Field embeddings which are conjugate under a $p$-adic classical group

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Abstract

Let $(V,h)$ be a Hermitian space over a division algebra $D$ which is of index at most two over a non-Archimedean local field $k$ of residue characteristic not 2. Let $G$ be the unitary group defined by $h$ and let $\sigma$ be the adjoint involution. Suppose we are given two $\sigma$-invariant but not $\sigma$-fixed field extensions $E_1$ and $E_2$ of $k$ in $\text{End}_D(V)$ which are isomorphic under conjugation by an element $g$ of $G$ and suppose that there is a point $x$ in the Bruhat–Tits building of $G$ which is fixed by $E_1$ and $E_2$ in the reduced building of $\text{Aut}_D(V)$. Then $E_1$ is conjugate to $E_2$ under an element of the stabilizer of $x$ in $G$ if $E_1$ and $E_2$ are conjugate under an element of the stabilizer of $x$ in $\text{Aut}_D(V)$ and a weak extra condition. In addition in many cases the conjugation by $g$ from $E_1$ to $E_2$ can be realized as conjugation by an element of the stabilizer of $x$ in $G$.

Further we give a concrete description of the canonical isomorphism from the set of $E_1$-times fixed points of the building of $G$ onto the building of the centralizer of $E_1$ in $G$.

1 Introduction

This article is about a Skolem–Noether kind of lemma in the framework of $p$-adic classical groups in the case of odd residue characteristic. For general linear groups these kinds of lemmas encode the invariants for analyzing the rigidity of certain irreducible representations on open compact subgroups of a $p$-adic group. These representations, called simple types, are used to construct and classify supercuspidal representations of the group of interest. The rigidity question for two simple types asks how they are related if they are contained in the same supercuspidal representation. For example the work of Broussous and Grabitz [BG00] is used in the above classification question for $\text{GL}_m(D)$. The latter is mainly done by Secherre, Stevens and Broussous in the framework of Bushnell-Kutzko theory. In this article we consider $p$-adic classical groups. Stevens constructed all supercuspidal representations for the case where $D$ is a field [Ste08]. To understand the rigidity question for the case where $D$ has index two we need a new Skolem–Noether lemma, i.e. a classical version of part one of [BG00].

To understand where the Skolem–Noether like proposition is involved let us give more details. Let $H$ be a $p$-adic group of the kind mentioned above. A simple type itself is constructed by a combinatorial algebraic object, called a simple stratum, which especially consists of a facet of the Bruhat–Tits building $\mathfrak{B}(H)$, represented by a certain hereditary order $a$ in an Azumaya algebra $A$ over a non-Archimedean local field $k$, and a field extension $E|k$ in $A$ such that $E^\times$ normalizes $a$. Note that $A$ always is the endomorphism ring of the vector space on which $H$ is defined. We call such a pair $(E,a)$ an embedding. An approach to the rigidity question for simple types is to classify the $n(a) \cap H$-conjugation classes of embeddings with same hereditary order, where $n(a)$ is the normalizer of $a$ in $A^\times$. In [BG00] the authors described these classes for the case that $H$ is $A^\times$, i.e. some $\text{GL}_m(D)$, using an equivalence relation on the set of embeddings which has the property that every equivalence class contains an embedding whose field is isomorphic to an unramified extension of $k$ in $D$. Moreover from their article it is easy to deduce the following proposition.

**Proposition 1.1** If two embeddings $(E_i,a)$ of $A$ with $k$-algebra isomorphic fields and the same hereditary order
Proposition 1.3 Let \( \phi_1 \) and \( \phi_2 \) be two \( \sigma' - \sigma \)-equivariant \( k \)-algebra homomorphisms from \( E[k] \) into \( A \). Then, \( \phi_1 \circ \phi_2^{-1} \) can be described as a conjugation under \( G \) if and only if \( (V, h^{\phi_1}) \) is isomorphic to \( (V, h^{\phi_2}) \) as signed Hermitian modules.
The signed Hermitian modules can be analyzed using Witt decompositions, and lattice chains give the possibility to consider symmetric or skew-symmetric bilinear forms on residue spaces over finite fields. I thank Prof. S. Stevens for giving me the hint to this topic and the DFG for financing my position at University of Muenster for that project.

2 The building of $GL_D(V)$ and centralizers

A good source for the preliminaries in this part is [BL02]. Let $D$ be a skew field of finite index $d$ whose center is a non-Archimedean local field $k$, and let $V$ be a finite dimensional right $D$-vector space. We write $\nu, o_D, p_D, K_D$ for the valuation, the valuation ring, the valuation ideal and the residue field of $D$, respectively. We use similar notation for other local fields. The valuation is normalized such that $\nu(k^\times)$ is $\mathbb{Z}$. We denote $\text{Aut}_D(V)$ by $\tilde{G}$.

In these notes we work with Bruhat–Tits buildings in terms of lattice functions. Let us briefly repeat the basic concept here.

Definition 2.1 An $o_D$-lattice of $V$ is a finitely generated $o_D$-submodule of $V$ containing a $D$-basis of $V$. An $o_D$-lattice function is a family $\Lambda = (\Lambda(t))_{t \in \mathbb{R}}$ of $o_D$-lattices of $V$ such that for any real numbers $t < s$ we have

1. $\Lambda(t) \supseteq \Lambda(s)$,
2. $\Lambda(t) = \bigcap_{r < t} \Lambda(r)$ and
3. $\Lambda(t)\pi = \Lambda(t + \frac{1}{d})$ for any prime element $\pi$ of $D$.

The set of all $o_D$-lattice functions is denoted by $\text{Latt}^1_{o_D}(V)$. Let $s$ be real number. The translation of $\text{Latt}^1_{o_D}(V)$ by $s$ is the map

$$\Lambda \mapsto \Lambda + s, \quad (\Lambda + s)(t) := \Lambda(t - s).$$

A translation class of an $o_D$-lattice function is denoted by $[\Lambda]$. A bijective $\subseteq$-decreasing map from $\mathbb{Z}$ to the image of some $o_D$-lattice function $\Lambda$ is called an $o_D$-lattice chain corresponding to $\Lambda$. The square lattice function of $\Lambda$ is an $o_k$-lattice function of $\text{End}_D(V)$ defined via

$$g_\Lambda(t) := \{ a \in \text{End}_D(V) \mid a(\Lambda(s)) \subseteq \Lambda(s + t) \text{ for all } s \in \mathbb{R} \},$$

which by definition only depends on $[\Lambda]$. The hereditary order $g_\Lambda(0)$ only depends on an $o_D$-lattice chain corresponding to $\Lambda$. In some arguments we need a right-limit of $\Lambda$ in $t$: We define

$$\Lambda(t+) := \bigcup_{r > t} \Lambda(t).$$

Theorem 2.2 ([BL02]) There is an affine and $\tilde{G}$-equivariant bijection from $\text{Latt}^1_{o_D}(V)$ to the Bruhat–Tits building $\mathcal{B}(\tilde{G})$ of $\tilde{G}$. Two such maps only differ by a translation. It induces the unique affine and $\tilde{G}$-equivariant bijection $f$ from the set of all translation classes of $\text{Latt}^1_{o_D}(V)$ to the reduced Bruhat–Tits building $\mathcal{B}_{\text{red}}(\tilde{G})$ of $\tilde{G}$. The facets of $\mathcal{B}_{\text{red}}(\tilde{G})$ correspond to the hereditary orders of $\text{End}_D(V)$, more precisely the point $f([\Lambda])$ is a point of the facet $\{ f([\Lambda']) \mid \text{im}(\Lambda') = \text{im}(\Lambda) \}$. For a point $x$ of $\mathcal{B}_{\text{red}}(\tilde{G})$ we denote by $a_x$ the hereditary order corresponding to $x$, i.e. $g_\Lambda(0)$ if $f([\Lambda]) = x$. We identify facets with hereditary orders. Let $E|k$ be a field extension in $\text{End}_D(V)$. We denote the centralizer of $E$ in a subgroup $H$ of $\tilde{G}$ by $H_E$. 

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Theorem 2.3 ([BL02, II.1.1.]) There is a unique map $j_E$ from the set of $E^\times$-fixed points of $\mathfrak{B}_{\text{red}}(\tilde{G})$, denoted by $\mathfrak{B}_{\text{red}}(\tilde{G})^{E^\times}$, to the reduced building $\mathfrak{B}_{\text{red}}(\tilde{G}_E)$ of $\tilde{G}_E$ such that

$$g_x(t) \cap \text{End}_{E \otimes_k D}(V) = g_{j_E(x)}(t),$$

for all $x \in \mathfrak{B}_{\text{red}}(\tilde{G})^{E^\times}$ and $t \in \mathbb{R}$. The map is

1. affine,
2. $\tilde{G}_E$-equivariant and
3. bijective.

$j_E^{-1}$ is the unique map from $\mathfrak{B}_{\text{red}}(\tilde{G}_E)$ to $\mathfrak{B}_{\text{red}}(\tilde{G})$ which satisfies 1. and 2.

Remark 2.4 1. The centralizer of $E$ in $\text{End}_D(V)$ is $E$-algebra isomorphic to a ring $\text{End}_\Delta(W)$ for some vector space $W$ over a skew-field $\Delta$. From Brauer theory it follows that the $\Delta$-dimension of $W$ is $\frac{\text{m \ gcd}([E:k],d)}{[E:k]}$ and $\Delta$ has index $\frac{d}{\text{gcd}([E:k],d)}$.

2. Let us recall how the map $j_E$ is constructed. The map is induced by a $\tilde{G}_E$-equivariant affine map $j_E^1$ from $\mathfrak{B}(\tilde{G}_E)^{E^\times}$ to $\mathfrak{B}(\tilde{G}_E)$, i.e. $j_E([\Lambda]) = [j_E^1(\Lambda)]$. For the sake of simplicity and this article we only need the description for the case where $E|k$ is unramified and embeddable in $D|k$. Choose a maximal unramified field extension $L|k$ in $D$ and a uniformizer $\pi_D$ which normalizes $L$. Denote by $\Delta$ the centralizer in $D$ of the intermediate field between $L$ and $k$ of degree $[E:k]$. Let $1^i, i \in \{1,\ldots,[E:k]\}$, be the idempotents of $E \otimes_k L$ and put $W := 1^1V$. The map $j_E^1$ is of the form

$$j_E^1(\Lambda) = \Theta \in \text{Latt}_{\Delta}(W), \text{ s.t. } \Lambda(t) = \bigoplus_{i=0}^{[E:k]-1} \Theta \left(t - \frac{i}{d}\right) \pi_D,$$

([BL02 II.3.1]) and $\tilde{G}_E$ is identified with $\text{Aut}_\Delta(W)$.

We now recall the concept of embeddings which is related to buildings of centralizers of $\tilde{G}$. We recommend [BG00] as a good introduction. Let us denote by $E_D$ the intermediate field of $E|k$ which is unramified over $k$ and whose degree is the greatest common divisor of $d$ and the residue class degree $f(E|k)$ of $E|k$. This is exactly the greatest field in $E$ which can be embedded into a maximal unramified field extension $L|k$ of $D$. Let us recall the definition of an embedding.

Definition 2.5 1. The normalizer of a hereditary order $\mathfrak{a}$ of $\text{End}_D(V)$ in $\tilde{G}$ is the set $\mathfrak{n}(\mathfrak{a})$ of all elements $g$ of $\tilde{G}$ for which $\mathfrak{a}$ is equal to $g\mathfrak{a}g^{-1}$.

2. An embedding is a pair $(E, \mathfrak{a})$ with a subfield $E$ of $\text{End}_D(V)$ which extends $k$ and a hereditary order $\mathfrak{a}$ normalized by $E$, i.e. $E^\times$ is a subset of $\mathfrak{n}(\mathfrak{a})$.

3. Two embeddings $(E_1, \mathfrak{a}_1)$ and $(E_2, \mathfrak{a}_2)$ are equivalent to each other if there is an element $g \in \tilde{G}$ such that $(gE_1Dg^{-1}, g\mathfrak{a}_1g^{-1})$ is equal to $(E_2D, \mathfrak{a}_2)$.

The importance of the equivalence of embeddings is described in the following theorem.

Proposition 2.6 (consequence of [BG00, 3.2]) 1. Suppose we are given two equivalent embeddings $(E_1, \mathfrak{a})$ and $(E_2, \mathfrak{a})$ of $\text{End}_D(V)$ and a $k$-algebra isomorphism $\phi$ from $E_1$ to $E_2$. Then $\phi$ can be realized as conjugation by an element of $\mathfrak{n}(\mathfrak{a})$. 

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2. Two equivalent embeddings \((E, a)\) and \((E, a')\) of \(\text{End}_D(V)\) are conjugate by an element \(g\) of the centralizer \(\hat{G}_E\) of \(E\) in \(\hat{G}\).

**Proof:**

1. The finite field extension \(E_1|k\) is primitive, because it is a tower of two primitive extensions where the first is unramified and thus separable. Fix a generator \(\alpha\) of \(E_1\). We apply the Skolem–Noether Theorem to realize \(\phi\) as a conjugation by an element \(g\) of \(G\). By [BG90, 3.2], there is an element \(g'\) of \(n(a)\) such that \(g'\alpha g'^{-1}\) is equal to \(g\alpha g^{-1}\). This proves 1.

2. There is an element \(g\) of \(\hat{G}\) such that
\[
gE_Dg^{-1} = E_D \quad \text{and} \quad ga'g^{-1} = a.
\]

The statement follows from 1. applied to \((gEg^{-1}, a), (E, a)\) and \(\phi\) defined by \(\phi(x) = g^{-1}xg, \ x \in gEg^{-1}\).

q.e.d.

In many cases the embeddings \((E, a)\) and \((E, a')\) are automatically equivalent if \(a\) and \(a'\) are conjugate under \(\hat{G}\) even if \(E_D\) is not \(k\). In fact we have the following stronger lemma.

**Lemma 2.7** Assume that \(\frac{d}{\gcd(d, e(E|k))}\) is prime to \(f(E|k)\). If two elements \(\Lambda, \Lambda'\) of \(\text{Latt}^1_{ae}(V)\) are conjugate under an element \(g \in \hat{G}\), i.e. \(g\Lambda = \Lambda'\), then there is an element \(\tilde{g}\) of \(\hat{G}_{E_D}\) such that \(\tilde{g}\Lambda = \Lambda'\).

**Proof:** We consider the map \(j_{E_D}^1\) from Remark 2.4. We denote \(j_{E_D}^1(\Lambda)\) by \(\Theta\) which is an \(\alpha\)-lattice function where \(\Delta\) is the centralizer of an embedding of \(E_D\) in \(D\), and it is still an \(\alpha\)-lattice function. More precisely there are elements \(g_1\) and \(g_2\) in \(\hat{G}_{E_D}\) such that
\[
g_1.\Theta = \Theta + \frac{1}{\gcd(e(E|k), d)} \quad \text{and} \quad g_2.\Theta = \Theta + \frac{[E_D : k]}{d}.
\]

The assumption of the proposition implies that there is a product \(g_3\) of powers of \(g_1\) and \(g_2\) such that \(g_3.\Theta\) is equal to \(\Theta + \frac{1}{d}\). We can conclude by the form of \(j_{E_D}^1\) in Remark 2.4 that the \(\kappa_k\)-dimension of \(\Theta(t)/\Theta(t+)\) is completely determined by \(\Lambda(t)/\Lambda(t+)\). The same is true for \(\Lambda'\) and \(\Theta' := j_{E_D}^1(\Lambda')\). The equality of the \(\kappa_k\)-dimension of \(\Theta(t)/\Theta(t+)\) with that of \(\Theta'(t)/\Theta'(t+)\), for all \(t \in \mathbb{R}\), implies that there is an element \(\tilde{g}\) of \(\hat{G}_E\) which satisfies \(\tilde{g}\Theta = \Theta'\) and thus \(\tilde{g}\Lambda = \Lambda'\).

q.e.d.

**Corollary 2.8** Assume that \(\frac{d}{\gcd(d, e(E|k))}\) is prime to \(f(E|k)\). Two embeddings \((E, a)\) and \((E, a')\) are conjugate under \(\hat{G}_E\) if and only if \(a\) and \(a'\) are conjugate under \(\hat{G}\).

**Proof:** We only have to proof the “if” part. We apply firstly Lemma 2.7 on the barycenters of \(a\) and \(a'\) in \(\mathcal{B}(G)\) and secondly Proposition 2.6. q.e.d.

**Definition 2.9** Let \(B\) be a Euclidean building and fix a labeling of the vertices. The (simplicial) type of \(x\) in \(B\) are the barycentric coordinates of \(x\) with respect to the vertexes of a chamber whose closure contains \(x\). The barycentric coordinates do not depend on the chosen chamber.

**Remark 2.10**

1. Two elements \(x, y\) of \(\mathcal{B}_{red}(\hat{G})\) have the same type if and only if there is an element of \(\text{SL}_D(V)\) which maps \(x\) to \(y\). Indeed, \(\text{SL}_D(V)\) acts strongly transitive on the simplicial structure of \(\mathcal{B}_{red}(\hat{G})\).

2. By 2. of Proposition 2.6, two facets of \(\mathcal{B}_{red}(\hat{G})\) give equivalent embeddings with respect to \(E\) if and only if the \(j_E\)-images of the barycenters have the same type in \(\mathcal{B}_{red}(\hat{G}_E)\) up to a rotation of the Coxeter diagram of \(\mathcal{B}_{red}(\hat{G}_E)\).
Here we come to a proposition similar to Corollary 2.8. The difference here is that we only consider type-preserving automorphisms of $\mathfrak{B}_{\text{red}}(\tilde{G})$ and of $\mathfrak{B}_{\text{red}}(\tilde{G}_E)$.

**Proposition 2.11** Assume that $d \mod \gcd(x,E(E))$ is prime to $f(E|k)$. Then two elements $x$ and $y$ of $\mathfrak{B}_{\text{red}}(\tilde{G})^{E^x}$ are of the same type in $\mathfrak{B}_{\text{red}}(\tilde{G})$ if and only if $jE(x)$ and $jE(y)$ are of the same type in $\mathfrak{B}_{\text{red}}(\tilde{G}_E)$.

The barycenter of a facet $F$ is denoted by $\text{bary}(F)$. Although the statement is geometric the given proof is algebraic and uses the reduced norm. For more details about reduced norms we recommend [Rei03]. Write $A$ for $\text{End}_D(V)$ and $A_E$ for the centralizer of $E$ in $A$. We write $Nrd_{\tilde{G}}$ for the reduced norm and $Nrd_{\tilde{G}}$ if $\tilde{G}$ is a field extension. We write $Nrd$ for the normalized valuation of $E$. In the proof we use the well-known fact that $g \in \tilde{G}$ is type-preserving if and only if $\nu(Nrd_{A,k}(g))$ is a multiple of $m$, and we use the tower law:

$$Nrd_{A,k}(g) = N_{E|k}(Nrd_{A_E}(g)),$$

for $g \in B$.

**Proof:** The proof is motivated by the proof of [BG00, 3.2] (see Proposition 2.6 above) which also uses results of [Gra99] on good continuations of hereditary orders, a terminology which we do not introduce here. The “if” part is trivial.

1. Case $\gcd(f(E|k),d) = 1$: The important part is the “only if” one. The embeddings $(E, a_x)$ and $(E, a_y)$ are equivalent because of the condition on $f(E|k)$ and therefore there is an element $g_1$ of $\tilde{G}_E$ which conjugates $a_x$ to $a_y$. By assumption there is also an element $g$ of $\text{SL}_D(V)$ which conjugates $g$ to $x$, i.e. $g_1$ is an element of $n(a_x)$. Using the condition on $f(E|k)$ again there is an element $z$ of $n(a_{jE(x)})$, the normalizer of $a_{jE(x)}$ in $A_E^*$, such that $n(a_x)$ is equal to $a_x^*_E(z)$, see [Gra99, 2.2]. Thus there is an element $g_2$ of $n(a_{jE(x)})$ such that $gg_1g_2$ lies in $a_x^*_E$, especially $Nrd_{A,k}(g_1g_2)$ is a unit of $\mathfrak{A}_k$ and thus $Nrd_{A,k}(g_1g_2)$ is a unit of $\mathfrak{A}_E$. This implies that $g_1g_2$ is type-preserving on $B_{\text{red}}(\tilde{G}_E)$. In other words there is an element $g_3$ of $\tilde{G}_E$ with reduced norm one such that $gg_3$ is a unit of $\mathfrak{A}_E$ and thus $Nrd_{A,k}(g_1g_2)$ is equal to $a_x^*_E$. We want to show $g_3a_x = y$. The condition on $f(E|k)$ to be prime to $d$ and [Gra99, 2.2] imply that there is only one facet $F$ of $\mathfrak{B}_{\text{red}}(\tilde{G})^{E^x}$ which satisfies that $jE(\text{bary}(F))$ is a point of $a_{jE(x)}$ (open facet). Thus $a_y$ is equal to $a_{g_3a_x}$, and thus $y = g_3x$ because $g_3$ is type-preserving on $B_{\text{red}}(\tilde{G})$.

2. Case $\gcd(f(E|k),d) > 1$: Let $\Lambda$ and $\Lambda'$ be two lattice functions such that $x$ corresponds to $[\Lambda]$ and $y$ to $[\Lambda']$ and such that there is an element $g$ of $\text{SL}_D(V)$ which satisfies $g\Lambda(t) = \Lambda'(t)$, for all $t \in \mathbb{R}$. Here we have used Remark 2.10 (1). There is an element $g' \in \tilde{G}_{E_D}$ such that $g'\Lambda = \Lambda'$ by Lemma 2.7. Thus $g'y$ is an element of $a_x^*_E$ and we deduce that $Nrd_{A,k}(g'y) = Nrd_{A,k}(g^{-1}y)$ is an element of $a_x^*_E$ and therefore $Nrd_{A,k}(E_D(g^{-1}y))$ to be prime to $d$. It follows that $j_{E_D}(x)$ and $j_{E_D}(y)$ have the same simplicial type in $\mathfrak{B}_{\text{red}}(\tilde{G}_{E_D})$ and the first case finishes the proof.

q.e.d.

**Lemma 2.12** Suppose that there is an isometric simplicial group action on $\mathfrak{B}_{\text{red}}(\tilde{G})$ by a cyclic group $T$ of order two. Assume that the generator $t$ of $T$ induces a reflection on the Coxeter diagram. Let $x$ and $y$ be two fixed points of $T$ in $\mathfrak{B}_{\text{red}}(\tilde{G})$ with same cyclic barycentric coordinates up to rotation of the Coxeter diagram, i.e. there is an element $g$ of $\tilde{G}$ which sends $x$ to $y$. Then we have:

1. The points $x$ and $y$ have the same type in $\mathfrak{B}_{\text{red}}(\tilde{G})$ if $\dim_D V$ is odd.

2. There are at most two different possible types for $y$ in $\mathfrak{B}_{\text{red}}(\tilde{G})$ if $\dim_D V$ is even.
The set \( H \) is equal to \( \alpha \), \( \alpha \) is cyclic, because \( Z/mZ \) is, and every element of \( H_1/H_2 \) has order 1 or 2. Thus \( H_1/H_2 \) has at most two elements this proves 2. The groups \( H_1 \) and \( H_2 \) equal if \( m \) is odd because the order of \( H_1/H_2 \) divides \( m \). This proves 1. q.e.d.

3 The Euclidean building of a classical group

From this article on, we consider the following situation. Let the residue characteristic of \( k \) be odd and \( \rho \) be an involution on \( D \), i.e. a bijective ring homomorphism from \( D \) to \( D^{op} \) of order two. The existence of the involution implies that the index is either one or two. We fix an element \( \epsilon \in \{1,-1\} \) and a non-degenerate \( \epsilon \)-Hermitian form \( h \) on \( V \) with adjoint involution \( \sigma \), i.e. \( h \) is a non-degenerate bi-additive map from \( V \times V \) to \( D \) such that

\[
h(va,wb) = \rho(a)\epsilon\rho(h(w,v))b,
\]

for all \( v, w \in V \) and all \( a, b \in D \). We denote the fixed field of \( k \) under \( \rho \) by \( k_0 \). The group of main interest in this article is the unitary group of \( k \), i.e.

\[
G := U(h) := U(\sigma) := \{g \in \tilde{G} \mid g\sigma(g) = id_V \}.
\]

We repeat the description of the Euclidean building \( B(G) \) of \( G \) in terms of lattice functions. Let us recall that the dual \( M^{\#}h \) of an \( oD \)-lattice \( M \) is defined via

\[
M^{\#}h := \{v \in V \mid h(v, M) \subseteq p_D \}.
\]

The dual \( \Lambda^{\#}h \) of an \( oD \)-lattice function \( \Lambda \) of \( V \) is defined via

\[
\Lambda^{\#}h(t) := (\Lambda((-t)+))^{\#}h.
\]

We call \( \Lambda \) self-dual with respect to \( h \) if it is equal to its dual. The set of self-dual lattice functions inherits an affine structure from \( \text{Latt}_{oD} \). (V).

Theorem 3.1 ([BTS7], 2.12, [BS09], 4.2, 3.3) \textbf{There is a unique} \( G \)-equivariant affine bijection from \( B(G) \) \textbf{to the set of self-dual } \( oD \)-lattice functions of \( V \).

In this article we always consider points of \( B(G) \) as self-dual \( oD \)-lattice functions.

4 Extending Hermitian forms

Suppose we are given a finite field extension \( E/k \) and a field automorphism \( \sigma' \) of order at most two on \( E \) which extends \( \rho|_k \). We consider the \( E \)-algebra \( E \otimes_k D \) together with the involution \( \sigma' \otimes_k \rho \), and in the manner of Broussous and Stevens given in [BS09] we fix a map \( \lambda \) from \( E \) to \( k \) which

\[
is non-zero, \( k \)-linear and \( \sigma' \)-\( \sigma \)-equivariant, and which satisfies \( p_{E_0} = \{e \in E_0 \mid \lambda(eo_E) \subseteq p_k \} \), \quad (2)
\]

where \( E_0 \) is the fixed field of \( \sigma' \) in \( E \). We extend \( \lambda \) to a map \( \tilde{\lambda} \) from \( E \otimes_k D \) to \( D \) via

\[
\tilde{\lambda}(x \otimes_k y) := \lambda(x)y.
\]
Let us now choose a suitable $E$-structure on $V$ and assume that there is a $\sigma'$-$\sigma$-equivariant $k$-algebra homomorphism $\phi$ from $E$ to $\text{End}_D(V)$.

We have that $V$ is in a canonical way a right-$E \otimes_k D$-module using $\phi$. We now define a Hermitian structure on that module.

**Proposition 4.1** There is a unique bi-additive map $h^\phi$ from $V \times V$ to $E \otimes_k D$ such that

1. $h^\phi(va,wb) = (\sigma' \otimes_k \rho)(a)h^\phi(v,w)b$ and $(\sigma' \otimes_k \rho)(h^\phi(v,w)) = ch^\phi(v,v)$, for all $a, b \in E \otimes_k D$ and $v, w \in V$, and
2. $\tilde{\lambda} \circ h^\phi = h$.

Assume we are given a second $\sigma'$-$\sigma$-equivariant $k$-algebra homomorphism $\phi'$ from $E$ into $\text{End}_D(V)$. Then, $\phi \circ \phi'^{-1}$ can be described as a conjugation by an element of $G$ if and only if $(V,h^\phi)$ is isomorphic to $(V,h^{\phi'})$ as signed Hermitian modules.

For the proof see [BS99, 5.2]. The proof also is valid if $E \otimes_k D$ is not a skew-field, because the theory of semisimple modules also implies that $\text{Hom}_{E \otimes_k D}(V,E \otimes_k D)$ and $\text{Hom}_D(V,D)$ have the same $k$-dimension. The second part is trivial and can be left to the reader.

For later purposes, we want to show that the passage from $h$ to $h^\phi$ respects the duality for lattice functions if $E \otimes_k D$ is a skew-field, (see [BS99, 5.5] for $D = k$). So, let us assume for the rest of this section that $E \otimes_k D$ is a skew-field and denote it by $\Delta$. Recall that this is equivalent to $[E:k]$ being odd. To study $o_\Delta$ we fix prime elements $\pi_E$ and $\pi_D$ of $E$ and $D$, respectively.

**Lemma 4.2** We abbreviate $e := e(E|k)$ and $f := f(E|k)$, and we extend the valuation $\nu$ on $k$ to a valuation, also denoted by $\nu$, on $o_\Delta$ and $\nu(t)$ it is equal to

$$(\nu \otimes_k \nu)(t) := \sup \{ \inf \{ \nu(e_i) + \nu(d_i) \mid i \in I \} \mid \sum_{i \in I} e_i \otimes_k d_i = t \}$$

for all $t \in \Delta$. Further we have:

(i) The element

$$\pi_\Delta := \pi_E^{\frac{1}{e}} \otimes_k \pi_D^{\frac{1}{f}}$$

is a prime element of $\Delta$.

(ii) The valuation ring of $\Delta$ is

$$o_\Delta \otimes_k o_D + \mathfrak{p}_E^{\frac{1}{e}} \otimes_k \mathfrak{p}_D^{\frac{1}{f}}.$$

(iii) The valuation ideal of $\Delta$ is

$$\mathfrak{p}_E^{\frac{1}{e}} \otimes_k o_D + \mathfrak{p}_E^{\frac{1}{e}} \otimes_k \mathfrak{p}_D^{\frac{1}{f}}.$$

**Proof:** We only need to consider the case $D \neq k$ because the other is trivial. We choose elements $a_1, a_2$ of $o_D$ and $b_i$, $1 \leq i \leq f$, of $o_E$ such that $(a_1, a_2)$ is a $\kappa_D$-basis of $\kappa_D$ and $(b_1, \ldots, b_f)$ a $\kappa_E$-basis of $\kappa_E$. Because of $f$ is odd, the tuple of products

$$(\bar{a}_i b_j)_{i,j}$$

is a $\kappa_D$-basis of $\kappa_\Delta$. The element $\pi_\Delta$ is a prime element of $\Delta$, because

$$\nu(\pi_\Delta) = \frac{1 - e}{2e} + \frac{1}{2} = \frac{1}{2e}.$$
and the tuple
\[(\pi l a b_i)_{0 \leq i \leq 2e-1, 1 \leq i \leq 2, 1 \leq j \leq f}\]
is a \(\nu\)-splitting basis of \(o\Delta|o_k\). From this follows easily assertion (ii), and we get (iii) from
\[
\pi\Delta(o_E \otimes o_k o_D) = p^{l_1-\epsilon}_E \otimes o_k p_D
\]
and
\[
\pi\Delta(p^{l_2}_E \otimes o_k p_D) = \pi_k p^{l_2}_E \otimes o_k o_D = p_E \otimes o_k o_D.
\]

By the Chinese remainder theorem the valuations
\[
\nu(\pi l_1 \otimes_k \pi l_2), \ l_1 \in \{0, \ldots, e - 1\}, \ l_2 \in \{0, 1\}
\]
form a system of representatives of \(\mathbb{Z}/(2e\mathbb{Z})\). And thus by [BT84, (21)] the valuation \(\nu\) on \(\Delta\) is as in the statement of the lemma. q.e.d.

**Proposition 4.3 (Analogous to [BS09], 5.5)** Let \(\phi\) be a \(k\)-algebra monomorphism from \(E\) to \(\text{End}_D(V)\). Let \(M\) be an \(o_D\)-lattice of \(V\) and assume \(M\) to be an \(o_E\)-lattice via \(\phi\), i.e. an \(o_\Delta\)-lattice. We then have
\[
M^\#_h = M^\#_h\phi.
\]

**Proof:** First we show
\[
p\Delta = \{ x \in \Delta \mid \hat{\lambda}(xo_\Delta) \subseteq p_D \}.
\]
The inclusion \(\subseteq\) follows from statement [(iii)] of Lemma 4.2 and the equality (2), more precisely:
\[
\hat{\lambda}(\pi\Delta) = \lambda(o_E)p_D + \lambda(p^{l_1-\epsilon}_E)p_D \subseteq p_D.
\]
The other inclusion \(\supseteq\) follows because the \(o_\Delta\)-module on the right hand side does not contain \(1_\Delta\) by (2).

Now we repeat the argument of [BS09] 5.5.
\[
M^\#_h = \{ v \in V \mid h(v, M) \subseteq p_D \} = \{ v \in V \mid \hat{\lambda}(h^\phi(v, M)) \subseteq p_D \} = \{ v \in V \mid h^\phi(v, M) \subseteq p_\Delta \} = M^\#_h\phi.
\]
q.e.d.

5 Skolem–Noether for \(U(h)\) and conjugate embeddings

Here we analyze when two points of \(\mathfrak{B}(G)\) have the same type. Afterwards we prove Theorem 1.2.

**Proposition 5.1 (Skol, 3.27)** Let \(E|k\) be a \(\sigma\)-invariant field extension in \(\text{End}_D(V)\). Then the group \(G_E\) is a unitary group, more precisely it coincides with \(U(\sigma|_{\text{End}_E \otimes_k D(V)})\). The map \(j_E\) is \(()^\#\)-equivariant, no matter which signed Hermitian form we choose for \(G_E\), and \(j_E(\mathfrak{B}(G) \cap \mathfrak{B}_{\text{red}}(G)E^x)\) is equal to \(\mathfrak{B}(G_E)\).

In the appendix we give a concrete description of \(j_E|_{\mathfrak{B}(G) \cap \mathfrak{B}_{\text{red}}(G)E^x}\). The next proposition gives a criteria for two points in \(\mathfrak{B}(G)\) to have the same type.
Proposition 5.2 Suppose we are given two elements \( x \) and \( y \) of \( B(G) \) with corresponding self-dual lattice functions \( \Lambda_x \) and \( \Lambda_y \), respectively. Then the following statements are equivalent.

1. The points \( x \) and \( y \) have the same type in \( B(G) \).
2. The points \( x \) and \( y \) have the same type in \( B_{red}(G) \).
3. For every real number \( t \) the quotient \( \Lambda_x(t)/\Lambda_x(t+) \) has the same \( \kappa_D \)-dimension as \( \Lambda_y(t)/\Lambda_y(t+) \).
4. There is an element \( g \) of \( G \) such that \( g.x \) is equal to \( y \).

Remark 5.3 There is only one case where \( B(G) \) is not \( B_{red}(G) \) it is the case of the isotropic orthogonal group over \( k \) of rank one. In this case we just take assertion 4. as definition for two points to have the same type in \( B(G) \).

Proof:
- The equivalence of 1. and 4. is general theory of Bruhat–Tits buildings which can be found in [Titt79]. Statement 2. follows from 4. because the reduced norm of an element \( g \) of \( G \) is an element of \( o_E^* \), i.e. \( g \) acts type-preservingly on \( B_{red}(G) \).

- We now prove that 3. follows from 2. The group \( SL_D(V) \) acts transitively on points of the same type in \( B_{red}(G) \), i.e. there is an element \( g \) of \( SL_D(V) \) which sends \( x \) to \( y \), i.e. in terms of lattice functions there is a real number \( s \) such that \( g.\Lambda_x \) is equal to \( \Lambda_y + s \). Without loss of generality we can assume that \( s \) is non-positive, because otherwise we can interchange \( x \) and \( y \). The self-duality of both lattice functions implies

\[
\sigma(g)^{-1}.\Lambda_x = \sigma(g)^{-1}.\Lambda_x^# = (g.\Lambda_x)^# = \Lambda_y^# - s = \Lambda_y - s.
\]

Thus \( g_1 := \sigma(g)g \) is an element of \( g_{\Lambda_x}(-2s) \cap SL_D(V) \). The reduced norm of an element of the radical \( g_{\Lambda_x}(0+) \) of \( g_{\Lambda_x}(0) \) is an element of \( p_k \), i.e. cannot be 1. In particular, \( s \) vanishes and we obtain the third assertion.

- The difficult part of the proposition is that 4. follows from 3. We consider the \( o_D \)-lattice chain \( L_x \) corresponding to \( \Lambda_x \) such that \( L_0 = \Lambda_y(0) \) and analogously \( L'_x \) for \( \Lambda_y \). There is a Witt decomposition \( \{W_i \mid i \in I\} \) of \( V \) with respect to \( h \) which splits both lattice chains. Without loss of generality we can assume that the anisotropic part \( W_0 \) of the Witt decomposition is trivial, because \( L \) and \( L' \) are equal on \( W_0 \) by [Bt87] 2.9]. Let \( r \) be the period of \( L \). We choose a decomposition of \( I \) into two disjoint sets \( I^+ \) and \( I^- \) such that

\[
\sigma(I^+) = I^-.
\]

Let us recall that \( \sigma(i) \) is defined to be \( i' \) if \( \sigma \) sends the projection onto \( W_i \) to the projection onto \( W_{i'} \). Further we define

\[
W^+ := \oplus_{i \in I^+} W_i, \quad W^- := \oplus_{i \in I^-} W_i
\]

and

\[
L^+ := L \cap W^+, \quad L^- := L \cap W^-.
\]

Let \( \mu(L, j) \) be the set of indices \( i \in I \) for which \( W_i \cap L_j \) differs from \( W_i \cap L_{j+1} \). Analogously we define \( \mu(L^+, j) \) and \( \mu(L^-, j) \). Caution: one of the latter sets can be empty.

Case 1 \( \{L_{0+}^+, L_{0-}^-\} = \{L_0, 2\} \): We choose, for \( 0 \leq j < \frac{r}{2} \), injective maps

\[
\phi_j^+ : \mu(L^+, j) \rightarrow \mu(L', j), \quad \phi_j^- : \mu(L^-, j) \rightarrow \mu(L', j) \setminus \text{im}(\phi_j^+).
\]
Proposition 5.4

Let $\phi_1$ and $\phi_2$ be two $k$-algebra monomorphisms from $E$ to $\text{End}_D(V)$ and $x$ be a point of $\mathfrak{B}(G) \cap \mathfrak{B}_{\text{red}}(\tilde{G})^{\phi_1(E)} \cap \mathfrak{B}_{\text{red}}(\tilde{G})^{\phi_2(E)}$. Assume that

1. $\gcd(d(E), d) \mid d(E)$ or
2. $\frac{m \cdot \gcd(d(E), d)}{\gcd([E:k], d)}$ is odd and $(\phi_1(E), a_x)$ and $(\phi_2(E), a_x)$ are equivalent embeddings.

Such a choice is possible because $L_i/L_{i+1}$ and $L'_i/L'_{i+1}$ have the same dimension over $\kappa_D$. We define

$$I^+ := \bigcup_{0 \leq j < \frac{r}{2}} (\text{im} \phi_j^+ \cup \sigma(\text{im} \phi_j^-)),$$

and we put $I^-$ to be the complement of $I^+$ in $I$. Because

$$i \in \mu(L', j)$$

if and only if $\sigma(i) \in \mu(L', -j - 1)$, for all $i \in I$, we have that $I^+ \cap \sigma(I^+)$ is empty, and by symmetry $I^- \cap \sigma(I^-)$ is empty too. Thus

$$\sigma(I^+) = I^-.$$

This new decomposition of $I$ defines

$$W'^+_i := \oplus_{i \in I^+} W_i, \quad W'^-_i := \oplus_{i \in I^-} W_i, \quad L'^+_i := L' \cap W'^+, \quad L'^-_i := L' \cap W'^-.$$

By construction, $L'^+$ and $L'^-$ are lattice sequences such that $L'^+/L'^+_{i+1}$ and $L'^-/L'^-_{i+1}$ have the same $\kappa_D$-dimension. We choose an isomorphism $u$ of $D$-vector spaces from $W'^+$ to $W'^+$ such that $uL'^+$ is equal to $L'^+$. The map

$$g := (u, 0) + \sigma((u^{-1}, 0)) : W'^+ \oplus W'^- \to W'^+ \oplus W'^-$$

is an element of $G$ and $gL$ is equal to $L'$.

Case 2 ($L_0^\# = L_0$ and $r$ is odd): We can construct $W'^+$ and $W'^-$ as in Case 1, but we have to change the definition of $\phi_{\frac{r}{2+1}}^+$. Because of (4), the set $\mu(L', \frac{r}{2+1})$ is invariant under the action of $\sigma$, i.e. we can choose $\phi_{\frac{r}{2+1}}^+$ such that

$$\sigma(\text{im} \phi_{\frac{r}{2+1}}^+) \cap \text{im} \phi_{\frac{r}{2+1}}^+ = \emptyset,$$

because $L_{\frac{r}{2+1}}/L_{\frac{r}{2+1}}$ and $L'_{\frac{r}{2+1}}/L'_{\frac{r}{2+1}}$ have the same $\kappa_D$-dimension. We now conclude as in Case 1.

Case 3 ($L_0^\# = L_1$): Unlike the cases before we have

$$i \in \mu(L', j)$$

if and only if $\sigma(i) \in \mu(L', -j)$, (5)

We follow the proof of the cases 1 and 2, but with the following differences:

1. We consider $0 \leq j \leq \frac{r}{2}$, i.e. if $r$ is even the index $\frac{r}{2}$ is considered too in all formulas.
2. The set $\mu(L', 0)$ is $\sigma$-equivariant.
3. If $r$ is even the set $\mu(L', \frac{r}{2})$ is $\sigma$-equivariant.

For the $\sigma$-equivariant sets we apply the procedure of Case 2 for the choice of the map $\phi_j^+$. After these preparations we conclude as in Case 1 to finish the proof.

q.e.d.

We now come to the proof of Theorem 112.
Then there is an element $g$ of the stabilizer of $x$ in $G$ such that
\[ \phi_1(x) = g\phi_2(x)g^{-1}, \]
for all $x \in E$, if and only if $(V, h^{\phi_1})$ is isomorphic to $(V, h^{\phi_2})$ as signed Hermitian $E \otimes_k D$-modules.

**Proof:** We only have to prove the "if"-part. The other direction is obvious. We write $E_1$ for $\phi_1(E)$. Let $g'$ be an isomorphism from $(V, h^{\phi_2})$ to $(V, h^{\phi_1})$. Then, $g'$ is an element of $G$ and the points $x$ and $g'x$ are elements of $\mathcal{B}(G) \cap \mathcal{B}_{\text{red}}(\tilde{G})^{E^\times}$ of the same type in $\mathcal{B}(G)$. We apply Proposition 2.11 on $x$ and $g'x$ in Case 1. and Lemma 2.12 on $j_{E_1}(x)$ and $j_{E_1}(g'x)$ in Case 2. We conclude that $j_{E_1}(x)$ and $j_{E_1}(g'x)$ have the same type in $\mathcal{B}_{\text{red}}(\tilde{G}_{E_1})$. There is an element $g''$ of $G_{E_1}$ such that $g''g'x = x$ by Proposition 5.2. The element $g := g''g'$ satisfies the desired assertion. q.e.d.

As a condition for part two of Theorem 1.2 we need a new property.

**Definition 5.5** A field extension of non-Archimedean local fields $E|k$ has the **extension property with respect to $D$** if the Galois generator $\tau$ of $E_D|k$ can be extended to an element of $\text{Aut}(E|k)$, the group of $k$-algebra automorphisms of $E$.

**Proposition 5.6** Let $\phi_1$ and $\phi_2$ be two $k$-algebra monomorphisms from $E$ to $E|k$ and $x$ be a point of $\mathcal{B}(G) \cap \mathcal{B}_{\text{red}}(\tilde{G})^{\phi_1(E)^\times}$ such that $\phi_1(E), \sigma_1$ and $\phi_2(E), \sigma_2$ are equivalent embeddings. Assume further that $\sigma'$ is non-trivial on $E$ and that $E|k$ has the extension property with respect to $D$. Then there is an element $g$ of the stabilizer of $x$ in $G$ such that
\[ \phi_1(E) = g\phi_2(E)g^{-1}, \]
if and only if $(V, h^{\phi_1})$ is isomorphic to $(V, h^{\phi_2})$ as signed Hermitian $E \otimes_k D$-modules.

For the proof of Proposition 5.6, we will need the following Lemma:

**Lemma 5.7** Assume that $e(E|k)$ is odd and $\gcd(f(E|k), d)$ is even. The extension $\sigma''$ of the non-trivial Galois element $\tau$ of $E_D|k$ to $E|k$ can be chosen to commute with $\sigma'$ and additionally in a way such that there is a map $\lambda''$ which satisfies (2) and $\sigma''$-invariant, i.e. $\lambda'' \circ \sigma''$ is $\lambda''$.

**Proof:** We denote the residue characteristic of $k$ by $p$ and the maximal unramified field extension of $E|k$ by $E_{\text{ur}}$. Let $\psi$ be an extension of $\tau$ to $E|k$. The group $< \sigma' >$ acts via conjugation on $\psi \text{Aut}(E|E_{\text{ur}})$ whose cardinality is odd by assumption. This action has a fixed point which we denote by $\sigma''$. We can assume that its order is a power of 2, because otherwise we take an appropriate odd power of $\sigma''$. Recall that $E_0$ is the fixed field of $\sigma'$ and denote the fixed field of $\sigma''$ in $E_0$ by $E_{00}$. Take a map $\lambda''_0$ from $E_{00}$ to $k$ which satisfies (2) if we substitute $E$ by $E_{00}$. Then,
\[ \lambda'' := \lambda''_0 \circ \text{tr}_{E|E_{00}} \]
has the desired properties because $E|E_{00}$ is separable and tamely ramified. q.e.d.

The conclusion of the lemma holds in many more cases, but we do not need this generality for our purposes, since for even ramification index we can use Proposition 5.4.

**Proof:** If $e(E|k)$ is odd and $f(E|k)$ and $\frac{m \gcd([E:k], d)}{|E:E|}$ are even, because all the others are covered by Proposition 5.4. Every element of $G$ which stabilizes a point of the facet associated to $\mathfrak{a}_x$ fixes the whole facet because $G$ acts type-preservingly on $\mathcal{B}_{\text{red}}(\tilde{G})$. Thus we can assume $x$ to be the barycenter of $\mathfrak{a}_x$ in $\mathcal{B}_{\text{red}}(\tilde{G})$. Fix an isomorphism $g_1$ from $(V, h^{\phi_1})$ to $(V, h^{\phi_1})$. Define $y$ to be $g_1 x$. In the remaining part we only work with $(V, h^{\phi_1})$ and $\phi_1(E)$, so without loss of generality let us assume that $\phi_1$ is the identity. We consider the description of $j_E$ given in Proposition 2.2, i.e. as a map
\[ j_E : \text{Latt}_{o_{\phi}}(V)^{E^\times} \to \text{Latt}_{o_{\phi}}(W), \]
which has a precise description in terms of self-dual lattice functions for its restriction to $\mathfrak{B}(G) \cap \mathfrak{B}_{\text{red}}(G^E)$. Let $\Theta_x$ and $\Theta_y$ be the self-dual lattice functions corresponding to $j_E(x)$ and $j_E(y)$ respectively. The condition on the embedding type forces the existence of an element $g'$ of $\hat{G}_E$ which sends $x$ to $y$ (and $y$ are barycenters!), i.e. there is a real number $s$ such that $\Theta_y + s$ is equal to $g' \Theta_x$. We take $g'$ such that $s$ is a minimal non-negative real number with the latter property. Such a minimum exists because if not then the set of points of discontinuity of $\Theta_y$ is dense in $\mathbb{R}$. We are done if $s$ is zero by Proposition 5.2. Thus assume $s$ to be positive. We have that $\sigma(g'g)\Theta_x$ is equal to $\Theta_x + 2s$ and the minimality of $s$ shows that there is no positive $t$ smaller than $2s$ together with an element $g''$ of $\hat{G}_E$ such that $g''\Theta_x$ is equal to $\Theta_x + t$. Thus $2s$ divides $1$, i.e. there is an integer $z$ such that $z2s = 1$.

Case 2/z: Then $2s$ divides $\frac{1}{2}$ implying that the quotients $\Theta_x(t)/\Theta_x(t+)$ and $\Theta_x(t + \frac{1}{2})/\Theta_x((t + \frac{1}{2})+)$ are $\kappa_k$-isomorphic. The formula in (c) of Proposition A.1 together with Proposition 5.2 shows that the type of $x$ and $y$ in $\mathfrak{B}(G)$ determines the type of $j_E(x)$ and $j_E(y)$ in $\mathfrak{B}(\hat{G}_E)$, respectively, i.e. $j_E(x)$ and $j_E(y)$ have the same type because $x$ and $y$ have, which leads to a contradiction to the positivity of $s$.

Case $z$ is odd: The fact that $zs$ is $\frac{1}{2}$ implies that $\Theta_x(t)/\Theta_x(t+)$ is $\kappa_k$-isomorphic to

$$\Theta_y(t - \frac{1}{2})\pi_D)/(\Theta_y((t - \frac{1}{2})+)(\pi_D).$$

Here we have $\lambda''$ of Lemma 5.6 and we take the description of $j_E$ from Proposition A.2 with $h^{\text{id}E}$ constructed from $\lambda'' \circ \phi^{-1}_1$ instead of $\lambda \circ \phi^{-1}_1$. In that proposition and afterwards we construct signed Hermitian spaces $(W, h_E)$ and $(W \pi_D, h'_E)$ whose isometry groups are isomorphic to $G_E$. Now, by Proposition A.4 $(W, h_E)$ is $E$-isomorphic to $(W, h'_E)$, because $\sigma'$ is non-trivial on $E$ and the $E$-dimension of $W$, which is $\frac{\max(d(E, k), d)}{[E:k]}$, is even. Thus, by Proposition 5.2 there is an isomorphism from $h_E$ to $h'_E$ which maps $\Theta_x$ to $(\Theta_y + \frac{1}{2})\pi_D$ and the result follows now from Corollary A.3. q.e.d.

The extension property of Proposition 5.6 is necessary, more precisely

**Proposition 5.8** Suppose we are given two embeddings $(E_i, a)$, $i = 1, 2$, with the same self-dual hereditary order and $\sigma$-invariant fields such that $E_1$ is conjugate to $E_2$ by an element $a \cap G$. Then at least one of the following two assertions is true:

1. For every $g \in G$ which conjugates $E_1$ to $E_2$ there is a $u \in a \cap G$ such that for all $x \in E_1$ we have $gxg^{-1} = uxx^{-1}$.

2. There is an automorphism of $E_1|k$ whose restriction to $(E_1)_D$ is non-trivial, in particular $d = 2$.

**Proof:** Assume that there are $g \in G$ and $u \in G \cap a$ such that both conjugate $E_1$ to $E_2$ such that the conjugation of $g$ on $E_1$ can not be witnessed by an element of $G \cap a$. Then $u^{-1}g$ normalizes $E_1$ and it does not fix $(E_1)_D$. Because if it would fix $(E_1)_D$ then we can apply Proposition 5.4 on $(E_i, b)$, $i = 1, 2$, and $G_{E_D}$, where $b$ is the centralizer of $E_D$ in $a$, i.e. we could verify the conjugation of $u^{-1}g$ on $E_1$ by an element of $a \cap G$. This is a contradiction. q.e.d.

### 6 The extension property

In this section we want to analyze the extension property of Proposition 5.6. For that we firstly study a stronger condition.
6.1 The strong extension property

We fix a finite field extension $E|k$. Let $e$ be the ramification index and $f$ be the residue class degree of $E|k$. We say that $E|k$ satisfies the strong extension property if the Frobenius automorphism of the maximal unramified subextension of $E|k$ can be extended to an automorphism of $E|k$. Results concerning the strong extension property were motivated by a communication with P. Schneider. We denote by $E_{ur}|k$ the maximal unramified field extension of $E|k$.

Proposition 6.1  (a) If $E|k$ is normal or isomorphic to a tensor product of a purely ramified and an unramified extension then $E|k$ satisfies the strong extension condition

(b) Let $f$ be prime to $e$. The following assertions are equivalent:

1. $E|k$ satisfies the strong extension property.
2. $f|\not\# \text{Aut}(E|k)$.
3. $\text{Aut}(E|k)$ is isomorphic to a semidirect product $\text{Aut}(E_{ur}|k) \rtimes \mathbb{Z}/(f\mathbb{Z})$.
4. There is subgroup of $\text{Aut}(E|k)$ of order $f$.
5. There is a subfield extension $E'|k$ of $E|k$ of degree $e$.
6. $E$ is $k$-algebra isomorphic to a tensor product of a purely ramified and an unramified extension over $k$.

Proof: We only need to concentrate on (b). We have immediatly:

$3.\Longrightarrow 4.\Longleftrightarrow 5.\Longrightarrow 6.\Longrightarrow 2.$

Let us denote $\text{Aut}(E|E_{ur})$ by $N$. It is group of order dividing $e$.

2. implies 1.: The order of $N$ is prime to $f$. Thus the restriction homomorphism from $\text{Aut}(E|k)$ to $\text{Gal}(E_{ur}|k)$ is surjective by 2. And we deduce 1.

1. implies 3.: Take the $e$th power of a lift of the Frobenius automorphism to get an element of order $f$ in $\text{Aut}(E|k)$. So we have a cyclic subgroup $C$ of order $f$. The restriction to $E_{ur}$ gives an isomorphism from $C$ to $\text{Gal}(E_{ur}|k)$ because both have the same order which is prime to the order of $N$. Thus $\text{Aut}(E|k)$ is isomorphic to $N \rtimes C$. q.e.d.

We now want to decide whether or not a tamely ramified extension is a tensor product of a purely ramified and an unramified extension over $k$. So let $E|k$ be tamely ramified with $f$ and $e$ not necessarily coprime. Define $\tilde{e} := \gcd(\frac{\#\kappa - 1}{\#\kappa_{E} - 1}, e)$. We denote by $\mu_i(E)$ the set of $i$th roots of unity in $E$.

$$\mu_{\text{tame}}(E|k) := \{ x \in \mu_{\#\kappa_{E} - 1}(E) \mid \exists \pi_E \in p_{E} \setminus p_{E}^{2} : \pi_E x^{-1} \in o_k \}.$$ 

Further we define an equivalence relation on $\mu_{\#\kappa_{E} - 1}(E)$ by: $x \sim_{E|k} y$ if and only if $(\frac{x}{\pi})^{\frac{\#\kappa - 1}{\#\kappa_{E} - 1}} = 1$. Let $\zeta$ be a primitive $\tilde{e}$th root of unity in $E$. Then an easy exercise shows that $\sim_{E|k}$ is under

$$\psi : \mathbb{Z}/(\#\kappa_{E} - 1)\mathbb{Z} \rightarrow \mu_{\#\kappa_{E} - 1}(E), \ 1 \mapsto \zeta$$

the push forward of the equivalence relation given by $\tilde{e}\mathbb{Z}/(\#\kappa_{E} - 1)\mathbb{Z}$.

Proposition 6.2 Let $E|k$ be tamely ramified. Then $\mu_{\text{tame}}(E|k)$ is non-empty and an equivalence class of $\sim_{E|k}$ and the following conditions are equivalent:

1. $E|k$ is a tensor product of a purely ramified and an unramified field extension over $k$.
2. $E|k$ contains an $e$th root of some uniformizer of $k$.
3. $1 \in \mu_{\text{tame}}(E|k)$.

4. There exists uniformizers $\pi_E$ and $\pi_k$ for $E$ and $k$, respectively, such that $(\frac{\pi_n}{\pi})^{\#E-1}$ is an element of $1 + p_E$.

5. For all uniformizers $\pi_E$ and $\pi_k$ for $E$ and $k$, respectively, the element $(\frac{\pi}{\pi})^{\#E-1}$ lies in $1 + p_E$.

The concreteness of the proposition allows to construct tamely ramified examples where the strong extension condition fails. It also implies an easy corollary which is a generalization of [Lan94 II.5.12]. The latter states that if $f$ is equal to 1 there is some uniformizer of $k$ which has an $e$th root in $E$.

**Corollary 6.3** Let $E|k$ be a field extension. Suppose that $e'$ is coprime to $\text{char}(k)$.



\[ \text{char}(k) \sum_{i=0}^{f-1} (\#k)^i. \]

Then there is a uniformizer $\pi_k$ of $k$ and an element $\alpha$ of $E$ such that $\alpha e'$ is equal to $\pi_k$.

For example the assumption of Corollary 6.3 is satisfied if $e'$ is a power of 2 and $f \text{char}(k)$ is odd.

**Proof:** Without loss of generality we assume that $E|k$ is tamely ramified because we can turn to the maximal tamely ramified subextension of $E|k$. We apply [Lan94 II.5.12] to find an $e'$th root $x$ of a uniformizer of $E_{ur}$ in $E$. We can now apply Proposition 6.2 on $E_{ur}|k$ because $e'$ is prime to $\sum_{i=0}^{f-1} (\#k)^i$ which is $\frac{\#E-1}{\#k-1}$.

**Proof:** [Prop. 6.2] The non-emptiness of $\mu_{\text{tame}}(E|k)$ comes from [Lan94 II.5.12] and the surjectivity of the $e$th power map from $1 + p_{E_{ur}}$ to $1 + p_{E_{ur}}$. For the next assertion let $x$ be an element of $\mu_{\text{tame}}(E|k)$. Then $y \in \mu_{\#\kappa -1}(E)$ is equivalent to $x$ if and only if $x = uy$ for some $u \in \mu_{\#\kappa -1}(E)$ if and only if $x = u'vy$ for some $u' = u$, $v \in \kappa$. The "only if"-part is a consequence of Bezout’s lemma: Let $\zeta$ be a primitive $\#\kappa -1$-root of unity. By the Henselian lemma: $\zeta \in E_{ur}$. Then $u$ is an $e'j$ power of $\zeta$ for some integer $j$. Now Bezout’s lemma states that there are integers $a$ and $b$ such that

\[ ae + b \frac{\#\kappa -1}{\#k -1} = \zeta. \]

Now the $\frac{\#E-1}{\#k-1}$th power of $\zeta$ is an element of $k$ which proves the "only if" part. The equivalence of the first three assertions is either trivial or part of [Lan94 II.5.12]. We have also $5 \implies 4$ and $3 \implies 4$. From 4. follows 5. because every element $u$ of $\alpha_E^\times$ satisfies $u^{\frac{\#E-1}{\#k-1}} \in 1 + p_E$. We finish the proof in showing that 5. implies 3. Take an element $x$ of $\mu_{\text{tame}}(E|k)$. 5. states that $x$ is equivalent to 1 and thus we get 3. q.e.d.

6.2 The extension property

We now combine the statements of Proposition 6.2 and Theorem 1.2 to get a statement when two appropriate field extensions are conjugate under $G$.

**Remark 6.4** For a field extension $E|k$ in $\text{End}_D(V)$ the extension property with respect to $D$ is equivalent to the strong extension property of $E|E'$ where $E'|k$ is the maximal unramified subextension of $E|k$ of degree prime to $d$.

**Theorem 6.5** Assume that $E_1|k$ and $E_2|k$ are two $\sigma$-invariant but not $\sigma$-fixed field extensions in $\text{End}_D(V)$ such that $E_1^\times$ is contained in the normalizer of a self-dual hereditary order $\mathfrak{a}$ and such that there are $g_1 \in \mathfrak{n}(\mathfrak{a})$ and $g_2 \in G$ which conjugate $E_1$ to $E_2$. Suppose that there is no element of $\mathfrak{a} \cap G$ which conjugates $E_1$ to $E_2$. Then $e(E_1|k)$ is odd and $f(E_1|k)$ and $d$ are even and $E_1|E'$ does not satisfy the strong extension property where $E'$ is the maximal unramified subextension of $E_1|k$ of odd degree. More precisely
(i) None of the statements from 1. to 6. in Proposition 6.1 hold for \( E_1|E' \).

(ii) None of statements from 1. to 5. in Proposition 6.2 hold for \( E_1|E' \) if \( E_1|k \) is tamely ramified.

A Embedding of buildings of classical groups

This appendix is devoted to Proposition 5.1. Its proof can be found in part one of the proof of Lemma 3.27 in [Sko10]. The existence of a signed Hermitian form for \( G_E \) can be found in [Sko10, 1.12]. In the latter dissertation the author did not mention the explicit formula for the restriction of \( j_E \) to \( \mathcal{B}(G) \cap \mathcal{B}_{red}(\tilde{G})^{E^\times} \). For this, one needs to construct a signed Hermitian form \( h_E \) whose unitary group is \( G_E \). The rest is stated in [BL02]. We assume the situation of section 4 with the restriction that \( \phi = \text{id}_E \). In this section we use extensively that the residue characteristic of \( k \) is different from 2. We denote by \( \text{Latt}_1^{o_D}(V) \) the set of \( o_D \)-lattice functions of \( V \) which are \( o_E \)-lattice functions. Let us recall how \( j_E \) is constructed in our situation.

Proposition A.1 ([BL02], II.3.1) There is a pair \((\Delta, W)\) consisting of a skew-field \( \Delta \) which is \( E \otimes_k D \) or \( E \) and a \( \Delta \)-subvector space \( W \) of \( V \) such that

1. \( \text{End}_{E \otimes_k D}(V) \) is \( E \)-algebra isomorphic to \( \text{End}_\Delta(W) \) via
   \[ a \mapsto a|_W, \]  
   (6)

and

2. the map \( j_E \) from \( \mathcal{B}_{red}(\tilde{G})^{E^\times} \) to \( \mathcal{B}_{red}(\tilde{G}_E) \) in terms of lattice functions has the form
   \[ j_E([\Lambda]) = [\Lambda \cap W]. \]

In more detail:

(a) We have that \( \Delta \) is \( E \otimes_k D \) if \( [E : k] \) is odd and \( E \) otherwise. Secondly \( V/W \) is \( \text{gcd}(d, [E : k]) \)-dimensional over \( \Delta \).

(b) Let \( L|k \) be a maximal unramified field extension in \( D \). If \( \text{gcd}(d, [E : k]) = 2 \) and \( f(E|k) \) is odd then \( V \) is \( E \otimes_k D \)-isomorphic to \( W \otimes_k L \) and \( \Lambda \in \text{Latt}_1^{o_D}(V)^{E^\times} \) is equal to \( (\Lambda \cap W) \otimes_{o_k o_L} \).

(c) If \( \text{gcd}(d, f(E|k)) = 2 \) then \( V \) is \( E \otimes_k D \)-isomorphic to \( W \otimes W \pi_D \) where \( \pi_D \) is a uniformizer of \( D \) and \( \Lambda \in \text{Latt}_1^{o_D}(V)^{E^\times} \) is equal to
   \[ (\Lambda \cap W) \oplus ((\Lambda \cap W) + \frac{1}{2}) \pi_D. \]

The point of finding \( h_E \) is the right choice of \( W \). The proof of Proposition A.1 is also included in the proof of:

Proposition A.2 We can choose \( W \) in Proposition (A.1) in a way such that

- the image \( \mathfrak{M} \) of \( h^{|_W \times W} \) is a one dimensional bi-\( \Delta \)-vector space, and
- there is a bi-\( \Delta \)-isomorphism \( \psi \) from \( \mathfrak{M} \) to \( \Delta \) such that \( h_E := \psi \circ h^{|_W \times W} \) satisfies the following.

1. The map (6) is \( \sigma \)-\( \sigma_{h_E} \)-equivariant. In particular \( G_E \) is isomorphic to \( U(h_E) \) via (6).
2. For an element $\Lambda$ of $\text{Latt}_{\rho}(V)^{E^x}$ we have:

$$\Lambda^* \cap W = \begin{cases} \\
(\Lambda \cap W)^{\#_{E^x},} + \frac{1}{2}, & \text{if } \gcd(d, f(E|k)) = 2 \text{ and } \text{tr}(\mathfrak{M}) = \{0\} \\
(\Lambda \cap W)^{\#_{E^x}}, & \text{otherwise} \\
\end{cases}$$

In the first case $(\Lambda \cap W) - \frac{1}{4}$ is self-dual if $\Lambda$ is.

In particular in terms of self-dual lattice functions $j_{E}^{G(W), \mathfrak{M} \in (G)^{E^x}}$ has the form

$$\Lambda \mapsto \begin{cases} \\
(\Lambda \cap W) - \frac{1}{4}, & \text{if } \gcd(d, f(E|k)) = 2 \text{ and } \text{tr}(\mathfrak{M}) = \{0\} \\
(\Lambda \cap W), & \text{otherwise} \\
\end{cases}$$

**Proof:**

**Case 1:** $(\gcd(d, [E : k]) = 1)$ Then $W$ is $V$ and $\Delta$ is $D$ and we apply Proposition 4.3.

**Case 2:** $(\gcd(d, [E : k]) = 2$ and $f(E|k)$ is odd.) Here two divides $e(E|k)$ and we can establish the following situation. There is a two-dimensional unramified and $\rho$-invariant field extension $L \cap D$, and an element $p_{E}$ of $E$ such that $p_{E}^{\sigma}$, which we denote by $\pi_{k}$, is a prime element of $k$, see Corollary 6.3. Further we find a square root $\pi_{D}$ of $\pi_{k}$ in $D$, which normalizes $L$, such that $\rho(\pi_{D})$ is either $+\pi_{D}$ or $-\pi_{D}$. We denote the non-trivial element of $\text{Gal}(L|k)$ by $\tau$. Let $W$ be an arbitrary $\frac{\text{dim}_{E}(V)}{2}$-dimensional $E$-vector space. We define a right-$D$-action on $W \otimes_{k} L$ via

$$(w \otimes_{k} l) . \pi_{D} := (p_{E} w \otimes_{k} l).$$

The space $W \otimes_{k} L$ is $E \otimes_{k} D$-module isomorphic to $V$ by the theory of semi-simple modules and we identify them.

**Step 2.1:** The algebra $E \otimes_{k} D$ is $E$-algebra isomorphic to $M_{2}(E)$ via

$$\pi_{D} \mapsto \begin{pmatrix} p_{E} & 0 \\ 0 & -p_{E} \end{pmatrix}, \quad l \mapsto \begin{pmatrix} 0 & l^{2} \\ 1 & 0 \end{pmatrix},$$

where $l^{2}$ is a unit in $L$ which satisfies $\tau(l^{2}) = -l^{2}$. We identify $E \otimes_{k} D$ with $M_{2}(E)$. Let $\mathfrak{M}$ be the image of $W \times W$ under $h^{id_{E}}$. For elements $A$ of $\mathfrak{M}$ we have

- $A p_{E} = A \pi_{D}$ and
- $\rho(\pi_{D}) A = \sigma^{-1}(p_{E}) A$,

especially if

$$\rho(\pi_{D}) \sigma^{-1}(p_{E}) = \pi_{D} p_{E} \quad (7)$$

we have

$$\mathfrak{M} = \left\{ \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \mid e \in E \right\}.$$ 

and if

$$\rho(\pi_{D}) \sigma^{-1}(p_{E}) = -\pi_{D} p_{E} \quad (8)$$

we have

$$\mathfrak{M} = \left\{ \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \mid e \in E \right\}.$$ 

We define now a signed hermitian form $h_{E}$ on $W$ to be $\psi \circ h^{id_{E}}|_{W \times W}$. Here $\psi$ is $a_{11}|_{\mathfrak{M}}$ for (7) and $a_{21}|_{\mathfrak{M}}$ for (8) where $a_{ij}$ is the map from $M_{2}(E)$ to $E$ which maps a matrix to its entry on position $(i, j)$.

**Step 2.2:** Here we prove

$$(\Lambda \cap W)^{h_{E}} = \Lambda^{\#} \cap W,$$
i.e. we have to prove that \( h_E(w, M) \) is a subset of \( p_E \) if and only if \( h(w, M \otimes o_L) \) is a subset of \( p_D \), for all \( w \in W \) and all full \( o_E \)-lattices \( M \) of \( W \). It follows from

\[
\psi^{-1}(p_E) = \mathfrak{M} \cap \bigcap_{u \in o_E \setminus \{0\}} u^{-1} \tilde{\lambda}^{-1}(p_D).
\]

To prove the last equation we use the decomposition

\[
M_2(E) = E \oplus El' \oplus E\pi_D \oplus El'\pi_D. \tag{9}
\]

For example for Case 7: Let \( A \) be an element of \( \mathfrak{M} \) with coefficients \( e_1, e_2, e_3, e_4 \) in the above decomposition, then we have

\[
e_2 = e_4 = 0, \quad e_1 - e_3 p_E = 0,
\]

and every such matrix is an element of \( \mathfrak{M} \). Thus \( \psi(A) \in p_E \) if and only if \( e_1 \in p_E \), i.e. if and only if

\[
\lambda(e_1 o_E) \subseteq p_k \quad \text{and} \quad \lambda(e_3 o_E) \subseteq o_k,
\]

i.e. if and only if

\[
\tilde{\lambda}(o_E A) \subseteq p_D.
\]

The Case 8 is analogous with the equations

\[
e_1 = e_3 = 0, \quad e_2 - e_4 p_E = 0.
\]

**Case 3:** (\( \gcd(d, f(E|k)) = 2 \)) We follow a similar strategy as in Case 2. We fix an unramified two-dimensional field-extension \( L \) of \( k \) in \( D \) which is \( \rho \)-invariant and a prime element \( \pi_D \) of \( D \) which normalizes \( L \), such that the square of \( \pi_D \) is a prime element of \( k \) denoted by \( \pi_k \) and \( \rho(\pi_D) = \pm \pi_D \). We can embed \( L \) into \( E \), because \( 2 \mid f(E|k) \), and we identify \( L \) with its image under the embedding. As in Case 2 we identify \( E \otimes_k D \) with \( M_2(E) \) but now via

\[
\pi_D \mapsto \begin{pmatrix} 0 & \pi_k \\ 1 & 0 \end{pmatrix}, \quad l' \mapsto \begin{pmatrix} l' & 0 \\ 0 & -l' \end{pmatrix}.
\]

**Step 3.1:** We consider the idempotents \( 1^1 \) and \( 1^2 \) in \( E \otimes_k L \), i.e. \( 1^1 \) is the matrix \( E_{1,1} \) and \( 1^2 \) is the matrix \( E_{2,2} \) in the standard notation of linear algebra. We define \( V^1 := 1^1 V \) and \( W := V^1 \). Recall that \( V^2 = V^1 \pi_D \).

Let \( \mathfrak{M} \) be the image of \( W \times W \) under \( h^{id_E} \). The equations

- \( A1^1 = A \) and
- \( (\sigma' \otimes \rho)(1^1) A = A, \)

for \( A \in \mathfrak{M} \) imply that

\[
\mathfrak{M} = \left\{ \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \mid e \in E \right\}, \tag{10}
\]

or

\[
\mathfrak{M} = \left\{ \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \mid e \in E \right\}. \tag{11}
\]

We take the same \( \psi \) as in Case 2 to define \( h_E. \)

**Step 3.2:** Consider the lattice function \( \Lambda \) under the decomposition \( V = V^1 \oplus V^2 \). We define \( (\Theta_A)(t) \) to be \( V^1 \cap \Lambda(t) \) for (10) and \( V^1 \cap \Lambda(t + \frac{1}{2}) \) for (11), i.e. \( \Lambda(t) \) (resp. \( \Lambda(t + \frac{1}{2}) \)) has the form

\[
\Theta_A(t) \oplus \Theta_A(t - \frac{1}{2}) \pi_D. \tag{12}
\]
Here $\Theta_\Lambda$ is an element of $\text{Latt}_{o_k}^1(W)$. We have to prove:

\[(\Theta_\Lambda)^{#,h_E} = \Theta_\Lambda^\#.\]

**Step 3.2a:** At first, we show the equivalence of the following two statements for an element $A$ of $\mathfrak{M}$.

1. $\psi(A) \in \mathfrak{p}_E$.
2. $\tilde{\lambda}(o_E A) \subseteq \mathfrak{p}_D^2$.

For this we look at decomposition (9) of $E \otimes_k D$, and we obtain for $A$ the relations

\[10 \hspace{1cm} e_3 = e_4 = 0 \text{ and } e_1 = l'e_2.\]

\[11 \hspace{1cm} e_1 = e_2 = 0 \text{ and } e_3 = -l'e_4.\]

From these relations the equivalences follow from (2).

**Step 3.2b:** We only consider (11). The other case is similar. We have to show

\[\Theta_\Lambda^{#,h_E}(t) = \Lambda^\#(t + \frac{1}{4}) \cap W.\]

$\supseteq$: This follows directly from 2.$\Rightarrow$1.$\Rightarrow$1.

\[\subseteq: \text{Let } w \text{ be an element of } \Theta_\Lambda^{#,h_E}(t), \text{ i.e. } h(w, \Theta_\Lambda((-t)+)) \text{ is a subset of } \mathfrak{p}_D^2, \text{ and more precisely it is a subset of } \mathfrak{p}_D^3 \text{ since it is contained in } \mathfrak{k}\mathfrak{p}_D. \text{ Thus}

\[h(w, 1^1\Lambda((-t + \frac{1}{4})+)) = h(w, \Theta_\Lambda((-t)+)) \subseteq \mathfrak{p}_D^3,
\]
i.e.

\[h(w, 1^2\Lambda((-t - \frac{1}{4})+)) \subseteq \mathfrak{p}_D^2 \text{ and } h(w, 1^1\Lambda((-t - \frac{3}{4})+)) \subseteq \mathfrak{p}_D\]

and especially

\[h(w, \Lambda((-t - \frac{1}{4})+)) \subseteq \mathfrak{p}_D.
\]

Thus $w$ is an element of $\Lambda^\#(t + \frac{1}{4})$. q.e.d.

The last proposition answers how to find an element of $G$ which centralizes $E$. But sometimes we look for an element of the normalizer of $E$ in $G$ which is not in the centralizer. This is what the last part of the section is about.

**For this assume that we are in Case 3 of the proof above, which means gcd$(f(E|k), d) = 2.$** Define $W' := W_{\pi_D}$.

\[h'_E := \left\{ \begin{array}{ll} a_{22} \circ h_{\text{id}}|_{W' \times W'}, & \text{if (10)} \\ \pi_k^{-1} a_{12} \circ h_{\text{id}}|_{W' \times W'}, & \text{if (11)} \end{array} \right. \] (13)

and

\[\Theta'_\Lambda(t) := \left\{ \begin{array}{ll} \Lambda(t) \cap W', & \text{if (10)} \\ \Lambda(t - \frac{1}{4}) \cap W', & \text{if (11)} \end{array} \right. \] (14)

for $\Lambda \in \text{Latt}_{o_D}^1(V)^E$. Analogous to Proposition $\Lambda2$, we have the equality $\Theta_\Lambda^{#,h_E} = \Theta_\Lambda^\#$ and affine isomorphisms

\[\mathcal{B}(U(h_E)) \cong \mathcal{B}(G) \cap \mathcal{B}_{\text{red}}(\tilde{G})^E \cong \mathcal{B}(U(h'_E)), \text{ } \Theta_\Lambda \mapsto \Lambda \mapsto \Theta'_\Lambda.\]
Corollary A.3 Suppose we are given two elements $\Lambda_1$ and $\Lambda_2$ of $\mathcal{B}(G) \cap \mathcal{B}_{\text{red}}(G)^{E^c}$ and an isomorphism from $(W', h'_{E})$ to $(W', h'_{E})$ which maps $\Theta_{\Lambda_1}$ to $\Theta'_{\Lambda_2}$. Then for every $\sigma'' \in \text{Aut}(E|k)$ which

- commutes with $\sigma'$ and
- which has nontrivial restriction to $E_D$ and
- for which $\lambda \circ \sigma''$ is $\lambda$

there is an element $g$ of $G$ such that $g\Lambda_1$ is equal to $\Lambda_2$ and $g\sigma g^{-1}$ is equal to $\sigma''(e)$ for all elements $e$ of $E$.

Proof: Consider instead of $h'_E$ the form $\sigma'' \circ h'_E$ and $W'$ with the structure

$$e \ast v := \sigma''^{-1}(e)v, \quad e \in E, \quad v \in W'.$$

If we look at Gram matrices we see immediately that both signed Hermitian spaces $(W', h'_E)$ and $(W', \sigma'' \circ h'_E)$ are isomorphic. Take a splitting basis $(v, w)$ for all $\pi E$. Then there is an isomorphism from $(W', h'_E)$ to $(W', \sigma'' \circ h'_E)$ which maps $w'_i$ into $\sigma''_E \ast w'_i$ for all $i$. This thus isomorphism fixes $\Theta_{\Lambda_2}$, i.e. if we use our assumption we get that there is an isomorphism $f$ from $(W, h_{E})$ to $(W', \sigma'' \circ h'_E)$ such that $\Theta_{\Lambda_1}$ is mapped to $\Theta'_{\Lambda_2}$. From the definition of $h_{E}$ and $h'_{E}$ made of $\text{id}_{E}$ and the fact that $\lambda \circ \sigma''$ is equal to $\lambda$ we deduce

$$h(f(v), f(w)) = h(v, w),$$

for all $v, w \in W$. We extend $f$ via

$$f(v + w\pi_D) := f(v) + f(w)\pi_D, \quad v, w \in W.$$

This $f$ is $D$-linear, because for $w, v \in W$ we have

$$f((v + w\pi_D)l') = f(l'v - l'(w\pi_D)) = f(l'v) - f(l'w)\pi_D = l' \ast f(v) - (l' \ast f(w))\pi_D = -l'f(v) + (l'f(w))\pi_D = f(v)l' - f(w)l'\pi_D = (f(v) + f(w)\pi_D)l' = (f(v + w\pi_D))l'.$$

q.e.d.

It is not clear in general that $(W, h_{E})$ is isomorphic to $(W', h'_{E})$. But:

Proposition A.4 The signed Hermitian space $(W, h_{E})$ is always isomorphic to $(W, \pi_k h'_{E})$. There is an isomorphism from $(W, h_{E})$ to $(W, h'_{E})$, i.e. anisotropic parts are isomorphic, if:

1. $\dim E W$ is even and the anisotropic parts have not dimension 2 or
2. $\dim E W$ is even and $\sigma'$ does not fix $E$ pointwise or
3. $\pi_k$ is a square in $E$. 

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Proof: Firstly assume that $\rho(\pi_D)$ is $\pi_D$. We define $f$ from $W$ to $W'$ via

$$f(w) := w\pi_D^{-1}, \ w \in W.$$ 

Then we have for (11)

$$\pi_kh'_E(f(v), f(w)) = a_{21}(h'_{idE}(f(v), f(w))) = a_{21}(\rho(\pi_D^{-1})h'_{idE}(v, w)\pi_D^{-1}) = a_{21}(\pi_D^{-1}h'_{idE}(v, w)\pi_D^{-1}) = a_{12}(h'_{idE}(v, w)) = h_E(v, w)$$

and for (10)

$$\pi_kh'_E(f(v), f(w)) = \pi_ka_{22}(h'_{idE}(f(v), f(w))) = a_{22}(\pi_Dh'_{idE}(v, w)\pi_D^{-1}) = a_{11}(h'_{idE}(v, w)) = h'_E(v, w).$$

If $\rho(\pi_D)$ is $-\pi_D$ we get that $h_E$ is isomorphic to $-\pi_kh'_E$. The latter is isomorphic to $\pi_kh'_E$ because $-1$ is a square in $E_D/k$. Thus we have that an anisotropic part of $h_E$ has the same $E$-dimension as one of $h'_E$. Recall that such a dimension is not greater than 4. If $\pi_k$ is a square in $E$ then it is also a norm with respect to $N_{E|E_0}$ because $-1$ is a square and the residue characteristic is odd. Thus $\pi_kh'_E$ is isomorphic to $h'_E$. The other cases are made the way such that the dimension of the anisotropic part determines its isomorphism class. q.e.d.

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