Twisting cocycles in fundamental representation and triangular bicrossproduct Hopf algebras

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Abstract

We find the general solution to the twisting equation in the tensor bialgebra $T(R)$ of an associative unital ring $R$ viewed as that of fundamental representation for a universal enveloping Lie algebra and its quantum deformations. We suggest a procedure of constructing twisting cocycles belonging to a given quasitriangular subbialgebra $H \subset T(R)$. This algorithm generalizes Reshetikhin’s approach, which involves cocycles fulfilling the Yang-Baxter equation. Within this framework we study a class of quantized inhomogeneous Lie algebras related to associative rings in a certain way, for which we build twisting cocycles and universal $R$-matrices. Our approach is a generalization of the methods developed for the case of commutative rings in our recent work including such well-known examples as Jordanian quantization of the Borel subalgebra of $sl(2)$ and the null-plane quantized Poincaré algebra by Ballesteros at al. We reveal the role of special group cohomologies in this process and establish the bicrossproduct structure of the examples studied.
1 Introduction

Quantum deformations of Lie groups and algebras are at present a subject of intensive studies from the viewpoints of collecting facts and crystallizing mathematical concepts as well as of searching for new physical applications. Among the established notions of the quantum group theory one should mention Drinfeld’s twisting [1, 2] and Majid’s bicrossproduct and doublecrossproduct constructions [3]. Twisting, realizing a specific equivalence between two Hopf algebras plays an important role for the geometrical and physical reasons because it controls deformation not only of the symmetry algebra of a manifold but of its whole geometry coherently Therefore a classification of quantum deformations of, say, a universal enveloping Lie algebra ought to provide the answer about twist-equivalence between its different types. Majid’s doublecrossproduct construction has close connection with twisting and the quantum double in particular [4, 5, 7]. As for the bicrossproduct, its relation to quasitriangularity and twisting is not so well understood, despite of numerous examples including quasitriangular Hopf algebras. The most significant step in that direction was made in Ref. [8], where the double of the algebra $\mathbb{C}(M) \bowtie \mathbb{C}G$, built on a matched pair of groups $M$ and $G$ was shown to be a bicrossproduct itself. As examples of bicrossproduct we would like to mention the $\kappa$-deformation of the Poincaré algebra [9, 10], the canonical example of the Jordanian quantization of the Borel subalgebra of $sl(2)$ [3], and the null-plane quantized Poincaré algebra [11, 12]. The last two are the results of twisting of classical universal enveloping algebras and are therefore triangular. As we have shown in Ref. [13], they are associated with commutative rings1 one-dimensional in the first case and

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1Throughout the paper by ring we mean a finite-dimensional algebra over a field $K$ so as to reserve the word ”algebra” for Hopf one.
that spanned by matrices

\[
\alpha_1 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \alpha_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad \alpha_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

in the second. Along this line we have found a generalization of the examples mentioned for an arbitrary commutative ring. The algebras studied in Ref. [13] were twisted classical universal enveloping algebras and also bicrossproduct Hopf algebras, although not considered in that context. In the present paper we formulate a generalization of our approach to an arbitrary associative ring providing a new class of quasitriangular bicrossproduct Hopf algebras. The Hopf operations are explicitly written out in terms of generators and the twisting 2-cocycles and universal \( R \)-matrices are presented. The class under investigation arises as an example of twisting which may be regarded as a generalized Reshetikhin’s procedure [14] involving a solution to the Yang-Baxter equation as a twisting cocycle. We come to this generalization analyzing solutions to the twisting equation in the tensor bialgebra \( T(R) \) of an associative unital ring \( R \), considered as that of the fundamental representation of a given Hopf algebra \( H \) (we call a homomorphism \( H \to R \) fundamental if its lifting to whole \( T(R) \), which always existits, is non-degenerate).

The paper is organized as follows. Section II is auxiliary and contains general description of the tensor bialgebra \( T(R) \) structure, its subbialgebras and homomorphisms. This part may be considered as the ”non-coordinate” formulation of the Faddeev-Reshetikhin-Takhtajan method [13] suitable for an arbitrary associative unital ring. Since rings with identities have exact matrix realizations, e. g. by the regular representations on themselves, such a reformulation does not supply with particular new information compared with the traditional matrix approach. Nevertheless, it provides certain technical convenience, so we find it possible to present this formulation here.
Section III is devoted to solving the twisting equation in $T(R)$. Therein we develop a procedure of constructing "universal" cocycles starting from elements of $R^\otimes 2$ obeying certain conditions. This algorithm is illustrated in Section IV on inhomogeneous Lie algebras related in a sense to associative rings. We build the deformed coproduct, quantum commutation relations, find twisting cocycles and universal $R$-matrices. In Section V the connection between the investigated algebras and the bicrossproduct construction is established.

2 Bialgebra $T(R)$

To perform algebraic manipulations it is convenient to formulate the algorithm by Faddeev, Reshetikhin, and Takhtajan [15] in the part of constructing quantum algebra $\text{Fun}_q(R)$ of functions on matrix rings for an arbitrary associative ring $R$. We need some information concerning its structure and the structure of homomorphisms from a Hopf algebra or, more generally, bialgebra $\mathcal{H}$ into $\text{Fun}_q(R)$. Let $\mu$ be the multiplication in $R$. We choose a basis $(x^\alpha) \subset R$ such that $x^0$ is the identity of $R$. The dual basis in $R^*$ will be marked with subscripts. Denote $F(R)$ the algebra over a field $K$ freely generated by 1 and $(x_\alpha) \in R^*$. Introduce the coproduct and the counit defining them on the generators as

$$\Delta(x_\alpha) = \mu^\rho_\sigma x_\rho \otimes x_\sigma, \quad \Delta(1) = 1 \otimes 1,$$

$$\epsilon(1) = 1, \quad \epsilon(x_0) = 1, \quad \epsilon(x_i) = 0, \quad i \neq 0$$

and extending over whole $F(R)$ homomorphically. The dual bialgebra $T(R) = F^*(R)$ appears to be a direct sum of its ideals $\sum_{n=0}^\infty R^\otimes n$, where $R^0$ coincides the field $K$ of scalars. The multiplication in $T(R)$ is characterized by the property $R^\otimes n R^\otimes m = 0$ for $m \neq n$. The identity of $T(R)$ is expanded as the sum $\sum_{n=0}^\infty e^n$ of idempotents, where
$e^0$ is the unity of $K$, and $e^n$ with $n > 0$ are the those of $R^\otimes n$. Multiplying by $e^n$ carries out the projection homomorphism

$$\pi^m: T(R) \to R^\otimes m,$$

and for $n = 0$ this is just the bialgebra counit. The coproduct in $T(R)$ is determined by the product of its dual algebra $F(R)$ and on the basis elements is defined by the formula

$$\Delta(x_{i_1..i_n}) = e^0 \otimes x_{i_1..i_n} + \ldots + x_{i_1..i_k} \otimes x_{i_{k+1}..i_n} + \ldots + x_{i_1..i_n} \otimes e^0, \quad x_{i_1..i_n} \in R^\otimes n.$$

It follows from here that the composite mapping

$$R^\otimes (i+j) \to T(R) \xrightarrow{\Delta} T(R) \otimes T(R) \to R^\otimes i \otimes R^\otimes j,$$

where the left arrow means the injection and the right one is the projection homomorphism, turns out to be a ring isomorphism. Quotient of $F(R)$ by the ideal $J$ generated by quadratic relations of the form $x_{\alpha}x_{\beta} = B_{\beta\gamma}^{\alpha}x_{\rho}x_{\sigma}$ inherits the coproduct if and only if the subspace $S \in R^\otimes 2$ of functionals annihilating these relations is a subalgebra in $R^\otimes 2$. In particular, such a subalgebra can be determined as the set of solutions to the equation

$$RzR^{-1} = \tau z, \quad z \in R^\otimes 2,$$

($\tau$ permutes the factors in $R^\otimes 2$), and then the relations in the dual algebra will look as

$$R_{\gamma\nu}\mu_{\alpha}^{\rho\sigma}\mu_{\beta}^{\sigma\nu}x_{\rho}x_{\sigma} = x_{\sigma}x_{\rho}\mu_{\alpha}^{\rho\gamma}\mu_{\beta}^{\sigma\nu}R_{\gamma\nu}.$$

The bialgebra $U$ dual to the factor-bialgebra $A \equiv F(R)/J$ is decomposed into the direct sum of its ideals $\sum_{n=0}^{\infty} \pi^n(U)$, where $\pi^1(U)$ is isomorphic to the ring $R$ itself and each addend $\pi^n(U)$ at $n > 1$ is a subring in $R^\otimes n$ characterized by the condition
Let us describe the structure of a homomorphism \( \phi \) of an arbitrary bialgebra \( \mathcal{H} \) to \( \mathcal{U} \). The composition of \( \phi \) with the projector \( \pi^n \) is an algebraic mapping. We set \( \rho \equiv \pi^1 \circ \phi \) and \( \rho^n \equiv \pi^n \circ \phi \) and also introduce the notations

\[
\Delta^1 \equiv id: \mathcal{H} \to \mathcal{H}, \quad \Delta^2 \equiv \Delta: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \quad \Delta^3 \equiv (\Delta \otimes id)\Delta: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}, \ldots
\]

Let \( h \in \mathcal{H} \) and \( x_{\vec{\alpha}} \in A \), where \( \vec{\alpha} \) is a multiindex of length \( n \). From the chain of equalities

\[
\langle \rho^n(h), x_{\vec{\alpha}} \rangle = \langle \pi^n \circ \phi(h), x_{\vec{\alpha}} \rangle = \langle \phi(h), x_{\vec{\alpha}} \rangle = \langle \Delta^n \circ \phi(h), x_{\alpha_1} \otimes \ldots \otimes x_{\alpha_n} \rangle =
\]

\[
\langle \phi \otimes^n \circ \Delta^n(h), x_{\alpha_1} \otimes \ldots \otimes x_{\alpha_n} \rangle =
\]

\[
\langle (\pi^1 \circ \phi) \otimes^n \circ \Delta^n(h), x_{\alpha_1} \otimes \ldots \otimes x_{\alpha_n} \rangle = \langle \rho \otimes^n \circ \Delta^n(h), x_{\alpha_1} \otimes \ldots \otimes x_{\alpha_n} \rangle
\]

we find that conditions

1. \( \rho^n = \rho \otimes^n \circ \Delta^n, \)
2. \( \rho^2(\mathcal{H}) \subset S = \pi^2(\mathcal{U}), \)

are satisfied, and these completely specify \( \phi \). Namely, if an algebraic mapping \( \rho: \mathcal{H} \to \mathbb{R} \) fulfills condition \( \mathbb{Q} \), there exists the unique bialgebra homomorphism of \( \phi: \mathcal{H} \to \mathcal{U} \) such that \( \pi^n \circ \phi = \rho^n \). Indeed, mapping \( \rho \otimes^n \circ \Delta^n \) respects the algebraic structure for every \( n \). Obviously, \( \rho \otimes(n+2) \circ \Delta^{n+2}(\mathcal{H}) \subset \mathbb{R} \otimes^i S \otimes \mathbb{R} \otimes(n-i) \) for every \( i \leq n \), because of the coassociativity of the coproduct. We define \( \phi \) via the formula

\[
\phi(h) = e^0 \varepsilon(h) + \sum_{n>0} \rho \otimes^n \circ \Delta^n(h)
\]

and verify that it is also a coalgebra homomorphism (its algebraic property is evident). For two multiindices \( \vec{\alpha} \) and \( \vec{\beta} \) of lengths \( m \) and \( n \) we have

\[
\langle (\phi \otimes \phi) \Delta(h), x_{\vec{\alpha}} \otimes x_{\vec{\beta}} \rangle = \langle (\rho \otimes \ldots \otimes \rho) \Delta^{m+n}(h), x_{\alpha_1} \otimes \ldots \otimes x_{\alpha_m} \otimes x_{\beta_1} \otimes \ldots \otimes x_{\beta_n} \rangle =
\]
\[ = \langle \phi(h), x_\alpha x_\beta \rangle = \langle \Delta \circ \phi(h), x_\alpha \otimes x_\beta \rangle \]

that proves what required. Conditions 1 and 2 are convenient practical criteria for checking out the homomorphic properties of mappings from \( \mathcal{H} \) to \( \mathcal{U} \).

3 Twisting equation in \( T(\mathbb{R}) \)

Algebras we are interested in, namely, universal enveloping Lie algebras and their quantum deformations appear to be embedded into \( T(\mathbb{R}) \) associated with their fundamental representation ring \( \mathbb{R} \). Since twisting of a subalgebra induces that of the whole algebra, the problem can be put forward of describing all solutions to the twisting equations in \( T(\mathbb{R}) \) with the hope of further selecting among them those belonging to required subalgebras. We can point out the following advantages of such an approach. As far as the composition of two twistings is concerned, thus we avoid inconvenience of dealing with a bialgebra different from original after performing the first deformation. Another remarkable feature of such a description is a possibility to reduce the elaborate task of constructing the universal twistor to the much easier problem of solving a system of equations, rather non-linear, in a finite-dimensional ring. Our study resulted in finding the general solution to the twisting equation in \( T(\mathbb{R}) \). We have managed to formulate conditions more general than those employed in Reshetikhin’s approach \([14]\) which ensure that the twisting cocycle built on its image in \( \mathbb{R}^{\otimes 2} \) would lie in the required subalgebra.

Consider the twisting equation

\[
(\Delta \otimes id)(\Phi)\Phi_{12} = (id \otimes \Delta)(\Phi)\Phi_{23},
\]

in the tensor cube of a bialgebra \( \mathcal{H} \) where the subscripts determine the embeddings \( \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 3} \). An invertible solution to this equation can be normalized in such a way...
that

$$(\varepsilon \otimes \text{id})(\Phi) = (\text{id} \otimes \varepsilon)(\Phi) = 1. \quad (2)$$

Such a solution takes part in transforming bialgebra $H$ into a new one with the same multiplication and the coproduct $\tilde{\Delta}(h) = \Phi^{-1}\Delta(h)\Phi$, $h \in H$. Other objects, e.g. the counit, antipode, universal $R$-matrix, if any, are connected with the old ones via the well-known formulas which can be found in [2]. We are going to prove the following assertion.

**Theorem 1** For every set of invertible elements $\Phi^{1,k} \in R \otimes R^k$, there exists the unique solution $\Phi \in T(R) \otimes T(R)$ to the twisting equation, such that $(\pi^1 \otimes \pi^k)(\Phi) = \Phi^{1,k}$.

Having applied the projector $\pi^m \otimes \pi^n \otimes \pi^k$ to both sides of equality (1), we come to the equation in $R \otimes (m+n+k)$:

$$\Phi^{m,n,k} \Phi^m_n = \Phi^{m,n+k} \Phi^n_k, \quad (3)$$

where $\Phi^{m,n}$ is the image of the twisting cocycle $(\pi^m \otimes \pi^n)(\Phi)$. Letters $b$ and $e$ indicate that the elements $\Phi^{m,n}_b$ and $\Phi^{n,k}_e$ are embedded into $R \otimes (m+n+k)$ from the "beginning" and from the "end", respectively: $\Phi^{m,n}_b \in R \otimes (m+n) \otimes e^k \subset R \otimes (m+n+k)$, $\Phi^{n,k}_e \in e^m \otimes R \otimes (n+k) \subset R \otimes (m+n+k)$. Because of (2), for every $k \geq 0$ we have $\Phi^{0,k}_e = e^0 \otimes e^k$ and $\Phi^{k,0}_b = e^k \otimes e^0$. Suppose now that the elements $\Phi^{1,k}$ are known. Then, using equation (3), $\Phi^{m,k}$ can be defined recursively for all $m$ and $k$ greater than 1:

$$\Phi^{m+1,k} = \Phi^{m,k} \Phi^{1,k}_e \Phi^{m,1}_b^{-1} \quad (4)$$

(the bar stands for the inverse). This implies the uniqueness of the solution. Obviously, equation (3) is true when one of the numbers $m$, $n$, and $k$ are equal to zero. By construction, it is fulfilled for $n = 1$ and all $m$ as well. So we must show that equation...
is satisfied for arbitrary \(m, n, \text{and } k\). Assume the required property proved for all \(m\) and \(n\), which sum is less than \(N > 2\). Then for \(m + n = N \text{ and } k > 1\) we have

\[
\Phi^{m + n, k} = \Phi^{m + n - 1, 1 + k} \Phi^{1, k}_e \Phi^{m + n - 1, 1}_b.
\]

Within the assumption made, we decompose the first factor on the right-hand side of this equality, according to (4), and rewrite (3) in the equivalent form

\[
\Phi^{m, n + k}_e \Phi^{n - 1, 1 + k}_b \Phi^{m, n - 1}_e \Phi^{1, k}_e \Phi^{m + n - 1, 1}_b = \Phi^{m, n + k}_e \Phi^{n, k}_e \Phi^{m, n}_b.
\]

Dividing both sides by the first factor we come to condition

\[
\Phi^{n - 1, 1 + k}_e \Phi^{m, n - 1}_b \Phi^{1, k}_e \Phi^{m + n - 1, 1}_b = \Phi^{n, k}_e \Phi^{m, n}_b.
\]

Again, decomposing the first factor on the right-hand side according to the recursion assumption we find

\[
\Phi^{n - 1, 1 + k}_e \Phi^{m, n - 1}_b \Phi^{1, k}_e \Phi^{m + n - 1, 1}_b = \Phi^{n - 1, 1 + k}_e \Phi^{1, k}_e \Phi^{n - 1, 1}_b \Phi^{m, n}_b,
\]

where the subscript in \(\Phi^{n - 1, 1}_{\{m + 1\}}\) means that it is embedded into \(\mathbb{R}^{\otimes (m+n+k)}\) beginning from \(\{m+1\}\)-th place. It is important that the factor \(\Phi^{m, n - 1}_b\) on the left-hand side can be permuted with \(\Phi^{1, k}_e\). Division by \(\Phi^{n - 1, 1 + k}_e \Phi^{1, k}_e\) yields

\[
\Phi^{m, n - 1}_b \Phi^{m + n - 1, 1}_b = \Phi^{n - 1, 1}_{\{m+1\}} \Phi^{m, n}_b
\]

or

\[
\Phi^{m + n - 1, 1}_b \Phi^{m, n - 1}_b = \Phi^{m, n}_b \Phi^{n - 1, 1}_{\{m+1\}}.
\]

This is exactly twisting equation (3) for \(k \to 1\) and \(n \to n - 1\). According to the induction principle we consider the theorem proved.

Thus, the family of solutions to equation (1) in the bialgebra \(T(\mathbb{R})\) turns out to be very large: it is parameterized by an arbitrary set of invertible elements \(\Phi^{1, k}, k > 0\).
On the other hand, we are interested only in those Φ which belong to the subalgebra \( \mathcal{U} \otimes \mathcal{U} \). We cannot propose the general method to build such solutions. However, we can point out an algorithm which can help to solve the problem at least for quasitriangular \( \mathcal{U} \) and which can be interpreted as a generalization of Reshetikhin’s approach. Starting form \( \Phi^{1,1} \) as known, set \( \Phi^{1,k} \equiv \Phi_{12}\Phi_{13} \ldots \Phi_{1(k+1)} \). Here \( \Phi_{ij} \) are the images of \( \Phi^{1,1} \) via the corresponding embeddings \( R \otimes^2 \to R \otimes R^k \). For \( \Phi^{1,k} \) to belong to \( \mathcal{U} \otimes \mathcal{U} \), it is necessary and sufficient to require

\[
R_{23}\Phi_{12}\Phi_{13} = \Phi_{13}\Phi_{12}R_{23}.
\]

(5)

Further, element \( \Phi^{2,1} \) lies in \( \mathcal{U} \otimes \mathcal{U} \) if and only if \( R_{12}\Phi^{2,1} = (\tau \otimes id)(\Phi^{2,1})R_{12} \). Having expressed \( \Phi^{2,1} \) through \( \Phi^{1,1} \) we come to equation \( R_{12}\Phi_{12}\Phi_{13}\Phi_{23}\Phi_{12} = \Phi_{21}\Phi_{23}\Phi_{13}\Phi_{21}R_{12} \) or, the matrix \( \tilde{R} = \tilde{\Phi}_{21}R\Phi \) introduced, to equation

\[
\tilde{R}_{12}\Phi_{13}\Phi_{23} = \Phi_{23}\Phi_{13}\tilde{R}_{12}.
\]

(6)

Verification of the condition \( R_{12}\Phi^{2,k} = (\tau \otimes id)(\Phi^{2,k})R_{12} \) for \( k > 1 \) boils down to equality \( \tilde{R}_{12}\Phi_{13} \ldots \Phi_{1 k+2}\Phi_{23} \ldots \Phi_{2 k+2} = \Phi_{23} \ldots \Phi_{2 k+2}\Phi_{13} \ldots \Phi_{1 k+2}\tilde{R}_{12} \) which is, as can be easily seen, follows from (6). Now, with further use of identity (6) one can see that \( \Phi^{m,n} \) belongs to \( \mathcal{U} \otimes \mathcal{U} \) for every \( m \) and \( n \), indeed.

4 Twisting of inhomogeneous Lie algebras

One can notice that the example of solution to the twisting equation built at the end of the previous section satisfies the identities

\[
(id \otimes \Delta)(\Phi) = \Phi_{12}\Phi_{13},
\]

(7)

\[
(\tilde{\Delta} \otimes id)(\Phi) = \Phi_{13}\Phi_{23}.
\]

(8)
where $\tilde{\Delta}$ is the twisted coproduct: $\tilde{\Delta}(h) = \Phi \Delta(h) \Phi$. Reshetikhin’s conditions are obtained from here if $\Phi$ solves the Yang-Baxter equation and, besides, $(\Delta \otimes \text{id})(\Phi) = \Phi_{23}\Phi_{13}$. This generalization of Reshetikhin’s twisting is non-trivial, the non-standard quantization of the Borel subalgebra of $sl(2)$ taken into account. The twisting cocycle for $U(b(2))$ has the form $\exp(X \otimes H)$, where $H$ is the primitive element in $U(b(2))$ and $X$ is the primitive element in $U_h(b(2))$ \[16\]. This example was generalized in Ref. \[13\] for an arbitrary commutative ring which in the case of $U(b(2))$ coincides with the field of scalars. In the present section we shall formulate the analogous generalization for an arbitrary associative ring $L$, not necessarily commutative. Let the multiplication in $L$ be defined by the structure constants $B^\sigma_{\mu\nu}$. Consider a Lie algebra built on $H_\mu \in L$ and $X^\nu \in L^*$ subjected to the commutation relations

$$[H_\mu, H_\nu] = (B^\sigma_{\mu\nu} - B^\sigma_{\nu\mu}) H_\sigma, \quad [H_\mu, X^\nu] = -B^\nu_{\mu\sigma} X^\sigma.$$ 

The subalgebra $L^*$ generated by $X^\nu$ is assumed to be commutative. It is easy to see that the element $H_\nu \otimes X^\nu - X^\nu \otimes H_\nu$ satisfies the classical Yang-Baxter equation. The subspace $L^*$ is a right module over ring $L$. Let us affiliate the identity to $L \triangleright L^*$ and denote the resulting ring $R$. The product in $R$ is evaluated according to the rules

$$\hat{H}_\mu \hat{H}_\nu = B^\sigma_{\mu\nu} \hat{H}_\sigma, \quad \hat{X}^\nu \hat{H}_\mu = B^\nu_{\mu\sigma} \hat{X}^\sigma, \quad \hat{X}^\mu \hat{X}^\nu = 0, \quad \hat{H}_\mu \hat{X}^\nu = 0,$$

plus evident expressions involving identity $\hat{E}$. Starting from this multiplication, one can see that the element

$$R = \hat{E} \otimes \hat{E} + \hat{H}_\nu \otimes \hat{X}^\nu - \hat{X}^\nu \otimes \hat{H}_\nu$$ \hspace{1cm} (9)

is a solution to the quantum Yang-Baxter equation, and the element

$$\Phi^{1,1} = \hat{E} \otimes \hat{E} - \hat{X}^\nu \otimes \hat{H}_\nu$$ \hspace{1cm} (10)
obeys (5) and (6) where $R$ should be set to $\hat{E} \otimes \hat{E}$. Correspondence $1 \rightarrow \hat{E}$, $X \rightarrow \hat{X}$, $H \rightarrow \hat{H}$ is extended to a homomorphism (non-degenerate) of the universal enveloping algebra $U(R)$ into $T(R)$.

**Theorem 2** Twisting cocycle $\Phi$ expanding $\Phi^{1,1}$ (Eq. (11)) by modified Reshetikhin’s procedure belongs to $U(L^*) \otimes U(L)$ and has the form $\exp(-\tilde{X}^\nu \otimes H_\nu)$, where $\tilde{X}^\nu$ are expressed by series in $X^\nu$.

Indeed, $\Phi^{1,k} \in U(L^*) \otimes U(L)$ by construction. From defining formula (3), recursively using the facts that $\text{Span}(\hat{E}, \hat{H}_\mu)$ and $\text{Span}(\hat{X}^\mu)$ form a subring and an ideal in $R$, respectively, we get the first assertion of the theorem. Now the announced form of the twisting cocycle follows from (7).

Unemployed so far identity (8) enables us to determine the twisted coproduct on the generators $\tilde{X}^\nu$. It turns out that $\tilde{\Delta}(\tilde{X}^\nu) = D^\nu(1 \otimes \tilde{X}, \tilde{X} \otimes 1)$, where $D(a,b)$ is the Campbell-Hausdorff series: $e^{D(a,b)} = e^a e^b$ for arbitrary $a$ and $b$ from the Lie algebra of the ring $L$. Thus, the commutative algebra $U_q(L^*)$ is isomorphic to the function algebra on the group $\exp(L)$ taken with the opposite coproduct. We have yet to evaluate the twisted coproduct on $H_\nu$ and to determine commutation relations $[H_\nu, \tilde{X}^\mu] = f(\tilde{X})^\nu_\mu$.

What can be said about functions $f(x)^\mu_\nu$ is that they are subject to the ”boundary” conditions

$$f(0)^\mu_\nu = 0, \quad \partial_\sigma f(0)^\mu_\nu = -B^\mu_{\nu\sigma}.$$  

(11)

It is accounted for the following. First of all, knowing the image $\Phi^{1,1}$ of the cocycle $\Phi$ in $R^\otimes 2$ we conclude that $\frac{\partial \tilde{X}^\mu}{\partial X^\sigma}|_{x=0} = \delta^\mu_\nu$. Now the required properties of $f(x)^\mu_\nu$ are conditioned by the homomorphism from $U(R)$ to $R$ and the strong nilpotence of $\tilde{X}^\mu$.

Having introduced matrices of the left and right regular representations $L(X)^\mu_\nu \equiv B^\mu_{\alpha\nu}X^\alpha$, $R(X)^\mu_\nu \equiv B^\mu_{\nu\sigma}X^\sigma$, from the definition of $\tilde{\Delta}$ we find

$$\tilde{\Delta}(H_\mu) = \exp(\tilde{X} \otimes H)(H_\mu \otimes 1 + 1 \otimes H_\mu) \exp(-\tilde{X} \otimes H).$$

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\[ H_\mu \otimes 1 - \left( \frac{e^{L(\tilde{X})-R(\tilde{X})} - 1}{L(\tilde{X})-R(\tilde{X})} \right)_\mu^{\nu} H_\nu \\
+ \left( e^{L(\tilde{X})-R(\tilde{X})} \right)_\mu^{\nu} H_\nu \\
= H_\mu \otimes 1 + g(\tilde{X})^{\nu}_{\mu} \otimes H_\nu. \]  

(12)

The coassociativity requirement imposed, formula (12) implies \( g(D(1 \otimes \tilde{X}, \tilde{X} \otimes 1)) = g(1 \otimes \tilde{X})g(\tilde{X} \otimes 1) \), and this with necessity entails \( g(\tilde{X}) = e^{A(\tilde{X})} \), where \( A \) represents a left action of \( L \) on itself. Resolving \( g(\tilde{X}) \) with respect to \( f(\tilde{X}) \) and using conditions (11) we come finally to \( A(\tilde{X})^{\mu}_{\nu} = L(\tilde{X})^{\mu}_{\nu} = B^{\mu}_{\sigma \nu} X^{\sigma} \). The resulting formulas describing the Hopf structure of the twisted algebra \( U_q(\mathbb{R}) \) read

\[
\begin{align*}
\tilde{\Delta}(\tilde{X})^{\mu} &= D(1 \otimes \tilde{X}, \tilde{X} \otimes 1), \\
\tilde{\Delta}(H) &= H_\mu \otimes 1 + \left( e^{L(\tilde{X})} \right)_\mu^{\nu} H_\nu, \\
[H_\mu, H_\nu] &= (B^\sigma_{\mu \nu} - B^\sigma_{\nu \mu}) H_\sigma, \\
[H_\nu, \tilde{X}]^{\mu} &= \left( \frac{L(\tilde{X})-R(\tilde{X})}{e^{L(\tilde{X})-R(\tilde{X})} - 1} \left( e^{L(\tilde{X})-R(\tilde{X})} - e^{L(\tilde{X})} \right) \right)_\nu^{\mu}.
\end{align*}
\]  

(13)

The antipode is easily found from the coproduct:

\[ S(\tilde{X})^{\mu} = -\tilde{X}^{\mu}, \quad S(H_\mu) = -(e^{-L(\tilde{X})})_\mu^{\nu} H_\nu. \]

The expressions obtained generalize formulas deduced for commutative ring in Ref. [13]. In the latter case the value of the commutator \([H_\nu, \tilde{X}]^{\mu} \) is simplified because of \( L(\tilde{X}) = R(\tilde{X}) \) and turns into \((1 - e^{L(\tilde{X})})\), as for the well-known example of the Borel subalgebra of \( sl(2) \). The universal \( R \)-matrix is expressed through the twisting cocycle by the standard formula \( R = \Phi^{-1} \Phi \) [14]:

\[ R = \exp(H_\nu \otimes \tilde{X}) \exp(-\tilde{X} \otimes H_\nu) \]  

(14)
and has the form familiar from the theories of the Jordanian quantization of \( sl(2) \) and the null-plane quantized Poincaré algebra. Thus we obtain a closed and complete description of the deformed algebra \( U_q(R) \), although it would be desirable to find the relation between \( \tilde{X}^\mu \) and the classical generators \( X^\mu \). To this end, let us calculate the twisted coproduct on elements \( X^\mu \):

\[
\tilde{\Delta}(X^\mu) = \exp(\tilde{X} \otimes H)(1 \otimes X^\mu + X^\mu \otimes 1)\exp(-\tilde{X} \otimes H) = X^\mu \otimes 1 + (e^{-L(\tilde{X})})^\mu_\nu \otimes X^\nu,
\]

that results in the functional equation

\[
\varphi(\Delta(\tilde{X})) = \varphi(\tilde{X}') + e^{-L(\tilde{X})}\varphi(\tilde{X}''),
\]

where the primes distinguish the tensor components, and \( \varphi \) is the transformation connecting the quantum and classical generators: \( X^\mu = \varphi^\mu(\tilde{X}) \). This equation is well known from the theory of the group cohomologies and its solution is

\[
\varphi(\tilde{X}) = \frac{e^{-L(\tilde{X})} - 1}{-L(\tilde{X})}\tilde{X}.
\]

This formula solves the problem of proceeding to the classical basis of \( U_q(R) \).

The analysis of the solution found allows us to perform a further generalization of the examples considered above in the following direction. Let \( G \) be a Lie group and \( L \) its Lie algebra with the basis elements \( H_\mu \). Assume a left action \( H_\mu \triangleright H_\nu = B^\mu_\nu H_\sigma \) of \( L \) on itself, which is, as a rule, does not coincide with the adjoint representation. Let function \( \varphi: G \to L \) be a group 1-cocycle, that is \( \varphi(ba) = \varphi(a) + a^{-1} \triangleright \varphi(b), a, b \in G \). It can be viewed as a mapping defined in some neighborhood of the origin in \( L \). We suppose \( \varphi \) to be invertible and denote its inverse \( \psi \). By the left conjugate action \( H_\mu \triangleright X^\nu = -B^\nu_\mu X^\sigma \) on the dual space we build the semidirect sum \( L \triangleright L^* \), where \( L^* \) is considered as an Abelian subalgebra.
Theorem 3 Element $\Phi = \exp(-\psi^\nu(X) \otimes H_\nu)$ is a twisting cocycle for the universal enveloping Hopf algebra $U(L \triangleright L^*)$.

Notice that $\exp(-\psi^\nu(X) \otimes H_\nu)$ satisfies identity (7). Hence, in order to prove the theorem the second identity (8) should be stated. Making use of the fact that $\tilde{\Delta}$ is an algebraic mapping for arbitrary $\Phi$, we evaluate $\tilde{\Delta} (\tilde{X})$ on the elements $\tilde{X}^\mu = \psi^\mu(X)$ and come to the equation (15), where operator $L(\tilde{X})$ is defined via tensor $B$ as above. Because of the invertibility of $\varphi$, this implies $\Delta(\tilde{X}^\mu) = D^\mu(1 \otimes \tilde{X}, \tilde{X} \otimes 1)$. Then identity (8) is obeyed as well.

5 Bicrossproduct structure

To exhibit the bicrossproduct structure of the examples considered let us use the FRT method and recover the quantum groups by the solution to the quantum Yang-Baxter equation (9). Modulo the order of the factors, they are isomorphic to the quantum algebras, and the isomorphism is realized via the universal $R$-matrix (14).

In terms of the basis $(e, x_\mu, h_\nu)$ dual to the basis $(\hat{E}, \hat{H}_\mu, \hat{X}^\nu)$ of the ring $R$ the coproduct, according to the scheme rendered in Section II, has the form

$$\Delta(h^\sigma) = h^\sigma \otimes e + e \otimes h^\sigma + B^\sigma_{\mu \nu} h^\mu \otimes h^\nu, \quad \Delta(x_\sigma) = x_\sigma \otimes e + e \otimes x_\sigma + B^\nu_{\mu \sigma} x_\nu \otimes h^\mu,$$

$$\Delta(e) = e \otimes e.$$ 

The counit is determined by the rule $\varepsilon(e) = 1, \varepsilon(x_\mu) = 0, \varepsilon(h^\mu) = 0$. Imposing RTT-type relations with the matrix $R$ given by (9), we come to the following permutation rules

$$[x_\mu, x_\nu] = (B^\sigma_{\nu \mu} - B^\sigma_{\mu \nu}) x_\sigma e, \quad (16)$$

$$[x_\mu, h^\nu] = B^\nu_{\mu \alpha} h^\alpha e + B^\nu_{\alpha \sigma} B^\sigma_{\beta \mu} h^\alpha h^\beta. \quad (17)$$
Other commutation relations are trivial and, in particular, the element \( e \) belongs to the center of the algebra. Note, that the ideal \((e - 1)\) is a Hopf one and set \( e = 1 \). Introduce quantities \( \eta^\mu \) starting from equality \( h^\nu \otimes \hat{H}_\mu = e^{\eta^\kappa \otimes \hat{H}_\nu - 1} \otimes \hat{E} \). In terms of new generators \((\eta^\mu, x_\mu)\) the coproduct turns out to be opposite to that of the algebra \( \hat{U}(\mathbf{L}) \), and that is seen through the substitution \( \eta^\mu \to \hat{X}^\mu, x_\mu \to -H_\nu \). Commutation relations (16) are thus recovered exactly. It has yet to be shown that relations (17) goes over into the last expression in (13). This is guaranteed by the uniqueness of the value of the commutator (17) as a function in \( h^\mu \) compatible with the given coproduct and fulfilling \( \frac{\partial [x_\mu, h^\nu]}{\partial h^\sigma} \big|_{h=0} = B^\nu_{\mu \sigma} \). This boundary condition is determined by the homomorphism from the quantum group into the ring \( \mathbf{L} \) via the ”square” matrix \( R \) [3].

According to [3], bicrossproduct \( \mathcal{A} \bowtie \mathcal{B} \) of Hopf algebras \( \mathcal{A} \) and \( \mathcal{B} \) is defined via a left action \( \triangleright \) of \( \mathcal{A} \) on \( \mathcal{B} \) and a right coaction \( \beta \) of \( \mathcal{B} \) on \( \mathcal{A} \). The latter is a mapping from \( \mathcal{A} \) into the tensor product \( \mathcal{A} \otimes \mathcal{B} \). The conjugate mapping to \( \beta \) realizes a right action \( \mathcal{A}^* \triangleleft \mathcal{B}^* \). These operations are subjected to the set of consistency conditions [3].

Multiplication and comultiplication on \( \mathcal{A} \bowtie \mathcal{B} \) are evaluated via

\[
(a \otimes h)(b \otimes g) = a(h^{(1)} \triangleright b) \otimes h^{(2)} g,
\]

\[
\Delta(a \otimes h) = (a^{(1)} \otimes h^{(1)}) \otimes (a^{(2)} h^{(2)} h^{(2)}),
\]

where \( a, b \in \mathcal{A}, \quad h, g \in \mathcal{B}, \quad \Delta(h) = h^{(1)} \otimes h^{(2)}, \) and \( \beta(h) = h^{(1)} \otimes h^{(2)} \). Turning to \( U_q(\mathbf{R}) \) constructed in the previous section we see that algebraically it has the same structure as \( U^*(\mathbf{L}) \triangleleft U(\mathbf{L}) \), whereas its dual quantum group has the form \( U^*(\mathbf{L}) \triangleright U(\mathbf{L}) \). In terms of generators \( h^\mu \) and \( x_\nu \) the left action is \( x_\mu \triangleright h^\nu = B^\nu_{\mu \sigma} h^\sigma + B^\nu_{\sigma \alpha} B^\sigma_{\beta \mu} h^\alpha h^\beta \), and the right coaction \( \beta \) can be computed from the coproduct formula

\[
\text{id} \otimes \beta(h) \otimes \text{id} = (\text{id} \otimes \text{id} \otimes \text{id} \otimes \varepsilon) \circ \Delta(1 \otimes h).
\]

This yields

\[
\beta(x_\sigma) = x_\sigma \otimes 1 + B^\nu_{\mu \sigma} x_\nu \otimes h^\mu
\]
for the generators $x^\mu$ thus stating the bicrossproduct structure of $U_q^*(R)$ and $U_q(R)$.

6 Discussions

The problem of explicitly calculating twisting 2-cocycles for Hopf algebras is a non-trivial one even if their existence is a priori known. Difficulties arise already in the simplest case of classical universal enveloping Lie algebras, despite of advanced Drinfeld’s theory on quantizing triangular Lie bialgebras [1]. This explains why examples of explicitly given twisting 2-cocycles are in relatively short supply. So, it seems quite natural to reduce the problem to studying ”quadratic” 2-cocycles which are images of ”universal” ones in fundamental representations, provided there exists some ”fusion” procedure to expand those matrix solutions over representations of higher spins. There are two algorithms of this kind [19, 20, 21], both based on factorisation properties of twisting elements [1, 4, 6, 22] $(\Delta \otimes id)(\Phi) = \Phi_{13}\Phi_{23}$ or $(\Delta \otimes id)(\Phi) = \Phi_{23}\Phi_{13}$ (and appropriate identities involving $id \otimes \Delta$). Depending on the order of the factors on the right hand side, additional requirements like $\Phi_{12}\Phi_{23} = \Phi_{23}\Phi_{12}$ or the Yang-Baxter equation are imposed on $\Phi$. Although most of explicitly known universal twisting 2-cocycles are due to these two options, it is clear that they cannot cover all possibilities. The idea of proceeding to fundamental representation in a finite-dimensional ring $R$ in studying twist-equivalences among various quantizations seems yet more fruitful because the associated bialgebra $T(R)$ plays the role of a container, in a general situation, for all the deformations of a Hopf algebra. Thus every twisting cocycle of a subbialgebra undergoing deformation remains so for whole $T(R)$. This makes it reasonable to consider twisting equation in $T(R)$ and then try to select solutions belonging to the given subbialgebra. The first part of this program has been completely solved in the present paper, while for the second we have suggested a new kind of fusion procedure which ap-
pears to be close to Reshetikhin’s twisting. The novelty is that the conditions imposed on Φ employ both twisted and untwisted coproducts. To demonstrate effectiveness of the proposed scheme we have considered a class of inhomogeneous universal enveloping Lie algebras related to associative rings in a special way and quantized them along that line. The technique can be viewed as a generalization of the theory developed in our previous work [13] dealing exclusively with commutative rings, which was motivated by the Jordanian quantization of sl(2) and the null-plane quantized Poincaré algebra. We have also exposed the bicrossproduct structure of the objects investigated thus providing new examples of quasitriangular bicrossproduct Hopf algebras. A remarkable fact is that the class considered may be treated directly with the use of special Lie group cohomologies which take part in building twisting elements. Relevance of Hopf algebra cohomologies to the twisting procedure and bicrossproduct construction was already pointed out in Ref. [3], so the present study gives a new insight to their role in the theory. It is interesting to generalize cohomological methods applied here for classical universal enveloping algebras to Hopf algebras of more general nature.

Acknowledgement

We are grateful to P. P. Kulish and V. D. Lyakhovsky for helpful and stimulating discussions.

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