Some possible approaches to the Riemann Hypothesis via the Li/Keiper constants

Donal F. Connon
dconnon@btopenworld.com

18 February 2010

Abstract

In this paper we consider some possible approaches to the proof of the Riemann Hypothesis using the Li criterion.

Some examples of the potentially useful formulae are set out below:

\[
\lambda_n = \frac{n}{2} (\gamma + \log \pi) + \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \zeta(m) + \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)]\bigg|_{s=0}
\]

\[
\lambda_n = n\lambda_1 + \sum_{m=2}^{n} \binom{n}{m} [2^{-m} \zeta(m) + (-1)^m b_{m-1}]
\]

where \( \lambda_n \) and \( b_n \) are the Li/Keiper constants and the Lehmer constants respectively.

CONTENTS

| CONTENTS                                                                 | Page |
|------------------------------------------------------------------------|------|
| 1. Introduction                                                       | 1    |
| 2. Application of the Lehmer constants \( b_n \)                       | 7    |
| 3. Application of the eta constants \( \eta_n \)                      | 25   |
| 4. A conjecture for the Riemann Hypothesis                            | 29   |
| 5. Using the positivity of \( \xi^{(n)}(1) \)                           | 32   |
| 6. Relative magnitudes of \( \xi^{(n)}(1) \)                           | 42   |
| 7. The sigma constants \( \sigma_n \)                                 | 49   |
| 8. Further applications of the (exponential) complete Bell polynomials | 58   |
| 9. The \( S_2(n) \) constants                                         | 61   |
| 10. Some aspects of the (exponential) complete Bell polynomials        | 63   |

1. Introduction

The Riemann xi function \( \xi(s) \) is defined as

\[
\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s)
\]
and we see from the functional equation for the Riemann zeta function $\zeta(s)$

\[(1.2) \quad \zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos(\pi s / 2) \zeta(s)\]

that $\xi(s)$ satisfies the functional equation

\[(1.3) \quad \xi(s) = \xi(1-s)\]

In 1996, Li [29] defined the sequence of numbers $(\lambda_n)$ by

\[(1.4) \quad \lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} [s^{n-1} \log \xi(s)] \bigg|_{s=1}\]

and proved that a necessary and sufficient condition for the non-trivial zeros $\rho$ of the Riemann zeta function to lie on the critical line $s = \frac{1}{2} + i\tau$ is that $\lambda_n$ is non-negative for every positive integer $n$. Earlier in 1991, Keiper [25] showed that if the Riemann hypothesis is true, then $\lambda_n > 0$ for all $n \geq 1$.

Using the Leibniz rule for differentiation we may write (1.4) as

\[(1.4.1) \quad \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log \xi(s) \bigg|_{s=1}\]

Li also showed that

\[(1.5) \quad \lambda_n = \sum_{\rho} \left[ 1 - \left( \frac{1}{\rho} \right)^n \right]\]

Taking logarithms of (1.1) and noting that $\frac{s}{2} \Gamma\left(\frac{s}{2}\right) = \Gamma\left(1 + \frac{s}{2}\right)$ we see that

\[(1.6) \quad \log \xi(s) = \log \Gamma\left(1 + \frac{s}{2}\right) - \frac{s}{2} \log \pi + \log[(s-1)\xi(s)]\]

We have the Maclaurin expansion about $s = 1$

\[(1.7.1) \quad \log \xi(s) = -\log 2 - \sum_{k=1}^{\infty} \frac{(-1)^k \sigma_k}{k} (s-1)^k\]

and about $s = 0$
(1.7.2) \[ \log \xi(s) = -\log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k \]

where the constant term in (1.7) arises because

\[ \lim_{s \to 1} \xi(s) = \frac{1}{2} \pi^{-1/2} \Gamma(1/2) \lim_{s \to 1} [(s-1)^{-1/2}] = \frac{1}{2} \]

We note that the coefficients \( \sigma_k \) are defined by

\[ \sigma_k = \left. \frac{(-1)^{k+1} d^k}{(k-1)! ds^k} \log \xi(s) \right| _{s=1} \]

and we see that

\[ \frac{d}{ds} \log \xi(s) = \frac{\xi'(s)}{\xi(s)} = -\sum_{k=1}^{\infty} (-1)^k \sigma_k (s-1)^{k-1} \]

and comparing this with (1.4) we immediately see that \( \lambda_1 = \sigma_1 \).

We see that \( \frac{\xi'(0)}{\xi(0)} = \sum_{k=1}^{\infty} \sigma_k \) and, since the series is convergent, we deduce that

\[ \lim_{k \to \infty} \sigma_k = 0 \]

Having regard to the definition of the Li/Keiper constants

\[ \lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} [s^{n-1} \log \xi(s)] \right| _{s=1} \]

we consider

\[ s^{n-1} \log \xi(s) = -s^{n-1} \log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^{k+n-1} = -s^{n-1} \log 2 - \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} s^{n-1} (s-1)^k \]

and using the Leibniz differentiation formula we immediately obtain

\[ (1.8) \quad \lambda_n = -\sum_{m=1}^{n} (-1)^m \binom{n}{m} \sigma_m \]

We have from Keiper’s paper [25]
(1.9) \[ \sigma_n = \sum_{\rho} \frac{1}{\rho^n} \]

and since

\[ \lambda_n = \sum_{\rho} \left[ 1 - \left(1 - \frac{1}{\rho}\right)^n \right] = -\sum_{\rho} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\rho^k} \]

\[ = -\sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{\rho} \frac{1}{\rho^k} \]

we deduce (1.8) again and we note that this also implies that \( \lambda_1 = \sigma_1 \).

The eta constants \( \eta_k \) are defined by reference to the logarithmic derivative of the Riemann zeta function

(1.10) \[ \frac{d}{ds} \left[ \log \zeta(s) \right] = \frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} - \sum_{k=1}^{\infty} \eta_{k-1} (s-1)^{k-1} \quad |s-1| < 3 \]

and we may also note that this is equivalent to

(1.11) \[ \frac{d}{ds} \log[(s-1)\zeta(s)] = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = -\sum_{k=1}^{\infty} \eta_{k-1} (s-1)^{k-1} \]

and, noting that \( \lim_{s \to 1} [(s-1)\zeta(s)] = 1 \), we obtain upon integration

(1.12) \[ \log[(s-1)\zeta(s)] = -\sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} (s-1)^k \]

We also have the Hadamard representation of the Riemann zeta function

(1.13) \[ \zeta(s) = \frac{e^{\gamma s}}{2(s-1)\Gamma\left(1+\frac{s}{2}\right)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \]

where \( c = \log(2\pi) - 1 - \gamma / 2 \).

It was shown by Zhang and Williams [38] in 1994 that (as corrected by Coffey [13])

(1.14) \[ \frac{d}{ds} \log[(s-1)\zeta(s)] = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \gamma + \sum_{n=1}^{\infty} (-1)^n \left[ \left(1 - \frac{1}{2^{n+1}}\right) \zeta(n+1) + \sigma_{n+1} - 1 \right] (s-1)^n \]
and we see that
\[
\lim_{s \to 1} \frac{d^{n+1}}{ds^{n+1}} \log[(s-1)\zeta(s)] = (-1)^n n! \left[ \sigma_{n+1} + \left(1 - \frac{1}{2^{n+1}}\right)\zeta(n+1) - 1 \right]
\]

Differentiating (1.11) gives us
\[
\lim_{s \to 1} \frac{d^{n+1}}{ds^{n+1}} \log[(s-1)\zeta(s)] = -n! n
\]

and hence we have for \( n \geq 1 \)
\[
(1.15) \quad \eta_n = (-1)^n n \left[ \sigma_{n+1} + \left(1 - \frac{1}{2^{n+1}}\right)\zeta(n+1) - 1 \right]
\]

We have the representation for \( \lambda_n \) given by Bombieri and Lagarias [6] in 1999 and by Coffey ([11] and [12]) in 2004
\[
(1.16) \quad \lambda_n = 1 - \frac{n}{2} \left[ \log \pi + \gamma + 2 \log 2 \right] + S_1(n) + S_2(n)
\]

where
\[
(1.16.1) \quad S_1(n) = \sum_{m=2}^{n} \binom{n}{m} (-1)^m \left(1 - \frac{1}{2^m}\right)\zeta(m)
\]

\[
(1.16.2) \quad S_2(n) = \sum_{m=1}^{n} \binom{n}{m} \left(1 \frac{n}{m} \frac{d^n}{ds^n} \log[(s-1)\zeta(s)] \right) \bigg|_{s=1}
\]

Maślanka [30] makes the decomposition
\[
\lambda_m = \tilde{\lambda}_m + \hat{\lambda}_m
\]

where the “trend” \( \tilde{\lambda}_m \) is strictly increasing
\[
\tilde{\lambda}_m = 1 - \frac{m}{2} \left[ \log \pi + \gamma + 2 \log 2 \right] + S_1(m)
\]

and the “oscillations” \( \hat{\lambda}_m \) are given by
\[ \tilde{\lambda}_m = -\sum_{n=1}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) \eta_{n-1} \]

From (1.11) we see that
\[ \lim_{s \to 1} \frac{1}{(n-1)!} \frac{d^n}{ds^n} \log[(s-1)\zeta(s)] = -\eta_{n-1} \]

and therefore we have
\[ -\sum_{n=1}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) \eta_{n-1} = \lim_{s \to 1} \sum_{n=1}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) \frac{1}{(n-1)!} \frac{d^n}{ds^n} \log[(s-1)\zeta(s)] \]

This then shows that

\[ (1.17) \quad \tilde{\lambda}_m = S_2(m) = -\sum_{n=1}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) \eta_{n-1} \]

The generalised Stieltjes constants \( \gamma_n(u) \) are the coefficients in the Laurent expansion of the Hurwitz zeta function \( \zeta(s,u) \) about \( s = 1 \)

\[ (1.18) \quad \zeta(s,u) = \sum_{n=0}^{\infty} \frac{1}{(n+u)^s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(u)(s-1)^n \]

and \( \gamma_0(u) = -\psi(u) \), where \( \psi(u) \) is the digamma function which is the logarithmic derivative of the gamma function \( \psi(u) = \frac{d}{du} \log \Gamma(u) \). It is easily seen from the definition of the Hurwitz zeta function that \( \zeta(s,1) = \zeta(s) \) and accordingly that \( \gamma_n(1) = \gamma_n \).

The generalised Euler-Mascheroni constants \( \gamma_n \) (or Stieltjes constants) are the coefficients of the Laurent expansion of the Riemann zeta function \( \zeta(s) \) about \( s = 1 \)

\[ (1.19) \quad \zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(s-1)^n \]

Since \( \lim_{s \to 1} \left[ \zeta(s) - \frac{1}{s-1} \right] = \gamma \) it is clear that \( \gamma_0 = \gamma \). It may be shown, as in [23, p.4], that
\[
\gamma_n = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{\log^k k}{k} - \frac{\log^{n+1} N}{n+1} \right] = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{\log^k k}{k} - \int_{1}^{N} \frac{\log^k t}{t} \, dt \right]
\]

where we define \( \log^0 1 = 1 \). The Stieltjes constants may be compared with the Sitaramachandrarao constants \( \delta_n \), which are considered later in (2.15).

It was previously shown in [16] that

\[
\gamma_n(u) = -\frac{1}{n+1} \sum_{i=0}^{n+1} \sum_{j=0}^{i} \binom{i}{j} (-1)^j \log^{n+i}(u+j)
\]

2. Application of the Lehmer constants \( b_n \)

Upon equating (1.6) and (1.7.2) we obtain

\[
\log \Gamma \left( 1 + \frac{s}{2} \right) - \frac{s}{2} \log \pi + \log[(s-1)\zeta(s)] = -\log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k
\]

and using the Maclaurin expansion [7, p.201]

\[
\log \Gamma(1+s) = -s \gamma + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} s^k, \quad -1 < s \leq 1
\]

we obtain

\[
\log[(s-1)\zeta(s)] = -\log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k + \frac{1}{2} \gamma s + \sum_{k=2}^{\infty} \frac{(-1)^{k+1} \zeta(k)}{k 2^k} s^k + \frac{s}{2} \log \pi
\]

This may be expressed as

\[
\log[(s-1)\zeta(s)] = -\log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\zeta(k)}{k 2^k} s^k + \frac{s}{2} \log \pi
\]

where, for convenience, we denote \( \zeta(1) = \gamma \).

We then have

\[
\log[(s-1)\zeta(s)] = -\log 2 + \sum_{k=1}^{\infty} \left[ \frac{(-1)^{k+1} \zeta(k) - \frac{1}{2} \delta_{k,1} \log \pi}{k} \right] s^k
\]

and differentiation results in
\[
\sum_{k=1}^{\infty} \left[ (-1)^{k+1} 2^{-k} \zeta(k) - \sigma_k + \frac{1}{2} \delta_{1,k} \log \pi \right] (k-1)(k-1)...(k-m+1) s^{k-m} = \frac{d^m}{ds^m} \log[(s-1)\zeta(s)]
\]

Letting \( s = 0 \) gives us

\[
\left[ (-1)^{m+1} 2^{-m} \zeta(m) - \sigma_m + \frac{1}{2} \delta_{1,m} \log \pi \right] (m-1)! = \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

and we make the finite summation

\[
\sum_{m=1}^{n} (-1)^m \binom{m}{n} \left[ (-1)^{m+1} 2^{-m} \zeta(m) - \sigma_m + \frac{1}{2} \delta_{1,m} \log \pi \right] = \sum_{m=1}^{n} (-1)^m \binom{m}{n} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

or equivalently

\[
-\sum_{m=1}^{n} \binom{n}{m} 2^{-m} \zeta(m) - \sum_{m=1}^{n} (-1)^m \binom{n}{m} \sigma_m - \frac{n}{2} \log \pi = \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

Then using (1.8) we obtain

\[
\lambda_n = \frac{n}{2} \log \pi + \sum_{m=1}^{n} \binom{n}{m} 2^{-m} \zeta(m) + \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

or equivalently

\[
\lambda_n = \frac{n}{2} (\gamma + \log \pi) + \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \zeta(m) + \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

(2.2)

Prima facie, this formula seems more attractive than (1.6) because we no longer have to deal with the alternating sum \( S_1(n) \) defined by (1.16.1).

Lehmer [28] considered the constants \( b_n \) defined by

\[
\frac{d}{ds} \log[2(s-1)\zeta(s)] = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \sum_{n=0}^{\infty} b_n s^n, \quad |s| < 2
\]

so that
(2.3.1) \[ \log[2(s-1)\zeta(s)] = \sum_{n=0}^{\infty} \frac{b_n}{n+1}s^{n+1} = \sum_{n=1}^{\infty} \frac{b_{n-1}}{n}s^n \]

and we note that

(2.3.2) \[ \frac{d^m}{ds^m}\log[2(s-1)\zeta(s)] \bigg|_{s=1} = (m-1)!b_{m-1} \]

We then have from (2.2)

\[ \lambda_n = \frac{n}{2}(\gamma + \log \pi) + \sum_{m=2}^{n} \binom{n}{m}2^{-m}\zeta(m) + \sum_{m=1}^{n} (-1)^m \binom{n}{m}b_{m-1} \]

and since \( b_0 = \log 2\pi - 1 \) we have

\[ \lambda_n = \frac{n}{2}(2 + \gamma - \log 4\pi) + \sum_{m=2}^{n} \binom{n}{m}[2^{-m}\zeta(m) + (-1)^m b_{m-1}] \]

We know that

\[ 2 + \gamma - \log 4\pi = 2\lambda_1 \approx 0.046... \]

and therefore we obtain

(2.4) \[ \lambda_n = n\lambda_1 + \sum_{m=2}^{n} \binom{n}{m}[2^{-m}\zeta(m) + (-1)^m b_{m-1}] \]

We have Lehmer’s relation [28] for \( m \geq 2 \)

(2.5) \[ b_{m-1} = (-1)^{m-1}2^{-m}\zeta(m) - \sigma_m \]

and noting that

\[ \sum_{m=2}^{n} (-1)^m \binom{n}{m}b_{m-1} = -\sum_{m=2}^{n} \binom{n}{m}2^{-m}\zeta(m) - \sum_{m=2}^{n} (-1)^m \binom{n}{m}\sigma_m \]

we easily see that

\[ \lambda_n = \frac{n}{2}(2 + \gamma - \log 4\pi) - \sum_{m=2}^{n} (-1)^m \binom{n}{m}\sigma_m \]

This may be written as
\[ \lambda_n = \frac{n}{2} (2 + \gamma - \log 4\pi) - n\sigma - \sum_{m=1}^{n} (-1)^m \binom{n}{m} \sigma_m \]

which simply leads us back to square one, i.e. equation (1.8)!

Coffey [13] has shown that \( b_m \) has strict sign alteration, i.e.

(2.6) \[ b_m = (-1)^m \mu_m \text{ where } \mu_m > 0 \]

This strict sign alteration was also reported in [17] where we considered the function \( L(s) = \log[(s - 1)\zeta(s)] \). We have from (1.12)

\[ L^{(1)}(s) = -\sum_{k=0}^{\infty} \eta_k(s - 1)^k \]

and hence

\[ L^{(1)}(1) = -\eta_0 = \gamma \]

\[ L^{(n+1)}(s) = -\sum_{k=0}^{\infty} \eta_k k(k - 1)\ldots(k - n)(s - 1)^{k-n} \]

As we shall see later in Section 3 of this paper, Coffey [13] has also shown that the sequence \( \eta_n \) has strict sign alteration

\[ \eta_n = (-1)^{n+1} \varepsilon_n \]

where \( \varepsilon_n \) are positive constants and therefore we have

\[ L^{(n+1)}(1) = -n! \eta_n = (-1)^n n! \varepsilon_n \]

We see that

\[ L^{(1)}(0) = -\sum_{k=0}^{\infty} (-1)^k \eta_k = \sum_{k=0}^{\infty} \varepsilon_k \]

is positive. In fact we have

\[ L^{(1)}(0) = -\sum_{k=0}^{\infty} (-1)^k \eta_k = \log(2\pi) - 1 \]
\[
L^{(n+1)}(0) = -\sum_{k=0}^{\infty} \eta_k (k-1) \ldots (k-n+1)(-1)^{k-n} = (-1)^n \sum_{k=0}^{\infty} \varepsilon_k (k-1) \ldots (k-n+1)
\]

and therefore we note that the signs of \( L^{(n+1)}(0) \) also strictly alternate.

Coffey [13] has indicated that for large values of \( m \) we have

\[
\mu_m \approx 2^{-m-1}
\]

However, since \( \lim_{m \to \infty} \sigma_m = 0 \), reference to (2.5) indicates that a better approximation would appear to be

(2.7) \[ \mu_{m-1} \approx 2^{-m} \zeta(m) \]

where of course we note that \( \lim_{m \to \infty} \zeta(m) = 1 \).

In order to verify the Riemann Hypothesis, equation (2.4) tells us that it would be sufficient to show that

\[
\sum_{m=2}^{n} \binom{n}{m} [2^{-m} \zeta(m) + (-1)^m b_{m-1}] > 0
\]

or

\[
\sum_{m=2}^{n} \binom{n}{m} [2^{-m} \zeta(m) - \mu_{m-1}] > 0
\]

and indeed it would be sufficient if we could show that \( 2^{-m} \zeta(m) > \mu_{m-1} \) for \( m \geq 2 \). The approximation (2.7) therefore prima facie lends a degree of support to the validity of the Riemann Hypothesis.

However, from (2.5) we have

\[
2^{-m} \zeta(m) - \mu_{m-1} = (-1)^{m+1} \sigma_m
\]

and, as reported by Coffey [12, p.16], the initial sign pattern of the \( (\sigma_m) \) sequence is simply \(- - + - + 
\ldots \) with \( \sigma_1 > 0 \). Therefore the inequality \( 2^{-m} \zeta(m) > \mu_{m-1} \) cannot be valid for all \( m \). The first 26 values of \( \sigma_m \) are reported in Lehmer’s paper [28]. The magnitude of \( \sigma_{26} \) is given as

\[
\sigma_{26} \approx -0.0000 0000 0000 0000 0000 0000 0000 01
\]
which gives an indication of just how small are the quantities we are dealing with.

Using (2.5) we see that

\[
\sum_{m=2}^\infty b_{m-1} = \sum_{m=2}^\infty \frac{(-1)^{m-1} \varsigma(m)}{2^m} - \sum_{m=2}^\infty \sigma_m
\]

and it is well known that [13]

\[
\sigma_1 = -\sum_{m=1}^\infty \sigma_m
\]

which gives us

\[
2\sigma_1 = -\sum_{m=2}^\infty \sigma_m
\]

We then have

\[
\sum_{m=2}^\infty \left[ b_{m-1} - \frac{(-1)^{m-1} \varsigma(m)}{2^m} \right] = 2\sigma_1
\]

or

\[
\sum_{m=2}^\infty (-1)^{m-1} \left[ \mu_{m-1} - \frac{\varsigma(m)}{2^m} \right] = 2\sigma_1
\]

which also demonstrates that

\[
\lim_{m \to \infty} \left[ \mu_{m-1} - \frac{\varsigma(m)}{2^m} \right] = 0
\]

\[\square\]

We now consider Cauchy’s inequality [35, p.84]. If

\[
f(s) = \sum_{m=0}^\infty a_m s^m \quad |s| < R
\]

and \( M(r) \) is the upper bound of \( |f(s)| \) on the circle \( |s| = r \ (r < R) \) then

\[
|a_m| r^m \leq M(r)
\]

for all values of \( m \).

We consider the particular function
\[ f(s) = \log[2(s-1)\zeta(s)] \quad f(0) = 0 \]

We have
\[ |(s-1)\zeta(s)| = |(s-1)|\zeta(s) |\]

and for \(|s| = r > 1\) we have
\[ |\zeta(s)| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \zeta(r) \]

With \(s = re^{i\theta}\) we have
\[ |(s-1)| = \sqrt{r^2 - 2r \cos \theta + 1} \]

so that on the circle \(|s| = r\)
\[ \max |(s-1)| \leq r + 1 \]

Hence we have
\[ |f(s)| \leq \log[2(r+1)\zeta(r)] \]

and, using \(x > \log x\), we obtain
\[ |f(s)| < 2(r+1)\zeta(r) \]
\[ \left| \left( -1 \right)^{m-1} \frac{\mu_{m-1}}{m} \right| \leq \frac{\mu_{m-1}}{m} < \frac{2}{r^m} (r+1)\zeta(r) \]
\[ \frac{\mu_{m-1}}{m} < \frac{2}{r^m} (r+1)\zeta(r) \text{ where } r < 2 \]

Unfortunately, this inequality is not sharp enough for our purposes.

\[ \square \]

In this section we show another more concise derivation of (2.4). We have from (1.4.1)
\[ \lambda_n = \sum_{m=1}^{n} \left( \frac{n}{m} \right) \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log \zeta(s) \]

13
Since \( \xi(s) = \xi(1 - s) \) this is equivalent to

\[
= \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log \xi(1 - s) \bigg|_{s=1}
\]

Making the substitution \( p = 1 - s \) this becomes

\[
= \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{dp^m} \log \xi(p) \bigg|_{p=0}
\]

and making another substitution \( p = -s \) we obtain

\[
(2.8) \quad \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log \xi(-s) \bigg|_{s=0}
\]

Letting \( s \to -s \) in (1.6) gives us

\[
\log \xi(-s) = \log \Gamma\left(1 - \frac{s}{2}\right) + \frac{s}{2} \log \pi + \log[-(s + 1)\xi(-s)]
\]

Differentiation gives us

\[
\frac{d}{ds} \log \Gamma\left(1 - \frac{s}{2}\right) = -\frac{1}{2} \psi\left(1 - \frac{s}{2}\right)
\]

and we see that

\[
\frac{d^m}{ds^m} \log \Gamma\left(1 - \frac{s}{2}\right) = (-1)^m \frac{1}{2^m} \psi^{(m-1)}\left(1 - \frac{s}{2}\right)
\]

Specifically we have

\[
\frac{d^m}{ds^m} \log \Gamma\left(1 - \frac{s}{2}\right) \bigg|_{s=0} = (-1)^m \frac{1}{2^m} \psi^{(m-1)}(1) = (-1)^m \frac{1}{2^m} \psi^{(m-1)}(1)
\]

We have for \( m \geq 2 \)

\[
\psi^{(m-1)}(1) = (-1)^m (m - 1)! \xi(m)
\]
\[
\frac{d^n}{ds^n} \log \Gamma \left(1 - \frac{s}{2}\right) \bigg|_{s=0} = \frac{1}{2^m} (m-1)! \zeta(m)
\]

\[
\frac{d}{ds} \log \Gamma \left(1 - \frac{s}{2}\right) \bigg|_{s=0} = \frac{1}{2} \gamma
\]

Letting \( s \to -s \) in (2.3.1) we see that

\[
\log[-2(s+1)\zeta(-s)] = \sum_{n=1}^{\infty} \frac{(-1)^n b_{n-1}}{n} s^n
\]

and we note that

\[
\frac{d^n}{ds^n} \log[-2(s+1)\zeta(-s)] \bigg|_{s=0} = (-1)^m (m-1)! b_{m-1}
\]

Dealing separately with the term involving \( m = 1 \) we obtain (2.4) again

\[
\lambda_n = \frac{n}{2} \log \pi + \frac{n}{2} \gamma - n(\log 2\pi - 1) + \sum_{m=2}^{\infty} \binom{n}{m} \frac{\zeta(m)}{2^m} (-1)^m b_{m-1}
\]

\[\square\]

From (2.3) we have

\[
\frac{d}{ds} [2(s-1)\zeta(s)] = 2(s-1)\zeta(s) \sum_{n=0}^{\infty} b_n s^n
\]

and hence referring to (10.10) we have

\[
(2.9) \quad \frac{d^n}{ds^n} [2(s-1)\zeta(s)] \bigg|_{s=0} = Y_m(0! b_0, 1! b_1, \ldots, (m-1)! b_{m-1})
\]

in terms of the (exponential) complete Bell polynomials.

We have the Maclaurin expansion

\[
(2.10) \quad \zeta(s) = \sum_{n=0}^{\infty} \zeta^{(n)}(0) \frac{s^n}{n!}
\]

and we see that
\[
\sum_{n=0}^{\infty} \zeta^{(n-1)} \left(0\right) \frac{n^{s^n}}{n!} = \sum_{n=1}^{\infty} \zeta^{(n-1)} \left(0\right) \frac{n^{s^n}}{n!} = s \sum_{n=1}^{\infty} \zeta^{(n-1)} \left(0\right) \frac{s^{n-1}}{(n-1)!}
\]

and in turn we have

\[
s \sum_{n=1}^{\infty} \zeta^{(n-1)} \left(0\right) \frac{s^{n-1}}{(n-1)!} = s \sum_{m=0}^{\infty} \zeta^{(m)} \left(0\right) \frac{s^m}{m!} = s \zeta \left(s\right)
\]

Hence we have

\[
(s - 1) \zeta \left(s\right) = \sum_{n=0}^{\infty} \left[n \zeta^{(n-1)} \left(0\right) - \zeta^{(n)} \left(0\right)\right] \frac{s^n}{n!}
\]

and differentiation shows that

\[
\frac{d^m}{ds^m} \left[2(s - 1) \zeta \left(s\right)\right]_{s=0} = 2 \left[m \zeta^{(m-1)} \left(0\right) - \zeta^{(m)} \left(0\right)\right]
\]

Therefore we obtain

\[
(2.11) \quad 2 \left[m \zeta^{(m-1)} \left(0\right) - \zeta^{(m)} \left(0\right)\right] = Y_m \left(0!b_0, 1!b_1, \ldots, (m-1)!b_{m-1}\right)
\]

and for example with \(m = 1\) we have

\[
2 \left[\zeta \left(0\right) - \zeta^{(1)} \left(0\right)\right] = Y_1 \left(0!b_0\right) = b_0
\]

so that

\[
b_0 = \log 2\pi - 1
\]

With \(m = 2\) we obtain

\[
2 \left[2 \zeta^{(1)} \left(0\right) - \zeta^{(2)} \left(0\right)\right] = Y_2 \left(0!b_0, 1!b_1\right) = b_0^2 + b_1
\]

so that

\[
(2.12) \quad b_1 = 2 \left[2 \zeta^{(1)} \left(0\right) - \zeta^{(2)} \left(0\right)\right] - 4 \left[\zeta \left(0\right) - \zeta^{(1)} \left(0\right)\right]^2
\]

Since \(b_1\) is negative we see that

\[
2 \left[\zeta \left(0\right) - \zeta^{(1)} \left(0\right)\right]^2 > \left[2 \zeta^{(1)} \left(0\right) - \zeta^{(2)} \left(0\right)\right]
\]
With \( s \to 1 - s \) in (2.10) we have

\[
\zeta(1 - s) = \sum_{n=0}^{\infty} \frac{\zeta^{(n)}(0)}{n!} (1 - s)^n
\]

and combining this with the geometric series

\[
\frac{1}{s} = \sum_{n=0}^{\infty} (1 - s)^n
\]

we obtain

(2.13) \[
\zeta(1 - s) + \frac{1}{s} = \sum_{n=0}^{\infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right] (1 - s)^n
\]

With \( 1 - s = p \) this becomes

\[
\zeta(p) + \frac{1}{1 - p} = \sum_{n=0}^{\infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right] p^n
\]

and we see the connection with the \( \delta_n \) constants considered by Sitaramachandrarao [32] in 1986

(2.14) \[
\zeta(s) + \frac{1}{1 - s} = \sum_{n=0}^{\infty} \frac{(-1)^n \delta_n s^n}{n!}
\]

where

(2.15) \[
\delta_n = \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \log^n k - \int_{1}^{m} \log^n x \, dx - \frac{1}{2} \log^m m \right] \]

\[
= (-1)^n \left[ \zeta^{(n)}(0) + n! \right]
\]

A derivation of (2.15) follows. Using the Euler-Maclaurin summation formula [3], Hardy [22, p.333] showed that the Riemann zeta function could be expressed as follows

(2.16) \[
\zeta(s) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k^s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} \right] \quad \text{Re}(s) > -1
\]

It may immediately be seen that this identity is trivially satisfied for \( \text{Re}(s) > 1 \) because
\[ \zeta(s) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k^s} - n^{1-s} - \frac{1}{2} n^{-s} \right] = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^s} + \lim_{n \to \infty} \left[ -n^{1-s} - \frac{1}{2} n^{-s} \right] \]

and the latter limit is clearly equal to zero.

Differentiating (2.16) results in for \( \Re(s) > -1 \)

\[ (2.17) \quad \zeta'(s) = \lim_{n \to \infty} \left[ -\sum_{k=1}^{n} \frac{\log k}{k^s} + \frac{n^{1-s} (1-s) \log n - n^{1-s}}{(1-s)^2} + \frac{1}{2} n^{-s} \log n \right] \]

and with \( s = 0 \) we obtain

\[ \zeta'(0) = \lim_{n \to \infty} \left[ -\sum_{k=1}^{n} \log k + \left( n + \frac{1}{2} \right) \log n - n \right] \]

Hence, using the Stirling approximation for \( n! \) we see that \( \zeta'(0) = -\frac{1}{2} \log(2\pi) \).

With regard to (2.16) we could determine \( \zeta''(0) \)

\[ \zeta''(s) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \left( \frac{\log^2 k}{k^s} + \frac{(1-s)^2 [n^{1-s} (1-s) \log^2 n] + 2[n^{1-s} (1-s) \log n - n^{1-s}]}{(1-s)^4} - \frac{1}{2} n^{-s} \log^2 n \right) \right] \]

so that

\[ (2.18) \quad \zeta''(0) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \log^2 k - n \log^2 n + 2n \log n - 2n - \frac{1}{2} \log^2 n \right] \]

and compare the result with

\[ \zeta''(0) = \gamma_1 + \frac{1}{2} \gamma^2 - \frac{1}{24} \pi^2 - \frac{1}{2} \log^2 (2\pi) \]

The equation (2.18) is contained in Ramanujan’s Notebooks [5, Part I, p.203].

In order to simplify the calculations, we write

\[ \zeta(s) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k^s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} \right] = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k^s} - \frac{n^{1-s} - 1}{1-s} - \frac{1}{2} n^{-s} \right] \]

and differentiation gives us
\[
\zeta^{(n)}(s) = \lim_{m \to \infty} \left[ (-1)^n \sum_{k=1}^{m} \log^k k - f^{(n)}(s) - \frac{n!}{(1-s)^{n+1}} - \frac{1}{2} (-1)^n m^{-s} \log^m m \right]
\]

where we have denoted \( f(s) \) as

\[
f(s) = \frac{m^{1-s} - 1}{s - 1}
\]

We can represent \( f(s) \) by the following integral

\[
f(s) = \frac{m^{1-s} - 1}{s - 1} = \int_{1}^{m} x^{-s} \, dx
\]

so that

\[
f^{(n)}(s) = (-1)^n \int_{1}^{m} x^{-s} \log^n x \, dx
\]

and thus

\[
f^{(n)}(0) = (-1)^n \int_{1}^{m} \log^n x \, dx
\]

Therefore, with \( s = 0 \) we obtain

\[
(2.19) \quad (-1)^n \left[ \zeta^{(n)}(0) + n! \right] = \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \log^k k - \int_{1}^{m} \log^n x \, dx - \frac{1}{2} \log^m m \right]
\]

As pointed out by Apostol [2], the power series expansion (2.13) converges for \( s = 0 \) and therefore we have

\[
(2.20) \quad \lim_{n \to \infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right] = 0
\]

Apostol [2] has calculated the first 18 values of \( \frac{\zeta^{(n)}(0)}{n!} \) and we note that they exhibit some small oscillations around the value \(-1\).

We also see that

\[
\lim_{n \to \infty} \left[ \frac{\zeta^{(n-1)}(0)}{(n-1)!} + 1 \right] - \lim_{n \to \infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right] = 0
\]

and thus
We note that
\[
\lim_{n \to \infty} \frac{1}{n!} \left[ n \zeta^{(n-1)}(0) - \zeta^{(n)}(0) \right] = 0
\]

We have the Laurent expansion of the zeta function $\zeta(s)$ about $s = 1$
\[
\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n
\]
where $\gamma_n$ are known as the Stieltjes constants. We have $\gamma_0 = -\psi(1) = \gamma$.

We see that
\[
\gamma = \sum_{n=0}^{\infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right] = \frac{1}{2} + \sum_{n=1}^{\infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right]
\]
and more generally we obtain by differentiation for $p \geq 1$
\[
(-1)^p \gamma_p = \sum_{n=p}^{\infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right] n(n-1)...(n-p+1)
\]
which may be written as
\[
(2.21) \quad \frac{(-1)^p}{p!} \gamma_p = \sum_{n=p}^{\infty} \left[ \frac{\zeta^{(n)}(0)}{n!} + 1 \right] \binom{n}{p}
\]

We showed in [16] that
\[
(2.22) \quad (-1)^r \zeta^{(r)}(s,u) = \frac{r!}{(s-1)^{r+1}} + \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} (s-1)^p \gamma_{p+r}(u)
\]
In 1995, Choudhury [10] mentioned that Ramanujan determined that for $r \geq 1$ and $\text{Re}(s) > 1$

\[
(-1)^r \zeta^{(r)}(s) = \sum_{p=1}^{\infty} \frac{\log^p p}{p^s} = \frac{r!}{(s-1)^{r+1}} + \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} (s-1)^p \gamma_{p+r}
\]
We have for \( s = 0 \)

\[
\zeta^{(n)}(0) = -n! + (-1)^n \sum_{p=0}^{\infty} \frac{\gamma_{p+n}}{p!}
\]

This shows that the Sitaramachandrarao constants \( \delta_n \) may be represented by

\[
(2.23) \quad \delta_n = (-1)^n \left[ \zeta^{(n)}(0) + n! \right] = \sum_{p=0}^{\infty} \frac{\gamma_{p+n}}{p!}
\]

We see that

\[
\zeta^{(n-1)}(0) = -(n-1)! + (-1)^{n-1} \sum_{p=0}^{\infty} \frac{\gamma_{p+n-1}}{p!}
\]

\[
-\zeta^{(n)}(0) + n\zeta^{(n-1)}(0) = (-1)^{n-1} \sum_{p=0}^{\infty} \frac{n \gamma_{p+n-1}}{p!} - (-1)^n \sum_{p=0}^{\infty} \frac{\gamma_{p+n}}{p!}
\]

\[
-\zeta^{(n)}(0) + n\zeta^{(n-1)}(0) = (-1)^{n+1} \sum_{p=0}^{\infty} \frac{n \gamma_{p+n-1} + \gamma_{p+n}}{p!}
\]

and reference to (2.11) shows that

\[
(2.24) \quad Y_n(0! b_0, 1! b_1, \ldots, (n-1)! b_{n-1}) = 2(-1)^{n+1} \sum_{p=0}^{\infty} \frac{n \gamma p+n-1 + \gamma_{p+n}}{p!}
\]

We may write (2.14) as

\[
(s-1) \zeta(s) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n \delta_n}{n!} \left[ s^{n+1} - s^n \right]
\]

We have

\[
\frac{d^m}{ds^m} (s-1) \zeta(s) = m \zeta^{(m-1)}(s) + (s-1) \zeta^{(m)}(s)
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n \delta_n}{n!} \left[ (n+1)...(n-m)s^{n+1-m} - n...(n-m+1)s^{n-m} \right]
\]

and evaluation at \( s = 0 \) results in
\[
m\zeta^{(m-1)}(0) - \zeta^{(m)}(0) = (-1)^{m-1}[m\delta_{m-1} + \delta_m]
\]

We recall Hasse’s formula for the zeta function that for \( \text{Re}(s) \neq 1 \) (see for example [15])
\[
(s-1)\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(1+k)^{s-1}}
\]

and we see that
\[
\frac{d^m}{ds^m} (s-1)\zeta(s) = (-1)^m \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \log^m(1+k)}{(1+k)^{s-1}}
\]

Evaluation at \( s = 0 \) gives us
\[
m\zeta^{(m-1)}(0) - \zeta^{(m)}(0) = (-1)^m \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+k) \log^m(1+k)
\]
\[
= (-1)^m \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \log^m(1+k) + (-1)^m \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k \log^m(1+k)
\]

We see from (1.21) that
\[
\gamma_{m-1} = -\frac{1}{m} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \log^m(1+k)
\]

and hence we have for \( m \geq 1 \)
\[
\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k \log^m(1+k) = (-1)^m \left[ m\zeta^{(m-1)}(0) - \zeta^{(m)}(0) \right] + \frac{\gamma_{m-1}}{m}
\]

\[\square\]

Adamchik [1] noted that the Hermite integral for the Hurwitz zeta function may be derived from the Abel-Plana summation formula [33, p.90]

\[
\sum_{k=0}^{\infty} f(k) = \frac{1}{2} f(0) + \int_{0}^{\infty} f(x) \, dx + i \int_{0}^{\infty} \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1} \, dx
\]
which applies to functions which are analytic in the right-hand plane and satisfy the convergence condition

\[
\lim_{y \to \infty} e^{-2\pi y} |f(x + iy)| = 0
\]

uniformly on any finite interval of \( x \). Derivations of the Abel-Plana summation formula are given in [36, p.145] and [37, p.108].

Letting \( f(k) = (k + u)^{-s} \) we obtain

\[
(2.26) \quad \zeta(s, u) = \sum_{k=0}^{\infty} \frac{1}{(k + u)^s} = \frac{u^{-s}}{2} + \frac{u^{1-s}}{s-1} + i \int_0^\infty \frac{(u + ix)^{-s} - (u - ix)^{-s}}{e^{2\pi x} - 1} \, dx
\]

Then, noting that

\[
(u + ix)^{-s} - (u - ix)^{-s} = (re^{i\theta})^{-s} - (re^{-i\theta})^{-s} = r^{-s} \left[ e^{-i\theta s} - e^{i\theta s} \right] = \frac{2}{i(u^2 + x^2)^{s/2}} \sin(s \tan^{-1}(x/u))
\]

we may write (2.26) as Hermite’s integral for \( \zeta(s, u) \)

\[
(2.27) \quad \zeta(s, u) = \frac{u^{-s}}{2} + \frac{u^{1-s}}{s-1} + 2 \int_0^\infty \frac{\sin(s \tan^{-1}(x/u))}{(u^2 + x^2)^{s/2} (e^{2\pi x} - 1)} \, dx
\]

We now take one step back and differentiate the intermediate equation (2.26) in the case where \( u = 1 \) with respect to \( s \) to obtain

\[
(2.28) \quad \zeta^{(n)}(s) = \frac{(-1)^n n!}{(s-1)^{n+1}} + i(-1)^n \int_0^\infty \frac{(1 + ix)^{-s} \log^n(1 + ix) - (1 - ix)^{-s} \log^n(1 - ix)}{e^{2\pi x} - 1} \, dx
\]

It may be noted that

\[
i(-1)^n \int_0^\infty \frac{(1 + ix)^{-s} \log^n(1 + ix) - (1 - ix)^{-s} \log^n(1 - ix)}{e^{2\pi x} - 1} \, dx = 2(-1)^{n+1} \text{Im} \int_0^\infty \frac{(1 + ix)^{-s} \log^n(1 + ix)}{e^{2\pi x} - 1} \, dx
\]

and with \( s = 0 \) we obtain
\[ \zeta^{(n)}(0) = -n! + 2(-1)^{n+1} \text{Im} \int_0^\infty \frac{\log^n(1+ix)}{e^{2\pi x} - 1} \, dx \]

\[ \frac{\zeta^{(n)}(0)}{n!} + 1 = 2 \frac{(-1)^{n+1}}{n!} \text{Im} \int_0^\infty \frac{\log^n(1+ix)}{e^{2\pi x} - 1} \, dx \]

In particular we have

\[ \zeta^{(1)}(0) + 1 = 2 \text{Im} \int_0^\infty \frac{\log(1+ix)}{e^{2\pi x} - 1} \, dx \]

\[ = 2 \int_0^\infty \frac{\tan^{-1} x}{e^{2\pi x} - 1} \, dx \]

which shows that

\[ \zeta^{(1)}(0) + 1 > 0 \]

We also have

\[ \frac{\zeta^{(2)}(0)}{2!} + 1 = -\text{Im} \int_0^\infty \frac{\log^2(1+ix)}{e^{2\pi x} - 1} \, dx \]

\[ = -\int_0^\infty \frac{\log(1+x^2)\tan^{-1} x}{e^{2\pi x} - 1} \, dx \]

which shows that

\[ 0 > \frac{\zeta^{(2)}(0)}{2!} + 1 \]

The above inequalities concur with the numerical values in [2]. However, it is not immediately obvious how this procedure may be extended to higher values of \( n \).

3. Application of the eta constants \( \eta_n \)

We have from (1.1)

\[ \frac{d^m}{ds^m} \log((s-1)\zeta(s)) = -\sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} k(k-1)(k-2) \ldots (k-m+1)(s-1)^{i-m} \]
so that
\[
\frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0} = -\sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} k(k-2)(k-1)...(k-m+1)(-1)^{k-m}
\]

\[
= (-1)^{m+1} \sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} (-1)^k k(k-1)(k-2)...(k-m+1)
\]

Therefore we have

\[
(3.1) \quad \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

\[
= -\sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{\infty} (-1)^k \eta_{k-1} (k-1)(k-2)...(k-m+1)
\]

Coffey [13] has shown that the sequence \((\eta_k)\) has strict sign alteration

\[
(3.2) \quad \eta_k = (-1)^{k+1} \varepsilon_k
\]

where \(\varepsilon_k\) are positive constants. This sign alteration was also noted by Israilov. With reference to (2.2) we may therefore write

\[
(3.3) \quad \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

\[
= -\sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{\infty} \varepsilon_{k-1} (k-1)(k-2)...(k-m+1)
\]

It is certainly a cruel twist of fate that a solitary negative sign appears at this stage of the analysis because reference to (2.2) shows that a plus sign here instead would have immediately resulted in the Riemann Hypothesis being verified!

Let us now consider a specific example.

\[
\lambda_i = \frac{1}{2} (\gamma + \log \pi) - \frac{d}{ds} \log[(s-1)\zeta(s)] \bigg|_{s=0}
\]

\[
= \frac{1}{2} (\gamma + \log \pi) - \frac{[-\zeta'(0) + \zeta(0)]}{-\zeta(0)}
\]
\[
\gamma = \frac{1}{2} (\gamma + \log \pi) - 2[-\zeta'(0) + \zeta(0)]
\]

Hence we have

(3.4) \[\lambda_i = \frac{1}{2} \gamma - \frac{1}{2} \log \pi + 1 - \log 2\]

where we have used the well-known values

(3.5) \[\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2} \log(2\pi)\]

We may write

\[
\log[(s-1)\zeta(s)] = \log[1 + (s-1)\zeta(s) - 1]
\]

and applying the logarithmic expansion we obtain

\[
\log[(s-1)\zeta(s)] = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} [(s-1)\zeta(s) - 1]^k
\]

Differentiation of (2.1) gives us

\[
\sum_{k=1}^{\infty} \left[ (-1)^{k+1} 2^{-k} \zeta(k) - \sigma_k + \frac{1}{2} \delta_{1,k} \log \pi \right] s^{k-1} = -\sum_{k=1}^{\infty} (-1)^k [(s-1)\zeta(s) - 1]^{k-1} [(s-1)\zeta'(s) + \zeta(s)]
\]

and with \( s = 0 \) we obtain

\[
\frac{1}{2} \gamma - \sigma_1 + \frac{1}{2} \log \pi = [-\zeta'(0) + \zeta(0)]\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}
\]

which simplifies to

\[
\sigma_1 = \frac{1}{2} \gamma + \frac{1}{2} \log \pi - 2[-\zeta'(0) + \zeta(0)]
\]

or equivalently

\[
\sigma_1 = \frac{1}{2} \gamma - \frac{1}{2} \log \pi + 1 - \log 2
\]
A further differentiation results in

\[
\sum_{k=1}^{\infty} \left[ (-1)^{k+1} 2^{-k} \zeta(k) - \sigma_k + \frac{1}{2} \delta_{1,k} \log \pi \right] (k-1) s^{k-2}
\]

\[
= -\sum_{k=1}^{\infty} (-1)^k (k-1) [(s-1)\zeta(s) - 1]^{k-2} [(s-1)\zeta'(s) + \zeta(s)]
\]

\[
-\sum_{k=1}^{\infty} (-1)^k [(s-1)\zeta(s) - 1]^{k-1} [(s-1)\zeta''(s) + 2\zeta'(s)]
\]

and with \( s = 0 \) we obtain

\[
-2^{-2}\zeta(2) - \sigma_2 = [-\zeta'(0) + \zeta(0)]^2 \sum_{k=1}^{\infty} \frac{k-1}{2^{k-2}} + [-\zeta''(0) + 2\zeta'(0)] \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}
\]

Since \( \frac{1}{1-x} = \sum_{k=1}^{\infty} x^{k-1} \) we have by differentiation

\[
\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} (k-1)x^{k-2}
\]

so that \( \sum_{k=1}^{\infty} \frac{k-1}{2^{k-2}} = 4 \). We then see that

\[
-2^{-2}\zeta(2) - \sigma_2 = 4[-\zeta'(0) + \zeta(0)]^2 + 2[-\zeta''(0) + 2\zeta'(0)]
\]

Using \( \lambda_2 = 2\sigma_1 - \sigma_2 \) we obtain

\[
\lambda_2 = \gamma + \log \pi + 2^{-2}\zeta(2) - 4[\zeta(0) - \zeta'(0)] + 4[\zeta(0) - \zeta'(0)]^2 + 2[2\zeta'(0) - \zeta''(0)]
\]

As previously derived by Ramanujan [5] and Apostol [2] we have

\[
\zeta''(0) = \gamma_1 + \frac{1}{2} \gamma^2 - \frac{1}{4} \zeta(2) - \frac{1}{2} \log^2 (2\pi)
\]

and therefore, by a rather circuitous route, we have obtained the second Li/Keiper constant

\[
\lambda_2 = \frac{3}{4} \zeta(2) + 1 + \gamma - \gamma^2 - 2 \log 2 - \log \pi - 2\gamma_1
\]

□
We recall (1.15)

\[ \eta_n = (-1)^{n+1} \left[ \sigma_{n+1} + \left( 1 - \frac{1}{2^{n+1}} \right) \zeta(n+1) - 1 \right] \]

and using (3.2) \( \eta_n = (-1)^{n+1} \varepsilon_n \) we see that

\[ \varepsilon_n = \sigma_{n+1} + \left( 1 - \frac{1}{2^{n+1}} \right) \zeta(n+1) - 1 \]

and hence we have

(3.9) \[ \sigma_{n+1} > 1 - \left( 1 - \frac{1}{2^{n+1}} \right) \zeta(n+1) \]

We recall Lehmer’s relation (2.5) for \( n \geq 1 \)

\[ \sigma_{n+1} = (-1)^n 2^{-n-1} \zeta(n+1) - b_n \]

and we deduce that for \( n \geq 1 \)

(3.10) \[ \zeta(n+1) - 1 - [1 + (-1)^{n+1}] \frac{\zeta(n+1)}{2^{n+1}} > b_n \]

With \( n \to 2n \) this inequality becomes

\[ \zeta(2n+1) - 1 > b_{2n} \]

With \( n \to 2n-1 \) we obtain

\[ \zeta(2n) \left[ 1 - \frac{1}{2^{2n-1}} \right] - 1 > b_{2n-1} \]

4. A conjecture for the Riemann Hypothesis

Coffey [12] noted that the radius of convergence of (1.11)

\[ \frac{d}{ds} \log[(s-1)\zeta(s)] = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = -\sum_{k=1}^{\infty} \eta_{k-1} (s-1)^{k-1} \]
is 3 because the first singularity encountered is the trivial zero of \( \zeta(s) \) at \( s = -2 \). Furthermore he remarked that \( |\eta_k| \) cannot increase faster than \( 3^{-k} \) for sufficiently large \( k \) and that \( \eta_k = -\gamma(-1/3)^k \) is a very good approximation.

The first 60 values of \( \eta_k \) are set out in [12, p.29] and, based on this limited data, it appears that

\[
\eta_k = (-1/3)^{k+1}
\]

is a better approximation for \( k \geq 1 \). For example, we have

\[
\begin{align*}
\eta_{10} &= -5.66605 \times 10^{-6} & (-1/3)^{11} &= -5.64503 \times 10^{-6} \\
\eta_{20} &= -9.56012 \times 10^{-11} & (-1/3)^{21} &= -9.55990 \times 10^{-11} \\
\eta_{30} &= -1.61898 \times 10^{-15} & (-1/3)^{31} &= -1.61897 \times 10^{-15} \\
\eta_{40} &= -2.74176 \times 10^{-20} & (-1/3)^{41} &= -2.74175 \times 10^{-20} \\
\eta_{50} &= -4.64318 \times 10^{-25} & (-1/3)^{51} &= -4.64318 \times 10^{-25} \\
\eta_{60} &= -7.86327 \times 10^{-30} & (-1/3)^{61} &= -7.86327 \times 10^{-30}
\end{align*}
\]

It is possible that \( \eta_k = (-1/3)^{k+1}\zeta(k+1) \) may be an even better approximation, but this aspect has not been pursued any further in this paper.

Referring to (3.1) we consider the summation

\[
S = \sum_{k=1}^{\infty} \varepsilon_{k-1}(k-1)(k-2)\ldots(k-m+1)
\]

and let us conjecture that for all \( k \geq 1 \)

\[
|\eta_k| \leq \alpha 3^{-k-1}
\]

where \( \alpha > 0 \), with the result that

\[
S \leq \alpha \sum_{k=1}^{\infty} \frac{1}{k!}(k-1)(k-2)\ldots(k-m+1)
\]

Since \( \frac{1}{1-x} = \sum_{k=1}^{\infty} x^{k-1} \) we have by differentiation

\[
\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} (k-1)x^{k-2}
\]
Higher derivatives result in

\[
\frac{(m-1)!}{(1-x)^m} = \sum_{k=1}^{\infty} (k-1)(k-2)...(k-m+1)x^{k-m}
\]

which may be written as

\[
\sum_{k=1}^{\infty} (k-1)(k-2)...(k-m+1)x^k = (m-1)! \left(\frac{x}{1-x}\right)^m
\]

With \( x = 1/3 \) we obtain

\[
\sum_{k=1}^{\infty} \frac{1}{3^k} (k-1)(k-2)...(k-m+1) = (m-1)! \frac{1}{2^m}
\]

and we therefore have

\[
S \leq \alpha (m-1)! \frac{1}{2^m}
\]

Hence we see that

\[
\sum_{m=1}^{n} \binom{n}{m} \frac{1}{2^m} \sum_{k=1}^{\infty} \varepsilon_{k-1} (k-1)(k-2)...(k-m+1) \leq \alpha \sum_{m=1}^{n} \binom{n}{m} \frac{1}{2^m}
\]

Substituting

\[
\sum_{m=1}^{n} \binom{n}{m} \frac{1}{2^m} = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{2^m} - 1 = \left(\frac{3}{2}\right)^n - 1
\]

gives us

\[
\sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{\infty} \varepsilon_{k-1} (k-1)(k-2)...(k-m+1) \leq \alpha \left[\left(\frac{3}{2}\right)^n - 1\right]
\]

We recall (3.1)

\[
\lambda_n = \frac{n}{2} (\gamma + \log \pi) + \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \zeta(m) - \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{\infty} \varepsilon_{k-1} (k-1)(k-2)...(k-m+1)
\]

and we therefore have
\[ \lambda_n \geq \frac{n}{2}(\gamma + \log \pi) + \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \zeta(m) - \alpha \left( \frac{3}{2} \right)^n + \alpha \]

Employing the inequality \( \zeta(m) > 1 \) we see that

\[ \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \zeta(m) > \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \]

and using

\[ \sum_{m=2}^{n} \binom{n}{m} \frac{1}{2^m} = \sum_{m=2}^{n} \binom{n}{m} \frac{1}{2^m} - 1 - \frac{1}{2} n = \left( \frac{3}{2} \right)^n - 1 - \frac{1}{2} n \]

we obtain

\[ (4.3) \quad \lambda_n > \frac{n}{2}(\gamma + \log \pi - 1) + (1 - \alpha) \left[ \left( \frac{3}{2} \right)^n - 1 \right] \]

where we note that \( \gamma + \log \pi - 1 = 0.72... \) is positive. Therefore the conjecture that \( |\eta_k| \leq \alpha 3^{-k-1} \), combined with \( 1 > \alpha > 0 \), immediately implies that \( \lambda_n > 0 \).

However, it should be noted that the factor \( \left( \frac{3}{2} \right)^n \) increases very rapidly; for example \( \left( \frac{3}{2} \right)^{100} \approx 10^{17} \) whereas we note from [12] that \( \lambda_{100} = 118.6038... \). It is therefore clear that our initial conjecture that \( |\eta_k| \leq \alpha 3^{-k-1} \) for all \( k \) can only be valid provided \( (1 - \alpha) \) is sufficiently close to zero. It is however clear that this condition is not valid because \( \eta_1 = 0.187546 \approx 1/3^2 \).

Further work is required to determine if the conjecture may be modified so that \( |\eta_k| \leq \alpha 3^{-k-1} \) for all \( k > N \) where \( N \) is a fixed integer.

The limited data in [12], also suggests that the terms \( |\eta_k| \) form a monotonic decreasing sequence.

5. **Using the positivity of \( \xi^{(n)}(1) \)**

Using the Jacobi theta function, Coffey [11] showed in 2003 that the even derivatives of \( \xi(s) \) are positive for all real values of \( s \), while the odd derivatives are positive for \( s \geq \frac{1}{2} \) and negative for \( s < \frac{1}{2} \). In particular, \( \xi^{(n)}(1) \) is positive for \( n \geq 1 \). This was also proved
by Freitas [20] in a different manner in 2005. It may also be noted from the functional equation (1.3) that

\[ (5.1) \quad \xi^{(n)}(0) = (-1)^n \xi^{(n)}(1) \]

We write

\[ \log[2\xi(s)] = \log\left(1 + [2\xi(s) - 1]\right) \]

and applying the Maclaurin expansion we have

\[ \log[2\xi(s)] = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} [2\xi(s) - 1]^k \]

Since

\[ \frac{d^n}{ds^n}s^{n-1} \log[2\xi(s)] = \frac{d^n}{ds^n}s^{n-1} \log \xi(s) \]

we note from (1.4) that

\[ \lambda_n = \frac{1}{(n-1)!} \left. \frac{d^n}{ds^n}[s^{n-1} \log 2\xi(s)] \right|_{s=1} \]

We have as before

\[ \frac{d^n}{ds^n}[s^{n-1} \log 2\xi(s)] = \sum_{m=1}^{n} \binom{n}{m} \frac{(n-1)!}{(m-1)!} \frac{d^m}{ds^m} \left[ \log 2\xi(s) \right] \]

and therefore we get

\[ \lambda_n = -\sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \left. \frac{d^m}{ds^m} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} [2\xi(s) - 1]^k \right|_{s=1} \]

which may be written as

\[ (5.2) \quad \lambda_n = -\sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left. \frac{d^m}{ds^m} [2\xi(s) - 1]^k \right|_{s=1} \]

For example with \( n = 1 \) we have
\[ \lambda_1 = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{d}{ds} \left[ 2\xi(s) - 1 \right]^k \bigg|_{s=1} \]

and we see that

\[ \frac{d}{ds} \left[ 2\xi(s) - 1 \right]^k = 2k \left[ 2\xi(s) - 1 \right]^{k-1} \xi'(s) \]

This gives us

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left. \frac{d}{ds} \right[ 2\xi(s) - 1 \right]^k \bigg|_{s=1} = 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} k\delta_k \xi''(1) \]

and hence we have

(5.3) \[ \lambda_1 = 2\xi''(1) \]

Then, using Coffey’s result [11] that \( \xi''(1) \) is positive, we see that

(5.4) \[ \lambda_1 > 0 \]

Using (1.7) we see that

\[ \frac{d}{ds} \log 2\xi(s) = \frac{\xi'(s)}{\xi(s)} = -\sum_{k=1}^{\infty} (-1)^k \xi_k (s-1)^{k-1} \]

which gives us

(5.5) \[ 2\xi'(1) = \xi_1 \]

With \( n = 2 \) we have

\[ \lambda_2 = -\sum_{m=1}^{\infty} \binom{2}{m} \frac{1}{(m-1)!} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{d^m}{ds^m} \left[ 2\xi(s) - 1 \right]^k \bigg|_{s=1} \]

\[ = -2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left. \frac{d}{ds} \right[ 2\xi(s) - 1 \right]^k \bigg|_{s=1} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{d^2}{ds^2} \left[ 2\xi(s) - 1 \right]^k \bigg|_{s=1} \]

\[ = 2\lambda_1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{d^2}{ds^2} \left[ 2\xi(s) - 1 \right]^k \bigg|_{s=1} \]

We have
\[
\frac{d^2}{ds^2}[2\xi(s) - 1]^k = 4k(k-1)[2\xi(s) - 1]^{k-2} \left[\xi''(s)\right]^2 + 2k [2\xi(s) - 1]^{k-1} \xi''(s)
\]

and thus

\[
\left. \frac{d^2}{ds^2}[2\xi(s) - 1]^k \right|_{s=1} = 4k(k-1)\delta_{2,1} \left[\xi''(1)\right]^2 + 2k\delta_{1,1} \xi''(1)
\]

so that

\[
\lambda_2 = 2\lambda_1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[4k(k-1)\delta_{2,1} \left[\xi''(1)\right]^2 + 2k\delta_{1,1} \xi''(1)\right] \\
= 4\xi''(1) - 4\left[\xi''(1)\right]^2 + 2\xi''(1)
\]

Hence we have

\[(5.6) \quad \lambda_2 = 4\xi''(1)\left[1 - \xi''(1)\right] + 2\xi''(1)\]

We know that \(1 > \xi''(1)\) and therefore we deduce that

\[(5.7) \quad \lambda_2 > 0\]

We now investigate whether the above process can be generalised.

Often in mathematics we look for divine inspiration but we do not usually expect to obtain it from a canonised saint. This is indeed the source of the next remark. The calculation of \(\lambda_n\) effectively involves the derivative of a composite function and the general formula for this was discovered by Francesco Faà di Bruno (1825-1888) who was declared a Saint by Pope John Paul II in St. Peter’s Square in Rome on the centenary of his death in 1988 [34].

In [24] di Bruno showed that

\[(5.8) \quad \frac{d^m}{dt^m} g(f(t)) = \sum_{b_1, b_2, \ldots, b_m} \frac{m!}{b_1!b_2!\ldots b_m!} g^{(k)}(f(t)) \left(\frac{f^{(1)}(t)}{1!}\right)^{b_1} \left(\frac{f^{(2)}(t)}{2!}\right)^{b_2} \cdots \left(\frac{f^{(m)}(t)}{m!}\right)^{b_m}\]

where the sum is over all different solutions in non-negative integers \(b_1, \ldots, b_m\) of \(b_1 + 2b_2 + \ldots + mb_m = m\), and \(k = b_1 + \ldots + b_m\).

As reported in [24] this may also be expressed in terms of the (exponential) partial Bell polynomials \(B_{m,j}\)
\[ (5.9) \quad \frac{d^m}{dt^m} g(f(t)) = \sum_{j=0}^{m} g^{(j)}(f(t))B_{m,j} \left( f^{(1)}(t), f^{(2)}(t), \ldots, f^{(m-j+1)}(t) \right) \]

where the (exponential) partial Bell polynomials \( B_{m,j} = B_{m,j} \left( x_1, x_2, \ldots, x_{m-j+1} \right) \) are defined by

\[
B_{m,j} = B_{m,j} \left( x_1, x_2, \ldots, x_{m-j+1} \right) = \sum \frac{m!}{b_1! b_2! \ldots b_m!} \left( \frac{x_1}{1!} \right)^{b_1} \left( \frac{x_2}{2!} \right)^{b_2} \cdots \left( \frac{x_1}{m!} \right)^{b_m}
\]

where the sum is over all different solutions in non-negative integers \( b_1, \ldots, b_m \) of \( b_1 + 2b_2 + \ldots + mb_m = m \), and \( j = b_1 + \ldots + b_m \).

We now consider \( \left[ \frac{d^m}{ds^m} \left[ 2e^\xi(s) - 1 \right]^k \right] \) so that

\[
\begin{align*}
g(x) &= x^k \\
f(s) &= 2e^\xi(s) - 1 \\
g^{(j)}(x) &= (k)(k-1)\ldots(k-j+1)x^{k-j} \\
g^{(j)}(0) &= \delta_{j,k}k! \\
f^{(j)}(s) &= 2e^{\xi^{(j)}(s)}
\end{align*}
\]

Using (5.9) gives us

\[
\left. \frac{d^m}{ds^m} \left[ 2e^\xi(s) - 1 \right]^k \right|_{s=1} = \sum_{j=0}^{m} \delta_{j,k}k!B_{m,j} \left( 2e^{\xi^{(1)}(1)}, 2e^{\xi^{(2)}(1)}, \ldots, 2e^{\xi^{(m-j+1)}(1)} \right) = k!B_{m,k} \left( 2e^{\xi^{(1)}(1)}, 2e^{\xi^{(2)}(1)}, \ldots, 2e^{\xi^{(m-k+1)}(1)} \right)
\]

Therefore we obtain from (5.2)

\[
\lambda_n = -\sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{m} \left( -1 \right)^{k} \sum_{j=0}^{k} \delta_{j,k}k!B_{m,k} \left( 2e^{\xi^{(1)}(1)}, 2e^{\xi^{(2)}(1)}, \ldots, 2e^{\xi^{(m-k+1)}(1)} \right)
\]

and, since \( B_{m,k} = 0 \) for \( k \geq m + 1 \), this becomes the finite double summation

\[ (5.10) \quad \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{m} \left( -1 \right)^{k-1} (k-1)!B_{m,k} \left( 2e^{\xi^{(1)}(1)}, 2e^{\xi^{(2)}(1)}, \ldots, 2e^{\xi^{(m-k+1)}(1)} \right) \]

35
We will see in (7.4) that
\[ 2\xi(s) = \sum_{n=0}^{\infty} Y_n \left(-\sigma_1, -1! \sigma_2, \ldots, -(n-1)! \sigma_n\right) \frac{s^n}{n!} \]

Hence it follows that
\[ 2(-1)^n \xi^{(m)}(1) = 2\varphi^{(m)}(0) = Y_m \left(-\sigma_1, -1! \sigma_2, \ldots, -(m-1)! \sigma_m\right) \]

We refer to the following inversion relation of Chou et al. [9]

(5.11) \[ Y_n = y_n = \sum_{k=1}^{n} B_{n,k} \left(x_1, x_2, \ldots, x_{n-k+1}\right) \Leftrightarrow x_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k} \left(y_1, y_2, \ldots, y_{n-k+1}\right) \]

We have seen that
\[ 2(-1)^n \xi^{(n)}(1) = Y_n \left(-\sigma_1, -1! \sigma_2, \ldots, -(n-1)! \sigma_n\right) = \sum_{k=1}^{n} B_{n,k} \left(x_1, x_2, \ldots, x_{n-k+1}\right) \]

where \( x_j = -(j-1)! \sigma_j \). Hence we deduce that
\[ -(n-1)! \sigma_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k} \left(2\varphi^{(1)}(1), 2\varphi^{(2)}(1), \ldots, (-1)^{m-k+1} 2\varphi^{(m-k+1)}(1)\right) \]

We have [8, p.412]
\[ B_{n,k} \left(2\varphi^{(1)}(1), 2\varphi^{(2)}(1), \ldots, (-1)^{m-k+1} 2\varphi^{(m-k+1)}(1)\right) = (-1)^n B_{n,k} \left(2\varphi^{(1)}(1), 2\varphi^{(2)}(1), \ldots, 2\varphi^{(m-k+1)}(1)\right) \]

and using (5.10) above
\[ \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{m} (-1)^{k-1} (k-1)! B_{m,k} \left(2\varphi^{(1)}(1), 2\varphi^{(2)}(1), \ldots, 2\varphi^{(m-k+1)}(1)\right) \]

we simply obtain (1.8) again, i.e.
\[ \lambda_n = -\sum_{m=1}^{n} \binom{n}{m} (-1)^m \sigma_m \]

\[ \square \]

The logarithmic partition polynomials \( L_m \) satisfy the following relations [8, p.424]
(5.12) \[ L_m(g_1, g_2, \ldots, g_m) = \sum_{k=1}^{m} (-1)^{k-1} (k-1)! B_{m,k} \left(g_1, g_2, \ldots\right) \text{ for } m \geq 1 \]

\[ L_0 = 0 \]

(5.13) \[ \log \left( \sum_{n=0}^{\infty} g_n \frac{s^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{s^n}{n!} \quad g_0 = 1 \]

and referring to (5.10) we see that

\[ \lambda_n = \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{1}{(m-1)!} L_m \left( 2\xi^{(1)}(1), 2\xi^{(2)}(1), \ldots, 2\xi^{(m)}(1) \right) \]

From (5.13) we have

\[ \frac{d^m}{ds^m} \log \left( \sum_{n=0}^{\infty} g_n \frac{s^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{L_n(n-1) \ldots (n-m+1)}{n!} \frac{s^{n-m}}{n!} \]

and hence

\[ \frac{d^m}{ds^m} \log \left( \sum_{n=0}^{\infty} g_n \frac{s^n}{n!} \right) \bigg|_{s=0} = L_m \]

This then gives us

\[ \lambda_n = \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log \left( \sum_{n=0}^{\infty} g_n \frac{s^n}{n!} \right) \bigg|_{s=0} \]

where \( g_n = 2\xi^{(n)}(1) \).

We see that

\[ \sum_{n=0}^{\infty} g_n \frac{s^n}{n!} = 2 \sum_{n=0}^{\infty} \xi^{(n)}(1) \frac{s^n}{n!} = 2 \sum_{n=0}^{\infty} \xi^{(n)}(0) \frac{(-s)^n}{n!} = 2\xi(-s) \]

and hence we have

\[ \lambda_n = \sum_{m=1}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log 2\xi(-s) \bigg|_{s=0} \]

37
Referring to (2.8) we see that this is equivalent to (1.4.1)

\[ \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{m!} \left( \frac{d^m}{dp^m} \right)_{s=1} \log 2 \xi(s) \]

and hence we have simply come full circle!

\[ \square \]

Equation (5.10) may be reindexed to

\[ \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{m!} \sum_{k=0}^{m-1} (-1)^k k! B_{m,k+1} \left( 2 \xi^{(1)}(1), 2 \xi^{(2)}(1), \ldots, 2 \xi^{(m-k)}(1) \right) \]

and, for example, we have

\[ \lambda_1 = B_{1,1} \left( 2 \xi^{(1)}(1) \right) \]

Since

\[ B_{1,1} = x_1 \]

we obtain as before

\[ \lambda_1 = 2 \xi^{(1)}(1) \]

With \( n = 2 \) we have

\[ \lambda_2 = \sum_{m=1}^{2} \binom{2}{m} \frac{1}{m!} \sum_{k=0}^{m-1} (-1)^k k! B_{m,k+1} \left( 2 \xi^{(1)}(1), 2 \xi^{(2)}(1), \ldots, 2 \xi^{(m-k)}(1) \right) \]

\[ \lambda_2 = 2B_{1,1} + \sum_{k=0}^{1} (-1)^k k! B_{2,k+1} \left( 2 \xi^{(1)}(1), 2 \xi^{(2)}(1), \ldots, 2 \xi^{(m-k)}(1) \right) \]

\[ \lambda_2 = 2B_{1,1} + B_{2,1} - B_{2,2} \quad x_{y} = 2 \xi^{(n)}(1) \]

\[ B_{2,1} = x_2 \quad B_{2,2} = x_1^2 \]

\[ \lambda_2 = 2x_1 + x_2 - x_1^2 \]

and thus we get
\[ \lambda_2 = 4 \xi^{(1)}(1) + 2 \xi''(1) - 4 \left[ \xi'(1) \right]^2 \]

We have

\[
\lambda_3 = \sum_{m=1}^{3} \left( \frac{3}{m} \right) \sum_{k=0}^{m-1} (-1)^k k! B_{m,k+1} \left( 2 \xi^{(1)}(1), 2 \xi^{(2)}(1), \ldots, 2 \xi^{(m-k)}(1) \right)
\]

\[
= \sum_{k=0}^{0} (-1)^k k! B_{1,k+1} + \sum_{k=0}^{1} (-1)^k k! B_{2,k+1} + \sum_{k=0}^{2} (-1)^k k! B_{3,k+1}
\]

\[ \lambda_3 = B_{1,1} + 3 B_{2,2} + 3 B_{3,3} - 6 B_{3,3} \]

And using

\[ B_{3,1} = x_3, \quad B_{3,2} = 3 x_1 x_2, \quad B_{3,3} = x_1^3 \]

we obtain

\[ \lambda_3 = x_1 + 3 x_2 - 3 x_1^2 + 3 x_3 - 9 x_1 x_2 + 6 x_1^3 \]

or

\[ \lambda_3 = 2 \xi^{(1)}(1) + 6 \xi^{(2)}(1) - 12 \left[ \xi'(1) \right]^2 + 6 \xi^{(3)}(1) - 18 \xi^{(1)}(1) \xi^{(2)}(1) + 48 \left[ \xi'(1) \right]^3 \]

Since

\[ \frac{d^n}{ds^n} \log[(s-1)\xi(s)] = \frac{d^n}{ds^n} \log[2(s-1)\xi(s)] \]

we may also include a factor of \( \log 2 \) so that

\[ \lambda_n = \frac{n}{2} (\gamma + \log \pi) + \sum_{m=1}^{n} \left( \frac{n}{m} \right) 2^{-m} \xi(m) + \sum_{m=1}^{n} (-1)^m \left( \frac{n}{m} \right) \frac{1}{(m-1)!} \frac{d^n}{ds^n} \log[2(s-1)\xi(s)]_{s=0} \]

Since

\[ \frac{d^n}{ds^n} \log[1 + 2(s-1)\xi(s) - 1] = - \frac{d^n}{ds^n} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} [2(s-1)\xi(s) - 1]^k \]

\[ = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{d^n}{ds^n} [2(s-1)\xi(s) - 1]^k \]

39
we obtain

\[
\lambda_n = \frac{n}{2} (\gamma + \log \pi) + \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \zeta(m) - \sum_{m=1}^{n} \frac{(-1)^m}{m} \binom{n}{m} - \frac{1}{\Gamma(m-1)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{d^m}{ds^m}[2(s-1)\zeta(s)-1]^{k} \\ \bigg|_{s=0}
\]

Including the factor of \(\log 2\) is convenient because

\[
\lim_{s \to 0} [2(s-1)\zeta(s)-1] = 0
\]

We now consider \(\frac{d^m}{ds^m}[2(s-1)\zeta(s)-1]^{k}\) and designating the composite function

\[
[2(s-1)\zeta(s)-1]^{k} \equiv g(f(s))
\]

so that

\[
g(x) = x^k
\]

\[
f(s) = 2(s-1)\zeta(s)-1
\]

Differentiation gives us

\[
g^{(j)}(x) = k(k-1)...(k-j+1)x^{k-j}
\]

\[
g^{(j)}(0) = \delta_{j,k}k!
\]

\[
f^{(j)}(s) = 2 \frac{d^j}{ds^j}(s-1)\zeta(s)
\]

\[
\frac{d^j}{ds^j}(s-1)\zeta(s) = (s-1)\zeta^{(j)}(s) + j\zeta^{(j-1)}(s)
\]

We then have using (5.9)

\[
\frac{d^m}{ds^m}[2(s-1)\zeta(s)-1]^{k} \bigg|_{s=0} = \sum_{j=0}^{m} \delta_{j,k}k!B_{m,j} \left( f^{(1)}(0), f^{(2)}(0), ..., f^{(m-j+1)}(0) \right)
\]

\[
= k!B_{m,k} \left( f^{(1)}(0), f^{(2)}(0), ..., f^{(m-k+1)}(0) \right)
\]

and from the logarithmic expansion we see that
\[
\frac{d^n}{ds^n}\log[(s-1)\zeta(s)] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k} k! B_{m,k} \left( f^{(1)}(0), f^{(2)}(0), \ldots, f^{(m-k+1)}(0) \right)
\]

Since \( B_{m,k} = 0 \) for \( k \geq m + 1 \) this becomes the finite double summation

\[
\frac{d^n}{ds^n}\log[(s-1)\zeta(s)] = \sum_{k=1}^{m} (-1)^{k-1} (k-1)! B_{m,k} \left( f^{(1)}(0), f^{(2)}(0), \ldots, f^{(m-k+1)}(0) \right)
\]

We then obtain

\[
\lambda_n = \frac{n}{2} (\gamma + \log \pi) + \sum_{m=2}^{n} \binom{n}{m} 2^{-m} \zeta(m)
\]

\[
+ \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{m} (-1)^{k-1} (k-1)! B_{m,k} \left( f^{(1)}(0), f^{(2)}(0), \ldots, f^{(m-k+1)}(0) \right)
\]

where

\[
f^{(j)}(s) = 2 \left[ j \zeta^{(j-1)}(0) - \zeta^{(j)}(0) \right]
\]

For example we have

\[
\lambda_1 = \frac{1}{2} (\gamma + \log \pi) - B_{1,1} \left( f^{(1)}(0) \right)
\]

and since \( B_{1,1} = x_1 \) we obtain again

\[
\lambda_1 = \frac{1}{2} (\gamma + \log \pi) - 2 \left[ \zeta(0) - \zeta^{(1)}(0) \right]
\]

We recall (2.11)

\[
2 \left[ m \zeta^{(m-1)}(0) - \zeta^{(m)}(0) \right] = Y_m \left( 0! b_0, 1! b_1, \ldots, (m-1)! b_{m-1} \right)
\]

which gives us

\[
\sum_{k=1}^{m} (-1)^{k-1} (k-1)! B_{m,k} \left( f^{(1)}(0), f^{(2)}(0), \ldots, f^{(m-k+1)}(0) \right) = \sum_{k=1}^{m} (-1)^{k-1} (k-1)! B_{m,k} \left( Y_1, \ldots, Y_{m-k+1} \right)
\]

Noting the inversion relation of Chou et al. [9]
\[ y_m = \sum_{k=1}^{m} B_{m,k} \left( x_1, x_2, \ldots, x_{m-k+1} \right) \Leftrightarrow x_m = \sum_{k=1}^{m} (-1)^{k-1} (k-1)! B_{m,k} \left( y_1, y_2, \ldots, y_{m-k+1} \right) \]

we see that

\[ x_m = (m-1)!b_{m-1} \]

and hence we simply recover (2.4).

6. Relative magnitudes of \( \zeta^{(n)}(1) \)

Riemann [19] showed that

\[ (6.1) \quad 2\zeta(s) = 1 + s(s-1)\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{x^{(1-s)/2} + x^{s/2}}{x} \, dx \]

and differentiation results in

\[ 2\zeta'(s) = s(s-1)\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{\log \sqrt{x}}{x} \left[ (-1)x^{(1-s)/2} + x^{s/2} \right] \, dx + (2s-1)\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{x^{(1-s)/2} + x^{s/2}}{x} \, dx \]

With \( s = 1 \) we have

\[ 2\zeta'(1) = \int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{1 + x^{1/2}}{x} \, dx \]

and we see that \( \zeta'(1) > 0 \). A further differentiation gives us

\[ 2\zeta''(s) = s(s-1)\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{\log^2 \sqrt{x}}{x} \left[ x^{(1-s)/2} + x^{s/2} \right] \, dx \]

\[ + 2(s-1)\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{\log \sqrt{x}}{x} \left[ (-1)x^{(1-s)/2} + x^{s/2} \right] \, dx + 2\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{x^{(1-s)/2} + x^{s/2}}{x} \, dx \]

and we have

\[ 2\zeta''(1) = 2\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{\log \sqrt{x}}{x} \left[ x^{1/2} - 1 \right] \, dx + 2\int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{1 + x^{1/2}}{x} \, dx \]

42
\[
= 2 \sum_1^{\infty} \int_1^{\infty} e^{-\pi k^2 x} \frac{\log \sqrt{x}}{x} \left[ x^{1/2} - 1 \right] dx + 4 \zeta''(1)
\]

Therefore we see that

(6.2) \quad \zeta''(1) > 2 \zeta'(1)

We write (6.1) as

\[
2 \xi(s) = 1 + s(s-1) f(s)
\]

where

\[
f(s) = \int_1^{\infty} \left( \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{x^{(1-s)/2} + x^{s/2}}{x} \right) dx
\]

Using the Leibniz rule we have

\[
2 \xi^{(n)}(s) = \sum_{j=0}^{n} \binom{n}{j} f^{(n-j)}(s) \frac{d^j}{ds^j} s(s-1)
\]

which immediately simplifies for \( n \geq 2 \) to

\[
2 \xi^{(n)}(s) = s(s-1) \left( \binom{n}{0} f^{(n)}(s) + (2s-1) \binom{n}{1} f^{(n-1)}(s) + 2 \binom{n}{2} f^{(n-2)}(s) \right)
\]

With \( s = 0 \) we obtain

\[
\xi^{(n)}(0) = \binom{n}{2} f^{(n-2)}(0) - \frac{1}{2} \binom{n}{1} f^{(n-1)}(0)
\]

We see that

\[
f^{(j)}(s) = \frac{1}{2^j} \int_1^{\infty} \left( \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{\log^j x}{x} \left[ (-1)^j x^{(1-s)/2} + x^{s/2} \right] \right) dx
\]

\[
f^{(0)}(0) = \frac{1}{2^j} \int_1^{\infty} \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{\log^j x}{x} \left[ (-1)^j x^{1/2} + 1 \right] dx
\]

\[
= \frac{(-1)^j}{2^j} \int \sum_{k=1}^{\infty} e^{-\pi k^2 x} \frac{\log^j x}{x} \left[ x^{1/2} + (-1)^j \right] dx
\]
Therefore we have

\[
\xi^{(n)}(0) = (-1)^n \frac{n}{2} \int_1^{\infty} \sum_{k=1}^{\infty} e^{-nk^2x} \frac{\log^{n-2} \sqrt{x}}{x} (n-1) \left[ \sqrt{x} + (-1)^n \right] + \left[ \sqrt{x} - (-1)^n \right] \log \sqrt{x} \, dx
\]

Since \( \xi^{(n)}(0) = (-1)^n \xi^{(n)}(1) \) we have

\[
(6.3) \quad \xi^{(n)}(1) = \frac{n}{2} \int_1^{\infty} \sum_{k=1}^{\infty} e^{-nk^2x} \frac{\log^{n-2} \sqrt{x}}{x} (n-1) \left[ \sqrt{x} + (-1)^n \right] + \left[ \sqrt{x} - (-1)^n \right] \log \sqrt{x} \, dx
\]

and, since \( \sqrt{x} \pm (-1)^n \geq 0 \) for \( x \geq 1 \), we see that for \( n \geq 2 \)

\[
(6.4) \quad \xi^{(n)}(1) > 0
\]

As reported by Keiper [25] we have

\[
2\xi(s) = \sum_{n=0}^{\infty} \alpha_n (s-1)^s = 2 \sum_{n=0}^{\infty} \frac{\xi^{(n)}(1)}{n!} (s-1)^n
\]

where

\[
\alpha_0 = 1 \quad \alpha_n = \beta_{n-2} + \beta_{n-1} \quad \text{for } n \geq 1
\]

\[
\beta_{-1} = 0 \quad \beta_0 = 1 + \frac{1}{2} \gamma - \log(2\sqrt{\pi})
\]

\[
\beta_n = \frac{1}{n!} \int_1^{\infty} \sum_{k=1}^{\infty} e^{-nk^2x} \frac{\log^n \sqrt{x}}{x} \left( \sqrt{x} + (-1)^n \right) dx \quad \text{for } n \geq 1
\]

\[
\xi^{(n)}(1) = \frac{1}{2} \alpha_n n!
\]

\[
\alpha_0 = 1
\]

\[
\alpha_1 = 1 + \frac{1}{2} \gamma - \log(2\sqrt{\pi})
\]

\[
\alpha_2 = 1 + \frac{1}{2} \gamma - \log(2\sqrt{\pi}) + \beta_1
\]

\[
\alpha_3 = \beta_1 + \beta_2
\]
It is easily seen that $\beta_n > 0$ for $n \geq 1$ which implies that $\alpha_n > 0$ for $n \geq 3$. Therefore we see again that $\zeta^{(n)}(1) > 0$ for (at least) $n \geq 3$. Curiously, this point was not specifically mentioned in Keiper’s paper [25]; this was first proved (and extended) by Coffey [11] in 2004 using a different formulation, namely

$$\zeta^{(2m)}(1) = \frac{2\pi}{2^{2m}} \sum_{n=1}^{\infty} \left( \pi n^2 x - \frac{3}{2} \right) n^2 e^{-n^2 x} \sqrt{x} \left\lfloor \sqrt{x} + 1 \right\rfloor \log^{2m} x \, dx$$

$$\zeta^{(2m+1)}(1) = \frac{\pi}{2^{2m}} \sum_{n=1}^{\infty} \left( \pi n^2 x - \frac{3}{2} \right) n^2 e^{-n^2 x} \sqrt{x} \left\lfloor \sqrt{x} - 1 \right\rfloor \log^{2m+1} x \, dx$$

Coffey’s result [11] may also be obtained by differentiating Riemann’s formula [19, p.17]

$$\frac{d}{dx} \left[ x^{3/2} \omega'(x) \right] x^{-1/4} \cosh \left\lfloor \frac{1}{2} \left( s - \frac{1}{2} \right) \log x \right\rfloor \, dx$$

where $\omega(x) = \sum_{k=1}^{\infty} e^{-x^2 k^2 x}$.

We immediately see that for $n \geq 3$

$$\frac{2}{n!} \left[ \zeta^{(n+1)}(1) - \zeta^{(n)}(1) \right] = n\beta_{n-1} + (n+1)\beta_n - \beta_{n-2}$$

so that

$$\frac{2}{n!} \left[ \zeta^{(n+1)}(1) - \zeta^{(n)}(1) \right] =$$

$$\frac{1}{n!} \sum_{k=1}^{\infty} e^{-x^2 k^2 x} \log^{n+2} x \sqrt{x} \left[ n^2 \log \sqrt{x} \left( \sqrt{x} - (-1)^{x} \right) + (n+1) \log^2 \sqrt{x} \left( \sqrt{x} + (-1)^{x} \right) - n(n-1) \left( \sqrt{x} + (-1)^{x} \right) \right] \, dx$$

With the substitution $t = \sqrt{x}$ the integral becomes

$$= \frac{2}{n!} \sum_{k=1}^{\infty} e^{-x^2 k^2 t} \log^{n+2} t \left[ n^2 \log t \left( t - (-1)^{x} \right) + (n+1) \log^2 t \left( t + (-1)^{x} \right) - n(n-1) \left( t + (-1)^{x} \right) \right] \, dt$$

For convenience we designate $f_n(t)$ for $n \geq 3$ as

$$f_n(t) = \left[ n^2 \log t \left( t - (-1)^{x} \right) + (n+1) \log^2 t \left( t + (-1)^{x} \right) - n(n-1) \left( t + (-1)^{x} \right) \right] \log^{n-2} t$$
and we first of all consider the specific case where \( n = 3 \), giving us

\[
f_3(t) = \left[9 \log t(t+1) + 4 \log^2 t(t-1) - 6(t-1)\right] \log t
\]

Differentiation results in

\[
f'_3(t) = \left[9 \log t(t+1) + 4 \log^2 t(t-1) - 6(t-1)\right] \frac{1}{t}
\]

\[
+ \left[9 \log t + 9 \frac{1}{t} (t+1) + 4 \log^2 t + \frac{1}{t} \log t(t-1) - 6\right] \log t
\]

\[
= \left[9 \log t(t+1) + 4 \log^2 t(t-1) - 6(t-1)\right] \frac{1}{t}
\]

\[
+ \left[9 \log t + 9 \frac{1}{t} + 4 \log^2 t + 8 \frac{1}{t} \log t(t-1) + 3\right] \log t
\]

where we note that the second term in parentheses is positive for \( t \geq 1 \). We now consider the first term

\[
g(t) = 9 \log t(t+1) + 4 \log^2 t(t-1) - 6(t-1) \hspace{1cm} g(1) = 0
\]

Differentiation results in

\[
g'(t) = 9 \log t + 9 \frac{1}{t} (t+1) + 4 \log^2 t + \frac{1}{t} \log t(t-1) - 6
\]

\[
= 9 \log t + 9 \frac{1}{t} + 4 \log^2 t + 8 \frac{1}{t} \log t(t-1) + 3
\]

and we note that \( g'(t) \) is positive for \( t \geq 1 \). We therefore see that \( f'_3(t) \geq 0 \) for all \( t \geq 1 \), thereby proving that \( f_3(t) \) is monotonic increasing for \( t \geq 1 \). Since \( f_3(1) = 0 \), we deduce that \( f_3(t) \geq 0 \) for all \( t \geq 1 \).

We now consider the case where \( n \geq 3 \)

\[
f'_n(t) = \left[n^2 \log t(t + (-1)^n) + (n+1) \log^2 t(t + (-1)^n) - n(n-1)(t + (-1)^n)\right] (n-2) \frac{1}{t} \log^{n-3} t
\]

\[
+ \left[n^2 \log t + n^2 \frac{1}{t} (t + (-1)^n) + (n+1) \log^2 t + 2(n+1) \log t \frac{1}{t} (t + (-1)^n) - n(n-1)\right] \log^{n-2} t
\]

and we write this as
\[ f_n'(t) = g(t)(n-2)\frac{1}{t}\log^{n-3} t + h(t)\log^{n-2} t \]

First of all, we see that \( h(t) \) is certainly positive when \( n \) is odd and \( t \geq 1 \).

We have

\[ g'(t) = n^2 \log t + n^2 \frac{1}{t} \left( t - (-1)^n \right) + (n+1) \log^2 t + (n+1) \frac{1}{t} \left( t + (-1)^n \right) - n(n-1) \]

\[ = n^2 \log t - (-1)^n n^2 \frac{1}{t} + (n+1) \log^2 t + (n+1) \frac{1}{t} \left( t + (-1)^n \right) + n \]

and we note that \( g'(t) \) is certainly positive when \( n \) is odd and \( n \geq 3 \). Therefore we conclude that \( f_n'(t) \geq 0 \) when \( n \geq 3 \) and \( n \) is odd. Hence we have determined that

\[ g^{(2m+2)}(1) > g^{(2m+1)}(1) \quad \text{for} \quad m \geq 1 \]

so that, for example, we have

\[ g^{(4)}(1) > g^{(3)}(1) \]

We shall determine below in (6.6) that

\[ g^{(2)}(1) > g^{(3)}(1) \]

\[ \Box \]

Using (6.3) we see that

\[ g^{(2)}(1) - g^{(3)}(1) = \int_1^\infty \sum_{k=1}^\infty e^{-\pi k^2 x} \frac{1}{x} \left( \sqrt{x} + 1 - 2\sqrt{x-1} \log \sqrt{x} - \frac{3}{2} \left[ \sqrt{x+1} \right] \log^2 \sqrt{x} \right)dx \]

With the substitution \( t = \sqrt{x} \) the integral becomes

\[ = 2 \int_1^\infty \sum_{k=1}^\infty e^{-\pi k^2 t^2} \frac{1}{t} \left( t+1 - 2[t-1] \log t - \frac{3}{2} [t+1] \log^2 t \right)dt \]

With the substitution \( u = 1/t \) the integral becomes

\[ = 2 \int_0^\infty \sum_{k=1}^\infty e^{-\pi k^2/u^2} u^2 \left( 1 + u + 2[1-u] \log u - \frac{3}{2} [1+u] \log^2 u \right)du \]

47
and we now consider that part of the integrand defined by

$$g(u) = u^2 \left( 1 + u + 2[1-u] \log u - \frac{3}{2}[1+u] \log^2 u \right)$$

where we note that $g(0) = 0$. We have the derivative

$$g'(u) = u^2 + 4u - 9u^2 \log u - 3u \log u - \frac{9}{2}u^2 \log^2 u$$

$$= u^2 + 4u - \frac{9}{2}u^2 \log u \left[ 2 + \log u \right] - 3u \log u$$

If $2 + \log u \geq 0$, then $g'(u) > 0$ because $\log u$ is negative in the interval $(0,1)$. Let us now consider the other possibility where $-\log u > 2$; therefore in the interval $[0,e^{-2}]$ we have the inequality

$$g'(u) > u^2 + 4u + 18u^2 + 6u - \frac{9}{2}u^2 \log^2 u$$

$$= uh(u)$$

where $h(u)$ is defined as

$$h(u) = 10 + 19u - \frac{9}{2}u \log^2 u$$

The smallest value of $h(u)$ arises when (i) $\frac{9}{2}u \log^2 u$ is at its maximum value and (ii) the other component, $10 + 19u$, is at its minimum value. We now consider the function $f(u)$ defined by

$$f(u) = \frac{9}{2}u \log^2 u$$

We note that $f(u) \geq 0$ and that $f(0) = f(1) = 0$ and we have

$$f'(u) = \frac{9}{2} \log u [2 + \log u]$$
It is easily seen that $f'(u) \geq 0$ in the interval $(0, e^{-2})$ and $f''(u) \leq 0$ in the interval $(e^{-2}, 1)$. Furthermore, $f'(u) = 0$ when $u = e^{-2}$. Therefore, the maximum value of $f(u)$ is attained when $u = e^{-2}$, i.e. the maximum value of $f(u)$ is $18e^{-2}$.

Accordingly, the smallest value of $h(u)$ is equal to $10 - 18e^{-2}$ (which is a positive number). Hence we have determined that $h(u) > 0$ and thereby deduce that $g'(u) > 0$. We have therefore proved that $g(u)$ is monotonic increasing and, since $g(0) = 0$, we see that $g(u) > 0$.

Hence, since the integrand is non-negative, we deduce that

\[(6.6) \quad \xi^{(2)}(1) > \xi^{(3)}(1)\]

This inequality is employed in Section 7 below.

Further work is required to determine whether similar inequalities exist for other combinations of $m$.

**7. The sigma constants $\sigma_n$**

It is well known that if

\[(7.1) \quad \log h(x) = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n} x^n\]

then we have in terms of the (exponential) complete Bell polynomials (see Section 10 of this paper)

\[(7.2) \quad h(x) = e^{b_0} \sum_{n=0}^{\infty} Y_n (b_1, 1!b_2, ..., (n-1)!b_n) \frac{x^n}{n!}\]

and

\[(7.3) \quad h^{\alpha}(x) = e^{\alpha b_0} \sum_{n=0}^{\infty} Y_n (\alpha b_1, 1!\alpha b_2, ..., (n-1)!\alpha b_n) \frac{x^n}{n!}\]

If we are only dealing with real values of $x$ then (7.1) implies that

\[(7.3.1) \quad h(x) > 0\]

Referring to (1.7.2)
\[ \log 2 \zeta(s) = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k \]

we then determine that

(7.4) \[ 2\zeta(s) = \sum_{n=0}^{\infty} Y_n \left(-\sigma_1, -1!\sigma_2, \ldots, -(n-1)!\sigma_n\right) \frac{s^n}{n!} \]

or equivalently

\[ 2\zeta(1-s) = \sum_{n=0}^{\infty} Y_n \left(-\sigma_1, -1!\sigma_2, \ldots, -(n-1)!\sigma_n\right) \frac{(1-s)^n}{n!} \]

Differentiation gives us

(7.5) \[ 2\zeta^{(n)}(0) = 2(-1)^n \zeta^{(n)}(1) = Y_n \left(-\sigma_1, -1!\sigma_2, \ldots, -(n-1)!\sigma_n\right) \]

Since \( \zeta^{(n)}(1) \) is positive, we see that \( Y_n \left(-\sigma_1, -1!\sigma_2, \ldots, -(n-1)!\sigma_n\right) \) has the same sign as \((-1)^n\).

With \( n = 1 \) we see that

(7.6) \[ 2\zeta^{(1)}(1) = -Y_1 \left(-\sigma_1\right) = \sigma_1 \]

Hence we see that \( \sigma_1 > 0 \). By calculation we find that \( 1 > \sigma_1 > 0 \) and hence we have

(7.7) \[ \sigma_1 > \sigma_1^2 \]

Referring to (1.8)

\[ \lambda_n = -\sum_{m=1}^{n} (-1)^{m} \binom{n}{m} \sigma_m \]

we see that

(7.8) \[ \lambda_1 = \sigma_1 > 0 \]

With \( n = 2 \) in (7.5) we obtain

\[ 2\zeta^{(2)}(1) = Y_2 \left(-\sigma_1, -1!\sigma_2\right) \]

and thus
This then tells us that
\[ \sigma_1^2 > \sigma_2 \]
We have from (1.8)
\[ \lambda_2 = 2\sigma_1 - \sigma_2 \]
\[ > 2\sigma_1 - \sigma_1^2 \]
and since \(-\sigma_1^2 > -\sigma_1\) we see that
\[ (7.10) \quad \lambda_2 > 2\sigma_1 - \sigma_1 = \sigma_1 \]
Hence we have
\[ (7.11) \quad \lambda_2 > \lambda_1 > 0 \]
We have seen in (6.2) that \( \xi^{(2)}(1) > 2\xi^{(1)}(1) \) and thus \( \xi^{(2)}(1) > \xi^{(1)}(1) \) implies that
\[ \sigma_1^2 - \sigma_2 > \sigma_1 \]
or equivalently
\[ \sigma_1(\sigma_1 - 1) - \sigma_2 > 0 \]
Since the first term on the left-hand side is negative, we deduce that \( \sigma_2 \) must be negative.

Furthermore, since \( \lambda_2 = 2\sigma_1 - \sigma_2 > 0 \) we determine that
\[ 2\sigma_1 > \sigma_2 \]
but this does not really provide any more useful information (because \( \sigma_1 \) and \( \sigma_2 \) are of opposite signs).

We now consider the case where \( n = 3 \). We have from (7.5)
\[ 2\xi^{(3)}(1) = -Y_3(-\sigma_1,-1!\sigma_2,-2!\sigma_3) \]
and since
\[ Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2x_3 \]
we obtain
\[ (7.12) \quad 2\xi^{(3)}(1) = \sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3 \]
Since \( \xi^{(3)}(1) \) is positive we have
\[ (7.13) \quad \sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3 > 0 \]
and (1.8) gives us
\[ \lambda_1 = 3\sigma_1 - 3\sigma_2 + \sigma_3 = 3(\sigma_1 - \sigma_2) + \sigma_3 \]
Then using (7.13) we see that
\[ \lambda_3 > 3\sigma_1 - 3\sigma_2 + \frac{3}{2}\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^3 \]
\[ = 3\sigma_1 + 3\sigma_2 \left( \frac{1}{2}\sigma_1 - 1 \right) - \frac{1}{2}\sigma_1^3 \]
\[ > \frac{5}{2}\sigma_1 + 3\sigma_2 \left( \frac{1}{2}\sigma_1 - 1 \right) \]
Therefore, since the second term on the right-hand side is also positive, we conclude that
\[ (7.14) \quad \lambda_3 > 0 \]
Referring back to
\[ \lambda_3 > \frac{5}{2}\sigma_1 + 3\sigma_2 \left( \frac{1}{2}\sigma_1 - 1 \right) \]
\[ = \frac{5}{2}\sigma_1 - \sigma_2 + \frac{3}{2}\sigma_2 \left( \sigma_1 - \frac{4}{3} \right) \]
\[ > 2\sigma_1 - \sigma_2 + \frac{3}{2}\sigma_2 \left( \sigma_1 - \frac{4}{3} \right) \]
Hence, since \( \lambda_2 = 2\sigma_1 - \sigma_2 \), we determine that

\[(7.15) \quad \lambda_3 > \lambda_2 \]

With reference to (1.8) we have the binomial transform

\[\sigma_n = -\sum_{m=1}^{n} (-1)^m \binom{n}{m} \lambda_m\]

and in particular we have

\[\sigma_2 = 2\lambda_1 - \lambda_2\]

Since \( \sigma_2 \) is negative, this implies that

\[(7.16) \quad \lambda_2 > 2\lambda_1 \]

Similarly we have

\[\sigma_3 = 3\lambda_1 - 3\lambda_2 + \lambda_3\]

Using (6.6) \( \xi^{(2)}(1) > \xi^{(3)}(1) \) we have

\[\sigma_1^2 - \sigma_2 > \sigma_3^3 - 3\sigma_1 \sigma_2 + 2\sigma_3\]

Equation (6.5)

\[\xi^{(2m+2)}(1) > \xi^{(2m+1)}(1) \text{ for } m \geq 1\]

may also be useful in investigating the properties of the higher orders of \( \lambda_n \), albeit the algebra will become increasingly tedious.

\[\square\]

Reference to (7.3) and (7.4) shows that

\[\left[2\xi(s)\right]^s = \sum_{n=0}^{\infty} Y_n \left((-j\sigma_1, -1! j\sigma_2, \ldots, -(n-1)! j\sigma_n), \frac{s^n}{n!}\right)\]

and using (1.3) \( \xi(s) = \xi(1-s) \) this becomes
\[
= \sum_{n=0}^{\infty} Y_n \left(-j\sigma_1, -1! j\sigma_2, ..., -(n-1)! j\sigma_n\right) \frac{(1-s)^n}{n!}
\]

The binomial theorem gives us

\[
\left[2\xi(s)-1\right]^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left[2\xi(s)\right]^j
\]

\[
= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{n=0}^{\infty} Y_n \left(-j\sigma_1, -1! j\sigma_2, ..., -(n-1)! j\sigma_n\right) \frac{(1-s)^n}{n!}
\]

We see that

\[
\frac{d^m}{ds^m} \left[2\xi(s)-1\right]^k
\]

\[
= (-1)^m \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{n=0}^{\infty} Y_n \left(-j\sigma_1, -1! j\sigma_2, ..., -(n-1)! j\sigma_n\right) n(n-1)\ldots(n-m+1) \frac{(1-s)^{n-m}}{n!}
\]

and thus

\[
\frac{d^m}{ds^m} \left[2\xi(s)-1\right]^k \bigg|_{s=1} = (-1)^m \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} Y_m \left(-j\sigma_1, -1! j\sigma_2, ..., -(m-1)! j\sigma_m\right)
\]

We recall (5.2)

\[
\lambda_n = -\sum_{m=1}^{\infty} \binom{n}{m} \frac{1}{(m-1)!} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{d^m}{ds^m} \left[2\xi(s)-1\right]^k \bigg|_{s=1}
\]

which gives us

\[
(7.17) \quad \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^{n+1}}{(m-1)!} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{j=0}^{k} \binom{k}{j} Y_m \left(-j\sigma_1, -1! j\sigma_2, ..., -(m-1)! j\sigma_m\right)
\]

For \( n = 1 \) we have

\[
\lambda_1 = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j Y_1 \left(-j\sigma_1\right)
\]

\[
= -\sigma_1 \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j
\]
where we used the identity

\[ \sum_{j=0}^{k} \binom{k}{j} (-1)^j j = -k\delta_{1,k} \]

This is easily derived using the binomial theorem

\[ (1-x)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j x^j \]

whereby differentiation results in

\[ -k(1-x)^{k-1} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j j x^{j-1} \]

and the result follows by letting \( x = 1 \).

Hence we obtain

\[ \lambda_1 = \sigma_1 \]

For \( n = 2 \) we have

\[ \lambda_2 = \sum_{m=1}^{2} \binom{2}{m} (-1)^{m+1} \sum_{k=1}^{m} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j Y_m (-j\sigma_1, -1! j\sigma_2, \ldots, -(m-1)! j\sigma_m) \]

\[ = 2\sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j Y_1 (-j\sigma_1) - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j Y_2 (-j\sigma_1, -1! j\sigma_2) \]

\[ = 2\sigma_1 - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j^2 \sigma_1^2 - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j \]

\[ = 2\sigma_1 - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j^2 + \sigma_1 \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j \]

\[ = 2\sigma_1 - \sigma_2 \]

We multiply (7.18) by \( x \) and differentiate to obtain
\[ k(k-1)x(1-x)^{k-2} - k(1-x)^{k-1} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \, j^2 \, x^{j-1} \]

and we see that

\[
\sum_{j=0}^{k} \binom{k}{j} (-1)^j j^2 = k(k-1)\delta_{2,k} - k\delta_{1,k}
\]

Therefore we have

\[
\sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j^2 = \sum_{k=1}^{\infty} \frac{1}{k} [k(k-1)\delta_{2,k} - k\delta_{1,k}] = 0
\]

and hence we obtain

\[ \lambda_2 = 2\sigma_1 - \sigma_2 \]

It is clear that the calculations may be extended to higher orders of \( \lambda_n \); however I suspect that this will simply lead to (1.8)

\[ \lambda_n = -\sum_{j=1}^{n} (-1)^j \binom{n}{j}\sigma_j \]

We may also note from [21] that the Bernoulli polynomials are given by

\[ B_p(u) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (u+k)^p \]

and hence the Bernoulli numbers are given by

\[ B_p = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^p \]

I initially thought that these numbers would feature in the above analysis but, as we shall see below, this is not the case.

\[ \square \]

We now refer to (10.18)

\[ Y_m(\alpha x_1, \ldots, \alpha x_m) = \sum_{i=1}^{m} \alpha^i B_{m,i}(x_1, \ldots, x_{m-i+1}) \]
which shows that we may write (7.17) as

\[ \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^{m+1}}{(m-1)!} \sum_{k=1}^{m} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{l=1}^{m} j^l B_{m,l} (-\sigma_1, -1! \sigma_2, ..., -(m-1)! \sigma_{m-1}) \]

We see that

\[ \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{l=1}^{m} j^l B_{m,l} = \sum_{l=1}^{m} B_{m,l} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \]

and hence we have

\[ \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^{m+1}}{(m-1)!} \sum_{l=1}^{m} B_{m,l} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \]

We have the well known representation for the Stirling numbers of the second kind [8, p.289]

\[ (-1)^j S(l,k) k! = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \]

and thus

\[ \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^{m+1}}{(m-1)!} \sum_{l=1}^{m} B_{m,l} (-1)^l \sum_{k=1}^{\infty} \frac{1}{k} S(l,k) k! \]

Since \( S(l,k) = 0 \) for \( k \geq l+1 \) we have

\[ (7.19) \quad \lambda_n = \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^{m+1}}{(m-1)!} \sum_{l=1}^{m} B_{m,l} (-1)^l \sum_{k=1}^{l} (k-1)! S(l,k) \]

and, for example, this gives us

\[ \lambda_1 = -B_{1,1} S(1,1) = -B_{1,1} = \sigma_1 \]

We note the recurrence relation [8, p.448]

\[ Y_n(x_1 + y_1, ..., x_n + y_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(x_1, ..., x_{n-k}) Y_k(y_1, ..., y_k) \]

which shows us that
\[
Y_n(-2x_1,\ldots,-2x_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(-x_1,\ldots,-x_{n-k}) Y_k(-x_1,\ldots,-y_k)
\]

and this may be generalised to the Bell polynomials with the following arguments
\[
Y_m(-j\sigma_1, -1!j\sigma_2,\ldots, -(m-1)!j\sigma_m)
\]

8. Further applications of the Bell polynomials

We recall (1.12)
\[
\log[(s-1)\zeta(s)] = -\sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} (s-1)^k
\]

and with \( s \to s \) this becomes
\[
\log[s\zeta(s+1)] = -\sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} s^k
\]

Applying (7.2) gives us
\[
s\zeta(s+1) = \sum_{n=0}^{\infty} Y_n \left(-\eta_0, -1!\eta_1,\ldots, -(n-1)!\eta_{n-1}\right) \frac{s^n}{n!}
\]

and with \( s \to s \) we obtain
\[
(s-1)\zeta(s) = \sum_{n=0}^{\infty} Y_n \left(-\eta_0, -1!\eta_1,\ldots, -(n-1)!\eta_{n-1}\right) \frac{(s-1)^n}{n!}
\]

so that
\[
\frac{d^m}{ds^m}[(s-1)\zeta(s)] \bigg|_{s=1} = Y_m \left(-\eta_0, -1!\eta_1,\ldots, -(m-1)!\eta_{m-1}\right)
\]

Using (1.19) we see that
\[
\frac{d^m}{ds^m}[(s-1)\zeta(s)] \bigg|_{s=1} = (-1)^{m-1} m\gamma_{m-1}
\]

and hence we have
(8.1) \((-1)^{m-1} m\gamma_{m-1} = Y_m \left(-\eta_0, -1!\eta_1,\ldots, -(m-1)!\eta_{m-1}\right)\)
which we originally reported in [17].

Referring to (3.2), i.e. $\eta_k = (-1)^{k+1}\varepsilon_k$ we obtain

$$(-1)^{m-1}m^{m-1}Y_m \left(\varepsilon_0, -1!\varepsilon_1, \ldots, (-1)^{m-1}(m-1)!\varepsilon_{m-1}\right)$$

and using (10.6) this becomes

$$= (-1)^{m}Y_m \left(-\varepsilon_0, -1!\varepsilon_1, \ldots, -(m-1)!\varepsilon_{m-1}\right)$$

However, no discernible sign pattern for $\gamma_{m-1}$ emerges from this representation.

We also see that

$$\log \frac{1}{(s-1)\xi(s)} = \sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} (s-1)^k$$

so that

$$\log \frac{1}{s\xi(s+1)} = \sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} s^k$$

Therefore applying (7.2) gives us

$$\frac{1}{s\xi(s+1)} = \sum_{n=0}^{\infty} \frac{Y_n \left(\eta_0, 1!\eta_1, \ldots, (n-1)!\eta_{n-1}\right)}{n!} \frac{s^n}{n!}$$

so that

$$\frac{1}{(s-1)\xi(s)} = \sum_{n=0}^{\infty} \frac{Y_n \left(\eta_0, 1!\eta_1, \ldots, (n-1)!\eta_{n-1}\right)}{n!} \frac{(s-1)^n}{n!}$$

We then obtain

$$\left. \frac{d^m}{ds^m} \frac{1}{(s-1)\xi(s)} \right|_{s=1} = Y_m \left(\eta_0, 1!\eta_1, \ldots, (m-1)!\eta_{m-1}\right)$$

Employing (3.2) we see that

$$\left. \frac{d^m}{ds^m} \frac{1}{(s-1)\xi(s)} \right|_{s=1} = Y_m \left(-\varepsilon_0, 1!\varepsilon_1, \ldots, (-1)^{m-1}(m-1)!\varepsilon_{m-1}\right)$$
\[ = (-1)^m Y_m (\varepsilon_0, 1! \varepsilon_1, \ldots, (m-1)! \varepsilon_{m-1}) \]

and hence we see that
\[
\frac{d^m}{ds^m} \left. \frac{1}{(s-1)\zeta(s)} \right|_{s=1} = (-1)^m d_m \text{ where } d_m > 0
\]

Similarly, referring to (2.3.1)
\[
\log \left( \frac{1}{2(s-1)\zeta(s)} \right) = -\sum_{n=1}^{\infty} \frac{b_{n-1} n^s}{n}
\]

and (7.2) gives us
\[
\frac{1}{2(s-1)\zeta(s)} = \sum_{n=0}^{\infty} Y_n \left( -b_0, -1! b_1, \ldots, -(n-1)! b_{n-1} \right) \frac{s^n}{n!}
\]

We then see that
\[
\frac{d^m}{ds^m} \left. \frac{1}{2(s-1)\zeta(s)} \right|_{s=0} = Y_m \left( -b_0, -1! b_1, \ldots, -(m-1)! b_{m-1} \right)
\]

and employing (2.6) this becomes
\[
= Y_m \left( -\mu_0, 1! \mu_1, \ldots, (-1)^m (m-1)! \mu_{m-1} \right)
\]

Using (10.5) we obtain
\[
\frac{d^m}{ds^m} \left. \frac{1}{2(s-1)\zeta(s)} \right|_{s=0} = (-1)^m Y_{-m} \left( \mu_0, 1! \mu_1, \ldots, (m-1)! \mu_{m-1} \right)
\]

which has the same sign as \((-1)^m\).

We also see that
\[
\frac{d}{ds} \frac{1}{f(s)} = -\frac{f'(s)}{f^2(s)} = -\frac{1}{f(s)} \frac{d}{ds} \log f(s)
\]

and applying (10.10) we obtain
\[
\frac{d^m}{ds^m} \frac{1}{f(s)} = \frac{1}{f(s)} Y_m \left( g(s), g^{(1)}(s), \ldots, g^{(m-1)}(s) \right)
\]
9. The $S_2(n)$ constants

We recall (1.16.2)

$$S_2(n) = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=1}$$

and we have

$$\frac{d^m}{ds^m} \log[(s-1)\zeta(s)] = \frac{d^{m-1}}{ds^{m-1}} \frac{f'(s)}{f(s)}$$

where $f(s) = (s-1)\zeta(s)$. The Leibniz rule for differentiation gives us

$$\frac{d^{m-1}}{ds^{m-1}} \frac{f'(s)}{f(s)} = \frac{1}{f(s)} \sum_{j=0}^{m-1} \binom{m-1}{j} f^{(m-j)}(s) \frac{d^j}{ds^j} \frac{1}{f(s)}$$

so that

$$\frac{d^m}{ds^m} \log[(s-1)\zeta(s)] \bigg|_{s=1} = \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^{m-j} (m-j)\gamma_{m-j-1} Y_{j}\left(\eta_0,1!\eta_1,\ldots,(j-1)!\eta_{j-1}\right)$$

We have

$$\frac{d^{n+1}}{ds^{n+1}}[(s-1)\zeta(s)] \bigg|_{s=1} = (-1)^n (n+1)\gamma_n$$

and

$$\frac{d^{m-j}}{ds^{m-j}}[(s-1)\zeta(s)] \bigg|_{s=1} = (-1)^{m-j-1} (m-j)\gamma_{m-j-1}$$

and we obtain

$$S_2(n) = \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^m}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^{j+1} (m-j)\gamma_{m-j-1} Y_{j}\left(\eta_0,1!\eta_1,\ldots,(j-1)!\eta_{j-1}\right)$$

Using (8.1) we get

$$(-1)^{m-j-1} (m-j)\gamma_{m-j-1} = Y_{m-j}\left(-\eta_0,-1!\eta_1,\ldots,-(m-j-1)!\eta_{m-j-1}\right)$$
and thus

\[
S_2(n) = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} Y_{m-j}(-\eta_0, -1! \eta_1, \ldots, -(m-j-1)! \eta_{m-j-1}) Y_j(\eta_0, 1! \eta_1, \ldots, (j-1)! \eta_{j-1})
\]

To simplify the notation we write

\[
Y_j = Y_j(\eta_0, 1! \eta_1, \ldots, (j-1)! \eta_{j-1})
\]

\[
Y_j^- = Y_j(-\eta_0, -1! \eta_1, \ldots, -(j-1)! \eta_{j-1})
\]

so that

\[
S_2(n) = \sum_{m=1}^{n} \binom{n}{m} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} Y_{m-j}^- Y_j
\]

For example, we see that

\[
S_2(1) = Y_0^- Y_1 = \eta_0
\]

\[
S_2(2) = 2Y_1^- Y_0^- + Y_2^- Y_0 + Y_1^- Y_1
\]

\[
= -2\eta_0 + \eta_0^2 - \eta_1 - \eta_0^2
\]

\[
= -2\eta_0 - \eta_1
\]

It should however be noted that this just adds complexity to the existing problem because we already have the simpler expression (1.17)

\[
S_2(n) = -\sum_{m=1}^{n} \binom{n}{m} \eta_{m-1}
\]

10. Some aspects of the (exponential) complete Bell polynomials

For ease of reference, some aspects of the (exponential) complete Bell polynomials are set out below; these may be useful in the earlier sections of this paper.

As noted, for example, by Köhlig [26] we have

\[
(10.1) \quad \frac{d^r}{dx^r} e^{f(x)} = e^{f(x)} Y_r\left(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(r)}(x)\right)
\]
where the (exponential) complete Bell polynomials may be defined by \( Y_0 = 1 \) and for \( r \geq 1 \)

\[
Y_r(x_1,\ldots,x_r) = \sum_{\pi(r)} \frac{r!}{k_1! k_2! \cdots k_r!} \left( \frac{x_1}{1!} \right)^{k_1} \left( \frac{x_2}{2!} \right)^{k_2} \cdots \left( \frac{x_r}{r!} \right)^{k_r}
\]

where the sum is taken over all partitions \( \pi(r) \) of \( r \), i.e. over all sets of integers \( k_j \) such that

\[ k_1 + 2k_2 + 3k_3 + \cdots + rk_r = r \]

The complete Bell polynomials have integer coefficients and the first five are set out below (Comtet [14, p.307])

\[
Y_1(x_1) = x_1
\]

\[
Y_2(x_1, x_2) = x_1^2 + x_2
\]

\[
Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3
\]

\[
Y_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4
\]

\[
Y_5(x_1, x_2, x_3, x_4, x_5) = x_1^5 + 10x_1^3x_2 + 10x_1^2x_3 + 15x_1x_2^2 + 5x_1x_4 + 10x_2x_3 + x_5
\]

The definition (10.2) immediately implies the following relation

\[
Y_n(a_1, a_2^2, \ldots, a^n x_n) = a^n Y_n(x_1,\ldots,x_n)
\]

and with \( a = 1 \) we have

\[
Y_n(-x_1, x_2,\ldots, (-1)^nx_n) = (-1)^n Y_n(x_1,\ldots,x_n)
\]

Hence if all of the \( x_j \) are positive numbers, we see that \( Y_n(-x_1, x_2,\ldots, (-1)^nx_n) \) has the same sign as \( (-1)^n \).

By letting \( x_j \to -x_j \) we also note that

\[
Y_n(x_1, -x_2,\ldots, (-1)^{n+1} x_n) = (-1)^n Y_n(-x_1,\ldots,-x_n)
\]

but no discernible sign pattern emerges here. We may note that
\[ Y_n \left( b_1, -1! b_2, \ldots, (-1)^{n+1} (n-1)! b_n \right) = (-1)^n Y_n \left( -b_1, -1! b_2, \ldots, -(n-1)! b_n \right) \]
\[ = (-1)^n e^{-b_0} \frac{d^n}{dx^n} \left. \frac{1}{h(x)} \right|_{x=0} \]

where \( h(x) \) is defined by (7.1).

The complete Bell polynomials are also given by the exponential generating function (Comtet [14, p.134])

\[ \exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} \]  

Since the exponential function is positive for real arguments, we see that the summation on the right-hand side must also be positive.

Using (10.1) we see that

\[ \frac{d^n}{dt^n} \exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) \bigg|_{t=0} = Y_n(x_1, \ldots, x_n) \]

and hence we note that (10.7) is simply the corresponding Maclaurin series.

We note that

\[ \sum_{n=0}^{\infty} Y_n(ax_1, \ldots, ax_n) \frac{t^n}{n!} = \exp \left( \sum_{j=1}^{\infty} ax_j \frac{t^j}{j!} \right) = \exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) \exp a \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = \left[ \exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) \right]^a \]

and thus we have

\[ \left[ \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} \right]^a = \sum_{n=0}^{\infty} Y_n(ax_1, \ldots, ax_n) \frac{t^n}{n!} \]

Let us now consider a function \( f(t) \) which has a Taylor series expansion around \( x \): we have

\[ e^{f(x+t)} = \exp \left( \sum_{j=0}^{\infty} f^{(j)}(x) \frac{t^j}{j!} \right) = e^{f(x)} \exp \left( \sum_{j=1}^{\infty} f^{(j)}(x) \frac{t^j}{j!} \right) \]
\[ = e^{f(x)} \left[ 1 + \sum_{n=1}^{\infty} Y_n \left( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x) \right) \frac{t^n}{n!} \right] \]
We see that
\[
\frac{d^m}{dx^m} e^{f(x)} = \frac{\partial^m}{\partial x^m} e^{f(x)} \bigg|_{x=t} = \frac{\partial^m}{\partial t^m} e^{f(x)} \bigg|_{x=0}
\]
and we therefore obtain a derivation of (10.1) above
\[
\frac{d^r}{dx^r} e^{f(x)} = e^{f(x)} Y_r \left( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(r)}(x) \right)
\]
Suppose that \( h'(x) = h(x)g(x) \) and let \( f(x) = \log h(x) \). We see that
\[
f'(x) = \frac{h'(x)}{h(x)} = g(x)
\]
and then using (10.1) above we have
\[
(10.10) \quad \frac{d^r}{dx^r} h(x) = \frac{d^r}{dx^r} e^{\log h(x)} = h(x) Y_r \left( g(x), g^{(1)}(x), \ldots, g^{(r-1)}(x) \right)
\]
In particular we have
\[
\frac{d^r}{dx^r} h(x) \bigg|_{x=0} = h(0) Y_r \left( g(0), g^{(1)}(0), \ldots, g^{(r-1)}(0) \right)
\]
Let us consider the function \( h(x) \) with the following Maclaurin expansion
\[
(10.11) \quad \log h(x) = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n} x^n
\]
and we wish to determine the coefficients \( a_n \) such that
\[
(10.12) \quad h(x) = \sum_{r=0}^{\infty} a_r x^r
\]
By differentiating (10.11) we obtain
\[
h'(x) = h(x) \sum_{n=1}^{\infty} b_n x^{n-1} = h(x)g(x)
\]
From (10.11) we have
\[
\frac{d^r}{dx^r} h(x) = \frac{d^r}{dx^r} e^{\log h(x)} = h(x) Y_r \left( g(x), g^{(1)}(x), \ldots, g^{(r-1)}(x) \right)
\]

and in particular we have

\[
\left. \frac{d^r}{dx^r} h(x) \right|_{x=0} = h(0) Y_r \left( g(0), g^{(1)}(0), \ldots, g^{(r-1)}(0) \right)
\]

Using (10.12) the Maclaurin series gives us

\[
a_r = \frac{1}{r!} \left. \frac{d^r}{dx^r} h(x) \right|_{x=0}
\]

\[
a_r = \frac{1}{r!} h(0) Y_r \left( g(0), g^{(1)}(0), \ldots, g^{(r-1)}(0) \right)
\]

We have

\[
g^{(j)}(x) = \sum_{n=1}^{\infty} b_n (n-1)(n-2) \cdots (n-j) x^{n-j}
\]

and thus

\[
g^{(j)}(0) = j! b_{j+1}
\]

Therefore we obtain

\[
a_r = \frac{1}{r!} e^{b_r} Y_r \left( g(0), g^{(1)}(0), \ldots, g^{(r-1)}(0) \right)
\]

\[
= \frac{1}{r!} e^{b_r} Y_r \left( b_1, b_2, \ldots, (r-1)! b_r \right)
\]

Since \( \log h(0) = \log a_0 = b_0 \) we have

\[
(10.13) \quad h(x) = e^{b_0} \sum_{n=0}^{\infty} Y_n \left( b_1, b_2, \ldots, (n-1)! b_n \right) \frac{x^n}{n!}
\]

Then referring to (10.7)

\[
\exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = \sum_{n=0}^{\infty} Y_n (x_1, \ldots, x_n) \frac{t^n}{n!}
\]

we see that
\[ h(x) = e^{bh} \exp \left( \sum_{j=1}^{\infty} \frac{b_i}{j} t^j \right) \]

and this is where we started from in (10.12).

\[ \log h(x) = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n} x^n \]

Multiplying (10.12) by \( \alpha \) it is easily seen that

\[ h^n(x) = e^{\alpha b} \sum_{n=0}^{\infty} Y_n \left( \alpha b_1, \alpha b_2, \ldots, (n-1)\alpha b_n \right) \frac{x^n}{n!} \]

and, in particular, with \( \alpha = -1 \) we obtain

\[ \frac{1}{h(x)} = e^{-b} \sum_{n=0}^{\infty} Y_n \left( -b_1, -1! b_2, \ldots, -(n-1)! b_n \right) \frac{x^n}{n!} \]

Differentiating (10.14) with respect to \( \alpha \) would give us an expression for \( h^n(x) \log h(x) \).

We have the recurrence relation [8, p.415]

\[ Y_{r+1}(x_1, \ldots, x_{r+1}) = \sum_{k=0}^{r} \binom{r}{k} Y_{r-k}(x_1, \ldots, x_{r-k}) x_{k+1} \]

Using this we find that

\[ (r+1)! e^{-b} a_{r+1} = \sum_{k=0}^{r} \binom{r}{k} (r-k)! e^{-b} a_{r-k} k! b_{k+1} \]

giving us the recurrence relation

\[ (r+1)a_{r+1} = \sum_{k=0}^{r} a_{r-k} b_{k+1} \]

or equivalently

\[ ra_r = \sum_{m=1}^{r} a_{r-m} b_m \]

Suppose that \( h'(x) = h(x) \frac{g(x)}{x} \) and let \( f(x) = \log h(x) \). We see that
\[ f'(x) = \frac{h'(x)}{h(x)} = g(x) \]

and then using (10.1) above we have

\[ \frac{d^r}{dx^r} h(x) = \frac{d^r}{dx^r} e^{\log h(x)} = h(x)Y_r \left( g(x), g^{(1)}(x), \ldots, g^{(r-1)}(x) \right) \]

Hence we have the Maclaurin expansion

\[ h(x) = h(0) \sum_{r=0}^{\infty} Y_r \left( g(0), g^{(1)}(0), \ldots, g^{(r-1)}(0) \right) \frac{x^r}{r!} \]

Reference to (10.7) gives us

\[ h(x) = h(0) \exp \left( \sum_{j=1}^{\infty} g^{(j-1)}(0) \frac{t^j}{j!} \right) \]

and hence we have

\[ \log h(x) = \log h(0) + \sum_{j=1}^{\infty} g^{(j-1)}(0) \frac{t^j}{j!} \]

Therefore, as expected, we obtain

\[ f(x) = f(0) + \sum_{j=1}^{\infty} f^{(j)}(0) \frac{t^j}{j!} \]

\[ \square \]

We refer to (7.3)

\[ h^a(x) = e^{ab_0} \sum_{n=0}^{\infty} Y_n(\alpha b_1, 1! \alpha b_2, \ldots, (n-1)! \alpha b_n) \frac{x^n}{n!} \]

which may be written as

\[ h^a(x) = e^{ab_0} + e^{ab_0} \sum_{n=1}^{\infty} Y_n(\alpha b_1, 1! \alpha b_2, \ldots, (n-1)! \alpha b_n) \frac{x^n}{n!} \]

We have [8, p.412]
\[ Y_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k} (x_1, \ldots, x_{n-k+1}) \]

so that
\[ Y_n(\alpha x_1, \ldots, \alpha x_n) = \sum_{k=1}^{n} B_{n,k} (\alpha x_1, \ldots, \alpha x_{n-k+1}) \]

We also have [8, p.412]
\[ B_{n,k} (\alpha x_1, \ldots, \alpha x_{n-k+1}) = \alpha^k B_{n,k} (x_1, \ldots, x_{n-k+1}) \]

and hence we see that
\[ (10.18) \quad Y_n(\alpha x_1, \ldots, \alpha x_n) = \sum_{k=1}^{n} \alpha^k B_{n,k} (x_1, \ldots, x_{n-k+1}) \]

Therefore we obtain
\[ (10.19) \quad h''(x) = e^{\alpha x_b} + e^{\alpha x_b} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \alpha^k B_{n,k} (b_1, 1! b_2, \ldots, (n-k)! b_{n-k+1}) \frac{x^n}{n!} \]

Differentiating (10.19) with respect to \( \alpha \) gives us
\[ h''(x) \log h(x) = b_0 e^{\alpha x_b} + b_0 e^{\alpha x_b} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \alpha^k B_{n,k} (b_1, 1! b_2, \ldots, (n-k)! b_{n-k+1}) \frac{x^n}{n!} \]
\[ + e^{\alpha x_b} \sum_{n=1}^{\infty} \sum_{k=1}^{n} k \alpha^{k-1} B_{n,k} (b_1, 1! b_2, \ldots, (n-k)! b_{n-k+1}) \frac{x^n}{n!} \]

and letting \( \alpha = 0 \) results in
\[ \log h(x) = b_0 + \sum_{n=1}^{\infty} B_{n,1} (b_1, 1! b_2, \ldots, (n-1)! b_n) \frac{x^n}{n!} \]

Comparing this with (7.1) shows that
\[ B_{n,1} (b_1, 1! b_2, \ldots, (n-1)! b_n) = (n-1)! b_{n-1} \]

A further differentiation gives us
\[ h^\alpha (x) \log^2 h(x) = b_0^2 e^{\alpha h_0} + b_0^2 e^{\alpha h_0} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \alpha^k B_{n,k} \left( b_1, 1! b_2, \ldots, (n-k)! b_{n-k+1} \right) \frac{x^n}{n!} \]

\[ + 2 b_0 e^{\alpha h_0} \sum_{n=1}^{\infty} \sum_{k=1}^{n} k \alpha^{k-1} B_{n,k} \left( b_1, 1! b_2, \ldots, (n-k)! b_{n-k+1} \right) \frac{x^n}{n!} \]

\[ + e^{\alpha h_0} \sum_{n=1}^{\infty} \sum_{k=1}^{n} (k-1) \alpha^{k-2} B_{n,k} \left( b_1, 1! b_2, \ldots, (n-k)! b_{n-k+1} \right) \frac{x^n}{n!} \]

and letting \( \alpha = 0 \) results in

\[ \log^2 h(x) = b_0^2 + 2 b_0 \sum_{n=1}^{\infty} B_{n,1} \left( b_1, 1! b_2, \ldots, (n-1)! b_{n} \right) \frac{x^n}{n!} + 2 \sum_{n=2}^{\infty} B_{n,2} \left( b_1, 1! b_2, \ldots, (n-2)! b_{n-1} \right) \frac{x^n}{n!} \]

or, since \( B_{1,2} = 0 \), we have

\[ (10.20) \quad \log^2 h(x) = b_0^2 + 2 b_0 \left[ \log h(x) - b_0 \right] + 2 \sum_{n=2}^{\infty} B_{n,2} \left( b_1, 1! b_2, \ldots, (n-2)! b_{n-1} \right) \frac{x^n}{n!} \]

Differentiating (10.18) gives us

\[ \frac{\partial^m}{\partial \alpha^m} Y_n(\alpha x_1, \ldots, \alpha x_n) = \sum_{k=1}^{n} k (k-1) \ldots (k-m+1) \alpha^{k-m} B_{n,k} (x_1, \ldots, x_{n-k+1}) \]

and we obtain

\[ \frac{\partial^m}{\partial \alpha^m} Y_n(\alpha x_1, \ldots, \alpha x_n) \bigg|_{\alpha=0} = m! B_{n,m} (x_1, \ldots, x_{n-m+1}) \]

Further applications of the (exponential) complete Bell polynomials are contained in [18]. For example, it is shown in [18] how may they be employed to prove that \( \Gamma^{(n)}(x) \) has the same sign as \((-1)^n\) for all \( x \) in the interval \((0, \alpha)\) where \( \alpha > 0 \) is the unique solution of \( \psi(\alpha) = 0 \).

REFERENCES

[1] V.S. Adamchik, On the Hurwitz function for rational arguments. Applied Mathematics and Computation, 187 (2007) 3-12.

[2] T.M. Apostol, Formulas for Higher Derivatives of the Riemann Zeta Function.
Math. of Comp., 169, 223-232, 1985.

[3] T.M. Apostol, An Elementary View of Euler’s Summation Formula. Amer. Math. Monthly, 106, 409-418, 1999.

[4] B.C. Berndt, Chapter eight of Ramanujan’s Second Notebook. J. Reine Agnew. Math, Vol. 338, 1-55, 1983. http://www.digizeitschriften.de/no_cache/home/open-access/nach-zeitschriftentiteln/

[5] B.C. Berndt, Ramanujan’s Notebooks. Parts I-III, Springer-Verlag, 1985-1991.

[6] E. Bombieri and J.C. Lagarias, Complements to Li’s criterion for the Riemann hypothesis. J. Number Theory 77, 274-287 (1999). [PostScript]  [Pdf]

[7] G. Boros and V.H. Moll, Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals. Cambridge University Press, 2004.

[8] C.A. Charalambides, Enumerative Combinatorics. Chapman & Hall/CRC, 2002.

[9] W.-S. Chou, Leetsch C. Hsu and Peter J.-S. Shiue, Application of Faà di Bruno’s formula in characterization of inverse relations. Journal of Computational and Applied Mathematics, 190 (2006), 151–169.

[10] B.K. Choudhury, The Riemann zeta function and its derivatives. Proc. Roy. Soc. Lond. A (1995) 450, 477-499.

[11] M.W. Coffey, Relations and positivity results for the derivatives of the Riemann \( \xi \) function. J. Comput. Appl. Math., 166, 525-534 (2004).

[12] M.W. Coffey, Toward verification of the Riemann Hypothesis: Application of the Li criterion. Math. Phys. Analysis and Geometry, 8, 211-255, 2005. arXiv:math-ph/0505052 [ps, pdf, other], 2005.

[13] M.W. Coffey, New results concerning power series expansions of the Riemann xi function and the Li/Keiper constants. Proc. R. Soc. A, (2008) 464, 711-731.

[14] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, 1974.

[15] D.F. Connon, Some series and integrals involving the Riemann zeta function, binomial coefficients and the harmonic numbers. Volume II(a), 2007. arXiv:0710.4023 [pdf]

[16] D.F. Connon, Some series and integrals involving the Riemann zeta function, binomial coefficients and the harmonic numbers. Volume II(b), 2007. arXiv:0710.4024 [pdf]
[17] D.F. Connon, A recurrence relation for the Li/Keiper constants in terms of the Stieltjes constants. arXiv:0902.1691 [pdf], 2009.

[18] D.F. Connon, Various applications of the (exponential) complete Bell polynomials. arXiv:1001.2835 [pdf], 2010.

[19] H.M. Edwards, Riemann’s Zeta Function. Academic Press, New York and London, 1974.

[20] P. Freitas, A Li-type criterion for zero-free half-planes of Riemann's zeta function. arXiv:math/0507368 [ps, pdf, other], 2005.

[21] J. Guillera and J. Sondow, Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent. 2005. math.NT/0506319 [abs, ps, pdf, other]

[22] G.H. Hardy, Divergent Series. Chelsea Publishing Company, New York, 1991.

[23] A. Ivić, The Riemann Zeta- Function: Theory and Applications. Dover Publications Inc, 2003.

[24] W.P. Johnson, The Curious History of Faà di Bruno’s Formula. Amer. Math. Monthly 109, 217-234, 2002. psu.edu [PDF]

[25] J.B. Keiper, power series expansions of Riemann’s \( \xi \) function. Math. Comp. 58, 765-773 (1992).

[26] K.S. Kölblig, The complete Bell polynomials for certain arguments in terms of Stirling numbers of the first kind. J. Comput. Appl. Math. 51 (1994) 113-116. Also available electronically at: A relation between the Bell polynomials at certain arguments and a Pochhammer symbol. CERN/Computing and Networks Division, CN/93/2, 1993 http://cdsweb.cern.ch/

[24] J.C. Lagarias, On a positivity property of the Riemann xi function. Acta Arithmetica, 89 (1999), No. 3, 217-234. [PostScript] [Pdf]

[28] D.H. Lehmer, the sum of like powers of the zeros of the Riemann zeta function. Math. Comp., 50, 265-273, 1988.

[29] X.-J. Li, The positivity of a sequence of numbers and the Riemann Hypothesis. J. Number Theory, 65, 325-333, 1997.

[30] K. Maślanka, Effective method of computing Li’s coefficients and their properties. math.NT/0402168 [abs, ps, pdf, other]
[31] K. Maślanka, An explicit formula relating Stieltjes constants and Li’s numbers. math.NT/0406312 [abs, ps, pdf, other]

[32] R. Sitaramachandrarao, Maclaurin Coefficients of the Riemann Zeta Function. Abstracts Amer. Math. Soc. 7, 280, 1986.

[33] H.M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions. Kluwer Academic Publishers, Dordrecht, the Netherlands, 2001.

[34] The Maclutor History of Mathematics archive. http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Faa_di_Bruno.html

[35] E.C. Titchmarsh, The Theory of Functions. 2nd Ed., Oxford University Press, 1932.

[36] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions. Fourth Ed., Cambridge University Press, Cambridge, London and New York, 1963.

[37] Z.X. Wang and D.R. Guo, Special Functions. World Scientific Publishing Co Pte Ltd, Singapore, 1989.

[38] N.-Y. Zhang and K.S. Williams, Some results on the generalized Stieltjes constants. Analysis 14, 147-162 (1994). www.math.carleton.ca/~williams/papers/pdf/187.pdf

Donal F. Connon
Elmhurst
Dundle Road
Matfield
Kent TN12 7HD
dconnon@btopenworld.com