Cosmological perturbations and classical change of signature

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Abstract

Cosmological perturbations on a manifold admitting signature change are studied. The background solution consists in a Friedmann-Lemaitre-Robertson-Walker (FLRW) Universe filled by a constant scalar field playing the role of a cosmological constant. It is shown that no regular solution exist satisfying the junction conditions at the surface of change. The comparison with similar studies in quantum cosmology

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is made.

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1 Introduction

The idea that the signature of spacetime could have been different in the very early Universe has first appeared in association with quantum cosmology [1, 2]. Quantum cosmology is based on the ADM version of general relativity [3] in which spacetime is sliced into three-dimensional spacelike hypersurfaces. The true degrees of freedom of the gravitational field are described by the components $h_{ij}$ of the metric on these hypersurfaces whereas the way in which they are matched remains essentially arbitrary. The main object of quantum cosmology is the wave function of the Universe $\Psi$, solution of the Wheeler-De Witt equation. The function $\Psi$ depends only on $h_{ij}$ and eventually on the matter fields. For minisuperspace models, the WKB approximation of $\Psi$ can be roughly described either by an oscillatory function $e^{iS}$ or by the exponential $e^{-S}$ corresponding respectively to classically allowed and classically forbidden behaviour of the Universe. Wave functions $e^{-S}$ are similar to those used in ordinary quantum mechanics for the study of the tunnel effect. Formally, it can be obtained from the oscillatory wave function by a Wick rotation (or a passage to imaginary time). In quantum cosmology, this suggests to interpret the wave functions $\Psi \sim e^{-S}$ as describing in fact a Riemannian manifold. Therefore, in this framework, change of
signature appears as one of possible quantum properties of spacetime. One of the main advantages of this formulation is that it gives a solution to the initial singularity problem.

Recently, it has been realized that change of signature can be also considered in the framework of classical general relativity [4]-[22]. This results from the fact that the Einstein equations do not fix the signature which thus appears as an extra assumption added *a posteriori* to the theory. Solutions very similar to the ”no-boundary” solution of quantum cosmology [1] have been obtained [4, 7]. A crucial point in the study of the classical change of signature is the choice of the matching conditions at the surface of change. In the literature, two different approaches have been adopted. The first one requires the continuity of the second fundamental form $K_{ij}$ on the surface of change $\Sigma$ [4, 6, 15, 16, 22] whereas the second, demands in addition that $K_{ij}$ should vanish on $\Sigma$ [7, 10, 11, 12, 20]. In what follows, we shall favour the second approach, in which the junction conditions can be obtained naturally as a consequence of the distributional parts of the field equations on $\Sigma$ [4, 10]. This approach is more restrictive than the first one and the class of solutions is smaller. Hayward [10] has emphasized that this restrictiveness can lead to predictions which are in agreement with the present status of
the cosmological observations. However, his conclusion is valid only for the background solution of Einstein equations.

There is no need to emphasize the importance of theoretical models of the cosmological perturbations [23, 24, 25]. They lead to significant results which can be compared with the observations. For example, the presence of small perturbations can be detected by their influence upon the angular variation of the cosmic background microwave radiation observed by the satellite COBE [24, 26]. Any model attempting to describe the primordial Universe must be able to explain these particular features of the background radiation.

The aim of this article is to address this question in the framework of the model in which the Universe is described by a manifold displaying a classical change of signature. More precisely, the background will consist in a FLRW metric and the matter will be represented by a constant scalar field playing the role of a cosmological constant.

The article is organized as follows. In the second section, we describe the background model. For the sake of completeness, we also describe briefly the later stages of evolution of the Universe after the inflation. The third section is devoted to the perturbed Einstein equations with signature change. These equations are solved in the Riemannian region, taking into account
all types of perturbations, namely the density perturbations, the rotational perturbations and gravitational waves. In the fourth section, we compare the solutions displaying a classical change of signature with those arising in the framework of quantum cosmology. Conclusions are presented in the fifth section.

2 The background model

The structure of the background manifold can be symbolized as follows:

\[ M = M^- \cup \Sigma \cup M^+ \]  

(1)

where \( M^- (M^+) \) is a Riemannian (Lorentzian) manifold endowed with a metric tensor whose the signature is \(+ + + + (- + + +)\). \( \Sigma \) is the three-dimensional spacelike hypersurface where the dynamical change of signature occurs. On this surface, the metric is either non degenerate but discontinuous (“discontinuous proposal”) or degenerate but continuous (“continuous proposal”). It can be shown that implementing the classical change of signature with any of these two proposals is equivalent \[20\]. Here, we will adopt the discontinuous approach. As previously emphasized in Ref. \[7, 10\], this requires the use of distributions in the sense of Schwarz.
In order to construct cosmological models, we will restrict our considerations to FRLW manifolds in which the spacelike sections are three-dimensional spheres. In that case, the metric tensor can be written as:

$$g = -N(t)dt \otimes dt + a^2(t) \left( d\chi \otimes d\chi + \sin^2 \chi (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi) \right)$$  \hspace{1cm} (2)

The sign of the lapse function $N(t)$ fixes the signature of $g$. The matter is described by a scalar field whose action is given by:

$$S[\phi] = \int d^4x - \sqrt{-g} \left( \frac{1}{2} g^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} + V(\phi) \right)$$  \hspace{1cm} (3)

Then, the system of Einstein-Klein-Gordon equations takes on the form (a dot means a derivative with respect to $t$):

$$\frac{\dot{a}^2}{a^2} + \frac{N}{a^2} = \frac{1}{6} \left( \frac{\dot{\phi}^2}{2} + NV(\phi) \right)$$  \hspace{1cm} (4)

$$\frac{\ddot{a}}{a} - \frac{\dot{N}\dot{a}}{2aN} = -\frac{1}{6} \left( \dot{\phi}^2 - NV(\phi) \right)$$  \hspace{1cm} (5)

$$\ddot{\phi} - \frac{\dot{N}\dot{\phi}}{2N} + 3\frac{\dot{a}}{a} \phi + N \frac{dV}{d\phi} = 0$$  \hspace{1cm} (6)

In the discontinuous proposal, the lapse function $N(t)$ is a distribution equal to:

$$N(t) = 2Y(t) - 1 = \epsilon$$  \hspace{1cm} (7)

where $Y(t)$ denotes the Heaviside distribution. $\epsilon$ is equal to $-1$ in the Riemannian region and to $+1$ in the Lorentzian one. The time derivative of
the lapse distribution is equal to \(2\delta(\Sigma)\) where \(\delta(\Sigma)\) is the Dirac distribution on the surface of change \(\Sigma\). Then, the Einstein-Klein-Gordon system of equations splits up into the regular part:

\[
\begin{align*}
\frac{\ddot{a}}{a^2} + \frac{\epsilon}{a^2} &= \frac{1}{6}\left(\frac{\dot{\phi}^2}{2} + \epsilon V(\phi)\right) \\
\frac{\ddot{\phi}}{a} &= -\frac{1}{6}\left(\dot{\phi}^2 - \epsilon V(\phi)\right) \\
\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \epsilon \frac{dV}{d\phi} &= 0
\end{align*}
\]

(8)

(9)

(10)

and, since we should treat the distributional part separately [7, 10], to the singular part:

\[
\begin{align*}
\dot{a}\delta(\Sigma) &= 0 \\
\dot{\phi}\delta(\Sigma) &= 0
\end{align*}
\]

(11)

(12)

which produces the following junction conditions at the surface of change: \(\dot{a} = 0\) and \(\dot{\phi} = 0\). In what follows in this article, we shall consider only the case where \(\dot{\phi} = 0\) everywhere. This implies that on the background level, the scalar field plays the role of the cosmological constant. Then, the solutions of field equations satisfying the matching conditions read:

\[
\begin{align*}
a(t) &= \frac{1}{H}\cos Ht & \frac{\pi}{2H} \leq t \leq 0 \\
a(t) &= \frac{1}{H}\cosh Ht & t \geq 0
\end{align*}
\]

(13)

(14)
where $H$ is defined as $\sqrt{\frac{V}{6}}$. In the framework of classical general relativity this solution was first discovered by Ellis et al. [4] and Hayward [7]. In the Riemannian region, the manifold is the sphere $S^4$ and in the Lorentzian region the Universe undergoes an inflationary phase described by a half of the De Sitter spacetime. We have thus constructed a cosmological model without the initial singularity. However, it is important to note that it does not violate the Hawking-Penrose theorem on singularities [27] since we have given up one of the fundamental assumptions necessary to prove this theorem, namely the causal structure of spacetime. It is also worth reminding that $M$ can be obtained from the entire De Sitter spacetime by performing a Wick rotation in the lower part of the hyperboloid. As the De Sitter spacetime is also a singularity-free manifold, this shows that the classical change of signature cannot be used in order to avoid any singularity. It simply enables us to construct singularity-free cosmological models which are, in a certain sense, "finite in time". This result is a particular case of a theorem proved in Ref. [20].

The solutions (13)-(14) are very similar to those obtained in quantum cosmology. The corresponding model is a one-dimensional minisuperspace for which the Wheeler-De Witt equation can be expressed in the following
The minisuperspace can be divided into two parts: one for which the potential $U(\phi) \equiv a^2(a^2 \frac{V(\phi)}{6} - 1)$ is negative, i.e. for which $a < \frac{1}{H}$ and one for which $U(\phi)$ is positive, i.e. for which $a > \frac{1}{H}$. The corresponding WKB solution of equation (15) is either an exponential or an oscillatory function, thus showing that the Riemannian region is classically forbidden in quantum cosmology. It is not clear if we should consider the model with the classical change of signature as a limit of a quantum solution when the Planck constant tends to zero, or if we should take the classical model into account seriously and then try to quantize it as it has been proposed in Ref. [19]. The matching conditions can be recovered by using the fact that at the surface of change $K_{ij}$ must vanish. In such case, it is equivalent to set $\dot{a} = 0$.

We can also construct a complete cosmological scenario which consists in radiation-dominated and matter-dominated eras following the inflationary stage. In the region $t \geq 0$, we use for our convenience the conformal time $\eta$ defined by $d\eta = dt/a$. The integration of the previous equation leads to the following relationship: $\eta = 2 \arctan(e^{Ht})$, $0 < \eta < \pi$. In terms of $\eta$-time, the
scale factor in the inflationary region can be written as:

\[ a(\eta) = \frac{1}{H \sin \eta} \]  

(16)

We assume that for \( \eta = \eta_1 \), the inflation stops and that the Universe enters a radiation-dominated era characterized by the equation of state \( p = \rho/3 \), for which the behaviour of the scale factor is given by:

\[ a(\eta) = a_r \sin(\eta - \eta_r) \]  

(17)

The constants \( a_r \) and \( \eta_r \) are chosen in such a way that the function \( a(\eta) \) is \( C^1 \) when \( \eta = \eta_1 \). This provides the following equations for \( a_r \) and \( \eta_r \):

\[ \eta_r = 2\eta_1 \]  

(18)

\[ a_r = -\frac{1}{H \sin^2 \eta_1} \]  

(19)

After the radiation-dominated era, the matter-dominated era begins at \( \eta = \eta_2 \). The equation of state is now \( p = 0 \) and the solution for the scale factor is:

\[ a(\eta) = \frac{a_m}{2} \left( 1 - \cos(\eta - \eta_m) \right) \]  

(20)

The requirement that \( a(\eta) \) and \( a'(\eta) \) must be continuous at \( \eta = \eta_2 \) is expressed by the following two equations:

\[ \eta_m = -\eta_2 + 2\eta_r \]  

(21)

\[ a_m = \frac{a_r}{\sin(\eta_2 - \eta_r)} \]  

(22)
The matter era represents the present state of our Universe. We have now at our disposal a complete description of the evolution of the Universe.

3 The perturbed model for the Riemannian region

3.1 General equations

This section is devoted to the study of perturbations around the Riemannian model described before. We shall assume that in the Riemannian region, the geometry could have fluctuated around the sphere $S^4$. These ”deformations” (probably a more appropriate term than ”perturbations” since, in this region, there is no notion of evolution) could constitute a possible mechanism explaining the origin of the perturbations in the usual (Lorentzian) Universe. Another interesting point is the following: the ordinary theory of cosmological perturbations enables to compute the evolution of the perturbed metric but does not fix the initial conditions. To specify them, the principles of quantum mechanics are usually evoked [24]: the initial state of the perturbations is taken to be the vacuum state. In the model of classical change of signature, we can also hope to say something about the initial data for
the perturbations. Indeed, we will have to solve the equations of motion which are second-order differential equations. The initial data for an ordinary second-order differential equation are \( f(t_0) \) and \( f'(t_0) \) for a given initial time \( t = t_0 \). As emphasized in Ref. [10], this is no longer the case for the singular equations which arise in the framework of the classical change of signature. The initial data are now \( f(t_0) \) with the requirement that \( f'(t_0) = 0 \) (\( t_0 \) is the value of \( t \) for which the change of signature occurs). In a certain sense, the change of signature fixes the initial conditions for the perturbations (however, we still have the freedom to choose \( f(t_0) \)). We will study later the consequences of this fact.

Let us consider a small perturbation \( \delta g_{\mu\nu} = h_{\mu\nu} \) of the background metric (2). The physical metric can be written as:

\[
g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}
\]

(23)

It is well known that \( h_{\mu\nu} \) contains unphysical degrees of freedom. Therefore, we will perform the computation using the synchronous gauge, that is, by requiring \( h_{0\mu} = 0 \). This choice has two main advantages. The first one is a technical reason: with this gauge, the calculations are easier; the second one is related to the interpretation of the solutions. Had we not have chosen the synchronous gauge, the frontier between the Riemannian and Lorentzian
regions would have been blurred due to the presence of a term $h_{00}$, possibly positive or negative. With this gauge, the frontier still remains distinct. The components of the perturbed Ricci tensor will be given in terms of $h^i_j = g^{(0)ik}h_{kj}$ and $h$ will denote the trace of the perturbed metric, $h^i_i = h$. The symbol $\tilde{\cdots}$ will refer to the three-dimensional metric $\tilde{g}_{ij}$ related to $g$ by the equation:

$$ g = -N(t)dt \otimes dt + a^2(t)\tilde{g}_{ij}dx^i \otimes dx^j $$

The components of the perturbed Ricci tensor can be written as:

$$ \delta R^0_0 = \frac{1}{2N}(\ddot{h} + 2\frac{\dot{a}}{a}\dot{h}) - \frac{\dot{N}}{2N^2}h $$

$$ \delta R^0_i = \frac{1}{2N} \frac{\partial}{\partial t}(\tilde{\nabla}_i h - \tilde{\nabla}_k h^k_i) $$

$$ \delta R^i_j = \frac{1}{2N} \ddot{h}^i_j - \frac{3}{2N} \frac{\dot{a}}{a} h^i_j + \frac{1}{2N} \frac{\dot{a}}{a} h^{k}h^i_j - \frac{2}{a^2} h^i_j - \frac{\dot{N}}{4N^2} h^i_j $$

$$ -\frac{1}{2a^2}(\tilde{\nabla}_j \tilde{\nabla}^i h^k_j - \tilde{\nabla}^i \tilde{\nabla}_j h^i_k - \tilde{\nabla}_i \tilde{\nabla}^i h^i_j + \tilde{\nabla}^k \tilde{\nabla}_k h^i_j) $$

It is interesting to note the presence of additional terms due to the lapse function. These terms will provide the junction conditions for the perturbed metric. According to Lifshitz and Khalatnikov [23], we can classified the perturbations as three types: scalar, vector and tensor ones. This classification is based on a theorem which states that any tensor of rank two can be
decomposed as:

\[ h_{ij} = h_{ij}^{(S)} + h_{ij}^{(V)} + h_{ij}^{(TT)} \]  

(28)

where \( \tilde{\nabla}_i h_{ij}^{(V)} = 0 \), \( h_{k}^{(TT)} = 0 \) and \( \tilde{\nabla}_i h_{ij}^{(TT)} = 0 \). The symbol ”(TT)” means transverse and trace-free. We can treat each type separately. The tensorial part \( h_{ij}^{(TT)} \) represents primordial gravitational waves whereas \( h_{ij}^{(S)} \) and \( h_{ij}^{(V)} \) represent respectively density and rotational perturbations, i.e. the perturbations accompanying fluctuations of the matter filling the Universe.

### 3.2 Density and rotational perturbations

Let us consider first the case of the density perturbations. They are constructed using the eigenfunction \( Q(\chi, \theta, \varphi) \) of the three-dimensional Laplacian \[23]\:

\[ \tilde{\nabla}^k \tilde{\nabla}_k Q(\chi, \theta, \varphi) = -(n^2 - k)Q(\chi, \theta, \varphi) \]  

(29)

where \( n \) is an integer greater or equal to one. Following Ref. \[23\], we can define the following tensors:

\[
Q^i_j = \frac{Q}{3} \delta^i_j
\]  

(30)

\[
P^i_j = \frac{\tilde{\nabla}^i \tilde{\nabla}_j Q}{n^2 - k} + Q^i_j
\]  

(31)

15
and express the scalar part of the perturbed metric $h_{ij}^{(S)}$ as:

$$h_{ij}^{(S)} = \lambda(\eta)P_{ij} + \mu(\eta)Q_{ij}$$  \hspace{1cm} (32)$$

In this section, for convenience, we do not write the indices and the sums. In fact $Q$ has to be understood as $Q_{lm}^n$ and $\lambda(\eta)$, $\mu(\eta)$ as $\lambda_{nlm}(\eta)$ and $\mu_{nlm}(\eta)$.

Then, it turns out that the components of $\delta R_{\mu \nu}^{(S)}$ are given by:

$$\delta R_{00}^{(S)} = \frac{1}{2N}(\ddot{\mu} + 2\dot{\mu} - \dot{N}/N\dot{\mu})Q$$  \hspace{1cm} (33)$$

$$\delta R_{0i}^{(S)} = \frac{1}{3N}\left((n^2 - k)\dot{\mu} + (n^2 - 4k)\dot{\lambda}\right)\frac{\nabla_i Q}{n^2 - k}$$  \hspace{1cm} (34)$$

$$\delta R_{ij}^{(S)} = \left(\frac{\ddot{\mu}}{2N} + \frac{3\dot{a}}{Na}\dot{\mu} - \frac{\dot{N}\dot{\mu}}{4N^2} + 2\left(n^2 - 4k\right)\left(\lambda + \mu\right)\right)Q_{ij}^i$$

$$+ \left(\frac{\ddot{\lambda}}{2N} + \frac{3\dot{a}}{2Na}\dot{\lambda} - \frac{\dot{N}\dot{\lambda}}{4N^2} - \frac{n^2 - k}{6a^2}\left(\lambda + \mu\right)\right)P_{ij}^i$$  \hspace{1cm} (35)$$

The components of the perturbed source tensor, defined as $S_{\mu \nu} = T_{\mu \nu} - T^2 g_{\mu \nu}$, are:

$$\delta S_{00} = \frac{dV}{d\varphi} \delta \varphi$$  \hspace{1cm} (36)$$

$$\delta S_{0i} = 0$$  \hspace{1cm} (37)$$

$$\delta S_{ij} = \frac{dV}{d\varphi} \delta \varphi \delta_{ij}$$  \hspace{1cm} (38)$$

If the function $\delta \varphi$ is decomposed according to $\delta \varphi = f(t)Q(\chi, \theta, \varphi)$, the regular part of the perturbed Einstein equations takes on the form (in fact $\epsilon = -1$
since we study the perturbations in the Riemannian region):

\[
\ddot{\mu} + \frac{\dot{a}}{a}\dot{\mu} = 2\epsilon \frac{dV}{d\varphi} f
\]  

(39)

\[
(n^2 - k)\dot{\mu} + (n^2 - 4k)\dot{\lambda} = 0
\]  

(40)

\[
\frac{\ddot{\lambda}}{2} + \frac{3\dot{a}}{2a} \dot{\lambda} - \epsilon \frac{n^2 - k}{6a^2} (\lambda + \mu) = 0
\]  

(41)

\[
\frac{\ddot{\mu}}{2} + \frac{3\dot{a}}{a} \dot{\mu} + 2\epsilon \frac{n^2 - 4k}{3a^2} (\lambda + \mu) = 3\epsilon \frac{dV}{d\varphi} f
\]  

(42)

and the distributional part reduces to:

\[
\dot{\mu} \delta(\Sigma) = \dot{\lambda} \delta(\Sigma) = 0
\]  

(43)

Therefore, the junction conditions for the density perturbations are \(\dot{\mu} = \dot{\lambda} = 0\) on the surface of change. From equations (39)-(42), one can compute the combination \((n^2 - k)(41) + n^2 - 4k(42) + 3(n^2 - k)(39)\). One obtains:

\[
\frac{1}{2} \left( (n^2 - k)\ddot{\mu} + (n^2 - 4k)\ddot{\lambda} \right) + \frac{3\dot{a}}{2a} \left( (n^2 - k)\dot{\mu} + (n^2 - 4k)\dot{\lambda} \right) = \frac{3}{2} \epsilon \frac{dV}{d\varphi} f
\]  

(44)

A comparison with equation (40) shows that \(V,\varphi = 0\). Then, the integration of equation (39) provides the following solution for \(\mu\):

\[
\mu = \alpha \int \frac{dt}{a^2} + \beta
\]  

(45)

where \(\alpha\) and \(\beta\) are arbitrary constants. Taking into account the matching conditions, we see that \(\alpha\) must vanish. Therefore, the solutions of the Einstein equations (39)-(42) satisfying the junction conditions are \(\mu = -\lambda = \beta\).
However, the relationship $h_{0\mu} = 0$ does not fix the gauge completely which is preserved under the following coordinate transformations:

$$
t \rightarrow t \\
x_i \rightarrow x_i + a^2 f_i(x^k)
$$

where $f_i(x^k)$ is an arbitrary function of the spacelike coordinates. Then, the scalar part of the perturbed metric transforms according to $h^{i j \text{(S)}} \rightarrow h^{i j \text{(S)}} + 2A \overset{\text{\nabla}}{\nabla}^i \overset{\text{\nabla}}{\nabla}^j Q$. This implies $\lambda \rightarrow \lambda + 2(n^2 - k)A$ and $\mu \rightarrow \mu + 2(n^2 - k)A$.

Therefore, with a proper choice of $A (A = \frac{\beta}{2(n^2 - k)})$ we conclude that the density perturbations do vanish identically.

Let us turn now to the case of rotational perturbations. Their construction implies the transverse eigenvector $S_i (\overset{\text{\nabla}}{\nabla}^i S_i = 0)$ of the three-dimensional Laplacian:

$$
\overset{\text{\nabla}}{\nabla}^k \overset{\text{\nabla}}{\nabla}_k S_i(\chi, \theta, \varphi) = -(n^2 - 2k)S_i(\chi, \theta, \varphi)
$$

where $n$ is an integer greater or equal to 2. If we introduce the tensor $S_{ij} = -\frac{1}{2}(\overset{\text{\nabla}}{\nabla}_i S_j + \overset{\text{\nabla}}{\nabla}_j S_i)$ and define $h_{i j \text{(V)}}$ as $h_{i j \text{(V)}} = \sigma S_{i j}$, then it turns out that the components of the perturbed Ricci tensor can be written as:

$$
\delta R^0_{0 \text{(V)}} = 0 \\
\delta R^0_{i \text{(V)}} = \frac{\dot{\sigma}}{2N(n^2 - 4k)} S_i
$$
\[ \delta R^{ij} = \left( \frac{\dot{\sigma}}{2N} + \frac{3\dot{\alpha}}{2Na} \dot{\hat{\sigma}} - \frac{\dot{N}}{4N^2} \dot{\hat{\sigma}} \right) S^i_j \] (51)

The perturbed stress-energy tensor does not contain any rotational terms, since a scalar field cannot support rotational oscillations. Therefore, the regular part of the Einstein equations reduces to:

\[ \ddot{\sigma} \left( n^2 - 4k \right) = 0 \] (52)

\[ \ddot{\sigma} + 3 \frac{\dot{\sigma}}{a} = 0 \] (53)

whereas the distributional part is:

\[ \dot{\sigma} \delta(\Sigma) = 0 \] (54)

The junction condition for the rotational perturbation is \( \dot{\sigma} = 0 \) at the surface of change. It is straightforward to integrate this equation; the solution is simply a constant:

\[ \sigma(t) = C \] (55)

Under the residual gauge (46)-(47), \( \sigma \) transforms according to \( \sigma \rightarrow \sigma + B \) where \( B \) is an arbitrary constant. The choice \( B = -C \) shows that the rotational perturbations can not exist either.

So, we have reached the conclusion that the perturbations associated with fluctuations within the matter do not survive in the Riemannian region.
This is not surprising and it is not a special feature of the solutions with change of signature. Indeed, this fact is already known for the De Sitter spacetime [29]. Here, we simply recover the same result because $S^4$ is nothing else but the De Sitter spacetime with imaginary time.

### 3.3 Gravitational waves

Contrary to the density and rotational perturbations, tensorial perturbations represent fluctuations of the geometry only. Therefore, it is for this type of perturbations that we can expect to obtain informations concerning the primordial Riemannian region. The tensor $h_{ij}^{TT}$ can be expressed by means of tensor spherical harmonics on $S^3$. Tensor spherical harmonics are the eigentensors of the Laplacian on the three-dimensional sphere and are defined by the equation [23]:

$$\tilde{\nabla}^k \tilde{\nabla}_k G_{ij}(\chi, \theta, \varphi) = -(n^2 - 3k)G_{ij}(\chi, \theta, \varphi)$$

with $\tilde{\nabla}^i G_{ij} = 0$, $\tilde{g}^{ij} G_{ij} = 0$. $n$ is an integer greater than three. The explicit form of the eigentensors can be found in Ref. [30]. They are normalized and obey to the relationship:

$$\int d^3 x \sqrt{\tilde{g}} (G_{ij})^{n}_{lm} (G^{ij})^{n'}_{m'n'} = \delta^{nn'} \delta_{ll'} \delta_{mm'}$$
As a consequence, we can develop $h_{ij}^{(TT)}$ in the basis of the eigentensors $(G_{j})_{lm}^{n}$:

$$h_{ij}^{(TT)}(t, \chi, \theta, \varphi) = \sum_{n=3}^{\infty} \sum_{l=2}^{l-1} \sum_{m=-l}^{l} \nu_{nlm}(t) (G_{j})_{lm}^{n}(\chi, \theta, \varphi)$$  \hspace{1cm} (58)

Putting this expression in equation (25)-(27) enables us to compute the components of the perturbed Ricci tensor:

$$\delta R_{00}^{(TT)} = \delta R_{ii}^{(TT)} = 0$$  \hspace{1cm} (59)

$$\delta R_{ij}^{(TT)} = \left( \frac{\ddot{\nu}_N}{2N} + 3\dot{a} \frac{\dot{\nu}_N}{a} + \frac{n^2 - 1}{2a^2} \nu_N - \frac{\dot{N}}{4N^2 \nu_N} \right) G_{ij}$$  \hspace{1cm} (60)

where the indices and the sums have been omitted for simplicity and the index $N$ denotes the whole set of indices $N \equiv (n, l, m)$. In what follows, we will consider each mode $(n, l, m)$ separately. Since the perturbed source tensor does not contain any tensorial part, the perturbed Einstein equations for the gravitational waves reduce themselves to:

$$\delta R_{\mu \nu}^{\nu} = 0$$  \hspace{1cm} (61)

Therefore, the time evolution of the amplitude of each mode is determined by the equations:

$$\ddot{\nu}_N + 3\frac{\dot{a}}{a} \dot{\nu}_N + \frac{n^2 - 1}{a^2} \nu_N = 0$$  \hspace{1cm} (62)

$$\nu_N \delta(\Sigma) = 0$$  \hspace{1cm} (63)
The distributional part of Einstein’s equations provides the junction condition for each mode:

$$\frac{d\nu_N}{dt} = 0$$  \hspace{1cm} (64)

The next step is to solve the equation in the Riemannian region, that is to say for $a(t) = \frac{1}{H} \cos Ht$ and $\epsilon = -1$. In that case, one obtains:

$$\frac{d^2\nu_N}{dt^2} - \frac{3H}{\cos Ht} \frac{d\nu_N}{dt} - \frac{H^2 n^2 - 1}{\cos^2 Ht} \nu_N = 0$$  \hspace{1cm} (65)

Let us study the features of this equation in more detail. For convenience, let us introduce the coordinate $\tau$ defined by $\tau = Ht + \frac{\pi}{2}$ such that the south pole is now given by $\tau = 0$. Near the south pole, for small values of $\tau$, the solution is $\nu_N \sim A\tau^{n-1} + B\tau^{-(n+1)}$ revealing that equation (62) possesses one regular and one divergent solution. However, since the system of coordinates is not well defined at the south pole, this does not imply anything in what concerns the real behaviour of the deformations at this point. To deal with this problem, let us introduce a new system of coordinates. Each section of the sphere $S^4$ is a sphere $S^3$ which can be embedded into the four-dimensional Euclidean space $R^4$. A point of $S^3$ can be located with the coordinate $\bar{x}^A$, $A = 1, \ldots, 4$, satisfying the constraint $\bar{x}^A \bar{x}_A = r^2$ where $r$ is the radius of $S^3$ (see figure 1). Then, the following formulae hold:

$$G_{ij}^{(n)}(\chi, \theta, \phi)dx^i dx^j = \frac{1}{r_{n-1}} T_{AC_1 BC_2 \ldots C_{n-1}}^{(n)} \bar{x}^{C_1} \ldots \bar{x}^{C_{n-1}} d\bar{x}^A d\bar{x}^B$$  \hspace{1cm} (66)
where $x^i = (\chi, \theta, \varphi)$ and $T_{AC_1,BC_2...C_{n+1}}^{(n)}$ is a constant tensor of rank $n+1$ with the following properties: [23, 31]. It is antisymmetric with respect to the pairs of indices $AC_1$ and $BC_2$, symmetric in the $C_n$ indices ($n \geq 3$), trace-free in any pair of indices and gives zero if we take the cyclic sum over any three different indices. $T_{AC_1,BC_2...C_{n+1}}^{(n)} \bar{x}^{c_1}...\bar{x}^{c_{n-1}}$ is an homogeneous polynomial of order $n-1$. The unit sphere $S^4$ can be projected perpendicularly onto the four-dimensional plane $P_S$. This corresponds to the following choice for the coordinate $\bar{x}^A$:

$$\bar{x}^A = r f(\chi, \theta, \varphi) = \sin \tau f^n(\chi, \theta, \varphi)$$

(67)

The resulting background metric is now regular at the south pole. Note that this new metric is no longer diagonal. To preserve the gauge, we could have used the stereographic projection ($M$ becomes $M''$ in figure 1):

$$\bar{x}^A = 2 \tan \frac{\tau}{2} f^A(\chi, \theta, \varphi)$$

(68)

However, as we are interested in the behaviour of the solution when $\tau$ tends to zero, the two previous changes of coordinates are equivalent for our purpose and we shall work with the first one. Under the transformation (67), the metric becomes:

$$g = g^{(0)} + \sum_{n \geq 3} \nu_n(\tau) T_{AC_1,BC_2...C_{n-1}}^{(n)} f^{C_1}...f^{C_{n-1}} d\bar{x}^A d\bar{x}^B$$

(69)
Equation (69) demonstrates explicitly that the "true" behaviour of the deformations is determined by the behaviour of the functions $\nu_N$. As a consequence, one solution for the deformations is indeed regular at the south pole whereas the other one actually diverges.

Let us now study the exact solutions of equation (62). The change of variable $x = \sin Ht$ and the change of function $\nu = \sqrt{1-x^2} g$ reduce our equation to:

$$(1-x^2)g'' - 2xg' + \left(2 - \frac{n^2}{1-x^2}\right)g = 0$$

(70)

which is a Legendre differential equation. The general solution is given in terms of the Legendre functions $P_1^{-n}$ and $Q_1^n$ [32, 33, 34]:

$$\nu_{nlm}(t) = A_{nlm} R_{nlm} \cos Ht P_1^{-n}(\sin Ht) + B_{nlm} R_{nlm} \cos Ht Q_1^n(\sin Ht)$$

(71)

where $A_{nlm}$ and $B_{nlm}$ are two arbitrary constants. We can also express the solution in a representation which is more convenient for the study of matching conditions:

$$\nu_{3lm}(t) = C_{3lm} R_{3lm} (-8 \tan^4 Ht - 12 \tan^2 Ht - 3) + D_{3lm} R_{3lm} \frac{\sin Ht}{\cos^4 Ht}$$

(72)

$$\nu_{4lm}(t) = C_{4lm} R_{4lm} (-24 \tan^5 Ht - 40 \tan^3 Ht - 15 \tan Ht) + D_{4lm} R_{4lm} \frac{1 + 5 \sin^2 Ht}{\cos^5 Ht}$$

(73)

where $C_{nlm}$ and $D_{nlm}$ are arbitrary constants. For higher values of $n$, the
general form is preserved, i.e a polynomial of order $n + 1$ in $\tan Ht$ for the first branch and a polynomial of order $n - 2$ in $\sin Ht$ divided by $\cos^{n+1} Ht$ for the second branch. For each value of $(n, l, m)$, both branches blow up at $t = -\pi/2H$. However, according to the previous discussion, a specific choice of $C_{nlm}^R$ and $D_{nlm}^R$ always enables us to construct a solution regular at the south pole. For example, for $n = 3$, the choice is $-8C_{3lm}^R = D_{3lm}^R$.

The next step is to take into account the matching conditions (63). They require $C_{2qlm}^R = D_{2p+1lm}^R = 0$ with $q \geq 2$ and $p \geq 1$. In other words, only one branch (not always the same according to if $n$ is a even or odd number) can cross the surface of change. Therefore, we have reached the conclusion that the requirement of regularity of the solution in the Riemannian region and the junction conditions are incompatible: every solution satisfying $\frac{d\nu}{dt} = 0$ is divergent at the south pole, except the trivial function $\nu_N = 0$. Note that it is known that the sphere $S^4$ is an isolated solution of the Riemannian Einstein equations [35]. In this paper, we recover this result using the formalism of cosmological perturbations. Before discussing the consequences of this result for the classical change of signature, we are going to study the equivalent problem in the framework of quantum cosmology.
4 Quantum cosmology and classical change of signature

In this section, we follow the treatment given by Halliwell and Hartle [36]. The basic formulae of the path integral formulation of quantum gravity is:

\[ \Psi[\bar{h}_{ij}, \Phi, B] = \sum_M \int \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi e^{-I[g_{\mu\nu}, \Phi]} \]

(74)

where the sum is taken over all manifolds \( M \) having \( B \) as part of their boundary and over all metrics \( g_{\mu\nu} \) and matter fields \( \Psi \) which induce \( \bar{h}_{ij} \) and \( \Phi \) on \( B \). \( I \) denotes the Euclidean Einstein-Hilbert action. Computing the wave function \( \Psi \) is a difficult task and this question has been discussed extensively in the literature. The action \( I \) is unbounded from below for real metric. Then, in order to give a meaning to \( \Psi \), we must perform the integration over complex metrics [36, 37]. This can be done explicitly for minisuperspace models. In that case, it has been shown by Halliwell [38] that the propagator between fixed three-geometry, in the gauge \( \dot{N} = 0 \), is given by:

\[ G(q'^\alpha|q'^\alpha) = \int dN \int \mathcal{D}q^\alpha e^{-I[q^\alpha(\tau), N(\tau)]} \]

(75)

where \( q^\alpha \) are the coordinates in the minisuperspace. \( G \) is computed over paths \( q^\alpha(\tau) \) satisfying \( q^\alpha(\tau') = q'^\alpha \) and \( q^\alpha(\tau'') = q''^\alpha \). The path integral
will be dominated by the saddle point \((N, q^\alpha)\), that is to say by complex configurations for which \(\frac{\delta I}{\delta q} = 0\), \(\frac{\delta I}{\delta N} = 0\). Solving these equations provides a solution for \(q^\alpha\) and \(N\). In general, \(N\) and \(q^\alpha\) are complex numbers. Now, we are interested in computing \(I\) for the saddle point. For minisuperspace models, \(I\) is given by:

\[
I = \int_{\tau'}^{\tau''} d\tau N \left( \frac{1}{2N^2} f_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta + U(q^\alpha) \right)
\]

(76)

The solution \(q^\alpha(\tau)\) has the property that it depends only on \(N\tau\). Then, in term of the complex variable \(T = N(\tau - \tau')\), \(I\) can be written as:

\[
I = \int_C dT \left( \frac{1}{2} f_{\alpha\beta} \frac{dq^\alpha}{dT} \frac{dq^\beta}{dT} + U(q(T)) \right)
\]

(77)

where \(C\) is the contour indicated on the figure 2, namely a straight line from 0 to \(\bar{T} = N(\tau'' - \tau')\). This contour can be deformed in such a way that:

\[
I = \int_{C_1} dT \left( \frac{1}{2} f_{\alpha\beta} \frac{dq^\alpha}{dT} \frac{dq^\beta}{dT} + U(q(T)) \right) + \int_{C_2} dT \left( \frac{1}{2} f_{\alpha\beta} \frac{dq^\alpha}{dT} \frac{dq^\beta}{dT} + U(q(T)) \right)
\]

(78)

Where the contours \(C_1\) and \(C_2\) are defined as in the figure 2. Along \(C_1\), \(T\) is real and the corresponding solution is Riemannian. Along \(C_2\), \(T\) can be chosen in such a way that \(T = \Re(\bar{T}) + it\Im(\bar{T})\), 0 \(\leq t \leq 1\), showing that the action will be purely imaginary if \(q^\alpha(\tau)\) is real. \(q^\alpha(\tau)\) is real if \(\dot{q}^\alpha(\Re(\bar{T})) = 0\) or, equivalently, if \(K_{ij} = 0\). In that case, the complex solution which
dominated $G(q''|q')$ can be viewed as a combination of a Riemannian and a Lorentzian manifolds. However, the point is that in general, $G(q''|q')$ is dominated by an intrinsically complex solution where all the $\dot{q}^\alpha$ do not vanish for $T = \Re(\bar{T})$.

The relationship with the classical change of signature is now clear. Solutions which are a combination of a Riemannian and a Lorentzian manifolds ("tunneling solutions") can be obtained classically by relaxing the assumption that the signature of the metric is always hyperbolic whereas the other ones cannot be conceived as a classical manifold with a change of signature. Our background model belongs to the first category whereas the perturbed solution belongs to the second. This is the reason why we found that the only solution satisfying the regularity condition and the matching conditions was $\nu_N = 0$. The junction conditions for the first category of solutions is $K_{ij} = 0$ because $K_{ij}$ is purely real in the Riemannian region and purely imaginary in the Lorentzian one \[36\]. For the complex solutions (second category) $K_{ij}$ has just to be continuous.
5 Discussion and conclusion

We are now in position to answer our main question with regard to the possible observable consequences of the primordial change of signature. The previous result shows that if we try to describe the primordial Universe as a FLRW signature-changing manifold with a cosmological constant (i.e. half of the sphere $S^4$ joined to half of the De Sitter spacetime), then it implies the absence of primordial gravitational waves. This result is a consequence of the fact that the sphere $S^4$ is an isolated solution of the Riemannian Einstein equations \[35\]. Primordial gravitational waves are present in the context of quantum cosmology. The reason is that the wave function of the perturbations is dominated by a configuration which cannot be thought as a combination of a Riemannian and a Lorentzian metric contrary to the wave function of the background. Clearly, if we consider seriously the classical change of signature as a possible model for describing the primordial Universe, that is to say if we assume that not only the background solution but also the perturbations can be represented by a signature-changing manifold (a priori, it seems to be the most logical attitude: indeed, if we believe that we can avoid the principles of quantum mechanics and replace them by what is called ”classical change of signature”, what should it be true only for the
background solution and not for the perturbed one?) it will lead to conflicts with observations. The absence of gravitational waves in this model is due to the restrictiveness of the junction condition $K_{ij} = 0$.

It is worth noticing that in the context of the classical change of signature, an alternative proposal for the junction conditions has been advocated \[1, 8, 13, 16, 22\]. For this proposal, the matching conditions require the continuity only of the second fundamental form. In our model, this means the continuity of $\dot{h}_{ij}$, or the continuity of $\frac{d\nu_N}{dt}$. Therefore, the regular solution of $\nu_N$ will be able to cross the surface of change $\Sigma$ leading to a non-vanishing amplitude for the gravitational waves in the inflationary era.

However, despite this possibility, it seems that a theory including both general relativity and quantum mechanics is more likely to provide a satisfactory description of the primordial Universe.

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Figures captions:

Figure 1: Different system of coordinates for the Riemannian region.

Figure 2: Contours for the propagator.