When are epsilon-nets small?

Andrey Kupavskii†, Nikita Zhivotovskiy†

Abstract

Given a range space \((X, \mathcal{R})\), where \(X\) is a set equipped with probability measure \(P\) and the family of measurable subsets \(\mathcal{R} \subset 2^X\), and \(\epsilon > 0\), an \(\epsilon\)-net is a subset of \(X\) in the support of \(P\), which intersects each \(R \in \mathcal{R}\) with \(P(R) \geq \epsilon\). In many interesting situations the size of \(\epsilon\)-nets depends only on \(\epsilon\) together with different complexity measures. The aim of this paper is to give a systematic treatment of such complexity measures arising in Discrete and Computational Geometry and Statistical Learning, and to bridge the gap between the results appearing in these two fields. As a byproduct, we obtain several new upper bounds on the sizes of \(\epsilon\)-nets that generalize/improve the best known general guarantees. In particular, our results work with regimes when small \(\epsilon\)-nets of size \(o(\frac{1}{\epsilon})\) exist, which are not usually covered by standard upper bounds. Inspired by results in Statistical Learning we also give a short proof of the Haussler’s upper bound on packing numbers \([16]\).

1 Introduction

In this section we define \(\epsilon\)-nets from both geometric and statistical learning point of view, and give the famous Vapnik-Chervonenkis-Haussler-Welzl result in several equivalent formulations. The structure of the rest of the paper is as follows.

In Section 2 we talk about Alexander’s capacity and the (upper) bounds for \(\epsilon\)-nets in the context of Statistical Learning and Active Learning. We then prove a new upper bound, which improves upon the previous bounds.

In Section 3 we introduce the doubling constant and give some of the results concerning it along with their improvements. Later, we prove a general theorem (which is one of our main contributions) giving bounds on \(\epsilon\)-nets in terms of both Alexander’s capacity and doubling...
constant, which improves upon many of the previously known results, and show that it is tight.

In Section 4 we revisit the bounds on packing numbers for VC classes. We also provide a very short proof of Haussler’s Lemma and generalize the result of Chazelle.

In Section 5 we discuss in detail the bounds on the doubling constant, in particular, in terms of the shallow cell complexity.

In Section 6 we compare in detail different upper bounds on $\epsilon$-nets.

We write $\log x$ for the natural logarithm (or in the case when the base is not important), and $\log_2 x$ for the binary logarithm. In many places we omit upper/lower integer parts when it does not affect the correctness of the statements.

$\epsilon$-nets. Combinatorial and geometric point of view

Consider a set $X$ equipped with probability measure $P$ and a family of measurable subsets $\mathcal{R} \subset 2^X$, where $2^X$ is the power set of $X$. The pair $(X, \mathcal{R})$ is called a range space. For a fixed $\epsilon > 0$, a subset $S \subset X \cap \text{supp}(P)$ is an $\epsilon$-net, if $R \cap S \neq \emptyset$ for each $R \in \mathcal{R}$ with $P(R) \geq \epsilon$. $\epsilon$-nets often arise in the context of Computational Geometry problems. In that context, the set $X$ is often finite and equipped with uniform measure, and we may speak about sizes instead of measures. It is usually simple to generalize the results that are valid for a uniform measure on a finite set to almost arbitrary measure. However, in this paper we focus on distribution dependent complexity measures of the range spaces and therefore choose to present everything for general probability measures.

We follow the standard geometric notation and will denote our domain by $X$. Although it may create some confusion in what follows we use the symbol $X_i$ to denote the random elements sampled from $X$ according to $P_X$.

A line of research, starting from a seminal work of Vapnik and Chervonenkis [26], is concerned with proving the existence of small $\epsilon$-nets in certain scenarios. The first historically and probably the most important such measure is VC-dimension. The VC-dimension of $(\mathcal{X}, \mathcal{R})$ is the maximal size of $Y \subseteq \mathcal{X}$ such that $Y$ is shattered: $\mathcal{R}|_Y := \{R \cap Y : R \in \mathcal{R}\} = 2^Y$. Building on the work [26], Haussler and Welzl [14] proved that one can find $\epsilon$-nets of size depending on $\epsilon$ and VC-dimension only. Moreover, the $\epsilon$-net can be constructed using an i.i.d. sample of points of $X$. Later their result was slightly sharpened in [17], and the matching lower bounds were proven. Given the VC-dimension of a range space, one can bound the size of the smallest $\epsilon$-net as follows.

**Theorem A** ([26], [14], [17]): Fix $\epsilon, \delta \in (0,1]$ and let $(X, \mathcal{R})$ be a range space of VC-dimension $d$ with probability distribution $P$. Then a set of size $O\left(\frac{d \log \frac{1}{\epsilon}}{\epsilon} + \log \frac{1}{\epsilon}\right)$ chosen i.i.d. from $X$ according to $P$ is an $\epsilon$-net for $(X, \mathcal{R})$ with probability at least $1 - \delta$.

In some cases it is more convenient to formulate this and analogous results in terms of an approximation, which may be obtained by a sample of fixed size. Here is such a reformulation of Theorem A, equivalent to the original.
**Theorem B.** Fix \( \varepsilon, \delta \in (0, 1] \) and let \((X, \mathcal{R})\) be a range space of VC-dimension \( d \) with probability distribution \( P \). A set of size \( n \) chosen i.i.d. from \( X \) according to \( P \) is an \( \varepsilon(n) \)-net for \((X, \mathcal{R})\) with probability at least \( 1 - \delta \) for \( \varepsilon(n) = O\left(\frac{d \log \frac{n}{d} + \log \frac{1}{\delta}}{n}\right) \).

In Computational Geometry \( X \) is typically a set of points in \( \mathbb{R}^d \), and the ranges are intersections of \( X \) with all objects from a certain class: lines, halfspaces, balls, etc. When providing upper bounds for \( \varepsilon \)-nets in this context, one searches for bounds that would hold for all range spaces of such type. In another common scenario of the so-called dual range spaces the roles of points and ranges are switched.

The applications of \( \varepsilon \)-nets cover several topics in Computational Geometry, including spatial partitioning and LP rounding. We refer to a very recent survey [25], which covers many of the recent developments in \( \varepsilon \)-nets, as well as their applications.

**\( \varepsilon \)-nets. Statistical point of view**

Similar ideas and notions were developed in Statistical Learning. Consider the following statistical model. We are given an instance space \( \mathcal{X} \) equipped with an unknown probability distribution \( P_X \) and a (known) family of classifiers \( \mathcal{F} \) consisting of functions \( f: \mathcal{X} \rightarrow \{\pm 1\} \). A learner observes \(((X_1, Y_1), \ldots, (X_m, Y_m))\), an i.i.d. training sample where \( X_i \) are sampled according to \( P_X \) and \( Y_i = f^*(X_i) \) for some fixed \( f^* \in \mathcal{F} \). This scenario is referred to as the realizable case classification. Sample-consistent learning algorithm (particular case of empirical risk minimization) refers to any learning algorithm with the following property: given a training sample of size \( m \), it outputs any classifier \( \hat{f} \in \mathcal{F} \) that is consistent with the sample (that is, \( \hat{f}(X_i) = f^*(X_i) \) for all \( i = 1, \ldots, m \)).

We say a set \( \{x_1, \ldots, x_k\} \subseteq \mathcal{X}^k \) is shattered by \( \mathcal{F} \) if there are \( 2^k \) distinct classifications of \( \{x_1, \ldots, x_k\} \) realized by classifiers in \( \mathcal{F} \). The VC-dimension of \( \mathcal{F} \) is the largest integer \( d \) such that there exists a set \( \{x_1, \ldots, x_d\} \) shattered by \( \mathcal{F} \).

The analogue of Theorem [A] in this context is the following classical result[1] of Vapnik and Chervonenkis:

**Theorem C ([26]).** Consider any sample-consistent learning algorithm over the i.i.d. sample of size \( m \), which outputs a classifier \( \hat{f} \in \mathcal{F} \). Assume that \( \mathcal{F} \) has VC dimension \( d \) and we are in the realizable case: there exists \( f^* \in \mathcal{F} \), such that \( Y = f^*(X) \). Then for any \( \varepsilon, \delta \in (0, 1] \) for \( m = O\left(\frac{d \log \frac{1}{\varepsilon} + \log \frac{1}{\delta}}{\varepsilon}\right) \) we have \( P_X(\hat{f}(X) \neq f^*(X)) \leq \varepsilon \) with probability at least \( 1 - \delta \).

It is actually easy to translate this problem into the combinatorial language: we just have to think of the instance space \( \mathcal{X} \) as our ground set, and the collection of sets \( \{i : f(X_i) \neq f^*(X_i)\} : f \in \mathcal{F} \} \) playing the role of ranges. Then Theorem C basically says that an i.i.d. sample gives an \( \varepsilon \)-net for such range space with high probability.

---

[1] Although this result is not presented explicitly in their book, it follows directly from their learning bounds for the realizable case classification.
2 Alexander’s capacity

Significant effort was put by many researchers in both Computational Geometry and Statistical Learning Theory to understand whether it is possible to improve the above bounds. In this context, several different measures of complexity were introduced. One of them is the VC-dimension, already mentioned above. Actually, to Prove Theorem A, one only needs the following property of the VC-dimension, implied by the famous Vapnik-Chervonenkis-Sauer-Shelah lemma: given a range space $(X, R)$ of VC-dimension $d$, for any $Y \subset X$ we have $|R|_Y \leq \sum_{i=0}^{d} \binom{|Y|}{i} = O(|Y|^d)$. Let us introduce the projection function $\pi_R(Y)$:

$$\pi_R(y) := \max\{|R|_Y : Y \subset X, |Y| = y\}. \tag{1}$$

Thus, the Vapnik-Chervonenkis-Sauer-Shelah lemma implies that

$$\pi_R(y) \leq \sum_{i=0}^{d} \binom{y}{i} \leq \left(\frac{e y}{d}\right)^d \tag{2}$$

for any $y = d, \ldots, |X|$, and Theorem A as well as Theorem C holds under this condition. Actually, Vapnik and Chervonenkis used a weaker requirement to obtain Theorem C: They required the projections to be small on average (see [6, 5] and the results related to a so-called VC entropy).

One of the measures coming from the side of Statistical Learning is Alexander’s capacity [1, 12] (similar to the disagreement coefficient in the Active Learning literature [19]). Initially it appeared in the work of Alexander [1] in the analysis of ratio-type empirical processes. For $\epsilon_0 > 0$ fix a set $F_{\epsilon_0} = \{ f \in F : P_X(f(X) \neq f^*(X)) \leq \epsilon_0 \}$. For $\epsilon \in (0,1]$ define Alexander’s capacity $\tau(\epsilon)$:

$$\tau(\epsilon) := \sup_{f^* \in F, \epsilon_0 \geq \epsilon} P_X\{ \{ x \in X : \exists f \in F_{\epsilon_0} \text{ s.t. } f(x) \neq f^*(x) \} \}/\epsilon_0$$

It is easy to see that $\tau(\epsilon) \leq 1/\epsilon$. We can also define the Alexander’s capacity for a range space $(X, R)$. It can be defined as follows:

$$\tau(\epsilon) := \sup_{\epsilon_0 \geq \epsilon} \frac{P(\cup_{R \in R_{\leq \epsilon_0}} R)}{\epsilon_0},$$

where $R_{\leq \epsilon_0} := \{ R \in R | P(R) \leq \epsilon_0 \}$. For the uniform measure on a finite set the last definition can be informally understood as ratio of the number of points of $X$ that lie in one of the sets of size at most $\epsilon_0 n$, over $\epsilon_0 n$ (which is maximized over $\epsilon_0 \geq \epsilon$).

**Theorem D** (Gine and Koltchinskii [12], Hanneke [19]). *Theorems A and C hold with a sample of size $m = O\left(\frac{d \log \tau(\epsilon) + \log \frac{1}{\delta}}{\epsilon}\right)$.*
Observe that, since \( \tau(\epsilon) \leq 1/\epsilon \), Theorem \( \text{[D]} \) is an improvement over Theorem \( \text{[C]} \). We refer to \([19, 20, 3, 12]\) where many examples when \( \tau(\epsilon) = o(1/\epsilon) \) are provided. The same result may be directly translated to the range spaces. However, we show that in many cases when \( \tau(\epsilon) \) is smaller than \( 1/\epsilon \) it is possible to construct nets of sizes significantly smaller than what is predicted by Theorem \( \text{[D]} \). The result of Theorem \( \text{[D]} \) is very specific to i.i.d. sampling and does not cover the situation when one is able to choose points in a more clever way.

2.1 Active Learning

Active Learning is a particular framework within Statistical Learning. As before there is an instance space \( \mathcal{X} \) and a label space \( \mathcal{Y} := \{-1, 1\} \) and a set \( \mathcal{F} \) of classifiers mapping \( \mathcal{X} \) to \( \mathcal{Y} \). In the the realizable case, there is a target function \( f^* \in \mathcal{F} \) and a sample \( (X_i, f^*(X_i))_{i=1}^n \). In the pool-based active learning, we define an active learning algorithm as an algorithm taking as input a budget \( n \in \mathbb{N} \), and proceeding as follows. The algorithm initially has access to the unlabeled independent data sequence \( X_1, X_2, \ldots \) distributed according to \( P_X \). The algorithm may select an index \( i_1 \) and request to observe the label \( Y_{i_1} \). The algorithm observes the value of \( Y_{i_1} \), and if \( n \geq 2 \), then based on both the unlabeled sequence and this new observation \( Y_{i_1} \), it may select another index \( i_2 \) and request to observe \( Y_{i_2} \). This continues for at most \( n \) rounds.

For the realizable case the algorithm named CAL (named after Cohn, Atlas, and Ladner) is by now the most studied. We refer to the exact definition of this learning procedure and the analysis to \([19]\).

**Theorem E** (Sample complexity bound for CAL \([19]\)). The exists an active learning algorithm (namely CAL), such that in the realizable case for any distribution \( P_X \), any \( \epsilon, \delta \in (0, 1] \) and a class of classifiers \( \mathcal{F} \) of VC dimension \( d \) after requesting

\[
n = O\left( \tau(\epsilon) [d \log \tau(\epsilon) + \log \frac{1}{\epsilon}] + \log \frac{1}{\delta} \right) \log \frac{1}{\epsilon}
\]

labels with probability at least \( 1 - \delta \) CAL returns a classifier \( \hat{f} \) with \( P(\hat{f}(X) \neq f^*(X)) \leq \epsilon \).

When \( \tau(\epsilon) \log \tau(\epsilon) = o(1/\epsilon) \) the above algorithm gives an improvement over the standard sampling strategy described in Theorem \( \text{[C]} \) In particular, a simple inspection of the proof of the result for the CAL algorithm gives the following bound for \( \epsilon \)-nets.

**Corollary 1.** Let \( (X, \mathcal{R}) \) be a range space of VC-dimension \( d \) and Alexander’s capacity \( \tau(\epsilon) \). Then there exists an \( \epsilon \)-net of size

\[
n = O\left( \tau(\epsilon) [d \log \tau(\epsilon) + \log \frac{1}{\epsilon}] \log \frac{1}{\epsilon} \right)
\]

The formal procedure behind the CAL algorithm, adapted to the case of range spaces is written below. The idea behind the algorithm applied to the the range spaces is very simple and natural. We sample random points according to \( P \) and add them to the \( \epsilon \)-net one by
one. But contrary to the strategy of Theorem A the new point is added if it is contained in a
least one of the sets which were not hit (by hitting we mean having a nonempty intersection)
by the points which were added to the \( \varepsilon \)-net on previous steps. In some sense, we never add
a new point if it does not hit any set that was not hit before.

**Algorithm 1 CAL for \( \varepsilon \)-nets**

1: procedure
2: \( m \leftarrow 0, t \leftarrow 0, \mathcal{R}_0 \leftarrow \mathcal{R} \)
3: while \( t < n \) and \( m < 2^n \) do
4: \( m \leftarrow m + 1 \)
5: if \( X_m \in \bigcup_{R \in \mathcal{R}_{m-1}} R \) then
6: Add \( X_m \) to \( \varepsilon \)-net,
7: \( \mathcal{R}_m \leftarrow \{ R \in \mathcal{R}_{m-1} | R \cap X_m = \emptyset \} \),
8: \( t \leftarrow t + 1. \)
9: else
10: \( \mathcal{R}_m \leftarrow \mathcal{R}_{m-1}. \)

### 2.2 New bound in terms of Alexander’s capacity

The following theorem could be shortly proved via an application of Theorem C.

**Theorem 2.** Let \((X, \mathcal{R})\) be a range space of VC-dimension \(d\). Fix \( \varepsilon > 0 \). Let \( \tau_i := \tau(2^i \varepsilon) \) and put \( z := \lceil \log_2 \frac{1}{\varepsilon} \rceil \). Then there exists an \( \varepsilon \)-net for \((X, \mathcal{R})\) of size

\[
O\left(d \sum_{i=1}^z \tau_i \log \tau_i \right). \tag{3}
\]

*Proof.*** We set \( \epsilon_i := 2^i \varepsilon \) and \( \mathcal{R}_i = \{ S \in \mathcal{R} : \epsilon_{i-1} \leq P(S) \leq \epsilon_i \}, i = 1, \ldots \) and define \( X^{(i)} := \bigcup_{R \in \mathcal{R}_i} R \) to be the support of \( \mathcal{R}_i \). Note that \( X = \bigcup_{i=1}^z X^{(i)} \setminus X^{(i-1)} \).

It is clearly sufficient to find an \( \varepsilon \)-net for each \((X^{(i)}, \mathcal{R}_i)\) of size \(O(d \tau_i \log \tau_i)\). Note that, by definition, \( P(X^{(i)}) \leq \tau_i \epsilon_i \), while for each \( R \in \mathcal{R}_i \) we have \( P(R) \geq \epsilon_{i-1} \). Therefore, \( P(R) \geq \frac{P(X^{(i)})}{2 \tau_i} \), and since for each \( R \in \mathcal{R}_i \) we have \( P(R \cap X^{(i)}) = P(R) \) and thus \( P(R \cap X^{(i)}) \geq \frac{P(X^{(i)})}{2 \tau_i} \) it is sufficient for us to find a \( \frac{1}{2 \tau_i} \)-net for \((X^{(i)}, \mathcal{R}_i)\) with respect to the conditional distribution \(P( |X^{(i)}|)\). But this could simply be done using the Vapnik-Chervonenkis-Haussler-Welzl Theorem C. This gives a net of size \(O(d \tau_i \log \tau_i)\) for each \((X^{(i)}, \mathcal{R}_i)\).

We immediately get the following Corollary.

**Corollary 3.** In the notation of Theorem 2 there exists an \( \varepsilon \)-net for \((X, \mathcal{R})\) of size

\[
O\left(d \tau(\varepsilon) \log \tau(\varepsilon) \log \frac{1}{\varepsilon} \right). \tag{4}
\]
Proof. There are only ⌈log₂ 1/ε⌉ summands in (3), each being \(O(\tau(\epsilon) \log \tau(\epsilon))\). \(\Box\)

Remark. Theorem 2 improves both Theorems A and D in many cases. Indeed, one just have to use \(\tau_i \leq 1/(2^i\epsilon)\). Corollary 3 is an improvement of Corollary 1. In particular, Corollary 3 implies that when \(\tau(\epsilon) = O(1)\) we have \(\epsilon\)-nets of size \(O(\log \frac{1}{\epsilon})\) which is significantly smaller than what is promised e.g., by Theorem A.

The \(\epsilon\)-net from Theorem 2 is based on a simple sampling strategy, although the probability of including different elements differs. The probabilities can be decided on before choosing a random sample quite easily. One should just find the sets \(X^{(i)}\), which could be done efficiently. However, Theorem A, as well as the CAL algorithm used for Corollary 1, gives a more natural sampling strategy to construct \(\epsilon\)-nets.

In general, it is not possible to get rid of the factor \(\log \frac{1}{\epsilon}\) since there are range spaces with \(\tau(\epsilon) = O(1)\) and with the smallest \(\epsilon\)-net of size \(\Omega(\log \frac{1}{\epsilon})\). See the remark after the proof of Theorem 4.

3 The doubling constant and \(\epsilon\)-nets

Another quantity of interest, which has wide applications in Statistical Learning is the notion of the doubling constant or the local covering number. For a class of classifiers \(\mathcal{G}\) let us define the distance \(\rho\) by \(\rho(f, g) = P(f(X) \neq g(X))\). We denote by \(\mathcal{M}(\mathcal{G}, \epsilon)\) the packing number of \(\mathcal{G}\) with respect to \(\rho\):

\[
\mathcal{M}(\mathcal{G}, \epsilon) := \max_{Q \subseteq \mathcal{G}} \{|Q| : \rho(f, g) \geq \epsilon \text{ for any distinct } f, g \in Q\}.
\]

Given any \(f^* \in \mathcal{F}\), define the set \(B(\mathcal{F}, \epsilon) = \{f \in \mathcal{F} : \rho(f, f^*) \leq \epsilon\}\) of all classifiers from \(\mathcal{F}\) at distance at most \(\epsilon\) from \(f^*\). Finally, define the doubling constant

\[
D_\epsilon(P, \mathcal{F}) = \sup_{\epsilon_0 \geq \epsilon} \mathcal{M}(B(\mathcal{F}, 2\epsilon_0), \epsilon_0).
\] (5)

We write \(D_\epsilon\) instead of \(D_\epsilon(P, \mathcal{F})\) when the class of functions is clear from the context. The logarithm of the doubling constant is referred to as the doubling dimension. It plays an important role in risk guarantees for some learning algorithms [8, 23, 28].

To motivate theorems we prove below, we give a preliminary known bound on \(D_\epsilon\).

**Theorem F** ([20], Theorem 17). We have \(D_\epsilon \leq (c\tau(\epsilon))^{c_1d}\) for some \(c, c_1 > 1\).

In this form it is usually sufficient for statistical applications, but in what follows we will need a tight bound in terms of \(c_1\). One of the corollaries of the results proved in Section 5 is that for a range space \((X, \mathcal{R})\) of VC dimension \(d\) it holds \(D_\epsilon \leq (c\tau(\epsilon))^d\) for some \(c > 0\).

Now we turn towards Computational Geometry. Let us reformulate (5) in terms of range spaces. Given a range space \((X, \mathcal{R})\) we denote by \(\mathcal{M}_H(\mathcal{R}, \epsilon)\) the maximal packing with respect to the distance \(\rho\) defined by \(\rho(R_1, R_2) = P(R_1 \Delta R_2)\). Then the doubling constant is

\[
D_\epsilon(P, \mathcal{R}) = \sup_{\epsilon_0 \geq \epsilon} \mathcal{M}(\mathcal{R}_{\leq 2\epsilon_0}, \epsilon_0).
\]
3.1 New bound for $\epsilon$-nets

The first part of the following Theorem is a generalization of a theorem from [24] (see also the related discussions there), and the structure of the proof is similar. We should note, however, that the technique of the proof is also closely related to the peeling technique which origins from the empirical processes theory and which is widely used in the Learning Theory [7, 8, 28]. The second part of the Theorem which complements the first bound has no known analogs.

**Theorem 4.** Let $(X, R)$ be a range space of VC-dimension $d$. Fix $\epsilon > 0$. Let $D_\epsilon$ be an upper bound on the doubling constant of $(X, R)$ and let $\tau_i := \tau(2^i \epsilon)$ and put $z := \lceil \log_2 \frac{1}{\epsilon} \rceil$.

1. If $D_\epsilon \geq 2 \tau_1$, then there exists an $\epsilon$-net for $(X, R)$ of size $O\left( \sum_{i=1}^{z} \left( \log_2 \frac{D_\epsilon}{\tau_i} + d \right) \tau_i \right)$.  

2. If $D_\epsilon \leq \frac{1}{2\epsilon}$, then there exists an $\epsilon$-net for $(X, R)$ of size $O(d \log \frac{1}{\epsilon D_\epsilon})$.

Moreover, for any $n, d, \epsilon > d/n$ and $D_\epsilon < \frac{1}{2\epsilon}$ there exists a range space on $n$ points with uniform measure and VC-dimension at most $d$, doubling constant at most $D_\epsilon$ and the smallest $\epsilon$-net of size $\Omega\left( d \log \frac{1}{\epsilon D_\epsilon} \right)$.

**Remark.** Since $\tau_1 \leq \frac{1}{2\epsilon}$, the two upper bounds from the theorem cover all possible range of $D_\epsilon$ except for $\tau_1 \leq D_\epsilon \leq 2 \tau_1$. But then we may substitute $2 \tau_1$ into the first bound instead of $D_\epsilon$, getting a valid bound. The upper bound in Part 1 of the theorem is sharp for constant $d$, since it is a strengthening of an upper bound from [24] (see Section 6), which was shown to be tight in some specific cases in [18]. The upper bound in Part 2 of the Theorem may be stated in terms of $\tau_i$, but the formulation gets rather complicated, so we decided to omit it.

Before proving the theorem, let us first obtain a handy corollary from Part 1.

**Corollary 5.** In the notation of Theorem 4, assume that $D_\epsilon$ is an upper bound on the doubling constant of $(X, R)$ and $\tau$ is an Alexander’s capacity. If $D_\epsilon \geq \epsilon/\epsilon$, then there exists an $\epsilon$-net for $(X, R)$ of size $O\left( \frac{1}{\epsilon} \left( d + \log_2 (\epsilon D_\epsilon) \right) \right)$.

Similarly, if instead $D_\epsilon \geq 2 \tau(\epsilon)$ there exists an $\epsilon$-net for $(X, R)$ of size $O\left( \tau(\epsilon) \left( d + \log_2 \frac{D_\epsilon}{\tau(\epsilon)} \right) \log \frac{1}{\epsilon} \right)$.  

**Proof.** The function $a \log_2 \frac{1}{a}$ is increasing for $a \in (0, 1/e)$. Then, recalling that $\tau_i \leq \frac{1}{2\epsilon}$ and $\frac{\tau_i}{D_\epsilon} \leq \frac{1}{D_\epsilon} \leq \frac{1}{2\epsilon}$, we may apply it in the form $\frac{\tau_i}{D_\epsilon} \log_2 \frac{D_\epsilon}{\tau_i} \leq \frac{1}{2\epsilon} \log_2 (2^i \epsilon D_\epsilon)$. We get that $\sum_{i=1}^{z} \tau_i \log_2 \frac{D_\epsilon}{\tau_i} \leq \frac{1}{\epsilon} \sum_{i=1}^{z} 2^{-i} \log_2 (2^i \epsilon D_\epsilon) \leq \frac{1}{\epsilon} \left[ \log_2 (\epsilon D_\epsilon) + \sum_{i=1}^{z} i 2^{-i} \right] = O\left( \frac{1}{\epsilon} \log_2 (\epsilon D_\epsilon) \right)$.
Moreover, we have \( \sum_{i=1}^{r} \tau_i \leq \frac{1}{D} \). We are left to substitute it into (9). The proof of the second bound is trivial given (9) and the fact that \( \tau(\epsilon) \) is nonincreasing.

Observe that the bound (7) is always not worse than the best known bound of Corollary 3 given in terms of Alexander’s capacity alone. This fact directly follows from the bound (9) below. The following weaker bound, which is nevertheless stronger than Theorem A, follows from Theorem 4. We sacrificed a factor in the logarithm in order to get a bound valid for any \( D_\epsilon \).

**Corollary 6.** In the notation of Theorem 4, there exists an \( \epsilon \)-net for \( (X, R) \) of size

\[
O\left( \frac{1}{\epsilon} (d + \log_2(D_\epsilon)) \right).
\]

To see that the corollary above is stronger than, e.g., Theorem A, one needs to use (9). Actually, substituting the bound (9), we are getting a bound in the spirit of Theorem 2. But (9) does not use the full strength of the doubling constant. We can get more interesting results using more fine-grained bounds. This is the subject of the next section.

### 3.2 Proof of Theorem 4

**Part 1. Upper bound**

We work in the notation of the proof of Theorem 2. By the definition of \( D_\epsilon \), there is a maximal \( \epsilon_{i-1} \)-packing \( Q_i \) of size at most \( D_\epsilon \) in \( R_i \). Note that for each \( R \in R_i \) there is such \( Q \in Q_i \), so that \( P(R \Delta Q) \leq \epsilon_{i-1} \), which, together with \( P(R), P(Q) \geq \epsilon_{i-1} \) implies that \( P(R \cap Q) \geq \epsilon_{i-2} \geq P(Q)/4 \). For each \( Q \in Q_i \) denote \( R_i(Q) := \{ R \in R_i : P(R \cap Q) \geq P(Q)/4 \} \). We have \( R \subseteq \cup_i R_i \subseteq \bigcup_i \cup_Q \in Q_i R_i(Q) \). Therefore, a set, which would be a 1/4-net (with respect to a conditional measure \( P( \cdot | Q) \)) for each of the families of ranges \( R_i(Q) \), would be an \( \epsilon \)-net for \( R \).

Recall that \( \tau_i := \tau(\epsilon_i) \). Thus, we have \( P(X^{(i)}) \leq \tau_i \epsilon_i \). Fix an absolute constant \( c \), which choice will be clear later, and put \( t_i = \log_2 \frac{D_i}{\tau_i} \). Note that \( 1 \leq \tau_i/\tau_{i+1} \leq 2 \) and \( D_\epsilon \geq 2\tau_i \) for any \( i \geq 1 \), therefore, \( t_i \geq \log_2 2 = 1 \), and \( 1 \leq t_{i+1}/t_i \leq 2 \).

Consider a random sample \( S_i \) of size \( c(t_i + d)\tau_{i-1} \), sampled from \( X \) according to the conditional distribution \( P( \cdot | X^{(i)}) \). Observe that we may think of \( S_i \cap Q \) as a sample with elements distributed according to a conditional distribution \( P( \cdot | Q) \). We also have that for any \( Q \in Q_i \) it holds \( P(Q|X^{(i)}) \geq \frac{\epsilon_{i-1}}{\tau_i} = \frac{1}{2 \tau_i} \). Using the Chernoff bound for an appropriately chosen \( c \) we have for a fixed \( Q \) on the event \( E_1 \) of probability at least \( 1 - 2^{-t_i - 1} \) that at least \( c(t_i + d)/8 \) of the elements of \( S \) belong to \( Q \). Then, by Theorem A once again since we are free to choose the value of the constant \( c \), when the event \( E_1 \) holds the set \( S_i \cap Q \) is a 1/4-net (with respect to a conditional measure \( P( \cdot | Q) \)) for \( R_i(Q) \) with probability at least \( 1 - 2^{-t_i - 1} \). Using a union bound we have that with probability at least \( 1 - 2^{-t_i} \) the set \( S_i \cap Q \) will be a 1/4-net for \( R_i(Q) \) (with respect to \( P( \cdot | Q) \)).
Put $S := \bigcup_{i=1}^z S_i$. Therefore, the expectation of the number $M_i$ of $Q \in Q_i$, for which $S$ is not a $1/4$-net, is

$$\mathbb{E}M_i \leq D_i 2^{-t_i} = \tau_i.$$  

On the other hand, the size $N$ of $S$ is

$$N \leq \sum_{i=1}^z c(t_i + d)\tau_{i-1}.$$  

Using Markov inequality, we get that, with positive probability, $\sum_{i=1}^z M_i \leq 3 \sum_{i=1}^z \tau_i$. We fix such a set $S$. Next, we manually find a $1/4$-net (with respect to conditional measure $P( |Q|)$) for each of the $\mathcal{R}_i(Q)$ that contribute to $M_i$. Using Theorem 1 again, we conclude that we need to add a set $A$ of additional $O(d \sum_{i=1}^z \tau_i)$ points to the $\epsilon$-net in order to cover the remaining sets that might be still uncovered. Therefore, in total we get an $\epsilon$-net of size

$$O\left( \sum_{i=1}^z (t_i + d)\tau_i \right).$$

**Part 2. Upper bound**

We work in the notation of Theorem 2 and the previous part. Put $\lceil \log \frac{1}{\tau_d} \rceil =: i_0$. Then all ranges in $\mathcal{R} := \mathcal{R} \setminus \bigcup_{i=1}^{i_0} \mathcal{R}_i$ have measure at least $\frac{1}{D_i} \geq 2\epsilon$. We know that the doubling constant of the range space $(X, \mathcal{R}')$ is not greater than $eD_{\epsilon}$, and, applying Corollary 3 with epsilon equal to $\frac{1}{D_i}$, we conclude that there is an $\epsilon$-net for $(X, \mathcal{R}')$ of size at most $O(d D_{\epsilon})$. Therefore, to conclude the proof of this part of the theorem, it is sufficient to show that for each $i = 1, \ldots, i_0$ the range space $\mathcal{R}_i$ has $\epsilon$-net of size $O(d D_{\epsilon})$.

Consider $Q_i$ and the corresponding $\mathcal{R}_i(Q)$ for $Q \in Q_i$. Then for a fixed $i$ we have $|Q_i| \leq D_i$ and for each $\mathcal{R}_i(Q)$ there is a $1/4$-net (with respect to $P( |Q|)$) of size $O(d)$.

Thus, there is an $\epsilon$-net for $\mathcal{R}_i$ of size $O(d D_{\epsilon})$. The total size of $\epsilon$-net is $O(d \log D_{\epsilon})$.

**Part 2. Lower bound**

To construct the lower bound we consider the finite set $X$ of $n$ elements and a uniform measure. For simplicity let us assume that $\epsilon n = k$ is an integer number. For each $i$ fix $X^{(i)}$ of cardinality $k2^i + d - 1$ and consider the following collection of ranges: $\mathcal{R}' := \{ R \subset X^{(i)} : |R| = k2^i \}$. Next, form a range space $(X^{(i)}, \mathcal{R}_i)$ by taking $l$ disjoint copies of $\mathcal{R}'$ on disjoint sets $X^{(i)}$. Finally, define $(X, \mathcal{R})$ to be the union of disjoint copies of $(X^{(i)}, \mathcal{R}_i)$ for $i = 0, \ldots, m-1$. Again, for simplicity we assume that $n = \sum_{i=0}^{m-1} |X^{(i)}| = (d-1)lm + lk \sum_{i=0}^{m-1} 2^i$. Knowing that $d$ is not too large, we get that $lk2^m < n < lk2^{m+1}$.

It is clear that the VC-dimension of $(X, \mathcal{R})$ is determined by each of the range spaces $(X^{(i)}, \mathcal{R}_i)$ and is equal to $d$. Next, the smallest $\epsilon$-net for $(X, \mathcal{R})$ has size $lm$ times the smallest $\epsilon$-net for each $(X^{(i)}, \mathcal{R}_i)$, which will give us $lm$. Let us calculate the doubling constant of $(X, \mathcal{R})$. For any $\gamma \geq \epsilon$, $\gamma n \in \mathbb{N}$, choose $j := \min \{ i : 2^j k > \gamma n \}$. How large can a packing of balls of radius $\gamma$ be in $(X, \mathcal{R}_{\leq 2^j})$? We should include in the packing exactly one set from each $\mathcal{R}_i$ for $i = j$ and $j - 1$, which gives $2l$ balls. All the sets of size from $\mathcal{R}_i$ for $i \leq j - 2$ will be covered by one ball of radius $\gamma$ with the center in any of those sets, and the sets from
for $i \geq j + 1$ are bigger than $2\gamma$ and are not present in the family. Therefore, $D_\epsilon \leq 2l + 1$ (actually, $D_\epsilon = 2l + 1$).

We have that $n = O(D_\epsilon k 2^m)$ and $\epsilon = \frac{k}{n} = \Omega(\frac{1}{D_\epsilon 2^m})$, which means that $\log \frac{1}{D_\epsilon \epsilon} = O(m)$. Therefore, the minimum size of an $\epsilon$-net is $lmd = \Omega(dD_\epsilon \log \frac{1}{D_\epsilon \epsilon})$.

**Remark.** The family that provides the lower bound in Part 2 of Theorem 4 may be used to show that Theorem 2 is tight at least for constant $\tau_i$. Putting $l = 1$ in the construction $(X, R_i)$ above, we get that $D_\epsilon$ is a constant, $\tau(\epsilon) < 2 + \frac{d}{en}$, and that the minimum size of an $\epsilon$-net is $\Omega(d \log \frac{1}{\epsilon})$.

It is likely that we may even show that the bound of Theorem 2 is tight for slowly growing $\tau_i$ (that is, that the factor $\log \tau_i$ is also necessary) by replacing $R_i$ with disjoint copies of families that provide lower bound in Theorem A.

### 4 Packing numbers for VC classes

In this section we discuss several packing results for VC classes of functions, which would be useful in getting upper bounds on the doubling constant. At first we recall the following classical result due to Haussler. As before, for a pair of binary functions define $\rho(f, g) = P(f(X) \neq g(X))$. Note that any result for the class of binary functions is directly translated to range spaces.

**Theorem G.** (Haussler [16]) Consider a class $F$ of binary functions of VC dimension at most $d$, such that for any distinct $f, g \in F$ it holds $\rho(f, g) \geq \epsilon$. Then

$$|F| \leq e(d + 1) \left(\frac{2\epsilon}{\epsilon}\right)^d$$

(8)

For a special case when distribution is uniform on a finite set, the next lemma directly follows from the result of Chazelle. This was observed in [24].

**Lemma 7** (Chazelle [10]). Consider the class $F$ of binary functions of VC dimension at most $d$ with $\rho(f, g) \geq \epsilon$ for any distinct $f, g \in F$. If the measure $P$ is uniform on a finite set then we have

$$|F| \leq 2E|F|_A|,$$

where $A$ is an i.i.d. sample of size $n = \frac{4d}{\epsilon}$ from $X$ sampled according to $P$ and $F|_A$ denotes the set of projections of $F$ on the sample $A$.

This Lemma directly implies the version of Haussler’s lemma for the uniform distribution. Using the Vapnik-Chervonenkis-Sauer bound (2) we have $|F| \leq 2 \sum_{i=0}^d \left(\begin{array}{c} n \\ i \end{array}\right) \leq 2 \left(\frac{4en}{d}\right)^d \leq 2 \left(4e\right)^d$. However, constants in this deduction are somewhat worse than in (8). In what follows we give a more general result, valid for any distribution\(^2\) and that directly implies the

\(^2\)Observe, however, that it is also easily possible to generalize the Lemma 7 in a way such that it will be valid for any distribution.
Haussler’s bound. As opposed to Lemma 7, our proof will be based on a purely statistical approach. In fact, the bound on the packing number will be derived as a byproduct of the minimax analysis of the learning rates of the so-called one-inclusion graph algorithm. The analysis is inspired by the minimax lower bounds provided in [4].

Lemma 8. Fix any $\delta \in (0, 1)$. Consider the class $F$ of binary function of VC dimension at most $d$ such that for any distinct $f, g \in F$ it holds $\rho(f, g) \geq \epsilon$. Then

$$|F| \leq \frac{\mathbb{E}|F|}{1 - \delta},$$

where $A$ is an i.i.d. sample of size $n = \frac{2d}{\epsilon \delta}$ from $X$ sampled according to $P$.

To prove this bound we need the following result from Learning Theory. Note that the proof of the following fact is not based on the bound on packing numbers.

Lemma 9. In the realizable case of classification there is a deterministic learning algorithm such that, given an i.i.d sample $((X_i, f^*(X_i)))_{i=1}^n$ of size $n = \frac{2d}{\epsilon \delta}$, it produces a classifier $\hat{f}$ with $\rho(\hat{f}, f^*) < \epsilon/2$ with probability at least $1 - \delta$ over the learning sample.

Proof. It follows from the fact that there is a strategy (namely one-inclusion graph algorithm [15]) with an expected error $\mathbb{E}\rho(\hat{f}, f^*) \leq \frac{d}{n+1} < \frac{d}{n}$, where expectation is taken with respect to the i.i.d random sample $((X_i, f^*(X_i)))_{i=1}^n$ for an arbitrary target function $f^* \in F$. Using Markov inequality we have $P(\rho(\hat{f}, f^*) \geq \epsilon/2) \leq \frac{2\mathbb{E}\rho(\hat{f}, f^*)}{\epsilon} < \frac{2d}{\epsilon n}$. We fix $n = \frac{2d}{\epsilon \delta}$ and get that with probability at least $1 - \delta$ it holds $\rho(\hat{f}, f^*) < \epsilon/2$.

Proof. of Lemma 8. For $n = \frac{2d}{\epsilon \delta}$ denote the output of the learning algorithm of Lemma 9 based on the sample $((X_i, f(X_i)))_{i=1}^n$ by $\hat{g}_f$. Define the uniform measure $\pi$ on $F$. Due to Lemma 9 we have $\mathbb{E}_{f \sim \pi} P(\rho(\hat{g}_f, f) < \epsilon/2) \geq 1 - \delta$. Assume that for a pair of distinct $f, h \in F$ it holds $f(X_i) = h(X_i)$ for $i = 1, \ldots, n$, i.e., they have the same projection on the sample. Since our prediction strategy is deterministic we have $\hat{g}_f = \hat{g}_h$. However, it is not possible that simultaneously we have $\rho(\hat{g}_f, f) < \epsilon/2$ and $\rho(\hat{g}_h, h) < \epsilon/2$ since in this case by the triangle inequality $\rho(f, h) \leq \rho(\hat{g}_f, f) + \rho(\hat{g}_h, h) < \epsilon$, but at the same time from the statement of the Lemma we have $\rho(f, h) \geq \epsilon$. Thus taking into account that for each random sample $A$ there are at most $|F|_A$ different functions $\hat{g}_f$ that may be output, and each corresponds to at most one function from $F$, we get

$$1 - \delta \leq \mathbb{E}_{f \sim \pi} P(\rho(\hat{g}_f, f) \leq \epsilon/2) = \frac{1}{|F|} \mathbb{E} \sum_{f \in F} 1[\rho(\hat{g}_f, f) < \epsilon/2] \leq \frac{1}{|F|} \mathbb{E}|F|_A.$$

When $\delta = \frac{1}{2}$ and $P$ is a uniform measure our Lemma 8 directly recovers Lemma 7. To recover the result of Haussler [3] we use the Vapnik-Chervonenkis-Sauer bound [2] again together with the Lemma 8

$$|F| \leq \inf_{\delta \in (0, 1)} \frac{\mathbb{E}|F|_A}{1 - \delta} \leq \inf_{\delta \in (0, 1)} \frac{1}{1 - \delta} \left(\frac{2e}{\epsilon \delta}\right)^d \leq e(d + 1) \left(\frac{2e}{\epsilon}\right)^d.$$
Remark. If instead of the algorithm of Lemma 9 we use a bound \( O\left( \frac{d \log \left( \frac{1}{\epsilon} \right)}{\epsilon} \right) \) for consistent learning algorithms (Theorem C) we obtain exactly \( |\mathcal{F}| = O \left( \left( \frac{1}{\epsilon} \right)^d (\log \left( \frac{1}{\epsilon} \right))^d \right) \) where constants in \( O \) depend on \( d \), which coincides with the bound of Dudley [11]. In fact, using our technique any deterministic learning algorithm with provable guarantees on probability of misclassification will provide an upper bound on packing numbers. For example, we may replace the algorithm in Lemma 9 by the recent result [21]. In this case the bounds will be the same up to absolute constants.

5 The bounds on the doubling constant

Let us start with defining another important measure of complexity, called the shallow-cell complexity. It was introduced recently [2], [9], [18], [27] for a more refined analysis of the projections of the range spaces, than the one that we can extract from \( \pi_R(y) \) and the VC-dimension. For the relation of shallow-cell complexity with the so-called union complexity, see [18]. Here we give a definition that slightly differs from the one given in [24], [18]: we do not isolate the term \(|Y|\) in the projections on \( Y \), but rather include it into the shallow cell complexity function.

A range space \((X, \mathcal{R})\) has shallow-cell complexity \( \varphi : [0, 1] \times \mathbb{N} \to \mathbb{N} \) if for every \( Y \subseteq X \), the number of sets of size at most \( \ell \) in the system \( \mathcal{R}|_Y \) is at most \( \varphi(|Y|, \ell) \). In all known geometric applications it is sufficient to consider the functions of the form \( \varphi(|Y|, \ell) = \varphi'(|Y|)^{c_R} \) for some constant \( c_R \), and, if this is the case for a range space, then we say that the range space has shallow cell complexity \((\varphi', c_R)\). Thus, the difference with the projection function is that \( \varphi \) bounds the number of sets of different sizes separately. We should note that for many known geometric scenarios very tight bounds on the shallow cell complexity are known.

Lemma H (Shallow-Packing Lemma [24]). If a range space \((X, \mathcal{R})\) of at most \( k \)-element sets has VC dimension \( d \), shallow-cell complexity \( \varphi \) and for any distinct \( R_1, R_2 \in \mathcal{R} \) it holds \( P(R_1, R_2) \geq \gamma \), then if \( P \) is a uniform measure on the finite set \( X \) of size \( n \)

\[
|\mathcal{R}| \leq 6\varphi \left( \frac{4d}{\gamma}, \frac{12kd}{n\gamma} \right).
\]

As a direct consequence of the definition we have the following.

Corollary 10. Assume that \( P \) is a uniform measure on set \( X \) of \( n \) elements. If \((X, \mathcal{R})\) has a shallow-cell complexity \( \varphi \) and VC dimension \( d \), then

\[
D_\epsilon(P, \mathcal{R}) \leq 6\varphi \left( \frac{4d}{\epsilon}, 24d \right).
\]

Proof. Indeed, substitute \( k := 2n\epsilon \) and \( \gamma := \epsilon \) into the formula in Lemma [11].

We provide two upper bounds on the doubling constant in terms of shallow-cell complexity and Alexander’s capacity. They are similar to Lemma [11] and Corollary [10] but generalize it.
to the case of arbitrary distribution on the ground set, as well as they make use of Alexander’s
capacity. The proof of the forthcoming results are postponed until the next subsection to
facilitate the reading.

**Lemma 11.** Assume that the range space \((X, \mathcal{R})\) has a shallow-cell complexity \((\varphi', c_R)\) such that \(\varphi'(x) \leq c_1 x^d\) for some \(c_1, d > 0\) and Alexander’s capacity \(\tau\). Then

\[
D_\epsilon \leq C(d, c_R) \tau^d(\epsilon) \log^{d+c_R} \tau(\epsilon),
\]

where \(C(d, c_R) = O((c(d + c_R) \log(d + c_R))^{d+c_R})\) for some \(c > 0\).

The next lemma is a generalization and strengthening of Corollary 10. It is better than
the previous one in many cases, but requires the control of the VC dimension.

**Lemma 12.** Assume that the range space \((X, \mathcal{R})\) has VC-dimension \(d\), shallow cell com-
plexity \(\varphi\) and Alexander’s capacity \(\tau\). Then

\[
D_\epsilon \leq 6\varphi(8d\tau(\epsilon), 24d).
\]

We immediately have the following result, already mentioned in Section 3.

**Corollary 13.** For a range space \((X, \mathcal{R})\) of VC dimension \(d\) it holds for some
\(c > 0\).

\[
D_\epsilon \leq (c\tau(\epsilon))^d. \tag{9}
\]

5.1 Proofs

**Proof of Lemma 11.** We follow the classic strategy [11]. In what follows we assume \(D_\epsilon > 10\).
Without loss of generality, assume that the supremum in the definitio
of \(D_\epsilon\) is achieved at \(\epsilon\). Denote the corresponding maximal packing by \(Q\), where \(|Q| = D_\epsilon\). We have \(P(Q_1 \Delta Q_2) \geq \epsilon\) for any two \(Q_1, Q_2 \in Q\) and \(P(Q) \leq 2\epsilon\) for any \(Q \in \mathcal{Q}\). By the definition of \(\tau\) we have

\[
P(X') \leq 2\tau(\epsilon) \epsilon \quad \text{for} \quad X' := \text{supp} \mathcal{Q} \subset \bigcup_{R \in \mathcal{R}} \mathcal{R}.
\]

Consider the conditional distribution \(P(\vert X')\) and denote it by \(P'\). Note that \(P'(Q_1 \Delta Q_2) > \frac{1}{2\tau(\epsilon)}\) for any distinct \(Q_1, Q_2 \in \mathcal{Q}\). Note that in what follows we work with \(P'\) only.
In particular, all expectations below are computed w.r.t. \(P'\).

Take a random i.i.d. sample \(A\) of size \(m\) according to \(P'\), where \(m := \frac{2\log D_\epsilon P(X')}{\epsilon}\). Note that

\[
m \leq 4\tau(\epsilon) \log D_\epsilon.
\]

Given any two \(Q_1, Q_2 \in \mathcal{Q}\) with this property we have that \((Q_1 \Delta Q_2) \cap A = \emptyset\) on some \(X_i\) with probability

\[
1 - \left(1 - \frac{\epsilon}{P(X')}\right)^m > 1 - \exp\left(-\frac{\epsilon m}{P(X')}\right) > 1 - \frac{1}{D_\epsilon}.
\]
Using a union bound and summing over all unordered pairs in the packing we conclude that with probability strictly bigger than $\frac{1}{2}$ each set in $Q$ has a unique projection on the random sample $A$.

At the same time for any $Q \in Q$ it holds $\mathbb{E}|A \cap Q| \leq mP'(Q) \leq 4 \log D_\epsilon$, since $P'(R) \leq \frac{2^k}{P(X')}$. We note that $|A \cap Q|$ is upper bounded by a random variable which counts the number of elements of $A$ which belong to $Q$. The last random variable has a binomial distribution and using the Chernoff bound together with the union bound we have

$$P[\exists Q \in Q : |Q \cap A| \geq 8 \log(D_\epsilon)] \geq 1 - D_\epsilon \exp \left( - \frac{4 \log(D_\epsilon)}{3} \right) = 1 - \frac{1}{D_\epsilon^{1/3}} > \frac{1}{2}.$$

Using union bound, we conclude that both displayed events hold with positive probability simultaneously. By the definition of shallow-packing we have for some absolute constant $C$

$$D_\epsilon = |Q|_A \leq \varphi'(m)(8 \log D_\epsilon)^{c_R} \leq C \tau^d(\epsilon)(8 \log D_\epsilon)^{d+c_R}.$$

It is straightforward to check that the last inequality implies

$$D_\epsilon \leq C' \tau^d(\epsilon)(c(d + c_R) \log(d + c_R) \log \tau(\epsilon))^{d+c_R},$$

for some $C', c > 0$.

**Proof of Lemma 12.** The proof follows the same logic as the previous one. Without loss of generality we assume that the supremum in the definition of doubling constant is achieved at $\epsilon_0 = \epsilon$. In particular, it means that the largest packing should contain sets of probability measure between $\epsilon$ and $2\epsilon$, and thus $\tau(\epsilon) \geq 1$. We work in the setting of the proof of Lemma 11 w.r.t. $Q$ and $P'$.

Applying Lemma 8 for $P'$ and $\delta = \frac{1}{2}$, we conclude that for an i.i.d. sample $A$ of size

$$n := \frac{4dP(X')}{\epsilon} \leq 8d\tau(\epsilon)$$

from $P'$ we have $|Q| \leq 2\mathbb{E}|Q|_A$.

At the same time for any $Q \in Q$ it holds $\mathbb{E}|A \cap Q| \leq nP'(Q) \leq 8d$, since $P'(Q) \leq \frac{2^k}{P(X')}$. Consider $Q_1 := \{Q \in Q : |A \cap Q| > 24d\}$. Using Markov inequality we have for any $Q \in Q$ that

$$P[Q \in Q_1] = P[|A \cap Q| > 24d] \leq \frac{8d}{24d} = \frac{1}{3}.$$

Finally, we have

$$|Q| \leq 2\mathbb{E}|Q|_A \leq 2\mathbb{E}|Q_1| + 2\mathbb{E}|(Q \setminus Q_1)|_A \leq \frac{2|Q|}{3} + 2\mathbb{E}|(Q \setminus Q_1)|_A.$$

Rearranging, we obtain $|Q| \leq 6\mathbb{E}|(Q \setminus Q_1)|_A \leq 6\varphi(8d\tau(\epsilon), 24d)$. 

\[\square\]
6 Comparison of upper bounds and discussions

In general, the \( \varepsilon \)-net theorems in this paper are arranged from the weaker to the stronger ones. Below we only discuss the strength of the bounds given, and mostly avoid discussing the algorithms. Every time we are focusing only on the implications of certain results on the existence of \( \varepsilon \)-nets.

Theorems A, B, C are weaker than any other result given in the paper. Theorem D is stronger than the previous ones, and its bound is implied for relatively small \( \tau(\varepsilon) \) by Corollary 1.

Theorem 2 implies both Theorem D and Corollary 1. Indeed, Theorem D follows easily from the fact that \( \tau_i \leq \frac{1}{2\varepsilon} \), and thus \( \sum_i \tau_i \leq \frac{1}{\varepsilon} \), and Corollary 1 follows from Corollary 3. We also note that the bounds in Theorem 2 are strictly stronger in many cases.

Speaking of bounds making use of the doubling constant, they are much stronger than all the previous ones. In particular, even the trivial Corollary 6 together with a weak bound on the doubling dimension, given in Theorem E, implies many of the previous bounds (except for Corollary 1 and Theorem 2), giving the bound \( O\left(\frac{d}{\varepsilon} \log \tau(\varepsilon)\right) \), and Corollary 5 combined with Theorem F implies Corollaries 1 and 3.

It is also easy to see that Theorem 4 combined with Theorem F implies Theorem 2. But its full strength becomes clear when the doubling constant is relatively small. Then the fact that we divide \( D_\varepsilon \) by \( \tau(\varepsilon) \) in the logarithm may play a crucial role, since it allows us to get rid of the logarithm in some cases. In this context, sharper bounds on the doubling constant that make use of the shallow-cell complexity come into play.

The previous best bound on the \( \varepsilon \)-nets, which used the notion of shallow-cell complexity, was as follows (see [9] and a simplified proof [24]).

**Theorem I** ([24]). Let \((X, \mathcal{R})\), \(|X| = n\) be a range space with uniform distribution of VC-dimension \( d \) and shallow-cell complexity \( \varphi(\cdot, \cdot) \), where \( \varphi(\cdot, \cdot) \) is a non-decreasing function in the first variable. Then there exists an \( \varepsilon \)-net for \( \mathcal{R} \) of size \( O\left(\frac{1}{\varepsilon} (\log \left(\varepsilon \varphi(\frac{8d}{\varepsilon}, 24d)\right) + d)\right) \). In particular, if \( \mathcal{R} \) has shallow-cell complexity \((\varphi', c_R)\) and finite VC-dimension, then there exists an \( \varepsilon \)-net for \( \mathcal{R} \) of size \( O\left(\frac{1}{\varepsilon} \log (\varepsilon \varphi_{\mathcal{R}}(\frac{1}{\varepsilon}))\right) \).

Substituting the bound from Corollary 10 into Corollary 5 we recover the same bound as in Theorem I above, but for an arbitrary distribution \( P \). Moreover, using other bounds from Section 5 we can get stronger statements, which replace \( 1/\varepsilon \) with \( \tau(\varepsilon) \). Using Lemma 12 and substituting it into the first part of Corollary 5 we get a strengthening of Theorem I which is comparable with it as Theorem D and Theorem A. Substituting the bound from Lemma 12 into the second part of Corollary 5 we get something like a Corollary 1 version of Theorem I. At the expense of an extra \( \log \log \) in the worst case, we may use Lemma 11 instead of Lemma 12 and then we do not require any bound on the VC-dimension of the range space, only on the shallow-cell complexity.

Overall, we feel that the doubling constant is a better general parameter to look at for \( \varepsilon \)-nets. From this perspective the notions like Alexander’s capacity and the shallow-cell complexity are simply the ways to control the doubling constant. The doubling constant
together with the Alexander’s capacity control almost all possible ranges of sizes of $\epsilon$-nets. Moreover, the extensions for the quantities like the doubling constant to the non-binary cases are straightforward (see [23] for these extensions related to the Learning Theory), while the notion of the shallow-cell complexity is very specific to the set systems.

We should also note that the effects of distribution-dependent improvements of the standard bounds were considered in the literature. In [3] authors prove and discuss the existence of $\epsilon$-nets of size $O(\frac{4}{\epsilon})$ for a set of regions of disagreement between all possible linear classifiers passing trough the origin in $\mathbb{R}^d$ and the linear fixed classifier, when the distribution of $X$ is zero-mean, isotropic and log-concave. Their bound is based on the improved version of Theorem C. However, it is not difficult to see that at least for some particular distributions (like uniform distribution on the unit sphere) even finite $\epsilon$-nets exist. Our bound (4) (given the fact the Alexander’s capacity is bounded in this case [19]) gives the result with the logarithmic dependence on $\frac{1}{\epsilon}$, which is significantly better than $O(\frac{4}{\epsilon})$.

References

[1] K. S. Alexander. Rates of growth and sample moduli for weighted empirical processes indexed by sets. Probability Theory and Related Fields, 75:379–423, 1987.

[2] B. Aronov, E. Ezra, and M. Sharir. Small-size $\epsilon$-nets for axis-parallel rectangles and boxes, SIAM J. Comput., 39(7):3248–3282, 2010.

[3] M.F. Balcan, P. M. Long. Active and passive learning of linear separators under log-concave distributions. In Proceedings of the 26th Conference on Learning Theory, 2013.

[4] G. Benedek, A Itai. Learnability with respect to a fixed distribution. Theoretical Computer Science, 86, 377–389, 1991

[5] O. Bousquet, S. Boucheron, G. Lugosi. Introduction to Statistical Learning Theory. Advanced Lectures on Machine Learning, LNCS, volume 3176, 2004

[6] S. Boucheron, O. Bousquet, G. Lugosi. Theory of classification: a survey of recent advances. ESAIM: Probability and Statistics, 9:323–375, 2005.

[7] P. L. Bartlett, O. Bousquet, S. Mendelson. Local Rademacher Complexities. The Annals of Statistics, 33(4):1497–1537, 08, 2005.

[8] N. H. Bshouty, Y. Li, P. M. Long, Using the doubling dimension to analyze the generalization of learning algorithms, Journal of Computer and System Sciences 75 (2009), N6, 323–335.

[9] T. M. Chan, E. Grant, J. Knemann, and M. Sharpe. Weighted capacitated, priority, and geometric set cover via improved quasi-uniform sampling, In Proceedings of Symposium on Discrete Algorithms (SODA), 2012.
[10] B. Chazelle. *A note on Hausslers packing lemma*. Geometric Discrepancy: An Illustrated Guide, 1992.

[11] R. M. Dudley. *A Course on Empirical Processes*. Ecole d’t de probabilits de St.-Flour, 1982

[12] E. Giné, V. Koltchinskii. *Concentration inequalities and asymptotic results for ratio type empirical processes*. The Annals of Probability, 34(3):1143–1216, 2006.

[13] A. Gupta, R. Krauthgamer, and J. R. Lee. *Bounded geometries, fractals, and low-distortion embeddings*, FOCS, 2003.

[14] D. Haussler, E. Welzl, *Epsilon-nets and simplex range queries*, Discrete & Computational Geometry, 127–151, 1987

[15] D. Haussler, N. Littlestone, M. Warmuth. *Predicting {0, 1}-functions on randomly drawn points*. Information and Computation, 115:248–292, 1994.

[16] D. Haussler. *Sphere packing numbers for subsets of the boolean n-cube with bounded Vapnik-Chervonenkis dimension*. J. Comb. Theory Ser. A, 69(2):217–232, 1995.

[17] J. Komlós, J. Pach, G.J. Woeginger, *Almost tight bounds for epsilon-nets*, Discrete & Computational Geometry 7 (1992), 163–173.

[18] A. Kupavskii, N. Mustafa, J. Pach, *Lower bounds for ε-nets*, SoCG’2016.

[19] S. Hanneke. *Theory of Disagreement-Based Active Learning*, Foundation and trends in machine learning, 2014.

[20] S. Hanneke, L. Yang, *Minimax Analysis of Active Learning*, Journal of Machine Learning Research 12, 3487–3602, 2015.

[21] S. Hanneke, *The Optimal Sample Complexity of PAC Learning*, Journal of Machine Learning Research 17, 1–15, 2016.

[22] L. M. Le Cam. *Convergence of estimates under dimensionality restrictions*. Annals of Statistics 1, 38–53, 1973.

[23] S. Mendelson. ‘Local’ vs. ‘global’ parameters – breaking the Gaussian complexity barrier. Annals of Statistics, 2017.

[24] N. Mustafa, K. Dutta, A. Ghosh, *A simple proof of optimal epsilon-nets*. Combinatorica, 2017.

[25] N. Mustafa, K. Varadarajan, *Epsilon-approximations and epsilon-nets* (2017), arXiv:1702.03676

[26] V. Vapnik, A. Chervonenkis. *Theory of Pattern Recognition*. Nauka, Moscow, 1974.
[27] K. Varadarajan, *Weighted geometric set cover via quasi uniform sampling*, In Proceedings of the Symposium on Theory of Computing (STOC), pages 641–648, New York, USA, 2010. ACM.

[28] N. Zhivotovskiy, S. Hanneke. *Localization of VC classes: beyond local Rademacher Complexities*, Algorithmic Learning Theory, 2016.