Excited states of holographic superconductors from massive gravity

Qian Xiang¹, Li Zhao¹,∗ and Yong-Qiang Wang²,∗

¹ Institute of Theoretical Physics, Lanzhou University, Lanzhou, Gansu 730000, China
² Lanzhou Center for Theoretical Physics and Key Laboratory of Theoretical Physics of Gansu Province, Lanzhou University, Lanzhou, Gansu 730000, China

E-mail: lizhao@lzu.edu.cn and yqwang@lzu.edu.cn

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Abstract
In this paper, we generalize the study of the model of holographic superconductors in excited states to the framework of massive gravity at the probe limit. By taking into account the effect of a massive graviton, we numerically present a family of solutions for holographic superconductors in excited states and find that the critical temperatures can be higher due to the effect of the massive graviton, in comparison with the superconductor in Einstein gravity. We also investigate the condensates and conductivities in the ground state and the excited states by studying various parameters that determine the framework of gravity background.

Keywords: AdS/CFT, massive gravity, holographic superconductors

(Some figures may appear in colour only in the online journal)

1. Introduction

In condensed matter physics, Landau-Ginzburg’s phenomenological theory and BCS theory suffer from some difficulties in fully explaining the mechanism of superconductivity. Due to the strong coupling, electrons can still pair up in some materials with higher critical temperatures, which the BCS theory failed to predict. Fortunately, as one of the greatest achievements of string theory, the AdS/CFT correspondence which establishes the relations between strongly correlated fields on the boundary and weak gravity in a one-dimensional higher bulk spacetime has provided a novel way to study condensed matter physics. In order to model superconductivity, Horowitz et al [1–4] coupled a complex scalar field to a U(1) gauge field. Considering a spontaneous U(1) symmetry breaking under the critical temperature, they successfully modeled the Cooper pair-like superconductor condensate with the scalar field, which provides us with a new way of understanding the mechanism of high temperature superconductivity. Afterward, the holographic superconductor condensates were realized in various ways. By replacing the complex field with a symmetric and traceless second-rank tensor, the holographic d-wave model was constructed in [5–7]. The holographic p-wave superconductor condensate can be modeled by a two-form field [8] or a complex, massive vector field with U(1) charge [9, 10]. In addition, the holographic p-wave superconductor can also be constructed by the SU(2) Yang-Mills field coupling to gravity [11]. For reviews of holographic superconductors, see [12–15].

Recently, inspired by the seminal paper of Gubser et al [1], Yong-Qiang Wang et al generalized the studies of holographic superconductors from ground states to excited states. In [16], the authors found a family of solutions of excited states for the s-wave holographic superconductor in the probe limit, where the scalar field possesses n nodes along the radial coordinate corresponding to the n-th excited state. Moreover, in [17], considering the holographic superconductor with the full backreaction, the author also numerically studied the condensate and conductivity of their ground state and excited states. It was found that the highly excited states of \( \langle O \rangle \) condensates converge to about 4.4 as the temperature approaches zero. It was also found that the condensation values of \( O_3 \) operators in the excited states are larger than in the ground state. Other studies on the excited state of holographic superconductor can be found in [18, 19], where in [18], the author proposed an analytic technique based on a variational method for the Sturm-Liouville eigenvalue problem, to investigate the excited states of the holographic superconductor in the probe limit. The
non-equilibrium process of the holographic s-wave superconductor with the excited states was investigated in [19]. In the above discussion, the Cooper pair-like superconductor condensate was modeled by matter fields coupling to Einstein gravity. As a holographically modeled superconductor, it is necessary to involve momentum dissipation to model features of a material in the experiment. In [20], the authors pointed out that there is a connection between momentum relaxation and graviton mass. Besides, as a natural curiosity, it is always interesting to ask how will such a system behaves, if the graviton is massive. However, this is not an easy task, since the diffeomorphism invariance in GR has already restricted the graviton to be the massless spin-2 boson [21]. Simple generalization would usually make the massive gravity unstable because of the problem of the Boulware-Deser ghost [22]. To avoid this, C. de Rham and G. Gabadadze (dRGT) introduced polynomial terms in a general action, successfully building a ghost-free nonlinear massive gravity [23–25]. Such a novel achievement arouses great concerns in the realm of holographic superconductivity. The holographic DC and Hall conductivity was studied in [26] in massive Einstein-Maxwell-Dilaton gravity providing a holographic method to explain strange metals. The nonequilibrium process of a holographic s-wave superconductor was investigated in the dRGT massive gravity in [27]. In [28], Cao H. Nam investigated the effect of massive gravity on the p-wave holographic superconductor and found that the critical temperature and condensate depend crucially on the sign of the massive gravity couplings. In addition, attempting to avoid complicated numerical calculation, a framework for translational symmetry breaking and momentum dissipation was set up in [29]. In [30], Ya-Peng Hu et al studied a holographic Josephson junction of its critical temperature, tunneling current and coherence length, and found that due to the graviton mass, the transition from a normal state to a superconducting state would be more difficult. The model of holographic superconductor with backreaction in massive gravity can be found at [31].

However, until now, the investigation of the holographic superconductor of its condensate and conductivity in the ground state and the excited states has not yet been studied in the framework of massive gravity. Therefore, in this paper, we will numerically investigate the holographic superconductor of its condensate and conductivity in the ground state and the excited states considering the effect of the dRGT massive gravity.

The paper is arranged as follows: in section 2 we give a brief review of the dRGT massive gravity. The construction of holographic superconductor in (3+1)-dimensional AdS spacetime with massive graviton can be found in section 3. The numerical results and properties of scalar condensate and optical conductivity are in section 4. The conclusion and discussion are in the last section.

2. Review of dRGT Massive Gravity

In order to construct a holographic superconductor in the framework of massive gravity we consider the following action in a 4-dimensional ghost-free dRGT massive gravity, which is the Hilbert-Einstein action including the nonlinear polynomial terms [23]

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{6}{\ell^2} \sum_i \mathcal{U}_i(g, f) \right],
\]

where \( f \) is the reference metric, the bulk metric is denoted by \( g \) and the nonzero graviton mass is represented by \( \lambda \). \( c_i \) are constants, \( \ell \) is the length scale of the AdS spacetime, and \( R \) is the usual Ricci scalar. The \( \mathcal{U}_i \) are nonlinear interaction terms of the eigenvalues of the 4 \( \times \) 4 matrix \( \mathcal{K} \equiv \sqrt{\text{det} R} \).

\[
\mathcal{K}_\mu = [\mathcal{K}], \quad \mathcal{K}_\mu = [\mathcal{K}]^2 - [\mathcal{K}], \quad \mathcal{K}_\mu = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}]^2 + 2[\mathcal{K}^2], \quad \\
\mathcal{K}_\mu = [\mathcal{K}]^4 - 6[\mathcal{K}][\mathcal{K}]^3 + 8[\mathcal{K}^2] - 6[\mathcal{K}],
\]

where the square root in \( \mathcal{K} \) is the matrix square root, for any positive tensor \( A^\mu_\nu \), the matrix square root has the form \( (\sqrt{\mathcal{A}})_\mu^\nu = A_\lambda^\nu \mathcal{K}_\mu^\lambda \), and \( [\mathcal{K}] = \mathcal{K}_\mu^\mu \) denotes the trace. By varying the above action (2.1) with respect to the metric \( g_{\mu\nu} \), the equations of motion (EoM) turn out to be

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{3}{\ell^2} g_{\mu\nu} + \lambda^2 \chi_{\mu\nu} = 0,
\]

where

\[
\chi_{\mu\nu} = -\frac{c_1}{2} \left( \mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu} \right) - \frac{c_2}{2} \left( \mathcal{U}_2 g_{\mu\nu} - 2 \mathcal{U}_2 \mathcal{K}_{\mu\nu} + 2 \mathcal{K}_{\mu\nu}^2 \right) - \frac{c_1}{2} \left( \mathcal{U}_3 g_{\mu\nu} - 3 \mathcal{U}_3 \mathcal{K}_{\mu\nu} \right) + 6 \mathcal{U}_4 \mathcal{K}_{\mu\nu}^2 - 6 \mathcal{K}_{\mu\nu}^2 - \frac{c_4}{2} \left( \mathcal{U}_4 g_{\mu\nu} - 4 \mathcal{U}_4 \mathcal{K}_{\mu\nu} \right) + 12 \mathcal{U}_6 \mathcal{K}_{\mu\nu}^2 - 24 \mathcal{U}_6 \mathcal{K}_{\mu\nu}^3 + 24 \mathcal{K}_{\mu\nu}^4.
\]

Since we study the holographic superconductor in a 4-dimensional background, we choose the reference metric as:

\[
f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_\rho),
\]

where \( c_0 \) is a positive constant that, for convenience, is set to one. Such that, a general black hole solution can be found [32, 33]

\[
dx^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 h_\rho dx^i dx^i,
\]
with
\[ f(r) = k - \frac{m_0}{r} + \frac{r^2}{\ell^2} + \frac{c_1 \lambda^2}{2} r + c_2 \lambda^2. \] (2.7)

Where \( k \) is a parameter usually used in topological black holes allowing one to study a hyperbolic, Ricci flat or sphere black hole horizon with its value chosen to be \(-1, 0, 1\). Here, we choose \( k = 0 \). The line element \( h_g \text{d}x^i \text{d}x^j \) is related to a 2-dimensional flat space. The \( m_0 \) is the mass of the black hole. In (2.7), two independent parameters \( c_1 \) and \( c_2 \) emerging from the ghost-free massive graviton self-interacting potential can be realized as couplings of massive graviton \( \lambda \) in the holographic superconductor. By tuning the product of \( c_1 \lambda^2/2 \) and \( c_2 \lambda^2 \) which serve as a linear term of the radius and a constant term in the metric solution, one can study the effect of the dRGT massive gravity on the holographic superconductor.

Besides, it is obvious that the solution will go back to a planar Schwarzschild-AdS black hole if one sets \( c_1 = c_2 = 0 \) or simply \( \lambda = 0 \). The temperature is introduced by the black hole temperature on the radius \( r_h = 1 \) horizon as below:
\[ T_{BH} = \left( \frac{f(r_h)}{4\pi} \right) \bigg|_{r=r_h} = \frac{1}{4\pi r_h} \left( k + \frac{3r_h^2}{\ell^2} + c_1 \lambda^2 r_h + c_2 \lambda^2 \right). \] (2.8)

In this paper, we will study the holographic condensate and conductivity via tuning the coupling factors \( c_1, c_2 \) and the graviton mass \( \lambda \).

3. Holographic setup

In order to model the scalar condensate at low temperature, we consider the following Maxwell and complex scalar field action in AdS_{4} as
\[ S = \int d^4x \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |\nabla \psi - iA\psi|^2 - \alpha^2 |\psi|^2 \right). \] (3.9)
in which \( A \) is the U(1) gauge field with the corresponding field strength \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) \( \alpha \) is the mass of the complex scalar field \( \psi \). Note that, we have rescaled the above action with the charge of the scalar field and therefore the backreaction of the matter field on the metric can be negligible if the charge is large enough.

From the above action, the EoMs can be obtained as
\[ (\nabla_{\mu} - iA_{\mu})(\nabla^{\mu} - iA^{\mu})\psi - \alpha^2 \psi = 0, \] (3.10)
\[ \nabla_{\mu} F^{\mu\nu} = i(\psi(\nabla^{\mu} - iA^{\mu})\psi - \psi(\nabla^{\mu} + iA^{\mu})\psi^{\ast}). \] (3.11)

On the two-dimensional flat space, an isotropic ansatz for the U(1) gauge field and the scalar field can be written as
\[ A = \phi(r) \text{d}t, \quad \psi = \psi(r). \] (3.12)

With the above ansatz, the equations of motion for the scalar field \( \psi(r) \) and electrical scalar potential \( \phi(r) \) in the background of the Schwarzschild-AdS black hole are
\[ \psi'' + \left( \frac{f'}{f} + \frac{2}{r} \right) \psi' + \frac{\alpha^2}{f^2} \psi = 0, \] (3.13)
\[ \phi'' + \frac{2}{r} \phi' - \frac{2\psi^2}{f} \phi = 0. \] (3.14)

Near the infinite boundary \( r \to \infty \), the asymptotic behaviors of the functions \( \psi(r) \) and \( \phi(r) \) have the following forms
\[ \psi = \psi^{(1)} \frac{r}{\Delta_r} + \psi^{(2)} \frac{r}{\Delta_s} + \cdots, \] (3.15)
\[ \phi = \mu - \frac{\rho}{r} + \cdots, \] (3.16)
with
\[ \Delta_{\pm} = \frac{3 \pm \sqrt{9 + 4\alpha^2}}{2}. \] (3.17)

Where the condensate is modeled by the expectation value of scalar operator \( \langle \mathcal{O}_1 \rangle \) dual to the field \( \psi \) on the \( r \to \infty \) boundary. \( \mu, \rho \) are the chemical potential and charge density of the holographic superconductor, respectively. Usually, by numerically setting one of the \( \psi^{(1)} \) to be zero on the \( r \to \infty \) boundary, the other can be used to describe the corresponding expectation value of the scalar operator with dimension \( \Delta_{\pm} \) without being sourced. In this paper, we choose \( \alpha^2 = -2 \) to meet the Breitenlohner-Freedman(BF) bound condition that \( \alpha^2 > -9/4 \) [34]. And hence, the dimensions of the condensate operators are \( \Delta_{+} = 2 \) and \( \Delta_{-} = 1 \), respectively. We will study both \( \langle \mathcal{O}_1 \rangle \) and \( \langle \mathcal{O}_2 \rangle \) condensates and conductivity corresponding to \( \langle \mathcal{O}_2 \rangle \) in the following section.

4. Numerical results

In condensed matter physics, when temperature \( T \) drops below a critical temperature \( T_c \), electrical resistivity will suddenly drop to zero leading to superconductivity. In the holographic approach, due to the spontaneously broken U(1) gauge symmetry, the scalar condensate will turn on when the chemical potential \( \mu \) (or \( \rho \)) is larger than \( \mu_c \) (or \( \rho_c \)). In this paper, the temperature is defined by the ratio between Hawking temperature on the horizon and \( \rho^{-1/2} \), such that \( T_c \) is represented by \( T_{BH} \) divided by \( \rho_c^{1/2} \). The dimensionless relation between temperature and condensate can be presented. In this section, we will numerically study the holographic superconductor in the framework of massive gravity of its condensate and conductivity in ground and excited states.

Firstly, we conduct a simple coordinate transformation, which reads:
\[ r = r_h/z. \] (4.18)

We will set the spatial infinity \( z_\infty = 0 \) and the black hole radius \( z_h = 1 \) for simplicity. After imposing the above transformation, we conduct all of our numerical calculations based on the spectral method. In the integration region \( 0 \leq z \leq 1 \), our choice of typical grids have sizes from 50 to 300.

In order to study the scalar condensate of its ground and excited states for both \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) operators, the coupled
equations (3.13) and (3.14) will be put into an iterative process by means of the Newton-Raphson method. We will give a good initial guess that leads to a solution for a ground or $n$-excited state for each time. For the ground state of $\langle C \rangle_1$ condensate, the initial guess for $\psi(z)$ is the profile that has no node along the $z$ coordinate. For the $n$-th excited state, we choose the initial guess that allows $\psi(z)$ to have $n$ nodes on the $z$ axis to reach the excited state. The estimated relative error for our numerical results is below $10^{-5}$.

Recall that the effect of massive gravity on the holographic superconductor is involved in the black hole solution (2.7) performed by the coupling coefficients $c_1$, $c_2$ and the graviton mass $\lambda$. Numerically changing their values, will enable us to study the behaviours of such a holographic system, its scalar condensate and its conductivity under different configurations of massive gravity framework. However, the contribution of massive gravity will disappear and the situation will go back to [2, 16] by setting these parameters to be $c_1 = 0$, $c_2 = 0$, or simply, $\lambda = 0$. Therefore, in the study of the effect of $c_1$ (or $c_2$) we change $c_1$ (or $c_2$) only with $c_2 = 0$ (or $c_1 = 0$) and $\lambda = 0$. As for studying graviton mass, we change $\lambda$ only while maintaining $c_1 = 1$, $c_2 = -0.5$.

4.1. Critical Temperature

In figure 1, we present our numerical results of critical temperatures for both $O_1$ (first row) and $O_2$ (second row) operators, where from left to right they correspond to the cases of tuning $c_1$ with fixed $c_2 = 0$, $\lambda = 1$, tuning $c_2$ with fixed $c_1 = 0$, $\lambda = 1$ and tuning $\lambda$ with fixed $c_1 = 1$, $c_2 = -0.5$, respectively.

From the figure, we can see that the $T_c$ of excited states (red and blue lines) corresponding to scalar field solutions that possess nodes along radius direction are all lower than ground states, while the $T_c$ of higher excited states is even lower than their former states. This means that by decreasing the temperature, the holographic superconductor will reach superconductivity firstly through the ground state, then new bound states will begin to appear with lower temperatures. Recall that the critical temperature is equal to the ratio of the Black hole temperature and $\mu_c^{1/2}$ or $\rho_c^{1/2}$ that both depend on $c_1$, $c_2$, $\lambda$. Thus, the value of $T_c$ is no longer proportional to $\mu_c^{-1}$ or $\rho_c^{-1/2}$ alone as it was in the Einstein-Maxwell framework. An interesting finding from the results is the monotonicity of the ground state of $T_c$ for $\langle O_1 \rangle$ (black lines in the first row of figure 1) will change during the process of $c_1$, $c_2$, $\lambda$ tuned from small to large, individually. However, it is notable that $T_c$ of ground states of $\langle O_1 \rangle$ cannot be infinitely high by simply lowering these parameters that determine the value of $T_{BH}$, because sufficiently small values of $c_1$ and $c_2$ will cause minus black hole temperature. On the other hand, except for the curves of ground states of $\langle O_1 \rangle$, we can see from the remaining curves, the $T_c$ of each state and each operator increases steadily when the parameters are tuned from small to large.

In tables 1–3, we present our results of critical chemical potential $\mu_c^\\gamma$ from ground state to fifth excited state. In tables 1 and 2, we investigate the coupling coefficients $c_1$ and $c_2$ changing from $-1$ to $3$ solely. In table 3, we investigate the graviton mass changing from $0.1$ to $0.5$ solely.

From the tables 1–3, we find that the $\mu_c$ of ground states, which corresponds to the ground states of $\psi$ that has no node along the radius, is the smallest for fixed coupling factors and graviton mass in both $\langle O_1 \rangle$ and $\langle O_2 \rangle$ condensates. It is the same for the results of Einstein gravity, the critical chemical potentials of higher excited states are all larger than their former states. By enlarging the effect of massive gravity, $\mu_c$ of each state and each operator will have a larger value.

Figure 1. Numerical results of critical temperatures for both kinds of condensates, where the three plots in the first row indicate the results of the $O_1$ operator and the second row corresponds to the $O_2$ operator. In all plots, the black, red and blue lines correspond to the critical temperatures of ground states, and first and second excited states, respectively.
Table 1. The table refers to critical chemical potential $\mu_c$ of the holographic superconductor from ground state to fifth excited state for the coupling coefficient $c_1$, where only $c_1$ changes its values and $c_2$ and $\lambda$ remain fixed.

| $\langle \Omega \rangle$ | $c_1$ | $c_2$ | $\lambda$ | $\mu_c$ | $\mu_c^2$ | $\mu_c^3$ | $\mu_c^4$ | $\mu_c^5$ |
|------------------------|-------|-------|-----------|---------|-----------|-----------|-----------|-----------|
| $\langle \Omega_1 \rangle$ | 1     | 0     | 0.2       | 0.668   | 6.383     | 11.585    | 16.759    | 21.927    | 27.092    |
| $\langle \Omega_2 \rangle$ | 3     | 0     | 0.2       | 1.884   | 6.820     | 12.045    | 17.312    | 22.593    | 29.710    |

| $\langle \Omega_3 \rangle$ | 1     | 0.2   | 1.429     | 6.603   | 11.816    | 17.037    | 22.262    | 27.488    |
| $\langle \Omega_4 \rangle$ | 0     | 1     | 0.2       | 1.142   | 6.530     | 11.759    | 16.980    | 22.199    | 27.418    |

| $\langle \Omega_5 \rangle$ | 0     | 1     | 0.2       | 1.184   | 6.603     | 11.875    | 17.141    | 22.406    | 27.672    |

Table 2. The table refers to critical chemical potential $\mu_c$ of the holographic superconductor from ground state to fifth excited state for the coupling coefficient $c_2$, where only $c_2$ changes its values and $c_1$ and $\lambda$ remain fixed.

| $\langle \Omega \rangle$ | $c_1$ | $c_2$ | $\lambda$ | $\mu_c$ | $\mu_c^2$ | $\mu_c^3$ | $\mu_c^4$ | $\mu_c^5$ |
|------------------------|-------|-------|-----------|---------|-----------|-----------|-----------|-----------|
| $\langle \Omega_1 \rangle$ | 0     | -1    | 0.2       | 1.098   | 6.456     | 11.642    | 16.817    | 21.989    | 27.161    |
| $\langle \Omega_2 \rangle$ | 0     | 3     | 0.2       | 1.142   | 6.530     | 11.759    | 16.980    | 22.199    | 27.418    |

| $\langle \Omega_3 \rangle$ | 0     | -1    | 0.2       | 4.037   | 9.140     | 14.287    | 19.445    | 24.681    | 29.775    |
| $\langle \Omega_4 \rangle$ | 0     | 1     | 0.2       | 4.091   | 9.236     | 14.527    | 19.632    | 24.842    | 30.055    |

| $\langle \Omega_5 \rangle$ | 0     | 3     | 0.2       | 4.144   | 9.329     | 14.565    | 19.815    | 25.072    | 30.332    |

According to the three tables, we fit the relations between state $n$ and $\mu_c$, the results are as follows:

The fittings of coupling coefficient $c_1$, listed in table 1

\[
\mu_c \approx \begin{cases} 
5.255 n + 0.932, & c_1 = -1, c_2 = 0, \lambda = 0.2, \\
5.260 n + 1.198, & c_1 = -1, c_2 = 0, \lambda = 0.2, \text{ for } \Omega_1, \\
5.216 n + 1.715, & c_1 = -1, c_2 = 0, \lambda = 0.2.
\end{cases}
\]

(4.19)

and the fittings of coupling coefficient $c_2$, listed in table 2

\[
\mu_c \approx \begin{cases} 
5.142 n + 3.986, & c_1 = -1, c_2 = 0, \lambda = 0.2, \\
5.204 n + 4.078, & c_1 = -1, c_2 = 0, \lambda = 0.2, \text{ for } \Omega_1, \\
5.266 n + 4.169, & c_1 = -1, c_2 = 0, \lambda = 0.2.
\end{cases}
\]

(4.20)

A previous study showed that, in the Einstein gravity model, the difference of $\mu_c$ between consecutive states is approximately 5 for both $\协会_f$ and $\协会_f$, indicating a linear relation between $\mu_c$ and $n$ [16]. This interesting finding was then found in a semi-analytical study [18]. Here, from (4.19) to (4.24), we can see that this property also appears when taking into account the effect of massive gravity.

4.2. Condensate

In this subsection, we study the condensates of our holographic superconductor for both $\Omega_1$ and $\Omega_2$ operators in the framework of massive gravity. According to AdS/CFT dictionary, the expectation value of $\langle \Omega_i \rangle$ dual to $\psi^{(i)}$ is

\[
\langle \Omega_i \rangle = \sqrt{2} \psi^{(i)}, i = 1, 2.
\]

(4.25)

As usual, we study the effects of a massive graviton on scalar condensates by numerically solving equation (3.13), equations (3.14) and (4.25) with changing parameters $c_1$, $c_2$ and $\lambda$ solely.

In figure 2, we present our condensation results in various configurations of a massive gravity background, where the $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ condensates are presented in the left and the right panel, respectively. In the first row, we fix $c_2 = 0$, $\lambda = 1$ and study the coupling coefficient $c_1$ in the cases of $c_1 = 0.2$, $c_1 = 0$ and $c_1 = 1$. In the middle row, we also study the coupling coefficient $c_2$ in the same way while setting $c_1 = 0$, $\lambda = 1$. In the last row, we fix $c_1 = 1$, $c_2 = -0.5$ while tuning the graviton mass from $\lambda = 0.1$ to $\lambda = 1.8$. 

(4.23)
From the three plots of \( \langle \mathcal{O}_1 \rangle \) condensates, we can see that the condensates of ground states rise rapidly at \( T/T_c = 1 \) and then go to infinity at lower temperatures. Such a singularity indicates that the charge of the scalar field is not large enough at low temperatures and consideration of the backreaction is needed. The excited states also begin to condense around \( T_c \), however, with the condensation values converge to limited values at low temperatures. We can also see that the condensation values of the ground states are larger than the excited states when the coupling coefficients \( c_1 \), \( c_2 \) and the graviton mass \( \lambda \) are small. However, when the effect of massive gravity is stronger, the condensation values of excited states can be equal or even less to or than ground states. Moreover, from the sub-figures shown inside the first and third plots in the left panel, one can find that, with larger coupling coefficients and graviton mass, the condensation values of the second excited states are higher than the first excited states when the temperature is not sufficiently low. At \( T \to 0 \), the scalar condensates of the second excited states converge to lower values compared to the first excited states.

For the \( \langle \mathcal{O}_2 \rangle \) condensates in the three plots of the right panel, the condensation values of all three plots eventually converge to constants at low temperatures. There exists a gravitational version of the Higgs mechanism which creates connections between the lattice or impurity of a modeled material and graviton mass leading to momentum relaxation rate \( \Gamma \sim \lambda^2 \) [20]. The results presented in figure 2 illustrate that the effect of dRGT massive gravity will lower condensation values of all states when couplings and graviton mass are tuned large. In addition, another feature is that the condensation values of higher excited states are larger than their former states which is similar to the massless gravity case. However, it is worth mentioning that even though the excited states have larger condensates since these are metastable and will last for a considerable time, they will eventually evolve to ground states [19].

It is obvious that our model also possesses a second order transition from normal state to superconducting state near the critical point, which is a square root behaviour predicted by the mean field theory. Therefore, we fit the condensation curves versus temperatures for both \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) operators near the critical temperatures according to our results shown in figure 2.

The formats of the fittings are as follows:

\[
\langle \mathcal{O}_1 \rangle \approx \zeta^{(\text{fit})} (T_c^{(\text{fit})})^2 (1 - T/T_c^{(\text{fit})})^{1/2},
\]

where \( \zeta^{(\text{fit})} \) and \( T_c^{(\text{fit})} \) are the fitting coefficient and critical temperature of the \( n \)-th state. The results of \( \zeta^{(\text{fit})} \) are presented in table 4. The corresponding critical temperatures are shown in figure 1 where we have also calculated other critical temperatures in different massive gravity backgrounds to show their tendencies.

From the fitting coefficients of \( \langle \mathcal{O}_2 \rangle \) condensate listed in table 4, one can easily find that, similar to the models in Einstein gravity, each \( \zeta^{(\text{fit})} \) of the \( n \)-th state is higher than its former state in every case. And, as we enlarge the coupling coefficients or the graviton mass, \( \zeta^{(n)} \) of each state decreases rapidly for both \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) operators. The difference between our model in the massive gravity framework and the model in Einstein gravity is reflected by the \( \langle \mathcal{O}_1 \rangle \) condensate, wherein in the latter model the fitting coefficient \( \zeta^{(n)} \) of the \( n \)-th state is smaller than its former state around the critical temperature. Whereas, on the contrary, in our model around the critical temperature, the values of the scalar field condense larger in the \( n \)-th excited state than its former state as shown in cases i to iii, the cases vii and ix. One can also directly discover such a tendency by looking at the first and the third plot around \( T_c \) in the left panel of figure 2. As for the remaining cases from iv to vii, we can see that the \( \zeta \) of excited states are smaller than the ground states with \( \zeta^{(2)} \) being slightly larger than \( \zeta^{(1)} \). In addition, as we extend our calculations to a fourth excited state for the cases vi to vii, \( \zeta^{(n)} \) of the excited state increases slightly with the \( n \)-th excited state.

### 4.3. Conductivity

In this subsection, we study the effects of massive gravity on the conductivity in the ground and the excited states of our holographic superconductor.

By turning on perturbations of the vector potential \( A_i \) in the bulk geometry of a Schwarzschild-AdS black hole, we consider the Maxwell equation with a time dependence of \( e^{-i \omega t} \), the linearized equations are given as

\[
A_i'' + \frac{f'}{f} A_i' + \left( \frac{\omega^2}{f^2} - 2 \frac{k^2}{f} \right) A_i = 0.
\]

When imposing the ingoing boundary conditions at the horizon, the asymptotic behaviour of the Maxwell field on the
The conductivity can be obtained according to the Ohm’s law

\[ \sigma(\omega) = -\frac{i A^{(1)}}{\omega A^{(0)}}. \]  

In figures 3 and 4, we plot the real and imaginary parts of optical conductivity as a function of frequency at low temperature \( T/T_c \approx 0.100 \) for the operator \( O_2 \) of ground states and excited states. In both figures, we use black lines, red lines and blue lines to represent the ground states, first and second excited states. The studied parameters from large to small sequencing are denoted by short dashed lines, long dashed lines and solid lines. In all plots, the parameters from large to small sequencing are marked by dotted lines, dashed lines and solid lines.

Figure 2. \( \langle O_1 \rangle \) and \( \langle O_2 \rangle \) condensates in various configurations of massive gravity where the \( \langle O_1 \rangle \) and \( \langle O_2 \rangle \) condensates are presented in left and right panel, respectively. From top to bottom the results correspond to changing parameters \( c_1, c_2 \) and \( \lambda \) accordingly, where the condensations of ground states, first and second excited states are denoted by black lines, red and blue lines. In all plots, the parameters from large to small sequencing are marked by dotted lines, dashed lines and solid lines.

The conductivity can be obtained according to the Ohm’s law

\[ A_\xi = A_\xi^{(0)} + \frac{A_\xi^{(1)}}{r} + \cdots. \]  

boundary is

In figures 3 and 4, we plot the real and imaginary parts of optical conductivity as a function of frequency at low temperature \( T/T_c \approx 0.100 \) for the operator \( O_2 \) of ground states and excited states. In both figures, we use black lines, red lines and blue lines to represent the ground states, first and second excited states. The studied parameters from large to small sequencing are denoted by short dashed lines, long dashed lines and solid lines. In both figure 3 and figure 4, plots, (a) and (b), we study the cases \( c_1 = -1, c_1 = 1, c_1 = 3 \) with \( c_2 \) and \( \lambda \) fixed at \( c_2 = 0, \lambda = 0.2 \) and
the cases $c_2 = -1$, $c_2 = 1$ $c_2 = 3$ with $c_1$ and $\lambda$ fixed at $c_1 = 0$, $m = 0.2$ respectively. In plot. (c), we study the cases $\lambda = 0.1$, $\lambda = 0.3$, $\lambda = 0.5$ with $c_1$ and $c_2$ fixed at $c_1 = 1$, $c_2 = -0.5$.

As it is known to all that in the real, dissipative part of conductivity, the horizontal lines at $\text{Re}(\sigma) = 1$ correspond to temperatures higher than the critical temperature $T_c$ where there is no scalar condensate. In the three plots of figure 3, the gaps of ground states (black lines) open at $\omega/T \approx 60$ until the $\text{Re}(\sigma)$ exponentially converges to 1. Comparing with the $\text{Re}(\sigma)$ in the ground states (black lines), we can see one additional peak for the first excited state (red lines) and two additional peaks (blue lines) for the second excited state. The appearance of additional peaks is the result of excited states which are bound states where $\psi$ has $n$ nodes along the radius.

Similarly, in Table 4, additional peaks of conductivity with the

![Figure 3](image.png)

**Figure 3.** The real part of optical conductivity, where (a), (b) and (c) correspond to the studies of $c_1$, $c_2$ and $\lambda$ respectively. The black, red and blue lines represent ground states, first and second excited states. In all three plots, the studied parameters from large to small sequencing are marked by short dashed lines, long dashed lines and solid lines.

| Case | $c_1$ | $c_2$ | $\lambda$ | $\zeta^{(0)}$ | $\zeta^{(1)}$ | $\zeta^{(2)}$ | $\zeta^{(0)}$ | $\zeta^{(1)}$ | $\zeta^{(2)}$ |
|------|------|------|------|-------|-------|-------|-------|-------|-------|
| i    | 0.2  | 0    | 1    | 7.71  | 7.87  | 7.97  | 134   | 314   | 497   |
| ii   | 1.0  | 0    | 1    | 4.21  | 6.21  | 6.69  | 111   | 240   | 368   |
| iii  | 2.0  | 0    | 1    | 2.58  | 4.55  | 5.37  | 82    | 182   | 293   |
| iv   | 0    | 0.2  | 1    | 8.92  | 7.84  | 7.90  | 138   | 305   | 486   |
| v    | 0    | 1.0  | 1    | 7.53  | 6.61  | 6.67  | 101   | 236   | 351   |
| vi   | 0    | 2.0  | 1    | 6.34  | 5.66  | 5.67  | 79    | 170   | 260   |
| vii  | 1    | -0.5 | 0.1  | 9.23  | 8.01  | 8.03  | 146   | 333   | 527   |
| viii | 1    | -0.5 | 1.0  | 4.51  | 6.82  | 7.37  | 123   | 285   | 457   |
| ix   | 1    | -0.5 | 1.8  | 1.92  | 3.89  | 5.11  | 98    | 209   | 330   |
scalar field mass equaling to $3/2$ were also found and explained that it is the result of interactions between quasi-particles that form new bound states excited beyond the ground state of BCS theory. Hence, the appearance of new bound states will lead to additional peaks. Besides, since we study the optical conductivity at $T/T_c \rightarrow 0.100$, the peaks of the ground states develop to almost delta functions as they are generated to excited states. Moreover, the number of the additional peaks of the $n$-th excited state is equal to $n$. To explain this, on the one hand, one can ascribe this to the monotonicity’s change of $\psi$ along a radius that affects $\frac{\partial^2}{\partial r^2} - \frac{2\nu^2}{r}$ in (4.28), where monotonicity of this term alters once for the ground state, three times for the first excited state and five times for the second excited state. On the other hand, by plotting the profile of $-\text{Re}(A_\nu)$ (whose value of the first derivative corresponds to the value of $\text{Im}(\sigma)$) for excited states around nodes of $\text{Im}(\sigma)$, one can indeed see the derivative value of $-\text{Re}(A_\nu)$ changes its sign around the nodes.

In both figures, compared with Einstein gravity, where conductivity, of course, develops only one curve in the frequency space with fixed temperature, under the influence of massive gravity, tuning these parameters will allow us to modify $\text{Re}[\sigma]$ and $\text{Im}[\sigma]$ on the frequency space.

4.4. Conclusions and discussions

In this paper, we investigated a holographic superconductor that was constructed by a Maxwell field coupled to a scalar field in dRGT massive gravity. The effects of massive gravity are involved in the black hole solution carried by coupling factors $c_1$, $c_2$ and graviton mass $\lambda$ in (2.7). By tuning their values, we numerically studied critical chemical potential, critical temperature, condensate, and conductivity of ground and excited states changing with the dRGT nonlinear massive gravity.

As we tuned $c_1$, $c_2$, $\lambda$ separately and thus tuned the effect of massive gravity, the change of $\mu_c$, $T_c$, condensation value and conductivity of ground and excited states are as follows:

- For $\mathcal{O}_4$ condensate, as couplings $c_1$, $c_2$ and graviton mass $\lambda$ were tuned from small to large in a range that guarantees a positive $T_{\text{BH}}$, the $T_c$ of ground states declined from high values to minimums and then went upward. The $T_c$ of higher excited states, lower than their
former states, however, rose steadily with the growth of these parameters. For $O_2$ condensate, the $T_c$ of ground and excited states all went higher monotonically with larger coupling factors and graviton mass. What’s more, the difference of $\mu_c$ between consecutive states is also about 5 for both $O_1$ and $O_2$ operators, similar to the discovery in Einstein gravity.

- Under the same configuration of dRGT massive gravity coefficients, condensation values of scalar fields in higher excited states are larger than their former states, for $O_2$. For $O_1$, when the effect of massive gravity is not manifest, the condensation value of its ground state is larger than its excited states, while the ground state’s condensation value can be smaller than the excited states with the coupling factors and the graviton mass becoming bigger. Moreover, the stronger effect of massive gravity will reduce the condensation values for both $O_1$ and $O_2$.

- When studying conductivity, we fixed temperature $T/T_c \approx 0.100$ and then tuned $c_1$, $c_2$, $\lambda$ from small to large. With bigger parameters, the optical conductivities of each state shift to lower frequencies. Besides, similar to the massless gravity case, we also find that the additional peaks (comparing with the $\text{Re}(\sigma)$ and $\text{Im}(\sigma)$ of ground state) of $n$-th excited state are equal to $n$.

There could be many interesting extensions of our work. First, as we have noticed in figure 2 the large scalar field charge limit could not suffice for the ground state of $O_1$, which required us to solve the coupled differential equations with an Einstein equation. Second, around the critical temperature, the condensation of the $O_1$ operator performs differently compared with the model in Einstein gravity and the mechanism is unclear. It is interesting to answer this question with a semi-analytical method [18]. In the end, we would like to extend our study of excited states in massive gravity to the p-wave and d-wave holographic superconductors in the future.

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ORCID iDs

Li Zhao https://orcid.org/0000-0002-0935-5671

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