a-NUMBERS IN ARTIN-SCHREIER COVERS

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ABSTRACT. Let \( \pi : Y \to X \) be a branched \( \mathbb{Z}/p\mathbb{Z} \)-cover of smooth, projective, geometrically connected curves over a perfect field of characteristic \( p > 0 \). We investigate the relationship between the \( a \)-numbers of \( Y \) and \( X \) and the ramification of the map \( \pi \). This is analogous to the relationship between the genus (respectively \( p \)-rank) of \( Y \) and \( X \) given the Riemann-Hurwitz (respectively Deuring-Shafarevich) formula. Except in special situations, the \( a \)-number of \( Y \) is not determined by the \( a \)-number of \( X \) and the ramification of the cover, so we instead give bounds on the \( a \)-number of \( Y \). We provide examples showing our bounds are sharp. The bounds come from a detailed analysis of the kernel of the Cartier operator.

1. Introduction

Let \( k \) be a field and \( \pi : Y \to X \) a finite morphism of smooth, projective, and geometrically connected curves over \( k \) that is generically Galois with group \( G \). The most fundamental numerical invariant of a curve is its genus, and the famous Riemann–Hurwitz formula says that the genus of \( Y \) is determined by that of \( X \) and the ramification of the cover \( \pi \): letting \( S \subset X(k) \) denote the branch locus,

\[
2g_Y - 2 = |G| \cdot (2g_X - 2) + \sum_{y \in \pi^{-1}(S)} \sum_{i \geq 0}(|G_i(y)| - 1).
\]

(1.1)

Here \( G_i(y) \subseteq G \) is the \( i \)-th ramification group (in the lower numbering) at \( y \).

When \( k \) is perfect of characteristic \( p > 0 \), which we will assume henceforth, there are important numerical invariants of curves beyond the genus coming from the existence of the Frobenius morphism. Writing \( \sigma \) for the \( p \)-power Frobenius automorphism of \( k \), attached to the finite flat absolute Frobenius \( F : X \to X \) is a natural trace map \( V := V_* : H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^1_{X/k}) \) which is additive and semilinear over \( \sigma^{-1} \) and, via Grothendieck-Serre duality, is dual to pullback by absolute Frobenius on \( H^1(X, \mathcal{O}_X) \). This map—called the Cartier operator—gives the \( k \)-vector space of holomorphic differentials on \( X \) the structure of a (left) module of finite length over the (non-commutative in general) polynomial ring \( k[V] \). Fitting’s Lemma provides a canonical direct sum decomposition of \( k[V] \)-modules

\[
H^0(X, \Omega^1_{X/k}) = H^0(X, \Omega^1_{X/k})^{\text{bij}} \oplus H^0(X, \Omega^1_{X/k})^{\text{nil}}
\]

with \( V \) bijective (respectively nilpotent) on \( H^0(X, \Omega^1_{X/k}) \) for \( * = \text{bij} \) (respectively \( * = \text{nil} \)). Let us write \( f_X \) for the \( k \)-dimension of \( H^0(X, \Omega^1_{X/k})^{\text{bij}} \); this integer is called the \( p \)-rank of \( X \), or more properly of the Jacobian \( J_X \) of \( X \), since one also has the description \( f_X = \dim_{\mathbb{F}_p} \text{Hom}(\mu_p, J_X[p]) \). When \( \pi : Y \to X \) is a branched \( G \)-cover with \( G \) a \( p \)-group, the Deuring-Shafarevich formula relates the \( p \)-ranks of \( X \) and \( Y \):

\[
f_Y - 1 = |G| \cdot (f_X - 1) + \sum_{y \in \pi^{-1}(S)} (|G_0(y)| - 1)
\]

(1.2)

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Like the Riemann–Hurwitz formula, \((1.2)\) says that the numerical invariant \(f_Y\) of \(Y\) is determined by \(f_X\) and the ramification of \(\pi\); unlike the Riemann–Hurwitz formula, it only applies when \(G\) has \(p\)-power order, and only incorporates tame ramification information. As Crew points out in [Cre84, Remark 1.8.1], there can be no version of the Deuring–Shafarevich formula if \(G\) is not assumed to be a \(p\)-group, since (for example) if \(p > 2\) any elliptic curve \(E\) over \(k\) is a \(\mathbb{Z}/2\mathbb{Z}\)-cover of the projective line branched at exactly 4 points (necessarily with ramification degree 2), but \(f_E\) can be 0 or 1, so that \(f_E\) is not determined by \(f_P\), and the ramification of \(\pi : E \to \mathbb{P}^1\). Of course, thanks to the solvability of \(p\)-groups, the essential case of \((1.2)\) is when \(G = \mathbb{Z}/p\mathbb{Z}\).

Since the \(k\)-dimension \(\delta_X\) of the nilpotent part \(H^0(X, \Omega^1_{X/k})^{\mathrm{nil}}\) satisfies \(\delta_X = g_X - \lfloor f_X \rfloor\), together the Riemann–Hurwitz and Deuring–Shafarevich formulae provide a similar formula relating \(\delta_X, \delta_Y,\) and the (wild) ramification of \(\pi\) for any \(p\)-group branched cover \(\pi : Y \to X\). Beyond this fact, very little seems to be understood about the behavior of the nilpotent part in \(p\)-group covers.

In this paper, we will study the behavior of the \(a\)-number of curves in branched \(\mathbb{Z}/p\mathbb{Z}\)-covers \(\pi : Y \to X\). By definition, the \(a\)-number of a curve \(C\) is

\[(1.3)\]

\[a_C := \dim_k \ker \left( V : H^0(C, \Omega^1_{C/k}) \to H^0(C, \Omega^1_{C/k}) \right) \]

Equivalently, \(a_C\) is the number of nonzero cyclic direct summands in the invariant factor decomposition of \(H^0(C, \Omega^1_{C/k})^{\mathrm{nil}}\) as a \(k[V]\)-module. Yet a third interpretation is \(a_C = \dim_k \Hom(a_p, J_C[p])\), where \(a_p\) denotes the group-scheme \(\ker(F : G_a \to G_a)\) over \(k\). Although this fundamental numerical invariant of curves in positive characteristic has been extensively studied (e.g. [WKS6], [KWS8], [Re01], [EP07], [Joh07], [Elk11], [FGM+13], [DF14], [MS18], [Pre18]), it remains rather mysterious.

When \(p = 2\) and \(X\) is ordinary (i.e. \(a_X = 0\)), Voloch [Vol88] establishes an explicit formula for \(a_Y\) in terms of the ramification of \(\pi\) and the genus of \(X\). If in addition \(X = \mathbb{P}^1\), Elkin and Pries [EP13] show that this data completely determines the Ekedahl-Oort type of \(J_Y[p]\); note that this situation is quite special, as every Artin-Schreier cover of \(\mathbb{P}^1\) in characteristic 2 is hyperelliptic. For general \(p\), Farnell and Pries [Far10] study branched \(\mathbb{Z}/p\mathbb{Z}\)-covers \(\pi : Y \to \mathbb{P}^1\), and prove that there is an explicit formula for \(a_Y\) in terms of the ramification of \(\pi\) whenever the unique break in the ramification filtration at every ramified point is a divisor of \(p - 1\). Unfortunately, there can be no such \(a\)-number formula in the spirit of \((1.2)\) in general: simple examples with \(p > 2\) show that there are \(\mathbb{Z}/p\mathbb{Z}\)-covers even of \(X = \mathbb{P}^1\) branched only at \(\infty\) which have identical ramification filtrations, but different \(a\)-numbers; cf. Example \((7.2)\).

Nonetheless, we will prove that the possibilities for the \(a\)-number of \(Y\) are tightly constrained by the \(a\)-number of \(X\) and the ramification of \(\pi\):

**Theorem 1.1.** Let \(\pi : Y \to X\) be a morphism of smooth, projective and geometrically connected curves over a perfect field \(k\) of characteristic \(p > 0\) that is generically Galois with group \(\mathbb{Z}/p\mathbb{Z}\). Let \(S \subseteq X(\overline{k})\) be the finite set of geometric closed points over which \(\pi\) ramifies, and for \(Q \in S\) let \(d_Q\) be the unique break in the lower-numbering ramification filtration at the unique point of \(Y\) over \(Q\). Then for any \(1 \leq j \leq p - 1,\)

\[
\sum_{Q \in S} \sum_{i=j}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} \right\rfloor \left(1 - \left\lfloor \frac{1}{p} \right\rfloor \right) \right) \leq a_Y \leq p a_X + \sum_{Q \in S} \sum_{i=1}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - (p - i) \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right).
\]

In fact, our main result (Theorem \((6.15)\)) features a slightly sharper—i.e. somewhat messier and less explicit in general — upper bound, and Theorem \((1.1)\) is an immediate consequence of this.

**Remark 1.2.** The lower bound is largest for \(j \approx p/2\), and in applications we will take \(j = \lceil p/2 \rceil\). Fixing \(X\) and \(S \subseteq X(\overline{k})\) and writing \(T := (p - 1) \sum_{Q \in S} d_Q\), elementary estimates show that our lower (respectively upper) bound is asymptotic to \((1 - \frac{1}{p^2}) T\) (respectively \((1 - \frac{1}{2p}) \frac{T}{2}\) as \(T \to \infty\);
Corollary 1.4. With hypotheses and notation as in Theorem 1.1 and \( Q \) and our lower bound with \( j \) odd is substantially better than the trivial bounds: see Example 7.1 for an illustration.

Remark 1.3. Elementary arguments give “trivial” bounds

\[
\dim_k \ker (V : H^0(X, \Omega^1_X(E_0)) \to H^0(X, \Omega^1_X(E_0))) \leq a_Y \leq p \cdot g_X - p \cdot f_X + \sum_{Q \in S} \frac{p-1}{2}(d_Q - 1)
\]

where \( E_0 = \sum_{Q \in S}(d_Q - \lfloor d_Q/p \rfloor)[Q] \). The trivial lower bound comes from the fact that there is an inclusion \( \Omega^1_X(E_0) \to \Omega^1_Y \) compatible with the Cartier operator: see Lemma 4.3. The trivial upper bound comes from the fact that \( a_Y + f_Y < g_Y \) and from applying the Riemann-Hurwitz formula for the genus and Deuring-Shafarevich formula for the \( p \)-rank to \( \pi : Y \to X \). Theorem 1.1 is substantially better than the trivial bounds: see Example 7.1 for an illustration.

Note that when \( p = 2 \) and \( a_X = 0 \), the upper and lower bounds of Theorem 5.1 (with \( j = 1 \)) coincide, and we recover Voloch’s formula \cite[Theorem 2]{Voloch88}; see Remark 6.19. Similarly, when \( p \) is odd, all \( d_Q \) divide \( p - 1 \), and \( a_X = 0 \), we prove in Corollary 6.21 that our (sharpest) upper bound and our lower bound with \( j = \lfloor p/2 \rfloor = (p + 1)/2 \) also coincide, thereby establishing the following “\( a \)-number formula:”

Corollary 1.4. With hypotheses and notation as in Theorem 1.1 and \( p \) odd, assume that \( d_Q|(p-1) \) for all \( Q \in S \) and that \( X \) is ordinary (i.e. \( a_X = 0 \)). Then

\[
a_Y = \sum_{Q \in S} a_Q \quad \text{where} \quad a_Q := \frac{(p-1)}{2}(d_Q - 1) - \frac{p-1}{d_Q} \left\lfloor \frac{(d_Q - 1)^2}{4} \right\rfloor.
\]

Specializing Corollary 1.4 to the case of \( X = \mathbb{P}^1 \) recovers the main result of \( \cite{FP13} \).

To get a sense of the bounds in Theorem 1.1 in \( \mathbb{F}_7 \) we work out a number of examples with \( X = \mathbb{P}^1 \) and with \( X \) the elliptic curve over \( \mathbb{F}_p \) with affine equation \( y^2 = x^3 - x \), which has \( a_X = 1 \) when \( p \equiv 3 \mod 4 \) and \( a_X = 0 \) when \( p \equiv 1 \mod 4 \). We show in particular that for \( X = \mathbb{P}^1 \) our upper bound of Theorem 6.15 is sharp, with the family of covers \( y^p - y = t^{-d} \) (for \( t \) a choice of coordinate on \( \mathbb{P}^1 \)) achieving the upper bound for all \( p > 2 \) and all \( d \); see Example 7.3. We similarly find in our specific examples that the lower bound is sharp and that “most” covers have \( a \)-number equal to the lower bound.

1.1. Outline of the Proof. Without loss of generality, we may assume that \( k \) is algebraically closed. A key idea in the proof is that the Cartier operator is not defined only on global differentials, but actually is a map of sheaves. Let \( X \) be a smooth projective curve over \( k \). Functorially associated to the finite flat absolute Frobenius map \( F : X \to X \) by Grothendieck’s theory of the trace \cite[2.7.36]{Con00} \( \mathcal{O}_X \)-linear map of sheaves

\[
V_X : F_* \Omega^1_X \to \Omega^1_X;
\]

the Cartier operator considered previously is obtained by taking global sections. (For the remainder of the paper, we include subscripts to clarify which curve/-ring we are working with.) The advantage of this perspective is that the Cartier operator admits a simple description on stalks, allowing local arguments. In particular, the Cartier operator for the ring \( k((t)) \) is given by

\[
V_{k((t))} \left( \sum_i a_i t^i \frac{dt}{t} \right) = \sum_j a_j^{1/p} t^{j\frac{dt}{t}}.
\]

To relate the kernels of \( V_X \) and \( V_Y \) on global differentials, we will combine an analysis over the generic point with an analysis at stalks at the points where the cover \( \pi : Y \to X \) is ramified. This strategy allows the use of geometric methods, and allows us to work with general Artin-Schreier
force the corresponding differential on the coefficients of the power series expansions of elements in the image at points $\pi^{-1}(0)$. However, (1.5) does not generalize to an isomorphism of sheaves. There are two technical problems:

- the short exact sequence (1.6) may not split. It does when $X = \mathbb{P}^1$, so at a first reading it is fine to ignore this issue. In [2], we show that we may produce maps which split the sequence over the generic point and introduce poles in a controlled manner. For the rest of the introduction, we ignore this;

- furthermore, letting $(\ker V_X)(F_\ast D)$ denote the sub-sheaf of differentials in $F_\ast (\Omega^1_X(D))$ that lie in the kernel of $V_X$, the natural map

$$\varphi : \pi_* \ker V_Y \to \bigoplus_{i=0}^{p-1} (\ker V_X)(F_\ast E_i)$$

obtained by restricting $\varphi_\eta$ is injective but not an isomorphism. The problem is that while (in the generic fiber) $V_X(\omega_i)$ is determined by $\omega_j$ for $j > i$, it is not automatic that the resulting form $\omega_i$ satisfies $\text{ord}_Q(\omega_i) \geq -\text{ord}_Q(E_i)$ when $\text{ord}_Q(\omega_j) \geq -\text{ord}_Q(E_j)$ for $j > i$. See Example [3] for an illustration.

However, $\pi_* \ker V_Y$ can be identified with the sub-sheaf of $\bigoplus_{i=0}^{p-1} (\ker V_X)(F_\ast E_i)$ subject to relations on the coefficients of the power series expansions of elements in the image at points $Q \in S$ that force the corresponding differential $\omega_i$ on $Y$ to be regular. In [3] we construct short exact sequence

$$
\ pi_* \Omega^1_Y \cong \bigoplus_{i=0}^{p-1} \Omega^1_X(\eta).
$$

This follows from the fact that the function field $K' = k(Y)$ of $Y$ is an Artin-Schreier extension of the function field $K = k(X)$ given by $y^p - y = f$ for some $f \in K$. This relation also gives a formula for $V_Y$ in terms of $V_X$ (Lemma [4.3]), which yields an isomorphism

$$\varphi_\eta : (\pi_* \ker V_Y)_\eta \cong \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta.$$
of sheaves to express these relations. In our simplified situation, Theorem 5.1 gives \( O_X \)-modules \( G_j \) and exact sequences

\[
0 \to G_j \to G_{j+1} \to M_j \to 0
\]

for \( 0 \leq j < p \) where \( M_j \) is an explicit skyscraper sheaf, \( G_0 = \pi_* \ker V_Y \), and \( G_p = \bigoplus_{i=0}^{p-1}(\ker V_X)(F_iE_i) \).

Each \( g'_j \) expresses the relations necessary to force \( \omega_j \) to satisfy \( \ord_Q(\omega_j) \geq -\ord_Q(E_j) \). These allow us to (rather indirectly) relate \( \pi_* \ker V_Y \) with twists of \( \ker V_X \).

The final step is to extract useful information about the \( a \)-number of \( Y \) (the dimension of the space of global sections of \( \pi_* \ker V_Y \)) from these short exact sequences. The key to doing so is knowledge about the dimension of the kernel of the Cartier operator on \( H^0(X, \Omega_X^1(D)) \) for various divisors \( D \), and related questions about the existence of elements of that space with specified behavior at points in \( S \). In §6 we analyze these questions using a theorem of Tango (Fact 6.5, the main theorem of [Tan72]) and obtain bounds on the \( a \)-number of \( Y \) by taking global sections of the exact sequences from Theorem 5.1. Tango’s theorem yields precise results only when the degree of the divisor is sufficiently large: to obtain our result we ensure that an auxiliary divisor introduced to deal with the technical issues mentioned previously is enlarged so that these methods give a clean answer.

**Example 1.5.** For example, when \( X = \mathbb{P}^1 \) Tango’s theorem applies to any effective divisor and we can deduce the following facts (which are elementary in this situation):

- For a divisor \( D = \sum_i n_i[P_i] \) with \( n_i \geq 0 \),

\[
\dim_k \ker V_X : H^0(X, \Omega_X^1(D)) \to H^0(X, \Omega_X^1(D)) = \sum_i \left( n_i - \left\lfloor \frac{n_i}{p} \right\rfloor \right).
\]

- For any positive integer \( n \) with \( n \not\equiv 1 \pmod{p} \) and closed point \( Q \) of \( X \), there is a differential \( \omega_{Q,n} \) in the kernel of \( V_X \) that is regular except at \( Q \), and such that at \( Q \)

\[
\omega_{Q,n} = t_Q^{-n}dt_Q + a_0dt_Q + \ldots
\]

where \( t_Q \) is a local uniformizer at \( Q \).

We use Tango’s theorem to generalize these statements to arbitrary \( X \) in §6.2.

**Remark 1.6.** The same arguments, with minor modifications, should yield bounds on the dimension of the kernel of powers of the Cartier operator. We leave that for future work.

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### 2. Producing Splittings

The Cartier operator gives a short exact sequence of \( O_X \)-modules

\[
(2.1) \quad 0 \to \ker V_X \to F_*\Omega_X^1 \to \text{Im} V_X \to 0.
\]

We are interested in when it splits.

**Lemma 2.1.** When \( X = \mathbb{P}^1_k \), \( (2.1) \) is a split exact sequence of sheaves.

**Proof.** Identify the generic fiber with \( k(t) \). From \( (1.3) \), we know that \( V_X(\frac{dt}{t}) = \frac{dt}{t} \) and \( V_X(t^i\frac{dt}{t}) = 0 \) if \( p \nmid i \). Thus we see that

\[
(\ker V_X)_\eta = \left\{ \sum_{i=1}^{p-1} h_i(t)t^i\frac{dt}{t} : h_i \in k(t^p) \right\} \quad \text{and} \quad (\text{Im} V_X)_\eta = \Omega_X^1_{\eta,\eta}.
\]
An explicit splitting of the generic fiber $s : (\text{Im} V_X)_\eta \to \Omega^1_{X,\eta}$ is given by

$$s \left( \sum_i a_i t^i \frac{dt}{t} \right) = \sum_i a_i^p p^i \frac{dt}{t}.$$  

For $Q \in \mathbb{P}^1_k$, a direct calculation shows that if $\text{ord}_Q(\omega) \geq 0$ then $\text{ord}_Q(s(\omega)) \geq 0$, so this defines a map of sheaves.

**Remark 2.2.** The corresponding splitting $r : F_* \Omega^1_X \to \ker V_X$ is given by

$$r \left( \sum_i a_i t^i \frac{dt}{t} \right) = \sum_i a_i \frac{dt}{t}.$$  

In general, (2.1) splits over the generic point for any smooth curve $X$, although it is not clear the sequence itself splits. However, it will split if we allow sections to introduce controlled poles.

Note that $(F_* \Omega^1_X)(D) = F_* (\Omega^1_X(pD))$, which complicates the relationship between twists and order of vanishing. To more closely connect to the order of vanishing, for a subsheaf $\mathcal{F} \subset F_* \Omega^1_X$ and divisor $D$ on $X$ we define a sheaf $\mathcal{F}(F_* D)$ via

$$\mathcal{F}(F_* D)(U) := F_* (\Omega^1_X(D))(U) \cap \mathcal{F}_\eta.$$  

**Example 2.3.** It is clear that $F_* (\Omega^1_X(D)) = (F_* \Omega^1_X)(F_* D)$. More generally,

$$\ker V_X (F_* D) \subset F_* (\Omega^1_X(D))$$

consists of differentials $\omega$ that lie in the kernel of $V_X$ and such that $\text{ord}_Q(\omega) \geq -\text{ord}_Q(D)$ for all $Q$. (Compare with $(\ker V_X)(D) = (\ker V_X) \otimes \mathcal{O}_X(D) = (\ker V_X)(F_* pD)$, which consists of differentials that lie in the kernel of $V_X$ and such that $\text{ord}_Q(\omega) \geq -p \text{ord}_Q(D)$.)

For any divisor $E = \sum_i n_i P_i \geq 0$, define $\overline{E} := \sum [n_i/p] P_i$.

**Lemma 2.4.** There is an exact sequence

$$0 \to \ker V_X (F_* E) \to F_* (\Omega^1_X(E)) \overset{V_X}{\to} \text{Im} V_X(\overline{E}) \to 0.$$  

Furthermore, each term is locally free.

**Proof.** For a section $\omega$ of $F_* \Omega^1_X(E)$ defined at $Q$, a local calculation shows that $\text{ord}_Q(V_X(\omega)) \geq \lfloor \text{ord}_Q(\omega)/p \rfloor$, so we can define the right map. The kernel is $\ker V_X(F_* E)$ by definition. We can check surjectivity on stalks, where as $-n_i \leq p \left\lfloor \frac{m_i}{p} \right\rfloor$ for a uniformizer $t_Q$ we have

$$V_X \left( \sum_i a_i^p i^n Q^i \frac{dt_q}{t_Q} \right) = \sum_i a_i t_Q \frac{dt_q}{t_Q}.$$  

Over a smooth curve, to check local freeness it suffices to check the sheaves are torsion-free. That is clear as the sheaves are subsheaves of $F_* \Omega^1_{X,\eta}$.

We will prove the following:

**Proposition 2.5.** Let $S$ be a finite set of points on a smooth projective curve over $k$ and

$$0 \to \mathcal{F}_1 \xrightarrow{i} \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

an exact sequence of locally free sheaves. There exists a divisor $D = \sum_i [P_i]$ with the $P_i$ distinct points of $X$ not in $S$ and a morphism $r : \mathcal{F}_2 \to \mathcal{F}_1(D)$ such that $r \circ i$ is the natural inclusion $\mathcal{F}_1 \to \mathcal{F}_1(D)$.

Applying this to the exact sequence of Lemma 2.4, we obtain:
Corollary 2.6. Let $S$ be a finite set of points of $X$. Fix a divisor $E$ supported on $S$. There is a divisor $D = \sum_i [P_i]$ with the $P_i$ distinct points of $X$ not in $S$ and a morphism
\[ r : F_*(\Omega^1_X(E)) \to \ker V_X(F_*(E + pD)) \]
such that $r \circ \iota$ is the natural inclusion $\ker V_X(F_*(E)) \to \ker V_X(F_*(E + pD))$, where $\iota$ is the inclusion $\ker V_X(F_*(E)) \to F_*(\Omega^1_X(E))$.

We will prove the Proposition in the remainder of this section. The key input is the following result from algebraic geometry:

Lemma 2.7. Let $S$ be a finite set of points of $X$, and $\mathcal{F}$ be a locally free sheaf on $X$. Then for any divisor $D = \sum_i [P_i]$ with the $P_i$ distinct points of $X$ not in $S$ with $\deg D \gg 0$, we have
\[ H^1(X, \mathcal{F} \otimes \mathcal{O}_X(D)) = 0. \]

Proof. Pick an ample line bundle $\mathcal{L} = \mathcal{O}_X(D')$. By Serre’s cohomological criterion for ampleness, we know that there is an $N$ such that
\[ H^1(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \]
for $n \geq N$.

Let $g$ be the genus of $X$. Riemann-Roch says that
\[ h^0(X, \mathcal{O}_X(D - ND')) - h^1(X, \mathcal{O}(D - ND')) = \deg(\mathcal{O}(D - ND')) - g + 1. \]
When $\deg(\mathcal{O}(D - ND')) > 2g - 2$, we know $h^1(X, \mathcal{O}(D - ND')) = 0$ by degree reasons. Thus when $\deg D \gg 0$ we conclude that
\[ h^0(X, \mathcal{O}_X(D - ND')) > 0. \]

Using a global section of $\mathcal{O}_X(D - ND')$, we obtain a short exact sequence
\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X(D' - ND) \to \mathcal{G} \to 0 \]
where $\mathcal{G}$ is a skyscraper sheaf supported on $D - ND'$. Tensoring with $\mathcal{O}_X(ND')$ is exact, as is tensoring with the locally free $\mathcal{F}$, so we obtain a short exact sequence
\[ 0 \to \mathcal{F} \otimes \mathcal{O}_X(ND') \to \mathcal{F} \otimes \mathcal{O}_X(D) \to \mathcal{G}' \to 0 \]
where $\mathcal{G}'$ is still a skyscraper sheaf supported on $D - ND'$. Part of the long exact sequence of cohomology is
\[ H^1(X, \mathcal{F} \otimes \mathcal{O}_X(ND')) \to H^1(X, \mathcal{F} \otimes \mathcal{O}_X(D)) \to H^1(X, \mathcal{G}') \]
The left term vanishes by our choice of $N$, and the right vanishes as $\mathcal{G}'$ is a skyscraper sheaf. Thus $H^1(X, \mathcal{F} \otimes \mathcal{O}_X(D)) = 0$. \hfill $\Box$

We now prove Proposition 2.5. By looking at stalks we see that the Hom-sheaf
\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1) \]
is locally free. Also, notice that for a divisor $D$
\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1(D)) \simeq \mathcal{F}_3' \otimes \mathcal{F}_1 \otimes \mathcal{O}_X(D) \simeq \text{Hom}(\mathcal{F}_3, \mathcal{F}_1) \otimes \mathcal{O}_X(D). \]

Applying the functor $\text{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{F}_1(D))$ to the exact sequence in Proposition 2.5 we obtain
\[ 0 \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1(D)) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{F}_1(1)) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_1(D)) \to 0. \]
This sequence is exact on the right as $\mathcal{F}_3$ is locally free. Taking global sections, part of the long exact sequence of cohomology is
\[ H^0(X, \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{F}_1(D))) \to H^0(X, \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_1(D))) \to H^1(X, \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1(D))) \]
Applying Lemma 2.7 with $\mathcal{F} = \text{Hom}_{O_S}(\mathcal{F}_S, \mathcal{F}_1)$, we may choose $D = \sum_i [P_i]$ where the distinct $P_i$ avoid $S$ such that $H^1(X, \mathcal{F} \otimes O_X(D)) = 0$. This means that we may find a preimage of the natural inclusion in the second term giving the desired splitting.

\[\square\]

**Corollary 2.8.** A choice of splitting $r : F_1 \to \text{ker} V_X(F_1 E + pD)$ as in Corollary 2.6 is equivalent to a choice of splitting

\[s : \text{Im} V_X(\mathcal{E}) \to F_1 \text{ker} V_X(E + pD)\]

such that $V_X \circ s$ is the natural inclusion $\text{Im} V_X(\mathcal{E}) \to \text{Im} V_X(D)$. Furthermore, we may choose $s$ so that for any point $Q \in \text{sup} E$, if $\text{ord}_Q(\omega) \geq d$ then

\[\text{ord}_Q(s(\omega)) \geq (d + 1)p - 1.\]

**Proof.** Given $r$, a section of

\[0 \to (\text{ker} V_X) \eta \to (F_1 \Omega^1_X) \eta \to (\text{Im} V_X) \eta \to 0\]

is given by $s(m) = \tilde{m} - vr(\tilde{m})$, where $\tilde{m}$ is any preimage of $m \in (\text{Im} V_X) \eta$. It is independent of lift, which allows us to make local calculations using (1.4) to check that this recipe defines a map $s : \text{Im} V_X(\mathcal{E}) \to F_1 \text{ker} V_X(E + pD)$. Conversely, given $s$ we define $r(m) = m - sV_X(m)$. This gives an equivalence between the two kinds of splittings.

To get the last claim, we will use Corollary 2.6 with the divisor $E' = \sum_{Q \in \text{sup} E} p\left\lfloor \frac{\text{ord}_Q(E)}{p} \right\rfloor - 1$. Note that $E' = E$. From that choice of $r'$, we obtain an

\[s' : \text{Im} V_X(\mathcal{E}) \to F_1 \text{ker} V_X(E' + pD).\]

Composing with the natural inclusion $F_1 \text{ker} V_X(E' + pD) \to F_1 \text{ker} V_X(E + pD)$, we obtain our map $s$: the condition on orders of vanishing follows by twisting. Let $r : F_1 \text{ker} V_X(E + pD) \to \text{ker} V_X(F_1 E + pD)$ be the corresponding map under the equivalence above. \[\square\]

**Remark 2.9.** Note that $r$ and $s$ give splittings of the generic fiber of (2.1) regardless of the choice of $E$ and $D$.

**Remark 2.10.** When $X = \mathbb{P}^1$, the explicit splitting in Lemma 2.1 shows we may take $D = 0$. This is a good simplification for subsequent arguments on a first reading.

### 3. Artin-Schreier Covers and Differential Forms

Let $\pi : Y \to X$ be an Artin-Schreier cover of smooth projective curves over an algebraically closed field $k$: this means that the function field $K'$ of $Y$ is an Artin-Schreier extension of the function field $K$ of $X$. Pick $f \in K$ such that $K' = K(y)$, where

\[y^p - y = f.\]

We may assume a generator $\tau \in G = \mathbb{Z}/p\mathbb{Z}$ sends $y$ to $y + 1$. The choice of $f$ is far from unique. In particular, by replacing $y$ with $y + g$ we may replace $f$ with $f + g^p - g$ but obtain the same extension. Let $\pi$ have branch locus $S \subset X$, and fix a geometric closed point $Q'$ not in $S$. Suppose $f$ has a pole of degree $p \cdot d$ at $Q \neq Q'$: by picking $g$ to have a pole of order $d$ at $Q$ and no poles except possibly at $Q'$, we may modify $f$ to decrease the order of the pole at $Q$ without introducing extra poles except at $Q'$. Repeating this procedure, we may arrange for $f$ to be minimal in the following sense.

**Definition 3.1.** We say $f \in K$ is minimal for $\pi : Y \to X$ (or the extension $K'$ of $K$) if

- for every $Q \in X$ not equal to $Q'$, $\text{ord}_Q(f) \geq 0$ or $p \nmid \text{ord}_Q(f)$;
- $Q'$, $p \nmid \text{ord}_{Q'}(f)$.

Let $d_Q$ denote the order of the pole of $f$ at $Q$. 

---

**End Note:** The content of this page appears to be an excerpt from a mathematical paper, discussing topics related to algebraic geometry, specifically focusing on Artin-Schreier covers and differential forms. The text includes definitions, lemmas, and corollaries, along with proofs and examples, typical of advanced mathematical exposition. The page contains complex mathematical expressions and concepts, indicating its target audience is students or researchers familiar with algebraic geometry.
Remark 3.2. When \( X = \mathbb{P}^1 \), it is easy to arrange for \( f \) to have poles only at the branch locus since there are functions on \( \mathbb{P}^1 \) with a single simple pole. In this case a separate analysis at \( Q' \) is unnecessary: this is a helpful simplification on a first reading. This reduction is not possible in general: see [Sha01 §7], especially the second example after Proposition 49.

Remark 3.3. There exists a function with a pole of order \( m \) at \( Q \) and a pole of order \( n \) at \( Q' \) provided that \( m + n > 2g - 2 \) using the Riemann-Roch theorem, where \( g \) is the genus of \( X \). When modifying \( f \), we may therefore assume that \( d_Q = -\text{ord}_Q(f) \) is at most \( p(2g-2) \).

Lemma 3.4. If \( \text{ord}_Q(f) < 0 \) and \( p \nmid d_Q \), then \( \pi \) is ramified above \( Q \) and \( d_Q \) is the unique break in the lower-numbering ramification filtration of \( G \) above \( Q \). Otherwise \( \pi \) is unramified above \( Q \).

Proof. If \( \text{ord}_Q(f) < 0 \) and \( p \nmid d_Q \), let \( P \) be a point of \( Y \) over \( Q \). Then the equation \( y^p - y = f \) and the fact that \( d_Q > 0 \) implies that \( \text{ord}_P(y^p - y) = p\text{ord}_P(y) = \text{ord}_P(f) = e_{P/Q}\text{ord}_Q(f) \). Since \( p \nmid d_Q \), we see that \( \pi \) is ramified over \( Q \).

Let \( t_Q \) be a uniformizer of \( \mathcal{O}_{X,Q} \), and \( P \) a point over \( Q \). Pick positive integers \( a \) and \( b \) such that 1 = \( ap - bd_Q \), using the fact that \( p \nmid d_Q \). Then \( u = t_Q^ay^b \) is a uniformizer of \( \mathcal{O}_{Y,P} \). We compute

\[
\text{ord}_P(\tau u - u) = \text{ord}_Q(t_Q^a ((y + 1)^b - y^b)) = \text{ord}_P(t_Q^a (by^{b-1} + \ldots)) = pa + (b-1)(-d_Q) = d_Q + 1.
\]

This shows that \( d_Q \) is the break in the lower-numbering filtration above \( Q \).

If \( \text{ord}_Q(f) < 0 \) and \( p|d_Q \) then since we chose \( f \) to be minimal we must have that \( Q = Q' \), and \( \pi \) is unramified at \( Q' \). Otherwise \( \text{ord}_Q(f) \geq 0 \), and direct inspections shows that there are \( p \) points above \( Q \) so \( \pi \) is unramified above \( Q \).

It is important to investigate when a differential is regular.

Lemma 3.5. Consider a rational differential \( \omega = \sum_i \omega_i y^i \in (\pi_s \Omega^1)_{\eta} \). It is regular at \( P \) over \( Q \in S \) provided

\[
\text{ord}_Q(\omega_i) \geq -n_{Q,i} := -\left\lfloor \frac{p - (p - 1 - i) d_Q}{p} \right\rfloor = -\left\lceil \frac{(p - 1 - i) d_Q}{p} \right\rceil.
\]

It is regular at \( P \) not lying over \( S \cup \{Q'\} \) provided \( \text{ord}_Q(\omega_i) \geq 0 \) for all \( i \).

Proof. Suppose \( Q \in S \). Let \( t_Q \) be a uniformizer of \( \mathcal{O}_{X,Q} \), and note that the fraction field of \( \mathcal{O}_{Y,P} \) can be obtained from the fraction field of \( \mathcal{O}_{X,Q} \) by adjoining a root of the polynomial \( y^p - y = f \) where \( f \) has a pole of order \( d_Q \) at \( Q \). We know that \( \text{ord}_P(y) = -d_Q \).

As before, pick positive integers \( a \) and \( b \) such that 1 = \( ap - bd_Q \), using the fact that \( p \nmid d_Q \), so that \( u = t_Q^ay^b \) is a uniformizer of \( \mathcal{O}_{Y,P} \). A direct computation shows that \( du = u^{-(p-1)(d_Q+1)} \beta dt_Q \) where \( \beta \) is a unit. Hence \( \text{ord}_P(dt_Q) = (p-1)(d_Q+1), \) and \( \text{ord}_P(\omega_i) = (p-1)(d_Q+1) + p\text{ord}_Q(\omega_i) \).

We can conclude that

\[
\text{ord}_P(\omega_i y^i) = p\text{ord}_P(\omega_i) + (p-1)(d_Q+1) - i d_Q.
\]

This is non-negative precisely when \( \text{ord}_Q(\omega_i) \geq -n_{Q,i} \). (An elementary argument shows the two expressions involving floor and ceiling functions are equal.) Since \( y \) has fractional valuation, all of the terms in \( \omega = \sum_i \omega_i y^i \) have different valuations, so \( \omega \) is regular above \( Q \) precisely when \( \text{ord}_Q(\omega_i) \geq -n_{Q,i} \) for all \( i \).

Now suppose \( \pi \) is étale over \( Q \), and \( Q \neq Q' \). Thus there are \( p \) points \( P_1, \ldots, P_p \) over \( Q \), and \( \mathcal{O}_{Y,P_i} \simeq \mathcal{O}_{X,Q} \) (and likewise for differentials).
The function $y$ corresponds to regular functions $h, h+1, \ldots, h+(p-1)$ in each of the $\mathcal{O}_{Y,P_1} \simeq \mathcal{O}_{X,Q}$. Thus $\omega = \sum_i \omega_i y^i$ is regular over $Q$ provided
\[
\text{ord}_Q \left( \sum_i \omega_i (h + j)^i \right) \geq 0
\]
for $0 \leq j \leq p - 1$. Define $\lambda_j = \sum_i \omega_i (h + j)^i \in \Omega^1_{X,Q}$, depending on $\omega$. They are related by the Vandermonde matrix $M = ((h + j)^i)_{0 \leq i, j \leq p-1}$:
\[
\begin{pmatrix}
(h + 0)^0 & (h + 0)^1 & \cdots & (h + 0)^{p-1} \\
(h + 1)^0 & (h + 1)^1 & \cdots & (h + 1)^{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
(h + (p - 1))^0 & (h + (p - 1))^1 & \cdots & (h + (p - 1))^{p-1}
\end{pmatrix}
\cdot
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_p
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_p
\end{pmatrix}.
\]
It is clear that if each $\omega_i$ is regular at $Q$ then the $\lambda_i$ are regular at $Q$. The determinant of $M$ is known to be
\[
\prod_{i \neq j} ((h + i) - (h + j)) = \prod_{i \neq j} (i - j) \in \mathbb{F}_p^*.
\]
Multiplying by $M^{-1}$, it follows that if the $\lambda_i$ are regular at $Q$ then so are the $\omega_i$. This completes the proof. \hfill \Box

**Remark 3.6.** The condition at $Q'$ is considerably more complicated because $\text{ord}_P(y) < 0$ for $P$ above $Q'$, so $g$ has a pole and the entries of $M$ are not in $\mathcal{O}_{X,Q}$. One simple case will be analyzed in the proof of Proposition 3.10 while a more complete analysis is deferred to Lemma 5.14.

As $G = \mathbb{Z}/p\mathbb{Z}$ acts on $Y$, for a fixed generator $\tau \in G$ we obtain a map of sheaves $\Omega^1_{Y} \rightarrow \tau_* \Omega^1_{Y}$. After pushing forward using the equivariant $\tau$, we obtain a map $\tau : \pi_* \Omega^1_Y \rightarrow \pi_* \Omega^1_Y$. Define a filtration $V_i := \pi_* \ker ((\tau - 1)^{i+1}) \subset \pi_* \Omega^1_Y$. Note that $V_{-1} = 0$ and $V_{p-1} = \pi_* \Omega^1_Y$.

**Proposition 3.7.** For $0 \leq i \leq p - 1$, we have exact sequences of $\mathcal{O}_Y$-modules
\[
0 \rightarrow V_{i-1} \rightarrow V_i \xrightarrow{\psi_i} \Omega^1_X(\tilde{E}_i) \rightarrow 0
\]
where $\tilde{E}_i = \sum_{Q \in S} n_{Q,i}[Q]$. In the generic fiber, a splitting is given by sending $\omega \in \Omega^1_X(\tilde{E}_i)$ to $\omega y^i$.

**Proof.** We compute that $(\tau - 1)$ reduces the degree in $y$:
\[
(\tau - 1)\omega y^i = \omega_i ((y + 1)^i - y^i) = \omega_j (j y^{i-1} + \ldots).
\]
Thus $(V_i)_\eta = \bigoplus_{j=0}^i (\Omega^1_X)_\eta y^j$. The map $\psi_i$ sending $\sum_{j=0}^i \omega_j y^j$ to $\omega_i$ restricts to a map of sheaves $V_i \rightarrow \Omega^1_X(\tilde{E}_i)$ with kernel $V_{i-1}$: we can check $Q \in S$ or $Q \notin S \cup \{Q'\}$ immediately using the previous Lemma. For $Q = Q'$, we compute that
\[
(\tau - 1)^{i-1} \omega = i! \cdot \omega_i
\]
and hence if $\omega$ is regular above $Q'$ then $\text{ord}_{Q'}(\omega_i) \geq 0$.

The kernel of $\psi_i$ is certainly $V_{i-1}$. In the generic fiber, it is split by the map sending $\omega \in \Omega^1_X(\tilde{E}_i)$ to $\omega y^i$: however, note that this may not be regular above $Q'$.

The lack of an integral splitting is a problem. We will modify the sheaf we are working with to obtain a splitting and to work with a simpler condition above $Q'$. Let $d_{Q'}$ be the order of the pole of $f$ above $Q'$: note that $p/d_{Q'}$ and that $y$ has a pole of order $d_{Q'}$ at any point above $Q'$. Define the sheaf $\mathcal{F}_0 \subset \pi_* \Omega^1_Y$ on $X$ which on an open set $U$ has sections
\[
\begin{cases}
\omega = \sum_i \omega_i y^i : \omega \text{ is regular above all } Q \in U \setminus \{Q'\} \text{ and } \text{ord}_{Q'} \omega_i \geq -(p - 1 - i)d_{Q'} \text{ if } Q' \in U
\end{cases}
\]
Lemma 3.8. We have that \( \pi_* \Omega^1_Y \) is a subsheaf of \( \mathcal{F}_0 \).

Proof. If \( \omega = \sum \omega_i y^i \) is regular above \( Q' \), it suffices to show that \( \text{ord}_{Q'}(\omega_i) \geq -(p - 1 - i)d_{Q'} \). We will do so by descending induction on \( i \). Observe that

\[
(g - 1)^{p-1} \omega = (p - 1)! \omega_{p-1}
\]

which must be regular above \( Q' \): this happens only if \( \text{ord}_{Q'} \omega_{p-1} \geq 0 \). To deal with \( i = n \) for the inductive step, note that

\[
(g - 1)^n \omega = n! \omega_n + f_{n+1}(y) \omega_{n+1} + \ldots + f_{p-1}(y) \omega_{p-1}
\]

where \( f_j(y) \) is a polynomial of degree at most \( j - n \) of \( y \). Let \( y \) correspond to functions \( h, h + 1, h + 2, \ldots, h + (p - 1) \) in the local rings above \( Q' \). Note that \( \text{ord}_{Q'}(h) = \frac{-d_{Q'}}{p} \geq -d_{Q'} \). If \( \omega \) is regular, we know that

\[
\text{ord}_{Q'}(n! \omega_n + f_{n+1}(h) \omega_{n+1} + \ldots + f_{p-1}(h) \omega_{p-1}) \geq 0
\]

Furthermore, \( \text{ord}_{Q'}(f_j(h) \omega_j) \geq (j - n)(-d_{Q'}) - (p - 1 - j)d_{Q'} = -(p - 1 - n)d_{Q'} \) by the inductive hypothesis. Thus \( \text{ord}_{Q'}(\omega_n) \geq -(p - 1 - n)d_{Q'} \).

\[\square\]

Remark 3.9. We could use \((p - 1 - i)\frac{d_{Q'}}{p}\) just as well in the argument, but our choice simplifies later arguments. This argument shows that the \( \omega_n \) usually must have a pole above \( Q' \) in order to cancel the poles introduced by \( y \).

We can now prove a variant of Proposition 3.7. As before, we can define a filtration \( W_i = \pi_* \ker((\tau - 1)^i + 1) \subset \mathcal{F}_0 \) where \( \tau \) is a generator of \( G = \mathbb{Z}/p\mathbb{Z} \). Note that \( W_{-1} = 0 \) and \( W_{p-1} = \mathcal{F}_0 \). For \( Q \in S \) we have already defined \( n_{Q,i} \). Let \( S' = S \cup Q' \), and define \( n_{Q',i} := (p - 1 - i)d_{Q'} \).

Proposition 3.10. For \( 0 \leq i \leq p - 1 \), we have split exact sequences of \( \mathcal{O}_Y \)-modules

\[
0 \rightarrow W_{i-1} \rightarrow W_i \rightarrow \Omega^1_X(E_i) \rightarrow 0
\]

where \( E_i = \sum_{Q \in S'} n_{Q,i}[Q] \). A splitting is given by sending \( \omega \in \Omega^1_X(E_i) \) to \( \omega y^i \).

Proof. The proof is the same as that of Proposition 3.7 with no special analysis needed at \( Q' \).

Let \( \text{gr}^* \mathcal{F}_0 \) denote the associated graded sheaf for the filtration \( \{W_i\} \). The splitting gives:

Corollary 3.11. For an Artin-Schreier cover \( \pi : Y \rightarrow X \) as above, \( \mathcal{F}_0 \simeq \text{gr}^* \mathcal{F}_0 \simeq \bigoplus_i \Omega^1_X(E_i) \).

Note that this splitting is as \( \mathcal{O}_X \)-modules, and is not compatible with the Cartier operator.

4. The Cartier Operator on Stalks

Continue the notation of the previous section, letting \( \pi : Y \rightarrow X \) be an Artin-Schreier cover. Let \( F \) denote the Frobenius, and let \( V_Y : F_* \Omega^1_Y \rightarrow \Omega^1_Y \) and \( V_X : F_* \Omega^1_X \rightarrow \Omega^1_X \) be the Cartier operators on \( Y \) and \( X \) respectively. In this section, we analyze the Cartier operator in the stalk over the generic point. Letting \( r_i : F_* \Omega^1_{X,\eta} \rightarrow (\ker V_X)_\eta \) be any splittings of \( \{2.1\} \), we will establish:

Proposition 4.1. The map \( \varphi_\eta \) coming from Corollary 3.11 and the splitting

\[
\varphi_\eta : (\pi_* \ker V_Y)_\eta \subset (\pi_* F_* \Omega^1_Y)_\eta \simeq (\text{gr}^* \mathcal{F}_0)_\eta \simeq \bigoplus_{i=0}^{p-1} F_* \Omega^1_{X,\eta} \stackrel{\oplus r_i}{\rightarrow} \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta
\]

is an isomorphism.

For \( \omega = \sum_i \omega_i y^i \), we will usually write \( \varphi_\eta(\omega) = (v_0, \ldots, v_{p-1}) \) with \( v_i \in (\ker V_X)_\eta \).
Example 4.2. Suppose $X = \mathbb{P}^1$ and the $r_i$ are all the simple splitting given in Lemma 2.4. Writing $\omega_i = \sum_j a_j t^j \frac{dt}{t}$, we see that

$$v_i = \sum_{p|j} a_j t^j \frac{dt}{t}.$$ 

We will see that the $a_j$ with $p|j$ are determined by the $v_n$ with $n > i$.

We begin the proof by obtaining a formula for $V_Y$. We will work in the stalk at the generic point: this will automatically give formulas in other stalks since they include into the stalk at the generic point. Recall that $y^p - y = f$ gave the Artin-Schreier extension $k(Y)$ of $k(X)$, and let $\eta$ and $\eta'$ denote the generic points of $X$ and $Y$ respectively.

Lemma 4.3. For $\omega \in \Omega^1_{Y, \eta'}$, write $\omega = \sum_{i=0}^{p-1} \omega_i y^i$ with $\omega_i \in \Omega^1_{X, \eta}$. Then

$$V_Y(\omega) = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} V_X \left( \binom{i}{j} \omega_i (-f)^{i-j} \right) y^j.$$ 

Proof. We use the relation $y = y^p - f$ to compute that

$$V_Y(\omega_i y^i) = V_Y \left( \omega_i (y^p - f)^i \right)$$

$$= V_Y \left( \sum_{j=0}^{i} \binom{i}{j} \omega_i y^j (-f)^{i-j} \right)$$

$$= \sum_{j=0}^{i} V_X \left( \omega_i \binom{i}{j} (-f)^{i-j} \right) y^j.$$ 

The result follows by additivity and collecting $y^j$ terms. \qed

Corollary 4.4. If $\omega = \sum_i \omega_i y^i \in (\ker V_Y)_{\eta'}$, then $V_X(\omega_j)$ is determined by $\omega_i$ for $i > j$ via

$$V_X(\omega_j) = - \sum_{i=j+1}^{p-1} V_X \left( \binom{i}{j} \omega_i (-f)^{i-j} \right).$$ 

Proof. Clear. \qed

Proof. To prove Proposition 4.1, consider an element $\omega = \sum_i \omega_i y^i \in (\pi_* \ker V_Y)_{\eta}$. Each $\omega_j$ is determined by $V_X(\omega_j)$ and $r_j(\omega_j) = v_j$ (see (2.1)): more concretely,

$$(4.1) \quad \omega_j = v_j + s_j(V_X(\omega_j))$$

where $s_j$ is the splitting $s_j(\text{Im } V_X)_\eta \to F_* \Omega^1_{X, \eta}$ from Corollary 2.4. Using induction and Corollary 4.3, this means $\omega \in (\pi_* \ker V_Y)_{\eta}$ is determined by the $\varphi_\eta(\omega)$. Note that $\omega_j$ depends only on $v_i$ for $i \geq j$. \qed

Unfortunately, the map $\varphi_\eta$ does not generalize to give a simple description of $\pi_* \ker V_Y$, as illustrated in the following example.

Example 4.5. Let $p = 5$, $X = \mathbb{P}^1$, $f_1(t) = t^{-3}$, and $f_2(t) = t^{-3} + t^{-2}$, ramified only over $Q = 0$. The Artin-Schreier cover determined by $f_1$ has $a$-number 4, while the cover determined by $f_2$ has $a$-number 3.
The natural map to consider is the restriction of $\varphi_\eta$ to $\pi_* \ker Y$. If we use the simple splittings from Lemma $2.1$ we obtain an analogous map

$$\varphi : \pi_* \ker Y \rightarrow 4 \ker V_{P_1}(F_{n_0,i}[0]).$$

The induced map on global sections is injective but not an isomorphism. Given an element $(v_0, \ldots, v_4) \in \bigoplus_i H^0(P^1, \ker V_{P_1}(F_{n_0,i}[0]))$, we investigate when it lies in the image of $\varphi$. We have that

$$n_{0,i} = \begin{cases} 0 & i = 4 \\ 1 & i = 3 \\ 2 & i = 2 \\ 3 & i = 0 \end{cases}$$

and

$$H^0(P^1, \ker V_{P_1}(F_{n_{0,i}})) = \begin{cases} 0 & i = 4 \\ 1 & i = 2 \\ 1 & i = 1 \\ 2 & i = 0 \end{cases}$$

so the elements of $\bigoplus_i H^0(P^1, \ker V_{P_1}(F_{n_{0,i}}[0]))$ are easy to describe. A basis is given by

$$v_{0,3} = (t^{-3}dt, 0, 0, 0, 0), \ v_{0,2} = (t^{-2}dt, 0, 0, 0, 0), \ v_{1,2} = (0, t^{-2}dt, 0, 0, 0), \ v_{2,2} = (0, 0, t^{-2}dt, 0, 0).$$

For the cover defined by $f_1(t) = t^{-3}$, using Corollary $4.4$ we see that

$$\varphi_\eta^{-1}(v_{0,3}) = t^{-3}dt, \ \varphi_\eta^{-1}(v_{0,2}) = t^{-2}dt, \ \varphi_\eta^{-1}(v_{1,2}) = t^{-2}dty, \ \varphi_\eta^{-1}(v_{2,2}) = t^{-2}dty^2.$$

In all cases, the term $V_X(\omega_j)$ was computed to be zero since no term of the form $t^{-6}dt$ appeared. In this case, all of the preimages are in $H^0(Y, \Omega_Y)$, and the $a$-number is 4.

On the other hand, consider the cover defined by $f_2(t) = t^{-3} + t^{-2}$. Then we compute that

$$\varphi_\eta^{-1}(v_{0,3}) = t^{-3}dt, \ \varphi_\eta^{-1}(v_{0,2}) = t^{-2}dt, \ \varphi_\eta^{-1}(v_{1,2}) = t^{-2}dty, \ \varphi_\eta^{-1}(v_{2,2}) = t^{-6}dt + t^{-2}dty^2.$$

The last is not regular: Corollary $4.4$ forced a $t^{-6}dt$ to appear in $\omega_0$ since $\omega_2(-f_2)^2$ had a term of the form $t^{-6}dt$. In this case, the $a$-number is 3.

The issue is that there are some relations which we must have on the coefficients of the $v_i$ in order to ensure that $\varphi_\eta^{-1}((v_i))$ is regular.

5. Short Exact Sequences and the Kernel

The goal of this section is to describe a set of relations that define the image of an integral version of $\varphi_\eta$, an example of which is illustrated in Example $4.5$. We continue the notation of the previous section. We will do so by constructing short exact sequences involving $\pi_* \ker Y$ that express the failure of $\varphi_\eta$ to induce an isomorphism integrally. For most of the argument, we will work with $\mathcal{F}_0$ and variants which have modified behavior at $Q'$: at the end we will properly account for the behavior at $Q'$. At a first reading, it could be useful to assume that $X = P^1$ to remove the need for $Q'$ and the divisors $D_i$ coming from the abstract splittings of Corollary $2.6$.

Using Corollary $2.8$ for each $0 \leq i \leq p - 1$ pick a divisor $D_i = \sum_j [P_{i,j}]$ consisting of distinct points $P_{i,j}$ of $X$ where $f$ does not have a pole or a zero (and so $\pi$ is unramified over $P_{i,j}$ by Lemma $6.1$) and maps

$$r_i : F_{*}(\Omega^1_X(E_i)) \rightarrow \ker V_{X}(F_{*}(E_i + pD_i)), \ s_i : \text{Im} V_{X}(E_i') \rightarrow F_{*}(\Omega^1_X(E_i + pD_i)).$$

We consider the map

$$\varphi : \pi_* \ker Y \rightarrow F_* \mathcal{F}_0 \simeq F_* \text{gr}^* \mathcal{F}_0 \simeq \bigoplus_{i=0}^{p-1} F_{*}(\Omega^1_X(E_i)) \oplus \bigoplus_{i=0}^{p-1} \ker V_{X}(F_{*}(E_i + pD_i)).$$

Using the following maps

$$r_i : F_{*}(\Omega^1_X(E_i)) \rightarrow \ker V_{X}(F_{*}(E_i + pD_i))$$

we obtain an analogous map

$$s_i : \text{Im} V_{X}(E_i') \rightarrow F_{*}(\Omega^1_X(E_i + pD_i)).$$

We consider the map

$$\varphi : \pi_* \ker Y \rightarrow F_* \mathcal{F}_0 \simeq F_* \text{gr}^* \mathcal{F}_0 \simeq \bigoplus_{i=0}^{p-1} F_{*}(\Omega^1_X(E_i)) \oplus \bigoplus_{i=0}^{p-1} \ker V_{X}(F_{*}(E_i + pD_i)).$$
coming from composing the natural inclusion with the identification of Corollary 3.11 and the splittings $r_i$. It induces an isomorphism of generic fibers, so is automatically injective. Over the course of this section, we will define a filtration

$$\mathcal{G}_j \subset \bigoplus_{i=0}^{p-1} \ker X_*(F_*(E_i + pD_i)),$$

explicit skyscraper sheaves $M_j$, and maps

$$g'_j: \mathcal{G}_{j+1} \to M_j$$

for $0 \leq j < p$ that express the desired relations on $\bigoplus_{i=0}^{p-1} \ker X_*(F_*(E_i + pD_i))$ which will describe the image of $\varphi$. We then prove the following:

**Theorem 5.1.** For $0 \leq j < p$, there exist exact sequences

$$0 \to \mathcal{G}_j \to \mathcal{G}_{j+1} \xrightarrow{g'_j} M_j \to 0.$$ 

We have that $\mathcal{G}_{-1} = \operatorname{Im}(\varphi) \simeq \pi_* \ker Y_\varphi$, $\mathcal{G}_p = \bigoplus_{i=0}^{p-1} \ker X_*(F_*(E_i + pD_i))$, and $M_j$ is a skyscraper sheaf supported on $S_j := S \cup \{Q\} \cup \sup(D_j)$. Furthermore, the cokernel of the natural inclusion $\pi_* (\ker Y_{\varphi}) \simeq \mathcal{G}_{-1} \hookrightarrow \mathcal{G}_0 \simeq \pi_* (\ker Y_{\varphi})$ is

$$\bigoplus_{i=0}^{p-1} \left( k[t_{Q,i}]/(t_{Q,i}^{(p-1-i)q_i}) \right)^{\oplus (p-1)}.$$

**Example 5.2.** In Example 4.5, the element $v_{2.2}$ will turn out to be in $H^0(P^1, \mathcal{G}_1)$ but not $H^0(P^1, \mathcal{G}_0)$ as $\varphi_{\eta}^{-1}(v_{2.2})$ is not regular at 0. We will check this by showing that $g'_0(v_{2.2}) \neq 0$. This is worked out in Example 7.3.

5.1. **Filtration and Skyscraper Sheaves.** The first step is to define the filtration $\mathcal{G}_j$. Recall we have fixed splittings $r_i$ and used them to define $\varphi$ in (5.1). For $Q \in S$, recall that $d_Q$ is the degree of the pole of $f$ at $Q$, and $n_{Q,i}$ is defined in Lemma 3.5 as

$$n_{Q,i} := \left\lfloor \frac{(p-1-i)d_Q}{p} \right\rfloor.$$

Recall that $n_{Q,i} := (p-1-i)d_Q$. For $Q \in \sup(D_1)$, let $n_{Q,i} = p$, and otherwise let $n_{Q,i} = 0$. In all cases, let $t_Q$ be a local uniformizer at $Q$.

**Definition 5.3.** Set $S_j := S \cup \{Q\} \cup \sup(D_j)$ and define $g_j$ to be the natural map

$$\bigoplus_{i=0}^{p-1} \ker X_*(F_*(E_i + pD_i)) \xrightarrow{\varphi_{\eta}^{-1}} (\pi_* \ker Y_{\varphi})_{\eta} \to \bigoplus_{i=0}^{p-1} F_\ast \Omega^1_{X,\eta} \xrightarrow{\pi_j} F_\ast \Omega^1_{X,\eta} \to F_\ast \bigoplus_{Q \in S_j} \Omega^1_{X,Q} \left( \left[ \frac{1}{t_Q} \right] / t_Q \right)^{\oplus (p+1)} \Omega^1_{X,Q}$$

where $\pi_j$ is the projection onto the $j$th factor.

An element $(v_i) \in \mathcal{G}_p$ is in the kernel of $g_j$ if and only if the coefficient of $y_i$ in $\varphi_{\eta}^{-1}((v_i))$ is regular above $S \cup \sup(D_j)$. The map $g_j$ is far from being surjective.

Now we define the filtration:

**Definition 5.4.** For $0 \leq j \leq p-1$, define the $\mathcal{O}_X$-modules

$$\mathcal{G}_j \subset \bigoplus_{i=0}^{p-1} \ker X_*(F_*(E_i + pD_i)).$$
to be sections \((v_0, \ldots, v_{p-1}) = \varphi_\eta(\sum_i \omega_i y^i)\) for which for each \(j \leq i \leq p-1\), \(\omega_i\) is a section of \(F_*(\Omega_X^1(E_i))\). Define \(\mathcal{G}_{-1} = \text{Im}(\varphi)\).

**Lemma 5.5.** We have that \(\mathcal{G}_j = \ker(g_{j+1}|_{\mathcal{G}_{j+1}})\) for \(j \geq 0\). Furthermore,

- \(\mathcal{G}_p = \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))\);
- \(\mathcal{G}_0 \simeq \pi_* \ker(V_Y|_{\mathcal{S}_0})\);
- \(\mathcal{G}_{-1} \simeq \pi_* \ker V_Y\);
- the action of \(G = \mathbb{Z}/p\mathbb{Z}\) on \(\bigoplus_i (\ker V_X)_\eta \simeq (F_* \pi_* \Omega_Y^1)_\eta\) preserves \(\mathcal{G}_j\).

**Proof.** Given a section

\[(v_0, \ldots, v_{p-1}) \in \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))(U)\]

for some open set \(U\) of \(X\), we view the elements in the generic fiber and consider

\[\omega = \sum_{i=0}^{p-1} \omega_i y^i = \varphi_\eta^{-1}(v_0, \ldots, v_{p-1}) \in (\pi_* \ker V_Y)_\eta.\]

When \((v_0, \ldots, v_{p-1}) \in \mathcal{G}_{j+1}(U)\), we know \(\omega_i \in F_*(\Omega_X^1(E_i + D_i))(U)\) for \(i > j\).

Now \(\omega_j\) will be in \(F_*(\Omega_X^1(E_j + pD_j))(U)\) provided that

\[\text{ord}_Q(\omega_j) \geq -\text{ord}_Q(E_j + pD_j)\]

for every point \(Q\) of \(X\). This is automatic except when \(Q \in S_j\). But for \(Q \in S_j\), being in the kernel of \(g_{j+1}\) imposes the condition that \(\omega_j \in t_Q^{-1} F_*(\Omega_X^1)\) for all \(Q \in S_j\). This is equivalent to the condition on valuations.

The description of \(\mathcal{G}_p\) is clear. **Lemma 3.5** says that \(\mathcal{G}_0\) is the subsheaf of \(\bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))\)

where the corresponding differential form \(\omega\) is regular except above \(Q'\). By definition, \(\omega\) lies in \(\mathcal{G}_0\) and the kernel of the Cartier operator, so \(\mathcal{G}_0 \simeq \pi_* \ker(V_Y|_{\mathcal{S}_0})\). As the map \(\varphi\) is an injection, the image \(\mathcal{G}_{-1}\) is automatically isomorphic to \(\pi_* \ker V_Y\).

Since \(E_i \geq E_j\) if \(i < j\) and the action of a generator of \(G\) is given by \(y \mapsto y + 1\), it is clear that the action preserves \(\mathcal{G}_j\). \(\Box\)

The main step is to analyze the quotient \(\mathcal{G}_j/\mathcal{G}_{j+1}\): it will be a skyscraper sheaf supported on \(S_j\).

We will do so by studying the image of \(g_j\). In particular, for \(Q \in S_j\) we will pick \(O_{X,Q}\)-modules \(M_{Q,j}\) and define a map

\[c_{Q,j} : \text{Im}(g_j)|_Q \to \bigoplus_{Q \in S_j} (M_{Q,j})_Q\]

where \(A_Q\) denotes the skyscraper sheaf at \(Q\) with stalk \(A\). By taking the direct sum over \(S_j\) and composing with \(g_j\), we obtain maps

\[g'_j = c_j \circ g_j : \mathcal{G}_j \to \bigoplus_{Q \in S_j} (M_{Q,j})_Q.\]

We will check they induce an isomorphism

\[g'_j : \mathcal{G}_j/\mathcal{G}_{j+1} \xrightarrow{\sim} \bigoplus_{Q \in S_j} (M_{Q,j})_Q\]

by checking on stalks, thereby identifying the image of \(g_i\) as being isomorphic to \(\bigoplus_{Q \in S_j} (M_{Q,i})_Q\).
We will now define the skyscraper sheaf $M_j$ in Theorem 5.1. For a $k[t_Q]$-module $M$, let $F_s M$ denote the $k[t_Q^n]$-module with underlying additive group $M$ and the action of $t_Q^n$ on $F_s M$ given by multiplication by $t_Q$ on $M$.

**Definition 5.6.** Suppose that $Q \in S$.

- Define $m_{Q,j}$ to be the largest integer such that $-m_{Q,j} \geq -(p-1-j)d_Q + 1$ and $p|m_{Q,j}$.
- For $0 \leq j < p$, define $r_{Q,j}$ to be the number of multiples of $p$ between $-m_{Q,j}$ and $-n_{Q,j}$ inclusive.
- Define $M_{Q,j} := k[t_Q^n]/(t_Q^{m_{Q,j}}) = F_s k[t_Q^n]/(t_Q^{r_{Q,j}})$.

For $Q \in \text{sup}(D_j)$, define $M_{Q,j} := k[t_Q^n]/(t_Q^{p-1}) = F_s k[t_Q^n]/(t_Q^{p-1})$. Define $M_{Q,j} := 0$. Putting these together, we define

$$M_j := \bigoplus_{Q \in S_j} M_{Q,j}.$$ 

In the next subsections, we will define $c_{Q,j}$ and hence $g_j^Q$ stalk by stalk.

### 5.2. Local Calculations above Ramified Points

We begin by analyzing $\mathcal{G}_{j,Q}$ and defining $c_{Q,j}$ for $Q \in S$. We first record a result about the size of $n_{Q,j} + (j-i)d_Q$.

**Lemma 5.7.** Fix $0 \leq i \leq p-1$ and $Q \in S$. On $i \leq j \leq p-1$, $n_{Q,j} + (j-i)d_Q$ is a non-decreasing function of $j$.

**Proof.** Recall that $n_{Q,j} = \left\lfloor \frac{p+1+(p-1-j)d_Q}{p} \right\rfloor = \left\lfloor \frac{(p-1-j)d_Q}{p} \right\rfloor$. Then the integer

$$n_{Q,j+1} + (j+1-i)d_Q - (n_{Q,j} + (j-i)d_Q) \geq d_Q + \frac{(p-1-(j+1))d_Q}{p} - \frac{(p-1-j)d_Q}{p} - 1 \geq d_Q(1-1/p) - 1 > -1.$$ 

Pick a point $Q \in S$ and uniformizer $t_Q$ at $Q$. Let $(v_0, \ldots, v_{p-1}) \in \mathcal{G}_{j,Q}$, and write $\omega = \sum_i \omega_i y^i = \varphi^{-1}_\eta(v_0, \ldots, v_{p-1})$. By definition of $\mathcal{G}_j$, we know that $\text{ord}_Q(\omega_i) \geq -n_{Q,i}$ for $i > j$. Now by (4.1)

$$\omega_j = v_j + s_j \left( V_X \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} \right) \right).$$

We write

$$\sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} = \sum_{i=-N+1}^{\infty} a_i t_Q^i \frac{dt_Q}{t_Q}.$$ 

Note that as $\text{ord}_Q(\omega_i) \geq -n_{Q,i}$ and $\text{ord}_Q(f) = -d_Q$,

$$\text{ord}_Q \left( \binom{i}{j} \omega_i (-f)^{i-j} \right) \geq -n_{Q,i} - (i-j)d_Q.$$ 

By Lemma 5.7 that this is most negative when $i = p-1$. In that case, $n_{Q,p-1} - (p-1-j)d_Q = -(p-1-j)d_Q$, and we can conclude that

$$\text{ord}_Q \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} \right) \geq -(p-1-j)d_Q.$$ 

Thus we may take $N = -(p-1-j)d_Q$ in (5.3). Then we compute that

$$s_j \left( V_X \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} \right) \right) = \sum_{i=-m_{Q,i}/p}^{\infty} a_i s_j \left( t_Q^i \frac{dt_Q}{t_Q} \right).$$
Proof. Furthermore we see that \( \text{ord}_Q s_j(t_Q^{d_Q}) \leq pi - 1 \) via the local description of the Cartier operator \((1.4)\) and the fact that \( s_j \) is a section. On the other hand, if \( pi - 1 \geq -n_{Q,j} \) then \( \text{ord}_Q(s_j(t_Q^{d_Q})) \geq -n_{Q,j} \) by choice of \( s_j \). Since we know that \( \text{ord}_Q(v_j) \geq -n_{Q,j} \), we see that

\[
g_j(v_0, \ldots, v_{p-1}) \in \text{Im}(g_j)_Q \subset F_* \left( \Omega^1_{X,Q} \left[ \frac{1}{t_Q} \right] / t_Q^{-n_{Q,j}} \Omega^1_{X,Q} \right)
\]

is determined by the \( a_{pi} \) for \( -n_{Q,j} \leq pi \leq -n_{Q,j} \). There are \( n_{Q,j} \) such integers by definition. We may identify the image with \( M_{Q,j} = k[t_P]/(t_P^{n_{Q,j}}) \) via

\[
\sum_{i = -n_{Q,j},/p}^{\infty} a_{pi} s_j \left( t_Q^{dt_Q} \right) \mapsto \sum_{i = 0}^{n_{Q,j} - 1} a_{-m_{Q,j}+pi} t_Q^i
\]

This is an \( \mathcal{O}_X \)-linear map, where the action of \( \mathcal{O}_X \) on the first factor is the natural one coming from localizing the action on \( F_* \Omega^1_X \) and the action of the second factor comes from the action of \( \mathcal{O}_{X,Q} \) on \( F_* k[t_Q] \). We denote the resulting maps by

\[c_{Q,j} : \text{Im}(g)_Q \to M_{Q,j} \quad \text{and} \quad g'_{Q,j} : \mathcal{G}_j \to M_{Q,j} .\]

**Proposition 5.8.** For \( Q \in S \), the map \( g'_{Q,j} \) is surjective.

We will prove this relying on the following technical result. Note that the completion of \( \Omega^1_{X,Q} \) is isomorphic to \( \Omega^1_{X,Q} \otimes_{\mathcal{O}_{X,Q}} k[[t_Q]] \cong k[[t_Q]]dt_Q \).

**Lemma 5.9.** Given \( \omega_0, \ldots, \omega_{p-2} \in k((t_Q))dt_Q \) with \( \text{ord}_Q(\omega_i) \geq -(p - 1 - i)d_Q \), there exists \( \omega_{p-1} \in k[[t_Q]]dt_Q \) such that \( \omega = \sum_i \omega_i y^i \in (\pi_\ast \ker V_Y)_Q \otimes_{\mathcal{O}_{X,Q}} k((t_Q)) \).

**Proof.** Assuming the Lemma, we can prove Proposition \( 5.8 \) easily. The key is that the completion \( \hat{\mathcal{O}}_{X,Q} \cong k[[t_Q]] \) is faithfully flat over \( \mathcal{O}_{X,Q} \), so it suffices to check surjectivity in a formal neighborhood: i.e. after tensoring with \( \hat{\mathcal{O}}_{X,Q} \).

Now pick arbitrary \( \omega_0, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_{p-2} \in k((t_Q))dt_Q \) with \( \omega_i \in F_* (\Omega^1_{X,Q}(-n_{Q,i})) \) for \( i \neq j, p - 1 \), and let

\[
\omega_j = s_j \left( t_Q^{-(m_{Q,j}+p)dt_Q} t_Q^i \right) .
\]

Note that \( \text{ord}_Q \left( s_j \left( t_Q^{-(m_{Q,j}+p)dt_Q} t_Q^i \right) \right) \geq -m_{Q,j} - 1 \geq -(p - 1 - i)d_Q \) by Corollary \( 2.8 \). As \( -n_{Q,i} \geq -(p - 1 - i)d_Q \) we can apply the Lemma, choosing \( \omega_{p-1} \) such that \( \omega = \sum_i \omega_i y^i \in (\pi_\ast \ker V_Y)_Q \otimes k((t_Q)) \) and \( \text{ord}_Q(\omega_{p-1}) \geq 0 \). Let \( (v_0, \ldots, v_{p-1}) = \varphi(\omega) \). Note that as \( v_i = r_i(\omega_i) \), for \( i \neq j \) we have that \( \text{ord}_Q(v_i) \geq -n_{Q,i} \). For \( i = j \), we see that \( v_j = 0 \) since \( r_j \circ s_j = 0 \) (see the proof of Corollary \( 2.8 \)). Thus \( (v_0, \ldots, v_{p-1}) \in \mathcal{G}_j \). By construction, we see that

\[
g'_{Q,j}(v_0, \ldots, v_{p-1}) = 1 \in M_{Q,j} \otimes \hat{\mathcal{O}}_{X,Q} = M_{Q,j} = k[t_P]/(t_P^{n_{Q,j}}) .
\]

This suffices to show that \( g'_{Q,j} \) is surjective, as it is a map of \( \mathcal{O}_X \)-modules. \( \square \)

We now prove Lemma \( 5.9 \).

**Proof.** Consider the formula

\[
0 = V_Y(\omega) = \sum_{j=0}^{p-1} \left( \sum_{i=j}^{p-1} V_X \left( (i/j) \omega_i (-f)^{i-j} \right) y^j \right) .
\]
from Lemma 4.3. So \( \omega_{p-1} = \sum_{i=1}^{\infty} a_i t_i^i \frac{dt}{t_Q} \) must satisfy

\[
V_X \left( \binom{p-1}{j} \omega_{p-1}(-f)^{p-1-j} \right) = -\sum_{i=j}^{p-2} V_x \left( \binom{i}{j} \omega_i(-f)^{i-j} \right)
\]

for \( 0 \leq j < p - 1 \). Let \( \text{coef}_i(h) \) denote the coefficient of \( t_i^i \) in \( h \), and \( \text{coef}_i(\omega) \) denote the coefficient of \( t_i^i \frac{dt}{t_Q} \) in a differential \( \omega \). Thus it suffices to check that for each \( j \) and every \( k \equiv 0 \pmod{p} \), the \( k \)-th coefficients agree:

\[
\sum_{i} \binom{p-1}{j} a_i \text{coef}_{k-1}((-f)^{p-1-j}) = \text{coef}_k \left( \sum_{i=j}^{p-2} \binom{i}{j} \omega_i(-f)^{i-j} \right).
\]

Note the right side is determined by the choice of \( \omega_0, \ldots, \omega_{p-2} \). We see that:

(A) \( \text{coef}_i((-f)^{p-1-j}) = 0 \) for \( i < (p - 1 - j)d_Q \);

(B) \( \text{coef}_{(p-1-j)d_Q}((-f)^{p-1-j}) \neq 0 \);

(C) \( \text{coef}_k \left( \sum_{i=j}^{p-2} \binom{i}{j} \omega_i(-f)^{i-j} \right) = 0 \) for \( k \leq -(p - 1 - j)d_Q \) with \( p|k \).

The first two are simple: the last follows from the fact that by hypothesis

\[
\text{ord}_Q(\omega_i(-f)^{i-j}) \geq -(p - 1 - i)d_Q + d_Q(j - i).
\]

Take \( a_i = 0 \) for \( i \leq 0 \). Then for any \( k \leq -(p - 1 - j)d_Q \) that is a multiple of \( p \), we see that \( \text{coef}_k \left( \binom{p-1}{j} \omega_{p-1}(-f)^{p-1-j} \right) = 0 \) for any \( j \), and likewise for the right side of (5.5) by (C). We will inductively define \( a_i \) for \( i > 0 \). For \( a_1 \) (the coefficient of \( t_Q \frac{dt}{t_Q} = dt_Q \)), choose \( 0 \leq j < p \) so \( -d_Q(p - 1 - j) \equiv -1 \pmod{p} \). In light of (A) and the fact that \( a_i \leq 0 \), (5.5) specifies that

\[
\ldots + 0 + \binom{p-1}{j} a_1 \text{coef}_{d_Q(p-1-j)}((-f)^{p-1-j}) + 0 + \ldots = \text{coef}_{-d_Q(p-1-j)+1} \left( \sum_{i=j}^{p-2} \binom{i}{j} \omega_i(-f)^{i-j} \right)
\]

where the right side is specified. By (B), there is a unique solution \( a_1 \).

In general, after having chosen \( a_i \) for \( i < N \), choose \( 0 \leq j < p \) so \( -d_Q(p - 1 - j) + N \equiv -1 \pmod{p} \). Then the right side of (5.5) with \( k = N - d_Q(p - 1 - j) \) is specified. The finitely many non-zero terms of the left side with \( i < N \) are determined by previous information, while the terms with \( i > N \) are zero by (A). By (B), we may uniquely solve for \( a_N \).

Since \( p \nmid d_Q \), as we carry out this procedure \( j \) runs through all of the residue classes modulo \( p \), and each pair \((j, k)\) is considered exactly once. Therefore there is a unique choice of \( \omega_{p-1} = \sum a_i t_i^i dt_Q \) that makes (5.5) hold for every \( j \) and every \( k \) that is a multiple of \( p \). This completes the proof. \( \square \)

**Remark 5.10.** If \( \omega_0, \ldots, \omega_{p-2} \) were rational (elements of \( k(t_Q)dt \)), it is not clear whether the power series for \( \omega_{p-1} \) constructed in the above argument gives a rational differential form or simply an element of \( k[[t_Q]]dt_Q \). We pass to the completion to avoid this issue.

Note that the process of reconstructing \( \omega_{p-1} \) from \( \omega_0, \ldots, \omega_{p-2} \) does not work globally. For example, suppose \( X = P^1 \) and we were writing differential forms as \( \sum a_i t^i dt \). Looking at how we reconstructed \( \omega_{p-1} \), it would likely have \( a_i \neq 0 \) for \( i \) large, which would mean that \( \text{ord}_\infty(\omega_{p-1}) \ll 0 \).
Lemma 5.11. We have that $Q$ and let $g$ such that

$$v = (v_0, \ldots, v_k, 0, \ldots, 0) = \varphi_Q(\omega) = \varphi_Q \left( \sum_{i=0}^{k} \omega_i y_i \right).$$

Furthermore, $\text{ord}_Q(p) = i$ since $v \in V_{j,k}$ we record the following fact about $V_{j,k}$, the kernel of $(\tau - 1)^{k+1}$ acting on $G_{j,Q}$. If $v \in V_{j,k}$ then

$$\text{ord}_Q(\omega_i) \geq -n_{Q,i}$$

Lemma 5.11. We have that $\dim g_{Q,j}(V_{j,k}) \leq c(j, k, Q)$, where $c(j, k, Q)$ is the number of multiples of $p$ between $-n_{Q,k} - (k - j)d_Q + 1$ and $-n_{Q,j}$ inclusive. In particular,

$$c(j, k, Q) = \left\lfloor \frac{n_{Q,k} + (k - j)d_Q - 1}{p} \right\rfloor - \left\lfloor \frac{n_{Q,j}}{p} \right\rfloor \leq r_{Q,j}.$$

Proof. From Lemma 5.1, we see that

$$\omega_j = v_j + s_j \left( V_X \left( - \sum_{i=j+1}^{k} \binom{i}{j} \omega_i (-f)^{i-j} \right) \right)$$

since $v_i$ and $\omega_i$ are 0 for $i > k$. Furthermore, we know that $\text{ord}_Q(\omega_i) \geq -n_{Q,i}$ for $i \geq j$. Then Lemma 5.1 shows that

$$\text{ord}_Q \left( \sum_{i=j+1}^{k} \binom{i}{j} \omega_i (-f)^{i-j} \right) \geq -n_{Q,k} - (k - j)d_Q.$$}

To define $g'_{Q,j}$, we expanded the inner sum in a power series $\sum a_i t_Q^{\frac{dt_Q}{t_Q}}$ and wrote

$$s_j \left( V_X \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} \right) \right) = \sum_{i=-M}^{\infty} a_{pi}s_{j} \left( t_Q^{\frac{dt_Q}{t_Q}} \right)$$

and let $g'_{Q,j}$ record the coefficients where $pi - 1 < -n_{Q,j}$. Since we have a better bound on $M$, we obtain a smaller image. In particular, the number of potentially non-zero coefficients is the number of multiples of $p$ between $-n_{Q,k} - (k - j)d_Q + 1$ and $-n_{Q,j}$. The rest is elementary. $\square$

5.3. Local Calculations at Poles of Sections. Now suppose that $Q \in \text{sup}(D_j)$, and pick a uniformizer $t_Q$ at $Q$. As before, let $(v_0, \ldots, v_{p-1}) \in G_{j,Q}$, and write $\omega = \sum_i \omega_i y_i^i = \varphi_{\eta}^{-1}(v_0, \ldots, v_{p-1})$. By definition of $G_j$, we know that $\text{ord}_Q(\omega_i) \geq 0$ for $i > j$. Furthermore, $\text{ord}_Q(v_j) \geq -p$ and

$$\omega_j = v_j + s_j \left( V_X \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} \right) \right).$$

Since $f$ has no pole at $Q$ by construction, $\text{ord}_Q \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} \right) \geq 0$, and hence we may write it as

$$- \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j} = \sum_{i=1}^{\infty} a_i t_Q^{\frac{dt_Q}{t_Q}}$$

Then

$$\omega_j = v_j + a_p t_Q^{p} s_{j} \left( t_Q^{\frac{dt_Q}{t_Q}} \right) + a_{2p} t_Q^{2p} s_{j} \left( t_Q^{\frac{dt_Q}{t_Q}} \right) + \ldots$$
Since $-p \leq \text{ord}_Q s_j(t_Q^{\infty})$, we see that

$$\text{Im}(g_i)_Q \subset F_* \left( t_Q^{-p} \Omega^1_{X,Q}/\Omega^1_{X,Q} \right)$$

Furthermore, since $V_X \circ s_j$ is the identity, $s_j(dt_Q)$ can have no $t_Q^{-1}dt_Q$ term. The subspace of $F_* \left( t_Q^{-p} \Omega^1_{X,Q}/\Omega^1_{X,Q} \right)$ with no $t_Q^{-1}dt_Q$ term is isomorphic to $k[t_Q]/(t_Q^{-p})$ (as $\mathcal{O}_X$-modules) under the identification

$$c_{Q,j} : \sum_{i=-p}^{p} b_i t_Q^i dt_Q \mapsto \sum_{i=0}^{p-2} b_{i-p} t_Q^i.$$

**Lemma 5.12.** For $Q \in \text{sup}(D_j)$, $g'_{Q,j} = c_{Q,j} \circ g_{Q,j}$ is surjective.

**Proof.** Note that $(\ker V_X)_Q$ can be generated by $t_Q^{j}dt_Q$ with $p \nmid i$ and

$$\omega_j = v_j + s_j \left( V_X \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i(-f)^{i-j} \right) \right) = \sum_{i=-p}^{\infty} b_i t_Q^i dt_Q^i.$$

The $\omega_i$ with $i > j$ are determined by $v_{j+1}, \ldots, v_{p-1}$ by Proposition 4.11 and $v_j \in (\ker V_X(D_j))_Q$ satisfies $\text{ord}_Q(v_j) \geq -p$. Since $g_{Q,j}$ just extracts the $t_Q^p dt_Q$ through $t_Q^{-2}dt_Q$ coefficients of $\omega_j$, it is clearly surjective. \hfill $\square$

### 5.4. Proof of the Theorem.

Putting the local maps $g'_{Q,j}$ together, we define

$$g'_{j} = \bigoplus_{Q \in S \cup \text{sup}(D_j)} g'_{Q,j} : \mathcal{G}_j \to M_j.$$

Note that we do not need a contribution from $Q'$ because of the next lemma.

**Lemma 5.13.** The natural inclusion $\mathcal{G}_j, Q' \to \mathcal{G}_{j+1}, Q'$ is an isomorphism.

**Proof.** We check surjectivity. Consider $(v_0, \ldots, v_{p-1}) \in \mathcal{G}_{j+1}, Q'$ corresponding to $\omega = \sum_i \omega_i y^i$, with $\text{ord}_{Q'}(\omega_i) \geq -(p - 1 - i)d_{Q'}$ for $i \geq j + 1$. As $\text{ord}_{Q'}(f) = -d_{Q'}$, the relation

$$\omega_j = v_j + s_j \left( V_X \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i(-f)^{i-j} \right) \right)$$

together with the properties of $s_j$ from Corollary 2.8 shows that $\text{ord}_{Q'}(\omega_j) \geq -(p - 1 - j)d_{Q'}$. Thus $(v_0, \ldots, v_{p-1}) \in \mathcal{G}_j, Q'$. \hfill $\square$

We can now prove Theorem 5.1.

**Proof.** We first establish the exact sequence. An element $(v_0, \ldots, v_{p-1}) = \varphi_{\eta}(\sum_i \omega_i y^i) \in \mathcal{G}_{j+1}$ is in the kernel of $g'_{j}$ exactly when $\omega_j \in F_* \left( \Omega^1_{X,Q}(E_j) \right)$ for every $Q \in X$. This is clear from the definition of $g'_{Q,j}$ for $Q \in S \cup \text{sup}(D_j)$ (since $c_{Q,j}$ maps the image of $g_{Q,j}$ to the skyscraper sheaf) and by Lemma 5.13 at $Q'$. For $Q \in X - S_j$ and $(v_0, \ldots, v_{p-1}) \in \mathcal{G}_j$, we claim that $\text{ord}_Q(\omega_j) \geq 0$. This follows from the formula

$$\omega_j = v_j + s_j \left( V_X \left( - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i(-f)^{i-j} \right) \right)$$

plus the facts that $\text{ord}_Q(v_j) \geq 0$ for all $j$, that $\text{ord}_Q(f) \geq 0$, that $\text{ord}_Q(\omega_i) \geq 0$ for $i > j$ by definition, and that $s_j$ induces a map $(\text{Im} V_X)_Q \to (F_* \Omega^1_X)_Q$. Combined with Lemma 5.5, this establishes exactness at the left and middle. Exactness on the right can be checked at stalks:
for $Q \in S_j$ this follows from Proposition $5.8$ and Lemmas $5.12$ and $5.13$. For $Q \in X - S_j$, the inclusion $\mathcal{G}_j \to \mathcal{G}_{j+1}$ is an isomorphism at $Q$ since it is automatic that the reconstructed $\omega_j$ satisfies $\text{ord}_Q(\omega_j) \geq 0$. This completes the main part of the proof.

The final statement is the content of Lemma $5.14$, where we finally analyze the relationship between $\mathcal{G}_0$ and $\mathcal{G}_1$ by imposing the correct regularity condition at points above $Q'$.

**Lemma 5.14.** There is an exact sequence

$$0 \to \mathcal{G}_-1 \to \mathcal{G}_0 \to \bigoplus_{i=0}^{p-1} \left( k[t^p_{Q_j}]/(t^i_{Q_j}d_Q') \right)^{(p-1)} \to 0$$

*Proof.* We know that the left map is an inclusion, and is an isomorphism away from $Q'$, so this is a local question at $Q'$. To simplify notation, we let $Q = Q'$. Since $\mathcal{G}_-1 \simeq \pi_* \ker V_Y$ and $\mathcal{G}_0 \simeq \mathcal{F}_0$, we can work in $F_0 \Omega^1_{X,Q} \left[ \frac{1}{t_Q} \right]$. Recall that $(\pi_* \ker V_Y)_Q$ consists of forms that are regular above $Q$ that lie in the kernel of $V_Y$, while $\mathcal{F}_{0,Q}$ consists of $\omega = \sum_i \omega_i y^i$ in the kernel of $V_Y$ and such that $\text{ord}_Q(\omega_i) \geq -(p - 1 - i)d_Q$. Despite the fact that $\pi$ is étale over $Q$, since the element $f$ defining the Artin-Schreier extension of function fields has a pole at $Q$ (as does $y$), the decomposition $\omega = \sum_i \omega_i y^i$ is tricky to analyze. We can fix this when working locally.

Pick a function $g \in \mathcal{O}_{X,Q} \left[ \frac{1}{t_Q} \right]$ such that $f + g^p - g = f'$ is regular at $Q$. We may assume that $\text{ord}_Q(g) \geq -\frac{d_Q}{p}$ since $\pi$ is étale over $Q$. Let $y' = y + g$, so that

$$(y')^p - y' = f'$$

is another element defining the same Artin-Schreier extension of fraction fields at $Q$.

Observe that $y'$ is regular above $Q$. Now write $\omega = \sum_i \omega_i y^i = \sum_i \omega_i'(y')^i$. While the condition of being regular above $Q$ is tricky to describe in terms of the $\omega_i$, it is simple to describe in terms of the $\omega_i'$:

$$\omega \text{ is regular above } Q \text{ if and only if } \text{ord}_Q(\omega_i') \geq 0 \text{ for all } 0 \leq i \leq p - 1.$$ 

The proof of Lemma $5.8$ carries over verbatim. Furthermore, by collecting $y^i$ terms we see that

$$\omega_j = \sum_{i=j}^{p-1} \binom{i}{j} \omega_i' g^{i-j}.$$ 

In particular, a descending induction on $i$ shows that $\text{ord}_Q(\omega_i) \geq -(p - 1 - i)d_Q$ if and only if $\text{ord}_Q(\omega_i') \geq -(p - 1 - i)d_Q$. Define

$$M_{-1} := \left\{ \omega = \sum_i \omega_i'(y')^i : \omega \in (\ker V_Y)_Q \text{ and } \omega_i' \in \Omega^1_{X,Q} \right\}$$

and likewise define

$$M_0 := \left\{ \omega = \sum_i \omega_i'(y')^i : \omega \in (\ker V_Y)_Q \text{ and } \omega_i' \in t_Q^{-(p-1-i)d_Q} \Omega^1_{X,Q} \right\}.$$ 

We view these as $k[t^p_{Q_j}]$-modules. To complete the proof, it suffices to show the cokernel of the natural inclusion $M_{-1} \hookrightarrow M_0$ is

$$\bigoplus_{i=0}^{p-1} \left( k[t^p_{Q_j}]/(t^{(p-1-i)d_Q'}) \right)^{(p-1)}.$$
By Corollary \[1.4\] applied at \(Q\), we see that \(V_X(\omega'_j)\) is determined by \(\omega'_i\) for \(i > j\). We have a simple section \(s\) to \(V_X\) given by

\[
s \left( \sum_i a_i t_Q^i \frac{dt_Q}{t_Q} \right) = \sum_i a_i^p t_Q^i \frac{dt_Q}{t_Q}.
\]

Let \(r : F_\ast \Omega^1_{X,Q} \to (\ker V_X)_Q\) be the splitting corresponding to \(s\): it satisfies

\[
s \left( \sum_i a_i t_Q^i \frac{dt_Q}{t_Q} \right) = \sum_i a_i^p t_Q^i \frac{dt_Q}{t_Q}.
\]

The fact that \(V_X(\omega'_j)\) is determined by \(\omega'_i\) for \(i > j\) means that the map

\[
\varphi_Q' \left( \sum_i \omega'_i(y')^i \right) = (r(\omega'_0), \ldots, r(\omega'_{p-1}))
\]

induces an isomorphism between \((\pi_\ast \ker V_Y)_Q \left[ \frac{1}{t_Q} \right]\) and \(\bigoplus_i (\ker V_X)_Q \left[ \frac{1}{t_Q} \right]\).

Furthermore, we see that \(s(V_X(\omega'_j)) \in \Omega^1_{X,Q}\) if \(\omega'_i \in \Omega^1_{X,Q}\) for \(i > j\), and likewise \(s(V_X(\omega'_j)) \in t_Q^{-(p-1-j)d_Q} \Omega^1_{X,Q}\) if \(\omega'_i \in t_Q^{-(p-1-i)d_Q}\) for \(i > j\). This relies on the fact that \(f'\) is regular at \(Q\), and thus \(\varphi_Q'\) identifies

\[
M_{-1} \simeq \bigoplus_i (\ker V_X)_Q \quad \text{and} \quad M_0 \simeq \bigoplus_i t_Q^{-(p-1-i)d_Q} (\ker V_X)_Q
\]

as submodules of \(\bigoplus_i (\ker V_X)_Q \left[ \frac{1}{t_Q} \right]\). The natural inclusion \(M_{-1} \hookrightarrow M_0\) becomes the natural inclusion of the direct summands. As \((\ker V_X)_Q\) is a free \(k[t^p_Q]\)-module generated by \(dt_Q, \ldots, t_Q^{p-1}dt_Q\) and as \(p|d_Q\), this completes the proof. \(\square\)

**Remark 5.15.** To streamline notation, define \(M_{Q',-1} = \bigoplus_{i=0}^{p-1} \left( k[t^p_{Q'}]/(t^{(p-1-i)d_{Q'}}_{Q'}) \right)^{(p-1)}\) and \(S_{-1} = \{Q'\}\), so that the cokernel of the natural inclusion \(\mathcal{G}_{-1} \hookrightarrow \mathcal{G}_0\) is \(\bigoplus_{Q \in S_{-1}} M_{Q,-1}\). Let \(g_{-1}'\) denote the natural map to the cokernel.

**Remark 5.16.** Tracing through the proof, we see that image of \(H^0(X, \ker ((r-1)^{j+1} : \mathcal{G}_0 \to \mathcal{G}_0))\) under \(H^0(g_{-1}')\) has dimension at most

\[
\sum_{i=0}^j \frac{(p-1-i)d_{Q'}}{p}(p-1).
\]

This is simply because \(\omega_i = \omega'_i = 0\) for \(i > j\).

6. \(a\)-Number Bounds

As before, we let \(\pi : Y \to X\) be an Artin-Schreier cover of smooth projective curves over \(k\). In this section we extract bounds on the \(a\)-number of \(Y\) in terms of the ramification data of \(\pi\) and the \(a\)-number of \(X\) from the short exact sequences in Theorem \[5.1\]. Recall that the \(a\)-number is defined to be

\[
a_Y = \dim \ker (V_Y : H^0(Y, \Omega^1_Y) \to H^0(Y, \Omega^1_Y)) = \dim H^0(Y, \ker V_Y) = \dim H^0(X, \mathcal{G}_{-1}).
\]
6.1. Abstract Bounds. Recall that \( G = \mathbb{Z}/p\mathbb{Z} \) acts on \( \pi_*\Omega_Y^1 \) by having a generator \( \tau \) send \( y \mapsto y + 1 \). By Lemma 5.3 \( G \) also acts on \( \mathcal{G}_j \).

Lemma 6.1. There is an isomorphism

\[
\ker \left( (\tau - 1)^{j+1} : H^0(X, \mathcal{G}_{j+1}) \to H^0(X, \mathcal{G}_{j+1}) \right) \cong \bigoplus_{i=0}^{j} H^0(X, \ker V_X(F_* (E_i + pD_i))).
\]

Proof. Both sides are isomorphic to

\[
\left\{ \omega = \sum_{i=0}^{j} \omega_i y^i \in (\pi_* \ker V_Y)_\eta : \varphi_\eta(\omega) \in H^0(X, \mathcal{G}_p) \right\}.
\]

This follows from the formulas in Lemma 4.3, the fact that \( \tau - 1 \) reduces the maximum power of \( y \) appearing in a differential by one, and the definition of \( \varphi \).

Definition 6.2. Define \( V_j = \ker \left( (\tau - 1)^{j+1} : H^0(X, \mathcal{G}_{j+1}) \to H^0(X, \mathcal{G}_{j+1}) \right) \).

- For \( Q \in S \) and integers \( 0 \leq i \leq j \), define \( c(i, j, Q) \) to be the number of integers \( n \) congruent to \(-1\) modulo \( p \) such that \(-nQ_{i,j} - dQ_{j-i} \leq n < -nQ_{i,j} \).
- For \( Q \in \sup(D_i) \), define \( c_{i,j,Q} = p - 1 \).
- Define \( c_{i,j,Q} = \frac{d_{Q'}(p-1)(p-1-i)}{p} \).
- Define \( L(X, \pi) \) to be

\[
L(X, \pi) := \max_j \left( \dim V_j - \sum_{i=0}^{j} \sum_{Q \in S_i} c(i, j, Q) \right)
\]

where \( 0 \leq j \leq p - 1 \).

Note that for \( 0 \leq i \leq p \), if \( v \in H^0(X, \mathcal{G}_i) \) then either \( g_{i-1}'(v) \neq 0 \) or \( v \in H^0(X, \mathcal{G}_{i-1}) \) by Theorem 5.1. In the latter case, when \( i - 1 \geq 0 \) we may consider \( g_{i-2}'(v) \), despite the fact that \( g_{i-2}' \) is not defined on the entire space \( H^0(X, \mathcal{G}_i) \). Thus for \( v \in H^0(X, \mathcal{G}_p) \), it makes sense to speak of whether \( g_i'(v) \neq 0 \) for some \(-1 \leq i < p \). Recall that \( g_{i-1}' \) has an ad-hoc definition: see Remark 5.13.

Definition 6.3. Let \( M(X, \pi) \) denote the maximal number of linearly independent elements \( \{v_j\} \) of \( H^0(X, \mathcal{G}_p) \) such that \( g_i'(v_j) \neq 0 \) for some \(-1 \leq i < p \). Define

\[
U(X, \pi) := \sum_{i=0}^{p-1} \dim H^0(X, \ker V_X(F_* (E_i + pD_i))) - M(X, \pi).
\]

Proposition 6.4. The \( a \)-number of \( Y \) satisfies

\[
L(X, \pi) \leq a_Y \leq U(X, \pi).
\]

Proof. The upper bound follows immediately from Theorem 5.1. \( H^0(Y, \ker V_Y) = H^0(X, \mathcal{G}_{-1}) \subset H^0(X, \mathcal{G}_p) \), each of the linearly independent \( v_j \in H^0(X, \mathcal{G}_p) \) does not lie in the kernel of some \( g_i' \), and

\[
\dim H^0(X, \mathcal{G}_p) \leq \sum_{i=0}^{p-1} \dim H^0(X, \ker V_X(F_* (E_i + pD_i))).
\]

The lower bound is slightly more involved. The key idea is to use the rank-nullity theorem to give a lower bound on the space of \( v \in V_j = \ker \left( (\tau - 1)^{j+1} : H^0(X, \mathcal{G}_{j+1}) \to H^0(X, \mathcal{G}_{j+1}) \right) \) such that \( g_i'(v) = 0 \) for some \(-1 \leq i \leq j \). Lemma 5.11 gives an upper bound on the dimension of \( g_{Q,i}(V_j) \) for \( Q \in S \) involving \( c(i, j, Q) \). The image of \( g_{Q,i} \) for \( Q \in \sup(D_i) \) is at most \( (p-1) \)-dimensional by definition. The image \( g_{-1}'(V_j) \) is at most \( \sum_{i=0}^{j} c(i, j, Q') \)-dimensional by Remark 5.16. Putting these together using the exact sequences in Theorem 5.1 gives the lower bound. \( \square \)
6.2. **Tools.** The quantities \( L(X, \pi) \) and \( U(X, \pi) \) are quite abstract. We now give some tools for estimating \( L(x, \pi) \) and \( U(X, \pi) \) in terms of the ramification data of \( \pi : Y \to X \) and the genus and a-number of \( X \).

The key is a theorem of Tango which allows us to compute the dimension of the kernel of the Cartier operator on global sections in favorable situations. Let \( \sigma : k \to k \) denote the \( p \)-power Frobenius automorphism of \( k \), and let \( C \) be a smooth, projective, and connected curve over \( k \) of genus \( g_C \). Attached to \( C \) is its **Tango number**:

\[
 n(C) := \max \left\{ \frac{\text{ord}_x(df)}{p} : f \in k(C) \setminus k(C)^p \right\},
\]

where \( k(C) \) is the function field of \( C \). In Lemma 10 and Proposition 14 of [Tan72], one sees that \( n(C) \) is well-defined and is an integer satisfying \(-1 \leq n(C) \leq \lfloor (2g - 2)/p \rfloor \), with the lower bound an equality if and only if \( g = 0 \).

**Fact 6.5** (Tango’s Theorem). Let \( \mathcal{L} \) be a line bundle on \( C \). If \( \deg \mathcal{L} > n(C) \) then the natural \( \sigma \)-linear map

\[
 F^* : H^1(C, \mathcal{L}^{-1}) \longrightarrow H^1(C, \mathcal{L}^{-p})
\]

induced by pullback by the absolute Frobenius of \( C \) is injective, and the \( \sigma^{-1} \)-linear Cartier operator

\[
 V_C : H^0(C, \Omega^1_{C/k} \otimes \mathcal{L}^p) \longrightarrow H^0(C, \Omega^1_{C/k} \otimes \mathcal{L})
\]

is surjective.

**Remark 6.6.** This is [Tan72, Theorem 15]; strictly speaking, Tango requires \( g > 0 \); however, by tracing through Tango’s argument—or by direct calculation—one sees easily that the result holds when \( g = 0 \) as well.

To simplify notation, let \( \delta(H^0(X, \Omega^1_X(E))) \) denote the dimension of the kernel of \( V_X \) on that space of differentials. Tango’s theorem tells us the following:

**Corollary 6.7.** Let \( D, R \) be divisors on \( X \) with \( R = \sum r_iP_i \) where \( 0 \leq r_i < p \). If \( \deg(D) > \max(n(X), 0) \), then

\[
 \delta(H^0(X, \Omega^1_X(pD + R))) = (p - 1) \deg(D) + \sum r_i - \left\lfloor \frac{r_i}{p} \right\rfloor.
\]

Otherwise we know that

\[
 0 \leq \delta(H^0(X, \Omega^1_X(pD + R))) - \left( (p - 1) \deg(D) + \sum r_i - \left\lfloor \frac{r_i}{p} \right\rfloor \right) \leq a_X.
\]

**Proof.** When \( R = 0 \), the first case follows from the surjectivity of the Cartier operator in Fact 6.5 taking \( \mathcal{L} = \mathcal{O}_X(D) \), plus the fact that

\[
 \dim H^0(X, \Omega^1_X(D)) = g - 1 + \deg(D) \quad \text{and} \quad \dim H^0(X, \Omega^1_X(pD)) = (g - 1) + p \deg(D)
\]

from the Riemann-Roch theorem.

We can build on this to prove the remaining cases of the first statement and establish the inequality. We know that for any divisor \( E \) with \( \deg(E) \geq 0 \),

\[
 \dim H^0(X, \Omega^1_X(E + [P])) = \dim H^0(X, \Omega^1_X(E)) + 1.
\]

with equality whenever \( \deg(E) > 0 \). Thus we know that

\[
 0 \leq \delta(H^0(X, \Omega^1_X(E + [P]))) - \delta(\dim H^0(X, \Omega^1_X(E))) \leq 1.
\]
Furthermore, if $p \mid \text{ord}_P(E)$ then this difference is 0, as a differential in $H^0(X, \Omega_X^1(E + [P]))$ not in $H^0(X, \Omega_X^1(E))$ must have a non-zero $t_p^{-\text{ord}_P(E)} \frac{dt}{t_p}$ term in the stalk at $P$, which forces the differential to not lie in the kernel of the Cartier operator.

When $\deg(D) > \max(n(x), 0)$ and $R \neq 0$, we proceed by induction on the number of points in the support of $R$. Assume the result for $R$, and pick another point $P$ (not in the support of $R$) and $0 < r < p$. Then

$$\delta(H^0(X, \Omega_X^1(pD + R))) \leq \delta(H^0(X, \Omega_X^1(pD + R + [P]))) \leq \ldots \leq \delta(H^0(X, \Omega_X^1(pD + [P] + R)))$$

and by the inductive hypothesis the last is $p - 1$ more than the first. This means that at each step after the first, the dimension of the kernel must increase by one. This shows that $\delta(H^0(X, \Omega_X^1(pD + R + r[P]))) = \delta(H^0(X, \Omega_X^1(pD + R))) + (r - 1)$, which completes the induction.

The second statement follows from the same type of reasoning. When passing from $a_X = \delta(H^0(X, \Omega_X^1(pD + R)))$ to $\delta(H^0(X, \Omega_X^1(pD + R)))$,

there are $(p - 1) \deg(D) + \sum(r_i - \lfloor r_i/p \rfloor)$ times the dimension might increase by one. When passing from $\delta(H^0(X, \Omega_X^1(pD + R)))$ to $\delta(H^0(X, \Omega_X^1(pD'))) = (p - 1) \deg(D')$

with $D'$ chosen so that $pD' \geq pD + R$ and $D' > n(X)$, there are $(p - 1) \deg(D' - D) + \sum(p - 1) - r_i)$ chances for the dimension to increase by one. This completes the proof. \hfill \square

**Remark 6.8.** Choosing $D' \geq D$ with $\deg(D') > n(X)$ and $pD' \geq pD + R$, we also obtain the bound $\delta(H^0(X, \Omega_X^1(pD + R))) \leq (p - 1) \deg D'$ from the inclusion $H^0(X, \Omega_X^1(pD + R)) \subset H^0(X, \Omega_X^1(pD'))$ and Tango’s theorem.

Fix a divisor $E = pD + R = \sum_j a_jQ_j$ with $R = \sum r_jQ_j$ and $0 \leq r_j < p$. For fixed $i$ and $1 \leq n \leq a_i$,

$$\delta(H^0(X, \Omega_X^1(\sum_{j < i} a_jQ_j + (n - 1)Q_i))) \leq \delta(H^0(X, \Omega_X^1(\sum_{j < i} a_jQ_j + nQ_i))).$$

If the dimension increases by one, let $\omega_{Q_i,n}$ be a differential in the larger space not in the smaller space. This differential satisfies the following properties:

1. $V_X(\omega_{Q_i,n}) = 0$;
2. $\text{ord}_{Q_i}(\omega_{Q_i,n}) \geq -a_j$ for $j < i$;
3. $\text{ord}_{Q_i}(\omega_{Q_i,n}) = -n$;
4. $\text{ord}_{Q_i}(\omega_{Q_i,n}) \geq 0$ for $j > i$.

Note that such a differential never exists if $n \equiv 1 \pmod p$.

**Corollary 6.9.** The differentials $\omega_{Q_i,n}$ are linearly independent. There are at least $(p - 1) \deg D + \sum(r_j - \lfloor r_j/p \rfloor) - a_X$ of them.

**Proof.** This follows from the proof of the previous Corollary. \hfill \square

**Remark 6.10.** We will use Corollary 6.7 to obtain a formula for $\dim V_k$ using Lemma 6.1 provided that the auxiliary divisors $D_i$ are sufficiently large. This will allow us to compute $L(X, \pi)$.

We can also use Corollary 6.9 to construct independent forms to give a lower bound on $M(X, \pi)$, and hence an upper bound on $U(X, \pi)$. The key is the following:

**Proposition 6.11.** Consider the set $T$ of triples $(Q, n, j)$ where:

- $0 \leq j \leq p - 1$ and $Q \in S_j$;
- $0 < n \leq \text{ord}_Q(E_j + pD_j)$;
- $n \equiv 1 \pmod p$;
- if $Q \in S$, there exists an integer $m$ with $0 \leq m \leq j$ such that $m \equiv j + (n - 1)d_Q^{-1} \pmod p$. 

Then $M(X, \pi) \geq \#T - M$, where $M$ is the number of triples where the form $\omega_{Q,n}$ does not exist.

Proof. For a triple $(Q, n, j) \in T$ such that $\omega_{Q,n}$ exists, we consider the element

$$v_{Q,n,j} = (0, \ldots, 0, \omega_{Q,n}, 0, \ldots, 0) \in H^0(\mathcal{G}_{j+1}),$$

where the element $\omega_{Q,n}$ from Corollary 6.9 occurs in the $j$-th component. When we write $\sum_i \omega_i y^i = \varphi^{-1}_y(v_{Q,n,j})$ we see that $\omega_i = 0$ for $i > j$ and $\omega_j = \omega_{Q,n}$. The conditions on $T$ will guarantee that $g_{Q,m}(v_{Q,n,j}) \neq 0$ for some $-1 \leq m \leq j$. These elements are certainly linearly independent, so the number of triples in $T$ for which $\omega_{Q,n}$ exists is a lower bound on $M(X, \pi)$.

First suppose that $Q \in S$ and $m \equiv j + (n-1)d_Q^{-1} \pmod{p}$. If $g'_Q(v_{Q,n,j}) = 0$ for $i > m$, we will show that $g'_{Q,m}(v_{Q,n,j}) \neq 0$. Looking at the definition of $g'_{Q,m}$ in (5.2) to compute $g'_{Q,m}(v_{Q,n,j})$ we work in the stalk at $Q$ and write

$$- \sum_{i=m+1}^j \binom{i}{m} \omega_i (-f)^{i-m} = \sum_{i=-N+1}^{\infty} a_i t_Q^i dt_Q.$$

$g'_{Q,m}$ records the $a_i$ for which $p|i$ and $i < -n_Q,m$. We know that $\text{ord}_Q(\omega_i) \geq -n_Q,i$ for $i > m$ since $g'_Q(v_{Q,n,j}) = 0$ for $i > m$, and that $\text{ord}_Q(f) = -d_Q$. One way to obtain the $t_Q^{1-(n-(j-m))d_Q} dt_Q$ term in the sum is to multiply the $t_Q^{-n} dt_Q$ term in $\omega_j$ with the $t_Q^{-d_Q(j-m)}$ term in $f^{j-m}$. Both terms are non-zero. There is no other way to obtain this term: for $i < j$ we see that

$$\text{ord}_Q(\omega_i (-f)^{i-m}) \geq -n_Q,i - (i-m)d_Q \geq -(j-m)d_Q > -n - (j-m)d_Q$$

since $n > 0$ and $n_Q,i \leq d_Q$. Thus $a_{1-n-(j-m)d_Q} \neq 0$. Since $1 - n - (j-m)d_Q \equiv 0 \pmod{p}$ by assumption, this is one of the terms $g'_{Q,m}$ records, so $g'_{Q,m}(v_{Q,n,j}) \neq 0$.

Secondly, suppose that $Q \in \text{sup}(D_j)$. Then $\text{ord}_Q(E_j + pD_j) = p$, and we assumed that $n \neq 1$. Looking at the definition of $g'_{Q,m}$ in (5.3) we see that $\omega_j = v_{Q,n,j} = \omega_{Q,n}$ and that $g'_{Q,m}$ extracts the coefficients of $t_Q^{-p} dt_Q$ through $t_Q^{-2} dt_Q$. Thus $g'_{Q,m}(v_{Q,n,j}) \neq 0$.

Example 6.12. When $X = \mathbb{P}^1$ and $Q = [\alpha]$ for $\alpha \in k$, then $\omega_{Q,n}$ can be taken to be $(t - \alpha)^{-n} dt$.

Remark 6.13. Using Corollary 6.7 and the observation that at most $a_X$ of the $\omega_{Q,n}$ do not exist, we will use Proposition 6.11 to bound $U(X, \pi)$, thereby obtaining bounds on $a_Y$. A precise statement will be given in Theorem 6.15.

Remark 6.14. Let $Q \in S$. This argument uses very limited information about the coefficients of $f$ in $O_{X,Q}$. In particular, it only uses that $f$ has a non-zero $t_Q^{-d_Q}$ term. If there were additional non-zero coefficients, it could be significantly “easier” to arrange for one of the $g'_{Q,m}$’s to be nonzero, allowing a relaxation of the congruence condition in the definition of $T$ and giving a sharper bound. However, Example 7.3 shows it is not possible to do better in general.

6.3. Bounds. Throughout this section, we fix an Artin Schreier cover $\pi : Y \to X$ with branch locus $S \subseteq X(\bar{k})$, and denote by $d_Q$ the unique break in the ramification filtration at the unique point of $Y$ above $Q$. For nonnegative integers $d, i$ with $p \nmid d$ let $\tau_p(d, i)$ be the number of positive integers $n \leq \lfloor id/p \rfloor$ with the property that $-n \equiv md \pmod{p}$ for some $m$ with $0 < m \leq p - 1 - i$.

Theorem 6.15. With notation as above,

$$a_Y \leq p a_X + \sum_{Q \in S} \sum_{i=1}^{p-1} \left( \left\lfloor \frac{id}{p} \right\rfloor - \left\lfloor \frac{id}{p^2} \right\rfloor - \tau_p(d_Q, i) \right)$$

(6.4)
and for any $j$ with $1 \leq j \leq p - 1$

\begin{equation}
(6.5) \quad a_Y \geq \sum_{Q \in S} \sum_{i=j}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{id_Q}{p} \right\rfloor \right).
\end{equation}

Moreover, for any nonnegative integers $d, i$ with $p \nmid d$ and $i \leq p - 1$,

\begin{equation}
(6.6) \quad \tau_p(d, i) \geq (p - 1 - i) \left\lfloor \frac{1}{p} \left\lfloor \frac{id_Q}{p} \right\rfloor \right\rfloor \geq (p - 1 - i) \left\lfloor \frac{id_Q}{p^2} \right\rfloor.
\end{equation}

**Proof.** Let $n(X)$ be the Tango number of $X$, and for $0 \leq i \leq p - 1$ let $D_i$ be as in (6.5). Adding in more points to each $D_i$ as needed, we may without loss of generality assume that $\deg(D_i) > n(X)$ for all $i$. Using this assumption, we will analyze the lower and upper bounds from Proposition 6.4 separately. As usual, for $0 \leq i \leq p - 1$ we set $n_{Q,i} : = [(p - 1 - i)d_Q/p]$, and define

$$A_i := \sum_{Q \in S_i} \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor [Q], \quad R_i := E_i - pA_i = \sum_{Q \in S} r(Q, i)[Q].$$

Note that $R_i$ is supported on $S$, as $n_{Q,i}$ is a multiple of $p$ if $Q \in \operatorname{sup}(D_i) \cup \{Q'\}$. Let $s(Q, i) = r(Q, i) - 1$ if $r(Q, i) > 0$, and 0 otherwise.

We will first establish (6.4). By proposition 6.4 we have

\begin{equation}
(6.7) \quad a_Y \leq \sum_{i=0}^{p-1} \dim H^0(X, \ker V_X(F_*(E_i + pD_i))) - M(X, \pi).
\end{equation}

We will understand the first sum using Corollary 6.7, and $M(X, \pi)$ using Proposition 6.11, treating the $i = p - 1$ case separately.

Suppose $i < p - 1$. As $\deg(D_i) > n(X)$, the first part of Corollary 6.7 implies that

$$\dim H^0(X, \ker V_X(F_*(E_i + pD_i))) = \dim H^0(X, \ker V_X(F_*(pA_i + D_i + R_i))) = (p - 1) \deg(A_i + D_i) + \sum_{Q \in S} s(Q, i)$$

Thus we see that

\begin{equation}
(6.8) \quad \dim H^0(X, \ker V_X(F_*(E_i + pD_i))) = (p - 1) \left( \sum_{Q \in S} \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor + \frac{d_Q}{p} (p - 1 - i) + \# \operatorname{sup}(D_i) \right) + \sum_{Q \in S} s(Q, i).
\end{equation}

We also want to count elements of the form $(Q, n, i)$ in the set $T$ of Proposition 6.11 for fixed $Q$ and $i$. If $Q \notin \operatorname{sup}(D_i)$, then there are $p - 1$ such element by definition since $\operatorname{ord}_Q(E_i + pE_i) = p$. If $Q = Q'$, there are $(p - 1)(p - 1 - i)d_Q/p$ such elements since $\operatorname{ord}_Q(E_i) = (p - 1 - i)d_Q$. When $Q \in S$, the number of elements of the form $(Q, n, i)$ in $T$ is precisely $\tau_p(d_Q, p - 1 - i)$ since the set of positive integers $n \leq n_{Q,i}$ with the property that $i + (n - 1)d_Q^{-1} \equiv m \mod p$ for some $m$ with $0 \leq m < i$ is in bijection with the set of positive integers $n' \leq \lfloor (p - 1 - i)d_Q/p \rfloor$ with $-n' \equiv m'd_Q \mod p$ for some $m'$ with $0 < m' \leq i$ via $n':= n - 1$ and $m':= i - m$. Putting this together, we find that there are exactly

\begin{equation}
(6.9) \quad \sum_{Q \in S} \tau_p(d_Q, p - 1 - i) + (p - 1) \# \operatorname{sup}(D_i) + (p - 1)(p - 1 - i)d_Q/p
\end{equation}

elements of $T$ of the form $(Q, n, i)$. 

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The case \( i = p - 1 \) is special: observe that \( E_{p-1} = 0 \), so \( \dim H^0(X, \Omega^1_X(F_*(E_i + pD_i))) = \dim H^0(X, \Omega^1_X(D_i)) \). Then consider the exact sequence
\[
0 \to H^0(X, \mathcal{G}_{p-1}) \to H^0(X, \mathcal{G}_p) \to H^0(M_{Q,p-1}) \oplus \bigoplus_{Q \in \sup D_{p-1}} H^0(M_{Q,p-1}).
\]
The tuple \((v_0, \ldots, v_{p-1})\) lies in the kernel of the right map if and only if the corresponding form \( \omega = \sum_i \omega_i y^i = \varphi^{-1}_n(v_0, \ldots, v_{p-1}) \) satisfies \( V_{\mathcal{G}}(\omega_{p-1}) \in H^0(X, \Omega^1_X(E_{p-1})) = H^0(X, \Omega^1_X) \). But we know that \( V_X(\omega_{p-1}) \) must be zero (Corollary 4.4), so \( \omega_{p-1} = v_{p-1} \). Thus there are
\[
\dim H^0(X, \ker V_X(D_{p-1})) = \dim H^0(X, \ker V_X).
\]
linearly independent elements of \( H^0(X, \mathcal{G}_p) \) not in the kernel of \( g'_{p-1} \).

Putting (6.7) together with (6.8) and the bound on \( M(X, \pi) \) coming from (6.9), Proposition 6.11 and the observation that at most \( a_X \) of the \( \omega_{Q,n} \) do not exist (and the special analysis when \( i = p-1 \)), we see that
\[
(6.10) \quad a_Y \leq p \cdot a_X + \sum_{i=0}^{p-2} (p-1) \left\lfloor \frac{nQ,i}{p} \right\rfloor + s(Q,i) - \tau_p(d_Q,p-1-i).
\]
Using the very definition of \( s(Q,i) \), one finds the formula
\[
(6.11) \quad s(Q,i) = nQ,i - p \cdot \left( \left\lfloor \frac{nQ,i}{p} \right\rfloor - \left[ \frac{nQ,i}{p} \right] \right) = nQ,i - \left\lfloor \frac{nQ,i}{p} \right\rfloor - (p-1) \left\lfloor \frac{nQ,i}{p} \right\rfloor.
\]
Substituting (6.11) into (6.10), re-indexing the sum \( i \mapsto p-1-i \) and using the equality \( \left\lfloor x \right\rfloor = [x] + 1 \) for \( x \notin \mathbb{Z} \) gives (6.4). To make this bound more explicit, we must bound \( \tau_p(d,i) \) from below. To do this, simply note that for any \( 0 < m \leq p-1-i \) and any interval of length \( p \), there is a unique \( n \) in that interval such that
\[
md \equiv -n \pmod{p}
\]
and that necessarily \( n \equiv 0 \pmod{p} \) as \( p \nmid d \). Thus,
\[
(6.12) \quad \tau_p(d_Q,i) \geq (p-1-i) \cdot \frac{1}{p} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor + 1 \right) = (p-1-i) \cdot \frac{1}{p} \left[ \frac{id_Q}{p} \right] \geq (p-1-i) \left[ \frac{id_Q}{p^2} \right],
\]
as \( 0 \leq n \leq \left[ \frac{id_Q}{p} \right] - 1 \).

We now turn to the lower bound (6.5). Proposition 6.4 gives that for each \( 0 \leq j < p-1 \),
\[
(6.13) \quad \dim V_j - \sum_{i=0}^{j} \sum_{Q \in S} c(i,j,Q) \leq a_Y.
\]
Using the definition, Lemma 6.1, Corollary 6.7 and the assumption that \( \deg D_i > n(X) \) for all \( i \) we calculate that
\[
\dim V_j = \sum_{i=0}^{j} \dim H^0(X, \Omega^1_X(F_*(E_i + pD_i))) = \sum_{i=0}^{j} \left( (p-1) \deg(D_i) + (p-1)(p-1-i) \frac{d_{Q,i}}{p} + \sum_{Q \in S} \frac{nQ,i}{p} + s(Q,i) \right).
\]
On the other hand, if \( Q \in \sup D_i \) then \( c(i,j,Q) = p-1 \) and if \( Q = Q' \) then \( c(i,j,Q') = (p-1)(p-1-i) \frac{d_{Q,i}}{p} \). If \( Q \in S \), then \( c(i,j,Q) \) is the number of integers \( n \) congruent to \(-1 \pmod{p} \) such that \(-nQ,j - d_Q(j-i) \leq n < -nQ,i \). We may express this last case by
\[
c(i,j,Q) = \left\lfloor \frac{nQ,j + d_Q(j-i)}{p} \right\rfloor - \left\lfloor \frac{nQ,i}{p} \right\rfloor.
\]
Thus in (6.13), the contributions from $D_i$ and $Q'$ cancel, so using the formula (6.11) for $s(Q,i)$ we obtain that

$$\sum_{i=0}^{j} \sum_{Q \in S} \left( n_{Q,i} - \left\lceil \frac{n_{Q,i} + d_Q(j-i)}{p} \right\rceil \right) \leq a_Y,$$

for all $0 \leq j < p - 1$. Using the equality

$$\left\lceil \frac{n_{Q,i} + (j-i)d_Q}{p} \right\rceil = \left\lceil \frac{(p-1-i)d_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{(p-1-j)d_Q}{p} \right\rceil,$$

changing variables $i \mapsto p - 1 - i$ and $j \mapsto p - 1 - j$, and employing again the equality $\lceil x \rceil = \lfloor x \rfloor + 1$ for $x \not\in \mathbb{Z}$ yields (6.5).

**Corollary 6.16.** Suppose $p$ is odd. With the same hypotheses of Theorem 6.15, we have

$$a_Y \leq p \cdot a_X + \left( \frac{(p-1)(p-2)}{2} + \left(1 - \frac{1}{p}\right)^2 \right) \cdot \#S + \left(1 - \frac{1}{p}\right) \sum_{Q \in S} \frac{(2p-1)}{6} d_Q$$

and

$$a_Y \geq \left(1 - \frac{1}{p}\right)^2 \left( \sum_{Q \in S} \frac{(p+1)}{4} d_Q - \#S \cdot \frac{p}{2} \right).$$

**Proof.** The upper bound follows easily from (6.4) and the first inequality in (6.6) by basic properties of the floor function and the well-known equality

$$\sum_{i=1}^{n-1} \left\lfloor \frac{id}{n} \right\rfloor = \frac{(n-1)(d-1)}{2}$$

for any positive and co-prime integers $d,n$. For the lower bound, we have

$$\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id}{p} - \left(1 - \frac{1}{p}\right) jd_Q \right\rfloor \geq \frac{id_Q - (p-1)}{p} - \left(1 - \frac{1}{p}\right) \frac{jd_Q}{p}$$

$$= \left(1 - \frac{1}{p}\right) \left( \frac{jd_Q}{p} - 1 \right).$$

Summing over $i$ with $j \leq i \leq p - 1$ and then $Q$ and using (6.5) gives the lower bound

$$\left(1 - \frac{1}{p}\right) \sum_{Q \in S} \left( \frac{d_Q}{p} (p-j)j - (p-j) \right)$$

which holds for all $1 \leq j \leq p - 1$; we then take $j = \frac{p+1}{2}$. Remark 6.17 motivates this choice.

**Remark 6.17.** The choice of $j = \frac{p+1}{2}$ in the proof of Corollary 6.16 is optimized for $\sum d_Q$ large relative to $\#S$. Indeed, (6.15) is a quadratic function in $j$ which attains its maximum when

$$j \approx \frac{p}{2} \left(1 - \frac{\#S}{\sum_{Q \in S} d_Q} \right)$$

Any nearby value of $j$ will give a similar bound. Supposing that $\sum d_Q$ is large relative to $\#S$ gives optimal choice $j = \lceil p/2 \rceil = (p+1)/2$. When all $d_Q$ are small and $p$ is large, one can get a better explicit lower bound by choosing a value of $j$ in accordance with (6.16); cf. Example 7.1.

**Remark 6.18.** For fixed $X$, $p$ and $S$ with $d_Q$ becoming large, the dominant terms of the lower and upper bounds in Corollary 6.16 are respectively

$$\sum_{Q \in S} d_Q \frac{p}{4} \quad \text{and} \quad \sum_{Q \in S} d_Q \frac{p}{3}$$
On the other hand, the dominant term in the Riemann–Hurwitz formula for the genus of $Y$ is $\sum_{Q \in S} d_Q p/2$, so that, for large $d_Q$, the $a$-number is approximately between $1/2$ and $2/3$ of the genus of $Y$.

**Remark 6.19.** When $p = 2$, the statement and proof of the Corollary do not work as written. If we take $j = 1$ in Theorem 6.15 we obtain

$$\sum_{Q \in S} \left\lfloor \frac{d_Q}{2} \right\rfloor - \left\lfloor \frac{d_Q}{4} \right\rfloor \leq a_Y \leq 2a_X + \sum_{Q \in S} \left\lfloor \frac{d_Q}{2} \right\rfloor - \left\lfloor \frac{d_Q}{4} \right\rfloor.$$  

In particular, when $X$ is ordinary (i.e. $a_X = 0$) we obtain an exact formula for $a_Y$. This recovers [Vol88 Theorem 2] (note that the formula there is for the rank of the Cartier operator, and that for $Q_i \in S$, our $d_{Q_i}$ is $2n_i - 1$).

Similarly, Corollary 6.21 will give an exact formula for $a_Y$ when $p$ is odd, $X$ is ordinary, and $d_Q | (p - 1)$. To derive it, we will need to investigate situations when it is possible to derive an exact formula for the quantity

$$\tau_p(d) := \sum_{i=0}^{p-1} \tau_p(d, i)$$

occurring in the upper bound (6.4):

**Proposition 6.20.** Let $p > 2$ and suppose that $d \equiv d' \mod p^2$. Then

$$\tau_p(d) = \tau_p(d') + (d - d')(p - 1)(p - 2)/6p.$$  

Moreover, $\tau_p(1) = 0$ and for $1 < d < p$ we have

$$\tau_p(d) = u_p(d) \cdot \frac{p}{d} + v_p(d)$$

where $u_p(d)$ and $v_p(d)$ are the integers (depending only on $p \mod d$) given by

$$u_p(d) := \sum_{j=0}^{b-1} \sum_{k \in S_j} ((j + 1)d - (b + 1)k) \quad v_p(d) := \sum_{j=0}^{b-1} \sum_{k \in S_j} \left( \frac{k}{d} \left\lfloor \frac{a(d - k)}{d} \right\rfloor \right)$$

where $a := p \mod d$ and $b := a^{-1} \mod d$ with $0 < a, b < d$ and $S_j := \{ k \in \mathbb{Z} : jd/b < k < (j + 1)d/(b + 1) \}$.

In particular, for $1 < d < p$ the quantity $\tau_p(d)$ depends only on $d$ and $p \mod d$.

Before proving Proposition 6.20 let us give some indication of its utility. For example, if $p \equiv 1 \mod d$, we have $a = b = 1$, whence

$$u_p(d) = \sum_{k=1}^{\lfloor d/2 \rfloor} (d - 2k) = \left\lfloor \frac{d}{2} \right\rfloor \cdot \left\lfloor \frac{d - 1}{2} \right\rfloor = \left\lfloor \frac{(d - 1)^2}{4} \right\rfloor$$

and

$$v_p(d) = \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{k}{d} \cdot \frac{d - k}{d} = \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{2k - d}{d} = -\left\lfloor \frac{(d - 1)^2}{4} \right\rfloor.$$  

Thus,

$$\tau_p(d) = \left\lfloor \frac{(d - 1)^2}{4} \right\rfloor \cdot \frac{p - 1}{d}.$$
whenever \( d < p \) and \( p \equiv 1 \mod d \). Another simple example is when \( p \equiv -1 \mod d \), so that \( a = b = d - 1 \). In this case, \( S_j := \{ k \in \mathbb{Z} : jd/(d - 1) < k < (j + 1) \} \) is the empty set for all \( j \), whence \( \tau_p(d) = 0 \).

**Proof.** The first assertion follows immediately from the fact that, as observed immediately prior to (6.12), for any \( 0 < m \leq p - 1 - i \) and any interval of length \( p \), there is a unique \( n \) in that interval such that

\[
md \equiv -n \pmod{p}
\]

and necessarily \( n \not\equiv 0 \pmod{p} \). So suppose that \( d < p \), and for integers \( m, i \), define

\[
\chi(m,i) := \begin{cases} 
1 & \text{if } m \leq i \\
0 & \text{otherwise}
\end{cases}
\]

We will use the convention that “\( x \mod y \)” denotes the unique integer \( x_0 \) with \( 0 \leq x_0 < y \) and \( x \equiv x_0 \mod y \), and we set \( a := p \mod d \) and \( b := a^{-1} \mod d \). By definition, we have

\[
\tau_p(d) = \sum_{i=0}^{p-1} \sum_{0 < j \leq n_i(d) \pmod{p}} \chi(-jd^{-1} \mod p, p - 1 - i),
\]

where \( n_i(d) := \lfloor id/p \rfloor \). For an integer \( k \), observe that \( k \leq n_i(d) < d \) if and only if \( i > kp/d \), and that \(-jd^{-1} \equiv j(bp - 1)/d \pmod{p}\) (note that \( d | (bp - 1) \)). Taking this into account, we swap the order of summation in (6.22), collecting all terms with \( j = k \) together

\[
\sum_{k=1}^{d-1} \sum_{i=\lfloor kp/d \rfloor}^{p-1} \chi(kbp - 1 \mod d \pmod{p}, p - 1 - i),
\]

Since \( k < d < p \) and \( kb \equiv kp^{-1} \not\equiv 0 \mod d \), we have

\[
kbp - 1 \mod d = (kb \mod d) \cdot p - k \pmod{d}
\]

and moreover \( kb \mod d = kb - \ell d \) for \( \ell d/b \leq k < (\ell + 1)d/b \). Breaking the sum over \( k \) up into these regions converts (6.23) into

\[
\sum_{\ell=0}^{b-1} \sum_{\ell d/b < k < (\ell + 1)d/b} \sum_{i=\lfloor kp/d \rfloor}^{p-1} \chi(kbp - 1 \mod d, p - 1 - i),
\]

The innermost sum has value the cardinality of the subset of positive integers

\[
S(k, \ell) := \left\{ i \in \mathbb{N} : \frac{(kb - \ell d)p - k}{d} \leq p - 1 - i \leq p - 1 - \left\lfloor \frac{kp}{d} \right\rfloor = \left\lfloor \frac{(d - k)p}{d} \right\rfloor - 1 \right\}.
\]

Now \( S(k, \ell) \) empty unless

\[
\frac{(d - k)p}{d} - 1 < \frac{(kb - \ell d)p - k}{d},
\]

in which case

\[
\#S(k, \ell) := \left\lfloor \frac{(d - k)p}{d} \right\rfloor - \frac{(kb - \ell d)p - k}{d} = \frac{(d - k)(p - a)}{d} + \frac{a(d - k)}{d} - \frac{(kb - \ell d)p - k}{d}
\]

\[
= ((\ell + 1)d - (b + 1)k) \frac{p}{d} + \left( k - \left\{ \frac{a(d - k)}{d} \right\} \right)
\]

The inequality (6.25) is equivalent to \(((\ell + 1)d - (b + 1)k)p > (d - k)\) which, as \( p > d > d - k \) is equivalent to \((\ell + 1)d > (b + 1)k\), or what is the same thing, \( k < (\ell + 1)d/(b + 1) \). In other words, the contribution from the innermost sum (6.23) for \( k \) in the ranges \((\ell + 1)d/(b + 1) < k < (\ell + 1)d/b\)
is zero so that these values of \( k \) in the middle sum may be omitted, and the expression (6.26) then substituted for the innermost sum, which yields the claimed formula for \( \tau_p(d) \) when \( d < p \). \[\square\]

Using the bounds of Theorem 6.15 we are able to generalize the main result of [FP13], which provides an \( a \)-number formula for branched \( \mathbb{Z}/p\mathbb{Z} \)-covers of \( X = \mathbb{P}^1 \) with all ramification breaks \( d \) dividing \( p - 1 \), to the case of arbitrary ordinary base curves \( X \):

**Corollary 6.21.** Let \( \pi : Y \to X \) be a branched \( \mathbb{Z}/p\mathbb{Z} \)-cover with \( a_X = 0 \), and suppose \( p \) is odd. If \( d_Q \) divides \( p - 1 \) for every branch point \( Q \), then

\[
(6.27) \quad a_Y = \sum_Q a_Q \quad \text{where} \quad a_Q := \left( \frac{p-1}{2} \right) (d_Q - 1) - \frac{p-1}{d_Q} \left( \frac{(d_Q - 1)^2}{4} \right).
\]

**Proof.** We will compute the upper and lower bounds for \( a_Y \) given by (6.4) and (6.5), and show that these bounds coincide when \( d_Q \mid (p-1) \) for all \( Q \) and \( a_X = 0 \), and agree with the stated formula. Using the hypothesis \( a_X = 0 \) together with the explicit formula for \( \tau_p(d) \) when \( d \equiv 1 \mod p \) provided by (6.21), the upper bound (6.4) becomes

\[
(6.28) \quad a_Y \leq \sum_{Q \in S} \sum_{i=1}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right) - \sum_{Q \in S} \frac{p-1}{d_Q} \left( \frac{(d_Q - 1)^2}{4} \right).
\]

Now \( d_Q < p \) for all \( Q \), so \( \left\lfloor \frac{id_Q}{p} \right\rfloor = 0 \) for \( 1 \leq i \leq p-1 \). Using (6.14), we conclude that

\[
(6.29) \quad \sum_{Q \in S} \sum_{i=1}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right) = \frac{(p-1)}{2} (d_Q - 1).
\]

Combining (6.28) and (6.29) gives the formula (6.27) as the upper bound of \( a_Y \). It remains to prove that \( a_Y \) is also bounded below by the same quantity.

Set \( j := (p+1)/2 \), and write \( d_Q \cdot k_Q = p - 1 \). For \( i < p \), we see that

\[
\left\lfloor \frac{id_Q}{p} \right\rfloor = \begin{cases} \left\lfloor \frac{i}{k_Q} \right\rfloor & \text{when } k_Q \nmid i \\ \left\lfloor \frac{i}{k_Q} \right\rfloor - 1 & \text{when } k_Q \mid i. \end{cases}
\]

Likewise we can check that

\[
\left\lfloor \frac{id_Q}{p} \right\rfloor - \left( 1 - \frac{1}{p} \right) \left\lfloor \frac{jd_Q}{p} \right\rfloor = \left\lfloor \frac{i-j}{k_Q} \right\rfloor.
\]

The inner sum from (6.5) becomes

\[
\sum_{i=(p+1)/2}^{p-1} \left\lfloor \frac{i}{k_Q} \right\rfloor - \sum_{i=0}^{(p-3)/2} \left\lfloor \frac{i}{k_Q} \right\rfloor - \# \left\{ (p+1)/2 \leq i \leq p-1 : k_Q \mid i \right\}.
\]

The term in the first sum can be rewritten as \( \left\lfloor d_Q/2 + (i - (p-1)/2) \right\rfloor \). When \( d_Q \) is even, \( d_Q/2 \) can be removed from the floor function, allowing cancellation with the second sum giving that the inner sum from (6.5) is

\[
\frac{(p-1)d_Q}{4}.
\]

This equals \( a_Q \) when \( d_Q \) is even. When \( d_Q \) is odd, a similar argument removing \( \left\lfloor d_Q/2 \right\rfloor = \frac{d_Q-1}{2} \) from the first sum shows that the lower bound is

\[
\left\lfloor d_Q/2 \right\rfloor \cdot \frac{p-1}{2} + k_Q \cdot \frac{d_Q}{2} - k_Q/2 = k_Q \cdot \frac{(d_Q+1)(d_Q-1)}{4}.
\]

Again, this matches the formula for \( a_Q \). Summing over \( Q \) completes the proof. \[\square\]
\[ d_Q = \begin{array}{cccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 14 & 15 & 32 & 128 & 1024 \\
\text{Trivial Lower} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 12 & 27 & 109 \\
L(P^1, \pi) & 0 & 6 & 8 & 12 & 15 & 18 & 21 & 24 & 26 & 30 & 33 & 36 & 42 & 45 & 96 & 382 & 3054 \\
U(P^1, \pi) & 0 & 6 & 8 & 12 & 16 & 18 & 36 & 30 & 34 & 36 & 38 & 36 & 78 & 60 & 120 & 488 & 3936 \\
\text{Trivial Upper} & 0 & 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 & 60 & 66 & 78 & 84 & 186 & 762 & 6138 \\
\end{array} \\
\text{Table 1. Bounds for a single pole with } p = 13
\]

Remark 6.22. Note that the proof shows that one has the alternative expression
\[
a_Q = \begin{cases} \frac{(p-1)d_Q}{d_Q} & \text{if } d_Q \text{ is even} \\ \frac{(p-1)(d_Q+1)(d_Q-1)}{4d_Q} & \text{if } d_Q \text{ is odd} \end{cases}
\]
for \( a_Q \) as in (6.27); cf. [FP13, Theorem 1.1].

7. Examples

As always, let \( \pi : Y \to X \) be an Artin-Schreier cover. We give some examples of the bounds given by Theorem 6.15 and the trivial bounds
\[
\dim H^0(X, \ker V_X(F_*E_0)) \leq a_Y \leq p \cdot g_X - p \cdot f_X + \sum_{Q \in S} \frac{p-1}{2} (d_Q - 1)
\]
discussed in Remark 1.3. When \( p = 3 \), one checks that the lower bound \( L(X, \pi) \) coincides with the trivial lower bound. Outside of this special case, the bounds in Theorem 6.15 are always better than the trivial bounds, and often sharp in the sense that there are Artin-Schreier covers \( \pi : Y \to X \) with \( a_Y \) realizing our bounds in many cases. We will give a number of examples illustrating these features. Magma programs which do the calculations in the following examples are available on the authors’ websites.

7.1. The Projective Line. Suppose \( X = P^1 \). Then \( n(X) = -1 \), \( g_X = 0 \), \( a_X = 0 \), and \( f_X = 0 \). Remark 6.2 shows we may choose a minimal polynomial \( f \) describing the cover \( \pi : Y \to P^1 \) ramified over \( S \), with break in the lower-numbering filtration above \( Q \in S \) equal to \( d_Q = -\ord_Q(f) \) and regular away from \( S \).

As always, \( n_{Q,i} = \left\lceil \frac{(p-1-i)d_Q}{p} \right\rceil \). In particular, \( n_{Q,0} = d_Q - \left\lfloor \frac{d_Q}{p} \right\rfloor \). Then the trivial bounds are
\[
\sum_{Q \in S} \left( n_{Q,0} - \left\lfloor \frac{n_{Q,0}}{p} \right\rfloor \right) \leq a_Y \leq \sum_{Q \in S} \frac{p-1}{2} (d_Q - 1).
\]

Example 7.1. Let \( p = 13 \), and suppose \( f \) has a single pole. Table 1 shows the trivial upper and lower bounds, as well as the bounds from Theorem 6.15 for various values of \( d_Q \). When \( d_Q > 4 \), an optimum value of \( j \) to use in the lower bound turns out to be \( \frac{p+1}{2} = 7 \). Notice that our bounds are substantially better than the trivial bounds.

Example 7.2. Using Magma (or a MAPLE program from Shawn Farnell’s thesis [Far10]), we can compute the \( a \)-number for covers of \( P^1 \). For example, let \( p = 13 \) and suppose \( f \) has a single pole of order 7. Our results show that the \( a \)-number of the cover is between 21 and 36. Table 2 lists the \( a \)-numbers for some choices of \( f \), and shows that our bounds are sharp in this instance.

Example 7.3. We continue with Example 4.5. In that situation, we compute that \( L(X, \pi) = 3 \) (obtained when \( j = 1 \)) and that \( \#T = 0 \) so Proposition 6.11 shows that the \( a \)-number is either 3 or 4, which is what we found in Example 4.5.
| Polynomial       | \( a_Y \) |
|------------------|-----------|
| \( t^{-1} + 2t^{-6} + 7t^{-9} \) | 21        |
| \( t^{-7} + t^{-2} + t^{-1} \)     | 23        |
| \( t^{-7} + 8t^{-2} \)             | 24        |
| \( t^{-7} + t^{-1} \)              | 27        |
| \( t^{-7} \)                        | 36        |

Table 2. \( a \)-Numbers for some select Artin-Schreier curves

In particular, the \( a \)-number of the cover is 3 if and only if there is an element of \( H^0(\mathbb{P}^1, \mathcal{G}_m) \) which is not in the kernel of some \( g_i' \). In example 4.5, the \( v_{i,j} \) were a basis for \( H^0(\mathbb{P}^1, \mathcal{G}_m) \). When we used Corollary 4.4 and the splitting \( s \) from Lemma 2.1, we saw that for both \( f_1 \) and \( f_2 \)

\[
\varphi^{-1}_\eta(v_{0,3}) = t^{-3} dt \quad \varphi^{-1}_\eta(v_{0,2}) = t^{-2} dt \quad \varphi^{-1}_\eta(v_{1,2}) = t^{-2} dt \cdot y.
\]

The interesting computation was \( \varphi^{-1}_\eta(v_{2,2}) \): it depended on the exact choice of Artin-Schreier equation \( y^p - y = f \). Using \( f_1 = t^{-3} \), we computed that

\[
\varphi^{-1}_\eta(v_{2,2}) = t^{-2} dt y^2
\]

because there are no terms of the form \( t^i dt \) with \( i \equiv -1 \pmod{5} \) appearing in \( V_{\mathbb{P}^1}(\omega_0) \) when using Corollary 4.4. On the other hand, using \( f_2 = t^{-3} + t^{-2} \) we see that

\[
\varphi^{-1}_\eta(v_{2,2}) = -t^{-6} dt + t^{-2} dt y^2
\]

since \( V_{\mathbb{P}^1}(\omega_0) = -s V_{\mathbb{P}^1}(t^{-2} - t^{-6} + 2t^{-5} + t^{-4}) \). This is not an element of \( H^0(Y, \Omega^1_Y) \) because of the \( t^{-6} dt \) term, recorded in the fact that \( g_0^*(v_{2,2}) = -1 \). Thus in the first case the \( a \)-number is 4, while in the second it is 3.

The set \( T \) in Proposition 5.11 is an attempt to produce differentials not in the kernel of some \( g_i' \). It only uses the leading terms of powers of \( f \) (since those are the only term guaranteed to exist). As seen with \( f_1(t) = t^{-3} \), the leading term doesn’t help. In the case \( f_2(t) = t^{-3} + t^{-2} \) we produced a differential not in \( \ker g_0^* \) making use of the \( t^{-4} \) term of \( f^2 \). This behavior illustrates why the \( a \)-number of the cover cannot depend only on the \( d_S \) and must incorporate finer information (in this case, expressed as whether certain coefficients of powers of \( f \) are non-zero).

**Example 7.4.** Let us now consider an example with multiple poles. Let \( p = 5 \), and suppose that \( f \) has two poles of order 7 (at \( \infty \) and \( -1 \)). Then our bounds say that \( 14 \leq a_Y \leq 16 \). Computing the \( a \)-number for a thousand random choices of \( f \) subject to the constraint on the poles, 942 of them had \( a \)-number 14, 41 had \( a \)-number 15, and 1 had \( a \)-number 16. The \( f \) giving \( a \)-number 16 was

\[
t^7 + 3t^6 + t^4 + t^3 + 2t^2 + t + 3(t + 1)^{-1} + 3(t + 1)^{-2} + 4(t + 1)^{-3} + 4(t + 1)^{-4} + 4(t + 1)^{-5} + 3(t + 1)^{-7}.
\]

**Example 7.5.** For general \( p \), take \( f(t) = t^{-d} \). We will show that this family achieves our upper bound. As discussed in [FP13, Remark 2.1] (which extracts the results from [Pri05]), the resulting \( a \)-number is

\[
\sum_{b=0}^{d-2} \min \left( h_b, \left[ \frac{pd - bp - p - 1}{d} \right] \right)
\]

where \( h_b \) is the unique integer in \([0, p - 1]\) such that \( h_b \equiv -(b - 1) d^{-1} \pmod{p} \). Note that if \( b \equiv -1 \pmod{p} \), \( h_b = 0 \). This counts the number of elements in the set

\[
T_1 = \{(b, j) : 0 \leq b \leq d - 2, 0 \leq j d \leq p(d - b - 1) - d - 1, j < h_b\}.
\]
On the other hand, our upper bound is \( \bigoplus_{i=0}^{p-1} \dim H^0(P^1, \ker V_{P^1}(F_*E_i)) - \# T \). For \( 0 \leq i \leq p - 1 \), the differentials \( t^{-n}dt \) for \( n \not\equiv 1 \pmod{p} \) and \( 0 < n \leq n_{Q,i} = \left\lfloor \frac{(p-1-i)d}{p} \right\rfloor \) form a basis for \( H^0(P^1, \ker V_{P^1}(F_*E_i)) \). The condition \( n \leq n_{Q,i} \) can be expressed as \( pn \leq (p - 1 - i)d + (p - 1) \), or equivalently
\[
i \cdot d \leq (p - 1)d + (p - 1) - pm = p(d - n + 1) - 1 - d.
\]
Represent each basis element as a triple \((Q, n, i)\). A triple does not lie in \( T \) if there does not exist an integer \( m \in [0, j] \) such that \( m \equiv j - (n - 1)d^{-1} \pmod{p} \): this can be rephrased as \( j < h_{n-2} \). Thus our upper bound is the size of the set
\[
T_2 = \{(n, j) : 2 \leq n \leq d, 0 \leq j \cdot d \leq p(d - n + 1) - 1 - d, j < h_{n-2}\}.
\]
(The condition \( n \not\equiv 1 \pmod{n} \) is implicit, as in that case \( h_{n-2} = 0 \).) But \( T_1 \) and \( T_2 \) have the same size: there is a bijection given by taking \( b = n - 2 \). Thus the covers given by \( f(t) = t^{-d} \) for \( p \nmid d \) realize the upper bound for a cover ramified at a single point.

### 7.2. Elliptic Curves

We now suppose that \( X \) is the elliptic curve over \( \mathbf{F}_p \) with affine equation \( y^2 = x^3 - x \) (recall that \( p > 2 \)). Of course, \( g_X = 1 \) and it is not hard to compute that the Tango number of \( X \) is \( n(X) = 0 \) and that \( X \) is ordinary (so \( a_X = 0 \) and \( f_X = 1 \)) when \( p \equiv 1 \pmod{4} \) and supersingular (i.e. \( a_X = 1 \) and \( f_X = 0 \)) when \( p \equiv 3 \pmod{4} \). In this simple case, it is simple to say more than what Tango’s theorem tells us.

**Lemma 7.6.** Let \( D = \sum n_Q \cdot Q > 0 \) be a divisor on \( X \) with some \( n_Q \geq 2 \), and set \( D' := \sum [n_Q/p] \cdot Q \). Then
\[
V_X : H^0(X, \Omega^1_X(D)) \to H^0(X, \Omega^1_X(D'))
\]
is surjective.

**Proof.** The Lemma is an immediate consequence of Corollary 6.7 when \( n_Q \geq p \) for some \( Q \) or when \( a_X = 0 \), so we suppose that \( n_Q < p \) for all \( Q \) and \( p \equiv 3 \pmod{4} \). A straightforward induction on the size of the support of \( D \) reduces us to the case that \( D = n \cdot Q \) with \( n \geq 2 \), and then as \( D' = Q \) it suffices to treat the case \( n = 2 \). Suppose that \( Q \) is not the point \( P \) at infinity on \( E \). Consider the commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(X, \Omega^1_X(2P)) & \longrightarrow & H^0(X, \Omega^1_X(2P + 2Q)) & \longrightarrow & (k[x]/x^3_Q)x^{-3}dx_Q \\
& & \downarrow V & & \downarrow V & & \downarrow V \\
0 & \longrightarrow & H^0(X, \Omega^1_X(P)) & \longrightarrow & H^0(X, \Omega^1_X(P + Q)) & \longrightarrow & (k[x]/x_Q)x^{-1}dx_Q \\
\end{array}
\]
The local description of \( V \) shows that the right vertical map is surjective, and it therefore suffices to prove the lemma in the special case \( D = 2P \). Now \( \{dx/y, xdx/y\} \) is a basis of \( H^0(X, \Omega^1_X(2P)) \) and one calculates
\[
V \left( \frac{dx}{y} \right) = \frac{1}{y} V(x(x^3 - x)^{(p-1)/2}dx) = (-1)^{(p+1)/4} \left( \frac{p-1}{2} \right) \neq 0.
\]
Since \( V(dx/y) = 0 \), the image of \( V \) is 1-dimensional. But \( H^0(X, \Omega^1_X(P)) = H^0(X, \Omega^1_X) \) because the sum of the residues of a meromorphic differential on a smooth projective curve is zero, and this space therefore has dimension one as well; the Lemma follows.

We will first consider Artin-Schreier covers \( Y \to X \) with defining equation of the form
\[
z^p - z = f
\]
with \( f \in H^0(E, \mathcal{O}_E(d_Q \cdot Q)) \) having a pole of exact order \( d_Q \) at \( Q \), for \( Q \) the point at infinity on \( E \), where \( d_Q \geq 2 \) is prime to \( p \). Such covers are branched only over \( Q \) with unique break in the ramification filtration \( d_Q \). Conversely, thanks to Remark 7.9, every \( \mathbb{Z}/p\mathbb{Z} \)-cover of \( X \) that is defined over \( \mathbb{F}_p \) and is ramified only above \( Q \) with unique ramification break \( d_Q \) occurs this way.

**Example 7.7.** For \( p = 5, 7 \) we compare the bounds given by Theorem 6.15 with the trivial upper and lower bounds, which are

\[
\left\lfloor \frac{p-1}{p} d_Q \right\rfloor \leq a_Y \leq \left\lfloor \frac{p-1}{p^2} d_Q \right\rfloor \leq a_Y \leq p \cdot a_X + \frac{p-1}{2} (d_Q - 1)
\]

thanks to Lemma 7.6 and the fact that \( \left\lfloor (p-1)\frac{d_Q}{p}\right\rfloor = d_Q - \lfloor d_Q/p \rfloor \geq 2 \). We summarize these computations in Tables 3 and 4.

**Example 7.8.** Again for \( p = 5, 7 \) and select \( d_Q \) in the tables above, we have computed \( a_Y \) for several thousand randomly selected \( f \in H^0(X, \mathcal{O}_X(d_Q \cdot Q)) \). In Tables 5 and 6, we record the values of \( a_Y \) that we found, as well a function \( f \) that produced it. These values of \( a_Y \) should be compared to the bounds in Tables 3 and 4.

**Remark 7.9.** Larger values of \( a_Y \) are more rare. For example, when \( p = 7, S = \{Q\} \) and \( d_Q = 6 \), among all 100842 = 6 \cdot 7^5 functions \( f \in H^0(X, \mathcal{O}_X(d_Q \cdot Q)) \) with a pole of exact order 6 at \( Q \), there are 86436 (= 85.71\%) with \( a_Y = 10 \), 11760 (= 11.66\%) with \( a_Y = 11 \), 2562 (= 2.54\%) with \( a_Y = 12 \) and 84 (= 0.08\%) with \( a_Y = 14 \). Curiously, none had \( a_Y = 13 \). In the spirit of [CLZB13], it would be interesting to investigate the limiting distribution of \( a \)-numbers in branched \( \mathbb{Z}/p\mathbb{Z} \)-covers of a fixed base curve with fixed branch locus, as the sum of the ramification breaks tends to infinity.

Note also that although \( d_Q = 6 \) divides \( p - 1 \) when \( p = 7 \), the \( a \)-number of \( Y \) can be 10, 11, 12 or 14; in particular, the ordinarity hypothesis in Corollary 6.21 is necessary.
Example 7.10. We now work out some examples with \( \pi : Y \to X \) branched at exactly two points. As before, let \( Q \) be the point at infinity on \( X \) and \( P \) be the point \((0,0)\). For \( p = 5 \) we considered Artin-Schreier covers \( Y \) of \( X \) branched only over \( P, Q \) with \( d_P = 4 \) and \( d_Q = 6 \). Our bounds are \( 10 \leq a_Y \leq 14 \), and among a sample of 10001 functions \( f \in H^0(X, \mathcal{O}_X(d_P \cdot P + d_Q \cdot Q)) \) with a pole of exact order \( d_\ast \) at \( \ast = P, Q \), we found 8021 with \( a_Y = 10 \), 1818 with \( a_Y = 11 \), 149 with \( a_Y = 12 \), and 13 with \( a_Y = 13 \). One of the 13 functions we found giving \( a \)-number 13 was

\[
f = -\frac{1}{xy} + \frac{x^5 - x^3 + 2x^2 - 2x + 1}{x^2}.
\]

Similarly, with \( p = 7 \) and \( d_Q = 6 \), \( d_P = 8 \) our bounds are \( 21 \leq a_Y \leq 37 \). Among a sample of 5001 random functions satisfying the required constraints we found 4318 with \( a_Y = 21 \), 668 with \( a_Y = 22 \), 14 with \( a_Y = 23 \), and 1 with \( a_Y = 24 \). The function producing a cover \( Y \) with \( a \)-number 24 was

\[
f = -\frac{x^4 - 3x^3 + 2x^2 - x + 1}{x^4 y} + \frac{-2x^7 + 3x^5 - 3x^4 + x^3 + x^2 + x + 3}{x^4}.
\]

7.3. Unramified Covers. Now suppose that \( \pi : Y \to X \) is an unramified Artin-Schreier cover. (Note that no such covers exist when \( X = \mathbb{P}^1 \).) Then the branch locus \( S \) is empty, and the bounds of Theorem 6.15 simply become

\[
0 \leq a_Y \leq p \cdot a_X.
\]

On the other hand, since \( E_0 = 0 \) the trivial bounds are

\[
a_X \leq a_Y \leq p \cdot (g_X - f_X).
\]

These are enough to re-derive the following fact (which is typically deduced from the Deuring-Shafarevich formula):

Corollary 7.11. Let \( \pi : Y \to X \) be an unramified Artin-Schreier cover. Then \( Y \) is ordinary if and only if \( X \) is ordinary.

Note that \( p \cdot (g_X - f_X) \geq p \cdot a_X \), so our upper bound is an improvement, but the trivial lower bound is better! The trivial lower bound is better because we lost some information in our applications of Tango’s theorem. In particular, in the proof of Theorem 6.15 the first step is to add more points to the auxiliary divisors \( D_i \) so that \( \deg(D_i) > n(X) \) for all \( i \) and we can apply Corollary 6.7 to exactly calculate

\[
\dim H^0(X, \ker V_X(F_i(E_i + pD_i))) = \dim H^0(X, \ker V_X(F_i(pD_i))) = (p - 1) \deg(D_i).
\]

The trivial bounds comes from the inclusion \( \ker V_X \hookrightarrow \pi_\ast \ker V_Y \) given by \( \omega \mapsto \omega y^0 \), which would correspond to taking \( j = 0 \) in Definition 6.2 and taking \( D_0 = 0 \). In this situation, the sum

\[
\sum_{i=0}^j \sum_{Q \in S_j} c(i, j, Q)
\]

is empty and \( \dim V_0 = a_X \). When we increase the size of the divisor \( D_0 \) to apply
Corollary 6.7 we increase \( \dim V_0 \) but we also add terms to this sum: in particular, \( c(0, 0, Q) = (p - 1) \) for \( Q \in D_0 \). Then \( \dim V_0 \) and the sum cancel, giving 0 as the lower bound. (In Theorem 6.15 we re-indexed so there we are taking \( j = p - 1 \).)

Theorem 6.15 is best when the ramification of \( \pi \) is large, in the sense that we do not need to increase the degree of the \( D_i \) in order to apply Corollary 6.7 to compute the dimension of \( H^0(X, \ker V_X(F_i(E_i + pD_i))) \). Once we have reached that point, increasing \( D_i \) further does not change the bound: the dimension of this space increases, but the increase is canceled by additional \( c(i, j, Q) \) terms for \( Q \in D_i \). This is why the divisor \( D_i \) makes no appearance in Theorem 6.15.

However, when the ramification is small and we increase the size of \( D_i \) we change the bounds for the worse as \( H^0(X, \ker V_X(F_i(D))) \) can be larger than expected until \( \deg(D) > n(X) \) (see the second part of Corollary 6.7). To obtain sharper bounds, one would need to be given

\[
\dim H^0(X, \ker V_X(F_*(D)))
\]

for all small divisors on \( X \).

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