Abstract

The intersection graph induced by a set $C$ of $n$ disks can be dense. It is thus natural to try and sparsify them, while preserving connectivity. Unfortunately, sparse graphs can always be made disconnected by removing a small number of vertices. In this work, we present a sparsification algorithm that maintains connectivity between two disks in the computed graph, if the original graph remains “well-connected” even after removing an arbitrary “attack” set $B \subseteq C$ from both graphs. Thus, the new sparse graph has similar reliability to the original disk graph, and can withstand catastrophic failure of nodes while still providing a connectivity guarantee for the remaining graph. The new graphs has near linear complexity, and can be constructed in near linear time.

The algorithm extends to any collection of shapes in the plane, such that their union complexity is near linear.

1. Introduction

Given a set $C$ of $n$ disks in the plane, their intersection graph is formed by connecting each pair of intersecting disks by an edge. While this graph has an implicit representation of linear size, its explicit graph representation might be of quadratic size. It is thus natural to try and replace this graph by a sparser graph that retains some desired properties, such as preserving distances (i.e., a spanner), or preserving connectivity.

Such questions becomes significantly more challenging if one want to preserve such properties under network failures. The main obstacle is that a sparse graph can always be made disconnected by deleting the neighbors of a low degree vertex. Thus, the minimum degree of a graph has to be high, failing to provide the desired sparsity, especially if the graph has to withstand large attacks. An alternative approach is to provide such a guarantee only to most of the remaining graph (after the failure), allowing some parts of the graph to be ignored. For geometric spanners there has been recent work on constructing such reliable spanners [BHO20a, BHO20b, HMO21].

Here we consider the case of geometric intersection graphs, and connectivity guarantees. Specifically, given a set of disks $C$ the corresponding intersection graph $G(C)$, our goal is to compute a sparse subgraph $H \subseteq G$, such that its connectivity is robust to vertex deletion. However, disk
intersection graphs might have cut vertices – that is, a single vertex whose removal disconnects the graph. So even a small number of disks being removed can dramatically effect the connectivity.

As such, a milder desired property is that for any attack set $B \subseteq \mathcal{C}$, the graphs $G - B$ and $H - B$ (that is, the graphs remaining after the vertices of $B$ are deleted) have similar connectivity. Even that is not achievable if the graph $H$ is sparse, as one can delete the neighbors of a disk in the sparse graph, which would leave it isolated in $H$ (while still connected to the remaining graph in $G$).

**ε-safe paths.** Instead, we seek to provide a more geometric guarantee – a point $p$ is $\varepsilon$-safe if the attack set $B$ removes, at most, a $(1 - \varepsilon)$-fraction of the disks covering $p$. A curve is thus $\varepsilon$-safe if all the points along it are $\varepsilon$-safe, and two disks are $\varepsilon$-safely connected if there is an $\varepsilon$-safe path connecting them. Our goal here is to pre-build a sparse graph $H$ that guarantees connectivity, for any attack set $B$, in the graph $H - B$, for any two disks that are $\varepsilon$-safely connected in $G - B$. Importantly, the graph $H$ is constructed before $B$ is known, and the required property should hold for any attack set $B$. A graph $H$ with this property is $\varepsilon$-safely connected.

**Our result**

Given a set $\mathcal{C}$ of $n$ disks in the plane, and a parameter $\varepsilon \in (0, 1)$, let $G = G(\mathcal{C})$ denote the intersection graph induced by $\mathcal{C}$. We present a near linear time construction of a sparse subgraph $H$ of $G$, that is $\varepsilon$-safely connected.

**Idea.** For a point $p \in \mathbb{R}^2$, let $H_p$ denote the induced subgraph of $H$ over the set of disks $\mathcal{C} \cap p = \{ \odot \in \mathcal{C} \mid p \in \odot \}$.

For any point $p$ in the plane, the graph $G_p$ is a clique. Our construction replaces this clique by the graph $H_p$, which is a “strong” expander. This property by itself is sufficient to guarantee that $H$ is safely connected. The challenge is how to construct $H$ such that it has the desired expander property for all points in the plane, while being sparse, and furthermore do this in near linear time (as building the graph explicitly takes quadratic time).

**Random coloring, sparsification, and expansion.** It is well known that random coloring of vertices can be used to sparsify a graph, by keeping only edges that connect certain colors. For example, if you randomly color a clique of $k$ vertices by $2k$ colors, and keep only edges that connect vertices that belong to consecutive colors, then the resulting graph has (in expectation) $\Theta(k)$ edges. This collection of edges is almost a matching. It is well known that the union of three random matchings form an expander with high probability [Pin73]. Thus, if one repeats the random coloring idea suggested above a sufficient number of times, the union of the collection of edges results in an expander.

**Stop in the shallow parts, before getting too deep.** The problem is that if we randomly color the given set $\mathcal{C}$, by $k$ colors, the deepest point $p$ in $\mathcal{A}(\mathcal{C})$ might be of depth $n$. As such, a single random coloring would replace $G_p$ by a graph that has $O(n^2/k)$ edges, which is way too many edges to be used in a sparse graph.
To avoid this problem, we only add edges of the coloring if they correspond to shallow regions (i.e., regions of depth $\approx k$). The Clarkson-Shor technique readily implies that the number of edges added by this is roughly $\tilde{O}(n)$, where $\tilde{O}$ hides polynomial terms in $1/\varepsilon$ and $\log n$. Repeating this sufficient number of times (i.e., polylogarithmic), provides the desired property for faces that are of depth in the range $k$ to $2k$. Repeating this for exponential scales, to cover all faces of the arrangement by good “depth” expanders.

**Generalizing to other families of objects.** The key property of disk graphs which we use to analyze our algorithm’s runtime and size of the resulting spanner is the bound on the union complexity $[89]$. As a result, our techniques immediately generalize to intersection graphs of other families of objects with near linear bounds on their union complexity – for example, the union complexity of $n$ fat triangles is $O(n \log^* n)$ [14], and our construction would work verbatim for this case.

Beyond the result itself, we believe our combination of techniques from traditional computational geometry and expanders is quite interesting, and should be useful for other problems.

2. Settings

2.1. Notations

For a positive integer $k$, let $\llbracket k \rrbracket = \{1, \ldots, k\}$.

For a graph $G = (C, E)$, for a set $X \subseteq C$, we denote by $G_X = (X, \{uv \in E \mid u, v \in X\})$ the **induced subgraph** of $G$ over $X$. For a set $Y \subseteq C$, let $G - Y = G(C - Y)$ denote the graph remaining from $G$ after deleting all the vertices of $Y$. Similarly, for $y \in C$, we use the shorthand $G - y = G - \{y\}$.

For a set of vertices $S \subseteq V(G)$, let $N(S) = \{x \in V \mid y \in S, xy \in E(G)\}$ be the **neighborhood** of $S$.

2.1.1. Intersection graph

For a set of regions $C$ in the plane, let

$$G = G(C) = (C, \{\circ_1 \cap \circ_2 \neq \emptyset, \circ_1, \circ_2 \in C\})$$

**denote the intersection graph** of $C$. Throughout this paper, we assume that the regions are in general position. For a point $p$ in the plane, let

$$C \cap p = \{\circ \in C \mid p \in \circ\}$$

be the set of disks of $C$ covering $p$. The induced subgraph is denoted by $G_p = G_{C \cap p}$.

2.2. Problem statement

Given a set of disks $C$ in the plane, consider the induced intersection graph $G = G(C) = (C, E)$, where $E = \{\circ_1 \cap \circ_2 \mid \circ_1 \cap \circ_2 \neq \emptyset, \circ_1, \circ_2 \in C\}$. We consider some arbitrary (unknown) **attack set** $B \subset C$ – the disks in this set are being deleted, and we are interested in the connectivity of the remaining graph $G(C - B)$ (that is, the induced subgraph of $G$ over $C - B$).
Definition 2.1. For a point \( p \in \mathbb{R}^2 \), its \textit{depth} is \( d(p) = d(p, C) = |C \cap p| \) is the number of disks in \( C \) that contain \( p \).

Definition 2.2. A point \( p \) is \( \epsilon \)-\textit{safe}, with respect to an attack set \( B \), if \( d(p, C - B) \geq \epsilon d(p, C) \). Namely, a point has at least an \( \epsilon \)-fraction of the disks originally covering it, even after the disks of the attack \( B \) are removed.

Definition 2.3. Given a set of disks \( C \), the \textit{arrangement} of \( C \), denoted by \( A(C) \) is the partition of the plane into faces, vertices and edges induced by \( C \), see [BCK08]. A face/edge/vertex is thus \( \epsilon \)-\textit{safe} if any point in it is \( \epsilon \)-safe. The union of all safe points forms the \( \epsilon \)-\textit{safe zone} \( Z = Z_\epsilon(C - B) \).

Two points \( p \) and \( q \) in the plane are \( \epsilon \)-\textit{safely connected}, if \( p \) and \( q \) belong to the same connected component of \( Z \).

\textbf{The problem.} The task at hand is to construct a sparse “expander-like” graph over a set \( V \) of \( \nu \) objects. Let \( \epsilon \in (0, 1) \) be a parameter. Let \( \xi \) be a fixed number, such that \( \nu \leq \xi \leq 2\nu \). For some sufficiently large constant \( c_\epsilon > 2 \), consider the following algorithm of generating a random graph.

The algorithm repeats the following \( M = c_\epsilon \lfloor \epsilon^{-2} \rfloor \) times:

It randomly colors the elements of \( V \) with \( \xi \) colors. For each such coloring, it connects two objects by an edge if their colors differ by 1 (modulo \( \xi \)).

The final graph \( G \) (over \( V \)) results from using all the computed edges in all these iterations.

\textbf{2.3. Expander construction via random coloring}

\textbf{2.3.1. Expander Construction}

Our purpose here is to build a sparse “expander-like” graph over a set \( V \) of \( \nu \) objects. Let \( \epsilon \in (0, 1) \) be a parameter. Let \( \xi \) be a fixed number, such that \( \nu \leq \xi \leq 2\nu \). For some sufficiently large constant \( c_\epsilon > 2 \), consider the following algorithm of generating a random graph.

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\textbf{2.3.2. Proving expansion properties}

\textbf{Lemma 2.4}. Let \( S \) be a set of \( \nu \) objects, and \( \chi : S \to [\xi] \) be a random coloring of \( S \), for \( \xi \geq \nu \). Let \( X = |\chi(S)| = |\{ \chi(v) \mid v \in S \}| \) be the number of different colors used in \( S \). Then, we have

\[ \mathbb{P}[X < \nu/e^2] < \exp(-\nu). \]

\textit{Proof.} Let \( T \subseteq [\xi] \) be a fixed set of \( t = \beta \nu \) colors. We let \( \gamma = \mathbb{P}[\chi(S) \subseteq T] = (t/\xi)^\nu \). As such, we have that the probability that \( |\chi(S)| \leq t \), is at most

\[ \sum_{T \subseteq [\xi] \mid |T| = t} \mathbb{P}[\chi(S) \subseteq T] \leq \left( \frac{\xi}{t} \right)^\nu \leq e^t \left( \frac{t}{\xi} \right)^{\nu-t} = e^t \beta^{\nu-t} \left( \frac{\nu}{\xi} \right)^{\nu-t}. \]

Since \( t \leq \beta \nu \) and letting \( \beta \leq 1/e^2 \), we have \( e^t \beta^{\nu-t} \leq \exp(t - 2(\nu - t)) \leq \exp(-\nu) \).
For a coloring $\chi_i$, and an object $s \in V$, we denote the set of all elements in $V$ that are connected to $s$ in this coloring by

$$N_i(s) = \{ t \in V \mid |\chi_i(t) - \chi_i(s)| \equiv 1 \mod \xi \}.$$ 

**Definition 2.5.** Let $\varepsilon \in (0, 1)$ be a parameter. A graph $G = (V, E)$ with $\nu$ vertices, is an $\varepsilon$-connector if

$$\forall S \subseteq V, |S| \geq \varepsilon \nu \implies |N(S)| > (1 - \varepsilon)\nu.$$

**Lemma 2.6.** Let $c_s, c_r$ be the two sufficiently large constants used in the above construction, and let $G$ be the graph built. Then, for $\nu \geq c_s/\varepsilon^2$, and $M \geq c_r/\varepsilon^2$, we have, with probability $\geq 1 - \exp(-4\nu)$, that $G$ is an $\varepsilon/4$-connector.

**Proof.** Fix the set $S$, and a “bad” set $T$ (disjoint from $S$) of size $(\varepsilon/4)\nu$. Here, the bad event is that $S$ is not connected to $T$ in $G$. For some $s \in S$, the probability that $s$ is not adjacent to any vertex $t \in T$, by an edge induced by a specific coloring $\chi_i$ is

$$\mathbb{P}[N_i(s) \cap T = \emptyset] \leq 1 - \frac{|\chi_i(T)|}{\xi}.$$

Conceptually, we first color the elements of $T$, and then color the elements of $S$. The case $|S| > (1 - \varepsilon/4)\nu$ is not possible as $S$ and $T$ are disjoint. Using the independence of the neighborhood of each vertex $s \in S$, for each coloring $i$, we have that $\mathbb{P}[N_i(S) \cap T = \emptyset]$ is

$$\prod_{s \in S} \mathbb{P}[N_i(s) \cap T = \emptyset] \leq \left( 1 - \frac{|\chi_i(T)|}{\xi} \right)^{|S|} \leq \exp\left( -\frac{\varepsilon \nu}{4\xi} |\chi_i(T)| \right) \leq \exp\left( -\frac{\varepsilon}{8} |\chi_i(T)| \right).$$

A coloring $\chi_i$ is good if $|\chi_i(T)| \geq |T|/\varepsilon^2 \geq \tau = \varepsilon \nu/40$. By Lemma 2.4, we have that

$$\mathbb{P}[^{\chi_i \text{ is bad}}] \leq \mathbb{P}[^{|\chi_i(T)| < \tau}] \leq \exp(-|T|) = \exp\left( -\frac{\varepsilon}{4} \right).$$

Thus, we have

$$\beta_i = \mathbb{P}[N_i(S) \cap T = \emptyset] \leq \mathbb{P}[N_i(S) \cap T = \emptyset \mid \chi_i \text{ is good}] + \mathbb{P}[\chi_i \text{ is bad}] \leq \exp\left( -\frac{\varepsilon}{8} \cdot \frac{\varepsilon \nu}{40} \right) + \exp\left( -\frac{\varepsilon^2}{4} \right) \leq 2 \exp\left( -\frac{\varepsilon^2}{320} \right) \leq \exp\left( -\frac{\varepsilon^2}{640} \right).$$

for $\nu > 640/\varepsilon^2$. As the colorings $\chi_1, \ldots, \chi_M$ are chosen independently, we have that

$$\beta = \mathbb{P}[N(S) \cap T = \emptyset] = \prod_{i=1}^{M} \beta_i \leq \exp\left( -M \varepsilon^2 \nu/640 \right).$$

There are $(\nu/\varepsilon v/4) (\nu/\varepsilon v/4) \leq 4^\nu$ choices for the sets $S$ and $T$. Thus, using the union bound over all of the choices for these sets, we have that

$$\mathbb{P}[\exists S, T : N(S) \cap T = \emptyset] \leq \sum_{S, T} \mathbb{P}[N(S) \cap T = \emptyset] \leq 4^n \beta \leq \exp\left( 2\nu - \frac{M \varepsilon^2 \nu}{640} \right) \leq \exp(-4\nu),$$

if $M \geq 2600/\varepsilon^2$. \hfill \blacksquare
3. The construction of the safely connected subgraph

3.1. Preliminaries

For a set $C$ of disks, and a parameter $k$, let $G_{\leq k}(C)$ be the subgraph of the intersection graph, where two disks $C_1, C_2 \in C$ are connected by an edge, if there exists a point $p$, such that $p \in C_1 \cap C_2$, and the depth of $p$ in $C$ at most $k$. Such an edge is $k$-shallow in $C$. Similarly, let $A_{\leq k}(C)$ be the arrangement formed by keeping only vertices, edges and faces of the arrangement $A(C)$ that are of depth at most $k$ (i.e., each connected region of all points of deeper depth form a “hole” face in this arrangement).

3.1.1. Computing the shallow parts of the intersection graph

The following is well known [CS89, BY98] – for the sake of completeness, we provide a proof.

**Lemma 3.1.** Let $C$ be a set of $n$ disks, and let $k$ be a parameter. We have that the combinatorial complexity of $A_{\leq k}(C)$ is $O(nk)$, and this also bounds $|E(G_{\leq k}(C))|$. Both the arrangement $A_{\leq k}(C)$ and the graph $G_{\leq k}(C)$ can be computed in $O(n \log n + nk)$ (expected) time.

**Proof.** The first part is a well known consequence of the Clarkson-Shor technique [CS89], as the union complexity of $n$ disks is linear. The construction algorithm for the arrangement is described by Boissonnat and Yvinec [BY98].

The second part (which is also known) follows by applying the Clarkson-Shor technique. Indeed, every face/vertex/edge of depth at most two of $A(C)$, can contribute one edge to $G_{\leq 2}(C)$. As such, the number of such edges is $O(n)$ as the complexity of $A_{\leq 2}(C)$ is $O(n)$.

Let $E = E(G_{\leq 2}(C))$. Consider an edge $e = C_1, C_2 \in E$, with a point $p \in e$ being the witness point of depth at most $k$ such that $p \in C_1 \cap C_2$. Let $R$ be a random sample of disks from $C$, where each disk is sampled with probability $\alpha = 1/k$. The probability of this edge to appear in $G_{\leq 2}(R)$ is

$$\mathbb{P}[e \in E(G_{\leq 2}(R))] \geq (1/k)^2(1 - 1/k)^{k-2} \geq 1/(10k^2),$$

Indeed, this is the probability of picking $C_1, C_2$ to the sample, and no other disks (of the at most $k - 2$ disks) that covers $p$.

The complexity of $A_{\leq 2}(R)$ is bounded by $O(|R|)$, and as $\mathbb{E}[|R|] = O(n/k)$, it follows that

$$\sum_{e \in E} \mathbb{P}[e \in E(G_{\leq 2}(R))] \leq O\left(\mathbb{E}[|A_{\leq 2}(R)|]\right) = O(\mathbb{E}[|R|]) = O(n/k).$$

As for the other direction, we have

$$\sum_{e \in E} \mathbb{P}[e \in E(G_{\leq 2}(R))] \geq \frac{|E|}{10k^2}.$$

Combining the above two, we have $|E|/k^2 = O(n/k)$, which implies $|E| = O(nk)$.

Having the arrangement $A_{\leq k}(C)$ is not by itself sufficient to compute efficiently the graph $G_{\leq k}(C)$. Instead, one can lift the disks to planes, and use $n/k$-shallow cuttings [CT16]. This results in a decomposition of the plane into $O(n/k)$ cells, such that each cell has a conflict-list of size $O(k)$. We compute the arrangement of the disks in the conflict list, and by tracing the boundary of each disk, it is straightforward to discover all the edges of $G_{\leq k}(C)$ that arise out of points in this cell. This takes $O(k^2)$ time per cell, and $O(nk + n \log n)$ time overall, since computing the shallow cuttings takes $O(n \log n)$ time. This also provides an alternative algorithm for computing $A_{\leq k}(C)$. ■
3.1.2. The bipartite case
Analogous to the previous section, for two sets of disks $C_1, C_2$ and a parameter $k$, we let $G_{\leq k}(C_1, C_2)$ be the intersection graph defined as before, but where edges are only present between disks $\odot_1 \in C_1$ and $\odot_2 \in C_2$. Using the preceding lemma and this definition, the following corollary is immediate.

**Corollary 3.2.** Let $F_1$ and $F_2$ be two disjoint sets of disks of total size $n$. We have that the number of edges of $G_{\leq k}(F_1, F_2)$ is bounded by $O(nk)$. Additionally, the edges of this graph can be computed, in $O(nk)$ time, for $k = \Omega(\log n)$.

3.2. The construction algorithm

The input is a set of $n$ disks $C$, and a parameter $\varepsilon \in (0, 1)$, where $c_s, c_r$ are the constants from Lemma 2.6, and $c_\alpha$ is a constant to be specified later. Initially, the algorithm starts with the empty graph over $C$. Let

$$
\alpha = \left\lceil c_\alpha c_s \left( \varepsilon^{-2} + 4 \ln n \right) \right\rceil.
$$

Next, using Lemma 3.1, the algorithm computes all the faces of depth $\leq \alpha$ in $A(C)$, and adds all the edges induced by $\alpha$-shallow intersections of the Lemma 3.1 to the graph.

This takes care of all the shallow faces of the arrangement. For deeper faces, in the $i$th round, for $i = 1, \ldots, 1 + \lceil \log_2 (n/\alpha) \rceil$, the algorithm sets $\alpha_i = 2^{i-1} \alpha$. The algorithm handles the faces with depth in the range $(\alpha_{i-1}, \alpha_i]$, as follows:

**ith round:** For $j = 1, \ldots, M = \lceil c_r/\varepsilon^2 \rceil$, the algorithm colors each disk of $C$ uniformly at random with $\alpha_i$ colors. Let $\chi_{i,j} : C \rightarrow [\alpha_i]$ be this coloring. Let $F_t = \chi_{i,j}^{-1}(t)$ be the set of disks of $C$ colored by color $t$, for $t \in [\alpha_i]$. Using Corollary 3.2 as a subroutine, the algorithm computes the edges of $E(G_{\leq \alpha}(F_{t-1} \cup F_t))$ and adds them to the resulting graph, for $t = 1, \ldots, \alpha_i$ (where $F_0 = F_{\alpha_i}$).

Namely, the algorithm computes $O(\log^2 n)$ random colorings in each round, and adds the shallow edges between consecutively colored disks, for each such coloring to the graph.

The final graph is denoted by $S = S(C)$.

3.3. Analysis

3.3.1. Construction time and size

**Lemma 3.3.** The construction algorithm runs in time $O(n\varepsilon^{-4}\log^2 n)$. This also bounds the number of edges in the computed graph.

**Proof.** At the start, the algorithm includes all edges from faces with depth $\leq \alpha$. By Lemma 3.1, this can be done in $O(n\alpha) = O(n/\varepsilon^2 + n \log n)$ time.

For inner iteration $j$, the disk coloring step can be performed in $O(n)$ time by simply randomly assigning each disk a color from $[\alpha_i]$. Let $n_t = |F_t|$, where $F_t \subseteq C$ is the set of disks assigned color $t$. The algorithm computes edges induced by $\alpha$-shallow faces in $A(F_{t-1} \cup F_t)$ for $t = 1, \ldots, \alpha_i$. By
Corollary 3.2, these edges can be computed in $O(\alpha(n_t + n_{t+1}))$ time. Summing over $t$ (for a fixed $j$), we have that edges of this iteration can be computed in

$$\sum_{t=1}^{\alpha_i} O(\alpha(n_t + n_{t+1})) = O(\alpha n) = O(n/\varepsilon^2 + n \log n),$$

time, as $\sum_{t=1}^{\alpha_i} n_t = n$. This is being performed $O(1/\varepsilon^2)$ times in each round, and there are $O(\log n)$ rounds. Thus, the total work of this algorithm is $O((n\varepsilon^{-2} + n \log n)\varepsilon^{-2} \log n) = O(n\varepsilon^{-4} \log^2 n)$.

### 3.3.2. Rejecting edges from deep faces

One issue that may arise is ignoring faces which are too deep in the execution of the construction algorithm. This can happen if there is some face in the arrangement $\mathcal{A}(C)$ with depth in the range $(\alpha_{i-1}, \alpha_i]$ that, in the $i$th round, has more than $\alpha$ disks intersecting it from a given color pair. So even under the random coloring, the face still has depth which is too large under some color pair, and some of its induced edges are ignored. Thus, we must upper bound the probability of any failure of this type for any color pair $(t - 1, t)$, and any coloring $\chi_{i,j}$ sampled in the $i$th round.

**Lemma 3.4.** Consider some face $f \in \mathcal{A}(C)$ such that $d(f) \in (\alpha_{i-1}, \alpha_i]$. In the $j$th coloring of the $i$th round of the algorithm (see Section 3.2), for any fixed color $t \in [\alpha_i]$, the probability that $f$ is a hole (i.e., has depth bigger than $\alpha$) in $\mathcal{A}_{\leq \alpha}(F_{t-1}, F_i)$ is bounded by $1/n^{O(1)}$.

**Proof.** The face $f$ is only a hole in $\mathcal{A}_{\leq \alpha}(F_{t-1}, F_i)$ during the $i$th round only if $d(f, F_{t-1} \cup F_i) > \alpha$. Since each disk is colored uniformly at random with $\alpha_i$ colors, the probability that some disk incident to $f$ has color $t - 1$ or $t$ is $2/\alpha_i$. Since there are $d(f) \leq \alpha_i$ disks incident to $f$, we can bound the probability of this event (taken over the choice of coloring $\chi_{i,j}$) by

$$\mathbb{P}[d(f, F_{t-1} \cup F_i) > \alpha] \leq (d(f) \alpha) \left(\frac{2}{\alpha_i}\right)^{\alpha} \leq \left(\frac{\alpha}{\alpha_i}\right)^{\alpha} \left(\frac{2e}{\alpha_i}\right)^{\alpha} \leq \exp(-\alpha) \leq \frac{1}{n^{O(1)}}.$$

as $(\frac{n}{\alpha})^k \leq \binom{n}{k} \leq (\frac{en}{\alpha})^k$, and by making $c_\alpha$ sufficiently large. $\blacksquare$

Now, we apply this lemma with a union bound to show that, with high probability, we never ignore any edges needed for our construction due to having a face colored with too much of a single color.

**Corollary 3.5.** With probability $\geq 1 - n^{-7}$, no faces are ever ignored during the iteration they are handled in the spanner construction from Section 3.2.

**Proof.** To never have any face ignored, every face must have depth at most $\alpha$ under any consecutive color pair $t-1, t$, under all $M$ colorings, in the iteration $i$ it is handled. There are $\alpha_i \leq n$ possibilities for the color $t$, and by Lemma 3.1, there are at most $O(n^2)$ distinct faces in $\mathcal{A}(C)$. Thus, applying Lemma 3.4 and union bounding, the probability any face is ever ignored during the iteration in which it is handled is at most

$$\sum_{f \in \mathcal{A}(C)} \sum_{j=1}^{M} \sum_{\alpha_i} \mathbb{P}[d(f, F_{t-1} \cup F_i) > \alpha] \leq O(n^2) \cdot \frac{1}{n^{O(1)}} = n^{-7},$$

which implies the desired lower bound on the probability of no faces ever being ignored. $\blacksquare$
3.3.3. The depth expander property

For a point \( p \) in the plane, the set \( C \cap p \) is the set of all disks of \( C \) that contains \( p \). For the intersection graph \( G = G(C) \), the induced subgraph \( G_{C \cap p} \) is a clique. We claim that the induced graph \( S_{C \cap p} \) is an expander.

**Lemma 3.6.** For any point in the plane \( p \), let \( \nu = \text{d}(p, C) = |C \cap p| \) be its depth in the set of disks \( C \). We have that \( S_{C \cap p} \) is an \( \varepsilon/4 \)-connector. This holds for all the points in the plane with probability \( \geq 1 - 1/n^{10} \).

**Proof.** If \( \nu \leq \alpha \), then \( S_{C \cap p} \) is a clique, and the claim holds. Otherwise, fix \( i \) such that \( \nu \leq \alpha_i < 2\nu \). Observe that for the disks of \( C \cap p \), the \( i \)th round of the algorithm, the construction is identical to the connector construction of Section 2.3.1. As such, the resulting graph (which might have more edges because of other iterations), has the properties of Lemma 2.6, with probability \( \geq 1 - \exp(-4\nu) \geq 1 - n^{10} \), since \( \nu > \alpha > 4 \ln n \). As the arrangement has at most \( O(n^2) \) faces, the claim follows by the union bound. \( \blacksquare \)

3.3.4. Safe connectivity

**Lemma 3.7.** For any attack set \( B \subseteq C \), let \( Z = Z_\varepsilon(C - B) \) be the safe zone (see Definition 2.3), and let \( p, q \) be two points that are in the same connected component of \( Z \). Then, there is a path in \( S - B \), between two disks \( C_p, C_q \in C - B \), where \( p \in C_p \) and \( q \in C_q \).

This property holds for all attack sets, with probability \( \geq 1 - n^4 \).

**Proof.** Let \( F \) be the connected component of \( Z \) that contains \( p \) and \( q \). The arrangement \( A(C) \) restricted to \( F \) is connected and has at most \( O(n^2) \) faces, edges and vertices, and as such, there is a path \( \gamma \) between \( p \) and \( q \), inside \( F \), that crosses at most \( O(n^2) \) faces and edges of \( C \). Indeed, we can assume without loss of generality that \( \gamma \) does not cross an edge of \( A(C) \) more than once, as otherwise it can be shortcut. Next, construct a sequence of points on \( \gamma \), as follows. Initially, let \( p_1 = p \). Then, continuously move along \( \gamma \) towards \( q \). Any time the traverser enters a new entity (i.e., face/edge/vertex) of \( A(C) \), the traversal places a new witness point on the new entity – specifically, in the connected component of \( \gamma \) formed with the intersection of this entity (in the interior of this intersection if possible) that contains that current location. The final point is \( p_m = q \). By the above, we have that \( m = O(n^2) \). Let \( R(i) \) be the set of all the disks of \( C - B \) that are reachable from \( p \) in \( S \) - \( B \). Here, the start set is \( R(1) = (C - B) \cap p \).

For all \( i \), let \( L_i = (C - B) \cap p_i - 1, \ell_i = |L_i|, \text{ and } d_i = \text{d}(p_i, C) \). We claim (indecisively), that for any \( i \), we have that \( |R(i)| \geq (\varepsilon/2) \text{d}(p_i, C) \).

The claim readily holds for \( i = 1 \), as \( |R(1)| \geq \varepsilon \text{d}(p, C) \), since \( p \) is in the safe zone, and there are \( \ell_1 \) disks of \( C - B \) that cover \( p \) and are thus reachable from \( p \). So assume this holds for all \( j < i \), and consider \( R(i - 1) \) and \( R(i) \). Observe that they differ by at most two disks, under general position assumption.

If \( d_{i-1} < \alpha \text{ and } d_i < \alpha \), then \( S \cap p_{i-1} \) and \( S \cap p_i \) are cliques. By induction, \( |R(i - 1)| \geq (\varepsilon/2)d_i \), which implies that all the disks of \( L_i \) are reachable from \( p \). By the general position assumption \( R(i - 1) \cap R(i) \) is not empty, which implies that there is at least one disk of \( R(i) \) that is reachable from \( p \). Since \( S_{C \cap p_{i-1}} \) is a clique, it follows that all the disks of \( L_i \) are reachable from \( p \), and since \( \ell_i \geq \varepsilon d_i \), the claim follows.

If \( d_{i-1} \geq \alpha \text{ and } d_i \geq \alpha \), then \( |R(i - 1)| \geq (\varepsilon/2)d_{i-1} \geq (\varepsilon/2)\alpha > 10/\varepsilon \), see Eq. (3.1). We have that at most two disks of \( R(i - 1) \) are not present in \( R(i) \), which implies that \( |R(i)| \geq |R(i - 1)| - 2 \geq \)
\( (\varepsilon/2)d_{i-1} - 2 \geq (\varepsilon/4)d_i \), as \(|d_i - d_{i-1}| \leq 2\). By the expansion property of Lemma 3.6, this implies that at least \((1-\varepsilon/4)d_i\) vertices of \(R(i-1)\) are connected to \(R(i)\) in \(S \cap p_i\). Let \(Z_i\) be the of vertices in \(C \cap p_i\) that are not connected to \(R(i-1)\), and observe that \(|Z_i| \leq (\varepsilon/4)d_i\). As such, we have that

\[
|R(i)| \geq \ell_i - |Z_i| = \varepsilon d_i - (\varepsilon/4)d_i \geq (3/4)\varepsilon d_i,
\]

which implies the claim.

The remaining cases, where \(d_i, d_{i-1} \in [\alpha - 2, \alpha + 2]\) are handled in similar fashion, and we omit the easy details. We thus conclude that the point \(p_i\), for all \(i\), has at least \((\varepsilon/2)d_i\) disks that are reachable to it from \(p\) in the graph \(S - B\).

The expansion property of Lemma 3.6 holds with probability \(\geq 1 - 1/n^{10}\), and so we obtain our final result by union bounding over all \(n^4\) possible pairs of adjacent faces.

### 3.4. The result

**Theorem 3.8.** Let \(C\) be a set of \(n\) disks in the plane, \(\varepsilon \in (0,1)\) be a parameter. The above algorithm constructs a sparse graph \(S\), which is a sparse subgraph of the intersection graph \(G(C)\), such that for any attack set \(B \subseteq C\), and any two points \(p, q\) in the plane, such that \(p\) and \(q\) are \(\varepsilon\)-safely connected, there is a path in \(S - B\) between a disk that contains \(p\) and a disk that contains \(q\).

This property holds for any attack set, and any two points, with probability \(\geq 1 - 1/n^3\). The construction time and the number of edges of \(S\) is bounded by \(O(\varepsilon^{-4}n\log^2 n)\).

**Proof.** Follows from the bounds on failure events in Lemma 3.7 and Corollary 3.5 and applying a union bound.

### 4. Conclusions

We presented a new technique for sparsifying the intersection graph of disks (or any shapes that their union complexity is near linear) – the resulting graph has the property of preserving connectivity in the regions that are still \(\varepsilon\)-covered after an attack (potentially involving a large fraction of the disks) are being deleted. There are other guarantees that one might want, for example, the reliable guarantee we have for spanners [BHO20a] – that is, that if an attack deletes \(b\) disks in the spanner, then deleting \((1 + \varepsilon)b\) disks in the original intersection graph would leave the original graph with similar connected components provided by the original graph. It is natural to conjecture that our construction (or a slightly modified construction) provides such a guarantee. We leave this as an open problem for further research.

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11