THE PROPORTION OF WEIERSTRASS SEMIGROUPS

NATHAN KAPLAN AND LYNNELLE YE

Abstract. We solve a problem of Komeda concerning the proportion of numerical semigroups which do not satisfy Buchweitz’ necessary criterion for a semigroup to occur as the Weierstrass semigroup of a point on an algebraic curve. We also show that the family of semigroups known to be Weierstrass semigroups using a result of Eisenbud and Harris, has zero density in the set of all semigroups. In the process, we prove several more general results about the structure of a typical numerical semigroup.

1. Introduction

A numerical semigroup $S$ is an additive submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. The complement is referred to as the gap set and is denoted by $H(S)$. Its size is called the genus of $S$ and is usually denoted by $g(S)$. The largest of these gaps is called the Frobenius number, denoted $F(S)$, and the smallest nonzero nongap is called the multiplicity, denoted $m(S)$. When it will not cause confusion we will omit the $S$ and write $g, F$ and $m$. A very good source for background on numerical semigroups is [9].

Let $C$ be a smooth projective algebraic curve of genus $g$ over the complex numbers. It is a theorem of Weierstrass that given any $p \in C$ there are exactly $g$ integers $\alpha_1(p) < \cdots < \alpha_g(p) \leq 2g - 1$ such that there does not exist a meromorphic function $f$ on $C$ which has a pole of order $\alpha_i(p)$ at $p$ and no other singularities [5]. This characterization makes it clear that the set $\mathbb{N} \setminus \{\alpha_1(p), \ldots, \alpha_g(p)\}$ is a numerical semigroup of genus $g$. We say that a semigroup $S$ is Weierstrass if there exists some curve $C$ and some point $p \in C$ such that $S$ is this semigroup. In the late 19th century, Hurwitz suggested studying which numerical semigroups are Weierstrass [8].

A point $p$ such that $(\alpha_1(p), \ldots, \alpha_g(p)) \neq (1, \ldots, g)$ is called a Weierstrass point of $C$, and it is known that there are at most $g^3 - g$ such points. It is an active area of research to consider a multiset $\mathcal{S}$ of at most $g^3 - g$ semigroups of genus $g$ and study the set of curves for which $\mathcal{S}$ is the collection of semigroups of the Weierstrass points of the curve. This multiset gives us important information about the geometry of the curve. We would like to better understand things like the dimension of the moduli space of curves with a fixed collection of semigroups attached to its Weierstrass points. For more on the history of this problem see the article of del Centina [5], or the book [1].

In this paper we focus on two criteria that address this problem of Hurwitz. The first is a simple combinatorial criterion of Buchweitz which is necessary for a
semigroup to occur as the Weierstrass semigroup of some point on some curve $C$ \cite{4}. This condition gave the first proof that not all semigroups are Weierstrass. In the final section of the paper we consider a criterion of Eisenbud and Harris \cite{6}, which shows that certain semigroups are Weierstrass. These two simple criteria cover much of what we know about this problem. The main result of this paper is to show that in some sense, both of the sets covered by these criteria have density zero in the entire set of numerical semigroups. The overall proportion of Weierstrass semigroups remains completely unknown.

Let $N(g)$ be the number of numerical semigroups of genus $g$. Recent work of Zhai \cite{16}, building on work of Zhao \cite{17}, gives a better understanding of the growth of $N(g)$. These papers build towards resolving a conjecture of Bras-Amorós \cite{3}.

**Theorem 1** (Zhai). The function $N(g)$ satisfies

$$
\lim_{g \to \infty} \frac{N(g)}{\varphi^g} = C,
$$

where $C$ is a constant, approximately $3.78$ and $\varphi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio.

This result will play an important role in some of our proofs.

We next recall the criterion of Buchweitz.

**Proposition 2** (Buchweitz). Let $S$ be a semigroup of genus $g$ and let $H(S)$ be the set of gaps of $S$. Suppose that there exists some $n > 1$ such that

$$
|nH(S)| > (2n - 1)(g - 1),
$$

where $nH(S)$ is the $n$-fold sum of the set $H(S)$. Then $S$ is not Weierstrass.

Let $NB(g)$ be the number of semigroups $S$ of genus $g$ for which there is some $n$ such that $S$ does not satisfy the Buchweitz criterion with this $n$. Komeda seems to be the first to have studied $\lim_{g \to \infty} \frac{NB(g)}{N(g)}$ \cite{11}.

The following is part of a table included in \cite{11}:

| $g$ | $N(g)$ | $NB(g)$ | $\frac{NB(g)}{N(g)}$ |
|-----|--------|---------|---------------------|
| 16  | 4806   | 2       | .000416             |
| 17  | 8045   | 6       | .000746             |
| 18  | 13467  | 15      | .001114             |
| 19  | 22464  | 31      | .001380             |
| 20  | 37396  | 67      | .001792             |
| 21  | 62194  | 145     | .002331             |
| 22  | 103246 | 293     | .002838             |
| 23  | 170963 | 542     | .003170             |
| 24  | 282828 | 1053    | .003723             |
| 25  | 467224 | 1944    | .004161             |

One main goal of this paper is to show that this limit is $0$. The key step in this argument will be a technical result building on work of Zhai \cite{16}.

In the final part of the paper we will focus on the proportion of semigroups which are known to occur as Weierstrass semigroups. Eisenbud and Harris \cite{6}, have shown that a certain class of semigroups with $F < 2m$ do occur as Weierstrass semigroups. We will show that the proportion of such semigroups is $0$ as $g$ goes to infinity.
2. Semigroups Satisfying the Buchweitz Criterion

We first show that certain classes of semigroups cannot possibly fail the Buchweitz criterion for any $n$. Fix $\varepsilon > 0$ and suppose that $S$ is a semigroup with $(2-\varepsilon)m < F < (2+\varepsilon)m$. We want to consider when $S$ fails the Buchweitz criterion for some chosen value of $n$. We have,

$$|nH| \leq (2 + \varepsilon)nm - (n - 1).$$

Therefore, $|nH| > (2n - 1)(g - 1)$ implies that

$$g \leq \frac{(2 + \varepsilon)nm + n}{2n - 1} = (2 + \varepsilon)\frac{nm}{2n - 1} + \frac{n}{2n - 1} < \left(2 + \frac{1}{20}\right)\frac{nm}{2n - 1},$$

for $\varepsilon < \frac{1}{21}$ and $m$ sufficiently large. We have $\frac{1}{20} - \varepsilon > \frac{1}{420}$ when $\varepsilon < \frac{1}{21}$, so for $m \geq 420$ we have $(2 + \varepsilon)nm + n < \left(2 + \frac{1}{20}\right)nm$.

We note that since $n \geq 2$ is an integer,

$$\left(2 + \frac{1}{20}\right)\frac{n}{2n - 1} \leq \frac{41}{30} < 1.3667.$$

We will state the results of the above paragraph as a proposition.

**Proposition 3.** Let $\varepsilon < \frac{1}{21}$ and $m \geq 420$.

Suppose that $S$ is a semigroup with $(2-\varepsilon)m < F < (2+\varepsilon)m$. Then $|nH| > (2n - 1)(g - 1)$ implies $g < 1.3667m$.

The main technical result of the rest of this paper is that the restriction on the genus in the proposition does not occur often.

**Theorem 4.**

1. Fix $\varepsilon > 0$. Let $A(g)$ be the number of semigroups of genus $g$ satisfying $(2-\varepsilon)m < F < (2+\varepsilon)m$. Then

$$\lim_{g \to \infty} \frac{A(g)}{N(g)} = 1.$$

2. Let $B(g)$ be the number of semigroups of genus $g$ with $m < 420$. Then

$$\lim_{g \to \infty} \frac{B(g)}{N(g)} = 0.$$

3. Let $C(g)$ be the number of semigroups of genus $g$ with $g < 1.3667m$. Then

$$\lim_{g \to \infty} \frac{C(g)}{N(g)} = 0.$$

The claim for $A(g)$ follows directly from Proposition and Theorem below. Suppose we have established this claim. The proportion of numerical semigroups of genus $g$ with Frobenius number at least $(2 + \varepsilon)m$ goes to 0 as $g$ goes to infinity. For any $\varepsilon$ the number of semigroups with Frobenius number at most $(2 + \varepsilon)420$ is finite. Therefore, as $g$ goes to infinity, almost all semigroups satisfying $(2-\varepsilon)m < F < (2+\varepsilon)m$ have $m > 420$. This establishes the claim for $B(g)$. The statement regarding $C(g)$ follows from Corollary and Proposition.

From Theorem it is easy to prove our main theorem.

**Theorem 5.** Let $NB(g)$ be the number of semigroups of genus $g$ which fail the Buchweitz criterion for some $n$. Then,

$$\lim_{g \to \infty} \frac{NB(g)}{N(g)} = 0.$$
Proof. Suppose Theorem 1 holds. Choose \( \varepsilon < \frac{1}{3m} \). Theorem 2 implies that almost all semigroups have Frobenius number and multiplicity in the range given in the statement of Proposition 3, but that almost no such semigroups satisfy \( g < 1.3667m \), completing the proof.  

3. Apéry Sets and Semigroups with \( F < 2m \)

The Apéry set of a numerical semigroup \( S \) with respect to its multiplicity \( m \), often just called the Apéry set, is a set of \( m \) nonnegative integers giving for each \( 0 \leq i \leq m - 1 \) the smallest integer in \( S \) congruent to \( i \) modulo \( m \). We will omit 0 from this set, and represent the Apéry set by \( \{k_1m + 1, \ldots, k_{m-1}m + m - 1\} \) where each \( k_i \in \mathbb{N} \). We note that there are exactly \( k_i \) gaps of \( S \) equivalent to \( i \) modulo \( m \), and therefore the genus of \( S \) is \( \sum_{i=1}^{m-1} k_i \). The Frobenius number is the largest Apéry set element minus \( m \).

From the definition of the Apéry set it is clear that certain inequalities must hold between the integers \( k_i \). In fact, a result of Branco, García-García, García-Sánchez and Rosales [2], gives a set of inequalities which completely determine whether the set \( \{k_1m + 1, \ldots, k_{m-1}m + m - 1\} \) is the Apéry set of a numerical semigroup of multiplicity \( m \).

Proposition 6 (Rosales et. al.). Consider the following set of inequalities:

\[
\begin{align*}
    x_i & \geq 1 \quad \text{for all } i \in \{1, \ldots, m-1\} \\
    x_i + x_j & \geq x_{i+j} + 1 \quad \text{for all } 1 \leq i \leq j \leq m-1, \ i + j \leq m - 1 \\
    x_i + x_j & \geq x_{i+j-m} \quad \text{for all } 1 \leq i \leq j \leq m-1, \ i + j > m \\
    x_i & \in \mathbb{Z} \quad \text{for all } i \in \{1, \ldots, m-1\}, \\
    \sum_{i=1}^{m-1} x_i & = g.
\end{align*}
\]

There is a one-to-one correspondence between semigroups with multiplicity \( m \) and genus \( g \) and solutions to the above inequalities, where we identify the solution \( \{k_1, \ldots, k_{m-1}\} \) with the semigroup that has Apéry set \( \{k_1m+1, \ldots, k_{m-1}m+m-1\} \).

Recent work has made use of this correspondence, counting semigroups by counting valid Apéry sets, for example [10]. This can be very useful in giving numerical results. We will separately consider two classes of semigroups, those with \( F < 2m \) and those with \( 2m < F < 3m \). The first case is much simpler.

Note that \( F < 2m \) is exactly equivalent to the condition that each \( k_i \) is equal to 1 or 2. Given \( m \), the above proposition implies that any set \( \{k_1, \ldots, k_{m-1}\} \) where each \( k_i \) is either 1 or 2 gives the Apéry set of some numerical semigroup.

Suppose we have a semigroup of genus \( g \) with \( F < 2m \) and Apéry set given by \( \{k_1m + 1, \ldots, k_{m-1}m + m - 1\} \). Let \( R \) be the number of \( k_i \) which are equal to 2. We see that \( m - 1 + R = g \) and that \( R \) can take on any value from 0 to \( m - 1 \). Therefore we have that \( m \) can take on any value from \( \lfloor \frac{g+2}{2} \rfloor \) to \( g + 1 \). Given a pair of \( m \) and \( R \) such that \( m - 1 + R = g \) we see that there are \( \binom{m-1}{R} = \binom{g-R}{R} \) choices for the \( R \) values of \( i \) such that \( k_i = 2 \). It is a straightforward inductive exercise to prove that

\[
\sum_{R=0}^{\lfloor \frac{g}{2} \rfloor} \binom{g-R}{R} = F_{g+1},
\]
the $g + 1$st Fibonacci number.

It is well-known that $F_{g+1}$ is asymptotic to $\frac{\phi^g}{\sqrt{5}} = \frac{5 + \sqrt{5}}{10} \phi^g$ as $g$ goes to infinity.

The above sum is very tightly clustered around its maximum value.

**Proposition 7.** Let $\alpha = \frac{5 - \sqrt{5}}{10}$ and fix $\varepsilon > 0$. We have

$$\sum_{R=0}^{[(\alpha - \varepsilon)g]} \frac{g - R}{R} = o(\phi^g).$$

We also have

$$\sum_{R=1}^{[\varepsilon g]} \frac{g - R}{R} = o(\phi^g).$$

**Proof.** Stirling’s approximation says that $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$. Therefore,

$$\frac{(1 - c)n}{cn} = \frac{(1 - c)n}{(cn)!(1 - 2c)n!} \sim \frac{\sqrt{2\pi(1 - c)n}}{\sqrt{2\pi(1 - 2c)n\sqrt{2\pi}} n^{1 - c}(1 - 2c)n^{1 - c}} \frac{(1 - c)n^{1 - c}}{(1 - 2c)n^{1 - c}(en)n} = \frac{1}{\sqrt{2\pi n}} \frac{\sqrt{1 - c}}{\sqrt{1 - 2c}} \left(\frac{1 - c}{c^{c(1 - 2c)^{1 - 2c}}}\right)^n.$$

We see that $\frac{(1 - c)n}{c^{(1 - 2c)^{1 - 2c}}}$ is asymptotic to a constant depending on $c$ divided by $\sqrt{n}$ times $\frac{c^{(1 - 2c)^{1 - 2c}}}{\sqrt{2\pi n}}$.

Let $f(c) = \frac{(1 - c)(1 - c)^{1 - c}}{c^{c(1 - 2c)^{1 - 2c}}}$. We claim that $f$ attains its maximum value in the range from 0 to $\frac{1}{2}$ at $\frac{5 + \sqrt{5}}{10}$. We instead find the maximum value of $\ln(f(c))$ in this range. We can see that the derivative of $\ln(f(c))$ is $2\ln(1 - 2c) - \ln(1 - c) - \ln(c)$. Taking an exponential, we see that this is equal to 0 when $\frac{(1 - 2c)^2}{(1 - c)c} = 1$. This gives $5c^2 - 5c + 1 = 0$, which has roots at $c = \frac{5 + \sqrt{5}}{10}$. Only one of these roots occurs in the range for which $c < 1 - \varepsilon$, meaning that this is our unique critical point in this interval. By choosing any value of $c$ between 0 and this critical point, for example $c = \frac{1}{4}$, we see that $f(c)$ is increasing in the range from 0 to our critical value, showing that $f(c)$ attains a maximum at $c = \frac{5 + \sqrt{5}}{10}$. At this value, $f(c) = \phi$, the golden ratio.

Therefore,

$$\sum_{R=0}^{[(\alpha - \varepsilon)g]} \frac{g - R}{R} \leq (g + 1) \left(\frac{1 - (\alpha - \varepsilon)}{(\alpha - \varepsilon)g}\right),$$

for $g$ sufficiently large, which is asymptotic to a constant depending on $\varepsilon$ times $\sqrt{g}$ times $\left(\frac{1 - (\alpha - \varepsilon)^{1 + \varepsilon}}{c^{(1 - 2c)^{1 - 2c}}}\right)^n$ for $c = \alpha - \varepsilon$. This last term is $r^g$ where $r < \phi$, showing that this sum is $o(\phi^g)$.

We also have,

$$\sum_{R=1}^{[\varepsilon g]} \frac{g - R}{R} \leq (g + 1) \left(\frac{1 - (\alpha + \varepsilon)}{(\alpha + \varepsilon)g}\right),$$
which is also asymptotic to a constant depending on $\varepsilon$ times $\sqrt{g}$ times some $r^g$ where $r < \varphi$. This sum is therefore also $o(\varphi^g)$. \qed

We first state a corollary related to the ratio of the multiplicity to the genus of a semigroup with $F < 2m$.

**Corollary 8.** Fix $\varepsilon > 0$ and $\gamma = \frac{5+\sqrt{5}}{10}$. Let $E_\varepsilon(g)$ be the number of numerical semigroups with $F < 2m$ and $(\gamma - \varepsilon)g < m < (\gamma + \varepsilon)g$. Let $I(g)$ be the number of numerical semigroups with $F < 2m$. Then $\lim_{g \to \infty} \frac{E_\varepsilon(g)}{I(g)} = 1$.

**Proof.** We have $g = m - 1 + R$ and have seen that almost all semigroups with $F < 2m$ have $(\alpha - \varepsilon)g < R < (\alpha + \varepsilon)g$ for $\alpha = \frac{5-\sqrt{5}}{10}$. Since $1 - \alpha = \gamma$, we see that almost all semigroups with $F < 2m$ have $(\gamma - \varepsilon)g + 1 < m < (\gamma + \varepsilon)g + 1$. Taking $g$ to infinity completes the proof. \qed

This proposition implies that for $g$ sufficiently large, almost all semigroups $S$ with genus $g$ and $F < 2m$ have genus $m - 1 + R$ close to $m - 1 + \alpha g$. We note that when $R > \frac{g}{4} + 1$, since $m - 1 + R = g$ we have $m - 1 < \frac{3g}{4} - 1$. Also note that $\frac{5-\sqrt{5}}{10} > \frac{1}{4}$. In this case, we see that since our largest gap is at most $2m - 1$, we have

$$|nH| < n(2m - 1) - (n - 1) = 2n(m - 1) + 1 < \frac{3n}{2}g - (2n - 1) < (2n - 1)g,$$

since $n \geq 2$. Therefore, we see that almost no semigroup with $F < 2m$ fails the Buchweitz criterion for any $n$. We have proven the following, the easy part of our main result.

**Proposition 9.** Let $D(g)$ be the number of semigroups of genus $g$ with $F < 2m$ and which fail the Buchweitz criterion for some $n$. Then

$$\lim_{g \to \infty} \frac{D(g)}{N(g)} = 0.$$

The following proposition will play a part in the proof of Theorem 4.

**Proposition 10.** Let $\varepsilon > 0$. Let $N^*_{\varepsilon}(g)$ be the number of semigroups of genus $g$ with $F < (2 - \varepsilon)m$. We have

$$\lim_{g \to \infty} \frac{N^*_{\varepsilon}(g)}{N(g)} = 0.$$

**Proof.** We note that $N^*_{\varepsilon}(g) = \sum_{R=0}^{\lceil \frac{g}{2} \rceil} \binom{(1-\varepsilon)(g-R)}{R}$. As above we let $R = cg$ and see that we are looking for the maximum value of $\binom{(1-\varepsilon)(g-R)}{cg}$. By the above argument the maximum value here is asymptotic to some constant depending on $\varepsilon$ divided by the square root of $g$, times $r^g$ for some $r < \varphi$. This shows that our sum is $o(\varphi^g)$. \qed

Finally, the following proposition will be convenient for the proof of Theorem 20.

**Proposition 11.** For any $\varepsilon > 0$, there exists a $\delta > 0$ so that

$$\sum_{R=0}^{\lceil \frac{g}{2} \rceil} \binom{g-R}{R} = o((1+\varepsilon)^g).$$
Proposition 14 (Zhao). Let \( m < F < g \) be a numerical semigroup with multiplicity \( m \) and Frobenius number \( F \). Every numerical semigroup with \( 2m < F < 3g \) must have more detail and show that for any \( \varepsilon > 0 \), \( \lim_{g \to \infty} \frac{L(g)}{N(g)} = 0 \).

We first recall a recent result of Zhai \cite{16}, building on work of Zhao \cite{17}, that

\[ \frac{c}{\ln c} \text{ divided by } \frac{\ln c}{1/c} \text{ approaches 0, and thus } c^c \text{ approaches 1. Then } f(c) \text{ approaches 1 as } c \text{ approaches 0. The proposition follows directly.} \]

4. Semigroups with \( F > 2m \)

We first recall a recent result of Zhai \cite{16}, building on work of Zhao \cite{17}, that shows that we can focus on semigroups with \( 2m < F < 3g \).

**Theorem 12** (Zhai). Let \( L(g) \) be the number of semigroups with \( F > 3m \) and genus \( g \). Then

\[ \lim_{g \to \infty} \frac{L(g)}{N(g)} = 0. \]

In the rest of this section we will focus on the semigroups with \( 2m < F < 3g \) in more detail and show that for any \( \varepsilon > 0 \), the proportion of them with \( (2+\varepsilon)m < F < 3m \) goes to zero as \( g \) goes to infinity.

**Theorem 13.** Let \( \varepsilon > 0 \) and let \( P_\varepsilon(g) \) be the number of semigroups with \((2+\varepsilon)m < F < 3m\). Then

\[ \lim_{g \to \infty} \frac{P_\varepsilon(g)}{N(g)} = 0. \]

We require the following concepts from Zhao \cite{17}. Let \( A_k = \{ A \in [0, k-1] \mid 0 \in A \text{ and } k \not\in A + A \} \). Let \( S \) be a numerical semigroup with multiplicity \( m \) and Frobenius number \( F \) satisfying \( 2m < F < 3g \). We say that \( S \) has type \((A; k)\), where \( 0 < k < m \) and \( A \in A_k \), if \( F = 2m + k \) and \( S \cap [m, m+k] = A + m \). Every numerical semigroup with \( 2m < F < 3g \) has a unique type \((A; k)\), since \( k = F - 2m \) and \( A = S \cap [m, m+k] - m \). Zhao proves the following.

**Proposition 14** (Zhao). Let \( k \) be a positive integer and let \( A \in A_k \). Then the number of numerical semigroups of genus \( g \) and type \((A; k)\) is at most

\[ F_{g-\lvert (A+A)\cap [0,k]\rvert + \lvert A \rvert - k-1}, \]

where \( F_a \) is the \( a \)th Fibonacci number.

We now note that if semigroup \( S \) of type \((A; k)\) satisfies \((2+\varepsilon)m < F < 3g \), it must have \( k > \varepsilon m > \varepsilon g/3 \), since \( g \leq 3(m-1) \). We also have the general fact that \( F_a \leq \frac{2}{\sqrt{5}} \varphi^a \) for all \( a \). Therefore

\[ P_\varepsilon(g) \leq \sum_{\varepsilon g/3 < k < g} \sum_{A \in A_k} F_{g-\lvert (A+A)\cap [0,k]\rvert + \lvert A \rvert - k-1} \]

implying that

\[ P_\varepsilon(g) \varphi^{-g} \leq \sum_{\varepsilon g/3 < k < g} \sum_{A \in A_k} \varphi^{g-\lvert (A+A)\cap [0,k]\rvert + \lvert A \rvert - k-1} \]

\[ \leq \sum_{k=\varepsilon g/3}^{\infty} \sum_{A \in A_k} \varphi^{g-\lvert (A+A)\cap [0,k]\rvert + \lvert A \rvert - k-1}. \]
Let \( T(g) \) be the number of numerical semigroups of genus \( g \) satisfying \( F < 3m \). We have the following theorem from Zhao [17].

**Theorem 15** (Zhao).

\[
\lim_{g \to \infty} T(g) \varphi^{-g} = \frac{\varphi}{\sqrt{\varphi}} + \frac{1}{\sqrt{\varphi}} \sum_{k=1}^{\infty} \sum_{A \in \mathcal{A}_k} \varphi^{-(A+A) \cap [0,k] + |A|-k-1}.
\]

We are now in a position to prove Theorem 13.

**Proof of Theorem 13.** By Theorem 1, we know that \( T(g) \varphi^{-g} \) is bounded above, so the sum

\[
\sum_{k=1}^{\infty} \sum_{A \in \mathcal{A}_k} \varphi^{-(A+A) \cap [0,k] + |A|-k-1}
\]

converges. It follows that \( \sum_{k=\lceil g/3 \rceil}^{\infty} \sum_{A \in \mathcal{A}_k} \varphi^{-(A+A) \cap [0,k] + |A|-k-1} \) approaches 0 as \( g \) goes to infinity. \( \square \)

**Proposition 16.** Let \( \epsilon > 0 \) and \( \gamma = \frac{5+\sqrt{10}}{10} \). Let \( \Phi_{\epsilon}(g) \) be the number of numerical semigroups with genus \( g \) and \( (\gamma - \epsilon)g < m < (\gamma + \epsilon)g \). Then \( \lim_{g \to \infty} \frac{\Phi_{\epsilon}(g)}{\Phi(g)} = 1 \).

**Proof.** By Theorem 12 it suffices to consider the cases \( m < F < 2m \) and \( 2m < F < 3m \). The first case is simply Corollary 8 so we now assume \( 2m < F < 3m \).

The Apéry set of a numerical semigroup with \( 2m < F < 3m \) is of the form \( \{k_1m + 1, \ldots, k_{m-1}m + (m-1)\} \) where each \( k_i \in \{1, 2, 3\} \) and at least one is equal to 3. Let \( a \) be maximal such that \( k_a = 3 \). By Proposition 6 the number of semigroups with multiplicity \( m \) and \( F = 2m + a \) is exactly equal to the number of sequences \( (k_1, \ldots, k_{m-1}) \) satisfying the following conditions:

1. For each \( 1 \leq i \leq a-1, \ k_i \in \{1, 2, 3\} \),
2. \( k_a = 3 \),
3. For each \( a+1 \leq j \leq m-1, \ k_j \in \{1, 2\} \),
4. For each \( i, j \) with \( i + j \leq m - 1 \) and \( k_i = k_j = 1 \) we have \( k_{i+j} \neq 3 \).

Let \( H(a, b) \) be the number of numerical semigroups with \( 2m < F < 3m \), multiplicity \( a + 1 \), and genus \( b \). Then the number of possibilities for the sequence \( (k_1, \ldots, k_a) \) is simply \( H(a, b) \). The remaining elements \( (k_{a+1}, \ldots, k_{m-1}) \) consist of \( g - b - (m - 1 - a) \) values of \( k_i \) equal to 2, with the rest equal to 1. These can be arranged in any order. Thus the total number of numerical semigroups with \( 2m < F < 3m \) is

\[
\sum_{b=3}^{g} \sum_{a=\lceil b/3 \rceil}^{b} \sum_{m=a+1}^{g} H(a, b) \left( m - 1 - a \right) \left( g - b - (m - 1 - a) \right).
\]

Applying Theorem 13 with \( \epsilon = \epsilon/6 \), we need only consider the case \( a < cm/6 \), for which \( b \leq 3a < cm/2 \leq eg/2 \). The number of such numerical semigroups is at most

\[
\sum_{b<eg/2} \sum_{a=\lceil b/3 \rceil}^{b} \sum_{m=a+1}^{g} H(a, b) \left( m - 1 - a \right) \left( g - b - (m - 1 - a) \right).
\]
We need to show that those terms in the above sum for which \( m \) is outside the range \((\gamma - \epsilon)g, (\gamma + \epsilon)g\) contribute \( o(\varphi^g) \) to the sum. For such \( m \), we have

\[
|m - \gamma g| \geq \epsilon g \\
|m - a - \gamma(g - b) + 1 + a - \gamma b| \geq \epsilon g.
\]

By the triangle inequality, we have

\[
|m - 1 - a - \gamma(g - b)| + 1 + a + \gamma b = |m - 1 - a - \gamma(g - b)| + |1 + a| + | - \gamma b| \\
\geq |m - 1 - a - \gamma(g - b) + 1 + a - \gamma b|.
\]

Therefore,

\[
|m - 1 - a - \gamma(g - b)| + 1 + a + \gamma b \geq \epsilon g \\
|m - 1 - a - \gamma(g - b)| \geq \epsilon g - 1 - a - \gamma b \\
\geq \epsilon g - 1 - \epsilon g/6 - \gamma \epsilon g/2 \\
\geq 0.472 \epsilon g \geq 0.472 \epsilon (g - b)
\]

for sufficiently large \( g \). As in the proof of Proposition \( 4 \) for such \( m \), there is some \( \psi < \varphi \) for which \( (g_{m-1-a}^{m-1-a}) = O(\varphi^{g-b}) \). Since the total number of numerical semigroups of genus \( b \) is asymptotic to \( \varphi^b \), we certainly have \( H(a, b) = O(\varphi^b) \). We conclude that \( H(a, b)(g_{m-1-a}^{m-1-a}) \leq c_\epsilon \psi^{g-b} \varphi^b \leq c_\epsilon (\psi^{1-\epsilon/2} \varphi^\epsilon/2)^g \), so the total contribution to the sum from such \( m \) is indeed \( o(\varphi^g) \).

Proposition \( 16 \) implies, in particular, that for fixed \( \epsilon > 0 \), as \( g \) approaches infinity, the proportion of numerical semigroups with \( m > (\gamma + \epsilon)g \) approaches 0. For \( \epsilon = \frac{1}{1.3667} - \gamma \approx 0.0081 > 0 \), the property \( m > (\gamma + \epsilon)g \) is precisely \( g < 1.3667m \). This implies statement (3) of Theorem \( 4 \) and thus completes the proof of Theorem \( 5 \).

5. SEMIGROUPS WHICH DO OCCUR AS WEIERSTRASS SEMIGROUPS

We first recall the definition of the weight of a numerical semigroup. This is another way of measuring a semigroup’s complexity.

**Definition.** Let \( S \) be a semigroup of genus \( g \) with gap set \( \{a_1, \ldots, a_g\} \). We define the *weight of \( S \),* \( W(S) = \sum_{i=1}^{g} a_i - \frac{g(g+1)}{2} \).

Note that for any \( g \geq 1 \), the semigroup containing all positive integers greater than \( g \) has weight zero. This definition plays an important role in the theory of semigroups and algebraic curves since for any curve \( C \) of genus \( g \) it is known that the sum of the weights of all of the semigroups of Weierstrass points of \( C \) is \( g^3 - g \).

The following difficult result of Eisenbud and Harris proves that certain semigroups do occur as the Weierstrass semigroup of a point on some curve.

**Theorem 17.** Let \( S \) be a semigroup with \( F < 2m \) and \( W(S) < g - 1 \). Then \( S \) is Weierstrass.

We will show that this condition on the weight of \( S \) is quite restrictive.
Proposition 18. Let $Q(g)$ be the number of semigroups with $F < 2m$ and $W(S) < g - 1$. Then

$$\lim_{g \to \infty} \frac{Q(g)}{N(g)} = 0.$$ 

This proposition is actually a consequence of a much stronger statement about the weights of numerical semigroups. To prove that statement, we first need the following lemma.

Lemma 19. Let $p(x, y, z)$ be the number of partitions of $x$ into at most $y$ parts, each of size at most $z$. Then the number of numerical semigroups with genus $g$, multiplicity $m$, and weight $w$ satisfying $m < F < 2m$ is exactly $p(w - (g - m + 1), g - m + 1, 2m - 2 - g)$.

Remark. Equivalently, this is the coefficient of $q^{w-(g-m+1)}$ in the $q$-binomial coefficient $\binom{m-1}{g-m+1}_q$. See [13] for details.

Proof. Suppose $N_0 \setminus S = \{1, 2, \ldots, m-1, m+i_1, \ldots, m+i_{g-m+1}\}$ with $i_a \in [1, m-1]$ for all $a$. We have

$$w = 1 + 2 + \cdots + m - 1 + (m + i_1) + \cdots + (m + i_{g-m+1}) - (1 + 2 + \cdots + m - 1 + m + \cdots + g) = \sum_{a=1}^{g-m+1} (i_a - a + 1),$$

which can be rearranged as

$$w - (g - m + 1) = \sum_{a=1}^{g-m+1} (i_a - a).$$

The $i_a - a$ are nonnegative because $i_1 \geq 1$ and $i_a > i_{a+1}$. They are non-decreasing since $i_{a+1} - (a + 1) \geq i_a + 1 - (a + 1) = i_a - a$. Finally, since $m - 1 \geq g - m + 1$ and $i_{g-m+1} - (g - m + 1) \geq i_a - a$, we have $2m - 2 + g \geq i_a - a$. Thus each distinct choice of these $i_a$ is associated with a unique partition of $w - (g - m + 1)$ into at most $g - m + 1$ parts, each of size at most $2m - 2 - g$. Furthermore, from any such partition $j_1 + \cdots + j_{g-m+1}$, where $0 \leq j_1 \leq \cdots \leq j_{g-m+1}$, it is easy to reconstruct $S$ by setting $i_a = j_a + a$; the resulting $i_a$ will be strictly increasing and bounded above by $m - 1$, as desired. This completes the proof of the lemma. \hfill \Box

We observe that $p(x, y, z) = p(yz - x, y, z)$, since if $j_1 + \cdots + j_y = x$ is a partition of $x$ into parts of size at most $z$, then $(z-j_1) + \cdots + (z-j_y) = yz - x$ is a partition of $yz - x$ into parts of size at most $z$, and vice versa. This simple fact will be useful later.

We now state and prove our main theorem for this section.

Theorem 20. Let $\beta_1 = \frac{3}{2} \left( \frac{\ln z}{\ln \gamma} \right)^2$, $\gamma = \frac{5 + \sqrt{5}}{10}$, $\beta_2 = (1 - \gamma)(2\gamma - 1) - \beta_1$ and $\epsilon > 0$. Let $Y_\gamma(g)$ be the number of numerical semigroups with genus $g$ and weight at most $(\beta_1 - \epsilon)g^2$ and $Z_\epsilon(g)$ be the number of numerical semigroups with genus $g$ and weight at least $(\beta_2 + \epsilon)g^2$. Then $\lim_{g \to \infty} \frac{Y_\gamma(g)}{N(g)} = \lim_{g \to \infty} \frac{Z_\epsilon(g)}{N(g)} = 0$.

In order to show this, we need the Hardy-Ramanujan formula [7]:

\begin{align*}
\phi(n) &= \sum_{d \mid n} \mu(d) \ln \left( \frac{d}{\phi(d)} \right), \\
\psi(n) &= \sum_{d \mid n} \mu(d) \ln d, \\
\chi(n) &= \sum_{d \mid n} \mu(d) \ln \left( \frac{n}{d} \right), \\
\Lambda(n) &= \sum_{d \mid n} \mu(d) \ln \ln \left( \frac{n}{d} \right), \\
\pi(n) &= \sum_{d \mid n} \mu(d) \left( \frac{n}{d} \right)^{1/2}, \\
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s}, \\
\zeta(1-s) &= \sum_{n=1}^{\infty} \frac{\lfloor n/\gamma \rfloor}{n^s}. \\
\end{align*}
Theorem 21 (Hardy, Ramanujan). Let \( p(n) \) be the total number of partitions of \( n \). Then as \( n \) grows large, \( p(n) \) is asymptotically equal to
\[
\frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.
\]

Proof of Theorem 21. We may assume that \( F < 3m \) by Theorem 12. We consider the cases \( m < F < 2m \) and \( 2m < F < 3m \) separately.

First suppose \( m < F < 2m \). Let \( K_2(w, m, g) \) be the number of numerical semigroups with genus \( g \), weight \( w \), multiplicity \( m \) and \( m < F < 2m \). We wish to bound \( Y_\epsilon(g) = \sum_{m=1}^g \sum_{w<(\beta_1-\epsilon)g^2} K_2(w, m, g) \). Now by Lemma 19 \( K_2(w, m, g) \) is equal to \( p(w - (g - m + 1), g - m + 1, 2m - 2 - g) \), so in particular \( K_2(w, m, g) \leq p(w - (g - m + 1)) \leq p(w) \) where \( p(w) \) is the total number of partitions of \( w \). But by Theorem 21 we have
\[
\sum_{m=1}^g \sum_{w<(\beta_1-\epsilon)g^2} p(w) = O(g^3 e^{\pi\sqrt{2/3}\sqrt{\beta_1-\epsilon}g^2}) = o(\varphi^g)
\]
for \( \beta_1 \) such that \( e^{\pi\sqrt{2/3\sqrt{m}}} = \varphi \), which gives \( \beta_1 = \frac{3}{2} \left( \frac{\ln \varphi}{\pi} \right)^2 \approx 0.035 \). This gives the desired lower bound on the weight of a typical numerical semigroup.

To show \( Z_\epsilon(g)/N(g) \) goes to zero as \( g \) goes to infinity, we transform the problem of bounding \( Z_\epsilon(g) \) into the problem of bounding \( Y_{\epsilon/2}(g) \). Let \( \gamma = \frac{5+\sqrt{5}}{10} \) and define \( \beta_2 := (1-\gamma)(2\gamma-1) - \beta_1 \), and \( I := ((\gamma - \delta)g, (\gamma + \delta)g) \), where \( \delta \) is so small that \( \delta + 2\delta^2 < \epsilon/2 \). From Proposition 11 it suffices to consider those semigroups for which \( m \in I \). We use Lemma 19 to write
\[
\sum_{m \in I} \sum_{w>(\beta_2+\epsilon)g^2} K_2(w, m, g) = \sum_{m \in I} \sum_{w>(\beta_2+\epsilon)g^2} p(w - (g - m + 1), g - m + 1, 2m - 2 - g).
\]
As noted above, we have \( p(w - (g - m + 1), g - m + 1, 2m - 2 - g) = p((g - m + 1)(2m - 2 - g) + (g - m + 1) - w, g - m + 1, 2m - 2 - g) \). Let \( w' = (g - m + 1)(2m - 2 - g) + (g - m + 1) - w \), so that the right-hand side of the above equation can be rewritten as
\[
\sum_{m \in I} \sum_{w>(\beta_2+\epsilon)g^2} p(w', g - m + 1, 2m - 2 - g)
\]
\[
= \sum_{m \in I} \sum_{w'<(g-m+1)(2m-2-g)+g-m+1-(\beta_2+\epsilon)g^2} p(w', g - m + 1, 2m - 2 - g)
\]
\[
= \sum_{m \in I} \sum_{w''<(g-m+1)(2m-2-g)+2(g-m+1)-(\beta_2+\epsilon)g^2} p(w'' - (g - m + 1), g - m + 1, 2m - 2 - g)
\]
\[
= \sum_{m \in I} \sum_{w''<(g-m+1)(2m-2-g)+2(g-m+1)-(\beta_2+\epsilon)g^2} K_2(w'', m, g)
\]
where we again used Lemma 19 in the last line. Assuming from Proposition 11 that \( m \in I \), we have \( g - m + 1 < g - (\gamma - \delta)g + 1 = (1 - \gamma + \delta)g + 1 \) and
$2m - 2 - g < 2(\gamma + \delta)g - g = (2\gamma - 1 + 2\delta)g$. Hence the last line above is at most

$$\sum_{m \in I} w'^{<((1-\gamma-\delta)g+1)((2\gamma-1+2\delta)g+2)((1-\gamma+\delta)g+2-(\beta_2+\epsilon)g^2})} K_2(w'', m, g)$$

$$\leq \sum_{m \in I} w'^{<((1-\gamma+\delta)((2\gamma-1+2\delta)g^2-(\beta_2+\epsilon)g^2+(1+4\delta)g+2}$$

$$= \sum_{m \in I} w'^{<((1-\gamma+\delta+2\delta^2-\epsilon)g^2+(1+4\delta)g+2}$$

$$\leq \sum_{m \in I} w'^{<((1-\gamma+\delta-\epsilon/2)g^2} K_2(w'', m, g)$$

for sufficiently large $g$, since $\epsilon - (\delta + 2\delta^2) > \epsilon/2$ by construction. But the last line is at most $Y_{1/2}(g) = o(\varphi^9)$, so we are done.

Next suppose $2m < F < 3m$, and suppose the Apéry set is given by \{\text{both}\}. Let $K_M(w, m, t, g)$ be the number of numerical semigroups with genus $g$, weight $w$, multiplicity $m$, exactly $t$ values of $a$ such that $k_0 = 3$, and $2m < F < 3m$. Using Theorem and Proposition above, we may assume that $F < (2 + \delta_0)m$, where $\delta_0$ will be chosen later. We first bound the number of semigroups with $w < (\beta_1 - \epsilon)g^2$. The intuition here is that the numerical semigroups with $2m < F < (2 + \delta_0)m$ and weight $w$ look fairly similar to the numerical semigroups with $F < 2m$ and weight $w$, and the ability to set some $k_i$ equal to 3 does not greatly increase the number of such semigroups.

Let $i_1 < \cdots < i_s$ be the set of indices such that $k_{i_a} \geq 2$, and let $i_{j_1} < \cdots < i_{j_t}$ be the set of indices such that $k_{i_{j_a}} = 3$. We have $s + t = g - m + 1$, or alternatively, $g = m + s + t - 1$. We can write the weight $w$ as follows:

$$w = \sum_{a=1}^{m-1} a + \sum_{a=1}^{s}(m + i_a) + \sum_{a=1}^{t}(2m + i_{j_a}) - \sum_{a=1}^{m+s+t-1} a$$

$$= \sum_{a=1}^{s}(m + i_a) - \sum_{a=1}^{s}(m - 1 + a) + \sum_{a=1}^{t}(2m + i_{j_a}) - \sum_{a=1}^{t}(m + s - 1 + a)$$

$$= \sum_{a=1}^{s}(i_a - a) + s + \sum_{a=1}^{t}(i_{j_a} - a) + t(m - s + 1).$$

Suppose that $\sum_{a=1}^{s}(i_a - a) = d$. We work in parallel to the case $F < 2m$. By the same reasoning as in the proof of Lemma the number of choices for the $i_a$ is precisely $p(d, s, m-1-s) < p(w)$. The total number of choices for the $i_{j_a}$ is at most $\binom{t}{t-1} = (2^{m+1-t}).$ Choose $e_0$ small enough that $\psi := (1 + e_0)e^{\sqrt{2/3}\sqrt{m-F}} < \varphi$. By Proposition there is some $\delta_0 > 0$ so that $\sum_{R=0}^{t} (q^R + R^R)$ is bounded above by $(1 + e_0)^9$ for sufficiently large $g$. Since we assumed that $F < (2 + \delta_0)m$, we have $t \leq i_{j_1} \leq \delta_0 m < \delta_0 g$. Then $\binom{t}{t-1}$ is bounded above by $(1 + e_0)^9$, and so of course $\binom{t}{t-1}$ is as well. Summing over $d$, we conclude that $K_M(w, m, t, g)$ is bounded by a polynomial in $g$ times $p(w)(1 + e_0)^9$, which is in turn bounded by $p((\beta_1 - \epsilon)g^2)(1 + e_0)^9 = O(\varphi^9)$ by the Hardy-Ramanujan formula. Summing over all $m, t$, and $w < (\beta_1 - \epsilon)g^2$ gives a count of possible such semigroups $S$ which is $o(\varphi^9)$, as desired.
For $w > (\beta_2 + \epsilon)g^2$, we proceed by reducing to the situation $w < (\beta_1 - \epsilon/4)g^2$, again in parallel with our strategy for $F < 2m$. Let $0 < \delta_0 < \epsilon/4$: then for sufficiently large $g$ and all $t < \delta_0 g$, we have $t(2m - s + 1) + s < 2nt + t + s \leq 2gt + t + s < 2\delta_0 g^2 + t + s < (\epsilon/2)g^2$. Now since $(1 - \gamma)\delta_0 < (1 - \gamma)\epsilon/4$, we can choose $\delta_1$ so small that $\delta_1 + 2\delta_1^2 + (1 - \gamma + \delta_1)\delta_0 < \epsilon/4$. Next, choose $\epsilon_0$ small enough such that $\psi := (1 + \epsilon_0)e^{\pi \sqrt{2/3} \sqrt{1 - \epsilon/4}} < \varphi$. Finally, choose $\delta_2$ small enough that $\sum_{R=0}^{\delta_2 g} (g-R)^\ast$ is bounded above by $(1 + \epsilon_0)^g$, and let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. Notice that for any $t$, we have $i_{j_a} - a < m$, so that $\sum_{a=1}^{t}(i_{j_a} - a) < tm$ and hence, for all $t < \delta g$,

$$\sum_{a=1}^{s}(i_{a}-a) = w - s - \sum_{a=1}^{t}(i_{j_a} - a) - t(m-s+1) > w - s - tm - (m-s+1) > (\beta_2 + \epsilon/2)g^2.$$ 

If $d = \sum_{a=1}^{s}(i_{a}-a)$, then for a fixed $d$ we have $p(d, s, m - 1 - s)$ ways of choosing the $i_a$ and no more than $(\binom{s}{t})$ ways of choosing the $i_{j_a}$ from the $i_a$. Therefore we have

$$K_3(w, m, t, g) \leq \sum_{d > (\beta_2 + \epsilon/2)g^2} p(d, s, m - 1 - s) \binom{s}{t}$$

and again using the fact that $p(x, y, z) = p(yz - x, y, z)$, the right-hand side can be rewritten as

$$\sum_{d < s(m-1-s)-(\beta_2+\epsilon/2)g^2} p(d, s, m - 1 - s) \binom{s}{t} \leq (1 + \epsilon_0)^g \sum_{d < s(m-1-s)-(\beta_2+\epsilon/2)g^2} p(d, s, m - 1 - s).$$

For $t < \delta g$, we have $s = g - m + 1 - t \geq (1 - \delta)g - m$. Then further assuming that $m \in ((\gamma - \delta)g, (\gamma + \delta)g)$, we have $s \leq g - m + 1 < (1 - \gamma + \delta)g + 1$ and $m - 1 - s \leq 2m - (1 - \delta)g < (2\gamma - 1 + 3\delta)g$. Thus the last line can be bounded above by

$$(1 + \epsilon_0)^g \sum_{d < ((1-\gamma+\delta)g+1)(2\gamma-1+3\delta)g-(\beta_2+\epsilon/2)g^2} p(d, s, m - 1 - s)$$

$$= (1 + \epsilon_0)^g \sum_{d < (\beta_1 - \epsilon/4)g^2 + (\beta_2 + 3\delta^2 + (1-\gamma+\delta)g^2 + (2\gamma - 1 + 3\delta)g} p(d, s, m - 1 - s).$$

Since we chose $\delta$ so that $(\delta + 3\delta^2 + (1-\gamma+\delta)g^2 < (\epsilon/4)g^2$, for sufficiently large $g$ this is at most

$$(1 + \epsilon_0)^g \sum_{d < (\beta_1 - \epsilon/4)g^2} p(d, s, m - 1 - s)$$

$$\leq (1 + \epsilon_0)^g \sum_{d < (\beta_1 - \epsilon/4)g^2} e^\pi \sqrt{2/3} \sqrt{\beta_1 - \epsilon/4} g = O(g^\epsilon \varphi^g)$$

by the Hardy-Ramanujan formula. Summing over $t$ and $m$ gives the desired bound.

\[\square\]

It would be interesting to see whether we can improve on the constants given in the statement of this result. A more careful analysis of the partitions occurring in this proof will probably yield better bounds.
Since \( g - 1 < (\alpha - \epsilon)g^2 \) for, say, \( \epsilon = \alpha/2 \) and sufficiently large \( g \), Proposition 18 follows immediately from Theorem 20. This result shows that although the Eisenbud and Harris family of semigroups are Weierstrass, they also have density zero in the entire set of numerical semigroups. The main theorem of the paper shows that the set of semigroups which are not Weierstrass because they fail the Buchweitz criterion for some \( n \) also has density zero. There are other known examples of semigroups which are known to be Weierstrass but do not fit into this Eisenbud and Harris family, and semigroups which are not Weierstrass but do not fail the Buchweitz criterion. See for example [14] [15].

It would be very interesting to show that a positive proportion of numerical semigroups are Weierstrass, or to show that a positive proportion are not Weierstrass. The problem of determining the density of the set of Weierstrass semigroups in the entire set of numerical semigroups remains completely open.

6. Acknowledgments

We would like to thank Nathan Pflueger, Alex Zhai, Gaku Liu and Yufei Zhao for helpful conversations. We would also like to thank Jonathan Wang and Alan Deckelbaum for assistance during this project. Sections of this work were done as part of the University of Minnesota Duluth REU program, funded by NSF/DMS grant 1062709 and NSA grant H98230-11-1-0224. Finally, we would like to thank Joseph Gallian for his advice and support.

References

[1] E. Arabello, M. Cornalba, P.A. Griffiths, and J. Harris, Geometry of algebraic curves. I, Springer-Verlang, New York, 1985.
[2] M. Branco, J. García-García, P.A. García-Sánchez, and J.C. Rosales, Systems of inequalities and numerical semigroups, J. London Math. Soc. (2) 65 (2002), no. 3, 611-623.
[3] M. Bras-Amorós, Fibonacci-like behavior of the number of numerical semigroups of a given genus, Semigroup Forum 76 (2008), no. 2, 379-384.
[4] R. O. Buchweitz, “Über deformation monomialer kurvensingularitäten und Weierstrasspunkte auf Riemannischen flächen”, Ph.D. Thesis, Hannover, 1976.
[5] A. del Centina, Weierstrass points and their impact in the study of algebraic curves: a historical account from the “Lückensatz” to the 1970s, Ann. Univ. Ferrara 54 (2008), no. 1, 37-59.
[6] D. Eisenbud and J. Harris, Existence, decomposition, and limits of certain Weierstrass points. Invent. Math. 87 (1987), no. 3, 495-515.
[7] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. (2) 17 (1918), 75-117. Also in Collected Papers of S. Ramanujan, Cambridge University Press, London and New York, 1927, reprinted by Chelsea, New York, 1962.
[8] A. Hürwitz, Über algebraische Gebilde mit eindeutigen transformationen in sich, Math. Ann. 41 (1892), no. 3, 403-442.
[9] P.A. García-Sánchez and J.C. Rosales, Numerical semigroups. New York: Springer, 2009.
[10] N. Kaplan, Counting numerical semigrous by genus and some cases of a question of Wilf. To appear in J. Pure and Appl. Algebra. Available online: http://dx.doi.org/10.1016/j.jpaa.2011.10.038
[11] J. Komeda, Non-Weierstrass numerical semigroups, Semigroup Forum, 57 (1998), no. 2, 157-185.
[12] N. Medeiros, Buchweitz’s criterion. Accessed May 20, 2010. http://w3.impa.br/~nivaldo/algebra/buchw.html
[13] R. Stanley, Enumerative combinatorics. Cambridge: Cambridge University Press, 2011. Vol. 1, 2nd ed., 65-67.
[14] F. Torres, On certain \( N \)-sheeted coverings of curves and numerical semigroups which cannot be realized as Weierstrass semigroups, Com. Alg. 23(1995), no. 11, 4211-4228.
[15] G. Oliveira, F. Torres, and J. Villanueva, On the weight of numerical semigroups, J. Pure Appl. Algebra 214 (2010), no. 11, 1955-1961.
[16] A. Zhai, Fibonacci-like growth of numerical semigroups with a given genus. arXiv:1111.3142v1.
[17] Y. Zhao, Constructing numerical semigroups of a given genus, Semigroup Forum, 80 (2010), no. 2, 242-254.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138
E-mail address: NKaplan@math.harvard.edu

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305
E-mail address: Lynelle@stanford.edu