QUASIHOMOGENEOUS \((\mathbb{C}^*)^k \times SL_2(\mathbb{C})\)-VARIETIES CONTAINING A FINITE NUMBER OF ORBITS

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Abstract. Let \(G := (\mathbb{C}^*)^k \times SL_2(\mathbb{C})\) act linearly on a vector space or its projectivisation. We obtain an effective criterion to detect whether a number of orbits in an orbit-closure is finite or not.

1. Introduction

In this work we operate over complex numbers \(\mathbb{C}\). Suppose \(G := (\mathbb{C}^*)^k \times SL_2(\mathbb{C})\), \(V\) is a rational finite-dimensional \(G\)-module, \(\mathbb{P}(V)\) is its projectivisation, \(X((\mathbb{C}^*)^k) \cong \mathbb{Z}^k\) is the lattice of characters of the torus \((\mathbb{C}^*)^k\). As each \(G\)-module is completely reducible, then \(V = \bigoplus_{i=1}^s V_i\), where \(V_i\) is a simple rational \(G\)-module. Let us denote by \(V_{\chi_i, n_i}\) the space of binary forms of degree \(n_i\) where \(G\) acts as follows:

\[(t, \left(\begin{array}{ll} \kappa & \lambda \\ \mu & \nu \end{array}\right))(f(x, y)) = \chi_i(t)f(\kappa x + \mu y, \lambda x + \nu y).\]

Here \(n_i \in \mathbb{Z}_+, \chi_i \in X((\mathbb{C}^*)^k)\). Then every \(V_i\) is isomorphic to some \(V_{\chi_i, n_i}\).

In this paper we consider the actions \(G : V\) and \(G : \mathbb{P}(V)\). Our aim is to obtain a criterion for detection whether a number of orbits in an orbit-closure is finite or not. In \([2]\) this problem was solved for actions \(SL_2(\mathbb{C}) : \mathbb{P}(V)\). In affine case there is always a finite number of orbits as it is shown in \([3]\).

A multiplicity is a map \(e_i : \mathbb{C} \cup \{\infty\} \to \mathbb{Z}_+\) such that \(e_i(\infty) := \sum_{a \in \mathbb{C}} e_i(a)\), and \(e_i(\infty) \leq n_i\). For any \(v_i \in V_i\) we have \(v_i = c_i x^{n_i - e_i(\infty)} \prod_{a \in \mathbb{C}} (ax + y)^{e_i(a)}\), and for any \(v \in V\) one has \(v = (c_i v_i)_{i=1}^s = (c_i x^{n_i - e_i(\infty)} \prod_{a \in \mathbb{C}} (ax + y)^{e_i(a)})^{s}_{i=1}, c_i \in \mathbb{C}\). One can assume that \(\forall i c_i \neq 0\). Since the map \(V \to V, (c_i v_i)_{i=1}^s \mapsto (v_i)_{i=1}^s\) is a \(G\)-automorphism we may assume \(c_i = 1\) for all \(i\). Denote by \((u_i)_{i \in I}\), \(I \subset \{1, \ldots, s\}\) the vector \(u \in V\) such that its \(i\)-th component is equal to \(u_i\) iff \(i \in I\) and to 0 in other case.

Here we recall some definitions and formulate the main results of the paper.

Suppose \(\langle \cdot, \cdot \rangle\) is the standard scalar product on \(\mathbb{Q}^{k+1}\). We take in \(\mathbb{Z}^k \times \mathbb{Z}_+ \subset \mathbb{Q}^{k+1}\) the set of points \((x_i, n_i) = \chi_i \in X((\mathbb{C}^*)^k \times \mathbb{C}^*)\), \(i = 1, \ldots, s\). This set is called characteristic and its points are called characteristic points. Let us consider the rays from zero to the characteristic points and denote by \(M\) the convex hull of these rays. One can see that \(M\) is a cone. To each face \(F\) of the cone \(M\) we assign the set of all characteristic points containing in this face. This set forms a face \(F\). Consider the family of vectors \(R = (r_1, \ldots, r_k, p)\) with \(p < 0\) which are orthogonal to a face \(F\) and non-orthogonal to any face containing \(F\). If there exists a vector such that for any characteristic point \(\chi_i^j\) we have \(\langle \chi_i^j, R \rangle \geq 0\), then the face \(F\) is called admissible. In this case we shall denote this face by \(I(R)\).

Theorem 1. The number of \(G\)-orbits in \(\overline{Gv}\), \(v \in V\) is finite iff for each admissible face \(I(R)\) of maximal dimension of the cone \(M\) and for any integer-valued vector \(\beta = (\beta_i)_{i \in I(R)}\) such that
\[ \sum_{i \in I(R)} \beta_i \chi_i = 0 \] the following conditions hold:

(1) \[ \sum_{i \in I(R)} e_i(a)\beta_i = 0 \quad \forall a \in \mathbb{C}. \]

Proceed to the projective case. As in affine version we take in \( \mathbb{Z}^k \times \mathbb{Z}_+ \subset \mathbb{Q}^{k+1} \) the points \( (\chi_i, n_i) = \chi_i' \in \mathcal{X}(\mathbb{C}^* \times \mathbb{C}^*) \) for \( i = 1, \ldots, s \) and denote their convex by \( C \). To each face \( F \) of \( C \) we assign the set of all characteristic points containing in this face. This set forms an affine face \( F \). Consider the family of vectors \( R = (r_1, \ldots, r_k, p) \) with \( p < 0 \) which are orthogonal to any face \( F \) and non-orthogonal to any face containing \( F \). If there exists a vector \( R = (r_1, \ldots, r_k, p) \) with \( p < 0 \) which is orthogonal to a face \( F \), non-orthogonal to any face containing \( F \), and directed into the polyhedron \( C \), then the affine face \( F \) is called admissible. In this case we shall denote this face by \( J(R) \).

**Theorem 2.** The number of \( G \)-orbits in \( \overline{G(v)} \), \( \langle v \rangle \in \mathbb{P}(V) \) is finite iff for each admissible face \( J(R) \) of maximal dimension of the polyhedron \( C \) and for any integer-valued vector \( \beta = (\beta_i)_{i \in J(R)} \) such that \( \sum_{i \in J(R)} \beta_i \chi_i = 0 \) and \( \sum_{i \in J(R)} \beta_i = 0 \) the following conditions hold:

(2) \[ \sum_{i \in J(R)} e_i(a)\beta_i = 0 \quad \forall a \in \mathbb{C}. \]

**Example 1.** Suppose \( k = 1, V = V_{1,1} \oplus V_{1,4} \oplus V_{2,4} \oplus V_{3,3} \oplus V_{4,2}, v = (v_1, x^2, x^2(y-x), xy(y-x), x^2), \) where \( v_1 \) is any linear form. Consider the orbit-closure \( \mathbb{C}^* \times SL_2(\mathbb{C})(v) \subset \mathbb{P}(V) \). The faces \( \{B, C\} \) and \( \{C, D, E\} \) are admissible faces of maximal dimension. For the face \( \{B, C\} \) all conditions of Theorem 2 hold. For the face \( \{C, D, E\} \) the vector \( \beta = (0, 0, 1, -2, 1) \) doesn’t satisfy the condition (2) for \( a = 0 \). Therefore, \( \overline{G(v)} \) contains infinitely many orbits. Actually, for the curves of the form \( \gamma(t) = (t^{-1}, (t^{-1}, \begin{pmatrix} t \quad -dt^{-1} \\ 0 \quad t \end{pmatrix}) \) with \( e_i(d) = 0 \) we have \( \lim_{t \to 0} \gamma(t) = (0, 0, (d+1)^2 x^4, d(d+1)x^3, x^2)). \) Let us prove that there is an infinite number of different \( G \)-orbits among orbits of this form. Suppose \( \Phi(v(d_1)) = (t, (\begin{pmatrix} \kappa & \alpha \\ \mu & \nu \end{pmatrix}) v(d_1)) = (v(d_2)). \) Since \( x \) is \( \Phi \)-invariant then we have \( \mu = 0, \nu = -1. \) As \( \Phi x^2 = x^2 \) then we obtain \( \kappa = \pm 1. \) The following conditions are: \( t^2(d_1 + 1)^2 = (d_2 + 1)^2 \) \( t^2d_1(d_1 + 1) = d_2(d_2 + 1). \) It’s easy to see that for any \( d_1 \) there exists only a finite number of \( d_2 \) such that \( \langle v(d_2) \rangle \) is in the same orbit. Therefore, the number of orbits in \( \overline{G(v)} \) is infinite.

![Diagram](image)

Also we describe all \( G \)-moduli for which the orbit-closure of any orbit in \( V \) (Corollary 1) and in \( \mathbb{P}(V) \) (Corollary 2) contains a finite number of \( G \)-orbits.

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2. Some Lemmas

Let \( B \) be the direct product of \( (\mathbb{C}^*)^k \) and the subgroup of upper-triangular matrices in \( SL_2(\mathbb{C}). \) Then \( B \) is the Borel subgroup of \( G. \)
We shall consider $B$-orbits in $Bv$. It is known that $Bv$ intersects any $G$-orbit in $Gv$ ([1] III.2.5, Cor. 1). Hence, if there is a finite number of $B$-orbits in $Bv$, then there is a finite number of $G$-orbits in $Gv$. In the sequel we shall see that the inverse statement is also true. Let us say that some condition holds for almost all orbits if it holds for all orbits except finitely many.

Two following lemmas generalize Lemma and Proposition 1 from [2].

Let $\gamma : \mathbb{C}^* \to B$ be a curve in $B$. We shall prove that any $B$-orbit in $Bv$ contains a point $\lim_{t \to 0} \gamma(t)v$ for some special curve $\gamma(t)$.

**Lemma 1.** Suppose $v \in V$, $w \in Bv$. Then there exist $p, q, r_1, \ldots, r_k \in \mathbb{Z}$ with $q < -p$, $c \in \mathbb{C}$, and a polynomial $h \in \mathbb{C}[t]$ with $h(0) = -1$ and $\deg h < -p - q$ such that

$$\lim_{t \to 0}(t^{r_1}, \ldots, t^{r_k}, \begin{pmatrix} t^p & ch(t)t^q \\ 0 & t^{-p} \end{pmatrix})v \in Bw.$$ 

**Proof.** Let $\gamma(t) \in B(\mathbb{C}(t))$ be an analytic curve in $B$ such that $\lim_{t \to 0} \gamma(t)v = w' \in Bw$. If $\delta(t) \in B(\mathbb{C}(t))$ such that $\lim_{t \to 0} \delta(t)v = \delta(0)w'$, and without loss of generality we can replace $\gamma(t)$ by $\delta(t)\gamma(t)$.

Consider $p, q, r_1, \ldots, r_k \in \mathbb{Z}, c \in \mathbb{C}, f_1, \ldots, f_k, g_1, g_2 \in \mathbb{C}[t], f_1(0) \neq 0, \ldots, f_k(0) \neq 0, g_1(0) \neq 0, g_2(0) = -g_1(0)$ and

$$\gamma(t) = (t^{r_1}f_1, \ldots, t^{r_k}f_k, \begin{pmatrix} t^p g_1 & c g_2 t^q \\ 0 & t^{-p} g_1^{-1} \end{pmatrix}).$$

If $= 0$ or $q \geq -p$ we put

$$\delta(t) := (f_1^{-1}, \ldots, f_k^{-1}, \begin{pmatrix} g_1^{-1} & -t^p + q c g_2 \\ 0 & g_1 \end{pmatrix}).$$

Then $\delta(t)\gamma(t) = (t^{r_1}, \ldots, t^{r_k}, \begin{pmatrix} t^p & 0 \\ 0 & t^{-p} \end{pmatrix}).$

If $\neq 0$ and $q < -p$ we put

$$\delta(t) := (f_1^{-1}, \ldots, f_k^{-1}, \begin{pmatrix} g_1^{-1} & ch(t) \\ 0 & g_1 \end{pmatrix}),$$

where $h' \in \mathbb{C}[t]$ such that $h := h' t^{-p-q}g_1^{-1} + g_2 g_1^{-1} \in \mathbb{C}[t]$ is a polynomial of degree less than $-p - q$. Then $h(0) = g_2(0) g_1(0)^{-1} = -1$ and

$$\delta(t)\gamma(t) = (t^{r_1}, \ldots, t^{r_k}, \begin{pmatrix} t^p & ch(t) \\ 0 & t^{-p} \end{pmatrix}).$$

Suppose $q < -p, \deg h < -p - q$. We shall compute $\lim_{t \to 0} \gamma(t)v$ and prove that in almost all $B$-orbits in $Bv$ a vector of some standard form is contained.

Now we give some notations:

$$p_i(d) = \prod_{a \in \mathbb{C}, a \neq d} (a - d)^{e_i(a)} \quad \text{as } d \in \mathbb{C} \text{ and } p_i(\infty) = 1;$$

$$I(d, R, A) := \{ i \mid \langle \chi_i', R \rangle + Ae_i(d) = 0 \}, d \in \mathbb{C} \cup \{\infty\}, A \in \mathbb{Z}.$$ (In particular, $I(R) := I(d, R, 0)$;)

$$v(d, R, A) := (p_i(d)x^{e_i})_{i \in I(d, R, A)}, \quad v(d, R) := v(d, R, 0).$$ Vectors of the form $v(d, R)$ are called standard.

**Lemma 2.** Consider a vector of the form $v(d, R)$, where $R = (r_1, \ldots, r_k, p)$, $p < 0$. If $\forall i \in I(R)\ e_i(d) = 0$, then $v(d, R) \in Bv$. Almost all $B$-orbits in $Bv$ contain a vector of this form.
Proof. We shall consider different types of the curve $\gamma$, where $\gamma$ is of the form described in Lemma 1. For an irreducible $G$-module $V = V_{\chi,n}$ we have

$$\gamma(t)v = t^{(\chi',R) - pe(\infty)}x^{n - e(\infty)} \prod_{a \in C} ((at^p + ct^q)x + t^{-p}y)^{e(a)},$$

where $R = (r_1, \ldots, r_k, p)$, $\chi' = (\chi, n)$.

1. $p = 0, c = 0$.
For an irreducible $G$-module $V$ we have

$$\lim_{t \to 0} \gamma(t)v = \lim_{t \to 0} t^{(\chi',R)-pe(\infty)}x^{n - e(\infty)} a^e(a).$$

The limit exists iff $\langle \gamma', R \rangle \geq 0$.

For a reducible $G$-module $V$ the limit exists iff $\forall i \langle \gamma'_i, R \rangle \geq 0$. In this case it is equal to $(v_i)_{i \in I(R)}$.

Since $\lim_{t \to 0} \gamma(t)v_i$ is either 0 or $x^{n_i - e_i(\infty)}y^{e_i(\infty)}$, there is a finite number of the vectors of this form.

2. $p > 0, c = 0$.
For an irreducible $G$-module $V$ we have

$$\lim_{t \to 0} \gamma(t)v = \lim_{t \to 0} t^{(\chi',R)-2pe(\infty)}x^{n - e(\infty)} y^{e(\infty)}.$$ 

The limit exists iff $\langle \gamma', R \rangle - 2pe(0) \geq 0$.

For a reducible $G$-module $V$ the limit exists iff $\forall i \langle \gamma'_i, R \rangle - 2pe_i(0) \geq 0$. In this case it is equal to $(x^{n_i - e_i(0)}y^{e_i(0)})_{i \in I(0,R,-2p)}$.

Since $\lim_{t \to 0} \gamma(t)v_i$ is either 0 or $x^{n_i - e_i(0)}y^{e_i(0)}$, there is a finite number of the vectors of this form.

3. $p < 0, c = 0$.
For an irreducible $G$-module $V$ we have

$$\lim_{t \to 0} \gamma(t)v = \lim_{t \to 0} t^{(\chi',R)-2pe(0)}x^{n - e(0)} y^{e(0)} \prod_{a \in C, a \neq 0} a^{e(a)}.$$ 

The limit exists iff $\langle \gamma', R \rangle - 2pe(0) \geq 0$.

For a reducible $G$-module $V$ the limit exists iff $\forall i \langle \gamma'_i, R \rangle - 2pe_i(0) \geq 0$. In this case it is equal to $(x^{n_i - e_i(0)}y^{e_i(0)}p_i(0))_{i \in I(0,R,-2p)}$.

Since $\lim_{t \to 0} \gamma(t)v_i$ is either 0 or $x^{n_i - e_i(0)}y^{e_i(0)}p_i(0)$, there is a finite number of the vectors of this form.

4. $p = q, h \equiv -1$ (then we have $p < 0$).
For an irreducible $G$-module $V$ we have

$$\lim_{t \to 0} \gamma(t)v = \lim_{t \to 0} t^{(\chi',R)-2pe(c)}x^{n - e(c)} y^{e(c)} \prod_{a \in C, a \neq c} (a - c)^{e(a)}.$$ 

The limit exists iff $\langle \gamma', R \rangle - 2pe(c) \geq 0$.

For a reducible $G$-module $V$ the limit exists iff $\forall i \langle \gamma'_i, R \rangle - 2pe_i(c) \geq 0$. In this case it is equal to $(x^{n_i - e_i(c)}y^{e_i(c)}p_i(0))_{i \in I(c,R,-2p)}$.

Since $\lim_{t \to 0} \gamma(t)v_i$ is either 0 or $x^{n_i - e_i(c)}y^{e_i(c)}p_i(e(c))$, there is a finite number of the vectors of this form with $e_i(e(c)) \neq 0$. If for any $i$ we have $e_i(c) = 0$, then this is a standard vector $v(c, R)$.

5. $p = q, c \neq 0, h \neq -1$ (then we have $p < 0$).
Suppose $h(t) = -1 + ht^l + \ldots + hm^m$.
For an irreducible $G$-module $V$ we have

$$\lim_{t \to 0} \gamma(t)v = \lim_{t \to 0} t^{(\chi',R)+le(c)}x^{n(ch_i)} y^{e(c)} \prod_{a \in C, a \neq c} (a - c)^{e(a)}.$$
The limit exists iff \( \langle \chi', R \rangle + le(c) \geq 0 \).
For a reducible \( G \)-module \( V \) the limit exists iff \( \forall i \ \langle \chi'_i, R \rangle + le_i(c) \geq 0 \). In this case it is equal to
\[
((ch_t)^{e_i(c)} p_i(x^{n_i}))/i_{i \in I(c, R, l)}.
\]
Let us act on the vector by the element
\[
((ch_t)^{r_1/l}, \ldots (ch_t)^{r_k/l}, \left( \begin{array}{rr}
(ch_t)^{p/l} & 0 \\
0 & (ch_t)^{-p/l}
\end{array} \right)
\)
and
\[
((ch_t)^{(\chi'_i, R)/l}+e_i(c) p_i(x^{n_i}))/i_{i \in I(c, R, l)} = (p_i(x^{n_i}))/i_{i \in I(c, R, l)} = v(c, R, l).
\]
There is a finite number of vectors of such form with \( e_i(c) \neq 0 \). If for any \( i \) we have \( e_i(c) = 0 \), then the vector is of the form \( v(c, R) \).

6. \( p > q, c \neq 0 \).

For an irreducible \( G \)-module \( V \) we have
\[
\lim_{t \to 0} \gamma(t)v = \lim_{t \to 0} (\chi'_i, R)+(q-p) e(\infty) x^n (-c)^e(\infty).
\]
The limit exists iff \( \langle \chi', R \rangle + (q-p) e(\infty) \geq 0 \).

For a reducible \( G \)-module \( V \) the limit exists iff \( \forall i \ \langle \chi'_i, R \rangle + (q-p) e_i(\infty) \geq 0 \). In this case it is equal to
\[
(((-c)^{e_i(\infty)} x^{n_i}))/i_{i \in I(\infty, R, q-p)}.
\]
Let us act on the vector by the element
\[
(((-c)^{r_1/(q-p)}, \ldots (-c)^{r_k/(q-p)}, \left( \begin{array}{rr}
(-c)^{p/(q-p)} & 0 \\
0 & (-c)^{-p/(q-p)}
\end{array} \right)
\)
and
\[
(p_i(\infty) x^{n_i}))/i_{i \in I(\infty, R, q-p)} = v(\infty, R, q-p).
\]

7. \( p < q, c \neq 0 \) (then we have \( p < 0 \)).

For an irreducible \( G \)-module \( V \) we have
\[
\lim_{t \to 0} \gamma(t)v = \lim_{t \to 0} (-c)^{e(0)} t (\chi'_i, R)+(q-p) e(0) x^n \prod_{a \in \mathbb{C}, a \neq 0} a^{e(0)}.
\]
The limit exists iff \( \langle \chi', R \rangle + (q-p) e(0) \geq 0 \).

For a reducible \( G \)-module \( V \) the limit exists iff \( \forall i \ \langle \chi'_i, R \rangle + (q-p) e_i(0) \geq 0 \). In this case it is equal to
\[
(p_i(0) (-c)^{e_i(0)} x^{n_i}))/i_{i \in I(0, R, q-p)}.
\]
Let us act on the vector by the element
\[
(((-c)^{r_1/(q-p)}, \ldots (-c)^{r_k/(q-p)}, \left( \begin{array}{rr}
(-c)^{p/(q-p)} & 0 \\
0 & (-c)^{-p/(q-p)}
\end{array} \right)
\)
and
\[
(p_i(0) x^{n_i}))/i_{i \in I(0, R, q-p)} = v(0, R, q-p).
\]

Let us remark that one \( B \)-orbit may contain more than one standard vector.
Consider the curves of the form 6 and 7 from Lemma 2. Each non-zero component of the limit vector is \( p_i(\infty) x^{n_i} \) in case 6 and \( p_i(0) x^{n_i} \) in case 7. However, each component has only a finite number of possible values. This means that the curves of the forms 6 or 7 have only a finite number of vectors \( v(c, R, A) \) as limits.

Consequently, \( Bv \) contains infinitely many \( B \)-orbits iff curves of the forms 4 and 5 from Lemma 2 with \( e_i(c) = 0 \) have \( \lim_{t \to 0} \gamma(t)v \) in an infinite number of different \( B \)-orbits.

**Lemma 3.** The number of \( G \)-orbits in \( \overline{Gv} \) is finite iff the number of \( B \)-orbits in \( \overline{Bv} \) is finite.

**Proof.** It is known that \( Bv \) intersects each \( G \)-orbit in \( \overline{Gv} \) ([Korollar III.2.5.1]). Hence, if the number of \( B \)-orbits is finite then the number of \( G \)-orbits is finite. On the other hand, suppose that the number of \( B \)-orbits is infinite. Then there exist infinitely many standard vectors in different \( B \)-orbits. Let the element \( g \in G \) move one standard vector to another. Then \( g \) must preserve \( \langle x \rangle \) and \( g \in B \). However, every \( G \)-orbit contains at most one \( B \)-orbit of the vector \( v(d, R) \) and the number of \( G \)-orbits in \( \overline{Gv} \) is also infinite. \( \square \)
Our aim is to check the finiteness of the number of $B$-orbits in $\overline{Bv}$. It is sufficient to check the finiteness of the number of $B$-orbits containing a vector of the form $v(c, R)$ such that $I(R)$ is an admissible face. We shall now find out when a vector $v(d_2, R_2)$ can be obtained from $v(d_1, R_1)$ by $B$-action. Let us note that in this case one has $I(R_1) = I(R_2)$ and thus we can consider each admissible face separately. The number of orbits in a given orbit-closure is finite iff for each admissible face there exists only a finite number of standard vector orbits.

Suppose that an algebraic group $H$ acts on an irreducible variety $X$. Denote the minimal codimension of $H$-orbit in $X$ by $d(X, H)$. For an arbitrary variety $Y$ the modality of the action $H : Y$ is

$$mod(Y, H) = \max_{X \subset Y} d(X, H).$$

**Proposition 1.** Under the action $G : V$ for any $v \in V$ we have $mod(\overline{Gv}, G) = mod(\overline{Bv}, B) \leq 1$.

**Proof.** Consider the space $\{(a; x^n)_{i \in I(R)}\}$ for each admissible face $I(R)$. The torus $(C^*)^{k+1}$ acts on this space as follows: $(\alpha_i)_{i \in I(R)} \mapsto (\chi'_i(t) \alpha_i)_{i \in I(R)}$. In this space there is the curve $v(c, R) = (\prod_{a \in C, a \neq c} (a - c) \chi(a) x^n)_{i \in I(R)}, c \in \mathbb{C}$. If this curve intersects an infinite number of $(C^*)^{k+1}$-orbits then $mod(\overline{Bv}, B) = 1$. If for all admissible faces $v(c, R)$ intersects only a finite number of orbits of the torus, then the modality is zero.

It is easy to prove that $mod(\overline{Gv}, G) = mod(\overline{Bv}, B)$ as in Lemma 3.\hfill $\square$

Let us note that the inequality from Proposition 1 follows also from the paper of E. B. Vinberg [5].

### 3. Affine case

We shall obtain a criterion to detect whether the number of $B$ orbits in $\overline{Bv}$ is finite or not. Almost all $B$-orbits in $\overline{Bv}$ contain $\lim_{t \to 0} \gamma(t)v$ for $\gamma(t)$ of the form described in Lemma 1 with $p = q < 0, e_i(c) = 0 \forall i$. By Lemma 2, for almost all orbits in $\overline{Bv}$ this limit is $v(c, R)$. We need to get the conditions which hold as we have $b \cdot v(d_1, R) = v(d_2, R)$ for some $b \in B, d_1, d_2 \in C, R = (r_1, \ldots, r_k, p)$, where $I(R)$ is an admissible face.

We shall use the following well-known fact (see, for example, [4]):

**Proposition 2.** Suppose $V = C^m$, the $k$-dimensional torus $T$ acts on $V$ multiplying ith coordinate by the character $\chi_i$. Consider two points $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ of $V$ such that $x_1 \ldots x_my_1 \ldots y_m \neq 0$. Then $x$ and $y$ are in the same $T$-orbit iff for any $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m$ such that $\beta_1 \chi_1 + \ldots + \beta_m \chi_m = 0$ the condition $x_1^{\beta_1} \ldots x_m^{\beta_m} = y_1^{\beta_1} \ldots y_m^{\beta_m}$ is fulfilled.

**Theorem 1.** The number of $G$-orbits in $\overline{Gv}$, $v \in V$ is finite iff for each admissible face $I(R)$ of maximal dimension of the cone $M$ and for any integer-valued vector $\beta = (\beta_i)_{i \in I(R)}$ such that $\sum_{i \in I(R)} \beta_i \chi_i = 0$ the following conditions hold:

$$\sum_{i \in I(R)} e_i(a) \beta_i = 0 \quad \forall a \in C.$$

**Proof.** By Lemma 3, it is sufficient to consider the number of $B$-orbits in $\overline{Bv}$. We consider $b \cdot v(d_1, R) = v(d_2, R)$ and we shall obtain the conditions which hold as there is an infinite number of $d_2$ for some $d_1$. We have

$$(t, \begin{pmatrix} \kappa & 1 \\ 0 & \kappa^{-1} \end{pmatrix})p_i(d_1)x^n = p_i(d_2)x^n.$$

$$\chi'_i(t, \kappa) \prod_{a \in C, a \neq d_1} (a - d_1) e_i(a) = \prod_{a \in C, a \neq d_2} (a - d_2) e_i(a).$$
Since the equation of the face \(I(R)\) is \(\langle \chi'_i, R \rangle = 0\), then we have \(\chi'_i(t, \kappa) = \chi_i(t')\).

\[
\chi_i(t') \prod_{a \in C, a \neq d_1} (a - d_1)^{e_i(a)} = \prod_{a \in C, a \neq d_2} (a - d_2)^{e_i(a)}.
\]

Hence, the action is reduced to the action of the torus and, using Proposition 2, we obtain that a \(B\)-orbit contains an infinite number of the vectors of the form \(v(d, R)\) iff \(\prod_{i \in I(R)} (a - c)^{e_i(a)\beta_i}\) is independent of \(c\) for any \(\beta\) such that \(\sum_{i \in I(R)} \beta_i \chi_i = 0\). And it is equivalent to the following condition:

for any \(a \in C\) and any \(\beta\) such that \(\sum_{i \in I(R)} \beta_i \chi_i = 0\) we have \(\sum_{i \in I(R)} e_i(a)\beta_i = 0\).

Finally, any admissible face is contained in an admissible face of maximal dimension. Hence, if condition \(\mathbb{II}\) is fulfilled for a bigger face then it is fulfilled for its subface. So, it is sufficient to deal only with faces of maximal dimension. \(\square\)

Now let us describe the \(G\)-moduli which have only a finite number of orbits in each orbit-closure.

Suppose that \(I(R)\) is an admissible face of maximal dimension. A character matrix for this face is the matrix with the coordinates of characters \(\chi'_i, i \in I(R)\) as its columns.

**Corollary 1.** Any orbit-closure in \(V\) contains a finite number of \(G\)-orbits iff for each admissible face \(I(R)\) of the cone \(M\) the characters \(\chi_i (i \in I(R), n_i \neq 0)\) are linearly independent over \(\mathbb{Q}\). Particularly, it is true if \(M\) does not contain admissible faces.

**Proof.** By Theorem 1, \(\overline{Gv}\) contains a finite number of \(G\)-orbits iff for each admissible face \(I(R)\) and for any \(c \in C\) the vector \((e_i(c))_{i \in I(R)}\) is a rational linear combination of the rows of character matrix. The vector \((e_i(c))_{i \in I(R)}\) is called the vector of multiplicities. Let us note that \(E_j\) (the vector with 1 for \(j\)th coordinate and 0 for others) is a vector of multiplicities for some \(c\) iff \(j \in I(R)\) \(n_j \neq 0\). On the other hand, any vector of multiplicities is a linear combination of \(E_j\). \(\square\)

### 4. Projective case

**Theorem 2.** The number of \(G\)-orbits in \(\overline{Gv}\), \(\langle v \rangle \in \mathbb{P}(V)\) is finite iff for each admissible face \(J(R)\) of maximal dimension of the polyhedron \(C\) and for any integer-valued vector \(\beta = (\beta_i)_{i \in J(R)}\) such that \(\sum_{i \in J(R)} \beta_i \chi_i = 0\) and \(\sum_{i \in J(R)} \beta_i = 0\) the following conditions hold:

\[
\sum_{i \in J(R)} e_i(a)\beta_i = 0 \quad \forall a \in \mathbb{C}.
\]

**Proof.** To obtain the criterion in the case of projective action \(G : \mathbb{P}(V)\), we shall consider the linear action of bigger group \(\mathbb{C}^* \times G\) on \(V\) with characters \(\chi_i = (1, \chi_i) \in \mathcal{X}((\mathbb{C}^*)^{k+1})\). Let the cone \(M\) correspond to this action.

**Lemma 4.** The number of \(G\)-orbits in \(\overline{Gv}\), \(\langle v \rangle \in \mathbb{P}(V)\) is finite iff for each admissible face \(I(R)\) of maximal dimension of the cone \(M\) and for any integer-valued vector \(\beta = (\beta_i)_{i \in I(R)}\) such that \(\sum_{i \in I(R)} \beta_i \chi_i = 0\) and \(\sum_{i \in I(R)} \beta_i = 0\) the following conditions hold:

\[
\sum_{i \in I(R)} e_i(a)\beta_i = 0 \quad \forall a \in \mathbb{C}.
\]

**Proof.** It follows from Theorem 1. \(\square\)
If we return from characters \( \hat{\chi}_i \) back to \( \chi_i \), then a set of the form \( J(R) := \{ i \mid (\chi'_j, R) = \min_j (\chi'_j, R) \} \) corresponds to each admissible face \( I(R) \) of the cone. This set forms an admissible affine face of the intersection of the cone and the plane \( x_1 = 1 \).

In the case \( G = SL_2 \) the polyhedron \( C \) is an arc and the only its admissible face is its right vertex. Theorem 2 shows that \( SL_2(v) \) contains a finite number of \( G \)-orbits iff all components of maximal degree of the vector \( v \) coincide. This fact was originally proved in [2, Prop. 4].

**Corollary 2.** Any orbit-closure in \( \mathbb{P}(V) \) contains a finite number of \( G \)-orbits iff for each admissible face \( J(R) \) of the polyhedron \( C \) the characters \( \hat{\chi}_i \) \( (i \in J(R), n_i \neq 0) \) are linearly independent over \( \mathbb{Q} \).

**Proof.** It easily follows from Corollary 1. \( \square \)

**Example 2.** Consider the action \( SL_2 : \mathbb{P}(V_{n_1}) \times \ldots \times \mathbb{P}(V_{n_m}) \). Then any orbit-closure contains only a finite number of orbits. To prove it, we take the action of the group \( SL_2 \times (\mathbb{C}^*)^m \) on \( V = V_{n_1} \oplus \ldots V_{n_m} \) with the character \( \chi_i \) on \( V_n \). Here \( \chi_1 = (1,0,\ldots,0), \ldots, \chi_m = (0,\ldots,0,1) \). Now we use Corollary 1.

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