Suppression of matter couplings with a vector field in generalized Proca theories

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In the context of generalized Proca theories, we derive the profile of a vector field \( A_\mu \) whose squared \( A_\mu A^\mu \) is coupled to the trace \( T \) of matter on a static and spherically symmetric background. The cubic Galileon self-interaction leads to the suppression of a longitudinal vector component due to the operation of the Vainshtein mechanism. For quartic and sixth-order derivative interactions, the solutions consistent with those in the continuous limit of small derivative couplings correspond to the branch with the vanishing longitudinal mode. We compute the corrections to gravitational potentials outside a compact body induced by the vector field in the presence of cubic, quartic, and sixth-order derivative couplings, and show that the models can be consistent with local gravity constraints under mild bounds on the temporal vector component. The quintic vector Galileon does not allow regular solutions of the longitudinal mode for a rapidly decreasing matter density outside the body.

I. INTRODUCTION

The problems of dark energy and dark matter imply that there may be additional dynamical degrees of freedom (DOFs) beyond those appearing in the Standard Model of particle physics [1]. The most studied case is a scalar field \( \phi \) with a potential energy \( V(\phi) \) [2]. Such a new scalar DOF can be responsible for the late-time cosmic acceleration or can mimic the dark matter property, depending on forms of the potential. There are also other scalar-field models with derivative self interactions—like Galileons [3, 4] and their extensions [5–7]. These derivative interactions can be the source of dark energy [8–11] while suppressing the propagation of fifth forces in the Solar System [11–19] through the operation of the Vainshtein mechanism [20].

The scalar field is not the only possibility for explaining the dark sector of the Universe; the vector field can also play a similar role [21]. The standard massless Maxwell field, which respects the \( U(1) \) gauge symmetry, has two transverse electric and magnetic polarizations. The gauge symmetry is explicitly broken by introducing a mass term or derivative interactions of the vector field, in which case the longitudinal propagation emerges. In Refs. [22–25], the four-dimensional action of a massive Proca field with nonminimal derivative couplings to gravity was constructed from the requirement of keeping two transverse and longitudinal modes besides two tensor polarizations arising from the gravity sector (see Refs. [26–33] for related works). In such generalized Proca theories, the equations of motion remain of second order, so there are no Ostrogradski instabilities. It is also possible to go beyond the second-order domain without increasing the number of propagating DOFs [34, 35].

If we apply generalized Proca theories and their extension to the isotropic and homogeneous cosmological background, there exists an interesting de Sitter attractor with a constant temporal vector component [36]. The spatial vector components can be treated as perturbations on such a background, which is consistent with the analysis on the anisotropic cosmological background with time-dependent spatial components [37]. There are dark energy models in which all the stability conditions of perturbations can be consistently satisfied [38]. Moreover, the presence of an intrinsic vector mode offers the possibility for realizing the gravitational interaction weaker than that in general relativity (GR) [38–40]. This allows one to distinguish dark energy models in generalized Proca theories from those in GR.

On a static and spherically symmetric background, the existence of hairy black hole solutions was recently studied in the context of generalized Proca theories [41–43]. For massless or massive vector fields without derivative interactions the spatial vector components vanish identically [51], so the background geometry is simply described by the Reissner-Nordström or the Schwarzschild space-time. The existence of derivative interactions gives rise to a variety of hairy black hole solutions with nonvanishing temporal and longitudinal vector components [44–46]. This leads to the difference between two metric components around the black hole horizon. The deviation from GR in the nonlinear regime of gravity can be potentially probed by future measurements of gravitational waves around black holes.

Inside the Solar System, the fifth force mediated by the vector field \( A_\mu \) nonminimally coupled to gravity should be screened for the consistency with local gravity experiments [51]. In Ref. [52], the propagation of fifth forces around a spherically symmetric and static body was studied in the presence of cubic and quartic vector Galileon interactions under the approximation of weak gravity. The cubic derivative interaction leads to a suppression of the longitudinal component \( A_1 \) thanks to the Vainshtein mechanism. The quartic derivative coupling gives rise to the branch \( A_1 = 0 \), so the gravitational potentials are subject to modifications only through the temporal vector component \( A_0 \). For both
cubic and quartic interactions, the models can be compatible with local gravity constraints under mild bounds on $A_0$. This property also persists even in beyond-generalized Proca theories [53].

The analysis of Ref. [52] assumed that a direct coupling between the vector field and matter is absent, so the vector-matter interaction arises indirectly from nonminimal gravitational couplings with the vector field. It is not yet clear whether the existence of direct vector-matter interactions leads to the screening of fifth forces at the level of being compatible with local tests of gravity. In this paper, we will address the issue of the screening mechanism in the presence of a matter coupling of the form $Q A_\mu A^\mu T$, where $Q$ is the coupling strength and $T$ is the trace of the energy-momentum tensor of matter. The quantum corrections to the generalized Proca action can be generated by matter loops, but the computation of one-loop corrections to the vector-field propagator shows that the theory remains healthy without the appearance of new ghosty DOFs in the domain of the effective theory [54]. We will derive the vector-field profile around a compact body in the presence of cubic, quartic, quintic, and sixth-order-derivative generalized Proca interactions and estimate the corrections to leading-order gravitational potentials in GR.

This paper is organized as follows: In Sec. II we present the full equations of motion on the static and spherically symmetric background in generalized Proca theories with the matter coupling. In Sec. III we derive the profiles of temporal and longitudinal vector components with cubic derivative interactions both inside and outside the body. We compute the corrections to gravitational potentials induced by the vector field and show how the Vainshtein mechanism is efficient to suppress the propagation of fifth forces even with matter couplings. In Secs. IV, V and VI we study the propagation of the vector field in the presence of quartic, quintic, and sixth-order derivative interactions, respectively. While the model with quintic derivative coupling does not possess regular solutions of the longitudinal mode for a rapidly decreasing density profile outside the body, the quartic and sixth-order interactions give rise to solutions with the vanishing longitudinal mode. The latter models can be consistent with local gravity constraints under mild bounds on the temporal vector component. Sec. VII is devoted to conclusions.

II. EQUATIONS OF MOTION

We consider a vector field $A_\mu$ with the field strength $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, where $\nabla_\mu$ is the covariant derivative operator. We introduce a matter perfect fluid given by the energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}},$$

where $g$ is a determinant of the metric tensor $g_{\mu\nu}$, and $L_m$ is the matter Lagrangian density. We assume that the vector field is coupled to the matter sector with the interacting Lagrangian density

$$L_{\text{coupling}} = \frac{Q}{M_{\text{pl}}} XT,$$

where $Q$ is a dimensionless coupling constant, $M_{\text{pl}}$ is the reduced Planck mass, $T$ is the trace of $T_{\mu\nu}$, and

$$X = -\frac{1}{2} A_\mu A^\mu.$$  

We assume that $|Q|$ is at most of the order 1.

The generalized Proca theories are the second-order vector-tensor theories given by the action [22, 25]

$$S = \int d^4x \sqrt{-g} \left( F + \sum_{i=2}^{6} \mathcal{L}_i + \mathcal{L}_m + \mathcal{L}_{\text{coupling}} \right),$$

with $F = -F_{\mu\nu}F^{\mu\nu}/4$, and

$$\mathcal{L}_2 = G_2(X) - 2g_4(X)F,$$

$$\mathcal{L}_3 = G_3(X)\nabla_\mu A^\mu,$$

$$\mathcal{L}_4 = G_4(X) R + G_{4,\alpha}(X) \left[ (\nabla_\mu A^\mu)^2 - \nabla_\mu A_\nu \nabla^\nu A^\mu \right],$$

$$\mathcal{L}_5 = G_5(X)G_{\mu\nu} \nabla^\mu A^\nu - \frac{1}{6} G_{5,X}(X) \left[ (\nabla_\mu A^\mu)^3 - 3\nabla_\mu A^\mu \nabla_\rho A_\sigma \nabla^\rho A^\sigma + 2\nabla_\rho A_\sigma \nabla^\rho A^\sigma A_\nu \right] - g_5(X)\hat{F}^{\mu\nu} \nabla_\mu A_\rho \nabla^\rho A_\beta,$$

$$\mathcal{L}_6 = G_6(X) L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{1}{2} G_{6,X}(X) \hat{F}^{\mu\nu} \nabla_\mu A_\alpha \nabla_\beta A_\nu.$$

where $G_{2,3,4,5,6}$ and $g_{4,5}$ are functions of $X$ with the notation $G_{i,X} \equiv \partial G_i / \partial X$, $R$ is the Ricci scalar, and $G_{\mu \nu}$ is the Einstein tensor. The tensors $\tilde{F}^{\mu \nu}$ and $L^{\mu \nu \alpha \beta}$ are defined, respectively, by

$$\tilde{F}^{\mu \nu} = \frac{1}{2} \mathcal{E}^{\mu \nu \alpha \beta} F_{\alpha \beta}, \quad L^{\mu \nu \alpha \beta} = \frac{1}{4} \mathcal{E}^{\mu \nu \rho \sigma} \mathcal{E}^{\alpha \beta \gamma \delta} R_{\rho \sigma \gamma \delta},$$

(2.10)

where $\mathcal{E}^{\mu \nu \rho \sigma}$ is the Levi-Civita tensor normalized by $\mathcal{E}^{\mu \nu \rho \sigma} \mathcal{E}_{\mu \nu \rho \sigma} = -4!$, and $R_{\rho \sigma \gamma \delta}$ is the Riemann tensor. The theory with constant $G_6(X)$ corresponds to the $U(1)$ gauge-invariant derivative interaction advocated by Horndeski [55]. The Lagrangians containing the functions $g_4(X)$ and $g_5(X)$ correspond to intrinsic vector modes.

We consider the static and spherically symmetric background described by the line element

$$ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(2.11)

where $f(r)$ and $h(r)$ are arbitrary functions of the distance $r$ from the center of symmetry. The vector-field profile compatible with the above background is given by [52]

$$A_\mu = (A_0(r), A_1(r), 0, 0).$$

(2.12)

The quantity $X$ can be expressed as $X = X_0 + X_1$, where

$$X_0 = \frac{A_0^2}{2f}, \quad X_1 = \frac{hA_1^2}{2}.$$

(2.13)

For the matter sector, we consider the perfect fluid with the energy-momentum tensor $T^{\mu \nu} = (\rho_m + P_m)u^\mu u_\nu + P_m \delta^{\mu \nu}$, where $\rho_m$ is the density, $P_m$ is the pressure, and $u^\mu = (f^{-1/2}, 0, 0, 0)$ is the fluid four-velocity in the rest frame. Thus, the trace $T$ is given by

$$T = T^{\mu \mu} = -\rho_m + 3P_m,$$

(2.14)

the interacting Lagrangian density [22] does not vanish except for the radiation ($\rho_m = 3P_m$).

Varying the action (2.4) with respect to $A_0$ and $A_1$, we obtain the equations for the motion of the proof and longitudinal vector components, respectively, as

$$r f [2fh(rA_0' + 2A_0' + r f (fh' - f'h)A_0'] (1 - 2g_4) - 2r^2 f^2 A_0 G_{2,X} - rf A_0 [2rh A_1' + (rf'h + rh' + 4fh)A_1] G_{3,X} + 4f^2 A_0 (rh' + h - 1) G_{4,X} - 8f A_0 [rfh A_1' - (r f'h + rh' + fh)X_1] G_{4,X} + 2r^2 A_0' (2rh A_1' + 2f'h X_0 - 2fh X_1 - hA_0 A_1') g_{4,4} + f A_0 [(3h - 1) h A_1' + h (h - 1) (fA_1' + 2fA_1')] G_{5,5} = 2f h A_0 X_1 [2f A_1' + f (f'h + h') A_1] G_{5,X} - 2f [f (3h - 1) h A_1' + h (h - 1) (2f A_1' - f'A_0')] G_{6} - 4fh A_0 X_1 [h A_0 A_1' - fh A_1' - 2f'h X_0 + 2fh X_1] G_{6,X} - 2f [4fh^2 X_1 A_1' - 2fh X_0 f A_0' + 2f (6h - 1) hX_1 A_0' + h (h - 1) A_0 A_1''] - 2fh^2 (3h - 1) A_0 A_1' G_{6,X} - 4fh [2f h A_1' A_0' - (rf'h - 3rf'h - 2f'h) A_0' - 2rf'h A_1'] g_{5,5} - 4rf h A_0 A_1' A_0' + 4fh X_1 A_0' - 2A_1 [f'h X_0 - f'h X_1] g_{5,5} = \frac{2Qr^2 f^2 A_0}{M_{\text{pl}}^2} (\rho_m - 3P_m),$$

(2.15)

$$A_1 [r^2 G_{2,X} - 2(r f'h + fh - f) G_{4,X} + 4h(r A_0 A_0' - r f X - f X) G_{4,X} - r^2 A_0 g_{4,4} X G_{6,L} - 2h A_0'^2 (3h - 1) G_{6,X} - 2fh^2 X_1 A_0'^2 G_{6,X} - \frac{f Q r^2}{M_{\text{pl}}^2} (\rho_m - 3P_m)] = r [f (r X - A_0 A_1') + 4f X] G_{3,X} + 2f'h X_1 G_{5,X} + (A_0 A_0' - f X) [(1 - h) G_{5,X} - 2h X_1 G_{5,X}] - 2rh A_0'^2 (g_{5} + 2X_{1,95,X}),$$

(2.16)

where a prime represents a derivative with respect to $r$. To derive the gravitational equations of motion, we write the metric (2.11) in the form $ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2$. Varying the action (2.4) with respect to $f$, $h$, $c_7$, and setting $\zeta = 0$ at the end, it follows that

$$\left( c_1 + \frac{c_2}{r} + \frac{c_3}{r^2} \right) h' + c_4 + \frac{c_5}{r} + \frac{c_6}{r^2} = A_1,$$

$$- \frac{h}{f} \left( c_1 + \frac{c_2}{r} + \frac{c_3}{r^2} \right) f' + c_7 + \frac{c_8}{r} + \frac{c_9}{r^2} = A_2,$$

$$\left( c_{10} + \frac{c_{11}}{r} \right) f'' + \left( c_{12} + \frac{c_{13}}{r} \right) f' + \frac{c_2}{2f} + \frac{c_4}{r} f'h' + \left( c_{15} + \frac{c_{16}}{r} \right) f' + \left( -\frac{c_8}{2h} + \frac{c_{17}}{r} \right) h' + c_{18} + \frac{c_{19}}{r} = A_3,$$

(2.17)

(2.18)

(2.19)
where \(c_i\)'s \((i = 1, 2, \ldots, 19)\) are given in the Appendix, and

\[
A_1 = \rho_m + \frac{Q}{M^2_{\text{pl}}} T(X_0 - X_1), \quad A_2 = P_m + \frac{Q}{M^2_{\text{pl}}} T(X_0 - X_1), \quad A_3 = -2P_m - \frac{2Q}{M^2_{\text{pl}}} T(X_0 + X_1).
\]  

(2.20)

The continuity equation of the matter sector is given by

\[
P' + \frac{f'}{2f} (\rho_m + P_m) - \frac{Q}{M^2_{\text{pl}}} (\rho'_m - 3P'_m) X = 0,
\]

(2.21)

which follows from Eqs. (2.15) – (2.19).

In the whole analysis of this paper, we take into account the Einstein-Hilbert term, such that

\[
G_4(X) \geq \frac{M^2_{\text{pl}}}{2},
\]

(2.22)

where \(M_{\text{pl}}\) is the reduced Planck mass. If there exist derivative couplings \(G_i(X)\) and \(g_i(X)\) with even indices \(i\) alone, Eq. (2.16) reduces to

\[
A_1 \left[ F(A_1, A_0, A'_0, f, h, f') - \frac{f Q r^2}{M^2_{\text{pl}}} (\rho_m - 3P_m) \right] = 0.
\]

(2.23)

The function \(F\) consists of the sum of terms written in the forms \(\beta_i \tilde{F}_i\) (with \(i = 2, 4, 6\)) and \(\gamma_4 \tilde{G}_4\), where \(\beta_i\) and \(\gamma_4\) are coupling constants in \(G_i(X)\) and \(g_4(X)\), respectively, and \(\tilde{F}\) and \(\tilde{G}_4\) are regular functions of \(A_0, A_1, f, h\) and their derivatives. From Eq. (2.23), the vector component \(A_1\) has two branches satisfying (i) \(A_1 = 0\), or (ii) \(F = f Q r^2 (\rho_m - 3P_m)/M^2_{\text{pl}}\). In the limit that \(\beta_i \to 0\) and \(\gamma_4 \to 0\), the function \(F\) vanishes, so branch (ii) does not exist for nonrelativistic matter \((P_m \ll \rho_m)\) with a nonvanishing coupling constant \(Q\). Hence, the branch consistent with the general-relativistic limit corresponds to \(A_1 = 0\). This property does not generally hold for the couplings \(G_i(X)\) and \(g_i(X)\) with odd indices \(i\), in which case the nonvanishing branch of \(A_1\) can arise from Eq. (2.16).

We have derived the field equations for general theories given by the action (2.4), but we will focus on the theories with

\[
g_4(X) = 0, \quad g_5(X) = 0,
\]

(2.24)

in the following discussion. This reflects the fact that the analysis with the derivative couplings \(G_{3,4,5,6}(X)\) is sufficiently general to understand basic properties of the screening mechanism in generalized Proca theories.

As we will see in subsequent sections, the temporal vector component \(A_0\) is generally dominated by a constant \(a_0\) with a small variation around it. In such cases, it is convenient to express the quantity \(X_0\) in the form

\[
X_0 = \tilde{X}_0 + \delta X_0, \quad \tilde{X}_0 = \frac{a_0^3}{2f(r)},
\]

(2.25)

where \(\delta X_0\) characterizes the deviation from \(\tilde{X}_0\). Unless \(a_0^3\) is very much smaller than \(M^2_{\text{pl}}\), the last term on the l.h.s of Eq. (2.21) gives rise to a large modification to the pressure \(P_m\) relative to the \(Q = 0\) case. By defining

\[
\rho \equiv \rho_m - \frac{Q}{M^2_{\text{pl}}} (\rho_m - 3P_m) \tilde{X}_0, \quad P \equiv P_m - \frac{Q}{M^2_{\text{pl}}} (\rho_m - 3P_m) \tilde{X}_0,
\]

(2.26, 2.27)

it is possible to rewrite Eq. (2.21) without having the contribution from \(\tilde{X}_0\). Expressing \(\rho_m\) and \(P_m\) in terms of \(\rho\) and \(P\) and substituting them into Eq. (2.21), it follows that

\[
P' + \frac{f'}{2f} (\rho + P) + \frac{Q [M^2_{\text{pl}} f T'_s + Q a_0^3 (fT_s)] (\delta X_0 + X_1)}{M^2_{\text{pl}} f + Q a_0^3} = 0,
\]

(2.28)

where

\[
T_s \equiv -\rho + 3P = \left(1 + 0 \frac{Q a_0^3}{M^2_{\text{pl}} f} \right) T.
\]

(2.29)
For nonrelativistic matter characterized by $P_m \ll \rho_m$, the matter-coupling term in Eq. (2.27) does not dominate over $P_m$ under the condition

$$\left| \frac{Q}{M_{pl}^2} \rho_m \dot{X}_0 \right| \lesssim P_m, \quad (2.30)$$

which is assumed in the following discussion. As long as the screening mechanism operates to suppress both $X_1$ and $\delta X_0$, the $Q$-dependent terms in Eq. (2.28) work as tiny corrections to the continuity equation $P' + f' (\rho + P)/(2f) = 0$ in GR. We also note that the terms on the rhs of Eqs. (2.17) – (2.19) are expressed, respectively, as

$$A_1 = \rho + \frac{\dot{Q}}{M_{pl}^2} T_1 (\delta X_0 - X_1), \quad A_2 = P + \frac{\dot{Q}}{M_{pl}^2} T_1 (\delta X_0 - X_1), \quad A_3 = -2P - \frac{2\dot{Q}}{M_{pl}^2} T_1 (\delta X_0 + X_1), \quad (2.31)$$

where

$$\dot{Q} \equiv \frac{Q}{1 + Qa_0^2/(M_{pl}^2 f)}. \quad (2.32)$$

Unlike Eq. (2.20), the matter-coupling terms in Eq. (2.31) do not contain $\dot{X}_0$. For nonrelativistic matter, the condition (2.30) translates to $|Qa_0^2| \ll M_{pl}^2 f$. Then, the coupling $\dot{Q}$ is approximately equivalent to $Q$ with $T_1 \simeq T$ in Eq. (2.29).

We will exploit the rescaled energy density $\rho$ and the pressure $P$ to discuss the profile of a spherically symmetric body with radius $r_\ast$. To derive solutions to the vector-field and gravitational potentials, we consider the following matter density profile:

$$\rho(r) = \begin{cases} \rho_0 & \text{(for } r < r_\ast) \\ \rho_0 \mu & \text{(for } r > r_\ast) \end{cases}, \quad (2.33)$$

where $\rho_0$ is a constant density, and $\mu$ is a dimensionless constant much smaller than 1. In Sec. III we will also numerically obtain solutions to the vector field for a varying matter density. The Schwarzschild radius of the source is defined by

$$r_g(r) = \frac{1}{M_{pl}^2} \int_0^r \rho(\hat{r}) \hat{r}^2 d\hat{r}. \quad (2.34)$$

Taking the vacuum limit $\mu \to 0$ outside the body, it follows that $r_g \simeq \rho_0 r_\ast^2/(3M_{pl}^2)$. We introduce the gravitational potentials $\Psi$ and $\Phi$, as

$$f(r) = e^{2\Psi(r)}, \quad h(r) = e^{-2\Phi(r)}, \quad (2.35)$$

and employ the weak gravity approximation

$$\Phi_\ast \equiv \frac{\rho_0 r_\ast^2}{M_{pl}^2} \ll 1, \quad (2.36)$$

which amounts to the condition $r_g \ll r_\ast$.

In GR without the coupling $Q$, the gravitational potentials $\Psi$ and $\Phi$ inside the body ($r < r_\ast$) are given by the internal Schwarzschild solution

$$e^{\Psi_{GR}} = \left( \frac{3}{2} - \frac{\Phi_\ast}{3} \right) \frac{3}{2} \frac{1 - \frac{\Phi_\ast r^2}{3}}{r^2}, \quad e^{\Phi_{GR}} = \left( 1 - \frac{\Phi_\ast r^2}{3} \right)^{-1/2}, \quad (2.37)$$

with the pressure

$$P = \sqrt{1 - \Phi_\ast r^2/(3r_\ast^2)} - \sqrt{1 - \Phi_\ast/3} \frac{3\sqrt{1 - \Phi_\ast/3} - \sqrt{1 - \Phi_\ast r^2/(3r_\ast^2)}}{3\sqrt{1 - \Phi_\ast/3} - \sqrt{1 - \Phi_\ast r^2/(3r_\ast^2)}} \rho_0. \quad (2.38)$$

Under the approximation (2.30), the ratio between $P$ and $\rho_0$ can be estimated as $P/\rho_0 \simeq \Phi_\ast (1 - r^2/r_\ast^2)/12 \ll 1$. In such cases, the condition (2.30) translates to

$$\left| \frac{Q a_0^2}{M_{pl}^2 f} \right| \lesssim \Phi_\ast. \quad (2.39)$$
In the presence of the vector field coupled to matter, the gravitational potentials \( \Psi_{GR} \) are subject to modifications due to the existence of the \( Q \)-dependent terms in Eq. (2.31). Provided that the two conditions

\[
\left| \frac{\ddot{Q}}{M_{pl}} \frac{\delta X_0}{M_{pl}} \right| \ll \Phi_*, \quad \left| \frac{\ddot{Q}}{M_{pl}} \frac{X_1}{M_{pl}} \right| \ll \Phi_* \tag{2.40}
\]

are satisfied, the contributions arising from the matter coupling can be treated as corrections to the leading-order gravitational potentials \( \Psi_{GR} \). Under the operation of the screening mechanism, we will show that it is possible to satisfy the conditions (2.40).

In the vacuum limit, the gravitational potentials outside the body in GR are given by

\[
e^{\Psi_{GR}} = \left( 1 - \frac{\Phi_*}{3} \frac{r_*}{r} \right)^{1/2}, \quad e^{\Phi_{GR}} = \left( 1 - \frac{\Phi_*}{3} \frac{r_*}{r} \right)^{-1/2} \tag{2.41}
\]

The presence of the vector field coupled to matter gives rise to modifications to \( \Psi_{GR} \) and \( \Phi_{GR} \), but they can be again treated as corrections to the leading-order gravitational potentials for the vector components obeying the conditions of Eq. (2.40).

For the comparison with the results derived for \( Q = 0 \) in Ref. 52, we will adopt the notations

\[
\phi(r) = -\frac{A_0(r)}{f(r)}, \quad \chi'(r) = A_1(r), \tag{2.42}
\]

where \( \chi(r) \) is a longitudinal scalar. The transverse vector mode in \( A_1 \) vanishes due to its regularity at \( r = 0 \). By using \( \rho \) and \( P \) defined by Eqs. (2.26) and (2.27) as well as the coupling \( \tilde{Q} \) given by Eq. (2.32), the terms containing \( Q \) in Eqs. (2.15) and (2.16) can be expressed as \(-2\tilde{Q}r^2 f^2 A_0(\rho - 3P)/M_{pl}^2\) and \(-f\tilde{Q}r^2 A_1(\rho - 3P)/M_{pl}^2\), respectively. We consider nonrelativistic matter satisfying \( P \ll \rho \) and employ the approximation \( \tilde{Q} \simeq Q \) under the condition (2.39).

### III. CUBIC VECTOR GALILEONS

Let us begin with the cubic Galileon model given by the functions

\[
G_2 = m^2 X, \quad G_3 = \beta_3 X, \quad G_4 = \frac{M_{pl}^2}{2}, \quad G_5 = 0, \quad G_6 = 0, \tag{3.1}
\]

where \( m \) is a constant having a dimension of mass, and \( \beta_3 \) is a dimensionless constant. From Eqs. (2.15) and (2.16), we obtain the following equations of motion:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \phi' \right) - \beta_3 \phi \frac{d}{dr} \left( r^2 \chi' \right) + 2 \phi \left( \Psi'' + \Psi^2 - \Psi' \Phi' \right) - \left( \beta_3 \phi \chi' - 3 \phi' - \frac{4\phi}{r} \right) \Psi' + \left( \beta_3 \phi \chi' - \phi' \right) \Phi' \\
= -e^{2\Psi} \phi \left( \frac{\tilde{Q} \rho}{M_{pl}^2} - m^2 \right), \tag{3.2}
\]

\[
\beta_3 \left[ e^{2\Psi} \phi' + \frac{2}{r} e^{-2\Psi} \chi'^2 + (e^{2\Psi} \phi^2 + e^{-2\Psi} \chi'^2) \Psi' \right] = \left( \frac{\tilde{Q} \rho}{M_{pl}^2} - m^2 \right) \chi'. \tag{3.3}
\]

For \( |m| \) smaller than the order of the today’s Hubble expansion rate \( H_0 \simeq 10^{-33} \text{ eV} \), the term \( |\tilde{Q} \rho/M_{pl}^2| \) is much larger than \( |m^2| \) for \( |\tilde{Q} \rho| \gg \rho_c \), where \( \rho_c \simeq 10^{-29} \text{ g/cm}^3 \) is today’s critical density. Unless \( |Q| \) is extremely smaller than 1, the condition \( |\tilde{Q} \rho| \gg \rho_c \) is well satisfied in the Solar System. Hence, we will ignore the term \( m^2 \) relative to \( \tilde{Q} \rho/M_{pl}^2 \) in the whole analysis of this paper. Under the condition (2.39), the coupling \( \tilde{Q} \) defined by Eq. (2.32) is at most of the order \( Q[1 + O(\Phi_*)] \). We neglect the contribution of the order \( \Phi_* \) in \( \tilde{Q} \) for the estimations of \( \phi \) and \( \chi' \), so that the term \( \tilde{Q} \rho/M_{pl}^2 \) appearing on the rhs of Eqs. (3.2) and (3.3) is approximated as \( Q \rho/M_{pl}^2 \).

We deal with the general-relativistic gravitational potentials \( \Psi_{GR} \) and \( \Phi_{GR} \) as leading-order contributions to \( \Psi \) and \( \Phi \), respectively, and substitute them into Eqs. (3.2) and (3.3) to obtain the solutions to \( \phi \) and \( \chi' \). We then plug the solutions of vector-field profiles into Eqs. (2.17) and (2.18) to derive corrections to the leading-order gravitational potentials outside the body.
A. \( r < r_* \)

We first derive the field profiles for the distance \( r \) smaller than \( r_* \). Substituting Eq. (2.37) into Eqs. (3.2) and (3.3), we obtain

\[
\frac{d}{dr} \left( r^2 \phi' \right) - \beta_3 \phi \frac{d}{dr} \left( r^2 \phi' \right) + (1 + Q) \phi \Phi_+ r^2 + \beta_3 \phi \chi' \Phi_+ r^3 \approx 0, \tag{3.4}
\]

\[
\beta_3 \left( \phi \frac{\phi'}{r} + \frac{\phi'^2}{6r^2} r \right) \approx Q \frac{\Phi_+}{r^2} \chi'. \tag{3.5}
\]

From Eq. (3.5), it follows that

\[
\chi'(r) = \frac{Q \Phi_+}{4 \beta_3 r^2} \left[ 1 - \sqrt{1 - \frac{8 \beta_3 r^2 \phi}{Q^2 \Phi_+^2} \left( \phi' + \frac{\phi \Phi_+}{6r^2} r \right)} \right] r, \tag{3.6}
\]

where we have chosen the branch recovering the solution \( \chi'(r) \to 0 \) in the continuous limit \( \beta_3 \to 0 \). We will focus on the positive derivative coupling

\[
\beta_3 > 0, \tag{3.7}
\]

but we will not restrict the signs of \( Q \).

Analogous to the discussion given in Ref. [52], we search for the solution where the temporal vector component \( \phi \) is close to a positive constant \( \phi_0 \), such that

\[
\phi(r) = \phi_0 + f(r), \quad |f(r)| \ll \phi_0. \tag{3.8}
\]

In what follows, we identify the constant \( a_0 \) in Eq. (2.25) with \( \phi_0 \). After deriving the solutions to \( \phi(r) \) and \( \chi'(r) \) under the assumption (3.8), we can confirm that the term \( \beta_3 \phi \chi' \Phi_+ r^3 / (6r^2) \) in Eq. (3.4) is negligible relative to other contributions. Integrating Eq. (3.4) after replacing \( \phi \) with \( \phi_0 \), we obtain

\[
r^2 \phi' - \beta_3 \phi_0 r^2 \chi' + (1 + Q) \phi_0 \Phi_+ r^3 = 0, \tag{3.9}
\]

where the integration constant is set to 0 to satisfy the boundary condition \( \phi'(0) = 0 \). Now, we substitute Eq. (3.6) into Eq. (3.9) by replacing \( \phi \) with \( \phi_0 \) and then solve Eq. (3.9) for \( \phi'(r) \). This process leads to

\[
\phi'(r) = -\frac{\phi_0 \Phi_+ \mathcal{F}_1}{3r^2}, \tag{3.10}
\]

\[
\phi(r) = \phi_0 \left( 1 - \frac{\Phi_+ r^2}{6 r_*^2} \right), \tag{3.11}
\]

where

\[
\mathcal{F}_1 = s_{\beta_3} + 1 + \frac{1}{4} Q - \sqrt{s_{\beta_3}^2 + \left( 1 + \frac{1}{2} Q \right) s_{\beta_3} + \frac{9}{16} Q^2}, \tag{3.12}
\]

\[
s_{\beta_3} = \frac{3(3 \beta_3 \phi_0 M_\parallel)^2}{4 \rho_0} = \frac{3(\beta_3 \phi_0 r^2)^2}{4 \Phi_+}. \tag{3.13}
\]

Substituting Eq. (3.10) into Eq. (3.9), we obtain

\[
\chi'(r) = \phi_0 \frac{Q}{8} \frac{3 \Phi_+}{s_{\beta_3}} \left[ 1 - \sqrt{1 + \frac{32 s_{\beta_3}}{9 Q^2} \left( \mathcal{F}_1 - \frac{1}{2} \right)} \right] \frac{r}{r_*}. \tag{3.14}
\]

In the limit that \( s_{\beta_3} \to 0 \), we have \( \mathcal{F}_1 = 1 - Q/2 \) for \( Q > 0 \) and \( \mathcal{F}_1 \simeq 1 + Q \) for \( Q < 0 \). From Eq. (3.11), the difference \( |\phi(r)/\phi_0 - 1| \) is of the order \( \Phi_+ \) around \( r = r_* \). Since the quantity \( \delta X_0 \) in Eq. (2.25) is at most of the order \( \phi_0^2 \Phi_+ \), the first condition of Eq. (2.40) is satisfied for \( |Q \phi_0^2 / M_\parallel^2| \ll 1 \). Indeed, this latter condition holds under the requirement (2.39). Taking the limit \( s_{\beta_3} \to 0 \), the longitudinal mode (3.14) reduces to

\[
\chi'(r) \simeq \phi_0 \sqrt{\Phi_+ s_{\beta_3} G_1} \frac{r}{r_*}, \tag{3.15}
\]
where \( G_1 = \sqrt{3}(Q - 1)/(9Q) \) for \( Q > 0 \) and \( G_1 = -\sqrt{3}(1 + 2Q)/(9Q) \) for \( Q < 0 \). In the small-coupling limits \( Q \to 0^+ \) and \( Q \to 0^- \), we have \( \chi'(r) < 0 \) and \( \chi'(r) > 0 \), respectively. The amplitude \( |\chi'(r)| \) reaches the maximum \( \phi_0 \sqrt{\frac{\Phi_{,s}s_{\beta_3}}{G_1}} \) at \( r = r_* \). Provided that \( s_{\beta_3} \lesssim Q^2 \), the second condition of Eq. (3.20) holds under the requirement of Eq. (3.39).

Taking another limit \( s_{\beta_3} \to \infty \), it follows that \( F_1 \simeq 1/2 + (1 - Q)(1 + 2Q)/(8s_{\beta_3}) \). In this case, the solutions (3.11) and (3.14) reduce, respectively, to

\[
\phi(r) \simeq \phi_0 \left( 1 - \frac{\Phi_{,s}r^2}{12r^2} \right),
\]

\[
\chi'(r) \simeq \phi_0 \sqrt{\frac{\Phi_{,s}g_2}{s_{\beta_3}}} \frac{r}{r_*},
\]

where \( g_2 = \sqrt{3}(Q - 1)/12 \) for \( Q > 0 \) and \( g_2 = \sqrt{3}(3Q + |Q + 2|)/24 \) for \( Q < 0 \). For \( s_{\beta_3} \gtrsim O(1) \), the cubic Galileon coupling leads to the strong suppression of the longitudinal mode due to the operation of the Vainshtein mechanism. Again, the two conditions (2.40) are consistently satisfied under the requirement (2.39). The value of \( s_{\beta_3} \) can be estimated as

\[
s_{\beta_3} \simeq 6 \times 10^9 \beta_3^2 \left( \frac{\phi_0}{M_{pl}} \right)^2 \left( \frac{1 \text{ g/cm}^3}{\rho_0} \right),
\]

and hence it can be naturally larger than unity for density of the order \( \rho_0 = 1 \text{ g/cm}^3 \) (like the Sun or the Earth).

**B. \( r > r_* \)**

Let us derive the solutions to \( \phi(r) \) and \( \chi'(r) \) outside the spherically symmetric body. On using the leading-order gravitational potentials (2.41), Eqs. (3.2) and (3.3) reduce, respectively, to\(^1\)

\[
\frac{d}{dr} \left( r^2 \phi' \right) - \beta_3 \phi \frac{d}{dr} \left( r^2 \chi' \right) \simeq - \frac{Q\phi_0 \rho(r)}{M_{pl}^2} r^2,
\]

\[
\beta_3 \left( \phi \phi' + \frac{2}{r} \chi' + \frac{\phi^2 \Phi_{,s}r_*}{6r^2} \right) \simeq \frac{Q\phi_0 \rho(r) \chi'}{M_{pl}^2}.
\]

We assume that the quantity \( \rho(r) \), which is defined by Eq. (2.33), is a constant \( \mu \) much smaller than 1. From Eq. (3.20), the longitudinal mode is expressed as

\[
\chi'(r) = \frac{Q\Phi_{,s} \mu}{4\beta_3 r_*^2} \left[ 1 - \sqrt{1 - \frac{8\beta_3^2 r_*^4 \phi}{\mu^2 Q^2 \Phi_{,s}^2} \left( \phi' + \frac{\phi \Phi_{,s}r_*}{6r^2} \right)} \right] r.
\]

On using the approximation scheme (3.8), we can integrate Eq. (3.19) to give

\[
r^2 \phi' - \beta_3 \phi \rho(r) - \frac{Q\phi_0 \Phi_{,s} \mu}{3r_*^2} r^3 \simeq C,
\]

where the integration constant \( C \) is fixed to be \( C = -\phi_0 \Phi_{,s} r_* (1 + Q - Q\mu)/3 \) by matching the solution (3.22) with Eq. (3.9) at \( r = r_* \). Then, we obtain the following relation:

\[
\phi' - \beta_3 \phi \rho(r) + \frac{Q\phi_0 \Phi_{,s} \mu}{3r_*^2} r - \frac{(1 + Q - Q\mu)\phi_0 \Phi_{,s} r_*}{3r^2}.
\]

Substituting Eq. (3.21) into Eq. (3.23) under the approximation (3.8), it follows that

\[
\phi'(r) = -\frac{\phi_0 \Phi_{,s} r_*}{3r^2} F_2(r),
\]

\(^1\) In Ref. [52], the approximate gravitational potentials \( \Psi_{GR} \simeq -\Phi_{,s} r_*/(6r) \) and \( \Phi_{GR} \simeq \Phi_{,s} r_*/(6r) \) were used instead of Eq. (2.41) for the derivation of the vector-field equations of motion. In this case, the extra term \( \Phi_{,s} r_*/(6r^2) \) arises on the lhs of Eq. (3.19), but this does not affect the discussion after Eq. (3.22).
where
\[ F_2(r) \equiv 1 + \xi(r) + Q(1 - \mu) + \frac{Q\mu}{4s_{\beta_3}} \xi(r) - \frac{1}{8s_{\beta_3}} \left( 1 + \xi(r) + 2Q + \frac{Q\mu}{2s_{\beta_3}} (\xi(r) - 4s_{\beta_3}) \right) \left( \frac{3Q\mu}{4s_{\beta_3}} \xi(r) \right)^2, \] (3.25)
\[ \xi(r) \equiv s_{\beta_3} \frac{r^3}{r_s^3}. \] (3.26)

1. \( s_{\beta_3} \gg 1 \)

We first study the case where \( s_{\beta_3} \gg 1 \). Since \( \xi(r) \gg 1 \) outside the body, we first take the limit \( \xi(r) \to \infty \) in Eq. (3.25). The ratio \( \mu \) is much smaller than 1, so we carry out the expansion of \( F_2(\xi(r) \to \infty) \) around \( \mu = 0 \). This process leads to
\[ F_2(r) \simeq \frac{1}{2} + \frac{1 + 2Q}{8s_{\beta_3}} \frac{r^3}{r_s^3} + \frac{1 + 2Q}{8s_{\beta_3}} Q\mu + \cdots, \] (3.27)
where we use the approximation \( \xi(r) \gg s_{\beta_3} \) (i.e., \( r \gg r_s \)) for deriving the third term on the rhs of \( F_2(r) \). For the distance \( r \) satisfying
\[ r \ll r_V \equiv \left| \frac{1 + 2Q}{Q\mu} \right|^{1/3} r_s, \] (3.28)
the second term on the rhs of Eq. (3.27) dominates over the third one. In this regime, Eqs. (3.24) and (3.21) reduce, respectively, to
\[ \phi'(r) \simeq - \frac{\phi_0 \Phi_* r_s}{3r^2} \left[ \frac{1}{2} + \frac{1 + 2Q}{8s_{\beta_3}} \frac{r_s^3}{r^3} \right], \] (3.29)
\[ \chi'(r) \simeq \frac{\phi_0 Q\mu}{8} \frac{3\Phi_*}{s_{\beta_3}} \frac{r_s}{r_s^3} \left[ 1 - \frac{1}{1 + \frac{4(1 + 2Q)^2}{9Q^2\mu^2} \frac{r_s^6}{r^6}} \right]. \] (3.30)

Since the second term in the square bracket of Eq. (3.29) rapidly approaches 0 for increasing \( r \), we obtain the approximate integrated solution
\[ \phi(r) \simeq \phi_0 \left( 1 + \frac{\Phi_*}{6} \frac{r_s}{r} - \frac{\Phi_*}{4} \right), \] (3.31)
where we have performed the matching of the solution with Eq. (3.16) at \( r = r_s \). This shows that \( \phi(r) \) is nearly constant around \( \phi_0 \). Under the condition (3.28), the second term in the square root of Eq. (3.30) is much larger than 1, so the longitudinal mode reduces to
\[ \chi'(r) \simeq - \frac{\phi_0}{12} \frac{3\Phi_*}{s_{\beta_3}} \frac{r_s^2}{r_s^3} \frac{\eta_1}{\eta_2^2}, \quad \eta_1 \equiv \frac{Q}{|Q|} \left| 1 + 2Q \right|. \] (3.32)

From Eqs. (3.31) and (3.32), we find that the two conditions of Eq. (3.40) are satisfied under the requirement (3.29).

The distance \( r_V \) can be regarded as the Vainshtein radius, within which the propagation of the longitudinal mode is suppressed due to the existence of cubic Galileon interactions. For \( |Q| \) of the order of unity, the Vainshtein radius can be estimated as \( r_V \simeq \mu^{-1/3} r_s \). The density \( \rho_0 \) is related to the Schwarzschild radius \( r_g \), as \( \rho_0 \simeq 3M_{pl}^2 r_g^2 / r_s^3 \). If \( \mu \rho_0 \) is of the order of the present cosmological density, it is related to today’s Hubble radius, \( r_H \simeq 10^{28} \text{ cm} \), as \( \mu \rho_0 \simeq 3M_{pl}^2 / r_H^2 \). Then, the Vainshtein radius is of the order of \( r_V \simeq (r_g r_H^2)^{1/3} \). For the Sun \( (r_g \simeq 10^5 \text{ cm}) \), we have \( r_V \simeq 10^{20} \text{ cm} \), which is much larger than the Solar System scale. Thus, the propagation of the longitudinal mode is suppressed inside the Solar System thanks to the Vainshtein mechanism.

2. \( s_{\beta_3} \ll 1 \)

We proceed to the case in which \( s_{\beta_3} \) is much smaller than 1. We introduce the critical distance
\[ r_c = \frac{r_s}{s_{\beta_3}^{1/3}}. \] (3.33)
For the distance $r_s < r \ll r_c$, the quantity $\xi(r)$ is much smaller than 1. Expanding Eq. (3.25) around $\xi(r) = 0$, it follows that $F(r) \approx 1 + Q(1 - \mu) - \sqrt{1 + 2Q(1 - \mu)}\xi(r)$. Ignoring the contribution of the $\xi(r)$-dependent term in $F(r)$, Eqs. (3.24) and (3.21) reduce, respectively, to

$$\phi'(r) \simeq -\frac{\phi_0 r_s}{3r^2}[1 + Q(1 - \mu)],$$

(3.34)

$$\chi'(r) \simeq \frac{\phi_0 Q\mu}{8\sqrt{3\beta_s}}r \left[ 1 - \sqrt{1 + \frac{16[1 + 2Q(1 - \mu)]}{9(\mu)^2}s_{\beta_s}^2 r_s^3} \right],$$

(3.35)

so that the temporal component stays nearly constant around $\phi_0$.

If the coupling $s_{\beta_s}$ satisfies the condition

$$s_{\beta_s} \ll (\mu)^2,$$

(3.36)

the magnitude of the second term in the square root of Eq. (3.35) is much smaller than 1. In this case, Eq. (3.35) yields

$$\chi'(r) \simeq -\frac{\phi_0}{9}[1 + 2Q(1 - \mu)]\sqrt{3\beta_s} \frac{\sqrt{s_{\beta_s}^2 r_s^2}}{Q\mu},$$

(3.37)

so the radial dependence of $|\chi'(r)|$ is similar to Eq. (3.32) with the suppression of the order $\sqrt{\Phi_s s_{\beta_s}^2/(Q\mu r_s^2})$ relative to $\phi_0$. In the limit that $\beta_s \rightarrow 0$, the longitudinal mode vanishes.

For the intermediate coupling strength $s_{\beta_s}$ satisfying

$$|Q|\mu \ll s_{\beta_s} \ll 1,$$

(3.38)

the magnitude of the second term in the square root of Eq. (3.35) is much larger than 1 for the distance $r_s < r < r_c$. Then, the longitudinal mode reduces to

$$\chi'(r) \simeq -\frac{\phi_0}{6}r^2 \sqrt{3\beta_s} \frac{r_s}{r}, \quad \eta_2 = \frac{Q}{|Q|} \sqrt{1 + 2Q(1 - \mu)},$$

(3.39)

which decreases more slowly relative to Eqs. (3.32) and (3.37). Note that the existence of the solution (3.39) requires the condition $1 + 2Q(1 - \mu) > 0$. The field profiles derived above satisfy the two consistency conditions of Eq. (2.40) under the requirement (2.39). For the coupling $s_{\beta_s}$ satisfying $(Q\mu)^2 \ll s_{\beta_s} \ll |Q|\mu$, the solution to $\chi'(r)$ is given by Eq. (3.39) for the distance $r_s < r \lesssim r_1 \equiv [s_{\beta_s}/(Q\mu)]^{1/3}$ and by Eq. (3.37) for $r_1 \lesssim r < r_c$.

Since $\xi(r) \gg 1$ for the distance $r_c \ll r \ll r_V$, the solutions to $\phi'(r)$ and $\chi'(r)$ are described by Eqs. (3.29) and (3.32), respectively.

C. Vector-field profile for varying matter density

While we have derived analytic solutions to the vector field for the constant densities inside and outside the body, we will also study the case of the varying matter density given by

$$\rho(r) = \rho_0 e^{-ar^2/r_s^2} + \mu \rho_0,$$

(3.40)

where $a$ and $\mu$ are constants, with $a = O(1)$ and $\mu = 1$. The density is nearly constant for the distance $r \ll r_s$, but it starts to decrease rapidly around $r = r_s$ to approach the asymptotic value $\mu \rho_0$. Numerically, we solve the full equations of motion (2.17)–(2.18), (2.21), and (2.22) by using Eqs. (3.11), (3.14), (2.37), and (2.38) as the boundary conditions of $\phi, \chi, \Psi, \Phi, P$ around the center of the body ($r = 10^{-3}r_s$).

We recall that the analytic vector-field profile was derived by employing the leading-order general-relativistic gravitational potentials. To check the consistency of this procedure, we also integrate Eqs. (2.17)–(2.18) by neglecting the contributions of the vector field and solve Eqs. (3.2)–(3.3) for $\phi(r)$ and $\chi(r)$, respectively, subtract the derived solutions of $\phi(r)$ and $\chi(r)$ into Eqs. (2.17)–(2.18) and confirm that the corrections to $\Psi_{\text{GR}}$ and $\Phi_{\text{GR}}$ induced by the vector field remain small. These approximate solutions exhibit good agreement with the full numerical results.

In the left panel of Fig. 11 we plot the numerically integrated solutions to $\phi(r)$, $-\phi'(r)$, and $\chi(r)$ versus $r/r_s$ for $s_{\beta_s} = 10^4, Q = 1.3, a = 3, \mu = 10^{-24}$, $\Psi_{\text{GR}} = 10^{-5}$, and $\phi_0 = 10^{-4}M_{\text{pl}}$. As estimated analytically from Eqs. (3.10) and (3.14), both $-\phi'(r)$ and $\chi(r)$ are in proportion to $r$ for the distance $r \lesssim r_s$. For $r$ larger than $r_s$, they start to
The distance $r$ chosen to match with $\Psi$ the derivatives $|v/r\regime$ (decrease according to Eqs. (3.29) and (3.32), i.e., $-v/r\propto$ consistent with Eqs. (3.11) and (3.14), respectively, at $\Phi$ and $\Psi$ for $r > r^\ast$. Where $\Delta$ decrease in proportion to $1/r^2$ consistent with Eqs. (3.11) and (3.14), respectively, at $r/r^\ast = 10^{-3}$. The boundary conditions of gravitational potentials are chosen to match with $\Psi_{GR}$ and $\Phi_{GR}$ given by Eq. (2.37).

Let us estimate the corrections to leading-order gravitational potentials (2.41) outside the body induced by the presence of the vector field coupled to matter. In doing so, we express Eqs. (2.17) and (2.18) in the forms

$$\frac{2M^2_{pl}}{r} \Phi' - \frac{M^2_{pl}}{r^2} (1 - e^{2\Phi}) = e^{2\Phi} \rho + \Delta \Phi,$$

(3.41)

$$\frac{2M^2_{pl}}{r} \Psi' + \frac{M^2_{pl}}{r^2} (1 - e^{2\Phi}) = \Delta \Psi,$$

(3.42)

where $\Delta \Phi$ and $\Delta \Psi$ are corrections to the gravitational equations in GR. Since we are interested in the behavior of $\Phi$ and $\Psi$ for $r > r^\ast$, we set $\rho = \mu \rho_0 = \mu \Phi, M^2_{pl}/r^2$ in Eq. (3.41). For the integrations of Eqs. (3.41) and (3.42), we employ the weak-gravity approximation (2.36) in such a way that the general-relativistic corrections to gravitational potentials higher than second order are neglected. In the following, we will consider the two different cases, (i) $s_{\beta_3} \gg 1$ and (ii) $s_{\beta_3} \ll 1$, separately.

**D. Gravitational potentials outside the body**

![Numerical solutions of $\phi(r)$, $-\phi'(r)$, and $\chi'(r)$ (normalized by $M_{pl}$, $M_{pl}/r_\ast$, and $M_{pl}$, respectively) for $\Phi = 10^{-5}$, $\phi_0 = 10^{-4} M_{pl}$, and the density profile (3.40) with $\alpha = 3$ and $\mu = 10^{-2}$. Each panel corresponds to the model parameters $s_{\beta_3} = 10^1$, $Q = 1.3$ (left) and $s_{\beta_3} = 10^{-1}$, $Q = -0.3$ (right). We choose the boundary conditions of $\phi(r)$ and $\chi'(r)$ to be consistent with Eqs. (3.11) and (3.14), respectively, at $r/r_\ast = 10^{-3}$. The boundary conditions of gravitational potentials are chosen to match with $\Psi_{GR}$ and $\Phi_{GR}$ given by Eq. (2.37).](image-url)

FIG. 1: Numerical solutions of $\phi(r)$, $-\phi'(r)$, and $\chi'(r)$ (normalized by $M_{pl}$, $M_{pl}/r_\ast$, and $M_{pl}$, respectively) for $\Phi = 10^{-5}$, $\phi_0 = 10^{-4} M_{pl}$, and the density profile (3.40) with $\alpha = 3$ and $\mu = 10^{-2}$. Each panel corresponds to the model parameters $s_{\beta_3} = 10^1$, $Q = 1.3$ (left) and $s_{\beta_3} = 10^{-1}$, $Q = -0.3$ (right). We choose the boundary conditions of $\phi(r)$ and $\chi'(r)$ to be consistent with Eqs. (3.11) and (3.14), respectively, at $r/r_\ast = 10^{-3}$. The boundary conditions of gravitational potentials are chosen to match with $\Psi_{GR}$ and $\Phi_{GR}$ given by Eq. (2.37).
$1. \ s_{\beta_3} \gg 1$

In this case, we exploit the solutions from Eqs. (3.31, 3.32) and substitute them into Eqs. (2.17–2.18). There exists the term of the form $\rho Q_0^2 \Phi_+ / M_{pl}^2$, which is at most of the order $\rho \Phi_+^2 = \mu M_{pl}^2 \dot{\Phi}_+^2 / r_+^2$ under the condition (3.38) by identifying $\phi_0$ with $\Phi_0$. We neglect such contributions relative to the term of the order $\phi_0^2 / (\rho_+ / M_{pl})^2$, which amounts to the condition $\rho_+ \Phi_+ \ll (\phi_0 / M_{pl})^2$. Then, the corrections up to the order of $\Phi_+^2$ can be estimated as

$$\Delta \Phi \simeq -\frac{(4\eta_0 - 1) \phi_0^2 \Phi_+^2}{72 r_+^4}, \quad \Delta \Phi \simeq -\frac{\phi_0^2 \Phi_+^2}{72 r_+^4}. \tag{3.43}$$

For the integration of Eqs. (3.41–3.42), we choose the integration constant in such a way that the solutions $\Phi = r_g / (2r) = r_+ \Phi_+ / (6r)$, $\Psi = -r_g / (2r) = -r_+ \Phi_+ / (6r)$ are recovered in the vacuum limit. Then, we obtain the integrated solutions corrected by the presence of the vector field, as

$$\Phi \simeq \frac{r_+ \Phi_+}{6r} \left[ 1 + \frac{(4\eta_0 - 1) \phi_0^2}{24 \Phi_+ / M_{pl}} \right] \frac{r_+}{r}, \tag{3.44}$$

$$\Psi \simeq -\frac{r_+ \Phi_+}{6r} \left[ 1 + \frac{(2\eta_0 - 1) \phi_0^2}{24 \Phi_+ / M_{pl}} \right] \frac{r_+}{r}. \tag{3.45}$$

The second terms in the square brackets of Eqs. (3.44) and (3.45) induce the difference between the gravitational potentials, as

$$\Phi + \Psi \simeq \beta_{PN} U^2, \tag{3.46}$$

where

$$\beta_{PN} \equiv \frac{\eta_0}{2} \left( \frac{\phi_0}{M_{pl}} \right)^2, \quad U \equiv \frac{r_+ \Phi_+}{6r}. \tag{3.47}$$

For $\phi_0 \ll M_{pl}$, we have $\Phi + \Psi \ll U^2$. The quantity $\beta_{PN}$ may be regarded as the second parametrized post-Newtonian parameter [56]. The experimental bound $|\beta_{PN}| \lesssim 2.3 \times 10^{-4}$ from the Nordtvedt effect [51] can be satisfied for $\phi_0 \lesssim 10^{-2} M_{pl}$. We note that the condition (3.39) translates to $\phi_0 \lesssim \sqrt{\Phi_+ / |Q| M_{pl}}$, so this can put a tighter limit of $\phi_0$ for $\Phi_+ / |Q| \ll 10^{-4}$.

$2. \ s_{\beta_3} \ll 1$

For the coupling $s_{\beta_3}$ satisfying $s_{\beta_3} \ll (Q/\mu)^2$, we employ the solutions from Eqs. (3.34) and (3.35) for the estimations of $\Phi$ and $\Psi$. Up to the order of $\Phi_+^2$, the correction terms $\Delta \Phi$ and $\Delta \Psi$ are given, respectively, by

$$\Delta \Phi \simeq \frac{Q^2 (1 - \mu)^2 \phi_0^2 \Phi_+^2}{18 r_+^4}, \quad \Delta \Psi \simeq -\frac{Q^2 (1 - \mu)^2 \phi_0^2 \Phi_+^2}{18 r_+^4}. \tag{3.48}$$

The integrated solutions to Eqs. (3.41) and (3.42) read

$$\Phi \simeq \frac{r_+ \Phi_+}{6r} \left[ 1 + \frac{Q^2 (1 - \mu)^2 \phi_0^2}{6 \Phi_+ / M_{pl}} \right] \frac{r_+}{r}, \tag{3.49}$$

$$\Psi \simeq -\frac{r_+ \Phi_+}{6r} \left[ 1 - \frac{Q^2 (1 - \mu)^2 \phi_0^2}{6 \Phi_+ / M_{pl}} \right] \frac{r_+}{r}. \tag{3.50}$$

The second terms in Eqs. (3.49) and (3.50) do not give rise to the difference between $\Phi$ and $-\Psi$, so $\beta_{PN} = 0$.

For the coupling satisfying $|Q/\mu | \ll s_{\beta_3} \ll 1$, we substitute the solutions from Eqs. (3.34) and (3.39) into Eqs. (2.17) and (2.18). The correction terms $\Delta \Phi$ and $\Delta \Psi$ yield

$$\Delta \Phi \simeq -\frac{\eta_0 \phi_0^2 \sqrt{2} \Phi_+}{2 \sqrt{r_+} r_+^{3/2}}, \quad \Delta \Psi \simeq O \left( \Phi_+^2 \right). \tag{3.51}$$
Then, the integrated solutions to gravitational potentials are approximately given by

\[ \Phi \simeq \frac{r_* \Psi}{6r} \left[ 1 - \eta_2 \sqrt{s_3 \beta_3} \left( \frac{\phi_0}{M_{\text{pl}}} \right)^2 \left( \frac{r}{r_*} \right)^{3/2} \right], \]

(3.52)

\[ \Psi \simeq -\frac{r_* \Phi}{6r} \left[ 1 + 2\eta_2 \sqrt{s_3 \beta_3} \left( \frac{\phi_0}{M_{\text{pl}}} \right)^2 \left( \frac{r}{r_*} \right)^{3/2} \right]. \]

(3.53)

The second terms in Eqs. (3.52) and (3.53) lead to the difference between \( \Phi \) and \( -\Psi \), with the relative ratio

\[ \gamma \equiv -\frac{\Phi}{\Psi} \simeq 1 - 3 \eta_2 \sqrt{s_3 \beta_3} \left( \frac{\phi_0}{M_{\text{pl}}} \right)^2 \frac{r}{r_*}. \]

(3.54)

At \( r = r_c = r_* \frac{1}{\sqrt{\beta_3}} \), the quantity \( |\gamma - 1| \) reaches the maximum value \( |\gamma - 1|_{\text{max}} = 3|\eta_2|(\phi_0/M_{\text{pl}})^2 \). The local gravity bound \( |\gamma - 1| < 2.3 \times 10^{-5} \) arising from the Cassini tracking [51] can be satisfied for \( \phi_0 \lesssim 10^{-3}M_{\text{pl}} \).

IV. QUARTIC DERIVATIVE COUPLINGS

We proceed to the case of quartic derivative interactions given by

\[ G_4 = \frac{M_{\text{pl}}^2}{2} + \beta_4 M_{\text{pl}}^2 \left( \frac{X}{M_{\text{pl}}^2} \right)^n, \quad G_{2,3,5,6} = 0, \]

(4.1)

where \( \beta_4 \) is a dimensionless constant, and \( n \) is a positive integer. As we already mentioned in Sec. III, we can express Eq. (2.16) in the form (2.23), where

\[ F = 2n \beta_4 \left( \frac{X}{M_{\text{pl}}} \right)^{n-1} \left[ f(1-h) + (f'-2nf')hr + 2(n-1)h A_0 A_0' r - fX \right], \]

(4.2)

which vanishes in the limit that \( \beta_4 \to 0 \). Hence, the branch consistent with this limit corresponds to

\[ \chi' = 0. \]

(4.3)

Since there is no longitudinal propagation of the vector field, the temporal component \( \phi \) alone can affect the solutions to gravitational potentials.

Let us consider the matter density profile given by Eq. (2.33). Inside the body, we substitute the leading-order gravitational potentials (2.37) into Eq. (2.15). Then, the temporal vector component obeys

\[ \frac{d}{dr} \left( r^2 \phi' \right) - \frac{\Phi_x}{\phi r^2} \left[ 4n \beta_4 M_{\text{pl}}^2 \left( \frac{\phi^2}{2M_{\text{pl}}^2} \right)^n - (1 + Q) \phi^2 \right] r^2 = 0. \]

(4.4)

Assuming the solution in the form (3.8) and imposing the boundary condition \( \phi'(0) = 0 \), the integrated solution to Eq. (4.4) reads

\[ \phi(r) \simeq \phi_0 \left[ 1 - \frac{\Phi_x(1 + Q - b_4)}{6} \frac{r^2}{r_*^2} \right], \]

(4.5)

where

\[ b_4 \equiv 2^{2-n} n \beta_4 \left( \frac{\phi_0}{M_{\text{pl}}} \right)^{2(n-1)}. \]

(4.6)

Provided that \( |b_4| \lesssim 1 \), the second term in the square bracket of Eq. (4.5) is at most of the order \( \Phi_x \). We will assume this condition in the following discussion.

Outside the body, we substitute the leading-order external gravitational potentials (2.41) into Eq. (2.15). This leads to

\[ \frac{d}{dr} \left( r^2 \phi' \right) \simeq -\frac{Q \phi \phi_x \mu}{r_*^2} r^2. \]

(4.7)
The integrated solutions to $\phi'(r)$ and $\phi(r)$, which match with those in the regime $r < r_\ast$, are given by

$$
\phi'(r) \simeq -\frac{\phi_0 r_\ast}{3r^2} \left( \mathcal{H} + Q\mu \frac{r^3}{r_\ast^3} \right),
$$

$$
\phi(r) \simeq \phi_0 \left[ 1 + \frac{\Phi_\ast}{6} \left( \frac{2r_\ast}{r} - 3 \right) - Q\mu \frac{r^2}{r_\ast^2} \right],
$$


where

$$
\mathcal{H} \equiv 1 + Q(1 - \mu) - b_4.
$$

For the distance

$$
r \ll \frac{r_\ast}{|Q\mu|^{1/3}},
$$

the last terms on the rhs of Eqs. (4.8) and (4.9), which contain $Q\mu$, can be neglected. The upper bound of Eq. (4.11) is similar to the Vainshtein radius $r_V$ given by Eq. (3.28). In this regime, the gravitational Eqs. (2.17) and (2.18) are expressed in the forms (3.41) and (3.42), respectively, with the correction terms (up to the order of $\Phi^2$)

$$
\Delta \Phi \simeq \frac{\phi_0^2 r_\ast^2 (H - 1)^2}{18r^4},
$$

$$
\Delta \Psi \simeq \frac{b_4 \phi_0^2 r_\ast^2 (2H - 1)}{3r^3} - \frac{\phi_0^2 r_\ast^2 (H - 1)^2}{18r^4},
$$

where the terms of the order $b_4 \phi_0^2 r_\ast^2 \Phi^2 / r^4$ have been neglected relative to the first term in $\Delta \Psi$. Then, the integrated solutions to $\Phi$ and $\Psi$ are given, respectively, by

$$
\Phi \simeq \frac{r_\ast}{6r} \left[ 1 - \frac{\Phi_\ast}{6} (H - 1)^2 \left( \frac{\phi_0}{M_{pl}} \right)^2 \frac{r_\ast}{r} \right],
$$

$$
\Psi \simeq -\frac{r_\ast}{6r} \left[ 1 + b_4 (2H - 1) \left( \frac{\phi_0}{M_{pl}} \right)^2 - \frac{\Phi_\ast}{6} (H - 1)^2 \left( \frac{\phi_0}{M_{pl}} \right)^2 \frac{r_\ast}{r} \right].
$$

While the last terms of Eqs. (4.13) and (4.14) do not give rise to the difference between $\Phi$ and $-\Psi$, the second term of Eq. (4.14) leads to the difference with the relative ratio

$$
\gamma = -\frac{\Phi}{\Psi} \simeq 1 - b_4 (2H - 1) \left( \frac{\phi_0}{M_{pl}} \right)^2.
$$

Since the constant $\mathcal{H}$ is of the order unity, the experimental bound $|\gamma - 1| < 2.3 \times 10^{-5}$ can be satisfied under the condition

$$
2^{2-n|\beta_4|} \left( \frac{\phi_0}{M_{pl}} \right)^{2n} \lesssim 10^{-5}.
$$

For $n = 1$ and $|\beta_4| = O(1)$, the bound (4.16) translates to $\phi_0 \lesssim 10^{-3} M_{pl}$. As in the case of cubic vector Galileons, the coupled vector-field model with quartic derivative interactions is also consistent with local gravity constraints for $\phi_0$ much smaller than $M_{pl}$.

### V. Quintic Vector Galileons

We will study whether or not quintic derivative interactions give rise to solutions of the vector field operated by the Vainshtein mechanism. For concreteness, we consider the quintic vector Galileon given by the functions

$$
G_5 = \beta_5 \frac{X^2}{M_{pl}^4}, \quad G_4 = \frac{M_{pl}^2}{2}, \quad G_{2,3,6} = 0,
$$

where $\beta_5$ is a dimensionless constant. We choose the matter density profile (2.33), but we will also discuss the case in which the density varies outside the body.
A. \( r < r_\ast \)

Inside the spherically symmetric body, we substitute the leading-order gravitational potentials of Eq. (2.37) into Eqs. (2.15)–(2.18) and then expand them up to the first order of \( \Phi_\ast \). Then, it follows that

\[
\frac{d}{dr} \left( r^2 \phi' \right) + (1 + Q) \phi \Phi_\ast \frac{r^2}{r_\ast} + \frac{\beta_5 \phi}{3 M_{\text{pl}}^4 r_\ast^2} \left[ 6 \phi'^2 - 3 r^2 \phi'' + r \Phi_\ast (r \phi^2 \phi'' + 2 \phi' \phi'') + 2 \Phi_\ast r \phi' \phi' \right] \approx 0 ,
\]

\[
3 M_{\text{pl}}^4 Q \Phi_\ast \phi' \Phi_\ast \beta_5 \left[ 3 \phi'' + 4 (1 + Q) \phi \phi'' + 6 r^2 \phi' \phi' + 2 \Phi_\ast r \phi' \phi' \right] \approx 0 .
\]

As in the case of cubic vector Galileons, we search for analytic solutions to \( \phi' (r) \) and \( \chi' (r) \) proportional to \( r \) around the center of the body. Provided that \( \phi \) is nearly constant around \( \phi_0 \), the first three terms on the lhs of Eq. (5.2) are in proportion to \( r^3 \), whereas the last term has the dependence \( 2 \beta_5 \Phi_\ast \phi' \phi' \propto r^5 \). Neglecting the last term of Eq. (5.3) around \( r = 0 \), we can solve Eq. (5.3) for \( \chi' \)

\[
\chi'(r) \approx \frac{-M_{\text{pl}}^4 Q \Phi_\ast \beta_5}{4 \beta_5 (1 + Q) \phi_0^2} \left[ 1 - \frac{1}{1 - (1 + Q) \phi_0^2} \right] ,
\]

where we have chosen the branch recovering \( \chi' \to 0 \) for \( \beta_5 \to 0 \). Provided that the longitudinal mode \( \chi' \) is suppressed relative to \( \phi \), the \( \beta_5 \)-dependent terms in Eq. (5.2) should work as corrections to the leading-order solution

\[
\phi_{\text{leading}} (r) = \phi_0 \left[ 1 - \frac{(1 + Q) \phi_0^2}{6} \right] ,
\]

which is nearly constant around \( \phi_0 \). Substituting the derivative of Eq. (5.5) into Eq. (5.4), the leading-order longitudinal mode reads

\[
\chi'_{\text{leading}} (r) \approx \frac{3 M_{\text{pl}}^4 Q r_{\text{pl}}}{4 \beta_5 (1 + Q) \phi_0^2} (1 - \sqrt{1 - \varepsilon_5}) , \quad \varepsilon_5 \equiv \frac{8 [(1 + Q)^2 \beta_5 \phi_0^2]}{2 Q r_{\text{pl}}^2} .
\]

For the existence of this solution, we require the condition \( \varepsilon_5 \leq 1 \), which translates to

\[
\beta_5^2 \leq \frac{2 Q}{8 (1 + Q)^2} M_{\text{pl}}^4 r_{\text{pl}}^2 ,
\]

so that the coupling \( \beta_5 \) is bounded from above.

Let us consider the case in which the condition

\[
\varepsilon_5 \ll 1
\]

is satisfied. Assuming that both \( \beta_5 \) and \( Q \) are positive, the longitudinal mode (5.6) reduces to

\[
\chi'_{\text{leading}} (r) \approx \frac{\phi_0}{12} \sqrt{6 \Phi_\ast \varepsilon_5 \frac{r}{r_\ast}} ,
\]

which means that \( \chi'_{\text{leading}} (r) \) is suppressed relative to \( \phi_0 \) by the factor \( (\Phi_\ast \varepsilon_5 / 24) (r / r_\ast)^2 \). By using the solution (5.7), the \( \beta_5 \)-dependent terms in Eq. (5.2) can be estimated as \( 3 Q \phi_0 \varepsilon_5 \Phi_\ast r_{\text{pl}}^2 / [(1 + Q) r_{\text{pl}}^2] \). Taking into account this contribution and integrating Eq. (5.2) with respect to \( r \), we obtain

\[
\phi (r) \approx \phi_0 \left[ 1 - \frac{(1 + Q)^2 + 3 Q \varepsilon_5 \Phi_\ast r_{\text{pl}}^2}{48 (1 + Q) r_{\text{pl}}^2} \right] ,
\]

which is close to the leading-order temporal component (5.5).

The solution (5.9) corresponds to the case in which the first two terms of Eq. (5.3) balance each other. On using Eq. (5.3), the third and fourth terms of Eq. (5.3) are suppressed relative to the first term by the factors \(-\varepsilon_5 / 4 \) and \( \varepsilon_5^2 \Phi_\ast r_{\text{pl}}^2 / [96 (1 + Q) r_{\text{pl}}^2] \), respectively. Then, the longitudinal vector component yields

\[
\chi' (r) \approx \frac{\phi_0}{12} \sqrt{6 \Phi_\ast \varepsilon_5 \frac{r}{r_\ast}} \left[ 1 + \frac{\varepsilon_5}{4} \left( 1 - \frac{\varepsilon_5}{24 (1 + Q) r_{\text{pl}}^2} \right) \right] ,
\]

and hence \( \chi' (r) \) is well described by the leading-order solution (5.9) for \( \varepsilon_5 \ll 1 \).
Substituting the leading-order gravitational potentials of Eq. (2.41) into Eqs. (2.15)–(2.16) and expanding them up to the first order of \( \Phi_* \), the field equations outside the body yield

\[
\frac{d}{dr} \left( r^2 \phi' \right) + Q \mu \Phi_* \frac{r^2}{r_*^2} + \frac{\beta_5 \phi}{3 M_{pl}^4 r^2} \left[ 6 r^2 \chi'' + r_* \Phi_* (r_0^2 \chi'' - \Phi' + 3 \chi^3) \right] \approx 0, \tag{5.12}
\]

\[
3 M_{pl}^4 Q \mu \Phi_* r^4 \chi' + \beta_5 r_*^2 \left[ \phi \phi' r \left( \Phi_* r_* \phi^2 + 6 r^2 \chi^2 \right) + 2 \Phi_* r^4 \chi' \right] \approx 0. \tag{5.13}
\]

Provided that the last term of Eq. (5.13) is much smaller than the other terms, Eq. (5.13) can be explicitly solved as

\[
\chi'(r) \approx - \frac{M_{pl}^4 Q \mu \Phi_* r^2}{4 \beta_5 \phi^2 r_*^2} \left[ 1 - \sqrt{1 - 8 \beta_5^2 \phi^2 r_*^2} \right] . \tag{5.14}
\]

In the vacuum limit \((\mu \to 0)\), the solution to Eq. (5.12) derived by neglecting the contributions of \( \chi' \) and \( \chi'' \) reads \( \phi'(r) \propto 1/r^2 \) and \( \phi(r) \) constant. Then, the second term in the square root of Eq. (5.14), which is always positive, exhibits the divergence for \( \mu \to 0 \). If we consider a rapidly decreasing density profile like Eq. (3.40), the solution to Eq. (5.14) becomes imaginary above the distance \( r_d \) at which the second term in the square root of Eq. (5.14) is equivalent to 1. Unless we consider slowly varying density profiles like \( \rho(r) = \rho_0 (r_*/r)^p \) with \( p < 9/2 \), it is not possible to realize the existence of regular solutions for the distance \( r > r_d \).

The solution in Eq. (5.14) has been derived by neglecting the contribution of the last term of Eq. (5.13). As we estimated in Eq. (5.11), this term works as a tiny correction to the leading-order longitudinal mode (5.6) around \( r = r_* \). For the density profile (3.40), we numerically integrate the vector-field equations of motion coupled to the gravitational equations and find that the last term of Eq. (5.13) remains small relative to the other terms for the distance \( r_* < r < r_d \). For \( r > r_d \), there are no real solutions to the longitudinal mode. Thus, the quintic vector Galileon does not allow the existence of consistent solutions to \( \chi'(r) \) for realistic density profiles that rapidly decrease for \( r > r_* \).

We also studied the model of the linear coupling \( G_5(X) = \beta_5 X/M_{pl}^2 \) and found that the solution to \( \chi' \) has a similar property to that of the model (5.1). In this case, the \( \chi'^4 \) term is absent unlike Eq. (5.13), so we have the exact solution to \( \chi'(r) \) analogous to Eq. (5.14). Hence, it is not possible to realize the solution of \( \chi'(r) \) regular throughout the region \( r > r_* \) for the rapidly decreasing density matter. Thus, the models with quintic derivative couplings are generally plagued by the absence of regular external solutions operated by the Vainshtein mechanism.

VI. SIXTH-ORDER DERIVATIVE COUPLINGS

Finally, we study the model with sixth-order derivative interactions given by

\[
G_6 = \frac{\beta_6}{M_{pl}^2} \left( \frac{X}{M_{pl}^2} \right)^n, \quad G_4 = \frac{M_{pl}^2}{2}, \quad G_{2,3,5} = 0, \tag{6.1}
\]

where \( \beta_6 \) is a dimensionless constant, and \( n \) is a positive integer of order 1. According to the discussion given in Sec. II, the branch consistent with the limit \( \beta_6 \to 0 \) corresponds to

\[
\chi' = 0. \tag{6.2}
\]

Let us consider the matter density profile (2.33). Inside the body, we substitute Eq. (2.37) into Eq. (2.15) and expand it up to second order in \( \Phi_* \) for the terms containing \( \beta_6 \). Then, the temporal vector component obeys

\[
\frac{d}{dr} \left( r^2 \phi' \right) + \phi \Phi_* \frac{r^2}{r_*^2} \left[ 1 + Q - \frac{2 b_6 (Q \Phi_* \phi^2 - n \phi^2 r_*^2)}{(1 + 2 b_6 \Phi_* \phi^2)} \right] \approx 0, \tag{6.3}
\]

where

\[
b_6 \equiv \frac{\beta_6}{3 M_{pl}^2 r_*^2} \left( \frac{\phi^2}{2 M_{pl}^2} \right)^n . \tag{6.4}
\]

In the limit that \( b_6 \to 0 \), we obtain the leading-order solution \( \phi_{\text{leading}}(r) = \phi_0 [1 - \Phi_*(1 + Q) r^2/(6 r_*^2)] \), so \( \phi(r) \) is nearly frozen around \( \phi_0 \). Assuming that \( |b_6 \Phi_*| \ll 1 \), the terms containing \( b_6 \) in Eq. (6.3) can be regarded as the
corrections to \( \phi_{\text{leading}}(r) \). Ignoring the terms higher than \( \Phi_s^2 \) inside the square bracket of Eq. (6.3) and dealing with \( b_6 \) as a constant with \( \phi(r) \simeq \phi_0 \), the resulting field derivative is given by

\[
\phi'(r) \simeq -\frac{\phi_0 \Phi_s}{3r^2} (1 + Q - 2Qb_6 \Phi_s) r,
\]

(6.5)

where we have dropped the terms of the order \( \Phi_s^2 \) which are not multiplied by \( b_6 \). Since \( |b_6 \Phi_s| \ll 1 \), the correction induced by the coupling \( b_6 \) to the leading-order solution of \( \phi'(r) \) is negligibly small.

Outside the body, we substitute Eq. (2.41) into Eq. (2.15), solve it for \( \phi''(r) \), and then expand it up to second order in \( \Phi_s \) for the terms containing \( b_6 \). This process leads to

\[
\frac{d}{dr}(r^2 \phi') \simeq -Q \mu \phi \Phi_s r^2/\phi r^4 + 2b_6 \Phi_s r^3/\phi r^4 \left[ Q \mu \phi^2 \Phi_s r^3/\phi r^4 + \Phi_s r_s \phi^2 + \frac{1}{3} \phi' (9 \phi + n^2 \phi\Phi_s \phi') r^2 - n \phi^2 \right].
\]

(6.6)

Provided that the terms multiplied by \( b_6 \) are negligibly small relative to the term \(-Q \mu \phi \Phi_s r^2/\phi r^4 \), the integrated solution for \( r > r_s \), which matches Eq. (6.5) at \( r = r_s \), reads

\[
\phi'(r) = -\frac{\phi_0 \Phi_s r_s}{3r^2} \left[ 1 + Q(1 - \mu) + Q \mu r_s^3 \right],
\]

(6.7)

where we have dropped the contribution \(-2Qb_6 \Phi_s \) in Eq. (6.3). The first term in the square bracket of Eq. (6.6) is suppressed by the factor \( b_6 \Phi_s r_s^3/\phi r^4 \) relative to the first term on the rhs of Eq. (6.3). For the distance \( r \ll r_s/(\sqrt{Q \mu})^{1/3} \), the leading-order field derivative is given by \( \phi_{\text{leading}}'(r) \simeq -\phi_0 \Phi_s r_s [1 + Q(1 - \mu)]/(3r^3) \), so the \( b_6 \)-dependent terms in Eq. (6.6) can be estimated as \(-2\phi_0 Q(1 - \mu) b_6 \Phi_s r_s^4/r^4 \). Then, the correction to \( \phi_{\text{leading}}'(r) \), which arises from the term containing \( b_6 \), yields

\[
\Delta \phi'(r) = \frac{2Q(1 - \mu) \phi_0 b_6 \Phi_s^2 r_s^4}{3r^5},
\]

(6.8)

which is suppressed by the factor \( Q b_6 \Phi_s r_s^3/\phi r^4 \) compared to Eq. (6.7).

On using the leading-order solution (6.7) outside the body, the correction terms in the gravitational Eqs. (2.17) and (2.18), expanded up to the order of \( \Phi_s^2 \), are given, respectively, by

\[
\Delta \phi \simeq \frac{Q^2(1 - \mu)^2 \phi_0^2 \Phi_s^2 r_s^2}{18r^4}, \quad \Delta \psi \simeq \frac{Q^2(1 - \mu)^2 \phi_0^2 \Phi_s^2 (r^2 - 12b_6 r_s^2)}{18r^6}.
\]

(6.9)

Then, the gravitational potentials induced by the vector field can be estimated as

\[
\Phi \simeq \frac{r_s \Phi_s}{6r} \left[ 1 - \frac{\Phi_s Q^2(\mu - 1)^2}{6} \left( \frac{\phi_0}{M_{\text{pl}}} \right)^2 \frac{r_s}{r} \right],
\]

(6.10)

\[
\Psi \simeq -\frac{r_s \Phi_s}{6r} \left[ 1 - \frac{\Phi_s Q^2(\mu - 1)^2}{6} \left( \frac{\phi_0}{M_{\text{pl}}} \right)^2 \frac{r_s}{r} \left( 1 - 3b_6 \frac{r_s^2}{r^2} \right) \right].
\]

(6.11)

The coupling \( b_6 \) induces the difference between \( \Phi \) and \( -\Psi \), such that

\[
\Phi + \Psi \simeq \beta_{\text{PN}} U^2, \quad \beta_{\text{PN}} = -3Q^2(1 - \mu)^2 b_6 \left( \frac{\phi_0}{M_{\text{pl}}} \right)^2 \frac{r_s^2}{r^2}.
\]

(6.12)

The parameter \( \beta_{\text{PN}} \) has a maximum at \( r = r_s \) and decreases for larger \( r \). The bound \(|\beta_{\text{PN}}| < 2.3 \times 10^{-4}\) is well satisfied for \(|Q| \sqrt{|b_6|} \phi_0/M_{\text{pl}} \lesssim 10^{-2}\).

**VII. Conclusions**

We studied the propagation of the vector field coupled to matter in the form (2.2) in generalized Proca theories with derivative self-interactions. The difference from the previous analysis (52) is the existence of matter-vector couplings in the form (2.2) with all the derivative interactions taken into account up to sixth order. On the spherically symmetric and static background, there exists a temporal vector component \( A_0 \) besides a longitudinal scalar \( \chi \). To extract the
constant mode $a_0$ in $A_0$ from the matter continuity equation, we defined the density $\rho$ and the pressure $P$ in the forms (2.25) and (2.27), respectively. The matter-coupling term induced by the constant mode of $A_0$ is smaller than the intrinsic pressure under the condition (2.30), which translates to the inequality (2.39).

In Sec. III, we derived the analytic vector-field profile both inside and outside a spherically symmetric body in the presence of cubic Galileon interactions by employing the general-relativistic gravitational potentials (2.37) and (2.41). This procedure can be justified under the conditions of Eq. (2.40), whose consistency was checked after deriving solutions to the vector field. We showed that both the longitudinal and temporal vector components are sufficiently suppressed due to the operation of the Vainshtein mechanism. This result was also numerically confirmed for the decreasing density profile (3.10). We computed corrections to the general-relativistic gravitational potentials outside the body and found that the model can be consistent with local gravity constraints even for the coupling $Q$ of order unity.

For the derivative couplings $G_i(X)$ with even indices $i$, the solution to the longitudinal mode, which is consistent with the continuous limit of small couplings, corresponds to $\chi' = 0$. In Sec. IV we obtained solutions to the temporal vector component and the gravitational potentials for quartic power-law couplings given by Eq. (4.1). This model is compatible with Solar-System constraints under the bound (4.16). In Sec. VI we also carried out the similar analysis for sixth-order power-law couplings (6.1) and showed the compatibility of solutions with local gravity tests.

In Sec. V we studied the vector-field profile in the presence of quintic Galileon interactions. We found that there are no consistent solutions of the longitudinal mode in the vacuum limit outside the body. This fact does not allow the existence of regular solutions for a realistic compact object whose density rapidly decreases outside the body. Thus, the quintic vector Galileon is the special case in which the screening mechanism for the longitudinal mode does not work. It remains to be seen whether this property also holds for general quintic derivative interactions with matter couplings other than Eq. (2.2).

While we focused on the behavior of the vector field on the spherically symmetric and static background within the Vainshtein radius, it will be of interest to explore how the effect of the matter-vector coupling leads to the modification to the cosmological dynamics in uncoupled generalized Proca theories studied in Ref. [35]. In particular, the signatures of weak gravity found in Refs. [38, 39] for scales relevant to the growth of large-scale structures may be compensated by the matter-coupling term. This may provide us with the possibility of distinguishing between the coupled and uncoupled dark energy models constructed in generalized Proca theories. This issue is left for a future work.

Appendix: Coefficients in the gravitational equations

In Eqs. (2.17)–(2.19) the coefficients $c_{1,2,\ldots,19}$ are given by

\begin{align*}
  c_1 &= -A_1 X G_{3,X}, \quad c_2 = -2G_4 + 4(X_0 + 2X_1)G_{4,X} + 8X_1 X G_{4,X,X}, \\
  c_3 &= -A_1 (3hX_0 + 5hX_1 - X)G_{5,X} - 2hA_1 X_1 X G_{5,X,X}, \\
  c_4 &= 2G_2 - 2X_0 G_{2,X} - \frac{h}{f} (A_0 A_1 A_0' + 2fX A_1') G_{3,X} + \frac{h A_0^2}{2f} (2g_4 - 1 + \frac{2A_0^2}{f} g_{4,X}), \\
  c_5 &= -4hA_1 X_0 G_{3,X} - 4h^2 A_1 A_1' G_{4,X} + \frac{8h}{f} (A_0 X_1 A_0' - fh A_1 X A_1') G_{4,X,X} + \frac{2h^2 A_1 A_1'^2 (g_5 + 2X_0 g_{5,X})}{f}, \\
  c_6 &= 2(1 - h)G_4 + 4(hX - X_0) G_{4,X} + 8hX_0 X_1 G_{4,X,X} - \frac{h}{f} [(h - 1) A_0 A_1 A_0' + 2f(3hX_1 + hX_0 - X) A_1'] G_{5,X} \\
  &\quad - \frac{2h^2 X_1}{f} (A_0 A_1 A_0' + 2fX A_1') G_{5,X,X} + \frac{h A_0^2}{f} [(h - 1) G_6 + 2(hX - X_0) G_{6,X} + 4hX_0 X_1 G_{6,X,X}], \\
  c_7 &= -G_2 + 2X_1 G_{2,X} - \frac{h A_0 A_1 A_0' G_{5,X}}{f} - \frac{h A_0^2}{2f} (2g_4 - 1 - 2h A_1^2 g_{4,X}), \\
  c_8 &= 4hA_1 X_1 G_{5,X} + \frac{4h A_0 A_0' (G_{4,X} + 2X_1 G_{4,X,X})}{f} - \frac{2h^2 A_1 A_1'^2 (3g_5 + 2X_1 g_{5,X})}{f}, \\
  c_9 &= 2(h - 1)G_4 - 4(2h - 1)X_1 G_{4,X} - 8hX_1^2 G_{4,X,X} - \frac{h A_0 A_1 A_0'}{f} [(3h - 1) G_{5,X} + 2hX_1 G_{5,X,X}] \\
  &\quad - \frac{h A_0^2}{f} [(3h - 1) G_6 + 2(6h - 1)X_1 G_{6,X} + 4hX_1^2 G_{6,X,X}], \\
  c_{10} &= -\frac{2h (G_4 - 2X G_{4,X})}{f}, \quad c_{11} = -\frac{2h A_1 X G_{5,X}}{f}, \quad c_{12} = \frac{h (G_4 - 2(2X_0 + X_1) G_{4,X} - 4X_0 X G_{4,X,X})}{f^2},
\end{align*}
\begin{equation}
\begin{aligned}
c_{13} &= \frac{h^2 A_1}{f^2} [3(3X_0 + X_1)G_{5,XX} + 2X_0 XG_{5,XX}] , \\
c_{14} &= - \frac{h A_1}{f} [(3X_0 + 5X_1)G_{5,XX} + 2X_1 XG_{5,XX}] , \\
c_{15} &= \frac{h}{f^2} \left[ 2f A_1 X_0 G_{3,XX} + 2(2A_0^2 - fhA_1 A'_1)G_{4,XX} + 4\left\{ A_0 (2X_0 + X_1) A'_0 - fhA_1 XA'_1 \right\} G_{4,XX} \\
&\quad - h A_1 A'_0 (g_5 + 2X_0 g_5) , \\
c_{16} &= - \frac{h}{f^2} \left[ 2f (G_4 - 2XG_{4,XX} + 4X_0 X_1 G_{4,XX}) + h \left\{ 3A_0 A_1 A'_0 + 2f (X_0 + 3X_1) A'_1 \right\} G_{5,XX} \\
&\quad + 2h \left\{ A_0 A_1 (X_1 + 2X_0) A'_0 + 2f X_1 XA'_1 \right\} G_{5,XX} + h A_0^2 (G_6 + 2XG_6, X + 4X_0 X_1 G_{6,XX}) , \\
c_{17} &= - 2G_2 + 8X_1 (G_{4,XX} + X_1 G_{4,XX}) \\
&\quad + \frac{h A_0}{f} \left\{ A_0 A_1 (3G_{5,XX} + 2X_1 G_{5,XX}) + A'_0 (3G_6 + 4X_1 (3G_{6,XX} + X_1 G_{6,XX})) \right\} , \\
c_{18} &= 2G_2 - \frac{2h}{f} \left[ (A_0 A_1 A'_0 + 2f X_1 A'_1) G_{3,XX} + 2(A_0 A'_0 + A_0^2) G_{4,XX} + 2A'_0 (2X_0 A'_0 - hA_0 A_1 A'_1) G_{4,XX} \right] \\
&\quad + \frac{2h^2 A_0}{f^2} \left[ f (A_0 A'_0 + A_1 A'_1) g_5 + A'_0 (A_0 A_1 A'_0 + 2f X_1 A'_1) g_5 \right] - \frac{h A_0^2}{f} (2g_4 - 1) , \\
c_{19} &= \frac{2h}{f} \left\{ - 2(A_0 A'_0 + fh A_1 A'_1) G_{4,XX} + 4X_1 (A_0 A'_0 - fh A_1 A'_1) G_{4,XX} + h(A_0 A'_0 + A_0 A'_0 A'_1 + A_0 A_1 A'_0) G_{5,XX} \right. \\
&\quad + 2h A'_0 (A_0 X_1 A'_1 + A_1 X_0 A'_0) G_{5,XX} + 2h A'_0 A'_0 G_6 + \frac{h A'_0}{f} \left\{ (A_0 A'_0^2 + 4f X_1 A'_0 - 3fh A_1 A'_1) G_{6,XX} \right. \\
&\quad + 2A'_0 X_1 (A_0 A'_0 - fh A_1 A'_1) G_{6,XX} \right\} .
\end{aligned}
\end{equation}

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