REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS RELATED TO THE REEB VECTOR FIELD

YOUNG JIN SUH

Abstract. In this paper we give a characterization of real hypersurfaces in noncompact complex two-plane Grassmannian $SU_{2,m}/SU_2U_m$, $m \geq 2$ with Reeb vector field $\xi$ belonging to the maximal quaternionic subbundle $Q$. Then it becomes a tube over a totally real totally geodesic $\mathbb{H}^n$, $m = 2n$, in noncompact complex two-plane Grassmannian $SU_{2,m}/SU_2U_m$, a horosphere whose center at the infinity is singular or another exceptional case.

Introduction

Let us denote by $SU_{2,m}$ the set of $(m + 2) \times (m + 2)$-indefinite special unitary matrices and $U_m$ the set of $m \times m$-unitary matrices. Then the Riemannian symmetric space $SU_{2,m}/SU_2U_m$, $m \geq 2$, which consists of positive definite complex two-planes in indefinite complex Euclidean space $\mathbb{C}_{m+2}^m$ (See page 315, Besse [2]), has a remarkable feature that it is a Hermitian symmetric space as well as a quaternionic Kähler symmetric space. In fact, among all Riemannian symmetric spaces of noncompact type the symmetric spaces $SU_{2,m}/SU_2U_m$, $m \geq 2$, are the only ones which are Hermitian symmetric and quaternionic Kähler symmetric. So we will say such a Hermitian symmetric space of noncompact type $SU_{2,m}/SU_2U_m$ a complex hyperbolic two-plane Grassmannian.

The existence of these two structures leads to a number of interesting geometric problems on $SU_{2,m}/SU_2U_m$, one of which we are going to study in this article. To describe this problem, we denote by $J$ the Kähler structure and by $\mathfrak{J}$ the quaternionic Kähler structure on $SU_{2,m}/SU_2U_m$. Let $M$ be a connected hypersurface in $SU_{2,m}/SU_2U_m$ and denote by $N$ a unit normal to $M$. Then a structure vector field $\xi$ defined by $\xi = -JN$ is said to be a Reeb vector field.

Now let us denote by $TM$ the tangent bundle of $M$. Then the maximal complex subbundle of $TM$ is defined by $\mathcal{C} = \{ X \in TM \mid JX \in TM \}$, and the maximal quaternionic subbundle $Q$ of $TM$ is defined by $Q = \{ X \in TM \mid \mathfrak{J}X \in TM \}$, where $\mathfrak{J} = \text{Span}\{J_1, J_2, J_3\}$, and $\{J_1, J_2, J_3\}$ denotes the quaternionic Kähler structure. The main subject we want to discuss in this paper is: What can we say about

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\end{itemize}
real hypersurfaces in $SU_{2,m}/S(U_2 U_m)$ with the Reeb vector field belonging to the maximal quaternionic subbundle $Q$?

Before going to do this, we introduce some hypersurfaces in $SU_{2,m}/S(U_2 U_m)$ with invariant maximal complex subbundle $C$ and quaternionic subbundle $Q$ of $M$. Let us denote by $\alpha \in SU_{2,m}/S(U_2 U_m)$ the unique fixed point of the action of the isotropy group $S(U_2 U_m)$ on $SU_{2,m}/S(U_2 U_m)$.

First we consider the conic (or geodesic) compactification of $SU_{2,m}/S(U_2 U_m)$. The points in the boundary of this compactification correspond to equivalence classes of asymptotic geodesics in $SU_{2,m}/S(U_2 U_m)$. Every geodesic in this non-compact Grassmannian lies in a maximal flat, that is, a two-dimensional Euclidean space embedded in $SU_{2,m}/S(U_2 U_m)$ as a totally geodesic submanifold. A geodesic in $SU_{2,m}/S(U_2 U_m)$ is called singular if it lies in more than one maximal flat in $SU_{2,m}/S(U_2 U_m)$. A singular point at infinity is the equivalence class of a singular geodesic in $SU_{2,m}/S(U_2 U_m)$. Up to isometry, there are exactly two singular points at infinity for $SU_{2,m}/S(U_2 U_m)$. The singular points at infinity correspond to the geodesics in $SU_{2,m}/S(U_2 U_m)$ which are determined by nonzero tangent vectors $X$ with $JX \in 3X$ or $JX \perp 3X$ respectively.

Motivated by the results mentioned above, recently Berndt and the author [4] have given a complete characterization of horospheres in $SU_{2,m}/S(U_2 U_m)$ whose center at infinity is singular as follows:

**Theorem A.** Let $M$ be a horosphere in $SU_{2,m}/S(U_2 U_m)$, $m \geq 2$. The following statements are equivalent:

(i) the center of $M$ is a singular point at infinity,

(ii) the maximal complex subbundle $C$ of $TM$ is invariant under the shape operator of $M$,

(iii) the maximal quaternionic subbundle $Q$ of $TM$ is invariant under the shape operator of $M$.

Next, we consider the standard embedding of $SU_{2,m-1}$ in $SU_{2,m}$. Then the orbit $SU_{2,m-1} \cdot \alpha$ of $SU_{2,m-1}$ through $\alpha$ is the Riemannian symmetric space $SU_{2,m-1}/S(U_2 U_{m-1})$ embedded in $SU_{2,m}/S(U_2 U_m)$ as a totally geodesic submanifold. Every tube around $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$ has the property that both maximal complex subbundle $C$ and quaternionic subbundle $Q$ are invariant under the shape operator.

Finally, let $m$ be even, say $m = 2n$, and consider the standard embedding of $Sp_{1,n}$ in $SU_{2,2n}$. Then the orbit $Sp_{1,n} \cdot \alpha$ of $Sp_{1,n}$ through $\alpha$ is the quaternionic hyperbolic space $\mathbb{H}^n$ embedded in $SU_{2,2n}/S(U_2 U_{2n})$ as a totally geodesic submanifold. Any tube around $\mathbb{H}^n$ in $SU_{2,2n}/S(U_2 U_{2n})$ has the property that both $C$ and $Q$ are invariant under the shape operator.

As a converse of the statements mentioned above, we assert that with one possible exceptional case there are no other such real hypersurfaces. Related to such a result, we introduce another theorem due to Berndt and Suh [4] as follows:

**Theorem B.** Let $M$ be a connected hypersurface in $SU_{2,m}/S(U_2 U_m)$, $m \geq 2$. Then the maximal complex subbundle $C$ of $TM$ and the maximal quaternionic subbundle
\(\mathcal{Q}\) of \(TM\) are both invariant under the shape operator of \(M\) if and only if \(M\) is congruent to an open part of one of the following hypersurfaces:

(A) a tube around a totally geodesic \(SU_{2,m-1}/S(U_2U_{m-1})\) in \(SU_{2,m}/S(U_2U_m)\);

(B) a tube around a totally geodesic \(\mathbb{HH}^n\) in \(SU_{2,2n}/S(U_2U_{2n})\), \(m = 2n\);

(C) a horosphere in \(SU_{2,m}/S(U_2U_m)\) whose center at infinity is singular; or the following exceptional case holds:

(D) The normal bundle \(\nu M\) of \(M\) consists of singular tangent vectors of type \(JX \perp \mathfrak{J}X\). Moreover, \(M\) has at least four distinct principal curvatures, three of which are given by

\[
\alpha = \sqrt{2}\ , \ \gamma = 0\ , \ \lambda = \frac{1}{\sqrt{2}}
\]

with corresponding principal curvature spaces

\[
T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q})\ , \ T_\gamma = J(TM \ominus \mathcal{Q})\ , \ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.
\]

If \(\mu\) is another (possibly nonconstant) principal curvature function, then we have \(T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}\), \(JT_\mu \subset T_\lambda\) and \(J^2T_\mu \subset T_\lambda\).

Usually, maximal complex subbundle of real hypersurfaces in a Kähler manifold is invariant under the shape operator when the Reeb vector field \(\xi = -JN\) is principal. Accordingly, the maximal complex subbundle \(\mathcal{C}\) of \(TM\) in Theorem B is invariant under the shape operator if and only if the Reeb vector field \(\xi\) becomes a principal vector field for the shape operator \(A\) of \(M\) in \(SU_{2,m}/S(U_2U_m)\). In this case we call \(M\) a Hopf hypersurface in \(SU_{2,m}/S(U_2U_m)\).

Besides of this, a real hypersurface \(M\) in \(SU_{2,m}/S(U_2U_m)\) also admits the maximal quaternionic subbundle \(\mathcal{Q}\) and the orthogonal complement \(\mathcal{Q}^\perp\), which is spanned by almost contact 3-structure vector fields \(\{\xi_1, \xi_2, \xi_3\}\), such that \(T_xM = \mathcal{Q} \oplus \mathcal{Q}^\perp\), \(x \in M\).

In order to give some characterizations for hypersurfaces given in Theorem B, we [15] have considered a geometric condition that the Reeb flow on \(M\) is isometric, that is, the shape operator of \(M\) in \(SU_{2,m}/S(U_2U_m)\) commutes with the structure tensor \(\phi\). By virtue of this condition, we gave a characterization of real hypersurfaces of type (A) or one of type (C) in \(SU_{2,m}/S(U_2U_m)\). Historically, many geometers considered such a notion on several kinds of manifolds. As a first the isometric Reeb flow on real hypersurfaces in complex projective space \(\mathbb{C}P^n\) was investigated by Okumura [12], and in complex hyperbolic space \(\mathbb{CH}^n\) by Montiel and Romero [11], and in compact complex two-plane Grassmannian \(G_2(\mathbb{C}^m+2)\) by Berndt and Suh [3] respectively. Moreover, for further investigating on commuting problems related to shape operator, Ricci tensor and the structure tensor are given. In complex projective space we want to mention some works due to Kimura [8], [9], in quaternionic projective space Martinez and Pérez [10], Pérez and Suh [13], [14], and in complex two-plane Grassmannian \(G_2(\mathbb{C}^{m+2})\) Pérez, Suh and Watanabe [15] and Suh [16], [17] respectively.

The complex hyperbolic two-plane Grassmannian \(SU_{2,m}/S(U_2U_m)\) has a remarkable geometrical structure. It is the unique noncompact irreducible Riemannian manifold being equipped with both a Kähler structure \(J\) and a quaternionic Kähler structure \(\mathfrak{J} = \text{Span} \{J_1, J_2, J_3\}\) not containing \(J\). In other words, \(SU_{2,m}/S(U_2U_m)\)
is the unique noncompact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold (See Berndt and Suh [4]).

Now in this paper we want to give a characterization of type (B), another one of type (C), that is, a horosphere whose center at infinity is singular, or of type (D) in noncompact complex two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$ related to the Reeb vector field $\xi$. Then we can assert a complete classification of all Hopf real hypersurfaces in $SU_{2,m}/S(U_2U_m)$ in terms of the Reeb vector field belonging to the maximal quaternionic subbundle $Q$ as follows:

**Main Theorem.** Let $M$ be a Hopf real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$ with the Reeb vector field belonging to the maximal quaternionic subbundle $Q$. Then one of the following statements holds,

- **(B)** $M$ is an open part of a tube around a totally geodesic $\mathbb{H}^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$;
- **(C)** $M$ is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp JN$,
  - or the following exceptional case holds:
- **(D)** The normal bundle $\nu M$ of $M$ consists of singular tangent vectors of type $JX \perp JX$. Moreover, $M$ has at least four distinct principal curvatures, three of which are given by
  $$\alpha = \sqrt{2}, \; \gamma = 0, \; \lambda = \frac{1}{\sqrt{2}},$$
  with corresponding principal curvature spaces
  $$T_\alpha = TM \ominus (C \cap Q), \; T_\gamma = J(TM \ominus Q), \; T_\lambda \subset C \cap Q \cap JQ.$$

If $\mu$ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset C \cap Q \cap JQ$, $JT_\mu \subset T_\lambda$ and $JT_\mu \subset T_\lambda$.

**Remark.** Real hypersurfaces of type (A) and (C) with $JN \in JN$ in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ mentioned in Theorem B are characterized by the geometric property that the Reeb flow is isometric (See Suh [18]). Of course, in these type of hypersurfaces the Reeb vector field $\xi$ belongs to the orthogonal complement $Q^\perp$ of the quaternionic maximal subbundle $Q$. But the other type of Hopf hypersurfaces in Theorem B are characterized in our main theorem as Hopf hypersurfaces in $SU_{2,m}/S(U_2U_m)$ with the Reeb vector field $\xi \in Q$.

1. **The Complex Hyperbolic Two-Plane Grassmannian $SU_{2,m}/S(U_2U_m)$**

In this section we summarize basic material about the noncompact complex two-plane Grassmann manifold $SU_{2,m}/S(U_2U_m)$, for details we refer to [4], [5], [6], [7] and [18].

The Riemannian symmetric space $SU_{2,m}/S(U_2U_m)$, which consists of all positive definite complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}^{m+2}$, becomes a connected, simply connected, irreducible Riemannian symmetric space of noncompact type with rank two. Let $G = SU_{2,m}$ and $K = S(U_2U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra. Let $B$ be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $B$. The
resulting decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $g$. The Cartan involution $\theta \in Aut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

and $I_2$ and $I_m$ is the identity $(2 \times 2)$-matrix and $(m \times m)$-matrix respectively. Then $< X, Y > = -B(X, \theta Y)$ becomes a positive definite $Ad(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a Riemannian metric $g$ on $SU_{2,m}/S(U_2U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2U_m)$ together with this particular Riemannian metric $g$.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{j}$ on $SU_{2,m}/S(U_2U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} m^{-1}I_2 & 0_{2,m} \\ 0_{m,2} & m^{-1}I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure $J$ on $SU_{2,m}/S(U_2U_m)$. By construction, $J$ commutes with each almost Hermitian structure $J_1$ in $\mathfrak{g}$. Recall that a canonical local basis $J_1, J_2, J_3$ of a quaternionic Kähler structure $\mathfrak{j}$ consists of three almost Hermitian structures $J_1, J_2, J_3$ in $\mathfrak{g}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index $\nu$ is to be taken modulo 3. The tensor field $JJ_\nu$, which is locally defined on $SU_{2,m}/S(U_2U_m)$, is selfadjoint and satisfies $(JJ_\nu)^2 = I$ and $tr(JJ_\nu) = 0$, where $I$ denotes the identity transformation. For a nonzero tangent vector $X$ we define $\mathbb{R}X = \{ \lambda X | \lambda \in \mathbb{R} \}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{j}X$.

Usually, the tangent space $T_oSU_{2,m}/S(U_2U_m)$ of $SU_{2,m}/S(U_2U_m)$ at $o$ can be identified with $\mathfrak{p}$. Let $a$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_oSU_{2,m}/S(U_2U_m) \simeq \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. In general this subspace is uniquely determined by $X$, in which case $X$ is called regular. If there exists more than one maximal abelian subspaces of $\mathfrak{p}$ containing $X$, then $X$ is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{j}X$ or $JX \perp \mathfrak{j}X$.

Up to scaling there exists a unique $SU_{2,m}$-invariant Riemannian metric $g$ on $SU_{2,m}/S(U_2U_m)$. Equipped with this metric $SU_{2,m}/S(U_2U_m)$ is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler. For computational reasons we normalize $g$ such that the minimal sectional curvature of $(SU_{2,m}/S(U_2U_m), g)$ is $-4$. The sectional curvature $K$ of the noncompact symmetric space $SU_{2,m}/S(U_2U_m)$ equipped with the Killing metric $g$ is bounded by $-4 \leq K \leq 0$. The sectional curvature $-4$ is obtained for all 2-planes $\mathbb{C}X$ when $X$ is a non-zero vector with $JX \in \mathfrak{j}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{CH}^2$ with constant holomorphic sectional curvature $-4$.

When $m = 2$, the isomorphism $SO(4,2) \simeq SU(2,2)$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space $\mathbb{R}^6$. For
this reason we assume \( m \geq 2 \) from now on, although many of the subsequent results also hold for \( m = 1, 2 \).

The Riemannian curvature tensor \( \bar{R} \) of \( SU_{2,m}/S(U_2U_m) \) is locally given by

\[
\bar{R}(X, Y)Z = -\frac{1}{2} \left[ g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\
- g(JX, Z)JY - 2g(JX, Y)JZ \\
+ \sum_{\nu=1}^{3} \{ g(J_{\nu}Y, Z)J_{\nu}X - g(J_{\nu}X, Z)J_{\nu}Y \\
- 2g(J_{\nu}X, Y)J_{\nu}Z \} \\
+ \sum_{\nu=1}^{3} \{ g(J_{\nu}JY, Z)J_{\nu}JX - g(J_{\nu}JX, Z)J_{\nu}JY \} \right],
\]

(1.1)

where \( J_1, J_2, J_3 \) is any canonical local basis of \( \mathfrak{J} \).

Recall that a maximal flat in a Riemannian symmetric space \( \mathring{M} \) is a connected complete flat totally geodesic submanifold of maximal dimension. A non-zero tangent vector \( X \) of \( \mathring{M} \) is singular if \( X \) is tangent to more than one maximal flat in \( \mathring{M} \), otherwise \( X \) is regular. The singular tangent vectors of \( SU_{2,m}/S(U_2U_m) \) are precisely the eigenvectors and the asymptotic vectors of the self-adjoint endomorphisms \( J_{J_1} \), where \( J_1 \) is any almost Hermitian structure in \( \mathfrak{J} \). In other words, a tangent vector \( X \) to \( SU_{2,m}/S(U_2U_m) \) is singular if and only if \( JX \in \mathfrak{J}X \) or \( JX \perp \mathfrak{J}X \).

In the previous paper of [13], we considered a singular vector of type \( JX \in \mathfrak{J}X \) and independently have given a characterization of hypersurfaces of type \( (A) \) and a horosphere of type \( (C_1) \). In this paper, we must compute explicitly Jacobi vector fields along geodesics whose tangent vectors are all singular of type \( JX \perp \mathfrak{J}X \). For this we need the eigenvalues and eigenspaces of the Jacobi operator \( \bar{R}_X := \bar{R}(., X)X \). Let \( X \) be a singular unit vector tangent to \( SU_{2,m}/S(U_2U_m) \) of type \( JX \perp \mathfrak{J}X \).

If \( JX \perp \mathfrak{J}X \) then the eigenvalues and eigenspaces of \( \bar{R}_X \) are given by (See Berndt and Suh [13])

\[
\begin{array}{ccc}
0 & \mathbb{R}X \oplus \mathfrak{J}X & 4 \\
-\frac{1}{2} & (\mathbb{R}X \oplus \mathbb{R}JX \oplus \mathfrak{J}X \oplus \mathfrak{J}JX)^\perp & 4m - 8 \\
-2 & \mathbb{R}JX \oplus \mathfrak{J}X & 4
\end{array}
\]

where \( \mathbb{R}X \), \( \mathbb{C}X \) and \( \mathbb{H}X \) denote the real, complex and quaternionic span of \( X \), respectively, and \( \mathbb{C}^\perp X \) the orthogonal complement of \( \mathbb{C}X \) in \( \mathbb{H}X \). The maximal totally geodesic submanifolds of \( SU_{2,m}/S(U_2U_m) \) are \( SU_{2,m-1}/S(U_2U_{m-1}) \), \( \mathbb{C}H^m \), \( \mathbb{C}H^k \times \mathbb{C}H^{m-k} \) (\( 1 \leq k \leq \lfloor m/2 \rfloor \)), \( G_2^\perp(\mathbb{R}^{m+2}) \) and \( \mathbb{H}^n \) (if \( m = 2n \)). The first three are complex submanifolds and the other two are real submanifolds with respect to the Kähler structure \( J \). The tangent spaces of the totally geodesic \( \mathbb{C}H^m \) are precisely the maximal linear subspaces of the form \( \{ X | JX = J_1X \} \) with some fixed almost Hermitian structure \( J_1 \in \mathfrak{J} \).
2. Real hypersurfaces in noncompact Grassmannian $SU_{2,m}/S(U_2U_m)$

Let $M$ be a real hypersurface in $SU_{2,m}/S(U_2U_m)$, that is, a hypersurface in $SU_{2,m}/S(U_2U_m)$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Levi Civita covariant derivative of $(M, g)$. We denote by $C$ and $Q$ the maximal complex and quaternionic subbundle of the tangent bundle $TM$ of $M$, respectively. Now let us put

\begin{equation}
JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N
\end{equation}

for any tangent vector field $X$ of a real hypersurface $M$ in $SU_{2,m}/S(U_2U_m)$, where $\phi X$ denotes the tangential component of $JX$ and $N$ a unit normal vector field of $M$ in $SU_{2,m}/S(U_2U_m)$. From the Kähler structure $J$ of $SU_{2,m}/S(U_2U_m)$ there exists an almost contact metric structure $(\phi, \xi, \eta, g)$ induced on $M$ in such a way that

\begin{equation}
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(X) = 0, \quad \text{and} \quad \eta(X) = g(X, \xi)
\end{equation}

for any vector field $X$ on $M$ and $\xi = -JN$.

If $M$ is orientable, then the vector field $\xi$ is globally defined and said to be the induced Reeb vector field on $M$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathfrak{J}$. Then each $J_\nu$ induces a local almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, on $M$. Locally, $C$ is the orthogonal complement in $TM$ of the real span of $\xi$, and $Q$ the orthogonal complement in $TM$ of the real span of $\{\xi_1, \xi_2, \xi_3\}$.

Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure $J_\nu$ of $SU_{2,m}/S(U_2U_m)$, together with the condition

\[ J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu \]

in section 1, induced an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$ as follows:

\begin{equation}
\begin{aligned}
\phi_\nu^2 X &= -X + \eta_\nu(\xi_\nu), \quad \phi_\nu \xi_\nu = 0, \quad \eta_\nu(\xi_\nu) = 1 \\
\phi_{\nu+1} \xi_\nu &= -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\
\phi_\nu \phi_{\nu+1} X &= \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\
\phi_{\nu+1} \phi_\nu X &= -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned}
\end{equation}

for any vector field $X$ tangent to $M$. The tangential and normal component of the commuting identity $J_\nu X = J_\nu J_\nu X$ give

\begin{equation}
\begin{aligned}
\phi_\nu \phi_\nu X - \phi_\nu \phi X &= \eta_\nu(X)\xi_\nu - \eta(X)\xi_\nu \\
\eta_\nu(\phi_{\nu+1} X) &= \eta_{\nu+2}(X) = -\eta_{\nu+1}(\phi_\nu X).
\end{aligned}
\end{equation}

The last equation implies $\phi_\nu \xi = \phi \xi_\nu$. The tangential and normal component of $J_\nu J_{\nu+1} X = J_{\nu+2} X = -J_{\nu+1}J_\nu X$ give

\begin{equation}
\begin{aligned}
\phi_\nu \phi_{\nu+1} X &= \eta_{\nu+1}(X)\xi_\nu = \phi_{\nu+2} X = -\phi_{\nu+1} \phi_\nu X + \eta_\nu(X)\xi_{\nu+1} \\
\eta_\nu(\phi_{\nu+1} X) &= \eta_{\nu+2}(X) = -\eta_{\nu+1}(\phi_\nu X).
\end{aligned}
\end{equation}

Putting $X = \xi_\nu$ and $X = \xi_{\nu+1}$ into the first of these two equations yields $\phi_{\nu+2} \xi_\nu = \xi_{\nu+1}$ and $\phi_{\nu+2} \xi_{\nu+1} = -\xi_\nu$ respectively. Using the Gauss and Weingarten formulas, the tangential and normal component of the Kähler condition $(\nabla_X J)Y = 0$ give $(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$ and $(\nabla_X \eta)Y = g(\phi AX, Y)$. The last equation implies $\nabla_X \xi = \phi AX$. Finally, using the explicit expression for the
Riemannian curvature tensor $\hat{R}$ of $SU_{2,m}/S(U_2U_m)$ in (1.1) the Codazzi equation takes the form

$$(\nabla_X A) Y - (\nabla_Y A) X = \frac{1}{2} \left[ \eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi \right. + \sum_{\nu=1}^{3} \{ \eta_{\nu}(X) \phi_{\nu} Y - \eta_{\nu}(Y) \phi_{\nu} X - 2g(\phi_{\nu} X, Y) \xi_{\nu} \}$$

$$(2.7) + \sum_{\nu=1}^{3} \{ \eta_{\nu}(\phi Y) \phi_{\nu} Y - \eta_{\nu}(\phi Y) \phi_{\nu} X \} + \sum_{\nu=1}^{3} \{ \eta(X) \eta_{\nu}(\phi Y) - \eta(Y) \eta_{\nu}(\phi X) \} T_{\nu} \right].$$

We now assume that the Reeb flow on $M$ in $SU_{2,m}/S(U_2U_m)$ is geodesic. Then, according to Proposition 3.1, there exists a smooth function $\alpha$ on $M$ so that $A \xi = \alpha \xi$. Taking an inner product of the Codazzi equation (2.7) with $\xi$ we get

$$g(\phi X, Y) - \sum_{\nu=1}^{3} \{ \eta_{\nu}(X) \eta_{\nu}(\phi Y) - \eta_{\nu}(Y) \eta_{\nu}(\phi X)$$

$$- g(\phi_{\nu} X, Y) \eta_{\nu}(\xi) \}$$

$$= g(\nabla_X A) Y - (\nabla_Y A) X, \xi$$

$$= g((\nabla_X A) \xi, Y) - g((\nabla_Y A) \xi, X)$$

$$= (X \alpha) \eta(Y) - (Y \alpha) \eta(X)$$

$$+ \alpha g((A \phi + \phi A) X, Y) - 2g(A \phi X, Y).$$

Substituting $X = \xi$ yields $Y \alpha = (\xi \alpha) \eta(Y) + 2 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \eta_{\nu}(\phi Y)$, and inserting this equation and the corresponding one for $X \alpha$ into the previous equation implies

**Proposition 2.1.** If $M$ is a connected orientable real hypersurface in $SU_{2,m}/S(U_2U_m)$ with geodesic Reeb flow, then

$$2g(A \phi X, Y) - \alpha g((A \phi + \phi A) X, Y) + g(\phi X, Y)$$

$$= \sum_{\nu=1}^{3} \{ \eta_{\nu}(X) \eta_{\nu}(\phi Y) - \eta_{\nu}(Y) \eta_{\nu}(\phi X) - g(\phi_{\nu} X, Y) \eta_{\nu}(\xi)$$

$$- 2\eta(X) \eta_{\nu}(\phi Y) \eta_{\nu}(\xi) + 2\eta(Y) \eta_{\nu}(\phi X) \eta_{\nu}(\xi) \}.$$
Proposition 3.1. Let $M$ be a connected orientable Hopf real hypersurface in $SU_{2,m}/S(U_{2}U_{m})$. If the Reeb vector field $\xi$ belongs to $Q$, then $g(AQ, Q^\perp) = 0$.

Proof. To prove this it suffices to show that $g(AQ, \xi_\nu) = 0$, $\nu = 1, 2, 3$. In order to do this, we put

$$Q = [\xi] \oplus [\phi_1 \xi, \phi_2 \xi, \phi_3 \xi] \oplus Q_0,$$

where the distribution $Q_0$ is an orthogonal complement of $[\xi] \oplus [\phi_1 \xi, \phi_2 \xi, \phi_3 \xi]$ in the distribution $Q$.

First, from the assumption $\xi \in Q$ we know $g(A\xi, \xi_\nu) = 0$, $\nu = 1, 2, 3$, because we have assumed that $M$ is Hopf.

Next we will show that $g(A\phi_i \xi, \xi_\nu) = 0$, for any indices $i$ and $\nu = 1, 2, 3$. In fact, by using (2.4) and $\xi \in Q$ we have the following

$$g(A\phi_i \xi, \xi_\nu) = -g(\phi_i A\xi, \xi_\nu) = -g(\nabla_{\xi_\nu} \xi_i, \xi) = g(\xi, \nabla_{\xi_\nu} \xi_i) = g(\xi, \phi_i A\xi_\nu) = -g(A\phi_i \xi, \xi_\nu),$$

which gives that $g(A\phi_i \xi, \xi_\nu) = 0$, $\nu = 1, 2, 3$.

Finally, we consider the case $X \in Q_0$, where the distribution $Q_0$ is denoted by $Q_0 = \{X \in Q | X \perp \xi \text{ and } \phi_i \xi, i = 1, 2, 3\}$.

By Proposition 2.1 and the assumption of $\xi \in Q$, we have

$$\alpha A\phi X + \alpha \phi AX - 2\phi A\phi AX - \phi X = 0,$$

for any tangent vector field $X \in Q_0$.

From now on, in order to show $g(AX, \xi_\nu) = 0$ for any $X \in Q_0$, we restrict $X \in T_pM$, $p \in M$ to $X \in Q_0$ unless otherwise stated. Now by taking the structure tensor $\phi$ into above equation and using the fact that $\xi \in Q$ we get

$$\alpha \phi A\phi X - \alpha AX - 2\phi A\phi AX + X = 0,$$

for any $X \in Q_0$.

Taking an inner product into (3.1) with $\xi_\mu$ we have

$$\alpha g(\phi A\phi X, \xi_\mu) - \alpha g(AX, \xi_\mu) - 2g(\phi A\phi AX, \xi_\mu) = 0,$$

that is,

$$\alpha g(AX, \xi_\mu) = \alpha g(\phi A\phi X, \xi_\mu) - 2g(\phi A\phi AX, \xi_\mu) \quad \text{for } X \in Q_0.$$
On the other hand, since \( g(\phi A \phi X, \xi_\mu) = g(\nabla_{\phi X} \xi, \xi_\mu) = -g(\xi, \nabla_{\phi X} \xi_\mu) \), we have
\[
g(\phi A \phi X, \xi_\mu) = -g(\xi, \phi_\mu A \phi X) = -g(\xi_\mu, \phi A \phi X)
\]
by virtue of (2.1) and (2.4). Accordingly, we get \( g(\phi A \phi X, \xi_\mu) = 0 \) for any \( X \in Q_0 \).

Next let us show that \( g(\phi A \phi AX, \xi_\mu) = 0 \).

In fact, (2.3) and (2.4) give
\[
g(\phi A \phi AX, \xi_\mu) = g(\nabla_{\phi AX} \xi, \xi_\mu) = -g(\xi, \nabla_{\phi AX} \xi_\mu)
\]
\[
= -g(\xi, \phi_\mu A \phi AX) = -g(\xi_\mu, \phi A \phi AX).
\]
It implies that \( g(\phi A \phi AX, \xi_\mu) = 0 \) for any \( X \in Q_0 \). Thus, from (3.2) we know that
\[
(3.3) \quad \alpha g(AX, \xi_\mu) = 0 \quad \text{for any} \quad X \in Q_0.
\]

From this we can divide two cases as follow:

**Case I.** Let \( \Omega = \{x \in M | \alpha(x) \neq 0\} \).

On such an open neighborhood \( \Omega \) we know that (3.3) gives \( g(AX, \xi_\mu) = 0 \) for any \( X \in Q_0 \).

**Case II.** Let \( \mathcal{W} = \text{Int} (M - \Omega) \), where \( \text{Int} \) denotes the interior set of the orthogonal complement of the open subset \( \Omega \) in \( M \).

In this case we consider two subcases. One is to consider that the function \( \alpha \) vanishes on a non-empty neighborhood \( \text{Int} (M - \Omega) \). The other subcase is to consider a point \( x \) such that \( \alpha(x) = 0 \) but the point \( x \) is the limit of a sequence of points where \( \alpha \neq 0 \). This subcase could be possible when the open subset \( \text{Int} (M - \Omega) \) is empty. Such a sequence necessary have an infinite subsequence. Then by the continuity we have \( g(AX, \xi_\mu) = 0 \) for any \( X \in Q_0 \) as in Case I.

Then we only focus on the first subcase that the function \( \alpha \) identically vanishes on some neighborhood of the point \( x \in M \). From this situation, the equation (3.1) can be given by
\[
X = 2\phi A \phi AX \quad \text{for any} \quad X \in Q_0.
\]

Taking the shape operator \( A \) into (3.4) we have
\[
(3.5) \quad AX = 2A \phi A \phi AX \quad \text{for any} \quad X \in Q_0.
\]

From this, let us take an inner product into (3.5) with \( \xi_\mu \), we have
\[
(3.6) \quad g(AX, \xi_\mu) = 2g(A \phi A \phi AX, \xi_\mu) \quad \text{for any} \quad X \in Q_0.
\]

On the other hand, we know the following
\[
g(A \phi A \phi AX, \xi_\mu) = -g(A \phi AX, \phi A \xi_\mu) = -g(A \phi AX, \nabla_{\xi_\mu} \xi)
\]
Then it follows that
\[
g(A \phi A \phi AX, \xi_\mu) = -g(A \phi AX, \nabla_{\xi_\mu} \xi)
\]
\[
= g((\nabla_{\xi_\mu} A) \phi AX, \xi) + g(A \phi (\nabla_{\xi_\mu} \phi) AX, \xi)
\]
\[
+ g(A \phi (\nabla_{\xi_\mu} A) X, \xi) + g(A \phi A (\nabla_{\xi_\mu} X), \xi)
\]
\[
= g((\nabla_{\xi_\mu} A) \phi AX, \xi)
\]
where we have used $g(AφAX, ξ) = 0$ and $Aξ = 0$. From this, together with $Aξ = 0$, it follows that

\[(3.7) \quad g(AφAφAX, ξ_µ) = g((∇_{ξ_µ} A)φAX, ξ).\]

On the other hand, by using the equation of Codazzi in section 2, we have the following

**Lemma 3.2.**

$$g((∇_{ξ_µ} A)φAX, ξ) = -g(Aξ_µ, φAφAX) + 2g(AX, ξ_µ), \quad µ = 1, 2, 3.$$  

**Proof.** By using the equation of Codazzi, it follows that for $ξ \in Q$

$$(∇_{ξ_µ} A)φAX = (∇_{φAX} A)ξ_µ - \frac{1}{2} \left[ η(ξ_µ)φ^2 AX - η(φAX)φξ_µ - 2g(φξ_µ, φAX)ξ + \sum_{ν=1}^{3} \left\{ η_ν(ξ_µ)φ_ν φAX - η_ν(φAX)φ_ν ξ_µ - 2g(φ_ν ξ_µ, φAX)ξ_ν \right\} \right]$$

$$\quad + \sum_{ν=1}^{3} \left\{ η_ν(φξ_µ)φ_ν φ^2 AX - η_ν(φ^2 AX)φ_ν φξ_µ \right\}$$

$$\quad + \sum_{ν=1}^{3} \left\{ η(ξ_µ)η_ν(φ^2 AX) - η(φAX)η_ν(φξ_µ) \right\} ξ_ν \right]$$

$$= (∇_{φAX} A)ξ_µ + g(ξ_µ, AX)ξ - \frac{1}{2} φ_µ φAX$$

$$\quad + \frac{1}{2} \sum_{ν=1}^{3} η_ν(φAX)φ_ν ξ_µ$$

$$\quad + \sum_{ν=1}^{3} g(φ_ν ξ_µ, φAX)ξ_ν - \frac{1}{2} \sum_{ν=1}^{3} η_ν(AX)φ_ν φξ_µ.$$  

Taking an inner product above equation with $ξ$ and using the fact that $φφ_µ ξ = -ξ_µ$, we have the following for any $X ∈ Q_0$

$$g((∇_{ξ_µ} A)φAX, ξ) = g((∇_{φAX} A)ξ_µ, ξ) + g(ξ_µ, AX) - \frac{1}{2} g(φ_µ φAX, ξ)$$

$$\quad + \frac{1}{2} \sum_{ν=1}^{3} η_ν(φAX)g(φ_ν ξ_µ, ξ) + \sum_{ν=1}^{3} g(φ_ν ξ_µ, φAX)g(ξ_ν, ξ)$$

$$\quad - \frac{1}{2} \sum_{ν=1}^{3} η_ν(AX)g(φ_ν φξ_µ, ξ)$$

$$= g((∇_{φAX} A)ξ_µ, ξ) + \frac{3}{2} g(AX, ξ_µ) + \frac{1}{2} η_µ(AX)$$

$$= g((∇_{φAX} A)ξ_µ, ξ) + 2g(AX, ξ_µ),$$

where we have used $g(φ_ν ξ_µ, ξ) = 0$ and $g(ξ_ν, ξ) = 0$ in the second equality.
On the other hand, since $g(A\xi_\mu, \xi) = g(\xi_\mu, A\xi) = \alpha g(\xi_\mu, \xi)$ and $\alpha = 0$, we have
\[
g((\nabla_{\phi A} A)\xi_\mu, \xi) = -g(A(\nabla_{\phi A} \xi_\mu), \xi) - g(A\xi_\mu, \phi A\phi A X) \\
= -\alpha g(\nabla_{\phi A} \xi_\mu, \xi) - g(A\xi_\mu, \phi A\phi A X) \\
= -g(A\xi_\mu, \phi A\phi A X).
\]
Therefore we have
\[
g((\nabla_{\xi_\mu} A)\phi A X, \xi) = -g(A\xi_\mu, \phi A\phi A X) + 2g(A X, \xi_\mu)
\]
for any $X \in \mathcal{Q}_0$. This completes the proof of our Lemma 3.2.

Consequently, from (3.7), Lemma 3.2 and the formula $g(A\xi_\mu, \xi) = 0$ we have
\[
g(A\phi A\phi A X, \xi_\mu) = g((\nabla_{\xi_\mu} A)\phi A X, \xi) \\
= -g(A\xi_\mu, \phi A\phi A X) + 2g(A X, \xi_\mu) \\
= -g(A\phi A\phi A X, \xi_\mu) + 2g(A X, \xi_\mu)
\]
that is,
\[
(3.8) \quad g(A\phi A\phi A X, \xi_\mu) = g(A X, \xi_\mu).
\]

Summing up (3.6) and (3.8) for $\alpha = 0$, we have $g(A X, \xi_\mu) = 0$. Then for any $X \in \mathcal{Q}_0$ we have $g(A X, \xi_\mu) = 0$, $\mu = 1, 2, 3$. This completes the proof of our Proposition 3.1.

By virtue of Proposition 3.1, the maximal complex subbundle $\mathcal{C}$ and the maximal quaternionic subbundle $\mathcal{Q}$ of Hopf real hypersurfaces $M$ in $SU_{2,m}/S(U_{2U_m})$ are invariant under the shape operator if the Reeb vector field $\xi$ of $M$ belongs to the subbundle $\mathcal{Q}$. Then naturally we get the result of Theorem B. But among the classifications given in Theorem B, the tube over a totally geodesic $SU_{2,m-1}/S(U_{2U_{m-1}})$ in $SU_{2,m}/S(U_{2U_m})$ and a horosphere in $SU_{2,m}/S(U_{2U_m})$ whose center at infinity with singular vector field of type $J X \in J X$ have the property that their Reeb vector field $\xi$ belong to the subbundle $\mathcal{Q}\perp$ which is orthogonal to the maximal quaternionic subbundle $\mathcal{Q}$. From such a point of view we complete the proof of our Main Theorem in the introduction.

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**Young Jin Suh**

**Kyungpook National University,**
**Department of Mathematics,**
**Taegu 702-701, Korea**

*E-mail address: yjsuh@knu.ac.kr*