A Functional Zip Operation On Streams

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Abstract

We answer an open question in the theory of transducer degrees initially posed in [1] on the existence of a diamond structure in the transducer hierarchy. Transducer degrees are the equivalence classes formed by word transformations which can be realized by a finite state transducer, which form an order based on which words can be transformed into other words. We provide a construction which proves the existence of a diamond structure, while also introducing a new function on streams which may be useful for proving more results about the transducer hierarchy.

1 Introduction

Finite state transducers (FSTs) are ubiquitous in computer science, and infinite streams are also common in many fields. Yet there are very few results on how to transform an arbitrary stream into another stream with an FST. We define a stream \( \sigma \) as being above another stream \( \tau \) if some FST \( T \) can transduce \( \sigma \) into \( \tau \), with \( \sigma \) and \( \tau \) being the same degree if they can both be transduced into each other. The structure of these degrees, called the transducer hierarchy, has many parallels with Turing degrees. The main results that have been done thus far mostly deal with streams determined by polynomials, and in this paper we will present a new result that comes from a new operation on streams. This new operation, called \( fzip \), may be useful in proving additional results, and allows us a new class of functions to consider the degrees of: piecewise polynomials.

2 Definitions

We will give some preliminary definitions with the goal of understanding the definition of a weight product, the key operation for all the results in this paper. For more definitions and background in this area, see [1, 2, 3, 4, 5]. We begin by setting \( \mathbb{2} = \{0, 1\} \), which we will use as our input and output alphabet for all of our transducers. We will focus only on finite state transducers of the following form:
Definition 2.1. A finite-state transducer is a tuple $T = (Q, q_0, \delta, \lambda)$ where $Q$ is a finite set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times 2 \to Q$ is the transition function, and $\lambda : Q \times 2 \to 2^*$ is the output function.

Note that $\delta$ and $\lambda$ can be extended ($\delta : Q \times 2^* \to Q, \lambda : Q \times 2^\infty \to 2^\infty$) as follows:

$$
\delta(q, 0) = q, \quad \delta(q, au) = \delta(\delta(q, a), u), \quad \text{where } q \in Q, a \in 2, u \in 2^*
$$

$$
\lambda(q, 0) = 0, \quad \lambda(q, au) = \lambda(q, a) \cdot \lambda(\delta(q, a), u), \quad \text{where } q \in Q, a \in 2, u \in 2^\infty
$$

The above equations correspond to inputting a (possibly infinite) stream of letters into $T$. This allows us to define a function $T$ on finite or infinite strings in $2$, by saying that $T(w)$ is equal to the output of the FST $T$ after inputting $w$.

Now we can make the following definition, which is the basis for the transducer hierarchy:

Definition 2.2. Let $T$ be an FST, and let $\sigma, \tau \in 2^\mathbb{N}$ be infinite sequences. We say that $T$ transduces $\sigma$ to $\tau$, or that $\tau$ is the $T$-transduct of $\sigma$, if $T(\sigma) = \tau$.

In general, for any two infinite sequences $\sigma, \tau$ we say that $\sigma \geq \tau$ if there exists some $T$ so that $T(\sigma) = \tau$.

This relation $\geq$ is reflexive, and can be shown to be transitive by composition of FSTs (See Lemma 8, [1]). If for some $\sigma, \tau$ we have $\sigma \geq \tau$ but not vice versa, we say $\sigma > \tau$. If we do have $\sigma \geq \tau$ and $\tau \geq \sigma$ then we say that $\sigma \equiv \tau$, and we use $[\sigma]$ to denote the equivalence class of $\sigma$. We call $[\sigma]$ the degree of $\sigma$.

Now that we have defined what a transducer degree is, we will focus our attention on a particular subset of streams, namely the streams which are generated by functions in the sense of the following definition.

Definition 2.3. For a function $f$ from $\mathbb{N}$ to $\mathbb{N}$, we define $\langle f \rangle$ to be the stream given by

$$
\langle f \rangle = \prod_{i=0}^{\infty} 10^{f(i)} = 10^{f(0)}10^{f(1)}10^{f(2)}\ldots
$$

We will often use $\langle f \rangle$ to mean both the stream determined by $f$, as well as the degree of that stream $[\langle f \rangle]$. We also refer to a part of the stream of the form $10^{f(i)}$ as a block.

Having defined $\langle f \rangle$ in this way, some relatively simple initial results have been obtained in [2], which we state here.

Lemma 2.4. Let $f : \mathbb{N} \to \mathbb{N}, a, b \in \mathbb{N}$. We have the following equivalences and inequalities:
1. \( \langle af(n) \rangle \equiv \langle f(n) \rangle \), for \( a > 0 \)

2. \( \langle f(n + a) \rangle \equiv \langle f(n) \rangle \)

3. \( \langle f(n) + a \rangle \equiv \langle f(n) \rangle \)

4. \( \langle f(n) \rangle \geq \langle f(an) \rangle \), for \( a > 0 \)

5. \( \langle f(n) \rangle \geq \langle af(2n) + bf(2n + 1) \rangle \)

One interesting consequence of the third equality is that any polynomial with a positive leading coefficient can be thought of as a stream and thus associated with a transducer degree, even if some of its values happen to be negative. For instance, the polynomial \((n - 2)^3\) can’t directly be interpreted as a stream, since it is negative for \( n = 0, 1 \). However, if we take \((n - 2)^3 + 8\), then this polynomial is nonnegative, and therefore corresponds to a stream (and thus a degree). So even though it’s technically incorrect, it will be convenient sometimes to refer to a degree such as \( \langle (n - 2)^3 \rangle \), when we mean more precisely the degree \( \langle (n - 2)^3 + k \rangle \) for any \( k \geq 8 \). Similarly, the first equality allows us to refer to the degree of a function with rational coefficients, where we really mean the degree of the corresponding function multiplied by the appropriate scalar to eliminate any fractional coefficients.

Now we are ready to start defining weight products. We will not provide the full proof of the main result we need (Theorem 2.6), but a more detailed explanation can be found in [2]. We begin by defining a weight.

**Definition 2.5.** A weight is a tuple \( \alpha = \langle a_0, a_1, \ldots, a_{k-1}, b \rangle \in \mathbb{Q} \) with each \( a_i \geq 0 \). If \( a_i = 0 \) for all \( i \) then we say the weight is constant. To distinguish between weights and tuples of weights, weights will not be bolded but tuples of weights will, except potentially in cases where there is only one weight in the tuple.

Given a weight \( \alpha \) as above and a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) we can define \( \alpha \cdot f \) as:

\[
\alpha \cdot f = a_0 f(0) + a_1 f(1) + \ldots + a_{k-1} f(k - 1) + b
\]

We are ready to define the weight product. Let \( \alpha = \langle \alpha_0, \alpha_1, \ldots, \alpha_{m-1} \rangle \) be a tuple of weights, with \( \alpha' \) being the cyclic shift \( \langle \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_0 \rangle \).

Then the weight product of \( \alpha \) with \( f \), written as \( \alpha \otimes f \), is defined in the following way:

\[
(\alpha \otimes f)(0) = \alpha_0 \cdot f \\
(\alpha \otimes f)(n + 1) = (\alpha' \otimes s^{[\alpha_0]}(f))(n)
\]
Here $S^k(f)(n) = f(n + k)$ and $|\alpha_0|$ indicates the length of the tuple $\alpha_0$. We call a weight product natural if $\alpha \otimes f(n) \in \mathbb{N}$ for all $n$. Note that since $\alpha$ is a finite tuple of finite tuples in $\mathbb{Q}$, we can take the LCM of all of the denominators and multiply through to make the product natural. Since this does not change the degree of the resulting function (by Lemma 2.4), from now on we will assume that all weight products are natural. We also define the length of a tuple of weights to be $||\alpha|| = \sum_{i=0}^{m-1} (|\alpha_i| - 1)$. (By $|\alpha_i|$ we mean simply the number of elements in that weight.)

The following image provides a more intuitive picture of how the weight product works, by showing pictorially how to compute the weight product of the tuple of weights $\alpha = \langle \alpha_0, \alpha_1 \rangle$ with an arbitrary function $f(n)$, where $\alpha_0 = \langle 2, 4, 6, 8 \rangle, \alpha_1 = \langle 1, 7, 4 \rangle$:

$$
\begin{align*}
\alpha \otimes f)(0) &= 2f(0) + 4f(1) + 6f(2) + 8 \\
\alpha \otimes f)(1) &= f(3) + 7f(4) + 4
\end{align*}
$$

$$
\begin{align*}
\alpha \otimes f)(2) &= 2f(5) + 4f(6) + 6f(7) + 8 \\
\alpha \otimes f)(3) &= f(8) + 7f(9) + 4
\end{align*}
$$

This also provides us with a better notion of what the length $||\alpha||$ represents. For this $\alpha$, we have $||\alpha|| = (4 - 1) + (3 - 1) = 5$, which is exactly how many values of $f$ we go through after applying every weight.

Now that we have defined weight products we can proceed to state the main result that we need for the rest of the paper. The full proof of this result can be found in (Theorem 21, [3]).

**Theorem 2.6.** Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be (possibly piecewise) polynomials. Then $\langle g \rangle \geq \langle f \rangle$ if and only if there exists a tuple of weights $\alpha$ and integers $n_0, m_0$ such that $S^{m_0}(f) = \alpha \otimes S^{n_0}(g)$.

This theorem tells us that if we want to show that one polynomial degree is above another, we can consider weight products, rather than trying to figure out a transducer directly. The following theorem will give us a useful result for comparing polynomial degrees with non-polynomial degrees.

**Theorem 2.7.** Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a (possibly piecewise) polynomial, and $\sigma \in 2^\mathbb{N}$. Then $\langle f \rangle \geq \sigma$ if and only if $\sigma = \langle \alpha \otimes S^{n_0}(f) \rangle$ for some integer $n_0 \geq 0$, and a tuple of weights $\alpha$. 

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By "possibly piecewise" we mean that $f$ can be a piecewise function composed only of polynomials. (In fact, it can belong to a more general class of functions called "spiralling functions", but we only need polynomials and piecewise polynomials for this paper.)

3 Basic Results

Let $f$ and $g$ be functions from $\mathbb{N} \rightarrow \mathbb{N}$.

We define $fzip(f, g)$ as the function

$$fzip(f, g)(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{n even} \\ g\left(\frac{n-1}{2}\right) & \text{n odd} \end{cases}$$

Therefore $\langle fzip(f, g) \rangle$ is

$$\langle fzip(f, g) \rangle = \prod_{i=0}^{\infty} 10^{fzip(f, g)(i)} = 10^{f(0)}10^{g(0)}10^{f(1)}10^{g(1)}\ldots$$

We can extend $fzip$ to have as many functions as inputs as we want, in the obvious way, but for this paper we will mostly focus on the $fzip$ of pairs of functions. We will provide some elementary properties of $fzip$ as well as some nontrivial properties, and conclude by showing that $fzip(n, n^2)$ is at the top of a diamond structure in the transducer hierarchy.

One property that is immediately obvious is that $\langle fzip(f, g) \rangle \geq \langle f \rangle, \langle g \rangle$. This inequality must be strict for both unless $f$ can be transduced into $g$ (or vice versa).

A less obvious property is that $fzip$ is not symmetric in general, so $\langle fzip(f, g) \rangle$ need not equal $\langle fzip(g, f) \rangle$. This was proven inadvertently in [5], where a careful reading of the main proof in terms of $fzip$ gives us a specific example of two functions $f, g$ which satisfy $\langle fzip(f, g) \rangle \neq (fzip(g, f))$. However, there are two classes of functions which do allow $fzip$ to be symmetric, as detailed in the following lemma.

**Lemma 3.1.** Let $f, g$ be functions from $\mathbb{N}$ to $\mathbb{N}$ and suppose that $f$ is linear or exponential. Then $\langle fzip(f, g) \rangle = \langle fzip(g, f) \rangle$.

**Proof.** We have two distinct cases: $f$ is linear or $f$ is exponential.

Case 1: $f$ is linear.

In this case, we begin with an $fzip$ of the form

$$fzip(f, g)(n) = \begin{cases} a\left(\frac{n}{2}\right) + b & \text{n even} \\ g\left(\frac{n-1}{2}\right) & \text{n odd} \end{cases}$$

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If we consider removing the first block from the stream $(f \operatorname{zip}(f, g))$, we end up with

$$h_1(n) = \begin{cases} g\left(\frac{n}{2}\right) & \text{n even} \\ a\left(\frac{n}{2} + \frac{1}{2}\right) + b & \text{n odd} \end{cases}$$

But $\frac{n}{2} + \frac{1}{2}$ is equal to $\frac{n - 1}{2} + 1$, and thus we have

$$h_1(n) = \begin{cases} g\left(\frac{n}{2}\right) & \text{n even} \\ a\left(\frac{n - 1}{2}\right) + a + b & \text{n odd} \end{cases}$$

Then simply removing $a$ 0’s from every other block will bring us to our desired result:

$$h_2(n) = \begin{cases} g\left(\frac{n}{2}\right) & \text{n even} \\ a\left(\frac{n - 1}{2}\right) + b & \text{n odd} \end{cases}$$

Each $h_i$ is in the same degree as $f \operatorname{zip}(f, g)$, since we have only used reversible operations, and clearly $h_2(n) = f \operatorname{zip}(g, f)(n)$. The exponential case will proceed similarly.

**Case 2: $f$ is exponential.**

Initially we have the following fzip:

$$f \operatorname{zip}(f, g)(n) = \begin{cases} a(b)^{\frac{n}{2}} & \text{n even} \\ g\left(\frac{n - 1}{2}\right) & \text{n odd} \end{cases}$$

As before, we can remove the first block to change our fzip to:

$$f \operatorname{zip}(f, g)(n) = \begin{cases} g\left(\frac{n}{2}\right) & \text{n even} \\ a(b)^{\frac{n + 1}{2}} & \text{n odd} \end{cases}$$
Again we use \( \frac{n+1}{2} = \frac{n-1}{2} + 1 \) and simplify:

\[
\text{zip}(f, g)(n) = \begin{cases} 
  g(\frac{n}{2}) & \text{n even} \\
  a(b) \frac{n-1}{2} b & \text{n odd}
\end{cases}
\]

Dividing each odd block by \( b \) will give us \( \text{zip}(g, f) \) and thus the proof is complete.

We observe that while it may be possible for functions which are not linear or exponential to have this property (of allowing \( \text{zip} \) to be symmetric) it is worth pointing out that the proof relies on the defining characteristics of linear and exponential functions, namely that for linear functions \( f(n+1) = f(n) + a \), and for exponential functions \( f(n+1) = bf(n) \). Since our operations on individual blocks are restricted to exactly these operations (addition/subtraction and division/multiplication) this may suggest that these are the only types of functions that could work for this lemma, or at least that expanding this lemma to other types of functions may be difficult. We also get a somewhat obvious corollary from the proof method of removing the first block of \( \text{zip}(f, g) \):

**Corollary 3.2.** For all functions \( f, g \) from \( \mathbb{N} \) to \( \mathbb{N} \), \( \langle \text{zip}(f, g) \rangle = \langle \text{zip}(g, f(n+1)) \rangle \)

This follows immediately from the definition of \( \text{zip} \), and the fact that adding or removing one block from a stream does not change its degree.

We conclude this section by considering a couple of natural questions about \( \text{zip} \): under what circumstances can we compare two different \( \text{zips} \), and how can we compare an \( \text{zip} \) of two functions \( f \) and \( g \) with some third function \( h \)? For the first question, intuitively we would think that if \( f_1(n), g_1(n) \) are above \( f_2(n), g_2(n) \) respectively in the transducer hierarchy, then \( \text{zip}(f_1(n), g_1(n)) \) should be above \( \text{zip}(f_2(n), g_2(n)) \). However, the main difficulty is that while \( \text{zip} \) intertwines two functions, the weight product can’t be easily intertwined in the same way. For the second question, we can provide a very weak result which may illustrate the difficulty of this problem.

**Lemma 3.3.** Let \( f, g, h \) be functions from \( \mathbb{N} \) to \( \mathbb{N} \). Suppose that there exist weights \( \alpha = (\alpha_0, \alpha_1, ..., \alpha_{n-1}) \) and \( \beta = (\beta_0, \beta_1, ..., \beta_{m-1}) \) with \( f = \alpha \otimes h \) and \( g = \beta \otimes h \). Further suppose that \( n = m \) and for all \( i \), \( |\alpha_i| = |\beta_i| \). Then \( \langle h(n) \rangle \geq \langle \text{zip}(f(2n), g(2n+1)) \rangle \).

**Proof.** We will begin by constructing a new weight \( \gamma \) from \( \alpha \) and \( \beta \), and showing that \( (\gamma \otimes h)(n) \) is equal to \( \text{zip}(f(2n), g(2n+1)) \). We have two different cases for constructing \( \gamma \) depending on if \( m \) is even or odd.
Case 1: $m$ is even

If $m$ is even, simply take $\alpha$ and replace each $\alpha_{2i+1}$ with the corresponding $\beta_{2i+1}$ to obtain $\gamma$. Then $\gamma = (\alpha_0, \beta_1, \alpha_2, \beta_3, ..., \beta_{m-1})$.

Case 2: $m$ is odd

Start as before, by replacing the $\alpha_{2i+1}$’s in $\alpha$ with the $\beta_{2i+1}$’s, but this time we will also be replacing the $\beta_{2i+1}$’s in $\beta$ with the $\alpha_{2i+1}$ weights from $\alpha$. Let’s call these new weight tuples $fzip(k)$.

Case 1: $m$ is even

Let’s start by assuming $n$ is even, simply take $\gamma$’s in $\gamma$ that satisfy $L$. Now that we have a definition of $\gamma$, we can proceed to proving that $(\gamma \otimes h)(n) = fzip(f(2n), g(2n + 1))$. We show this by proving that for even $n$ we have $(\gamma \otimes h)(n) = (\alpha \otimes h)(n) = f(n)$, and for odd $n$ we have $(\gamma \otimes h)(n) = (\beta \otimes h)(n) = g(n)$. Let’s start by assuming $n$ is even, and from the proof for even $n$ it will be clear that the proof for odd $n$ is identical.

If $n$ is even, then by construction of $\gamma$ and the definition of the weight product, we have for some $k$, $(\gamma \otimes h)(n) = \alpha_m \cdot S^k(f)$. Similarly, for some $k'$, $(\alpha \otimes h)(n) = \alpha_m \cdot S^{k'}(f)$. Letting $L = ||\gamma||$, $m$ is equal to $n$ mod $L$ (intuitively, $m$ is the number of times we go through all of the weights in $\gamma$). We will show that $k = k'$.

The key to this proof is the condition that each $\alpha_i$ has the same length as the corresponding $\beta_i$. Because of this condition, whenever we apply the recursive part of the weight product definition, we are shifting $f$ by the same amount whether we are applying the weights from $\gamma$ (which are half $\alpha_i$’s and half $\beta_i$’s) or weights from $\alpha$ only. More rigorously, we can see this by calculating the exact values of $k$ and $k'$.

We have that $k$ is equal to $(n - m)\frac{h}{m} + \sum_{i=0}^{m-1} (|\gamma_i| - 1)$, where $\gamma_i$ is the $i$th weight in $\gamma$ ($\gamma_i$ alternates between $\alpha_i$ and $\beta_i$). Similarly, $k' = (n - m)\frac{h}{m} + \sum_{i=0}^{m-1} (|\alpha_i| - 1)$. From the length condition on the weights in $\alpha$ and $\beta$, $|\gamma_i| = |\alpha_i|$ for all $i$, and therefore $k = k'$. This means that for even $n$, $(\gamma \otimes h)(n) = (\alpha \otimes h)(n)$, and by definition of $\alpha$, $g(\alpha \otimes h)(n) = f(n)$. Therefore $(\gamma \otimes h)(n) = f(n)$ for even $n$. The case where $n$ is odd proceeds mostly identically, replacing $\alpha$ with $\beta$ (and $f$ with $g$) where appropriate. So now we have shown that $(\gamma \otimes h)(n) = f(n)$ for even $n$ and $(\gamma \otimes h)(n) = g(n)$ for odd $n$, which is exactly the definition of $fzip(f(2n), g(2n + 1))$. Therefore $(\gamma \otimes h)(n) = fzip(f(2n), g(2n + 1))$ and thus $f(n)$ is equal to $(fzip(f(2n), g(2n + 1)))$. 

$\square$
4 A Diamond Structure in the Transducer Hierarchy

Now we proceed to our main result. We need one pair of lemmas before moving on to the main proof.

Lemma 4.1. For all polynomials $f$ of degree 2, we can find a weight $\alpha$ of length 2 and integers $k, m$ such that $\alpha \otimes S^k(n^2) = f(n + m)$.

Proof. Let $f(n) = an^2 + bn + c$. We assume for simplicity that $c = 0$, since the constant term is irrelevant for transducer degrees. We can also assume that $b > 0$, since if not we can simply shift $f$ (i.e. choose a positive $m$) until this holds.

Then we claim that the weight $\alpha = \frac{1}{4}(a-b+a[\frac{a}{b}], b-a[\frac{b}{a}], -(2b[\frac{b}{a}]+b-a[\frac{a}{b}])$.

Indeed, this is simply a matter of verifying this computationally. From the definition of a weight product, since $\alpha$ only contains one weight, we have $$(\alpha \otimes S^k(n^2))(n) = \frac{1}{4}((a-b+a[\frac{a}{b}])((2n+1)+\frac{a}{b})) + \frac{1}{4}((b-a[\frac{b}{a}])((2n+1)+\frac{a}{b})) - \frac{1}{4}(2b[\frac{b}{a}]+b-a[\frac{a}{b}])$$

Now we prove the second lemma, which gives us essentially the inverse statement of Lemma 3.1. The basic computations behind this lemma were modified from Theorem 5.2 in [2] to better suit the purposes of this paper.

Lemma 4.2. Let $f$ be a quadratic function of the form $a(n + 1)^2 + b(n + 1)$, with $2a > b > 0$. Then there is a weight $\alpha$ of length 2 such that $\alpha \otimes f = (n + 1)^2$.

Proof. We claim that the weight $\alpha = \frac{1}{4}(b, 2a - b, b^2 + ab + 6a^2 + 1)$ satisfies $(\alpha \otimes f)(n) = (n + 1)^2$. This can be verified computationally, by simplifying the
expression \( \frac{1}{\mu_2}(bf(2n) + (2a - b)f(2n + 1)) \), and noting that since \( 2a > b > 0 \), both \( b \) and \( 2a - b \) are positive, making \( \alpha \) a valid weight.

**Theorem 4.3.** The degree \( \langle f_{zip}(n, n^2) \rangle \) is strictly greater than \( \langle n \rangle \), and there are no intermediate degrees.

**Proof.** The fact that \( \langle f_{zip}(n, n^2) \rangle \) is strictly greater than \( \langle n \rangle \) is trivial, since if \( n \) could be transduced into \( f_{zip}(n, n^2) \), it could also be transduced into \( n^2 \), which has been shown to be impossible. Proving that there are no intermediate degrees is the nontrivial part.

We set out to prove this statement by assuming that there is some intermediate degree. Since \( f_{zip}(n, n^2) \) is a piecewise polynomial function, the degree of anything below it is equivalent to \( \langle g \rangle \) for some (piecewise) polynomial function \( g \), and in particular \( g \) is a weight product of \( f_{zip}(n, n^2) \). Because \( f_{zip}(n, n^2) \) is a piecewise polynomial function, any weight product will also be a piecewise polynomial function, and the pieces will all be linear or quadratic polynomials. (We can remove constant functions without any loss of degree). So then there are three cases:

- **Case 1:** \( g \) has only quadratic polynomials as its pieces.
- **Case 2:** \( g \) has only linear polynomials as its pieces.
- **Case 3:** \( g \) has both linear and quadratic polynomials as its pieces.

To be more precise about "pieces", we mean that since \( g \) can be written as a function which is defined piecewise by \( N \) different functions, we call these functions the "pieces" of \( g \).

Now we consider each case. For Case 1, \( g \) cannot be transduced into \( n \) if each piece is quadratic, since the block size grows too fast to allow this. For Case 2, where \( g \) is a piecewise linear function, suppose \( g \) is equal to:

\[
g(n) = \begin{cases} 
a_1n + b_1 & n \equiv 0 \mod N \\
 a_2n + b_2 & n \equiv 1 \mod N \\
  \vdots \\
 a_Nn + b_N & n \equiv N-1 \mod N
\end{cases}
\]

Then taking the product of the weight \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \), where \( \alpha_i = (a_i, b_i) \), with \( n \) shows that in this case \( \langle n \rangle \geq \langle g \rangle \), and since \( g \) is clearly transducible to \( n \) they must be of equal degree.

Now we have only Case 3, where \( g \) has both linear and quadratic pieces. We will show that \( g \) can be transduced back into \( f_{zip}(n, n^2) \), and this will complete the proof.

First, we can assume that the first piece of \( g \) is linear by simply deleting blocks from the beginning of \( \langle g \rangle \).
We can then combine all of the other pieces of \(g\) via a weight of the form
\[
\langle (1, 0), (1, 1, \ldots, 1, 0) \rangle,
\]
which preserves the first piece and sums up all of the other pieces. So now \(g\) has the form
\[
g_1(n) = \begin{cases} 
  an + b & \text{n even} \\
  An^2 + Bn + C & \text{n odd}
\end{cases}
\]

We can easily construct a transducer to subtract \(b\) from only the even-numbered blocks and \(C\) from the odd-numbered blocks, and also a transducer to divide even-numbered blocks by \(a\). Then \(g_2\) has the form
\[
g_2(n) = \begin{cases} 
  n & \text{n even} \\
  An^2 + Bn & \text{n odd}
\end{cases}
\]

We can shift \(g_2\) to the right by 1, and then subtract 1 from all even blocks to obtain:
\[
g_3(n) = \begin{cases} 
  n & \text{n even} \\
  (n + 1)^2 + B(n + 1) & \text{n odd}
\end{cases}
\]

By Lemma 3.2, \(A(n + 1)^2 + B(n + 1)\) can be transduced into \((n + 1)^2\) via a weight product with only one weight of length 2. Let’s call this weight \(\alpha = (\alpha_1, \alpha_2, \beta)\) (The lemma tells us exactly what these are, but it’s not important).

Then take the product of the weight \(\langle (1, 0), (\alpha_1, 0, \alpha_2, \beta) \rangle\) with \(g_3\):
\[
g_4(n) = \begin{cases} 
  n & \text{n even} \\
  (n + 1) & \text{n odd}
\end{cases}
\]

Now shift to the left by 2 and add 2 to the even blocks, then divide the odd blocks by 4 and the even blocks by 2.
\[
g_5(n) = \begin{cases} 
  \frac{n}{2} & \text{n even} \\
  (\frac{n-1}{2})^2 & \text{n odd}
\end{cases}
\]

which is exactly \(fzip(n, n^2)\). Since each \(g_i\) was formed by transducing \(g_{i-1}\), we have that \(\langle g \rangle \geq \langle fzip(n, n^2) \rangle\), and thus \(\langle g \rangle = \langle fzip(n, n^2) \rangle\).

Therefore there are no intermediate degrees between \(fzip(n, n^2)\) and \(n\).

Now we proceed to prove the same result for \(n^2\).

**Theorem 4.4.** The degree \(\langle fzip(n, n^2) \rangle\) is strictly greater than \(\langle n^2 \rangle\), and there are no intermediate degrees.

**Proof.** The proof will proceed in a similar manner to the previous theorem. Again the fact that \(\langle fzip(n, n^2) \rangle\) is strictly greater than \(\langle n^2 \rangle\) is trivial, since
otherwise $n^2$ could be transduced into $n$. So we only need to prove that there are no intermediate degrees.

Letting $g$ be a potential intermediate degree, we have the same three cases as before:

Case 1: $g$ has only quadratic polynomials as its pieces.

Case 2: $g$ has only linear polynomials as its pieces.

Case 3: $g$ has both linear and quadratic polynomials as its pieces.

Case 2 is not possible because we showed in the previous theorem that such a $g$ would be the same degree as $n$. For Case 3, the previous theorem proved that $\langle g \rangle$ would be equal to $\langle fzip(n, n^2) \rangle$. So we need to turn our attention to Case 1. For this case, we will prove that $g$ is the same degree as $n^2$, and this will complete the proof.

If $g$ has only quadratic polynomials as its pieces, it has the form:

$$g(n) = \begin{cases} 
  a_1n^2 + b_1n + c_1 & n \equiv 0 \mod N \\
  a_2n^2 + b_2n + c_2 & n \equiv 1 \mod N \\
  \vdots \\
  a_Nn^2 + b_Nn + c_N & n \equiv N-1 \mod N 
\end{cases}$$

We can remove all of the constant terms:

$$g(n) = \begin{cases} 
  a_1n^2 + b_1n & n \equiv 0 \mod N \\
  a_2n^2 + b_2n & n \equiv 1 \mod N \\
  \vdots \\
  a_Nn^2 + b_Nn & n \equiv N-1 \mod N 
\end{cases}$$

Now using the result of Lemma 3.1, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ where $\alpha_i = \frac{1}{4}(a_i - b_i + a_i|\frac{b_i}{a_i}|, b_i - a_i|\frac{b_i}{a_i}|, -(2b_i|\frac{b_i}{a_i}| + b_i - a_i|\frac{b_i}{a_i}|)^2)$. If we take the product of this weight with $n^2$, then we obtain $g$, and therefore $\langle n^2 \rangle \geq \langle g \rangle$. But $g$ was a degree between $fzip(n, n^2)$ and $n^2$, and therefore $\langle n^2 \rangle = \langle g \rangle$. Since this was the last case for $g$, there are no intermediate degrees between $fzip(n, n^2)$ and $n^2$.

5 Conclusion

We have shown that $fzip(n, n^2)$ lies strictly above both $n$ and $n^2$, with no intermediate degrees between them. From earlier results, we know that the degrees of both $n$ and $n^2$ are atoms, that is, there is nothing between them and the bottom degree 0. Therefore $fzip(n, n^2)$ forms a diamond structure with $n, n^2$ and 0, and this is the first such structure that has been found. This result sheds more light on the structure of the transducer hierarchy, and also raises
some further questions about the potential use of \textit{fzip} to find new results. We state a few of these questions here:

1. We know that \( \langle fzip(f, f) \rangle \geq \langle f \rangle \). In general, is this inequality strict, or can \( \langle fzip(f, f) \rangle = \langle f \rangle \) for more than just linear or quadratic \( f \)'s?

2. What can we say about \( \langle fzip(f, g) \rangle \) when \( f, g \) are both cubic polynomials?

3. For some \( f, g \) is it possible to find degrees between \( f \) and \( fzip(f, g) \)?

4. If the degrees of \( f_1, g_1 \) are above the degrees of \( f_2, g_2 \) respectively, then is \( \langle fzip(f_1, g_1) \rangle \geq \langle fzip(f_2, g_2) \rangle \)?

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\[ \text{□} \]