The vertex leafage of chordal graphs

Steven Chaplick\textsuperscript{a}, Juraj Stacho\textsuperscript{b}

\textsuperscript{a}Department of Physics and Computer Science, Wilfrid Laurier University, 75 University Ave. West, Waterloo, Ontario N2L 3C5, Canada
\textsuperscript{b}DIMAP and Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

Abstract

Every chordal graph $G$ can be represented as the intersection graph of a collection of subtrees of a host tree, a so-called tree model of $G$. The leafage $\ell(G)$ of a connected chordal graph $G$ is the minimum number of leaves of the host tree of a tree model of $G$. The vertex leafage $v\ell(G)$ is the smallest number $k$ such that there exists a tree model of $G$ in which every subtree has at most $k$ leaves. The leafage is a polynomially computable parameter by the result of [11]. In this contribution, we study the vertex leafage.

We prove for every fixed $k \geq 3$ that deciding whether the vertex leafage of a given chordal graph is at most $k$ is NP-complete by proving a stronger result, namely that the problem is NP-complete on split graphs with vertex leafage of at most $k + 1$. On the other hand, for chordal graphs of leafage at most $\ell$, we show that the vertex leafage can be calculated in time $n^{O(\ell)}$. Finally, we prove that there exists a tree model that realizes both the leafage and the vertex leafage of $G$. Notably, for every path graph $G$, there exists a path model with $\ell(G)$ leaves in the host tree and it can be computed in $O(n^2)$ time.

Key words: chordal graph, leafage, tree model, clique tree, path graph

1. Introduction

In the following text, a graph is always finite, simple, undirected, and loopless. We write that a graph $G = (V, E)$ has vertex set $V(G)$ and edge set $E(G)$. We write $uv$ for the edge $(u, v) \in E(G)$. We write $N_G(v)$ to denote the neighbourhood of $v$ in $G$, and write $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v$ in $G$ is denoted by $\deg_G(v) = |N_G(v)|$. Where appropriate, we drop the index $G$ and write $N(v)$, $N[v]$, and $\deg(v)$, respectively. We write $G[X]$ to denote the subgraph of $G$ induced by $X \subseteq V(G)$, and write $G - X$ for the graph $G[V(G) \setminus X]$. We write $G - v$ for $G - \{v\}$. We say that $X$ is a clique of $G$ if $G[X]$ is a complete graph, and $X$ is an independent set of $G$ if $G[X]$ has no edges.

A tree model of a graph $G = (V, E)$ is a pair $\mathcal{T} = (T, \{T_u\}_{u \in V})$ where $T$ is a tree, called a host tree, each $T_u$ is a subtree of $T$, and a pair $uv$ is in $E$ if and only if $V(T_u) \cap V(T_v) \neq \emptyset$. In other words, $\mathcal{T}$ consists of a host tree and a collection of its subtrees whose vertex intersection graph is $G$.

A graph is chordal if it does not contain an induced cycle of length four or more. It is well-known [1, 6, 20] that a graph is chordal if and only if it has a tree model.

For a tree $T$, let $\mathcal{L}(T)$ denote the set of its leaves, i.e., vertices of degree one. If $T$ consists of a single node, we define $\mathcal{L}(T) = \emptyset$. In other words, we consider such a tree to have no leaves.

The leafage of a chordal graph $G$, denoted by $\ell(G)$, is defined as the smallest integer $\ell$ such that there exists a tree model of $G$ whose host tree has $\ell$ leaves (see [14]). It is easy to see that $\ell(G) = \ell$ if and only if $G$ is a complete graph, and otherwise $\ell(G) \geq 2$. Moreover the case $\ell(G) = 2$ corresponds precisely to interval graphs (intersection graphs of intervals of the real line) [4]. In this sense, the leafage of a chordal graph $G$ measures how close $G$ is to being an interval graph.

In this paper, we study a similar parameter.

\textsuperscript{*}The main work on this project was done during the second author’s visit at the University of Toronto in 2009 and later during the first author’s visit at the Caesarea Rothschild Institute of the University of Haifa in 2011. Both visits were supported by Prof. Derek Corneil of the University of Toronto via his NSERC Grant. The second author also gratefully acknowledges support from EPSRC, award EP/I01795X/1.

\textsuperscript{\textbullet}Email addresses: chaplick@cs.toronto.edu (Steven Chaplick), j.stacho@warwick.ac.uk (Juraj Stacho)

Preprint submitted to Elsevier

January 19, 2013
Definition 1. For a chordal graph $G = (V, E)$, the vertex leafage of $G$, denoted by $\nu(G)$, is the smallest integer $k$ such that there exists a tree model $(T, \{T_u\}_{u \in V})$ of $G$ where $|\mathcal{L}(T_u)| \leq k$ for all $u \in V$.

In other words, the vertex leafage of $G$ seeks a tree model of $G$ where each of the subtrees (corresponding to the vertices of $G$) has at most $k$ leaves and the value of $k$ is smallest possible.

As in the case of leafage, the vertex leafage is a natural parameter related to some subclasses of chordal graphs previously studied in the literature. We note that $\nu(G) \geq 2$ unless $G$ is a complete graph (in which case $\nu(G) = 0$), and the case $\nu(G) \leq 2$ corresponds precisely to the so-called path graphs (intersection graphs of paths in trees) [7] (see also [2, 13, 15, 17]). Thus, the vertex leafage of a chordal graph $G$ can be seen as a way to measure how close $G$ is to being a path graph. In [10], it is further observed that in $O(kn)$ time one can find: an optimal colouring, a maximum independent set, a maximum clique, and an optimal clique cover of an $n$-vertex chordal graph $G$ with vertex leafage $\ell$ if a representation of $G$ (a tree model realizing vertex leafage) is given.

In [7] it is shown that path graphs can be recognized in polynomial time. Currently, the best known recognition algorithms for path graphs run in $O(nm)$ time [2, 17], where $n = |V(G)|$ and $m = |E(G)|$. In other words, for a graph $G$, testing whether $\nu(G) \leq 2$ can be performed in $O(nm)$ time.

Some other restrictions/variations on the standard tree model have also been studied. One such family of these variations is captured by the $[h, s, t]$ graphs (introduced in [12]) defined as follows: $G = (V, E)$ is an $[h, s, t]$ graph if there is a tree model $(T, \{T_u\}_{u \in V})$ of $G$ such that the maximum degree of $T$ is at most $h$, the maximum degree of each of $\{T_u\}_{u \in V}$ is $s$, and $uv$ is an edge of $G$ if and only if $Tu$ and $Tv$ have at least $t$ vertices in common. For more information on these graphs see [3, 9].

We summarize the results of our paper in the following theorems.

Theorem 2. For every $k \geq 3$, it is NP-complete to decide, for a split graph $G$ whose vertex leafage is at most $k + 1$, if the vertex leafage of $G$ is at most $k$.

Theorem 3. For every $\ell \geq 2$, there exists an $n^{O(\ell)}$ time algorithm that, given an $n$-vertex chordal graph $G$ with $\ell(G) \leq \ell$, computes the vertex leafage of $G$ and construct a tree model of $G$ that realizes the vertex leafage of $G$.

Theorem 4. There exists an $O(n^3)$ time algorithm that, given an $n$-vertex chordal graph $G = (V, E)$ and a tree model $(T, \{T_u\}_{u \in V})$ of $G$, computes a tree model $(T^*, \{T^*_u\}_{u \in V})$ of $G$ such that

(i) $|\mathcal{L}(T^*_u)| \leq |\mathcal{L}(T_u)|$ for all $u \in V$,
(ii) $|\mathcal{L}(T^*)| = \ell(G)$.

Corollary 5. For every chordal graph $G = (V, E)$, there exists a tree model $(T^*, \{T^*_u\}_{u \in V})$ such that

(i) $|\mathcal{L}(T^*_u)| \leq \nu(G)$ for all $u \in V$.
(ii) $|\mathcal{L}(T^*)| = \ell(G)$.

In other words, such a tree model is optimal with respect to the leafage and also the vertex leafage of $G$.

This paper is structured as follows. First, in §2, we discuss some technical details related to tree models. After that, in §3, we prove for every fixed $k \geq 3$ that deciding whether the vertex leafage of a chordal graph is at most $k$ is NP-complete (i.e., we prove Theorem 2). In light of theorem 2, in §4, we discuss calculating vertex leafage subject to bounded leafage. More specifically, for bounded leafage $\ell$, we show how to compute the vertex leafage in time $n^{O(\ell)}$ (i.e., we prove Theorem 3). Finally, in §5, we show that the vertex leafage and leafage of any chordal graph $G$ can be realized simultaneously in a tree model of $G$ (i.e., we prove Theorem 4 and Corollary 5). We close the paper in §6 with a summary and a discussion of possible extensions of this work.

2. Minimal Tree Models and Clique Trees

We need to discuss a particular type of tree models of chordal graphs. Most of this section is rather technical and a reader experienced with tree models can easily skip this part. However, we include it for completeness as some of the subtle transformations involved may not be clear to every reader.
Let $G = (V, E)$ be a chordal graph. We say that two tree models $\mathcal{T} = (T, \{T_u\}_{u \in V})$ and $\mathcal{T}' = (T', \{T'_u\}_{u \in V})$ of $G$ are isomorphic, and write $\mathcal{T} \simeq \mathcal{T}'$, if there exists an isomorphism $\phi$ between $T$ and $T'$ that induces an isomorphism between $T_u$ and $T'_u$ for all $u \in V$, namely $\phi(V(T_u)) = V(T'_u)$.

A tree model $\mathcal{T} = (T, \{T_u\}_{u \in V})$ of $G$ is minimal if $|V(T)|$ is smallest possible among all tree models of $G$. A clique tree of $T$ is a tree $T'$ whose nodes are the maximal cliques of $G$ such that for all $C, C' \in V(T)$, every $C''$ on the path between $C$ and $C'$ in $T$ satisfies $C'' \supseteq C \cap C'$. Every clique tree $T$ of $G$ defines a tree model $\mathcal{T}_T$ of $G$, where $\mathcal{T}_T = (T, \{T_u\}_{u \in V})$ and $T_u$ is defined as $T \left\{ \{ C \in V(T) \mid u \in C \} \right\}$ for all $u \in V$.

**Fact 6.** Let $\mathcal{T} = (T, \{T_u\}_{u \in V})$ be a tree model of $G$. Then the following statements are equivalent.

1. $\mathcal{T}$ is a minimal tree model of $G$.
2. $\mathcal{T} \simeq \mathcal{T}_T$ for some clique tree $T$ of $G$.
3. For all $XY \in E(T)$, contracting $XY$ in $T$ and all subtrees $T_u$ containing it yields a tree model of $G' \neq G$.
4. The mapping $\psi$ defined for $X \in V(T)$ as $\psi(X) = \{ u \in V \mid X \in V(T_u) \}$ is a bijection between the vertices of $T$ and the maximal cliques of $G$.

**Proof.** (i)⇒(ii) and (ii)⇐(iv) are clear, while (iii)⇒(iv)⇒(i) follow from the Helly property of subtrees.

Note that (iv) in the above claim states, in other words, that the set of all vertices of $G$ whose subtrees contain $X$ is a maximal clique of $G$. In particular, for any tree model, the set of such vertices is always a clique of $G$, but it is not always necessarily a maximal clique. This is only true for minimal tree models.

It follows from Fact 6(i)⇐(iii) that every tree model $(T, \{T_u\}_{u \in V})$ of $G$ can be transformed (by contracting some edges of the host tree and the subtrees) into a minimal tree model $(T', \{T'_u\}_{u \in V})$. Notably, as this transformation involves only contracting edges, it follows that this does not increase the number of leaves both in the host tree and the subtrees, namely $|\mathcal{L}'(T')| \leq |\mathcal{L}(T)|$ and $|\mathcal{L}'(T_u)| \leq |\mathcal{L}(T'_u)|$ for all $u \in V$.

This observation allows us to focus exclusively on minimal tree models. Namely, it shows that if there exists a tree model with minimum number of leaves in the host tree (subtrees), then there also is a minimal tree model with minimum number of leaves in the host tree (subtrees). Consequently, in the remainder of the paper, all tree models are assumed to be minimal tree models unless otherwise specified.

Furthermore, using Fact 6(i)⇐(ii), we shall view minimal tree models of $G$ as tree models defined by clique trees of $G$. We shall switch between the two viewpoints as needed.

### 3. Vertex Leafage is NP-complete

In this section, we prove Theorem 2 stating that calculating the vertex leafage of a chordal graph is NP-complete. We describe a polynomial time reduction from the problem NOT-ALL-EQUAL-$k$-SAT which is well-known to be NP-complete [5].

**Proof of Theorem 2.** The problem is clearly in NP as one can easily compute in polynomial time the number of leaves in subtrees of a given tree model. To prove NP-hardness, we show a reduction from NOT-ALL-EQUAL-$k$-SAT. By standard arguments [16], we may assume, without loss of generality, that the instances to this problem contain no repeated literals and no negated variables. Thus we can phrase the problem as follows.

**NOT-ALL-EQUAL-$k$-SAT**

**Instance** $\mathcal{I}$: a collection $C_1, C_2, \ldots, C_m$ of $k$-element subsets of $\{v_1, \ldots, v_n\}$.

**Solution** to $\mathcal{I}$ (if exists): a set $S \subseteq \{v_1, \ldots, v_n\}$ such that each $j \in \{1 \ldots m\}$ satisfies $C_j \setminus S \neq \emptyset$ and $S \setminus C_j \neq \emptyset$.

In addition, we may assume the following property of any instance $\mathcal{I}$.

(*) There are no distinct indices $i, i^+$ such that $v_{i^+} \in C_j$ whenever $v_i \in C_j$.

Indeed, if there exist $i \neq i^+$ with $v_{i^+} \in C_j$ whenever $v_i \in C_j$, then we replace $\mathcal{I}$ by another instance $\mathcal{I}^+$ constructed from $\mathcal{I}$ by removing $v_i$ and all clauses $C_j$ that contain $v_i$. If there is a solution to $\mathcal{I}$, then clearly $S \setminus \{v_i\}$ is a solution to $\mathcal{I}^+$. Conversely, if $S$ is a solution to $\mathcal{I}^+$, then either $S$ is a solution to $\mathcal{I}$ if $v_{i^+} \in S$, or $S \cup \{v_i\}$ is a solution to $\mathcal{I}$ if otherwise.

Now, for the reduction, we consider an instance $\mathcal{I}$ satisfying (*) and construct a graph, denoted by $G_\mathcal{I}$, as follows:
(i) the vertex set of \( G_T \) consists of \( n + m + 2 \) vertices: \( V(G_T) = \{v_1, \ldots, v_n, y_1, \ldots, y_m, z_1, z_2\} \).
(ii) the vertices \( \{y_1, \ldots, y_m\} \) form a clique,
(iii) the vertices \( \{v_1, \ldots, v_n, z_1, z_2\} \) form an independent set,
(iv) each vertex \( v_i \) is adjacent to all vertices \( y_j \) such that \( v_i \in C_j \),
(v) the vertices \( z_1, z_2 \) are adjacent to each vertex of the clique \( \{y_1, \ldots, y_m\} \).

We observe that \( G_T \) is a split graph with partition into clique \( \{y_1, \ldots, y_m\} \) and independent set \( \{v_1, \ldots, v_n, z_1, z_2\} \).

We prove that the vertex leafage of \( G_T \) is:

(a) at most \( k + 1 \), and
(b) is at most \( k \) if and only if there is a solution to \( T \).

To do this we analyze the cliques of \( G_T \). This is easy, since \( G_T \) is a split graph; all its maximal cliques are formed by taking a vertex of the independent set with its neighbourhood. In particular, the maximal cliques of \( G_T \) are \( A = \{z_1, y_1, \ldots, y_m\}, B = \{z_2, y_1, \ldots, y_m\}, \) and \( Q_i = \{v_i \cup \{y_j \mid v_i \in C_j\}\} \) for each \( i \in \{1 \ldots n\} \).

We first prove (a). Recall that \( \{A, B, Q_1, \ldots, Q_n\} \) is the set of all maximal cliques of \( G_T \), and hence, the vertex set of every clique tree of \( G_T \). Each of the vertices \( z_1, z_2, \) and \( v_i \), \( i \in \{1 \ldots n\} \), belongs to exactly one of these cliques, namely \( A, B, \) and \( Q_i \), respectively. Also, each \( y_j \), for \( j \in \{1 \ldots m\} \), belongs to exactly \( k + 2 \) cliques, namely \( A, B, \) and \( \{Q_i_1, \ldots, Q_i_k\} \) where \( C_j = \{v_{i_1}, \ldots, v_{i_k}\} \). So, as \( k \geq 3 \), every tree spanning these cliques has at most \( k + 1 \) leaves. We thus conclude that in every clique tree of \( G_T \), each subtree corresponding to a vertex of \( G_T \) has at most \( k + 1 \) leaves. In other words, any clique tree of \( G_T \) certifies that \( \nu(T) \leq k + 1 \) which proves (a).

We now prove (b). Let \( S \) be a solution to \( T \). Construct a tree \( T \) with vertex set \( \{A, B, Q_1, \ldots, Q_n\} \) and edge set \( \{AB\} \cup \{AQ_i \mid v_i \in S\} \cup \{BQ_i \mid i \notin S\} \). Let us verify that \( T \) is a clique tree of \( G_T \). Its vertex set is the set of all maximal cliques of \( G_T \). For distinct \( i, i^+ \in \{1 \ldots n\} \), the path between \( Q_i \) and \( Q_i^+ \) contains \( A \) or \( B \) or both, and no other vertex. Note that \( Q_i \cap Q_i^+ \subseteq \{y_1, \ldots, y_m\} = A \cup B \). This verifies the path between \( Q_i \) and \( Q_i^+ \). Similarly, the path between \( Q_i \) and \( A \) or \( B \) additionally contains only \( A \) or \( B \) and we have \( Q_i \cap A = Q_i \cap B \) which verifies this path. Thus both \( A \) and \( B \) have at least two neighbours in \( T \) and are therefore not leaves of \( T \). Consequently, \( |\nu(T)| = (|Q_1, \ldots, Q_n|) = k \) which implies \( \nu(T) \leq k \) as certified by the tree model \( T \).

Conversely, suppose that \( \nu(T) \leq k \). Then there exists a clique tree \( T \) of \( G_T \) such that the corresponding model \( T = \{T \} \cup V(G_T) \) satisfies \( |\nu(T)| \leq k \) for all \( v \in V(G_T) \). We analyze the structure of \( T \). First, we observe that \( AB \) must be an edge of \( T \). If otherwise, the path between \( A \) and \( B \) in \( T \) contains some clique \( Q_i, i \in \{1 \ldots n\} \). As \( T \) is a clique tree, we conclude \( \{y_1, \ldots, y_m\} = A \cup B \subseteq Q_i = \{v_i \cup \{y_j \mid v_i \in C_j\}\} \). But then \( v_i \) belongs to each \( C_i, j \in \{1 \ldots m\} \), and since \( n \geq k \geq 3 \), this contradicts (*). Similarly, we show that each \( Q_i, i \in \{1 \ldots n\} \) is a leaf of \( T \). If otherwise, some \( Q_i \) has at least two neighbours in \( T \). These cannot be \( A \), \( B \) as this would imply a triangle in \( T \), since \( AB \) is an edge of \( T \). Thus \( Q_i \) is adjacent to \( Q_{i^+} \) for some \( i^+ \in \{1 \ldots n\} \). As \( T \) is a tree, we have that either \( Q_{i^+} \) lies on the path from \( A \) to \( Q_i \) or \( Q_i \) lies on the path from \( A \) to \( Q_{i^+} \). By symmetry, we may assume the former. Thus, since \( T \) is a clique tree, we conclude \( \{y_j \mid v_i \in C_j\} = A \cap Q_i \subseteq Q_{i^+} = \{y_{i^+} \} \cup \{y_j \mid v_{i^+} \in C_j\} \). So \( y_{i^+} \in C_j \) whenever \( v_i \in C_j \), contradicting (*).

Now, we are ready to construct a set \( S \subseteq \{v_1, \ldots, v_n\} \) as follows: for each \( i \in \{1 \ldots n\} \), put \( v_i \) in \( S \) if \( AQ_i \) is an edge of \( T \). We show that \( S \) is a solution to \( T \). If not, there exists \( i \in \{1 \ldots m\} \) such that either \( S \supseteq C_j \) or \( S \cap C_j = \emptyset \). We look at the subtree \( T_{y_i} \) corresponding to the vertex \( y_i \). Recall that \( y_i \) belongs to cliques \( A, B, \) and \( k \) cliques \( Q_{i_1}, \ldots, Q_{i_k} \) where \( C_k = \{v_{i_1}, \ldots, v_{i_k}\} \). The cliques \( Q_{i_1}, \ldots, Q_{i_k} \), are leaves of \( T \), because they are leaves of \( T \) (as proved above). If \( S \subseteq C_j \), we have, by construction, that \( A \) is the unique neighbour of each of the cliques \( Q_{i_1}, \ldots, Q_{i_k} \) in \( T \). Consequently, none of the cliques \( Q_{i_1}, \ldots, Q_{i_k} \) is adjacent to \( B \) in \( T \). This shows that \( B \) is
only adjacent to \( A \) in \( T_y \), and hence, is a leaf. But then \( \mathcal{L}(T_y) = |\{Q_1, \ldots, Q_k, B\}| = k + 1 \), contradicting our assumption about \( T \). Similarly, if \( S \cap C_i = \emptyset \), the cliques \( Q_1, \ldots, Q_k \) are only adjacent to \( B \) and not to \( A \), in which case, \( A \) is a leaf of \( T_y \), leading to the same contradiction.

Therefore, \( S \) must indeed be a solution to \( \mathcal{I} \) and that concludes the proof. \( \square \)

### 4. Vertex Leafage Parameterized by Leafage

In this section, we discuss calculating vertex leafage in chordal graphs of bounded leafage. Namely, we prove Theorem 3, that is, for a fixed \( \ell \), we demonstrate how to calculate the vertex leafage of an \( n \)-vertex chordal graph \( G \) with \( \ell(G) \leq \ell \) in polynomial time, namely, in time \( n^{O(\ell)} \). We do this by enumerating clique trees of \( G \) with respect to high (\( \geq 3 \)) degree nodes. The enumeration is based on the observation that the number of high-degree nodes in a tree is directly related to the number of leaves. This goes as follows.

For a tree \( T \), let \( \mathcal{H}(T) \) denote the set of nodes of \( T \) of degree \( \geq 3 \), and let \( \mathcal{E}(T) \) denote the set of edges of \( T \) incident to the nodes in \( \mathcal{H}(T) \). Further, let \( n_t \) denote the number of nodes of degree \( i \) in \( T \). Then

\[
|\mathcal{H}(T)| = \sum_{i \geq 3} n_t \leq |\mathcal{E}(T)| \leq \sum_{i \geq 3} (i - 2)n_t = 2|E(T)| - 2|V(T)| + n_1 = |\mathcal{L}(T)| - 2
\]

In particular, if \( |\mathcal{L}(T)| \) is bounded, then so is \( |\mathcal{H}(T)| \) and \( |\mathcal{E}(T)| \). This will become useful later.

Recall that the vertex set of every clique tree of \( G \) is the set of all maximal cliques of \( G \). Notably, all clique trees have the same vertex set. Let \( \mathcal{E}(G) \) denote the clique graph of \( G \), i.e., the graph whose nodes are the maximal cliques of \( G \) and where two nodes are adjacent if and only if the corresponding maximal cliques are adjacent. It is well-known [8, 18] that every clique tree of \( G \) is a spanning tree of \( \mathcal{E}(G) \).

Our algorithm is based on the following lemma.

**Lemma 7.** There is an \( O(n^3) \) time algorithm that, given an \( n \)-vertex chordal graph \( G \) and a set \( F \subseteq E(\mathcal{E}(G)) \), decides if there exists a clique tree \( T \) of \( G \) with \( \mathcal{E}(T) = F \) and constructs such a tree if one exists.

**Proof.** We describe an algorithm for the problem as follows.

**Algorithm 1:**

**Input:** A chordal graph \( G \) and a set \( F \subseteq E(\mathcal{E}(G)) \).

**Output:** A clique tree \( T \) of \( G \) with \( \mathcal{E}(T) = F \), or report that no such tree exists.

1. Construct a graph \( G' \) as follows:
   
   \[
   V(G') = V(G) \cup \{v_e \mid e \in F\} \\
   E(G') = E(G) \cup \{u v_e \mid e = CC', u \in C \cup C'\} \cup \{v_e v_{e'} \mid e \cap e' \neq \emptyset\}
   \]

2. **If** \( G' \) is chordal **then**

3. Construct a clique tree \( T' \) of \( G' \) with minimum number of leaves.

4. Construct a tree \( T \) from \( T' \) by renaming each node \( C' \in V(T') \) to \( C' \cap V(G) \)

5. **If** \( T \) is a clique tree of \( G \) and \( \mathcal{E}(T) = F \) **then**

6. **return** \( T \)

7. return “no such tree exists”

We now prove correctness of the above algorithm. For simplicity, we shall refer to any clique tree \( T \) with \( \mathcal{E}(T) = F \) as a “solution”. First, observe that if the algorithm returns the tree \( T \) in line 6, then this is indeed a solution. This proves that if there is no solution, the algorithm provides the correct answer in line 7.

Thus, for the rest of the proof, we may assume that a solution exists. Namely we shall assume there is a clique tree \( T^* \) of \( G \) satisfying \( \mathcal{E}(T^*) = F \). For every maximal clique \( C \) of \( G \), define \( \phi(C) = C \cup \{v_e \mid e \in \epsilon\} \).

In the following claim, we discuss the properties of the graph \( G' \) constructed in line 1.

\((1)\) \( G' \) is chordal, satisfies \( \ell(G') \leq |\mathcal{L}(T^*)| \), and \( \phi \) is a bijection between the maximal cliques of \( G \) and \( G' \).

To prove the claim, we construct a minimal tree model of \( G' \) as follows. Let \( \mathcal{T}^* = \{T^*, \{T^*_e \}_{e \in V(G')}\} \) be the minimal tree model of \( G \) that is defined by the clique tree \( T^* \), namely \( T^*_e = T(\{C \in V(T^*) \mid u \in C\}) \). For each edge \( e = CC' \in F \), define \( T^*_e = T^*(\{C, C'\}) \). Finally, let \( \mathcal{T}^* = \{T^*, \{T^*_e \}_{e \in F}\} \).
It is easy to verify that $\mathcal{P}^+$ is a tree model of $G'$. In particular, each subtree in the collection is a connected subgraph of $T^*$. This follows from the fact that $T^*$ is a clique tree and that $F = \mathcal{E}(T^*) \subseteq E(T^*)$. Further, for each edge $e = CC' \in F$, we see that the subtree $T^*_{v_e}$ intersects only subtrees $T^*_u$ where $u \in C \cup C'$. Moreover, $T^*_{v_e}$ only intersects subtrees $T^*_v$ where $C \cap C' = V(T^*_v)$, i.e., those where $e \cap e' \neq \emptyset$. This corresponds precisely to the definition of $G'$.

Thus, we conclude that $G'$ is indeed a chordal graph, and $\ell(G') \leq |\mathcal{L}(T^*)|$ as $\mathcal{P}^+$ is a particular tree model of $G'$ and $T^*$ is its host tree. Moreover, we see that $\mathcal{P}^+$ is actually a minimal tree model of $G'$. Indeed, if there were a tree model of $G'$ with less than $|V(T^*)|$ nodes in its host tree, then by removing subtrees corresponding to the vertices $\{v_e \mid e \in F\}$ we would obtain a tree model of $G$ whose host tree has less than $|V(T^*)|$ nodes. But this would contradict the minimality of $\mathcal{P}_T$.

This implies, by Fact 6(ii), that there exists a clique tree $T^+$ of $G'$ that defines $\mathcal{P}^+$, i.e., $\mathcal{P}^+ = \mathcal{P}_T^+$. Namely, there is an isomorphism between $T^+$ and the host tree $T^*$ of $\mathcal{P}^+$ where each node $C \in V(T^*)$ corresponds to the set of all vertices of $G'$ whose subtrees contain $C$, i.e., the set $\{u \in V(G) \mid C \subseteq V(T_u)\}$ which is exactly $\varphi(C)$. In other words, $\mathcal{P}(T^*) = \{\varphi(C) \mid C \in V(T^*)\}$, and consequently, $\varphi$ constitutes an isomorphism between $T^*$ and $T^*$. As one is a clique tree of $G$ and the other a clique tree of $G'$, we conclude that $\varphi$ is a bijection between the maximal cliques of $G$ and $G'$. This proves (1).

This proves that the test in Line 2 succeeds. Now, consider the trees $T'$ and $T$ constructed in line 3 and 4. Notably, $T'$ is a clique tree of $G'$ with $|\mathcal{L}(T')| = \ell(G')$.

(2) $T$ is a clique tree of $G$.

Recall that $T$ is obtained from $T'$ by renaming each node $C'$ of $T'$ to $C' \cap V(G)$. Moreover, by (1), the mapping $\varphi$ is a bijection between the maximal cliques of $G$ and $G'$. Namely, for each $C' \in V(T)$, the set $C = \varphi^{-1}(C')$ is a maximal clique of $G$. Therefore, we can write $C' \cap V(G) = \varphi(C) \cap V(G) = \{C \cap \{v_e \mid e \in e\}\} \cap V(G) = C = \varphi^{-1}(C')$.

This proves that the vertex set of $T$ is precisely the set of maximal cliques of $G$, and $\varphi$ is an isomorphism between $T$ and $T'$, by the construction of $T$. To see that $T$ is indeed a clique tree of $G$, it remains to prove the “connectivity condition” for $T$. Namely, consider nodes $C_1, C_2 \in V(T)$ and a node $C_3$ in the path in $T$ between $C_1$ and $C_2$. Since $\varphi$ is an isomorphism between $T$ and $T'$, we have $\varphi(C_i) \in V(T')$ for $i = 1, 2, 3$ and $\varphi(C_3)$ lies on the path in $T'$ between $\varphi(C_1)$ and $\varphi(C_2)$. Thus, we conclude $\varphi(C_3) \supseteq \varphi(C_1) \cap \varphi(C_2)$ because $T'$ is a clique tree. So we write $C_3 = \varphi(C_3) \cap V(G) \supseteq \varphi(C_1) \cap \varphi(C_2) \cap V(G) = C_1 \cap C_2$. This proves (2).

This proves that $T$ is a clique tree of $G$. Notably, as $T^*$ is also a clique tree of $G$, we have that both $T$ and $T^*$ have the same vertex set, i.e., $V(T) = V(T^*)$. We now look at the edges of $T$.

(3) $F \subseteq E(T)$

Consider an edge $e = CC' \in F$, and recall the definition of $\varphi$ and the claim (1). From this it follows that $\varphi(C)$ and $\varphi(C')$ are the only maximal cliques of $G'$ that contain $v_e$. As $\varphi(C)$ and $\varphi(C')$ are also nodes of $T^*$ which is a clique tree of $G'$, we conclude that every maximal clique on the path in $T'$ between $\varphi(C)$ and $\varphi(C')$ also contains $v_e$. But, as $v_e$ is in no other maximal clique of $G'$, this is only possible if $\varphi(C)$ and $\varphi(C')$ are adjacent in $T'$. Consequently, $C$ and $C'$ are adjacent in $T$, namely $e \in E(T)$. This proves (3).

(4) $\mathcal{H}(T^*) \subseteq \mathcal{H}(T)$ and each $C \in \mathcal{H}(T^*)$ satisfies $N_T(C) \subseteq N_T(C)$.

Consider $C \in \mathcal{H}(T')$, namely $C$ is a node of $T^*$ with at least three neighbours in $T^*$. Then, by the definition of $\mathcal{E}(T^*)$, all edges incident to $C$ in $T^*$ belong to $\mathcal{E}(T^*)$. As $\mathcal{E}(T^*) = F$ and $F \subseteq E(T)$ by (3), the edges incident to $C$ in $T^*$ are also edges of $T$. In other words, every neighbour of $C$ in $T^*$ is a neighbour of $C$ in $T$, namely $N_T(C) \supseteq N_{T^*}(C)$. Thus $C$ has at least three neighbours in $T$ implying $C \in \mathcal{H}(T)$. This proves (4).

(5) $\mathcal{H}(T) = \mathcal{H}(T^*)$ and $\mathcal{E}(T) = \mathcal{E}(T^*)$.

By (3), we conclude $\mathcal{H}(T) \supseteq \mathcal{H}(T^*)$. For the converse, we calculate using (1) and ($*$) as follows.

$$\ell(G') \leq |\mathcal{L}(T^*)| = 2 + \sum_{C \in \mathcal{H}(T^*)} (\deg_{T^*}(C) - 2) \leq 2 + \sum_{C \in \mathcal{H}(T)} (\deg_T(C) - 2) = |\mathcal{L}(T)| = \ell(G)$$
If this is not so, we can safely report that no solution exists. Thus, since the two sums are equal, the only possibility is that \( \mathcal{H}(T) = \mathcal{H}(T^*) \) and each \( C \in \mathcal{H}(T^*) \) contributes to the sum on the right at least as much as to the sum on the left, since \( N_T(C) \geq N_{T^*}(C) \) by (4). Further, every \( C \in \mathcal{H}(T) \) has a positive contribution to the sum on the right as \( \deg_{\mathcal{T}}(C) \geq 3 \) by the definition of \( \mathcal{H}(T) \). Thus, since the two sums are equal, the only possibility is that \( \mathcal{H}(T) = \mathcal{H}(T^*) \) and that each \( C \in \mathcal{H}(T^*) \) satisfies \( N_T(C) = N_{T^*}(C) \) as claimed.

To conclude the proof, recall that \( \delta(T) \), resp. \( \delta(T^*) \), is the set of edges of \( T \), resp. \( T^* \), incident to the nodes in \( \mathcal{H}(T) \), resp. \( \mathcal{H}(T^*) \). As \( \mathcal{H}(T) = \mathcal{H}(T^*) \) and each \( C \in \mathcal{H}(T) = \mathcal{H}(T^*) \) is incident to the same set of edges in \( T \) and \( T^* \) for it satisfies \( N_T(C) = N_{T^*}(C) \), we conclude that \( \delta(T) = \delta(T^*) \). This proves (5).

This and (2) prove that \( T \) is indeed a solution, namely \( T \) is a clique tree of \( G \) with \( \delta(T) = \delta(T^*) = F \). Hence, the test in Line 5 succeeds and the algorithm correctly return a solution in Line 6.

That concludes the proof of correctness of the algorithm. To address the complexity, let \( n = |V(G)| \) as usual. First, we note that we may assume that \( F \) contains at most \( n - 1 \) edges as no clique tree of \( G \) has more than \( n \) edges. If this is not so, we can safely report that no solution exists. Thus, as \( G' \) has \( |V(G)| + |F| = O(n) \) vertices, we conclude that step 3 takes \( O(n^3) \) time using the algorithm of [14]. All other steps clearly take at most \( O(n^2) \) time. Notably, in step 2 we use a linear time algorithm from [19].

Thus the total complexity is \( O(n^3) \) as promised. That concludes the proof.

Finally, we are ready to prove Theorem 3.

**Proof of Theorem 3.** Let \( G \) be a chordal graph with \( \ell(G) \leq \ell \). By Corollary 5 (proven in §5), there exists a tree model of \( G \) that simultaneously minimizes both the leafage and the vertex leafage. By the remarks in §2, there is also a clique tree of \( G \) with this property; let \( T^* \) denote this clique tree. Note that \( |\mathcal{L}(T^*)| = |\mathcal{L}(G)| \) and \( |\mathcal{L}(T_u)| \leq \nu(G) \) for all \( u \in V(G) \) where \( T_u^* = T^* \{\{C \in V(T^*) \mid u \in C\}\} \).

We show that it suffices to know the set \( \delta(T^*) \) to find a tree model that minimizes the vertex leafage.

(6) If \( T \) is a clique tree of \( G \) with \( \delta(T) = \delta(T^*) \), then \( T \) minimizes the vertex leafage.

Consider \( u \in V(G) \). We need to show that \( |\mathcal{L}(T_u)| \leq \nu(G) \) where \( T_u = T \{\{C \in V(T) \mid u \in C\}\} \).

First, we observe that \( \delta(T) = \delta(T^*) \) implies \( \mathcal{H}(T) = \mathcal{H}(T^*) \) and each \( C \in \mathcal{H}(T) = \mathcal{H}(T^*) \) has the same neighbourhood in both \( T \) and \( T^* \), i.e., \( N_T(C) = N_{T^*}(C) \). Next, we remark that if a node has degree \( \geq 3 \) in \( T_u \), then also its degree in \( T \), since \( T_u \) is an induced subgraph of \( T \). In other words, we have \( \mathcal{H}(T_u) \subseteq \mathcal{H}(T) \). Further, we observe that each \( C \in V(T_u) \) satisfies \( N_T(C) = N_{T^*}(C) \cap V(T_u) \), since \( T_u \) is an induced subgraph of \( T \). By the same token, \( N_{T^*}(C) = N_{T^*}(C) \cap V(T_u) \) for each \( C \in V(T_u) \). Finally, we note that \( V(T_u) = V(T_u^*) \), since \( V(T) = V(T^*) \). Thus, for each \( C \in \mathcal{H}(T_u) \), we can write

\[
N_{T^*}(C) = N_{T^*}(C) \cap V(T_u) = N_{T^*}(C) \cap V(T_u^*) = N_{T_u^*}(C).
\]

This implies \( C \in \mathcal{H}(T_u^*) \) and \( \deg_{T^*}(C) = \deg_{T_u^*}(C) \) for all \( C \in \mathcal{H}(T_u^*) \). Thus, we calculate by (5).

\[
|\mathcal{L}(T_u)| = 2 + \sum_{C \in \mathcal{H}(T_u)} (\deg_{T_u}(C) - 2) \leq 2 + \sum_{C \in \mathcal{H}(T_u^*)} (\deg_{T_u^*}(C) - 2) = |\mathcal{L}(T_u^*)| \leq \nu(G)
\]

For the inequality to hold, also note that \( \deg_{T_u^*}(C) \geq 3 \) for each \( C \in \mathcal{H}(T_u^*) \), by definition. This proves (6).

This claim allows us to finally formulate our algorithm. We need to introduce additional of notation. Let \( F \subseteq E(\mathcal{G}(G)) \). If there exists a clique tree \( T \) with \( \delta(T) = F \), then define \( \nu_F = \max_{u \in V(G)} |\mathcal{L}(T_u)| \) where \( T_u = T \{\{C \in V(T) \mid u \in C\}\} \). If such a tree does not exist, define \( \nu_F = +\infty \). Observe that \( \nu_{\delta(T^*)} \leq \nu(G) \).

Our algorithm tries all possible sets \( F \subseteq E(\mathcal{G}(G)) \) of size at most \( \ell - 2 \) as candidates for \( \delta(T^*) \) and chooses one that that minimizes \( \nu_F \). If \( F_{\text{opt}} \) is this set, the algorithm outputs a clique tree \( T_{\text{opt}} \) of \( G \) with \( \delta(T_{\text{opt}}) = F_{\text{opt}} \).

We claim that this algorithm correctly finds a clique tree of \( G \) that minimizes the vertex leafage. By (5), we observe that \( \delta(T^*) \leq |\mathcal{L}(T^*)| \leq \nu(G) \). Thus, the algorithm must, at some point, consider as \( F \) the set \( \delta(T^*) \). For this \( F \), we have \( \nu_F = \nu_{\delta(T^*)} \leq \nu(G) \). By the minimality of \( F_{\text{opt}} \), we conclude \( \nu_F \leq \nu_{\delta(T^*)} \leq \nu(G) \).

Hence, \( \nu_{F_{\text{opt}}} < \infty \) and so the tree \( T_{\text{opt}} \) exists. Moreover, \( \nu_F \geq \nu(G) \) for all sets \( F \), by the definition of \( \nu(G) \).
Thus, we conclude $\ell_{F,\text{opt}} = \ell(G)$ and consequently by (6), $T_{\text{opt}}$ is a clique tree of $G$ that minimizes the vertex leafage. This proves the correctness of the algorithm.

Finally, let us analyze the complexity. Let $n = |V(G)|$ as usual. Recall that $G$ has at most $n$ maximal cliques. Thus there are at most $n^2$ edges in $\mathcal{C}(G)$, and hence, at most $n^{2^l-4}$ choices for the set $F$. For each choice of $F$, we use Lemma 7 to find a clique tree $T$ with $\mathcal{O}(T) = F$ if it exists. This takes $O(n^3)$ for each $F$, including the calculation of $\ell_F$. This yields, altogether, running time $O(n^{2^l-1}) = n^{O(l)}$ as promised.

We have shown how to calculate vertex leafage in polynomial time when the input graph has bounded leafage. It remains open whether this problem is fixed parameter tractable (FPT) with respect to leafage.

5. Vertex Leafage with Optimum Leafage

In this section, we prove Theorem 4 and Corollary 5. Namely, we demonstrate that the algorithm from [11], solving the leafage problem, satisfies the claim of Theorem 4. This algorithm, given a chordal graph $G$, outputs a clique tree of $G$ with minimum possible number of leaves. This is done by starting from an arbitrary clique tree $T$ of $G$, and iteratively decreasing the number of leaves of $T$ as long as possible.

We observe (and formally prove later in this section) that this process has the additional property that it never increases the number of leaves in the subtrees of the tree model $\mathcal{T}$ defined by $T$. In other words, if $T^*$ is the clique tree resulting from this process, then $\mathcal{T}^* = \mathcal{T}_T$ satisfies the claim of Theorem 4. This will imply that if the starting clique tree $T$ realizes the vertex leafage of $G$, then $\mathcal{T}^* = \mathcal{T}_T$ satisfies the claim of Corollary 5.

For the proof of the above, we need to explain the inner workings of the algorithm from [11]. This algorithm, in place of clique trees, operates on the so-called token assignments defined as follows.

For a chordal graph $G$, a token assignment of $G$ is a function $\tau$ that assigns to every maximal clique $C$ of $G$, a multiset $\tau(C)$ of subsets of $C$. We use the word token for the members of $\tau(C)$. Note that the same subset may appear in $\tau(C)$ many times. We focus on special token assignment that arise from clique trees.

The token assignment defined by a clique tree $T$ of $G$, and denoted by $\tau_T$, assigns to every maximal clique $C$ of $G$, the multiset $\tau_T(C) = \{C \cap C' \mid CC' \in E(T)\}$. In other words, $\tau_T(C)$ consists of the intersections of $C$ with its neighbours in $T$. A token assignment $\tau$ is realizable if there is a clique tree $T$ of $G$ such that $\tau = \tau_T$.

(See Figure 1 for an illustration of these concepts.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_chordal_graph}
\caption{a) Example chordal graph $G$, b) clique tree $T$ of $G$, c) token assignment $\tau = \tau_T$.}
\end{figure}

Notice that the token assignment $\tau = \tau_T$ contains all the information needed to determine the number of leaves in $T$ and also the number of leaves in the subtrees of the corresponding model $\mathcal{T}_T$. We summarize this as follows.

Lemma 8. Let $G$ be a chordal graph, let $T$ be a clique tree of $G$, and let $\mathcal{T}_T = \{T, \{T_u\}_{u \in V(G)}\}$ denote the tree model of $G$ defined by $T$. Let $\tau = \tau_T$, and define $\tau_u(C) = \{S \mid S \in \tau(C), u \in S\}$ for each $u \in V(G)$. Then

- $\deg_T(C) = |\tau(C)|$ for all $C \in V(T)$, and
- $\deg_{\tau_u}(C) = |\tau_u(C)|$ for all $u \in V(G)$ and all $C \in V(T_u)$.

Consequently, $\mathcal{L}(T) = \{C \mid |\tau(C)| = 1\}$ and $\mathcal{L}(T_u) = \{C \mid |\tau_u(C)| = 1\}$ for all $u \in V(G)$. 

8
In particular, while there can be multiple clique trees defining the same token assignment, these clique trees will have the same sets of leaves and consequently we do not need to distinguish them from one another. In other words, it suffices to maintain that the token assignment we consider corresponds to some clique tree of $G$. This can be tested easily by applying four particular conditions as described in [11]. As we do not use this test here directly, we omit further details. (For more, see [11, Theorem 6].)

Now, we are finally ready to explain the main steps of the algorithm from [11]. The algorithm is given a chordal graph $G$ and a clique tree $T$ of $G$. It starts by constructing the token assignment $\tau = \epsilon_T$. Then it proceeds iteratively. During each iteration step, a current token assignment $\tau$ is examined to determine if there exists a different token assignment corresponding to a clique tree with fewer leaves. This is done by checking for an augmenting path in $\tau$, which is a specific sequence of token moves (see definitions below). If an augmenting path exists, we pick the shortest such path and exchange tokens along the path. This results in a new token assignment $\tau$ that corresponds to a clique tree with fewer leaves. If no augmenting path exists, we arrive at an optimal solution (i.e., a token assignment whose $\ell(G)$ leaves) and we output this solution. We summarize the above procedure as Algorithm 2. Below we provide the missing definitions.

Let $G$ be a chordal graph and $\tau$ be a token assignment of $G$. A token move is an ordered triple $(C_1, C_2, S)$ where $C_1, C_2$ are maximal cliques of $G$ and $S \in \tau(C_1)$. For a token move $(C_1, C_2, S)$, we write $\tau \div (C_1, C_2, S)$ to denote the token assignment $\tau'$ that is the result of moving $S$ from $\tau(C_1)$ to $\tau(C_2)$. Namely, we have $\tau'(C_1) = \tau(C_1) \setminus \{S\}$ and $\tau'(C_2) = \tau(C_2) \cup \{S\}$, while $\tau'(C) = \tau(C)$ for all other $C \not\in \{C_1, C_2\}$.

A sequence of token moves $(C_1, C_2, S_1), (C_2, C_3, S_2), \ldots, (C_{k-1}, C_k, S_{k-1})$ where $k > 1$ is an augmenting path of $\tau$ if $|\tau(C_j)| = 1$ and each $j \in \{1 \ldots k-1\}$ satisfies

(i) $\tau \div (C_j, C_{j+1}, S_j)$ is a realizable token assignment$^2$, and

(ii) $|\tau(C_j)| = \begin{cases} 3 & \text{if } j = 1 \\ 2 & \text{otherwise} \end{cases}$

See Figure 2 for an example of an augmenting path of a token assignment $\tau$ and its application to $\tau$.

![Figure 2](image-url) a) token assignment $\tau$, b) augmenting path $(abc, adf, a)$, $(adf, cdk, d)$ – directed edges, c) $\tau$ after applying the path.

It is easy to see that the application of an augmenting path decreases the number of leaves in the resulting token assignment. This, however, does not guarantee that the resulting assignment corresponds to a clique tree of $G$. Fortunately, it can be proved that a shortest augmenting path has this property, and moreover, there always exists an augmenting path unless $\tau$ corresponds to an optimal clique tree. The details can be found in [11]. We only remark the following invariant which is maintained throughout the algorithm.

**Lemma 9.** [11] In line 2 of Algorithm 2, the variable $\tau$ is a realizable token assignment.

After this introduction, we are ready to prove Theorem 4.

**Proof of Theorem 4.** We prove the theorem by showing that each application of an augmenting path in Algorithm 2 does not increase the number of leaves in the subtrees of the corresponding tree model.

$^1$Note that as both $\tau(C_1)$ and $\tau'(C_1)$ are multisets, to obtain $\tau'(C_1)$ we only remove one instance of $S$ from $\tau(C_1)$ in case $S$ appears in $\tau(C_1)$ several times. This is consistent with the semantics of the set difference for multisets.

$^2$I.e., it corresponds to a clique tree of $G$. 

9
Consider \( \tau \) of \( G \).

Algorithm 2: Leafage(\( G, T \))

**Input:** A chordal graph \( G \), and a clique tree \( T \) of \( G \).

**Output:** A clique tree \( T^* \) of \( G \) with \( |L'(T^*)| = \ell(G) \).

1. Initialize \( \tau \leftarrow \varepsilon_T \) /* initialize the token assignment with the given clique tree. */
2. while there exists an augmenting path of \( \tau \) do
3. Let \((C_1, C_2, S_1), \ldots, (C_{k-1}, C_k, S_{k-1})\) be a shortest augmenting path of \( \tau \) considered in this iteration (Line 3).
4. for all \( i \) from 1 to \( k - 1 \) do
5. \( \quad \tau \leftarrow \tau \div (C_i, C_{i+1}, S_i) \)
6. return \( T^* \) where \( \varepsilon_{T^*} = \tau \).

In other words, let \( \tau \) be the token assignment considered at the start of some iteration (Lines 2-5) of Algorithm 2, and let \((C_1, C_2, S_1), \ldots, (C_{k-1}, C_k, S_{k-1})\) be the shortest augmenting path of \( \tau \) considered in this iteration (Line 3). Let \( \tau' \) denote the value of \( \tau \) after applying the token moves of this path (Lines 4-5).

By Lemma 9, both \( \tau \) and \( \tau' \) are realizable token assignments of \( G \). In other words, there exist clique trees \( T \) and \( T' \) of \( G \) such that \( \tau = \varepsilon_T \) and \( \tau' = \varepsilon_{T'} \). Let \( \delta_T = (T, \{T_u\}_{u \in V(G)}) \) and \( \delta_{T'} = (T', \{T'_u\}_{u \in V(G)}) \) be the corresponding tree models of \( G \). In other words, for each \( u \in V(G) \), we have \( T_u = T\{C \in V(T) \mid u \in C\} \) and \( T'_u = T'\{C \in V(T') \mid u \in C\} \). Moreover, just like in Lemma 8, we define for each \( u \in V(G) \) and each maximal clique \( C \) of \( G \), the sets \( \tau_u(C) = \{S \mid S \in \tau(C), u \in S\} \) and \( \tau'_u(C) = \{S \mid S \in \tau'(C), u \in S\} \).

Now, to prove the theorem, it suffices to demonstrate that \( |L'(T_u)| \geq |L'(T'_u)| \) for every \( u \in V(G) \). Consider \( u \in V(G) \) and define two sequences of integers \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \) where \( a_j = |\tau_u(C_j)| \) and \( b_j = |\tau'_u(C_j)| \) for all \( i \in \{1, \ldots, k\} \). Note that \( \tau_u(C) = \tau'_u(C) \) for all \( C \not\in \{C_1, \ldots, C_k\} \), and by Lemma 8, we have \( L'(T_u) = \{C \mid |\tau_u(C)| = 1\} \) and \( L'(T'_u) = \{C \mid |\tau'_u(C)| = 1\} \). This implies the following.

\[
|L'(T_u)| - |L'(T'_u)| = \left| \{C \mid |\tau_u(C)| = 1\} \right| - \left| \{C \mid |\tau'_u(C)| = 1\} \right| = \left| \left\{ i \mid a_i = 1 \right\} \right| - \left| \left\{ i \mid b_i = 1 \right\} \right|
\]

In other words, this boils down to showing that \( \left| \left\{ i \mid b_i = 1 \right\} \right| \) does not have more elements than \( \left| \left\{ i \mid a_i = 1 \right\} \right| \).

Recall that, by the definition of the augmenting path, \( |\tau(C_k)| = 1 \) and \( |\tau(C_i)| = 2 \) for all \( i \in \{2 \ldots k - 1\} \). Notably, since the path is shortest, \( C_1, \ldots, C_k \) are distinct maximal cliques of \( G \). Thus, as \( \tau_u(C) \subseteq \tau(C) \) for all \( C \), we conclude that \( a_k \leq 1 \) and \( a_i \leq 2 \) for all \( i \in \{2 \ldots k - 1\} \). Further, note that \( |\tau'_u(C)| = 2 \) when \( |\tau(C)| = 2 \) for all \( i \in \{2 \ldots k - 1\} \). In other words, we have \( b_i \leq 2 \) for all \( i \in \{2 \ldots k\} \).

We shall use the following two claims to show that \( \left| \left\{ i \mid b_i = 1 \right\} \right| \leq \left| \left\{ i \mid a_i = 1 \right\} \right| \).

(7) If \( b_j = 1 \), then \( a_i \geq 1 \).

Consider \( i \in \{1 \ldots k\} \) such that \( b_j = 1 \). First, we show that \( a_i \geq 1 \). Suppose that \( a_i = 0 \). Since \( b_j = 1 \), we have by Lemma 8 that \( 1 = b_j = |\tau'_u(C_j)| = \deg_{T'_u}(C_j) \). In other words, \( C_j \) is a leaf of \( T'_u \), and thus \( T'_u \) contains at least 2 vertices. Recall that \( V(T'_u) = V(T'_u) \), and note that \( 0 = a_i = |\tau_u(C_j)| = \deg_{T_u}(C_j) \) by Lemma 8. This means that \( C_j \) is a vertex of \( T_u \) with no neighbour in \( T_u \). This is clearly impossible, since \( T_u \) is connected and \( |V(T_u)| = |V(T'_u)| \geq 2 \). Thus we must conclude that \( a_i \geq 1 \). This proves (7).

(8) If \( b_j = 1 \) and \( a_i \geq 2 \), then there exists \( j \geq i \) such that \( a_j = 1, b_j = 2, \) and \( a_r = b_r \) for all \( r \in \{i + 1, \ldots, j - 1\} \).

To see this, first recall the construction of \( \tau' \) from \( \tau \) by moving the tokens \( S_1, \ldots, S_{k-1} \) as follows.

\[
\tau'(C_i) = \begin{cases} 
\tau(C_i) \setminus \{S_i\} & \text{if } i = 1 \\
(\tau(C_i) \setminus \{S_i\}) \cup \{S_{i-1}\} & \text{if } 1 < i < k \\
\tau(C_i) \cup \{S_{i-1}\} & \text{if } i = k 
\end{cases}
\]

Also recall that \( a_i = |\tau_u(C_i)| = \left| \{S \mid S \in \tau(C_i), u \in S\} \right| \) and \( b_i = |\tau'_u(C_i)| = \left| \{S \mid S \in \tau'(C_i), u \in S\} \right| \).

From these two facts we conclude the following relationship between the values of \( a_i \) and \( b_i \) (\( 1 < i < k \)).
Also, applying (8) for recall that $i$
model with $a_i$.

$G$

Section 6. Concluding Remarks

For each $i$, $\phi(i)$ is $a_i$ by (8), we have [a_i ≤ 1 and thus $a_i = 2$. Further, $i < k$ since $b_i ≥ a_i$ by (8), but $b_i = 1 < 2 = a_i = 1$. Moreover, $u ∈ S_i$ since $i < k$ and $b_i = a_i = 1$ if $i < k$.

First, we observe that $u ∈ S_j$ for each $r ∈ \{i + 1, \ldots, k - 2\}$. Indeed, if otherwise, we let $r$ be the smallest index in $\{i, \ldots, j - 2\}$ with $u ∉ S_r$. As we just argued, we have $u ∈ S_j$, and so $r > i$. Therefore, $u ∈ S_{j - 1}$ by the minimality of $r$. But then $b_r = a_r + 1$ by (8), since $1 ≤ r < j - 1 ≤ k$, a contradiction.

This also implies that $j ≤ k$. Indeed, if $j = k + 1$, then $i ≤ j - 2 = k - 1$ since $i < k$. Thus $u ∈ S_{j - 2} = S_{k - 1}$ which yields $b_k = a_k + 1$ by (8). However, $k ∈ \{i + 1, \ldots, j - 1\}$ and so $b_k = a_k$ by the choice of $j$.

We can now also conclude that $u ∈ S_{j - 1}$. Indeed, if $i = j - 1$, then we use the fact that $u ∈ S_j$. Otherwise, $i ≤ j - 2$ in which case $u ∈ S_{j - 2}$ as argued above, and we conclude $u ∈ S_{j - 1}$ by (8), since $1 ≤ i < j - 1 < k$.

Finally, we consider the value of $j$. First, suppose that $j = k$. Then $b_k = a_k + 1$, since $u ∈ S_{j - 1} = S_{k - 1}$. We recall that $a_k ≤ 1$ and so $b_k ∈ \{1, 2\}$. If $b_k = 1$, we have $a_k ≥ 1$ by (7), but then $a_k ≥ b_k = a_k + 1 > a_k$, a contradiction. Thus, we must conclude $b_k = 2$ and $a_k = 1$. Thus, as $j = k$, we have $b_j = 2$, $a_j = 1$, and $b_r = a_r$ for all $r ∈ \{i + 1, \ldots, j - 1\}$ as required. Thus we may assume that $j < k$. By the maximality of $j$, we have $a_j ≠ b_j$. Also, $u ∈ S_{j - 1}$ and $1 ≤ i < j < k$. So by (8) we conclude that $b_j = a_j + 1$. We recall that $b_j ≤ 2$ as $j > 1$. Thus $b_j ∈ \{1, 2\}$ as $a_j ≥ 0$. Again, if $b_j = 1$, we conclude $a_j ≥ 1$ by (7) in which case $a_j ≥ b_j > a_j$, a contradiction. Thus $b_j = 2$, $a_j = 1$, and $a_r = b_r$ for all $r ∈ \{i + 1, \ldots, j - 1\}$, as required. This proves (8).

We are now ready to conclude the proof. Denote $A = \{i | a_i = 1\}$ and $B = \{i | b_i = 1\}$. We show that $|B| ≤ |A|$ which will imply the present theorem as argued above the claim (7).

For each $i ∈ B$, if $a_i = 1$, we define $\phi(i) = i$; otherwise, we define $\phi(i) = j$ where $j$ is the index obtained by applying (8) for $i$; note that $a_j = 1$ and $b_j = 2$. It follows that $\phi$ is a mapping from $B$ to $A$. We show that $\phi$ is, in fact, an injective mapping. For otherwise, and let $i, i^+ \neq j$ be distinct elements of $B$ be such that $\phi(i) = \phi(i^+)$. Recall that $b_i = b_{i^+} = 1$ and note that $i ≤ \phi(i)$ and $i^+ ≤ \phi(i^+)$. If $i = \phi(i)$, then $i^+ ≤ \phi(i^+) = \phi(i) = i$ implying $i^+ ≤ \phi(i^+)$ as $i$ and $i^+$ are distinct. So $a_{i^+} ≠ 1$ by the definition of $\phi$, and hence $b_{\phi(i^+)} = 2$ as $\phi(i^+) = 2$ was obtained by applying (8) for $i^+$. But then $1 = b_i = b_{\phi(i)} = b_{\phi(i^+)} = 2$, a contradiction. Thus we must conclude that $i < \phi(i)$ and, by symmetry, also $i^+ < \phi(i^+)$. Now, without loss of generality, assume $i < i^+$. Since $i^+ < \phi(i^+)$, we must have $a_{i^+} ≠ 1$ by the definition of $\phi$. However, $b_{i^+} = 1$ as $i^+ ∈ B$, and hence, $a_{i^+} ≠ b_{i^+}$. Recall that the choice of $\phi(i)$ using (8) for $i$ guarantees that $a_i = b_i$ for all $r ∈ \{i + 1, \ldots, \phi(i) - 1\}$. In particular, $i < i^+ < \phi(i^+) = \phi(i)$ and so $a_{i+} = b_{i+}$ which is a contradiction. This verifies that $\phi$ is indeed an injective mapping from $B$ to $A$, which yields $|B| ≤ |A|$. This concludes the proof of Theorem 4.

It follows that $\phi$ is indeed an injective mapping from $B$ to $A$, which yields $|B| ≤ |A|$. This concludes the proof of Theorem 4.

6. Concluding Remarks

In this paper we have studied the vertex leafage of chordal graphs. Specifically, a chordal graph $G = (V, E)$ has vertex leafage $k$ when it has a tree model $T = \{T_u\}_{u ∈ V}$ such that each subtree $T_u$ has at most $k$ leaves. We have shown that, for every fixed $k ≥ 3$, it is NP-complete to decide if a split graph $G$ has vertex leafage at most $k$ even when $G$ is known to have vertex leafage at most $k + 1$. Additionally, we have demonstrated an $n^{O(k)}$ algorithm to compute the vertex leafage of a chordal graph whose leafage is bounded by $k$. It remains open whether the vertex leafage is FPT with respect to leafage (or any other graph parameter).

Finally, we have shown that every chordal graph $G$ has a tree model which simultaneously realizes $G$’s leafage and vertex leafage. In proving this final result we have also shown that, for every path graph $G$, there exists a path model with $\ell(G)$ leaves in the host tree and that such a path model can be computed in $O(n^3)$ time.
References

[1] Buneman, P. A characterization of rigid circuit graphs. *Discrete Mathematics* 9 (1974), 205–212.
[2] Chaplick, S. *Path Graphs and PR-trees*. PhD thesis, University of Toronto, 2012.
[3] Cohen, E., Golombic, M., Lipshteyn, M., and Stern, M. What is between chordal and weakly chordal graphs? In *Graph-Theoretic Concepts in Computer Science (WG 2008)*, Lecture Notes in Computer Science 5344 (2008), pp. 275–286.
[4] Fulkerson, D. R., and Gross, O. A. Incidence matrices and interval graphs. *Pacific Journal of Mathematics* 15 (1965), 835–855.
[5] Garey, M. R., and Johnson, D. S. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.
[6] Gavrili, F. The intersection graphs of subtrees of trees are exactly the chordal graphs. *Journal of Combinatorial Theory Series B* 16 (1974), 47–56.
[7] Gavril, F. A recognition algorithm for the intersection graphs of paths in trees. *Discrete Mathematics* 23 (1978), 211–227.
[8] Gavril, F. Generating the maximum spanning trees of a weighted graph. *Journal of Algorithms* 8 (1987), 592–597.
[9] Golombic, M. C., Lipshteyn, M., and Stern, M. Equivalences and the complete hierarchy of intersection graphs of paths in a tree. *Discrete Applied Mathematics* 156 (2008), 3203–3215.
[10] Habib, M., and Stacho, J. Linear algorithms for chordal graphs of bounded directed vertex leafage. *Electronic Notes in Discrete Mathematics* 32 (2009), 99–108.
[11] Habib, M., and Stacho, J. Polynomial-time algorithm for the leafage of chordal graphs. In *Algorithms - ESA 2009*, Lecture Notes in Computer Science 5757 (2009), pp. 290–300.
[12] Jamison, R. E., and Mulder, H. M. Constant tolerance intersection graphs of subtrees of a tree. *Discrete Mathematics* 290 (2005), 27–46.
[13] Lévêque, B., Maffray, F., and Preissmann, M. Characterizing path graphs by forbidden induced subgraphs. *Journal of Graph Theory* 62 (2009), 369–384.
[14] Lin, I. J., McKee, T. A., and West, D. B. The leafage of a chordal graph. *Discuss. Math. Graph Theory* 18 (1998), 23–48.
[15] Monma, C. L., and Wei, V. K.-W. Intersection graphs of paths in a tree. *Journal of Combinatorial Theory B* 41 (1986), 141–181.
[16] Schaefer, T. J. The complexity of satisfiability problems. In *STOC* (1978), pp. 216–226.
[17] Schaefer, T. J. A faster algorithm to recognize undirected path graphs. *Discrete Applied Mathematics* 43, 3 (1993), 261–295.
[18] Shibata, Y. On the tree representation of chordal graphs. *Journal of Graph Theory* 12 (1988), 421–428.
[19] Tjaras, R. E. Depth first search and linear graph algorithms. *SIAM Journal on Computing* 1 (1972), 146–160.
[20] Walter, J. R. Representations of chordal graphs as subtrees of a tree. *Journal of Graph Theory* 2 (1978), 265–267.