HARMONIC AND SPECTRAL ANALYSIS OF
ABSTRACT PARABOLIC OPERATORS IN
HOMOGENEOUS FUNCTION SPACES

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Abstract. We use methods of harmonic analysis and group representation theory to study the spectral properties of the abstract parabolic operator \( L = -d/dt + A \) in homogeneous function spaces. We provide sufficient conditions for invertibility of such operators in terms of the spectral properties of the operator \( A \) and the semi-group generated by \( A \). We introduce a homogeneous space of functions with absolutely summable spectrum and prove a generalization of the Gearhart-Prüss Theorem for such spaces. We use the results to prove existence and uniqueness of solutions of a certain class of non-linear equations.

1. Introduction

In this paper we study the spectral properties of a differential operator
\[
L = -d/dt + A : D(L) \subset \mathcal{F}(\mathbb{R}, X) \to \mathcal{F}(\mathbb{R}, X)
\]

in homogeneous Banach spaces \( \mathcal{F}(\mathbb{R}, X) \) of functions with values in a complex Banach space \( X \). The operator \( A : D(A) \subset X \to X \) in (1.1) is assumed to be the infinitesimal generator of a \( C_0 \)-semigroup \( T : \mathbb{R}_+ = [0, \infty) \to B(X) \). The homogeneous spaces \( \mathcal{F} = \mathcal{F}(\mathbb{R}, X) \) and the operator \( L \) are identified precisely in Definitions 2.1 and 3.1, respectively.

The properties of the homogeneous spaces \( \mathcal{F} \) allow us to correctly define the Howland semigroup \( \mathcal{S} : \mathbb{R}_+ \to \mathcal{B}(\mathcal{F}) \) by
\[
(\mathcal{S}(t)x)(s) = T(t)x(s-t), \ s \in \mathbb{R}, x \in \mathcal{F}, t \in \mathbb{R}_+.
\]

In some homogeneous spaces such as \( C_0(\mathbb{R}, X) \) and \( L^p(\mathbb{R}, X) \), see Example 2.1 for the definitions, it was proved [4, 5] that the operator...
\( \mathcal{L} \) is the infinitesimal generator of the semigroup \( \mathcal{T} \). Moreover, in a large class of homogeneous spaces the following result holds.

**Theorem 1.1.** The following are equivalent:

- The operator \( \mathcal{L} \) is invertible, that is, the equation
  \[
  \frac{dx}{dt} = Ax + y
  \]
  has a unique (mild) solution \( x \in \mathcal{F}(\mathbb{R}, X) \) for any \( y \in \mathcal{F}(\mathbb{R}, X) \);
- The semigroup \( T \) is hyperbolic, that is, the spectrum \( \sigma(T) \) of the semigroup \( T \) satisfies \( \sigma(T(1)) \cap \mathbb{T} = \emptyset \), \( \mathbb{T} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \);
- The Howland semigroup \( \mathcal{T} \) is hyperbolic.

For \( \mathcal{F} \in \{ C_0, L^p \}, 1 \leq p < \infty \), the above result was proved in [30, 31] and [17, Theorem 2.39]. For a larger class of spaces, including \( L^\infty \) and \( C_0 \), the theorem appears in [4, 5, 9]. We remark that in many homogeneous spaces the result still holds even though the Howland semigroup \( \mathcal{T} \) is not strongly continuous (see [9]).

In this paper we prove that one of the implications in the above theorem holds for all homogeneous spaces in Definition 2.1:

**Theorem 1.2.** If \( T \) is a hyperbolic semigroup, then the operator \( \mathcal{L} \) is invertible.

If \( X \) is a Hilbert space, classical results of L. Gearhart and J. Prüss [24, 35, 20, 34] provide another equivalence to the statements in Theorem 1.1.

**Theorem 1.3.** The operator \( \mathcal{L} \) is invertible if and only if the imaginary axis \( i\mathbb{R} \) is a subset of the resolvent set \( \rho(A) \) of the generator \( A \), i.e.
\[
\sigma(A) \cap (i\mathbb{R}) = \emptyset,
\]
and the resolvent operator \( R(\lambda, A) = (A - \lambda I)^{-1} \) satisfies
\[
M = \sup_{\lambda \in i\mathbb{R}} \| R(\lambda, A) \| < \infty.
\]

If \( X \) is not a Hilbert space, this equivalence does not hold in general (see [25] and [20, Counterexample IV.2.7]). Typical generalizations of the above result to Banach spaces [32, and references therein] would impose additional restrictions on the resolvent \( R(\lambda, A) \) or the Banach space \( X \). In this paper, we pursue a different kind of generalization, where we deal with an arbitrary (complex) Banach space \( X \) and impose no additional restrictions on the resolvent. Instead, we define a class \( \mathcal{F}_{aw} \) of functions in \( \mathcal{F}(\mathbb{R}, X) \) which is a Banach function space where the differential operator \( \mathcal{L} \) is invertible provided that (1.3) and (1.4) hold.
The space $\mathcal{F}_{\text{as}}$ of functions with the absolutely summable spectrum (see Definition 2.5) is defined using the spectral theory of Banach modules [7, 28].

We refer to [1, 17, 20] for more information on the background and history of research related to this paper.

The remainder of the paper is organized as follows. In Section 2 we define homogeneous function spaces and introduce the space $\mathcal{F}_{\text{as}}$ of functions with the absolutely summable spectrum. We do the latter by means of the spectral theory of Banach $L^1(\mathbb{R})$-modules, basic notions of which are also discussed in Section 2. In Section 3 we study the basic properties of the differential operator $\mathcal{L}_F$ and prove the first of our two main results: a sufficient condition for invertibility of $\mathcal{L}_F$ in a homogeneous Banach function space, which appears in Theorem 3.3. Our other main result, Theorem 4.1, appears in Section 4 and establishes sufficient conditions for invertibility of the operator $\mathcal{L}$ in any homogeneous space of functions with the absolutely summable spectrum. We also provide an estimate for $\|\mathcal{L}_F^{-1}\|$. In the final section of the paper we illustrate our results with a counterexample to the Gearhart-Prüss Theorem and an application to a special kind of non-linear differential equations.

2. Preliminaries

In this section we introduce the notation, define homogeneous function spaces, and survey the necessary tools from the spectral theory of Banach $L^1(\mathbb{R})$-modules. We also introduce the space of functions with the absolutely summable spectrum.

The symbol $X$ will denote a complex Banach space and $B(X)$ will be the Banach algebra of all bounded linear operators on $X$. By $T : \mathbb{R}_+ \to B(X)$ we shall denote a $C_0$-semigroup of operators in $B(X)$ and $A : D(A) \subset X \to X$ will be its infinitesimal generator [20].

By $L^1(\mathbb{R}, X)$ we shall denote the Banach space of all (equivalence classes) of Bochner integrable $X$-valued functions with the standard $L^1$-norm:

$$\|f\| = \|f\|_1 = \int_{\mathbb{R}} \|f(t)\|_X dt, \quad f \in L^1(\mathbb{R}, X).$$

If $X = \mathbb{C}$, we shall use the notation $L^1 = L^1(\mathbb{R})$ for the standard group algebra of Lebesgue integrable functions. For $f \in L^1(\mathbb{R})$, we shall denote by $\hat{f}$ the Fourier transform of $f$ given by

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(t)e^{-it\lambda} dt, \quad \lambda \in \mathbb{R}.$$
The space $L^1(\mathbb{R}, B(X))$ defined as above, however, may be too small for our purposes. Occasionally, we will use the space $L^1_s(\mathbb{R}, B(X))$ that consists of all functions $F : \mathbb{R} \to B(X)$ with the following properties:

1. For all $x \in X$ the function $s \mapsto F(s)x : \mathbb{R} \to X$ is measurable;
2. There is $f \in L^1(\mathbb{R})$ such that

\[(2.1) \quad \|F(s)\| \leq f(s) \text{ a.e.} \]

For $F \in L^1_s(\mathbb{R}, B(X))$ we let

\[\|F\| = \|F\|_1 = \inf \|f\|,\]

where the infimum is taken over all functions $f$ that satisfy (2.1).

The space $L^1_s(\mathbb{R}, B(X))$ is a Banach algebra with the multiplication given by

\[(F_1 * F_2)(t) = \int_{\mathbb{R}} F_1(s)F_2(t - s)ds, \quad F_1, F_2 \in L^1_s(\mathbb{R}, B(X)).\]

In particular, $\|F_1 * F_2\| \leq \|F_1\| \|F_2\|_2$.

We shall also use the space of locally integrable $X$-valued functions $L^1_{loc}(\mathbb{R}, X)$, which consists of all measurable functions $f : \mathbb{R} \to X$ such that

\[\int_K \|f(t)\|_X dt < \infty\]

for any compact set $K \subset \mathbb{R}$.

For $p \in [1, \infty)$, the Stepanov space $S^p = S^p(\mathbb{R}, X)$ consists of all functions $x \in L^1_{loc}(\mathbb{R}, X)$ such that

\[\|x\|_{S^p} = \sup_{t \in \mathbb{R}} \left( \int_0^1 \|x(s + t)\|^p ds \right)^{1/p} < \infty.\]

2.1. Homogeneous Banach function spaces.

In this paper we consider Banach function spaces $\mathcal{F}(\mathbb{R}, X)$ that are homogeneous according to the following definition.

**Definition 2.1.** A Banach function space $\mathcal{F} = \mathcal{F}(\mathbb{R}, X)$ is homogeneous if it has the following properties:

1. $\mathcal{F}$ is continuously embedded into $S^1$;
2. For all $t \in \mathbb{R}$ and $x \in \mathcal{F}$ we have $S(t)x \in \mathcal{F}$, where

\[S(t)x(s) = x(t + s),\]

and the translation operator $S(t)$ is an isometry in $B(\mathcal{F})$;
3. Given $x \in \mathcal{F}$ and $C \in B(X)$, the function

\[y(t) = C(x(t))\]

belongs to $\mathcal{F}$ and $\|y\| \leq \|C\| \|x\|$.
(4) Given \( x \in \mathcal{F} \) and \( F \in L^1_s(\mathbb{R}, B(X)) \), the convolution

\[
(F \ast x)(t) = \int_{\mathbb{R}} F(s)x(t-s)ds
\]

belongs to \( \mathcal{F} \) and \( \|F \ast x\| \leq \|F\|_1 \|x\| \).

(5) If \( x \in \mathcal{F} \) is such that \( f \ast x = 0 \) for all \( f \in L^1(\mathbb{R}) \) then \( x = 0 \).

**Example 2.1.** The following Banach spaces are homogeneous or have an equivalent norm that makes them homogeneous:

1. The spaces \( L^p = L^p(\mathbb{R}, X), p \in [1, \infty] \), of functions \( x \in L^1_{loc}(\mathbb{R}, X) \) such that

\[
\|x\|_{L^p} = \left( \int_{\mathbb{R}} \|x(s)\|^p ds \right)^{1/p} < \infty, \ p \in [1, \infty),
\]

or \( \|x\|_{\infty} = \text{ess sup}_{t \in \mathbb{R}} |x(t)| < \infty \);

2. Stepanov spaces \( S^p = S^p(\mathbb{R}, X), p \in [1, \infty) \);

3. Wiener amalgam spaces \( L^{p,q} = L^{p,q}(\mathbb{R}, X), p, q \in [1, \infty) \), of functions \( x \in L^1_{loc}(\mathbb{R}, X) \) such that

\[
\|x\|_{L^{p,q}} = \left( \sum_{k \in \mathbb{Z}} \left( \int_0^1 \|x(s+k)\|^p ds \right)^{q/p} \right)^{1/q} < \infty, \ p, q \in [1, \infty);
\]

4. The space \( C^b = C^b(\mathbb{R}, X) \) of bounded continuous \( X \)-valued functions with the norm

\[
\|x\|_{\infty} = \sup_{t \in \mathbb{R}} \|x(t)\|, \ x \in C^b;
\]

5. The subspace \( C_{ub} = C_{ub}(\mathbb{R}, X) \subset C^b \) of uniformly continuous functions;

6. The subspace \( C_0 = C_0(\mathbb{R}, X) \subset C_{ub} \) of continuous functions vanishing at infinity: \( x \in C_0 \) if \( \lim_{|t| \to \infty} \|x(t)\| = 0 \);

7. The subspace \( C_{sl,\infty} = C_{sl,\infty}(\mathbb{R}, X) \subset C_{ub} \) of slowly varying at infinity functions: \( x \in C_{sl,\infty} \) if \( \lim_{|\tau| \to \infty} \|x(\tau + t) - x(\tau)\| = 0 \) for all \( t \in \mathbb{R} \) (see [11]);

8. The subspace \( C_{\omega} = C_{\omega}(\mathbb{R}, X) \subset C_{ub} \) of \( \omega \)-periodic functions, \( \omega \in \mathbb{R} \);

9. The subspace \( AP = AP(\mathbb{R}, X) \subset C_{ub} \) of (Bohr) almost periodic functions \([2, 33]\);

10. The subspace \( AP_{\infty} = AP_{\infty}(\mathbb{R}, X) \subset C_{ub} \) of almost periodic at infinity functions \([9]\) defined by

\[
AP_{\infty} = \text{span} \{ e^{i\lambda}x : \lambda \in \mathbb{R}, x \in C_{sl,\infty} \}.\]
The spaces $C^k = C^k(\mathbb{R}, X)$, $k \in \mathbb{N}$, of $k$ times continuously differentiable functions with a bounded $k$-th derivative and the norm

$$||x||_{(k)} = ||x||_\infty + ||x^{(k)}||_\infty < \infty;$$

(12) The Hölder spaces $C^{k,\alpha} = C^{k,\alpha}(\mathbb{R}, X)$, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 1]$:

$$C^{k,\alpha} = \left\{ x \in C^k : ||x^{(k)}||_{C^{0,\alpha}} = \sup_{t \neq s \in \mathbb{R}} \frac{|x(t) - x(s)|}{|t - s|^\alpha} < \infty \right\},$$

$$||x||_{C^{k,\alpha}} = ||x||_{C^k} + ||x^{(k)}||_{C^{0,\alpha}}.$$

**Remark 2.1.** We note that Definition 2.1 in this paper differs from Definition 2.1 in [9], which is more narrow. In particular, here we do not assume that the space $\mathcal{F}$ is solid and this allows us to consider the spaces of periodic and almost periodic functions.

**Definition 2.2.** A homogeneous space $\mathcal{F} = \mathcal{F}(\mathbb{R}, X)$ is called spectrally homogeneous if for all $\lambda \in \mathbb{R}$ and $x \in \mathcal{F}$ we have $e^{i\lambda x} \in \mathcal{F}$ and $||e^{i\lambda x}|| = ||x||$.

Among all of the homogeneous spaces in Example 2.1 the only class of spaces that are not spectrally homogeneous are the spaces of $C_\omega$ of $\omega$-periodic functions.

In a spectrally homogeneous space $\mathcal{F}$ there is a well-defined isometric representation $V : \mathbb{R} \rightarrow B(\mathcal{F})$ given by

$$V(\lambda)x(t) = e^{i\lambda t}x(t), \quad \lambda, t \in \mathbb{R}, x \in \mathcal{F}. \quad (2.3)$$

### 2.2. Banach $L^1$-modules and the Beurling spectrum.

Properties (4) and (5) in Definition 2.1 ensure that a homogeneous Banach space is a non-degenerate Banach $L^1(\mathbb{R})$-module [7, 28]. In this subsection we present the necessary definitions and results from the spectral theory of such Banach modules. The proofs omitted in the presentation and further details can be found in [7, 12, 14, 28].

**Definition 2.3.** A Banach space $\mathcal{X}$ is a *Banach $L^1(\mathbb{R})$-module* if there is a bilinear map $(f, x) \mapsto fx : L^1(\mathbb{R}) \times \mathcal{X} \rightarrow \mathcal{X}$ such that

1. $(f * g)x = f(gx), f, g \in L^1(\mathbb{R}), x \in \mathcal{X}$;
2. $||fx|| \leq ||f|| ||x||, f \in L^1(\mathbb{R}), x \in \mathcal{X}$.

The module structure is non-degenerate if, in addition,

3. $fx = 0$ for all $f \in L^1(\mathbb{R})$ implies $x = 0 \in \mathcal{X}$.

We say that the structure of a Banach $L^1(\mathbb{R})$-module $\mathcal{X}$ is associated with a representation $U : \mathbb{R} \rightarrow B(\mathcal{X})$ if

$$U(t)(fx) = f_tx = f(U(t)x), \quad f \in L^1(\mathbb{R}), x \in \mathcal{X}, \quad f_t(s) = f(s + t). \quad (2.4)$$
As we mentioned above, any homogeneous function space \( X = \mathcal{F}(\mathbb{R}, X) \) is a non-degenerate Banach \( L^1(\mathbb{R}) \)-module. Its structure is given by
\[
(2.5) \quad (fx)(t) = (f \ast x)(t) = \int_{\mathbb{R}} f(t - s)x(s)ds = \int_{\mathbb{R}} f(s)x(t - s)ds,
\]
f \( \in \) \( L^1(\mathbb{R}) \), \( x \in \mathcal{F} \), and is associated with the translation representation \( S \) defined by (2.2).

**Definition 2.4.** The *Beurling spectrum* of an element \( x \in X \) is the subset \( \Lambda(x) \subseteq \mathbb{R} \) the complement of which is given by
\[
\{ \lambda \in \mathbb{R} : \text{there is } f \in L^1(\mathbb{R}) \text{ such that } \hat{f}(\lambda) \neq 0 \text{ and } fx = 0 \}.
\]

**Remark 2.2.** In homogeneous Banach spaces the Beurling spectrum \( \Lambda(x) \) coincides with the support of the (distributional) Fourier transform of \( x \in \mathcal{F} \).

In the next lemma we present basic properties of the Beurling spectrum that will be used throughout the paper. We refer to [7, 12, 36] and references therein for the proof.

**Lemma 2.1.** Let \( \mathcal{X} \) be a non-degenerate Banach \( L^1(\mathbb{R}) \)-module. Then
\begin{itemize}
  \item[(i)] \( \Lambda(x) \) is closed for every \( x \in \mathcal{X} \) and \( \Lambda(x) = \emptyset \) if and only if \( x = 0 \);
  \item[(ii)] \( \Lambda(Ax + By) \subseteq \Lambda(x) \cup \Lambda(y) \) for all \( A, B \in B(\mathcal{X}) \) that commute with all operators \( x \mapsto fx, f \in L^1(\mathbb{R}) \);
  \item[(iii)] \( \Lambda(fx) \subseteq (\text{supp } \hat{f}) \cap \Lambda(x) \) for all \( f \in L^1(\mathbb{R}) \) and \( x \in \mathcal{X} \);
  \item[(iv)] \( fx = 0 \) if \( (\text{supp } \hat{f}) \cap \Lambda(x) \) is countable and \( \hat{f}(\lambda) = 0 \) for all \( \lambda \in (\text{supp } \hat{f}) \cap \Lambda(x), f \in L^1(\mathbb{R}), x \in \mathcal{X} \);
  \item[(v)] \( fx = x \) if \( \Lambda(x) \) is a compact set, the boundary of \( \Lambda(x) \) is countable, and \( \hat{f} \equiv 1 \) on \( \Lambda(x), f \in L^1(\mathbb{R}), x \in \mathcal{X} \).
\end{itemize}

Given a closed set \( \Delta \subseteq \mathbb{R} \) we shall denote by \( \mathcal{X}(\Delta) \) the (closed) *spectral submodule* of all vectors \( x \in \mathcal{X} \) such that \( \Lambda(x) \subseteq \Delta \). The symbol \( \mathcal{X}_{\text{comp}} \) will stand for the set of all vectors \( x \) such that \( \Lambda(x) \) is compact. If the module structure is associated with a representation \( U \), by \( \mathcal{X}_U \) we shall denote the submodule of \( U \)-continuous vectors, i.e., the set of all vectors \( x \in \mathcal{X} \) such that the function \( t \mapsto U(t)x : \mathbb{R} \rightarrow \mathcal{X} \) is continuous.

**Remark 2.3.** Observe that any spectral submodule \( \mathcal{F}(\Delta) \) of a homogeneous function space \( \mathcal{F}(\mathbb{R}, X) \) is itself a homogeneous function space. It may happen that \( \mathcal{F}(\Delta) = \{0\} \) even if \( \Delta \neq \emptyset \). For example, if \( \mathcal{F} = L^p(\mathbb{R}, X), 1 \leq p < \infty \), and \( \Delta \) is finite then \( \mathcal{F}(\Delta) = \{0\} \). If \( \Delta \) is compact then \( \mathcal{F}(\Delta) \subseteq C_{\text{ub}} \) and each \( x \in \mathcal{F}(\Delta) \) extends to an entire function of exponential type \( \omega = \max\{|\lambda|, \lambda \in \Delta\} \) [11].
The following lemma opens up the possibility of applying our main results to non-linear equations.

**Lemma 2.2.** Let \( \mathcal{X} \) be a non-degenerate Banach \( L^1(\mathbb{R}) \)-module. Assume that the module structure is associated with a representation \( U \) as in (2.4). Let \( F : \mathcal{X}^n \to \mathcal{X}, \mathcal{X}^n = \mathcal{X} \times \cdots \times \mathcal{X}, \) be an \( n \)-linear map such that for any \( t \in \mathbb{R} \) we have

\[
U(t)(F(x_1, \ldots, x_n)) = F(U(t)x_1, \ldots, U(t)x_n).
\]

Then

\[
\Lambda(F(x_1, \ldots, x_n)) \subseteq \Lambda(x_1) + \cdots + \Lambda(x_n).
\]

**Proof.** In case \( \mathcal{X} \) is a Banach algebra and \( F(x_1, x_2) = x_1x_2 \), this result appears in [12, 14]. In the general case the proof may be repeated nearly verbatim with obvious modifications. \( \square \)

### 2.3. Vectors with the absolutely summable spectrum.

In this subsection we define the class \( \mathcal{X}_{as} \) of vectors in a Banach \( L^1(\mathbb{R}) \)-module \( \mathcal{X} \) that have absolutely summable spectrum. In the scalar case this class was introduced in [6]. In this exposition we follow [14] where the general definition appears. We use the family of functions \( (\phi_\alpha), \alpha \in \mathbb{R} \), defined via the Fourier transform by

\[
\hat{\phi}_a(\lambda) = \hat{\phi}(\lambda - a), \ a \in \mathbb{R},
\]

where

\[
\hat{\phi}(\lambda) \equiv \hat{\phi}_0(\lambda) = (1 - |\lambda|)\chi_{[-1,1]}(\lambda),
\]

and \( \chi_E \) is, as usually, the characteristic function of the set \( E \).

**Definition 2.5.** The class \( \mathcal{X}_{as} \) of vectors in a Banach \( L^1(\mathbb{R}) \)-module \( \mathcal{X} \) that have absolutely summable spectrum is

\[
\mathcal{X}_{as} = \left\{ x \in \mathcal{X} : \|x\|_{\tilde{\alpha}s} = \int_{\mathbb{R}^d} \|\phi_\alpha x\|da < \infty \right\},
\]

where the functions \( \phi_\alpha \) are defined by (2.6) and (2.7).

In [14], one can find plenty of examples of the spaces with absolutely summable spectrum. Classical Wiener amalgam spaces [21, 23] are among the well-studied spaces that arise in such a way.

**Remark 2.4.** In the above definition, instead of the family of functions \( (\phi_\alpha) \) one can use just about any bounded uniform partition of unity [21, 22]. One would obtain the same space as a result [14]. This is analogous to using different window functions in the short time Fourier transform [26].
In [14] we have shown that $X_{as}$ is a Banach space with the norm $\| \cdot \|_{as}$. For our purposes, however, it is often more convenient to use an equivalent norm given by
\[
\|x\|_{as} = 5 \sum_{n \in \mathbb{Z}} \|\phi_n x\|.
\]
In [14] we obtained the inequalities
\[
\|x\|_{as} \leq \|x\|_{as} \leq 20 \|x\|_{as}, \quad x \in X_{as}.
\]
In [14] we have also shown that if $X$ is a Banach algebra and the module structure is associated with a group of algebra automorphisms, then $X_{as}$ is also a Banach algebra and, due to the choice of the constant in (2.9),
\[
\|xy\|_{as} \leq \|x\|_{as} \|y\|_{as}, \quad x, y \in X_{as}.
\]
Moreover, the key result of [14] states that the algebra $X_{as}$ is inverse closed, i.e. if $x \in X_{as}$ is invertible in $X$, then $x^{-1} \in X_{as}$. The following analog of (2.11) can be proved in exactly the same way.

**Proposition 2.3.** Let $\mathcal{X}$ be a non-degenerate Banach $L^1(\mathbb{R})$-module. Assume that the module structure is associated with a representation $U$ as in (2.4). Let $F : \mathcal{X}^n \to \mathcal{X}$, be an $n$-linear map such that for any $t \in \mathbb{R}$ we have
\[
U(t)(F(x_1, \ldots, x_n)) = F(U(t)x_1, \ldots, U(t)x_n).
\]
Then
\[
\|F(x_1, \ldots, x_n)\|_{as} \leq \|F\| \cdot \prod_{k=1}^n \|x_k\|_{as}.
\]

It has been observed by many people, see, e.g., [27] and [14, Remark 3.5], that smoothness of the function $x$ is closely related to the spectral decay of $x$. In particular, we have $C^{1, \alpha} \subset X_{as}$, $\alpha > 0$. Below we prove a slightly weaker sufficient condition that uses the following modulus of continuity.

**Definition 2.6.** Let $\mathcal{X}$ be a non-degenerate Banach $L^1(\mathbb{R})$-module with the structure associated with an isometric representation $U : \mathbb{R} \to B(\mathcal{X})$. For $x \in \mathcal{X}$, its *modulus of continuity* $\omega_x$ is defined by
\[
\omega_x(t) = \sup_{|s| \leq t} \|U(s)x - x\|, \quad t \geq 0.
\]

**Remark 2.5.** The basic properties of the modulus of continuity can be found, for example in [16, 29]. Here we mention the obvious facts that
\[
\lim_{t \to 0} \omega_x(t) = 0, \quad x \in \mathcal{X}_U,
\]
and \( \omega_x \) is subadditive, that is
\[
\omega_x(t + s) \leq \omega_x(t) + \omega_x(s), \quad x \in \mathcal{X}.
\]
As an immediate consequence of subbaditivity we get
\[
(2.13) \quad \omega_x(ks) \leq k\omega_x(s) \quad \text{and} \quad \frac{\omega_x(1)}{k} \leq \frac{\omega_x(1)}{k}, \quad x \in \mathcal{X}, k \in \mathbb{N}, s \geq 0.
\]
Using the monotonicity of \( \omega_x \) we also get
\[
(2.14) \quad \omega_x(\lambda s) \leq (\lambda + 1)\omega_x(s), \quad \lambda, s > 0, \quad x \in \mathcal{X}.
\]

**Lemma 2.4.** For \( x \in \mathcal{X} \) we have
\[
(2.15) \quad \| \phi_k x \| \leq \text{Const} \cdot \omega_x\left(\frac{1}{|k|}\right), k \in \mathbb{Z} \setminus \{0\}.
\]

**Proof.** Observe that for \( k \neq 0 \)
\[
\phi_k x = \int_{\mathbb{R}} \phi(t) e^{ikt} U(-t) x dt = -\int_{\mathbb{R}} \phi(\tau + \frac{\pi}{k}) e^{ikt} U(-\tau - \frac{\pi}{k}) x d\tau,
\]
where we made the change of variables \( t = \tau + \pi/k \). Averaging the above two expressions we get
\[
\phi_k x = \frac{1}{2} \int_{\mathbb{R}} e^{ikt} (\phi(t) U(-t) - \phi(t + \frac{\pi}{k}) U(-t - \frac{\pi}{k})) x dt
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} e^{ikt} (\phi(t) - \phi(t + \frac{\pi}{k})) U(-t) x dt
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}} e^{ikt} \phi(t + \frac{\pi}{k}) (U(-t) - U(-t - \frac{\pi}{k})) x dt.
\]
Hence,
\[
\| \phi_k x \| \leq \frac{1}{2} \| S\left(\frac{\pi}{k}\right) \phi - \phi \|_1 \| x \| + \frac{1}{2} \omega_x\left(\frac{\pi}{|k|}\right).
\]
Direct computation (as well as [12, Theorem 3.7]) implies that
\[
\| S\left(\frac{\pi}{k}\right) \phi - \phi \|_1 \leq \text{Const} \cdot \frac{1}{|k|}.
\]
Finally, using (2.13) and (2.14) we get (2.15). \( \square \)

**Theorem 2.5.** Assume that \( \mathcal{X} \) is a non-degenerate Banach \( L^1(\mathbb{R}) \)-module with the structure associated with a representation \( U : \mathbb{R} \to B(\mathcal{X}) \). Let \( B \) be the infinitesimal generator of \( U \) and assume that \( x \in D(B) \). Assume also that \( y = Bx \) satisfies
\[
(2.16) \quad \sum_{k \in \mathbb{N}} \frac{\omega_y(1/k)}{k} < \infty.
\]
Then \( x \in \mathcal{X}_{as} \).
Proof. For $k \in \mathbb{N} \setminus \{1\}$, let $f_k \in L^1(\mathbb{R})$ be such that
\[
\hat{f}_k(\lambda) = \begin{cases} \frac{1}{i\lambda}, & \lambda \geq k - 1; \\ \frac{1}{i(2k-2-\lambda)}, & \lambda < k - 1. \end{cases}
\]
Similarly, for $-k \in \mathbb{N} \setminus \{1\}$, let $f_k \in L^1(\mathbb{R})$ be such that
\[
\hat{f}_k(\lambda) = \begin{cases} \frac{1}{i\lambda}, & \lambda \leq k + 1; \\ \frac{1}{i(2k+2-\lambda)}, & \lambda > k + 1. \end{cases}
\]
Observe that for $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$ we have $\|f_k\|_1 = \frac{1}{|k| - 1}$ and the function $\varphi_k = f_k \ast \phi_k \in L^1(\mathbb{R})$ satisfies
\[
\hat{\varphi}_k(\lambda) = \frac{\hat{f}_k(\lambda)}{i\lambda}, \lambda \in \mathbb{R} \setminus \{0\}.
\]
Hence, for $x \in D(B)$ integration by parts yields $\phi_k x = \varphi_k (Bx)$, $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Finally, using (2.15) for $y = Bx$, we get
\[
\|\phi_k x\| \leq \|f_k\|_1 \|\varphi_k y\| \leq \frac{\text{Const}}{|k| - 1} \cdot \omega_y\left(\frac{1}{|k|}\right).
\]
and the result follows. \qed

Observe that condition (2.16) is satisfied automatically if $x \in D(B^{1+\alpha})$, $\alpha > 0$. We also have the following corollary.

**Corollary 2.6.** Let $\mathcal{X} = C_b(\mathbb{R}, X)$. Then for any $\alpha > 0$ we have $C^{1,\alpha} \subset \mathcal{X}_{as}$.

**Proof.** In this case $B = d/dt$. Since, $x \in C^{1,\alpha}$ implies
\[
\omega_x'(t) \leq \|x\|_{C^{1,\alpha}} \cdot t^\alpha, \ t > 0,
\]
the series (2.16) converges. \qed

**Remark 2.6.** If $x \in \mathcal{F} = C_0(\mathbb{R}, X)$ and $x = \hat{y}$ for some $y \in L^1(\mathbb{R})$, then $x \in \mathcal{F}_{as}$ and $\|x\|_{as} \leq 5\|y\|_1$.

We conclude the section with the following useful result.

**Lemma 2.7.** If $\mathcal{F}$ is a (spectrally) homogeneous space then $\mathcal{F}_{as}$ is also a (spectrally) homogeneous space.

**Proof.** Since convolution operators commute with translation, the verification of the properties of a (spectrally) homogeneous space is straightforward and is left to the reader. \qed
3. Basic properties of the operator $L$

In this section we collect and enhance some of the known spectral properties of abstract parabolic operators.

Recall that by $A : D(A) \subseteq X \to X$ we denote the infinitesimal generator of a $C_0$-Semigroup $T$. In a homogeneous space $\mathcal{F}(\mathbb{R}, X)$ we define the differential operator $L = L_F = -d/dt + A$ in (1.1) as follows [5, 9, 33].

**Definition 3.1.** A function $x \in \mathcal{F}$ belongs to the domain $D(L)$ of the operator $L$ if there is a function $y \in \mathcal{F}$ such that for all $s \leq t$ in $\mathbb{R}$ we have

\[(3.1) \quad x(t) = T(t-s)x(s) - \int_s^t T(t-\tau)y(\tau)d\tau.\]

For $x \in D(L)$ we let $Lx = y$, if $x$ and $y$ satisfy (3.1).

We remark that the operator $L$ is well defined as it is not hard to see that for $x \in D(L)$ there is a unique $y$ such that (3.1) is satisfied.

We also remark that $\mathcal{F} \subset S^1$ implies $D(L) \subset C_{ub}$.

The operator $L$ is invertible if it is injective, i.e. ker $L = \{0\}$, and surjective, i.e. its range $\text{Im} L = L D(L)$ satisfies $\text{Im} L = \mathcal{F}$.

We begin studying the spectral properties of the operator $L_F$ with the following key property of its kernel. It was originally proved in [3] for $\mathcal{F} = C_b$.

**Lemma 3.1.** Assume $x \in \ker L_F$. Then

\[(3.2) \quad i\Lambda(x) \subseteq \sigma(A) \cap (i\mathbb{R}).\]

**Proof.** Let $x \in \ker L_F$. Then from (3.1) we get $x(t) = T(t-s)x(s)$, $s \leq t$. Then in view of this equality, for any $f \in L^1(\mathbb{R})$, we have

\[
T(t-s)(f*x)(s) = \int_{\mathbb{R}} f(s-\tau)T(t-s)x(\tau)d\tau = \int_{\mathbb{R}} f(s-\tau)x(t-s+\tau)d\tau = (f*x)(t).
\]

Hence, (3.1) implies $f*x \in \ker L_F$.

Let $i\lambda_0 \notin \sigma(A)$ and $f \in L^1(\mathbb{R})$ be such that $\hat{f}(\lambda_0) = 0$, supp $\hat{f}$ is compact, and $i$ supp $\hat{f} \subset \rho(A)$ Then, according to the above, $y = f*x$ satisfies $y(s+t) = T(t)y(s)$ for all $s \in \mathbb{R}$ and $t \geq 0$. Observe that since $f \in C^\infty$, i.e. $f$ is differentiable infinitely many times, we have $y \in C^\infty$. Moreover,

\[
y'(s+t) = T(t)Ay(s) = AT(t)y(s), \quad s \in \mathbb{R}, t \geq 0,
\]
and, therefore, plugging in $t = 0$, we get $y' - Ay = 0$. 
Let $\psi \in L^1(\mathbb{R})$ be such that $\hat{\psi} = 1$ in a neighborhood of $\text{supp} \hat{f}$, $\text{supp} \hat{\psi}$ is compact, and $i\text{supp} \hat{\psi} \subset \rho(A)$. Consider $F \in L^1(\mathbb{R}, B(\mathcal{F}))$ defined by

$$F(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\psi}(\lambda)R(i\lambda, A)e^{i\lambda t}d\lambda, \quad t \in \mathbb{R};$$

the above integral makes sense because $i\text{supp} \hat{\psi} \subset \rho(A)$. Observe that

$$\hat{F}(\lambda) = \begin{cases} \hat{\psi}(\lambda)R(i\lambda, A), & i\lambda \in \rho(A); \\ 0, & i\lambda \notin \rho(A); \end{cases}$$

is compactly supported and infinitely differentiable. Then, since the operator $A$ is closed,

$$0 = F * (Ay - y') = A(F * y) - F' * y = F_0 * y,$$
where $F_0 = AF - F'$ has Fourier transform

$$\hat{F}_0(\lambda) = \hat{\psi}(\lambda)AR(i\lambda, A) - i\lambda\hat{\psi}(\lambda)R(i\lambda, A) = \hat{\psi}(\lambda)I.$$

Hence, $f * x = y = F_0 * y = 0$ and, therefore, $\lambda_0 \notin \Lambda(x).$ \hfill $\square$

**Corollary 3.2.** If the generator $A$ satisfies (1.3) then the operator $\mathcal{L}_F$ is injective.

**Proof.** Assume that $\mathcal{L}_F x = 0$. Immediately from Lemma 3.1 we get $\Lambda(x) = \emptyset$. Hence, $x = 0$ and, the operator $\mathcal{L}_F$ is injective. \hfill $\square$

Next we proceed to use the above result to obtain invertibility conditions for $\mathcal{L}_F$ in the case when the semigroup $T$ is hyperbolic, i.e. it satisfies

$$\sigma(T(1)) \cap \mathbb{T} = \emptyset.$$  \hfill (3.4)

For such semigroups we have

$$\sigma(T(1)) = \sigma_{in} \cup \sigma_{out},$$

where $\sigma_{in}$ is the spectral component inside the unit disc and $\sigma_{out} = \sigma(T(1)) \setminus \sigma_{in}$. We let $P_{in}$ and $P_{out}$ be the corresponding spectral projections

$$P_{in} = \frac{1}{2\pi i} \int_{\mathbb{T}} (T(1) - \lambda I)^{-1}d\lambda, \quad P_{out} = I - P_{in},$$

and represent the space $X$ as a direct sum

$$X = X_{in} \oplus X_{out}, \quad X_{in} = P_{in}X, \quad X_{out} = P_{out}X.$$

From the definition of the spectral projections it follows that $P_{in}$ and $P_{out}$ commute with the operators $T(t), \quad t \geq 0$. Therefore, $X_{in}$ and $X_{out}$
are invariant subspaces for these operators and we can consider the (restriction) semigroup $T_{in}$ and the group $T_{out}$ defined by

$$T_{in} : \mathbb{R}_+ \to B(X_{in}), \quad T_{in}(t) = T(t)|_{X_{in}};$$

$$T_{out} : \mathbb{R} \to B(X_{out}), \quad T_{out}(t) = \begin{cases} T(t)|_{X_{out}}, & t \geq 0; \\ (T(-t)|_{X_{out}})^{-1}, & t < 0. \end{cases}$$

The following theorem is one of the main results of this paper. As we mentioned in the introduction, its special cases appear in [5, 9, 15, 17].

**Theorem 3.3.** If $T$ is a hyperbolic semigroup, the operator $L_F$ from Definition 3.1 is invertible. The inverse $L_F^{-1} \in B(F)$ is defined by

$$L_F^{-1} y(t) = (G \ast y)(t) = \int_{\mathbb{R}} G(t - \tau)y(\tau)d\tau, \quad t \in \mathbb{R}, \ y \in F,$$

where the Green function $G \in L^1_s(\mathbb{R}, B(F))$ is given by

$$G(t) = \begin{cases} -T(t)P_{in}, & t \geq 0; \\ T_{out}(t)P_{out}, & t < 0. \end{cases}$$

**Proof.** It is immediate from Definition 2.1(4) that the right hand side of (3.5) defines a bound operator in $B(F)$. We need to check that this operator is, indeed, the inverse of $L_F$.

Since the semigroup $T$ is hyperbolic, the spectral inclusion theorem [20, Theorem IV.3.6] implies that the operator $A$ satisfies (1.3). Therefore, the operator $L_F$ is injective by Corollary 3.2.

It remains to show that given $y \in F$ and $x = G \ast y$ we have $L_F x = y$. We get

$$x(t) - T(t-s)x(s) = (G \ast y)(t) - T(t-s)(G \ast y)(s)$$

$$= \int_t^\infty T_{out}(t-\tau)P_{out}y(\tau)d\tau - \int_{-\infty}^{t \wedge s} T(t-\tau)P_{in}y(\tau)d\tau$$

$$- \int_s^\infty T(t-\tau)T_{out}(s-\tau)P_{out}y(\tau)d\tau - \int_{s \wedge -\infty}^s T(t-\tau)P_{out}y(\tau)d\tau$$

$$= -\int_s^t T(t-\tau)P_{out}y(\tau)d\tau + \int_{-\infty}^{t \wedge s} T(t-\tau)P_{in}y(\tau)d\tau$$

$$= -\int_s^t T(t-\tau)y(\tau)d\tau,$$

and the result follows from (3.1). \qed

**Corollary 3.4.** If $T$ is a hyperbolic semigroup, then the generator $A$ satisfies (1.3) and (1.4).
Proof. As we mentioned above (1.3) follows from the spectral mapping theorem. The inequality (1.4) follows since the Fourier transform $\hat{G}$ of the Green function $G$ satisfies $\hat{G}(\lambda) = R(i\lambda, A)$, see [15] for details.

Corollary 3.5. The operator $L_F$ in Definition 3.1 is closed.

Proof. Let $\omega \in \mathbb{R}$ be such that the semigroup $T_\omega$, $T_\omega(t) = T(t)e^{-\omega t}$, satisfies $\|T_\omega(t)\| \to 0$ as $t \to \infty$. The value $\omega_0 = \min \omega$, where the minimum is taken over all $\omega$ with the above property is usually called the growth bound of the semigroup $T$ [20, Definition I.5.6]. Theorem 3.3 applied to $T_\omega$ implies that the operator $L_F - \omega I$ is invertible and $(L_F - \omega I)^{-1}y = G_{\omega} \ast y$, where $G_{\omega}(t) = \begin{cases} -T_\omega(t), & t \geq 0; \\ 0, & t < 0. \end{cases}$

Hence, $\rho(L_F) \neq \emptyset$ and the operator $L_F$ is closed.

In what follows we shall denote by $\tilde{S} : L^1(\mathbb{R}) \to B(F)$ the algebra homomorphism given by $\tilde{S}(f)x = f \ast x$, $f \in L^1(\mathbb{R}), x \in F$.

The next result asserts that in any homogenous space $F$ the operator $L_F$ commutes with the operators $\tilde{S}(f)$, $f \in L^1(\mathbb{R})$.

Lemma 3.6. For all $f \in L^1(\mathbb{R})$ and $x \in D(L_F)$ we have $L_F \tilde{S}(f)x = \tilde{S}(f)L_Fx$.

Proof. Assume $\lambda > \omega_0$ where $\omega_0$ is the growth bound of the semigroup $T$. From (3.5) we deduce that $R(\lambda, L_F)\tilde{S}(f) = \tilde{S}(f)R(\lambda, L_F)$. Let $x \in D(L_F)$ and $y = (L_F - \lambda I)x$. Then

$$\tilde{S}(f)x = \tilde{S}(f)R(\lambda, L_F)(L_F - \lambda I)x = R(\lambda, L_F)\tilde{S}(f)y$$

implies $\tilde{S}(f)x \in D(L_F)$ and

$$(L_F - \lambda I)\tilde{S}(f)x = \tilde{S}(f)y,$$

from where the result immediately follows.

Corollary 3.7. Assume $\Delta \subset \mathbb{R}$ is closed. Then the spectral submodule $F(\Delta)$ is an invariant subspace of the operator $L_F$ and the restriction of $L_F$ to $F(\Delta)$ coincides with the operator $L_{F(\Delta)}$ given by Definition 3.1.

Proof. Assume $x \in F(\Delta)$ and $y = L_Fx$. Let $f \in L^1(\mathbb{R})$ be such that supp $\hat{f} \cap \Delta = \emptyset$. Then $0 = f \ast x = L_F(f \ast x) = f \ast (L_Fy)$ and, therefore, $y \in F(\Delta)$. 

Next, we use the above commutativity relation to extend the result of Lemma 3.1 to the non-homogeneous case.

**Lemma 3.8.** Assume $x \in D(L_F)$ and $y = L_F x$. Then

\[ \Lambda(x) \subseteq \Lambda(y) \cup \{ \lambda \in \mathbb{R} : i\lambda \in \sigma(A) \}. \]

**Proof.** Assume $\lambda \notin \Delta_0 = \Lambda(y) \cup \{ \lambda \in \mathbb{R} : i\lambda \in \sigma(A) \}$ and let $f \in L^1(\mathbb{R})$ be such that $\hat{f}(\lambda) \neq 0$, $\text{supp} \hat{f}$ is compact, and $\text{supp} \hat{f} \cap \Delta_0 = \emptyset$. From Definition 3.1, Lemma 3.6, and Lemma 2.1(iv) we deduce that $f \ast x \in D(L_F)$ and $L_F(f \ast x) = f \ast y = 0$. Hence, Lemma 3.1 implies $i\Lambda(f \ast x) \subseteq \sigma(A) \cap (i\mathbb{R})$. On the other hand, $\Lambda(f \ast x) \subseteq \text{supp} \hat{f} \cap \Lambda(x)$ from Lemma 2.1(iii). Hence, $\Lambda(f \ast x) = \emptyset$, $f \ast x = 0$, and $\lambda \notin \Lambda(x)$. \(\square\)

The following corollary is immediate in view of Lemma 2.1(i).

**Corollary 3.9.** Assume $\Delta \subset \mathbb{R}$ is closed and

\[ i\Delta \cap \sigma(A) = \emptyset. \]

Then $\ker L_{F(\Delta)} = \{0\}$.

**Theorem 3.10.** Assume $\Delta \subset \mathbb{R}$ is compact and the generator $A$ satisfies (3.7). Then the operator $L_{F(\Delta)}$ is invertible.

**Proof.** In view of Corollary 3.9 we only need to prove that the operator $L_{F(\Delta)}$ is onto. Let $y \in F(\Delta)$, i.e. $\Lambda(y) \subseteq \Delta$. Let also $f \in L^1(\mathbb{R})$ be such that $\hat{f} = 1$ in a neighborhood of $\Delta$, $i\text{supp} \hat{f} \subset \rho(A)$ is compact, and $\hat{f} \in C^\infty$, i.e. $\hat{f}$ is differentiable infinitely many times. Consider the function $F \in L^1(\mathbb{R}, B(F))$ defined by

\[ F(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\lambda) R(i\lambda, A)e^{i\lambda t} d\lambda, \quad t \in \mathbb{R}; \]

the above integral makes sense because $i\text{supp} \hat{f} \subset \rho(A)$. Observe that

\[ \hat{F}(\lambda) = \begin{cases} \hat{f}(\lambda) R(i\lambda, A), & i\lambda \in \rho(A); \\ 0, & i\lambda \notin \rho(A); \end{cases} \]

is compactly supported and infinitely differentiable, and $F \ast y \in F(\Delta)$ by Lemma 2.1(iii). Moreover, since the operators $L$ and $A$ are closed, we can write

\[ -\frac{d}{dt}(F \ast y)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} i\lambda \hat{F}(\lambda)e^{i\lambda(t-s)} y(s) d\lambda ds \]

and

\[ A(F \ast y)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\lambda)(I + i\lambda R(i\lambda, A))e^{i\lambda(t-s)} y(s) d\lambda ds. \]
Hence, \( F \ast y \in D(L) \) and \( L(F \ast y) = f \ast y = y \), where the last equality follows from Lemma 2.1(iv).

We conclude this paragraph with a result on the spectrum of the operator \( L_F \) in a spectrally homogeneous space. For several specific homogeneous spaces this result was proved in [5].

**Theorem 3.11.** Assume \( F \) is a spectrally homogeneous space. Then
\[
\sigma(L_F) = \sigma(L_F) + i\mathbb{R}.
\]

**Proof.** Directly from Definition 3.1 and (2.3) we have
\[
V(\lambda) L_F V(-\lambda) = L_F + i\lambda I, \quad \lambda \in \mathbb{R},
\]
and the result follows. □

4. INVERTIBILITY OF THE OPERATOR \( L \) IN \( \mathcal{F}_{as} \)

The other main result of this paper is the following theorem.

**Theorem 4.1.** Let \( \mathcal{F} = \mathcal{F}(\mathbb{R}, X) \) be a homogeneous function space and \( \mathcal{F}_{as} \subset \mathcal{F} \) be the space of functions with the absolutely summable spectrum. Assume that an operator \( L = L_F : D(L) \subseteq \mathcal{F}_{as} \to \mathcal{F}_{as} \) from Definition 3.1 is such that the generator \( A \) of the semigroup \( T \) satisfies (1.3) and (1.4). Then the operator \( L \) is invertible, \( L^{-1} \in B(\mathcal{F}_{as}) \), and
\[
\|L^{-1}\| \leq \frac{18}{\pi} M (4 + 4M + 2M^2)^{1/2}, \quad M = \sup_{\lambda \in i\mathbb{R}} \|R(\lambda, A)\|.
\]

The proof of the above result is based on several lemmas.

**Lemma 4.2.** Assume that \( \Phi \in C^2(\mathbb{R}, B(X)) \) and \( \phi \in L^1(\mathbb{R}) \) is defined by (2.7), i.e.
\[
\hat{\phi}(\lambda) = \hat{\phi}_0(\lambda) = (1 - |\lambda|)\chi_{[-1,1]}(\lambda), \quad \lambda \in \mathbb{R}.
\]

Let \( \Phi_0 = (\Phi \hat{\phi})' \), i.e. \( \Phi_0 \) is the inverse Fourier transform of the function \( \Phi \hat{\phi} \). Then \( \Phi_0 \in L^1(\mathbb{R}, B(X)) \) and
\[
\|\Phi_0\|_1 \leq \frac{2}{\pi} \|\Phi\|_{\infty} (4\|\Phi\|_{\infty} + 4\|\Phi'\|_{\infty} + \|\Phi''\|_{\infty})^{1/2}.
\]

**Proof.** Observe that the definition of \( \Phi_0 \) implies that \( \Phi_0 \in C_b(\mathbb{R}, B(X)) \) and
\[
\|\Phi_0(t)\| \leq \frac{1}{2\pi} \|\Phi\|_{\infty}, \quad t \in \mathbb{R}.
\]
We also have
\[
\Phi_0(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \Phi(\lambda) \hat{\phi}(\lambda) e^{i\lambda t} d\lambda = \frac{1}{2\pi} \int_{-1}^{1} \Phi(\lambda)(1 - |\lambda|) e^{i\lambda t} d\lambda
\]
\[ \Phi_0(t) = \frac{1}{2\pi} \left[ \Phi(0) - \int_{-1}^{0} (\Phi(\lambda) + \Phi'(\lambda)(1+\lambda)) e^{i\lambda t} d\lambda - \Phi(0) - \int_{0}^{1} (-\Phi(\lambda) + \Phi'(\lambda)(1-\lambda)) e^{i\lambda t} d\lambda \right] \]

Applying integration by parts in the above integrals, we get
\[ \Phi_0(0)(t) = \frac{1}{2\pi i t} \left[ \Phi(0) - \int_{-1}^{0} (\Phi(\lambda) + \Phi'(\lambda)(1+\lambda)) e^{i\lambda t} d\lambda - \Phi(0) - \int_{0}^{1} (-\Phi(\lambda) + \Phi'(\lambda)(1-\lambda)) e^{i\lambda t} d\lambda \right] \]

Applying integration by parts once again, we get
\[ \Phi_0(0)(t) = \frac{1}{2\pi t^2} \left[ 2\Phi(0) - \Phi(1)e^{it} - \Phi(-1)e^{-it} + 2 \int_{0}^{1} \Phi'(\lambda) e^{i\lambda t} d\lambda - 2 \int_{-1}^{0} \Phi'(\lambda) e^{i\lambda t} d\lambda - \int_{-1}^{1} \Phi''(\lambda)(1-|\lambda|) e^{i\lambda t} d\lambda \right] . \]

Hence, for \( t \neq 0 \), we have
\[ \|\Phi_0(t)\| \leq \frac{1}{2\pi t^2} \left[ 4\|\Phi\|_{\infty} + 4\|\Phi'\|_{\infty} + \|\Phi''\|_{\infty} \right] . \]

Using (4.3) and (4.4) we get \( \Phi_0 \in L^1(\mathbb{R}, B(X)) \) since for any \( \alpha > 0 \)
\[ \|\Phi_0\|_1 = \int_{|t| \leq \alpha} \|\Phi_0(t)\|dt + \int_{|t| \geq \alpha} \|\Phi_0(t)\|dt \]
\[ \leq \frac{1}{\pi} \left( \alpha\|\Phi\|_{\infty} + (4\|\Phi\|_{\infty} + 4\|\Phi'\|_{\infty} + \|\Phi''\|_{\infty}) \int_{\alpha}^{\infty} \frac{dt}{t^2} \right) \]
\[ = \frac{1}{\pi} \left( \alpha\|\Phi\|_{\infty} + \frac{1}{\alpha} (4\|\Phi\|_{\infty} + 4\|\Phi'\|_{\infty} + \|\Phi''\|_{\infty}) \right) . \]

Plugging in \( \alpha = \left( \frac{4\|\Phi\|_{\infty} + 4\|\Phi'\|_{\infty} + \|\Phi''\|_{\infty}}{\|\Phi\|_{\infty}} \right) \) we get (4.2).

**Corollary 4.3.** Assume that \( \Phi \in C^2(\mathbb{R}, B(X)) \) and \( \phi_n \in L^1(\mathbb{R}) \) is defined by (2.6) and (2.7). Let \( \Phi_n = (\Phi \hat{\phi}_n)' \). Then \( \Phi_n \in L^1(\mathbb{R}, B(X)) \) and
\[ \|\Phi_n\|_1 \leq \frac{2}{\pi} \|\Phi\|_{\infty} (4\|\Phi\|_{\infty} + 4\|\Phi'\|_{\infty} + \|\Phi''\|_{\infty})^{1/2} . \]

**Proof.** The result follows by applying the lemma to the function \( S(-n)\Phi \), where \( S \) is the translation representation (2.2). \( \Box \)
Corollary 4.4. Assume that the generator $A$ of a semigroup $T$ satisfies (1.3) and (1.4). Let $R_n = (R(i\cdot, A)\hat{\phi}_n)^\nu$. Then $R_n \in L^1(\mathbb{R}, B(X))$ and

\begin{equation}
\|R_n\|_1 \leq \frac{2}{\pi} M \left( 4 + 4M + 2M^2 \right)^{1/2}.
\end{equation}

Proof. The result follows from (4.5) since $d/d\lambda R(i\lambda, A) = iR^2(i\lambda, A)$ and $d^2/d\lambda^2 R(i\lambda, A) = 2R^3(i\lambda, A)$. \hfill $\square$

We are now ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Since $\mathcal{F}_{as}$ is itself a homogeneous space, Corollary 3.2 applies and the operator $\mathcal{L} = \mathcal{L}_{\mathcal{F}_{as}}$ is injective.

To prove surjectivity, consider $y \in \mathcal{F}_{as}$. Let $y_n = \phi_n * y$ so that $\Lambda(y_n) \subseteq [n - 1, n + 1] = \Delta(n)$. Let $R_n$ be defined as in Corollary 4.4. Then from the proof of Theorem 3.10 and Lemma 2.1 (v) we see that $x_n = R_n * y = R_n * (y_{n-1} + y_n + y_{n+1}) \in \mathcal{F}(\Delta(n))$ satisfies $\mathcal{L}x_n = y_n$. Let $x = \sum_{n \in \mathbb{Z}} x_n$, where the series converges absolutely since $y \in \mathcal{F}_{as}$ and $R_n, n \in \mathbb{Z}$, satisfy (4.6). Since the operator $\mathcal{L}$ is closed, we conclude that $\mathcal{L}x = y$.

It remains to estimate $\|x\|_{as}$. Observe that

$$
\phi_n * x = \phi_n * (x_{n-1} + x_n + x_{n+1}).
$$

Hence, (4.6) implies

\begin{equation}
\|\phi_n * x\| \leq \frac{2}{\pi} M \left( 4 + 4M + 2M^2 \right)^{1/2} \sum_{k=n-1}^{n+1} (\|y\|_{k-1} + \|y\|_k + \|y\|_{k+1}),
\end{equation}

and the postulated estimate for $\|\mathcal{L}\|^{-1}$ follows. \hfill $\square$

Remark 4.1. Observe that if $\mathcal{F} = C_{ub}$, and $\mathcal{L}_x$ satisfies the conditions of Theorem 4.1, then $\mathcal{L}_x \mathcal{F} \supseteq \mathcal{L}_x \mathcal{F}_{as} \supseteq C^{1,\alpha}, \alpha > 0$. Moreover, for any function $y \in C^{1,\alpha}$ the equation $\mathcal{L}_x x = y$ has a unique solution $x \in \mathcal{F}_{as}$.

5. Examples

We begin this section with an example of an operator $A$ and a homogeneous space $\mathcal{F}(\mathbb{R}, X)$ such that $A$ satisfies (1.3) and (1.4) but the operator $\mathcal{L} = \mathcal{L}_x$ is not invertible. This example appears in [20, Counterexample IV.2.7], we provide it in order to point out a feature that seems to be common for all such examples.

We let

$$
X = C_0(\mathbb{R}^+) \cap L^1_{\nu}(\mathbb{R}^+), \quad \nu(s) = e^s,
$$

where $L^1_\nu$ is the Beurling algebra of all measurable functions that are summable with the weight $\nu$. The norm in $L^1_\nu(\mathbb{R}^+)$ is given by
\[
\|x\|_\nu = \int_0^\infty |x(s)| e^{s} ds
\]
so that $\|x\|_X = \|x\|_\infty + \|x\|_\nu$. We let $A = \frac{d}{ds}$ be the generator of the semigroup $T : \mathbb{R}_+ \to X$ given by $T(t)x(s) = x(s + t)$, $x \in X$, $s,t \in \mathbb{R}_+$. Observe that $\|T(t)\| = 1$, $t \geq 0$, the growth bound of $T$ is equal to 0, and the spectral bound $s(A) = \sup \{\Re \lambda, \lambda \in \sigma(A)\}$ satisfies $s(A) \leq -1$. Since
\[
R(i\lambda, A)x(s) = \int_0^\infty e^{-\lambda t} x(s + t) dt,
\]
the operator $A$ indeed satisfies (1.3) and (1.4). However, if we let $F$ be, for example, $C_0(\mathbb{R}, X)$, Theorem 1.1 would imply that $L_F$ is not invertible since the spectral radius of the operator $T(1)$ is equal to 1. Observe that in this case $\sigma(T(1)) = T$. To the best of our knowledge, the semigroups in all of the known examples of this kind have this property.

We conclude the paper with an application of our results to the following non-linear equation in $F = C_b$:
\[
(5.1) \quad x'(t) = (Ax)(t) + y(t) + F(x(t)), \quad y \in F_{as},
\]
where $F$ is the polynomial
\[
F(z) = F_1(z) + F_2(z, z) + \ldots + F_n(z, z, \ldots, z), \quad z \in X,
\]
and each $F_k$, $k = 1, \ldots, n$, is a $k$-linear map.

We assume that the operator $A$ satisfies (1.3) and (1.4). Then $x \in F_{as}$ is a mild solution of the equation (5.1) if it satisfies
\[
x = z + \Phi x,
\]
where $z = L_{F_{as}}^{-1} y$ and the non-linear map $\Phi : F_{as} \to F_{as}$ is given by $\Phi = L_{F_{as}}^{-1} \circ F$. Observe that the map $\Phi$ is Lipschitz in any ball $B_\beta(0)$ of radius $\beta$ centered at $0 \in F_{as}$, that is
\[
\|\Phi(x) - \Phi(y)\|_{as} \leq L_{F,A}(\beta) \|x - y\|_{as}, \quad x, y \in B_\beta(0).
\]
Moreover, because of (4.1), the Lipschitz constant satisfies
\[
(5.2) \quad L_{F,A}(\beta) \leq \frac{18}{\pi} M (4 + 4M + 2M^2)^{1/2} \sum_{k=1}^n k \|F_k\| \beta^{k-1}.
\]

**Theorem 5.1.** Assume that $\beta > 0$, $y \in F_{as}$, and $F$ are such that $L_{F,A}(\beta) < 1$ and $\|\Phi(z)\|_{F_{as}} < \beta(1 - L_{A,F}(\beta))$. Then the non-linear equation (5.1) has a unique mild solution $x \in F_{as}$ and $\|x - z\|_{as} \leq \beta$. 

Proof. The solution is obtained by the method of simple iterations as in [19, Theorem 10.1.2]. □

Remark 5.1. Results of this paper can be extended in a straightforward way to the case of differential inclusions

\[ \frac{dx}{dt} \in A x + y, \; x, y \in F_{as}(\mathbb{R}, X), \]

where \( A \) is a linear relation on \( X \) [18, 8, 10, 13].

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