Solving the Subset Sum Problem with Heap-Ordered Subset Trees

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Abstract

In the field of algorithmic analysis, one of the more well-known exercises is the subset sum problem. That is, given a set of integers, determine whether one or more integers in the set can sum to a target value. Aside from the brute-force approach of verifying all combinations of integers, several solutions have been found, ranging from clever uses of various data structures to computationally-efficient approximation solutions. In this paper, a unique approach is discussed which builds upon the existing min-heap solution for positive integers, introducing a tree-based data structure influenced upon the existing min-heap solution for positive integers, termed the subset tree, in which all subsets of \( O \) length are generated as needed during the binary search phase of the algorithm rather than building the complete subset tree upfront. To account for sets with negative integers when building the subset tree, each value is offset by a value that produces a set of strictly positive integers. This value is the absolute value of the minimum negative integer in the set plus one. The complexity of this algorithm is \( O(N^3 k \log k) \), where \( N \) is the length of the set and \( k \) is the index of the subset tree being located during the binary search.

A sample implementation of this algorithm has been provided. The provided implementation includes sample code for each section discussed in this paper.

Prior Work

A dynamic programming solution exists in pseudo-polynomial time \( \tilde{O}(sN) \), where \( s \) is the target value and \( N \) is the length of the set. This approach maintains an array of Boolean values \( Q(N, s) \) and employs recursive arithmetic operations for each element in the set from the index of the target value. The value of the array at each index computed by the algorithm is set to true, indicating that there exists a subset in the set which sums to the value of the index. The solution to whether a subset exists which sums to the target value can then be found by accessing the Boolean value of \( Q(N, s) \). The complexity of this algorithm has since been improved to \( \tilde{O}(s \sqrt{N}) \) (Koiliaris and Xu, 2015).

Improvements to the exponential time algorithm have been explored, bringing the running time down to \( O(2^{N/2}) \) (Horowitz and Sahni, 1974). The algorithm takes a set of length \( N \) and partitions the elements into two sorted sets of \( N/2 \) each. One list is stored for each of the two sets, with each list consisting of the sums for all \( 2^{N/2} \) possible subsets. The algorithm maintains two pointers: one pointer starting at the smallest sum in the first list and the other pointer starting at the largest sum in the second list. If the sum of these two sums equals the target value, then the search terminates. If the sum is less than the target value, then the first pointer is incremented by one position. If the sum is greater than the target value, then the second pointer is decremented by one position.
Algorithm 1: Binary min-heap for subsets of positive integers

Input: S, t
Output: O
1 Sort S into increasing order
2 Define a tree O where:
   (I) The root is the singleton consisting only of the first element
   (II) The left child of a set whose maximum element is $i$th in the sorted input list is obtained by replacing that element by element $i + 1$
   (III) The right child is obtained by including element $i + 1$ without removing any existing element
3 Return $O$

one position. This process continues until the target value is found or the search space has been exhausted.

This $O(2^{N/2})$ solution has been improved through the use of min-heaps (Schroeppe1 and Shamir, 1981). Rather than evaluating and sorting the list of sums for all subsets in the divided set, these sums are generated in order through the use of the min-heap data structure. The first list is divided into two additional lists $a$ and $b$, resulting in a length of $N/4$. The subsets from each list are generated and sorted in increasing order. A min-heap is then constructed from the pairs of subsets from $a$ and $b$. As each subset from the min-heap is needed, it is popped from the top of the heap and the head element $(a_i, b_j)$ is replaced with the subset $(a_{i+1}, b_j)$. Subsequently, all pairs of subsets from $a$ and $b$ are generated in order of their total sum. Likewise, the second list is divided into two additional lists and the subsets are generated in order. However, these subsets are generated in decreasing order via a max-heap. The aforementioned exponential algorithm is then applied in which the subsets of the first list are incremented and the subsets of the second list are decremented as necessary until the target value is found or both heaps become empty. The running time of this method is $O(N2^{N/2})$.

Min-Heap Solution with Positive Integers

Building upon the prior work with min-heaps of subsets, a binary min-heap may be used to extend the solution to the initial input set as a whole when the input set contains only positive integers. A binary min-heap is a binary tree whose parent nodes are less than or equal to their child nodes. Because it makes no guarantee of total sorting (i.e. the root node is always the smallest in the tree), it is considered a partially-ordered data structure. Assuming all positive integers, Algorithm 1 is used to generate a tree of subsets in order of increasing sum, where $S$ is the input set of positive integers, $t$ is the target value, and $O$ is the subset that sums to the target value.

Figure 1 visualizes the complete min-heap binary tree for the set $\{2, 5, 7\}$. By repeating Step 2 of Algorithm 1 for each node, the heap-ordered binary tree is generated as demonstrated.

The $k$ smallest elements of the heap-ordered tree can be found in time $O(k)$ (Frederickson, 1993). This is done through the hierarchical grouping of particular elements in the heap and maintaining them in a recursively defined data structure. An alternate solution to finding the $k$th smallest element of a min-heap is shown in Algorithm 2, where the input $T$ is a min-heap binary tree of subsets. In this case, the lookup is performed in time $O(k \log k)$.

Because each child node is trivially derived from its parent node, the search may be performed in a lazy manner and child nodes may be generated only as required for the search. In the aforementioned example $\{2, 5, 7\}$, if the 4th smallest subset is desired, then the root node of $\{2\}$ and its children, $\{5\}$ and $\{2, 5\}$, are generated. Per Algorithm 2, the subsequent child nodes for both of the root node’s child nodes are generated in the same step as adding the root node’s children to $M$. Thus, the lookup is effectively performed on a virtual array without the need to generate all nodes in the power set of the input.

A binary search can be performed on the sorted list of subsets that can be made from the input set. Since there are $2^N$ subsets in total, a binary search to find a target sum $t$ can be performed in time $O(\log 2^N)$. This reduces to $O(N \log 2)$, or $O(N)$. Since each search for the $k$th largest element takes $O(k \log k)$ time, a binary search on the virtual sorted list of all subsets to find the sum will take $O(Nk \log k)$ time, where $k \leq 2^N$.

However, if negative integers are included in the initial set alongside positive integers, then the approach taken to construct a heap-ordered binary tree will not suffice, as Algorithm 1 would place negative integers in the child nodes of subsets of strictly positive integers and violate the properties of the min-heap. Therefore, a modification to Algorithm 1 is necessary to handle cases of sets with both positive and negative integers.

Inclusion of Negative Integers

Assume a set of integers with both positive and negative values. To continue using a heap-ordered tree constructed in a similar fashion as previously discussed, all values in the set must be scaled by some offset value to be made positive. This can be done by adding the absolute value of the least element plus one to all elements. The offset applied to each element must also be stored for future use in the algorithm.
As an example, take the set \{-7, -3, -2, 5, 8\} and the target sum 0. The scaled input is \{1, 5, 6, 13, 16\}. The offset value 8 must also be stored, as it will be applied to the target sum later.

Despite this new set consisting of all positive values, Algorithm 1 cannot be applied as is. This is due to the lack of a consistent target value for subsets of different lengths. For example, although \{-3, -2, 5\} sums to 0, the scaled subset \{5, 6, 13\} will only sum to the target value by adding the offset to the target three times. In this case, the target value must be scaled from 0 to 24. Generally, the target sum can only be compared to a scaled subset of size \(n\) if \((\text{offset } n)\) is added to the target value itself. Therefore, \(N\) min-heaps must be generated, with each one consisting of all subsets of equal length. A binary search may then be performed on each min-heap.

Maintaining an ordered min-heap of subsets of the same length can be accomplished through the use of a unique data structure similar to the binomial heap, introduced here as the heap-ordered subset tree.

### The Heap-Ordered Subset Tree

The subset tree is a heap-ordered tree structure consisting of \(n\)-length subsets of a set \(S\) of length \(N\). The process of constructing a subset tree \(T\) of order \(n\) from set \(S\) is detailed in Algorithm 3.

For a subset tree of order \(n\), the total number of nodes is bounded at \(\binom{N}{n}\), where \(N\) is the total number of elements in the input set. Each node has at most \(N\) children and the height of the tree is at most \(N\).

To demonstrate a simple example, Figure 2 displays the 4-subset tree for the set \{1, 2, 3, 4, 5, 6\}.

As Algorithm 3 must be performed with subsets consisting of only positive elements, this operation is performed on the scaled set as opposed to the initial set.

Although this operation generates the tree in its entirety, it is trivial to generate nodes up to the \(k\)th smallest element due to the partial ordering in which the subset tree is constructed. This optimization minimizes the space needed for subset trees to be stored, which is especially useful for subsets of an arbitrarily large input set.

### Subset Sum Solution with the Subset Tree

The complete algorithm for the subset sum problem for sets of both positive and negative integers is provided in Algorithm 4, where \(S\) is a given input set of length \(N\), \(t\) is the target value, and \(R\) is the returned subset.

The solution begins by sorting the set and uniformly scaling each member of the set to a positive value. For 1 to \(N\), where \(N\) is the length of the input set, a binary search is performed on all subsets of the appropriate length. The subsets are generated in order of their sum with the use of the subset tree. This iteration is continued until a subset is found which sums to the target value or the search space has been exhausted. In the case that no subset of a given set exists for a given target value, an empty subset is returned.

The sorting step of the algorithm is independent to its implementation. The binary search is performed in time logarithmic to the size of the binomial coefficient \(\binom{N}{n}\), where \(N\) is the length of the input set and \(n\) is the order of the
The subset sum problem continues to offer unique approaches to its solution through the use of various methodologies deviating from the naïve brute-force approach, ranging from dynamic programming solutions to uses of data structures such as the min-heap to provide an iterative search space of subsets of positive integers. Building upon existing heap-based solutions, this paper introduced the subset tree data structure, allowing for proper heap-ordering of all subsets of an arbitrary length despite whether the input set consists of positive elements, negative elements, or both. Standard kth-minimum lookup procedures can then be performed through a binary search on all subsets of a particular length \( n \). Iterating through all possible lengths of subsets from a given set, target values may be found in time \( O(N^3k\log k) \), where \( N \) is the length of the set and \( k \) is the index of the list of subsets that is being searched.

Future work in this area may include improving the algorithm to allow for heap-ordering of subsets from a set of both positive and negative elements in a single tree, removing the need for the iteration step over subset trees of a particular order. This improvement would also have the auxiliary benefit of removing the need to apply and store an offset value to the initial set, further optimizing the space required for the algorithm to find a solution.

### References

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### Algorithm 4: Subset sum solution with subset trees

**Input:** \( S, t \)  
**Output:** \( R \)

1. Sort \( S \)
2. Offset each element in \( S \) by the absolute value of the least element in \( S \) plus one
3. Store this \( offset \) value
4. Define a variable \( O \) and set it to \( 1 \). This is the order of the virtual subset heap being searched.
5. While \( O \leq N \) and a subset \( R \) that sums to \( S + (offset * O) \) has not been found:
   (I) Perform a binary search for \( S + (offset * O) \) on the \( O \)-subset tree \( T \)
   (II) If a subset \( R \) is found, then terminate the loop. Else, increment \( O \) by \( 1 \) and repeat Step 5(I). Do this until a subset has been found or \( O > N \)
6. If a subset \( R \) has been found, then subtract \( offset \) from each element in \( R \) and return \( R \). Else, return an empty subset.