Functional renormalization group approach to the sine-Gordon model

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The renormalization group flow is presented for the two-dimensional sine–Gordon model within the framework of the functional renormalization group method by including the wave-function renormalization constant. The Kosterlitz–Thouless–Berezinskii type phase structure is recovered as the interpolating scaling law between two competing IR attractive area of the global renormalization group flow.

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I. Introduction.— The two-dimensional (2D) sine-Gordon (SG) model, defined by the bare action

\[ S = \int_x \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + u \cos(\beta \varphi) \right] \]  

in Euclidean spacetime, has already received a considerable amount of attention since it is the simplest non-trivial quantum field theory with compact variables. This feature is common with non-Abelian gauge theories and is supposed to be the key to their confinement mechanism. In two dimensions this is the driving force to form a non-trivial phase structure. The SG model is known to belong to the universality class of the \( \beta > 2 \) model which has received important applications in condensed matter systems. The perturbative RG results beyond the local potential approximation (LPA) can account for the KTB point of the XY model. Recently, by using the flow equation approach a different IR limit is obtained for the frequency, i.e. \( \beta^2 \rightarrow 4 \pi \). However, the latter method is not able to recover the leading order perturbative UV results for \( \beta^2 < 4 \pi \), due to the wrong sign of the evolution equation derived for the frequency. Functional RG approaches have also been used to map the phase structure of the SG model but their description is not complete since on the one hand the LPA is used and on the other hand, the SG model is mapped onto other models belonging to the same universality class. Therefore, the analysis of the SG model is still incomplete.

Our aim with this work is to determine the complete phase structure of the original SG model by extending the functional RG analysis beyond the LPA, by including the field-independent wave-function renormalization, as well. We use the functional RG method for the effective average action which enables us to treat the wave-function renormalization. The evolution arises as the result of the gradual turning on of field fluctuations according to their increasing amplitude by decreasing the control parameter \( \Lambda \ll k_0 \) (with \( k_0 \) the UV cut-off which goes to infinity) to zero.

The phase structure is found to be the global result of a competition between an IR fixed line and an IR fixed point. The traditional KTB scaling law is actually an interpolation between these two effects.

II. The sine-Gordon model.— The functional renormalization group equation for the effective action of an Euclidean field theory is

\[ k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \frac{k \partial_k R_k}{R_k + \Gamma''_k} \]  

where the notation \( ' = \partial / \partial \varphi \) is used and the trace Tr stands for the integration over all momenta. We use a power-law type regulator function

\[ R_k = p^2 \left( \frac{k^2}{p^2} \right)^b \]  

with the parameter \( b \geq 1 \). Eq. (2) has been solved over the functional subspace defined by the ansatz

\[ \Gamma_k = \int_x \left[ \frac{z}{2} (\partial_\mu \varphi_x)^2 + V_k(\varphi_x) \right], \]
with the field-independent wave-function renormalization \( z(k) \). Eq. (2) leads to the evolution equations \[ (2 + k\partial_k)\tilde{u}_1 = \frac{1}{4\pi z} \tilde{u}_1, \] \[ k\partial_k z = -\frac{\tilde{u}_1^2}{z^{2-2/b}} c_b, \] which is the typical scaling law for KTB type phase transitions, modified by analytic corrections vanishing for \( \tilde{u}_1^* \to 0 \). It is worthwhile mentioning that only the quantitative details depend on the choice of the parameter \( b \) in the formula (11). Using Eqs. (7) and (8) the critical exponent \( \eta \) can also be calculated via the vortex-vortex correlation function \[ \langle \tilde{u}_1^* \rangle^2 \] and it is proved to take the value \( \eta = 1/4 \) independently of the parameter \( b \).

IV. Coleman point, revisited as the dual KTB point.— Let us now take into account the higher-order terms of the Taylor expansion in \( V''_k \), as well as the higher harmonics of the local potential \( \tilde{u}_n \). We choose \( b = 1 \) corresponding to the Callan-Symanzik RG scheme \[ [13] \] which is free of UV divergences for \( d = 2 \) and ulralocal. The evolution equations assume a simpler form rendering easier the handling of the higher Fourier modes.

The Fourier transform of Eqs. (11) and (12) produces a set of coupled equations for \( \tilde{u}_n \) and \( z \). We refer to the solution of these equations with 10 Fourier modes as the full solution. According to our experience the retaining of more Fourier harmonics modifies the flow in a negligible manner. By restricting the solution to a single Fourier mode one obtains the evolution equations

\[
(2 + k\partial_k)\tilde{u}_1 = \frac{1}{2\pi \tilde{u}_1 z} \left[ 1 - \sqrt{1 - \tilde{u}_1^2} \right],
\]

\[
k\partial_k z = -\frac{1}{24\pi (1 - \tilde{u}_1^2)^2},
\]

whose solution will be referred as exact in \( \tilde{u}_1 \). The first two terms in the Taylor expansion of Eq. (12) can be identified with the approximation used in [3] by a proper transformation of the parameters in Callan-Symanzik RG scheme. The higher-order terms make negligible effect on the evolution even in the neighbourhood of the turning point \( (\tilde{u}_1^*, z^*) \) of the RG trajectory. We refer to the solution of Eqs. (7) and (8) as the linearized solution. The RG trajectories are plotted in Fig. 1. They move to the left as \( k \) is decreased. This picture is reminiscent of the usual

![FIG. 1: Phase diagram of the SG model. The solid/dashed/dotted lines show the RG trajectories for the linearized/exact in \( \tilde{u}_1 \)/full solutions, respectively. The wide solid line depicts the separatrix.](image-url)
KTB phase structure. What we see here is actually the vicinity of the dual of the KTB point of the XY model \[10\]. One can see that the higher harmonics modify the RG trajectories rather slightly as compared to both the linearized and the exact in \(\tilde{u}_1\) solutions. Although the Coleman point lies at a crossover scale between the UV and IR scaling regions for the trajectories above the separatrix, the UV scaling remains valid in its vicinity.

Note the smallness of the \(z\)-interval covered. The \(z\)-dependence of Eq. \((\ref{eq:z-dependence})\) is weak under the two solid lines of the separatrix, bordering three different regions, where the regions from the left to the right correspond to the renormalizable, the non-renormalizable and the asymptotically free regimes of the SG model, respectively.

\section{Non-perturbative scaling at the crossover.}

Let us now turn our attention to the new phase which is opened up by the evolution of the wave function renormalization constant in the middle of the figure, above the separatrix. The \(z\)-dependence is crucial here, it prevents the system to come to standstill where the renormalized trajectory is stationary in \(\tilde{u}_1\), just above the Coleman point. It has been established that all Fourier modes are irrelevant (decrease with \(k\)) before the crossover and they turn to relevant (starts to increase as \(k\) is further decreased) at \(k^*_n\), showing very weak \(n\)-dependence at the location of the turning point, \(k^*_n \approx k^*\). We find \(z(k^*) = z^* = 1/(8\pi)\), in a manner similar to the case \(z = 1\) \[7\].

The vertical line \(z = z^*\) appears to be a single IR stable fixed point as far as the evolution of the potential is considered only. In fact, the values of the coupling constants, \(\tilde{u}^*_n = \tilde{u}_n(k^*)\), determined by the Fourier transform of the evolution equation Eq. \((\ref{eq:evolution})\) satisfy at this line the condition that the ratios

\begin{equation}
  c_n = \frac{\tilde{u}^*_n}{\tilde{u}^*_2 n^2},
\end{equation}

are universal constants, \(c_2 = 1/12, c_3 = 1/96, c_4 = 13/8640, c_5 = 97/414720\), etc. \[8\]. We recover renormalizability and asymptotical freedom in this phase because the dynamics is characterized by a single coupling strength, \(\tilde{u}_1\) at and below the crossover scale. This is a non-perturbative phenomenon because the crossover "fixed point" is not Gaussian.

We see furthermore the subtle meaning of the "KTB fixed point". As soon as one goes beyond the LPA the Coleman point ceases to be a fixed point and is separating different phases only under the separatrix of Fig \[11\] where the beta functions have a common analytic structure \[7\] and something irregularity shows up in the deep IR region of the symmetry broken phase only.

A more detailed and explicit similarity with the KTB scaling of the XY model is found by introducing the correlation length \(\xi\) by identifying it with the inverse cutoff at the crossover. The numerical results are shown in Fig \[2\]. The various approximations, i.e. the full solution, the solution exact in \(\tilde{u}_1\), and the linearized one give the same critical behaviour as Eq. \((\ref{eq:linearized})\), showing that neither the inclusion of higher-order terms in Eqs. \((\ref{eq:linearized})\) and \((\ref{eq:full})\) nor that of the higher harmonics affect the type of the phase transition, their effects are negligible. The reduced temperature is given formally by the wave-function renormalization constant at the UV cutoff as \(t = [z(\Lambda) - z_s(\Lambda)]/z_s(\Lambda)\), where \((1/8\pi z_s(\Lambda), \tilde{u}^*_1(\Lambda))\) is a point of the separatrix. The turning point \(\tilde{u}^*_1\) is shown in the inset of Fig. \[2\] as the function of the reduced temperature \(t\) for the linearized solution, giving

\begin{equation}
  \tilde{u}^*_1 t = gt + O(t^2)
\end{equation}

as in the XY model \[8\], and in the Coulomb gas \[13\]. The same relation is recovered for the exact solution in \(\tilde{u}_1\) and for the full solution, as well. The critical scaling
remains finite. One can get a clear picture of the phase structure of the IR scaling regime by omitting the effect of the higher harmonics, a frequently used approximation [3, 4]. After introducing \( \omega = \sqrt{1 - \mu^2} \), \( \chi = 1/2\omega \) and \( \partial_t = \omega^2 k \partial_k \) we arrive at the evolution equations

\[
\begin{align*}
\partial_t \omega &= 2\omega(1 - \omega^2) - \frac{\omega^2 \chi}{2\pi}(1 - \omega), \\
\partial_t \chi &= \frac{\chi^2(1 - \omega^2)}{24\pi} - 2\chi(1 - \omega^2) + \frac{\omega\chi^2}{2\pi}(1 - \omega), \quad (16)
\end{align*}
\]

possessing two lines of Gaussian fixed points separated by the well-known Coleman (alias KTB) fixed point, \((1/8\pi z, \tilde{u}_1) = (1, 0)\), and an additional (IR) fixed point \((1/8\pi z, \tilde{u}_1) = (0, 1)\) (see Fig. 3 for the RG trajectories).

Such a modification of the scaling laws which is believed to preserve the qualitative features of the RG flow makes the IR fixed points explicit in the complete phase diagram. It also demonstrates that the hyperbolic nature of the flow in the vicinity of the KTB-Coleman point stems from global effects, the competition between two regions of the phase diagram. This is the attraction of the line of Gaussian IR fixed points of the symmetrical phase, dominated by the kinetic energy on the one hand and of the non-Gaussian, IR fixed point which is dominated by the potential energy on the other.

The feature lost in this approximation is that the effective potential, built up in the IR region makes the evolution equation singular, which is typical for phases with spontaneously broken symmetry [17], automatically guaranteeing the superuniversality for the potentials. In fact, the precise treatment with no expansion should give the superuniversal potential \( \tilde{V}_k \rightarrow 0 = -\frac{1}{2} \beta^2 \) due to the Maxwell cut [7, 18, 19] and \( 1/z(k \rightarrow 0) = 0 \).

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