AN APPLICATION OF A MOMENT PROBLEM TO COMPLETELY MONOTONIC FUNCTIONS

VLADIMIR JOVANOVIĆ 1 AND MILANKA TREML 1

Abstract. We consider the following question: if a function of the form \( \int_0^\infty \varphi(t) e^{-xt} dt \) is completely monotonic, is it then \( \varphi \geq 0 \)? It turns out that the question is related to a moment problem. In the end we apply those results to answer some questions concerning complete monotonicity of certain functions raised in F. Qi and R. Agarwal, On complete monotonicity for several classes of functions related to ratios of gamma functions, J Inequal Appl (2019).

1. Introduction

At the beginning we review basic notions and facts related to completely monotonic functions. An infinitely differentiable function \( f : (0, \infty) \to \mathbb{R} \) is called completely monotonic, if

\[
(-1)^n f^{(n)} \geq 0, \quad n = 0, 1, 2, \ldots
\]

The crucial fact concerning this class of functions is Bernstein theorem: a function \( f \) is completely monotonic is and only if there exists a positive Borel measure \( \mu \) on \([0, \infty)\), such that

\[
(1.1) \quad f(x) = \int_{[0, \infty)} e^{-xt} d\mu(t),
\]

for all \( x > 0 \). Furthermore, the measure \( \mu \) is uniquely determined (see [1], p. 61). In many applications one comes up to the situation that a function of the form \( \int_0^\infty \varphi(t) e^{-xt} dt \) is completely monotonic. Usually, in view of Bernstein theorem, it is tacitly assumed that the function \( \varphi \) is then necessarily non-negative. Our aim here is to clarify this question: we give a sufficient condition on \( \varphi \) which guarantees the claim and provide a complete proof. It turns out that our question has to do with

2010 Mathematics Subject Classification. Primary: 26A48. Secondary: 30E20.

Key words and phrases. Complete monotonicity, digamma function, moment problem.
uniqueness of measures in a moment problem which we consider in the
next section. In the sequel we apply those results in order to answer
the question (in a slightly more general form) raised in [4] on page 34
whether the functions \( \psi'(x + 1) - \sinh \frac{1}{x+1} \) and \( \frac{1}{2} \sinh \frac{2}{x} - \psi'(x + 1) \) are
completely monotonic, where \( \psi \) is digamma function.

2. A moment problem

As we previously mentioned, our considerations are tightly related
to the uniqueness question for measures in a moment problem, which
we state in Theorem 2.1 (see below). It resembles the Stieltjes moment
problem: if two non-negative measures \( \mu \) and \( \nu \) with support on \([0, \infty)\)
have the same moments, that is, if \( \int_0^\infty t^n \, d\mu(t) = \int_0^\infty t^n \, d\nu(t) \), for all
\( n = 0, 1, \ldots \), is it then \( \mu = \nu \)? In our case, we use a substitution
and reduce it to the Hausdorff moment problem, where the support of
measures is \([0, 1]\). Now, we turn to our moment problem.

\textbf{Theorem 2.1.} Assume \( \mu \) and \( \nu \) are complex Borel measures on \([0, \infty)\)
with the property
\[
\int_{(0, \infty)} e^{-nt} \, d\mu(t) = \int_{(0, \infty)} e^{-nt} \, d\nu(t), \quad n = 0, 1, 2, \ldots.
\]
Then, \( \mu = \nu \).

We need the following change of variables formula.

\textbf{Proposition 2.1.} Let \((X, \mathcal{M}, \mu)\) be a measure space, \((Y, \mathcal{N})\) a mea-
surable space and \( F : X \to Y \) a measurable map. Then for every
measurable function \( f : Y \to \mathbb{C} \) and every \( E \in \mathcal{N} \) we have
\[
\int_E f(y) \, dF_\ast \mu(y) = \int_{F^{-1}(E)} f(F(x)) \, d\mu(x),
\]
in the case either of two sides is defined. Here \( F_\ast \mu = \mu \circ F^{-1} \) is a
measure on \((Y, \mathcal{N})\), the so-called push-forward of \( \mu \).

For the proof, see [3, p. 30-31].

\textbf{Remark 2.1.} We notice that the change of variable formula also holds
for complex Borel measures.

\textbf{Proof of Theorem 2.1.} We notice that the change of variable formula also holds
for complex Borel measures.

\textit{Proof of Theorem 2.1:} Recall that the complex measures \( \mu \) and \( \nu \) are of bounded variation,
\( M_\mu := |\mu|([0, \infty)) < \infty \) and \( M_\nu := |\nu|([0, \infty)) < \infty \) (see [3]). Let us
define a homeomorphism $F : [0, \infty) \to (0, 1]$, $F(t) = e^{-t}$. Applying Proposition 2.1 (more precisely Remark 2.1), we obtain

$$
\int_{[0, \infty)} e^{-nt} d\mu(t) = \int_{(0,1]} s^n dF_\ast \mu(s), \quad \int_{[0, \infty)} e^{-nt} d\nu(t) = \int_{(0,1]} s^n dF_\ast \nu(s).
$$

From the assumptions of Theorem 2.1, we have

$$
\int_{(0,1]} s^n dF_\ast \mu(s) = \int_{(0,1]} s^n dF_\ast \nu(s),
$$

for all $n = 0, 1, 2 \ldots$. Hence

$$
(2.1) \quad \int_{(0,1]} P(s) \, dF_\ast \mu(s) = \int_{(0,1]} P(s) \, dF_\ast \nu(s),
$$

for all polynomials $P$. Notice

$$
|F_\ast \mu|((0,1]) = F_\ast |\mu|((0,1]) = |\mu|([0, \infty)) = M_\mu
$$

and similarly $|F_\ast \nu|((0,1]) = M_\nu$. Therefore, each bounded and measurable (in Borel sense) function on $(0, 1]$ is integrable with respect to the both measures $F_\ast \mu$ and $F_\ast \nu$. In view of

$$
\left| \int_{(0,1]} g(s) \, dF_\ast \mu(s) \right| \leq M_\mu \|g\|_\infty, \quad \left| \int_{(0,1]} g(s) \, dF_\ast \nu(s) \right| \leq M_\nu \|g\|_\infty,
$$

for all bounded measurable functions $g : (0, 1] \to \mathbb{R}$, where $\|g\|_\infty = \sup\{|g(x)| : x \in (0, 1]\}$, we conclude from Stone - Weierstrass theorem that

$$
(2.2) \quad \int_{(0,1]} g(s) \, dF_\ast \mu(s) = \int_{(0,1]} g(s) \, dF_\ast \nu(s),
$$

for all $g \in C[0, 1]$. For small $\delta > 0$ introduce a continuous, piecewise linear function $I_\delta : (0, 1] \to \mathbb{R}$,

$$
I_\delta(t) = \begin{cases} 
0, & t < a - \delta \\
\frac{t - (a - \delta)}{\delta}, & a - \delta \leq t \leq a \\
1, & a \leq t \leq b \\
\frac{b + \delta - t}{\delta}, & b \leq t \leq b + \delta \\
0, & b + \delta \leq t,
\end{cases}
$$

where $[a, b] \subset (0, 1]$. From (2.2), we have

$$
(2.3) \quad \int_{(0,1]} I_\delta(s) \, dF_\ast \mu(s) = \int_{(0,1]} I_\delta(s) \, dF_\ast \nu(s).
$$
Taking into account that $I_\delta \to \chi_{[a,b]}$ pointwise as $\delta \to 0^+$ (here $\chi$ denotes characteristic function) and $0 \leq I_\delta \leq 1$, one infers, applying Lebesgue dominant convergence theorem to integrals in (2.3), that
$$\int_{(0,1]} \chi_{[a,b]}(s) dF_*\mu(s) = \int_{(0,1]} \chi_{[a,b]}(s) dF_*\nu(s),$$
or equivalently $F_*\mu([a,b]) = F_*\nu([a,b])$, for all $[a,b] \subset (0,1]$. Following a similar procedure one can also deduce $F_*\mu((0,b]) = F_*\nu((0,b])$, for all $(0,b] \subset (0,1]$. Therefore
$$F_*\mu(E) = F_*\nu(E)$$
for all Borel sets $E \subset (0,1]$, which implies $F_*\mu = F_*\nu$. Finally, we obtain $\mu = \nu$, since $F$ is a homeomorphism. □

**Proposition 2.2.** Let $\varphi : [0, \infty) \to \mathbb{R}$ be a continuous function with the property

$$(2.4) \quad \int_0^\infty |\varphi(t)| dt < \infty.$$  

If $f(x) = \int_0^\infty \varphi(t) e^{-xt} dt$ is completely monotonic, then $\varphi \geq 0$.

**Proof.** Since $f$ is completely monotonic, then according to Bernstein theorem, there exists a non-negative Borel measure $\mu$ on $[0, \infty)$ satisfying (1.1) for all $x > 0$. Due to (2.4), we have
$$\mu([0, \infty)) = \int_{[0, \infty)} d\mu = f(0) = \int_0^\infty \varphi(t) dt < \infty,$$
and consequently, $\mu$ is a finite measure. Again, thanks to (2.4), we conclude that $\nu(E) = \int_E \varphi(t) dt$ is a Borel measure of bounded variation $|\nu|([0, \infty)) = \int_0^\infty |\varphi(t)| dt < \infty$. Taking into account that
$$\int_{[0, \infty)} e^{-xt} d\nu(t) = \int_0^\infty \varphi(t) e^{-xt} dt = f(x) = \int_{[0, \infty)} e^{-xt} d\nu(t),$$
for all $x \geq 0$, we see that the assumptions of Theorem 2.1 are fulfilled. Therefore, $\mu = \nu$. This implies
$$\int_a^b \varphi(t) dt = \nu([a,b]) = \mu([a,b]) \geq 0,$$
for all $[a,b] \subset [0, \infty)$. However, $\varphi$ is continuous, whence $\varphi \geq 0$. □

3. **Applications**

We apply the results from the previous section with the aim to answer two questions stated in [4] on page 34, which concern complete monotonicity of functions $\psi'(x+1) - \sinh \frac{1}{x+1}$ and $\frac{1}{2} \sinh \frac{2}{x+1} - \psi'(x+1)$. 
We will actually prove slightly more general assertions. Here $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

**Proposition 3.1.** For all $m > 0$ the function $f(x) = \psi'(x+1) - \frac{1}{m} \sinh \frac{m}{x+1}$ is not completely monotonic.

**Proof.** We employ the following representations

\[(3.1) \quad \frac{1}{x^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-xt} dt, \quad \psi'(x) = \int_0^\infty \frac{t}{1-e^{-t}} e^{-xt} dt,
\]

for all $x > 0$ and $n \in \mathbb{N}$. The latter one is due to S. Ramanujan (see [2, p. 374]). From

\[\frac{1}{m} \sinh \frac{m}{x+1} = \sum_{n=0}^\infty \frac{m^{2n}}{(2n+1)!} \frac{1}{(x+1)^{2n+1}},\]

we conclude that

\[\psi'(x+1) - \frac{1}{m} \sinh \frac{m}{x+1} = \int_0^\infty \left( t \frac{1}{1-e^{-t}} - \sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} \right) e^{-(x+1)t} dt = \int_0^\infty \varphi(t) e^{-xt} dt,
\]

where $\varphi(t) = \left( t \frac{1}{1-e^{-t}} - \sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} \right) e^{-t}$. Owing to $\frac{t}{1-e^{-t}} \sim t$ as $t \to \infty$ and $\sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} \geq \frac{m^2}{4n^2}$, one obtains that $\varphi$ is negative for large $t$. It is easy to see that $\int_0^\infty |\varphi(t)| dt < \infty$ and using Proposition 2.2 we infer that $f$ is not completely monotonic. \(\square\)

**Proposition 3.2.** For all $m > 0$ function the $f(x) = \frac{1}{m} \sinh \frac{m}{x} - \psi'(x+1)$ is completely monotonic.

**Proof.** Using (3.1), we have

\[\frac{1}{m} \sinh \frac{m}{x} = \sum_{n=0}^\infty \frac{m^{2n}}{(2n+1)!} \frac{1}{x^{2n+1}} = \int_0^\infty \sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} e^{-tx} dt,
\]

and

\[\frac{1}{m} \sinh \frac{m}{x} - \psi'(x+1) = \int_0^\infty \left( \sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} - \frac{te^{-t}}{1-e^{-t}} \right) e^{-xt} dt.
\]
for all $x > 0$. Hence

\begin{equation}
(3.2) \quad f(x) = \int_{0}^{\infty} \varphi(t) e^{-xt} \, dt,
\end{equation}

where $\varphi(t) = \sum_{n=0}^{\infty} \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} - \frac{te^{-t}}{1-e^{-t}}$. Further, it is

$$\varphi(t)(e^{t} - 1) = (e^{t} - 1) \sum_{n=0}^{\infty} \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} - t \geq t \cdot 1 - t = 0,$$

for all $t \geq 0$. Consequently, $\varphi \geq 0$ on $[0, \infty)$ and by (3.2) one concludes the proof. □

REFERENCES

[1] C. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer-Verlag, Berlin Heidelberg New York, 1975.
[2] B. C. Berndt, *Ramanujan’s Notebook. Part IV*, Springer, New York, 1994.
[3] R. Durrett, *Probability. Theory and Examples*, Cambridge University Press, 2010.
[4] F. Qi and R. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J Inequal Appl (2019), 1–42.
[5] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987.

1 Faculty of Sciences and Mathematics, University of Banja Luka, Banja Luka, Republic of Srpska, Bosnia and Herzegovina

Email address: vladimir.jovanovic@pmf.unibl.org
Email address: milanka.treml@pmf.unibl.org