Rigidity of the Hopf fibration

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Abstract
In this paper, we study minimal maps between euclidean spheres. The Hopf fibrations provide
explicit examples of such minimal maps. Moreover, their corresponding graphs have second
fundamental form of constant norm. We prove that a minimal submersion from $S^3$ to $S^2$
whose Gauss map satisfies a suitable pinching condition must be weakly conformal with
totally geodesic fibers. As a consequence, we obtain that an equivariant minimal submersion
from $S^3$ to $S^2$ coincides with the Hopf fibration. Furthermore, we prove that a minimal
map $f : U \subset S^3 \to S^2$ with constant singular values and constant norm of the second
fundamental form is either constant or, up to isometries, coincides with the Hopf fibration.

Keywords  Minimal graphs · Hopf fibrations · Weakly conformal maps · Maximum
principle

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1 Introduction

Let $f : M \to N$ be a smooth map between two manifolds $M$ and $N$. It is a fundamen-
tal problem to find canonical representatives in the homotopy class of $f$. By a canonical
representative is usually meant a map in the homotopy class of the given map $f$ which is
a critical point of a suitable functional. In the mid-1960’s, Eells and Sampson [12] intro-
duced the harmonic maps as critical points of the energy functional, in order to attack the
aforementioned problem. However, the existence of non constant harmonic maps between

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Riemannian manifolds, especially between spheres, is a highly non trivial issue; see for example [4,13,14,31,46,47,53].

There is another important functional that we may consider in the space of smooth maps. Given a smooth map $f : M \to N$ between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$, let us denote its graph in the product space $(M \times N, g_M \times g_N)$ by

$$\Gamma(f) := \{(x, f(x)) \in M \times N : x \in M\}.$$  

Following the terminology introduced by Schoen [45], the map $f$ is called minimal if it is a critical point of the volume functional

$$\text{Vol}(f) = \int_M \sqrt{\det g_M + f^*g_N} \, dvol_{g_M}.$$  

Thus, a map $f$ is minimal, if and only if the corresponding graph is a minimal submanifold of the Riemannian product manifold $M \times N$.

There is a very interesting class of non trivial maps from odd-dimensional spheres to projective spaces, the so-called Hopf fibrations [29,30]; see Sect. 3. These maps are described by the actions

$$S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n, \quad S^3 \hookrightarrow S^{4n+3} \to \mathbb{Q}P^n \quad \text{and} \quad S^7 \hookrightarrow S^{15} \to \mathbb{O}P^1.$$  

The fibers of the fibration $S^{2n+1} \to \mathbb{C}P^n$ are great circles. Such submersions are called great circle fibrations. The space of all great circle fibrations (not necessarily harmonic or minimal) of odd-dimensional euclidean spheres is infinite dimensional. For example, according to a beautiful result of Gluck and Warner [21] any great circle fibration of $S^3$ arise from a graphical surface of $S^2 \times S^2$ that is generated by a strictly length decreasing map. Hence, there is an abundance of such maps that are not obtained by linear repositioning of the Hopf fibration. Additionally, Gluck and Warner [21], Yang [54,55], and McKay [38], proved that for any submersion of $S^{2n+1}$ into $\mathbb{C}P^n$ with totally geodesic fibers, there is a diffeomorphism of the sphere $S^{2n+1}$ carrying it to the Hopf fibration. For more details in this topic we refer to [6,18–24].

One aim of this paper is to understand the Hopf fibrations in geometric terms, to explore their properties, and to show that their graphs are rigid in a suitable sense. It turns out that the Hopf fibrations are minimal Riemannian submersions and their graphs have second fundamental form of constant norm; see Sect. 3. In the theorem below, we show the following rigidity result:

**Theorem A.** Let $f : U \to S^2$ be a minimal map, where $U$ is an open subset of $S^3$. If $f$ has constant singular values and $\Gamma(f)$ has constant norm of the second fundamental form, then its singular values are equal and so the map $f$ is either constant or weakly conformal. If $f$ is non-constant, then there exists an isometry $T : S^3 \to S^3$ such that $f = \Phi \circ \Pi \circ T$, where $\Pi$ is the Hopf fibration and $\Phi : S^2(1/2) \to S^2$ is the dilation from the sphere $S^2(1/2)$ of radius $1/2$.

A related classical problem is the Bernstein problem, which asks under what conditions a graphical minimal submanifold is totally geodesic. In the last years there has been a lot of research in this direction; for example, see [2,10,26,27,33–36,44,48,50]. A result in the same spirit as our Theorem A was obtained by Jost, Xin and Yang [32]. More precisely, they proved that a minimal, coassociative, 4-dimensional graph with constant singular values in $\mathbb{R}^7$ either is flat or an open piece of the Lawson-Osserman cone. However, the result of Jost et al. does not follow from our Theorem A and the methods they use are different from ours.
Let us mention a similarity of Theorem A with the Chern problem [8,9], which asks to determine the compact (or complete) Riemannian manifolds with constant scalar curvature that can be isometrically immersed as minimal submanifolds in a euclidean sphere. From the Gauss equation, it follows that a minimal submanifold of the sphere has constant scalar curvature if and only if it has second fundamental form of constant norm. In general, the conjecture is still open and is settled only in the case of 3-dimensional hypersurfaces in $S^4$ by the efforts of Peng and Terng [39,40], de Almeida and Brito [1], and Chang [7]. More precisely, it is shown that a compact 3-dimensional minimal hypersurface in $S^4$ is either totally geodesic, a minimal Clifford torus, or a minimal Cartan hypersurface. It is conjectured by Bryant that the same conclusion should hold even without the compactness assumption; for more details see the article [7, page 524].

There is an abundance of minimal maps from odd-dimensional spheres into the complex projective space that does not coincide with the Hopf fibration. One easy way to construct such maps is by composing the Hopf fibration with a holomorphic automorphism of the complex projective space; see for details Proposition 2. It turns out that these minimal maps are weakly conformal, i.e. the restriction of the differential of $f$ on the orthogonal complement of its kernel is conformal. Hence, another interesting problem is to find conditions under which a minimal map $f : S^{2n+1} \to \mathbb{CP}^n$ is weakly conformal. Motivated by a conjecture of Eells in harmonic map theory, we would expect that any minimal map $f : S^3 \to S^2$ must be weakly conformal; see [4, Note 10.4.1, page 422] and [49, page 730].

Let us point out here that weakly conformal (not necessarily minimal) submersions of spheres with totally umbilical fibers were studied recently in [28,56,57]. Due to a nice result of Heller [28, Theorem 3.7], up to conformal transformations of $S^2$ and $S^3$, every weakly conformal fibration of $S^3$ by circles (not necessarily great circles) is the Hopf fibration.

In the next theorem, we provide a partial answer to the problem of when a minimal submersion $f : S^3 \to S^2$ is weakly conformal. The quantity which will play a crucial role in our analysis is the Gauss map $G$ of the submersion. According to Baird [3], $G$ associates to each point $x \in S^3$ the line in the bundle $G_1(S^3)$ over $S^3$, whose fiber at each point $x \in S^3$ is the Grassmannian of oriented lines in $T_x S^3$. Note that, the orthogonal complement $\mathcal{H}$ of the kernel of $df$ is a real plane bundle and so it can be regarded as a complex line bundle. By fixing at each point $x \in S^3$ an oriented orthonormal basis of the singular value decomposition of $df$ we may represent $\mathcal{H}_x$ as the direct sum $\text{Re}(\mathcal{H}_x) \oplus \text{Im}(\mathcal{H}_x)$ of two real lines, which we call real and imaginary parts of $\mathcal{H}_x$ with respect to the given frame. Observe that at points where the non-zero singular values $\lambda_2 \leq \lambda_3$ of $df$ are distinct, the choice of the aforementioned frame is unique.

**Theorem B.** Let $f : S^3 \to S^2$ be a minimal submersion whose Gauss map $G$ (with respect to the graphical metric) satisfies the condition

$$(\lambda_3 - \lambda_2)(\lambda_3 |\text{Im } \nabla G|^2 - \lambda_2 |\text{Re } \nabla G|^2) \geq 0,$$

where $0 < \lambda_2 \leq \lambda_3$ are the non-zero singular values of the map $f$. Then, $f$ is weakly conformal with totally geodesic fibers. Moreover, there exists an isometry $T : S^3 \to S^3$ and a conformal diffeomorphism $\Phi : S^2 \to S^2$ such that $f = \Phi \circ \Pi \circ T$, where $\Pi$ is the standard Hopf fibration.

Let $f : S^3 \to S^2$ be a weakly conformal minimal submersion with totally geodesic fibers. Then, for any choice of an orthonormal frame of the singular value decomposition we have that

| Re $\nabla G$ | = | Im $\nabla G$ |.
Moreover, the condition on Theorem B can be equivalently expressed in terms of of the second fundamental form of the graph; see Lemma 8 in Sect. 5 and equation (38) in Sect. 6.

Next, we turn our attention to equivariant minimal maps with respect to the standard isometric actions $S^1 \times S^1 \hookrightarrow S^3$ and $S^1 \hookrightarrow S^2$. It turns out that the minimality of such maps is expressed in terms of a degenerate ODE; see Sect. 4. As an application of Theorem B, we deduce the following uniqueness result.

**Theorem C.** Let $f : S^3 \to S^2$ be an equivariant minimal submersion. Then $f$ is the composition of the standard Hopf fibration with the dilation from the radius $1/2$ sphere into the unit sphere.

## 2 Maps between Riemannian manifolds

In this section, we briefly discuss the geometry of maps between Riemannian manifolds following the notation in [41–44].

### 2.1 Notation

Let $(M, g_M)$ and $(N, g_N)$ be two Riemannian manifolds of dimension $m$ and $n$, respectively. The induced metric on the product $M \times N$ will be denoted by $g_{M \times N} = g_M \times g_N$, which often we denote by $\langle \cdot, \cdot \rangle$. A smooth map $f : M \to N$ defines an embedding $F : M \to M \times N$, by $F(x) = (x, f(x))$, for any $x \in M$ The graph of the map $f$ is defined to be the submanifold $F(f) = F(M)$. Since $F$ is an embedding, it induces another Riemannian metric $g = F^* g_{M \times N}$ on $M$. The two natural projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are submersions, that is they are smooth and have maximal rank. Note that the tangent bundle of the product manifold $M \times N$, splits as a direct sum $T(M \times N) = TM \oplus TN$. The four metrics $g_M, g_N, g_{M \times N}$ and $g$ are related by

\[ g_{M \times N} = \pi_M^* g_M + \pi_N^* g_N, \tag{1} \]

\[ g = F^* g_{M \times N} = g_M + f^* g_N. \tag{2} \]

The Levi-Civita connection $\nabla_{g_{M \times N}}$ associated to $g_{M \times N}$ is related to the Levi-Civita connections $\nabla_{g_M}$ on $(M, g_M)$ and $\nabla_{g_N}$ on $(N, g_N)$ by

\[ \nabla_{g_{M \times N}} = \pi_M^* \nabla_{g_M} \oplus \pi_N^* \nabla_{g_N}. \]

The corresponding curvature operator $\tilde{R}$ on the product $M \times N$ is related to the curvature operators $R_M$ on $(M, g_M)$ and $R_N$ on $(N, g_N)$ by

\[ \tilde{R} = \pi_M^* R_M \oplus \pi_N^* R_N. \]

Denote the Levi-Civita connection on $M$ with respect to the induced metric $g = F^* g_{M \times N}$ by $\nabla^g$ and the curvature tensor by $R$. The differential $dF$ of the map $F$ is a section in $F^* T(M \times N) \otimes T^* M$. The covariant derivative of it is called the second fundamental tensor $A$ of the graph. That is,

\[ A(v_1, v_2) = (\nabla^F_{v_1} dF) v_2 = \nabla^F_{dF(v_1)} dF(v_2) - dF(\nabla^g_{v_1} v_2) \]

for $v_1, v_2 \in \mathfrak{X}(M)$, where $\nabla^F$ is the induced connection on $F^* T(M \times N) \otimes T^* M$ and $\nabla$ is the Levi-Civita connection associated with $g$. 

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The trace of $A$ with respect to the metric $g$ is called the mean curvature vector field of $\Gamma(f)$ and it will be denoted by

\[ H = \text{tr}_g A. \]

Note that $H$ is a section in the normal bundle of the submanifold. If $H$ vanishes identically the graph is called minimal. Following Schoen’s [45] terminology, a map $f : M \to N$ is called a minimal map if its graph $\Gamma(f)$ is a minimal submanifold of the product space.

By Gauss equation the tensors $R$ and $\tilde{R}$ are related by the formula

\[ (R - F^* \tilde{R})(v_1, v_2, v_3, v_4) = \langle A(v_1, v_3), A(v_2, v_4) \rangle - \langle A(v_1, v_4), A(v_2, v_3) \rangle \]

for any $v_1, v_2, v_3, v_4 \in \mathcal{X}(M)$. Moreover, the second fundamental form satisfies the Codazzi equation

\[ (\nabla_{v_1}^A)(v_2, v_3) - (\nabla_{v_2}^A)(v_1, v_3) = (\tilde{R}(dF(v_1), dF(v_2), dF(v_3)))^\perp \]

for any $v_1, v_2, v_3 \in \mathcal{X}(M)$.

### 2.2 Singular value decomposition

Consider the non-negative definite symmetric 2-tensor $f^* g_N$ on $M$. Fix now a point $x \in M$ and assume that $r = \text{rank} df_x$. Obviously, $r \leq \min\{m, n\}$. Suppose further that at $x$ and with respect to $g_M$, the 2-tensor $f^* g_N$ has $r$ zero eigenvalues and $q$ distinct positive eigenvalues

\[ \lambda_1^2 \leq \cdots \leq \lambda_q^2 \]

with multiplicities $m_1, \ldots, m_q$ and with eigenspaces $E_1, \ldots, E_q$, respectively. Hence,

\[ m_1 + \cdots + m_q = r. \]

The corresponding values $\{0, \lambda_1, \ldots, \lambda_q\}$ are called singular values of the differential $df$ of $f$. At the point $x$ consider an orthonormal basis

\[ \{\xi_s, \alpha_{i_1}, \ldots, \alpha_{i_q}\}, \]

with respect to $g_M$ which diagonalizes $f^* g_N$, where $s \in \{1, \ldots, m - r\}$, $i_j \in \{1, \ldots, m_j\}$ and $j \in \{1, \ldots, q\}$. Note that $\{\xi_1, \ldots, \xi_{m-r}\}$ span the kernel $\mathcal{V}$ of $df$. Moreover, at the point $f(x)$ consider an orthonormal basis $\{\beta_k, \beta_{i_1}, \ldots, \beta_{i_q}\}$ with respect to the metric $g_N$, where $k \in \{1, \ldots, n - r\}$, $i_j \in \{1, \ldots, m_j\}$ and $j \in \{1, \ldots, q\}$, such that

\[ df(\xi_s) = 0, \quad df(\alpha_{i_1}) = \lambda_1 \beta_{i_1}, \ldots, df(\alpha_{i_q}) = \lambda_q \beta_{i_q}. \quad (3) \]

The above procedure is called the singular value decomposition of $df$.

It is a well-known fact that, with the above ordering, the singular values give rise to continuous functions. In a matter of fact, they are even smooth on an open and dense subset of $M$. In particular, they are smooth on open subsets where the corresponding multiplicities are constant and the corresponding eigenspaces are smooth distributions.

Now we are going to define a special basis for the tangent and the normal space of the graph in terms of the singular values. The vectors

\[ \left\{ e_s = \xi_s, e_{i_1} = \frac{\alpha_{i_1}}{\sqrt{1 + \lambda_1^2}}, \ldots, e_{i_q} = \frac{\alpha_{i_q}}{\sqrt{1 + \lambda_q^2}} \right\}, \quad (4) \]
where \( s \in \{1, \ldots, m - r\} \), \( i_j \in \{1, \ldots, m_j\} \) and \( j \in \{1, \ldots, q\} \), form an orthonormal basis with respect to the metric \( g \) of the tangent space \( T_x M \). Moreover, the vectors

\[
\left\{ \xi_k = \beta_k, \xi_{k_1} = \frac{-\lambda_k \alpha_i \oplus \beta_i}{\sqrt{1 + \lambda_k^2}}, \ldots, \xi_{i_1} = \frac{-\lambda_{i_1} \alpha_q \oplus \beta_q}{\sqrt{1 + \lambda_{i_1}^2}} \right\}, \tag{5}
\]

where \( k \in \{1, \ldots, n - r\} \), \( i_j \in \{1, \ldots, m_j\} \) and \( j \in \{1, \ldots, q\} \), give rise to an orthonormal basis of the normal space of the graph \( \Gamma(f) \) at \( f(x) \).

We discuss now the structure of the singular values of holomorphic maps \( f : M \to N \), where \( M \) and \( N \) are Kähler manifolds of the same dimension \( 2m \). Let us denote the complex structures of \( M \) and \( N \) by \( J_M \) and \( J_N \), respectively. Then, \( df \circ J_M = J_N \circ df \). Moreover, the product manifold is again a Kähler manifold with complex structure given by

\[
J_{M \times N} = J_M \oplus J_N.
\]

It turns out that the metric \( g \) is a Kähler metric for the complex structure \( J_M \) of \( M \). Moreover, the map \( F : (M, g, J_M) \to (M \times N, g_{M \times N}, J_{M \times N}) \) given by

\[
F(x) = (x, f(x))
\]

is a holomorphic isometric immersion. Consequently, any holomorphic map between Kähler manifolds is minimal; see also [11]. Fix again a point \( x \in M \). Since the map \( f \) is holomorphic, there exists an orthonormal basis \( \{\alpha_i, J_M \alpha_i\}, i \in \{1, \ldots, m\} \), of \( T_x M \) with respect to \( g_M \) and an orthonormal basis \( \{\beta_i, J_N \beta_i\}, i \in \{1, \ldots, m\} \), of \( T_{f(x)} M \) with respect to \( g_N \), such that

\[
df(\alpha_i) = \lambda_i \beta_i \quad \text{and} \quad df(J_M \alpha_i) = \lambda_i J_N \beta_i \tag{6}
\]

for any \( i \in \{1, \ldots, m\} \). Note that here we regard the singular values without taking into account their multiplicities. Then, the vectors

\[
\left\{ e_i = \frac{\alpha_i}{\sqrt{1 + \lambda_i^2}}, e_{m+i} = \frac{J_M \alpha_i}{\sqrt{1 + \lambda_i^2}} \right\}, \tag{7}
\]

where \( i \in \{1, \ldots, m\} \), form an orthonormal basis with respect to the metric \( g \) of the tangent space \( T_x M \) and the vectors

\[
\left\{ \xi_i = \frac{-\lambda_i \alpha_i \oplus \beta_i}{\sqrt{1 + \lambda_i^2}}, \xi_{m+i} = \frac{-\lambda_{m+i} \alpha_{m+i} \oplus \beta_{m+i}}{\sqrt{1 + \lambda_{m+i}^2}} \right\}, \tag{8}
\]

where \( i \in \{1, \ldots, m\} \), form an orthonormal basis of the normal space of the graphical submanifold \( \Gamma(f) \) at \( f(x) \).

### 2.3 Second fundamental form of graphs

We will see now how the second fundamental form \( A \) of the graph \( \Gamma(f) \) can be written in terms of the differential \( df \) and the Hessian

\[
B(v_1, v_2) = \nabla^f_{v_1} df(v_2) - df(\nabla^g_{v_1} v_2)
\]

of the map \( f : (M, g_M) \to (N, g_N) \).

In the following lemma, we see that the difference between the connections induced by \( g_M \) and \( g \) is given in terms of the singular values and the Hessian \( B \) of the map \( f \).

**Lemma 1** The following relation holds

\[
\nabla^g_{v_1} v_2 - \nabla^{g_M}_{v_1} v_2 = (I_m + df^t df)^{-1} df^t B(v_1, v_2)
\]

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for any pair of vector fields \( v_1, v_2 \in \mathcal{X}(M) \), where \( df^t \) is the transpose of the differential \( df : (TM, \mathfrak{g}_M) \to (TN, \mathfrak{g}_N) \) and \( I_m : TM \to TM \) the identity transformation.

**Proof** Consider a local coordinate system \( \{ \partial_1, \ldots, \partial_m \} \) on \( M \). From the Koszul formula, we obtain that

\[
2g\left( \nabla^\mathfrak{g}_{\partial_i} \partial_j, \partial_k \right) - 2g_M\left( \nabla^\mathfrak{g}_{\partial_i} \partial_j, \partial_k \right) = \partial_i g(\partial_j, \partial_k) + \partial_j g(\partial_i, \partial_k) - \partial_k g(\partial_i, \partial_j)
\]

\[
= -\partial_i g_M(\partial_j, \partial_k) - \partial_j g_M(\partial_i, \partial_k) + \partial_k g_M(\partial_i, \partial_j)
\]

\[
= \partial_i g_N(df(\partial_j), df(\partial_k)) + \partial_j g_N(df(\partial_i), df(\partial_k)) - \partial_k g_N(df(\partial_i), df(\partial_j))
\]

\[
= 2g_N(\nabla^f_{\partial_i} df(\partial_j), df(\partial_k)).
\]

By the definition of \( B \) and the transpose of \( df \), the above equation yields

\[
g_M(df^t B(\partial_i, \partial_j), \partial_k) = g_N(B(\partial_i, \partial_j), df(\partial_k))
\]

\[
= g\left( \nabla^\mathfrak{g}_{\partial_i} \partial_j, \partial_k \right) - g_M\left( \nabla^\mathfrak{g}_{\partial_i} \partial_j, \partial_k \right) = g_N(df(\nabla^\mathfrak{g}_{\partial_i} \partial_j), df(\partial_k))
\]

Since the difference between two connections is tensorial, we deduce that

\[
\nabla^\mathfrak{g}_{v_1} v_2 - \nabla^\mathfrak{g}_{v_1} v_2 = (I_m + df^t df)^{-1} df^t B(v_1, v_2)
\]

for any pair \( v_1, v_2 \in \mathcal{X}(M) \). This completes the proof.

In the next lemma, we express the second fundamental form of the graph in terms of the singular values and the Hessian of \( f \).

**Lemma 2** The second fundamental form \( A \) of the graph is given by

\[
A(v_1, v_2) = \left( \nabla^\mathfrak{g}_{v_1} v_2 - \nabla^\mathfrak{g}_{v_1} v_2, df(\nabla^\mathfrak{g}_{v_1} v_2 - \nabla^\mathfrak{g}_{v_1} v_2) + B(v_1, v_2) \right)
\]

for any \( v_1, v_2 \in \mathcal{X}(M) \). Equivalently, the second fundamental form \( A \) can be written in the form

\[
A(v_1, v_2) = \left( - (I_m + df^t df)^{-1} df^t B(v_1, v_2), (I_n + df df^t)^{-1} B(v_1, v_2) \right)
\]

where \( I_m : TM \to TM \) and \( I_n : TN \to TN \) are the identity transformations.

**Proof** For any \( v_1, v_2 \in \mathcal{X}(M) \), we have

\[
A(v_1, v_2) = \nabla^\mathfrak{g}_{v_1} F(v_2) - df(\nabla^\mathfrak{g}_{v_1} v_2)
\]

\[
= \left( \nabla^\mathfrak{g}_{v_1} v_2 - \nabla^\mathfrak{g}_{v_1} v_2, df(\nabla^\mathfrak{g}_{v_1} v_2) \right)
\]

\[
= \left( \nabla^\mathfrak{g}_{v_1} v_2 - \nabla^\mathfrak{g}_{v_1} v_2, df(\nabla^\mathfrak{g}_{v_1} v_2 - \nabla^\mathfrak{g}_{v_1} v_2) + B(v_1, v_2) \right).
\]

Note that

\[
I_n - df (I_m + df^t df)^{-1} df^t = (I_n + df df^t)^{-1}.
\]

Combining the last formula with Lemma 1, we get the result.
As an immediate consequence of Lemma 2, we obtain the following well-known characterization of minimality; see for example [11].

**Corollary 1** Suppose that \( f : (M, g_M) \to (N, g_N) \) is a smooth map. Then, the following two statements are equivalent:

(a) The graph \( \Gamma(f) \) is a minimal submanifold of product space \( M \times N \).

(b) The trace of \( B \) with respect to \( g \) is zero, that is \( \text{tr}_g B = 0 \).

Moreover, at points where \( \text{rank } df = \dim N \), the minimality of the graph is equivalent with the vanishing of the trace, with respect to \( g \), of the \( (2, 1) \)-tensor \( W \) given by

\[
W(v_1, v_2) = \nabla^g_{v_1} v_2 - \nabla^g_{v_2} v_1,
\]

for any \( v_1, v_2 \in \mathcal{X}(M) \).

**Proof** From Lemma 2, it follows that

\[
H = \text{tr}_g A = (\text{tr}_g W, df(\text{tr}_g W)) + (0, \text{tr}_g B) = dF(\text{tr}_g W) + (0, \text{tr}_g B).
\]

Hence, the equivalence of (a) and (b) is immediate. Suppose now that the trace of \( W \) with respect to \( g \) is zero. In this case, we have \( H = (0, \text{tr}_g B) \). Taking into account the equations (3), we deduce that at a fixed point \( x \in M \) we have

\[
0 = g_{M \times N}(H, dF(\alpha_{ij})) = \lambda_j g_N(\text{tr}_g B, \beta_{ij}),
\]

If we assume that \( \text{rank } df_x = n \), it follows that \( \text{tr}_g B = 0 \). This completes the proof. \( \square \)

### 3 Geometry of the Hopf fibrations

According to Hurwitz’s Theorem, there are precisely four normed division algebras over the real numbers; the real numbers \( \mathbb{R} \), the complex numbers \( \mathbb{C} \), the quaternions \( \mathbb{H} \), and the octonions \( \mathbb{O} \). Hopf [29,30] showed that there exist natural submersions of unit euclidean spheres over corresponding projective spaces formed by \( n \)-dimensional real normed division algebras. Here we explore the geometric properties of graphs generated from the Hopf fibrations.

#### 3.1 Complex Hopf fibrations

Let us regard the unit euclidean sphere \( S^{2n+1} \) as a subset of \( \mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2} \), with center at the origin, and denote by \( J \) its standard complex structure, i.e. left multiplication with \( i \in \mathbb{C} \). The vector field

\[
\zeta = -Jp,
\]

where \( p \) is the position vector of \( S^{2n+1} \), is globally defined, and is called the *Reeb vector field* of the sphere. Moreover, for any vector field \( v \) on \( S^{2n+1} \), the decomposition in tangent and normal components determines a \( (1, 1) \)-tensor field \( \varphi \) and a 1-form \( \eta \) on \( S^{2n+1} \) such that

\[
Jv = \varphi(v) + \eta(v)p = \varphi(v) + \langle v, \zeta \rangle p,
\]

where \( \langle \cdot, \cdot \rangle \) is the euclidean metric. One can easily check that \( \varphi \) and \( \eta \) satisfy the following properties:

\[
\varphi(\zeta) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\zeta) = 1 \quad \text{and} \quad \varphi^2 = -I + \eta \otimes \zeta,
\]
where $I$ is the identity transformation. Moreover, it holds
\[
\langle \varphi(v_1), \varphi(v_2) \rangle = \langle v_1, v_2 \rangle - \eta(v_1)\eta(v_2),
\]
for $v_1, v_2 \in \mathcal{X}(S^{2n+1})$. Note that $(\eta, \zeta, \varphi)$ is the standard contact structure of the euclidean sphere $S^{2n+1}$.

By a direct computation we see that for any tangent vector field on the sphere, we have
\[
\nabla_{v}\zeta = -D_v Jp - \langle v, Jp \rangle p = -JD_v p + \langle Jv, p \rangle p = -Jv + \eta(v) p.
\]
where $D$ is the euclidean Levi-Civita connection. Hence, from (9) we get
\[
\varphi(v) = -\nabla_{v}^{2n+1} \zeta,
\]
from where we deduce that the integral curves of the Reeb vector field are geodesics of the sphere. Let now $\mathcal{V}$ be the sub-bundle generated by the Reeb vector field and by $\mathcal{H}$ its orthogonal complement. $\mathcal{V}$ is called the vertical bundle and $\mathcal{H}$ the horizontal bundle. Hence, for any $x \in S^{2n+1}$, we have the orthogonal decomposition
\[
T_x S^{2n+1} = \mathcal{V}_x \oplus \mathcal{H}_x.
\]
Hence, tensor $\varphi$ is measuring how the distribution $\mathcal{H}$ is twisted within the tangent bundle of the sphere.

The complex projective space $\mathbb{CP}^n$ is defined as the set of equivalence classes of $\mathbb{C}^{n+1} - \{0\}$ under the equivalence relation $\sim$ defined by $z \sim w$ if there is a nonzero element $\lambda \in \mathbb{C}$ that $z = \lambda w$. Equivalently, we may regard $\mathbb{CP}^n$ as the quotient of $S^{2n+1}$ under the group action of $S^1 \subset \mathbb{C}$ given by
\[
(z_1, \ldots, z_{n+1})e^{i\theta} = (z_1e^{i\theta}, \ldots, z_{n+1}e^{i\theta}).
\]
The natural quotient map $f : S^{2n+1} \rightarrow \mathbb{CP}^n$ is the so-called Hopf fibration. The map $f$ is a submersion and clearly its fibers are great circles. As a matter of fact, the kernel of the differential $df$ is spanned by the Reeb vector field. We can endow $\mathbb{CP}^n$ with a Riemannian metric $g_{\mathbb{CP}^n}$, the so-called Fubini-Study metric, which makes $f$ a Riemannian submersion. With this Riemannian metric, the complex projective space $\mathbb{CP}^n$ becomes a Kähler manifold with complex structure $J_{\mathbb{CP}^n}$ given by the formula
\[
df \circ \varphi = J_{\mathbb{CP}^n} \circ df.
\]
Due to a theorem of Escobales [15,16] (see also [17, Chapter 2, Theorem 2.6 and Theorem 2.7]) the complex Hopf fibration is rigid among all Riemannian submersions with totally geodesic fibers in the following sense:

**Theorem 1** Let $g : S^{2n+1} \rightarrow \mathbb{CP}^n$ be a Riemannian submersion with totally geodesic fibers and $f : S^{2n+1} \rightarrow \mathbb{CP}^n$ the complex Hopf fibration. Then, there exists an isometry $T_1$ of the sphere $S^{2n+1}$ and an isometry $T_2$ of $\mathbb{CP}^n$ such that $g = T_2 \circ f \circ T_1$.

Let us consider now the graph over Hopf fibration and explore its geometry.

**Proposition 1** Let $f : S^{2n+1} \rightarrow \mathbb{CP}^n$ be the complex Hopf fibration. Then, the following statements hold:

(a) The second fundamental form $A$ of the graph $\Gamma(f)$ is zero along the kernel of $df$ as well as along its orthogonal complement on the sphere.
(b) The graph $\Gamma(f)$ is a minimal submanifold of the product $S^{2n+1} \times \mathbb{C}P^n$ with squared norm of the second fundamental form $|A|^2 = n$.

**Proof** Recall that the kernel of $df$ is generated by the Reeb vector field $\xi$. Moreover, the singular values of $df$ are $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_{2n+1} = 1$. Using the Koszul formula and the fact that $f$ is a Riemannian submersion we deduce that for any pair of horizontal vector fields $v_1, v_2 \in \mathcal{H}$ we have

$$B(v_1, v_2) = \nabla^f_{v_1} df(v_2) - df(\nabla^g_{v_1} v_2) = 0,$$

for details see also [4, Lemma 4.5.1, page 119]. Hence, $B$ vanishes on the horizontal bundle. Since the Reeb vector field has geodesic integral curves and it belongs to the kernel of $df$, we obtain that

$$B(\xi, \xi) = 0.$$

and

$$B(\xi, v) = -df(\nabla^g_v \xi) = df(\varphi(v)),$$

for any horizontal vector $v$.

From Lemma 2 now, we deduce that $A$ vanishes also in the horizontal bundle $\mathcal{H}$ as well as in the vertical bundle $\mathcal{V}$. Therefore, $\Gamma(f)$ is a minimal submanifold. For the mixed terms, again from Lemma 2, we get that

$$A(\xi, v) = \frac{-\varphi(v) \oplus df(\varphi(v))}{2}.$$

Recall that the restriction of $\varphi$ on the horizontal bundle is an isometry with respect to $g_{S^{2n+1}}$. Since $f$ is a Riemannian submersion, $\varphi$ is an isometry also with respect to the graphical metric. Indeed, for any horizontal vectors $v_1$ and $v_2$ we have

$$g(\varphi(v_1), \varphi(v_2)) = g_{S^{2n+1}}(\varphi(v_1), \varphi(v_2)) + g_{\mathbb{C}P^n}(df(\varphi(v_1)), df(\varphi(v_2)))$$

$$= g_{S^{2n+1}}(v_1, v_2) + g_{S^{2n+1}}(\varphi(v_1), \varphi(v_2)) = 2 g_{S^{2n+1}}(v_1, v_2)$$

$$= g(v_1, v_2).$$

(14)

Let $\{\xi; e_1, \ldots, e_n; e_{n+1} = \varphi(e_1), \ldots, e_{2n} = \varphi(e_n)\}$ be a local orthonormal frame with respect to $g$, where $\{e_1, \ldots, e_{2n}\}$ span the horizontal space. Then, from (13) and (14) we obtain that

$$|A|^2 = 2 \sum_{i=1}^{2n} |A(\xi, e_i)|^2 = n.$$

This completes the proof. \[\square\]

**Remark 1** In the special case where $n = 1$, the Hopf fibration is a minimal Riemannian submersion between spheres. By scaling properly the target in order to become a unit sphere, we obtain a minimal submersion $f : S^3 \to S^2$ with constant singular values $\lambda_1 = 0$ and $\lambda_2 = 2 = \lambda_3$. In this case, there exists an adapted orthonormal frame $\{e_1 = \xi, e_2, e_3; \xi_2, \xi_3\}$, with respect to which the shape operators $A^{\xi_2}$ and $A^{\xi_3}$ are given by

$$A^{\xi_2} = \begin{pmatrix} 0 & 0 & 2/5 \\ 0 & 0 & 0 \\ 2/5 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^{\xi_3} = \begin{pmatrix} 0 & -2/5 & 0 \\ -2/5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Moreover, the squared norm of the second fundamental form is equal to $|A|^2 = 16/25$. 

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We show now that there are plenty of weakly conformal minimal maps \( f : S^{2n+1} \to \mathbb{CP}^n \) other than the complex Hopf fibration.

**Proposition 2** Let \( f : S^{2n+1} \to \mathbb{CP}^n \) be the complex Hopf fibration and let \( g : \mathbb{CP}^n \to \mathbb{CP}^n \) be a holomorphic map. Then, the composition \( G = g \circ f \) is again a weakly conformal minimal map. If, additionally, \( g \) is a local diffeomorphism then, \( G \) has great circle fibers. In the special case \( n = 1 \), the squared norm of the second fundamental form \( A_G \) of the graph \( \Gamma(G) \) is

\[
|A_G|^2 = 4 \frac{\lambda^2 (1 + \lambda^2) + |\nabla_{\mathbb{CP}^n} \lambda|^2}{(1 + \lambda^2)^3},
\]

where \( \lambda \) is the conformal factor of \( g \).

**Proof** At first, note that the Reeb vector field \( \zeta \) is unit with respect to the graphical metric too. Moreover, the spaces \( V \) and \( H \) are again perpendicular with respect to the graphical metric. Furthermore, the differential of the map \( G \) commutes with \( \varphi \) and the complex structure of \( \mathbb{CP}^n \). From this observation, it follows that the restriction of \( \varphi \) on the horizontal bundle \( H \) is an isometry also with respect to the induced metric \( g \). By a straightforward computation, the Hessian of \( G \) is given by

\[
B_G(v_1, v_2) = B_g(df(v_1), df(v_2)) + dg(B_f(v_1, v_2)), \quad (15)
\]

for any vector fields \( v_1 \) and \( v_2 \) of the sphere \( S^{2n+1} \). Hence,

\[
B_G(\zeta, \zeta) = B_g(df(\zeta), df(\zeta)) + dg(B_f(\zeta, \zeta)) = 0.
\]

From Lemma 2 we obtain that \( A(\zeta, \zeta) = 0 \). Consequently,

\[
\nabla_{\zeta}^g \zeta = \nabla^{S^{2n+1}}_{\zeta} \zeta = 0,
\]

from where it follows that the integral curves of \( \zeta \) are geodesics also with respect to the graphical metric. Moreover, because the map \( g \) is holomorphic, \( f \) is a Riemannian submersion, and (10), we have that

\[
B_G(v, v) + B_G(\varphi(v), \varphi(v)) = B_g(df(v), df(v)) + B_g(df(\varphi(v)), df(\varphi(v)))
\]

\[
= B_g(df(v), df(v)) + B_g(J_{\mathbb{CP}^n} df(v), J_{\mathbb{CP}^n} df(v))
\]

\[
= B_g(df(v), df(v)) - B_g(df(v), df(v)) = 0,
\]

for any \( v \in H \). Consequently, from Lemma 2 it follows that

\[
A_G(v, v) + A_G(\varphi(v), \varphi(v)) = 0,
\]

for any horizontal vector \( v \). Since \( \varphi \) satisfies \( \varphi^2 = -I_{2n} \) on \( H \), we can always find a local orthonormal frame of the form

\[
\{ e_1 = \zeta, \ e_i, \ e_{n+i} = \varphi(e_i) \}_{i \in \{2, \ldots, n+1\}}
\]

with respect to the graphical metric \( g \). Hence,

\[
A_G(e_i, e_i) + A_G(e_{n+i}, e_{n+i}) = 0,
\]

for any \( i \in \{2, \ldots, n+1\} \). Hence, the composition \( G = g \circ f \) is again a minimal map.

Let us examine now the case where \( f : S^3 \to \mathbb{CP}^1 \) and \( g : \mathbb{CP}^1 \to \mathbb{CP}^1 \) is a holomorphic map. In this case, the map \( g \) is conformal. Let us denote by \( \lambda \) its conformal factor. Consider
at a fixed point $x \in \mathbb{S}^3$ an orthonormal with respect to $g_{\mathbb{S}^3}$ frame $\{\alpha_1 = \zeta, \alpha_2, \alpha_3 = \varphi(\alpha_2)\}$ and at the point $f(x)$ an orthonormal with respect to $g_{\mathbb{CP}^1}$ frame $\{\beta_2, \beta_3\}$ that diagonalizes the differential of $f$. Moreover, consider at $G(x)$ an orthonormal, with respect to $g_{\mathbb{CP}^1}$, frame $\{\tilde{\beta}_2, \tilde{\beta}_3\}$ such that

$$dg(\beta_2) = \lambda \tilde{\beta}_2 \quad \text{and} \quad dg(\beta_3) = \lambda \tilde{\beta}_3.$$ 

Note that the frames described above satisfy $J_{\mathbb{CP}^1} \beta_2 = \beta_3$ and $J_{\mathbb{CP}^1} \tilde{\beta}_2 = \tilde{\beta}_3$. Moreover, the vectors

$$e_1 = \zeta, \quad e_2 = \frac{\alpha_2}{\sqrt{1 + \lambda^2}} \quad \text{and} \quad e_3 = \frac{\alpha_3}{\sqrt{1 + \lambda^2}},$$

an orthonormal frame with respect to the graphical metric. Since,

$$dG(e_1) = 0, \quad dG(e_2) = \frac{\lambda \tilde{\beta}_2}{\sqrt{1 + \lambda^2}} \quad \text{and} \quad dG(e_3) = \frac{\lambda \tilde{\beta}_3}{\sqrt{1 + \lambda^2}},$$

the map $G$ is weakly conformal. Assume that $\lambda$ is not identically zero, since otherwise we have nothing to show. By Koszul formula we get that, away from the zero set of $\lambda$, it holds

$$B_g(\varepsilon_1, \varepsilon_2) = \varepsilon_1 (\log \lambda) dg(\varepsilon_2) + \varepsilon_2 (\log \lambda) dg(\varepsilon_1) - g_{\mathbb{CP}^1}(\varepsilon_1, \varepsilon_2) dg(\nabla_{\mathbb{CP}^1} \log \lambda),$$

where $\varepsilon_1, \varepsilon_2$ are tangent vectors of $\mathbb{CP}^1$. Hence, from (15), (10), (11) and (12) and we deduce that

$$B_G(\alpha_1, \alpha_1) = 0, \quad B_G(\alpha_1, \alpha_2) = \lambda \tilde{\beta}_3 \quad \text{and} \quad B_G(\alpha_1, \alpha_3) = -\lambda \tilde{\beta}_2.$$ 

Furthermore,

$$B_G(\alpha_2, \alpha_2) = \beta_2(\lambda) \tilde{\beta}_2 - \beta_3(\lambda) \tilde{\beta}_3 = -B_G(\alpha_3, \alpha_3)$$

and

$$B_G(\alpha_2, \alpha_3) = \beta_3(\lambda) \tilde{\beta}_2 + \beta_2(\lambda) \tilde{\beta}_3.$$ 

From Lemma 2, it follows that

$$A_G(e_1, e_1) = 0, \quad A_G(e_1, e_2) = \frac{\lambda \xi_3}{1 + \lambda^2} \quad \text{and} \quad A_G(e_1, e_3) = \frac{-\lambda \xi_2}{1 + \lambda^2}.$$ 

Moreover,

$$A_G(e_2, e_2) = \frac{\beta_2(\lambda) \xi_2 - \beta_3(\lambda) \xi_3}{(1 + \lambda^2)^{3/2}} = -A_G(e_3, e_3)$$

and

$$A_G(e_2, e_3) = \frac{\beta_3(\lambda) \xi_2 + \beta_2(\lambda) \xi_3}{(1 + \lambda^2)^{3/2}}.$$ 

Therefore,

$$|A_G|^2 = \frac{4\lambda^2}{(1 + \lambda^2)^2} + \frac{4|\nabla_{\mathbb{CP}^1} \lambda|^2}{(1 + \lambda^2)^2}.$$ 

This completes the proof. □
Remark 2 There is a plethora of surjective holomorphic maps $g : \mathbb{C}P^n \to \mathbb{C}P^n$ which are not isometries. On the other hand, it is a well-known fact in Algebraic Geometry that there are no non-constant holomorphic maps $g : \mathbb{C}P^n \to \mathbb{C}P^n$, if $n > m$. Moreover, there are many examples of conformal diffeomorphisms of $S^2 \simeq \mathbb{C}P^1$ which are not rotations. In fact, the subgroup of orientation preserving conformal diffeomorphisms of $S^2$ is isomorphic with the projective linear group $PGL_2(\mathbb{C})$, which contains $SO(3)$ as a maximal compact subgroup.

Remark 3 We expect that any minimal submersion $f : S^3 \to S^2$ should have totally geodesic fibers and arise as the composition of the Hopf fibration with a holomorphic automorphism. This is the case where the minimal map is already weakly conformal. Another interesting question is under which conditions a minimal submersion $f : S^{2n+1} \to \mathbb{C}P^n$ coincides with the Hopf fibration.

3.2 Quaternionic Hopf fibrations

As a vector space, the quaternions are

$$\mathbb{Q} = \{a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$  

They become an associative algebra with 1 as the multiplicative unit via

$$i_1^2 = i_2^2 = i_3^2 = -1, \quad i_1 i_2 = i_3 = -i_2 i_1$$
and cyclic permutations. Consider now $S^{4n+3}$ as a hypersurface of $\mathbb{Q}^{n+1} \simeq \mathbb{R}^{4n+4}$ with center at the origin. Let us consider the complex structures $J_1, J_2, J_3$ on $\mathbb{R}^{4n+4}$, respectively, induced by the left multiplication for $i_1, i_2, i_3$ in $\mathbb{Q}$. Each of these complex structures when applied to the position vector $p$ of the sphere gives a globally defined vector field

$$\zeta_i = -J_i p,$$
where $i \in \{1, 2, 3\}$, on $S^{4n+3}$. As in the complex case, we denote by $\eta_i$ the dual form associated with $\zeta_i$ and by $\varphi_i$ the $(1, 1)$-tensors given by

$$J_i v = \varphi_i(v) + \eta_i(v) p,$$
for any $i \in \{1, 2, 3\}$ and $v \in \mathcal{X}(S^{4n+3})$. Clearly, each tensor $\varphi_i$ satisfies

$$\varphi_i^2 = -I_{4n+3} + \eta_i \otimes \zeta_i$$

The pairs $\{\eta_i, \zeta_i, \varphi_i\}_{i \in \{1, 2, 3\}}$, give rise to the standard 3-Sasakian structure on $S^{4n+3}$. The 1-forms $\eta_i$ and the $(1, 1)$-tensors $\varphi_i$ are related through the identities

$$\varphi_k = \varphi_i \circ \varphi_j - \eta_j \otimes \zeta_i = -\varphi_j \circ \varphi_i + \eta_i \otimes \zeta_j,$$
$$\zeta_k = \varphi_i(\zeta_j) = -\varphi_j(\zeta_i),$$
$$\eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i,$$

for an even permutation $(i, j, k)$ of $(1, 2, 3)$. Moreover, from the Weingarten formula we obtain

$$\varphi_i(v) = -\nabla_v^{S^{4n+3}} \zeta_i,$$
for any \( i \in \{1, 2, 3\} \) and \( v \in \mathcal{X}(\mathbb{S}^{4n+3}) \). By using (16), we get

\[
\nabla_{\xi_1}^{\mathbb{S}^{4n+3}} \xi_1 = 0, \quad \nabla_{\xi_1}^{\mathbb{S}^{4n+3}} \xi_2 = \xi_3, \quad \nabla_{\xi_1}^{\mathbb{S}^{4n+3}} \xi_3 = -\xi_2, \\
\nabla_{\xi_2}^{\mathbb{S}^{4n+3}} \xi_1 = -\xi_3, \quad \nabla_{\xi_2}^{\mathbb{S}^{4n+3}} \xi_2 = 0, \quad \nabla_{\xi_2}^{\mathbb{S}^{4n+3}} \xi_3 = \xi_1, \\
\nabla_{\xi_3}^{\mathbb{S}^{4n+3}} \xi_1 = \xi_2, \quad \nabla_{\xi_3}^{\mathbb{S}^{4n+3}} \xi_2 = -\xi_1, \quad \nabla_{\xi_3}^{\mathbb{S}^{4n+3}} \xi_3 = 0,
\]

(17)

and so the integral curves of the vector fields \( \xi_i, \ i \in \{1, 2, 3\} \), are geodesic circles.

The distribution \( \mathcal{V} \) formed by \( \xi_1, \xi_2 \) and \( \xi_3 \) is integrable and its leaves are geodesic 3-spheres of \( \mathbb{S}^{4n+3} \). As usual, vector fields on \( \mathcal{V} \) are called vertical and vector fields on the orthogonal complement \( \mathcal{H} \) of \( \mathcal{V} \) are called horizontal.

The quaternionic projective space \( \mathbb{QP}^n \) is defined as vectors in \( \mathbb{Q}^{n+1} \setminus \{0\} \) modulo left scalar multiplication. The space \( \mathbb{QP}^n \) can be obtained also by identifying the leaves of the vertical distribution \( \mathcal{V} \) on the sphere \( \mathbb{S}^{4n+3} \). As in the complex case, we obtain a natural projection \( f : \mathbb{S}^{4n+3} \rightarrow \mathbb{QP}^n \) which can be made into a Riemannian submersion. The map \( f \) is called quaternionic Hopf fibration. Equivalently, \( \mathbb{QP}^n \) can be regarded as the set of orbits of the natural group action of \( \mathbb{S}^3 \subset \mathbb{Q} \) on the sphere \( \mathbb{S}^{4n+3} \subset \mathbb{Q}^{n+1} \). Let us mention here that \( \mathbb{QP}^1 \) with its canonical metric is isometric to the round 4-sphere of radius 1/2.

**Proposition 3** Let \( f : \mathbb{S}^{4n+3} \rightarrow \mathbb{QP}^n \) be the quaternionic Hopf fibration. Then, the following statements hold:

(a) The second fundamental form \( A \) of the graph \( \Gamma(f) \) is zero along the kernel of \( df \) as well as along its orthogonal complement on the sphere.

(b) The graph \( \Gamma(f) \) is a minimal submanifold of the product \( \mathbb{S}^{4n+3} \times \mathbb{QP}^n \) with squared norm of the second fundamental form \( |A|^2 = 6n \).

**Proof** As in the complex case, we can show that the spaces \( \mathcal{V} \) and \( \mathcal{H} \) are again perpendicular with respect to the graphical metric \( g \). Moreover, the restriction of each \( \varphi_i, \ i \in \{1, 2, 3\} \), on \( \mathcal{H} \) is an isometry with respect to both metrics \( g_{\mathbb{S}^{4n+3}} \) and \( g \). Since \( f \) is Riemannian submersion, from the Koszul formula, it follows that the Hessian \( B \) of \( f \) vanishes on the horizontal bundle \( \mathcal{H} \). Additionally, since \( \mathcal{V} \) form the kernel of \( df \), from (17) we get

\[
B(\xi_i, \xi_j) = 0 \quad \text{and} \quad B(\xi_i, v) = df(\varphi_i(v))
\]

for any \( i, j \in \{1, 2, 3\} \) and \( v \in \mathcal{H} \). According to Lemma 2, we obtain that \( A \) vanishes on \( \mathcal{V} \) and \( \mathcal{H} \) and that

\[
A(\xi_i, v) = \frac{-\varphi_i(v) \oplus df(\varphi_i(v))}{2}
\]

(18)

for any \( i \in \{1, 2, 3\} \) and \( v \in \mathcal{H} \). Consider an orthonormal, with respect to the graphical metric \( g \), frame of the form

\[
\{e_i = \xi_i, \ e_l, \ \varphi_l(e_l)\}_{i \in \{1, 2, 3\}; l \in \{4, \ldots, n+3\}}
\]

on the sphere. Then, from (18) we obtain that

\[
|A|^2 = 6n.
\]

This completes the proof. \( \square \)
3.3 Octonionic Hopf fibration

Let us describe here the octonionic Hopf fibration. Recall at first that as a vector space, the octonions \( \mathbb{O} \) are described by

\[
\mathbb{O} = \{ a_0 i_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 + a_4 i_4 + a_5 i_5 + a_6 i_6 + a_7 i_7 : a_0, a_1, \ldots, a_7 \in \mathbb{R} \}.
\]

We can turn the space \( \mathbb{O} \) of octonions into a nonassociative algebra with \( i_0 \) as multiplicative unit. The standard canonical basis \( \{ i_0, i_1, \ldots, i_7 \} \) for \( \mathbb{O} \) satisfies the following multiplication rules:

\[
i_m i_n = \begin{cases} 
i_n, & \text{if } m = 0, \\
i_m, & \text{if } n = 0, \\
-\delta_{mn} i_0 + \epsilon_{mnk} i_k, & \text{otherwise},
\end{cases}
\]

where \( \epsilon_{mnk} \) is the completely antisymmetric tensor with value 1 when

\[(m, n, k) = (1, 2, 3), (1, 4, 5), (1, 7, 6), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 6, 5).\]

For \( n \geq 2 \), the naive definition of \( \mathbb{O} \mathbb{P}^n \) as vectors in \( \mathbb{O}^{n+1} - \{0\} \) modulo left scalar multiplication has the problem that the equality up to left scalar multiplication fails to be an equivalence relation.

Consider now the unit sphere \( \mathbb{S}^{15} \) as a hypersurface of \( \mathbb{O}^2 \simeq \mathbb{R}^{16} \). Let us also consider the complex structures \( J_1, J_2, \ldots, J_7 \) on \( \mathbb{R}^{16} \), respectively, induced by the left multiplication for \( i_1, i_2, \ldots, i_7 \) in \( \mathbb{O} \). Each of these complex structures when applied to the position vector \( p \) of the sphere gives a globally defined vector field

\[
\zeta_i = -J_i p,
\]

where \( i \in \{1, 2, \ldots, 7\} \), on \( \mathbb{S}^{15} \). As in the complex and quaternionic case, we denote by \( \eta_i \) the dual form associated with \( \zeta_i \) and by \( \varphi_i \) the \((1, 1)\)-tensor given by

\[
J_i v = \varphi_i(v) + \eta_i(v) p,
\]

for \( i \in \{1, 2, \ldots, 7\} \) and \( v \in \mathcal{X}(\mathbb{S}^{15}) \). Since \( \mathbb{O} \) is a nonassociative algebra, the tensors \( \varphi_i \) do not satisfy similar relations as in the quaternionic case.

The distribution \( \mathcal{V} \) formed by \( \zeta_1, \zeta_2, \ldots, \zeta_7 \) is integrable and its leaves are geodesic 7-spheres of \( \mathbb{S}^{15} \). As usual, vector fields on \( \mathcal{V} \) are called vertical and vector fields on the orthogonal complement \( \mathcal{H} \) of \( \mathcal{V} \) are called horizontal.

**Proposition 4** Let \( f : \mathbb{S}^{15} \to \mathbb{O} \mathbb{P}^1 \simeq \mathbb{S}^8(1/2) \) be the octonionic Hopf fibration. Then, the following statements hold:

(a) The second fundamental form \( A \) of the graph \( \Gamma(f) \) is zero along the kernel of \( df \) as well as along its orthogonal complement on the sphere.

(b) The graph \( \Gamma(f) \) is a minimal submanifold of the product \( \mathbb{S}^{15} \times \mathbb{S}^8(1/2) \) with squared norm of the second fundamental form \( |A|^2 = 28 \).

**Proof** As in the quaternionic case, we can show that the spaces \( \mathcal{V} \) and \( \mathcal{H} \) are again perpendicular with respect to the graphical metric \( g \). Moreover, the restriction of each \( \varphi_i \), \( i \in \{1, 2, \ldots, 7\} \), on \( \mathcal{H} \) is an isometry with respect to both metrics \( g_{\mathbb{S}^{15}} \) and \( g \). Since \( f \) is Riemannian submersion with totally geodesic fibers, we get

\[
B(\zeta_i, \zeta_j) = 0 \quad \text{and} \quad B(\zeta_i, v) = df(\varphi_i(v))
\]
for any $i, j \in \{1, 2, \ldots, 7\}$ and $v \in \mathcal{H}$. According to Lemma 2, we obtain that $A$ vanishes on $\mathcal{V}$ and $\mathcal{H}$ and that

$$A(\zeta_i, v) = -\varphi_i(v) \oplus df(\varphi_i(v))$$

for any $i \in \{1, 2, \ldots, 7\}$ and $v \in \mathcal{H}$. Consider an orthonormal, with respect to the graphical metric $g$, frame of the form $\{\zeta_i, e_1, e_2, \ldots, e_8\}_{i \in \{1, \ldots, 7\}}$ on $S^3$, where $\{e_1, \ldots, e_8\} \in \mathcal{H}$. Then, we obtain that $|A|^2 = 28$. 

\[\square\]

4 Equivariant minimal maps

Let us regard $S^3$ as a subset of $\mathbb{C} \times \mathbb{C}$, i.e.

$$S^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1|^2 + |z_2|^2 = 1\}$$

and $S^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e.

$$S^2 = \{(z, \tau) \in \mathbb{C} \times \mathbb{R} : |z|^2 + \tau^2 = 1\}.$$

Recall that the standard action of $S^1 \times S^1$ on $S^3$ is given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = (e^{i\theta_1} \cdot z_1, e^{i\theta_2} \cdot z_2)$$

and the standard action of $S^1$ on $S^2$ is given by

$$e^{i\theta} \cdot (z, \tau) = (e^{i\theta} \cdot z, \tau).$$

Define now the multiplication $\varrho_{kl} : S^1 \times S^1 \to S^1$, $(k, l) \in \mathbb{Z} \times \mathbb{Z}$, given by

$$\varrho_{kl}(e^{i\theta_1}, e^{i\theta_2}) = e^{i(k\theta_1 + l\theta_2)}.$$  

A map $f : S^3 \to S^2$ is called equivariant with respect to $\varrho_{kl}$ if

$$f(e^{i\theta_1}, e^{i\theta_2})(z_1, z_2) = \varrho_{kl}(e^{i\theta_1}, e^{i\theta_2}) f(z_1, z_2).$$

Let see now how we can describe such equivariant maps. To this end, we parametrize points in the sphere $S^3$ by $(\sin s \cdot e^{i\xi}, \cos s \cdot e^{i\eta})$, where $(\xi, \eta) \in (0, 2\pi) \times (0, 2\pi)$ and $s \in [0, \pi/2]$, and points in $S^2$ by $(\sin a \cdot e^{i\sigma}, \cos a)$, where $\sigma \in (0, 2\pi)$ and $a \in (0, \pi)$. Then, one can easily see that such an equivariant map $f$ can be represented in the form

$$f_{kl}(\xi, \eta, s) = f(\sin s \cdot e^{i\xi}, \cos s \cdot e^{i\eta}) = (\sin a(s) \cdot e^{i(k\xi + l\eta)}, \cos a(s)),$$  \hspace{1cm} (19)

where $a : [0, \pi/2] \mapsto [0, \pi]$ is a function satisfying the boundary conditions $a(0) = 0$ and $a(\pi/2) = \pi$. The function $a$ is called generating function. Maps of the form (19) are also called $a$-Hopf constructions; see [14, Chapter X]. In the case $a(s) = 2s$, up to a dilation of $S^2$, we obtain the Hopf fibration.

Lemma 3 Let $f_{kl} : S^3 \to S^2$ be an $a$-Hopf construction. Then, the singular values of $df_{kl}$ at a point $(\sin s \cdot e^{i\xi}, \cos s \cdot e^{i\eta})$ are

$$\lambda_1 = 0, \quad \lambda_2 = \sqrt{\frac{k^2 \sin^2 a}{\sin^2 s} + \frac{l^2 \sin^2 a}{\cos^2 s}} \quad \text{and} \quad \lambda_3 = |a_s|.$$
In particular, the map $f_{kl}$ is weakly conformal if and only if

$$a(s) = \begin{cases} 
2 \arctan(c \tan^k s), & \text{if } l = k, \\
2 \arctan \left( c \frac{\cosec - \sqrt{l^2 \cot^2 s + k^2}}{\sqrt{k^2 \tan^2 s + l^2 - k \cosec}} \right), & \text{if } l > k,
\end{cases}$$

where $c$ is a positive constant and $s \in [0, \pi/2]$.

**Proof** Note at first that with respect to the coordinate systems introduced above, the metrics of $\mathbb{S}^3$ and $\mathbb{S}^2$ are given by the expressions

$$g_{\mathbb{S}^3} = \sin^2 s d\xi^2 + \cos^2 s d\eta^2 + ds^2 \quad \text{and} \quad g_{\mathbb{S}^2} = \sin^2 \tau d\sigma^2 + d\tau^2,$$

respectively. One can easily verify that the vector fields

$$v_1 = \frac{\partial_\xi}{\sin s}, \quad v_2 = \frac{\partial_\eta}{\cos s} \quad \text{and} \quad v_3 = \partial_s$$

constitute an orthonormal basis of $T(\xi, \eta, s)\mathbb{S}^3$. Similarly, the vector fields

$$w_1 = \frac{\partial_\sigma}{\sin \tau} \quad \text{and} \quad w_2 = \partial_\tau$$

constitute an orthonormal basis of $T(\sigma, \tau)\mathbb{S}^2$. Let us compute the differential of $f_{kl}$ now. We have,

$$df_{kl}(v_1) = \frac{df_{kl}(\partial_\xi)}{\sin s} = \frac{k \partial_\sigma}{\sin s} = \frac{k \sin a}{\sin s} w_1,$$

$$df_{kl}(v_2) = \frac{df_{kl}(\partial_\eta)}{\cos s} = \frac{l \partial_\sigma}{\cos s} = \frac{l \sin a}{\cos s} w_1,$$

and

$$df_{kl}(v_3) = df_{kl}(\partial_s) = a_s \partial_\tau = -a_s (-\partial_\tau).$$

Diagonalizing $f_{kl}^* g_{\mathbb{S}^2}$ with respect to $g_{\mathbb{S}^3}$ we see that the squares of the singular values of the differential $df_{kl}$ at an arbitrary point $\Pi(\xi, \eta, s)$ are

$$\lambda_1^2 = 0, \quad \lambda_2^2 = \frac{k^2 \sin^2 a}{\tan^2 s} + \frac{l^2 \sin^2 a}{\cos^2 s} \quad \text{and} \quad \lambda_3^2 = a_s^2$$

with eigendirections

$$\alpha_1 = \frac{l (\sin s) v_1 - k (\cos s) v_2}{\sqrt{l^2 \sin^2 s + k^2 \cos^2 s}}, \quad \alpha_2 = \frac{k (\cos s) v_1 + l (\sin s) v_2}{\sqrt{l^2 \sin^2 s + k^2 \cos^2 s}}, \quad \alpha_3 = \partial_s,$$

respectively.

Taking into account that the values of $a$ are on $[0, \pi]$ that $a(0) = 0$ and $a(\pi/2) = \pi$, we deduce that $f_{kl}$ is weakly conformal if and only if

$$a_s = \sin a \sqrt{\frac{k^2 \cos^2 s + l^2 \sin^2 s}{\cos^2 s \sin^2 s}}.$$

Integrating, it follows that $f_{kl}$ is weakly conformal if and only if

$$a(s) = \begin{cases} 
2 \arctan(c \tan^k s), & \text{if } l = k, \\
2 \arctan \left( c \frac{\cosec - \sqrt{l^2 \cot^2 s + k^2}}{\sqrt{k^2 \tan^2 s + l^2 - k \cosec}} \right), & \text{if } l > k,
\end{cases}$$
where \( c \) is a positive constant and \( s \in [0, \pi/2] \). This completes the proof. \( \square \)

**Proposition 5** An \( a \)-Hopf construction \( f_{kl} : S^3 \to \mathbb{S}^2 \) is minimal if and only if the generating function \( a : [0, \pi/2] \to [0, \pi] \) satisfies the equation

\[
0 = \frac{a_{ss}}{1 + a_s^2} + \frac{\cos s \sin s (\cos 2s + (l^2 - k^2) \sin^2 a)}{\sin^2 s \cos^2 s + \sin^2 a (l^2 \sin^2 s + k^2 \cos^2 s)} a_s - \frac{\sin a \cos a (k^2 \cos^2 s + l^2 \sin^2 s)}{\sin^2 s \cos^2 s + \sin^2 a (l^2 \sin^2 s + k^2 \cos^2 s)},
\]

with boundary conditions \( a(0) = 0 \) and \( a(\pi/2) = \pi \). Moreover, an \( a \)-Hopf construction \( f_{kl} \) is weakly conformal if and only if \( k^2 = l^2 \).

**Proof** Let us consider the frames \( \{v_1, v_2, v_3\} \), \( \{\alpha_1, \alpha_2, \alpha_3\} \) and \( \{w_1, w_2\} \) as in (20) and (23) and (21), respectively. Note that

\[
[v_1, v_2] = 0, \quad [v_1, v_3] = (\cot s) v_1, \quad [v_2, v_3] = -(\tan s) v_2,
\]

and

\[
[w_1, w_2] = (\cot \tau) w_1.
\]

From the Koszul formula and the above Lie brackets, we easily get that

\[
\nabla_{v_1}^S v_1 = -(\cot s) v_3, \quad \nabla_{v_1}^S v_2 = 0, \quad \nabla_{v_1}^S v_3 = (\cot s) v_1,
\]

\[
\nabla_{v_2}^S v_1 = 0, \quad \nabla_{v_2}^S v_2 = (\tan s) v_3, \quad \nabla_{v_2}^S v_3 = -(\tan s) v_2,
\]

\[
\nabla_{v_3}^S v_1 = 0, \quad \nabla_{v_3}^S v_2 = 0, \quad \nabla_{v_3}^S v_3 = 0,
\]

and

\[
\nabla_{w_1}^S w_1 = -(\cot \tau) w_2, \quad \nabla_{w_1}^S w_2 = (\cot \tau) w_1, \quad \nabla_{w_1}^S w_1 = 0, \quad \nabla_{w_2}^S w_2 = 0.
\]

By a straightforward computation, we obtain the connection forms of the frame \( \{\alpha_1, \alpha_2, \alpha_3\} \) with respect to the Levi-Civita connection of \( S^3 \), namely

\[
\nabla_{\alpha_1}^S \alpha_1 = \frac{(k^2 - l^2) \sin s \cos s}{l^2 \sin^2 s + k^2 \cos^2 s} \alpha_3,
\]

\[
\nabla_{\alpha_1}^S \alpha_2 = -\frac{kl}{l^2 \sin^2 s + k^2 \cos^2 s} \alpha_3,
\]

\[
\nabla_{\alpha_1}^S \alpha_3 = \frac{k^2 \cos^2 s + l^2 \sin^2 s}{k^2 \cos^2 s + l^2 \sin^2 s} \alpha_2,
\]

and

\[
\nabla_{\alpha_2}^S \alpha_1 = \frac{(l^2 - k^2) \cos s \sin s}{l^2 \sin^2 s + k^2 \cos^2 s} \alpha_1 + \frac{kl}{l^2 \sin^2 s + k^2 \cos^2 s} \alpha_2,
\]

\[
\nabla_{\alpha_2}^S \alpha_2 = \frac{k^2 \cos^4 s - l^2 \sin^4 s}{k^2 \cos^2 s + l^2 \sin^2 s \cos s (l^2 \sin^2 s + k^2 \cos^2 a)} \alpha_2,
\]

\[
\nabla_{\alpha_3}^S \alpha_3 = 0.
\]

Now let us compute the second fundamental form of \( \Gamma(f_{kl}) \). According to (22), it suffices to compute at points where \( a_s > 0 \). Recall that, at a fixed point of the graph, the vectors

\[
e_1 = \alpha_1, \quad e_2 = \frac{\alpha_2}{\sqrt{1 + \lambda_2^2}}, \quad e_3 = \frac{\alpha_3}{\sqrt{1 + \lambda_3^2}}
\]
forms an orthonormal basis of \( \Gamma(f_{kl}) \) and
\[
\xi_4 = -\frac{\lambda_2 \alpha_2 \oplus w_1}{\sqrt{1 + \lambda_2^2}}, \quad \xi_5 = -\frac{\lambda_3 \alpha_3 \oplus w_2}{\sqrt{1 + \lambda_3^2}}
\]
forms an orthonormal basis in the normal bundle of the graph. From
\[
\mathbf{h}_{ij}^\alpha = \langle \nabla_{e_i} \nabla^F \mathbf{e}_j, \xi_\alpha \rangle_{\mathbb{S}^3 \times \mathbb{S}^2} = \langle \nabla_{e_i} \mathbf{e}_j, \nabla^\alpha \mathbf{f}_{kl}(\mathbf{e}_j), \xi_\alpha \rangle_{\mathbb{S}^3 \times \mathbb{S}^2}
\]
we get that
\[
\mathbf{h}_{411}^4 = \mathbf{h}_{112}^4 = \mathbf{h}_{22}^4 = \mathbf{h}_{33}^4 = 0,
\]
\[
\mathbf{h}_{413}^4 = -k l \lambda_2 \frac{(l^2 \sin^2 s + k^2 \cos^2 s)\sqrt{(1 + \lambda_2^2)(1 + \lambda_3^2)}}{(l^2 \sin^2 s + k^2 \cos^2 s)(1 + \lambda_2^2)(1 + \lambda_3^2)},
\]
\[
\mathbf{h}_{23}^4 = \frac{a_s \cos a \sin s \cos s(k^2 \cos^2 s + l^2 \sin^2 s) + \sin a(l^2 \sin^4 s - k^2 \cos^4 s)}{\sqrt{k^2 \cos^2 s + l^2 \sin^2 s}(\cos^2 s \sin^2 s + \sin^2 a(k^2 \cos^2 s + l^2 \sin^2 s))\sqrt{1 + \lambda_3^2}}.
\]
Additionally,
\[
\mathbf{h}_{11}^5 = \frac{a_s (l^2 - k^2) \cos s \sin s}{\sqrt{1 + \lambda_3^2(l^2 \sin^2 s + k^2 \cos^2 s)}},
\]
\[
\mathbf{h}_{12}^5 = \frac{k l \alpha S}{\sqrt{(1 + \lambda_2^2)(1 + \lambda_3^2)(l^2 \sin^2 s + k^2 \cos^2 s)}},
\]
\[
\mathbf{h}_{13}^5 = 0, \quad \mathbf{h}_{23}^5 = 0, \quad \mathbf{h}_{33}^5 = \frac{a_s S}{(1 + \alpha S)(1 + \lambda_3^2)},
\]
and
\[
\mathbf{h}_{22}^5 = \frac{a_s \sin s \cos s(k^2 \cos^4 s + l^2 \sin^4 s) - \sin a \cos a(k^2 \cos^2 s + l^2 \sin^2 s)^2}{\sqrt{1 + \lambda_3^2(l^2 \sin^2 s + k^2 \cos^2 s)(\cos^2 s \sin^2 s + \sin^2 a(k^2 \cos^2 s + l^2 \sin^2 s))}}.
\]
Since
\[
\mathbf{h}_{11}^4 = \mathbf{h}_{22}^4 = \mathbf{h}_{33}^4 = 0
\]
it follows that \( f_{kl} \) is a minimal map if and only if
\[
0 = \frac{a_s S}{1 + \alpha S} + \frac{\cos s \sin s(\cos 2s + (l^2 - k^2) \sin^2 a)}{\sin^2 s \cos^2 s + \sin^2 a(l^2 \sin^2 s + k^2 \cos^2 s)}a_s \sin a(k^2 \cos^2 s + l^2 \sin^2 s) - \frac{\cos s \sin s(\cos 2s + (l^2 - k^2) \sin^2 a)}{\sin^2 s \cos^2 s + \sin^2 a(l^2 \sin^2 s + k^2 \cos^2 s)}.
\]
Let \( f_{kl} \) be a weakly conformal minimal map. Then the function \( a \) satisfies
\[
a_s = \sin a \sqrt{\frac{k^2}{\sin^2 s} + \frac{l^2}{\cos^2 s}}.
\]
Differentiating (25), we get
\[ a_{ss} = \sin a \cos a \left( \frac{k^2 \cos^2 s + l^2 \sin^2 s}{\sin^2 s \cos^2 s} \right) + \sin a \left( \frac{l^2 \sin s}{\cos^2 s} - \frac{k^2 \cos s}{\sin^3 s} \right). \]  
(26)

Substituting (25) and (26) in (24), we get
\[ 0 = (l^2 - k^2) \left( \frac{1}{4} - \frac{\cos^2 2s}{4} + \left( k^2 \cos^2 s + l^2 \sin^2 s \right) \sin^2 a \right) \sin a \]
\[ = (l^2 - k^2) \left( \frac{\sin^2 2s}{4} + \left( k^2 \cos^2 s + l^2 \sin^2 s \right) \sin^2 a \right) \sin a. \]

Hence a weakly conformal map \( f_{kl} \) is minimal only if \( l^2 = k^2 \).

\[ \square \]

5 Structure equations of maps from \( S^3 \) to \( S^2 \)

We consider now maps \( f : S^3 \to S^2 \) between unit euclidean spheres. Assume that \( U \) is an open neighbourhood of \( S^3 \) where the kernel \( \mathcal{V} \) of \( df \) is a line bundle. Hence, in \( U \) two singular values of \( f \) are non-zero. For simplicity, let us denote them by \( 0 < \lambda \leq \mu \). Fix \( x \in U \) and consider an orthonormal basis \( \{ \alpha_1, \alpha_2, \alpha_3 \} \) of \( T_x S^3 \) with respect to the spherical metric and an orthonormal basis \( \{ \beta_2, \beta_3 \} \) of \( T_{f(x)} S^2 \) again with respect to the spherical metric such that \( df(\alpha_1) = 0, df(\alpha_2) = \lambda \beta_2, df(\alpha_3) = \mu \beta_3 \).

Then, the vectors
\[ e_1 = \alpha_1, \quad e_2 = \frac{\alpha_2}{\sqrt{1 + \lambda^2}}, \quad e_3 = \frac{\alpha_3}{\sqrt{1 + \mu^2}} \]
form an orthonormal basis of \( T_x S^3 \) with respect to \( g \). Moreover, the vectors
\[ \xi_4 = \frac{-\lambda \alpha_2 \oplus \beta_2}{\sqrt{1 + \lambda^2}} \quad \text{and} \quad \xi_5 = \frac{-\mu \alpha_3 \oplus \beta_3}{\sqrt{1 + \mu^2}} \]
constitute an orthonormal basis of the normal space of the graph \( F(f) \) at \( f(x) \). In the following, we adopt the following terminology
\[ b'^{ij}_{\alpha} = \langle B(\alpha_i, \alpha_j), \beta_{\alpha-2} \rangle_{S^2} \quad \text{and} \quad h'^{ij}_{\alpha} = \langle A(e_i, e_j), \xi_{\alpha} \rangle_{S^3 \times S^2} \]
for all \( i, j \in \{1, 2, 3\} \) and \( \alpha \in \{4, 5\} \).

5.1 Projections and Jacobians

There are several quantities that encode information about the geometry of the graph \( F : U \subset S^3 \to S^3 \times S^2 \). The first one is the Jacobian of the projection into the first factor. Namely, let \( \Omega_{S^3} \) be the volume form of \( S^3 \). We can extend to a parallel form on \( S^3 \times S^2 \) by pulling it back via the natural projection map \( \pi_{S^3} : S^3 \times S^2 \to S^3 \). That is, we define \( \Omega_1 = \pi_{S^3}^* \Omega_{S^3} \). Then, the Jacobian of the projection from the graph into \( S^3 \) is \( u_1 = * (F^* \Omega_1) \) where \( * \) stands for the Hodge star operator with respect to \( g \). In terms of the singular values of the map \( f \) the function \( u_1 \) has the form
\[ u_1 = \frac{1}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}}. \]
There is another important function that will play a crucial role in our analysis. Let us denote by $\mathcal{H}$ the orthogonal, with respect to the spherical metric, complement of $\mathcal{V}$ in $TU$. Observe that $\mathcal{H}$ is orthogonal to $\mathcal{V}$ also with respect to the graphical metric. Since $\mathcal{H}$ is a two-dimensional sub-bundle it posses a complex structure $\Omega\mathcal{H}$. Let $\Omega_{S^2}$ the volume form $S^2$ and $\Omega_2 = \pi^*_S \Omega_{S^2}$. One can readily check that the 2-form $F^* \Omega_2$ is non-zero only on $\mathcal{H}$. Consequently, it must be a multiple of the volume form $\Omega_{\mathcal{H}}$ of $(\mathcal{H}, g)$. This means that there exists a smooth function $u_2$ on $U$ such that

$$F^* \Omega_2 = u_2 \Omega_{\mathcal{H}}.$$  

Without loss of generality, we may assume that $u_2$ is positive on $U$. Then, in terms of the singular values of $f$ we have

$$u_2 = \frac{\lambda \mu}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}}.$$  

Note that the function $w = u_1 + u_2$ is less or equal than 1. In particular, if at some point $x_0 \in U$ it holds $w(x_0) = 1$ then $\lambda(x_0) = \mu(x_0)$. Hence, $w$ measures how much $f$ deviates from being weakly conformal.

### 5.2 The Gauss map of a submersion

Following Baird [3], the Gauss map $G : (S^3, g) \rightarrow G_1(S^3)$ of the submersion $f : (S^3, g) \rightarrow S^2$ associates to each point $x \in S^3$ the line spanned by $e_1$, where $G_1(S^3)$ denotes the Grassmann bundle over $S^3$. Consider the tensor $\varphi : TS^3 \rightarrow \mathcal{H} \subset TS^3$, given by

$$\varphi(v) = -\nabla_v e_1,$$

for all $v \in TS^3$. The tensor $\varphi$ is (minus) the covariant derivative of the Gauss map $G$ and describes how the complex line bundle $\mathcal{H}$ is twisted within $T S^3$.

Fix now an oriented orthonormal frame $\{e_1, e_2, e_3 = J_\mathcal{H} e_2\}$ of the singular value decomposition of $df$. Then, with respect to this frame, we may introduce the tensors $Re \varphi$, $Im \varphi : TS^3 \rightarrow \mathcal{H}$ given by

$$(Re \varphi)v = g(\varphi(v), e_2) e_2 \quad \text{and} \quad (Im \varphi)v = g(\varphi(v), J_\mathcal{H} e_2) J_\mathcal{H} e_2$$

for any $v \in TS^3$. Note that, at points where $\lambda \neq \mu$ the orthonormal frame $\{e_1, e_2, e_3\}$ is unique.

### 5.3 Local formulas

We will see now how all the notions defined above are related together.

**Lemma 4** Let $f : U \subset S^3 \rightarrow S^2$ be a submersion with non-zero singular values $\lambda$ and $\mu$. Then, on the open and dense subset of $U$ where $\lambda$ and $\mu$ are smooth, we have

$$b^4_{11} = \sqrt{1 + \lambda^2} h^4_{11}, \quad b^4_{12} = \alpha_1(\lambda), \quad b^4_{13} = \sqrt{(1 + \lambda^2)(1 + \mu^2)} h^4_{13},$$

$$b^4_{22} = \alpha_2(\lambda), \quad b^4_{23} = \alpha_3(\lambda), \quad b^4_{33} = (1 + \mu^2) \sqrt{1 + \lambda^2} h^4_{33},$$

$$b^5_{11} = \sqrt{1 + \mu^2} h^5_{11}, \quad b^5_{13} = \alpha_1(\mu), \quad b^5_{12} = (1 + \lambda^2)(1 + \mu^2) h^5_{12},$$

$$b^5_{33} = \alpha_3(\mu), \quad b^5_{23} = \alpha_2(\mu), \quad b^5_{22} = (1 + \lambda^2) \sqrt{1 + \mu^2} h^5_{22},$$
and
\[ h_1^2 = e_1(\arctan \lambda), \quad h_2^2 = e_2(\arctan \lambda), \quad h_3^2 = e_3(\arctan \lambda), \]
\[ h_1^3 = e_1(\arctan \mu), \quad h_2^3 = e_2(\arctan \mu), \quad h_3^3 = e_3(\arctan \mu), \]
where all indices are with respect to the orthonormal frames of the singular value decomposition.

**Proof** Differentiating with respect to \( \alpha \) the identity
\[ \langle df(\alpha_2), df(\alpha_2) \rangle_{\mathbb{S}^2} = \lambda^2, \]
and making use that \( \alpha \) spans the kernel of \( df \), we get
\[ \lambda \alpha(\lambda) = \langle B(\alpha_1, \alpha_2) + df(\nabla_{\alpha_1} \alpha_2), df(\alpha_2) \rangle_{\mathbb{S}^2} \]
\[ = \langle B(\alpha_1, \alpha_2), df(\alpha_2) \rangle_{\mathbb{S}^2} + \langle \nabla_{\alpha_1} \alpha_2, \alpha_3 \rangle_{\mathbb{S}^2} \langle df(\alpha_2), df(\alpha_3) \rangle_{\mathbb{S}^2} \]
\[ = \lambda b_{12}^4. \]
Therefore,
\[ b_{12}^4 = \alpha(\lambda). \quad (28) \]

Similarly, differentiating with respect to \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) the equations
\[ \langle df(\alpha_2), df(\alpha_2) \rangle_{\mathbb{S}^2} = \lambda^2 \quad \text{and} \quad \langle df(\alpha_3), df(\alpha_3) \rangle_{\mathbb{S}^2} = \mu^2 \]
we obtain
\[ b_{13}^5 = \alpha(\mu), \quad b_{22}^5 = \alpha(\lambda), \quad b_{23}^5 = \alpha(\lambda), \quad b_{23}^5 = \alpha(\mu), \quad b_{33}^5 = \alpha(\mu). \quad (29) \]

From Lemma 2, equations (28) and (29) we get the expression relating the second fundamental form of the graph with the singular values and the Hessian of the map \( f \). This completes the proof. \( \square \)

**Lemma 5** Let \( f : U \subset \mathbb{S}^3 \to \mathbb{S}^2 \) be a submersion with distinct singular values 0, \( \lambda \) and \( \mu \). Then,
\[ \nabla_{\alpha_1}^3 \alpha_1 = -\sqrt{1 + \lambda^2} \frac{h_1^4 \alpha_2}{\lambda} - \frac{\sqrt{1 + \mu^2}}{\mu} h_1^5 \alpha_3, \]
\[ \nabla_{\alpha_1}^3 \alpha_2 = \frac{\sqrt{1 + \lambda^2}}{\lambda} h_1^4 \alpha_2 - \frac{\mu h_1^5 + \lambda h_1^4}{(\mu^2 - \lambda^2)u_1} \alpha_3, \]
\[ \nabla_{\alpha_1}^3 \alpha_3 = \frac{\sqrt{1 + \mu^2}}{\mu} h_1^5 \alpha_2 + \frac{\mu h_1^5 + \lambda h_1^4}{(\mu^2 - \lambda^2)u_1} \alpha_2. \]

Moreover,
\[ \nabla_{\alpha_2}^3 \alpha_1 = -\alpha(\log \lambda) \alpha_2 - \frac{h_1^5}{\mu u_1} \alpha_3, \]
\[ \nabla_{\alpha_2}^3 \alpha_2 = \alpha(\log \lambda) \alpha_1 - \frac{u_1 \lambda \alpha_3(\lambda) + \mu \sqrt{1 + \lambda^2} h_2^5}{(\mu^2 - \lambda^2)u_1} \alpha_3, \]
\[ \nabla_{\alpha_2}^3 \alpha_3 = \frac{h_1^5}{\mu u_1} \alpha_1 + \frac{u_1 \lambda \alpha_3(\lambda) + \mu \sqrt{1 + \lambda^2} h_2^5}{(\mu^2 - \lambda^2)u_1} \alpha_2. \]
Additionally,
\[
\nabla_{\alpha_1} S^3 = -\frac{h^4_{13}}{\lambda u_1} \alpha_2 - \alpha_1 (\log \mu) \alpha_3
\]
\[
\nabla_{\alpha_1} S^3 = \frac{h^4_{13}}{\lambda u_1} \alpha_1 - \frac{u_1 \mu \alpha_2 (\mu) + \lambda \sqrt{1 + \mu^2} h^4_{33}}{(\mu^2 - \lambda^2) u_1} \alpha_3,
\]
\[
\nabla_{\alpha_1} S^3 = \alpha_1 (\log \mu) \alpha_1 + \frac{u_1 \mu \alpha_2 (\mu) + \lambda \sqrt{1 + \mu^2} h^4_{33}}{(\mu^2 - \lambda^2) u_1} \alpha_2.
\]

**Proof** By a straightforward computation, we have
\[
h^4_{11} = \langle \nabla^F_{e_1} dF(e_1), \xi_4 \rangle_{S^3 \times S^2} = \frac{1}{\sqrt{1 + \lambda^2}} \langle \nabla^S_{\alpha_1 \alpha_1} 0, -\lambda \alpha_2 \oplus \beta_3 \rangle_{S^3 \times S^2}
\]
\[
= -\frac{\lambda}{\sqrt{1 + \lambda^2}} \langle \nabla^M \alpha_1, \alpha_2 \rangle_{S^3}.
\]
Similarly, we deduce
\[
h^5_{11} = -\frac{\mu}{\sqrt{1 + \mu^2}} \langle \nabla^S_{\alpha_1 \alpha_1}, \alpha_3 \rangle_{S^3}.
\]
As a consequence, we get
\[
\langle \nabla^S_{\alpha_1 \alpha_2}, \alpha_2 \rangle_{S^3} = -\frac{\sqrt{1 + \lambda^2}}{\lambda} h^4_{11} \quad \text{and} \quad \langle \nabla^S_{\alpha_1 \alpha_1}, \alpha_3 \rangle_{S^3} = -\frac{\sqrt{1 + \mu^2}}{\mu} h^5_{11}.
\] (30)
Using (28) and (29), we have
\[
\alpha_1 (\lambda) = \langle B(\alpha_1, \alpha_2), \beta_2 \rangle_{S^2}
\]
\[
= \langle \nabla^f_{e_2} f(\alpha_1) \rangle = \langle d (\nabla^S_{\alpha_2} \alpha_1), \beta_2 \rangle_{S^2} = -\langle d (\nabla^S_{\alpha_2} \alpha_1), \beta_2 \rangle_{S^2}
\]
\[
= -\lambda \langle \nabla^S_{\alpha_2} \alpha_1, \alpha_2 \rangle_{S^3},
\]
and so
\[
\langle \nabla^S_{\alpha_2} \alpha_1, \alpha_2 \rangle_{S^3} = -\alpha_1 (\log \lambda). \quad (31)
\]
Similarly, we get
\[
\langle \nabla^S_{\alpha_1} \alpha_3, \alpha_3 \rangle_{S^3} = -\alpha_1 (\log \mu). \quad (32)
\]
Furthermore,
\[
h^5_{12} = \langle \nabla^F_{e_2} dF(e_1), \xi_5 \rangle_{S^3 \times S^2}
\]
\[
= -\frac{1}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}} \langle \nabla^S_{\alpha_2} \alpha_1 \oplus 0, -\mu \alpha_3 \oplus \beta_3 \rangle_{S^3 \times S^2}
\]
\[
= -\frac{\mu}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}} \langle \nabla^S_{\alpha_2} \alpha_1, \alpha_3 \rangle_{S^3} \quad (33)
\]
and in the same way we obtain
\[
h^4_{13} = -\frac{\lambda}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}} \langle \nabla^S_{\alpha_3} \alpha_1, \alpha_2 \rangle_{S^3}. \quad (34)
Differentiating with respect to $\alpha_1$ the identity
\[ \langle df(\alpha_2), df(\alpha_3) \rangle_{S^2} = 0, \]
we get
\[ 0 = \langle \nabla^f_{\alpha_1} df(\alpha_2), df(\alpha_3) \rangle_{S^2} + \langle df(\alpha_2), \nabla^f_{\alpha_1} df(\alpha_3) \rangle_{S^2} \]
\[ = \langle B(\alpha_1, \alpha_2) + df(\nabla^S_{\alpha_1} \alpha_2), df(\alpha_3) \rangle_{S^2} + \langle df(\alpha_2), B(\alpha_1, \alpha_3) + df(\nabla^S_{\alpha_1} \alpha_3) \rangle_{S^2} \]
\[ = \mu b_{12}^2 + \lambda b_{13}^4 + (\mu^2 - \lambda^2)(\nabla^S_{\alpha_1} \alpha_2, \alpha_3)_{S^3}. \]

Using the expressions of Lemma 4, we obtain
\[ \langle \nabla^S_{\alpha_1} \alpha_2, \alpha_3 \rangle_{S^3} = -\sqrt{(1 + \lambda^2)(1 + \mu^2)} \frac{\mu h_{12}^5 + \lambda h_{13}^4}{\mu^2 - \lambda^2}. \] (35)

Differentiating with respect to $\alpha_2$ and $\alpha_3$ the identity
\[ \langle df(\alpha_2), df(\alpha_3) \rangle_{S^2} = 0 \]
and proceeding in the same way, we get
\[ \langle \nabla^S_{\alpha_2} \alpha_2, \alpha_3 \rangle_{S^3} = \frac{\lambda \alpha_3(\lambda) + \mu(1 + \lambda^2)\sqrt{1 + \mu^2}h_{22}^5}{\lambda^2 - \mu^2} \] (36)
and
\[ \langle \nabla^S_{\alpha_3} \alpha_2, \alpha_3 \rangle_{S^3} = \frac{\mu \alpha_2(\mu) + \lambda(1 + \mu^2)\sqrt{1 + \lambda^2}h_{33}^4}{\mu^2 - \lambda^2}. \] (37)

Combining (30), (31), (32), (33), (34), (35), (36) and (37), we get the desired result. \qed

**Lemma 6** Let $f : U \subset S^3 \to S^2$ be a submersion with distinct singular values $0, \lambda$ and $\mu$. Then, the normal connection of the graphical submanifold $\Gamma(f)$ is given by
\[
\nabla^\perp_{e_1} \xi_4 = \frac{\lambda(1 + \mu^2)h_{12}^5 + \mu(1 + \lambda^2)h_{13}^4}{\lambda^2 - \mu^2} \xi_5, \\
\nabla^\perp_{e_2} \xi_4 = \frac{\mu \alpha_3(\lambda) + \lambda(1 + \mu^2)h_{22}^5}{\lambda^2 - \mu^2} \xi_5, \\
\nabla^\perp_{e_3} \xi_4 = \frac{\lambda \alpha_2(\mu) + \mu(1 + \lambda^2)h_{33}^4}{\lambda^2 - \mu^2} \xi_5,
\]
where all the frames are with respect to the singular value decomposition.

**Proof** By a straightforward computation, we obtain
\[
\langle \nabla^\perp_{e_1} \xi_4, \xi_5 \rangle_{S^3 \times S^2} = \langle \nabla^F_{\alpha_1} \frac{-\lambda \alpha_2 \oplus \beta_2}{\sqrt{1 + \lambda^2}} - \mu \alpha_3 \oplus \beta_3}{\sqrt{1 + \mu^2}} \rangle_{S^3 \times S^2} \\
= \langle \nabla^F_{\alpha_1} (-\lambda \alpha_2 \oplus \beta_2) - \mu \alpha_3 \oplus \beta_3 \rangle_{S^3 \times S^2} \sqrt{(1 + \lambda^2)(1 + \mu^2)} \\
= \lambda \mu \langle \nabla^S_{\alpha_1} \alpha_2, \alpha_3 \rangle_{S^3} + \lambda^{-1} \langle \nabla^f_{\alpha_1} df(\alpha_2), \beta_3 \rangle_{S^2} \sqrt{(1 + \lambda^2)(1 + \mu^2)} \\
= \lambda \mu \langle \nabla^S_{\alpha_1} \alpha_2, \alpha_3 \rangle_{S^3} + \lambda^{-1} b_{12}^5 + \lambda^{-1} \mu \langle \nabla^S_{\alpha_1} \alpha_2, \alpha_3 \rangle_{S^3} \sqrt{(1 + \lambda^2)(1 + \mu^2)}.
\]
Using the formulas from Lemmas 4 and 5, we get

\[
\langle \nabla_{e_1} \xi_4, \xi_5 \rangle_{\mathbb{S}^3 \times \mathbb{S}^2} = \lambda \mu \langle \nabla_{a_1}^{\mathbb{S}^3} a_2, a_3 \rangle_{\mathbb{S}^3} + \lambda^{-1} b_{12}^5 + \lambda^{-1} \mu \langle \nabla_{a_1}^{\mathbb{S}^3} a_2, a_3 \rangle_{\mathbb{S}^3}
\]

\[
= -\lambda \mu \frac{\mu h_{12}^5 + \lambda h_{13}^4}{\mu^2 - \lambda^2} + \frac{h_{12}^5}{\lambda} - \frac{\mu (\mu h_{12}^5 + \lambda h_{13}^4)}{\lambda (\mu^2 - \lambda^2)}
\]

\[
= -\frac{\lambda (1 + \mu^2) h_{12}^5 + \mu (1 + \lambda^2) h_{13}^4}{\mu^2 - \lambda^2}.
\]

Exactly in the same way, we compute the other terms. This completes the proof. \(\square\)

Combining Lemmas 1, 4, and 5 we can express the connection of the induced graphical metric \(g\) in terms of the second fundamental form, the singular values and their derivatives. In particular, the following holds:

**Lemma 7** Let \(f : U \subset \mathbb{S}^3 \rightarrow \mathbb{S}^2\) be a submersion with distinct singular values \(0, \lambda, \text{and} \mu\). Then,

\[
\nabla^g_{e_1} e_1 = -\frac{h_{11}^4}{\lambda} e_2 - \frac{h_{12}^5}{\mu} e_3,
\]

\[
\nabla^g_{e_1} e_2 = \frac{h_{11}^4}{\lambda} e_1 - \frac{(1 + \lambda^2) \mu h_{12}^5 + \mu (1 + \mu^2) h_{13}^4}{\mu^2 - \lambda^2} e_3,
\]

\[
\nabla^g_{e_1} e_3 = \frac{h_{11}^4}{\mu} e_1 + \frac{(1 + \lambda^2) \mu h_{12}^5 + \lambda (1 + \mu^2) h_{13}^4}{\mu^2 - \lambda^2} e_2.
\]

Moreover,

\[
\nabla^g_{e_2} e_1 = -\frac{e_1 (\arctan \lambda)}{\lambda} e_2 - \frac{h_{12}^5}{\mu} e_3,
\]

\[
\nabla^g_{e_2} e_2 = \frac{e_1 (\arctan \lambda)}{\lambda} e_1 - \frac{\mu (1 + \lambda^2) h_{12}^5 + \lambda (1 + \mu^2) e_3 (\arctan \lambda)}{\mu^2 - \lambda^2} e_3,
\]

\[
\nabla^g_{e_2} e_3 = \frac{h_{11}^4}{\mu} e_1 + \frac{\mu (1 + \lambda^2) h_{12}^5 + \lambda (1 + \mu^2) e_3 (\arctan \lambda)}{\mu^2 - \lambda^2} e_2.
\]

Additionally,

\[
\nabla^g_{e_3} e_1 = -\frac{h_{13}^4}{\lambda} e_2 - \frac{e_1 (\arctan \mu)}{\mu} e_3,
\]

\[
\nabla^g_{e_3} e_2 = \frac{h_{13}^4}{\lambda} e_1 - \frac{\mu (1 + \lambda^2) e_2 (\arctan \mu) + \lambda (1 + \mu^2) h_{33}^4}{\mu^2 - \lambda^2} e_3,
\]

\[
\nabla^g_{e_3} e_3 = \frac{e_1 (\arctan \mu)}{\mu} e_1 + \frac{\mu (1 + \lambda^2) e_2 (\arctan \mu) + \lambda (1 + \mu^2) h_{33}^4}{\mu^2 - \lambda^2} e_2,
\]

and

\[
[e_1, e_2] = \frac{h_{11}^4}{\lambda} e_1 + \frac{e_1 (\arctan \lambda)}{\lambda} e_2 + \frac{\lambda (1 + \mu^2)}{\mu (\mu^2 - \lambda^2)} (\lambda h_{12}^4 + \mu h_{13}^4) e_3,
\]

\[
[e_1, e_3] = \frac{h_{11}^4}{\mu} e_1 + \frac{\mu (1 + \lambda^2)}{\lambda (\mu^2 - \lambda^2)} (\lambda h_{12}^4 + \mu h_{13}^4) e_2 + \frac{e_1 (\arctan \mu)}{\mu} e_3,
\]
where all the frames are with respect to the frames arising from the singular value decomposition.

As a consequence of the above formulas, we may express the differential $\varphi$ of the Gauss map of the submersion $f : S^3 \to S^2$ in terms of the singular values and the second fundamental form.

**Lemma 8** Let $f : U \subset S^3 \to S^2$ be a submersion with smooth and non-zero singular values $\lambda$ and $\mu$. Then, the tensor $\varphi$ satisfies

\[
\varphi_{12} = g(\varphi(e_1), e_2) = \lambda^{-1} h_{11}^4,
\]
\[
\varphi_{22} = g(\varphi(e_2), e_2) = \lambda^{-1} h_{12}^4,
\]
\[
\varphi_{32} = g(\varphi(e_3), e_2) = \lambda^{-1} h_{13}^4.
\]

Moreover,

\[
\varphi_{13} = g(\varphi(e_1), e_3) = \mu^{-1} h_{11}^5,
\]
\[
\varphi_{23} = g(\varphi(e_2), e_3) = \mu^{-1} h_{12}^5,
\]
\[
\varphi_{33} = g(\varphi(e_3), e_3) = \mu^{-1} h_{13}^5,
\]

where the frames are arising from the singular value decomposition.

**5.4 Bochner–Weitzneböck formulas**

In the sequel, we compute the gradients and the Laplacians of $u_1$ and $u_2$ in the case where $f$ is minimal. Let us point out here that the function $u_1$ is smooth and well defined globally on $M$, even at points where $f$ is not a submersion. The proofs are straightforward and we omit them; for details, we refer to [52].

**Lemma 9** Let $f : U \subset S^3 \to S^2$ be a minimal map. The gradient and the Laplacian of the functions $u_1$ are given by

\[
\nabla u_1 = -u_1 \sum_{k=1}^{3} \left( \lambda h_{k2}^4 + \mu h_{k3}^5 \right) e_k
\]

and

\[
\Delta u_1 = -|A|^2 u_1 + 2u_2 \sum_{k=1}^{3} \left( h_{k2}^4 h_{3k}^5 - h_{k3}^5 h_{3k}^4 \right) - 2(\lambda^2 + \mu^2) u_1^3,
\]

where $\{e_1, e_2, e_3\}$ is the frame arising from the singular value decomposition.

**Lemma 10** Let $f : U \subset S^3 \to S^2$ be a minimal submersion. The gradient and the Laplacian of the function $u_2$ are given by:

\[
\nabla u_2 = u_1 \sum_{k=1}^{3} \left( \mu h_{k2}^4 + \lambda h_{k3}^5 \right) e_k
\]

and

\[
\Delta u_2 = -u_2 \sum_{k=1}^{3} \left( |h_{k2}|^2 + |h_{k3}|^2 \right) + 2u_1 \sum_{k=1}^{3} \left( h_{k2}^4 h_{3k}^5 - h_{k3}^5 h_{3k}^4 \right) + 4u_1^2 u_2
\]
\[
+ \mu u_1 \frac{(h_{11}^4)^2 + (h_{12}^4)^2 + (h_{13}^4)^2}{\lambda} + \lambda u_1 \frac{(h_{11}^5)^2 + (h_{12}^5)^2 + (h_{13}^5)^2}{\mu}.
\]
Let us prove at first that
\[ \text{Proof} \]
standard Hopf fibration.
\[ x \]
condition
\[ w \]
Now from the formula for the Laplacian of
\[ \{ \]
where as usual \( \{ e_1, e_2, e_3 \} \) is the orthonormal frame arising from the singular value decomposition.

6 Proofs of our main theorems

Here we give the proofs of Theorem A, Theorem B and Theorem C. We follow the notation introduced in the previous sections.

Theorem B. Let \( f : S^3 \rightarrow S^2 \) be a minimal submersion whose Gauss map \( G \) satisfies the condition
\[ (\mu - \lambda)(\mu |\text{Im}\, G|^2 - \lambda |\text{Re}\, G|^2) \geq 0, \]
where \( 0 < \lambda \leq \mu \) are the non-zero singular values of \( df \). Then, the map \( f \) is weakly conformal with totally geodesic fibers. Moreover, there exists an isometry \( T : S^3 \rightarrow S^3 \) and a conformal diffeomorphism \( \Phi : S^2(1/2) \rightarrow S^2 \) such that \( f = \Phi \circ \Pi \circ T \), where \( \Pi \) is the standard Hopf fibration.

Proof Let us prove at first that \( f \) is weakly conformal. On the contrary, we suppose that \( f \) is not weakly conformal. Let \( x_0 \in S^3 \) be a point where the globally defined function
\[ w = \frac{1 + \lambda \mu}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}} \leq 1 \]
attains its minimum. Clearly at the point \( x_0 \) it holds \( 0 < w(x_0) < 1 \) and so \( \lambda(x_0) < \mu(x_0) \). Moreover, in a sufficiently small neighbourhood of \( x_0 \) the function \( w \) is smooth. From Lemma 8 and our assumption, we get that
\[ C = (\mu - \lambda)(\frac{1}{\mu} \sum_{k=1}^3 (h_{1k}^5)^2 - \frac{1}{\lambda} \sum_{k=1}^3 (h_{1k}^4)^2) \]
\[ = (\mu - \lambda)(\mu |\text{Im}\, \varphi|^2 - \lambda |\text{Re}\, \varphi|^2) \]
\[ \geq 0. \] (38)

Now from the formula for the Laplacian of \( w \) in Lemma 11, we obtain
\[ 0 \leq \Delta w(x_0) \leq -2(\mu(x_0) - \lambda(x_0))u_3^3(x_0) < 0, \]
which leads to a contradiction. Therefore, \( f \) must be weakly conformal and \( w = 1 \). Going back to the Laplacian of \( w \), we get that
\[ h_{22}^4 = h_{23}^5 = -h_{33}^4 \quad \text{and} \quad h_{22}^5 = -h_{23}^4 = -h_{33}^5. \]
Because of the minimality, it follows that $h_{11}^4 = 0$ and $h_{11}^5 = 0$. Hence, $A(e_1, e_1) = 0$ and from Lemma 2 we get that

$$\nabla_{e_1}^S e_1 = \nabla_{e_1}^{\mathbb{S}^3} e_1 \quad \text{and} \quad B(e_1, e_1) = 0.$$  

Since $df(e_1) = 0$, we have that  

$$0 = B(e_1, e_1) = \nabla_{e_1}^f df(e_1) - df(\nabla_{e_1}^{\mathbb{S}^3} e_1) = -df(\nabla_{e_1}^{\mathbb{S}^3} e_1).$$

Consequently, the integral curves of the vector field $e_1$ are great circles.

According to Heller’s Theorem [28, Theorem 3.7], there exist conformal diffeomorphisms $T : \mathbb{S}^3 \to \mathbb{S}^3$ and $\Phi : \mathbb{S}^2(1/2) \to \mathbb{S}^2$ such that $f = \Phi \circ \Pi \circ T$, where $\Pi$ is the standard Hopf-fibration. By the classical Liouville’s Theorem [5, Theorem 6.3], each conformal transformation $T : \mathbb{S}^3 \to \mathbb{S}^3$ can be written as $T = T_1 \circ T_2$ where $T_1 \in O(4)$, and after identifying $\mathbb{S}^3$ with $\mathbb{R}^3 \cup \{\infty\}$ via stereographic projection, $T_2$ takes the form $T_2(x) = \varrho x - b$ for some $\varrho > 0$ and $b \in \mathbb{R}^3$; see for example [25, Proposition 1.1.7] or [37, page 48]. Because the fibers of $f$ and $\Pi$ are great circles and $T$ maps the fibers of $f$ onto the fibers of $\Pi$, we deduce that $\varrho = 1, b = 0$ and $T$ must be an isometry. This completes the proof. \qedhere

**Theorem C.** Let $f_{kl} : \mathbb{S}^3 \to \mathbb{S}^2$ be an equivariant minimal submersion. Then $f_{kl}$ is the composition of the standard Hopf-fibration with the dilation from the radius 1/2 sphere into the unit sphere.

**Proof** From Proposition 5 the generating function $a : [0, \pi/2] \to [0, \pi]$ satisfies the differential equation

$$0 = \frac{a_{ss}}{1 + a_s^2} + \frac{\cos s \sin s (\cos 2s + (l^2 - k^2) \sin^2 a)}{\sin^2 s \cos^2 s + \sin^2 a (l^2 \sin^2 s + k^2 \cos^2 s)} a_s$$

$$- \frac{\sin a \cos a (k^2 \cos^2 s + l^2 \sin^2 s)}{\sin^2 s \cos^2 s + \sin^2 a (l^2 \sin^2 s + k^2 \cos^2 s)}.$$  \hfill (39)

Since $f_{kl}$ is a submersion, $a(0) = 0$ and $a(\pi/2) = \pi$, from Lemma 3 we deduce that $a$ must be a strictly increasing smooth function. In particular, close to 0, the function $a$ must have a Taylor expansion of the form

$$a(s) = a_s(0)s + \cdots$$

with $a_s(0) > 0$. Observe that close to 0, the denominator $D$ of the last two terms in the right-hand side of (39) has an asymptotic behavior of the form

$$D(s) = \sin^2 s \cos^2 s + \sin^2 a (l^2 \sin^2 s + k^2 \cos^2 s) \sim O(s^2).$$

By Taylor expanding the nominator $N$ around the point 0, we see that

$$N(s) = a_s \cos s \sin s (\cos 2s + (l^2 - k^2) \sin^2 a) - \sin a \cos a (k^2 \cos^2 s + l^2 \sin^2 s)$$

$$\sim O(a_s(0)(1 - k^2)s).$$

Therefore, $k^2 = 1$. Similarly, we deduce that $l^2 = 1$. Let us now denote by $\lambda_2$ and $\lambda_3$ the non-zero singular values of $f_{kl}$. Following the notation and the formulas of Proposition 5
for the second fundamental form of an equivariant submersion, we obtain that \( f_{k,l} \) has totally geodesic fibers and that
\[
C = (\lambda_3 - \lambda_2) \left( \frac{1}{\lambda_3} \sum_{k=1}^{3} (h_{1k}^4)^2 - \frac{1}{\lambda_2} \sum_{k=1}^{3} (h_{1k}^4)^2 \right)
= \frac{(\lambda_3 - \lambda_2)^2}{(1 + \lambda_2^2)(1 + \lambda_3^2)}
\geq 0.
\]

From Theorem B we deduce that \( f_{k,l} \) is a weakly conformal minimal submersion with \( k^2 = l^2 = 1 \). From Lemma 3, we get that \( a(s) = 2s \) and so \( f_{k,l} \) is the standard Hopf-fibration. This completes the proof. \( \square \)

**Theorem A.** Let \( f : U \to \mathbb{S}^2 \) be a minimal map, where \( U \) is an open subset of \( \mathbb{S}^3 \). If \( f \) has constant singular values and \( \Gamma(f) \) has constant norm of the second fundamental form, then its singular values are equal and so the map \( f \) is either constant or weakly conformal. If \( f \) is non-constant, then there exists an isometry \( T : \mathbb{S}^3 \to \mathbb{S}^3 \) such that \( f = \Phi \circ \Pi \circ T \), where \( \Pi \) is the Hopf fibration and \( \Phi : \mathbb{S}^2(1/2) \to \mathbb{S}^2 \) is the dilation from the sphere \( \mathbb{S}^2(1/2) \) of radius \( 1/2 \).

**Proof** We will prove at first that the singular values \( \lambda \) and \( \mu \) of \( f \) are equal. If \( \mu \) is zero then also \( \lambda \) is zero and \( f \) is a constant map. Suppose now that \( \lambda < \mu \) and that \( \mu \) is positive. We distinguish the following cases:

**Case A:** Suppose at first that \( \lambda = 0 \). Using Lemma 9 we get
\[
0 = \Delta u_1 = -|A|^2 u_1 - 2\mu^2 u_1^3.
\]
Hence, \( \mu = 0 \) and \( f \) must be constant.

**Case B:** Suppose that the singular value \( \lambda \) is also positive. From (27), we get
\[
h_{12}^4 = h_{13}^5 = h_{22}^4 = h_{33}^5 = h_{23}^4 = h_{23}^5 = 0.
\]
Hence, from minimality, we have that
\[
h_{33}^5 = -h_{11}^4 \quad \text{and} \quad h_{22}^5 = -h_{11}^5.
\]
For the sake of convenience, we set
\[
h_{12}^5 = \chi, \ h_{13}^4 = \psi, \ h_{11}^5 = z, \ h_{11}^4 = \tau.
\]
Using Lemma 9, we get
\[
|A|^2 = -2\lambda \mu \chi \psi - \frac{2(\lambda^2 + \mu^2)}{(1 + \lambda^2)(1 + \mu^2)}.
\]
Since by assumption, the norm of the second fundamental form is constant, it follows that \( \chi \psi \) is a negative constant function on \( U \subset \mathbb{S}^3 \). Applying the Codazzi equation to the triples of vectors \( \{e_1, e_2, e_1\}, \{e_1, e_3, e_1\}, \{e_1, e_2, e_2\}, \{e_1, e_1, e_3\}, \{e_1, e_2, e_3\}, \{e_2, e_1, e_2\}, \{e_2, e_3, e_3\} \) and \( \{e_2, e_3, e_2\} \), and making use of Lemma 6 and Lemma 7, we obtain
\[
e_2(\tau) = \frac{\lambda(1+\mu^2)}{\mu^2-\lambda^2} (\chi^2 + \psi^2) + \frac{2\lambda^2(1+\mu^2)}{\mu(\mu^2-\lambda^2)} \chi \psi - \frac{\lambda}{\mu} \tau^2 + \frac{\lambda(1+\mu^2)}{\mu^2-\lambda^2} \frac{\tau^2}{2} - \frac{\lambda}{2}, \quad (40)
\]
\[
e_2(\tau) = \frac{\lambda(1+\mu^2)}{\mu(\mu^2-\lambda^2)} - \frac{2}{\mu} \chi \psi + \frac{1}{\mu} \psi^2 - \frac{\lambda(1+\mu^2)}{\mu(\mu^2-\lambda^2)} \tau^2, \quad (41)
\]
\[
e_3(\tau) = \frac{\mu(1+\lambda^2)}{\lambda^2-\mu^2} (\chi^2 + \psi^2) - \frac{2\mu^2(1+\lambda^2)}{\lambda(\lambda^2-\mu^2)} \chi \psi - \frac{\mu}{\lambda} \tau^2 + \frac{\mu(1+\lambda^2)}{\lambda^2-\mu^2} \frac{\tau^2}{2} - \frac{\mu}{2}, \quad (42)
\]
\[
e_3(\tau) = \frac{\mu(1+\lambda^2)}{\lambda^2-\mu^2} (\chi^2 + \psi^2) - \frac{2\mu^2(1+\lambda^2)}{\lambda(\lambda^2-\mu^2)} \chi \psi - \frac{\mu}{\lambda} \tau^2 + \frac{\mu(1+\lambda^2)}{\lambda^2-\mu^2} \frac{\tau^2}{2}. \quad (43)
\]
Moreover,
\[ e_1(\chi) - e_2(z) = \frac{2\mu^2 - \lambda^2 + \lambda^2 \mu^2}{\lambda(\mu^2 - \lambda^2)} \tau z, \quad (44) \]
\[ e_1(z) + e_2(\chi) = -\frac{2}{\lambda} \tau \chi, \quad (45) \]
and
\[ e_1(\psi) - e_3(\tau) = \frac{2\lambda^2 - \mu^2 + \lambda^2 \mu^2}{\mu(\lambda^2 - \mu^2)} \tau z \]
\[ e_1(\tau) + e_3(\psi) = -\frac{2}{\mu} \psi z. \quad (47) \]

Additionally,
\[ e_2(\psi) = \frac{2\lambda^2 - \mu^2 + \lambda^2 \mu^2}{\mu(\lambda^2 - \mu^2)} \chi \tau + \frac{\mu^2(1 + \lambda^2)}{\lambda(\lambda^2 - \mu^2)} \tau \psi, \quad (48) \]
\[ e_3(\chi) = \frac{2\mu^2 - \lambda^2 + \lambda^2 \mu^2}{\lambda(\mu^2 - \lambda^2)} \psi z + \frac{\lambda^2(1 + \mu^2)}{\mu(\mu^2 - \lambda^2)} \chi z. \quad (49) \]

Substituting (40) in (41) and (42) in (43), we get the following equations:
\[ \lambda^2(1 + \mu^2)|A^5|^2 + (\lambda^2 \mu^2 + 2\lambda^2 - \mu^2)|A^4|^2 = \frac{4\lambda^2(\mu^2 - \lambda^2)}{(1 + \lambda^2)(1 + \mu^2)} - 4\lambda \mu(1 + \lambda^2) \chi \psi, \]
\[ (\lambda^2 \mu^2 + 2\mu^2 - \lambda^2)|A^5|^2 + \mu^2(1 + \lambda^2)|A^4|^2 = \frac{4\mu^2(\lambda^2 - \mu^2)}{(1 + \lambda^2)(1 + \mu^2)} - 4\lambda \mu(1 + \mu^2) \chi \psi. \]

Let us regard the above as a system with respect to \(|A^5|^2\) and \(|A^4|^2\). Then, its main determinant is
\[ D = 2(\lambda^2 - \mu^2)^2. \]

As a consequence, the system has a unique solution which is given by
\[ \chi^2 + z^2 = \frac{1}{\mu^2 - \lambda^2} \left( \frac{3\lambda^2 \mu^2 + \lambda^4 \mu^2 + \lambda^2 \mu^4 - \mu^4}{(1 + \lambda^2)(1 + \mu^2)} + \lambda \mu(\lambda^2 \mu^2 - \mu^2 - 2) \chi \psi \right), \quad (50) \]
\[ \psi^2 + \tau^2 = \frac{1}{\lambda^2 - \mu^2} \left( \frac{3\lambda^2 \mu^2 + \lambda^4 \mu^2 + \lambda^2 \mu^4 - \mu^4}{(1 + \lambda^2)(1 + \mu^2)} + \lambda \mu(\lambda^2 \mu^2 - \lambda^2 - 2) \chi \psi \right). \]

Differentiating the first equation of (50) with respect to \(e_1\) and \(e_2\) and using (44) and (45), we obtain the following equations:
\[ \chi e_1(\chi) - z e_2(\chi) = \frac{2}{\lambda} \tau \chi z, \quad (51) \]
\[ z e_1(\chi) + \chi e_2(\chi) = \frac{2\mu^2 - \lambda^2 + \lambda^2 \mu^2}{\lambda(\mu^2 - \lambda^2)} z^2 \tau. \]

Since \(\chi \psi < 0\), the main determinant of (51) is
\[ D_1 = \chi^2 + z^2 \neq 0. \]

As a consequence, the system (51) has a unique solution which is given by
\[ e_1(\chi) = \left(2\chi^2 + \frac{2\mu^2 - \lambda^2 + \lambda^2 \mu^2}{\mu^2 - \lambda^2} z^2 \right) \frac{\tau \chi}{\lambda(\chi^2 + z^2)} \quad \text{and} \quad e_2(\chi) = \frac{\lambda(1 + \mu^2) \chi^2 \tau z}{(\mu^2 - \lambda^2)(\chi^2 + z^2)}. \quad (52) \]

As a consequence, we get
\[ e_2(\zeta) = e_1(\chi) - \frac{2\mu^2 - \lambda^2 + \lambda^2 \mu^2}{\lambda(\mu^2 - \lambda^2)} z \tau = \frac{\lambda(1 + \mu^2) \chi^2 \tau z}{(\mu^2 - \lambda^2)(\chi^2 + z^2)}. \quad (53) \]
Since the function $\chi \psi$ is constant on $U$, we have $e_2(\chi \psi) = 0$. Using (48) and (52), we get
\[
\frac{\tau}{\mu^2 - \lambda^2} \left( \frac{\lambda(1 + \mu^2)}{\chi^2 + z^2} \chi \psi z^2 - \frac{\mu^2(1 + \lambda^2)}{\lambda} \chi \psi - \frac{2\lambda^2 - \mu^2 + \lambda^2 \mu^2}{\mu} \chi^2 \right) = 0.
\]
If $\tau = 0$ on $U$, then (41) gives that the function $\psi$ is constant on $U$. Using (47), we obtain that $z = 0$ on $U$. In the sequel, we consider the open subset
\[W_1 = \{ p \in U : \tau(p) \neq 0 \}\]
of $U$. Then,
\[
\frac{\lambda(1 + \mu^2)}{\chi^2 + z^2} \chi \psi z^2 - \frac{\mu^2(1 + \lambda^2)}{\lambda} \chi \psi - \frac{2\lambda^2 - \mu^2 + \lambda^2 \mu^2}{\mu} \chi^2 = 0. \tag{54}
\]
on $W_1$. Differentiating (54) with respect to $e_2$ and using (52) and (53), we obtain
\[
\tau z^2 \chi^2 \left( \frac{\lambda(1 + \mu^2)}{\chi^2 + z^2} \chi \psi + \frac{2\lambda^2 - \mu^2 + \lambda^2 \mu^2}{\mu} \right) = 0.
\]
If $z = 0$ on $W_1$, then
\[
e_1(z) = e_3(z) = 0
\]
on $W_1$. Using (43), we easily conclude that the function $\chi$ is constant on $W_1$. Consequently, (45) gives $\tau = 0$ on $W_1$ which is a contradiction. In the sequel, we consider the open subset
\[W_2 = \{ p \in W_1 : z(p) \neq 0 \}\]
of $W_1$. Then,
\[\chi \psi = -\frac{2\lambda^2 - \mu^2 + \lambda^2 \mu^2}{\lambda \mu(1 + \mu^2)} (\chi^2 + z^2) \tag{55}\]
on $W$. Substituting (55) in (54), we get
\[-\frac{1}{\mu} + \frac{\mu(1 + \lambda^2)}{\lambda^2(1 + \mu^2)} = 0,
\]
from where we get $\lambda = \mu$, a contradiction. As a consequence, $\tau = z = 0$ on $U$. Then, (40) and (42) give
\[
\lambda \mu (\chi^2 + \psi^2) + 2\lambda^2 \chi \psi = \frac{\lambda \mu(\mu^2 - \lambda^2)}{(1 + \lambda^2)(1 + \mu^2)}, \tag{56}
\]
\[
\lambda \mu (\chi^2 + \psi^2) + 2\mu^2 \chi \psi = \frac{\lambda \mu(\lambda^2 - \mu^2)}{(1 + \lambda^2)(1 + \mu^2)}. \tag{57}
\]
Adding and subtracting (56) with (57), we get
\[
\chi^2 + \psi^2 = \frac{\mu^2 + \lambda^2}{(1 + \lambda^2)(1 + \mu^2)} \text{ and } \chi \psi = -\frac{\lambda \mu}{(1 + \lambda^2)(1 + \mu^2)}.
\]
Equivalently,
\[
\chi + \psi = \pm(\mu - \lambda)u_1 \text{ and } \chi - \psi = \pm(\mu + \lambda)u_1. \tag{58}
\]
The solution of (58) are the pairs:
\[(\chi, \psi) = (\mu u_1, -\lambda u_1), (\chi, \psi) = (\mu u_1, -\lambda u_1),\]
\[(\chi, \psi) = (\lambda u_1, -\mu u_1), (\chi, \psi) = (-\mu u_1, \lambda u_1).\]

We suppose that $\chi = \mu u_1$ and $\psi = -\lambda u_1$. Then, (41) and (43) give $\lambda^2 = \mu^2 = 4$ which is a contradiction. If $\chi = -\lambda u_1$ and $\psi = \mu u_1$, (41) and (43) give
\[3\lambda^2 + \mu^2 = \lambda^2 \mu^2 \quad \text{and} \quad 3\mu^2 + \lambda^2 = \lambda^2 \mu^2.
\]
Therefore, $\lambda = \mu$ which is again a contradiction. The other two cases for the values $\chi$ and $\psi$ are treated similarly.

Let us suppose now that $f$ is non-constant. As in the proof of Theorem B, from the minimality and the Laplacian of $w \equiv 1$, we deduce that $f$ is a conformal submersion, with totally geodesic fibers and constant singular values $0, \lambda, \lambda$. Hence, by [15, Theorem 1.1], it follows that there exists an isometry $T$ of the euclidean unit sphere $S^3$, which maps the kernel of $df$ into the kernel of the differential of the standard Hopf fibration $\Pi$, and a conformal diffeomorphism $\Phi : S^2(1/2) \rightarrow S^2$ such that $f = \Phi \circ \Pi \circ T$; compare also with the results in [21, Theorem A and Proposition 5.10]. Denote by $\varrho$ the conformal factor of $\Phi$. Since $T$ is an isometry, it will map the horizontal space of $f$ into the horizontal space of the Hopf fibration. Consequently, if $v$ is a horizontal unit vector of $f$, then
\[\lambda^2 = g_{S^2} (d\Phi \circ d\Pi \circ dT(v), d\Phi \circ d\Pi \circ dT(v)) = \varrho^2.
\]
Because $\lambda$ is constant, the conformal factor $\varrho$ must be constant. Since $\Phi$ is conformal, we get that $\varrho = 2$. This completes the proof. \(\square\)

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