STATIONARY DYNAMICAL SYSTEMS

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Abstract. Following works of Furstenberg and Nevo and Zimmer we present an outline of a theory of stationary (or $m$-stationary) dynamical systems for a general acting group $G$ equipped with a probability measure $m$. Our purpose is two-fold: First to suggest a more abstract line of development, including a simple structure theory. Second, to point out some interesting applications; one of these is a Szemerédi type theorem for $SL(2,\mathbb{R})$.

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INTRODUCTION

Classical ergodic theory was developed for the group of real numbers $\mathbb{R}$ and the group of integers $\mathbb{Z}$. Later generalizations to $\mathbb{R}^d$ and $\mathbb{Z}^d$ actions evolved and more recently the theory has been vastly extended to handle more general concrete and abstract amenable groups. There however the theory finds a natural boundary, since by definition it deals with measure preserving actions on measurable or compact spaces, and these need not exist for a non-amenable group. Of course semi-simple Lie groups or non-commutative free groups admit many interesting measure preserving actions, but for many other natural actions of these groups no invariant measure exists.

Following works of Furstenberg (e.g. [7], [8], [9], [11]) and Nevo and Zimmer (e.g. [23], [24], [25]), we present here an outline of a theory of stationary (or $m$-stationary) dynamical systems for a general acting group $G$ equipped with a probability measure $m$. A preliminary version of this work has been in circulation as a preprint for several years now but for technical reasons was not previously submitted for publication.

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m. By definition such a system comprises a compact metric space X on which $G$ acts by homeomorphisms and a probability measure $\mu$ on $X$ which is $m$ stationary; i.e. it satisfies the convolution equation $m \ast \mu = \mu$. The immediate advantage of stationary systems over measure preserving ones is the fact that, given a compact $G$-space $X$, an $m$-stationary measure always exists and often it is also quasi-invariant.

The aforementioned works, as well as e.g. [19] and the more recent works [2] and [3], amply demonstrate the potential of this new kind of theory and our purpose here is two-fold. First to suggest a more abstract line of development, including a simple structure theory, and second, to point out some interesting applications.

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1. Stationary dynamical systems

Definitions: Let $G$ be a locally compact second countable topological group, $m$ an admissible probability measure on $G$. I.e. with the following two properties: (i) For some $k \geq 1$ the convolution power $\mu^\ast k$ is absolutely continuous with respect to Haar measure. (ii) the smallest closed subgroup containing $\text{supp}(m)$ is all of $G$. Let $(X, \mathcal{B})$ be a standard Borel space and let $G$ act on it in a measurable way. A probability measure $\mu$ on $X$ is called $m$-stationary, or just stationary when $m$ is understood, if $m \ast \mu = \mu$. As shown by Nevo and Zimmer, every $m$-stationary probability measure $\mu$ on a $G$-space $X$ is quasi-invariant; i.e. for every $g \in G$, $\mu$ and $g\mu$ have the same null sets.

Given a stationary measure $\mu$ the quintuple $\mathcal{X} = (X, \mathcal{B}, G, m, \mu)$ is called an $m$-dynamical system, or just an $m$-system. (Usually we omit the $\sigma$-algebra $\mathcal{B}$ from the notation of an $m$-system, and often also the group $G$ and the measure $m$). An $m$-system $\mathcal{X}$ is called measure preserving if the stationary measure is in fact $G$-invariant. With no loss of generality we may assume that the Borel space $X$ is a compact metric space and that the action of $G$ on $X$ is by homeomorphisms. For a compact metric space $X$, the space of probability Borel measures on $X$ with the weak* topology will be denoted by $M(X)$; it is a compact convex metric space. When $G$ acts on $X$ by homeomorphisms the closed convex subset of $M(X)$ consisting of $m$-stationary measures will be denoted by $M_m(X)$. By the Markov-Kakutani fixed point theorem $M_m(X)$ is non-empty. We say that the $m$-system $(X, \mu)$ is ergodic if $\mu$ is an extreme point of $M_m(X)$ and that it is uniquely ergodic if $M_m(X) = \{\mu\}$. It is easy to see that when $\mu$ is ergodic every $G$-invariant measurable subset of $X$ has $\mu$ measure 0 or 1. Unless we say otherwise we will assume that an $m$-system is ergodic.

When $\mathcal{X} = (X, \mathcal{B}, G, m, \mu)$ and $\mathcal{Y} = (Y, \mathcal{A}, G, m, \nu)$ are two $m$-dynamical systems, a measurable map $\pi : X \to Y$ which intertwines the $G$-actions and satisfies $\pi_* \mu = \nu$ is called a homomorphism of $m$-stationary systems. We then say that $\mathcal{Y}$ is a factor of $\mathcal{X}$, or that $\mathcal{X}$ is an extension of $\mathcal{Y}$.

Let $\Omega = G^\mathbb{N}$ and let $P = m^\mathbb{N} = m \times m \times m \ldots$ be the product measure on $\Omega$, so that $(\Omega, P)$ is a probability space. We let $\xi_n : \Omega \to G$, denote the projection onto the $n$-th coordinate, $n = 1, 2, \ldots$. We refer to the stochastic process $(\Omega, P, \{\eta_n\}_{n \in \mathbb{N}})$, where $\eta_n = \xi_1 \xi_2 \cdots \xi_n$ as the $m$-random walk on $G$. 
A real valued function \( f(g) \) for which \( \int f(gg') \, dm(g') = f(g) \) for every \( g \in G \) is called **harmonic**. For a harmonic \( f \) we have
\[
E(f(g\xi_1\xi_2\cdots\xi_n|\xi_1\xi_2\cdots\xi_n)) = \int f(g\xi_1\xi_2\cdots\xi_ng') \, dm(g') = f(g\xi_1\xi_2\cdots\xi_n),
\]
so that the sequence \( f(g\xi_1\xi_2\cdots\xi_n) \) forms a **martingale**.

For \( F \in C(X) \) let \( f(g) = \int F(gx) \, d\mu(x) \), then the equation \( m \ast \mu = \mu \) shows that \( f \) is harmonic. It is shown (e.g.) in [8] how these facts combined with the martingale convergence theorem lead to the following:

**1.1. Theorem.** The limits

\[
\text{lim}_{n \to \infty} \eta_n \mu = \lim_{n \to \infty} \xi_1\xi_2\cdots\xi_n \mu = \mu_\omega,
\]

exist for \( P \) almost all \( \omega \in \Omega \).

The measures \( \mu_\omega \) are the **conditional measures** of the \( m \)-system \( X \). We let \( \Omega_0 \) denote the subset of \( \Omega \) where the limit \( \{1\} \) exists. The fact that \( \mu \) is \( m \)-stationary can be expressed as:
\[
\int \xi_1(\omega) \mu dP(\omega) = m \ast \mu = \mu.
\]

By induction we have
\[
\int \xi_1(\omega)\xi_2(\omega)\cdots\xi_n(\omega) \mu dP(\omega) = \mu,
\]
and passing to the limit we also have the **barycenter equation**:

\[
\text{lim}_{n \to \infty} \int \mu_dP(\omega) = \mu.
\]

There is a natural “action” of \( G \) on \( \Omega \) defined as follows. For \( \omega = (g_1, g_2, g_3, \ldots) \in \Omega \) and \( g \in G \), \( g\omega \in \Omega \) is given by \( g\omega = (g, g_1, g_2, g_3, \ldots) \). (This is not an action in the usual sense; e.g. \( g^{-1}(g\omega) \neq \omega \).) It is easy to see that for every \( g \in G \) and \( \omega \in \Omega_0 \), \( \mu_{g\omega} = g\mu_\omega \), so that \( \Omega_0 \) is \( G \)-invariant. The map \( \zeta : \Omega \to M(X) \) given \( P \) a.s. by \( \omega \mapsto \mu_\omega = \lim_n \xi_1\xi_2\cdots\xi_n \mu \), sends the measure \( P \) onto a probability measure, \( \zeta_\ast P = P^* \in M(M(X)) \); i.e. \( P^* \) is the distribution of the \( M(X) \)-valued random variable \( \zeta(\omega) = \mu_\omega \). Clearly for each \( k \geq 1 \), the random variable \( \zeta_k = \lim_n \xi_k\xi_{k+1}\cdots\xi_{k+n} \mu \) has the same distribution \( P^* \) as \( \zeta(\omega) \). We also have \( \zeta_k = \zeta_k\zeta_{k+1} \). The functions \( \{\zeta_k\} \) therefore satisfy:

(a) \( \zeta_k \) is a function of \( \xi_k, \xi_{k+1}, \ldots \)
(b) all the \( \zeta_k \) have the same distribution,
(c) \( \xi_k \) is independent of \( \zeta_{k+1}, \zeta_{k+2}, \ldots \)
(d) \( \zeta_k = \zeta_k\zeta_{k+1} \).

In other words, the \( M(X) \)-valued stochastic process \( \{\zeta_k\} \) is an **\( m \)-process** in the sense of definition 3.1 of [8] and it follows that the measure \( P^* \) is \( m \)-stationary (condition (d)) and that \( \Pi(X) = (M(X), G, m, P^*) \) is an **\( m \)-system**.

1 The “barycenter” equation \([12]\) is what makes the “quasifactor” \( \Pi(X) \) meaningful in the general measure theoretical setup, where \( X \) is just a standard Borel space; see e.g. [10].
Definitions: We call the $m$-system $X = (X, G, m, \mu)$, $m$-proximal (or a “boundary” in the terminology of [8]) if $P$ a.s. the conditional measures $\mu_\omega \in M(X)$ are point masses. Clearly a factor of a proximal system is proximal as well. Let $\pi : (X, G, m, \mu) \to (Y, G, m, \nu)$ be a homomorphism of $m$-dynamical systems. We say that $\pi$ is a measure preserving homomorphism (or extension) if for every $g \in G$ we have $g \mu_y = \mu_{gy}$ for $\nu$ almost all $y$. Here the probability measures $\mu_y \in M(X)$ are those given by the disintegration $\mu = \int \mu_y d\nu(y)$. It is easy to see that when $\pi$ is a measure preserving extension then also (with obvious notations), $P$ a.s. $g(\mu_\omega)_y = (\mu_\omega)_{gy}$ for $\nu$ almost all $y$. Clearly, when $\mathcal{Y}$ is the trivial system, the extension $\pi$ is measure preserving iff the system $X$ is measure preserving. We say that $\pi$ is an $m$-proximal homomorphism (or extension) if $P$ a.s. the extension $\pi : (X, \mu_\omega) \to (\mathcal{Y}, \nu_\omega)$ is a.s. 1-1, where $\nu_\omega$ are the conditional measures for the system $\mathcal{Y}$. Clearly, when $\mathcal{Y}$ is the trivial system, the extension $\pi$ is $m$-proximal iff the system $X$ is $m$-proximal. When there is no room for confusion we sometimes say proximal rather than $m$-proximal.

Proposition 3.2 of [8] can now be formulated as:

1.2. Proposition. For every $m$-dynamical system $X$ the system $\Pi(X) = (M(X), P^*)$ is $m$-proximal. It is a trivial, one point system, iff $X$ is a measure preserving system.

Given the group $G$ and the probability measure $m$, there exists a unique universal $m$-proximal system $(\Pi(G, m), \eta)$ called the Poisson boundary of the pair $(G, m)$. Thus every $m$-proximal system $(X, \mu)$ is a factor of the system $(\Pi(G, m), \eta)$.

Given an $m$-system $(X, \mu)$ let

$$h_m(X, \mu) = -\int_G \int_X \log(dg\mu \over dm\mu) dm(g),$$

or

$$h_m(X, \mu) = -\sum g(m) \int_X \log\left(\frac{dg\mu \over dm\mu}\right) dm(g),$$

when $G$ is discrete. This nonnegative number is the $m$-entropy of the $m$-system $(X, \mu)$. We have the following theorem (see [6], [24]).

1.3. Theorem. (1) The $m$-system $(X, \mu)$ is measure preserving iff $h_m(X, \mu) = 0$.

(2) More generally, an extension of $m$-systems $\pi : (X, \mu) \to (Y, \nu)$ is a measure preserving extension iff $h_m(X, \mu) = h_m(X, \nu)$.

(3) An $m$-proximal system $(X, \mu)$ is isomorphic to the Poisson system $(\Pi(G, m), \eta)$ iff

$$h_m(X, \mu) = h_m(\Pi(G, m), \eta).$$

Typically the conditional measures $\mu_\omega$ are singular to the measure $\mu$. In fact we have the following statement.

1.4. Theorem. Let $X = (X, G, \mu)$ be an $m$-system with the property that a.s. the conditional measures $\mu_\omega$ are absolutely continuous with respect to $\mu$ ($\mu_\omega \ll \mu$). Then $\mu$ is $G$-invariant; i.e. $X$ is measure preserving.

Proof. We consider the usual unitary representation of $G$ on $H = L_2(X, \mu)$ given by

$$U_gf(x) = f(g^{-1}x)u(g^{-1}, x), \quad \text{with} \quad u(g, x) = \sqrt{dg^{-1}\mu \over d\mu}.$$
For \( \omega \in \Omega_0 \) let \( f_\omega = \frac{dg}{dg\mu} \in L_1(\mu) \) denote the Radon-Nikodym derivative of \( \mu_\omega \) w.r.t. \( \mu \), and put \( h_\omega = \sqrt{f_\omega} \). Then for \( \omega \in \Omega_0 \), \( g \in G \) and \( f \in L_2(X, \mu) \), denoting \( v(g, x) = \frac{dg^{-1}\mu}{dg\mu} \), we get

\[
\int f(x) dg \mu_\omega(x) = \int f(gx) f_\omega(x) d\mu(x) \\
= \int f(x) f_\omega(g^{-1}x) d\mu(x) \\
= \int f(x) f_\omega(g^{-1}x) v(g^{-1}, x) d\mu(x).
\]

Hence \( f_{g\omega} = (f_\omega \circ g^{-1}) \cdot v(g^{-1}, \cdot) \) and

\[
h_{g\omega} = (h_\omega \circ g^{-1}) \cdot u(g^{-1}, \cdot) = U_g h_\omega.
\]

It is now easy to see that the map \( \mu_\omega \mapsto h_\omega \) from \( \Omega_0 \) into the unit ball \( B \) of \( H = L_2(X, \mu) \), is a Borel isomorphism which intertwines the \( G \)-action on \( \Omega_0 \) with the unitary action of \( G \) on \( B \). If we let \( Y \) be the weak closure of the set of functions \( \{ h_\omega : \omega \in \Omega_0 \} \) in \( B \), we get a compact \( G \)-space \((Y, G)\) by restricting the unitary representation \( g \mapsto U_g \) to \( Y \). Such a \( G \)-space is WAP and our theorem follows from theorem 7.4 in section 7 below, which asserts that every \( m \)-stationary measure on \( Y \) is \( G \)-invariant. (For the definition and basic properties of weakly almost periodic (WAP) \( G \)-systems we refer e.g. to [16, Chapter 1].)

\[\square\]

2. \textbf{Examples}

1. Let \( G = SL(2, \mathbb{R}) \) and let \( m \) be any absolutely continuous right and left \( K \)-invariant probability measure on \( G \) such that \( \text{supp}(m) \) generates \( G \) as a semigroup. \( G \) acts on the compact space \( X \) of rays emanating from the origin in \( \mathbb{R}^2 \)—which is homeomorphic to the unite circle in \( \mathbb{R}^2 \). Normalized Lebesgue measure \( \mu \) is the unique \( m \)-stationary measure on \( X \). \( G \) acts as well on the space \( Y = \mathbb{P}^1 \) of lines in \( \mathbb{R}^2 \) through the origin (the projective line) and the natural map \( \pi : X \to Y \), that sends a ray in \( X \) to the unique line that contains it in \( Y \), is a 2 to 1 homomorphism of \( m \)-systems, where we take \( \nu = \pi(\mu) \). It is easy to see that \((Y, \nu)\) is \( m \)-proximal and that \( \pi \) is a measure preserving extension. It can be shown that \((Y, \nu)\) is the unique \( m \)-proximal system so that in particular \((Y, \nu)\) is the Poisson boundary \( \Pi(G, m) \).

2. ([7]) Let \( G \) be a connected semisimple Lie group with finite center and no compact factors. Let \( G = KNA \) be an Iwasawa decomposition, \( S = AN \) and \( P = MAN \), the corresponding minimal parabolic subgroup. Set \( X = G/S, Y = G/P \) and let \( m \) be an admissible probability measure on \( G \). More specifically we assume that \( m \) is absolutely continuous with respect to Haar measure, right and left \( K \)-invariant, and \( \text{supp}(\mu) \) generates \( G \) as a semigroup. Then

(1) There exists on \( Y \) a unique \( m \)-stationary measure \( \nu \) (which is the unique \( K \)-invariant probability measure on \( Y \)) such that the \( m \)-system \((Y, \nu)\) is \( m \)-proximal. In fact \((Y, \nu)\) is the Poisson boundary \( \Pi(G, m) \) and the collection of \( m \)-proximal systems coincides with the collection of homogeneous spaces \( G/Q \) with \( Q \) a parabolic subgroup of \( G \).
(2) For any \( m \)-stationary measure \( \mu \) on \( X \) the natural projection \( (X, \mu) \xrightarrow{\pi} (Y, \nu) \) is a measure preserving extension.

3. (23) Let \( G \) be a connected semisimple Lie group with finite center, no compact factors, and \( \mathbb{R} \)-rank \( (G) \geq 2 \). Let \( m \) be an admissible probability measure on \( G \) and let \( (X, G) \) be a compact metric \( G \)-space. Let \( P \) be a minimal parabolic subgroup of \( G \) and \( \lambda \) a \( P \)-invariant probability on \( X \). Let \( \nu_0 \) be the unique \( m \)-stationary probability measure on \( G/P \). Let \( \nu_0 \) be any probability measure on \( G \) which projects onto \( \nu_0 \) under the natural projection of \( G \) onto \( G/P \), and put \( \mu = \nu_0 \ast \lambda \) (it follows from \([7]\), that \( (X, \mu) \) is an \( m \)-system, and moreover that any \( m \)-stationary measure on \( X \) is of this form). Suppose further that the measure preserving \( P \)-action \( (X, \lambda) \) is mixing. Then there exists a parabolic subgroup \( Q \subset G \), a \( Q \)-space \( Y \), and a \( Q \)-invariant probability measure \( \eta \) on \( Y \) such that the \( m \)-system \( (X, \mu) \) is isomorphic to the “induced” \( m \)-system \( Y \times G/Q = ((Y \times G)/Q, \tilde{\eta}) \), where \( \tilde{\eta} \) is an \( m \)-stationary measure. In particular \( (X, \mu) \) is a measure preserving extension of an \( m \)-proximal system \( G/Q \), and \( \mu \) is \( G \)-invariant iff \( Q = G \).

In the following examples let \( G \) be the free group on two generators, \( G = F_2 = \langle a, b \rangle \), and \( m = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}) \).

4. (See [8]) Let \( Z \) be the space of right infinite reduced words on the letters \( \{a, a^{-1}, b, b^{-1}\} \). \( G \) acts on \( Z \) by concatenation on the left and reduction. Let \( \eta \) be the probability measure on \( Z \) given by
\[
\eta(C(\epsilon_1, \ldots , \epsilon_n)) = \frac{1}{4 \cdot 3^{n-1}},
\]
where for \( \epsilon_j \in \{a, a^{-1}, b, b^{-1}\} \), \( C(\epsilon_1, \ldots , \epsilon_n) = \{z \in Z : z_j = \epsilon_j, \ j = 1, \ldots , n\} \). The measure \( \eta \) is \( m \)-stationary and the \( m \)-system \( Z = (Z, \eta) \) is \( m \)-proximal. In fact \( Z \) is the Poisson boundary \( \Pi(F_2, m) \).

5. Let \( Y = \{0, 1\} \), \( \nu = \frac{1}{2}(\delta_0 + \delta_1) \), and the action be defined by \( a\epsilon = \bar{\epsilon} \), \( b\epsilon = \bar{\epsilon} \) for \( \epsilon \in \{0, 1\} \), where \( \bar{0} = 1 \) and \( \bar{1} = 0 \). \( Y = (Y, \nu) \) is a measure preserving system.

6. Let \( X = Y \times Z \), \( \mu = \nu \times \eta \), where \( Y, Z, \nu, \eta \) are as above, and let the action of \( G \) on \( X \) be defined as follows:
\[
\begin{align*}
a(\epsilon, z) &= (\bar{\epsilon}, a\epsilon z), & a^{-1}(\epsilon, z) &= (\epsilon, a\epsilon^{-1} z), \\
b(\epsilon, z) &= (\bar{\epsilon}, b\epsilon z), & b^{-1}(\epsilon, z) &= (\epsilon, b\epsilon^{-1} z),
\end{align*}
\]
where for \( g \in G \) we let \( g_0 = e \) and \( g_1 = g \). Finally let \( \pi : X \to Y \) be the projection on the first coordinate. One can check that \( m \ast \mu = \mu \) so that \( X \) is an \( m \)-system, and that the extension \( \pi \) is a relatively proximal extension. We claim that the following system is a description of \( \Pi(X) = (M, P^*) \). Let \( M = \{(\epsilon, z), (\bar{\epsilon}, z') : \epsilon \in \{0, 1\}, z, z' \in Z\} \), here \( \langle \cdot , \cdot \rangle \) denotes the unordered pair. The measure \( P^* \) is given by
\[
P^*((\{\epsilon\} \times A) \times ((\bar{\epsilon}) \times B) \cup ((\bar{\epsilon}) \times B) \times ((\epsilon) \times A)) = \eta(A)\eta(B),
\]
for \( A, B \subset Z \) and \( \epsilon \in \{0, 1\} \). It is not hard to see that, although the \( m \)-system \( X \) is not measure preserving, it admits no nontrivial \( m \)-proximal factor.
7. A small variation on example 6 gives an example of a similar nature, with the conditional measures $\mu_\omega$ being continuous. Take $Y$ to be the diadic adding machine $Y = \{0,1\}^\mathbb{Z} = \{\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \ldots ) : \epsilon_i \in \{0,1\}\}$, let $X = Y \times Z$, and define the action of $F_2$ on $X$ by:

$$a(\epsilon, z) = (\epsilon + 1, a_\epsilon z), \quad a^{-1}(\epsilon, z) = (\epsilon + 1, a_{\epsilon}^{-1}z),$$
$$b(\epsilon, z) = (\epsilon + 1, b_\epsilon z), \quad b^{-1}(\epsilon, z) = (\epsilon + 1, b_{\epsilon}^{-1}z),$$

where $1 = (1, 0, 0, \ldots)$ and $a_\epsilon = e$ when $\epsilon_1 = 0$, $a_\epsilon = a$ when $\epsilon_1 = 1$, and $b_\epsilon$ is defined similarly.

8. Let $G$ be the closed subgroup of the Lie group $GL(4, \mathbb{R})$ consisting of all $4 \times 4$ matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

with $A, B \in GL(2, \mathbb{R})$. We let $G$ act on the subspace $X$ of the projective space $\mathbb{P}^3$ consisting of the disjoint union of the two one dimensional projective spaces $\mathbb{P}^1$, which are naturally embedded in $\mathbb{P}^3$, the quotient space of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Call these two copies $X_1$ and $X_2$ respectively. There is a natural projection from $(X, G)$ onto the two-point $G$-system $(Y, G) = (\{X_1, X_2\}, G)$. Let $m$ be an admissible probability on $G$ and $\mu$ an $m$-stationary measure on $X$. Then it is easy to see that the $m$-system $(X, \mu)$ is an $m$-proximal extension of the (measure preserving) two-point system $Y$. Moreover the $m$-system $(X, \mu)$ has no nontrivial $m$-proximal factor. If we let $Z \subset M(X)$ be the collection of measures of the form:

$$Z = \{\frac{1}{2}(\delta_{x_1} + \delta_{x_2}) : x_i \in X_i, \ i = 1, 2\},$$

then one can check that the elements of $Z$ are the conditional measures $\mu_\omega$ of the $m$-system $(X, \mu)$. It follows that $(M(X), P^s)$ is isomorphic as an $m$-system to the symmetric product $\mathbb{P}^1 \times \mathbb{P}^1 / \{\text{id, flip}\}$.

9. (24) Let $G = SL(2, \mathbb{R})$ and fix an admissible $K$-invariant measure $m$ on $G$. In [24, Theorem 3.1] Nevo and Zimmer construct a co-compact lattice $\Gamma < G = SL(2, \mathbb{R})$, a $\Gamma$-space $Z$ and an $m$-stationary measure $\eta$ on the induced $G$-space $X = G/\Gamma \times Z$, with the property that $0 < h_\eta(X) < h_\nu(Y)$, where $Y = \Pi(G, m)$ and $\nu$ is the unique $m$-stationary probability measure on $Y$ (see example 1 above).

Claim: The $m$-system $(X, \eta, G)$ admits no nontrivial $m$-proximal factors.

Proof. There is a unique $m$-proximal $G$-system, namely the Poisson boundary $(\Pi(G, m), \nu)$. Since the entropy of the $m$-system $(G/\Gamma \times Z, \eta, G)$ is strictly lower than the entropy of $(\Pi(G, m), \nu)$, the former cannot admit the latter as a factor. \(\square\)

3. Joinings

Definitions: Let $X$ and $Y$ be two $m$-systems. We say that a probability measure $\lambda$ on $X \times Y$ is an $m$-joining of the measures $\mu$ and $\nu$ if it is $m$-stationary and its marginals are $\mu$ and $\nu$ respectively. In contrast to the situation in the class of measure preserving dynamical systems, the product measure $\mu \times \nu$ is usually not $m$-stationary
and therefore not an \( m \)-joining. On the other hand we have the following natural construction. We let the probability measure \( \lambda \in M(X \times Y) \) be defined by

\[
\lambda = \mu \upharpoonright \nu = \int \mu_\omega \times \nu_\omega dP(\omega).
\]

The equation

\[
g\lambda = \int g_{\mu_\omega} \times g_{\nu_\omega} dP(\omega),
\]

for each \( g \in G \), implies

\[
\int g\lambda dm(g) = \int \int g\mu_\omega \times g\nu_\omega dP(\omega) dm(g)
= \int \int \mu_{g_\omega} \times \nu_{g_\omega} dP(\omega) dm(g)
= \int \mu_\omega \times \nu_\omega dP(\omega) = \lambda;
\]

i.e. \( \lambda \) is \( m \)-stationary. We call the \( m \)-system \( \mathcal{X} \upharpoonright \mathcal{Y} = (X \times Y, \lambda) \), the \( m \)-join of the two \( m \)-systems \( \mathcal{X} \) and \( \mathcal{Y} \). We use the notation \( \mathcal{X} \lor \mathcal{Y} \) to denote any joining of the systems \( \mathcal{X} \) and \( \mathcal{Y} \); e.g. when they are both factors of a third \( m \)-system \( \mathcal{Z} \) then we usually mean \( \mathcal{X} \lor \mathcal{Y} \) to be the factor of \( \mathcal{Z} \) defined by the smallest \( \sigma \)-algebra containing \( \mathcal{X} \) and \( \mathcal{Y} \).

3.1. \textbf{Proposition.} Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two \( m \)-systems,

(1) if \( \mathcal{X} \) is measure preserving then \( \mu \upharpoonright \nu = \mu \times \nu \);
(2) if \( \mathcal{X} \) is \( m \)-proximal then

\[
\mu \upharpoonright \nu = \int \delta_{\mu_\omega} \times \nu_\omega dP(\omega)
\]

is the unique \( m \)-joining of the two systems.

\textbf{Proof.} (1) Since the conditional measures for \( \mathcal{X} \) satisfy \( \mu_\omega = \mu \) a.s.,

\[
\mu \upharpoonright \nu = \int \mu_\omega \times \nu_\omega dP(\omega)
= \int \mu_\omega \times \nu_\omega dP(\omega)
= \mu \times \nu.
\]

(2) Let \( \lambda \) be any \( m \)-joining of \( \mu \) and \( \nu \). Our assumption now is that the conditional measures of \( \mathcal{X} \) are a.s. point masses \( \delta_{\mu_\omega} \), whence the conditional measures \( \lambda_\omega = \lim \xi_1 \xi_2 \cdots \xi_n \lambda \) have marginals \( \delta_{\mu_\omega} \) and \( \nu_\omega \) on \( X \) and \( Y \) respectively. This means \( \lambda_\omega = \delta_{\mu_\omega} \times \nu_\omega \) and therefore

\[
\lambda = \int \lambda_\omega dP(\omega) = \int \delta_{x_\omega} \times \nu_\omega dP(\omega) = \mu \upharpoonright \nu.
\]

\( \square \)
3.2. **Proposition.**  
(1) The only endomorphism of a proximal system is the identity automorphism.  
(2) For every \( m \)-system \( (X, \mu) \) there is a unique maximal proximal factor.

*Proof.* (1) Let \( \alpha : X \to X \) be an endomorphism of the proximal system \( (X, \mu) \). Consider the map \( \phi : x \mapsto \theta_x = \frac{1}{2}(\delta_x + \delta_{\alpha(x)}) \) of \( X \) into \( M(X) \). This induces a quasifactor \( (M(X), \lambda) \) where \( \lambda = \phi_* (\mu) \). Now the conditional measures of the proximal system \( (M(X), \lambda) \) are point masses of the form \( \delta_{\theta_x} \). On the other hand applying the barycenter map \( b \) to the limits:

\[
\xi_1(\omega) \cdots \xi_n(\omega) \lambda \to \delta_{\theta_x(\omega)},
\]

we get

\[
\xi_1(\omega) \cdots \xi_n(\omega) \mu \to \delta_{x(\omega)}.
\]

Thus \( b(\delta_{\theta_x(\omega)}) = \theta_x(\omega) = \delta_{x(\omega)} \) a.e.; i.e. \( \alpha(x) = x \) a.e.

(2) It is easy to check that the join of all proximal factors of an \( m \)-system \( (X, \mu) \) is a maximal proximal factor of \( (X, \mu) \). \( \square \)

4. **A structure theorem for stationary systems**

4.1. **Proposition.** Let

\[
\begin{array}{ccc}
(X, \mu) & \xrightarrow{\sigma} & (Z, \eta) \\
\pi & \downarrow & \downarrow \\
(Y, \nu) & \xleftarrow{\rho} & \\
\end{array}
\]

be a commutative diagram of \( m \)-systems

(1) if \( \pi \) is a measure preserving extension then so are \( \rho \) and \( \sigma \).

(2) if \( \pi \) is a proximal extension then so are \( \rho \) and \( \sigma \).

*Proof.* (1) Let

\[
\mu = \int \mu_y d\nu(y) = \int \mu_z d\eta(z), \quad \eta = \int \eta_y d\nu(y),
\]

be the disintegrations of \( \mu \) over \( Y \) and \( Z \) and of \( \eta \) over \( Y \) respectively. We assume that for all \( g \) and \( \nu \) almost every \( y \), \( g\mu_y = \mu_{gy} \), hence

\[
g\eta_y = g\sigma\mu_y = \sigma g\mu_y = \sigma\mu_{gy} = \eta_{gy},
\]

so that \( \rho \) is a measure preserving extension. Now, since

\[
\mu = \int \mu_y d\nu(y) = \int \mu_z d\eta(z) = \int (\int \mu_z d\eta_y(z)) d\nu(y),
\]

the uniqueness of disintegration shows that

\[
\mu_y = \int \mu_z d\eta_y(z).
\]
Thus for \( g \in G \) we have:

\[
g \mu_y = g \left( \int \mu_z \, d\eta_y(z) \right) = \int g \mu_z \, d\eta_y(z),
\]

and also

\[
g \mu_y = \mu_{gy} = \int \mu_z \, d\eta_{gy}(z) = \int \mu_{gz} \, d\eta_y(z).
\]

Again the uniqueness of disintegration yields \( g \mu_z = \mu_{gz} \), so that also \( \sigma \) is a measure preserving extension.

(2) This is a straightforward consequence of the definition of \( m \)-proximal extension. \( \square \)

Let us call an \( m \)-system \((X, \mu)\) **standard** if there exists a homomorphism \( \pi : (X, \mu) \to (Y, \nu) \) with \((Y, \nu)\) proximal and the homomorphism \( \pi \) a measure preserving extension. Note that with this terminology the results described in the examples 2 and 3 above can be stated as saying that the stationary systems \((X, \mu)\) described there are standard (of a very particular kind, namely measure preserving extensions of boundaries of the form \( G/Q \) with \( Q \subset G \) a parabolic subgroup).

**4.2. Proposition.**

(1) The structure of a standard system as a measure preserving extension of a proximal system is unique.

(2) Let \((X, \mu)\) be a standard \( m \)-system: \( \pi : (X, \mu) \to (Y, \nu) \) with \((Y, \nu)\) proximal and the homomorphism \( \pi \) a measure preserving extension. If \( \alpha : (X, \mu) \to (Z, \eta) \) is a measure preserving homomorphism then there is a commutative diagram:

```
X  \alpha \downarrow \pi \downarrow \beta \rightarrow Z
   \downarrow \pi
   \downarrow \beta
Y  \rightarrow Z
```

**Proof.** (1) Let \((X, \mu)\) be a standard \( m \)-system: \( \pi : (X, \mu) \to (Y, \nu) \) with \((Y, \nu)\) proximal and the homomorphism \( \pi \) a measure preserving extension. If \( \pi' : (X, \mu) \to (Y', \nu') \) is another factor with \((Y', \nu')\) proximal, then the system \( Y \vee Y' \) is also \( m \)-proximal and we have the diagram:

```
X  \sigma \downarrow \pi \downarrow \beta \rightarrow Z
   \downarrow \pi
   \downarrow \beta
Y  \rightarrow Z
```

Now \( \rho \) is clearly a proximal extension and by proposition 4.1 it is also a measure preserving extension. Thus \( \rho \) is an isomorphism, so that \( Y' \) is a factor of \( Y \). We now
have the diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\downarrow{\pi'} & & \downarrow{\alpha} \\
Y' & \xleftarrow{\alpha} & Z \\
\end{array} \]

If \( \pi' \) is a measure preserving homomorphism then by proposition 4.1 so is \( \alpha \) and being also a proximal homomorphism it is necessarily an isomorphism.

(2) Consider the diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{\phi} & Y \lor Z \\
\downarrow{\alpha} & & \downarrow{\psi} \\
Z & \xleftarrow{\sigma} & Z \\
\end{array} \]

Since \( \alpha \) is a measure preserving homomorphism so is \( \psi \) (proposition 4.1). On the other hand, since \( Y \) is proximal it follows that \( \psi \) is a proximal extension. Thus \( \psi \) is an isomorphism and we deduce that \( Y \) is a factor of \( Z \) as required.

\[
\Box
\]

4.3. Theorem (A structure theorem for stationary systems). Let \( X = (X, \mu) \) be an \( m \)-system, then there exist canonically defined \( m \)-systems \( X^* = (X^*, \mu^*) \), and \( \Pi(X) = (M, P^*) \), with \( X^* \) standard and \( \Pi(X) \) \( m \)-proximal, and a diagram

\[ \begin{array}{ccc}
X^* & \xrightarrow{\pi} & X \\
\downarrow{\sigma} & & \downarrow{\Pi(X)} \\
\Pi(X) & \xleftarrow{\pi} & X \\
\end{array} \]

where \( \pi \) is an \( m \)-proximal extension, and \( \sigma \) is a measure preserving extension. Thus every \( m \)-system admits an \( m \)-proximal extension which is standard. The \( m \)-system \( X \) is measure preserving iff \( \Pi(X) \) is trivial. The \( m \)-system \( X \) is \( m \)-proximal iff both \( \pi \) and \( \sigma \) are isomorphisms. We call \( X^* \) the standard cover of \( X \).

Proof. We let \( X^* = X \times M(X) \) and the measure \( \mu^* = \mu \times P^* \) is defined by the integral

\[
(4.1) \quad \mu^* = \int \mu_\omega \times \delta_{\mu_\omega} dP(\omega).
\]

The assertions of the theorem now follow from propositions 1.2 and 3.1, however for clarity and completeness we give below a more detailed proof. Denote by \( \pi \) and \( \sigma \) the projections on the first and second coordinates respectively. Clearly \( X^* = (X^*, \mu^*) \) is a joining of the systems \( (X, \mu) \) and \( (M(X), P^*) \) in the sense that \( \pi(\mu^*) = \mu \), and \( \sigma(\mu^*) = P^* \). We show next that \( \mu^* \) is \( m \)-stationary. For \( g \in G \) we have a.s.

\[
(4.2) \quad g \mu_\omega = \mu_{g \omega},
\]
Theorem. \( \pi \) is a proximal extension. Since

\[
g \mu^* = \int \mu_{g \omega} \times \delta_{\mu_{g \omega}} dP(\omega),
\]

hence

\[
\int_G g \mu^* dm(g) = \int_G \int_{\Omega} \mu_{g \omega} \times \delta_{\mu_{g \omega}} dP(\omega) dm(g)
\]

\[
= \int \mu_{\xi_1 \omega} \times \delta_{\mu_{\xi_1 \omega}} dP(\omega)
\]

\[
= \int \mu_\omega \times \delta_{\mu_\omega} dP(\omega) = \mu^*.
\]

Now (4.1) gives the disintegration of \( \mu^* \) with respect to \( P^* \), i.e. w.r.t. \( \sigma \), and (4.2) shows that \( \sigma \) is a measure preserving extension.

Next we mimic the proof of proposition 3 in [8] in order to show that the measures \( \theta_\omega = \mu_\omega \times \delta_{\mu_\omega} \) are the conditional measures of the \( m \)-system \( (X^*, \mu^*) \), i.e. we will show that a.s.

\[
\lim \xi_1 \xi_2 \cdots \xi_n \mu^* = \theta_\omega.
\]

First observe that

\[
\theta_\omega = \lim_{n \to \infty} \xi_1 \xi_2 \cdots \xi_n (\mu \times \delta_\mu).
\]

Write \( \theta_1(\omega) := \theta_\omega \) and let

\[
\theta_k = \lim_{l \to \infty} \xi_{k+1} \xi_{k+2} \cdots \xi_{k+l} (\mu \times \delta_\mu),
\]

so that \( \xi_1 \xi_2 \cdots \xi_n \theta_{n+1} = \theta_1 \). For a bounded continuous function \( f \) on \( X^* \) and a measure \( \tau \in M(X^*) \) we write \( f(\tau) = \int_{X^*} f(x^*) d\mu^*(x^*) \). Now for any such \( f \) we have

\[
\int_{X^*} f(\xi_1 \xi_2 \cdots \xi_n x^*) d\mu^*(x^*)
\]

\[
= \int_{\Omega} \int_X f(\xi_1 \xi_2 \cdots \xi_n (x, \delta_{\omega'})) d\mu_{\omega'}(x) dP(\omega')
\]

\[
= \int_{\Omega} f(\xi_1 \xi_2 \cdots \xi_n (\mu_\omega \times \delta_{\omega'})) dP(\omega')
\]

\[
= E(f(\xi_1 \xi_2 \cdots \xi_n (\theta_{n+1}))|\xi_1 \xi_2 \cdots \xi_n)
\]

\[
= E(f(\theta_1)|\xi_1 \xi_2 \cdots \xi_n) \to f(\theta_1) = f(\mu_\omega \times \delta_{\mu_\omega}),
\]

where the convergence in the last line follows from the martingale convergence theorem. Since clearly a.s. \( \pi : (X \times M, \mu_\omega \times \delta_{\mu_\omega}) \to (X, \mu_\omega) \) is 1-1, we see that \( \pi \) is an \( m \)-proximal extension. This completes the proof of the theorem. \( \square \)

4.4. **Theorem.** If \( (X, \mu) \) is an \( m \)-system with maximal entropy (i.e. \( h_m(X, \mu) = h_m(\Pi(G, m)) \)); then \( (X, \mu) \) is standard and it admits the Poisson boundary \( \Pi(G, m) \) as its maximal proximal factor.

**Proof.** Let \( \pi : X^* \to X \) be the standard cover of \( (X, \mu) \), so that in particular \( \pi \) is a proximal extension. Since \( \pi \) does not raise entropy it is a measure preserving extension (theorem 3). Thus \( \pi \) is an isomorphism and \( X \) is a standard system whose
maximal proximal factor has maximal entropy. Again theorem 1.3 implies that this factor is isomorphic to $\Pi(G, m)$. □

4.5. **Theorem.** Let $\mathcal{X} = (X, \mu)$ be an $m$-system which admits a strict tower of proximal and measure preserving extensions

$$X \cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0.$$ 

Assume that $X_0$ is the maximal proximal factor of $\mathcal{X}$, that the proximal and measure preserving maps alternate, and that each such map is maximal. Thus $X_{2n+1} \rightarrow X_{2n}$ is measure preserving and if $X \rightarrow Y \rightarrow X_{2n+1} \rightarrow X_{2n}$ is such that $Y \rightarrow X_{2n}$ is also measure preserving then $Y = X_{2n+1}$. Likewise $X_{2n+2} \rightarrow X_{2n+1}$ is proximal and if $X \rightarrow Y \rightarrow X_{2n+2} \rightarrow X_{2n+1}$ is such that $Y \rightarrow X_{2n+1}$ is also proximal then $Y = X_{2n+2}$. Let

\begin{equation}
(4.4) \quad \Pi(X) \cdots \rightarrow \Pi(X_{n+1}) \rightarrow \Pi(X_n) \cdots \rightarrow \Pi(X_2) \rightarrow \Pi(X_1) \rightarrow \Pi(X_0) = X_0
\end{equation}

be the corresponding sequence of homomorphisms. (Since for every $n$ the map $\phi : X_{2n+1} \rightarrow X_{2n}$ is measure preserving, $\Pi(X_{2n+1}) \rightarrow \Pi(X_{2n})$ is an isomorphism.) If at any stage in this sequence we have that $\Pi(X_{2n+2}) \rightarrow \Pi(X_{2n+1})$ is an isomorphism, then $X = X_{2n+1}$.

**Proof.** For convenience we write $m = 2n + 1$. In the diagram

\begin{align*}
\begin{tikzpicture}

\node (Xm) at (0,0) {$X_m$};
\node (Xm+1) at (2,2) {$X_{m+1}$};
\node (Xm_times_Pi) at (4,2) {$\Pi(X_{m+1})$};
\node (Pi Xm) at (4,-2) {$\Pi(X_m)$};
\node (Xm_times_Pi_m) at (2,-2) {$X_m \vee \Pi(X_m)$};
\node (Pi Xm_times_Pi_m) at (0,2) {$\Pi(X_m) \vee \Pi(X_{m+1})$};

\draw[->, above] (Xm+1) to node[above]{$\text{prox}$} (Xm_times_Pi_m);
\draw[->, below] (Xm) to node[below]{$\text{prox}$} (Xm_times_Pi_m);
\draw[->, above] (Xm_times_Pi_m) to node[above]{$\text{mp}$} (Xm+1);
\draw[->, below] (Pi Xm) to node[below]{$\phi$} (Pi Xm_times_Pi_m);
\draw[->, below] (Xm+1) to (Pi Xm_times_Pi_m);
\draw[->, below] (Pi Xm) to (Pi Xm_times_Pi_m);
\draw[->, above] (Xm_times_Pi) to node[above]{$\text{mp}$} (Xm_times_Pi_m);
\end{tikzpicture}
\end{align*}

by assumption, $\phi$ is an isomorphism and therefore all the maps on the right of the central vertical arrow $X_{m+1} \vee \Pi(X_{m+1}) \rightarrow X_m \vee \Pi(X_m)$ are measure preserving maps. On the other hand all the arrows on the left of this arrow are proximal maps. We conclude that $X_{m+1} \vee \Pi(X_{m+1}) \rightarrow X_m \vee \Pi(X_m)$ is both measure preserving and proximal, hence an isomorphism. However this implies that also $X_{m+1} \rightarrow X_m$ is an isomorphism. Since we assumed that at each stage the extension is maximal we now realize that the whole tower above $X_m$ collapses, i.e. $X = X_m = X_{2n+1}$. □

4.6. **Corollary.** For $G = SL(n, \mathbb{R})$ and $K$-invariant admissible $m$, every strict maximal tower is of height $\leq n$.

**Proof.** As was shown in [7] the Poisson $(G, m)$-space $\Pi(G, m)$ is the flag manifold on $\mathbb{R}^n$. Since every proximal $G$-system is a factor of $\Pi(G, m)$, every sequence of the form (4.4) is defined by a nested sequence of parabolic subgroups, whence of length at most $n$. □
Examples: 10. Applying the construction of the structure theorem to example 3. in section 1, we obtain the following description for the $m$-system $(X^*, \mu^*) = (X \times M, \mu \gamma \nu)$. $X^*$ can be taken as the subset of $X \times X$ consisting of all (ordered) pairs $((\epsilon, z), (\overline{\epsilon}, z'))$, $\epsilon \in \{0, 1\}$, $z, z' \in Z$, with the diagonal action $g((\epsilon, z), (\overline{\epsilon}, z')) = (g(\epsilon, z), g(\overline{\epsilon}, z'))$. The measure $\mu^*$ is then given by

$$\frac{1}{2} (\delta_0 + \delta_1) \times \eta \times \eta.$$

11. As we have seen (proposition 3.2), for every $m$-system $(X, \mu)$ there is a uniquely defined maximal proximal factor. This is not always the case with respect to measure preserving factors. We produce next an example of a product system $(X, \mu) = (Z \times Y, \eta \times \nu)$ where $(Z, \eta)$ is $m$-proximal and $(Y, \nu)$ is measure preserving—so that $(X, \mu)$ is standard—with a factor $(Y', \nu')$ which is also measure preserving but such that the factor $Y \vee Y'$ of $X$ is not measure preserving.

We let $G = F_2$, the free group on two generators $a$ and $b$, $m = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}})$. Let $Z$ be the Poisson boundary $\Pi(F_2, m)$ which we can take as the space of right infinite reduced words on the letters $\{a, a^{-1}, b, b^{-1}\}$ with the natural Markov measure $\eta$ as in example (4) above. The system $(Y, \nu)$ will be the Bernoulli system $Y = \{0, 1\}^F_2$ with product measure $\nu = \{\frac{1}{2}, \frac{1}{2}\}^F_2$. Thus $\nu$ is an invariant measure under the natural action of $F_2$ on $Y$ by translations. Clearly $\mu = \eta \times \nu = \eta \gamma \nu$ is $m$-stationary, so that $(X, \mu)$ is an $m$-system.

Next let $A$ be the subset $\{z \in Z : a$ is the first letter of $z\}$, and let $\phi : Z \to Y$ be the continuous function defined by $(\phi(z))_g = 1_A(gz)$. We observe that the map $\Phi : X \to Y$ defined by $\Phi(z, y) = z + y \pmod{1}$ is an equivariant continuous map. Let $Y' = \Phi(X)$ and $\nu' = \Phi_* (\mu)$. It is now easy to check that $(Y', \nu')$ is a measure preserving factor of the $M$-system $(X, \mu)$, which is isomorphic to the Bernoulli system $(Y, \nu)$. However it is also clear that the factor $Y \vee Y'$ of $(X, \mu)$ is a non-measure preserving $m$-system. In fact $Y \vee Y'$ admits the non-trivial proximal factor $Z' = \phi(Z)$.

4.7. Remark. For ergodic probability measure preserving transformations there is a more satisfactory structure theorem (due to Furstenberg [10], [12] and independently to Zimmer [27], [28]) according to which every such system is canonically presented as a weakly mixing extension of a measure-distal system (the latter is defined as a tower, possibly of infinite height, of compact extensions). In topological dynamics there is an analogous theorem for a minimal dynamical system $(X, G)$ (see [4], [26]). However, as in our theorem 4.3 one is forced in this setting to first associate with $X$ a proximal extension $X^* \to X$ so that only $X^*$ has the required structure of a weakly mixing extension of a PI-system (where the latter is a tower of alternating proximal and isometric extensions). In [15] there is an example of a minimal dynamical system $(X, T)$ which does not admit nontrivial factors that are either proximal or incontractible (this is the analogue of a measure preserving system in topological dynamics). We do not know how to construct a similar example for stationary systems. Such an example will show that in some sense one can not do better than what one gets in theorem 4.3.
4.8. **Problem.** Are there a group \( G \), a probability measure \( m \) on \( G \), and an ergodic \( m \)-stationary system \( \mathcal{X} = (X, \mu, G) \) such that \( \mathcal{X} \) does not admit nontrivial factors that are either proximal or measure preserving?

5. **Nevo-Zimmer theorem in an abstract setup**

**Definitions:**

1. Notations as in theorem 4.3, we say that the quasifactor \( \Pi(\mathcal{X}) = (M, P^*) \) is **mixingly embedded** in the \( m \)-system \( \mathcal{X} = (X, \mu) \), if the measure preserving extension \( \sigma : \mathcal{X}^* \to \Pi(\mathcal{X}) \) is a mixing extension; i.e. if for every \( f \in L_\infty(\mu^*) \) and every sequence \( g_n \to \infty \) in \( G \),

\[
  w^* \text{-} \lim g_n(f - E_{\Pi(\mathcal{X})} f) = 0,
\]

where \( E_{\Pi(\mathcal{X})} f \) is the conditional expectation of \( f \) with respect to the factor \( \Pi(\mathcal{X}) \).

2. For an \( m \)-system \( (X, \mu) \) and a subset \( V \) of \( L_\infty(\mu) \), let

\[
  \mathcal{F}(V) = \{ w^* \text{-} \lim g_n f : f \in V, g_n \to \infty \},
\]

the set of all weak* limit points of sequences \( g_n f \) where \( f \in V \) and \( g_n \to \infty \). Let \( \mathcal{F}(V) \) be the smallest \( \sigma \)-algebra with respect to which all members of \( \mathcal{F}(V) \) are measurable. Call the \( m \)-system \( (X, \mu) \) **reconstructive** with respect to \( V \) if \( \mathcal{F}(V) \) is the full \( \sigma \)-algebra of measurable sets on \( X \).

5.1. **Theorem.** Let \( \mathcal{X} = (X, \mu) \) be an \( m \)-system such that

1. The canonical \( m \)-proximal quasifactor \( \Pi(\mathcal{X}) = (M, P^*) \) is mixingly embedded in \( \mathcal{X} \).
2. \( \Pi(\mathcal{X}) \) is a reconstructive \( m \)-system with respect to the subspace

\[
  V = E_{\Pi(\mathcal{X})}(C(X)).
\]

Then the \( m \)-proximal quasifactor \( \Pi(\mathcal{X}) \) is actually a factor of \( \mathcal{X} \).

**Proof.** Consider an arbitrary continuous function \( f \) on \( X \), \( f \in C(X) \subset L_\infty(\mu) \subset L_\infty(X \times M(X), \mu^*) \), and the corresponding function \( \tilde{f} \in L_\infty(P^*) \) on \( M(X) \) defined by:

\[
  \tilde{f}(\mu_\omega) = \int_X f(x) d\mu_\omega = E_{\Pi(\mathcal{X})} f.
\]

By assumption (1), for every sequence \( g_n \to \infty \) in \( G \) for which \( w^* \text{-} \lim g_n \tilde{f} \) exists, we have

\[
  \hat{f} = w^* \text{-} \lim g_n f = w^* \text{-} \lim g_n \tilde{f},
\]

hence \( \hat{f} \) is in the \( w^* \)-closed subspace \( L_\infty(\mu) \cap L_\infty(P^*) \).

On the other hand, by assumption (2), with the subspace \( V = \{ \tilde{f} : f \in C(X) \} = E_{\Pi(\mathcal{X})}(C(X)) \), the smallest \( \sigma \)-algebra with respect to which all the functions:

\[
  \{ \tilde{f} = w^* \text{-} \lim g_n \tilde{f} : f \in C(X), g_n \to \infty \}
\]

are measurable is the full \( \sigma \)-algebra of measurable sets on \( \Pi(\mathcal{X}) \). It thus follows that with respect to \( \mu^* \), \( L_\infty(P^*) \subset L_\infty(\mu) \) and the proof is complete. \( \Box \)
5.2. Corollary. Let $X = (X, \mu)$ be an $m$-system such that the canonical $m$-proximal quasifactor $\Pi(X) = (M, P^*)$ is mixingly embedded in $X$. Then for every $f \in L_\infty(X, \mu)$, for a.e. $\omega$

$$w^* \text{-lim } \xi_1(\omega)\xi_2(\omega) \cdots \xi_n(\omega) f \equiv \tilde{f}(\mu_\omega).$$

Proof. In the proof of theorem 5.1 taking $g_n = \eta_n(\omega) = \xi_1(\omega)\xi_2(\omega) \cdots \xi_n(\omega)$ we have, for every $h \in L_1(\mu^*)$ by Lebesgue’s dominated convergence theorem,

$$\lim_{n \to \infty} \int_{X^*} \tilde{f}(\xi_1(\omega)\xi_2(\omega) \cdots \xi_n(\omega)\mu_\omega) h(x, \mu_\omega) \, d\mu^*(x, \mu_\omega)$$

$$= \tilde{f}(\mu_\omega) \int_{X^*} h(x, \mu_\omega) \, d\mu^*(x, \mu_\omega);$$

i.e. $w^* \text{-lim } \xi_1(\omega)\xi_2(\omega) \cdots \xi_n(\omega) \tilde{f} \equiv \tilde{f}(\mu_\omega)$, $\omega$-a.s. In view of (5.1) we deduce that $\omega$-a.s.

$$w^* \text{-lim } \xi_1(\omega)\xi_2(\omega) \cdots \xi_n(\omega) f \equiv \tilde{f}(\mu_\omega),$$

□

Example: Let $T$ and $S$ be two discrete countable groups, $m_S$ and $m_T$ probability measures on $S$ and $T$ respectively such that the corresponding Poisson spaces $\Pi(S, m_S)$ and $\Pi(T, m_T)$ are nontrivial. We form the product group $G = T \times S$ and the product measure $m = m_T \times m_S$.

5.3. Theorem. For $G = T \times S$ as above the Poisson spaces for the couples $(G, m)$, $(T, m_T)$ and $(S, m_S)$ satisfy:

$$\Pi(G, m) = \Pi(T, m_T) \times \Pi(S, m_S).$$

Proof. Clearly the systems $\Pi(T, m_T)$ and $\Pi(S, m_S)$ can be viewed as $G$ $m$-systems and as such they are proximal. Thus these systems are factors of the $m$-system $\Pi(G, m)$. It is now easy to check that if $\eta_T$ and $\eta_S$ are the $m$-stationary measures on $\Pi(T, m_T)$ and $\Pi(S, m_S)$ respectively then the measure $\eta_T \gamma \eta_S = \eta_T \times \eta_S$. Whence $\Pi(T, m_T) \times \Pi(S, m_S)$ is a factor of the system $\Pi(G, m)$. Since the entropy of both systems is $h_m(\Pi(T, m_T), \eta_T) + h_m(\Pi(S, m_S), \eta_S)$ we can now apply theorem 1.3 to conclude that $\Pi(G, m) = \Pi(T, m_T) \times \Pi(S, m_S)$.

5.4. Remark. Another proof of this fact follows directly from the characterization of the Poisson boundary of $(G, m)$ as the space of ergodic components of the time shift in the path space of the random walk due to Kaimanovich and Vershik, [21].

5.5. Lemma. Let $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{F}, \nu)$ be two probability spaces, $\mathcal{A}$ a sub-$\sigma$-algebra of $\mathcal{F}$ and $f \in L_\infty(X \times Y, \mu \times \nu)$. If for $\mu$ a.e. $x \in X$ the function $f_x(y) = f(x, y)$ is $\mathcal{A}$ measurable, then $f$ is $\mathcal{B} \times \mathcal{A}$ measurable.

5.6. Theorem. Let $(X, \mu, G)$ be an $m$-system. If the canonical $m$-proximal quasifactor $\Pi(X) = (M(X), P^*)$ is mixingly embedded in $X$. Then $\Pi(X)$ is a factor of $(X, \mu)$.

Proof. In view of theorem 5.1 all we have to show is that $\Pi(X)$ is a reconstructive $m$-system with respect to the subspace $V = E^{\Pi(X)}(C(X))$. For $f \in C(X)$ the function $\tilde{f}$ is $\Pi(X)$ measurable. Since $\Pi(X)$ is a factor of $\Pi(G)$, by theorem 5.3 lifting $\tilde{f}$ to $\Pi(G)$
we can write \( \tilde{f} \) as a function of two variables \( \tilde{f}(u,v) \), with \( u \in \Pi(S) \) and \( v \in \Pi(T) \). Now for almost every \( v_0 \in \Pi(T) \) there exists a sequence \( t_n^{v_0} \in T \) with \( \lim t_n^{v_0} v = v_0 \) for \( \mu_T \) almost every \( v \in \Pi(T) \). Thus, by (5.1), we see that

\[
\tilde{f}^{v_0}(u) = \omega^* - \lim t_n^{v_0} f = \omega^* - \lim t_n^{v_0} \tilde{f}(u,v) = \tilde{f}(u,v_0)
\]

is \( \mathcal{X} \) measurable. Similarly for almost every \( u_0 \in \Pi(S) \) the function \( \tilde{f}_{u_0}(v) = f(u_0,v) \) is \( \mathcal{X} \) measurable. By lemma 5.5 \( \tilde{f} \) is \( \mathcal{X} \) measurable and since the subspace \( V = \{ \tilde{f} : f \in C(X) \} \) generates \( C(\Pi(X)) \) as an algebra, we conclude that \( \Pi(X) \) is a reconstructive \( m \)-system with respect to \( V \).

6. A Szemerédi type theorem for \( SL(2,\mathbb{R}) \).

In this section \( G \) will denote the Lie group \( SL(2,\mathbb{R}) \) and we write \( G = KAN \) for the standard Iwasawa decomposition of \( G \); in particular \( K \) is the subgroup of 2 by 2 orthogonal matrices.

Recall that a mean on a topological group \( G \) is a positive linear functional \( \rho \) on \( LUC(G) \) with \( \rho(1) = 1 \). Here \( LUC(G) \) denotes the commutative \( C^* \)-algebra of bounded, complex valued, left uniformly continuous functions on \( G \). (\( f : G \to \mathbb{C} \) is left uniformly continuous if for every \( \epsilon > 0 \) there exists a neighborhood \( V \) of the identity element \( e \) such that \( \sup_{g \in G} |f(vg) - f(g)| < \epsilon \) for every \( v \in V \).) The set of means on \( G \) forms a \( \omega^* \)-closed convex subset of \( LUC(G)^* \) and we say that an element of this set is \( m \)-stationary if \( m * \rho = \rho \). By the Markov-Kakutani fixed point theorem the set of \( m \)-stationary means is nonempty.

Let \( Z \) be the (compact Hausdorff) Gelfand space corresponding to the \( C^* \)-algebra \( \mathcal{L} = LUC(G) \). Recall that \( Z \) can be viewed as the space of non-zero continuous \( C^* \)-homomorphisms of the \( C^* \)-algebra \( \mathcal{L} \) into \( \mathbb{C} \). In particular, for each \( g \in G \) the evaluation map \( z_g : F \to F(g) \) is an element of \( Z \). The fact that \( \mathcal{L} \) is \( G \)-invariant (i.e. for \( f \in \mathcal{L} \) and \( g \in G \) also \( f_g \in \mathcal{L} \), where \( f_g(h) = f(gh) \)) implies that there is a naturally defined \( G \)-action on \( Z \). We have \( g z_e = z_g \) for every \( g \in G \), and it follows directly that the \( G \)-orbit of the point \( z_e \) is dense in \( Z \). Also by the construction of the Gelfand space we obtain a natural isomorphism of the commutative \( C^* \)-algebras \( \mathcal{L} \) and \( C(Z) \). Let \( \tilde{f} \) denote the element of \( C(Z) \) which corresponds to \( f \) under this isomorphism. Now according to Riesz’ representation theorem we identify \( LUC(G)^* \) with the Banach space of complex regular Borel measures on \( Z \). In this setting a mean on \( G \) is identified with a probability measure on \( Z \). If \( L \) is a subset of \( G \) and \( \rho \) is a mean on \( G \), we say that \( L \) is charged by \( \rho \) and write \( \rho(L) > 0 \) if

\[
\mu_\rho(\text{cls} \{ z_g : g \in L \}) > 0.
\]

Here \( \mu_\rho \) is the probability measure on \( Z \) which corresponds to the mean \( \rho \). It is easy to check that with respect to the natural \( G \)-action on \( Z \) the mean \( m * \rho \) (defined by \( m * \rho(f) = \int_G \rho(f_g) dm(g) \)) corresponds to the measure \( m * \mu_\rho \), so that \( \rho \) is \( m \)-stationary if and only if the measure \( \mu_\rho \) is \( m \)-stationary.

6.1. Theorem. Let \( m \) be an admissible probability measure on \( G \) and let \( \rho \) a \( K \)-invariant \( m \)-stationary mean on \( G = SL(2,\mathbb{R}) \). If \( \rho(L) > 0 \) for a subset \( L \subset G \) then for every \( \epsilon > 0 \), \( k \geq 1 \) and a compact set \( Q \subset G \), there exist \( a_0 \) and \( h \) in \( G \setminus Q \) such that

\[
a_0, ha_0, \ldots, h^k a_0 \in L_\epsilon,
\]
where \( L_\epsilon = \{ g \in G : d(g, L) < \epsilon \} \).

6.2. Lemma. (A correspondence principle) Let \( G \) be a locally compact group. Given a nonempty subset \( L \subset G \), there exists a compact metric \( G \)-space \( X \) and an open subset \( A \subset X \) such that

\[
g_1^{-1}A \cap g_2^{-1}A \cap \cdots \cap g_k^{-1}A \neq \emptyset \implies g_1a, \ldots, g_k a \in L_\epsilon,
\]

for some \( a \in G \). If, moreover, \( m \) is a probability measure on \( G \) and \( \rho \) an \( m \)-stationary mean on \( G \) with \( \rho(L) > 0 \), then there exists an \( m \)-stationary probability measure \( \mu \) on \( X \) with \( \mu(A) \geq \rho(L) \).

Proof. Let \( f : G \to [0, 1] \) be a left uniformly continuous function such that \( f(g) = 1 \) for every \( g \in L_{\epsilon/2} \) and \( f(g) = 0 \) for every \( g \notin L_\epsilon \). Let \( A \) be the uniformly closed subalgebra of the algebra \( LUC(G) \) of complex valued bounded left uniformly continuous functions on \( G \) generated by the orbit \( \{f_g : g \in G\} \) of \( f \), where \( f_g(h) = f(gh) \). Let \( X \) be the (compact metric) Gelfand space corresponding to \( A \). The fact that \( A \) is \( G \)-invariant implies that there is a naturally defined \( G \)-action on \( X \). Clearly the restriction map \( \pi : Z \to X \), where with the above notation \( Z \) is the Gelfand space corresponding to \( L \), is a homomorphism of the corresponding dynamical systems. We denote \( x_g = \pi(z_g) \), so that \( gx_e = x_g \) and \( \text{cls} \{gx_e : g \in G\} = X \).

By the construction of the Gelfand space we obtain a natural isomorphism of the commutative \( C^* \)-algebras \( A \) and \( C(X) \). Let \( \hat{f} \) denote the element of \( C(X) \) which corresponds to \( f \) under this isomorphism.

Clearly the restriction of \( \rho \) to \( A \) defines an \( m \)-stationary probability measure \( \mu \) on \( X \), so that the system \((X, \mu, G)\) is an \( m \)-system. In fact \( \mu = \pi_* (\mu_\rho) \).

Let \( A = \{ x \in X : \hat{f}(x) > 1/2 \} \) and consider the set \( N(x_e, A) = \{ g \in G : gx_e \in A \} \). We clearly have \( \{ g \in G : f(g) > 1/2 \} = \{ g \in G : gx_e \in A \} \). Note that indeed

\[
\mu(A) = \int_X 1_A \, d\mu = \int_X 1_{\{f > 1/2\}} \, d\mu \geq \int_X 1_{\{f = 1\}} \, d\mu = \int_Z 1_{\{f = 1\}} \, d\mu_\rho \\
\geq \rho(L_{\epsilon/2}) \geq \rho(L).
\]

Now assume \( g_1^{-1}A \cap g_2^{-1}A \cap \cdots \cap g_k^{-1}A \neq \emptyset \) and let \( x \) be a point in this intersection. Then \( g_i x \in A \) for \( i = 1, \ldots, k \) and choosing \( a \in G \) so that \( d(ax_0, x) \) is sufficiently small, we also have \( g_i ax_0 \in A \) for \( i = 1, \ldots, k \). Thus \( f(g_i a) > 1/2 \) and we conclude that \( g_1a, g_2a, \ldots, g_k a \in L_\epsilon \). \( \square \)

Proof of the theorem. By the correspondence principle, lemma 6.2 we can associate with \( L \) an \( m \)-system \((X, \mu, G)\) and an open subset \( A \subset X \) with \( \mu(A) \geq \rho(L) > 0 \) such that

\[
(6.1) \quad \mu(g_1^{-1}A \cap g_2^{-1}A \cap \cdots \cap g_k^{-1}A) > 0 \implies g_1 a, g_2 a, \ldots, g_k a \in L_\epsilon,
\]

for some \( a \in G \).
Let
\[ \pi \xrightarrow{(X^*, \mu^*)} (X, \mu) \xrightarrow{\sigma} \Pi(X) = (Y, \lambda), \]
be the canonical standard cover of \((X, \mu)\), given by theorem 4.3 and set \(A^* = \pi^{-1}(A), B = \sigma(A^*)\).

Let
\[ \mu^* = \int_Y \mu_y \times \delta_y \, d\lambda(y) \]
be the decomposition of \(\mu^*\) over \((Y, \lambda)\). If we write \(A^* = \bigcup\{A_y \times \{y\} : y \in B\}\) then, by reducing \(B\) if necessary, we can assume that \(\mu_y(A_y) \geq \delta\) for some \(\delta > 0\).

In the present situation \(Y = \mathbb{P}^1\), the projective line, and \(\sigma(\mu^*) = \lambda\) is Lebesgue measure. Let \(y_0 \in B\) be a Lebesgue density point of \(B\).

Claim: If \(g\) is a parabolic element of \(SL(2, \mathbb{R})\) with fixed point \(y_0\) then for every \(N\)
\[ \lambda(B \cap gB \cap g^2B \cap \cdots \cap g^N B) > 0. \]

Proof of claim: Of course \(y_0 = gy_0\) is a Lebesgue density point of each of the sets \(g^j B\) and therefore we can choose an interval \(J \subset Y\) such that
\[ \frac{|J \cap g^j B|}{|J|} \geq (1 - \frac{1}{2N})|J| \quad j = 1, \ldots, N, \]
whence \(\lambda(J \cap \bigcap_{j=1}^N g^j B) \geq \frac{1}{2}|J|\).

Now back to the proof of theorem 6.1.

Szemerédi’s theorem yields, for every positive integer \(k\) and \(\delta > 0\), a positive integer \(N = N(k, \delta)\) such that every subset of \(\{1, 2, \ldots, N\}\) of size \(\delta N\) contains an arithmetic progression of length \(k\).

Take \(g\) as in the above claim, and such that \(\{g^n : n = 1, 2, \ldots\} \cap Q = \emptyset\). Denoting \(B_0 = B \cap gB \cap g^2B \cap \cdots \cap g^N B\), we have for \(\lambda\) almost every point \(y \in B_0\),
\[ \sum_{i=1}^N \int_X 1_{A_i}(x) \, d\mu_{g^i y}(x) \geq \delta N, \]
where \(A_i = A_{g^i y}\). Thus for some \(A_{ij}, 1 \leq j \leq [\delta N]\) the intersection \(\bigcap_{j=1}^{[\delta N]} A_{ij} \neq \emptyset\). If we take \(N = N(k, \delta)\) then we can find \(s\) and \(d\) such that
\[ \{s, s + d, s + 2d, \ldots, s + kd\} \subset \{i_1, i_2, \ldots, i_{[\delta N]}\}, \]
and some \(x^* \in X^*\) with
\[ g^s x^*, g^{s+d} x^*, g^{s+2d} x^*, \ldots, g^{s+kd} x^* \]
all in \(A^*\), hence, with \(x = \pi(x^*)\),
\[ g^s x, g^{s+d} x, g^{s+2d} x, \ldots, g^{s+kd} x \]
all in \(A\).

Since there are only finitely many possible \(s\) and \(d\) we conclude that for some pair \(s, d\)
\[ \mu(g^s A \cap g^{s+d} A \cap \cdots \cap g^{s+kd} A) > 0. \]
(6.2)
In fact
\[
\mu(g^s A \cap g^{s+d} A \cap \cdots \cap g^{s+kd} A)
\]
\[
= \mu^*(g^s A^* \cap g^{s+d} A^* \cap \cdots \cap g^{s+kd} A^*)
\]
\[
= \int_{B_0} \int_X \prod_{j=0}^k 1_{A_{g^{s+jd}y}}(x) \, dg^{s+jd} \mu_y(x) \, d\lambda(y)
\]
\[
= \int_{B_0} \int_X \prod_{j=0}^k 1_{A_{g^{s+jd}y}}(x) \, d\mu_{g^{s+jd}y}(x) \, d\lambda(y).
\]
Thus, if \( \mu(g^s A \cap g^{s+d} A \cap \cdots \cap g^{s+kd} A) = 0 \) then also
\[
\int_X \prod_{j=0}^k 1_{A_{g^{s+jd}y}}(x) \, d\mu_{g^{s+jd}y}(x) \, d\lambda(y) = 0
\]
for \( \lambda \) a.e. \( y \in B_0 \), contradicting our assumption on \( s \) and \( d \).

From \((6.2)\), by the correspondence principle \((6.1)\), we can find \( a \in G \) with
\[
g^{-s}a, g^{-(s+d)}a, \ldots, g^{-(s+kd)}a \in L_\epsilon.
\]
Setting \( a_0 = g^{-s}a \) and \( h = g^{-d} \) we finally get
\[
a_0, ha_0, \ldots, h^k a_0 \in L_\epsilon.
\]

### 6.3. Remark.
Independently of our work T. Meyerovich applies in a recent work \[22\], similar ideas in order to obtain multiple and polynomial recurrence lifting theorems for infinite measure preserving systems.

### 7. WAP actions are stiff

A compact topological dynamical system \((X, G)\) is called **weakly almost periodic** or **WAP**, if for every \( f \in C(X) \), the set \( \{f_g : g \in G\} \) is relatively compact in the weak topology on \( C(X) \) (where \( f_g \in C(X) \) is defined by \( f_g(x) = f(gx) \)). Ellis and Nerurkar \[5\] showed that \((X, G)\) is weakly almost periodic if and only if every element \( p \) in the enveloping semigroup \( E \) of the system \((X, G)\) is a continuous map. (Recall that \( E = E(X, G) \), the **enveloping semigroup** of the compact topological dynamical system \((X, G)\) is, by definition, the closure of the set of maps \( \{g : X \to X : g \in G\} \) in the compact product space \( X^X \), where the semigroup structure is defined by composition of maps.) As shown in \[5\] the enveloping semigroup \( E = E(X, G) \) of a WAP system contains a unique minimal left ideal \( I \) which is in fact a compact topological group. Consequently, in a topologically transitive WAP system there is a unique minimal subset and the action of \( G \) on this minimal set is equicontinuous.

A topological dynamical system \((X, G)\) is called **stiff with respect to** \( m \) or **\( m \)**-stiff if every \( m \)-stationary measure on \( X \) is \( G \)-invariant, (see \[13\]). Our goal in this section is to show that WAP systems are stiff (theorem \[7.4\] below).

#### 7.1. Lemma.
Let \((X, G)\) be a WAP dynamical system. Every element \( p \in E \) defines an element \( p_* \in E(M(X), G) \) and the map \( p \mapsto p_* \) is an isomorphism of \( E = E(X, G) \) onto \( E(M(X), G) \). In particular the dynamical system \((M(X), G)\) is also WAP.
Proof. If \( g_i \to p \) is a net of elements of \( G \) converging to \( p \in E = E(X,G) \), then by Grothendieck’s theorem, for every \( f \in C(X) \), \( f \circ g_i \to f \circ p \) weakly in \( C(X) \). Therefore, we have for every \( \nu \in M(X) \) and \( f \in C(X) \):

\[
g_i \nu(f) = \nu(f \circ g_i) \to \nu(f \circ p) := p_* \nu(f).
\]

It is easy to see that \( p \mapsto p_* \) is an isomorphism of flows, whence a semigroup isomorphism. Finally as \( G \) is dense in both enveloping semigroups, it follows that this isomorphism is onto. \( \square \)

In the sequel we will identify the two enveloping semigroups and will write \( p \) for both \( p \) and \( p_* \).

We recall the following theorem of R. Azencott ([1], theorem I.2, page 11).

7.2. Theorem. Let \((X,G)\) be a topological dynamical system with \( X \) a compact metric space. Let \( \mu \) be a probability measure on \( X \). The following properties are equivalent:

1. For every \( x \in X \), the measure \( \delta_x \) is a weak * limit point of the set \( \{g \mu : g \in G\} \) in \( M(X) \).
2. For every countable dense subset \( D \) of \( X \) there exists a Borel subset \( A \) of \( X \) with \( \mu(A) = 1 \) and with the property that for every \( x \in D \) there exists a sequence \( g_n \in G \) such that

\[
\lim g_n y = x \quad \forall y \in A.
\]

We call a measure \( \mu \) satisfying the equivalent conditions of theorem 7.2 a \textbf{contractible} measure, we then call the dynamical system \((X,\mu,G)\) a \textbf{contractible system}. Note that a contractible system is necessarily topologically transitive and moreover every point of the subset \( A \) belongs to the dense \( G_\delta \) subset \( X_{tr} \) of the transitive points of \( X \). Also note that every \( m \)-proximal system \((X,\mu,G)\), with \( X = \text{supp}(\mu) \), is contractible.

7.3. Lemma. Let \((X,G)\) be a WAP system. Let \( \mu \) be an \( m \)-stationary probability measure on \( X \) with \( X = \text{supp}(\mu) \) such that the \( m \)-system \((X,\mu)\) is \( m \)-proximal. Then \((X,G)\) is the trivial one point system.

Proof. Let \( X_{tr} \) be the dense, \( G \)-invariant, \( G_\delta \) subset of \( X \) consisting of all points with dense \( G \)-orbit. Let \( D \) and \( A \) be the subsets of \( X \) given by theorem 7.2 (2). Since \( D \) can be any countable dense subset of \( X \), we can assume that \( D \subset X_{tr} \). Fix a point \( x_0 \in D \) and let \( g_n \in G \) satisfy \( \lim g_n y = x_0 \) for every \( y \in A \). We can assume that the limit \( p = \lim g_n \) exists in the enveloping semigroup \( E = E(X,G) \), and then \( \lim g_n y = py = x_0 \) for every \( y \in A \). Since \( p \) is a continuous map and since clearly \( A \) is a dense subset of \( X \), it follows that \( px = x_0 \) for every \( x \in X \). The elements of the left ideal \( I = Ep \) are in 1-1 correspondence with the points of \( X \) (with \( q_x \in I \) defined by \( q_x y = x, \forall y \in X \)) and it follows that \( I \) is the unique minimal left ideal in \( E \). It is now clear that \((X,G)\) is a minimal proximal system. However in a WAP system the group action on the unique minimal subset is equicontinuous and we conclude that \( X \) consists of a single point. \( \square \)
7.4. Theorem. Let $G$ be a locally compact second countable topological group, $m$ a probability measure on $G$ with the property that the smallest closed subgroup containing $\text{supp}(m)$ is all of $G$. Then every WAP dynamical system $(X,G)$ is $m$-stiff.

Proof. Let $E = E(X,G)$ be the enveloping semigroup of the WAP system $(X,G)$. Let $\mu$ be an $m$-stationary ergodic probability on $X$; we will show that $g\mu = \mu$ for every $g \in G$. As in section 1 we let

$$\lim_{n \to \infty} \eta_n \mu = \mu, \quad \omega \in \Omega_0,$$

be the conditional measures of the $m$-system $\mathcal{X}$, and let $P^* \in M(M(X))$ be the distribution of the $M(X)$-valued random variable $\mu(\omega) = \mu_\omega$. Let now $Z = \text{supp}(P^*) \subset M(X)$. Clearly $Z$ is a closed $G$-invariant subset of $M(X)$, and by proposition 1.2 the $m$-dynamical system $(Z,P^*,G)$ is $m$-proximal. By lemma 7.1 the dynamical system $(Z,G)$ is WAP and therefore, by lemma 7.3, it is the trivial one point system. Since the barycenter of $P^*$ is $\mu$, we have $P^* = \delta_\mu$, and it follows that $P^*$ as well as $\mu$ are $G$-invariant measures. □

8. The SAT property

The notion of SAT (strongly approximately transitive) dynamical systems was introduced by Jaworsky in [18], where he developed their theory for discrete groups. For these groups he shows that the stationary measure on the Poisson boundary is SAT. It was later used in a slightly stronger version (SAT*) by Kaimanovich [20] in order to study the horosphere foliation on a quotient of a CAT($-1$) space by a discrete group of isometries $G$, using the SAT* property on the boundary of $G$.

Definitions Let $G$ be a locally compact second countable topological group. We fix some right Haar measure $m = m_G$ and let $e$ be the identity element of $G$.

8.1. Definitions.

(1) A Borel $G$-space is a standard Borel space $(X, \mathcal{X}, G)$ with a Borel action $G \times X \to X$.

(2) A $G$-system is a Borel $G$-space $(X, \mathcal{X}, G)$ equipped with a probability measure $\mu$ whose measure class is preserved by each element of $G$.

(3) We say that a Borel probability measure $\mu$ on a Borel $G$-space is strongly approximately transitive (SAT) if the measure class of $\mu$ is preserved by each element of $G$ and:

For every $A \in \mathcal{X}$ with $\mu(A) > 0$ there is a sequence $g_n \in G$ such that

$$\lim_{n \to \infty} \mu(g_n A) = 1.$$ 

When $\mu$ is SAT we will say that the system $(X, \mathcal{X}, \mu, G)$ is SAT.

(4) If $Y$ is a compact metric space, $G$ acts on $Y$ via a continuous representation of $G$ into Homeo($Y$) and $\nu$ is a Borel probability measure whose measure class is preserved by each element of $G$, we will say that the dynamical system $(Y, \mathcal{B}(Y), \nu, G)$ is topological ($\mathcal{B}(Y)$ denotes the Borel field on $Y$).

(5) Let $(X, \mathcal{X}, \mu, G)$ be a $G$-system. A topological $G$-system $(Y, \mathcal{B}(Y), \nu, G)$ is a topological model for $(X, \mathcal{X}, \mu, G)$ if $\text{supp}(\nu) = Y$ and $(X, \mathcal{X}, \mu, G)$ and $(Y, \mathcal{B}(Y), \nu, G)$ are isomorphic as $G$ measure boolean algebras; i.e. there is an equivariant isomorphism between the corresponding measure algebras.
(6) Recall that for a topological system \((Y, G)\), we say that a probability measure \(\nu\) on \(Y\) is **contractible** if for every \(y \in Y\) there exists a sequence \(g_n \in G\) such that, in the weak* topology, \(\lim_{n \to \infty} g_n\nu = \delta_y\).

(7) Given a Borel system \((X, \mathcal{X}, G)\) we say that a probability measure \(\mu\) on \(X\) is **absolutely contractible** if for each topological model \((Y, \nu, G)\) of \((X, \mathcal{X}, \mu, G)\) the measure \(\nu\) on \(Y\) is contractible.

Our goal is to show that a measure \(\mu\) is **SAT** iff it is absolutely contractible.

**Contractible topological systems**

We now use a second characterization of contractible measures ([1], theorem I.2, page 11).

8.2. **Theorem.** Let \((Y, G)\) be a topological dynamical system with \(Y\) a compact metric space. Let \(\nu\) be a probability measure on \(Y\). The following properties are equivalent:

1. The measure \(\nu\) is contractible; i.e., for every \(y \in Y\), the measure \(\delta_y\) is a weak* limit point of the set \(\{g\nu : g \in G\}\) in \(M(Y)\).
2. The linear operator \(P_\nu : C(Y) \to \text{LUC}(G)\) defined by

   \[
   P_\nu f(g) = \int_Y f(gy) \, d\nu(y),
   \]

   is an isometry of the Banach space \(C(Y)\) of continuous functions on \(Y\) into the Banach space \(\text{LUC}(G)\) of bounded left uniformly continuous functions on \(G\) (with sup-norm).

We note that the operator \(P_\nu\) can be extended to the larger Banach space \(L^\infty(Y, \nu)\) using the same formula (8.1), and since for \(f \in L^\infty(Y, \nu)\) and \(g, h \in G\)

\[
|P_\nu f(g) - P_\nu f(h)| = |\langle f, g\nu - h\nu \rangle| \\
\leq \|f\|_\infty \|g\nu - h\nu\|_{\text{total variation}} \\
= \|f\|_\infty \|\nu^{-1}g\nu - h\nu\|,
\]

we conclude that \(P_\nu(L^\infty(Y, \nu)) \subset \text{LUC}(G)\).

**G-continuous functions**

Recall the following definition and representation theorem from [17].

8.3. **Definition.** Given a \(G\)-system \((X, \mathcal{X}, \mu, G)\), a function \(f \in L^\infty(X, \mu)\) is called \(G\)-**continuous** if \(f \circ g_n\) converges in norm to \(f\) in \(L^\infty(X, \mu)\) whenever \(g_n \to e\).

8.4. **Theorem.** Let \(G\) be a Polish topological group. A boolean \(G\) system admits a topological model if and only if there exists a sequence of \(G\)-continuous functions that generates the \(\sigma\)-algebra (equivalently: separates points).

We first remark that although in [17] the boolean system is assumed to be measure preserving the proof, in fact, goes through if one assumes only that the **measure class** is preserved. Next we note that the condition in the theorem of admitting a sequence of \(G\)-continuous functions that generates the \(\sigma\)-algebra, is always satisfied when the group \(G\) is, in addition, locally compact (see corollary 8.7 below). Thus, in this case, one recovers the classical result that ensures the existence of a topological
model for every measure class preserving boolean action of a locally compact second countable group.

More importantly, we observe that an immediate corollary of the proof of theorem 2.2 in [17] is the following version of the theorem (still for a general Polish topological group $G$).

8.5. Theorem. Let $(X, \mathcal{X}, \mu, G)$ be a boolean system which satisfies the condition of theorem 8.4 and let $f$ be a function in $L^\infty(X, \mu)$. Then there exists a topological model $(Y, \nu, G)$ such that the function $F \in L^\infty(Y, \nu)$ corresponding to $f$ is in $C(Y)$ iff $f$ is $G$-continuous.

Thus when $G$ is a locally compact second countable group, theorem 8.5 applies for every $G$ system.

8.6. Lemma. Let $(X, \mathcal{X}, \mu, G)$ be a $G$-system and $f \in L^\infty(X, \mu)$. Let $\psi : G \to \mathbb{R}$ be a non-negative continuous function with compact support. Define $\hat{f} = f * \psi$ by

$$\hat{f}(x) = \int_G f(hx)\psi(h)\,dm_G(h).$$

The function $\hat{f}$ is $G$-continuous.

Proof. For $g \in G$ we have

$$\|\hat{f} \circ g - \hat{f}\|_\infty = \text{ess- sup}_{x \in X} \left| \int_G f(hgx)\psi(h)\,dm_G(h) - \int_G f(hx)\psi(h)\,dm_G(h) \right|$$

$$= \text{ess- sup}_{x \in X} \left| \int_G f(hx)\psi(hg^{-1})\,dm_G(h) - \int_G f(hx)\psi(h)\,dm_G(h) \right|$$

$$\leq \text{ess- sup}_{x \in X} \int_G |f(hx)||\psi(hg^{-1}) - \psi(h)|\,dm_G(h)$$

$$\leq \|f\|_\infty \int_G |\psi(hg^{-1}) - \psi(h)|\,dm_G(h).$$

Thus $g \to e$ implies $\|\hat{f} \circ g - \hat{f}\|_\infty \to 0$ and $\hat{f}$ is $G$-continuous. \square

We let $\{\psi_n : n = 1, 2, \ldots\}$ be a fixed approximate identity. This means that there is a decreasing sequence $V_n$ of precompact neighborhoods of $e$ in $G$ with $\cap_{n=1}^\infty V_n = \{e\}$, and $\psi_n : G \to \mathbb{R}$ is a sequence of nonnegative continuous functions with $\text{supp} \, \psi_n \subset V_n$ and $\int_G \psi_n\,dm_G = 1$ for $n = 1, 2, \ldots$.

8.7. Corollary. The bounded $G$-continuous functions are dense in $L^2(X, \mu)$.

Proof. It is easy to check that a sequence $\{\psi_n : n = 1, 2, \ldots\}$ as above is an approximate identity in $L^2(X, \mu)$; i.e. $\|f * \psi_n - f\|_2 \to 0$ for every bounded $f \in L^2(X, \mu)$. Now apply lemma 8.6. \square

8.8. Proposition. Let $(X, \mathcal{X}, \mu, G)$ be a $G$-system and $0 \neq f = 1_A \in L^\infty(X, \mu)$. Let $\psi_n : G \to \mathbb{R}$ be an approximate identity in $L^2(X, \mu)$ as above. Then

$$\lim_{n \to \infty} \|f * \psi_n\|_\infty = \|f\|_\infty = 1.$$
Proof. With no loss in generality we can assume that \((X, \mu, G)\) is a topological model. By the regularity of the measure \(\mu\) we can also assume (by passing to a subset) that \(A\) is closed. Again by regularity of \(\mu\), given \(\epsilon > 0\) we can choose an open neighborhood \(U\) of \(A\) in \(X\) such that \(\mu(U \setminus A) < \epsilon\). Since
\[
\lim_{h \to e} \mu(hA \triangle A) = \lim_{h \to e} \|1_{hA} - 1_A\|_1 = 0,
\]
we can choose a neighborhood \(V = V^{-1}\) of \(e\) in \(G\) such that for all \(h \in V\)
\[(i) \quad \|1_{hA} - 1_A\|_1 = \mu(hA \triangle A) < \epsilon, \quad \text{and} \quad (ii) \quad hA \subset U.\]
Let \(\psi = \psi_n\) be a member of the approximate identity which satisfies \(\text{supp } (\psi) \subset V\).
Set
\[
\hat{f}(x) = f * \psi = \int_G f(hx)\psi(h) \, dm_G(h) = \int_G f(hx) \, dp(h),
\]
where \(dp = \psi \cdot dm_G\), a probability measure on \(G\). By (ii), if \(x \not\in U\) then \(f(hx) = 1_A(hx) = 0\) for every \(h \in V\) and it follows that \(\hat{f}(x) = 0\). Thus
\[(8.2) \quad \int_X \hat{f} \, d\mu(x) = \int_U \hat{f} \, d\mu(x).\]
By Fubini and the estimation (i),
\[(8.3) \quad \int_X \hat{f} \, d\mu(x) = \int_X \mu(hA) \, dp(h) \geq \mu(A) - \epsilon.\]
For \(\delta > 0\) let
\[
D = D_\delta = \{x \in U : \hat{f}(x) < 1 - \delta\}.
\]
Then
\[
\mu(A) - \epsilon \leq \int_X \hat{f} \, d\mu(x) = \int_U \hat{f} \, d\mu(x) = \int_D \hat{f} \, d\mu(x) + \int_{U \setminus D} \hat{f} \, d\mu(x) \leq (1 - \delta)\mu(D) + \mu(U \setminus D) = -\delta\mu(D) + \mu(U) \leq -\delta\mu(D) + \mu(A) + \epsilon,
\]
hence \(\mu(D) \leq \frac{2\epsilon}{\delta}\). Fixing \(\delta\) at the outset we choose \(\epsilon\) so that, say, \(\frac{2\epsilon}{\delta} \leq \frac{1}{2}\mu(A)\), and then for sufficiently large \(n\),
\[
\hat{f}(x) = f * \psi_n(x) \geq 1 - \delta,
\]
for every \(x\) in the set \(U \setminus D\) whose measure \(\mu(U \setminus D) \geq \frac{1}{2}\mu(A) > 0\). This completes the proof of the proposition. \(\square\)

Sat and absolute contractibility are equivalent

8.9. Theorem. Let \((X, \mathcal{X}, \mu, G)\) be a \(G\) system, then \(\mu\) is SAT iff it is absolutely contractible.
Proof. Let \((Y, \nu, G)\) be a compact model and let \(f \in L_\infty(\nu)\) with \(\|f\|_\infty = 1\) and \(\epsilon > 0\) be given. Set \(A = \{y \in Y : |f(y)| \geq 1 - \epsilon\}\). Then \(\nu(A) > 0\) and we now assume that also \(\nu(A^+) > 0\) where \(A^+ = \{y \in Y : f(y) \geq 1 - \epsilon\}\). By assumption there is a sequence \(g_n \in G\) such that \(\lim_{n \to \infty} \nu(g_n^{-1}A^+) = 1\). Hence

\[
P_\nu f(g_n) = \int_{g_n^{-1}A^+} f(g_ny) \, d\nu(y) + \int_{g_n^{-1}A^c} f(g_ny) \, d\nu(y)
\]

\[
\geq (1 - \epsilon)\nu(g_n^{-1}A^+) - \nu(g_n^{-1}A^+) \to 1 - \epsilon.
\]

Hence \(\limsup_{n \to \infty} P_\nu f(g_n) \geq 1 - \epsilon\). Similarly when \(\nu(A^-) > 0\) with \(A^- = \{y \in Y : f(y) \leq -1 + \epsilon\}\) we get \(\limsup_{n \to \infty} |P_\nu f(g_n)| \geq 1 - \epsilon\). Thus the LUC(G) norm \(\|P_\nu f\| \geq 1 - \epsilon\) and as this holds for every \(\epsilon\) we get \(\|P_\nu f\| = 1\). Thus \(P_\nu : L_\infty(Y, \nu) \to \text{LUC}(G)\) is an isometry. In particular \(P_\nu : C(Y) \to \text{LUC}(G)\) is an isometry and by theorem 8.2 \((Y, \nu, G)\) is contractible.

Conversely, assume that \(\mu\) is absolutely contractible and let \(A \in \mathcal{X}\) be a set with positive \(\mu\) measure. Write \(f = 1_A\). By proposition 8.8 given \(\epsilon > 0\), we can choose \(\psi = \psi_n : G \to \mathbb{R}\), a function in the approximate identity, such that for \(\hat{f} = f * \psi\),

\[
\|\hat{f}\|_\infty - 1 = \|\hat{f}\|_\infty - \|f\|_\infty < \epsilon.
\]

By lemma 8.6 the function \(\hat{f}\) is \(G\)-continuous and by theorem 4.3 there is a topological model \((Y, \nu, G)\) for \((X, \mathcal{X}, \mu, G)\) in which the \(L_\infty(Y, \nu)\) function corresponding to \(\hat{f}\), say \(F\), is in \(C(Y)\). By assumption the measure \(\nu\) on \(Y\) is contractible and thus by theorem 8.2 \(P_\nu(F) = P_\mu(\hat{f}) \in \text{LUC}(G)\) satisfies

\[
\|P_\mu(\hat{f})\| = \|F\| = \|\hat{f}\|_\infty.
\]

Let \(g \in G\) satisfy

\[
\|P_\mu(\hat{f})\| < P_\mu(\hat{f})(g) + \epsilon.
\]

Now

\[
P_\mu(\hat{f})(g) = \int_X \hat{f}(gx) \, d\mu(x)
\]

\[
= \int_X \int_G f(hgx)\psi(h) \, dm(h) \, d\mu(x)
\]

\[
= \int_G \psi(h) \left(\int_X f(hgx) \, d\mu(x)\right) \, dm(h)
\]

\[
= \int_G \psi(h)P_\mu(f)(hg) \, dm(h),
\]

and, since \(\psi \geq 0\) and \(\int_G \psi \, dm = 1\), it follows that for some \(h \in G\)

\[
P_\mu(f)(hg) > P_\mu(\hat{f})(g) - \epsilon.
\]

Collecting the estimations (8.4), (8.5) and (8.6) we get

\[
P_\mu(f)(hg) > P_\mu(\hat{f})(g) - \epsilon > \|P_\mu(\hat{f})\| - 2\epsilon = \|\hat{f}\|_\infty - 2\epsilon > 1 - 3\epsilon.
\]

Explicitly

\[
P_\mu(f)(hg) = \int_X f(hgx) \, d\mu(x) = \mu(hgA) > 1 - 3\epsilon
\]
and the proof is complete. □

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