Preconditioned MHSS iterative algorithm and its accelerated method for solving complex Sylvester matrix equations *

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\textbf{Abstract}

This paper introduces and analyzes a preconditioned modified of the Hermitian and skew-Hermitian splitting (PMHSS). The large sparse continuous Sylvester equations are solved by PMHSS iterative algorithm based on non-Hermitian, complex, positive definite/semidefinite, and symmetric matrices. We prove that the PMHSS is converged under suitable conditions. In addition, we propose an accelerated PMHSS method consisting of two preconditioned matrices and two iteration parameters $\alpha, \beta$. Theoretical analysis showed that the convergence speed of the accelerated PMHSS is faster compared to the PMHSS. Also, the robustness and efficiency of the proposed two iterative algorithms were demonstrated in numerical experiments.

\textit{Keywords:} PMHSS iterative algorithm, APMHSS iterative algorithm, Continuous Sylvester equations.

\section{1. Introduction}

The continuous Sylvester equations are formulated as follows:

$$AX + XB = F$$

(1.1)

where complex matrices $A \in \mathbb{C}^{m \times m}, B \in \mathbb{C}^{n \times n}, F \in \mathbb{C}^{m \times n}$, are sparse and large. Let $W \in \mathbb{R}^{m \times m}$ be a positive definite real symmetric matrix, and...
$T \in \mathbb{R}^{m \times m}$, $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{n \times n}$ be positive semidefinite real symmetric matrices. Then, $A = W + iT$ and $B = U + iV$. Under the assumption $T \neq 0$, indicating that $A$ is not Hermitian, it is easily proved the existence of a unique solution of (1.1) [21] due to the lack of an eigenvalue shared between $-B$ and $A$.

Let assume $A = I_n \otimes A + B^T \otimes I_m$ where $I_m$ and $I_n$ represent the $m$ and $n$ order identity matrices, respectively, $\otimes$ indicates the Kronecker product, and $A^T$ represents a conjugate transpose of matrix. Then, (1.1) can be rewritten to the following linear system:

$$Ax = f$$

(1.2)

where vectors $f$ and $x$ are composed of the concatenated columns of $F$ and $X$. However, obtaining the solution of (1.1) is ill-posed problem and expensive.

In past decades, active research has been focused on the efficient solving method for this problem. Direct solver algorithms such as Bartels-Stewart and Hessenberg-Shur methods [35, 36] are only applicable to small-sized matrices. Recently, many iterative approaches were studied to solve large continuous Sylvester equation [1, 2, 3, 17, 18, 20, 22]. Bai et al. firstly proposed the Hermitian and skew-Hermitian splitting (HSS) iterative algorithm. Since then, its variants were actively explored and studied, proving that the HSS-based iteration can solve a large continuous Sylvester equation.

In 2005, a modified HSS iterative algorithm was proposed [1]. In 2019, Dehghan and Shirilord proposed to use GMHSS for solving the Sylvester equation [10].

This paper focuses on solving the Sylvester equation that is complex, non-Hermitian, positive definite/semidefinite, and symmetric matrices. This paper first investigates HSS iterative algorithm in solving the Sylvester equation to the best of our knowledge.

The Hermitian part $\mathcal{H}$ and skew-Hermitian parts $\mathcal{S}$ of $A$ and $B$ are defined as follows:

$$\mathcal{H}(A) = \frac{1}{2}(A + A^*) = W,$$
$$\mathcal{S}(A) = \frac{1}{2}(A - A^*) = iT,$$
$$\mathcal{H}(B) = \frac{1}{2}(B + B^*) = W,$$
$$\mathcal{S}(B) = \frac{1}{2}(B - B^*) = iV,$$

Then, according to the HSS, $A$ and $B$ are split into $\mathcal{H}(A)/\mathcal{S}(A)$ and
\(\mathcal{H}(B)/\mathcal{S}(B)\), respectively. The HSS iterative algorithm is conducted as:

**HSS iterative algorithm:**

The HSS is conducted until \(\{X^{k+1}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m \times n}\) meets the termination condition where \(\alpha > 0, \beta > 0\) are constants:

\[
\begin{aligned}
(\alpha I + W)X^{(k + \frac{1}{2})} + X^{(k + \frac{1}{2})}(\beta I + U) &= (\alpha I - iT)X^{(k)} + X^{(k)}(\beta I - iV) + F, \\
(\alpha I + iT)X^{(k+1)} + X^{(k+1)}(\beta I + iV) &= (\alpha I - W)X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I - U) + F.
\end{aligned}
\]  

(1.3)

In (1.3), two sub-steps are solved under the conditions that \(\alpha I + W, \beta I + U\) are positive definite and symmetric coefficient matrices, \(\alpha I + iT, \beta I + iV\) are shifted skew-Hermitian coefficient matrices. However, complex arithmetic is required to actually solve the complex and non-Hermitian matrices \(\alpha I + iT, \beta I + iV\). The modified HSS iteration (MHSS) was proposed to address this problem in [1].

**MHSS iterative algorithm:**

With an initialized \(X^{(0)} \in \mathbb{C}^{m \times n}\), \(X^{(k+1)} \in \mathbb{C}^{m \times n}\) where \(k \in \mathbb{Z}\) is computed via the following scheme until \(\{X^{k+1}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m \times n}\) meets the stopping condition:

\[
\begin{aligned}
(\alpha I + W)X^{(k + \frac{1}{2})} + X^{(k + \frac{1}{2})}(\beta I + U) &= (\alpha I - iT)X^{(k)} + X^{(k)}(\beta I - iV) + F, \\
(\alpha I + iT)X^{(k+1)} + X^{(k+1)}(\beta I + iV) &= (\alpha I + iW)X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I + iU) - iF.
\end{aligned}
\]  

(1.4)

where \(\alpha\) is a positive constant.

From (1.1), Zhou derived an MHSS iteration equation (1.4) that is used to solve the Sylvester equation [1], where the involved two sub-steps are directly or arithmetically solved in an efficiency way. The MHSS iterative algorithm is converged to obtain the unique solution of (1.1) without any condition.

This paper proposes the preconditioned MHSS (PMHSS) iterative algorithm and its accelerated variant to solve the Sylvester equation that is complex, non-Hermitian, positive definite/semidefinite, and symmetric matrices. Two methods both efficiently converge since they are preconditioned iterative algorithms.
2. The proposed PMHSS

2.1. PMHSS

Based on the MHSS iterative algorithm for Sylvester equation, we derive a preconditioned MHSS iteration to reduce convergence time of (1.4) by preconditioning the equations (1.1) and (1.2) with two proper positive definite and symmetric matrices, \( P_1 \in \mathbb{R}^{(m \times m)} \), \( P_2 \in \mathbb{R}^{(n \times n)} \).

Method 2.1 (The PMHSS iterative algorithm)

With an initial matrix \( X^{(0)} \in \mathbb{C}^{m \times n} \), \( X^{(k+1)} \in \mathbb{C}^{m \times n} \) for \( k \in \mathbb{Z} \) is calculated through the following equations until \( \{X^{(k+1)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m \times n} \) meets the termination condition:

\[
\begin{align*}
(\alpha P_1 + W)X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\alpha P_2 + U) & = (\alpha P_1 - iT)X^{(k)} + X^{(k)}(\alpha P_2 - iV) + F, \\
(\alpha P_1 + T)X^{(k+1)} + X^{(k+1)}(\alpha P_2 + V) & = (\alpha P_1 + iW)X^{(k+\frac{3}{2})} + X^{(k+\frac{3}{2})}(\alpha P_2 + iU) - iF.
\end{align*}
\]

(2.1)

where \( \alpha > 0 \) is a constant, and \( P_1 \in \mathbb{R}^{(m \times m)} \), \( P_2 \in \mathbb{R}^{(n \times n)} \) are prescribed positive definite and symmetric matrices.

Let \( W \in \mathbb{R}^{m \times m} \) be positive definite and symmetric real matrix, \( T \in \mathbb{R}^{m \times m} \) be positive semidefinite and symmetric real matrix, \( U, V \in \mathbb{R}^{n \times n} \) be positive semidefinite and symmetric real matrices, and \( \alpha > 0 \) is a constant. Then, \( \alpha P_1 + W, \alpha P_2 + U, \alpha P_1 + T \) and \( \alpha P_2 + V \) become positive definite and symmetric. Thus, no common eigenvalue exists between \( \alpha P_1 + W \) and \( -(\alpha P_2 + U) \), and between \( \alpha P_1 + T \) and \( -(\alpha P_2 + V) \). Accordingly, the unique solution is obtained for the two fixed-point equations. Mostly real arithmetic or efficient direct solver can be applied to solve the two half-steps in the PMHSS iterative algorithm.

The theorem for the convergence of PMHSS is established in solving (1.1) in the following.

Theorem 2.1. Let denote \( A = H + iD \), where

\[
H = I \otimes W + U \otimes I, \quad D = I \otimes T + V \otimes I,
\]

(2.2)

and represent by

\[
P = I \otimes \alpha P_1 + \alpha P_2 \otimes I,
\]

(2.3)

where positive definite and symmetric matrices \( P_1 \in \mathbb{R}^{(m \times m)} \), \( P_2 \in \mathbb{R}^{(n \times n)} \) are prescribed, and \( PH = HP, PD = DP \). Then, the unique solution
$X_* \in \mathbb{C}^{m \times n}$ of (1.1) is obtained as PMHSS is converged. The spectral radius $\rho(M(\alpha))$ of $M(\alpha)$ determines the convergence factor bounded by $\sigma(\alpha)$.

It holds that $\rho(M(\alpha)) \leq \sigma(\alpha) := \max_{\tilde{\lambda}_j \in \text{sp}(P^{-1}H)} \frac{\sqrt{\alpha^2 + \tilde{\lambda}_j^2}}{\alpha + \tilde{\lambda}_j} \cdot \max_{\bar{\mu}_j \in \text{sp}(P^{-1}D)} \frac{\sqrt{\alpha^2 + \bar{\mu}_j^2}}{\alpha + \bar{\mu}_j} \leq \max_{\tilde{\lambda}_j \in \text{sp}(P^{-1}H)} \frac{\sqrt{\alpha^2 + \tilde{\lambda}_j^2}}{\alpha + \tilde{\lambda}_j} < 1, \forall \alpha > 0$. Here and in the sequel, we use $\rho(\cdot)$ to denote the spectral radius of the corresponding matrix.

**Proof:** By using Kronecker product, we can reformulate (2.1) in the matrix-vector form as follows:

\[
\begin{align*}
(I \otimes (\alpha P_1 + W) + (\alpha P_2 + U)^T \otimes I)\text{vec}(X^{(k+\frac{1}{2})}) &= (I \otimes (\alpha P_1 - iT) + (\alpha P_2 - iV)^T \otimes I)\text{vec}(X^{(k)}) + \text{vec}(F), \\
(I \otimes (\alpha P_1 + T) + (\alpha P_2 + V)^T \otimes I)\text{vec}(X^{(k+1)}) &= (I \otimes (\alpha P_1 + iT) + (\alpha P_2 + iU)^T \otimes I)\text{vec}(X^{(k+\frac{1}{2})}) - \text{ivec}(F),
\end{align*}
\]

(2.4)

Denote by (2.2) and (2.3), then we obtain

\[
\begin{align*}
(\alpha P + H)\text{vec}(X^{(k+\frac{1}{2})}) &= (\alpha P - iD)\text{vec}(X^{(k)}) + \text{vec}(F), \\
(\alpha P + D)\text{vec}(X^{(k+1)}) &= (\alpha P + iH)\text{vec}(X^{(k+\frac{1}{2})}) - \text{ivec}(F),
\end{align*}
\]

(2.5)

equation (2.5) is the PMHSS iterative algorithm for solving the linear system (1.2). As $P, H \in \mathbb{R}^{(m \times n)(m \times n)}$ are positive definite and symmetric, $D \in \mathbb{R}^{(m \times n)(m \times n)}$ is positive semidefinite and symmetric, and $\alpha \in \mathbb{R}$ is positive. Removing $\text{vec}(X^{(k+\frac{1}{2})})$ from (2.5) yields $\text{vec}(X^{(k+1)}) = M(\alpha)\text{vec}(X^{(k)}) + G(\alpha)\text{vec}(F)$, where $M(\alpha)$ is the iteration matrix in (2.4), and $G(\alpha) = (1 - i\alpha)(\alpha P + D)^{-1}\bar{P}(\alpha P + H)^{-1}$. We can get the conclusion straightforwardly according the Theoretical results from [12], define $\widetilde{H} = P^{-1/2}HP^{-1/2}, \widetilde{D} = P^{-1/2}DP^{-1/2}$, then $\widetilde{H}, \widetilde{D} \in \mathbb{R}^{(m \times n)(m \times n)}$ are positive semidefinite and symmetric, and real matrices, when $\widetilde{H}$ is positive definite. Since $\widetilde{H}, \widetilde{D}$ are similar to $P^{-1}H$ and $P^{-1}D$, respectively, analogously to Theorem 2.1 in [4], it is proved that the unique solution of (1.2) is obtained as (2.5) is converged for any initial matrices. The convergence rate is bounded by

\[
\sigma(\alpha) \equiv \max_{\tilde{\lambda}_j \in \text{sp}(P^{-1}H)} \frac{\sqrt{\alpha^2 + \tilde{\lambda}_j^2}}{\alpha + \tilde{\lambda}_j} \cdot \max_{\bar{\mu}_j \in \text{sp}(P^{-1}D)} \frac{\sqrt{\alpha^2 + \bar{\mu}_j^2}}{\alpha + \bar{\mu}_j} \leq \max_{\tilde{\lambda}_j \in \text{sp}(P^{-1}H)} \frac{\sqrt{\alpha^2 + \tilde{\lambda}_j^2}}{\alpha + \tilde{\lambda}_j} < 1, \forall \alpha > 0.
\]
Moreover, referred from Theorem 2.1 in [4], the minimum point $\alpha^*$ and the corresponding value $\sigma(\alpha^*)$ of the upper bound are respectively as $\alpha^* = \arg \min_{\alpha} \sqrt{\alpha^2 + \tilde{\lambda}_j} = \sqrt{\tilde{\lambda}_{\min}^2 \tilde{\lambda}_{\max}^2}$, with $\tilde{\lambda}_{\min}$ and $\tilde{\lambda}_{\max}$ being the smallest and largest eigenvalues of the matrix $P^{-1}H$, and $\sigma(\alpha^*) = \sqrt{\kappa_2^2 (P^{-1}H)^{-1} + 1}$.

**Remark 2.1.** Particularly, when it holds that $P=H$, $M(\alpha) = (\alpha H + D)^{-1}(\alpha H + iH)(\alpha H + H)^{-1}(\alpha H - iD)$, then $\rho(M(\alpha)) \leq \frac{\sqrt{\kappa_2^2 (P^{-1}H)^{-1} + 1}}{\alpha + 1} < 1$.

Actually, in our experiments, we take $P_1 = W, P_2 = U$ to see the results under the condition $P=H$ with convenience, and it outperforms MHSS iterative algorithm in solving complex Sylvester matrix equations.

### 3. Accelerated PMHSS (APMHSS) iterative algorithm

We propose and analyze accelerated preconditioning modified of the HSS (APMHSS) iterative algorithm to solve complex Sylvester equations. Based on the PMHSS iterative algorithm (2.1), it involves a parameter $\alpha$ and two preconditioned matrices which are positive definite and symmetric, we focus on the improvement on the value of the parameter to accelerate its convergence and give relevant theoretical analyses.

Firstly, we consider the accelerated PMHSS algorithm involved two iteration parameters $\alpha, \beta$, which are both positive constants in each step of the iterative algorithm.

**Method 3.1: The APMHSS iterative algorithm**

With an initial matrix $X^{(0)} \in \mathbb{C}^{m \times n}$, $X^{(k+1)} \in \mathbb{C}^{m \times n}$ for $k \in \mathbb{Z}$ is calculated through the following equations until $\{X^{(k+1)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m \times n}$ meets the termination condition:

$$
\begin{align*}
(\alpha P_1 + W)X^{(k+1)} + X^{(k+1)}(\alpha P_2 + U) &= (\alpha P_1 - iT)X^{(k)} + X^{(k)}(\alpha P_2 - iV) + F, \\
(\beta P_1 + T)X^{(k+1)} + X^{(k+1)}(\beta P_2 + V) &= (\beta P_1 + iW)X^{(k+1)} + X^{(k+1)}(\beta P_2 + iU) - iF.
\end{align*}
$$

where $\alpha > 0, \beta > 0$ are constants and $P_1 \in \mathbb{R}^{m \times m}, P_2 \in \mathbb{R}^{n \times n}$ are prescribed positive definite and symmetric matrices.

It is easy to see that, when $\alpha = \beta$, the APMHSS is same as the PMHSS, further, when it also holds that $P_1 = I \in \mathbb{R}^{m \times m}, P_2 = I \in \mathbb{R}^{n \times n}$, which reduces to MHSS method. The two half-steps in the APMHSS is effectively
solved via a direct solver since $\alpha P_1 + W, \alpha P_2 + U, \beta P_1 + T$ and $\beta P_2 + V$ are positive definite and symmetric. Then the emphasis is to choose suitable $\alpha, \beta$ and $P_1, P_2$ to possess an efficient computation and quick convergence. This kind of GMHSS has also been studied in [10], despite its iteration process with four-parameters and the lack of preconditioned matrix.

**Theorem 3.1.** Let denote $A = H + iD$, where

$$H = I \otimes W + U \otimes I, \quad D = I \otimes T + V \otimes I,$$

$$P = I \otimes \alpha P_1 + \alpha P_2 \otimes I,$$

and represent by

$$\Theta(\alpha, \beta) = (\beta P + D)^{-1}(\beta P + iH)(\alpha P + H)^{-1}(\alpha P - iD),$$

where positive definite and symmetric matrices $P_1 \in \mathbb{R}^{m \times m}, P_2 \in \mathbb{R}^{n \times n}$ are prescribed, and $PH = HP, PD = DP$. If $0 < \beta \leq \alpha$, $\alpha^2 - \beta^2 \leq 2\beta \mu_{\min}$, the unique solution $X^* \in \mathbb{C}^{m \times n}$ of (1.1) is obtained as APMHSS iterative algorithm is converged, and the spectral radius $\rho(\Theta(\alpha, \beta))$ determines the convergence factor, bounded by $\sigma(\alpha, \beta)$. It holds that

$$\rho(\Theta(\alpha, \beta)) \leq \sigma(\alpha, \beta) := \max_{\lambda_j \in \text{sp}(P^{-1}H)} \frac{\beta + i\lambda_j}{\alpha + \lambda_j} \cdot \max_{\mu_j \in \text{sp}(P^{-1}D)} \frac{\alpha + i\mu_j}{\beta + \mu_j} \leq \max_{\lambda_j \in \text{sp}(P^{-1}H)} \frac{\sqrt{\beta^2 + \lambda_j^2}}{\alpha + \lambda_j} < 1.$$

**Proof:** By using Kronecker product, (2.1) is reformulated in the matrix-vector form as follows:

$$
\begin{aligned}
&\begin{cases}
(I \otimes (\alpha P_1 + W) + (\alpha P_2 + U)^T \otimes I)vec(X^{(k+\frac{1}{2})}) \\
= (I \otimes (\alpha P_1 - iT) + (\alpha P_2 - iT)^T \otimes I)vec(X^{(k)}) + vec(F),
\end{cases} \\
&\begin{cases}
(I \otimes (\beta P_1 + T) + (\beta P_2 + V)^T \otimes I)vec(X^{(k+1)}) \\
= (I \otimes (\beta P_1 + iW) + (\beta P_2 + iU)^T \otimes I)vec(X^{(k+\frac{1}{2})}) - i vec(F),
\end{cases}
\end{aligned}
$$

Denote by (3.3) and (3.4), then we obtain

$$
\begin{aligned}
&\begin{cases}
(\alpha P + H)vec(X^{(k+\frac{1}{2})}) = (\alpha P - iD)vec((X^{(k)}) + vec(F), \\
(\beta P + D)vec(X^{(k+1)}) = (\beta P + iH)vec(X^{(k+\frac{1}{2})}) - i vec(F),
\end{cases}
\end{aligned}
$$

By straightforward derivations, we can reformulate (3.6) into the standard form
\( \text{vec}(X^{(k+1)}) = \Theta(\alpha, \beta)\text{vec}(X^{(k)}) + \Psi(\alpha, \beta)\text{vec}(F), \)

where

\[ \Theta(\alpha, \beta) = (\beta P + D)^{-1}(\beta P + iH)(\alpha P + H)^{-1}(\alpha P - iD), \]

and

\[ \Psi(\alpha, \beta) = (\beta P + D)^{-1}(\beta P - i\alpha P)(\alpha P + H)^{-1}, \]

Through the similarity invariance of the matrix spectrum, we obtain

\[ \rho(\Theta(\alpha, \beta)) = \rho((\beta P + D)^{-1}(\beta P + iH)(\alpha P + H)^{-1}(\alpha P - iD)) \]

\[ = \rho((\beta I + iP^{-1}H)(\alpha I + P^{-1}H)^{-1}(\alpha I - iP^{-1}D)(\beta I + P^{-1}D)^{-1}) \]

\[ \leq \| (\beta I + iP^{-1}H)(\alpha I + P^{-1}H)^{-1}(\alpha I - iP^{-1}D)(\beta I + P^{-1}D)^{-1} \| \]

\[ \leq \| (\beta I + iP^{-1}H)(\alpha I + P^{-1}H)^{-1} \|_2 \| (\alpha I - iP^{-1}D)(\beta I + P^{-1}D)^{-1} \|_2 \]

Then it is analogous to Theorem 2.1 in [11], thus the following is obtained:

\[ \rho(\Theta(\alpha, \beta)) \leq: \max_{\lambda_j \in \text{sp}(P^{-1}H)} \frac{\beta + i\lambda_j}{\alpha + \lambda_j} \cdot \max_{\mu_j \in \text{sp}(P^{-1}D)} \frac{\alpha + i\mu_j}{\beta + \mu_j} \]

\[ \leq \max_{\lambda_j \in \text{sp}(P^{-1}H)} \frac{\sqrt{\beta^2 + \lambda_j^2}}{\alpha + \lambda_j} \cdot \max_{\mu_j \in \text{sp}(P^{-1}D)} \frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j} \]

\( P^{-1}H, P^{-1}D \) are symmetric because \( PH = HP, PD = DP \) and \( H, D, P \in \mathbb{R}^{(m \times n) \times (m \times n)} \) are symmetric, and because \( \alpha^2 - \beta^2 \leq 2\beta \mu_{\text{min}}, \) then \( \alpha^2 + \mu_j^2 \leq \beta^2 + 2\beta \mu_{\text{min}} + \mu_j^2 \leq \beta^2 + 2\beta \mu_j + \mu_j^2 \leq (\beta + \mu_j)^2. \) So \( \frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j} \leq 1, \rho(\Theta(\alpha, \beta)) \leq \max_{\lambda_j \in \text{sp}(P^{-1}H)} \sqrt{\frac{\beta^2 + \lambda_j^2}{\alpha + \lambda_j}} < 1. \)

The proof is completed.

**Remark 3.1.** When it holds that \( P = H \) in Theorem 3.1, it is equally to take \( P_1 = W, P_2 = U, \) the spectral radium \( \rho(\Theta(\alpha, \beta)) \leq \frac{\beta^2 + 1}{\beta + 1}. \)

In Theorem 3.1, we define \( \beta \) is less than \( \alpha \) and \( \alpha^2 - \beta^2 \leq 2\beta \mu_{\text{min}}, \) then after computations we can compute the value of spectral radium which only involve the part \( \lambda_j \in \text{sp}(P^{-1}H), \) otherwise, we can obtain Theorem 3.2 similarly with the assumption when \( 0 < \alpha \leq \beta. \)

**Theorem 3.2.** Similar to the previous sections, let denote \( A = H + iD, \)
where

\( H = I \otimes W + U \otimes I, D = I \otimes T + V \otimes I, \)

and \( P = I \otimes \alpha P_1 + \alpha P_2 \otimes I, \)
and represent by
\[ \Theta(\alpha, \beta) = I \otimes \alpha P_1 + \alpha P_2 \otimes I, \]
where positive definite and symmetric matrices \( P_1 \in \mathbb{R}^{(m \times m)} \), \( P_2 \in \mathbb{R}^{(n \times n)} \) are prescribed, and \( PH = HP, PD = DP \). If \( 0 < \alpha \leq \beta, \beta^2 - \alpha^2 \leq 2\alpha \lambda_{\min} \), then the unique solution \( X_+ \in \mathbb{C}^{m \times n} \) of (1.1) is obtained as APMHSS iterative algorithm is converged, and the spectral radius \( \rho(\Theta(\alpha, \beta)) \) determines the convergence factor, bounded by \( \sigma(\alpha, \beta) \). It holds that
\[
\rho(\Theta(\alpha, \beta)) \leq \sigma(\alpha, \beta) := \max_{\lambda_j \in \text{sp}(P^{-1}H)} \left| \frac{\beta + \lambda_j}{\alpha + \lambda_j} \right| \cdot \max_{\mu_j \in \text{sp}(P^{-1}D)} \left| \frac{\alpha + i\mu_j}{\beta + \mu_j} \right| < 1.
\]

**Proof:** It is similarly proved as Theorem 3.1.

**Corollary 3.1.** If the conditions of the Theorem 3.1 and Theorem 3.2 is satisfied, the upper bound of the convergence rate of the APMHSS iterative algorithm is smaller than the PMHSS iterative algorithm (2.1).

**Proof:** We can see the upper bound of the APMHSS iterative algorithm in Theorem 3.1 is 
\[
\max_{\lambda_j \in \text{sp}(P^{-1}H)} \sqrt{\frac{\beta^2 + \lambda_j^2}{\alpha + \lambda_j}}, \quad (0 < \beta \leq \alpha),
\]
and the upper bound of PMHSS method in Theorem 2.1 we obtained is 
\[
\max_{\lambda_j \in \text{sp}(P^{-1}H)} \sqrt{\frac{\alpha^2 + \lambda_j^2}{\alpha + \lambda_j}}, \quad \text{apparently.}
\]

By Corollary 3.1, it implies that we can choose suitable parameters \( \alpha, \beta \) to accelerate the convergence of PMHSS. It is implemented in our experiments in the following section.

4. **Numerical experiments**

The PMHSS and APMHSS are compared with MHSS for a few numerical examples in approximating the solution of the complex Sylvester matrix equation. As in [1], the robustness of the MHSS method is compared to the HSS method. We compare all of MHSS, PMHSS and APMHSS method in this section. The experiments were implemented in Matlab on an Intel dual-core processor (2.5GHZ) with 8GB RAM. A zero matrix was used as an initial matrix for all the methods, and the termination condition was \( \| R^{(k)} \|_F \| R^{(0)} \|_F \leq 10^{-6} \), where \( R^{(k)} = F - AX^{(k)} - X^{(k)}B \). The performance is evaluated in terms of the iteration number \( (n_i) \) and the running time \( (t_r \text{ sec}) \).
We denote the example in [1] to take a comparison. Assume that \( A = W + iT = U + iV = B \), and

\[
W = K + (3 - \sqrt{3}) (m + 1) I, T = K + (3 + \sqrt{3}) (m + 1) I,
\]

Such that \( K = I \otimes V_m + V_m \otimes I \), with \( V_m = (m + 1)^2 \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m} \), then the matrix \( K \in \mathbb{R}^{m \times m} \) is an \( m \times m \) blocktridiagonal matrix, with \( n = m^2 \).

Moreover, in our experiments, we choose the preconditioner \( P_1 = W, P_2 = U \). We search the optimal parameter \( \alpha \) in MHSS and PMHSS iterative algorithms, which minimize the iteration number. In the APMHSS iterative algorithm correspondingly, we take the parameter \( \alpha \) invariably with the optimal parameter experimentally found in the PMHSS method, then find the corresponding optimal parameter \( \beta \) minimizing the iteration steps.

| \( n \) | MHSS \( \alpha_{opt} \) | \( n_i \) | \( t_r \) [sec] | PMHSS \( \alpha_{opt} \) | \( n_i \) | \( t_r \) | APMHSS \( \alpha_{opt} \) | \( \beta_{opt} \) | \( n_i \) | \( t_r \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 71.023 | 26 | 0.014 | 1.051 | 18 | 0.004 | 1.051 | 0.582 | 17 | 0.002 |
| 16 | 140.231 | 31 | 0.034 | 1.052 | 18 | 0.007 | 1.052 | 0.641 | 17 | 0.004 |
| 64 | 270.127 | 38 | 0.093 | 1.037 | 17 | 0.043 | 1.037 | 0.671 | 17 | 0.029 |
| 100 | 361.643 | 41 | 2.220 | 1.011 | 17 | 0.167 | 1.011 | 0.782 | 17 | 0.063 |
| 256 | 608.662 | 46 | 8.413 | 1.012 | 17 | 0.470 | 1.012 | 0.800 | 17 | 0.264 |
| 400 | 810.543 | 48 | 13.264 | 1.016 | 17 | 1.187 | 1.016 | 0.623 | 17 | 0.864 |

Table 4.1 shows that the PMHSS outperforms the MHSS in terms of \( t_r \). Also, the APMHSS iterative algorithm performs better than the PMHSS iterative algorithm both in \( n_i \) and \( t_r \). It proves that, in actual experiments, we can accelerate the MHSS method by choosing suitable preconditioner and parameters \( \alpha, \beta \).

5. Conclusion

This paper establishes a PMHSS iterative algorithm to solve complex Sylvester equations and its accelerated variant APMHSS, which provides a more general condition with more parameters. The convergence of the PMHSS iterative algorithm was shown, and it was proved that the APMHSS iterative algorithm is converged. The upper bound of the spectral radius
of APMHSS is smaller than PMHSS. APMHSS employed two parameters. Future works include the exploration of preconditioner and four-parameters to search more probabilities.

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