UNSTABLE ENTROPY OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS AND GENERIC POINTS OF ERGODIC MEASURES

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Abstract. Given a partially hyperbolic diffeomorphism $f : M \to M$ defined on a compact Riemannian manifold $M$, in this paper we define the concept of unstable topological entropy of $f$ on a set $Y \subset M$ not necessarily compact. Using recent results of J. Yang [15] and H. Hu, Y. Hua and W. Wu [5] we extend a theorem of R. Bowen [2] proving that, for an ergodic $f$-invariant measure $\mu$, the unstable measure theoretical entropy of $f$ is upper bounded by the unstable topological entropy of $f$ on any set of full $\mu$-measure. We define a notion of unstable topological entropy of $f$ using a Hausdorff dimension like characterization and we prove that this definition coincides with the definition of unstable topological entropy introduced in [5]. At last, extending another result of R. Bowen, we show that the unstable topological entropy of the set of generic points of an ergodic invariant measure $\mu$ is equal to the unstable metric entropy with respect to $\mu$.

1. Introduction

Given a smooth compact, connected Riemannian manifold $M$ without boundary we say that a $C^1$ diffeomorphism $f : M \to M$ is partially hyperbolic if for every point $x \in M$ there is a splitting

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

and a Riemannian metric on $M$ such that for all unit vectors $v_s \in E^s(x), v_c \in E^c(x), v_u \in E^u(x)$ we have

$$||Df(x) \cdot v_s|| < ||Df(x) \cdot v_c|| < ||Df(x) \cdot v_u||,$$

and

$$\max\{||Df(x)||_{E^s(x)}||, ||Df(x)^{-1} \cdot E^u(x)||\} < 1.$$ We call $E^s$ and $E^u$ the stable and unstable bundles of $TM$ respectively. From results of [4] there are $f$-invariant foliations $F^\tau$ tangent to $E^\tau$, $\tau = s, u$, called the stable foliation (when $\tau = s$) and the unstable foliation of $f$ (when $\tau = u$).

From the definition we can say that a partially hyperbolic diffeomorphism is composed of a hyperbolic component, which is the dynamics induced by $f$ along the subbundles $E^s$ and $E^u$, and a “central” component which may or may not have contracting or expanding characteristics. A starting point to understand the dynamics of such diffeomorphisms is then try to understand how much influence does the hyperbolic part of $f$ exerts on the dynamics of $f$. A good example of such situation is the use of the accessibility property, which essentially says that any two points can be connected by a path tangent to the hyperbolic components of $f$, to obtain ergodicity for certain partially hyperbolic diffeomorphisms (see for example [3]). In the seminal papers [9, 8] F. Ledrappier and L.S. Young were able to give a characterization of the metric entropy of a $C^2$ diffeomorphism $f$ in terms of the “unstable characteristics” of $f$, that is, in terms of the contribution of the unstable direction of $f$ to the entropy of $f$. In particular they were able to characterize the measures for which Pesin’s entropy formula occurs. A central

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tool in their results is the concept of unstable metric entropy, which is defined via a certain conditional entropy using an increasing partition subordinated to the unstable foliation. Given the importance of such tool, some attention has been directed to the study of the so called unstable entropy of a partially hyperbolic dynamical system. For example, much more recently, J. Yang [15] used such type of entropies to show that the set of Gibbs $u$-states of $C^{1+\alpha}$ partially hyperbolic diffeomorphism is an upper semi-continuous function of the map in the $C^1$ topology and that the sets of partially hyperbolic diffeomorphisms with either mostly contracting or mostly expanding center are $C^1$ open. Another very interesting example of an application of such tools was given by J. Yang and A. Tahzibi in [13] where they establish a beautiful criterium for $u$-invariance of an $f$-invariant measure based on how large is the entropy (compared to the unstable entropy of $f$) of the dynamics induced on the space of central leaves. Also very recently R. Saghin and J. Yang [12] used the entropy along expanding foliations as a tool to obtain Gibbs property of certain measures and then applied it to establish a local rigidity result of linear Anosov diffeomorphisms.

With the idea of putting these “unstable entropy tools” altogether in a framework similar to the one already existing for the classical entropy, H. Hu, H. Hua and W. Wu [5] redefined the concept of unstable metric entropy $h^u_\mu(f)$ (see Definition 2.3), defined the concept of topological unstable entropy (see Definition 2.7) and proved several results in the direction of the classical theorems of entropy theory. For example they have proved that their definition of metric unstable entropy coincides with the definition given in [9, 8], proved a version of the Shannon-McMillan-Breiman theorem for such entropies and also a variational principle relating the metric and topological unstable entropies. While the unstable metric entropy is defined in [9, 8] through the conditional entropy $H_\mu(\xi|f\xi)$, where $\xi$ is an increasing partition subordinated to the unstable manifolds, the topological unstable entropy is defined by taking a Bowen entropy-like definition via refinements of open covers, in analogy to [1], restricted to a compact subset of the unstable manifold and is proved to be equal to the unstable volume growth of $f$ (see [6]).

In [2] R. Bowen defined the topological entropy of a homeomorphism $f : X \to X$ on a subset $Y \subset X$ using a Hausdorff dimension like approach and proved that restricted to compact sets such entropy coincides with the standard entropy. Furthermore it is also proved in [2] that, for an $f$-invariant measure $\mu$, the metric entropy of $f$ is upper bounded by the topological entropy restricted to any subset of $X$ with full measure. Inspired by these ideas we define the concept of unstable topological entropy along a non-compact subset $Y \subset M$ (see Definition 3.1) and we define the H-unstable topological entropy of a partially hyperbolic diffeomorphism $f$ as being the unstable entropy along the whole manifold $M$. We denote the H-unstable topological entropy of $f$ by $h^u_H(f)$. Then we extend Bowen’s Theorem on the upper bound of the metric entropy to the context of unstable entropies (Theorem A(2)) and, using these results and the unstable variational principle we show that the $H$-unstable topological entropy of $f$ coincides with the unstable topological entropy defined via open covers (Theorem A(3)). This provides a characterization of the unstable topological entropy via a Hausdorff dimension approach.

**Theorem A.** Let $f : M \to M$ be a $C^1$ partially hyperbolic diffeomorphism defined on a compact Riemannian manifold $M$. The following are true

1) for any $x \in M$, if $Y \subset \mathcal{F}^u(x)$ is a compact subset then $h^u_H(f, Y) = h^u_B(f, Y)$;
2) if $\mu$ is an ergodic $f$-invariant probability measure then
   \[
   h^u_\mu(f) \leq h^u_H(f, Y)
   \]
   for every measurable subset $Y \subset X$ with $\mu(Y) = 1$;
3) $h^u_B(f) = h^u_{top}(f)$. 

Given a homeomorphism \( f : X \to X \) on a compact space \( X \) and an ergodic invariant measure \( \mu \) on \( X \), the main results in [2] states that the set of generic points of \( \mu \) have topological entropy equal to the metric entropy of \( f \) with respect to \( \mu \). Using results of J. Yang [15], which allows us to approximate the unstable metric entropy by conditional entropies between finite partitions, and a concept of conditional entropy of distributions we are able to prove that such result is also true in the context of the unstable entropies.

**Theorem B.** Let \( f : M \to M \) be a \( C^{1+\alpha} \) partially hyperbolic diffeomorphism defined on a compact Riemannian manifold \( M \), \( \mu \) an ergodic \( f \)-invariant measure and \( G(\mu) \) be the set of generic points of \( \mu \). Then

\[
h^u_\mu(f) = h^u_{G}(f, G(\mu)).
\]

## 2. Preliminaries

Along all the exposition \( M \) is taken to be a smooth compact, connected Riemannian manifold without boundary, \( f : M \to M \) is a \( C^1 \) (sometimes we require it to be \( C^{1+\alpha} \)) partially hyperbolic diffeomorphism and \( \mathcal{F}^u \) and \( \mathcal{F}^s \) are the unstable and stable foliations of \( f \) respectively.

### 2.1. Measure entropy for unstable foliation of partially hyperbolic diffeomorphisms.

In this section we recall the definition of unstable metric entropy of a partially hyperbolic diffeomorphism as defined in [5] and state some properties which will be useful along the rest of the exposition.

Given a partition \( \alpha \) of \( X \) we will denote by \( \alpha(x) \), \( x \in X \), the element of \( \alpha \) which contains \( x \).

**Definition 2.1.** We say that a partition \( \alpha \) of \( X \) is measurable with respect to \( \mu \) if there exist a family \( \{A_i\}_{i \in \mathbb{N}} \) of measurable sets and a measurable set \( F \) of full \( \mu \)-measure such that if \( B \in \alpha \), then there exists a sequence \( \{B_i\}_{i \in \mathbb{N}} \), where \( B_i \in \{A_i, A_i^c\} \) such that

\[
B \cap F = \bigcap_i B_i \cap F.
\]

For a certain \( \varepsilon_0 > 0 \) small enough denote by \( \mathcal{P} = \mathcal{P}_{\varepsilon_0} \) the set of all finite measurable partitions of \( M \) whose elements have diameter at most equal to \( \varepsilon_0 \). For each \( \beta \in \mathcal{P} \) we can define a partition \( \eta \) given by

\[
\eta(x) = \beta(x) \cap W^u_{loc}(x)
\]

where \( W^u_{loc}(x) \) denotes the local unstable manifold at \( x \) whose size is greater than \( \varepsilon_0 \). This partition \( \eta \) is then a measurable partition and \( \eta \) is finer than \( \beta \). Let \( \mathcal{P}^u = \mathcal{P}_{\varepsilon_0}^u \) be the set of all partitions \( \eta \) obtained in this manner.

**Definition 2.2.** A partition \( \xi \) of \( M \) is said to be subordinated to the unstable manifolds of \( f \) with respect to a measure \( \mu \) if for \( \mu \)-almost every \( x, \xi(x) \subset W^u(x) \) and \( \xi(x) \) contains an open neighborhood of \( x \) in \( W^u(x) \).

Let \( \mu \) be a probability measure on \( X \) and \( \alpha \) and \( \eta \) two measurable partitions of \( X \). The classical Rokhlin’s Theorem (see [14]) guarantees the existence of a canonical system of conditional measures which disintegrates \( \mu \), that is, there exists a family of probability measures \( \{\mu^\eta_x : x \in X\} \) such that

- \( \mu^\eta_x(\eta(x)) = 1 \);
- for every measurable subset \( B \subset X \) the function \( x \mapsto \mu^\eta_x(B) \) is a measurable function and;
- for \( B \subset X \) measurable,

\[
\mu(B) = \int_X \mu^\eta_x(B) d\mu(x).
\]

...
The conditional entropy of $\alpha$ given $\eta$ with respect to $\mu$ is defined by
\[
H_\mu(\alpha|\eta) = -\int_M \log \mu_\alpha^\eta(x) d\mu(x)
\]
where $\{\mu_\alpha^\eta : x \in M\}$ is a family of conditional measures of $\mu$ relative to $\eta$ as defined above.

**Definition 2.3.** The conditional entropy of $f$ with respect to a measurable partition $\alpha$ given $\eta \in \mathcal{P}^u$ is defined as
\[
h_\mu(f, \alpha|\eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta),
\]
where $\alpha_0^{n-1} = \bigvee_{i=0}^{n-1} f^{-i}\alpha$. The conditional entropy of $f$ given $\eta \in \mathcal{P}^u$ is defined by
\[
h_\mu(f|\eta) = \sup_{\alpha \in \mathcal{P}} h_\mu(f, \alpha|\eta),
\]
and the unstable metric entropy of $f$ is defined setting
\[
h_\mu^u(f) = \sup_{\eta \in \mathcal{P}^u} h_\mu(f|\eta).
\]

The following are standard properties in entropy theory so that we state them without a proof.

**Lemma 2.4.**

a) For $\beta, \eta$ measurable partitions and $n \in \mathbb{N}$ we have
\[
h_\mu(f^n, \beta_0^{n-1}|\eta) = n \cdot h_\mu(f, \beta|\eta).
\]

b) For $\beta$ and $\eta$ measurable partitions we have
\[
h_\mu(f, \beta|\eta) \leq h_\mu(f, \alpha|\eta) + H_\mu(\beta|\alpha).
\]

**Remark 2.5.** Given an $f$-invariant splitting $E^s \oplus E^c \oplus E^u$ of $M$, $f^{-1}$ is also partially hyperbolic with splitting $E^s_{f^{-1}} \oplus E^c \oplus E^u_{f^{-1}}$ where $E^s_{f^{-1}} := E^u$ and $E^u_{f^{-1}} := E^s$. However it is not true that, with respect to these splittings, $h^u_\mu(f) = h^u_\mu(f^{-1})$. A simple example is given by the following. Fix any $k_0 \geq 5$ and take $f : \mathbb{T}^3 \to \mathbb{T}^3$ be the linear automorphism of $\mathbb{T}^3$ induced by
\[
A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & k_0 \end{pmatrix}.
\]
An easy calculation (see [10] Lemma 4.1) shows that $A$ has three real distinct eigenvalues $0 < \lambda^s < \lambda^c < 1 < \lambda^u$. Write $E^\tau$ the subspace of $T_x \mathbb{T}^3$ induced by the eigenspace of $A$ with respect to $\lambda^\tau$, $\tau = s, c, u$. As the Lebesgue measure $m$ on $\mathbb{T}^3$ is $f$ invariant and is clearly $u$-Gibbs we have by [7] Theorem 3.4] (also stated in [15] Proposition 5.3) that
\[
h^u_\mu(f) = \int \log \text{Jac}^u_f(x) dm(x) = \log \lambda^u.
\]
Now, if we regard $f^{-1}$ as a partially hyperbolic diffeomorphism with splitting $E^s_{f^{-1}} \oplus E^c_{f^{-1}} \oplus E^u_{f^{-1}}$, given by $E^s_{f^{-1}}(x) := E^u(x)$, $E^c_{f^{-1}}(x) = E^c(x)$ and $E^u_{f^{-1}}(x) = E^s(x)$ we have, by the same argument,
\[
h^u_\mu(f^{-1}) = \int \log \text{Jac}^u_{f^{-1}}(x) dm(x) = -\log \lambda^s = \log \lambda^u + \log \lambda^c < h^u_\mu(f).
\]

From Theorem A from [5] and Propositions 4.1 and 4.5 from [15] we have the following:
Proposition 2.6. For a $C^{1+\alpha}$ partially hyperbolic diffeomorphism $f : M \to M$. For any $\varepsilon > 0$ there exists two finite partitions $\xi$ and $\xi_0$ of $M$ by measurable sets and $m_0 = m_0(\varepsilon) \in \mathbb{N}$ big enough so that
\[
\left| \frac{1}{m_0} H_\mu(\xi|\xi_0) - h^u_\mu(f) \right| \leq \varepsilon.
\]

2.2. Bowen unstable topological entropy of a compact subset. Let $C^0_M$ denote the set of all finite open covers of $M$. Given $U \in C^0_M$ denote $U^n_m := \bigvee_{i=m}^n f^{-i}U$. For any $K \subset X$ denote
\[
N(U|K) := \min\{\text{card}(V) : V \subset U, \bigcup_{V \subset V} V \ni K\},
\]
and
\[
H(U|K) := \log N(U|K),
\]
where card($V$) denotes the cardinality of the family of sets $V$.

Definition 2.7 ([5]). Let $d^u_x$ be the metric induced by the Riemannian structure on the unstable manifold $F^u(x)$. With respect to the metric $d^u_x$ we denote $W^u(x, \delta)$ the open ball of radius $\delta$ inside $F^u(x)$ centered at $x$. Given a compact subset $K \subset F^u(x)$ we define the unstable Bowen entropy of $K$ by
\[
h_B^u(f, K) = \sup_{U \in C^0_M} \limsup_{n \to \infty} \frac{1}{n} H(U_0^n|K).
\]
The unstable topological entropy of $f$ is defined by:
\[
h^u_{\text{top}}(f) = \lim_{\delta \to 0} \sup_{x \in M} h_B^u(f, W^u(x, \delta)).
\]

Theorem 2.8. ([5] Theorem D) Let $f : M \to M$ be a $C^1$-partially hyperbolic diffeomorphism. Then
\[
h^u_{\text{top}}(f) = \sup\{h^u_\mu(f) : \mu \in M_f(M)\} = \sup\{h^u_\mu(f) : \mu \in M^\text{f}_{\text{erg}}(M)\}
\]
where $M_f(M)$ (resp. $M^\text{f}_{\text{erg}}(M)$) denote the space of $f$-invariant probability measures (resp. $f$-invariant ergodic probability measures) on $M$.

3. Unstable topological entropy of non-compact subsets

For each $x \in M$ we denote by $M_x$ the union of all the unstable leaves along the orbit of $x$, that is,
\[
M_x := \bigcup_{j=-\infty}^{\infty} F^u(f^j(x)).
\]
Given any open cover $A$ of $M$ and a subset $E \subset M$ we say that $E$ is thinner than $A$, and we denote it by $E \subset A$, if there exists a set $A \subset A$ such that $E \subset A$. Fixed a finite open cover $A$ of $M$, for each $E \subset M_x$ we denote:
\[
n_{f,A}(E) = \text{the biggest nonnegative integer for which } f^k(E) \subset A \text{ for all integer } k \in [0, n_{f,A}(E))
\]
and
\[
D_A(E) := e^{-n_{f,A}(E)}.
\]
If $\mathcal{E} = \{E_i : i = 1, 2, \ldots\}$ is a family of sets with $E_i \subset M_x$ for every $i \geq 1$ we define
\[
D_A(\mathcal{E}, \lambda) := \sum_{i=1}^{\infty} D_A(E_i)^\lambda.
\]
Similar to the idea of the definition of Hausdorff measure we define a measure $m^x_{A,\lambda}$ in $M_x$ in the following way: for each $Y \subset M_x$

$$m^x_{A,\lambda}(Y) = \liminf_{\varepsilon \to 0} \left\{ D_A(\mathcal{E}, \lambda) : \mathcal{E} = \{ E_i \subset M_x : i = 1, 2, \ldots \}, \cup_{i=1}^{\infty} E_i \supset Y, D_A(E_i) < \varepsilon \right\}.$$ 

Now, in analogy to the definition of Hausdorff dimension, we define

$$h^n_{H,A}(f,Y) = \inf\{ \lambda : m^x_{A,\lambda}(Y) = 0 \}, \quad Y \subset M_x.$$ 

**Definition 3.1.** For $Y \subset M$ and $x \in M$ we define

$$h^n_{H}(f,Y,x) = \sup_{A} h^n_{H,A}(f,Y \cap M_x)$$

where the sup ranges over all finite open covers of $M$. Finally we define the $H$-unstable entropy of $Y$ by

$$h^n_{H}(f,Y) = \sup_{x \in Y} h^n_{H}(f,Y,x).$$

We define the $H$-unstable entropy of $f$, and we denote it by $h^n_H(f)$, by taking $Y = M$, that is,

$$h^n_H(f) := h^n_H(f,M).$$

The following also follows from the definition and standard arguments of entropy theory.

**Lemma 3.2.** For $f : M \to M$ a partially hyperbolic diffeomorphism of a compact manifold $M$ the following are true:

a) $h^n_H(f, f(Y)) = h^n_H(f,Y)$.

b) $h^n_H(f, \cup_{i=1}^{\infty} Y_i) = \sup_{x \in Y} h^n_H(f, Y_i)$.

c) $h^n_H(f^m, Y) = m \cdot h^n_H(f, Y)$ for $m > 0$.

4. PROOF OF THEOREM A

The arguments used to prove items (1) and (2) of Theorem A are similar to those given in [2] but, as the proof of the second item relies on the Shannon-McMillan-Breiman, which for the case of the unstable entropy, is given in terms of a conditional information function, we need to overcome the issue of always dealing with conditional measures instead of the original one. This is done in Lemma 1. Item (3) follows as a consequence of the two first items and the variational principle for unstable entropy.

**Proof of item(1).** The proof of the first item is identical to the proof of Proposition 1 in [2]. As we use $h^n_H(f, Y) \leq h^n_N(f, Y)$ in the proof of the last item, we only repeat the proof of this side of the inequality here.

Let $Y \subset \mathcal{F}^u(x)$ be a compact set and let $A$ be any finite open cover of $Y$. Let $\mathcal{E}_n$ be a subcover of $Y$ with $N(A_{0}^{n-1}|Y)$ members. Thus, if we can consider $\mathcal{E}_n$ to be the collection of sets $E \cap \mathcal{F}^u(x)$ with $E \in \mathcal{E}_n$ we have a family of $N(A_{0}^{n-1}|Y)$ subsets of $\mathcal{F}^u(x)$ covering $Y$. Consequently

$$D_A(\mathcal{E}_n, \lambda) \leq N(A_{0}^{n-1}|Y)e^{-n\lambda},$$

which implies

$$m^x_{A,\lambda}(Y) \leq \lim_{n \to \infty} \left[ e^{-\lambda + \frac{1}{n} H(A_{0}^{n-1}|Y)} \right]^n.$$

If $\lambda > h^n_H(f,Y)$ then for $n$ large enough we have $-\lambda + \frac{1}{n} H(A_{0}^{n-1}|Y) < 0$ which implies $m^x_{A,\lambda}(Y) = 0$ and consequently

$$h^n_{A,\lambda}(f,Y,x) \leq h^n_{H}(f,Y) \Rightarrow h^n_{H}(f,Y) \leq h^n_{H}(f,Y).$$

$\square$
The proof of the second item follows from the following lemmas.

**Lemma 4.1.** Assume that there exists a finite Borel partition $\alpha$ of $M$ such that every $x \in M$ is in the closure of at most $c$ sets of $\alpha$. Then, if $\mu(Y) = 1$ we have

$$h^u_\mu(f) \leq h^u_\mu(f, Y, x_0) + \log c,$$

for almost every $x_0 \in Y$. In particular,

$$h^u_\mu(f) \leq h^u_\mu(f, Y) + \log c.$$

**Proof.** Let $\eta \in \mathcal{P}^n$ be any fixed partition. For each $y \in M$ consider

$$I_n(y) := I_\mu(\alpha_0^{n-1}\vert \eta)(y),$$

where $I_\mu(\xi\vert \beta)$ denotes the conditional information function of $\xi \in \mathcal{P}$ with respect to a measurable partition $\beta$ of $M$ defined by $I_\mu(\xi\vert \beta)(x) := -\log \mu_\beta^\mu(\xi(x))$. By the Shannon-McMillan-Breiman Theorem for the unstable entropy [5, Theorem 2], we have

$$\lim_{n \to +\infty} \frac{1}{n} I_n(x) = h^u_\mu(f) = h^u_\mu(f, \eta) =: a$$

(4.1)

for $\mu$-almost every point $x \in M$. Let $\tilde{Y} \subset Y$ be the subset of $Y$ for which (4.1) occurs and take an arbitrary point $x_0 \in \tilde{Y}$. For $\delta > 0$ and $N \in \mathbb{N}$ denote

$$Y_{\delta,N} := \left\{ y \in \tilde{Y} : \frac{1}{n} I_n(y) \geq a - 2\delta, \quad \forall n \geq N \right\}.$$

Observe that

$$\tilde{Y} = \bigcup_{m \in \mathbb{N}, N \in \mathbb{N}} Y_{1/m,N}$$

so that we can take $m, N \in \mathbb{N}$ for which $\mu(Y_{1/m,N}) > 0$. Now, let $\mathcal{B}$ be a finite open cover of $M$ such that each set of $\mathcal{B}$ intersects at most $c$ elements of $\alpha$. Suppose that $\mathcal{E} = \{E_i\}_{i=1}^t$, $E_i \subset M_{x_0}$, covers $Y_0 = Y \cap M_{x_0}$ with $D_{\mathcal{B}}(E_i) \leq \epsilon^{-N}$.

If $\beta \in \alpha_{n_{f,A}(E_i)}^{m_{f,A}(E_i)} = \bigvee_{i=0}^{m_{f,A}(E_i)} f^{-i}\alpha$ intersects $Y_{1/m,N}$, say $y_0 \in \beta \cap Y_{1/m,N}$, then $\beta = \alpha_{n_{f,A}(E_i)}^{m_{f,A}(E_i)}(y_0)$ and

$$-\frac{1}{n_{f,A}(E_i) + 1} \log \mu_{y_0}^\mu(\beta) \geq a - \frac{2}{m} \Rightarrow \mu_{y_0}^\mu(\beta) \leq \exp(-(a - 2/m)n_{f,A}(E_i)).$$

Thus, given any $y_0 \in \beta \cap Y_{1/m,N}$ and any $y \in \eta(y_0)$ we have

$$\mu_{y}^\mu(\beta) \leq \exp(-(a - 2/m)n_{f,A}(E_i)).$$

(4.2)

Denote by $\eta(F)$ the $\eta$ saturation of a set $F \subset M$, that is, $\eta(F) := \bigcup_{y \in F} \eta(y)$. From the definition of the system of conditional measures we obtain

$$\int_{M \setminus \eta(\beta \cap Y_{1/m,N})} \mu_{y}^\mu(\beta \cap Y_{1/m,N})d\mu(y) = 0.$$  

(4.3)

Now, by (4.2) and (4.3) we have

$$\mu(\beta \cap Y_{1/m,N}) = \int_M \mu_{y}^\mu(\beta \cap Y_{1/m,N})d\mu(y)$$

$$= \int_{\eta(\beta \cap Y_{1/m,N})} \mu_{y}^\mu(\beta \cap Y_{1/m,N})d\mu(y) + \int_{M \setminus \eta(\beta \cap Y_{1/m,N})} \mu_{y}^\mu(\beta \cap Y_{1/m,N})d\mu(y)$$

$$\leq \exp(-(a - 2/m)n_{f,A}(E_i))$$
Thus Since $E_i \cap Y_{1/m,N}$ is covered by at most $c_{\alpha_i}^{n,A(E_i)}$ elements of the form $\beta \in \alpha_0^{n,A(E_i)}$ we have
\[
\mu(E_i \cap Y_{1/m,N}) \leq c_{\alpha_i}^{n,A(E_i)} \exp(-(a - 2/m) n_f, A(E_i)) = \exp((\log c - a + 2/m) n_f, A(E_i)).
\]
Thus, for $\lambda = -\log c + a - 2/m$ we have
\[
D_A(\mathcal{E}, \lambda) = \sum_i \exp(-\lambda n_f, A(E_i)) \geq \sum_i \mu(E_i \cap Y_{1/m,N}) \geq \mu(Y_{1/m,N}).
\]
Letting the cover $\mathcal{E}$ vary we have $m_{\alpha_i}^{n, \lambda}(Y) \geq \mu(Y_{1/m,N})$ which implies
\[
h^u_H(f, Y, x_0) \geq h^u_{H,A}(f, Y \cap M_{x_0}) \geq \lambda = -\log c + a - 2/m.
\]
Taking $m \to \infty$ we obtain
\[
h^u_{\mu}(f) \leq h^u_H(f, Y, x_0) + \log c
\]
and
\[
h^u_{\mu}(f) \leq h^u_H(f, Y) + \log c.
\]
as we wanted to show. \hfill \square

To prove the second item we need the following Lemmas from [2].

**Lemma 4.2.** [2] Lemma 2] Let $\mathcal{A}$ be a finite open cover of $X$. For each $n > 0$ there is a finite Borel partition $\alpha_n$ of $X$ such that $f^k \alpha_n < \mathcal{A}$ for all $k \in [0, n]$ and at most $n \cdot \text{card}(\mathcal{A})$ sets in $\alpha_n$ can have a point in all their closures.

**Lemma 4.3.** [Lemma 3] Given a finite Borel partition $\beta$ and $\varepsilon > 0$ there is an open cover $\mathcal{A}$ so that $H_{\mu}(\beta|\alpha) < \varepsilon$ whenever $\alpha$ is a finite Borel partition with $\alpha < \mathcal{A}$.

**Proof of (2).** Let $\beta$ be a finite Borel partition of $M$ and $\varepsilon > 0$. Let $\mathcal{A}$ be as in Lemma 4.3 and $\alpha_n$ as in Lemma 4.2. Then, using the properties stated in Lemma 2.4 we have
\[
h^u_{\mu}(f) = h_{\mu}(f, \beta|\eta) \leq h^u_{H}(f, \beta_n|\eta)
\]
\[
\leq n^{-1} h_{\mu}(f^n, \beta_n|\eta) + n^{-1} H_{\mu}(\beta_n|\alpha_n)
\]
\[
\leq n^{-1} h^u_{H}(f^n, Y) + n^{-1} H_{\mu}(\beta|\alpha_n)
\]
\[
\leq h^u_{H}(f, Y) + n^{-1} \log(n \cdot \text{card}(\mathcal{A})) + n^{-1} \sum_{k=0}^{n-1} H_{\mu}(f^{-k}|\alpha_n)
\]
\[
\leq h^u_{H}(f, Y) + n^{-1} \log(n \cdot \text{card}(\mathcal{A})) + \varepsilon.
\]
Above we have also used the classical fact from entropy theory $H_{\mu}(f^{-1}\beta|f^{-1}\alpha) = H_{\mu}(\beta|\alpha)$. Taking $n \to \infty$ and $\varepsilon \to 0$ we get
\[
h^u_{\mu}(f) \leq h^u_{H}(f, Y).
\]
\hfill \square

**Proof of (3).** By (1) we have:
\[
h^u_{H}(f, \overline{W}(x, \delta)) = h^u_{\text{top}}(f, \overline{W}(x, \delta)), \quad \text{for any } \delta > 0.
\]
Now we can write $M_x$ as a countable union of sets of the form $W(x, \delta)$ and, consequently, by item (c) of Lemma 3.2 we conclude that

$$h^u_\mu(f) = h^u_M(f, M) \leq h^u_{\text{top}}(f).$$

On the other hand, by item (2) it follows that for any ergodic invariant measure $\mu$ we have $h^u_\mu(f) \leq h^u_M(f, M)$, thus

$$\sup_{\mu \in \mathcal{M}^0(M)} h^u_\mu(f) \leq h^u_M(f, M).$$

Finally, by Theorem 2.8 we conclude that $h^u_{\text{top}}(f) \leq h^u_M(f)$. Thus $h^u_M(f) = h^u_{\text{top}}(f)$ as we wanted to show. □

5. The unstable topological entropy of generic points

Let $\mathcal{M}(M)$ denote the set of all probability measures on $M$ with the weak topology (in the literature it is also called weak-* topology). It is well known that $\mathcal{M}(M)$ is a compact metrizable space [14]. We denote the convergence of a sequence $(\mu_n)_n$ in this topology simply by $\mu_n \to \mu$.

**Definition 5.1.** We say that a point $x \in M$ is a generic point for a continuous transformation $f : M \to M$ preserving a probability measure $\mu$ if the only accumulation point of the sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \right\}_{n \in \mathbb{N}}$$

is $\mu$ itself.

In [2] Bowen proved that for a continuous map $f : X \to X$ on a compact space $X$, the set of generic points of an $f$-invariant ergodic measure $\mu$ has topological entropy equal to the metric entropy of $f$ with respect to $\mu$. In what follows we extend this result to the setting of the unstable topological entropy. The main ingredients of the proof, which allows us to extend Bowen’s Theorem, is the concept of conditional entropy of a distribution with respect to an extended vector (Definitions 5.2 and 5.3) and Proposition 2.6 which approximates the unstable metric entropy by conditional entropies between certain finite partitions.

**Definition 5.2.** i) A vector $p = (p_1, \ldots, p_n)$ is called a probability vector, or an $n$-distribution, if $p_i \geq 0$ for all $1 \leq i \leq n$ and

$$\sum_{i=1}^{n} p_i = 1;$$

ii) An extended vector is a vector $q = (N; q_1, \ldots, q_s)$ such that $N \in \mathbb{N}$, $q_i > 0$ for all $1 \leq i \leq s$ and

$$\sum_{i=1}^{s} q_i = N;$$

iii) Given a probability vector $p = (p_1, \ldots, p_n)$ and an extended probability vector $q = (N; q_1, \ldots, q_s)$ we say that $q$ is compatible with $p$ if $N \cdot s = n$.

**Definition 5.3.** Given a probability vector $p = (p_1, \ldots, p_n)$ and an extended probability vector $q = (N; q_1, \ldots, q_s)$ compatible with $p$, the conditional entropy of $p$ with respect to $q$ is the quantity given by

$$H(p|q) = - \sum_{j=1}^{s} \sum_{i=1}^{j \cdot N} p_i \cdot \log \left( \frac{p_i}{q_j} \right).$$
where we are using the convention $0 \cdot \log 0 = 0$. For $p = (p_1, \ldots, p_n)$ and $q_0 := (n; 1)$ we denote $H(p|q_0)$ simply by $H(p)$, that is, $H(p) = -\sum_i p_i \log p_i$.

Given $a = (a_1, \ldots, a_m) \in \{1, \ldots, n\}^m$ we define the distribution vector associated to $a$ as

$$\text{dist}(a) := \frac{1}{m} \# \{1 \leq j \leq m : a_j = i\}, \quad 1 \leq i \leq n.$$ 

For $p$ and $q$ two $n$-distributions we define the distance between $p$ and $q$ by

$$|p - q| = \max_i |p_i - q_i|.$$ 

Given a $n$-distribution $p$ and an extended vector $\gamma$ compatible with $p$, it is easy to see from the definition that given any $\varepsilon > 0$ we can find $\delta > 0$ such that if $q$ is a $n$-distribution with $|p - q| < \delta$ then $|H(p|\gamma) - H(q|\gamma)| < \varepsilon$. It is also a classical fact that $H(p|\gamma) \leq H(p)$.

Given a cover $\mathcal{B} = \{B_1, \ldots, B_N\}$ of $X$, an $n$-choice for $x$, with respect to $\mathcal{B}$ and $f$, is a $n$-uple

$$\mathfrak{B} = (B_{i_0}, \ldots, B_{i_{n-1}}) \in \mathcal{B}^n$$

such that $f^k(x) \in B_{i_k}$ for $k = 0, 1, \ldots, n - 1$. Fixed such $\mathcal{B}$, associated to an $n$-choice $\mathfrak{B}$ we have the $N$-distribution

$$q(\mathfrak{B}) := \text{dist}(i_0, i_1, \ldots, i_{n-1}).$$

The set of all such distributions (that is, the set of all distributions associated to an $n$-choice for $x$ with respect to $\mathcal{B}$ and $f$) is denoted by $\text{Dist}_\mathcal{B}(x, n)$. Since every element in $\text{Dist}_\mathcal{B}(x, n)$ is an $N$-distribution an extended vector $q = (N^0; q_1, \ldots, q_S)$ is compatible with every element of $\text{Dist}_\mathcal{B}(x, n)$ if, and only if, $N^0 \cdot S = N$. When this is the case we will say that $q$ is compatible with $\text{Dist}_\mathcal{B}(x, n)$.

In what follows we denote by $h_\mathcal{A}(f, Z)$ the entropy of a subset $Z$ with respect to a finite cover $\mathcal{A}$ as given in [2]. We omit the full definition of such entropy here but we remark that the procedure to define it is the same as the procedure made in Section 3 but without taking the intersection of $Z$ with some $M_x$.

**Lemma 5.4.** [2] Lemma 5 Suppose $f : X \to X$ is a continuous map of a topological space, $\mathcal{A}$ an open cover of $X$, $\mathcal{B}$ a finite cover of $X$ and $k$ a positive integer so that $f^i(\mathcal{B}) < \mathcal{A}$ for all $i \in \{0, 1, \ldots, k - 1\}$. For $t \geq 0$ define

$$Q(t, \mathcal{B}) = \{x \in X : \liminf_{n \to \infty} (\inf \{H(q) : q \in \text{Dist}_\mathcal{B}(x, n)\}) \leq t\}.$$ 

Then $h_\mathcal{A}(f, Q(t, \mathcal{B})) \leq t/k$.

**Corollary 5.5.** Let $\mathcal{A}$ be any open cover of $M$, $\mathcal{B}$ a finite cover of $M$ and $k$ a positive integer such that $f^i(\mathcal{B}) < \mathcal{A}$ for all $i \in \{0, 1, \ldots, k - 1\}$. Given an extended probability vector $\gamma$ compatible with $\text{Dist}_\mathcal{B}(x, n)$, for $t \geq 0$ define

$$Q(t, \mathcal{B}, \gamma) = \{x \in M : \liminf_{n \to \infty} (\inf H(q|\gamma) : q \in \text{Dist}_\mathcal{B}(x, n) \leq t)\}.$$ 

Then $h_\mathcal{A}^u(f, Q(t, \mathcal{B}, \gamma)) \leq t/k$.

**Proof.** For each $y \in M$ let $X := M_y$ and $f|M_y$ play the role of $f$ in Lemma 5.4 Then

$$h_\mathcal{A}^u(f, Q(t, \mathcal{B}) \cap M_y) = h_\mathcal{A}(f|M_y, Q(t, \mathcal{B}) \cap M_y) \leq t/k.$$ 

Now, since $H(q|\gamma) \leq H(q)$, we have

$$h_\mathcal{A}^u(f, Q(t, \mathcal{B}, \gamma) \cap M_y) \leq h_\mathcal{A}(f, Q(t, \mathcal{B}) \cap M_y).$$
Thus,
\[
\sup_{y \in M} h^n_\mu(f, Q(t, B, \gamma) \cap M_y) \leq t/k \Rightarrow h^n_\mu(f, Q(t, B, \gamma)) \leq t/k.
\]
\[\square\]

**proof of Theorem 3** By item (2) of Theorem A we already have \( h^n_\mu(f) \leq h^n_H(f, G(\mu)) \). Let us prove the other hand of the inequality.

Let \( A \) be a finite open cover of \( M \) and let \( \varepsilon > 0 \) be fixed. By Proposition 2.10 there exist finite partitions \( \xi \) and \( \xi_0 \) of \( M \) such that
\[
\left| \frac{1}{k} H(\xi|\xi_0) - h^n_\mu(f) \right| \leq \varepsilon. \tag{5.1}
\]

Denote \( \xi = \{E_1, \ldots, E_N\} \) and \( \xi_0 = \{F_1, \ldots, F_S\} \), and take the extended vector \( \gamma = (N, \mu(F_1), \ldots, \mu(F_S)) \). Consider \( q' \) the NS-distribution whose entries are \( \mu(E_i \cap F_j), 1 \leq i \leq N, 1 \leq j \leq S \) listed in lexicographical order. By (5.1) we have
\[
\left| \frac{1}{k} H(q'|\gamma) - h^n_\mu(f) \right| \leq \varepsilon.
\]
Thus we can choose \( \delta > 0 \) such that if \( q \) is an NS-distribution with \( |q - q'| < \delta \) then
\[
\left| \frac{1}{k} H(q|\gamma) - h^n_\mu(f) \right| \leq 2\varepsilon.
\]

Take a family of open sets \( U_{ij} \) such that
- \( E_i \cap F_j \subset U_{ij} \), and
\[
\mu((E_i \cap F_j) \setminus U_{ij}) < \frac{\delta}{2N},
\]
for \( 1 \leq i \leq N, 1 \leq j \leq S \);
- \( f^n(U_{ij}) \subset A \) for \( 0 \leq n < k \).

Put \( \beta := \{U_{ij} : 1 \leq i \leq N, 1 \leq j \leq S\} \). Now, for each pair \((i, j)\) with \( 1 \leq i \leq N \) and \( 1 \leq j \leq S \) take a compact set \( K_{ij} \subset E_i \cap F_j \) such that
\[
\mu((E_i \cap F_j) \setminus K_{ij}) < \frac{\delta}{2N}.
\]
Consider a family of disjoint open sets \( \{V_{ij} : 1 \leq i \leq N, q \leq j \leq S\} \) such that \( K_{ij} \subset V_{ij} \subset U_{ij} \) for every such \( i \) and \( j \). Let \( x \in G(\mu) \) be any point and let \( B_n(x) \in \beta^n \) be an \( n \)-choice for \( x \) such that
\[
B_{ip} = U_{ij} \quad \text{if} \quad f^n(x) \in V_{ij}.
\]
Since \( x \) is a generic point of \( \mu \), there exists a sequence \((n_w)_{w \in N} \subset N \) with \( n_w \to \infty \) such that
\[
\frac{1}{n_w} \sum_{i=0}^{n_w-1} \delta f^i(x) \to \mu.
\]

Now recall that the convergence is in the weak topology satisfies the following properties: if \( \nu_n \) converges to \( \nu \) in the weak topology then
- for every open subset \( U \subset X \) we have
  \[
  \liminf \nu_n(U) \geq \nu(U);
  \]
- for every closed subset \( C \subset X \) we have
  \[
  \limsup \nu_n(C) \leq \nu(C).
  \]
Thus, for $w$ large enough we have
\[
\frac{1}{n_w} \sum_{i=0}^{n_w-1} \delta f^i(x)(V_{ij}) \geq \mu(K_{ij}) - \frac{\delta}{2N} \geq \mu(E_i \cap F_j) - \frac{\delta}{N}
\]
for all $1 \leq i \leq N$ and all $1 \leq j \leq S$. Let
\[
q^w := \text{dist} B_{n_w}(x) = (q_{11}^w, \ldots, q_{NS}^w).
\]
Therefore we have
\[
q_{ij}^w \geq \mu(E_i \cap F_j) - \frac{\delta}{N} \quad (5.2)
\]
As each $V_{ij}$ is compact, for $w$ large enough, we also have
\[
\mu(U_{ij}) + \frac{\delta}{2N} \geq \mu(V_{ij}) + \frac{\delta}{2N} \geq \frac{1}{n_w} \sum_{i=0}^{n_w-1} \delta f^i(x)(V_{ij}) \geq q_{ij}^w \quad (5.3)
\]
for all $1 \leq i \leq N$ and all $1 \leq j \leq S$. Thus (5.3) yields
\[
\mu(E_i \cap F_j) - q_{ij}^w \geq \mu(U_{ij}) - q_{ij}^w - \frac{\delta}{2N} \geq - \frac{\delta}{N}
\]
which, together with (5.2), implies $|q^w - q'| \leq \delta$. From the choice of $\delta$ we have $H(q^w|\gamma) \leq k(t + 2\varepsilon)$. Hence $x \in Q(k(h^u_{\mu}(f) + 2\varepsilon), \beta, \gamma)$ and, by Corollary 5.5 it follows that
\[
h_H^u(f, G(\mu)) \leq h_A^u(f, Q(k(h^u_{\mu}(f) + 2\varepsilon), \beta, \gamma)) \leq h^u_{\mu}(f) + 2\varepsilon.
\]
By taking $\varepsilon \to 0$ we get $h_H^u(f, G(\mu)) \leq h^u_{\mu}(f)$ as we wanted to show. \hfill \Box

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