Rates of convergence for constrained deconvolution problem

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Abstract

Let $X$ and $Y$ be two independent identically distributed random variables with density $p(x)$ and $Z = \alpha X + \beta Y$ for some constants $\alpha > 0$ and $\beta > 0$. We consider the problem of estimating $p(x)$ by means of the samples from the distribution of $Z$. Non-parametric estimator based on the sync kernel is constructed and asymptotic behaviour of the corresponding mean integrated square error is investigated.

1 Introduction

Let $Z_1, ..., Z_n$ be i.i.d observations, where $Z_i = X_i + Y_i$ and $X_i$ and $Y_i$ are independent random variables. Assume that the distribution of $Y$’s is known. The ordinary deconvolution problem is the problem of estimating the distribution of a random variables $X_i$ from the observations $Z_i$. In some situations (for instance, in signal processing) we don’t know the distribution of $Y_i$ but rather can assume that $X_i$ and $Y_i$ belong to the same class of distributions (constrained deconvolution). For example, $X_i$ and $Y_i$ may be of the same multiplicative type (see Belomestny (2002, 2003)), that is

$$Z_i = \alpha X_i + \beta Y_i, \quad \alpha > 0, \beta > 0,$$

where $X_i$ and $Y_i$ are now independent identically distributed random variables. Let distribution of $Z_i$ be absolutely continuous with density $p_Z$ then the distribution of $X_i$ (and $Y_i$) is also continuous with some density, say $p(x)$. Our aim is to construct a non-parametric estimator for $p(x)$ based on the sample $Z_1, ..., Z_n$ and to study its asymptotic behaviour.

**Example** Multiple Access FH SS Radio Networks systems have been considered for a variety of applications such as military ground-based communications and cellular radio (see Simon et al (1994) and Steele (1994)). In these wireless networks with the random multiple access protocol, because of channel reuse, each terminal interferes with signals transmitted by other terminals. This interference is usually referred to as multiple access or self-interference. In the system model, a receiver is located at the center of a plane where there are $N$ transmitters (terminals). The distance between the receiver and interfering terminals is denoted as $r_i$. The signal amplitude loss function over distance $r$ is given by

$$a(r) = \frac{K}{r^m},$$

where the constant $K$ depends on the transmitted power, and the attenuation factor $m$ characterizes the environment. The received passband signal is

$$Y(t) = \sum_{i=1}^{N} a(r_i) X_i(t),$$
where $X_i(t)$ is the signal from the $i$th interferer. Because all terminals use the same modulation scheme and power, it is reasonable to assume that $\{X_i(t)\}_{i=1}^N$ for every $t$ are independent and identically distributed (i.i.d.). The problem in this case can be formulated as one of reconstructing the distribution of $X_i$ from the sampled distribution of $Y$.

2 Main results

Let us denote by $f(t)$ the characteristic function of $X$ and suppose that $0 < \alpha < \beta$. The characteristic function $g(t)$ of the random variable $Z/\beta$ can be expressed as

$$g(t) = f(t)f(\gamma t), \quad 0 < \gamma = \alpha/\beta < 1.$$ 

If the infinite product $\prod g(\gamma^{2k}t)/g(\gamma^{2k+1}t)$ converges then

$$f(t) = \prod_{k=0}^{\infty} g(\gamma^{2k}t)/g(\gamma^{2k+1}t)$$

and a natural estimator for $f(t)$ can be given as

$$\hat{f}_n(t) = \prod_{k=0}^{\infty} g_n(\gamma^{2k}t)/g_n(\gamma^{2k+1}t),$$

provided that $g_n(t) \neq 0$ on $(0, t]$ where $g_n(t)$ is the empirical characteristic function corresponding to $g(t)$:

$$g_n(t) = \frac{1}{n} \sum_{k=1}^{n} e^{itZ_k/\beta}.$$ 

First of all we establish some asymptotic properties of this estimator.

**Theorem 1** If $E|Z|^r < \infty$ for some $r > 0$ and $g(u) \neq 0$ on $(0, t]$ then

$$\sqrt{n}(\hat{f}_n(t) - f(t)) \xrightarrow{D} f(t) \sum_{k=0}^{\infty} \frac{(-1)^k}{g(\gamma^k t)} Y_F(\gamma^k t),$$

where $Y_F = U(t) + iV(t)$ is a complex valued Gaussian process with the $EY_F(t) = 0$ and with the cross-covariance matrix

$$C(t, s) = \begin{pmatrix}
EU(t)U(s) & EU(t)V(s) \\
EV(t)U(s) & EV(t)V(s)
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{4}[u(t-s) + u(t+s)] - u(t)u(s) & \frac{1}{2}[-v(t-s) + v(t+s)] - u(t)v(s) \\
\frac{1}{2}[v(t-s) + v(t+s)] - v(t)u(s) & \frac{1}{2}[u(t-s) - u(t+s)] - v(t)v(s)
\end{pmatrix},$$

where $u(t)$ and $v(t)$ denote real and imaginary part of $g(t)$.

Turn now to the original problem of estimating $p(x)$. Taking into account (1), we can define the estimator for $p(x)$ as follows

$$\hat{p}_n(x) = \frac{1}{2\pi} \int_{-1/\hbar}^{1/\hbar} e^{-ixt} \left[ \prod_{k=0}^{\infty} g_n(\gamma^{2k}t)/g_n(\gamma^{2k+1}t) \right] dt,$$
with
\[
1/h_n = \min\{\max\{\theta > 0 : |g_n(\theta)| > \varepsilon_n\}, c_n\},
\]
where \(\varepsilon_n > 0\) and \(c_n > 0\) are two sequences of real numbers tending correspondingly to zero and infinity as \(n \to \infty\). The exact form of \(\varepsilon_n\) and \(c_n\) will be defined later. Let us further put
\[
d_n = \inf_{|s| < c_n} |g(s)|.
\]
In the following theorem we give some results concerning the behaviour of mean integrated square error corresponding to the estimator (3)

**Theorem 2** Let the following conditions be satisfied
1. \(\int_{-\infty}^{\infty} |x|^r p_Z(x) dx < \infty, \quad r > 0\)
2. \(p(x) \in L_2(\mathbb{R})\)
3. \(\phi(t) = |g(t)|\) does not increase on \((0, \infty)\)

There exist \(A \equiv A(r) > 0\) and \(D \equiv D(r) > 0\) such that for
\[
\varepsilon_n = An^{-1/2} \log^{1/2} n, \quad c_n \leq \phi^{-1}(2\varepsilon_n),
\]
where \(\phi^{-1}(\cdot)\) is inverse function to \(\phi(\cdot)\)
\[
\text{MISE}(\hat{p_n}N) = \int_{-\infty}^{\infty} |p(x) - \hat{p_n}N(x)|^2 dx.
\]

**Corollary** If \(p_Z(x)\) is a density of stable distribution and
\[
|\mathbb{E}e^{itZ}| = e^{-b|t|^a}, \quad b > 0, \quad 0 < a \leq 2,
\]
then \(p(x)\) is also a stable density and for the estimator \(\hat{p_n}N\) with
\[
N = \nu \log n, \quad c_n = (\zeta \log n)^{1/a}, \quad \nu > 0, \quad \zeta > 0
\]
\[
\text{MISE}(\hat{p_n}N) \leq C(\log n)^{(1+a)/a} \times
\]
\[
x n^{-1/(2+\gamma^1/\gamma-\ln 2/((\zeta-\delta)\ln \gamma))}, \quad a > \delta > 0.
\]

Results of the simulation for Cauchy density are presented in Fig.1

**Remark 1** As has been shown by Stefanski and Carroll (1990) the best possible rates of convergence for the MISE in general deconvolution problem are usually (including normal and Cauchy distribution) \(\log^{-p} n\) for some \(p > 0\). We see that in the case of the constrained deconvolution the situation is better.

**Remark 2** The above method can produce estimates which are not probability density functions i. e. may take negative values or/and do not integrate to one. It happens due to the finiteness of \(n\). For this reason, some methods of modification of density estimators (all estimators, not only kernel estimators) has been constructed (see Glad et al (1999, 2003)) in such a way that the resulting estimator always produces estimates which are almost surely probability density functions, and, in addition, the resulting estimator is better or at least almost as good as the initial one.
3 Auxiliary results

Lemma 1 For any \( r \in (0, 2) \) and any real characteristic function \( f(t) \) of a distribution with finite absolute moment \( \beta_r \) of the order \( r \) the following inequality holds

\[
f(t) \geq 1 - c(r) \beta_r |t|^r,
\]

where

\[
c(r) = \frac{2}{(2r)^{r/2}}.
\]

Proof. First of all we prove the inequality

\[
\cos(x) > 1 - c(r)x^r, \quad x > 0.
\]

From the equality

\[
\left( \frac{1 - \cos(x)}{x^r} \right)' = \frac{x^r \sin x - rx^{r-1}(1 - \cos(x))}{x^{2r}} = 0
\]

we have

\[
\frac{1 - \cos(x)}{x^r} = x \sin(x), \quad \frac{1 - \cos(x)}{x^r} \leq \frac{x^{2-r}}{r}, \quad x > 0
\]

On the other hand

\[
\frac{1 - \cos(x)}{x^r} \leq \frac{2}{x^r}, \quad x > 0.
\]

Combining (8) and (9), we get (7)

\[
\frac{1 - \cos(x)}{x^r} \leq \frac{2}{(2r)^{r/2}}, \quad x > 0.
\]

Finally,

\[
f(t) = \int_{-\infty}^{\infty} \cos(tx) \, dF(x) \geq \int_{-\infty}^{\infty} (1 - c(r)|tx|^r) \, dF(x) = 1 - c(r)\beta_r |t|^r.
\]
Lemma 2 Let \( \xi \) be a random variable with characteristic function \( f \) and finite absolute moment \( \beta_r \) of order \( r \in (0, 2) \), then for \( |t| < 1/2 \sqrt{10c(r)\beta_r} \) the following inequality holds

\[
| \ln f(t) | \leq 2\pi \sqrt{2r |t| c(r)\beta_r}, \tag{10}
\]

where \( c(r) \) is given by \( 10c(r)\beta_r \).

**Proof.** Since \( |f(t)|^2 \) and \( \Re^2 f(t) \) are two real c.f. we have

\[ 0 \leq 1 - |f(t)|^2 \leq 2^{r+1}|t|^r c(r)\beta_r, \quad 0 \leq 1 - \Re^2 f(t) \leq 2^{r+1}|t|^r c(r)\beta_r \]

and

\[
\sin^2 \psi(t) = \frac{\Im^2 f(t)}{|f(t)|^2} < \frac{1 - |f(t)|^2 + 1 - \Re^2 f(t)}{|f(t)|^2} \leq \frac{2^{r+2}|t|^r c_\beta_r}{1 - 2^{r+1}|t|^r c_\beta_r},
\]

where \( \psi(t) \equiv \arg f(t) \) satisfying \( \psi(0) = 0 \). The elementary inequality \( |x| \leq \frac{\pi}{3} |\sin x| \) that holds for \( |x| < \pi/6 \) entails

\[
|\psi(t)| \leq \frac{\pi}{3} \sqrt{\frac{2^{r+2}|t|^r c_\beta_r}{1 - 2^{r+1}|t|^r c_\beta_r}} \tag{11}
\]

if \( |t| < 1/2 \sqrt{10c(\beta_r)} \). Further, using the inequality \( |\log(1 - x)| \leq 2|x|, 0 < x \leq 1/2 \)

\[- \log |f(t)| \leq -\frac{1}{2} \log (1 - 2^{r+1}|t|^r c(\beta_r)) \leq 2^{r+1}|t|^r c(\beta_r). \tag{12}\]

The combination of (11) and (12) gives (10).

\[ \square \]

Lemma 3 Let us have \( n \) i.i.d random variables \( X_1, \ldots, X_n \) with common characteristic function \( f(t) \) and the underlying distribution possesses finite absolute moment \( \beta_r \) of order \( r > 0 \). Then, for \( a > 0, \ 0 < b \leq 2 \)

\[
\Pr \left( \sup_{|\theta| < a} |f_n(\theta) - f(\theta)| > b \right) \leq 2 \left( 1 + a \Theta(n, b, r) \right) e^{-n^2/144} + \frac{\nu_r}{n},
\]

where \( f_n(\theta) \) is the corresponding empirical characteristic function

\[ f_n(t) = \frac{1}{n} \sum_{k=1}^{n} e^{itX_k}, \]

\( \nu_r \) is a constant not depending on \( \beta_r, \ a, b \) and

\[
\Theta(n, b, r) = \begin{cases} 
\frac{b_1^{1/r} n^{(2-r)/r}}{b^{1/r}}, & 0 < r \leq 1 \\
\frac{b_2^{2/r} n^{1/r}}{b^{2/r}}, & 1 < r \leq 2.
\end{cases}
\]

**Proof.** Let us prove the inequality for \( 0 < r \leq 1 \). Define

\[ \gamma = \left( \frac{b}{15n^{2-r}\beta_r} \right)^{1/r}. \]
We find numbers $t_1 < t_2 < \ldots < t_k$ with the property that $t_1 = -a$, $t_k = a$, $|t_i - t_{i+1}| \leq \gamma$. Clearly, we can assure this with $k \geq 1 + 2a/\gamma$. We begin with

$$\Pr \left( \sup_{|t| < a} |f_n(t) - f(t)| > b \right) \leq \Pr \left( \sup_{|t-s| < \gamma} |f(t) - f(s)| > b/3 \right)$$

$$+ \Pr \left( \sup_{|t-s| < \gamma} |f_n(t) - f_n(s)| > b/3 \right) + \sum_{i=1}^{k} \Pr(|f_n(t_i) - f(t_i)| > b/3).$$

Denote the three summands on the right hand side by $T_1$, $T_2$ and $T_3$ and estimate them. Due to the inequality $|1 - e^{ix}| \leq c(r)|x|^r$ that holds for $0 < r \leq 1$ with $c(r) = \sqrt{4/(2r)^r + 1/r^{2r}} \leq 5$ we have

$$|f(t) - f(s)| \leq E|1 - e^{i(t-s)X}| \leq c(r)E|(t-s)X|^r \leq c(r)\gamma^r \beta_r \leq b/3,$$

when $|t-s| \leq \gamma$. Therefore, $T_1 = 0$. Next, we let $Y$ be the random variable that puts mass $1/n$ at each of the $X_i$'s, then

$$|f_n(t) - f_n(s)| \leq E|1 - e^{i(t-s)Y}| \leq E|(t-s)Y| = |t-s| \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|.$$

Using the inequality

$$\mathbb{E} \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|^r \leq n^{1-r} \mathbb{E}|X_1|^r, \quad 0 < r \leq 1,$$

we get

$$T_2 \leq \Pr \left( \gamma \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq b/3 \right) \leq \frac{5b^{1-r}3^{r+1}}{n} \leq \frac{\nu_r}{n}$$

by the Chebyshev inequality. Finally, for fixed $t_i$,

$$\Pr(|f_n(t_i) - f(t_i)| > b/3) \leq \Pr(|u_n(t_i) - u(t_i)| > b/6)$$

$$+ \Pr(|v_n(t_i) - v(t_i)| > b/6) \leq 2e^{-n6^2/144},$$

by Berneistein’s inequality for bounded random variables (see, for example, Bosoq(1998)), where as usually $u(t)$ and $v(t)$ are the real and imaginary parts of $f(t)$, and $u_n(t)$ and $v_n(t)$ are those of $f_n(t)$. The proof in the case $1 < r \leq 2$ can be conducted in a similar way using the inequality

$$\mathbb{E} \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|^r \leq \mathbb{E}|X_1|^r, \quad r > 1$$

instead of (13).
4 Proofs of the main results

Proof of Theorem 1 Let us consider the process

\[ Y_n(t) = \sqrt{n}(g_n(t) - g(t)) \]

with \( EY_n(t) = 0 \), the cross-covariance matrix \( C(t, s) \) and

\[ EY_n(t)Y_n(s) = f(t - s) - f(t)f(-s). \]

The finite-dimensional distributions of \( Y_n(t) \) converge by the multidimensional central limit theorem to those of \( Y_F(t) \) as \( n \to \infty \) (Ushakov (1999), Chapter 3). Further, it is known (see, for example, Billingsley (1968), Theorem 4.2) that if

\[ \zeta_{un} \overset{D}{\to} \zeta \quad \text{as} \quad n \to \infty \]

for each \( u \),

\[ \zeta_u \overset{D}{\to} \zeta \quad \text{as} \quad u \to \infty, \]

and

\[ \lim_{n \to \infty} \lim_{u \to \infty} \sup \Pr(|\zeta_{un} - \eta_u| \geq \varepsilon) = 0, \quad (14) \]

then

\[ \eta_n \overset{D}{\to} \zeta \quad \text{as} \quad n \to \infty. \]

Put

\[ \zeta_N = \sum_{k=0}^{N} \frac{(-1)^k}{g(\gamma^k t)} Y_F(\gamma^k t), \quad \zeta = \sum_{k=0}^{\infty} \frac{(-1)^k}{g(\gamma^k t)} Y_F(\gamma^k t) \]

\[ \zeta_{Nn} = \sum_{k=0}^{N} (-1)^k Y_n^L(\gamma^k t), \quad \eta_n = \sum_{k=0}^{\infty} (-1)^k Y_n^L(\gamma^k t), \]

where \( Y_n^L = \sqrt{n}(\ln g_n(t) - \ln g(t)) \). We have

\[ \Pr(|\zeta_{Nn} - \eta_n| \geq \varepsilon) = \Pr\left( \left| \sum_{k=N+1}^{\infty} (-1)^k Y_n^L(\gamma^k t) \right| \geq \varepsilon \left| \max_{k>N} \frac{g(\gamma^k t) - g_n(\gamma^k t)}{g(\gamma^k t)} \right| \leq \frac{1}{2} \right) + \]

\[ \Pr\left( \max_{k>N} \left| \frac{g(\gamma^k t) - g_n(\gamma^k t)}{g(\gamma^k t)} \right| > \frac{1}{2} \right) = P_1 + P_2 \quad (15) \]

Using the Markov and Cauchy-Schwarz inequality, we have

\[ P_1 \leq \frac{1}{\varepsilon^2} \sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} (-1)^{i+j} \mathbb{E} \left\{ Y_n^L(\gamma^i t) Y_n^L(\gamma^j t) \left| \max_{k>N} \left| \frac{g(\gamma^k t) - g_n(\gamma^k t)}{g(\gamma^k t)} \right| \leq \frac{1}{2} \right\} \right\} \]

\[ \leq \frac{1}{\varepsilon^2} \left( \sum_{j=N+1}^{\infty} \mathbb{E} \left\{ \left| Y_n^L(\gamma^j t) \right|^2 \left| \max_{k>N} \left| \frac{g(\gamma^k t) - g_n(\gamma^k t)}{g(\gamma^k t)} \right| \leq \frac{1}{2} \right\} \right\} \right)^{1/2}. \quad (16) \]
Since \( g(u) \neq 0 \) for \(|u| < t\) there exists \( b > 0 \) such that \( g(u) > 2b \) on \((-t, t)\) and due to Lemma 3
\[
P_2 \leq \Pr \left( \max_{k > N} |g_n(\gamma k^t) - g_n(\gamma k^t)| > b \right) \leq 4 \left( 1 + \gamma^{2N+1} |t| \Theta(n, b, r) \right) e^{-n^2} + \frac{\nu r}{n}. \tag{17}
\]
Further, using the elementary inequality \(|\ln(1 - z)| \leq 2|z|\) that holds for \(|z| \leq 1/2\) we get
\[
\mathbb{E} \left\{ \ln \left( 1 - \frac{g(\gamma k^t) - g_n(\gamma k^t)}{g(\gamma k^t)} \right) \right\}^2 \leq \frac{1}{2}
\]
\[
\leq \frac{4}{|g(\gamma k^t)|^2} \mathbb{E}|g(\gamma k^t) - g_n(\gamma k^t)|^2 = \frac{4}{n|g(\gamma k^t)|^2} (1 - |g(\gamma k^t)|^2).
\]
Combining \((15)\), \((16)\), \((17)\) and using inequality \(|g(t)|^2 > 1 - c(r)\beta r |t|^r\) (see Lemma 1) we get \((14)\). Analogously
\[
\Pr(|\zeta_N - \zeta| \geq c) \leq \frac{1}{c^2} \sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} (-1)^{i+j} g((\gamma^i - \gamma^j)t) - g(\gamma^i t)g(-\gamma^j t) \leq \sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} C((\gamma^i + \gamma^j)|t| + \gamma^{r(i+j)}|t|^2), \quad N > N_0
\]
that implies \(\zeta_N \overset{D}{\rightarrow} \zeta\), \(N \rightarrow \infty\). Thus,
\[
\sqrt{n}(\ln \hat{f}_n(t) - \ln f(t)) \overset{D}{\rightarrow} \sum_{k=0}^{\infty} \frac{(-1)^k}{g(\gamma k^t)} Y_F(\gamma k^t)
\]
and
\[
\sqrt{n}(\hat{f}_n(t) - f(t)) \overset{D}{\rightarrow} f(t) \sum_{k=0}^{\infty} \frac{(-1)^k}{g(\gamma k^t)} Y_F(\gamma k^t)
\]
as \(n \rightarrow \infty\).
\[\square\]

**Proof of Theorem 2** Plancherel-Parseval formula entails
\[
\int_{-\infty}^{\infty} |p(x) - \hat{p}_N(x)|^2 \, dx = \frac{1}{2\pi} \int_{-1/h_n}^{1/h_n} \left| \prod_{k=0}^{N} \frac{g_n(\gamma^{2k} t)}{g_n(\gamma^{2k+1} t)} - \prod_{k=0}^{\infty} \frac{g(\gamma^{2k} t)}{g(\gamma^{2k+1} t)} \right|^2 \, dt + \frac{1}{2\pi} \int_{|t| > 1/h_n} |f(t)|^2 \, dt.
\]
Let us now estimate the first summand on the right-hand side. Using lemma 2 we have for some constants \(C \equiv C(r)\) and \(D \equiv D(r)\) not depending on \(N\) and \(\gamma\)
\[
\left| \ln \left( \prod_{k=N+1}^{\infty} \frac{g(\gamma^{2k} t)}{g(\gamma^{2k+1} t)} \right) \right| \leq \sum_{k=2(N+1)}^{\infty} |\ln g(\gamma^k t)| \leq C \frac{\gamma^r(N+1) |t|^r/2}{1 - \gamma r/2}, \quad |t| < \frac{D}{\gamma^{2(N+1)}}
\]

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The elementary inequality $|z| \leq 2|\ln(1 - z)|$ that holds for $|z| \leq 1/3$ implies
\[1 - \prod_{k=N+1}^{\infty} \frac{g(\gamma^{2kt})}{g(\gamma^{2k+1}t)} \leq C \frac{\gamma^r(N+1)|t|^{r/2}}{1 - \gamma^{r/2}}, \quad |t| < \frac{D}{\gamma^{2(N+1)}}\]

Further,
\[
\prod_{k=0}^{N} \frac{g_n(\gamma^{2kt})}{g_n(\gamma^{2k+1}t)} - \prod_{k=0}^{N} \frac{g(\gamma^{2kt})}{g(\gamma^{2k+1}t)} = \sum_{i=0}^{N} H(\gamma^{2it}) \prod_{k \neq i t}^{N} \frac{g(\gamma^{2kt})}{g(\gamma^{2k+1}t)} + \sum_{i_1 \neq i_2} H(\gamma^{2i_1}t)H(\gamma^{2i_2}t) \prod_{k \neq i_1, i_2}^{N} \frac{g(\gamma^{2kt})}{g(\gamma^{2k+1}t)} + \ldots + \prod_{i=0}^{N} H(\gamma^{2it}),
\]
where
\[H(t) = \frac{g_n(t)}{g_n(\gamma t)} - \frac{g(t)}{g(\gamma t)}\]

Since $|g(t)|$ is non-increasing
\[
\left| \frac{g(t)}{g(\gamma t)} \right| \leq 1
\]
and for $|t| < 1/h_n$
\[
|H(t)| \leq \frac{\Delta(t)}{\epsilon_n},
\]
where
\[
\Delta(t) = |g_n(t) - g(t)| + |g_n(\gamma t) - g(\gamma t)|.
\]

Further, using the equality $E|g_n(t) - g(t)|^2 = \frac{1-|g(t)|^2}{n}$, one gets
\[
E \int_{-1/h_n}^{1/h_n} \left| \prod_{k=1}^{N} \frac{g_n(\gamma^{k-1}t)}{g_n(\gamma^kt)} - \prod_{k=1}^{N} \frac{g(\gamma^{k-1}t)}{g(\gamma^kt)} \right|^2 dt \leq 4c_n \Pr(\sup_{|t| < c_n} \Delta(t) > \epsilon_n) + \frac{4Nc_n}{n\epsilon_n^2}.
\]

The combination of the previous estimates and some simple calculations yield
\[
E \int_{-1/h_n}^{1/h_n} \left| \prod_{k=1}^{N} \frac{g_n(\gamma^{k-1}t)}{g_n(\gamma^kt)} - \prod_{k=1}^{N} \frac{g(\gamma^{k-1}t)}{g(\gamma^kt)} \right|^2 dt \leq 8c_n \Pr(\sup_{|t| < c_n} \Delta(t) > \epsilon_n) + \frac{4N+1c_n}{n\epsilon_n^2} + \frac{C(\gamma^{2rN\epsilon_n^2+1})}{(1 - \gamma^{r/2})^2}, \quad C > 0
\]
where according to lemma 3
\[
\Pr(\sup_{|t| < c_n} \Delta(t) > \epsilon_n) \leq \Pr(\sup_{|t| < c_n} |g_n(t) - g(t)| > \epsilon_n/2) + \Pr(\sup_{|t| < c_n} |g_n(\gamma t) - g(\gamma t)| > \epsilon_n/2) \leq \frac{4(1 + c_n \Theta(n, \epsilon_n/2, r)) e^{-\frac{n\epsilon_n^2}{r}}}{n} + \frac{\nu_r}{n}
\]
The condition $p(x) \in L_2(\mathbb{R})$ implies $\phi(\cdot) \in L_2(\mathbb{R})$ that in its turn means
\[
\phi^{-1}(x) < \frac{1}{x^2}, \quad x < \delta
\]
and therefore \( c_n = O(n) \) as \( n \to \infty \). So, one has for some \( \kappa = \kappa(r) > 0 \) and \( C > 0 \)

\[
\Pr(\sup_{|t| < c_n} \Delta(t) > \varepsilon_n) \leq 4 \left( 1 + \frac{C}{A} n^{1 + \kappa} \right) n^{-4A/144} = O(1/n), \quad n \to \infty
\]

for large enough \( A \).

Thus, there exist constant \( D > 0 \) such that

\[
\int_{-\infty}^{\infty} |p(x) - \hat{p}_n(x)|^2 dx \leq D \left[ \frac{4Nc_n}{nd_n^2} + \frac{\gamma A r}{(1 - \gamma r)^2} \right] + \frac{1}{2\pi} \int_{|t| > 1/h_n} |f(t)|^2 dt.
\]

Let us now estimate the last term in (18). We have

\[
E \left[ \int_{|t| \geq 1/h_n} |f(t)|^2 d\theta \right] \leq \Pr \left[ 1/h_n < c_n \right] \int_{\mathbb{R}} |f(\theta)|^2 d\theta + \int_{|\theta| \geq c_n} |f(\theta)|^2 d\theta. \quad (19)
\]

Since \( d_n - \varepsilon_n > \phi(c_n) - \varepsilon_n \geq \varepsilon_n \), lemma 3 yields

\[
\Pr \left[ 1/h_n < c_n \right] = \Pr(\inf_{|\theta| < c_n} |g_n(\theta)| \leq \varepsilon_n) \leq \Pr(\sup_{|\theta| < c_n} |g_n(\theta) - g(\theta)| > d_n - \varepsilon_n) \leq 2 \left( 1 + c_n \Theta(n, \varepsilon_n, r) \right) e^{-n^2/144} + \frac{\ln n}{n} = O(1/n), \quad n \to \infty
\]

\[
\Box
\]

**Proof of Corollary 1.** Let us put

\[
N = \nu c_n^\alpha, \quad c_n = (\zeta \ln n)^{1/\alpha}, \quad \zeta > 0
\]

Since \( X \) possesses moments of \( r \) order for \( r < \alpha \)

\[
\int_{-\infty}^{\infty} |p(x) - \hat{p}_n(x)|^2 dx \leq D \left[ (\zeta \ln n)^{1/\alpha} n^{2\zeta(1 + \gamma 1/\alpha) + \nu \zeta \ln 2 - 1} + (\zeta \ln n)^{(r+1)/\alpha} n^{r\nu \zeta \ln \gamma} \right] + (\ln n)^{2\alpha} n^{-2\zeta \beta}
\]

Taking \( \nu = 2\beta/r \ln(1/\gamma) \) and \( \zeta = 1/2\beta(1 + (1 + \gamma 1/\alpha) - \ln 2/r \ln \gamma) \) we come to (13).

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