Nonperturbative Effects in Noncritical Strings with Soliton Backgrounds

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Abstract

We explicitly construct soliton operators in $D < 2$ (or $c < 1$) string theory, and show that the Schwinger-Dyson equations allow solutions with these solitons as backgrounds. The dominant contributions from 1-soliton background are explicitly evaluated in the weak coupling limit, and shown to agree with the nonperturbative analysis of string equations. We suggest that fermions should be treated as fundamental dynamical variables since both macroscopic loops and solitons are constructed in their bilinear forms.

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1 Introduction

Solitons in string theory have played important roles in understanding the nonperturbative dynamics of strings [1]. Such solitons in the weak coupling region are expected to have the typical dependence on the string coupling constant $g$. In fact, in the string perturbation theory, perturbation series diverge as $(2h)!$ (not $h!$) with genus $h$, and thus the leading nonperturbative effect is generally expected to have the form $e^{-\text{const}/g}$ [2]. D-branes (especially D-instantons) [3] actually have this behavior, since they are defined as the contribution from the (Dirichlet-)boundaries of world sheets, and the leading contribution comes from a disk amplitude of order $g^{-1}$, which will be exponentiated through a special combinatorics in space-time picture [4]. This observation, however, makes difficult the constructive approach to closed string field theory with simple covariant actions, since if some action exists and yields genus expansion with weight $g^2$ for each loop (genus), then natural nonperturbative effects will have the form $e^{-\text{const}/g^2}$. One possibility to solve this problem is, that there are more fundamental dynamical variables which describe both elementary strings and these solitons in a unified manner.

In the present letter, we point out that this possibility is actually realized in $D < 2$ string theory, which can be constructed as the double scaling limits [5] of matrix models. In particular, we identify the fundamental dynamical variables from which both elementary strings and solitons are constructed as their bilinears. We further evaluate the nonperturbative effects due to 1-soliton background, and show that they have the form $e^{-\text{const}/g}$ with the correct exponent, which are consistent with the nonperturbative analysis of string equations. Further investigation of the dynamics for these variables will be reported in our future communication.

We start our discussion with recapitulating the Schwinger-Dyson equation approach to noncritical string theory [6, 7, 8], aiming to fix our notation. For strings in $D < 2$, or equivalently 2D quantum gravity coupled to some conformal fields with central charge $c < 1$, string variables should be invariant for reparametrizations of the world sheet. Such variables that appear naturally in matrix models are the macroscopic-loop operators [9]. Their connected $N$-point correlation functions $v(l_1, \ldots, l_N; g)$ are defined as the sum over fluctuating (connected) surfaces with $N$ boundaries of length $l_1, \ldots, l_N$ along which spins of conformal matters have the same state. For later use, it is convenient to introduce their
Laplace transforms, which have an asymptotic expansion in genus \( h \) of the form

\[
w(\zeta_1, \cdots, \zeta_N; g) = \int_0^\infty dl_1 \cdots dl_N e^{-\zeta_1 l_1 - \cdots - \zeta_N l_N} v(l_1, \cdots, l_N; g) = \sum_{h \geq 0} g^{-2+2h+N} w^{(h)}(\zeta_1, \cdots, \zeta_N).
\]

Here \( g \) is the renormalized string coupling constant.

If the conformal field theory is the minimal \((p, q)\) model with central charge \( c = 1 - 6(p - q)^2/pq \), then \( w(\zeta_1, \cdots, \zeta_N; g) \) has the following form of Laurent expansion around \( \zeta = \infty \):

\[
w(\zeta_1, \cdots, \zeta_N; g) = \frac{1}{p^N} \sum_{n_1, \cdots, n_N} \zeta_1^{-n_1/p - 1} \cdots \zeta_N^{-n_N/p - 1} \; w_{n_1, \cdots, n_N}(g).
\]

For positive \( n_i \), the coefficient \( w_{n_1, \cdots, n_N}(g) \) is identified with the connected correlation function of the \textit{microscopic-loop} operators \( \mathcal{O}_{n_i} \):

\[
\langle \mathcal{O}_{n_1} \cdots \mathcal{O}_{n_N} \rangle = \sum_{h \geq 0} g^{-2+2h+N} \langle \mathcal{O}_{n_1} \cdots \mathcal{O}_{n_N} \rangle^{(h)}.
\]

These operators \( \mathcal{O}_n \) \((n = 1, 2, 3, \cdots)\) correspond to the physical operators in the Liouville gravity \([10]\) obtained as the surface integral of the gravitationally-dressed, spinless primary fields with (undressed) conformal dimension \( \Delta_{r,s}^{(0)} = [(qr - ps)^2 - (p - q)^2]/4pq \) and \( n = |qr - ps| \).

For this \((p, q)\) case, any correlation function including \( \mathcal{O}_n \) with \( n \) multiple of \( p \) always vanishes. Furthermore, solving the Schwinger-Dyson equations of the matrix models with appropriate double scaling limit, we find infinitely many relations among the correlation functions. They are compactly expressed as the \( W_p \) constraint \([7, 8]\) on the exponential of the generating function \( F(j) \) of connected correlation functions:

\[
F(j) = \left\langle \exp \left\{ \sum_{n \geq 1; n \not\equiv 0 \; (\text{mod} \; p)} j_n \mathcal{O}_n \right\} \right\rangle^{(c)} = \log \left\langle 0 \left| \exp \left\{ \sum_{n \geq 1; n \not\equiv 0 \; (\text{mod} \; p)} (j_n - B_n/g) \alpha_n \right\} \right| \Phi \right\rangle.
\]

Here \( \alpha_n \) \((n \geq 1)\) are the positive part of the bosonic oscillators \( \alpha_n \) \((n \in \mathbb{Z})\) with commutation relation \( [\alpha_n, \alpha_m] = n \delta_{n+m, 0} \), and \( \langle 0 | \) is the vacuum satisfying \( \langle 0 | \alpha_n = 0 \) \((n \leq 0)\). The state \( | \Phi \rangle \) satisfies the vacuum condition of the \( W_p \) algebra made from the \( \mathbb{Z}_p \)-twisted bosons \( \tilde{\varphi}_k(\zeta) = (1/p) \sum_{n \equiv k \; (\text{mod} \; p)} \zeta^{-n/p - 1} \alpha_n \) \((k = 1, 2, \cdots, p - 1)\), whose explicit form is not important here and can be found in refs. \([7, 8, 11]\). The \( B_n \)'s are backgrounds which characterize the theory, and the \((p, q)\) case is realized by \( B_n = (B_1, B_2, \cdots, B_{p+q}, 0, 0, \cdots) \) with nonvanishing \( B_{p+q} \). With the help of string equations \([12]\), it can be further shown \([11, 13]\).
that $\tau(x) = \langle 0 \exp\{\sum_{n \geq 1} x_n \alpha_n\} | \Phi \rangle$ is a $\tau$ function of the $p$th reduced KP hierarchy, and also that the $W_p$ constraint is automatically enhanced to the $W_{1+\infty}$ constraint; We will fully use the latter fact later, giving an explicit basis of the $W_{1+\infty}$ algebra [14]. We have found that the string coupling constant $g$ can be placed in the generating function in this simple manner. It is easy to check that this form actually reproduces correct genus expansion of the Schwinger-Dyson equations.

For one- and two-point functions ($N = 1, 2$), however, the right-hand side of (1.2) includes the terms with negative $n_i$ of order $g^{-1}$ and $g^0$, respectively. (Note that both come from spherical topology.) Such terms are difficult to be identified with the correlation functions of microscopic-loop operators, and (misleadingly) termed “non-universal parts” although they do scale properly in the continuum limit. They are found to be [7, 15, 16, 17]

$$w_{\text{non}}(\zeta) = - \frac{1}{pg} \sum_n n B_n \zeta^{n/p-1}$$

(1.4)

$$w_{\text{non}}(\zeta_1, \zeta_2) = \frac{1}{p^2 (\zeta_1 \zeta_2)^{1-1/p}} \left( \frac{\zeta_1^{1/p} - \zeta_2^{1/p}}{2} \right)^2 - \frac{1}{(\zeta_1 - \zeta_2)^2}$$

$$= \partial_{\zeta_2} \left[ \frac{1}{p} \sum_{i=0}^{p-1} \left( \frac{\zeta_2/\zeta_1}{i/p} - 1 \right) \right].$$

(1.5)

Since it is those macroscopic-loop operators that are obtained naturally in the scaling limit of matrix models (see, e.g., [16] for further investigation on this point), we will regard the expression including the non-universal terms as being fundamental.

This paper is organized as follows. In section 2 we first introduce the twist operator $\sigma(\zeta)$ to “untwist” the twisted bosons, and then construct free fermions with which the Schwinger-Dyson equations are most simply expressed. We identify the string field that describes macroscopic loops. In section 3, we first construct soliton fields from the fermions, and show that the Schwinger-Dyson equations have solutions with backgrounds of these solitons. We then make an explicit calculation of the nonperturbative effects for 1-soliton background and show that they agree with the value appearing in the nonperturbative analysis of string equations. Section 4 will be devoted to conclusions with some speculations.
2 Fermionic Representation of the Schwinger-Dyson Equations with Twist Operator

In order to simplify the discussion on the $W_{1+\infty}$ constraint, we first introduce a twist operator $\sigma(\zeta)$ \cite{[15]} which enables us to deal with $\partial\hat{\phi}_k(\zeta)$ as being untwisted. The twist operator $\sigma(\zeta)$ thus has the property that $\partial\hat{\phi}_k(\zeta)$ has a definite monodromy when going around the point at which $\sigma(\zeta)$ is inserted. This is ensured by the following OPE for the operators $\partial\hat{\phi}_k(\zeta)$ ($k = 0, 1, \cdots, p - 1$) and $\sigma(\zeta)$:

\[
\partial\hat{\phi}_k(\zeta) \partial\hat{\phi}_l(\zeta') \sim \frac{g_{kl}/p}{(\zeta - \zeta')^2}, \quad (k, l = 0, 1, \cdots, p - 1), \quad (2.1)
\]

\[
\partial\hat{\phi}_k(\zeta) \sigma(\zeta') = \frac{1}{p} \sum_{m \leq -1} (\zeta - \zeta')^{-m-k/p-1} (\alpha_{mp+k})\sigma(\zeta'), \quad (2.2)
\]

where $g_{kl} \equiv \delta_{k+l=0(\text{mod } p)}$. We have introduced a monodromy-free boson $\partial\hat{\phi}_0(\zeta)$, which is necessary to reproduce the correct non-universal terms in later discussions. By denoting the $SL(2, \mathbb{C})$ invariant vacuum by $\langle \text{vac} \rangle$, the state $\langle 0 \rangle$ in the preceding section is now interpreted as the state $\langle \sigma \rangle \equiv \langle \text{vac} \rangle \sigma(\infty)$, and $\partial\hat{\phi}_k(\zeta)$ has a monodromy $\omega^{-k}$ around $\zeta = \infty$:

\[
\langle \sigma | \partial\hat{\phi}_k(e^{2\pi i} \zeta) = \omega^{-k} \langle \sigma | \partial\hat{\phi}_k(\zeta). \quad (2.3)
\]

To express the $W_{1+\infty}$ algebra, we introduce “orthonormal basis” $\partial\varphi_a(\zeta)$ ($a = 0, 1, \cdots, p - 1$) from the monodromy-diagonalizing basis $\partial\hat{\phi}_k(\zeta)$ by

\[
\partial\varphi_a(\zeta) = \sum_{k=0}^{p-1} \omega^{-ka} \partial\hat{\phi}_k(\zeta) \quad \left( \Leftrightarrow \partial\hat{\phi}_k(\zeta) = \frac{1}{p} \sum_{a=0}^{p-1} \omega^{ka} \partial\varphi_a(\zeta) \right), \quad (2.4)
\]

\[
\partial\varphi_a(\zeta) \partial\varphi_b(\zeta') \sim \frac{\delta_{ab}}{(\zeta - \zeta')^2}, \quad (2.5)
\]

which has the following monodromy around $\langle \sigma \rangle$:

\[
\langle \sigma | \partial\varphi_a(e^{2\pi i} \zeta) = \langle \sigma | \partial\varphi_{[a+1]}(\zeta) \quad ([a] \equiv a (\text{mod } p)). \quad (2.6)
\]

The $W_{1+\infty}$ currents $W^k(\zeta)$ ($k = 1, 2, \cdots$) are then written \cite{[11]} as

\[
W^k(\zeta) = -\frac{1}{k} \sum_{a=0}^{p-1} e^{\varphi_a(\zeta)} \partial^k \varphi_a(\zeta), \quad (2.7)
\]

Note that since we make a diagonal sum over $a$, there is no monodromy for $W^k(\zeta)$ even under the twisted vacuum, and thus it has an expansion of the form $W^k(\zeta) = \sum_{n \in \mathbb{Z}} \zeta^{-n-k} W^k_n.$
The coefficients form a (linear) basis of the $W_{1+\infty}$ algebra of central charge $p$. We will see later in this section, that $\partial \varphi_0(\zeta)$ is the string field that creates a loop boundary with $\zeta$.

This $W_{1+\infty}$ currents are simply represented if we introduce free fermions $c_a(\zeta)$ and $\bar{c}_a(\zeta)$ ($a = 0, 1, \cdots, p - 1$) through bosonization:

$$c_a(\zeta) = K_a : e^{-\varphi_a(\zeta)} : , \quad \bar{c}_a(\zeta) = K_a : e^{\varphi_a(\zeta)} : .$$  \hfill (2.8)

The cocycle factor $K_a$ is necessary to be introduced in order to ensure the anticommutation relations with two different indices ($a \neq b$). An explicit form may be $K_a = (-1)^a \prod_{b=0}^{a-1} (1 - 1)^{p_b}$, where $p_a$ is the momentum of $\partial \varphi_a(\zeta)$. Thus, $c_a(\zeta)$ and $\bar{c}_a(\zeta)$ ($a = 0, 1, \cdots, p - 1$) are free fermions satisfying the following OPE and monodromy:

$$\bar{c}_a(\zeta) \ c_b(\zeta') \sim \frac{\delta_{ab}}{\zeta - \bar{\zeta}} ; \quad \bar{c}_a(\zeta) \ c_b(\zeta') = - c_b(\zeta') \ \bar{c}_a(\zeta) \quad (a \neq b)$$  \hfill (2.9)

$$c_a(\zeta) \ c_b(\zeta') = - c_b(\zeta') c_a(\zeta), \quad \bar{c}_a(\zeta) \ \bar{c}_b(\zeta') = - \bar{c}_b(\zeta') \ \bar{c}_a(\zeta)$$  \hfill (2.10)

$$\langle \sigma | \ c_a(e^{2\pi i} \zeta) \rangle = \begin{cases} - \langle \sigma | \ c_{a+1}(\zeta) \rangle & (0 \leq a \leq p - 2) \\ (-1)^{a-1} \langle \sigma | \ c_0(\zeta) \rangle & (a = p - 1) \end{cases}$$  \hfill (2.11)

$$\langle \sigma | \ \bar{c}_a(e^{2\pi i} \zeta) \rangle = \begin{cases} - \langle \sigma | \ \bar{c}_{a+1}(\zeta) \rangle & (0 \leq a \leq p - 2) \\ (-1)^{a-1} \langle \sigma | \ \bar{c}_0(\zeta) \rangle & (a = p - 1). \end{cases}$$  \hfill (2.12)

It is easy to see that the $W_{1+\infty}$ currents can be rewritten in the following simple form:

$$W^k(\zeta) = \sum_{a=0}^{p-1} \bar{c}_a(\zeta) \partial_{\zeta}^{k-1} c_a(\zeta) : = \sum_{a=0}^{p-1} \lim_{\zeta' \to \zeta} \partial_{\zeta}^{k-1} \left( \bar{c}_a(\zeta') c_a(\zeta) - \frac{1}{\zeta' - \zeta} \right).$$  \hfill (2.13)

Now that we have necessary machinery, we can make more explicit statement on the state $| \Phi \rangle = \Phi(0) | \text{vac} \rangle$ as such that gives the same monodromy with $| \sigma \rangle$ to $\partial \varphi_k(\zeta)$, $\partial \varphi_a(\zeta)$, $c_a(\zeta)$, $\bar{c}(\zeta)$, and that satisfies the following constraint: \footnote{We comment that this condition can be restated that for any contour integral $\oint_C \frac{d\zeta}{2\pi i} \zeta^\alpha W^k(\zeta)$ ($\alpha$: nonnegative integer) with a closed path $C$ surrounding the origin, where $\Phi(0)$ is inserted, we can move the path freely in such a way that it does not surround the origin. In fact, this observation can be used to calculate various correlation functions, although we do not describe it in the present paper.}

$$W^k_n | \Phi \rangle = 0 \quad (n \geq -k + 1).$$  \hfill (2.14)
This constraint is independent of which basis we choose for the \( W_{1+\infty} \) algebra. In fact, any other basis can be obtained from \( W^k(\zeta) \) as \( W'^{k}(\zeta) \equiv \sum_{n} \zeta^{-n-k} W_n^k = \sum_{l=0}^{k-1} \beta_l \partial_\zeta W^{k-l}(\zeta) \) (\( \beta_l \) constant), and thus the same form of constraint also holds for the \( W_n^k \)'s.

Since \( \partial \phi_a(\zeta) = : c_\alpha(\zeta) c_\alpha(\zeta) : (a = 1, \cdots, p - 1) \) can be obtained from \( \partial \phi_0(\zeta) \) by letting \( \zeta \) go around \( \zeta = \infty \) \( a \) times, we only have to consider \( \partial \phi_0(\zeta) \). It is easy to see that the background \( \langle -B/g | \equiv \langle \sigma | \exp \{ -(1/g) \sum_{n} B_n \alpha_n \} \rangle \) can be written as \( \langle \sigma | \exp \{- (1/g) \sum_{n} B_n \zeta^n/p \} \rangle \). Note that we here have to make contour integrals \( p \) times around \( \zeta = \infty \), since \( \partial \phi_0(\zeta) \) has a nontrivial monodromy. We thus define the generating functional for the connected correlation functions of \( \partial \phi_0(\zeta) \) as

\[
F[j(\zeta)] = \log \left( -\frac{B}{g} \right) \langle : \exp \left\{ \int \frac{d\zeta}{2\pi i} j(\zeta) \partial \phi_0(\zeta) \right\} : \Phi \rangle \left( \partial \phi_0(\zeta) \right) \quad (2.15)
\]

We claim that \( F[j(\zeta)] \) is exactly the generating functional of \( w(\zeta_1, \cdots, \zeta_N; g) \) in the previous section:

\[
F[j(\zeta)] = \sum_{N \geq 0} \frac{1}{N!} \int \frac{d\zeta_1}{2\pi i} \cdots \frac{d\zeta_N}{2\pi i} j(\zeta_1) \cdots j(\zeta_N) w(\zeta_1, \cdots, \zeta_N; g)
\]

\[
= \sum_{N \geq 0} \left( \sum_{h \geq 0} \frac{g^{-2+2h+N}}{N!} \left( \int \frac{d\zeta_1}{2\pi i} \cdots \frac{d\zeta_N}{2\pi i} j(\zeta_1) \cdots j(\zeta_N) w^{(h)}(\zeta_1, \cdots, \zeta_N) \right) \right). \quad (2.16)
\]

To prove this, we first note that the symbol \( : \) in \( \langle 2.13 \rangle \) is the normal ordering that respects the \( SL(2, \mathbb{C}) \) invariant vacuum \( \langle \text{vac} \rangle \). On the other hand, the normal ordering in terms of creation (\( \alpha_{-n} \)) and annihilation (\( \alpha_{+n} \)) operators respects the twisted vacuum \( \langle \sigma \rangle \). The difference between these two normal orderings gives rise to a finite renormalization, and can be calculated explicitly:

\[
: \exp \left\{ \int j(\zeta) \partial \phi_0(\zeta) \right\} : = \exp \left\{ \frac{1}{2} \int \frac{d\zeta_1}{2\pi i} \frac{d\zeta_2}{2\pi i} N_2(\zeta_1, \zeta_2) j(\zeta_1) j(\zeta_2) \right\} \times \quad (2.17)
\]

\[
\exp \left\{ \int j(\zeta) \partial \phi_0^{(-)}(\zeta) \right\} \exp \left\{ \int j(\zeta) \partial \phi_0^{(+)}(\zeta) \right\}.
\]

Here \( \partial \phi_0^{(\pm)}(\zeta) \) is the part of \( \partial \phi_0(\zeta) \) consisting only of \( \alpha_{\pm n} \), respectively, and \( N_2(\zeta_1, \zeta_2) \) is given by

\[
N_2(\zeta_1, \zeta_2) = \frac{1}{p^2 (\zeta_1 \zeta_2)^{1-1/p} (\zeta_1^{1/p} - \zeta_2^{1/p})^2} - \frac{1}{(\zeta_1 - \zeta_2)^2}. \quad (2.18)
\]
Furthermore, picking up the contribution coming from the commutation between the background term and the source term with negative modes, we obtain the following expression:

\[ F[j(\zeta)] = -\frac{1}{g} \oint_p \frac{d\zeta}{2\pi i} N_1(\zeta) j(\zeta) + \frac{1}{2} \oint_p \frac{d\zeta_1}{2\pi i} \frac{d\zeta_2}{2\pi i} N_2(\zeta_1, \zeta_2) j(\zeta_1) j(\zeta_2) \]

\[ + \log \left\langle \sigma \left| \exp \left\{ \oint_p \frac{d\zeta}{2\pi i} \left( j(\zeta) - \frac{1}{g} B(\zeta) \right) \partial \varphi_0^+(\zeta) \right\} \right| \Phi \right\rangle, \quad (2.19) \]

where \( N_1(\zeta) = \partial B(\zeta) \), and we have used \( \langle \sigma \left| \partial \varphi_0^{-}(\zeta) \right| = 0 \). Note also that \( \langle \sigma \left| \partial \varphi_0^{(+)}(\zeta) \right| = (1/p) \sum_{n \geq 1} \zeta^{-n/p-1} \langle \sigma \left| \alpha_n \right. \right. \). Since the positive-power part \( j^{(+)}(\zeta) \equiv \sum_{n \geq 1} j_n \zeta^n \) of \( j(\zeta) \) only comes up in the bracket, we finally obtain

\[ \langle \partial \varphi_0(\xi_1) \cdots \partial \varphi_0(\xi_N) \rangle_c \equiv \frac{\delta}{\delta j(\xi_1)} \cdots \frac{\delta}{\delta j(\xi_N)} F[j(\zeta)] \bigg|_{j=0} \]

\[ = -\frac{1}{g} N_1(\xi_1) \delta_{N,1} + N_2(\xi_1, \xi_2) \delta_{N,2} \]

\[ + \frac{1}{p^N} \sum_{n_1, \cdots, n_N \geq 1} \zeta_1^{-n_1/p-1} \cdots \zeta_N^{-n_N/p-1} \langle O_{n_1} \cdots O_{n_N} \rangle_c. \quad (2.20) \]

The first two terms gives the non-universal terms, (1.4) and (1.5), and thus we have established the desired relation:

\[ w(\zeta_1, \cdots, \zeta_N) = \langle \partial \varphi_0(\xi_1) \cdots \partial \varphi_0(\xi_N) \rangle_c. \quad (2.21) \]

### 3 Soliton Operators and Their Nonperturbative Effects

Since the string fields \( \partial \varphi_a(\zeta) \) (with \( a = 0 \) being enough as explained in the previous section) are composites of the fermions \( c_a(\zeta) \) and \( \bar{c}_a(\zeta) \), we can interpret \( c_a(\zeta) \) and \( \bar{c}_a(\zeta) \) as more fundamental dynamical variables in noncritical string field theories. If we adopt this point of view, it is quite natural to consider other bilinears of fermions with vanishing fermion-number:

\[ S_{ab}(\zeta) \equiv \bar{c}_a(\zeta) c_b(\zeta) \quad (a \neq b) \]

\[ = \begin{cases} 
- K_a K_b e^{\varphi_0(\zeta)-\varphi_b(\zeta)} & (a < b) \\
+ K_a K_b e^{\varphi_0(\zeta)-\varphi_a(\zeta)} & (a > b)
\end{cases} \quad (3.1) \]

Note that no normal ordering is required since \( a \neq b \). In this section, we show that this is a soliton field which generates a state with the expectation value of order \( e^{-\text{const}/g} \) in the weak coupling region \( (g \sim 0) \).
The most important property of this soliton field is that its commutation with the generators of $W_{1+\infty}$ gives a total derivative:

$$\left[ W^k_n, S_{ab}(\zeta) \right] = \partial_\zeta K^k_n, ab(\zeta) \quad (k \geq 1, \, n \in \mathbb{Z}), \quad (3.2)$$

where $K^1_{n,ab}(\zeta) = 0$ and $K^k_{n,ab}(\zeta) = -\sum_{l=0}^{k-2} (-1)^l \partial^l_{\zeta} \left( \zeta^{n+k-1} \bar{c}_a(\zeta) \right) \partial^{k-l-2}_{\zeta} c_b(\zeta)$ $(k \geq 2)$.

Thus, if we introduce the global soliton operator $D_{ab}$ as the contour integral of $S_{ab}(\zeta)$

$$D_{ab} = \oint p d\zeta \frac{e^{2\pi i \zeta}}{2\pi i} S_{ab}(\zeta), \quad (3.3)$$

we see that $D_{ab}$ commutes with the generators $W^k_n$:

$$\left[ W^k_n, D_{ab} \right] = 0 \quad (k \geq 1, \, n \in \mathbb{Z}). \quad (3.4)$$

Note that the contour must surround the origin $p$ times, since the $S_{ab}(\zeta)$ has a monodromy around that point: $S_{ab}(e^{2\pi i \zeta}) \Phi) = S_{[a+1][b+1]}(\zeta) |\Phi\rangle$. Eq. (3.4) implies that if a state $|\Phi\rangle$ satisfies the Schwinger-Dyson equations $W^k_n |\Phi\rangle = 0$ $(k \geq 1, \, n \geq -k+1)$, then $|D_{ab}\rangle \equiv D_{ab} |\Phi\rangle$ also satisfies the same equations $W^k_n |D_{ab}\rangle = 0$. In this sense, we have found that the Schwinger-Dyson equations do not determine the vacuum uniquely.

Interpreting this state $|D_{ab}\rangle$ as 1-soliton background,\footnote{Multi-instanton background will be obtained by repeatedly inserting the operators $D_{ab}$.} we can calculate its nonperturbative effect explicitly for the unitary case $(p,q) = (p,p+1)$. This will be defined by the ratio

$$A_{ab} \equiv \left\langle \frac{-B}{g} \middle| D_{ab} \right\rangle \sim \left\langle \oint p \frac{d\zeta}{2\pi i} e^{\varphi_a(\zeta) - \varphi_b(\zeta)} \right\rangle. \quad (3.5)$$

Here we have neglected the irrelevant contribution from the cocycles. In the weak coupling region, the leading contribution comes from the disk amplitude:

$$A_{ab} = \oint p \frac{d\zeta}{2\pi i} e^{(1/g)\Gamma_{ab}(\zeta) + O(g^0)}, \quad (3.6)$$

where the “effective action” $\Gamma_{ab}(\zeta)$:

$$\Gamma_{ab}(\zeta) = \langle \varphi_a(\zeta) \rangle^{(0)} - \langle \varphi_b(\zeta) \rangle^{(0)} \quad (3.7)$$
Can be read off from the disk amplitude $w^{(0)}(\zeta) = \langle \partial \varphi_0(\zeta) \rangle^{(0)}$. In fact, $w^{(0)}(\zeta)$ was calculated in ref. [10], and has the following form:

$$w^{(0)}(\zeta) = \beta_p \left[ (\zeta + \sqrt{\zeta^2 - t})^r + (\zeta - \sqrt{\zeta^2 - t})^r \right], \quad (3.8)$$

where $r = q/p = (p + 1)/p$ and $\beta_p = 2^{(p-1)/p} / (p + 1)$. Since $\langle \partial \varphi_0(\zeta) \rangle^{(0)} = \sum_{k=0}^{p-1} \langle \partial \hat{\varphi}_k(\zeta) \rangle^{(0)}$, and $\langle \partial \hat{\varphi}_k(\zeta) \rangle^{(0)}$ behaves as $(1/p) \sum_{n \equiv k \mod p} \zeta^{-n/p-1} \langle \mathcal{O}_n \rangle^{(0)}$ around $\zeta = \infty$ up to non-universal part, we find that

$$\langle \partial \hat{\varphi}_1(\zeta) \rangle^{(0)} = \beta_p \left( \zeta - \sqrt{\zeta^2 - t} \right)^r,$$
$$\langle \partial \hat{\varphi}_{p-1}(\zeta) \rangle^{(0)} = \beta_p \left( \zeta + \sqrt{\zeta^2 + t} \right)^r,$$
$$\langle \partial \hat{\varphi}_k(\zeta) \rangle^{(0)} = 0 \quad (k \neq 0, p - 1). \quad (3.9)$$

We can in turn calculate $\langle \partial \varphi_a(\zeta) \rangle^{(0)}$ as

$$\langle \partial \varphi_a(\zeta) \rangle^{(0)} = \sum_{k=0}^{p-1} \omega^{-ka} \langle \partial \hat{\varphi}_k(\zeta) \rangle^{(0)}$$
$$= \beta_p \left[ \omega^{-a} \left( \zeta - \sqrt{\zeta^2 - t} \right)^r + \omega^a \left( \zeta + \sqrt{\zeta^2 - t} \right)^r \right], \quad (3.10)$$

and $\Gamma_{ab}$ is obtained by integrating $\langle \partial \varphi_a(\zeta) \rangle^{(0)} - \langle \partial \varphi_b(\zeta) \rangle^{(0)}$.

We here notice that for $\partial \varphi_a(\zeta)$ with $a \neq 0$, there is an extra cut of second order along $-\sqrt{t} \leq \zeta \leq \sqrt{t}$, in addition to the “physical” cut of order $p$ on the negative real $\zeta$ axis which $\partial \varphi_0(\zeta)$ has originally. It might seem strange that there appears a function which gives a cut with a leg on the positive real axis. However, in the situation we consider, such function is always integrated on some contour, and thus this kind of cut will never be observed in final results. To cope with this “unphysical” cut, we introduce a new coordinate $s$ which is defined by

$$s(\zeta) = \frac{1}{\sqrt{t}} \left( \zeta + \sqrt{\zeta^2 - t} \right). \quad (3.11)$$

Here $s(\zeta)$ maps two sheets of the $\zeta$-plane with cut, onto the outside and the inside of the unit circle in the $s$-plane, respectively. Note that the original physical cut is now mapped

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3 The present normalization corresponds to the background $B_1 = t \ (> 0)$, $B_{2p+1} = -4p/(p + 1)(2p + 1)$ and $B_n = 0 \ (n \neq 1, 2p + 1)$. $B_{2p+1}$ must be negative when $t > 0$ in order for $w^{(0)}(\zeta)$ to have a cut only on the negative real axis.
to $-\infty < s < 0$, and also that two points on different sheets with the same \( \zeta \) coordinate, are related by the map \( s \mapsto 1/s \) in the \( s \)-plane.

Making use of this \( s \)-coordinate, the first derivative of the effective action \( \Gamma_{ab} \) is found to be

\[
\partial_s \Gamma_{ab}(s) = \frac{d\zeta}{ds} \left( \langle \partial \varphi_a (\zeta) \rangle^{(0)} - \langle \partial \varphi_b (\zeta) \rangle^{(0)} \right) = \frac{1-s^{-2}}{2} \beta_p t^{(r+1)/2} \left[ (\omega^a - \omega^b) s^r + (\omega^{-a} - \omega^{-b}) s^{-r} \right].
\]

By integrating this with respect to \( s \), \( \Gamma_{ab}(s) \) is easily obtained to be

\[
\Gamma_{ab}(s) = \gamma_p \left[ \frac{1}{r+1} \left\{ (\omega^a - \omega^b) s^{r+1} + (\omega^{-a} - \omega^{-b}) s^{-(r+1)} \right\} - \frac{1}{r-1} \left\{ (\omega^a - \omega^b) s^{r-1} + (\omega^{-a} - \omega^{-b}) s^{-(r-1)} \right\} \right],
\]

where \( \gamma_p = 2^{-1/p} t^{(2p+1)/2p} / (p+1) \).

Saddle points \( s_0 \) of \( \Gamma_{ab} \) for \( g \to +0 \) are obtained from the equation \( \partial_s \Gamma_{ab}(s) = 0 \). Among them, the points \( s_0 = \pm 1 \), coming from \( d\zeta/ds = \sqrt{t} (1 - s^{-3})/2 \) in (3.12), do not contribute to the contour integral due to the Jacobian factor: \( \oint \frac{ds}{2\pi i} = \oint \frac{d\zeta}{2\pi i} ds \). Thus, it is enough to consider the saddle points coming only from the latter factor in (3.12), and we find

\[
s_0^2 = - (\omega^{-a} - \omega^{-b}) / (\omega^a - \omega^b) = \omega^{-a-b} = \omega^{np-a-b},
\]

namely,

\[
s_0 = e^{(np-a-b)\pi i / (p+1)}. \quad (3.14)
\]

Here the integer \( n \) was introduced to take into account on which (physical) sheet \( s_0 \) lives. Then \( \Gamma_{ab}(s_0) \) and \( \partial_s^2 \Gamma_{ab}(s_0) \) are easily evaluated to be

\[
\Gamma_{ab}(s_0) = \frac{8p}{21p \cdot (2p+1)} t^{2p+1} \sin \left( \frac{a+b+n}{p+1} \pi \right) \sin \left( \frac{a-b}{p} \pi \right)
\]

\[
\frac{d^2}{ds^2} \Gamma_{ab}(s_0) = \frac{2p+1}{p^2} s_0^{-2} \Gamma_{ab}(s_0). \quad (3.15)
\]

In order to investigate the nonperturbative effects of stable solitons which give small and finite contributions in the weak coupling limit, \( g \to +0 \), we choose a contour along which \( \text{Re}(\Gamma_{ab}(s)) \) only takes negative values. The maximum contribution to \( A_{ab} \) can then be picked up from saddle points if the contour passes these points in the direction of steepest descent, along which \( \partial_s^2 \Gamma_{ab}(s_0)(s - s_0)^2 \) is negative. According to (3.13), such direction is
Figure 1: The heavy solid line in the left (right) figure denotes a path of integration $C_1$ ($C_2$) on the first physical sheet for $p = 2$ ($p = 3$). The blobs describe the saddle points, and the physical cuts are on the negative real axis.

given by $\arg((s - s_0)^2/s_0^2) = 0$ when $\Gamma_{ab}(s_0) < 0$, namely, the path goes through the unit circle perpendicularly at $s = s_0$.

As an example, we first consider $A_{01}$ in the $p = 2$ (pure gravity) case. Then, the saddle points $s_0$ satisfying $\Gamma_{01}(s_0) < 0$ are only on the first physical sheet ($-\pi \leq \arg(s) < \pi$): $s_0 = e^{\pm\pi i/3}$. As a path of integration passing these saddle points on the first physical sheet, we find the one depicted as $C_1$ in Fig. 1. On the other hand, since a path on the second physical sheet ($\pi \leq \arg(s) < 3\pi$) must be also chosen so that the $\Gamma_{01}(s)$ only takes negative values on it, the path turns out to go around the point of infinity in the region $9\pi/5 < \arg(s) < 11\pi/5$.

Thus, the contribution to $A_{01}$ comes only from the integration around the saddle points on the first physical sheet, and we get $A_{01} \sim g^{1/2} t^{-1/8} e^{-4\sqrt{5}g^{1/4}/5}$. The exponent in the expression coincides with the value [3, 2, 19] obtained in the nonperturbative analysis of the string equation (Painlevé equation): $4u^2 + (2g^2/3)\partial_t^2 u = t$, where $u = \langle O_1 O_1 \rangle_c$. A similar consideration can be applied to $A_{10}$, giving the same value with $A_{01}$, but now the contribution comes from the second physical sheet with $s_0 = e^{5\pi i/3}$, $e^{7\pi i/3}$.

We finally consider the $p = 3$ (Ising) case. Here we have three physical sheets. For $A_{01}$, there are two saddle points satisfying $\Gamma_{01}(s_0) < 0$ on the first physical sheet ($-\pi \leq$
arg(s) < π): s₀ = e⁻πı/4 and s₀ = eπı/2. The path of integration passing these saddle points can be taken as C₂ depicted in Fig. 1. The saddle point at s₀ = e⁻πı/4 gives the leading nonperturbative effect A₀₁ ∼ g¹/₂ t⁻¹/₁₂ e⁻⁶√π⁹/₁₂/²¹/³₇g. A smaller contribution to A₀₁ also appears from the saddle point s₀ = eπı/₂ as ∼ g¹/₂ t⁻¹/₁₂ e⁻¹²√π⁹/₁₂/²¹/³₇g. Paths on the second physical sheet (π ≤ arg(s) < 3π) and the third one (3π ≤ arg(s) < 5π) can be taken as going around the point of infinity in the region where Re(Γ₀₁(s)) < 0, giving no contribution to A₀₁. The values of the exponent in A₀₁ coming from the saddle points agree with those appearing as the nonperturbative effects in the string equation $4u^3 + (3g^2/2)(∂₁u)^2 + 3g^2ω_0^2u + (g^4/6)∂₄u = -t$ where u is again $⟨ O_1O_1 ⟩_c$. A similar consideration can be applied to other A₁₂ and A₂₀ come from the saddle points on the third and the second physical sheet, respectively.

4 Conclusions

In this paper we first rewrite the Schwinger-Dyson equations in terms of free fermions, $c_a(ζ), \bar{c}_a(ζ)$, with twist operator $σ(ζ)$. We then find that the Schwinger-Dyson equations allow solutions with soliton backgrounds constructed from the soliton fields $S_{ab}$, which are composites of the fermions: $S_{ab}(ζ) = \bar{c}_a(ζ)c_b(ζ) (a ≠ b)$. Since elementary strings (macroscopic loops) are also given as their composites, $∂ϕ_a(ζ) = :c_a(ζ)\bar{c_a}(ζ):$; we suggest that these fermions should be regarded as the fundamental dynamical variables in noncritical string theory. We further evaluate the contributions from the stable solitons, and show that they are consistent with the nonperturbative analysis of string equations.

We, however, have the impression that the formalism presented here is still not satisfactory, because we cannot determine the weights in the summation over multi-instantons, while they seem to be fixed according to the analysis of string equations. We expect that these weights will be determined automatically once the actions for these fermions are constructed. In fact, actions would be constructed for such fundamental dynamical variables, while it seems almost impossible to construct simple string-field actions only with elementary string fields (macroscopic loops). The investigation in this direction is now in progress, and will be reported elsewhere.
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