Duality of Super D-brane Actions
in
General Type II Supergravity Background

Tadahiko Kimura

Department of Physics, Faculty of Science, Chiba University, Chiba 263-8522, JAPAN

and

Ichiro Oda

Edogawa University, 474 Komaki, Nagareyama City, Chiba 270-0198, JAPAN

Abstract

We examine duality transformations of supersymmetric and $\kappa$-symmetric Dp-brane actions in a general type II supergravity background where in particular the dilaton and the axion are supposed to not be zero or a constant but a general superfield. Due to non-constant dilaton and axion, we can explicitly show that the dilaton and the axion as well as the two 2-form gauge potentials transform as doublets under the $SL(2, R)$ transformation from the point of view of the world-volume field theory.

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1 E-mail address: kimura@cuphd.nd.chiba-u.ac.jp
2 E-mail address: ioda@edogawa-u.ac.jp
1 Introduction

Notably with the advent of the 'Dirichlet brane', which is nowadays referred to as the D-brane, discovered by Polchinski [1], the D-brane theory in various world-volume dimensions has been widely recognized as an essential ingredient in which to discuss a variety of non-perturbative aspects of superstring theory and M-theory, especially the ones concerning the intricate web of string dualities [2]. It is also remarkable that D-branes not only explain the origin of a black hole entropy [3, 4] but also provide the elementary building blocks of matrix models [5, 6]. Thus it is desirable to achieve as thorough an understanding of physical properties of D-branes as possible.

In a previous paper [7] we have shown that the supersymmetric and $\kappa$-symmetric Dp-brane actions [8, 9, 10] in a type II supergravity background have the same duality transformation properties as those in a flat Minkowski background [11]. Specially, it is shown that super D-string transforms in a covariant way while super D3-brane is self-dual under the $SL(2, Z)$ duality. Also, D2-brane and D4-brane transform in ways expected from the relation between type IIA superstring theory and M-theory.

However, in the study [7] we have confined ourselves to a restricted background geometry where the dilaton and the axion are set to zero or a constant although the other fields are considered to be general superfields. This restriction is obviously unsatisfactory from the following three reasons.

First, the dilaton and the axion are ordinary interacting particles in the low energy theory of superstring and thus should be treated on same footing as the other low energy fields such as the graviton, the antisymmetric second rank tensor and fermions.

Second, in a type IIB superstring or supergravity it is well known that the dilaton and the axion as well as the two 2-form antisymmetric tensor fields, those are, the NS-NS 2-form and the R-R 2-form, are doublets of the $SL(2, R)$ Mobius group whereas the graviton and the 4-form gauge field are singlets in the Einstein metric. Thus, if we wish to show that they transform covariantly under the $SL(2, R)$ transformation these local fields must be treated as not constants but fields. In this context, notice that the vanishing or constant dilaton and axion imply the vanishing 3-form field strengths, so in order to understand the transformation rules under the $SL(2, R)$ we are led to consider the non-constant dilaton and axion.

Finally, a study of general dilaton and axion may shed light on F-theory [12], a conjectured 12-dimensional quantum field theory underlying a type IIB superstring. F-theory is expected to give a systematic description of the IIB superstring with non – trivial dilaton and axion background to which the complex structure of a two-dimensional torus would correspond. According to these reasons, the extension of our previous work [7] to the non-constant dilaton and axion system is not only valuable but also quite non-trivial.

In this paper, we will remove our previous restriction on the dilaton and the axion completely and prove that various duality transformations found in a type II supergravity background with the constant or vanishing dilaton and axion [7] can be extended to a more general

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$^3$One of authors (I. O.) would like to thank M. Tonin for pointing out this issue.
situation where the dilaton and the axion are superfields depending on the supercoordinates \(Z^M = (X^m, \theta^\mu)\). Therefore, the present study offers a complete proof that various duality relations in the super D-brane actions, which were constructed in [8, 9, 10], exist in the general on-shell type II supergravity background geometry.

Here let us summarize the results that will be obtained in this paper. As in the cases of a flat Minkowski [11] and a type II supergravity backgrounds [7] with the constant dilaton and axion, we can show that the super D-string action is the \((m, 1)\) string action with the NS-NS charge \(m\) and the R-R charge 1. We prove that the super D3-brane action in the general background is self-dual under the \(SL(2, Z)\) duality transformation in both the classical and the quantum-mechanical exact approaches. Furthermore, it is shown that under a duality transformation the super D2-brane and D4-brane actions transform in ways expected from the relation between type IIA superstring theory and M-theory. Namely, the dual action of the super D2-brane action is identified as the M2-brane action with a circular 11th dimension. As for D4-brane, the double-dimensional reduction of the M5-brane action gives rise to the dual action of the super D4-brane action.

It is worthwhile to point out the technical differences between the case of the constant (or zero) dilaton and axion and that of the non-constant dilaton and axion. Moving the former on to the latter, we would encounter at least two complications. The one is that we have to introduce a new superfield \(\Lambda\), i.e., the dilatino superfield, associated with the dilaton superfield in a theory, which gives rise to rather complicated constraint structures [8, 9]. Indeed, due to this complication we have confined our consideration to the constant or zero dilaton and axion in our previous work [7]. The second complication that arises is that the non-constant axion makes it impossible to use a convenient technique; in the case of the constant axion we are free to add a theta term to the dual action at the end of calculations by hand to derive the \(SL(2, Z)\) covariance of the super D-string tension and the \(SL(2, Z)\) self-duality of the super D3-brane [11] while in the case of the non-constant axion we have to deal with the corresponding Wess-Zumino term from the beginning. As mentioned in the above, this convenient prescription is far from complete from the point of view of proof of the \(SL(2, R)\) symmetry.

At this point let us comment briefly several previous articles relating to the study at hand. The duality transformations of the bosonic D-brane action have been investigated in Refs. [14, 15, 16, 17]. Afterwards, supersymmetric D-brane actions with local kappa symmetry were constructed in a flat background [18] and in a type II supergravity background [8, 1, 10]. Various duality transformations in a flat background have been clarified in the context of the super D-brane actions in Ref. [11]. (See also an important paper [19].) Recently, motivated by AdS/CFT correspondence [20], the duality transformations of super D-string action [21] and super D3-brane action [22, 23, 24] were clarified in the \(AdS_5 \times S^5\) background. More recently, we have generalized these works to a type II supergravity background [13, 4]. However, these works are not completely general in that we have limited our consideration to the constant or zero dilaton and axion except super D-string [13]. The main motivation in this paper is to present a full detail of the duality transformations of super D-brane actions constructed
in Refs. [8, 9, 10] in a general type II supergravity background. Incidentally, there are several interesting papers which attempt to construct actions of super D-branes with the manifest $SL(2, Z)$-duality and a dynamical tension [23, 29]. It would be of interest to ask possible relations between these works and the present study in future.

This article is organized as follows. Section 2 reviews super D-brane actions in a general II supergravity background [8, 9, 10]. In Section 3 it is then proved in both classically and quantum-mechanically exact manner that the super D-string action in a type IIB on-shell supergravity background is transformed to the type IIB Green-Schwarz superstring action [27] in a background with the NS-NS charge $-\lambda$ and the RR-charge 1. Section 4 deals with the super D2-brane in a type IIA on-shell supergravity background and presents that the super D2-brane action can be transformed to the super M2-brane action with a circular eleventh dimension by a duality transformation. In Section 5 we show that the super D3-brane action in a type IIB on-shell supergravity background is mapped into itself by an S-duality transformation, thereby verifying the $SL(2, Z)$ self-duality of the action. We shall offer both classical and quantum-mechanical proofs here. In Section 6 it is shown that the super D4-brane action becomes identical to the supersymmetric action which is obtained in terms of double-dimensional reduction of the super M5-brane action in the eleven dimensional space-time through a duality transformation. The final section will be devoted to discussions. In Appendix a brief review of the $SL(2, R)/SO(2)$ coset description of the dilaton and the axion in the type IIB supergravity is given and the type IIB supergravity constraint equations are shown to be invariant under the $SL(2, R)$ duality transformation.

2 Super D-brane actions

Before describing our results for various duality transformations of super D-brane actions [8, 9, 10] in a general type II supergravity background, we shall review the salient points of the superspace formulation of the super Dp-brane theories. It is well known that super Dp-brane actions are divided into two pieces, namely, the Dirac-Born-Infeld action and the Wess-Zumino action in $(p+1)$-dimensional world-volume. The former includes the NS-NS two-form, dilaton and world-volume metric in addition to Abelian gauge field while the latter contains the coupling of the D-brane to the R-R fields. The existence of this Abelian gauge field is a peculiar feature of D-brane theories. The two terms are separately invariant under type II superspace reparametrizations as well as $(p + 1)$-dimensional general coordinate transformations. However, local $\kappa$ symmetry is achieved by a suitable conspiracy between the two pieces.

Then, the super Dp-brane actions in a general type II on-shell supergravity background which we consider in this paper are given by

$$S = S_{DBI} + S_{WZ},$$

(1)
with

\[
S_{DBI} = -\int_{M^{p+1}} d^{p+1}\sigma e^{\frac{4}{g^2} \sqrt{-\det(G_{ij} + e^{-\frac{1}{2} \phi} F_{ij})}},
\]

\[
S_{WZ} = \int_{M^{p+1}} e^F \wedge C = \int_{M^{p+1}} \Omega_{p+1} = \int_{M^{p+2}} I_{p+2},
\]

where \( \sigma^i \) \((i = 0, 1, \ldots, p)\) are the world-volume coordinates, \( \phi \) the dilaton superfield, and \( G_{ij} \) the induced Einstein metric of the world-volume. We have defined various quantities as follows:

\[
\begin{align*}
F &= F - b_2, \\
F &= dA, \\
C &= \bigoplus_{n=0}^9 C(n), \\
I_{p+2} &= d\Omega_{p+1} = d(e^F \wedge C), \\
M_{p+1} &= \partial M_{p+2},
\end{align*}
\]

where \( F \) is the Maxwell field strength 2-form, and the 2-form \( b_2 \) is introduced in \( F \) such that \( F \) is invariant under supersymmetry. And the R-R \( n \)-form fields \( C(n) \) are collected in \( C \) with \( n \) taking odd integers for type IIA and even integers for type IIB. The integration of the integrand \( \mathbb{2} \) involves forms of various rank; the integral picks out precisely the terms that are proportional to the volume form of the p-brane world-volume.

In addition, in order to describe the curved target superspace geometry we have to introduce the superspace vielbein 1-form \( E^A \) defined by

\[
E^A = dZ^M E^A_M,
\]

with \( dZ^M \) denoting the superspace differential \( (dX^m, d\theta^\mu) \), and the torsion 2-form \( T^A = DE^A \) as well as the curvature 2-form defined in terms of the spin connection \( \omega_A{}^B \) as

\[
R_A{}^B = d\omega_A{}^B + \omega_A{}^C \wedge \omega_C{}^B.
\]

Note that we have also defined as \( M = (m, \mu) \) in curved superspace while \( A = (a, \alpha) \) in flat superspace as usual. The superspace vielbein 1-form \( E^A \) in the tangent space decomposes under the action of the Lorentz group into a vector \( E^a \) and a spinor \( E^\alpha \). We shall follow the conventions that the Lorentz spinor \( E^\alpha \) is a 32-component Majorana spinor for the type IIA superspace, on the other hand, a pair of 16-component Majorana-Weyl spinors for the type IIB superspace so that the latter may be written as \( E^{I\alpha} \) with \( I \) being the \( N = 2 \) index \((I = 1, 2)\).

Then the world-volume metric \( G_{ij} \) is represented by

\[
G_{ij} = E_i{}^a E_j{}^b \eta_{ab},
\]
where $E_i A = \partial_i Z^M E_M A$ and $\eta_{ab} = \text{diag}(\ldots, +, \ldots, +)$.

Throughout this paper we use following conventions for superspace forms. Firstly, a general $n$-form superfield $\Omega_{(n)}$ is expanded as

$$
\Omega_{(n)} = \frac{1}{n!} dZ^M \wedge \ldots \wedge dZ^M \Omega_{M_1 \ldots M_n},
$$

$$
= \frac{1}{n!} E^{A_n} \wedge \ldots \wedge E^{A_1} \Omega_{A_1 \ldots A_n},
$$

(7)

where the superspace differential $dZ^M$ and the superspace vielbein $E^A$ are antisymmetric with respect to bosonic coordinates while they are symmetric with respect to fermionic coordinates.

Secondly, we define the exterior derivative as an operator acting from the right

$$
d(\Omega_{(m)} \wedge \Omega_{(n)}) = \Omega_{(m)} \wedge d\Omega_{(n)} + (-)^n d\Omega_{(m)} \wedge \Omega_{(a)}. 
$$

(8)

Now, following the paper [9], let us define the NS-NS 3-form superfield $H_3$ and the R-R $n$-form superfield $R$ as

$$
H_3 = db_2,
$$

$$
R = e^{b_2} \wedge d(e^{-b_2} \wedge C) = \bigoplus_{n=1}^{10} R_{(n)}. 
$$

(9)

It is obvious that from these definitions the field strengths obey the following Bianchi identities

$$
dH_3 = 0,
$$

$$
e^{b_2} \wedge d(e^{-b_2} \wedge R) = dR - R \wedge H_3 = 0. 
$$

(10)

In order to reduce the enormous unconstrained field content included in the superfields to the field content of the on-shell type II supergravity theory, one has to impose the constraints on the torsion and the field strengths by hand, which make various Bianchi identities coincide with the equations of motion of type II supergravity. The nontrivial constraints imposed on the torsion and field strength components [9] take the following forms for type IIA:

$$
T_{\alpha\beta}^c = 2i\gamma_{\alpha\beta}^c,
$$

$$
T_{\alpha\gamma} = \frac{3}{2} \delta_{(\alpha} \gamma_{\beta)} + 2(\gamma_{11})_{(\alpha} \gamma(\gamma_{11})_{\beta)} - \frac{1}{2} (\gamma_a)_{\alpha\beta} (\gamma^a \Lambda)^\gamma
$$

$$
+ (\gamma_a \gamma_{11})_{\alpha\beta} (\gamma^a \gamma_{11} \Lambda)^\gamma + \frac{1}{4} (\gamma_{ab})_{(\alpha} \gamma(\gamma^{ab} \Lambda)_{\beta)},
$$

$$
H_{aa\beta} = -2i e^{\frac{a}{2}}(\gamma_{11} \gamma_a)_{\alpha\beta},
$$

$$
H_{aba} = e^{\frac{a}{2}}(\gamma_{ab} \gamma_{11} \Lambda)_a,
$$

$$
R_{(n)a_1 \ldots a_{n-2}\alpha\beta} = 2i e^{\frac{a}{2}}(\gamma_{a_1 \ldots a_{n-2}}(\gamma_{11})^\frac{a}{2})_{\alpha\beta},
$$

$$
R_{(n)a_1 \ldots a_{n-1}\alpha} = -\frac{n-5}{2} e^{\frac{a}{4}}(\gamma_{a_1 \ldots a_{n-1}}(-\gamma_{11})^\frac{a}{2} \Lambda)_a,
$$

(11)

4See the ref.[10] for type IIA massive supergravity, i.e., $R_0 = m$.
and for IIB:

\[
T_{\alpha\beta}^c = 2i\gamma_{\alpha\beta}^c, \\
T_{\alpha\beta}^\gamma = -(\mathcal{I}(\alpha)^\gamma(\mathcal{K}\Lambda)_{\beta}) + (\mathcal{K}(\alpha)^\gamma(\mathcal{K}\Lambda)_{\beta}) \\
+ \frac{1}{2}(\gamma_{\alpha}\mathcal{I})_{\alpha\beta}(\gamma^a\mathcal{I}\Lambda)^\gamma - \frac{1}{2}(\gamma_{\alpha}\mathcal{K})_{\alpha\beta}(\gamma^a\mathcal{K}\Lambda)^\gamma,
\]

\[
H_{\alpha\alpha\beta} = -2ie^{\frac{1}{2}\phi}(\mathcal{K}\gamma_{\alpha})_{\alpha\beta}, \\
H_{\alpha\beta\alpha} = e^{\frac{1}{2}\phi}(\gamma_{\alpha}\mathcal{K}\Lambda)_{\alpha}, \\
R_{(n)a_1...a_{n-2}\alpha\beta} = 2ie^{\frac{n-5}{2}\phi}(\gamma_{a_1...a_{n-2}}\mathcal{K}^{n-1}\mathcal{E})_{\alpha\beta}, \\
R_{(n)a_1...a_{n-1}\alpha} = -\frac{n-5}{2}e^{\frac{n-5}{2}\phi}(\gamma_{a_1...a_{n-1}}\mathcal{K}^{n-1}\mathcal{E}\Lambda)_{\alpha},
\]

where the round brackets enclosing indices denote symmetrization with 'strength one', and then \( \Lambda_{\alpha} \equiv \frac{1}{2}\partial_{\alpha}\phi \), and \( \mathcal{E}, \mathcal{I}, \) and \( \mathcal{K} \) describing the \( SL(2, R) \) matrices are defined in terms of the conventional Pauli matrices \( \sigma_i \) as follows:

\[
\mathcal{E} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{I} = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{K} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Based on this formulation of the super Dp-brane actions in a type II on-shell supergravity background, we shall explore various duality symmetries in subsequent sections.

### 3 The super D-string

In this section we would like to consider the super D-string (i.e. the super D1-brane) first. The super D2, D3 and D4-branes will be treated in order in subsequent sections. In these sections we shall prove various duality symmetries of the super D-brane actions in a type II on-shell supergravity background. The corresponding proofs have been already done in a flat Minkowskian background [11] and a type II supergravity background with the constant or zero dilaton and axion [7]. A partial result was also presented in a type IIB supergravity background with the non-constant dilaton and axion [13]. The purpose of this section is to offer the full detail and address the issue of the \( SL(2, Z) \) symmetry in such a most general background. Our study appears to be quite important for future development in string theory and M-theory since the global discrete symmetries such as the \( SL(2, Z) \) S-duality are nowadays believed to be exact symmetries in still mysterious underlying theory [28, 29] so that these symmetries should be valid even in a curved background geometry.

In the case at hand, by making use of the superspace convention [7] the action (1), (2) and the relevant constraints (12) for type IIB reduce to

\[
S = S_{DBI} + S_{WZ}, \\
S_{DBI} = -\int_{M_2} d^2\sigma e^{-\frac{1}{2}\phi}\sqrt{-\det(\mathcal{G}_{ij} + e^{-\frac{1}{2}\phi}\mathcal{F}_{ij})},
\]
\begin{equation}
S_{WZ} = \int_{M_2=\partial M_3} \left( C_2 + C_0 F \right) = \int_{M_3} I_3,
\end{equation}

\begin{align*}
H_3 &= db_2 = ie^{\frac{i}{2}\phi} \bar{E} \wedge \hat{E} \wedge \mathcal{K} E + \frac{1}{2} e^{\frac{i}{2}\phi} \bar{E} \wedge \gamma_{ab} \mathcal{K} E \wedge E^a,
R_{(1)} &= dC_0 = 2e^{-\phi} \bar{E} \mathcal{E} \Lambda,
R_{(3)} &= dC_2 - H_3 C_0
\end{align*}

\begin{equation}
= -ie^{-\frac{i}{2}\phi} E \wedge \hat{E} \wedge \mathcal{I} E + \frac{1}{2} e^{-\frac{i}{2}\phi} E \wedge \gamma_{ab} \mathcal{I} E \Lambda E^b \wedge E^a,
\end{equation}

where \( \bar{E}, \hat{E} \) and \( E \) represent the Dirac conjugate of \( E^{I\alpha}, E^a \gamma_a \) and \( E^{I\alpha} \), respectively. The equations in (14) and (15) will serve as our basis for all subsequent considerations in this section.

### 3.1 The classical analysis

In this subsection, we turn our attention to a duality transformation of the super D-string action in a classical approach. Next subsection will provide a quantum-mechanical exact treatment of the same model.

Having exhibited the explicit forms of the classical action and the constraints of the super D-string, the next task is to construct a dual action out of them. The strategy for this purpose is now standard, but for definiteness let us present the detailed exposition. As a first step, one needs to incorporate a Lagrange multiplier field \( \tilde{H}_{ij} = -\tilde{H}_{ji} \) into the classical action as follows [15]:

\begin{equation}
S = -\int_{M_2} d^2 \sigma e^{-\frac{i}{2}\phi} \sqrt{-\det(G_{ij} + e^{-\frac{i}{2}\phi} F_{ij})} + \int_{M_2} \left( C_2 + C_0 F \right) \epsilon_{ij} F_{ij} + \int_{M_2} d^2 \sigma \frac{1}{2} \tilde{H}^{ij} (F_{ij} - 2 \partial_i A_j),
\end{equation}

and then regard \( F_{ij} \) as an independent superfield. The equation of motion of \( A_i \) gives rise to \( \partial_i \tilde{H}^{ij} = 0 \) whose classical solution is found to be \( \tilde{H}^{ij} = \varepsilon^{ij} \lambda \) with a constant scalar superfield \( \lambda \).

Next, substituting this classical solution into the original action (14) gives us an action \( S' = S_1 + S_{2D} \), where

\begin{align*}
S_1 &= \int_{M_2} d^2 \sigma \left[ -e^{-\frac{i}{2}\phi} \sqrt{-\det(G_{ij} + e^{-\frac{i}{2}\phi} F_{ij})} + \frac{1}{2} (\lambda + C_0) \varepsilon^{ij} F_{ij} \right],
S_{2D} &= \int_{M_2} (C_2 + \lambda \varepsilon_2).
\end{align*}

The important procedure to get a dual action is to solve the equation of motion obtained by varying \( F_{ij} \) to express the action \( S' \) in terms of \( \lambda \) instead of \( F_{ij} \). Note that since \( S_{2D} \) does not
include $F_{ij}$ in it, this part of the action is manifestly unaffected by the duality transformation. A simple calculation leads to

$$F_{01} = -e^{\frac{1}{2}\phi} \frac{\lambda + C_0}{\sqrt{e^{-2\phi} + (\lambda + C_0)^2}} \sqrt{-\det G_{ij}},$$

where we have used the formula holding for $2 \times 2$ matrices

$$\det(G_{ij} + e^{-\frac{1}{2}\phi} F_{ij}) = \det G_{ij} + e^{-\phi} F_{01}^2 = \det G_{ij} + \frac{1}{4} e^{-\phi} (\varepsilon_{ij} F_{ij})^2.\hspace{1cm}(19)$$

It follows that inserting Eq.(18) to $S_1$ together with $S_2$ yields the dual action of the super D-string action

$$S_D = -\int_{M^2} d^2 \sigma e^{\frac{1}{2}\phi} \sqrt{e^{-2\phi} + (\lambda + C_0)^2} \sqrt{-\det G_{ij}} + \int_{M^2} (C_2 + \lambda b_2).\hspace{1cm}(20)$$

We may interpret the dual action (20) in the following way. If we regard the Wess-Zumino term in (20), $\int_{M^2} (C_2 + \lambda b_2)$, as the source term for the NS-NS and the R-R 2-form potentials, it turns out that this dual action carries the NS-NS charge $-\lambda$ and the R-R charge $1$. Then provided that we identify the constant scalar superfield $-\lambda = m$ with the NS-NS charge corresponding to the $(m, 1)$ string, the result obtained above, $\sqrt{e^{-2\phi} + (m - C_0)^2}$, agrees with the tension formula for the $SL(2, Z)$ S-duality spectrum of strings in the type IIB superstring [31]. This identification means that the D-string action may be actually the action for an arbitrary number of 'fundamental' IIB strings bound to a single D-string. This singleness of D-string of course reflects the $U(1)$ character of the Abelian gauge field since a system of $N$ coincident Dp-branes would be described by the non-abelian version of the Dirac-Born-Infeld action [32].

Now let us examine the implication of Eq.(20) in more detail and consider the exact connection with the type IIB Green-Schwarz superstring action. For that purpose we rewrite the Wess-Zumino term in (20) in the form

$$d\Omega_D \equiv d(C_2 + \lambda b_2) = R_{(3)} + (\lambda + C_0) H_3 = i \tilde{E} \wedge \gamma_a \left[ e^{-\frac{1}{2}\phi} (\lambda + C_0) e^{\frac{1}{2}\phi} \mathcal{K} \right] E \wedge E^a + \frac{1}{2} \tilde{E} \wedge \gamma_{ab} \left[ e^{-\frac{1}{2}\phi} (\lambda + C_0) e^{\frac{1}{2}\phi}, \mathcal{K} \right] \Lambda E^b \wedge E^a.\hspace{1cm}(21)$$

---

5Here we have used the fact that $(b_2, -C_2)$ is the $SL(2, R)$ doublet as shown around the end of this section.

6See the next subsection about the reason why the scalar $\lambda$ becomes an integer $m$.

7For comparison of the string tension, we need to consider the string metric defined as $G_{ij}^{(s)} = e^{\frac{1}{2}\phi} G_{ij}$ where $G_{ij}$ is the Einstein metric.
where Eq.(13) was used. Although the two matrices in the square brackets which are $2 \times 2$ hermitian matrices, have the same eigenvalues $\pm e^{-\frac{1}{2} \phi} \sqrt{1 + (\lambda + C_0)^2} e^{2\phi}$, these matrices are not mutually commutable so that they cannot be diagonalized simultaneously.

In the case of constant dilaton and axion the last term in (21) vanishes and the first term can be diagonalized by a suitable $SO(2)$ spinor-rotation. As a result, (21) can be written as

$$d\Omega_D = e^{\frac{1}{2} \phi} \sqrt{e^{-2\phi} + (\lambda + C_0)^2} i E' \wedge \gamma_a E^a \wedge K E'. $$

From this form of the dual action, it has been concluded previously [11, 13, 7] that the dual action of D-string is equivalent to the type IIB Green-Schwarz superstring with the $SL(2, Z)$ covariant string tension $\sqrt{e^{-2\phi} + (\lambda + C_0)^2} [31]$. But this argument is a little naive because in this case we can change the string tension at will through the field redefinitions.

In the present formulation, however, we have to take notice of at least two problems in this argument. Firstly in the process of the diagonalization the $SO(2)$ spinor-rotation must be accompanied by some associated $SL(2, R)$ transformation of the background fields in order to preserve the type IIB supergravity constraint equations; namely type IIB supergravity equations of motion. Secondly since in the general case of non-constant dilaton and axion the last term in (21) does not vanish, two terms in (21) cannot be simultaneously diagonalized by some $SO(2)$ spinor-rotation.

It is remarkable to see that these problems are closely related and resolved in the following way. To do so let us rewrite (21) in the form

$$S_D' = - \int_{M_2} d^2 \sigma e^{\frac{1}{2} \phi'} \sqrt{- \det G_{ij}} + \int_{M_2} b'_2, $$

where

$$e^{\frac{1}{2} \phi'} = e^{\frac{1}{2} \phi} \sqrt{e^{-2\phi} + (\lambda + C_0)^2}, \quad b'_2 = C_2 + \lambda b_2. $$

According to the $SL(2, R)/SO(2)$ coset description of dilaton and axion in type IIB supergravity reviewed in Appendix, Eq.(23) is just the $SL(2, Z)$ transformation $\tau \to \tau' = \frac{1}{\tau + \lambda}$ associated with the $SL(2, Z)$ matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}, \quad (S^T)^{-1} = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}, $$

where $(b_2, -C_2)$ belongs to an $SL(2, R)$ doublet which transforms like $B$ in (A.11).

As discussed in Appendix the type IIB supergravity constraints are invariant under the $SL(2, R)$ transformation combined with some suitable $SO(2)$ spinor-rotation and therefore $b'_2$ satisfies the type IIB supergravity constraint equation for the NS-NS 2-form. To compare with the Wess-Zumino term of the type IIB Green-Schwarz superstring action, let us take an
exterior derivative of the integrand in the Wess-Zumino term, which is calculated as follows:

\[
\begin{align*}
\mathrm{d}\Omega'_D & \equiv \mathrm{db}_2' \\
& = e^{i\frac{1}{2}\phi'} \left[ i\bar{E}' \wedge \gamma_a E^a \wedge \mathcal{K} E' + \frac{1}{2} E' \wedge \gamma_{ab} \mathcal{K} \Lambda'E^b \wedge E^a \right], \tag{25}
\end{align*}
\]

where \(E'\) and \(\Lambda'\) are the \(SO(2)\) spinor-rotated supervielbein and dilatino superfield, respectively. The spinor-rotation angle is given by (A.17) and (24). Thus it follows that the dual action (22) is the type IIB Green-Schwarz superstring action with the unit string tension, \(T = 1\) in the \(SL(2, R)\) transformed type IIB supergravity background \([30]\). (Note that this \(SL(2, Z)\) transformed background has the NS-NS charge \(-\lambda\) and the RR-charge 1.) In this way the dual action (20) is precisely transformed into the type IIB Green-Schwarz superstring action by a \(SL(2, Z)\) duality transformation (combined with a suitable \(SO(2)\) spinor-rotation).

Of course, this does not imply that the dual action (20) of the super D-string is equivalent to the Green-Schwarz superstring action with the unit NS-NS charge, for the \(SL(2, Z)\) transformation is not a symmetry in the case of string theory, as opposed to D3-brane case. Instead, the above demonstration means that the dual action (20) is indeed nothing but the type IIB superstring action in a background with the NS-NS charge \(-\lambda\) and the RR-charge 1.

As alluded in Section 1, in order to obtain this result, we have not adopted the convenient technique for which the Wess-Zumino action containing the constant axion, which is a topological theta term, is taken into account at the end of calculations in order to produce the \(SL(2, Z)\) covariant tension, instead treated its non-trivial Wess-Zumino action due to the non-constant character of the axion field from the beginning in a direct manner.

### 3.2 The quantum analysis

The results we have obtained in the previous subsection unfortunately rely on the classical analysis in the sense that we have made use of the field equations at least in the two stages, namely, we have used the field equations of \(A_i\) and \(F_{ij}\). Now we are ready to present a quantum-mechanical exact proof of \(SL(2, Z)\) S-duality of the super D-string action in a general ten dimensional IIB supergravity background. (The results in this subsection were briefly reported in a short article \([13]\).)

To this aim, let us utilize the first-order Hamiltonian form of path integral following the techniques developed in \([16, 33]\). As a first step of the Hamiltonian formalism, let us introduce the canonical conjugate momenta \(\pi^i\) corresponding to the gauge field \(A_i\) defined as

\[
\pi^i \equiv \frac{\partial S}{\partial \dot{A}_i}, \tag{26}
\]

where the dot denotes the time derivative. Then the canonical conjugate momenta \(\pi^i\) are calculated to be

\[
\begin{align*}
\pi^0 & = 0, \quad \pi^1 = e^{-\frac{1}{2}\phi} \frac{\mathcal{F}_{01}}{\sqrt{-\det(G_{ij} + e^{-\frac{1}{2}\phi} \mathcal{F}_{ij})}} + C_0, \tag{27}
\end{align*}
\]
where the former equation implies the existence of the $U(1)$ gauge invariance. From the latter equation $\dot{A}_1$ can be expressed in terms of $\pi^1$ whose result is given by

$$\dot{A}_1 = e^{\frac{i}{2}\phi} \frac{\pi^1 - C_0}{\sqrt{(\pi^1 - C_0)^2 + e^{-2\phi}}} \sqrt{-\det G_{ij} + b_{01} + \partial_1 A_0}. \tag{28}$$

Then we will see that the Hamiltonian density takes the form

$$\mathcal{H} = e^{\frac{i}{2}\phi} \sqrt{e^{-2\phi} + (\pi^1 - C_0)^2} \sqrt{-\det G_{ij} - A_0 \partial_1 \pi^1 + \partial_1 (A_0 \pi^1) + \pi^1 b_{01} - C_{01}}. \tag{29}$$

Now the partition function is defined by the first-order Hamiltonian form with respect to only the gauge field as follows:

$$Z = \frac{1}{\mathcal{D}\pi^0} \int \mathcal{D}\pi^0 \mathcal{D}\pi^1 \mathcal{D}A_0 \mathcal{D}A_1 \exp i \int d^2 \sigma (\pi^1 \partial_0 A_1 - \mathcal{H})$$

$$= \int \mathcal{D}\pi^1 \mathcal{D}A_0 \mathcal{D}A_1 \exp i \int d^2 \sigma$$

$$\times \left[ -A_1 \partial_0 \pi^1 + A_0 \partial_1 \pi^1 - e^{\frac{i}{2}\phi} \sqrt{e^{-2\phi} + (\pi^1 - C_0)^2} \sqrt{-\det G_{ij} - \pi^1 b_{01} + C_{01}} \right], \tag{30}$$

where we have taken the boundary condition for $A_0$ such that the surface term identically vanishes. Then we can carry out the integrations over $A_i$ explicitly, which gives rise to $\delta$ functions

$$Z = \int \mathcal{D}\pi^1 \delta(\partial_0 \pi^1) \delta(\partial_1 \pi^1) \exp i \int d^2 \sigma$$

$$\times \left[ -e^{\frac{i}{2}\phi} \sqrt{e^{-2\phi} + (\pi^1 - C_0)^2} \sqrt{-\det G_{ij} + C_{01} - \pi^1 b_{01}} \right]. \tag{31}$$

The existence of the $\delta$ functions reduces the integral over $\pi^1$ to the one over only its zero-modes. If we require that one space component is compactified on a circle, these zero-modes are quantized to be integers $[34]$. As a consequence, the partition function becomes

$$Z = \sum_{m \in \mathbb{Z}} \exp i \int d^2 \sigma \left[ -e^{\frac{i}{2}\phi} \sqrt{e^{-2\phi} + (m - C_0)^2} \sqrt{-\det G_{ij} + C_{01} - mb_{01}} \right], \tag{32}$$

from which we can read off the effective action

$$S = \int d^2 \sigma \left( -e^{\frac{i}{2}\phi} \sqrt{e^{-2\phi} + (m - C_0)^2} \sqrt{-\det G_{ij} + C_{01} - mb_{01}} \right). \tag{33}$$

Note that if we replace $m$ with $-\lambda$ this action precisely becomes equivalent to (20). In this way, we have derived the dual action of the super D-string action in a type IIB on-shell supergravity background geometry in quantum-mechanical exact manner.

To close this section let us present an interesting observation which will play an important role especially in proving the self-duality of the super D3-brane action in Sec.5. Notice that the original action (14) possesses the following two symmetries:

$$C_0 \rightarrow C'_0 = C_0 + 1,$$

$$b_2 \rightarrow b'_2 = b_2, \quad C_2 \rightarrow C'_2 = C_2 - \mathcal{F} = C_2 + b_2 - F, \tag{34}$$
and

\[
C_0 \to C_0' = C_0 + 1, \quad b_2 \to b_2' = b_2, \quad C_2 \to C_2' = C_2 + b_2,
\]  

which stem from the structure of the Wess-Zumino action. The super D-string action (14) is exactly invariant under (34) and invariant only up to the topological term F under (35). These symmetries do not affect the type IIB supergravity constraints (15), which implies that these are real symmetries of the theory. Of course, it is obvious that the partition function (32) is also invariant under these symmetries. Noticing \(F = dA\) the transformation (34) may be interpreted as the transformation (35) combined with a gauge transformation of \(C_2\) whose gauge parameter is the world-volume Abelian gauge field itself.

Then it is natural to ask what the meaning of these symmetries is. In the following we shall follow the notations in Appendix. First, when we introduce the complex variable \(\tau\) by 

\[
\tau = C_0 + ie^{-\phi},
\]

the shift of \(C_0\) corresponds to \(\tau \to \tau + 1\), being an element of \(SL(2, R)\) associated with \(SL(2, R)\) matrix

\[
S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (S^T)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},
\]

Next we should also investigate how this symmetry acts on the other superfields in the theory. Comparing the transformation (35) with (A.11), \((b_2, -C_2)\) is expected to form an \(SL(2, R)\) doublet which transforms like \(B\) in (A.11). This statement is consistent with the result obtained in the previous subsection and also supported in Sec.5 by the fact that under the weak-strong duality \(\tau \to -\frac{1}{\tau}\), \((b_2, -C_2)\) correctly transforms as an \(SL(2, R)\) doublet.

It is illuminating that super D-brane actions in the general type IIB supergravity background has a shift symmetry \(\tau \to \tau + 1\) as expected from the fact that this is a symmetry of perturbation theory. This fact has been thus far understood in the target space approach where the R-R scalar \(C_0\) appears only through its field strength. But in the present world-volume approach this comes from the non-trivial symmetry of the super D-brane action. We wish to stress that it is possible to prove this interesting observation only in the present formalism within the context of the world-volume theory as mentioned in Sec.1.

4 The super D2-brane

Next we turn to the classical derivation of a duality transformation between the super D2-brane (i.e., the super D-membrane) in a type IIA supergravity background and the super M2-brane in eleven dimensional supergravity. The authors in a paper [10] have already dealt with this problem from a different viewpoint. The method adopted there is to start with the super M2-brane in eleven dimensions, achieve the dimensional reduction to ten dimensions a la Kaluza-Klein ansatz, then perform a duality transformation for the purpose of getting
the super D2-brane action and its $\kappa$-symmetry. Our method is similar to that of Aganagic et al. \cite{11} where the above arguments were reversed, namely, the super M2-brane action was obtained from starting with the super D2-brane action through a duality transformation of the world-volume gauge field. The case of the constant or zero dilaton was already considered in our previous work \cite{7}. Making use of the formulation explained in Sec.2, we focus on the problem of how our previous work should be extended to the case of the non-constant dilaton superfield.

From Eqs.\((1)\) and \((2)\), the super D2-brane action is of the form

\[
S = S_{DBI} + S_{WZ} + S_{\bar{H}},
\]

\[
S_{DBI} = - \int_{M_3} d^3\sigma e^{-\frac{1}{4}\phi} \sqrt{-\det(G_{ij} + e^{-\frac{1}{2}\phi}F_{ij})},
\]

\[
S_{WZ} = \int_{M_3} (C_3 + C_1 \wedge F) = \int_{M_4} I_4,
\]

\[
S_{\bar{H}} = \int_{M_3} d^3\sigma \frac{1}{2} \bar{H}^{ij}(F_{ij} - 2\partial_iA_j),
\]

(37)

where we have added $S_{\bar{H}}$ to the original action in order to perform a duality transformation. Moreover, in this case the constraints \((11)\) on the field strengths reduce to

\[
H_3 = db_2 = ie^{\frac{1}{2}\phi}E \wedge \gamma_{11}E \wedge E + \frac{1}{2}e^{\frac{1}{2}\phi}E \wedge \gamma_{ab}\gamma_{11}\Lambda E^b \wedge E^a,
\]

\[
R_{(2)} = dC_1 = ie^{-\frac{1}{2}\phi}E \wedge \gamma_{11}E - \frac{3}{2}e^{-\frac{1}{2}\phi}E \wedge \gamma_a\gamma_{11}\Lambda E^a,
\]

\[
R_{(4)} = dB_3 + H_3 \wedge C_1 + \frac{i}{2}e^{-\frac{1}{4}\phi}E \wedge \gamma_{ab}E \wedge E^b \wedge E^a + \frac{1}{12}e^{-\frac{1}{4}\phi}E \wedge \gamma_{abc}\Lambda E^c \wedge E^b \wedge E^a,
\]

(38)

where \((7), (8)\) and \((9)\) were also utilized. Of course, $C_1$ and $C_3$ are determined by solving Eq.\((38)\).

Varying $A_i$ in the action \((37)\) gives the equation of motion, $\partial_i\bar{H}^{ij} = 0$ like the super D-string case. Then the solution is obviously given by $\bar{H}^{ij} = \epsilon^{ijk}\partial_kB$ with $B$ being a scalar superfield. Next, substituting this classical solution into the original action \((37)\) leads to an action $S' = S_1 + S_{2D}$, where

\[
S_1 = \int_{M_3} d^3\sigma \left[ -e^{-\frac{1}{4}\phi} \sqrt{-\det(G_{ij} + e^{-\frac{1}{2}\phi}F_{ij})} + \frac{1}{2} \epsilon^{ijk}(\partial_kB + C_k)F_{ij} \right],
\]

\[
S_{2D} = \int_{M_3} (C_3 + b_2 \wedge dB).
\]

(39)

Since $S_{2D}$ does not include $F_{ij}$, it is manifestly duality invariant. Solving the equation of motion for $F_{ij}$ in order to rewrite the action $S_1$ in terms of $B$ instead of $F_{ij}$, we arrive at the dual action $S_D$ of \((37)\)

\[
S_D = - \int_{M_3} d^3\sigma e^{-\frac{1}{4}\phi} \sqrt{-\det G_{ij}} + \int_{M_3} (C_3 + b_2 \wedge dB),
\]

(40)
where we have defined as
\[ G'_{ij} = G_{ij} + e^{\frac{2}{3}\phi}(\partial_i B + C_i)(\partial_j B + C_j). \]  

Incidentally, in order to derive the dual action we have used the mathematical formulas holding for 3 × 3 matrices
\[
\begin{align*}
\det(G_{ij} + A_i A_j) &= (\det G_{ij}) \times (1 + G'^{ij} A_i A_j), \\
\det(G_{ij} + \mathcal{F}_{ij}) &= (\det G_{ij}) \times (1 + \frac{1}{2}G^{ijk} G_{kl} \mathcal{F}_{ik} \mathcal{F}_{jl}),
\end{align*}
\]
where \( \mathcal{F}_{ij} = -\mathcal{F}_{ji} \).

Eq.\((\text{II})\) implies the identification \( dB + C_1 = e^{-\frac{2}{3}\phi} E^{11} \), in other words, identifying the world-volume scalar with the coordinate of a compact extra target-space dimension. Consequently, the Dirac-Born-Infeld action in Eq.\((\text{I})\) takes the standard form for the induced metric of the M2-brane. The remaining work is to show that the second term in the right hand side of Eq.\((\text{II})\) equals to the expression for the Wess-Zumino term of the super M2-brane. Taking the exterior derivative and using the relation \((\text{III})\) we can evaluate this term as follows:

\[ d\Omega_D \equiv d(C_3 + b_2 \wedge dB) = R_4 + (dB + C_1) \wedge H_3, \]
\[ = \frac{i}{2} e^{-\frac{2}{3}\phi} E \wedge \gamma_{ab} E \wedge E^b \wedge E^a + \frac{1}{12} e^{-\frac{2}{3}\phi} E \wedge \gamma_{abc} E \wedge E^b \wedge E^a \]
\[ + (ie^{-\frac{2}{3}\phi} E \wedge \gamma_{11} \gamma_a E \wedge E^a + \frac{1}{2} e^{-\frac{2}{3}\phi} E \wedge \gamma_{ab} \gamma_{11} E^b \wedge E^a) \wedge E^{11} \]
\[ = \frac{i}{2} e^{-\frac{2}{3}\phi} E \wedge \gamma_{\hat{a}b} E \wedge E^b \wedge E^{\hat{a}} \]
\[ + \frac{1}{12} e^{-\frac{2}{3}\phi} E \wedge \gamma_{abc} \Lambda E^c \wedge E^b \wedge E^a - \frac{1}{2} e^{-\frac{2}{3}\phi} E \wedge \gamma_{ab} \gamma_{11} E^b \wedge E^a \wedge E^{11} \]
\[ \equiv d\Omega_1 + d\Omega_2, \]

where \( \hat{a} \equiv (a, 11) \) denotes 11 dimensional index, and \( d\Omega_1 \) and \( d\Omega_2 \) are respectively defined as
\[ d\Omega_1 = \frac{i}{2} e^{-\frac{2}{3}\phi} E \wedge \gamma_{\hat{a}b} E \wedge E^b \wedge E^{\hat{a}} \]
and \( d\Omega_2 = \frac{1}{12} e^{-\frac{2}{3}\phi} E \wedge \gamma_{abc} \Lambda E^c \wedge E^b \wedge E^a - \frac{1}{2} e^{-\frac{2}{3}\phi} E \wedge \gamma_{ab} \gamma_{11} E^b \wedge E^a \wedge E^{11} \). Accordingly, the dual action \((\text{IV})\) of the super D2-brane can be written as

\[ S_D = - \int_{M_3} d^3 \sigma e^{-\frac{2}{3}\phi} \sqrt{-\det G'_{ij}} + \int_{M_3} \Omega_1 + \int_{M_3} \Omega_2, \]

where \( G'_{ij} = E_{\hat{a}i} E_{\hat{b}j} \eta_{\hat{a}\hat{b}} \). In order to make \((\text{IV})\) coincide with the super M2-brane action where there is no dilaton, we should remove the dilaton superfield except \( \Omega_2 \) by rescaling the other superfields. It then turns out that the following rescaling makes a job

\[ E^{\hat{a}} \to e^{\frac{1}{3}\phi} E^{\hat{a}}, \ E^a \to e^{\frac{1}{3}\phi} E^a, \ \Lambda \to e^{\frac{2}{3}\phi} \Lambda. \]

\( ^8 \eta_{\hat{a}\hat{b}} \) is the 11 dimensional flat metric defined as diag\((-,-,\ldots,+,+\)).
After this rescaling, the dual action takes the form

$$S_D = - \int_{M_3} d^3 \sigma \sqrt{- \det G_{ij}^{11}} + \int_{M_3} \Omega^{11} + \int_{M_3} \Omega', \quad (46)$$

where $d\Omega^{11} = \frac{1}{2} \bar{E} \wedge \gamma_{\hat{a}\hat{b}} E \wedge E^\hat{c} \wedge E^\hat{a}$ and $d\Omega' = \frac{1}{12} e^{\frac{4}{3} \Phi} \bar{E} \wedge \gamma_{abc} E^c \wedge E^b \wedge E^a - \frac{1}{2} e^{\frac{2}{3} \Phi} \gamma_{ab} \gamma_{11} \Lambda E^b \wedge E^a \wedge E^{11}$. Provided that we neglect the last term $\int_{M_3} \Omega'$ in the Wess-Zumino action for the time being, this dual action is nothing but the standard form of the M2-brane action \[37\]. Thus, we have proved that the super D2-brane action in a type IIA supergravity background is transformed to the super M2-brane action with a circular compactified 11th dimension in eleven dimensional supergravity background through a duality transformation of the world-volume gauge field as expected from IIA/M-duality.

In fact, we can show the rescaling (45) that was required arises the well-known relation between the dilaton and the radius of a compactified circle in the 11th direction as follows. From Eq.(41) and the rescaling, we have a relation

$$G_{ij}^{11} = e^{-\frac{1}{6} \Phi} G_{ij} + e^{\frac{4}{3} \Phi} (\partial_i B + C_i)(\partial_j B + C_j). \quad (47)$$

Next in order to get the desired relation we have to rewrite the Einstein metric $G_{ij}$ in terms of the string metric $G_{ij}^{(s)} \equiv e^{\frac{2}{3} \Phi} G_{ij}$ whose result is given by

$$G_{ij}^{11} = e^{-\frac{2}{3} \Phi} G_{ij}^{(s)} + e^{\frac{4}{3} \Phi} (\partial_i B + C_i)(\partial_j B + C_j). \quad (48)$$

This correctly yields the relation between the 11 dimensional metric and the 10 dimensional string metric \[29\]. Particularly, the coefficient in front of $(\partial B)^2$ gives us the relation that $R_{11} = e^{\frac{4}{3} \Phi}$ \[29\], where $R_{11}$ is the radius of a compactified circle in the 11th direction.

Finally, let us comment the term $\int_{M_3} \Omega'$ in the Wess-Zumino action. Since $\Lambda_\alpha \equiv \frac{1}{2} \partial_\alpha \phi$ as defined in Sec.2, this term never decouples from the theory unless the dilaton is a constant. This fact is physically reasonable from the fact that $\Lambda_\alpha$ is the dilatino which is a superpartner of the dilaton. Thus, at first sight, since the dilaton and the dilatino are contained, the dual action (46) appears more general than the standard M2-brane action formulated in an 11-dimensional supergravity background whose bosonic fields are the metric and a 3-form gauge potential. But this is illusory since it has been already shown \[38\] that by suitable redefinitions of the superconnection and parts of the supervielbein the terms involving the dilatino can be set to zero in 11 dimensions. Accordingly, we have precisely shown that the dual of the super D2-brane action can be identified as the M2-brane action with a circular 11th dimension.

5 The super D3-brane

In this section let us show the $SL(2, Z)$ self-duality of the super D3-brane action in a general type IIB supergravity background with non-constant dilaton and axion in both classically and
quantum mechanically exact manner. In the case of non-constant dilaton and axion there appear several new features as we will discuss in this section.

From Eqs.(1), (2) and (3), the super D3-brane action is given by

\[ S = S_{DBI} + S_{WZ}, \]
\[ S_{DBI} = - \int_{M_4} d^4\sigma \sqrt{- \text{det}(G_{ij} + e^{-\Phi} F_{ij})}, \]
\[ S_{WZ} = \int_{M_4} (C_4 + C_2 \wedge F + \frac{1}{2} C_0 F \wedge F) = \int_{M_4} \Omega_4. \] (49)

The constraints (12) on the field strengths of NS-NS and R-R form-potentials are given by

\[ H_3 = db_2 = ie^{\Phi} \hat{E} \wedge \hat{E} \wedge K E + \frac{1}{2} e^{\Phi} \hat{E} \wedge \gamma_{ab} \mathcal{K} \Lambda E^b \wedge E^a, \]
\[ R_{(1)} = dC_0 = 2e^{-\Phi} \hat{E} \mathcal{E} \Lambda, \]
\[ R_{(3)} = dC_2 - H_3 C_0 = -ie^{-\Phi} \hat{E} \wedge \hat{E} \wedge \mathcal{I} E + \frac{1}{2} e^{-\Phi} \hat{E} \wedge \gamma_{ab} \mathcal{I} \Lambda E^b \wedge E^a, \]
\[ R_{(5)} = dC_4 - H_3 \wedge C_2 = \frac{i}{6} \hat{E} \wedge \gamma_{abc} \mathcal{E} E \wedge E^c \wedge E^b \wedge E^a. \] (50)

Note that in the case of the non-constant dilaton and axion the Abelian worldvolume gauge field enters into the axion term in the Wess-Zumino term only through the gauge invariant and supersymmetric \( F \). Therefore this term is no longer a topological term and the symmetry under the constant shift of the axion seems to be lost. However, as we will see in the next subsection, it is this form of the Wess-Zumino action that ensures the NS-NS and the R-R 2-form potentials transform as a doublet under the \( SL(2, \mathbb{R}) \) duality transformation.

5.1 The classical analysis

In this subsection we show that the super D3-brane action in a general type IIB supergravity background is classically self-dual. For this we first add a Lagrangian multiplier term \([15]\)

\[ S_{\tilde{H}} = \int_{M_4} d^4\sigma \frac{1}{2} \tilde{H}^{ij} (F_{ij} - 2 \partial_i A_j), \] (51)

to the above action \([19]\) and solve the equation of motion for \( A_i \) by \( \tilde{H}^{ij} = \epsilon^{ijkl} \partial_k B_l \) with a dual vector potential \( B_i \). Then substitution of this solution into the action \([19]\) leads to an action \( S' = S_1 + S_{2D} \), where

\[ S_1 = \int_{M_4} \left[ -d^4\sigma \sqrt{- \text{det}(G_{ij} + e^{-\Phi} F_{ij})} + (C_2 + \tilde{F}) \wedge F + \frac{1}{2} C_0 F \wedge F \right], \]
\[ S_{2D} = \int_{M_4} (C_4 + \tilde{F} \wedge b_2), \] (52)
where $\tilde{F} = dB$.

Next we solve the equation of motion for $F_{ij}$ and substitute its solution into the action $S'$. Following [13], we perform the calculation in the specific local Lorentz frame where $G_{ij} = \eta_{ij} = \text{diag}(-1, 1, 1, 1)$ and $F$ has the following block diagonal form

$$F_{ij} = \begin{pmatrix} 0 & F_{01} & 0 & 0 \\
-F_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & F_{23} \\
0 & 0 & -F_{23} & 0 \end{pmatrix}. \quad (53)$$

Then the action (52) is written in this local Lorentz frame as

$$S' = \int_{M^4} d^4\sigma \left[ -\sqrt{\det(G_{ij} + 1/\sqrt{e^{-\phi} + e^{\phi}C_0^2} \tilde{F}_{ij} + e^{\phi}C_0^2}) + (C_2 + \tilde{F})_{01}F_{01} + (C_2 + \tilde{F})_{01}F_{23} + C_0F_{01}F_{23} + C_{0123} + \tilde{F}_{01}b_{23} + \tilde{F}_{23}b_{01} \right]. \quad (54)$$

Solving the equation of motion for $F_{01}$ and $F_{23}$ yields

$$F_{01} = -\frac{1}{A} \left( e^{\phi}C_0b + a\sqrt{\frac{A-b^2}{A+a^2}} \right), \quad F_{23} = -\frac{1}{A} \left( e^{\phi}a - b\sqrt{\frac{A+a^2}{A-b^2}} \right), \quad (55)$$

where

$$A \equiv e^{-\phi} + e^{\phi}C_0^2, \quad a \equiv (C_2 + \tilde{F})_{23}, \quad b \equiv (C_2 + \tilde{F})_{01}. \quad (56)$$

Inserting these solutions into the action (54) and returning to the general coordinate reference frame, we arrive at the dual action $S_D$ in a simple form

$$S_D = -\int_{M^4} \sqrt{-\det(G_{ij} + 1/\sqrt{e^{-\phi} + e^{\phi}C_0^2} (\tilde{F}_{ij} + e^{\phi}C_0^2))} + \int_{M^4} \Omega_D, \quad (57)$$

Now if we define a new primed dilaton, axion and form-potentials which are of the form,

$$e^{-\phi'} = \frac{1}{e^{-\phi} + e^{\phi}C_0^2},$$

$$C_0' = \frac{e^{\phi}C_0}{e^{-\phi} + e^{\phi}C_0^2}, \quad (58)$$

and

$$b_2' = -C_2, \quad C_2' = b_2, \quad C_4' = C_4 - b_2 \wedge C_2, \quad (59)$$

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then the dual action can be written as

\[ S_D = -\int_{M_4} d^4\sigma \sqrt{-\det(G_{ij} + e^{-\phi} \tilde{F}'_{ij})} \]

\[ + \int_{M_4} (C'_4 + C'_2 \wedge \tilde{F}' + \frac{1}{2} C'_0 \tilde{F}' \wedge \tilde{F}'), \]

where \( \tilde{F}' = \tilde{F} - b'_2 \). The resulting action is completely the same form as the original action (49) with which we have started. This means that under the transformations (58), (59) and

\[ \tilde{F} \rightarrow F, \quad F \rightarrow \tilde{F}, \]

the action (49) and (60) are classically equivalent.

In order to establish the \( SL(2, R) \) self-duality of the super D3-brane action in a general type IIB supergravity background, we have to answer the following two problems. The first problem is to clarify the transformation properties of form-potentials under the \( SL(2, R) \) duality transformation. The second one is whether the type IIB constraint equations (50) are preserved under the \( SL(2, R) \) duality transformation.

First of all, let us introduce complex variable \( \tau \) as in Sec.3,

\[ \tau = C_0 + ie^{-\phi}, \]

then the transformation (58) is written as \( \tau \rightarrow \tau' = -\frac{1}{\tau} \), which is an element of the \( SL(2, R) \) with the transformation matrix \( S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). According to the transformation rules (A.10) and (A.11) the transformations (59) and (61) imply that \( (b_2, -C_2) \) and \( (\tilde{F}, -F) \) are expected to form \( SL(2, R) \) doublets which transform like B in (A.11) and A in (A.10), respectively.

Next we must find the \( SL(2, R) \) element \( \tau \rightarrow \tau' = \tau + 1 \) which corresponds to the shift of the axion \( C_0 \rightarrow C_0 + 1 \). It is easily shown that as in the super D-string action the super D3-brane action (49) is invariant under the following two transformations

\[ C_0 \rightarrow C_0 + 1, \]

\[ b_2 \rightarrow b_2, \quad C_2 \rightarrow C_2 - F = C_2 + b_2 - F, \]

\[ C_4 \rightarrow C_4 + \frac{1}{2} F \wedge F, \]

\[ F \rightarrow F, \]

and

\[ C_0 \rightarrow C_0 + 1, \]

\[ b_2 \rightarrow b_2, \quad C_2 \rightarrow C_2 + b_2, \]

\[ C_4 \rightarrow C_4 + \frac{1}{2} b_2 \wedge b_2, \]

\[ F \rightarrow F. \]
The super D3-brane action (49) is exactly invariant under (63) and invariant only up to the topological term \(\frac{1}{2} F \wedge F\) under (64). It is also easily shown that the type IIB constraint equations (50) are invariant under these transformation. Thus the transformations (63) and (64) are real symmetries of the theory. These transformations are also consistent with the postulate that \((b_2, -C_2)\) and \((\tilde{F}, -F)\) form \(SL(2, R)\) doublets which transform like \(B\) in (A.11) and \(A\) in (A.10), respectively.

The transformation (63) is an exact symmetry of the theory and seems to be more attractive than another one (64). It may be interpreted in such a way that there may exist a manifestly self-dual formulation of the super D3-brane which are broken by some gauge fixing to the present formulation of super D3-brane action. Unfortunately, however, as discussed in the next subsection, the transformation (63) does not satisfy the consistency condition of the constructive relation (71) in the next subsection. Therefore we will not consider this transformation any more in this paper.

At this point we should stress the following fact. In the case of constant or vanishing dilaton and axion the following form is sometimes taken as the Wess-Zumino action;

\[
S_{WZ} = \int (C_4 + C_2 \wedge F + \frac{1}{2} C_0 F \wedge F),
\]

Although this action is trivially invariant up to the topological term \(\frac{1}{2} F \wedge F\) under the shift of the axion \(C_0\), the R-R 2-form potential \(C_2\) can not transform in such a way to form \(SL(2, R)\) doublet with the NS-NS 2-form potential \(b_2\) and preserve the constraint equation on the R-R 5-form strength \(R^{(5)}\). Therefore with this Wess-Zumino action the \(SL(2, R)\) duality symmetry would be lost even in the case of constant or zero dilaton and axion.

The general \(SL(2, R)\) transformations are generated by \(\tau \rightarrow -\frac{1}{\tau}\) and \(\tau \rightarrow \tau + 1\). Here we recapitulate the \(SL(2, R)\) transformation rule for various fields in the theory for later use in the next subsection. Corresponding to the \(SL(2, R)\) transformation of the complex variable \(\tau\)

\[
\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \tag{65}
\]

which in terms of \(C_0\) and \(\phi\) is equivalent to

\[
C_0' = \frac{(aC_0 + b)(cC_0 + d) + ace^{-2\phi}}{(cC_0 + d)^2 + c^2 e^{-2\phi}},
\]

\[
e^{-\phi'} = \frac{e^{-\phi}}{(cC_0 + d)^2 + c^2 e^{-2\phi}}, \tag{66}
\]

we define the matrix \(S\) as

\[
S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (S^T)^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}. \tag{67}
\]
Then it is expected that $\tilde{F}, F, b_2, C_2$ and $C_4$ transform in the following manner;

$$\tilde{F} \rightarrow \tilde{F}' = a\tilde{F} - bF, \quad F \rightarrow -c\tilde{F} + dF,$$

$$b_2 \rightarrow b'_2 = db_2 + cC_2, \quad C_2 \rightarrow C'_2 = bb_2 + aC_2,$$  \hspace{1cm} (68)

and

$$C_4 \rightarrow C'_4 = C_4 + \frac{bd}{2}b_2 \wedge b_2 + bcb_2 \wedge C_2 + \frac{ac}{2}C_2 \wedge C_2.$$ \hspace{1cm} (70)

The transformation of $C_4$ is determined by the requirement that $R_{(5)}$ is invariant under the $SL(2, R)$ transformations (64) and (68). The $SL(2, R)$ transformations (63)-(70) are just the same as those in [42], but the authors of [42] gave neither the transformation of $C_4$ nor $SO(2)$ spinor-rotation (A.16) since they considered only the bosonic D3-brane action. The $SL(2, R)$ transformation (70) of $C_4$ and the $SO(2)$ spinor-rotation (A.16) are indispensable ingredients for the type IIB supergravity constraints to be invariant under the $SL(2, R)$ duality transformation.

Next we turn to the problem of the invariance (or covariance) of the type IIB supergravity constraints under the $SL(2, R)$ duality transformation. In Appendix it is shown that all the type IIB supergravity constraint equations (of course including the torsion constraints) are invariant under general $SL(2, R)$ duality transformations (65), (69), (70) and the $SO(2)$ spinor-rotation (A.16).

Finally we note that, since the invariance under $\tau \rightarrow \tau + 1$ holds only up to the topological term $\frac{1}{2}F \wedge F$, the $SL(2, R)$ invariance of the theory would be broken to the $SL(2, Z)$ invariance in a quantum theory.

### 5.2 The exact analysis

Let us turn to the other analysis of duality condition which was initiated by Gaillard and Zumino [38] and extended by several authors [11, 12, 13].

Let us define the dual field strength $K_{ij}$ by the following constructive relation;

$$* K_{ij} = \frac{\partial L}{\partial F_{ij}}, \quad \frac{\partial F_{kl}}{\partial F_{ij}} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k,$$  \hspace{1cm} (71)

where the Hodge dual components $*K_{ij}$ for the antisymmetric tensor $K_{ij}$ are defined by

$$*K_{ij} \equiv \frac{1}{2} \epsilon_{ijkl} K_{kl}, \quad *K_{ij} = -K_{ij},$$ \hspace{1cm} (72)

where $\epsilon_{ijkl}$ is the Levi-Civita symbol in 4 dimensions, $\epsilon^{0123} = 1$ and the signature of $G_{ij}$ is $(-, +, +, +)$. At the classical level $K_{ij}$ introduced here is equivalent to the dual variable $-\tilde{F}_{ij}$.
in the last subsection. Therefore it is natural to assume that \((K_{ij}, F_{ij})\) transforms as Eq.(68); for the infinitesimal transformation with 
\[ S = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 - \alpha \end{pmatrix} \]

it is given by
\[ \delta K_{ij} = +\alpha K_{ij} + \beta F_{ij}, \quad \delta F_{ij} = -\alpha F_{ij} + \gamma K_{ij}, \]

(73)

The sufficient condition for the consistency of the constructive relation (71), the invariance of the field equation and the invariance of the (world-volume) energy-momentum tensor under the \(SL(2, R)\) duality transformation is that the following equation is satisfied for arbitrary \(SL(2, R)\) parameters \(\alpha, \beta\) and \(\gamma\);
\[ \frac{\gamma}{4} * K^{ij} K_{ij} - \frac{\beta}{4} * F^{ij} F_{ij} - \frac{\alpha}{2} * K^{ij} F_{ij} + \delta \Phi L = 0, \]

(74)

where \(\delta \Phi\) means to take the \(SL(2, R)\) variation for all fields except the world volume Abelian gauge field. This equation is derived by the similar argument in [43] where a similar equation was derived for the special case of \(\alpha = 0\) and \(\beta = -\gamma\) (\(SO(2)\) duality transformation). We call Eq.(74) the Gaillard-Zumino (G-Z) duality condition.

In [43] it has been shown that the G-Z condition is actually the necessary and sufficient condition in order that one can define off-shell (non-local) duality transformation for the \(U(1)\) gauge potential itself under which the action is invariant up to some surface terms. Therefore if one can show for an action to satisfy the G-Z condition, then one establishes the exact self-duality of the theory described by this action without resort to any classical approximation. It has been shown that the super D3-brane action on a flat [44], an \(AdS_5 \times S^5\) [23] and a general type IIB supergravity backgrounds with constant dilaton and axion [7] indeed satisfies the G-Z condition.

In this subsection we show that the D3-brane action on the most general type IIB supergravity background described by the action (49) and constraints (50) satisfies the G-Z duality condition under the \(SL(2, R)\) duality transformations (65)-(70) and the \(SO(2)\) spinor-rotation (A.16).

Before we enter the detailed discussions we should note that the transformation (63) does not satisfy the consistency condition for the constructive relation (71). The consistency condition of the constructive relation demands the equation
\[ \delta * K^{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial F_{kl} \partial F_{ij}} \delta F_{kl} + \frac{\partial^2 L}{\partial \Phi \partial F_{ij}} \delta \Phi, \]

to be satisfied. A simple calculation leads us to a contradictory result; \(* F^{ij} = 0\). It seems to be curious that the exact symmetry of Lagrangian does not satisfy the consistency condition.

The super D3-brane Lagrangian (49) in terms of component fields is written as
\[ L = L_{DBI} + \epsilon^{ijkl} \left( \frac{1}{24} C_{ijkl} + \frac{1}{4} C_{ij} F_{kl} + \frac{1}{8} C_{0} F_{ij} F_{kl} \right), \]

(75)
\[ L_{DBI} = -\sqrt{-\det(G_{ij} + e^{-\phi/2}F_{ij})} \]
\[ = -\sqrt{-G}\sqrt{1 + \frac{e^{-\phi}}{2}F_{ij}F_{ij} - \frac{e^{-2\phi}}{16}(F_{ij} \ast F_{ij})^2}, \quad (76) \]
where \( G = \det G_{ij} \). Then the dual field strength defined by \((71)\) is given by
\[ *K^{ij} = \frac{\partial L_{DBI}}{\partial F_{ij}} + *C^{ij} + C_0 \ast F_{ij}, \]
\[ K_{ij} = -(*\frac{\partial L_{DBI}}{\partial F})_{ij} + C_{ij} + C_0 F_{ij}. \quad (77) \]

The infinitesimal \( SL(2, R) \) transformations with \( S = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 - \alpha \end{pmatrix} \) of various fields in our theory are given by
\[ \delta C_0 = 2\alpha C_0 + \beta - \gamma(C_0^2 - e^{-2\phi}), \]
\[ \delta \phi = 2\gamma C_0 - 2\alpha, \quad (78) \]
\[ \delta K_{ij} = +\alpha K_{ij} + \beta F_{ij}, \quad \delta F_{ij} = -\alpha F_{ij} + \gamma K_{ij}, \quad (79) \]
\[ \delta b_{ij} = -\alpha b_{ij} + \gamma C_{ij}, \quad \delta C_{ij} = +\alpha C_{ij} + \beta b_{ij}, \quad (80) \]
and
\[ \delta C_4 = \frac{\beta}{2}b_2 \wedge b_2 + \frac{\gamma}{2}C_2 \wedge C_2. \quad (81) \]

The infinitesimal \( SO(2) \) spinor-rotation is given by
\[ \delta \theta = -\frac{\gamma e^{-\phi}}{2}\mathcal{E}\theta, \quad \delta \partial_\alpha = \frac{\gamma e^{-\phi}}{2}(\mathcal{E}\partial_\alpha), \]
\[ \delta E = \frac{\gamma e^{-\phi}}{2}\mathcal{E}E, \quad \delta \bar{E} = -\frac{\gamma e^{-\phi}}{2}\bar{E}\mathcal{E}. \quad (82) \]

The infinitesimal \( SO(2) \) spinor-rotation \((82)\) ensures the invariance of the type IIB supergravity constraints.

Under these infinitesimal \( SL(2, R) \) transformations \( \delta_\phi L \) is given by
\[ \delta_\phi L = \frac{1}{2} \frac{\partial L}{\partial b_{ij}} \delta b_{ij} + \frac{1}{2} \frac{\partial L}{\partial C_{ij}} \delta C_{ij} + \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial C_0} \delta C_0 + \frac{1}{24} \frac{\partial L}{\partial C_{ijkl}} \delta C_{ijkl}. \quad (83) \]

The G-Z duality condition demands that Eq.\((74)\) must hold for arbitrary variations \( \alpha, \beta \) and \( \gamma \). Therefore coefficients of \( \alpha, \beta \) and \( \gamma \) should identically vanish, respectively. Substituting \((83)\) with \((78)-(81)\) into \((73)\) we obtain following three equations for \( \alpha, \beta, \gamma \) coefficients;
\[ -\frac{1}{4} *K^{ij}K_{ij} - \frac{1}{2} \frac{\partial L}{\partial b_{ij}} b_{ij} + \frac{1}{2} \frac{\partial L}{\partial C_{ij}} C_{ij} - 2 \frac{\partial L}{\partial \phi} + 2C_0 \frac{\partial L}{\partial C_0} = 0, \quad (84) \]
\[-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \frac{\partial L}{\partial C_{ij}} b_{ij} + \frac{\partial L}{\partial C_0} + \frac{1}{8} \epsilon^{ijkl} b_{ij} b_{kl} = 0, \tag{85}\]

and
\[\frac{1}{4} K^{ij} K_{ij} + \frac{1}{2} \frac{\partial L}{\partial b_{ij}} C_{ij} + 2 C_0 \frac{\partial L}{\partial \phi} - (C_0^2 - e^{-2\phi}) \frac{\partial L}{\partial C_0} + \frac{1}{8} \epsilon^{ijkl} C_{ij} C_{kl} = 0, \tag{86}\]

respectively.

Eq. (85) (\(\beta\) coefficient) is almost trivially satisfied. Eq. (84) (\(\alpha\) coefficient) is also easily shown to be satisfied, using the following identity
\[\frac{\partial L_{DBI}}{\partial \phi} = -\frac{1}{4} \frac{\partial L_{DBI}}{\partial F_{ij}} F_{ij}, \tag{87}\]

which is derived by the fact that the dilaton \(\phi\) is contained in \(L_{DBI}\) in the form of \(e^{-2\phi} F\).

The \(\gamma\) coefficient (86) is reduced after straightforward but somewhat lengthy calculations to the following equation
\[\frac{1}{2} \epsilon^{ijkl} \left( \frac{\partial L_{DBI}}{\partial F_{ij}} \frac{\partial L_{DBI}}{\partial F_{kl}} + e^{-2\phi} F_{ij} F_{kl} \right) = 0. \tag{88}\]

This is the only equation which explicitly depends on the specific form of \(L_{DBI}\). Using the explicit expression (76) for \(L_{DBI}\), it is easily shown that the equation (87) is satisfied. Therefore we have shown that the super D3-brane action (49) in the most general type IIB supergravity background indeed satisfies the Gaillard-Zumino duality condition.

In [44] it has been shown that the super D3-brane action in a flat background with constant dilaton and axion which satisfies the Gaillard-Zumino condition is pseudo-invariant under the \(SL(2, R)\) (non-local) duality transformation of the world-volume Abelian gauge field in both the Lagrangian and the Hamiltonian formalism. According to the theorem proved in [43] our above result strongly suggests that the super D3-brane action in the most general type IIB supergravity background is also exactly self-dual without resort to any semi-classical approximation. It would be an interesting problem to establish it in both the Lagrangian and the Hamiltonian formalism.

6 The super D4-brane

In this section let us start with the super D4-brane action and perform a duality transformation of the world-volume gauge field to reach the action obtained by the double-dimensional reduction of the super M5-brane [45, 46]. The method we consider is analogous to the one adopted in Section 4, so we shall follow a similar path of argument as in the super D2-brane. Like the super D2-brane, the analysis in this section is purely classical.
This time, from (1) and (2) the super D4-brane action with a Lagrange multiplier term becomes

\[ S = S_{DBI} + S_{WZ} + S_H, \]

\[ S_{DBI} = - \int_{M_5} d^5 \sigma e^{\frac{1}{2} \phi} \sqrt{- \det(G_{ij} + e^{-\frac{1}{2} \phi} F_{ij})}, \]

\[ S_{WZ} = \int_{M_5 = \partial M_6} \left( C_5 + C_3 \wedge F + \frac{1}{2} C_1 \wedge F \wedge F \right) = \int_{M_6} I_6, \]

\[ S_H = \int_{M_5} d^5 \sigma \frac{1}{2} \tilde{H}^{ij}(F_{ij} - 2 \partial_i A_j). \]  

(89)

And the constraints (11) for type IIA on the field strengths are given by

\[ H_3 = db_2 = ie^{\frac{1}{2} \phi} \tilde{E} \wedge \gamma_{11} \tilde{E} \wedge E + \frac{1}{2} e^{\frac{1}{2} \phi} \tilde{E} \wedge \gamma_{ab} \gamma_{11} E^b \wedge E^a, \]

\[ R_{(2)} = dC_1 = ie^{-\frac{1}{2} \phi} \tilde{E} \wedge \gamma_{11} E - \frac{3}{2} e^{-\frac{1}{2} \phi} \tilde{E} \wedge \gamma_a \gamma_{11} E^a, \]

\[ R_{(4)} = dC_2 + H_3 \wedge C_1 \]

\[ = \frac{i}{2} e^{-\frac{1}{2} \phi} \tilde{E} \wedge \gamma_{ab} E \wedge E^b \wedge E^a + \frac{1}{12} e^{-\frac{1}{2} \phi} \tilde{E} \wedge \gamma_{abc} E^c \wedge E^b \wedge E^a, \]

\[ R_{(6)} = dc_5 + H_3 \wedge C_3 \]

\[ = \frac{i}{24} e^{\frac{1}{2} \phi} \tilde{E} \wedge \gamma_{abcd} \gamma_{11} E \wedge E^d \wedge E^c \wedge E^b \wedge E^a \]

\[ + \frac{1}{240} e^{\frac{1}{2} \phi} \tilde{E} \wedge \gamma_{a_1 \ldots a_5} \gamma_{11} \Lambda E^{a_5} \wedge \ldots \wedge E^{a_1}, \]  

(90)

from which \( C_5, C_3 \) and \( C_1 \) are determined.

As usual, we take the variation with respect to \( A_i \), which gives rise to the solution \( \tilde{H}^{ij} = \frac{1}{6} e^{ijklm} K_{klm} \) with \( K = dB \) with \( B \) being a second rank tensor superfield. After substituting this solution into the action, we obtain the action \( S' = S_1 + S_{2D} \) where \( S_1 \) and \( S_{2D} \) are defined as

\[ S_1 = - \int_{M_5} d^5 \sigma e^{\frac{1}{2} \phi} \sqrt{- \det(G_{ij} + e^{-\frac{1}{2} \phi} F_{ij})} + \int_{M_5} (\mathcal{H} \wedge F + \frac{1}{2} C_1 \wedge F \wedge F), \]

\[ S_{2D} = \int_{M_5} (C_5 + K \wedge b_2), \]  

(91)

with \( \mathcal{H} = K + C_3 \). As before, \( S_{2D} \) is unaffected under a duality transformation. Therefore a duality transformation amounts to solving the equation of motion for \( F_{ij} \) in \( S_1 \) in order to rewrite the action in terms of \( B \) (or its field strength \( K \)) instead of \( F_{ij} \). Following the formula in ref. [11], it is tedious but straightforward to derive the dual action \( S_D = S_{1D} + S_{2D} \) where \( S_{1D} \) is given by

\[ S_{1D} = - \int_{M_5} d^5 \sigma \left[ e^{\frac{1}{2} \phi} \sqrt{-G} \sqrt{1 + z_1 + \frac{z_2^2}{2} - z_2} - \frac{e^\phi}{8(1 + e^\phi C_4^2)} \epsilon_{ijklm} C^i \mathcal{H}^{jk} \mathcal{H}^{lm} \right], \]  

(92)
where

\[
\begin{align*}
z_1 &= \frac{1}{2(-G)(1 + C_1^2)} \text{tr}(\tilde{G}\tilde{H}\tilde{G}\tilde{H}), \\
z_2 &= \frac{1}{4(-G)^2(1 + C_1^2)^2} \text{tr}(\tilde{G}\tilde{H}\tilde{G}\tilde{H}\tilde{G}\tilde{H}\tilde{G}\tilde{H}), \\
G &= \det G_{ij}, \\
\tilde{G}_{ij} &= G_{ij} + C_i C_j, \\
\tilde{\mathcal{H}}_{ij} &= \frac{1}{6} \epsilon^{ijklm} \mathcal{H}_{klm}.
\end{align*}
\]

(93)

Now let us turn our attention to $S_{2D}$. The conditions (90) yield the equation

\[
d\Omega_D \equiv d(C_5 + K \wedge b_2) \\
= R_{(6)} + (K + C_3) \wedge H_3 \\
= \frac{i}{24} e^{\phi/2}\bar{E} \wedge \gamma_{abcd}\gamma_{111} E \wedge E^d \wedge E^c \wedge E^b \wedge E^a - i\epsilon^{\phi/2}\bar{E} \wedge \gamma_{a_1} \gamma_{11} E \wedge E^a \wedge \mathcal{H} \\
+ \frac{1}{240} \epsilon^{\phi/2}\bar{E} \wedge \gamma_{a_1 \ldots a_5} \Lambda E^{a_1} \wedge \ldots \wedge E^{a_5} - \frac{1}{2} \epsilon^{\phi/2}\bar{E} \wedge \gamma_{a_b} \gamma_{11} \Lambda E^b \wedge E^a \wedge \mathcal{H} \\
\equiv d\Omega_1 + d\Omega_2,
\]

(94)

where $d\Omega_1$ and $d\Omega_2$ are respectively defined as the first plus second terms independent of the dilatino $\Lambda$ and the third plus forth terms dependent on the dilatino in the third line of the above equation.

As a result, we have the dual action of the super D4-brane in type IIA supergravity background

\[
S_D = -\int_{M_5} d^5\sigma \left[ e^{\phi/2} \sqrt{-G} \sqrt{1 + z_1 + \frac{z_1^2}{2} - z_2} - \frac{e^{\phi}}{8(1 + e^{\phi} C_1^2)} \epsilon^{ijklm} \mathcal{H}^i \mathcal{H}^j \mathcal{H}^k \mathcal{H}^l \right] \\
+ \int_{M_5} (\Omega_1 + \Omega_2).
\]

(95)

As in the case of the super D2-brane in Sec.4, we can also set $\int_{M_5} \Omega_2$ to be zero in terms of redefinitions of the superconnection and the supervielbein. With this situation, if the dilaton is vanishing, this dual action of (95) exactly reduces to our previous action [7], while if the dilaton is vanishing and the background is in a flat Minkowski spacetime, it becomes the action considered in Ref.[11]. Then it is straightforward to show that each action is identical to its corresponding action which is obtained by the double-dimensional reduction of the super M5-brane [45, 46] after rearranging the constant dilaton factor in a suitable way. Following the similar line of argument, it turns out that the double-dimensional reduction of the super M5-brane action coincides with the dual super D4-brane action even in a general type IIA supergravity background under consideration as suggested by the duality between M-theory and IIA superstring theory.
7 Discussions

In this paper, we have studied the properties of a duality transformation of super Dp-brane actions ($p = 1, 2, 3, 4$) in a general type II on-shell supergravity background, which have been constructed in Refs. [8, 9, 10]. They have already been investigated in the case of a flat background with the zero or constant dilaton and axion [11] and in a type II on-shell supergravity background with the zero or constant dilaton and axion [7]. Our presentation in this paper is most general compared to the previous approaches so we believe that we have succeeded in showing that various duality symmetries in the super D-brane actions are indeed valid in a general type II supergravity background geometry.

The main motivation of the present paper was to take account of the non-constant dilaton and axion superfields in carrying out a duality transformation of the super D-brane actions. To the best of our knowledge, this is the first attempt and consequently gives us several new fruitful results and insights as shown thus far. In particular, from the viewpoint of the world-volume field theory we have succeeded in showing that the dilaton and the axion as well as the two 2-form gauge fields, those are, the NS-NS 2-form and the R-R 2-form, are doublets of the $SL(2, R)$ Mobius group whereas the graviton and the 4-form gauge potential are singlets in the Einstein metric. These facts are of course familiar to us but have been so far proved only in the target space formulation while our presentation is purely from the world-volume field theory. The main new issue in the theory at hand is the existence of a non-trivial symmetry corresponding to a shift of the modulus field $\tau$. The discovery of this symmetry enables us to prove the expected duality relations of the super D-brane actions under the $SL(2, R)$ transformation, in particular, the self-duality of the super D3-brane action.

Moreover, it was shown that the dual actions of the super D2-brane and D4-brane actions on an IIA supergravity background with the constant dilaton, respectively, have the standard forms which are expected from the dimensional reduction of the M2-brane and the M5-brane actions on the 11 dimensional supergravity background. Note that the dilaton and the dilatino appear in 10 dimensional spacetime through the Kaluza-Klein compacification on a circle from the M2- and M5-brane actions in an 11 dimensional supergravity background.

Of course, in this paper our considerations have been focused on only the super D-brane actions [8, 9, 10], thus a detailed comparison between the formulation at present and the other approaches treating different forms of the p-brane actions [25, 26] certainly merits further investigation in trying to clarify a number of non-perturbative aspects of D-brane theory.

In the last section in a paper [11], it is stated that "...... For the most part, our analysis has been classical and limited to flat backgrounds. The results should not depend on these restrictions, however." In our previous paper [4], we have relaxed these restrictions to some extent, while in this paper, we have removed such restrictions completely for the super D1-brane and D3-brane. On the other hand, for the super D2-brane and D4-brane we have removed the restriction of 'flat background', but we have presented only the classical analysis. This restriction should be also removed in future. But as emphasized in the previous paper [4], it is not always clear for us whether it is necessary to remove this restriction or not since the
Dp-brane actions with \( p > 1 \) are in essence unrenormalizable so these actions might describe the low energy effective theory of some underlying renormalizable theory. This problem still deserves further investigation.

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Appendix

**SL(2,R) coset construction of dilaton and axion**

In type IIB supergravity the dilaton and the axion describe the coordinates of \( SL(2, R)/SO(2) \) coset manifold. In this appendix we briefly review the \( SL(2, R)/SO(2) \) coset description of the dilaton and the axion in the type IIB supergravity.

Let us parametrize the \( SL(2, R)/SO(2) \) coset space by

\[
V = e^{\frac{\phi}{2}} \begin{pmatrix} e^{-\phi} & C_0 \\ 0 & 1 \end{pmatrix},
\]

(A.1)

where \( \phi \) and \( C_0 \) are the dilaton and the axion fields, respectively. Then \( V \) transforms under the \( SL(2, R) \) transformation in the following way

\[
V \rightarrow V' = e^{\frac{\phi'}{2}} \begin{pmatrix} e^{-\phi'} & C'_0 \\ 0 & 1 \end{pmatrix} = S V O(S)^{-1}
\]

\[
= S e^{\frac{\phi}{2}} \begin{pmatrix} e^{-\phi} & C_0 \\ 0 & 1 \end{pmatrix} O(S)^{-1},
\]

(A.2)

where \( S \) is an \( SL(2, R) \) matrix given by

\[
S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (S^T)^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad ad - bc = 1,
\]

(A.3)

and \( O(S)^{-1} \) is an \( SO(2) \) matrix defined as

\[
O(S)^{-1} = \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix}.
\]

(A.4)
Eqs. (A.2) and (A.3) determine the transformation rule of $C'_0$ and $\phi'$, and $\lambda$;

$$\begin{align*}
C_0 & \rightarrow C'_0 = \frac{(aC_0 + b)(cC_0 + d) + ace^{-2\phi}}{(cC_0 + d)^2 + c^2e^{-2\phi}}, \\
e^{-\phi} & \rightarrow e^{-\phi'} = \frac{e^{-\phi}}{(cC_0 + d)^2 + c^2e^{-2\phi}},
\end{align*}$$

(A.5)

and

$$\begin{align*}
\cos \lambda &= \frac{cC_0 + d}{\sqrt{(cC_0 + d)^2 + c^2e^{-2\phi}}}, \\
\sin \lambda &= \frac{ce^{-\phi}}{\sqrt{(cC_0 + d)^2 + c^2e^{-2\phi}}}.
\end{align*}$$

(A.6)

When we define the complex variable $\tau$ by

$$\tau = C_0 + ie^{-\phi},$$

(A.7)

Eq. (A.3) gives the transformation rule for $\tau$ under the $SL(2, R)$ transformation;

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}.$$  

(A.8)

By construction $V$ in (A.1) is the $SL(2, R)$ matrix which transforms the origin of the coset space to the point $(\phi, C_0)$; equivalently $\tau_0 = i \rightarrow \tau = C_0 + ie^{-\phi}$.

Next we introduce a symmetric $SL(2, R)$ matrix

$$M = VV^T = e^\phi \begin{pmatrix} |\tau|^2 & C_0 \\ C_0 & 1 \end{pmatrix},$$

(A.9)

which transforms covariantly under $SL(2, R)$ transformation $M \rightarrow M' = SMST^T$. Then one can define two types of $SL(2, R)$ doublet which transform linearly under $SL(2, R)$ transformation. The one is $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ which transforms like

$$A \rightarrow A' = SA,$$

(A.10)

and another is $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ which transforms like $(MB) \rightarrow (MB)' = S(MB)$, that is,

$$B \rightarrow B' = (S^T)^{-1}B.$$ 

(A.11)
Invariance of type IIB supergravity constraints under $SL(2,R)$

As discussed in the text the NS-NS 2-form $b_2$ and the R-R 2-form $C_2$ form an $SL(2,R)$ doublet of $B$-type in $[A,11]$;

$$
\begin{pmatrix}
  b_2' \\
  -C_2'
\end{pmatrix} = (S^T)^{-1}
\begin{pmatrix}
  b_1 \\
  -C_2
\end{pmatrix},
$$

(A.12)

and the R-R 4-form $C_4$ transforms under the $SL(2,R)$ transformation like

$$
C_4 \rightarrow C'_4 = C_4 + \frac{bd}{2} b_2 \wedge b_2 + bcb_2 \wedge C_2 + \frac{ac}{2} C_2 \wedge C_2,
$$

(A.13)

where the $SL(2,R)$ matrix $(S^T)^{-1}$ is given by $[A,3]$. Then it is explicitly shown after straightforward and somewhat lengthy calculations that the type IIB supergravity constraints

$$
H_3 = db_2 = ie^{\frac{\phi}{2}} \hat{E} \wedge \hat{E} \wedge \mathcal{K} E + \frac{1}{2} e^{\frac{\phi}{2}} \hat{E} \wedge \gamma_{ab} \mathcal{K} \Lambda E^b \wedge E^a,
$$

$$
R_{(1)} = dC_0 = 2e^{-\phi} \hat{E} \mathcal{E} \Lambda,
$$

$$
R_{(3)} = dC_2 - H_3 C_0 = -ie^{-\frac{\phi}{2}} \hat{E} \wedge \hat{E} \wedge \mathcal{T} E + \frac{1}{2} e^{-\frac{\phi}{2}} \hat{E} \wedge \gamma_{ab} \mathcal{T} \Lambda E^b \wedge E^a,
$$

$$
R_{(5)} = dC_4 - H_3 \wedge C_2 = \frac{i}{6} \hat{E} \wedge \gamma_{abc} \mathcal{E} E \wedge E^c \wedge E^b \wedge E^a,
$$

(A.14)

are transformed to

$$
H'_{3} = db_2' = ie^{\frac{\phi'}{2}} \hat{E}' \wedge \hat{E}' \wedge \mathcal{K} E' + \frac{1}{2} e^{\frac{\phi'}{2}} \hat{E}' \wedge \gamma_{ab} \mathcal{K} \Lambda' E^b \wedge E^a,
$$

$$
R'_{(1)} = dC'_0 = 2e^{-\phi'} \hat{E}' \mathcal{E} \Lambda',
$$

$$
R'_{(3)} = dC'_2 - H'_3 C'_0 = -ie^{-\frac{\phi'}{2}} \hat{E}' \wedge \hat{E}' \wedge \mathcal{T} E' + \frac{1}{2} e^{-\frac{\phi'}{2}} \hat{E}' \wedge \gamma_{ab} \mathcal{T} \Lambda' E^b \wedge E^a,
$$

$$
R'_{(5)} = dC'_4 - H'_3 \wedge C'_2 = \frac{i}{6} \hat{E}' \wedge \gamma_{abc} \mathcal{E} E' \wedge E'^c \wedge E'^b \wedge E^a,
$$

(A.15)

where $\Lambda'_{\alpha'} = \frac{1}{2} \partial_{\alpha'} \phi'$, and $E'$ and $\partial_{\alpha'}$ are the $SO(2)$ spinor-rotated $N = 2$ supervielbein and supercoordinate derivative defined by

$$
\theta' = \exp\left(-\frac{\lambda}{2} \mathcal{E}\right) \theta, \quad \partial_{\alpha'} = (\exp\left(\frac{\lambda}{2} \mathcal{E}\right) \partial)_{\alpha}
$$

$$
E' = \exp\left(\frac{\lambda}{2} \mathcal{E}\right) E, \quad E' = \hat{E} \exp\left(-\frac{\lambda}{2} \mathcal{E}\right),
$$

(A.16)

where

$$
\cos \lambda = \frac{cC_0 + d}{\sqrt{(cC_0 + d)^2 + c^2 e^{-2\phi}}}, \quad \sin \lambda = \frac{ce^{-\phi}}{\sqrt{(cC_0 + d)^2 + c^2 e^{-2\phi}}}
$$

(A.17)

Therefore all the type IIB supergravity constraint equations (of course including the torsion constraints) are invariant under general $SL(2,R)$ duality transformations $[A,8], (A.12)$,
(A.13) and the SO(2) spinor-rotation (A.16). We note that the rotation angle $\lambda$ is just the same as given in Eq. (A.6) as expected.

References

[1] J. Polchinski, Phys.Rev.Lett.75 (1995) 4724, hep-th/9510017.

[2] J. Polchinski, *String Theory*, Cambridge University Press (1998) and references cited therein.

[3] A. Strominger and C. Vafa, Phys.Lett.B379 (1996) 99, hep-th/9601029.

[4] G.T. Horowitz, gr-qc/9704072 and references cited therein.

[5] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, Phys.Rev.D55 (1997) 5112, hep-th/9610043.

[6] N. Ishibashi, H. Kawai, Y.Kitazawa and A. Tsuchiya, Nucl.Phys.B498 (1997) 467, hep-th/9612115.

[7] T. Kimura and I. Oda, hep-th/9811134.

[8] M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, Nucl.Phys.B490 (1997) 163, hep-th/9610148.

[9] M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, Nucl.Phys.B490 (1997) 179, hep-th/9611159.

[10] E. Bergshoeff and P.K. Townsend, Nucl.Phys.B490 (1997) 145, hep-th/9611173.

[11] J. Aganagic, J. Park, C. Popescu, and J.H. Schwarz, Nucl.Phys.B496 (1997) 215, hep-th/9702133.

[12] C. Vafa, Nucl.Phys.B469 (1996) 403, hep-th/9602022.

[13] I. Oda, hep-th/9810203.

[14] C. Schmidhuber, Nucl.Phys.B467 (1996) 146, hep-th/9601003.

[15] A.A. Tseytlin, Nucl.Phys.B469 (1996) 51, hep-th/9602063.

[16] S.P. de Alwis and K. Sato, Phys.Rev.D53 (1996) 7187, hep-th/9601167.

[17] Y. Lozano, Phys.Lett.B399 (1997) 233, hep-th/9701186.
[18] J. Aganagic, C. Popescu, and J.H. Schwarz, Phys.Lett.\textbf{B393} (1997) 311, \texttt{hep-th/9610243}; Nucl.Phys.\textbf{B495} (1997) 99, \texttt{hep-th/9612080}.

[19] P.K. Townsend, Phys.Lett.\textbf{B373} (1996) 68, \texttt{hep-th/9512062}.

[20] J. Maldacena, Adv. Theor. Math. Phys.\textbf{2} (1998) 231, hep-th/9711200.

[21] I. Oda, Phys.Lett.\textbf{B444} (1998) 127, \texttt{hep-th/9809076}.

[22] I. Oda, J.High Energy Phys.\textbf{10} (1998) 015, \texttt{hep-th/9810024}.

[23] T. Kimura, Mod.Phys.Lett.\textbf{A14} (1999) 327, \texttt{hep-th/9810136}.

[24] J. Park and S.J. Rey, hep-th/9810154.

[25] P. K. Townsend, Phys.Lett.\textbf{B409} (1997) 131, \texttt{hep-th/9705160}; P. K. Townsend and M. Cederwall, J.High Energy Phys.\textbf{99} (1997) 003, \texttt{hep-th/9709002}; E. Bergshoeff and P. K. Townsend, Nucl.Phys.\textbf{B531} (1998) 226, \texttt{hep-th/9804011}.

[26] M. Cederwall and A. Westerberg, J.High Energy Phys.\textbf{01} (1998) 004, \texttt{hep-th/9710007}.

[27] M.B. Green and J.H. Schwarz, Phys.Lett.\textbf{B136} (1984) 367.

[28] C.M. Hull and P.K. Townsend, Nucl.Phys.\textbf{B348} (1995) 109.

[29] E. Witten, Nucl.Phys.\textbf{B443} (1995) 85, \texttt{hep-th/9503124}.

[30] M. Grisaru, P. Howe, L. Mezincescu, B. Nilsson and P.K. Townsend, Phys.Lett.\textbf{B162} (1985) 116.

[31] J.H. Schwarz, Phys.Lett.\textbf{B360} (1995) 13; ERRATUM ibid. \textbf{B364} (1995) 252.

[32] A.A. Tseytlin, Nucl.Phys.\textbf{B501} (1997) 41, \texttt{hep-th/9701123}.

[33] I. Oda, Phys.Lett.\textbf{B430} (1998) 242, \texttt{hep-th/9802152}.

[34] E. Witten, Nucl.Phys.\textbf{B460} (1996) 335, \texttt{hep-th/9510135}.

[35] P.K. Townsend, Phys.Lett.\textbf{B350} (1995) 184, \texttt{hep-th/9501068}.

[36] J.H. Schwarz, hep-th/9607201 and references cited therein.

[37] E. Bergshoeff, E. Sezgin and P.K. Townsend, Phys.Lett.\textbf{189B} (1987) 75; Ann.of Phys.\textbf{185} (1988) 330.

[38] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, Phys.Lett.\textbf{B191} (1987) 70.

[39] M.K. Gaillard and B. Zumino, Nucl.Phys.\textbf{B193} (1981) 221.
[40] M.K. Gaillard and B. Zumino, hep-th/9705226, hep-th/9712103.

[41] G. W. Gibbons and D.A. Rasheed, Phys.Lett. B365 (1996) 46, hep-th/9509141.

[42] M.B. Green and M. Gutperle, Phys.Lett. B377 (1996) 28, hep-th/9602074.

[43] Y. Igarashi, K. Itoh and K. Kamimura, Nucl.Phys. B536 (1998) 454, hep-th/9806160.

[44] Y. Igarashi, K. Itoh and K. Kamimura, Nucl.Phys. B536 (1998) 469, hep-th/9806161.

[45] J. Aganagic, J. Park, C. Popescu, and J.H. Schwarz, Nucl.Phys. B496 (1997) 191, hep-th/9701166.

[46] P. Pasti, D. Sorokin and M. Tonin, Phys.Lett. B398 (1997) 41, hep-th/9701037; Phys.Rev. D52 (1995) 4277, hep-th/9506109. I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, Phys.Rev.Lett. 78 (1997) 4332, hep-th/9701148.