I. INTRODUCTION

Observations of gravitational waves by advanced LIGO [1] open a new window for probing black hole physics. Any deviation of black holes in alternative theories of gravity from those in the Einstein’s general relativity can be potentially tested by precise measurements of gravitational waves. In this new age of gravity studies, it is of great importance to seek for new black hole solutions in alternative theories of gravity.

It is well known that the Schwarzschild black hole has one event horizon. The Reissner-Nordström black hole (the Kerr black hole and the Kerr-Newman black hole) has two horizons. The Reissner-Nordström-de Sitter (the Kerr-de Sitter and the Kerr-Newman-de Sitter black hole) black hole has three horizons. Note that all these black holes have a singularity in its center. Some interesting questions one would ask are as follows: 1) do black hole solutions with $N$ horizons exist? 2) can black holes with multiple singularities be constructed? In fact, regular multi-horizon black holes in General Relativity, Lovelock theory and modified gravity with nonlinear electrodynamics have been presented in Ref. [18]. The spacetime with multiple singularities and multiple horizons is highly likely to be produced in the merger process of multiple black holes. But it is rather involved to obtain the analytic solutions describing this process.

A few of the studies are as follows. The Majumdar-Papapetrou solution describes multiple extreme Reissner-Nordström black holes [2]. The Kastor-Traschen solution [3, 4] generalized the Majumdar-Papapetrou solution into the background of de Sitter universe. So it can describe the collisions of several black holes in asymptotically de Sitter space-time. It is worth noting that the solution is the first exact solution that describes black holes collisions. The dilaton version of the Kastor-Traschen solution is given in [5] and the spinning version of the dilatonic Kastor-Traschen solution is given in [6]. Within five-dimensional supergravity, Ref. [7] provides examples of multi-centered charged black holes in asymptotic de Sitter space. Using the compactification of intersecting brane solution in higher dimensional unified theory, a time-dependent cosmological black hole system was found in [8]. The global picture of dynamical solution describes a multi-black hole system in the expanding Universe filled by “stiff matter” is clarified in [9]. Finally, the multiple black hole solution of general relativity with a scalar field and two Maxwell-type U(1) fields is found in [10].

In this paper, we present some toy models for the black holes spacetime with multiple horizons and multiple singularities. As toy models, the solutions are all static and spherically symmetric. Therefore, the multiple horizons are concentric spheres and the singularities consists of one singular point and multi-singular spheres. These spacetimes can be viewed as the generalization of Reissner-Nordström spacetime. As a by-product, the spacetime with one singularity and one singular sphere can also be modeled as a black hole with a firewall [11].

We shall construct our solutions in an especially attractive class of generalized Proca theories [12, 13]. The es-
II. EQUATIONS OF MOTION IN NONLINEAR ELECTRODYNAMICS THEORIES

We start from the action of nonlinear electrodynamics theories which are minimally coupled to gravity

$$ S = \frac{1}{16\pi} \int \sqrt{-g} \left[ R + K (X) \right] d^4x , \quad (1) $$

with

$$ X = F_{\mu\nu} F^{\mu\nu} , \quad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu . \quad (2) $$

Here $R$ is the Ricci scalar and $A_\mu$ is the Maxwell field. $K(X)$ is the function of $X$ to be specified. Variation of the action with respect to the metric gives the Einstein equations

$$ G_{\mu\nu} = -2K_{,X} F_{\mu\lambda} F^{\lambda\nu} + \frac{1}{2} g_{\mu\nu} K , \quad K_{,X} \equiv \frac{dK}{dX} . \quad (3) $$

Variation of the action with respect to the field $A_\mu$ gives the generalized Maxwell equations

$$ \nabla_\mu \left( K_{,X} F^{\mu\nu} \right) = 0 . \quad (4) $$

In the background of static and spherically symmetric spacetime which can always be parameterized as

$$ ds^2 = -U (r) \, dt^2 + \frac{1}{U (r)} \, dr^2 + f (r)^2 \, d\Omega_2^2 . \quad (5) $$

Here $d\Omega_2^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$. The non-vanishing component of Maxwell field $A_\mu$ is uniquely to be

$$ A_0 = \phi (r) , \quad (6) $$

by resorting to a gauge transformation of $A_\mu \to A_\mu + \nabla_\mu \psi$. Then we obtain the Einstein equations and the generalized Maxwell equation

$$ \frac{U' f'}{f} - \frac{2U f''}{f^2} + \frac{1}{f^2} \frac{U f''}{f^2} = 2K_{,X} \phi'^2 + \frac{1}{2} K , \quad (7) $$

$$ - \frac{U' f'}{f} + \frac{1}{f^2} \frac{U f''}{f^2} = 2K_{,X} \phi'^2 + \frac{1}{2} K , \quad (8) $$

$$ \left( f^2 K_{,X} \phi' \right)' = 0 . \quad (9) $$

The prime denotes the derivative with respect to $r$. Eqs. (7-9) comes from $G_0^0 = \rho$, $G_1^1 = p_r$, and $G_2^2 = p_\theta$, respectively.

Due to the Bianchi identities, only three of the four equations are independent. One usually assume the expression of $K(X)$ initially. Then they are left with three unknown functions, $U$, $f$ and $\phi$ and the system of equations are closed. However, we shall not follow this way in this paper because the equations of motion are rather involved. Instead, we shall let $K(X)$ keep open and assume the expression of the metric component $U(r)$ initially. Then we solve for the Maxwell field $\phi$ and the unknown function $K(X)$. The method is quite a success and is widely used in the construction of the black hole solutions. In the next subsections, we shall seek for some interesting spacetimes by using this method. Before the presentation of some interesting spacetimes, we observe the difference of Eq. (7) and Eq. (8) which gives

$$ f'' = 0 . \quad (11) $$

Thus we obtain the physical solution for $f$,

$$ f = r . \quad (12) $$

Therefore, the static and spherically symmetric spacetime with the source of nonlinear electrodynamic field is

$$ ds^2 = -U (r) \, dt^2 + \frac{1}{U (r)} \, dr^2 + r^2 \, d\Omega_2^2 . \quad (13) $$

III. BLACK HOLES WITH N HORIZONS

In section III the function $f = r$ is obtained. So we could assume that the metric for static and spherically
symmetric black holes with $N$ horizons can be written as Eq. (13) with

$$U(r) = \prod_{i}^{N} \left(1 - \frac{a_i}{r}\right).$$

Here $a_i$ are $N$ positive constants ($i = 1, 2, 3, \ldots, N$). If $0 < a_1 < a_2 < \cdots < a_N$, there are $N$ horizons in the spacetime. The spacetime is singular at $r = 0$ and asymptotically flat at $r = +\infty$. Substituting the metric into the Einstein equations with the energy momentum tensor of anisotropic fluid,

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

we find the energy density $\rho$ and the pressures, $p_r, p_\theta, p_\phi$ of the fluid

$$\rho = \frac{1}{8\pi} \left[ \frac{1}{r^7} \sum_{i \neq j}^{N} a_i a_j - \frac{2}{r^5} \sum_{i \neq j \neq k}^{N} a_i a_j a_k + \frac{3}{r^6} \sum_{i \neq j \neq k \neq l}^{N} a_i a_j a_k a_l \right] - \cdots + \frac{(-1)^N}{r^{N+2}} (N - 1) a_1 a_2 a_3 \cdots a_N,$$

$$p_r = -\rho,$$

$$p_\theta = \frac{1}{8\pi} \left[ \frac{1}{r^7} \sum_{i \neq j}^{N} a_i a_j - \frac{3}{r^5} \sum_{i \neq j \neq k}^{N} a_i a_j a_k + \frac{6}{r^6} \sum_{i \neq j \neq k \neq l}^{N} a_i a_j a_k a_l \right] - \cdots + \frac{(-1)^N}{r^{N+2}} \frac{N (N - 1)}{2} a_1 a_2 a_3 \cdots a_N,$$

$$p_\phi = p_\theta.$$

If $N$ is odd, the energy density would be negative for sufficiently small $r$. By contrast, if $N$ is even, the energy density is always positive for both large and small $r$. The positive energy theorem claims that the energy density can not be negative. Therefore we shall be interested in the even number of $N$ in the next subsection. As an example, we focus on the $N = 4$ case.

### A. Black holes with 4 horizons

The metric for static and spherically symmetric black holes with 4 horizons assumes the form of Eq. (13) with

$$U(r) = \prod_{i}^{4} \left(1 - \frac{a_i}{r}\right).$$

It is asymptotically flat in space. We assume $0 < a_1 < a_2 < a_3 < a_4$. There are four horizons, $r = a_i$ and a singularity, $r = 0$ in the spacetime. The energy density $\rho$ and the pressures, $p_r, p_\theta, p_\phi$ of the anisotropic fluid have the form

$$\rho = \frac{1}{8\pi} \left(\frac{\alpha}{r^4} - \frac{2\beta}{r^6} + \frac{3\gamma}{r^8}\right),$$

$$p_r = -\rho,$$

$$p_\theta = \frac{1}{8\pi} \left(\frac{\alpha}{r^4} - \frac{3\beta}{r^6} + \frac{6\gamma}{r^8}\right),$$

$$p_\phi = p_\theta,$$

with

$$\alpha = a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_2 a_4,$$

$$\beta = a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4,$$

$$\gamma = a_1 a_2 a_3 a_4.$$

The energy density is always positive provided that

$$\beta^2 - 3\alpha\gamma < 0.$$

This requirement is not hard to be satisfied. Substituting Eq. (29) into Eq. (14), we obtain the corresponding electric potential $\phi$, the square of field strength $F_{\mu\nu} F^{\mu\nu}$ and the Lagrangian function $K$:

$$\phi = \phi_0 \left(\frac{2\alpha}{r^4} - \frac{5\beta}{2r^6} + \frac{3\gamma}{r^8}\right),$$

$$F_{\mu\nu} F^{\mu\nu} = -2\phi_0^2 \left(\frac{2\alpha}{r^4} - \frac{5\beta}{2r^6} + \frac{9\gamma}{r^8}\right)^2,$$

$$K (F_{\mu\nu} F^{\mu\nu}) = -\frac{2\alpha}{r^4} + \frac{6\beta}{r^6} - \frac{12\gamma}{r^8}.$$

Here $\phi_0$ is an integration constant. Actually, $\phi_0$ is related to the electric charge of the spacetime as follows

$$Q_e \equiv r^2 K_{X} F^{01} = \frac{1}{2\phi_0}.$$

When

$$a_1 = m - \sqrt{m^2 - Q^2}, \quad a_3 = a_4 = 0,$$

$$a_2 = m + \sqrt{m^2 - Q^2}, \quad \phi_0 = \frac{1}{2Q},$$

we have

$$Q_e = Q, \phi = \frac{Q}{r}, F_{\mu\nu} F^{\mu\nu} = -\frac{2Q^2}{r^4}, K = F_{\mu\nu} F^{\mu\nu}.$$

It is exactly the Maxwell theory for the Reissner-Nordström spacetime. $m, Q$ are the mass and the electric charge of the black hole, respectively. Eq. (29) tells us the electric potential $\phi$ is endowed with two extra terms, $\beta/r^2, \gamma/r^3$ except for the usual Maxwell term, $\phi = a/r$. With the presence of extra terms, the electric force exerted on a charged particle with unit electric charge becomes

$$F_e = \phi_0 \left(\frac{-2a}{r^2} + \frac{5\beta}{r^4} - \frac{9\gamma}{r^6}\right).$$
which differs from the Coulomb’s law.

Eq. (20) can be written as a biquadratic algebraic equation of \( r \). By solving this equation we obtain \( r = r(F^2) \). Then substituting this expression into Eq. (31), we find \( K \) can be explicitly expressed as the function of \( F^2 \).

**B. Penrose diagram**

In this subsection, we give the Penrose diagram of the 4-horizon spacetime. One of the easy ways of constructing Penrose diagrams for any spherical (and not only spherical) space-times can be found in the book of Bronnikov and Rubin [35] (see also [36]). But here we follow the ways of Plebanski and Krasinski [37].

Following [37], we define the tortoise coordinate \( r_* \) as follows

\[
    r_\ast = r + \sum_i a_i^4 \ln |r - a_i| \frac{a_i^4}{(a_i - a_j)(a_i - a_k)(a_i - a_l)}. \tag{37}
\]

Here \( i \) runs over from 1 to 4 and \( i \neq j \neq k \neq l \). We make coordinates transformation \((t, r) \rightarrow (v, u)\)

\[
    v = e^{\gamma r} \sinh \gamma t, \quad u = e^{\gamma r} \cosh \gamma t, \tag{38}
\]

where \( \gamma \) is a constant. Then the metric Eq. (13) becomes

\[
    ds^2 = F(v, u)(-dv^2 + du^2) + r^2 d\Omega_2^2, \tag{39}
\]

with

\[
    F = \frac{1}{\gamma^4 e^{2\gamma r}} \prod_i |r - r_i|^{\frac{1}{2}} \frac{2\gamma a_i^4}{(a_i - a_j)(a_i - a_k)(a_i - a_l)}. \tag{40}
\]

Here \( i \) runs over from 1 to 4 and \( i \neq j \neq k \neq l \). In order to remove the coordinate singularity \( a_i \), we should let

\[
    1 - \frac{2\gamma a_i^4}{(a_i - a_j)(a_i - a_k)(a_i - a_l)} = 0, \tag{41}
\]

namely,

\[
    \gamma_i = \frac{1}{2a_i^4}(a_i - a_j)(a_i - a_k)(a_i - a_l). \tag{42}
\]

From Eq. (31), we obtain

\[
    u^2 - v^2 = e^{2\gamma r_*} \frac{e^{2\gamma r} \prod_i (r - r_i)^{2\gamma a_i^4}}{\prod_i |r - r_i|^{\frac{1}{2}} (a_i - a_j)(a_i - a_k)(a_i - a_l)}. \tag{43}
\]

Substituting \( r = a_i \) and \( \gamma = \gamma_i \) into above equation, we have

\[
    u^2 - v^2 = 0. \tag{44}
\]

So in the \((v, u)\) coordinate system, the horizon \( r = a_i \) is not singular. Furthermore, the horizon consists of two lines \( u = \pm v \) in the \((v, u)\) plane. At the true singularity \( r = 0 \), we have

\[
    u^2 - v^2 = \text{constant} > 0. \tag{45}
\]

Therefore in the \((v, u)\) coordinates, they are a pair of hyperbolae and the hyperbolae are timelike. After employing the Penrose transformation, we arrive at the conventional coordinates system by

\[
    \nabla = \frac{1}{2} [\tanh (u + v) - \tanh (u - v)] \tag{46}
\]

\[
    \nabla = \frac{1}{2} [\tanh (u + v) + \tanh (u - v)] \tag{47}
\]

Figure 1 shows the Penrose diagram of the black hole spacetime with four horizons. The singularities are timelike.

**FIG. 1:** The Penrose diagram of a black hole with four horizons. The wavy lines are the singularities. 1, 2, 3, 4 stand for the number of horizons, 0 stands for the null spatial infinity and a stands for for \( r = -\infty \).

**C. Observational aspect 1: quasinormal modes**

When a black hole is perturbed it evolves into three stages by emitting gravitational waves: 1) a relatively short period of initial outburst of radiation, 2) a long period of damping proper oscillations, dominated by the so-called quasinormal modes, 3) at very large time the
quasinormal modes are suppressed by power-law or exponential late-time tails. The dominating contribution to gravitational waves is the quasinormal mode with lowest frequency: the fundamental mode. So in this subsection we study the quasinormal modes generated by the propagation of a test minimally coupled massless scalar field in the 4-horizon black hole spacetime. To this end, we start from the well-known Klein-Gordon equation
\[ \nabla^2 \Psi = 0 , \] (48)
which is the general perturbation equation for the massless scalar field in the curve spacetime. Here \( \nabla^2 \) is the four dimensional Laplace operator and \( \Psi \) the massless scalar field. Making the standard decomposition
\[ \Psi = e^{-i\omega t}Y_{lm}(\theta, \phi)\frac{\Phi(r)}{r} , \] (49)
we obtain the radial perturbation equation
\[ \frac{d^2\Phi}{dr^2} + (\omega^2 - V) = 0 , \] (50)
where the effective potential is given by
\[ V = U\left(\frac{l(l+1)}{r^2} + \frac{U_r}{r}\right) , \] (51)
and \( r_\ast \) is the tortoise coordinate defined by
\[ r_\ast = r + \sum_i \frac{a_i^2 \ln(r - a_i)}{(a_i - a_j)(a_i - a_k)(a_i - a_l)} . \] (52)

The effective potential \( V \) as a function of the tortoise coordinate \( r_\ast \) can be seen in Fig. 2.

![FIG. 2: The effective potential \( V(r_\ast) \) as a function of the tortoise coordinate \( r_\ast \), assuming \( a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4 \) for three different cases \( l = 3, 2, 1 \), from up to down, respectively.](image)

We shall evaluate the quasinormal frequencies for the massless scalar field by using the third-order WKB approximation, a numerical and perhaps the most popular method, devised by Schutz, Will and Iyer [38–40]. This method has been used extensively in evaluating quasinormal frequencies of various black holes. For an incomplete list see [11] and references therein.

The quasinormal frequencies are given by
\[ \omega^2 = V_0 + \Lambda \sqrt{-2V''_0} - i\nu (1 + \Omega) \sqrt{-2V''_0} , \] (53)
where \( \Lambda \) and \( \Omega \) are
\[ \Lambda = \frac{1}{\sqrt{-2V_0}} \left\{ \frac{V_0^{(4)}}{V_0^{(2)}} \left( \frac{1}{32} + \frac{1}{8} l^2 \right) - \left( \frac{V_0''}{V''_0} \right)^2 \left( \frac{7}{288} + \frac{5}{24} l^2 \right) \right\} , \] (54)
\[ \Omega = \frac{1}{\sqrt{-2V_0}} \left\{ \frac{5}{6912} \left( \frac{V''_0}{V_0''} \right)^4 (77 + 188\nu^2) \right. \]
\[ - \frac{1}{384} \left( \frac{V_0''^2 V_0^{(4)}}{V_0^{(3)}} \right) (51 + 100\nu^2) \]
\[ + \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0} \right)^2 (67 + 68\nu^2) \]
\[ + \frac{1}{288} \left( \frac{V_0'' V_0^{(4)}}{V_0^{(2)}} \right) (19 + 28\nu^2) \]
\[ - \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0} \right) (5 + 4\nu^2) \} , \] (55)
and
\[ \nu = n + \frac{1}{2} , \quad V_0^{(n)} = \frac{d^n V}{dr^*_n} |_{r_\ast=r_\ast} , \] (56)
n is overtone number and \( r_\ast \) corresponds to the peak of the effective potential. It is pointed that [12] that the accuracy of the WKB method depends on the multipole number \( l \) and the overtone number \( n \). The WKB approach is consistent with the numerical method very well provide that \( l > n \). Therefore we shall present the quasinormal frequencies of scalar perturbation for \( n = 0 \) and \( l = 1, 2, 3 \), respectively. In order to satisfy the constraint equation Eq. (28), we set \( a_1 = 1, a_2 = 2, a_3 = 3, 3 \leq a_4 < 4 \). The fundamental quasinormal frequencies of the massless scalar perturbation field for fixed \( l = 1, 2, 3 \) are given by in Table I. From the table we see that with the decreasing of the outermost horizon, both the real part and the imaginary part of the frequencies are increasing. When \( a_4 = a_3 = 3 \), we obtain an extreme black hole.
D. Observational aspect 2: black hole shadow

As another potential observational aspect, we consider the black hole shadow in this subsection. When a black hole is in front of a bright background which is produced by faraway radiating object, it casts a black shadow. The apparent shape of the black hole is just defined by the boundary of the black shadow. The ability of very long baseline interferometry (VLBI) observation has been improved significantly. This opens up the possibility to make observations on the black hole shadow \[13, 14\]. A significant attention has been payed to the study of black hole shadow and it has become a quite active research field \[15\] (for a review, see \[16\]).

In order to obtain the boundary of the shadow of the black hole, we need to study the radial motion

$$r^2 + V_{eff} = 0,$$  \hspace{1cm} (57)

where the dot denotes the derivative respect to the proper time and the effective potential is

$$V_{eff} = \frac{1}{r^2} U (\eta + \xi^2) E^2 - E^2.$$  \hspace{1cm} (58)

Here $\mathcal{K}$ is the Carter constant \[47\], $E$ and $L$ are energy and angular momentum of the photon, respectively. $\eta = \mathcal{K}/E^2$ and $\xi = L/E$ are two impact parameters which character the motion of photon near the black hole. The boundary of the shadow is mainly determined by the circular photon orbit, which satisfies the following conditions

$$V_{eff} = 0, \quad \partial_r V_{eff} = 0.$$  \hspace{1cm} (59)

In order to find the shadow of the 4-dimensional black hole, we introduce the celestial coordinates

$$\alpha = \lim_{r \to \infty} \left( \frac{rP(\phi)}{P(t)} \right), \quad \beta = \lim_{r \to \infty} \left( \frac{rP(\theta)}{P(t)} \right),$$  \hspace{1cm} (60)

where $(P(t), P(\theta), P(\phi))$ are the vi-tetrad components of angular momentum. Supposing the observer is located in the equatorial plane of the black hole, one has $\theta = \pi/2$. Then the celestial coordinates $\alpha$ and $\beta$ takes the form

$$\alpha = -\xi, \quad \beta = \pm \sqrt{\eta}. \hspace{1cm} (61)$$

Let the parameters $(\xi, \eta)$ run over all possible values. Then we obtain the shadow of black hole in the parameter space $(\alpha, \beta)$. In Fig. 3, we plot the shadow of a 4-dimensional black hole for different radii. The figure shows that the size of the shadow increases with the increasing of the inner horizons. In Fig. 4, we plot the shadow of a Reissner-Nordstrom (solid line) and a 4-dimensional black hole (dotted lines) for different radii. It shows that the size of the shadow becomes greater due to the presence of inner horizons. We hope this difference may be confirmed by the future VLBI observations.

![FIG. 3: Shadow of a 4-horizon black hole for different radii, from inner to outer. (1). $a_4 = 4, a_3 = 3, a_2 = 2, a_1 = 1$; (2). $a_4 = 4, a_3 = 3, a_2 = 2, a_1 = 1.5$ ; (3). $a_4 = 4, a_3 = 3, a_2 = 2, a_1 = 2$; (4). $a_4 = 4, a_3 = 3, a_2 = 2.5, a_1 = 2.5$; (5). $a_4 = 4, a_3 = 3, a_2 = 3, a_1 = 3$; (6). $a_4 = 4, a_3 = 3.5, a_2 = 3.5, a_1 = 3.5$; (7). $a_4 = 4, a_3 = 4, a_2 = 4, a_1 = 4$.](image)

![FIG. 4: Shadow of a Reissner-Nordstrom (solid line) and 4-horizon black hole (dotted lines) for different radii, from inner to outer. (1). $a_4 = 4, a_3 = 3, a_2 = 2, a_1 = 1$; (2). $a_4 = 4, a_3 = 3, a_2 = 2, a_1 = 1.5$ ; (3). $a_4 = 4, a_3 = 3, a_2 = 2, a_1 = 2$; (4). $a_4 = 4, a_3 = 3, a_2 = 2.5, a_1 = 2.5$; (5). $a_4 = 4, a_3 = 3, a_2 = 3, a_1 = 3$; (6). $a_4 = 4, a_3 = 3.5, a_2 = 3.5, a_1 = 3.5$; (7). $a_4 = 4, a_3 = 4, a_2 = 4, a_1 = 4$.](image)

IV. COSMOS WITH N HORIZONS

In previous section, we present the black hole space-time with multiple horizons. In this section, we turn to the case of cosmos. We know both the de Sitter universe and the Friedmann-Robertson-Walker Universe have one apparent horizon. Then can we construct a cosmos with
multiple horizons? We focus on this question in this section. We assume the metric for static and spherically symmetric spacetimes with $N$ cosmic horizons could be written as Eq. (13) with

$$U(r) = (1 - h_1 r) (1 - h_2 r) (1 - h_3 r) \cdots (1 - h_N r) = \prod_i (1 - h_i r). \quad (62)$$

Here $h_i$ are $N$ positive constants ($i = 1, 2, 3, \cdots, N$). If $0 < h_1 < h_2 < \cdots < h_N$, there are $N$ cosmic horizons. When $h_i = 0$, the metric reduces to the Minkowsky one.

The energy density $\rho$ and the pressures, $p_r, p_\theta, p_\phi$ of the source are given by

$$\rho = \frac{1}{8\pi} \left[ \frac{2}{r} \sum_i^n h_i - 3 \sum_{i \neq j} h_i h_j + 4r \sum_{i \neq j \neq k} h_i h_j h_k - \cdots \right] $$

$$+ (-1)^{N-1} (N + 1) r^{N-2} h_1 h_2 h_3 \cdots h_N, \quad (63)$$

$$p_r = -\rho, \quad (64)$$

$$p_\theta = \frac{1}{8\pi} \left[ \frac{1}{r} \sum_i^n h_i + 3 \sum_{i \neq j} h_i h_j - 6r \sum_{i \neq j \neq k} h_i h_j h_k - \cdots \right] $$

$$+ (-1)^N \frac{N (N + 1)}{2} r^{N-2} h_1 h_2 h_3 \cdots h_N, \quad (65)$$

$$p_\phi = p_\theta. \quad (66)$$

It seems the spacetime is singular at $r = 0$ because of the divergence of density at $r = 0$ when

$$\sum_i^n h_i \neq 0. \quad (67)$$

This is not the case. We shall illustrate this point below by using an example with $N = 3$.

If $N$ is even, the energy density can be negative for sufficiently large $r$. In contrast, if $N$ is odd, the energy density is always positive at both large and small $r$. Therefore, we are only interested in cases with $N$ being odd numbers. As an example, we shall only focus on the case with $N = 3$ below.

### A. Cosmos with 3 horizons

The metric for static and spherically symmetric spacetimes with 3 cosmic horizons could be written as Eq. (13) with

$$U(r) = (1 - h_1 r) (1 - h_2 r) (1 - h_3 r). \quad (68)$$

We assume $0 < h_1 < h_2 < h_3$. There are three cosmic horizons in the spacetime. The energy density $\rho$ and the pressures, $p_r, p_\theta, p_\phi$ of the anisotropic fluid have the form

$$\rho = \frac{1}{8\pi} \left( \frac{2\alpha}{r} - 3\beta + 4\gamma r \right), \quad (69)$$

$$p_r = -\rho, \quad (70)$$

$$p_\theta = \frac{1}{8\pi} \left( -\frac{\alpha}{r} + 3\beta - 6\gamma r \right), \quad (71)$$

$$p_\phi = p_\theta. \quad (72)$$

with

$$\alpha = h_1 + h_2 + h_3, \quad (73)$$

$$\beta = h_1 h_2 + h_1 h_3 + h_2 h_3, \quad (74)$$

$$\gamma = h_1 h_2 h_3. \quad (75)$$

The energy density is always positive provided that

$$9\beta^2 - 32\alpha \gamma < 0. \quad (76)$$

Substituting Eq. (68) into Eq. (74), we obtain the consistent solution

$$\phi = \phi_0 \left( -\alpha r^2 + \gamma r^4 \right), \quad (77)$$

$$F_{\mu\nu} F^{\mu\nu} = -8\phi_0^2 r^2 \left( 2\gamma r^2 - \alpha \right)^2, \quad (78)$$

$$K (F_{\mu\nu} F^{\mu\nu}) = \frac{2\alpha}{r} - 6\beta + 12\gamma r. \quad (79)$$

Here $\phi_0$ is an integration constant, and it is related to the electric charge of the spacetime

$$Q_e \equiv r^2 K, X_{F^{01}} = \frac{1}{4\phi_0}. \quad (80)$$

If

$$h_3 = 0, \quad h_1 = -\sqrt{\frac{\lambda}{3}}, \quad h_2 = \sqrt{\frac{\lambda}{3}}, \quad (81)$$

we have

$$\phi = 0, \quad F_{\mu\nu} F^{\mu\nu} = 0, \quad K (F_{\mu\nu} F^{\mu\nu}) = -2\lambda. \quad (82)$$

It is exactly the de Sitter (or Anti-de Sitter) solution.

Eq. (78) can be written as a cubic algebraic equation of $r$. Solving this equation for $r = r(F^2)$ and substituting the expression into Eq. (74), we find $K$ can be also explicitly expressed as the function of $F^2$.

### B. Penrose diagram

To plot the Penrose diagram, we introduce the tortoise coordinate $r_*$ as

$$r_* = -\sum_i \frac{\ln |1 - h_i r|}{(h_i - h_j) (h_i - h_k)}. \quad (83)$$

Here $i$ runs over from 1 to 3 and $i \neq j \neq k$. We make coordinates transformation $(t, r) \to (v, u)$

$$v = e^{r_*} \sinh \gamma t, \quad u = e^{r_*} \cosh \gamma t. \quad (84)$$
Then the metric Eq. (13) becomes
\[ ds^2 = F(v, u) \left( -dv^2 + du^2 \right) + r^2 d\Omega^2, \]
with
\[ F = \frac{1}{\gamma^2} \prod_i \left( 1 - h_i r \right)^{\frac{2 \gamma h_i}{(h_i - h_j)(h_i - h_k)}}. \]

Now we see the spacetime is regular at \( r = 0 \). In order to remove the coordinate singularity \( r_i \), we should let
\[ 1 + \frac{2 \gamma h_i}{(h_i - h_j)(h_i - h_k)} = 0, \]
namely,
\[ \gamma_i = -\frac{1}{2h_i} (h_i - h_j)(h_i - h_k). \]

From Eq. (83), we obtain
\[ u^2 - v^2 = e^{2\gamma r} \cdot \frac{\gamma \prod_i (1 - h_i r)^{\frac{1}{(h_i - h_j)(h_i - h_k)}}}{\prod_i (1 - h_i r)^{\frac{1}{(h_i - h_j)(h_i - h_k)}}}. \]

Substituting \( r = 1/h_i \) and \( \gamma = \gamma_i \) into above equation, we have
\[ u^2 - v^2 = 0. \]

So in the \( (v, u) \) coordinate system, the horizon \( r = 1/h_i \) is not a singularity. Furthermore, it consists of two lines \( u = \pm v \) in the \( (v, u) \) plane. Figure 5 shows the Penrose diagram of the cosmos with 3 horizons.

V. BLACK HOLES WITH \( N \) SINGULARITIES AND \( M \) HORIZONS

As argued in the introduction, the spacetime with multiple singularities and multiple horizons is highly likely to be produced in the merger process of multiple black holes. But it is rather involved to construct the analytic spacetime solutions describing this process. So far only a few solutions are present [2–6, 8, 10, 54]. In this section, we give the exact but toy-model spacetime with multiple singularities and multiple horizons. As toy models, the solutions are all static and spherically symmetric. Therefore, the multiple horizons are concentric spheres and the singularities consists of one singular point and multi-singular spheres. Some of these spacetimes may be constructed by using the Majumdar-Papapetrou solution [2].

In Majumdar-Papapetrou solution, a system of extreme Reissner-Nordstrom black holes (ERN) (each having electric charge equal to its mass) can remain in static equilibrium. No matter how the black holes are arranged in space, the electrostatic repulsions exactly balance the gravitational attractions. In other words, the black holes in Majumdar-Papapetrou solution ignore each other. Now suppose one ERN black hole is in the center and a lot of ERNs on a sphere as shown in Fig. 6. We expect a spacetime with a central singularity, a singular sphere (wavy circle) and three horizons (three solid circles) is present.

The metric with multiple horizons and multiple singu-
larities can be assumed to be Eq. (13) with

\[ U(r) = \frac{(1 - \frac{b_i}{r}) (1 - \frac{b_2}{r}) (1 - \frac{b_3}{r}) \cdots (1 - \frac{b_M}{r})}{(1 - \frac{a_1}{r}) (1 - \frac{a_2}{r}) (1 - \frac{a_3}{r}) \cdots (1 - \frac{a_{N-1}}{r})} \]

\[ = \frac{\prod_i^M (1 - \frac{b_i}{r})}{\prod_i^{N-1} (1 - \frac{a_i}{r})}. \quad (91) \]

We assume \( 0 < a_1 < a_2 < a_3 < \cdots < a_{N-1} < b_1 < b_2 < \cdots < b_M \) and \( 0 < N - 1 < M \). Then there are \( M \) black hole horizons, \( r_i = b_i \) and \( N \) singularities, \( r_i = a_i \) together with \( r = 0 \). All the singularities are concealed by the black hole horizons. We emphasize that the number \( M \) and \( N \) cannot be arbitrary in order that the energy density is positive. As for Fig. 6, the metric can be assumed to be Eq. (13) with

\[ U(r) = \frac{(1 - \frac{b_1}{r})^2 (1 - \frac{b_2}{r})^2 (1 - \frac{b_3}{r})^2}{(1 - \frac{a_1}{r})^2}. \quad (92) \]

In principle, one could also construct black hole solutions with one or more singularities concealed by two horizons. For example, one could assume \( b_1 < a_{N-1} < b_2 \) rather than \( a_{N-2} < a_{N-1} < b_1 \) in Eq. (77). In this case, the singularity \( a_{N-1} \) is concealed by two horizons \( b_1 \) and \( b_2 \).

A. Black hole with 2 singularities and 2 horizons

As an example, we focus on the case with \( N = 2 \) and \( M = 2 \) which is for the Reissner-Nordström black hole with a singular sphere inside the horizons. Then the metric is given by

\[ ds^2 = \frac{(1 - \frac{b_1}{r}) (1 - \frac{b_2}{r})}{1 - \frac{a_1}{r}} dt^2 + \frac{1 - \frac{a_2}{r}}{(1 - \frac{b_1}{r}) (1 - \frac{b_2}{r})} dr^2 + r^2 d\Omega_2^2. \quad (93) \]

When \( a_1 = 0 \), it reduces to the Reissner-Nordström black hole. When \( a_1 \neq 0 \), a singular sphere is present inside the black hole. The energy density \( \rho \) and the pressures, \( p_r, p_\theta, p_\phi \) of the matter source are

\[ \rho = \frac{1}{8\pi} \frac{(b_2 - a_1)(b_1 - a_1)}{r^2 (r - a_1)^2}, \quad (94) \]

\[ p_r = -\rho, \quad (95) \]

\[ p_\theta = -\frac{1}{8\pi} \frac{(b_2 - a_1)(b_1 - a_1)}{r (r - a_1)^2}, \quad (96) \]

\[ p_\phi = p_\theta. \quad (97) \]

It is apparent the energy density is always positive provided that \( 0 < a_1 < b_1 < b_2 \). When \( b_2 = 0 \) or \( b_1 = 0 \), the energy density is negative which violates the positive energy theorem. On the other hand, when \( b_1 = b_2 = 0 \), the singularity \( r = a_1 \) would be naked which violates the cosmic censorship conjecture. Therefore, in order to obey both the positive energy theorem and the cosmic censorship conjecture, there are at least two horizons to conceal the two singularities. We note that both the energy density and pressures are divergent at the singularity and the singular sphere. The solution of Eq. (93) corresponds to the generalized Maxwell theories as follows

\[ \phi = \phi_0 \frac{3a_1 - 4r}{r - a_1} \quad (98) \]

\[ F_{\mu\nu}F^{\mu\nu} = \frac{8\phi_0^2 (a_1 - 2r)^2}{(r - a_1)^2}, \quad (99) \]

\[ K (F_{\mu\nu}F^{\mu\nu}) = -\frac{2(b_2 - a_1)(b_1 - a_1)}{r (r - a_1)^3}, \quad (100) \]

where \( \phi_0 \) is an integration constant. The electric charge of the spacetime is

\[ Q_e \equiv i^2 K, X F^{01} = -\frac{(b_2 - a_1)(b_1 - a_1)}{4\phi_0}. \quad (101) \]

When

\[ b_1 = m - \sqrt{m^2 - Q^2}, \quad a_1 = 0, \quad (102) \]

\[ b_2 = m + \sqrt{m^2 - Q^2}, \quad \phi_0 = -\frac{Q}{4}, \quad (103) \]

they reduce to

\[ Q_e = Q, \quad K (F_{\mu\nu}F^{\mu\nu}) = F_{\mu\nu}F^{\mu\nu}, \quad (104) \]

which is exactly the Maxwell theory for the Reissner-Nordström spacetime.

Eq. (100) can be written as a cubic algebraic equation of \( r \). Solving this equation for \( r = r(F^2) \) and substituting the expression into Eq. (93), we find \( F^2 \) can be explicitly expressed as the function of \( K \).

B. Penrose diagram

Define the tortoise coordinate \( r_* \) as follows

\[ r_* = r + \frac{1}{b_1 - b_2} \left[ (a_1 b_2 - b_2) \ln |r - b_2| - (a_1 b_1 - b_1) \ln |r - b_1| \right]. \quad (105) \]

We make coordinates transformation \((t, r) \rightarrow (v, u)\)

\[ v = e^{\gamma r} \sinh \gamma t, \quad u = e^{\gamma r} \cosh \gamma t. \quad (106) \]

Then the metric Eq. (13) becomes

\[ ds^2 = F(v, u) (-dv^2 + du^2) + r^2 d\Omega_2^2, \quad (107) \]

with

\[ F = \frac{1}{\gamma^2 r (r - a_1) e^{2\gamma r} \prod_i |r - b_i|^{1 - \frac{2a_i}{b_i - b_j}}}. \quad (108) \]
Here \( i \) runs over from 1 to 2 and \( i \neq j \). In order to remove the coordinate singularity \( r_i \), we should let

\[
1 - \frac{2\gamma b_i (b_i - a_1)}{b_i - b_j} = 0 ,
\]

namely,

\[
\gamma_i = \frac{b_i - b_j}{2\gamma b_i (b_i - a_1)} .
\]

From Eq. (106), we obtain

\[
u^2 - v^2 = e^{2\gamma r} \cdot \frac{e^{2\gamma r} \prod_{i} (r - b_i)}{\prod_{i} |r - b_i|^{2+2b_i (b_i - a_1)/r - \gamma_i}} .
\]

Substituting \( r = b_i \) and \( \gamma = \gamma_i \) into above equation, we have

\[
u^2 - v^2 = 0 .
\]

So in the \((v, u)\) coordinate system, the horizon \( r = b_i \) is not nonsingular. It consists of two lines \( u = \pm v \) in the \((v, u)\) plane. At the true singularity \( r = a_1 \) and \( r = 0 \), we have

\[
u^2 - v^2 = \text{constant} > 0 , \quad u^2 - v^2 = \text{constant} < 0 ,
\]

respectively. Therefore in the \((v, u)\) coordinates, they are a pair of hyperbolae and the hyperbolae are timelike and spacelike, respectively. Figure 3 shows the Penrose diagram of the black hole spacetime with two horizons.

VI. MULTI-HORIZON BLACK HOLE IN MULTI-HORIZON UNIVERSE

The black hole spacetime with multiple singularities and multiple horizons given in the above section is asymptotically flat. In this section, we show that it is possible to construct a black hole spacetime with multiple black hole horizons and multiple cosmic horizons, which is not asymptotically flat. A metric for multi-horizon black hole in multi-horizon universe can be given by Eq. (13) with

\[
U(r) = \left(1 - \frac{a_1}{r}\right) \times \left(1 - \frac{a_2}{r}\right) \times \cdots \times \left(1 - \frac{a_N}{r}\right) \times \left(1 - \frac{r}{b_1}\right) \times \left(1 - \frac{r}{b_2}\right) \times \cdots \times \left(1 - \frac{r}{b_M}\right).
\]

Here we assume \( 0 < a_1 < a_2 < a_3 < \cdots < a_N < b_1 < b_2 < b_3 < \cdots < b_M < \infty \). Then the spacetime has \( M \) cosmic horizon \( r_i = b_i, N \) black hole horizons \( r_i = a_i \) and a singularity \( r = 0 \). The reconstruction of its source in the framework of nonlinear electromagnetic field theories can be done following the procedures shown in previous sections. Here we do not go into the detailed calculations.

VII. BLACK HOLE FIREWALL

In this section, we construct a black hole spacetime with the so-called firewall by using the nonlinear electromagnetic field theories. In order to resolve the black hole information loss paradox [13], Almheiri, Marolf, Polchinski and Sully (hereinafter, AMPS) raised the black hole firewall proposal [11]. In essence, AMPS argued that the local quantum field theory, unitary, and no-drama (the infalling observers could not experience anything unusual when crossing the event horizon) cannot all be consistent with each other. They found that the most conservative resolution to this inherent inconsistency is to give up no-drama. Instead, there is a firewall with considerable high energy density on or near the event horizon. As a result, an infalling observer would be terminated once he/she hits the firewall. Below we construct the model for a black hole with firewall by using the nonlinear electromagnetic field theories. So let’s start from Eq. (13).

A. Thin wall

The massive shell scenario of black hole firewall has been examined by a number of authors (e.g., [16, 22]), the calculations are a standard application of Einstein’s equations. In this subsection, we assume the firewall is a shell of matter positioned at \( r = r_0 \) and the thickness of
the shell is zero. We note that an exact solution describing a black hole surrounded by a thin massive shell has been investigated by Frauendiener et al. \[53\].

Then the expression of density for the wall (or shell) is given by the Dirac function

$$\rho = \frac{m_1}{4\pi r^2} \delta (r - r_0) , \quad (115)$$

such that the energy of the wall is

$$\int_0^\infty 4\pi r^2 \rho dr = m_1 . \quad (116)$$

Here the background metric Eq. (113) is taken into considerations. Substituting Eq. (115) into the Einstein equations, we obtain

$$U = 1 - \frac{2m_1}{r} \text{Heaviside} (r - r_0) - \frac{2m_2}{r} , \quad (117)$$

where the Heaviside function is related to the Dirac function by

$$[\text{Heaviside} (r - r_0)]' = \delta (r - r_0) , \quad (118)$$

and can be expressed as

$$\text{Heaviside} (r - r_0) = \begin{cases} 1 , & \text{if } r > r_0 ; \\
\text{undefined} , & \text{if } r = r_0 ; \\
0 , & \text{if } r < r_0 . \end{cases} \quad (119)$$

Here $m_2$ is the mass of the black hole and $m_1$ is the mass (or energy) of the firewall. The metric is then given by

$$ds^2 = - \left[ 1 - \frac{2m_1}{r} \text{Heaviside} (r - r_0) - \frac{2m_2}{r} \right] dt^2$$

$$+ \left[ 1 - \frac{2m_1}{r} \text{Heaviside} (r - r_0) - \frac{2m_2}{r} \right]^{-1} dr^2$$

$$+ r^2 d\Omega^2 . \quad (120)$$

It is the same as the solution in \[53\]. The firewall is supposed in the vicinity of event horizon which corresponds to $r_0 \geq 2m_2$. We show in the next section that the energy $m_1$ of the firewall could come from the electric field.

**B. Thick wall**

The Dirac function can be expressed as

$$\delta (r) = \lim_{\epsilon \to 0} \frac{1}{\pi r^2 + \epsilon^2} , \quad (121)$$

where $\epsilon$ is a constant. Then the density of the wall can be assumed as

$$\rho = \frac{m_1}{4\pi r^2} \frac{\epsilon}{(r - r_0)^2 + \epsilon^2} . \quad (122)$$

Substituting Eq. (122) into the Einstein equations, we obtain

$$U = 1 - \frac{2m_1}{r} \left[ \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{r - r_0}{\epsilon} \right) \right] - \frac{2m_2}{r} . \quad (123)$$

The metric is then given by

$$ds^2 = - \left\{ 1 - \frac{2m_1}{r} \left[ \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{r - r_0}{\epsilon} \right) \right] - \frac{2m_2}{r} \right\} dt^2$$

$$+ \left\{ 1 - \frac{2m_1}{r} \left[ \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{r - r_0}{\epsilon} \right) \right] - \frac{2m_2}{r} \right\}^{-1} dr^2$$

$$+ r^2 d\Omega^2 . \quad (124)$$

Here $m_2$ is the mass of the black hole and $m_1$ is the mass (or energy) of the firewall because $\int_0^\infty 4\pi r^2 \rho dr = m_1$. The solution corresponds to the generalized Maxwell theory as follows:

$$\phi = \frac{1}{2\pi} \frac{\phi_0 r\epsilon}{(r - r_0)^2 + \epsilon^2}$$

$$- \frac{3\phi_0}{2\pi} \arctan (r\epsilon^{-1} - r_0 \epsilon^{-1}) , \quad (125)$$

$$F_{\mu\nu}F^{\mu\nu} = - \frac{2\phi_0^2 \epsilon^2 (2r^2 - 3rr_0 + r_0^2 + \epsilon^2)^2}{\pi^2 (r - r_0)^2 + \epsilon^2} , \quad (126)$$

$$K (F_{\mu\nu}F^{\mu\nu}) = - \frac{4m_1 \epsilon (r - r_0)}{\pi r (r - r_0)^2 + \epsilon^2} , \quad (127)$$

The electric charge on the wall is

$$Q_\epsilon = r^2 K \epsilon F^{01} = \frac{m_1}{\phi_0} . \quad (128)$$

It is proportional to the total energy, $m_1$ of the wall. Thus we can conclude that the energy of the firewall could come from the electric field.

Eq. (128) can be written as a biquadratic algebraic equation of $r$. Solving this equation for $r = r (F^2)$ and substituting the expression into Eq. (127), we find $K$ can be be explicitly expressed as the function of $F^2$.

**VIII. DISCUSSION AND CONCLUSIONS**

By using the non-linear electromagnetic field theories, Ayon-Beato and Garcia proposed a powerful method to generate black hole solutions \[17\]. The method was successfully applied to build up the modified Reissner-Nordström solution and the Bardeen solution. It was later generalized by Dymnikova \[24\] for spherical and static solutions. More researches can be found in \[18–34\].

In this paper, we generalize the black hole and cosmos spacetime to the scenarios of multiple horizons and multiple singularities. They are solutions of general relativity with the non-linear electromagnetic field. The
solutions have regular event horizons, approaches asymptotically flat or non-flat, and all its singularities are concealed by the horizons. Since the spacetimes are all static and spherically symmetric, we could calculate the surface gravity (see [54] for details) and such that the temperatures. For example, we find the 4-horizon black hole temperature on the outermost horizon by the surface gravity as

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{(a_4 - a_1)(a_4 - a_2)(a_4 - a_3)}{a_4^2}.$$  \hspace{1cm} (129)

We have assumed $a_1 < a_2 < a_3 < a_4$. But when $a_3 = a_4$, we have $T = 0$ which is for the extreme black hole case. It is therefore interesting to discuss the thermodynamical properties since we can define the entropy and temperature in the spacetime. As an example, Ref. [56] investigates the thermodynamic instability problem of higher dimensional topological black holes in the presence of nonlinear electrodynamics. The extreme 4-horizon black holes have the vanishing surface gravity which means their gravitational attraction is exactly balanced by the Coulomb repulsion. So, motivated by the Kastor-Traschen solution, can we construct multiple 4-horizon black holes in the de Sitter universe? This is an open question. On the other hand, the orbits of test particles in the background of these spacetimes may be interesting since the causal structure of these spacetimes is rich. Continuum spectrum from black hole accretion disc holds enormous information regarding the black hole physics [55]. The signatures sculptured on these black hole might be distinct by the observations of continuum spectrum. Finally, by calculating the perturbations to the metric or the quasinormal modes of test fields, we hope the deviation of these black hole solutions may be potentially tested by future measurements of gravitational waves.

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| $a_4$ | $\omega$ ($l = 1$) | $\omega$ ($l = 2$) | $\omega$ ($l = 3$) |
|------|--------------------|--------------------|--------------------|
| 4.0  | 0.085240-0.018464i | 0.141767-0.017561i | 0.198349-0.017305i |
| 3.9  | 0.086245-0.018599i | 0.143464-0.017689i | 0.200732-0.017429i |
| 3.8  | 0.087263-0.018736i | 0.145186-0.017821i | 0.203150-0.017559i |
| 3.7  | 0.088304-0.018911i | 0.146933-0.017957i | 0.205604-0.017693i |
| 3.6  | 0.089358-0.019066i | 0.148707-0.018104i | 0.208093-0.017832i |
| 3.5  | 0.090424-0.019219i | 0.150504-0.018254i | 0.210616-0.017980i |
| 3.4  | 0.091508-0.019378i | 0.152325-0.018405i | 0.213173-0.018130i |
| 3.3  | 0.092608-0.019555i | 0.154172-0.018551i | 0.215763-0.018290i |
| 3.2  | 0.093721-0.019715i | 0.156042-0.018740i | 0.218385-0.018458i |
| 3.1  | 0.094851-0.019920i | 0.157935-0.018919i | 0.221037-0.018630i |
| 3.0  | 0.095997-0.020132i | 0.159849-0.019106i | 0.223721-0.018819i |

TABLE I: The fundamental ($n = 0$) quasinormal frequencies of scalar field for the 4-horizon black hole for $l = 1, 2, 3$. 