Upper and Lower Bounds on the Smoothed Complexity of the Simplex Method

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May 16, 2024

Abstract

The simplex method for linear programming is known to be highly efficient in practice, and understanding its performance from a theoretical perspective is an active research topic. The framework of smoothed analysis, first introduced by Spielman and Teng (JACM ’04) for this purpose, defines the smoothed complexity of solving a linear program with \(d\) variables and \(n\) constraints as the expected running time when Gaussian noise of variance \(\sigma^2\) is added to the LP data. We prove that the smoothed complexity of the simplex method is \(O(\sigma^{-3/2}d^{13/4} \log^{7/4} n)\), improving the dependence on \(1/\sigma\) compared to the previous bound of \(O(\sigma^{-2}d^2 \sqrt{\log n})\). We accomplish this through a new analysis of the shadow bound, key to earlier analyses as well. Illustrating the power of our new approach, we moreover prove a nearly tight upper bound on the smoothed complexity of two-dimensional polygons.

We also establish the first non-trivial lower bound on the smoothed complexity of the simplex method, proving that the shadow vertex simplex method requires at least \(\Omega\left(\min\left(\sigma^{-1/2}d^{-1/2} \log^{-1/4} d, 2^d\right)\right)\) pivot steps with high probability. A key part of our analysis is a new variation on the extended formulation for the regular \(2^k\)-gon. We end with a numerical experiment that suggests our lower bound could be further improved.

1 Introduction

Introduced by Dantzig [Dan47], the simplex method is one of the primary methods for solving linear programs (LP’s) in practice and is an essential component in many software packages for combinatorial optimization. It is a family of local search algorithms which begin by finding a vertex of the set of feasible solutions and iteratively move to a better neighboring vertex along the edges of the feasible polyhedron until an optimal solution is reached. These moves are known as pivot steps. Variants of the simplex method can be differentiated by the choice of pivot rule, which determines which neighboring vertex is chosen in each iteration, as well as by the method for obtaining the initial vertex. Some well-known pivot rules are the most negative reduced cost rule, the steepest edge rule, and an approximate steepest edge rule known as the devex rule. In theoretical work, the parametric objective rule, also known as the shadow vertex rule, plays an important role.

Empirical evidence suggests that the simplex algorithm typically takes \(O(d + n)\) pivot steps, see [Sha87, And04, Gol94]. However, obtaining a rigorous explanation for this excellent performance has proven challenging. In contrast to the practical success of the simplex method, all studied variants are known to have super-polynomial or even exponential worst-case running times. For deterministic variants, many published

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Bad inputs are based on deformed cubes, see \cite{KM72, Jer73, AC78, GS79, Mur80, Gol76} and a unified construction in \cite{AZ98}. For randomized and history-dependent variants, bad inputs have been constructed based on Markov Decision Processes \cite{Kal92, MSW96, HZ15, FHZ11, Fri11, DHF22}. The fastest provable (randomized) simplex algorithm takes $O(2^{\sqrt{\log n}})$ pivot steps in expectation \cite{Kal92, MSW96, HZ15}.

Average-case analyses of the simplex method have been performed for a variety of random distributions over linear programs \cite{Bor82, Bor87, Bor99, Sma83, Hai83, Meg86, AKS87, Tod86, AM85}. While insightful, the results from average-case analyses might not be fully realistic due to the fact that “random” linear programs tend to have certain properties that “typical” linear programs do not.

To better explain why simplex algorithm performs well in practice, while avoiding some of the pitfalls of average-case analysis, Spielman and Teng \cite{ST04} introduced the smoothed complexity framework. For programs tend to have certain properties that “typical” linear programs do not.

The motivation for smoothed analysis lies in the observation that the above-mentioned worst-case instances are very “brittle” to perturbations, and computer implementations require great care in handling numerical inaccuracies to obtain the theorized running times even on problems with a small number of variables. When implemented with a larger number of variables, the limited accuracy of floating-point numbers make it impossible to reach the theorized running times.

An algorithm is said to have polynomial smoothed complexity if under the perturbation of constraints, it has expected running time $\mathrm{poly}(n, d, \sigma^{-1})$, and \cite{ST04} proved that the smoothed complexity of a specific simplex method based on the shadow vertex simplex method (which we will describe next) is at most $O(d^{25} n^{86} \sigma^{-30} + d^7 n^{86})$. The best bound available in the literature is $O(\sigma^{-2} d^7 \sqrt{\log n})$ pivot steps due to \cite{DH20}, assuming $\sigma \leq 1/\sqrt{d \log n}$. We note that assuming an upper bound on $\sigma$ can be done without loss of generality; its influence can be captured as an additive term in the upper bound that does not depend on $\sigma$.

This work improves the dependence on $\sigma$ of the smoothed complexity, obtaining an upper bound of $O(\sigma^{-3/2} d^{13/4} \log^{7/4} n)$ for $\sigma \leq 1/d \sqrt{\log n}$. As a second contribution, we prove the first non-trivial lower bound on the smoothed complexity of a simplex method, finding that the shadow vertex simplex method requires $\Omega(\min(1/\sqrt{\sigma d \sqrt{\log n}}, 2^d))$ pivot steps.

**Shadow Vertex Simplex Algorithm** One of the most extensively studied simplex algorithms in theory is the shadow vertex simplex algorithm \cite{GS55, Bor82}. Given an LP

$$\max_{x \in \mathbb{R}^d} c^T x, \quad Ax \leq b,$$

for $A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n, c \in \mathbb{R}^d$, let $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ denote the feasible polyhedron of the linear program. The algorithm starts from an initial vertex $x_0 \in P$ that optimizes an initial objective $c_0^T x$. During

1 There are many standard methods of finding such initialization with at most multiplicative $O(d)$ overhead in running time, so we can assume that both $x_0$ and $c_0$ are already given. See the discussion in \cite{DH20}.
the smoothed complexity of the shadow vertex simplex algorithm satisfies $c_{\lambda} = \lambda c + (1 - \lambda)c_0$ and a vertex that optimizes $c_{\lambda}$. Thus by slowly increasing $\lambda$ from 0 to 1 during different pivot steps, the temporary objective gradually changes from $c_0$ to $c$, revealing the desired solution at the end. Since each pivot step requires $poly(d,n)$ computational work, theoretical analysis has focused on analyzing the number of pivot steps.

The algorithm is called shadow vertex simplex method because, when performing orthogonal projection of the feasible set onto the two-dimensional linear subspace $W = \text{span}(c_0,c)$, the vertices visited by the algorithm project onto the boundary of the projection ("shadow") $\pi_W(P)$. Assuming certain non-degeneracy conditions, which will hold with probability 1 for the distributions we consider, this projection gives an injective map from iterations of the method to vertices of the shadow, meaning that we can upper bound the number of pivot steps in the algorithm by the number of vertices of the shadow polygon. This characterization makes the shadow vertex simplex method ideally suited for probabilistic analysis.

To analyze the “shadow size”, the number of vertices of the shadow polygon, we follow earlier work of [Ver09] and reduce to the case that $b = 1_n$. In this case, well-established principles of polyhedral duality show that we can bound the number of vertices of a convex polygon by the number of edges of its dual polygon:

$$\text{vertices}(\pi_W(P)) \leq \text{edges}(W \cap \text{conv}(0, a_1, \ldots, a_n)) \leq \text{edges}(W \cap \text{conv}(a_1, \ldots, a_n)) + 1.$$  

Here, $a_1, \ldots, a_n$ denote the rows of the matrix $A$ used to define $P = \{x \in \mathbb{R}^d : Ax \leq 1_n\}$. Note that the second inequality holds because $0 \in W$.

The smoothed complexity of shadow vertex simplex algorithm can thus be reduced to the smoothed complexity of a two-dimensional slice of a convex hull. For this reason, let us define the maximum smoothed shadow size as

$$S(n, d, \sigma) = \max_{a_1, \ldots, a_n \in \mathbb{R}^d} \mathbb{E}_{a_1, \ldots, a_n \sim \mathcal{N}(0,\sigma^2)} \left[ \text{edges} \left( \text{conv}(\bar{a}_1 + \bar{a}, \ldots, \bar{a}_n + \bar{a}) \cap W \right) \right]$$  

(1)

The following upper bound we take from [DH20], which states that the analysis of [Ver09] can be strengthened to obtain the claimed bound. This upper bound should be understood as stating that there exists a shadow vertex rule based simplex algorithm which satisfies that smoothed complexity bound. The lower bound is due to [Bor87] and shows that the shadow vertex simplex rule, with two adversarially specified objectives $c_0, c$, can be made to follow paths of this length.

**Theorem 1** (Smoothed Complexity of Shadow Vertex Simplex Algorithm). *Given any $n \geq d \geq 2, \sigma > 0$, the smoothed complexity of the shadow vertex simplex algorithm satisfies*

$$S(n, d, \sigma)/4 \leq SC_{\text{ShadowSimplex}, n, d, \sigma} \leq 2 \cdot S \left( n + d, d, \min(\sigma, \frac{1}{\sqrt{d \log d}}, \frac{1}{\sqrt{d \log n}}) \right) + 4.$$  

With this reduction, analyzing the smoothed complexity of the simplex method comes down to bounding the smoothed shadow size $S(n, d, \sigma)$. As such, that will be the focus of the remainder of this paper.

### 1.1 Our Results

The previous best shadow bound is due to [DH20], who prove that $S(n, d, \sigma) \leq O(d^2 \sqrt{\log n} \sigma^{-2})$. Our first main result strengthen this result for small values of $\sigma$.

**Theorem 2.** *For $n \geq d \geq 3$ and $\sigma \leq \frac{1}{nd \sqrt{\log n}}$, the smoothed shadow size satisfies*

$$S(n, d, \sigma) = O \left( \sigma^{-3/2} d^{13/4} \log^{7/4} n \right).$$

A full overview of bounds on the smoothed shadow size, including previous results in the literature, can be found in Table 1.

Second, we prove the first non-trivial lower bound on the smoothed shadow size, establishing that $S(4d - 13, d, \sigma) \geq \Omega(\min(\frac{1}{\sqrt{\sigma d \sqrt{\log d}}}, 2^d))$ for $d > 5$. This lower bound is proven by constructing a polyhedron
Theorem 3. For any $d > 5$ and $\sigma \leq \frac{1}{360d \log(4d)}$, the smoothed shadow size satisfies

$$S(4d - 13, d, \sigma) = \Omega \left( \min \left( \frac{1}{\sqrt{\sigma d \log d}}, 2^d \right) \right).$$

It is possible that the exponent of $\sigma$ in our bound can be further optimized. In Section 6.7 we describe numerical experiments which suggest that the actual shadow size for random perturbations of our constructed polytope might be as high as $\Omega(\min(\sigma^{-3/4}, 2^d))$. We leave open the question whether having $n/d > 4$ can lead to stronger lower bounds in the regime of $\sigma < 2^{-d}$.

| Reference  | Shadow size                           | Model   |
|------------|--------------------------------------|---------|
| Bor87      | $\Theta(d^{3/2} \sqrt{\log n})$    | Average-case, Gaussian |
| ST04       | $O(\sigma^{-6} d^3 n + d^6 n \log^3 n)$ | Smooth |
| DS05       | $O(\sigma^{-2} d n^2 \log n + d^2 n^2 \log^2 n)$ | Smooth |
| Ver09      | $O(\sigma^{-4} d^3 + d^5 \log^2 n)$   | Smooth |
| DH20       | $O(\sigma^{-2} d^2 \sqrt{\log n} + d^3 \log^{1.5} n)$ | Smooth |
| This paper | $O(\sigma^{-3/2} d^{13/4} \log^{7/4} n + d^{19/4} \log^{13/4} n)$ | Smooth |
| This paper | $\Omega(\min(\frac{1}{\sqrt{\sigma d \log d}}, 2^d))$ | Smooth |
| Mur80 [Gol83] | $2^d$                   | Worst   |

Table 1: Bounds of expected number of pivots in previous literature, assuming $d \geq 3$. Logarithmic factors are simplified. The lower bound of [Bor87] holds in the smoothed model as well.

Two-dimensional polygons To better understand the smoothed complexity of the intersection polygon $\text{conv}(a_1, \ldots, a_n) \cap W$, we also analyze its two-dimensional analogue introduced by [DS04]. Taking $\bar{a}_1, \ldots, \bar{a}_n \in \mathbb{R}^2$, each satisfying $\|a_i\|_2 \leq 1$, we are interested in the number of edges of the smoothed polygon $\text{conv}(\bar{a}_1 + \bar{a}_1, \ldots, \bar{a}_n + \bar{a}_n)$, where $\bar{a}_1, \ldots, \bar{a}_n \sim N(0, \sigma^2)$ are independent. The previous best upper bound on the smoothed complexity of this polygon is $O(\sigma^{-1} + \sqrt{\log n})$, due to [DH21]. Their analysis is based on an adaptation of the shadow bound by [DH20]. In Section 4 we improve this upper bound, obtaining the following theorem.

Theorem 4 (Two-Dimensional Upper Bound). Let $\bar{a}_1, \ldots, \bar{a}_n \in \mathbb{R}^2$ be $n > 2$ vectors with norm at most 1. For each $i \in [n]$, let $a_i$ be independently distributed as $N(\bar{a}_i, \sigma^2 I_2)$. Then

$$\mathbb{E} [\text{edges } (\text{conv}(a_1, \ldots, a_n))] \leq O \left( \sqrt{\log n} + \frac{\sqrt{\log n}}{\sqrt{\sigma}} \right).$$

To confirm that the above upper bound is stronger than that of [DH21], one may use the AM-GM inequality to verify that $2^{\frac{\sqrt{\log n}}{\sqrt{\sigma}}} \leq \sqrt{\log n} + \sigma^{-1}$. Combined with the trivial upper bound of $n$ vertices, our bound nearly matches the lower bound of $\Omega(\min(\sqrt{\log n} + \frac{\sqrt{\log n \sqrt{\sigma}}}{\sqrt{\sigma}}, n))$ proven by [DGGT16]. A full overview of previous results on the smoothed complexity of the two-dimensional convex hull can be found in Table 2.
| Reference | Smoothed polygon complexity |
|-----------|-----------------------------|
| DS04      | $O(\log(n)^2 + \sigma^{-2} \log n)$ |
| Sch14     | $O(\log n + \sigma^{-2})$ |
| DGGT16    | $O(\sqrt{\log n} + \frac{1}{\sqrt{\sigma}})$ |
| DH21      | $O(\sqrt{n} + \frac{1}{\sqrt{\sigma}})$ |
| This paper| $O(\sqrt{\log n} + \frac{1}{\sqrt{\sigma}})$ |
| DGGT16    | $\Omega(\min(\sqrt{\log n} + \frac{1}{\sqrt{\sigma}}, n))$ |

Table 2: Bounds on the smoothed complexity of a two-dimensional polygon.

1.2 Related work

**Shadow Vertex Simplex Method**  The shadow vertex simplex algorithm has played a key role in many analyses of simplex and simplex-like algorithms. On well-conditioned polytopes, such as those of the form $\{x \in \mathbb{R}^d : Ax \leq b\}$ where $A$ is integral with subdeterminants bounded by $\Delta$, the shadow vertex method has been studied in [DH16; BR13]. The shadow vertex method on polytopes all whose vertices are integral was studied in [BDKS21; Bla].

On random polytopes of the form $\{x \in \mathbb{R}^d : Ax \leq 1\}$, assuming the constraint vectors are independently drawn from any rotationally symmetric distribution, the expected iteration complexity of the shadow vertex simplex method was studied by [Bor82; Bor87; Bor99]. In the case when the rows of $A$ arise from a Poisson distribution on the unit sphere, concentration results for the shadow size and diameter bounds were proven in [BDGHL22]. The diameter of smoothed polyhedra was studied by [NSS22], who used the shadow bound of [DH20] to show that most vertices, according to some measure, are connected by short paths.

A randomized algorithm for solving linear programs in weakly polynomial time, using the shadow vertex simplex method as a subroutine, was proposed in [KS06]. The shadow vertex algorithm was recently used as part of the analysis of an interior point method by [ADLNV22].

**Extended Formulations**  For a polyhedron $P \subseteq \mathbb{R}^d$, an extended formulation is any polyhedron $Q \subseteq \mathbb{R}^{d'}$, $d' \geq d$, such that $P$ can be obtained as the orthogonal projection of $Q$ to some $d$-dimensional subspace. Importantly, $Q$ can have much fewer facets than $P$. While there is a wider literature on extended formulations, here we describe only what is most relevant to the construction in Section 6.

The construction in our lower bound is based on an adaptation of the extended formulation by [BN01] of the regular $2^k$-gon. They used this construction to obtain a polyhedral approximiation of the second order cone $\{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 \leq x_{n+1}^2\}$. A variant on their construction using fewer variables and inequalities was given by [Gli00]. A more general construction based on reflection relations is used to construct extended formulation for the regular $2^k$-gon, as well as other polyhedra, in [KP13]. Extended formulations for regular $n$-gons, when $n$ is not a power of 2, can be found in [VGG17].

Approximations of the second order cone based on the work of [BN01; Gli00] have been used to solve second order conic programs, see, e.g., [BHMPT15]. These approximations were included in the open-source MIP solver SCIP until version 7.0 [Gam+20a; Gam+20b].

1.3 Proof Overview

1.3.1 Smoothed Complexity Upper Bound

We write the random polytope from (1) as $Q = \text{conv}(a_1, \ldots, a_n)$ where each $a_i$ is sampled independently from $\mathcal{N}_d(\bar{a}_i, \sigma^2 I)$ such that $\|\bar{a}_i\| \leq 1$. Our goal is to upper bound the expected number of edges of the polygon $Q \cap W$ for fixed two-dimensional plane $W \subseteq \mathbb{R}^d$ and $\bar{a}_1, \ldots, \bar{a}_n$. This will immediately give us an upper bound of $S(n, d, \sigma)$. 


A fact used since the early days of smoothed analysis [ST04] states that the intersection polygon $Q \cap W$ is non-degenerate with probability measure 1: every edge on $Q \cap W$ is uniquely given by the intersection between $W$ and a facet of $Q$, and every facet of $Q$ is spanned by exactly $d$ vertices. For any index set $I \in \binom{[n]}{d}$, write $E_I$ as the event that $\text{conv}(a_i : i \in I) \cap W$ is an edge of $Q \cap W$. Non-degeneracy implies that every edge of $Q \cap W$ uniquely corresponds to an index set $I \in \binom{[n]}{d}$ such that $E_I$ holds.

Before sketching our proof, we first review the approach of [DH20], then discuss the main technical challenges in achieving an upper bound with better dependence on $\sigma$.

As a first step in [DH20], the authors replace the Gaussian distribution with a Laplace-Gaussian distribution. The latter distribution approximates the probability density of the former, in particular having nearly equivalent smoothed shadow size, while being $O(\sigma^{-1}\sqrt{d \log n})$-log-Lipschitz for any point on its domain. A probability distribution with probability density function $\mu$ is $L$-log-Lipschitz for some $L > 0$ if, for any $x, y \in \mathbb{R}^d$, the condition $|\log(\mu(x)) - \log(\mu(y))| \leq L\|x - y\|$ holds.

Next, define $\ell_I$ as the length of the edge on $Q \cap W$ that corresponds to $I$, i.e. when $\text{conv}(a_i : i \in I) \cap W$ is an edge of $Q \cap W$ then $\ell_I$ gives the length of this line segment, and otherwise $\ell_I = 0$. [DH20] showed that, for any family $S \subseteq \binom{[n]}{d}$, the expected number of edges of $Q \cap W$ coming from $S$ is at most

$$
\mathbb{E}\left[\sum_{I \in S} 1[E_I]\right] \leq \frac{\mathbb{E}[\text{perimeter}(Q \cap W)]}{\min_{I \in S} \mathbb{E}[\ell_I]}. 
$$

Therefore, by taking $S = \{I \in \binom{[n]}{d} : \mathbb{P}[E_I] \geq \binom{n}{d}^{-1}\}$, the expected number of edges of $Q \cap W$ is at most

$$
\mathbb{E}[\text{edges}(Q \cap W)] = \mathbb{E}\left[\sum_{I \in \binom{[n]}{d}} 1[E_I]\right] \leq 1 + \mathbb{E}\left[\sum_{I \in S} 1[E_I]\right] \leq 1 + \frac{\mathbb{E}[\text{perimeter}(Q \cap W)]}{\min_{I \in S} \mathbb{E}[\ell_I]}. 
$$

To upper bound the numerator of (3), notice that $Q \cap W$ is a convex polygon contained in the two-dimensional disk centered at $0$ with radius $\max_{i \in [n]} \|\pi_W(a_i)\|$. It follows that the perimeter of $Q \cap W$ is at most the perimeter of this disk, namely,

$$
\mathbb{E}[\text{perimeter}(Q \cap W)] \leq 2\pi \mathbb{E}[\max_{i \in [n]} \|\pi_W(a_i)\|] \leq 2\pi \cdot (1 + 4\sigma \sqrt{\log n}),
$$

where the last step comes from a Gaussian tail bound. For the denominator of (3), [DH20] showed that for any $I \in \binom{[n]}{d}$ with $\mathbb{P}[E_I] \geq \binom{n}{d}^{-1}$ that, conditional on $E_I$, the expected edge length is at least

$$
\mathbb{E}[\ell_I | E_I] \geq \Omega\left(\frac{\sigma^2}{d^2 \sqrt{\log n}} \cdot \frac{1}{1 + \sigma \sqrt{d \log n}}\right).
$$

Combining the (3), (4) and (5), they obtain an upper bound of $O(\sigma^{-2}d^2 \sqrt{\log n} + d^3 \log^{1.5} n)$.

**New Strategy for Counting Edges** While [DH20] made the best possible analysis based on their edge-counting strategy as outlined in (3), the strategy itself is sub-optimal. The main drawback is that using the minimum expected length of edge $\min_{I \in \binom{[n]}{d}} \mathbb{E}[\ell_I | E_I]$ at the denominator of (3) is too pessimistic when the edges of $Q \cap W$ are long. For instance, consider the case where an edge on $Q \cap W$ has length $\Omega(1)$ without perturbation. After the perturbation, it is very likely that the length of this edge is still $\Omega(1)$, but [DH20] uses a lower-bound of $\Omega\left(\frac{\sigma^2}{\sqrt{\log n}}\right)$.

To improve this, we developed a new edge-counting strategy that can handle the long and short edges separately with two different ways of counting the edges. Take any index set $I \in \binom{[n]}{d}$: conditional on $E_I$, we write $e_I$ for its edge $\text{conv}(a_i : i \in I) \cap W$. We call the next edge in clockwise direction as $e_{I^+}$ and say it has length $\ell_{I^+}$. We say $\ell_{I^+}$ is likely to be long if $\mathbb{P}[\ell_{I^+} \geq t | E_I] \geq 0.05$ for some parameter $t > 0$ to be specified later. Let $S_0 \subseteq \binom{[n]}{d}$ denote the set of $I \in \binom{[n]}{d}$ such that $\ell_{I^+}$ is likely to be long. Following (2), the expected number of edges in $Q \cap W$ that are likely to be long is at most

$$
\mathbb{E}\left[\sum_{I \in S_0} 1[E_I]\right] \leq \frac{\mathbb{E}[\text{perimeter}(P \cap W)]}{0.05t} \leq \frac{(2\pi + O(\sigma \sqrt{\log n}))}{0.05t}.
$$
Here the second step uses the exact same upper bound of $\mathbb{E}[\text{perimeter}(P \cap W)]$ as in (4).

In the other case, $e_{I^+}$ is unlikely to be long, i.e. $\Pr[\ell_{I^+} \geq t \mid E_I] < 0.05$. Now we will upper bound the number of such edges by claiming that their exterior angles each are large in expectation. Let $\theta_I$ be the exterior angle at the endpoint of $\text{conv}(a_i : i \in I) \cap W$ that comes last in clockwise order, i.e., the vertex where $e_I$ and $e_{I^+}$ meet. Our key observation is that $\sin(\theta_I) \cdot \ell_{I^+}$ equals the distance from the affine hull of $e_I$ to the second vertex of $e_{I^+}$ in clockwise order. Therefore, by establishing a universal lower bound for such line-to-vertex distances, we can derive a lower bound for $E[\theta_I \mid E_I]$ for any edge $e_{I^+}$ that is unlikely to be long. See Figure 1 for an illustration.

More formally, let $p_{I^+}$ denote the next vertex of $Q \cap W$ after $e_I$ in clockwise order. For a specific value of $\gamma > 0$, let $S_I \subseteq \binom{[n]}{d}$ be the collection of index sets $I \in \binom{[n]}{d}$ for which $\Pr[E_I] \geq 10(n)^{-1}$. Suppose for each $I \in S_I$,

$$\Pr[\text{dist}(p_{I^+}, \text{affhull}(e_I)) \geq \gamma \mid E_I] \geq 0.1. \tag{7}$$

Then, for $I \in S_I \setminus S_0$, conditional on $E_I$, we have $\theta_I \geq \sin(\theta_I) \geq \frac{\gamma}{\ell_{I^+}}$ with probability at least 0.05, and the expectation of the exterior angle at the shared endpoint of $e_I$ and $e_{I^+}$ is at least $\frac{\gamma}{20t}$.

![Figure 1: Illustration of the case when $e_{I^+}$ is short. In brown is the edge $e_I$ with its extension line dashed. The next edge in clockwise direction, drawn in black, has length $\ell_{I^+}$. In orange is the line-to-vertex distance $\text{dist}(p_{I^+}, \text{affhull}(e_I))$, and in blue is the angle $\theta_I$. If $\text{dist}(p_{I^+}, \text{affhull}(e_I)) \geq \gamma$ then $\theta_I \geq \sin(\theta_I) \geq \gamma/\ell_{I^+}$.

On the other hand, the sum of exterior angles of a polygon equals to $2\pi$. Therefore we can upper bound the expected number of edges that are not likely to be long by at most

$$\mathbb{E}[\sum_{I \in S_I \setminus S_0} 1[E_I]] \leq \frac{2\pi}{\min_{I \in S_I \setminus S_0} \mathbb{E}[\theta_I \mid E_I]} \leq \frac{2\pi \cdot 20t}{\gamma}. \tag{8}$$

We will select $\gamma > 0$ as large as possible subject to the fact that every $I \in \binom{[n]}{d}$ with $\Pr[E_I] \geq 10(n)^{-1}$ satisfies $I \in S_I$.

Summing up the number of edges induced by sets in $S_0, S_I \setminus S_0$ and $\binom{[n]}{d} \setminus S_I$, we get an upper bound on the expected edge-count of $Q \cap W$ by at most

$$\mathbb{E}[\text{edges}(Q \cap W)] \leq \frac{2\pi + O(\sigma \sqrt{\log n})}{t} + \frac{40\pi t}{\gamma} + 10 = O\left(\sqrt{\frac{1 + \sigma \sqrt{\log n}}{\gamma}}\right) \tag{9}$$

where the final step follows from optimizing $t > 0$ to get the strongest possible bound. We summarize our result in Theorem 28. For details of the edge-counting strategy, see Section 3.

**Two-dimensional Upper Bound** In the second part of our proof, we need to show a lower bound of the expected distance from the affine hull of an edge of $Q \cap W$ to the next vertex in clockwise order, which is the quantity $\gamma$ mentioned in (7).

As a warm-up, we first discuss the case two-dimensional case $d = 2$, which will be explained in detail in Section 4. In this case, $W$ will become the entire two-dimensional space and will disappear from consideration.
In two-dimensional case, each edge is the convex hull of two vertices among \(J\) next to the edge interest. Consider the second endpoint with two vertices almost surely, so we can assume without loss of generality that \(\gamma > 0\) such that for any \(I \in \binom{[n]}{2}\),
\[
\Pr\{\text{dist}(\{a_j : j \notin I\}, \text{affhull}(e_I)) \geq \gamma | E_I \geq 0.1.
\]

We can obtain a lower bound on this quantity for any \(L\)-log-Lipschitz distribution. Through an appropriate coordinate transformation we prove that, irrespective of the values of \(a_j, j \notin I\), the distance \(\text{dist}(\text{conv}(a_j, j \notin I), \text{affhull}(e_I))\), conditional on being non-zero, follows from a 2\(L\)-log-Lipschitz distribution. We will calculate that we may choose \(\gamma = \Omega(1/L)\). This result can be applied with small changes to our Gaussian random variables \(a_1, \ldots, a_n\) by substituting for the Laplace-Gaussian distribution following [DH20]. Plugging into (9), we get that in the two-dimensional case, \(E[\text{dist}(Q)] \leq O(\sqrt{(1 + \sigma \sqrt{\log n})/(\sigma / \sqrt{\log n})}) = O(\sqrt{\log n} / \sqrt{\sigma} + \sqrt{\log n})\) as stated in Theorem 4.

**Multi-Dimensional Upper Bound** As in the two-dimensional case, we must lower-bound of the line-to-vertex distance \(\text{dist}(p_{1+}, \text{affhull}(e_I))\) (see [4]) of \(Q \cap W\). Analyzing this, however, becomes more challenging. In two-dimensional case, each edge is the convex hull of two vertices among \(a_1, \ldots, a_n\) and is independent to the other vertices on \(Q \cap W\). In contrast, if \(d \geq 3\) then each edge on \(Q \cap W\) will be the intersection between \(W\) and a \((d - 1)\)-dimensional facet of \(Q\) (which is the convex hull of \(d\) vertices), and each vertex will be the intersection between \(W\) and a \((d - 2)\)-dimensional ridge of \(Q\) (which is the convex hull of \(d - 1\) vertices). So the distribution of \(e_I\) and \(p_{1+}\) are correlated.

To overcome these difficulties, we first factor \(\text{dist}(p_{1+}, \text{affhull}(e_I))\) into the product of separate parts which are easier to analyze, and then use log-Lipschitzness of \(a_1, \ldots, a_n\) to lower-bound each part with good probability. Fix without loss of generality \(e = e_d = \text{conv}(a_1, \ldots, a_d) \cap W\), as the potential edge of interest. Consider the second endpoint \(p\) on \(e\) in clockwise direction and let \(J \in \binom{[d]}{d-1}\) be the index set such that \(\{p\} = \text{conv}(a_j : j \in J) \cap W\). Let \(p' = \text{conv}(a_{i} : i \in J') \cap W\) (with \(J' \in \binom{[d]}{d-1}\)) be the vertex next to the edge \(e\) in clockwise direction. From the non-degeneracy conditions, we know that \(J'\) only differs to \(J\) with two vertices almost surely, so we can assume without loss of generality that \(J = \{2, \ldots, d\}\) and \(J' = \{3, \ldots, d\} \cup \{k\}\) for some \(k \in \{d + 1, \ldots, n\}\).

The main idea of our analysis is the observation that if the radius of \(Q\) is bounded above by \(O(1)\) (which happens with overwhelming probability due to Gaussian tail bound), then we can lower bound the two-dimensional line-to-vertex distance \(\text{dist}(p', \text{affhull}(e))\) by the product of two distances \(\Omega(\delta \cdot r)\), where

- \(\delta\) is the \(d\)-dimensional distance from the facet-defining hyperplane \(\text{affhull}(a_1, \ldots, a_d)\) containing \(e\), to the vertices that are not in the facet, i.e.
  \[
  \delta = \text{dist}(\text{conv}(a_{d+1}, \ldots, a_n), \text{affhull}(a_1, \ldots, a_d));
  \]
- \(r\) is the distance from the boundary of the ridge \(\partial \text{conv}(a_2, \ldots, a_d)\) to the one-dimensional line \(\text{affhull}(e)\), i.e. \(r = \text{dist}(\text{affhull}(e), \partial \text{conv}(a_2, \ldots, a_d))\).

We will give the formal statement of the distance splitting lemma in Lemma 36.

It then remains to show that \(r\) and \(\delta\) are both unlikely to be too small. Similar to the two-dimensional case, we will also use log-Lipschitzness of \(a_1, \ldots, a_d\) as our main tool.

- After specifying \(\text{affhull}(a_1, \ldots, a_d)\), the lower bound on \(\delta\) is derived from the remaining randomness in \(a_{d+1}, \ldots, a_n\). Here we use both the \(L\)-log-Lipschitz of the distributions of \(a_{d+1}, \ldots, a_n\) as well as the knowledge that we only need to consider hyperplanes \(\text{affhull}(a_1, \ldots, a_d)\) which are likely to have all points \(a_{d+1}, \ldots, a_n\) on its one side. This is made precise in Section 5.3.
- The lower bound of \(r\) resembles a more technical version of the proof of the "distance lemma" of [ST04]. Write \(\pi : \text{affhull}(a_1, \ldots, a_d) \to \text{affhull}(e)\) for the orthogonal projection sending \(e\) to a single point \(p = \pi(e)\). With this notation we have \(r = \text{dist}(p, \partial \text{conv}(\pi(a_2), \ldots, \pi(a_d)))\).

First we show that each vertex of the ridge \(\text{conv}(\pi(a_2), \ldots, \pi(a_d))\) is \(\Omega(1/(d^2 L))\)-far away from the hyperplane spanned by its other vertices. That means that the projected ridge \(\text{conv}(\pi(a_2), \ldots, \pi(a_d))\) is wide in every direction.
In the second step, we show that \( W \) intersects \( \text{conv}(a_2, \ldots, a_d) \) “through the center”. Specifically, we show that if we write \( p = \sum_{i \in [d]} \lambda_i \pi(a_i) \) as the convex combination of \( a_2, \ldots, a_d \), then with constant probability \( \min_{i \in [d]} \lambda_i \geq \Omega(1/(d^2L)) \).

The product of the lower bounds \( \Omega(1/(d^2L)) \) and \( \Omega(1/(d^2L)) \) for the individual quantities will yield a lower bound for \( r \) with good probability. This is included in Section 5.4.

We conclude our main result of the line-to-vertex distance lower bound in Lemma 38. Readers are referred to Section 5 for detailed discussions.

### 1.3.2 Smoothed Complexity Lower Bound

Our smoothed complexity lower bound (Theorem 49) is based on two geometric observations using the inner and outer radius of the perturbed polytope. For a polytope \( P \) and a unit norm ball \( B \), its outer radius with center \( x \) is the smallest \( R \) such that there exists \( P \subseteq R \cdot B + x \). Its inner radius with center \( x \) is the largest \( r \) such that \( r \cdot B + x \subseteq P \).

The first observation is that, if a two-dimensional polygon \( T \) has inner \( \ell_2 \)-radius of \( r \) and outer \( \ell_2 \)-radius of \( (1 + \varepsilon) \cdot r \) with respect to the same center, then \( T \) has at least \( \Omega(\varepsilon^{-1/2}) \) edges (Lemma 58). This comes from the fact that every edge of \( T \) has length at most \( O(r/) \), whereas the perimeter of \( T \) is at least \( 2\pi r \).

Second, if two polytopes \( Q, \tilde{Q} \subseteq \mathbb{R}^d \) each with inner radius \( t \), have Hausdorff distance \( \varepsilon < t/2 \) to each other, then \( Q \) will approximate \( \tilde{Q} \) in the way that (Lemma 57)

\[
(1 - 2\varepsilon/t) \cdot Q \subseteq \tilde{Q} \subseteq (1 + \varepsilon/t) \cdot Q.
\]

In particular, for any two-dimensional linear subspace \( W \) we have

\[
(1 - 2\varepsilon/t) \cdot Q \cap W \subseteq \tilde{Q} \cap W \subseteq (1 + \varepsilon/t) \cdot Q \cap W.
\]

To prove our lower bound, we construct a polytope \( \tilde{Q} = \text{conv}(\tilde{a}_1, \ldots, \tilde{a}_n) \subseteq \mathbb{R}^d \) and a two-dimensional linear subspace \( W \) such that \( \Omega(1) \cdot B^d_1 \subseteq Q \subseteq B^d_1 \), and \( Q \cap W \) has outer \( \ell_2 \)-radius \( r > 0 \) and inner \( \ell_2 \)-radius \( \frac{r}{(1 + 4 - \varepsilon)} \). Perturbing the vertices of \( Q \), we obtain \( \tilde{Q} = \text{conv}(\tilde{a}_1, \ldots, \tilde{a}_n) \), where \( a_i \sim N(\bar{a}_i, \sigma^2 I_{d \times d}) \) for each \( i \in [n] \). Note that \( Q \subseteq B^d_1 \) implies that \( \bar{a}_1, \ldots, \bar{a}_n \) satisfy the normalization requirement in [12]. With high probability the Hausdorff distance in \( \ell_1 \) between \( Q \) and \( \tilde{Q} \) is bounded by \( \max_{i \in [n]} \|a_i - \bar{a}_i\|_1 \leq O(\sigma d \sqrt{\log n}) \). Using (10), we bound the inner and outer radius of \( \tilde{Q} \cap W \). A lower bound on the number of edges of \( \tilde{Q} \cap W \) then follows from Lemma 58 as described above.

We remark that the polytopes \( Q = \text{conv}(a_1, \ldots, a_n) \subseteq \mathbb{R}^d \) with \( n = O(d) \) and two-dimensional subspaces \( W \) such that \( Q \cap W \) has inner \( \ell_2 \)-radius \( \frac{r}{1 + \frac{1}{4} - \varepsilon} \) and outer \( \ell_2 \)-radius \( r > 0 \) were first obtained by [BN01] as an extended formulation for a regular \( 2^k \)-gon with \( O(k) \) variables and \( O(k) \) inequalities. Their polytope, however, has an outer and inner radius that differ by a factor \( 2^{O(k)} \), meaning that we cannot apply Lemma 57 for \( \sigma > 2^{-k} \). We construct an alternative extended formulation where the ratio between inner and outer \( \ell_1 \)-radius is only \( O(1) \). With an appropriate scaling to get \( Q \subseteq B^d_1 \), we find that the perturbed polytope \( \tilde{Q} \) will have intersection \( \tilde{Q} \cap W \) with inner radius \( \frac{r}{1 + \frac{1}{4} - \varepsilon} (1 - 2\varepsilon/t) \) and outer radius \( (1 + \varepsilon/t) r \), where \( \varepsilon = O(\sigma d \sqrt{\log n}) \), and thus has \( \Omega(\min(\frac{1}{\sqrt{\varepsilon}}, 2^d)) \) edges, with high probability.
2 Preliminaries

We write \( \mathbf{1}_n \) for the all-ones vector in \( \mathbb{R}^n \), \( \mathbf{0}_n \) for the all-zeroes vector in \( \mathbb{R}^n \), and \( I_{n \times n} \) for the \( n \) by \( n \) identity matrix. The standard basis vectors are denoted by \( e_1, \ldots, e_n \in \mathbb{R}^n \). For a linear subspace \( W \subseteq \mathbb{R}^n \) we denote the orthogonal projection onto \( W \) by \( \pi_W \). The subspace of vectors orthogonal to a given vector \( \omega \in \mathbb{R}^n \) is denoted \( \omega^\perp \).

For a vector \( x \in \mathbb{R}^n \), the \( \ell_1 \) norm is \( \|x\|_1 = \sum_{i \in [n]} |x_i| \), the \( \ell_2 \)-norm is \( \|x\|_2 = \sqrt{\sum_{i \in [n]} x_i^2} \) and the \( \ell_\infty \)-norm is \( \|x\|_\infty = \max_{i \in [n]} |x_i| \). A norm without a subscript is always the \( \ell_2 \)-norm. Given \( p \geq 1, d \in \mathbb{Z}_+ \), define \( \mathbb{B}_p^d = \{ x \in \mathbb{R}^d : \|x\|_p \leq 1 \} \) as the \( d \)-dimensional unit ball of \( \ell_p \) norm.

We write \( [n] := \{1, \ldots, n\} \). The convex hull of vectors \( a_1, \ldots, a_n \) is denoted \( \text{conv}(a_1, \ldots, a_n) = \text{conv}(a_i : i \in [n]) \), and similarly the affine hull as \( \text{affhull}(a_i : i \in [n]) \). For sets \( A, B \subseteq \mathbb{R}^d \), the distance between the two is \( \text{dist}(A, B) = \inf_{a \in A, b \in B} \|a - b\| \). For a point \( x \in \mathbb{R}^d \) we write \( \text{dist}(x, A) = \text{dist}(A, x) = \text{dist}(A, \{x\}) \).

We say a random event happens almost surely if it occurs with probability 1.

For a convex body \( K \subseteq \mathbb{R}^d \), we define \( \partial K \subseteq \text{span}(K) \) as the boundary of \( K \) in linear subspace spanned by the vectors in \( K \).

2.1 Polytopes

Definition 5 (Polytope). A convex set \( P \subseteq \mathbb{R}^d \) is a polyhedron, if it can be expressed as \( P = \{ x \in \mathbb{R}^d : Ax \leq b \} \), for some \( A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n \) where \( n \in \mathbb{Z}_+ \). A bounded polyhedron is also called a polytope.

Definition 6 (Valid Condition and Facet). Given a polytope \( P \subseteq \mathbb{R}^d \), vector \( c \in \mathbb{R}^d \) and \( d \in \mathbb{R} \), we say the linear condition \( x^\top c \leq d \) is valid for \( P \) if the condition holds for all \( x \in P \).

A subset \( F \subseteq P \) is called a face of \( P \), if \( F = P \cap \{ x \in \mathbb{R}^d : x^\top c = d \} \neq \emptyset \), for some valid condition \( x^\top c \leq d \). A facet is a \( d-1 \)-dimensional face, a ridge is a \( d-2 \)-dimensional face, an edge is a 1-dimensional face and a vertex is a 0-dimensional face.

Definition 7 (Polar dual of a convex body). Let \( P \subseteq \mathbb{R}^d \) be a convex set. Define the polar dual of \( P \) as

\[
P^\circ = \{ y \in \mathbb{R}^d : y^\top x \leq 1, \forall x \in P \}.
\]

We state some basic facts from duality theory:

Fact 8 (Polar dual of polytope). Let \( P \subseteq \mathbb{R}^d \) be a polytope given by the linear system \( P = \{ x \in \mathbb{R}^d, Ax \leq 1_n \} \subseteq \mathbb{R}^d \) for some \( A \in \mathbb{R}^{n \times d} \). Then the polar dual of \( P \) equals to

\[
P^\circ := \text{conv}(\mathbf{0}_d, a_1, a_2, \ldots, a_n).
\]

where \( a_1, \ldots, a_n \in \mathbb{R}^d \) are the row vectors of \( A \). Moreover, \( P \) is bounded if and only if \( \mathbf{0}_d \in \text{int}(\text{conv}(a_1, \ldots, a_n)) \).

Fact 9. Let \( P, Q \subseteq \mathbb{R}^d \) be two convex sets such that \( P \subseteq Q \). Then \( Q^\circ \subseteq P^\circ \).

Fact 10. Let \( P \subseteq \mathbb{R}^d \) be a polytope, and let \( W \subseteq \mathbb{R}^d \) be any \( k \leq d \)-dimensional linear subspace. Then the polar dual of \( \pi_W(P) \), considered as a subset of the linear space \( W \), is equal to \( P^\circ \cap W \).

2.2 Probability Distributions

All probability distributions considered in this paper will admit a probability density function with respect to the Lebesgue measure.

Definition 11 (Gaussian distribution). The \( d \)-dimensional Gaussian distribution \( N_d(\bar{a}, \sigma^2 I) \) with support on \( \mathbb{R}^d \), mean \( \bar{a} \in \mathbb{R}^d \), and standard deviation \( \sigma \), is defined by the probability density function

\[
(2\pi)^{-d/2} \cdot \exp \left( -\|s - \bar{a}\|^2/(2\sigma^2) \right).
\]

at every \( s \in \mathbb{R}^d \).

A basic property of Gaussian distribution is the following strong tail bound:
Lemma 12 (Gaussian tail bound). Let \( x \in \mathbb{R}^d \) be a random vector sampled from \( \mathcal{N}_d(0, \sigma^2 I) \). For any \( t \geq 1 \) and any \( \theta \in S^{d-1} \) where \( S^{d-1} \) is the unit sphere in the \( d \)-dimensional space, we have

\[
\Pr[\|x\| \geq t \sigma \sqrt{d}] \leq \exp(-d/2(t-1)^2).
\]

From this, one can upper-bound the maximum norm over \( n \) Gaussian random vectors with mean \( 0_d \) and variance \( \sigma^2 \) by \( 4 \sigma \sqrt{d \log n} \) with dominating probability.

Corollary 13 (Global diameter of Gaussian random variables). For any \( n \geq 2 \), let \( x_1, \ldots, x_n \in \mathbb{R}^d \) be random variables where each \( x_i \sim \mathcal{N}_d(0_d, \sigma^2 I) \). Then with probability at least \( 1 - \binom{n}{d}^{-1}, \max_{i \in [n]} \|x_i\| \leq 4 \sigma \sqrt{d \log n} \).

Proof. From Lemma 12 we have for each \( i \in [n] \) that

\[
\Pr[\|x_i\| > 4 \sigma \sqrt{d \log n}] \leq \exp(-d/2(4 \log n - 1)^2) \leq \exp(-2d \log n) \leq n^{-1}. \binom{n}{d}^{-1}.
\]

Then the statement follows from the union bound. \( \square \)

A helpful technical substitute for the Gaussian distribution was introduced by [DH20]:

Definition 14 ((\( \sigma, r \))-Laplace-Gaussian distribution). For any \( \sigma, r > 0, \bar{a} \in \mathbb{R}^d \), define the \( d \)-dimensional \((\sigma, r)\)-Laplace-Gaussian distribution with mean \( \bar{a} \), or \( LG_d(\bar{a}, \sigma, r) \), if its density function is proportional to

\[
f(x) = \begin{cases} 
\exp\left(-\frac{\|x-\bar{a}\|^2}{2r^2}\right), & \text{if } \|x-\bar{a}\| \leq r \sigma \\
\exp\left(-\frac{\|x-\bar{a}\| r}{\sigma} + \frac{r^2}{2}\right), & \text{if } \|x-\bar{a}\| > r \sigma.
\end{cases}
\]

The Laplace-Gaussian random variables satisfies many desirable properties. Like the Gaussian distribution, the distance to its mean is bounded above with high probability. Moreover, its probability density is log-Lipschitz throughout its domain (as a contrast, the probability density of Gaussian distribution is only log-Lipschitz close to the expectation). The definition of L-log-Lipschitz is as follows:

Definition 15 (L-log-Lipschitz random variable). Given \( L > 0 \), we say a random variable \( x \in \mathbb{R}^d \) with probability density \( \mu \) is L-log-Lipschitz (or \( \mu \) is L-log-Lipschitz), if for all \( x, y \in \mathbb{R}^d \), we have

\[
|\log(\mu(x)) - \log(\mu(y))| \leq L \|x-y\|,
\]
or equivalently, \( \mu(x)/\mu(y) \leq \exp(L \|x-y\|) \).

Lemma 16 (Properties of Laplace-Gaussian random variables, Lemma 45 of [DH20]). Given any \( n \geq 1 \) and \( \sigma > 0 \). Let \( a_1, \ldots, a_n \in \mathbb{R}^d \) be independent random variables each sampled form \( LG_d(\bar{a}, \sigma, \sigma \sqrt{d \log n}) \) (see Definition 14). Then \( a_1, \ldots, a_n \) satisfy the follows:

1. (Log-Lipschitzness) For each \( i \in [n] \), the probability density of \( a_i \) is \((4\sigma^{-1}\sqrt{d \log n})\)-log-Lipschitz.
2. (Bounded maximum norm) With probability at least \( 1 - \binom{n}{d}^{-1}, \max_{i \in [n]} \|a_i - \bar{a}_i\| \leq 4\sigma \sqrt{d \log n} \cdot \max_{i \in [n]} \|\bar{a}_i\| \).
3. (Bounded expected radius of projection) For any \( k \leq d \), any fixed \( k \)-dimensional linear subspace \( H \subseteq \mathbb{R}^d \), we have \( \mathbb{E} \left[ \max_{i \in [n]} \|\pi_H(a_i - \bar{a}_i)\| \right] \leq 4\sigma \sqrt{k \log n} \cdot \max_{i \in [n]} \|\bar{a}_i\| \).

Most importantly, Laplace-Gaussian perturbations lead to nearly the same shadow size as Gaussian perturbations.

Lemma 17 (Lemma 46 of [DH20]). Given any \( n \geq 1 \), \( d \geq 2 \), \( \sigma > 0 \), any two-dimensional linear subspace \( W \subseteq \mathbb{R}^d \), and any \( \bar{a}_1, \ldots, \bar{a}_n \in \mathbb{R}^2 \). For every \( i \in [n] \), let \( a_i \sim \mathcal{N}_d(\bar{a}_i, \sigma) \) and \( \bar{a}_i \sim LG_d(\bar{a}_i, \sigma, 4\sigma \sqrt{d \log n}) \) be independently sampled. Then the following holds

\[
\mathbb{E} \left[ \text{edges}((\text{conv}(a_1, \ldots, a_n) \cap W)) \right] \leq 1 + \mathbb{E} \left[ \text{edges}((\text{conv}(\bar{a}_1, \ldots, \bar{a}_n) \cap W)) \right].
\]

Although [DH20] state the above lemma only for \( d \geq 3 \), their proof applies without change to the case \( d = 2 \).
2.3 Change of Variables

We will use the following change of variables, which is a standard tool in stochastic and integral geometry.

**Definition 18** (Change of variables). Let \( a_1, \ldots, a_d \) be \( d \) affine independent vectors in \( \mathbb{R}^d \). Let \( \theta \in S^{d-1}, t \in \mathbb{R} \) be such that \( \forall i \in [d], \theta^\top a_i = t \) and suppose \( \theta^\top e_1 > 1 \) without loss of generality where \( e_1 = (1, 0, \ldots, 0)^\top \in \mathbb{R}^d \) is the unit vector that has nonzero element on the first coordinate.

Fix \( h \) as any isometric embedding from \( \mathbb{R}^d \to \mathbb{R}^d \). Let \( R_\theta : \mathbb{R}^d \to \mathbb{R}^d \) denote the rotation that rotates \( e_1 \) to \( \theta \) in the two-dimensional subspace \( \text{span}(e_1, e_2) \), and is the identity transformation on \( \text{span}(e_1, \theta)^\perp \). Define \( R_\theta = R_\theta \circ h \) to be the resulting isometric embedding from \( \mathbb{R}^{d-1} \), identified with \( e_1^\perp \), to \( \theta^\perp \). Now define the transformation \( \phi \) from \( \theta, t, b \) for some constant \( \mu \) and let \( W \) be such that \( \forall b \in \mathbb{R}^d \).

\[
\phi(\theta, t, b_1, \ldots, b_d) = (R_\theta(b_1) + t\theta, \ldots, R_\theta(b_d) + t\theta) = (a_1, \ldots, a_d).
\]

**Lemma 19** (Jacobian of the transformation). Let \( \phi : S^{d-1} \times \mathbb{R} \times \mathbb{R}^{(d-1)\times d} \to \mathbb{R}^{d\times d} \) be the transformation defined in Definition 18. The inverse transformation of \( \phi \) is defined almost everywhere and has Jacobian determinant that equals to

\[
\left| \det \left( \frac{\partial \phi(a)}{\partial a} \right) \right| = C_d \cdot (d-1)! \cdot \text{vol}_{d-1}(\text{conv}(a_1, \ldots, a_d))
\]

for some constant \( C_d \) depending only on the dimension. As a consequence, if \( a_1, \ldots, a_d \) are points with probability density \( \mu(a_1, \ldots, a_d) \) and if \( \theta \in S^{d-1}, t \in \mathbb{R}, b_1, \ldots, b_d \in \mathbb{R}^{d-1} \) have probability density proportional to

\[
\text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \mu(t\theta + R_\theta(b_1), \ldots, t\theta + R_\theta(b_d))
\]

then \( \mathbb{E}[f(a_1, \ldots, a_d)] = \mathbb{E}[f(\phi(\theta, t, b_1, \ldots, b_d))] \) for any measurable function \( f \).

In particular, we will use this transformation to condition on the value of \( \theta \) and consider events in the variables \( t, b_1, \ldots, b_d \). For this purpose, we have the following fact.

**Lemma 20** (Log-Lipschitzness of the Position of Affine Hull). Let \( a_1, \ldots, a_d \in \mathbb{R}^d \) be \( d \) independent L-log-Lipschitz random variables, and let \( (\theta, t, b_1, \ldots, b_d) = \phi^{-1}(a_1, \ldots, a_d) \), where \( \phi : S^{d-1} \times \mathbb{R} \times \mathbb{R}^{(d-1)\times d} \to \mathbb{R}^{d\times d} \) is defined in Definition 18. Then conditional on the values of \( \theta, b_1, \ldots, b_d \), the random variable \( t \) is (dL-log-Lipschitz).

**Proof.** By Lemma 19, the joint probability density of \( (a_1, \ldots, a_d) \) is proportional to

\[
\text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \prod_{i=1}^d \mu_i(R_\theta(b_i) + t\theta)
\]

where \( \mu_i \) is the probability density of \( a_i \). Conditioning on \( b_1, \ldots, b_d \in \mathbb{R}^{d-1} \) and \( \theta \in S^{d-1} \), the volume \( \text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \) is fixed. The statement then follows from the fact that for each \( i \in [d], \mu_i(R_\theta(b_i) + t\theta) \) is L-log-Lipschitz in \( t \) for any \( b_i \).

2.4 Non-Degenerate Conditions

**Definition 21** (Non-degenerate polytope). A polytope \( Q = \text{conv}(a_1, \ldots, a_n) \subseteq \mathbb{R}^d \) is called non-degenerate, if it is simplicial (every facet is a simplex) and if, for \( i \in [n] \), \( a_i \in \partial Q \) implies that \( a_i \) is a vertex of \( Q \).

**Definition 22** (Non-degenerate intersection with a 2D-plane). Let \( Q \subseteq \mathbb{R}^d \) be a non-degenerate polytope and let \( W \subseteq \mathbb{R}^d \) be a two-dimensional linear subspace. We say \( Q \) has non-degenerate intersection with \( W \), if

1. the edges of the two-dimensional polygon \( Q \cap W \) have one-to-one correspondence to the facets of \( Q \) that have non-empty intersection with \( W \); and
2. the vertices of \( Q \cap W \) have one-to-one correspondence to the \( (d-2) \)-dimensional faces (ridges) of \( Q \) that have non-empty intersection with \( W \).
Fact 23 (Non-degenerate conditions of random polytope). Given any \( n \geq d \geq 2 \) and any fixed two-dimensional plane \( W \subseteq \mathbb{R}^d \), for \( a_1, \ldots, a_n \in \mathbb{R}^d \), the polytope \( Q = \text{conv}(a_1, \ldots, a_n) \) satisfies the following properties everywhere except for a set of measure 0:

1. \( Q \) is non-degenerate;
2. \( Q \) has non-degenerate intersection with \( W \);
3. For every normal vector \( v \) to any facet of \( Q \), \( e_1^\top v \neq 0 \).

Assume the polytope \( Q = \text{conv}(a_1, \ldots, a_n) \) and the two-dimensional linear subspace \( W \subseteq \mathbb{R}^d \) satisfy the non-degenerate conditions in Fact 23. Each edge of the two-dimensional polygon formed by the intersection \( W \cap Q \) can be described by a set of \( d \) vertices, where the edge is equivalent to the intersection of \( W \) with the convex hull of these \( d \) vertices. Furthermore, each vertex of \( W \cap W \) corresponds to a set of \((d - 1)\) vertices. The following lemma characterizes the relation of these sets for adjacent vertices and edges:

Fact 24 (Properties of neighboring vertices on non-degenerate intersection polygon). Let \( W \subseteq \mathbb{R}^d \) be a two-dimensional linear subspace, \( Q = \text{conv}(a_1, \ldots, a_n) \subseteq \mathbb{R}^d \) is simplicial and has non-degeneracy intersection with \( W \). Given \( J_1, J_2 \in \binom{[n]}{d-1} \), \( I \in \binom{[n]}{d} \), suppose (1) \( V_{J_1} = \text{conv}(a_j : j \in J_1) \cap W \) and \( V_{J_2} = \text{conv}(a_j : j \in J_2) \cap W \) are two adjacent vertices of \( Q \cap W \), and (2) \( \text{conv}(a_i : i \in I) \cap W \) is an edge of \( Q \cap W \) that contains \( V_{J_1} \) but not contains \( V_{J_2} \). Then we have \( |J_1 \setminus J_2| = |J_2 \setminus J_1| = 1 \) and \( |I \setminus J_2| = 2 \).

Proof. Let \( I' = J_1 \cup J_2 \). Then \( \text{conv}(V_{J_1}, V_{J_2}) = \text{conv}(a_i : i \in I) \cap W \) is an edge of the polygon \( Q \cap W \). Since \( Q \) has non-degenerate intersection with \( W \), we have that \( |I'| = d \). Combining with \( |J_1| = |J_2| = d - 1 \) gives us that \( |J_1 \setminus J_2| = |J_2 \setminus J_1| = 1 \).

Next we consider \( |I \setminus J_2| \). Since \( J_1 \subseteq I \) and \( |J_1 \setminus J_2| = 1 \), it could only be the case that \( |I \setminus J_2| \in \{1, 2\} \). If \( |I \setminus J_2| = 1 \), then by \( |I| = |J_2| + 1 \) we must have \( J_2 \subseteq I \), but this contradicts to the fact that \( \text{conv}(J_2) \not\subseteq \text{conv}(I) \). Therefore we could only have \( |I \setminus J_2| = 2 \). \( \square \)
3 Smoothed Complexity Upper Bound

In this section, we establish our key theorem for upper bounding the number of edges of a random polygon \( \text{conv}(a_1, \ldots, a_n) \cap W \) for \( W \) a fixed 2-dimensional linear subspace and \( a_1, \ldots, a_n \in \mathbb{R}^d \). We demonstrate that if for any edge on the shadow polygon \( \text{conv}(a_1, \ldots, a_n) \cap W \), the expected distance between the affine hull of the edge and the next vertex on the shadow is sufficiently large in expectation, then the expected number of edges of \( \text{conv}(a_1, \ldots, a_n) \cap W \) can be upper-bounded.

**Definition 25** (Facet and edge event). For \( I \subseteq [n] \), we write \( F_I = \text{conv}(a_i : i \in I) \). Define \( E_I \) to be the event that both \( F_I \) is a facet of \( \text{conv}(a_1, \ldots, a_n) \) and \( F_I \cap W \neq \emptyset \).

**Remark 26.** Any edge \( e \) of \( \text{conv}(a_1, \ldots, a_n) \cap W \) can be written as \( e = F_I \cap W \) for some \( I \subseteq [n] \) for which \( E_I \) holds. Assuming non-degeneracy, this relation between edges and index sets is a one-to-one correspondence, and moreover every \( I \subseteq [n] \) for which \( E_I \) holds satisfies \( |I| = d \).

To state the key theorem’s assumption, we need a concept of ‘clockwise’ to characterize the order of edges and vertices on the shadow polygon.

**Definition 27** (Clockwise order of edges and vertices). For any given two-dimensional linear subspace \( W \subseteq \mathbb{R}^d \) we denote an arbitrary but fixed rotation as “clockwise”. For the polygon \( \text{conv}(a_1, \ldots, a_n) \cap W \) of our interest, let \( p_1, \ldots, p_k \) denote its vertices in clockwise order and write \( p_{k+1} = p_1, p_{k+2} = p_2 \). Then for any edge \( e = [p_{i-1}, p_i] \), we call \( p_i \) its second vertex in clockwise order and we call \( p_{i+1} \) the next vertex after \( e \) in clockwise order. The edge \( [p_i, p_{i+1}] \) is the next edge after \( e \) in clockwise order.

Note that the above terms are well-defined in the sense that they depend only on the polygon and the orientation of the subspace, not on the vertex labels. With this definition in place, we can now state the theorem itself:

**Theorem 28** (Smoothed complexity upper bound for continuous perturbations). Fix any \( n, d \geq 2, \sigma \geq 0 \), and any two-dimensional linear subspace \( W \subseteq \mathbb{R}^d \). Let \( a_1, \ldots, a_n \in \mathbb{R}^d \) be independently distributed each according to a continuous probability distribution.

For any \( I \in \binom{[n]}{d} \), conditional on \( E_I \), define \( y_I \in W \) as the outer unit normal of the edge \( F_I \cap W \). For \( \gamma > 0 \), suppose that for each \( I \in \binom{[n]}{d} \) such that \( \Pr[E_I] \geq 10\left(\frac{n}{d}\right)^{-1} \) we have

\[
\Pr[y_I^\top p_2 - y_I^\top p_3 \geq \gamma \mid E_I] \geq 0.1,
\]

where we write \( [p_1, p_2] = F_I \cap W \) and \( p_3 \in \text{conv}(a_1, \ldots, a_n) \cap W \) as the next vertex after \( F_I \cap W \) in clockwise order. Then we have

\[
\mathbb{E}[\text{edges}(\text{conv}(a_1, \ldots, a_n))] \leq 10 + 80\pi \sqrt{\mathbb{E}[\max_{i \in [n]} \|\pi_W(a_i)\|]} / \gamma = O\left(\sqrt{\mathbb{E}[\max_{i \in [n]} \|\pi_W(a_i)\|]} / \gamma\right).
\]

Note that, assuming non-degeneracy, \( y_I \) is well-defined if and only if \( E_I \) happens. In this case, we are guaranteed that \( y_I^\top p_2 - y_I^\top p_3 > 0 \).

To prove the above theorem, we show that any \( I \in \binom{[n]}{d} \) with \( \Pr[E_I] \geq \left(\frac{n}{d}\right)^{-1} \) can be charged to either a portion of the perimeter of the polygon \( \text{conv}(a_1, \ldots, a_n) \cap W \) or to a portion of its sum \( 2\pi \) of exterior angles at its vertices.

**Definition 29** (Exterior angle and length of the next edge). Given any \( I \in \binom{[n]}{d} \), we define two random variables \( \theta_I, \ell_I^+ \geq 0 \). If \( E_I \) happens, write \( v \in F_I \cap W \) for the second endpoint of \( F_I \cap W \) in clockwise order. Let \( \theta_I \) to be the (two-dimensional) exterior angle of \( \text{conv}(a_1, \ldots, a_n) \cap W \) at \( v \); if \( E_I \) doesn’t happen then let \( \theta_I = 0 \).

Let \( \ell_I^+ \) denote the following random variable: If \( E_I \) happens, then \( \ell_I^+ \) equals the length of the next edge after \( F_I \cap W \) in clockwise order, i.e., the other edge of \( \text{conv}(a_1, \ldots, a_n) \cap W \) containing \( v \). If \( E_I \) doesn’t happen then let \( \ell_I^+ = 0 \).
Proof of Theorem 28. Since we have non-degeneracy with probability 1, by Fact 23 and linearity of expectation we find

\[ E[\text{edges} (\text{conv}(a_1, \ldots, a_n) \cap W)] = \sum_{t \in \binom{n}{d}} \Pr[E_t]. \]

We can give an upper bound on the expected number of edges of \( \text{conv}(a_1, \ldots, a_n) \cap W \) by upper-bounding each \( \Pr[E_t] \). Fix any \( t \in \binom{n}{d} \) and let \( t > 0 \) be a parameter to be determined later. We consider three different possible upper bounds on \( \Pr[E_t] \), at least one of which will always hold:

**Case 1:** \( \Pr[E_t] \leq 10 \binom{n}{d}^{-1} \).

Since \( \sum_{t \in \binom{n}{d}} 10 \binom{n}{d}^{-1} = 10 \), one can immediately see that the total contribution of edges counted in this case is at most 10.

**Case 2:** \( \Pr[E_t] > 10 \binom{n}{d}^{-1} \) and \( \Pr[\ell_{t+} \geq t \mid E_t] \geq \frac{1}{20} \).

In this case, \( \mathbb{E}[\ell_{t+} \mid E_t] \geq \frac{1}{20} \), therefore we obtain from \( \mathbb{E}[\ell_{t+} \mid E_t] \Pr[E_t] \) that

\[ \Pr[E_t] = \frac{\mathbb{E}[\ell_{t+}]}{\mathbb{E}[\ell_{t+} \mid E_t]} \leq \frac{20}{t} \cdot \mathbb{E}[\ell_{t+}]. \]

**Case 3:** \( \Pr[E_t] > 10 \binom{n}{d}^{-1} \) and \( \Pr[\ell_{t+} \leq t \mid E_t] \geq \frac{10}{20} \).

Conditional on \( E_t \), without loss of generality we write \( [p_1, p_2] = E_t \cap W \) and let \( p_3 \) denote the next vertex after \( E_t \cap W \) in clockwise direction. From the theorem’s assumption we have \( \Pr[\text{dist}(\text{affhull}(p_1, p_2), p_3) \geq \gamma \mid E_t] \geq \frac{1}{10} \). Then from the union bound,

\[ \Pr[\left( \ell_{t+} \leq t \right) \land (\text{dist}(\text{affhull}(p_1, p_2), p_3) \geq \gamma) \mid E_t] \]

\[ \geq 1 - \Pr[\ell_{t+} > t \mid E_t] - \Pr[\text{dist}(\text{affhull}(p_1, p_2), p_3) < \gamma \mid E_t] \geq \frac{1}{20}. \]

Since \( \theta_t \geq 0 \) and

\[ \theta_t \geq \sin(\theta_t) = \frac{\text{dist}(\text{affhull}(p_1, p_2), p_3)}{\ell_{t+}} \]

we have \( \mathbb{E}[\theta_t \mid E_t] \geq \frac{1}{20} \cdot \frac{7}{7} \), and therefore we can upper bound \( \Pr[E_t] \) by

\[ \Pr[E_t] = \frac{\mathbb{E}[\theta_t]}{\mathbb{E}[\theta_t \mid E_t]} \leq \frac{20t}{\gamma} \cdot \mathbb{E}[\theta_t]. \]

Readers are referred to Figure 1 for more illustration of the proof.

Combining the upper bounds for each \( \Pr[E_t] \) for the above three cases, we get that

\[ \mathbb{E}[\text{edges} (\text{conv}(a_1, \ldots, a_n) \cap W)] = \sum_{t \in \binom{n}{d}} \Pr[E_t] \]

\[ \leq \sum_{t \in \binom{n}{d}} \left( 10 \binom{n}{d}^{-1} + \frac{20}{t} \cdot \mathbb{E}[\ell_{t+}] + \frac{20t}{\gamma} \cdot \mathbb{E}[\theta_t] \right) \]

\[ = 10 + \frac{20}{t} \cdot \mathbb{E}[\sum_{t \in \binom{n}{d}} \ell_{t+}] + \frac{20t}{\gamma} \cdot \mathbb{E}[\sum_{t \in \binom{n}{d}} \theta_t] \]

(13)

To upper bound the second term of (13), we notice that \( \sum_{t \in \binom{n}{d}} \ell_{t+} \) exactly equals the perimeter of \( \text{conv}(a_1, \ldots, a_n) \cap W \). Since the shadow polygon \( \text{conv}(a_1, \ldots, a_n) \cap W \) is contained in the two-dimensional disk of radius \( \max_{i \in [n]} ||\pi_W(a_i)|| \), by the monotonicity of surface area for convex sets we have

\[ \mathbb{E}[\sum_{t \in \binom{n}{d}} \ell_{t+}] \leq 2\pi \cdot \mathbb{E}[\max_{i \in [n]} ||\pi_W(a_i)||]. \]
To upper bound the third term of (13), we notice that the sum of exterior angles for any polygon always equals $2\pi$. Thus

$$\mathbb{E}[\sum_{I \in \binom{[n]}{3}} \theta_I] = 2\pi$$  \hfill (14)

Finally, we combine (13 - 14) and minimize over all $t \geq 0$:

$$\mathbb{E}[\text{edges (conv}(a_1, \ldots, a_n) \cap W)] \leq \min_{t > 0} \left( 10 + \frac{40\pi \mathbb{E}[\max_{i \in [n]} \|\pi_W(a_i)\|]}{t} + \frac{40\pi t}{\gamma} \right)$$

$$= 10 + 80\pi \sqrt{\frac{\mathbb{E}[\max_{i \in [n]} \|\pi_W(a_i)\|]}{\gamma}}.$$

where in the final step, we set $t = \sqrt{\gamma \mathbb{E}[\max_{i \in [n]} \|\pi_W(a_i)\|]}$. \hfill $\square$

In the subsequent sections, we will show a lower bound for the edge-to-vertex distance $\gamma$ assuming the independently distributed vectors $a_1, \ldots, a_n$ follow Laplace-Gaussian distributions. This allows us to directly apply Theorem 28 to derive an upper bound on the expected number of edges of $\text{conv}(a_1, \ldots, a_n) \cap W$. Furthermore, by using Lemma 17, we can further reduce our upper bound to the case when $a_1, \ldots, a_n$ are Gaussian distributed vectors.
4 Upper Bound for Two Dimension

In this section, we establish the smoothed complexity upper bound for \( d = 2 \). For this scenario, the shadow plane \( W \) encompasses the entire two-dimensional Euclidean space, and \( P \cap W \) is identical to \( P = \text{conv}(a_1, \ldots, a_n) \). From Theorem 28 and Lemma 17, it remains to lower bound the distance from the affine hull of an edge to its neighboring vertex in clockwise order (denoted by \( \gamma \) in Theorem 28), where the polygon \( \text{conv}(a_1, \ldots, a_n) \) is under Laplace-Gaussian perturbation. We will demonstrate a slightly stronger result: a lower bound for the distance between the affine hull of an edge to all of the remaining \( (n - 2) \) vertices.

Lemma 30 (Edge-to-vertex distance in Two Dimension). Let \( a_1, \ldots, a_n \in \mathbb{R}^2 \) be \( n \) independent L-log-Lipschitz random variables. Then for any \( I \subseteq \binom{[n]}{2} \), conditional on \( E_I \) happening, the outer unit normal \( y \in W \) of the edge \( \text{conv}(a_i : i \in I) \) satisfies

\[
\Pr[y^\top a_i - \max_{j \notin I} y^\top a_j \geq \frac{1}{L} \mid E_I] \geq 0.1,
\]

for any \( i \in I \).

Together with Theorem 28, Lemma 17 and the Laplace-Gaussian tail bound Lemma 16, we find the upper bound for two-dimensional polygons under Gaussian perturbation:

Theorem 31 (Two-Dimensional Upper Bound). Let \( \tilde{a}_1, \ldots, \tilde{a}_n \in \mathbb{R}^2 \) be \( n > 2 \) vectors with norm at most 1. For each \( i \in [n] \), let \( a_i \) be independently distributed as \( \mathcal{N}_2(\hat{a}_i, \sigma^2 I) \). Then

\[
E[\text{edges (conv}(a_1, \ldots, a_n))] \leq O\left(\frac{\sqrt{\log n}}{\sqrt{\sigma}} + \sqrt{\log n}\right).
\]

Proof. For each \( i \in [n] \), let \( \hat{a}_i \) be independently sampled form the 2-dimensional Laplace-Gaussian distribution \( LG_2(\hat{a}_i, \sigma, 4\sigma\sqrt{2\log n}) \). It follows from Lemma 16 that \( \hat{a}_i \) is \((4\sigma^{-1}\sqrt{2\log n})\)-log-Lipschitz and \( E[\max_{i \in [n]} \|\hat{a}_i\|] \leq 1 + 4\sigma\sqrt{2\log n} \). We use Lemma 30 by setting \( L = 4\sigma^{-1}\sqrt{2\log n} \), and Theorem 28 by setting \( \gamma = \frac{1}{L} = \frac{\sigma}{4\sqrt{2\log n}} \), to find

\[
E[\text{edges (conv}(\tilde{a}_1, \ldots, \tilde{a}_n))] \leq O\left(\frac{\sqrt{\log n}}{\sqrt{\sigma}} + \sqrt{\log n}\right).
\]

Finally, from Lemma 17, we conclude that \( E[\text{edges (conv}(a_1, \ldots, a_n))] \leq 1 + O\left(\frac{\sqrt{\log n}}{\sqrt{\sigma}} + \sqrt{\log n}\right) \). \( \square \)

Proof of Lemma 30. Fix any set \( I = \{i, i'\} \subseteq [n] \). Define \( z \in S^1 \) and \( t \) to satisfy \( z^\top a_i = z^\top a_{i'} = t \) and \( z^\top e_1 > 0 \). Both are well-defined with probability 1.

Note that \( E_I \) is now equivalent to either having \( z^\top a_j < t \) for all \( j \notin I \) or having \( z^\top a_j > t \) for all \( j \notin I \). Write \( E_I^+ \) for the former case and \( E_I^- \) for the latter. The vector \( z \) is always defined, assuming non-degeneracy, and is equal to the outer normal unit vector \( y \) if \( \tilde{E}_i^+ \) and equal to \( -y \) if \( \tilde{E}_i^- \).

Using Fubini’s theorem, we condition on the values of \( a_j, j \notin I \) and \( z \) using Lemma 19. Let \( \mu : \mathbb{R} \to \mathbb{R}_{\geq 0} \) denote the induced density of \( t = y^\top a_i = y^\top a_{i'} \). Then from Lemma 20, \( \mu \) is \((2L)\)-log-Lipschitz.

In the first case, for \( E_I^+ \), we have, still only considering the randomness over \( t \),

\[
\Pr[(t - \max_{j \notin I} z^\top a_j \geq \frac{1}{L}) \land E_I^+ \mid a_j, j \notin I, z] = \int_{\max_{j \notin I} z^\top a_j + 1/L}^{\infty} \mu(x)dx = \int_{\max_{j \notin I} z^\top a_j}^{\infty} \mu(z + 1/L)dx \geq \int_{\max_{j \notin I} z^\top a_j}^{\infty} e^{-2\mu(z)}dz \quad (\text{By (2L)-log-Lipschitzness of } \mu)
\]

\[
= e^{-2} \Pr[E_I^+].
\]
Similarly for the other case, $E_I^-$, we find

$$\Pr[(\min_{j \not\in I} z^\top a_j - t \geq \frac{1}{L}) \land E_I^-] \geq e^{-2} \Pr[E_I^-].$$

Now observe that, for $i \in I$,

$$\Pr[y^\top a_i - \max_{j \not\in I} y^\top a_j \geq \frac{1}{L} \land E_I \mid a_j, j \not\in I]$$

$$= \Pr[(t - \max_{j \not\in I} z^\top a_j \geq \frac{1}{L}) \land E_I] + \Pr[(\min_{j \not\in I} z^\top a_j - t \geq \frac{1}{L}) \land E_I^-]$$

$$\geq e^{-2} \Pr[E_I^+] + e^{-2} \Pr[E_I^-] = e^{-2} \Pr[E_I].$$

This finishes the proof since

$$\Pr[y^\top a_i - \max_{j \not\in I} y^\top a_j \geq \frac{1}{L} \mid E_I] = \Pr[y^\top a_i - \max_{j \not\in I} y^\top a_j \geq \frac{1}{L} \land E_I] / \Pr[E_I] \geq e^{-2} \geq 0.1.$$
5 Multi-Dimensional Upper Bound

In this section, we establish the upper bound for the higher-dimensional case (i.e. \(d \geq 3\)):

Theorem 32 (Multi-dimensional Upper Bound). Given any \(d > 2, n \geq d\), and \(\sigma \leq \frac{1}{8d\sqrt{\log n}}\). Let \(\tilde{a}_1, \ldots, \tilde{a}_n\) be \(n\) vectors with \(\max_{i \in [n]} \|\tilde{a}_i\| \leq 1\). For each \(i \in [n]\), let \(a_i\) be independently distributed as \(\mathcal{N}_d(\tilde{a}_i, \sigma^2 I)\). Then

\[
E[\text{edges}(\text{conv}(a_1, \ldots, a_n) \cap W)] = O\left(\sigma^{-3/2}d^{13/4} \log^{7/4} n\right).
\]  

Similar to the two-dimensional case (see Section 4), the main technical ingredient of Theorem 32 is a lower-bound of the edge-to-vertex distance (the quantity \(E\)).

Lemma 33 (Edge-to-vertex distance of shadow polygon in multi-dimension). For any \(d \geq 3\), let \(a_1, \ldots, a_n \in \mathbb{R}^d\) be independent L-log-Lipschitz random variables. For any \(I \in \binom{[n]}{d}\) that satisfies \(\Pr[E_I] \geq 10^{-\gamma} \cdot \sqrt{n}\), (where \(E_I\) is defined in Definition 27), we have

\[
\Pr[y^\top p - y^\top p' \geq \Omega\left(\frac{1}{L^3d^5 \log n}\right) | E_I] \geq 0.1,
\]

where \(p\) is any point in \(F_I \cap W\), and \(p' \in \text{conv}(a_1, \ldots, a_n) \cap W\) is the next vertex after \(F_I \cap W\) in clockwise direction. Here \(y \in W\) is the outer unit normal to the edge \(F_I \cap W\) on \(\text{conv}(a_1, \ldots, a_n) \cap W\).

Theorem 32 then immediately follows from Lemma 33, Theorem 28, and Lemma 17.

Proof of Theorem 32. For each \(i \in [n]\), let \(\hat{a}_i\) be independently sampled from the Laplace-Gaussian distribution \(LG_d(\tilde{a}_i, \sigma, 4\sqrt{d} \log n)\). From Lemma 16, we know that

1. Each \(\hat{a}_i\) is \(L = (4\sigma^{-1}\sqrt{d} \log n)\)-log-Lipschitz;
2. \(\mathbb{E}[\max_{i \in [n]} \|\pi_W(\hat{a}_i)\|] \leq 1 + 4\sigma \sqrt{2\log n} \leq 1.5\).

From Lemma 33, we get that for any \(p \in F_I \cap W\), if \(p'\) is the next vertex after the edge \(F_I \cap W\) in clockwise order, then

\[
\Pr[y^\top p \geq y^\top p' + \Omega\left(\frac{1}{L^3d^5 \log^2 n}\right)] \geq 0.1.
\]

Here \(y_I \in W\) is the outer unit normal vector of the polygon \(\text{conv}(\hat{a}_1, \ldots, \hat{a}_n) \cap W\) on the edge \(F_I \cap W\). Then we can use Theorem 28 by setting \(L = 4\sigma^{-1}\sqrt{d} \log n\), \(\gamma = \Omega\left(\frac{1}{L^3d^5 \log n}\right)\) and \(\mathbb{E}[\max_{i \in [n]} \|\pi_W(a_i)\|] = 1.5\), to find

\[
\mathbb{E}[\text{edges}(\text{conv}(\hat{a}_1, \ldots, \hat{a}_n))] \leq 10 + O\left(\sqrt{\sigma^{-3}d^{13/2} \log^{7/2} n}\right).
\]

Finally, from Lemma 17 we conclude that

\[
\mathbb{E}[\text{edges}(\text{conv}(a_1, \ldots, a_n) \cap W)] \leq 11 + O\left(\sqrt{\sigma^{-3}d^{13/2} \log^{7/2} n}\right)
= O\left(\sigma^{-3/2}d^{13/4} \log^{7/4} n\right).
\]

The rest of this section will be structured as follows. In section 5.1 we define some basic notations that will be used in the proof. In section 5.2 we provide a deterministic argument to establish sufficient criteria for the conclusion of Lemma 33 to hold. In section 5.3 and section 5.4 we prove that these conditions hold with good probability conditional on \(E_I\). The proof of Lemma 33 is then finished in section 5.5.
5.1 Notations

Since we assume that \(a_1, \ldots, a_n\) each have a continuous probability density function, \(\text{conv}(a_1, \ldots, a_n)\) and \(W\) satisfy the non-degenerate conditions (see Fact 23) almost surely. In this case, each edge of the polygon \(\text{conv}(a_1, \ldots, a_n) \cap W\) is given by a \(F_I \cap W = \text{conv}(a_i : i \in I)\) for which \(I \in \binom{[n]}{d}\) and \(E_I\) holds (where \(F_I\) and \(E_I\) are defined in Definition 25). In addition, each vertex of the polygon \(\text{conv}(a_1, \ldots, a_n) \cap W\) is given by the intersection between \(W\) and \((d-2)\)-dimensional ridges of \(\text{conv}(a_1, \ldots, a_n)\), which are convex hulls of \((d-1)\) vertices of \(\text{conv}(a_1, \ldots, a_n)\). We define the following notations of ridge and the corresponding vertex:

**Definition 34** (Ridge and vertex event). For any \(J \subseteq [n]\), write \(R_J = \text{conv}(a_j : j \in J)\). Define \(A_J\) to be the event that \(R_J\) is a ridge of \(\text{conv}(a_1, \ldots, a_n)\) and \(R_J \cap W \neq \emptyset\).

**Remark 35.** Any vertex \(v\) of \(\text{conv}(a_1, \ldots, a_n)\) can be written as \(v = R_J \cap W\) for some \(J \subseteq [n]\) for which \(A_J\) holds. Assuming non-degeneracy, each \(J\) for which \(A_J\) holds satisfies \(|J| = d-1\) and the relation between vertices and index sets \(J \in \binom{[n]}{d-1}\) with \(A_J\) is a one-to-one correspondence.

5.2 Deterministic Conditions for a Good Edge-to-Vertex Separator

In this subsection, we present a series of sufficient conditions that an edge on the polygon \(\text{conv}(a_1, \cdots, a_n) \cap W\) maintains a significant distance to its next vertex in the clockwise order.

**Lemma 36.** Let \(W \subseteq \mathbb{R}^d\) be a two-dimensional linear subspace, \(Q = \text{conv}(a_1, \ldots, a_n) \subseteq \mathbb{R}^d\) be a non-degenerate polytope with a non-degenerate intersection with \(W\) such that \(\max_{1 \leq j \leq n} |a_j-a_1| \leq 3\) and \(W \cap Q \neq \emptyset\). Fix any facet \(F\) of \(Q\) such that \(F \cap W \neq \emptyset\) and any ridge \(R \subseteq F\) of \(F\) such that \(W \cap R\) is a singleton set \(\{p\}\). Let \(\delta, r \geq 0\) be such that

1. (distance between \(F\) and other vertices) \(\forall a_k \notin F, \text{dist}(\text{aff}(F), a_k) \geq \delta;\)
2. (Inner radius of \(R\)) \(\text{dist}(F \cap W, \partial R) \geq r.\)

Then the outer unit normal vector \(\bar{\theta} \in W\) to the edge \(F \cap W\) satisfies

\[
\bar{\theta}^\top p - \bar{\theta}^\top p' \geq \delta r/3
\]

for any \(p \in F \cap W\), here \(p' \in Q \cap W\) is the next vertex after \(F \cap W\) in clockwise order.

We remark that Lemma 36 only provides a deterministic characterization of \((a_1, \cdots, a_n)\)'s that result in large edge-to-vertex distance. However, to show that such edge-to-vertex distance exist with good probability, we need to further explore the characteristics of \((a_1, \cdots, a_n)\)’s distributions.

**Proof.** Write \(R\) for the ridge of \(Q\) such that \(\{p'\} = R \cap W\). Since \(p' \in Q \cap W\) is adjacent to vertex \(p\) and the edge \(F \cap W\), by Fact 24 we may relabel the \(a_i\) such that \(R' = \text{conv}(a_1, \ldots, a_{d-1})\), \(R = \text{conv}(a_2, \ldots, a_d)\), and \(F = \text{conv}(a_2, \ldots, a_{d+1})\) without loss of generality. Let \(\bar{\theta} \in S^{d-1}\) denote the outward unit normal to \(F\). This normal vector satisfies

\[
\delta \leq \min_{i \in [n]} \theta^\top (p-a_i) \leq \theta^\top (p-a_1).
\]

Let \(s \in S^{d-1}\) be the unit vector indicating the direction of the (one-dimensional) line \(F \cap W\). This vector is unique up to sign. Also, let \(\bar{s} = \pi_W(\theta)/\|\pi_W(\theta)\|\) be the outward unit normal to \(F \cap W\) in the two-dimensional plane \(W\). Notice that \(\bar{\theta}\) and \(s\) form an orthonormal basis of \(W\). Therefore we get

\[
\bar{\theta}^\top (p-p') = \bar{\theta}^\top \pi_{s^\perp} (p-p') = \|\pi_{s^\perp} (p-p')\|
\]

(16)

Here the last equality comes from \((p-p') \in W = \text{span}(\bar{\theta}, s)\), thus \(\pi_{s^\perp} (p-p') = \pi_{\bar{\theta}} (p-p')\).

Now we focus on the \((d-1)\)-dimensional space \(s^\perp\), and consider the projections \(\pi_{s^\perp}(a_1), \ldots, \pi_{s^\perp}(a_d)\). Since the diameter of \(\text{conv}(a_1, \ldots, a_d)\) is at most 3, we have \(\max_{1 \leq j \leq d} \|\pi_{s^\perp}(a_j) - \pi_{s^\perp}(a_j)\| \leq 3\). Because \(\bar{\theta}\) is a unit normal vector of \(R\) and \(\theta \in s^\perp\), we know that \(\bar{\theta}\) is also a unit normal of \(\pi_{s^\perp}(R) = \pi_{s^\perp}(\text{conv}(a_2, \ldots, a_d))\). This gives

\[
\text{dist}(\pi_{s^\perp}(a_1), \text{aff}(\pi_{s^\perp}(R))) = \theta^\top (p-a_1) \geq \delta.
\]
Also, since \( \text{dist}(F \cap W, \partial R) \geq r \) where \( F \cap W = \{ p + st : t \in \mathbb{R} \} \) is one-dimensional. After the projection to \( s^\perp \) we have 
\[
\text{dist}(\pi_{s^\perp}(p), \partial \pi_{s^\perp}(R)) = \text{dist}(F \cap W, \partial R) \geq r.
\]
Therefore we can use Lemma 37 to get 
\[
\|\pi_{s^\perp}(p) - \pi_{s^\perp}(p')\| \geq \text{dist}(\pi_{s^\perp}(p), \text{affhull}(\pi_{s^\perp}(R'))) \geq r\delta/3,
\]
where the first step comes from \( \pi_{s^\perp}(p') \in \text{affhull}(\pi_{s^\perp}(R')) \). The lemma then follows from (16).

**Lemma 37.** Given \( b_1, \ldots, b_d \in \mathbb{R}^{d-1} \) such that \( \text{conv}(b_1, \ldots, b_d) \) is non-degenerate. Suppose
\begin{enumerate}
\item \( \forall i, j \in [d], \|b_1 - b_j\| \leq 3; \)
\item \( \text{dist}(b_1, \text{affhull}(b_2, \ldots, b_d)) \geq \delta; \)
\item There exists \( q \in \text{conv}(b_2, \ldots, b_d) \) such that \( \text{dist}(q, \partial(\text{conv}(b_2, \ldots, b_d))) \geq r. \)
\end{enumerate}
Then we have \( \text{dist}(q, \text{affhull}(b_1, \ldots, b_{d-1})) \geq r\delta/3. \)

**Proof.** For simplicity, write \( B = \text{conv}(b_2, \ldots, b_d) \) and \( B' = \text{conv}(b_1, \ldots, b_{d-1}) \). Let \( q' = \pi_{B'}(q) \) be the point closest to \( q \) on \( \text{affhull}(B') \), i.e. \( \|q - q'\| = \text{dist}(q, \text{affhull}(b_1, \ldots, b_{d-1})) \).

Let \( x = (B \cap B') \cap \text{affhull}(b_1, q, q') \) be its intersection between the two-dimensional plane \( \text{affhull}(b_1, q, q') \) and the \((d - 3)\)-dimensional ridge \( B \cap B' \) (which gives a unique point). (See Figure 2 for an illustration). Consider the triangle \( \text{conv}(b_1, q, x) \) and calculate its area in two different ways. On one hand, it has base \( \text{conv}(b_1, x) \) of length \( \|b_1 - x\| \leq 3 \) with height \( \text{dist}(q, \text{affhull}(b_1, x)) = \|q - q'\| \), which gives that the area of the triangle is at most \( 3\|q - q'\|/2 \). On the other hand, this triangle has base \( \text{conv}(x, q) \) of length \( \|x - q\| \geq \text{dist}(q, \partial(B)) \geq r \) with height \( \text{dist}(b_1, \text{affhull}(x, q)) \geq \text{dist}(b_1, B) \geq \delta \), which gives that the area of the triangle is at least \( r\delta/2 \).

Combining the above two ways of determining the area of triangle \( \text{conv}(b_1, q, x) \), we have \( 3\|q - q'\|/2 \geq r\delta/2 \). Therefore we have \( \text{dist}(q, \text{affhull}(B')) = \|q - q'\| \geq r\delta/3 \) as desired.

![Figure 2: Illustration of Lemma 37 when \( d - 1 = 3 \). In gray is the intersection between the two-dimensional plane \( \text{affhull}(b_1, q, q') \) and \( \text{conv}(b_1, \ldots, b_d) \). The red triangle is \( \text{conv}(b_1, q, x) \). The bottom face is \( B \) and the back face is \( B' \).](image)

### 5.3 Randomized Lower-Bound for \( \delta \): Distance between vertices and facets

In this section, we show that the affine hull of a given facet \( F \) of the polytope \( \text{conv}(a_1, \ldots, a_n) \) is \( \Omega(\frac{1}{Ld \log n}) \)-far away to other vertices with good probability, or in other words, the distance \( \delta \) in Lemma 30 is at least \( \Omega(\frac{1}{Ld \log n}) \) with good probability. Our main result of this section is as follows:
Lemma 38 (Randomized lower-bound for δ). Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be independent $L$-log-Lipschitz random vectors. For any $I \in \binom{[n]}{d}$ such that $\Pr[E_I] \geq 10(n_d)^{-1}$, we have

$$
\Pr[\min_{k \in [n] \setminus I} \text{dist}(\text{aff}(F_I), a_k) \geq \frac{1}{10e^3dL \log n}] \geq 0.72.
$$

To show Lemma 38, we fix any $I \in \binom{[n]}{d}$ of consideration. Without loss of generality, assume $I = [d]$ and write $E = E_{[d]}$. We define the following event $B_\varepsilon$ indicating that the distance from $F_{[d]}$ to other vertices is at least $\varepsilon$.

Definition 39 (Separation by the margin of a facet). Let $\theta \in \mathbb{S}^{d-1}, t \in \mathbb{R}$ be as in Definition 18. For any $\varepsilon > 0$, let $B_\varepsilon^+$ denote the event that $\theta^\top a_i < t - \varepsilon$ for all $i \in [n] \setminus [d]$ and $B_\varepsilon^-$ denote the event that $\theta^\top a_i > t + \varepsilon$ for all $i \in [n] \setminus [d]$. We write $B_\varepsilon = B_\varepsilon^+ \lor B_\varepsilon^-$. In the following lemma, we show that for sufficiently small $\varepsilon$, $\Pr[E \land B_\varepsilon]$ is still a constant fraction of $\Pr[E]$.

Lemma 40. For any $0 < \varepsilon \leq \frac{1}{10e^3dL \log n}$ it holds that

$$
\Pr[E] \leq \left(\frac{n}{d}\right)^{-1} + \frac{5}{4} \Pr[E \land B_\varepsilon].
$$

Proof. Writing random variables as subscripts to denote which expectation is over which variables, we start by using Fubini’s theorem to write

$$
\Pr_{a_1, \ldots, a_n}[E] = \mathbb{E}_{a_1, \ldots, a_d}[\Pr_{a_{d+1}, \ldots, a_n}[E]].
$$

Fix any $a_1, \ldots, a_d \in \mathbb{R}^n$ subject to the non-degeneracy assumptions in Fact 23 and $\text{conv}(a_1, \ldots, a_d) \cap W \neq \emptyset$. Define $\theta \in \mathbb{S}^{d-1}, t > 0$ as described in Definition 18, i.e. $\theta^\top a_i = t$ for each $i \in [d]$. Write $s_i = \theta^\top a_i$ for each $i \in [n] \setminus [d]$. We note that $s_i$ is an $L$-log-Lipschitz random variable for all $i \in [n] \setminus [d]$. Moreover, over the remaining randomness in $a_{d+1}, \ldots, a_n$, we have $\Pr[E] = \Pr[B_0^+] + \Pr[B_0^-]$ and $\Pr[B_\varepsilon] = \Pr[B_\varepsilon^+] + \Pr[B_\varepsilon^-]$. We will show that $\Pr[B_\varepsilon^+] \leq \frac{1}{2(n_d)} + \frac{3}{4} \Pr[B_0^+]$, and the appropriate statement will follow for $B_\varepsilon^-$ analogously. Putting together this will prove the lemma.

If $\Pr[B_0^+] \leq \frac{1}{2}(n_d)^{-1}$ then the desired inequality holds directly. Otherwise, fix any $i \in [n] \setminus [d]$ and let $\mu_i$ denote the induced probability density function of $s_i$. We then have

$$
\Pr[s_i \geq t - \varepsilon \mid s_i \leq t] = \frac{\int_0^t \mu_i(t + s)ds}{\int_{-\infty}^0 \mu_i(t + s)ds} = \frac{\varepsilon L \int_{-1/L}^0 \mu_i(t + \varepsilon Ls)ds}{\int_{-\infty}^0 \mu_i(t + s)ds} \leq e L \int_{-1/L}^0 \mu_i(t + s)ds \leq e L \int_{-1/L}^0 \mu_i(t + s - 1/L)ds \leq e^2 \varepsilon L \Pr[s_i \geq t \mid s_i \leq t + 1/L] \leq e^3 \varepsilon L \Pr[s_i \geq t].
$$

(17)
The first two inequalities above follow from $L$-log-Lipschitzness of $\mu_i$. The third follows from the fact that $s_i \geq t + 1/L$ implies $s_i \geq t$. As such we can, for fixed $t, \theta$, upper-bound the probability over $s_1, \ldots, s_d$ that, conditional on $B_0^+$, there exists a vertex being $\varepsilon$-close to affhull($F_t$):

$$\Pr[-B_\varepsilon^+ \cap B_0^+] = \Pr[\exists i \in [n] \setminus [d] : s_i \geq t - \varepsilon]$$

(By union bound)

$$\leq \sum_{i \in [n] \setminus [d]} \Pr[s_i \geq t - \varepsilon]$$

$$\leq \sum_{i \in [n] \setminus [d]} e^3\varepsilon L \Pr[s_i \geq t]$$

(By (17))

$$= e^3\varepsilon L \Pr[\{i \in [n] \setminus [d] : s_i \geq t\}]$$

(18)

To interpret the last equality above, we observe that $\#\{i \in [n] \setminus [d] : s_i \geq t\} = 0$ if and only if $B_0^+$ happens. The upper bound $\Pr[B_0^+] \leq \frac{3}{4} \Pr[B_\varepsilon^+]$ then follows from (18) together with Claim 41 and our choice of $\varepsilon$. 

\textbf{Claim 41.} Conditional on $\theta, t$, if $\Pr[B_0^+ \cap \theta, t] \geq n^{-d}$ then $\Pr[\#\{i \in [n] \setminus [d] : s_i \geq t\}] \leq 2d \log n$. If $\Pr[B_0^+ \cap \theta, t] \geq n^{-d}$ then $\Pr[\#\{i \in [n] \setminus [d] : s_i \leq t\}] \leq 2d \log n$.

\textbf{Proof.} We prove the first implication, and the second follows analogously. For each $i \in [n] \setminus [d]$, let $X_i \in \{0, 1\}$ have value 1 if and only if $s_i \geq t$. Since $\theta, t$ are fixed and depend only on $a_1, \ldots, a_d$, the random variables $X_{d+1}, \ldots, X_n$ are independent. Write $X = \sum_{i=d+1}^{n} X_i$. The Chernoff bound gives

$$\Pr[X = 0] \leq \exp \left( -\frac{\mathbb{E}[X]}{2} \right).$$

As such, $\mathbb{E}[X] > 2d \log n$ would imply $\Pr[X = 0] < n^{-d}$, contradicting the original assumption that $\Pr[X = 0] \geq n^{-d}$. It follows that $\mathbb{E}[X] \leq 2d \log n$.

Now we can prove Lemma 38 using Lemma 40.

\textbf{Proof of Lemma 38.} Fix any $i \in [n] \setminus [d]$. By Lemma 40, we have that $\Pr[E_i] \leq \binom{n}{d}^{-1} + \frac{1}{4} \Pr[E_i \cap (\delta \geq \varepsilon)]$ for $\varepsilon = \frac{1}{40e^3Ld \log n}$. This gives that

$$\frac{\Pr[E_i \cap (\delta \geq \varepsilon)]}{\Pr[E_i]} \geq \frac{4}{5} - \binom{n}{d}^{-1} \cdot \frac{4}{5 \Pr[E_i]}.$$

Moreover, since $\Pr[E_i] \geq 10 \binom{n}{d}^{-1}$, we have

$$\Pr[(\delta \geq \varepsilon) \mid E_i] = \frac{\Pr[E_i \cap (\delta \geq \varepsilon)]}{\Pr[E_i]} \geq \frac{4}{5} - \binom{n}{d}^{-1} \cdot \frac{4}{5 \Pr[E_i]} \geq 0.72,$$

as desired.

\textbf{5.4 Randomized Lower-Bound for $r$: Inner Radius of a Ridge Projected onto $(d-1)$-Dimensional Space}

In this section, we demonstrate that for any ridge $R$ of the polytope $P$, wherein $R \cap W$ is a vertex of $P \cap W$, its inner radius—after projection onto the subspace orthogonal to the adjacent edge of $P \cap W$—is at least $\Omega(d^{-4}L^{-2})$ with high probability. Essentially, this establishes that the parameter $r$, as referred to in Lemma 36, is at least $\Omega(d^{-4}L^{-2})$ with good probability. We remark that Lemma 42 has no analogue when $d = 2$ and will require more technical effort. Its proof is similar to Lemma 4.1.1 (Distance bound) in [ST04].
**Lemma 42** (Randomized Lower-bound for r). Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be independent $L$-log-Lipschitz random vectors. Let $D$ denote the event that $\forall i, j \in [n], \|a_i - a_j\| \leq 3$. Fix any $I \in \binom{[n]}{d-1}$ and any $J \in \binom{[d]}{d-1}$, we have

$$\Pr[\text{dist}(W \cap \text{affhull}(a_i : i \in I), \partial \text{conv}(a_j : j \in J)) \leq \frac{1}{19200d^4L^2} \mid E_I \wedge A] \leq 0.1 + \Pr[\neg D \mid E_I \wedge A].$$

*Proof.* We may assume without loss of generality that $I = [d]$ and $J = [d-1]$. Apply the change of variables $\phi$ as in Definition 18 to $\{a_i : i \in [d]\}$ and obtain

$$\phi(\theta, t, b_1, \ldots, b_d) = (a_1, \ldots, a_d),$$

where $\theta \in \mathbb{S}^{d-1}, t \in \mathbb{R}, b_1, \ldots, b_d \in \mathbb{R}^{d-1}$. For any $i \in [n]$, let $\mu_i$ denote the probability density function of $a_i$. Writing the conditioning to $(E_{[d]} \wedge A_{[d-1]})$ as part of the pdf, we find that the joint probability density of $t, \theta, b_1, \ldots, b_d, a_{d+1}, \ldots, a_n$ is proportional to

$$\text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \prod_{i=1}^{d} \mu_i(t, \theta, b_i) \cdot \prod_{i=d+1}^{n} \mu_i(a_i) \cdot 1[E_{[d]} \wedge A_{[d-1]}],$$

(19)

where $\text{vol}_{d-1}(\cdot)$ is the volume function of $(d-1)$-dimensional simplex, $\mu_i(t, \theta, b_i) = \mu_i(t + \theta R_0(b_i))$ is the induced probability density of $b_i$, which is $L$-log-Lipschitz, and $1[\cdot]$ denotes the indicator function. Write $S$ for the event that

$$\text{dist}(W \cap \text{affhull}(a_i : i \in I), \partial \text{conv}(a_j : j \in J)) \leq \frac{1}{19200d^4L^2}.$$  

In this language, our goal is to prove that $\Pr[S] \leq 0.1 + \Pr[\neg D]$. Let $D'$ denote the event that $\|b_i - b_j\| \leq 3$ for all $i, j \in [d]$. Each of the events $E_I, A_J, S, D', D$ are functions of the random variables $\theta, t, b_1, \ldots, b_d, a_{d+1}, \ldots, a_n$. We then use Fubini’s theorem to write

$$\Pr_{\theta, t, b_1, \ldots, b_d, a_{d+1}, \ldots, a_n}[S] = \mathbb{E}_{\theta, t, a_{d+1}, \ldots, a_n}[\Pr_{b_1, \ldots, b_d}[S]]$$

With probability 1 over the choice of $\theta, t, a_{d+1}, \ldots, a_n$, the inner term satisfies all the conditions of Lemma 43. Specifically, since the value of $1[E_{[d]}]$ is already fixed, the intersection $(t\theta + \theta^\perp) \cap W$ is a line. Let $\ell \subseteq \mathbb{R}^{k-1}$ be the image of such line under the inverse change of variables $\phi^{-1}$, i.e. $(t\theta + \theta^\perp) \cap W = t\theta + R_\theta(\ell)$. Then the event $A_{[d-1]}$ is equivalent to $\ell \cap \text{conv}(b_1, \ldots, b_{d-1}) \neq \emptyset$. From Lemma 13 the joint probability distribution of $b_1, \ldots, b_d$ is thus proportional to

$$\text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \prod_{i=1}^{d} \mu_i(b_i) \cdot 1[\ell \cap \text{conv}(b_1, \ldots, b_{d-1}) \neq \emptyset]$$

Applying Lemma 45 to the term $\Pr_{b_1, \ldots, b_d}[S]$ we find

$$\mathbb{E}_{\theta, t, a_{d+1}, \ldots, a_n}[\Pr_{b_1, \ldots, b_d}[S]] \leq \mathbb{E}_{\theta, t, a_{d+1}, \ldots, a_n}[0.1 + \Pr_{b_1, \ldots, b_d}[\neg D']]$$

$$= 0.1 + \Pr[\neg D'] \leq 0.1 + \Pr[\neg D],$$

using Fubini’s theorem for the equality and the fact that $\neg D'$ implies $\neg D$ for the final inequality.  

**Lemma 43.** Let $\ell \subseteq \mathbb{R}^k$ be an affine line and $x \in \mathbb{R}^k$ be point. Suppose $w \in \mathbb{S}^{k-1} \setminus \{0\}$ points in the direction of $\ell$, i.e., that $\ell + w = \ell$. Then $\text{dist}(x, \ell) = \text{dist}(\pi_{w^\perp}(x), \pi_{w^\perp}(\ell))$.

*Proof.* Let $\{z\} = \ell \cap (x + w^\perp)$. This intersection is non-empty because $\dim(\ell) + \dim(x + w^\perp) = k$, and is a singleton because if $z, z'$ were two distinct points in this set then $z - z' \in w \cap w^\perp = \{0\}$.

Since $x - z \in w^\perp$, we have $\text{dist}(x, z) = \text{dist}(\pi_{w^\perp}(x), \pi_{w^\perp}(\ell))$. Also notice that for any $z' \in \ell, z' \neq z$,

$$||x - z'||^2 = ||\pi_w(x - z')||^2 + ||\pi_{w^\perp}(x - z')||^2 \geq ||\pi_{w^\perp}(x - z')||^2 = ||x - z||^2.$$  

Therefore, $\text{dist}(x, \ell) = ||x - z|| = \text{dist}(\pi_{w^\perp}(x), \pi_{w^\perp}(\ell))$. 

\[\square\]
Lemma 44. Let $H \subseteq \mathbb{R}^k$ be a co-dimension 1 hyperplane. Let $p_1, \ldots, p_k \in H$ and $p_{k+1} \in \mathbb{R}^k$, and assume $\lambda_1, \ldots, \lambda_{k+1} \geq 0$ are such that $\sum_{i=1}^{k+1} \lambda_i = 1$.

Then

$$\text{dist}(\sum_{i=1}^{k+1} \lambda_i p_i, H) = \lambda_{k+1} \text{dist}(p_{k+1}, H).$$

Proof. Let $y \in \mathbb{R}^k, t \in \mathbb{R}$ be such that $H = \{x \in \mathbb{R}^k : y^T x = t\}$ and $\|y\| = 1$.

Now we have

$$\text{dist}(\sum_{i=1}^{k+1} \lambda_i p_i, H) = |t - y^T (\sum_{i=1}^{k+1} \lambda_i p_i)|$$

$$= |\sum_{i=1}^{k+1} \lambda_i (t - y^T p_i)|$$

$$= |\lambda_{k+1} (t - y^T p_{k+1})|$$

$$= \lambda_{k+1} \text{dist}(p_{k+1}, H),$$

using that $y^T p_i = t$ for all $i = 1, \ldots, k$. \hfill \square

Lemma 45 (Randomized lower bound for $r$ after change of variables). Let $b_1, \ldots, b_d \in \mathbb{R}^{d-1}$ be random vectors with joint probability density proportional to

$$\text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \prod_{i=1}^{d} \bar{\mu}_i(b_i)$$

where $\bar{\mu}_i$ is $L$-log-Lipschitz for each $i \in [d]$. Let $D'$ denote the event that the set $\{b_1, \ldots, b_d\}$ has Euclidean diameter of at most 3. Given any fixed one-dimensional line $\ell \subseteq \mathbb{R}^{d-1}$, we have that

$$\Pr\left[\left(\text{dist}(\ell, \partial \text{conv}(b_1, \ldots, b_{d-1})) < \frac{1}{19200d^3L^2} \right) \mid \ell \cap \text{conv}(b_1, \ldots, b_{d-1}) \neq \emptyset\right]$$

$$\leq 0.1 + \Pr[\neg D' \mid \ell \cap \text{conv}(b_1, \ldots, b_{d-1}) \neq \emptyset].$$

Proof. We can bound the distance from $\ell$ to $\partial \text{conv}(b_1, \ldots, b_{d-1})$.

$$\text{dist}(\ell, \partial \text{conv}(b_1, \ldots, b_{d-1}))$$

$$= \text{dist}(\pi_{w^+}(\ell), \pi_{w^+}(\partial \text{conv}(b_1, \ldots, b_{d-1})))$$

$$= \text{dist}(\pi_{w^+}(\ell), \partial \text{conv}(\pi_{w^+}(b_1), \ldots, \pi_{w^+}(b_{d-1})))$$

$$= \min_{i \in [d-1]} \text{dist}(\pi_{w^+}(\ell), \partial \text{conv}(\pi_{w^+}(b_i), \ldots, \pi_{w^+}(b_{d-1})))$$

$$\geq \min_{i \in [d-1]} \lambda_i \cdot \text{dist}(\pi_{w^+}(b_i), \partial \text{affhull}(\pi_{w^+}(b_j) : j \in [d-1], j \neq i))$$

$$\geq \min_{i \in [d-1]} \lambda_i \cdot \min_{k \in [d-1]} \text{dist}(\pi_{w^+}(b_k), \partial \text{affhull}(\pi_{w^+}(b_j) : j \in [d-1], j \neq k))$$

where in the fifth step, $\lambda \in \mathbb{R}_{\geq 0}^{d-1}$ is the unique solution to $\sum_{i=1}^{d-1} \lambda_i b_i = \ell \cap \text{conv}(b_1, \ldots, b_{d-1})$ and $\sum_{i=1}^{d-1} \lambda_i = 1$. Additionally, assume $\ell = w\mathbb{R}$ for a non-zero $w \in \mathbb{S}^{d-2}$. Abbreviate, for $k \in [d-1]$,

$$r_k = \text{dist}(\pi_{w^+}(b_k), \partial \text{affhull}(\pi_{w^+}(b_j) : j \in [d-1], j \neq k)).$$

Let $T$ denote the event that $\ell \cap \text{conv}(b_1, \ldots, b_{d-1}) \neq \emptyset$. We now find using the union bound, for any $\alpha, \beta > 0$,

$$\Pr[\text{dist}(\ell, \partial \text{conv}(b_1, \ldots, b_{d-1})) < \alpha \beta \mid T]$$
\[ \lambda \geq \text{observe that } l \quad \lambda \text{ is defined to satisfy } P \quad \text{which is } w \quad \text{Fix any } \pi \quad \text{we prove the result using the randomness in } \pi \quad \text{Lemma 46} \quad \text{Lemma 46 to show that the every convex parameter } \lambda \text{ is at least } \Omega(1/d^2L) \text{ with constant probability.} \]

**Lemma 46 (Lower-bound for Convex Parameters of Vertices on the Ridge).** Let \( b_1, \ldots, b_d \in \mathbb{R}^{d-1} \) be random vectors with joint probability density proportional to

\[ \text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \prod_{i=1}^{d} \bar{\mu}_i(b_i) \]

where each \( \bar{\mu}_i : \mathbb{R}^{d-1} \to \mathbb{R}_+ \) is L-log-Lipschitz. Given any one-dimensional line \( \ell \subseteq \mathbb{R}^{d-1} \) and conditional on \( \ell \cap \text{conv}(b_i : i \in [d-1]) \neq \emptyset \). Let \( \lambda \in \mathbb{R}^{d-1}_+ \) be the unique solution to \( \sum_{i=1}^{d-1} \lambda b_i = \ell \cap \text{conv}(b_i : i \in [d-1]) \). Let \( D' \) denote the event that \( \forall i, j \in [d], ||b_i - b_j|| \leq 3 \). Then we have

\[ \Pr \left[ \forall i \in [d-1] : \lambda_i \geq \frac{1}{120d^2L} \left| D' \cap \ell \cap \text{conv}(b_i : i \in [d-1]) \neq \emptyset \right. \right] \geq 0.95. \]

**Proof.** By using the union bound, it suffices to prove for each \( i \in [d-1] \) that

\[ \Pr[\lambda_i < \frac{1}{120d^2L} \mid D' \cap \ell \cap \text{conv}(b_j : j \in [d-1]) \neq \emptyset] \leq \frac{1}{20(d-1)}. \]

Fix any \( i \in [d-1] \), without loss of generality \( i = 1 \). We can assume \( \ell = wR \) for a non-zero \( w \in \mathbb{R}^{d-2} \). Thus \( \lambda \) is defined to satisfy \( \sum_{j=1}^{d-1} \lambda_j \pi_{w^\perp}(b_j) = 0 \).

For any given values of \( b_1 - b_j \), \( j \in [d] \), which determine the shape of the simplex \( \text{conv}(b_j : j \in [d]) \), we prove the result using the randomness in \( \pi_{w^\perp}(b_1) \), the position of the simplex in the space \( w^\perp \). For the remainder of this proof, we can consider all \( b_j, j \in [d] \) to be functions of \( b_1 \). If we furthermore fix any value for \( w^\top b_1 \) then \( \text{vol}(\text{conv}(b_j : j \in [d]) \) is fixed, hence \( \pi_{w^\perp}(b_1) \) has probability density \( \mu'(\pi_{w^\perp}(b_1)) \propto \prod_{j=1}^{d} H_j(b_j) \), which is \( dL \)-log-Lipschitz in \( \pi_{w^\perp}(b_1) \) with respect to the \( d-2 \)-dimensional Lebesgue measure on \( w^\perp \).

Write \( M = \text{conv}(\pi_{w^\perp}(b_1 - b_j) : j \in [d-1]) \subseteq w^\perp \), for which we can see that \( \pi_{w^\perp}(b_1) \in M \) if and only if \( \lambda \geq 0 \). It remains to show that

\[ \Pr[\lambda_1 < \frac{1}{120d^2L} \mid D' \cap \pi_{w^\perp}(b_1) \in M] < \frac{1}{20d}. \quad (20) \]

For any \( j \in [d-1] \), let \( l_j : M \to [0, 1] \) be the function sending any point to its \( j \)-th convex coefficient, i.e., the functions satisfy \( \sum_{j=1}^{d-1} l_j(x) = 1 \) and \( \sum_{j=1}^{d-1} l_j(x) \cdot \pi_{w^\perp}(b_1 - b_j) = x \) for every \( x \in M \). For any \( 1 \geq \alpha \geq 0 \), observe that \( l_1 \) takes values in the interval \( [\alpha, 1] \) on the set \( (1 - \alpha)M \). Hence we get

\[ \Pr[\lambda_1 \geq \alpha \mid \pi_{w^\perp}(b_1) \in M] = \frac{\int_M \mu'(x) \cdot l_1(x) \geq \alpha \mid x \mid dx}{\int_M \mu'(x) dx} \geq \frac{\int_{(1-\alpha)M} \mu'(x) dx}{\int_M \mu'(x) dx} \geq \frac{(1-\alpha)^{d-2} \int_M \mu'(1-(1-\alpha)x) dx}{\int_M \mu'(x) dx} \geq (1-\alpha)^{d-2} \max_{x \in M} e^{-dL\cdot \|ax\|}. \quad (\forall x \in (1-\alpha)M, l_1(x) \geq \alpha) \]

(By \( d \)-log-Lipschitzness of \( \mu' \))
By definition of $D'$, we know that $M$ has Euclidean diameter at most $3$. Thus we can bound $\|\alpha x\| \leq 3\alpha$ for any $x \in M$. Now take $\alpha = \frac{1}{120d^2L}$, we find

$$\Pr[(\lambda_i < \frac{1}{120d^2L}) \mid D' \land \pi_{w^+}(b_i) \in M] \leq 1 - \Pr[\lambda_i \geq \frac{1}{120d^2L} \mid D' \land \pi_{w^+}(b_i) \in M]$$

$$\leq 1 - (1 - \frac{1}{120d^2L})^{d^2/20d} \leq \frac{1}{20(d-1)},$$

where the last line comes from $L \geq 1$. Thus (20) holds as desired. 

In the second part, we lower bound the distance between each vertex $b_j$ (where $j \in [d-1]$) and the $(d-3)$-dimensional hyperplane spanned by the other vertices affhull$(b_j : j \in [d-1], j \neq i)$. Specifically, we show the following lemma:

**Lemma 47.** Let $b_1, \ldots, b_d \in \mathbb{R}^{d-1}$ be random vectors with joint probability density proportional to $\nu_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \prod_{i=1}^d \mu_i(b_i)$ where each $\mu_i : \mathbb{R}^{d-1} \rightarrow [0,1]$ is $L$-log-Lipschitz. Given any one-dimensional line $\ell \subseteq \mathbb{R}^{d-1}$, and let $w \in S^{d-2}$ be any unit direction of $\ell$. For any $i \in [d-1]$ we have

$$\Pr\left[ \text{dist} (\pi_{w^+}(b_i), \text{affhull}(\pi_{w^+}(b_j) : j \in [d-1], j \neq i)) \geq \frac{1}{160d^2L} \mid \ell \cap \text{conv}(b_i : i \in [d-1]) \neq \emptyset \right] \geq 1 - \frac{1}{20d}.$$

**Proof.** In the following arguments, we condition on $\ell \cap \text{conv}(b_i : i \in [d-1]) \neq \emptyset$. Write $\pi = \pi_{w^+}$ for the orthogonal projection onto $\ell^\perp$. Without loss of generality, set $i = d-1$ and assume that $\ell$ is a linear subspace, i.e., $\pi(\ell) = 0$.

We start with a coordinate transformation. Let $\phi \in w^+ \cap S^{d-2}$ denote the unit vector satisfying $\phi^T b_1 = \phi^T b_1 > 0$ for all $j = 1, \ldots, d-1$. Note that $\phi$ is uniquely defined almost surely: $w^+$ is a $(d-2)$-dimensional linear space and we impose $(d-3)$ linear constraints $\{\phi^T a_j = \phi^T a_j, \forall j \in [d-2]\}$. Almost surely, these give an one-dimensional linear subspace which, after adding the unit norm and $b_1^\perp \phi > 0$ constraint, leaves a unique choice of $\phi$.

Now define $h \in \mathbb{R}$ by $h = \phi^T b_1$ and define $\alpha \in \mathbb{R}$ by $\alpha h = -\phi^T b_{d-1}$. Since $0 \in \text{conv}(\pi(b_i) : i \in [d-1])$ but $\phi^T b_1 > 0$ for all $i \in [d-2]$, we must have $\alpha \geq 0$ for otherwise $\phi$ would separate $\text{conv}(\pi(b_i) : i \in [d-1])$ from 0. Again from almost-sure non-degeneracy we get $\alpha > 0$ and $h \neq 0$. We define the following coordinate transformation:

$$b_j = h\phi + c_j, \quad \forall j \in [d-2]$$

$$b_{d-1} = -\alpha h\phi + c_{d-1}$$

where for each $j \in [d-1]$, $c_j \in \phi^\perp \cap \text{affhull}(b_1, \ldots, b_{d-1})$ has $(d-3)$ degrees of freedom. From here on out, we consider the vertices $(b_1, \ldots, b_{d-1})$ to be a function of $(h, \alpha, \phi, c_1, \ldots, c_{d-1})$. Again by Lemma 19, the induced joint probability density on the random variables $(h, \alpha, \phi, c_1, \ldots, c_{d-1}, b_d)$, is proportional to

$$\nu_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \nu_{d-3}(\text{conv}(c_1, \ldots, c_{d-2})) \cdot \prod_{j=1}^d \mu_j(b_j)$$

$$\propto \nu_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \nu_{d-3}(\text{conv}(\pi_{w^+}(c_1), \ldots, \pi_{w^+}(c_{d-2}))) \cdot \prod_{j=1}^d \mu_j(b_j).$$

Condition on the exact values of $(\alpha, \phi, c_1, \ldots, c_{d-1}, b_d)$. Note that the event $0 \in \text{conv}(\pi(b_i) : i \in [d-1])$ depends only on these variables and not on $h$, and the same is true for $\nu_{d-3}(\text{conv}(\pi_{w^+}(c_1), \ldots, \pi_{w^+}(c_{d-2})))$. By Lemma 19 the induced probability density on $h$ is now proportional to

$$\nu_{d-1}(\text{conv}(b_1, \ldots, b_d)) \cdot \prod_{j=1}^{d-1} \mu_j(h),$$

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where \( \tilde{\mu}_j(h) = \mu_j(h \phi + c_j), j \in [d - 2] \) and \( \tilde{\mu}_{d-1}(h) = \mu_{d-1}(\alpha h \phi + c_{d-1}) \). Since each \( \tilde{\mu}_j \) is \( L \)-log-Lipschitz, it follows that the product \( \prod_{j=1}^{d-1} \tilde{\mu}_j(h) \) is \((d - 2 + \alpha)L \leq d(1 + \alpha)L\)-log-Lipschitz in \( h \).

Next, consider the volume term. We can write \( \text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) \) as a constant depending on \( d \) times the absolute value of the determinant of matrix

\[
\begin{bmatrix}
(b_1 - b_d)^T \\
\vdots \\
(b_{d-1} - b_d)^T
\end{bmatrix} = \begin{bmatrix}
(h \phi + c_1 - b_d)^T \\
\vdots \\
(h \phi + c_{d-2} - b_d)^T \\
(-\alpha h \phi + c_{d-1} - b_d)^T
\end{bmatrix}
\]

where \( \theta \in S^{d-1} \) denotes a normal vector to \( \text{aff} h^\perp \). Define

\[
B := \begin{bmatrix}
(c_1 - b_d)^T \\
\vdots \\
(c_{d-2} - b_d)^T \\
(c_{d-1} - b_d)^T
\end{bmatrix},
\quad v := \begin{bmatrix}
1 \\
\vdots \\
1 \\
-\alpha \\
0
\end{bmatrix}.
\]

Then by the matrix determinant lemma, we can write the volume as the absolute value of an affine function of \( h \) (which is a convex function):

\[
k(h) := \text{vol}_{d-1}(\text{conv}(b_1, \ldots, b_d)) = |\det(B) + hv^\top| = |\det(B)(1 + \phi^\top B^{-1}v \cdot h)|
\]

Hence, we have found a convex function \( k : \mathbb{R} \to \mathbb{R}_{\geq 0} \) and a \((d + 1)\)-log-Lipschitz function \( \nu : \mathbb{R} \to \mathbb{R}_{\geq 0} \) such that \( h \) has probability density proportional to \( k(h) \cdot \nu(h) \).

To finalize the argument, we write \( \text{dist}(\pi(b_i), \text{aff}(b_j)) : j \in [d - 1], j \neq i \rangle = |(1 + \alpha)h| \). It follows that the signed distance \((1 + \alpha)h \) has a probability density function proportional to the product of a \( d \)-log-Lipschitz function and a convex function. The result follows from Lemma 48 by plugging in the signed distance \((1 + \alpha)h \) and \( K = dL, \varepsilon = \frac{1}{\text{conv} e} \).

**Lemma 48.** Assume that \( h : \mathbb{R} \to \mathbb{R}_{\geq 0} \) is a \( K \)-log-Lipschitz function and \( g : \mathbb{R} \to \mathbb{R}_{\geq 0} \) is a convex function such that \( \int_{-\infty}^{\infty} g(x) \cdot h(x) \, dx = 1 \). Suppose that \( X \in \mathbb{R} \) is distributed with probability density \( g(X) \cdot h(X) \). For any \( \varepsilon > 0 \) we have \( \Pr[X \in [-\varepsilon, \varepsilon]] \leq 8\varepsilon K \).

**Proof.** We can assume that \( \varepsilon < 1/(8K) \), for otherwise the bound is trivial. First, we use the rudimentary upper bound

\[
\Pr[X \in [-\varepsilon, \varepsilon]] \leq \Pr[X \in [-\varepsilon, \varepsilon] \mid X \in [-1/K, 1/K]] = \frac{\int_{-1/K}^{1/K} g(x) \cdot h(x) \, dx}{\int_{-1/K}^{1/K} g(x) \cdot h(x) \, dx}.
\]

Log-Lipschitzness implies that for any \( \gamma > 0 \) we have

\[
e^{-\gamma K} h(0) \int_{-\gamma}^{\gamma} g(x) \, dx \leq \int_{-\gamma}^{\gamma} g(x) \cdot h(x) \, dx \leq e^{\gamma K} h(0) \int_{-\gamma}^{\gamma} g(x) \, dx;
\]

and hence we get

\[
\Pr[X \in [-\varepsilon, \varepsilon]] \leq e^{(1/K + \varepsilon)K} \frac{\int_{-\varepsilon}^{\varepsilon} g(x) \, dx}{\int_{-1/K}^{1/K} g(x) \, dx} \leq e^{1+\varepsilon K} \frac{2\varepsilon \max_{x \in [-\varepsilon, \varepsilon]} g(x)}{\int_{-1/K}^{1/K} g(x) \, dx}
\]

Since \( g(x) \) is convex, at least one of

\[
\max_{x \in [-\varepsilon, \varepsilon]} g(x) \leq \min_{x \in [-1/K, -\varepsilon]} g(x) \quad \text{or} \quad \max_{x \in [-\varepsilon, \varepsilon]} g(x) \leq \min_{x \in [\varepsilon, 1/K]} g(x)
\]

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holds. Without loss of generality, assume the second case holds. Then we bound
\[
\frac{\max_{x \in [-\varepsilon, \varepsilon]} g(x)}{\int_{-1/K}^{1/K} g(x) dx} \leq \frac{\max_{x \in [-\varepsilon, \varepsilon]} g(x)}{\int_{-\varepsilon}^{\varepsilon} g(x) dx} \leq \frac{1}{1/K - \varepsilon}.
\]
To summarize, we find \( \Pr[X \in [-\varepsilon, \varepsilon]] \leq e^{3+\varepsilon K} \cdot \frac{2\varepsilon}{1/K - \varepsilon}. \) Since \( \varepsilon < 1/(8K) \) this implies
\[
\Pr[X \in [-\varepsilon, \varepsilon]] \leq 2e^{9/8} \cdot \frac{8}{7} \cdot \varepsilon K \leq 8\varepsilon K.
\]

5.5 Combining Together and Proof of Lemma 33

In this section, we combine the deterministic argument in Lemma 36 and the randomized arguments in Lemma 38 and Lemma 42. We can finally show the main technical lemma (Lemma 33).

**Proof of Lemma 33.** Without loss of generality, let \( I = [d] \) and write \( E = E_I \). Suppose \( p = A_J = \text{conv}(a_j : j \in J) \cap W \) is the next vertex after the edge \( F_I \cap W \). Here \( J \in \binom{[n]}{d-1} \) and \( R_J \) is the \((d-2)\)-dimensional ridge. With probability 1, the polytope \( \text{conv}(a_1, \ldots, a_n) \) is non-degenerate and \( W \cap R' \) is a single point for any ridge \( R' \) of \( \text{conv}(a_1, \ldots, a_n) \) that intersects with \( W \). We will show that conditional on \( E \), each of the following conditions in the deterministic argument (Lemma 36) is satisfied with good probability:

1. (Bounded diameter) \( \forall i, j \in [n] \), \( ||a_i - a_j|| \leq 3; \)
2. (Lower bound of \( \delta \)) \( \min_{i \notin [n] \setminus I} \text{dist} (\text{affhull}(F_I), a_k) \geq \Omega(\frac{1}{Ld\log n}); \)
3. (Lower bound of \( r \)) \( \forall J \in \binom{[d-1]}{d-1} \) for which the ridge \( R_J = \text{conv}(a_j : j \in J) \) has nonempty intersection with \( W \), we have \( \text{dist}(F_I \cap W, \partial R_J) \geq \Omega(\frac{1}{\sqrt{dL^2}}). \)

Note for the last point that Lemma 36 only requires this for the set \( J \) which indexes the second vertex of \( F_I \cap W \) in clockwise direction, but we prove it for both of the sets \( J \) for which \( R_J \cap W \neq \emptyset \).

First, we write \( D \) as the event that \( \forall i, j \in [n] \) for which \( ||a_i - a_j|| \leq 3 \). From Lemma 16 for any \( \sigma \leq \frac{1}{8d\sqrt{\log n}} \), with probability at least \( 1 - \binom{n}{d-1}^{-1} \), we have \( \max_{i \notin [n]} ||a_i|| \leq 1 + 4\sigma \sqrt{d \log n} \leq \frac{3}{2} \), i.e. \( \Pr[D] \geq 1 - \binom{n}{d-1}^{-1} \). Using the assumption that \( \Pr[E_I] \geq 10(\frac{n}{d})^{-1} \), we have
\[
\Pr[-D \mid E] = \Pr[-D \wedge E] / \Pr[E] \leq \frac{\Pr[-D]}{\Pr[E]} \leq 0.1,
\]
This immediately implies \( \Pr[D \mid E] \geq 0.9. \)

Next, we consider \( \delta := \text{dist}(\text{affhull}(a_1, \ldots, a_d), \{a_{d+1}, \ldots, a_n\}) \). Using Lemma 38 we have \( \Pr[\delta \geq \frac{1}{10eLd\log n} \mid |E|] \geq 0.72. \)

Finally, we consider \( r := \max_J \text{dist}(\text{affhull}(a_1, \ldots, a_d) \cap W, \partial R_J) \) subject to all \( J \in \binom{[d-1]}{d-1} \) such that \( A_J \) happens (in other words, \( R_J = \text{conv}(a_j : j \in J) \) is a ridge of \( F_I \) such that \( R_J \cap W \neq \emptyset \)). By union bound,
\[
\Pr \left[ \exists J \in \binom{[d-1]}{d-1}, A_J \wedge \text{dist}(\text{affhull}(F \cap W), \partial R_J) \geq \frac{1}{19200d^4L^2} \mid |E| \right] \\
\geq 1 \sum_{J \in \binom{[d-1]}{d-1}} \Pr[A_J \wedge \text{dist}(\text{affhull}(F \cap W), \partial R_J) < \frac{1}{19200d^4L^2} \mid |E|] \\
= 1 \sum_{J \in \binom{[d-1]}{d-1}} \Pr[\text{dist}(\text{affhull}(F \cap W), \partial R_J) < \frac{1}{19200d^4L^2} \mid E \wedge A_J] \Pr[A_J \mid E].
\]

From Lemma 42 for each \( J \in \binom{[d-1]}{d-1} \), we know that
\[
\Pr[\text{dist}(\text{affhull}(F \cap W), \partial R_J) < \frac{1}{19200d^4L^2} \mid E \wedge A_J] \leq 0.1 + \Pr[-D \mid E \wedge A_J],
\]
Notice that when $E$ happens, there are exactly two distinct ridges $R_J, R_J'$ that has nonempty intersection with $W$ (or $A_J$ happens), thus $\sum_{J \in (\mathcal{J}_{d-1})} \Pr[A_J \mid E] = 2$. Therefore

$$\sum_{J \in (\mathcal{J}_{d-1})} \Pr[\text{dist}(\text{affhull}(F \cap W), \partial R_J < \frac{1}{19200d^4L^2} \mid E \land A_J)] \Pr[A_J \mid E]$$

$$\leq \sum_{J \in (\mathcal{J}_{d-1})} (0.1 + \Pr[\lnot D \mid E \land A_J]) \Pr[A_J \mid E]$$

$$\leq 0.1 \cdot 2 - 2 \cdot \Pr[\lnot D \mid E],$$

and (21) becomes

$$\Pr\left[ \exists J \in \left(\mathcal{J}_{d-1}\right), A_J \land \text{dist}(\text{affhull}(F \cap W), \partial R_J \geq \frac{1}{19200d^4L^2} \mid E) \right]$$

$$\geq 1 - 0.1 \cdot 2 - 2 \cdot \Pr[\lnot D \mid E] \geq 0.6.$$  

Therefore by union bound, the three conditions hold with probability at least $1 - (1 - 0.9) - (1 - 0.72) - (1 - 0.6) \geq 0.1$, and the lemma directly follows from Lemma [36].
6 Smoothed Complexity Lower Bound

In this section, we present the lower bound of the smoothed complexity by studying the intersection between the smoothed dual polytope $Q = \text{conv}(a_1, \ldots, a_n) \subseteq \mathbb{R}^d$ (where each $a_i$ is under Gaussian perturbation), and the two-dimensional shadow plane $W \subseteq \mathbb{R}^d$. Our main result is as follows:

**Theorem 49.** For any $d > 5, n = 4d - 13$, there exists a two-dimensional linear subspace $W \subseteq \mathbb{R}^d$ and vectors $\bar{a}_1, \ldots, \bar{a}_n \in \mathbb{R}^d$, $\max_{i \in [n]} \|\bar{a}_i\|_1 \leq 1$ such that the following holds. Let $a_1, \ldots, a_n$ be independent Gaussian random variables where each $a_i \sim \mathcal{N}(\bar{a}_i, \sigma^2 I), \sigma \leq \frac{1}{7200d\sqrt{\log n}}$. Then with probability at least $1 - \left(\frac{n}{d}\right)^{-1}$, we have

$$\text{edges}(\text{conv}(a_1, \ldots, a_n) \cap W) \geq \Omega \left(\min \left(\frac{1}{\sqrt{d\sigma\sqrt{\log n}}}, 2^d\right)\right).$$

Theorem 49 is the direct consequence of the next theorem, which is a lower bound for adversarial perturbations of bounded magnitude:

**Theorem 50.** For any $d > 5, n = 4d - 13$, there exists a two-dimensional linear subspace $W \subseteq \mathbb{R}^d$ and vectors $\bar{a}_1, \ldots, \bar{a}_n \in \mathbb{R}^d$, $\max_{i \in [n]} \|\bar{a}_i\|_1 \leq 1$ such that the following holds. For any $\varepsilon < \frac{1}{150}$, if $a_1, \ldots, a_n \in \mathbb{R}^d$ satisfy $\|a_i - \bar{a}_i\|_1 \leq \varepsilon$ for all $i \in [n]$ then we have

$$\text{edges}(\text{conv}(a_1, \ldots, a_n) \cap W) \geq \Omega \left(\min \left(\frac{1}{\sqrt{\varepsilon}}, 2^d\right)\right).$$

The rest of this section is organized as follows: In Section 6.1, we construct the polytope $P$ by a set of constraints, and the two-dimensional shadow plane $W$. An informal intuition behind these inequalities is described in Section 6.2. In Section 6.3, we show that $\pi_W(P)$ approximates the unit disk $B_2$. In Section 6.4, we analyze the largest $\ell_\infty$-ball contained in $P$ and the smallest $\ell_\infty$-ball containing $P$. Section 6.5 investigates the polar polytope $Q = (P - x)^\circ$ of a shift of $P$ and derives bounds on the radius of its largest contained $\ell_1$-ball and smallest containing $\ell_1$-ball. Finally, Section 6.6 shows that the bounded ratio between these radii imply that any perturbation $Q$ still has $Q \cap W$ approximate the unit disk $B_2$ well and uses this fact to prove Theorem 49.

6.1 Construction of the Primal Polytope

In this subsection, we first construct the primal polytope and the two-dimensional plane $W$. For $k \in \mathbb{N}$, we construct a $(k + 5)$-dimensional polytope. We will use the following vectors in the definition:

- Define $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- For every $i \in \{0, 1, \ldots, k\}$, define the pair of orthogonal unit vectors $w_i = \begin{bmatrix} \cos(\pi/2^{i+2}) \\ \sin(\pi/2^{i+2}) \end{bmatrix} \in \mathbb{R}^2$ and $v_i = \begin{bmatrix} -\cos(\pi/2^{i+2}) \\ \sin(\pi/2^{i+2}) \end{bmatrix} \in \mathbb{R}^2$.

With these definitions in mind, let $P' \subseteq \mathbb{R}^{3k+5}$ denote the set of points $(x, y, p_0, \ldots, p_k, t, s)$, where $x, y \in \mathbb{R}, p_0, p_1, \ldots, p_k \in \mathbb{R}^2, t \in \mathbb{R}^k, s \in \mathbb{R}$, satisfying the following system of linear inequalities:

$$e_1^T p_0 \geq |x|, e_2^T p_0 \geq |y|$$

$$w_i^T p_i = w_{i-1}^T p_{i-1}, \forall i \in [k]$$

$$t_i + is = v_i^T p_i \geq |v_{i-1}^T p_{i-1}|, \forall i \in [k]$$

$$e_1^T p_k \leq 1$$

$$0_k \leq t \leq 1_k$$

(22)-(26)
6.2 Intuition of the Construction

To explain the intuition behind the equations (22 - 27), let us consider the simpler system of inequalities in variables $x, y \in \mathbb{R}, r_0, \ldots, r_k \in \mathbb{R}^2$:

\begin{align*}
    e_1^\top r_0 &\geq |x|, e_2^\top r_0 \geq |y| \tag{29} \\
    w_i^\top r_i &\geq w_i^\top r_{i-1}, \forall i \in [k] \tag{30} \\
    v_i^\top r_i &\geq |v_i^\top r_{i-1}|, \forall i \in [k] \tag{31} \\
    e_1^\top r_k &\leq 1 \tag{32} \\
    e_2^\top r_i &\geq 0, \forall i \in [k] \tag{33}
\end{align*}

Let $R \subseteq \mathbb{R}^{2k+4}$ denote the set of vectors $(x, y, r_0, \ldots, r_k)$ satisfying the above inequalities. For each $i = 0, \ldots, k$, write $R_i = \{r_i : \exists x, y, r_0, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k \in R \} \subseteq \mathbb{R}^2$ for the projections of $R$ onto the two-dimensional coordinate subspace of $r_i$. Also, let $W$ be the two-dimensional plane spanned by the $x$ and $y$ directions, so that $\pi_W(R) = \{(x, y) : \exists r_0 \in R_0, r_0 \geq (|x|, |y|)^\top\}$. The vertices of these sets are depicted in Figure 4 for $k = 2$.

For these sets, we have the following observations. We will skip the proof since they only give illustrations of our analysis and will not be functional to the proof of Theorem 49.

1. The set $R_k$ can be described by the inequalities $e_2^\top r_k \geq 0, e_1^\top r_k \leq 1, v_k^\top r_k \geq 0$. These inequalities describe a small slice of the regular $2^{k+1}$-gon.

Figure 3: Vertices of the projected primal polytope $\pi_W(P)$ (see (28)) without perturbation for $k = 4$. 

We remark that $p_0, t, s$ uniquely define the values of $p_1, p_2, \ldots, p_k$ via (23) and (24). As such, define the polytope $P \subseteq \mathbb{R}^{k+5}$ as the projection of $P'$ onto the subspace spanned by the variables $(x, y, p_0, t, s)$:

$$P = \{(x, y, p_0, t, s) : \exists p_1, \ldots, p_k, s.t. (x, y, p_0, \ldots, p_k, t, s) \in P'\}. \tag{28}$$

The plane $W$ of interest is that spanned by the unit vectors in the $x$ and $y$ directions. An illustration of the vertices of the projected polytope $\pi_W(P)$ can be found in Figure 3 for $k = 4$. Note that the figure appears to depict a regular polygon with $2^{k+1}$ vertices.
2. For each $i = 0, \ldots, k - 1$, the set $R_i$ is obtained by taking the union of $R_{i+1}$ and its mirror image in the line spanned by $w_i$.

3. For each $i = 1, \ldots, k$, the set $R_i$ can be described as the restriction of a regular $2^{k+1}$-gon intersected with the set $\{ r_i : e_2^\top r_i \geq 0, v_i^\top r_i \geq 0 \}$.

4. The set $R_0$ can be described as a regular $2^{k+1}$-gon intersected with the non-negative orthant.

5. The set $\pi_W(R) = \{(x, y) : \exists r_0 \in R_0, r_0 \geq (|x|, |y|)^\top \}$ is a regular $2^{k+1}$-gon.

More careful inspection allows for the following additional observations, which we also state without proof:

1. Turning (30) from an inequality to an equality constraint does not change any of the sets $R_0, \ldots, R_k$.

2. Removing the constraint (33) does not change $R_0$.

3. Adding upper bounds $v_i^\top r_i \leq 2$ does not change $R_0$.

Each of the above-mentioned changes either serves to increase the radius of the largest ball (in the affine hull of $R$) contained in the relative interior of $R$ or to decrease the radius of the smallest ball containing $R$. The addition of the variable $s$ in the construction of $P$ serves to further increase the radius of the largest ball contained in $P$ without increasing the radius of the smallest ball containing $P$.

### 6.3 Projected Primal Polytope Approximates Two-Dimensional Unit Disk

In this subsection, we will show that the polytope $P$ we constructed in (28) has a projection $\pi_W(P)$ which approximates the two-dimensional unit disk $\mathbb{B}^2 = \{ x, y \in \mathbb{R} : x^2 + y^2 \leq 1 \}$ within exponentially small error.
Lemma 51 (Projected primal polytope approximates the two-dimensional disk). For any \( k \in \mathbb{N} \), let \( P \subseteq \mathbb{R}^{k+5} \) be the polytope defined by the linear system (28) with variables \( x, y, s \in \mathbb{R} \), \( p_0 \in \mathbb{R}^2 \), \( t \in \mathbb{R}^k \). Let \( W \) be the two-dimensional subspace spanned by the directions of \( x \) and \( y \). Then we have

\[
\mathbb{B}_2 \subseteq \pi_W(P) \subseteq \cos(\pi/2k+2)^{-1}\mathbb{B}_2.
\]

Lemma 51 directly follows from the next two lemmas. First, we show that the two-dimensional unit disk is contained in \( \pi_W(P) \):

Lemma 52 (Inner radius of the projected primal polytope). For every \( x, y \in \mathbb{R} \) with \( x^2 + y^2 \leq 1 \) there exist \( p_0 \in \mathbb{R}^2 \), \( t \in \mathbb{R}^k \) and \( s \in \mathbb{R} \) such that \( (x, y, p_0, t, s) \in P \).

Proof. Given such \( x, y, s \) set \( s = 0 \) and set \( p_0, p_1, \ldots, p_k, t \) such that (22) and (24) are satisfied with equality. This will result in \( \sqrt{x^2 + y^2} = \|p_0\| = \|p_i\| \) for every \( i \in [k] \). Since \( t = [v_i^\top p_{i-1} | \|v_i\| \cdot \|p_{i-1}\| \leq 1, \) we know that (26) is satisfied. Furthermore, we have \( \varepsilon_i^\top p_k \leq \|e_i\| \cdot \|p_k\| = \|e_i\| \cdot \sqrt{x^2 + y^2} \leq 1 \) which ensures that (25) is satisfied.

In the next lemma, we will show that \( \pi_W(P) \) is contained in the two-dimensional disk \( \cos(\pi/2k+2)^{-1}\mathbb{B}_2 \):

Lemma 53 (Outer radius of the projected primal polytope). For every \( x, y \in \mathbb{R} \) such that \( \sqrt{x^2 + y^2} \geq \cos(\pi/2k+2)^{-1} \), there exist \( p_0 \in \mathbb{R}^2 \), \( t \in \mathbb{R}^k \) and \( s \in \mathbb{R} \) such that \( (x, y, p_0, t, s) \in P \).

Proof. Fix any \( (x, y) \in \mathbb{R}^2 \) and \( p_0 \in \mathbb{R}^2 \), such that \( \sqrt{x^2 + y^2} > \cos(\pi/2k+2)^{-1} \) and \( p_0 \geq [\|x\|, \|y\|]^\top \). Also fix any \( p_1, \ldots, p_k \in \mathbb{R}^2 \) satisfying (25) and (24). We will show that such \( p_1, \ldots, p_k \) would violate (23), i.e. \( \varepsilon_i^\top p_k > 1 \). To simplify our notation, for all \( i \in \{0, 1, \ldots, k\} \), let \( (p_i)_v = v_i^\top p_i \in \mathbb{R} \) and \( (p_i)_w = w_i^\top p_i \in \mathbb{R} \). Then \( p_i = (p_i)_v v_i + (p_i)_w w_i \).

Notice that for all \( i \in [k] \), the increment of the first coordinate from \( p_{i-1} \) to \( p_i \) is

\[
\varepsilon_i^\top p_i - \varepsilon_{i-1}^\top p_{i-1} = \varepsilon_i^\top ((p_i)_v w_i + (p_i)_w v_i - (w_i^\top p_{i-1}) w_i - (v_i^\top p_{i-1}) v_i) \\
= \varepsilon_i^\top (p_i)_v v_i - (v_i^\top p_{i-1}) v_i \\
\geq \varepsilon_i^\top (v_i^\top p_{i-1} - v_i^\top p_{i-1}) \\
\geq 0
\]

where the inequality in (34) is tight when \( v_i^\top p_i = |v_i^\top p_{i-1}| \). Let \( p_0, p_1^*, \ldots, p_k^* \in \mathbb{R}^2 \) be the (unique) sequence defined by

\[
p_0^* = p_0 \\
(w_i^\top p_i^*) = (v_i^\top p_{i-1}^*)^\top, \quad \forall i \in [k] \quad \text{(Tight for (23))} \\
(v_i^\top p_i^*) = (|v_i^\top p_{i-1}|)^\top, \quad \forall i \in [k] \quad \text{(Tight for (24))}
\]

Then \( \varepsilon_i^\top p_i - \varepsilon_{i-1}^\top p_{i-1} \geq \varepsilon_i^\top p_i^* - \varepsilon_{i-1}^\top p_{i-1}^* \) for each \( i \in [k] \). Also, notice that \( \varepsilon_i^\top p_0 = \varepsilon_i^\top p_0^* \), therefore for each \( i \in [k] \), \( \varepsilon_i^\top p_i \geq \varepsilon_i^\top p_i^* \).

It remains to show that \( \varepsilon_i^\top p_i^* > 1 \). For all \( i \in \{0, 1, \ldots, k\} \), let \( \theta_i \in [-\pi, \pi] \) denote the angle between \( p_i^* \in \mathbb{R}^2 \) and \( e_1 \). Then since \( \varepsilon_0^\top p_0 \geq 0 \) and \( e_2^\top p_0 \geq 0 \), we have \( 0 \leq \theta_0 \leq \frac{\pi}{2} \). For any \( i \in [k] \), notice that \( p_i^* \) equals to \( p_{i-1}^* \) (if \( \theta_{i-1} \leq \frac{\pi}{2} \)), or equals to the mirror of \( p_{i-1}^* \) with respect to the line spanned by \( w_i \) (if \( \theta_{i-1} \geq \frac{\pi}{2} \)). By induction, this gives

\[
\|p_i^*\|_2 = \|p_{i-1}^*\|_2 = \ldots = \|p_0^*\|_2 = \|p_0\|_2,
\]

and

\[
\theta_i = \frac{\pi}{2i+2} - \left| \theta_{i-1} - \frac{\pi}{2i+2} \right| \leq \frac{\pi}{2i+2}.
\]

Therefore, we get

\[
\varepsilon_i^\top p_i^* = \|p_i^*\| \cdot \cos(\theta_i) \geq \|p_0\| \cdot \cos\left(\frac{\pi}{2k+2}\right) > 1.
\]

Thus we have shown \( x^2 + y^2 > \cos(\pi/2k+2)^{-1} \) implies that \( \varepsilon_i^\top p_k \geq \varepsilon_i^\top p_k^* > 1 \) as desired. \( \square \)
6.4 Inner and Outer Radius of the Primal Polytope

In this subsection, we will show that the primal polytope $P$ has large inner radius and small outer radius.

**Lemma 54** (Inner radius of the primal polytope). For $k \in \mathbb{N}$, let $P \subseteq \mathbb{R}^{k+5}$ be the polytope defined by the linear system \([23]\). Then for or $\bar{x} = \bar{y} = 0, \bar{p}_0 = (1/6, 1/6)^\top, t = 1_k/30, \bar{s} = \frac{1}{3}$, it holds that

$$\frac{1}{30} \cdot \mathbb{B}^{k+5}_\infty \subseteq P - (\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s}) \subseteq \frac{3}{2} \cdot \mathbb{B}^{k+5}_\infty.$$

**Proof.** In Lemma [25] we construct a point $(\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s})$ such that $\frac{1}{30} \mathbb{B}^{k+5}_\infty \subseteq P - (\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s})$.

For the second inclusion, we show that $P \subseteq (\cos(\pi/2k^2))^{-1} \cdot \mathbb{B}^{k+5}_\infty$. Suppose $(x, y, p_0, t, s) \in P$ is arbitrary. From Lemma [53] we know that $\|x, y\|_\infty \leq \sqrt{x^2 + y^2} \leq \|p_0\|_2 \leq \cos(\pi/2k^2)^{-1}$. Since $0 \leq t \leq 1_k$ we get $\|v\|_\infty \leq 1$, and lastly we have $0 \leq s \leq 1$. Put together, we find that $\|x, y, p_0, t, s\|_\infty \leq \cos(\pi/2k^2)^{-1}$. Since $(x, y, p_0, t, s) \in P$ was arbitrary, we find that $P \subseteq (\cos(\pi/2k^2))^{-1} \cdot \mathbb{B}^{k+5}_\infty$. By the triangle inequality we find that $P - (\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s}) \subseteq (\cos(\pi/2k^2)^{-1} + \|x, y, p_0, t, s\|_\infty) \cdot \mathbb{B}^{k+5}_\infty$ and we will see in Lemma [55] that $\|x, y, p_0, t, s\|_\infty = 1/3$.

**Lemma 55** (Inner Radius of the Polytope). For $\bar{x} = \bar{y} = 0, \bar{p}_0 = (1/6, 1/6)^\top, t = 1_k/30, \bar{s} = \frac{1}{3}$, we have $(\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s}) + r \cdot \mathbb{B}^{k+5}_\infty \subseteq P$ for $r = \frac{1}{30}$.

**Proof.** Fix any $(x, y, p_0, t, s) \in \mathbb{R}^{k+5}$ such that $\|(x - \bar{x}, y - \bar{y}, p_0 - \bar{p}_0, t - \bar{t}, s - \bar{s})\|_\infty \leq r$. Let $p_1, \ldots, p_k \in \mathbb{R}^2$ be uniquely defined by \([23]\) and the equality in \([24]\), i.e., $w_i^T p_i = w_i^T p_{i-1}$ and $t_i + s_i = v_i^T p_i$ with our fixed $p_0, s$ and $t$. We will show that $(x, y, p_0, t, s) \in P$ by verifying \([22]-[27]\). To simplify our notation, let $(p_i)_v = v_i^T p_i \in \mathbb{R}$ and $(p_i)_w = w_i^T p_i \in \mathbb{R}$ for all $i \in \{0, 1, \ldots, k\}$. Then $p_i = (p_i)_v v_i + (p_i)_w w_i$.

First, observe that $e_i^T p_0 \geq \frac{1}{3} - r \geq r \geq |x|$ and $e_i^T p_0 \geq \frac{1}{3} - r \geq r \geq |y|$, confirming that \([22]\) holds. Also, notice that $t_i \in [0, \bar{t} - r, \bar{t} + r] \subseteq [0, 1]$ and $s \in [\bar{s} - r, \bar{s} + r] \subseteq [0, 1]$. Thus \([26]\) and \([27]\) hold. The equality constraint \([23]\) holds by definition of $p_1, \ldots, p_k$.

It remains to show \([24]\) and \([25]\), i.e., $(p_i)_v \geq |v_i^T p_i|_{-1}$ for all $i \in [k]$ and $e_i^T p_k \leq 1$. To aid in the remaining steps of the proof, we show $(p_i)_w \geq 0$ for all $i \in \{0, 1, \ldots, k\}$. Observe that $w_i^T p_0 \geq w_i^T \bar{p}_0 - \|w_0\| \geq \sqrt{\frac{2}{\pi}} - \sqrt{2} \geq 0$. Also, for all $i \in [k]$,

$$p_i)_w = w_i^T p_i = w_i^T ( (p_{i-1})_w w_{i-1} + (p_{i-1})_v v_{i-1} ) = (p_{i-1})_w \cdot \cos(\frac{\pi}{2t_i+2}) + (p_{i-1})_v \cdot \sin(\frac{\pi}{2t_i+2}) \geq (p_{i-1})_w \cdot \cos(\frac{\pi}{2t_i+2}).$$

(by \([23]\))

It then follows by induction that $(p_i)_w \geq 0$ for all $i \in \{0, 1, \ldots, k\}$.

Next, we verify the inequality listed in \([24]\), i.e., $(p_i)_v \geq |v_i^T p_i|_{-1}$ for all $i \in [k]$. Notice that for all $i \in [k]$,

$$|v_i^T p_i|_{i-1} = |v_i^T ((p_{i-1})_v v_{i-1} + (p_{i-1})_w w_{i-1})| \leq |(p_{i-1})_v v_{i-1} + (p_{i-1})_w w_{i-1} | \cdot |v_i^T w_{i-1}| \leq (t_{i-1} + (i-1)s) \cdot \cos(\frac{\pi}{2t_i+2}) + (p_{i-1})_w \cdot \sin(\frac{\pi}{2t_i+2}) \geq (p_{i-1})_w + (t_{i-1} + (i-1)s) \cdot \sin(\frac{\pi}{2t_i+2}) \geq (p_{i-1})_w \geq 0 \text{ and } \geq 0 \quad \text{by } (p_{i-1})_w \geq 0$$

Next, we require an upper bound on $(p_{i-1})_w$. For all $i \in [k]$, from \([35]\)

$$(p_{i-1})_w = (p_{i-1})_w \cdot \cos(\frac{\pi}{2t_i+2}) + (p_{i-1})_v \cdot \sin(\frac{\pi}{2t_i+2}) \leq (p_{i-1})_w + (t_{i-1} + (i-1)s) \cdot \sin(\frac{\pi}{2t_i+2}) \geq (p_{i-1})_w \geq 0 \text{ and } \geq 0 \quad \text{by } (p_{i-1})_w \geq 0 \text{ and } (p_{i-1})_v \geq 0$$

Let $t_0 = v_0^T p_0$ and $\bar{t}_0 = v_0^T \bar{p}_0 = 0$. By applying the above inequality to $1, 2, \ldots, i-1$, we have

$$(p_i)_w \leq w_0^T p_0 + \sum_{j=0}^{i-1} (t_j + js) \cdot \sin(\frac{\pi}{2j+3})$$

35
where the last step comes from Plugging (37) back into (36), we have for all $i$
that in (38), the third term in the brackets is at most the fourth term:
\[
|v_i^\top p_{i-1}| \leq (t_i + (i-1)s) \cdot \cos(\frac{\pi}{2^{i+2}}) + (0.263 + 2.226r + 0.898s) \cdot \sin(\frac{\pi}{2^{i+2}})
\]
\[
\leq \left(\frac{1}{30} + r + (i-1)s\right) + (0.263 + 2.226r + 0.898s) \cdot \sin(\frac{\pi}{8})
\]
\[
\leq 0.134 + 1.852r + (i - 0.656)s
\]
\[
\leq 0.134 + 1.852r + is - 0.656(s - r)
\]
\[
\leq is
\]
\[
\leq (p_i)_v.
\]
Finally, we verify (25). The increment of the first coordinate from $p_{i-1}$ to $p_i$ is
\[
e_i^\top p_i - e_i^\top p_{i-1} = e_i^\top ((p_i)_v v_i - (v_i^\top p_{i-1}) v_i)
\]
\[
= \sin(\frac{\pi}{2^{i+2}}) \cdot (t_i + is - v_i^\top p_{i-1})
\]
\[
= \sin(\frac{\pi}{2^{i+2}}) \cdot \left(t_i + is - v_i^\top \left((p_{i-1})_w w_{i-1} + (p_{i-1})_v v_{i-1}\right)\right)
\]
\[
= \sin(\frac{\pi}{2^{i+2}}) \cdot \left(t_i + is + (p_{i-1})_w \cdot \sin(\frac{\pi}{2^{i+2}}) - (t_{i-1} + (i-1)s) \cdot \cos(\frac{\pi}{2^{i+2}})\right)
\]
where the last step comes from $v_i^\top v_{i-1} = \cos(\frac{\pi}{2^{i+2}})$ and $v_i^\top w_{i-1} = -\sin(\frac{\pi}{2^{i+2}})$. For all $i \geq 2$, we can show that in (38), the third term in the brackets is at most the fourth term:
\[
(p_{i-1})_w \cdot \sin(\frac{\pi}{2^{i+2}}) \leq (0.263 + 2.226r + 0.898s) \cdot \sin(\frac{\pi}{2^{i+2}})
\]
\[
\leq (0.263 + 2.226r + 0.898s) \cdot \tan(\frac{\pi}{8}) \cdot \cos(\frac{\pi}{2^{i+2}})
\]
\[
\leq (0.338 + 0.898 \cdot \frac{11}{30}) \cdot \tan(\frac{\pi}{8}) \cdot \cos(\frac{\pi}{2^{i+2}})
\]
\[
\leq 0.277 \cdot \cos(\frac{\pi}{2^{i+2}})
\]
\[
\leq (\tilde{t}_{i-1} - r + (i - 1)s) \cdot \cos(\frac{\pi}{2^{i+2}})
\]
\[
\leq (t_{i-1} + (i-1)s) \cdot \cos(\frac{\pi}{2^{i+2}}).
\]
Plugging back into (38), we have
\[
e_i^\top p_i - e_i^\top p_{i-1} \leq \sin(\frac{\pi}{2^{i+2}}) \cdot (t_i + is)
\]
\[
\leq \sin(\frac{\pi}{2^{i+2}}) \cdot (\tilde{t}_i + i\tilde{s} + (i+1)r)
\]
\[
\leq \sin(\frac{\pi}{8}) \cdot 1.9^{-(i-1)} \cdot (\frac{1}{30} + \frac{i}{3} + (i+1)r)
\]
\[
\leq \sin(\frac{\pi}{8}) \cdot 1.9^{-(i-1)}
\]
Therefore,
\[ e_1^T pk \leq e_1^T p_0 + \sum_{i=1}^{k} (e_1^T p_i - e_1^T p_i-1) \]
\[ \leq \left( \frac{1}{6} + r \right) + \sin\left( \frac{\pi}{8} \right) \cdot \sum_{i=1}^{k} 1.9^{-(i-1)} \cdot \left( \frac{1}{30} + \frac{i}{3} + (i+1)r \right) \]
\[ \leq 0.763 + 3.514r \leq 1. \]

where the last inequality holds for any \( r \leq \frac{1}{30} \). Therefore (25) holds and \((x, y, p_0, t, s) \in P\).

### 6.5 Properties of the Dual Polytope

In this section, we will analyze the scaled polar dual polytope \( Q = \frac{1}{30}(P - (\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s}))^\circ \). From well-known duality properties, we will find that \( Q \) satisfies the following desirable properties:

1. \( Q \cap W \) approximates a two-dimensional disk;
2. The inner radius of \( Q \) is at least \( \frac{1}{45} \) when centered at 0;
3. The outer radius of \( Q \) is at most 1 when centered at 0.

**Lemma 56.** For any \( k \in \mathbb{N} \), there exists a two-dimensional linear subspace \( W \subseteq \mathbb{R}^{k+5} \) and \( n = 4k + 7 \) points \( a_1, \ldots, a_n \in \mathbb{B}^{k+5} \) such that \( Q := \text{conv}(a_1, \ldots, a_n) \) satisfies

\[ \cos\left( \frac{\pi}{2k+2} \right) \cdot \mathbb{B}^{k+5} \cap W \subseteq Q \cap W \subseteq \frac{1}{30} \cdot \mathbb{B}^{k+5} \cap W \]

and

\[ \frac{1}{45} \cdot \mathbb{B}^{k+5} \subseteq Q \subseteq \mathbb{B}^{k+5}. \]

**Proof.** Let \( P \subseteq \mathbb{R}^{k+5} \) be the polytope defined by the linear system in (28), and let

\[ \tilde{P} = P - (\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s}) \]

denote the polytope obtained from shifting its center \((\bar{x}, \bar{y}, \bar{p}_0, \bar{t}, \bar{s})\) to 0. Here, as in Lemma 55

\[ \bar{x} = \bar{y} = 0, \bar{p}_0 = (1/6, 1/6)^T, \bar{t} = k/30, \bar{s} = 1/3. \]

By applying row rescalings we transform the constraints (22 - 27) into a matrix \( A \in \mathbb{R}^{(4k+7) \times (k+5)} \) such that \( \tilde{P} = \{ z \in \mathbb{R}^{k+5} : Az \leq 1 \} \).

Let \( \tilde{Q} = (\tilde{P})^\circ \subseteq \mathbb{R}^d \) denote the polar body of \( \tilde{P} \). Since \( \tilde{P} \) is bounded, \( \tilde{Q} \) is the convex hull of the rows of the matrix \( A \), i.e.

\[ \tilde{Q} = \{ A^T \lambda : \lambda \in [0,1]^{4k+7} \text{ s.t. } \sum_{i=1}^{4k+7} \lambda_i = 1 \}. \]

Then by Lemma 54 and Fact 9 the inner and outer ball of \( \tilde{Q} \) satisfy

\[ \frac{2}{3} \cdot \mathbb{B}^{k+5} \subseteq \tilde{Q} \subseteq 30 \cdot \mathbb{B}^{k+5}. \]

Also, by Lemma 51, Fact 10 and Fact 11 the inner and outer ball of \( \tilde{Q} \cap W \) satisfy

\[ \cos(\pi/2k+2) \cdot \mathbb{B}^{k+5} \cap W \subseteq \tilde{Q} \cap W \subseteq \mathbb{B}^{k+5} \cap W. \]

The lemma then follows from taking \( Q = \frac{1}{30} \tilde{Q} \). \( \Box \)
6.6 Perturbation Analysis and Proof of the Lower Bound

In this subsection, we study the number of edges of the intersection polygon $Q \cap W$ after perturbation and prove our main lower bound theorem (Theorem 49). To show that our construction has many edges even after perturbation, we require the following two statements:

**Lemma 57.** Let $\bar{a}_1, \ldots, \bar{a}_n \in \mathbb{R}^d$ be points with $r\mathbb{B}_1^d \subseteq \text{conv}(\bar{a}_1, \ldots, \bar{a}_n)$ for some $r > 0$. If $\epsilon \leq r/2$ and points $a_1, \ldots, a_n \in \mathbb{R}^d$ satisfy $||a_i - \bar{a}_i||_1 \leq \epsilon$ for all $i \in [n]$ then it follows that

$$(1 - \frac{2\epsilon}{r}) \text{conv}(\bar{a}_1, \ldots, \bar{a}_n) \subseteq \text{conv}(a_1, \ldots, a_n) \subseteq (1 + \frac{\epsilon}{r}) \text{conv}(\bar{a}_1, \ldots, \bar{a}_n).$$

**Proof.** Write $Q = \text{conv}(a_1, \ldots, a_n)$ and $\tilde{Q} = \text{conv}(\bar{a}_1, \ldots, \bar{a}_n)$. The second inclusion follows by $Q \subseteq \tilde{Q} + \epsilon \mathbb{B}_1^d \subseteq \tilde{Q} + \frac{\epsilon}{r} Q$. For the first inclusion, we observe that $r \mathbb{B}_1^d \subseteq Q \subseteq \mathbb{B}_1^d \subseteq Q + \frac{\epsilon}{r} \mathbb{B}_1^d$. This implies that $\frac{\epsilon}{r} \mathbb{B}_1^d \subseteq Q$, for if there were to exist $x \in \mathbb{B}_1^d$ such that $x \notin Q$ then, since $Q$ is closed and convex, then we could find $y \in \mathbb{R}^d$ such that $y^\top x > y^\top z$ for all $z \in Q$. Writing $f(S) = \max_{z \in S} y^\top z$ for $S \subseteq \mathbb{R}^d$, this would give

$$f(r \mathbb{B}_1^d) \geq f(x + \frac{r}{2} \mathbb{B}_1^d) = y^\top x + f(\frac{r}{2} \mathbb{B}_1^d) > f(Q) + f(\frac{r}{2} \mathbb{B}_1^d) \geq f(Q + \frac{r}{2} \mathbb{B}_1^d) \geq f(r \mathbb{B}_1^d).$$

By contradiction it follows that $\frac{\epsilon}{r} \mathbb{B}_1^d \subseteq Q$.

Now the desired result follows by $Q \subseteq Q + \epsilon \mathbb{B}_1^d \subseteq Q + \frac{\epsilon}{r} \mathbb{B}_1^d$ and the fact that $(1 + x)^{-1} \geq 1 - x$ for every $x > -1$.

**Lemma 58.** If a polygon $T \subseteq \mathbb{R}^2$ satisfies $\alpha \cdot \mathbb{B}_2^d \subseteq T \subseteq \beta \cdot \mathbb{B}_2^d$ for some $\alpha, \beta > 0$ then $T$ has at least $\sqrt{\alpha / (\beta - \alpha)}$ edges.

**Proof.** If $\beta > 2\alpha$ then the bound is trivially true, so assume that $\beta \leq 2\alpha$.

Without loss of generality, re-scale $T$ so that $\mathbb{B}_2^d \subseteq T \subseteq (1 + \epsilon) \cdot \mathbb{B}_2^d$, where $\epsilon = \beta / \alpha - 1 > 0$.

Consider any edge $[q_1, q_2] \subseteq T$ and let $p \in [q_1, q_2]$ denote the minimum-norm point in this edge. Then we have $||q_1 - p||^2 = ||q_2||^2 + ||p||^2 - 2\langle q_1, p \rangle$. Since $p$ is the minimum-norm point, we have $\langle q_1, p \rangle \geq ||p||^2$, and hence $||q_1 - p||^2 \leq ||q_2||^2 - ||p||^2 \leq (1 + \epsilon)^2 - ||q_2||^2$. Since $p$ lies on the boundary of $T$ we have $||p|| \geq 1$, which implies that $||q_1 - p||^2 \leq (1 + \epsilon)^2 - 1 = 2\epsilon + \epsilon^2$. The analogous argument for $||q_2 - p||$ and the triangle inequality tell us that $||q_1 - q_2|| \leq 2\sqrt{2\epsilon + \epsilon^2} \leq 4\sqrt{\epsilon}$. The choice of the edge $[q_1, q_2]$ was arbitrary, hence every edge of $T$ has length at most $4\sqrt{\epsilon}$.

But $T$ has perimeter at least $2\pi$. Since the perimeter of a polygon is equal to the sum of the lengths of its edges, this implies that $T$ has at least $\frac{2\pi}{4\sqrt{\epsilon}} > \frac{1}{\sqrt{\epsilon}}$ edges.

Now, we can prove our generic lower bound Theorem 50 on the shadow size under adversarial $\ell_1$-perturbations.

**Proof of Theorem 49.** Fix any $d > 5$, let $k = d - 5$ and observe that $n = 4k + 7$. Let $\bar{a}_1, \ldots, \bar{a}_n$ be as constructed in Lemma 56. Then we have

$$\frac{\cos(\pi/2k+2)}{30} \cdot \mathbb{B}_2^{k+5} \cap W \subseteq \text{conv}(\bar{a}_1, \ldots, \bar{a}_n) \cap W \subseteq \frac{1}{30} \cdot \mathbb{B}_2^{k+5} \cap W,$$

and

$$\frac{1}{45} \cdot \mathbb{B}_1^{k+5} \subseteq \text{conv}(\bar{a}_1, \ldots, \bar{a}_n) \subseteq \mathbb{B}_1^{k+5}.$$
Therefore, we can bound the inner and outer radius of \( \text{conv}(a_1, \ldots, a_n) \cap W \) by

\[
\frac{\cos(\pi/2^{k+2})}{30 \cdot (1 + 90\varepsilon) \cdot 2^{k+5}} \cap W \subseteq \text{conv}(a_1, \ldots, a_n) \cap W \subseteq \frac{1}{30 \cdot (1 - 90\varepsilon) \cdot 2^{k+5}} \cap W.
\]

It follows from Lemma 58 that the polygon \( \text{conv}(a_1, \ldots, a_n) \cap W \) has at least \( \Omega\left(\frac{1}{\sqrt{\varepsilon + 4 - k}}\right) \) edges.

Finally, we can prove our main result using Gaussian tail bound:

**Proof of Theorem 49** Using concentration of Gaussian distribution in Corollary 13, we find that if \( \sigma \leq 1/(360d\sqrt{\log n}) \), then with probability at least \( 1 - (\tfrac{n}{d})^{-1} \), we have \( \max_{i \in [n]} \|a_i - \bar{a}_i\|_2 \leq 4\sigma\sqrt{d\log n} \leq \frac{1}{90\sqrt{d}} \).

The result follows from Theorem 50 and the fact that \( \|x\|_1 \leq \sqrt{d}\|x\|_2 \) for every \( x \in \mathbb{R}^d \).

### 6.7 Experimental Results

To measure whether analysis in Theorem 49 is tight or not, we ran numerical experiments. Using Python and Gurobi 10.0.3, we constructed a matrix \( A \) such that

\[
P - (\bar{x}, \bar{y}, \bar{p}_0, \bar{l}, \bar{s}) = \{ z \in \mathbb{R}^{k+5} : Az \leq 1_{4k+7} \},
\]

as described earlier in this section. Writing \( R \) as the maximum Euclidean norm among the row vectors of \( A \), we sampled \( \bar{A} \) with independent Gaussian distributed entries with standard deviation \( \sigma R \) and \( \mathbb{E}[\bar{A}] = A \). To approximate the shadow size, we optimized the objective vectors \( \cos(\tfrac{(i+0.3)\pi}{2^{k+1}})x + \sin(\tfrac{(i+0.3)\pi}{2^{k+1}})y \), with \( i = 0, \ldots, 2^{k+5} - 1 \), over the polyhedron \( \{ z \in \mathbb{R}^{k+5} : \bar{A}z \leq 1 \} \) and counted the number of distinct values \( (x,y) \) found among the solutions. Two consecutive solutions were counted as distinct if their \( x, y \) coordinates differed in \( \ell_1 \) norm by at least \( 10^{-11} \).

When \( \sigma = 0 \), our code found \( 2^{k+1} \) such distinct points. For \( \sigma > 0 \), Theorem 49 shows that we expect to find at least \( \Omega\left(\min\left(\frac{1}{\sqrt{d}\sqrt{\log d}}, 2^{k}\right)\right) \) distinct pairs \((x,y)\).

For \( k = 10, 15, 20 \), we measured the shadow size for 20 different values of \( \sigma \) ranging from 0.01 to 0.0001/\( 2^k \). The resulting data is depicted in Figure 5 along with a graph of the function \( \sigma \mapsto \sigma^{-\frac{3}{4}} \). We observe that for each \( k \), the measured shadow size appears to follow the graphed function up to a point, plateauing slightly above \( 2^{k+1} \) when \( \sigma \) is small. The fact that some measurements come out higher than \( 2^{k+1} \), the shadow size for \( \sigma = 0 \), is not unexpected: the polytope \( P \) is highly degenerate, whereas the perturbed polytope is simple and can thus have many more vertices.

The measured shadow sizes appear to grow much faster than \( 1/\sqrt{\sigma} \) as \( \sigma \) gets small, closer to the \( \sigma^{-3/4} \) line that we plotted. These results suggest that the behaviour of the shadow size is substantially different in \( d = 2 \), where we have an upper bound of \( O\left(\frac{\sqrt{\log(n)}}{\sqrt{\sigma}} + \sqrt{\log n}\right) \), and \( d > 3 \), where one might expect a lower bound with a higher dependence on \( \sigma \).

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\(^{2}\)The code used for this experiment is available at [https://github.com/sophiehuiberts/shadow-size/tree/main/HLZ23](https://github.com/sophiehuiberts/shadow-size/tree/main/HLZ23)
Figure 5: Measured shadow sizes for sampled perturbations of our construction, for different values of $k$ and $\sigma$.

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