Stratified Large-Amplitude Steady Periodic Water Waves with Critical Layers

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Abstract: By means of a conformal mapping and bifurcation theory, we prove the existence of large-amplitude steady stratified periodic water waves, with a density function depending linearly on the streamfunction, which may have critical layers and overhanging profiles. We also provide certain conditions for which these waves cannot overturn.

1. Introduction

We will study two-dimensional steady water waves which propagate on the surface of an inviscid fluid and over a flat, impermeable bed. These waves are solutions to the incompressible Euler equations and the only exterior force acting on them is gravity. In this paper, we will only study periodic waves; for work on solitary waves, we refer to [1,2]. The term stratification here refers to the presence of a non-constant density function $\rho(X, Y)$, and throughout this paper, we will choose $\rho$ to depend linearly on the streamfunction. One additional assumption we will make is that the variation of energy along streamlines is a constant. This last condition is mathematically equivalent to the case of constant vorticity for constant density waves (see discussion in [3]). We point out that in our case, the expression for vorticity is more complicated, and we will not be working with it directly. The main particularity of the waves we will be constructing is that we allow for internal stagnation points, critical layers, and overturning profiles, which to our knowledge, has not be investigated analytically before.

1.1. Historical considerations. Let us talk briefly about the history of the problem. The main difficulty is that we are working in a domain where the top boundary, the free surface, is an unknown. In order to overcome this, the usual approach is to choose suitable variables that fix the domain, and this has been done in various ways. Dubreil-Jacotin [4] derived an elegant change of variable, reformulating the problem in a rectangle using what is sometimes known as semi-lagrangian coordinates: the $x$ variable remains...
unchanged and $\gamma$ is transformed into the streamfunction. Back in the 80s, Amick and Turner (see [5,6]) used this approach to construct large-amplitude stratified waves in a strip bounded above and below by rigid walls. In [3], this approach was then used in a free-surface setting. Walsh was able to prove the existence of small- and large-amplitude stratified waves with the added novelty of allowing for general Bernoulli and non-negative density functions. However, in order for this change of variables to produce valid solutions in the physical plane, the gradient of the streamfunction can never be zero and therefore the waves constructed cannot admit stagnation points or critical layers. Moreover, the free surface needs to be a graph of a function, thus also excluding the possibility of waves with an overturning profile. Such waves were observed in field data and were simulated numerically (see the discussion in [7]).

In [8], the author used a flattening technique to fix the free surface, and then successfully constructs small-amplitude homogeneous waves with constant vorticity function, which have critical layers. For works with a more general vorticity distribution, see [9]. Recently, [10] used this technique to construct large-amplitude homogeneous waves with general vorticity. However, in that formulation, allowing for the presence of internal stagnation points makes it difficult to prove suitable nodal properties. Moreover, these waves cannot have overhanging profiles. Finally, also of note-worthy mention is [11], where global bifurcation has been achieved for capillary-gravity waves with variable density and vorticity, and allowing for stagnation points in the flow.

In [7,12] the authors managed to construct small- and large-amplitude water waves with constant vorticity, by considering the fluid domain as the image of a strip via a conformal map. This map makes no assumption on the geometry of the physical domain, or on the streamfunction, and so the solutions obtained in this way may contain critical layers, stagnation points and overturning profiles. This is the approach we will use in this paper. The strength of this theory derived in [7] becomes apparent as it carries over very nicely to our setting of stratified waves. Moreover, a lot of the proofs will present similarities to the methods developed in [7].

1.2. Main result. The main result is a combination of Theorem 3.2, located at the end of Sect. 3 and Theorem 4.4, in Sect. 4. However, the first one is very long and both are in parameterized form and can seem obscure at first glance. For the benefit of the reader, we somewhat crudely restate the results in plainer terms and in physical quantities.

Theorem 1.1. Fix a Hölder exponent $\alpha \in (0,1)$, a wave speed $c > 0$, a wave number $k > 0$, the gravitational constant $g > 0$, a non-negative, density function $\rho$, linear in terms of the pseudo-streamfunction $\psi$, and a constant energy variation $\gamma \in \mathbb{R}$. Then there exists a family of global curves of periodic waves. Solutions $(\Omega, \psi)$ on these curves are symmetric and of class $C^{1,\alpha}$ in the sense that the surface $S$ of the domain $\Omega$ is parameterized by functions of class $C^{1,\alpha}$ and $\psi \in C^\infty(\Omega) \cap C^{1,\alpha}(\partial \Omega)$. In addition, one of two alternatives occurs:

(i) the regularity of the free boundary blows up,
(ii) the solutions on the curve exist until the wave profile self-intersects strictly above the trough line.

The second alternative in the above theorem is proved in Theorem 4.4. We point out that the main particularities of these waves is that they can have overhanging profiles and internal stagnation points and critical layers.
1.3. Plan of the paper. The plan of this paper is as follows. In Sect. 2, we set up the problem. In particular, we reformulate it by viewing the fluid domain as the image of a strip via the holomorphic function $U + iV$. The connection between the conformal map and the original water wave problem is given by the fact that the free surface is parameterized by the functions $U$ and $V$, evaluated on the top of strip, which in our case is the real-axis. We then boil down the whole problem to studying Bernoulli’s boundary condition at the surface of the fluid domain by reformulating it in terms of the periodic Hilbert transform for the strip, applied to the imaginary conformal coordinate. This will give us a nonlinear pseudo-differential equation containing several parameters. However, this equation does not seem to satisfy certain compactness criteria we need in order to construct large-amplitude solutions. To this end, using Riemann–Hilbert theory, we will prove that the equation is equivalent to a quasilinear pseudo-differential equation, coupled with a scalar constraint. Although we will then be able to apply the real-analytic global bifurcation theorem by Dancer [13] and Buffoni and Toland [14] to this generalized Babenko formulation by using commutator properties of the Hilbert transform, this reformulated problem will be more general than the original water wave problem, an issue which we will address in Sect. 4.

In Sect. 3, we begin by constructing small-amplitude waves, which bifurcate from laminar flows. We then extend that local curve of solutions to prove the existence of the large-amplitude waves mentioned in the previous paragraph. In particular, we prove the existence of a sequence of continuous curve of solutions $K_{n, \pm}$ each representing waves of period $\frac{2\pi n}{n}$ for all $n \in \mathbb{N}$, and for both signs $\pm$. We obtain two possible alternatives for these curves of solutions: either they loop back to the bifurcation point, or they either go off to infinity or hit the boundary of the domain, at which point the solution in the physical plane would admit a stagnation point at the crest.

In Sect. 4, we study the physical relevance of the previously constructed solution curves. In particular, we need to take into account the fact that the free surface cannot be self-intersecting and cannot cross the flat bottom, which in our case, reduces to verifying that it stays in the upper-half plane. Moreover, by showing that the vertical coordinate of the parametrized surface is strictly decreasing between each crest and trough, we eliminate the loop alternative in the previous theorem. The main result is shown using a continuation argument: first one must check that the qualitative properties are satisfied in the neighborhood of the bifurcation point, and then use a series of arguments involving the strong maximum principle, and the Hopf and Serrin inequalities, to check whether these properties hold along the entire curve. We once again arrive at two alternatives, either the solutions exist at all time, or they exist until the surface self-intersects right above the trough. The proof of the main result in this section is similar to the one in [7].

Finally, in Sect. 5, we study properties of the horizontal coordinate of the parametrized free surface. In particular, since in the previous section we only put a strict monotonicity constraint on the vertical coordinate, the waves may still have overhanging profiles. Here, we prove that under certain assumptions on the density function and on the sign of the variation of the energy in terms of the streamfunction, the waves cannot overhang. The proof, a continuation argument, closely follows that in [15]. We argue by contradiction, first proving that there exist solutions with overturning profiles and then use a maximum principle argument involving the pressure of the fluid to show that, for our assumptions, such solutions cannot exist in the physical plane.
2. Formulation of the Problem

2.1. The parameterized domain. We formulate the problem in an unbounded domain \( \Omega \) in the \((X, Y)\)-plane. We denote the impermeable flat bottom by \( B := \{(X, 0) : X \in \mathbb{R}\} \).

Moreover, we define the unknown parameterized surface, representing the water’s free surface, as

\[
S := \{(u(s), v(s)) : s \in \mathbb{R}\}
\]

where the map

\[
s \mapsto (u(s) - s, v(s))
\]

is periodic of period \( L := \frac{2\pi}{k} \), with \( k \in \mathbb{N} \) denoting the wave number. In addition, we require that

\[
u'(s)^2 + v'(s)^2 \neq 0 \quad \forall s \in \mathbb{R}
\]

and that \( u, v \) are functions of class \( C^{1,\alpha} \). For the remainder of the paper, let us fix \( \alpha \in (0, 1) \). From this point onwards, for any integer \( p \geq 0 \), we denote by \( C^{p,\alpha} \) the space of functions whose partial derivatives up to order \( p \) are Holder continuous with exponent \( \alpha \) over their domain of definition. Such a domain with the above properties is called an \( L \)-periodic strip-like domain of class \( C^{1,\alpha} \).

2.2. Pseudo-streamfunction formulation. Since we have an incompressibility condition, the velocity vector field of the fluid is divergence free. We can therefore introduce a (relative) pseudo-streamfunction, \( \psi = \psi(X, Y) \), \( L \)-periodic in \( X \) throughout \( \Omega \). It is defined such that it provides the (relative) pseudo-velocity field \((\psi_Y - \psi_X)\) in a moving frame at constant wave speed, thus eliminating the time-dependence. This pseudo-streamfunction differs from the streamfunction used for homogeneous fluids in the sense that it is scaled by a factor of \( \sqrt{\rho} \). This enables it to capture the effects of stratification which the regular streamfunction doesn’t.

Let us briefly recall that Euler’s equations for steady stratified waves in the interior of the domain \( \Omega \) expressed in terms of the pseudo-streamfunction are given by

\[
\begin{align*}
\psi_Y \psi_{XY} - \psi_X \psi_{YY} &= -P_X \\
\psi_X \psi_{XY} - \psi_Y \psi_{XX} &= -P_Y - \rho g \\
\psi_Y \rho_X - \psi_X \rho_Y &= 0.
\end{align*}
\]

The last equation (continuity equation) indicates that the density \( \rho \) is transported, and therefore constant along streamlines. As a result, it can be expressed in terms of the pseudo-streamfunction. We thus denote

\[
\rho(X, Y) := \rho(\psi).
\]

By eliminating the pressure term in Euler’s equations, we can show that

\[
\nabla \cdot (\Delta \psi + gY\rho'(\psi)) = 0
\]

which enables us to conclude that \((\Delta \psi + gY\rho'(\psi))\) and \( \psi \) have the same level sets. We can therefore write:

\[
\Delta \psi + gY\rho'(\psi) = \beta(\psi).
\]
This equation is known as the Yih-Long Equation (for more details, see [3]). Note that since we are not assuming that $\psi_Y \neq 0$, $\beta(\psi)$ is not necessarily a single-valued function!

In our case, and for the remainder of this paper, we will set

$$\beta(\psi) = \gamma$$

where $\gamma \in \mathbb{R}$, is an arbitrary constant. What this means physically is that the variation of energy as a function of streamlines is constant. Moreover, we will choose a density function which depends linearly on streamlines:

$$\rho(\psi) := A\psi + B$$

with arbitrary $A \in \mathbb{R}$. The constant $B$ is also arbitrary but since the density function needs to be non-negative at all times, and $\psi = 0$ at the surface, we will require that $B \in \mathbb{R}^+$. We recover the above eliminated pressure term through Bernoulli’s theorem which states that the quantity

$$E := P + \frac{|\nabla \psi|^2}{2} + gY\rho$$

is constant along streamlines. In plain terms, $E$ represents the energy of the fluid particles at a point $(X, Y)$ in the domain. In particular, (2) indicates that $E$ is equal to the sum of the energy due to internal pressure, the kinetic energy and the gravitational potential energy. On the free surface, where we have $\psi \equiv 0$ and constant (atmospheric) pressure $P = P_{\text{atm}}$, (2) becomes

$$|\nabla \psi|^2 + 2YgB = Q.$$  

Here the hydraulic head $Q := (E - P_{\text{atm}}) |S|$ groups together all constant quantities on the surface.

As a result, we obtain the following problem in the physical plane:

$$\begin{cases}
\Delta \psi = \gamma + aY & \text{in } \Omega \\
|\nabla \psi|^2 + bY = Q & \text{on } S \\
\psi = 0 & \text{on } S \\
\psi = -m & \text{on } B
\end{cases}$$

where the constants $a := -Ag$, $b := 2gB$. The quantity $m$ is the pseudo-relative mass flux. In this paper, $m$ and $Q$ will be the parameters which will vary along the solution curve.

The goal is to find solutions $(\Omega, \psi)$ to the above water wave problem with $\psi \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

2.3. Physical relevance. Before we begin, we would like to briefly discuss the physical relevance of the choices we have made for the density function and the energy variation along streamlines. As we have pointed out in the introduction, the density function needs to be positive at all times, which necessarily calls for $B$ to be non-negative. The sign of the constant $A$ is however a more delicate question. It is customary to work with what is known as stably stratified fluids. This simply means that the density function is non-decreasing with depth, implying that the derivative of $\rho$ with respect to $Y$ should be non-positive. Notice that

$$\rho_Y = A\psi_Y.$$  

(5)
However, in this paper the waves we construct can have internal stagnation points and critical layers, implying that $\psi_Y$ doesn’t necessarily have a fixed sign. Moreover, we are also allowing for overhanging profiles: when a wave overturns, it will be unstably stratified. With these considerations in mind, we have opted to choose $A \in \mathbb{R}$, rather than assigning it a fixed sign.

The main motivation for choosing a density function which depends linearly on the pseudo-streamfunction and a constant energy variation, has been from a mathematical standpoint. By removing all $\psi$ dependence on the right-hand side of the first equation in (4), we are able to reformulate the problem in terms of a harmonic function. We are still working to develop a version of the approach used in this paper to a more general setting. That being said, as we will see later on, even in this relatively “simple” setting, the analysis is already quite involved and the equations long and complicated.

Keeping the above considerations in mind, in Sect. 5 we prove that for $A$ negative and $\gamma$ positive, we cannot have internal stagnation points, critical layers, or overhanging profiles. As we will see, with these choices of $A$ and $\gamma$, the pseudo-streamfunction is non-negative and $\psi_Y < 0$ in the fluid domain. Our first concern is ensuring that the density function is non-negative. Recall that a necessary and sufficient condition for our density function to be positive is

$$-Am + B \geq 0. \quad (6)$$

Since $A$ and $B$ are fixed we set the additional requirement that

$$m \geq \frac{B}{A}. \quad (7)$$

Moreover, using (5), we see that we are in an unstably stratified setting. Although $A$ positive would be more reasonable physically, the results we have obtained were the best we could get with our methods.

2.4. The conformal mapping. The way in which (4) is formulated above in the physical plane is disadvantageous in the sense that the unknown is one of the boundaries: the free surface. In order to handle this difficulty, we will reformulate the problem via a conformal mapping.

Before we begin, let us define the following horizontal strip:

$$\mathcal{R}_d := \{(x, y) \in \mathbb{R}^2 : -d < y < 0\}. \quad (8)$$

The idea is to regard the unknown fluid domain as the conformal image of a strip $\mathcal{R}_d$. Consider any $L$-periodic strip-like domain $\Omega$ of class $C^{1,\alpha}$, whose boundary consists of $\mathcal{B}$ and an $L$-periodic curve $\mathcal{S}$. Then it was shown in [7] that there exists a unique positive constant $h$ (the conformal mean depth) such that we can find a conformal mapping $U+iV$ from $\mathcal{R}_h$ onto $\Omega$. This mapping extends as a homeomorphism between the closures of these domains and is defined such that

$$\left\{ \begin{array}{ll}
U(x + 2\pi/k, y) = U(x, y) + \frac{2\pi}{k} & (x, y) \in \mathcal{R}_h \\
V(x + 2\pi/k, y) = V(x, y) & (x, y) \in \mathcal{R}_h
\end{array} \right.$$
As a result, the top of the strip is mapped to the free surface, and the bottom, to the impermeable flat bottom of the fluid domain. Since \( \Omega \) has a periodic structure, we need some periodicity in the strip as well. The natural choice is for the horizontal periodicity to be identical to the one in the original domain: \( L = \frac{2\pi}{k} \). The vertical dimension, the conformal mean depth \( h \), is a number that is determined by the geometry of the fluid domain, and thus cannot be known yet (we refer to [12] for a proof of its existence and uniqueness). We therefore get a rectangular box of height \( h \) and length \( \frac{2\pi}{k} \) whose image via the conformal map corresponds to one period in \( \Omega \).

This rectangle is certainly not unique, but the ratio of its length over its width must always be constant. And therefore, since we prefer to work in a setting of period \( 2\pi \), we can dilate the top of the rectangle and thus the vertical coordinate at the bottom will be \( -kh \). Following definition (8) we denote rectangular strip by \( R_{kh} \). This rectangle is periodic in the sense that \( x \) lies on a suitable one-dimensional torus.

Let us now take a closer look at the conformal map \( U + iV \), and in particular, at the boundary conditions the function \( V \) has to satisfy. We define

\[
v(x) := V(x, 0) \quad \forall x \in \mathbb{R}.
\]

Since \( V \) maps the strip into \( \Omega \), we must have

\[
\begin{cases}
\Delta V = 0 & \text{in } R_{kh} \\
V(x, 0) = v(x) & \forall x \in \mathbb{R} \\
V(x, -kh) = 0 & \forall x \in \mathbb{R}
\end{cases}
\]

and therefore, knowing \( v(x) \) gives us the entire conformal mapping (\( U \) being recovered as the harmonic conjugate of \( -V \)), which in turn then gives us the fluid region. Recall that the connection between \( v(x) \) and the unknown free surface is given by the parametrization

\[
S := \{ (u(s), v(s)) : s \in \mathbb{R} \}.
\]

We would like to point out that such a conformal mapping exists for any geometry of the free surface and we can therefore allow for waves with an overhanging profile.

2.5. Nonlocal reformulation via the periodic Hilbert transform. One important consideration we have to take into account is that we have boundary conditions on \( \partial \Omega \) which need to be satisfied, in particular, the non-linear Bernoulli condition on the surface. We therefore need to consider which conditions \( v(x) \) needs to satisfy such that this boundary condition in the fluid domain is verified, which will lead us to derive the equations in terms of \( v(x) \).

To begin with, because of the consideration on the conformal mean depth (see [7]), we have that

\[
[v] = h,
\]

where square brackets, throughout this entire paper, will represent the mean over one period of \( 2\pi \)-periodic function.

From (9), we have

\[
V_y(x, 0) = G_{kh}(v)(x)
\]

for all \( x \in \mathbb{R} \) where \( w \mapsto G_{kh}(w) \) denotes the periodic Dirichlet-Neumann operator for the strip.
Since $U$ is a harmonic conjugate of $-V$ on $\mathcal{R}_{kh}$ we obtain

$$U(x, 0) = \frac{x}{k} + (C_{kh}(v - h))(x)$$

for all $x \in \mathbb{R}$, where $C_{kh}$ is the periodic Hilbert transform on the strip. And therefore, using the Cauchy–Riemann equations, we get

$$V_y(x, 0) = U_x(x, 0) = \frac{1}{k} + C_{kh}(v')(x)$$

since $C_{kh}$ commutes with derivatives.

**Remark 1.** The periodic Hilbert transform for the strip is given by:

$$C_{kh}(w) = \sum_{n=1}^{\infty} a_n \coth(nkh) \sin(nx) - \sum_{n=1}^{\infty} b_n \coth(nkh) \cos(nx) \quad (10)$$

for all $w \in L^2_{2\pi}$ with $\|w\| = 0$, where $w$ has the Fourier series expansion

$$w(x) = \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \quad x \in \mathbb{R}.$$ 

Moreover, $C_{kh}$ is a bounded linear operator from $C^{p,\alpha}_{2\pi}$ into itself. This will be needed for the construction of small-amplitude solutions, in Sect. 3. For more details on the periodic Hilbert transform, see the Appendix in [7].

Using the Hilbert transform, we can now rewrite the problem in terms of the function $v(x)$. Indeed, since from classical elliptic theory we know that the problem

\[
\begin{cases}
\triangle \psi = \gamma + aY 
& \text{in } \Omega \\
\psi = 0 
& \text{on } S \\
\psi = -m 
& \text{on } B
\end{cases}
\quad (11)
\]

admits a unique solution, it suffices to find a suitable $v(x)$ which in turn will determine a suitable domain $\Omega$ in which the solution to (11) also satisfies the Bernoulli boundary condition.

Notice from (11) that the mapping $(X, Y) \mapsto \psi(X, Y) - \frac{\gamma}{2} Y^2 - \frac{a}{6} Y^3 + m$ is harmonic in $\Omega$ and so our function in the strip will also be by the invariance of harmonicity under conformal mappings.

More specifically, if $\psi$ is the unique solution to (11), we define $\xi : \mathcal{R}_{kh} \to \mathbb{R}$ by

$$\xi(x, y) := \psi(U(x, y), V(x, y))$$

for all $(x, y) \in \mathcal{R}_{kh}$, and $\zeta : \mathcal{R}_{kh} \to \mathbb{R}$ by:

$$\zeta(x, y) := \xi(x, y) + m - \frac{1}{2} \gamma V^2(x, y) - \frac{a}{6} V^3(x, y).$$

Using the fact that the Bernoulli boundary condition in terms of $\xi$ is given by

$$|\nabla \xi|^2 = (Q - bY)|\nabla V|^2$$
we can see that (4) becomes
\[
\begin{cases}
\Delta \zeta = 0 & \text{in } \mathcal{R}_{kh} \\
\zeta(x, 0) = m - \frac{1}{2} \gamma v^2(x) - \frac{a}{6} v^3(x) & \forall x \in \mathbb{R} \\
\zeta(x, -kh) = 0 & \forall x \in \mathbb{R} \\
(\zeta_y + \gamma V V_y + \frac{a}{2} V^2 V_y)^2 = (Q - bV)(V_x^2 + V_y^2) & \text{on } y = 0.
\end{cases}
\]
(12)

From the definition of the Dirichlet-Neumann operator \( G_{kh} \), we get:
\[
\begin{cases}
\zeta_y = \frac{m}{kh} - \frac{\gamma}{2} G_{kh}(v^2) - \frac{a}{6} G_{kh}(v^3) & \text{on } y = 0 \\
V_y = G_{kh}(v) & \text{on } y = 0
\end{cases}
\]
(13)

where \( V \) satisfies (9). We can therefore rewrite the last condition of (12) as
\[
\left( \frac{m}{kh} - \gamma \left( G_{kh} \left( \frac{v^2}{2} \right) - v G_{kh}(v) \right) - \frac{a}{2} \left( G_{kh} \left( \frac{v^3}{3} \right) - v^2 G_{kh}(v) \right) \right)^2 = (Q - bV) \left( (v')^2 + (G_{kh}(v))^2 \right).
\]
(14)

We have thus reduced the problem (4) in the physical plane to a problem of finding a positive number \( h \) and a function \( v \in C^{1,2\pi}_{2\pi}(\mathbb{R}) \) which satisfy
\[
\begin{cases}
\left( \frac{m}{kh} - \gamma \left( G_{kh} \left( \frac{v^2}{2} \right) - v G_{kh}(v) \right) - \frac{a}{2} \left( G_{kh} \left( \frac{v^3}{3} \right) - v^2 G_{kh}(v) \right) \right)^2 = (Q - bV) \left( (v')^2 + (G_{kh}(v))^2 \right) \\
[v] = h \\
v(x) > 0 & \text{for all } x \in \mathbb{R} \\
x \mapsto \left( \frac{x}{k} + C_{kh}(v - h)(x), v(x) \right) & \text{is injective on } \mathbb{R} \\
(v'(x))^2 + (G_{kh}(v)(x))^2 \neq 0 & \forall x \in \mathbb{R}
\end{cases}
\]
(15)

where \( C^{p,\alpha}_{2\pi}(\mathbb{R}) \) denotes the space of \( 2\pi \)-periodic functions whose partial derivatives up to order \( p \) are Hölder continuous with exponent \( \alpha \) over their domain of definition.

However, since \( [v] = h \), (13) can be rewritten as
\[
\begin{cases}
V_y = \frac{1}{k} + C_{kh}(v') \\
\zeta_y = \frac{m}{kh} - \frac{\gamma}{2kh} [v^2] - \gamma C_{kh}(vv') - \frac{a}{6kh} [v^3] - \frac{a}{2} C_{kh}(v^2 v')
\end{cases}
\]
(15)

and therefore, the first condition in (14) becomes
\[
\left( \frac{m}{kh} - \frac{\gamma}{2kh} [v^2] - \gamma C_{kh}(vv') - \frac{a}{6kh} [v^3] - \frac{a}{2} C_{kh}(v^2 v') \right)
\]
\[
+ \gamma v \left( \frac{1}{k} + C_{kh}(v') \right) + \frac{a}{2} v^2 \left( \frac{1}{k} + C_{kh}(v') \right)^2
\]
\[
- (Q - bV) \left( (v')^2 + \left( \frac{1}{k} + C_{kh}(v') \right)^2 \right) = 0.
\]
(16)
2.6. Reformulation using Riemann–Hilbert theory. The above formulation would be sufficient for proving the existence of small-amplitude waves by means of local bifurcation theory. However, for the existence of large-amplitude waves, some compactness criteria need to be satisfied, which doesn’t seem to be the case for (16). We therefore need to reformulate the problem, and we will do so using Riemann–Hilbert theory.

We begin by introducing the following class of function, see [7] for more details.

**Definition 2.1.** A $2\pi$-periodic function $z + iw : \mathbb{R} \to \mathbb{C}$ is said to belong to the class $A_{d}^{p,\alpha}$ for some $d > 0$, $\alpha \in (0, 1)$ and some integer $p \geq 0$, if $[w] = 0$ and there exists a holomorphic function $Z + iW : \mathbb{R}_d \to \mathbb{C}$ such that $Z, W \in C_{2\pi}^{p,\alpha}(\mathbb{R}_d)$, where $W$ satisfies (9) with $W(x, 0) = w(x)$, and $Z(x, 0) = z(x)$ for all $x \in \mathbb{R}$.

Moreover, given a $2\pi$-periodic function $z + iw : \mathbb{R} \to \mathbb{C}$ with $[w] = 0$, we have that

$$z + iw \in A_{d}^{p,\alpha} \iff z = [z] + C_{d}(w).$$

Finally, it can also be easily checked (or looked up in [7]) that $A_{d}^{p,\alpha}$ is an algebra.

We will show that a certain mapping (see (22) below) being in this class of functions is equivalent to (18)–(19) below. This will then enable us to show that (18) coupled with (19) is a reformulation of (16). This new formulation is of Babenko type and is given as follows:

$$C_{kh}\left(\left(Q - bv - \gamma^2 v^2 - \gamma av^3 - \frac{a^2}{4} v^4\right)v'\right)$$

$$+ \left(Q - bv - \gamma^2 v^2 - \gamma av^3 - \frac{a^2}{4} v^4\right)\left(\frac{1}{k} + C_{kh}(v')\right)$$

$$- (2\gamma v + av^2)\left(\frac{m}{kh} - \frac{\gamma}{2kh}[v^2] - \gamma C_{kh}(vv') - \frac{a[v^3]}{6kh} - \frac{a}{2} C_{kh}(v^2 v')\right) - \kappa = 0 \quad (18)$$

with

$$\kappa := -[vC_{kh}(v^2 v')] - \gamma a[v^2 C_{kh}(vv')] - \frac{a^2}{2} [v^2 C_{kh}(v^2 v')]
$$

$$+ b[vC_{kh}(v')] + \frac{a^2}{2} [v^4 C_{kh}(v')] + \gamma a[v^3 C_{kh}(v')]$$

$$- \frac{1}{k} \left(Q - bh - \frac{a^2}{4} [v^4] - \frac{2\gamma a}{3} [v^3] - 2\gamma m - \frac{am}{h} [v^2] + \frac{\gamma a[v^2]}{2h} + \frac{a^2[v^3][v^2]}{6h}\right)$$

coupled with the scalar constraint

$$\left[\left(\frac{m}{kh} - \frac{\gamma}{2kh}[v^2] - \gamma C_{kh}(vv') - \frac{a}{6kh}[v^3] - \frac{a}{2} C_{kh}(v^2 v')\right)
$$

$$+ \gamma v\left(\frac{1}{k} + C_{kh}(v')\right) + \frac{a}{2} v^2\left(\frac{1}{k} + C_{kh}(v')\right)\right)^2\right]$$

$$- \left[(Q - bv)\left((v')^2 + \left(\frac{1}{k} + C_{kh}(v')\right)^2\right)\right] = 0 \quad (19)$$

for unknowns $(m, Q, v) \in \mathbb{R} \times \mathbb{R} \times C_{2\pi}^{1,\alpha}(\mathbb{R})$. 
**Theorem 2.2.** Let \((m, Q, v) \in \mathbb{R} \times \mathbb{R} \times C_{2\pi}^{1,\alpha}(\mathbb{R})\), with \([v] = h\). Then we have the following:

(i) If (18)–(19) hold, then (16) holds.
(ii) If (16) holds and 

\[ V_x^2 + V_y^2 \neq 0 \, \text{in} \, \mathcal{R}_{kh}, \]

where \(V\) is a solution of (13), then (18)–(19) hold.

**Proof.** First off, notice that for any \(\zeta\) solution of (12), we have

\[ \zeta_x(x, 0) = -\gamma V(x, 0) V_x(x, 0) - \frac{a}{2} V^2 V_x(x, 0) \]  

for all \(x \in \mathbb{R}\). Moreover, (16) is equivalent to the last line in (12). Therefore, (19) is equivalent to

\[ \left[ (Q - bV(., 0))(V_x^2(., 0) + V_y^2(., 0)) \right. \]

\[ \left. - \left( \zeta_y + \gamma V(., 0)V_y(., 0) + \frac{a}{2} V^2(., 0)V_y(., 0) \right)^2 \right] = 0. \]  

(21)

Since \(A_{kh}^{0,\alpha}\) is an algebra, it is easy to see that

\[ V_y(., 0) + iV_x(., 0) \in A_{kh}^{0,\alpha}, \]

\[ \zeta_y(., 0) + i\zeta_x(., 0) \in A_{kh}^{0,\alpha}, \]

\[ (\zeta_y^2 - \zeta_x^2 + 2i\zeta_y\zeta_x)|_{(.,0)} \in A_{kh}^{0,\alpha}. \]

We would now like to prove that (18) can be expressed as:

\[ \left( x \mapsto \left( \left( Q - bV - \gamma^2 V^2 - \gamma a V^3 - \frac{a^2}{4} V^4 \right) (V_y - iV_x) - 2\gamma \zeta_y V - a\zeta_y V^2 \right) \right|_{(.,0)} \right) \in A_{kh}^{0,\alpha}. \]  

(22)

Indeed, the imaginary part of (22) is

\[ w = -\left( Q - bV - \gamma^2 v^2 - \gamma av^3 - \frac{a^2}{4} v^4 \right) v' \]

which satisfies the condition \([w] = 0\).

On the other hand, the real part of (22) is, using (15), equal to:

\[ z = \left( Q - bV - \gamma^2 v^2 - \gamma av^3 - \frac{a^2}{4} v^4 \right) \left( \frac{1}{k} + C_{kh}(v') \right) \]

\[ - (2\gamma v + av^2) \left( \frac{m}{kh} - \frac{\gamma}{2kh} [v^2] - \gamma C_{kh}(vv') - \frac{a[v^3]}{6kh} - \frac{a}{2} C_{kh}(v^2v') \right) \]
and therefore, using the fact that \( f \mapsto C_{kh}(f') \) is a self-adjoint linear operator to obtain some simplifications, we get:

\[
[z] = [vC_{kh}(v^2v')] + \gamma a[v^2C_{kh}(vv')] + \frac{a^2}{2}[v^2C_{kh}(v^2v')]
\]

\[
- b[vC_{kh}(v')] - \frac{a^2}{2}[v^4C_{kh}(v')] - \gamma a[v^3C_{kh}(v')]
\]

\[
+ \frac{1}{k} \left( Q - bh - \frac{a^2}{4}[v^4] - \frac{2\gamma a}{3}[v^3] - 2\gamma m - \frac{am}{h}[v^2] + \frac{\gamma a[v^2]^2}{2h} + \frac{a^2[v^3][v^2]}{6h} \right)
\]

which is equal to what we defined earlier as \( \kappa \). As a result, by (17), the formulation (22) is equivalent to

\[
[z] = \left[ z + C_{kh}(w) \right]
\]

from which we can clearly see that (22) is equivalent to (18).

(i) Let us first suppose that (18)–(19) hold. Then naturally, (22) and (21) hold. Therefore, since \( A_{kh}^{0,\alpha} \) is an algebra, we can clearly see that

\[
\left( x \mapsto \left( (Q - bV - \gamma^2V^2 - \gamma aV^3 - \frac{a^2}{4}V^4) (V_x^2 + V_y^2) - (2\gamma \zeta_x V + a\zeta_x V^2)(V_y + iV_x) \right) \right|_{(x,0)} \in A_{kh}^{0,\alpha}.
\]

Moreover, using (20), we have

\[
\left( x \mapsto \left( (Q - bV - \gamma^2V^2 - \gamma aV^3 - \frac{a^2}{4}V^4) (V_x^2 + V_y^2) - 2\gamma \zeta_y VV_y - a\zeta_y V^2V_y - \left( \zeta_y^2 - \zeta_x^2 \right) \right) \right|_{(x,0)} \in A_{kh}^{0,\alpha}.
\]

Since this function is real-valued and in \( A_{kh}^{0,\alpha} \), it must be constant. Therefore, using (20) once more, we can write

\[
\left( (Q - bV)(V_x^2 + V_y^2) - \left( \zeta_y + \gamma VV_y + \frac{a}{2}V^2V_y \right)^2 \right) \big|_{(x,0)} \equiv C_0
\]

and from the scalar equation, we know that \( C_0 \equiv 0 \) and therefore (16) holds.

(ii) Let us assume now that (16) holds. Clearly, (19) holds by just taking averages. Moreover, (16) is equivalent to (25) with \( C_0 = 0 \). Therefore, (24) holds and using (20), (23) must hold. Since we are assuming that

\[
V_x^2 + V_y^2 \neq 0 \quad \text{in} \quad \overline{R_{kh}},
\]

it is clear that

\[
\left( x \mapsto \frac{1}{V_y(x,0) + iV_x(x,0)} \right) \in A_{kh}^{0,\alpha}.
\]

Since \( A_{kh}^{0,\alpha} \) is an algebra, one can deduce from (23) that (22) holds, thus completing the proof.

Although we will address this matter later on in the paper again, we would like to make the reader aware of the fact that the problem (18)–(19) is more general than the water wave problem and that solutions we find to this problem might not exist in the physical plane. We will have to find solutions such that, in addition, the parameterized free surface is injective, non-self-intersecting and contained in the upper-half plane. We will address this matter in Sect. 4.
3. Existence Theory

3.1. Parameters and family of trivial solutions. The goal of this section is to prove the existence of solutions \((m, Q, v)\) of (18)–(19) in the space \(\mathbb{R} \times \mathbb{R} \times C_{2\pi, e}^{2,\alpha}(\mathbb{R})\), such that \([v] = h\) and where we defined

\[
C_{2\pi, e}^{2,\alpha}(\mathbb{R}) := \{ f \in C_{2\pi}^{2,\alpha} : f(x) = f(-x), \ \forall x \in \mathbb{R} \}.
\]

We will begin by looking at small-amplitude solutions, by using the local bifurcation theorem of Crandall and Rabinowitz (see [16]). We stated the theorem in the appendix, combined in one with the real-analytic global bifurcation theorem we will use later on. At this time, we only verify assumptions \((H1)-(H2)\).

Given \(v \in C_{2\pi}^{2,\alpha}\), we separate the average \(h\) of \(v\) by setting

\[
v = w + h \quad \text{with} \quad [w] = 0.
\]

We begin by noting that we have a family of trivial solutions for which \(v = h\), and \(Q\) and \(m\) are related by

\[
Q = bh + \left( \frac{m}{h} + \frac{\gamma h^2}{2} + \frac{ah^2}{3} \right)^2
\]

for \(m \in \mathbb{R}\) arbitrary. In particular, this family represents a curve

\[
\mathcal{K}_{\text{triv}} = \left\{ \left( m, bh + \left( \frac{m}{h} + \frac{\gamma h^2}{2} + \frac{ah^2}{3} \right)^2 \right) : m \in \mathbb{R} \right\}
\]

in the space \(\mathbb{R} \times \mathbb{R} \times C_{2\pi, e}^{2,\alpha}(\mathbb{R})\). These solutions are the laminar flows in the fluid domain bounded below by \(\mathcal{B}\) and above by the free surface \(Y \equiv h\) with streamfunction

\[
\psi(X, Y) = \frac{\gamma}{2} Y^2 + \frac{a}{6} Y^3 + \left( \frac{m}{h} - \frac{\gamma h^2}{2} - \frac{ah^2}{3} \right) Y - m.
\]

Since one must have \([v] = h\), we will work with the function \(w = v - h\) for which \([w] = 0\). Then \(w\) satisfies:

\[
\mathcal{C}_{kh} \left( \left( Q - b(w + h) - \gamma^2 (w + h)^2 - \frac{a^2}{4} (w + h)^4 - \gamma a (w + h)^3 \right) w' \right)
\]

\[
+ \left( \left( Q - b(w + h) - \gamma^2 (w + h)^2 - \frac{a^2}{4} (w + h)^4 - \gamma a (w + h)^3 \right) \left( \frac{1}{k} + \mathcal{C}_{kh}(w') \right) \right)
\]

\[
- (2\gamma (w + h) + a(w + h)^2) \left( \frac{m}{kh} - \frac{\gamma}{2kh} (|w|^2 + h^2) \right)
\]

\[
- \gamma \mathcal{C}_{kh}((w + h)w') - \frac{a}{6kh} (|w|^3 + 3h|w^2| + h^3) - \frac{a}{2} \mathcal{C}_{kh}((w + h)^2 w') - \kappa(w) = 0
\]

(26)

where \(\kappa(w)\) denotes \(\kappa\) in terms of \(w\). For the sake of readability, and since the exact formulation is not important here, we don’t write it out.
Moreover, \((26)\) is coupled with the following scalar constraint:

\[
\left[ \left( \frac{m}{kh} - \frac{\gamma}{2kh} [w^2 + h^2] - C_{kh}((w + h)w') - \frac{a}{6kh} ([w^3] + 3h[w^2] + h^3) \right) - \frac{a}{2} C_{kh}((w + h)^2 w') + (\gamma(w + h) + \frac{a}{2} (w + h)^2) \left( \frac{1}{k} + C_{kh}(w') \right) \right]
\]
\[
- \left[ (Q - b(w + h) (w')^2 + \left( \frac{1}{k} + C_{kh}(w') \right)^2 ) \right] = 0.
\]

And finally, we also have:

\[
\begin{cases}
[w] = 0 \\
w(x) > -h \\
x \mapsto \left( \frac{x}{k} + C_{kh}(w)(x), w(x) + h \right) \text{ is injective on } \mathbb{R} \\
w'(x)^2 + \left( \frac{1}{k} + C_{kh}(w')(x) \right)^2 \neq 0 \text{ for all } x \in \mathbb{R}
\end{cases}
\]

for \(w \in C_{2\pi}^{1,\alpha}, \ m \in \mathbb{R} \) and \(h > 0\).

Note that \(w = 0 \in C_{2\pi}^{1,\alpha} \) is a solution if and only if

\[
Q = bh + \left( \frac{m}{h} + \frac{\gamma h}{2} + \frac{ah^2}{3} \right)^2.
\]

\(Q\) and \(m\) are related as such for trivial solutions, but this is not necessarily the case in general. This therefore suggests setting

\[
\mu := Q - bh - \left( \frac{m}{h} + \frac{\gamma h}{2} + \frac{ah^2}{3} \right)^2.
\]

We introduce the parameter

\[
\lambda := \frac{m}{h} + \frac{\gamma h}{2} + \frac{ah^2}{3},
\]

which is a reasonable choice since the above expression is the horizontal velocity at the free surface of laminar flows, a customary bifurcation parameter for small-amplitude waves.

3.2. Small-amplitude solutions. We choose the Banach spaces

\[
\mathcal{X} = \mathbb{R} \times C^{p+1,\alpha}_{2\pi,e} \quad \text{and} \quad \mathcal{Y} = C^{p,\alpha}_{2\pi,e} \times \mathbb{R}.
\]

In terms of the new parameters \(\lambda\) and \(\mu\), \((26)-(27)\) may be written as:

\[
F(\lambda, (\mu, w)) = 0
\]
where \( F : \mathbb{R} \times X \to Y \) with \( X \) and \( Y \) defined by (29). Here \( F := (F_1, F_2) \) is given by
\[
\begin{align*}
F_1(\lambda, (\mu, w)) &= 2(\mu + \lambda^2)C_{kh}(w') + C_{kh}\left(\left(-bw - \gamma^2(w + h)^2 - \frac{a^2}{4}(w + h)^4 - \gamma a(w + h)^3\right)w'\right) \\
&\quad + \left(\left(-bw - \gamma^2(w + h)^2 - \frac{a^2}{4}(w + h)^4 - \gamma a(w + h)^3\right)\left(\frac{1}{k} + C_{kh}(w')\right)\right) \\
&\quad - 2\gamma(w + h) + a(w + h)^2 \left(\frac{\lambda}{k} - \frac{\gamma h}{k} - \frac{ah^2}{2k} - \frac{\gamma}{2kh}[w^2]\right) \\
&\quad - \gamma C_{kh}((w + h)w') - \frac{a}{6kh}([w^3] + 3h[w^2]) - \frac{a}{2} C_{kh}((w + h)^2w') - \kappa(w)
\end{align*}
\]
and
\[
F_2(\lambda, (\mu, w)) \begin{align*}
&= \left[\frac{\lambda}{k} - \gamma \left(\frac{[w^2]}{2kh} - \frac{w}{k} + C_{kh}(ww') - wC_{kh}(w')\right)\right] \\
&\quad - \frac{a}{2} \left(\frac{[w^3]}{3kh} - \frac{[w^2]}{k} - \frac{w^2 + 2hw}{k} + C_{kh}(w^2w') - w^2C_{kh}(w') + 2hC_{kh}(ww') - 2hwC_{kh}(w')\right)^2 \\
&\quad - \left[\left(\lambda^2 + \mu - bw\right)\left(w^2 + \left(\frac{1}{k} + C_{kh}(w')^2\right)\right)\right]
\end{align*}
\]
for \( w \in C_{2\pi}^1(\mathbb{R}) \), \( \mu \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \). Clearly, \( F = (F_1, F_2) \) is real-analytic on \( \mathbb{R} \times X \).

Let us define the following open set in \( \mathbb{R} \times X \):
\[
\mathcal{O} = \{(\lambda, (\mu, w)) \in \mathbb{R} \times X : \mu + \lambda^2 - bw(x) > 0 \text{ for all } x \in \mathbb{R}\}.
\]

We now prove that (H1)-(H2) hold.

We calculate:
\[
\partial_{(\mu, w)} F(\lambda, (0, 0))(\nu, f) = \left(\left(2\lambda^2 C_{kh}(f') - \frac{2}{k} \left(\gamma \lambda + ah\lambda + \frac{b}{2}\right) f\right), -\nu \right)
\]
for \((\nu, f) \in X\).

The function \( f \) is even, 2\pi-periodic and has zero average. We can therefore expand it into the Fourier series
\[
f(x) = \sum_{n=1}^{\infty} a_n \cos(nx). \tag{30}
\]

Therefore, using the definition of the Hilbert transform (10) we get
\[
\partial_{(\mu, w)} F(\lambda, (0, 0))(\nu, f) = \left(\sum_{n=1}^{\infty} a_n \left(2\lambda^2 \text{coth}(nk\nu) - \frac{2}{k} \left(\gamma \lambda + ah\lambda + \frac{b}{2}\right) f\right) \cos(nx), -\nu \right) \tag{31}
\]

We can therefore deduce that the bounded linear operator \( \partial_{(\mu, w)} F(\lambda, (0, 0)) : X \to Y \) is invertible whenever \( \lambda \) is not a solution of:
\[
\gamma \lambda + ah\lambda + \frac{b}{2} = \lambda^2 nk \text{coth}(nk\nu), \quad \forall n \in \mathbb{N}. \tag{32}
\]
Hence, by the implicit function theorem, for any integer \( n \geq 1 \), the \( \lambda \) satisfying (32) are potential bifurcation points. We denote these by \( \lambda^*_{n, \pm} \) and denote the set of all these points by \( \{ \lambda^*_{n, \pm} : n \in \mathbb{N} \} \).

It now suffices to show that (H2) holds for all \( \lambda^* \in \{ \lambda^*_{n, \pm} : n \in \mathbb{N} \} \). We pick any such \( \lambda^* \). We can immediately see that the kernel \( \text{Ker}(\partial_{(\mu,w)} F(\lambda^*, (0, 0))) \) is 1-dimensional and is generated by \((0, w^*) \in \mathbb{K} \) where \( w^*(x) = \cos(nx) \).

We now claim that the co-range \( \mathbb{Y} \setminus \text{Ran}(\partial_{(\mu,w)} F(\lambda^*, (0, 0))) \) is also 1-dimensional. Indeed, notice that the range \( \text{Ran}(\partial_{(\mu,w)} F(\lambda^*, (0, 0))) \) is the closed subset of \( \mathbb{Y} \) consisting of the elements \((\varphi, c) \in \mathbb{Y} \), where \( c \in \mathbb{R} \) is arbitrary and \( \varphi \) satisfies
\[
\int_{-\pi}^{\pi} \varphi(x) \cos(nx) \, dx = 0.
\]

As a result, \( \mathbb{Y} \setminus \text{Ran}(\partial_{(\mu,w)} F(\lambda^*, (0, 0))) \) is generated by \((w^*, c) \) where \( w^*(x) = \cos(nx) \) and the claim is proven.

Finally, we check that the transversality condition is satisfied. To this end, we compute:
\[
\begin{align*}
\partial_{\lambda} \partial_{(\mu,w)} F(\lambda^*, (0, 0))(1, (w^*, 0)) &= \left( 4\lambda^* n \coth(nkh) - \frac{2\gamma}{k} - \frac{2ah}{k} \right) w^*, 0 \right).
\end{align*}
\]

This is not in the range of \( \partial_{(\mu,w)} F(\lambda^*, (0, 0))(1, (w^*, 0)) \) since (32) yields
\[
4\lambda^* n \coth(nkh) - \frac{2\gamma}{k} - \frac{2ah}{k} = 2\lambda^* \left( n \coth(nkh) + \frac{b}{k(\lambda^*)^2} \right) \neq 0.
\]

Therefore, the conditions of the Crandall-Rabinowitz theorem are satisfied for every \( \lambda^* \in \{ \lambda^*_{n, \pm} : n \in \mathbb{N} \} \) and the existence of water waves of small-amplitude is now immediate.

3.3. Global continuum. We now want to extend these local solution curves to global ones. For this, we will use the global bifurcation theorem for real-analytic operators due to Dancer [13] and Buffoni and Toland [14] (see the full theorem recalled in the Appendix). In particular, we need to verify the compactness conditions (H3)-(H4). To this end, we reorganize \( F_1 \) in the following form
\[
F_1(\lambda, (\mu, w)) = 2(\mu + \lambda^2 - bw)C_{kh}(w') + 4[wC_{kh}(w')] + J(\lambda, w).
\]

Here \( J(\lambda, w) := J_1(w) + J_2(\lambda, w) \) with
\[
J_1(w) := -b(C_{kh}(w)w') - wC_{kh}(w')
- \gamma^2(C_{kh}((w + h)((w + h)w')) - (w + h)C_{kh}((w + h)w'))
- \gamma^2((w + h)((w + h)C_{kh}(w') - C_{kh}((w + h)w')))
- \frac{a^2}{4}(C_{kh}((w + h)^2((w + h)^2 w')) - (w + h)^2C_{kh}((w + h)^2 w'))
- \frac{a^2}{4}(w + h)^2((w + h)^2C_{kh}(w') - C_{kh}((w + h)^2 w'))
- \gamma a(C_{kh}((w + h)^2((w + h)w')) - (w + h)^2C_{kh}((w + h)w'))
- \gamma a((w + h)^2C_{kh}(w') - C_{kh}((w + h)^2 w'))
\]
and \( J_2(\lambda, w) \) gathers up all the remaining terms. Notice that \( J_2(\lambda, w) \) is just a combination of a polynomial in \( w \) and scalar terms which depend on \( w \).

We will now need the following theorem. For the proof, we refer to [12].

**Theorem 3.1.** [12] If \( f \in C_{2\pi}^{j,\alpha}(\mathbb{R}) \) and \( g \in C_{2\pi}^{j-1,\alpha}(\mathbb{R}) \), with \( j \in \mathbb{N} \) and \( \alpha \in (0, 1) \), then \( fC_d(g) - C_d(fg) \in C_{2\pi}^{j,\delta}(\mathbb{R}) \) for all \( \delta \in (0, \alpha) \), and there exists a constant \( C := C(j, \alpha, \delta) \) such that

\[
\| fC_d(g) - C_d(fg) \|_{j,\delta} \leq C \| f \|_{j,\alpha} \| g \|_{j-1,\alpha}.
\]

As a consequence of the above theorem, the continuous non-linear mapping \( J_1 \) maps bounded sets of \( C_{2\pi}^{2,\alpha}(\mathbb{R}) \) into bounded sets of \( C_{2\pi}^{2,\delta}(\mathbb{R}) \) and therefore into relatively compact subsets of \( C_{2\pi}^{1,\alpha} \).

We now notice that

\[
\partial w F_1(\lambda, (\mu, w)) f = 2(\mu + \lambda^2 - 2bw)C_{kh}(f') - 4bC_{kh}(w') f + 4b[wC_{kh}(f')] + 4b[C_{kh}(w') f] + \partial_w J_1(\lambda, (\mu, w)) f.
\]

Since, by the definition of \( \mathcal{O} \), we have \( \mu + \lambda^2 - 2bw(x) > 0 \) for all \( x \in \mathbb{R} \), it follows that \( \partial_w F_1(\lambda, (\mu, w)) : C_{2\pi}^{2,\alpha}(\mathbb{R}) \rightarrow C_{2\pi}^{1,\alpha}(\mathbb{R}) \) is the sum of an invertible linear mapping and a compact one. It is therefore a Fredholm operator of index 0 (see Theorem 2.7.6 in [14]).

We now notice that

\[
\partial(\mu, w) F(\lambda, (\mu, w)) = \begin{pmatrix} \partial_\mu F_1 & \partial_w F_1 \\ \partial_\mu F_2 & \partial_w F_2 \end{pmatrix} : \mathbb{X} \rightarrow \mathbb{Y}.
\]

Since \( \partial_w F_1 \) is Fredholm with index 0 and all the other entries are bounded, using the Fredholm bordering lemma (see Lemma 2.3 in [17] for full statement and clever proof), the operator matrix (36) is also Fredholm with index 0. We have thus verified condition (H3).

We are now ready to verify (H4). We define the sequence \( \{Q_j\}_{j \geq 1} \) by:

\[
Q_j = \left\{ (\lambda, (\mu, w)) \in \mathcal{O} : \| (\lambda, (\mu, w)) \| \leq j \text{ and } \mu + \lambda^2 - bw(x) \geq \frac{1}{j} \text{ for all } x \in \mathbb{R} \right\}
\]

and therefore, each \( Q_j \) is bounded and closed with \( \bigcup_{j \in \mathbb{N}} Q_j = \mathcal{O} \).

We choose an arbitrary \( j \in \mathbb{N} \) and consider \( (\lambda, (\mu, w)) \in Q_j \) such that \( F(\lambda, (\mu, w)) = 0 \). Therefore, from (34), since

\[
\mu + \lambda^2 - bw \geq \frac{1}{j} > 0,
\]

we can invert the linear operator \( w \mapsto C_{kh}(w') \) to get

\[
w = -(C_{kh}\partial_x)^{-1}\left(\frac{1}{\mu + \lambda^2 - bw}(J(\lambda, w) + 4[wC_{kh}(w')])\right).
\]

This enables us to get a uniform upper bound for \( w \) in the space \( C_{2\pi}^{3,\delta}(\mathbb{R}) \), which in turn implies that \( \{(\lambda, (\mu, w)) \in \mathcal{O} : F(\lambda, (\mu, w)) = 0\} \cap Q_j \) is compact in \( \mathbb{R} \times \mathbb{X} \). All the assumptions of the analytic global bifurcation theorem are now satisfied and we have therefore proved the following theorem.
Theorem 3.2. Let $h, k > 0$ and the constants $\gamma, a \in \mathbb{R}$ and $b \in \mathbb{R}^2$ be given. For each $n \in \mathbb{N}$, let

\[ m_{n, \pm}^* = -\frac{\gamma h^2}{2} - \frac{ah^3}{6} + h \left( \gamma + \frac{ah}{2} \right) \tanh(nkh) \left( 1 + \sqrt{\frac{1}{4} + \frac{b}{2} \tanh(nkh) \tanh(3nk)} \right) \]

and

\[ Q_{n, \pm}^* = bh + \left( \frac{m_{n, \pm}^*}{h} + \frac{\gamma h}{2} + \frac{ah^2}{3} \right)^2. \]

For any $m \in \mathbb{R} \setminus \{m_{n, \pm}^* : n \in \mathbb{N}\}$, there exists a neighborhood in $\mathbb{R} \times \mathbb{R} \times C_{2\pi, e}^2(\mathbb{R})$ of the point $(m, Q, h)$ on $\mathcal{K}_{\text{triv}}$, where $Q$ is related to $m$ by

\[ Q = bh + \left( \frac{m_{n, \pm}^*}{h} + \frac{\gamma h}{2} + \frac{ah^2}{3} \right)^2 \]

in which the only solutions of (18)–(19) are those on $\mathcal{K}_{\text{triv}}$.

Moreover, consider the points $m_{n, \pm}^*$ for each integer $n \geq 1$ and each choice of sign $\pm$, there exists in the space $\mathbb{R} \times \mathbb{R} \times C_{2\pi, e}^2(\mathbb{R})$ a continuous curve

\[ \mathcal{K}_{n, \pm} = \{(m(s), Q(s), v_s) : s \in \mathbb{R}\} \]

of solutions of (18)–(19) such that the following properties hold:

(I) $(m(0), Q(0), v_0) = (m_{n, \pm}^*, Q_{n, \pm}^*, h)$ where $m_{n, \pm}^*$ and $Q_{n, \pm}^*$ are given by (i) and (ii);

(II) $v_s(x) = h + s \cos(nx) + o(s)$ in $C_{2\pi, e}^2(\mathbb{R})$ if $0 \leq |s| < \epsilon$ for some $\epsilon > 0$, sufficiently small,

(III) There exists a neighborhood $\mathcal{W}_{n, \pm}$ of $(m_{n, \pm}^*, Q_{n, \pm}^*, h)$ in $\mathbb{R} \times \mathbb{R} \times C_{2\pi, e}^2(\mathbb{R})$ and $\epsilon > 0$ sufficiently small such that

\[ \{(m, Q, v) \in \mathcal{W}_{n, \pm} : v \neq h \text{ and } (18)–(19) \text{ hold}\} = \{(m(s), Q(s), v_s) : 0 < |s| < \epsilon\}; \]

(IV) $Q(s) - bv_s(x) > 0$ for all $s, x \in \mathbb{R}$;

(V) $\mathcal{K}_{n, \pm}$ has a real-analytic reparametrization locally around each of its points;

(VI) One of the following alternatives occurs:

(a) either

\[ \min \left\{ \frac{1}{1 + \|m(s), Q(s), v_s\|_{\mathbb{R} \times \mathbb{R} \times C_{2\pi, e}^2(\mathbb{R})}}, \min_{x \in \mathbb{R}} (Q(s) - bv_s(x)) \right\} \rightarrow 0 \text{ as } s \rightarrow \pm \infty \]

(b) or, there exists a $T > 0$ such that

\[ (m(s + T), Q(s + T), v_{s+T}) = (m(s), Q(s), v_s), \quad \forall s \in \mathbb{R}. \]

Moreover, for each integer $n \geq 1$, and both choices of sign $\pm$, such a curve of solutions of (18)–(19) with the properties (I)-(VI) is unique up to reparametrization.
4. Nodal Analysis

The problem (18)–(19) is more general than the water wave problem, and therefore the solutions we have found above do not necessarily provide physical water wave solutions. In particular, the solutions are only physically relevant if the reparametrization of the free surface satisfies some non-self-intersecting condition. In other words, the solutions give rise to water waves if and only if the mapping

$$x \mapsto \left( \frac{x}{k} + (C_{kh}(v - h)(x), v(x)) : x \in \mathbb{R} \right)$$

is injective and

$$S = \left\{ \left( \frac{x}{k} + (C_{kh}(v - h)(x), v(x)) : x \in \mathbb{R} \right) \right\}$$

is contained in the upper-half plane.

4.1. Periodicity. Since the solutions we have constructed in the previous section are in the function space $C_{2\pi/\alpha}^2(\mathbb{R})$, they are symmetric and periodic, and it therefore suffices to study what happens in a half-period of a wave.

We will now look in more detail at the solutions along the curve $K_{1, \pm}$. From Theorem 3.2, we know that near the bifurcation point these solutions are of the form

$$v(x) = h + s \cos(x) + o(s),$$

which are monotone functions on a half-period. This added monotonicity makes these solutions easier to study than the ones for integers $n \geq 2$, where $\cos(nx)$ oscillates within the half-period.

However, from Lemma 4.1(i) recalled below, similar results to the ones we will provide in this section, can be shown for the curves $K_{n, \pm}$ for all $n \geq 2$ by working in a function space of period $2\pi/n$.

Moreover, by denoting

$$K_{1, \pm} = \mathcal{K}_{1, \pm}^\leq \cup \{(m^*_1, Q^*_1, h) : s \in (-\infty, 0)\},$$

where

$$\mathcal{K}_{1, \pm}^\leq = \{(m(s), Q(s), v_s) : s \in (-\infty, 0)\},$$

and

$$\mathcal{K}_{1, \pm}^\geq = \{(m(s), Q(s), v_s) : s \in (0, \infty)\},$$

from Lemma 4.1(ii) below, it suffices to study the properties of solutions along $\mathcal{K}_{1, \pm}^\geq$.

**Lemma 4.1.** Let $h, k > 0$ and the constants $\gamma, a \in \mathbb{R}$ and $b \in \mathbb{R}^+$ be given. For each integer $n \geq 1$ and both choices of sign $\pm$, denote by

$$\mathcal{K}_{n, \pm} = \{(m(s), Q(s), v_s) : s \in \mathbb{R}\}$$

the continuous curve of solutions of (18)–(19) in the space $\mathbb{R} \times \mathbb{R} \times C_{2\pi/\alpha}^2$ given by the previous theorem. Then the following additional properties hold along $\mathcal{K}_{n, \pm}$:

(i) $v_s$ is periodic of period $2\pi/n$ for each $s \in \mathbb{R}$;

(ii) $m(-s) = m(s), Q(-s) = Q(s)$, and $v_{-s}(x) = v_s(x + /n)$ for all $x \in \mathbb{R},$ for each $s \in \mathbb{R}$.

We refer to (Lemma 10, [7], Sect. 4) for the proof, choosing

$$\tilde{\mathbb{X}} = \mathbb{R} \times C_{2\pi/n, e}^{2, \alpha}(\mathbb{R})$$

and

$$\tilde{\mathbb{Y}} = C_{2\pi/n, e}^{1, \alpha}(\mathbb{R}) \times \mathbb{R}.$$
4.2. Injectivity. That our solutions near the bifurcation point are monotonic on \((0, \frac{\pi}{k})\) enables us to assume that along \(K_{\pm, \text{loc}}\), we have \(v'(x) < 0\) for all \(x \in (0, \frac{\pi}{k})\). If we can prove that this property is preserved along the entire curve \(K_{\pm}\), we not only rule out the possibility of the curve looping back to the bifurcation point (Theorem 3.2(b)) but also obtain that the solution must be injective between the trough and the crest. In particular, it will then be sufficient to merely check the injectivity condition along the crest line and the trough line.

In order to show injectivity, we will use the reformulation presented in the following Lemma, stated without proof (see Lemma 11 in [7]).

**Lemma 4.2.** If

\[ V_\alpha(x, 0) \neq 0 \quad \text{for all} \quad x \in (0, \pi), \]

then the injectivity condition holds if and only if

\[ 0 < U(x, 0) < \frac{\pi}{k} \quad \text{for all} \quad x \in (0, \pi). \tag{38} \]

Using the definition of the periodic Hilbert transform for the strip, (38) is equivalent to

\[ 0 < \frac{x}{k} + (C_{kh}(v - h))(x) < \frac{\pi}{k} \quad \text{for all} \quad x \in (0, \pi). \]

4.3. Monotonicity properties. We are now ready to present a set of monotonicity properties for the pair \((m, v) \in \mathbb{R} \times C_{2\pi,e}^2\). These are motivated by the form of the small-amplitude solutions and will be shown to hold globally along the curve:

\[
egin{align*}
  v(x) &> 0 \quad \text{for all} \quad x \in \mathbb{R}, \tag{39} \\
  v &\neq h, \tag{40} \\
  v'(x) &< 0 \quad \text{for all} \quad x \in (0, \pi), \tag{41} \\
  v''(0) &< 0 \quad \text{and} \quad v''(\pi) > 0, \tag{42} \\
  0 &< \frac{x}{k} + (C_{kh}(v - h))(x) < \frac{\pi}{k} \quad \text{for all} \quad x \in (0, \pi), \tag{43} \\
  \frac{1}{k} + (C_{kh}(v'))(0) &> 0 \quad \text{and} \quad \frac{1}{k} + (C_{kh}(v'))(\pi) > 0, \tag{44} \\
  \pm \left( \frac{m}{kh} - \frac{\gamma}{2kh} [v^2] - \gamma C_{kh}(vv') - \frac{a}{6kh} [v^3] - \frac{a}{2} C_{kh}(v^2v') \right) \\
  + \left( \gamma v + \frac{a}{2} v'^2 \left( \frac{1}{k} + C_{kh}(v') \right) \right) &> 0 \quad \text{for all} \quad x \in \mathbb{R}. \tag{45}
\end{align*}
\]

We would like to point out that for solutions \((m, Q, v)\) of (16), condition (45) is equivalent to

\[ Q - bv(x) \neq 0 \quad \text{and} \quad (v'(x))^2 + \left( \frac{1}{k} + (C_{kh}(v'))(x) \right)^2 \neq 0 \quad \text{for all} \quad x \in \mathbb{R}. \]
4.4. Local properties. The question is now, to what extent are these properties satisfied along $K^\pm_0$? We introduce the following notation:

$$
V^\pm_\pm = \{(m, Q, v) \in \mathbb{R} \times \mathbb{R} \times C^{2, \alpha}_{2\pi, \epsilon} (\mathbb{R}) : (39)-(45) \text{ hold}\}.
$$

It is not difficult to show that these properties hold for solutions on $K^\pm_0$ that are near enough to the trivial one. We therefore get the following lemma. We refer the reader to Lemma 12 in [7] for a proof.

**Lemma 4.3.** For either choice of sign $\pm$, let (37) be the curve of solutions of (18)–(19). Then there exists $\epsilon > 0$ sufficiently small such that

$$
\{(m(s), Q(s), v_s) : s \in (0, \epsilon)\} \subset V^\pm_\pm.
$$

4.5. Main result. We now state and prove the main result of this section.

**Theorem 4.4.** For either choice of sign $\pm$, let (37) be the curve of solutions of (18)–(19). Then one of the following alternatives occurs:

(A1) $K^\pm_0 \subset V^\pm_\pm$, in which case alternative (a) in Theorem 3.2 occurs;

(A2) there exists some $s^* \in (0, \infty)$ such that $\{(m(s), Q(s), v_s) : s \in (0, s^*)\} \subset V^\pm_\pm$, while $(m(s^*), Q(s^*), v_{s^*})$ satisfies (39)–(42), (44) and (45), and instead of (43), it satisfies

$$
0 < \frac{x}{k} + (C_{kh}(v - h))(x) \leq \frac{\pi}{k} \quad \text{for all } x \in (0, \pi),
$$

$$
\frac{x_0}{k} + (C_{kh}(v - h))(x_0) = \frac{\pi}{k} \quad \text{for some } x_0 \in (0, \pi).
$$

In other words, the first alternative (A1) means that all solutions on $K^\pm_0$ correspond to physical water waves with the qualitative properties described above, whereas alternative (A2) means that such solutions exist until $s = s^*$, at which point the wave profile self-intersects on the line strictly above the trough.

Since the qualitative properties hold in a neighborhood of the bifurcation point, the idea is to use a continuation argument: we need to check to what extent these properties are satisfied along the global curve of solutions. We are therefore faced with two possible options: either the properties hold along the entire curve, in which case (39)–(45) hold with strict inequalities. Or, there exists a first point at which at least one of the properties fails, and thus that particular property must then have a non-strict inequality.

Therefore, we rewrite (39)–(45), replacing all the strict inequalities with non-strict ones and argue by contradiction. We work with the harmonic functions $U$ and $V$ from the conformal map $U + iV$ described in the beginning of the paper, evaluated at the surface of $R_{kh}$. Notice that from the evenness of $v$, we can deduce the function $V(x, y)$ is even in the $x$ variable and the function $U(x, y)$, being the harmonic conjugate of $-V$ up to a constant, is odd in the $x$ variable.

Exploiting the harmonicity and parity of $U$ and $V$ and using the Cauchy–Riemann equations, we then use a series of strong maximum principle arguments, along with the Hopf and Serrin inequalities at the boundary, to show that the inequalities in (39)–(42), (44)–(45) and the first inequality in (43) must be strict.

The proof of this theorem presents very strong similarities to the proof of the analogous theorem in [7]. However, for the sake of readability, we will provide most of the details: the maximum principle arguments are rather delicate and it is essential to
we will consider. Choosing the \( \epsilon \) excluded and
\( s \).

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Proof of Theorem 4.4. We begin by fixing the choice of sign \( \pm \): we choose the + sign but all arguments follow identically for the – sign.

First off, we define the set
\[
I = \{ s \in (0, \infty) : (m(s), Q(s), v_s) \in \mathcal{V}_+ \}.
\]

Notice that \( \mathcal{V}_+ \) is an open set in \( \mathbb{R} \times \mathbb{R} \times C_{2\pi, \mathbb{R}}^a(\mathbb{R}) \). This implies that \( I \) is an open subinterval of \( (0, \infty) \). We therefore have two options. The first one is \( I = (0, \infty) \), in which case \( \mathcal{K}_{\infty} \subset \mathcal{V}_+ \). This would imply that the alternative \( b \) in Theorem 3.2 is excluded and \( a \) must be valid. In this case, there is nothing more to show.

The other option is that \( I \) is not equal to the entire interval \( (0, \infty) \) and this is the case we will consider. Choosing the \( \epsilon \) from the local properties lemma, we have \( (0, \epsilon) \subset I \). We now denote by \( s^* \) the upper end-point of the largest interval containing \( (0, \epsilon) \) and contained in \( I \). In other words, we have \( (0, s^*) \subset I \) but \( s^* \notin I \). We will investigate the properties of the solution \( (m(s^*), Q(s^*), v_{s^*}) \).

Claim 1: \( v_{s^*} \neq h \)

The proof of this claim is quite technical and completely identical to the one in [7], so we omit it here.

For notational simplicity, for the remainder of the proof, we will denote \( (m(s^*), Q(s^*), v_{s^*}) \) by \( (m(s), Q(s), v_s) \). Note that this is the limit of the solutions satisfying the monotonicity properties.

From the definition of \( I \), we therefore have the following inequalities:

\[
\begin{align*}
v(x) & \geq 0 \quad \text{for all } x \in \mathbb{R}, \quad \text{and } v \text{ is even and of period } 2\pi, \\
v'(x) & \leq 0 \quad \text{for all } x \in [0, \pi], \quad \text{so that } v' \geq 0 \quad \text{in } [\pi, 2\pi], \\
0 & \leq \frac{x}{k} + (C_{kh}(v - h))(x) \leq \frac{\pi}{k} \quad \text{for all } x \in [0, \pi], \\
\frac{m}{kh} - \frac{\gamma}{2kh}[v^2] - \gamma C_{kh}(vv') - \frac{a}{6kh}v^3 - \frac{a^2}{2}C_{kh}(v^2v') + \left( \gamma v + \frac{a}{2}v^2 \right) \left( \frac{1}{k} + C_{kh}(v') \right) & \geq 0 \quad \text{for all } x \in \mathbb{R}.
\end{align*}
\]

In other words, either (39)–(45) are satisfied and the inequalities are strict, or they aren’t, in which case we have equality signs. Using sharp forms of maximum principles, we now prove that all the above inequalities, except for the second one in (48) are strict. To this end, we define the rectangular domain
\[
\mathcal{R} := \{(x, y) : 0 < x < \pi, \quad \text{and} \quad -kh < y < 0\}
\]

with boundary \( \partial \mathcal{R} \) divided up into the following segments:
\[
\begin{align*}
\partial \mathcal{R}_t &= \{(x, 0) : 0 \leq x \leq \pi\} \quad \partial \mathcal{R}_b = \{(x, -kh) : 0 \leq x \leq \pi\} \\
\partial \mathcal{R}_l &= \{(0, y) : -kh \leq y \leq 0\} \quad \partial \mathcal{R}_r = \{(-\pi, y) : -kh \leq y \leq 0\}.
\end{align*}
\]
Claim 2: \( v(x) > 0 \) for all \( x \in \mathbb{R} \).

Assume that the claim does not hold. Then (46) and (47) imply that \( v(\pi) = 0 \). Since \( V \) is harmonic in \( R_{kh} \), it would therefore have a global minimum in \( R_{kh} \) at \( (\pi, 0) \). Using the Hopf boundary point lemma, this would imply that \( V_y(\pi, 0) < 0 \). However, since (48) is equivalent to

\[
0 = U(0, 0) \leq U(x, 0) \leq U(\pi, 0) = \frac{\pi}{k}, \quad \forall x \in [0, \pi],
\]

using the Cauchy–Riemann equations, we get

\[
V_y(\pi, 0) = U_x(\pi, 0) \geq 0.
\]

This is a contradiction, and thus the claim has been proven.

Claim 3: (45) with the + sign holds.

Note that this is equivalent to

\[
\begin{align*}
\zeta_y + \gamma V V_y + \frac{a}{2} V^2 V_y = & \frac{m}{kh} - \frac{\gamma}{2kh} [v^2] - \gamma C_{kh}(vv') - \frac{a}{6kh} [v^3] \\
& - \frac{a}{2} C_{kh}(v^2v') + \left( \gamma v + \frac{a}{2} v^2 \right) \left( \frac{1}{k} + C_{kh}(v') \right) \geq 0 \quad \text{for all } x \in \mathbb{R}.
\end{align*}
\]

We know from Theorem 3.2 that

\[
Q - 2bv(x) > 0 \quad \forall x \in \mathbb{R}.
\]

Therefore, since we have

\[
\left( \zeta_y + \gamma V V_y + \frac{a}{2} V^2 V_y \right)^2 = (Q - 2bV)(V_x^2 + V_y^2) \quad \text{on } y = 0
\]

the proof of the claim boils down to verifying that

\[
\left( \zeta_y + \gamma V V_y + \frac{a}{2} V^2 V_y \right)^2 = (Q - 2bV)(V_x^2 + V_y^2) > 0 \quad \text{on } y = 0.
\]

This is equivalent to

\[
v'(x)^2 + \left( \frac{1}{k} + C_{kh}(v')(x) \right)^2 > 0, \quad \forall x \in \mathbb{R}
\]

or

\[
V_x^2(x, 0) + V_y^2(x, 0) > 0, \quad \forall x \in \mathbb{R}.
\]

We argue by contradiction.

We assume that there exists a point \( x_0 \in \mathbb{R} \) such that \( V_x^2(x_0, 0) + V_y^2(x_0, 0) = 0 \). This would imply that

\[
\zeta_y(x_0, 0) + \gamma V(x_0, 0) V_y(x_0, 0) + \frac{a}{2} V^2(x_0, 0) V_y(x_0, 0) = 0.
\]

Notice that (49) is equivalent to

\[
\zeta_y(x, 0) + \gamma V(x, 0) V_y(x, 0) + \frac{a}{2} V^2(x, 0) V_y(x, 0) \geq 0 \quad \text{for all } x \in \mathbb{R}.
\]
Combining (54) with (55) ensures that \( \zeta_y + \gamma V V_y + \frac{\partial}{\partial y} V^2 V_y = O((x - x_0)^2) \) as \( x \to x_0 \) at \( y = 0 \). From (52), we thus have that \( V_x^2(x, 0) + V_y^2(x, 0) = O((x - x_0)^4) \) as \( x \to x_0 \). Moreover, from the second derivative of the right-hand side of (52), we see that
\[
V_{xx}(x_0, 0) = V_{xy}(x_0, 0) = 0.
\] (56)

From the evenness and periodicity of \( v \), we can assume, without loss of generality that \( 0 \leq x_0 \leq \pi \). We now examine the three different possibilities for \( x_0 \).

(i) \( x_0 = 0 \) : this would imply, using (47), that \( V \) has a global maximum in \( \overline{R_{kh}} \). By the Hopf boundary point lemma, we therefore get \( V_y(0, 0) > 0 \). However this would mean that
\[
1/k + C_{kh}(v')(0) = V_y(0, 0) > 0,
\]
which is a contradiction and so \( x_0 \neq 0 \).

(ii) \( 0 < x_0 < \pi \) : this would imply that \( V_x \) has its global maximum in \( \overline{R} \) at \( (x_0, 0) \). Indeed, \( V_x = 0 \) on the vertical sides by oddness and on the bottom by the boundary condition, whereas (47) indicates that \( V_x \leq 0 \) on the top. By the Hopf boundary point lemma, we get \( V_{xy}(x_0, 0) > 0 \), a contradiction to (56).

(iii) \( x_0 = \pi \) : Since the mapping \( x \mapsto V(x, y) \) is even about \( \pi \), for every \( y \in [-kh, 0] \), we have
\[
V_{xxx}(\pi, 0) = V_{xyy}(\pi, 0) = 0.
\] (57)

As above, \( V_x \) has its global maximum in \( \overline{R} \) at \( (\pi, 0) \). Taking into account (56) and (57), the Serrin edge-point lemma gives us that \( V_{xyy}(\pi, 0) < 0 \). The Cauchy–Riemann equations thus imply that \( U_{xxx}(\pi, 0) < 0 \). Similarly, we can deduce that \( U_x(\pi, 0) = U_{xx}(\pi, 0) \) since \( V_y(\pi, 0) = 0 \) by assumption and (56) holds. As a result, we obtain
\[
U(x, 0) - U(\pi, 0) = \frac{1}{6} U_{xxx}(\pi, 0)(x - \pi)^3 + O((x - \pi)^4) > 0 \quad \text{as} \quad x \nearrow \pi.
\]

This is a contradiction to (50).

From points (i)-(iii), we can deduce that \( x_0 \) doesn’t exist and we have shown that
\[
V_x^2(x, 0) + V_y^2(x, 0) > 0 \quad \text{for all} \quad x \in \mathbb{R}.
\]

Since we have \( V_x(\pi, 0) = 0 \), we necessarily have \( V_y(\pi, 0) \neq 0 \). From (50) and the Cauchy–Riemann equations, we have \( V_y(\pi, 0) = U_x(\pi, 0) \geq 0 \). As a result, we obtain
\[
1/k + (C_{kh}(v'))(\pi) = V_y(\pi, 0) > 0.
\]

This concludes the proof of the claim. □

Remark 2. Notice that we have also proved (44) and (53).

Remark 3. For the next part of the proof, we will need \( m \in \mathbb{R} \times \mathbb{R} \times C^3_{2\pi, e}(\mathbb{R}) \). However, notice that by bootstrapping the improvement of regularity that is based on Theorem 3.1 in the proof of Theorem 3.2, it is not difficult to show that for any solution \( (m, Q, v) \) of (18)–(19), the function \( v \) is necessarily of class \( C^\infty \) on \( \mathbb{R} \).

In addition, we will need the fact that
\[
V_x^2 + V_y^2 > 0 \quad \text{in} \quad \overline{R_{kh}}.
\] (58)

We omit the proof here as it is slightly technical, and is completely identical to the one in [7], located directly before Lemma 15.
Remark 4. Notice that from the dynamic boundary condition, we have
\[ \zeta_y = -\gamma V V_y - \frac{a}{2} V^2 V_y \pm (Q - 2bV)^{1/2}(V_x^2 + V_y^2)^{1/2} \quad \text{at } (x, 0), \forall x \in \mathbb{R}. \quad (59) \]

In what follows, we will be applying maximum-principle arguments to the function \( f : \overline{R}_{kh} \to \mathbb{R} \) given by
\[ f := \frac{V_x \zeta_y - V_y \zeta_x}{V_x^2 + V_y^2}, \]
which is well-defined thanks to (58). Notice that \( f \) is in fact the (pseudo) horizontal velocity of the fluid, \(-\psi_X\). Moreover, \( f \) is a harmonic function in \( R_{kh} \) as it is the imaginary part of 
\[ - (\zeta_y + i \zeta_x) \]
\[ V_y + i V_x, \]
a quotient of two holomorphic functions. In addition, we clearly have \( f = 0 \) on \( \partial R_l \cup \partial R_b \cup \partial R_r \). On the top, \( \partial R_t \) we have
\[ f = \frac{V_x (\zeta_y + \gamma V V_y + \frac{a}{2} V^2 V_y) \pm (Q - 2bV)}{(V_x^2 + V_y^2)^{1/2}}, \quad (60) \]
where we used (20) to get the left-hand side and (59) to obtain the right-hand side.

Moreover, notice that on \( \partial R_t \), \( f \) does not vanish identically and has a constant sign (which depends on the sign \( \pm \)). By the maximum principle, \( f \) therefore has a strict sign in \( R \).

Claim 4: \( v'(x) < 0 \) for all \( x \in (0, \pi) \).
Arguing by contradiction, we assume that for some \( x_0 \in (0, \pi) \), we have \( v'(x_0) = 0 \). Since, as a result, \( x_0 \) is a global maximum for \( v' \) on \( (0, \pi) \), we have \( v''(x_0) = 0 \). We can clearly see that we must then also have \( f(x_0, 0) = 0 \) and therefore \( f \) has at \( (x_0, 0) \) a global extremum in \( R \cup \partial R \). Applying the Hopf boundary point lemma, we get \( f_y(x_0, 0) \neq 0 \). The idea is now to show that we actually have \( f_y(x_0, 0) = 0 \), which would lead to a contradiction.

For simplicity, we write
\[ f = \frac{p}{q} \quad \text{where} \quad p = V_x \zeta_y - V_y \zeta_x \quad \text{and} \quad q = V_x^2 + V_y^2. \]

Clearly, for every point in \( \overline{R}_{kh} \), we have
\[ f_y = \frac{p_y q - p q_y}{q^2}. \]

In particular, since at \( (x_0, 0) \) we have \( p = V_x = V_{xx} = 0 \), we obtain
\[ f_y = \frac{p_y}{q} = \frac{V_{xy} \zeta_y - V_y \zeta_{xy}}{V_y^2}. \quad (61) \]

Differentiating (59) with respect to \( x \), we get:
\[ \zeta_{xy} = -\gamma(V_x V_y + V V_{xy}) - \frac{a}{2}(2VV_y + V^2 V_{xy}) \]
\[ \pm \left( -bV_x(V_x^2 + V_y^2)^{1/2} + (Q - 2bV)^{1/2}(V_x V_{xx} + V_y V_{xy}) \right) \]
at \((x, 0)\) for all \(x \in \mathbb{R}\). Therefore, at the point \((x_0, 0)\), where we already know that \(V_x = V_{xx} = 0\), we have
\[
\zeta_y = -\gamma VV_y - \frac{a}{2} V^2 V_y \pm (Q - 2bV)^{1/2}|V_y|, \tag{62}
\]
\[
\zeta_{xy} = -\gamma VV_{xy} - \frac{a}{2} V^2 V_{xy} \pm \frac{(Q - 2bV)^{1/2} V_y V_{xy}}{|V_y|}. \tag{63}
\]
We can now clearly see that at the point \((x_0, 0)\), we have
\[
V_{xy} \zeta_y = V_y \zeta_{xy}.
\]
Consequently, using (61) we get that \(f_y(x_0, 0) = 0\), thus obtaining the desired contradiction which proves the claim.

**Claim 5:** \(v''(0) < 0\) and \(v''(\pi) > 0\).

Since \(v'(0) = v'(\pi) = 0\), and \(v'(0) \leq 0\) on \([0, \pi]\) we get
\[
v''(0) \leq 0 \quad \text{and} \quad v''(\pi) \geq 0.
\]
We therefore only need to prove that these two inequalities are strict. Again, we argue by contradiction.

Let us assume that for some \(x_0 \in \{0, \pi\}\), we have \(v''(x_0) = 0\). Then we clearly have \(V_x = V_{xx} = 0\) at \((x_0, 0)\). Since \(f\) admits a global extremum at \((x_0, 0)\) in \(\mathcal{R} \cup \partial R\), the Serrin edge-point lemma ensures that not all first and second derivatives of \(f\) can be equal to zero at the point \((x_0, 0)\). However, we will now show that they all do in fact vanish, which will be a contradiction.

Since \(f = 0\) on \(\partial R_l \cup \partial R_r\) and \(f\) is harmonic, we must therefore have \(f_y = f_{yy} = f_{xx} = 0\) on \(\partial R_l \cup \partial R_r\). From the definition (60) of \(f\) on \(\partial R_t\), we can see that \(f_x(x_0, 0)\) contains a factor of \(V_x\) or \(V_{xx}\) for every term, and therefore, \(f_x(x_0, 0) = 0\).

We now check the mixed derivative \(f_{xy}(x_0, 0)\). Using the same notation as for the previous claim, we get
\[
f_{xy} = f_{yx} = \frac{(p_{xy}q + p_yq_x - p_xq_y - pq_{xy})q^2 - (p_yq - pq_y)2qq_x}{q^4}.
\]
Since at the point \((x_0, 0)\) we have
\[
\zeta_x = V_x = \zeta_{xy} = V_{xy} = V_{xx} = V_{yy} = p = p_x = p_y = 0,
\]
we get
\[
f_{xy} = \frac{p_{xy}}{q}.
\]
Differentiating \(p\) and evaluating it at \((x_0, 0)\), we get
\[
p_{xy} = V_{xxy} \zeta_y - V_y \zeta_{xxy}.
\]
We calculate \(\zeta_{xxy}(x_0, 0)\) by taking the derivative with respect to \(x\) of (63). Taking into account that \(V_x = V_{xx} = V_{xy} = 0\) at \((x_0, 0)\), we are left with
\[
\zeta_{xxy} = -\gamma VV_{xxy} - \frac{a}{2} V^2 V_{xxy} \pm \frac{(Q - 2bV)^{1/2} V_y V_{xxy}}{|V_y|}.
\]
Since (62) is also valid at \((x_0, 0)\) we clearly have

\[ V_{xy} \zeta_y = V_y \zeta_{xy} \]

and therefore, \(f_{xy}(x_0, 0) = 0\). As a result, all first and second derivatives of \(f\) vanish at the point \((x_0, 0)\) and we have obtained the desired contradiction and proven the claim.

**Claim 6:** condition (A2) in Theorem 4.4 holds.

Recall that (48) is valid. We first assume that the first part of (A2) fails. This means, there exists an \(x_0 \in (0, \pi)\) such that

\[ 0 = \frac{x_0}{k} + C_k h (v - h)(x_0) = U(x_0, 0). \]

This would imply that the harmonic function \(U\) in \(\mathcal{R}\) has at \((x_0, 0)\) a global minimum in \(\overline{\mathcal{R}}\). Therefore, by the Hopf boundary point lemma, we get \(U_y(x_0, 0) < 0\). However, from the Cauchy–Riemann equations, we would then have \(V_x(x_0, 0) > 0\), a contradiction to (47). As a result, the first part of (A2):

\[ 0 < \frac{x}{k} + (C_k h (v - h))(x) \leq \frac{\pi}{k} \quad \text{for all} \quad x \in (0, \pi), \]

is true.

The second part of (A2):

\[ \frac{x_0}{k} + (C_k h (v - h))(x_0) = \frac{\pi}{k} \quad \text{for some} \quad x_0 \in (0, \pi), \]

must also necessarily be true. Otherwise, taking into account everything we have proved thus far, we would have \((m(s^*), Q(s^*), v_{s^*}) \in \mathcal{V}_+,\) which would be a contradiction to the definition of \(s^*\). This completes the proof of the claim and of the theorem.

## 5. Overhanging Waves

Notice from the qualitative properties in the previous section that the symmetric physically relevant water wave solutions we have constructed have a wave profile whose vertical coordinate strictly decreases between each consecutive global maxima and minima. However, no monotonicity requirements were made on the horizontal coordinate and therefore, the waves could have overhanging profiles.

In this section, we will show that for certain assumptions on \(\gamma\) and \(a\), the physical water waves from the previous section cannot have overhanging profiles. In particular, we would like to prove that for \(\gamma > 0\) and for \(a > 0\), the waves do not overhang. We also set the additional requirement

\[ m \geq -\frac{b}{2a} \quad (64) \]

to ensure that the density function remains non-negative (see discussion in Sect. 2.3). Note that for the case \(A = 0\), we are in the homogeneous case, studied in [15].

Before we begin, we would like to note that (18)–(19) are invariant under the change of parity \(\gamma \to -\gamma, \ a \to -a, \ m \to -m\). Since in this section we are only considering the case in which \(a\) and \(\gamma\) have the same sign, it therefore suffices to only consider either \(\mathcal{K}_-\) with \(\gamma, a \geq 0\) or \(\mathcal{K}_+\) with \(\gamma, a \leq 0\). We will work with the former. Waves whose
surface profile do not overhang have not only a strictly monotone vertical coordinate but also a strictly monotone horizontal coordinate. Our aim is to show that

$$u'(x) = \frac{1}{k} + (C_{kh}(v'))(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}. \quad (65)$$

The idea is now to show that property (65) is preserved along all of $K_-$. Notice that (65) cannot occur in alternative (A2), and therefore for our choice of $\gamma$ and $a$, the validity of (65) would imply that the curve of solutions $K_-$ must satisfy (A1).

The first part of the proof is similar to the one in [15] and can be easily adapted to our case, but for the benefit of the reader, we recall it here. We will study the validity of (65) in conjunction with the properties (39)–(45). This leads us to introduce the set

$$\mathcal{U}_- := \{(m, Q, v) \in \mathcal{V}_- : (65) \text{ holds}\},$$

and to define the set $J$ as

$$J = \{s \in (0, \infty) : (m(s), Q(s), v_s) \in \mathcal{U}_-\}.$$ 

Since $\mathcal{U}_-$ is an open set in $\mathbb{R} \times \mathbb{R} \times C_{2\pi,e}^2$, $J$ is an open set in $(0, \infty)$. We will prove, by contradiction, that $J = (0, \infty)$.

We assume that $J$ is not the whole of $(0, \infty)$. It is immediate that there exists an $\epsilon > 0$ such that $(0, \epsilon) \subset J$. We denote by $s^*$ the upper end-point of the largest interval that contains $(0, \epsilon)$ and is contained in $J$. In other words

$$(0, s^*) \subset J \quad \text{and} \quad s^* \notin J.$$ 

We will from now on look at the properties of the solution $(m^*, Q^*, v_s)$. For simplicity of notation, we will denote this solution simply by $(m, Q, v)$.

Following the same arguments as in [7], we necessarily have $v \neq h$. Moreover, as in the previous section, the definition of $s^*$ implies that the following non-strict inequalities hold

$$v(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}, \quad (66)$$

$$v'(x) \leq 0 \quad \text{for all} \quad x \in (0, \pi), \quad (67)$$

$$0 \leq \frac{x}{k} + (C_{kh}(v - h))(x) \leq \frac{\pi}{k} \quad \text{for all} \quad x \in (0, \pi), \quad (68)$$

$$\left(\frac{m}{kh} - \gamma \frac{v^2}{2kh} - \gamma C_{kh}(vv') - \frac{a}{6kh}[v^3] - \frac{a}{2}C_{kh}(v^2v') + \left(\gamma v + \frac{a}{2}v^2\right)\left(\frac{1}{k} + C_{kh}(v')\right)\right) \leq 0 \quad \text{on} \quad \mathbb{R}, \quad (69)$$

$$\frac{1}{k} + (C_{kh}(v'))(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}. \quad (70)$$

Following the same arguments as in the proof in the previous section, we can deduce that the inequalities (66)–(69), coupled with the fact that $(m, Q, v)$ is the limit of a sequence of solutions that correspond to solutions of the original water waves problem in the physical plane, ensure the validity of (39)–(42), and (44) and the validity of (45) in the form

$$\left(\frac{m}{kh} - \gamma \frac{v^2}{2kh} - \gamma C_{kh}(vv') - \frac{a}{6kh}[v^3] - \frac{a}{2}C_{kh}(v^2v') + \left(\gamma v + \frac{a}{2}v^2\right)\left(\frac{1}{k} + C_{kh}(v')\right)\right) < 0 \quad \text{on} \quad \mathbb{R}. \quad (71)$$

However, if (65) were also valid, this would imply that (43) would hold, which would be a contradiction to the definition of $s^\star$. We can therefore conclude that there exists an $x_0 \in (0, \pi)$ such that
\[
\frac{1}{k} + (C_{kh}(v'))(x_0) = 0.
\]
Finally, in order for $(m(s^\star), Q(s^\star), v^\star)$ to correspond to a solution of the original water wave problem, we need to check that the free surface is injective and non-self-intersecting (we have already shown that $v(x) > 0$ for all $x \in \mathbb{R}$). To begin with, the validity of (44) and (70) implies that (43) holds. Coupling this with the evenness and monotonicity of $v$, from the discussion in the previous section, we clearly have:
\[
\text{the mapping } x \mapsto \left( \frac{x}{k} + C_{kh}(v - h)(x), v(x) \right) \text{ is injective on } \mathbb{R}.
\]
Moreover, from (71) and (16), we clearly have
\[
(u'(x))^2 + \left( \frac{1}{k} + C_{kh}(v')(x) \right)^2 \neq 0 \text{ for all } x \in \mathbb{R}
\]
and therefore $(m(s^\star), Q(s^\star), v^\star)$ corresponds to a solution $(\Omega, \psi)$ of (4).
We now show the contradiction: in the physical plane, a solution satisfying all of the properties mentioned above, cannot exist. Recall that
\[
u(x) = \frac{x}{k} + (C_{kh}(v - h))(x) \text{ for all } x \in \mathbb{R}.
\]
Then the free surface $S = \{(u(x), v(x)) : x \in \mathbb{R}\}$ has the following properties
\begin{align*}
v & \text{ is } 2\pi\text{-periodic and even} \quad (72) \\
v'(x) & < 0 \text{ for all } x \in (0, \pi) \quad (73) \\
u & \text{ is odd, } x \mapsto \left( u(x) - \frac{2\pi}{k} x \right) \text{ is } 2\pi\text{-periodic} \quad (74) \\
(u'(x))^2 + (v'(x))^2 & > 0 \text{ for all } x \in \mathbb{R} \quad (75) \\
u'(0) & > 0, \quad u'(\pi) > 0 \quad (76) \\
u'(x) & \geq 0 \text{ for all } x \in \mathbb{R} \quad (77) \\
\text{there exists } x_0 \in (0, \pi) \text{ such that } u'(x_0) = 0. \quad (78)
\end{align*}
We begin by examining the sign of $\psi_Y$ in $\overline{\Omega}$.
On $y = 0$, we have
\[
\zeta_{x}(x, 0) = -\gamma v(x) v_{x}(x) - \frac{a}{2} v^2(x) v_{x}(x)
\]
and from the Cauchy–Riemann equations, we get
\[
\psi_Y(u(x), v(x)) = \frac{u'(x) \left\{ \xi_Y(x, 0) + \gamma V(x, 0) V_y(x, 0) + \frac{a}{2} V^2(x, 0) V_y(x, 0) \right\}}{(u'(x))^2 + (v'(x))^2} \quad (79)
\]
\[
-\psi_X(u(x), v(x)) = \frac{v'(x) \left\{ \xi_Y(x, 0) + \gamma V(x, 0) V_y(x, 0) + \frac{a}{2} V^2(x, 0) V_y(x, 0) \right\}}{(u'(x))^2 + (v'(x))^2} \quad (80)
\]
Moreover, on $y = 0$ we have

$$
\zeta_y + \gamma V V_y + \frac{a}{2} V^2 V_y =
\left( \frac{m}{kh} - \frac{\gamma}{2kh} [v^2] - \gamma C_{kh} (v v') - \frac{a}{6kh} [v^3] - \frac{a}{2} C_{kh} (v^2 v') + \left( \gamma v + \frac{a}{2} v^2 \right) \left( \frac{1}{k} + C_{kh} (v') \right) \right)
$$

which is strictly negative by (71). Therefore, it follows from (79) that

$$
\psi_Y \leq 0 \quad \text{on } S
$$

and from (78), we get that

$$
\psi_Y(u(x_0), v(x_0)) = 0.
$$

In addition, $\psi_Y$ doesn’t vanish identically on $S$.

However, since $a > 0$, we know that $\psi_Y$ is subharmonic in $\Omega$.

At the bottom, we have

$$
(\psi_Y)_Y = \gamma \geq 0
$$

since $Y = 0$ on $\mathcal{B}$. Therefore, by the strong maximum principle and the Hopf boundary point lemma, the maximum of $\psi_Y$ over $\Omega$ cannot be attained on $\Omega$ or on $\mathcal{B}$. As a result,

$$
\psi_Y < 0 \quad \text{on } \Omega \cup \mathcal{B}. \quad (81)
$$

Moreover, from (79), (80), (81), we can conclude that

$$
|\nabla \psi| \neq 0 \quad \text{in } \overline{\Omega}.
$$

Let us now consider in $\overline{\Omega}$ the function $R$ given by:

$$
R := \frac{1}{2} |\nabla \psi|^2 + g \rho(\psi) Y - \frac{1}{2} Q - \gamma \psi
$$

which is of the form $R = -P - \psi M$ where $P$ is the pressure and we choose $M := \gamma$.

Using Yih’s equation we find that in $\Omega$, the function $R$ satisfies:

$$
\triangle R - s_1 R_x - s_2 R_y = \frac{2 g^2 \rho - b \rho (3 \gamma + a Y) \psi Y}{|\nabla \psi|^2} + (3 \gamma + a Y) \gamma + g \rho'(\psi) \psi Y \quad (82)
$$

where

$$
s_1 = 2 \frac{\psi_X (3 \gamma + a Y)}{|\nabla \psi|^2} \quad \text{and} \quad s_2 = 2 \frac{\psi_Y (3 \gamma + a Y) - 2 g \rho}{|\nabla \psi|^2}.
$$

Note, this is the equation derived in [1], Proposition 4.1.

However, since $\psi_Y < 0$ on $\Omega \cup \mathcal{B}$, the righthand side of (82) is strictly positive.

Moreover,

$$
R_Y > 0 \quad \text{on } \mathcal{B}.
$$

Therefore, the maximum of $R$ over $\overline{\Omega}$ cannot be attained on either $\mathcal{B}$ or in $\Omega$, and is therefore attained all along $S$ since $R = 0$ there.

Thus, by the Hopf boundary point lemma, the normal derivative of $R$ has a strict sign along all of $S$. In particular, at $(u(x_0), v(x_0))$, we have $R_x \neq 0$. 

Recalling that \( \rho(\psi) = A\psi + B \) and \( a = -Ag \) and \( b = 2gB \), notice that we can rewrite \( R \) in the following way:

\[
R = \frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} bY - \frac{1}{2} Q - (\gamma + aY) \psi
\]

and therefore, at \((u(x_0), v(x_0))\), we get

\[
0 \neq R_x = \psi_X \psi_{XX} + \psi_Y \psi_{XY} - \psi_X (\gamma + aY) = \psi_X (\psi_{XX} - (\gamma + aY)) = \psi_X \psi_{YY}
\]

since \( \psi_Y \) is equal to zero at \( x_0 \). We can therefore conclude that

\[
\psi_{YY} \neq 0 \text{ at } (u(x_0), v(x_0)). \tag{83}
\]

However, if we differentiate \( \psi(u(x), v(x)) = 0 \) and evaluate it at the point \( x_0 \), since \( \psi_Y(u(x_0), v(x_0)) = 0 \) and \( u'(x_0) = 0 \), we get

\[
\psi_X(u(x_0), v(x_0))u''(x_0) + \psi_{YY}(u(x_0), v(x_0))(v'(x_0))^2 = 0.
\]

Moreover, from (77) and (78), we can deduce that \( u''(x_0) = 0 \) and since we know that \( v'(x_0) \neq 0 \), we necessarily have

\[
\psi_{YY}(u(x_0), v(x_0)) = 0
\]

which is a contradiction to (83). The only possible source of contradiction could be the assumption that \( J \neq (0, \infty) \) which must thus be false. Therefore, we have proven that

\[
K_{-} \subset \mathcal{U}_{-}.
\]

We would now like to prove that the property (81) is in fact valid for all solutions on \( K_{-} \).

Indeed, let us assume that \((m, Q, v)\) is an arbitrary solution and \((\Omega, \psi)\) is the associated solution to the original problem in the physical plane. Then the top boundary \( S \) of \( \Omega \) admits the parametrization (1) and the properties (72)–(74) hold and instead of (75)–(78) we have the condition

\[
u'(x) > 0 \text{ for all } x \in \mathbb{R}.
\]

By applying the maximum principle for \( \psi_Y \) in \( \Omega \) as done above, we achieve the desired result. We have therefore proven the following theorem:

**Theorem 5.1.** Let \( \gamma \geq 0 \), \( a \geq 0 \) and \( m \geq -b/(2a) \). Then the bifurcating curve \( K_{-} \) of solutions of (18)–(19) satisfies alternative (\( A\Omega \)). Moreover, any solution \((m, Q, v)\) on \( K_{-} \) satisfies

\[
\frac{1}{k} + (C_{kh}(v'))(x) > 0 \text{ for all } x \in \mathbb{R}
\]

and if \((\Omega, \psi)\) denotes the corresponding solution of (4), then

\[
\psi_Y < 0 \text{ in } \overline{\Omega}.
\]

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A. Appendix

We use this appendix to state the real-analytic bifurcation theorem we use in the paper. This result is originally by Dancer and was later improved by Buffoni and Toland. We draw the reader’s attention to the fact that the version below is not completely identical to the one in [14]. Instead, we state it as it is written in [7], where it has been modified to better suit the problem. Remark 8 in [7] discusses certain aspects of how the two formulations differ. In addition, notice that the non-vanishing assumption on $\Lambda'$ in Theorem 9.1.1 of [14] is dropped in Theorem 6 in [7]. As in [7], we combine the theorem with the Crandall and Rabinowitz theorem (see [16]) for local bifurcation. In other words, conditions (H1)–(H2) are those for local bifurcation and (H3)–(H4) the compactness conditions from the global bifurcation theorem.

**Theorem A.1.** Let $X$ and $Y$ be Banach spaces, $\mathcal{O}$ be an open subset of $\mathbb{R} \times X$ and $F : \mathcal{O} \to Y$ be a real-analytic function. Suppose that

(H1) $(\lambda, 0) \in \mathcal{O}$ and $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$;

(H2) for some $\lambda^* \in \mathbb{R}$, the kernel $\text{Ker}(\partial_u F(\lambda^*, 0))$ and the co-range $Y \setminus \text{Ran}(\partial_u F(\lambda^*, 0))$ are 1-dimensional, with the kernel generated by $u^*$ and the transversality condition

$$\partial^2_{\lambda, u} F(\lambda^*, 0)(1, u^*) \notin \text{Ran}(\partial_u F(\lambda^*, 0))$$

holds;

(H3) $\partial_u F(\lambda, u)$ is a Fredholm operator of index zero for any $(\lambda, u) \in \mathcal{O}$ such that $F(\lambda, u) = 0$;

(H4) for some sequence $(Q_j)_{j \in \mathbb{N}}$ of bounded closed subsets of $\mathcal{O}$ with $\mathcal{O} = \bigcup_{j \in \mathbb{N}} Q_j$, the set $\{(\lambda, u) \in \mathcal{O} : F(\lambda, u) = 0\} \cap Q_j$ is compact for each $j \in \mathbb{N}$.

Then there exists in $\mathcal{O}$ a continuous curve $K = \{(\lambda(s), u(s)) : s \in \mathbb{R}\}$ of solutions to $F(\lambda, u) = 0$ such that:

(C1) $$(\lambda(0), u(0)) = (\lambda^*, 0);$$

(C2) $u(s) = su^* + o(s)$ in $X$, $|s| < \epsilon$ as $s \to 0$;

(C3) there exist a neighborhood $\mathcal{W}$ of $(\lambda^*, 0)$ and $\epsilon > 0$ sufficiently small such that

$$\{(\lambda, u) \in \mathcal{W} : u \neq 0 \text{ and } F(\lambda, u) = 0\} = \{(\lambda(s), u(s)) : 0 < |s| < \epsilon\};$$

(C4) $K$ has a real-analytic reparametrization locally around each of its points;

(C5) one of the following alternatives occurs:

(a) for every $j \in \mathbb{N}$, there exists $s_j > 0$ such that $(\lambda(s), u(s)) \notin Q_j$ for all $s \in \mathbb{R}$ with $|s| > s_j$.
there exists $T > 0$ such that $(\lambda(s + T), u(s + T)) = (\lambda(s), u(s))$ for all $s \in \mathbb{R}$.

Moreover, such a curve of solutions to $F(\lambda, u) = 0$ having the properties (C1)–(C5) is unique up to reparametrization.

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