1. Introduction

In this paper, we adopt the common notation and terminology for basic hypergeometric series as in Refs. [10, 15]. Throughout this paper, we assume that $q$ is a fixed nonzero real or complex number and $|q| < 1$. The $q$-shifted factorial and its compact factorial are defined [10, 15], respectively by:

$$ (a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k) \quad \text{(1.1)} $$

and $(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_r; q)_n$, $n \in \{0, 1, 2, \ldots\}$. We will use frequently the following relation

$$ (aq^{-n}; q)_n = (q/a; q)_n(-a)^n q^{-n-\binom{n}{2}}. \quad \text{(1.2)} $$

The generalized $q$-binomial coefficient is defined as [10]

$$ \binom{\alpha}{k} = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} (-1)^k q^{\binom{k}{2}}, \quad \alpha \in \mathbb{C}. \quad \text{(1.3)} $$

Similarly, by replacing $q$ by $-q$ in the above relation, we get the following:

$$ \binom{\alpha}{k}_{-q} = \frac{(-q^{-\alpha}; q)_k}{(-q; q)_k} q^{\binom{k}{2}}, \quad \alpha \in \mathbb{C}. \quad \text{(1.4)} $$

The basic or $q$-hypergeometric function in the variable $z$ (see Slater [21, Chap. 3], Srivastava and Karlsson [24, p. 347, Eq. (272)] for details) is defined as:

$$ \Phi_x \left[ \begin{array}{cc} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s; \end{array} \right] q; z = \sum_{n=0}^{\infty} \left[ (-1)^n q^{\binom{n}{2}} \right] (a_1, a_2, \ldots, a_r; q)_n \left( \frac{z^n}{(b_1, b_2, \ldots, b_s; q)_n (q; q)_n} \right). \quad \text{(1.5)} $$
when \( r > s + 1 \). Note that, for \( r = s + 1 \), we have:

\[
\Phi_{r+1} \left\{ a_1, a_2, \ldots, a_{r+1}; \frac{q^n}{b_1, b_2, \ldots, b_r} \right\} = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} z^n.
\]

We remark in passing that, in a recently-published survey-cum-expository review article, the so-called \((p, q)\)-calculus was exposed to be a rather trivial and inconsequential variation of the classical \(q\)-calculus, the additional parameter \( p \) being redundant or superfluous (see, for details, [22, p. 340]).

Chen et al. [8] introduced homogeneous \(q\)-difference operator \(D_{xy}\) as

\[
D_{xy} \{f(x, y)\} := \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y},
\]

which turn out to be suitable for dealing with the Cauchy polynomials.

Wang and Cao [26] presented two extensions of Cigler’s (see [3]) polynomials

\[
C_n^{(a-n)}(x, y, b) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{\alpha}{k}_q b^k (q; q)_n p_{n-k}(x, y)
\]

and

\[
D_n^{(a-n)}(x, y, b) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{\alpha}{k}_q b^k (q; q)_n (-1)^{n+k} q^{-n} Q_{n-k}(y, x),
\]

where \( p_n(x, y) := (x-y)(x-qy) \cdots (x-q^{n-1}y) = (y/x; q)_n x^n \) are the Cauchy polynomials.

Recently, Jia et al. [14] introduced the following

\[
L_{\tilde{m}, \tilde{n}}(\alpha, x, z, a) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{\alpha}{k}_q b^k (q; q)_n z^{\tau(\tilde{m}, \tilde{n})+\binom{k}{2}}(a; q)_k z^{\tilde{m} - k},
\]

with

\[
\tau(\tilde{m}, \tilde{n}) = \tilde{m} \binom{1}{2} - \tilde{n} \binom{1}{2},
\]

where \( \tilde{m} \) and \( \tilde{n} \) are real numbers.

Our present investigation is essentially motivated by the earlier works by Jia et al [14]. Our aim here is to introduce and study some further extensions of the above-mentioned \(q\)-polynomials

\[
\tilde{L}_n^{(p, \tilde{r}, \tilde{s})}(\alpha, x, y, z, a) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{\alpha}{k}_q b^k (q; q)_n z^{\tau(\tilde{r}, \tilde{s})+\binom{k}{2}}(a; q)_k z^{\tilde{m} - k},
\]

where \( \tau(\tilde{r}, \tilde{s}) \) is defined as in (1.9).

The main task in this paper is to show how the \(q\)-polynomials (1.10) are related to other known polynomials. For example, the generalized \(q\)-polynomials \(\tilde{L}_n^{(p, \tilde{r}, \tilde{s})}(\alpha, x, y, z, a)\) defined in (1.10) are generalized and unified forms of Hahn polynomials and Al-Salam-Carlitz polynomials. Particular cases of known results in the literature are given in Remark 1.

**Remark 1.**

1. Upon setting \( y = 0 \), the generalized \(q\)-polynomials \(\tilde{L}_n^{(p, \tilde{r}, \tilde{s})}(\alpha, x, y, z, a)\) defined in (1.10) reduce to (1.8) (see [14]):

\[
\tilde{L}_n^{(p, \tilde{r}, \tilde{s})}(\alpha, x, 0, z, a) = L_{\tilde{r}, \tilde{s}}(\alpha, x, z, a).
\]

2. For \((\alpha, \tilde{r}, \tilde{s}, x, y, z, a) = (\infty, 0, 0, y, x, -z, -q)\), the generalized \(q\)-polynomials \(\tilde{L}_n^{(p, \tilde{r}, \tilde{s})}(\alpha, x, y, z, a)\) reduce to trivariate \(q\)-polynomials \(F_n(x, y, z; q)\) [1]

\[
\tilde{L}_n^{(0,0,0)}(\infty, y, x, -z, -q) = (-1)^n q^{\binom{n}{2}} F_n(x, y, z; q).
\]
(3) Upon setting \( \alpha = n \in \mathbb{Z} \) and \((\tilde{r}, \tilde{s}, a, x, y, z) = (0, -1, -yz, 1, 0, x)\), the \(q\)-polynomials \( \tilde{L}_n^{(\tilde{r}, \tilde{s})}(a, x, y, z, a) \) reduce to \( \rho_c(n, y, x, q) \) (see [14]):

\[
\tilde{L}_n^{(0, -1)}(n, 1, 0, x, -qy) = \rho_c(n, y, x, q). \tag{1.13}
\]

(4) Upon setting \((\alpha, \tilde{r}, \tilde{s}, a, x, y, a) = (\infty, -1, 0, 1, -q)\), the generalized \(q\)-polynomials \( \tilde{L}_n^{(\tilde{r}, \tilde{s})}(a, x, y, z, a) \) reduce to the homogeneous Rogers-Szegő polynomials \( h_n(x, y | q) \) (see [18]):

\[
\tilde{L}_n^{(-1, 0)}(\infty, x, y, 1, -q) = h_n(x, y | q). \tag{1.14}
\]

(5) By choosing \((\alpha, \tilde{r}, \tilde{s}, a, x, y) = (\infty, -1, 0, -q, xq^{-n}, 0)\), the generalized \(q\)-polynomials \( \tilde{L}_n^{(\tilde{r}, \tilde{s})}(a, x, y, z, a) \) reduce to the Rogers-Szegő polynomials \( g_n(z, x | q) \) (see [2]):

\[
\tilde{L}_n^{(-1, 0)}(\infty, xq^{-n}, 0, z, -q) = g_n(z, x | q). \tag{1.15}
\]

The rest of the paper is organized as follows. In section 2, we deduce the main results of \(q\)-difference equations with seven-variable for generalized \(q\)-polynomials. In section 3, we obtain the generating function of generalized \(q\)-polynomials by the method of \(q\)-difference equations. In section 4, we derive Rogers formula for generalized \(q\)-polynomials by using the \(q\)-difference equations. In section 5, we gain a mixed generating function for generalized \(q\)-polynomials by \(q\)-difference equations.

### 2. Main Results

In this section, we give the following fundamental theorem.

**Theorem 2.** Let \( f(\alpha, x, y, a, z, \tilde{r}, \tilde{s}) \) be a seven-variable analytic function in a neighborhood of \((\alpha, x, y, a, z, \tilde{r}, \tilde{s}) = (0, 0, 0, 0, 0, 0, 0) \in \mathbb{C}^7 \). Then, \( f(\alpha, x, y, a, z, \tilde{r}, \tilde{s}) \) can be expanded in terms of \( \tilde{L}_n^{(\tilde{r}, \tilde{s})}(a, x, y, z, a) \) if and only if \( f \) satisfies the following \(q\)-difference equation:

\[
(x - q^{-1}y) \begin{bmatrix} f(\alpha, x, y, a, z, \tilde{r}, \tilde{s}) - f(\alpha, x, y, a, q^2 z, \tilde{r}, \tilde{s}) \\ q^{\alpha - \tilde{r}} \left[ f(\alpha, q x, y, a, zq^{1-\tilde{s}}, \tilde{r}, \tilde{s}) - f(\alpha, x, q^{-1}y, a, zq^{1-\tilde{s}}, \tilde{r}, \tilde{s}) \right] \\ q^{-\tilde{s}}(1 - aq^\alpha) \left[ f(\alpha, q x, y, a, zq^{1-\tilde{s}}, \tilde{r}, \tilde{s}) - f(\alpha, x, yq^{-1}, a, zq^{1-\tilde{s}}, \tilde{r}, \tilde{s}) \right] \\ -aq^{-\tilde{s}} \left[ f(\alpha, q x, y, a, zq^{1+\tilde{s}}, \tilde{r}, \tilde{s}) - f(\alpha, x, yq^{-1}, a, zq^{1+\tilde{s}}, \tilde{r}, \tilde{s}) \right] \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ q \alpha \\ \alpha q \end{bmatrix} = 0. \tag{2.1}
\]

**Remark 3.** For \( y = 0 \) in Theorem 2, we get the concluding remarks of Jia et al [14].

To determine whether a given function is an analytic function in several complex variables, we often use the following Hartogs’s theorem. For more information, please refer to [16, 25].

**Lemma 4** ([11, Hartogs’s theorem]). If a complex-valued function is separately holomorphic (analytic) in each variable in an open domain \( D \in \mathbb{C}^n \), then it is holomorphic (analytic) in \( D \).

In order to prove Theorem 2, we need the following fundamental property of several complex variables.

**Lemma 5** ([17, Proposition 1]). If \( f(x_1, x_2, \ldots, x_k) \) is analytic at the origin \((0, 0, \ldots, 0) \in \mathbb{C}^k \), then, \( f \) can be expanded in an absolutely convergent power series,

\[
f(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k=0}^{\infty} \alpha_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.
\]

**Proof of Theorem 2.** From the Hartogs’s theorem and the theory of several complex variables, we assume that

\[
f(\alpha, x, y, a, z, \tilde{r}, \tilde{s}) = \sum_{k=0}^{\infty} A_k(\alpha, x, y, a, \tilde{r}, \tilde{s}) z_k^k. \tag{2.2}
\]
On one hand, substituting (2.2) into (2.1) yields

\[(x - q^{-1}y) \sum_{k=0}^{\infty} \left(1 - q^{2k}\right) A_k(\alpha, x, y, a, \tilde{r}, \tilde{s}) z^k\]

\[= \sum_{k=0}^{\infty} \left[q^{\alpha - 3k(\tilde{r} - \tilde{s})} + q^{\alpha - k(\tilde{r} - \tilde{s})} (1 - aq^\alpha) - aq^{\alpha - k(2 + \tilde{r} - 3)} \right] \left[A_k(\alpha, x, q^{-1}y, a, \tilde{r}, \tilde{s}) - A_k(\alpha, qx, y, a, \tilde{r}, \tilde{s}) \right] z^{k+1}, \quad (2.3)\]

which is equal to

\[(x - q^{-1}y) \sum_{k=0}^{\infty} \left(1 - q^{2k}\right) A_k(\alpha, x, y, a, \tilde{r}, \tilde{s}) z^k\]

\[= \sum_{k=0}^{\infty} \left[q^{\alpha - 3k(\tilde{r} - \tilde{s})} + q^{\alpha - k(\tilde{r} - \tilde{s})} (1 - aq^\alpha) - aq^{\alpha - k(2 + \tilde{r} - 3)} \right] \left[A_k(\alpha, x, q^{-1}y, a, \tilde{r}, \tilde{s}) - A_k(\alpha, qx, y, a, \tilde{r}, \tilde{s}) \right] z^{k+1}. \quad (2.4)\]

Equating coefficients of $z^k$, $k \geq 1$ on both sides of equation (2.4), we see that

\[(x - q^{-1}y)(1 - q^{k}) (1 + q^k) A_k(\alpha, x, y, a, \tilde{r}, \tilde{s})\]

\[= q^{\alpha - 3(k-1)(\tilde{r} - \tilde{s})} (q^{\alpha} + q^{k-1})(1 - aq^{k-1}) [A_{k-1}(\alpha, x, q^{-1}y, a, \tilde{r}, \tilde{s}) - A_{k-1}(\alpha, qx, y, a, \tilde{r}, \tilde{s})], \quad (2.5)\]

or

\[A_k(\alpha, x, y, a, \tilde{r}, \tilde{s}) = q^{(k-1)\tilde{r} - 3\tilde{s}} (q^{\alpha} + q^{-1}) (1 - aq^{-k-1}) - \frac{A_{k-1}(\alpha, x, q^{-1}y, a, \tilde{r}, \tilde{s}) - A_{k-1}(\alpha, qx, y, a, \tilde{r}, \tilde{s})}{(1 - q^k)(1 + q^{k-1})}, D_{xy}[A_{k-1}(\alpha, x, y, a, \tilde{r}, \tilde{s})].\]

The iteration then reveals that:

\[A_k(\alpha, x, y, a, \tilde{r}, \tilde{s}) = q^{(k-1)\tilde{r} - 3\tilde{s}} \frac{(q^{\alpha} + q^{k-1})(1 - aq^{k-1})}{(1 - q^{k})(1 + q^{k-1})} \cdot D_{xy}[A_{k-1}(\alpha, x, y, a, \tilde{r}, \tilde{s})].\]

Letting $f(\alpha, x, y, a, 0, \tilde{r}, \tilde{s}) = A_0(\alpha, x, y, a, \tilde{r}, \tilde{s}) = \sum_{n=0}^{\infty} \mu_n p_n(x, y)$ yields

\[A_k(\alpha, x, y, a, \tilde{r}, \tilde{s}) = q^{(k-1)\tilde{r} - 3\tilde{s}} \frac{(q^{\alpha} + q^{k-1})(1 - aq^{k-1})}{(1 - q^{k})(1 + q^{k-1})} \cdot \sum_{n=0}^{\infty} \mu_n \frac{(q; q)_n}{(q^2; q^2)_k} p_{n-k}(x, y), \quad (2.6)\]

and we have

\[f(\alpha, x, y, a, z, \tilde{r}, \tilde{s}) = \sum_{k=0}^{\infty} q^{(k-1)\tilde{r} - 3\tilde{s}} \frac{(q^{\alpha} + q^{k-1})(1 - aq^{k-1})}{(1 - q^{k})(1 + q^{k-1})} \cdot \sum_{n=0}^{\infty} \mu_n \frac{(q; q)_n}{(q^2; q^2)_k} p_{n-k}(x, y) z^k\]

\[= \sum_{n=0}^{\infty} \mu_n \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] a^k \frac{(q^{\alpha} + q^{k-1})(1 - aq^{k-1})}{(1 - q^{k})(1 + q^{k-1})} \cdot p_{n-k}(x, y) z^k\]

\[= \sum_{n=0}^{\infty} \mu_n L^{(\tilde{r}, \tilde{s})}_n(\alpha, x, y, z, a).\]

On the other hand, if $f(\alpha, x, y, a, z, \tilde{r}, \tilde{s})$ can be expanded in terms of $L^{(\tilde{r}, \tilde{s})}_n(\alpha, x, y, z, a)$, we verify that $f(\alpha, x, y, a, z, \tilde{r}, \tilde{s})$ satisfies equation (2.1). The proof of Theorem 2 is complete.
3. Generating function of generalized $q$-polynomials

In this section, we give the generating function of generalized $q$-polynomials by the method of $q$-difference equations.

**Theorem 6.** Let $f^{(r)}(x,y,z,a)$ be the polynomials defined as in (1.10).

- For $|xt| < 1$, and $\bar{r}, \bar{s} \in \mathbb{Z}^*$, we have:
  \[
  \sum_{n=0}^{\infty} f^{(r)}(x,y,z,a) \frac{t^n}{(q; q)_n} = \left( \frac{yt; q}{(xt; q)_\infty} \right) \sum_{k=0}^{\infty} \frac{(-q^{-\alpha}, a; q)_k}{(q; q^2)_k} q^{k\alpha + \bar{r}(\bar{k}) - \bar{s}(\bar{k})} (zt)^k.
  \]
  \[
  \sum_{n=0}^{\infty} \frac{p_n(x,y)t^n}{(q; q)_n} = \frac{\left( \frac{yt; q}{(xt; q)_\infty} \right)}{-q^{-\alpha}, a; q; ztq^\alpha}.
  \]

**Proof of Theorem 6.** By the Weierstrass $M$-test, the series \( \sum_{n=0}^{\infty} M_n \) is convergent when $\lim_{n \to +\infty} \frac{M_{n+1}}{M_n} < 1$. We check that both sides of (3.1) are convergent if $|xt| < 1$, that is

- For $\bar{r} = \bar{s} = 0$ in (3.1) and $\max\{||zt||, |xt||} < 1$, we recover the Cauchy polynomials as follows:

\[
\begin{aligned}
\sum_{n=0}^{\infty} p_n(x,y)t^n \frac{n!}{(q; q)_n} &\sum_{n=0}^{\infty} \frac{\alpha}{k!} \frac{(-1)^k q^{k(n)}(a; q)_k}{(q^2; q^2)_k} q^{-\alpha} \frac{p_n(x,y)xt^{n+1}}{(x; q)_\infty} \Phi_1 \left[ \frac{-q^{-\alpha}, a; q; ztq^\alpha}{-q^{-\alpha}, a; q; ztq^\alpha} \right].
\end{aligned}
\]

Denoting by $f(x,y,a,z,\bar{r},\bar{s})$ the right-hand side of equation (3.1), it can equivalently be written by

\[
\begin{aligned}
\sum_{k=0}^{\infty} \frac{\alpha}{k!} \frac{(-1)^k q^{k(n)}(a; q)_k}{(q^2; q^2)_k} q^{-\alpha} \frac{p_n(x,y)xt^{n+1}}{(x; q)_\infty} \Phi_1 \left[ \frac{-q^{-\alpha}, a; q; ztq^\alpha}{-q^{-\alpha}, a; q; ztq^\alpha} \right].
\end{aligned}
\]

Letting $f(x,y,a,z,\bar{r},\bar{s}) = \sum_{k=0}^{\infty} A_k(x,y,a,\bar{r},\bar{s})z^k$ and

\[
A_k(x,y,a,\bar{r},\bar{s}) = q^{k\alpha + \bar{r}(\bar{k}) - \bar{s}(\bar{k})} \frac{(-q^{-\alpha}, a; q)_k}{(q^2; q^2)_k} D_{xy} \left( \frac{yt; q}{(xt; q)_\infty} \right).
\]

we obtain

\[
A_0(x,y,a,\bar{r},\bar{s}) = \frac{yt; q}{(xt; q)_\infty},
\]

and $f(x,y,a,0,\bar{r},\bar{s}) = A_0(x,y,a,\bar{r},\bar{s})$. Taking (3.5) into (3.4), we get:

\[
A_k(x,y,a,\bar{r},\bar{s}) = q^{k\alpha + \bar{r}(\bar{k}) - \bar{s}(\bar{k})} \frac{(-q^{-\alpha}, a; q)_k}{(q^2; q^2)_k} D_{xy} \left[ A_0(x,y,a,\bar{r},\bar{s}) \right].
\]

Through above identities, $f(x,y,a,z,\bar{r},\bar{s})$ satisfies the equation (2.1). So, we have

\[
\begin{aligned}
f(x,y,a,z,\bar{r},\bar{s}) &= \sum_{k=0}^{n} \frac{(-q^{-\alpha}, a; q)_k}{(q^2; q^2)_k} q^{k\alpha + \bar{r}(\bar{k}) - \bar{s}(\bar{k})} \frac{p_n(x,y)t^n}{(q; q)_n} \sum_{n=0}^{\infty} \frac{p_n(x,y)t^n}{(q; q)_n}
\end{aligned}
\]
Lemma 8.

\[
\text{Proof.} \text{ By means of the Leibniz rule (4.3), the left-hand side of (4.5) equals}
\]
\[
\sum_{n=0}^{\infty} \frac{r^n}{(q; q)_n} \sum_{k=0}^{n} \frac{\omega^{k(j-n)}}{(q; q)_k} \frac{\omega^k}{\omega} \sum_{j=0}^{k} \frac{(-1)^j q^{j-j}\omega q^j}{(q; q)_j} \frac{\omega^k}{\omega} \cdot \text{Proof.} \text{ By means of the Leibniz rule (4.3), the left-hand side of (4.5) equals}
\]

which is the left-hand side of (3.1). The proof is complete. 

Remark 7. Setting \( y = 0 \), in (3.1), we get the concluding remarks of [14]:

\[
\sum_{n=0}^{\infty} L_{j, k}(x, y, z, a) \frac{r^n}{(q; q)_n} = \frac{1}{(xt; q)_\infty} \sum_{k=0}^{\infty} \frac{(-q^{a}; a; q)_k}{(q^2; q^2)_k} q^{kz(q^{(j+1)}-y^{(j+1)})} (zt^k), \quad |xt| < 1. \tag{3.7}
\]

For \( a \to \infty, r = \delta = 0, x \to y, y \to x, z = -z \) and \( a = -q \), in (3.1), we get the concluding remarks of [1]:

\[
\sum_{n=0}^{\infty} F_n(x, y, z, q) \frac{r^n}{(q; q)_n} = \frac{(xt; zq)_\infty}{(yt; q)_\infty}, \quad |yt| < 1. \tag{3.8}
\]

4. Rogers formula for generalized \( q \)-polynomials

In this section, we give and prove the Rogers formula for generalized \( q \)-polynomials by using the \( q \)-difference equations, so that we derive Rogers formula for the trivariate \( q \)-polynomials:

Chen and Liu [7] have defined the \( q \)-exponential operator as follows (see [18]):

\[
T(bD_a) = \sum_{n=0}^{\infty} \frac{(bD_a)^n}{(q; q)_n}, \tag{4.1}
\]

where the usual \( q \)-differential operator, or \( q \)-derivative, is defined by

\[
D_a f(a) = \frac{f(a) - f(qa)}{a}. \tag{4.2}
\]

The Leibniz rule for \( D_a \) is the following identity, which is a variation of \( q \)-binomial theorem [19]:

\[
D_a^n(f(a))g(a) = \sum_{k=0}^{n} \binom{n}{k} D_a^{n-k} f(a) D_a^k g(a). \tag{4.3}
\]

where \( D_q^0 \) is understood as the identity. The following property of \( D_a \) is straightforward, but important.

Lemma 8.

\[
D_a^n \left( \frac{(as; q)_\infty}{(as; q)_\infty} \right) = \omega^n \left( \frac{(s/\omega; q)_\infty}{(as; q)_\infty} \right) \tag{4.4}
\]

Lemma 9. For \( k \in \mathbb{N}_0 \) and \( \max\{|x_0|, |xt|\} < 1 \), we have:

\[
T(tD_\omega) \left( \frac{(yw; q)_\infty}{(x_0; q)_\infty} \omega^k \right) = \frac{(yw; q)_\infty}{(x_0; q)_\infty} \omega^k \sum_{j=0}^{k} \frac{(-1)^j q^{j-j}(q^{-k}, x_0; q)_j (t/\omega)^j}{(y; q)_j} \Phi_1 \left[ \frac{y/x, 0; q; xt}{ywq^j; \omega} \right]. \tag{4.5}
\]

Proof. By means of the Leibniz rule (4.3), the left-hand side of (4.5) equals

\[
\sum_{n=0}^{\infty} \frac{r^n}{(q; q)_n} \sum_{j=0}^{n} \frac{\omega^{k(j-n)}}{(q; q)_k} \frac{\omega^k}{\omega} \sum_{j=0}^{k} \frac{(-1)^j q^{j-j}\omega q^j}{(q; q)_j} \frac{\omega^k}{\omega} \cdot \text{Proof. By means of the Leibniz rule (4.3), the left-hand side of (4.5) equals}
\]
\[ \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{q^{(j-n)n}}{(q; q)_n (q; q)^{n-j}} \omega^n \text{D}_{\omega}^{n-j} \left\{ \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \right\} \]

\[ = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{-jn} n^j}{(q; q)_n (q; q)^{n-j}} \omega^n \text{D}_{\omega}^{n-j} \left( \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \right) \] (by (4.4))

\[ = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{-jn} n^j}{(q; q)_n (q; q)^{n-j}} \omega^n \text{D}_{\omega}^{n-j} \left( \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \right) \]

which is the right-hand side of (4.5).

The generalized Rogers-Szegö polynomials [9, 20] are defined as

\[ r_n(x, y) = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \] (4.6)

where [20]

\[ r_n(x, y) = T(xD_x) \{y^n\}. \] (4.7)

Now, we are in position to give and prove the following Rogers formula for generalized \( q \)-polynomials by using the \( q \)-difference equations.

**Theorem 10.** Let \( \tilde{L}_n^{(\tilde{\alpha}, \tilde{\beta})}(\alpha, x, y, z, a) \) be the polynomials defined as in (1.10).

- For \( |\omega x| < 1 \), we have:

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{L}_n^{(0,0)}(x, y) \frac{t^n}{(q; q)_n} \omega^m \frac{t^m}{(q; q)_m} \]

\[ = \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} D_{\omega}^{n-j} \left( \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \right) \frac{y/x, 0; q, xt}{y_0q}; q_0 \]. (4.8)

- For \( \max(|\omega x|, |xt|) < 1 \), and \( \tilde{\beta} = \tilde{\gamma} = 0 \), we have:

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{L}_n^{(0,0)}(x, y) \frac{t^n}{(q; q)_n} \omega^m \frac{t^m}{(q; q)_m} \]

\[ = \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} D_{\omega}^{n-j} \left( \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \right) \frac{y/x, 0; q, xt}{y_0q}; q_0 \]. (4.9)

**Proof of Theorem 10.** The right-hand side of equation (4.8) can equivalently be written as:

\[ f(\alpha, x, y, a, z, \tilde{\alpha}, \tilde{\beta}) = \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} D_{\omega}^{n-j} \left( \frac{(y_0; q)_\infty}{(x_0; q)_\infty} \right) \frac{y/x, 0; q, xt}{y_0q}; q_0 \]
The Hahn polynomials \( \Phi_n(x|q) \) are defined as
\[
\phi_n^{(\sigma)}(x|q) = \sum_{k=0}^{n} \binom{n}{k} (\sigma;q)_k x^k.
\]

While Cao [6] used the technique of exponential operator decomposition, Srivastava and Agarwal [23] adopted the method of transformation theory to deduce the following results (For more information, please refer to [12, 13, 23, 24, 5, 6, 6]):

**Lemma 12 ([23, Eq. (3.20)])**. For max\([|t|, |xt|] < 1\), we have:
\[
\sum_{n=0}^{\infty} \phi_n^{(\sigma)}(x|q)t^n = \frac{\lambda t; q^n}{(\alpha; q)_n} \frac{\lambda, \sigma; q}{\lambda t; q} \Phi_1 \left[ \frac{\lambda, \sigma; q}{\lambda t; q} \right].
\]

By setting \( m \to m + n \), we get the left-hand side of (4.8). This completes the proof.

As a special case of Theorem 10, if we take \( \alpha \to \infty, \tilde{t} = 0, x \to y, y \to x, z = -z \), and \( a = -q \), in (4.8), we obtain the following corollary:

**Corollary 11 ([1, Theorem 3.1]).** For max\([|y|, |x|] < 1\), we have:
\[
\Phi_n^{(\sigma)}(x|q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{m+n}(x, y, z; q)(-1)^{m+n} q^{m+n} \frac{\omega^m}{(q; q)_n} \frac{\omega^m}{(q; q)_m} \Phi_1 \left[ \frac{x, y, z; q}{y, z; q} \right].
\]
In Theorem 13 below, we give and prove a mixed generating function for generalized \( q \)-polynomials by applying the \( q \)-difference equations.

**Theorem 13.** Let \( \bar{r}, \bar{s} \in \mathbb{Z} \). For \(|ut| < 1\), we have:

\[
\sum_{n=0}^{\infty} \phi_n^{(r)}(x) L_n^{(r,\bar{s})}(\alpha, u, v, z, a) \frac{t^n}{(q; q)_n} = \frac{(vt; q)_\infty}{(ut; q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\sigma; q)_m x^n (q^{-m}; ut; q)_j q^j}{(q; q)_m} \frac{(-q^{-\alpha}; q)_k (\alpha; q)_k (tzq^j)^k}{(q^2; q^2)_k} q^{k\bar{r}+\bar{s}+\bar{k}+\bar{z}+\bar{k}+1}. \tag{5.3}
\]

In the proof of Theorem 13, the following \( q \)-Chu-Vandermonde formula will be needed:

**Lemma 14** (\( q \)-Chu-Vandermonde [10, Eq. (II.6)].)

\[
\Phi_1 \left[ \frac{q^{-\alpha}; a}{(q; q)_n} \right] = \frac{(c/a; q)_n}{(c; q)_n} a^n. \tag{5.4}
\]

**Proof of Theorem 13.** Eq. (5.3) can equivalently be written as follows:

\[
\sum_{n=0}^{\infty} \phi_n^{(r)}(x) L_n^{(r,\bar{s})}(\alpha, u, v, z, a) \frac{t^n}{(q; q)_n} = \sum_{m=0}^{\infty} \frac{(\sigma; q)_m x^n}{(q; q)_m} \sum_{j=0}^{\infty} \frac{(q^{-m}; q)_j q^j}{(q; q)_j} \sum_{k=0}^{\infty} q^{k\bar{r}+\bar{s}+\bar{k}+\bar{z}+\bar{k}+1} \frac{(-q^{-\alpha}; q)_k (\alpha; q)_k z^k}{(q^2; q^2)_k} D^k_{uv} \left\{ (vtq^j; q)_\infty \right\}. \tag{5.5}
\]

If we use \( g(\alpha, u, v, a, z, \bar{r}, \bar{s}) \) to denote the right-hand side of (5.5), it is easy to see that \( g(\alpha, u, v, a, z, \bar{r}, \bar{s}) \) satisfies (2.1). Letting \( g(\alpha, u, v, a, z, \bar{r}, \bar{s}) = \sum_{k=0}^{\infty} B_k(\alpha, u, v, a, \bar{r}, \bar{s}) z^k \) and

\[
B_k(\alpha, u, v, a, \bar{r}, \bar{s}) = q^{k\bar{r}+\bar{s}+\bar{k}+\bar{z}+\bar{k}} \left( \frac{-q^{-\alpha}; q)_k (\alpha; q)_k}{(q^2; q^2)_k} D^k_{uv} \left\{ \sum_{m=0}^{\infty} \frac{(\sigma; q)_m x^n}{(q; q)_m} \sum_{j=0}^{\infty} \frac{(q^{-m}; q)_j q^j}{(q; q)_j} \frac{(vtq^j; q)_\infty}{(utq^j; q)_\infty} \right\} \right),
\]

we obtain

\[
B_0(\alpha, u, v, a, \bar{r}, \bar{s}) = \frac{(vt; q)_\infty}{(ut; q)_\infty} \sum_{m=0}^{\infty} \frac{(\sigma; q)_m x^n}{(q; q)_m} \sum_{j=0}^{\infty} \frac{(q^{-m}; ut; q)_j q^j}{(q; q)_j} \Phi_1 \left[ \frac{q^{-m}; ut; a}{(q; q)_n} \right] \frac{q^j}{vt}; \tag{by (5.4)}
\]

\[
= \frac{(vt; q)_\infty}{(ut; q)_\infty} \sum_{m=0}^{\infty} \frac{(\sigma; q)_m x^n}{(q; q)_m} \frac{(v/ut; q)_m (ut)^m}{(vt; q)_m}
\]

\[
= \frac{(vt; q)_\infty}{(ut; q)_\infty} \Phi_1 \left[ \frac{v/ut; a}{(v; q)_n} \right] \frac{q^{utx}}{vt}; \tag{5.7}
\]

and \( g(\alpha, u, v, a, 0, \bar{r}, \bar{s}) = B_0(\alpha, u, v, a, \bar{r}, \bar{s}) \). Taking equation (5.7) into (5.6), we get:

\[
B_k(\alpha, u, v, a, \bar{r}, \bar{s}) = q^{k\bar{r}+\bar{s}+\bar{k}+\bar{z}+\bar{k}} \frac{(-q^{-\alpha}; a; q)_k}{(q^2; q^2)_k} D^k_{uv} \left\{ B_0(\alpha, u, v, a, \bar{r}, \bar{s}) \right\}. \tag{5.8}
\]
Through above identities, \( g(\alpha, u, v, a, z, \tilde{r}, \tilde{s}) \) satisfies the equation (2.1). So, we have
\[
g(\alpha, u, v, a, z, \tilde{r}, \tilde{s}) = \sum_{n=0}^{\infty} \frac{(-q^{-\alpha}a; q)_k}{(q^2; q^2)_k} q^{ka+\tilde{r}_k+\tilde{e}_k+1} z^k D_{uv}^k \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q) \frac{p_n(u, v)}{(q; q)_n}^{\alpha}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-q^{-\alpha}a; q)_k}{(q^2; q^2)_k} q^{ka+\tilde{r}_k+\tilde{e}_k+1} z^k \phi_n^{(\alpha)}(x|q) \frac{p_{n-k}(u, v)}{(q; q)_{n-k}}^{\alpha}
\]
\[
= \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q) \frac{p_n(u, v)}{(q; q)_n}^{\alpha}
\]
which is the left hand side of (5.3). The proof is complete.

As a special case of Theorem 13, if we take \( \alpha \to \infty, a = -q, u \to v, v \to u, z = -z \) and \( \tilde{r} = \tilde{s} = 0 \), we have the following corollary:

Corollary 15 (Mixed generating function for trivariate \( q \)-polynomials \( F_n(x, y, z; q) \)). For max\(|vt|, |x|\) < 1, we have:
\[
\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q) F_n(u, v, z; q) \frac{(-1)^n q^{\frac{n(n+1)}{2}} p_n^{\alpha}}{(q; q)_n} = \frac{(x, ut, zt; q)_\infty}{(vt, x; q)_\infty} q\Phi_3 \left[ \begin{array}{c} \alpha, vt, 0, 0; \\ ut, zt, q/x; \\ q, q; \end{array} \right].
\] (5.9)

6. Concluding remarks

In this paper, by using the method of \( q \)-difference equations, we have systematically deduced several types of generating functions for certain \( q \)-polynomials. In fact, \( q \)-Laguerre polynomials and Cigler’s polynomials have double \( q \)-binomial coefficients. Computing their generating functions reveals to be too difficult. We have noticed that the \( q \)-difference equations are important tools to calculate generating functions for \( q \)-polynomials. It is necessary to construct \( q \)-difference equations satisfied by the general \( q \)-polynomials with double \( q \)-binomial coefficients. We have then focused on the expansion of a function of many variables and on some \( q \)-polynomials. Therefore, we have searched for the generalized \( q \)-difference equations for general \( q \)-polynomials with double \( q \)-binomial coefficients. We believe that this work will be a motivation to study other \( q \)-polynomials and their applications.

Acknowledgments

The authors are grateful to the referees and editor for their useful comments and suggestions to improve the paper. This work was supported by the Zhejiang Provincial Natural Science Foundation of China (No. LY21A010019). The ICMP-UNESCO Chair is in partnership with the Association pour la Promotion Scientifique de l’Afrique (APSA), France, and Daniel Iagolnitzer Foundation (DIF), France, supporting the development of mathematical physics in Africa.

References

[1] M. A. Abdullhusein, Two operator representations for the trivariate \( q \)-polynomials and Hahn polynomials, Ramanujan J. 40 (2016), pp. 491–509.
[2] W. A. Al-Salam and L.Carlitz, Some orthogonal \( q \)-polynomials, Math. Nachr. 30 (1965), pp. 47–61.
[3] S. Arjika, Certain Generating Functions for Cigler’s Polynomials, Montes Taurus J. Pure Appl. Math. 3 (3) (2021), pp. 284–296.
[4] J. Cao, New proofs of generating functions for Rogers-Szegö polynomials, Appl. Math. Comput. 207 (2009), pp. 486–492.
[5] J. Cao, Bivariate generating functions for Rogers-Szegö polynomials, Appl. Math. Comput. 217 (2010), pp. 2209–2216.
[6] J. Cao, \textit{Generalizations of certain Carlitz's trilinear and Srivastava--Agarwal type generating functions}, J. Math. Anal. Appl. \textbf{396} (2012), pp. 351–362.

[7] W. Y. C. Chen and Z.-G. Liu, \textit{Parameter augmenting for basic hypergeometric series}, II. J. Comb. Theory A \textbf{80} (1997), pp. 175–195.

[8] W. Y. C. Chen, A. M. Fu, and B. Zhang, \textit{The homogeneous \(q\)-difference operator}, Adv. Appl. Math. \textbf{31} (2003), pp. 659–668.

[9] J. Cigler, \textit{Elementare \(q\)-identit"aten}, Publication de L’institute de recherche Mathématique Avancée, Strasbourg (1982), pp. 23–57.

[10] G. Gasper and M. Rahman, \textit{Basic Hypergeometric Series}, 2nd edn, Cambridge University Press, Cambridge, 2004.

[11] R. Gunning, \textit{Introduction to Holomorphic Functions of Several Variables. In: Function theory \textbf{1}}, Wadsworth and Brooks/Cole, Belmont, 1990.

[12] W. Hahn, \textit{Uber orthogonalpolynome, die \(q\)-differenzengleichungen}, Math. Nachr. \textbf{2} (1949), 434.

[13] W. Hahn, \textit{Beitrage zur Theorie der Heineschen Reihen; Die 24 Integrale der hypergeometrischen \(q\)-Differenzengleichung; Das \(q\)-Analogon der Laplace-Transformation}, Math. Nachr. \textbf{2} (1949), pp. 340–379.

[14] Z. Jia, B. Khan, Q. Hu and D.-W. Niu, \textit{Applications of generalized \(q\)-difference equations for general \(q\)-polynomials}, Symmetry \textbf{2021} (13), 1222.

[15] R. Koekock and R. F. Swarttouw, \textit{The Askey-scheme of hypergeometric orthogonal polynomials and its \(q\)-analogue report}, Tech. Rep. 98–17, Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, 1998.

[16] Z.-G. Liu, \textit{On the \(q\)-partial differential equations and \(q\)-series}, In: \textit{The legacy of Srinivasa Ramanujan}, Ramanujan Mathematical Society Lecture Note Series, Mysore \textbf{20} (2013), pp. 213–250

[17] B. Malgrange, \textit{Lectures on the Theory of Functions of Several Complex Variables}, Springer, Berlin 1984.

[18] H. L. Saad, A. A. Sukhi, \textit{The \(q\)-exponential operator}, Applied Mathematical Sciences, \textbf{7} (128) (2013), pp. 6369–6380.

[19] S. Roman, \textit{More on the umbral calculus, with emphasis on the \(q\)-umbral calculus}, J. Math. Anal. Appl. \textbf{107} (1985), pp. 222–254.

[20] H.L. Saad, M.A. Abdilhusien, \textit{The \(q\)-exponential operator and generalized Rogers-Szegő polynomials}, J. Adv. Math. \textbf{8} (2014), 1440–1455.

[21] L. J. Slater, \textit{Generalized Hypergeometric Functions}, Cambridge Univ. Press, Cambridge, London, New York, 1966.

[22] H. M. Srivastava, \textit{Operators of basic (or \(q\)) calculus and fractional \(q\)-calculus and their applications in geometric function theory of complex analysis}, Iran. J. Sci. Technol. Trans. A: Sci. \textbf{44} (2020), pp. 327–344.

[23] H. M. Srivastava and A. K. Agarwal, \textit{Generating functions for a class of \(q\)-polynomials}, Ann. Mat. Pure Appl. \textbf{154} (4) (1989), pp. 99–109.

[24] H. M. Srivastava and P. W. Karlsson, \textit{Multiple Gaussian Hypergeometric Series}, Halsted Ellis Horwood, Chichester; Wiley, New York, 1985.

[25] J. Taylor, \textit{Several complex variables with connections to algebraic geometry and lie groups}, Graduate Studies in Mathematics, American Mathematical Society, Providence, \textbf{46}, 2002.

[26] X.-F. Wang and J. Cao, \textit{\(q\)-Difference equations for the generalized Cigler’s polynomials}, J. Difference Equ. Appl. \textbf{24} (2018), pp. 479–502.