Distributed computations in fully-defective networks

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Abstract
We address fully-defective asynchronous networks, in which all links are subject to an unlimited number of alteration errors, implying that all messages in the network may be completely corrupted. Despite the possible intuition that such a setting is too harsh for any reliable communication, we show how to simulate any algorithm for a noiseless setting over any fully-defective setting, given that the network is 2-edge connected. We prove that if the network is not 2-edge connected, no non-trivial computation in the fully-defective setting is possible. The key structural property of 2-edge-connected graphs that we leverage is the existence of an oriented (non-simple) cycle that goes through all nodes (Robbins, Am. Math. Mon., 1939). The core of our technical contribution is presenting a construction of such a Robbins cycle in fully-defective networks, and showing how to communicate over it despite total message corruption. These are obtained in a content-oblivious manner, since nodes must ignore the content of received messages.

1 Introduction

Faults are a main hurdle in a large variety of distributed systems. Faults manifest themselves in several different manners, ranging from nodes that crash due to malfunctions to environmental disruptions that affect the communication channels connecting distant nodes. In the last few decades, research has focused on developing fault-tolerant algorithms, as nodes crashes and channel noise are utterly inevitable. See, e.g., recent books and surveys on fault-tolerant systems [12, 27] and algorithms [4, 37], and references within.

In this work, we consider the case of channel noise within asynchronous distributed networks, where messages communicated between nodes are subject to corruption. When dealing with channel noise, some restrictions must be imposed on its power. Clearly, if noise can affect channels arbitrarily without any restrictions, then it could, for instance, delete all the communication and prevent any non-trivial computation over the network. Previous work either limited the number of channels that may suffer (arbitrary) noise [11, 22, 23, 32, 39, 40] or the total amount of corruptions (usually, alterations) the channels are allowed to make altogether [2, 8, 18, 24].

Throughout this work, we consider noisy channels that may arbitrarily change the content of transmitted messages, but can neither delete nor inject messages. This is known in the literature as alteration noise. Yet, we do not bound the amount of noise nor the number of noisy channels in any way. That is, we ask the following question:

Can one design fault-tolerant algorithms robust to an unlimited amount of corruption on all communication channels?

On its surface, the above task seems doomed. However, we answer the question in the affirmative for the large family of 2-edge-connected networks. We further show that if the network is not 2-edge-connected, the noise can destroy any non-trivial computation.

Towards this goal, we develop content-oblivious algorithms, that is, algorithms that do not rely on the content of communicated messages [8]. Instead, the actions of a node depend on the specific links and the order in which messages
are received. In particular, we devise a method that compiles any distributed algorithm into a content-oblivious version that computes the same task over 2-edge-connected graphs.

A folklore approach (see, e.g., in [8, 25]) is to send a message along a certain path from $u$ to $v$ to signify a 0 bit, and to send a message along a different path to signify a 1 bit, where the existence of two different paths is promised by the 2-edge-connectivity property. This approach conceals many challenges. First, the edges along these two paths are also edges in paths between other nodes in the network, and so the nodes must somehow be able to associate each such “bit” with its correct origin, in order to be able to decode each original piece of information and avoid mixing up bits of different ones. Second, in order to know where to forward the message to, the nodes need to extract the sender/receiver information from these “bit” messages, yet those might be fully corrupted. Third, some guarantee needs to be obtained on the order in which different 0/1 “bits” arrive at their destination, in order for them to faithfully represent the encoded message, a caveat on which the asynchrony of the network imposes another obstacle.

Before elaborating on how we overcome all these issues and stating our main results, let us explain our setting and noise model in more detail. We abstract the network as a graph $G = (V, E)$ where every node $v \in V$ is a computing device and every edge $e \in E$ is a noisy bi-directional communication channel. Once $u$ sends a message $m$ over some link $(u, v)$, the channel guarantees that after some arbitrary yet finite time, $v$ receives some message $m' \in \{0, 1\}^+$. Note that $m'$ may or may not equal $m$. In other words, the noise over the channel can corrupt the content of any transmitted $m$ into any $m'$, but it cannot completely delete it, nor can it inject new messages. We say that $G$ is a fully-defective network if all its channels are noisy in the above manner.

### 1.1 Our contribution and techniques

**Intuition: content-oblivious encoding with parallel channels** Let us begin with a simple toy example that illustrates some of our techniques. Suppose $u$ and $v$ are directly connected by two separate noisy channels, which we name DATA and END. The basic idea is to communicate the information over the DATA channel by sending, “bit-by-bit”, a unary encoding of the original message. In order to communicate the end of the unary encoding, a single message is sent on the END channel. Note, however, that timing is crucial: if the message sent on END is received before all the messages sent on DATA reach their destination, the receiver decodes incorrect information. To avoid this confusion, the receiver sends one message over END as an acknowledgment for each received DATA message. The sender waits until all its DATA messages are acknowledged and only then sends the termination message on END. Sending the terminating END message has an additional effect: it switches the roles of the nodes. If $u$ is the sender, then after sending the END message it takes the role of the receiver and vice versa. We call the sender at each point the **token holder**.

**Main result** Since we do not wish to assume two separate channels between any two nodes, we ask whether they can be replaced with two separate **paths** between any two nodes, i.e., can we constitute reliable communication between any two neighbors in 2-edge-connected graphs?

We answer this question affirmatively and show a method that takes any asynchronous message-passing distributed algorithm $\pi$ for a noiseless network $G$, and simulates it over the fully-defective $G$, given that $G$ is 2-edge connected. By simulating we mean that every node has a black-box interface to $\pi$ through which the node can deliver messages to $\pi$ and (asynchronously) receive messages to be communicated to some neighbor. The simulation guarantees that, at any given moment, all the nodes behave similarly to some valid execution of $\pi$ over the noiseless $G$.

**Theorem 1 (main, informal)** There exists a simulator for any asynchronous algorithm $\pi$ such that executing the simulator over a 2-edge connected fully-defective network $G$ simulates an execution of $\pi$ over the noiseless network $G$.

Once we establish that such a simulator even exists, a natural question is, what is the best that could be aimed for in terms of its message overhead? To avoid excessive clutter in the presentation, we delay the complete statement of our main theorem that includes its overhead, to Theorem 2 at the end of this section.

**Warm-up: resilient computations over a simple cycle** To describe our approach for proving Theorem 1, we begin with the much simpler case of cycle graphs. In a cycle graph, any node $u$ is connected to only two neighbors. Our goal is to simulate $u$’s communication with its two neighbors over a fully-defective cycle. In this special case, every two neighbors have exactly two separate paths between them: the direct link, and the rest of the cycle. The difference from the two-channel toy example illustrated above is that the paths of each two certain neighbors intersect the paths of other neighbors and we need to coordinate between the nodes so that each message reaches its correct destination and is interpreted correctly.

We address this difficulty by guaranteeing that only a single node is the sender (token holder) at any given time. All the other nodes are passive and only forward messages along the cycle. In this way, the sender can communicate with its neighbor using both paths of the cycle—one of them, say, the clockwise path, replaces the DATA channel, and the other one, the counterclockwise path, replaces the END channel.

In fact, $u$ can use the same method to communicate with any other node on the cycle, since all the nodes see the same sequence of clockwise and counterclockwise messages. In a
sense, \( u \) broadcasts information over the cycle, and all the
nodes learn this information. Next, we design a method to
change the roles such that another node may become the
token holder, i.e., the sender: After forwarding the message
initiated by the sender in the END path, the nodes enter a token
delivery phase. During this phase, a counterclockwise TOKEN
message, initiated by the previous token holder, is forwarded
along the cycle. When it reaches a node, if the node does
not have a message to send, then it forwards the token along
by propagating it counterclockwise. Otherwise, it becomes
the new token holder and initiates a clockwise message for-
warded along the entire cycle which denotes the end of the
token phase, so all the nodes go back to the stage of inter-
preting messages as DATA and END.

The above method has one significant drawback. If it
takes some time for the nodes to produce a message to send,
then the TOKEN message will keep circulating in the cycle,
cause many superfluous transmissions. We circumvent this
situation of wasteful transmissions by introducing a request
mechanism: the token transfer is performed only if some
node issues a request, which is done by sending a clockwise
REQUEST message. The requesting node can be far away
from the current token holder, thus, each node, upon receiving a
REQUEST message, propagates it clockwise. We note that sev-
eral nodes might issue a request at the same time at different
locations on the cycle. Eventually, all nodes will have sent
and received a REQUEST message, and after it reaches the
current token holder, it issues the counterclockwise TOKEN
message described above.

This simulator for simple cycles, in which messages are
interpreted as DATA, END, TOKEN or REQUEST based only on
their direction and order of transmissions, is formally given
and proved in Sect. 3.

**Main result: resilient computations over 2-edge-connected
graphs** To apply our approach for simple cycles to more com-
plex graphs, we mimic it over a (not necessarily simple) cycle
that goes through all the graph nodes. Such a cycle needs to
be chosen carefully, because of the crucial role that the direc-
tion of messages plays in our approach. Robbins’s theorem
[38] states that any 2-edge-connected graph \( G \) is orientable.
That is, there exists a way to orient all edges in \( G \) so that the
implied directed graph is strongly connected. This implies
that there exists a cycle that goes through all nodes, possi-
bly with multiple occurrences of some of the nodes, where all
instances of any edge along the cycle bear the same orienta-
tion. We leverage the existence of such a Robbins cycle by
mimicking our approach for the simple cycle over the Rob-
bins cycle. To this end, we must first construct a Robbins
cycle, as the nodes are unaware of the topology of the net-
work. Then, we need to communicate over the Robbins cycle.
Both steps are highly non-trivial and pose many challenges,
as we now describe. During the first step—the construction
of a Robbins cycle—we need the nodes to start communi-
cating over partial pieces of the cycle (which are cycles by
themselves), for which we need to already use the second
step. For this reason, we describe the two steps in reverse
order: we begin with describing the second step of commu-
nicating over a non-simple cycle, given that each node knows
its previous and next neighbors along the cycle for each of
its occurrences (Sect. 4). Then, we show the first step of how
to construct the Robbins cycle and produce this information
(Sect. 5).

**Second step: communicating over a non-simple cycle**
The input of each node for this step, as will be guaranteed
by our construction for the first step, is the previous and
next nodes along the cycle for each of its occurrences. These
inputs are consistent with some Robbins cycle, so that, in
particular, each edge in the cycle has a unique orientation,
and a single orientation of the edges is considered by all
nodes as the clockwise direction.

Mimicking our approach for a simple cycle over a Robbins
cycle brings along several challenges. Consider, for instance,
the network \( G \) and its induced Robbins cycle depicted in
Fig. 1. Suppose a clockwise message is received at node \( d \)
along the edge \((c, d)\). Should this message be propagated
over the edge \((d, e)\) or over the edge \((d, a)\), or maybe over
both? Note that both these options are in the clockwise direc-
tion, however, they belong to different segments of the cycle.
Further, note that some messages are initiated in an asyn-
chronous manner, e.g., the REQUEST message. Thus, when
the node \( d \) receives a REQUEST message from node \( c \), it is
possible that the request originated at node \( a \) and should be
propagated to node \( e \) or it originated at node \( e \) and should be
propagated to node \( a \).

We cope with these issues using two separate mechanisms.
The first mechanism makes sure that the TOKEN, DATA, and
END messages are propagated correctly along the Robbins
cycle. This mechanism consists of two main ingredients.
First, we guarantee that these three message types are for-
warded in a sequential manner, in the sense that the token
holder issues the next message among them only after receiv-
ing the previous one from the other direction of the cycle.
Second, we assure that at any given moment, each node \( u \)
knows “where the token is”, that is, on which segment of the cycle (i.e., between which two occurrences of \( u \)) the token resides. Since the token holder is the only node to initiate the above three message types, knowing the relative position of the token holder resolves the above and allows each node to track each message along the Robbins cycle. Indeed, each such message must first arrive from the segment in which the token holder resides and then be propagated by \( u \) to the next segment in the respective direction of the cycle. We prove that since at any given moment only one message travels through the cycle, there can be no confusion at \( u \) regarding what the message type is, which one of \( u \)’s occurrences has received a message and where a message should be forwarded to.

The second mechanism we employ is for request messages. These have no pre-specified origin, and they can be initiated by any node and even by multiple nodes at the same time. The mechanism for these messages is as follows. Whenever a node receives a request message or when a node wishes to initiate one, it sends a clockwise message to all of its clockwise neighbors along the Robbins cycle at the same time. Then, the node waits to receive a request message from each of its counterclockwise neighbors and only then it continues with executing the cycle algorithm described above. We prove that this guarantees that all nodes send and receive a request message regardless of their position(s) on the Robbins cycle.

**First step: a content-oblivious construction of a Robbins cycle** Our Robbins cycle construction follows an ear-decomposition technique by Whitney [47], claiming that any 2-edge-connected graph \( G \) can be decomposed into edge-disjoint parts, \( G = C_0 \cup E_0 \cup E_1 \cup \cdots \cup E_k \), where \( C_0 \) is a simple cycle, and for any \( 0 \leq i \leq k \), \( E_i \) is an ear—a simple path or cycle whose endpoints belong to \( C_0 \cup E_0 \cup \cdots \cup E_{i-1} \). Following Whitney’s work, we iteratively decompose a 2-edge-connected graph \( G \) into some \( C_0 \), \( E_0 \), \( \ldots \), \( E_k \), part by part, and combine them into a Robbins cycle. The main obstacle we face is that our construction must be content-oblivious and cannot rely on the content of messages sent by the nodes.

The first stage of our construction is performing a DFS-like search starting from a specified root node. The DFS search progresses by sending a message (a DFS-token) sequentially, i.e., each node propagates this message to one of its unexplored adjacent edges. This DFS-token message propagates through the network until it reaches the root node again. At this stage, the path the DFS-token has taken defines a cycle \( C_0 \).

The key challenge in this stage is that the DFS-token might reach some node \( u \) twice before reaching the root. This might cause the DFS to “get stuck”, e.g., if \( \deg(u) = 3 \). We overcome this pitfall by insisting on \( C_0 \) being a simple cycle that starts and closes at the root. If some \( u \neq \text{root} \) receives the DFS-token for the second time, it sends that message back on the same edge on which it was received. This has the effect of “backtracking” that edge so it is excluded from the constructed cycle. Nodes that backtrack all their adjacent edges go back to their initial state and are added to the Robbins cycle at a later step.

Once \( C_0 \) is established, the nodes on it switch to the second stage, in which they use our resilient communication approach of the above second step, in order to coordinate exploring further ears. One node on \( C_0 \) that has adjacent edges that do not belong to \( C_0 \) gets selected to initiate another DFS-like search, which again propagates in \( G \) until reaching a node on \( C_0 \), possibly different from the initiator. The path the DFS-token takes defines the ear \( E_0 \). Then, the nodes on \( C_0 \) and \( E_0 \) jointly coordinate to form a new non-simple cycle \( C_1 \) that includes all the edges in \( C_0 \) and \( E_0 \). The nodes on \( C_1 \) switch to communicate over this cycle using the above resilient communication of the second step. The nodes iterate this process, until a Robbins cycle is formed. A crucial aspect of these iterations of adding ears is that much coordination is required among the nodes for switching in a timely manner from communication on \( C_i \) to communication on \( C_{i+1} \). The technical specification of this mechanism is given in Sect. 5.

We emphasize that the nodes do not know \(|V|\), and hence they do not know when a Robbins cycle is already formed, i.e., when each node already appears on the current \( C_i \) at least once. Instead, they keep adding edges to the constructed cycle, until no node has an adjacent edge that is not in \( C_i \), which is a state they can detect. At that point, the construction ends.

**Putting it all together** With the above two steps, our result can now be formally stated. Given any 2-edge-connected fully-defective network \( G \) and an asynchronous algorithm \( \pi \) designed to work on the noiseless \( G \), we show how to compute \( \pi \) over the fully defective \( G \) by first constructing a Robbins cycle \( C \) on \( G \) using a resilient content-oblivious algorithm, and then simulating \( \pi \) over the Robbins cycle \( C \) in a resilient content-oblivious manner.

**Theorem 2 (main)** There exists a simulator for any asynchronous algorithm \( \pi \), such that executing the simulator over a 2-edge connected fully-defective network \( G \) simulates an execution of \( \pi \) over the noiseless network \( G \).

The simulator has a pre-processing phase that constructs a Robbins cycle \( C \) (which depends only on \( G \)) and an online phase that simulates the communication of \( \pi \) over \( C \). The pre-processing step communicates \( C_{\text{init}} = |C|^0(1) \) bits. In the online phase, any message \( m \) communicated by \( \pi \) is simulated by communicating \( C_{\text{overhead}}(m) = O(|C| \cdot |m| + |C| \log |V|) \) bits.

We note that, in the worst case, \(|C| = O(|V|^2)\); see Sect. 5.3 for a detailed discussion. We do not strive to optimize the polynomial overhead of our schemes, as their mere existence is the focus of this paper. Nevertheless, unary encoding as explained above imposes an exponential overhead in the
length of the message. In Sect. 3.3 we offer a binary encoding method that reduces the communication complexity to the polynomial terms stated above.

**Impossibility result** We complement the above result and show that if $G$ is not 2-edge-connected, then there is no way to conduct non-trivial computations over a fully-defective $G$. To this end, we prove the following impossibility for two-party computation over a fully-defective channel. The impossibility for a non 2-edge-connected $G$ follows since it contains a bridge, and we can reduce the two sides of the bridge to the two-party case.

**Theorem 3** Fix a non-constant function $f(x, y)$. No two-party deterministic algorithm that gives output or terminates can compute $f$ over a fully-defective channel.

The theorem requires the nodes to either terminate or irrevocably give an output. Note that the above theorem differs from the famous two generals coordinated-attack impossibility [21], since our noise model does not allow deleting messages. See Sect. 6 for complete details.

### 1.2 Related work

There are two common ways to deal with channel corruptions. One is by adding redundancy, i.e., coding the information, an approach that is known in the literature as Interactive Coding. The other is by diverting the communication so it would not pass through corrupted edges, which are known as Byzantine edges.

We review some related work in these areas, but we stress that neither approach can be used in fully-defective networks: Interactive coding must assume some bound on the errors, either per channel or globally, while solutions for networks with Byzantine edges must assume a bound on the number of noisy channels.

Interactive coding was initiated by the seminal work of Schulman [35, 43, 44], see [15] for a recent survey on this field. In this setting, communication channels either suffer from stochastic noise [3, 6, 17, 19, 35] or from some bounded amount of adversarial noise. E.g., if limiting the overhead of the coding scheme to be linear, [18, 19, 24, 25, 28] develop schemes resilient to up to a fraction $O(1/|E|)$ of the total communication. Without any restriction on the overhead, schemes can cope with noise up to a fraction $O(1/|V|)$ of the total communication, and such a fraction is shown to be maximal [25]—otherwise, the adversarial noise could completely corrupt all the outgoing communication of the node that communicates the least. The above works assume synchronous networks. Censor-Hillel, Gelles, and Haeupler [8] developed the first coding scheme for asynchronous networks that suffer from up to a fraction $O(1/|V|)$ of adversarial noise. Communication with an unbounded (yet, finite) amount of noise was examined in [1, 9, 16] for the two-party case and in [2] for the multiparty case. In a work by Efremenko, Haramaty, and Kalai [13], the noise model is similar to the one we consider here in the sense that it can corrupt the content of messages but not their existence. However, the amount of bit-corruptions in [13] (measured as the edit distance between sent and received messages) is bounded to a constant fraction out of the entire communication. Furthermore, their work considers only two parties.

Networks with Byzantine edges do not restrict the amount of noise per link, and even allow insertion/deletion errors, but allow only a bounded number of links to be noisy. In asynchronous settings, Fisher, Lynch, and Paterson [14] exclude the existence of consensus algorithms when a single node may crash, or equivalently, when all the links connected to some single node may crash. In synchronous networks, certain tasks are also impossible with arbitrary link failures [21, 41]. On the other hand, Santoro and Widmayer [39] considered distributed function evaluation when (a large number of) links suffer either corruptions, insertions, deletions, or their combination. In a sense, the synchrony guarantee allows simpler solutions, e.g., encoding information via the time in which messages are sent. Pelc [32] shows that if the number of Byzantine links is bounded by $f$, robust communication is achievable only over graphs whose edge-connectivity is more than $2f$. This is also implied by the work of Dolev [11]. Additional works [5, 10, 20, 34, 45] consider the case of mixed node and link failures.

Recent work by Hitron and Parter [22, 23] gives a compiler that turns any algorithm in the noise-free setting into an algorithm that works correctly even if the adversary controls $f$ edges in a $(2f + 1)$-edge-connected network. The above is for the synchronous Congest setting. Their approach is to construct a family of low-congestion cycle-covers (see also [30, 31]), which are structures in which for every edge $(u, v)$, there are at least $f + 1$ cycles that contain no adversarial edges. We stress that low-congestion cycle-covers do not seem to be helpful for our setting: Even if we were promised only two cycles that share a single edge, it is not clear how to communicate over them in a way that distinguishes one from the other.

### 2 Preliminaries

**Notations** We use $a \parallel b$ or $a \cdot b$ for the concatenation of $a$ and $b$. For a positive integer $k \in \mathbb{N}$ and a string $b$, we let $b^k = b \cdot b \cdot \cdots \cdot b$ denote $b$ concatenated to itself for $k$ times; $b^0 = \varepsilon$ is the empty string. For a string $b$ and an integer $0 \leq i \leq |b| - 1$, we let $b_i$ denote the $i$-th bit of $b$, i.e., $b = b_0b_1 \cdots b_{|b|-1}$.

**Networks and protocols** A protocol $\pi$ over an undirected network $G = (V, E)$ with $n = |V|$ nodes is an asynchronous event-driven distributed algorithm, in which nodes conduct
some computation by sending messages to their neighbors in \( G \) (for simplicity, we assume only deterministic algorithms in this paper). Upon the reception of a message, \( \pi \) instructs the recipient node what message(s) to send next, as a function of the node’s input and all the messages it has received so far. Specifically, each node \( v \) begins with a private input \( x_v \) (which may be empty), and knowledge of the IDs of its neighboring nodes, \( N(v) = \{ u \mid (u, v) \in E \} \) (we can remove this assumption, see Remark 6). According to the input to \( v \), \( \pi \) generates messages to send to zero or more of \( v \)'s neighbors (possibly different messages to different neighbors). Afterwards, the protocol behaves in an event-driven manner, i.e., nodes act only upon receiving messages: whenever a node \( v \) receives a message, it performs some computation and produces messages designated to zero or more of its neighbors. We impose no assumption on the computation time of \( \pi \) except that it is finite. We additionally assume a preselection of one designated node (which will function as a root node in our Robbins cycle construction), and assume that every node knows whether it is the designated node.

Communicating a message over some link of \( G \) takes arbitrary positive finite time. Channels are not assumed to be FIFO. Incoming messages are kept in an incoming buffer until processed by the node.

The protocol’s transcript \( \tau \) of a given execution, is the sequence of messages sent and received during the execution. Each item in \( \tau \) indicates the message sent or received, the sending or receiving node and the link on which the message was communicated. Events that happen in different nodes at the same time are assumed to be ordered in some arbitrary order. The local transcript \( \tau_v \) of a node \( v \), is the ordered sequence of messages sent and received by \( v \). Note that \( \tau_v \) can be derived from \( \tau \) as the sub-sequence in which \( v \) is the sending or receiving node.

We say that \( \pi \) gives an output if every node eventually writes an output to its write-only output register. This action is irrevocable. If needed, the node may remain active and send and receive messages after giving an output; that is, we do not require termination, but our result also applies to protocols that terminate. We say that the protocol has reached quiescence at some time, if no message is still in transit and from that time on, no new messages are sent over the network.

Fully-defective networks and noise-resilient simulations

We work in networks with noisy channels exposed to alteration noise, which can corrupt the content of any message communicated over any channel. That is, once a message \( m \in \{0, 1\}^+ \) is sent over some link, the received message may be any \( m' \in \{0, 1\}^+ \). However, the noise cannot completely delete a message nor can it inject a message on a link in which no \( m \) was sent. We stress that, except for inserting and deleting messages, the noise has no restrictions at all. In particular, it can apply to all channels and corrupt all messages in a given execution. We call networks that suffer noise as specified above fully-defective networks. Equivalently, one can think about such a network as one in which nodes communicate only by means of sending pulses to their neighbors, which could be the case, for instance, when the nodes have very basic communication hardware.

A noise-resilient simulator designed for a noiseless network \( G = (V, E) \) is a protocol \( \hat{\pi} \) which is given as an input an asynchronous black-box interface to some \( \pi \). When \( \hat{\pi} \) is executed on a fully-defective network \( G \), it produces for each node \( v \in V \) a string \( \hat{\tau}_v \), such that there exists some execution of \( \pi \) over the noiseless network \( G \) that generates a transcript \( \tau_v \) for which \( \tau_v = \hat{\tau}_v \) for each node \( v \). We allow a simulator to perform some pre-processing before simulating \( \pi \). We define \( \text{CC}_{\text{init}} \) to be the communication complexity in bits of the simulator during the pre-processing, and \( \text{CC}_{\text{overhead}}(m) \) to be the communication complexity for simulating the delivery of a message \( m \). Note that \( \text{CC} \) accounts only for the length of sent messages, even if later their content is corrupted by the noise.

Distributed representation of cycles A (directed) cycle can be represented in a distributed network in two manners: locally and globally. A local representation of some cycle \( C \) means that every node on \( C \) knows its two neighbors on the cycle along with their respective direction, clockwise or counterclockwise, usually held in the local variables next and prev, respectively. In case \( C \) is not a simple cycle, then every node knows its clockwise and counterclockwise neighbors for each of its occurrences on \( C \). This information is consistent across all nodes in the sense that an outside observer who follows the neighbors and directions of each node would see a consistent directed cycle.

A global representation of a directed cycle means that every node \( v \in C \) holds the string \( C = (v_1, v_2, \ldots) \) of the IDs of the nodes on \( C \) in their clockwise order.

3 Simulating computations over a fully-defective simple cycle

As discussed in Sect. 1, we can establish a resilient connection between two nodes connected by two separate links, sending content-less messages between them, which we will call pulses throughout this paper. Our goal is to implement this idea for any two nodes in a 2-edge-connected graph, since in such a graph any two nodes are connected by two separate paths. As a stepping stone, in this section we consider the special case of simple cycles.

Theorem 4 (A simulator for a simple cycle) There exists a noise-resilient simulator for any asynchronous protocol \( \pi \) and any fully-defective simple cycle \( G \) in which each node knows its clockwise and counterclockwise neighbors. The
simulator features $C_{\text{init}} = 0$ and $C_{\text{overhead}}(m) = O(|V| \cdot |m| + |V| \log |V|)$ pulses.

Let $G$ be a simple cycle on $V = \{v_i\}_{0 \leq i \leq n-1}$ with $E = \{(v_i, v_{i+1})\}_{0 \leq i \leq n-1}$, where indices are taken mod $n$. The main idea is to imitate the two-channel idea described in Sect. 1 above over the cycle. That is, suppose $v_i$ wishes to send a message to its neighbor $v_{i+1}$. We can think of the link $(v_i, v_{i+1})$ as the DATA channel, and on the path $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i+1}$ as the END channel. For this to work, all the nodes beside $v_i$ and $v_{i+1}$ need to simply forward each pulse they receive along the same direction. However, the above description supports only a single fixed sender and a single fixed receiver. Thus, we need a method that allows different nodes to become the sender. For this we use a token mechanism, where only a single node holds the token at any given time.

Our simulator can be split into two separate phases per message transfer: the first one is the token phase which handles transferring the token between the nodes, and the second one is the data phase that handles communication between the current token-holder and the rest of the nodes.

The token phase works as follows. At the starting point, there exists only a single token holder. During the token phase, pulses carry one out of two possible meanings: either they are a REQUEST pulse or a TOKEN pulse. The meaning of a pulse is dictated by the direction in which the pulse progresses along the cycle: REQUEST is a counterclockwise pulse while TOKEN is a clockwise pulse. A node that receives the token issues a REQUEST pulse. Every node that receives such a REQUEST pulse, propagates it in the same direction, unless it has already sent a REQUEST pulse previously in this phase, so eventually every node sends and receives a single REQUEST pulse.

Upon receiving a REQUEST pulse, the current (single) token holder releases the token by sending a counterclockwise TOKEN pulse. This pulse propagates along the cycle until it reaches one of the nodes which requested the token. A node that receives the TOKEN pulse and wishes to become a token holder does not propagate the TOKEN pulse but instead sets itself as the new token holder. Then, the new token holder switches to its data phase and begins sending clockwise pulses, which are interpreted as DATA pulses. The first of these pulses propagates throughout the entire cycle and informs all the other nodes that the token phase has completed. This first pulse cannot be confused with a REQUEST pulse since we guarantee that every node sends and receives exactly a single REQUEST pulse in each token phase. In other words, the second clockwise pulse received during a token phase must be a DATA pulse, which triggers its recipient to switch to its data phase.

In the data phase, the token holder delivers its message via a unary coding. That is, it sends a number of clockwise DATA pulses that equals the length of the unary encoding of the information. Each node other than the token holder forwards each received DATA pulse clockwise, so these pulses propagate along the cycle until they reach the token holder back from the other side of the cycle. Then, the token holder sends a single counterclockwise END pulse that signals the end of the message and the end of the data phase. Note that once the token holder receives the END pulse from the other direction, all nodes know that the data phase is over, and are back in the token phase. Note also that due to the asynchrony, nodes that already moved to the next token phase might send a REQUEST pulse before the END pulse arrives at the token holder. Our design promises that these REQUEST pulses are not confused with pulses of the current data phase: END pulses are sent in the other direction, and as for DATA pulses—the token holder does not proceed to sending a REQUEST pulse before it receives the END pulse of the data phase, so REQUEST pulses of the new token phase can only reach nodes that have already received the END pulse for this phase and therefore do not interpret them as additional DATA pulses.

A phase is a local concept, in the sense that each node runs a specific data or token phase in any given time, and different nodes might be in different phases in a certain time. We denote each token phase and its subsequent data phase an epoch. An epoch is a local concept too, viewed by each node according to the phase it is currently running. Different nodes might be in different epochs in a certain time: some nodes might already send a REQUEST pulse in the new epoch while others have still not received an END pulse for the previous epoch.

3.1 Formal description

We now formally describe our simulator over fully-defective simple cycles, where each node is given the identities of its clockwise and counterclockwise neighbors. Our simulator receives as an input an asynchronous protocol $\pi$ for noiseless communication channels. Messages to be sent are generated by $\pi$, and any message received by a node in our simulator is delivered and processed by $\pi$. Our simulator thus treats $\pi$ as an asynchronous black box that interfaces with the simulator by sending and receiving messages, internally at each node. We stress that $\pi$‘s actions take finite arbitrary time unknown to and independent of the simulator algorithm.

Our simulator appears in Algorithms 1(a) and 1(b). All nodes begin executing the token phase (Algorithm 1(a)). Each node $u$ has an internal isTokenHolder$_u$ variable that indicates whether it is the token holder. Moreover, each node $u$ keeps a queue $Q_u$ of messages generated by $\pi$, which should be broadcast over the cycle. Messages in $Q_u$ are of the form $(m, u, v)$, where $m$ is a message that $\pi$ instructs $u$ to send to $v$. At the onset, isTokenHolder$_u$ is True for a single node, and each $Q_u$ is empty. When $\pi$ gives an output,
the respective node gives the same output in the simulator but keeps executing the communication algorithm over the cycle. If in a certain time all the queues \( \{Q_v\} \) are empty and remain empty, then the simulator stops sending messages and reaches quiescence.

The simulator is content-oblivious, and as such it communicates by sending pulses (content-less messages). Note that in our algorithms we write next to each pulse its meaning (DATA, END, REQUEST, TOKEN), however, this is only for the analysis; the nodes assign this meaning according to their current state and the clockwise/counterclockwise direction of the pulse, and not by the pulse content, which is ignored.

### Algorithm 1(a) A simulator for simple cycles: token phase (node \( u \))

**Init:** A single node has \( isTokenHolder_u = True \). Node \( u \) holds a (possibly empty) input \( x_u \) for \( \pi \).

**Handling messages sent by \( \pi \):** During the execution of the algorithm, node \( u \) enqueues to \( Q_u \) any new message \( \pi \) asks \( u \) to send, in the form (message, source, destination). The actions of \( \pi \) occur in parallel to the execution of this algorithm.

1: \[ \text{wait until } Q_u \text{ is not empty or a clockwise REQUEST pulse is received} \]
2: \[ \text{send a clockwise REQUEST pulse} \]
3: \[ \text{if no clockwise REQUEST pulse was received then wait until receiving a clockwise REQUEST pulse} \text{ end if} \]
4: \[ \text{if } isTokenHolder_u \text{ then} \]
5: \[ \text{isTokenHolder}_u \leftarrow False \]
6: \[ \text{send a counterclockwise TOKEN pulse} \]
7: \[ \text{end if} \]
8: \[ \text{wait until receiving a pulse} \]
9: \[ \text{if the pulse is a counterclockwise TOKEN pulse then } \] Else, the pulse is a clockwise DATA pulse
10: \[ \text{if } Q_u \text{ is not empty then} \]
11: \[ \text{isTokenHolder}_u \leftarrow True \]
12: \[ \text{else} \]
13: \[ \text{forward the counterclockwise TOKEN pulse} \]
14: \[ \text{go to Line 8} \]
15: \[ \text{end if} \]
16: \[ \text{end if} \]
17: \[ \text{continue with Algorithm 1(b)} \]

### Algorithm 1(b) A simulator for simple cycles: data phase (node \( u \))

18: \[ \text{if } isTokenHolder_u \text{ then} \]
19: \[ \text{dequeue a message from } Q_u, \text{ denote it by } (m, u, v) \text{ and let } 1^d \text{ be its unary encoding} \]
20: \[ \text{send } d \text{ clockwise DATA pulses} \]
21: \[ \text{wait until receiving } d \text{ clockwise DATA pulses} \]
22: \[ \text{send a counterclockwise END pulse} \]
23: \[ \text{wait until a counterclockwise END pulse is received} \]
24: \[ \text{else} \]
25: \[ \text{forward any received clockwise DATA pulse until receiving a counterclockwise END pulse} \]
26: \[ \text{end if} \]
27: \[ \text{decode } 1^d \text{ as the unary encoding of } (m', u', v') \]
28: \[ \text{if } u = v' \text{ then deliver } m' \text{ to } \pi \text{ (as if received from } u' \text{)} \text{ end if} \]
29: \[ \text{forward the counterclockwise END pulse} \]
30: \[ \text{end if} \]
31: \[ \text{continue with Algorithm 1(a)} \]

### 3.2 Analysis

Let us set some notation for the analysis of Algorithm 1. Let ‘1’ indicate a pulse sent clockwise, and let ‘0’ indicate a pulse sent counterclockwise. Recall that an *epoch* is the execution of consecutive token and data phases. We say that a node has completed its \( k \)-th epoch once it has executed Line 31 for the \( k \)-th time. Let \( T_k \) be the \( k \)-th node to have set its \( isTokenHolder_u \) to True in Line 11, whereas \( T_0 \) is the node whose \( isTokenHolder_u \) variable is initialized to True. (We will show that \( T_k \) sends, in its \( k \)-th epoch, the \( k \)-th simulated message in the system.) Let \( s_k \), for \( k \geq 1 \), be the time in which \( T_k \) sets its \( isTokenHolder \) \( \leftarrow \) True \( (s_k = \infty \text{ if } T_k \) is undefined). Finally, let \( t_k \) be the time in which \( T_k \) completes its \( k \)-th epoch \( (t_k = \infty \text{ if } T_k \) is undefined or never ends the \( k \)-th epoch). We let \( s_0 = t_0 = 0 \). Let \([u \sim v]\) denote the clockwise path from \( u \) to \( v \) along the cycle including both ends, and similarly let \([u \sim v]\) denote the counterclockwise path from \( u \) to \( v \). In the special case of identical endpoints, \([u \sim v]\) denotes the path through the whole cycle. To exclude an endpoint, we use a round bracket in place of a square bracket, e.g., \([u \sim v]\) denotes the clockwise path excluding \( v \); the path can be empty, i.e., \((u \sim v)\) for \( u, v \) neighbors.

Our analysis is based on the following technical lemma, which provides us with three important properties satisfied by Algorithm 1 in every epoch: (1) *progress*, which says that as long as there is a message to send, the next epoch will eventually start and complete; (2) *single token holder*, which says that at most a single node holds the token at any moment (there is no such node during the time in which the token is being passed), and \( T_k \) is the only one to hold it during the data phase of the \( k \)-th epoch; and (3) *global consistency*, which says that in any given epoch \( k \), exactly one message is being communicated—sent by \( T_k \) and received by all other nodes, and the pattern of pulses every node sends has a distinct structure. We now formalize these ideas as follows.

**Lemma 5** Consider an execution of Algorithm 1 and consider any \( k \geq 1 \), for which \( t_{k-1} < \infty \). If from time \( t_{k-1} \) and forward, all queues \( \{Q_v\}_{v \in V} \) are always empty, then \( t_k = \infty \). Otherwise, the following hold:

1. **Progress** All nodes eventually complete their \( k \)-th epoch.
   In particular, \( t_k < \infty \), and at time \( t_k \), all nodes have not yet passed the END pulse (of epoch \( k \)) but have not yet passed Line 8 in epoch \( k + 1 \) (they are either
waiting in Lines 1 or 3 for a REQUEST pulse, or waiting in Line 8 for either a DATA or a TOKEN pulse).

(2) **Single token holder** It holds that $t_{k-1} < s_k < t_k$. At each moment in $(t_{k-1}, t_k)$, there is at most a single node for which $\text{isTokenHolder} = \text{True}$. More specifically, within this time frame, the token is passed as follows: the node $T_{k-1}$ releases the token at some time in $[t_{k-1}, s_k)$ and the node $T_k$ is the next node that gains the token at time $s_k$. The node $T_k$ (solely) holds the token in $[s_k, t_k]$.

(3) **Global consistency** There exist integers $d_1, \ldots, d_k > 0$ and for any $u \in V$ there are $b^n_u, \ldots, b^k_u \in \{0, 1\}$, such that when the node $u$ completes its $k$-th epoch, its sent transcript (the overall pulses sent so far by $u$) is $P_{u,k} = 10^{b^k_u}1^{d_k}0 \cdot 10^{b^{k-1}_u}1^{d_{k-1}}0 \ldots 10^{b^1_u}1^{d_1}0$. In addition, the message each node decodes and processes (Lines 26–28) in its $k$-epoch is the unary decoding of $1^{d_k}$, which is the message sent by $T_k$ (Line 19) in its $k$-th epoch.

**Proof** We prove the statement by induction on the epoch number $k$. We start with proving the base case, $k = 1$. The proof for the general case is very similar. The analysis follows the progress of the protocol and shows that each pulse sent with a certain meaning (i.e., DATA, END, TOKEN, REQUEST) is correctly interpreted by its recipient.

**Base case.** $k = 1$ Note that $t_0 = 0$, hence, $n_{k-1} < \infty$. All the nodes begin by executing Algorithm 1(a), with a single node $T_0$ having $\text{isTokenHolder} = \text{True}$. While all nodes have empty queues $Q_v$, they all wait in Line 1 and thus, if the queues remain empty indefinitely, we have $t_1 = \infty$.

Otherwise, at some time there is at least one node $u$ that queues to $Q_u$ a message to be simulated. Each such $u$ sends a REQUEST pulse in Line 2 and waits to receive a REQUEST pulse in Line 3, unless it has already received such a pulse in Line 1. As there is at least one such node, at least one REQUEST pulse is sent. The rest of the nodes first wait to receive a REQUEST pulse and then forward it. It follows that all nodes eventually receive and send a single REQUEST pulse. Let us denote by $\tilde{P}_u$ the partial transcript of a node $u$ at the “current” time (which evolves with the proof), then $\forall u \in V$, we have $\tilde{P}_u = 1$ after sending the REQUEST. To prove Property (3), we keep track of the partial transcript $\tilde{P}_u$, recording the pulses sent by each node.

After sending and receiving a REQUEST pulse, any node $u \neq T_0$ waits to receive another pulse (Line 8). The node $T_0$ sets $\text{isTokenHolder}_{T_0} = \text{False}$, sends a counterclockwise TOKEN pulse (Line 6) and then waits for another pulse like all other nodes. The TOKEN pulse triggered by $T_0$ propagates counterclockwise until it reaches a node $v$ with a non-empty $Q_v$, which must exist. The node $v$ subsequently sets $\text{isTokenHolder}_v$ to True (Line 11). Thus, by the above definitions, we get that $T_1 = v$ and $s_1$ is the time when $v$ executes Line 11. Note that $T_1$ might get the TOKEN pulse before getting a REQUEST pulse, in which case it delays its actions until a REQUEST pulse is received. This has no effect on the proof.

In case $T_1 \neq T_0$, at time $s_1$, all the nodes on $[T_0 \rightsquigarrow T_1]$ have sent a TOKEN pulse and are now waiting for a DATA pulse (Line 8) that would switch them to their data phase of epoch $k = 1$. The nodes on $(T_0 \rightsquigarrow T_1)$ could be in two possible stages: either they are still waiting for a REQUEST pulse (Lines 1 or 3) as described above, or they are waiting in Line 8. Hence at time $s_1$, every node $u \in [T_0 \rightsquigarrow T_1]$ has $\tilde{P}_u = 10$, while every node $u \in (T_0 \rightsquigarrow T_1)$ has $\tilde{P}_u = 0$ if it has not yet sent a REQUEST pulse, or $\tilde{P}_u = 1$ otherwise. Eventually, perhaps at a different time per node, each node $u$ thus reaches the partial transcript $\tilde{P}_u = 10^{b^1_u}$ with $b^1_u \in \{0, 1\}$ being the indicator of whether $u$ has sent a TOKEN pulse (namely, whether $u \in [T_0 \rightsquigarrow T_1]$).

In the special case where $T_1 = T_0$, at time $s_1$, the token has just reached back at $T_1$; all nodes have sent a TOKEN pulse, and all nodes but $T_1$ are now waiting for a DATA pulse (Line 8) that would switch them to their data phase of epoch $k = 1$. Hence at time $s_1$, every node $u$ has the partial transcript $\tilde{P}_u = 10^{b^1_u}$ with $b^1_u = 1$ indicating that $u$ has sent a TOKEN pulse.

When the node $T_1$ switches to the data phase (Algorithm 1(b)), its queue $Q_{T_1}$ is non-empty and so it sends $d \geq 1$ clockwise DATA pulses (Line 20). We define $d_1 = d$. These DATA pulses propagate clockwise through all nodes, after the first received DATA pulse in each node but $T_1$ triggers it to switch to its data phase, after it has previously received a REQUEST pulse. Note that each such node must have received a REQUEST pulse before it receives the first DATA pulse. This is because its counterclockwise neighbor, who sends the DATA pulse, moves to the data phase only after it has sent a clockwise REQUEST pulse.

Once a node $u \neq T_1$ is in its data phase of epoch $k = 1$, it records each received DATA pulse. The node propagates the pulse clockwise and eventually the pulse arrives back at $T_1$. Thus, since $T_1$ sends $d_1$ DATA pulses, after propagating them, each node $u$ has $\tilde{P}_u = 10^{b^1_u}1^{d_1}$. Once the $d_1$ clockwise pulses reach back at $T_1$, and only then, it issues a counterclockwise END pulse (Line 22). $T_1$ does not generate nor does it propagate any additional pulses before receiving the propagated END from the other side of the cycle. This implies that any $u \neq T_1$ receives exactly $d_1$ clockwise DATA pulses followed by a counterclockwise END pulse. Upon receiving the END pulse, $u$ processes the message $1^{d_1}$ (Lines 26–28) and forwards the END pulse (Line 29). It then completes its $k$-th data phase and its $k$-th epoch, with $\tilde{P}_u = 10^{b^1_u}1^{d_1}0$. The node $T_1$ also has $\tilde{P}_{T_1} = 10^{b^1_u}1^{d_1}0$ when it receives the END pulse and switches to the next token phase, at time $t_1$. At that time, all the other nodes have already processed the END pulse. This proves the first part of Property (1).
Next, we need to prove that at time $t_1$, none of the nodes has passed Line 8. In order for a node to pass Line 8, it must be the case that after the node has switched to the token phase, it has received a REQUEST pulse followed by one additional pulse (in any direction). We argue this cannot happen. Indeed, at the time where some node $v$ receives the END pulse and switches to its second token phase, only the nodes in $\{T_1 \rightsquigarrow v\}$ have received the END pulse and only these nodes have switched to their (second) token phase. In their token phase, they may or may not have sent a clockwise REQUEST pulse by this time. Thus, only nodes in $\{T_1 \rightsquigarrow v\}$ might have received a REQUEST pulse. However, it is impossible that they received an additional pulse by time $t_1$, as we next show. Each node in $\{T_1 \rightsquigarrow v\}$ that has received a REQUEST pulse is waiting to receive another pulse (Line 8) and is not generating any pulse. The node $T_1$, if receiving a REQUEST pulse, does not process it and does not send a REQUEST pulse before receiving an END pulse in Line 23. It also never sends a pulse in the counterclockwise direction before receiving its END pulse back. Furthermore, each node in $\{T_1 \rightsquigarrow v\}$ (for $v \neq T_1$) is still executing Line 25, so it only forwards pulses and never generates pulses. Finally, $v$ has just started its token phase and is waiting to receive a REQUEST. We conclude that no additional pulse (beyond the REQUEST pulse, if sent) can be received by the nodes in $\{T_1 \rightsquigarrow v\}$. This holds for any $v$ at the time it transitions to its second token phase. It thus holds for all nodes at time $t_1$, when the END pulse eventually reaches back at $T_1$.

Next we prove Property (2) based on the above description of the first epoch. At the onset (at time $t_0$), $T_0$ is the only node with $isTokenHolder_{T_0} = True$. As mentioned above, $T_0$ sets $isTokenHolder_{T_0} = False$ and sends a TOKEN pulse during its token phase. The propagated TOKEN pulse is the one that triggers $T_1$ to set $isTokenHolder_{T_1} = True$ later, at time $s_1$. Thus, it is clear that $s_1 > t_0 = 0$, and that $T_0$ releases the token before $s_1$ and $T_1$ becomes a token holder at $s_1$. Later, at time $t_1$, the node $T_1$ completes the first epoch, hence, $t_1 > s_1$. The node $T_1$ does not set $isTokenHolder = False$ during the time frame $[s_1, t_1]$, and it remains to show that it is the only token holder throughout this time frame.

It is clear that no node in $\{T_0 \rightsquigarrow T_1\}$ has set itself as a token holder as otherwise, that node would have been the node we indicate as $T_1$. After time $s_1$, no more TOKEN pulses are sent in the first token phase. Further, $T_1$ sends a clockwise DATA pulse that transitions all other nodes into their data phase. This implies that no node besides $T_1$ can execute Line 11 and set $isTokenHolder = True$ in this token phase. As for the second token phase, each node that reaches it before $t_1$ does not pass Line 8 before time $t_1$, as we have shown above, thus in particular, it does not receive a TOKEN pulse and does not reach Line 11.

Finally, we prove Property (3), that is, that all nodes reach a global consistency regarding the sent message of the first epoch. This follows from the above analysis: As we argued, at the time some node $u$ completes its first epoch, it holds that $P_u = 10^{d_u} 1^{d_u} 0$. The part $1^{d_u}$ corresponds to the $d_u$ DATA pulses initiated by $T_1$, which form the encoding of the message communicated in this epoch. This completes the proof of Property (3).

**Induction step** To complete the proof, we need to prove the induction step. Most of the above proof holds as is for $k > 1$, if we replace $s_1, t_1, T_1$ with $s_k, t_k, T_k$, etc.

Fix $k > 1$ with $t_{k-1} < \infty$ (otherwise, the lemma holds vacuously). We use the induction hypothesis on epoch $k-1$. We are allowed to do so since $t_{k-1} < \infty$, which implies that at or after time $t_{k-2}$ there is at least one non-empty $Q_v$ and the three properties of the lemma apply to epoch $k - 1$. The differences between proving the base case and the step are as follows:

In the case where all nodes have an empty $Q_v$, it is immediate in the base case that no node ever passes Line 1; we prove the same happens here. However, all the induction hypothesis gives us is that at $t_{k-1}$ all nodes are waiting either in Line 1 or 3, or 8. Clearly, nodes cannot be in Line 3 since their queue is empty. We now prove they cannot be in Line 8 as well.

Assume towards a contradiction that $v$ is the first to pass Line 1 in its $k$-th epoch. Let $i$ be the time when $v$ receives the END pulse of its epoch number $k - 1$. After time $i$, the node $v$ completes its epoch and transitions to its $k$-th token phase. Since $Q_v$ is empty, $v$ gets to Line 1 and awaits there for a REQUEST pulse. Because $v$ eventually reaches Line 8, it must have received a clockwise pulse from its neighbor $u$, which caused $v$ to pass Line 1.

For $v \neq T_{k-1}$, recall that at time $i$, the nodes $\{T_{k-1} \rightsquigarrow v\}$ are still in their data phase after receiving $d_{k-1}$ DATA pulses. Recall also that they do not generate new pulses but only propagate pulses, and they do not propagate any additional clockwise pulses because $T_{k-1}$ does not generate any clockwise pulses until the counterclockwise END reaches it. The above-mentioned neighbor $u$ belongs to $\{T_{k-1} \rightsquigarrow v\}$, hence, it does not propagate further clockwise pulses from time $i$ until $u$ gets the END pulse. In case $v = T_{k-1}$, $u$ gets the END pulse before time $i$.

In any case, after $u$ gets the END pulse, $u$ transitions to its $k$-th token phase and reaches Line 1. Therefore, if $u$ did send a REQUEST pulse that causes $v$ to pass Line 1, then $u$ would have also passed Line 1 prior to sending this REQUEST pulse, in contradiction to our assumption that $v$ is the first node to pass Line 1.

If some node has a non-empty $Q_v$, in the base case we had that all nodes begin the token phase at the same time $t_0$, and then send a REQUEST pulse if their queue is non empty or if they receive a REQUEST pulse. When considering the induction step at time $t_{k-1}$, some nodes may have already started their $k$-th epoch, and have already sent a REQUEST
pulse before time $t_{k-1}$, as given by Property (1) of the induction hypothesis. The behavior from this point on remains the same as described above for the base case.

For proving the global consistency property in the induction step, let $P_{u,k-1}$ be the transcript of $u$ at the end of its epoch $k-1$. By Property (3) of the induction hypothesis, we know that there exist $d_1, \ldots, d_{k-1}$ and $b_1^u, \ldots, b_{k-1}^u$ for any $u \in V$ such that $P_{u,k-1} = \pi = 10^{d_1^u}1^{d_2^u}0 \cdot 10^{d_2^u}1^{d_3^u} \ldots 10^{d_k^u}1^{d_{k-1}^u}0$ for any $u \in V$. Furthermore, the above analysis shows that there exist an integer $d_k^u > 0$ and an indicator $b_k^u \in \{0, 1\}$ per $u$, such that the pulses sent by node $u$ during its $k$-th epoch can be described by $10^{d_k^u}1^{d_{k-1}^u}0$, where $1^{d_k}$ signifies the DATA pulses sent by $T_k$, which encodes the communicated message of this epoch. This gives Property (3).

Next, we show that Lemma 5 implies the correctness of the simulator (Theorem 6). We then analyze its overhead (Sect. 7). Finally, we discuss in Sect. 3.3 ways to improve the obtained complexity (Lemma 9). Together, these three prove Theorem 4.

**Theorem 6** Let $G = (V, E)$ be a cycle. For any asynchronous protocol $\pi$, let $\hat{\pi}$ be the protocol defined by Algorithm 1 with the input $\pi$. Then, executing $\hat{\pi}$ on the fully-defective $G$ simulates an execution of $\pi$ on the noiseless network $G$.

**Proof** Let $E_{\hat{\pi}}$ be an execution of $\hat{\pi}$ over the fully-defective cycle $G$. We derive a transcript $\tau$ from $E_{\hat{\pi}}$, and claim that $\tau$ corresponds to a valid transcript of some execution of $\pi$ in the noiseless network $G$, which we denote $E_{\pi}$. The reader should distinguish between the simulated $\pi$ which is the black-box interface used as an input of Algorithm 1, and the protocol $\pi$ that generates the execution $E_{\pi}$ on the noiseless network $G$.

In order to avoid confusion, we will use the term *simulated* $\pi$ to denote the former and refer to $E_{\pi}$ when discussing the latter.

Let us specify the structure of the transcript $\tau$. We can think about it as an ordered sequence of events $\tau = \tau_1 \tau_2 \cdots$, where $\tau_i$ is either the event that some node $u$ sent a message $m$ to $v$, i.e., $\tau_i = (\text{send}, u, v, m)$ or the event that some node $u$ received a message $m$ from $v$, i.e., $\tau_i = (\text{receive}, u, v, m)$. To derive $\tau$ from $E_{\hat{\pi}}$, we follow the execution of $\hat{\pi}$ as the time evolves. We add a send event every time the simulated $\pi$ instructs node $u$ to send a new message $m$. Specifically, when node $u$ queues $M = (m, u, v)$ to $Q_u$, we add the event $(\text{send}, u, v, m)$ to $\tau$. Additionally, every time some node $v$ delivers the message $M = (m, u, v)$ to the simulated $\pi$ (Line 28), we add the event $(\text{receive}, u, v, m)$ to $\tau$. Recall that $\pi$ generates messages to $u$ sequentially and $\tau$ maintains this order; events that happen at the same time in different nodes are ordered arbitrarily in $\tau$.

Given $E_{\hat{\pi}}$ and its derived $\tau$, we prove that there exists an execution $E_{\pi}$ of $\pi$ on the noiseless network $G$ that produces these exact same events in the same order, i.e., such that $\tau$ is exactly the transcript of $E_{\pi}$.

The execution $E_{\pi}$ is obtained by executing $\pi$ over $G$ with the following scheduler that imitates the behavior of $E_{\hat{\pi}}$. Every time a node sends a message in $E_{\pi}$, the message is delayed at the channel and delivered only at the time the respective message is received in $E_{\hat{\pi}}$. That is, our scheduler “follows” the execution of $E_{\hat{\pi}}$, and delays each message until the time its corresponding message is delivered in $E_{\hat{\pi}}$. Specifically, whenever a receive event is registered in $\tau$ (in $E_{\hat{\pi}}$), we deliver the corresponding message in $E_{\pi}$. The scheduler also controls the execution time of all nodes, which enables it to control the timing of the send events $\pi$ initiates in $E_{\pi}$ so they correspond to the same order of send events in $E_{\hat{\pi}}$. We now show that the above defines a valid scheduler, and that the resulting $E_{\pi}$ has the transcript $\tau$.

Define time($j$) to be the time in $E_{\hat{\pi}}$ when the event $\tau_j$ is registered (note, for multiple events that occur at the same time, we let time($j$) refer only to the events up to $\tau_j$). We prove the following statement by induction on $j$: (1) The scheduler is valid: whenever instructed to deliver a message $m$, this $m$ was issued to the channel and hasn’t been delivered yet. (2) The transcript $\tau$ derived from $E_{\hat{\pi}}$ is a prefix of the transcript of $E_{\pi}$.

For the base case, time(0), the transcript $\tau$ is empty. It is clear that the scheduler is (vacuously) valid, and that $\tau$ is a prefix of the transcript of $E_{\pi}$.

We proceed with the induction step. Assume that the induction statement holds at time($j-1$), that is, at this point in time, the events $\tau_1 \cdots \tau_{j-1}$ are a prefix of the events in $E_{\pi}$, and all the actions of the scheduler so far are valid. Now consider the next event recorded to $\tau$. There are two options here, either it is a send event or a receive event.

In the first case, let $\tau_j = (\text{send}, u, v, m)$. Consider $E_{\pi}$ right after the event $\tau_{j-1}$, i.e., at time($j-1$). Every node $u$ in $E_{\pi}$ has exactly the same state as the simulated $u$ in the simulated $\pi$, which follows from the induction hypothesis. Therefore, if the simulated $\pi$ instructs node $u$ to send a message $m$ to $v$ (which triggers $\tau_j$), the same (eventually) happens at node $u$ in $E_{\pi}$. The scheduler delays all other nodes until the same message is sent in $E_{\pi}$, and the claim thus holds after event $\tau_j$, i.e., at time($j$) as well.

The other case is when the $j$-th event is a receive event, say, $\tau_j = (\text{receive}, u, v, m)$. Consider the node $u$ that executes Line 28 which corresponds to this event. By Property (3) of Lemma 5, this message is sent by the message sender of that epoch, $v$. Denote this epoch by $k$. This means that at the beginning of epoch $k$ the message $m$ appeared in $Q_v$, and was dequeued by $v$ at the beginning of the data phase of epoch $k$; note that dequeued messages are never enqueued back to $Q_v$. This means that at some point in time before time($j$), node $v$ enqueued this message to $Q_v$ and it was never dequeued before the $k$-th epoch; let $\tau_i$ with $i < j$ be the corresponding
event of enqueuing $m$ to $Q_v$. Now consider $E_\pi$. By the induction hypothesis we know that up till event $\tau_{j-1}$ at time($j-1$), the transcript $\tau$ describes the execution $E_\pi$, thus, the message $m$ was sent at time($i$) and is currently being delayed by the channel (since the first and only delivery of $m$ in $E_\pi$ occurs at time($j$)). The scheduler then instructs the channel to deliver this message, which is a valid action as this message was already issued to the channel and never delivered before. This completes the inductive proof.

The above proves that at any point in time the execution $E_{\hat{\pi}}$ over the fully-defective $G$ simulates a prefix of a valid execution of $\pi$ over the noiseless $G$. It remains to show liveness, namely, that the prefix keeps growing. This follows from Properties (1) and (3) of Lemma 5: the simulation makes progress as long as some $Q_v$ is non-empty or eventually becomes non-empty. Progress means that all nodes begin and complete their next epoch. In each epoch one message (from some $Q_v$) is being delivered to its destination. If the simulated $\pi$ of some node gives an output, the same node will give the same output in $E_{\hat{\pi}}$. If we consider the point in time where all nodes have given output, then all these outputs are valid since $\tau$ at that time is a prefix of the execution $E_{\pi}$, which also gives the same outputs, by definition.

The only situation where the simulation could reach quiescence is when all the queues $Q_v$ are empty and remain empty indefinitely. But $\tau$ up to that time, as argued above, is a transcript of some $E_\pi$ on the noiseless $G$, where no messages are currently delayed by any channel, and no new messages are going to be sent since the nodes in $E_\pi$ are at the same state as in the simulated $\pi$. Thus, $E_{\hat{\pi}}$ has reached quiescence as well. $\square$

Let us point out a couple of additional remarks about our simulation.

**Remark 1** **FIFO** The scheduler derived from our simulator maintains FIFO: Consider an execution $E_{\hat{\pi}}$ of the simulator $\hat{\pi}$. If multiple messages from $u$ to $v$ exist in the simulation, they are enqueued and communicated by their order. These enqueues translate in $E_\pi$ to messages sent over the same link. However, the scheduler for $E_\pi$ will deliver them according to their order in $E_{\hat{\pi}}$’s transcript, which is their order in $Q_u$, that maintains a FIFO property. This strengthens our result, that is, the simulation works correctly both with or without FIFO assumptions for the simulated protocol.

**Remark 2** **No starvation** Our proof shows that as long as some $u$ has a message to send, then some message will be sent during the next epoch. Since the TOKEN pulse travels counterclockwise sequentially in the cycle, there can be at most $n-1$ epochs until $u$ becomes the token holder. Thus, our simulator actually satisfies the stronger notion of no starvation.

**Remark 3** **Broadcast** We note that by design, our simulator offers an additional broadcast operation. That is, a node can send a message whose destination is all other nodes. To provide this functionality, we utilize the fact that every message arrives at all nodes, regardless of its original destination. To broadcast a message $m$, a node simply fixes its destination to $\epsilon$. Each node that decodes a message delivers it to $\pi$ if its destination is either that node (as before) or $\epsilon$. We will use this feature in our Robbins cycle construction in Sect. 5.

**Lemma 7** The overhead of simulating a single message $m$ in Algorithm 1 is $\text{CC}_{\text{overhead}}(m) = |V|O(1) \cdot 2^{|m|}$.

**Proof** Let $n = |V|$ be the length of the cycle. Suppose the message $M = (m, u, v)$ is dequeued in some epoch and is being communicated (i.e., $m$ is being communicated by the simulated $\pi$ over the link $(u,v)$). We can write $|M| = |m| + O(\log n)$. Communicating $M$ over the cycle results in the following pulses sent by each node during this epoch (Property (3) of Lemma 5): a single REQUEST pulse, at most a single TOKEN pulses, $2^{\lceil |M| \rceil}$ DATA pulses and a single END pulse. Since there are $n$ nodes, where each node sends at most $3 + 2^{\lceil |M| \rceil}$ pulses, we conclude that $\text{CC}_{\text{overhead}}(m) = O(n \cdot 2^{|m|+O(\log n)}) = n^O(1) \cdot 2^{|m|}$. $\square$

### 3.3 Reducing the communication via binary encoding

Encoding each message via a unary encoding leads to a pulse overhead that is exponential in the message size: $\text{CC}_{\text{overhead}}(m) = \text{poly}(n, 2^{|m|})$, with $n = |V|$. We now show how to send messages over a simple cycle via a binary encoding of the message. This binary encoding leads to a much improved communication complexity of $\text{CC}_{\text{overhead}}(m) = O(n \cdot |m| + n \log n)$.

Let $M = (m, u, v) \in \{0, 1\}^*$ be the message that the token holder $u$ wishes to send. The idea is to encode the bits of $M$ so that a clockwise pulse denotes the bit 1, and a counterclockwise pulse denotes the bit 0; we denote these as DATA(1) and DATA(0), respectively. Since the order of the bits is important, the token holder sends the next bit only after receiving the previous bit from the other direction of the cycle. As an optimization, all the pulses of consecutive same-bit sequences may be sent concurrently, and then the token holder should wait to receive all the pulses of the same direction before sending pulses in the other direction. For clarity of the presentation, we do not delve into the details.) However, now that counterclockwise pulses signify a 0 data bit, the challenge is that we need a different way to indicate the end of transmitting $M$, that is, we need a way to encode an END pulse.

We overcome this challenge by having the nodes agree on a fixed parameter $L$. In order to communicate that $M$’s transmission has completed (replacing the END pulse), the token
holder sends $L \geq 2$ consecutive counterclockwise pulses. In addition, the bitstring $M$ is padded with a 1 after every $L - 1$ consecutive 0s (when read from its first symbol and onward). An additional trailing 1 is sent after $pad(M)$ and guarantees that, even if $M$ has ended with a 0 or a sequence of 0s, then $pad(M) \cdot 1 \cdot 0^L$ has $L$ consecutive 0s only at its suffix. Furthermore, we add a preceding 1 before $pad(M)$: Recall that the token holder must initiate the sending protocol with a clockwise DATA pulse, as otherwise, the sender’s first counterclockwise pulse might be mistaken for a TOKEN pulse in those nodes that have not yet forwarded a TOKEN pulse and are still in the token phase. To summarize, in order to communicate in those nodes that have not yet forwarded a pulse and are still in the token phase. To summarize, in order to communicate the message $M$, the token holder communicates pulses according to the encoded message $Z = 1 \cdot pad(M) \cdot 1 \cdot 0^L$.

The revised data phase algorithm is given in Algorithm 2.

| Algorithm 2 Data phase: Broadcasting a message, binary version (node $u$) |
|---|
| 1: if isTokenHolder$_u$ then |
| 2: dequeue a message from $Q_u$ and denote it as $M = (m, u, v)$ |
| 3: let $pad(M)$ be the string obtained from $M$ by inserting a 1 after every $L - 1$ consecutive 0s of $M$ |
| 4: $Z \leftarrow 1 \cdot pad(M) \cdot 1 \cdot 0^L$ |
| 5: for $j = 0$ to $|Z| - 1$ do |
| 6: if $Z_j = 1$ then |
| 7: send a clockwise DATA(1) pulse |
| 8: wait until a clockwise pulse is received |
| 9: else |
| 10: send a counterclockwise DATA(0) pulse |
| 11: wait until a counterclockwise pulse is received |
| 12: end if |
| 13: end for |
| 14: else |
| 15: repeat |
| 16: forward every incoming pulse along the cycle, according to its original direction |
| 17: record each clockwise pulse as a 1 and each counterclockwise pulse as a 0 |
| 18: until $L$ consecutive 0s have been recorded |
| 19: let $Z$ be the recorded string. Parse $Z = 1 \cdot P \cdot 1 \cdot 0^L$ |
| 20: let $M' \leftarrow pad^{-1}(P)$ be the string obtained by removing any 1 that appears after a sequence of $L - 1$ consecutive 0s. Parse $M' = (m', u', v')$ |
| 21: if $u = u'$ then |
| 22: deliver $m'$ to $\pi$ (as if received from $u'$) end if |
| 23: continue with Algorithm 1(a) |

We argue that replacing Algorithm 1(b) with Algorithm 2 does not change the premise of Theorem 6. For the rest of this section, we change Line 17 in Algorithm 1(a) to say “continue with Algorithm 2”.

We show that an execution of Algorithm 1(a) along with Algorithm 2 satisfies a Global consistency property similar to the one of Lemma 5. One can easily verify that the Progress property and the Single token holder property hold as well, with the same proof as before.

**Lemma 8** Consider the following modification of the Global consistency property:

There exist strings $z_1, \ldots, z_k$, where $z_i = 1m_i10^L$ for some $m_i \in \{0, 1\}^+$, and for any $u \in V$ there are $b^u_1, \ldots, b^u_k \in \{0, 1\}$, such that when the node $u$ completes its $k$-th epoch, its sent transcript (the overall pulses sent so far by $u$) is $P_{u,k} \triangleq 10^{b^u_1}z_1 \cdot 10^{b^u_2}z_2 \cdot \cdots \cdot 10^{b^u_k}z_k$. In addition, the message each node decodes and processes (Line 20 in Algorithm 2) in epoch $k$ is exactly the message $pad^{-1}(m_k)$ sent by $T_k$.

Then the statement of Lemma 5 holds for the simulator given by Algorithms 1(a) and 2.

In order to avoid excessive repetition, we sketch below only the differences from the proof of Lemma 5 that stem from replacing Algorithm 1(b) with Algorithm 2.

**Proof** Recall from the proof of Lemma 5, that $T_k$ gains the token during its $k$-th token phase (at time $s_k$) since its $Q_{Tk}$ is non-empty and a TOKEN pulse arrives from its counterclockwise neighbor. The node $T_k$ then switches to its data phase (Algorithm 2).

The node $T_k$ dequeues a message $M$ from its queue $Q_{Tk}$ and sets $z_k = 1 \cdot pad(M) \cdot 10^L$ in Line 4. Thus, its first pulse is a clockwise DATA(1) pulse, which causes every other node $u$ to switch to its data phase and execute the code with $isTokenHolder_u = \text{False}$, similarly to the case in the proof of Lemma 5, upon receiving the first DATA.

Note that $T_k$ transmits bits sequentially and proceeds to the next bit only after the previous pulse is received from its other side of the cycle. That is, once the first DATA(1) pulse arrives back at $T_k$, it continues to communicating $pad(M) \cdot 10^L$, bit after bit.

The padding $pad(M)$ and the trailing 1 following it guarantee that there exists only a single substring of $L$ consecutive 0s in $z_k$, which resides at the suffix of $z_k$. It follows that all other nodes receive the string $z_k$: they record the message communicated by $T_k$ bit by bit, until they see $L$ consecutive 0s. This sequence appears only at the suffix of $z_k$ and signifies its termination. We can thus deduce that the message $Z$ recorded by each node has the structure $Z = 1 \cdot P \cdot 10^L$ so each node continues to extracting the part $P$ (whose length is unknown beforehand) and decodes $M' = pad^{-1}(P)$ to obtain the correct message $M' = M$ communicated by $T_k$ in Line 20. If the node is the recipient of $M$ it delivers it to its simulated $\pi$ (Line 21). Each node then updates its $k$-th epoch, and transitions to its token phase $k + 1$ (Algorithm 1(a)).
Since every received pulse is forwarded along the same direction it was received, during the $k$-th epoch each node $u$ transmits exactly the sequence of pulses described by $10^{2L} z_k$, and thus its overall sent transcript is $P_{u,k} = P_{u,k-1} \cdot 10^{2L} z_k$, which has the correct structure using the induction hypothesis. The rest of the proof follows the one of Lemma 5 as is.

Theorem 10 (A simulator for a Robbins cycle) Let $C$ be a Robbins cycle (over $G$) and let each node know its clockwise direction it was received, during the $k$-th epoch each node $u$ transmits exactly the sequence of pulses described by $10^{2L} z_k$, which has the correct structure using the induction hypothesis. The rest of the proof follows the one of Lemma 5 as is.

\[ \text{Lemma 9} \quad \text{The overhead of simulating a single message } m \text{ by Algorithm 1(a) and Algorithm 2 over the simple cycle } G \text{ is } \text{C} \text{Coverhead}(m) = O(n \cdot |m| + n \log n). \]

\[ \text{Proof} \quad \text{Let } n = |V| \text{ be the length of the cycle } G. \text{ Suppose the message } M = (m, u, v) \text{ is dequeued in some epoch and being communicated (i.e., } m \text{ is being communicated by the simulated } \pi \text{ over the link } (u, v)). \text{ Communicating } M \text{ over the cycle results in the following pulses sent by each node during this epoch (Property (3) of Lemma 8): a single REQUEST pulse, at most a single TOKEN pulses, and at most } 2 + L + \left(1 + \frac{1}{L-1}\right) |M| \text{ DATA pulses (a preceding and trailing } 1\text{s, } L \text{ trailing } 0\text{s, and } |pad(M)| \leq \left(1 + \frac{1}{L-1}\right) |M| \text{ “content” pulses). Since } L \geq 2 \text{ is a constant, each of the } n \text{ nodes sends } O(|M|) = O(|m| + \log n) \text{ pulses. We conclude that } \text{C} \text{Coverhead}(m) = n \cdot O(|M|) = O(n \cdot |m| + n \log n). \quad \square \]

4 Simulating computations over fully-defective 2-edge connected networks

In this section we show how to perform resilient computations over any 2-edge-connected fully-defective network, given a Robbins cycle.

Theorem 10 (A simulator for a Robbins cycle) Let $C$ be a Robbins cycle (over $G$) and let each node know its clockwise and counterclockwise neighbors for each of its occurrences on $C$. There exists a noise-resilient simulator over the fully-defective $G$ for any asynchronous protocol $\pi$. The simulator features $\text{CCinit} = 0$ and $\text{C} \text{Coverhead}(m) = O(|C| \cdot |m| + |C| \log |V|)$ pulses.

Let $G$ be a 2-edge-connected graph, and assume the nodes are given a Robbins cycle $C$, namely, a directed cycle that passes through each vertex at least once, and that does not use any edge in both of its directions. (We stress that this assumption is later removed by showing how to construct the Robbins cycle from scratch, in Sect. 5.) As a node $u$ may appear in $C$ more than once, we denote by $k_u$ the number of its occurrences on $C$. The initial knowledge of each node $u$ about $C$ is the value of $k_u$ and its clockwise and counterclockwise neighbors along the cycle. That is, every node $u$ knows the nodes $\text{prev}_{u,i}$ and $\text{next}_{u,i}$ for every $0 \leq i \leq k_u - 1$, such that the nodes along the cycle $C$ correspond to the $\text{prev}$ and $\text{next}$ variables of all nodes in a consistent manner. We refer to the sequence of nodes between two successive occurrences of $u$ on $C$ (including the ending occurrence of $u$) as a segment $(u \rightarrow \cdots \rightarrow u)$. The high level approach for the algorithm is built upon Algorithm 1 of the simple cycle, with pulses forwarded across the Robbins cycle $C$. By way of mimicking the protocol for the simple cycle, $u$ views each of its occurrences along $C$ as a different node along the cycle. Accordingly, when a node $u$ is the token holder, it has exactly one occurrence on $C$ which is associated with holding the token, and when we refer to a token holder in this section, we refer to that precise occurrence. When any node $u$ forwards a pulse in some direction, it forwards it to the node along $C$ that follows its occurrence that received the pulse. There are several challenges in this generalization.

Challenge 1: edge repetition along $C$ Perhaps the main challenge is for $u$ to keep track of its occurrences and distinguish between them: it could be that multiple occurrences of $u$ have the same incoming edge. Still, this node needs to be able to associate each pulse it receives with its appropriate segment, even when pulses that belong to different segments arrive from the same neighbor.

For instance, consider the node $d$ in Fig. 1, and suppose it has just started its data phase and received a clockwise DATA pulse from node $c$. This data pulse could have originated at node $e$ and should be forwarded to node $a$, or it could have originated at node $a$ and should be forwarded to node $e$.

To avoid this type of confusion, each node $u$ tracks throughout the execution in which of its segments the token is located. Specifically, the node $u$ maintains the invariant that $\text{prev}_{u,i}$ and $\text{next}_{u,i}$ reflect the previous and next nodes of its occurrence number $i$, for $0 \leq i \leq k_u - 1$, in the specific rotation of the cycle that starts with the token segment considered by $u$, i.e., the token always resides within segment 0, (locally) for all nodes. To achieve this, node $u$ applies a local rotation function upon receiving information about the token holder, namely, upon receiving a TOKEN pulse which we show that can be traced correctly to a specific segment.

Challenge 2: distinguishing between different DATA pulses Another challenge that arises is how to distinguish between different DATA pulses. Recall that in the simulator for the simple cycle, the $d$ DATA pulses are forwarded concurrently, in the sense that the token holder issues all $d$ DATA pulses and only then waits to receive them. However, once an edge appears more than once in $C$, its endpoint $u$ needs to tell apart the case in which it receives two different DATA pulses on that edge from the case in which it receives the same DATA pulse on that edge but from different segments. This is crucial because $d$ is not known in advance (and in fact the value of $d$ is the exact piece of information that needs to be learned).
We overcome this challenge by making sure that the DATA pulses get forwarded in a sequential manner as follows. The node-occurrence that is the token holder does not issue all $d$ DATA pulses, but rather waits to receive DATA pulse number $\ell$ from its counterclockwise neighbor before issuing DATA pulse $\ell + 1$, for $1 \leq \ell \leq d - 1$. A node $u$ that receives the DATA pulse for the $i$-th time since the last reception of a counterclockwise END pulse, forwards it to $next_{u,i-1}$ (where the index is taken mod $k_u$).

**Challenge 3** REQUEST pulses have no guaranteed structure While our approach for overcoming Challenges 1 and 2 allows the nodes to have consistent rotations of the cycle and the token segments for streamlining the DATA, END, and TOKEN pulses, it is insufficient for handling REQUEST pulses. The reason for this is that each of the other three types of pulses traverses the cycle sequentially (or partially traverses in case of a TOKEN pulse), but REQUEST pulses could be initiated by different nodes, so that a node that receives a REQUEST pulse does not have any particular promise about its origin and hence cannot tell which neighbor to forward this pulse to.

We remedy this uncertainty by having each node disseminate REQUEST pulses to all of its clockwise neighbors, regardless of their origin (which is not known to the node). We show that in the case of REQUEST pulses, this coarse action satisfies the conditions that are needed in order for the simulator to work correctly, despite its somewhat more aggressive and unstructured nature.

### 4.1 Formal description

The main idea of the simulator, as mentioned above, is to let each node mimic Algorithm 1 while simulating each one of its occurrences on $C$ as if it were a separate node on a simple cycle. Nevertheless, some actions are performed by the node and apply for all its occurrences. We expand on this shortly.

In particular, each node $u$ has the internal variables $isTokenHolder_u$ and $Q_u$, for holding the token and queuing its simulated messages. These will serve all its occurrences. Recall that a segment $(u \rightarrow \cdots \rightarrow u)$ is a sub-path of the cycle $C$ between two consecutive occurrences of $u$. Each node $u$ holds the variables $prev_{u,i}$ and $next_{u,i}$ that reflect the previous and next nodes of its occurrence number $i$, for $0 \leq i \leq k_u - 1$, see Fig. 2.

Further, each node $u$ tracks throughout the execution in which of its segments the token is located and calls this its token segment (segment 0). The node $u$ applies a local rotation function $\text{RotateEdges}(\cdot)$ upon receiving information about the token holder, namely, upon receiving a TOKEN pulse, which maintains this invariant. The procedure $\text{RotateEdges}(\cdot)$ is formally defined as follows.

**Algorithm 3(a)** A simulator for fully-defective networks given a Robbins cycle: token phase (node $u$)

1. wait until $Q_u$ is not empty or a clockwise REQUEST pulse is received from some $prev_{u,i}$
2. send a REQUEST pulse to $next_{u,i}$ for all $0 \leq i \leq k_u - 1$
3. wait until a REQUEST pulse is received on each $prev_{u,i}$ for all $0 \leq i \leq k_u - 1$  
   $\triangleright$ Including REQUEST pulses received in Line 1, if any
4. if $isTokenHolder_u$ then
5. \hspace{1em} $isTokenHolder_u \leftarrow \text{False}$
6. \hspace{1em} send a counterclockwise TOKEN pulse to $prev_{u,0}$
7. end if
8. wait until receiving a pulse  
   $\triangleright$ Or process any second pulse received in Line 3 from $prev_{u,i}$ for some $0 \leq i \leq k_u - 1$
9. if the pulse is a counterclockwise TOKEN pulse then  
   $\triangleright$ Else, the pulse is a clockwise DATA pulse
10. \hspace{1em} $\text{RotateEdges}()$
11. \hspace{1em} if $Q_u$ is not empty then
12. \hspace{2em} $isTokenHolder_u \leftarrow \text{True}$
13. \hspace{1em} else
14. \hspace{2em} forward the counterclockwise TOKEN pulse to $prev_{u,0}$
15. \hspace{2em} go to Line 8
16. \hspace{1em} end if
17. \hspace{1em} end if
18. continue with Algorithm 3(b)

### 4.2 Analysis

Similar to the analysis in Sect. 3.2, we begin by proving the technical Lemma 11 that specifies the behavior of the simulation and replaces Lemma 5. This technical lemma is then used to argue the correctness of our simulation over a Robbins cycle (Theorem 12). We prove the complexity of our simulation in Lemmas 13 and 14. Together, these prove Theorem 10.
Consider an execution of Algorithm 3 and consider any \( k \geq 1 \), for which \( t_{k-1} < \infty \). If from time \( t_{k-1} \) and forward, all queues \( \{Q_v\}_{v \in V} \) are always empty, then \( t_k = \infty \). Otherwise, the following hold:

1. **Progress** All nodes eventually complete their \( k \)-th epoch. In particular, \( t_k < \infty \), and at time \( t_k \), all nodes have already processed the END pulse (of epoch \( k \)) but have not yet passed Line 8 in epoch \( k + 1 \) (they are either waiting in Lines 1 or 3 for a REQUEST pulse, or waiting in Line 8 for either a DATA or a TOKEN pulse).

2. **Single token holder** It holds that \( t_{k-1} < s_k < t_k \). At each moment in \((t_{k-1}, t_k)\), there is at most a single node \( u \) for which \( \text{isTokenHolder}_u = \text{True} \) and \( u \) associates this with a single occurrence on \( C \). More specifically, within this time frame, the token is passed as follows: the node-occurrence \( T_k \) releases the token at some time \((t_{k-1}, s_k)\) and the node-occurrence \( T_k \) is the next node that gains the token at time \( s_k \). The node-occurrence \( T_k \) (solely) holds the token in \([s_k, t_k)\).

3. **Global consistency** There exist integers \( d_1, \ldots, d_k > 0 \) and for any \( u \in V \) there are \( b^u_1, \ldots, b^u_k \in [0, 1) \), such that when the node \( u \) completes its \( k \)-th epoch, the sent transcript by each of its occurrences is \( P_u \triangleq 10^{b^u_1} 1^{d_1} 0, 10^{b^u_2} 1^{d_2} 0 \ldots 10^{b^u_k} 1^{d_k} 0 \).

In addition, the message each node decodes and processes (Line 39) at its \( k \)-epoch is the unary decoding of \( 1^{d_k} \), which is the message sent by \( T_k \) (Line 20).

**Proof** In essence, we wish to follow the line of proof of Lemma 5. The high-level observation is that in the general case, every occurrence of a node on \( C \) behaves as in the case of the simple cycle, rather than every node behaving that way. There are a few subtle exceptions to this, which do not harm the proof but are rather essential for allowing it to go through. We elaborate as follows.

**Token phase** For the token phase, if a node \( u \) has a non-empty queue \( Q_u \), then it reaches Line 2 and sends a REQUEST pulse to each of its clockwise neighbors, \( \text{next}_u \), for all \( 0 \leq i \leq k_u - 1 \), and waits in Line 3 to receive a REQUEST pulse from each of its counterclockwise neighbors, \( \text{prev}_u \), for all \( 0 \leq i \leq k_u - 1 \). This is equivalent to saying that every occurrence of \( u \) on \( C \) sends a single clockwise REQUEST pulse and waits to receive a single counterclockwise REQUEST pulse. Thus, Lines 1–3 are equivalent to Lines 1–3 of Algorithm 1 for each occurrence of \( u \).

Similarly, any node \( u \) that receives a REQUEST pulse from \( \text{prev}_u \), for some \( 0 \leq j \leq k_u - 1 \) in Line 1, forwards it to each of its neighbors \( \text{next}_u \), for all \( 0 \leq i \leq k_u - 1 \), and waits in Line 3 to receive a REQUEST pulse from each of its neighbors.
prev_u,i for all 0 ≤ i ≤ k_u − 1 for which i ≠ j. For occurrence 

j of u, this is equivalent to Lines 1–3 of Algorithm 1. For other occurrences of u, this is slightly different, as they first 
forward the REQUEST pulse and only then wait to receive it. 

However, this still satisfies that if some REQUEST pulse is sent 
in an epoch, then each occurrence of every node sends and 
receives exactly one REQUEST pulse in that epoch, which is 
all that is needed for the proof of Lemma 5. Notice that in 
Line 3 a node may receive a second clockwise pulse from 
prev_u,i for some single 0 ≤ i ≤ k_u − 1, in which case this is 
a DATA pulse that is processed in Line 8.

Consider now the node u which has isTokenHolder_u set 
to True. In Lines 4–6, this node sends a counterclockwise 
token pulse to prev_u,0. This corresponds to having only 
occurrence 0 of u on C send a TOKEN pulse, which is the 
same as Lines 4–6 in Algorithm 1 and indeed the proof of 
Lemma 5 needs that only a single TOKEN pulse traverses the 
cycle.

It remains to show that each occurrence of a node u 
on C that receives a TOKEN pulse forwards it to its 
counterclockwise neighbor in C if the queue Q_u is empty, or 
sets isTokenHolder_u to True otherwise, and that there is 
exactly one occurrence of one node among those with a 
non-empty queue which receives a TOKEN pulse. For this, 
we need to show that each node correctly keeps track of its 
token segment. We rely on the local RotateEdges() pro-
cedure by showing that the following invariant holds in any 
time throughout the execution: Let v* be the node-occurrence 
of v that is associated with v having isTokenHolder_v set to 
True, or the node-occurrence that most recently received a 
TOKEN pulse if no such v exists. Then for every node u, 
the occurrence v* is located in segment 0 of u on C (i.e., 
v* is located in [next_u,k_u−1,...,prev_u,0,u]). This invariant 
is assumed to hold at the onset of the execution. Since the 
invariant holds, once a TOKEN pulse reaches a node u in 
Line 9, it must reach it from next_u,k_u−1. Then, u invokes 
RotateEdges() in Line 10 before setting isTokenHolder_u 
to True in Line 12 or forwarding the TOKEN pulse in 
Line 14, depending on whether its queue Q_u is empty. In 
either case, the invocation of RotateEdges() guarantees the 
invariant is maintained.

Now, consider the case where a node v sets its isTokenHolder_v 
to True. For the node-occurrence v*, Lines 8–17 correspond 
to Lines 8–16 in Algorithm 1, that is, v* receives a TOKEN 
pulse and switches to its data phase. For the other occur-
cences of v this is slightly different, as the node v along with 
all its occurrences, switches to its data phase once v* obtains 
the token and node v executes Line 18 after completing 
Line 12. This is fine, since all the occurrences at this point 
have sent and received a REQUEST pulse, and can switch to the 
data phase. Indeed, some of v’s node-occurrences might have 
already forwarded a TOKEN pulse before and remained in the 
token phase (e.g., if Q_v was empty once the token had reached 
them); these occurrences switch to the data phase “late” in 
comparison to Algorithm 1. Additionally, some of v’s node-
occurrences might have not received any pulse in this token 
phase and they switch “early” compared to respective node 
in Algorithm 1 (i.e., before they receive the first DATA pulse). 

Nevertheless, they are all in the data phase when they need to 
send and receive DATA pulses. This essentially corresponds 
to Lines 8–17 in Algorithm 1. The same reasoning applies 
to every other node: once one of its occurrences receives the 
DATA pulse originated at v* and switches to the data phase, 
then all its occurrences do so at the same time, but they all wait 
for the first DATA pulse to arrive (Line 34) and thus behave 
similarly to Algorithm 1, despite the “early” transition to the 
data phase.

**Data phase** For the data phase, if node u has its 
isTokenHolder_u set to True, then in Line 19 it dequeues 
a message from Q_u and denotes by 1^u its unary encoding. 
Next, in Lines 20–30, for each of the d pulses of DATA that 
need to be forwarded, each occurrence of u according to their 
order on C receives and sends the clockwise DATA pulse and 
then receives and sends the counterclockwise END pulse. This 
corresponds to Lines 19–23 in Algorithm 1, with two sub-
tleties.

The main subtlety is that Line 19 is invoked only once 
by u, which corresponds to Line 19 in Algorithm 1 being 
invoked only by occurrence 0 of u in its rotation of C. This 
is essential, as otherwise if each occurrence of u initiates d 
separate DATA pulses then clearly there will be too many in 
the system and the message will not be correctly decoded. 
We emphasize that the queue Q_u is a single queue used by 
all occurrences of u, and hence once a message is dequeued 
from Q_u, other occurrences cannot dequeue it again later 
in further epochs if they become the occurrence of u that 
is associated with the isTokenHolder_u variable being set to 
True.

The second subtlety is that each DATA pulse begins its 
traversal over C only after the previous one is received back 
at occurrence 0 of u. The latter does not harm the proof as it 
is only a stronger requirement compared to Algorithm 1.

Similarly, for every node u whose isTokenHolder_u variable 
is set to False, each of its occurrences according to their 
order on C receives and sends the clockwise DATA pulse (in 
Lines 32–37) and the counterclockwise END pulse (in Lines 
41–44), for d times. This corresponds to Lines 25 and 29 in 
Algorithm 1.

Finally, Lines 38–40 are also invoked by any node u only 
one, in order to avoid delivering to π duplicates of the 
received message, corresponding to Lines 26–28 in Algo-

rithm 1. Note that this also means that every node u moves 
to the token phase of the next epoch in Algorithm 3(a) only 
after all of its occurrences finish the current data phase in 
Algorithm 3(b).
The above establishes that we can now repeat the proof of Lemma 5 for obtaining a proof of Lemma 11.

Lemma 11 allows proving the correctness of our simulation, as follows.

**Theorem 12** Let \( G = (V, E) \) be some graph and let \( C \) be a Robbins cycle in it. Given any asynchronous protocol \( \pi \), let \( \pi' = \) the Algorithm 3 given the input \( \pi \). Then, executing \( \pi' \) on the fully-defective network \( G \) simulates an execution of \( \pi \) on the noiseless \( G \).

**Proof** The proof of Theorem 12 is exactly the same as the proof of Theorem 6, with the modifications that (i) it uses Lemma 11 instead of Lemma 5, (ii) instead of referring to a node in the network, it refers to its occurrences along \( C \), and (iii) it adjusts the line numbers that reflect the delivery of a message to the protocol \( \pi \) (Line 40 in Algorithm 3(b) instead of Line 28 in Algorithm 1(b)).

Finally, the following lemma states the message overhead of our simulator. Its proof is identical to that of Lemma 7, except that we consider pulses sent by each of the occurrences of nodes on \( C \), whose length \(|C|\) can be greater than the number of nodes \( n \).

**Lemma 13** Given a Robbins cycle \( C \), the overhead of simulating a single message \( m \) in Algorithm 3 is \( \overline{C_{\text{overhead}}}(m) = O(|C| \cdot 2^{|m|} + O(\log n)) \).

A direct application of the binary encoding described in Sect. 3.3 yields the following optimization.

**Lemma 14** Given a Robbins cycle \( C \), the overhead of simulating a single message \( m \) in Algorithm 3, replacing the unary encoding with a binary encoding, is \( \overline{C_{\text{overhead}}}(m) = O(|C| \cdot |m| + |C| \log n) \).

We omit the details as they repeat the proofs in Sect. 3.3 for reducing the communication complexity in the simple cycle.

## 5 Constructing a Robbins cycle in a fully-defective 2-edge connected network

The simulator of Sect. 4 assumes the nodes are given a Robbins cycle on which they communicate. In this section, we show how the nodes can construct such a cycle on any 2-edge-connected fully-defective network \( G \).

Whitney [47] proved that any 2-edge-connected graph \( G \) can be decomposed into

\[
G = C_0 \cup E_0 \cup E_1 \cup \cdots \cup E_k,
\]

where \( C_0 \) is a simple cycle, and for any \( i \geq 0 \), \( E_i \) is an *ear*—a simple path or cycle whose endpoints belong to \( C_0 \cup E_0 \cup \cdots \cup E_{i-1} \). Moreover, the process of decomposing \( G \) into ears can be performed by starting from a single node, and constructing \( C_0 \) and \( E_i \) in an increasing order \( i = 0, 1, 2, \ldots \). See also [26, 29, 42, 46] for further details and several distributed ear-decomposition algorithms in noiseless settings. Our Robbins cycle construction essentially performs a distributed and content-oblivious version of Whitney’s ear-decomposition process (see, e.g., Lemma 2.1 in [36] for the centralized algorithm), where nodes form the cycle \( C_0 \) and the ears \( E_0, E_1, \ldots \) sequentially. A newly constructed ear is incorporated with the previous constructions to form a non-simple cycle that includes them all.

We start at a designated root node and perform a content-oblivious DFS by sending a token over edges in a sequential manner; see, e.g., [33, Section 5.4]. This process continues until the token returns to the root which signifies that a cycle is closed at the root. We require the constructed cycle to be simple. Indeed, if the token reaches some node \( v \neq \text{root} \) twice, then \( v \) sends the token back to where it came from, which is equivalent to backtracking in the standard DFS algorithm. Backtracked edges do not participate in the constructed cycle, and they are left for future ears.

We denote the simple cycle constructed by the above procedure by \( C_0 \). The order the DFS-token progresses along \( C_0 \) defines the clockwise direction on the cycle. Nodes on \( C_0 \) employ Algorithm 3 to communicate over \( C_0 \) in a noise-resilient manner with the root being the first token holder.

Recall that a directed cycle can be represented by the nodes either locally, i.e., each node knows its clockwise and counterclockwise neighbor(s), or globally, i.e., knowing the sequence of IDs that defines the cycle. Our algorithm will use both representations, however, this is done only to simplify the analysis and reduce the length of the constructed Robbins cycle. In Remark 5 we sketch how to remove this assumption.

Before the nodes on \( C_0 \) continue with adding ears to \( C_0 \), they first broadcast their IDs and achieve a global representation of the cycle. The root sends its ID to its neighbor, who appends its own ID and transfers the message to its next neighbor and so on. When the message reaches the root again, it contains the sequence of IDs of the cycle \( C_0 = (\text{root}, \ v_1, \ v_2, \ldots \ \text{root}) \). The root broadcasts this information; it will be used towards continuing the Robbins cycle construction.

Next, the nodes on \( C_0 \) select a new root, denoted by \( \text{root}_0 \), to be one of the nodes on \( C_0 \) that still has unexplored edges, which are edges that do not participate in \( C_0 \). The construction proceeds by constructing a new ear, \( E_0 \), starting from \( \text{root}_0 \). Again, the nodes perform a sequential DFS by sending a DFS-token over unexplored edges, until the DFS-token reaches some node \( z_0 \) that belongs to \( C_0 \). As before, we require the path of the DFS-token to be simple, and back-
track whenever the token reaches twice the same node that
does not lie on $C_0$.

The simple path that the DFS-token has undergone from
$root_0$ to $z_0$, excluding edges that have backtracked in
the DFS search, becomes the newly constructed ear $E_0$. A new
car can be a simple cycle if it is a closed ear with $z_0 = root_0$,
or it can be a simple path if it is an open ear with $z_0 \neq root_0$.

Based on $C_0$ and the ear $E_0$, we define a new cycle $C_1$
that contains all the edges of $C_0$ and of $E_0$, possibly multiple
times, so that $C_1$ is a closed (non-simple) cycle. Recall that in
a Robbins cycle, each edge has a unique orientation and the
cycle is not allowed to cross the same edge in both directions.
Thus, we let $C_1$ be the cycle

$$
root_0 \rightarrow root_0 \rightarrow E_0 \rightarrow z_0 \rightarrow root_0.
$$

The notation $a \rightarrow b$ here means that we take the complete
path $P$. The notation $a \Rightarrow b$ means the shortest path from
$a$ to $b$ implied by the clockwise orientation of edges in $P$.
If multiple such paths exist, we take the first one by lexico-
graphic order. Note that this path might not be a sub-path of $P$,\(^1\) however, it is uniquely defined and can be retrieved
by any node that holds the sequence of IDs that defines $P$.

It follows that the paths $root_0 \rightarrow root_0$ and $z_0 \Rightarrow root_0$
are well defined and known by all nodes on $C_0$, since all
these nodes know the sequence of IDs that lie on $C_0$, in their
respective order. However, the nodes still need to know the
IDs on $root_0 \rightarrow z_0$ in order to obtain the sequence of IDs
in the new cycle $C_1$. Towards this end we do the following.

The nodes on $P_0 \equiv (z_0 \Rightarrow root_0)$ along with the nodes
on $E_0$ form a simple cycle $E_0 \parallel P_0$ (recall that $\parallel$ denotes con-
catenation). This cycle is locally defined: the nodes on $E_0$
define their neighbors when they first obtain the DFS-token.
Each node on $P_0$ belongs to $C_0$ and, as argued above, can
locally define its neighbors on $P_0$. Then, the root starts com-
municating over this cycle using Algorithm 3. As before, the
first thing the nodes do is communicating their IDs. In fact,
only the new nodes that are on $E_0$ but not in $C_0$ need to broad-
cast their IDs in their respective order, similarly to the way
it was done after the completion of $C_0$. After this part, $root_0$
can simply construct the string of IDs of the nodes in $C_1$
and communicate it over $E_0 \parallel P_0$.

Then, the root communicates the sequence of IDs of $C_1$
to all the nodes in cycle $C_0$. That is, the root and the nodes on
$P_0$ stop communicating on the cycle $P_0 \parallel E_0$ and switch back
to communicating over the cycle $C_0$. Next, the root sends a
message to instruct all the nodes in $C_0$ to switch to the new
cycle $C_1$. Note that this message need not reach the nodes

\(^1\) For instance, let $P = (a \rightarrow b \rightarrow c \rightarrow b \rightarrow e)$, then $a \Rightarrow e$ is the
path $(a \rightarrow b \rightarrow e)$ which is not a sub-path of $P$. in $E_0$, as they are already “set” to the correct $C_1$. Since the
other nodes are set to communicate over $C_0$, the nodes in
$E_0$ are excluded from this communication and these nodes
remain idle until the first pulse arrives, which happens once
the rest of the nodes switch to communicate over $C_1$.

The process then repeats: for any $i > 0$, $root_i$ is selected to
be a node on $C_i$ that still has edges that do not belong
to $C_i$. The nodes construct a new ear $E_i$ whose endpoints,
$root_i$ and $z_i$, belong to $C_i$. The nodes then locally define the
non-simple cycle $C_{i+1} = root_i \rightarrow root_i \rightarrow z_i \Rightarrow root_i,$
and start communicating over it. Next, the nodes globally
learn the sequence of IDs included in $C_{i+1}$, which is required
for the next iteration, and so on. This process ends when the
cycle $C_{i+1}$ contains all the edges of $G$. See Fig. 3 for a
demonstration.

### 5.1 Formal description

We now formally define our construction. Each node holds
a variable named cycle that contains a global representa-
tion of the current $C_i$. At the same time, the simple cycle
$root_i \rightarrow z_i \Rightarrow root_i$, is represented locally, using the vari-
bles next$v$ for the clockwise neighbor of $v$ and prev$v$ for its
clockwise neighbor.

In our algorithms, the first ID in the variable cycle is the
current root. When the root node changes, each node $v$ locally
rotates the sequence of IDs in cycle$v$, (say, clockwise), so that
the new root becomes the first ID in the string.

The pseudo-code for our content-oblivious protocol for
constructing a Robbins cycle appears in Algorithms 4(a)
and 4(b). These use as sub-procedures the protocols $\Pi_{learnD}$
and $\Pi_{NextRoot}$, which are the content-oblivious versions of
Algorithms 5 and 6, obtained by simulating them through
Theorem 12. Note that all these algorithms share the same
variables, i.e., cycle$v$, prev$v$, and next$v$ of node $v$.

Our protocols use the ability to broadcast a message on
a cycle defined either locally or globally. To be more accu-
rate, the instruction “broadcast $M$” and “wait for message
$M$” are to be understood as sending the message $M$ with
destination $*$ and receiving any message with destination $*$,
respectively, using the method of Remark 3. The sender also
receives the broadcast message after all other nodes receive
it and acts upon the pseudo-code for processing it. This guar-
antees synchronization, i.e., that the sender does not continue
before all other nodes receive the broadcast message, which
is crucial, for example, when we switch the underlying cycle
we communicate over. Indeed, in the noise-resilient proto-
col, the sender holds the token and does not release it before
it gets the END pulse for that message, and by this time all
other nodes receive that message as well. If now all nodes
change their cycle$v$, then the next pulse sent by the root goes
through the new cycle.
5.2 Analysis

Our main theorem in this section shows that Algorithm 4 constructs a Robbins cycle that includes all the edges in $G$ despite a fully-defective environment.

Theorem 15 For any 2-edge-connected graph $G$, Algorithm 4 constructs a sequence of cycles $C_0, \ldots, C_k$, where $C_0$ is a simple cycle that includes the root, and $C_k$ is a Robbins cycle that contains all the edges $E$ of $G$.

For the ease of the analysis, we define iterations of Algorithm 4. We say that iteration $i+1$ begins when the $\Pi_{\text{NextRoot}}$ is being executed for the $i$-th time by a node which is currently marked as root, i.e., when such a node reaches either Line 33 or 30. Note that by the code, there can only be one root for each iteration. We start with some helping lemmas.

Lemma 16 Suppose Algorithm 4(a) is executed by all nodes in a 2-edge-connected graph $G$, where a single node is marked as a root. Then, the root node eventually reaches Line 29, and at that time, there exists a single simple cycle $C_0$, locally represented by the nodes on it. Furthermore, root $\in C_0$.

Proof It is immediate from the pseudo-code that Algorithm 4(a) performs a sequential depth first traversal starting from the root and using marked edges to avoid repeating already visited edges. We can think of the DFS as sending a DFS-token that progresses over non-visited edges until reaching a visited node $v$. The DFS-token advances by sending a single pulse.

Suppose the DFS-token reaches an already-visited node $v$, this node is either the root, in which case we are done, or it is not the root. In the latter case, the node $v$ sends the DFS-token back to where it came from, causing the DFS to backtrack that edge and continue with the DFS from the parent of $v$ in the induced DFS tree. Since the graph is 2-edge-connected, there exists a simple cycle that begins and ends at the root. A DFS search, once completed, explores all the edges in $G$. Therefore, the DFS must eventually reach the root again and
Algorithm 4(a) Content-oblivious Ear-Decomposition: Closing an ear for the first time

1: **Init:** Set $\Pi_{\text{learnID}}$ and $\Pi_{\text{NextRoot}}$ to be the content-oblivious versions of Algorithms 5 and 6, respectively, obtained via Theorem 12.

node $v$, upon initialization:

2: $state_v \leftarrow \text{init}$, $next_v \leftarrow \perp$, $prev_v \leftarrow \perp$, $cycle_v \leftarrow \epsilon$. All edges unmarked.

3: if $v$ is the root then

4: choose an arbitrary edge $(v, u)$

5: send a pulse to $u$ and mark the edge $(v, u)$ as used.

6: $next_v \leftarrow u$, $state_v \leftarrow \text{DFS root}$

7: end if

node $v$, upon receiving a pulse from $w$:

8: if $state_v = \text{init}$ then

9: $prev_v \leftarrow w$, mark $(w, v)$ as used

10: choose an arbitrary neighbor $u \neq w$ where $(v, u)$ is unmarked

11: send a pulse to $u$ and mark $(v, u)$ as used

12: $next_v \leftarrow u$, $state_v \leftarrow \text{DFS}$

13: else if $state_v = \text{DFS$}$ then

14: if $w = next_v$ then $\triangleright$ This is a cancellation pulse

15: choose an arbitrary neighbor $u'$ where $(v, u')$ is unmarked

16: send a pulse to $u'$, set $next_v \leftarrow u'$ and mark $(v, u')$ as used

17: if no such $u'$ exists then

18: send a pulse to $prev_v$. Send a cancellation pulse to parent

19: $state_v \leftarrow \text{init}$, $prev_v \leftarrow \perp$, $next_v \leftarrow \perp$, unmark all edges

20: end if

21: else if $w \neq prev_v$, then $\triangleright$ A cycle is closed at $v$, but $v$ is not the root

22: send a pulse to $w$ and mark $(v, w)$ as used.

23: else $(w = prev_v)$ $\triangleright$ This is a second pulse—node is on a cycle

24: send a pulse to $next_v$

25: $cycle_v \leftarrow \Pi_{\text{learnID}}$, executed over the cycle locally defined by $prev_v$, $next_v$; initialize as non token-holder.

26: execute $\Pi_{\text{NextRoot}}$ over $cycle_v$; initialize as non token-holder.

27: end if

28: else if $state_v = \text{DFS root}$ then

29: $prev_v \leftarrow w$

30: send a pulse to $next_v$ $\triangleright$ A cycle is closed, start communicating on it

31: wait until a pulse is received from $prev_v$

32: $cycle_v \leftarrow \Pi_{\text{learnID}}$, executed over the simple cycle locally defined by $prev_v$, $next_v$; initialize as token holder.

33: execute $\Pi_{\text{NextRoot}}$ over $cycle_v$; initialize as token holder.

34: end if

close a simple cycle, defined by the progress of the DFS-token while ignoring any backtracked edges. Indeed, each node sets its $prev_v$ variable to the first node from which the DFS-token is received and sets its $next_v$ variable to be the node to which the DFS-token progresses. Backtracking an edge resets $prev_v$, $next_v$, accordingly in Lines 16 or 19.

Denote the above constructed cycle as $C_0$. We note that nodes that are not on $C_0$ are either never reached by the DFS or the DFS reaches them and backtracks since it does not reach the root from that path. In either case, their status at the time when the root reaches Line 29, and also at the end of Algorithm 4(a), is $\text{init}$ with no marked edges, and with Algorithm 4(b) Content-oblivious Ear-Decomposition: Ear extension

node $v$, marked as root, upon initialization:

35: choose an edge $(v, u) \notin cycle_v$ and send a pulse to $u$

36: $next_v \leftarrow u$

node $v$, upon receiving a pulse on $(v, u) \notin cycle_v$:

37: $prev_v \leftarrow u$

38: broadcast “(EarClosedAt),$ $v$” over $cycle_v$

$\triangleright$ In parallel to the above, pulses from $cycle_v$ are interpreted as messages of a noise-resilient protocol

node $v$, upon receiving “(EarClosedAt),$ $w$” on $cycle_v$:

39: $P_v \leftarrow$ the simple path $w \Longrightarrow root$ $\triangleright P_v = \emptyset$ if $w = root$

40: if $v \in P_v$ then

41: set $prev_v$, $next_v$ according to $P_v$

$\triangleright$ The root sets $prev$ and $next$ sets $root$ (unless root $= 0$

42: $\text{identifier}$ nodes set both

43: if $v$ is the root then

44: send a pulse to $next_v$

45: if root $= w$ then

46: wait to receive a pulse from $prev_v$. A closed ear, the pulse will reach back the root

47: broadcast (ready) on $cycle_v$

48: end if

49: else if $w = root$ then $\triangleright w \neq root$

50: wait to receive a pulse from $prev_v$

51: broadcast (ready) on $cycle_v$

52: end if

53: wait to receive (ready) on $cycle_v$

54: if $prev_v$, $next_v \neq \perp$ then $\triangleright v$ is on $P_v$

55: execute $\Pi_{\text{learnID}}$ over the simple cycle locally defined by $prev_v$, $next_v$; root is token holder.

56: $prev_v \leftarrow \perp$, $next_v \leftarrow \perp$

57: end if

58: if $v$ is the root then

59: broadcast “(NewCycle),$ C_{i+1}$” over $cycle_v$, where $C_{i+1}$ is the output of $\Pi_{\text{learnID}}$

60: else

61: wait to receive the message “(NewCycle),$ C_{i+1}$” over $cycle_v$.

62: end if

63: $cycle_v \leftarrow C_{i+1}$ $\triangleright$ All nodes in $C_i$ switch to $C_{i+1}$; nodes on $E_i$ were set at line 25

64: execute $\Pi_{\text{NextRoot}}$ over $cycle_v$; The root initializes as the token holder

prev $= next \leftarrow \perp$. Therefore, $C_0$ is the only cycle defined at this point. $\square$

Next, we observe that the nodes on $C_0$ switch to a global representation of their cycle.

Lemma 17 Once the root completes Line 32, all the nodes on $C_0$ hold a global representation string of $C_0$.

Proof Lemma 16 establishes that once the root reaches Line 29, then $C_0$ is locally well-defined, i.e., every node that belongs to $C_0$ knows the previous and subsequent nodes in the cycle. The root then sends a second pulse which progresses over $C_0$ and causes all the nodes on $C_0$ to execute $\Pi_{\text{learnID}}$, where the root is the token holder (Line 32) and other nodes are non token holders (Line 25). Note that the root awaits...
Algorithm 5 $\text{learnID}$, learning the IDs on a newly constructed ear (noiseless setting)

node $v$, upon initialization:
1: If $v$ is the root then
2: send $id(v)$ to $next_v$.
3: end if

node $v$, upon receiving $m = (id_1, id_2, \ldots)$:
4: if $id_1 \neq id(v)$ then $\{next_v\}_{v \in V}$ is guaranteed to induce a simple cycle
5: $m' = m \cup \{id(v)\}$
6: send $m'$ to $next_v$.
7: else $\Rightarrow$ Back to root, $m$ contains all the nodes on $\{next_v\}_{v \in V}$
8: new_cycle $\leftarrow cycle_u \cup m$
9: broadcast “(done), new_cycle”
10: end if

node $v$, upon receiving “(done), $C$”:
11: return $C$

Algorithm 6 $\text{NextRoot}$, choosing a new root (noiseless setting)

node $v$, upon initialization:
1: If $v$ is the root then
2: broadcast “(check edges)”
3: wait to receive $\{|id(v')| v' \in cycle_v\}$ many replies
4: if received “(has unexplored edges), id(v)” then $\Rightarrow$ Choose arbitrarily, if no unique
5: broadcast “(new root), id(u)”
6: else $\Rightarrow$ All edges are explored
7: broadcast “(completed)”
8: end if
9: end if

node $v$, upon receiving (check edges):
10: if $v$ has unexplored edges then
11: broadcast “(has unexplored edges), id(v)”
12: else
13: broadcast “(no unexplored edges), id(v)”
14: end if

node $v$, upon receiving “(new root), id(u)”:
15: rotate cycle, clockwise until it starts with an occurrence of $u$. The node $u$ is now marked root
16: execute Algorithm 5.1

node $v$, upon receiving (completed):
17: terminate $\Rightarrow$ A Robbins cycle is constructed

until the second pulse reaches it back (Line 31). By that time, all the other nodes on $C_0$ start executing $\text{learnID}$, but they are not token holders, so they remain idle. Only once the root starts executing $\text{learnID}$, pulses are sent over $C_0$ and the content-oblivious computation of Algorithm 5 initiates.

The execution of Algorithm 5 produces the sequence of IDs in $C_0$ according to the clockwise direction of the cycle: the root begins by sending its ID to its next (clockwise) neighbor, which concatenates its ID, and so on. Once the message reaches the root again, it contains all the IDs of the nodes in $C_0$ according to the clockwise direction of the cycle. This string is then broadcast to all $C_0$, so all the nodes now possess the global representation of $C_0$ as required.

Note that after the construction of $C_0$ completes, the nodes that belong to $C_0$ continue to execute Algorithm 4(b), while the rest of the nodes are still executing Algorithm 4(a). We now argue that the algorithm keeps adding edges to the currently-constructed cycle.

For a cycle $C$, let us denote by $Edge(C)$ the set of edges in $C$. We prove that each iteration of Algorithm 4 constructs a larger cycle. That is, assuming the nodes on $C$ execute Algorithm 4(b) while the rest of the nodes execute Algorithm 4(a), then at the end of that iteration, there is a globally defined cycle $C'$ such that all the nodes on $C'$ know this cycle (the other nodes keep executing Algorithm 4(a)), and $C'$ is strictly larger than $C$, that is, $Edge(C) \subset Edge(C')$.

Lemma 18 Let $G$ be a 2-edge-connected graph and let $C_i$ be a cycle, such that $E \setminus Edge(C_i) \neq \emptyset$. Let the root be a single marked node on $C_i$ that is adjacent to an edge in $E \setminus Edge(C_i)$. Suppose nodes on $C_i$ all start executing Algorithm 4(b) while other nodes in $G$ run Algorithm 4(a) and their state is init. At the end of this iteration, there exists a cycle $C_{i+1}$ with $Edge(C_i) \subset Edge(C_{i+1})$, all the nodes on $C_{i+1}$ know its global representation, and all the other nodes continue executing Algorithm 4(a) and their state is init. Further, if all the occurrences of any edge in $Edge(C_i)$ have the same orientation, the same holds for $C_{i+1}$.

Proof Note that the nodes basically perform a DFS search over the unused edges, i.e., over all the edges except edges that belong to $C_i$. The root initiates the DFS search (Line 1). Since the root has at least one edge which does not belong to $C_i$, denote the edge to which the root sends a pulse in Line 1 by $(root, v)$.

We argue that the DFS, after passing the DFS-token over $(root, v)$, must reach a node that belongs to $C_i$ before it backtracks the edge $(root, v)$. Suppose not, then there is no path between $v$ and any node in $C_i$ that does not go through $(root, v)$. Hence, $(root, v)$ is a bridge, yet this is a contradiction since $G$ is 2-edge-connected.

Once the DFS reaches some node $z$ on $C_i$ in Line 3, the path $E_i$ is well defined: it is the new ear—the path the token has taken from $root$ to $z$, disregarding any backtracked edge. Note that $E_i$ is not empty and $Edge(E_i) \subseteq E \setminus Edge(C_i)$, i.e., $E_i$ contains at least one new edge that does not belong to $C_i$. Additionally, the path $P_i$ constructed in Line 5 is well defined: it is the shortest path between $z$ and $root$ that uses only the directed edges in $Edge(C_i)$. We know at least one such path exists since $z$ and $root$ are both nodes on the cycle $C_i$, and take the lexicographic-first such path if multiple shortest-paths exist. Since all nodes on $C_i$ know $Edge(C_i)$ then $P_i$ is agreed upon all of them. Hence
\[ C_{i+1} = C_i \| E_i \| P_i \] is a well defined cycle from root to root for which \( \text{Edge}(C_i) \subseteq \text{Edge}(C_{i+1}) \). It is easy to verify that all the occurrences of any edge in \( \text{Edge}(C_{i+1}) \) have the same orientation: edges in \( E_i \) appear only once in \( C_i \), and all the other edges obey their orientation in \( C_i \), which is unique by assumption.

We now show that at the end of the iteration, all the nodes on \( C_{i+1} \) hold a global representation of \( C_{i+1} \) while the rest of the nodes remain in state \( \text{init} \), executing Algorithm 4(a).

Remark 4 In order to communicate over any intermediate (non-simple) cycle \( C_i \) via Algorithm 3, a single node-occurrence must be defined as the token holder. Furthermore, all other nodes must know the segment in \( C_i \) that contains that designated node-occurrence. Recall that in Algorithm 3, each node maintains the invariant that the token resides in its segment 0 (see Sect. 4). Our construction indeed provides the

Finally, the root broadcasts \( \langle \text{NewCycle} \rangle, C_{i+1} \rangle \) over \( C_i \) which causes all the nodes in \( C_i \) to change their cycle variable to \( C_{i+1} \). The root is the last to finish the procedure of the broadcast invocation, and by that time, all nodes of \( C_{i+1} \) are set to the cycle \( C_{i+1} \) and idle. The root is the token holder and is expected to send the next message on \( C_{i+1} \).

The proof of Theorem 15 can now easily be obtained as a corollary of the above lemma. Multiple invocations of Algorithm 4(b) eventually yield a Robbins cycle \( C_k \) with \( \text{Edge}(C_k) = E \).

Proof of Theorem 15 By Lemma 16, we know that after the first iteration of Algorithm 4(a) we obtain a simple cycle \( C_0 \). If \( C_0 \) consists of all the edges of \( G \), we are done—the nodes run \( \Pi_{\text{NextRoot}} \) to find out that all edges are exhausted, and the algorithm terminates in Line 17 of Algorithm 6. Otherwise, we keep executing Algorithm 4(b) with a new root that has an adjacent unused edge. This is done by Algorithm 6: each node broadcasts whether or not it has unused edges adjacent to it, along with its ID. The current root arbitrarily picks one node with unused edges (Line 4) and broadcasts this choice to all the nodes of \( C_i \). Since all the nodes possess a global representation of \( C_i \), they can rotate it so that the new root becomes first in the global representation, which is consistent among all nodes and allows, for example, to determine \( P_1 \) in a consistent manner. Then, Algorithm 4(b) is invoked again with this chosen node as the new root (Line 16). At this point, the statement of Lemma 18 holds: there is a cycle \( C_i \) globally represented by all the nodes in it, there is a single root on \( C_i \) and it has adjacent unused edges, and all the nodes in \( G \setminus C_i \) are in state \( \text{init} \) in the execution of Algorithm 4(a).

By Lemma 18, every iteration of the algorithm starting on \( C_i \) produces a cycle \( C_{i+1} \) with at least one additional edge in \( E \) that does not appear in \( C_i \). It is easy to verify that, as long as some edge is still unused, at the end of constructing \( C_{i+1} \), i.e., after executing Line 30 but before the nodes re-iterate Algorithm 4(b) (Line 16 of Algorithm 4(b)), the requirements for Lemma 18 hold with respect to the newly constructed cycle. Thus, after at most \( |E| - |\text{Edge}(C_0)| \) iterations of Algorithm 4(b), the obtained cycle consists of all the edges \( E \) in \( G \). Since each edge has a single orientation induced by the cycle (this clearly holds for the simple cycle \( C_0 \), and inductively throughout the construction), and since all the nodes in \( G \) appear in the obtained cycle, it is a Robbins cycle.

\[ \square \]
nodes with this information, which can be retrieved from the global representation of \( C_i \). The first node-occurrence in \( C_i \) is defined to be the token holder, and each other node can re-number its occurrences along \( C_i \) in the natural manner, so it is consistent with having the token at its segment 0. The above also holds also for the Robbins cycle \( C_k \) constructed in Theorem 15.

**Remark 5 Avoiding global knowledge** In the above construction, the nodes obtain a global representation of the cycles \( C_i \) they construct. We remark that this knowledge helps in simplifying the construction and reducing the length of the constructed cycle. However, it is not necessary, and a similar construction can be designed in which each node only holds local information about \( C_i \), i.e., only its clockwise and counterclockwise neighbors for each of its occurrences on \( C_i \). We provide here the main differences in such a construction.

1. The general representation of \( C_i \) is used to determine the path \( P_i \) between the end points (root, \( z \)) of the newly constructed ear \( E_i \). For the above construction to work, we need every node to know whether or not it belongs to \( P_i \); if it is part of \( P_i \), then it should appear one more time in \( C_{i+1} \). Now, suppose that every node \( v \) on \( C_i \) knows only a local representation of \( C_i \), namely, its next and prev neighbors for each occurrence of \( v \) on \( C_i \). The path \( P_i \) can be determined in the following way. Once the endpoint \( z \) of the ear \( E_i \) broadcasts the message “\( \langle \text{EarClosedAt}, z \rangle \)” over \( C_i \), all the nodes in \( C_i \) switch to a new state of “detecting \( P_i \)”. In this state, if a node-occurrence receives a clockwise pulse, it means that this occurrence belongs to \( P_i \). A counterclockwise pulse signifies that the node-occurrence should quit this new state and continue executing Algorithm 4(b). In both cases, each pulse is propagated by the node-occurrence along the same direction it is received.

   The nodes use the above mechanism as follows. Once the broadcast of “\( \langle \text{EarClosedAt}, z \rangle \)” completes at \( z \), it sends a single clockwise pulse. This pulse propagates along \( C_i \) until it reaches a node-occurrence of the root; denote by \( P_i \) the path that this pulse has taken. The root does not propagate the pulse, but instead sends a single counterclockwise pulse, which travels along the entire \( C_i \) until reaching that same root node-occurrence again. At this point, all the node-occurrences that belong to \( P_i \) have received a clockwise pulse, and all the node-occurrences on \( C_i \) have received a counterclockwise pulse, so all nodes can continue with the construction as above. Note that this method also allows the nodes to track the segment in which the root lies, so that at the end of the construction they can infer the token segment at any step.

2. The other place our construction uses the global representation is in \( \pi_{\text{NextRoot}} \), where the root awaits to receive a message from every node on \( C_i \) to know whether the construction is done. However, without a global representation, the root does not know how many nodes are in \( C_i \) and thus it cannot know how many messages to expect. The remedy for this issue utilizes the token delivery method of Algorithm 3. Namely, we replace Algorithm 6 with the following method. The root begins by broadcasting \( \langle \text{check edges} \rangle \). Every node that still has an unexplored edge requests the token, and if it receives the token, it sends its ID. The first node to do so becomes the new root. If no such node exists, the token propagates until it reaches the (old) root again. In this case, the root acquires the token and broadcasts \( \langle \text{completed} \rangle \) to indicate that the Robbins construction is done.

**Remark 6 Coping with \( KT_0 \)** Algorithm 5 and its noise-resilient form \( \Pi_{\text{learnID}} \) are \( KT_1 \) algorithms, in which each node knows the IDs of its neighbors. We remark that we can establish the learn-ID functionality, and thus the construction of the Robbins cycle, even in \( KT_0 \) networks, in which the IDs of the neighbors of a node are not known to it upon initialization. Note that Algorithm 5 as stated cannot work in a \( KT_0 \) network since a node does not know which node comes immediately next to it in the cycle. In other words, after the root sends its ID as the first message, this message reaches all other nodes and none of them knows they are the next one on \( C_0 \).

We can solve this issue by relying on the order in which the token holder shifts in the underlying simulator. A \( KT_0 \) protocol for learning the IDs starts by instructing all the nodes to broadcast their ID. Thus, all nodes request to be token holders. Once the root sends its own ID and releases the token, its immediate counterclockwise neighbor becomes the new token holder. Thus, the IDs are broadcast exactly in their counterclockwise order on \( C_0 \). Once the root becomes a token holder again, this process is done.

We also note that the simulator of Sect. 4 only requires local knowledge of a Robbins cycle and thus can run on \( KT_0 \) networks with the above pre-processing step. Thus, Theorem 2 holds for \( KT_0 \) networks as well.

### 5.3 The length of the obtained Robbins cycle

We complete this section with a crude analysis of the size of Robbins cycle our construction obtains and the communication complexity of the construction.

**Lemma 19** Let \( G \) be a 2-edge-connected graph, and let \( C \) be the Robbins cycle constructed by Theorem 15. Then \( |C| = O(n^3) \). Further, Algorithm 4 communicates \( O(n^8 \log n) \) pulses altogether.

**Proof** Given some \( C_i \), it holds that \( |C_{i+1}| = |C_i| + |E_i| + |P_i| \). Since \( P_i \) is a shortest (simple) path between two nodes, we have \( |P_i| < n \), for all iterations \( i \). A bound on the worst-case length of the Robbins cycle is obtained by considering
O(n^2) iterations of Algorithm 4, in each of which, adding only a single edge to the current $C_i$. In this case, the cycle’s length extends by $O(n)$ in each of the $O(n^2)$ iterations, yielding a total length of $O(n^3)$.

Let us now bound the communication complexity. Consider the iteration where the nodes begin with $C_i$ and construct $C_{i+1}$. The $\pi_{\text{teamID}}$ algorithm communicates at most $\alpha_i = |E_i| + |P_i|$ messages, each of length at most $O(\alpha_i \log n)$, except for the (done) message whose length is $O(|C_i| \log n)$. The $\pi_{\text{NextRoot}}$ algorithm communicates $|C_{i+1}|$ messages of length $O(\log n)$. The rest of Algorithm 4(b) makes $O(1)$ broadcasts of messages of length $O(\log n)$, and a single $\langle$NewCycle$\rangle$ message whose length is $\alpha_i$. Recall that by Lemma 14, broadcasting a message of length $m$ over the cycle $C_i$ takes $O(|C_i|(m + \log n))$ pulses.

Next, we argue that the DFS search within a single iteration of Algorithm 4 sends $O(n^2)$ pulses. To see that, recall that each edge is marked as used once the DFS-token passes through it. Additionally, the token might backtrack that edge, but no more pulses should be sent on that edge, leading to a total of at most $2|E| = O(n^2)$ pulses overall. The above does not hold for nodes that have backtracked all their edges and reset their state to \textit{init}, because they also unmark all their edges and might re-send pulses over edges that were already explored in this iteration. We argue, however, that such nodes will never get the DFS-token again during that iteration. Indeed, assume towards contradiction that $u$ is a node that has reset its state during the current iteration and is the \textit{first} node that receives the DFS-token after resetting its state, say, over the edge $(u, v)$. Since $u$ has explored and backtracked all its edges, the DFS-token must have already passed through the edge $(u, v)$ previously in this iteration. Therefore, it is marked used by $v$, and it is impossible that $v$ sends a DFS-token over this edge, unless $v$ resets its state and unmarks all its edges. However, if $v$ resets its state and then sends a DFS-token over $(u, v)$, then $v$ must have received the DFS-token after resetting and before $u$ did, contradicting our choice of $u$.

We then conclude that the complexity of constructing the Robbins cycle in Algorithm 4 is bounded by

$$\sum_i \left[ \alpha_i \cdot O(\alpha_i \cdot \alpha_i \log n) + O(\alpha_i \cdot |C_i| \log n) ight. + \left. |C_{i+1}| \cdot O(|C_{i+1}| \log n) + O(|C_i| \log n) + O(n^2) \right]$$

pulses. Bounding $\alpha_i = O(n)$ and $|C_i|, |C_{i+1}| = O(n^3)$, and the number of iterations $i \leq |E| = O(n^2)$, we conclude that the complexity of constructing the Robbins cycle is $O(n^8 \log n)$ pulses.

Note that the complexity can be reduced if we assume $KT_1$ networks and global representation of the constructed cycle.

Instead of terminating when all the adjacent edges of all the nodes were explored, we terminate when all nodes see that all their neighbors appear on the current $C_i$. Each node can determine this information assuming $KT_1$ knowledge and a global representation of the cycle. This guarantees that at least one node is added at each iteration of Algorithm 4, which reduces the number of iterations to $i \leq n$. This method leads to a Robbins cycle of total length $O(n^2)$ and a communication complexity of $O(n^8 \log n)$.

### 6 Impossibility of resilient communication in fully-defective networks which are not 2-edge connected

In this section we complement our simulator for 2-edge-connected graphs, with a proof showing that 2-edge connectivity is required for communication in fully-defective networks. The intuitive argument is that if the communication network is not 2-edge connected, then a bridge exists, and corrupting messages over that edge will lead to disconnecting the network, preventing the correct computation of any non-trivial function. Towards that goal we show the impossibility of asynchronous computation with two parties in the presence of fully-defective channel noise. The two-party impossibility implies a general impossibility result for any network that contains a bridge since the two connected components over the two sides of the bridge can be reduced to the two parties case.

Formalizing the above intuition is slightly more subtle. For the impossibility to hold, we must require the protocol to give output (or explicitly terminate). To see why, consider the case of two parties (say, Alice and Bob) that hold the private inputs $x$ and $y$, respectively, and need to compute some fixed known function $f(x, y)$. Suppose that, instead of requiring the protocol to give a non-revocable output, we only require that there exists a time $t$ after which both parties hold $f(x, y)$ and never change it again. Then, the following protocol succeeds in computing $f$ in the fully-defective two-party network (stated for Alice; Bob’s protocol is symmetric):

(a) Send $x$ messages to Bob; (b) $\text{count} \leftarrow 0$; (c) Upon the reception of a message, $\text{count} \leftarrow \text{count} + 1$; update the output variable to $f(x, \text{count})$.

Nevertheless, if we require the parties to terminate or to give an output, no protocol for non-trivial functions $f$ exists.

**Theorem 20** Consider a fully-defective network of two parties connected via a single noisy channel, and let $f(x,y)$ be any non-constant function. Any two-party deterministic protocol that computes $f$ and gives an output, is incorrect.

**Proof** Let $f$ be some non-constant function and assume, without loss of generality, that its input and output domains are the natural numbers. We can restrict the discussion to
protocols in which each message sent by any of the parties contains a single ‘1’ bit. This is without loss of generality, since we can equivalently consider the case where the adversary corrupts the content of any message to be ‘1’. Since the setting is asynchronous, a party can send zero or more messages as a function of its input and the number of messages it has received so far. A party is assumed to be idle between the time it sends a batch of messages until the time a new message arrives (which may trigger the transmission of new messages). In particular, once a new message arrives, the party immediately decides upon the number \( k \geq 0 \) of new messages to send, transmits them, and then goes back to being idle (or terminates).

Consider some inputs \((x, y)\) and \((x', y')\) for which \( f(x, y) \neq f(x', y') \), if no such inputs exist then a symmetric proof holds for a pair of inputs \((x, y)\) and \((x', y')\). Fix Bob’s input to \( y \). Note that once \( y \) is fixed, Bob’s actions depend only on the number of messages he has received so far. That is, we can completely describe Bob’s protocol by the sequence \( B_y = (0, \text{action}_0)(1, \text{action}_1)(2, \text{action}_2) \cdots \), where for any \( t \geq 0 \), the item \((r, \text{action})\) is to be interpreted as the action Bob performs after seeing \( r \) messages from Alice. The value \( \text{action}_t \in \{\text{send}_k, \text{SendAndOutput}_{k,r}\}_{k,r \geq 0} \) describes the action Bob takes at that step of the protocol: \( \text{send}_k \) means that Bob transmits \( k \) messages to Alice, and \( \text{SendAndOutput}_{k,r} \) means that Bob sends \( k \) messages to Alice and sets its output register (irrevocably) to \( r \), i.e., Bob commits to the output \( r \). Note that this is a complete characterization of Bob’s protocol. We may assume that Bob continues to send and receive messages after setting its output, however, if in a later step Bob performs the action \( \text{SendAndOutput}_{k,r} \), then Bob will only send \( k \) messages but the output register will not change.

Also note that Bob progresses sequentially. That is, Bob first performs \( \text{action}_0 \), then \( \text{action}_1 \), etc. Once Bob receives no further messages from Alice, he stops making any further progress. Thus, in order to give an output, Bob must reach some \( t \geq 0 \) where \( \text{action}_t = \text{SendAndOutput}_{k,r} \). Consider \( B_y \) and set \( \hat{r} = \arg \min_t \{\text{action}_t \in \{\text{SendAndOutput}_{k,r}\}_{k,r \geq 0}\} \); we know that \( \hat{r} < \infty \) and \( \text{action}_t = \text{SendAndOutput}_{k,r} \), with some \( k, \hat{r} \geq 0 \), or otherwise Bob never gives an output on input \( y \). Finally, we note that Bob acts as described regardless of Alice’s input: Bob advances sequentially until seeing \( \hat{r} \) messages from Alice, after which it commits on the output \( \hat{r} \).

Now consider an execution of the protocol on the input \((x, y)\). As described above, Bob commits on output when performing \( \text{action}_t = \text{SendAndOutput}_{\hat{r},\hat{r}} \). If Bob does not give the correct output, we are done. Otherwise, \( \hat{r} = f(x, y) \). Next, consider the execution of the protocol on the input \((x', y)\). If Bob receives less than \( \hat{r} \) messages overall (and the protocol then reaches quiescence), Bob does not give an output. Otherwise, upon receiving the \( \hat{r} \)-th message, Bob outputs \( \hat{r} = f(x, y) \). As both these options are incorrect for the input \((x', y)\), we have reached a contradiction.

### 7 Conclusion and open questions

We showed that content-oblivious computation, where message content is empty or invalid, is possible in any 2-edge connected network. Further, this condition is necessary: No non-trivial computation can be performed in networks that are not 2-edge connected and have a bridge. Content-oblivious computation may have applications in systems suffering from very harsh noise or in systems where communication is extremely limited, so nodes can only signal each other by sending pulses.

We conclude with some open questions and research directions.

- **Overhead** Our main motivation was to show the feasibility of content-oblivious simulation, and we did not care too much about the resulting overhead caused by our compiler. Finding the minimum overhead for content-oblivious simulations and the exact relationship between topology properties and overhead is an interesting open question. For example, any graph that can be decomposed into (short) disjoint cycles will have overhead proportional to its longest cycle length. Can other properties be exploited to achieve a faster compiler, for example, when the network is \( k \)-edge-connected, with \( k > 2 \)? Is it possible to simulate computations using an infrastructure other than cycles, such as (a family of) spanning trees? The overhead of the pre-processing phase can potentially be further reduced. As mentioned in Sect. 5.3, our Robbins Cycle construction exhausts all edges in the network, even when an edge does not add new nodes to the cycle. Constructing cycles over directed graphs or when edges have weights (which we try to minimize) might lead to compelling applications.

- **A root** The preprocessing phase of our construction assumes that a particular node has been preselected as the root. We conjecture that this assumption is necessary for any non-trivial content-oblivious computation.

- **Probabilistic protocols** Our construction assumes deterministic algorithms. If the noiseless protocol is probabilistic, then our compiler will still simulate one specific execution of the noiseless protocol. However, this will affect the success probability and might have further undesired consequences. Extending the simulation and analysis to probabilistic algorithms remains an interesting open direction.

- **Content-oblivious computation of specific tasks** Our compiler can be used to compute any task (that has a noiseless protocol) in a content-oblivious manner, but...
with high overhead. Can some specific tasks be solved more efficiently directly (i.e., without applying our general compiler)? In this paper, we give direct constructions for Robbins Cycle, DFS, and ear-decomposition. In [8], a direct BFS construction was given. What other tasks can be computed (fast!) in this setting?

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Declarations

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