The 2-Parametric Extension of $h$ Deformation of $GL(2)$, and The Differential Calculus on Its Quantum Plane

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Abstract

We present an alternative 2-parametric deformation $GL(2)_{h,h'}$, and construct the differential calculus on the quantum plane on which this quantum group acts. Also we give a new deformation of the two dimensional Heisenberg algebra.
I. Introduction

Recently quantum matrices in two dimensions, admitting left and right quantum spaces, are classified\(^1\). They fall into two families. One of them is the 2-parametric extension of \(q\) deformation of \(GL(2)\), which is well studied\(^2\). There is an alternative case which its \(R\) matrix is given in reference 1, and we denote it by \(R_{h,h'}\). In this paper we construct the quantum group associated with the \(R_{h,h'}\). On the other hand, it is also shown that\(^3\), up to isomorphism, there exist just two quantum deformation of \(GL(2)\) which admit a central determinant, the well known \(q\) deformation and recently constructed \(h\) deformation.

The 2-parametric deformation \(GL(2)_{q,p}\) is studied in ref. 2. \(R\) matrix associated with this quantum group which solves quantum Yang Baxter equation

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
\]

is

\[
R_{qp} = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q-p & q/p & 0 \\
0 & 0 & 0 & q
\end{pmatrix}
\]

Using the above \(R\) matrix and the method developed in ref.4, the algebra of the elements of quantum matrix \(T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) can be obtained. For the two parametric case, the quantum determinant is

\[
\mathcal{D} = ad - pbc = ad - qcb = da - p^{-1}cb = da - q^{-1}bc
\]

The crucial difference with the one parametric case, is that the quantum determinant is not central but satisfies the following relations
\[ [\mathcal{D}, a] = [\mathcal{D}, d] = 0, \quad q\mathcal{D} = pb\mathcal{D}, \quad p\mathcal{D}c = qc\mathcal{D} \quad (4) \]

There is an alternative \( R \) matrix

\[
R = \begin{pmatrix}
1 & -h' & h' & hh' \\
0 & 1 & 0 & -h \\
0 & 0 & 1 & h \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (5)

which solves (1). The algebra of polynomials on the quantum \( GL(2) \) associated with the above \( R \) matrix for the special cases \( h = h' = 1 \) and \( h = h' \) are studied in references 5 and 6. The universal enveloping algebra \( U_h(sl(2)) \) has also been constructed\(^7\) and the quantum de Rham complexes associated with \( h \) deformation of \( sl(2) \) is given in ref. 8.

The quantum groups which are associated with two matrices \( R \) and \( R' = (S \otimes S)R(S \otimes S)^{-1} \) are equivalent. This is the case for the matrix \( R \) as given in eq. (5) when \( h \) equals \( h' \),

\[
R_{h=h'=1} = (S \otimes S)R_{h=h'}(S \otimes S)^{-1}, \quad S = \begin{pmatrix}
h^{-1/2} & 0 \\
0 & h^{1/2}
\end{pmatrix}
\] (6)

So for the special case \( h = h' \), all the Hopf algebras for \( h \neq 0 \) are isomorphic to the case \( h = 1 \). Thus \( h \) is not a continuous parameter of deformation. However for the general case \( h \neq h' \), there is no such \( S \) which simultaneously fixes \( h \) and \( h' \) to one. Of course one can always fix \( h \) to one, but since we are interested in the classical limit \( h \to 0, \ h' \to 0 \), we do not fix it to be one. In this paper we study 2-parametric deformation of \( GL(2) \), \( GL(2)_{h,h'} \), the quantum plane on which it acts and differential calculus on that plane.
II. The Algebra of Functions

Following the method of ref. 4 and using the $R$ matrix (5), we arrive at the commutation relations of the quantum matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

\[ [a, c] = h c^2 \quad [b, c] = h c d + h' a c \quad [a, b] = h' (D - a^2) \quad (7) \]

\[ [d, c] = h' c^2 \quad [a, d] = h c d - h' c a \quad [d, b] = h (D - a^2) \quad (8) \]

where $D$ is

\[ D = ad - cb - h c d = ad - bc + h' a c \quad (9) \]

This algebra can then be made a bialgebra $A_{h, h'}(2)$ by definition of co-product and co-unit

\[ \Delta(T_{ij}) = T_{ik} \otimes T_{kj}, \quad \epsilon(T_{ij}) = \delta_{ij} \quad (10) \]

So

\[ \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix} \quad (11) \]

\[ \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (12) \]

For turning $A_{h, h'}(2)$ into a Hopf algebra we should be able to write down explicitly the inverse of the quantum matrix $T$. $D$, which is defined in (9) is quantum determinant and by direct verification it can be shown that

\[ D(T T') = D(T) D(T'), \quad \text{if} \quad [T_{ij}, T'_{kl}] = 0 \quad (13) \]

\[ \Delta(D) = D \otimes D \quad (14) \]

\[ \epsilon(D) = 1 \quad (15) \]
For the general case $\mathcal{D}$ is not central and its commutation relations with elements of $T$ is

$$[\mathcal{D}, a] = [d, \mathcal{D}] = (h' - h)\mathcal{D}c$$  \hspace{1cm} (16)

$$[\mathcal{D}, c] = 0, \quad [\mathcal{D}, b] = (h' - h)(\mathcal{D}d - a\mathcal{D})$$  \hspace{1cm} (17)

This is reminiscent of the other 2-parametric deformation, $GL(2)_{qp}$, where the quantum determinant is not central. If $\mathcal{D} \neq 0$ one extends the algebra by an inverse of $\mathcal{D}$ which obeys

$$\mathcal{DD}^{-1} = \mathcal{D}^{-1}\mathcal{D} = 1$$  \hspace{1cm} (18)

from which it follows that:

$$[\mathcal{D}^{-1}, a] = [d, \mathcal{D}^{-1}] = (h - h')\mathcal{D}^{-1}c$$  \hspace{1cm} (19)

$$[\mathcal{D}^{-1}, c] = 0, \quad [\mathcal{D}^{-1}, b] = (h - h')(d\mathcal{D}^{-1} - \mathcal{D}^{-1}a)$$  \hspace{1cm} (20)

It is easy to show that

$$MT = TM' = \mathcal{D} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (21)

where

$$M = \begin{pmatrix} d + hc & -b + h(d - a) + h^2 c \\ -c & a - hc \end{pmatrix}$$  \hspace{1cm} (22)
\[ M' = \begin{pmatrix}
  d + h'c & -b + h'(d - a) + h'^2c \\
  -c & a - h'c
\end{pmatrix} \] (23)

So a consistent definition of inverse can be given by

\[ T^{-1} = D^{-1} M = M'D^{-1} \] (24)

Clearly for the case \( h = h' \), \( M \) and \( M' \) coincide and determinant \( D \) is in the center of algebra. The quantum group \( GL(2)_{h,h'} \) is defined as the Hoph algebra obtained from the bialgebra \( A_{h,h'}(2) \) extended by the element \( D^{-1} \) and the antipode given by

\[ S(T) = T^{-1} = \begin{pmatrix}
  d + h'c & -b + h'(d - a) + h'^2c \\
  -c & a - h'c
\end{pmatrix} D^{-1} \] (25)

### III. Differential Calculus on The Quantum Plane

In this section we will construct a covariant differential calculus on the quantum plane. General formalism for constructing differential calculus on the quantum plane has been given by Wess and Zumino\(^9\). In this paper we use the formalism of ref. 10. Consider quadratic relations between coordinates \( x_i \) of a non-commutative space

\[ C_{kl}^i x^k x^l = 0 \] (26)

Introducing partial derivatives

\[ \partial_i = \partial/\partial x_i, \quad (\partial_i x^k) = \delta_i^k \] (27)

and assuming deformed Leibniz rule for partial derivatives

\[ \partial_i (fg) = (\partial_i f)g + O_i^j(f)\partial_j g, \quad \text{where} \quad O_i^j(x^k) = Q_i^j_{mn}x^n \] (28)

one arrives at
\[ \partial_i x^k = \delta_i^k + Q_{in}^k x^n \partial_m. \] (29)

If we now differentiate the commutation relations (26)
\[ 0 = \partial_k C_{ij} x^j = C_{ij} (\delta_k^i \delta_n^j + Q_{kn}^j) x^n. \] (30)

Since there are no linear relation among the variables \( x^i \), we have the following commutation relations
\[ C_{ij} (\delta_k^i \delta_n^j + Q_{kn}^j) = 0. \] (31)

By defining an exterior derivative
\[ d = \xi^i \partial_i \] (32)

which satisfy the undeformed Leibniz rule and the co-boundary condition
\[ d(fg) = (df)g + f dg \] (33)
\[ d^2 f = 0 \] (34)

one can obtain the following commutation relations (see ref. 10 for more details)
\[ x^i x^j = (\delta^i_k \delta^j_l - C_{ij}^k) x^k x^l = B_{kl}^i x^k x^l \] (35)
\[ x^i \xi^j = Q_{kl}^i \xi^k x^l \] (36)
\[ \xi^i \xi^j = -Q_{kl}^i \xi^k \xi^l \] (37)
\[ \partial_i x^j = \delta_i^j + Q_{im}^j x^m \partial_n \] (38)
\[ \partial_i \xi^j = (Q^{-1})_{im}^j \xi^m \partial_n \] (39)
\[ \partial_i \partial_j = \delta_{ij} \partial_m \partial_n \] (40)
with the following consistency relations

\[(δ_r^i δ_j^s + B_{ij}^{rs})(δ_k^i δ_n^j + Q_{kn}^{ij}) = 0\] (41)

\[(δ_m^k δ_n^l + Q_{mn}^{kl})(δ_j^m δ_i^n - S_{ji}^{mn}) = 0\] (42)

and \(Q\) should satisfy the Braid group equation

\[Q_{12}Q_{23}Q_{12} = Q_{23}Q_{12}Q_{23}\] (43)

if we choose

\[
Q = \begin{pmatrix}
1 & -h' & h' & hh' \\
0 & 0 & 1 & h \\
0 & 1 & 0 & -h \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad B = Q, \quad S = PQP
\] (44)

where \(P\) is the permutation matrix, then all the consistency relations (41-43) are satisfied. By inserting (44)in (35) we obtain the relation of quantum plane \(R_h(2)\) with the coordinates \(x\) and \(y\)

\[[x, y] = hy^2\] (45)

Similarly one can obtain the relations of dual quantum plane \(R_{h'}(2)\) with the coordinates \(\xi\) and \(\eta\)

\[\eta^2 = \xi \eta + \eta \xi = 0, \quad \xi^2 = h' \xi \eta\] (46)

This means that \(T\) acts on the \(h\) plane and \(h'\) exterior plane. The relations between coordinates of quantum plane and its dual are

\[[x, \eta] = h \eta y, \quad [x, \xi] = h'(x \eta - \xi y)\] (47)

\[[y, \eta] = 0, \quad [y, \xi] = -h \eta y\] (48)
and the deformed Leibniz rule is given by:

\[
[\partial_x, x] = 1 - h'y\partial_x, \quad [\partial_y, y] = 1 - hy\partial_x
\]  
(49)

\[
[\partial_x, y] = 0 \quad [\partial_y, x] = h'x\partial_x + hh'y\partial_x + hy\partial_y
\]  
(50)

and for the derivatives we have

\[
[\partial_x, \partial_y] = h'\partial_x^2
\]  
(51)

Finally to complete the set of relations we give the relations among \(\partial_i\) and \(\xi^k\)

\[
[\partial_x, \xi] = -h'\eta\partial_x, \quad [\partial_y, \eta] = -h\eta\partial_x
\]  
(52)

\[
[\partial_x, \eta] = 0 \quad [\partial_y, \xi] = h'\xi\partial_x + hh'\eta\partial_x + h\eta\partial_y
\]  
(53)

### IV. Deformed Heisenberg Algebra

Now we will give a new deformed two dimensional Heisenberg algebra. It is interesting to note that for the general case (and also for the one parametric case \(h = h'\)) identifying \(\partial_x\) and \(\partial_y\) with \(ip_x\) and \(ip_y\) is not compatible with the hermiticity of coordinates and momenta (see (51)). To identify \(\partial_x\) and \(\partial_y\) with the momenta \(ip_x\) and \(ip_y\), one must care about hermiticity of coordinates and momenta. This can be done by taking \(h\) as a pure imaginary parameter and \(h' = -h\). Then the hermiticity of \(x, y, p_x\) and \(p_y\) are compatible with the relations (50-52). The final form of the deformed Heisenberg algebra is

\[
[p_x, x] = -i + hyp_x \quad [p_y, y] = -i - hyp_x
\]  
(54)

\[
[p_x, x] = 0 \quad [p_y, x] = -hyp_x - h^2yp_x + hyp_y
\]  
(55)

\[
[p_x, p_y] = -hp_x^2 \quad [x, y] = hy^2
\]  
(56)
This gives a deformed Heisenberg algebra which can be used to study a two dimensional quantum space.

One of the interesting problems is constructing $U(gl(2))_{h,h'}$. In the case of $q$ deformation, universal enveloping algebra of multiparametric case can be obtained, by simply twisting\(^{11}\), but for the $h$ deformation, it should be clarified how to multiparametrize the universal enveloping algebra.

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References

1. H. Ewen, O. Ogievetsky, J. Wess, Lett. Math. Phys. 22, 297 (1991).

2. A. Schirrmacher, J. Wess, B. Zumino, Z. Phys. C 49, 317 (1991); O. Ogievetsky, J. Wess, Z. Phys. C 50, 123 (1991); V. K. Dobrev, J. Math. Phys. 33, 3419 (1992).

3. B A Kupershmidt, J. Phys. A 25, L1239 (1992).

4. L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan, Leningrad Math. J. 1, 193 (1990).

5. E. Demidov, Yu. I. Manin, E. E. Mukhin, D. V. Zhdanovich, preprint RIMS-701 (1990).
6. S. Zakrzewski, Lett. Math. Phys. 22, 287 (1991).

7. Ch. Ohn, Lett. Math. Phys. 25, 89 (1992).

8. V. Karimipour, Sharif Univ. Preprint (1993).

9. J. Wess, B. Zumino, Nuclear Phys. B 18, 302 (1990).

10. J. Schwenk, Proceeding of The Argonne Workshop (1990).

11. N. Yu. Reshetikhin, Lett. Math. Phys. 20, 331 (1990).