Dynamics of Wilson Observables in Non-Commutative Gauge Theory

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Abstract

An equation for the quantum average of the gauge invariant Wilson loop in non-commutative Yang-Mills theory with gauge group $U(N)$ is obtained. In the 't Hooft limit, the equation reduces to the loop equation of ordinary Yang-Mills theory. At finite $N$, the equation involves the quantum averages of the additional gauge invariant observables of the non-commutative theory associated with open contours in space-time. We also derive equations for correlators of several gauge invariant (open or closed) Wilson lines. Finally, we discuss a perturbative check of our results.

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1 Introduction

In the approach to gauge theory pioneered by Mandelstam [1] and Wilson [2], the basic dynamical object is the phase factor (or parallel transporter)

\[ U[C, A] = P \exp i \int_C A_\mu (x(\tau)) dx^\mu (\tau). \]  

(1)

Here \( P \) denotes the Dyson path ordering operation, the gauge field \( A_\mu \) is represented by matrices lying in the Lie algebra of the gauge group and the line integral runs over a closed loop \( C \) parametrised by \( x(\tau), 0 \leq \tau \leq 1 \). In the quantum theory, one would like to compute the gauge-field average of the (gauge-invariant) trace of \( U[C, A] \) weighted by the gauge field action \( 3 \),

\[ \langle W[C, A] \rangle = \frac{1}{Z} \int [DA] e^{-\frac{1}{g^2} \int F_{\mu\nu} F^{\mu\nu} \frac{1}{N} \text{tr} U[C, A]} \]  

(2)

\( (F_{\mu\nu} \text{ denotes the Yang-Mills field strength, and } Z \text{ is a normalisation factor}). \) In the strong coupling limit, the loop average \( \langle W[C] \rangle \) determines whether quarks are confined according to Wilson’s area law [2]. It also turns out to satisfy the so-called loop equations [3, 4], which are closed functional equations from which the Feynman perturbation expansion can be shown to arise [5]. Thus quantum gauge theory can be reformulated in terms of such gauge-invariant Wilson observables. The loop equations in the ’t Hooft limit \( N \to \infty \) with \( \lambda \equiv g^2 N \) kept fixed can be written as

\[ \partial^\mu \frac{\delta \langle W[C] \rangle}{\delta \sigma^{\mu\nu}(x)} = -\lambda \int_C dy \delta(x - y) \langle W[C_{xy}] \rangle \langle W[C_{yx}] \rangle, \]  

(3)

where \( C_{yx} \) is defined as the part of \( C \) from \( x \) to \( y \), \( \partial^\alpha (x) \) is the path derivative at \( x \) and \( \delta \sigma^{\mu\nu}(x) \) is the area derivative in the \( \mu, \nu \) plane at \( x \in C \). See e. g. [3, 4] for a detailed derivation and further explanation of these equations.

In the present note, we will derive equations analogous to (3) for Wilson observables in non-commutative gauge theory. The latter theory is a spatially non-local, higher derivative relative of ordinary gauge theory which exhibits interesting perturbative behaviour [8, 10]. Interestingly, the theory can be obtained in a zero-slope limit from open bosonic string theory in a constant background 2-form gauge potential [11]. Using this connection to string theory, it was shown that the ordinary and non-commutative gauge theories are classically equivalent [11], but it is not clear whether this persists at the quantum level. It is therefore important to reformulate

\[ ^3\text{Below, we will often omit indicating the dependence on } A \text{ for } U \text{ and } W. \]
the quantum non-commutative theory in terms of gauge invariant observables satisfying geometric equations generalising (3). This formulation, presumably, is suitable for the confining (strong coupling) phase of non-commutative gauge theory.

Consider a $D$-dimensional Euclidean space with non-commutative coordinates, $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ (we assume the matrix $\theta$ to be nondegenerate). The non-commutative Yang-Mills theory is obtained by working with commuting coordinates and replacing all ordinary products of gauge fields by star products. The action for a $U(N)$ theory is

$$S = \frac{1}{g^2} \int dx \text{tr} (F_{\mu\nu}(x) \ast F^{\mu\nu}(x)),$$

where the non-commutative field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i (A_\mu \ast A_\nu - A_\nu \ast A_\mu),$$

and $\ast$ is the star product defined by

$$f(x) \ast g(x) \equiv e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu \partial_\nu} f(x + y)g(x + z)|_{y = z = 0.}$$

The key properties of this product are associativity and cyclicity of the integrals of $\ast$-products over space. The cyclic property implies that the action (4) is invariant under non-commutative gauge transformations. The classical field equations of the non-commutative gauge theory obtained by varying (4) are

$$D^\mu F_{\mu\nu} = 0,$$

where $D_\mu = \partial_\mu - i[A_\mu, \cdot]_\ast$ is the covariant derivative. The Jacobi identity

$$\varepsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0$$

holds whether the field equations are satisfied or not.

Consider an arbitrary (open or closed) contour $C$ in non-commutative space-time parametrised by $x + \xi(\tau)$ with $0 \leq \tau \leq 1$. The generalisation of the Wilson factor (1) in the non-commutative theory is

$$U[C, A] = P_\ast \exp \left( i \int_C A_\mu(x + \xi(\tau)) d\xi^\mu(\tau) \right)$$

where $P_\ast$ denotes path ordering along $x + \xi(\tau)$ from right to left with respect to increasing $\tau$ of $\ast$-products of functions. The star multiplication is performed with respect to the variable $x$. The result in (9) is independent of the splitting of the

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4Henceforth, $A_\mu$ will denote the non-commutative gauge field which was denoted by $\hat{A}_\mu$ in [1].
constant mode, i.e. as long as \( x + \xi(\tau) = x' + \xi'(\tau) \) one can replace \( x \) and \( \xi \) in (9) by \( x' \) and \( \xi' \).

Note that (9) is not gauge invariant, even after taking the trace over Lie algebra-valued matrices. If the contour \( C \) is closed, this contrasts with the situation in ordinary gauge theory, where a closed Wilson line is gauge invariant. For closed contours the gauge invariant Wilson loop is defined by

\[
W_c[C] = \int dx \frac{1}{N} \text{tr} U[x + C].
\] (10)

Here \( x + C \) denotes the contour obtained by translating \( C \) by \( x \). In addition, it was found in \([12]\) that for an open contour \( C \) the following open Wilson line with momentum \( k_{\xi} \)

\[
W_o[C] = \int dx \frac{1}{N} \text{tr} U[x + C] \star e^{-ik_{\xi}x}
\] (11)
is gauge invariant provided the momentum \( k_{\xi} \) and the distance \( l = \xi(1) - \xi(0) \) between the end-points of \( C \) satisfy the condition

\[
l^\nu = \theta^{\mu\nu} k_{\mu}, \quad k_{\xi} = \theta^{-1}(\xi(1) - \xi(0)).
\] (12)

In the quantum field theory, the expectation value of a given functional \( H[A] \) of the gauge field is understood in the sense of

\[
\langle H \rangle = \int [DA] e^{-S[A]} H[A],
\] (13)

with the normalisation \( \langle 1 \rangle = 1 \). Although on the classical level and in a given gauge field configuration generically \( U[x + C] \neq U[C] \), the gauge field quantisation restores translation invariance:

\[
\langle U[x + C] \rangle = \langle U[C] \rangle.
\] (14)

Note that for any two \( \star \)-multiplied factors the order of factors under the \( x \)-integration can be interchanged. Moreover, for factors satisfying suitable boundary conditions, the star multiplication can be replaced by ordinary multiplication,

\[
\int dx f(x) \star g(x) = \int dx f(x)g(x).
\] (15)

Using these facts, we find

\[
\langle W_o[C] \rangle = (2\pi)^D \det \theta \delta(\xi(1) - \xi(0)) \langle \frac{1}{N} \text{tr} U[C] \rangle,
\] (16)
as well as \( (V \) is the volume of \( D \)-dimensional space-time)

\[
\frac{1}{V} \langle \int dx \frac{1}{N} \text{tr} U[C] \rangle = \frac{\int dx}{V} \langle \frac{1}{N} \text{tr} U[C] \rangle = \langle \frac{1}{N} \text{tr} U[C] \rangle.
\] (17)
Due to the δ-function in (13) it looks as if \( \langle W_o[\xi, A] \rangle \) would be nontrivial in the limit of closed contours only. However, there are nontrivial correlation functions for more than one contour [13, 14]. Such correlation functions will indeed appear below. Similarly, for the use in correlation functions it is crucial to keep track of the \( x \)-integration for closed loops.

**2 The loop equations**

We now turn to the derivation of the loop equations. The starting point for this is the result

\[
\partial^{\mu} \frac{\delta W[C]}{\delta \sigma^{\mu\nu}(\xi(\tau))} = \frac{i}{N} \int dx \text{tr} P_\ast \left( D^{\mu} F_{\mu\nu}(x + \xi(\tau)) \exp \left( i \int_C A_\mu(x + \xi(\sigma)) d\xi^\mu(\sigma) \right) \right) .
\]

(18)

Since the derivation of this formula is based on purely geometrical considerations together with the algebraic properties of the multiplication of gauge fields, we can take it directly from the case of ordinary Yang-Mills; see also [14].

Thus the operator \( \partial^{\mu} \delta/\delta \sigma^{\mu\nu}(\xi) \) inserts the field equation (7) at the point \( x + \xi(\tau) \).

To compute the action of \( \partial^{\mu} \delta/\delta \sigma^{\mu\nu}(\xi) \) on the quantum average \( \langle W_C \rangle \), we use the quantum field equation

\[
\epsilon \left( \delta S \right) \delta A^a(Y) H + O(\epsilon^2) = \langle H[A + \delta A]\rangle - \langle H[A] \rangle ,
\]

(19)

with \( A_\mu = A^a_\mu T_a \), \( \delta A^b_\mu(X) = \epsilon \delta_{\mu\nu} \delta^{ab} \delta(X - Y) \). In the following we will take

\[
H[A] = \frac{ig^2}{N} \text{tr} P_\ast (T_a(\tau) \exp(i \int A_\mu(x + \xi) d\xi^\mu))
\]

and consequently

\[
\delta A^b_\mu(x + \xi(\sigma)) = \epsilon \delta_{\mu\nu} \delta^{ab} \delta(x + \xi(\sigma) - x' - \xi(\tau)) .
\]

(20)

The notation \( T_a(\tau) \) indicates that the matrix \( T_a \) representing one of the normalised generators of \( U(N) \) is inserted at the parameter value \( \tau \).

Let us denote for a moment \( X(\tau) = (x + \xi(\tau)) \in x + C \). The δ-function \( \delta(X(\sigma) - X(\tau)) \) as a function of the argument \( X(\sigma) \) is a special case of a variation of the gauge field as a function of \( X(\sigma) \), as indicated in (20). On the other hand, the star product in (18) refers to the constant part in \( X(\sigma) \). The second entry \( X(\tau) \) is a parameter.

\[\text{For quantisation one has to add gauge fixing and ghost terms to the action. In the case of ordinary Yang-Mills theory, it has been shown in [17] that the contribution of these terms to the field equation inserted into the Wilson loops cancel. We assume this to hold here, too.}\]
which only determines where the δ-function is peaked. It is therefore not involved in the evaluation of the star product. This we indicate by our notation. Only after evaluating the star product one has to put \( x' = x \). Making this identification from the very beginning would factor the δ-function out of the star product and lead to incorrect results.

Note that reparametrisation invariance and the cyclic property of the \(*\) product imply that we can choose the point \( \xi(\tau) \) where \( \partial^\mu \delta/\delta \sigma^{\mu\nu}(\xi) \) acts to be the endpoint \( \zeta = \xi(1) \) of the contour. Then the insertion of the field equation must be made at the position mostly to the left. Using (15), we can write

\[
\int dx \frac{\delta S}{\delta A^\mu_\tau(x + \zeta)} P_\tau \left( T_a(1)e^{i \int A_\nu(x + \xi)d\xi} \right) = \frac{1}{g^2} \int dx P_\tau \left( D^\mu F_{\mu\nu}(x + \zeta)e^{i \int A_\nu(x + \xi)d\xi} \right).
\]  

(21)

Thus we find\(^6\)

\[
\frac{1}{V} \partial^\mu \frac{\delta}{\delta \sigma^{\mu\nu}(\zeta)} \langle W_\xi[C] \rangle = -\frac{g^2}{NV} \int d\xi_\nu(\tau) \langle \int dx \partial^\mu P_\tau \left( T_a(1)e^{i \int A_\nu(x + \xi(d\xi)} \right) \delta(x + \xi(\tau) - x' - \zeta) \mid_{x=x'} \rangle \].
\]

(22)

The \( U(N) \) colour group indices can be rearranged using the factorisation of the matrices \( T_a \),

\[
(T_a)_{kl}(T_a)_{mn} = \delta_{kn}\delta_{lm}.
\]

(23)

Then, applying formula (38) of the appendix with \( f \) and \( g \) represented by the Wilson factors in the integrand of (22) yields

\[
\frac{1}{V} \partial^\mu \frac{\delta}{\delta \sigma^{\mu\nu}(\zeta)} \langle W_\xi[C] \rangle = -\frac{g^2}{NV} \int d\xi_\nu(\tau) \langle \int dx \partial^\mu P_\tau \left( e^{i \int A_\nu(x + \xi)d\xi} \right) \ast e^{-ik\xi(\tau)x} \ast \int dy P_\tau \left( e^{i \int A_\nu(y + \xi)d\xi} \right) \rangle.
\]

(24)

Using the notation introduced above and reinstating differentiation at points \( \xi(\tau) \) corresponding to arbitrary parameter values, (24) can be written as

\[
\frac{1}{V} \partial^\mu \frac{\delta}{\delta \sigma^{\mu\nu}(\zeta)} \langle W_\xi[C] \rangle = -\frac{g^2 N}{V} \frac{1}{(2\pi)^D \det \theta} \int d\eta_\nu \langle W_\xi[C_\eta]\rangle \langle W_\xi[C_\eta] \rangle.
\]

(25)

This is the loop equation for the quantum average of the gauge-invariant closed Wilson loop. An interesting reformulation of this equation is obtained by writing

\[
\langle W_\xi[C_\eta]\rangle \langle W_\xi[C_\eta] \rangle = \langle W_\xi[C_\eta]\rangle \langle W_\xi[C_\eta] \rangle + \langle W_\xi[C_\eta]\rangle \langle W_\xi[C_\eta] \rangle \rangle_{\text{conn}}
\]

(26)

\(^6\)This equation was previously derived in [14], however the key issue of star product versus constant mode dependence was not discussed there.
(where \( \langle \ldots \rangle_{\text{conn}} \) denotes the connected part of a correlation function) and applying eq. (16) to both \( \langle W_o[C_{\xi\eta}] \rangle \) and \( \langle W_o[C_{\eta\xi}] \rangle \). The product of delta functions which arises is dealt with in the usual way: writing

\[
\left[ \delta(\theta^{-1}(\xi - \eta)) \right]^2 = \delta(\theta^{-1}(\xi - \eta))\delta(0) = \frac{V}{(2\pi)^D} \det \theta \delta(\xi - \eta),
\]

we recover a delta function \( \delta(\xi - \eta) \) as in the ordinary loop equation (3) multiplied with a factor of \( \det \theta \). Thus the final result for the loop equation in the non-commutative theory at finite \( N \) takes the form:

\[
\frac{1}{V} \partial^{\mu} \frac{\delta}{\delta \sigma^{\mu}(\xi)} \langle W_c[C] \rangle = -\frac{\lambda}{V^2} \int_C d\eta_\nu \delta(\xi - \eta) \langle W_c[C_{\xi\eta}] \rangle \langle W_c[C_{\eta\xi}] \rangle
\]

\[
-\frac{g^2 N}{(2\pi)^D V \det \theta} \int_C d\eta_\nu \langle W_o[C_{\xi\eta}] W_o[C_{\eta\xi}] \rangle_{\text{conn}},
\]

for all points \( \xi(\tau) \in C \). In the ’t Hooft limit, the second term on the r. h. s. involving the connected part \( \langle \ldots \rangle_{\text{conn}} \) of the two-point function vanishes and the equation looks like (3). Moreover for finite \( N \) it is possible to argue that, in the limit in which the non-commutativity parameter \( \theta \) is taken to zero, the oscillatory behaviour of the exponential factors in this connected part conspires to yield a smooth limit in spite of the apparently divergent prefactor of \( 1/\det \theta \). Thus in this limit the equation is just the same as for standard Yang-Mills theory.

It is remarkable that for finite \( N \) the new gauge invariant objects for open contours appear to be necessary for the description of the dynamics of closed loops. In the non-commutative case for finite \( N \) there is no overall \( \delta \)-function on the r. h. s. of the loop equation.

In order to solve eq. (28) or eq. (3), one should supplement it with the condition

\[
\varepsilon^{\mu\nu\rho\sigma} \partial_\nu \frac{\delta}{\delta \sigma^{\rho\lambda}(\xi(\tau))} \langle W_c[C] \rangle = 0,
\]

which follows from the Mandelstam formula [1] and the Bianchi identity (8).

### 3 Loop equations for correlators

Since the quantum average for a single observable \( W_o \) vanishes as a result of eq. (16), there is no nontrivial quantum dynamics for these new objects by themselves. The

\footnote{The explicit powers of the (infinite) volume \( V \) conspire to cancel trivial divergencies produced by overall translation invariance of quantum averages.}
situation changes if one considers correlation functions of several closed and/or open contours. Here we consider as a first step the correlation function of two Wilson loops. The generalisation to higher correlation functions and to the inclusion of Wilson functionals for open contours is straightforward. If one acts with the differentiation on loop $C^1$ the equation analogous to (22) is 

\[ \langle W_c[C^1]W_c[C^2] \rangle = \frac{1}{V} \frac{\partial^\mu}{\partial \sigma^{\mu\nu}(\xi(1))} \left( \int_{C^1} d\xi_{\nu} \langle \int dx dy \text{tr} P_{\tau}^a T_a(\tau) \right) \]

This term simplifies to

\[ \cdot U[C^1 + x] \delta(x + \xi(\tau) - x' - \xi(1)) \right)_{x'=x} \text{tr} U[C^2 + y] \right) \]

The first term looks like the r.h.s of (24) with $W_c[C^2]$ being a spectator only. Using the $\delta$-function to perform the $y$-integration and rearranging the group indices using (23), the second term simplifies to

\[ \text{term 2} = - \frac{g^2}{N^2V} \int d\eta_{\nu}(\sigma) \langle \int dx \left( P_{\sigma} e^{i \int_0^1 A(x+\xi)d\xi} \right)_{nm} \left( P_{\sigma} e^{i \int_0^1 A(x+\xi)-\eta(\sigma)+\eta(\omega)d\eta(\omega)} \right)_{mp} \rangle. \]

A priori the first multiplication is no star product, but again due to the $x$-integration we can replace it by a star multiplication. Then both the star and matrix multiplications line up in such a way as to form the Wilson factor for first going along a shifted version of $C^2$ (from $\eta$ back to $\eta$) and then along $C^1$ (from $\xi$ back to $\xi$), i.e.

\[ \frac{1}{V} \frac{\partial^\mu}{\partial \sigma^{\mu\nu}(\xi)} \langle W_c[C^1]W_c[C^2] \rangle = \frac{-g^2N}{(2\pi)^D V \text{det} \theta} \int_{C^1} d\chi_{\nu} \langle W_c[C^1] W_c[C^1] \rangle W_c[C^2] \]

\[ - \frac{g^2}{NV} \int_{C^2} d\eta_{\nu} \langle W_c[C^1] \circ (C^2 + \xi - \eta) \rangle, \]

which is our equation for the correlation function of two Wilson loops. Note that $W_c[C^1] \circ (C^2 + \xi - \eta)$ can be written as

\[ \int dx \frac{1}{N} \text{tr} \left( U[x + C^1] \star e^{i\theta^{-1}(\xi-\eta)} \star U[x + C^2] \star e^{-i\theta^{-1}(\xi-\eta)} \right). \]

\[ ^8 \text{For open contours, one has to be careful with possible subtleties in the case where the variations hit one of the endpoints. There are however no such problems if the difference vector } l^\mu \text{ is held fixed.} \]
4 Perturbative check

Being very careful with the \( \delta \)-function under the star-multiplication was the reason for obtaining a loop equation (before the \( N \to \infty \) limit) differing substantially from that in the usual Yang-Mills case for commutative space-time. To illustrate this mechanism from a point of view slightly different from that above, let us look at perturbation theory. For this purpose we consider a particular diagram contributing in order \( g^4 \) to \( \frac{1}{\sqrt{N}} \int dx \text{tr}(P_s(D^\mu F_{\mu\nu}U)) \). We will show that a \( \delta \)-function, forcing the two contours on the r.h.s. of the loop equation to be closed, at finite \( N \) appears in the commutative limit \( \theta \to 0 \) only. Considering all diagrams contributing to order \( g^4 \) is beyond the scope of this paper.

The insertion of the equation of motion at some point of the contour gives a vertex which is the sum of contributions with one, two and three gauge field legs. One has

\[
D^\mu F_{\mu\nu} = \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu + \text{ terms quadratic and cubic in } A. \tag{33}
\]

We consider the insertion of the first summand at \( \tau_4 = 1 \) into the expansion of the Wilson loop up to order \( A^3 \) (with the \( \tau \)-integration restricted by \( 0 \leq \tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4 = 1, \ \xi_i = \xi(\tau_i) \))

\[
I = \frac{i^3}{V} \int dx \int \prod_{i=1}^{3} d\xi_i^{\lambda_i} \langle \partial^2 A_\nu(x + \xi_4) \star A_{\lambda_3}(x + \xi_3) \star \cdots \star A_{\lambda_1}(x + \xi_1) \rangle_{g^4} \tag{34}
\]

\[
= -\frac{i}{V} \int dx \int \prod_{i=1}^{3} d\xi_i^{\lambda_i} \exp\left(\frac{i}{2} \sum_{1 \leq i < j \leq 4} \partial_{\xi_i} \theta \partial_{\xi_j}\right) \langle \partial^2 A_\nu(x + \xi_4) A_{\lambda_3}(x + \xi_3) \cdots A_{\lambda_1}(x + \xi_1) \rangle_{g^4} + g^4 \int \prod_{i=1}^{3} d\xi_i^{\lambda_i} \exp\left(\frac{i}{2} \sum_{1 \leq i < j \leq 4} \partial_{\xi_i} \theta \partial_{\xi_j}\right) \{ (\partial^2 G_{\nu\lambda_2}(\xi_4 - \xi_2) G_{\lambda_3\lambda_1}(\xi_3 - \xi_1) + \partial^2 G_{\nu\lambda_3}(\xi_4 - \xi_3) G_{\lambda_2\lambda_1}(\xi_2 - \xi_2) \}
\]

We have denoted the gauge field propagator by \( g^2 G_{\mu\nu} \). Choosing Feynman gauge we have \( \partial^2 G_{\mu\nu}(x) = -g_{\mu\nu}\delta(x) \). Denoting by \( I_1 \) the contribution from the first summand in the curly bracket of the last line of (34), we find

\[
I_1 = ig^4 g_{\nu\lambda_2} \int \prod_{i=1}^{3} d\xi_i^{\lambda_i} \exp\left(\frac{i}{2} \sum_{1 \leq i < j \leq 4} \partial_{\xi_i} \theta \partial_{\xi_j}\right) \delta(\xi_4 - \xi_2) G_{\lambda_3\lambda_1}(\xi_3 - \xi_1)
\]

\[
= \frac{ig^4}{(2\pi)^D/2} g_{\nu\lambda_2} \int \prod_{i=1}^{3} d\xi_i^{\lambda_i} \int dk \delta(\xi_4 - \xi_2 + \theta k) \tilde{G}_{\lambda_3\lambda_1}(k) e^{ik(\xi_3 - \xi_1)} \tag{35}
\]

To simplify notation we take \( N = 1 \) here.
\( \tilde{G} \) is the Fourier transform of \( G \).

For \( \theta = 0 \) eq. (35) becomes

\[
I_1 = i g^4 g_{\nu \lambda_2} \int \prod_{i=1}^3 d\xi_i^\lambda i \delta(\xi_4 - \xi_2)G_{\lambda_3 \lambda_1}(\xi_3 - \xi_1) .
\] (36)

Then \( I_1 \) gets contributions only from points of the contour coinciding with the point where the equation of motion is inserted by the contour variation.

For \( \theta \neq 0 \) eq.(35) can be written as

\[
I_1 = \frac{ig^4}{(2\pi)^{D/2} \det \theta} \int \prod_{i=1}^3 d\xi_i^\lambda_i \tilde{G}_{\lambda_3 \lambda_1}(\theta^{-1}(\xi_2 - \xi_4)) e^{i(\xi_3 - \xi_1)\theta^{-1}(\xi_2 - \xi_4)} .
\] (37)

Obviously, now all points of the contour contribute to \( I_1 \).

Since both eq. (36) (for \( \theta = 0 \)) and eq. (37) are simple reformulations of eq. (35) we see that the limit \( \theta \to 0 \) is smooth.

Throughout this paper we have omitted questions of renormalisation. In ordinary Yang-Mills theory, the renormalisation of Wilson loops is completely understood [17, 18, 19, 20, 21]. On the other hand loop equations have been derived in a satisfactory manner in the presence of some regularisation only, although some remarks were made on the problems of such equations for renormalised Wilson loops [17, 19, 22, 23]. In the present context, addressing these issues will require a better understanding of the perturbative behaviour found in [8, 9, 10].

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Appendix

For any two functions \( f, g \) with suitable boundary conditions, the identity

\[
\int dx f(x) \delta(x + \xi(\tau) - x' - \zeta) g(x)|_{x=x'} = \frac{1}{(2\pi)^D \det \theta} \int dx f(x) e^{-ikx} \int dy g(y) e^{iky} ,
\] (38)
where \( k_{\xi(\tau)} \equiv \theta^{-1}(\zeta - \xi(\tau)) \), is established as follows. Taking the star products on the l. h. s. yields

\[
\int dx \frac{dp_1}{(2\pi)^{D/2}} \frac{dp_2}{(2\pi)^D} \frac{dp_3}{(2\pi)^{D/2}} e^{-\frac{i}{2}(p_1 \theta p_2 + p_2 \theta p_1 + p_1 \theta p_3)} e^{ip_2(x + \xi(\tau) - x' - \zeta)} \tilde{f}(p_1) \tilde{g}(p_3).
\]

(39)

Setting \( x = x' \) and integrating over \( x \) imposes \( p_1 = -p_3 \). Upon further integration over \( p_3 \) one finds

\[
\int \frac{dp_1}{(2\pi)^{D/2}} \frac{dp_2}{(2\pi)^{D/2}} e^{-ip_1 \theta p_2} e^{ip_2(\xi(\tau) - \zeta)} \tilde{f}(p_1) \tilde{g}(-p_1).
\]

(40)

Integrating over \( p_2 \) imposes \( \theta p_1 = -(\xi(\tau) - \zeta) \), so we are left with

\[
\int dp_1 \delta(\theta p_1 + \xi(\tau) - \zeta) \tilde{f}(p_1) \tilde{g}(-p_1) = \frac{1}{\det \theta} \tilde{f}(\theta^{-1}k_{\xi}) \tilde{g}(-\theta^{-1}k_{\xi}),
\]

(41)

which is identical to the r. h. s. of the desired result (38).

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