Neutrinos and SU(3) Family Gauge Symmetry

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We include the standard-model (SM) leptons in a recently proposed framework for the generation of quark mass ratios and Cabibbo-Kobayashi-Maskawa (CKM) mixing angles from an SU(3) family gauge interaction. The set of SM-singlet scalar fields describing the spontaneous breaking is the same as employed for the quark sector. The imposition at tree-level of the experimentally correct Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix, in the form of a tri-bi maximal structure, fixes several of the otherwise free parameters and renders the model predictive. The normal hierarchy among the neutrino masses emerges from this scheme.

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INTRODUCTION

In a recent paper 1, we introduced an effective-field-theory (EFT) framework for the computation of quark mass ratios and CKM mixing angles based on an SU(3) family gauge symmetry. Here we show that this framework can accommodate the lepton masses and PMNS mixing angles through the addition of only the three families of SM leptons, and that this fixes several of the parameters of the model.

The EFT includes operators respecting the global family symmetries, coupling SM-fermion bilinears and the Higgs-doublet to a set of SM-singlet scalars. Spontaneous breaking of the family symmetries at a high scale then yields a set of Goldstone-boson (GB) and pseudo-Goldstone-boson (PGB) degrees of freedom, the symmetry being realized nonlinearly in the EFT below this scale 1. The spontaneous breaking also leads to the Yukawa interactions of the SM. Radiative corrections to these couplings arising from the SU(3) family gauge interaction play a key role in generating realistic mass ratios and mixing angles.

In addition, we include here a set of sub-leading operators which explicitly break the global family symmetries. These operators affect small quantities such as the up-quark mass and the CKM angle $\theta_{13}$. Most importantly, they generate realistically small Majorana masses for the neutrinos.

We first establish some notation and review the current experimental data. We next describe our model, and discuss spontaneous breaking, masses, and mixing angles at tree level (in the absence of family-gauge interactions). We then include family-gauge radiative corrections, compare with experimental data, and discuss our results.

FERMION MASSES AND MIXING ANGLES

Notation

Below the electroweak breaking scale, the quark and charged-lepton mass operators are $-\bar{\psi}_L^{(i)} M^{(i)} \psi_R^{(i)}$, where $i = u, d, e$. The $\psi_{L,R}^{(i)}$ are chiral fields for the quarks and charged leptons and the $M^{(i)}$ are $3 \times 3$ matrices. Family indices are understood. Similarly, with only the three left-handed neutrinos present, the (Majorana) neutrino mass operator is $-(1/2) \nu_L^T C M^{(\nu)} \nu_L$, where $C$ is the charge conjugation matrix.

All matrices are non-diagonal in flavor space and symmetric in our specific case. One can diagonalize them with appropriate transformations,

$$\text{diag } M^{(i)} = L^{(i)\dagger} M^{(i)} L^{(i)*},$$
$$\text{diag } M^{(\nu)} = L^{(\nu)T} M^{(\nu)} L^{(\nu)},$$

where $L^{(i)}$ and $L^{(\nu)}$ are $3 \times 3$ matrices in flavor space. The mixing matrices appearing in the charged-current weak interactions are then given by

$$V_{CKM} = L^{(u)\dagger} L^{(d)},$$
$$V_{PMNS} = L^{(\nu)T} L^{(\nu)},$$

for quarks and leptons, respectively. We use the standard definitions of the mixing matrices, in which one writes the down-type quark (neutrino) flavor eigenstates $d (\nu)$ in terms of the mass eigenstates $d (\nu)$—in the basis in which up-type quarks (charged leptons) are diagonal—as

$$d = V_{CKM} \hat{d},$$
$$\nu = V_{PMNS} \hat{\nu}.$$
In these expressions, \( U \) where \( \beta \) mass differences, respectively.

\[
V_{CKM} = \begin{pmatrix}
  c_{12}c_{13} & -s_{12}c_{13} & s_{13}
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}
\end{pmatrix}
\]

(7)

where \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \).

An analogous expression in terms of \( \theta_{ij} \) and \( \delta_i \) is valid for the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix \( V_{PMNS} \), neglecting the flavor-diagonal Majorana phases. The complete expression for the PMNS matrix, including the Majorana phases \( \phi_1 \) and \( \phi_2 \), reads \( U_{PMNS} = V_{PMNS} K \) with

\[
K = \text{diag}\{1, e^{i\phi_1}, e^{i(\phi_2+\delta)}\}.
\]

In our model, Dirac and Majorana phases of order unity arise naturally, but all phases are neglected here.

**Experimental data**

The quark masses in GeV units are \( m_t(M_Z) = 176 \pm 5 \), \( m_b(M_Z) = 2.95 \pm 0.15 \), \( m_s(M_Z) = 0.65 \pm 0.12 \), \( m_u(M_Z) = 0.062 \pm 0.015 \), \( m_d(M_Z) = 0.0017 \pm 0.0005 \), \( m_d(M_Z) = 0.0032 \pm 0.0009 \).

The CKM mixing angles measured in tree-level processes, and as defined in Ref. [2], are:

\[
\sin^2 \theta_{12} = 0.2243 \pm 0.0016, \quad \sin^2 \theta_{23} = 0.0413 \pm 0.0015, \quad \sin^2 \theta_{13} = 0.0037 \pm 0.0009.
\]

The masses in GeV of the charged leptons are \( m_e = 1.78 \), \( m_\mu = 0.106 \) and \( m_\tau = 0.511 \times 10^{-3} \), where we have neglected the small errors. At present, combined fits of neutrino oscillation data give [2], at the 3\( \sigma \) level:

\[
\Delta m_{12}^2 = (7.1 - 8.9) \times 10^{-5} \text{eV}^2,
\]

\[
|\Delta m_{23}^2| = (1.4 - 3.3) \times 10^{-3} \text{eV}^2,
\]

\[
\sin^2 2\theta_{12} = 0.70 - 0.94,
\]

\[
\sin^2 \theta_{13} < 0.051,
\]

\[
\sin^2 2\theta_{23} = 0.87 - 1.0,
\]

(8)

where \( \Delta m_{12}^2 \) and \( \Delta m_{23}^2 \) are the solar and atmospheric mass differences, respectively.

Two other observables, for which we have only upper bounds at present, are the effective masses entering \( \beta \)-decay

\[
m_\beta^2 = \sum_i |U_{ei}|^2 m_i^2,
\]

(9)

\[
m_\beta < 2.3 \text{eV},
\]

(10)

and neutrinoless double \( \beta \)-decay

\[
m_{ee} = \sum_i U_{ei}^2 m_i,
\]

(11)

\[
|m_{ee}| < 0.9 \text{eV},
\]

(12)

where \( m_i \) are the eigenvalues of the neutrino mass matrix. In these expressions, \( U_{ei} \) are the elements of the complete \( U_{PMNS} \). Hence \( m_{ee} \) depends explicitly on the phases, which are neglected here.

Each of these experimental quantities is measured at the electroweak scale or below, while the parameters of our EFT are defined naturally at the much higher family breaking scale. The parameters must be evolved to the lower scales through SM interactions for a precise comparison with experiment. These renormalization group (RG) effects are expected to be small, and they depend on the choice of the family breaking scale which we do not make here. We disregard them and compare our expressions directly with the above quantities.

**THE MODEL**

As in Ref. [1], we introduce an \( SU(3) \) family gauge symmetry, taking it to be broken at some scale \( F \), large enough to suppress flavor-changing neutral currents. We employ an effective field theory (EFT) including the SM interactions and the \( SU(3) \) family gauge interaction to describe physics below the cutoff \( M_F \approx 4\pi F \). We take electroweak breaking to be described by a Higgs-doublet field, requiring some additional mechanism to stabilize the Higgs mass. We do not address this problem here.

With the family gauge coupling weak enough, the family gauge bosons are part of the EFT, and their effects can be computed perturbatively. The fermion fields of the EFT are those of the SM quarks and leptons, together with a set of partners with the quantum numbers of the up-type quarks. We assume that any additional new physics, for example SM-singlet neutrinos or fields associated with grand unification, appear only above the cutoff \( M_F \).

The goal of Ref. [1] was to compute the quark mass ratios \( m_d/m_b, m_s/m_b, m_u/m_t, m_e/m_t \), and the CKM mixing angles radiatively in the family gauge interaction. These quantities were arranged to vanish in its absence by introducing two global symmetries, \( SU(3)_1 \times SU(3)_2 \), with the standard model fermions and their partners transforming according to \( SU(3)_1 \), and additional fields of a “hidden” sector transforming according to \( SU(3)_2 \). The \( SU(3) \) family gauge interaction arose from gauging the diagonal subgroup of \( SU(3)_1 \times SU(3)_2 \). We also made use of an additional \( Z_3 \) symmetry to classify the operators of the EFT.

This goal was only partly realized. While the quark mass ratios and mixing angles were calculable perturba-
We first discuss the structure of the model at tree level, i.e., in the absence of the $SU(3)$ family gauge interaction. The tree-level Yukawa Lagrangian for the fermion masses and mixing angles consists, first of all, of terms that respect the global symmetries of the model. In Ref. [1], only such terms were considered. Here, in addition to incorporating the leptons, we also include smaller, symmetry breaking terms in the tree-level Lagrangian. These affect quantities that were estimated to be very small in Ref. [1], for example the CKM angle $\theta_{13}$ which was well below the experimental range. They also play a central role in the generation of neutrino masses and mixing angles.

The symmetry-preserving terms are given by

$$\mathcal{L}_Y = \frac{y_d}{F} q_d h S^c + \frac{y_\ell}{F} q_\ell h S^c + y_2 \chi_S u^c + y_3 \chi_\Sigma u^c + \frac{y_e}{F} \ell h S^c + \text{h.c.}$$

They are invariant under $SU(3)_1 \times SU(3)_2$ and $Z_3$, as well as $U(1)_q \times U(1)_\ell$ describing quark and lepton number conservation, and the SM gauge symmetries. The scalars $S$ and $\Sigma$ are neutral under $U(1)_q \times U(1)_\ell$. (We assume that $U(1)_S$ and $U(1)_\Sigma$ symmetries associated with these complex fields are broken explicitly by the underlying dynamics [2].) Each of the $y_i$ couplings is a dimensionless parameter determined by physics above $M_F$. Each except for $y_1$ is small compared to the family gauge coupling $g$, which will be $O(1)$, that is, $\alpha/\pi \equiv g^2/4\pi^2 = O(1/40)$. This will allow using the $y_i$ couplings at only first order, with quantum corrections arising from the family gauge interactions alone.

A comment is in order to explain why Eq. (13) contains only five interactions, all linear in $S$ and $\Sigma$. After all, there are many invariant operators bilinear in the fermion fields, but with higher powers of $S$ and $\Sigma$. With the scale of these operators taken to be the same as the VEV’s of $S$ and $\Sigma$ as we do here, they are not suppressed by power counting. In the limit $y_i \to 0$, however, this Lagrangian preserves an additional, global $U(1)$ for each of the $y_i$. It is hence technically natural to assume that each of the $y_i$ is a small parameter. At the loop level, combinations of the couplings in Eq. (13) together with scalar self-interactions generate all other possible operators compatible with the exact symmetries, containing higher powers of $S$ and $\Sigma$, and still bilinear in the fermion fields. But the coefficients of operators generated in this way are suppressed by products of the $y_i$, and are hence very small. Their effects are comfortably below those induced by the gauge interaction discussed here. Eq. (13) represents the first few terms of an expansion, truncated at the leading order in the global $U(1)$ symmetry-breaking, small parameters $y_i$.
A word about anomalies of global symmetries is in order. First of all, we disregard the familiar $B + \ell$ anomaly induced by the electroweak interactions since its effects are very small. There is also a global $U(1)_t$ anomaly induced by the $SU(3)$ family gauge interaction. While this could be canceled by physics above $M_F$ (for example, a set of SM-singlet neutrinos), the $U(1)_t$ symmetry is broken in the EFT itself.

We turn next to the additional Yukawa operators that explicitly break the global symmetries $SU(3)_1 \times SU(3)_2$ and/or $Z_3$ by small amounts. These terms are taken to preserve $U(1)_q$. (We assume that the $U(1)_t$ symmetry associated with the complex field $H$ is broken explicitly by the underlying dynamics.)

We include the following small symmetry-breaking operators:

$$-\mathcal{L}_Y' = y_u' \frac{q h \Sigma u^c}{F} + y_e' \frac{\ell h \Sigma e^c}{F} + \frac{y_e' \ell h H h^c}{2 F^2} + \text{h.c.} \quad (14)$$

The first contributes to the up-quark masses and (importantly) to $\theta_{13}^{\ell}$, and the second contributes to the charged-lepton masses. The last term is the source of the (small) Majorana mass matrix for the neutrinos. The first two terms break $Z_3$, while the third breaks $SU(3)_1 \times SU(3)_2$ to its diagonal subgroup. The final breaking is also triggered by the $SU(3)$ family gauge interaction. Each of the dimensionless coefficients in $\mathcal{L}_Y'$ will be of $O(10^{-4})$ or smaller, well below those in $\mathcal{L}_Y$.

Additional symmetry-breaking operators can also be included. An obvious example is $q h \Sigma d^c / F$, similar in structure to the first two terms above, and contributing to the down-quark masses. The coefficients of these operators can consistently be taken to be very small, since, when they are generated from quantum loops based on the interactions of Eqs. (13) and (14), they arise with very small coefficients. These operators then produce very small physical effects.

**Spontaneous Breaking**

Here, as in Ref. [1], we assume that the global symmetries $SU(3)_1 \times SU(3)_2$ and $Z_3$ are broken spontaneously at the scale $F$ by vacuum expectation values (VEVs) of the scalar fields $S$, $\Sigma$ and $H$. The VEV of $H$ also breaks $U(1)_t$. The VEV’s are taken to be

$$\langle S \rangle = F \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

$$\langle \Sigma \rangle = F \begin{pmatrix} 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (16)$$

$$\langle H \rangle = F \begin{pmatrix} b_1^2 & b_2 & b_3 \\ b_2 & a_1 & a_2 \\ b_3 & a_2 & 1 \end{pmatrix}, \quad (17)$$

where $s$, $\sigma$ and the $a_i$ will be $O(1)$, while the $b_i$ will be of order the Cabibbo angle $\theta_{12}^\ell$. The overall scale of these dimensionless parameters can be absorbed into the definition of $F$, so we have arbitrarily set one element of $\langle H \rangle / F$ to unity. We neglect all phases, taking each of the dimensionless parameters to be positive. It was argued in Ref. [1] that the above pattern in the visible sector (the $S$ and $\Sigma$ fields), and the hierarchical structure in the “hidden” sector (the $H$ field), emerge naturally from a class of potentials.

In the visible sector, since both $\langle S \rangle$ and $\langle \Sigma \rangle$ are diagonal, two $Z_2$ subgroups of $SU(3)_1$ are preserved. The quark mixing angles vanish in the absence of the family gauge interaction. The breaking pattern in the hidden sector, being described by a single sextet $H$ field, automatically preserves two $Z_2$ subgroups of $SU(3)_2$. (This is most evident in a frame in which $\langle H \rangle$ is made diagonal.) The alignment of the visible and hidden sectors will be determined dominantly by the family gauge interaction which links them. We assume here that these interactions misalign the two sectors such that no $Z_2$ symmetry remains when the sectors are gauge coupled. In the limit $b_i \to 0$, an exact $Z_2$ remains in tact.

The spontaneous breaking of $SU(3)_1 \times SU(3)_2$ leads to a set of 8 PGB’s along with the massive gauge bosons. The PGB’s acquire mass from explicit symmetry breaking, dominantly due to the gauge interaction, of $O(g^2 F/4\pi)$, and are thus part of the EFT. But they couple weakly to the fermions, and are neglected here [1].

**Mass Matrices**

After electroweak symmetry breaking, the mass matrices for the fermions are generated by $\mathcal{L}_Y$, Eq. (13) and $\mathcal{L}_Y'$, Eq. (14) with the scalar fields replaced by their VEVs. For down-type quarks,

$$M^{(d)} = y_d v \langle S \rangle / F = y_d v \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (18)$$

where $v \simeq 250 GeV$ is the VEV of Higgs doublet $h$. At this level, only the $b$ quark develops a mass, of the right order for $y_d \simeq 10^{-2}$.

The up-type quark mass matrix is $6 \times 6$:

$$(u \chi) \tilde{M}^{(u)} \begin{pmatrix} u^c \\ \chi^c \end{pmatrix} = (u \chi) \begin{pmatrix} y_u v \langle \Sigma \rangle & y_u v \langle S \rangle & y_u v \langle \Sigma \rangle \\ y_\ell v \langle S \rangle & y_\ell v \langle S \rangle & y_\ell v \langle \Sigma \rangle \end{pmatrix} \begin{pmatrix} u^c \\ \chi^c \end{pmatrix} \quad (19)$$

The squares of the eigenvalues of this (non-symmetric) matrix can be read off from the diagonal matrix
There are four non-vanishing eigenvalues. Two are small \((y_1^2v^2\) and \(y_3^2v^2\)), and two are large \((y_2^2F^2\) and \(y_3^2F^2\)) providing that \(v/F \ll y_2, y_3\). When the family gauge interactions are included, another large eigenvalue is generated, and a seesaw mechanism leads to masses and mixing angles for the up-type quarks.

For charged leptons, the mass matrix is

\[
M^{(e)} = y_\ell v \frac{(S)}{F} + y_\ell v \frac{(\Sigma)}{F} = y_\ell v \begin{pmatrix}
0 & 0 & 0 \\
0 & y_\ell^2 \sigma & 0 \\
0 & 0 & s
\end{pmatrix}, \quad (20)
\]

from which one can read off the (tree-level) masses \(m_\ell = y_\ell sv\) and \(m_\mu = y_\ell^2 \sigma v\). At this level, the electron is massless and the lepton mixing matrix \(L^{(e)}\) is the identity matrix.

For neutrinos, the Majorana mass matrix is proportional to the VEV of \(H\):

\[
M^{(\nu)} = y_\nu v^2 \frac{\langle H \rangle}{F} = \frac{y_\nu v^2}{F} \begin{pmatrix}
b_1^2 & b_2 & b_3 \\
b_2 & a_1 & a_2 \\
b_3 & a_2 & 1
\end{pmatrix}. \quad (21)
\]

Thus the scalar field \(H\) plays a more central role here than in Ref. [1] where it entered only through its family-gauge coupling.

**Neutrinos**

We next observe that the pattern of neutrino masses and mixing angles can be accommodated at tree level, placing a further restriction on the above parameters. We first recall that there will be a hierarchical structure to \(\langle H \rangle\): \(b_i = \mathcal{O}(b) < a_j = \mathcal{O}(1)\). A simple approach is to impose further on \(\langle H \rangle\) a form that leads to tri-bi-maximal mixing [8]. This so far provides a good fit to the experimentally measured angles. At tree level, since the charged lepton matrix \(L^{(e)}\) remains the identity, the tri-bi form corresponds to

\[
L^{(\nu)} = \begin{pmatrix}
\sqrt{\frac{1}{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}}
\end{pmatrix}, \quad (22)
\]

leading to the PMNS mixing angles \(\theta_{13} = \pi/4\), \(\theta_{12} = \sin^{-1}\sqrt{1/3}\), and \(\theta_{13} = 0\).

The tri-bi form emerges by imposing on the parameters of \(\langle H \rangle\) the three conditions

\[
b_3 = -b_2, \quad a_1 = 1, \quad a_2 = 1 - b_2 - b_1^2. \quad (23)
\]

These relations do not correspond to a symmetry limit of our model, and we have not obtained them by a vacuum alignment analysis. We impose them simply to accommodate the current experimental data on the PMNS angles, reducing the number of parameters used to describe the quarks, charged leptons, and neutrino-mass hierarchy. With these conditions, \(\langle H \rangle\) takes the form

\[
\frac{\langle H \rangle}{F} = \begin{pmatrix}
b_1^2 & b_2 & -b_2 \\
b_2 & 1 & 1 - b_2 - b_1^2 \\
-b_2 & 1 - b_2 - b_1^2 & 1
\end{pmatrix} = L^{(\nu)} \begin{pmatrix}
b_1^2 - b_2 & 0 & 0 \\
0 & b_1^2 + 2b_2 & 0 \\
0 & 0 & 2 - b_1^2 - b_2
\end{pmatrix} L^{(\nu)\dagger}. \quad (24)
\]

With \(b_i \ll 1\), we then have

\[
m_1 \simeq -b_2 \frac{y_\nu^2 v^2}{F} \quad (25)
\]

\[
m_2 \simeq 2b_2 \frac{y_\nu^2 v^2}{F} \quad (26)
\]

\[
m_3 \simeq 2 \frac{y_\nu^2 v^2}{F}, \quad (27)
\]

and therefore

\[
\Delta m_{23}^2 \simeq 4 \frac{y_\nu^2 v^4}{F^2} \quad (28)
\]

\[
\Delta m_{12}^2 \simeq 3 \frac{y_\nu^2 v^4}{F^2}. \quad (29)
\]

Thus, the normal neutrino hierarchy emerges from the imposition of the tri-bi form on \(\langle H \rangle\).

There is a simple reason for this. Without the tri-bi conditions, Eq. (23), \(\langle H \rangle\) has one small (\(\mathcal{O}(b)\)) eigenvalue and two large eigenvalues, with the size of \(b\) being set by the Cabibbo angle \(\theta_{12}^q\). The tri-bi conditions (together with the hierarchy \(b_i < a_j\)) restrict the 2–3 sub-matrix to have a small determinant: \(a_1 - a_3^2 = \mathcal{O}(b)\). There are then two small eigenvalues and one large eigenvalue, a consequence of only this restriction, not requiring the full imposition of the tri-bi conditions. The hierarchy between solar and atmospheric mass-squared differences follows automatically.

The restriction \(a_1 - a_3^2 = \mathcal{O}(b)\), by itself, gives also \(\theta_{13}^{\nu} = \mathcal{O}(b)\), that is, \(\sin^2 \theta_{13}^{\nu} = \mathcal{O}(b^2)\) at tree level. The full set of tri-bi conditions, Eq. (23), leads to the vanishing of \(\theta_{13}^{\nu}\), but this tree-level result will be lifted by the radiative corrections, leading to an \(\mathcal{O}(b^2)\) estimate. We note finally that the single restriction \(a_1 - a_3^2 = \mathcal{O}(b)\) also leads to an approximate \(U(2)\) invariance of \(\langle H \rangle\).
RADIATIVE CORRECTIONS TO THE MASS MATRICES

The tree-level theory with the conditions Eq. (28) can accommodate a realistic neutrino mass matrix while yielding vanishing CKM angles and vanishing masses for the first-family quarks. We next discuss the effects of the SU(3) family gauge interaction. Perturbation theory will be valid since $g^2/4\pi^2 \ll 1$. The loop corrections to the mass matrices may be viewed as corrections to \langle S \rangle, \langle \Sigma \rangle and \langle H \rangle.

At one loop, we find

$$\delta(S)_{ij} = -\frac{1}{\alpha} s_F \log \left( \frac{M_F^2}{M_F^2} \right) \frac{1}{(t_a)^2} \{t_b\}^2 O_{ac} O_{bc},$$

where $i, j = 1, 2, 3$ are the family indices and $a, b, c = 1, \cdots, 8$ label the 8 gauge bosons. $M_F \equiv 4\pi F$ is the cutoff scale and the $M_F^2$ are the mass eigenvalues of the family gauge bosons. The matrix $O$ is the orthogonal transformation diagonalizing the gauge boson mass matrix. The small parameters $b_1$ and $b_2$ enter this matrix. The $M_F$ dependence survives in only the 33 element. Similar expressions obtain for $\delta(\Sigma)_{ij}$ and $\delta(H)_{ij}$. In $\delta(S)_{ij}$, the $M_F$ dependence enters only the 22 element. In $\delta(H)_{ij}$, the $M_F$-dependent term is proportional to $\langle H \rangle_{ij}$. Thus, at one loop level in the gauge interaction, the cutoff ($M_F$) dependence is universal, and can be absorbed into a renormalization of the coupling constants in $\cal{L}_V$ and $\cal{L}_\nu$.

To derive these expressions, it is convenient to work in a renormalizable gauge. Eq. (29) is derived in Feynman gauge. There are also corrections due to wave function renormalization (kinetic energy mixing) arising from the family gauge interaction, as well as contributions from emission and re-absorption of the GB degrees of freedom. They lead to corrections of the same general form with no new parameters, and we do not exhibit them explicitly.

The corrected form of the $S$ matrix thus includes $O(\alpha/\pi)$ entries replacing the 0’s in Eq. (15). The presence of the small parameters $b_1$ and $b_2$ in the first row and column of $\langle H \rangle$ leads through its contribution to the gauge-boson masses to a similar presence in the corrected $\langle S \rangle$. Its general form is

$$\langle S \rangle' = \langle S \rangle + \delta(S) = F \left( \begin{array}{ccc} O(\pi b^2) & O(\pi b) & O(\pi b) \\ O(\pi b) & O(\pi) & O(\pi) \\ O(\pi b) & O(\pi) & s \end{array} \right)$$

(31)

Similarly, the general form of the corrected $\langle \Sigma \rangle$ matrix is

$$\langle \Sigma \rangle' = \langle \Sigma \rangle + \delta(\Sigma) = F \left( \begin{array}{ccc} O(\pi b^2) & O(\pi b) & O(\pi b) \\ O(\pi b) & \sigma & O(\pi) \\ O(\pi b) & O(\pi) & O(\pi) \end{array} \right)$$

(32)

Here again, each entry in the first row and column carries a suppression factor of $O(b)$.

The precise form of $\langle S \rangle'$, used in place of its tree-level counterpart in Eq. (13), then leads to the corrected down-type quark mass matrix $M^{(\nu)}$. Similarly, the precise $\langle S \rangle'$ and $\langle \Sigma \rangle'$, used in Eq. (15), lead to a corrected $6 \times 6$ up-type matrix. After integrating out the 3 heavy up-type fermions (implementing the up-sector seesaw), we obtain the $3 \times 3$ matrix $M^{(\nu)/2}$ for the $u$, $c$, and $t$ quarks. The masses of the heavy up-type fermions are of order $y_2 F$, and $y_3 F$, and $(\alpha/\pi)y_3 F$. These are below the cutoff $M_F$ but well above the electroweak scale.

The matrices $\langle S \rangle'$ and $\langle \Sigma \rangle'$ also enter the charged-lepton mass matrix, which takes the general form

$$M^{(\nu)} = y_e v \frac{\langle S \rangle'}{F} + y_{\nu} v \frac{\langle \Sigma \rangle'}{F}$$

$$= y_e v s \left( \begin{array}{ccc} O(\pi b^2) & O(\pi b) & O(\pi b) \\ O(\pi b) & O(\pi) & y_{\nu} \sigma \\ O(\pi b) & O(\pi) & 1 \end{array} \right)$$

(33)

We note that the matrix $L^{(\nu)}$ which diagonalizes $M^{(\nu)}$ and enters $V_{PMNS}$, contributes terms of $O(b)$, not suppressed by $O(\alpha/\pi)$, to the mixing in the $1-2$ sub-sector. Its general form, neglecting terms of $O(\alpha/\pi)$, is

$$L^{(\nu)} \simeq \left( \begin{array}{ccc} 1 & 0 & 0 \\ -O(b) & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

This same feature appears in the CKM matrix, that is, $\theta_{12}^d = O(b)$. In the present case, it relies on the fact that the quantity $y_{\nu} \sigma/y_{\nu} s$ turns out to be no larger than $O(\alpha/\pi)$.

The approximate form Eq. (31) leads to an approximation for the PMNS matrix $V_{PMNS} = L^{(\nu)/2} L^{(\nu)}$, the neutrino matrix $M^{(\nu)}$ is given by Eq. (21) together with its one-loop radiative corrections. As we have noted, these can be viewed as corrections $\delta(H)_{ij}$ to $\langle H \rangle_{ij}$, and are all of $O(\alpha/\pi)$. If they are neglected, then with the tri-bi conditions, Eq. (28), $L^{(\nu)}$ continues to have the form Eq. (22), and $V_{PMNS}$ has the same form together with corrections of $O(b)$. This leads to small shifts to the tri-bi values $\sin^2 \theta_{12}^d = 8/9$ and $\sin^2 \theta_{23}^d = 1$, and to

$$\sin^2 \theta_{13}^d = O(b^2).$$

(35)

This result emerged already at tree level if only the normal neutrino hierarchy was imposed on $\langle H \rangle$.

PHENOMENOLOGY

There are four adjustable Yukawa couplings, determined by physics above the cutoff $M_F$, that set the scale for the up-type quarks, the down-type quarks, the charged leptons, and the neutrinos. The first three are $y_1 = O(1)$, $y_3 = O(10^{-2})$, and $y_e = O(10^{-2})$, which enter the symmetry-preserving Lagrangian Eq. (19). The
coupling $y'$ enters the symmetry-breaking Lagrangian Eq. (1). The expression for $\Delta m^2_23$, Eq. (28) leads to $y' = \mathcal{O}(F/10^{15} \text{GeV})$. Thus $y' \ll 1$ providing that $F$, the family-breaking scale, is well below the grand unified theory scale. We won’t need to commit to a particular value for $F$, except, as noted above, that it must be large enough to suppress flavor-changing neutral currents.

We focus here on the remaining parameters of the model (Table II), employing them to reproduce the fermion mass ratios and mixing angles. There are 2 $\mathcal{O}(1)$ parameters $s$ and $\sigma$, entering the VEV’s $\langle S \rangle$ and $\langle \Sigma \rangle$. There are the 2 parameters $b_1$ and $b_2$, entering $\langle H \rangle$, which will be comparable and small ($\mathcal{O}(b)$), describing the small spontaneous breaking of a $Z_2$ symmetry. The size of $b$ is set by the Cabibbo angle $\theta_{12}^q$. The gauge coupling $\alpha/\pi$ is even smaller, breaking $SU(3)_1 \times SU(3)_2$ to its diagonal subgroup. The two Yukawa couplings $y_2$ and $y_3$ in $\mathcal{L}_Y$ enter only as the ratio $z = y_2/y_3$ in the see-saw expression for the up-type quark masses. We will find $z$ to be a small parameter. Finally, there are the two small ratios $z_u = y_u/y_1$ and $z_e = y_e/y_e$ arising from the symmetry-breaking Lagrangian $\mathcal{L}_Y$, Eq. (13).

The restriction to just these parameters is due to the imposition of the conditions Eq. (28) leading at tree-level to the form of the PMNS matrix. There are then 10 experimental quantities (Table III) to be accommodated after the inclusion of the radiative corrections. They are the up-type mass ratios $m_u/m_t$ and $m_c/m_t$, the down-type ratios $m_d/m_b$ and $m_s/m_b$, the CKM angles $\theta_{12}^d$, $\theta_{23}^d$, and $\theta_{13}^d$, the charged-lepton mass ratios $m_e/m_\tau$ and $m_\mu/m_\tau$, and the neutrino mass ratio $\Delta m^2_{12}/\Delta m^2_{23}$.

To check that the model effectively reproduces the experimental data, we perform a numerical study by sampling the parameter space. An example of a set of parameters that reproduces the data well is reported in Table IV. For the study it is sufficient to use the tree-level neutrino mass matrix. Radiative corrections are crucial, however, for the quarks and charged leptons. The radiative expressions in terms of the 8 parameters, are in general somewhat complicated. For orientation, we exhibit the approximate algebraic form of these expressions in the quark sector for the parameter values that emerge from the numerical study (Table III). For values in this range, it can be shown that

$$\theta_{12}^q = \mathcal{O}(b), \quad \theta_{23}^q = \mathcal{O}(\frac{\alpha}{\pi}), \quad \theta_{13}^q = \mathcal{O}(\frac{b \Delta m^2_{23}}{z}),$$

$$\frac{m_c}{m_t} = \mathcal{O}(\frac{z \alpha}{b}), \quad \frac{m_u}{m_t} = \mathcal{O}(b \sqrt{zz_e} \frac{\alpha}{\pi}),$$

$$\frac{m_s}{m_b} = \mathcal{O}(\frac{\alpha}{\pi}), \quad \frac{m_u}{m_t} = \mathcal{O}(\frac{\alpha}{\pi} b^2).$$

The expressions for $\theta_{13}^q$ and $m_u/m_t$ depend directly on the small parameter $z_u$ which measures the strength of the symmetry-violating operator $q \bar{h} \Sigma u e / F$ in $\mathcal{L}_Y$. It was neglected in Ref. [1].

Two similar algebraic formulas obtain for the charged-lepton mass ratios $m_e/m_\tau$ and $m_\mu/m_\tau$. Their dependence on the parameter $z_e = y_e/y_e$ is essential to accommodate the fact that they differ from the down-type ratios $m_d/m_b$ and $m_s/m_b$.

The radiative corrections $\delta(S)_{ij}$ and $\delta(\Sigma)_{ij}$ depend on the cutoff $M_F$, but this dependence enters only the 33 and 22 elements respectively. As already noted, it can be interpreted as a renormalization of the Yukawa couplings in Eqs. (13) and (14). It does not enter the expressions for the mass ratios and mixing angles. The one-loop calculations, with $\alpha/\pi = 0.05$, are no more accurate than about $5\%$. In addition, we are neglecting RG running effects from the symmetry breaking scale $F$ to the electroweak scale, which can affect the comparison with experiment at the same level. Some further uncertainties affecting the first family arise from having neglected phases.

| $b_1$ | $b_2$ | $s$ | $\sigma$ |
|------|------|-----|--------|
| 0.16 | 0.17 | 0.62 | 0.55 |

\[ \alpha/\pi = z = z_e = z_u = 0.053, 0.014, 0.055, 0.00028 \]

TABLE II: Input values of the 8 parameters, which are taken to be real and positive.

| $\Delta m^2_{ij}$ | $\sin \theta_{12}^d$ | $\cos \theta_{13}^d$ | $\sin \theta_{23}^d$ | $\sin \theta_{12}^e$ | $\Delta m^2_{23}$ | $\Delta m^2_{23}$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $9.8 \pm 5.4 \times 10^{-6}$ | $0.2234 \pm 0.0016$ | $0.0413 \pm 0.0015$ | $0.0037 \pm 0.0005$ | $2.88 \times 10^{-4}$ | $2.9 \times 10^{-4}$ | $2.9 \times 10^{-4}$ |
| $5.7 \times 10^{-6}$ | $0.0003 \pm 0.0008$ | $0.00011 \pm 0.0006$ | $0.0021$ | $0.014$ | $5.7 \times 10^{-6}$ | $0.060$ |

TABLE III: Comparison to the 10 experimental quantities. The accuracy of the values from the model are limited by uncertainties associated with perturbation theory, RG evolution to the electroweak scale and the neglect of phases. The small experimental errors for $m_e/m_\tau$ and $m_\mu/m_\tau$ are neglected.
DISCUSSION

Most of the 10 small experimental quantities of Table 11 are reproduced well by the 3 small parameters $b_1$, $a/\pi$, and $z \equiv y_2/y_3$, together with the 3 $O(1)$ parameters $s$, $\sigma$ and $b_2/b_1$. The smallest angle $\theta_{13}$ and the smallest mass ratio $m_u/m_t$ require the inclusion of the small parameter $y'$, $z' = y'/y$, from the symmetry-breaking Lagrangian $\mathcal{L}_Y$, Eq. (12). The CKM angle $\theta_{13}$, which was much too small in Ref. [1] where the small symmetry breaking operators were neglected, is now in the experimentally allowed range. A more accurate treatment of $\theta_{13}$ would call for the inclusion of the CP-violating phase $\delta^\beta$ in $V_{CKM}$. The ratio $m_u/m_t$ turns out to be somewhat large, but this is the smallest experimental quantity considered, and therefore most sensitive to additional small corrections. The charged-lepton mass ratios are both moderately sensitive to $y'_e$ ($z_e = y'_e/y_e$), also entering $\mathcal{L}_Y$.

We note that the numerical values of $z_e$ and $z_u$ correspond to $y'_e = O(10^{-4})$ and $y'_u = O(10^{-5})$. Since $y'_e = O(F/10^{15} \text{GeV})$, this coupling constant, too, will be at least this small providing only $F \leq O(10^{15} \text{GeV})$. Then, for a range of values of $y_3$ in the symmetric Lagrangian $\mathcal{L}_Y$, Eq. (13), each of the couplings in the symmetry-breaking Lagrangian $\mathcal{L}_Y$, Eq. (14), will be small compared to those in $\mathcal{L}_Y$.

Neutrinoless Double $\beta$ Decay

The structure of our neutrino mass matrix leads to predictions for the parameters measured in both neutrinoless double $\beta$-decay experiments and $\beta$-decay experiments. From Eq. (22) we have $m_3 \approx \sqrt{\Delta m^2_{31}} \approx 0.05 \text{ eV}$. Eqs. (25) and (26) then give $m_2 \approx -2m_1 \approx b_2 m_3 \approx 0.01 \text{ eV}$.

For neutrinoless double $\beta$-decay, the sum in Eq. (11) is dominated by the $i = 1, 2$ terms, with $|U^2_{e1}| = O(1)$ in each case. We estimate the experimental parameter $|m_{ee}|$, Eq. (11), roughly, since we have neglected phases. Assuming no cancellation, we have

$$|m_{ee}| = O(|m_2|) = O(\sqrt{\Delta m^2_{31}}) \approx 10^{-2} \text{ eV}.$$  (39)

This is well below the current experimental upper bound 0.9 eV, Eq. (12), and unlikely to be accessible in next-generation experiments. This result is not unique to our model. It would emerge in any model leading to the normal neutrino mass hierarchy with the lightest mass eigenvalue $m_1^2 \leq O(\Delta m^2_{31})$, along with the approximate measured values of the PMNS mixing angles. The discovery of neutrinoless double $\beta$-decay, at a higher rate, would rule out this class of models.

For $\beta$-decay, a similar argument gives $m_\beta \approx 0.01 \text{ eV}$. This, too, is well below the current bound, 2.3 eV, Eq. (10), and even less likely to be accessible in future experiments.

The Small Leptonic (PMNS) Mixing Angle

The imposition of only the normal neutrino mass hierarchy within our model has led to the $O(b^2)$ estimate Eq. (35) for $\sin^2 \theta_{13}$. Using the numerical values of $b_1$ and $b_2$ (Table 11), we find

$$\sin^2 \theta_{13} \approx 0.02,$$  (40)

a value somewhat below the current experimental bound, Eq. (8), but within the anticipated range of planned accelerator and reactor experiments. This is a rough estimate, valid to within an $O(1)$ multiplicative factor, but we stress that there is no reason within our model to expect $\sin^2 \theta_{13}$ to be parametrically smaller than this.

The large PMNS angles have been set at tree level to their tri-bi values $\sin^2 \theta_{12} = 8/9$ and $\sin^2 \theta_{23} = 1$. The radiative corrections then produce shifts, of $O(b)$ in the case of $\sin^2 \theta_{12}$, and of $O(b^4)$ in the case of $\sin^2 \theta_{23}$, keeping them within the experimental range.

SUMMARY AND CONCLUSIONS

Within the general framework of an $SU(3)$ family gauge symmetry we have described the hierarchical structure of the Yukawa couplings of the standard-model fermions as being generated by the interplay of spontaneous and explicit breaking of a set of global family symmetries of the standard model, as well as the spontaneous breaking of the $SU(3)$ gauge symmetry.

Below a high scale $F$ associated with the spontaneous breaking, the model is described by an effective field theory (EFT) containing a small number of tree-level couplings. These determine the overall mass scales for the fermions, breaking the separate global $U(1)$ symmetries associated with each (chiral) fermion species. These couplings preserve a $G = SU(3)^2 \times Z_3$ family symmetry, large enough to ensure the vanishing of all the mixing angles and the ratios between the masses of fermions with the same SM quantum numbers.

Operators that explicitly break $G$ are present, but with small coefficients. The gauging of an $SU(3) \subset G$ plays the leading role in the explicit breaking. It is the only coupling that gives significant loop corrections. All the mixing angles and mass hierarchies in the quark and charged lepton sectors are computable in terms of these corrections, restricted here to the one-loop level. Neutrinos play a special role in the construction. They are assumed to couple directly (at tree-level) with a "hidden" sector of the theory, which otherwise communicates only through the family gauge interaction. The generation of flavor structure (mixing matrices and mass ratios)
is controlled ultimately by the hidden sector through the
gauge interactions. Requiring that the neutrino mass ma-
trix is compatible with present experimental data largely
determines the internal structure of this hidden sector
and renders the model predictive.

This effective field theory approach to flavor physics
has intrinsic limitations, and cannot be used to predict
(or explain) some physical parameters such as the top
mass, bottom mass, and the large leptonic mixing an-
gles, which are controlled by physics at energies above the
cutoff $M_F \equiv 4\pi F$. But it constitutes a systematic tool
for understanding most of the other parameters entering
the physics of flavor changing processes in the standard
model.

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