Topological Dynamics and Grand Unified Theory

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A formalism is presented to construct a non-perturbative Grand Unified Theory when gravitational Planck-scale phenomena are included. The fundamental object on the Planck scale is the three-torus $T^3$ from which the known properties of superstrings, such as the geometric action and duality, follow directly. The low energy theory is 11-dimensional and compactification to a Lorentzian four-manifold is an automatic feature of the unified model. In particular, the simply-connected K3 Calabi-Yau manifold follows naturally from the model and provides a direct link with M-theory. The high energy theory is formulated on a $T^3$ lattice with handles which exhibits the necessary symmetry groups for the standard model, and yields a consistent amplitude for the cosmic microwave background fluctuations. The equation of motion for the supersymmetric, unified theory is derived and leads to the Higgs field. This formulation predicts a remnant, scalar topological defect of mass $\leq \frac{9m_{\text{Planck}}}{46}$ from the Planck epoch, which is a candidate for dark matter.

Finally, it is shown that if the universe is quantum-mechanical, its spatial dimension is equal to three and the laws of nature are Lorentz invariant when gravity can be neglected.

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1 Introduction

One of the outstanding questions in quantum cosmology and particle physics is the unification of gravity with the electro-weak and strong interactions. Much effort has been devoted in the past years to formulate a purely geometrical and topological theory for both types of interactions[1,2]. This work aims

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at extending the topological dynamics approach presented in [3] (paper I from here on) to include non-gravitational interactions and to construct a Quantum-gravitational Grand Unified Theory (QGUT).

The main results of paper I are as follows: The properties of space-time topology are governed by homotopically inequivalent loops in the prime manifolds $S^3$, $S^1 \times S^2$, and $T^3$. This is the only set which assures Lorentz invariance and the superposition principle for all times, and thus provides a natural boundary condition for the universe. The dynamics of the theory are determined by the loop creation ($T^\dagger$) and loop annihilation ($T$) operators. On the Planck scale the quantum foam has the unique structure of a lattice of three-tori with 4 homotopically inequivalent paths joining where the $T^3$ are connected through three-ball surgery. The existence of this four-fold symmetry is called *multiplication* and leads to $O(n)$ and $SU(n)$ gauge groups as well as black hole entropy. The number of degrees of freedom, i.e. the prime quanta, associated with the three-sphere, the handle manifold, and the three-torus are 1, 3, and 7, respectively. During the Planck epoch, $S^1 \times S^2$ handles can attach themselves to the $T^3$ lattice and lead to supersymmetric interactions.

In a way one can say that the quantum-mechanical formulation of the Equivalence Principle is found in the occurrence of multiplication, which accounts for the homotopic Planck scale degrees of freedom not treated in general relativity. Also note that Wheeler's argument for a "defective gene" in $T^3$ related to the initial value problem of general relativity[4] is avoided, because metrics and connections are defined on the intersection points of the lattice. Finally, it was found that the cosmological constant is very small and proportional to the number of macroscopic black holes at the current epoch. This result follows from the generalization of Mach's principle in which local topology change is a consequence of global changes in the matter degrees of freedom and vice versa. Such a result is consistent with 't Hooft's "Naturalness Principle" and the fact that the symmetry of the Einstein equations is not increased when $\Lambda = 0$. Since life forms can only exist in a universe where the bulk of matter is not in the form of black holes, it follows from the Anthropic Principle that we observe a small cosmological constant. A further consequence is that massive isolated black holes still grow due to quantum fluctuations. This steady increase in entropy, independent of the matter degrees of freedom, yields a natural arrow of time.

This paper is organized as follows. Section 2 presents a derivation of the dimension of the universe and its link with Lorentz invariance. Section 3 discusses the construction of a QGUT based on the $T^3$ lattice with handles. Section 4 contains the discussion and suggestions for further investigations.
2 The Dimension of the Universe, Lorentz Invariance and the Superposition Principle

The invariance of the speed of light should hold in any quantum field theory. In paperI it was shown that the loop as representative of the superposition principle leads to $T^3$ as the fundamental Planck scale object. Since the three-torus bounds a Lorentz four-manifold with SL(2;C) spin structure, i.e. is nuclear, the superposition principle implies Lorentz invariance.

Rather than impose the dimension of space-time one would want to derive it from first principles. In a loop-topological sense, field interactions correspond to loops which are linked. Clearly one can exclude spatial dimensions $\leq 2$ on these grounds. Furthermore, any two linked loops embedded in a space of dimension $\geq 4$ are homeomorphic to two separate loops. Given that Lorentz invariance implies time dilation of the arbitrary intersection point of two loops, it follows that the only spatial dimension of the universe consistent with quantum mechanics is 3. As a corollary one finds that the occurrence of spin $1/2$ in the universe, a property of the three-torus, follows from the superposition principle. Since both Lorentz invariance and the dimension of the universe derive from the superposition principle, the fact that life depends on quantum-mechanical processes in an essential way provides a natural formulation of the Anthropic Principle. Conversely, the sum of the prime quanta, $1 + 3 + 7 = 11$, implies that the group theoretic dimension of the universe is eleven. This dimension pertains to those scales and energies for which the GUT is realized and direct interactions between three-tori and handles through matter degrees of freedom can be neglected (paperI).

3 Quantum Gravity, Superstrings and Topological Dynamics

3.1 $K3$ as a Homotopically Trivial Internal Space

The prime quantum of $T^3$ equals 7 which implies that 4 dimensional degrees of freedom exist on the Planck scale which are dynamically trivial under the action of the loop algebra, generated by $T^\dagger$ and $T$, which acts on the three-torus. This leads to the decomposition

$$G^7 = T^3 \times M,$$

where $M$ has dimension four and $G^7$ is a manifold with a group theoretic dimension of seven. Furthermore, $M$ is not necessarily confined to the Planck scale, just like the homotopically trivial three-sphere $S^3$ and the handle manifold $S^1 \times S^2$. Because there is only one type of loop and $T^4 T^3 = T^4$ is the
only four-manifold which can be constructed from the three-torus under the loop algebra (paperI), \( M \) must be of the form \( S(\oplus_i T^4_i) \). Here \( S \) denotes a “simplification” which renders \( M \) simply-connected. The operator \( S \) is local and should be a projection, \( S^2 = S \). This amounts to identifying all points \( x \) and \(-x\) generated by the modular group \( SL(4;Z) \) of \( T^4 \), where the \( x \) are coordinates local to each four-torus.

The manifold thus constructed is the compact, simply-connected Kummer K3 quartic surface in \( CP^3 \) with an Euler number of 24. This space is known to be Kähler with vanishing first Chern class and to admit an (anti-)self dual metric. In fact, K3 has no symmetries at all. If one views interacting superstrings as being built up from the surfaces of three-tori, then heterotic string duality in seven dimensions results from the fact that the effective compactification space changes from K3 in M-theory to \( T^3 \). Also, the internal space \( K3 \times T^3 \) has an \( SU(2) \) holonomy, and supports \( N = 4 \), Osp(4/4), supersymmetries with 192+192 massless degrees of freedom, i.e. compactified 11-dimensional super gravity[5]. The natural scale for \( K3 \times T^3 \) is therefore the GUT scale.

3.2 QGUT Phenomenological Preliminaries

In paperI it was shown that the three-torus is the fundamental Planck scale object. It was also found that the non-irreducible handle manifold (a black hole) can exist on all length scales and that interactions between \( T^3 \) and \( S^1 \times S^2 \) occur as matter traverses the wormholes. The three-torus has 3 branches and it is easy to see that individual branches yield spin 1 particles and pairs of branches lead to spin 1/2[4]. As a handle attaches itself to two singlets or connects two branches of a doublet it effectively converts 1-bosons into 1/2-fermions and vice versa. The same effect occurs through the junction points of two three-tori connected through three-ball surgery. The three branches then yield three pairs of leptons and three doublets of quarks. It follows that the maximum non-integer spin carried by the three-torus, and therefore by \( K3 \times T^3 \), is 3/2 (i.e. a gravitino).

In fact, in paperI it was shown that the lattice junctions support O(n) and \( SU(n) \) symmetry groups. From the fact that the effective, group theoretic dimension of the GUT is 11, it follows that the dimension of these junction groups is 10, i.e. \( SU(5) \) and \( O(10) \). That is, one dimension is lost through projection onto an \( S^1 \) space obtained through connecting junction points on the \( T^3 \) lattice along closed paths. These local closed paths are needed to define metric connections and covariant derivatives.

For supersymmetry in the presence of quantum-gravitational effects, i.e. \( Q \)-supersymmetry, the \( S^1 \times S^2 \) number density on the \( T^3 \) lattice should be close
to unity. It was already shown in paper I that this density depends on the formation, merging and evaporation of black holes as well as the rate of expansion of the universe. From the discussion on the cosmological constant in the Introduction it then follows that the $T^3$ field theory without handles yields $\Lambda = 0$, as found in conventional superstring theory. In summary, the fundamental structure for the QGUT is a $T^3$ lattice with handles. This structure will be referred to as being charged by $S^1 \times S^2$.

### 3.3 QGUT Construction

#### 3.3.1 The Fundamental Topological Manifold

Only odd sums of nuclear primes bound Lorentz manifolds (paper I). Note that every orientable manifold in two and three dimensions is a spin manifold and that one requires equal numbers of bosonic and fermionic fields. Supersymmetry also requires the action to be symmetric in fermion-boson and boson-fermion interchange processes. This requires three handle manifolds per pair of three-tori. That is, two branches of each $T^3$ are connected internally and the third handle links the remaining two branches of the three-tori. So at the QGUT scale a supersymmetric manifold can be written as

$$Q = 2T^3 \oplus 3S^1 \times S^2,$$

which assures nuclearity and Lorentz invariance.

If one works on the conformally invariant boundary $\partial Q = TQ$ of this structure, then $\partial T^3 = T_3$ is identified with a cubically interacting, gauge string field $A$. The loop annihilation operator reduces the handles to $S^2$ spheres, which are trivial in the connected sum. The string connection $\omega$ lives on the junction of $T_3 \oplus T_3$. This yields a covariant derivative $D = d + \omega f$ with $f$ the structure constants. The effective geometric string action, the Chern-Simons form, immediately follows, $S = A \times DA + 2/3A \times A \times A$, where the Verma indices and the anti-symmetric three-tensor $\epsilon$ have been suppressed. This suggests that M-theory on $S^1$ is dual to the 10-dimensional type IIa superstring.

#### 3.3.2 The Equation of Motion for Constant Charge

In paper I an elaborate derivation was given for the topological dynamics of space-time in terms of the $T^3$, $S^1 \times S^2$, and $S^3$ prime manifolds when the loop is the fundamental object. The derivation in paper I was based on 1) the existence of homotopic classes of paths in a topological manifold as representatives of the linear superposition principle, and 2) the validity of CPT represented by loop creation and annihilation processes. The homotopy and
geometry of paths through the $T^3$ lattice, modified by the handle interactions, are here seen as different realizations of the fundamental topological object $Q$. As such, the derivation of paperI is equally applicable to the loop properties of the topological four-manifold $Q \times R$. The QGUT equation of motion then follows from the single time topological dynamics equation derived in paperI for a fixed $Q \times R$ topology described by the paths in a set of connected prime manifolds $T_j$, 

$$
\sum_j [T_i, T_j^\dagger]/T_j = R(TT^\dagger + T^\dagger T)T_i, \hspace{1cm} (3)
$$

with the identifications $(T, T^\dagger \rightarrow \partial_\mu, \partial^\mu)$, $T_i \rightarrow q_\lambda$. Because the topology is fixed now, the dynamics of the theory, the left hand side in Equation (3), must be a pure self-interaction with no reference to any topological basis set (see paperI). This yields for the four vector $q_\lambda$

$$
q^\mu[q_\lambda \Box q_\mu - q_\mu \Box q_\lambda] = 2R \Box q_\lambda, \hspace{1cm} (4)
$$

with $\Box$ the four-dimensional Laplace operator or d’Alembertian. Here $R$ is a constant associated with the dynamical changes in the topology of the charged $T^3$ lattice (see paperI). It follows immediately through multiplication with $q^\lambda$ that a QGUT is only self-consistent if the charged $T^3$ lattice has $R = 0$. One finds that the resulting self-interaction is cubic as, demanded by the triple loop structure on the three-torus, and linear in the second derivative operators. A natural interpretation for the quadruplet is that it describes the excited states of $Q$, analogous to the vibration modes of the superstring. The possible functional forms of $q_\lambda$ are determined by the quadruplet solutions to the equation of motion, and the cyclic structure of $Q$.

The tensor $q$ of rank one results as follows. The three loop homotopic degrees of freedom of the seven-dimensional internal space $K3 \times T^3$ are equivalent to a tensor $V$ of rank 3 and dimension 7. This tensor should be symmetric in two of its indices because it describes three-point interactions and two of the $T^3$ legs are connected internally through $S^1 \times S^2$ handles. So $V$ supports 196 algebraically independent components. From the 196 degrees of freedom, 192 (on each three-torus) can be associated directly with 11-dimensional, N=1 supergravity compactified on $K3 \times T^3$, i.e. the low energy limit Osp(1/4). The remaining four degrees of freedom are purely quantum-gravitational in nature and correspond to the four currents (wave amplitudes) which can flow along the homotopically inequivalent paths associated with the charged $T^3$ lattice.

The equation of motion for the quadruplet $q_\lambda$ on $Q \times R$ is now

$$
q_\lambda q^\mu \Box q_\mu = q^\mu q_\mu \Box q_\lambda, \hspace{1cm} (5)
$$

in the constant charge limit (see §3.3.3 below). A large class of solutions of (5) is determined by the wave equations $\Box q_\lambda = 0$. This equation of motion is of the form “the boundary of the boundary is zero”, in analogy with the
sourceless Maxwell equations. Its solutions are represented by the well-known traveling wave forms. Another class of solutions is determined by the Klein-Gordon equations $\Box q_\lambda + m^2 q_\lambda = 0$. Because these solutions are applicable to a constant charge system, they also hold for the neutral $T^3$ lattice.

Note that Equation (5) is invariant under CPT, a property which is conserved in the full equation of motion derived below. In PaperI it was shown that the SL(2;C) gauge group follows naturally from a nuclear manifold. The theory is therefore manifestly Lorentz invariant with $\partial^\mu = \eta^{\mu\nu} \partial_\nu$. Furthermore, the Dirac matrices can be constructed from the hermitian Pauli matrices contained in SL(2;C), and yield a local formulation of spin 1/2.

### 3.3.3 Symmetry Breaking and the Equation of Motion in the Presence of Charge Fluctuations

The properties of the resulting QGUT depend on both the charge on the $T^3$ lattice as well as the energy density of the matter degrees of freedom. The interactions in the GUT on $T^3$, in the low charge and low energy limit, correspond to 11-dimensional N=1 super gravity compactified on $K3 \times T^3$. That is, $\text{Osp}(N/4) \supset \text{O}(N) \times \text{Sp}(4)$ with $N=4$ for the massless sector in dimension 4. For small charge and large energy (above the GUT scale), M-theory results on $K3$. That is, M-theory corresponds to the small charge quadruplet solutions on the $T^3$ lattice. Membranes appear to have the correct limiting behavior for small energy and encompass all the known types of superstrings. For decreasing charge and energy density, the QGUT domains will therefore become interspersed with regions described by M-theory and 11-dimensional super gravity.

For large charge densities, $Q_h \approx 1$, but with quantum perturbations in the local number of handles, which are subject to evaporation, an additional field is generated. These fluctuations in the charge on the $T^3$ lattice cause phase changes in the currents flowing through $Q$. The fundamental object to solve for is therefore $e^{i\phi} q_\lambda$, with $\phi$ a function of time and position. The presence of this U(1) multiplier then leads to the full QGUT equation of motion

$$2i\partial_{\nu} \phi [(\partial^\nu q_\lambda) q^\mu q_\mu - (\partial^\nu q_\mu) q^\mu q_\lambda] = q_\lambda q^\mu \Box q_\mu - q^\mu q_\mu \Box q_\lambda,$$

(6)

with the scalar constraint $q^\mu q_\mu = \phi^2$ to preserve locally the internal phase information of the system. PaperI discusses the topological formulation of this constraint in terms of multiplication and the effective interaction potential. This conservation law remains in effect as the QGUT on $Q \times \mathbb{R}$ is broken down to an M-theory. The internal $K3 \times T^3$ space only becomes accessible for $Q_h << 1$ and small energy. Recall here that $K3$ has no symmetries at all, and therefore $K3 \times T^3$ supports non-trivial holonomies only when $Q$-supersymmetry is replaced by the gauge group $\text{Osp}(N/4)$. Because the evo-
olution of $\phi$ is driven by the handle degrees of freedom, it follows that in M-
theory the field $\phi$ obeys the condition $\partial_\nu \phi = 0$, and is effectively frozen in. Note
that the handle quantum fluctuations lead to a fourth-order interaction term
which couples the quadruplet and the scalar field. Note that there are precisely
enough degrees of freedom and constraints in the theory to support both the
quadruplet and a scalar field. In fact, multipliers other than $U(1)$ would yield
the system of equations underdetermined. The full solution is now determined
by Equation (6), the scalar constraint, and the cyclic properties of $Q$.

The junction potential has the form $V(q^\mu q_\mu) = \lambda (q^\mu q_\mu)^2 + \mu^2 q^\mu q_\mu$. This poten-
tial follows from the four-fold homotopic symmetry associated with the $T^3$
lattice and reduces to the form of paperI under the scalar constraint. In the
low charge limit, the field $\phi$ has a constant, Lorentz invariant vacuum expecta-
tion value $\langle 0 | \phi(x) | 0 \rangle = c \neq 0$, defined on the junctions of the $T^3$ lattice. The additional Poincaré scalar is, in fact, the Higgs field. The vacuum expectation
value of the Higgs field is non-zero for $\mu^2 < 0$ which leads to spontaneous
symmetry breaking, as first suggested by Nambu and co-workers.

3.3.4 Topological Defects

As the charge on the $T^3$ lattice decreases it need not do so in a homogeneous
manner. Therefore, topological defects can result where the phase $\phi$ differs
between charged and neutral $T^3$ lattice sites. These topological defects have
no electrical charge and spin 0. It is proposed here that these topological de-
defects contribute to the so-called dark matter necessary in models of large scale
structure formation.

The number of degrees of freedom of $Q$ is given by

$$N_Q = (TT^\dagger + T^\dagger T)(2T^3 \oplus 3S^1 \times S^2) = 23.$$  \hspace{1cm} (6)

In the loop homotopic approach adopted here, these degrees of freedom are
all equivalent and equipartition should apply. For the neutral lattice manifold
$P = T^3 \oplus T^3$, one has $N_P = 14$. The latent heat is therefore

$$H = (N_Q - N_P)m_{\text{Planck}}/N_Q = 9m_{\text{Planck}}/23.$$  \hspace{1cm} (7)

Thus, one finds

$$m_q \leq H/2 = 9m_{\text{Planck}}/46$$  \hspace{1cm} (8)

for the mass of a topological defect $q$ produced by the phase difference across
the junction of $P$. The determination of the precise mass distribution of the
defects, which is important for the small scale power spectrum, requires a more
detailed computation and will be presented elsewhere. The efficiency of this
defect mechanism depends on the number of (mini) black holes in the universe
at early epochs as discussed in paperI. Furthermore, the characteristic energy
scale for a supersymmetric GUT is of the order of $m_{\text{GUT}} = m_{\text{Planck}}/N_Q$, which is smaller than the typical mass of a defect. It follows that approximately $H/m_{\text{Planck}} \approx 40\%$ of the dark matter in the universe can be in the form of scalar topological defects. Additional GUT phase transitions can increase the percentage of non-baryonic dark matter. Still, it appears that current estimates of the fraction of baryonic matter in the universe [6] suggest that not all dark matter in the universe is non-baryonic if $\Omega = 1$.

### 3.3.5 Quantum Gravity Effects and Large Scale Structure

From the above discussion it follows that the QGUT energy scale corresponds to a size and matter density of the universe where mini black holes are formed rapidly enough to sustain the topological manifold $Q$. During the QGUT epoch the formation and evaporation of mini black holes leads to spatial variations in the scalar field $\phi$ which are frozen in when the charge on the $T^3$ lattice becomes small. The multiplication effect of paperI then assures that the associated (thermal) fluctuations in the quantum fields on the charge-free $T^3$ lattice are suppressed and scale free.

The strength of these initial, large scale perturbations can be computed rigorously with very few assumptions. The quadruplet interaction is intrinsically cubic. Therefore, the 23 equipartitioned degrees of freedom on $Q$ lead to a maximum amplitude which is given by

$$\frac{\delta T}{T} = N_Q^{-3} = 8.2 \times 10^{-5}. \tag{9}$$

This amplitude is the first non-zero correction term to the average energy of $Q$ on purely thermodynamic grounds. If the $1\sigma$ topological correction due to multiplication (paperI) is applied one finds a final amplitude $\delta T/T = 3.7 \times 10^{-5}$ on the horizon scale. This amplitude is consistent with Cosmic Microwave Background measurements if the soft equation of state during recombination at $z \sim 1000$ is included[7].

### 4 Discussion

It has been shown that the requirement of Lorentz invariance follows from the superposition principle and leads to a spatial dimension of the universe equal to three. An equation of motion has been derived for a QGUT on the supersymmetric manifold $Q = 2T^3 \oplus 3S^1 \times S^2$, i.e. a charged $T^3$ lattice. M-theory results for the low charge limit. For small energy densities, the theory is described by 11-dimensional super gravity and the compactification procedure is prescribed by the QGUT. It has been found that the non-baryonic mass
content of the universe is directly linked with Planck scale dynamics through
the formation and evaporation of mini black holes and the production of scalar
topological defects of mass $m_q \leq 9m_{\text{Planck}}/46$ during the QGUT era.

The general supersymmetric solution for the QGUT and M-theory, including
the cosmological evolution of the defects and the choice of initial conditions
for the equation of motion on an evolving $T^3$ lattice (paperI), will be presented in [8]. That is, the state space will be constructed from the quadruplet
solutions and the cyclic properties of $Q$. The nature of black hole singular-
ities can then be addressed as well, since they are quadruplet solutions in
the low charge limit. In particular, such solutions should provide a statistical
mechanical formulation of black hole entropy.

The link between black holes and Planck scale phenomena was already sug-
gested in paperI, and it has been investigated here how very small and very
large scale physics are linked. The existence of a charge threshold at which the
QGUT symmetry is broken, is equivalent to a quiescent and stable quantum
foam which derives from a relatively small set of macroscopic black holes. This
macroscopic set of black holes, as a measure of the topology of the universe,
fixes the Planck scale quantum fluctuations, and the nature of field interac-
tions.

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References

[1] S. Kaku, Introduction to Superstrings (Springer-Verlag, 1990).

[2] S.W. Hawking and A. Strominger 1991 in Quantum Cosmology and Baby
Universes, eds. S. Coleman, J.B. Hartle, T. Piran & S. Weinberg, (World Scientific,
1991) p. 245, p. 272.

[3] M. Spaans, Nuc. Phys. B in press (1997, paperI).

[4] J.A. Wheeler in Quantum Cosmology, eds. L.Z. Fang & R. Ruffini, (World
Scientific, 1987) p. 27.

[5] M.J. Duff, B.E.W. Nilsson and C.N. Pope, Phys. Lett. B 129 (1983) 39.

[6] P.J.E. Peebles, Principles of Physical Cosmology (Princeton:PUP, 1993).

[7] C.A. Norman and M. Spaans, ApJ submitted.
[8] M. Spaans, in preparation.