The entanglement of some non-two-colorable graph states

Xiao-yu Chen, Li-zhen Jiang

College of Information and Electronic Engineering, Zhejiang Gongshang University, Hangzhou 310018, People’s Republic of China

Abstract

We exactly evaluate the entanglement of a six vertex and a nine vertex graph states which correspond to non "two-colorable" graphs. The upper bound of entanglement for five vertices ring graph state is improved to 2.9275, less than upper bound determined by LOCC. An upper bound of entanglement is proposed based on the definition of graph state.

PACS number(s): 03.67.Mn, 03.65.Ud

Keyword(s): graph state; closest separable state; multipartite entanglement

1 Introduction

Entanglement is one of the most important concepts and resources in quantum information theory. However, the quantification of the entanglement of a given quantum state is quite difficult except for bipartite pure state, where the distance-like measure of the entanglement (Relative Entropy of Entanglement) and the operational measures of entanglement (Entanglement of Formation, Distillable Entanglement) are all equal to the entropy of the reduced state obtained by tracing out one part of the pure state. In bipartite system, apart from pure state, the entanglement of a mixed state is not easy to calculate in general if not impossible. The situation becomes even worse for multipartite system where the basic states corresponding to Bell basis are not clearly recognized. Thus the extension of operational entanglement measures to multipartite system is not available. Nevertheless, a variety of different entanglement measure have been proposed for multipartite setting. Among them are the (Global) Robustness of Entanglement, the Relative Entropy of Entanglement, and the Geometric Measure. The robustness measures the minimal noise (arbitrary state) that we need to added to make the state separable. The geometric measure of entanglement is upper bounded by the local operational entanglement measure to multipartite system. For a state of ring graph with odd n vertices, we have $\frac{n}{2} \leq E \leq \frac{n}{2}$.

If the lower bound coincides with the upper bound, the the entanglement of the graph state can be obtained. This is the case for 'two-colorable' graph states such as multipartite GHZ states, Steane code, cluster state, and state of ring graph with even vertices. For a state of ring graph with odd n vertices, we have $\frac{n}{2} \leq E \leq \frac{n}{2}$.

We in this paper will concern with the entanglement of a graph state whose graph is not "two-colorable". A new upper bound based directly on the definition of graph state is proposed. The symmetry of the graph is utilized to further reduce the upper bounds for some highly symmetric graph states, including the five vertices ring and Peterson graph.

2 Graph state

A graph $G = (V; \Gamma)$ is composed of a set $V$ of n vertices and a set of edges specified by the adjacency matrix $\Gamma$, which is an $n \times n$ symmetric matrix with vanishing diagonal entries and $\Gamma_{ab} = 1$ if vertices $a, b$ are connected and $\Gamma_{ab} = 0$ otherwise. The neighborhood of a vertex $a$ is denoted by $N_a = \{ v \in V | \Gamma_{av} = 1 \}$, i.e., the set of all the vertices that are connected to $a$. Graph states are useful multipartite entangled states that are essential resources for the one-way computing and can be experimentally demonstrated. To associate the graph state to the underlying graph, we assign each vertex with a qubit, each edge represents the interaction between the corresponding two qubits. More physically, the interaction may be Ising interaction of spin qubits. Let us denote the Pauli matrices.
at the qubit $a$ by $X_a, Y_a, Z_a$ and identity by $I_a$. The graph state related to graph $G$ is defined as

$$ |G| = \prod_{\Gamma \in G} U_{ab} |+\rangle^V_z = \frac{1}{\sqrt{2^n}} \sum_{\mu = 0}^{1} (-1)^{\frac{1}{2} \mu \Gamma^T \mu} |\mu\rangle_z \tag{3} $$

where $|\mu\rangle_z$ is the joint eigenstate of Pauli operators $Z_a$ ($a \in V$) with eigenvalues $(-1)^{\mu_a}$, $|+\rangle^V_z$ is the joint +1 eigenstate of Pauli operators $X_a$ ($a \in V$), and $U_{ab}$ ($U_{ab} = \text{diag}\{1, 1, 1, -1\}$ in the Z basis) is the controlled phase gate between qubits $a$ and $b$. Graph state can also be viewed as the result of successively performing 2-qubit Control-Z operations $U_{ab}$ to the initially unconnected $n$ qubit state $|+\rangle^V_z$. It can be shown that graph state is the joint +1 eigenstate of the $n$ vertex stabilizers

$$ K_a = X_a \prod_{b \in N_a} Z_b := X_a Z_{N_a}, \ a \in V. \tag{4} $$

Meanwhile, the graph state basis are $|G_{k_1, k_2, \ldots, k_n}\rangle = \prod_{a \in V} Z_a^{k_a} |G\rangle$, with $k_a = 0, 1$. Since all of the graph basis states are local unitary equivalent, they all have equal entanglement, so we only need to determine the entanglement of graph state $|G\rangle$. Once the entanglement of a graph state is obtained, the entanglement of all the graph basis states are obtained.

### 3 The upper bound of graph state

The fidelity $F_0 = |\langle G| \phi \rangle|^2$ plays a crucial role in calculating the entanglement. For a graph state, we have

$$ E = \min_{\phi} \log_2 |\langle G| \phi \rangle|^2 = -\log_2(\max_{\phi} F_0). \tag{5} $$

Denote $F = \max_{\phi} F_0$ as the fidelity between the graph state and the closest pure separable state. One of the ways to obtain the upper bound of the entanglement is to relax the maximization. For two-colorable graph, the set with majority vertices is colored with Amber, the set with minority vertices is colored with Blue. Without loss of generality, the Amber colored vertices are labelled as $a = 1, \ldots, |A|$, the Blue vertices are labelled as $b = |A| + 1, \ldots, n$. Since all Amber vertices are not adjacent with each other, we perform $X_a$ ($a = 1, \ldots, |A|$) measurements to all Amber qubits simultaneously. And applying $Z_b$ ($b = |A| + 1, \ldots, n$) measurements to all Blue qubits at the same time. Thus all Amber stabilizers $K_a$ can be measured simultaneously by LOCC. The maximal number of states that can be discriminated by LOCC then is $2^{|A|}$ according to the theory of graph state basis [12]. Applying the inequality on the relationship of LOCC discrimination of states and the entanglement [3], one has $|A| \leq n - E$, that is,

$$ E \leq n - |A|. \tag{6} $$

This upper bound of LOCC may be extended to graphs that are not two-colorable by some modification. However, it is possible to obtain the upper bound without the LOCC state discrimination.

We will obtain the upper bound of the entanglement with the definition of the graph state. The graph state may not be two-colorable. Suppose the maximal non-adjacent vertices set $A$ has $|A|$ vertices. As before, we label these vertices with $a = 1, \ldots, |A|$. The other vertices are in the set $B = V - A$, the vertices are labelled with $b = |A| + 1, \ldots, n$. Note that the vertices within set $B$ may connect with each other, for the graph may not be two-colorable. The adjacency matrix $\Gamma$ now is

$$ \Gamma = \begin{bmatrix} \Gamma_A & \Gamma_{AB} \\ \Gamma_{AB}^T & \Gamma_B \end{bmatrix}. \tag{7} $$

Since any vertices pairs are not adjacent in set $A$, the adjacency matrix of set $A$ is an all zero $|A| \times |A|$ matrix,

$$ \Gamma_A = 0. \tag{8} $$

Denote $\mu = (\mu_A, \mu_B)$, where the binary vectors $\mu_A = (\mu_1, \ldots, \mu_{|A|})$, $\mu_B = (\mu_{|A|+1}, \ldots, \mu_n)$, then the graph state can be written as $|G\rangle = |G_1\rangle + |G_2\rangle$, the unnormalized states (in Z basis)

$$ |G| = \frac{1}{\sqrt{2^n}} \sum_{\mu_A = 0}^1 \sum_{\mu_B = 0}^1 \langle \mu_A, \mu_B \rangle^{\mu_A \mu_B} |\mu_A, \mu_B\rangle. \tag{9} $$

where we have used the fact that

$$ \frac{1}{2} \langle \mu_A, \mu_B \rangle^{\mu_A \mu_B} \begin{bmatrix} 0 & \Gamma_{AB} \\ \Gamma_{AB}^T & \Gamma_B \end{bmatrix} (\mu_A, \mu_B)^T = 0. \tag{10} $$

And

$$ |G_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mu_A = 0, \mu_B \neq 0} \langle \mu_A, \mu_B \rangle^{\mu_A \mu_B} |\mu_A, \mu_B\rangle. \tag{11} $$

To obtain a lower bound of the extremal fidelity $F$, we can choose

$$ |\phi\rangle = \bigotimes_a (\sqrt{p_a} |0\rangle + \sqrt{1 - p_a} e^{i\phi_a} |1\rangle) \otimes |0\rangle^{\otimes (n - |A|)}. \tag{12} $$

Since $\mu_B \neq 0$ in the state $|G_2\rangle$ and the last $(n - |A|)$ qubits of $|\phi\rangle$ are all in $|0\rangle$, we have $\langle G_2 | \phi \rangle = 0$. Thus one has

$$ \langle G | \phi \rangle = \langle G_1 | \phi \rangle = \frac{1}{\sqrt{2^n}} \sum_{\mu_A = 0}^{\mid A\mid} \langle \mu_A \rangle^{\mu_A} \bigotimes_a (\sqrt{p_a} |0\rangle \\
+ \sqrt{1 - p_a} e^{i\phi_a} |1\rangle) = \frac{1}{\sqrt{2^n}} \prod_{a=1}^{\mid A\mid} (\sqrt{p_a} + \sqrt{1 - p_a} e^{i\phi_a}). \tag{13} $$

The maximal fidelity for separable state [12] is

$$ F_0 = \max_{p_a, \phi_a} |\langle G | \phi \rangle|^2 = 2^{-(n - |A|)}. \tag{14} $$
For a graph state of a ring graph with odd entanglement of the graph state is 3. The upper bound and the lower bound coincide, thus the have $C - U\{12\}$ is also 3 adjacent vertices of the graph is 3 (i.e. vertices 1 and 2, 3, 4, 5). The maximal number of non-adjacent vertices is 6, and apply operation which is local in subgraph 45. With a numerical calculation, the entanglement upper bound is obtained. Let us consider the fidelity $F_{G} = |\langle G| \phi \rangle|^2$.

The upper bound is obtained by $F_{G} = \frac{1}{16} |x^3 - 5xy^4|^2$. (19)

The fidelity $F_{G}$ can be rewritten as $F_{G} = \frac{1}{16}(p^5 + 25p(1 - p)^4 - 10p^2(1 - p)^2 \cos 4\varphi)$. The maximal will be achieved when $\cos 4\varphi = -1$, so $F_{G_{\max}} = \frac{1}{16}[p^2 + 5(1 - p)^2]^2$. The derivative $\frac{dF_{G}}{dp} = 0$ reduces to $6p^2 - 6p + 1 = 0$, which is $p = \frac{1}{2}(1 \pm \frac{1}{\sqrt{3}})$. For $p = \frac{1}{2}(1 - \frac{1}{\sqrt{3}})$, $\frac{d^2F_{G}}{dp^2} = -\frac{1}{4}(1 + \sqrt{3}) < 0$; For $p = \frac{1}{2}(1 + \frac{1}{\sqrt{3}})$, $\frac{d^2F_{G}}{dp^2} = \frac{3}{4}(\sqrt{3} - 1) > 0$. Thus the fidelity reaches its maximal $F_{G_{\max}} = \frac{3 + \sqrt{3}}{4}$ when $\varphi = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$, and $p = \frac{1}{2}(1 - \frac{1}{\sqrt{3}})$. It is interesting that when at $p = \frac{1}{2}(1 - \frac{1}{\sqrt{3}})$, $\varphi = \pm \frac{\pi}{4}$, we have the the maximal fidelity $F_{G_{\max}} = \frac{3 + \sqrt{3}}{4}$. We may calculate the derivatives of $F_{G}$ at points $(p, \varphi) = (\frac{1}{2}(1 - \frac{1}{\sqrt{3}}), \pm \frac{\pi}{4})$, the first order derivatives are $\frac{\partial F_{G}}{\partial p} = 0$, $\frac{\partial F_{G}}{\partial \varphi} = 0$, the second order derivatives are $\frac{\partial^2 F_{G}}{\partial p^2} = -\frac{3}{4} < 0$, $\frac{\partial^2 F_{G}}{\partial p \partial \varphi} = \frac{\pi}{16} < 0$, $\frac{\partial^2 F_{G}}{\partial \varphi^2} = -\frac{\pi}{16}(3 + 2\sqrt{3}) < 0$. The Jacobian is

$$J = \begin{vmatrix} \frac{\partial^2 F_{G}}{\partial p^2} & \frac{\partial^2 F_{G}}{\partial p \partial \varphi} \\ \frac{\partial^2 F_{G}}{\partial p \partial \varphi} & \frac{\partial^2 F_{G}}{\partial \varphi^2} \end{vmatrix} = \frac{25\sqrt{3}}{216} > 0.$$ (20)

Thus $(p, \varphi) = (\frac{1}{2}(1 - \frac{1}{\sqrt{3}}), \pm \frac{\pi}{4})$ are the points of maximal fidelity $F_{G}$. We can also prove that $(p, \varphi) = (\frac{1}{2}(1 + \frac{1}{\sqrt{3}}), \pm \frac{\pi}{4})$ are the points of maximal fidelity $F_{G}$.

The entanglement upper bound is

$$\min_{p, \varphi} - \log_2 F_{G} = - \log_2 \frac{3 + \sqrt{3}}{36} \approx 2.9275.$$ (21)

4 Improving the upper bound with symmetry

For a graph state of a ring graph with odd $n$ vertices, we have $\left\lfloor \frac{n}{2} \right\rfloor \leq E \leq \left\lceil \frac{n}{2} \right\rceil$. The upper bound is obtained by LOCC and also by our non-adjacent vertices set method. We will show that this upper bound can be further improved for five vertices ring graph state by making use of the symmetry of the graph. Our symmetrical consideration also shows that the Peterson graph [18] state has entanglement upper bound of 5, while LOCC and non-adjacent vertices set method can only give the upper bound of 6.
It is less than 3, the best upper bound by LOCC or non-adjacent vertices set method.

Although we use the identical product state to obtain the upper bound of the entanglement, and this upper bound is far from the lower bound which is 2, a random search calculation indicates that this upper bound is possibly the entanglement itself.

4.2 Peterson graph

For Peterson graph $G_P$ in Fig. 1 (b), the lower bipartite bound for the entanglement of the graph state is easily obtained to be 5, which is the number of Bell pairs between the subgraph $C$ with $V_C = \{1, 2, 3, 4, 5\}$ and subgraph $D$ with $V_D = \{6, 7, 8, 9, 10\}$. The number of maximal non-adjacent vertices set is 4, thus the entanglement upper bound is $10 - 4 = 6$.

Suppose the separable state be

$$|\phi\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}e^{i\varphi}|1\rangle)\otimes^{10}. \quad (22)$$

Denote $x = \sqrt{p}$, $y = \sqrt{1-p}e^{i\varphi}$. We have

$$\langle G_P | \phi \rangle = 2^{-5} \sum_{j=0}^{10} c_j x^{10-j} y^j. \quad (23)$$

The coefficient $c_j = \sum_{\mu \in \Lambda_j} (-1)^j 2^{\mu T} \mu^T$, where $\Lambda_j = \{ \mu | \sum_{k=1}^{10} \mu_k = j \}$. The coefficient vector is

$$c = (1, 10, 15, 0, -50, 108, 50, 0, -15, 10, -1). \quad (24)$$

A rather special closest separable state is with $p = \frac{1}{2}$, $\varphi = \frac{\pi}{4}$. The maximal fidelity is

$$F = 2^{-3}(-1 + i)^2 = \frac{1}{32}. \quad (25)$$

The entanglement upper bound coincides with its lower bound. The entanglement of the Peterson graph state is 5.

5 Conclusions

We have proposed an upper bound for the entanglement of a graph state. The bound is based on the definition of the graph state. We obtain the bound by calculating the fidelity of the graph state with respect to some separable state. The vertices of the graph are divided into two subsets, one with all its vertices that are not adjacent with each other, and we make this subset as large as possible and it has $|A|$ vertices. Then the entanglement of the $n$ vertices graph state is upper bounded by $n - |A|$. The entanglement measure can be the (Global) Robustness of Entanglement, the Relative Entropy of Entanglement, and the Geometric Measure. These measures are all equal for graph states. Using this bound, we find the entanglement of graph state which $[[6,1,3]]$ code based on to be 3. The upper bound of the graph state has been further improved for some highly symmetric states. These states are five vertices ring graph state and Peterson graph state. With the product of identical qubit states, we find that the entanglement upper bound for five vertices ring graph state is about 2.9275, which is less than 3, the bound given by LOCC and our non-adjacent vertices set method. We also determine the entanglement of Peterson graph state to be 5 (less than 6 given by LOCC) by using the product of identical qubit states as the closest separable state.

References

[1] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 Phys. Rev. Lett. 78 2275
[2] Vedral V and Plenio M B 1998 Phys. Rev. A 57 1619
[3] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[4] Dur W, Vidal G and Cirac J I 2000 Phys. Rev. A 62 062314
[5] Miyake A 2003 Phys. Rev. A 67 012108
[6] Vidal G and Tarrach R 1999 Phys. Rev. A 59 141
[7] Wei T-C and Goldbart P M 2003 Phys. Rev. A, 68 042307
[8] Wei T-C, Ericsson M, Goldbart P M and Munro W J 2004 Quant. Inform. Comp. 4 252
[9] Hayashi M, Markham D, Murao M, Owari M and Virmani S 2006 Phys. Rev. Lett. 96 040501
[10] Wei T-C 2008 Phys. Rev. A, 78 012327
[11] Hayashi M, Markham D, Murao M, Owari M and Virmani S 2008 Phys. Rev. A, 77 012104
[12] Markham D, Miyake A and Virmani S 2007 New. J. Phys. 9 194
[13] Hein M, Eisert J, and Briegel H J 2004 Phys. Rev. A 69 062311
[14] Schlingemann D and Werner R F 2002 Phys. Rev. A 65 012308
[15] Raußendorf R and Briegel H J 2001 Phys. Rev. Lett. 86 5188
[16] Walther P et.al. 2005 Nature 434 169
[17] Lu C Y et.al. 2007 Nature Physics 3 91
[18] Hein M, Dur W, Eisert J, Raußendorf R, van den Nest M and Briegel H J 2006 eprint: quant-ph/0602096
[19] Gottesman D 2007 Caltech Ph. D. Thesis, eprint: quant-ph/9705052