EQUIVARIANT FORMALITY IN K-THEORY

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Abstract. In this note we present an analogue of equivariant formality in K-theory and show that it is equivalent to equivariant formality à la Goresky-Kottwitz-MacPherson. We also apply this analogue to give alternative proofs of equivariant formality of conjugation action on compact Lie groups, left translation action on generalized flag manifolds, and compact Lie group actions with maximal rank isotropy subgroups.

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1. Introduction

Equivariant formality, first defined in [GKM], is a special property of group actions on topological spaces which allows for easy computation of their equivariant cohomology. A \( G \)-action on a space \( X \) is said to be equivariantly formal if the Leray-Serre spectral sequence for the rational cohomology of the fiber bundle \( X \to X \times_G EG \to BG \) collapses on the \( E_2 \)-page. The latter is also equivalent to \( H^*_G(X; \mathbb{Q}) \cong H^*_G(\text{pt}; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}) \) as \( H^*_G(\text{pt}; \mathbb{Q}) \)-modules. There are various examples of interest which are known to be equivariantly formal, e.g. Hamiltonian group actions on compact symplectic manifolds and linear algebraic torus actions on smooth complex projective varieties (cf. [GKM Section 1.2 and Theorem 14.1]).

Though equivariant formality was first defined in terms of equivariant cohomology, in some situations working with analogous notions phrased in terms of other equivariant cohomology theories may come in handy. The notion of equivariant formality in K-theory was introduced and explored by Harada and Landweber in [HL], where they instead used the term ‘weak equivariant formality’ and exploited this notion to show equivariant formality of Hamiltonian actions on compact symplectic manifolds.

Definition 1.1 (cf. [HL Def. 4.1]). Let \( k \) be a commutative ring, \( G \) a compact Lie group and \( X \) a \( G \)-space. We use \( K^*(X) \) (resp. \( K^*_G(X) \)) to denote the \( \mathbb{Z}_2 \)-graded complex (equivariant) K-theory of \( X \), and \( K^*(X; k) \) (resp. \( K^*_G(X; k) \)) to denote \( K^*(X) \otimes k \) (resp. \( K^*_G(X) \otimes k \)). We denote the complex representation ring of \( G \) by \( R(G) \), and write \( R(G; k) := R(G) \otimes k \), and \( I(G; k) = I(G) \otimes k \), where \( I(G) \) is the augmentation ideal of \( R(G) \). Let

\[ f_G : K^*_G(X) \to K^*(X) \]

be the forgetful map. A \( G \)-action on a space \( X \) is \( k \)-weakly equivariantly formal if \( f_G \) induces an isomorphism

\[ K^*_G(X; k) \otimes_{R(G; k)} k \to K^*(X; k) \]

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1Recall that one can use \( \mathbb{Z}_2 \)-grading in defining complex K-theory thanks to Bott periodicity.
We simply say the action is weakly equivariantly formal in the case \( k = \mathbb{Z} \).

Harada and Landweber settled for weakly equivariant formality as in Definition 1.1 as the \( K \)-theoretic analogue of equivariant formality, instead of the seemingly obvious candidate \( K^*_G(X) \cong K^*_G(\text{pt}) \otimes K^*(X) \), citing the lack of the Leray-Serre spectral sequence for Atiyah-Segal’s equivariant \( K \)-theory. The term ‘weak’ is in reference to the condition in Definition 1.1 being weaker than \( K^*_G(X) \cong K^*_G(\text{pt}) \otimes K^*(X) \) because of the possible presence of torsion. We would like to define the following version of \( K \)-theoretic equivariant formality in exact analogy with another cohomological equivariant formality condition that the forgetful map \( H^*_G(X) \to H^*(X) \) be onto.

**Definition 1.2.** We say that \( X \) is a rational \( K \)-theoretic equivariantly formal (RKEF for short) \( G \)-space if the forgetful map

\[
f_G \otimes \text{Id}_\mathbb{Q} : K^*_G(X; \mathbb{Q}) \to K^*(X; \mathbb{Q})
\]

is onto.

Recall that \( K^0(X) \) (resp. \( K^{-1}(X) \)) is the Grothendieck group of the commutative monoid of isomorphism classes of (resp. reduced) complex vector bundles over \( X \) (resp. \( \Sigma X \)) under Whitney sum, and \( K^*_G(X) \) can be similarly defined using equivariant vector bundles. The above condition then admits a natural interpretation in terms of vector bundles: for every vector bundle \( V \) over \( X \) and its suspension \( \Sigma X \), there are natural numbers \( p, q \) such that \( V^\oplus p \oplus \mathbb{Q}^q \) admits an equivariant \( G \)-structure.

In this note, we will prove the following theorem, which asserts the equivalence of RKEF and equivariant formality in the classical sense.

**Theorem 1.3.** Let \( G \) be a compact and connected Lie group which acts on a finite CW-complex \( X \). The following are equivalent.

1. \( X \) is a RKEF \( G \)-space.
2. \( X \) is an equivariantly formal \( G \)-space.
3. \( X \) is a \( \mathbb{Q} \)-weakly equivariantly formal \( G \)-space.

We will also give alternative proofs of equivariant formality of certain group actions which were proved in cohomological terms. These are conjugation action on compact Lie groups, left translation action on generalized flag manifolds, and compact Lie group actions with maximal rank isotropy subgroups. In the remainder of this note, the coefficient ring of any cohomology theory is always \( \mathbb{Q} \).

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## 2. The Proof

From now on, unless otherwise specified, \( X \) is a finite CW-complex equipped with an action by a torus \( T \) or more generally a compact connected Lie group \( G \). The following \( K \)-theoretic abelianization result enables us to prove \( K \)-theoretic results in this Section in the \( T \)-equivariant case first and then generalize to the \( G \)-equivariant case.
Theorem 2.1 (cf. [HLS Theorem 4.9(ii)]). Let $T$ be a maximal torus of $G$ and $W$ the Weyl group. The map $r^*: K_G^*(X; \mathbb{Q}) \to K_T^+(X; \mathbb{Q})$ restricting the $G$-action to the $T$-action is an injective map onto $K_T^+(X; \mathbb{Q})^W$. Here if $w \in W$ and $V$ is an equivariant $T$-vector bundle, $w$ takes $V$ to the same underlying vector bundle with $T$-action twisted by $w$, and this $W$-action on the set of isomorphism classes of equivariant $T$-vector bundles induces the $W$-action on $K_T^*(X)$.

Definition 2.2. Let $H^{**}(X)$ be the completion of $H^*_G(X)$ as a $H^*_G(pt)$-module at the augmentation ideal $J := H^*_G(pt)$ (cf. the paragraph preceding [RK Proposition 2.8]).

The equivariant Chern character for a finite CW-complex with a $G$-action is the map

$$\text{ch}_G : K_G^*(X; \mathbb{Q}) \to H_G^{**}(X)$$

which is defined by applying the Borel construction to the non-equivariant Chern character (cf. the discussion before [RK Lemma 3.1]). Like the non-equivariant Chern character, $\text{ch}_G$ maps $K_G^0(X; \mathbb{Q})$ to the even degree part of $H_G^*(X)$ and $K_G^{-1}(X; \mathbb{Q})$ to the odd degree part. The image of $\text{ch}_G$ lies in $H_G^{**}(X)$ for the following reason which is borrowed from the proof of [RK Lemma 3.1]: as $X$ is a finite CW-complex, we can choose $a_1, a_2, \cdots, a_m \in H_G^*(X)$ which generate $H_G^*(X)$ as a $H_G^*(pt)$-module. Let

$$a_i \cdot a_j = \sum_{k=1}^m f_{ij}^k a_k$$

for $f_{ij}^k \in H_G^*(pt)$, and $c$ be $c_1^G(L)$ for some $G$-equivariant line bundle $L$ such that

$$c = \sum_{i=1}^m g_i a_i$$

for $g_i \in H_G^*(pt)$. So

$$\text{ch}_G(L) = e^c = 1 + \sum_i g_i a_i + \frac{1}{2} \sum_{i,j,k} g_i g_j f_{ij}^k a_k + \frac{1}{6} \sum_{i,j,k,l,p} g_i g_j g_l f_{ij}^k f_{jk}^l f_{kl}^p a_p + \cdots.$$ 

Write $\text{ch}_G(L) = 1 + \sum_{i=1}^m p_i a_i$, where $p_i$ are power series in $g_i$ and $f_{ij}^k$. Identifying $g_i$ and $f_{ij}^k$ with $W$-invariant polynomials on $t$ through the identification $H_G^*(pt) \cong H_T^*(pt)^W \cong S(t^*)^W$ and using the estimate for $p_i$ given in the proof of [RK Lemma 3.1], we have that $p_i$ are in $H_G^{**}(pt)$ and hence $\text{ch}_G(L) \in H_G^{**}(X)$. The assertion $\text{ch}_G(E) \in H_G^{**}(X)$ for general equivariant $G$-vector bundle $E$ follows from the splitting principle.

Proposition 2.3. Let $G$ be a compact connected Lie group acting on a finite CW-complex $X$. Then the equivariant Chern character

$$\text{ch}_G : K_G^*(X; \mathbb{Q}) \to H_G^{**}(X)$$

is injective, and $\text{ch}_G^{-1}(J) = I(G; \mathbb{Q})$ when $X$ is a point.

Proof. By [AS Theorem 2.1], $K^*(X \times_G EG) \cong K_G^*(X \times EG)$ is the completion of $K_G^*(X)$ at $I(G)$. The map $\iota : K_G^*(X) \to K^*(X \times G EG)$ induced by the projection map $X \times EG \to X$ is injective because the $I(G)$-adic topology of the completion is Hausdorff if $G$ is connected (cf. the Note immediately preceding [AH Section 4.5]). It follows that the rationalized
Lemma 2.4. Let $\mathcal{H}$ be the Milnor join of $n$ copies of $G$. Then $X \times G \mathcal{H}$ is compact and the ordinary Chern character map $\mathrm{ch}_n : K^*(X \times G \mathcal{H}; \mathbb{Q}) \to H^*(X \times G \mathcal{H}; \mathbb{Q})$ is an isomorphism. Note that

$$K^*_G(X \times G \mathcal{H}; \mathbb{Q}) \cong \lim_{\longleftarrow} K^*_G(X \times G \mathcal{H}_n; \mathbb{Q})$$

(see [AS Corollary 2.4, Proposition 4.1 and proof of Proposition 4.2]). It follows that the map $\mathrm{ch} : K^*(X \times G \mathcal{H}; \mathbb{Q}) \to H^*_G(X)$ is the inverse limit of the isomorphisms $\mathrm{ch}_n$ and injective by the left-exactness of inverse limit. The map $\mathrm{ch}_G$ is the composition of the two injective maps $\iota \otimes \mathbb{Q}$ and $\mathrm{ch} : K^*(X \times G \mathcal{H}; \mathbb{Q}) \to H^*_G(X)$. Therefore $\mathrm{ch}_G$ is injective. Next, consider the commutative diagram

$$
\begin{array}{ccc}
R(G) & \longrightarrow & K^*(pt; \mathbb{Q}) \\
\downarrow \mathrm{ch}_G & & \downarrow \mathrm{ch} \\
H^*_G(pt) & \longrightarrow & H^*(pt)
\end{array}
$$

where the two horizontal maps are forgetful maps. Since $J$ is the kernel of the bottom map and both $\mathrm{ch}_G$ and $\mathrm{ch}$ are injective, $\mathrm{ch}_G^{-1}(J)$ is the kernel of the top map, which is precisely $I(G; \mathbb{Q})$.

Under the condition of weak equivariant formality, [HL Proposition 4.2] asserts that the kernel of $f$ is $I(G) \cdot K^*_G(X)$. In fact, we also have

**Lemma 2.4.** Let $X$ be a finite CW-complex which is acted on by a compact connected Lie group $G$ equivariantly formally. Then the kernel of the forgetful map

$$f_G \otimes \mathrm{Id}_\mathbb{Q} : K^*_G(X; \mathbb{Q}) \to K^*(X; \mathbb{Q})$$

is $I(G; \mathbb{Q}) \cdot K^*_G(X; \mathbb{Q})$.

**Proof.** In the following diagram,

$$
\begin{array}{ccc}
K^*_G(X; \mathbb{Q}) & \xrightarrow{f_G \otimes \mathrm{Id}_\mathbb{Q}} & K^*(X; \mathbb{Q}) \\
\downarrow \mathrm{ch}_G & & \downarrow \mathrm{ch} \\
H^*_G(X) & \xrightarrow{g_G \otimes \mathrm{Id}_\mathbb{Q}} & H^*(X)
\end{array}
$$

where $g_G \otimes \mathrm{Id}_\mathbb{Q}$ is the forgetful map, $H^*_G(X)$ is the completion of $H^*_G(X)$ at the augmentation ideal $J$ of $H^*_G(pt)$. Since $X$ is an equivariantly formal $G$-space, $H^*_G(X)$ is isomorphic to $H^*_G(pt) \otimes H^*(X)$ as a $H^*_G(pt)$-module, and the forgetful map

$$g_G \otimes \mathrm{Id}_\mathbb{Q} : H^*_G(X) \to H^*(X)$$

has $J \cdot H^*_G(X)$ as the kernel. Since $H^*_G(X)$ is a finitely generated module over the Noetherian ring $H^*_G(pt)$, a simple result on completions (cf. [Ma Theorem 55]) implies that $H^*_G(X) \cong H^*_G(X) \otimes_{H^*_G(pt)} H^*_G(pt)$. So the kernel of $g_G \otimes \mathrm{Id}_\mathbb{Q}$ is $J \cdot H^*_G(X)$. By Proposition 2.3, the
preimage $\text{ch}_{G}^{-1}(J)$ is $I(G; \mathbb{Q})$ and $\text{ch}_{G}$ is injective. It follows that the kernel of $f_{G} \otimes \text{Id}_{\mathbb{Q}}$ is $\text{ch}_{G}^{-1}(J \cdot H_{G}^{*}(X)) = I(G; \mathbb{Q}) \cdot K_{G}^{*}(X; \mathbb{Q})$.

\textbf{Proof of Theorem 1.3} (1) $\iff$ (2). We first deal with the $T$-equivariant case, where $T$ is a maximal torus of $G$. We claim that, if $X$ is an equivariantly formal $T$-space, we have the following string of (in)equalities.

$$\dim_{\mathbb{Q}}K^{*}(X^{T}; \mathbb{Q}) = \dim_{\mathbb{Q}}K^{*}(X; \mathbb{Q}) = \dim_{\mathbb{Q}}K^{*}(X; \mathbb{Q})/I(T; \mathbb{Q}) \cdot K_{T}^{*}(X; \mathbb{Q}) \leq \dim K^{*}(X; \mathbb{Q}).$$

Applying Segal’s localization theorem to the case of torus group actions (cf. [Sc, Proposition 4.1]), we have that the restriction map $K_{T}^{*}(X; \mathbb{Q}) \to K_{T}^{*}(X^{T}; \mathbb{Q})$ becomes an isomorphism after localizing at the zero prime ideal, i.e., to the field of fraction of $R(T; \mathbb{Q})$. So rank$_{R(T; \mathbb{Q})}K_{T}^{*}(X; \mathbb{Q}) = \dim_{\mathbb{Q}}K^{*}(X^{T}; \mathbb{Q})$. By [Sc, Proposition 2.2], $K_{T}^{*}(X^{T}; \mathbb{Q})$ is isomorphic to $R(T; \mathbb{Q}) \otimes K^{*}(X^{T}; \mathbb{Q})$, whose rank over $R(T; \mathbb{Q})$ equals $\dim_{\mathbb{Q}}K^{*}(X^{T}; \mathbb{Q})$. The first equality then follows. Next, by [Sc, Proposition 5.4] and the discussion thereafter, we have that $K_{T}^{*}(X; \mathbb{Q})$ is a finite $R(T; \mathbb{Q})$-module. After localizing $K_{T}^{*}(X; \mathbb{Q})$ at $I(T; \mathbb{Q})$ and reduction modulo the same ideal, we have that $K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})} \cong K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})}$ is a finite dimensional $\mathbb{Q}$-vector space. We let $n$ be the dimension of this vector space, and $x_{1}, \ldots, x_{n} \in K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})}$ be its basis. Finite generation of $K_{T}^{*}(X; \mathbb{Q})$ as a module over the Noetherian ring $R(T; \mathbb{Q})$ enables us to invoke Nakayama lemma, and have that there exist lifts $\hat{x}_{1}, \ldots, \hat{x}_{n} \in K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})}$ that generate $K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})}$ as a $R(T; \mathbb{Q})_{I(T; \mathbb{Q})}$-module. It follows, after further localization to the field of fraction of $R(T; \mathbb{Q})$, that $\hat{x}_{1}, \ldots, \hat{x}_{n}$ span $K_{T}^{*}(X; \mathbb{Q})_{(0)}$ as a $R(T; \mathbb{Q})_{(0)}$-vector space, and that

$$\dim_{R(T; \mathbb{Q})_{(0)}}K_{T}^{*}(X; \mathbb{Q})_{(0)} = \dim_{\mathbb{Q}}K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})} \otimes_{R(T; \mathbb{Q})} K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})} = n.$$ 

Noting the isomorphism $K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})} \otimes_{R(T; \mathbb{Q})} K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})} \cong K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})} \otimes_{R(T; \mathbb{Q})} K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})}$, we arrive at the first inequality. Finally, the last inequality follows from Lemma 2.3.

If $X$ is an equivariantly formal $T$-space, then $\dim H^{*}(X) = \dim H^{*}(X^{T})$ (see [Hsi p. 46]). The Chern character implies that $\dim K^{*}(X^{T}; \mathbb{Q}) = \dim K^{*}(X; \mathbb{Q})$ which, together with the (in)equalities in the above claim, yields $\dim K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})} \otimes_{R(T; \mathbb{Q})} K_{T}^{*}(X; \mathbb{Q})_{I(T; \mathbb{Q})}$, we prove the claim.

Assume on the other hand that $X$ is RKEF. Consider the commutative diagram (2.1). Since $f_{T} \otimes \text{Id}_{\mathbb{Q}}$ is onto and $h$ is an isomorphism, $\tilde{g}_{T} \otimes \text{Id}_{\mathbb{Q}}$ is onto. By [Ma, Theorem 55], we have that $H_{T}^{*}(X) \cong H_{T}^{*}(X \otimes H_{T}^{*}(pt))$ $H_{T}^{*}(pt)$. Applying $\tilde{g}_{T} \otimes \text{Id}_{\mathbb{Q}}$ gives $H^{*}(X) = \text{Im}(\tilde{g}_{T} \otimes \text{Id}_{\mathbb{Q}}) = \text{Im}(g_{T} \otimes \text{Id}_{\mathbb{Q}}) = \text{Im}(g_{T} \otimes \text{Id}_{\mathbb{Q}})$. Hence $X$ is $T$-equivariantly formal.

With the equivalence of equivariant formality and RKEF for $T$-action we have just proved and the fact that, if $T$ is a maximal torus of $G$ which is compact and connected, $T$-equivariant formality is equivalent to $G$-equivariant formality (cf. [GR, Proposition 2.4]), it suffices to show that $f_{T} \otimes \text{Id}_{\mathbb{Q}}$ is onto if and only if $f_{G} \otimes \text{Id}_{\mathbb{Q}}$ is onto in order to establish the equivalence of equivariant formality and RKEF for $G$-action. One direction is easy: if $f_{G} \otimes \text{Id}_{\mathbb{Q}}$ is onto, so is $f_{T} \otimes \text{Id}_{\mathbb{Q}}$ because $f_{G} \otimes \text{Id}_{\mathbb{Q}} = (f_{T} \otimes \text{Id}_{\mathbb{Q}}) \circ \tau^{*}$. Conversely, suppose that $f_{T} \otimes \text{Id}_{\mathbb{Q}}$ is onto. Then any $x \in K^{*}(X; \mathbb{Q})$ admits a lift $\tilde{x} \in K_{T}^{*}(X; \mathbb{Q})$. Note that for
any \( w \in W \), \((f_T \otimes \text{Id}_Q) (w \cdot \tilde{x}) = x \). It follows that the average
\[
\overline{x} := \frac{1}{|W|} \sum_{w \in W} w \cdot \tilde{x}
\]
is also a lift of \( x \). Moreover, by Theorem 2.1, \( \overline{x} \in r^*K_G(X; \mathbb{Q}) \). So \((r^*)^{-1}(\overline{x}) \in K^*_G(X; \mathbb{Q})\)
is a lift of \( x \) and \( f_G \otimes \text{Id}_Q \) is onto as well. \( \square \)

**Proof of Theorem 1.3** \((1) \iff (3)\). That \( \mathbb{Q} \)-weakly equivariant formality implies RKEF is immediate (cf. [HL, Definition 4.1]). On the other hand, if \( X \) is a RKEF \( G \)-space, then by Theorem 1.3, \((1) \implies (2)\), \( X \) is an equivariantly formal \( G \)-space. The map
\[
K^*_G(X; \mathbb{Q}) \otimes_{R(G; \mathbb{Q})} \mathbb{Q} \to K^*(X; \mathbb{Q})
\]
\[
\alpha \otimes z \mapsto f_G(\alpha)z
\]
is injective by Lemma 2.4 and surjective by RKEF. Hence \( X \) is a \( \mathbb{Q} \)-weakly equivariantly formal \( G \)-space. This completes the proof. \( \square \)

3. Some applications

In this Section, we shall demonstrate the utility of Theorem 1.3 by giving alternative proofs of some previous results.

3.1. Conjugation action on compact Lie groups. Let \( G \) be a compact connected Lie group with conjugation action by itself. It is well-known that this action is equivariantly formal. See, for example, [GS, Sect. 11.9, Item 6] for a sketch of proof for the case \( G = U(n) \), and [J] for an explicit construction of equivariant extensions of the generators of \( H^*(G) \). We will show equivariant formality of conjugation action by proving that \( G \) is a RKEF \( G \)-space. By [Ho, II, Theorem 2.1],
\[
K^*(G; \mathbb{Q}) \cong \bigwedge^*_Q (R \otimes \mathbb{Q}),
\]
where \( R \) is the image of the map
\[
\delta : R(G) \to K^{-1}(G)
\]
which sends \( \rho \in R(G) \) to the following complex of vector bundles:
\[
0 \to G \times \mathbb{R} \times V \to G \times \mathbb{R} \times V \to 0
\]
\[
(g, t, v) \mapsto \begin{cases} (g, t, -t\rho(g)v), & \text{if } t \geq 0, \\ (g, t, v), & \text{if } t \leq 0. \end{cases}
\]
For any \( \rho \), \( \delta(\rho) \) admits an equivariant lift in \( K^*_G(G) \) because \( G \times \mathbb{R} \times V \) can be equipped with the \( G \)-action given by
\[
g_0 \cdot (g, t, v) = (g_0gg_0^{-1}, t, \rho(g_0)v),
\]
with respect to which the middle map of the above complex of vector bundles is \( G \)-equivariant. Thus \( f_G \otimes \text{Id}_Q : K^*_G(G; \mathbb{Q}) \to K^*(G; \mathbb{Q}) \) is onto, i.e., \( G \) is a RKEF \( G \)-space.

\footnote{The map \( \delta \), which was defined in [?] and corrected in [F], is the same as the map \( \beta \) defined in [Ho].}
3.2. Left translation action on $G/K$ where rank $G = \text{rank} \ K$. Let $G$ be a compact connected Lie group and $K$ a connected Lie subgroup of the same rank. The left translation action on $G/K$ by $G$ is well-known to be equivariantly formal, which can be proved by noting that $G/K$ satisfies the sufficient condition for equivariant formality that its odd cohomology vanish (cf. [GHV, Chapter XI, Theorem VII]). Alternatively, by the rationalized version of [Sn, Theorem 4.2] and the remark following it, its odd cohomology vanish (cf. [GHV, Chapter XI, Theorem VII]).

Proposition 3.2. Let $G$ be a compact connected Lie group and $X$ a finite $G$-CW complex. Suppose that the $G$-action on $X$ has maximal rank connected isotropy subgroups. Then $X$ is an equivariantly formal $G$-space.

Remark 3.3. In fact, Proposition 3.2 follows from [GR] Corollary 3.5, where connectedness of isotropy subgroups is not assumed. Though the space under consideration in [GR] Corollary 3.5 is the subset of a compact $G$-manifold consisting of those points with maximal rank isotropy subgroups, its proof does not make use of this assumption and can be easily adapted to the more general case of $G$-CW complexes. Indeed the proof hinges on the observation that for any compact space $X$ with maximal rank isotropy subgroups and a maximal torus $T$, the map $G \times_{N_G(T)} X^T \to X$ given by $[g, x] \mapsto gx$ is onto and that the fibers of the map are acyclic. This enables one to assert the isomorphism $H_G^*(X) \cong H^*_{N_G(T)}(X^T)$. The latter, by abelianization, is $H_T^*(X^T)^W$, which in turn by a commutative algebra result ([GR Lemma 2.7]) is a free module over $H_T^*(pt)^W \cong H_G^*(pt)$. Hence $X$ is an equivariantly formal $G$-space.

We would like to give a different proof of this result by using Theorem 1.3 and induction on the dimension of $X$. We shall point out that the group actions considered in Sections 3.1 and 3.2 are examples of group actions we discuss in this section. However, equivariant formality of left translation actions on generalized flag manifolds as in Section 3.2 is used in the following proof.

Proof. Consider the $n$-skeleton $X_n$. It is obtained by gluing the cells $G/K_i \times \mathbb{D}^n$, $1 \leq i \leq k$, to the $(n-1)$-skeleton $X_{n-1}$ through some $G$-equivariant attaching maps

$$f_i : G/K_i \times \partial \mathbb{D}^n \to X_{n-1}.$$ 

Let

$$F_i : G/K_i \times \mathbb{D}^n \to X_n$$

be a continuous map that sends the boundary of $\mathbb{D}^n$ to $f_i$. The map $F_i$ can be glued together to form an equivariant map $F : G/K \times \mathbb{D}^n \to X$. This shows that $X$ is equivariantly formal.
be the inclusion of the cell $G/K_i \times D^n_-$ into $X_i$, and $V$ be any given vector bundle over $X_i$. We shall show that, for some $p$ and $q$, $V^{\oplus p} \oplus C^q$ admits an equivariant structure, assuming by induction hypothesis that $V_0$, which is the restriction of $V$ to $X_{n-1}$, satisfies the condition that $V^{\oplus p_0} \oplus C^{q_0}$ admits an equivariant structure for some $p_0$ and $q_0$. Note that $V$ can be obtained by gluing $V_0 \to X_{n-1}$ and $V_i \to G/K_i \times D^n$, where $V_i := F^i_* V$, through the clutching maps, i.e. vector bundle homomorphisms

$$h_i : V_{i|G/K_i \times \partial D^n} \to V_0$$

which cover the maps $f_i$ and send fiber to fiber isomorphically. By the induction hypothesis, $V^{\oplus p_0} \oplus C^{q_0}$ admits an equivariant structure for some $p_0$ and $q_0$. By the discussion in Section 3.2 and the contractibility of $D^n$, there exist $p_i$ and $q_i$ such that $V^{\oplus p_i} \oplus C^{q_i}$ is isomorphic to a certain homogeneous vector bundle which is obviously $G$-equivariant. For simplicity we may take $p = \text{LCM}(p_0, p_i)$ and $q = \max \{q_0, q_i\}$ such that $V^{\oplus p} \oplus C^q \cong (G \times D^n)_K W_i$ for some $K_i$-representation $W_i$ (and thus is $G$-equivariant) and $V^{\oplus p} \oplus C^q$ admit an equivariant structure. Consider the clutching maps

$$h_i^{\oplus p} + \text{Id}_{C^q} : V^{\oplus p}|_{G/K_i \times \partial D^n} \oplus C^q \to V^{\oplus p} \oplus C^q$$

for the vector bundles $V^{\oplus p} \oplus C^q$ and $V^{\oplus p} \oplus C^q$. The vector bundle $V^{\oplus p} \oplus C^q$ admits an equivariant structure if $h_i^{\oplus p} + \text{Id}_{C^q}$ is homotopy equivalent to another clutching map which is $G$-equivariant. Note that $h_i^{\oplus p} + \text{Id}_{C^q}$ is the composition of

$$h_i^{\prime \oplus p} + \text{Id}_{C^q} : V^{\oplus p}|_{G/K_i \times \partial D^n} \oplus C^q \to f_i^* (V_0^{\oplus p} \oplus C^q)$$

and

$$f_i^{\prime \oplus p} + \text{Id}_{C^q} : f_i^* (V_0^{\oplus p} \oplus C^q) \to V^{\oplus p} \oplus C^q.$$
and the fact that \( K^{-1}(G/K_i) = 0 \) (because \( G/K_i \) can be given a CW-complex structure consisting of only even dimensional cells) imply that \( K^0(G/K_i \times S^n, G/K_i \times \{pt\}) \) is the kernel of the restriction map \( K^0(G/K_i \times S^n) \to K^0(G/K_i) \). Note that by the Künneth formula for \( K \)-theory ([A]) and the torsion-freeness of both \( K^*(G/K_i) \) and \( K^*(S^n) \), \( K^0(G/K_i \times S^n) \cong K^0(G/K_i) \otimes K^0(S^n) \). We shall consider the following two cases.

1. When \( n \) is odd: In this case, the restriction map is an isomorphism, and \( K^0(G/K_i \times S^n, G/K_i \times \{pt\}) = 0 \). So \([E_\alpha] - [E_{Id}] = 0\) and \([E_\alpha] - [E_{Id}] = 0\) in \( K^0(G/K_i \times S^n) \).

   In other words, if \( q \) is sufficiently large, \( E_\alpha \) is isomorphic to \( E_{Id} \). It follows that \( \alpha \) is homotopy equivalent to the identity automorphism, which is obviously \( G \)-equivariant.

2. When \( n \) is even: In this case, \( K^0(G/K_i \times S^n, G/K_i \times \{pt\}) \cong K^0(G/K_i) \otimes \tilde{K}^0(S^n) \cong K^0(G/K_i) \otimes \tilde{K}^0(S^n) \). Since \( \tilde{K}^0(S^n-1) \cong \mathbb{Z} \), we may assume that the restrictions of both \( H_1 \) and \( H_2 \) to \( \partial \mathbb{D}^n \) (the equator of \( S^n \)) are trivial vector bundles \( \mathbb{C}^r \) by taking Whitney sum with trivial vector bundles of sufficiently high rank if necessary. There exist vector bundles \( U_1 \) and \( U_2 \) over \( G/K_i \) such that

\[
[E_\alpha] - [E_{Id}] = ([U_1] - [U_2]) \otimes ([H_1] - [H_2]).
\]

Rearranging and for \( q \) sufficiently large enough, we have the vector bundle isomorphism

\[
E_\alpha \oplus U_1 \otimes H_2 \oplus U_2 \otimes H_1 \cong E_{Id} \oplus U_1 \otimes H_1 \oplus U_2 \otimes H_2.
\] (3.1)

Let \( Y_\pm \) be the restriction of the vector bundles in (3.1) to \( G/K_i \times \mathbb{D}^n \), and \( Z \) the restriction to \( G/K_i \times \partial \mathbb{D}^n \). Note that \( Y_+ \cong Y_- \cong (G \times \mathbb{D}^n) \times_{K_i} W_i \oplus (U_1 \oplus U_2) \otimes \mathbb{C}^r \), and \( Z \cong (G \times \partial \mathbb{D}^n) \times_{K_i} W_i \oplus (U_1 \oplus U_2) \otimes \mathbb{C}^r \). The vector bundle on the right-hand side of (3.1) is obtained by gluing \( Y_+ \) and \( Y_- \) along \( G/K_i \times \partial \mathbb{D}^n \) through the vector bundle automorphism \( \theta := \text{Id}_{(G \times \partial \mathbb{D}^n) \times_{K_i} W_i} \oplus \text{Id}_{U_1} \otimes \beta_1 \oplus \text{Id}_{U_2} \otimes \beta_2 \) on \( Z \), where \( \beta_i : \partial \mathbb{D}^n \to GL(r, \mathbb{C}) \) is a clutching function for \( H_i \), \( i = 1, 2 \). Similarly the vector bundle automorphism used to obtain the vector bundle on the left-hand side of (3.1) is \( \phi := \alpha \oplus \text{Id}_{U_1} \otimes \beta_2 \oplus \text{Id}_{U_2} \otimes \beta_1 \), which is homotopy equivalent to \( \theta \) because of the vector bundle isomorphism (3.1).

Let

\[
\phi' := (\text{Id}_{(G \times \partial \mathbb{D}^n) \times_{K_i} W_i} \oplus \text{Id}_{U_1} \otimes \beta_2^{-1} \oplus \text{Id}_{U_2} \otimes \beta_1^{-1}) \circ \phi
\]

\[
= \alpha \oplus (\text{Id}_{U_1 \oplus U_2}) \otimes \text{Id}_{\mathbb{C}^r}
\]

\[
\theta' := (\text{Id}_{(G \times \partial \mathbb{D}^n) \times_{K_i} W_i} \oplus \text{Id}_{U_1} \otimes \beta_2^{-1} \oplus \text{Id}_{U_2} \otimes \beta_1^{-1}) \circ \theta
\]

\[
= \text{Id}_{(G \times \partial \mathbb{D}^n) \times_{K_i} W_i} \oplus \text{Id}_{U_1} \otimes \beta_2^{-1} \beta_1 \oplus \text{Id}_{U_2} \otimes \beta_1^{-1} \beta_2.
\]

Then \( \phi' \) and \( \theta' \) are homotopy equivalent. Let \( U_3 \) be a vector bundle over \( G/K_i \) such that \( U_1 \oplus U_2 \oplus U_3 \cong \mathbb{C}^s \) for some positive integer \( s \) (for the existence of such a vector bundle see [Hat], Proposition 1.4) and its complex version. Let

\[
Z := Z \oplus U_3 \otimes \mathbb{C}^r
\]

\[
= (G \times \partial \mathbb{D}^n) \times_{K_i} W_i \oplus \mathbb{C}^s.
\]
\[ \tilde{\varphi} := \varphi' \oplus \text{Id}_{U_3} \otimes \text{Id}_{C^r} \]
\[ = \alpha \oplus \text{Id}_{C^r}, \quad \text{and} \]
\[ \tilde{\theta} := \theta' \oplus \text{Id}_{U_3} \otimes \text{Id}_{C^r} \]
\[ = \text{Id}_{(G \times \partial \mathbb{D}^n) \times K_i W_i} \oplus \text{Id}_{U_1} \otimes \beta_2^{-1} \beta_1 \oplus \text{Id}_{U_2} \otimes \beta_1^{-1} \beta_2 \oplus \text{Id}_{U_3} \otimes \text{Id}_{C^r} \]

Note that both \( \tilde{\theta} \) and \( \tilde{\varphi} \) are vector bundle automorphisms of \( \tilde{Z} \). As the action of \( \tilde{\theta} \) on the fiber \( \tilde{Z}(x,y) \) over \( (x,y) \in G/K_i \times \partial \mathbb{D}^n \) only depends on \( y \) and the action of \( G \) on \( \partial \mathbb{D}^n \) is trivial, \( \tilde{\theta} \) is a \( G \)-equivariant vector bundle automorphism on \( \tilde{Z} \). Moreover, \( \tilde{\theta} \) is homotopy equivalent to \( \tilde{\varphi} = h'^{\oplus p} \oplus \text{Id}_{\Sigma q+rs} \). Replacing \( q+rs \) with \( q \), we have that \( h'^{\oplus p} \oplus \text{Id}_{\Sigma q} \) is homotopy equivalent to a \( G \)-equivariant map.

We have shown that, by induction on the dimension of \( X \), for any given vector bundle \( V \to X \), \( V^{\oplus p} \oplus q^q \) admits an equivariant structure for some \( p \) and \( q \). The same is true for the suspension \( \Sigma X \) because it is also a \( G \)-CW complex with maximal rank connected isotropy subgroups. It follows that the \( G \)-action on \( X \) is equivariantly formal by Theorem 1.3.

\[ \square \]

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