Analysis of the Faddeev model

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Abstract

In this paper we consider a generalization of the Faddeev model for the maps from a closed three-manifold into the two-sphere. We give a novel representation of smooth \(S^2\)-valued maps based on flat connections. This representation allows us to obtain an analytic description of the homotopy classes of \(S^2\)-valued maps that generalizes to Sobolev maps. It also leads to a new proof an old theorem of Pontrjagin. For the generalized Faddeev model, we prove the existence of minimizers in every homotopy class.

1 Introduction

In 1975 L. D. Faddeev introduced an interesting nonlinear sigma-model motivated by the Hopf invariant and the Skyrme model, [5]. The Hopf invariant is an integer associated to any continuous map from the 3-sphere to the 2-sphere. Roughly, it counts the linking number of the inverse images of two generic points on \(S^2\). Homotopy classes of such maps are completely classified by the Hopf invariant, [3, 8]. Faddeev’s model is now often called the Faddeev - Hopf model. It is also referred to as the Faddeev - Skyrme model. The fields in this model are maps \(n\) from \(\mathbb{R}^3\) to \(S^2\) asymptotically constant at infinity. The energy functional is

\[
E(n) = \int_{\mathbb{R}^3} |dn|^2 + |dn \wedge dn|^2 d^3x.
\]

The Hopf invariant can be evaluated analytically as follows, [3, 8],

\[
Q(n) = \int_{\mathbb{R}^3} \alpha \wedge d\alpha,
\]

where \(\alpha\) is the unique \(\delta\)-closed 1-form vanishing at infinity such that \(d\alpha\) equals the pull-back, \(n^*\omega_{S^2}\), of the volume 2-form on \(S^2\). In coordinates, \(\omega_{S^2} = \frac{1}{4\pi} \left( n^1 dn^2 \wedge dn^3 + n^2 dn^3 \wedge dn^1 + n^3 dn^1 \wedge dn^2 \right)\). In the homotopy class corresponding to \(Q(n) \in \mathbb{Z}\), the energy is bounded from below (see [11]):

\[
E(n) \geq const \cdot |Q(n)|^{\frac{3}{2}}.
\]

This estimate leads one to believe that each sector (homotopy class) should have a ground state – a minimizer of \(E(n)\). Faddeev expected the minimizers with \(Q(n) \neq 0\) to have interesting knot-like structures, [5], and recently several numerical investigations have provided some support to this conjecture, see [6, 7] and the

∗The first author was partially supported by NSF grant DMS-0204651.
† The second author was partially supported by NSF grant DMS-0200670.
references therein. Mathematically, one problem with a domain of \( \mathbb{R}^3 \) is that it is not compact; intuitively, one may imagine a minimizing sequence with a concentrated lump sliding to infinity. When the domain is a closed three-manifold, this particular difficulty is avoided. However, all of the other interesting physical and mathematical features remain. If one only wished to consider minimizers on \( S^3 \), the Hopf invariant gives the homotopy classification and has an analytic expression analogous to \( \mu_{S^2} \).

When \( \mathbb{R}^3 \) or \( S^3 \) is replaced by a general Riemannian three-manifold, \( M^3 \), the homotopy classification of maps to \( S^2 \) is more complicated. The classification result is due to Pontrjagin. In \( \mu_{S^2} \). Pontrjagin, in fact, classifies the maps from general three-complexes to the 2-sphere. For the special case of three-manifolds, we repeat his result in the proposition below. Compared to the \( S^3 \) case, two new features arise. First of all, there is a new invariant given by the induced map on second cohomology. Second, the Hopf invariant generalizes into a secondary invariant that sometimes takes values in a finite cyclic group.

Theorem 1 (Pontrjagin) Let \( M \) be a closed, connected, oriented three-manifold. To any continuous map \( \phi \) from \( M \) to \( S^2 \) one associates a cohomology class \( \phi^* \mu_{S^2} \in H^2(M; \mathbb{Z}) \), where \( \mu_{S^2} \) is a generator of \( H^2(S^2; \mathbb{Z}) \). Every class may be obtained from some map, and two maps with different classes lie in different homotopy classes. The homotopy classes of maps with a fixed class \( \alpha \in H^2(M; \mathbb{Z}) \) are in bijective correspondence with \( H^3(M; \mathbb{Z})/(2\alpha \cup H^1(M; \mathbb{Z})) \).

The known proofs of Pontrjagin’s proposition provides a pretty picture of the homotopy classification. These proofs are geometric and combinatorial in nature, and cannot be easily used in our minimization problem. To circumvent this difficulty, we give a novel description of smooth maps from closed three-manifolds into the 2-sphere. Namely, an \( S^2 \)-valued map will be represented by a flat connection and a smooth reference map. Here we rely on our earlier work \( \mu_{S^2} \) and on the concurrent research in \( \mu_{S^2} \). This description allows us to obtain an analytically friendly picture of the homotopy classification and a new proof of Pontrjagin’s result. At the same time, this presentation fits in well with Faddeev’s functional, which we rewrite using the connection and reference map. In \( \mu_{S^2} \), the developing map associated with the connection is used to compute the fundamental group and rational cohomology of the configuration space.

The first part of Section 2 relates maps into \( S^2 \) to maps into \( S^3 \), which is the basis of our proof of the Pontrjagin theorem. We then introduce flat connections to encode \( S^3 \)-valued maps. To encode an \( S^3 \)-valued map by a flat connection, a framing is required. Without the framing, the map is determined up to an orientation preserving isometry of \( S^2 \). Since the Faddeev functional is invariant under isometries, we ignore framings. Our description of orientation preserving isometry classes of smooth \( S^2 \)-valued maps in terms of special flat connections is given in Theorem 2. In Section 3 we rewrite the Faddeev energy functional in terms of flat connections. At this point we turn from smooth connections to connections with finite energy. Our main result, Theorem 3, is that the analytical conditions fixing the homotopy class are well-defined for finite energy connections. In addition, we prove the existence of a minimizer of the energy functional in every class. When the primary obstruction vanishes, there is an alternate approach to minimization that may be interesting in its own right. This is discussed in Section 4.

2 \( S^2 \)-valued maps and homotopy classification

Let \( M \) be a closed, orientable 3-manifold and \( \mu_{S^2} \) be a generator of \( H^2(S^2; \mathbb{Z}) \). The image of \( \mu_{S^2} \) in the de Rham cohomology is the equivalence class of the form \( \omega_{S^2} \) given previously. For smooth (or just continuous) maps \( \psi : M \to S^2 \), it is well known that \( \psi^* \mu_{S^2} \) is a homotopy invariant. Every class \( \alpha \in H^2(M; \mathbb{Z}) \) arises from some map \( \psi \). Here is the standard construction: Let \( \gamma \) be a 1-cycle in \( M \) dual to \( \alpha \). Since \( M \) is orientable, the normal bundle to \( \gamma \) is trivial. Using a trivialization of the normal bundle, each fiber may be identified with \( \mathbb{R}^2 \) and mapped via stereographic projection onto the 2-sphere. Finally, map the complement of the normal bundle to the North pole. Now, there are many trivializations of the normal bundle. Any two trivializations are related by some number of twists (full rotations of the fiber when moving along \( \gamma \)). The number of twists is the secondary invariant described in the second part of Proposition \( \mu_{S^2} \). To describe the secondary invariant analytically, we will need a few constructions. We start with notation.
Notation.

- $Sp_1$ is the group of unit quaternions, $q^*$ denotes the quaternionic conjugation of $q$. The Lie algebra of $Sp_1$ is the purely imaginary quaternions, denoted $sp_1$.
- $S^1$ is the group of unit complex numbers regarded as a subgroup of $Sp_1$.
- $S^2$ will be identified with the unit sphere in the purely imaginary quaternions.
- The usual dot-product may be expressed using quaternionic multiplication as $\langle p, q \rangle = \frac{1}{2}(p^*q + q^*p)$.
- $C^\infty(X,Y)$ denotes the space of smooth maps from $X$ to $Y$.
- $W^{s,p}$ denotes the usual Sobolev space of functions with $s$ derivatives in $L^p$; $W^{s,p}(M,Sp_1)$ denotes the subset of quaternion-valued $W^{s,p}$ functions on $M$ which take values in $Sp_1$ almost everywhere; $W^{s,p}(M,S^2)$ is defined analogously.

Given any map $\varphi : M \to S^2$ and any map $u : M \to Sp_1$, we construct a new map $\psi : M \to S^2$ by $\psi(x) = u(x)\varphi(x)u(x)^*$. We will show that $\psi$ has the same associated cohomology class as $\varphi$. Conversely, we will see that any map $\psi$ with the same associated cohomology class may be represented in this way.

To prove these facts we will need several maps (compare with the discussion in [1]). The most important map is $q : S^2 \times S^1 \to Sp_1$ defined by $q(z,\lambda) = q\lambda q^*$, where $z = q i q^*$. This bizarre looking map will later encode the gauge freedom when we describe $Sp_1$ associated line bundle. We have $\sigma$ trivial. Hence, it admits a section $\psi$.

Lemma 1  There exists a smooth map $u : M \to Sp_1$ such that $\psi$ and $\varphi$ are related by $\psi = u \varphi u^*$ if and only if $\psi^*\mu_{S^2} = \varphi^*\mu_{S^2}$.

Proof. Denote $P = S^2 \times Sp_1$ considered as a principal bundle over $S^2 \times S^2$ with bundle map $f$. Let $f_1, f_2 : P \to S^1$ be the first and second components of $f$, i.e., $f_1(z,q) = z$ and $f_2(z,q) = q z q^*$. Notice, that $f_2^*\mu_{S^2} = f_1^*\mu_{S^2}$ because $f_2^*\mu_{S^2}[S^2 \times \{1\}] = f_1^*\mu_{S^2}[S^2 \times \{1\}]$ and $[S^2 \times \{1\}]$ generates $H_2(S^2 \times Sp_1;\mathbb{Z})$. If $\psi = u \varphi u^*$, then $\psi^*\mu_{S^2} = (\varphi, u)^* f_2^*\mu_{S^2} = (\varphi, u)^* f_1^*\mu_{S^2} = \varphi^*\mu_{S^2}$.

In the opposite direction, assume that $\psi^*\mu_{S^2} = \varphi^*\mu_{S^2}$. We will prove that the bundle $Q_{\varphi,\psi}$ is then trivial. Hence, it admits a section $\sigma : M \to Q_{\varphi,\psi}$. The composition of $\sigma$ with the projection on the second component of $Q_{\varphi,\psi}$ is the desired $u$. To see that $Q_{\varphi,\psi}$ is trivial, we compute the first Chern class of the associated line bundle. We have

$$c_1(Q_{\varphi,\psi}) = (\varphi, \psi)^* c_1(P) = (\varphi, \psi)^*(pr_1^* \mu_{S^2} - pr_2^* \mu_{S^2}) = \varphi^* \mu_{S^2} - \psi^* \mu_{S^2} = 0.$$  \hspace{1cm} (3)\

In the above computation, $pr_k$ is the projection of $S^2 \times S^2$ onto the $k$-th factor. We have $c_1(P)[\{1\} \times S^2] = -1$, since $f$ restricted to $\{1\} \times S^2$ is the Hopf fibration. In addition, for the diagonal map $\Delta : S^2 \to S^2 \times S^2$ we have $c_1(P)[\Delta(S^2)] = 0$, since $\Delta^*P$ admits the section $\sigma(z) = (z,1)$. Recalling that $[\Delta(S^2)] = [\{1\} \times S^2] + [S^2 \times \{1\}]$, we conclude that $c_1(P) = pr_1^* \mu_{S^2} - pr_2^* \mu_{S^2}$, as used above. \hfill $\square$

We can now complete our proof of Pontrjagin’s theorem. Fix a reference map $\varphi$. Let $\mathfrak{D}(M, S^2)$ be the set of smooth maps from $M$ to $S^2$ with the same associated cohomology class as $\varphi$. Using the covering
homotopy property of the fibration $f$, we obtain a fibration $C^\infty(M, Sp_1) \to \mathcal{D}(M, S^2)$ with homotopy fiber $C^\infty(M, S^1)$. The fiber is included by $\lambda \mapsto q(\varphi, \lambda)$. The fibration is given by $u \mapsto u \varphi u^*$. This fibration induces a short exact sequence at the level of path components:

$$\pi_0(C^\infty(M, S^1)) \to \pi_0(C^\infty(M, Sp_1)) \to \pi_0(\mathcal{D}(M, S^2)).$$

It is well known that $\pi_0(C^\infty(M, S^1))$ is isomorphic to $H^1(M; \mathbb{Z})$ by $\lambda \mapsto \lambda^* \mu_{S^1}$, and $\pi_0(C^\infty(M, Sp_1))$ is isomorphic to $H^3(M; \mathbb{Z})$ by $u \mapsto u^* \mu_{Sp_1}$. Now, let $\psi \in \pi_0(\mathcal{D}(M, S^2)) \cong H^3(M; \mathbb{Z})/ (2 \varphi^* \mu_{S^2} \cup H^1(M; \mathbb{Z})$. This completes our proof of Pontrjagin’s theorem.

Fix a reference map $\varphi : M \to S^2$. We have just shown that any map $\psi$ with the same associated cohomology class $\varphi^* \mu_{S^2}$ may be represented in the form $\psi = u \varphi u^*$ for some $u : M \to Sp_1$. If $\psi$ is homotopic to $\varphi$, then, in addition, $u$ may be chosen homotopic to a constant map. This follows by a simple application of the covering homotopy property of the fibration $f$. In fact, there are many such maps $u$. For any map $\lambda : M \to S^1$, the map $u \mapsto (\varphi, \lambda)$ may also be used to represent $\psi$. However, $u \mapsto (\varphi, \lambda)$ will not necessarily be null-homotopic. Varying the map $\lambda$ one obtains all $Sp_1$-valued maps representing $\psi$. Using equation (4) we see that

$$\deg u q(\varphi, \lambda) = (2 \varphi^* \mu_{S^2} \cup \lambda^* \mu_{S^1}) [M] + \deg u,$$

(5)

The term $(2 \varphi^* \mu_{S^2} \cup \lambda^* \mu_{S^1}) [M]$ in (5) is an even integer, because $\mu_{S^2}$ and $\mu_{S^1}$ are integral classes. The map $\eta \mapsto (\varphi^* \mu_{S^2} \cup \eta) [M]$ from $H^1(M; \mathbb{Z})$ to $\mathbb{Z}$ is a group homomorphism, and therefore has image $m \mathbb{Z}$ for some $m$ depending on the class $\varphi^* \mu_{S^2}$. Since the degree of a null-homotopic map is zero, we conclude, that the degree of any map $u$ corresponding to a map $\psi$ homotopic to $\varphi$ lies in $2m \mathbb{Z}$.

**Remark 1** All homotopy classes of maps $\psi : M \to S^2$ with the same second cohomology class $\varphi^* \mu_{S^2} = \varphi^* \mu_{S^2}$ are obtained in the form $\psi = u \varphi u^*$. The maps $u_1 \varphi u_1^*$ and $u_2 \varphi u_2^*$ are homotopic if and only if $\deg u_1 \equiv \deg u_2 (\text{mod } 2m)$.

Every map $u : M \to Sp_1$ is the developing map of a flat connection $a = u^{-1} du$. Connections arising from such $u$’s have trivial holonomy. Conversely, given any flat connection, $a$, with trivial holonomy, one can find a map $u : M \to Sp_1$ so that $a = u^{-1} du$. Such $u$ is unique up to left multiplication by a constant in $Sp_1$. The degree of the map $u$ is given by $\frac{1}{4\pi^2} \int_M \text{Re} (a \wedge a \wedge a)$. Notice that $da + a \wedge a = 0$ implies that this integral is exactly the Chern-Simons invariant,

$$\text{cs}(a) = \frac{1}{4\pi^2} \int_M \text{Re} (a \wedge da + \frac{2}{3} a \wedge a \wedge a),$$

of the flat connection $a$. Recall from the previous paragraph that $u \varphi u^*$ is homotopic to $\varphi$ if and only if $\deg u = -\frac{1}{4\pi^2} \int_M \text{Re} (a \wedge a \wedge a) \in 2m \mathbb{Z}$.

For any $\psi = u \varphi u^*$ homotopic to $\varphi$, we consider the Hodge decomposition of the $\varphi$-component, $(a, \varphi)$, of the associated connection $a = u^{-1} du$. Let $\mathcal{H}(a, \varphi)$ be the harmonic component of $(a, \varphi)$. The space of harmonic 1-forms on $M$ is identified with $H^1(M; \mathbb{R})$. Let $\{\eta_1, \ldots, \eta_k\}$ be an integral basis for $H^1(M; \mathbb{R})$ and write

$$\mathcal{H}(a, \varphi) = h_1 \eta_1 + \ldots + h_k \eta_k.$$

Recall, that every class $\eta \in H^1(M; \mathbb{Z})$ is a pull-back of $\mu_{S^1}$ under a smooth map $\lambda : M \to S^1$. Given such $\lambda$, let $a^\lambda = q^{-1} a q + q^{-1} dq$ be the flat connection corresponding to $u q(\varphi, \lambda)$. Notice, that $\langle a^\lambda, \varphi \rangle - \langle a, \varphi \rangle$ is a de Rham representative of the image of $\lambda^* \mu_{S^1}$ in $H^1(M; \mathbb{R})$. We conclude, that by an appropriate choice of gauge, $\lambda$, we can make each coefficient $h_k$ assume values in the interval $[0,1)$. After this is achieved, turn
Thus, by an appropriate choice of gauge, we may further assume that the flat connection corresponding to $uq(\varphi,e^{i\theta})$ is $\exp(i\theta) = q^{-1} a q + q^{-1} dq$. Now, $\langle q^{\exp(i\theta)},\varphi \rangle - \langle a,\varphi \rangle = d\theta$. Thus, by an appropriate choice of gauge, we may further assume that the $d$-component of $\langle a,\varphi \rangle$ is zero. We summarize all of these observations in the following theorem.

**Theorem 2** Any orientation preserving $S^2$-isometry class of a smooth map from $M$ to $S^2$ homotopic to $\varphi$ is uniquely represented by a smooth flat connection $a$, which has trivial holonomy and satisfies the conditions

1. $cs(a) = -\frac{1}{12\pi^2} \int_M \text{Re}(a \wedge a \wedge a) \in 2m\mathbb{Z} = \{2(\varphi^*\mu_{S^2} \cup \eta)[M] | \eta \in H^1(M;\mathbb{Z})\}$.
2. $\mathcal{H}(a,\varphi) = h_1 \eta_1 + \ldots + h_b \eta_b$ with $h_1,\ldots,h_b \in [0,1)$.
3. $\delta \langle a,\varphi \rangle = 0$ (here and below $\delta$ is the adjoint of the exterior derivative, $d$).

### 3 Minimizers of the Faddeev functional, (I)

The Faddeev energy of a map $\psi : M \to S^2$, $\psi(x) = \psi^1(x)i + \psi^2(x)j + \psi^3(x)k$, is given by

$$E(\psi) = \int_M |d\psi|^2 + |d\psi \wedge d\psi|^2 \text{dvol},$$

where

$$|d\psi|^2 = |d\psi^1|^2 + |d\psi^2|^2 + |d\psi^3|^2,$$

and

$$|d\psi \wedge d\psi|^2 = |d\psi^1 \wedge d\psi^2|^2 + |d\psi^2 \wedge d\psi^3|^2 + |d\psi^3 \wedge d\psi^1|^2 = \frac{1}{4}||d\psi, d\psi||^2.$$

We begin this section by rewriting $E(\psi)$ using the representation of $S^2$-valued maps in Theorem 2. Fix a smooth reference map $\varphi : M \to S^2$. If $\psi$ is smooth and homotopic to $\varphi$, it can be represented as $\psi = u \varphi u^*$ with $u : M \to \text{Sp}_1$. Substitute this expression into the energy functional, use $\text{Ad}$-invariance of the norm and the Lie bracket, and the notation $a = u^* da$ to obtain

$$E(\psi) = E_\varphi[a] := \int_M |D_a\varphi|^2 + |D_a\varphi \wedge D_a\varphi|^2,$$

where $D_a\varphi = da + [a,\varphi]$. There are several advantages in using $E_\varphi[a]$ as the primary expression for the energy functional, the two main advantages are: the conditions fixing the homotopy class can be expressed analytically in terms of $a$; the primary field, $a$, takes values in a linear space. The natural space for our minimization problem is the space $\mathfrak{A}_\varphi$ of finite energy flat connections $a$ satisfying the conditions of Theorem 2. More precisely, we assume that $a \in L^2(M, \text{sp}_1)$, that $da + a \wedge a = 0$ in the sense of distributions, and that $a$ has trivial holonomy, $\rho_a = 1$ (see [1], Section 3, Lemma 4). In addition, we assume that $E_\varphi[a] < \infty$ and $cs(a) = -\frac{1}{12\pi^2} \int_M \text{Re}(a \wedge a \wedge a) \in 2m\mathbb{Z}$. Also, we require that $\mathcal{H}(a,\varphi) = h_1 \eta_1 + \ldots + h_b \eta_b$ with $h_k \in [0,1]$. Note, that we now allow $h_k = 1$. By doing so, we lose the unique representation of the orientation preserving isometry class of $\psi$, but this is more convenient for taking limits, and we can always return to the case $h_k \in [0,1]$ by a smooth gauge transformation.

**Theorem 3** The functional $E_\varphi[a]$ has a minimum in the class $\mathfrak{A}_\varphi$.

**Proof.** Let $a_n$ be a minimizing sequence in $\mathfrak{A}_\varphi$. Our first step is to show that $a_n$ are uniformly bounded in $L^2$. For each of the forms $a_n$ we use the orthogonal decomposition

$$a_n = \langle a_n,\varphi \rangle \varphi + \frac{1}{2} \varphi [a_n,\varphi].$$
Accordingly, the curvature condition, \( da_n + a_n \wedge a_n = 0 \), decomposes into

\[
d (a_n, \varphi) = \frac{1}{4} [\varphi d\varphi, [a_n, \varphi]] + \frac{1}{4} \varphi [a_n, \varphi] \wedge [a_n, \varphi],
\]

(7)

\[
d \left( \frac{1}{2} \varphi[a_n, \varphi] \right) = \langle a_n, \varphi \rangle \wedge D_{a_n} \varphi + \frac{1}{4} d\varphi \wedge [a_n, \varphi] + \frac{1}{4} [a_n, \varphi] \wedge d\varphi.
\]

(8)

Recall that \( \varphi \) is a smooth \( S^2 \)-valued function. The terms \([a_n, \varphi]\) and \([a_n, \varphi] \wedge [a_n, \varphi]\) are uniformly bounded in \( L^2 \):

\[
\| [a_n, \varphi] \|_{L^2} \leq \| D_{a_n} \varphi \|_{L^2} + \| d\varphi \|_{L^2},
\]

\[
\| [a_n, \varphi] \wedge [a_n, \varphi] \|_{L^2} \leq \| D_{a_n} \varphi \wedge D_{a_n} \varphi \|_{L^2} + 2 \| d\varphi \|_{L^\infty} \| [a_n, \varphi] \|_{L^2} + \| d\varphi \wedge d\varphi \|_{L^2}.
\]

The harmonic part of \( \langle a_n, \varphi \rangle \) is uniformly bounded in \( L^\infty \) by the assumptions of our class \( \mathfrak{A}_\varphi \). From equation (7) and \( \delta \langle a_n, \varphi \rangle = 0 \) we obtain an elliptic estimate

\[
\| \langle a_n, \varphi \rangle \|_{W^{1,2}} \leq C (\langle \| d (a_n, \varphi) \|_{L^2} + \| \mathcal{H}(a_n, \varphi) \|_{L^2} \rangle).
\]

Thus, the sequence \( a_n \) is uniformly bounded in \( L^2 \). Choose a subsequence, called also \( a_n \), that converges weakly in \( L^2 \), and let the limit be \( a \). Note that \( \langle a_n, \varphi \rangle \varphi \rightharpoonup \langle a, \varphi \rangle \varphi \) and \( \varphi [a_n, \varphi] \rightharpoonup \varphi [a, \varphi] \) in \( L^2 \). In fact, \( \langle a_n, \varphi \rangle \) converges to \( \langle a, \varphi \rangle \) strongly in \( L^p \) for \( 2 \leq p < 6 \) by the compactness of the embedding \( W^{1,2} \subset L^p \). Consider the wedge products \( \varphi [a_n, \varphi] \wedge [a_n, \varphi] \). By sparsing the sequence \( a_n \), we will assume that \( \varphi [a_n, \varphi] \wedge [a_n, \varphi] \) converges weakly in \( L^2 \) to some 2-form \( \xi \). Let us show that \( \xi = \varphi [a, \varphi] \wedge [a, \varphi] \).

We need the following version of the \( \text{div} - \text{curl} \) lemma. [14]

Lemma 2 Let \( M \) be a smooth 3-dimensional Riemannian manifold. Let \( \omega_1^m \in L^2 \) be a sequence of matrix-valued differential forms and \( \omega_2^m \in L^2 \) be a sequence of matrix-valued differential forms on \( M \). If \( \omega_1^m \) converges weakly to \( \omega_2^m \) converges weakly in \( L^2 \) to a form \( \omega_1 \) and \( \omega_2^m \) converges weakly in \( L^2 \) to a form \( \omega_2 \), and if each sequence \( d\omega_1^m \) and \( d\omega_2^m \) is precompact in \( W_{loc}^{-1,2}(M) \), then \( \omega_1^m \wedge \omega_2^m \) converges to \( \omega_1 \wedge \omega_2 \) in the sense of distributions.

In our case, \( d (\varphi [a_n, \varphi]) \) is given by equation (8). We have

\[
\| \langle a_n, \varphi \rangle \wedge D_{a_n} \varphi \|_{L^2} \leq \| \langle a_n, \varphi \rangle \|_{L^6} \| D_{a_n} \varphi \|_{L^2},
\]

\[
\| d\varphi \wedge [a_n, \varphi] + [a_n, \varphi] \wedge d\varphi \|_{L^2} \leq 2 \| [a_n, \varphi] \|_{L^2} \| d\varphi \|_{L^\infty},
\]

and \( \| \langle a_n, \varphi \rangle \|_{L^6} \lesssim C \| \langle a_n, \varphi \rangle \|_{W^{1,2}} \) by the embedding \( W^{1,2} \subset L^6 \). We note that \( L^2 \) and \( L^6 \) are compactly embedded in \( W_{loc}^{-1,2}(M) \) by the Sobolev embedding theorem. Using estimate (9), we conclude that the sequence \( d (\varphi [a_n, \varphi]) \) is precompact in \( W_{loc}^{-1,2}(M) \). Applying the \( \text{div} - \text{curl} \) lemma, we obtain that \( \xi = \varphi [a, \varphi] \wedge [a, \varphi] \). At this point we can conclude that \( E_\varphi[a] \leq \lim \inf E_\varphi[a_n] \). It remains to prove that \( a \in \mathfrak{A}_\varphi \).

We start by showing that the limit form, \( a \), is distributionally flat. We know, that \( da_n \) converges to \( da \) in the sense of distributions since \( a_n \to a \) in \( L^2 \). Next, \( a_n \wedge a_n \) converges to \( a \wedge a \) in distributions since

\[
a_n \wedge a_n = \frac{1}{4} [a_n, \varphi] \wedge [a_n, \varphi] - \langle a_n, \varphi \rangle \wedge [a_n, \varphi],
\]

and the right hand side converges in the sense of distributions by the previous arguments.

We next note that the holonomy of \( a \) is trivial by Lemma 8 of [14]. We have already seen that the energy of \( a \) is finite. The harmonic part of \( \langle a_n, \varphi \rangle \) converges to the harmonic part of \( \langle a, \varphi \rangle \) because the space of harmonic forms is finite dimensional.

Finally, consider the degree,

\[
- \frac{1}{4\pi^2} \int \text{Re} \langle a_n \wedge a_n \wedge a_n \rangle = - \frac{1}{36\pi^2} \int \text{Re} (\varphi [a_n, \varphi] \wedge [a_n, \varphi] \wedge [a_n, \varphi])
\]

\[
+ - \frac{1}{10\pi^2} \int \text{Re} (\langle a_n, \varphi \rangle \varphi \wedge [a_n, \varphi] \wedge [a_n, \varphi])
\]

6
We already know that $\langle a_n, \varphi \rangle$ converges strongly in $L^2$ to $\langle a, \varphi \rangle$ and $\varphi[a_n, \varphi] \wedge \varphi[a_n, \varphi] \rightarrow \varphi[a, \varphi] \wedge \varphi[a, \varphi]$. This implies the convergence of the second term. We are going to use the $\text{div} - \text{curl}$ lemma on the first term. We know that $\varphi[a_n, \varphi] \rightarrow \varphi[a, \varphi]$ and $d (\varphi[a_n, \varphi])$ is precompact in $W^{-1,2}$. Now,

$$d([a_n, \varphi] \wedge [a_n, \varphi]) = 2 \langle a_n, \varphi \rangle \wedge \varphi[D_{a_n} \varphi, D_{a_n} \varphi] + 2 \langle a_n, \varphi \rangle \wedge [\varphi d \varphi, D_{a_n} \varphi] + \varphi d \varphi \wedge [a_n, \varphi] \wedge [a_n, \varphi].$$

The first 1-form factor in each term is uniformly bounded in $L^6$ and the following 2-forms are uniformly bounded in $L^2$. It follows that the entire expression is uniformly bounded in $L^2$, hence precompact in $W^{-1,2}$. The $\text{div} - \text{curl}$ lemma allows us to conclude that, after taking a subsequence, $cs(a_n) \rightarrow cs(a)$. This completes the proof of the theorem. \hfill \square

## 4 Minimizers of the Faddeev functional, (II)

Given a smooth reference map $\varphi$, we now know that there is a minimizer of $E_\varphi$ in the class $\mathfrak{A}_\varphi$. The smooth connections in this class correspond exactly to $\text{SO}(3)$-equivalence classes of smooth maps from $M$ to $S^2$ homotopic to $\varphi$. A general connection, $a$, in the class $\mathfrak{A}_\varphi$ may also be represented as $a = u^* du$, but now $u$ is only in $W^{1,2}(M, S^2)$. This follows from Lemma 6 of \[1\]. The corresponding map $\psi = u \varphi u^*$ lives in $W^{1,2}(M, S^2)$ and has finite energy $E(\psi)$. We believe that minimizers of $E_\varphi$ are smooth, but this is an open problem. If a minimizer, $a$, is smooth, then the corresponding $u$ and $\psi$ are smooth as well. In the smooth case, by Lemma \[1\] $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$ independent of $u$. Even stronger, $\psi$ is homotopic to $\varphi$. Thus, such a $\psi$ would be a minimizer of the original Faddeev functional, $E$, in the class of smooth maps homotopic to $\varphi$. Even without any additional regularity result, $\psi$ is a minimizer of $E$ in the class of finite energy Sobolev maps of the form $u \varphi u^*$ with $u \in W^{1,2}(M, S^2)$ and $cs(u^* du) \in \{2(\varphi^* \mu_{S^2} \cup \eta)[M] \mid \eta \in H^1(M; \mathbb{Z})\}$. It is an open problem to extend obstruction theory to the class of finite energy Sobolev maps. In particular, a reasonable extension of the definition of pull-back for Sobolev maps with finite energy would imply $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$. The pull-back by a $W^{1,2}(M, S^2)$ map is well-defined in the de Rham theory.

**Lemma 3** For any $u \in W^{1,2}(M, S^2)$, we have

$$(u \varphi u^*)^* \omega_{S^2} = \varphi^* \omega_{S^2} + \frac{1}{2\pi} d \text{Re}(\varphi u^* du)$$

almost everywhere on $M$.

**Proof.** In quaternionic notation, the standard volume form on $S^2$ is given by $\omega_{S^2} = -\frac{1}{8\pi} \text{Re}(z dz \wedge dz)$. The proof follows by straightforward computation. \hfill \square

Notice, that the integrals $\int_M \psi^* \omega_{S^2} \wedge \eta_k$ specify $\varphi^* \mu_{S^2}$ up to torsion. The secondary homotopy invariant works even better. To wit: $\int_M \text{Re}(u^* du \wedge u^* du \wedge u^* du)$ is well defined. In addition to Theorem \[2\] we prove the following result.

Let $\mathfrak{F}_k$ denote the class of all finite energy maps $\psi \in W^{1,2}(M, S^2)$ for which there exist a 1-form $\theta_\psi \in W^{1,2}$

$$\psi^* \omega_{S^2} = d \theta_\psi$$

and

$$\int_M \theta_\psi \wedge d \theta_\psi = k.$$

The argument from \[3\] shows, that there exists a constant $c_M > 0$ so that

$$E(\psi) \geq c_M |k|^\frac{2}{3},$$

for all $\psi \in \mathfrak{F}_k$. \hfill 7
Theorem 4 In every nonvoid class \( \mathcal{F}_k \) there exists a minimizer of \( E \).

Proof. Choose a minimizing sequence, \( \psi_n \), and let \( \theta_n \) denote the corresponding \( \theta^{\psi_n} \). By Hodge theory, we may assume that each \( \theta_n \) is co-closed (\( \delta \theta_n = 0 \)), and has trivial harmonic part. Taking a subsequence, \( \psi_n \) converges weakly in \( W^{1,2}(M, S^2) \) and almost everywhere to some \( \psi \in W^{1,2}(M, S^2) \). A direct computation shows that pointwise \( |\psi^* \omega_{S^2}| = (8\pi)^{-1} |d\psi \wedge d\psi| \). Since \( E(\psi_n) \) is bounded, \( \|\psi^* \omega_{S^2}\|_{L^2} \) is bounded. Taking another subsequence, \( \psi_n^* \omega_{S^2} \rightarrow \xi \) in \( L^2 \), for some \( \xi \). To show that \( \xi = \psi^* \omega_{S^2} \), we use the div - curl lemma. We have \( \psi_n^* \omega_{S^2} = -\frac{1}{8\pi} \text{Re}(\psi_n d\psi_n \wedge d\psi_n) \). Certainly, \( d\psi_n \rightarrow d\psi \) and \( \psi_n d\psi_n \rightarrow \psi d\psi \) in \( L^2 \). Their differentials are 0 and \( d\psi_n \wedge d\psi_n \), respectively, which are both precompact in \( W^{-1,2} \). Thus, \( \xi = \psi^* \omega_{S^2} \).

This implies that \( E(\psi) \leq \inf_{\mathcal{F}_k} E \). The 1-forms \( \theta_n \) are uniformly bounded in \( W^{1,2} \), hence, by taking a subsequence, converge weakly in \( W^{1,2} \) and strongly in \( L^2 \) to some \( \theta \). At the same time, \( d\theta_n \rightarrow d\theta \) in \( L^2 \). Recalling that \( d\theta_n = \psi_n^* \omega_{S^2} \rightarrow \psi^* \omega_{S^2} \), we conclude, that \( d\theta = \psi^* \omega_{S^2} \). Since \( \theta_n \rightarrow \theta \) and \( d\theta_n \rightarrow d\theta \), we obtain \( \int_M \theta \wedge d\theta = k \). Thus, \( \psi \in \mathcal{F}_k \) and \( \psi \) is the minimizer of \( E \).

5 Concluding remarks

One can pose many minimization problems for the functionals

\[
E(\psi) = \int_M |d\psi|^2 + |d\psi \wedge d\psi|^2 \mathrm{d} \text{vol},
\]

and

\[
E_\varphi[a] := \int_M |D_\varphi|^2 + |D_\varphi \wedge D_\varphi|^2.
\]

For example, one could minimize the first functional over all maps with fixed primary obstruction only; one could minimize the second functional over the classes of flat connections with fixed nontrivial holonomy, or fixed Chern-Simons invariant, or arbitrary holonomy. The arguments that we used in Theorems 3 and 4 apply equally well to each of these problems.

There are several interesting open questions related to this model. We have already mentioned, that we expect the minimizers to be smooth. What about maps in general: Do \( W^{1,2} \) maps with finite Faddeev energy have extra regularity? Note that the second term in \( E(\psi) \) rules out local singularities of the form \( x \mapsto \frac{1}{x} \). How does one extend obstruction theory to finite energy maps? In particular, what is the appropriate definition of homotopy of finite energy maps? Is there a cohomology theory that agrees with integral singular theory for which pull-back is defined for finite energy maps? Pull-backs in such a theory should also be homotopy invariant and analogues of Lemma 1 and Proposition 1 should hold. (Note that the proofs of Lemma 1 and Proposition 1 only require continuity. Thus, the above questions only make sense if there exist honestly discontinuous finite energy maps.) The Hopf invariant given in equation (2) or \( k \) in (10) should be an integer. This has not been verified for Sobolev maps. In another direction, it would be very interesting to see the structure of the minimizers. There may very well exist explicit minimizers on special Riemannian manifolds such as the three-sphere, three-torus, lens spaces, etc. There may be new phenomena for closed domains that could be discovered by numerical experimentation. The closed case has the additional advantage that one does not have to worry about behavior at infinity.

References

[1] Auckly, D., Kapitanski, L.: Holonomy and Skyrme’s model. Commun. Math. Phys. 240, 97–122 (2003)
[2] Auckly, D., Speight, M.: Fermionic quantization and configuration spaces for the Skyrme and Faddeev- Hopf models. in preparation
[3] Bott, R., Tu, L. W.: Differential forms in algebraic topology. Springer-Verlag, New York-Berlin, 1982
[4] Bredon, G. E.: *Introduction to compact transformation groups*. Academic Press, New York - London, 1972

[5] Faddeev, L. D.: Quantization of solitons. Preprint IAS print-75-QS70 (1975)

[6] Faddeev, L. D.: Knotted solitons and their physical applications. Phil. Trans. R. Soc. Lond. A 359, 1399–1403 (2001)

[7] Faddeev, L. D., Niemi, A. J.: Stable knot-like structures in classical field theory. Nature 387, 58–61 (1997)

[8] Hopf, H.: Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. Math. Annalen 104, 637–665 (1931)

[9] Kapitanski, L.: On Skyrme’s model, in: Nonlinear Problems in Mathematical Physics and Related Topics II: In Honor of Professor O. A. Ladyzhenskaya, Birman et al., eds. Kluwer, 2002, pp.229-242

[10] Pontrjagin, L.: A classification of mappings of the three-dimensional complex into the two-dimensional sphere. Rec. Math. [Mat. Sbornik] N. S. 9 (51), 331–363 (1941).

[11] Vakulenko, A. F., Kapitanski, L.: On $S^2$-nonlinear $\sigma$-model. Dokl. Acad. Nauk SSSR 248, 810–814 (1979)

[12] Kobayashi, S., Nomizu, K.: *Foundations of differential geometry, Vols. I, II*. John Wiley & Sons, Inc., New York, 1996

[13] Munkres, J. R.: *Elementary differential topology*, Revised edition. Annals of Mathematics Studies, No. 54 Princeton University Press, Princeton, N.J. 1966

[14] Robbin, J. W., Rogers, R. C., Temple, B.: On weak continuity and the Hodge decomposition, Trans. AMS, 303 (2), 609-618 (1987)

[15] Skyrme, T. H. R.: A non-linear field theory, Proc. R. Soc. London, A 260 (1300), 127-138 (1961)

[16] Skyrme, T. H. R.: A unified theory of mesons and baryons, Nuclear Physics, 31, 556-569 (1962)

[17] Skyrme, T. H. R.: The origins of Skyrmions, Int. J. Mod. Phys., A3, 2745-2751 (1988)

[18] Spanier, E. H.: *Algebraic topology*. New York: Springer, 1966