A VARIATIONAL PROBLEM ASSOCIATED TO A HYPERBOLIC CAFFARELLI–KOHN–NIRENBERG INEQUALITY

HARDY CHAN, LUIZ FERNANDO DE OLIVEIRA FARIA, AND SHAYA SHAKERIAN

Abstract. We prove a Caffarelli–Kohn–Nirenberg inequality in the hyperbolic space. For a semi-linear elliptic equation involving the associated weighted Laplace–Beltrami operator, we establish variationally the existence of positive radial solutions in the subcritical regime. We also show a non-existence result in star-shaped domains when the exponent is supercritical.

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1. Introduction

1.1. The Caffarelli–Kohn–Nirenberg inequality. In [8], Caffarelli, Kohn and Nirenberg proved the following celebrated interpolation inequality which states that in any dimension \(N \geq 3\), there is a constant \(C = C(a, b, N) > 0\) such that for all \(u \in C_c^\infty(\mathbb{R}^N)\), the following holds:

\[
\left( \int_{\mathbb{R}^N} |x|^{-b} |u|^p \, dx \right)^{\frac{2}{p}} \leq C \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx,
\]

(1.1)

where

\(-\infty < a < \frac{N - 2}{2}, \quad 0 \leq b - a \leq 1\)

and

\(p = \frac{2N}{N - 2 + 2(b - a)}\).

The best constant in the above inequality is defined as

\[
S(a, b, \mathbb{R}^N) := \inf_{u \in D^{1,2}_a(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |x|^{-b} |u|^p \, dx \right)^{\frac{2}{p}}},
\]

(1.2)

where \(D^{1,2}_a(\mathbb{R}^N)\) is the completion of \(C_c^\infty(\mathbb{R}^N)\) with respect to the norm \(\|u\|_a^2 = \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx\).

It was computed explicitly by Aubin [4], Talenti [31] for the Sobolev inequality when \(a = b = 0\), by Lieb [22] for the case \(a = 0, \, 0 < b < 1\), then by Chou–Chu [11] in the full \(a\)-non-negative region \(0 \leq a < \frac{N - 2}{2}, \quad 0 \leq b \leq a + 1\). Minimizers, which are radial, were given explicitly using classical ODE analysis. The picture was completed by Catrina–Wang [10], who investigated the parameter region with \(a < 0\). They observed, among other things, a symmetry breaking phenomenon. Namely, the best constant is attained by a non-radial minimizer.
The existence or non-existence of minimizers and their qualitative properties as well as improved versions with remainders have been extensively studied over the last two decades. See, for instance, Abdellaoui–Colorado–Peral [1], Catrina–Wang [10], Dolbeault–Esteban–Loss–Tarantello [13], Felli [14], Nápoli–Drelichman–Durán [24], Sano–Takahashi [28], Shen–Chen [29] and references therein.

In this paper, we prove this family of Caffarelli–Kohn–Nirenberg (C–K–N) inequalities (1.1) on the disc model of the Hyperbolic space \( \mathbb{H}^N \).

**Theorem 1.1** (A C–K–N inequality on the Hyperbolic space). For any \(-\infty < a < \frac{N-2}{2}, 0 \leq b-a \leq 1 \) and \( p = \frac{2N}{N-2+2(b-a)} \), there exists a constant \( C = C(a, b, N) > 0 \) such that
\[
\left( \int_{\mathbb{B}^N} d^{-bp}|u|^p \, dV \right)^{\frac{2}{p}} \leq C \int_{\mathbb{B}^N} d^{-2a}|\nabla_{\mathbb{B}^N} u|^2 \, dV, \quad \text{for } u \in C_c^\infty(\mathbb{B}^N),
\]
where we denote the hyperbolic distance function from the origin by \( d \), the hyperbolic gradient by \( \nabla_{\mathbb{B}^N} \) and the hyperbolic volume element by \( dV \), as introduced in Section 2.

The proof is given in Section 3. Some remarks are in order.

**Remark 1.** One can deduce from the proof that inequality (1.3) holds for weights other than \( d \), which are radially decreasing and have a quadratic singularity at the origin.

**Remark 2.** The Dirichlet integral \( \int_{\mathbb{B}^N} d\alpha|\nabla_{\mathbb{B}^N} u|^2 \, dV \), as in the Euclidean case, corresponds to a weighted Laplace–Beltrami operator
\[
\Delta_{\mathbb{B}^N} u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( \sqrt{gg^{ij}} d^\alpha \frac{\partial u}{\partial x_j} \right),
\]
with \( g \) the canonical metric on \( \mathbb{B}^N \). It has been considered in [9].

**Remark 3.** One may put \( \alpha = -2a \) and \( \beta = -bp \) to state (1.3) in the form
\[
\left( \int_{\mathbb{B}^N} d\alpha\beta|u|^{2N+\beta} \, dV \right)^{\frac{N-2+\alpha}{N+\beta}} \leq C \int_{\mathbb{B}^N} d\alpha|\nabla_{\mathbb{B}^N} u|^2 \, dV,
\]
for \(-N - \alpha - 2 \leq \beta \leq \frac{N\alpha}{N-2} \). This motivates the definition of the critical exponent \( \frac{2(\alpha+\beta)}{N-2+\alpha} \).

As in the Euclidean case, the C–K–N inequalities (1.3) contain the Hardy inequality \( (a = 0, b = 1) \) and the Sobolev inequality \( (a = b = 0) \) on Hyperbolic space as special cases. It is worth mentioning that our proof additionally makes use of the hyperbolic Poincaré inequality [23] to accommodate the lower order terms. The best constant
\[
S(a, b, \mathbb{B}^N) := \inf_{u \in D^1_{\text{loc}}(\mathbb{B}^N) \setminus \{0\}} \frac{\int_{\mathbb{B}^N} d^{-2a}|\nabla_{\mathbb{B}^N} u|^2 \, dV}{\left( \int_{\mathbb{B}^N} d^{-bp}|u|^p \, dV \right)^{\frac{2}{p}}},
\]
is known to equal \( S(a, b, \mathbb{B}^N) \) in the cases of weighted Hardy inequality \( (a < \frac{N-2}{2} \text{ and } b = a + 1, [21]) \) and non-weighted Sobolev inequality \( (a = b = 0, [19]) \). To our knowledge, the problem of determining \( S(a, b, \mathbb{B}^N) \) is largely open. While it is tempting to look for radial extremals, the resulting non-linear ODE cannot be transformed to one with constant coefficients, making the analysis very difficult.

**1.2. Existence and non-existence.** We now concentrate on the study of the equation
\[
-\Delta_{\mathbb{B}^N} u = \lambda d^{\alpha-2} u + d^\beta |u|^{q-2} u, \quad u \in H^1(\mathbb{B}^N),
\]
for \(-N < \alpha - 2 \leq \beta \leq \frac{\alpha}{N-2}, 2 \leq q \leq \frac{\alpha}{\alpha}\beta := \frac{2(\alpha+\beta)}{N-2+\alpha} \) and \( \lambda < \left(\frac{N-2+\alpha}{2}\right)^2 \). This arises as the Euler–Lagrange equation of an energy functional associated to the weighted Hardy–Sobolev inequality in the hyperbolic space,
\[
c \left( \int_{\mathbb{B}^N} d^\beta u^q \, dV \right)^{\frac{2}{q}} + \lambda \int_{\mathbb{B}^N} d^\beta u^2 \, dV \leq \int_{\mathbb{B}^N} d^\alpha |\nabla_{\mathbb{B}^N} u|^2 \, dV \quad \text{for all } u \in C_c^\infty(\mathbb{B}^N).
\]
Indeed, (1.5) can be obtained by interpolating (1.3) and the following weighted Hardy’s inequality with sharp constant due to Kombe–Özaydin [21]

$$\left(\frac{N - 2 + \alpha}{\alpha} \right)^2 \int_{B_N} d^\alpha \frac{u^2}{d^2} dV \leq \int_{B_N} d^\alpha |\nabla_{B_N} u|^2 dV, \text{ for all } u \in C_c^\infty(\mathbb{B}^N)$$

(1.6)

via Hölder’s inequality. An inequality of flavor similar to (1.5) is known in [21], where the authors showed that for certain smaller exponent \( q \), one can in fact take \( \lambda = (\frac{N-2+\alpha}{\alpha})^2 \).

In the absence of the Hardy potential and the weight \( d^\alpha \) in the Laplacian (i.e. \( \lambda = \alpha = 0 \)), when \( \beta > 0 \), the equation (1.4) of interest originate from the study of stellar structures as proposed by Hénon. Gidas–Spruck [16] classified the non-negative solutions of (1.4) in the full Euclidean space \( \mathbb{R}^n \) with a Liouville-type theorem so that no non-trivial solution exists for \( 2 < q < 2\beta = \frac{2(N+\beta)}{N-2} \). In the hyperbolic space, in contrast, He [18] established the existence of solutions in the same range of exponents. This was subsequently generalized by Carrião, Miyagaki and the second author [9] to the weighted case \( \alpha \neq 0 \), who showed that problem (1.4) possesses a positive radial solution.

In a bounded domain, it is easily shown that ground state solutions exist when \( 2 < q < 2^* = \frac{2N}{N-2} \) and \( \beta > 0 \). Ni [25] found radial solutions when \( 2 < q < 2\beta \) which, in the case of Hénon equations \( (\beta > 0) \), extend the existence result to Sobolev supercritical exponents. On the other hand, non-existence in star-shaped domains has been proved using generalized versions of the Pohožaev identity [26] in both Hardy and Hénon type problems.

In this paper, we address the remaining cases by considering problem (1.4) which involves the singular potential (i.e. when \( \beta < 0 \)) and the Hardy term (i.e., \( \lambda d^\alpha d^\alpha u \)). By studying the compactness properties of radial \( H^1(\mathbb{B}^N) \) functions, we establish the following existence result.

**Theorem 1.2.** Assume that \( N \geq 3, \beta > \alpha - 2 > -N \), \( \lambda < \left(\frac{N-2+\alpha}{2}\right)^2 \) and \( q \in (2, 2\beta) \). Then, problem (1.4) has (at least) a positive solution.

**Remark 4.** The critical exponent \( 2^* \) is Sobolev critical or supercritical (i.e. \( 2^* \geq 2^* \)) if and only if \( \beta \geq \frac{N\alpha}{N-2} \).

**Remark 5.** We emphasize that with the same arguments as in [18, 9], our result is still valid for

$$-\Delta_{B_N}^\alpha u - \lambda [d(x)]^\alpha - 2 u = K(d(x))f(u), \quad u \in H^1(\mathbb{B}^N),$$

which involves more general non-linearities under suitable growth conditions on \( K \) and \( f \).

On the other hand, we establish non-existence of an equation associated to inequality (1.3) in a star-shaped domain with respect to the origin, in terms of which the distance function \( d \) is defined. More precisely, we say that \( \Omega \subset \mathbb{B}^N \) is a star-shaped domain if \( x \cdot \nu \geq 0 \) where \( \nu \) denotes the outward normal of \( \partial \Omega \).

For semilinear elliptic equations in hyperbolic space, the Pohožaev identity has been applied, for example, [15, 5].

**Theorem 1.3.** Let \( \Omega \subset \mathbb{B}^N \) be a star-shaped domain and \( \alpha, \beta \in \mathbb{R} \) satisfy \( -N < \alpha - 2 \leq \beta \). If \( p \geq \max \{2*, 2^*_\} \), then there does not exist any non-trivial weak solution to the Dirichlet problem

$$\begin{cases} -\Delta_{B_N}^\alpha u = d^\beta |x|^{\alpha-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega. \end{cases} \quad (1.7)$$

We give the proof in Section 6. The first ingredient is, as expected, a Pohožaev type identity (Proposition 6.1). As opposed to the Euclidean case whose corresponding result holds for all \( p \geq 2^* \), special care has to be taken because of the lower order terms arising from the derivatives of the distance function and the metric tensor. Under the restrictions \( p \geq 2^* \), we are able to conclude the proof using the global estimate on \( d \),

$$2|x| \leq \log \frac{1 + |x|}{1 - |x|} \leq \frac{2|x|}{1 - |x|^2} \quad \text{for all } |x| < 1.$$
A careful study of the quantity in Lemma 6.1(2) reveals that it cannot be controlled in a sufficiently large domain whenever $2^\frac{N}{d} \leq p < 2^\ast$. On this interval of $p$, the question of existence remains open.

2. Preliminaries

We start by recalling and introducing a suitable function space (on the hyperbolic space) and its properties for the variational principles that will be needed in the sequel. Our main sources for this section are the papers [7, 23] and the book [27].

Let $B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$ be the unit disc in $\mathbb{R}^N$. The Poincaré ball model of the hyperbolic space, $\mathbb{B}^N$, is the set $B_1(0)$ endowed with the Riemannian metric $g = (g_{ij})$, where

$$(g_{ij}(x))_{i,j=1...N} = (\rho^2(x) \delta_{ij})_{i,j=1...N} = \begin{cases} \rho^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$\rho : B_1(0) \rightarrow \mathbb{R} \quad x \mapsto \frac{2}{1-|x|^2}$$

We denote by $g^{ij}$ the components of the inverse matrix of the metric tensor $(g_{ij})$. Using this notation, we can write the weighted Laplace–Beltrami type operator as

$$-\Delta_{\mathbb{B}^N} u = -\frac{1}{\rho^N} \sum_{i=1}^{N} \partial_{x_i} \left( \rho^{N-2} (d(x))^2 \partial_{x_i} u \right),$$

for $u \in H^1(\mathbb{B}^N)$, where the space $H^1(\mathbb{B}^N)$ denotes the Sobolev space on $\mathbb{B}^N$ with the metric $g$. We introduce important quantities in the hyperbolic space which will be used freely in this paper:

- The hyperbolic gradient $\nabla_{\mathbb{B}^N}$ is given by

$$\nabla_{\mathbb{B}^N} = \frac{\nabla}{\rho(x)}.$$

- The hyperbolic laplacian $\Delta_{\mathbb{B}^N}$ is defined as

$$\Delta^0_{\mathbb{B}^N} = \Delta_{\mathbb{B}^N} = \rho^{-2} \Delta + (N-2) \rho^{-1} \langle x, \nabla \rangle.$$

- The hyperbolic distance $d_{\mathbb{B}^N}(x, y)$ between $x, y \in \mathbb{B}^N$ in the Poincaré ball model is given by the formula

$$d_{\mathbb{B}^N}(x, y) = \text{Arccosh} \left( 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)} \right).$$

From this, we immediately obtain for $x \in \mathbb{B}^N$,

$$d(x) = d_{\mathbb{B}^N}(x, 0) = \ln \left( \frac{1+|x|}{1-|x|} \right).$$

- The associated hyperbolic volume element by $dV$ and it is given by

$$dV = \left( \frac{2}{1-|x|^2} \right)^N dx.$$
Another important space that will be needed for our variational setting is the weighted Hyperbolic space. In order to characterize such space, we first recall the standard norm in the weighted Lebesgue space $L^q(\mathbb{B}^N, d\beta)$ is

$$
\|u\|_{\beta,q} = \left( \int_{\mathbb{B}^N} (d(x))^\beta |u(x)|^q dV \right)^{\frac{1}{q}}.
$$

We consider now the Hilbert space $H^1_0(\mathbb{B}^N; d\beta dV)$, the completion of $C_c^\infty(\mathbb{B}^N)$ with respect to the norm

$$
\|u\|_{H^1_0(\mathbb{B}^N; d\beta dV)} = \left( \int_{\mathbb{B}^N} d\beta |\nabla_{\mathbb{B}^N} u|^2 dV \right)^{\frac{1}{2}}.
$$

It is easy to verify that $\|u\|_{H^1_0(\mathbb{B}^N; d\beta dV)}$ is a norm in $H^1_0(\mathbb{B}^N; d\beta dV)$. Indeed, one can notice that $\|u\|_{H^1_0(\mathbb{B}^N; d\beta dV)} = 0$ implies that almost everywhere $\nabla_{\mathbb{B}^N} u(x) = 0$. Now, we recall a basic information that the bottom of the spectrum of $-\Delta_{\mathbb{B}^N}$ on $\mathbb{B}^N$ is

$$
\lambda_1(-\Delta_{\mathbb{B}^N}) := \inf_{u \in H^1(\mathbb{B}^N) \setminus \{0\}} \frac{\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 dV_{\mathbb{B}^N}}{\int_{\mathbb{B}^N} |u|^2 dV_{\mathbb{B}^N}} = \frac{(N - 1)^2}{4}. \quad (2.1)
$$

It then follows from (2.1) that $u = 0$. We finally define the space $H^1_r(\mathbb{B}^N; d\beta dV)$ as the subspace of radially symmetric functions endowed with the induced norm $\|u\|_{H^1_r(\mathbb{B}^N; d\beta dV)} = \|u\|_{H^1_0(\mathbb{B}^N; d\beta dV)}$.

Remark 6. The hyperbolic sphere with centre $0 \in \mathbb{B}^N$ is also a Euclidean sphere with centre $0 \in \mathbb{B}^N$, therefore $H^1_r(\mathbb{B}^N; d\beta dV)$ can also be seen as the subspace of $H^1_0(\mathbb{B}^N; d\alpha dV)$ consisting of Hyperbolic radial functions.

We point out an relation between the distance $d$ and the function $\rho$ that will be useful in Section 6. For all $r \in [0, 1)$ it holds that

$$
2r \leq \log \frac{1 + r}{1 - r} \leq \frac{2r}{1 - r^2}.
$$

3. THE CAFFARELLI–KOHN–NIRENBERG INEQUALITY: PROOF OF THEOREM 1.1

Our idea is simple and has two ingredients. Firstly we will use a change of variable which relates the Caffarelli–Kohn–Nirenberg inequality and the Hardy inequality, which is not weighted. Then we interpolate between the following Poincaré–Sobolev inequality [23]

$$
\left( \int_{\mathbb{B}^N} |u|^{\frac{2N}{N-2}} dV \right)^{\frac{N-2}{N}} \leq C \int_{\mathbb{B}^N} \left( |\nabla_{\mathbb{B}^N} u|^2 - \frac{(N - 1)^2}{4} u^2 \right) dV \leq (N - 1)^2 \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 dV, \quad (3.1)
$$

and the Hardy inequality [21]

$$
\frac{(N - 2)^2}{4} \int_{\mathbb{B}^N} \frac{u^2}{d^2} dV \leq \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 dV, \quad (3.2)
$$

which are valid for all $u \in C_c^\infty(\mathbb{B}^N)$.

We state a general lemma of changing variables.

Lemma 3.1. Let $\gamma_1, \gamma_2 > 0$ and $u = \rho^{\alpha/2} w$. Then

$$
\int_{\mathbb{B}^N} \left( |\nabla_{\mathbb{B}^N} u|^2 - \gamma_1 \frac{u^2}{d^2} - \gamma_2 u^2 \right) dV
\quad = \int_{\mathbb{B}^N} d^\alpha \left( |\nabla_{\mathbb{B}^N} w|^2 - \alpha (\alpha - 2) \frac{w^2}{d^2} + 2 \alpha (N - 1) \frac{d}{\rho |x|} \frac{w^2}{d^2} + \gamma_1 \right) dV
\quad \leq \frac{\alpha (N - 1)}{2} \int_{\mathbb{B}^N} d^{n-1} w^2 |x| dV. \quad (3.3)
$$
In particular,
\[ \int_{B} \left( |\nabla_{B}^{N} u|^{2} - \gamma_{1} \frac{u^{2}}{d^{2}} - \gamma_{2} u^{2} \right) \, dV \leq \int_{B} d^{\alpha}|\nabla_{B}^{N} u|^{2} \, dV. \quad (3.4) \]

The proof is a direct computation and is postponed to the end of this section. Now we give a

**Proof of Theorem 1.1.** Interpolating (3.1) and (3.2), we have for \(-2 \leq \beta \leq 0,\)
\[
\left( \int_{B} d^{\beta} \frac{u^{2(a+\beta)}}{d^{2}} \, dV \right)^{\frac{N-2}{N+2}} \leq C \left( \int_{B} |\nabla_{B}^{N} u|^{2} \, dV \right)^{\frac{N-2}{N+2}}. \quad (3.5)
\]

For any \(\gamma_{1} < \frac{(N-2)^{2}}{4}, \gamma_{2} < \frac{(N-1)^{2}}{4}\) such that \(1 - \frac{4}{(N-2)^{2}} \gamma_{1} - \frac{4}{(N-1)^{2}} \gamma_{2} > 0,\) we have
\[
\left( 1 - \frac{4}{(N-2)^{2}} \gamma_{1} - \frac{4}{(N-1)^{2}} \gamma_{2} \right) \int_{B} |\nabla_{B}^{N} u|^{2} \, dV
\leq \left( \int_{B} |\nabla_{B}^{N} u|^{2} \, dV - \gamma_{1} \int_{B} \frac{u^{2}}{d^{2}} \, dV - \gamma_{2} \int_{B} u^{2} \, dV \right).
\]

Putting together (3.4), (3.5) and (3.6), we have for \(u = d^{\alpha/2}w,\)
\[
\left( \int_{B} d^{\beta+\alpha 2 \beta/2} w^{2 \beta} \, dV \right)^{2/2 \beta} = \left( \int_{B} d^{\beta} u^{2 \beta} \, dV \right)^{2/2 \beta}
\leq C \int_{B} \left( |\nabla_{B}^{N} u|^{2} - \gamma_{1} \frac{u^{2}}{d^{2}} - \gamma_{2} u^{2} \right) \, dV
\leq C \int_{B} d^{\alpha}|\nabla_{B}^{N} w|^{2} \, dV.
\]

If we put \(\tilde{\beta} = \beta + \alpha 2 \beta/2,\) then
\[
\left( \int_{B} d^{\tilde{\beta}} u^{2 \beta} \, dV \right)^{2/2 \beta} \leq C \int_{B} d^{\alpha}|\nabla_{B}^{N} w|^{2} \, dV.
\]

It remains to rename \(\alpha = -2a\) and solve the elementary equations
\[
\tilde{\beta} = -bp, \quad 2 \beta \alpha = p
\]
which gives in particular \(p = \frac{2N}{N-2(1-b/a)}.\)

**Proof of Lemma 3.1.** We observe that the non-differentiated terms simply becomes
\[ -\gamma_{2} \int_{B} d^{\alpha} w^{2} \, dV - \gamma_{2} \int_{B} d^{\alpha} w^{2}. \quad (3.7) \]

The gradient term is expressed as
\[
\int_{B} |\nabla_{B}^{N} u|^{2} \, dV = \int_{B} \left| \nabla (d^{\frac{\beta}{2}} w) \right|^{2} \rho^{N-2} \, dx
\]
\[ = \int_{B} \left| d^{\frac{\beta}{2}} \nabla w + \frac{\alpha}{2} w d^{\frac{\beta}{2}-1} \nabla d \right|^{2} \rho^{N-2} \, dx
\]
\[ = \int_{B} \left( d^{\alpha} |\nabla w|^{2} + \frac{\alpha^{2}}{4} w^{2} d^{\alpha-2} |\nabla d|^{2} + \alpha w \nabla w \cdot d^{\alpha-1} \nabla d \right) \rho^{N-2} \, dx
\]
\[ = \int_{B} d^{\alpha} \left( |\nabla_{B}^{N} w|^{2} + \frac{\alpha^{2}}{4} \frac{w^{2}}{d^{2}} \right) \, dV + \frac{1}{2} \int_{B} \nabla w^{2} \cdot \nabla d^{\alpha} \rho^{N-2} \, dx. \quad (3.8) \]
For the last integral, we perform an integration by parts to yield
\[
\frac{1}{2} \int_B \nabla w^2 \cdot \nabla \rho^{N-2} \, dx = -\frac{1}{2} \int_B w^2 \nabla \cdot (\rho^{N-2} \nabla \rho) \, dx
\]
\[= -\frac{\alpha}{2} \int_B w^2 \nabla \cdot \left( \rho^{N-1} d^{N-1} \frac{x}{|x|} \right) \, dx. \tag{3.9} \]
Observing that
\[
\nabla \cdot \left( \rho^{N-1} \frac{x}{|x|} \right) = (N-1) \rho^{N-2} \frac{x}{|x|} \frac{x}{|x|} + \rho^{N-1} \frac{N-1}{|x|} = (N-1) \rho^N \left( |x| + \frac{1}{|\rho|} \right)
\]
\[= \frac{N-1}{2} \rho^N \frac{1 + |x|^2}{|x|}, \]
we may expand the divergence in (3.9) as
\[
\frac{1}{2} \int_B \nabla w^2 \cdot \nabla \rho^{N-2} \, dx
\]
\[= -\frac{\alpha}{2} \int_B w^2 \nabla \cdot \left( \rho^{N-1} d^{N-1} \frac{x}{|x|} \right) \, dx
\]
\[= -\frac{\alpha}{2} \int_B w^2 \left( d^{N-1} \frac{N-1}{2} \rho^N \frac{1 + |x|^2}{|x|} + \rho^{N-1} \frac{N-1}{|x|} (\alpha - 1) d^{N-2} \frac{x}{|x|} \right) \, dx \tag{3.10} \]
\[= -\frac{\alpha(N-1)}{4} \int_B d^{N-1} w^2 \rho^N \frac{1 + |x|^2}{|x|} \, dx - \frac{\alpha(\alpha - 1)}{2} \int_B d^{N-2} w^2 \rho^N \, dx. \]
Combining (3.7), (3.8) and (3.10), we have
\[
\int_{\mathbb{B}^N} \left( |\nabla_{\mathbb{B}^N} u|^2 - \gamma_1 \frac{w^2}{d^2} - \gamma_2 u^2 \right) \, dV
\]
\[= \int_{\mathbb{B}^N} \left( |\nabla_{\mathbb{B}^N} u|^2 + \left( \frac{\alpha - 2}{4} \right) \frac{w^2}{d^2} - \gamma_2 u^2 \right) \, dV + \frac{1}{2} \int_B \nabla w^2 \cdot \nabla \rho^{N-2} \, dx
\]
\[= \int_{\mathbb{B}^N} d^{N} \left( |\nabla_{\mathbb{B}^N} w|^2 + \left( \frac{\alpha - 2}{4} \right) \frac{w^2}{d^2} - \gamma_2 u^2 \right) \, dV
\]
\[= \frac{\alpha(N-1)}{4} \int_B d^{N-1} u^2 \rho^N \frac{1 + |x|^2}{|x|} \, dx
\]
\[= \frac{\alpha(N-1)}{4} \int_{\mathbb{B}^N} d^{N-1} u^2 \frac{1 + |x|^2}{|x|} \, dV
\]
\[= \frac{\alpha(N-1)}{2} \int_{\mathbb{B}^N} d^{N-1} u^2 |x| \, dV. \]
This proves (3.3). To prove (3.4), we note that the second integral in (3.3) can be absorbed by the “Cauchy–Schwarz inequality with $\varepsilon$”, namely
\[
\int_{\mathbb{B}^N} d^{N-1} u^2 |x| \, dV \leq \varepsilon \int_{\mathbb{B}^N} d^{N} u^2 \, dV + C_\varepsilon \int_{\mathbb{B}^N} d^{N} u^2 |x| \, dV
\]
The proof is completed by taking $0 < \varepsilon \leq \gamma_2$ and using (1.6).
4. A COMPACT EMBEDDING

In this section we prove a weighted compact Sobolev embedding result, which extends a former result of Ni [25] made for an unit ball in \( \mathbb{R}^N \).

Let \( H^1_0(\mathbb{B}^N; d^\alpha dV) \) be the completion of \( C_c^\infty(\mathbb{B}^N) \) with respect to the norm

\[
\|u\|_{H^1_0(\mathbb{B}^N; d^\alpha dV)} = \left( \int_{\mathbb{B}^N} d^\alpha |\nabla_{\mathbb{B}^N} u|^2 dV \right)^{1/2}.
\]

We denote by \( H^1_r(\mathbb{B}^N; d^\alpha dV) \) the subspace of radially symmetric functions endowed with the induced norm \( \|u\|_{H^1_r(\mathbb{B}^N; d^\alpha dV)} = \|u\|_{H^1_0(\mathbb{B}^N; d^\alpha dV)} \).

By Theorem 1.1, the embedding

\[
H^1(\mathbb{B}^N; d^\alpha dV) \hookrightarrow L^p(\mathbb{B}^N; d^\beta dV)
\]

is continuous for \( 2 \leq p \leq 2^\beta_\alpha \). We will show that it is in fact compact for \( 2 < p < 2^\beta_\alpha \), when restricted to radial functions. When \( \beta > 0 \), a proof can be found in [9, Lemma 1], in the spirit of Strauss [30]. We present a unified and simplified proof.

In the following, \( f \sim g \) means that \( f/g \) is a bounded positive function; and \( \log^a(x) \) means \( (\log x)^a \).

**Proposition 4.1** (Compact embedding). Let \( n \geq 2 \) and \( \beta > \alpha - 2 > -N \). The embedding

\[
H^1_r(\mathbb{B}^N; d^\alpha dV) \hookrightarrow L^p(\mathbb{B}^N; d^\beta dV)
\]

is compact when \( 2 < p < 2^\beta_\alpha \), or when \( p = 2 \) and \( \beta < \alpha - 1 \).

**Proof.** Let \( (u_m) \subset H^1_r(\mathbb{B}^N; d^\alpha dV) \) be a bounded sequence. Up to a subsequence, we can assume \( u_m \rightharpoonup u \) weakly in \( H^1_r(\mathbb{B}^N; d^\alpha dV) \) and \( u_m \rightarrow u \) a.e. in \( \mathbb{B}^N \). It suffices to show that

\[
\int_{\mathbb{B}^N} d^\beta |u_m|^p dV \rightarrow \int_{\mathbb{B}^N} d^\beta |u|^p dV,
\]

that is,

\[
\int_0^1 d^\beta \rho^N r^{N-1} |u_m(r)|^p dr \rightarrow \int_0^1 d^\beta \rho^N r^{N-1} |u(r)|^p dr.
\]

By Lebesgue dominated convergence theorem, it suffices to verify that the integrand on the left is dominated by some function \( h \in L^1(0, 1) \).

Proceeding as above, we have

\[
d^\beta \rho^N r^{N-1} |u_m(r)|^p \leq C \left\| u_m \right\|^p_{H^1_r(\mathbb{B}^N; d^\alpha dV)} \max \left\{ \frac{1}{1-r}; e \right\}^{\frac{\alpha}{2}\beta - \frac{\beta}{2} p} \max \left\{ \frac{1}{1-r}; e \right\}^{\frac{\alpha}{2}\beta - \frac{\beta}{2} p + \frac{\alpha}{2} p - 2\beta - 1}\times
\]

\[
\log^{-\frac{\alpha}{2} p} \max \left\{ \frac{1}{1-r}; e \right\}^{\frac{\alpha}{2}\beta - \frac{\beta}{2} p + \frac{\alpha}{2} p - 2\beta - 1}
\]

\[
\leq C r^{\frac{N-2\alpha + (2\beta_\alpha - \beta) - 1}{2} p - 2\beta - 1} \log^{-\frac{\alpha}{2} p} \max \left\{ \frac{1}{1-r}; e \right\}^{\frac{\alpha}{2}\beta - \frac{\beta}{2} p + \frac{\alpha}{2} p - 2\beta - 1}
\]

since \( (u_m) \) is bounded in \( H^1_r(\mathbb{B}^N; d^\alpha dV) \). Let \( h \) denotes the right hand side. When \( 2 < p < 2^\beta_\alpha \), we can ignore the logarithmic factor and we see that \( h \) is integrable near both \( r = 0 \) and \( r = 1 \). When \( p = 2 \), we have

\[
h(r) \sim (1-r)^{-1} \log^{\frac{\beta}{2} - \frac{\alpha}{2}} \frac{1}{1-r} \quad \text{near} \quad r = 1,
\]
Lemma 5.1. Let $i \frac{dr}{1-r} = dt$,
\[
\int_{1-1/e}^1 h(r) \, dr \leq C \int_{1-1/e}^1 \frac{1}{1-r} \log^{\beta-\alpha} \frac{1}{1-r} \, dr
\]
\[
\leq C \int_1^\infty e^{\beta-\alpha} \, dt
\]
provided that $\beta - \alpha < -1$.  

Remark 7. Note that inequality (1.6) asserts that $H^1_0(B^N; d^\alpha dV)$ is embedded in the weighted space $L^2(B^N, d^{\alpha-2} dV)$ and that this embedding is continuous. If $\lambda < \left(\frac{N-2+\alpha}{2}\right)^2$, it follows from (1.6) that
\[
\|w\| := \left(\int_{B^N} d^\alpha |\nabla u|^2 dV - \lambda \int_{B^N} d^{\alpha-2} |u|^2 dV\right)^{\frac{1}{2}}
\]
is well-defined on $H^1_0(B^N; d^\alpha dV)$ and is equivalent to the norm $\|\cdot\|_{H^1(B^N,d^\alpha dV)}$.

5. Existence of the weighted Hardy–Hénon equation: Proof of Theorem 1.2

The proof of the existence result rely on the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [2]. Associated with the problem (1.4) we define the functional $I : H^1_0(B^N; d^\alpha dV) \rightarrow \mathbb{R}$ given by
\[
I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{q} \int_{B^N} d^\beta |u|^q dV,
\]
which is $C^1$ and its derivative is, for all $u, v \in H^1_0(B^N; d^\alpha dV)$, given by
\[
I'(u)v = \int_{B^N} d^\alpha \nabla u \cdot \nabla v dV - \lambda \int_{B^N} d^{\alpha-2} uv dV
\]
\[- \int_{B^N} d^\beta |u|^q-2 uv dV.
\]

We are going to prove that $I$ verifies the Mountain Pass Geometry conditions.

Lemma 5.1. Let $N \geq 3$, $\alpha > 2 - N$, $\beta < 0$ and $q \in (2, 2^*_d)$. Then

i) there exist positive constants $\rho, \gamma$ such that $I(u) \geq \gamma$, for all $\|u\|_{H^1_0(B^N; d^\alpha dV)} = \rho$;

ii) there exist a constant $R > \rho$ and $e \in H^1_0(B^N; d^\alpha dV)$ with $\|e\|_{H^1_0(B^N; d^\alpha dV)} > R$, verifying $I(e) \leq 0$.

Proof. By Theorem 1.1 and Remark 7, we get
\[
I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{q} \int_{B^N} d^\beta |u|^q dV
\]
\[
\geq C_1 \|u\|^2_{H^1_0(B^N; d^\alpha dV)} - C_2 \|u\|^q_{H^1_0(B^N; d^\alpha dV)}.
\]
Since $q > 2$, there exists $\gamma > 0$ such that
\[
I(u) \geq \gamma, \text{ for } \|u\|_{H^1_0(B^N; d^\alpha dV)} = \rho \text{ sufficiently small.}
\]
This proves (i).

Now, choose any $u \in H^1_0(B^N; d^\alpha dV) \setminus \{0\}$, then
\[
I(tu) = \frac{t^2}{2} \|u\|^2 - \frac{t^q}{q} \int_{B^N} d^\beta |u|^q dV.
\]
Therefore
\[
I(tu) \rightarrow -\infty, \text{ as } t \rightarrow \infty.
\]
This proves (ii).
Lemma 5.2. According to our previous notation, $I$ satisfies the Palais-Smale condition (PS), that is, if whenever $\{u_n\}$ is a sequence in $H^1_w(\mathbb{B}^N; d^\alpha dV)$ such that

$$I(u_n) \text{ is bounded, and } I'(u_n) \to 0, \text{ as } n \to \infty,$$

then $\{u_n\}$ has a convergent subsequence in $H^1_w(\mathbb{B}^N; d^\alpha dV)$.

Proof. For some positive constant $M > 0$, we get

$$M + \|u_n\|_{H^1_w(\mathbb{B}^N; d^\alpha dV)} \geq I(u_n) - \frac{1}{q} I'(u_n)u_n = C \left( \frac{1}{2} - \frac{1}{q} \right) \|u_n\|^2_{H^1_w(\mathbb{B}^N; d^\alpha dV)}.$$

This implies that $\|u_n\|_{H^1_w(\mathbb{B}^N; d^\alpha dV)}$ is bounded. So that, there exists $u \in H^1_w(\mathbb{B}^N; d^\alpha dV)$ such that, passing to a subsequence if necessary,

$$u_n \rightharpoonup u \text{ weakly in } H^1_w(\mathbb{B}^N; d^\alpha dV) \text{ and pointwise, as } n \to \infty.$$

By Proposition 4.1, we get $u_n \to u$, in $L^q(\mathbb{B}^N; d^\beta dV)$, as $n \to \infty$, $q \in (2, \frac{2(N+\beta)}{N-2+\alpha})$. Since

$$C\|u_n - u\|^2_{H^1_w(\mathbb{B}^N; d^\alpha dV)} = I'(u_n)(u_n - u) - I'(u)(u_n - u) + o(1) = o(1) \text{ as } n \to \infty,$$

we have

$$u_n \to u, \text{ in } H^1_w(\mathbb{B}^N; d^\alpha dV), \text{ as } n \to \infty.$$

By Mountain Pass Theorem, there exists a solution $u \in H^1_w(\mathbb{B}^N; d^\alpha dV)$. By testing $u^- = \max\{-u, 0\}$ in $I'(u)u^- = 0$, we reach that $u^- = 0$. That is, $u = u^+ = \max\{u, 0\}$. By invoking a version of the maximum principle due to Antonini, Mugnai and Pucci [3], we can conclude the positivity of the solution.

Now, we will use standard arguments to conclude that the function $u$ is a critical point of $I$ in $H^1_w(\mathbb{B}^N; d^\alpha dV)$. Since $u$ is a critical point of $I$ in $H^1_w(\mathbb{B}^N; d^\alpha dV)$, then $I'(u)v_r = 0$, for all $v_r \in H^1_w(\mathbb{B}^N; d^\alpha dV)$. Notice that $H^1_w(\mathbb{B}^N; d^\alpha dV)$ is a closed subspace of the Hilbert space $H^1_w(\mathbb{B}^N; d^\alpha dV)$, so that, we can write

$$H^1_w(\mathbb{B}^N; d^\alpha dV) = H^1_w(\mathbb{B}^N; d^\alpha dV) \oplus H^1_w(\mathbb{B}^N; d^\alpha dV)^\perp.$$

Therefore, for any $v \in H^1_w(\mathbb{B}^N; d^\alpha dV)$ we can split it as

$$v = v_r + v^\perp, \text{ with } v_r \in H^1_w(\mathbb{B}^N; d^\alpha dV) \text{ and } v^\perp \in H^1_w(\mathbb{B}^N; d^\alpha dV)^\perp.$$

On the other hand, since $H^1_w(\mathbb{B}^N; d^\alpha dV)$ is also a Hilbert space, we can identify, through the duality, $I'(u)$ with an element in $H^1_w(\mathbb{B}^N; d^\alpha dV)$. Then

$$\langle I'(u), v^\perp \rangle_{H^1_w(\mathbb{B}^N; d^\alpha dV)} = 0.$$

Hence, for all $v \in H^1_w(\mathbb{B}^N; d^\alpha dV)$

$$\langle I'(u), v \rangle_{H^1_w(\mathbb{B}^N; d^\alpha dV)} = \langle I'(u), v_r \rangle_{H^1_w(\mathbb{B}^N; d^\alpha dV)} + \langle I'(u), v^\perp \rangle_{H^1_w(\mathbb{B}^N; d^\alpha dV)} = 0.$$

This proves that the function $u$ is a critical point of $I$ in $H^1_w(\mathbb{B}^N; d^\alpha dV)$. Therefore, $u$ is a positive solution of (1.4). \hfill \Box

6. Non-existence in Sobolev-supercritical case: Proof of Theorem 1.3

We employ a Pohožaev type identity, taking into account the extra terms arising from the hyperbolic metric and the weight. Specifically, we test (1.7) against $\nabla u \cdot x$ as well as a weighted version of $u$, namely $\nabla (d^{\alpha \beta \gamma})_{x^\eta} u$. 
Proposition 6.1. If \( u \) solves (1.7), then
\[
- \frac{1}{2} \int_{\partial \Omega} d^p \rho^{n-2} |\nabla u|^2 x \cdot \nu \, d\sigma(x) \\
= \int_{\Omega} d^p |\nabla_{\!B} u|^2 \left( -1 + \frac{\nabla \cdot (d^p \rho^{N-2} x)}{2d^p \rho^{N-2}} - \frac{\nabla \cdot (d^p \rho^{N} x)}{p d^p \rho^{N}} \right) \, dx + \frac{1}{2p} \int_{\Omega} u^2 \Delta_{\!B}^p \left( \frac{\nabla \cdot (d^p \rho^{N} x)}{d^p \rho^{N}} \right) \, dx
\]
(6.1)

The proof is completed by a straightforward computation showing the positivity of the two brackets on the right hand side of (6.1).

Lemma 6.1. We have the following.
1. For \( |x| < 1 \),
\[
\frac{2|x|}{1 + |x|^2} \leq 2|x| \leq d(x) \leq \frac{2|x|}{1 - |x|^2}.
\]
This implies, in particular, that the non-negative quantities \( A = 1 + \rho |x|^2 - \frac{\rho |x|^2}{d} \) and \( B = \frac{\rho |x|^2}{d} \) satisfy
\[
d\rho |x| \geq A \quad \text{and} \quad d\rho |x| B - (B^2 - 1) \geq 0.
\]
2. For \( p \geq \frac{2N}{N-2} \geq 2^p \alpha \),
\[
-1 + \frac{\nabla \cdot (d^p \rho^{N-2} x)}{2d^p \rho^{N-2}} - \frac{\nabla \cdot (d^p \rho^{N} x)}{p d^p \rho^{N}} \geq 0.
\]
3. For \( N \geq \alpha - 1 \) and \( \beta > -N \),
\[
\Delta_{\!B}^p \left( \frac{\nabla \cdot (d^p \rho^{N} x)}{d^p \rho^{N}} \right) \geq 0.
\]

Proof of Theorem 1.3. This follows immediately from Proposition 6.1 and Lemma 6.1 which force all the integrals in (6.1) to vanish.

To prove Proposition 6.1 we recall the following facts.

Lemma 6.2. We have the following.
1. A Bochner-type identity
\[
\nabla u \cdot \nabla (\nabla u \cdot x) = |\nabla u|^2 + \frac{1}{2} \nabla |\nabla u|^2 \cdot x.
\]
2. The boundary derivatives
\[
\nabla u \cdot \nu = |\nabla u| \quad \text{and} \quad \nabla u \cdot x = (x \cdot \nu)|\nabla u|.
\]

Proof. (1) is a straightforward computation. For (2) we observe that from the Dirichlet boundary condition, \( \nu = \frac{\nabla u}{|\nabla u|} \).

Proof of Proposition 6.1. The equation (1.7) can be rewritten as
\[
- \nabla \cdot (d^p \rho^{N-2} \nabla u) = d^p \rho^N |u|^{p-2} u \quad \text{in} \ \Omega.
\]
(6.2)

Multiplying both sides by \( \nabla u \cdot x \) and integrating over \( \Omega \), we find
\[
- \int_{\partial \Omega} d^p \rho^{N-2} (\nabla u \cdot \nu)(\nabla u \cdot x) \, d\sigma(x) + \int_{\Omega} d^p \rho^{N-2} \nabla u \cdot \nabla (\nabla u \cdot x) \, dx = \int_{\Omega} d^p \rho^N |u|^{p-2} u \nabla u \cdot x \, dx.
\]
(6.3)

By Lemma 6.2(2), the boundary integral is equal to
\[
- \int_{\partial \Omega} d^p \rho^{N-2} |\nabla u|^2 (x \cdot \nu) \, d\sigma(x)
\]
as appears in the left hand side of (6.1).
Following the standard argument we will integrate by parts several times. Using Lemma 6.2(1), the integral on $\Omega$ of the left hand side of (6.3) is equal to

$$\int_{\Omega} d^\alpha \rho^{N-2} \nabla u \cdot \nabla (\nabla u \cdot x) \, dx$$

$$= \int_{\Omega} d^\alpha \rho^{N-2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} d^\alpha \rho^{N-2} \nabla |\nabla u|^2 \cdot x \, dx$$

$$= \int_{\Omega} d^\alpha \rho^{N-2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} \nabla \cdot \left( d^\alpha \rho^{N-2} |\nabla u|^2 \cdot x \right) \, dx - \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \nabla \cdot (d^\alpha \rho^{N-2} x) \, d\sigma(x)$$

For the right hand side of (6.3) we use the Dirichlet boundary condition to obtain

$$\int_{\Omega} d^\beta \rho^N |u|^{p-1} u \nabla u \cdot x \, dx = \frac{1}{p} \int_{\Omega} d^\beta \rho^N \nabla (|u|^p) \cdot x \, dx$$

$$= \frac{1}{p} \int_{\Omega} \nabla \cdot (d^\beta \rho^N |u|^p x) \, dx - \frac{1}{p} \int_{\Omega} |u|^p \nabla \cdot (d^\beta \rho^N x) \, dx$$

$$= \frac{1}{p} \int_{\partial \Omega} d^\beta \rho^N |u|^p (x \cdot \nu) \, d\sigma(x) - \frac{1}{p} \int_{\Omega} |u|^p \nabla \cdot (d^\beta \rho^N x) \, dx$$

$$= -\frac{1}{p} \int_{\Omega} |u|^p \nabla \cdot (d^\beta \rho^N x) \, dx.$$

Putting things together, (6.3) becomes

$$-\frac{1}{2} \int_{\partial \Omega} d^\alpha \rho^{N-2} |\nabla u|^2 (x \cdot \nu) \, d\sigma(x)$$

$$= -\int_{\Omega} d^\alpha \rho^{N-2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \nabla \cdot (d^\alpha \rho^{N-2} x) \, dx - \frac{1}{p} \int_{\Omega} |u|^p \nabla \cdot (d^\beta \rho^N x) \, dx. \quad (6.4)$$

It remains to rewrite the last integral. To this end we test (6.2) with $\frac{-1}{p} \nabla \cdot (d^\beta \rho^N x) u$ which yields, using again that $u = 0$ on $\partial \Omega$,

$$-\frac{1}{p} \int_{\Omega} |u|^p \nabla \cdot (d^\beta \rho^N x) \, dx$$

$$= \frac{1}{p} \int_{\Omega} \nabla \cdot (d^\beta \rho^N x) u \nabla \cdot (d^\alpha \rho^{N-2} \nabla u) \, dx$$

$$= \frac{1}{p} \int_{\Omega} \nabla \cdot \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) d^\alpha \rho^{N-2} \nabla u \, dx - \frac{1}{p} \int_{\Omega} d^\alpha \rho^{N-2} \nabla u \cdot \nabla \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) u \, dx \quad (6.5)$$

$$= \frac{1}{p} \int_{\Omega} \nabla \cdot (d^\beta \rho^N x) d^\alpha \rho^{N-2} u \nabla u \cdot \nu \, d\sigma(x)$$

$$- \frac{1}{p} \int_{\Omega} d^\alpha \rho^{N-2} |\nabla|^2 \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) \, dx - \frac{1}{p} \int_{\Omega} d^\alpha \rho^{N-2} u \nabla u \cdot \nabla \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) \, dx.$$
Here the boundary integral vanishes. In the last integral we write
\( u \nabla u = \frac{1}{2} \nabla (u^2) \) and integrate by parts, giving
\[
- \frac{1}{2p} \int_\Omega d^\alpha \rho^{N-2} \nabla (u^2) \cdot \nabla \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) \, dx
\]
\[
= - \frac{1}{2p} \int_{\partial \Omega} d^\alpha \rho^{N-2} u^2 \nabla \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) \cdot \nu \sigma(x) + \frac{1}{2p} \int_\Omega u^2 \nabla \cdot \left( d^\alpha \rho^{N-2} \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) \, dx
\]
\[
= \frac{1}{2p} \int_\Omega u^2 \Delta_{\rho^N} \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) \, dx.
\]
(6.6)

The proposition now follows from (6.4), (6.5) and (6.6).

We conclude the section by supplying the calculations of the explicit functions.

**Proof of Lemma 6.1.** (1) The first global estimate is well-known and can be proved by the equality at \( r = \|x\| = 0 \) and the corresponding inequalities of the radial derivatives

\[
(2r)' \leq d'(r) = \frac{2}{1-r^2} \leq \left( \frac{2r}{1-r^2} \right)' = \frac{2(1+r^2)}{(1-r^2)^2}.
\]

Then we have

\[
d|x|\rho - A = \rho \left( |x|d - \frac{1-|x|^2}{2} - |x|^2 + \frac{|x|}{d} \right)
\]
\[
= \rho \left( |x|d + \frac{|x|}{d} - \frac{1+|x|^2}{2} \right)
\]
\[
\geq \rho \left( 2|x|^2 + \frac{1-|x|^2}{2} - \frac{1+|x|^2}{2} \right)
\]
\[
= \rho|x|^2 \geq 0
\]

and

\[
d\rho|x|B - B^2 + 1 = \rho^2|x|^2 \left( 1 - \frac{1}{d^2} + \frac{1}{\rho^2|x|^2} \right)
\]
\[
\geq 0
\]

for

\[
d^2 \geq \frac{1}{1 + \rho^2|x|^2} = \frac{4|x|^2}{4|x|^2 + (1-|x|^2)^2} = \left( \frac{2|x|}{1 + |x|^2} \right)^2
\]

which is known to be true.
(2) We compute, using part (1),

\[
-1 + \frac{\nabla \cdot (d^\alpha \rho^{N-2}x)}{2d^\alpha \rho^{N-2}} - \frac{\nabla \cdot (d^\beta \rho^N x)}{pd^\beta \rho^N}
\]

\[
= -1 + \frac{1}{2} \left( N + (N - 2)\rho|x|^2 + \alpha \rho \frac{|x|}{d} \right) \rho \frac{|x|}{d}
\]

\[
= \left( \frac{N - 2 - N}{p} \right) (1 + \rho|x|^2) + \left( \frac{\alpha}{2} - \frac{\beta}{p} \right) \rho \frac{|x|}{d}
\]

\[
\geq \left( \frac{N - 2 - N}{p} + \frac{\alpha}{2} - \frac{\beta}{p} \right) \rho \frac{|x|}{d}
\]

since \( p \geq \frac{2N}{N - 2} \)

\[
= \left( \frac{N - 2 + \alpha}{p} - \frac{N + \beta}{p} \right) \rho \frac{|x|}{d}
\]

\[
\geq 0 \quad \text{since} \quad p \geq 2\beta.
\]

(3) From

\[
\frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} = N + N\rho|x|^2 + \beta \rho \frac{|x|}{d},
\]

it suffices to compute the functions when weighted Laplace–Beltrami operator is acted upon \( \rho|x|^2 \) and \( \rho|x|/d \). We have

\[
\Delta^\alpha_{B^N}(\rho|x|^2) = \nabla \cdot \left( d^\alpha \rho^{N-2}(\rho^2|x|^2x + 2\rho^2x) \right)
\]

\[
= \nabla \cdot \left( d^\alpha \rho^N(|x|^2 + (1 - |x|^2))x \right)
\]

\[
= \nabla \cdot (d^\alpha \rho^N x)
\]

\[
= d^\alpha \rho^N \left( N + N\rho|x|^2 + \alpha \rho \frac{|x|}{d} \right)
\]

On the other hand,

\[
\Delta^\alpha_{B^N} \left( \frac{\rho|x|}{d} \right) = \nabla \cdot \left( d^\alpha \rho^{N-2} \left( \frac{\rho x}{d|x|} + \rho^2 |x| \frac{x}{d} - \rho^2 \frac{x^2}{d^2} \right) \right)
\]

\[
= \nabla \cdot \left( d^{\alpha-1} \rho^N x \left( \frac{1}{\rho|x|} + |x| - \frac{1}{d} \right) \right)
\]

\[
= \nabla \cdot \left( d^{\alpha-1} \rho^N x \left( \frac{1}{\rho|x|} + |x| - \frac{1}{d} \right) + d^{\alpha-1} \rho^N x \cdot \left( -\frac{x}{|x|} - \frac{x}{\rho|x|^3} + |x| + \rho \frac{x}{d^2|x|} \right) \right)
\]

\[
= d^{\alpha-1} \rho^N \left( N + N\rho|x|^2 + (\alpha - 1)\rho \frac{|x|}{d} \right) \left( \frac{1}{\rho|x|} + |x| - \frac{1}{d} \right) + d^{\alpha-1} \rho^N \left( \rho \frac{|x|}{d^2} - \frac{1}{\rho|x|} \right)
\]

With \( A, B \) as in Lemma 6.1, we have

\[
\frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} = N\Delta^\alpha_{B^N}(\rho|x|^2) + \beta \Delta^\alpha_{B^N} \left( \frac{\rho|x|}{d} \right)
\]

\[
\frac{1}{d^{\alpha-1} \rho^N} \Delta^\alpha_{B^N} \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) = Nd \frac{(NA + (N + \alpha)B)}{\rho|x|} + \frac{\beta}{\rho|x|} \left( (NA + (N - 1 + \alpha)B)A + B^2 - 1 \right)
\]

\[
= (NA + (N - 1 + \alpha)B) \left( Nd + \frac{\beta}{\rho|x|} A \right) + NdB + \frac{\beta}{\rho|x|}(B^2 - 1).
\]

Since \( N - 1 + \alpha > 1 > 0 \), \( A \geq 0 \) and \( B^2 - 1 \geq 0 \), we may use \( \beta > -N \) to finally yield

\[
\frac{\rho|x|}{Nd^{\alpha-1} \rho^N} \Delta^\alpha_{B^N} \left( \frac{\nabla \cdot (d^\beta \rho^N x)}{d^\beta \rho^N} \right) \geq (NA + (N - 1 + \alpha)B)(d\rho|x| - A) + d\rho|x|B - (B^2 - 1) \geq 0,
\]
in view of (1).
This completes the proof. □

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Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2
E-mail address, H. Chan: hardy@math.ubc.ca
E-mail address, S. Shakerian: shaya@math.ubc.ca

Departamento de Matemática, Universidade Federal de Juiz de Fora
E-mail address, L.F.O. Faria: lfofaria@gmail.com