Modelling time variations of root diameter and elongation rate as related to assimilate supply and demand: Supplementary material

Study of the model’s response curve

The root tip diameter time variations are given by the following equation:

\[
\frac{\partial d}{\partial t} = K f\left(\frac{d}{d_{\text{min}}}\right) \tag{1}
\]

with \( f(u) = ak\frac{(u-1)^2}{u^2} - (u - 1)^{1+\varepsilon} \). The diameter \( d \) increases when \( f \) is positive and decreases when \( f \) is negative. The zeros of \( f \) correspond to equilibrium values of \( d \). We thus need to study \( f \) to understand the diameter dynamic.

First, notice that \( f(1) = 0 \). Looking for the zeros of \( f \) in \([1, +\infty[\), we have :

\[
\forall u > 1, f(u) = 0 \iff ak\frac{(u-1)^2}{u^2} = (u - 1)^{1+\varepsilon} \iff ak = u^2(u - 1)^{\varepsilon-1} \tag{2}
\]

Let us note \( g(u) = u^2(u - 1)^{\varepsilon-1} \). We can rewrite:

\[
\forall u > 1, f(u) = 0 \iff ak = g(u) \tag{3}
\]

Let us study \( g \). We have \( \lim_{u \to 1} g(u) = +\infty \) and \( \lim_{u \to +\infty} g(u) = +\infty \). The first derivative of \( g \) is \( g'(u) = u(u-1)((\varepsilon+1)u-2) \), which is negative for \( u < \frac{2}{1+\varepsilon} \) and positive otherwise. Thus \( g \) varies according to the following table, where we denote \( k_l = g\left(\frac{2}{1+\varepsilon}\right) = \frac{4}{(1+\varepsilon)^2(\frac{1+\varepsilon}{1+\varepsilon})^{\varepsilon-1}} \):

| \( u \) | 1 | \( \frac{2}{1+\varepsilon} \) | +\infty |
|---|---|---|---|
| \( g'(u) \) | | \( \theta \) | + |
| \( g(u) \) | +\infty | \( k_l \) | +\infty |

The zeros of \( f \) are the fiber of \( ak \) under \( g \). We also have \( \forall u > 1, f(u) > 0 \iff ak > g(u) \) and \( \forall u > 1, f(u) < 0 \iff ak < g(u) \) According to the variations of \( g \) we thus have three distinct cases:
• if $ak < k_l$, $f$ have no zero in $]1, +\infty[$, and it is always negative.

• if $ak = k_l$, $f$ have one zero in $]1, +\infty[$, which is $\frac{2}{1+e}$, and $f$ is always negative.

• if $ak > k_l$, $f$ have two zeros in $]1, +\infty[$, one is lower than $\frac{2}{1+e}$ and the other is greater. $f$ is negative before its first zero, positive between its two zeros and negative after its second zero.

The time variations of $d$ depends on the sign of $f$. From the study of $f$, we can thus conclude on the dynamic of the root tip diameter. We have three distinct cases depending on whether the product $ak$ is lower, equal or greater than $k_l = \frac{4}{(1+e)^2} \left( \frac{1-e}{1+e} \right) e^{-1}$.

• If $ak < k_l$, regardless of its initial value, the root tip diameter will decrease toward $d_{min}$, which is the only equilibrium value.

• If $ak = k_l$, the root tip diameter has two equilibrium values: $d_{min}$ and $\frac{2d_{min}}{1+e}$. If the initial value of the diameter is lower than $\frac{2d_{min}}{1+e}$, it will decrease toward $d_{min}$. If the initial value is greater than $\frac{2d_{min}}{1+e}$ it will decrease toward $\frac{2d_{min}}{1+e}$.

• If $ak > k_l$, the root tip diameter has two stable equilibrium values : $d_{min}$ and $d_{eq}$, and one unstable : $d_r$. If the initial value of the diameter is lower than $d_r$, it will decrease toward $d_{min}$. If the initial value is in the interval $]d_r, d_{eq}[$ it will increase toward $d_{eq}$. If it is greater than $d_{eq}$, it will decrease toward $d_{eq}$. 

2