Duality Between the Weak and Strong Interaction Limits for Randomly Interacting Fermions

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We establish the existence of a duality transformation for generic models of interacting fermions with two-body interactions. The eigenstates at weak and strong interaction $U$ possess similar statistical properties when expressed in the $U = 0$ and $U = \infty$ eigenstates bases respectively. This implies the existence of a duality point $U_d$ where the eigenstates have the same spreading in both bases. $U_d$ is surrounded by an interval of finite width which is characterized by a non-Lorentzian spreading of the strength function in both bases. Scaling arguments predict the survival of this intermediate regime as the number of particles is increased.

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In noninteracting systems, many-body fermionic states are totally antisymmetrized products of one-body states, so-called Slater determinants. For strongly interacting fermions on the other hand, the eigenstates expressed in the basis of Slater determinants have a large number of nonzero components, since then the one-orbital occupation is in general no longer a good quantum number. As the interaction is turned on, the number of such components increases, and this crossover from the weak to the strong interaction regime has been investigated both from the point of view of the statistical properties of the spectrum and the eigenstates [1, 2, 3, 4, 5, 6]. This crossover determines the threshold in excitation energy above which Dyson’s random matrix theory applies to the strong interaction basis has been investigated (TBRE) for $n$ interacting spinless fermions [11, 12]

$$H = H_0 + U H_1 = \Lambda \left( \sum c_\alpha c^\dagger_\alpha + U \sum Q^\gamma_\alpha_\beta c^\dagger_\alpha c^\dagger_\beta \right). \quad (1)$$

The $m$ different one-body energies are distributed as $e_\alpha \in [-m/2, m/2]$ so as to fix the mean level spacing $\Delta \equiv 1$. The interaction matrix elements $Q^\gamma_\alpha_\beta$ are independent random variables with a zero-centered Gaussian distribution of unit variance. The parameter $\Lambda = \Delta/(U + \Delta)$ has been introduced to keep the density of states roughly constant as $U$ varies. In the $U = 0$ eigenstate basis, the Hamiltonian is represented by a $N \times N$ matrix of size $N = m(m+1)/2$ with $K = 1 + n(m-n) + n(n-1)(m-n)(m-n-1)/4$ nonzero matrix elements per row. For a sufficiently large number of particles, the many-body Density of States (DOS) is well approximated by a Gaussian of width $\delta/E_0 \sim \sqrt{\Lambda}$ [11, 12]. Below we will investigate the properties of levels in the middle of the DOS.

In the $U = 0$ basis (superscripts $(0)$ and $(\infty)$ indicate the corresponding basis) the eigenstates structure is conveniently described by the strength function (SF)

$$\rho^{(0)}(E) = \langle \sum A | \psi_A(I) |^2 \delta(E + E_I - E_A) \rangle_I, \quad (2)$$

and the inverse moments

$$\xi^{(0)}_p = \langle \left( \sum I | \psi_A(I) |^{2p} \right)^{-1} \rangle_A, \quad (3)$$

where the averages $\langle \ldots \rangle_I,A$ are taken either over the eigenstates $\phi_I$ (with eigenvalues $E_I$) of $H_0$ or the eigenstates $\psi_A$ (with eigenvalues $E_A$) of $H$. The moment $p = 2$ is the participation ratio (PR) and gives the typical number of nonzero components $\psi_A(I)$. For model (1) one has $\xi^{(0)}_p \in$...
For each realization of the Hamiltonian (1). Since the $U$ decreases we now turn our attention to the reversed problem and eigenstates structure as the interaction is switched on, the width $\Gamma(U)$ of the strength function $\Gamma(E)$ for $m$-$\rho$ crossover of $2.15$ ($\times$), $4.26$ ($\times$), $8.43$ ($\Delta$), $33$ ($\varnothing$), and $129.15$ ($\circ$). The solid and dashed lines give a Lorentzian (solid) and a Gaussian (dashed) fit respectively. Lower inset: strength functions solid and dashed lines give a Lorentzian (solid line) nor Gaussian (dashed line).

\[ p^{(\infty)}(E) = p^{(\infty)}(E^*) \]

| $U/\Delta$ | $\Gamma(E)$ | $E/\Delta$ |
|-----------|-------------|-----------|
| -30       | 10          | -30       |
| -15       | 10          | -15       |
| 0         | 10          | 0         |
| 15        | 10          | 15        |
| 30        | 10          | 30        |

FIG. 1: Width of the strength function $\rho^{(\infty)}$ as a function of $U/\Delta$ for $n \in [3, 6]$ and $m \in [10, 14]$. The dotted line indicates the Golden rule predicted behavior $\Delta/\sqrt{nN}$. Upper inset: crossover of $\rho^{(\infty)}$ from delta peak to Lorentzian to Gaussian shape for $m = 12$, $n = 5$ and $U/\Delta = 0.55$ ($\square$), $1.09$ ($\triangle$), $2.15$ ($\times$), $4.26$ ($\times$), $8.43$ ($\Delta$), $33$ ($\varnothing$), and $129.15$ ($\circ$). The solid and dashed lines give a Lorentzian (solid) and a Gaussian (dashed) fit respectively. Lower inset: strength functions $\rho^{(\infty)}$ ($\square$) and $\rho^{(\infty)}(+) at the dual point $\xi^2(0)$ and $\Gamma^{(0)}=\xi^2(0) \propto \sqrt{\rho^{(\infty)}} \approx \Gamma^{(0)} \propto \xi^2(0)/\Delta^{(0)}$. The Lorentzian regime is defined by the two conditions $\xi^2(0) \ll 1$ and $\Gamma^{(0)} < \Lambda^{(0)}$. In the dilute limit $1 \ll n \ll m$, these conditions translate into

\[ \Delta/\sqrt{nN} \ll U < \Delta/\sqrt{m}. \] (4)

When the Gaussian regime is entered, the SF spreads over the full bandwidth so that the eigenstates have a finite fraction of nonzero components $\xi^2(0) = O(N)$. Having summarized some of the known results for the eigenstates structure the interaction is switched on, we now turn our attention to the reversed problem and decrease $U$ starting from $U = \infty$. Both the SF $\rho^{(\infty)}$ and the moments $\xi^2(\infty)$ can be defined in the same way as above provided the $U = \infty$ basis is fixed individually for each realization of the Hamiltonian (1). Since the occupation operator $n_\alpha = c_\alpha^\dagger c_\alpha$ does not commute with $H_1$, $H_0$ induces one-body transitions between different eigenstates of $H_1$, leading to an increase of $\xi^2(\infty)$ and a broadening of $\rho^{(\infty)} as $U$ decreases. The $H_0$-induced transitions also lead to two successive crossovers of the SF $\rho^{(\infty)}$, first from a delta peak to a Lorentzian shape, then to a Gaussian shape, and this is shown on the upper inset to Fig. 3. As is the case for $\Gamma^{(0)}$, the Golden rule gives a good estimate of the width $\Gamma^{(\infty)}$ of the SF in the Lorentzian regime, as we now proceed to show. Under the assumption that for $U = \infty$, the TBBRE has random ergodic eigenstates in the middle of its spectrum [13], all eigenstates are directly connected to each other, as can be seen on Fig. 2. Remarkably enough, we have found that at $U$, the width $\Gamma^{(\infty)} \propto \Delta/\sqrt{nN}$ and $\Gamma^{(\infty)} \propto \xi^2(\infty)$, the Lorentzian regime in the $U = \infty$ basis is bounded by the inequalities

\[ \Delta < U < \sqrt{\Lambda}/\Delta. \] (6)

This prediction is confirmed by numerical data presented in Fig. 3. These data, as well as those to be presented below, have been obtained via exact diagonalization of systems of up to $N = 3432$ (corresponding to $n = 7$ and $m = 14$), performing averages over 20 (for the largest systems) to 500 realizations of Hamiltonian (1) for each parameter set and avoiding the tails of the DOS by keeping only 33% of the states in the middle of the spectrum.

In the Golden rule regime, the PR is given by $\xi^2(\infty) = \Gamma^{(0)}/\Delta^{(\infty)} \propto N(U/\Delta)^2$ and according to the conditions $\xi^2(\infty) > 1$ and $\Gamma^{(\infty)} < \Lambda^{(\infty)}$, the Lorentzian regime in the $U = \infty$ basis is bounded by the inequalities

\[ \Delta < U < \sqrt{\Lambda}/\Delta. \] (6)

This is confirmed qualitatively by the upper inset to Fig. 3 and by numerical data to be published elsewhere [10].

The remarkable fact that $\Delta^{(\infty)}$ and $\Delta^{(0)}$ have the same parametric dependence in $n$ and $m$ implies that $\Gamma^{(0)} = \Gamma^{(\infty)}$ and $\xi^2(\infty) = \xi^2(\infty)$ can be both satisfied for $U_d/\Delta \sim 1/n^{1/4}$. This is illustrated on the lower inset to Fig. 3 and the inset to Fig. 4. Remarkably enough, we have found that at $U_d$, the PR’s take a universal value $\xi^2(\infty) = \xi^2(\infty) \approx 0.8 \xi^2(\infty)$ independently on $n$ and $m$. This calls for a rescaling $U \rightarrow U/\Delta$ for $\xi^2(\infty)/\xi^2(\infty)$ and $\Delta \rightarrow \Delta/\sqrt{nN}$ after which all data for the PR fall on top of each other, as can be seen on Fig. 4.

The inset to Fig. 4 shows that at $U_d$, the higher moments $\xi^2(\infty)$ with $p > 2$ also cross. This behavior has been
found to hold for all \( m \) and \( n \). It implies that the eigenstates have exactly the same spreading over both bases, and that the SF’s have not only the same width, but also exactly the same shape at \( U_d \) (\( \xi^{-1} \) are moments of the SF), as shown in the lower inset to Fig. 4. Moreover, we found that at the dual point the SF satisfies the scaling \( \Gamma(U_d) \propto mn^{3/2}\Delta \) as follows from (4) and \( U_d \propto \Delta/n^{1/4} \). We then calculate the PR \( \xi^{(d)}_2 \) in the \( U = U_d \) (dual) basis, and numerical results are shown in Fig. 3. The symmetry around \( U = U_d \) is evident, and for both \( U \to 0 \) and \( U \to \infty \) limits, there is saturation at \( \xi^{(d)}_2 \approx 0.8 \xi^{\text{GOE}}_2 \). These results establish the existence of a duality transformation \( U^* = U_d^2/U \) connecting the strong and weak interaction regimes, so that the PR’s and the SF satisfy \( \xi^{(0)}_2(U) = \xi^{(\infty)}_2(U^*), \xi^{(d)}_2(U) = \xi^{(d)}_2(U^*), \rho^{(0)}(U) = \rho^{(\infty)}(U^*) \) and \( \rho^{(d)}(U) = \rho^{(d)}(U^*) \).

The above estimate for \( U_d \sim \Delta/n^{1/4} \) relies on the assumptions that the Golden rule correctly estimates the width of the SF in both bases, and that \( \xi^{(0,\infty)}_2 = \Gamma(0,\infty)/\Delta^{(0,\infty)} \). Equivalently, this requires to be in the Lorentzian regime, which is however impossible as the two conditions (4) and (3) are mutually exclusive, and the predicted \( U_d \sim \Delta/n^{1/4} \) lies outside both Lorentzian regimes. One may thus wonder if its predicted parametric dependence in \( m \) and \( n \) makes any sense at all. \( U_d \) as a function of both \( m \) and \( n \) is shown in the insets to Fig. 3. First, it is seen that the \( m \)-dependence of \( U_d \) is very weak (we found \( U_d \sim \Delta m^{-\alpha} \) with \( \alpha < 0.08 \) for a range \( m \in [8,20] \)). Even though quite weak, this \( m \)-dependence presumably indicates that we are not deep enough in the dilute limit. Second we extracted the \( n \)-dependence of \( U_d \) for \( m = 14 \) where we had the largest range \( n \in [2,7] \). We got \( U_d \sim \Delta n^{-\beta} \), with an exponent \( \beta \in [0.3;0.5] \). These bounds on \( \beta \) are compatible with the Golden rule estimates which exclude \( U_d \) from both Lorentzian regimes, i.e.

\[
\Delta/\sqrt{n} < U_d < \Delta,
\]

giving \( \beta \in [0,0.5] \). We conclude that \( U_d \sim \Delta/n^{1/4} \) is in good qualitative agreement with our numerical results, and that the inequalities (3) define an intermediate regime which is nonperturbative in the two bases, and whose width increases parametrically with \( n \).

Additional structures in the wave functions can be captured by the structural entropy \( S_{\text{str}} \), which is defined by subtracting from the Shannon entropy \( S = (\sum_I |\psi_A(I)|^2 \ln |\psi_A(I)|^2)_A \) the log of the PR: \( S_{\text{str}}^{(0,\infty)} = \)
$S^{(0,\infty)} - \ln \xi_{p}^{(0,\infty)}$ \[17\]. It measures the contribution to the Shannon entropy which is not contained in the PR, and thus not in the bulk of the SF. Numerical results are presented in Fig. 4. Expectedly, the crossing of $S_{\text{str}}^{(0,\infty)}$ occurs at $U_{d}$ (since all the moments $\xi_{p}$ determine $S_{\text{str}}$) but the crucial feature of Fig. 4 is that almost immediately after the crossing, $S_{\text{str}}$ takes on its asymptotic GOE value $S_{\text{GOE}}^{(0,\infty)} = 0.3689 \ldots$ - and therefore $U_{d}$ is the crossover point to the regime with GOE-like behavior in both bases. The second crucial feature of Fig. 4 are the peaks in $S_{\text{str}}$, located on both sides of $U_{d}$. Such peaks have already been observed in various models where they indicate unusually large fluctuations of eigenstates \[18\]. It is expectable that these peaks extend over the intermediate, non-Lorentzian regime $\Delta/\sqrt{n} < U < \Delta$, which is borne out by the data plotted in Fig. 4, where once $S_{\text{str}}$ is plotted against $U/U_{d} \simeq Un^{1/4}/\Delta$, the peaks broaden symmetrically as more particles are added.

Examples of models where such a duality between localized (with $\xi_{2} = 1$) and delocalized (with $\xi_{2} = \mathcal{O}(N)$) asymptotic regimes is related to a sharp metal-insulator transitions are provided by one-dimensional lattices with quasi-periodic potential (where the duality connects momentum and spatial eigenfunction coordinates) \[13\] and tight-binding models for strongly correlated fermions \[20\]. Our model is however fundamentally different in that the duality point $U_{d}$ is protected by a finite-sized interval characterized by a non-Lorentzian spreading of the eigenstates over a finite fraction of both the strong and weak interaction eigenstate. According to \[15\], this intermediate regime survives as $m$ and $n$ increase and this results in a smooth crossover (and not a sharp transition) to fully developed chaos, characterized by the collapse of all the data points over a single curve shown on Fig. 4 (and not a single crossing of the curves at $U/U_{d} = 1$).

This is to be put in perspective with the recent controversy over whether or not particle-hole excitations undergo a delocalization transition in Fock space similar to the Anderson transition as their excitation energy increases \[1,12\] (increasing the excitation energy reduces the level spacing and is thus equivalent to increasing the interaction strength $U$). The results we presented corroborate the conclusions drawn in \[1\] of the existence of a smooth crossover and not a sharp transition.

In summary we have established the existence of a duality transformation between the weak and strong interaction regimes of deformed TBRE. At the duality point $U_{d}$, the eigenstates have GOE fluctuations (for $U > (\leq)U_{d}$ in the $U = 0$ (\infty) basis) and $U_{d}$ is surrounded by a finite-sized, intermediate regime where both limiting bases fail to describe the eigenstates. Together with the Lorentzian form of $\rho^{(\infty)}$ at large $U \gg \Delta$, this duality suggests the existence of quasiparticle excitations at $U_{d} \ll U \leq \infty$, which we were however not able to identify. The existence of this duality transformation is related to the two-body nature of the interaction. For $k$-body interaction, $n \gg k > 2$, the $U = \infty$ bandwidth is given by $B_{k}^{(\infty)} \sim \Delta n^{k/2}/m^{k/2} > B_{0}^{(0)}$ \[1,12\], and the PR in both bases are equal to each other for $U_{d} \propto \Delta^{n-k/2}/m^{3/4-k/2}$. We then have $\Gamma^{(0)}(U_{d}) \propto \Delta n^{3/2}$ while $\Gamma^{(\infty)}(U_{d}) \propto \Delta n^{3/2}/m^{1/2+k/2}$ so that the SF’s should differ at the intersection point of the PR’s for $k \neq 2$.

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