \circ \mu_\tau)^2_\ast = (\text{id}_\Gamma, \text{int}(g_\tau))^2_\ast \circ (\lambda_\tau, \mu_\tau)^2_\ast,$$

which is equal to $(\lambda_\tau, \mu_\tau)^2_\ast$ given Proposition 2.7. Thus $(\lambda_\tau, \mu_\tau)^2_\ast$ does not depend on the choice of $(f, g)$ and there’s even more.

**Proposition 2.8.** $(\lambda_\tau, \mu_\tau)^2_\ast$ is the identity map on $H^2(\Gamma, G, \kappa)$ for every $\tau \in \Gamma$.

**Remark.**

In the context of Galois cohomology, this result is a sort of analog of the classic result in abelian Galois cohomology stating that, for a field $k$, the groups $H^i(k_s/k, A)$ do not depend on the choice of the separable closure $k_s$ of $k$ for $A$ a commutative $k$-group (c.f. [Ser02 II.1.1] for this classic result and section 2.5 below for the corresponding non-abelian result).

**Proof.** From the definition of $(\lambda_\tau, \mu_\tau)$ one deduces that it sends $[f', g']$ onto $[f'', g'']$, with

$$f''_\sigma = f_\tau \circ f'_{\tau^{-1}\sigma \tau} \circ f^{-1}_\tau,$$

$$g''_{\sigma, \tau} = f_\tau(g'_{\tau^{-1}\sigma \tau}) = f'_{\tau^{-1}\sigma \tau},$$

where one can verify that $(f'', g'')$ is actually a cocycle if $(f', g')$ is. Now, if we recall that $f'_\tau = \text{int}(g_\tau) \circ f_\tau$ for some $g_\tau \in G$, then using equation (2) for $(f', g')$, we see that

$$f''_\sigma = f_\tau \circ f'_{\tau^{-1}\sigma \tau} \circ f^{-1}_\tau =$$

$$\text{int}(g^{-1}_\tau) \circ f'_\tau \circ f'_{\tau^{-1}\sigma \tau} \circ f^{-1}_\tau \circ \text{int}(g_\tau)$$

$$= \text{int}(g^{-1}_\tau) \circ \text{int}(g'_{\tau^{-1}\sigma \tau}) \circ f'_\tau \circ f^{-1}_\tau \circ \text{int}(g_\tau)$$

$$= \text{int}(g^{-1}_\tau) \circ \text{int}(g'_{\tau^{-1}\sigma \tau}) \circ \text{int}(g'_{\tau^{-1}\sigma \tau}) \circ \text{int}(f'_\sigma(g_\tau)) \circ f'_\sigma$$

$$= \text{int}(g^{-1}_\tau) \circ \text{int}(g'_{\tau^{-1}\sigma \tau}) \circ f'_\sigma(g_\tau) \circ f'_\sigma.$$

Setting $c_\sigma = g^{-1}_\tau g'_{\tau^{-1}\sigma \tau} f'_{\tau^{-1}\sigma \tau} f'_\sigma(g_\tau)$, one can show that $(f'', g'') = c \cdot (f', g')$ and thus $[f', g'] = [f'', g'']$. Indeed, it is by definition of $c$ that we have $f'' = c \cdot f'$. As for $g'' = c \cdot g'$, this is a (long!) direct computation that uses equations (2) and (3). \qed
2.4 \( k \)-kernels

We pass now to Galois cohomology. We will define thus the notion of \( k \)-kernel in a smooth algebraic \( k_s \)-group \( \bar{G} \), which is nothing but a \( \Gamma \)-kernel (in the sense of Definition 2.1) with \( \Gamma = \text{Gal}(k_s/k) \) and \( k_s \) a separable closure of \( k \). However, in order to “respect” the algebraic group structure of \( \bar{G} \), we need to add some restrictions to this \( \Gamma \)-kernel.

Let \( \bar{G} \) be a smooth algebraic group over \( k_s \). We denote \( p : \bar{G} \to \text{Spec} (k_s) \) its structure morphism and \( \text{Aut}(\bar{G}) \) its group of automorphisms (of \( k_s \)-group schemes). For \( \sigma \in \Gamma = \text{Gal}(k_s/k) \), we denote \( \sigma^e : \text{Spec} (k_s) \to \text{Spec} (k_s) \) the automorphism induced by \( \sigma^{-1} \). Notice that \((\sigma\tau)^e = \sigma^e \tau^e \). Finally, let \( \sigma \cdot \bar{G} \) be the group obtained by base change from \( \sigma^e \). Note that as a \( k_s \)-scheme, it is isomorphic to \( \sigma^e \circ p : \bar{G} \to \text{Spec} k_s \).

**Definition 2.9.** A \( \sigma \)-semialgebraic automorphism of \( \bar{G} \) is a \( k_s \)-group isomorphism \( s : \sigma^e \cdot \bar{G} \to \bar{G} \). A \( k \)-semialgebraic automorphism is simply a \( \sigma \)-semialgebraic automorphism for some \( \sigma \in \Gamma \). This last set is denoted \( \text{SAut}(\bar{G}/k) \), or simply \( \text{SAut}(\bar{G}) \) when there is no ambiguity on \( k \).

**Remark.**

Note that a \( \sigma \)-semialgebraic morphism is in the end nothing but a morphism of schemes \( s : \bar{G} \to \bar{G} \) such that the diagram

\[
\begin{array}{ccc}
\bar{G} & \xrightarrow{s} & \bar{G} \\
p & & p \\
\text{Spec} (k_s) & \xrightarrow{\sigma^e} & \text{Spec} (k_s),
\end{array}
\]

commutes, to which one asks further that the morphism between the \( k_s \)-algebraic group \( \sigma^e \cdot \bar{G} \) (whose structure morphism is given by the composition \( \sigma^e \circ p \)) and the \( k_s \)-algebraic group \( \bar{G} \) is an actual isomorphism of algebraic groups. In particular, we can see that \( \text{Aut}(\bar{G}) \) is a subgroup of \( \text{SAut}(\bar{G}) \) by taking \( \sigma = \text{id}_{k_s} \).

Given the remark above, one sees that for any \( k \)-semialgebraic automorphism \( s \), there is a unique element \( \sigma \in \Gamma \) such that \( s = \sigma \)-semialgebraic isomorphism. Indeed, thanks to diagram (5) one can see that \( \sigma^e = p \circ s \circ e \), where \( e : \text{Spec} k_s \to \bar{G} \) denotes the identity point of \( \bar{G} \), and this clearly defines \( \sigma \) uniquely. In particular, the \( \text{id}_{k} \)-semialgebraic automorphisms correspond precisely to the subgroup \( \text{Aut}(\bar{G}) \). We will denote \( \gamma : \text{SAut}(\bar{G}) \to \Gamma \) the map we have thus obtained.

Furthermore, it is now easy to see that the set \( \text{SAut}(\bar{G}) \) is a group. Indeed, one can verify that if \( s, t \in \text{SAut}(\bar{G}) \) are respectively a \( \sigma \)-semialgebraic automorphism and a \( \tau \)-semialgebraic automorphism, then \( s \circ t \) is a \( \sigma\tau \)-semialgebraic automorphism. This amounts in fact to the commutativity of the following diagram:

\[
\begin{array}{ccc}
\bar{G} & \xrightarrow{t} & \bar{G} \\
p & & p \\
\text{Spec} (k_s) & \xrightarrow{\tau^e} & \text{Spec} (k_s),
\end{array} \quad \begin{array}{ccc}
\bar{G} & \xrightarrow{s} & \bar{G} \\
p & & p \\
\text{Spec} (k_s) & \xrightarrow{\sigma^e} & \text{Spec} (k_s).
\end{array}
\]

\[
\begin{array}{ccc}
\bar{G} & \xrightarrow{t \circ s} & \bar{G} \\
p & & p \\
\text{Spec} (k_s) & \xrightarrow{(\sigma\tau)^e} & \text{Spec} (k_s).
\end{array}
\]
Moreover, we get that $\gamma$ is a group morphism giving the following exact sequence:

$$1 \to \text{Aut}(\bar{G}) \to \text{SAut}(\bar{G}) \overset{\gamma}{\to} \Gamma,$$

where we immediately remark that $\gamma$ need not be surjective in general.

Now, we know that $\text{Int}(\bar{G})$ is normal in $\text{Aut}(\bar{G})$ and it is easy to see that it is still normal in $\text{SAut}(\bar{G})$. Indeed, for $s \in \text{SAut}(\bar{G})$ and $g \in \bar{G}(k_s)$ (i.e. $g : \text{Spec}(k_s) \to \bar{G}$), we have $s \circ \text{int}(g) \circ s^{-1} = \text{int}(s(g))$, where $s(g) = s \circ g \circ \gamma(s)^{-1}$. We thus define

$$\text{Out}(\bar{G}) := \text{Aut}(\bar{G})/\text{Int}(\bar{G}),$$

$$\text{SOut}(\bar{G}) = \text{SOut}(\bar{G}/k) := \text{SAut}(\bar{G}/k)/\text{Int}(\bar{G}).$$

The exact sequence (6) gives then the new exact sequence:

$$1 \to \text{Out}(\bar{G}) \to \text{SOut}(\bar{G}) \overset{q}{\to} \Gamma.$$  (7)

One can actually verify (using diagram (5) for example) that the formula defining $s(g)$ defines an action of $\text{SAut}(\bar{G})$ on $\bar{G}(k_s)$, which means that there is a group morphism $r : \text{SAut}(\bar{G}) \to \text{Aut}(\bar{G}(k_s))$.

Remark that this homomorphism need not be injective, as it is the case for example for finite algebraic groups. We will nevertheless make an abuse of notation by identifying a map $\Gamma \to \text{SAut}(\bar{G})$ with the map $\Gamma \to \text{Aut}(\bar{G}(k_s))$ obtained by composition with $r$. In fact, since $\text{Ker}(r)$ and $\text{Int}(\bar{G})$ are both normal in $\text{SAut}(\bar{G})$ and have a trivial intersection, one can easily see that the subgroup generated by them is isomorphic to their direct product. In particular, we see that $r$ factors through the quotient by $\text{Int}(\bar{G})$, giving a morphism $\bar{r} : \text{SOut}(\bar{G}) \to \text{Out}(\bar{G}(k_s))$.

Let us now recall the notion of $k$-form. A $k$-form of a $k_s$-group $\bar{G}$ is a $k$-group $G$ which is isomorphic to $\bar{G}$ after base change to $k_s$, i.e. such that $G \times_k k_s \cong \bar{G}$. A $k_s$-group admitting a $k$-form is said to be defined over $k$. Remark that any $k$-form $G$ of $\bar{G}$ defines a splitting $f_G : \Gamma \to \text{SAut}(\bar{G})$ of sequence (6) by sending $\sigma \in \Gamma$ to the morphism $\sigma_s := \text{id}_G \times \sigma : G \times_k k_s \to G \times_k k_s$. Recall finally that for every smooth $k_s$-group $\bar{G}$ there exists a finite separable extension $K/k$ such that $\bar{G}$ admits a $K$-form $\tilde{G}$. If we note $\Gamma_K = \text{Gal}(k_s/K) \subset \Gamma$, this gives us a “partial splitting” $f_{\tilde{G}} : \Gamma_K \to \text{SAut}(\tilde{G}/K) \subset \text{SAut}(\bar{G}/k)$ of sequence (6).

**Definition 2.10.** A $k$-kernel in $\bar{G}$ is a group morphism $\kappa : \Gamma \to \text{SOut}(\bar{G})$ such that

(i) $\kappa$ splits sequence (7), i.e. $q \circ \kappa$ is the identity on $\Gamma$, and

(ii) there exists a section $f : \Gamma \to \text{SAut}(\bar{G})$ of (6) lifting $\kappa$ such that $r \circ f$ is continuous in the sense of Section 2.1.
A pair $L = (\bar{G}, \kappa)$ as above is simply called a $k$-kernel. It will be said to be trivial if there exists such an $f$ which moreover is a group morphism (which will then split $\varphi$).

**Remarks.**

1. If we equip $G(k_s)$ with the discrete topology and $SAut(\bar{G})$ with the weak topology with respect to the evaluation maps (maps which factor through $Aut(\bar{G}(k))$ via $r$), then one easily sees that a $k$-kernel $L = (\bar{G}, \kappa)$ defines a $\Gamma$-kernel in $\bar{G}(k_s)$ via $\bar{r} \circ \kappa$ and $r \circ f$. However, it is not true that any $\Gamma$-kernel in $\bar{G}(k_s)$ will induce a $k$-kernel. This is precisely what we stated at the beginning of this section: the action of $\Gamma$ must “respect” the algebraic structure of $\bar{G}$ by only acting via semi-algebraic automorphisms. Noting moreover that the mere existence of a $K$-form for a finite extension $K/k$ tells us that every $k$-kernel is locally trivial as a $\Gamma$-kernel, this also tells us that we can apply all the results in the sections above to $k$-kernels.

2. One can in fact prove that condition 2 in this last definition can be replaced by

$$(ii)' \text{ there exists a continuous map } f : \Gamma \to Aut(\bar{G}(k_s)) \text{ lifting } \bar{r} \circ \kappa.$$  

Indeed, knowing that the subgroup of $SAut(\bar{G})$ generated by $Ker(r)$ and $Int(\bar{G})$ is isomorphic to their direct product, one can easily see that there is a unique way of lifting such a map $f : \Gamma \to Aut(\bar{G}(k_s))$ to a map $\Gamma \to SAut(\bar{G})$ which will also lift $\kappa$. The second property will be trivially verified. Moreover, it is easy to see that if $f$ is a group morphism then this unique lift will also be a morphism. One can thus (re)define the notions of trivial and locally trivial $k$-kernels in the same way than for $\Gamma$-kernels without any ambiguity.

3. Our definition of $k$-kernel differs from that given by Flicker, Scheiderer and Sujatha in [FSS98, 1.11], ours being slightly more general. We give below a more precise version of $k$-kernel which coincides with their definition and which we will use further on.

### 2.5 Non abelian Galois 2-cohomology

Having already established the fact that all we have said about $\Gamma$-kernels will apply to $k$-kernels, one is lead to the following definition.

**Definition 2.11.** For $L = (\bar{G}, \kappa)$ a $k$-kernel, we set

$$Z^2(k, L) = Z^2(k, \bar{G}, \kappa) := Z^2(\Gamma, \bar{G}(k_s), \bar{r} \circ \kappa),$$

$$H^2(k, L) = H^2(k, \bar{G}, \kappa) := H^2(\Gamma, \bar{G}(k_s), \bar{r} \circ \kappa),$$

$$N^2(k, L) = N^2(k, G, \kappa) := N^2(\Gamma, G(k_s), \bar{r} \circ \kappa).$$

Let us now resume the remark at the end of section 2.3. Let $k'_s$ be another separable closure of $k$ and write $\Gamma' = Gal(k'_s/k)$. For a $k$-group $G$ we thus have the $\Gamma$-kernel $L_G = (G_{k_s}, \kappa_G)$ and the $\Gamma'$-kernel $L'_G = (G_{k'_s}, \kappa'_G)$, both trivial. Let $\varphi : k'_s \to k_s$ be an isomorphism. As such it induces isomorphisms $\lambda_{\varphi} : \Gamma' \to \Gamma$ and $\mu_{\varphi} : G(k_s) \to G(k'_s)$. Now, clearly $\mu_{\varphi}(\lambda_{\varphi}(\sigma')(g)) = \sigma'_*(\mu_{\varphi}(g))$ for $\sigma' \in \Gamma'$ and $g \in G(k_s)$, so that $\mu_{\varphi}$ is a
morphism from $\kappa_G$ onto $\kappa'_G$ compatible with $\lambda_\varphi$. Moreover, by Proposition 2.6 the surjectivity of $\mu_\varphi$ tells us that the relation

$$(\lambda_\varphi, \mu_\varphi)^2 : H^2(\Gamma, G(k_s), \kappa_G) \twoheadrightarrow H^2(\Gamma', G(k'_s), \kappa'_G),$$

is in fact a map, and one can show that:

**Lemma 2.12.** $(\lambda_\varphi, \mu_\varphi)^2$ is a bijection independent of $\varphi$.

**Proof.** It suffices to prove the result when $k'_s = k_s$ and $\varphi \in \text{Aut}_{k_s}(k_s) = \Gamma$. In this case, we have

$$\lambda_\varphi(\sigma) = \varphi^{-1}\sigma \varphi \quad \text{and} \quad \mu_\varphi(g) = \varphi\ast(g) = (f_G)_\varphi(g).$$

The result thus follows from Proposition 2.8.

Lemma 2.12 tells us then that for any $k$-group $G$ we can identify the sets $H^2(\Gamma, L_G)$ and $H^2(\Gamma', L'_G)$ in a canonical way, allowing us to introduce the (now non-abusive) notation $H^2(k, G) := H^2(k, L_G)$. As we mentioned in section 2.2 this set has a base point corresponding to the class $[f_G, 1]$.

**Remark.** When $G$ is abelian, our $H^2(k, G)$ coincides with the classic (abelian) $H^2(k, G)$. Note that this last group is also well defined if $G$ is non-smooth and it admits the same cocycle description: it is indeed nothing but the étale cohomology group $H^2_{\text{ét}}(k, G)$. This remark will be useful later, when we will describe the (non-abelian) 2-cohomology of an algebraic group via the 2-cohomology of its center, which may not be smooth (cf. Section 3.3).

### 2.6 Some continuity issues in positive characteristic

If $G$ is a $k$-form of $\bar{G}$, then the morphism $f_G : \Gamma \to \text{SAut}(\bar{G})$ defined in section 2.4 above is clearly seen to be continuous in the sense of Definition 2.10. Moreover, this morphism gives by composition a morphism $\kappa_G : \Gamma \to \text{SOut}(\bar{G})$ which clearly splits (7), giving thus a trivial $k$-kernel. On the other hand, it would be natural to expect that a continuous splitting of sequence (6) should induce an action of $\Gamma$ on $\bar{G}(k_s)$ which, by a Galois descent argument, would account for the existence of a $k$-form of $\bar{G}$. This, however, may not be the case if our base field has positive characteristic. This issue was overcome by Flicker, Scheiderer and Sujatha in [FSS98] with their definition of $k$-kernel, which uses the following stronger notion of continuity (cf. [FSS98, 1.10]).

**Definition 2.13.** Let $\bar{G}$ be a smooth $k_s$-group, let $\bar{G}$ be a $K$-form of $\bar{G}$ for some finite separable extension $K/k$ and let $\phi : \Gamma_K \to \text{SAut}(\bar{G}/K)$ be the partial splitting thus obtained. A map $\Gamma \to \text{SAut}(\bar{G})$ will be said to be **alg-continuous** if, for every $\sigma \in \Gamma$, the map

$$\Gamma_K \to \text{Aut}(\bar{G}) : \tau \mapsto \phi^{-1}_\tau f^{-1}_\sigma f_{\sigma \tau},$$

is locally constant.

Let now $\kappa$ be a $k$-kernel in $\bar{G}$. We will say that $\kappa$ is **admissible** if it admits an alg-continuous lift $f : \Gamma \to \text{SAut}(\bar{G})$. 

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Remark.
Note that an alg-continuous map \( f \) coincides up to translation with \( \phi \) over an open subgroup of \( \Gamma \) (take \( \sigma = 1 \) above). In particular, the definition of alg-continuous is independent of the field \( K \) and of the \( K \)-form \( \bar{G} \), since we know that any two forms of \( \bar{G} \) over finite separable extensions of \( k \) will become isomorphic over a suitable finite extension \( K' \) and hence the induced morphisms \( \Gamma_{K'} \rightarrow \text{SAut}(\bar{G}/K') \) will coincide.

If \( \bar{G}(k_s) \) is equipped with the discrete topology and \( \text{SAut}(\bar{G}) \) with the weak topology with respect to the evaluation maps as above, then the notion of alg-continuity is easily seen to imply continuity of the maps \( \Gamma \rightarrow \text{SAut}(\bar{G}) \) and \( \Gamma \rightarrow \text{Aut}(\bar{G}(k_s)) \) in the topological sense. These two notions actually coincide in characteristic zero, but this is not the case in positive characteristic (cf. \cite[1.6–10]{FSS98}).

The notion of admissible \( k \)-kernel is what Flicker, Scheiderer and Sujatha simply call \( k \)-kernel (cf. \cite[1.11]{FSS98}). The utility of these new definitions is that one can show that every trivial admissible \( k \)-kernel admits a \( k \)-form (cf. \cite[1.15]{FSS98}) and, conversely, it is easy to see that every \( k \)-kernel coming from a \( k \)-form is admissible. This is hence the good generalization to any smooth algebraic group of the definition given by Borovoi in \cite[1.3]{Bor93} for linear groups in characteristic 0. In \cite{Spr66}, Springer does not even take into account the continuity problem for defining his \( k \)-kernels. Neither does he take precautions in the abstract context of \( \Gamma \)-kernels for \( \Gamma \) a topological group, so one must be careful when using his results. For further discussion on these issues, we send the reader to \cite[1.6–1.14]{FSS98}.

From now on, we will assume that every \( k \)-kernel is admissible. If we gave first the more general definition of \( k \)-kernels is simply because we think that it is more intuitive and because it gives anyway the good notion in characteristic 0, so that one can forget about “admissibility” in this context.

Remark.
We must remark that the definition of 2-cocycles given in \cite[1.17]{FSS98} differs also from ours since they consider pairs \( (f, g) \) where \( f \) takes values in \( \text{SAut}(\bar{G}) \) and is alg-continuous\(^2\). However, by \cite[1.13]{FSS98}, we know that if \( \kappa \) is admissible, then any continuous lift \( f \) of \( \kappa \) is alg-continuous. Then, once we know that we can lift continuous maps \( \Gamma \rightarrow \text{Aut}(\bar{G}(k_s)) \) lifting \( \bar{r} \circ \kappa \) to continuous maps \( \Gamma \rightarrow \text{SAut}(\bar{G}) \) lifting \( \kappa \), there is no harm in sticking to our definition as long as we keep the assumption on admissibility.

2.7 Description of \( H^2(k, L) \) via the center of \( L \)
Let us now translate Proposition 2.5 to the language of Galois cohomology. The first thing one should remark then, is that any (admissible) \( k \)-kernel \( L = (\bar{G}, \kappa) \) such that \( \bar{G} \)
is a commutative \( k_s \)-group is trivial and admits a canonical \( k \)-form \( G \) by what we have stated before.

Thus, if we let \( L = (\tilde{G}, \kappa) \) be a \( k \)-kernel and let \( \tilde{Z} \) be the center of \( \tilde{G} \), then in the same fashion as in section 2.2 there is a canonical morphism \( \text{SAut}(\tilde{G}) \to \text{SAut}(\tilde{Z}) = \text{SOut}(\tilde{Z}) \) that is trivial over inner morphisms. Note that \( \tilde{Z} \) need not be smooth a priori. However, the definition of \( \text{SAut}(\tilde{G}) \) does not use the smoothness assumption, hence \( \text{SAut}(\tilde{Z}) \) is well defined even in this case. Composing then this morphism with \( \kappa \), we get an alg-continuous section of (6) for \( \tilde{Z} \) which, accordingly with what we just said, defines a canonical \( k \)-form \( Z \) of \( \tilde{Z} \). We will call \( Z \) the center of \( L \). Note that if \( L = (G_{k_s}, \kappa_G) \) for a given \( k \)-group \( G \), then \( Z \) will coincide with the center of \( G \).

Having said this, Proposition 2.5 gives us the following result.

**Proposition 2.14.** Let \( L = (\tilde{G}, \kappa) \) be a \( k \)-kernel and let \( Z \) be its center (note that it may not be smooth). Then \( H^2(k, Z) \) acts freely and transitively on \( H^2(k, L) \). In particular, when \( \tilde{G} \) comes from a \( k \)-group \( G \), there is a canonical way of identifying \( H^2(k, G) \) with \( H^2(k, Z) \) by sending the trivial element in \( H^2(k, Z) \) to the base point in \( H^2(k, G) \).

**Proof.** It is clear that \( \tilde{Z}(k_s) \) is the center of \( \tilde{G}(k_s) \) and that \( \Gamma \) has property (1) since it is profinite. Then in order to apply Proposition 2.5 we must show that for any two 2-cocycles

\[(f, g), (f', g') \in Z^2(k, \tilde{G}, \kappa) = Z^2(\Gamma, \tilde{G}(k_s), \tilde{r} \circ \kappa),\]

there exists an open subgroup \( \Delta \subset \Gamma \) over which \( f \) and \( f' \) coincide up to translation by an element in \( \text{Int}(\tilde{G}) \). Now, this is an easy consequence of the definition of alg-continuity (recall that continuous cocycles are alg-continuous if \( \kappa \) is admissible) and the fact that both \( f \) and \( f' \) lift \( \tilde{r} \circ \kappa \).

Finally, for the sake of completeness, let us give the argument which proves that an alg-continuous section of (6) for \( \tilde{G} \) defines a canonical \( k \)-form \( G \) of \( \tilde{G} \) (even when \( \tilde{G} \) is not assumed to be smooth). Let \( f \) be such an alg-continuous splitting. Then \( \Gamma \) acts on \( \text{Aut}(\tilde{G}) \) by conjugation in \( \text{SAut}(\tilde{G}) \). Let now \( K/k \) be a finite separable extension such that \( \tilde{G} \) admits a \( K \)-form \( \tilde{G} \). Let \( K'/k \) be a Galois extension such that \( f \) coincides with \( f_{\tilde{G}} \) over \( \Gamma_{K'} \) (such a \( K' \) exists by definition of alg-continuity) and consider the subgroup of \( \text{Aut}(\tilde{G}) \) given by its \( \Gamma_{K'} \)-invariant elements (which is nothing but \( \text{Aut}_{K'}(\tilde{G}_{K'}) \)). It is easily seen that this subgroup is stable by the action of \( \Gamma \) and hence we can consider the subgroup \( E \) of \( \text{SAut}(\tilde{G}) \) generated by \( \text{Aut}_{K'}(\tilde{G}_{K'}) \) and \( \Gamma_K \) which clearly fits into a split exact sequence:

\[1 \to \text{Aut}_{K'}(\tilde{G}_{K'}) \to E \to \Gamma \to 1.\]

Moreover, since the group in the left is \( \Gamma_{K'} \)-invariant, it is easy to see that \( \Gamma_{K'} \) is normal in \( E \) and taking quotients we get then the split exact sequence:

\[1 \to \text{Aut}_{K'}(\tilde{G}_{K'}) \to E/\Gamma_{K'} \to \Gamma_{K'/k} \to 1,\]

where \( \Gamma_{K'/k} = \text{Gal}(K'/k) \). It is now easy to see that this sequence corresponds to the data satisfying condition (** in [Ser75, V.20] and hence the existence of a \( k \)-form is ensured by [Ser75, V.20, Cor. 2.ii].
3 Extensions of algebraic groups and “mixed” non-abelian 2-cohomology

Let $k$ be a field. Suppose now that we wish to study extensions of a finite smooth algebraic $k$-group $F$ by another smooth algebraic $k$-group $G$. If one were to work over a separably closed field $k$, it would certainly suffice to study the extensions of $F(k_s)$ by $G(k_s)$ as abstract groups and one would thus use non-abelian cohomology in the group theoretical setting. However, if $k$ is not separably closed, it is easy to see that the study of extensions of $F(k_s)$ by $G(k_s)$ can’t suffice since we don’t know if any such extension admits a $k$-group structure (even less if we ask for the $k$-structure to be compatible with the obvious morphisms linking the extension with $F$ and $G$). One would then feel compelled to call for non-abelian Galois cohomology in order to have a certain control on $k$-forms.

In this section, we define a non-abelian set of 2-cohomology set that “almost” classifies extensions of $F$ by $G$ in this setting. Here, “almost” means that there may be classes of extensions that fall into the same element of this set. This shortcoming can be overcomed (see Proposition 3.5 below): we will see in fact that two classes of extensions will fall into the same element if and only if one of them is a very particular twist of the other one.

3.1 $F$-kernels

Let us fix some notation. We will note $\Gamma = \text{Gal}(k_s/k)$ as always. For any $k$-group $G$ it is clear that we get an action of $\Gamma$ on $G(k_s)$. We will use the notation $G\Gamma$ for the semi-direct product $G(k_s) \rtimes \Gamma$ and we will denote by $\gamma_G$ the natural section $\Gamma \to G\Gamma$.

Let $G$ be an algebraic $k$-group and let $G_S$ be the group sheaf (on the étale topology) associated to it. We will denote by $\underline{\text{Aut}}(G)$ the group sheaf given by

$$\underline{\text{Aut}}(G)(S) = \text{Aut}_{S\text{-gp-sh}}(G_S),$$

where $S\text{-gp-sh}$ means group sheaves over $S$ and $G_S$ is the sheaf associated to the $S$-group $G_S := G \times_k S$. This sheaf is not always representable by a smooth $k$-group scheme.

Recall that a $k$-action of a $k$-group $G$ on a $k$-scheme $V$ is a $k$-morphism $a : G \times_k V \to V$ such that the following diagrams commute

$$
\begin{array}{ccc}
G \times_k G \times_k V & \xrightarrow{m_G \times \text{id}_V} & G \times_k V \\
\downarrow{\text{id}_G \times a} & & \downarrow{a} \\
G \times_k V & \xrightarrow{a} & V,
\end{array}
\quad
\begin{array}{ccc}
\text{Spec } k \times_k V \cong V & \xrightarrow{\text{id}_V} & V \\
\downarrow{\epsilon \times \text{id}_V} & & \downarrow{a} \\
G \times_k V & \xrightarrow{a} & V,
\end{array}
$$

where $\epsilon$ is the neutral element in $G$ and $m_G : G \times_k G \to G$ denotes the multiplication morphism. We will denote by $a_2$ the natural $k$-action $G \times_k V \times_k V \to V \times_k V$ of $G$ on $V \times_k V$ obtained by acting via $a$ on each component of $V \times_k V$. 

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Definition 3.1. Let $F, G$ be arbitrary algebraic $k$-groups. We will say that $F$ acts algebraically on $G$ if there is a $k$-action $a : F \times_k G \rightarrow G$ such that the following diagram commutes

$$
\begin{array}{ccc}
F \times_k G \times_k G & \xrightarrow{\text{id}_F \times m_G} & F \times_k G \\
\downarrow{a_2} & & \downarrow{a} \\
G \times_k G & \xrightarrow{m_G} & G.
\end{array}
$$

Proposition 3.2. Let $F, G$ be smooth algebraic $k$-groups and let $\mathcal{F}$ be the group sheaf associated to $F$. Denote by $f_G$ the section of sequence (6) naturally associated to the trivial $k$-kernel $L = (G_{k_s}, \kappa_G)$ (see section 2.4). Assume that $F$ is finite. Then the following are equivalent

1. there is a morphism of group sheaves $\mathcal{F} \rightarrow \text{Aut}(G)$,
2. there is a morphism of $\Gamma$-groups $F(k_s) \rightarrow \text{Aut}(G_{k_s})$, where $\Gamma$ acts on $\text{Aut}(G_{k_s})$ by conjugation via $f_G$,
3. there is a morphism of groups $F_\Gamma \rightarrow \text{SAut}(G_{k_s}/k)$ sending $\gamma_F(\Gamma)$ identically to $f_G(\Gamma)$.
4. $F$ acts algebraically on $G$.

Proof. 1 $\Rightarrow$ 2: The morphism of group sheaves gives us group morphisms $F(K) \rightarrow \text{Aut}(G_K)$ for every extension $K/k$. The sheaf structure tells us moreover that these morphisms are compatible with the action of $\Gamma$ for finite separable extensions, hence a $\Gamma$-group morphism $F(k_s) \rightarrow \text{Aut}(G_{k_s})$.

2 $\Leftrightarrow$ 3: This follows basically from the definition of a semi-direct product.

2 $\Rightarrow$ 4: Let us assume that $F$ is a constant group. The existence of the $k$-action in the general case follows by Galois descent from the $\Gamma$-equivariance of the morphism (cf. [Ser75 V.20]). Under this assumption, the scheme $F \times_k G$ is nothing but a finite set of copies $G_f$ of $G$ indexed by the $k$-points $f$ of $F$. Hence, in order to define the $k$-morphism $a$ it will suffice to define a $k$-morphism for each copy. We send then each $G_f = G$ to $G$ via the image of $f$ in $\text{Aut}(G_{k_s})$ (which is actually a $k$-morphism by $\Gamma$-equivariance). One can easily check that this morphism verifies diagrams (8) and (9).

4 $\Rightarrow$ 1: Let $f : S \rightarrow F$ be an $S$-point of $F$ and consider the pullback of $a$ by this point. This is a morphism $a_f : G_S \rightarrow G_S$ which is seen to be an automorphism of $G_S$. Thanks to diagram (9). Moreover, diagram (8) tells us that $f \mapsto a_f$ defines a morphism of group sheaves.

Remark.
The equivalence 1 $\Leftrightarrow$ 4 is actually valid for any smooth algebraic group $F$ (not necessarily finite) and any algebraic group $G$ (not necessarily smooth). Indeed, to prove that 1 $\Rightarrow$ 4 it suffices to define the map $a : F \times_k G \rightarrow G$ via the evident sheaf maps $\mathcal{F} \times \mathcal{G} \rightarrow \text{Aut}(G) \times \mathcal{G} \rightarrow \mathcal{G}$. 

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Let us now try to define an outer algebraic action of $F$ on $G$. This naturally leads us to the following definition.

**Definition 3.3.** Let $F$ be a finite smooth algebraic $k$-group and $G$ a smooth algebraic $k_s$-group. An $(F,k)$-kernel (or $F$-kernel, for short) in $\bar{G}$ is a group morphism $\kappa : F_\Gamma \to \text{SOut}(\bar{G})$ such that the restriction of $\kappa$ to $\gamma_F(\Gamma)$ is a $k$-kernel. This restriction will be called the $k$-kernel associated to $\kappa$. We define an outer algebraic action of $F$ on $\bar{G}$ to be an $F$-kernel such that its associated $k$-kernel is trivial. An $F$-kernel will be said to be trivial if $\kappa$ can be lifted to a continuous morphism $F_\Gamma \to \text{SAut}(\bar{G}/k)$.

**Remarks.**

1. It is clear from the definition that an $F$-kernel in $\bar{G}$ corresponds to a $F_\Gamma$-kernel in $\bar{G}(k_s)$ in the sense of Definition 2.1. In particular, an $F$-kernel is trivial if and only if the corresponding $F_\Gamma$ kernel is. Note moreover that such an $F$-kernel gives an algebraic action of $F$ on a $k$-form of $\bar{G}$ by Proposition 3.2.

2. Note that there are no new continuity issues in here since $F_\Gamma$ has a natural structure of profinite group which depends completely on that of $\Gamma$. To be more precise, a basis of open neighbourhoods of the neutral element in $F_\Gamma$ is given by taking a basis of neighbourhoods in $\Gamma$ and taking their image via $\gamma_F(\Gamma)$. Another way of seeing this is to note that $F_\Gamma$ is nothing but a finite product of copies of $\Gamma$ (the translates of $\gamma_F(\Gamma)$) and as such it has a natural product topology. Having said this, we will omit any comments on continuity in what follows.

3. Given Proposition 3.2, one could try to define an outer algebraic action in a more general setting (say, for any smooth $k$-group $F$) by looking at the “sheaf of outer automorphisms” $\text{Out}(G)$ which can be defined as the quotient of $\text{Aut}(G)$ by the image of the morphism $\bar{G} \to \text{Aut}(G)$ given by the natural $k$-action of $G$ on itself by conjugation (we are using here the equivalence $1 \Leftrightarrow 4$ in Proposition 3.2). It would be interesting to know if such an approach would work. In particular, it would be interesting to know what $\text{Out}(G)(S)$ looks like for a given $k$-scheme $S$.

Consider now an extension of smooth algebraic $k$-groups

$$1 \to G \to H \to F \to 1,$$

with finite $F$. We want to show that such an extension defines an outer algebraic action of $F$ on $G_{k_s}$. Since $G$ is smooth, we have $H^1_{\text{fppf}}(k_s, G) = 0$ and hence $H(k_s)$ surjects onto $F(k_s)$. Choose then arbitrary preimages $\hat{f} \in H(k_s)$ for $f \in F(k_s)$ and set, for $(f, \sigma) \in F_\Gamma = F(k_s) \rtimes \Gamma$,

$$\hat{f}_{(f,\sigma)} := \text{int}(\hat{f}) \circ (f_G)_{\sigma} \in \text{SAut}(G_{k_s}),$$

$^3$We still assume all $k$-kernels to be admissible.
where \( f_G : \Gamma \to \text{SAut}(G_{k_s}) \) is the natural section of sequence \((\text{6})\) given by the \( k \)-group structure of \( G \). Define \( \kappa : F_\Gamma \to \text{SOut}(G_{k_s}) \) as the composition of this map with the natural projection \( \text{SAut}(G_{k_s}) \to \text{SOut}(G_{k_s}) \). One easily verifies then that \( \kappa \) is a group morphism restricting to \( \kappa_G \) on \( \gamma_F(\Gamma) \) and hence an \( F \)-kernel whose associated \( k \)-kernel is the trivial \( k \)-kernel defined by \( G \).

The \( F \)-kernel given by \( H \) is rapidly seen to be independent of the choice of the preimages. We will say then, by analogy with the previous setting, that \( H \) extends \( \kappa \) algebraically.

### 3.2 Non abelian 2-cohomology and extensions

Since \( F \)-kernels are a particular case of \( F_\Gamma \)-kernels, one can always define the following.

**Definition 3.4.** For \( \kappa \) an \( F \)-kernel in \( \bar{G} \), we set

\[
\begin{align*}
Z^2(F, \bar{G}, \kappa) &:= Z^2(F_\Gamma, \bar{G}(k_s), \bar{r} \circ \kappa), \\
H^2(F, \bar{G}, \kappa) &:= H^2(F_\Gamma, \bar{G}(k_s), \bar{r} \circ \kappa), \\
N^2(F, \bar{G}, \kappa) &:= N^2(F_\Gamma, \bar{G}(k_s), \bar{r} \circ \kappa),
\end{align*}
\]

where \( \bar{r} : \text{SOut}(\bar{G}/k) \to \text{Out}(\bar{G}(k_s)) \) is the natural morphism mentioned in section \(2.4\).

If moreover \( \bar{G} = G_{k_s} \) for \( G \) a \( k \)-group, we denote by \( \text{Ext}(F, G, \kappa) \) the set of \( k \)-isomorphism classes of extensions \( H \) of \( F \) by \( G \) extending \( \kappa \) algebraically.

We will assume in the sequel that the set \( \text{Ext}(F, G, \kappa) \) is always non empty. In other words, we assume that \( \kappa \) is algebraically extendable.

Consider now a class \( \xi \in \text{Ext}(F, G, \kappa) \) and, as before, an extension

\[
1 \to G \to H_\xi \to F \to 1,
\]

representing \( \xi \). Again, by smoothness of \( G \), we get the same exact sequence at the level of \( k_s \)-points:

\[
1 \to G(k_s) \to H_\xi(k_s) \to F(k_s) \to 1,
\]

where the morphisms are moreover clearly \( \Gamma \)-equivariant. Consider now the extension

\[
1 \to H_\xi(k_s) \to E_\xi \to \Gamma \to 1,
\]

given by the semi-direct product \( E_\xi = H_\xi(k_s) \rtimes \Gamma \) associated to the natural action of \( \Gamma \) on \( H_\xi(k_s) \). It is clear that the subgroup \( \bar{G}(k_s) \) of \( H_\xi(k_s) \) is normal in \( E_\xi \). Therefore, we get an extension

\[
1 \to G(k_s) \to E_\xi \to F(k_s) \rtimes \Gamma \to 1,
\]

which defines, as in section \(2.1\), an \( F_\Gamma \)-kernel in \( G(k_s) \). Now it is easy to verify that this \( F_\Gamma \)-kernel is nothing but the one induced by the \( F \)-kernel \( \kappa \). By Proposition \(2.4\) we get
From these, one easily deduces that the image in nothing but étale cohomology and since I.5.3.\textsuperscript{4} We have thus defined a map
\[ \varphi : \text{Ext}(F, G, \kappa) \to H^2(F, G_{k_s}, \kappa), \]
which clearly sends split extensions into neutral elements.

Consider again an arbitrary element \( \xi \in \text{Ext}(F, G, \kappa) \) and its associated extension
\[ 1 \to G \to H_\xi \to F \to 1. \]
Let \( Z \) denote the center of \( G \). The group \( Z \) acts on \( H_\xi \) naturally by conjugation and thus, if we consider a class \( \alpha \in H^1(k, Z) \) and a 1-cocycle \( \zeta \) representing the image of this class in \( H^1(k, G) \), we can consider the twisted group \( \tilde{H}_\xi \) (see for example \textsuperscript{[Ser02 I.5.3]}).\textsuperscript{5} This group is a \( k \)-form of \( H_\xi \) and has \( G \) as a normal \( k \)-subgroup since the action of \( Z \) on \( G \) by conjugation is trivial and hence so is the twisting. Moreover, the quotient \( \tilde{H}_\xi / G \) is clearly \( k \)-isomorphic to \( F \) since the action of the whole group \( G \) on \( H_\xi \) by conjugation gets trivialized when one passes to the quotient \( H_\xi / G = F \). We have then a new extension
\[ 1 \to G \to \tilde{H}_\xi \to F \to 1, \]
which actually extends \( \kappa \) as well. Indeed, since \( H_\xi(k_s) = \tilde{H}_\xi(k_s) \), we see that the outer action of \( F \) (or \( F \)-kernel) \( \gamma \kappa \) induced by \( \tilde{H}_\xi \) is the same as \( \kappa \) when restricted to \( F(k_s) \subset F_\Gamma \), whereas its restriction to \( \gamma_F(\Gamma) \subset F_\Gamma \) is \( \kappa \) simply by the definition of twisting: we have just modified the action of \( \Gamma \) by inner automorphisms.

The extension \( H_\xi \) represents then an element \( \alpha \cdot \xi \) in \( \text{Ext}(F, G, \kappa) \). Note that this notation is not abusive, since the choice of another cocycle \( \zeta' \) would give a twisted group \( \tilde{H}_\xi \) which would be isomorphic to \( \tilde{H}_\xi \) and one can easily see that the isomorphism will reduce to identity on \( G \) and on \( F \), giving thus an isomorphism of extensions. One can then verify that this construction defines an action of \( H^1(k, Z) \) on \( \text{Ext}(F, G, \kappa) \).

Finally, if we abusively still note \( \kappa \) its restriction to the subgroup \( \gamma_F(\Gamma) \subset F_\Gamma \) we can consider the natural restriction map
\[ H^2(F, G_{k_s}, \kappa) = H^2(F_\Gamma, G(k_s), \bar{\rho} \circ \kappa) \xrightarrow{\text{Res}} H^2(\Gamma, G(k_s), \bar{\rho} \circ \kappa) = H^2(k, G). \]
Note that this is indeed a map since it corresponds to the relation \( (\gamma_F, \text{id}_G)^2 \) of \( F_\Gamma \)-kernels and \( \text{id}_G \) is trivially surjective, c.f. Proposition \textsuperscript{2.6}. Recall that \( H^2(k, G) \) admits a natural neutral element corresponding to the semi-direct product \( G(k_s) \rtimes \Gamma \) for the natural action of \( \Gamma \) over \( G(k_s) \). We can thus consider \( H^2(k, G) \) as a pointed set by taking this class as the base point.\textsuperscript{6}

\textsuperscript{4}Note that the center \( Z \) may a priori not be smooth and one would hence want to consider the group \( H^1_{\text{fppf}}(k, Z) \) in order to have access to all possible twists. However, since Galois cohomology over \( k \) is nothing but étale cohomology and since \( G \) is smooth, one has the equality \( H^1_{\text{ét}}(k, G) = H^1_{\text{fppf}}(k, G) \). From these, one easily deduces that the image in \( H^1(k, G) \) of the fppf cohomology and that of Galois cohomology over \( Z \) is the same.

\textsuperscript{5}Note that there may be other neutral elements in \( H^2(k, G) \). However, they define other actions of \( \Gamma \) over \( G(k_s) \) (or, if one wishes, other \( k \)-forms of \( G_{k_s} \)) and hence our base point is uniquely defined.
Proposition 3.5. Let $F$ be a finite smooth $k$-group and let $G$ be a smooth algebraic $k$-group. Assume that there is an outer algebraic action of $F$ on $G_{k_s}$ given by an $F$-kernel $\kappa$. Then the map $\varphi$ defined above passes to the quotient of $\text{Ext}(F, G, \kappa)$ by the action of $H^1(k, Z)$. Moreover, there is an isomorphism (which we abusively still note $\varphi$)

$$H^1(k, Z) \setminus \text{Ext}(F, G, \kappa) \xrightarrow[\varphi]{\cong} \text{Ker}[H^2(F, G_{k_s}, \kappa) \xrightarrow{\text{Res}} H^2(k, G)],$$

(10)

where $\text{Ker(Res)}$ means the preimage of the base point in $H^2(k, G)$.

Proof. We start by showing that this restriction of $\varphi$ is actually well defined.

The map [10] is well defined: It is clear by the construction of $\varphi$ that the composition $\text{Res} \circ \varphi$ gives us the trivial extension $G(k_s) \rtimes \Gamma$ for any $\xi \in \text{Ext}(F, G, \kappa)$, thus $\varphi$ actually falls into $\text{Ker(Res)}$. We must now show that for $\alpha \in H^1(k, Z)$ and $\xi \in \text{Ext}(F, G, \kappa)$ we have $\varphi(\alpha \cdot \xi) = \varphi(\xi)$.

Let $\zeta \in Z^1(k, Z)$ represent $\alpha$, let

$$1 \to G \to H_{\xi} \to F \to 1,$$

(resp. $1 \to G \to \zeta H_{\xi} \to F \to 1$),

be an extension representing $\xi$ (resp. $\alpha \cdot \xi$) and let

$$1 \to G(k_s) \to E_{\xi} \to F(k_s) \rtimes \Gamma \to 1,$$

(resp. $1 \to G(k_s) \to \zeta E_{\xi} \to F(k_s) \rtimes \Gamma \to 1$),

be the extension, representing $\varphi(\xi)$ (resp. $\varphi(\alpha \cdot \xi)$), obtained by taking the semi-direct product $H_{\xi}(k_s) \rtimes \Gamma$ (resp. $\zeta H_{\xi}(k_s) \rtimes \Gamma$).

In order to prove the assertion, we will show that there exists an isomorphism of extensions:

$$
\begin{array}{ccccccc}
1 & \longrightarrow & G(k_s) & \longrightarrow & E_{\xi} & \longrightarrow & F_{\Gamma} & \longrightarrow & 1 \\
\phantom{1} & & \downarrow & & \sim & & \downarrow & & \phantom{1} \\
1 & \longrightarrow & G(k_s) & \longrightarrow & \zeta E_{\xi} & \longrightarrow & F_{\Gamma} & \longrightarrow & 1 \\
\end{array}
$$

For an element $e \in E_{\xi} = H_{\xi}(k_s) \rtimes \Gamma$, we note $e = (h, \sigma)$ with $h \in H_{\xi}(k_s)$ and $\sigma \in \Gamma$. We use the same notation for $e' \in \zeta E_{\xi}$ (the sets $E_{\xi}$ and $\zeta E_{\xi}$ are in fact the same, but remark that the group laws are different). For $\sigma \in \Gamma$, we note $\zeta \sigma \in Z(k_s)$ its image by the 1-cocycle $\zeta$. Define:

$$\phi : E_{\xi} \longrightarrow \zeta E_{\xi}$$

$$(h, \sigma) \mapsto (h\zeta^{-1} \sigma, \sigma).$$

Let us show first that $\phi$ restricts to identity on $G(k_s)$ and $F_{\Gamma}$. For the latter, this is evident since $\zeta \sigma \in Z(k_s) \subset G(k_s)$ and hence it disappears in the quotient. As for the
former, it suffices to remark that \( z_1 = 1 \) and thus \( \phi(h, 1) = (h, 1) \) for any \( h \in H(k_s) \), so in particular for \( g \in G(k_s) \).

Let us finally show that \( \phi \) is an isomorphism. Given the identities we have just shown, we only have to prove that \( \phi \) is a group morphism. We use the notation \( \sigma h \) for the action of \((1, \sigma)\) on \((h, 1)\) by conjugation in \( E_\xi \) and \( \sigma^* h \) for the action in the twisted case (i.e. the action of \((1, \sigma)\) on \((h, 1)\) by conjugation in \( _1 E_\xi \)). Recall that \( \sigma^* h = z_\sigma \sigma h z_\sigma^{-1} \) by the definition of twisting. Then, on one side, we have:

\[
\phi((h_1, \sigma_1)(h_2, \sigma_2)) = \phi(h_1 \sigma_1^1 h_2, \sigma_1 \sigma_2) = (h_1 \sigma_1^1 h_2 \delta_{\sigma_1 \sigma_2}, \sigma_1 \sigma_2),
\]

whereas, noting that \( \sigma g = \sigma^* g \) for \( g \in G(k_s) \) and \( \sigma \in \Gamma \), we have

\[
\phi((h_1, \sigma_1)) \phi((h_2, \sigma_2)) = (h_1 \delta_{\sigma_1}, \sigma_1)(h_2 \delta_{\sigma_2}, \sigma_2)
= (h_1 \delta_{\sigma_1}^{-1} \sigma_1^1, \sigma_1 \sigma_2)
= (h_1 \delta_{\sigma_1}^{-1} \sigma_1^1 h_2 \delta_{\sigma_2}, \sigma_1 \sigma_2)
= (h_1 \delta_{\sigma_1}^{-1} \sigma_1 h_2 \delta_{\sigma_1} \delta_{\sigma_2}, \sigma_1 \sigma_2)
= (h_1 \sigma_1^1 h_2 \delta_{\sigma_1} \delta_{\sigma_2} \delta_{\sigma_1}^{-1}, \sigma_1 \sigma_2)
= (h_1 \sigma_1^1 h_2 \delta_{\sigma_1 \sigma_2}, \sigma_1 \sigma_2).
\]

This proves that \( \phi \) is indeed a group morphism and concludes the proof of \( \varphi(\alpha \cdot \xi) = \varphi(\xi) \).

In particular, the map (11) is well defined.

The map (11) is surjective: Take a class in \( H^2(F, G_{k_s}, \kappa) \) such that its image in \( H^2(k, G) \) is the base point. If we note \((f, \sigma)\) an arbitrary element of \( F_\Gamma \) with \( f \in F(k_s), \sigma \in \Gamma \), then this amounts to consider a 2-cocycle \((f, g) \in Z^2(F, G_{k_s}, \kappa)\) such that

\[
g_{(1, \sigma_1), (1, \sigma_2)} = 1, \quad \forall \sigma_1, \sigma_2 \in \Gamma,
\]

and such that \( f_{(1, \sigma)} \) corresponds to the natural action of \( \sigma \) on \( G(k_s) \). With such a cocycle, one can always define an extension

\[
1 \to G(k_s) \to E \to F_\Gamma \to 1,
\]

where \( E = G(k_s) \times F_\Gamma \) as a set and the group law on \( E \) is given by

\[
(g_1, (f_1, \sigma_1)) \cdot (g_2, (f_2, \sigma_2)) = (g_1 f_{(f_1, \sigma_1)}(g_2) g_{(f_1, \sigma_1), (f_2, \sigma_2)}, (f_1 \sigma_1^1 f_2, \sigma_1 \sigma_2)),
\]

for \( g_i \in G(k_s), (f_i, \sigma_i) \in F_\Gamma \) and \( i = 1, 2 \) (see section 2.2). Consider now the preimage of \( F(k_s) \) (as a subgroup of \( F_\Gamma \)) in \( E \). This gives us an extension

\[
1 \to G(k_s) \to \bar{H} \to F(k_s) \to 1,
\]

that fits into an exact sequence

\[
1 \to \bar{H} \to E \to \Gamma \to 1.
\]
We claim that this sequence is split. Indeed, consider the following map:

\[ s_0 : \Gamma \to E, \sigma \mapsto (1, (1, \sigma)). \]

One can thus verify that

\[ s_0(\sigma_1)s_0(\sigma_2) = (1, (1, \sigma_1)) \cdot (1, (1, \sigma_2)) = (1f(1,\sigma_1)(1)g(1,\sigma_1),(1,\sigma_2), (1^{\sigma_1}1, \sigma_1\sigma_2)) \]
\[ = (1, (1, \sigma_1\sigma_2)) = s_0(\sigma_1\sigma_2). \]

We see then that there is a natural action of \( \Gamma \) on \( \bar{H} \), by conjugation in \( E \), which clearly restricts to actions on \( G(k_s) \) and \( F(k_s) \) that coincide with the natural action of \( \Gamma \) given by the \( k \)-group structure of \( G \) and \( F \) respectively. In other words, the extension \eqref{eq:extension1} is \( \Gamma \)-equivariant. Moreover, since \( F(k_s) \) is finite and smooth, \( \bar{H} \) can be naturally given the structure of a smooth \( k \)-algebraic group: as a \( k \)-variety, it is a finite union of copies of \( G(k_s) \) (one per element of \( F(k_s) \)) and the morphisms giving the group structure can be easily defined using the \( k \)-automorphisms \( f_{(f,1)} \) for \( f \in F(k_s) \). The action of \( \Gamma \) over \( \bar{H} \) is then seen to define semialgebraic automorphisms: this is evident for the copy of \( G(k_s) \) containing the neutral element (since the action of \( \Gamma \) over \( G \) is by semialgebraic automorphisms) and it can be deduced for the other components from the \( k \)-group structure of \( \bar{H} \) by translation. We conclude that \( \bar{H} \) descends into a smooth algebraic \( k \)-group representing an element in \( \text{Ext}(F, G, \kappa) \). The fact that this element is a preimage of the given class in \( H^2(F, G_{k_s}, \kappa) \) is obvious.

**The map \eqref{eq:extension1} is injective:** Let \( \xi_1, \xi_2 \in \text{Ext}(F, G, \kappa) \) be elements such that \( \varphi(\xi_1) = \varphi(\xi_2) \). Since we have \( \varphi(\xi_i) \in \text{Ker}(\text{Res}) \), this means that we have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & G(k_s) & \xrightarrow{\iota_1} & E_{\xi_1} & \xrightarrow{\iota_1} & F_{\Gamma} & \longrightarrow & 1 \\
\phantom{1} & \searrow & \phi & \nearrow & \gamma_F & \phantom{1} & \phantom{1} & \phantom{1} & \phantom{1} \\
1 & \longrightarrow & G(k_s) & \xrightarrow{\iota_2} & E_{\xi_2} & \xrightarrow{\iota_2} & F_{\Gamma} & \longrightarrow & 1 \\
\end{array}
\]

(12)

where, for \( i = 1, 2 \), \( E_{\xi_i} \) is the semi-direct product \( H_{\xi_i}(k_s) \rtimes \Gamma \) for the natural action of \( \Gamma \) on the \( k \)-group \( H_{\xi_i} \) representing \( \xi_i \) and where the action of \( \Gamma \) on \( G(k_s) \) by conjugation in \( E_0 \) (via the homomorphism \( s_0 \)) corresponds to its natural action given by the \( k \)-group structure on \( G \). For \( i = 1, 2 \), let \((f^i, g^i) \in Z^2(F, G_{k_s}, \kappa) \) be cocycles corresponding to the \( E_{\xi_i} \)'s. The diagram tells us that we may choose them in such a way that we have

\[
\begin{align}
& f^1_{(1,\sigma)} = f^2_{(1,\sigma)} = \text{int}(s_0(\sigma)); \\
& g^1_{(1,\sigma_1),(1,\sigma_2)} = g^2_{(1,\sigma_1),(1,\sigma_2)} = 1,
\end{align}
\]

(13) (14)
where, in order to lighten notation, we have identified $E_{\xi_1}$ and $E_{\xi_2}$ via $\phi$ and also $s_0(\sigma)$ with its image in $E_{\xi_i}$ via $\iota_i$. Now, since these two 2-cocycles represent the same class in $H^2(F, G_{k_s}, \kappa)$, there exists a continuous map $\hat{\mathcal{Z}} : F_\Gamma \to G(k_s)$ such that

$$f_1^2(f, \sigma) = \text{int}(\hat{\mathcal{Z}}(f, \sigma)) \circ f_1^1(f, \sigma), \quad (15)$$

$$g_1^2((f_1, \sigma_1), (f_2, \sigma_2)) = \hat{\mathcal{Z}}(f_1, \sigma_1)(f_2, \sigma_2), \quad (16)$$

Let us recall how to obtain such a map $\hat{\mathcal{Z}}$. The cocycles $(f^i, g^i)$ are actually dependant of the choice of respective (non necessarily homomorphic) sections $s_i : F_\Gamma \to E_{\xi_i}$, both compatible with $s_0$ in order to verify (13) and (14), for which $E_{\xi_i}$ can be considered as the set $G(k_s) \times F_\Gamma$ with the group law given by

$$(g_1, (f_1, \sigma_1)) \cdot (g_2, (f_2, \sigma_2)) = (g_1 f_1^i(f_1, \sigma_1)(g_2), (f_1, \sigma_2), (f_2, \sigma_2)), \quad \text{for } g_1, g_2 \in G(k_s), \quad (f_1, \sigma_1), (f_2, \sigma_2) \in F_\Gamma.$$ 

In order to respect the commutativity of diagram (12), the isomorphism $\phi$ must then verify, for $g \in G(k_s), \quad (f, \sigma) \in F_\Gamma$,

$$\phi(g, (1, 1)) = (g, (1, 1)),$$

$$\phi(1, (f, \sigma)) = (\hat{\mathcal{Z}}(f, \sigma), (f, \sigma)),$$

with $\hat{\mathcal{Z}}(f, \sigma) \in G(k_s)$. This is precisely our map $\hat{\mathcal{Z}} : F_\Gamma \to G(k_s)$. Note that in particular we have

$$\phi(\iota_1(s_0(\sigma))) = \hat{\mathcal{Z}}_\sigma \cdot \iota_2(s_0(\sigma)), \quad \forall \sigma \in \Gamma. \quad (17)$$

Now, applying the different equalities on our 2-cocycles above, we get that

1. $\hat{\mathcal{Z}}|_{F_\Gamma}$ has its values in $Z$, since (13) and (15) imply $\text{int}(\hat{\mathcal{Z}}(1, \sigma)) = \text{id}_G$.

2. $\hat{\mathcal{Z}}|_{F_\Gamma}$ is a cocycle, since (14) and (16) imply

$$\hat{\mathcal{Z}}(1, \sigma_1)(1, \sigma_2) f_1^i(1, \sigma_1) \hat{\mathcal{Z}}(1, \sigma_2) = 1,$$

i.e. $\hat{\mathcal{Z}} \sigma_1 \sigma_2 = \hat{\mathcal{Z}} \sigma_1 \hat{\mathcal{Z}} \sigma_2$. 

Otherwise stated, we get that $\hat{\mathcal{Z}}|_{F_\Gamma} \in Z^2(k, Z)$. We claim that this cocycle will do the trick, i.e. if we denote

$$1 \to G \to H_{\xi_i} \to F \to 1, \quad i = 1, 2,$$

the extensions representing respectively $\xi_1$ and $\xi_2$, we have $H_{\xi_2} \cong \hat{\mathcal{Z}} H_{\xi_1}$. 

In order to prove this last assertion, it will suffice to prove that $\Gamma$ acts in the same way on $\hat{\mathcal{Z}} H_{\xi_1}(k_s)$ and $H_{\xi_2}(k_s)$. Notice that these two groups are identified via the isomorphism $\phi$ of the diagram (12) as the respective preimages of $F(k_s) \subset F_{\Gamma}$ in $E_{\xi_i}$ and $E_{\xi_2}$ using the evident identification $\hat{\mathcal{Z}} H_{\xi_i}(k_s) = H_{\xi_i}(k_s)$. Moreover, the action of $\Gamma$ on $E_{\xi_i}$ is clearly given by conjugation when one considers $\Gamma$ as a subgroup via $\iota_i \circ s_0$ in diagram (12). Thus, $\sigma \in \Gamma$ acts on $H_{\xi_i}(k_s)$ via $\text{int}(\iota_i(s_0(\sigma)))$, hence on $\hat{\mathcal{Z}} H_{\xi_i}(k_s)$ via $\text{int}(\hat{\mathcal{Z}}(\sigma)) \circ \text{int}(\iota_i(s_0(\sigma)))$ by the definition of twisting. Recalling then equation (17), we see immediately that these actions are the same on $H_{\xi_2}(k_s) = \hat{\mathcal{Z}} H_{\xi_1}(k_s)$, which concludes the proof.
In the particular case of a smooth commutative group $A$, we have $Z = A$ and the outer action of $F$ on $A$ becomes an algebraic action since the $F$-kernel $\kappa$ is now clearly trivial. It is well known that in this setting the sets $\text{Ext}(F, A, \kappa)$, $H^2(F, A_{k_s}, \kappa)$ and $H^1(k, A)$ have a natural group structure (see for example [SGA3, XVII, App. I] for the first one, [Ser02, I.§2 and II.§1] for the second and third ones). One can easily see then that the map

$$H^1(k, A) \xrightarrow{\psi} \text{Ext}(F, A, \kappa)$$

$$\alpha \mapsto \alpha \cdot 0,$$

where 0 represents the trivial element in $\text{Ext}(F, A, \kappa)$ (i.e. the class of the semidirect product $A \rtimes F$), and the maps

$$\text{Ext}(F, A, \kappa) \xrightarrow{\varphi} H^2(F, A_{k_s}, \kappa),$$

$$H^2(F, A_{k_s}, \kappa) \xrightarrow{\text{Res}} H^2(k, A),$$

defined as above, are group morphisms. Proposition 3.5 then becomes:

**Proposition 3.6.** Let $A, F$, be smooth algebraic $k$-groups such that $A$ is commutative, $F$ is finite of order $n$ and $F$ acts algebraically on $A$ (via $\kappa$, as above). Then the sequence

$$H^1(k, A) \xrightarrow{\psi} \text{Ext}(F, A, \kappa) \xrightarrow{\varphi} H^2(F, A_{k_s}, \kappa) \xrightarrow{\text{Res}} H^2(k, A),$$

(18)

is exact. In particular, $\text{Ext}(F, A, \kappa)$ is a torsion group and, if $H^1(k, A)$ is $d$-torsion, then $\text{Ext}(F, A, \kappa)$ is an nd-torsion group.

**Proof.** We only have to prove the last two statements. Now this is easy: the classic restriction-corestriction argument tells us that the image of $\varphi$ is killed by $n$. On the other side, it is well known that $H^1(k, A)$ is torsion simply because $\Gamma$ is profinite (cf. [Ser02, I.2.2, Cor. 3]). The last assertion is then evident.

**Remark.**

Note that in both results (i.e. Propositions 3.5 and 3.6), the action of $H^1(k, Z)$ (where $Z = A$ in the second case) factors through its image in $H^1(k, Z/Z^F)$, where $Z^F$ denotes the biggest $k$-subgroup of $Z$ over which $F$ acts trivially (we will prove that $F$ actually acts algebraically on $Z$ even if $Z$ is not smooth right below). Indeed, for $H_\xi$ an extension representing $\xi \in \text{Ext}(F, G, \kappa)$, inner twists of $H_\xi$ are defined by cocycles with values in $\text{Int}(H_\xi) = H_\xi/Z(H_\xi)$ and it is easy to see that the center $Z(H_\xi)$ of $H_\xi$ is nothing but an extension of a central subgroup of $F$ by $Z^F$. Indeed, the intersection of $Z(H_\xi)$ with $G$ must be $Z^F$ since it is contained in $Z$ and it is invariant by the action of $H_\xi$ by conjugation, hence $F$-invariant. One sees then that the action of $H^1(k, Z)$ must factor through $H^1(k, Z/Z^F)$ which, using the classic long exact sequence in Galois cohomology, can be restated as saying that the action of $H^1(k, Z)$ is actually defined over the quotient $H^1(k, Z)/H^1(k, Z^F)$.

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Moreover, in the commutative case, one may actually add the term $H^1(k, A^F)$ to the left of the exact sequence in Proposition 3.6 meaning that the action of the quotient $H^1(k, A)/H^1(k, A^F)$ is free in this case. Indeed, two inner twists of $A \times F$ will be isomorphic if and only if they are given by cocycles representing the same class in $H^1(k, (A \times F)/Z(A \times F))$. Now, here the center $Z(A \times F)$ is the direct product of $AF$ with a central subgroup $C$ of $F$, whence the split exact sequence

$$1 \longrightarrow A/A^F \longrightarrow (A \times F)/(A^F \times C) \longrightarrow F/C \longrightarrow 1,$$

from which one readily deduces that the map $H^1(k, A/A^F) \rightarrow H^1(k, (A \times F)/(A^F \times C))$ is injective. Using once again the classic long exact sequence in Galois cohomology, we get that the image of $H^1(k, A)$ in $H^1(k, (A \times F)/A^F \times C)$ is isomorphic to the quotient $H^1(k, A)/H^1(k, A^F)$.

### 3.3 Description of $\text{Ext}(F, G, \kappa)$ via the center of $G$

Considering what we did in section 2.7, one would be tempted now to relate these two last results using an analog of Proposition 2.14 (see Proposition 3.7 here below). That is, one would like to describe extensions of $F$ by $G$ via the extensions of $F$ by the center $Z$ of $G$.

In order to do so, we need to find a way of getting around the non smooth case for commutative groups, since we have already remarked before that the center $Z$ of a smooth $k$-group $G$ may not be smooth and hence the application $\varphi$ on the last propositions would not be well defined a priori.

Let then $F$ be a finite algebraic smooth $k$-group, let $G$ be a smooth algebraic $k$-group and let $Z$ be its center. Assume that there is an outer algebraic action of $F$ on $G_{k_s}$ given by an $F$-kernel $\kappa$. Then it is easy to deduce that $F$ acts algebraically on $Z$. Indeed, it suffices to compose $\kappa$ with the natural morphism $S\text{Out}(G_{k_s}) \rightarrow S\text{Out}(Z_{k_s})$ induced by the same morphism at the level of $S\text{Aut}$. Recalling then that $S\text{Out}(Z_{k_s}) = S\text{Aut}(Z_{k_s})$ simply because $Z$ is commutative, and that $\kappa$ is a splitting of (7) when restricted to $\gamma_F(\Gamma)$, we get the desired algebraic action thanks to the remark after Proposition 3.2.

A natural consequence is that the group $H^2(F, Z, \kappa)$ can be defined in the same way than in the smooth case (where $\kappa$ denotes here the algebraic action as a morphism $\Gamma \rightarrow S\text{Aut}(Z_{k_s})$). Indeed, recall that in the smooth case $H^2(F, Z, \kappa)$ is by definition the set $H^2(F_{\Gamma}, Z(k_s), \bar{\varphi} \circ \kappa)$, where $\bar{\gamma}$ is the morphism $S\text{Out}(Z_{k_s}) \rightarrow \text{Out}(Z(k_s)) = \text{Aut}(Z(k_s))$. Now, this last set is clearly well defined in the sense of Definition 2.3 for a non smooth $Z$. We define thus $H^2(F, Z, \kappa)$ in this way in all generality. Moreover, since $Z(k_s)$ is still the center of $G(k_s)$, we have the following analog of Proposition 2.14 (whose proof is word by word the same).

**Proposition 3.7.** Let $F$ be a finite algebraic smooth $k$-group and let $G$ be a smooth algebraic $k$-group. Assume that there is an outer algebraic action of $F$ on $G_{k_s}$ given by an $F$-kernel $\kappa$. Let $Z$ be the center of $G$ and denote also by $\kappa$ the algebraic action induced on $Z$. Then $H^2(F, Z_{k_s}, \kappa)$ acts freely and transitively on $H^2(F, G_{k_s}, \kappa)$.
In the same fashion, we can always define the group \( \text{Ext}(F, Z, \kappa) \) in the non smooth case. Note however that the construction given at the end of section \( \S 3.1 \) which extracts an outer algebraic action from a given extension \( E \) of \( F \) by \( Z \), will not work here since it uses the smoothness assumption. However, one can recover the algebraic action of \( F \) on \( Z \) by looking at the algebraic action of \( E \) on \( Z \) by conjugation, which factors through \( F \) since \( Z \) is commutative (one may use the first assertion in Proposition \( \S 3.2 \) to see this).

Finally, it is clear that the groups \( H^i(k, Z) \) are also well defined in this context, as well as the morphisms \( \psi \) and \( \text{Res} \) defined above (cf. Proposition \( \S 3.6 \)). The only construction that causes problems then is the map \( \varphi \), for whose construction we do use the smoothness assumption.

Let us recall now the notion of maximal separable \( k \)-subscheme (cf. \cite[Lemma C.4.1]{CGP10}).

**Lemma 3.8.** Let \( X \) a scheme locally of finite type over a field \( k \). There exists a unique geometrically reduced closed subscheme \( X_{\text{gr}} \subset X \) such that \( X_{\text{gr}}(k') = X(k') \) for all separable extensions \( k'/k \). The construction is moreover functorial and commutes with products over \( k \) and separable extensions of the base field. In particular, if \( G \) is a \( k \)-group scheme, then \( G_{\text{gr}} \) is a smooth \( k \)-subgroup of \( G \).

**Remark.**

In the case where \( k \) is perfect, one can see that \( X_{\text{gr}} \) is nothing but the closed subscheme \( X_{\text{red}} \) of \( X \) defined by taking the reduced structure of \( X \) as a scheme. In particular, if \( G \) is a \( k \)-group scheme then \( G_{\text{red}} \) is always a smooth \( k \)-subgroup (although not necessarily normal in \( G \), cf. \cite[VI, A, 0.2]{SGA3}). In the general case, \( G_{\text{red}} \) is not always a \( k \)-subgroup of \( G \), and, if it ever is a \( k \)-subgroup, then it is not necessarily a smooth \( k \)-subgroup. Examples of this are given in \cite[VI, A, 1.3.2]{SGA3}.

This object will prove to be the one we need. First of all, because of the following:

**Proposition 3.9.** Let \( A, F \) be algebraic \( k \)-groups such that \( A \) is commutative and \( F \) is smooth, finite and acts algebraically on \( A \). Then \( \kappa \) restricts naturally to \( A_{\text{gr}} \) and there are isomorphisms

\[
H^2(F, (A_{\text{gr}})_{k_s}, \kappa) \xrightarrow{\sim} H^2(F, A_{k_s}, \kappa),
\]

\[
H^i(k, A_{\text{gr}}) \xrightarrow{\sim} H^i(k, A).
\]

Moreover, if \( k \) is perfect, we have also the isomorphism

\[ \text{Ext}(F, A_{\text{gr}}, \kappa) \xrightarrow{\sim} \text{Ext}(F, A, \kappa). \]

**Proof.** The restriction of \( \kappa \) to \( A_{\text{gr}} \) is a simple consequence of Lemma \( \S 3.8 \). Indeed, the map \( a : F \times A \to A \) induces a map \( F_{\text{gr}} \times_k A_{\text{gr}} \to A_{\text{gr}} \) by functoriality and commutativity.

---

*I must thank Philippe Gille for bringing up this notion to our knowledge.*
with products over $k$. A straightforward verification shows that this map defines indeed an action of $F_{\text{gr}} = F$ on $A_{\text{gr}}$.

The isomorphisms for $H^2(F, \cdot, \kappa)$ and $H^i(k, \cdot)$ are evident since the $k_s$-points of both groups are the same.

Assume now that $k$ is perfect. Denote by $A_0$ the quotient $A/A_{\text{gr}}$. It is an infinitesimal group by definition. By [SGA3, XVII, App. I, Prop. 2.1], Note first that we get an exact sequence

$$H^1_0(F, A) \to \text{Ext}(F, A_{\text{red}}, \kappa) \to \text{Ext}(F, A, \kappa) \to \text{Ext}(F, A_0, \kappa),$$

where $H^1_0(F, A)$ denotes the Hochschild cohomology group. Since $A_0$ is infinitesimal, it is clear that for any extension $H$ of $F$ by $A_0$ the $k$-subgroup $H_{\text{red}}$ surjects onto (and is actually isomorphic to) $F$. This tells us that every such extension admits a schematic section and hence the group $\text{Ext}(F, A_0, \kappa)$ can be calculated via the Hochschild cohomology group $H^0_0(F, A_0)$ (cf. [SGA3, XVII, App. I, Prop. 3.1]). The isomorphism follows then from the lemma here below.

**Lemma 3.10.** Let $k$ be a field and let $A$, $F$, be algebraic $k$-groups such that $A$ is commutative and infinitesimal and $F$ is smooth, finite and acts algebraically on $A$. Then $H^0_0(F, A_0)$ is trivial for every $i \geq 0$.

**Proof.** Recall that Hochschild cohomology groups are obtained as the derived functors of the functor $B \mapsto B^F(k)$ (cf. [DG70, II.3.1.3]). The same is true for the functor $B_L \mapsto B^F_L(L) = B^F(L)$ for $L/k$ a finite extension such that $F_L$ is constant. We can see thus the former functor as the composition

$$B \mapsto B_L \mapsto B^F_L(L) = B^F(L) \mapsto B^F(L)\Gamma_{L/k} = B^F(k),$$

where the last functor is simply the one of fixed-$\Gamma_{L/k}$-points (where $\Gamma_{L/k}$ denotes of course the corresponding Galois group). Since the first of these three functors is clearly exact, Grothendieck’s spectral sequence for the other two functors gives then

$$H^p(\Gamma_{L/k}, H^q_0(F_L, A_L)) \Rightarrow H^{p+q}(F, A).$$

Now, since $F_L$ is a constant group, we know (cf. [DG70, III.6, Prop. 4.2]) that $H^i_0(F_L, A_L) = H^q(F(L), A(L))$ and hence this group is trivial for every $q$ since $A$ is infinitesimal. We deduce then that $H^0_0(F, A)$ is trivial for every $i \geq 0$.

Proposition 3.9 tells us then that we could have also defined the groups $H^2(F, A_{k_s}, \kappa)$ and $H^i(k, A)$ by taking the smooth group $A_{\text{gr}}$. Moreover, when $k$ is perfect, it allows us to define the morphism $\varphi : \text{Ext}(F, A, \kappa) \to H^2(F, A_{k_s}, \kappa)$ by passing through the same map for $A_{\text{gr}} = A_{\text{red}}$. Hence the corollary:

---

$^7$Note that in the case $i = 2$, not only have we proved above that every extension of $F$ by $A_0$ admits a schematic section, but actually we proved that they are always split, hence the triviality of $H^0_0(F, A_0)$, cf. also [DG70, III.6, Cor. 4.9]).
Corollary 3.11. When $k$ is a perfect field, Proposition 3.6 is also valid for non smooth $A$. 

If we consider again our $k$-group $G$ with center $Z$, we see then that $Z_{gr}$ is a smooth $k$-subgroup of $G$ whose $k_s$-points correspond to the center of $G(k_s)$. Let us now define an action of $\text{Ext}(F, Z_{gr}, \kappa)$ on $\text{Ext}(F, G, \kappa)$. Consider $\xi \in \text{Ext}(F, G, \kappa)$ and $\zeta \in \text{Ext}(F, Z_{gr}, \kappa)$. Take extensions

$$1 \to G \to H_\xi \to F \to 1,$$

and

$$1 \to Z_{gr} \to H_\zeta \to F \to 1,$$

representing $\xi$ and $\zeta$ respectively and consider their direct product. This gives an extension

$$1 \to G \times_k Z_{gr} \to H_\xi \times_k H_\zeta \to F \times_k F \to 1.$$

Consider now the diagonal subgroup $F$ of $F \times_k F$ and take preimages in order to get

$$1 \to G \times_k Z_{gr} \to H \to F \to 1.$$

Consider now the multiplication $k$-morphism $G \times_k Z_{gr} \to G$. This is clearly $H$-equivariant and hence the kernel of such a morphism is normal in $H$. One may thus take quotient by this kernel in order to get the extension

$$1 \to G \to H_\xi' \to F \to 1,$$

representing a class $\xi' \in \text{Ext}(F, G, \kappa)$. We set then $\xi \cdot \xi := \xi'$. This definition is clearly independent of the choice of the extensions representing $\xi$ and $\zeta$.

Fix now a class $\xi \in \text{Ext}(F, G, \kappa)$ and define the following maps:

$$\theta_\xi : \text{Ext}(F, Z_{gr}, \kappa) \to \text{Ext}(F, G, \kappa) : \zeta \mapsto \zeta \cdot \xi,$$

$$\psi_\xi : H^1(k, Z) \to \text{Ext}(F, G, \kappa) : \alpha \mapsto \alpha \cdot \xi$$

The lower map is immediately seen to be bijective by Proposition 3.7.

**Proposition 3.12.** Under the hypotheses of Proposition 3.6, and denoting by $Z$ the center of $G$, there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
H^1(k, Z^F) & \longrightarrow & H^1(k, Z) & \psi & \text{Ext}(F, Z_{gr}, \kappa) & \varphi & H^2(F, Z_{k_s}, \kappa) & \text{Res} & H^2(k, Z) \\
\downarrow & & \downarrow & \theta_\xi & \downarrow & \varphi & \downarrow & \theta & \downarrow \\
H^1(k, Z) & \longrightarrow & \text{Ext}(F, G, \kappa) & \varphi & H^2(F, G_{k_s}, \kappa) & \text{Res} & H^2(k, G),
\end{array}
$$

where $\theta$ denotes the bijection given by Proposition 2.14 and the base point in $H^2(F, G_{k_s}, \kappa)$ is $\varphi(\xi)$.

In particular, the action of $\text{Ext}(F, Z_{gr}, \kappa)$ on $\text{Ext}(F, G, \kappa)$ is transitive and with stabilizer isomorphic to the quotient $(H_\xi/Z^F)(k)\backslash(H_\xi/Z)(k)$. 

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Remark.
Of course, given Proposition \textbf{3.9} one sees that we may replace $Z_{gr}$ by $Z$ and vice-versa everywhere if $k$ is perfect, and everywhere except at the central term in the general case. This must actually be done if one wishes the upper $\varphi$ and $\psi$ maps to make any sense.

Proof. The exactness of both rows follows from Proposition \textbf{3.5}, Proposition \textbf{3.6} and the remark right after it and Proposition \textbf{3.9}.

Commutativity in the square on the right side is trivial when one looks at it at the level of cocycles. The same is true for the square on the left, since the twisting procedure is functorial. We must prove then commutativity in the middle square, which is also done at the cocycles level. Indeed, it suffices to follow the definition of the action of $\text{Ext}(F, Z_{gr}, \kappa)$ over $\text{Ext}(F, G, \kappa)$ defined above. Let $(f, g)$ be a cocycle obtained from the extension $H_\xi$ as in the construction of $\varphi$. Then $f$ restricts to $Z_{gr}(k_s) = Z(k_s)$ as the natural action of $F_\Gamma$. In particular, we may denote by $(f, g)$ the cocycle obtained from $\zeta \in \text{Ext}(F, Z_{gr}, \kappa)$ by the same procedure. It is then easy to see that the cocycle obtained from the extension $H$ of $F$ by $G \times_k Z_{gr}$ above is nothing but $(f \times f, g \times g)$ and hence the cocycle representing $H_\psi$ is $(f, g)$ since $f_\psi$ is compatible with the multiplication morphism. This is precisely the cocycle representing $\theta_{\varphi(\xi)}(\varphi(\zeta)) = \varphi(\zeta) \cdot \varphi(\xi)$, which proves commutativity.

The transitivity of the action of $\text{Ext}(F, Z_{gr}, \kappa)$ on $\text{Ext}(F, G, \kappa)$ is easily proved by diagram chasing. Let now $\zeta \in \text{Ext}(F, Z_{gr}, \kappa)$ be an element stabilizing $\xi \in \text{Ext}(F, G, \kappa)$. The diagram tells us then that there exists $\alpha \in H^1(k, Z)$ such that $\varphi(\alpha) = \zeta$. In other words, $\zeta$ is represented by a twist of the semi-direct product $Z_{gr} \rtimes F$ by a cocycle representing the image $\alpha_0$ of $\alpha$ in $H^1(k, Z/Z^F)$. Now, since $\theta_{\xi}(\psi(\alpha)) = \psi_{\zeta}(\alpha) = \alpha \cdot \xi = \xi$, we know that the image of $\alpha$ in $H^1(k, H_\xi/Z(H_\xi))$ must be trivial. In other words, we have

$$\alpha_0 \in \text{Ker}[H^1(k, Z/Z^F) \rightarrow H^1(k, H_\xi/Z(H_\xi))].$$

Moreover, since $Z(H_\xi)$ is an extension of a central subgroup $C$ of $F$ by $Z^F$, it is not difficult to see that

$$\text{Ker}[H^1(k, Z/Z^F) \rightarrow H^1(k, H_\xi/Z(H_\xi))] = \text{Ker}[H^1(k, Z/Z^F) \rightarrow H^1(k, H_\xi/Z^F)],$$

since $H_\xi/Z(H_\xi) = (H_\xi/Z^F)/C$ and the subgroups $Z/Z^F$ and $C$ of $H_\xi/Z^F$ commute and have trivial intersection.

To conclude, we claim that this last kernel is isomorphic to the quotient $(H_\xi/Z^F)(k) \backslash (H_\xi/Z)(k)$ and that the image of $H^1(k, Z)$ in $H^1(k, Z/Z^F)$ covers the whole kernel. Both assertions in fact are easily seen from the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \rightarrow & Z & \rightarrow & H_\xi & \rightarrow & H_\xi/Z & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z/Z^F & \rightarrow & H_\xi/Z^F & \rightarrow & H_\xi/Z & \rightarrow & 1,
\end{array}$$

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and from the commutative diagram it induces in cohomology:

\[
\begin{array}{c}
(H_\xi/Z)(k) \xrightarrow{\delta} H^1(k, Z) \xrightarrow{} H^1(k, H_\xi) \\
\downarrow \downarrow \downarrow \\
(H_\xi/Z^F)(k) \xrightarrow{\delta} H^1(k, Z/Z^F) \xrightarrow{} H^1(k, H_\xi/Z^F).
\end{array}
\]

4 Reduction of extensions

In this section, we follow the "dévissage" ideas in section 3 of [Spr66], in which Springer proves the following theorem in the context of \(k\)-kernels (cf. [Spr66, Thm. 3.4]):

**Theorem 4.1** (Springer). Let \(k\) be a perfect field, let \(\bar{G}\) be a (smooth) algebraic \(k_s\)-group and let \(\kappa\) be a \(k\)-kernel in \(\bar{G}\). Then for every \(\eta \in H^2(k, \bar{G}, \kappa)\) there exists a finite nilpotent subgroup \(\bar{H}\) of \(\bar{G}\) and a \(k\)-kernel \(\lambda\) in \(\bar{H}\), compatible with \(\kappa\), such that \(\eta \in (\text{id}_\Gamma, \iota)_*^*(H^2(k, \bar{H}, \lambda))\), where \(\iota : \bar{H} \to \bar{G}\) denotes the inclusion.

Recall from section 2.3 that the notation \(\eta \in (\text{id}_\Gamma, \iota)_*^*(H^2(k, \bar{H}, \lambda))\) means that there exists a class \(\xi \in H^2(k, \bar{H}, \lambda)\) that is related to \(\eta\) via the relation \((\text{id}_\Gamma, \iota)_*\). Once again, this amounts to the existence of a commutative diagram

\[
\begin{array}{c}
1 \xrightarrow{} \bar{H}(k_s) \xrightarrow{\iota} E_\xi \xrightarrow{} \Gamma \xrightarrow{\text{id}_\Gamma} 1 \\
1 \xrightarrow{} \bar{G}(k_s) \xrightarrow{\iota} E_\eta \xrightarrow{} \Gamma \xrightarrow{\text{id}_\Gamma} 1,
\end{array}
\]

where \(\Gamma\) denotes \(\text{Gal}(k_s/k)\) as always, \(E_\xi\) represents \(\xi\) and \(E_\eta\) represents \(\eta\).

**Remark.**

Springer’s theorem asserts moreover that \(\bar{H}\) is defined over \(k\). This assertion however does not have much sense unless the group \(\bar{G}\) itself is defined over \(k\) (a finite group can always be given the structure of a constant group over any field, even over \(\mathbb{Z}\)). We decided then to take this assertion as a typo, since there is no mention of it in the proof (Proposition 3.1 and Lemmas 3.2, 3.3, which are used in the proof, do not have this assertion). One could wonder if Springer actually had a proof of the existence of finite \(k\)-groups factoring extensions for an arbitrary algebraic \(k\)-group. All the more since we show here below that the same techniques can actually give such a result.

We restrict from now on to a perfect field \(k\).

**Theorem 4.2.** Let \(k\) be a perfect field of characteristic \(p \geq 0\). Let \(F\) be a smooth finite \(k\)-group of order \(n\), and \(G\) an arbitrary smooth \(k\)-group. Then, for any extension

\[1 \to G \to H \to F \to 1,\]
there exists a finite smooth $k$-subgroup $S$ of $G$ and a commutative diagram with exact rows

$$
\begin{array}{ccc}
1 & \longrightarrow & S \\
\downarrow & & \downarrow \\
1 & \longrightarrow & H' \\
H' & \longrightarrow & F & \longrightarrow & 1 \\
\end{array}
$$

Moreover, if $G$ is linear, let $T$ be a maximal torus in $G$, $W$ be the Weyl group of $G$ (that is, the finite group of connected components of the normalizer of $T$) and $K/k$ be a separable algebraic extension splitting $T$. Denote by $r$ the rank of $T$, $w$ the order of $W$ and $d$ the degree of $K/k$. Assume that either $nw$ is prime to $p$ or that $G^o$ is reductive. Then one can take $S$ to be contained in an extension of $W$ by the $ndw$-torsion subgroup of $T$, hence of order dividing $nrdw^2$.

Remarks.

1. The first part of this result has been recently proved by Brion (cf. [Bri15, Thm. 1]) for any field $k$ and with no smoothness assumption on $G$ or $F$. However, it had already been claimed to be true for any perfect field $k$ more than 50 years ago by Borel and Serre (cf. [BS64, Lem. 5.11, footnote on p. 152]), although they only gave the proof for linear $G$ and $k = k_s$ of characteristic zero. Platonov gave shortly after a proof of this fact for linear groups over a perfect field (cf. [Pla66, Lem. 4.14]). The second part has been already proved by the author in the particular case in which $G$ is itself a torus and $nd$ is prime to $p$. Other particular cases had been treated for example in [Vin96, Prop. 7] and [LMMR13, Lem. 5.2].

2. The extra hypotheses in the second part are (almost) strict, i.e. one cannot control the order of the finite group obtained if, for example, $G$ is not linear, or when the order of $F$ is divisible by $p$ and $G^o$ is not reductive. See the very last remark of this article (after the proof of Theorem 5.1) for more details.

Before we get started, let us prove a technical lemma that will be used several times in the proof.

**Lemma 4.3.** Let $k$ be a field. Let $F$ be a smooth finite $k$-group, let $G$ be an arbitrary smooth $k$-group and let $Z$ be its center. Let

$$
1 \rightarrow G \rightarrow H \rightarrow F \rightarrow 1,
$$

be an extension of $k$-groups, $\kappa$ the $F$-kernel in $G_{k_s}$ induced by this extension and $\xi \in \text{Ext}(F, G, \kappa)$ the class corresponding to this extension. Finally, let $(f, g) \in Z^2(F, G_{k_s}, \kappa)$ be a cocycle representing $\varphi(\xi) \in H^2(F, G_{k_s}, \kappa)$ (see section 3.2) such that the restriction of $\varphi$ to $\gamma_F(\Gamma)$ gives the natural action of $\Gamma$ over $G(k_s)$.

Assume that there exists a smooth $k_s$-subgroup $\bar{M}$ of $G_{k_s}$ invariant by $f_{(f, \sigma)}$ for all $(f, \sigma) \in F_{\Gamma}$ and such that the image of $g$ is contained in $\bar{M}$. Then there exists a smooth $k$-subgroup $M$ of $G$ inducing the inclusion of $\bar{M}$ and a commutative diagram with exact
for some \( \zeta \in \mathbb{Z}^1(k, Z) \), where \( \zeta H \) is the twist of \( H \) by the cocycle \( \zeta \). In particular, if \( H^1(k, Z) = 0 \) or if \( Z \subset \mathcal{M} \), then one may replace \( \zeta H \) by \( H \) above.

**Proof.** Since \( f \) defines semialgebraic automorphisms of \( G \) and preserves \( \mathcal{M} \), one easily sees that its restriction to \( \mathcal{M} \) defines semialgebraic automorphisms of \( \mathcal{M} \), hence we may consider \( f \) as taking values in \( \text{SAut}(\mathcal{M}) \). Moreover, since the restriction of \( f \) to \( \gamma_F(\Gamma) \) is a morphism by hypothesis (recall that it represents the action of \( \Gamma \) on \( G(k_s) \)), we get by restriction a morphism \( \Gamma \to \text{SAut}(\mathcal{M}) \) defining a \( k \)-form \( \mathcal{M} \) of \( \mathcal{M} \) which is clearly a smooth \( k \)-subgroup of \( G \). The cocycle \( (f, g) \) can then be seen as an element of \( Z^2(F, M_{k_s}, \kappa') \) where \( \kappa' \) is the \( F \)-kernel induced by the restriction of \( f \). This amounts to the existence of a commutative diagram with exact rows

\[
1 \to M(k_s) \to E' \to F_{\Gamma} \to 1,
\]

\[
1 \to G(k_s) \to E \to F_{\Gamma} \to 1.
\]

Consider the preimage of \( F(k_s) \subset F_{\Gamma} \) in the upper row of the diagram. This gives a subgroup \( \tilde{H}' \) of \( H(k_s) \) which can easily be given a smooth \( k_{s^{-}} \)-group structure (it already has it over \( M(k_s) \subset \tilde{H}' \) and it can be defined elsewhere by translation since \( F \) is finite and smooth and acts via semi-algebraic automorphisms). Taking now the preimages of \( \gamma_F(\Gamma) \) in both rows, we get the rear part of the following commutative diagram

\[
1 \to M(k_s) \to E''' \to \gamma_F(\Gamma) \to 1,
\]

\[
1 \to \tilde{H}'(k_s) \to E' \to \Gamma \to 1,
\]

\[
1 \to G(k_s) \to E'' \to \gamma_F(\Gamma) \to 1,
\]

\[
1 \to H(k_s) \to E \to \Gamma \to 1,
\]

where the compatible splittings on the rear are given by the fact that (the same!) \( (f, g) \) restricts to a neutral cocycle over \( \gamma_F(\Gamma) \). We see then that there are natural compatible splittings of the exact sequences on the front. This splitting on the lower part of the diagram defines an action of \( \Gamma \) on \( H(k_s) \) which is easily seen to define a \( k \)-form \( \tilde{H} \) of \( H \). Now, this action being induced by our cocycle \( (f, g) \), \( \tilde{H} \) must be an extension of \( H \) by \( G \) inducing the class of \( \varphi(\xi) \) and hence, by Proposition 3.5, it must correspond to a
twist \( \hat{\imath}H \) by a cocycle \( \hat{\imath}z \in Z^1(k, Z) \) (note that it is precisely with this construction that we prove surjectivity in Proposition 3.5). The diagram is then basically telling us that this action is compatible with the inclusion of \( \bar{H}'(k) \subset H(k) \) in \( H(k) \). In other words, there is a \( k \)-subgroup \( H' \subset \hat{\imath}H \) inducing the inclusion \( \bar{H}' \subset \hat{\imath}H(k) \) (recall that the action of \( \Gamma \) on \( \bar{H}' \) is given by actual semialgebraic automorphisms: this has already been stated for the subgroup \( \bar{M} = M(k) \subset \bar{H}' \) and can be deduced elsewhere by translation). Finally, this group is clearly an extension of \( F \) by \( M \), i.e. we have the commutative diagram

\[
\begin{array}{cccccc}
1 & \to & M & \to & H' & \to & F & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & G & \to & \hat{\imath}H & \to & F & \to & 1.
\end{array}
\]

The last assertions are easy to prove. If \( H^1(k, Z) = 0 \), then the twist \( \hat{\imath}H \) is \( k \)-isomorphic to \( H \) and hence the image of \( H' \) by this \( k \)-isomorphism (which reduces to identity over \( F \) and \( G \)) will give the same diagram with \( H \) instead of \( \hat{\imath}H \). If \( Z \subset M \), then \( Z \subset H' \) and it is easy to see that we can twist the whole diagram by the inverse \( \hat{\imath}^{-1} \) of \( \hat{\imath} \), which will give a \( k \)-subgroup \( \hat{\imath}^{-1}H' \) of \( H \) that is another \( k \)-form of \( \bar{H}' \) and an extension of \( F \) by \( M \) (since, once again, the twisting reduces to identity over \( F \) and \( M \)).

**Proof of Theorem 4.2.** An extension such as the one given in the statement of the theorem induces an outer algebraic action of \( F \) on \( G(k) \), i.e. an \( F \)-kernel \( \kappa \). The proof follows the reasoning of Springer for this \( F \)-kernel and goes by successive generalizations.

**Step 1: the case of tori and abelian varieties.** Assume that \( G \) is either a torus or an abelian variety. Then the result is a consequence of Proposition 3.6. Indeed, since Ext\((F, G, \kappa)\) is a torsion group, then for any \( \xi \in \text{Ext}(F, G, \kappa) \) there exists some \( m \in \mathbb{N} \) such that \( m \cdot \xi = 0 \). Moreover, the sequence

\[
1 \to G[m] \to G \to G[k] \to 1,
\]

where \( G[m] \) denotes the \( m \)-torsion of \( G \), is exact for every \( m \in \mathbb{N} \). Then we can easily see that \( \kappa \) defines by restriction an algebraic action of \( F \) over \( G[m] \) and we finally get an exact sequence (cf. [SGA3, XVII, App. I, Prop. 2.1])

\[
\text{Ext}(F, G[m], \kappa) \to \text{Ext}(F, G, \kappa) \to \text{Ext}(F, G, \kappa).
\]

We see thus that \( \xi \) comes from \( \text{Ext}(F, G[m], \kappa) \), which clearly gives us the commutative diagram

\[
\begin{array}{cccccc}
1 & \to & G[m] & \to & H' & \to & F & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & G & \to & H & \to & F & \to & 1.
\end{array}
\]

It suffices then to recall that \( S = G[m] \) is finite for \( G \) a torus or an abelian variety. Moreover, in the case of a torus, we know that \( H^1(k, G) \) is \( d \)-torsion by the classic
restriction-corestriction argument and Hilbert’s theorem 90. We deduce again from Proposition 3.6 that we may take \( m = nd \) independently of \( \xi \) and hence the order of \( S \) is \( nrd \).

Assume that the order of \( S \) is prime to the characteristic. Then \( S \) is clearly smooth and we are done. If this is not the case, then one may always consider the smooth \( k \)-subgroup \( H'_{gr} \) of \( H' \) (see section 3.3). This is easily seen to be an extension of \( F \) by \( S_{gr} \) which is a \( k \)-subgroup of \( S \) and hence of order dividing \( nrd \) in the case where \( G \) is a torus.

**Step 2: the connected unipotent case.** Since \( k \) is perfect, we know that \( G \) admits a characteristic decomposition series in which every quotient is isomorphic to \( \mathbb{G}_a \) (cf. SGA3 XVII, Cor. 4.1.3)).

Let us treat then first the case in which \( G \cong \mathbb{G}_a \). In this case, Proposition 3.6 tells us that \( \text{Ext}(F, G, \kappa) \) is isomorphic to \( H^2(F_{\Gamma}, G(k_s)) \) since \( H^i(k, G) = 0 \) for \( i = 1, 2 \) (cf. [Ser02 III.2.1 Prop. 6]). Then, if \( g : F^2 \rightarrow G(k_s) \) is a cocycle representing the class of our extension, by continuity we know that it has a finite image.

If \( k \) has positive characteristic, then the \( F_{\Gamma} \)-invariant subgroup \( \bar{S} \) of \( G(k_s) \) generated by the image of \( g \) is also finite since \( G(k_s) \) is \( p \)-torsion and the action of \( F_{\Gamma} \) over \( G(k_s) \) is also continuous. Moreover, since \( \bar{S} \) is finite, it may be considered as a smooth \( k_s \)-subgroup in a canonical way. Lemma 4.3 applies then (recall that \( G \) is commutative and \( H^1(k, G) = 0 \)), giving us a \( k \)-form \( S \) of \( \bar{S} \) and the commutative diagram with exact rows

\[
\begin{array}{cccccc}
1 & \rightarrow & S & \rightarrow & H' & \rightarrow & F & \rightarrow & 1 \\
| & | & \downarrow & & \downarrow & | & | \\
1 & \rightarrow & G & \rightarrow & H & \rightarrow & F & \rightarrow & 1.
\end{array}
\]

Assume now that \( n \) is prime to \( p \) and consider the Hochschild-Serre spectral sequence

\[
E_2^{p,q} = H^p(\Gamma, H^q(F(k_s), G(k_s))) \Rightarrow H^{p+q}(F_{\Gamma}, G(k_s)).
\]

Then \( H^q(F(k_s), G(k_s)) = 0 \) for every \( q > 0 \) since multiplication by \( n \) in \( G(k_s) \) is an isomorphism and \( H^q(F(k_s), G(k_s)) \) is \( n \)-torsion. Thus \( H^2(F_{\Gamma}, G(k_s)) \) is isomorphic to \( H^2(\Gamma, H^0(F(k_s), G(k_s))) \). Since \( n \) is prime to \( p \), it is easy to see that the action of \( F \) on \( G \) must be linear and hence \( H^0(F(k_s), G(k_s)) \) is either \( G(k_s) \) itself or 0 and thus \( H^2(\Gamma, H^0(F(k_s), G(k_s))) \) is also trivial. We deduce that in this case the group \( \text{Ext}(F, G, \kappa) \) is trivial and hence our extension is a semi-direct product.

The general case is easily deduced from the previous one by induction. Indeed, let

\[
1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1,
\]

with \( G'' \cong \mathbb{G}_a \) the first quotient obtained from the characteristic decomposition of \( G \). We know then that the sequence is \( F \)-equivariant and we may assume that the result is true for \( G' \). The extension \( H \) of \( F \) by \( G \) gives us then, by taking quotients by \( G' \), an extension

\[
1 \rightarrow G'' \rightarrow H/G' \rightarrow F \rightarrow 1,
\]

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from which we can get a finite $k$-subgroup $S''$ of $G''$ and a commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & S'' \rightarrow H'' \rightarrow F \rightarrow 1 \\
\downarrow & & \downarrow \\
1 & \rightarrow & G'' \rightarrow H/G'' \rightarrow F \rightarrow 1.
\end{array}
\]

Consider then the preimage of $H''$ in $H$: it is an extension of the finite $k$-group $H''$ by $G'$ and hence by assumption it admits a finite $k$-subgroup $H'$ extension of $H''$ by a finite subgroup $S'$ of $G'$. This $H'$ is clearly an extension of $F$ by a finite subgroup, itself extension of $S''$ by $S'$. This concludes the proof in this case.

Assume at last once again that $n$ is prime to $p$. Then it is clear that in each induction step above we may assume the groups $S'$ and $S''$ to be trivial and hence the finite group $H'$ obtained will be isomorphic to $F$. We deduce that $H$ is in this case a semi-direct product.

**Step 3: the linear case.** Let $T$ be a maximal torus of $G$ and let $N = N_G(T)$ be the normalizer of $T$ in $G$. Such subgroups do exist and are smooth and defined over $k$ as it can be seen by [SGA3, XIV, Thm. 1.1; VI_B, Prop. 6.2.5; XI, Cor. 2.4]. Moreover, the neutral connected component $N^0$ of $N$ centralizes $T$ (cf. [Bri13, Lem. 2.4]) and it is actually isomorphic to a direct product $T \times_k U$ with $U$ a unipotent group since it is linear, connected and nilpotent (cf. [SGA3, XII, Cor. 6.7; XVII, Thm. 6.1.1]). We will show that we can always pull back an extension of $F$ by $G$ to an extension of a finite group by $U \times_k T$, the finite group being itself an extension of $F$ by the Weyl group $W = N/N^0$.

Consider an extension

\[
1 \rightarrow G \rightarrow H \rightarrow F \rightarrow 1,
\]
and a class $\eta \in H^2(F, G_{k_s}, \kappa)$ representing the corresponding extension

\[
1 \rightarrow G(k_s) \rightarrow E \rightarrow F_\Gamma \rightarrow 1.
\]

Let $(f, g)$ be a cocycle representing $\eta$. Since the restriction of $\eta$ to $H^2(k, G)$ is the trivial cocycle representing the natural action of $\Gamma$ over $G(k_s)$ (this follows from Proposition 3.5), we may assume that $g_{(1, \sigma), (1, \tau)} = 1$ for every $\sigma, \tau \in \Gamma$ and that $f$ restricted to $\gamma_F(\Gamma) \subset F_\Gamma$ describes precisely this action.

Consider now, for $(f, \sigma) \in F_\Gamma$ the $k_s$-subgroup $f_{(f, \sigma)}(T)$ of $G$. This is a maximal torus and hence it is conjugate to $T$ over $k_s$ (cf. for example SGA3 XII, Thm. 1.7), i.e. there exists $c_{(f, \sigma)} \in G(k_s)$ such that $f_{(f, \sigma)} = c_{(f, \sigma)}^{-1} T c_{(f, \sigma)}$. Now, from our assumption on $(f, g)$ it is easy to see that one may take $c_{(1, \sigma)} = 1$ for every $\sigma \in \Gamma$. Since $\gamma_F(\Gamma)$ is clearly open in $F_\Gamma$, this implies that one may choose the $c_{(f, \sigma)}$ in order to get a continuous map $c : F_\Gamma \rightarrow G(k_s)$ which is trivial on $\gamma_F(\Gamma)$ and hence, up to changing $(f, g)$ by $c \cdot (f, g)$, we may further assume that $f_{(f, \sigma)}$ fixes $T$ for all $(f, \sigma) \in F_\Gamma$.

It suffices then to look at equation (24) to see that $g$ must take values in $N(k_s)$ and that $f$ may be considered as taking values in SAut($N_{k_s}$). Lemma 13 applies again (note
that by definition $N$ contains the center of $G$), giving us the following commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & H_0 & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & F & \longrightarrow & 1.
\end{array}
\]

One should note by the way that the $k$-form of $N_k$ given by Lemma 4.3 is nothing but $N$ itself since the modifications done above on $f$ do not change its restriction to $\gamma_F(\Gamma)$ and hence do not change the natural action of $\Gamma$.

Recall now that $N$ sits in an extension

\[1 \to T \times_k U \to N \to W \to 1,\]

with $U$ a smooth connected unipotent group. In particular, since $T \times_k U = N^\circ$ is clearly normal in $H_0$, we get an extension

\[1 \to N^\circ \to H_0 \to F' \to 1,\]

where $F'$ is an extension of $F$ by $W$ and hence a finite group. Consider now this last extension and take quotients by $U$ and by $T$. We get extensions

\[1 \to R \to H_R \to F' \to 1,\]

where $R$ denotes either $T$ or $U$. Steps 1 and 2 tell us then that there exist a finite smooth $k$-subgroup $S_R \subset R$ and a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & S_R & \longrightarrow & H'_R & \longrightarrow & F'' & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & R & \longrightarrow & H_R & \longrightarrow & F'' & \longrightarrow & 1.
\end{array}
\]

Consider then the direct product of $H_T$ and $H_U$. One easily sees that there is a commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & S_T \times_k S_U & \longrightarrow & H'_T \times_k H'_U & \longrightarrow & F' \times_k F' & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T \times_k U & \longrightarrow & H_T \times_k H_U & \longrightarrow & F' \times_k F' & \longrightarrow & 1.
\end{array}
\]

Consider now the diagonal subgroup $F'$ of $F' \times_k F'$ and take the preimages of this subgroup in both rows. Since $N^\circ = T \times_k U$, it is then easy to see that in the lower row we recover our group $H_0$ (it suffices to write down the cocycles representing these extensions). Setting then $S' := S_T \times_k S_U$ we get the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & S' & \longrightarrow & H' & \longrightarrow & F'' & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N^\circ & \longrightarrow & H_0 & \longrightarrow & F' & \longrightarrow & 1.
\end{array}
\]
from which we get

\[
\begin{array}{ccccccccc}
1 & \rightarrow & S & \rightarrow & H' & \rightarrow & F & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & N & \rightarrow & H_0 & \rightarrow & F & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & G & \rightarrow & H & \rightarrow & F & \rightarrow & 1.
\end{array}
\]

where \( S \) is the preimage of \( W \) (as a subgroup of \( F' \)) in \( H' \) and hence is smooth since \( W \) and \( S' \) are. Moreover, if \( G^\circ \) is reductive or if \( nw \) is prime to \( p \), step 2 tells us that we may assume \( S_U \) to be trivial and hence \( S \) becomes an extension of \( W \) by \( S_T \) whose order divides \( nrdw \) by step 1. The order of \( S \) divides then \( nrdw^2 \).

**Step 4: the general case.** This is simply an application of the previous steps. Indeed, for \( G \) an arbitrary smooth \( k \)-group there is always an exact sequence

\[
1 \rightarrow G^\circ \rightarrow G \rightarrow F' \rightarrow 1,
\]

where \( G^\circ \) is connected and characteristic in \( G \) and both \( F' = G/G^\circ \) and \( G^\circ \) are smooth. In particular, the extension \( H \) of \( F \) by \( G \) gives rise to an extension

\[
1 \rightarrow G^\circ \rightarrow H \rightarrow F'' \rightarrow 1,
\]

where \( F'' \) is also finite and smooth, since it is an extension of \( F \) by \( F' \). By an argument similar to that given in last step, it is easy to see that we may assume \( G \) to be connected.

Now, for \( G \) a smooth connected \( k \)-group, there is a unique exact sequence

\[
1 \rightarrow L \rightarrow G \rightarrow A \rightarrow 1,
\]

where \( L \) is a smooth connected linear \( k \)-group and \( A \) is an abelian variety. (cf. \textbf{Con02}). Consider then the extension

\[
1 \rightarrow G \rightarrow H \rightarrow F \rightarrow 1.
\]

Since \( L \) is characteristic in \( G \), it is normal in \( H \) and hence one gets the following extension by taking quotients

\[
1 \rightarrow A \rightarrow H/L \rightarrow F \rightarrow 1.
\]

Step 1 then tells us that there exists a smooth finite group \( S_0 \subset A \) (whose order we cannot control a priori) and a commutative diagram

\[
\begin{array}{ccccccccc}
1 & \rightarrow & S_0 & \rightarrow & H_0 & \rightarrow & F & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & A & \rightarrow & H/L & \rightarrow & F & \rightarrow & 1.
\end{array}
\]
Let $H_1$ be the preimage of $H_0$ in $H$. It is an extension of the finite smooth $k$-group $H_0$ by the smooth linear group $L$. Step 3 then tells us that there exists a finite smooth $k$-subgroup $S'$ of $L$ and a commutative diagram

$$
1 \longrightarrow S' \longrightarrow H' \longrightarrow H_0 \longrightarrow 1 \\
1 \longrightarrow L \longrightarrow H_1 \longrightarrow H_0 \longrightarrow 1.
$$

Finally, since $H_0$ is an extension of $F$ by $S_0$, one readily sees that $H'$ is an extension of $F$ by $S$, itself an extension of $S_0$ by $S'$. In other words, we have the commutative diagram

$$
1 \longrightarrow S \longrightarrow H' \longrightarrow F \longrightarrow 1 \\
1 \longrightarrow G \longrightarrow H \longrightarrow F \longrightarrow 1.
$$

This concludes the proof.

**Remark.**

One can even give a more general (although less useful) version of Springer’s theorem by an almost word-by-word copy of his proof (see [Spr66, 3.1-3.4]):

**Theorem 4.4.** Let $k$ be a perfect field, let $F$ be a finite smooth $k$-group, let $\widehat{G}$ be a smooth $k_s$-group and let $\kappa$ be an $F$-kernel in $\widehat{G}$. Then for every $\eta \in H^2(F, \widehat{G}, \kappa)$ there exists a finite nilpotent subgroup $\bar{H}$ of $\widehat{G}$ and an $F$-kernel $\lambda$ in $\bar{H}$, compatible with $\kappa$, such that $\eta \in (\text{id}_F, \iota)_2^*(H^2(F, \bar{H}, \lambda))$, where $\iota : \bar{H} \rightarrow \widehat{G}$ denotes the inclusion.

Note that here we can demand the finite group to be nilpotent, something we can’t do a priori in the case of extensions of $k$-groups, the obstruction being the fact that the reduction from finite groups to nilpotent ones (i.e. [Spr66, Prop. 3.1]) need not give a group defined over $k$.

## 5 Some finiteness results

Notations are as above. Given Proposition 3.12 and Theorem 4.2, one can prove the finiteness of $\text{Ext}(F, G, \kappa)$ under convenient hypotheses.

Following Serre (cf. [Ser02, III.4.2]), we say that a field $k$ is of type (F) if it is perfect and if, for every $n \geq 1$, there exist only a finite number of subextensions of $k_s$ of degree $n$ over $k$. Examples of such fields are $\mathbb{R}$, $\mathbb{C}((T))$, finite fields and $p$-adic fields. For such fields, one can state the following result.

**Theorem 5.1.** Let $k$ be a field of type (F), $F$ be a finite algebraic smooth $k$-group of order $n$ and let $G$ be a smooth $k$-group. Assume there is an outer algebraic action of $F$ on $G_{k_s}$ given by an $F$-kernel $\kappa$. Assume moreover that one of the following holds:

1. $k$ is finite;
2. \( G \) is linear and \( n \) is prime to \( p \);

3. \( G \) has a reductive neutral component.

Then the set \( \text{Ext}(F,G,\kappa) \) is finite.

The first step of the proof is the following lemma, which is an easy corollary of Proposition 3.12.

**Lemma 5.2.** Let \( k \) be a field of type \((F)\), \( F \) be a finite algebraic smooth \( k \)-group and let \( G \) be a smooth \( k \)-group with center \( Z \). Assume there is an outer algebraic action of \( F \) on \( G \) given by an \( F \)-kernel \( \kappa \) inducing thus an algebraic action of \( F \) on \( Z \) (see section 3). Then the set \( \text{Ext}(F,G,\kappa) \) is finite if and only if the group \( \text{Ext}(F,Z,\kappa) \) is.

**Proof.** This is a corollary of Proposition 3.12 and [Ser02, III.4.4, Thm. 5], which tells us that the quotient \((H\xi/ZF)(k)\setminus(H\xi/Z)(k)\) is finite.

**Proof of Theorem 5.1.** In all three cases, Lemma 5.2 and Proposition 3.9 allow us to reduce to the case where \( G \) is commutative. Moreover, using the fact (cf. [SGA3, XVII. App. I, Prop. 2.1]) that an exact sequence of commutative \( k \)-groups with an \( F \)-action

\[
1 \to A_1 \to A_2 \to A_3 \to 1,
\]

gives an exact sequence of groups

\[
\text{Ext}(F,A_1,\kappa_1) \to \text{Ext}(F,A_2,\kappa_2) \to \text{Ext}(F,A_3,\kappa_3),
\]

we easily reduce ourselves to the case where \( G \) is itself finite or connected.

Assume then first that \( G \) is finite and commutative. Using Proposition 3.6 we see that we only need to show finiteness of \( H^1(k,G) \) and of \( \text{Im}(\varphi) = \text{Ker}(\text{Res}) \), where \( \text{Res} \) is the restriction map \( H^2(F,G_{k_s},\kappa) \to H^2(k,G) \). The first group is finite if \( G \) is (this is equivalent to the type \((F)\) hypothesis, cf. [Ser02, III.4.1, Prop. 8]). As for \( H^2(F,G_{k_s},\kappa) = H^2(F\Gamma,G(k_s)) \), it can be calculated via the spectral sequence

\[
E^{p,q}_2 = H^p(\Gamma, H^q(F(k_s),G(k_s))) \Rightarrow H^{p+q}(F\Gamma,G(k_s)) = E^{p+q}_\infty.
\]

Let us look at the terms with \( p + q = 2 \). The term \( E^{0,2}_2 \) is finite simply because \( F(k_s) \) and \( G(k_s) \) are finite groups. This is also the case for the term \( E^{1,1}_2 \) once again by finiteness of \( F(k_s) \) and \( G(k_s) \) plus the finiteness of \( H^1(k,A) \) for \( A \) a finite \( k \)-group. Finally, recall that for the term \( E^{2,0}_2 \) there is a natural morphism

\[
E^{2,0}_2 = H^2(\Gamma, H^0(F(k_s),G(k_s))) \to H^2(F\Gamma,G(k_s)) = E^2,
\]

which is nothing but inflation for the quotient \( \Gamma \) of \( F\Gamma \). Composing this morphism with \( \text{Res} \), this corresponds to the natural morphism \( H^2(k,G^F) \to H^2(k,G) \), whose kernel is a quotient of the finite group \( H^1(k,G/G^F) \) (recall that \( G \) is finite and \( k \) of type \((F)\)). One sees then that the part of the image of \( E^{2,0}_2 \) that actually falls into the kernel of \( \text{Res} \).
is a finite group. This proves that the whole group $\text{Ker}(\text{Res})$ is finite (note by the way that the term $E_2^{2,0}$ is actually trivial in case 1 (i.e. $k$ finite) by cohomological dimension).

Assume now that $G$ is connected and commutative. In cases 2 and 3, the result is then a corollary of Theorem 4.2. Indeed, then we know that there exists a positive integer $m$ such that the finite $k$-subgroup $S_m$ of $G$ given by the $m$-torsion of a (actually, the) maximal torus of $G$ is such that the map

$$\text{Ext}(F, S_m, \kappa) \to \text{Ext}(F, G, \kappa),$$

is surjective. We are then reduced to the finite case which we have already proved.

Finally, in case 1, every extension of $F$ by $G$ induces a natural $G$-torsor over $F$ and hence an element of $H^1_{\text{et}}(F, G)$. This element is clearly trivial since $F$ as a variety is nothing but a product of separable extensions of $k$ and hence of finite fields, whence the triviality of the whole $H^1_{\text{et}}(F, G)$ by Lang’s theorem (cf. [Lan56]). We deduce that every extension is a trivial torsor and hence admits a schematic section and thus $\text{Ext}(F, G, \kappa)$ is isomorphic to the Hochschild cohomology group $H^2_0(F, G)$ (cf. [SGA3, XVII. App. I, Prop. 3.1]). Let us show then that this last group is finite in this context.

Recall from the proof of Lemma 3.10 that we have the spectral sequence

$$H^p(\Gamma_{L/k}, H^q_0(F_L, G_L)) \Rightarrow H^{p+q}_0(F, G),$$

and that, since $F_L$ is a constant group, we have $H^0_0(F_L, G_L) = H^0(F(L), G(L))$ (cf. [DG70, III.6, Prop. 4.2]) and hence this group is finite for every $q$ since both $F(L) = F(k_s)$ and $G(L)$ are finite for a finite field $L$. The finiteness of $\Gamma_{L/k}$ gives then the finiteness of every term of the spectral sequence and hence in particular of $H^2_0(F, G)$.

Remarks.

1. The type (F) hypothesis is necessary already in the case of a finite group $G$. Indeed, in that case one could have an infinity of terms in $\text{Ext}(F, G, \kappa)$ coming from $H^1(k, Z)/H^1(k, Z^F)$, which can’t be assumed to be finite anymore. For an explicit counterexample, take $k$ a number field, $G = (\mathbb{Z}/2\mathbb{Z})^2$ and $F = \mathbb{Z}/2\mathbb{Z}$ acting by permutation of the two components of $G$. Then we have $Z = G \cong Z^F \times \mathbb{Z}/2\mathbb{Z}$ and hence there is a non trivial twist of the constant group $G \times Z$ for each quadratic extension of $k$, which are known to be infinite. It would be interesting to see if one could create such counterexamples with semisimple groups (whose center is known to be finite).

2. One could wonder if the more general result for finite fields holds over other fields of cohomological dimension $\leq 1$. Unfortunately, linearity of $G$ is in general necessary, since already for $k = \mathbb{C}(T), \mathbb{C}((T))$ the group $H^1(k, A)$ may have a non trivial divisible component for $A$ an abelian variety (cf. [Ogg62]) and hence, once again, one could have such a component in $\text{Ext}(F, G, \kappa)$ coming from $H^1(k, Z)/H^1(k, Z^F)$ if $Z$ is non linear. Again, taking $G$ to be the direct product of two copies of a well chosen elliptic curve permuted by $F = \mathbb{Z}/2\mathbb{Z}$ gives an explicit counterexample.

\footnote{Note that the Weyl group is trivial for $G$ connected and commutative, so that in case 2 the hypotheses of Theorem 4.2 are well met.}
3. The extra hypotheses for linear $G$ also seem to be necessary. Indeed, Proposition 3.6 and [Ser02, III.2.1, Prop. 6] tell us that for connected unipotent $G$ we have that $\text{Ext}(F,G,\kappa)$ is equal to $H^2(F,G_{k_s},\kappa)$. Take then for example $G = G_a$ and $F = \mathbb{Z}/p\mathbb{Z}$ with trivial action and consider the group $H^0(\Gamma, H^2(F(k_s), G(k_s)))$. Since $F$ is cyclic, we have $H^2(F(k_s), G(k_s)) = H^0(F(k_s), G(k_s)) = G(k_s)$ and hence $H^0(\Gamma, H^2(F(k_s), G(k_s))) = G(k) = k$, that is, an infinite group if $k$ is not finite. Now, by the spectral sequence used above, this group appears as a quotient of $H^2(F,G_{k_s},\kappa)$. Indeed, since the $p$-cohomological dimension of $\Gamma$ is $\leq 1$ (cf. [Ser02, II.2.2, Prop. 3]), it is easy to see that this infinite group does not disappear in the spectral sequence and hence our $H^2(F,G_{k_s},\kappa)$ actually surjects onto it.

4. These last remarks also tell us that the situations in which one can always reduce extensions to the same finite $k$-subgroup as in Theorem 4.2 cannot be much more general, otherwise we would get a similar proof of finiteness of $\text{Ext}(F,G,\kappa)$ in these cases too.

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