Quantum Coherence Quantiﬁers Based on Rényi α-Relative Entropy*

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Abstract The resource theories of quantum coherence attract a lot of attention in recent years. Especially, the monotonicity property plays a crucial role here. In this paper we investigate the monotonicity property for the coherence measures induced by the Rényi α-relative entropy, which present in [Phys. Rev. A 94 (2016) 052336]. We show that the Rényi α-relative entropy of coherence does not in general satisfy the monotonicity requirement under the subselection of measurements condition and it also does not satisfy the extension of monotonicity requirement, which present in [Phys. Rev. A 93 (2016) 032136]. Due to the Rényi α-relative entropy of coherence can act as a coherence monotone quantifier, we examine the trade-off relations between coherence and mixedness. Finally, some properties for the single qubit of Rényi 2-relative entropy of coherence are derived.

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Key words: quantum coherence, Rényi α-relative entropy, monotonicity, mixedness

1 Introduction
Coherence arising from quantum superposition rule, is an important resources in quantum information theory. Coherence is discussed in the interference phenomena and it is know due to the role of phase coherence in optical phenomena.[1] A rigorous framework for quantifying coherence was proposed by Baumgratz et al.[2] and they proposed several measures of coherence, which are based on information distance measures including relative entropy and 1l norm.[2] The quantification framework of quantum coherence stimulated many further considerations, which include other coherence measures,[3–5] the operational interpretations of quantum coherence,[6–8] the relationship among quantum entanglement, quantum discord and quantum deﬁcit,[9–13] quantiﬁcation of coherence in inﬁnite dimensional system,[14–15] the other properties are similar to quantum entanglement theory.[16–29]

From the view point of the deﬁnition, one can straightforwardly quantify the coherence in a given basis by measuring the distance between the quantum state ρ and its nearest incoherent state. Baumgratz et al. gave four necessary criteria,[2] which any quantity should fulﬁll them. Given a finite-dimensional Hilbert space H with d = dim (H). We note that I is the set of quantum states, which is called incoherent state that are diagonal in a ﬁxed basis { |i⟩}d i=1, K 1 is a set of Kraus operators, and satisﬁes ∑n K nIK n I = 1 with K nIK n I ⊂ I. Then any proper measure of the coherence C must satisfy the following conditions:

(C1) C (ρ) ≥ 0 for all quantum states ρ, and C (ρ) = 0 if and only if ρ ∈ I.
(C2a) Monotonicity under all the incoherent completely positive and trace preserving (ICPTP) maps Φ: C (ρ) ≥ C (Φ (ρ)), where Φ (ρ) = ∑n K nρK † n .
(C2b) Monotonicity for average coherence under sub-selection based on measurements outcomes:

C (ρ) ≥ ∑n ρnC (ρn),

where ρn = K nρK † n /pn and pn = Tr (K nρK † n ).
(C3) Non-increasing under mixing of quantum states:

∑n ρnC (ρn) ≥ C (∑n ρn pn) for any ensemble {ρn, pn}.

The Rényi entropy is important in quantum information theory. It can be used as a measure of entanglement.[30] In Ref. [31], Mosonyi and Hiai deﬁned the Rényi α-relative entropy, which can act as an information distance measure. In Ref. [32], Chitambar et al. proposed that the Rényi α-relative entropy of coherence fulﬁlls condition C1 and C2a for α ∈ [0, 2], then we call the Rényi α-relative entropy of coherence is a coherence monitone.[34] As we know, the condition C2b is important as it allows for sub-selection based on measurement outcomes, a process available in well controlled quantum experiments and it is also diﬃcult to verify.[2] A natural question arises immediately, is the condition C2b satisﬁed for the Rényi α-relative entropy of coherence?

In this paper, we will resolve the above question. In Sec. 2, we review basic points for the Rényi α-relative entropy of coherence. In Sec. 3, we prove that the Rényi

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α-relative entropy of coherence doesn’t fulfill the condition C2b and it also does not fulfill the extension condition C2b presented in Ref. [35]. We give the tradeoff relation between the Rényi α-relative entropy of coherence and mixedness in Sec. 4. The case of the Rényi α-relative entropy of coherence for a single qubit is discussed in Sec. 5. In Sec. 6 we give the summary of results.

2 The Rényi α-Relative Entropy of Coherence

In this section, we recall basic points of the Rényi α-relative entropy of coherence present in Ref. [32]. For α ∈ [0, ∞) the Rényi α-relative entropy of the states ρ by δ is defined by[31]

$$S_α(ρ|δ) := \frac{1}{α−1} \log \text{tr}(ρ^α δ^{1−α}) .$$

This quantity is contractive for all α ∈ (0, 2]. Since the Rényi α-relative entropy can act as an information distance measure,[31] then, we can define the Rényi α-relative entropy of coherence as:

$$C_α(ρ) := \min_{δ ∈ Z} S_α(ρ|δ) .$$

Note that in the limit α → 1, $S_{α→1}(ρ|δ)$ gives the relative entropy $S(ρ|δ) = \text{tr}(ρ \log ρ) − \text{tr}(ρ \log δ)$. Let δ = $\sum_i q_i |i⟩⟨i|$ be some incoherent states, then the analytical expression of $C_α(ρ)$ can be obtained as[32]

$$C_α(ρ) := \min_{i} \frac{1}{α−1} \log \sum_i (q_i |i⟩⟨i|)^{1−α} .$$

Equation (3) can be further simplified as[32]

$$C_α(ρ) = \frac{α}{α−1} \log \left(\sum_i (q_i |i⟩⟨i|)^{1−α}\right)^{1/α} .$$

In this paper, we do not consider the cases for α = 0 and the limit α → 1. For α = 0, the Rényi relative entropy of coherence is always equal to 0. For the limit α → 1, a detailed study for the standard relative entropy of coherence is presented in Ref. [2].

3 The Monotonicity Property

First we show that $C_α(ρ)$ fulfills the condition C3 for α ∈ [0, 1]. In Ref. [31], it is shown that $S_α(ρ|δ)$ is convexity for α ∈ [0, 1). For any ensemble $\{p_i, ρ_i\}$, we assume the incoherent states $δ_i$ are closet with respect to $ρ_i$, then we have

$$C_α\left(\sum_i p_i ρ_i\right) \leq \min_{δ ∈ Z} S_α\left(\sum_i p_i ρ_i|δ\right) \leq \sum_i p_i S_α(ρ_i|δ_i) \geq \sum_i p_i C_α(ρ_i) ,$$

where the second inequality using the convexity of $S_α(ρ|δ)$. We conclude that for α ∈ [0, 1), $C_α(ρ)$ cannot increase under mixing of quantum states, then $C_α(ρ)$ fulfills the condition C3 for α ∈ [0, 1). $S_α(ρ|δ)$ is not convexity anymore for α ∈ (1, 2],[31] there may exist some cases, which lead to $C_α(\sum_i p_i ρ_i) > \sum_i p_i C_α(ρ_i)$. The condition C3 combined with C2b, implies C2a. In Ref. [32], Chitambar et al. studied the Rényi α-relative entropy of coherence fulfills C2a for different kinds of incoherent operations. They do not consider whether the Rényi α-relative entropy of coherence fulfills the condition C2b. This motivates us to study whether C2b is satisfied for $C_α(ρ)$.

Now we use the example, which presented in Ref. [2] to show that condition C2b is violated. We choose

$$|0⟩ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1⟩ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |2⟩ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are the prescribed orthonormal basis. The two Kraus operators are written as

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} ,$$

where the complex numbers $a$ and $b$ obey $|a|^2 + |b|^2 = 1$. This condition guarantees that $\sum_n K_n^† K_n = 1$. The density matrix is presented as

$$ρ = \frac{1}{4} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

After applying this channel to the density matrix $ρ$, we obtain the output states:

$$ρ_1 = \frac{1}{2 + |a|^2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |a|^2 \end{pmatrix} , \quad ρ_2 = \frac{1}{1 + |b|^2} \begin{pmatrix} 1 & b^* & 0 \\ b & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

With the probabilities:

$$p_1 = \frac{2 + |a|^2}{4} , \quad p_2 = \frac{1 + |b|^2}{4} .$$

By using Eq. (4), we obtain the Rényi α-relative entropy of coherence for $ρ$ as

$$C_α(ρ) = \frac{α}{α−1} \log \left[\frac{1}{2} + \left(\frac{1}{2}\right)^{1/α}\right] .$$

Note that the operator $K_1$ makes $C_α(ρ_1) = 0$, we only need to calculate $C_α(ρ_2)$.

$$C_α(ρ_2) = \frac{α}{α−1} \log \left[\left(\frac{1}{1 + |b|^2}\right)^{1/α} + \left(\frac{|b|^2}{1 + |b|^2}\right)^{1/α}\right] .$$

We choose $b = 1$, substituting it into Eqs. (9) and (10), we then get

$$p_2 C_α(ρ_2) = \frac{1}{2} \frac{α}{α−1} \log \left[\left(\frac{1}{2}\right)^{1/α} + \left(\frac{1}{2}\right)^{1/α}\right] .$$

Using the inequality $x + y ≥ 2\sqrt{xy}$, we obtain

$$\frac{1}{2} + \left(\frac{1}{2}\right)^{1/α} ≥ 2\sqrt{\frac{1}{2} × \left(\frac{1}{2}\right)^{1/α}} = \sqrt{2} × \left(\frac{1}{2}\right)^{1/α} .$$


The equality holding if and only if $\alpha = 1$. Thus, for $\alpha \in (0, 1)$

$$C_\alpha(\rho) = \frac{\alpha}{\alpha - 1} \log \left[ C \left( \frac{1}{2} + \left( \frac{1}{2} \right)^{1/\alpha} \right) \right]$$

$$< \frac{1}{2} \frac{\alpha}{\alpha - 1} \log \left[ \left( \frac{1}{2} \right)^{1/\alpha} + \left( \frac{1}{2} \right)^{1/\alpha} \right]$$

$$= p_2 C_\alpha(\rho_2).$$

If we choose $b = 1/2$, substituting it into Eqs. (9) and (10), we get

$$p_2 C_\alpha(\rho_2) = \frac{3 \alpha}{8 \alpha - 1} \log \left[ \left( \frac{2}{3} \right)^{1/\alpha} + \left( \frac{1}{3} \right)^{1/\alpha} \right].$$

We plot $C_\alpha(\rho)$ and $p_2 C_\alpha(\rho_2)$ in Fig. 1. From Fig. 1, we can also find some cases to illustrate $C_\alpha(\rho) < \sum_n p_n C_\alpha(\rho_n)$.

![Fig. 1](image1)

(Color online) Comparison between $C_\alpha(\rho)$ and $p_2 C_\alpha(\rho_2)$ for $b = 1/2$. The black line shows $p_2 C_\alpha(\rho_2)$. The red line shows $C_\alpha(\rho)$.

Recently, Yu et al. proposed an alternative framework for quantifying coherence, which is more flexible and convenient for applications than the original one.\[33\] Their framework can be expressed as follows. Any proper measure of the coherence $C$ must satisfy the following three conditions:

(B1) Nonnegativity: $C(\rho) \geq 0$ for all quantum states $\rho$, and $C(\rho) = 0$ if and only if $\rho \in I$.

(B2) Monotonicity: $C(\rho) \geq C(\Phi(\rho))$, where $\Phi(\rho) = \sum_n K_n \rho K_n^\dagger$ is an incoherent operation.

(B3) Additivity of coherence for subspace-independent states: $C(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$ for block-diagonal states $\rho$ in the incoherent basis, where density operators $\rho_1$ and $\rho_2$ are defined on the two independent subspaces, $p_1$ and $p_2$ are two possibility coefficients with $p_1 + p_2 = 1$ and

$$p_1 \rho_1 \oplus p_2 \rho_2 = \begin{pmatrix} p_1 \rho_1 & 0 \\ 0 & p_2 \rho_2 \end{pmatrix}.$$

The above three conditions (B1, B2, B3) are fulfilled by all the coherence measures based on the original four conditions (C1, C2a, C2b, C3). Thus, this framework provides us an alternative method to illustrate that the measure of coherence induced by Rényi $\alpha$-relative entropy must violate C2b. We consider a state $\rho = p_1 \rho_1 \oplus p_2 \rho_2$, with $p_1 = (1/2)(|0\rangle + |1\rangle)(0\rangle + 1\langle)$ and $p_2 = (1/3)(|2\rangle + |3\rangle + |4\rangle)(0\rangle + 3\langle + 4\langle)$.\[33\] We choose the computational basis $|i\rangle_{i=0}$ as the reference basis, then we have

$$C_\alpha(\rho_1) = 1, \quad C_\alpha(\rho_2) = \log 3,$$

$$C_\alpha(\rho) = \frac{\alpha}{\alpha - 1} \log \left[ \left( \frac{1}{2} \right)^{1/\alpha} + \frac{3}{2} \left( \frac{1}{3} \right)^{1/\alpha} \right].$$

We plot $C_\alpha(\rho)$ and $C(p_1 \rho_1 \oplus p_2 \rho_2)$ in Fig. 2. It is shown that

$$C_\alpha(\rho) = C(p_1 \rho_1 \oplus p_2 \rho_2) \neq p_1 C(\rho_1) + p_2 C(\rho_2).$$

We note that for the limit $\alpha \to 1$, the Rényi $\alpha$-relative entropy of coherence will become the standard relative entropy of coherence, thus we have $C_{\alpha\to1}(\rho) = p_1 C_{\alpha\to1}(\rho_1) + p_2 C_{\alpha\to1}(\rho_2)$. Therefore, the Rényi $\alpha$-relative entropy of coherence must violate C2b in general.

![Fig. 2](image2)

(Color online) Comparison between $C_\alpha(\rho)$ and $C(p_1 \rho_1 \oplus p_2 \rho_2)$. The black solid line shows $C_\alpha(\rho)$. The red dotted line shows $C(p_1 \rho_1 \oplus p_2 \rho_2)$. The black solid line shows $C_\alpha(\rho)$ and $C_\alpha(\rho_1) + C_\alpha(\rho_2)$ in Fig. 2. It is shown that

$$C_\alpha(\rho) = C(p_1 \rho_1 \oplus p_2 \rho_2) \neq p_1 C_\alpha(\rho_1) + p_2 C_\alpha(\rho_2).$$

From the above examples, we then conclude that condition C2b, i.e., $C_\alpha(\rho) \geq \sum_n p_n C_\alpha(\rho_n)$ is not generally true for the measure of coherence induced by Rényi $\alpha$-relative entropy.

In Ref. [35], Rastegin studied the Tsallis relative $\alpha$-entropies of coherence

$$C_\alpha^T(\rho) = \min \frac{1}{\alpha - 1} [\text{tr} (\rho^\alpha \sigma^{1-\alpha}) - 1]$$

and give an extension of condition C2b. The extension of condition C2b can be represented as

$$\sum_n p_n q_n^{-\alpha} C_\alpha^T(\rho) \leq C_\alpha^T(\rho),$$

$$\text{where} \quad p_n = \text{tr} (K_n \rho K_n^\dagger), \quad q_n = \text{tr} (K_n \sigma K_n^\dagger).$$

We compare the Tsallis relative $\alpha$-entropies of coherence with the Rényi $\alpha$-relative entropy of coherence

$$C_\alpha(\rho) = \min \frac{1}{\alpha - 1} \log \text{tr} (\rho^\alpha \sigma^{1-\alpha}),$$

$C_\alpha^T(\rho)$ and $C_\alpha(\rho)$ are different with the log function. We recall that the fidelity of coherence $F_\rho(\rho) = 1 - \max_{\delta \in I} \sqrt{F(\rho, \delta)}$, fulfills conditions C1, C2a, and C3, it
violates condition C2b. Another related quantity is geometric coherence $C_\alpha(p) = 1 - \max_{\delta \in \mathcal{F}} F(p, \delta)$, fulfills conditions C1, C2a, C2b, and C3.\textsuperscript{3} Although $C_F(p)$ and $C_\alpha(p)$ are different with the square root function, but $C_\alpha(p)$ fulfills the conditions C2b and $C_F(p)$ does not fulfill condition C2b. Another question arises immediately, is the extension of condition C2b satisfied for the Rényi $\alpha$-relative entropy of coherence $C_\alpha(p)$?

We also use the above example to solve this problem. According to Ref. \cite{32}, when we use the Rényi $\alpha$-relative entropy to quantify coherence, the optimal incoherent state for $p$ is

$$\delta = \frac{1}{2 + \sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (20)$$

With the corresponding probabilities:

$$q_1 = \text{tr}(K_1 \delta K_1^{\dagger}) = \frac{\sqrt{2} + |a|^2}{2 + \sqrt{2}},$$

$$q_2 = \text{tr}(K_2 \delta K_2^{\dagger}) = \frac{1 + |b|^2}{2 + \sqrt{2}}. \quad (21)$$

We also choose $b = 1$, substituting it into Eqs. (10), (12), and (21), we can obtain the expression of $p_2 q_2^{1-\alpha} C_\alpha(p_2)$

$$p_2 q_2^{1-\alpha} C_\alpha(p_2) = \left(\frac{1 + |b|^2}{2 + \sqrt{2}}\right)^{1-\alpha} \left(\frac{1 + |b|^2}{2 + \sqrt{2}}\right)^{1-\alpha} \frac{\alpha}{\alpha - 1} \times \log \left[\frac{1 + |b|^2}{2 + \sqrt{2}} \right]^\alpha \times \log \left[\frac{1 + |b|^2}{2 + \sqrt{2}} \right]^{\alpha - 1}$$

$$\times \log \left[\frac{1 + |b|^2}{2 + \sqrt{2}} \right]^{\alpha - 1}.$$ (22)

Compare Eq. (13) with Eq. (22), after some simple algebraic operation, we obtain

$$\left(\frac{1}{2}\right)^{\alpha} \times \left(\frac{2}{2 + \sqrt{2}}\right)^{1-\alpha} \geq \frac{1}{2}. \quad (23)$$

The equality holding if and only if $\alpha = 1$. Thus, for $\alpha \in (0, 1)

$$C_\alpha(p) < p_2 C_\alpha(p_2) < p_2 q_2^{1-\alpha} C_\alpha(p_2). \quad (24)$$

From the above example, we then conclude that the extension of condition C2b, i.e.,

$$\sum_n p_n q_n^{1-\alpha} C_\alpha(p) \leq C_\alpha(p)$$

is not generally true for the measure of coherence induced by Rényi $\alpha$-relative entropy.

4 The Rényi $\alpha$-Relative Entropy of Coherence and Mixedness

In order to be a meaningful resource quantum quantifier for coherence, the minimal requirements are the conditions C1 and C2a for any quantity $C$.\textsuperscript{34} We have proved that the Rényi $\alpha$-relative entropy of coherence does not fulfill the condition C2b and the extension of condition C2b, but the Rényi $\alpha$-relative entropy of coherence is satisfied the minimal requirements to be a coherence quantifier for $\alpha \in [0, 2]$, thus the Rényi $\alpha$-relative entropy of coherence can act as a coherence monotone quantifier.\textsuperscript{34}

An important problem for quantifying coherence is the relationship between quantum coherence quantities and mixedness. The trade-off between some quantities and mixedness have been discussed in Refs. [37–38]. Here we focus on the trade-off between the Rényi $\alpha$-relative entropy of coherence and mixedness.

**Theorem 1** For $0 < \alpha \leq 2$ and $\alpha \neq 1$, the upper bound of the Rényi $\alpha$-relative entropy of coherence is given by

$$C_\alpha(p) \leq \frac{\ln 2}{d - 1} C_\alpha(p) + M(p) \leq 1,$$ \quad (25)

and the trade-off between $C_\alpha(p)$ and mixedness can be expressed as

$$\frac{\ln 2}{d - 1} C_\alpha(p) + M(p) \leq 1,$$ \quad (26)

where $M(p) := [d/(d - 1)][1 - \text{tr}(\rho^2)]$.

**Proof** We only prove the case of $\alpha \in (0, 1)$, $\alpha \in (1, 2]$ is completely analogous. For $\alpha \in (0, 2]$ and $\alpha \neq 1$, we choose $\delta$ is the completely mixed state $\delta = \sum_i (1/d)|i\rangle\langle i|$, then we can obtain

$$C_\alpha(p) = \min_{\delta \in \mathcal{F}} S_\alpha(p||\delta) \leq S_\alpha(p|| \sum_i 1/d |i\rangle\langle i|)$$

$$= \frac{1}{\alpha - 1} \log \text{tr} \left[\rho^\alpha \left(\sum_i 1/d |i\rangle\langle i|\right)^{1-\alpha}\right]$$

$$= \frac{1}{\alpha - 1} \log \text{tr}(d^{\alpha-1} \rho^\alpha)$$

$$= \frac{1}{\alpha - 1} \log d^{\alpha - 1} + \log \text{tr}(\rho^\alpha). \quad (27)$$

The inequality holding is that the completely mixed state $\sum_i (1/d)|i\rangle\langle i|$ may not be the optimal incoherent state for $p$. According to Ref. [35], for $\alpha \in (0, 2]$ and $\alpha \neq 1$, the function $\varepsilon \rightarrow \varepsilon^{\alpha - 1}/(\alpha - 1)$ is concave, applying Jensen’s inequality, we then have

$$\text{tr}(\rho^\alpha) \leq \frac{\sum_i \lambda_i \lambda_i^{\alpha - 1}}{\alpha - 1} \leq \frac{\text{tr}(\rho^2)^{\alpha - 1}}{\alpha - 1}. \quad (28)$$

Here $\lambda_i$ are the eigenvalues of $\rho$ and obey the normalization condition. For $\alpha \in (0, 1)$, $\alpha - 1 < 0$, then

$$\text{tr}(\rho^\alpha) \geq \left[\text{tr}(\rho^2)^{\alpha - 1}\right]^{1/\alpha - 1},$$

so

$$\frac{\log \text{tr}(\rho^\alpha)}{\alpha - 1} \leq \frac{\log \text{tr}(\rho^2)^{\alpha - 1}}{\alpha - 1},$$

then

$$C_\alpha(p) \leq \frac{1}{\alpha - 1} \left(d^{\alpha - 1} + \log \text{tr}(\rho^2)\right) \leq \log d^{\alpha - 1} + \log \text{tr}(\rho^2) \leq \frac{\log \text{tr}(\rho^2)\alpha - 1}{\alpha - 1} = \log d + \log \text{tr}(\rho^2). \quad (29)$$

Using $\ln x \leq x - 1$, we then have

$$C_\alpha(p) \leq \log d + \log \text{tr}(\rho^2)$$
Combining Eqs. (4) and (32), we can obtain
\[
\ln \frac{2}{d-1} C_\alpha (\rho) + M (\rho) \leq 1. 
\] (31)

Equation (25) provides an upper bound on Rényi α-relative entropy of coherence in terms of the purity \( \text{tr} (\rho^2) \). Equation (26) shows that when mixedness increases, an upper bound on Rényi α-relative entropy of coherence decreases.

### 5 The Rényi α-Relative Entropy of Coherence for a Single Qubit

Due to the analytical expression of \( C_\alpha (\rho) \) for qubit is complicated, so we consider a simple case that we choose \( \alpha = 2 \) for the coherence quantity. The qubit states can write as
\[
\rho = \begin{pmatrix} a & b^* \\ b & 1-a \end{pmatrix}. 
\] (32)
The eigenvalues of \( \rho \) are expressed as
\[
\lambda_1 = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4|b|^2 + 4a^2 - 4a} ,
\]
\[
\lambda_2 = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4|b|^2 + 4a^2 - 4a}. 
\] (33)
Combine Eqs. (4) and (32), we can obtain
\[
C_2 (\rho) = 2 \log (\sqrt{a^2 + |b|^2} + \sqrt{|b|^2 + (1-a)^2}). 
\] (34)
Due to \( 0 \leq \lambda_1, \lambda_2 \leq 1 \), thus we have
\[
|b|^2 \leq a(1-a). 
\] (35)
For the given \( a \), the minimum of \( C_2 (\rho) \) is zero. The maximum of \( C_2 (\rho) \) can be expressed as
\[
C_2^{\text{max}} (\rho) = 2 \log (\sqrt{a} + \sqrt{1-a}). 
\] (36)
Now, we consider the precise trade-off between \( C_2 (\rho) \) and mixedness for qubit case.

**Theorem 2** For the given \( a \), the trade-off between \( C_2 (\rho) \) and mixedness \( M (\rho) \) can express as
\[
\ln 2C_2 (\rho) + M (\rho) < 2 \sqrt{a(1-a)} \leq 1, 
\] (37)
where \( M (\rho) = |d/(d-1)|[1 - \text{tr} (\rho^2)] \).

**Proof** Combine Eqs. (31) and (34), we can obtain
\[
\ln 2C_2 (\rho) + M (\rho) = \ln \frac{2}{d-1} C_\alpha (\rho) + M (\rho)
\]
\[
= \ln (\sqrt{a^2 + |b|^2} + \sqrt{|b|^2 + (1-a)^2})^2 + M (\rho)
\]
\[
\leq (\sqrt{a^2 + |b|^2} + \sqrt{|b|^2 + (1-a)^2})^2 - 1
\]
\[
+ 2[1 - (a^2 + |b|^2) - (|b|^2 + (1-a)^2)]
\]
\[
= 1 - (\sqrt{a^2 + |b|^2} - \sqrt{|b|^2 + (1-a)^2})^2, 
\] (38)
where the inequation uses in \( x \leq x-1 \). Next, we illustrate that \( 1 - (\sqrt{a^2 + |b|^2} - \sqrt{|b|^2 + (1-a)^2})^2 \) is a monotone increasing function of \( |b|^2 \) for the given \( a \) and \( |b|^2 \leq a(1-a) \). We take derivative with respect to \( |b|^2 \), then we have
\[
d(1 - (\sqrt{a^2 + |b|^2} - \sqrt{|b|^2 + (1-a)^2})^2)
\]
\[
d(|b|^2)
\]
\[
= -2(\sqrt{a^2 + |b|^2} - \sqrt{|b|^2 + (1-a)^2})
\]
\[
\times \left( \frac{1}{\sqrt{a^2 + |b|^2}} - \frac{1}{\sqrt{|b|^2 + (1-a)^2}} \right)
\]
\[
= 2(\sqrt{a^2 + |b|^2} - \sqrt{|b|^2 + (1-a)^2})^2 
\]
\[
\geq 0. 
\] (39)
The maximum value of
\[
1 - (\sqrt{a^2 + |b|^2} - \sqrt{|b|^2 + (1-a)^2})^2
\]
is obtained when \( |b|^2 = a(1-a) \). Thus,
\[
\ln 2C_2 (\rho) + M (\rho)
\]
\[
\leq 1 - (\sqrt{a^2 + |b|^2} - \sqrt{|b|^2 + (1-a)^2})^2
\]
\[
\leq 1 - (\sqrt{a} - \sqrt{1-a}) = 2 \sqrt{a(1-a)}. 
\] (40)
The equality holding if and only if
\[
\ln (\sqrt{a^2 + |b|^2} + \sqrt{|b|^2 + (1-a)^2})^2
\]
\[
= (\sqrt{a^2 + |b|^2} + \sqrt{|b|^2 + (1-a)^2})^2 + 1
\]
and \( |b|^2 = a(1-a) \). After some simple algebraic operation, we can get \( a = 0, b = 1 \) or \( a = 1, b = 0 \). Those two solutions can not be represented quantum states. Then we can obtain
\[
\ln 2C_2 (\rho) + M (\rho) < 2 \sqrt{a(1-a)}. 
\] (41)

In Fig. 3, we plot the left-hand side of Eq. (41) by a red line and the right-hand side of Eq. (41) by a black line. From Fig. 3, we can see \( \ln 2C_2 (\rho) + M (\rho) \) is smaller than \( 2 \sqrt{a(1-a)} \). It can ensure the correctness of the Theorem 2.

![Fig. 3](image_url)

Fig. 3 The values of \( \ln 2C_2 (\rho) + M (\rho) \) and \( 2 \sqrt{a(1-a)} \) as functions of \( a \).

### 6 Summary

In this paper, we show that the Rényi α-relative entropy of coherence does not satisfy condition C2b and extension of C2b for \( \alpha \in (0, 1) \) by presenting examples. Thus the measure of coherence induced by the Rényi α-relative entropy can not be called coherence measure.[34] Due to the Rényi α-relative entropy of coherence fulfill the condition C1 and C2a, so the Rényi α-relative entropy of coherence can be called as a coherence monotone quantifier.
The Rényi $\alpha$-relative entropy of coherence fulfills the minimal requirements to be a meaningful resource quantum quantifier for coherence,[34] then we examine the trade-off relations between coherence and mixedness. Some properties are further exemplified with a single qubit for $\alpha = 2$. Our findings complement the results present in Ref. [32].

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