Abstract

For indices $p$ and $q$, $1 < p \leq q < +\infty$ and a sublinear operator $L$ satisfying some weak-type boundedness conditions on suitable function spaces, we give in the Dunkl setting sufficient conditions on non-negative pairs of weight functions $u, v$ on $\mathbb{R}^d$ in order that $L$ satisfies a weighted strong-type $(p,q)$ inequality. We apply our results to obtain weighted $L^p \rightarrow L^q$ boundedness of the Riesz potentials and of the related fractional maximal operators associated to the Dunkl transform. We deduce finally a weighted generalized Sobolev inequality.

Keywords: Dunkl operators, Dunkl transform, Riesz potentials, fractional maximal operators.

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1 Introduction

Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^d$. The Dunkl operators introduced by C.F. Dunkl in [9] are differential-difference operators $T_i, 1 \leq i \leq d$ associated to an arbitrary finite reflection group $W$ on $\mathbb{R}^d$. These operators attached with a root system $R$ and a non negative multiplicity function $k$, can be considered as perturbations of the usual partial

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derivatives by reflection parts. They provide a useful tool in the study of special functions with root systems. Moreover, the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Moser-Sutherland models, which deal with systems of identical particles in a one-dimensional space (see [21]). Dunkl theory was further developed by several mathematicians (see [8, 11, 15, 17]), later was applied and generalized in different ways by many authors (see [1, 2, 3, 4, 18]) and has gained considerable interest in mathematical physics. The Dunkl kernel $E_k$ has been introduced by C.F. Dunkl in [10]. For a family of weight functions $w_k$ invariant under a reflection group $W$, we use the Dunkl kernel and the measure $w_k(x)dx$ to define the Dunkl transform $F_k$, which enjoys properties similar to those of the classical Fourier transform $F$. If the parameter $k \equiv 0$ then $w_k(x) = 1$, so that $F_k$ becomes $F$ and the $T_i, 1 \leq i \leq d$ reduce to the corresponding partial derivatives $\frac{\partial}{\partial x_i}, 1 \leq i \leq d$. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis (see next section, Remark 2.1).

Given the important role of the Fourier transform in analysis, one naturally asks if it is possible to extend results obtained for the classical Fourier analysis to the Dunkl theory. The classical Fourier transform behaves well with the translation operator $f \mapsto f(\cdot - y)$, which leaves the Lebesgue measure on $\mathbb{R}^d$ invariant. However, the measure $w_k(x)dx$ is no longer invariant under the usual translation. One ends up with the Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, introduced by K. Trimèche in [20] on the space of infinitely differentiable functions on $\mathbb{R}^d$. An explicit formula for the Dunkl translation $\tau_x(f)$ of a radial function $f$ is known. In particular, the boundedness of $\tau_x$ is established in this case. As a result one obtains a formula for the convolution $*_k$ (see next section).

In this paper, we consider a sublinear operator $\mathcal{L}$ satisfying some weak-type boundedness conditions on suitable function spaces. Our aim is to give for $1 < p \leq q < +\infty$, sufficient conditions on the decreasing rearrangement of non-negative locally integrable weight functions $u, v$ on $\mathbb{R}^d$, such that $\mathcal{L}$ satisfies the weighted inequality

$$\left( \int_{\mathbb{R}^d} |\mathcal{L}(f)(y)|^q u(y) d\nu_k(y) \right)^{\frac{1}{q}} \leq c \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) d\nu_k(x) \right)^{\frac{1}{p}},$$

(1.1)

for $f \in L^p_{k,v}(\mathbb{R}^d)$ with $L^p_{k,v}(\mathbb{R}^d)$ is the space $L^p(\mathbb{R}^d, v(x) d\nu_k(x))$ and $\nu_k$ the weighted measure associated to the Dunkl operators defined by

$$d\nu_k(x) := w_k(x)dx \quad \text{where} \quad w_k(x) = \prod_{\xi \in \mathbb{R}^d^+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.$$
\(\langle \cdot, \cdot \rangle\) being the standard Euclidean scalar product on \(\mathbb{R}^d\) and \(R_+\) a positive root system (see next section). As applications of our results, we consider for \(0 < \alpha < 2\gamma + d\) with \(\gamma = \sum_{\xi \in R_+} k(\xi),\) the Riesz potential \(I_\alpha^k\) associated to the Dunkl transform, defined on the Schwartz space \(S(\mathbb{R}^d)\) by

\[
I_\alpha^k f(x) = 2^{\gamma + \frac{d}{2} - \alpha} \frac{\Gamma(\gamma + \frac{d - \alpha}{2})}{\Gamma(\frac{d}{2})} \int_{\mathbb{R}^d} \frac{\tau_y f(x)}{\|y\|_2^{\gamma + d - \alpha}} d\nu_k(y),
\]

(1.2)

We apply the inequality (1.1) to obtain weighted \(L^p \to L^q\) boundedness of the Riesz potential \(I_\alpha^k\) and of the related fractional maximal operator \(M_{k, \alpha}\) given by

\[
M_{k, \alpha} f(x) = \sup_{r > 0} \frac{1}{m_k r^{d+2\gamma - \alpha}} \int_{\mathbb{R}^d} |f(y)| \tau_x \chi_{B_r}(y) d\nu_k(y), \quad x \in \mathbb{R}^d,
\]

(1.3)

for \(f \in L^2_k(\mathbb{R}^d)\) with \(L^2_k(\mathbb{R}^d)\) is the space \(L^2(\mathbb{R}^d, d\nu_k(x))\) and

\[
m_k = \left( \frac{c_k 2^{\gamma + \frac{d}{2} + 1}}{2^{d+2\gamma} \Gamma(\gamma + \frac{d}{2} + 1)} \right) \frac{\alpha^{-\alpha}}{d+2\gamma - \alpha} \quad \text{with} \quad c_k^{-1} = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} w_k(x) dx.
\]

\(\chi_{B_r}\) denotes the characteristic function of the ball \(B_r\) of radius \(r\) centered at 0 (see next section). These results are generalizations of those obtained by Heinig for the classical case in [14]. Finally, we prove a weighted generalized Sobolev inequality.

The contents of this paper are as follows.
In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.
The section 3 is devoted to the proof of the weighted \(L^p \to L^q\) boundedness of the sublinear operator \(L\) when the weights \(u, v\) satisfy some conditions.
In section 4, as examples, we apply our results to the Riesz potential operators and the related fractional maximal operators. We obtain finally a weighted generalized Sobolev inequality.

Along this paper, we use \(c\) to denote a suitable positive constant which is not necessarily the same in each occurrence and we write for \(x \in \mathbb{R}^d, \|x\| = \sqrt{\langle x, x \rangle}\). Furthermore, we denote by

- \(\mathcal{E}(\mathbb{R}^d)\) the space of infinitely differentiable functions on \(\mathbb{R}^d\).
- \(\mathcal{S}(\mathbb{R}^d)\) the Schwartz space of functions in \(\mathcal{E}(\mathbb{R}^d)\) which are rapidly decreasing as well as their derivatives.
- \(\mathcal{D}(\mathbb{R}^d)\) the subspace of \(\mathcal{E}(\mathbb{R}^d)\) of compactly supported functions.
2 Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the surveys [16].

Let $W$ be a finite reflection group on $\mathbb{R}^d$, associated with a root system $R$. For $\alpha \in R$, we denote by $\mathbb{H}_\alpha$ the hyperplane orthogonal to $\alpha$. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} \mathbb{H}_\alpha$, we fix a positive subsystem $R_+ = \{ \alpha \in R : \langle \alpha, \beta \rangle > 0 \}$. We denote by $k$ a nonnegative multiplicity function defined on $R$ with the property that $k$ is $W$-invariant. We associate with $k$ the index

$$ \gamma = \sum_{\xi \in R_+} k(\xi), $$

and a weighted measure $\nu_k$ given by

$$ d\nu_k(x) := w_k(x)dx \quad \text{where} \quad w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d, $$

Further, we introduce the Mehta-type constant $c_k$ by

$$ c_k = \left( \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} w_k(x)dx \right)^{-1}. $$

For every $1 \leq p \leq +\infty$, we denote respectively by $L^p_k(\mathbb{R}^d)$, $L^p_{k,u}(\mathbb{R}^d)$, $L^p_{k,v}(\mathbb{R}^d)$ the spaces $L^p(\mathbb{R}^d, d\nu_k(x))$, $L^p(\mathbb{R}^d, u(x)d\nu_k(x))$, $L^p(\mathbb{R}^d, v(x)d\nu_k(x))$ and we use respectively $\| \|_{p,k}$, $\| \|_{p,k,u}$, $\| \|_{p,k,v}$ as a shorthand for $\| \|_{L^p_k(\mathbb{R}^d)}$, $\| \|_{L^p_{k,u}(\mathbb{R}^d)}$, $\| \|_{L^p_{k,v}(\mathbb{R}^d)}$.

By using the homogeneity of degree $2\gamma$ of $w_k$, it is shown in [15] that for a radial function $f$ in $L^1_1(\mathbb{R}^d)$, there exists a function $F$ on $[0, +\infty)$ such that $f(x) = F(|x|)$, for all $x \in \mathbb{R}^d$. The function $F$ is integrable with respect to the measure $r^{2\gamma + d - 1}dr$ on $[0, +\infty)$ and we have

$$ \begin{align*}
\int_{\mathbb{R}^d} f(x) d\nu_k(x) &= \int_0^{+\infty} \left( \int_{S^{d-1}} f(ry)w_k(ry)d\sigma(y) \right)r^{d-1}dr \\
&= \int_0^{+\infty} \left( \int_{S^{d-1}} w_k(ry)d\sigma(y) \right)F(r)r^{d-1}dr \\
&= d_k \int_0^{+\infty} F(r)r^{2\gamma + d - 1}dr,
\end{align*} $$

(2.1)
where $S^{d-1}$ is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$d_k = \int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma + \frac{d}{2}} - 1 \Gamma(\gamma + \frac{d}{2})}. \tag{2.2}$$

The Dunkl operators $T_j$, $1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group $W$ and the multiplicity function $k$ are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\rho_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in \mathcal{E}(\mathbb{R}^d), \quad x \in \mathbb{R}^d,$$

where $\rho_\alpha$ is the reflection on the hyperplane $\mathbb{H}_\alpha$ and $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$.

**Remark 2.1** In the case $k \equiv 0$, the weighted function $w_k \equiv 1$ and the measure $\nu_k$ associated to the Dunkl operators coincide with the Lebesgue measure. The $T_j$ reduce to the corresponding partial derivatives. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis.

For $y \in \mathbb{C}^d$, the system

$$\begin{cases}
T_j u(x, y) = y_j u(x, y), & 1 \leq j \leq d, \\
 u(0, y) = 1.
\end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $E_k(x, y)$ and called the Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. We have for all $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$, $E_k(z, z') = E_k(z', z)$, $E_k(\lambda z, z') = E_k(z, \lambda z')$ and for $x, y \in \mathbb{R}^d$, $|E_k(x, iy)| \leq 1$.

The Dunkl transform $\mathcal{F}_k$ is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) d\nu_k(y), \quad x \in \mathbb{R}^d.$$

We list some known properties of this transform:

i) The Dunkl transform of a function $f \in L^1_k(\mathbb{R}^d)$ has the following basic property

$$\|\mathcal{F}_k(f)\|_{\infty,k} \leq \|f\|_{1,k}.$$
ii) The Dunkl transform is an automorphism on the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \).

iii) When both \( f \) and \( F_k(f) \) are in \( L^1_k(\mathbb{R}^d) \), we have the inversion formula

\[
f(x) = \int_{\mathbb{R}^d} F_k(f)(y) E_k(ix, y) d\nu_k(y), \quad x \in \mathbb{R}^d.
\]

iv) (Plancherel’s theorem) The Dunkl transform on \( \mathcal{S}(\mathbb{R}^d) \) extends uniquely to an isometric automorphism on \( L^2_k(\mathbb{R}^d) \).

v) For \( f \in \mathcal{S}(\mathbb{R}^d) \) and \( 1 \leq j \leq d \), we have

\[
F_k(T_j f)(\xi) = i\xi_j F_k(f)(\xi), \quad \xi \in \mathbb{R}^d.
\] (2.3)

K. Trimèche has introduced in [20] the Dunkl translation operators \( \tau_x \), \( x \in \mathbb{R}^d \), on \( \mathcal{E}(\mathbb{R}^d) \). For \( f \in \mathcal{S}(\mathbb{R}^d) \) and \( x, y \in \mathbb{R}^d \), we have

\[
F_k(\tau_x(f))(y) = E_k(ix, y) F_k(f)(y).
\]

Notice that for all \( x, y \in \mathbb{R}^d \), \( \tau_x(f)(y) = \tau_y(f)(x) \) and for fixed \( x \in \mathbb{R}^d \)

\( \tau_x \) is a continuous linear mapping from \( \mathcal{E}(\mathbb{R}^d) \) into \( \mathcal{E}(\mathbb{R}^d) \).

As an operator on \( L^2_k(\mathbb{R}^d) \), \( \tau_x \) is bounded. A priori it is not at all clear whether the translation operator can be defined for \( L^p \)-functions with \( p \) different from 2. However, according to ([18], Theorem 3.7), the operator \( \tau_x \) can be extended to the space of radial functions \( L^p_k(\mathbb{R}^d)^{rad} \), \( 1 \leq p \leq 2 \) and we have for a function \( f \) in \( L^p_k(\mathbb{R}^d)^{rad} \),

\[
\| \tau_x(f) \|_{p,k} \leq \| f \|_{p,k}. \quad (2.4)
\]

The Dunkl convolution product \( \ast_k \) of two functions \( f \) and \( g \) in \( L^2_k(\mathbb{R}^d) \) is given by

\[
(f \ast_k g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y) g(y) d\nu_k(y), \quad x \in \mathbb{R}^d.
\]

The Dunkl convolution product is commutative and for \( f, g \in \mathcal{D}(\mathbb{R}^d) \), we have

\[
F_k(f \ast_k g) = F_k(f) F_k(g). \quad (2.5)
\]
Weighted norm inequalities

It was shown in ([18], Theorem 4.1) that when \( g \) is a bounded radial function in \( L^1_k(\mathbb{R}^d) \), then

\[
(f *_k g)(x) = \int_{\mathbb{R}^d} f(y) \tau_x(g)(-y) d\nu_k(y), \quad x \in \mathbb{R}^d,
\]

initially defined on the intersection of \( L^1_k(\mathbb{R}^d) \) and \( L^2_k(\mathbb{R}^d) \) extends to \( L^p_k(\mathbb{R}^d) \), \( 1 \leq p \leq +\infty \) as a bounded operator. In particular,

\[
\|f *_k g\|_{p,k} \leq \|f\|_{p,k} \|g\|_{1,k}.
\]

From (1.2), it was shown in [13] that we can write the Riesz potential \( I_k^\alpha \) in the form

\[
I_k^\alpha f(x) = \frac{2^{\gamma + \frac{d}{2} - \alpha}}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^{+\infty} s^{\gamma + \frac{d-\alpha}{2}} \int_{\mathbb{R}^d} f(y) \tau_x(e^{-s\|y\|}) d\nu_k(y) \frac{ds}{s}.
\]

According to (2.4) and using this identity, the authors in [13] extend the mapping \( f \mapsto I_k^\alpha f \) to the space \( L^r_k(\mathbb{R}^d) \), \( r \geq 1 \). More precisely, they obtain the following Hardy-Littlewood-Sobolev theorem.

**Theorem 2.1** Let \( 0 < \alpha < 2\gamma + d \). Then

i) for \( f \in L^r_k(\mathbb{R}^d) \), \( 1 < r < \frac{2\gamma + d}{\alpha} \), the mapping \( f \mapsto I_k^\alpha f \) is of strong-type \((r, \ell = \frac{1}{r - \frac{\alpha}{2\gamma + d}})\) and one has

\[
\|I_k^\alpha f\|_{k,\ell} \leq c \|f\|_{k,p}.
\]

ii) for \( f \in L^1_k(\mathbb{R}^d) \), the mapping \( f \mapsto I_k^\alpha f \) is of weak-type \((1, \frac{1}{1 - \frac{\alpha}{2\gamma + d}})\) and one has for any \( \lambda > 0 \),

\[
\int_{\{x \in \mathbb{R}^d : |I_k^\alpha f(x)| > \lambda\}} d\nu_k(x) \leq c \left( \frac{\|f\|_{k,1}}{\lambda} \right)^{\frac{1}{1 - \frac{\alpha}{2\gamma + d}}},
\]

**Remark 2.2** The boundedness of Riesz potentials can be used to establish the boundedness properties of the fractional maximal operator given by (1.3). This follows from the fact that

\[
M_{k,\alpha} f(x) \leq c I_k^\alpha (|f|)(x), \quad x \in \mathbb{R}^d.
\]

Hence we deduce for \( M_{k,\alpha} \), the same results obtained in Theorem 2.1.
3 Weighted norm inequalities

In this section, we prove the weighted $L^p \rightarrow L^q$ boundedness of a sublinear operator when the weights satisfy some conditions. As examples, we apply our results to the Riesz potential operators and the related fractional maximal operators. We denote by $p'$ the conjugate of $p$ for $1 < p < +\infty$. The proof requires a useful well-known facts which we shall now state in the following remark.

Remark 3.1

1/ (see [6]) (Hardy inequalities) If $\mu$ and $\vartheta$ are locally integrable weight functions on $(0, +\infty)$ and $1 < p \leq q < +\infty$, then there is a constant $c > 0$ such that for all non-negative Lebesgue measurable function $f$ on $(0, +\infty)$, the inequality

$$
\left( \int_0^{+\infty} \left[ \int_0^t f(s)ds \right]^q \mu(t)dt \right)^{\frac{1}{q}} \leq c \left( \int_0^{+\infty} (f(t))^p \vartheta(t)dt \right)^{\frac{1}{p}}
$$

(3.1)

is satisfied if and only if

$$
\sup_{s>0} \left( \int_s^{+\infty} \mu(t)dt \right)^{\frac{1}{q}} \left( \int_0^s (\vartheta(t))^{1-p'} dt \right)^{\frac{1}{p'}} < +\infty.
$$

(3.2)

Similarly for the dual operator,

$$
\left( \int_0^{+\infty} \left[ \int_t^{+\infty} f(s)ds \right]^q \mu(t)dt \right)^{\frac{1}{q}} \leq c \left( \int_0^{+\infty} (f(t))^p \vartheta(t)dt \right)^{\frac{1}{p}}
$$

(3.3)

is satisfied if and only if

$$
\sup_{s>0} \left( \int_0^s \mu(t)dt \right)^{\frac{1}{q}} \left( \int_s^{+\infty} (\vartheta(t))^{1-p'} dt \right)^{\frac{1}{p'}} < +\infty.
$$

(3.4)

2/ Let $f$ be a complex-valued $\nu_k$-measurable function on $\mathbb{R}^d$. The distribution function $D_f$ of $f$ is defined for all $s \geq 0$ by

$$
D_f(s) = \nu_k(\{x \in \mathbb{R}^d : |f(x)| > s\}).
$$

The decreasing rearrangement of $f$ is the function $f^*$ given for all $t \geq 0$ by

$$
f^*(t) = \inf \{s \geq 0 : D_f(s) \leq t\}.
$$

We list some known results:
• Let \( f \in L^p_k(\mathbb{R}^d) \) and \( 1 \leq p < +\infty \), then
\[
\int_{\mathbb{R}^d} |f(x)|^p d\nu_k(x) = p \int_0^{+\infty} s^{p-1} D_f(s) ds = \int_0^{+\infty} (f^*(t))^p dt.
\]

• (see \cite{12}) (Hardy-Littlewood rearrangement inequality)
Let \( f \) and \( \nu \) be non negative \( \nu_k \)-measurable functions on \( \mathbb{R}^d \), then
\[
\int_{\mathbb{R}^d} f(x) \nu(x) d\nu_k(x) \leq \int_0^{+\infty} f^*(t) \nu^*(t) dt \tag{3.5}
\]
and
\[
\int_0^{+\infty} f^*(t) \frac{1}{\left(\frac{t}{\lambda}\right)^*} dt \leq \int_{\mathbb{R}^d} f(x) \nu(x) d\nu_k(x). \tag{3.6}
\]

• (see A. P. Calderón \cite{7}) Let \( 1 \leq p_1 < p_2 < \infty \) and \( 1 \leq q_1 < q_2 < +\infty \). A sublinear operator \( \mathcal{L} \) satisfies the weak-type hypotheses \((p_1, q_1)\) and \((p_2, q_2)\) if and only if
\[
(\mathcal{L}f)^*(t) \leq c \left( t^{-\frac{1}{q_1}} \int_0^{\frac{t}{\lambda_2}} s^{\frac{1}{p_1} - 1} f^*(s) ds + t^{-\frac{1}{q_2}} \int_{\frac{t}{\lambda_2}}^{+\infty} s^{\frac{1}{p_2} - 1} f^*(s) ds \right) \tag{3.7}
\]
where \( \lambda_1 = \frac{1}{p_1} - \frac{1}{q_1} \) and \( \lambda_2 = \frac{1}{p_2} - \frac{1}{q_2} \).

**Example 3.1** Let \( \delta < 0 \) and \( \beta > 0 \). Take \( u(x) = ||x||^\delta \), \( v(x) = ||x||^\beta \), and \( f(x) = \chi_{(0,r)}(||x||) \), \( x \in \mathbb{R}^d, r > 0 \). Then using (2.1) and (2.2), we have for \( s \geq 0 \),
\[
D_u(s) = \nu_k \left( \{ x \in \mathbb{R}^d : ||x||^\delta > s \} \right) = \nu_k \left( B(0, s^{\frac{1}{\Delta_1}}) \right) = \frac{d_k}{2\gamma + d} s^{\frac{2\delta + d}{2}},
\]
\[
D_v^\beta(s) = \nu_k \left( \{ x \in \mathbb{R}^d : ||x||^{-\beta} > s \} \right) = \nu_k \left( B(0, s^{-\frac{1}{\Delta_2}}) \right) = \frac{d_k}{2\gamma + d} s^{-\frac{2\beta + d}{2}}.
\]
and

\[ D_f(s) = \nu_k \left( \{ x \in \mathbb{R}^d : \chi_{(0,1)}(\|x\|) > s \} \right) \]
\[ = \nu_k \left( B(0,1) \right) r^{2\gamma+d} \chi_{(0,1)}(s) \]
\[ = \frac{d_k}{2\gamma + d} r^{2\gamma+d} \chi_{(0,1)}(s). \]

Note that from (2.2), \( d_k = \frac{c_k^{-1}}{2^{\gamma+d-1}\Gamma(\gamma + \frac{d}{2})} \), this yields

\[ \frac{d_k}{2\gamma + d} = \frac{c_k^{-1}}{2^{\gamma+d}\Gamma(\gamma + \frac{d}{2} + 1)}. \]

This gives for \( t \geq 0 \),

\[ u^*(t) = \inf \{ s \geq 0 : D_u(s) \leq t \} = \left( \frac{2\gamma + d}{d_k} \right)^{\frac{\beta}{2\gamma+d}} t^{\frac{\beta}{2\gamma+d}}, \]

\[ \left( \frac{1}{v} \right)^*(t) = \inf \{ s \geq 0 : D_\frac{1}{v}(s) \leq t \} = \left( \frac{2\gamma + d}{d_k} \right)^{-\frac{\beta}{2\gamma+d}} t^{-\frac{\beta}{2\gamma+d}}, \]

and

\[ f^*(t) = \chi_{(0,R)}(t) \text{ where } R = \frac{d_k}{2\gamma + d} r^{2\gamma+d}. \]

Hence, using (2.1) and (2.2) again, we obtain for \(-(2\gamma+d) < \delta\)

\[ \int_{\mathbb{R}^d} f(x)u(x)d\nu_k(x) = \frac{d_k}{\delta + 2\gamma + d} r^{\delta+2\gamma+d} \]
\[ = \int_0^{+\infty} f^*(t)u^*(t)dt, \]

and

\[ \int_0^{+\infty} f^*(t) \frac{1}{\left( \frac{1}{v} \right)^*(t)} dt = \frac{d_k}{\beta + 2\gamma + d} r^{\beta+2\gamma+d} \]
\[ = \int_{\mathbb{R}^d} f(x)v(x)d\nu_k(x), \]

giving equalities for (3.5) and (3.6) in these cases.
Our main result on the weighted norm inequalities for a sublinear operator is the following theorem.

**Theorem 3.1** Let $u, v$ be non negative $\nu_k$-locally integrable weight functions on $\mathbb{R}^d$ and $1 \leq p_1 < p_2 < +\infty$, $1 \leq q_1 < q_2 < +\infty$. Suppose $\mathcal{L}$ is a sublinear operator defined on $\mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{L}$ is simultaneously of weak-type $(p_i, q_i)$, $i = 1, 2$, then for $1 < q < +\infty$, $\mathcal{L}$ can be extended to a bounded operator from $L^p_{k, u}(\mathbb{R}^d)$ to $L^q_{k, u}(\mathbb{R}^d)$ and the inequality

$$
\|\mathcal{L}f\|_{p, k, u} \leq c \|f\|_{p, k, v}
$$

holds with the following conditions on $u$ and $v$:

$$
\sup_{s > 0} \left( \int_{s}^{+\infty} u^*(t) t^{-\frac{q}{q_1}} dt \right)^{\frac{1}{q}} \left( \int_{0}^{t} s^{\lambda_2} \left[ \left( \frac{1}{v} \right)^*(t) \right] (p' - 1) p' \left( \frac{1}{p_2} - 1 \right) dt \right)^{\frac{1}{p'}} < +\infty \tag{3.8}
$$

and

$$
\sup_{s > 0} \left( \int_{0}^{s} u^*(t) t^{-\frac{q}{q_2}} dt \right)^{\frac{1}{q}} \left( \int_{0}^{+\infty} s^{\lambda_1} \left[ \left( \frac{1}{v} \right)^*(t) \right] (p' - 1) p' \left( \frac{1}{p_1} - 1 \right) dt \right)^{\frac{1}{p'}} < +\infty \tag{3.9}
$$

where $\lambda_1 = \frac{1}{q_1} - \frac{1}{q_2}$ and $\lambda_2 = \frac{1}{p_1} - \frac{1}{p_2}$.

**Proof.** Let $f \in \mathcal{S}(\mathbb{R}^d)$. Using (3.7) and applying Minkowski's inequality, we have

$$
\left( \int_{0}^{+\infty} \left[ (\mathcal{L} f)^*(t) \right]^q u^*(t) dt \right)^{\frac{1}{q}}
$$

\[
\leq c \int_{0}^{+\infty} u^*(t) t^{-\frac{q}{q_1}} \left( \int_{0}^{t} s^{\lambda_2} \left[ \left( \frac{1}{v} \right)^*(s) \right] ds \right)^{\frac{q}{d}} ds dt
\]

\[
+ c \int_{0}^{+\infty} u^*(t) t^{-\frac{q}{q_2}} \left( \int_{t}^{+\infty} s^{\lambda_1} \left[ \left( \frac{1}{v} \right)^*(s) \right] ds \right)^{\frac{q}{d}} ds dt.
\]

By means of change of variable in the right side, we obtain

$$
\left( \int_{0}^{+\infty} \left[ (\mathcal{L} f)^*(t) \right]^q u^*(t) dt \right)^{\frac{1}{q}}
$$

\[
\leq c \int_{0}^{+\infty} u^*(t) t^{\frac{\lambda_2}{p_2} \left( \frac{1}{q_2} - \frac{1}{p_2} \right)} \left[ \int_{0}^{t} \frac{1}{s^{\lambda_2}} \left[ \left( \frac{1}{v} \right)^*(s) \right] ds \right]^{\frac{1}{q}} ds dt
\]

\[
+ c \int_{0}^{+\infty} u^*(t) t^{\frac{\lambda_1}{p_1} \left( \frac{1}{q_1} - \frac{1}{p_1} \right)} \left[ \int_{t}^{+\infty} \frac{1}{s^{\lambda_1}} \left[ \left( \frac{1}{v} \right)^*(s) \right] ds \right]^{\frac{1}{q}} ds dt.
\]

\[
= I_1 + I_2. \tag{3.10}
\]
Applying (3.1) and (3.2) for $I_1$, we can assert that

$$I_1 \leq \left( \int_0^{+\infty} \left[ \left( \frac{1}{v} \right)^s(t) \right]^{-1} \left[ f^s(t) \right]^p dt \right)^\frac{1}{p} \tag{3.11}$$

if and only if

$$\sup_{s>0} \left( \int_s^{+\infty} u^s(t) \left( t^{\frac{\lambda_2}{\alpha_2}} \right)^{\frac{\lambda_2}{\alpha_2} \left( 1 - \frac{\lambda_1}{\alpha_1} \right) - 1} dt \right)^\frac{1}{q} \left( \int_s^{+\infty} \left[ \left( \frac{1}{v} \right)^s(t) \right]^{(p'-1)q} t^{p'(\frac{\lambda_2}{\alpha_2} - 1)} dt \right)^\frac{1}{p'} \leq +\infty.$$  

Then if we replace $s$ by $s \frac{\lambda_2}{\alpha_2}$ in this condition, it’s easy to see that if we use a change of variable in the first integral of the expression, we obtain (3.8). Similarly by applying (3.3) and (3.4) for $I_2$, we get

$$I_2 \leq \left( \int_0^{+\infty} \left[ \left( \frac{1}{v} \right)^s(t) \right]^{-1} \left[ f^s(t) \right]^p dt \right)^\frac{1}{p} \tag{3.12}$$

if and only if

$$\sup_{s>0} \left( \int_s^{+\infty} u^s(t) \left( t^{\frac{\lambda_2}{\alpha_2}} \right)^{\frac{\lambda_2}{\alpha_2} \left( 1 - \frac{\lambda_1}{\alpha_1} \right) - 1} dt \right)^\frac{1}{q} \left( \int_s^{+\infty} \left[ \left( \frac{1}{v} \right)^s(t) \right]^{(p'-1)q} t^{p'(\frac{\lambda_2}{\alpha_2} - 1)} dt \right)^\frac{1}{p'} \leq +\infty,$$

which is equivalent to (3.9). Combining (3.10), (3.11) and (3.12), it yields

$$\left( \int_0^{+\infty} \left[ (Lf)^s(t) \right]^q u^s(t) dt \right)^\frac{1}{q} \leq c \left( \int_0^{+\infty} \left[ \left( \frac{1}{v} \right)^s(t) \right]^{-1} \left[ f^s(t) \right]^p dt \right)^\frac{1}{p}. \tag{3.13}$$

Using (3.5) on the left side and (3.6) on the right side of (3.13), we obtain by density of $S(\mathbb{R}^d)$ in $L^p_{k,v}(\mathbb{R}^d)$, $1 \leq p < +\infty$,

$$\left( \int_{\mathbb{R}^d} \left[ (Lf)(x) \right]^q u(x) d\nu_k(x) \right)^\frac{1}{q} \leq c \left( \int_{\mathbb{R}^d} \left[ f(x) \right]^p v(x) d\nu_k(x) \right)^\frac{1}{p}.$$  

This completes the proof. □

4 Applications

Our first application is for the Riesz Potential $I^k_\alpha$. We have the following weighted Hardy-Littlewood-Sobolev theorem.
Theorem 4.1 Let $0 < \alpha < 2\gamma + d$, $1 < r < \frac{2\gamma + d}{\alpha}$ and $u,v$ be non negative $\nu_k$-locally integrable weight functions on $\mathbb{R}^d$. Then for $1 < p < q < +\infty$, $I^k_\alpha$ can be extended to a bounded operator from $L^p_{k,v}(\mathbb{R}^d)$ to $L^q_{k,u}(\mathbb{R}^d)$ and the inequality

$$\|I^k_\alpha f\|_{q,k,u} \leq C \|f\|_{p,k,v}$$

holds with the following conditions on $u$ and $v$:

$$\sup_{s>0} \left( \int_s^{+\infty} u^*(t)t^{-q(1-\frac{\alpha}{2\gamma+d})}dt \right)^\frac{1}{q} \left( \int_0^s \left( \frac{1}{v} \right)^*(t)(p'-1)dt \right)^\frac{1}{p'} < +\infty \quad (4.1)$$

and

$$\sup_{s>0} \left( \int_{0}^{s} u^*(t) t^{-q\left(\frac{1}{r} - \frac{\alpha}{2\gamma+d}\right)}dt \right)^\frac{1}{q} \left( \int_s^{+\infty} \left[\left( \frac{1}{v}\right)^*(t)(p'-1)t^{r'-1}\right] dt \right)^\frac{1}{p'} < +\infty \quad (4.2)$$

Proof. From Theorem 2.1, $I^k_\alpha$ is of weak-type $(p_1,q_1) = (1,\frac{1}{2\gamma+d})$ and strong-type $(p_2,q_2) = (r,\frac{1}{\frac{1}{r} - \frac{\alpha}{2\gamma+d}})$, then the result follows immediately from Theorem 3.1 with $\lambda_1 = \lambda_2 = 1 - \frac{1}{r}$.

Remark 4.1 Note that if $u = v \equiv 1$, the boundedness conditions (4.1) and (4.2) are valid if and only if $1 < p < r < \frac{2\gamma + d}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2\gamma+d}$. Therefore, Theorem 4.1 reduces to Theorem 2.1, i).

As consequence of Theorem 4.1 for power weights, we obtain the result below.

Corollary 4.1 Let $0 < \alpha < 2\gamma + d$, $1 < p < \frac{2\gamma + d}{\alpha}$, $u(x) = \|x\|^\delta$, $v(x) = \|x\|^\beta$, $x \in \mathbb{R}^d$ with $\delta < 0$ and $0 < \beta = \delta + \alpha p < (2\gamma + d)(p-1)$, then for $f \in L^p_{k,u}(\mathbb{R}^d)$, we have

$$\left( \int_{\mathbb{R}^d} |I^k_\alpha f(x)|^p \|x\|^\delta d\nu_k(x) \right)^\frac{1}{p} \leq C \left( \int_{\mathbb{R}^d} |f(x)|^p \|x\|^\beta d\nu_k(x) \right)^\frac{1}{p}$$

Proof. From Example 3.1, we have for $\delta < 0$ and $\beta > 0$

$$u^*(t) = \left( \frac{2\gamma + d}{d_k} \right)^{\delta (2\gamma + d)} t^{\delta (2\gamma + d)} \quad \text{and} \quad \left( \frac{1}{v} \right)^*(t) = \left( \frac{2\gamma + d}{d_k} \right)^{-\beta (2\gamma + d)} t^{-\beta (2\gamma + d)}$$

then if we take $p = q = r$ in Theorem 4.1, the boundedness conditions (4.1) and (4.2) are valid if and only if

$$\begin{cases} 0 < \beta < (2\gamma + d)(p-1), \\ \beta = \delta + \alpha p. \end{cases}$$

Under these conditions and from Theorem 4.1, we obtain our result. \(\square\)
Remark 4.2 From Remark 2.2, the sublinear fractional maximal operator $M_{k,\alpha}$ is of weak-type $(p_1,q_1) = (1, \frac{1}{1-\frac{2\gamma+d}{d}})$ and strong-type $(p_2,q_2) = (r, \frac{1}{r-\frac{2\gamma+d}{d}})$, then we get for $M_{k,\alpha}$ the same results obtained in Theorem 4.1 and Corollary 4.1.

Our next application concerns the weighted Generalized Sobolev inequality. Before, we need some useful results that we state in the following remark.

Remark 4.3

1/ (see [20]) In Dunkl setting the Riesz transforms are the operators $R_j$, $j = 1 \ldots d$ defined on $L^2_k(\mathbb{R}^d)$ by

$$R_j(f)(x) = 2^{\frac{\ell_k}{2}} \Gamma\left(\frac{\ell_k}{2}\right) \lim_{\varepsilon \to 0} \int_{\|y\| > \varepsilon} \frac{\tau_x(f)(-y)}{\|y\|^\frac{\ell_k}{2}} y_j \, d\nu_k(y), \quad x \in \mathbb{R}^d$$

where

$$\ell_k = 2\gamma + d + 1.$$

- The Riesz transform $R_j$ is a multiplier operator with

$$\mathcal{F}_k(R_j(f))(\xi) = \frac{-i\xi_j}{\|\xi\|} \mathcal{F}_k(f)(\xi), \quad 1 \leq j \leq d, \quad f \in S(\mathbb{R}^d). \quad (4.3)$$

- Let $0 < \alpha < 2\gamma + d$. The identity

$$\mathcal{F}_k(I_{\alpha}^k f)(x) = \|x\|^{-\alpha} \mathcal{F}_k(f)(x) \quad (4.4)$$

holds in the sense that

$$\int_{\mathbb{R}^d} I_{\alpha}^k f(x) g(x) \, d\nu_k(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(x) \|x\|^{-\alpha} \mathcal{F}_k(g)(x) \, d\nu_k(x),$$

whenever $f, g \in S(\mathbb{R}^d)$.

2/ (see [5]) The Riesz transform $R_j$, $1 \leq j \leq d$, can be extended to a bounded operator from $L^p_k(\mathbb{R}^d)$ into itself for $1 < p < +\infty$ and we have

$$\|R_j(f)\|_{p,k} \leq c \|f\|_{p,k}. \quad (4.5)$$

Theorem 4.2 (Weighted generalized Sobolev inequality) Let $u$ be a non-negative $\nu_k$-locally integrable function on $\mathbb{R}^d$ and $1 < r < 2\gamma + d$. Then for $1 < p \leq q < +\infty$ such that $p < r$ and $f \in S(\mathbb{R}^d)$, the inequality

$$\|f\|_{q,u,k} \leq c \|\nabla_k f\|_{p,k},$$
Weighted norm inequalities

holds with the following conditions on \( u \):

\[
\left( \int_s^\infty u^*(t)t^{-q(1-\frac{1}{2\gamma_d})}dt \right)^{\frac{1}{q}} \leq c s^{\frac{1}{p'}-1} \tag{4.6}
\]

and

\[
\left( \int_0^s u^*(t)t^{-q(\frac{1}{2}-\frac{1}{p'})}dt \right)^{\frac{1}{q}} \leq c s^{\frac{1}{p}-\frac{1}{r}}, \tag{4.7}
\]

for all \( s > 0 \). Here \( \nabla_k f = (T_1 f, ..., T_d f) \) and \( |\nabla_k f| = \left( \sum_{j=1}^d |T_j f|^2 \right)^{\frac{1}{2}} \).

**Proof.** For \( f \in S(\mathbb{R}^d) \), we write

\[
\mathcal{F}_k(f)(\xi) = \frac{1}{\|\xi\|} \sum_{j=1}^d -i\xi_j \mathcal{F}_k(f)(\xi),
\]

then by (2.3) and (4.3), we get

\[
\mathcal{F}_k(f)(\xi) = \frac{1}{\|\xi\|} \sum_{j=1}^d -i\xi_j \mathcal{F}_k(T_j f)(\xi) = \frac{1}{\|\xi\|} \sum_{j=1}^d \mathcal{F}_k(\mathcal{R}_j(T_j f))(\xi) = \frac{1}{\|\xi\|} \mathcal{F}_k \left( \sum_{j=1}^d \mathcal{R}_j(T_j f) \right)(\xi).
\]

This yields from (4.4) that

\[
\mathcal{F}_k(f)(\xi) = \mathcal{F}_k \left[ I_t^k \left( \sum_{j=1}^d \mathcal{R}_j(T_j f) \right) \right](\xi),
\]

which gives the following identity,

\[
f = I_t^k \left( \sum_{j=1}^d \mathcal{R}_j(T_j f) \right).
\]
Now, observe that the conditions (4.6) and (4.7) are equivalent to (4.1) and
(4.2) with \( v \equiv 1 \) and \( \alpha = 1 \), then using Theorem 4.1, we obtain
\[
\|f\|_{k,u,q} = \|I_1^k \left( \sum_{j=1}^{d} \mathcal{R}_j(T_j f) \right)\|_{k,u,q} \\
\leq c \|\mathcal{R}_j \left( \sum_{j=1}^{d} (T_j f) \right)\|_{k,p},
\]
which gives from (4.5) that
\[
\|f\|_{k,u,q} \leq c \sum_{j=1}^{d} (T_j f)\|_{k,p} \\
\leq c \|\nabla_k f\|_{k,p}.
\]
Our result is proved. \( \Box \)

**Corollary 4.2** Let \( 1 < r < 2\gamma + d \) and \( 1 < p \leq q < +\infty \) such that \( p < r \). Then for \( \delta < 0 \) such that \( \delta = q \left( (2\gamma + d) \left( \frac{1}{p} - \frac{1}{q} \right) - 1 \right) \), we have for \( f \in S(\mathbb{R}^d) \)
\[
\left( \int_{\mathbb{R}^d} |f(x)|^q \|x\|^d d\nu_k(x) \right)^{\frac{1}{q}} \leq c \|\nabla_k f\|_{p,k}.
\]

**Proof.** For \( \delta < 0 \), if we take \( u(x) = \|x\|^\delta \), \( x \in \mathbb{R}^d \) in Theorem 4.2, the boundedness conditions (4.6) and (4.7) are valid if and only if
\[
\delta = q \left( (2\gamma + d) \left( \frac{1}{p} - \frac{1}{q} \right) - 1 \right).
\]
Under this condition and from Theorem 4.2, we obtain our result. \( \Box \)

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