The Lipkin Model in Many-Fermion System as an Example of the $su(1, 1) \otimes su(1, 1)$-Algebraic Model

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Abstract

Following the idea, recently, proposed by the present authors for the two-level pairing model, the Lipkin model is reexamined. It is a natural generalization of the method already developed by the present authors with Kuriyama. This model is a schematic model for many-fermion system and obeys the $su(2)$-algebra. It is shown that the use of the Schwinger boson representation, the model is expressed in terms of the $su(1, 1) \otimes su(1, 1)$-algebra and with the aid of the MYT mapping method, it is disguised from the original form in terms of the Holstein-Primakoff representation. Further, under various coherent states, the classical counterparts are derived. It is concluded that the Lipkin model can be treated in the common ring as that of the two-level pairing model.
§1. Introduction

The Lipkin model was originally proposed by Lipkin, Meshkov and Glick\(^1\) and has been widely used to test several kinds of many-body theories of strongly interacting fermion systems, for instance, the time-dependent Hartree-Fock method,\(^2\) finite temperature dynamics,\(^3,4\) collective dynamics of many-fermion system,\(^5,6\) phase transitions, spin tunnelling,\(^7\) etc. This statement can be found in the paper by the present authors with Kuriyama.\(^8\) Further, they have investigated the Lipkin model for thermal effect in Ref. 9). With the aim of participating in these researches under a certain new viewpoint for the Lipkin model, Ref. 8) was published. On the basis of a new boson realization proposed by the present authors with Kuriyama,\(^10\) various numerical results were presented in Ref. 8)

It is well known that, as was already mentioned, the Lipkin model is a schematic model for understanding the dynamics of many-fermion systems and it obeys the $su(2)$-algebra. The interaction term in the Hamiltonian is expressed in terms of the quadratic form with respect to the raising and the lowering operator in the $su(2)$-generators. On the other hand, we know that the $su(2)$-generators can be expressed in terms of the bilinear forms of two kinds of boson creation and annihilation operator, i.e., the Schwinger boson representation. Therefore, we can apply the Schwinger boson representation to the Lipkin model and the interaction term can be re-formed in terms of the product of the boson-pair creation and annihilation. The boson pairs and the boson number operator form the $su(1, 1)$-algebra, and then, the Lipkin model is formulated in the frame of the $su(1, 1) \otimes su(1, 1)$-algebra. With the aid of the MYT mapping method,\(^11\) we can transcribe the system in the Holstein-Primakoff representation for the $su(1, 1)$-algebra, and in Refs. 8) and 10), a certain special case was treated.

In response to the above situation, very recently, the present authors proposed a viewpoint that the two-level pairing model in many-fermion systems is also described in terms of the $su(1, 1) \otimes su(1, 1)$-algebra.\(^12,13\) Hereafter, Ref. 13) is referred to as (I). The two-level pairing model is a kind of the $su(2) \otimes su(2)$-algebraic model, and with the aid of four kinds of bosons, we can treat this model in the Schwinger boson representation. In this sense, also, by applying an idea similar to that adopted in Refs. 8) and 10), we can show that the present system is reduced to that obeying the $su(1, 1) \otimes su(1, 1)$-algebra.\(^12\)

A main aim of the present paper is to show again that the Lipkin model is a kind of the $su(1, 1) \otimes su(1, 1)$-algebraic model. Of course, the basic idea is similar to that adopted in Ref.12). In contrast to the case of the two-level pairing model, the $su(1, 1)$-generators in the Lipkin model consist of boson-pairs and boson number operators. Therefore, the structures of the $su(1, 1)$-algebra are different from each other. The present case is characterized by
the quantum numbers \( t = 1/4 \) and 3/4 and in Refs. 8) and 10), only the case \( t = 1/4 \) was treated. As was shown in Ref. 12), the two-level pairing model is characterized by the quantum numbers \( t = 1/2, 1, 3/2, \ldots \). This difference is not so important for applying the MYT boson mapping method to obtain the Holstein-Primakoff representation. However, it influences us into constructing boson coherent states which lead to the classical counterpart of the models. The two-level pairing and the Lipkin model are expressed in terms of four and two kinds of bosons, respectively. Therefore, compared with the two-level pairing model, a possibility to construct boson coherent state in the Lipkin model is limited and a special device is necessary. Under the above-mentioned scheme, we arrive at the following conclusion: We can describe the Lipkin model and the two-level pairing model in common ring. Of course, the comparison with the result based on the \( su(2) \otimes su(2) \)-coherent state is necessary.

In §2, the Lipkin model in many-fermion systems is recapitulated with some new aspects and in §3, the Schwinger boson representation is described. Section 4 is devoted to reforming the Lipkin model in the Schwinger boson representation. Of course, the MYT mapping method and a certain boson coherent state, which is called the \( su(1,1) \otimes su(1,1) \)-coherent state, play a central role and the model can be expressed in terms of two kinds of new bosons. In §5, the model is formulated in the frame of one kind of boson. Finally, in §6, the comparison with the result based on the \( su(2) \otimes su(2) \)-coherent state is sketched. In Appendix A, a possible idea for constructing the orthogonal set for the Lipkin model in the fermion space is demonstrated. In Appendix B, the boson-pair coherent state and a certain new boson coherent state are formulated.

\section*{§2. The original form of the Lipkin model in the fermion space}

The model discussed in this paper is the Lipkin model, which enables us to obtain a schematic understanding of the problems mentioned in the opening paragraph of §1. The framework of this model consists of two single-particle levels with the same degeneracies \( \Omega = 2j + 1 \) (\( \Omega \); even integer, \( j \); half-integer), in which \( N \) fermions are moving a certain type of the interaction. We specify the upper and the lower level by \( \sigma = + \) and \( \sigma = - \), respectively. The Hamiltonian \( \hat{H} \) is expressed in the form

\[ \hat{H} = \epsilon \hat{S}_0 - (G/4) \left[ (\hat{S}_+)^2 + (\hat{S}_-)^2 \right] . \]  

(2.1)

Here, \( \hat{S}_{\pm,0} \) are defined as

\[ \hat{S}_+ = \hbar \sum_{m=-j}^{j} \hat{c}_m^* (+) \hat{c}_m (-) , \quad \hat{S}_- = \hbar \sum_{m=-j}^{j} \hat{c}_m^* (-) \hat{c}_m (+) , \]
\[ \hat{S}_0 = \left(\frac{\hbar}{2}\right) \sum_{m=-j}^{j} (\hat{c}_m^*(+)\hat{c}_m(+) - \hat{c}_m^*(-)\hat{c}_m(-)) . \] (2.2)

The parameters \( \epsilon \) and \( G \) denote the difference of the single-particle energies between the two levels and the strength of the interaction, respectively. The set \((\hat{c}_m(\sigma), \hat{c}_m^*(\sigma))\) represents fermion operator in the single-particle state \((\sigma, j, m; m = -j, -j + 1, \cdots, j - 1, j)\). Total fermion number operator \( \hat{N} \) is given as

\[ \hat{N} = \sum_{m=-j}^{j} (\hat{c}_m^*(+)\hat{c}_m(+) + \hat{c}_m^*(-)\hat{c}_m(-)) . \] (2.3)

We know that the set \((\hat{S}_\pm, 0)\) obeys the \( su(2) \)-algebra, and then, the Hamiltonian \( \hat{H} \) is diagonalized in the frame of the orthogonal set obtained in the well-known procedure. In this sense, it may be not necessary to repeat the discussion on the eigenvalue problem for the Lipkin model. However, as was mentioned in \( \S 1 \), we will describe this model in the framework of the Schwinger boson representation and the quantum numbers characterizing this model formally do not connect with those appearing in this model as a many-fermion system we are discussing in this section. One of the typical examples may be total fermion number, which is not contained in the Schwinger boson representation. Therefore, inevitably, focusing on the quantum numbers, we have to repeat the discussion on the eigenvalue problem.

If the state with the minimum weight, which we denote \(|m\rangle\), is derived, we can construct the orthogonal set by operating the raising operator \( \hat{S}_+ \) successively on \(|m\rangle\). In the present case, \(|m\rangle\) should satisfy the condition

\[ \hat{S}_-|m\rangle = 0 , \] (2.4a)
\[ \hat{S}_0|m\rangle = -S|m\rangle , \quad (S = \hbar s; s = 0, 1/2, 1, \cdots) \] (2.4b)
\[ \hat{N}|m\rangle = N|m\rangle , \quad (N = 0, 1, 2, \cdots, 2\Omega) \] (2.4c)

In order to understand the structure of \(|m\rangle\) obeying the condition (2.4), first, we set up the following form of \(|m\rangle\):

\[ |m\rangle = \hat{X}_\nu^*\hat{Y}_s^*|0\rangle . \] (2.5)

Here, \( \hat{X}_\nu^* \) and \( \hat{Y}_s^* \) are operators which should satisfy the condition

\[ [\hat{S}_\pm, \hat{X}_\nu^*] = 0 , \quad [\hat{S}_0, \hat{X}_\nu^*] = 0 , \] (2.6a)
\[ [\hat{N}, \hat{X}_\nu^*] = 2\nu\hat{X}_\nu^* , \quad (\nu; \text{positive integer}) , \] (2.6a)
\[ [\hat{S}_-, \hat{Y}_s^*] = 0 , \] (2.6b)
\[ [\hat{S}_0, \hat{Y}_s^*] = -\hbar s\hat{Y}_s^* , \] (2.6b)
\[ [\hat{N}, \hat{Y}_s^*] = 2s\hat{Y}_s^* , \] (2.6b)
\[ \hat{X}_\nu\hat{Y}_s^*|0\rangle = 0 . \] (2.6c)
If $\hat{X}_\nu^*$ and $\hat{Y}_s^*$ obey the condition (2.6), we can prove easily that the form (2.5) satisfies the condition (2.4). In order to investigate $\hat{X}_\nu^*$ and $\hat{Y}_s^*$ in rather concrete form, we note

\[
[ \hat{S}_\pm, \hat{c}_m^*(\pm) ] = 0, \quad [ \hat{S}_\mp, \hat{c}_m^*(\pm) ] = \hbar \hat{c}_m^*(\mp),
\]

\[
[ \hat{S}_0, \hat{c}_m^*(\pm) ] = \pm (\hbar/2) \hat{c}_m^*(\pm), \quad [ \hat{N}, \hat{c}_m^*(\pm) ] = \hat{c}_m^*(\pm). \tag{2.7}
\]

We can see that the operator $\hat{c}_m^*(\pm)$ is spinor with respect to ($\hat{S}_\pm, 0$). With the use of the relation (2.7), we are permitted to set up

\[
\hat{X}_\nu^* = \sum_{(k,l)} c(k_1, \ldots, k_\nu; l_1, \ldots, l_\nu) \hat{c}_{k_1}^*(+) \cdots \hat{c}_{k_\nu}^*(+) \hat{c}_{l_1}^*(-) \cdots \hat{c}_{l_\nu}^*(-), \tag{2.8a}
\]

\[
\hat{Y}_s^* = \hat{c}_{m_1}^*(-) \cdots \hat{c}_{m_2}^*(-). \tag{2.8b}
\]

Here, $c(k_1, \ldots, k_\nu; l_1, \ldots, l_\nu)$ may be determined by appropriate method, which is sketched in Appendix A, but, in this paper, the explicit form is not necessary. Clearly, there exists the relation

\[
\nu = N/2 - s. \tag{2.9}
\]

Then, we have the restriction

\[
0 \leq \nu \leq \Omega, \quad 0 \leq \nu + 2s \leq \Omega. \tag{2.10}
\]

The relation (2.10) gives us $0 \leq \nu \leq \Omega - 2s$ and using the relation (2.9), we have

\[
0 \leq s \leq N/2, \quad 0 \leq s \leq \Omega - N/2. \tag{2.11}
\]

The relation (2.11) is equivalent to

\[
\begin{align*}
\text{if} & \quad 0 \leq N \leq \Omega, & \quad 0 \leq s \leq N/2, \\
\text{if} & \quad \Omega \leq N \leq 2\Omega, & \quad 0 \leq s \leq \Omega - N/2. \tag{2.12}
\end{align*}
\]

Noting that $\nu$ is a positive integer, the relation (2.12) can be expressed as follows:

\[
\begin{align*}
\text{if} & \quad N = \text{even and } 0 \leq N \leq \Omega, & \quad s = N/2, N/2 - 1, \ldots, 1, 0, \\
\text{if} & \quad N = \text{even and } \Omega \leq N \leq 2\Omega, & \quad s = \Omega - N/2, \Omega - N/2 - 1, \ldots, 1, 0, \\
\text{if} & \quad N = \text{odd and } 0 < N < \Omega, & \quad s = N/2, N/2 - 1, \ldots, 3/2, 1/2, \\
\text{if} & \quad N = \text{odd and } \Omega < N < 2\Omega, & \quad s = \Omega - N/2, \Omega - N/2 - 1, \ldots, 3/2, 1/2. \tag{2.13}
\end{align*}
\]

The relation (2.9) tells us that $2\nu$ denotes the seniority number, and usually, the Lipkin model is treated in the case $N = \Omega$ with $s = \Omega/2$ ($\nu = 0$). The above means that the lower
level is fully occupied by $\Omega$ fermions and the upper is empty if the interaction is switched off.

As a final discussion in this section, we will contact with the problem how to construct the orthogonal set. Since $|m\rangle$ satisfies the conditions (2.4a) and (2.4b), formally, $(\hat{S}_\pm)^\sigma|m\rangle$ $(\sigma = s+s_0)$ gives us the state with the eigenvalue $(s, s_0; s_0 = -s, -s+1, \cdots s-1, s)$. However, further consideration is necessary. The Hamiltonian (2.1) consists of $\hat{S}_0^2, \hat{S}_+^2$ and $\hat{S}_-^2$. Therefore, the matrix elements of $\hat{H}$ between two states with $\Delta s_0 = \pm 1$ always vanish. This means that the whole space specified by a given value of $s$ is divided into two groups, i.e., $(\hat{S}_+)^{2n}|m\rangle$ and $(\hat{S}_+)^{2n+1}|m\rangle$. Here, $n$ denotes $n = 0, 1, 2, \cdots$ and the normalization constants are omitted. For the group $(\hat{S}_+)^{2n}|m\rangle$, $n$ is restricted to

\[ -s \leq -s + 2n \leq s, \quad \text{i.e.,} \quad 0 \leq n \leq s. \]  

(2.14a)

On the other hand, for the group $(\hat{S}_+)^{2n+1}|m\rangle$, we have

\[ -s + 1 \leq -s + 2n + 1 \leq s, \quad \text{i.e.,} \quad 0 \leq n \leq s - 1/2. \]  

(2.14b)

Noting that $s$ is integer or half-integer, the orthogonal set for diagonalizing $\hat{H}$ is classified into four groups:

(i) $\{(\hat{S}_+)^{2n}|m\rangle; s = \text{integer}, n = 0, 1, 2, \cdots, s\}$,  

(2.15a)

(ii) $\{(\hat{S}_+)^{2n+1}|m\rangle; s = \text{integer}, n = 0, 1, 2, \cdots, s-1\}$,  

(2.15b)

(iii) $\{(\hat{S}_+)^{2n}|m\rangle; s = \text{half-integer}, n = 0, 1, 2, \cdots, s - 1/2\}$,  

(2.15c)

(iv) $\{(\hat{S}_+)^{2n+1}|m\rangle; s = \text{half-integer}, n = 0, 1, 2, \cdots, s - 1/2\}$.  

(2.15d)

Through the above-mentioned procedure, we can prepare the orthogonal set for the original form of the Lipkin model.

§3. The Lipkin model in the Schwinger boson representation

It is well known that the $su(2)$-generators in the Schwinger boson representation are expressed in terms of two kinds of bosons $(\hat{b}_+, \hat{b}_+^*)$ and $(\hat{b}_-, \hat{b}_-^*)$:

\[ \tilde{S}_+ = \hbar \hat{b}_+^* \hat{b}_-, \quad \tilde{S}_- = \hbar \hat{b}_+ \hat{b}_+^* - \hat{b}_- \hat{b}_-^*, \quad \tilde{S}_0 = (\hbar/2)(\hat{b}_+^* \hat{b}_+ - \hat{b}_-^* \hat{b}_-). \]  

(3.1)

The operator expressing the magnitude of the $su(2)$-spin is given in the form

\[ \tilde{S} = (\hbar/2)(\hat{b}_+^* \hat{b}_+ + \hat{b}_-^* \hat{b}_-). \]  

(3.2)
In the fermion space, it may be impossible to describe such a simple expression. The state with the minimum weight, which we denote as $|m\rangle$, is given as

$$|m\rangle = \left(\sqrt{(2s)!}\right)^{-1}(\hat{b}_-^*)^{2s} |0\rangle.$$  \hspace{1cm} (3.3)

The state $|m\rangle$ satisfies

$$\tilde{S}_- |m\rangle = 0,$$  \hspace{1cm} (3.4a)
$$\tilde{S}_0 |m\rangle = -S|m\rangle, \hspace{1cm} (S = \hbar s).$$  \hspace{1cm} (3.4b)

In the Schwinger boson representation, there does not exist any operator which corresponds to $\hat{N}$, and then, we cannot set up the relation such as (2.4c). The successive operation of $\tilde{S}_\pm$ on the state $|m\rangle$ gives us the orthogonal set in the Schwinger boson representation. Later, we will discuss this problem.

The Hamiltonian $\tilde{H}$ which corresponds to $\hat{H}$ shown in the form (2.1) is, of course, expressed as follows:

$$\tilde{H} = \epsilon \tilde{S}_0 - (G/4) \left[ \left(\tilde{S}_+\right)^2 + \left(\tilde{S}_-\right)^2 \right].$$  \hspace{1cm} (3.5)

Any matrix element of $\tilde{H}$ for the orthogonal set derived from $|m\rangle$ is of the same form as that of $\hat{H}$ for the orthogonal set derived from $|m\rangle$. However, the original form of the Lipkin model is characterized by the quantities $\Omega$ and $N$, which give us the restriction (2.13). On the other hand, the Lipkin model in the Schwinger boson representation does not contain such quantities. Then, introducing $\Omega$ and $N$ from the outside, we require the relation (2.13) as the condition that the magnitude of the $su(2)$-spin $s$, the eigenvalue of $\tilde{S}$ shown in the relation (3.2), should satisfy in the Schwinger boson representation. Through this requirement, both frameworks connect with each other.

Now, let us construct the orthogonal set for diagonalizing $\tilde{H}$ shown in the relation (3.5). The Hamiltonian $\tilde{H}$ also consists of $\tilde{S}_0$, $\tilde{S}_+^2$ and $\tilde{S}_-^2$. Therefore, the same argument as that given in §2 is possible. We can divide the whole space into two groups, $\{(\tilde{S}_+)^{2n}|m\rangle\}$ and $\{(\tilde{S}_+)^{2n+1}|m\rangle\}$. Except the normalization constants, both states can be rewritten as

$$\left(\tilde{S}_+\right)^{2n} |m\rangle = (\hat{b}_+^*)^{2n}(\hat{b}_+^*)^{2(s-n)} |0\rangle,$$  \hspace{1cm} (3.6a)
$$\left(\tilde{S}_+\right)^{2n+1} |m\rangle = (\hat{b}_+^*)^{2n+1}(\hat{b}_+^*)^{2(s-n)-1} |0\rangle.$$  \hspace{1cm} (3.6b)

The form (3.6a) gives us

$$2n \geq 0, \hspace{1cm} 2(s-n) \geq 0, \hspace{1cm} \text{i.e.,} \hspace{1cm} 0 \leq n \leq s.$$  \hspace{1cm} (3.7a)
Also, the form \( (3.6b) \) gives us
\[
2n + 1 \geq 0, \quad 2(s - n) - 1 \geq 0, \quad \text{i.e.,} \quad 0 \leq n \leq s - 1/2. \tag{3.7b}
\]
The relation \( (3.7) \) is identical to the relation \( (2.14) \). Under the same idea as that used in §2, the orthogonal set is also classified into four groups:

(i) \( \{ (\tilde{S}_+)^{2n} | m \rangle \} = (\hat{b}_+^2)^n (\hat{b}_-^2)^{s-n} |0\rangle \);

\[
s = \text{integer}, \ n = 0, 1, 2, \ldots, s, \tag{3.8a}
\]

(ii) \( \{ (\tilde{S}_+)^{2n+1} | m \rangle \} = (\hat{b}_+^2)^n (\hat{b}_-^2)^{s-n-1} \cdot \hat{b}_+ \hat{b}_- |0\rangle \);

\[
s = \text{integer}, \ n = 0, 1, 2, \ldots, s - 1, \tag{3.8b}
\]

(iii) \( \{ (\tilde{S}_+)^{2n} | m \rangle \} = (\hat{b}_+^2)^n (\hat{b}_-^2)^{s-n-1/2} \cdot \hat{b}_+^* |0\rangle \);

\[
s = \text{half-integer}, \ n = 0, 1, 2, \ldots, s - 1/2, \tag{3.8c}
\]

(iv) \( \{ (\tilde{S}_+)^{2n+1} | m \rangle \} = (\hat{b}_+^2)^n (\hat{b}_-^2)^{s-n-1/2} \cdot \hat{b}_+^* |0\rangle \);

\[
s = \text{half-integer}, \ n = 0, 1, 2, \ldots, s - 1/2. \tag{3.8d}
\]

From the relation \( (3.8) \), we can learn that for a given value of \( s \), the maximum number of the operation of \( \tilde{S}_+^2 \) is fixed. We also mentioned that the quantities \( \Omega \) and \( N \) are introduced from the outside and they should obey the condition \( (2.13) \). However, it may be convenient for the practical aim to rewrite the condition \( (2.13) \) in the form so as to be able to know the values of \( N \) permitted for a given value of \( s \). This can be done in the form

(a) if \( s = \text{integer} \), \( N = 2s, 2s + 2, \ldots, \Omega \),

\[
2\Omega - 2s, 2\Omega - 2s - 2, \ldots, 2\Omega, \tag{3.9a}
\]

(b) if \( s = \text{half-integer} \), \( N = 2s, 2s + 2, \ldots, \Omega - 1 \),

\[
2\Omega - 2s, 2\Omega - 2s - 2, \ldots, 2\Omega - 1. \tag{3.9b}
\]

The above is the outline of the Lipkin model in the Schwinger boson representation.

**§4. Reformulation in the form of the \( su(1,1) \otimes su(1,1) \)-algebra**

In Refs. 12) and 13), we showed that the two-level pairing model was re-formed in terms of the \( su(1,1) \otimes su(1,1) \)-algebraic model in the Schwinger boson representation. The present authors, with Kuriyama, already showed that the Lipkin model can be also re-formed in terms of the \( su(1,1) \otimes su(1,1) \)-algebraic model in the Schwinger boson representation.8),10) In this section, we recapitulate this re-formation in the notations used in Refs. 12) and 13). Of course, the formalism given in Refs. 8) and 10) is supplemented with newly added features.
First, we note the following re-form:

\[
\left(\tilde{S}_+\right)^2 = 4(\hbar/2)\hat{b}^2_+ \cdot (\hbar/2)\hat{b}^2_+ , \quad (4.1a)
\]
\[
\left(\tilde{S}_-\right)^2 = 4(\hbar/2)\hat{b}^2_- \cdot (\hbar/2)\hat{b}^2_+ , \quad (4.1b)
\]
\[
\tilde{S}_0 = (\hbar/4 + (\hbar/2)\hat{b}_+\hat{b}_-) - (\hbar/4 + (\hbar/2)\hat{b}_+\hat{b}_-) . \quad (4.1c)
\]

Then, we define the boson-pairs in the set \((\tilde{T}_{\pm,0}(\sigma))\):

\[
\tilde{T}_+(\sigma) = (\hbar/2)\hat{b}_\sigma^2 , \quad \tilde{T}_-(\sigma) = (\hbar/2)\hat{b}_\sigma^2 , \quad \tilde{T}_0(\sigma) = \hbar/4 + (\hbar/2)\hat{b}_\sigma^2 \cdot (\sigma = \pm) \quad (4.2)
\]

It is easily verified that the set \((\tilde{T}_{\pm,0}(\sigma))\) obeys the \(su(1,1)\)-algebra. With the use of the set \((\tilde{T}_{\pm,0}(\sigma);\sigma = \pm)\), the Hamiltonian \((3.5)\) can be rewritten as

\[
\tilde{H} = \epsilon(\tilde{T}_0(+) - \tilde{T}_0(-)) - G \left(\tilde{T}_+(+)\tilde{T}_-(-) + \tilde{T}_+(--)\tilde{T}_-(-)\right) . \quad (4.3)
\]

The form \((4.3)\) tells that the Lipkin model is a possible model obeying the \(su(1,1) \otimes su(1,1)\)-algebra.

Our concern is to compare the Lipkin model with the two-level pairing model in the framework of the \(su(1,1) \otimes su(1,1)\)-algebra. For this purpose, further, we re-form the present frame. The state with the minimum weight, \(|m(\sigma)\rangle\rangle\), for the algebra \((\tilde{T}_{\pm,0}(\sigma))\) is determined under the condition

\[
\tilde{T}_-(\sigma)|m(\sigma)\rangle\rangle = 0 , \quad (4.4a)
\]
\[
\tilde{T}_0(\sigma)|m(\sigma)\rangle\rangle = T(\sigma)|m(\sigma)\rangle\rangle , \quad (T(\sigma) = \hbar t_\sigma) . \quad (4.4b)
\]

The condition \((4.4)\), with the definition \((4.2)\), gives

\[
|m(\sigma)\rangle\rangle = (\hat{b}_\sigma^*2t_\sigma-1/2|0\rangle\rangle = \begin{cases} |0\rangle\rangle , & \text{for } t_\sigma = 1/4 , \\ \hat{b}_\sigma^*|0\rangle\rangle , & \text{for } t_\sigma = 3/4 . \end{cases} \quad (4.5)
\]

It should be noted that there does not exist any other type. Then, the state with the minimum weight in the present \(su(1,1) \otimes su(1,1)\)-algebra is specified by \((t_+, t_-)\) and given in the form

\[
|t_+, t_-\rangle\rangle = (\hat{b}_+^*2t_+-1/2(\hat{b}_-^*2t_-1/2|0\rangle\rangle . \quad (4.6)
\]

Then, we can construct the orthogonal set by operating \(\tilde{T}_+(+)\) and \(\tilde{T}_+(-)\) successively in the form

\[
|\kappa_+, \kappa_-; t_+, t_-\rangle\rangle = \left(\sqrt{2\kappa_+ + 2t_+ - 1/2}!(2\kappa_- + 2t_- - 1/2)!\right)^{-1} \times (\hat{b}_+^*2\kappa_-|t_+, t_-\rangle\rangle , \quad \kappa_+, \kappa_- = 0, 1, 2, \cdots . \quad (4.7)
\]
It may be interesting to compare the state (4.7) with the form (3.8). We can see that both are identical with each other under the following correspondence:

(i) \( t^+ = 1/4 \), \( t^- = 1/4 \), \( \kappa^+ = n \), \( \kappa^- = s - n \), \( (4.8a) \)
(ii) \( t^+ = 3/4 \), \( t^- = 3/4 \), \( \kappa^+ = n \), \( \kappa^- = s - n - 1 \), \( (4.8b) \)
(iii) \( t^+ = 1/4 \), \( t^- = 3/4 \), \( \kappa^+ = n \), \( \kappa^- = s - n - 1/2 \), \( (4.8c) \)
(iv) \( t^+ = 3/4 \), \( t^- = 1/4 \), \( \kappa^+ = n \), \( \kappa^- = s - n - 1/2 \). \( (4.8d) \)

We know that the quantum number \((t^+, t^-)\) plays a role of classifying the four cases (i) ∼ (iv).

Next, as was done in Ref. 13), we apply the MYT mapping method\(^{11}\) to the present system. For this purpose, we prepare the other boson space constructed by new boson \((\hat{c}_\sigma, \hat{c}_\sigma^*; \sigma = \pm)\). The orthogonal set is given as

\[
|\kappa^+, \kappa^-\rangle = \left( \sqrt{\kappa^+! \kappa^-!} \right)^{-1} (\hat{c}_+^\kappa (\hat{c}_-^\kappa)^\kappa |0\rangle .
\] \( (4.9) \)

Then, the MYT mapping operator \(\hat{U}\) is defined in the form

\[
\hat{U} = \sum_{\kappa^+, \kappa^-} |\kappa^+, \kappa^-\rangle \langle \kappa^+, \kappa^-; t^+, t^-| .
\] \( (4.10) \)

With the use of \(\hat{U}\), we have the following relations:

\[
\hat{U} \left( \hat{T}_0(+) - \hat{T}_0(-) \right) \hat{U}^\dagger = T(+(-) - T(-) + \hbar \hat{c}_+^\kappa \hat{c}_-^\kappa - \hbar \hat{c}_-^\kappa \hat{c}_+^\kappa ,
\] \( (4.11) \)
\[
\hat{U} \tilde{T}_0(+) \tilde{T}_0(-) \hat{U} = \sqrt{2T(-) + \hbar \hat{c}_-^\kappa \hat{c}_-^\kappa \cdot \hbar \hat{c}_+^\kappa \hat{c}_+^\kappa \cdot \sqrt{2T(+)} + \hbar \hat{c}_+^\kappa \hat{c}_+^\kappa ,
\] \( (4.12) \)
\[
T(+) = \hbar t^+ , \quad T(-) = \hbar t^- .
\] \( (4.13) \)

With the help of the forms \( (4.11) \) and \( (4.12) \), the Hamiltonian \( (4.3) \) can be mapped to the following:

\[
\hat{H} = \epsilon(T(+(-) - T(-) + \epsilon(\hbar \hat{c}_+^\kappa \hat{c}_-^\kappa - \hbar \hat{c}_-^\kappa \hat{c}_+^\kappa )
\]
\[
- G \left( \sqrt{2T(-) + \hbar \hat{c}_-^\kappa \hat{c}_-^\kappa \cdot \hbar \hat{c}_+^\kappa \hat{c}_+^\kappa \cdot \sqrt{2T(+)} + \hbar \hat{c}_+^\kappa \hat{c}_+^\kappa } + \sqrt{2T(+)} + \hbar \hat{c}_+^\kappa \hat{c}_+^\kappa \cdot \hbar \hat{c}_-^\kappa \hat{c}_-^\kappa \cdot \sqrt{2T(-)} + \hbar \hat{c}_-^\kappa \hat{c}_-^\kappa ) .
\] \( (4.14) \)

The magnitude of the \( su(2)\)-spin, \( \tilde{S} \), shown in the relation \( (3.2) \) is mapped to

\[
\tilde{S} = \hat{U} \tilde{S} \hat{U}^\dagger = T(+) + T(-) - \hbar/2 + 2 \cdot (\hbar/2)(\hat{c}_+^\kappa \hat{c}_+^\kappa + \hat{c}_-^\kappa \hat{c}_-^\kappa ) .
\] \( (4.15) \)
Thus, we can construct the Lipkin model in the framework of the $su(1,1) \otimes su(1,1)$-algebra in the Holstein-Primakoff representation.

Comparison of the form (1.14) with the Hamiltonian (1.13) is interesting. Except a certain term related to $\hat{h}\hat{c}_+^\dagger \hat{c}_+$ and $\hat{h}\hat{c}_-^\dagger \hat{c}_-$, both coincide with each other. Of course, the form (1.13) is characterized by

$$T(\sigma) = \hbar/2, \hbar, 3\hbar/2, \cdots .$$

The present case is characterized by $T(\sigma) = \hbar/4$ and $3\hbar/4$. From the above argument, we can conclude that the two-level pairing and the Lipkin model can be treated in the common ring. The Hamiltonian is expressed as

$$\hat{H} = \epsilon(T(+) - T(-)) + \epsilon(\hbar\hat{c}_+^\dagger \hat{c}_+ - \hbar\hat{c}_-^\dagger \hat{c}_-)$$

$$-2g(T(+) \cdot \hbar\hat{c}_+^\dagger \hat{c}_+ + T(-) \cdot \hbar\hat{c}_-^\dagger \hat{c}_-) - 2g \cdot \hbar\hat{c}_+^\dagger \hat{c}_+ \cdot \hbar\hat{c}_-^\dagger \hat{c}_-$$

$$-G\left(\sqrt{2T(-)} + \hbar\hat{c}_-^\dagger \hat{c}_- \cdot \hbar\hat{c}_-^\dagger \hat{c}_- \cdot \sqrt{2T(+) + \hbar\hat{c}_+^\dagger \hat{c}_+} \right)$$

$$+ \sqrt{2T(+) + \hbar\hat{c}_+^\dagger \hat{c}_+ \cdot \hbar\hat{c}_-^\dagger \hat{c}_- \cdot \sqrt{2T(-) + \hbar\hat{c}_-^\dagger \hat{c}_-} \right) .$$

The cases $g = G$ and $g = 0$ correspond to the two-level pairing and the Lipkin model, respectively. As was shown in (1), the Hamiltonian (4.17) can be treated in the $su(2)$-algebra.

Finally, we will show that a certain wave packet defined in the form (4.21), which we call the $su(1,1) \otimes su(1,1)$-coherent state, presents a classical counterpart of the Lipkin model re-formed in this section. First, extending the state (3.10), we introduce the following state:

$$|c_0\rangle = \left(\frac{\sqrt{U_+}}{\hbar U_+} \right)^{-1} \exp \left(\frac{V_+}{\hbar U_+} \tilde{T}_+(+)\right) \cdot \left(\frac{\sqrt{U_-}}{\hbar U_-} \right)^{-1} \exp \left(\frac{V_-}{\hbar U_-} \tilde{T}_+(-)\right) |0\rangle , \quad (4.18)$$

$$U_\sigma = \sqrt{1 + |V_\sigma|^2} . \quad (\sigma = \pm) \quad (4.19)$$

Here, $V_\sigma$ denotes a complex parameter. The state $|c_0\rangle$ is a vacuum of boson operator $(\hat{\beta}_\sigma, \hat{\beta}_\sigma^*)$, i.e., $\hat{\beta}_\sigma |c_0\rangle = 0$:

$$\hat{\beta}_\sigma = U_\sigma \hat{b}_\sigma - V_\sigma \hat{b}_\sigma^* ,$$

$$\hat{\beta}_\sigma^* = -V_\sigma^* \hat{b}_\sigma + U_\sigma \hat{b}_\sigma^* , \quad (4.20a)$$

i.e.,

$$\hat{b}_\sigma = U_\sigma \hat{\beta}_\sigma - V_\sigma \hat{\beta}_\sigma^* ,$$

$$\hat{b}_\sigma^* = V_\sigma^* \hat{\beta}_\sigma + U_\sigma \hat{\beta}_\sigma^* . \quad (4.20b)$$

With the use of $|c_0\rangle$ and $\hat{\beta}_\sigma^*$, we define the $su(1,1) \otimes su(1,1)$-coherent state $|c^0\rangle$ in the form

$$|c^0\rangle = \left(\sqrt{3/2 - 2t_+} + \sqrt{2t_+ - 1/2\hat{\beta}_+^*}\right) \left(\sqrt{3/2 - 2t_-} + \sqrt{2t_- - 1/2\hat{\beta}_-^*}\right) |c_0\rangle . \quad (4.21)$$
The state (4.21) is a possible extension of the state (3.13) to the case of the \( su(1, 1) \otimes su(1, 1) \)-algebra. By calculating the expectation value of \( \hat{H} \) given in the relation (4.3) for \( |c^0\rangle \) and using various relations shown in Appendix, we have the following form:

\[
H^0 = \epsilon(T(+)-T(-)) + \epsilon(hc^*_+,c_- - hc^*_-,c_+)
- G(\sqrt{2T(-)+hc^*_-,c_- \cdot hc^*_+,c_-} \cdot \sqrt{2T(+)+hc^*_+,c_+})
+ 2\sqrt{2T(+)+hc^*_+,c_+ \cdot hc^*_-,c_-} \cdot \sqrt{2T(-)+hc^*_-,c_-}).
\]

(4.22)

We can see that \( H^0 \) is equivalent to \( \hat{H} \) shown in the relation (4.14). Combining the above result with that derived in (I), we obtain the classical Hamiltonian of \( \hat{H} \) given in the relation (4.17). Of course, for the present case, \( T(\sigma) \) should be used as \( T(\sigma) = \hbar/4 \) or \( 3\hbar/4 \).

**§5. Re-formation in terms of one kind of boson operator**

As was discussed in (I), the Hamiltonian (4.14) is expressed in terms of the Schwinger boson representation for the \( su(2) \)-algebra:

\[
\hat{M}_+ = \hbar \hat{c}^*_+ \hat{c}_- , \quad \hat{M}_- = \hbar \hat{c}^*_- \hat{c}_+ , \quad \hat{M}_0 = (h/2)(\hat{c}^*_+ \hat{c}_+ - \hat{c}^*_- \hat{c}_-) ,
\]

(5.1)

\[
\hat{M} = (h/2)(\hat{c}^*_+ \hat{c}_+ + \hat{c}^*_- \hat{c}_-).
\]

(5.2)

Then, in the same idea as that shown in (I), we can re-form our present system in terms of one kind of boson. For example, we obtain the following form, which is the same as that shown in the relation (1-30):

\[
\Rightarrow \hbar \hat{c}^* \sqrt{2T(+)+hc^*_+,c_+} \cdot \sqrt{2T(-)+hc^*_-,c_-} \cdot \sqrt{2T(+)+hc^*_+,c_+}
\]

(5.3)

Using the canonical transformation (\( \sqrt{hc_+} = \sqrt{hc e^{-i\chi/2}} \), \( \sqrt{hc_-} = \sqrt{2M - hc e^{-i\chi/2}} \)) adopted in (I), the classical counterpart obtained by the state \( |c^0\rangle \) shown in the relation (4.21) can be re-formed in terms of one kind of boson-type canonical variable. For example, we have

\[
\sqrt{2T(-)+hc^*_-,c_-} \cdot hc^*_+,c_- \cdot \sqrt{2T(+)+hc^*_+,c_+}
\]

\[
\Rightarrow \hbar \hat{c}^* \sqrt{2T(+)+hc^*_+,c_+} \cdot \sqrt{2T(-)+hc^*_-,c_-} \cdot \sqrt{2T(+)+hc^*_+,c_+}
\]

(5.4)

We can see that the form (5.4) does not agree with that obtained by the \( c \)-number replacement of the relation (5.3). This is also in the same situation as that in the case of the two-level pairing model. The form (5.4) is obtained by the \( su(1, 1) \otimes su(1, 1) \)-coherent state...
and the other by the \( su(2) \otimes su(1, 1) \)-coherent state. Then, we have a question: What type of the coherent state reproduces the \( c \)-number replaced form of the relation (5.3)?

In order to give an answer to the above question, let us remember the case of the two-level pairing model. In this case, four kinds of bosons are divided into two groups. The \( su(2) \otimes su(1, 1) \)-coherent state is constructed under the following idea: This state consists of the product of two parts. First consists of the exponential form for the raising operator of the \( su(1, 1) \)-algebra and the second is expressed in terms of the Glauber form for two kinds of bosons. In the present case, the first part can be expressed in the same form as the above. However, if adopting the \( su(2) \otimes su(2) \)-coherent state for the second, we cannot derive the expected form. Under the above mentioned background, we adopt the following coherent state:

\[
|c_0'\rangle = \left( \sqrt{U_+} \right)^{-1} \exp \left( \frac{V_+}{\hbar U_+} \tilde{T}_+(+) \right) \left( \sqrt{U_-} \right)^{-1} \exp \left( \frac{\gamma_-}{|\gamma_-| \hbar U_-} \tilde{T}_+(-) + \sqrt{\frac{\gamma_-}{U_-}} b_+^* \right) |0\rangle .
\] (5.5)

Here, \( V_+ \) and \( \gamma_- \) are complex, but \( V_- \) is real and \( U_+ \) and \( U_- \) are given as

\[
U_+ = \sqrt{1 + |V_+|^2} , \quad U_- = \sqrt{1 + V_-^2} .
\] (5.6)

The detail properties of the state \( |c_0'\rangle \) are discussed separately in Appendix B. Further, we define the state \( |c_0'\rangle \) in the form

\[
|c_0'\rangle = \left( \sqrt{3/2 - 2t_+} + \sqrt{2t_+ - 1/2\beta_+^*} \right) \left( \sqrt{3/2 - 2t_-} + \sqrt{2t_- - 1/2\beta_-^*} \right) |c_0'\rangle .
\] (5.7)

Using the various relations shown in §4 and Appendix B and noting the relation \( M = (\hbar/2)(c_+^* c_+ + c_-^* c_-) \), we get the following result for the Hamiltonian \( H_0' \):

\[
\begin{align*}
H_0' &= &\langle c_0'|\tilde{H}|c_0'\rangle \\
&= &\epsilon(T(+)-T(-)-M) + 2\epsilon \cdot \hbar c^* c \\
& &-G \left( \sqrt{\hbar c^*} \cdot \sqrt{2T(+)} + \hbar c^* c \sqrt{2M - \hbar c^* c} \sqrt{2T(-)} + 2M - \hbar - \hbar c^* c \\
& &+ \sqrt{2T(-)} + 2M - \hbar - \hbar c^* c \sqrt{2M - \hbar c^* c} \sqrt{2T(+)} + \hbar c^* c \cdot \sqrt{\hbar c} \right) .
\end{align*}
\] (5.8)

Of course, the relation (5.3) presents us the quantized form of \( H_0' \):

\[
\begin{align*}
\tilde{H}' &= &\epsilon(T(+)-T(-)-M) + 2\epsilon \cdot \hbar \tilde{c}^* \tilde{c} \\
& &-G \left( \sqrt{\hbar \tilde{c}^*} \cdot \sqrt{2T(+)} + \hbar \tilde{c}^* \tilde{c} \sqrt{2M - \hbar \tilde{c}^* \tilde{c}} \sqrt{2T(-)} + 2M - \hbar - \hbar \tilde{c}^* \tilde{c} \\
& &+ \sqrt{2T(-)} + 2M - \hbar - \hbar \tilde{c}^* \tilde{c} \sqrt{2M - \hbar \tilde{c}^* \tilde{c}} \sqrt{2T(+)} + \hbar \tilde{c}^* \tilde{c} \cdot \sqrt{\hbar \tilde{c}} \right) .
\end{align*}
\] (5.9)
The Hamiltonians $H^0$ and $\hat{H}'$ contain the quantity $M$, which is an eigenvalue of $\hat{M}$ defined in the relation (5·2). Further, we note the relation (4·15) and $\hat{S}$ is related to $\tilde{S}$ defined in the relation (3·2). Therefore, we have the following relation:

$$ S = T(+) + T(-) - \hbar/2 + 2M. \quad (5·10) $$

Through the relation (3·9), $M$ connects with $\Omega$ and $N$ in the original fermion system. For example, the most popular condition ($s = N, N = \Omega, T(+) = T(-) = \hbar/4$) gives

$$ M = (\hbar/2)\Omega. \quad (5·11) $$

As was mentioned in §2, the case (5·11) shows that $\Omega$ fermions occupy fully in the lower level, if the interaction is switched off.

§6. Discussion

Finally, in this discussion, we will sketch the results obtained in the frame of the $su(2) \otimes su(2)$-coherent state $|c''_0\rangle \rangle$, which is expressed as follows:

$$ |c''_0\rangle \rangle = e^{-|c_+|^2} \exp\left(\sqrt{2c_+}c_+|b_+^*\rangle\right) \cdot e^{-|c_-|^2} \exp\left(\sqrt{2c_-}c_-|b_-^*\rangle\right) |0\rangle \rangle. \quad (6·1) $$

Here, $(c_\sigma, c_\sigma^*)$ plays the same role as that in the two cases we already discussed; the boson-type canonical variables. The state $|c''_0\rangle \rangle$ satisfies

$$ \hat{b}_\sigma|c''_0\rangle \rangle = \sqrt{2c_\sigma}c_\sigma|c''_0\rangle \rangle \quad (\sigma = \pm) \quad (6·2) $$

Differently from the previous two cases, we must pay a special attention on the present one. The expectation values of $\tilde{T}_\pm(\sigma)$ are given by

$$ \langle \langle c''_0|\tilde{T}_+(\sigma)|c''_0\rangle \rangle = \sqrt{\hbar}c_\sigma^* \sqrt{\hbar}|c_\sigma|^2, \quad \langle \langle c''_0|\tilde{T}_-(\sigma)|c''_0\rangle \rangle = \sqrt{\hbar}|c_\sigma|^2 \cdot \sqrt{\hbar}c_\sigma. \quad (6·3) $$

However, for $\langle \langle c''_0|\tilde{T}_0(\sigma)|c''_0\rangle \rangle$, formally, we have

$$ \langle \langle c''_0|\tilde{T}_0(\sigma)|c''_0\rangle \rangle = \hbar/4 + \hbar|c_\sigma|^2. \quad (6·4) $$

The problem concerning the expression is that we should include the term $\hbar/4$ into the expression (6·4) or not. In Ref. 14), quantal fluctuation around the expectation value calculated by $|c'_0\rangle$ was discussed. If we follow the conclusion of Ref. 14), the term $\hbar/4$ should be included in the quantal fluctuations. Then, we should set up

$$ \langle \langle c''_0|\tilde{T}_0(\sigma)|c''_0\rangle \rangle = \hbar|c_\sigma|^2. \quad (6·5) $$
However, for the calculation of the expectation value of the Hamiltonian $\tilde{H}$, we can escape from this problem, because $\tilde{H}$ consists of the difference $(\hbar \hat{b}_+^* \hat{b}_+ - \hbar \hat{b}_-^* \hat{b}_-)$ and the term $\hbar/4$ disappears.

From the above argument, we have the Hamiltonian

$$H'' = \langle \langle c''_0 | \tilde{H} | c''_0 \rangle \rangle = \epsilon \cdot (\hbar c^*_+ c_- - \hbar c^*_- c_+) - G \left( \sqrt{\hbar} c^*_- c_- \sqrt{\hbar} c^*_+ c_+ + \sqrt{\hbar} c^*_+ c_+ \sqrt{\hbar} c^*_- c_- \right).$$ \hspace{1cm} (6.6)

If taking into account the relation $M = (\hbar/2)(c^*_+ c_+ + c^*_- c_-)$, we have another form

$$H'' = -\epsilon M + 2\epsilon \cdot \hbar c^* c - G \left( \sqrt{\hbar} c^* \cdot \sqrt{\hbar} c \left( 2M - \hbar c^* c \right) \right).$$ \hspace{1cm} (6.7)

As was done in Ref. 12), we introduce the canonical variable ($\psi, K$) in the form

$$\sqrt{\hbar} c = \sqrt{K} e^{-i\psi}, \quad \sqrt{\hbar} c^* = \sqrt{K} e^{i\psi}. \hspace{1cm} (6.8)$$

Then, $H''$ in the form (6.7) can be rewritten as

$$H'' = -2M + 2\epsilon K - 2G \cdot K (2M - K) + 4G K (2M - K) (\sin \psi/2)^2.$$

(6.9)

On the other hand, $H'$ shown in the form (5.8) is rewritten in the form

$$H' = \epsilon (T(+) - T(-) - M) + 2\epsilon K - 2G \sqrt{K(2T(+) + K)} \cdot \sqrt{2M - K} \sqrt{2T(-) + 2M - \hbar - K} - 4G \sqrt{K(2T(+) + K)} \cdot \sqrt{2M - K} \sqrt{2T(-) + 2M - \hbar - K} (\sin \psi/2)^2.$$

(6.10)

Both Hamiltonians are valid only in the region

$$0 \leq K \leq 2M - \hbar/2 \quad \text{or} \quad 2M \quad (6.11)$$

The minimum point for both Hamiltonians (energies) appears at least at

$$\psi = 0 \quad (6.12)$$

Therefore, it may be interesting to investigate at which points for $K$ the minimum points appear. For the Hamiltonian (the energy) (6.9), the behavior near $K = 0$ is determined by the factor $2(\epsilon - 2GM)$. If $\epsilon > 2GM$, the energy increases in any region of $K$, and then, the minimum point appears at $K = 0$. If $\epsilon < 2GM$, the energy decreases and afterward
increases. Then, the minimum point appears at a certain value of $K > 0)$. This means that, at $\epsilon = 2GM$, the phase change appears. In the case of the Hamiltonian (the energy), the behavior near $K = 0$ is determined by the factor $2\epsilon K - 2G\sqrt{K(T^+ + K)}$. In this case, the energy decreases at any point of $K$ near $K = 0$ and afterward increases. Therefore, the minimum point appears at a certain point ($> 0$) except $K = 0$ in the case $G = 0$. From this argument, we cannot expect the sharp phase change. This is in the same situation as that in the case of the two-level pairing model. In the succeeding paper, we will repeat the above-mentioned point qualitatively including other various features.

Appendix A

Further interpretation of $\hat{X}_\nu^*$ and $\hat{Y}_s^*$ defined in the relation (2.8)

We introduce the following operator:

$$\hat{T}_{l}^*(n_1, \cdots, n_{2l}) = \hat{c}_{n_1}^*(-) \cdots \hat{c}_{n_{2l}}^*(-) \cdot \quad (2l; \text{positive integer}) \quad (A.1)$$

It may be clear from the relation (2.7) that $(\hat{c}_{m}^*(\pm))$ is spinor, and then, the operator (A.1) is the $(-l)$-th component of the tensor operator with rank $l$ for $(\hat{S}_\pm, 0)$ and we have

$$[ \hat{N}, \hat{T}_{l}^*(n_1, \cdots, n_{2l}) ] = 2l\hat{T}_{l}^*(n_1, \cdots, n_{2l}) \cdot \quad (A.2)$$

The operator $\hat{Y}_{s}^*$ given in the relation (2.8b) is nothing but

$$\hat{Y}_{s}^* = \hat{T}_{s}^*(m_1, \cdots, m_{2s}) \cdot \quad (A.3)$$

It may be self-evident that $\hat{Y}_{s}^*$ satisfies the condition (2.6b).

The $m$-th component of the present tensor operator is expressed in the form

$$\hat{T}_{lm}^*(n_1, \cdots, n_{2l}) = \sqrt{\frac{(l - m)!}{(2l)!(l + m)!}} \left( \hat{S}^* \right)^{l+m} \hat{T}_{l}^*(n_1, \cdots, n_{2l}) \cdot \quad (A.4)$$

Here, for any operators $\hat{A}$ and $\hat{B}$, $\hat{A} \hat{B}$ is defined as

$$\hat{A} \hat{B} = [ \hat{A}, \hat{B} ] \cdot \quad (A.5)$$

With the use of the form (A.4), we can construct the scalar operator as follows:

$$\hat{T}_{00}^*(n_1, \cdots, n_{2\lambda}; n'_1, \cdots, n'_{2\lambda}) = \sum_{\mu=-\lambda}^{\lambda} (-)^{\lambda - \mu} \hat{T}_{\lambda\mu}^*(n_1, \cdots, n_{2\lambda}) \hat{T}_{\lambda-\mu}^*(n'_1, \cdots, n'_{2\lambda}) \cdot \quad (A.6)$$
Clearly, $\hat{T}_{00}^*(n_1, \cdots, n_{2\lambda}; n'_1, \cdots, n'_{2\lambda})$ commutes with $\hat{S}_{\pm 0}$ and symbolically it is expressed in the form

$$\hat{T}_{00}^*(n_1, \cdots, n_{2\lambda}; n'_1, \cdots, n'_{2\lambda}) = \sum_{(\rho, \rho')} D(n_1, \cdots, n_{2\lambda}; n'_1, \cdots, n'_{2\lambda}|\rho_1, \cdots, \rho_{2\lambda}; \rho'_1, \cdots, \rho'_{2\lambda})$$

$$\times \hat{c}_{\rho_1}^*(+) \cdots \hat{c}_{\rho_{2\lambda}}^*(+) \hat{c}_{\rho'_1}^*(-) \cdots \hat{c}_{\rho'_{2\lambda}}^*(-).$$

(A.7)

Of course, we have

$$\hat{T}_{00}^*(n_1, \cdots, n_{2\lambda}; n'_1, \cdots, n'_{2\lambda}) \hat{Y}_{s}^*|0\rangle = 0.$$  \hspace{1cm} (A.8)

From the above argument, $\hat{X}_{\nu}^*$ can be constructed in the form which satisfies $[\hat{N}, \hat{X}_{\nu}^*] = 2\nu \hat{X}_{\nu}^*.$

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**Appendix B

The Holstein-Primakoff boson representation of the $su(1,1)$-algebra for boson pair and its classical counterpart**

In this Appendix, we summarize the $su(1,1)$-algebra for boson pair which is a base of the $su(1,1) \otimes su(1,1)$-algebra for boson pairs appearing in the text of this paper. In the present case, the generators are given in the form

$$\hat{\tau}_+ = (\hat{b}^*)^2/2, \quad \hat{\tau}_- = (\hat{b})^2/2, \quad \hat{\tau}_0 = 1/4 + (\hat{b}^*\hat{b})/2.$$  \hspace{1cm} (B.1)

Here, $(\hat{b}, \hat{b}^*)$ denotes boson operator, and for simplicity, $\hbar = 1$ is taken. The state with the minimum weight $|t\rangle$ is obtained under the condition

$$\hat{\tau}_-|t\rangle = 0, \quad \hat{\tau}_0|t\rangle = t|t\rangle. \quad (t = 1/4, 3/4)$$  \hspace{1cm} (B.2)

For $t$, we have the form

$$|t\rangle = |0\rangle \quad \text{for} \quad t = 1/4, \quad |t\rangle = \hat{b}^*|0\rangle \quad \text{for} \quad t = 3/4.$$  \hspace{1cm} (B.3)

Here, $|0\rangle$ denotes the vacuum for $(\hat{b}, \hat{b}^*)$. The state $|t\rangle$ is expressed as

$$|t\rangle = (\hat{b}^*)^{2t-1/2}|0\rangle.$$  \hspace{1cm} (B.4)

Then, by denoting the eigenvalue of $\hat{\tau}_0$ as $k + t$ ($k = 0, 1, 2, \cdots$), the orthogonal set $\{|k; t\rangle\}$ is expressed as follows:

$$|k; t\rangle = 2^k \left(\sqrt{(2k + 2t - 1/2)!}\right)^{-1} (\hat{\tau}_+)^k |t\rangle$$

$$= \left(\sqrt{(2k + 2t - 1/2)!}\right)^{-1} (\hat{b}^*)^{2k+2t-1/2}|0\rangle.$$  \hspace{1cm} (B.5)
In order to obtain the Holstein-Primakoff representation, we adopt the MYT mapping method.\(^{11}\) We prepare a new boson space spanned by boson \((\hat{c}, \hat{c}^*)\):
\[
|k\rangle = \left(\sqrt{k!}\right)^{-1} (\hat{c}^*)^k |0\rangle . \quad (\hat{c}|0\rangle = 0)
\] (B.6)

Then, the mapping operator \(\hat{U}\) is defined as
\[
\hat{U} = \sum_{k=0}^{\infty} |k\rangle \langle k; t|.
\] (B.7)

With the use of \(\hat{U}\), we get the Holstein-Primakoff boson representation in the form
\[
\begin{align*}
\hat{\tau}_+ &= \hat{U} \hat{\tau}_+ \hat{U}^\dagger = \hat{c}^* \sqrt{2t + \hat{c}^* \hat{c}} , \\
\hat{\tau}_- &= \hat{U} \hat{\tau}_- \hat{U}^\dagger = \sqrt{2t + \hat{c}^* \hat{c}} \cdot \hat{c} , \\
\hat{\tau}_0 &= \hat{U} \hat{\tau}_0 \hat{U}^\dagger = t + \hat{c}^* \hat{c} .
\end{align*}
\] (B.8)

Here, \(t\) denotes
\[
t = 1/4 , \quad 3/4 .
\] (B.9)

The form (B.8) is also valid for the case \(t = 1/2, 1, 3/2, \cdots\), which is treated in (I). The above is our first interest.

As a second interest, we investigate the classical counterpart of the form (B.1) or (B.8). For this purpose, we, first, introduce the following state:
\[
|c_0\rangle \rangle = \left(\sqrt{U}\right)^{-1} \exp \left(\frac{V}{U} \hat{\tau}_+ \right) |0\rangle \rangle
\]
\[
= \left(\sqrt{U}\right)^{-1} \exp \left(\frac{V}{2U} (\hat{b}^*)^2 \right) |0\rangle \rangle .
\] (B.10)

Here, \(V\) denotes a complex parameter and \(U\) is given as
\[
U = \sqrt{1 + |V|^2} .
\] (B.11)

The state \(|c_0\rangle \rangle\) is a vacuum of the boson operator \((\hat{\beta}, \hat{\beta}^*)\) defined as
\[
\begin{align*}
\hat{\beta} &= U \hat{b} - V \hat{b}^* , \\
\hat{\beta}^* &= -V^* \hat{b} + U \hat{b}^* , \quad \text{(B.12a)}
\end{align*}
\]
\[
i.e., \quad \hat{b} = U \hat{\beta} + V \hat{\beta}^* , \\
\hat{b}^* = V^* \hat{\beta} + U \hat{\beta}^* . \quad \text{(B.12b)}
\]
In relation to $|c_0\rangle$, we introduce the state $|c^0\rangle$ in the form

$$|c^0\rangle = \left( \sqrt{3/2 - 2t} + \sqrt{2t - 1/2} \hat{\beta}^* \right) |c_0\rangle . \quad (t = 1/4, 3/4) \quad (B.13)$$

Of course, we have $(\sqrt{3/2 - 2t})^2 + (\sqrt{2t - 1/2})^2 = 1$ and $\sqrt{3/2 - 2t} \cdot \sqrt{2t - 1/2} = 0$. With the use of the state $|c^0\rangle$, we have the relations

$$\tau_+^0 = \langle c^0 | \hat{\tau}_+ | c^0 \rangle = 2tUV^* ,$$

$$\tau_-^0 = \langle c^0 | \hat{\tau}_- | c^0 \rangle = 2tUV ,$$

$$\tau_0^0 = \langle c^0 | \hat{\tau}_0 | c^0 \rangle = t + 2t|V|^2 ,$$

$$\langle c | \frac{\partial}{\partial z} | c \rangle = \frac{1}{2} \cdot 2t \left( V^* \frac{\partial V}{\partial z} - V \frac{\partial V^*}{\partial z} \right) . \quad (B.14)$$

Instead of $(V, V^*)$, we introduce new parameter $(c, c^*)$ obeying

$$\langle c^0 | \frac{\partial}{\partial c} | c^0 \rangle = c^* / 2 , \quad \langle c^0 | \frac{\partial}{\partial c^*} | c^0 \rangle = -c / 2 . \quad (B.16)$$

With the use of the relation (B.15), $(V, V^*)$ can be expressed as

$$V = c / \sqrt{2t} , \quad V^* = c^* / \sqrt{2t} . \quad (B.17)$$

The relation (B.16) tells that $(c, c^*)$ plays a role of canonical variable in boson type in classical mechanics. Then, we have the following form for $(\tau_{\pm,0}^0)$:

$$\tau_+^0 = c^* \sqrt{2t} + c^* c , \quad \tau_-^0 = \sqrt{2t} + c^* c , \quad \tau_0^0 = t + c^* c . \quad (B.18)$$

We can see that the form (B.18) is a classical counterpart of the form (B.8), i.e., (B.1) under the replacement

$$\hat{c} \longrightarrow c , \quad \hat{c}^* \longrightarrow c^* . \quad (B.19)$$

Further, we can show that the Poisson bracket for the expression (B.18) gives us the same form as the commutation relation in Dirac’s sense.

Our third interest is related to the state

$$|c'_0\rangle = \left( \sqrt{U} \right)^{-1} \exp \left( \gamma \frac{V}{|\gamma|} \frac{\sqrt{\gamma \hat{b}^*}}{2U} \right) |0\rangle . \quad (B.20)$$

Here, $\gamma$ is a complex parameter and $V$ is real and $U$ is given by

$$U = \sqrt{1 + V^2} . \quad (B.21)$$
The state \(|c_0\rangle\rangle\) is a vacuum for the boson operator \((\hat{\beta}, \hat{\beta}^*)\) defined as
\[
\hat{\beta} = U\hat{b} - \frac{\gamma}{|\gamma|} V\hat{b}^* - \sqrt{\gamma}, \\
\hat{\beta}^* = -\frac{\gamma^*}{|\gamma|} V\hat{b} + U\hat{b}^* - \sqrt{\gamma^*},
\]
i.e.,
\[
\hat{b} = U\hat{\beta} + \frac{\gamma}{|\gamma|} V\hat{\beta}^* + \sqrt{\gamma(U + V)}, \\
\hat{b}^* = \frac{\gamma^*}{|\gamma|} V\hat{\beta} + U\hat{\beta}^* + \sqrt{\gamma^*}(U + V).
\]

In relation to the state \(|c_0\rangle\rangle\), we introduce the state \(|c^0\rangle\rangle\) in the form
\[
|c^0\rangle\rangle = \left(\sqrt{3/2 - 2t} + \sqrt{2t - 1/2}\hat{\beta}^*\right)|c_0\rangle\rangle \quad (t = 1/4, 3/4)
\]

Of course, we have \((\sqrt{3/2 - 2t})^2 + (\sqrt{2t - 1/2})^2 = 1\) and \(\sqrt{3/2 - 2t} \cdot \sqrt{2t - 1/2} = 0\). The state \(|c^0\rangle\rangle\) gives the following relations:
\[
\tau_+^{0'} = \langle c^0|\hat{\tau}_+|c^0\rangle = 2t \frac{\gamma^*}{|\gamma|} UV + \gamma^*(U + V)^2, \\
\tau_-^{0'} = \langle c^0|\hat{\tau}_-|c^0\rangle = 2t \frac{\gamma}{|\gamma|} UV + \gamma(U + V)^2, \\
\tau_0^{0'} = \langle c^0|\hat{\tau}_0|c^0\rangle = t + 2tV^2 + \frac{1}{2} |\gamma|(U + V)^2, \\
\langle c^0|\frac{\partial}{\partial z}|c^0\rangle = \frac{1}{2} \left(2t \frac{V^2}{|\gamma|^2} + \frac{1}{2} |\gamma|(U + V)^2\right) \left(\gamma^* \frac{\partial \gamma}{\partial z} - \gamma \frac{\partial \gamma^*}{\partial z}\right).
\]

Instead of \((\gamma, \gamma^*)\), we introduce new parameter \((c, c^*)\) obeying
\[
\langle c^0|\frac{\partial}{\partial c}|c^0\rangle = c^*/2, \quad \langle c^0|\frac{\partial}{\partial c^*}|c^0\rangle = -c/2.
\]

The relation (B.26) with (15.25) gives us
\[
\gamma = \frac{c}{|c|} \cdot \frac{2(|c|^2 - 2tV^2)}{(U + V)^2}, \quad \gamma^* = \frac{c^*}{|c|} \cdot \frac{2(|c|^2 - 2tV^2)}{(U + V)^2}.
\]

In the same meaning as that in the previous case, \((c, c^*)\) plays a role of the canonical variable in classical mechanics. Then, the form (B.24) is expressed in the following form:
\[
\tau_+^{0'} = \frac{c^*}{|c|}(|c|^2 + 2t(V - U)) , \\
\tau_-^{0'} = \frac{c}{|c|}(|c|^2 + 2t(V - U)) , \\
\tau_0^{0'} = t + |c|^2.
\]
By changing $V$ in various forms as a function of $|c|^2$, we get various forms for $(\tau_{0,0}^0)$. For instance, under the form (B.28), we have

$$\tau_0' = \frac{c}{|c|} \sqrt{|c|^2(2t - 1 + |c|^2)} = \sqrt{2t - 1 + |c|^2} \cdot c .$$  \hspace{1cm} (B.29)$$

The condition (B.26) gives us

$$V = \sqrt{\frac{|c|^2}{2t}} \frac{|2t - 1|}{\sqrt{(2t - 1)^2 + (\sqrt{2t - 1 + |c|^2} + |c|^2)^2}} .$$  \hspace{1cm} (B.30)$$

This form is used in §5. Generally, we have the following form:

$$\tau_- = \sqrt{2\tau + |c|^2} \cdot c ,$$  \hspace{1cm} (B.31)$$

$$V = \sqrt{\frac{|c|^2}{2t}} \frac{|2\tau|}{\sqrt{(2\tau)^2 + 2(t - \tau)(\sqrt{2\tau + |c|^2} + |c|^2)^2}} .$$  \hspace{1cm} (B.32)$$

Various boson-pair coherent states were investigated in Ref. 15) under the viewpoint of the deformed boson scheme.

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