Abstract. Let \( p \) be an odd prime number. We study the problem of determining the module structure over the mod \( p \) Steenrod algebra \( A(p) \) of the Dickson algebra \( D_n \) consisting of all modular invariants of general linear group \( GL(n, \mathbb{F}_p) \). Here \( \mathbb{F}_p \) denotes the prime field of \( p \) elements. In this paper, we give an explicit answer for \( n = 2 \). More precisely, we explicitly compute the action of the Steenrod-Milnor operations \( St_{S,R} \) on the generators of \( D_n \) for \( n = 2 \) and for either \( S = \emptyset, R = (i) \) or \( S = (s), R = (i) \) with \( s, i \) arbitrary nonnegative integers.

1. Introduction

Let \( p \) be an odd prime number and let \( GL_n = GL(n, \mathbb{F}_p) \) be the general linear group over the prime field \( \mathbb{F}_p \) of \( p \) elements. This group acts naturally on the algebra \( P_n := E(x_1, x_2, \ldots, x_n) \otimes P(y_1, y_2, \ldots, y_n) \). Here and in what follows, \( E(\ldots, \ldots) \) and \( P(\ldots, \ldots) \) are the exterior and polynomial algebras over \( \mathbb{F}_p \) generated by the indicated variables. We grade \( P_n \) by assigning \( \deg x_i = 1 \) and \( \deg y_i = 2 \).

Dickson showed in \cite{1} that the invariant algebra \( P(y_1, y_2, \ldots, y_n)^{GL_n} \) is a polynomial algebra over \( \mathbb{F}_p \) generated by the Dickson invariants \( Q_{n,s}, 0 \leq s < n \). In \cite{6}, Huynh Mui proved that the Dickson algebra \( D_n = P_n^{GL_n} \) of invariants is generated by the Dickson invariants \( Q_{n,s}, 0 \leq s < n \), and Mui invariants \( R_{n,s_1, \ldots, s_k}, 0 \leq s_1 < \ldots < s_k < n \).

It is well known that \( P_n \) is a module over the Steenrod algebra \( A(p) \). The action of \( A(p) \) on \( P_n \) is determined by the formulas

\[
\beta x_j = y_j, \quad \beta y_j = 0,
\]

\[
P^i(x_j) = \begin{cases} x_j, & i = 0, \\ 0, & i > 0, \end{cases} \quad P^i(y_j) = \begin{cases} y_j, & i = 0, \\ y^p_j, & i = 1, \\ 0, & i > 1, \end{cases}
\]

and subject to the Cartan formulas

\[
\beta(xy) = \beta(x)y + (-1)^{\deg x}\beta(y),
\]

\[
P^r(xy) = \sum_{i=0}^{r} P^i(x)P^{r-i}(y),
\]

for \( x, y \in P_n \) and \( \beta \) is the Bockstein homomorphism (see Steenrod \cite{9}).
Since this action commutes with the one of $GL_n$, it induces an action of $A(p)$ on Dickson algebra $D_n$. So $D_n$ is a submodule of $P_n$. Note that the polynomial algebra $P(y_1, y_2, \ldots, y_n)$ is a submodule of $P_n$ and $P(y_1, y_2, \ldots, y_n)^{GL_n}$ is a submodule of the algebra $D_n$.

Let $\tau_s, \xi_i$ be the Milnor elements of degrees $2p^s - 1$, $2p^t - 2$ respectively in the dual algebra $A(p)^*$ of $A(p)$. In [3], Milnor showed that as an algebra,

$$A(p)^* = E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots).$$

Then $A(p)^*$ has a basis consisting of all monomials

$$\tau_{\Sigma S} = \tau_{s_1} \cdots \tau_{s_k} \xi_1^{r_1} \cdots \xi_m^{r_m},$$

with $S = (s_1, \ldots, s_k)$, $0 \leq s_1 < \ldots < s_k$, $R = (r_1, \ldots, r_m)$, $r_i \geq 0$. Let $St^{S,R} \in A(p)$ denote the dual of $\tau_{S} \xi_{R}$ with respect to that basis. Then $A(p)$ has a basis consisting of all operations $St^{S,R}$. For $S = \emptyset$, $R = (r)$, $St^{\emptyset,(r)}$ is nothing but the Steenrod operation $P^r$. So, we call $St^{S,R}$ the Steenrod-Milnor operation of type $(S, R)$.

The operations $St^{S,R}$ have the following fundamental properties:
- $St^{\emptyset,(0)} = 1$, $St^{(0),0} = \beta$.
- $St^{S,R}(z) = 0$ if $z \in P_n$ and $\deg z < k + 2(r_1 + r_2 + \ldots + r_m)$.
- The Cartan formula

$$St^{S,R}(zt) = \sum_{S_1 \cup S_2 = S \atop R_1 + R_2 = R} (-1)^{\deg z + \ell(S_1) + \ell(S_2)} (S : S_1, S_2) St^{S_1,R_1}(z) St^{S_2,R_2}(t),$$

where $R_1 = (r_1), R_2 = (r_2), R_1 + R_2 = (r_1 + r_2), S_1 \cap S_2 = \emptyset, z, t \in P_n, \ell(S_j)$ means the length of $S_j$, and

$$(S : S_1, S_2) = \text{sign} \left( \begin{array}{cccc} s_1 & \ldots & s_h & \ldots & s_k \\ s_{1,1} & \ldots & s_{h,1} & \ldots & s_{k,1} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ s_{1,h} & \ldots & s_{h,h} & \ldots & s_{k,h} \\ s_{1,k} & \ldots & s_{h,k} & \ldots & s_{k,k} \end{array} \right),$$

with $S_1 = (s_{1,1}, \ldots, s_{1,h}), s_{1,1} < \ldots < s_{1,h}$, $S_2 = (s_{2,1}, \ldots, s_{2,k-h}), s_{2,1} < \ldots < s_{2,k-h}$ (see Mui [7]).

The action of $St^{S,R}$ on Dickson invariants $Q_{n,s}$ has partially been studied by many authors. This action for $S = \emptyset, R = (i)$ was explicitly determined by Madsen-Milgram [4], Smith-Switzer [9], Hung-Minh [2], Kechagias [3], Sum [12], Wilkerson [14]. This action for either $S = (s), R = (0)$ or $S = \emptyset, R = (0, \ldots, 0, 1)$ with 1 at the $i$-th place, was studied by Wilkerson [14], Neusel [8], Sum [13].

In this paper, we explicitly determine the action of the Steenrod-Milnor operations $St^{S,R}$ on Dickson invariants $Q_{2,0}, Q_{2,1}$ and Mui invariants $R_{2,0}, R_{2,1}, R_{2,0,1}$ for either $S = \emptyset, R = (i)$ or $S = (s), R = (i)$.

In Section 2 we recall some results on the modular invariants of the general linear group $GL_2$ and the action of the Steenrod-Milnor operations on the generators of $P_2$. In Section 3, we compute the action of the Steenrod operations on Dickson-Mui invariants. Finally, in Section 4, we explicitly determine the action of the Steenrod-Milnor operations $St^{(s),(i)}$ on $Q_{2,0}, Q_{2,1}, R_{2,0}, R_{2,1}$ and $R_{2,0,1}$.
2. Preliminaries

**Definition 2.1.** Let \( u, v \) be nonnegative integers. Following Dickson \[1\], Mui \[6\], we define

\[
[u; v] = \begin{vmatrix} y_1^p & y_2^p \\ y_1 & y_2 \end{vmatrix}, \quad [1; u] = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}.
\]

In particular, we set

\[
L_2 = [0, 1], \quad L_{2, 0} = [1, 2], \quad L_{2, 1} = [0, 2], \quad M_{2, 0} = [1, 1], \quad M_{2, 1} = [1, 0], \quad M_{2, 0, 1} = x_1 x_2.
\]

The polynomial \([u, v]\) is divisible by \( L_2 \). Then, Dickson invariants \( Q_{2, 0}, Q_{2, 1} \) and Mui invariants \( R_{2, 0}, R_{2, 1}, R_{2, 0, 1} \) are defined by

\[
Q_{2, 0} = L_{2, 0}/L_2, \quad Q_{2, 1} = L_{2, 1}/L_2, \quad R_{2, 0} = M_{2, 0} L_2^{-2}, \quad R_{2, 1} = M_{2, 1} L_2^{p-2}, \quad R_{2, 0, 1} = M_{2, 0, 1} L_2^{p-2}.
\]

Now we prepare some data in order to prove our main results. First, we recall the following which will be needed in the next sections.

Let \( \alpha_i(a) \) denote the \( i \)-th coefficient in \( p \)-adic expansion of a non-negative integer \( a \). That means

\[
a = \alpha_0(a)p^0 + \alpha_1(a)p^1 + \alpha_2(a)p^2 + \ldots,
\]

for \( 0 \leq \alpha_i(a) < p, i \geq 0 \).

Denote by \( I(u, v) \) the set of all integers \( a \) satisfying

\[
\alpha_i(a) + \alpha_{i+1}(a) \leq 1, \quad \text{for any } i,
\]

\[
\alpha_i(a) = 0, \quad \text{for either } i < u \text{ or } i \geq v - 2.
\]

**Proposition 2.2 (Sum \[13\]).** Under the above notations, we have

\[
[u, v] = \sum_{a \in I(u, v)} (-1)^a L_2^{u+p(p-1)a} Q_{2, 1}^{p^{v-1}-a-(p+1)a}.
\]

**Lemma 2.3 (Sum \[12\]).** Let \( b \) be a nonnegative integer and \( \varepsilon = 0, 1 \). We have

\[
\text{St}^S, R(x_k^\varepsilon y_k^b) = \begin{cases} \binom{b}{R} x_k^\varepsilon y_k^{b+|R|}, & S = \emptyset, \\ \varepsilon \binom{b}{R} x_k^\varepsilon y_k^{p b+|R|}, & S = (s), \quad s \geq 0, \\ 0, & \text{otherwise.} \end{cases}
\]

Here \( \binom{b}{R} = \frac{b!}{(b-r_1-r_2-\ldots-r_m)!r_1!\ldots r_m!} \) for \( r_1 + r_2 + \ldots + r_m \leq b \) and \( \binom{b}{R} = 0 \) for \( r_1 + r_2 + \ldots + r_m > b \) and \( |R| = (p-1)r_1 + (p^2-1)r_2 + \ldots + (p^m-1)r_m \).

Note that for \( R = (i) \), \( \binom{b}{R} = \binom{b}{i} \) is the binomial coefficient. By convention, we set \( \binom{0}{i} = 0 \) for \( i < 0 \).

Applying Lemma 2.3 to \( P^i = \text{St}^{0, (i)} \), we get

**Corollary 2.4 (Steenrod \[10\]).** Let \( b, i \) be nonnegative integers. Then we have

\[
P^i y_k^b = \binom{n}{i} y_k^{b+(p-1)i}.
\]
Since \((p^r_i) = 0\) in \(F_p\) for \(1 < i < p^e\), we get

**Corollary 2.5.** For any nonnegative integers \(e, i,\)

\[ P^i y^e = \begin{cases} y^e, & i = 0, \\ y^{e+1}, & i = p^e, \\ 0, & \text{otherwise}. \end{cases} \]

Applying Corollary 2.5 and the Cartan formula to \([u, v] = y^1 y^1 y^2 - y^2 y^1 y^2\), we obtain

**Lemma 2.6** (Mui [6]). Let \(u, v, i\) be nonnegative integers. Then we have

\[ P^i [u, v] = \begin{cases} [u, v], & i = 0, \\ [u + 1, v], & i = p^u, \\ [u, v + 1], & i = p^v, \\ [u + 1, v + 1], & i = p^u + p^v, \\ 0, & \text{otherwise}. \end{cases} \]

Since \(L_2 = [0, 1], L_{2,0} = [1, 2], L_{2,1} = [0, 2]\), from Lemma 2.6 we get

**Corollary 2.7.** For any nonnegative integers \(i,\)

\[ P^i L_2 = \begin{cases} L_2, & i = 0, \\ L_2 Q_{2,1}, & i = p, \\ L_2 Q_{2,0}, & i = p + 1, \\ 0, & \text{otherwise}, \end{cases} \]

\[ P^i L_{2,0} = \begin{cases} [0, 1] = L_2 Q_{2,0}, & i = 0, \\ [1, 3] = L_2 Q_{2,0} Q_{2,1}, & i = p^2, \\ [2, 3] = L_2 Q_{2,0}^{p+1}, & i = p^2 + p, \\ 0, & \text{otherwise}, \end{cases} \]

\[ P^i L_{2,1} = \begin{cases} [0, 2] = L_2 Q_{2,1}, & i = 0, \\ [1, 2] = L_2 Q_{2,0}, & i = 1, \\ [0, 3] = L_2 (Q_{2,1}^{p+1} - Q_{2,0}^p), & i = p^2, \\ [1, 3] = L_2 Q_{2,0} Q_{2,1}, & i = p^2 + 1, \\ 0, & \text{otherwise}. \end{cases} \]

Combining Lemma 2.3, Corollary 2.5 and the Cartan formula gives

**Lemma 2.8.** Let \(s, i\) be nonnegative integers. Then we have

\[ S^{(s)(i)}[1; u] = \begin{cases} [s, u], & i = 0, \\ [s, u + 1], & i = p^u, \\ 0, & \text{otherwise}. \end{cases} \]

Applying Lemma 2.3 and Corollary 2.5 to \(M_{2,0} = [1; 1], M_{2,1} = [1; 0]\), we obtain
Corollary 2.9. For any nonnegative integer $i$,

$$P^i(M_{2,0}) = \begin{cases} [1; 1] = M_{2,0}, & i = 0, \\ [1; 2] = M_{2,0}Q_{2,1} - M_{2;1}Q_{2,0}, & i = p, \\ 0, & \text{otherwise}, \end{cases}$$

$$P^i(M_{2;1}) = \begin{cases} [1; 0] = M_{2;1}, & i = 0, \\ [1; 1] = M_{2;0}, & i = 1, \\ 0, & \text{otherwise}. \end{cases}$$

3. The action of the Steenrod operations on Dickson-Mui invariants

First of all, we prove the following which was proved in Hung-Minh [2], by another method.

Theorem 3.1 (Hung-Minh [2]). For any nonnegative integer $i$ and $s = 0, 1$, we have

$$P^iQ_{2,s} = \begin{cases} (-1)^{k+i}Q_{2,0}^{r+1-s}Q_{2,1}^{k+s-r}, & i = kp + r, 0 \leq r - s \leq k < p, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Recall that $\deg Q_{2,s} = 2(p^2 - p^s) < 2p^2$. Hence, $P^iQ_{2,s} = 0$ for $i \geq p^2$. Suppose that $i < p^2$. Then, using the $p$-adic expansion of $i$, we have

$$i = kp + r \text{ for } 0 \leq k, r < p.$$

We prove the theorem by induction on $k$. We have $P^0Q_{2,s} = Q_{2,s}$. According to Corollary 2.7

$$0 = P^1L_{2,0} = P^1(L_2Q_{2,0}) = L_2P^1Q_{2,0},$$

$$L_2Q_{2,0} = P^1L_{2,1} = P^1(L_2Q_{2,1}) = L_2P^1Q_{2,1}.$$ These equalities imply $P^1Q_{2,0} = 0, P^1Q_{2,1} = Q_{2,0}$. For $1 < r < p$, $P^rL_{2,s} = 0$ and $P^rL_2 = 0$. Using the Cartan formula and Corollary 2.7, we have

$$0 = P^rL_{2,s} = P^r(L_2Q_{2,s}) = L_2P^rQ_{2,s}.$$ This implies $P^rQ_{2,s} = 0$. So the theorem holds for $k = 0$ and any $0 \leq r < p$. Suppose $0 < k < p$ and the theorem is true for $k - 1$ and any $0 \leq r < p$. Using the Cartan formula, Corollary 2.7 and the inductive hypothesis, we get

$$0 = P^iL_{2,s} = P^i(L_2Q_{2,s})$$

$$= L_2P^iQ_{2,s} + L_2Q_{2,1}P^{i-p}Q_{2,s} + L_2Q_{2,0}P^{i-p-1}Q_{2,0}$$

$$= L_2P^iQ_{2,s} + L_2Q_{2,1}(-1)^{k-1}\binom{k - 1 + s}{r}Q_{2,0}^{r+1-s}Q_{2,1}^{k+s-r-1}$$

$$+ L_2Q_{2,0}(-1)^{k-1}\binom{k - 1 + s}{r - 1}Q_{2,0}^{r-s}Q_{2,1}^{k+s-r}$$

$$= L_2P^iQ_{2,s} + (-1)^{k-1}\left(\binom{k - 1 + s}{r} + \binom{k - 1 + s}{r - 1}\right)L_2Q_{2,0}^{r+1-s}Q_{2,1}^{k+s-r}.$$
Lemma 3.2. Let $i$ be a nonnegative integer. Then we have

$$P^i L_2^{p-2} = \begin{cases} (-1)^k(k+1)(k)L_2^{p-2}Q_{2,0}^{k-r}Q_{2,1}^{r-1}, & i = kp + r, 0 \leq r < p, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Note that $\deg L_2^{p-2} = 2(p-2)(p+1) < 2p^2$. So $P^i L_2^{p-2} = 0$ for $i \geq p^2$. Hence, it suffices to prove the theorem for $i = kp + r$ with $0 \leq k, r < p$.

Since $P^0 = 1$, we have $P^0 L_2^{p-2} = L_2^{p-2}$. If $0 < r < p$ then from Theorem 3.1 the Cartan formula and the relation $Q_{2,0}^p = L_2^{p-1} = L_2 L_2^{p-2}$, we get

$$0 = \delta^r Q_{2,0} = L_2 \delta^r L_2^{p-2}.$$  

This implies $\delta^r L_2^{p-2} = 0$. The lemma is true for $k = 0$ and $0 \leq r < p$.

Suppose that $0 < k < p$ and the lemma holds for $k - 1$ and any $0 \leq r < p$. Using the Cartan formula, Theorem 3.1 Corollary 2.9 and the inductive hypothesis, we have

$$(-1)^k \binom{k}{r} Q_{2,0}^{r+1} Q_{2,1}^{k-r} = P^i Q_{2,0} = P^i(L_2 L_2^{p-2}) = L_2 P^i L_2^{p-2} + L_2 Q_{2,1} P^{i-p} L_2^{p-2} + L_2 Q_{2,0} P^{i-p-1} L_2^{p-2} = L_2 P^i L_2^{p-2} + L_2 Q_{2,1} (-1)^{k-1} k \binom{k-1}{r} L_2^{p-2} Q_{2,0}^{k-r} + L_2 Q_{2,0} (-1)^{k-1} k \binom{k-1}{r} L_2^{p-2} Q_{2,0}^{k-r} = L_2 P^i L_2^{p-2} + (-1)^{k-1} k \binom{k-1}{r} + \binom{k-1}{r-1} Q_{2,0}^{r+1} Q_{2,1}^{k-r}.$$  

This equality and the relation $\binom{k-1}{r} + \binom{k-1}{r-1} = \binom{k}{r}$ imply the lemma for $k$ and any $0 \leq r < p$.

Theorem 3.3. Let $i$ be a nonnegative integer. We have

$$P^i R_{2,0} = \begin{cases} (-1)^k((r+1) \binom{k}{r} R_{2,0} Q_{2,0}^{r} Q_{2,1}^{k-r} + k \binom{k-1}{r} R_{2,1} Q_{2,0}^{r+1} Q_{2,1}^{k-r-1}), & i = kp + r, 0 \leq r \leq k < p, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Note that $\deg R_{2,0} = 2p^2 - 3 < 2p^2$. So $P^i R_{2,0} = 0$ for $i \geq p^2$. We prove the theorem for $i = kp + r$ with $0 \leq k, r < p$.

For $k = r = 0$, $P^0 R_{2,0} = R_{2,0}$. For $k = 0, 0 < r < p$, applying the Cartan formula and Corollary 2.9 we get

$$P^r R_{2,0} = P^r (M_{2,0} L_2^{p-2}) = M_{2,0} P^r L_2^{p-2} = 0.$$  

The theorem holds for $k = 0$ and $0 \leq r < p$. 

Suppose that $0 < k < p$. Using the Cartan formula, Corollary 2.9 and Lemma 2.2, we obtain

$$P^i R_{2,0} = P^i (M_{2,0} L_2^{p-2}) = P^0 M_{2,0} P^i L_2^{p-2} + P^p M_{2,0} P^{i-p} L_2^{p-2}$$

$$= M_{2,0} P^i L_2^{p-2} + (M_{2,0} Q_{2,1} - M_{2,1} Q_{2,0}) P^{i-p} L_2^{p-2}$$

$$= M_{2,0} (-1)^k (k+1) \binom{k}{r} L_2^{p-2} Q_{2,0} Q_{2,1}^{k-r}$$

$$+ (M_{2,0} Q_{2,1} - M_{2,1} Q_{2,0})(-1)^{k-1} \binom{k-1}{r} L_2^{p-2} Q_{2,0} Q_{2,1}^{k-r-1}$$

$$= (-1)^k ((k+1) \binom{k}{r} - k (\binom{k-1}{r})) R_{2,0} Q_{2,0} Q_{2,1}^{k-r}$$

$$+ k (\binom{k-1}{r}) R_{2,1} Q_{2,0}^{r+1} Q_{2,1}^{k-r-1}.$$

This equality and the relation $(k+1) \binom{k}{r} - k \binom{k-1}{r}$ imply the theorem for $k$ and $0 \leq r < p$. \qed

By an analogous argument as given in the proof of Theorem 3.3, we can easily obtain the following

**Theorem 3.4.** For any nonnegative integer $i$, we have

$$P^i R_{2,1} = \begin{cases} (-1)^k (k+1) \binom{k}{r} R_{2,1} Q_{2,0}^{k-r} + (\binom{k-1}{r}) R_{2,0} Q_{2,0}^{r+1} Q_{2,1}^{k-r+1}, & i = kp + r, 0 \leq r \leq k < p, \\ 0, & \text{otherwise,} \end{cases}$$

$$P^i R_{2,0,1} = \begin{cases} (-1)^k (k+1) \binom{k}{r} R_{2,0,1} Q_{2,0}^{k-r}, & i = kp + r, 0 \leq r \leq k < p, \\ 0, & \text{otherwise.} \end{cases}$$

4. **On the action of the Steenrod-Milnor operations on Dickson-Mui invariants**

In this section, we compute the action of $S^t(s, i)$ on Dickson-Mui invariants. It is easy to see that $S^t(s, i) Q_{2,0} = 0$. So we need only to compute the action of $S^t(s, i)$ on $R_{2,0}, R_{2,1}$ and $R_{2,0,1}$.

First, we recall the following

**Lemma 4.1** (Sum [11]). For any nonnegative integers $s, i$,

$$S^t(s, i) (M_{2,0}) = \begin{cases} [s, 1], & i = 0, \\ [s, 2], & i = p, \\ 0, & \text{otherwise.} \end{cases}$$

$$S^t(s, i) (M_{2,1}) = \begin{cases} [s, 0], & i = 0, \\ [s, 1], & i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This lemma can easily be proved by using the Cartan formula and Lemma 2.3.
Theorem 4.2. Let $s, i$ be nonnegative integers. Then we have

$$St^{(s),(i)}R_{2;0} = \begin{cases} 
(-k)^{(r+1)} \binom{k}{r+1} Q_{2,0}^{k+1} Q_{2,1}^{k-r+1}, & s = 0, i = kp + r, 0 \leq r \leq k < p, \\
(-1)^{k+1}(-1)^{r+1} \binom{k}{r} Q_{2,0}^{k+1} Q_{2,1}^{k-r}, & s = 1, i = kp + r, 0 \leq r \leq k < p, \\
(-1)^{k+1}(-1)^{r+1} \binom{k}{r} Q_{2,0}^{k+1} Q_{2,1}^{k-r}, & s = 2, i = kp + r, 0 \leq r \leq k < p, \\
- (k - r) \sum_{a \in I(2,s)} (-1)^a Q_{2,0}^{p(a+1)+r+2} Q_{2,1}^{(p+1)a+k-r}, & s > 2, i = kp + r, 0 \leq r \leq k < p, \\
0, & \text{otherwise.}
\end{cases}$$

Proof. Since $\deg R_{2;0} = 2p^2 - 3$, $St^{(s),(i)}R_{2;0} = 0$ for $i \geq p^2$. Suppose $i < p^2$, then using the $p$-adic expansion of $i$, we have $i = kp + r$ for $0 \leq r < p$.

We have $St^{(s),(0)}R_{2;0} = St^{(s),(0)}(M_{2;0} L_{2;2}^{p-2}) = St^{(s),(0)} M_{2;0} P^0 L_{2;2}^{p-2} = [s, 1] L_{2;2}^{p-2}$.

For $0 < r < p$, using the Cartan formula, Lemma 3.2 and Lemma 4.1, we get

$$St^{(s),(r)}R_{2;0} = St^{(s),(r)}(M_{2;0} L_{2;2}^{p-2}) = St^{(s),(0)}(M_{2;0}) P^r L_{2;2}^{p-2} = 0.$$

The above equalities and Proposition 2.2 imply the theorem for $k = 0$.

For $0 < k < p$, using the Cartan formula, Lemma 3.2 and Lemma 4.1, we obtain

$$St^{(s),(i)}R_{2;0} = St^{(s),(i)}(M_{2;0} L_{2;2}^{p-2})$$

$$= St^{(s),(0)} M_{2;0} P^i L_{2;2}^{p-2} + St^{(s),(p)} M_{2;0} P^{i-p} L_{2;2}^{p-2}$$

$$= [s, 1] (-k)^{k+1} \binom{k}{r+1} Q_{2,0}^{k+1} Q_{2,1}^{k-r}$$

$$+ [s, 2] (-1)^{k+1} \binom{k}{r} Q_{2,0}^{k+1} Q_{2,1}^{k-r+1}.$$

Combining this equality and Proposition 2.2, we obtain the theorem. $\square$

Theorem 4.3. For any nonnegative integers $s, i,$

$$St^{(s),(i)}R_{2;1} = \begin{cases} 
(-1)^k(-1)^{r+1} \binom{k}{r} Q_{2,0}^{k+1} Q_{2,1}^{k-r+1}, & s = 0, i = kp + r, 0 \leq r \leq k < p, \\
(-1)^{k+1}(-1)^{r+1} \binom{k}{r} Q_{2,0}^{k+1} Q_{2,1}^{k-r}, & s = 1, i = kp + r, 0 \leq r \leq k < p, \\
(-1)^{k+1}(-1)^{r+1} \binom{k}{r} \sum_{a \in I(0, s)} (-1)^a Q_{2,0}^{p(a+1)+r+1} Q_{2,1}^{(p+1)a+k-r}$$

$$+ \binom{k}{r+1} \sum_{a \in I(1, s)} (-1)^a Q_{2,0}^{p(a+1)+r+1} Q_{2,1}^{(p+1)a+k-r+1}, & s > 1, i = kp + r, 0 \leq r \leq k < p, \\
0, & \text{otherwise.}
\end{cases}$$

Proof. Since $\deg R_{2;1} = 2(p^2 - p) - 1$, $St^{(s),(i)}R_{2;1} = 0$ for $i \geq p^2$. Suppose $i < p^2$ and $i = kp + r$ with $0 \leq r < p$. Using the Cartan formula and Lemma 4.1, we have

$$St^{(s),(0)}R_{2;1} = St^{(s),(0)}(M_{2;1}) L_{2;2}^{p-2} = [s, 0] L_{2;2}^{p-2}.$$
From this and Proposition 2.2 we see that the theorem is true for $i = 0$.

For $i > 0$, using the Cartan formula, Lemma 3.1 and Lemma 3.2 we obtain

$$St^{(s),(i)}_{2:1} R_{2:1} = St^{(s),(0)}_{2:1} M_{2:1}^{i} L_{2}^{p-2} + St^{(s),(1)}_{2:1} M_{2:1}^{i-1} L_{2}^{p-2}$$

$$= [s, 0](-1)^{k}(k + 1) \binom{k}{r} L_{2}^{p-2} Q_{2,0}^{k-r}$$

$$+ [s, 1](-1)^{k}(k + 1) \binom{k}{r-1} L_{2}^{p-2} Q_{2,0}^{r-1} Q_{2,1}^{k-r+1}$$

$$= (-1)^{k}(k + 1) \binom{k}{r} [s, 0] L_{2}^{p-2} Q_{2,0}^{k-r}$$

$$+ \binom{k}{r-1} [s, 1] L_{2}^{p-2} Q_{2,0}^{r-1} Q_{2,1}^{k-r+1}.$$ 

Now the theorem follows from this equality and Proposition 2.2

**Theorem 4.4.** Suppose $s, i$ are nonnegative integers. We have

$$St^{(s),(i)}_{2:0,1} = \begin{cases} (-1)^{k+1}(k+1) \binom{k}{r} R_{2:1} Q_{2,0}^{k-r}, & s = 0, i = kp + r, 0 \leq r \leq k < p, \\ (-1)^{k+1}(k+1) \binom{k}{r} R_{2:0} Q_{2,0}^{k-r}, & s = 1, i = kp + r, 0 \leq r \leq k < p, \\ (-1)^{k}(k+1) \binom{k}{r} \left(R_{2:1} \sum_{a \in I(1,s)} (-1)^{a} Q_{2,0}^{p^{a+1}+r} Q_{2,1}^{p^{a+1}-(p+1)a+k-r} - R_{2:0} \sum_{a \in I(0,s)} (-1)^{a} Q_{2,0}^{p^{a+r}} Q_{2,1}^{p^{a+r}-(p+1)a+k-r} \right), & s > 1, i = kp + r, 0 \leq r \leq k < p, \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** Since $\deg R_{2:0,1} = (p - 2)(p + 1) + 2$, $St^{(s),(i)}_{2:0,1} = 0$ for $i \geq p^2$. Suppose $i < p^2$ and $i = kp + r$ with $0 \leq k, r < p$. Using the Cartan formula and Lemma 3.2 we have

$$St^{(s),(0)}_{1} (x_1 x_2) = y_1^{p^r} x_2 - x_1 y_2^{p^r} = -[1; s] = (M_{2:1}[1, s] - M_{2:0}[0, s]) / L_2.$$ 

Since $R_{2:0,1} = x_1 x_2 L_{2}^{p-2}$, using the Cartan formula, the above equality and Lemma 3.2 we get

$$St^{(s),(i)}_{2:0,1} = St^{(s),(0)}_{1} (x_1 x_2) L_{2}^{p-2}$$

$$= (M_{2:1}[1, s] - M_{2:0}[0, s]) (-1)^{k}(k + 1) \binom{k}{r} L_{2}^{p-3} Q_{2,0}^{k-r}. $$

Combining this equality and Proposition 2.2 we get the theorem. 

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