Solutions to the Einstein-Maxwell-Current System with Sasakian manifolds

Hideki Ishihara\textsuperscript{a*} and Satsuki Matsuno\textsuperscript{b†}

\textsuperscript{a,b}Department of Mathematics and Physics, Graduate School of Science, Nambu Yoichiro Institute of Theoretical and Experimental Physics (NITEP), Osaka City University Advanced Mathematical Institute (OCAMI), Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

Abstract

We construct stationary solutions to the Einstein-Maxwell-current system by using the Sasakian manifold for the three-dimensional space. Both the magnetic field and the electric current in the solution are specified by the contact form of the Sasakian manifold. The solutions contain an arbitrary function that describes inhomogeneity of the number density of the charged particles, and the function determines the curvature of the space.

\*ishihara@sci.osaka-cu.ac.jp
\†smatsuno@sci.osaka-cu.ac.jp
## Contents

1 Introduction 3

2 Contact Metric Manifolds and Sasakian Manifolds 5

3 Solutions to the Einstein-Maxwell-Current System 7
   3.1 Equations of the system 7
   3.2 Contact magnetic field and contact current 8
   3.3 Gas model 8
   3.4 Energy-momentum tensors 10
   3.5 Solutions 10

4 Examples of Solutions 13

5 Summary 14

6 Acknowledgement 14
1 Introduction

The origin of magnetic fields observed in various scales in the universe is one of the most important problems in astrophysics [1]. The strong magnetic fields that cause energetic physical phenomena become sources of gravitational field in the general relativity. It is an interesting task to clarify the structure of space-time with the magnetic field, but it is not easy to solve the coupled system of the Maxwell equation and the Einstein equation.

The Bertotti-Robinson solutions [2, 3] and the Melvin solutions [4] are well-known as the solutions to the Einstein-Maxwell system, where magnetic fields are sources of the gravity. In these solutions, the magnetic fields are sourceless vacuum solutions to the Maxwell equation. In contrast, we present, in this paper, a family of solutions to the coupled system of the Einstein equation, the Maxwell equation and equation of motion of the electric current by using the Sasakian manifolds as a three-dimensional space.

Recently, Sasakian manifolds [5, 6] gather much attention in the context of M-theory [7, 8]. On the other hand, magnetic fields on the Sasakian manifold and motion of charged particles in the magnetic fields are extensively studied [9, 10]. We present a new application of the Sasakian manifold in the Einstein-Maxwell-current system.

The Reeb vector, $\xi$, of the contact form, $\eta$, of a contact metric space is divergence-free, parallel to its rotation, and geodesic tangent [5]. We consider a direct product of time and a three-dimensional contact metric manifold as a four-dimensional space-time, and set a magnetic field and an electric current field that generates the magnetic field. We assume both the magnetic field and the current are characterized by $\xi$, and call them contact magnetic field and contact current, respectively. The contact current is assumed to be carried by a gas consisting many charged particles moving along $\xi$. The energy-momentum tensor of the coupled system of the gas and the contact magnetic field requires the three-dimensional space is $\eta$-Einstein. Therefore, the Sasakian manifold, which is a contact metric manifold and $\eta$-Einstein space [5], together with the contact magnetic field and the contact current solves the Einstein-Maxwell-current system.

We present stationary solutions to the coupled equations of the Einstein-Maxwell-current with a cosmological constant. The solutions have an arbitrary function that describes the number density of the charged particles, and the function determines the sectional curvature of the Sasakian manifold. It is shown that if the three-dimensional space of the solution is compact and simply connected, it is homeomorphic to three-dimensional sphere. In the limit of the particles at rest, the space-time reduces to Einstein’s static universe, $\mathbb{R} \times S^3$.

The organization of this paper is as follows. In Section 2, we recall the contact metric
geometry and the Sasakian manifold, and review some properties of the Reeb vector of the contact form. In Section 3 we present stationary solutions with the Sasakian manifold to the Einstein-Maxwell-current system, and, in Section 4 we show solutions explicitly by using the Sasakian space forms, as examples. Section 5 is devoted to the summary.
2 Contact Metric Manifolds and Sasakian Manifolds

We recall the definition of the contact metric manifold and some known facts. A contact metric manifold is a 5-tuple \((\mathcal{M}, \phi, \xi, \eta, g)\) of a \((2n+1)\)-dimensional Riemannian manifold \((\mathcal{M}, g)\), a 1-form field \(\eta\), a vector field \(\xi\) and a \((1,1)\)-type tensor field \(\phi\) such that

\[
\eta \wedge (d\eta)^n = \eta \wedge d\eta \wedge \cdots \wedge d\eta \neq 0, \quad \iota_\xi d\eta = 0, \quad \eta(\xi) = 1,
\]

\[
\phi^2 = -I + \eta \otimes \xi, \quad g(\phi(X), \phi(Y)) = g(X,Y) - \eta(X)\eta(Y), \quad g(X, \phi(Y)) = d\eta(X,Y)
\]

holds.\(^\dagger\) The 1-form \(\eta\) and the vector \(\xi\) are called the contact form and the Reeb vector field, respectively. Through the metric \(g\), \(\eta\) and \(\xi\) are related by \(g(\xi, X) = \eta(X)\), or simply, \(\xi = \#\eta, \quad \eta = \flat \xi\) by the musical isomorphism notation.

A sectional curvature invariant under the action of \(\phi\) is called a \(\phi\)-sectional curvature, and a sectional curvature of the plane spanned by \(\xi\) and any vector field \(X\) is called a \(\xi\)-sectional curvature. A contact metric manifold admits the \(\phi\)-sectional curvature.

There are some equivalent definitions of the Sasakian manifold. We adopt the following definition: a contact metric manifold \((\mathcal{M}, \phi, \xi, \eta, g)\) is Sasakian if

\[
R(X,Y)\xi = \eta(Y)X - \eta(X)Y
\]

holds, where \(R\) is the Riemann curvature tensor of \(\mathcal{M}\). It follows that the \(\xi\)-sectional curvature is 1. A contact metric manifold \((\mathcal{M}, \phi, \xi, \eta, g)\) is called K-contact if \(\xi\) is a Killing vector field. In general, a Sasakian manifold is K-contact, furthermore in three dimensions, a K-contact manifold is Sasakian. A Sasakian manifold that has a constant \(\phi\)-sectional curvature is called a Sasakian space form.

A contact structure of \(\mathcal{M}\) is called regular if \(\xi\) is a regular vector field, that is, an arbitrary point \(p \in \mathcal{M}\) has a neighborhood \(U\) such that the number of components of the intersection of \(U\) and an integral curve of \(\xi\) through any point \(q \in U\) is one. We consider only compact and regular contact manifolds in this paper. In general, a compact regular contact manifold is a \(S^1\) bundle on a compact symplectic manifold, and its fibers are integral curves of the Reeb vector field \(\xi\) (Boothby-Wang fibration). As a particular case, a Sasakian manifold is a \(S^1\) bundle on a Kähler manifold. In more detail, let \((\mathcal{M}, \phi, \xi, \eta, g)\) be a compact regular Sasakian manifold, then there exists the Riemmanian submersion \(\pi : (\mathcal{M}, g) \to (\mathcal{N}, h)\) whose fibers are integral curves of \(\xi\), where \((\mathcal{N}, h)\) is the Kähler manifold with the metric \(h\) and the Kähler form \(\omega\) satisfying \(d\eta = \pi^* \omega\).

\(^\dagger\)For 1-forms \(\alpha^i\) and vector fields \(X_j\), \((i, j = 1, 2 \cdots, k)\), we adopt the convention

\[
(\alpha^1 \wedge \cdots \wedge \alpha^k)(X_1, \cdots, X_k) := \det(\alpha^i(X_j))/k!.
\]
Proposition 2.1. On a contact metric manifold \((M, \phi, \xi, \eta, g)\), \(\nabla_\xi \xi = 0\) holds.

Proof. For \(X \in \Gamma(TM)\),

\[
0 = (\iota_\xi d\eta)(X) = (\mathcal{L}_\xi \eta)(X) = \xi g(\xi, X) - g(\xi, [\xi, X]) = g(\nabla_\xi \xi, X),
\]

where we have used \(\eta(\xi) = g(\xi, \xi) = 1\).

Proposition 2.2. On a three-dimensional contact metric manifold \((M, \phi, \xi, \eta, g)\), \(*d\eta = 2\eta\) holds, where the symbol * denotes the Hodge star operation with respect to \(g\).

Proof. Let \(\Omega\) be the volume form of \(M\). For \(X, Y, Z \in \Gamma(TM)\), we define the vector product \(X \times Y \in \Gamma(TM)\) by \(g(X \times Y, Z) = \Omega(X, Y, Z)\). Using the local orthonormal basis \(\{\xi, e, \phi e\}\), called a \(\phi\)-basis, we define an orientation of \(M\) such that \(\Omega(\phi e, e, \xi) = 1 > 0\). Using the \(\phi\)-basis, we represent the volume form as \(\Omega(X, Y, Z) = \varepsilon_{ijk} X^i Y^j Z^k\), where \(\varepsilon_{ijk}\) is the totally anti-symmetric tensor field.

For an arbitrary vector field \(X \in \Gamma(TM)\), we have

\[
\phi X = X \times \xi. \tag{2.4}
\]

Indeed, it is sufficient to show (2.4) for \(\phi\)-basis. First, at any point on \(M\), \(\phi e = a(e \times \xi)\) trivially holds, where \(a\) is a real number, and \(1 = g(\phi e, \phi e) = ag(e \times \xi, \phi e) = a\Omega(e, \xi, \phi e)\), thus \(a = 1\). Second, we have \(\phi(\phi e) = b(\phi e \times \xi)\), where \(b\) is a real number, and \(-1 = g(e, \phi^2 e) = bg(e, \phi e \times \xi) = b\Omega(\phi e, \xi, e) = -b\), then \(b = 1\). It is apparent that (2.4) holds for \(X = \xi\). Therefore, (2.4) holds for an arbitrary \(X\).

From (2.2), we have

\[
d\eta(X, Y) = g(X, \phi Y) = g(X, Y \times \xi) = \Omega(X, Y, \xi) = \varepsilon_{ijk} X^i Y^j Z^k = 2(*\eta)(X, Y), \tag{2.5}
\]

then \(*d\eta = 2\eta\).

From Proposition 2.2, on a three-dimensional contact metric manifold \((M, \phi, \xi, \eta, g)\), we have

\[
\text{div } \xi := *d(*\eta) = \frac{1}{2} *d(d\eta) = 0, \tag{2.6}
\]

\[
\text{rot } \xi := \#(*d\eta) = 2 \#\eta = 2\xi, \tag{2.7}
\]

namely, the Reeb vector \(\xi\) is divergence-free, and is parallel to its own rotation\(^\text{§}\).

\(^\text{§}\)A vector field with this property is known as the Beltrami field\([11, 12, 13]\).
3 Solutions to the Einstein-Maxwell-Current System

3.1 Equations of the system

Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be a three-dimensional contact metric manifold, and we consider a four-dimensional space-time $(\tilde{\mathcal{M}} = \mathbb{R} \times \mathcal{M}, \tilde{g} = -dt^2 + g)$, where $g$ is independent on $t$. We extend $\eta, \xi, \phi$ to the fields on $\tilde{\mathcal{M}}$ by the following way; that is, let $p$ be the Riemannian submersion $p : \tilde{\mathcal{M}} = \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, then $\tilde{\eta} := p^*\eta$ is the 1-form field on $\tilde{\mathcal{M}}$. For the decomposition $T_x\tilde{\mathcal{M}} = \mathbb{R} \oplus T_{p(x)}\mathcal{M}$, we extend $\xi$ to the vector field $\tilde{\xi} := (0, \xi)$, and we define the $(1,1)$-type tensor field $\tilde{\phi}$ on $\tilde{\mathcal{M}}$ by $\tilde{\phi}(X) := (0, \phi(p^*X))$. Hereafter, we drop ‘tilde’ and use $\eta, \xi, \phi$ for the four-dimensional quantities on $\tilde{\mathcal{M}}$, for simplicity.

We consider a Maxwell field $F = dA$ where $A$ is a potential form, and a collision-less many-particle system that carries an electric current $j$. We construct solutions to the Einstein-Maxwell-current system with a cosmological constant governed by the coupled equations

\begin{align*}
\tilde{\text{Ric}} - \frac{1}{2}\tilde{\text{R}}\tilde{g} + \Lambda\tilde{g} &= T, \\
*\text{d}*_\text{d}F &= j, \\
m_{(i)}\nabla u_{(i)}u_{(i)} &= f_{(i)},
\end{align*}

where $\tilde{\text{Ric}}, \tilde{\text{R}}$ are the Ricci tensor and the scalar curvature of $(\tilde{\mathcal{M}}, \tilde{g})$, $T$ is the energy momentum tensor of the coupled system of the Maxwell field and the many-particle system, and $*$ is the Hodge star operator with respect to $\tilde{g}$. The equation (3.3) denotes the equation of motion for a particle, labeled by $i$, where $m_{(i)}$ and $u_{(i)}$ are mass and 4-velocity of the particle. The right hand side of (3.3) is the Lorentz force acting on the charged particle written by

\[ f = -e \#(\iota_u F), \]

where $e$ is the electric charge of the particle. The current $j$ carried by the particles is defined by

\[ \#j = \sum_i e_{(i)}u_{(i)}. \]

Since the space-time geometry is assumed to be static, we consider the Maxwell field and all quantities that characterize the particles: number densities, 4-velocities, and so on are time independent.
3.2 Contact magnetic field and contact current

Using the contact form as a vector potential \( A = B \eta \), we define the contact magnetic field \([9, 10]\) as

\[
F_B := B d \eta, \quad (3.6)
\]

where \( B \) is a constant. For the Maxwell 2-form field \( F \) on \( \tilde{\mathcal{M}} \), the electric vector field, \( \tilde{E} \), and magnetic vector field, \( \tilde{B} \), are defined by

\[
g(\tilde{E}, X) := F(X, \partial_t), \quad g(\tilde{B}, X) := *F(\partial_t, X). \quad (3.7)
\]

Then, for the contact magnetic field, we see \( \tilde{B} = 2B \xi \) from Proposition \([2.2]\) and \( \tilde{E} = 0 \). Immediately, it holds that \( \text{div} \, \tilde{B} = 0 \). From \([2.2]\), \( \phi \) is related to the contact magnetic field by

\[
F_B(X, Y) = g(X, B \phi(Y)). \quad (3.8)
\]

We also have \( *d *F_B = 4B \eta \), then the Maxwell equation \([3.2]\) requires

\[
j = 4B \eta. \quad (3.9)
\]

We call the current \( j \) contact current. It should be noted that the contact magnetic vector field, \( \tilde{B} \), is parallel to the contact current, \( \# j \), that generates the magnetic field. Therefore, the Lorentz force dose not act on the contact current\(^*\). From Proposition \([2.1]\) on \( \mathcal{M} \), we also have \( \nabla_\xi \xi = 0 \) on \( \tilde{\mathcal{M}} \), where \( \nabla \) is the Riemman connection with respect to \( \tilde{g} \). Then, the contact current flows along the geodesic curves.

In the case that the magnetic vector field is not parallel to the current field that generates the magnetic field, usually this case occurs, the Lorentz force by the magnetic field acts on the current. It would be a complicated task to find a consistent solution to the equations of motion of the current and the Maxwell equation if the back-reaction on the current field from the magnetic field is taken into account. In contrast, the contact current that generates the contact magnetic field flows along geodesic curves without the back-reaction.

3.3 Gas model

We consider a gas as the many-particle system. The gas is assumed to contain a huge number of particles, then the 4-velocity \( u \) is treated as a smooth vector field on \( \tilde{\mathcal{M}} \). We assume the

\[^*\text{The case that the magnetic vector field is parallel to the current is studied in various setups}\([13, 14, 15]\).\]
gas consists of three-components of particles: particles with mass $m$ and charge $e = +1$, particles with mass $m$ and charge $e = -1$, and neutral particles with mass $m_n$. The collision of particles and the thermal motion are neglected.

We set the number densities of charged particles, $n_+$ and $n_-$ take the same value $n \in C^\infty(M)$, so that the gas is electrically neutral, and the 4-velocities of the charged particles are

$$u_\pm = u^0 \partial_t \pm u^3 \xi,$$

where $u_\pm$ are unit timelike vector fields, i.e.,

$$-(u^0)^2 + (u^3)^2 = -1.$$  

The number density of the neutral particles is denoted by $n_n \in C^\infty(M)$, and the neutral particles are assumed to be at rest, namely the 4-velocity is $u_n = \partial_t$.

The charged particles move along the contact magnetic vector field, then the Lorentz force acting on the particles vanishes. Therefore, all particles move along geodesics,

$$\nabla_{u_\pm} u_\pm = 0 \text{ and } \nabla_{u_n} u_n = 0.$$  

Since $\partial_t$ and $\xi$ are geodesic tangents, we further assume that

$$\xi(u^0) = 0, \text{ and } \xi(u^3) = 0,$$

so that (3.12) holds.

The electric current (3.5) carried by the charged particles with the 4-velocity (3.10) is given by

$$\# j = nu_+ - nu_- = 2nu^3 \xi,$$

and the Maxwell equation (3.9) requires

$$nu^3 = 2B.$$  

The contact magnetic field and the gas as the many-particle system that carry the contact current solve the coupled system of the Maxwell equation and the equations of motion of the particles.
3.4 Energy-momentum tensors

The energy momentum tensor of a Maxwell field $F$ defined by

$$T_{EM}(X,Y) = \tilde{g}^*(\iota_XF, \iota_YF) - \frac{1}{4}\|F\|^2\tilde{g}(X,Y)$$  \hspace{1cm} (3.16)$$

reduces to

$$T_{EM} = \frac{1}{2}B^2(dt \otimes dt + g - 2\eta \otimes \eta)$$  \hspace{1cm} (3.17)$$

for the contact magnetic field \(3.6\).

The energy momentum tensor of the gas that consists of charged particles and neutral particles is given by

$$T_F = nm\left(\left(^b u_+\right) \otimes \left(^b u_+\right) + \left(^b u_-\right) \otimes \left(^b u_-\right)\right) + n_n m_n \left(^b u_n\right) \otimes \left(^b u_n\right),$$  \hspace{1cm} (3.18)$$

where

$$^b u_\pm = -u^0 dt \pm u^3 \eta, \quad \text{and} \quad ^b u_n = -dt.$$  \hspace{1cm} (3.19)$$

Insert (3.19) into (3.18), we have

$$T_F = \rho \ dt \otimes dt + P \ \eta \otimes \eta,$$  \hspace{1cm} (3.20)$$

where $\rho$ is the energy density of the gas and $P$ is the effective pressure caused by the motion of particles. These are given by

$$\rho = 2nm(u^0)^2 + n_n m_n,$$  \hspace{1cm} (3.21)$$

$$P = 2nm(u^3)^2.$$  \hspace{1cm} (3.22)$$

It is apparently that $\partial_\tau \rho = 0$ and $\partial_\tau P = 0$.

3.5 Solutions

We inspect the condition that $(\tilde{\mathcal{M}}, \tilde{\mathcal{g}})$ solves the Einstein-Maxwell-current system with the contact current and the contact magnetic field as sources of the gravitational field. The Einstein equation \(3.1\) is rewritten in the form

$$\tilde{\text{Ric}} = T - \frac{1}{2}\text{tr}T \ \tilde{g} + \Lambda \tilde{g},$$  \hspace{1cm} (3.23)$$

\footnote{The symbol $\tilde{g}^*$ gives the inner product of $\Omega^1(M)$, and the inner product for $\Omega^p(M)$ is defined by $\tilde{g}^*(\alpha^1 \wedge \cdots \wedge \alpha^k, \beta^1 \wedge \cdots \wedge \beta^k) = \det(\tilde{g}^*(\alpha^i, \beta^j))/p!$.}
where \( T := T_{EM} + T_F \). We decompose this equation into the time component and the space components of \( \mathcal{M} \), and using (3.17) and (3.20), we obtain

\[
\tilde{\text{Ric}}(\partial_t, \partial_t) = 0 = \frac{1}{2} (\rho + P + B^2) - \Lambda, \tag{3.24}
\]

\[
\text{Ric} = \frac{1}{2} (\rho - P + B^2 + 2\Lambda) \, g + (P - B^2) \, \eta \otimes \eta, \tag{3.25}
\]

where \( \text{Ric} \) denotes the Ricci tensor of \((\mathcal{M}, g)\). Equation (3.25) shows the Ricci tensor of the contact metric manifold \((\mathcal{M}, \phi, \xi, \eta, g)\) has the form of \( \text{Ric} = a g + b \eta \otimes \eta, \quad a, b \in \mathbb{C}^\infty(\mathcal{M}) \). Therefore, the manifold should be an \( \eta \)-Einstein.

The space \((\mathcal{M}, g)\) being \( \eta \)-Einstein requires the Ricci tensor are diagonal in the \( \phi \)-basis, \( \{e, \phi e, \xi\} \), on \( \mathcal{M} \). It is easy to derive the diagonal components of the Ricci tensor in the forms:

\[
\text{Ric}(e, e) = K(e, \xi) + K(e, \phi e), \tag{3.26}
\]

\[
\text{Ric}(\phi e, \phi e) = K(\phi e, \xi) + K(e, \phi e), \tag{3.27}
\]

\[
\text{Ric}(\xi, \xi) = K(e, \xi) + K(\phi e, \xi), \tag{3.28}
\]

where \( K \) denotes the sectional curvature. As mentioned before, a three-dimensional contact metric manifold \( \mathcal{M} \) admits a \( \phi \)-sectional curvature \( \mathcal{H} = K(\mathcal{e}, \phi e) \), and furthermore, if the manifold \( \mathcal{M} \) is \( \eta \)-Einstein, it is known that there are three possibilities for \( \mathcal{M} \): (a) a contact metric manifold that has constant \( \xi \)-sectional curvature, say \( k \), less than 1, and \( \mathcal{H} = -k \), (b) a three-dimensional flat contact metric manifold, and (c) a Sasakian manifold [17].

In the case of (a), substitute \( K(\xi, e) = K(\xi, \phi e) = k \) and \( K(e, \phi e) = H = -k \) into (3.27), the Ricci tensor reduces to \( \text{Ric} = 2k \, \eta \otimes \eta \). The Einstein equations (3.24) and (3.25) lead to \( \rho = -B^2 \). We exclude the case (a) because the negative energy is unphysical. In the case of (b), since \( \text{Ric} = 0 \), we have \( \rho = -B^2 \) from (3.24) and (3.25) as same as the case (a). Then we also exclude the case (b).

In the case of (c), since \((\mathcal{M}, \phi, \xi, \eta, g)\) is a three-dimensional Sasakian manifold, the \( \xi \)-sectional curvature \( K(\xi, e) = K(\xi, \phi e) = 1 \). Then, the Ricci tensor is given by

\[
\text{Ric} = (1 + \mathcal{H})g + (1 - \mathcal{H})\eta \otimes \eta. \tag{3.29}
\]

Then, the Einstein equations (3.24) and (3.25) become

\[
\Lambda = 1 + \frac{B^2}{2}, \tag{3.30}
\]

\[
\rho = 1 - B^2 + \mathcal{H}, \tag{3.31}
\]

\[
P + \rho = 2. \tag{3.32}
\]
The equation (3.30) means that the cosmological constant is positive. Since the Reeb vector \( \xi \) is a Killing vector in the Sasakian manifold, \( \xi(\rho) = 0 \) and \( \xi(P) = 0 \) hold.

Equations (3.31) and (3.32) with the help of (3.11), (3.15), (3.21) and (3.22) lead to

\[
H = 1 + B^2 - 8\frac{B^2m}{n},
\]

\[
m_n n_n = 2 - 16\frac{B^2m}{n} - 2mn.
\]

The functions \( n, n_n \) should satisfy \( \xi(n) = \xi(n_n) = 0 \), then these are functions on the base space \( \mathcal{N} \) of Boothby-Wang fibration for the Sasakian manifold \( \mathcal{M} \). The functions \( H \) and \( n_n \) are determined by the function \( n \). Because the number density of the particles should be positive, we see that \( B \) and \( mn \) should be in the range

\[
0 < (Bm)^2 < \frac{1}{32}, \quad \text{and} \quad 1 - \sqrt{1 - 32B^2m^2} < 2mn < 1 + \sqrt{1 - 32B^2m^2}.
\]

The function \( H \) is expressed by

\[
H = B^2 + mn + \frac{1}{2} m_n n_n > 0,
\]

and it can vary on \( \mathcal{M} \) in the range

\[
\frac{1}{2} + B^2 - \frac{1}{2} \sqrt{1 - 32B^2m^2} < H \leq \frac{1}{2} + B^2 + \frac{1}{2} \sqrt{1 - 32B^2m^2}.
\]

Here, we specify the class of Sasakian manifolds of the solution. We concentrate on a compact, simply connected and regular Sasakian manifold. Namely, the manifold \( \mathcal{M} \) is a \( S^1 \) bundle on a compact symplectic manifold, \( \mathcal{N} \), and its fibers are integral curves of the Reeb vector field \( \xi \). It is known that the \( \phi \)-sectional curvature of \( \mathcal{M} \) and the sectional curvature \( K_* \) of \( \mathcal{N} \) are related as \( H = K_* - 3 \). Since we have \( H > 0 \), then \( K_* > 3 \). Any two-dimensional complete Riemannian manifolds admitting everywhere positive sectional curvature is homeomorphic to \( S^2 \), then \( \mathcal{N} \approx S^2 \).

Therefore, \( (\mathcal{M}, g) \) is a \( S^1 \) bundle on a Kähler manifold \( (S^2, h, \omega) \), where Riemannian metric \( h \) satisfies the condition (3.37) for its sectional curvature \( K_* = H + 3 \), and \( \omega \) is a Kähler form. We can take locally a 1-form \( \tau \) such that \( d\tau = \omega \), and let \( z \) be a fiber coordinate, then the metric \( g \) is given by

\[
g = \pi^* h + \eta \otimes \eta, \tag{3.38}
\]

\[
\eta = dz + \tau. \tag{3.39}
\]

We can regard \( \eta \) as a connection form of the \( S^1 \)-bundle, and we have

\[
\int_{S^2} d\eta = \int_{S^2} \omega \neq 0, \tag{3.40}
\]

12
thus the Euler number of $\mathcal{M}$ is not zero. Then, $\mathcal{M}$ is not a direct product bundle. Since $\mathcal{M}$ is assumed to be simply connected, there exists a bundle isomorphism from $\mathcal{M}$ to the Hopf fibration. Therefore, $\mathcal{M}$ with this metric is homeomorphic to $S^3$.

4 Examples of Solutions

A compact and simply connected three-dimensional Sasakian space form with a positive constant $\phi$-sectional curvature is homeomorphic to $S^3$. Then the metric can be written in the form of a $S^1$ bundle on $S^2$:

$$g = \frac{\alpha}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{\alpha^2}{4} (d\psi + \cos \theta d\phi)^2,$$

(4.1)

where $\alpha$ is a positive constant, and $\psi$ is a fiber coordinate. The contact form and the Reeb vector are

$$\eta = \frac{\alpha}{2} (d\psi + \cos \theta d\phi), \quad \xi = \frac{2}{\alpha} \partial_{\psi},$$

(4.2)

respectively.

The space-time is homeomorphic to $\mathbb{R} \times S^3$ and the metric is

$$\tilde{g} = -dt^2 + g,$$

(4.3)

and the constant $\phi$-sectional curvature is given by $H = \frac{4}{\alpha} - 3$. The contact magnetic field $F_B$ and the contact current $j$ are

$$F_B = -\frac{\alpha B}{2} \sin \theta d\theta \wedge d\phi,$$

(4.4)

$$j = \frac{8B}{\alpha} \partial_{\psi}.$$

(4.5)

From (3.33), the parameter $\alpha$ is expressed by the number density of the charged particles, a constant on $\tilde{\mathcal{M}}$ in this case, in the form

$$\frac{1}{\alpha} = 1 + \frac{B^2}{4} \left(1 - \frac{8m}{n}\right),$$

(4.6)

In order that $H$ satisfies (3.37), there are varieties of solutions with $\alpha$ in the range

$$\frac{14 + 4B^2 - 2\sqrt{1 - 32B^2m^2}}{12 + B^4 + B^2(7 + 8m^2)} < \alpha < \frac{14 + 4B^2 + 2\sqrt{1 - 32B^2m^2}}{12 + B^4 + B^2(7 + 8m^2)}.$$  

(4.7)

If we set $u^3 = 0$, we have $j = 0, B = 0, \rho = 2, P = 0, \Lambda = 1, \alpha = 1$. This is Einstein’s static universe. Therefore, the solutions based on the Sasakian space forms (4.3) are generalizations of Einstein’s static universe by the existence of the contact magnetic field and the gas that carries the contact electric current.
5 Summary

We constructed exact stationary solutions to the Einstein-Maxwell-current system by using a direct product of time and a three-dimensional Sasakian manifold. We considered that a gas of collision-less charged particles that carries stationary electric current, and the current generates a magnetic field. Taking both the gas and the magnetic field as the source of gravitational field, we have presented solutions to the coupled system of the equations of particles in the gas, the Maxwell equation and the Einstein equation with a positive cosmological constant.

The three-dimensional space was assumed to be a contact metric manifold, which has the contact form, $\eta$, and the Reeb vector, $\xi$. The key properties are the followings: $\xi$ is divergence-free, parallel to its own rotation, and geodesic tangent. We considered that both the electric current and the magnetic vector field are parallel to $\xi$.

The electric current is assumed to be carried by moving charged particles in a gas. Through the Einstein equation, the energy momentum tensor of the fluid and the Maxwell field generated by the current requires the three-dimensional contact metric manifold to be a Sasakian manifold, which has $\eta$-Einstein metric. As the result, the Einstein equations are reduced to a coupled system of algebraic equations that is easy to solve.

The solution contains a function that describes inhomogeneity of the number density of particles. Accordingly, $\phi$-sectional curvature of the Sasakian manifold of the solution is positive and can vary in the range such that the number density of the particles is positive. It was also shown that if three-dimensional space is compact and simply connected, the space-time topology of the solution is $\mathbb{R} \times S^3$.

We presented explicitly the solution with anisotropy of the three-dimensional space by using the Sasakian space form as an example. In the case that the particles in the gas are at rest, the solution reduces to Einstein’s static universe. Then, the solutions obtained in the present paper are generalizations of it by introducing the gas that carries the electric current and magnetic field that are characterized by the contact form.

6 Acknowledgement

This work was partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JP-MXP0619217849).
References

[1] L. M. Widrow, “Origin of galactic and extragalactic magnetic fields”, Rev. Mod. Phys. 74, 775-823 (2002).

[2] B. Bertotti, “Uniform electromagnetic field in the theory of general relativity”, Phys. Rev. 116, 1331 (1959).

[3] I. Robinson, “A Solution of the Maxwell-Einstein Equations”, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. 7, 351-352 (1959).

[4] M. A. Melvin, “Pure magnetic and electric geons”, Phys. Lett. 8, 65-70 (1964).

[5] Blair, David E., “Riemannian geometry of contact and symplectic manifolds”. Springer Science Business Media, (2010).

[6] Boyer, C., and Galicki, K., “Sasakian geometry”. Oxford Univ. Press. (2008).

[7] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity”, Int. J. Theor. Phys. 38, 1113 (1999).

[8] I. R. Klebanov and E. Witten, “Superconformal field theory on three-branes at a Calabi-Yau singularity”, Nucl. Phys. B 536, 199 (1998).

[9] Cabrerizo, J. L., M. Fernández, and J. S. Gómez. “The contact magnetic flow in 3D Sasakian manifolds.” Journal of Physics A: Mathematical and Theoretical 42.19 (2009): 195201.

[10] Druță-Romaniuc, S. L., Inoguchi, J. I., Munteanu, M. I., Nistor, A. I., “Magnetic curves in Sasakian manifolds”. Journal of Nonlinear Mathematical Physics, 22(3), 428-447 (2015).

[11] E. Beltrami, “Considerations on Hydrodynamics”, Rendiconti del Reale Instituto Lombardo... Series II, 22, (1889).

[12] Z. Yoshida and Y. Giga, “Remarks on Spectra of Operator Rot”, Math. Z. 204, 235-245 (1990).
[13] D. Reed, “Foundational electrodynamics and Beltrami vector fields in Advanced Electromagnetism”. Foundations, Theory and Applications, pp. 217-249 (1995).

[14] G.E. Marsh, “Force-free magnetic fields: solutions, topology and applications.” World Scientific, Singapore, (1996).

[15] S. M. Mahajan and Z. Yoshida, “Double Curl Beltrami Flow”, Phys. Rev. Lett. 81, 4863 (1998).

[16] Z. Yoshida, Eigenfunction Expansions Associated with the Curl Derivatives in Cylindrical Geometries: Completeness of the Chandrasekhar-Kendall Eigenfunctions, J. Math. Phys., 33(4), pp. 1252-1256, 1992.

[17] Blair, David E., Themis Koufogiorgos, and Ramesh Sharma, “A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$”. Kodai Mathematical Journal 13.3 (1990): 391-401.