Conventionally, the strong-CP problem is assumed to be a naturalness puzzle, with the axion solution sometimes viewed as an ad hoc fix. Gravity is either ignored or taken as a threat for the global Peccei-Quinn symmetry. We explain that the situation is fundamentally different. In gravity, axion is a matter of consistency imposed by the $S$-matrix: Each gauge sector must include axion with exact relaxation of the corresponding $\theta$. We show that this favors an alternative and remarkably simple formulation of the axion, fully fixed by the gauge redundancy of QCD, without involvement of a global symmetry. The axion mechanism is a Higgs effect for the QCD 3-form, ensuring that physics is independent of $\theta$ to all orders in operator expansion. A near-future experimental detection of the neutron EDM will be an unambiguous signal of CP-violating physics beyond the Standard Model. The axion coupling is tied to the scale of gravity.

A. The message

In the standard discussion of the strong-CP puzzle, gravity plays no useful role. The puzzle is formulated as a naturalness problem of the following essence. QCD has a continuum of vacua \cite{1, 2}, conventionally labelled by the CP-violating vacuum angle $\theta$. These vacua belong to different superselection sectors. In fact, the physically measurable parameter is the quantity $\bar{\theta} = \theta + \arg \det M_q$, where $\arg \det M_q$ is the phase of the determinant of the quark mass matrix.

In quantum theory, $\bar{\theta}$ induces the electric dipole moment of neutron (EDMN) \cite{3, 4}. The comparison of the resulting theoretical value with the current experimental limit, $|d_n| < 2.9 \times 10^{-26} \text{ cm }$ \cite{5}, gives the bound,

$$|\bar{\theta}| \lesssim 10^{-9}. \quad (1)$$

Notice that there exists an additional contribution to EDMN, coming from the breaking of CP-symmetry by the weak interaction \cite{6, 7}. However, this correction is too small for affecting the bound \cite{11}.

Thus, the observations indicate that we live in a sector with a minuscule or zero $\bar{\theta}$. This is puzzling.

Axion \cite{8, 10} eliminates this vacuum structure by making $\bar{\theta}$ dynamical and by relaxing it to a CP-invariant ground state. In the original model by Peccei and Quinn \cite{11}, the axion emerges as a pseudo-Nambu-Goldstone boson of an anomalous global symmetry $U(1)_{PQ}$. However, this scenario is subjected to the following potential criticism.

First, it appears rather artificial, since axion is designated to do a single job. Its introduction is not justified by any visible consistency requirement.

Secondly, one may argue that the original puzzle is replaced by a new one, since the phenomenological bound \cite{11} requires an extraordinary precision of the global $U(1)_{PQ}$-symmetry, modulo the chiral anomaly of QCD. This is a mystery, since the global symmetries are not protected by any known fundamental principle. The puzzle is sharpened by the fact that the axion mechanism is based on the explicit breaking of this very symmetry by the QCD anomaly. It is then totally unclear what prevents the explicit breaking of $U(1)_{PQ}$ by other sources.

Thirdly, the theory suffers from the lack of predictivity. Due to the arbitrariness of the explicit breaking of $U(1)_{PQ}$-symmetry, the quantity $\bar{\theta}$ is uncalculable. Correspondingly, the EDMN cannot be predicted.

In the above picture gravity is either ignored or is viewed as a potential threat. This is because, at the level of an effective field theory (EFT), gravity has no obligation to respect global symmetries. Due to this, it is usually assumed that $U(1)_{PQ}$ is broken by the high-dimensional operators of gravitational origin. If so, they can jeopardise the axion mechanism.

The goal of the present paper is to argue that the situation is fundamentally different.

First, in gravity the absence of the strong-CP-violating $\bar{\theta}$-vacua is a consistency requirement. The most transparent reason is the $S$-matrix formulation of gravity \cite{12}. This formulation is incompatible with the existence of a non-degenerate landscape of CP-violating $\bar{\theta}$-vacua. This demands the existence of axions that relax the landscape towards the $S$-matrix vacua \cite{13}. As we shall show, the relaxation must be exact, persistent to all orders in the operator expansion.

Thus, at first glance, gravity provides the two contradictory messages. On one hand, it demands the exactness of the axion mechanism. On the other hand, it exhibits no respect for global symmetries, such as $U(1)_{PQ}$.

Within the standard Peccei-Quinn formulation of axion, the reconciliation of these two tendencies is difficult to understand.

In the present work, we show that this is accomplished by an alternative theory of axion, as previously discussed in \cite{14}. In this formulation, there exist no global symmetry. Instead, the axion, introduced as a 2-form $B_{\mu\nu}$, shares the gauge redundancy of gluons. With no further assumptions, the gauge redundancy fixes the structure of the theory making it fully insensitive to deformations caused by arbitrary operators.

In this theory, the axion mechanism is described as a Higgs-like effect for the Chern-Simons 3-form of QCD. Eating up the $B_{\mu\nu}$-axion, the 3-form becomes massive.
Correspondingly, the topological susceptibility of the vacuum (TSV) vanishes and $\tilde{\vartheta}$ becomes unphysical.

Due to gauge protection, the axion vacuum remains exactly $CP$-invariant under any continuous deformation of the theory. It is insensitive towards arbitrary sort of $UV$-physics \[14\]. We show that while the gravitational effects, such as the virtual black holes, may contribute into the masses of axion and $\eta'$-meson, they are unable to destabilize the axion mechanism. This guarantees the fulfillment of the constraint imposed by the gravitational $S$-matrix to all orders. No such protection exist in the ordinary Peccei-Quinn realization in which the axion represents a Goldstone phase of a complex scalar.

We explain this difference carefully. As shown in \[14\], the $B_{\mu\nu}$ formulation can be dualized to a theory of a pseudo-scalar axion $a$ with arbitrary shape of the potential. However, the caveat is that in the dualized theory the axion enters with an integration constant. This constant encodes the information about the gauge structure of the $B_{\mu\nu}$-theory.

Such an integration constant is absent in a generic EFT of a pseudo-Goldstone axion with arbitrary explicit breaking of $U(1)_{PQ}$. This is the key for understanding of how the protection by the gauge redundancy of the $B_{\mu\nu}$-theory is transferred to its exact duals. At the same time, this explains the lack of protection in the ordinary Peccei-Quinn case.

In the alternative theory, the axion emerges as an inseparable part of the gauge redundancy of QCD. This redundancy is the fundamental reason for the elimination of $\vartheta$-vacua. However, at the level of EFT, the axion decay constant $f_a$ enters as a free parameter. Although this formulation is independent of gravity, it is strongly motivated by it. Therefore, the painted picture hints towards the gravitational origin of the scale $f_a$.

The exactness of the axion mechanism has important phenomenological consequences. First, it endows the theory with the predictive power. Due to vanishing of $\vartheta$, the EDMN becomes calculable in terms of other parameters. For example, within the minimal setup, a sole source of EDMN is predicted to be the breaking of $CP$-invariance by the weak interaction. This contribution is calculable and is well below the sensitivity of current and prospective experiments (see, e.g., \[16\]).

This may sound disappointing but it is not. As a flip side of the coin, the EDMN becomes a direct probe for new physics. An experimental detection of EDMN will be an unambiguous sign of a $CP$-violating physics coming from beyond the Standard Model.

The discussion below is structured as follows. We first discuss the essence of the strong-$CP$ puzzle in the absence of gravity. We do this in the language of \[14\], which is most suitable for preparing the stage for gravity. Next, we discuss the role of gravity with various implications and conclude. In particular, we compare the effects of gravity on the Peccei-Quinn formulation of QCD axion and on alternative $B_{\mu\nu}$-formulation.

Although, it is assumed that the reader is familiar with the standard description of the strong-$CP$ problem and of its Peccei-Quinn solution, we try to make the presentation self-contained.

We note that, although the exactness of the axion mechanism is justified by the $S$-matrix formulation of gravity, most of our analysis and conclusions are independent of it. For example, the $3$-form formulation of the axion mechanism and its protection by the gauge redundancy \[14\] are universal. Likewise, the subsequent phenomenological implications for EDMN are valid regardless of gravity.

Note that throughout the paper all irrelevant numerical factors will be either set to one or rescaled in redefinitions of fields and parameters with each step. In this way we avoid carrying them around. The scale of the $3$-form theory, such as the QCD scale, unless shown explicitly, will also be set to one.

### B. 3-form vacua

The deep meaning of the strong-$CP$ problem and of its solution by axion is especially transparent in the language of the $3$-form gauge theory \[14\]. The advantage of this formulation is that it relies on basic properties of QFT, such as, the power of gauge redundancy and the spectral representation of correlators. Following \[14\], we first discuss the effect for a generic $3$-form and later focus on gauge theories such as QCD.

A massless $3$-form field $C_{\alpha\beta\gamma}$ propagates no degrees of freedom. The Lagrangian can be chosen as a generic algebraic function ($K$) of the gauge-invariant field strength $E_{\mu} \equiv \epsilon^{\alpha\beta\gamma} \partial_{\mu} C_{\alpha\beta\gamma}$,

$$ L = K(E) \, . \quad (2) $$

For example, in the simplest case we can take $K(E) = E^2$. In the absence of sources, there is no restriction on the $K$-function. In the presence of sources that satisfy a quantization condition, this function is restricted accordingly.

The massless $3$-form exhibits the following gauge redundancy,

$$ C_{\alpha\beta\gamma}(x) \rightarrow C_{\alpha\beta\gamma}(x) + \partial_{\alpha} \Omega_{\beta\gamma}(x) \, , \quad (3) $$

where $\Omega_{\beta\gamma}(x)$ is an antisymmetric $2$-form.

The equation of motion,

$$ \partial_{\mu} \frac{\partial K(E)}{\partial E_{\mu}} = 0 \, , \quad (4) $$

is solved by an arbitrary constant,

$$ E = E_0 = \text{constant} \, . \quad (5) $$

This represents a $3 + 1$ dimensional analog of the electric field in $1 + 1$ dimensional Schwinger model. The constant “electric field” $E_0$ is also analogous to an ordinary Maxwell electric field created by a charged capacitor.
plate located at the boundary. Of course, in case of the
3-form, the field \( E_0 \) is a Lorentz pseudo-scalar, with no
preferred direction. Due to these analogies, one can say
that a massless 3-form is in the “Coulomb” phase.

The constant electric field contributes into the vacuum
energy density \( \sim E_0^2 \). Thus, a massless 3-form creates a
continuum landscape of vacua with different values of the
vacuum energy density. These vacua define the distinct
super-selection sectors. No transitions between them are
possible.

The presence of dynamical sources allows some jumps.
By gauge redundancy (3), the source must be conserved.
The examples are the 2-branes or the axionic domain
walls. The part of the action describing a gauge invariant
coupling of \( C \) to such a source is

\[
Q \int dX^\alpha \wedge dX^\beta \wedge dX^\gamma C_{\alpha\beta\gamma} \equiv Q \int_{2+1} C ,
\]

where \( Q \) is the charge and the integration is performed
over the world volume of a 2-brane with the embedding
coordinates \( X^\mu \).

It is easy to see that across the source, \( E \) experiences
a jump determined by \( Q \). For example, for a static brane
coincident with \( z = 0 \) plane, the change is \( \Delta E \equiv E(z > 0) - E(z < 0) = Q \).

In general, the sources can obey the quantization con-
ditions, a la Dirac. This puts certain restrictions on the
kinetic function \( K(E) \).

In the presence of sources, some discrete transitions
among the vacua are possible. This splits the vacuum
landscape into families of vacua that can be connected via
quantum tunnelling. However, as long as the 3-form is
massless, the vacua are not degenerate and a continuum
of super-selection sectors is maintained.

In summary, a massless 3-form creates a continuous
landscape of non-degenerate vacua. These vacua are
physically distinct as they differ by the VEV of a gauge-
invariant order parameter \( E_0 \). This quantity is parity-
odd. Correspondingly, it breaks both \( P \) and \( CP \). Due
to this, different vacua have different strengths of \( CP \)-
violation. This is a physically measurable effect.

Notice that breaking of \( CP \) comes from the VEV and
therefore is spontaneous. This is important because a
spontaneous \( CP \)-violation cannot be eliminated by im-
posing a symmetry on the Lagrangian. Indeed, imposing
the exact parity symmetry implies that \( K(E) \) is an even
function of \( E \). For example, \( K(E) = \frac{1}{4} E^2 \). Neverthe-
less, the vacuum equation (11) is solved by an arbitrary
non-zero VEV \( E_0 \). All solutions are the legitimate vacua.
One can of course demand that \( E_0 = 0 \). However, this
is merely a choice of the vacuum, not more justified than
any other choice.

C. Higgsing of 3-form by axion

There exists an unique way for eliminating the vacua
with \( E_0 \neq 0 \). For this it is necessary to make the 3-
form massive. That is, the 3-forms must be in Proca (or
Higgs) phase. A massive 3-form propagates one degree
of freedom and is equivalent to a massive pseudoscalar.
The VEV of the corresponding electric field \( E_0 \), is strictly
zero.

The only known gauge invariant (and therefore con-
sistent) way of endowing the 3-form with the mass, is
via coupling it to an axion. At the level of the low en-
ergy EFT, the axion can be introduced in two ways. We
discuss the formulation in terms of a pseudoscalar field,
\( a(x) \), first.

As shown in [14], this theory is equivalent to a theory
of a massive axion with the potential that depends on an
integration constant. This integration constant is crucial
for keeping in tact the global structure of the theory.

The form of the potential is determined by the kinetic
function of the 3-form. The Lagrangian describing the
generation of mass for a 3-form is,

\[
L = \mathcal{K}(E) + \frac{1}{2} (\partial_a a)^2 - \frac{a}{f_a} E ,
\]

where \( f_a \) is an axion decay constant. Notice that the
axion shift by an arbitrary constant \( \alpha \),

\[
a \rightarrow a + \alpha ,
\]

results in a total derivative in the Lagrangian. This can
be written as the shift of the action by a boundary term,

\[
\delta S = \int_{3+1} a E = \alpha \int_{2+1} C ,
\]

where the last integration is taken over the boundary.
For generating a mass of \( C \), it is crucial that this global
shift symmetry is not broken by any additional source
(see later).

The equations of motion are,

\[
\partial_a \frac{\partial \mathcal{K}(E)}{\partial E} = \partial_\mu \frac{a}{f_a} ,
\]

\[
\Box a + \frac{1}{f_a} E = 0 .
\]

Integrating the first equation, we get,

\[
\frac{\partial \mathcal{K}(E)}{\partial E} = \frac{a - a_0}{f_a} ,
\]

where \( a_0 \) is an arbitrary integration constant. This is an
algebraic equation which expresses \( E \) as a function of the
r.h.s. of (12). Thus, we can write,

\[
E(z) = \text{inv} \frac{\partial \mathcal{K}(z)}{\partial z} ,
\]

where \( \text{inv} \) stands for the inverse-function and \( z \equiv (a - a_0)/f_a \). From the equation (11), it is clear that \( E(z) \) at
the same time represents the first derivative of the axion
potential with respect to axion,

\[
E(z) = \frac{\partial V(z)}{\partial z} .
\]
This allows to rewrite the axion equation in a conventional form,
\[ \Box a + \frac{\partial V(a - a_0)}{\partial a} = 0. \]  
(15)

The theory (17) is thus equivalent to a theory of a pseudoscalar with the following Lagrangian,
\[ L_a = \frac{1}{2} (\partial_{\mu} a)^2 - V \left( \frac{a - a_0}{f_a} \right). \]  
(16)

The potential is determined from the equations (13) and (14) which for convenience we combine in one equation
\[ \frac{\partial V(z)}{\partial z} = \text{inv} \frac{\partial K(z)}{\partial z} = E(z). \]  
(17)

Very important: The equation (14) guarantees that \( E \) vanishes at any extremum of the axion potential for any choice of \( a_0 \). Thus, the would-be superselection sectors disappear and there is an unique vacuum with \( E = 0 \). Notice, a discrete degeneracy among the vacua is still possible, but all of them have \( E = 0 \) and therefore conserve CP.

As emphasized in (14), the dependence of \( V(a - a_0) \) on the integration constant \( a_0 \) is crucial for carrying along the information about the invariance of the original theory (7) with respect to the axion shift symmetry (9).

In other words, the theory of a massive 3-form represents a theory of a pseudo-scalar with a special global structure of the vacuum. This structure is encoded in the integration constant \( a_0 \).

This global structure is not present for the generic pseudo-scalars. For example, the pseudo-Goldstone bosons obtained by introduction of arbitrary explicit breaking operators do not possess the integration constants. Correspondingly, such explicit-breaking operators cannot emerge from the couplings to 3-forms.

As we show later, this difference has important implications for the UV-completion of the theory of axion and for its dual formulation.

1. Examples

Let us demonstrate the above on examples with particular forms of the 3-form kinetic function.

For the simplest choice, \( K(E) = \frac{1}{2} E^2 \), the axion potential has the form, \( V(a) = m_a^2 (a - a_0)^2 / f_a \), with \( m_a^2 = f_a^{-2} \). The electric field is given by \( E = f_a (a - a_0) / m_a^2 \). The vacuum is at \( a = a_0 \), which implies \( E = 0 \).

Another example discussed in (14) is,
\[ K(E) = E \arcsin(E) + \sqrt{1 - E^2}. \]  
(18)

This form is important as it generates a periodic axion potential of the form,
\[ V(a) = - \cos \left( \frac{a - a_0}{f_a} \right). \]  
(19)

Again, in both cases, the presence of the integration constant \( a_0 \) is important for embedding such pseudo-scalars in a theory (17) with a 3-form and with the shift symmetry (9).

D. Dual formulation: Axion as gauge theory

As shown in (14), for a theory of axion (16) with an arbitrary potential \( V(a - a_0) \), there exists an exact dual formulation, in which axion \( a \) is replaced by an antisymmetric Kalb-Ramond 2-form \( B_{\mu \nu} \). The Lagrangian is,
\[ L = K(E) + m_a^2 (C - dB)^2, \]  
(20)

where \( dB \equiv \partial(a B_{\mu \nu}) \). This is equivalent to a theory of axion with the equation of motion (15), where the potential \( V(a - a_0) \) is determined from the function \( K(E) \) via (17).

This theory exhibits a gauge redundancy under which \( C \) and \( B \) transform as,
\[ C_{\alpha \beta \gamma} \rightarrow C_{\alpha \beta \gamma} + \partial_{[\alpha} \Omega_{\beta \gamma]}, \]
\[ B_{\alpha \beta} \rightarrow B_{\alpha \beta} + \Omega_{\alpha \beta}. \]  
(21)

On top of this, the theory exhibits an additional gauge redundancy, acting solely on \( B_{\mu \nu} \),
\[ B_{\mu \nu} \rightarrow B_{\mu \nu} + \partial_{[\mu} \xi_{\nu]}, \]  
(22)

where \( \xi_{\nu}(x) \) is an arbitrary one-form. Due to this gauge redundancy, \( B_{\mu \nu} \) propagates a single degree of freedom.

As explained in (14), the gauge symmetry ensures the protection of axion mechanism against arbitrary UV-physics. In particular, no continuous deformation of the theory can un-Higgs the 3-form.

This may sound rather puzzling, since it appears that in dual theory (17) the continuous deformation has a different effect. For example, it is easy to show that an explicit breaking of the axion shift symmetry (9) by an arbitrary additional potential \( V_{\text{exp}}(a) \), renders the 3-form massless.

The key to the puzzle, is in paying attention to the integration constant. A continuously deformed a-theory, without an accompanying integration constant \( a_0 \), cannot be dualized into a gauge invariant theory of \( B_{\mu \nu} \). That is, the only deformations of the axion potential permitted by duality are the deformations by the functions of \( a - a_0 \). Such deformations leave the 3-form in the Higgs phase. Correspondingly, they have gauge duals.

In order to see this, we perform the dualization as its done in (14). As the first step, we treat the field strength of \( B \) as a fundamental 3-form, \( X_{\alpha \mu \nu} \equiv \partial_{[\alpha} B_{\beta \mu \nu]} \), and impose the constraint,
\[ \tilde{d}X = e^{\alpha \beta \gamma \mu} \partial_{\alpha} X_{\beta \gamma \mu} = 0, \]  
(23)

via a Lagrange multiplier field \( a(x) \). The Lagrangian thus becomes,
\[ L = K(E) + m_a^2 (C - X)^2 + a \tilde{d}X. \]  
(24)
Integrating out $X$ through its equation of motion, we arrive to the theory (7), which describes the axion $a$ coupled to the 3-form $C$.

As the next step, we integrate out $C$ through the equation of motion (10). As we have already seen, this gives an effective theory of axion (16) with the potential $V(a-a_0)$ determined through the equation (17).

The above two-step process shows that the theory of $B_{\mu\nu}$ is dual to a theory of axion $a$ with the potential $V(a-a_0)$ that depends on $a-a_0$. The integration constant $a_0$ carries the information about the shift symmetry (9) of (7) and the gauge symmetry of $B_{\mu\nu}$-formulation. In this way, the global structure of the theory of $a$-axion, obtained via the dualization from the gauge theory of $\mu\nu$, differs from a generic theory of axion with the potential $V(a)$.

To conclude this section, let us summarize. A massless 3-form creates the superselection sectors parameterised by the VEV of its field strength $E_0$. The vacua have different energies. All except $E_0=0$ break parity and CP-symmetry. The breaking of symmetry is spontaneous, as it comes from the VEV. Due to this, the vacua cannot be forbidden by imposing $P$, $CP$ or any other discrete symmetry on the Lagrangian. As usual, the spontaneous breaking implies that the vacuum does not respect the symmetry of the Hamiltonian.

The only consistent way for eliminating the CP-violating superselection sectors, is via making the 3-form massive. This requires an axion.

The formulation (20) in terms of $B_{\mu\nu}$ makes the analogy with the Higgs effect very transparent. The 3-form gets a mass by “eating up” the 2-form $B_{\mu\nu}$ which plays the role of the St"uckelberg field.

Upon dualization and integration out of the 3-form, we arrive to a theory of a pseudo-scalar axion $a$ (16) with the potential $V(a-a_0)$ that is a function of $a-a_0$. This global dependence on the integration constant $a_0$ reflects the gauge redundancy of the $B_{\mu\nu}$-theory (20) and the respective shift symmetry (9) of the $a$-theory (7). The shape of the potential is determined by the kinetic function $K(E)$.

E. 3-form description of $\vartheta$ vacua of gauge theories

The superselection sectors (9) of a massless 3-form theory (2) are of direct relevance for the $\vartheta$-vacua of non-abelian gauge theories. As discussed in (14), the $\vartheta$-vacua in $SU(N)$ gauge theory, represent a particular case of the 3-form vacuum. Their structure is fully captured by the theory (2).

In $SU(N)$ gauge theory, the massless 3-form $C$ of (2) originates from the Chern-Simons 3-form,

$$C^{(CS)} = \text{tr}(A_{[\mu} \partial_{\nu]} A_{\alpha]} + \frac{2}{3} A_{[\mu} T_{\nu}^b A_{\alpha]}),$$  \hspace{1cm} (25)

where $A_{\mu} \equiv A^b_{\mu} T^b$ is an $N \times N$ gluon matrix and $T^b$ are the generators of $SU(N)$ with $b = 1, 2, \ldots, N^2 - 1$ the adjoint index. The trace is taken over the color indices. Under the $SU(N)$ gauge transformation, $A_{\mu} \rightarrow U(x) A_{\mu} U^\dagger(x) + U^\dagger \partial_{\mu} U$ with $U(x) \equiv e^{-i\vartheta(x) T^b}$, the 3-form shifts as (9) with $\Omega_{\mu\nu} = \text{tr} A_{[\mu} \partial_{\nu]} \vartheta$. The gauge invariant field strength is $E^{(CS)} \equiv dC^{(CS)}$.

In more traditional notations, $E^{(CS)} \equiv F_{\mu\nu} \tilde{F}^{\mu\nu}$, where $F$ is the gluon field strength and $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is its dual.

The choice of a particular $\vartheta$-vacuum is parameterized by the introduction of the following term in the Lagrangian,

$$\int_{3+1} \vartheta E^{(CS)} = \vartheta \int_{2+1} C^{(CS)}.$$  \hspace{1cm} (26)

In the last expression the integral is taken over the world-volume coordinates of the boundary. This form shows that $\vartheta$ represents a charge of the boundary under $C^{(CS)}$.

The correspondence between the $\vartheta$-vacuum and the 3-form vacuum goes through the TSV,

$$\langle C^{(CS)}, E^{(CS)} \rangle_{p \to 0} = \lim_{p \to 0} \int d^4x e^{ipx} (T[ E^{(CS)}, (E^{(CS)})]) = \text{const},$$  \hspace{1cm} (27)

where $T$ stands for time-ordering and $p$ is a four-momentum. As it is well-known, the existence of the $\vartheta$-vacua is equivalent to a non-vanishing value of the above correlator.

The language of TSV is most convenient for our purposes, since it is insensitive to particularities of physics that makes this correlator non-zero. In the original proposal by ’t Hooft, these are instantons. However, our discussion is equally receptive to any other source within QCD or beyond.

The important thing is that the existence of $CP$-violating $\vartheta$-vacua is equivalent to having a non-zero TSV, and vice versa. This information is sufficient for reaching our conclusions. The language of TSV makes is very transparent that the $\vartheta$-vacua represent the vacua with different VEVs of the Chern-Simons electric field $E^{(CS)}$.

In order to see this, let us first notice that the expression (27) implies the existence of a pole at $p^2 = 0$ in the correlator of the two Chern-Simons 3-forms. That is, the Källén-Lehmann representation of this correlator has the following form,

$$\langle C^{(CS)}, C^{(CS)} \rangle = \frac{\rho(0)}{p^2} + \sum_{m \neq 0} \frac{\rho(m^2)}{p^2 - m^2},$$  \hspace{1cm} (28)

where $\rho(m^2)$ is a spectral function. The key point here is that $\rho(0)$ is non-zero. Without loss of generality, we set it to one. The poles $m^2 \neq 0$ are separated from $m^2 = 0$ by finite gaps. This follows from the fact that, in the absence of massless quarks, the $SU(N)$ gluodynamics is a theory with a mass gap.

The entry with $m = 0$ does not violate this rule. This is due to the fact that it corresponds to a massless 3-form which contains no propagating degrees of freedom.
Despite this feature, the \( m = 0 \) component is very important, since it is the one responsible for the vacuum structure of QCD.

To summarize, from non-zero TSV \( \vartheta \neq 0 \) it follows that the Chern-Simons 3-form operator \( C^{(CS)} \) contains a massless 3-form field \( C \), plus the tower of massive fields,

\[
C^{(CS)} = C + \sum_{\text{massive modes}}.
\]

(29)

It is this massless 3-form \( C \) that gives rise to \( \vartheta \)-vacua.

For determining the vacuum state, the massive modes are irrelevant, as their contributions vanish in the zero momentum limit. That is, the massive modes (glueballs, or mesons) have zero occupation numbers in the vacuum state. This directly follows from the Poincaré invariance of the vacuum \( \vartheta = 0 \).

We thus reach a very important conclusion: The vacuum of the \( SU(N) \) theory is fully determined by the massless 3-form \( C \). The corresponding vacuum equation is given by \( (\ref{eq:2}) \) and is solved by an arbitrary constant \( E_0 \). This constant represents an order parameter of \( CP \)-violation by the vacuum of QCD. This is an exact statement.

Notice that we can think of a constant \( E_0 = \vartheta \) as of an electric field of a massless 3-form sourced by a boundary “brane” \( (\vartheta) \) with the charge \( Q = \vartheta \). The existence of a massless pole in \( (\ref{eq:26}) \) also makes it clear how a boundary term \( \vartheta F \tilde{F} \) can have a local physical effect.

In order to carry any information from the boundary to the local bulk, the existence of a massless 3-form field is necessary. In the opposite case, the boundary term would have no physical effect, since in a theory with only massive fields at large distances all correlators die exponentially.

As already discussed, each solution \( E_0 \neq 0 \) represents a separate vacuum state with no possibility of a transition into others. All of these vacua preserve the Poincaré invariance in form of the space-time translations, rotations and Lorentz boosts. However, the vacua with \( E_0 \neq 0 \) break \( P \) and \( CP \). It is clear that this breaking is spontaneous since it comes from the VEV.

The \( E_0 \)-vacuum landscape is physically equivalent to \( \vartheta \)-vacua. In particular, the CP conserving vacuum \( \vartheta = 0 \) is the one with \( E_0 = 0 \). In pure glue, \( \vartheta \sim E_0 \), in units of QCD scale, which for convenience we have set equal to one.

Obviously, the language of the 3-form \( [14] \) does not change the physics of \( \vartheta \)-vacua. However it provides an understanding through a different prism. This prism relies solely on TSV. This allows to better formulate the essence of the strong-\( CP \) puzzle and of the axion mechanism. This formulation is also very convenient for understanding the effect of gravity on strong \( CP \).

In this language, the puzzle originates from the fact that among all possible values of \( E_0 \), offered by QCD, we happen to live in the vacuum with an extraordinarily small \( E_0 \) (\( E_0 \lesssim 10^{-9} \), in QCD units). The theory with pure glue offers no proper explanation to this fact.

The situation changes upon the introduction of axion, which renders the 3-form massive. The effect is in Higgsing the 3-form. At the level of EFT, axion can be incorporated in two equivalent ways of \( a \) and \( B_{\mu\nu} \). These are described by the Lagrangians \( (\ref{eq:7}) \) and \( (\ref{eq:20}) \) respectively. The explicit nature of underlying physics that gives rise to these Lagrangians is unimportant for the efficiency of the mechanism. For example, in the original Peccei-Quinn realization, the coupling between the axion and \( E \) is generated through the fermion chiral anomaly. However, it may as well originate at more fundamental level, as shall be discussed later in the context of gravity.

One way or the other, the Higgsing of the 3-form eliminates all vacua with non-zero \( E \). As a result, the true vacuum in QCD is the \( CP \)-conserving state \( E_0 = 0 \).\(^2\)

We must note that the existence of the above vacuum structure has an experimental verification from the mass of the \( \eta' \)-meson. This will be discussed in a separate section. For now, we shall ignore the effect of the \( \eta' \)-meson.

1. \( P, CP \) and Strong-\( CP \)

Although this is secondary to our main topic, before moving to gravity, let us discuss whether the observed smallness of \( \vartheta \) can be explained by imposing a discrete symmetry on the Lagrangian. Such approaches have been proposed in the literature \([25–29]\). The general idea is to start with initial \( \vartheta = 0 \) by imposing a symmetry, usually \( P \) or \( CP \). Of course, these symmetries are broken in the electroweak sector. That breaking is assumed to be spontaneous, or at least soft. The model then is constructed in such a way that the dangerous contribu-

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\(^2\) The 3-form language of \([14]\) clarifies the claim of \([21]\) that by changing the order of limits in ordinary instanton calculation, one ends up with \( \vartheta = 0 \). In this approach one performs calculation in the finite volume and then takes it to infinity. In 3-form language the meaning of this is rather transparent. The finite volume is equivalent of introducing an infrared cutoff in form of a shift of the massless pole in \( (\ref{eq:28}) \) away from zero. This effectively gives a small mass to the 3-form. For any non-zero value of the cutoff, the unique vacuum is \( E_0 = 0 \) which is equivalent to \( \vartheta = 0 \). Other states \( E \neq 0 \) (corresponding to \( \vartheta \neq 0 \)) have finite lifetimes which tend to infinity when cutoff is taken to zero. In this way the \( \vartheta \neq 0 \) vacua are of course present but one is constrained to \( \vartheta = 0 \) by the prescription of the calculation. Thus, changing the order of limits by no means eliminates the \( \vartheta \)-vacua. As usual, when taking the limit properly, one must keep track of states that become stable in that limit. These are the states with \( \vartheta \neq 0 \) \( (E \neq 0) \), which become the valid vacua in the infinite volume limit. The effect is in certain sense equivalent to introducing an auxiliary axion and then decoupling it.
tions, such as the phase of the determinant of the quark mass matrix, are under control and are small.

This approach does not take into account the fact that \( \vartheta \) itself breaks the “protective” symmetry spontaneously. As we have seen, the order parameter of this breaking is the VEV of the “electric” field \( E_0 \). This VEV is arbitrary, regardless of the symmetry of the original Lagrangian. In this light, even if we put aside gravity, the validity of the above solution is questionable. One can come up with the following arguments pro and contra.

Contra:
The spontaneous breaking of a symmetry cannot be restrained by the same symmetry, since a solution of the theory need not respect the symmetry of the Hamiltonian. The physical \( \vartheta \) is a solution that depends on an integration constant \( E_0 \). Due to this, the imposition of a symmetry, such as \( P \) or \( CP \), on the Lagrangian, does not eliminate vacua with arbitrary \( \vartheta \), including \( \vartheta \sim 1 \). Thus, selecting \( \vartheta = 0 \) has no particular justification, since other vacua are equally legitimate.

Pro:
Due to absence of the transitions, one may as well argue that a choice of the vacuum is equivalent to a choice of the theory. One can therefore choose \( \vartheta = 0 \) by a symmetry.

The above ambiguity is a peculiarity of the superselection. As we shall see, gravity eliminates it, leaving the axion as the only viable solution.

F. Gravity

It has already been proposed that gravity necessitates the existence of axion \cite{13}. The key argument was that, in the opposite situation, the set of \( \vartheta \)-vacua would contain members with positive energy. In a hypothetical world without gravity this is not an issue. The problem is gravity. As argued previously \cite{12}, such vacua are inconsistent with its quantum effects. From here, the conclusion was reached that gravity is incompatible with \( \vartheta \)-vacua. Hence, the presence of axion is a necessary condition for a consistent embedding of a gauge theory in gravity. Below, we shall support this conclusion and shall further elaborate on it. However, we shall rely on a more general and powerful argument based on the \( S \)-matrix \cite{12}.

The current understanding of quantum gravity is fundamentally based on its \( S \)-matrix formulation. This puts the severe constraints on the vacuum landscape of the theory. In particular, it eliminates the possibility of de Sitter vacua, i.e., the vacua with positive energy densities. However, other cosmological space-times that do not asymptote to Minkowski vacuum, are equally problematic from the \( S \)-matrix perspective. This list includes cosmologies with either collapsing or the eternally inflating Universes.

The detailed arguments can be found in \cite{12}. Here, we shall take this knowledge as our starting point. The necessity of the \( S \)-matrix vacuum has far-reaching consequences for the \( \vartheta \)-vacua and the axion physics.

Let us first specify our framework. In what follows, we shall be working within an effective low energy description of some unspecified fundamental theory of gravity. We shall assume that this fundamental theory is formulated through the \( S \)-matrix. We shall restrict the asymptotic \( S \)-matrix vacuum to be Minkowski. No other assumptions will be made.

This framework puts severe restrictions on the field content of EFT. In particular, it excludes the massless 3-form fields. The reason is very general. As already explained, a massless 3-form leads to the existence of an infinite set of vacua. They are labeled by the VEV of the electric field \( E_0 \). These vacua are non-degenerate in energy and belong to distinct superselection sectors. Due to this, a massless 3-form produces an infinite number of superselection sectors of non-Minkowski vacua. Correspondingly, such a theory does not satisfy our \( S \)-matrix criterion and cannot be embedded in gravity.

Thus, only the massive 3-forms are compatible with gravity. This requires an axion. In other words, the \( S \)-matrix formulation demands that every 3-form is accompanied by an axion which Higgses it \cite{46}. Obviously, this restriction extends to a Chern-Simons 3-form of any non-abelian gauge theory.

We thus reach a surprisingly powerful conclusion. When coupled to gravity, \( \vartheta \)-vacua must be eliminated. That is, gravity demands the existence of axions, one per each gauge sector with a non-trivial topological structure. Among them, is of course QCD.

G. Necessity of exactness of axion mechanism

It is important to understand that the absence of \( \vartheta \)-vacua implies that the axion mechanism is exact. In the opposite case, the unwanted superselection sectors cannot be avoided.

In other words, the axion mass must come strictly from its mixing with the corresponding Chern-Simons 3-form as in \cite{7}. Any additional explicit breaking of axion shift symmetry, no matter how weak, is excluded. As shown in \cite{46}, such an explicit breaking “un-Higgses” the 3-form and re-introduces the set of non-degenerate \( \vartheta \)-vacua. This violates the \( S \)-matrix constraint.

It is easy to see this explicitly. Let the mass of the axion be generated entirely from Higgsing the QCD 3-form. This is described by the effective Lagrangian \cite{7}. In this case, the vacuum is given by the \( CP \)-invariant state \( E_0 = 0 \). In the ordinary language, this implies that \( \vartheta \) is unphysical.

\(^{3}\) The existence of a chiral fermion accomplishes the same goal, with axion being a composite of fermions. The example would be an \( \eta' \)-meson in QCD with a massless quark.
Following [46], let us now allow for some external contribution to the mass of the axion. We denote this increment by \( \mu^2 \). The corrected Lagrangian has the following form,
\[
L = \frac{1}{2} E^2 + \frac{1}{2} (\partial_a \vartheta)^2 - \frac{a}{f_a} E - \frac{1}{2} \mu^2 a^2 ,
\]
where for simplicity we have taken \( K(E) = \frac{1}{2} E^2 \).

The equation of motion for the 3-form is the same as in (10) and is solved by,
\[
E = \frac{a - a_0}{f_a} .
\]

As previously, \( a_0 \) is an integration constant which plays the role of the \( \vartheta \). In the absence of axion it would set the VEV of \( CP \)-odd field \( E_0 \). We can write \( \vartheta = \frac{a_0}{f_a} \).

From (31) we get the following effective potential for the axion,
\[
V(a) = \frac{1}{2} f_a^{-2} (a - a_0)^2 + \frac{1}{2} \mu^2 a^2 .
\]

Minimizing \( V(a) \), we get \( a = a_0 \frac{1}{1 + \mu^2 f_a^2} \). Plugging this into (31), we obtain the following expression for the VEV of the \( CP \)-violating electric field,
\[
E_0 = - \frac{a_0}{f_a} \frac{\mu^2 f_a^2}{1 + \mu^2 f_a^2} = - \frac{\vartheta}{1 + \mu^2 f_a^2} .
\]

We observe that for non-zero \( \mu^2 \), the VEV is proportional to the integration constant \( a_0 \).

In other words, for \( \mu \neq 0 \), the 3-form field becomes un-Higgsed. As a consequence, the superselection sectors with different VEVs \( E_0 \), re-emerge. This VEV measures the amount of \( CP \)-violation. Translated in the standard Peccei-Quinn language, for \( \mu \neq 0 \), the physical vacua with different values of \( \vartheta \) re-emerge.

At the same time, the energy densities in these vacua also depend on the integration constant, \( a_0 \), and correspondingly on \( E_0 \) or \( \vartheta \).

\[
V_{\min} = \frac{1}{2} \vartheta^2 \frac{\mu^2 f_a^2}{1 + \mu^2 f_a^2} = \frac{1}{2} E_0^2 \frac{1 + \mu^2 f_a^2}{\mu^2 f_a^2} .
\]

The lowest energy is in the \( CP \)-conserving vacuum. There is no possibility to have all the vacua in Minkowski. Thus, such a theory cannot satisfy our \( S \)-matrix constraint.

The absence of non-degenerate vacua requires \( \mu = 0 \). In this case, only the \( CP \)-invariant vacuum \( E_0 = 0 \) is present. Thus, we see that gravity excludes any amount of explicit breaking of axion symmetry.

We are learning that not only gravity demands the existence of axion per each 3-form, it also requires sort of a “fidelity” among the two. That is, the axion shift symmetry [53] must not be broken by any external source.

This is a rather curious result. Ordinarily, gravity is assumed not to respect global symmetries. However, the situation we are encountering here is more radical. It appears that not only gravity respects the axion mechanism, but also protects it from any external disturbance.

We shall argue that this is an indication that gravity favors the formulation of QCD axion in terms of \( B_{\mu \nu} \) field, described by the Lagrangian [7] [14]. In this formulation the protection is guaranteed by the QCD gauge redundancy. This gives a protective power to the theory which is lacking in the standard Peccei-Quinn formulation. Correspondingly, this also makes the theory predictive. Let us explain this point carefully.

**H. Gravity and the theory of axion**

The EFT of axion (20) formulated in terms of \( B_{\mu \nu} \) is dual to \( a \)-formulation given by (7) [14]. However, these EFTs are only valid below the scale \( f_a \). Above this scale, the axion requires an UV-completion. In particular, both theories must properly merge with gravity. In general, the UV-completions of \( a \) and \( B_{\mu \nu} \) are different. A clarification of this difference is important for understanding the influence of gravity on axion.

A generic EFT that includes gravity is characterized by a gravitational cutoff \( M_{gr} \). This is the scale above which the quantum gravitational effects become strong. In particular, gravity starts to dominate the scattering processes with the momentum-transfer per particle exceeding \( M_{gr} \). In general, the scale \( M_{gr} \) is lower than the Planck mass, \( M_P \). For now, we shall keep it otherwise free.

Above the scale \( M_{gr} \), any EFT faces the urgency of merging with quantum gravity. This is regardless whether other interactions are still weak at this scale. Obviously, the need for an ultimate completion in gravity applies to both formulations of axion. However, the two differ by the possible forms of an intermediate completion.

If we assume that \( f_a < M_{gr} \), then, above \( f_a \) and below \( M_{gr} \), the pseudo-scalar \( a \) can be UV-completed into a phase of a complex scalar field. This is the case in the original Peccei-Quinn model. However, in this completion the explicit breaking of axion symmetry can be achieved by a continuous deformation of the theory. On the other hand, this would be incompatible with the \( S \)-matrix constraint of gravity. Thus, such deformations must be absent in Peccei-Quinn. This must hold to all orders in field expansion. However, EFT provides no visible reason for such a protection.

In contrast, the \( B_{\mu \nu} \)-formulation is protected from such deformations by the power of the gauge redundancy [14]. The renormalizable completion of \( B_{\mu \nu} \) theory below the scale \( M_{gr} \) is not known. This implies that in this theory the axion constant \( f_a \) cannot be below \( M_{gr} \). This is the good news for the theory, since it gives a new correlation between the scales. At the same time, there is no harm to calculability, since below \( f_a \) the theory is weakly-interacting anyway. Quite the contrary, the bonus is the
explicit protection of axion by the gauge redundancy already at the level of EFT. This enables to predict $\tilde{\vartheta} = 0$.

Due to above, we conclude that gravity favors the $B_{\mu\nu}$-formulation of axion. Below, we consider the stability of the two formulations separately and compare them.

I. UV-completion in Peccei-Quinn

Let us assume that above the scale $f_a$ the axion $a$ is UV-completed in form of a Nambu-Goldstone phase of a complex scalar field $\Phi = \rho(x)e^{i\varphi(x)/f_a}$, where $\rho(x)$ is the modulus. The potential can be chosen as,

$$V(\Phi) = \lambda^2(\Phi^\dagger\Phi - f_a^2)^2.$$  

(35)

The axion scale $f_a$ is set by the VEV $\langle \rho \rangle = f_a$. In this theory the axion shift symmetry (9) is realized as a global $U(1)_{\text{PQ}}$ transformation of the complex field,

$$\Phi \rightarrow e^{i\alpha} \Phi,$$  

(36)

by an arbitrary constant phase $\alpha$.

The gravitational consistency condition demands that the sole source of the explicit breaking of $U(1)_{\text{PQ}}$ is the coupling of axion with $E$, as it is given in the effective Lagrangian (7). In the Peccei-Quinn scenario, this coupling is generated through the fermion anomaly. This is accomplished via coupling $\Phi$ to $N_f$ flavors of quarks $\psi_j$ (with $j = 1, 2, ..., N_f$), which transform under the chiral symmetry (36) as

$$\psi \rightarrow e^{-i\frac{1}{2}\alpha\gamma_5}\psi.$$  

(37)

In addition, the theory can include the set of $N_f'$ quark flavors, $\psi'_r$ (with $r = 1, 2, ..., N_f'$), which do not transform under the $U(1)_{\text{PQ}}$-symmetry. These quarks acquire their masses, $M'_r$, irrespective of the spontaneous breaking of $U(1)_{\text{PQ}}$.

In different realizations of the Peccei-Quinn scenario [42–45], the ordinary and/or exotic quarks are assigned differently to the two sets. This is a matter of the model-building which does not change the essence of the story. The Lagrangian that captures the key physics of the Peccei-Quinn framework has the following form,

$$L_\psi = i\bar{\psi}_j \gamma^\mu D_\mu \psi_j - g_j \bar{\psi}_j \bar{\psi}_j \psi_j +$$
$$+ i\bar{\psi}_r' \gamma^\mu D_\mu \psi'_r - M'_r \bar{\psi}_r' \psi'_r -$$
$$- F_{\mu\nu} F^{\mu\nu} - \vartheta F_{\mu\nu} F^{\mu\nu},$$  

(38)

where $D_\mu$ is the standard covariant derivative and $g_j$ are the Yukawa coupling constants. The quantity $\vartheta$ is the “initial” value of the boundary term which can be shifted by anomalous $U(1)_{\text{PQ}}$ transformation. As previously, all irrelevant coefficients are set equal to one.

After taking into account the chiral fermion anomaly and integrating over the instantons, the theory reduces to the following EFT of the QCD 3-form and the axion,

$$L = E \arcsin(E) + \sqrt{1 - E^2} + \frac{1}{2}(\partial_\mu a)^2 -$$
$$- \left(\frac{a}{f_a} N_f - \tilde{\vartheta}\right) E,$$  

(39)

where, $\tilde{\vartheta} = \vartheta + \arg det(M')$. We have chosen the form of the kinetic function (18) which gives the cos-type axion potential (19). In the standard computations, this form of the potential is obtained in the dilute instanton gas approximation. As already discussed, the 3-form picture shows why the mechanism works beyond this approximation, as the vacuum of the theory is totally insensitive to the form of $K(E)$. However, for the illustrative purposes, we shall work with (18) in order to make the full parallel with the standard picture. Any other form of $K(E)$, gives the exact same outcome.

The equation of motion (10) obtained from (39),

$$\partial_\mu \arcsin(E) = N_f \frac{\partial_\mu a}{f_a},$$  

(40)

is solved by,

$$E = \sin\left(\frac{N_f a - a_0}{f_a}\right).$$  

(41)

After plugging this into the equation (11) for $a$,

$$\Box a + \frac{N_f}{f_a} E = 0,$$  

(42)

we arrive to the following EFT of the axion,

$$L_a = \frac{1}{2}(\partial_\mu a)^2 + \cos\left(\frac{N_f a - a_0}{f_a}\right).$$  

(43)

Notice that the entire information about the parameter $\tilde{\vartheta}$, got absorbed into an over-all integration constant, $a_0$,

$$N_f \frac{a_0}{f_a} = \tilde{\vartheta}.\quad (44)$$

The 3-form language shows very clearly how the axion makes this quantity unphysical: The VEV of $E$ vanishes regardless of its value. Of course, this is fully confirmed by the minimization of the potential (43) which gives,

$$\tilde{\vartheta}_{eff} = N_f \frac{a}{f_a} - \tilde{\vartheta} = 0.$$

(45)

Obviously, the same outcome persists for a potential generated by an arbitrary form of $K(E)$.  

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4 We comment that this result can be viewed as a generalization of Vafa-Witten theorem (11), which shows that the global minimum of energy in QCD is $CP$-conserving. The 3-form language extends this conclusion to every extremum of the axion potential, since $E = 0$ at each.
The potential in (43) respects the anomaly-free $Z_{N_f}$ subgroup of $U(1)_{PQ}$, under which the axion transforms as,
\[ \frac{a}{f_a} \to \frac{a}{f_a} + \frac{2\pi}{N_f}, \] (46)
with integer $l$. Again, from (42) and (41) it is clear that, irrespective of the value of $a_0$, the vacuum is CP-invariant. In particular, all Poincaré-invariant solutions give the theory of $\Phi$-field, this contradicts to which the $\Phi$-field is coupled. At every scale, this is incompatible with the $S$-matrix constraint. Thus, due to the consistency requirement of gravity, all such $U(1)_{PQ}$-violating operators must vanish.

However, the dualization from the theory of axion formulated as the gauge theory of $B_{\mu \nu}$ field [14]. We have already discussed the $B_{\mu \nu}$-formulation of axion for a generic 3-form $C_{\mu \nu \alpha}$. The formulation for the QCD axion is very similar, with the role of $C$ played by the massless component of the Chern-Simons 3-form of gluons (24).

The idea is that, at the level of a fundamental theory, the QCD gauge fields are accompanied by a Kalb-Ramond field $B_{\mu \nu}$. This field shifts under the QCD gauge transformation as (21). This simple starting point fully dictates the theory and determines its predictive power.

Indeed, the same transformation acts on $C_{CS}$ as (43). Due to this, $B_{\mu \nu}$ enters the Lagrangian exclusively through a gauge invariant combination with the corresponding Chern-Simons 3-form $C_{CS}$

\[ \tilde{C}_{\alpha \beta \mu} \equiv (C_{CS} - f_a dB)_{\alpha \beta \mu}, \] (49)

where $f_a$ is a scale. Notice, the field $B$ is assigned a canonical dimensionality, whereas $C_{CS}$ has the dimensionality 3. Correspondingly, under the QCD gauge symmetry the two transform as

\[ C_{CS} \rightarrow C_{CS} + d\Omega, \quad B \rightarrow B + \frac{1}{f_a} \Omega, \] (50)

where, as previously, $\Omega_{\mu \nu} = tr A_{\mu} \partial_\nu z$. Of course, there also exists an additional redundancy (22) acting on $B_{\mu \nu}$.

Thus, in the dual formulation, the axion emerges as an intrinsic part of QCD that fully shares the gauge redundancy of gluons. This is the key to understanding the insensitivity of the QCD vacuum towards arbitrary local deformations of the theory.

Notice that we are making no assumption other than adding a single degree of freedom $B_{\mu \nu}$ to QCD. The gauge redundancy fixes the Lagrangian to be an arbitrary Poincaré invariant function of the gauge invariant quantities $\tilde{C}$ and $E \equiv \epsilon^{\mu \nu \alpha} \partial_\nu C_{CS}^{\mu \alpha}$. That is, one can simply start at some cutoff scale scale $M_{gr}$ with the ordinary QCD Lagrangian, $L_{QCD}$, with a single additional term fixed by the gauge symmetry (50),

\[ L = L_{QCD} + \tilde{\partial} F \tilde{F} + \frac{1}{f_a^2} (C_{CS} - f_a dB)^2. \] (51)

With no further input, this is a fully accomplished theory of axion which makes $\tilde{\partial}$ unphysical. The mechanism

\[ \tilde{\partial} F \tilde{F} \rightarrow \tilde{\partial} F \tilde{F} + \frac{1}{f_a^2} (C_{CS} - f_a dB)^2. \] (51)
is immune against the addition of arbitrary set of local gauge invariant operators. Correspondingly, it is stable against UV quantum corrections, coming from gravity or any other physics.

As already discussed, the high derivative operators, as well as, all the massive modes contained in the spectral representation of the correlator (28), can be set equal to zero without the loss of any information about the vacuum structure. Therefore, after integrating out all additional physics, the relevant part of the Lagrangian is fixed as,

\[ L = \mathcal{K}(E) + \tilde{\vartheta}E + \frac{1}{f_a^2}(C_{CS} - f_a dB)^2. \] (52)

The algebraic function \( \mathcal{K}(E) \) is taken to be a generic function. It is therefore free to accommodate all possible QCD and gravitational contributions that are algebraic in \( E \).

However, in order to make a clear connection with the standard language, we have singled out the linear term in \( E \). Of course, this is simply the standard boundary term, \( \tilde{\vartheta}E = \tilde{\vartheta}FF \), of the original theory (61).

In addition, we can add arbitrary powers and/or the mixed products of \( C \) and \( E \). Due to the gauge invariance, they leave the vacuum unchanged. In short, the vacuum of the theory is insensitive to the addition of arbitrary local operators to (52).

The above theory makes it very clear why physical observables, such as EDM, are independent of \( \tilde{\vartheta} \). Due to the gauge-redundancy, the 3-form is in the Higgs phase regardless the value of \( \tilde{\vartheta} \). Correspondingly, all the \( CP \)-odd observables, such as the VEV of \( E \), vanish. Let us make this more explicit.

As already explained, at low energies, the above theory reduces to an EFT of a would-be massless 3-form \( C \), contained in the operator expansion of \( C_{CS} \), coupled to the axion 2-form \( B \). Therefore, in order to find the vacuum of theory, we must replace \( C_{CS} \rightarrow C \).

All Poincaré-invariant solutions of the equations of motion,

\[ \partial_\mu \frac{\partial \mathcal{K}(E)}{\partial E} + \frac{1}{f_a^2} \epsilon_{\mu\nu\alpha}(C - f_a dB)^\nu\alpha = 0, \]
\[ \partial^\mu (C - f_a dB)_{\mu\nu\alpha} = 0, \] (53)

give \( E = 0 \). Correspondingly, the vacuum is exactly \( CP \)-conserving. The gauge invariance guarantees that QCD 3-form is in the Higgs phase regardless the form of the function \( \mathcal{K}(E) \). No gauge-invariant deformation of the theory can change this result.

For example, it is easy to check that the addition of arbitrary higher order terms in \( C \), leaves \( E = 0 \) as the vacuum state. This is in accordance with the general arguments of (14) showing that in \( B_{\mu\nu} \) formulation the axion vacuum is insensitive to arbitrary heavy physics. That is, the 3-form Higgs effect cannot be abolished by an arbitrary set of massive modes. This is the key for explaining the axion protection "miracle" in \( B_{\mu\nu} \)-theory.

One can explicitly check this statement by analysing the effect of arbitrary heavy sources on the 3-form propagator (14) (see, (32), (23), for more recent analysis of explicit examples.)

As shown in (14) the only gauge-invariant way of un-Higgsing the QCD 3-form, is the coupling of axion \( B_{\mu\nu} \) to an additional massless 3-form. In such a case, one linear superposition of the two 3-forms remains massless. This would leave one physical family of \( CP \)-violating \( \tilde{\vartheta} \)-vacua intact.

That is, in order to undo the axion mechanism, we must change the theory discontinuously, via coupling the axion to an additional 3-form \( C' \). In this way, the axion \( B_{\mu\nu} \) is "shared" among the two 3-forms,

\[ L = \mathcal{K}(E) + \mathcal{K}'(E') + \frac{1}{f_a^2}(C_{CS} + C' - f_a dB)^2, \] (54)

where \( E' \) is the field strength of \( C' \) and \( \mathcal{K}'(E') \) is its kinetic function. Of course, \( B_{\mu\nu} \) now transforms under the two independent gauge symmetries.

As a result, only one linear superposition, \( C_{CS} + C' \), of the two 3-forms is Higgsed. The orthogonal superposition, \( C_{CS} - C' \), remains massless. Correspondingly, in such a theory the \( \tilde{\vartheta} \)-vacua are physical.

By dualizing \( B \) to \( a \) and integrating out \( C' \), it is easy to see that we arrive to a theory of a pseudoscalar axion \( a \) coupled to \( E \), with the additional explicit breaking potential \( V'(a - a'_0) \). This potential is determined from the equation (17) applied to \( \mathcal{K}'(E') \). This gives,

\[ \frac{\partial \mathcal{K}'(E')}{\partial E'} = \frac{a - a'_0}{f_a}. \] (55)

However, it is very important that the information about the integrated-out 3-form is encoded in the integration constant \( a'_0 \) (14).

Instead, if one starts with the pseudo-Goldstone formulation of Peccei-Quinn axion with a fixed explicit-breaking potential (17), the additional integration constant would be abolished. This distinction is crucial for understanding why, unlike \( B_{\mu\nu} \) case, there is no explicit protection mechanism for the Peccei-Quinn formulation of the axion \( a \).

One of the consequences of the above is that the axion potentials with random explicit breaking terms (48) cannot be dualized to \( B_{\mu\nu} \) theory.

This shows a fundamental difference between the \( B_{\mu\nu} \)-formulation and the formulation of axion in form of the phase of a complex scalar \( \Phi \). In the latter completion, the operators (17) that explicitly break \( U(1)_{PQ} \) symmetry, are obtained by a continuous deformation of the theory, without changing its field content. In this light, the absence of such operators to all orders in \( \Phi \)-expansion looks mysterious.

In contrast, in \( B_{\mu\nu} \)-formulation the axion mechanism can only be jeopardised by the introduction of a new massless Chern-Simons 3-form and the respective change
of the gauge properties of the axion. However, this represents a discontinuous change of the theory already at the fundamental level.

The bottomline is that the dual formulation makes the protection of axion mechanism by gravity rather transparent. Gravity respects the 3-form Higgs effect of QCD due to gauge redundancy. This fulfills the necessary condition for the S-matrix formulation of gravity.

K. Confronting the two theories of gravity

Notice that no renormalizable UV-completion in terms of a complex scalar field exist for the $B$-formulation of axion. Instead, such a formulation likely requires the UV-completion directly in quantum theory of gravity. Thus, for $B_{\mu\nu}$-formulation the gravitational origin of axion is rather organic.

It is important to understand that the lack of known renormalizable UV-completion for $B_{\mu\nu}$ cannot be taken as a disadvantage as compared to the pseudo-scalar realization which can be embedded in the Peccei-Quinn model.

First, we must remember that the Peccei-Quinn theory is not UV-complete either. It can certainly remain in weak-coupling regime up until the gravitational cutoff, but the same is true about the $B_{\mu\nu}$-theory. In addition, the question of embedding in gravity cannot be avoided by either of the two theories.

Thus, the relevant question to be asked is by how much the UV-completion into gravity (or other physics) impairs the predictive power of the theory. Here, $B_{\mu\nu}$-theory is at a clear advantage. Being equipped by the gauge redundancy, its prediction $\vartheta = 0$ is respected by an arbitrary UV-physics.

No such predictive power is exhibited by the standard Peccei-Quinn realization which, with or without gravity, is fully sensitive to the explicit-breaking operators of the type $[\vartheta]$. This makes it impossible to predict the value of $\vartheta$.

L. Harmless gravitational contribution to axion potential

As we showed, in $B_{\mu\nu}$-theory, the axion mechanism is protected by the gauge symmetry to all orders in operator expansion. By no means this implies that the gravitational contribution to the axion mass is zero. Rather, it only implies that the gravitational contribution is not Higgsing the QCD 3-form and thereby respects the axion mechanism. Nevertheless, this contribution can play an important role in phenomenology as we now explain.

The relevant corrections can come in two possible forms:

- The non-derivative functions of the invariants $E$ and $\tilde{C}$;
- A direct gravitational contribution into the TSV of QCD.

The first category effectively reduces to a correction of the form of the kinetic function $\mathcal{K}(E)$. Without loss of generality, we can split this function as,

$$\mathcal{K}(E) = \mathcal{K}_{\text{QCD}}(E) + \mathcal{K}_{\text{Gr}}(E),$$

where $\mathcal{K}_{\text{QCD}}(E)$ accounts for pure-QCD contributions, whereas $\mathcal{K}_{\text{Gr}}(E)$ accounts the gravitational effects.

The contribution $\mathcal{K}_{\text{Gr}}(E)$ means that gravity generates some additional operators of $E$. This contribution can be non-zero, even if a direct gravitational contribution into the TSV of QCD vanishes. It is clear that $\mathcal{K}_{\text{Gr}}(E)$ will correct the axion potential without jeopardising the 3-form Higgs effect. The corrections can be estimated to the leading order in field expansion.

For example, a gravitationally generated operator $E^2$ suppressed by $M_0^4$ generates a correction $\sim \Lambda^4/M_0^4$ into the kinetic term of a canonically normalized 3-form. This translates as a relative correction to the axion mass of order $\Lambda^4/M_0^4$, which is phenomenologically insignificant.

However, potentially there exists a more important gravitational correction to the axion mass in form of the gravitational contribution into the TSV of QCD. Under this we mean a gravitational contribution into the correlator (27). This is important due to the fact that it corrects the spectral weight of the massless pole in the Källén-Lehmann representation of the 3-form correlator (28). This affects the mass of the axion as well as the mass of the $\eta'$-meson. As we shall see, this contribution can be more substantial than the one coming from the high dimensional operators contributing into (56).

One potential source of gravitational contribution in TSV can come from the tower of virtual black holes [46],

$$\langle E | E \rangle_{\text{Gr}} = \sum_{BH} \langle E |BH\rangle \langle BH|E \rangle + \ldots.$$  

This expression shares some similarity with the contribution to TSV from the glueball tower in large-$N$ QCD in the framework of the Witten-Veneziano mechanism [52, 53]. In this context, Witten showed that the non-zero TSV in pure glue can be understood as a result of integrating out a tower of glueballs.

Once we couple QCD to gravity, the black holes with all possible glueball quantum numbers are present. Since an off-shell glueball must have a non-zero overlap with a corresponding black hole, gravity is expected to produce the counterparts of the matrix elements of the pure QCD.

We can estimate the semi-classical part of this contribution using the arguments of [46]. A virtual black hole of entropy $S$ is expected to contribute $\sim e^{-S}$. Therefore, the dominant contribution comes from the black holes of the smallest entropy.

However, for the validity of the semi-classical estimate, the black hole tower must be cutoff by the scale $M_{\text{Gr}}$ beyond which the semi-classical picture is not applicable.
In a theory with $N_{sp}$ particle species, this cutoff is given by the species scale \cite{47},

$$M_{gr} = \sqrt{\frac{N_{sp}}{M_{P}}}.$$  \hspace{1cm} (58)

Correspondingly, the entropy of a cutoff-size black hole in Einstein gravity is $S = N_{sp}$.

Thus, in a theory with $N_{sp}$, the gravitational contribution to the TSV of QCD is expected to be exponentially suppressed as,

$$\langle E, E \rangle_{Gr} \propto e^{-N_{sp}}.$$ \hspace{1cm} (59)

 Independently, there exist a phenomenological bound which comes from the mass of the $\eta'$-meson. This mass would exceed the experimental value, if TSV would exceed the scale of QCD. Thus, the gravitational contribution must be below the QCD scale. The fact that the estimated contribution \cite{59} is exponentially small, goes in the right direction. Beyond this, it is hard to put a constraint on \cite{57}.

Thus, at the current stage of our understanding, gravity could contribute non-negligibly into the mass of the $\eta'$-meson. It is hard to better bound this contribution due to the fact that the QCD contribution into the TSV has not been computed fully.

Through the same channel, gravity would simultaneously contribute into the mass of the QCD axion. Of course, none of the above contributions affect the Higgs mechanism of the QCD 3-form. Correspondingly, they have no effect on exact CP-invariance of the QCD vacuum. $\vartheta$ remains unphysical.

M. Gravitational TSV

One of the implications of the S-matrix criterion is that the TSV must vanish in each gauge sector of the theory. In particular, this applies to the TSV of gravity. This is the case of special interest due to the universal nature of gravity. The correlator has the following form,

$$\langle R \tilde{R}, R \tilde{R} \rangle_{p \to 0} \equiv \lim_{p \to 0} \int d^4 x e^{ipx} \langle T[R \tilde{R}(0), R \tilde{R}(x)] \rangle,$$ \hspace{1cm} (60)

where $R$ is Riemann tensor and $\tilde{R}$ its dual. The invariant $RR$,

$$R \tilde{R} = \epsilon^{\alpha\beta\mu\nu} \partial_\alpha C^{(G)}_{\beta\mu\nu},$$ \hspace{1cm} (61)

represents a field strength of the gravitational Chern-Simons 3-form,

$$C^{(G)}_{\mu\nu\alpha} \equiv \text{tr}(\Gamma_\mu \partial_\nu \Gamma_\alpha) + \frac{2}{3} \Gamma_{[\mu} \Gamma_\nu \Gamma_{\alpha]}.$$ \hspace{1cm} (62)

As explained in \cite{14}, if the correlator \cite{60} were a non-zero constant, this would imply that $C^{(G)}_{\mu\nu\alpha}$ contains a massless 3-form. The argument is identical to the one given in the case of QCD and will not be repeated here.

As already discussed, such a 3-form would produce the superselection sectors with different VEVs of $R \tilde{R}$. These vacua would have different energies and would violate CP-symmetry spontaneously. They would serve as the gravitational analogs of the QCD $\vartheta$-vacua. Their existence would conflict with the S-matrix criterion. Thus, the necessary condition is that the gravitational TSV is zero.

As discussed in \cite{46}, one potential source contributing to the correlator \cite{60} is a tower of virtual black holes. The reasoning parallels the one given in the previous section in the estimate of black hole contribution in the TSV of QCD. Similarly to that estimate, the semi-classical black hole contribution to \cite{60} is expected to be exponentially suppressed by $e^{-N_{sp}}$. Of course, there may exist other sources for the TSV of gravity. The important message is that due to the S-matrix constraint, the sum over all contributions must be zero.

That is, if it happens that the contribution to \cite{60} from non-perturbative effects such as the virtual black holes is non-zero, this contribution must be exactly cancelled by some other physics. For this purpose, gravity must be accompanied by a designated axion or a chiral fermion.

An interesting candidate is neutrino. A neutrino with zero tree-level Yukawa coupling would make the gravitational $\vartheta$ unphysical due to the anomalous chiral transformation \cite{14, 20}. This would be similar to the way a massless chiral quark makes $\vartheta$ unphysical in QCD. In this process, the neutrino would get an effective mass, similarly to the mass of $\eta'$-meson in QCD. It has been speculated \cite{20} that this phenomenon can be the origin of small neutrino masses in the Standard Model.

The need for an additional axion, due to a mixed instanton contribution, was also discussed in \cite{22}. Again, this source can be potentially problematic, provided it contributes to \cite{60}. In such a case it must be removed either by an additional axion or by a chiral rotation of a would-be massless neutrino, as described in \cite{13, 20}.

It also has been argued that gravity may contain a candidate for extra axion in form of torsion \cite{21}.

One way or the other, the S-matrix formulation demands that TSV of gravity is strictly zero. This is a prediction from quantum gravity.

N. Gravitational origin of $f_a$

From the fact that the presence of axion is imposed by gravity, we are naturally lead to the conclusion that $f_a$ is not independent of the scale of gravity. This expectation is especially supported by the $B_{\mu\nu}$-formulation of axion, which has no renormalizable UV-completion. As already discussed, in this formulation $f_a$ is around or above the gravitational cutoff $M_{gr}$. On the other hand, the relation
between \( M_g \) and \( M_P \) is theory-dependent. For example, as we know, the particle species impose an universal upper bound on \( M_g \) given by \( \lesssim 10^{12} \text{GeV} \).

Taking this as a crude guideline, we can make some estimates. In the minimal case, counting only the number of the Standard Model particle species plus graviton (around 120), we put \( f_a \) not far below the Planck mass. Of course, at the level of the present discussion it is hard to come up with a precise relation.

In this respect, two comments are in order.

First, \( f_a \) can easily be as large as \( M_P \) without any conflict with cosmology. As shown in [48], the so-called standard cosmological upper bound, \( f_a \lesssim 10^{12} \text{GeV} \), is removed within the inflationary scenario due to the fact that QCD could become strong in the early epoch. This would lead to an efficient dilution of the energy of the axion’s coherent oscillations. In this light, the axion can be cosmologically harmless, or even be a viable dark matter candidate, for \( f_a \) as high as \( \sim M_P \).

Of course, having \( f_a \) around the Planck scale would make the direct searches of the QCD axion very difficult. However, in a generic theory, the scale \( M_g \), and therefore \( f_a \), can be much stronger suppressed relative to \( M_P \). The important message however is that \( f_a \) cannot be below the cutoff of EFT.

\[ \theta \]

**O. Implications for non-axion approaches to strong-\( CP \)**

We briefly comment on the impact of our analysis on non-axion approaches to the strong-\( CP \) problem. Even though they may offer a way of selecting a vacuum with small \( \theta \), the non-degenerate \( \theta \)-vacua of the full theory are still maintained. As explained, this structure is incompatible with the \( S \)-matrix.

We can split non-axion approaches in two categories:

- The models based on symmetries;
- The models based on cosmological selection of the desired \( \theta \)-vacuum via dynamical and/or statistical mechanisms.

The earliest proposals from the first category are based on \( P \) or \( CP \) symmetry \[ 25, 30 \]. In these models the vacuum with small \( \theta \) is selected by imposing a discrete symmetry, \( P \) or \( CP \), on the initial Lagrangian.

As we discussed, already at non-gravitational level, the solution is ambiguous, since \( \theta \) breaks parity and \( CP \) spontaneously. However, gravity introduces a whole new dimension into the issue. Even if we somehow select a vacuum with favored value of \( \theta \), the other vacua are still part of the full theory. Such a theory cannot be embedded in gravity due to the \( S \)-matrix constraint. The additional vacua must be eliminated and this requires an axion. Correspondingly, the non-axion selection mechanism becomes redundant.

A qualitatively different approach \[ 31 \], is based on permutation symmetry between the large number of the hidden copies of the Standard Model. This framework was introduced earlier \[ 47 \] as the solution to the Hierarchy Problem, using the fact that a large number of hidden copies lowers the cutoff \( M_g \) to the scale of species \[ 58 \]. It was argued in \[ 31 \] that, as a byproduct, this setup simultaneously lowers the value of \( \theta \).

Notice that the permutation symmetry imposed on the initial Lagrangian does not imply the equality of the \( \theta \)-terms in different copies of QCD. This is because - just as they do with \( P \) and \( CP \) - these terms break the permutation symmetry spontaneously. However, the sum of the \( \theta \)-parameters over all sectors is bounded from above by the universal cutoff \( \lesssim 10^{12} \text{GeV} \). It was then shown that statistically most probable value of the \( \theta \)-term comes out to be close to its phenomenological bound \[ 11 \]. Nevertheless, this approach does not eliminate the non-degenerate \( \theta \)-vacua fully. It is therefore incompatible with the \( S \)-matrix.

The second category of non-axion models, is based on a dynamical cosmological selection of the \( \theta \)-term. To our knowledge, the earliest representative of this idea is \[ 32 \]. This model employs a so-called “attractor” mechanism, originally introduced in \[ 33, 34 \] for the cosmological relaxation of the Higgs mass. Although, in the scenario of \[ 32 \] \( \theta \) is a dynamical field, unlike axion, its evolution is dominated by heavy physics. However, this physics is back-controlled by \( \theta \) in the way that makes \( \theta \to 0 \) into an attractor of the cosmological evolution. Unfortunately, this scenario does not eliminate vacua with other values of \( \theta \). Many of these are long lived (eternally inflating) de Sitter vacua. Due to this, they are in conflict with the \( S \)-matrix constraint.

Another representative of the cosmological selection is the scenario of \[ 35 \]. This model is free from the presence of the eternal de Sitter vacua. However, instead it incorporates the cosmological branches with asymptotically-collapsing anti de Sitter space-times. These too are incompatible with the \( S \)-matrix.

We must comment that despite the above difficulties, the issue with cosmological relaxation is not closed. One could envisage some modification of the scenarios of the type \[ 32 \] or \[ 35 \] in which the unwanted vacua would only exist temporary and would not affect the asymptotic future. This can in principle be reconciled with the \( S \)-matrix. An interesting question is whether such “refined” scenarios, effectively become equivalent to an ordinary axion solution.

There exist number of other interesting representatives from both categories, which are not reviewed here due to lack of space. The important thing is to draw a general lesson: The approaches that somehow select a desired vacuum with a specific value of \( \theta \), while maintaining the vacua with other values of \( \theta \), are problematic for coupling to gravity.
P. Role of $\eta'$

We wish to touch upon the role of the $\eta'$-meson. Let us imagine a situation in which at least one of the quarks, in the Standard Model possesses an axial symmetry \([37]\), exclusively broken by the QCD anomaly. Of course, this would require a strictly zero Yukawa coupling with the Higgs field, as well as, the absence of all non-invariant higher dimensional operators. This appears to be excluded phenomenologically \([54]\) (for more recent analysis, see, \([55]\), and references therein). Nevertheless, the regime of massless chiral quarks is very useful for the theoretical understanding.

In such a situation there would be no need for the axion. Or, to be more precise, the role of the axion would be taken-up by the $\eta'$-meson \([14]\). The axial symmetry of the massless quark would play the role of $U(1)_{PQ}$-symmetry. This symmetry is spontaneously broken by the quark condensate.

As ’t Hooft showed \([15]\), the corresponding would-be Goldstone boson, $\eta'$ gets a mass from instantons through its anomalous coupling with $FF$. As it is well-known, this solves the $U(1)$-problem. However, if one of the quarks were massless, simultaneously, the $\eta'$ would solve the strong-CP problem, exactly the way this is done by the axion.

The generation of $\eta'$ mass can also be understood in terms of TSV \([52, 53]\), without an explicit reference to instantons.

In the real world description of the Standard Model, the axial symmetry of quarks is explicitly broken by their Yukawa couplings. Correspondingly, $\eta'$ cannot eliminate $\vartheta$-vacua and axion is necessary. However, it still plays an important role for our analysis, as it represents an experimental proof of the existence of the $\vartheta$-vacua.

This fact is very important for eliminating a potential loophole in our statement that axion is necessary for the $S$-matrix formulation of gravity. Without the $\eta'$-meson, the following loophole can be considered\([16]\). Let us restrict the Hilbert space of the theory to a particular $\vartheta$-vacuum which satisfies the $S$-matrix criterion. At the same time, we exclude all others. In other words, we imagine that, instead of explaining their absence dynamically, via axion relaxation, gravity instructs us to simply forget all values of $\vartheta$ that do not correspond to Minkowski vacua.

The mass of the $\eta'$ is the evidence that gravity did not take this path. It represents a proof of the existence of the $\vartheta$-vacua as part of the theory. The coherent excitations of the $\eta'$-field, are nothing but the excursions into the neighbouring $\vartheta$-vacua.

Notice that $\eta'$ somewhat reduces the amount of $CP$-violation triggered by the $\vartheta$-vacua. In the language of the 3-form description, the effect of $\eta'$ can be understood as a “partial-Higgsing”. This is accounted by the effective Lagrangian of the type \([30]\) in which $a$ is replaced by $\eta'$,

\[
L = \frac{1}{2}E^2 + \frac{1}{2}(\partial_\mu \eta')^2 - \frac{\eta'}{f_\eta}E - \frac{1}{2}\mu^2 \eta'^2. \tag{63}
\]

The decay constant is of order the QCD scale $f_\eta \sim 1$. The quantity $\mu^2$ accounts for explicit breaking of chiral symmetry by non-zero quark masses. It vanishes in the chiral limit when at least one of the quarks is taken massless. Since some quarks are lighter that the QCD scale, we have $\mu^2 \ll f_\eta$.

Integrating out the 3-form through its equation of motion, we get,

\[
E = \frac{\eta'}{f_\eta} - \vartheta, \tag{64}
\]

where $\vartheta$ is an integration constant. Plugging this into the equation for $\eta'$, we obtain the following expressions for its VEV,

\[
\eta' = \frac{\vartheta}{1 + \mu^2 f_\eta^2}, \tag{65}
\]

and the mass,

\[
m^2_{\eta'} = f_\eta^{-2} + \mu^2. \tag{66}
\]

It is clear that the contribution of the explicit breaking-term is subdominant as compared to the contribution from the 3-form.

Due to this, the contribution from $\vartheta$ into the electric field is partially cancelled. This can be easily seen by plugging \((65)\) into \((64)\), which gives,

\[
E_0 = -\vartheta \frac{\mu^2 f_\eta^2}{1 + \mu^2 f_\eta^2}. \tag{67}
\]

It is clear from both \((64)\) and \((67)\) that in the limit of a chiral quark, $\mu^2 = 0$, the $\eta'$ fully eliminates the dependence on $\vartheta$ and we end up with an unique $CP$-invariant vacuum, $E = 0$. In this limit $\eta'$ becomes an axion.

We would like to remark that the above generation of the mass of $\eta'$ can also be understood in purely topological terms \([52]\).

In the real world, due to non-zero quark masses, the $\eta'$-meson cannot fully eliminate the $CP$-odd electric field $E$. Correspondingly, the $\vartheta$-vacua persist and $\vartheta$ is physical. As a result, the S-matrix criterion requires an axion.

Q. EDM of neutron

Within the Standard Model, the QCD contribution to the electric dipole moment of neutron (EDMN) is controlled by the quantity $\vartheta$ \([2, 4]\). The role of this parameter in 3-form formulation is taken up by the integration constant, $a_0/f_a \equiv \vartheta$. The axion renders this quantity unphysical, provided the axion shift symmetry \([19]\) is broken.

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\(\text{\footnote{We thanks Otari Sakhelashvili for raising this question.}}\)
exclusively by the QCD anomaly. In such a case the QCD contribution to EDMN vanishes.

Within the Standard Model, there exists an additional contribution to EDMN coming from the electroweak breaking of CP-invariance \( \bar{\theta} \). However, the predicted value, \( |d_n| < 2.9 \times 10^{-32}\text{cm} \), is unmeasurably small. It is minuscule as compared to the current experimental limit \( |d_n| < 2.9 \times 10^{-26}\text{cm} \). It is also hopeless to be detected in a foreseeable future (e.g., see \( \bar{\theta} \)).

However, within the standard Peccei-Quinn approach the above cannot be taken as a prediction, since the global \( U(1)\) symmetry can be explicitly broken, arbitrarily. This introduces an uncalculable \( \bar{\theta} \)-dependent contribution to EDMN.

We have argued that gravity makes such a breaking impossible. That is, in gravity, the axion mechanism must remain fully undisturbed. Correspondingly, \( \bar{\theta} \) must be strictly unphysical.

Somewhat surprisingly, this emerges as an absolute statement. However, it is a direct consequence of a more fundamental absolute statement: The exactness of the axion mechanism imposed by gravity. We have explained that this exactness can be understood as a consequence of the QCD gauge redundancy \( \bar{\theta} \). This is intrinsic in the gauge formulation of the axion mechanism described as the 3-form Higgs effect \( \bar{\theta} \). In this theory, the independence of physics on \( \bar{\theta} \) is a prediction.

This implies that there is no contribution to EDMN from \( \bar{\theta} \). The phenomenological lesson from here is that any experimental indication of non-zero EDMN will be a probe of new physics beyond the Standard Model. In particular, such evidence will be explainable neither in terms of the effective \( \bar{\theta} \) nor in terms of the weak CP-violation. The former is strictly zero, whereas the latter \( \bar{\theta} \) is by four orders of magnitude below the prospective experimental sensitivity \( \bar{\theta} \). Due to this, the non-zero signature of EDMN can only come from new physics.

The bottomline is that EDMN represents an unambiguous experimental probe of new physics. In this light, it is important to analyse the contributions to EDMN from the motivated extensions of the Standard Model. Such analysis can be performed \( \bar{\theta} \) directly within the EFT of chiral Lagrangian in the spirit of \( \bar{\theta} \). For a reader less interested in gravity, the results of this chapter can still be useful. It addresses the question of EDMN in the presence of exact axion mechanism, irrespective of its origin. The arguments indicating that

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are insensitive to what guarantees the protection of the axion mechanism.

\(^7\) We are grateful to Goran Senjanovic who for a long time was insisting on the need of such a clarification. We thank Otari Sakhelashvili and Akaki Rusetsky for valuable discussions.

R. Summary and outlook

The present paper brings the two main points. First, gravity demands the existence of an “undisturbed” axion per each gauge sector. Secondly, due to this, gravity favors the theory of the QCD axion based exclusively on the gauge redundancy of gluons, without involvement of a global \( U(1)\) symmetry.

This solution to strong-CP puzzle was studied earlier in \( \bar{\theta} \), where it was shown to be stable against deformations by arbitrary local operators. It therefore emerges as a viable setup that reconciles the two seemingly-conflicting tendencies of gravity:

1) The absence of \( \bar{\theta} \)-vacua, required for the consistency of the S-matrix formulation;

and

2) no respect towards the global symmetries.

The first requirement demands the presence of axions with exact relaxation mechanisms. This is difficult to achieve in the standard Peccei-Quinn setup, since the global \( U(1)\) symmetry need not be respected by gravity.

In contrast, as shown in \( \bar{\theta} \) and discussed at length here, the \( B_{\mu\nu}\)-theory, which does not rely on any global symmetry, is fully immune to the gauge redundancy. In this theory the axion relaxation is substituted by the Higgs-like effect in which the QCD Chern-Simons 3-form becomes massive by eating-up the \( B_{\mu\nu}\)-axion. This relaxes the \( \bar{\theta} \)-vacua to the ground state in which all the CP-odd observables vanish. Stability of the 3-form Higgs phase is guaranteed by the gauge redundancy of QCD. This ensures the exactness of the mechanism to all orders in operator expansion.

Besides, the gauge theory of \( B_{\mu\nu}\) has number of advantages over the standard approach. First, it is remarkably simple. All one needs to do, is to introduce a single degree of freedom, \( B_{\mu\nu}\), with a proper gauge charge under QCD.

With no further assumptions, the theory takes care of itself. Due to the gauge redundancy, the structure and the outcome of the theory are determined unambiguously.

Secondly, the theory has a predictive power, implying \( \bar{\theta} = 0 \). This goes in contrast with the standard Peccei-Quinn setup in which \( U(1)\) is not protected by any fundamental principle. Correspondingly, there \( \bar{\theta} \) is not calculable.

Due to the exactness of the axion mechanism, an important phenomenological prediction is that the QCD contribution to EDMN, which is set by \( \bar{\theta} \), is exactly zero. At the same time, the contribution from the electroweak CP-violation is by six orders of magnitude below the current experimental limit \( \bar{\theta} \) and by four orders magnitude below the aim of the planned experiments (see, e.g., \( \bar{\theta} \)). Due to this, unless the precision is improved substantially, the electroweak contribution to EDMN cannot be measured. Within such sensitivity, any detection of
EDMN will be a signal of a new CP-violating physics, coming from beyond QCD and weak interactions.

There are number of other implications. Through its $S$-matrix formulation, quantum gravity predicts that TSV must vanish in each sector of the theory. In particular, the TSV of gravity must also be zero. In this light, it is important to understand whether some non-perturbative entities, such as virtual black holes \cite{Gonzalez-Tarrago2011}, contribute non-trivially to this correlator. If the answer is positive, this would predict the existence of a new degree of freedom - the second (gravitational) axion or a chiral fermion - designated for an exact cancellation of such contributions into the gravitational TSV \cite{Gonzalez-Tarrago2010}.

The standard model contains an interesting candidate in form of neutrino \cite{Gonzalez-Tarrago2010, Gonzalez-Tarrago2011}. In the absence of a tree-level Yukawa couplings for one of the neutrino species, the gravitational TSV would identically vanish due to the gravitational anomaly in the chiral neutrino current.

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