On the Dirichlet problem for the CMC graph equation on multiply connected domains of a Riemannian manifold

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Abstract

We establish existence and uniqueness of compact graphs of constant mean curvature in $M \times \mathbb{R}$ over bounded multiply connected domains of $M \times \{0\}$ with boundary lying in two parallel horizontal slices of $M \times \mathbb{R}$.

1 Introduction

Let $M$ be a complete $n$-dimensional Riemannian manifold, $n \geq 2$. Let $\Lambda$ and $\Lambda_i$, $i = 1, \ldots, m$, be bounded, simply connected domains of class $C^{2,\alpha}$ of $M$ such that $\overline{\Lambda} \subset \Lambda$ and $\overline{\Lambda}_i \cap \overline{\Lambda}_j = \emptyset$ if $i \neq j$. Consider the multiply connected domain $\Omega = \Lambda \setminus (\bigcup_{i=1}^m \Lambda_i)$ and set $\Gamma := \partial \Lambda$, $\Gamma_i := \partial \Lambda_i$. With this notation

$$\partial \Omega = \Gamma \cup (\bigcup_{i=1}^m \Gamma_i).$$

Let $h, H \geq 0$ be given. In this paper we investigate the Dirichlet problem

$$\begin{cases}
Q_H(u) := \text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} + nH = 0 \text{ in } \Omega, \ u \in C^{2,\alpha}(\overline{\Omega}) \\
u|_{\Gamma} = 0, \ u|_{\Gamma_i} = h, \ i = 1, \ldots, m
\end{cases} \tag{1}$$

where div and $\nabla$ are the divergence and the gradient in $M$. If $u$ is a solution of (1) then the graph of $u$ is a compact constant mean curvature $H$ hypersurface of $M \times \mathbb{R}$ oriented by a unit normal vector $N$ such that $\langle N, d/dt \rangle \leq 0$ and whose boundary lies in the slices $M \times \{0\} \cup M \times \{h\}$.

The Dirichlet problem (1) was studied in the work [7] for $M = \mathbb{R}^2$, in [11] when $M = \mathbb{H}^2$ and $\partial \Omega$ has only two connected components and in [4]
when \( M = \mathbb{H}^n \) and \( H = 0 \). In these works existence results are obtained when the height \( h \) is less than or equal to a constant which depends on the mean curvature of \( \partial \Omega \), the distance between the connected component of \( \partial \Omega \), the dimension \( n \), and on the diameter of \( \Omega \) and \( H \) if \( H > 0 \). In \cite{7} some nonexistence results also are established. In the case \( H = 0 \) we observe that Theorem 1 of \cite{3}, an extension of the classical Jenkins-Serrin result - Theorem 2 of \cite{10} -, gives us an existence result with the upper bound of \( h \) depending on \( n \), on the mean curvature and on the injectivity radius of \( \partial \Omega \).

The main motivation to study the problem (1) is that, for \( H > 0 \), we did not find in the literature, even for \( M = \mathbb{R}^n \), a result where the hypersurface \( \Gamma \) is not assumed to be mean convex. We explore this situation when \( M \) is a Hadamard manifold (Theorem 1.2). We observe that in \cite{6} the authors conclude, for \( M = \mathbb{R}^n \), the existence and uniqueness of \( H \)-graphs for a large class of prescribed boundary data over \( \Gamma \) where \( \Gamma \) is not necessarily mean convex but, however, \( \Gamma = \partial \Omega \) with \( \Omega \) simply connected.

Relatively to the minimal case, we obtain Theorem 1.1 whose estimate on \( h \) is more in line with that in Theorem 2.1 of \cite{7} than that in Theorem 1 of \cite{3}. Despite being difficult to say if our result improves or not the estimate of \( h \) given in \cite{3} in general, for some domains \( \Omega \) we got some gain (see Remark 3.1).

An extra motivation to our work is the problem proposed by A. Ros and H. Rosenberg in Remark 4 of \cite{13}: Given two convex Jordan curves in distinct parallel planes of \( \mathbb{R}^3 \), is there a CMC annulus having such curves as boundary? Besides of the aforementioned works, this situation was also investigated in \cite{8}, \cite{2} and \cite{11} (for some characterization results, see \cite{15} and \cite{12}). The results obtained so far do not give a complete answer to this problem. Our results give some contribution in the \( M \times \mathbb{R} \) context.

We fix some notations: the mean curvature of \( \partial \Omega \) with respect to the unit normal vector field \( \eta \) to \( \partial \Omega \) pointing to \( \Omega \) will be denoted by \( H^{\partial \Omega} \) and

\[
H^\partial_{\Omega} := \inf_{\partial \Omega} H^{\partial \Omega}.
\]

Let \( d \) be the Riemannian distance in \( M \). Denote by \( R_\Gamma, R_{\Gamma_i} \), the biggest positive numbers such that the exponential maps

\[
\exp_\Gamma : \Gamma \times [0, R_\Gamma) \longrightarrow U_\Gamma := \{ z \in \Omega; \ d(z, \Gamma) < R_\Gamma \} \subset \Omega \quad (2)
\]

and

\[
\exp_{\Gamma_i} : \Gamma_i \times [0, R_{\Gamma_i}) \longrightarrow U_{\Gamma_i} := \{ z \in \Lambda \setminus \Lambda_i; \ d(z, \Gamma_i) < R_{\Gamma_i} \} \subset \Lambda \setminus \Lambda_i \quad (3)
\]
are diffeomorphisms (here, \( \text{exp} (p, s) := \text{exp}_p (s \eta (p)) \)) and set
\[
R = \min \{ R_\Gamma, R_\Gamma^1, R_\Gamma^2, \ldots, R_\Gamma^m \}.
\]  \hspace{1cm} (4)

Figure 1: Examples of \( U_\Gamma \) and \( U_{\Gamma_j} \) domains.

Denote by \( \delta (\approx 1.8102) \) the solution of the equation
\[
x = \cosh \left( \frac{x}{\sqrt{x^2 - 1}} \right), \quad x > 1.
\]

We prove:

**Teorema 1.1** Let \( M \) be a complete \( n \)-dimensional Riemannian manifold, \( n \geq 2 \). If
\[
h < \frac{1}{\delta (n - 1) |H_{\inf}^{\partial \Omega}|} \cosh^{-1} \left( 1 + \tau \delta (n - 1) |H_{\inf}^{\partial \Omega}| \right)
\]  \hspace{1cm} (5)

where
\[
\tau = \min \left\{ R, \frac{\delta - 1}{\delta (n - 1) |H_{\inf}^{\partial \Omega}|} \right\},
\]
then the Dirichlet problem (1) has a unique solution for \( H = 0 \).
Teorema 1.2 Assume that $M$ is a Hadamard manifold (complete, simply connected Riemannian manifold with nonpositive sectional curvature) and that $\Omega$ is contained in a geodesic ball of radius $\mathcal{R}$ of $M$. Then there is a positive constant $C = C (n, H^{\partial \Omega}, \mathcal{R})$, explicitly given in (31) such that, given $H \in [0, C)$ we have

$$h_H := \frac{\cosh^{-1}(1 + \delta [(n - 1) |H^{\partial \Omega}_{\inf}| + nH (1 + \delta)] \sigma)}{\delta [(n - 1) |H^{\partial \Omega}_{\inf}| + nH (1 + \delta)]} = \frac{H \mathcal{R}^2}{1 + \sqrt{1 - H^2 \mathcal{R}^2}} > 0,$$

where

$$\sigma = \min \left\{ R, \frac{\mathcal{R} (\delta - 1)}{\delta [\mathcal{R} (n - 1) |H^{\partial \Omega}_{\inf}| + n (\delta + 1)]} \right\},$$

and the Dirichlet Problem (11) has a unique solution if $h \leq h_H$.

2 Barriers for the Dirichlet problem (1)

We shall use the continuity method from Elliptic PDE theory in the proof of the main results. Then, we need to construct local barriers relatively to the Dirichlet problem (11) (see [9], p. 333).

We work with the construction of the local barriers relatively to the points in $\Gamma$ and in $\Gamma_i$ at same time, using $U$ to means both $U_\Gamma$ and $U_{\Gamma_i}$ and having in mind that, for $z \in U$, $d(z) = d(z, \Gamma)$ if $U = U_\Gamma$ and $d(z) = d(z, \Gamma_i)$ if $U = U_{\Gamma_i}$.

We consider, at $z \in U$, an orthonormal referential frame $\{E_j\}$ of $T_z M$, $j = 1, \ldots, n$, where

$$E_n := \nabla d. \tag{6}$$

Lemma 2.1 Given $\psi \in C^2 ([0, \infty))$, set $s = d(z)$, $z \in \overline{U}$, and consider $w \in C^2 (\overline{U})$ given by

$$w(z) = \psi(s) \tag{7}$$

Suppose $H \geq 0$. Then $Q_H (w) \leq 0$ in $U$ if

$$\psi'' + \left( \psi' + [\psi']^3 \right) \Delta d + nH \left( 1 + [\psi']^2 \right)^{3/2} \leq 0, \tag{8}$$

where $\Delta$ is the Laplacian in $M$. 4
Proof. We have $Q_H (w) = Q_0 (w) + nH$. After some calculus we see that $Q_0 (w)$ is

$$(1 + |\nabla w|^2)^{-1/2} \sum_{i=1}^{n} \langle \nabla_{E_i}^w, E_i \rangle - \frac{1}{2} (1 + |\nabla w|^2)^{-3/2} \sum_{i=1}^{n} E_i (|\nabla w|^2) E_i (w).$$

(9)

Notice that $\nabla w = \psi E_n$ and so

$$E_i (|\nabla w|^2) = \begin{cases} 2\psi' \psi'' & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases},$$

(10)

$$E_i (w) = \begin{cases} \psi' & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

(11)

and

$$\nabla_{E_i}^w = \begin{cases} \psi'' E_n + \psi' \nabla_{E_i} E_n & \text{if } i = n \\ \psi' \nabla_{E_i} E_n & \text{if } i \neq n \end{cases}.$$  

(12)

It follows from (12) that

$$\sum_{i=1}^{n} \langle \nabla_{E_i}^w, E_i \rangle = \psi' \text{ div } (E_n) + \psi'' = \psi' \Delta d + \psi''$$

(13)

and from (10) and (11) that

$$\sum_{i=1}^{n} E_i (|\nabla w|^2) E_i (w) = 2 [\psi']^2 \psi''.$$  

(14)

Plugging (13) and (14) in (9), as $|\nabla w|^2 = [\psi']^2$, we obtain

$$Q_0 (w) = \left(1 + [\psi']^2\right)^{-3/2} \left[\psi'' + \left(\psi' + [\psi']^3\right) \Delta d\right]$$

and, therefore, $Q_H (w) \leq 0$ if

$$\psi'' + \left(\psi' + [\psi']^3\right) \Delta d + nH \left(1 + [\psi']^2\right)^{3/2} \leq 0.$$  

Lemma 2.2 If $H^{0\Omega} \geq -c$, $c > 0$, then $\Delta d \leq (n - 1) c$ in $U$. 


Proof. Since $\partial U$ is compact, there is $0 < k < c$ such that

$$\text{Ric}_M (\eta, \eta) \geq - (n - 1) k^2,$$

where $\eta$ is the normal unit vector to $\partial \Omega$ pointing to $\Omega$. Consider $f : [0, R^*] \to (0, +\infty)$ in $C^2 ([0, R^*])$ defined by

$$f (t) = k \sinh \left[ \text{arccoth} \left( \frac{c}{k} \right) + kt \right],$$

where $R^*$ is $R_\Gamma$ or $R_{\Gamma_i}$, according $U$ is $U_\Gamma$ or $U_{\Gamma_i}$. We have

$$\frac{f'' (t)}{f (t)} = k^2, t \in [0, R^*].$$

(16)

Setting $H_t$ the mean curvature of $\partial U_t$, where $U_t := \{ z \in \overline{U}; d(z) \leq t \}$. Since $H_{\partial \Omega} \geq -c$ and $U_0 \subset \partial \Omega$, we have

$$H_0 \geq H_{\text{inf}}^{\partial \Omega} \geq -c \geq - \frac{f' (0)}{f (0)}.$$

(17)

Let $\gamma : [0, R^*] \to U$ be the arc-length parametrized geodesic such that $\gamma (0) \in U_0$ and $\gamma' (t) = \nabla d (\gamma (t))$. We have from (15) and (16) that

$$\text{Ric}_M (\gamma' (t), \gamma' (t)) \geq - (n - 1) \frac{f'' (t)}{f (t)},$$

(18)

for all $t \in [0, R^*]$. Since $\nabla d$ is a extension of $\eta$ to $U$ and (17), (18) occurs, it follows from Theorem 5.1 of [11] that

$$- H_t (\gamma (t)) = \frac{\Delta d (\gamma (t))}{(n - 1)} \leq \frac{f' (t)}{f (t)}.$$

Then

$$\Delta d \leq (n - 1) \frac{f' (t)}{f (t)} = (n - 1) k \coth \left[ \text{arccoth} \left( \frac{c}{k} \right) + kt \right] \leq (n - 1) c.$$  (19)

Proposition 2.3 Suppose $H_{\partial \Omega} \geq -c$, $c > 0$. Given $\lambda > \alpha > 1$ and $H \geq 0$

set $\mu := (n - 1) c$ and define $\psi_{\alpha, \lambda} (s) \equiv \psi (s)$ by

$$\psi (s) = \frac{1}{\lambda [\mu + nH (1 + \lambda)]} \left[ \cosh^{-1} (\alpha + \lambda [\mu + nH (1 + \lambda)] s) - \cosh^{-1} (\alpha) \right],$$

(20)
\( s = d(z), \ z \in U. \) Then \( w \) as in (7) satisfies \( Q_H(w) \leq 0 \) in \( U_\varepsilon := \{z \in U; d(z) < \varepsilon\} \) where
\[
\varepsilon = \min \left\{ R, \frac{1}{\mu + nH (1 + \lambda)} \left( \frac{\lambda - \alpha}{\lambda} \right) \right\}. \tag{21}
\]

**Proof.** From Lemma 2.1 and Lemma 2.2 we have \( Q_H(w) \leq 0 \) in \( U \) if
\[
\psi'' + [(n - 1) c + nH] (\psi')^3 + nH (\psi')^2 + [(n - 1) c + nH] \psi' + nH \leq 0 \tag{22}
\]
where \( w(z) = \psi(s), \ s = d(z) \). Notice that (22) can be rewritten as
\[
\psi'' + B \left( \psi' + [\psi']^3 \right) + nH \left( 1 + [\psi']^2 \right) \leq 0, \tag{23}
\]
where \( B = (n - 1) c + nH = \mu + nH, \) with \( \mu = (n - 1) c. \)

We choose
\[
\psi(s) = a \cosh^{-1}(\alpha + bs) - a \cosh^{-1}(\alpha), \ \alpha > 1
\]
where \( a, b \) are positive constants to be determined and \( \alpha > 1 \).

Setting \( u(s) = \alpha + bs \), since \( u'(s) = b \) it follows that
\[
\psi' = \frac{ab}{(u^2 - 1)^{1/2}},
\]
and
\[
\psi'' = -\psi' \left( \frac{bu}{u^2 - 1} \right).
\]

Then, from (23) we see that \( Q_H(w) \leq 0 \) if
\[
[-bu + B \left( u^2 - 1 + a^2b^2 \right)] \psi' + nH \left( u^2 - 1 + a^2b^2 \right) \leq 0
\]
that is, if
\[
[-bu + B \left( u^2 - 1 + a^2b^2 \right)] ab + nH \left( u^2 - 1 + a^2b^2 \right) (u^2 - 1)^{1/2} \leq 0.
\]
We assume \( a, b \) such that \( ab = 1 \). Then the last inequality is true if
\[
-b + Bu + nHu (u^2 - 1)^{1/2} \leq 0
\]
which is true if
\[-b + Bu + nHu^2 \leq 0.\]

As \( u(s) = \alpha + bs \), the last inequality is
\[nHb^2 s^2 + b [2nH\alpha + B] s + B\alpha + nH\alpha^2 - b \leq 0. \tag{24}\]

Let \( \lambda > \alpha \). We assume
\[b := \lambda [\mu + nH (1 + \lambda)] = B\lambda + nH\lambda^2 > B\alpha + nH\alpha^2.\]

We see that (24) is true for \( s \in [0,\varepsilon] \), where \( \varepsilon \) is given by (21). This concludes the proof of the proposition. 

**Lemma 2.4** Under the hypothesis of Proposition 2.3,
\[\psi_{\alpha,\lambda}(\varepsilon) < \frac{\cosh^{-1}(\delta)}{\delta (\mu + 2nH)} \]
where \( \delta (\approx 1.8102) \) is the solution of the equation \[x = \cosh \left( x \left( x^2 - 1 \right)^{-1/2} \right), \] \( x > 1 \). In particular,
\[\lim_{\alpha \to 1, \lambda \to \delta} \psi_{\alpha,\lambda}(\varepsilon) = \frac{\cosh^{-1} \left[ 1 + \delta (\mu + nH (1 + \delta)) \rho \right]}{\delta (\mu + nH (1 + \delta))} \tag{25}\]
where
\[\rho = \min \left\{ R, \frac{1}{\mu + nH (1 + \delta)} \left( \frac{\delta - 1}{\delta} \right) \right\}\]

**Proof.** Notice that, since \( 1 < \alpha < \lambda \),
\[\psi_{\alpha,\lambda}(\varepsilon) = \frac{\cosh^{-1} (\alpha + \lambda (\mu + nH (1 + \lambda)) \varepsilon) - \cosh^{-1} (\alpha)}{\lambda (\mu + nH (1 + \lambda))} \]
\[< \frac{\cosh^{-1} (\alpha + \lambda (\mu + nH (1 + \lambda)) \varepsilon)}{\lambda (\mu + nH (1 + \lambda))} \]
\[\leq \frac{\cosh^{-1} (\lambda)}{\lambda (\mu + nH (1 + \lambda))} \leq \frac{\cosh^{-1} (\lambda)}{\lambda (\mu + 2nH)}, \lambda > 1.\]

Set
\[f(\lambda) = \frac{\cosh^{-1} (\lambda)}{\lambda (\mu + 2nH)}, \lambda > 1.\]
We have $f'(\lambda) = 0$ iff

$$\lambda = \cosh\left(\frac{\lambda}{\sqrt{\lambda^2 - 1}}\right), \lambda > 1.$$  \hspace{1cm} (26)

The equation (26) has a unique solution $\delta \approx 1.8102$, and $\delta$ is the maximum (global) point for $f$. Then

$$\psi_{\alpha,\lambda}(\varepsilon) < \frac{1}{\delta (\mu + 2nH)} \cosh^{-1}(\delta).$$

The equality (25) follows immediately from definition of $\psi$ and $\varepsilon$ (observing that both depend on $\alpha$ and $\lambda$).

**Lemma 2.5** Let $M$ be a Hadamard manifold. Let $G \subset M \times \mathbb{R}$ be a compact graph of constant mean curvature $H > 0$ over a domain $\Omega \subset M$ and such that $\partial G = \partial \Omega$ and let $h$ be the height of $G$. If $\Omega$ is contained in a normal ball in $M$ of radius $\Re \leq 1/H$ then

$$h \leq \frac{H\Re^2}{1 + \sqrt{1 - (H\Re)^2}}.$$  \hspace{1cm} (27)

**Proof.** Let $p$ the center of the normal ball and set $d_p(z) = d(z, p)$, $z \in \Omega$. Consider at $z \in \Omega$ an orthonormal referential frame $\{E_j\}$ of $T_zM$, $j = 1, ..., n$, where $E_n = \text{grad } d_p$.

Let $v := g \circ d_p : \Omega \rightarrow \mathbb{R}$, where

$$g(s) = -\sqrt{\frac{1}{H^2} - \Re^2 + \sqrt{\frac{1}{H^2} - s^2}}, s = d_p(z).$$

We have $Q_H(v)$ given by

$$Q_H(v) = (1 + (g')^2)^{-\frac{3}{2}} \left[ g'' + (g' + (g')^3) \Delta d_p + nH (1 + (g')^2)^{\frac{3}{2}} \right].$$

As the sectional curvature of $M$ is $K_M \leq 0$, by the Laplacian Comparison Theorem we have $\Delta d_p \geq \Delta d_E$, where $d_E$ is the Euclidean distance. Then

$$Q_H(v) \leq (1 + (g')^2)^{-\frac{3}{2}} \left[ g'' + (g' + (g')^3) \Delta d_E + nH (1 + (g')^2)^{\frac{3}{2}} \right] = 0,$$
where the equality is due that, for $M = \mathbb{R}^n$, the graph of $g(s)$ is a spherical cap.

The result follow now of the fact that the maximum height of the graph of $v$ is

$$\frac{H\mathcal{R}^2}{1 + \sqrt{1 - (H\mathcal{R})^2}}.$$

3 Proof of Theorems

Proof of Theorem 1.1

Proof. Since 

$$h < \frac{1}{\delta (n - 1) |H_{\text{inf}}^{\partial \Omega}|} \cosh^{-1} \left( 1 + \tau \delta (n - 1) |H_{\text{inf}}^{\partial \Omega}| \right)$$

where 

$$\tau = \min \left\{ R, \frac{\delta - 1}{\delta (n - 1) |H_{\text{inf}}^{\partial \Omega}|} \right\},$$

from Lemma 2.4 there is $1 < \alpha < \delta$, $\alpha$ close enough to 1, such that, setting 

$$\psi(s) = \frac{1}{\delta (n - 1) |H_{\text{inf}}^{\partial \Omega}|} \left[ \cosh^{-1} \left( \alpha + \delta (n - 1) |H_{\text{inf}}^{\partial \Omega}| s \right) - \cosh^{-1}(\alpha) \right], \quad (28)$$

$s = d(z)$, the function $w(z) = c + \psi(s)$, $c$ constant, satisfies $Q_0(w) \leq 0$ in $U_\epsilon = \{ z \in U; d(z) < \epsilon \}$ where 

$$\epsilon = \min \left\{ R, \frac{1}{(n - 1) |H_{\text{inf}}^{\partial \Omega}|} \left( \frac{\delta - \alpha}{\delta} \right) \right\}.$$

Moreover, 

$$h \leq \psi(\epsilon) < \frac{1}{\delta (n - 1) |H_{\text{inf}}^{\partial \Omega}|} \cosh^{-1} \left( 1 + \tau \delta (n - 1) |H_{\text{inf}}^{\partial \Omega}| \right).$$

Notice that 

$$U_\epsilon = \left\{ 
\begin{array}{ll}
U_\Gamma^\epsilon := \{ z \in U_\Gamma; d(z, \Gamma) < \epsilon \} & \text{if } U = U_\Gamma \\
U_{\Gamma_i}^\epsilon := \{ z \in U_{\Gamma_i}; d(z, \Gamma_i) < \epsilon \} & \text{if } U = U_{\Gamma_i}
\end{array} \right.,$$

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and then we can consider the supersolutions \( w_\Gamma \in C^2 (\bar{U}_\Gamma) \) and \( w_{\Gamma, t} \in C^2 (\bar{U}_\Gamma) \), \( t \in [0, 1] \), relatively to the operator \( Q_0 \), given by \( w_\Gamma (z) = (\psi \circ d) (z) \) and \( w_{\Gamma, t} (z) = th + (\psi \circ d) (z) \) respectively, where \( \psi \circ d \) is given by (28). It follows that \( w_\Gamma = 0 \) in \( \Gamma = \partial U_\epsilon \cap \partial \Omega \) and \( h \leq w_\Gamma (\varepsilon) \) in \( \partial U_\epsilon \setminus \Gamma \). Moreover, \( w_{\Gamma, t} \geq th \) in \( \bar{U}_\Gamma \) and, then, in \( \bar{U}_{\Gamma_i} \cap \Omega \). Let \( \varphi \in C^\infty (\partial \Omega) \) given by

\[
\varphi (p) = \begin{cases} 
0 \text{ if } p \in \Gamma \\
\, h \text{ if } p \in \Gamma_i, \; i = 1, ..., m \end{cases}.
\]  

(29)

From the Maximum Principle, it follows that \( w_\Gamma \) and \( w_{\Gamma, t} \) are upper local barriers relatively to the boundary data \( t \varphi \) and, for lower local barriers relatively to \( t \varphi \), just take \( w^-_{\Gamma} = 0 \) and \( w^-_{\Gamma, t} = th - \psi \circ d \) in the domains \( \bar{U}_\Gamma \) and \( \bar{U}_{\Gamma_i} \), respectively.

Now, set

\[ V = \{ t \in [0, 1]; \exists u_t \in C^{2,\alpha} (\bar{\Omega}) \text{ satisfying } Q_0 (u_t) = 0, \; u_t|_{\partial \Omega} = t \varphi \}. \]

We have \( V \neq \emptyset \) since \( t = 0 \in V \). Moreover, since \( Q_0 \) is a uniformly elliptic operator on \( C^{2,\alpha} (\bar{\Omega}) \) we can apply the implicit function theorem in Banach spaces to conclude that \( V \) is an open. Now, we apply a standard sequence of arguments to conclude that \( V \) is closed. Let \( (t_n) \subset V \) a sequence with \( t_n \to t \in [0, 1] \). For each \( n \), let \( u_{t_n} \in C^{2,\alpha} (\bar{\Omega}) \) satisfying \( Q_0 (u_{t_n}) = 0, \; u_{t_n}|_{\partial \Omega} = t_n \varphi \). From the barriers above it follows that the sequence \( (u_{t_n}) \) has uniformly bounded \( C^0 \) norm. Moreover

\[
\max_{\partial \Omega} |\nabla u_{t_n}| \leq \max_{\partial \Omega} |\nabla w_\Gamma| < \infty.
\]

It follows of Section 5 of [5] that there is \( K > 0 \) such that \( \max_{\Omega} |\nabla u_{t_n}| \leq K \) and, consequently, \( |u_{t_n}| \leq K \) \( \leq \infty \) with the constant \( K \) independent of \( n \). Hölder estimates and PDE linear elliptic theory - see [9] - give us that \( (u_{t_n}) \) is equicontinuous in the \( C^{2,\beta} \) norm for some \( \beta > 0 \). It follows that \( (u_{t_n}) \) contains a subsequence converging uniformly on the \( C^2 \) norm to a solution \( u \in C^2 (\bar{\Omega}) \). Regularity theory of linear elliptic PDE (9) implies that \( u \in C^{2,\alpha} (\bar{\Omega}) \). Therefore, \( V \) is closed, that is, \( V = [0, 1] \) and this gives us the existence result. The uniqueness of the solution is a consequence of the Maximum Principle for the difference of two solutions. ■

**Remark 3.1 :** Denote by \( r \) the biggest positive number such that the normal exponential map

\[
\exp_{\partial \Omega} : \partial \Omega \times [0, r) \longrightarrow U_{\partial \Omega} := \{ z \in \Omega; d(z, \partial \Omega) < r \} \subset \bar{\Omega},
\]  

(30)
is a diffeomorphism. We call $r$ the "injectivity radius of $\partial \Omega$". Notice that $r \leq R$ and we can have $r < R$. The estimative of $h$ in Theorem 1 of [3], relatively the Dirichlet problem (1), is $h \leq B - \frac{1}{\ln (1 + B \varepsilon)}$, where $B = 6 (1 + r') (n - 1) \left| H_{\inf}' \right|$ and $\varepsilon = \min \{ r', 1/(2B) \}$, for some $0 < r' \leq r$. Then, depending of the domain $\Omega$, we have some improvement on the estimate of $h$ in Theorem 1.1 when we compare with that in Theorem 1 of [3]. For example, if $\Omega$ is such that $1/(2B) < r'$ and

$$\frac{\delta - 1}{\delta (n - 1) |H_{\inf}'|} < R.$$  

Moreover, using $R$ instead of $r$ we are more in line with Theorem 2.1 of [7] and Theorem 1.1 of [4].

We pass now to the proof of Theorem 1.2.

**Proof.** Set $\mu = (n - 1) |H_{\inf}'|$, and

$$C = \frac{2\delta \mu \cosh^{-1}(1 + \delta \sigma)}{\left[ \cosh^{-1}(1 + \delta \left[ \mu + \frac{\delta}{\mathcal{R}} (1 + \delta) \right]) \right]^2 + \mu \delta^2 \mathcal{R} [\mu \mathcal{R} + 2n(1 + \delta)] + \delta n(1 + \delta) \left[ \delta n(1 + \delta) - 2 \cosh^{-1}(1 + \delta \sigma) \right]},$$

where

$$\sigma = \min \left\{ R, \frac{\delta - 1}{\delta \left[ \mu + \frac{\delta}{\mathcal{R}} (\delta + 1) \right]} \right\}.$$  

We first notice that

$$0 < C < \frac{1}{\mathcal{R}}.$$  

In fact, $C > 0$ since

$$n \delta (1 + \delta) - 2 \cosh^{-1}(1 + \delta \mu \sigma) \geq n \delta (1 + \delta) - 2 \cosh^{-1} \delta \geq 5.087 - 2.3994 > 0.$$  

On the other hand, $C < 1/\mathcal{R}$ since we have (32) and

$$\frac{2\delta \mu \cosh^{-1}(1 + \delta \mu \sigma)}{\mu \delta^2 \mathcal{R} [\mu \mathcal{R} + 2n(1 + \delta)]} \leq \frac{2\delta \mu \cosh^{-1} \delta}{\mu \delta^2 \mathcal{R} [2n(1 + \delta)]} = \left( \frac{1}{\mathcal{R}} \right) \cosh^{-1} \delta \frac{1}{\delta n(1 + \delta)} < \frac{1}{\mathcal{R}}.$$  

Let $0 \leq H \leq C < 1/\mathcal{R}$. We claim that

$$\frac{\cosh^{-1}(1 + \delta [\mu + nH (1 + \delta)] \rho)}{\delta [\mu + nH (1 + \delta)]} - \frac{H \mathcal{R}^2}{1 + \sqrt{1 - H^2 \mathcal{R}^2}} > 0,$$  

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where
\[
\rho = \min \left\{ R, \frac{\delta - 1}{\delta [\mu + nH (\delta + 1)]} \right\}.
\]
Indeed, notice that, if \( H = 0 \) then (33) is true. Suppose \( H > 0 \). After some calculations, we have (33) iff
\[
\frac{2\delta \cosh^{-1}(1 + \delta [\mu + nH (1 + \delta)] \rho)}{H} - 2\delta^2 \mu n \Re (1 + \delta) - \delta^2 n^2 H^2 \Re^2 (1 + \delta)^2
\geq \left[ \cosh^{-1}(1 + \delta [\mu + nH (1 + \delta)] \rho) \right]^2 + 2\Re^2 \delta^2 \mu^2 - 2\delta n (1 + \delta) \cosh^{-1}(1 + \delta [\mu + nH (1 + \delta)] \rho).
\] (34)
Notice that, since \( 0 < H < 1/\Re \),
\[
cosh^{-1}(1 + \delta \mu \rho) < \cosh^{-1}(1 + \delta [\mu + nH (1 + \delta)] \rho) \leq \cosh^{-1}(1 + \delta \left[ \mu + \frac{n}{\Re} (1 + \delta) \right] \rho)
\] (35)
and then, from (35), we have that if
\[
\frac{2\delta \cosh^{-1}(1 + \delta \mu \rho)}{H} - 2\delta^2 \mu n \Re (1 + \delta) - \delta^2 n^2 (1 + \delta)^2
> \left[ \cosh^{-1}(1 + \delta \left[ \mu + \frac{n}{\Re} (1 + \delta) \right] \rho) \right]^2 + \Re^2 \delta^2 \mu^2 - 2\delta n (1 + \delta) \cosh^{-1}(1 + \delta \mu \rho)
\] (36)
then (34) is true. Notice that since \( 0 < H < 1/\Re \) we have \( \sigma \leq \rho \) and, then, (36) occurs if
\[
\frac{2\delta \cosh^{-1}(1 + \delta \mu \sigma)}{H} - 2\delta^2 \mu n \Re (1 + \delta) - \delta^2 n^2 (1 + \delta)^2
> \left[ \cosh^{-1}(1 + \delta \left[ \mu + \frac{n}{\Re} (1 + \delta) \right] \sigma) \right]^2 + \Re^2 \delta^2 \mu^2 - 2\delta n (1 + \delta) \cosh^{-1}(1 + \delta \mu \sigma).
\] (37)
Notice that (37) is equivalent to \( H < C \) and this concludes the proof of the claim.

As \( \sigma \leq \rho \), it follows that
\[
0 < h_H \leq \frac{\cosh^{-1}(1 + \delta [\mu + nH (1 + \delta)] \rho)}{2\delta [\mu + nH (1 + \delta)]} - \frac{H \Re^2}{1 + \sqrt{1 - H^2 \Re^2}},
\] (38)
and, as \( H \leq h_H \), from Lemma 2.4 there is \( 1 < \alpha < \delta \), \( \alpha \) close enough to 1, such that, setting
\[
\psi(s) = \frac{\cosh^{-1}(\alpha + \delta [\mu + nH (1 + \delta)] s) - \cosh^{-1}(\alpha)}{\delta [\mu + nH (1 + \delta)]},
\] (39)
s = d(z), the function \( w(z) = c + \psi(s), \) \( c \) constant, satisfies \( Q_H(w) \leq 0 \) in \( U_\varepsilon = \{ z \in U; d(z) < \varepsilon \} \), where

\[
\varepsilon = \min \left\{ R, \frac{1}{\mu + nH(1 + \lambda)} \left( \frac{\delta - \alpha}{\delta} \right) \right\}
\]

and, moreover, from (38),

\[
\psi(\varepsilon) \geq h + \frac{H\Re^2}{1 + \sqrt{1 - H^2\Re^2}}.
\]

As

\[
U_\varepsilon = \begin{cases} U_\Gamma^\varepsilon := \{ z \in U_\Gamma; d(z, \Gamma) < \varepsilon \} & \text{if } U = U_\Gamma \\ U_{\Gamma_i}^\varepsilon := \{ z \in U_{\Gamma_i}; d(z, \Gamma_i) < \varepsilon \} & \text{if } U = U_{\Gamma_i} \end{cases}
\]

we can consider the supersolutions \( w_\Gamma \in C^2(U_\Gamma^\varepsilon) \) and \( w_{\Gamma_i,t} \in C^2(U_{\Gamma_i}^\varepsilon) \), \( t \in [0, 1] \), relatively to the operator \( Q_{ttH} \), given by \( w_\Gamma(z) = (\psi \circ d)(z) \) and \( w_{\Gamma_i,t}(z) = th + (\psi \circ d)(z) \) respectively, where \( \psi \circ d \) is given by (39). It follows that \( w_\Gamma = 0 \) in \( \Gamma = \partial U_\Gamma^\varepsilon \cap \partial \Omega \) and

\[
w_\Gamma(\varepsilon) \geq h + \frac{H\Re^2}{1 + \sqrt{1 - H^2\Re^2}}
\]
in \( \partial U_\Gamma^\varepsilon \setminus \Gamma \). Moreover, \( w_{\Gamma_i,t} = th \) in \( \Gamma_i = \partial U_{\Gamma_i}^\varepsilon \cap \partial \Omega \) and

\[
w_{\Gamma_i,t}(\varepsilon) \geq h(1 + t) + \frac{H\Re^2}{1 + \sqrt{1 - H^2\Re^2}}
\]
in \( (\partial U_{\Gamma_i}^\varepsilon \cap \Omega) \setminus \Gamma_i \). Let \( \varphi \) as in (29). From Lemma 2.5 and from the Maximum Principle, it follows that \( w_\Gamma \) and \( w_{\Gamma_i,t} \) are upper local barriers relatively to the boundary data \( t\varphi \in C^\infty(\partial \Omega) \). From the Theorem 1.1 since \( \sigma \leq \tau \), there is \( u \in C^{2, \alpha}(\Omega) \) satisfying \( Q_0(u) = 0, u|_\Gamma = 0, u|_{\Gamma_i} = h, i = 1, ..., m \). As vertical translation in \( M \times \mathbb{R} \) are isometries, we use vertical translations of \( \text{graf}(u) \) to obtain lower barrier relatively to the boundary data \( t\varphi \).

Now, set

\[
V = \{ t \in [0, 1]; \exists u_t \in C^{2, \alpha}(\Omega) \text{ satisfying } Q_{ttH}(u_t) = 0, u_t|_{\partial \Omega} = t\varphi \}.
\]

We have \( V \neq \emptyset \) since \( t = 0 \in V \). Moreover, \( V \) is open by the Implicit Function Theorem in Banach spaces. From the barriers above, we obtain a priori uniform \( C^1 \) estimates for the family of Dirichlet problems \( Q_0(u_t) = 0, u_t|_{\partial \Omega} = t\varphi \), which give us that \( V \) is closed (by similar sequence of arguments exposed in the last paragraph of proof of Theorem 1.1). The uniqueness of the solution is a consequence of the Maximum Principle for the difference of two solutions. \( \blacksquare \)
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