Study of Results of Katugampola Fractional Derivative and Chebyshev Inequalities

Nazakat Nazeer¹ · Muhammad Imran Asjad² · Muhammad Khursheed Azam² · Ali Akgül³

Accepted: 18 July 2022
© The Author(s), under exclusive licence to Springer Nature India Private Limited 2022

Abstract
In this work, we use generalized form of Caputo-type fractional derivative and Riemann–Liouville fractional Integral which is known as Katugampola fractional derivative. This work deals with some results having applications of Katugampola fractional derivative. We discuss commutative and inverse property of Katugampola fractional derivative. We have also introduced Chebyshev inequalities and some other integrals inequalities applying the Katugampola fractional derivative.

Keywords Katugampola fractional derivative · Mellin transformation · Synchronous function · Chebyshev inequalities

Introduction
Calculus has vital role in the field of sciences. Fractional calculus gives complete expansion of derivative and integration. We know about the derivative and anti-derivative with positive integers but in fractional calculus studied about derivative and anti-derivative of fractional order. Machado [1] discuss some application of fractional calculus. Some important applications and useful techniques of fractional calculus, we refer the reader [2–10]. Sun et al. [11] to know the several known form and modification of fractional calculus. In [12–14] had studied the behavior of fractional calculus in different fields of sciences. Machado [15] in their work told that system had memory reason being well overview upon fractional calculus. Caputo and Fabrizio [16] also tell about the nature and useful application of fractional calculus. In [17, 18] told about the true value and use of fractional calculus. Rodriguezes [19] showed the application FC in the field Bio-Mathematics and also described the modeling of covid-19 pandemic. In [18, 20, 21] told about important operator of fractional calculus. K-Gamma and extention of the gamma function introduced by Daiz and pariguan [22].
Some Ostrowski type integral inequalities discussed by Huseyin et al. [23]. In this paper application of Ostrowski type integral derived with help of Taylor expansion and Trapezoid type inequalities studied by Fuat et al. [24]. They discussed Trapezoid type inequalities by the contemporary theory of inequalities. Fuat Usta and Mehmet Sarikaya [25] made the extension of Gronwall, Pachpatte and Volterra results which are suitable for the differential equations. The study if generalized of midpoint type inequalities by application of generalized fractional integral operators done by Huseyin et al. [26]. Fuat Usta and Mehmet Sarikaya [27] tried to set up and conformable some latest inequalities. They also worked on old inequalities like Grownwall type which has huge application area in differential equations.

Preliminaries

In this section, we will discuss notions of k-gamma function, k-beta function, Millen transformation, Katugampola fractional integral, and Katugampola fractional derivative.

Definition 1 [22] The k-gamma function written as:

\[
\Gamma_k(n) = \int_0^\infty r^{n-1}e^{-\frac{r}{k}}dr, \quad \text{Re}(n) > 0, \quad k > 0.
\]  

Definition 2 [22] The k-beta function written as:

\[
\beta_k(e,f) = \int_0^1 (1-x)^{\frac{e}{k}-1}x^{\frac{f}{k}-1}dx, \quad \text{Re}(f) > 0, \quad k > 0.
\]

Definition 3 [6] The Katugampola Fractional Integral can be written as:

Let \( h(v) \) be continuous function \([0,\infty)\) and \( \alpha \in (0,1), p > -1 \) Then \( \forall 0 < a < t \).

\[
(pI_a^\alpha h)(v) = \frac{p^{1-a}}{\Gamma(a)} \int_a^v \frac{t^{p-1}}{(v^p - t^p)^{1-a}}h(t)dt.
\]

Definition 4 [6] The Katugampola Fractional Derivative can be written as:

Let \( h(v) \) be continuous function \([0,\infty)\) and \( \alpha \in (0,1), p > -1 \) Then \( \forall 0 < a < t \).

\[
(pD_a^\alpha h)(v) = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(v^{1-p} \frac{d}{dv} \right)^n \frac{1}{v^p - t^p} \int_a^v \frac{t^{p-1}}{(v^p - t^p)^{\alpha-n+1}}h(t)dt.
\]

Definition 5 [8] The Mellin Transformation can be written as:

The Mellin Transformation of a real function \( h(z) \) is write as

\[
h^*(p) = M(h(v)) = \int_0^\infty v^{p-1}h(v)dv,
\]

Whenever \( h^*(p) \) exist \( p \in C, Re(p) > 0 \).
Main Results and Discussion

**Theorem 1**  Let \( g \) be continuous on \([0, \infty)\) and let \( a, p \in \mathbb{R} \) and \( p \in \mathbb{N} \). Then \( \forall 0 < a < t \)

\[
M(p \, D^\alpha_a g) v = \frac{\Gamma(\alpha - n + 1 - \frac{p_1}{p})}{\Gamma(1 - \frac{p_1}{p})} p^{a-n} \left( x^{1-p} \frac{dx}{dx} \right)^n w^*(p_1 + pn - p\alpha). \tag{2.1}
\]

**Proof** Using Eq. (1.5)

\[
M(p \, D^\alpha_a g) v = \int_0^\infty x^{p_1-1} (p \, D^\alpha_a g) x \, dx,
\]

Now using Eq. (1.4)

\[
M(p \, D^\alpha_a g) v = \frac{p^{a-n+1}}{\Gamma(n - \alpha)} \left( x^{1-p} \frac{dx}{dx} \right)^n \int_0^\infty x^{p_1-1} \left[ \int_a^t t^{p-1} g(t) \left( x^{p_1-1} (x^{p} - t^{p})^{a+n-1} \right) dt \right] dx, \tag{2.2}
\]

By using Fubini’s theorem

\[
M(p \, D^\alpha_a g) v = \frac{p^{a-n+1}}{\Gamma(n - \alpha)} \left( x^{1-p} \frac{dx}{dx} \right)^n \int_0^\infty \int_a^t t^{p-1} g(t) \left[ \int_a^t z^{a-n+1 - \frac{p_1}{p}} - (1 - z)^{a-1} \right] dz dt, \tag{2.3}
\]

Substituting

\[
x^p = \frac{t^p}{z}, \tag{2.4}
\]

Using (2.4)

\[
M(p \, D^\alpha_a g) v = \frac{p^{a-n}}{\Gamma(n - a)} \left( x^{1-p} \frac{dx}{dx} \right)^n \int_0^\infty t^{p_1+pn-pn-1} g(t) \left[ \int_0^1 z^{a-n+1 - \frac{p_1}{p}} - (1 - z)^{a-1} \right] dz dt,
\]

\[
M(p \, D^\alpha_a g) v = \frac{\Gamma(\alpha - n + \frac{p_1}{p} + 1)}{p_1 - 1} p^{a-n} \left( x^{1-p} \frac{dx}{dx} \right)^n \int_0^\infty t^{p_1+pn-pn-1} g(t) dt.
\]

Using Eq. (1.5), we get result. \(\Box\)

**Theorem 2**  Let \( f \) be continuous function on \([0, \infty)\) and let \( \alpha, p \in \mathbb{R}, n \in \mathbb{N} \). Then \( \forall 0 < \alpha < t \)

\[
M[p \, D^\alpha_a (t^{p\alpha} f)(v)] = \frac{\Gamma(\alpha - n + \frac{p_1}{p} + 1)}{\Gamma(1 - \frac{p_1}{p})} p^{a-n} f^*(p\alpha + p_1). \tag{2.5}
\]

**Proof** Using Eq. (1.5)

\[
M[p \, D^\alpha_a (t^{p\alpha} f)(v)] = \int_0^\infty x^{p_1-1} [p \, D^\alpha_a (x^{p\alpha} f(x))] dx,
\]

Using the Eq. (1.4)
\[
M^{[p\,D_a^n]}\left[t^{p\alpha} f(v)\right] = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-p} \frac{d}{dx}\right)^n \int_0^\infty \int_0^{\infty} \frac{1}{\int_0^{\infty} x^{p-1}(x^p - t^{p})^{-\alpha+n+1} \, dx} \, dr,
\]

\[
M^{[p\,D_a^n]}\left[t^{p\alpha} f(v)\right] = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-p} \frac{d}{dx}\right)^n \int_0^\infty \int_0^{\infty} \frac{1}{\int_0^{\infty} x^{p-1}(x^p - t^{p})^{-\alpha+n+1} \, dx} \, dr.
\]

Substituting
\[
x^p = \frac{t^p}{w},
\]

Using Eq. (2.7)
\[
p^{[p\,D_a^n]}\left[t^{p\alpha} f(v)\right] = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-p} \frac{d}{dx}\right)^n \int_0^\infty \int_0^{\infty} \frac{1}{\int_0^{\infty} x^{p-1}(x^p - t^{p})^{-\alpha+n+1} \, dx} \, dr,
\]

\[
M^{[p\,D_a^n]}\left[t^{p\alpha} f(v)\right] = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-p} \frac{d}{dx}\right)^n \int_0^\infty \int_0^{\infty} \frac{1}{\int_0^{\infty} x^{p-1}(x^p - t^{p})^{-\alpha+n+1} \, dx} \, dr.
\]

After simplification, we get result.

**Theorem 3** Let \( \alpha \in R, \, p \in Rn \in N, \, \forall 0 < \alpha < v. \) Then we have
\[
M^{[p\,D_a^n]}\left[v^{p(\alpha-n)}\right] = \frac{\Gamma(\alpha - n - \frac{p_1}{p} - 1) p^{\alpha-n}}{\Gamma(1 - \frac{p_1}{p})} \left(x^{1-p} \frac{d}{dx}\right)^n g^*(p_1).
\]

**Proof** Using Eq. (1.5)
\[
M^{[p\,D_a^n]}\left[v^{p(\alpha-n)}\right] = \int_0^\infty x^{p_1-1} p^{[p\,D_a^n]}\left[x^{p(\alpha-n)}\right] \, dx.
\]

Using Eq. (1.4)
\[
M^{[p\,D_a^n]}\left[v^{p(\alpha-n)}\right] = \int_0^\infty x^{p_1-1} \left[\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-p} \frac{d}{dx}\right)^n \int_0^{\infty} \frac{1}{\int_0^{\infty} x^{p-1}(x^p - t^{p})^{-\alpha+n+1} \, dx} \, dt\right] \, dx.
\]

By fubinis theorem
\[
M^{[p\,D_a^n]}\left[v^{p(\alpha-n)}\right] = \frac{p^{\alpha-n+1}}{\Gamma(\alpha - n)}\left(x^{1-p} \frac{d}{dx}\right)^n \int_0^{\infty} \int_0^{\infty} \frac{1}{\int_0^{\infty} x^{p-1}(x^p - t^{p})^{-\alpha+n+1} \, dx} \, dr.
\]
Substitution

\[ x^p = \frac{t^p}{y}, \quad (2.10) \]

Using Eq. (2.10)

\[
M[p D^\alpha_a v^{p(\alpha-n)}] = \frac{p^{\alpha-n+1}}{\Gamma(\alpha-n)} \left( x^{1-p} \frac{d}{dx} \right)^n \int_0^\infty t^{p-1+\alpha-\alpha n} \left[ \int_0^1 \left( \frac{1}{y} - t \right)^{\alpha-n+1} \left( ty^{\frac{1}{p}} \right)^{(p-1)p} t^{\frac{1}{p} y dy} \right] dt,
\]

\[
M[p D^\alpha_a v^{p(\alpha-n)}] = \frac{p^{\alpha-n}}{\Gamma(\alpha-n)} \left( x^{1-p} \frac{d}{dx} \right)^n \int_0^\infty t^{p-1+\alpha-\alpha n+p-1+\alpha-\alpha n} f(t) dt \left[ \int_0^1 (1-y)^{\alpha-n+1} y^{\alpha-n+1-p1/p-1} dy \right].
\]

Using Eq. (1.2), we get result.

\[ \square \]

**Theorem 4** Let \( f \) be continuous on \([0, \infty)\) and let \( \alpha \in (0, 1) \), \( P \geq 0 \). Then

\[ p D^\alpha_a [p D^\beta_a f(x)] = p D^{\alpha+\beta}_a f(x). \quad (2.11) \]

**Proof** Using Eq. (1.4)

\[
p D^\alpha_a [p D^\beta_a f(x)] = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( x^{1-p} \frac{d}{dx} \right)^n \int_a^x \frac{v^{p-1}}{(x^p - v^p)^{n-\alpha+1}} [p D^\beta_a f(v)] dv, \quad (2.12)
\]

\[
p D^\alpha_a [p D^\beta_a f(x)] = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( x^{1-p} \frac{d}{dx} \right)^n \int_a^x \frac{v^{p-1}}{(x^p - v^p)^{n-\alpha+1}} \left[ \frac{p^{\beta-n+1}}{\Gamma(n-\beta)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{u^{p-1}}{u^p - \tau^p} f(\tau) d\tau \right] dv.
\]

Using fubini’s theorem

\[
p D^\alpha_a [p D^\beta_a f(x)] = \frac{p^{\alpha-n+1+\beta-n+1}}{\Gamma(n-\beta) \Gamma(n-\alpha)} \left( x^{1-p} \frac{d}{dx} \right)^n \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \tau^{n-1} f(\tau) \times \left[ \int_{\tau}^v (x^p - v^p)^{1-\alpha+n-1} (v^{p-1} - \tau^{p-1})^{-\beta+n-1} dv \right] d\tau, \quad (2.13)
\]

Substitution

\[ y = \frac{v^p - \tau^p}{x^p - \tau^p}, \quad (2.14) \]
Using (2.14)
\[
\int_{x}^{\infty} (x^p - v^p)^{\alpha-1}(v^p - \tau^p)^{\beta-1} v^p dv = \frac{(x^p - \tau^p)^{2n-\alpha-\beta-1}}{p} \int_{0}^{1} (1 - y)^{n-\alpha-1} y^{n-\beta-1} dy,
\]
Using (2.15) in (2.13)
\[
p D_a^\beta [p D_a^\alpha f(x)] = \frac{p^{\alpha+\beta-2n+2}}{\Gamma(n - \alpha) \Gamma(n - \beta)} \left( x^{1-p} \frac{d}{dx} \right)^n \left( v^{1-p} \frac{d}{dv} \right)^n \frac{1}{p} \int_{a}^{x} (x^p - \tau^p)^{2n-\alpha-\beta-1} \tau^{p-1} f(\tau) d\tau,
\]
\[
p D_a^\alpha [p D_a^\beta f(x)] = \frac{p^{\alpha+\beta-2n+1}}{\Gamma(n - \alpha) \Gamma(n - \beta)} \left( x^{1-p} \frac{d}{dx} \right)^n \left( v^{1-p} \frac{d}{dv} \right)^n \int_{a}^{x} (x^p - \tau^p)^{2n-\alpha-\beta-1} \tau^{p-1} f(\tau) d\tau,
\]
\[
p D_a^\alpha [p D_a^\beta f(x)] = p D_a^{\alpha+\beta} f(x).
\]
This is completes proof.

\textbf{Corollary 5} Let \( f(x) \) be continuous on \([0, \infty) \) and let \( \alpha \in (0, 1), P \geq 0. \) Then
\[
p D_a^\alpha[p D_a^\beta f(x)] = p D_a^\beta[p D_a^\alpha f(x)].
\]
(Commutative result).

\textbf{Theorem 6} Let \( f \) be continuous on \([0, \infty) \) and let \( \alpha, \beta > 0, \) and \( P \geq 0. \) Then\( x > a > 0. \)
\[
p D_a^\alpha [v^p - a^p]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(n - \alpha + \beta)} p^{\alpha-n} v \left( v^{1-p} \frac{d}{dv} \right)^n (v^p - a^p)^{n-\alpha+\beta-1}.
\]
\textbf{Proof} Using Eq. (1.4)
\[
p D_a^\alpha [v^p - a^p]^{\beta-1} = \frac{p^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_{a}^{x} (v^p - t^p)^{\alpha-n+1} (t^p - a^p)^{\beta-1} f(t) dt,
\]
\[
p D_a^\alpha [v^p - a^p]^{\beta-1} = \frac{p^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_{a}^{x} (v^p - t^p)^{\alpha-n+1} (t^p - a^p)^{\beta-1} t^{p-1} dt,
\]
Substitution
\[
y = (v^p - t^p)/(v^p - a^p),
\]
Using (2.18)
\[
p D_a^\alpha [(v^p - a^p)^{\beta-1}] = \frac{p^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_{0}^{1} ((v^p - a^p)(1 - y))^{\beta-1} ((v^p - a^p)y)^{n-\alpha-1} (v^p - a^p) dy,
\]
\[
p D_a^\alpha [(v^p - a^p)^{\beta-1}] = \frac{p^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n (v^p - a^p)^{\alpha+\beta-1} \int_{0}^{1} (1 - y)^{\beta-1} y^{n-\alpha-1} dy,
\]
\[
p D_a^\alpha [(v^p - a^p)^{\beta-1}] = \frac{\Gamma(\beta)}{\Gamma(n - \alpha + \beta)} p^{\alpha-n} \left( v^{1-p} \frac{d}{dv} \right)^n (v^p - a^p)^{n-\alpha+\beta-1}.
\]
This is completing proof.
**Theorem 8** Let \( f \) be a continuous on \([0, \infty)\) and let \( \alpha, \beta, p \in \mathbb{R}, n \in \mathbb{N}. \) Then \( \forall x > a > 0. \)

\[
p^{\beta}D^\alpha_a[p^{\beta}I^\alpha_a f](x) = f(x). \tag{2.19}
\]

**Inverse property**

**Proof** Using Eq. (1.4)

\[
p^{\beta}D^\alpha_a[p^{\beta}I^\alpha_a f](x) = \frac{p^{\alpha - n + 1}}{\Gamma(n - a)} \left( x^{1 - p} \frac{d}{dx} \right)^n \int_a^x \frac{y^{p - 1}}{(x^p - y^p)^{\alpha - n + 1}} (p^{\beta}I^\alpha_a f)(y) dy,
\]

Using Eq. (1.3) and Fubini’s theorem

\[
p^{\beta}D^\alpha_a[p^{\beta}I^\alpha_a f](x) = \frac{p^{\alpha - n + 1}}{\Gamma(n - a)} \left( x^{1 - p} \frac{d}{dx} \right)^n \int_a^x \frac{y^{p - 1}}{(x^p - y^p)^{\alpha - n + 1}} \left[ \frac{p^{\alpha - n + 1}}{\Gamma(\alpha)} \int_a^t (y^p - t^p)^{\alpha - 1} f(t) dt \right] dy.
\]

Substitution

\[
z = \frac{(y^p - t^p)}{(x^p - t^p)}, \tag{2.20}
\]

Using equation (2.20)

\[
p^{\beta}D^\alpha_a[p^{\beta}I^\alpha_a f](x) = \frac{p^{\alpha - n + 1}}{\Gamma(n - a)} \left( x^{1 - p} \frac{d}{dx} \right)^n \int_a^x \frac{y^{p - 1}}{(x^p - y^p)^{\alpha - n + 1}} \left[ \frac{p^{\alpha - n + 1}}{\Gamma(\alpha)} \int_a^t (1 - z^p)^{1 - (\alpha - n + 1)} dz \right] dr,
\]

Using the definition, we get result.

**Theorem 8** Let \( f \) a continuous on \([0, \infty)\) and let \( \alpha, \beta, p \in \mathbb{R}, n \in \mathbb{N}. \) Then \( \forall x > a > 0. \)

\[
p^{\beta}I^\alpha_a(p^{\beta}D^\alpha_a f)(x) = p^{\beta}D^\alpha_a[p^{\beta}f](x). \tag{2.21}
\]

**Proof** Using Eqs. (1.3) and (1.4).

\[
p^{\beta}I^\alpha_a(p^{\beta}D^\alpha_a f)(x) = \frac{p^{\alpha - n + 1}}{\Gamma(\beta)} \int_a^x (x^p - y^p)^{\beta - 1} y^{1 - p} \frac{d}{dy} \left( y^{1 - p} \frac{d}{dy} \right)^n \int_a^y (y^p - t^p)^{\alpha - 1} f(t) dt dy,
\]

By Fubini’s theorem we get

\[
p^{\beta}I^\alpha_a(p^{\beta}D^\alpha_a f)(x) = \frac{p^{\alpha - n + 1}}{\Gamma(\beta)\Gamma(n - a)} \left( y^{1 - p} \frac{d}{dy} \right)^n \int_a^x t^{p - 1} f(t) \int_a^y (x^p - y^p)^{\beta - 1} (y^p - t^p)^{\alpha - 1} y^{p - 1} dy dt.
\]

Substitution

\[
z = \frac{y^p}{x^p - t^p}, \tag{2.22}
\]

Using equation (2.22)

\[
p^{\beta}I^\alpha_a(p^{\beta}D^\alpha_a f)(x) = \frac{p^{\alpha - n + 1}}{\Gamma(\beta)\Gamma(n - a)} \left( y^{1 - p} \frac{d}{dy} \right)^n \int_a^x t^{p - 1} f(t) \int_a^y (x^p - t^p)^{\alpha - 1 + \beta - 1} f(t) dt.
\]
\[ p I_a^a \left( p D_a^a f(x) \right) = \frac{p^\alpha - \beta + 1}{\Gamma(n - \alpha + \beta)} \left( y^{1-p} \frac{d}{dy} \right)^n \int_a^x (x^{1-p} - t^{1-p})^{n-\alpha+\beta-1} t^{p-1} f(t) dt. \]

This gives the required result. \( \square \)

**Theorem 9** Let \( f \) be continuous on \([0, \infty)\) and let \( \alpha, p \in R, n \in N \).
Then \( \forall x > a > 0 \).
\[ p I_a^a \left( p D_a^a f \right) (x) = f(x) - f(a). \quad (2.23) \]

**Proof** Using Eqs. (1.3) and (1.4)
\[ p I_a^a \left( p D_a^a f \right) (x) = \frac{p^{1-a}}{\Gamma(a)} \int_a^x (x^p - y^p)^{a-1} \left( \frac{p^{a-1}}{\Gamma(n - \alpha)} \int_a^y (x^p - v^p)^{n-\alpha-1} v^{p-1} f(v) dv \right) dy. \]

Using Fubini’s theorem we get
\[ p I_a^a \left( p D_a^a f \right) (x) = \frac{p^{2-a}}{\Gamma(n - \alpha) \Gamma(a)} \left( y^{1-p} \frac{d}{dy} \right)^n \int_a^x v^{p-1} f(v) \int_a^x (x^p - y^p)^{a-1} (y^p - v^p)^{n-\alpha-1} dy dv. \]

Using Eq. (2.24)
\[ p I_a^a \left( p D_a^a f \right) (x) = \frac{p^{1-a}}{\Gamma(a) \Gamma(n - \alpha)} \left( y^{1-p} \frac{d}{dy} \right)^n \int_a^x (x^p - v^p)^{n-\alpha-1} v^{p-1} f(v) dv. \]

This gives the required result. \( \square \)

**Corollary 10** Let \( g \) and \( h \) be continuous on \([0, \infty)\) and let \( \alpha, \beta, p \in R, n \in N \).
Then \( \forall x > a > 0 \).
\[ p D_a^a [c_1 g(x) + c_2 h(x)] = [c_1 p D_a^a g(x) + c_2 p D_a^a h(x)]. \quad (2.25) \]

**Theorem 11** Let \( l \) and \( m \) be synchronous functions over \([0, \infty)\) then \( \forall v > a > 0, p > 0 \).
The given inequalities hold for Katugampola Fraction Derivative of order \( \alpha \in C Re(\alpha) > 0 \).
\[ p D_a^\alpha \left( p D_a^\alpha l \right) m \geq \left( p D_a^\alpha l \right) v \left( p D_a^\alpha m \right) v. \quad (2.26) \]
\[ \left( p D_a^\alpha l \right) v \left( p D_a^\alpha m \right) + p D_a^\alpha \left( 1 + p D_a^\alpha l \right) m \geq \left( p D_a^\alpha l \right) v \left( p D_a^\alpha m \right) v + \left( p D_a^\alpha l \right) v \left( p D_a^\alpha m \right) v. \quad (2.26A) \]

**Proof** We have
\[ [ l(x) - l(y) ][ m(x) - m(y) ] \geq 0, \quad (2.27) \]
Arranging form
\[ l(x) m(x) + l(y) m(y) \geq l(x) m(y) + l(y) m(x), \quad (2.28) \]
Multiplying Eq. (2.28) by $\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}(v^{1-p} \frac{d}{dv})^n \frac{x^{n-1}}{(v^{p}-x^{p})^{\alpha-n+\beta}}$ and integrating w.r.t ‘x’ over [a, v] on both sides

\[
\begin{align*}
\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}
&\left(v^{1-p} \frac{d}{dv}\right)^n \int_a^v \frac{x^{p-1}}{(v^{p}-x^{p})^{\alpha-n+1}} l(x)m(x)dx \\
+ \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}
&\left(v^{1-p} \frac{d}{dv}\right)^n \int_a^v \frac{x^{p-1}}{(v^{p}-x^{p})^{\alpha-n+1}} l(y)m(y)dy \\
\geq \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}
&\left(v^{1-p} \frac{d}{dv}\right)^n \int_a^v \frac{x^{p-1}}{(v^{p}-x^{p})^{\alpha-n+1}} m(x)l(x)dx,
\end{align*}
\]

Using Eq. (1.4), we get

\[
\begin{align*}
(p D_a^\alpha l)(v) + l(y)m(y)p D_a^\alpha (1) \\
\geq m(y)(p D_a^\alpha l)v + (p D_a^\alpha m)v l(y),
\end{align*}
\]

Multiplying Eq. (2.30) by $\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}(v^{1-p} \frac{d}{dv})^n \frac{y^{p-1}}{(v^{p}-y^{p})^{\alpha-n+\beta}}$ and integrating w.r.t ‘y’ over [a, v] on both sides

\[
\begin{align*}
(p D_a^\alpha l)(v) &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}
&\left(v^{1-p} \frac{d}{dv}\right)^n \int_a^v \frac{y^{p-1}}{(v^{p}-y^{p})^{\alpha-n+1}} dy \\
+ &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}
&\left(v^{1-p} \frac{d}{dv}\right)^n \int_a^v \frac{y^{p-1}}{(v^{p}-y^{p})^{\alpha-n+1}} l(y)m(y)dy \\
\geq (p D_a^\alpha l) &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}
&\left(v^{1-p} \frac{d}{dv}\right)^n \int_a^v \frac{y^{p-1}}{(v^{p}-y^{p})^{\alpha-n+1}} m(y)dy \\
+ (p D_a^\alpha m) &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)}
&\left(v^{1-p} \frac{d}{dv}\right)^n \int_a^v \frac{y^{p-1}}{(v^{p}-y^{p})^{\alpha-n+1}} l(y)dy.
\end{align*}
\]

Using Eq. (1.4) and simplifying we get result.

And similarly get (2.26A).

\[
\square
\]

**Theorem 12** Let land mbe synchronous functions over \([0, \infty]\) then $\forall v > a > 0, p > 0$. The given inequalities hold for Katugampola Fraction Derivative of order $\alpha \in CRe(a) > 0$.

\[
\begin{align*}
(p D_a^\alpha l^2) p D_a^\alpha (1) + (p D_a^\alpha m^2) p D_a^\alpha (1) \geq 2(p D_a^\alpha l)(v)(p D_a^\alpha m)v. \\
(p D_a^\alpha l^2) v(p D_a^\alpha m^2) v + (p D_a^\alpha l^2) (p D_a^\alpha m^2) v \geq 2(p D_a^\alpha lm)v(p D_a^\alpha lm)v.
\end{align*}
\]

**Proof** We have

\[
[l(x) - m(y)]^2 \geq 0,
\]

\[
\square
\]
\[ l(x)^2 + m(y)^2 \geq 2l(x)m(y). \] (2.34)

Multiplying (2.34) by \( \frac{\Gamma(\alpha - n + 1)}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \frac{x^{p-1}}{(v^p - x^p)^{\alpha-n+1}} \) and integrating w.r.t \( x' \) over \([a, v]\) on both sides

\[
\begin{align*}
\frac{p^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^p - x^p)^{\alpha-n+1}} l(x)^2 \, dx \\
+ \frac{p^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^p - x^p)^{\alpha-n+1}} m(y)^2 \, dx \\
\geq 2 \frac{p^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^p - x^p)^{\alpha-n+1}} l(x)m(y) \, dx,
\end{align*}
\]

\[(p D_a^\alpha l^2) v + m^2(y) p D_a^\alpha (1) \geq 2m(y)^p D_a^\alpha l(v). \] (2.35)

Multiplying Eq. (2.35) by \( \frac{\Gamma(\beta-n+1)}{\Gamma(n - \beta)} \left( v^{1-p} \frac{d}{dv} \right)^n \frac{y^{p-1}}{(v^p - y^p)^{\beta-n+1}} \) and integrating w.r.t \( y' \) over \([a, v]\) on both sides

\[
\begin{align*}
(p D_a^\alpha l^2) v^p \frac{p^{\beta-n+1}}{\Gamma(n - \beta)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{y^{p-1}}{(v^p - y^p)^{\beta-n+1}} dy \\
+ \frac{p^{\beta-n+1}}{\Gamma(n - \beta)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{y^{p-1}}{(v^p - y^p)^{\beta-n+1}} m(y) \, dy \left( p D_a^\alpha (1) \right) \\
\geq 2 \frac{p^{\beta-n+1}}{\Gamma(n - \beta)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{y^{p-1}}{(v^p - y^p)^{\beta-n+1}} m(y) \, dy \left( p D_a^\alpha l(v) \right). \] (2.36)

Using Eq. (1.4)

\[
(p D_a^\alpha l^2) v^p D_a^\alpha (1) + (p D_a^\beta m^2) v^p D_a^\alpha (1) \geq 2(p D_a^\alpha l) v^p D_a^\beta m(v). \]

This is completes proof. \(\square\)

Similarly, (2.33) can also be proof. \(\square\)

**Theorem 13** Let \( l \) and \( m \) be synchronus functions over \([0, \infty]\) then \( \forall v > a > 0, p > 0. \) The given inequalities hold for Katugampola Fraction Derivative of order \( \alpha \in C Re(\alpha) > 0. \)

\[
\begin{align*}
(p D_a^\alpha lmn) v^p D_a^\alpha (1) + & p D_a^\alpha (1) (p D_a^\alpha lmn) v \\
\geq & (p D_a^\alpha lmn) v (p D_a^\alpha n) v - (p D_a^\alpha ln) v (p D_a^\alpha m) v \\
+ & (p D_a^\alpha mn) v (p D_a^\alpha l) v + (p D_a^\beta nm) v (p D_a^\alpha l) v \\
- & (p D_a^\beta ln) v (p D_a^\beta m) v + (p D_a^\beta lm) v (p D_a^\beta n) v. \] (2.37)
\]

**Proof** We have

\[
\begin{align*}
[l(x) - l(y)][m(x) - m(y)][n(x) - n(y)] & \geq 0, \\
l(x)m(x)n(x) + l(y)m(y)n(y) & \geq l(x)m(x)n(y) - l(x)m(y)n(x)
\end{align*}
\]
\[ + l(x)m(y)n(y) + l(y)m(x)n(x) \]
\[ - l(y)m(x)n(y) + l(y)m(y)n(x), \quad (2.38) \]

Multiplying Eq. (2.38) by \( \frac{\alpha-n+1}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv})^n \left( \frac{x^{p-1}}{(v^{p-x} y)^{\alpha-n+1}} \right) \) and Integrating w.r.t ‘x’ over \([a, v]\) on both sides

\[
\begin{align*}
&\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^{p-x} y)^{\alpha-n+1}} l(x)m(x)n(x)dx \\
+ &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^{p-x} y)^{\alpha-n+1}} l(y)m(y)n(y)dx \\
\geq &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^{p-x} y)^{\alpha-n+1}} l(x)m(x)n(y)dx \\
- &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^{p-x} y)^{\alpha-n+1}} l(y)m(x)n(y)dx \\
+ &\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^{p-x} y)^{\alpha-n+1}} l(y)m(y)n(x)dx, \quad (2.39)
\end{align*}
\]

Using Eq. (1.4)

\[
\begin{align*}
&\left( p D_a^\alpha l m n \right) v + l(y)m(y)n(y) (p D_a^\alpha) (1) \\
\geq &\ n(y) \left( p D_a^\alpha l m \right) v - m(y) \left( p D_a^\alpha l n \right) v \\
+ &\ m(y)n(y) \left( p D_a^\alpha l \right) v + l(y) \left( p D_a^\alpha m \right) v \\
- &\ l(y)n(y) \left( p D_a^\alpha m \right) v + l(y) m(y) \left( p D_a^\alpha n \right) v. \quad (2.40)
\end{align*}
\]

Multiplying (2.40) by \( \frac{\beta-n+1}{\Gamma(n-\beta)} (v^{1-p} \frac{d}{dv})^n \left( \frac{y^p}{(v^{p-y} x)^{\alpha-n+1}} \right) \) and Integrating w.r.t ‘y’ over \([a, v]\) on both sides

\[
\begin{align*}
&\left( p D_a^\alpha l m n \right) v \frac{p^{\beta-n+1}}{\Gamma(n-\beta)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{y^{p-1}}{(v^{p-x} y)^{\beta-n+1}} dy \\
+ &\ p D_a^\alpha (1) \frac{p^{\beta-n+1}}{\Gamma(n-\beta)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{y^{p-1}}{(v^{p-x} y)^{\beta-n+1}} l(y)m(y)n(y)dy
\end{align*}
\]
Corollary 14  Let $f$, $g$ and $h$ be synchronous functions over $[0, \infty[$ then $\forall t > a > 0$, $p > 0$. The following inequalities holds for Katugampola Fraction Derivative of order $\alpha \in C Re(\alpha) > 0$.

\[
(P D^\alpha_a fgh) v^p D^\alpha_a (1) \geq (P D^\alpha_a f) v^p D^\alpha_a gh)v - (P D^\alpha_a f) v^p D^\alpha_a g)v + (P D^\alpha_a f) v^p D^\alpha_a h)v.
\]

(2.42)

Theorem 15  Let $l$ and $n$ be synchronous functions over $[0, \infty[$ then $\forall v > a > 0$, $p > 0$. The given inequalities hold for Katugampola Fraction Derivative of order $\alpha \in C Re(\alpha) > 0$.

\[
(P D^\alpha_a lmn) v^p D^\alpha_a (1) - (P D^\alpha_a lmn) v^p D^\alpha_a (1) \geq (P D^\alpha_a l) v^p D^\alpha_a n)v + (P D^\alpha_a l) v^p D^\alpha_a m)v - (P D^\alpha_a l) v^p D^\alpha_a mn) v + (P D^\alpha_a l) v^p D^\alpha_a l)v - (P D^\alpha_a l) v^p D^\alpha_a ln) v - (P D^\alpha_a mn) v^p D^\alpha_a l)v.
\]

(2.43)

Proof  We have

\[
[l(x) - l(y)][n(x) - n(y)][m(x) - m(y)] \geq 0,
\]

\[
l(x) m(x) n(x) - l(y) m(y) n(y) \geq -l(x) m(x) n(y) + l(x) m(y)n(x)
\]
\[ -l(x) m(y) n(y) + l(y) m(x) n(x) \\
- l(y) m(x) n(y) - l(y) m(y) n(x) \],

(2.44)

Multiplying Eq. (2.44) by \( \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \) and Integrating w.r.t ‘x’ over \([a, v]\) on both sides

\[
- l(x) m(y) n(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{x^{p-1}}{(v^{p} - x^{p})^{\alpha-n+1}} l(x)m(x)n(x)dx \\
- l(y) m(x) n(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{x^{p-1}}{(v^{p} - x^{p})^{\alpha-n+1}} l(x)m(x)n(x)dx \\
+ m(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{x^{p-1}}{(v^{p} - x^{p})^{\alpha-n+1}} m(x)l(x)dx \\
- m(y)n(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{x^{p-1}}{(v^{p} - x^{p})^{\alpha-n+1}} l(x)m(x)dx \\
+ l(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{x^{p-1}}{(v^{p} - x^{p})^{\alpha-n+1}} m(x)n(x)dx \\
- l(y) n(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{x^{p-1}}{(v^{p} - x^{p})^{\alpha-n+1}} m(x)dx \\
- l(y) m(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{x^{p-1}}{(v^{p} - x^{p})^{\alpha-n+1}} n(x)dx,
\]

(2.45)

Using Eq. (1.4)

\[
\left( { }^{p} D_{a}^{\alpha} l m n \right) v + l(y) m(y) n(y) { }^{p} D_{a}^{\alpha} (1) \\
\geq n(y) { }^{p} D_{a}^{\alpha} l m + m(y) { }^{p} D_{a}^{\alpha} n(v) \\
- m(y)n(y) { }^{p} D_{a}^{\alpha} l + l(y) { }^{p} D_{a}^{\alpha} m n(v) \\
- l(y) n(y) { }^{p} D_{a}^{\alpha} m v - l(y) m(y) { }^{p} D_{a}^{\alpha} n v.
\]

(2.46)

Multiplying equation (2.46) by \( \frac{p^{\beta-n+1}}{\Gamma(n-\beta)} (v^{1-p} \frac{d}{dv}) \) and Integrating w.r.t ‘y’ over \([a, v]\) on both sides

\[
\left( { }^{p} D_{a}^{\alpha} l m n \right) v \frac{p^{\beta-n+1}}{\Gamma(n-\beta)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{y^{p-1}}{(v^{p} - y^{p})^{\beta-n+1}} dy \\
+ { }^{p} D_{a}^{\alpha} (1) \frac{p^{\beta-n+1}}{\Gamma(n-\beta)} (v^{1-p} \frac{d}{dv}) \right]_{a}^{v} \frac{y^{p-1}}{(v^{p} - y^{p})^{\beta-n+1}} l(y)m(y)n(y)dy
\]
Theorem 16 Let \( l \) and \( m \) be synchronous functions on \([0, \infty]\) then \( \forall t > 0, \; \Re(a) > 0 \).

The following inequalities hold for Katugampola Fraction Derivative of order \( \Re(a) > 0 \).

\[
p_D^\alpha(1)[(p D_a^\alpha l^2) v + (p D_a^\alpha m^2) v] \geq 2(p D_a^\alpha l) v (p D_a^\alpha m) v. \tag{2.48}
\]

Proof We have

\[
[l(x) - m(y)]^2 \geq 0,
\]

\[
l^2(x) + m^2(y) \geq 2l(x)m(y), \tag{2.49}
\]

Multiplying (2.49) by \( \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (1-p^D_{a\alpha}) v \frac{d}{dv} \frac{x^{p-1}}{(v^p-x^p)^{\alpha-n+1}} \) and Integrating w.r.t \( x \) over \([a,v]\) on both sides

\[
\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (1-p^D_{a\alpha}) v \frac{d}{dv} \int_a^v \frac{x^{p-1}}{(v^p-x^p)^{\alpha-n+1}} l^2(x) dx
\]

\[
+ m^2(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (1-p^D_{a\alpha}) v \frac{d}{dv} \int_a^v \frac{x^{p-1}}{(v^p-x^p)^{\alpha-n+1}} dx
\]
\[
\geq 2m(y) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( v^{1-p} \frac{d}{dv} \right)^n \int_a^v \frac{x^{p-1}}{(v^p - x^p)^{\alpha-n+1}} I(x) dx,
\]
(2.50)

Using Eq. (1.4)

\[
(p D_a^\alpha l^2) v + m^2(y) p D_a^\alpha (1) \geq 2 m(y) D_a^\alpha l(v).
\]
(2.51)

Multiplying (2.51) by \[\frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv})^n \frac{y^{p-1}}{(v^p - y^p)^{\alpha-n+1}}\] and Integrating w.r.t ‘y’ over [a, v] on both sides

\[
(p D_a^\alpha l^2) v \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv})^n \int_a^v \frac{y^{p-1}}{(v^p - y^p)^{\alpha-n+1}} dy
\]

\[+ p D_a^\alpha (1) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv})^n \int_a^v \frac{y^{p-1}}{(v^p - y^p)^{\alpha-n+1}} m^2(y) dy
\]

\[\geq 2 D_a^\alpha l(v) \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} (v^{1-p} \frac{d}{dv})^n \int_a^v \frac{y^{p-1}}{(v^p - y^p)^{\alpha-n+1}} m(y) dy.
\]
(2.52)

Using Eq. (1.4)

\[
p D_a^\alpha (1) [ (p D_a^\alpha l^2) v + (p D_a^\alpha m^2(v)] \geq 2 (p D_a^\alpha l) v (p D_a^\alpha m) v.
\]

\[\square\]

**Conclusion**

In this study, we make awareness about an operator Katugampola fractional derivative. Important results of Katugampola fractional derivative are found. Some useful results of Mellin transformation are found by applying the definition of Katugampola fractional derivative. New results of Katugampola fractional derivative will be applied in future works.

**Funding** The authors have not disclosed any funding.

**Data availability** Enquiries about data availability should be directed to the authors.

**Declarations**

**Competing interests** The authors have not disclosed any competing interests.

**References**

1. Machado, J.A.: What a fractional world, Fractional Calculus Applied. Analysis **14**(4), 635–654 (2011)
2. Sloane, N.J.A.: The On-line Encyclopedia of Integer Sequences, [http://oeis.org/](http://oeis.org/) (2014)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
4. Katugampola, U.N.: New approach to generalized fractional derivatives. Bull. Math. Anal. Appl. **6**(4), 1–15 (2014)
5. Azam, M.K., Ahmad, F., Sarikaya, M.Z.: Applications of Integral Transforms on some k-fractional Integrals. J. Appl. Environ. Biol. Sci. 6(12), 127–132 (2017)
6. Katugampola, U.N.: New approach to a generalized fractional integral. Appl. Math. Comput. 218(3), 860–865 (2011)
7. Butkovskii, A.G., Postnov, S.S., Postnova, E.A.: Fractional integro-differential calculus and its control-theoretical applications, – Mathematical fundamentals and the problem of interpretation. Autom. Remote Control 74(4), 543–574 (2013)
8. Debnath, L., Bhatta, D.: Integral Transforms and Their Applications. CRC Press, Boca Raton (2011)
9. Hussain, A., Alsanad, A., Ullah, K., Ali, Z., Jamil, M.K., Mosleh, M.A.: Investigating the Short-Circuit Problem Using the Planarity Index of Complex q-Rung Orthopair Fuzzy Planar Graphs, Complexity, (2021).
10. Ullah, K., Hussain, A., Mahmood, T., Ali, Z., Alabrah, A., Rahman, S.M.M.: Complex q-rung orthopair fuzzy competition graphs and their applications. Electron. Res. Arch. 30(4), 1558–1605 (2022)
11. Sun, H., Zhang, Y., Baleanu, D., Chen, W., Chen, Y.: A new collection of real world applications of fractional calculus in science and engineering. Commun. Nonlinear Sci. Numer. Simul. 64, 213–232 (2018)
12. Ren, F.Y., Yu, Z.G., Su, F.: Fractional integral associated to the self-similar set or the generalized self-similar set and its physical interpretation. Phys. Lett. A 219, 59–68 (1996)
13. Azam, M.K., Farid, G., Rehman, M.A.: Study of Generalized type k-Fractional Derivatives. Adv. Diff. Equ. 24(2017).
14. Gaboury, S., Tremblay, R., and Fugre, B., Some relations involving a generalized fractional derivative operator. J. Inequal. Appl. 167(2013).
15. Machado, J.T., Kiryakova, V., Mainardi, F.: Recent history of fractional calculus. Commun. Nonlinear Sci. Numer. Simul. 16, 1140–1153 (2011)
16. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Progress Fract. Diff. Appl. 1, 73–85 (2015)
17. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations, Smarter scholar, (1993).
18. Azam, M.K., Rehman, M.A., Ahmad, F., Imran, M., Yaqoob, M.T.: Integral transforms of k-weyl fractional integrals. Sci. Int. 28, 3287–3290 (2017)
19. Kilbas, A.A., Saigo, M.: Theory and Applications. Chapman & Hall/CRC, Boca Raton (2004)
20. Dalir, M., Bashour, M.: Applications of fractional calculus. Appl. Math. Sci. 4(21), 1021–1032 (2010)
21. Podlubny, I.: Fractional differential equations, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Mathematics in Science and Engineering, Academic Press, Elsevier, (1999).
22. Diaz, R., Pariguan, E.: On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. 15, 179–192 (2017)
23. Budak, H., Usta, F., Sarikaya, M.Z.: New upper bounds of Ostrowski type integral inequalities utilizing Taylor expansion. Hacetettepe J. Math. Stat. 47(3), 567–578 (2018)
24. Usta, F., Budak, H., Sarikaya, M.Z., Set, E.: On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators. Filomat 32, 2153–2171 (2018)
25. Usta, F., Sarikaya, M.Z.: On generalization conformable fractional integral inequalities. Filomat 32(16), 5519–5526 (2018)
26. Budak, H., Usta, F., Sarikaya, M.Z., Ozdemir, M.E.: On generalization of midpoint type inequalities with generalized fractional integral operators. Revista de la real academia de ciencias exactas fisicas y naturales serie a-matematicas 113(2), 769–779 (2019)
27. Usta, F., Sarikaya, M.Z.: Some Improvements of Conformable Fractional Integral Inequalities. Int. J. Anal. Appl. 14(2), 162–166 (2017)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.