Robust optimization of structures subjected to frictionless unilateral contact with uncertain initial gaps

Yoshihiro KANNO*
Mathematics and Informatics Center, The University of Tokyo
Hongo 7-3-1, Tokyo 113-8656, Japan
E-mail: kanno@mist.i.u-tokyo.ac.jp

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Abstract
This short paper studies a robust optimization problem of a structure subjected to frictionless unilateral contacts. We suppose that initial gaps possess non-probabilistic uncertainty, and attempt to maximize the worst-case stiffness of the structure. It is often that a structural optimization problem involving contact conditions is formulated as a mathematical programming problem with complementarity constraints (an MPCC problem). Since any feasible solution of an MPCC problem does not satisfy standard constraint qualifications, special treatment is required to solve an MPCC problem. In contrast, the formulation developed in this paper is free from complementarity constraints, and hence can be handled as a standard nonlinear programming problem. It is shown that this formulation is readily derived as a natural extension of the recently proposed optimal design problem formulation that does not consider uncertainty.

Keywords: Structural optimization, Uncertainty, Robust optimization, Stiffness maximization, Contact mechanics

1. Introduction
The frictionless unilateral contact condition can be formulated with a complementarity condition (Wriggers and Panagiotopoulos, 1999). Accordingly, a static problem of elasticity with frictionless unilateral contacts is formulated as a complementarity problem. Therefore, a design optimization problem of structure subjected to frictionless unilateral contacts is often formulated as a mathematical programming problem with complementarity constraints (an MPCC problem). Such formulations can be found in, e.g., Tin-Loi (1999), Strömberg and Klarbring (2010), and the references compiled in Kanno (2020a). Since any feasible solution of an MPCC problem does not satisfy any standard constraint qualification (Luo et al., 1996), conventional nonlinear programming (NLP) techniques are likely to fail to find a local optimal solution. Therefore, special treatment is required to solve an MPCC problem (Luo et al., 1996). Typically, reformulation and smoothing methods are used to approximate the MPCC problem as a standard NLP problem (Tin-Loi, 1999). For a stiffness maximization problem of a structure subjected to frictionless unilateral contacts, Kanno (2020a) presented a formulation different from MPCC. This formulation involves no complementarity constraint, and hence we can apply a conventional NLP approach directly.

This brief paper is a sequel to Kanno (2020a). Specifically, we consider a robust counterpart, in which the initial gaps of contact conditions are supposed to be uncertain. As a natural extension of Kanno (2020a), we see that this robust optimization problem can also be reduced to an NLP problem, which does not involve complementarity constraints.

Robust optimization of structures have been studied extensively over the years; see, e.g., Beyer and Sendhoff (2007) and Kanno (2020b) for survey. Among some different formulations of robust structural optimization, in this paper we adopt the notion of worst-case optimization: We attempt to optimize structural performance in the worst case, when a set of possible realizations of uncertain parameters is given. There exists a vast literature on the worst-case compliance minimization, where uncertainties in the external load (Ben-Tal and Nemirovski, 1997; Yonekura and Kanno, 2010; Takezawa et al., 2011; Kanno, 2018) and in the structural geometry (Sigmund, 2009; Guo et al., 2009; Hashimoto and
Kanno (2015) are often considered. To the best of the author’s knowledge, a standard problem setting of robust compliance minimization against uncertainty in initial gaps has not been studied. A related but more specific topic of study can be found in optimal design of gears (Gabiccini et al., 2010; Korta and Mundo, 2018), where teeth of gears are subjected to frictional contacts and robustness against misalignment variations (Gabiccini et al., 2010) or manufacturing errors (Korta and Mundo, 2018) is taken into consideration. Gao et al. (2020) studied a design optimization problem of a crank-slider mechanism, where robustness against increase of joint clearances and uncertainties of design variables is considered. Kim et al. (2007) proposed a six-sigma design method for a paper feeding mechanism, where the mechanism involves frictional contacts between paper and rollers. Also, reliability-based design optimization of structures subjected to contact conditions was studied by Park et al. (2000) and Cui et al. (2003). Evgrafov et al. (2003) presented a stochastic programming approach to structural optimization with frictionless contacts. Thus, literature on robust optimization of structures with contact conditions is limited, compared with other problem settings in robust structural optimization.

The paper is organized as follows. In section 2, we briefly recall the formulation for a problem without considering uncertainties. Section 3 states the main result. Section 4 presents some numerical examples. Section 5 concludes the paper.

2. Existing results

In this section, we recall a problem formulation for structural optimization with frictionless unilateral contacts (Kanno, 2020a). Uncertainty is not taken into account here. For ease of notation and presentation, we consider truss topology optimization. Throughout the paper, we assume linear elasticity and small deformation.

We use the same notation as in Kanno (2020a, section 2). Let \( \mathbf{u} \in \mathbb{R}^n \) denote the nodal displacement vector, where \( n \) is the number of degrees of freedom. The non-penetration conditions are written in the form

\[
C_\mathbf{u} \mathbf{u} \leq g,
\]

where \( g_j \geq 0 \ (j = 1, \ldots, c) \) is the initial gap between the \( j \)-th contact candidate node and the corresponding rigid obstacle surface, \( C_\mathbf{u} \in \mathbb{R}^{c \times n} \) is a constant matrix, and \( c \) is the number of contact candidate nodes.

Let \( m \) denote the number of members. We use \( x_e \geq 0 \ (e = 1, \ldots, m) \) to denote the cross-sectional area of member \( e \), which is a design variable. For given nodal external force \( \mathbf{f} \in \mathbb{R}^n \) and initial gap \( \mathbf{g} \), the compliance of truss design \( \mathbf{x} \) is defined by

\[
\pi(\mathbf{x}; \mathbf{g}) = \sup_{\mathbf{u}} \{ 2\mathbf{f}^T \mathbf{u} - \mathbf{u}^T K(\mathbf{x}) \mathbf{u} \mid C_\mathbf{u} \mathbf{u} \leq \mathbf{g} \},
\]

where \( K(\mathbf{x}) \) is the stiffness matrix. Note that the stiffness matrix of a truss has a form

\[
K(\mathbf{x}) = \sum_{e=1}^{m} \frac{E}{l_e} x_e \mathbf{b}_e \mathbf{b}_e^T,
\]

where \( E \) is the Young modulus, \( l_e \) is the undeformed member length, and \( \mathbf{b}_e \in \mathbb{R}^n \) is a constant vector. The topology optimization problem that maximizes the structural stiffness is formulated as follows:

\[
\begin{align*}
\text{Minimize} & \quad \pi(\mathbf{x}; \mathbf{g}) \\
\text{subject to} & \quad \mathbf{x} \geq 0, \quad (1a) \\
& \quad \mathbf{l}^T \mathbf{x} \leq v. \quad (1b)
\end{align*}
\]

Here, \( v \ (> 0) \) is a specified upper bound for the structural volume, and \( \mathbf{x} \) is the decision variable.

For notational simplicity, define a set \( D(\mathbf{x}) \subseteq \mathbb{R}^m \times \mathbb{R}^c \) by

\[
D(\mathbf{x}) = \{ (\mathbf{w}, \mathbf{r}) \in \mathbb{R}^m \times \mathbb{R}^c \mid \exists q \in \mathbb{R}^m : w_e x_e \geq \frac{r_j}{E} q_e^2 \ (e = 1, \ldots, m), \sum_{e=1}^{m} q_e \mathbf{b}_e = \mathbf{f} + C_\mathbf{u}^T \mathbf{r}, \ w \geq 0, \ r \geq 0 \},
\]

where \( q_e \) is the axial force of member \( e \), \( r_j \) is the normal contact reaction at node \( j \), and \( w_e \) is a subsidiary variable corresponding to the twice of the complementary strain energy stored in member \( e \). Theorem 1 in Kanno (2020a) shows

\[
\pi(\mathbf{x}; \mathbf{g}) = \inf_{\mathbf{w}, \mathbf{r}} \{ \mathbf{1}^T \mathbf{w} - 2g^T \mathbf{r} \mid (\mathbf{w}, \mathbf{r}) \in D(\mathbf{x}) \},
\]

where \( \mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^m \). Substitution of (2) into (1) results in a formulation that does not include complementarity constraints.
3. Robust optimization and its reformulation

We are now in position to consider uncertainty in the initial gap, \( \mathbf{g} \). We continue to consider truss topology optimization, for ease of notation and presentation. Application of the result established in this section to continuum-based topology optimization is straightforward; see Remark 1.

Let \( \hat{\mathbf{g}} \geq 0 \) denote the nominal value, or the best forecast, of \( \mathbf{g} \). Assume that the set of all possible realizations of \( \mathbf{g} \) is given as

\[
G = \{ \hat{\mathbf{g}} + \mathbf{A} \mathbf{\theta} \mid \| \mathbf{\theta} \| \leq \alpha \},
\]

where \( \alpha \geq 0 \) is a parameter representing the magnitude of uncertainty, \( \mathbf{A} \in \mathbb{R}^{c \times a} \) is a constant matrix with rank \( \mathbf{A} = a \), \( \mathbf{\theta} \in \mathbb{R}^a \) is an unknown vector reflecting the uncertainty in \( \mathbf{g} \), and \( \| \mathbf{\theta} \| \) is the Euclidean norm of \( \mathbf{\theta} \), i.e., \( \sqrt{\mathbf{\theta}^\top \mathbf{\theta}} \). It is worth noting that the level of uncertainty becomes larger as \( \alpha \) gets larger, and \( \alpha = 0 \) means the absence of uncertainty. We call \( G \) the uncertainty set of \( \mathbf{g} \).

Since a smaller value of the compliance is preferable, the worst-case compliance of a truss design \( \mathbf{x} \) corresponds to the maximum value of \( \pi(\mathbf{x}; \mathbf{g}) \) among \( \mathbf{g} \in G \). With referring to the nominal optimization problem in (1), we can formulate the robust optimization problem as follows:

\[
\begin{align*}
\text{Minimize} & \quad \sup_{\mathbf{g}} \left\{ \pi(\mathbf{x}; \mathbf{g}) \mid \mathbf{g} \in G \right\} \quad (4a) \\
\text{subject to} & \quad \mathbf{x} \geq \mathbf{0}, \quad (4b) \\
& \quad \mathbf{l}^\top \mathbf{x} \leq \nu. \quad (4c)
\end{align*}
\]

The following theorem is crucial to recasting problem (4) as a tractable form.

**Theorem 1** \( x \in \mathbb{R}^m \) satisfying \( x \geq 0 \), we have

\[
\sup_{\mathbf{g}} \left\{ \pi(\mathbf{x}; \mathbf{g}) \mid \mathbf{g} \in G \right\} = \inf_{\mathbf{w}, \mathbf{r} \in D(x)} \left\{ \mathbf{1}^\top \mathbf{w} - 2 \hat{\mathbf{g}}^\top \mathbf{r} + 2 \alpha \| \mathbf{A}^\top \mathbf{r} \| \right\}.
\]

**Proof.** Observe that \( D(x) \) is convex for any \( x \geq 0 \). Also, \( G \) is compact and convex. Application of the minimax theorem (Sion, 1958; Komiya, 1988; Bertsekas, 2009), following the use of (2), yields

\[
\sup_{\mathbf{g} \in G} \left\{ \min_{\mathbf{w}, \mathbf{r} \in D(x)} \left\{ \mathbf{1}^\top \mathbf{w} - 2 \hat{\mathbf{g}}^\top \mathbf{r} \right\} \right\} = \inf_{\mathbf{w}, \mathbf{r} \in D(x)} \left\{ \sup_{\mathbf{g} \in G} \left\{ \mathbf{1}^\top \mathbf{w} - 2 \hat{\mathbf{g}}^\top \mathbf{r} \right\} \right\}.
\]

Substitution of (3) into the last expression results in

\[
\begin{align*}
\min_{\mathbf{w}, \mathbf{r} \in D(x)} & \left\{ \max_{\mathbf{g} \in G} \left\{ \mathbf{1}^\top \mathbf{w} - 2 \hat{\mathbf{g}}^\top \mathbf{r} \right\} \right\} \\
= & \left\{ \max_{\mathbf{g} \in G} \left\{ \mathbf{1}^\top \mathbf{w} - 2 \hat{\mathbf{g}}^\top \mathbf{r} \right\} \right\} = \inf_{\mathbf{w}, \mathbf{r} \in D(x)} \left\{ \mathbf{1}^\top \mathbf{w} - 2 \hat{\mathbf{g}}^\top \mathbf{r} + 2 \alpha \| \mathbf{A}^\top \mathbf{r} \| \right\},
\end{align*}
\]

where the last equality follows the Cauchy–Schwarz inequality. \( \square \)

It follows from Theorem 1 that problem (4) can be recast as follows:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{e=1}^{m} w_e - 2 \hat{\mathbf{g}}^\top \mathbf{r} + 2 \alpha \| \mathbf{A}^\top \mathbf{r} \| \quad (5a) \\
\text{subject to} & \quad w_e x_e \geq \frac{f_e}{E} q_e^2, \quad e = 1, \ldots, m, \quad (5b) \\
& \quad \sum_{e=1}^{m} q_e b_e = f + C_0^r \mathbf{r}, \quad (5c) \\
& \quad \mathbf{w} \geq \mathbf{0}, \quad (5d) \\
& \quad \mathbf{r} \leq \mathbf{0}, \quad (5e) \\
& \quad \mathbf{x} \geq \mathbf{0}, \quad (5f) \\
& \quad \mathbf{l}^\top \mathbf{x} \leq \nu. \quad (5g)
\end{align*}
\]

It should be noted that this formulation has no complementarity constraint. Therefore, we can apply a standard NLP approach to problem (5).
Comparing to the formulation presented in Kanno (2020a) for nominal optimization (1), we see that problem (5) can be obtained by replacing the term $-2\hat{g}^T r$ in the objective function with $-2\tilde{g}^T r + 2\alpha \|A^T r\|$. Roughly speaking, the additional term, $2\alpha \|A^T r\|$, for robust optimization can be viewed as a penalty on the magnitude of the contact reactions, $r$. Indeed, in the numerical examples presented in section 4, we shall see that increase of $\alpha$ (i.e., increase of the magnitude of uncertainty) yields a structural design that mitigates concentration of large contact reactions.

It is also worth noting that problem (5) can be reduced to an SOCP problem; hence, we can compute its global optimal solution in an efficient manner with a primal-dual interior-point method (Anjos and Lasserre, 2012). To perform this reduction, we first observe that the term $2\alpha \|A^T r\|$ in (5a) can be converted to minimization of an additional decision variable $z \in \mathbb{R}$ under the second-order cone constraint

$$z \geq \|2\alpha A^T r\|.$$  

Moreover, for each $e = 1, \ldots, m$, the constraint in (5b), together with (5d) and (5f), can be reduced to a second-order cone constraint

$$w_e + x_e \geq \left\| \frac{w_e - x_e}{\sqrt{e/Eq}} \right\|$$

in terms of $w_e$, $x_e$, and $q_e$. Thus, problem (5) is reduced to an SOCP problem.

**Remark 1** Theorem 1 can be applied also to a continuum-based topology optimization problem in a similar manner. Namely, we can obtain robust counterpart to the formulation presented in Kanno (2020a), by simply replacing the term $-2\hat{g}^T r$ in the objective function with $-2\tilde{g}^T r + 2\alpha \|A^T r\|$. It should be noted that the robust optimization problem for continuum is nonconvex due to the SIMP (solid isotropic material with penalization) penalization.

### 4. Numerical examples

This section presents numerical examples of continuum-based robust topology optimization using the formulation described in section 3. We adopt the SIMP approach (Bendsøe and Sigmund, 1999) together with the density filter (Bourdin, 2001). The following numerical experiments were carried out on a 2.6 GHz Intel Core i7 processor with 32 GB RAM. The sequential SOCP implemented in Kanno (2020a) was run with SeDuMi ver. 1.1 (Sturm, 1999; Pólik, 2005) and Matlab ver. 9.8.

Consider a problem setting outlined in Figure 1. The elastic body is discretized as a uniform $100 \times 50$ finite element mesh. For simplicity, we omit units of quantities. The Young modulus and Poisson ratio are 1 and 0.3, respectively. A downward vertical external force of 1 is applied at the bottom right corner. We set $A$ to the identity matrix of size $a = e = 203$. The specified upper bound for the volume fraction is 0.35.

Figure 2 collects the results obtained when the nominal values of the initial gaps are $\tilde{g}_j = 0$ ($j = 1, \ldots, e$), where $\alpha$ represents the magnitude of uncertainty (see (3)). Small open circles, accompanied by dimension lines, in the figures indicate the contact candidate nodes that are in contact at the equilibrium state with the nominal initial gaps. It is observed that, in a robust optimal design, the width of the structure increases compared with the nominal one. Therefore, the reaction at each contact node is reduced as the magnitude of uncertainty, $\alpha$, increases. This is consistent with the observation, made in section 3, that assuring robustness corresponds to adding the penalty term $2\alpha \|A^T r\|$ to the objective function; a larger penalty is imposed on the magnitude of the contact reactions as $\alpha$ increases.

Fig. 1 Problem setting for numerical examples.
Figure 3, Figure 4, and Figure 5 show the solutions obtained with $\tilde{g}_j = 2$, $\tilde{g}_j = 4$, and $\tilde{g}_j = 6$ ($j = 1, \ldots, c$), respectively. Like the case in Figure 2, it is observed that the width of structural design increases as the magnitude of uncertainty, $\alpha$, increases. Also, the structural topology slightly changes as $\alpha$ increases. When $\alpha$ is fixed, the width of structural design increases as the nominal initial gaps, $\tilde{g}_1, \ldots, \tilde{g}_c$, increase.

For the robust optimization problems with $\alpha = 1, 1.5, 2,$ and $4$, Table 1 reports the objective value (i.e., the worst-case compliance) of the solution obtained for each problem instance, as well as the nominal-case compliance (i.e., $\pi(x; \tilde{g})$). For example, in the case of the solution shown in Figure 2c, the worst-case compliance and the nominal-case compliance are 25.8469 and 23.5776, respectively. As expected, the worst-case compliance value increases as $\alpha$ increases. For
Fig. 4 The solutions obtained with $\hat{g}_j = 4$. (a) $\alpha = 0$; (b) $\alpha = 1$; (c) $\alpha = 2$; (d) $\alpha = 4$.

Fig. 5 The solutions obtained with $\hat{g}_j = 6$. (a) $\alpha = 0$; (b) $\alpha = 1$; (c) $\alpha = 2$; (d) $\alpha = 4$.

| $\alpha$ | Worst-case compliance | | Nominal-case compliance |
|----------|----------------------|----------|------------------------|
|          | $\hat{g}_j = 0$ | $\hat{g}_j = 2$ | $\hat{g}_j = 4$ | $\hat{g}_j = 6$ | $\hat{g}_j = 0$ | $\hat{g}_j = 2$ | $\hat{g}_j = 4$ | $\hat{g}_j = 6$ |
| 1.0      | 24.7462            | 33.8544  | 42.1670                | 49.3638              | 23.4244           | 32.4261           | 40.6304              | 47.7909              |
| 1.5      | 23.4771            | 34.4525  | 42.5665                | 50.0396              | 23.4771           | 32.4617           | 40.4698              | 47.8947              |
| 2.0      | 25.8469            | 34.9667  | 43.0511                | 50.5752              | 23.5776           | 32.4655           | 40.4901              | 47.8900              |
| 4.0      | 27.5116            | 36.6639  | 44.7919                | 52.4455              | 23.7477           | 32.5824           | 40.5561              | 47.9938              |
Table 2  Worst-case compliance of the obtained solutions for nominal optimization problems.

| $\alpha$ | $\check{g}_j = 0$ | $\check{g}_j = 2$ | $\check{g}_j = 4$ | $\check{g}_j = 6$ |
|----------|------------------|------------------|------------------|------------------|
| 0        | 23.4019          | 32.4826          | 40.5259          | 47.8461          |
| 1.0      | 24.8966          | 34.1057          | 42.2539          | 49.6456          |
| 1.5      | 25.5623          | 34.8085          | 42.9875          | 50.3952          |
| 2.0      | 26.1733          | 35.4378          | 43.6327          | 51.0435          |
| 4.0      | 28.0768          | 37.2382          | 45.3600          | 52.6421          |

the nominal optimization problem (i.e., the problem with $\alpha = 0$), Table 2 lists the objective value (i.e., the nominal-case compliance) of the obtained solution, as well as the worst-case compliance values with $\alpha = 1, 1.5, 2,$ and 4. For example, in the case of the solution shown in Figure 2a, the nominal-case compliance is 23.4019. When we consider the uncertainty in $g$ with $\alpha = 2$, the worst-case compliance of this solution is 26.1733. Thus, comparing the solutions in Figure 2a and Figure 2c, we observe that Figure 2a has a smaller nominal-case compliance value, while Figure 2c has a smaller worst-case compliance value for $\alpha = 2$, as expected. Exceptions are the solutions obtained for the nominal optimization problems with $\check{g}_j = 4$ and 6. Namely, the nominal-case compliance values of these solutions (i.e., the values corresponding to $\alpha = 0$ in Table 2) are larger than the nominal-case compliance values of the solutions obtained for the robust optimization problems with $\alpha = 1$ (i.e., the values corresponding to $\alpha = 1$ in Table 1). Thus, the solutions shown in Figure 4a and Figure 5a are local optimal solutions inferior to the solutions shown in Figure 4b and Figure 5b, respectively.

5. Conclusions

In this paper, we have studied robust design optimization of structures subjected to frictionless unilateral contact conditions, where the initial gaps are supposed to be uncertain and the structural stiffness in the worst case is maximized.

A direct formulation of a structural optimization problem involving unilateral contacts is an optimization problem with complementarity constraints, because the equilibrium analysis problem involving unilateral contacts is formulated as a complementarity problem. In contrast, the formulation presented in this paper does not contain complementarity constraints. Therefore, unlike formulations with complementarity constraints, the presented formulation can be directly handled via a standard nonlinear programming approach. Exploring an efficient algorithm for the continuum-based topology optimization problem, rather than application of a sequential second-order cone programming approach (Kanno, 2020a), remains to be studied.

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