Solving System of Nonlinear Equations with the Genetic Algorithm and Newtons Method

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Abstract. An implementation and an application of the combination of the genetic algorithm and Newton’s method for solving a system of nonlinear equations is presented. The method first uses the advantage of the robustness of the genetic algorithm for guessing the rough location of the roots, then it uses the advantage of a good rate of convergence of Newton’s method. An effective application of the method for the positioning problem of multiple small rovers proposed for the use in asteroid exploration is shown.

Keywords: Nonlinear equations · Newton’s method · Genetic Algorithm.

1 Introduction

Solving a system of nonlinear equations is one of the fundamental problems in science and technology. If the given system consists of algebraic equations, it can be solved with algebraic techniques such as triangularization of the system using Gröbner bases [1]. However, if the given system has non-algebraic equations, algebraic methods may not be applicable.

In such a case, other numerical methods including the genetic algorithm or Newton’s method are used for computing the approximate roots ([2], [7]). Newton’s method [2] has a good rate of convergence, although the performance of the method depends on the initial values, and it may be difficult to find good initial values in some cases. On the other hand, a method based on the genetic algorithm [6] is sufficiently robust to find global solutions, although the convergence rate may not be high, compared with Newton’s method. Furthermore, a combination of both methods has been proposed [3]: first by guessing good
initial values by the genetic algorithm, then by using Newton’s method for fast convergence.

In this paper, we present an implementation and an application of the combination of the genetic algorithm and Newton’s method for solving a system of nonlinear equations. We demonstrate that the present method finds the solutions effectively in the positioning problem of multiple small rovers proposed for the use in asteroid exploration. A comparison with the root-finding with the genetic algorithm is presented.

The rest of this paper is organized as follows. In Section 2, an application of the present method to the positioning problem of multiple rovers and the formulation of a system of nonlinear equations is presented. In Section 3, the result of experiments along with comparison with a method using only the genetic algorithm is shown.

2 Application of the present method to the positioning problem of multiple rovers to be used in asteroid exploration

In this section, we explain an application of the present method to the positioning problem of multiple rovers, proposed by one of the present authors [5], for asteroid exploration.

2.1 Specification of the rovers

The rover has a rectangular shape with approximately 50 mm in length, width, and height, and multiple rovers are placed on the asteroid for exploration. Each rover communicates with others using radio wave with configuring a wireless mesh network on a surface of the asteroid. In the radio wave communication, the rover obtains the received signal strength indicator (RSSI), which is used for estimating relative distances among the rovers.

For communicating with other rovers under good condition even if the rover is placed with unusual position (such as upside down), the rover has a pair of two antennas on each side, and it uses a pair of antennas on the top side for communicating with other rovers (see Fig. 1). Let antennas $a_i$ and $b_i$ be the pair of the two on the top side, respectively.

2.2 Estimating relative distances among the rovers by using the RSSI

For estimating relative distances among the rovers by using the RSSI, for rover $i$ ($i = 1, \ldots, n$), we define a coordinate system $\Sigma_i$ as shown in Fig. 2.

Since each rover communicates with others using antennas on the top side, let $a_i$ and $b_i$ be the antennas $a$ and $b$ on the top side, respectively, and let $\Sigma_{a_i}$.
Fig. 1. The rover equipped with twelve antennas [5, Fig. 3].

and $\sum_{b_i}$ be coordinate systems with the origin placed at the center of $a_i$ and $b_i$, respectively. Let

$$a_i \mathbf{p}_{b_j} = \begin{pmatrix} a_i x_{b_j} \\ a_i y_{b_j} \\ a_i z_{b_j} \end{pmatrix} \in \mathbb{R}^3$$

denote the position of the antenna $b_j$ with respect to the coordinate system $\sum_a$. Especially, $\mathbf{p}_a$, denote the position of the antenna $a_i$ with respect to the coordinate system $\sum_1$. Furthermore, let $R_{a_i} \in \mathbb{R}^{3 \times 3}$ denote the orientation of the antenna $a_i$ with respect to the coordinate system $\sum_1$. Then, by using the relationship

$$\mathbf{p}_a = \mathbf{p}_{b_j} + R_{a_i} a_i \mathbf{p}_{b_j},$$

we derive the relative position of the antenna $b_j$ with respect to the coordinate system $\sum_a$, and vice versa, as

$$b_j \mathbf{p}_a = (\mathbf{p}_{b_j} - \mathbf{p}_a) R_{b_j}^{-1}, \quad a_i \mathbf{p}_{b_j} = (\mathbf{p}_{b_j} - \mathbf{p}_a) a_i R_{b_j},$$

noting that $b_j R_{b_j} = (R_{b_j})^{-1}$ and $a_i R_{b_j} = (R_{a_i})^{-1}$.

Next, let $a_i \phi_{b_j}$ and $a_i \theta_{b_j}$ denote the horizontal and the elevation angle of the antenna $b_j$ with respect to the coordinate system $\sum_a$, respectively, as shown in fig. 3. Then, by using the relative position $a_i \mathbf{p}_{b_j} = \begin{pmatrix} a_i x_{b_j} \\ a_i y_{b_j} \\ a_i z_{b_j} \end{pmatrix}$, $a_i \phi_{b_j}$ and $a_i \theta_{b_j}$ are derived as

$$a_i \phi_{b_j} = \arctan \left( \frac{a_i y_{b_j}}{a_i x_{b_j}} \right), \quad a_i \theta_{b_j} = \arctan \left( \frac{a_i z_{b_j}}{\sqrt{(a_i x_{b_j})^2 + (a_i y_{b_j})^2}} \right).$$

The horizontal and the elevation angle of the antenna $a_i$ with respect to the coordinate system $\sum_{b_i}$, respectively, are derived similarly.

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Let $r_{a_i b_j}$ be the RSSI between the antenna $a_i$ and $b_j$. By the discussion above, the mathematical model of $r_{a_i b_j}$ are expressed as

$$r_{a_i b_j}(a_{x_i b_j}, a_{y_i b_j}, a_{z_i b_j}, a_i \theta_{b_j}, b_j \phi_{a_i}, b_j \theta_{a_i}) = r'_{a_i b_j}(a_{x_i b_j}, a_{y_i b_j}, a_{z_i b_j}) + r_h(a_i \phi_{b_j}) + r_v(b_j \phi_{a_i}) + r_v(b_j \theta_{a_i}),$$

where

$$r'(x, y, z) = -14.69 \log_{10}(\sqrt{x^2 + y^2 + z^2 + 0.31}) - 49.17,$$

$$r_h(\phi) = \frac{5}{2}(\cos(2\phi) - 1),$$

$$r_v(\theta) = 25 \left( \frac{\cos\left(\frac{5\pi}{2}\phi\right) \cos\left(\frac{5\pi}{2} - |\theta|\right)}{\sin\left(\frac{5\pi}{2} - |\theta|\right)} \right) - 1.$$

We find the values of $a_{x_i b_j}, a_{y_i b_j}, a_{z_i b_j}, a_i \theta_{b_j}, b_j \phi_{a_i}, b_j \theta_{a_i}$ by solving eq. (1) with respect to these variables. In the original article \[5\], the author solves the equation by using the genetic algorithm in whole using GAlib \[8\]. In the genetic algorithm, the evaluation function is defined as follows. Let $\bar{r}_{a_i b_j}$ and $\hat{r}_{a_i b_j}$ be the measured and the estimated RSSI values, respectively. Then, the evaluation function $f(r)$ is defined as

$$f(r) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left\{ (\bar{r}_{a_i b_j} - \hat{r}_{a_i b_j})^2 + (\bar{r}_{a_i b_j} - \hat{r}_{a_i b_j})^2 ight\}.$$

### 2.3 Setting up nonlinear equations with Newton’s method and the genetic algorithm

In this paper, let us assume that all the rovers have the same direction and they are placed on the $xy$ plane so that the elevation angle $a_i \theta_{b_j}$ are omitted.

Nonlinear equations for estimating relative distance and the rotation angle between two rovers are derived, as shown in fig. 4. Let $\varphi$ be the acute angle between lines $C_1 C_{12}$ and $C_1 C_2$. Note that triangles $A_1 A_2 A_{12}$, $B_1 B_2 B_{12}$
and $C_1C_2C_{12}$ are congruent. Since $\angle C_2C_1C_{12} = \angle A_2A_1A_{12}$ and $\angle A_1A_2C_{12}$ and $\angle A_1A_2C_{21}$ are the complex angle, we have $\angle A_1A_2C_{21} = \varphi$. Furthermore, $\angle B_2B_1B_{12} = \varphi$ shows that $\angle B_1B_2B_{12} = \frac{\pi}{2} - \varphi$. Thus, the RSSI is expressed as

$$r_{a_1,a_2}(a_1x_{a_2}, a_1y_{a_2}, a_1\varphi_{a_2}, a_2\varphi_{a_1}) = -14.69 \log_{10}(\sqrt{(a_1x_{a_2})^2 + (a_1y_{a_2})^2} + 0.31) - 49.17 + 5(\cos 2\varphi - 1),$$

(3)

$$r_{b_1,b_2}(b_1x_{b_2}, b_1y_{b_2}, b_1\varphi_{b_2}, b_2\varphi_{b_1}) = -14.69 \log_{10}(\sqrt{(b_1x_{b_2})^2 + (b_1y_{b_2})^2} + 0.31) - 49.17 + 5(\cos 2(\frac{\pi}{2} - \varphi) - 1).$$

(4)

By using $A_1A_2 = B_1B_2$ and rewriting $\cos 2(\frac{\pi}{2} - \varphi)$ as

$$\cos \left(2\left(\frac{\pi}{2} - \varphi\right)\right) = \cos^2 \left(\frac{\pi}{2} - \varphi\right) - \sin^2 \left(\frac{\pi}{2} - \varphi\right)$$

$$= \sin^2 \varphi - \cos^2 \varphi = -\cos(2\varphi),$$

we have

$$r_{a_1,a_2} - r_{b_1,b_2} = 10\cos(2\varphi),$$

(5)

which gives the value of $\varphi$. Then, by solving a system of nonlinear equations

$$r_{a_1,a_2}(a_1x_{a_2}, a_1y_{a_2}, \varphi) = -14.69 \log_{10}(\sqrt{(a_1x_{a_2})^2 + (a_1y_{a_2})^2} + 0.31) - 49.17 + 5(\cos 2\varphi - 1),$$

(6)

$$\varphi = \arctan \left(\frac{a_1y_{a_2}}{a_1x_{a_2}}\right),$$

we find the relative position of $(x_2, y_2)$ with respect to $(x_1, y_1)$. 

Fig. 4. Positions of two rovers.
In this paper, we assume that there exists a rover on the origin and we estimate the position of the other rovers by computing the distance and the orientation angle of those from the one on the origin. In the genetic algorithm, the evaluation function is defined as in eq. (2). In Newton’s method applied to eq. (6), the Jacobian matrix becomes

\[
J = \begin{pmatrix}
-14.69(a_1 x_{a_2}) \ln 10 \\
(\frac{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2 + 0.31 \sqrt{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2}}{a_1 x_{a_2}}) \\
(\frac{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2}{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2}) \\
(\frac{-14.69(a_1 y_{a_2}) \ln 10}{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2 + 0.31 \sqrt{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2}}) \\
(\frac{0.31 \sqrt{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2}}{(a_1 x_{a_2})^2 + (a_1 y_{a_2})^2})
\end{pmatrix}.
\]

(7)

3 Experiments

We have tested the method for estimating positions of rovers placed on the xy plane as shown in fig. [5]. For the implementation of the method, GAlib [8] was used for the genetic algorithm and our implementation was used for Newton’s method.

Position of Rover \(i\) \((i = 1, \ldots, 8)\) is estimated by computing the distance and the horizontal angle between Rover 0 placed on the origin. In the experiments, the genetic algorithm was executed with the following settings: the number of population was set to 100, the number of generation was set to 200, single-point crossover and roulette wheel selection were used with the crossover rate set to 0.9, and the mutation rate was set to 0.01. After evolving the population for repeating the computation for \(10 \times 10 = 100\) times (which is denoted by \(10 \times 10\)), the best individuals which minimize \(f(r)\) were used as the initial values for Newton’s method. Newton’s method was terminated either when the magnitude of the updated value became less than \(1.0 \times 10^{-10}\) or the number of iterations reached 100.

The computing environment is as follows: Intel Xeon E5607 2.27GHz, RAM 48 GB, Linux 3.16.0, GCC 9.2.1.

3.1 Experiment 1: estimation of rovers with the present method

Table 1 shows the result of the genetic algorithm computation. The columns are as follows: ‘i’ is the index of the rover, ‘Actual position’ and ‘Estimated position’ is the actual and the estimated position of Rover \(i\), respectively, ‘Relative error’ is

\[
|d_i - d'_i|/d_i,
\]

where \(d_i\) and \(d'_i\) is the actual and the estimated distance of Rover \(i\) from the origin, respectively. The bottom row (with ‘Average’ in the column of ‘Actual position’) shows the average of the relative errors. Note that the estimated position is used as the initial value of Newton’s method.
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Fig. 5. Positions of the rovers (unit: mm).

Table 2 shows the result of Newton’s method after the genetic algorithm. The columns ‘i’, ‘Actual position’, ‘Estimated position’ and ‘Relative error’ are the same as those in Table 1. The column ‘#iterations’ is the number of iterations executed for estimating the position. For Rover 5, Newton’s method has terminated after the number of iteration becomes 100, while, for the other rovers, it has terminated the magnitude of the updated value has become less than $1.0 \times 10^{-10}$. We see that for all the rovers except for Rover 5, the relative error decreases after applying Newton’s method. Especially, for Rovers 2, 3 and 8, the relative error decreases significantly.

3.2 Experiment 2: estimation of rovers using only the genetic algorithm and discussions

For comparison, we have conducted another experiment for estimating the position of the rovers using only the genetic algorithm. In this experiment, after executing the genetic algorithm for $n \times n = n^2$ times (which is denoted by $n \times n$), the best population which minimizes $f(r)$ in eq. (2) was chosen. We have solved the equations by this method for $n = 20, 30, 40, 50, 60, 70, 80$. All other settings of the genetic algorithm are the same as those in the previous experiment. Table 3 shows the average of relative errors and the computing time of the experiment. Note that, for comparison, the result for $n = 10$ is taken from Table 1 and the bottom row shows the result of Experiment 1.

Comparison of the average of relative errors of the estimated positions of the rovers in both experiments (Table 3) shows that the present method can estimate
Table 1. The result of estimation by the genetic algorithm with the setting of $10 \times 10$.

| $i$ | Actual position | Estimated position | Relative error |  
|-----|-----------------|--------------------|----------------|
| 1   | (20000.0, 30000.0) | (18411.987305, 33178.985596) | 0.052413 |
| 2   | (10000.0, 10000.0) | (10122.985841, 10196.990967) | 0.016006 |
| 3   | (10000.0, 20000.0) | (11677.993774, 20789.993286) | 0.066396 |
| 4   | (30000.0, 40000.0) | (26173.995972, 43239.990234) | 0.010896 |
| 5   | (80000.0, 50000.0) | (77340.995789, 55555.999756) | 0.009400 |
| 6   | (90000.0, 60000.0) | (89379.997253, 64238.998413) | 0.071642 |
| 7   | (50000.0, 70000.0) | (50185.989380, 75285.999146) | 0.018041 |
| 8   | (0.0, 60000.0)    | (2766.006470, 64238.998413) | 0.037075 |

Table 2. The result of estimation by Newton’s method.

| $i$ | Actual position | Estimated position | Relative error | #iterations |
|-----|-----------------|--------------------|----------------|-------------|
| 1   | (20000.0, 30000.0) | (19511.732516, 29804.663731) | 0.01985 | 99 |
| 2   | (10000.0, 10000.0) | (10000.000000, 10000.000000) | $< 1.0 \times 10^{-11}$ | 95 |
| 3   | (10000.0, 20000.0) | (10000.000000, 20000.000000) | $< 1.0 \times 10^{-11}$ | 98 |
| 4   | (30000.0, 40000.0) | (29120.352787, 39684.351546) | 0.01552 | 98 |
| 5   | (80000.0, 50000.0) | (76602.187691, 50147.903356) | 0.02946 | 100 |
| 6   | (90000.0, 60000.0) | (91596.396916, 59963.917465) | 0.01213 | 88 |
| 7   | (50000.0, 70000.0) | (51871.537800, 70688.990493) | 0.019247 | 96 |
| 8   | (0.0, 60000.0)    | (0.0, 60000.000000) | $< 1.0 \times 10^{-11}$ | 88 |
| Average | — | — | 0.011051 | — |

the position of the rovers with better accuracy on average than the method using only the genetic algorithm. Furthermore, we see that the present method is significantly more efficient than the method using only the genetic algorithm. Thus, we conclude that the method of combining the genetic algorithm and Newton’s method estimates the position of rovers more efficiently than methods using only the genetic algorithm, with better accuracy on average.

4 Concluding remarks

In this paper, we have demonstrated a numerical method combining the genetic algorithm and Newton’s method for solving a system of nonlinear equations. Our method uses the genetic algorithm for a global search of approximate roots, then it uses Newton’s method for fast convergence. We have formulated the problem of estimating the position of rovers used in asteroid exploration into a system of nonlinear equations for the use of the present method. The experiments have shown that the present method computes the roots with almost the same
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Table 3. The result of estimation by the genetic algorithm with the setting of \( n \times n \), comparison with the result of the method of combining the genetic algorithm and Newton's method.

| \( n \) | The average of relative errors | Computing time (sec.) |
|-------|-------------------------------|-----------------------|
| 10    | 0.037075                      | 2516.31199            |
| 20    | 0.024174                      | 10047.422038          |
| 30    | 0.024484                      | 8614.607373           |
| 40    | 0.020342                      | 15288.596504          |
| 50    | 0.024484                      | 23874.125484          |
| 60    | 0.024605                      | 34370.2462            |
| 70    | 0.032624                      | 46743.015786          |
| 80    | 0.016325                      | 61102.5245            |
| GA+Newton | 0.011051                        | 2516.31389          |

accuracy and significantly better efficiency than a method using only the genetic algorithm.

Future research direction includes the following.

1. Since our formulation of a system of equations\(^5\) uses arctangent function, we can estimate the position of rovers located only in the first and the 4th quadrant. Another formulation of equations for estimating the position of rovers located in the second and the third quadrant will be needed.

2. In this paper, we assumed that the rover is placed horizontally on the \( xy \) plane. However, in the actual exploration, the surface of the asteroid is probably not flat, a formulation of the estimation problem in the \( xyz \) space with the elevation angle will be needed.

3. In our method, we first estimated the orientation angle of the rovers from the RSSI values as in\(^5\), then solved eq. (3). Since the error in the measured RSSI values affects the coefficients in eq. (3), estimation of errors in the roots of eq. (3) caused by the error in the measured RSSI values will be needed.

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