TOPOLOGICAL CONJUGACY AND ITS RELATIONS FOR SYMBOLIC MATRICES

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Abstract. In 1988 Boyle and Krieger defined sub-matrices for representation matrices of sofic shift. This paper presents some details of relations between integral sub-matrices and representation matrices. Besides, we express a new version of the Decomposition Theorem by sub-matrices. Generally, strong shift equivalence (conjugacy) of sub-matrices does not apply to representation matrices, but we show that this result can be achieved by the fixed diagonal integral sub-matrix.

1. Introduction

The classification of shifts of finite types up to conjugacy is one of the most critical problems in symbolic dynamics which introduced for certain smooth dynamical systems (1-dimensional systems), and some open questions remain. State splitting was introduced by R. Williams when he tried to prove the conjugacy for shifts of finite type. However, earlier than Williams, the definition of state splitting was introduced in information theory by Even (1965) (See (KOHAVI, 1962; Patel, 1975; Kohavi and Jha, 2009)). The concept of state splitting for sofic shifts is called the finite-state coding Theorem – Transforming an uncontrolled sequence to a controlled sequence by changing the label of the graph from road-coloring (input labeling) to right-closing (output labeling). But getting a situation for the finite-state coding Theorem to apply is not always possible. Marcus (1985), and then Karabed and Marcus (1988) showed that under some assumptions one can get the finite-state code.

Williams proved the Decomposition Theorem and the Classification Theorem for shifts of finite type. Williams (1970) introduced strong shift equivalence (SSE) and shift equivalence (SE) for shifts of finite type alongside with the well-known decomposition Theorem which states that every conjugacy from one edge shift to another is, in fact, the composition of some splitting and amalgamation codes. He figured out that determining shift equivalence is easier than strong shift equivalence, and that strong shift equivalence implies shift equivalence. But the converse is not true. The question is when shift equivalence implies strong shift equivalence. According to several works, Boyle and Krieger (1987); Wagoner (1992); Kim et al. (1992, 1997); Kim and Roush (1999) proved that for shifts of finite type and even for irreducible matrices, shift equivalence does not imply strong shift equivalence. Therefore, this question has remained a conjecture for about forty years. Although

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there exist examples for which shift equivalence implies strong shift equivalence (Williams, 1970).

Later the generalization of strong shift equivalence to sofic shifts was done by Nasu (1986). Using bipartite codes, Nasu proved that two sofic shifts are strong shift equivalent if and only if they are conjugate — Classification Theorem for sofic shifts.

Boyle and Krieger (1988) defined two sofic systems to be shift equivalent if their canonical resolving covers are shift equivalent. Then they proved that two sofic systems are shift equivalent if and only if they are eventually conjugate (i.e. their $n$-th power are conjugate for all $n$ sufficiently large). They used a representation (symbolic) matrix as the corresponding matrix of a sofic shift. The representation matrix is constructed by a set of finite matrices over a semi-ring of polynomials. The variables are non-commutative and coefficients are taken from $\mathbb{Z}_+$. Therefore, the coefficients of sub-matrices are symbols of a representation matrix. Afterwards, Boyle and Krieger (1988) extended shift equivalence for sofic shifts. Also, they defined the dimension group for sofic shifts and showed that the dimension group is a complete invariant for shift equivalence.

Kim and Roush (1990) showed that shift equivalence for resolving maps is decidable. The question of whether shift equivalence implies strong shift equivalence remains for sofic shifts. Thereafter, Kim and Roush (1991) developed path methods for strong shift equivalence for positive matrices on $\mathbb{Q}^+ \subseteq \mathbb{R}^+$. Boyle et al. (2013) extended the results for any dense subring $\mathcal{U} \subseteq \mathbb{R}$ where the entries of the matrices come from $\mathcal{U}_+ = \mathcal{U} \cap \mathbb{R}_+$, and the constituent matrices have non-zero eigenvalues.

Here, we investigate how path methods for strong shift equivalence can be extended to symbolic matrices by integral sub-matrices. In Sec. 2, we define new versions of splitting and amalgamation for symbolic matrices. Without considering the vertices of the edge graph of a symbolic matrix we assume the set of labels. Using these definitions of splitting and amalgamation we present a Decomposition Theorem for symbolic matrices, Sec. 4.

In Sec. 5, we redefine strong shift equivalent and shift equivalence for symbolic matrices by their integral sub-matrices. We prove that there is no relation between strong shift equivalence of symbolic matrices and strong shift equivalence of their sub-matrices in each symbol. In Theorem 15, we prove that with a diagonal assumption, strong shift equivalence of sub-matrices implies strong shift equivalence of symbolic matrices.

Right-closing codes were introduced to the general definition of finite-to-one codes by Kitchens. Using finite-to-one codes, right-resolving and consequently right-closing codes were introduced. In automata theory, right-closing means lossless of finite order (Even, 1965; Kohavi and Jha, 2009). Also, Kitchens proved that right-closing labeling can be re-coded to right-resolving labeling. Krieger (1984) investigated the relation between sofic shift and topological Markov shift. By extension Boyle and Krieger (1987) expressed that sofic shift is almost Markov if it is the image of a topological Markov shift under a bi-closing factor map. Therefore, the bi-closing factor map is an important tool for the classification of sofic shifts. Our next interest is to study, in Sec. 6, the closing factor codes for symbolic matrices. Kitchens (2012) shows that for a finite-to-one factor map $\phi : \Sigma_A \to \Sigma_B$ between two irreducible SFTs, the following are equivalent:

1. $\phi$ is constant-to-one,
(2) \( \phi \) is bi-closing,
(3) \( \phi \) is an open map.

Jung (2009, 2011) also considers this problem and shows that for sofic shifts any two of the above three conditions implies the other. By introducing a numerical computation on entries of integral sub-matrices of sofic shifts, we determine whether each finite graph is right- or left- closing. Using a Python program implementing the algorithm we can decide whether finite symbolic matrices of size \( n, n \geq 1 \) are bi-closing. Finding a bi-closing graph is not easy in many cases; therefore, using a bi-closing algorithm which works for any size of the matrix is convenient.

2. Splitting and amalgamation in symbolic matrices

By assuming the vertices of matrices, splitting and amalgamation for edge graphs were defined, (Williams, 1970; Lind et al., 1995). Williams defined one of the important theorems of symbolic dynamics, “Decomposition Theorem (DT)”. According to the Decomposition Theorem, every conjugacy between two edge shifts is a decomposition of splitting and amalgamation codes. Then he represented “Classification Theorem for shifts of finite type”. We can form the labeled graph corresponding to the sofic system. The labeled graph can be represented by a symbolic matrix in which the entries are the labels of the graph. In this section, we investigate a new algorithm for splitting and amalgamation of symbolic matrices, based on splitting and amalgamation on integral sub-matrices of a symbolic matrix.

Symbolic matrix of a graph. Nasu (1986) defined symbolic (representation) matrix for the labeled graph corresponding to sofic cover. Assume \( G \) is a graph with vertex set \( V \) and the edge set \( E \). For each pair of vertices \( (I,J) \), \( I,J \in V \), let \( A(I,J) \) be the number of edges from vertex \( I \) to \( J \) in graph \( G \). The integral adjacency matrix \( A_G = [A(I,J)] \) is defined for graph \( G \). Consider a labeled graph \( G \). For each edge of graph \( G \), there is a symbol from a finite alphabet \( \mathcal{A} \). These symbols can be repeated for different edges. The labeled graph \( G = (G, P) \) is the graph \( G \) with the label set \( \mathcal{P} \). Similar to the integral adjacency matrix for a graph \( G \), the symbolic adjacency matrix \( A_G \) for a labeled graph \( G \) is defined. For each pair of vertices \( (I,J) \), the entry \( A(I,J) \) is a formal sum of labels of edges from vertex \( I \) to \( J \). If there is no edges from vertex \( I \) to vertex \( J \), assume \( \emptyset \) in the entry of symbolic adjacency matrix \( A_G \).

For instance, assume the labeled graph \( \mathcal{G} \) in Figure 1, with the label set \( \mathcal{P} = \{a_1,a_2,a_3,a_4\} \). The symbolic adjacency matrix \( A_G \) is expressed as

\[
A_G = \begin{bmatrix}
2a_1 + a_2 & a_2 + a_3 \\
 a_4 & a_4
\end{bmatrix}.
\]

(2.1)

Algebraic properties such as commutative of multiplication does not work for symbolic matrices in general. Boyle and Krieger (1988) showed that rewriting the symbolic matrices into a formal sum of symbolic monomials is more tractable for algebraic settings. Using this approach every symbolic matrix is identified by integral sub-matrices; we do this for \( A_G \). Then, \( A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \), \( A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \), \( A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \) identify \( A_G \) as below:
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Figure 1. Labeled graph $\mathcal{G}$.

(2.2) $A_\mathcal{G} = \begin{bmatrix} 2a_1 + a_2 & a_2 + a_3 \\ a_4 & a_4 \end{bmatrix} = a_1 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Let $\mathcal{P} := \{a_1, a_2, \ldots, a_k\}$ be the finite set of labels. Assume $A_i$ is an integral sub-matrix corresponding to label $a_i \in \mathcal{P}$. Generally the symbolic matrix $A_\mathcal{G}$ corresponding to labeled graph $\mathcal{G}$ is expressed as

(2.3) $A_\mathcal{G} = \sum_{a_i \in \mathcal{P}} a_i A_i$.

Two symbolic matrices $M_1$ and $M_2$ are considered to be equal if there is a bijection between their entries, and by an abuse of notation we represent it as $M_1 = M_2$.

3. LABEL SPLITTING OF SYMBOLIC MATRIX

Let $e_a(I)$ be an edge labeled $a$ and terminating at $I$ which is not a cycle and $e_a^c(I)$ if it is a cycle. Let $\mathcal{P}(I) = \{I_1, \ldots, I_\ell\}$ be the set of vertices of a graph whose adjacency matrix is $A$. Each sub-matrix $A_i$ is an $\ell \times \ell$ matrix $[i_p^i_q]$ where $i_p^i_q$ is the number of edges labeled $a_i$ from $I_p$ to $I_q$.

Assume an splitting due to a partition is applied on the set of edges terminating at some vertex in $\mathcal{V}(A)$ and by a possible rearranging let that vertex be $I_1$ with $\mathcal{P}(I_1) = \{I_1, \ldots, I_m\}$. This forces $I_1$ to split to $m$ vertices $I_1^1, \ldots, I_1^m$ and let $B_j$ be the $(m + \ell - 1) \times (m + \ell - 1)$ adjacency matrix for the splitted graph with $\mathcal{V}(B_j) = \{I_1^1, \ldots, I_1^m, I_2, \ldots, I_\ell\}$.

Let $M(I, J)$ denote the entry of a matrix $M$ at row $I$ and column $J$. Then for $1 \leq k \leq m$, $B_j$ is determined as follows

1. $B_j(I_1^1, I_1^1) = \#\{e_a^c(I) \in \mathcal{P}_k\} \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $1 \leq \beta \leq m$,
2. $B_j(I_\alpha, I_1^k) = \#\{e_a(I) \in \mathcal{P}_k\} \in \mathbb{N}_0$, $1 < \alpha \leq \ell$,
3. $B_j(I_1^k, I_\alpha) = A_j(I_1^1, I_\alpha)$, $1 \leq \alpha \leq \ell$,
4. $B_j(I_\alpha, I_\beta) = A_j(I_\alpha, I_\beta)$, $1 < \alpha, \beta \leq \ell$.

Next we give two examples to demonstrate different splitting on a vertex.

Consider Fig. 1 and let $A$ be the corresponding matrix. Let the label set be $\mathcal{P}(I_1) = \{a_1, a_2\}$, $\mathcal{P}_2 = \{a_1, a_4\}$. Then the in-splitting symbolic matrix $A'$ is
3.1. Elementary splitting for symbolic matrix. Labeled graphs and matrices are closely related. In this respect, Boyle et al. (2013) defined elementary row and column splitting for a unital sub-ring of \( \mathbb{R} \). We redefine the term for symbolic matrices.

**Definition 1.** Suppose \( A \) is a symbolic matrix of the sofic system \( X \) and \( B \) is the out-split symbolic matrix of \( A \) by a partition \( \mathcal{P} \). Let \( \mathcal{V} \) be the vertex set of \( A \) and \( \mathcal{W} \) the vertex set of \( B \). The division matrix \( S \) corresponding to \( \mathcal{P} \) is the \( \mathcal{V} \times \mathcal{W} \) integral matrix defined by

\[
S(I, J^k) = \begin{cases}
1, & \text{if } I = J, \\
0, & \text{otherwise}.
\end{cases}
\]
$J^k$ is one of the vertices obtained from splitting the vertex $J$ by $P$.

Recall that an amalgamation matrix is a rectangular integral matrix with exactly one 1 in each row and at least one 1 in each column. The amalgamation matrix is the transpose of a division matrix (Lind et al., 1995).

**Definition 2.** A symbolic matrix $B$ is an elementary row splitting of the symbolic matrix $A$ if there is a pair of matrices $(R, S)$ where $S$ is an integral division matrix and $R$ a symbolic matrix such that $A = SR$ and $B = RS$.

For a pair of matrices $(R, S)$ where $S$ is an integral amalgamation matrix and $R$ a symbolic matrix, such that $A = RS, B = SR$, then $B$ is an elementary column splitting of matrix $A$.

The symbolic edge matrix $R$ for a partition $P$ is a matrix of size $W \times V$ with entries

\[ R^k(I, J) = \sum_{a_i \in P} a_i R_i^k(I, J), \]

where $R_i^k(I, J) = |E_i^k \cap E_j^k|$. Subscript $i$ is used for $i$-th sub-matrix.

**Proposition 3.** Let $A$ and $B$ be symbolic matrices of two sofic systems. Then $B$ is an out-splitting matrix of $A$ if and only if there exists a fixed division matrix $S$ and a symbolic rectangular matrix $R = \sum_{a_i \in P} a_i R_i$ with $A_i = SR_i, B_i = R_i S$ for all sub-matrices of $A$ such that

\[ (SR)(I, J) = \sum_{k=1}^{m(I)} S(I, I^k) R(I^k, J) = \sum_{k=1}^{m(I)} R(I^k, J) \]

\[ = \sum_{k=1}^{m(I)} (\sum_{a_i \in P} a_i R_i(I^k, J)) = \sum_{a_i \in P} a_i \sum_{k=1}^{m(I)} |E_i^k \cap E_j^k|_i \]

\[ = \sum_{a_i \in P} a_i (\bigcup_{k=1}^{m(I)} E_i^k) \cap E_j^k |_i \]

\[ = \sum_{a_i \in P} a_i A_i(I, J) = A(I, J) \] (3.6)

Also, we have
\[(RS)(I^\ell, J^k) = R(I^\ell, J)S(J, J^k)\]
\[= R(I^\ell, J) = \sum a_i R_i(I^\ell, J)\]
\[= \sum a_i |\mathcal{E}_I^\ell \cap \mathcal{E}_J^k|_i = \sum a_i |\mathcal{E}_I^\ell \cap \mathcal{E}_J^k|_i\]
\[= \sum a_i B_i(I^\ell, J^k) = B(I^\ell, J^k).\] (3.7)

Similarly, for in-splitting matrix \(B\) of \(A\), there is an amalgamation integral matrix \(R\) and a symbolic matrix \(S = \sum a_i \in \mathcal{P} a_i S_i\) such that \(A_i = RS_i\) and \(B_i = S_i R\).

\[\begin{array}{c|c|c|c}
& a_1 & a_2 & a_3 \\
\hline
a_1 & 1 & 1 & 0 \\
a_2 & 1 & 1 & 0 \\
a_3 & 0 & 0 & 0 \\
a_4 & 0 & 0 & 1 \\
a_5 & 0 & 0 & 0 \\
\end{array} = a_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.
\]

is an out-splitting matrix for \(A\) and:

\[S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \sum_{i=1}^{4} a_i R_i = \begin{bmatrix} a_1 & a_2 \\ a_1 & a_3 \\ a_4 & a_4 \end{bmatrix}.\]

Here
\[R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.
\]

Assume \(X_A\), \(X_{B_1}\) and \(X_{B_2}\) are shifts of finite type with adjacency matrices \(A\), \(B_1\) and \(B_2\) respectively. If \(B_1\) and \(B_2\) are obtained from \(A\) through division (resp. amalgamation) matrices, then there exists \(X_C\) a shift of finite type which can be obtained by divisions (resp. amalgamations) from \(X_{B_1}\) and \(X_{B_2}\) (Kitchens, 2012). Next lemma shows that similar result holds for symbolic matrices.

**Lemma 5.** Suppose \(A = \sum_{a_i \in \mathcal{P}} a_i A_i\) is a symbolic matrix, and \(B_1\) and \(B_2\) are two other symbolic matrices obtained from \(A\) by elementary amalgations. Then, there exists a symbolic matrix \(C\) obtained by amalgamations from \(X_{B_1}\) and \(X_{B_2}\).

We give the proof of Lemma 5 in the Appendix.

**4. Decomposition Theorem for sofic shifts by submatrices**

Williams (1973) proved any conjugacy between two edge shifts, is a decomposition of the sequence of splitting and amalgamation codes. The symbolic decomposition has the same result as the Williams Decomposition Theorem. Here, all decompositions on integral matrices are assumed to be the same as the vertex decomposition. Notice that we do not introduce a new argument for decomposition theorem. We investigate the relation of decomposition theorem from symbolic matrices to their integral sub-matrices. Decomposition theorem for submatrices is
same as Williams’s theorem. We show that using the decomposition theorem for sub-matrices, decomposition theorem for their symbolic matrices can be defined.

In Section 3, label splitting and amalgamation were defined. Now we assume a set of labels and then use these label splittings and amalgamations on integral sub-matrices. In the next step, using the sequence of label splitting and amalgamation for each sub-matrices, we present decomposition Theorem for each label \( a_i \). We use the term “label overlapping”: Considering all splitting and amalgamation sequences at each vertex, i.e. for each vertex \( I \) we consider splitting and amalgamation sequences in each sub-graphs corresponding to sub-matrices at the same time.

**Theorem 6 (Decomposition Theorem).** Every conjugacy from one sofic shift to another is layers overlapping of composition splitting and amalgamation codes between shifts of finite type corresponding to sub-matrices of symbolic matrices.

Williams assumed a partition depending upon vertices and then defined a vertex decomposition theorem. Suppose the pair \((X_A, X_B)\) of edge shifts of sofic systems. Assume \((A, B)\) is a pair of symbolic matrices corresponding to \((X_A, X_B)\). Then similar to (2.3),

\[
A = \sum_{a_j \in P} a_j A_j, \quad B = \sum_{a_j \in P} a_j B_j.
\]

Suppose \(((X_A)_j, (X_B)_j)\) are the corresponding edge shifts of pair \((A_j, B_j)\).

By William’s decomposition theorem, for edge shifts \(X_A\) and \(X_B\), there exists a sequence of out-splitting codes \(\psi_i\), out-amalgamation \(\alpha_i\) and 1-block conjugacies \(\tilde{\phi}_i\) where \(1 \leq i \leq n\) (Lind et al. (1995)). Note that we can record \(\phi\) to be a 1-block code \(\tilde{\phi}\) from a higher block shift of \(X_A\) to \(X_B\). This recording amounts to writing \(\phi\) as the composition of \(\tilde{\phi}\) with a succession of splitting codes, obtained from complete splitting.

\[
(4.1) \quad X_A \xrightarrow{\psi_1} X_{A_1} \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-1}} X_{A_{n-1}} \xrightarrow{\psi_n} X_{A_n} \\
X_B \xleftarrow{\alpha_1} X_{B_1} \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{n-1}} X_{B_{n-1}} \xleftarrow{\alpha_n} X_{B_n}
\]

Let \((X_{A_i})_j\) be edge shift corresponds to \(i\)-th splitting and the \(j\)-th symbolic matrix \(1 \leq i, j \leq n\). Define codes;

(2) \( (\phi_j)_j : (X_{A_i})_j \to (X_{B_i})_j \), \hspace{1em} (1-block conjugacy in each step \(j\))

(3) \( (\psi_j)_j : (X_{A_{i-1}})_j \to (X_{A_i})_j \), \hspace{1em} (out-splitting code in each step \(j\))

(4) \( (\alpha_j)_j : (X_{B_i})_j \to (X_{B_{i-1}})_j \), \hspace{1em} (out-amalgamation code in each step \(j\))

where \(1 \leq i, j \leq n\). By separating the labels of edge shift \(X_A\) into edge subshifts \((X_{A_i})_j\) corresponds to label \(a_j\), and then using the above codes we investigate the decomposition theorem for submatrices of symbolic matrices in each step \(j\). In Figure 3 we show the decomposition theorem for each step \(j\) and for sofic shifts.

Note that the converse of the symbolic decomposition theorem is not true; i.e., if for each pair \((A_j, B_j)\), \(A_j\) conjugates \(B_j\), then \(A\) is not necessarily conjugate to \(B\). Next example presents two symbolic matrices whose integral sub-matrices are
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Figure 3. Decomposition Theorem for sofic systems. William’s Theorem (Williams (1973)) can be extended by horizontal lines (overlapping) from edgie shifts of sofic systems to their edge shifts of shifts of finite type. The $X_A$ and $X_B$ are edge shifts of sofic systems and $\phi$ is 1-block conjugacy between them. and $(X_A)_j$ and $(X_B)_j$ are edge sub-shifts of shifts of finite type. Parallelogram is William’s Decomposition Theorem. Top diagonal arrows are in-splitting codes and bottom diagonal arrows are in-amalgamation codes. Also, the vertical arrows $\phi_j$ and $(\tilde{\phi})_j$, are relabeling codes.

Example 7. Let

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & a_1 + a_2 \end{bmatrix} = a_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Here, $A_1 \simeq B_1$ and $A_2 = B_2$; however, $A$ is not conjugate to $B$.

5. Symbolic matrices and path methods for conjugacy

In the previous topic, by defining Decomposition Theorem, conjugacy of symbolic matrices implies conjugacy of integral submatrices in each step (each symbol). Generally, there is no relation from SSE (or SE) of integral submatrices in each symbol to SSE (or SE) of their symbolic matrices. In this section, using diagonal matrix, we investigate SSE of submatrices implies SSE symbolic matrices.

We extend some results of Boyle et al. (2013) for symbolic matrices. We show in some cases, a diagonal matrix can obtain more results.

Definition 8. Let $\mathcal{U} \subset \mathbb{R}$ be a non-discrete unital sub-ring. Two square symbolic matrices $A$ and $B$ of size $n \geq 1$ are conjugate if there exists a matrix $W \in GL(n, \mathcal{U})$ such that $AW = WB$. (Also, see Campbell and Trouy (1991).)
Notice that if we have square symbolic matrices $A$ and $B$ of sizes $m$ and $n$ respectively with $m < n$, then by applying some suitable splitting on $A$ one will have $A'$ with the same size as $B$.

**Definition 9.** Two square symbolic matrices $A$ and $B$ of size $n$ are elementary strong shift equivalent ($A \sim B$) if there exists a pair $(R, S)$ where $S$ is a division integral matrix and $R$ is a symbolic matrix such that $A = RS$ and $B = SR$.

Also, symbolic matrices $A$ and $B$ are strong shift equivalent ($A \approx B$) with lag $\ell$ if there is a sequence $(R^0, S^0), (R^1, S^1), \cdots (R^\ell, S^\ell)$ where $S^i$ is a division matrix and $R^i$ is a symbolic matrix for $1 \leq i \leq \ell$ such that

\begin{equation}
A \approx R^0 S^0, \quad S^0 R^0 \approx R^1 S^1, \quad \cdots \quad S^{\ell-1} R^{\ell-1} \approx R^\ell S^\ell, \quad S^\ell R^\ell \approx B, \tag{5.1}
\end{equation}

where superscripts are the lag of strong shift equivalence.

**Definition 10.** A diagonal refactorization for symbolic matrices $A$ and $B$ is an elementary strong shift equivalence such that $A = DX$ and $B = XD$, where $D$ is a non-degenerate symbolic diagonal matrix and $X$ is an integral matrix.

The canonical factorization of a non-degenerate integral matrix was introduced by Williams as the “Williams factorization”, Williams (1973). Also, Boyle et al. (2013) expressed diagonal refactorization and canonical refactorization for matrices over semiring $\mathcal{U}$. Here, we recall the Williams factorization for symbolic matrices with a new approach.

Suppose $M$ is a non-degenerate symbolic matrix of row index set $\mathcal{I}$ and column index set $\mathcal{J}$. Suppose $\mathcal{E}$ is the set of entries $(i, j)$ such that $M(i, j) \neq \emptyset$. We define an integral division matrix $U_M$ of dimension $|\mathcal{I}| \times |\mathcal{E}|$ such that $U_M(i', (i, j)) = 1$ iff $i' = i$. Also, an integral amalgamation matrix $V_M$ of dimension $|\mathcal{E}| \times |\mathcal{J}|$ is defined such that $V_M((i, j), j') = 1$ iff $j = j'$. In this case, a symbolic diagonal matrix $D_M$ will be defined where $D_M((i, j), (i, j)) = M(i, j)$. Hence, the symbolic matrix $M$ is expressed by $M = U_M D_M V_M$, see (Boyle et al., 2013, Definition 2.8).

**Definition 11.** The Williams factorization for symbolic matrices is $M = U_M D_M V_M$ is defined as above. $M$ is a symbolic matrix, $D_M$ is symbolic diagonal matrix, and $U_M, V_M$ are integral matrices.

To make the statement clear, we bring an example here.

Suppose

\begin{equation}
M = \begin{bmatrix}
a & 0 & c \\
b & a & 0 \\
0 & 0 & d
\end{bmatrix}. \tag{5.2}
\end{equation}

For matrix $M$, we have $\mathcal{E} = \{(1, 1), (1, 3), (2, 1), (2, 2), (3, 3)\}$. For the matrix $U_M$, the rows are from index set $\mathcal{I} = \{1, 2, 3\}$ and columns are from $\mathcal{E}$. Therefore,

\begin{equation}
U_M = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}. \tag{5.3}
\end{equation}
Also, we can define for $V_M$ that index set $J = \{1, 2, 3\}$ as below:

\begin{equation}
V_M = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\end{equation}

(5.4)

It is sufficient to consider $D_M$ as a symbolic diagonal matrix of size 5.

\begin{equation}
D_M = \begin{bmatrix}
a & 0 & 0 & 0 & 0 \\
0 & c & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & d \\
\end{bmatrix},
\end{equation}

(5.5)

so that $M = U_M D_M V_M$.

**Proposition 12** (Boyle et al. (2013)). Suppose $A = RS$ and $B = SR$ is an elementary strong shift equivalence over a semiring $\mathcal{U}$ containing $\{0, 1\}$; $\mathcal{U}$ has no zero divisors; and the matrices $A, B$ are non-degenerate. Then there are non-degenerate matrices $C_1, C_2, D$ over $\mathcal{U}$ such that $D$ is diagonal and

1. $C_1$ is an elementary row splitting of $A$,
2. There is a matrix $X$ over $\mathcal{U}$ such that $DX = C_1$ and $XD = C_2$,
3. $C_2$ is an elementary column splitting of $B$.

**Corollary 13.** Proposition 12 applies for the strong shift equivalence assumption.

In the next proposition, we prove that if all conditions of Corollary 13 are satisfied for integral sub-matrices, then we have Corollary 13 for symbolic matrices too.

**Proposition 14.** Let $A = \sum_{a_i \in P} a_i A_i$ and $B = \sum_{a_i \in P} a_i B_i$ be two non-degenerate symbolic matrices. If Proposition 12 is satisfied for adjacency matrices $A_i$ and $B_i$ for each $a_i \in P$, and if for symbolic matrices $A$ and $B$, with $A = RS$ and $B = SR$, then the result of Proposition 12 is satisfied for the symbolic pair $(A, B)$.

**Proof.** By Prop. 12 for each pair of $(A_i, B_i)$ there exists diagram $(A_i, B_i, C_{i1}, C_{i2}, D_i)$ as follows:

We prove that we have the diagram for pair $(A, B)$. By refactorization, we have

\begin{equation}
A_i = (U_RD_R V_R)_i (U_S D_S V_S)_i,
\end{equation}

(5.6)

\begin{equation}
B_i = (U_S D_S V_S)_i (U_R D_R V_R)_i,
\end{equation}

such that we define

\begin{equation}
C_{i1}^1 = (D_R V_R U_S D_S V_S)_i (U_R)_i,
\end{equation}

(5.7)

\begin{equation}
C_{i2}^1 = (V_R)_i (U_S D_S V_S U_R D_R)_i,
\end{equation}

\begin{equation}
X_{i1}^1 = (D_R V_R U_S D_S V_S)_i,
\end{equation}

(5.8)

\begin{equation}
X_{i2}^1 = (U_S D_S V_S U_R D_R)_i,
\end{equation}

in which $U_R, U_S$ are division matrices and $V_R, V_S$ are amalgamation matrices.
Let $A = RS$ and $B = SR$. By symbolic refactorization we have

$$A = (U_R D_R V_R)(U_S D_S V_S),$$
$$B = (U_S D_S V_S)(U_R D_R V_R),$$

(5.9)

in which, $D_R$ and $D_S$ are symbolic matrices

$$D_R = \sum_{a_i \in P} a_i (D_R)_i,$$
$$D_S = \sum_{a_i \in P} a_i (D_S)_i.$$

(5.10)

Also, there is a matrix $X_i$ such that $C^1_i = X_i D_i$ and $C^2_i = X_i D_i$. Set

$$D_i = (D_R)_i,$$
$$X_i = (V_R U_S D_S V_S U_R)_i,$$

(5.11)

in which $U_R$, $U_S$ are integral division matrices and $V_R$ and $V_S$ are integral amalgamation matrices. Also, $(D_R)_i$ and $(D_S)_i$ are diagonal integral matrices, and clearly $D_R$ and $D_S$ are symbolic diagonal matrices. We show that by assumptions (ERS: $A_i \leftrightarrow C^1_i$), there exists symbolic matrix $C^1$ an elementary row splitting of $A$. Suppose there is a pair $(U_i, X_i)$ such that for elementary row splitting, $A_i = U_i X_i$.
and $C^1_i = X_i U_i$ where $U_i$ is an integral division matrix. By Eqs. (5.6)-(5.10),

$$A = \sum_{a_i \in P} a_i A_i = \sum_{a_i \in P} a_i U_i X_i = \sum_i U_i \sum_{a_i \in P} a_i X_i$$

$$= \sum_i U_i \sum_{a_i \in P} a_i (D_R V_R U S D_S V_S) i$$

$$= \sum_i U_i \left( \sum_{a_i \in P} a_i (D_R) i \right) (V_R U S) \left( \sum_{a_i \in P} a_i (D_S) i \right) (V_S)$$

$$= \sum_i U_i \left( \sum_{a_i \in P} a_i (D_R) i \right) (V_R U S) \sum_{a_i \in P} a_i (D_S) i (V_S)$$

$$= (D_R V_R U S D_S V_S) U = X^1 U.$$

Similarly, for pair $(V, X')$ there exists an ECS from $C^2$ to $B$ in which $V$ is an amalgamation matrix. Finally, we prove that there is a symbolic diagonal refactorization between $C^1$ and $C^2$. There is a pair $(D_i, X_i)$ with $C^1_i = D_i X_i$ and $C^2_i = X_i D_i$. Now, it is sufficient to use Eqs. (5.7), (5.11), we have $C^1 = DX$ and $C^2 = XD$. □

Boyle et al. (2013) involved strong shift equivalence of matrices over a dense sub-ring $\mathcal{U}$ of $\mathbb{R}$. For symbolic matrices in general, this is not the case anymore. In Proposition 14, we proved that elementary strong shift equivalence on integral sub-matrices of symbolic matrices implies diagonal refactorization between symbolic matrices. Note that generally, since there is the equivalence relation from a bijection of symbols, which involves not just the entries of a matrix but its partition into the matrices, one for each monomial. Consequently, strong shift equivalence (conjugacy) on sub-matrices $(A_i \approx B_i)$ does not imply Strong shift equivalence on symbolic matrices. In Theorem 15 we show that diagonal refactorization between sub-matrices implies strong shift equivalence on matrices. Then in Proposition 16, we investigate how a diagonal matrix can relate conjugacy between sub-matrices to a conjugacy between their symbolic matrices. The matrix being diagonal is our main assumption. In the general case, Proposition 16 is not true.

**Theorem 15.** Let $A = \sum_{a_i \in P} a_i A_i$ and $B = \sum_{a_i \in P} a_i B_i$ be two symbolic matrices. Assume for sub-matrices $A_i$, $B_i$ with size $n \times n$ over unital ring $\mathcal{U}$, there are non-degenerate fixed diagonal matrices $D_i$ and matrices $C_i$ with $A_i = D_i C_i$ and $B_i = C_i D_i$.

Then there are symbolic matrices $A'$, $B'$ such that $A'$ is an elementary row splitting of matrix $A$ and $B'$ is an elementary column splitting of matrix $B$ which $A'$ and $B'$ are conjugate.

**Proof.** Suppose $A_i = D_i C_i$ and $B_i = C_i D_i$ with

$$D_i = \begin{bmatrix} (E) i & 0 \\ 0 & (I_{n-k}) i \end{bmatrix}, C_i = \begin{bmatrix} (c_1) i & (c_2) i \\ (c_3) i & (c_4) i \end{bmatrix}.$$


We express matrix $A_i$ as below:

$$A_i = \begin{pmatrix} (E_i) & 0 \\ 0 & (I_{n-k}) \end{pmatrix} \begin{pmatrix} (c_1) & (c_2) \\ (c_3) & (c_4) \end{pmatrix} = \begin{pmatrix} (Ec_1) & (Ec_2) \\ (c_3) & (c_4) \end{pmatrix}.$$  

(5.15)

Now $A'$, the ERS for each $A_i$, will be expressed by a division matrix $U_i$:

$$A_i = U_iX_i = \begin{pmatrix} (I_k) & 0 \\ 0 & (I_{n-k}) \end{pmatrix} \begin{pmatrix} (c_1) & (c_2) \\ (c_3) & (c_4) \end{pmatrix} = \begin{pmatrix} (I_k) & 0 \\ 0 & (I_{n-k}) \end{pmatrix} \begin{pmatrix} (Ec_1) & (Ec_2) \\ (c_3) & (c_4) \end{pmatrix}.$$  

(5.16)

Similarly, there is an amalgamation matrix $V_i$ such that

$$B_i = Y_iV_i = \begin{pmatrix} (c_1) & (c_1(-I_k + E)) \\ (c_3) & (c_3(-I_k + E)) \end{pmatrix} \begin{pmatrix} (I_k) & 0 \\ 0 & (I_{n-k}) \end{pmatrix} \begin{pmatrix} (Ec_1) & (Ec_2) \\ (c_3) & (c_4) \end{pmatrix}.$$  

(5.17)

Now, by definition of symbolic ERS and ECS we have

$$A = \sum_{a_i \in P} a_i A_i = \sum_{a_i \in P} a_iU_iX_i = (\sum_{i} U_i)(\sum_{a_i \in P} a_iX_i) = UX$$

$$A' = XU.$$  

Also, $A' = XU$. Also, we have

$$B = \sum_{a_i \in P} a_iB_i = \sum_{a_i \in P} a_iY_iV_i = (\sum_{a_i \in P} a_iY_i)(\sum_{i} V_i) = YV,$$

and similarly, $B' = VY$. Now for integral matrix

$$W = \begin{pmatrix} I_k & 0 \\ I_k & E - 2I_k \\ 0 & 0 \end{pmatrix}$$

(5.22)

we will have conjugacy of the symbolic matrices, $A'W \simeq WB'$.  

\[ \square \]

**Proposition 16.** Suppose $U$ as a nondiscrete unital subring of $\mathbb{R}$. Let $A = \sum_{a_i \in P} a_iA_i$ and $B = \sum_{a_i \in P} a_iB_i$ be two $n \times n$ symbolic matrices where $A_i$, $B_i$ are their respective adjacency sub-matrices. Then there are non-degenerate diagonal matrices $W_i$ in $GL(n, U)$ such that $W_i^{-1}A_iW_i = B_i$ for each $i \in I$ if and only if there is a non-degenerate diagonal matrix $W$ with $AW \simeq WB$, an equality mod bijection of words between symbolic matrices $AW$ and $WB$. In this case, $A$ and $B$ are conjugate.
Proof. Let \((E)_i\) be \(k \times k\) matrices with \(W_i = \begin{bmatrix} (E)_i & 0 \\ 0 & (I_{n-k})_i \end{bmatrix}\). For sub-matrices 
\[ A_i = \begin{bmatrix} (a_1)_i \\ (a_3)_i \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} (a_1)_i \\ (Ea_3)_i \end{bmatrix} \]
we have \(A_iW_i = W_iB_i\).

Now for two symbolic matrices \(A\) and \(B\) we define the integral matrix \(W\) as below:
\[
W = \begin{bmatrix} I_{n-k} & 0 \\ 0 & (\sum E_i)^{-1} \end{bmatrix},
\]
which shows that \(AW \simeq WB\) (\(\simeq\) is defined as an equality mod bijection of words).

Conversely, it is sufficient to assume for each \(i \in I\), \(W_i = W\). \(\square\)

Note that above theorem is not true for any \(W_i \in Gl(n, \mathbb{Z})\). There exists a counterexample when \(W_i\) has diagonal entries of \(\pm 1\).

6. RIGHT-CLOSING FACTOR CODE IN SYMBOLIC MATRICES

Jung (2009) proved that for \(\phi : X \to Y\) is a factor code of two shifts of finite type with equal entropy; right-closing, open and constant-to-one factor codes are equivalent. In this section, we show the relation between the properties of factor code on symbolic matrices and factors on their integral sub-matrices. In general, right-closing factor codes on integral sub-matrices do not imply right-closing factor code on symbolic matrices. Therefore, having a bi-closing factor code is not easily possible. In the last part of this section, we introduced a numerical computation for right-closing graphs. By symbolic matrix with size \(n \times n\), \(n \geq 1\), by evaluating the entries of sub-matrices, we can figure out the right-closing property of the corresponding graph. Although right-closing is easy to distinguish for small size matrices (see Lind et al. (1995)), for the large size of matrices, this is a very painful task.

Right-closing graph is a weak variant of the right-resolving graph. A right-resolving graph is a right-closing with delay zero (D=0). In the first proposition of this section, we give the relation of right-resolving factor code between symbolic matrices and their integral sub-matrices.

**Proposition 17.** Let \(A = \sum_{a_i \in P} a_iA_i\) and \(B = \sum_{a_i \in P} a_iB_i\) be two symbolic matrices of sofic systems \(X\) and \(Y\), respectively. Then for sub-matrices \(A_i\) and \(B_i\), \(\phi_i : X_{A_i} \to X_{B_i}\) right-resolving factor map if and only if there is a right-resolving factor map \(\phi : X \to Y\). In this case, \(h(X) = h(Y)\).

**Proof.** Suppose \(\phi_i = (\Phi_i)_\infty : X_{A_i} \to X_{B_i}\) is a right-resolving factor code for pair \((A_i, B_i)\). Suppose \(G_i\) is labeled graph of matrix \(A_i\) and \(H_i\) is labeled graph of matrix \(B_i\), for each \(i\). There exists amalgamation matrix \(S_i\) such that \(A_iS_i = S_iB_i\). In this case, \(h(X_{A_i}) = h(X_{B_i})\) (Lind et al., 1995, Theorem 8.2.6). Since \(S_i\) is amalgamation matrix, for each \(I \in V(G_i)\) define \(\Phi_i(I)\) to be the unique \(J \in V(H_i)\) for which \(S_i^{J} I = 1\), so
\[
\bigcup_{k \in \partial \Phi_i^{-1}(J)}^K \epsilon_i^K (G_i) \to \epsilon_i^J (H_i)
\]
is a bijection, implies a right-covering code between \(X_{G_i}\) and \(X_{H_i}\). Since the factor code \((6.1)\) is defined for each label \(a_i \in P\), for all the labels the graph preserves right-resolving. It shows that there is a right-covering code between \(X_G\) and \(X_H\) and consequently \(\phi\) is a right-resolving factor code.
Let \( \phi : X \to Y \) is a right-resolving factor code for symbolic pair \((A, B)\). There exists an integral amalgamation matrix \( S \) which \( AS = SB \). By definition \((2.3)\), For symbolic matrices \( A \) and \( B \),

\[
(\sum_{a_{i} \in P} a_{i}A_{i})S = S(\sum_{a_{i} \in P} a_{i}B_{i}).
\]

Therefore, for each \( a_{i} \in P \), \( A_{i}S = SB_{i} \). Therefore, for integral pair \((A_{i}, B_{i})\), there exists an integral amalgamation matrix \( S \) such that \( A_{i}S = SB_{i} \). Hence, there is a right-resolving factor code \( \phi_{i} : X_{A_{i}} \to X_{B_{i}} \).

**Proposition 18.** Suppose \( A = \sum_{a_{i} \in P} a_{i}A_{i} \) and \( B = \sum_{a_{i} \in P} a_{i}B_{i} \) are two symbolic matrices of sofic systems \( X \) and \( Y \), respectively. For pair of \((A_{i}, B_{i})\), \( \psi_{i} : X_{A_{i}} \to X_{B_{i}} \) is a constant-to-one factor map if and only if there is a constant-to-one factor map \( \psi : X \to Y \).

**Proof.** suppose \( \psi_{i} : X_{A_{i}} \to X_{B_{i}} \) is a constant-to-one factor map. For each point \( y_{i} \in Y \), there exist \( d_{i} \) pre-images. Therefore, for each symbol \( a_{i} \in P \), there exists \( d_{i} \) pre-images. Suppose the factor \( \psi : X \to Y \) is not constant-to-one. Therefore, there is no constant which is the number of pre-image of any point in \( Y \). Contradicting our assumption that for each symbol there exist \( d_{i} \) pre-images.

Converse follows directly from \( \phi_{i} \) is constant-to-one for each point in \( Y \).

Recall that Prop. 17 is not satisfied for right-closing factor code. Therefore, we can not assume assumptions of Jung’s Theorem (Jung, 2009, Theorem 4.1) for integral sub-matrices and have same result for symbolic matrices.

In the following, we present an algorithm to identify right-closing matrices by their integral sub-matrices. Hence, recognition of right-closing will be more convenient for the matrix with sufficiently large size.

**Numerical computation of symbolic matrices:** Let \( A \) be a symbolic matrix with symbols \( a_{i} \in P \). By definition \((2.3)\), assume \( A_{m} \) is a sub-matrix of \( A \). For any sub-matrix \( A_{i} \), we consider non-zero entries. Let entry \((i, j)\) of sub-matrix \( A_{m} \) be non-zero. Consider the columns of \((i, j)\) in sub-matrices \( A_{n} \), \( n \neq m \). Write the labels of sub-matrices with non-zero entries in column \((i, j)\). Suppose \( M_{1} \) is the set of all labels that they have non-zero \((i, j)\) entry. Now, we assume another non-zero entry \((i', j')\), in sub-matrix \( A_{m} \). We do the same argument for the entry. Suppose \( M_{2} \) is the set of label for the entry \((i', j')\).

If the set of labels \( M_{1} \) is different from the set \( M_{2} \), the graph of matrix \( A \) is right-closing. Otherwise, if the sets are completely same the graph is not right-closing. If one set is a subset of another one, we consider the same labels and again check the same argument for their sub-matrices.

For left-closing graph just check the labels of the rows instead of the column in the algorithm above.

**Example 19.** Assume two matrices \( A \) and \( B \) with alphabets \( \mathcal{A} = \{a, b, c, d\} \) as below.

\[
A = \begin{bmatrix} a & b & c & d \\ \emptyset & \emptyset & b & a \\ \emptyset & \emptyset & a & \emptyset \end{bmatrix}, \quad B = \begin{bmatrix} b & d & \emptyset \\ a & \emptyset & a \\ \emptyset & c & b \end{bmatrix}.
\]

Matrix \( A \) is a bi-closing matrix with delay \( D = 1 \). Matrix \( B \) is not right-closing but left-closing with delay \( D = 1 \).
7. Appendix

Lemma 5. Suppose $A = \sum_{a_i \in P} a_i A_i$ is a symbolic matrix and $B_1$, $B_2$ are two symbolic matrices obtained from $A$ by elementary amalgamations. Then, there exists a symbolic matrix $C$, obtained by amalgamations from $X_{B_1}$ and $X_{B_2}$.

Proof. By (Kitchens, 2012, Lemma 2.1.2) for each integral sub-matrix $A_i$ there is different ways of amalgamations. There are matrices $B_1^i$ and $B_2^i$ obtained by amalgamations from $A_i$. So these amalgamations apply in columns which are named $i_1$ and $i_2$. Thus, $P_{A_1}(1_1) = P_{A_1}(1_2)$ and $f_{A_1}(1_1) \cap f_{A_1}(1_2) = \emptyset$. These show that for each integral sub-matrix $A_i$ we have different rows and same columns.

We prove that the argument above is satisfied for symbolic matrices $A$ and $B$. For symbolic matrix $A = \sum_{a_i \in P} a_i A_i$, for each column of $A$ we have $c(A) := \sum_{a_i \in P} a_i c(A_i)$. So

\begin{equation}
    P_{A}(1_1) = \sum_{a_i \in P} a_i P_{A_i}(1_1) = \sum_{a_i \in P} a_i P_{A_i}(1_2) = P_{A}(1_2).
\end{equation}

Hence, $A$ has same columns. If $f_{A_1}(1_1) \cap f_{A_1}(1_2) = \emptyset$, $i \in P$ then, $\sum(f_{A_1}(1_1)) \cap \sum(f_{A_1}(1_2)) = \emptyset$. Therefore, $f_{A_1}(1_1) \cap f_{A_1}(1_2) = \emptyset$ which it shows that $A$ has different rows. Similarly, we have

\begin{equation}
    P_{A}(2_1) = P_{A}(2_2) \text{ and } f_{A_1}(2_1) \cap f_{A_1}(2_2) = \emptyset.
\end{equation}

For $r = 1, 2$, if $1_r \in P_{A_1}(2_1), P_{A_1}(2_2)$ then, $1 \in P_{A_1}(2_1), P_{A_1}(2_2)$. So $1 \in P_{B_1 \times}(2_1, P_{B_1 \times}(2_2), r = 1, 2$. Hence, $1 \notin P_{A_1}(2_1), P_{A_1}(2_2)$. Therefore, $P_{B_1 \times}(2_1) = P_{B_1 \times}(2_2)$. If $1_r \in P_{A_1}(2_1), P_{A_1}(2_2)$ for all $i \in I$ then, $1_r \in \sum_{i \in I} P_{A_1}(2_1), \sum_{i \in I} P_{A_1}(2_2)$. So $1_r \in P_{A_1}(2_1), P_{A_1}(2_2)$ for $r = 1, 2$.

Now assume these vertices have one common vertex and $1_2 = 2_1$ is their common vertex. We have $P_{B_1}(2_1) = P_{B_1}(1)$. Since $P_{A_1}(1_1) = P_{A_1}(1_2) = P_{A_1}(2_1)$. By applying summation on predecessors we have $P_{A}(1_1) = P_{A}(1_2) = P_{A}(2_1)$ and then $P_{B_1 \times}(2_1) = P_{B_1}(1)$. Also, $f_{B_1 \times} \cap f_{B_1} = \emptyset$. Since $f_{A_1}(1_1) \cap f_{A_1}(2_1) = \emptyset$ and $\sum(f_{A_1}(1_1)) \cap \sum(f_{A_1}(2_1)) = \emptyset$, $f_{A_1}(1_1) \cap f_{A_1}(2_1) = \emptyset$. \hfill \Box

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