Optimistic-Conservative Bidding in Sequential Auctions

Avinatan Hassidim∗ Yishay Mansour†

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Abstract

In this work we consider selling items using a sequential first price auction mechanism. We generalize the assumption of conservative bidding to extensive form games (henceforth optimistic conservative bidding), and show that for both linear and unit demand valuations, the only pure subgame perfect equilibrium where buyers are bidding in an optimistic conservative manner is the minimal Walrasian equilibrium.

In addition, we show examples where without the requirement of conservative bidding, subgame perfect equilibria can admit a variety of unlikely predictions, including high price of anarchy and low revenue in markets composed of additive bidders, equilibria which elicit all the surplus as revenue, and more. We also show that the order in which the items are sold can influence the outcome.

∗Bar-Ilan University and Google. avinatanh@gmail.com
†Blavatnik School of Computer Science, Tel Aviv University. This research was supported in part by The Israeli Centers of Research Excellence (I-CORE) program, (Center No. 4/11), by a grant from the Israel Science Foundation, by a grant from United States-Israel Binational Science Foundation (BSF), and by a grant from the Israeli Ministry of Science (MoS). mansour@cs.tau.ac.il
1 Introduction

In everyday economy almost all items are sold at a price set by the seller, given her belief on the demand. In some cases, especially when the seller does not have any credible information regarding the demand, items are auctioned between multiple potential buyers. For example, rare art artifacts are often auctioned in auction houses, such as Sotheby’s or Christie’s. While in theory the auction houses could have conducted a huge combinatorial auction, this is rarely the road taken. Items are simply auctioned individually at some order (decided by the auctioneer). Even when there are multiple items sold at once (e.g., in US spectrum auctions, or off-shore drilling rights), all the participants know that in the future there will be more auctions for the good, and strategies accordingly.

We consider selling multiple items sequentially by holding a separate first price auction with no reserve price for each item. We stress that our goal is not to propose new mechanisms and analyze their benefits, but rather concentrate on the simple existing per item auctions and analyze their weaknesses and strengths.

Our solution concept is a sub-game perfect equilibrium (SPE), where each buyer has a strategy that depends on the history. We assume a full information model, where the seller and each buyer knows everyone else’s valuation. Note that although the seller knows the buyers’ valuations, she is limited in her ability to influence the outcome since her only action is to select the order in which the items are sold.

The starting point of this work is those of Leme et al. [13, 14], who showed the following results:

1. Existence: Every sequential auction has a pure equilibrium.

2. Sort of good news: If all buyers are unit demand, then the price of anarchy of pure equilibria is 2.

3. Bad news: If all buyers are submodular, then the price of anarchy is $\Omega(m)$, where $m$ is the number of items.

Our first contribution is to show a necessary and sufficient condition for an allocation and prices to be a pure SPE for the sequential auction. This condition is very weak, and we use it to show that there are many equilibria that do not make sense, e.g., an equilibrium for submodular bidders in which the social welfare is half of the optimal welfare, but the seller takes it all as revenue.

Worse, we show an example with two additive bidders in which the price of anarchy is $\Theta(m)$, where $m$ is the number of items. Finally, we show that we can not hope to have any non-trivial revenue guarantees on the auction, even when there are two additive bidders.

Motivated by the bad examples of the additive bidders, we generalize the notion of conservative bidding [3, 6] to sequential auctions. The challenge here is twofold. First, conservative bidding is a natural notion when players are additive, and requires adjustment for non-additive bidders. Fortunately, there are several previous works that extended this notion, see e.g. [2, 9]. The bigger challenge is that in a sequential auction part of the reason a buyer may want to buy an item is that it leads to a more desired branch of the game tree, and we do not want to rule that out. Therefore, we define optimistic conservative bidding, which allows a bidder to place a bid such that if he miraculously wins this bid, he is no worse than he is on the equilibrium path. This notion coincides with conservative bidding for a single item. See Appendix for a discussion on why we chose this solution concept, and its relation to trembling hand equilibrium.

The notion of optimistic conservative bidding already allows us to get that the unique equilibrium when all buyers are additive is the “correct” one, where the buyer with the highest valuation for each item gets it and pays the second price.

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1There are numerous other examples of items which are auctioned sequentially, such as the Japanese Fish auctions, the Dutch flower auctions, and Internet ad auctions, such as the ones performed by Yahoo’s Right Media and Google’s Ad-Exchange. Bidders may have combinatorial valuations for the objects at sale, such as an advertiser who wants a campaign in which a user doesn’t see the same ad twice, or a restaurant which needs to buy enough fish for the day.

2The reason we do not allow for reserve price is this full information model. While clearly the more interesting case is the Bayesian model, solving the full information model is a necessary step.

3We also show that the PoA is at most $\Theta(m)$ for subadditive bidders, but we are more worried about the lower bound for additive bidders here.
Unfortunately, already for unit demand buyers optimistic conservative bidding is not enough to guarantee the optimal outcome, and indeed in the bad example of Leme et al.\cite{14} the players bid in an optimistic conservative way. It turns out that the order of selling the items has a crucial effect on both the revenue and the welfare. This immediately raises the algorithmic challenge, of choosing the order: \textit{Given the valuations of the players, and the outcomes of previous auctions, what is the optimal way to choose which item to sell next?}

We solve this challenge for unit demand bidders, showing that if the selling order is chosen correctly, the only pure optimistic conservative equilibrium is the optimal outcome, and (perhaps more important), the minimal Walrasian revenue (equal to the VCG revenue in this case). In Appendix \ref{app} we discuss the selling order, and give an example where it can have a factor of $m$ on the revenue, for unit demand bidders.

\subsection{Related Work}

The equivalence between Walrasian pricing and parallel first price auctions is presented in Bikhchandani \cite{4}, showing that there is a one-to-one correspondence between Walrasian equilibrium and pure Nash equilibria of selling the items using parallel first price auctions.

The work of Hasidim et al.\cite{11} discusses parallel first price auctions and Walrasian pricing. The main motivation there was to understand the case of mixed equilibria in the parallel first price auctions, in the cases where there is no Walrasian equilibrium. This work was later improved by \cite{7}.

The study of sequential auctions in the CS literature was initiated by Leme et al.\cite{13}, which we already discussed. Feldman et al.\cite{8} improved the lower bound on the price of anarchy, and showed that it is at least $\min(n, m)$ in a market where the bidders are either unit demand or additive. Syrgkanis and Tardos\cite{17} extended the model to the Bayesian case. In later work \cite{18} they proposed using composeable mechanisms to study a series of auctions performed together (in parallel or sequentially).

Sequential auctions were also studied extensively in the economic literature\cite{5,12}, where they discuss different strategies that auction houses use to order the goods for sale\cite{11,12}, and the change of price between identical items sold one after the other\cite{10,15}. The results there are not algorithmic, but are somewhat aligned with the selling order we present.

\subsection{Paper Organization}

The paper is organized as follows. We start with some preliminaries and notation. Then in Section \ref{sec:bad} we show the bad SPE for additive players, and also that there is a single optimistic conservative equilibrium. After this motivating example, we show the characterization of the space of all possible equilibria (and mention some bad examples deferring proofs to the appendix). Finally, we move the more involved part of the paper, showing the optimistic conservative equilibrium for unit demand players, and proving that it is the unique pure equilibrium.

Following the advice in the call for papers, in addition to a motivating example we have included discussions in the potential weak spots of the paper as well. One may not like the extension of conservative bidding to extensive form games, and we point to Appendix \ref{app} for a discussion. Another question revolves around extending the result to more complicated valuations. One of the challenges here is demand reduction, which is one of the pain points in auction theory\cite{16}. See Appendix \ref{app} for a discussion on why demand reduction is relevant here, and what is needed to analyze it. Finally, one may be worried that the seller has full information. We are less worried about this, since the seller can only choose the next item to sell, and studying the effect of the order on the revenue is an important question even if the seller is ignorant.

\footnote{Note that we allow the seller to look at the outcome of the previous auction before choosing what to sell next. This is not a problem in the solution concept - imagine that the seller first commits to a function which determines which item is sold next and then an optimistic conservative SPE is chosen.}
2 Model and Preliminaries

Basic setting: We assume that there is a single seller with a set $M$ of $m$ items to sell. There is a set $N$ of $n$ buyers who would like to buy the items. The items are sold sequentially in first price auctions. We allow the seller to choose which item to sell next, and how to set the tie breaking rule, as a function of the allocation of the previous items sold. For an item $j$, we will usually denote $\text{win}(j)$ the buyer who won item $j$.

Each buyer $i \in N$ has a valuation $v_i(S)$ for every set $S \subseteq M$. We assume that $v_i(\emptyset) = 0$, for normalization. Given a set of items $S \subseteq M$ for price $p(S)$ the utility of buyer $i$ is quasi-linear, $u_i(S) = v_i(S) - p(S)$. Also, we assume that buyers are risk neutral, namely, given a distribution over outcomes, their utility is their expected utility.

A buyer is additive (also called linear), if there are non-negative values $v_{i,1}, \ldots, v_{i,m}$ such that the valuation is $v_i(S) = \sum_{j \in S} v_{i,j}$ and $v_i(\emptyset) = 0$.

A buyer $i$ has a unit demand valuation function, if there are non-negative values $v_{i,1}, \ldots, v_{i,m}$ such that the valuation is $v_i(S) = \max_{j \in S} v_{i,j}$ and $v_i(\emptyset) = 0$. If a unit demand buyer already received a set of items $S$, we say that his value for an additional item $f$ is the marginal value of that item $v_i(S \cup \{f\}) - v_i(S)$, and denote this by $v_{i,f|S}$.

Solution concepts. We assume a full information model, where both the seller and each buyer know everyone’s valuation functions. Note that although the seller knows everything, the influence she has is limited to choosing the selling order.

We consider subgame perfect equilibrium. A subgame perfect equilibrium (SPE) specifies a strategy for each buyer, such that the set of strategies are in equilibrium from any possible state (i.e., history), including states which are not reachable by the combined set of strategies. It was shown by that every sequential auction has a pure SPE. For the most part, we consider in this work strategies that depend on the history only through the allocation of items (and not, for example, on specific bids).

Walrasian Equilibrium was proposed by Leon Walras as early as 1874 as a solution concept, and is define as follows:

Definition 1 A Walrasian Equilibrium is a sets of prices $p_j$ for $j \in [1,m]$ and allocation $\text{win}(j) \in [1,n]$ where $\text{win}(j)$ is the winner of item $j$, such that: (1) Each buyer $i$ does a best response, i.e., $v_{i,k} - p_k \geq v_{i,j} - p_j$, where $\text{win}(i) = k$ or if $i$ does not receive any items then $v_{i,j} \leq p_j$ for any item $j$. (2) For each item $j$ we have $\text{win}(j) = i$ for some buyer $i$. We define the sum of the prices $\sum_j p_j$ as the Walrasian revenue.

Conservative bidding. When characterizing auctions, one of the main issues is our assumptions regarding the buyers that would submit ‘loosing bids’. A common way to handle this issue in a one-shot auction is to assume that the buyers bid conservatively, namely, they always bid below their valuation. This notion does not immediately fit sequential auctions. For example, consider an auction with two unit demand buyers $A$, $B$ and two items 1, 2, where the selling order is 1 and then 2. If buyer $A$ is not interested in 2, and buyer $B$ prefers 2 to 1, then a non zero bid of buyer $B$ on 1 is not a conservative move - he can anticipate that he will get item 2, and does not benefit to win 1 as well.

We define the optimistic conservative as a guarantee to the buyer that winning with his current bid would not result in a lower utility than that on the equilibrium path. This is ‘optimistic’ since the buyer is assigning high likelihood that the equilibrium path would be realized. Formally, assuming that we have a subset $S$ of items already allocated, then buyer $i$ would not bid for item $j$ more than $v_{i,j|S} - u_{i|S}$, where $u_{i|S}$. (Note that the allocation $S$ specifies a node in the game tree, since we assume that the nodes of the game depend only on the outcomes of the auctions, i.e., who wins.) An optimistic conservative bidding guarantees that a buyer would never be worse-off winning an item (which he is not suppose to win).
**Price of Anarchy** is the ratio of the maximum social welfare to the worse equilibrium allocation, where the valuation of an allocation is the sum of buyers valuation.

## 3 Additive Valuations

It is often the case that when all buyers have linear (additive) valuation functions, one can look at each item independently, and the analysis becomes simple, inheriting many of the nice properties of single item auctions. Surprisingly, this is not the case for sequential auctions.

**Theorem 1** When all buyers have additive valuation functions, the Price of Anarchy of a sequential auction is $\Theta(m)$, where $m$ is the number of items. The lower bound on the PoA holds even when there are two buyers and all the items are identical (so the selling order is irrelevant).

The above theorem would follow from Theorem 2 and Lemma 1. It is interesting to compare Theorem 1 with the results of [14], which showed a PoA of 2 for unit demand bidders, and showed that the price of anarchy is $\Omega(m)$ for sub-modular buyers.

The upper bound on the PoA in Theorem 1 actually holds for sub-additive buyers, as the following theorem shows:

**Theorem 2** When all buyers have sub-additive valuations, the Price of Anarchy of a sequential auction is $O(m)$, where $m$ is the number of items.

**Proof:** Let $v_{i,j}$ be the valuation buyer $i$ assigns to item $j$, if this is the only item that she gets. Let $M = \max_{i,j} v_{i,j}$, and let $i^*, j^*$ be such that $v_{i^*,j^*} = M$. Since the buyers are sub-additive, the social welfare is bounded by $m \cdot M$.

Consider the equilibrium path. If the valuation of buyer $i$ bundle is more than $M/2$, we are done. Else, it must be that the price of $j^*$ is at least $M/2$ (since buyer $i^*$ prefers the equilibrium path to snatching $j^*$ when it comes up for sell). In this case, the revenue is lower bounded by $M/2$, and so is the social welfare. □

**Lemma 1** The Price of Anarchy for additive buyers is $\Omega(m)$ even for identical items and for any selling order.

**Proof:** Consider a market with two additive buyers $A, B$, and $m$ items, denoted $1, \ldots, m$. We have that for any $i$

$$V_{A,i} = m \quad \text{but} \quad V_{B,i} = 1$$

As the items are identical, the selling order is irrelevant, and we therefore assume that they are always sold in the order $1$ to $m$.

It is enough to present a bad equilibrium. While buyer $B$ received all items that were on sale so far, both buyers bid 1 and ties are broken in favor of $B$ on the first $m - 1$ items and in favor of $A$ in the last item. If $A$ has at least one item both buyers bid $m$, and ties are broken in favor of $A$.

In the described strategy $A$ has a utility of $m - 1$ while $B$ has a utility of 0. Observe that if $A$ outbids $B$ on any item, he wins all the remaining items and has a utility of zero for the additional items. This implies that $A$ will end with a utility strictly less than $m - 1$. Clearly $B$ has no incentive to deviate since he will always have utility 0. It is easy to see that it obtains a social welfare of $2m - 1$ where the maximal social welfare is $m^2$. □

In the market we presented, at least the revenue behaves correctly, in the sense that the revenue is the Walrasian revenue. The following example shows that this is not always the case.

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8The fact that this is a SPE also follows from our characterization of subgame perfect equilibria (Theorem 5).
Theorem 3 There exists a sub-game perfect equilibrium for additive buyers where the ratio of its revenue to the Walrasian revenue is vanishing.

Proof: As before, consider a market with two additive buyers $A, B$, and $m$ items, denoted $1, \ldots, m$. We have that $V_{A,i} = m$ for any item $i$ and $V_{B,i} = 1$ for $i \leq m - 1$ and $V_{B,m} = \epsilon$. Given that item $m$ is always sold last, the following is an equilibrium:

While buyer $B$ receives all items that are on sale so far, both buyers bid $\epsilon$ (i.e., $b_A = b_B = \epsilon$) and ties are broken in favor of $B$ on the first $m - 1$ items and in favor of $A$ in the last item. If $A$ has at least one item both buyers bid $m$, and ties are broken in favor of $A$.

In this equilibrium, the revenue is $\epsilon m$, which is arbitrarily lower than $m - 1 + \epsilon$, which is the Walrasian revenue.

In contrast to the negative results in the general case, we show that when the buyers are bidding optimistic conservative, we get a PoA of 1, and the revenue the minimal Walrasian revenue. Let the max tie-breaking rule break a tie between a subset of buyers that bid the same value for a given item $i$ as selecting $\arg\max_{j \in S} v_{i,j}$. Note that the max tie-breaking has a different priority for each item.

Theorem 4 For the max tie breaking rule, for any selling order the only pure optimistic conservative equilibrium is the one in which each item $i$ is allocated to a buyer in $\arg\max_j v_{j,i}$ and the price is the second highest valuation.

Proof: The proof is by induction.

Base case Let $f$ denote the last item being sold. Regardless of the allocation of the previous items, we have a first price auction for a single item. Since in the last item optimistic conservative bidding coincides with conservative bidding, no buyer $j$ can bid more than $v_{j,f}$. Let $w$ be the buyer that maximizes $v_{j,f}$, and let $s$ be the buyer with the second highest valuation, that is

$$s = \arg\max_{j \neq w} v_{j,f}$$

Since buyers are conservative, buyer $j$ can not bid over $v_{j,f}$. Therefore, the equilibrium price is at most $v_{s,f}$.

If the equilibrium price is $p < v_{s,f}$, then at least one of $s$ or $w$ would be better-off bidding $\frac{p + v_{s,f}}{2}$ than their current bid. Therefore, we have $p = v_{s,f}$.

If the price is $v_{s,f}$, then either

1. Buyer $w$ values $f$ strictly more than $s$, that is $v_{w,f} > v_{s,f}$ and $w$ always buys
2. There are several (more than one buyer) with value $v_{w,f}$ for the item, and one of them always buys

If we assume no ties, then the first option is the only equilibrium. If ties are possible, then there are several equilibria, but they are all equivalent (item $f$ has the same price and goes to a buyer in $\arg\max_j v_{j,f}$ who desires it the most).

Inductive step In the inductive step we assume that no matter who gets item $f$, for any subsequent item $g$, the buyer who gets $g$ is one of the buyers who want it the most, i.e., $\arg\max_j v_{j,g}$, and would pay the second valuation of $g$.

In this case, the buyer which is allocated $f$ is independent of the rest of the auction, and an argument very similar to the base case shows that one of the buyers in $\arg\max_j v_{j,f}$ will be allocated $f$ and pay the second highest valuation.

Remark: The uniqueness also holds for mixed equilibria, via an inductive argument. The base case appears in Appendix A. For the inductive step we note that no matter who wins the item the auction stays the same, and therefore we invoke the base case again (just like the inductive step in the proof of Theorem 3 applies the base case).
4 Equilibrium Characterization

To motivate the definition of optimistic conservative bidding, we show the large variety of possible pure subgame perfect equilibria in sequential first price auctions. To do this, we provide a characterization of this space.

Let \( \text{win} : M \to N \) be an allocation function, and let \( p(1), \ldots, p(m) \) be prices. It would be convenient to define the set of items buyer \( i \) wins out of the first \( j \) items sold using the permutation \( \pi \) as \( x_i(\pi, j) = \{ k : k \in \{ \pi_1, \ldots, \pi_j \}, \text{win}(k) = i \} \). Let

\[
    u_i(\text{win}, p) = v_i(x_i(\pi, m)) - \sum_{k \in x_i(\pi, m)} p(k) = \sum_{k \in x_i(\pi, m)} v_{i,k} - p(k)
\]

be the utility of buyer \( i \) at the end of the auction assuming everyone played according to \( \text{win} \) and \( p \). One possible deviation for buyer \( i \) is to play according to the equilibrium up to item \( j \), to snatch item \( j \) by bidding \( p(j) + \epsilon \), and to bid zero on all subsequent items. The following theorem shows that there is an SPE with allocation \( \text{win} \) and prices \( p \) if and only if this deviation is not profitable for some permutation \( \pi \) of the items. Formally,

**Theorem 5** Fix an allocation \( \text{win} \) and prices \( p \) such that for any buyer \( i \) we have \( u_i(\text{win}, p) \geq 0 \). Suppose that there exists a permutation \( \pi \) (which describes the selling order assuming everyone plays the equilibrium path) such that for every buyer \( i \) and for every item \( j \)

\[
    u_i(\text{win}, p) \geq v_i(x_i(\pi, j - 1) \cup \{ \pi_j \}) - p(\pi_j) - \sum_{k \in x_i(\pi, j - 1)} p(k) .
\]

Then there is an SPE with allocation \( \text{win} \) and prices \( p \).

On the other hand, if for every permutation \( \pi \) there exists a buyer \( i \) and an item \( j \) such that Equation (1) does not hold, then there is no pure SPE with \( \text{win} \) and \( p \).

**Proof:** We present the strategies for the SPE assuming (1) holds where \( \pi \) is the identity. Items are always sold in the order \( 1, \ldots, m \) regardless of who bought previous items. On the equilibrium path, at step \( j \) (that is when auctioning item \( j \)), all buyers bid \( p(\pi_j) \) and ties are broken in favor of buyer \( \text{win}(j) \). Off the equilibrium path, at some state \( G \) of the game let \( X_i(G) \) denote the set of items buyer \( i \) has at state \( G \). Given that at state \( G \) we are selling item \( j \), we define the marginal valuation of buyer \( i \) at state \( G \) as,

\[
    v_{i,j|G} = v_i(\{X_i(G)\} \cup \{j\}) - v_i(\{X_i(G)\}) .
\]

We prove an inductive claim on the utilities of the buyers in the off equilibrium part of the game tree:

**Claim 1** Consider an off-equilibrium state \( G \) in which item \( j \) is sold. If the buyers play according to the SPE rooted at \( G \) no buyer will gain positive utility, and all deviations give either zero or negative utility.

**Proof:** The proof is by induction from the leaves to the root. Consider a leaf state \( G \) where item \( m \) is sold. Let \( k \) be the buyer which maximizes \( v_{k,m|G} \). In state \( G \) we have a first price auction for item \( m \), where buyer \( k \) wins the item and pays \( v_{k,m|G} \), which gives her zero utility. If buyer \( k \) bids less, she does not get the item (as everyone else bids \( v_{k,m|G} \) and therefore has zero utility, and if she bids more she has negative utility. For any buyer \( r \neq k \), underbidding does not affect the outcome, and overbidding buyer \( k \) would give negative marginal utility.

\[\footnote{Note that the buyers may have gained positive utility before reaching \( G \), we only claim that from \( G \) onwards no utility will be gained.}\]
Now we need to prove the inductive step. By induction hypothesis all the children vertices of state \( G \) give zero utility to all the buyers. Similar to the leaf case, that there is no local action which gives a positive utility in state \( G \) to any buyer.

If Condition (1) does not hold for any permutation, then there is no SPE, since at least one of the buyers would deviate. \( \square \)

We present two surprising corollaries of this theorem. We view this weird equilibria as more evidence that one must refine the set of possible equilibria in a sequential first price auction, and not as plausible outcomes.

Define non-singleton valuation, where \( v_i(\{j\}) = 0 \) for any \( j \in M \). Note that this includes as a special case, single minded buyers with sets of size at least two.

**Corollary 1** Assume that we have \( n \) buyers who have non-singleton valuations then there is a sub-game perfect equilibrium, where all the prices are zero and one buyer receives all items.

The proof of this corollary is simple, and appears in the full version.

In contrast to the previous example, where the prices were very low, one can build equilibria in which the prices are high, and the auctioneer gets a lot of revenue. Consider a market with sub-modular buyers. Let \( \pi \) be the following selling order, defined inductively. Let \( i_1, j_1 \) the buyer and item that maximize \( \max_{i,j} v_i(\{j\}) \). Now at step \( k \), let \( i_k, j_k \) be the pair of buyer and item who maximize

\[ v_{i,j|G} = v_i(\{X_i(G)\} \cup \{j\}) - v_i(\{X_i(G)\}) \]

where \( G \) is the state of the game where buyer \( i_r \) wins item \( j_r \) for every \( r < k \).

**Corollary 2** Assume all buyers have a submodular valuation. If the items are sold according to \( \pi \), there is an SPE which elicits at least half the optimal social welfare as revenue. Moreover, in this SPE the players have no utility, and all the welfare goes to the auctioneer.

This corollary is proven in Appendix E.

5 Unit Demand valuations

5.1 Walrasian Equilibrium for Unit Demand Buyers

Walrasian equilibrium exists when all buyers are gross substitute. We focus in this section on Walrasian equilibrium when all the buyers are unit demand, to enable us to state the order in which we sell the items. We begin with the definition of supporting a price. Consider a Walrasian equilibrium defined by the allocation \( \text{win} \) and a price vector \( p_1, \ldots, p_m \).

**Definition 2** Buyer \( i \) supports the price of item \( j \) if \( \text{win}(j) \neq i \) and either: (1) there is an item \( k \) such that \( \text{win}(k) = i \) and \( v_{i,k} - p_k = v_{i,j} - p_j \), (2) otherwise (buyer \( i \) gets no item) \( v_{i,j} = p_j \).

Note that the definition of support almost implies conservativeness. Next we define a minimal Walrasian Equilibrium.

**Definition 3** A minimum Walrasian Equilibrium is a set of price \( p_1, \ldots, p_m \) such that for any other Walrasian equilibrium \( p_1', \ldots, p_m' \), we have \( p_i \leq p_i' \) for any item \( i \).

We show that there exists a minimum Walrasian equilibrium when the buyers are unit demand.\(^{10}\)

**Theorem 6** For unit demand buyers there always exist minimum Walrasian prices.

\(^{10}\)This is true in general but not necessary for this work.
In this section we describe the strategy Unit-Wlrs-Eq each node which item is sold and what are the bids.) each sub-tree the on-equilibrium strategy. This defines the strategy in each node. (We need to define for each node which item is sold and what are the bids.)

Claim 2 A minimum Walrasian equilibrium has for each item \( j \) with a strictly positive price \( (p_j > 0) \) a buyer \( i \) that supports it.

Proof: Assume that there is an item \( j \) with \( p_j > 0 \) and no supporter. Let \( u_i \) be the utility of buyer \( i \) in the equilibrium. Let \( \epsilon = \min_{i \neq \text{win}(j)} u_i - (v_{i,j} - p_j) \). Since item \( j \) does not have any supporter \( \epsilon > 0 \). Consider an \( \epsilon/2 \) decrease in the price of item \( j \). Buyer \( \text{win}(j) \) definitely still prefers item \( j \). Any other buyer \( i \), since it was not a supporter of item \( j \), then after the \( \epsilon/2 \) decrease it still does not prefer item \( j \). Therefore we showed that the original Walrasian equilibrium is not minimum.

We can now define the order of the support in a minimum Walrasian equilibrium.

Definition 4 A support order of a Walrasian equilibrium is a permutation \( \pi \) of the items, such that for any item \( j \) the buyer \( i \) that supports it receives a later item or no item.

Theorem 7 A minimum Walrasian equilibrium has a support order \( \pi \).

Proof: We build the permutation \( \pi \) from the end to the start. The last items are items that are priced at zero (if there are any) in an arbitrary order. All the remaining items have strictly positive prices, and therefore have at least one supporter (by Claim 2).

Let \( D \) be the set of buyers that either buy at price zero or do not buy and \( I \) be the set of items they buy. If there is an item \( i \) such that \( \text{win}(i) \notin D \) and \( \text{support}(i) \in D \), we add item \( i \) to \( I \) and to the permutation \( \pi \), and add \( \text{win}(i) \) to \( D \). The process can terminate either if we exhaust all the items, in which case we have a permutation \( \pi \), or the set of remaining items do not have supporters in \( D \), i.e., for any item \( i \notin I \) and any supporter we have \( \text{support}(i) \notin D \). We will show that such an event will contradict the fact that we have a Walrasian equilibrium.

In such a case for any \( j \in D \) we have that \( v_{j,k} - p_k > v_{j,i} - p_i \) for the item \( k \) buyer \( i \) buys (\( \text{win}(k) = j \)) and any item \( i \notin I \). Let

\[
\epsilon = \min_{j \in D, i \notin I, \text{win}(k) = j} (v_{j,k} - p_k) - (v_{j,i} - p_i)
\]

We are guarantee that \( \epsilon > 0 \). Consider reducing the prices of all items not in \( I \) by \( \epsilon/2 \). The preference between items not in \( I \) by buyers not in \( D \) does not change. No buyer in \( D \) will prefer an item not in \( I \), by definition of \( \epsilon \). A buyer not in \( D \) will not prefer an item in \( I \) since he has an item he prefers not in \( I \) and therefore after lowering the price his preference only strengthen.

This results in a Walrasian equilibrium with lower prices, which contradict that we started with a minimum Walrasian equilibrium.

5.2 Walrasian sub-game prefect equilibrium (Unit-Wlrs-Eq)

In this section we describe the strategy Unit-Wlrs-Eq. We basically define the tree strategy by defining for each sub-tree the on-equilibrium strategy. This defines the strategy in each node. (We need to define for each node which item is sold and what are the bids.)
Given the set of all items \( I \) we compute the minimum Walrasian equilibrium and its support order. The on-equilibrium strategy is the support order of the items. Let \( u_i \) be the utility of buyer \( i \) in the Walrasian equilibrium, using the marginal utilities. The bid of buyer \( i \) on item \( j \) on the equilibrium path is \( v_{i,j} - u_i \), if \( i \) did not win an item yet and 0 otherwise. Note the winning buyer and the supporting buyer both bid the Walrasian equilibrium price.

We now complete the off-equilibrium path strategy. If a non-winner buyer bids below his expected bid or the winner buyer bids above his expected bid we stay on the equilibrium path (the allocation did not change). If either the winner bids below the price (losing the item, which we call underbidding) or another bidder bids higher than the price (winning the item, which we call overbidding) we basically construct recursively a strategy for the remaining items.

Before we complete the off-equilibrium strategy, we first define the residual valuation of a buyer. Given the already allocated items \( S \) we define for each buyer a marginal valuation,

\[
v_{i,j|S} = \max(0, v_{i,j} - \max_{j' \in S, \text{win}(j') = i} v_{i,j'})
\]

Give the set of items already sold, we compute the minimum Walrasian equilibrium \( W_{I-S} \) on \( I-S \) with valuation \( v_{i,j|S} \), where \( \text{win}^{I-S} \) is the allocation \( p^{I-S} \) are the prices. Let \( u_{i|S} \) be the utility of agent \( i \) in \( W_{I-S} \) with respect to \( v_{i,j|S} \). (This is essentially the marginal utility from the item he receives in \( W_{I-S} \).) Give \( W_{I-S} \) we compute the support order of \( W_{I-S} \) and sell items in that order. The bid of buyer \( i \) which has already received his item in \( W_{I-S} \) is zero. The bid of buyer \( i \) which has not received yet his item in \( W_{I-S} \) is \( v_{i,j|S} - u_{i|S} \). Again, note that the winner and supporter of item \( j \) bid the new Walrasian price, i.e., \( p^{I-S}_{j} \).

We continue this recursive process until we define the entire strategy tree. In Section 5.3 We prove that this is indeed an equilibrium.

**Theorem 8** The resulting strategy is sub-game perfect.

The equilibrium strategies are optimistic conservative by construction.

### 5.3 Proof of Theorem 8

The proof is by induction on the height of the node in the strategy tree from the leaf (which is also equal to the number of items left). The following is the inductive claim.

**Claim 3** Every subtree of height \( \ell \) (from the leaves) defines a subgame perfect equilibrium.

We start by proving the base of the inductive claim (Claim 4).

**Claim 4** The leaves of the strategy tree (subtrees of height 1) define an equilibrium strategy.

**Proof:** In a leaf node \( r \) where an item \( j \) is sold. The minimal Walrasian equilibrium is the second highest valuation \( v_{i,j|S} \) where \( S = I - \{j\} \). Since there is no continuation, each buyer \( i \), except for the highest valuation buyer, bids \( v_{i,j|S} \). Let \( i_r \) be the highest valuation buyer and he bids \( b_{i_r} = \max_{i \neq i_r} \{v_{i,j|S}\} \).

This is an equilibrium, since any buyer \( i \neq i_r \) bidding higher than \( b_{i_r} \) would result in negative utility. Since this is a leaf, a subgame perfect equilibrium coincide with equilibrium. \( \square \)

In the inductive step, some subset of items \( S \) has been sold. The current item on sale is \( f \). We show two lemmas.

**Lemma 2** Assume that Claim 5 holds for any subtree of height at most \( \ell - 1 \). Then in any node of height \( \ell \) underbidding is weakly dominated.

Lemma 2 follows from the following lemma
**Lemma 3** Buyer \( \text{win}(f) \) weakly prefers getting \( f \) for \( p_f \) to any minimal Walrasian equilibrium generated after \( f \) is given to any other buyer.

If there are no ties, then this is a strong preference.

**Proof:** Denote by taker the buyer who received the item, and let the new Walrasian equilibrium be denoted as \( \text{WIN} \) and \( \mathcal{P} \).

If buyer \( \text{win}(f) \) does not buy any item at \( \text{WIN} \), or buys an item \( g_1 \) such that \( \mathcal{P}_{g_1} \geq p_{g_1} \), then \( \text{win}(f) \) does not strictly prefer the new equilibrium. Therefore, we only need to consider the case that for some item \( g_1 \) we have \( \text{WIN}(g_1) = \text{win}(f) \) and \( \mathcal{P}(g_1) = p(g_1) - \delta \) for some \( \delta > 0 \).

To reach a contradiction, we assume that there is a maximum size set \( D \) of \( k \geq 1 \) items, such that for each \( j \in D \) we have \( \mathcal{P}_j \leq p_j - \delta \), and show that there must be a set of buyers \( B(D) \) with the following properties:

1. Each buyer \( i \in B(D) \) bought an item in \( \text{WIN} \) whose price dropped by at least \( \delta \). Formally, letting \( \text{WIN}(j) = i \) we have \( \mathcal{P}_j \leq p_j - \delta \).
2. The set is large: \( |B(D)| \geq k + 1 \)

The existence of this set implies a contradiction. If \( k + 1 \) buyers bought items cheaper by \( \delta \), how can only \( k \) items be cheaper by \( \delta \)?

To show that \( B(D) \) exists, let

\[
B(D) = (\{\text{win}(f)\} \cup \{\text{sup}(x) : x \in D\} \cup \{\text{win}(x) : x \in D\}) \setminus \{\text{taker}\}
\]

We show that the first condition holds. Note that \( \text{win}(f) \) bought an item whose price dropped \( \delta \) by our assumption. Consider an item \( j \in D \) and a buyer \( i \) with \( \text{win}(j) = i \) or \( \text{sup}(j) = i \). If buyer \( i \) did not get an item in \( \text{WIN} \), then we have a contradiction, since \( v_{i,j} \geq p_j \geq \mathcal{P}_j + \delta \). This implies that each buyer in \( \{\text{sup}(x) : x \in D\} \cup \{\text{win}(x) : x \in D\} \) received an item which maximizes their utility given the price vector \( \mathcal{P} \). Letting \( \text{WIN}(j') = i \), it must be that

\[
v_{i,j'}|S - \mathcal{P}_j' \geq v_{i,j}|S - \mathcal{P}_j
\]

but we also know that

\[
v_{i,j}|S - \mathcal{P}_j \geq v_{i,j'}|S - \mathcal{P}_j' \quad \text{and} \quad \mathcal{P}_j \leq p_j - \delta,
\]

which implies that \( \mathcal{P}_j' \leq p_j' - \delta \).

We now show the second condition, that \( |B(D)| \geq k + 1 \). Since we are selling items using a support order, we have that

\[
|\{\text{sup}(x) : x \in D\} \cup \{\text{win}(x) : x \in D\}| \geq k + 1,
\]

since the last item that is sold in \( D \) has the supporter not winning any item in \( D \).

In addition, \( \text{win}(f) \notin \{\text{win}(x) : x \in D\} \) since all the items in \( D \) had strictly positive price in \( p \), and a buyer cannot buy two items with positive price in an equilibrium. Finally, \( \text{win}(f) \notin \{\text{sup}(x) : x \in D\} \) since the items are sold in support order.

Together this implies that \( |\{\text{win}(f)\} \cup \{\text{sup}(x) : x \in D\} \cup \{\text{win}(x) : x \in D\}| \geq k + 2 \), and hence \( |B(D)| \geq k + 1 \).

\( \square \)

**Proof of Lemma 2** Consider a node \( r \) of height \( \ell \) where an item \( f \) is sold, and previously sold items are allocated using \( S \). Assume that from the \( r \) the on equilibrium path is \( (\text{win}, p) \).

If any player except \( \text{win}(f) \) underbids, the payment and allocation do not change, and the claim follows from the inductive hypothesis.

If \( \text{win}(f) \) underbids, he no longer gets the item and the buyer \( \text{sup}(f) \) wins the item. The strategy tree continue to node \( r' \), in which \( \text{sup}(f) \) wins the item. According to Lemma 3 this is not desirable for \( \text{win}(f) \). \( \square \)
Lemma 4 Assume that Claim 3 holds for any subtree of height at most $\ell - 1$. Then in any node of height $\ell$ overbidding is weakly dominated.

Proof: Consider a node $r$ of height $\ell$ where an item $f$ is sold, and previously sold items are allocated using $S$. Assume that from the $r$ the on equilibrium path is $(\text{win}, p)$.

If buyer $\text{win}(f)$ overbids, the allocation does not change and the price goes up, so it is strictly dominated. A buyer $i \neq \text{win}(f)$ overbids and does not get the item nothing changes (since it is a first price auction we do not need to worry about payments).

The case we need to consider is that buyer $i$ overbids $\text{win}(f)$ and wins the item. The strategy tree continue to node $r'$ with a new Walrasian equilibrium define by $\text{winn}$ and $p_n$. Let $S_n$ be the allocation $S$ with the addition that $\sup(f)$ is allocated $f$.

If buyer $i$ does not get an item in the new equilibrium, then the deviation was not profitable, since by the inductive hypothesis (Claim 3) following deviations after $r'$ are not profitable. Let item $j$ be such that $\text{winn}(j) = i$. If $p_n_j \geq p_j$, then the deviation was not profitable since win is an equilibrium (so either $v_{i,j} \leq p_j$ or buyer $i$ received an item he prefers under $p$). So it must be that $p_n_j = p_j - \delta$ for some $\delta > 0$.

Again we let $D$ be a maximum size set such that for each $j \in D$ we have $p_n_j \leq p_j - \delta$, and denote $|D| = k$. We show that there must be a set of buyers $B(D)$ with the following properties:

1. Each buyer $i' \in B(D)$ bought an item in $\text{winn}$ whose price dropped by at least $\delta$. Formally, letting $\text{winn}(j) = i'$ we have $p_n_j \leq p_j - \delta$.
2. The set is large: $|B(D)| \geq k + 1$

We define

$$B(D) = \{i\} \cup \{\sup(x) : x \in D\} \cup \{\text{win}(x) : x \in D\}$$

Since we sell items in a support order, we have that $|B(D)|$ is at least $k + 1$, establishing the second condition. An in Lemma 2 each buyer in $B(D)$ has to win an item whose price dropped. \qed

Ruling out underbidding and overbidding finishes the proof of Theorem 8.

5.4 Optimistic Conservative and Uniqueness

We would like to show that our strategy Unit-Wlrs-Eq is the unique pure optimistic conservative bidding strategy, given our game tree. More precisely, we fix the tree order of selling the items and the tie-breaking rules in each node. Given this, the only pure sub-game perfect optimistic conservative bidding strategy is Unit-Wlrs-Eq. We establish the following main result.

Theorem 9 The equilibrium Unit-Wlrs-Eq is the unique pure optimistic conservative sub-game perfect, up to degeneracies.

In this extended abstract we assume that valuations are generic, so there are no degeneracies. Handling ties in the valuations requires a bit more care, but doesn’t change the proof fundamentally.

Proof: The proof is based on induction from the leaves of the tree to the root.

Base case The base case is the leaves of the tree. In each leaf $(S, f)$, we have a set of items $S$ which was already allocated, and the last item $f$ is being on sale. Recall that $v_{i,f|S}$ is the residual valuation of buyer $i$ for item $f$ given the allocation $S$. Therefore we have a single item first price auction with valuations $v_{i,f|S}$. In this case, optimistic conservative bidding and conservative bidding coincide. In particular, buyer $i$ can bid above $v_{i,f|S}$, the residual value for buyer $i$ for item $f$ given the allocation of the items in $S$. The minimal revenue Walrasian equilibrium has the highest valuation buyer winning the item at the second highest valuation price.

Let $w = \text{winn}(f) = \arg\max_i v_{i,f|S}$ be the buyer that maximizes the residual valuation. According to the tie breaking rules, ties are always broken in favor of $w$. Let $s = \sup(f)$ be the player which maximizes $v_{i,f|S}$ where $i \neq \text{winn}(f)$, i.e., $s = \arg\max_{i \neq w} v_{i,f|S}$. That is, $s$ has the second highest valuation for the item,
given the allocation $S$. Note that since only $w$ can bid above $v_{s,f|S}$, and since ties are broken in favor of $w$, no player would ever bid strictly above $v_{s,f|S}$.

Now if the price is strictly below $v_{s,f|S}$, at most one of $s$ and $w$ would win the item, and the other would deviate and take it. Therefore the price is $v_{s,f|S}$.

**Inductive step** For the inductive step, an item $f$ is being sold, after a set $S$ was already sold. The inductive hypothesis is that any extension of the allocation $S$ by allocating the item $f$ to one of the buyers, the only optimistic conservative sub-game perfect equilibrium is the minimal Walrasian equilibrium. Let $(\text{winn}, p_n)$ be a minimal Walrasian equilibrium with respect to $v_{-,|S}$. Suppose now that $\text{winn}(f)$ is not allocated item $f$ and item $f$ is allocated to a buyer $i$. In this case, due to the inductive hypothesis, buyer $\text{winn}(f)$ utility is determined by a minimal Walrasian equilibrium, in which $S$ is extended to $S'$ by allocating $f$ to $i$.

Lemma 3 shows that this is not an equilibrium, since $\text{winn}(f)$ would be better off bidding $v_{\sup(f),f|S}$ and winning $f$ for sure.

Suppose that $\text{winn}(f)$ is allocated the item $f$, but for some price $p < p_n$. Due to the inductive hypothesis, we know that after $f$ is allocated to $\text{winn}(f)$, the rest of the minimal Walrasian equilibrium plays out. The definition of supporting buyer gives that $u_{\sup(f)} = v_{\sup(f),f|S} - p_n$, and therefore buyer $\sup(f)$ is better off bidding $(p + p_n)/2$. Therefore, player $\text{winn}(f)$ is allocated the item $f$ for the price $p_n$, and the theorem holds.

□

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Enumerate the four cases:

• Case $F_w(b_w) = 0, F_s(b_s) = 0$: In this case $w$ is not best responding. Note that there is an $\epsilon_0 > 0$, such that for any $\epsilon < \epsilon_0$ there is a bid $b'_w > b_w$ which has $F_w(b'_w) \geq \epsilon$ and $F_s(b'_w) < 1$. This implies that the utility of $w$ is at most $\epsilon$ for arbitrary small $\epsilon$ and therefore it is 0. However (due to the non-generic valuations) by bidding $v_{s,f}$ buyer $w$ is guaranteed a positive utility. Contradiction to the assumption that this was an equilibrium.

• Case $F_w(b_w) > 0, F_s(b_s) = 0$: This case $w$ is not BR and would do better to shift the distribution mass from $b_w$ to $v_{s,f}$.

• Case $F_s(b_s) > 0$: In this case $s$ has zero utility when it bids $b_s$, hence it has overall zero utility. Since $s$ bids conservatively it implies that it either bids $v_{s,f}$ or losses with any other bid. This implies that $h_s = \sup_b \{ b : F_s(b) < 1 \}$ has $h_s \leq b_w$. Therefore we need that $b_w = v_{s,f}$ otherwise buyer $s$ can bid
(b_w + v_{s,f})/2 and secure a positive utility. Since w bids at most v_{s,f} we have established that w always bids v_{s,f}.

B The Effect of the Selling Order

Part of the contribution of this work is to raise the algorithmic question of choosing the selling order. In this appendix, we present an example where the selling order leads to a factor of m in the revenue, of the worse equilibrium. We use the same market to show an example in which the worse revenue of a bad order is far from the Walrasian revenue.

**Theorem 10** There is a market with m items and m + 1 unit demand buyers and maximal social welfare of m + o(1) where for some ordering of items there exists an SPE whose revenue is 1, and for a different ordering of items every SPE has revenue at least m.

And also

**Theorem 11** There is a market with m items and m + 1 unit demand buyers where the minimal Walrasian revenue is m, and there exists an order of items and an SPE whose revenue is 1.

**Proof:** [of Theorems 10 and 11] We begin by describing the market, denoting the items as 1, . . . , m, and the buyers as \{b_i : 0 ≤ i ≤ m\}. Each buyer b_i will have a set of items which interest it, and will have the same value of 1 + \epsilon_i for all the items in the set (we later take \epsilon = 1/m^{3/2} to guarantee the bound on the social welfare). Buyer b_0 is interested in item 1, and values it for 1. Buyer b_i for 1 ≤ i ≤ m − 1 is an OR between items i and i + 1, and values each of them 1 + \epsilon_i · \epsilon. Buyer b_m is only interested in item m, and values it 1 + m · \epsilon.

We start with a bad SPE equilibrium for the order 1, . . . , m.

**Claim 5** There exists an order of items and an SPE whose revenue is 1.

**Proof:** We apply Theorem 3 with the allocation \omega(i) = i and prices \( p(1) = 1 \) and \( p(i) = 0 \), for \( i ≥ 2 \). That is, item 1 goes to buyer 1 for 1, item i goes to buyer i for 0, and buyer 0 gets no items.

Intuitively, buyers 0 and 1 compete for item 1, and buyer 1 wins. Now there is no more competition in the market, and all items have price zero.

Since in equilibrium at least one buyer won’t get any items, it easy to see that the revenue of any equilibrium is at least 1.

We now consider what would happen if we sell the items in the order m, m − 1, . . . , 1.

**Claim 6** In any SPE, for any buyer b_i with i ≥ 1, we have \( \omega(i) = b_i \).

**Proof:** The proof is by induction. Buyer \( b_m \) values m more than \( b_{m-1} \) does, and \( b_m \) only values item m. Therefore, in any equilibrium \( \omega(m) = b_m \). Now if buyer \( b_{i+1} \) gets item \( i + 1 \) in step \( m - i \) of the auction, then in the leftover market buyer \( b_i \) is only interested in item i. Since \( b_i \) values item i more than buyer \( b_{i-1} \) does, buyer \( b_i \) must get it, and \( \omega(i) = b_i \).

We now turn to the prices

**Claim 7** In any SPE we have \( p_1 ≤ p_2 ≤ . . . ≤ p_m \)

**Proof:** Suppose that \( p_i > p_{i+1} \) for some i. But since we know buyer i gets item i, and item i + 1 is sold before item i, that would make buyer i regret not bidding more for item i + 1.

Finally, \( p_1 ≥ 1 \), as buyer \( b_0 \) gets no items, and we have \( v_{b_0}(\{1\}) = 1 \). This gives a total revenue of at least m, and proves Theorem 11.
To prove Theorem 11, we show that the Walrasian revenue is \( m \). It is easy to see that it is at most \( m \), by using the price vector \( p_i = 1 \) for every \( i \), and \( \omega(b_i) = i \).

To show that it is at least \( m \), we give a more general claim. Suppose that all the buyers are unit demand and given a buyer \( i \) and an item \( j \) either \( v_i(j) \geq 1 \) or \( v_i(j) = 0 \). Let \( S_i = \{ j : v_i(j) \geq 1 \} \).

Given a set of items \( T \) let \( B(T) \) be the set of buyers that want items in \( T \), i.e., \( B(T) = \{ i : S_i \cap T \neq \emptyset \} \). A set of buyers is “complete” if for every set of items \( T \) we have \( |T| < |B(T)| \).

Claim 8 For a complete set of buyers the Walrasian prices for all items price are at least \( 1 \).

Proof: Assume that item \( j \) is sold at a price \( p < 1 \). Then any buyer in \( B(j) \) buys an item for price at most \( p \). Let \( T_1 \) be the set of items bought by the buyers in \( B(j) \). Now we can consider \( B(T_1) \). Again, we can continue the process. Since the system is “complete” we are guaranteed that we can add more buyers in each iteration. At the end we will have the set of items include all items, and there are still buyers that did not buy any item. Those buyers can bid \( p' \in (p, 1) \) and have a positive utility. A contradiction that some item was sold at a price \( p < 1 \).

Since the set of all items in the theorem is complete, the minimal Walrasian revenue is \( m \). This completes the proof of Theorem 11.

C Demand Reduction and More Complex Valuations

The definition of optimistic conservative bidding allows us to disqualify threats which are not credible (see [11]), and choosing the order wisely lets us build up auction pressure. However, these two are not enough to guarantee (say) the optimal outcome for large families of valuations. One challenge that we see is demand reduction. Indeed, the simplest market beyond unit demand buyers and additive buyers is a market that contains both additive and unit demand buyers.

Consider such a market with two buyers OR (unit demand) and SUM (additive), and two identical items 1, 2. Player OR is an or player, and values each item for 4 (but also values the items together for 4). Player SUM is additive, and values each item for 5, and both of them for 10. Since the items are identical, selling order is not an issue.

The Walrasian equilibrium is for SUM to buy both items, and pay 4 on each.

To us the only SPE that makes sense is that both players bid 1 on the first item, and OR gets it[11]. Then, the buyer SUM gets the second item for free. In the off path, if somehow SUM got the first item, SUM also gets the second item, but pays 4 for it.

In any (optimistic) conservative equilibrium, the price of the first item can not be strictly less than 1, since then SUM would want to buy it, and get the second item for 4.

Note that there are many possible equilibria, but any prediction which does not give rise to this one seems problematic to us. Also note that the equilibrium we presented is the only optimistic conservative equilibrium.

In contrast to the bad equilibrium with the additive players presented in Section 8 (which was a problem in equilibrium selection), or the bad example presented in [13] (which was a problem in the selling order), here the problem is with the benchmark - we should not expect optimal welfare, nor any decent revenue[12]. The solution to this problem would need to consider new techniques and new benchmarks, which could be of wider interest for understanding demand reduction.

[11] If tie breakers are for SUM then SUM can bid a distribution which has supermum 1 and is very close to it

[12] Note that as the value of OR increases the revenue decreases. The reason for that is that as the value rises, OR can place a more credible threat on the second item, which incentivizes SUM not to take the first item
D Optimistic Conservative Bidding

One of the greatest challenges for us was to define the correct generalization of conservative bidding to sequential first price auctions. Recall the original motivation for conservative bidding. Consider a second price auction for a single item, with two buyers $A, B$, where $A$ values the item for 10, and $B$ values the item for 5. One possible equilibrium is that $A$ bids 0, $B$ bids 100, and $B$ wins paying nothing. This is an equilibrium, since $A$ doesn’t want to outbid $B$. We can see this as a type of threat (which is not credible) which $B$ poses to $A$, allowing him to buy the item only if $A$ is willing to pay a large amount.

One could hope that in first price auctions there would be no need for this assumption, since this particular bad example can no longer happen. However, we see something similar in sequential first price auctions, where buyers can pose threats which are not really credible, but are still formally allowed by the definition of the SPE - take the funny equilibrium in Section 3 for example.

It is known in economics that SPE is a problematic solution concepts, and one of the natural refinements is trembling hand perfect equilibrium. In a trembling hand equilibrium, each player needs to play best response even if there is a vanishing probability that other players will play some random move.

The problem is that using this notion in auctions as is doesn’t work well with the fact that prices are continuous. Indeed take the example we had before, with $A$ valuing the item for 10, and $B$ valuing the item for 5, and suppose that the item is sold in first price auction, with ties broken in favor of $A$.

The “right” equilibrium, namely $A$ and $B$ both bid 5 and $A$ wins is not a trembling hand equilibrium, since if $A$ lowers his bid in any way (or doesn’t participate in the auction or any other definition of tremble), then bidding 5 is never a best response for $B$. Indeed, bidding 5 guarantees that $B$ walks away with zero utility, no matter what $A$ does. Worse, since the equilibrium where $A$ pays 5 and wins is the only equilibrium under these conditions, the auction has no trembling hand perfect equilibrium. Part of the problem is that $B$ has zero utility - but in any auction someone is just holding the price and has zero utility.

As an alternative, we wanted to choose a definition of conservativeness which would prevent buyers from posing threats which are composed of bids that they do not want to win (in the spirit of not allowing threats which are not credible), and also allow for buyers who have zero utility in the equilibrium (such as buyer $B$ in the previous example). Optimistic conservative bidding does just that - it treats every bid as a “threat” and requires that if the buyer actually (in some weird off path equilibrium) needs to implement the threat, the buyer is not worse off than in the on path.

For the leaves of the tree, this notion coincides with the usual notion of conservative bidding.

We note that it is enough for all our results that the second highest buyer (who holds the price) is conservative optimistic, but there is no reason to limit the notion only to the second highest bidder.

E Submodular valuations - Highest Revenue Equilibrium

In this appendix we prove Corollary 2. We consider the best equilibrium from the seller’s perspective, and analyze how much revenue it elicits, where our benchmark here will be the full revenue which is defined as the social welfare of the optimal allocation (denoted as OPT).

Consider a market with sub-modular buyers. Let $\pi$ be the following selling order, defined inductively. Let $i_1, j_1$ the buyer and item that maximize $\max_{i,j} v_i(j)$. Now at step $k$, let $i_k, j_k$ be the pair of buyer and item who maximize

$$v_i(j|G) = v_i(\{x_i(G)\} \cup \{j\}) - v_i(\{x_i(G)\})$$

where $G$ is the state of the game where buyer $i_r$ wins item $j_r$ for every $r < k$.

We remind Corollary 2

Assume all buyers have a submodular valuation. If the items are sold according to $\pi$, there is an SPE which elicits half of the full revenue, or OPT/2.

\[\text{13}\text{One could remedy this by letting the players bid only integer numbers, and limit the possible ways in which $A$ can tremble based on the valuations. This creates its own problems.}\]
Proof: Define the equilibrium path inductively. In the first vertex, let $i_1$ be the buyer who maximizes $v_i(\pi(1))$. Player $i_1$ gets $\pi(1)$ and pays $v_{i_1}(\pi(1))$. In step $j$, let $G_j$ denote the vertex of the tree. Let $i_j$ be the buyer who maximizes $v_i(\pi(j)|G_j)$. Player $i_j$ wins item $\pi(j)$ and pays $v_{i_j}(\pi(j)|G_j)$.

Using Theorem 5 one can complete this to an SPE, as for any $i \neq i_j$ buying item $\pi(j)$ for strictly more than $v_{i_j}(\pi(j)|G_j)$ would result in negative utility.

Finally, as the greedy algorithm gives a 2 approximation to maximizing a submodular function, this allocation achieves half the social welfare. As the buyers end up with zero utility, the revenue must be at least $OPT/2$ as required. \qed