LAGRANGIAN TRACES FOR THE JOHNSON FILTRATION OF THE HANDLEBODY GROUP

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Abstract. In this paper, we define trace-like operators on a subspace of the space of derivations of the free Lie algebra generated by the first homology group $H$ of a surface $\Sigma$. This definition depends on the choice of a Lagrangian of $H$, and we call these operators the Lagrangian traces. We suppose that $\Sigma$ is the boundary of a handlebody with first homology group $H'$, and we show that the Lagrangian traces corresponding to the Lagrangian $\ker(H \to H')$ vanish on the image by the Johnson homomorphisms of the elements of the Johnson filtration that extend to the handlebody.

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Let $\Sigma := \Sigma_{g,1}$ be a compact connected oriented surface of genus $g$ with one boundary component, and denote by $\mathcal{M} := M(\Sigma)$ the mapping class group of $\Sigma$. Let $V_g$ be a handlebody whose boundary minus a disk is identified with $\Sigma$. The handlebody group $A$ is the subgroup of mapping classes of $\mathcal{M}$ that admit an extension to the handlebody. It is a non-normal subgroup of the mapping class group with infinite index, hence its study as a subgroup of $\mathcal{M}$ is difficult.

It is known, by the Dehn-Nielsen theorem, that $\mathcal{M}$ can be studied via its action on the fundamental group $\pi := \pi_1(\Sigma_{g,1})$ of the surface, which is a free group on $2g$ generators. Let us denote $H := H_1(\Sigma_{g,1})$. Similarly, let $\pi' := \pi_1(V_g)$ and $H' := H_1(V_g)$ denote respectively the fundamental group and the first homology group of the handlebody. The abelian groups $H$ and $H'$ are the abelianization of $\pi$ and $\pi'$, respectively. We also denote $\mathcal{A}$ the kernel of the projection $\pi \to \pi'$ and $A$ the kernel of the projection $H \to H'$. We have the following commutative exact diagram, induced by the inclusion $\Sigma_{g,1} \subset V_g$:

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{A} \\
& \downarrow & \downarrow \\
0 & \longrightarrow & A \\
& \downarrow & \downarrow \\
& \longrightarrow & H \\
& \downarrow & \downarrow \\
& \longrightarrow & H' \\
& \downarrow & \downarrow \\
& \longrightarrow & 0.
\end{array}
$$

The group $\mathcal{A}$ actually coincides with the subgroup of $\mathcal{M}$ that preserves $\mathcal{A}$ [3].

Recall that the intersection form of $\Sigma$ induces a symplectic form $\omega : H \otimes H \to \mathbb{Z}$, hence $H$ is naturally isomorphic to $H^*$ via the map $x \mapsto \omega(x, -)$. The action of $\mathcal{M}$ on the homology of the surface preserves the symplectic form $\omega$, inducing a representation $\rho_0 : \mathcal{M} \to \text{Sp}(H)$ whose kernel is, by definition, the Torelli group $\mathcal{I}$. The duality $H \simeq H^*$ actually restricts to an isomorphism between the lagrangian $A$ and $H'^*$ (see [1, Section 4.2]). We fix a basis $(\alpha_1, \alpha_2 \ldots, \alpha_g, \beta_1, \beta_2 \ldots, \beta_g)$ of $\pi$, such that:

- it induces a symplectic basis of $H$,
- $\mathcal{A}$ is normally generated by the $\langle \alpha_i \rangle_{1 \leq i \leq g}$,
- the image $\langle \beta'_i \rangle_{1 \leq i \leq g}$ of the family $\langle \beta_i \rangle_{1 \leq i \leq g}$ in $\pi'$ form a basis of $\pi'$.

Such a choice of basis for $\pi$ is induced by the curves drawn in Figure 1. Notice that the curves $\alpha_i$ bound disks in the handlebody.
Figure 1. A basis of $\pi$ adapted to the inclusion $\Sigma \subset V_g$.

The $k$-th term $J_k$ of the Johnson filtration is defined as the subgroup of $M$ of elements acting trivially on $\pi$ modulo the commutators of length $k+1$. In particular the first term of this filtration is $I$. The Johnson filtration $(J_k)_{k \geq 1}$ of the mapping class group is a separating filtration of $M$ and its associated graded space $Gr^J := \bigoplus_{k \geq 1} J_k / J_{k+1}$ has been extensively studied. The latter is in particular a graded Lie ring. The so-called Johnson homomorphisms $(\tau_k)_{k \geq 1}$ induce an embedding $\tau$ of the graded space associated to the Johnson filtration to the graded Lie ring $D(H) := \bigoplus_{k \geq 1} D_k(H)$ of symplectic derivations of $\mathcal{L}(H)$, the free graded Lie algebra generated by $H$. Recall that a derivation of the free Lie algebra can be identified via the restriction to $H$, with an element of $H^k \otimes \mathcal{L}(H) \simeq H \otimes \mathcal{L}(H)$. The space $D_k(H)$ then corresponds to the kernel of the bracket from $H \otimes \mathcal{L}_{k+1}(H)$ to $\mathcal{L}_{k+2}(H)$. The determination of the cokernel of $\tau$ is a question of major importance. In [9], Morita defined a family of trace-like operators that detect a part of this cokernel (but not all of it).

In this paper, we are interested in the intersection of the Johnson filtration with $A$. We set $A_k := A \cap J_k$. The $AJ$ filtration $(A_k)_{k \geq 1}$ is also a separating filtration of $A$. Hence we would like to understand its associated graded $Gr^AJ$, or, equivalently, the subspace $\tau_k(A_k) \subset D_k(H)$. Note that the study of the $AJ$ filtration is important for the study of Heegaard splittings and finite-type invariants of integral homology 3-spheres [11, 12, 2]. In [10], Omori gave a generating set of the first term of the $AJ$ filtration. In [6], Levine noticed that the image of the restriction of $\tau_k$ to $A_k$ is included in the kernel $\mathcal{G}$ of the natural projection $D_k(H) \to D_k(H')$, where $D_k(H') := \text{Ker}(H' \otimes \mathcal{L}_{k+1}(H') \to \mathcal{L}_{k+2}(H'))$. Hence the Johnson homomorphisms induce a well-defined embedding $Gr^AJ \to \bigoplus_{k \geq 1} \mathcal{G}_k$. A computation of Morita in [8] shows that $\mathcal{G}_1 = (\tau_1(A_1))$. We will show that this equality does not hold in general. In [1], the author defined a family of trace-like operators $(Tr^A_k)_{k \geq 1}$, defined on $\mathcal{G}_k$, and valued in $S^0(H')$. We shall refer to these maps as the Lagrangian traces. In the same paper, using a variation formula for the Casson invariant from [8], the author shows that $\tau_2(A_2) = \text{Ker}(Tr^A_2) \cap \text{Im}(\tau_2)$. Here, adapting the strategy of Morita in [9] to the handlebody case, we prove the following theorem:

**Theorem A.** Let $g \geq 2$, and $k \geq 2$, then the map $Tr^A_k$ vanishes on $\tau_k(A_k) \subset \mathcal{G}_k$.

The proof of Theorem A uses the Magnus representation of the mapping class group, and also its reduction to the handlebody case. We will recall only what is needed here, and refer to [13] for a survey on the subject. We define in Section 2 two Magnus “representations” $r^A : A \to \text{GL}(g, \mathbb{Z}[[\pi]])$ and its abelian version $r^{A,\pi} : A \to \text{GL}(g, \mathbb{Z}[H'])$. These are representations only in a twisted sense. For any group $G$, we shall denote $I_G$ the augmentation ideal of $\mathbb{Z}[G]$. For short, we set $I := I_\pi$.

**Theorem B.** Let $g \geq 2$, and $\varphi \in A_1$, then $\det(r^{A,\pi}(\varphi)) \in H' \subset \mathbb{Z}[H']$ and

$\det(r^{A,\pi}(\varphi)) = Tr^1(A_1(\varphi))$.

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Consequently, the determinant of the Magnus representation is trivial on \( A_k \), for \( k \geq 2 \).

In the first section, we review the definition of the trace-like operators \((\Tr_k^A)_{k \geq 1}\), and study their properties. In the second section, we will prove Theorem B and Theorem A by considering the magnus representation of the handlebody group, and relating it to the \( A \)-traces. Throughout the paper, we shall detail the connections with Morita’s work in [9].

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1. The Lagrangian traces

Before recalling the definition of the Lagrangian trace \( \Tr_k^A \), let us recall a few facts about the Johnson homomorphisms \((\tau_k)_{k \geq 1}\) and the spaces \( D_k(H) \). The \( k \)-th Johnson homomorphism of \( \varphi \in J_k \) associates to the class \( \{x\} \in H \) of an element \( x \) in \( \pi \), the class of \( \phi(x)x^{-1} \) in \( \Gamma_{k+1}\pi/\Gamma_{k+2}\pi \simeq \mathcal{L}_{k+1}(H) \). This defines a map \( \tau_k : J_k \to \text{Hom}(H, \mathcal{L}_{k+1}(H)) \). The target space can be identified with the space of derivations of degree \( k \) of the free Lie algebra. Furthermore, one can prove that such a derivation sends \( \omega \), the bivector dual to the symplectic form, to 0. Such an element will be called a symplectic derivation. These form a graded subalgebra \( D(H) \). Any element in \( D_k(H) \) has a Morita trace, which coincides with the evaluation:

\[
D_k(H) \subset H^* \otimes \mathcal{L}_{k+1}(H) \longrightarrow H^* \otimes T_{k+1}(H) \longrightarrow H^* \otimes H \otimes T_k(H) \longrightarrow T_k(H) \longrightarrow S^k(H)
\]

where \( \mathcal{L}(H) \) embeds in the associative tensor algebra \( T(H) \) via the rule \([a, b] = a \otimes b - b \otimes a\), and \( H^* \otimes H \to \mathbb{Z} \) is the evaluation. Morita’s trace map actually vanishes on the image of the Johnson homomorphism [9]. Let us recall the definition of the trace-like operators defined in [1].

As we recalled, there is a canonical isomorphism between \( A \) and \((H')^*\) given by the intersection pairing

\[
\omega' : A \otimes H' \longrightarrow \mathbb{Z}
\]

which is induced by the symplectic form \( \omega \). We denote \((\omega')^{1,2}\) the contraction of the first two tensors in \( A \otimes T(H') \) by \( \omega' \). An element of \( \mathcal{G}_k \) naturally induces an element of \( A \otimes \mathcal{L}_{k+1}(H') \).

Indeed \( A \otimes \mathcal{L}_{k+1}(H') \) is the kernel of \( H \otimes \mathcal{L}_{k+1}(H') \to H' \otimes \mathcal{L}_{k+1}(H') \). Then the map \( \Tr_k^A \) is defined by

\[
\Tr_k^A : \mathcal{G}_k \longrightarrow A \otimes \mathcal{L}_{k+1}(H') \overset{\iota}{\longrightarrow} A \otimes T_{k+1}(H') \overset{\omega'^{1,2}}{\longrightarrow} T_k(H') \overset{\omega'}{\longrightarrow} S^k(H')
\]

**Remark 1.1.** We actually have \( A \simeq H'^* \), hence an element of \( \mathcal{G}_k \) naturally induces an element of \( H'^* \otimes \mathcal{L}_{k+1}(H') \), i.e. a derivation of \( \mathcal{L}(H') \). The Lagrangian traces that we define here are hence a handlebody version of the traces defined by Morita.

Note that \( \mathcal{G} \) is a graded Lie subalgebra of \( D(H) \). This subalgebra is not stable by the action of \( \text{Sp}(H) \), but it is stable by \( \rho_0(\mathcal{A}) \), the image of \( \mathcal{A} \) in \( \text{Sp}(H) \). This is precisely the subgroup of \( \text{Sp}(H) \) that preserves \( A \subset H \), hence it acts on \( H' \), and on \( S^k(H') \).

**Lemma 1.2.** The linear maps \((\Tr_k^A)_{k \geq 1}\) are \( \rho_0(\mathcal{A}) \)-equivariant.

**Proof.** It follows from the fact that an element of \( \rho_0(\mathcal{A}) \) preserves the intersection form \( \omega' \). \( \square \)

Note also that the graded space associated to the \( AJ \)-filtration is stable by conjugation by elements of \( \mathcal{A} \), and the imbedding \( \text{Gr}^{AJ} \to \bigoplus_{k \geq 1} \mathcal{G}_k \) induced by \( \tau \) is \( \mathcal{A} \)-equivariant with respect to the actions we described.

For the sake of completeness, we describe how the Lagrangian traces behave with respect to the brackets in \( \mathcal{G} \). As a consequence of [7, Lemma 3.2], we have the following result:

**Proposition 1.3.** The map \( \Tr^A \) vanishes on brackets of derivations. For any \( \delta, \eta \in \mathcal{G} \), we have \( \Tr^A([\delta, \eta]) = 0 \).

We shall not use this proposition in the sequel.

**Remark 1.4.** Unlike the trace-like operators defined by Morita, the Lagrangian traces do not necessarily vanish in even degree.
2. Magnus representations for the handlebody group

2.1. Morita’s trace map and the Magnus representation. We begin by reviewing the case of the mapping class group, following [9]. In this section we work with use the notation $\gamma_i = \alpha_i$ and $\gamma_{g+1} = \beta_1$ for $1 \leq i \leq g$. Recall that the projections $\beta_i^s$ of the $\beta_i$’s in $\pi'$ induce a basis of $\pi'$ and that the elements $\alpha_i$ normally generate the kernel of the projection $\pi \to \pi'$. We consider Fox derivatives $(\frac{\partial}{\partial \gamma_i} : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi])_{1 \leq i \leq g}$ with respect to the elements $\gamma_i$ of the basis. By definition, these are $\mathbb{Z}$-linear maps which verify $\frac{\partial \gamma_j}{\partial \gamma_i} = \delta_{i,j}$ for $1 \leq i, j \leq 2g$ and $\frac{\partial u}{\partial \gamma_i} = \frac{\partial v}{\partial \gamma_i} + u \frac{\partial w}{\partial \gamma_i}$ for any $u, v, w \in \pi$. We also consider Fox derivatives for the basis chosen for $\pi'$.

To an element $\varphi$ in $\mathcal{M}$, we associate the Fox matrix $r(\varphi)$:

$$r : \mathcal{M} \longrightarrow \text{GL}(2g, \mathbb{Z}[\pi])$$

$$\varphi \longmapsto \left( \frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)_{1 \leq i, j \leq 2g}$$

where for any $x \in \mathbb{Z}[\pi]$, $\pi$ is the element obtained by applying the anti-automorphism induced by the inversion in $\pi$. The abelianization map from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[H]$ then gives a map

$$r^a : \mathcal{M} \longrightarrow \text{GL}(2g, \mathbb{Z}[H])$$

usually called the Magnus representation of the mapping class group. Nevertheless, this is not a representation in the usual sense because it is a crossed homomorphism. When the map $r^a$ is restricted to the Torelli group $\mathcal{I}$, it becomes a representation. Recall that $D_1(H) \simeq \mathbb{A}^3(H)$ and define $C$ to be the contraction map sending $a \wedge b \wedge c$ to $\omega(a, b) c + \omega(b, c) a + \omega(c, a) b$. Morita [9, Prop. 6.15] proved the following proposition:

**Proposition 2.1.** Let $g \geq 2$, and $\varphi \in \mathcal{I}$, then $\det(r^a(\varphi)) \in H \subset \mathbb{Z}[H]$ and

$$\det(r^a(\varphi)) = 2C(\tau_1(\varphi)).$$

Consequently, the determinant of the Magnus representation is trivial on $J_k$, for $k \geq 2$.

Furthermore, Morita proved that the reduction $r_k(\varphi)$ of the Fox matrix of $\varphi$ in $\text{GL}(2g, \mathbb{Z}[\pi]/I^{k+1})$ can be recovered from the $k$-th Johnson homomorphism. Any element in $D_k(H)$ sends an element of $H$ to an element in $\Gamma_{k+1} \pi / \Gamma_{k+2} \pi$. The Fox derivatives of such an element relatively to the $\gamma_i$’s yield a well-defined vector in $(I^k/I^{k+1})^3$. Indeed a Fox derivation sends an element of $\Gamma_k \pi$ to an element of $I^{k-1}$. For $d \in D_k(H)$, we set

$$\|d\| := \left( \frac{\partial d(\{\gamma_j\})}{\partial \gamma_i} \right) \in \text{GL}(2g, I^k/I^{k+1})$$

where $\{\gamma_j\}$ is the class of $\gamma_j$ in $H$. The following formula was proven by Morita:

$$\forall \varphi \in J_k, r_k(\varphi) = \text{Id} + \|\tau_k(\varphi)\|$$

(2.1)

For an element $d \in D_k(H)$, Morita’s trace map was originally defined as the trace of $\|d\|$ after reduction in $I^k/I^{k+1}$ (the projection from $\pi$ to $H$ sends $I$ to $I_H$). From formula (2.1) it is clear that

$$\det(r^a(\varphi)) \equiv 1 + \text{Tr}(\|\tau_k(\varphi)\|) \mod I^{k+1}_H.$$ 

Then it follows from Proposition 2.1 that Morita’s trace map vanishes on the image of the Johnson homomorphisms. Note here that $\text{Tr}$ is valued in $I^k/I^{k+1}$ which can be identified with $S^k(H)$.

2.2. The handlebody case. We will now adapt Morita’s strategy of proof to the case of the handlebody group. We first give an equivalent definition for the $A$-trace $T^A_k$. Let $d$ be an element of $\mathcal{G}_k := \text{Ker}(D_k(H) \to D_k(H'))$. As explained in Section 1, the class of $d$ in $H^* \otimes L_{k+1}(H')$ sends $A$ to $0$. Equivalently, an element of $\mathcal{G}_k$ induces an element of $H^* \otimes L_{k+1}(H')$. Similarly to the definition of $\|d\|$, we define

$$\|d\|^A := \left( \frac{\partial d(\{b_j\})}{\partial \gamma_i} \right) \in \text{GL}(g, I^k_{H'}/I^{k+1}_{H'}).$$

It is clear that this is actually the reduction in $\pi'$ of the $g \times g$ right-down submatrix of $\|d\|$. We denote $\|d\|^A \in \text{GL}(g, I^k_{H'}/I^{k+1}_{H'})$. Once again, note that $I^k_{H'}/I^{k+1}_{H'}$ can be identified with $S^k(H')$. 


As a consequence [9, Prop. 6.3], we have:

**Lemma 2.2.** For any \( k \geq 1 \), for any \( d \in G_k \), \( \text{Tr}_k^A(d) = \text{Tr}(d \parallel^A \epsilon) \in S^k(H') \).

We now define the Magnus representation in the handlebody case. We are in fact interested in the right-down \( g \times g \) part of the usual Fox matrix, after reduction in \( \mathbb{Z}[\pi'] \). Set

\[
\begin{align*}
  r^A : A &\rightarrow \text{GL}(g, \mathbb{Z}[\pi']) \\
  \varphi &\mapsto \left( \frac{\partial \psi(\alpha)}{\partial \alpha_i} \right)_{1 \leq i, j \leq 2g}
\end{align*}
\]

and \( r^{A,\epsilon} : A \rightarrow \text{GL}(g, \mathbb{Z}[H']) \) its abelian version. It is actually the composition of \( \text{Aut}(\pi') \) with the usual Magnus representation of the group \( \text{Aut}(\pi') \). The map \( r^A \) is a crossed homomorphism in the following sense:

\[
\forall \varphi, \psi \in A, r^A(\varphi \psi) = r^A(\varphi) (r^A(\psi))
\]

where \( \varphi \cdot r^A(\psi) \) is the matrix obtained from \( r^A(\psi) \) after applying \( \varphi \) to all the coefficients.

**Remark 2.3.** The mapping \( \text{GL}(2g, \mathbb{Z}[\pi]) \rightarrow \text{GL}(g, \mathbb{Z}[\pi']) \) (resp. its abelian version), which consist in projecting in \( \pi' \) (resp. \( H' \)) the coefficients of the righ-down matrix is of course not a homomorphism. This explains why the Lagrangian traces are not reductions of the Morita trace maps (see [1, Remark 4.1]).

We are now ready to prove Theorem B.

**Proof of Theorem B.** First, the map \( \det \circ r^{A,\epsilon} \) is \( A \)-equivariant. Indeed, if \( \psi \) is an element of \( A \), and \( \varphi \) is at least in \( A_1 \), we deduce from (2.2) that

\[
r^{A,\epsilon}(\psi \varphi \psi^{-1}) = r^{A,\epsilon}(\psi)(\psi \cdot r^{A,\epsilon}(\varphi))(\psi \cdot r^{A,\epsilon}(\psi^{-1})),
\]

because \( \varphi \) acts trivially on \( H' \). We conclude by observing that

\[
\text{Id} = r^{A,\epsilon}(\psi \psi^{-1}) = r^{A,\epsilon}(\psi \cdot r^{A,\epsilon}(\psi^{-1})),
\]

hence \( r^{A,\epsilon}(\psi \varphi \psi^{-1}) \) is conjugated to \( \psi \cdot r^{A,\epsilon}(\varphi) \). The map \( \text{Tr}_1^A \circ \tau_1 \) is also \( A \)-equivariant, by Lemma 1.2. According to [10], the subgroup \( A_1 \) is normally generated by the map \( \varphi := T_c \circ T_d^{-1} \) described in Figure 2. Note that \( \varphi \) is an *annulus twist* in the sense of [3] and a BP map in the sense of [5]. Indeed \( c \) and \( d \) cobound an annulus in the handlebody, and a surface in \( \Sigma \), which ensures that \( \varphi \) is both an element of \( A \) and \( I \). We set \( \hat{c} := [\beta_2, \alpha_1, \beta_1] \) a lift of the curve \( c \) in \( \pi \).

The action of \( \varphi \) on \( \pi \) is

\[
\begin{align*}
  \alpha_1 &\mapsto \hat{c} \alpha_1 \hat{c}^{-1} \\
  \alpha_2 &\mapsto \alpha_2 \beta_2 \hat{c}^{-1} \\
  \beta_1 &\mapsto \hat{c} \beta_1 \hat{c}^{-1} \\
  \beta_2 &\mapsto \hat{c} \beta_2 \hat{c}^{-1},
\end{align*}
\]

hence we compute directly that

\[
r^{A,\epsilon}(\varphi) = \begin{pmatrix} b_2^{-1} & 0 \\ 1 - b_2^{-1} & 1 \end{pmatrix}.
\]

We deduce that \( \det(r^{A,\epsilon}(\varphi)) = b_2^{-1} \in \mathbb{Z}[H'] \), and, switching to the additive notation in \( H' \), we obtain an element \( -b_2 \in H' \). By equivariance, as the map \( \det \circ r^{A,\epsilon} \) is a multiplicative map, we deduce that it sends the subgroup \( A_1 \) in \( H' \). Furthermore, by [4], we know that \( \tau_1(\varphi) = a_1 \wedge b_1 \wedge b_2 \). Hence, \( \text{Tr}_1^A(\tau_1(\varphi)) = -b_2 \), and we deduce that, on \( A_1 \),

\[
\det \circ r^{A,\epsilon} = \text{Tr}_1^A \circ \tau_1.
\]

As the first Johnson homomorphism vanishes on \( A_k \), for \( k \geq 2 \), we deduce that the map \( \det \circ r^{A,\epsilon} \) is trivial on these subgroups.

We can now prove Theorem A.

**Proof of Theorem A.** The argument is similar to [9]. Indeed, we set \( r_k^A \) the reduction of \( r^A \) modulo \( I_{2g+1}^k \). Clearly, as in [9][Prop. 6.13], we have the following equality:

\[
\forall \varphi \in A_k, r_k^A(\varphi) = \text{Id} + \| \tau_k(\varphi) \| A
\]
Recall that $\varphi \in A_k$ implies that $\tau_k(\varphi) \in G_k$, hence the right-hand side of the equation makes sense. We can deduce equation (2.3) from equation (2.1), by reducing the coefficients of the right-down $g \times g$ matrix of $\tau_k(\varphi)$, or directly using the very definition of the Johnson homomorphisms.

After projection of the coefficients in $\mathbb{Z}[H'_k]$, we obtain

$$\det(r^a - A(\varphi)) \equiv 1 + \text{Tr}(\|\tau_k(\varphi)\|^4) \mod I_k^{k+1}.$$ 

Using Theorem B and Lemma 2.2, we infer that, for any $\varphi \in A_k$, $\text{Tr}_k^A(\tau_k(\varphi)) \equiv 0 \mod I_k^{k+1}$. □

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