Shaping nonlinear optical response using nonlocal forward Brillouin interactions

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Abstract
In most practical scenarios, optical susceptibilities can be treated as a local property of a medium. For example, in the context of nonlinear optics we can typically treat the Kerr and Raman response as local, such that optical fields at one location do not produce a nonlinear response at distinct locations in space. This is because the electronic and vibrational disturbances produced within the material are confined to a region that is smaller than an optical wavelength. By comparison, Brillouin interactions, mediated by traveling-wave acoustic phonons, can result in a highly nonlocal nonlinear response as the elastic waves generated in the process can occupy a region in space much larger than an optical wavelength. The unique properties of these interactions can be exploited to engineer new types of processes, where highly delocalized phonon modes serve as an engineerable channel that mediates scattering processes between light waves propagating in distinct optical waveguides. These types of nonlocal optomechanical responses have recently been demonstrated as the basis for information transduction, however the nontrivial dynamics of such systems has yet to be explored. In this work, we show that the third-order nonlinear process resulting from spatially extended Brillouin-active phonon modes involves mixing products from spatially separated, optically decoupled waveguides, yielding a nonlocal susceptibility. Building on these concepts, we illustrate how nontrivial multi-mode acoustic interference can produce a nonlocal susceptibility with a multi-pole frequency response, as the basis for new optical and microwave signal processing schemes within traveling wave systems.

1. Introduction

Optical nonlinear processes such as four-wave mixing and harmonic generation are usually described in terms of a frequency dependent, spatially local susceptibility. Namely, the optical fields in one location do not alter the nonlinear response in another point in space [1]. In Raman scattering, the short mean-free path of the THz-frequency optical phonons participating in the nonlinear process also results in a local susceptibility [1]. By comparison, nonlocal nonlinearities require a mechanism by which the optical fields in one location affect the optical response in another location. Nonlocal response has been studied in the context of thermally induced effects [2–4], as well as more exotic systems such as nematic liquid crystals [5], trapped atoms [6], Rydberg gases [7], plasmonic systems [8, 9], and graphene [10]. All of these interactions are the result of transport mechanisms, such as heat, electric charge, or atoms, that mediate the optical response over an extended distance.

Alternatively, the acoustic phonons that participate in a Brillouin scattering process can serve as the transport mechanism for long-range interactions. These elastic modes can be long-lived and propagate many optical wavelengths before decaying, yielding nonlocal dynamics [11–13]. This acousto-optic coupling is a three-wave mixing process producing a coherent interaction of optical waves and acoustic phonons [1, 14, 15]. More specifically, in a forward Brillouin scattering process the optical fields are co-propagating, while the phonons produced by the scattering process are emitted perpendicular to the direction of optical wave propagation [12, 13, 16]. The transverse
nature of the phonons, combined with their long lifetime, enables them to explore a space much larger than the acousto-optic overlap region [12]. This allows the design of structures where the acoustic fields extend the distance between distinct optical guided waves, which are otherwise optically decoupled [17, 18].

In this work, we analyze these forms of nonlocal interactions in the context of Brillouin-active superstructures supporting acoustic modes extending multiple optical waveguides. Because the phonon mode participating in the Brillouin interaction has an overlap with multiple optical fields, the scattering processes in the different waveguides are no longer independent. Hence, the light scattering in one waveguide will affect the dynamics in another, resulting in a nonlocal susceptibility between the two spatially separated waveguides. We show that the intensity envelope of light propagating through a waveguide in the device results in pure phase modulation of light in a spatially separate optical guided wave. This phase modulation is determined by the acoustic and optical properties in both waveguides, revealing the nonlocal nature of the interaction. We further extend our analysis to the case of multiple acoustic modes participating in the acousto-optic process. Specifically, we show that coupling multiple acoustic modes results in phonon super-modes, all occupying the extended space and interacting with the optical fields. The coherent interference of these phonon super-modes yields a multi-pole frequency response for the nonlocal susceptibility, showing a faster frequency roll-off compared to a typical acoustic resonance Lorentzian line shape.

Such nonlocal Brillouin interactions have been recently demonstrated both in optical fiber and in chip-scale photonic devices. In multi-core fibers, light guided within spatially distinct cores can be coupled to acoustic modes occupying the entire fiber cross section [19]. By comparison, integrated photonic systems allow additional structural degrees of freedom which can be used to tailor both the optical and acoustic modes that participate in the nonlocal interaction [17, 18]. These devices can be the basis for new signal processing schemes such as filters [18], transducers [17], oscillators [20], and modulators [21] for both optical and microwave applications. The description and analysis of the underlying processes will enable the design and optimization of future technologies based on these devices.

2. Theoretical study

We begin our analysis by considering a system consisting of two optical waveguides, and supporting a single acoustic mode overlapping with both optical waveguides. While the two waveguides are optically decoupled, the light in each waveguide is acousto-optically coupled to the acoustic mode through a forward stimulated-Brillouin scattering (FSBS) process. We assume the optical fields are co-propagating, and that the photon-phonon coupling is a consequence of electrostrictive forces and radiation pressure [22, 23], which can be tailored through the design of the device geometry [12, 24]. Examples of chip-scale and fiber-based devices that can produce such interactions are illustrated in figure 1(b).

The FSBS process in each of the waveguides can be described as a three-wave interaction involving two photons and a phonon, as illustrated in the phase-matching diagrams in figure 1(a). The phase matching condition required in both waveguides is given by \( q(\Omega_0) = k(\omega_0) - k(\omega_{-1}) \), where \( k(\omega) \) is the optical wave-vector at optical frequency \( \omega \) and \( q(\Omega_0) \) is the wave-vector of the acoustic wave with frequency \( \Omega_0 \). Since the optical waves have similar wave-vectors, this requires a cut-off phonon mode, with a vanishing axial wave-vector, such that the acoustic wave is nearly perpendicular to the direction of the optical wave propagation [25–27]. The transverse nature of these phonons allows them to extend much further than the optical waveguide cross-section. This is in contrast with backward Brillouin scattering processes utilizing bulk acoustic modes, where the phonon typically occupies a similar region as the optical waves, illustrated in figure 1(d).

The wavelengths of the light in the two optical waveguides, denoted A and B, can be different as long as energy is conserved in the process, as illustrated in figure 1(c). This requires \( h\Omega_0 = h\omega^{(A)}_0 - h\omega^{(A)}_{-1} \) and \( h\Omega_0 = h\omega^{(B)}_0 - h\omega^{(B)}_{-1} \), such that the phonon frequency matches the frequency difference of the two optical tones in both waveguides. Phase matching and energy conservation require that the optical modes in both waveguides have a similar optical group velocity [12].

In the absence of strong optical dispersion, an FSBS process enables light to be cascaded to multiple optical frequencies [13, 25, 27, 28], and we describe the optical fields in each waveguide as a sum of discrete tones with field amplitude \( a_n \), and spaced by frequency \( \Omega \) such that \( \omega_n - \omega_{n-1} = \Omega \). The equations of motion describing this system can be derived using quantum operators following [23, 27] and are outlined in appendix A. An alternative formulation in terms of classical variables is provided in appendix G.

We denote the steady-state optical field amplitudes in the two waveguides as \( a^{(A)}_n \) and \( a^{(B)}_n \), with frequencies \( \omega^{(A)}_n \) and \( \omega^{(B)}_n \), and the steady-state acoustic field amplitude as \( b \), with an acoustic dissipation rate \( \Gamma \). We assume a constant optical group velocity over the frequency range of interest, such that the optical tones can cascade to an arbitrary number of sidebands. The equations of motion for the optical fields in each waveguide and the acoustic field are given by
where we denote the frequency response as

\[
\frac{\partial a_n^{(A)}}{\partial z} = -\frac{i}{\nu^{(A)}} [g^{(A)} b a_{n-1}^{(A)} + g^{(A)^*} b^\dagger a_{n+1}^{(A)}],
\]

\[
\frac{\partial a_n^{(B)}}{\partial z} = -\frac{i}{\nu^{(B)}} [g^{(B)} b a_{n-1}^{(B)} + g^{(B)^*} b^\dagger a_{n+1}^{(B)}],
\]

\[
b = -i \left( \frac{1}{i\Delta + \Gamma/2} \right) \sum_n \left[ g^{(A)} b a_{n}^{(A)^\dagger} a_{n+1}^{(A)} + g^{(B)} b a_{n}^{(B)^\dagger} a_{n+1}^{(B)} \right]_{z=0}.
\]

Next, in order to study the dynamics of the system, we analyze the effect of light scattering in waveguide A on the spectral evolution of a separate optical wave propagating in waveguide B. To do this, we assume two tones in the input of waveguide B, the second term of equation (1) is phase matched in both waveguides. The phonon couples multiple tones spaced by frequency \( \Omega_a \), as long as the dispersion relation is linear. Bottom inset: the acoustic dispersion diagram, illustrating the acoustic mode taking part in the interaction is near its cutoff frequency with a vanishing axial wave-vector. To do this, we assume two tones in

\[
\frac{\partial a_n^{(A)}}{\partial z} = -\frac{1}{\nu^{(A)}} [g^{(A)} b a_{n-1}^{(A)} a_{n}^{(A)^\dagger} (0) a_{n+1}^{(A)} (0) - a_{n+1}^{(A)} \chi a_{n-1}^{(A)} (0) a_{n}^{(A)^\dagger} (0)],
\]

\[
\frac{\partial a_n^{(B)}}{\partial z} = -\frac{1}{\nu^{(B)}} [g^{(B)} b a_{n-1}^{(B)} a_{n}^{(B)^\dagger} (0) a_{n+1}^{(B)} (0) - g^{(B)^*} \chi a_{n+1}^{(B)} (0) a_{n-1}^{(B)^\dagger} (0) a_{n}^{(B)^\dagger} (0)],
\]

where we denote the frequency response as \( \chi = (i\Delta + \Gamma/2)^{-1} \).
In order to understand the nonlinear susceptibility produced by the forward Brillouin process, we examine the \( n = -1 \) tone in each waveguide (the Stokes wave), and consider a small signal analysis, such that we can neglect tones with \( |n| \geq 2 \), yielding

\[
\frac{\partial a^{(A)}_n}{\partial z} = i \gamma^{(A)}(a^{(A)}_n a^{(A)*}_{n+1} a^{(A)*}_0 + a^{(A)*}_n a^{(A)}_{n+1} a^{(A)}_0),
\]

\[
\frac{\partial a^{(B)}_n}{\partial z} = i \gamma^{(B)}(a^{(B)}_0 a^{(B)}_n a^{(B)*}_{n+1} a^{(B)*}_0 + a^{(B)*}_n a^{(B)}_{n+1} a^{(B)}_0),
\]

where we have defined the susceptibilities in the two waveguides

\[
\gamma^{(A)} = - \frac{i}{\nu^{(A)}} |g^{(A)}|^2 \chi^*, \quad \gamma^{(B)} = - \frac{i}{\nu^{(B)}} g^{(B)} s^{(B)} A \chi^*.
\]

We see that this is a third order susceptibility, where we have three field amplitudes driving the field amplification, similar to the form derived for four-wave mixing and backward stimulated Brillouin scattering [1]. Examining the equation for waveguide B reveals the nonlocal nature of the susceptibility, where field amplitudes in waveguide A determine the response in the optically decoupled, spatially separated waveguide B. Further, the susceptibility \( \gamma^{(B)} \) depends on the Brillouin coupling rate in both waveguides A and B. Similar expressions can be written for the anti-Stokes (\( n = 1 \)) tone in the waveguides.

Next, we assume the two waveguides are similar, such that \( \nu^{(A)} = \nu^{(B)} = \nu \) and \( g^{(A)} = g^{(B)} = g \), valid in many physical systems [17, 18, 21]. The equations describing the fields in waveguides A and B are now

\[
\frac{\partial a^{(A)}_n}{\partial z} = - \frac{1}{\nu} |g|^2 [a^{(A)}_{n-1} a^{(A)}_0 a^{(A)*}_0 + a^{(A)*}_n a^{(A)}_{n+1} - a^{(A)}_n e^{i(\phi + \Lambda) n}],
\]

\[
\frac{\partial a^{(B)}_n}{\partial z} = - \frac{1}{\nu} |g|^2 [a^{(B)}_0 a^{(B)}_n a^{(B)*}_{n+1} a^{(B)*}_0 + a^{(B)*}_n a^{(B)}_{n+1} - a^{(B)}_n e^{i(\phi + \Lambda) n}],
\]

where we denote the relative phase between the two tones at the input to waveguide A as \( \Lambda = \arg(a^{(A)}_0 a^{(A)*}_0) \) and the phase of the frequency response \( \phi = \arg(\chi) \). We now solve the differential equation by rotating the field operators \( \tilde{a}_n = a_n e^{-i(\phi + \Lambda) n} \), giving us the form

\[
\frac{\partial \tilde{a}_n}{\partial z} = - \frac{1}{\nu} G_B \sqrt{P^{(A)} P^{(A)*}} \sum_m \frac{1}{2} |\chi| (\tilde{a}_{n-1} - \tilde{a}_{n+1}) + \nu \tilde{a}_n
\]

for both waveguides, and we have expressed the field inputs in terms of optical power using \( P_0 = h \omega_0 v_0 a_0^2 \), and the acousto-optic coupling in terms of Brillouin gain \( G_B = 4 |g|^2 / (2 \omega_0 \nu^2) \) [14, 27]. This recurrence relation is consistent with the Bessel function identity \( J_\nu(\nu) = 2 i J_{\nu+1}(\nu) \), such that the optical fields can be written as a linear combination \( \tilde{a}_n(z) = \sum_n c_{n,m}(\xi) \tilde{a}_{m,n}(z) \), where \( \xi = G_B (P^{(A)} P^{(A)*} / 2)(\nu / 2) |\chi| \). We can find the coefficients \( c_{n,m} \) using \( J_\nu(\nu) = \tilde{e}_{m,n}(0) \) such that \( c_{n,m} = \tilde{a}_{n,m}(0) \), and using the identity \( J_\nu(\nu) = \tilde{J}_\nu(\nu) / \tilde{\nu} \), we arrive at

\[
a_n(z) = \sum_m a_{n+m}(0) J_m(\xi z) e^{-i(\phi + \Lambda) m},
\]

where we have rotated the operators back to the field envelope frame. Plugging in the initial conditions for each waveguide gives us

\[
a^{(A)}_n(z) = a^{(A)}_0(0) J_n(\xi z) e^{i(\phi + \Lambda) n} + a^{(A)*}_{n-1}(0) J_{n+1}(\xi z) e^{i(\phi + \Lambda) (n+1)},
\]

\[
a^{(B)}_n(z) = a^{(B)}_0(0) J_{-n}(\xi z) e^{-i(\phi + \Lambda) n},
\]

where we have denoted the amplitude at the input to waveguide B as \( a^{(B)}_0(0) \), with optical frequency \( \omega^{(B)}_0 \). Evaluating the optical field amplitude at the output of waveguide B, i.e. the sum of the amplitudes at equally spaced frequencies \( s_{\text{out}}^{(B)}(t) \propto \sum_n a^{(B)}_n e^{-i\omega^{(B)}_n t + n\Omega t} \), we arrive at

\[
s_{\text{out}}^{(B)}(t) = \sqrt{P_0^{(B)}} e^{-i\phi^{(B)}_0} \sum_n J_n \left( \frac{1}{2} |\chi| G_B P^{(A)} P^{(A)*} \right) e^{-i(\Omega t - (\phi + \Lambda) n)},
\]

where we have used the identity \( J_{-n} = (-1)^n J_n \), and neglected a global phase of the input field. The field amplitudes are normalized such that the total power entering waveguide B is \( P_0^{(B)} \). Using the Jacobis–Anger expansion, \( J_n e^{i\phi} = e^{i\phi} e^{i\pi n} \), we can directly see that the field is purely phase modulated

\[
s_{\text{out}}^{(B)}(t) = \sqrt{P_0^{(B)}} e^{-i\phi^{(B)}_0} \left( \frac{1}{2} |\chi| G_B P^{(A)} P^{(A)*} \right) \sin(\Omega t - (\phi + \Lambda)),
\]

where the modulation depth is determined by the Brillouin gain, the optical powers in waveguide A and the modulation length. The frequency response \( \chi(\Omega) \) of this phase-modulated field follows a Lorenzian lineshape, determined by the acoustic resonant frequency and dissipation rate, with a magnitude given by \( |\chi|^2 = \{\Omega_0 - \Omega^2 + (\Omega^2 / 2)^2\}^{-1} \).

Summarizing the results of this section, we see how the optical intensity envelope generated by the two tones in waveguide A drives the acoustic mode, as seen in equation (3) and illustrated in figure 2(a). This acoustic field modulates an optical tone in waveguide B as it propagates through the device (figure 2(e)) resulting in pure phase
modulation, as seen in figure 2(f). The optical field in waveguide A also experiences phase modulation as it interacts with the acoustic field, illustrated in figures 2(b) and (c), however the intensity envelope remains unchanged (discussed further in appendix B).

Our analysis shows an overall red-shift of the light in waveguide A as it propagates, illustrated in figures 2(b) and (c), consistent with previous forward Brillouin scattering studies [29]. The phonons generated through the Brillouin process have a finite lifetime, and their dissipation takes energy out of the system, which in the absence of optical dissipation represents the only loss mechanism. The red-shift can be derived from the mode amplitudes (equation (10)), yielding a power-loss given by $P_{\text{loss}}(z) = k\Omega |b(z)|^2 \Gamma z$, showing the energy dissipated from the phonon field (further details can be found in appendix B.1).

Throughout our analysis we have assumed vanishing optical group velocity dispersion (GVD) in the frequency range over which the optical tones are cascaded. This is a valid approximation in many practical systems [13, 25, 27] and this situation can be realized using dispersion engineering in optical waveguides [30, 31]. In the presence of GVD, the output field will no longer be purely phase modulated, and will exhibit residual amplitude modulation (RAM) [28] (discussed further in appendix F).

### 2.1. Multiple coupled phonon modes

The frequency response of the Brillouin-induced nonlinear susceptibility is determined by the properties of the phonon taking part in the interaction. This response can be further engineered by utilizing multiple acoustic modes, such that they all contribute to the susceptibility. To explore this extended system, we consider two identical spatially-separated waveguides, each guiding an optical mode and each supporting a phonon mode with a resonant frequency $\Omega_0$ and an amplitude $b_0^{(A)}$, $b_0^{(B)}$ for waveguides A and B respectively. The two waveguides are designed such that they are optically decoupled, while the two acoustic fields are coupled with a rate $\mu$, as illustrated in figure 3(a).

An example of such a device, implementing the acoustic coupling using a phononic crystal, was demonstrated in [17] and is illustrated in figure 3(c). In this scheme, the acoustic modes are guided using line-
defects in the phononic crystal structure while the central region between the waveguides acts as a Bragg-reflector that enables evanescent coupling of the two phonon modes [17]. The acoustic coupling gives rise to symmetric and anti-symmetric acoustic super-modes, denoted $b_+$ and $b_-$, which are linear combinations of the spatial acoustic modes and have frequencies determined by the acoustic coupling rate $\mu$. This is demonstrated by a finite element method (FEM) simulation, solving for the acoustic eigen-modes of such a geometry, shown in figures 3(d) and (e). The two super-modes are decoupled, as they are eigen-modes of the system, but are both coupled to the optical modes in both waveguides, illustrated in figure 3(b).

Following the derivation presented in appendix C, the equations of motion are now given by

$$b_\pm = \frac{1}{\sqrt{2}} \left( b^{(A)} \pm \frac{1}{\sqrt{2}} \right)$$

$$\frac{\partial a^{(A)}_n}{\partial z} = \frac{i}{\sqrt{2}} \left( g^{(A)}_- a^{(A)}_{n-1} b_+ + g^{(A)}_+ a^{(A)}_{n+1} b_- + g^{(A)}_- a^{(A)}_{n-1} b_- + g^{(A)}_+ a^{(A)}_{n+1} b_+ \right),$$

$$\frac{\partial a^{(B)}_n}{\partial z} = -\frac{i}{\sqrt{2}} \left( g^{(B)}_- a^{(B)}_{n-1} b_+ + g^{(B)}_+ a^{(B)}_{n+1} b_- + g^{(B)}_- a^{(B)}_{n-1} b_- + g^{(B)}_+ a^{(B)}_{n+1} b_+ \right),$$

where $b_\pm = (b^{(A)} \pm b^{(B)}) / \sqrt{2}$ are the amplitudes of the acoustic super-modes with frequencies $\Omega_{\pm}$, and $g^{(\pm)}_\chi$ is the Brillouin coupling rate of acoustic mode $b_\pm$ to the optical field in waveguide $\chi$ (for $\chi = \{ A, B \}$). By projecting
the ‘±’ super-modes on waveguides A and B we see that \( g_{\pm}^{(A)} = g / \sqrt{2} \), and \( g_{\pm}^{(B)} = \pm g / \sqrt{2} \), a result of the symmetries of the super-modes. Since we assume the waveguides are identical, we set the group velocities to the same value \( v = v^{(A)} = v^{(B)} \), yielding

\[
b_{\pm} = - \frac{i}{\sqrt{2}} \left( i(\Omega_+ - \Omega) + 1/2 \right) g^{\pm} \sum_n (a_{n-1}^{(A)})^\dagger \pm a_{n}^{(B)} a_{n-1}^{(B)} \dagger,
\]

\[
\frac{\partial a_{n}^{(A)}}{\partial z} = - \frac{i}{\sqrt{2} v} (ga_{n-1}^{(A)} b_{\pm} + g^* a_{n}^{(A)} b_{\pm}^\dagger + ga_{n}^{(A)} b_{-} + g^* a_{n+1}^{(A)} b_{-}^\dagger),
\]

\[
\frac{\partial a_{n}^{(B)}}{\partial z} = - \frac{i}{\sqrt{2} v} (ga_{n-1}^{(B)} b_{\pm} + g^* a_{n}^{(B)} b_{\pm}^\dagger - ga_{n}^{(B)} b_{-} - g^* a_{n+1}^{(B)} b_{-}^\dagger).
\]

(14)

These equations have the same form as we saw earlier in equation (2), such that \( \partial_z b_{\pm} = 0 \), and the phonon super-modes are only dependent on the initial conditions of the optical fields. We assume again the optical inputs to be two tones in waveguide A, separated in frequency by \( \Omega \), and a single tone in waveguide B giving us

\[
b_{\pm} = - \frac{i}{\sqrt{2}} \left( i(\Omega_+ - \Omega) + 1/2 \right) g^{\pm} a_{0}^{(A)}(0) a_{-1}^{(B)}(0),
\]

(15)

where we use the notation \( \Delta_{\pm} = \Omega_{\pm} - \Omega \). Plugging the phonon fields into the optical equations of motion yields

\[
\frac{\partial a_{n}^{(A)}}{\partial z} = - \frac{|g|}{v} \left| a_{n-1}^{(A)}(0) a_{0}^{(A)}(0) \right| \frac{\chi^{(A)}}{\chi^{(A)}}^{\dagger} \frac{e^{i\phi^{(A)}}}{e^{i\phi^{(A)}}} - a_{n-1}^{(A)} e^{-i\phi^{(A)}} e^{-i\phi^{(A)}},
\]

\[
\frac{\partial a_{n}^{(B)}}{\partial z} = - \frac{|g|}{v} \left| a_{n-1}^{(B)}(0) a_{0}^{(B)}(0) \right| \frac{\chi^{(B)}}{\chi^{(B)}}^{\dagger} \frac{e^{i\phi^{(B)}}}{e^{i\phi^{(B)}}} - a_{n-1}^{(B)} e^{-i\phi^{(B)}} e^{-i\phi^{(B)}},
\]

(16)

where we have defined the frequency response in each waveguide \( \chi^{(A)} = (1/2)(i\Delta_{\pm} + \Gamma/2)^{-1} + (i \Delta_{\pm} - \Gamma/2)^{-1} \)

\( \chi^{(B)} = (1/2)(i\Delta_{\pm} + \Gamma/2)^{-1} - (i \Delta_{\pm} + \Gamma/2)^{-1} \), with phases \( \phi^{(A)} = \arg(\chi^{(A)}) \), \( \phi^{(B)} = \arg(\chi^{(B)}) \), and the phase difference of the input tones \( \Lambda = \arg(a_{-1}^{(A)}(0) a_{0}^{(A)}(0)) \).

The frequency response \( \chi^{(B)} \) of the nonlocal susceptibility induced in waveguide B by the optical fields in waveguide A is dramatically changed in this multi-phonon case, a result of the phase difference between the two complex terms. As demonstrated in figure 3(f), the coupling of multiple acoustic fields yields a high-order frequency response, showing a sharp frequency roll-off. This can also be seen by the interference of the two phonon super-modes, shown in figure 3(b), yielding a multi-pole function. The susceptibility in waveguide A is altered around the acoustic resonance, as seen in figure 3(g), but decays as a Lorentzian away from the center frequency, similar to the single-phonon case.

Solving the equations as described in the previous section, yields the field amplitude at the output of waveguide B

\[
s_{w_{\pm}}^{(B)}(t) = \sqrt{P_{0}^{B}} e^{-i\phi_{0}^{B}} \exp \left[ \frac{1}{2} \left| \chi^{(B)} \right| G_{B} \sqrt{P_{0}^{A} P_{-1}^{(A)}} z \sin (\Omega t - (\phi^{(B)} + \Lambda)) \right],
\]

(17)

which now has the modified frequency response \( \chi^{(B)} \). Practically, this permits the design of opto-acoustic filter responses with a high-order line shape, enabling high-performance RF-photonic applications [32].

The coupling of acoustic modes can be further extended to an arbitrary number of phonons taking part in the transduction. For \( N \) identical waveguides, each supporting an acoustic mode with resonant frequency \( \Omega_0 \) and a nearest neighbor coupling with rate \( |\mu|e^{i\theta} \), the nonlocal susceptibility frequency response is given by

\[
\chi^{(B)} = \sum_{m=1}^{N} V_{m}^{(N)} V_{m}^{(1)} \left( \frac{1}{i\Delta_{m} + \Gamma/2} \right)
\]

(18)

where the detuning term for the \( m \)th phonon super-mode is \( \Delta_{m} = (\Omega_0 - \Omega) + 2|\mu| \cos(\pi m/2) \), and the phonon super-modes coefficients are given by \( V_{m}^{(N)} = \sqrt{2/(N+1)} \sin(\pi m/2) (\sin(\pi m/2)) e^{im\theta} \) as described in detail in appendix D. The function described by equation (18) yields sharper frequency responses with the addition of coupled acoustic modes, as demonstrated in figure 3(f).

2.2. Spontaneous scattering

Up to this point, we have considered acousto–optic scattering from coherently driven phonons produced in waveguide A. However, thermal occupation of the acoustic modes results in spontaneous Brillouin scattering [1, 27]. This can also be understood as the noise associated with the dissipation of the acoustic mode, through the fluctuation-dissipation theorem [33, 34]. If we seek to utilize such nonlocal susceptibilities to transduce information, as the basis for new signal processing technologies [17–19, 21], it is essential to understand how this noise is imparted from the elastic field onto the optical fields.
Assuming a single acoustic mode with frequency $\Omega_0$, an optical tone at frequency $\omega_0$ with a field amplitude $a_0$ will be scattered to sidebands at frequencies $\omega_0 \pm \Omega_0$. The amplitudes of the spontaneously scattered light are given by [27, 34]

$$a_{-\ell}(z, \tau) = -\frac{G}{v} a_0 \int_0^\tau d\tau' \int_0^\infty dz' \eta'(z', \tau') e^{-i(\tau' - \tau)},$$

$$a_{\ell}(z, \tau) = -\frac{G}{v} a_0 \int_0^\tau d\tau' \int_0^\infty dz' \eta'(z', \tau') e^{-i(\tau - \tau')},$$

(19)

where $\eta(z, \tau)$ is the Langevin force corresponding to the phonon dissipation rate, with statistics $\langle \eta(z, \tau) \rangle = 0$ and $\langle \eta(z, \tau) \eta(z', \tau') \rangle = \bar{n} \delta(z - z') \delta(t - t')$ when evaluating ensemble averages [27]. $\bar{n}$ is the average number of thermally occupied phonons following a Bose–Einstein distribution $\bar{n} = \exp(\hbar \Omega / k_B T) - 1^{-1}$, with $T$ denoting the temperature, and $k_B$ the Boltzmann constant. These expressions apply for the scattering in both waveguides A and B. Using these equations, the spectral density of the spontaneous scattering can be calculated [27]

$$S(\omega) = \hbar \omega_0 G_B v \left( \frac{\Gamma}{2} \right)^2 \frac{\bar{n} + 1}{(\omega - (\omega_0 - \Omega_0))^2 + (\Gamma/2)^2} + \frac{\bar{n}}{(\omega - (\omega_0 + \Omega_0))^2 + (\Gamma/2)^2},$$

(20)

showing a Lorentzian lineshape with a full-width at half-maximum $\Gamma$. In the case of coupled acoustic modes, as discussed in the previous section, equation (19) can be generalized

$$a_{-\ell}^{(m)}(z, \tau) = -\frac{G}{v} a_0 \sum_{m=1}^N V_m^{(m)} \int_0^\tau d\tau' \int_0^\infty dz' \eta_m'(z', \tau') e^{-i(\tau' - \tau)},$$

$$a_{\ell}^{(m)}(z, \tau) = -\frac{G}{v} a_0 \sum_{m=1}^N V_m^{(m)} \int_0^\tau d\tau' \int_0^\infty dz' \eta_m'(z', \tau') e^{-i(\tau - \tau')},$$

(21)

where $\eta_m$ is the Langevin force associated with the $m$th phonon super-mode, and $V_m^{(m)}$ is the coefficient used in equation (18). Since the acoustic super-modes are orthogonal, we use the fact that the thermal phonons in each eigenmode are uncorrelated and follow $\langle \eta_m(z, \tau) \rangle = 0$ and $\langle \eta_m(z, \tau) \eta_m(z', \tau') \rangle = \bar{n}_m \Gamma_b \delta(z - z') \delta(t - t')$. The spectral density is now given by

$$S(\omega) = \hbar \omega_0 G_B v \left( \frac{\Gamma}{2} \right)^2 \sum_{m=1}^N |V_m^{(m)}|^2 \frac{\bar{n}_m + 1}{(\omega - (\omega_0 - \Omega_m))^2 + (\Gamma/2)^2} + \frac{\bar{n}_m}{(\omega - (\omega_0 + \Omega_m))^2 + (\Gamma/2)^2},$$

(22)

where $\bar{n}_m$ is the average thermal phonon occupation of the $m$th phonon super mode. We see that the spontaneous scattering results in a noise spectrum that is comprised of a sum of Lorentzian line shapes. This is distinct from the multi-pole response observed for a transduced signal (equation (18)) which results from coherent interference. In the case of weakly coupled acoustic modes, we can readily see how the noise spectrum takes on a Lorentzian form. In this case, the frequency differences of the super-modes will be small and we can approximate $\bar{n}_m \approx \bar{n}$ and $\omega_0 \approx \Omega_m \approx \omega_0 \pm \Omega_0$. Using $\sum_{m=1}^N |V_m^{(m)}|^2 = 1$, we see that the spontaneous spectrum is consistent with a Lorentzian frequency response obtained from a single acoustic mode, given by equation (20) (more details can be found in appendix D). Interestingly, the presence of spontaneously scattered photons in either waveguide has no impact on the acoustic field [27]. The scattering produced from the thermally occupied phonons only adds phase fluctuations which do not change the forcing function driving the coherent acoustic field, as it is determined only by the intensity envelope of the light field. This can be seen directly from equation (1) where the contribution to the driven phonon field is the sum of terms $a_{-\ell}^{(A)} a_{\ell}^{(A)} + a_{-\ell}^{(A)} a_{\ell}^{(A)} + a_{-\ell}^{(B)} a_{\ell}^{(B)} + a_{-\ell}^{(B)} a_{\ell}^{(B)}$ which equals zero when plugging in the mode amplitudes from equation (19).

3. Discussion and conclusion

The theoretical model presented in this work describes the nonlinear dynamics of forward Brillouin active devices which utilize the nonlocal nature of the acoustic modes taking part in the process. We have shown that the optical intensity envelope of light in waveguide A induces a nonlinear response in the spatially-separated waveguide B. In the absence of optical dispersion, this process results in a phase modulation, where the modulation depth is set by the Brillouin gain, acoustic properties, interaction length, and optical powers. This nonlocal susceptibility is enabled by the long lifetime of the phonons, which allows them to propagate a large distance—spanning many optical wavelengths—to connect optically decoupled regions of the device. This phenomenon can be readily measured within integrated photonic devices, where the tight confinement of the optical and acoustic fields to micrometer-scale waveguides yields high Brillouin gain [17, 18, 21]. Multi-core optical fibers can also demonstrate nonlocal susceptibility between the separated cores, where the long interaction length and high-power handling can produce strong nonlocal effects [19].
We have shown that this nonlocal susceptibility can be mediated by multiple coupled phonons, yielding distinct spectral features. In this case, the phonons can be treated as acoustic super-modes, i.e. the eigen-modes of the coupled system, which are all interacting with the optical fields. The coherent interference of these acoustic super-modes yields a nonlocal susceptibility corresponding to a multi-pole frequency response, which offers a sharp frequency roll-off and high out-of-band suppression. The nonlocal susceptibility can be utilized to perform filtering operations by encoding information in the form of intensity-modulation on the light propagating in waveguide A, and detecting the phase modulation experienced by the light in waveguide B. The frequency response of the output from such a filter follows the nonlocal susceptibility $\chi^{(B)}$ (equations (12) and (18)), which can have a narrow-band multi-pole response, as illustrated in figure 3(f). Such hybrid photonic-phononic signal processing strategies yield transfer functions that would be challenging to realize using all-optical techniques, providing technological value in many microwave and photonic applications [32, 35]. RF-photonic filters utilizing this nonlocal response have recently been demonstrated and analyzed [17, 18, 36]. By comparison, the local nonlinear susceptibility ($\chi^{(A)}$) produces a single-pole response, regardless of the number of acoustic modes that mediate Brillouin coupling (further details can be found in appendix D). Interestingly, the same super-modes that produce a multi-pole response through nonlocal susceptibility yield a single-pole, Lorentzian-like noise spectrum when driven by incoherent thermal fluctuations. When considering practical applications such as filtering operations, this noise spectrum needs to be taken into account as it contributes to the total noise power level at the device output [18, 36].

The forward Brillouin dynamics utilized in these devices have interesting advantages when considering noise and scalability. The acousto-optic interaction results in phase modulation alone, such that the intensity envelope of the light propagating through the system remains unchanged, regardless of the strength of the nonlinear interaction. The intensity modulation of the propagating light corresponds to the optically induced forcing function, hence the driven coherent acoustic phonons remain unaffected. Similarly, the scattering produced by thermally occupied phonons result in phase fluctuations alone, which do not affect the intensity envelope and the driven coherent phonon field. This results in no degradation of information encoded in the form of optical intensity modulation, as the light field propagates through the device. These properties can enable cascading of multiple devices in series without losing signal fidelity, further broadening their technological impact.

The analysis presented in this work can be readily adapted to describe nonlocal susceptibilities involving other scattering processes such as inter-band Brillouin scattering [37, 38]. In this scattering process, light is scattered between two different optical guided modes. The acoustic modes that mediate the interaction are still mostly transverse in nature and can extend a large space outside the optical guiding region. This enables interaction with spatially separated optical waveguides, yielding nonlocal susceptibility. The inter-band scattering process and the phonons taking part in the interaction yield different dynamics compared to the analysis in this work, such as single-sideband modulation and non-reciprocity. A wide-band nonreciprocal modulator utilizing nonlocal susceptibility produced by a single phonon mode has been recently demonstrated experimentally [21]. Further details of how the nonlocal susceptibility is formulated for an inter-modal process from the model presented here are provided in appendix E.

In conclusion, tailorable, nonlocal, nonlinear responses can be achieved by interfacing signals in different domains, coupling light and coherent long-lived acoustic waves. These interactions are useful in many practical applications, such as filtering, coherent signal addition, opto-acoustic storage and spectral analysis [15]. Further, the unique properties of the forward Brillouin process can enable applications utilizing cascaded devices for spectral awareness, channelizing and sensing. The understanding of the dynamics of the fields participating in these processes is essential for the design of future devices and systems.

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Appendix A. Deriving the equations of motion

We start our analysis by considering a system of two spatially separated optical waveguides, denoted A and B, both coupled to a single acoustic mode. We can separate the Hamiltonian of the system to terms describing the optical fields in waveguides A and B, the acoustic field, and the acousto-optic interaction in each waveguide.
\[ H = H_{\text{opt}}^{(A)} + H_{\text{opt}}^{(B)} + H_{\text{ac}} + H_{\text{int}}^{(A)} + H_{\text{int}}^{(B)}. \]  

(A1)

Following the treatment of [23, 27], the different terms are given by

\[ H_{\text{opt}}^{(A)} = \sum_n \hbar \int \text{d}z \ a_n^{(A)}(z) \hat{\omega}_n^{(A)} a_n^{(A)}(z), \]
\[ H_{\text{opt}}^{(B)} = \sum_n \hbar \int \text{d}z \ a_n^{(B)}(z) \hat{\omega}_n^{(B)} a_n^{(B)}(z), \]
\[ H_{\text{ac}} = \hbar \int \text{d}z \ b^\dagger(z) \hat{\Omega}_0 b(z), \]
\[ H_{\text{int}}^{(A)} = \sum_n \hbar \int \text{d}z \ g_n^{(A)e} a_n^{(A)}(z) a_n^{(A)}(z) b^\dagger(z) e^{-i\Delta k_n^{(A)} z} + \text{h.c.}, \]
\[ H_{\text{int}}^{(B)} = \sum_n \hbar \int \text{d}z \ g_n^{(B)e} a_n^{(B)}(z) a_n^{(B)}(z) b^\dagger(z) e^{-i\Delta k_n^{(B)} z} + \text{h.c.}, \]  

(A2)

where \( \Delta k_n^{(A)} = k_n^{(A)} - k_n^{(A)} \) and \( \Delta k_n^{(B)} = k_n^{(B)} - k_n^{(B)} - 1 \) is the phase mismatch between the phonon (with wave-vector \( q_0 \)) and the photons (with wave vectors \( k_n \) and \( k_{n+1} \)) in each of the two waveguides. We assume each waveguide supports a single optical spatial mode, and the index \( n \) sums over all optical tones coupled through the acousto-optic interaction. The coupling rate \( g \) can have both photo-elastic and radiation pressure components, which can be evaluated following [27].

We can calculate the dynamics of the fields using the Heisenberg equation of motion

\[ \dot{a}_n^{(A)}(t) = \frac{1}{i\hbar} [a_n^{(A)}, H], \quad \dot{a}_n^{(B)}(t) = \frac{1}{i\hbar} [a_n^{(B)}, H], \quad \dot{b}(t) = \frac{1}{i\hbar} [b, H], \]  

(A3)

and using the commutation relations

\[ [a_n^{(A)}(z, t), a_{m}^{(A)}(z', t)] = \delta(z - z') \delta_{nm}, \]
\[ [a_n^{(B)}(z, t), a_{m}^{(B)}(z', t)] = \delta(z - z') \delta_{nm}, \]
\[ [b(z, t), b^\dagger(z', t)] = \delta(z - z'), \]  

(A4)

yields

\[ \dot{a}_n^{(A)}(z, t) = -i \hat{\omega}_n a_n^{(A)}(z, t) - i g_n^{(A)e} a_{n+1}^{(A)}(z, t) b(z, t) e^{i\Delta k_n^{(A)} z} + g_n^{(A)e} a_n^{(A)}(z, t) b^\dagger(z, t) e^{-i\Delta k_n^{(A)} z}, \]
\[ \dot{a}_n^{(B)}(z, t) = -i \hat{\omega}_n a_n^{(B)}(z, t) - i g_n^{(B)e} a_{n-1}^{(B)}(z, t) b(z, t) e^{i\Delta k_n^{(B)} z} + g_n^{(B)e} a_n^{(B)}(z, t) b^\dagger(z, t) e^{-i\Delta k_n^{(B)} z}, \]
\[ \dot{b}(z, t) = -i \hat{\Omega}_0 b(z, t) - \sum_n (g_n^{(A)e} a_n^{(A)}(z, t) a_{n-1}^{(A)}(z, t) e^{-i\Delta k_n^{(A)} z} + g_n^{(B)e} a_n^{(B)}(z, t) a_{n+1}^{(B)}(z, t) e^{-i\Delta k_n^{(B)} z}). \]  

(A5)

We keep the dispersion operators to first order for the optical and acoustic fields, \( \hat{\omega}_n \approx \omega_n - i v_n \partial_z \), \( \hat{\Omega}_0 \approx \Omega_0 - iv_\text{ac} \partial_z \), and add phonon dissipation by including an imaginary part to the acoustic frequency, \( \Omega_0 \to \Omega_0 - \frac{i}{2} v_\text{ac} \). Rewriting the equations in the rotating frame by factoring out the fast oscillating term

\[ a_n(z, t) \to a_n(z, t) e^{-i \hat{\omega}_n t}, \quad b(z, t) \to b(z, t) e^{-i \Omega_0 t}, \]  

(A6)

where \( \Omega = \omega_n - \omega_{n-1} \), we now have

\[ \dot{a}_n^{(A)}(z, t) + v_n \partial_z a_n^{(A)}(z, t) = -i g_n^{(A)e} a_{n-1}^{(A)}(z, t) b(z, t) e^{-i\Delta k_n^{(A)} z} + g_n^{(A)e} a_n^{(A)}(z, t) b^\dagger(z, t) e^{-i\Delta k_n^{(A)} z}, \]
\[ \dot{a}_n^{(B)}(z, t) + v_n \partial_z a_n^{(B)}(z, t) = -i g_n^{(B)e} a_{n-1}^{(B)}(z, t) b(z, t) e^{-i\Delta k_n^{(B)} z} + g_n^{(B)e} a_n^{(B)}(z, t) b^\dagger(z, t) e^{-i\Delta k_n^{(B)} z}, \]
\[ \dot{b}(z, t) + \frac{v_\text{ac}}{2} \partial_z b + i(\Omega_0 - \Omega - \frac{\Gamma}{2}) b = -\sum_n (g_n^{(A)e} a_n^{(A)}(z, t) a_{n-1}^{(A)}(z, t) e^{-i\Delta k_n^{(A)} z} + g_n^{(B)e} a_n^{(B)}(z, t) a_{n+1}^{(B)}(z, t) e^{-i\Delta k_n^{(B)} z}). \]  

(A7)

In the case of forward Brillouin scattering (FSBS) the phonon field \( b(z) \) is close to its cut-off frequency, with a small axial wave-vector and vanishing group velocity [27], such that we can set \( v_\text{ac} \to 0 \). We further assume that the optical mode has a constant group velocity in the frequency range of interest for each of the two waveguides, equivalent to no optical group velocity dispersion (GVD), such that \( v_n = v \) and \( \Delta k_n = 0 \) [24, 27]. We also set the opto-mechanical coupling rates to be equal for all optical frequencies, \( g_n = g \). The steady state phonon field envelope now has the form

\[ b = -i \left( \frac{1}{\Gamma + i \Delta} \right) \sum_n (g_n^{(A)e} a_n^{(A)}(z) a_{n-1}^{(A)} + g_n^{(B)e} a_n^{(B)} a_{n-1}^{(B)}), \]  

(A8)
where $\Delta = \Omega_0 - \Omega$, and the optical field envelopes are given by
\[
\frac{\partial a_n^{(A)}}{\partial z} = -\frac{i}{\gamma} (g^{(A)\dagger} a_n^{(A)\dagger} b + g^{(A)\dagger} a_n^{(A)\dagger} b^\dagger),
\]
\[
\frac{\partial a_n^{(B)}}{\partial z} = -\frac{i}{\gamma} (g^{(B)\dagger} a_n^{(B)\dagger} b + g^{(B)\dagger} a_n^{(B)\dagger} b^\dagger).
\]
Calculating the spatial derivative of the phonon field in equation (A8) results in
\[
\frac{\partial b}{\partial z} = -i \left( \frac{1}{\Delta + \Gamma/2} \right) \sum_n \left[ g^{(A)\dagger} b (a_n^{(A)\dagger} a_n^{(A)\dagger} + a_n^{(A)\dagger} a_n^{(A)\dagger}) + g^{(B)\dagger} b (a_n^{(B)\dagger} a_n^{(B)\dagger} + a_n^{(B)\dagger} a_n^{(B)\dagger}) \right] = 0.
\]
We can see directly from the above equation that the spatial derivative vanishes when we sum over all $n$ for which the fields are non-zero. Hence the phonon field is constant in space and can be determined by its value at $z = 0$
\[
b = -i \left( \frac{1}{\Delta + \Gamma/2} \right) \sum_n \left[ g^{(A)\dagger} b (a_n^{(A)\dagger} a_n^{(A)\dagger} + a_n^{(A)\dagger} a_n^{(A)\dagger}) + g^{(B)\dagger} b (a_n^{(B)\dagger} a_n^{(B)\dagger} + a_n^{(B)\dagger} a_n^{(B)\dagger}) \right] \big|_{z=0}.
\]

**Appendix B. Dynamics in waveguide A**

Restating equation (10), we saw that the equations of motion describing the field amplitudes in waveguide A, which has a pump and a Stokes wave at its input, are given by
\[
a_n(z) = a_n(0) J_{-n}(\xi z) e^{i(\phi + \lambda_n z)} + a_{-1}(0) J_{-(n+1)}(\xi z) e^{i(\phi + \lambda_{n+1} z)},
\]
where $\xi = (\gamma_0 P_0)^{1/2} G_B (\Gamma/2)|\chi|$, $\chi = [i(\Omega_0 - \Omega) + \Gamma/2]^{-1}$, $\phi = \arg(\chi)$, and $\lambda = \arg(a_{-1}(0) a_n(0))$. The total field amplitude propagating through the waveguide is the sum of all the amplitudes, $s(z, t) \propto \sum_n a_n(z) e^{-i(\omega_0 + \Omega t)z}$, yielding
\[
s(z, t) \propto e^{-i\omega_0 t} \sum_n \left[ J_{-n}(\xi z) e^{i(\phi + \lambda_n z)} + a_{-1}(0) \sum_{n'} J_{-(n+1)}(\xi z) e^{i(\phi + \lambda_{n+1} z)} \right] e^{-i\omega_0 t}.
\]
By using the index substitutions $m = -n$ and $m' = -(n + 1)$ in the two sums respectively and rearranging the equation, we have
\[
s(z, t) \propto (a_0(0) e^{-i\omega_0 t} + a_{-1}(0) e^{-i(\omega_0 - \Omega) t}) \sum_{m'} J_m(\xi z) e^{-i(\phi + \Lambda - \Omega t) m'},
\]
which can also be expressed as a phase modulation by using the Jacobi–Anger expansion
\[
s(z, t) \propto (a_0(0) e^{-i\omega_0 t} + a_{-1}(0) e^{-i(\omega_0 - \Omega) t}) \exp \left[ -i \frac{\Omega}{2} \chi \right] \frac{G_B}{P_0 P_{-1}} z \sin (\Omega t - (\phi + \Lambda)),
\]
where we have plugged back the argument of the Bessel function.

We see that the two input tones are phase modulated as they traverse the device, with the modulation index increasing linearly with the propagation length. However, the intensity profile is unaffected by the forward Brillouin process. Examining the power as would be measured by a photo-detector yields
\[
|s(z, t)|^2 \propto |a_0(0)|^2 + |a_{-1}(0)|^2 + a_0(0) a_{-1}(0) e^{-i\Omega t} + a_0(0) a_{-1}(0) e^{i\Omega t},
\]
and in terms of optical powers
\[
|s(z, t)|^2 = P_0 + P_{-1} + 2 \sqrt{P_0 P_{-1}} \cos (\Omega t - \Lambda),
\]
revealing the DC terms and the oscillation at the beat-note frequency $\Omega$, the frequency separation between the two tones. We can see directly that the propagation through the device and the Brillouin scattering that takes place does not affect the intensity envelope.
B.1. Optical red-shift
We calculate the total optical power in waveguide A by summing the power contribution of all tones

\[ P^{(A)}_{\text{tot}}(z) = \hbar \sum_{n} (\omega^{(A)}_n + n\Omega) |a^{(A)}_n(z)|^2, \quad (B1) \]

where we are following the definitions from [27], \( P_n = \hbar \omega_n v |a_n|^2 \). Plugging in the mode amplitudes of waveguide A, given in equation (B1), gives us

\[ P^{(A)}_{\text{tot}}(z) = \hbar \left[ |a^{(A)}_0(0)|^2 \left( e^{i\omega^{(A)}_0 z \sum_n J^2_n + \sum_n n J^2_{n+1}} + |a^{(A)}_1(0)|^2 \left( e^{i\omega^{(A)}_1 z \sum_n J^2_{n+1}} + \sum_n n J^2_{n+1} \right) \right) + 2 \Re \left\{ a^{(A)}_{-1}(0) a^{(A)}_0(0) e^{-i(\Delta \Lambda)} (e^{i\omega^{(A)}_0 z \sum_n J^2_{n+1}} + \sum_n n J^2_{n+1}) \right\} \right], \]

(B8)

where for clarity of notation we have omitted the arguments of the Bessel functions, which are all \((\xi z)\). We can calculate the infinite summations from equation (B8)

\[ \sum_n J^2_n = J_0(0) = 1, \]
\[ \sum_n J^2_{n+1} = J_1(0) = 0, \]
\[ \sum_n n J^2_n = \sum_{n=1}^{n=\infty} n J^2_n + \sum_{n=1}^{\infty} n J^2_{n+1} = \sum_{n=1}^{\infty} (n+1) J^2_n + \sum_{n=1}^{\infty} n J^2_{n+1} = 0, \]
\[ \sum_n J^2_{n+1} = \sum_{n+1}^{n+1} n J^2_{n+1} - 1 = \sum_{n+1}^{\infty} n J^2_{n+1} - 1 = -1, \]
\[ \sum_n J^2_{n+1} = \sum_{n+1}^{n+1} n J^2_{n+1} = n J^2_{n+1} = -n(\xi^2/2) \sum_{n=1}^{\infty} J_{n+1}(\xi) J_{n-1}(\xi) = -(\xi/2), \]

(B9)

where we have used the Bessel-function identities \( J_0 = J_{-n}(x) \), \( J_{n+1}(x) = J_1(x) \), and \( J_n(x) = \delta_{n,0} \) and the Bessel addition theorem \( \sum_n h_n(x) h_{n+1}(y) = h_n(x+y) \). Using these results as well as our definitions \( \Lambda = \arg(a^{(A)}_0(0)) \), \( \phi = \arg(\chi) \) and \( \xi = (P^{(A)}_{\text{tot}} P^{(A)}_{\text{tot}})^{1/2} G_\text{B}(\Gamma/2)|\chi| \), we arrive at

\[ P^{(A)}_{\text{tot}}(z) \mid \text{red-shift} = \frac{\hbar \omega^{(A)}_0 v |a^{(A)}_0(0)|^2 + \hbar (\omega^{(A)}_0 - \Omega) v |a^{(A)}_1(0)|^2 - \Omega J^{(A)}_0 G_\text{B} \left[ \frac{\Omega J^{(A)}_0}{\hbar^2} \right] 2 \Re \chi [1 - (\Omega/\omega^{(A)}_0)^{-1/2}] \}

(B10)

The first two terms are the power of the tones at the input to waveguide A, and the third term reveals the red-shift scaling linearly with the propagation length \( z \). We can express this red-shift in terms of the driven phonon number by using equation (3) and the Brillouin gain definition [27], yielding

\[ P^{(A)}_{\text{tot}}(z) = P^{(A)}_{\text{tot}}(0) - \hbar \Omega |b|^2 \Gamma z, \]

(B11)

where we have used \( \Re \chi = (\Gamma/2)|\chi|^2 \) for \( \chi = [i(\Omega - \Omega + \Gamma/2)]^{-1} \). This enables us to interpret the energy lost as the energy in the phonon field \( \hbar \Omega |b|^2 \), with a constant dissipation rate \( \Gamma \) over a length \( z \). The red-shift can be seen in figures 2(b), (c) and B1(a), where the there is more power in the lower optical sidebands. Interestingly, since the phonon number is larger for high-Q acoustic modes (\( |b|^2 \propto \Gamma^{-2} \)) the red-shift is inversely proportional to the dissipation rate \( P^{(A)}_{\text{tot}} \propto \Gamma^{-1} \).

We can carry out a similar calculation for the optical power in waveguide B

\[ P^{(B)}_{\text{tot}}(z) = \hbar v \sum_n (\omega^{(B)}_0 + n\Omega) |a^{(B)}_n(z)|^2, \]

(B12)

and plugging in the expression for the mode amplitudes from equation (B1), this yields

\[ P^{(B)}_{\text{tot}}(z) = \hbar v |a^{(B)}_0(0)|^2 \left( e^{i\omega^{(B)}_0 z \sum_n J^2_n + \sum_n n J^2_{n+1}} \right), \]

(B13)

Using the calculations from equation (B9) we are left with

\[ P^{(B)}_{\text{tot}}(z) = \hbar v |a^{(B)}_0(0)|^2 = P^{(B)}_{\text{tot}}(0), \]

(B14)

revealing that in waveguide B there is no red-shift, and the scattering is symmetric to the upper and lower sidebands, also seen in figures 2(e), (f) and B1(b).

B.2. Small signal forward Brillouin gain
Using the equations we have derived, we can also examine the gain experienced by a small Stokes signal as it propagates through the device along with a pump wave. Using equation (B1), which assumes pump and Stokes waves at its input, we examine the Stokes \((n = -1)\) field amplitude

\[ a_{-1}(z) = a_0(0) j_0(\xi z) e^{-i(\Delta \Lambda)} + a_{-1}(0) j_0(\xi z), \]

(B15)

where \( \xi = \sqrt{P_0(0) P_{-1}(0) G_\text{B}(\Gamma/2)} |\chi| \), \( \phi = \arg(\chi) \), and \( \Delta = \arg(a_{-1}(0)) \). Rearranging the equation, we have
and in terms of the optical powers $P_n \propto |a_n|^2$, this yields

$$P_{\pm 1}(z) = P_{-1}(0) \left( \frac{P_{\pm 1}(0)}{P_{-1}(0)} \right) \sqrt{f_1(z) e^{-i\phi} + j_0(z)}.$$  \hspace{1cm} (B17)

We now take the small signal limit, where we assume the pump power is much larger than the input signal and pump depletion can be neglected, equivalent to $(\xi z) \ll 1$. We can expand the Bessel functions to first order, $j_0(x) \approx 1$, $j_1(x) \approx x/2$, yielding

$$P_{\pm 1}(z) = P_{-1}(0) \left[ 1 - \frac{1}{2} G B P_0 \frac{\Gamma}{2} \chi(\Omega) z \right]^2,$$  \hspace{1cm} (B18)

showing the quadratic growth of power in the Stokes tone in the small signal limit, consistent with the results derived in [27].

### Appendix C. Two coupled phonons

We now consider a device consisting of two identical separate Brillouin active waveguides, denoted A and B, each supporting an optical mode $a(z)$ and an acoustic mode $b(z)$. The acoustic modes both have a resonant frequency $\Omega_0$ and are coupled with a rate $\mu$. The acoustic term of the Hamiltonian from equation (A2) now has the form

$$H_{ac} = \hbar \int dz \left( b^{(A)\dagger}(z) \hat{\Omega}_0 b^{(A)}(z) + b^{(B)\dagger}(z) \hat{\Omega}_0 b^{(B)}(z) - b^{(A)\dagger}(z) \mu b^{(B)}(z) - b^{(A)}(z) \mu b^{(B)\dagger}(z) \right),$$  \hspace{1cm} (C1)

which can also be written in matrix form

$$H_{ac} = \hbar \int dz \left( \begin{array}{l} b^{(A)\dagger} \\ b^{(B)\dagger} \end{array} \right) \left( \begin{array}{l} \hat{\Omega}_0 \\ -\mu \end{array} \right) \left( \begin{array}{l} b^{(A)} \\ b^{(B)} \end{array} \right).$$  \hspace{1cm} (C2)

Assuming phase matching conditions are met, the interaction Hamiltonian terms are now

$$H_{int}^{(A)} = \sum_n \hbar \int dz \ g_n^{(A)\dagger} a_n^{(A)}(z) a_{n-1}^{(A)\dagger}(z) b^{(A)\dagger}(z) + h.c.,$$

$$H_{int}^{(B)} = \sum_n \hbar \int dz \ g_n^{(B)\dagger} a_n^{(B)}(z) a_{n-1}^{(B)\dagger}(z) b^{(B)\dagger}(z) + h.c.,$$  \hspace{1cm} (C3)

where we see the three wave process in each waveguide, involving two optical tones and an acoustic mode. The optical Hamiltonian terms stay unchanged from equation (A2). We can now calculate the equations of motion of the phonons using equation (A3), yielding

$$\dot{b}^{(A)}(z, t) = -i \hat{\Omega}_0 b^{(A)}(z, t) + i \mu b^{(B)}(z, t) - i \sum_m g_m^{(A)\dagger} a_m^{(A)}(z, t) a_{m-1}^{(A)\dagger}(z, t),$$

$$\dot{b}^{(B)}(z, t) = -i \hat{\Omega}_0 b^{(B)}(z, t) + i \mu b^{(A)}(z, t) - i \sum_m g_m^{(B)\dagger} a_m^{(B)}(z, t) a_{m-1}^{(B)\dagger}(z, t).$$  \hspace{1cm} (C4)
This result is consistent with a temporal coupled-mode theory approach such as described in [17, 39]. Alternatively, we can diagonalize equation (C2) to the eigen-basis such that we have two decoupled phonon modes

\[ H_{ac} = \hbar \int dz \left( b_+^* b_+ + b_-^* b_- \right) \begin{pmatrix} \hat{\Omega}_0 & 0 \\ 0 & \hat{\Omega}_0 \end{pmatrix} \begin{pmatrix} b_+ \\ b_- \end{pmatrix}. \]  

(C5)

These phonon 'super-modes' extend spatially to both waveguides, and can be written as a superposition of the spatial phonon modes \( b_{\pm}(z) = (b^{(A)} \pm b^{(B)})/\sqrt{2} \), and their respective frequencies \( \pm \hat{\Omega}_0 \pm \mu \), which retain the commutation relations \([b_\pm(z), t), b_\pm^\dagger(z', t)] = \delta(z - z') \) and \([b_\pm(z), t), b_\pm^\dagger(z', t)] = 0 \). The interaction Hamiltonian terms in this basis are now

\[
H_{int}^{(A)} = \sum_n \hbar \int dz \left( g^{(A)\mu}_{n} a_{n}^{(A)}(z) a_{n-1}^{(A)\dagger}(z) b_+^\dagger(z) + g^{(A)\mu}_{n} a_{n-1}^{(A)}(z) a_{n}^{(A)\dagger}(z) b_+^\dagger(z) + h.c., \right),
\]

\[
H_{int}^{(B)} = \sum_n \hbar \int dz \left( g^{(B)\mu}_{n} a_{n}^{(B)}(z) a_{n-1}^{(B)\dagger}(z) b_-^\dagger(z) + g^{(B)\mu}_{n} a_{n-1}^{(B)}(z) a_{n}^{(B)\dagger}(z) b_-^\dagger(z) + h.c., \right),
\]

where \( g^{(\mu)}_{n} \) is the coupling rate between the phonon super-modes and the optical tones in waveguide \( \ell \), where \( \ell = [A, B] \).

We can calculate the equations of motion using equation (A3), and keep the dispersion operator to first order for the optical modes, and to zero order for the acoustic modes which have a vanishing group velocity. Additionally, we factor out the fast oscillating term, and add a dissipation rate \( \Gamma \) to the phonon modes, yielding

\[
\begin{align*}
\dot{a}_{n}^{(A)} &= -i \left( g^{(A)\mu}_{n} a_{n-1}^{(A)}(z) b_+ + g^{(A)\mu}_{n} a_{n}^{(A)}(z) b_+^\dagger + g^{(A)\mu}_{n} a_{n}^{(A)}(z) b_- + g^{(A)\mu}_{n} a_{n-1}^{(A)}(z) b_-^\dagger \right), \\
\dot{a}_{n}^{(B)} &= -i \left( g^{(B)\mu}_{n} a_{n-1}^{(B)}(z) b_+ + g^{(B)\mu}_{n} a_{n}^{(B)}(z) b_+^\dagger + g^{(B)\mu}_{n} a_{n}^{(B)}(z) b_- + g^{(B)\mu}_{n} a_{n-1}^{(B)}(z) b_-^\dagger \right), \\
\dot{b}_\pm &= i \left( \hat{\Omega}_0 - \Omega \pm i \frac{\Gamma}{2} \right) b_\pm = -i \sum_n \left( g^{(A)\mu}_{n} a_{n}^{(A)}(z) a_{n-1}^{(A)\dagger}(z) + g^{(B)\mu}_{n} a_{n}^{(B)}(z) a_{n-1}^{(B)\dagger}(z) \right). 
\end{align*}
\]

(C7)

Appendix D. Generalizing to multiple coupled phonons

Using the same approach, we can expand the previous derivation to an arbitrary number of acoustic coupled modes. Assuming \( N \) identical waveguides supporting the same phonon resonance and nearest neighbor coupling, the acoustic Hamiltonian is now given by

\[ H_{ac} = \hbar \int dz \left( b^{(A)} b^{(A)*} + b^{(B)} b^{(B)*} + \cdots + b^{(N)} b^{(N)*} \right) \begin{pmatrix} \hat{\Omega}_0 & \mu & \mu^* & \cdots & \mu \\ \mu^* & \hat{\Omega}_0 & \mu & \cdots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu & \mu^* & \cdots & \hat{\Omega}_0 & \mu \\ \mu^* & \mu & \cdots & \mu & \hat{\Omega}_0 \end{pmatrix} \begin{pmatrix} b^{(A)} \\ b^{(B)} \\ b^{(C)} \\ \vdots \\ b^{(N)} \end{pmatrix}, \]  

(D1)

where \( \mu = |\mu|e^{i\phi} \), and we assume 0 in all matrix elements not on the three main diagonals. The interaction Hamiltonian in the \( \ell \) th waveguide is now given by

\[ H_{int}^{(\ell)} = \sum_n \hbar \int dz \ g^{(\ell)\mu}_{n} a_{n}^{(\ell)}(z) a_{n-1}^{(\ell)\dagger}(z) b^{(\ell)*}(z) + h.c. \]  

(D2)

The tri-diagonal matrix of equation (D1) can be diagonalized, yielding \( N \) distinct eigenvalues [40]

\[ \hat{\Omega}_m = \hat{\Omega}_0 + 2|\mu| \cos \left( \frac{m \pi}{N + 1} \right), \]  

(D3)

and the phonon fields can be decomposed into phonon eigen-modes

\[ b^{(\ell)} = \sum_{m=0}^{N} V_m^{(\ell)} b_m, \quad V_m^{(\ell)} = \sqrt{\frac{2}{N + 1}} \sin \left( \frac{m \ell \pi}{N + 1} \right) e^{im\phi}, \]  

(D4)

where we have chosen the notation such that \( m \) is an index for phonon super-modes, \( \ell \) enumerates the different waveguides, and \( n \) sums over the different optical tones spaced by frequency \( \Omega \). We can now rewrite the acoustic and interaction terms of the Hamiltonian.
\[ H_{\text{opt}} = \sum_{m=1}^{N} \hbar \int dz \ b_m^\dagger \hat{\Omega}_m b_m \]
\[ H_{\text{int}}^{(c)} = \sum_{m=1}^{N} \hbar \int dz \ g_m^{(c)}(z) a_m^{(c)}(z) a_{m-1}^{(c)}(z) b_m^\dagger(z) + \text{h.c.}, \]
(D5)
where the rate \( g_m^{(c)} \), denoting the coupling of the \( m^{th} \) phonon eigen-modes to the optical tones in the \( c^{th} \) waveguide, can be described in terms of the single-phonon coupling \( g_m^{(c)} = g_m^{(c)}(z) \) using equation (D4).

Calculating the equations of motion using equation (A3), under the same assumptions as in the previous section, we can write the field envelopes for the phonon eigen-modes and the optical fields in the \( c^{th} \) membrane

\[
b_m = -i \left( \frac{1}{i\Delta_m + \Gamma_m/2} \right) \sum_{n=1}^{N} g_m^{(c)}(z) a_{n-1}^{(c)}(z) b_m^\dagger(z) + \text{h.c.},
\]
(E6)
Assuming two tones at the input of waveguide A with a frequency separation \( \Omega \), and a single tone in the input of waveguide B, the phonon field from equation (D6) is now

\[
b_m = -i \left( \frac{1}{i\Delta_m + \Gamma_m/2} \right) s_m^{(A)}(z) a_{0}^{(A)}(0) a_{-1}^{(A)}(0),
\]
(E7)
where \( \Delta_m = \Omega_m = \Omega \) and \( \Gamma_m \) are the detuning and the loss of the \( m^{th} \) phonon eigen-mode respectively.

Following the same steps as described in section 2.1, we can find the equation of motion for the optical tones in waveguide \( \ell \)

\[
\frac{\partial a_m^{(c)}}{\partial z} = -\frac{1}{\nu} [g^{(c)}(z) a_{m-1}^{(c)}(z)][\chi^{(c)}(z)][a_{m}^{(c)}(z)e^{i\phi(z)}e^{i\Omega z} - a_{m}^{(c)}(z)e^{-i\phi(z)}e^{-i\Omega z}],
\]
(D8)
where the frequency response is denoted \( \chi^{(c)} = \sum_{n=1}^{N} \nu^{(c)}(z)\nu^{(A)(B)} (i\Delta_n + \Gamma_m/2)^{-1} \), \( \phi^{(c)} = \arg(\chi^{(c)}) \), and the phase difference between the two input tones to waveguide A is given by \( \Lambda = \arg(a_{m}^{(c)}(0) a_{n}^{(A)(0)}) \). This equation has an identical form to equation (7), and following the steps presented in section 2 we can find the optical field envelopes

\[
a_{m}^{(A)}(z) = a_{m}^{(A)}(0)/\nu^{(A)} + a_{m}^{(A)}(0)J_{-(a+1)}(\xi^{(A)}z)e^{i(a+1)z} + a_{m}^{(A)}(0)J_{-(a+1)}(\xi^{(A)}z)e^{i(a+1)z},
\]
(D9)
where \( \xi^{(A)} = (\Gamma/2) |\chi^{(A)}|G^{(A)}(P^{(A)}_{-1}P^{(A)})^{1/2} \) and \( \xi^{(B)} = (\Gamma/2) |\chi^{(B)}|G^{(B)}(P^{(B)}_{-1}P^{(B)})^{1/2} \).

We note that waveguides A and B do not have to be in specific spatial positions along the coupled-waveguide array, and are defined only by the optical inputs. The field amplitude at the output of waveguide B is now

\[
s_{\text{out}}^{(B)}(t) = \sqrt{\nu^{(B)}} e^{-i\omega_0 t} \sum_{n} P_n^{(B)}/2 \chi^{(B)}(z) G^{(B)} \sqrt{P^{(A)}_{-1}P^{(A)}} e^{i(\Omega(t-(\phi^{(B)} + \Lambda(t))\pi/\nu)}
\]
(D10)
and by using the Jacobi–Anger expansion, this can also be expressed as

\[
s_{\text{out}}^{(B)}(t) = \sqrt{\nu^{(B)}} e^{-i\omega_0 t} \exp \left[ i \frac{\Gamma}{2} \chi^{(B)}(z) G^{(B)} \sqrt{P^{(A)}_{-1}P^{(A)}} z \sin(\Omega t - (\phi^{(B)} + \Lambda)) \right].
\]
(D11)
We can examine the frequency response at the output of waveguide B for a few cases.

- **Single acoustic mode:**

\[
\chi^{(B)} = \frac{1}{\Omega - (\Omega_0 - i\Gamma/2)}.
\]
(D12)
This is a single pole function, with a maximum amplitude response at \( \Omega = \Omega_0 \).

- **Two coupled acoustic modes:**

\[
\chi^{(B)} = \frac{i\mu}{[\Omega - (\Omega_0 - i\Gamma/2)][\Omega - (\Omega_0 - i\Gamma/2)]}.
\]
(D13)
This function has two poles, with a maximum amplitude response at \( \Omega = \Omega_0 \pm \sqrt{\mu^2 - (\Gamma/2)^2} \).

- **Three coupled acoustic modes:**

\[
\chi^{(B)} = \frac{-\mu^2}{[\Omega - (\Omega_0 - i\Gamma/2)][\Omega - (\Omega_0 - i\Gamma/2)]}.
\]
(D14)
Now we have a function with three poles, which will have maximum amplitude at frequencies
\[ \Omega = \Omega_0, \]
\[ \Omega = \Omega_0 \pm \left( \frac{\Gamma}{4} \right) \mu^2 + \frac{\mu}{2} \sqrt{\mu^2 - 6(\Gamma/2)^2} - \left( \Gamma/2 \right)^2, \]
for different numbers of coupled acoustic modes. Now, the frequency response is modified around the
corresponding to a Lorentzian function. (c) Normalized spectrum of thermally induced spontaneous-
scattering, for different numbers of coupled acoustic modes, following equation (22). (d)–(f) Magnified view of panels (a)–(c),
respectively. All of the simulated frequency responses were calculated using \( \mu = \Gamma/2 \).

**Appendix E. Limited optical cascading**

We now examine the case where there is no optical cascading, such that only two optical tones propagate in each of the waveguides A and B. This can be a result of high optical dispersion leading to phase matching of the nonlinear process only between two tones [27], or when an inter-modal Brillouin process drives the acoustic field [37, 38]. The different geometries and conditions for Brillouin scattering, and the resulting phase matched nonlinear processes are illustrated in figure E1.

Starting with equation (1), we consider two optical tones in waveguides A and B by keeping only \( n = \{-1, 0\} \), leaving us with

\[
\begin{align*}
\frac{\partial a_n^{(A)}}{\partial z} &= -i \frac{1}{\nu^{(A)}} g^{(A)} b a_{n+1}^{(A)}, \\
\frac{\partial a_{n+1}^{(A)}}{\partial z} &= -i \frac{1}{\nu^{(B)}} g^{(B)} b^{\dagger} a_n^{(A)}, \\
\frac{\partial a_n^{(B)}}{\partial z} &= -i \frac{1}{\nu^{(B)}} g^{(B)} b a_{n+1}^{(B)}, \\
\frac{\partial a_{n+1}^{(B)}}{\partial z} &= -i \frac{1}{\nu^{(B)}} g^{(B)} b^{\dagger} a_n^{(B)},
\end{align*}
\]

(E1)

and the acoustic field

\[ b = -i \left( \frac{1}{i \Delta + \Gamma/2} \right) \left( g^{(A)} g^{(B)} a_{n+1}^{(A)} a_n^{(B)} + g^{(A)} g^{(B)} a_{n+1}^{(B)} a_n^{(A)} \right). \]

(E2)

We note that in this inter-modal process, the phonons do not have a vanishing group velocity as in the FSBS case. However, the axial spatial evolution of the acoustic field is very slow compared to the optical fields and can be adiabatically eliminated, such that equation (E2) is still valid [21, 41]. The number of photons in each waveguide is conserved, which can be seen from the derivatives

Figure D1. (a) Frequency response of the nonlocal susceptibility in waveguide B for different numbers of coupled acoustic modes, following equation (18), showing a sharper response with the addition of coupled phonon modes to the system. (b) Brillouin induced susceptibility in waveguide A for different numbers of coupled acoustic modes. Now, the frequency response is modified around the resonance, but decays in a similar manner to a Lorentzian function. (c) Normalized spectrum of thermally induced spontaneous-scattering, for different numbers of coupled acoustic modes, following equation (22). (d)–(f) Magnified view of panels (a)–(c), respectively.
The phonon field is not constant in space, and is in fact amplified or damped along the propagation direction \([28]\). However, assuming two tones in the input to waveguide A and a single tone to waveguide B, to leading order (\(|a_1^{(A)} a_0^{(A)}|^2 \gg |a_{-1}^{(B)} a_0^{(B)}|^2\)), we can drop the second term in equation (E2), yielding

\[
\frac{\partial}{\partial z}(a_0^{(A)} a_{-1}^{(A)} a_0^{(A)} + a_{-1}^{(A)} a_{-1}^{(A)}) = 0, \quad \frac{\partial}{\partial z}(a_0^{(B)} a_{-1}^{(B)} a_0^{(B)} + a_{-1}^{(B)} a_{-1}^{(B)}) = 0.
\]

(E3)

The phonon field is not constant in space, and is in fact amplified or damped along the propagation direction \([28]\). However, assuming two tones in the input to waveguide A and a single tone to waveguide B, to leading order (\(|a_1^{(A)} a_0^{(A)}|^2 \gg |a_{-1}^{(B)} a_0^{(B)}|^2\)), we can drop the second term in equation (E2), yielding

\[
b = -i\chi a_0^{(A)} a_{-1}^{(A)} a_0^{(A)},
\]

(E4)

where \(\chi = (i\Delta + \Gamma/2)^{-1}\). We can use our notation for the nonlinear susceptibilities

\[
\chi^{(A)} = -\frac{i}{\gamma^{(A)}} |g^{(A)}|^2 \chi^*, \quad \gamma^{(B)} = -\frac{i}{\gamma^{(B)}} |g^{(B)}|^2 g^{(A)} \chi^*,
\]

giving us the equations of motion for the optical waves

\[
\frac{\partial a_0^{(A)}}{\partial z} = i\gamma^{(A)} a_{-1}^{(A)} a_0^{(A)} a_{-1}^{(A)}, \quad \frac{\partial a_0^{(B)}}{\partial z} = i\gamma^{(B)} a_{-1}^{(B)} a_0^{(B)} a_{-1}^{(B)},
\]

\[
\frac{\partial a_{-1}^{(A)}}{\partial z} = i\gamma^{(A)} a_0^{(A)} a_{-1}^{(A)} a_0^{(A)}, \quad \frac{\partial a_{-1}^{(B)}}{\partial z} = i\gamma^{(B)} a_0^{(B)} a_{-1}^{(B)} a_0^{(B)},
\]

(E6)

Figure E1. (a) Forward SBS (stimulated Brillouin scattering), where the acoustic mode is perpendicular to the optical propagation direction, and can explore a large space overlapping with another waveguide. (b) The dispersion curve of forward SBS shows the single optical mode taking part in the process. In the case of no optical dispersion (constant group velocity), a cascaded array of optical tones can all couple to the same phonon. In the limit of high dispersion, the cascading is limited to a finite number of optical tones, determined by the phase-matching conditions. (c) The acoustic dispersion curve, illustrating the cutoff mode, with a vanishing axial wave-vector taking part in forward SBS. (d) Inter-modal SBS, where two different spatial optical modes take part in the nonlinear process, and an acoustic mode which can extend far beyond the optical mode region. (e) The optical dispersion curves show that the phonon is phase-matched to a transition between two different optical spatial modes. (f) The acoustic mode taking part in the nonlinear process is a cutoff mode with a small axial wave-vector. This results in an acoustic mode with a non-zero axial wave-vector. (g) In backward SBS, the optical tones are counter propagating. This geometry can be phase-matched to a bulk acoustic mode confined spatially to the same region as the optical waves, propagating along the same axis. (h) The dispersion diagram of the backward SBS process shows the phase-matching conditions for the scattering between forward and backward propagating optical tones. (i) In backward SBS, a bulk acoustic mode with a large axial wave-vector, propagating parallel to the optical waves, is phase-matched in this scattering process.
illustrating the nonlocal nonlinear susceptibility in the two-tone case. A description of the spatial dynamics of this case can be found in [21].

For completeness, we mention that a backward Brillouin process, where the two optical modes are counter propagating, requires a phonon with a large wave-vector to take part in the process [1]. This is commonly achieved by using a bulk-acoustic mode, as illustrated in figure E1(i). In this case, the acoustic field typically has a similar spatial extent as the optical waves. However, if a cutoff acoustic mode takes part in the three wave process, it could extend in space further than the overlap region, similar to the inter-modal SBS case.

**Appendix F. Effects of dispersion**

In our analysis, we have considered a constant optical group velocity for all optical tones, such that the optical group velocity dispersion (GVD) is zero. This is a good approximation in many systems [13, 25, 27] and can further be avoided through the design of zero-GVD waveguides [30, 31]. Hence, we have used a single value for the group velocity, and assumed an infinite summation in equation (1) and the following derivations.

When considering non-zero GVD, the modulation experienced by the light in waveguide B will no longer be purely phase modulated and will exhibit some residual intensity modulation [28]. We numerically demonstrate the effect of GVD by keeping the second order term of the optical dispersion operator in equation (A5) in both waveguides A and B

$$\begin{align*}
    b &= -i \left( \frac{1}{i\Delta + \Gamma/2} \right) \sum_n \left( g_n^g a_n^{(A)} a_n^{(A)\dagger} e^{-i\Delta k_n z} + g_n^b a_n^{(B)} a_n^{(B)\dagger} e^{-i\Delta k_n z} \right), \\
    &= -i \left( \frac{\partial^2 \omega}{\partial^2 \omega} \right) \frac{\partial^2 a_n}{\partial^2 z} + \nu_a \frac{\partial a_n}{\partial z} = -i \left( g_n^g a_{n-1} b e^{i\Delta k_n z} + g_n^b a_{n+1} b e^{-i\Delta k_n z} \right),
\end{align*}
$$

where we now have frequency dependent optical group velocity $\nu_a = \partial \omega_{\nu_a} / \partial (\nu_0 + \nu_0)$ and a phase mismatch term $\Delta k_n = q_n - (k_n - k_{n-1}) \approx \Omega (\nu_0^{-1} - \nu_0^{-1})$, where we have used a piecewise linear approximation. The dispersion leads to non-symmetric sidebands around the carrier and now the optical field is not only phase modulated, but also has residual amplitude modulation (RAM). We quantify the RAM by looking at the components of the intensity at frequency $\Omega$, normalized to the total scattered power

$$\text{RAM} = \frac{\sum_n \left(a_n^{(B)\dagger} a_n^{(B)} - a_n^{(A)\dagger} a_n^{(A)}\right)}{\sum_n \left(a_n^{(B)\dagger} a_n^{(B)} + a_n^{(A)\dagger} a_n^{(A)}\right)}.$$

The numerically simulated RAM is shown in figure F1, where the dispersion is given both in terms of GVD, and a commonly used dispersion parameter $D$, defined by

$$GVD = \left. \frac{\partial^2 k}{\partial \omega^2} \right|_{\omega = \omega_0}, \quad D = \frac{2\pi c}{\lambda^2} \text{GVD}.$$

When using numerical values consistent with common materials and waveguide design, the simulation results in a residual intensity modulation on the order of $-50 \text{ dB}$. In many applications, the frequency response of other optical elements in the system, such as detectors, couplers and modulators, can yield similar or more dominant effects on the signal.

In our analysis, we have studied the case where both waveguides A and B have the same group velocity, based on the assumption that they are similar in design and operating at similar wavelengths. However, when using vastly different optical wavelengths in waveguides A and B, the group velocity in each wavelength range can be different, and can lead to a phase mismatch between the phonons driven in waveguide A and those needed for efficient phase modulation in waveguide B. This phase mismatch can be expressed in terms of the driving frequency $\Omega$ and the frequency dependent group velocity $\nu(\omega)$

$$\Delta q = \Omega \left( \frac{1}{\nu(\omega_0^{(A)})} - \frac{1}{\nu(\omega_0^{(B)})} \right),$$

where we assume the optical dispersion is identical in both waveguides A and B. For the phase mismatch to be small over the device length, i.e. $\Delta q L < \pi/2$, this gives us the condition

$$\left( \frac{1}{\nu(\omega_0^{(A)})} - \frac{1}{\nu(\omega_0^{(B)})} \right) < \frac{\pi}{2\Omega L}.$$
Appendix G. Alternative derivation of the equations of motion

We can derive the equations of motion describing the dynamics of a forward Brillouin process (FSBS) using classical variables as an alternative to the derivation given earlier in appendix A. We start by considering an acoustic eigen-mode \( \psi_0 \) at frequency \( \omega_0 \) using
\[
\rho \frac{\partial^2 \psi_0}{\partial t^2} - \omega_0^2 \psi_0 = -\rho \Omega_0^2 \psi_0, \tag{G1}
\]
where \( x \) is a spatial coordinate, \( \rho \) the mass density of the waveguide material, \( c_{ijkl} \) the elastic tensor, and summation is implied for repeated indexes. The guided acoustic eigen-modes relevant to FSBS are transversely polarized, such that the acoustic mode profile can be written as
\[
\psi(x,t) = \psi_0(x) \exp(i\mathbf{q} \cdot \mathbf{r}) + \text{c.c.},
\]
where \( \mathbf{q} \) is the acoustic wave-vector. We can write the displacement field in this waveguide as
\[
\mathbf{u}(\mathbf{r}, t) = \frac{1}{2} \left( \mathbf{U}(\mathbf{r}) - \mathbf{U}(\mathbf{r}) \right) + \text{c.c.}, \tag{G2}
\]
where \( b \) is the mode amplitude.

The total energy density, \( \rho \mathbf{u}^2 \), associated with this displacement field is the sum of the potential and kinetic energy terms. Integrating the total energy density over volume and making the rotating wave approximation, we get the following Hamiltonian
\[
\langle H_{\text{ac}} \rangle = \Omega_0 \int d^3x \mathbf{U}^*(\mathbf{r}) \cdot \rho(\mathbf{r}) \mathbf{U}(\mathbf{r}) b^* b. \tag{G3}
\]

We normalize the acoustic eigen-mode profile such that \( \int d^3x \mathbf{U}^*(\mathbf{r}) \cdot \rho(\mathbf{r}) \mathbf{U}(\mathbf{r}) = 1 \), leaving us with a simplified expression for the time-averaged Hamiltonian.

Figure F1. (a) Numerically calculated residual amplitude modulation (RAM) as a function of dispersion, for a 2.5 cm long device with Brillouin gain \( G_{1B} = 700 \) (W m\(^{-1}\)) and an acoustic Q-factor of 1000. The input powers are 100 mW in each of the two tones into waveguide A and 100 mW into a single tone in waveguide B. The parameters have been chosen similar to those demonstrated in [18].
(b) The same results shown on a logarithmic scale. Values of GVD in bulk silicon and optical fiber are shown for reference. (c) Normalized optical field envelope in waveguide A as a function of time shows the intensity modulation that drives an acoustic field at 4.5 GHz. (d) The normalized optical field envelope in waveguide B shows a constant envelope in the absence of dispersion (green). When some dispersion is taken into account, a small intensity modulation can be seen (red).
\[ \langle H_{ac} \rangle = \Omega_0 b^\dagger b. \]  

Photoelastic coupling of the light and sound fields result in an interaction term in the Hamiltonian. To calculate this term, we look at the change in the energy density \( \delta U = \sigma_{mn} \delta S_{mn} \). Here, \( \sigma_{mn} \) is the stress and \( \delta S_{mn} = 1/2(\partial u_m/\partial x_n + \partial u_n/\partial x_m) \) is the strain. For cubic and isotropic media, the electrostrictive stress is given by

\[ \sigma_{mn} = -\frac{1}{2} \varepsilon_0 \varepsilon_0^2 P_{k\ell m} E_k E_{k\ell}, \]

where \( \varepsilon_0 \) is the relative permittivity, \( P_{k\ell m} \) is the photo-elastic tensor, and \( E \) is the total electric field inside the waveguide (summation is implied for repeated indexes).

The resulting interaction Hamiltonian from the change in energy density is given by

\[ H_{int} = \int d^3x \frac{1}{2} \varepsilon_0 \varepsilon_0^2 P_{k\ell m} E_k E_{k\ell} \delta S_{mn}, \]

and when we consider electric fields co-polarized in the \( x \)-direction, this yields

\[ H_{int} = -\frac{1}{2} \varepsilon_0 \int d^3x \varepsilon_0^2 \left( p_{11} \frac{\partial u_x}{\partial x} + p_{12} \frac{\partial u_y}{\partial y} + p_{12} \frac{\partial u_z}{\partial z} \right) E_x E_x. \]  

We describe an electric field propagating in the waveguide

\[ E_n(\mathbf{r}, t) = \frac{1}{2} \sum_n (E_n(\mathbf{r}) a_n e^{i(k_n x - \omega_n t)} + c.c.) \hat{x}, \]

where \( E_n(\mathbf{r}) \) is the transverse optical mode profile, \( a_n \) the optical mode, \( k_n \) the wave vector and \( \omega_n \) the angular frequency of the different optical frequencies, spaced by \( \Omega (\Omega = \omega_n - \omega_{n-1}) \). We further assume a phase-matched process, such that \( q = k_n - k_{n-1} \).

Substituting \( E_n(\mathbf{r}, t) \) and \( u_n(\mathbf{r}, t) \) into the interaction Hamiltonian from equation (G7) we have

\[ \langle H_{int} \rangle = \frac{1}{16} \varepsilon_0 \frac{2}{\Omega_0} \int d^3x \varepsilon_0^2 \left\{ p_{11} \frac{\partial u_x}{\partial x} e^{i q b} + p_{12} \frac{\partial u_y}{\partial y} e^{i n b} + c.c. \right\} \left( \sum_n E_n(\mathbf{r}) a_n e^{i(k_n x - \omega_n t)} + c.c. \right)^2. \]  

Making the rotating wave approximation, where we only consider the terms in \( H_{int} \) that vary with frequency \( \Omega \) (i.e. terms close to \( \Omega_0 \) that can drive the acoustic mode) we get

\[ \langle H_{int} \rangle = -\frac{1}{8} \varepsilon_0 \frac{2}{\Omega_0} \int d^3x \varepsilon_0^2 \sum_n S^*(\mathbf{r}) E_{n-1}(\mathbf{r}) E_n(\mathbf{r}) b^\dagger a^\dagger_n a_n e^{-i \Delta t} + c.c., \]

where the strain profile is given by

\[ S(\mathbf{r}) = p_{11} \frac{\partial u_x}{\partial x} + p_{12} \frac{\partial u_y}{\partial y}. \]

Carrying out the integration in the axial direction over the waveguide length \( L \), and defining the following coupling term

\[ g = \int dx dy \varepsilon_0^2 \sum_n \left( S(\mathbf{r}) E_{n-1}(\mathbf{r}) E_n(\mathbf{r}) S(\mathbf{r}) \right), \]

the interaction Hamiltonian resulting from the photoelastic coupling of light and sound is given by

\[ \langle H_{int} \rangle = -\frac{1}{8} \varepsilon_0 L \frac{2}{\Omega_0} \sum_n g b^\dagger a^\dagger_n a_n e^{-i \Delta t} + c.c. \]

The total time-averaged Hamiltonian of the system, including the acoustic and interaction terms, is

\[ \langle H \rangle = \langle H_{ac} \rangle + \langle H_{int} \rangle = \Omega_0 b^\dagger b - \gamma \sum_n (g b^\dagger a^\dagger_n a_n e^{-i \Delta t} + g a^\dagger_n a_n b e^{i \Delta t}), \]

where we have defined \( \gamma = \frac{1}{4} \varepsilon_0 L \sqrt{2/\Omega_0} \). Now that we have derived the Hamiltonian of the system, we can calculate the time evolution of the acoustic field using [43]

\[ \dot{b}(t) = i \{ H, b \}, \]
where the Poisson bracket is defined \[ \{H, b\} = [(\partial H / \partial b)(\partial b / \partial b^*) - (\partial H / \partial b^*)(\partial b / \partial b)] \], yielding

\[
b(t) = -\frac{\partial H}{\partial b^*} = -i\Omega_0 b + i\sum_n g_n^a a_n^* e^{-i\Omega t}.
\] (G16)

We now look at the envelope of the field, \( b(t) = \bar{b}(t) e^{-i\Omega t} \), leaving us with the following equation of motion

\[
\dot{\bar{b}}(t) = i(\Omega - \Omega_0) \bar{b} + i\sum_n g_n^a a_n^*.
\] (G17)

We account for phonon dissipation by adding an imaginary part to the frequency \( \Omega_0 \rightarrow \Omega_0 - i\Gamma/2 \), and see the steady state mode amplitude, i.e. \( \bar{b}(t) = 0 \), is given by

\[
\bar{b} = -\gamma \frac{\sum_n g_n^a a_n^*}{(\Omega - \Omega_0 + i\Gamma/2)}.
\] (G18)

Plugging back into equation (G2), the steady state displacement field associated with this phonon mode is

\[
\mathbf{u}(\mathbf{r}, t) = \frac{1}{2} \left( -\frac{\epsilon_n \epsilon_0}{4\Omega_0} \mathbf{U}(\mathbf{r}_s) \sum_n g_n^a a_n^* \frac{\epsilon(\Omega - \Omega_0 + i\Gamma/2)}{(\Omega - \Omega_0 + i\Gamma/2)^2} + c.c. \right).
\] (G19)

A nonlinear polarization generated by this displacement field can scatter photons from pump to Stokes or anti-Stokes. The spatial and temporal evolution of the optical field in a medium with nonlinear polarization is described by [1]

\[
\frac{\partial^2 E}{\partial z^2} + \frac{1}{v_p^2} \frac{\partial^2 E}{\partial t^2} = \frac{1}{\epsilon_0 c^2} \frac{\partial^2 P_{NL}}{\partial t^2},
\] (G20)

where \( v_p \) is the phase velocity of the light in the waveguide, \( c \) is the velocity of light in vacuum and \( P_{NL} \) is the nonlinear polarization. Since all the electric fields are polarized along \( \mathbf{S} \), \( P_{NL} = \epsilon_0 \epsilon_1^2 \mathbf{P}_{1\text{int}} \delta S_m \mathbf{E}_s \) reduces to

\[
P_{NL} = \epsilon_0 \epsilon_1^2 \frac{\epsilon_0}{16\Omega_m} \left( \sum_n g_n^a a_n^* S_m \right) e^{i(\Omega - \Omega_0) t} + c.c. \left( \sum_{n} g_n^a a_n^* e^{i(\Omega - \Omega_0 + i\Gamma/2) t} + c.c. \right),
\] (G21)

after substituting the displacement field. Considering only the terms of nonlinear polarization that are phase matched to drive the field we get

\[
P_{NL} = \epsilon_0 \epsilon_1^2 \frac{\epsilon_0}{16\Omega_m} \left( \sum_n g_n^a a_n^* S_{m+1} \right) e^{i(\Omega - \Omega_0 + i\Gamma/2) t} + c.c.
\] (G22)

Substituting \( E_n(r, t) \) and \( P_{NL} \) into equation (G20) and making the slowly varying envelope approximation results in

\[
E(r_s) \left( 2ik_n \frac{\partial a_n}{\partial z} + \frac{2ikn}{v_p^2} \frac{\partial a_n}{\partial t} \right) = \epsilon_0 \epsilon_1^2 \frac{\epsilon_0}{8\Omega_0 \epsilon_0 c^2} \left( \sum_n g_n^a a_n^* S_{m+1} \right) \frac{\epsilon(\Omega - \Omega_0 + i\Gamma/2)}{(\Omega - \Omega_0 + i\Gamma/2)^2} a_{n-1}
\]

\[
+ E(r_s) S_{n+1} \frac{\epsilon(\Omega - \Omega_0 + i\Gamma/2)}{(\Omega - \Omega_0 + i\Gamma/2)} a_{n+1}.
\] (G23)

We examine the steady state and integrate the transverse dimension after multiplying both sides by \( E^*(r_s) \) to get

\[
\frac{\partial a_n}{\partial z} = -i\alpha |g|^2 \left( \frac{\sum_n g_n^a a_n^* a_{n+1}^*}{(\Omega - \Omega_0 + i\Gamma/2)} a_{n-1} + \frac{\sum_n g_n^a a_n^* a_{n+1}^*}{(\Omega - \Omega_0 + i\Gamma/2)} a_{n+1} \right),
\] (G24)

where \( \alpha = \epsilon_0 \omega_0^2 / (16\pi, \Omega_0 c^2, \langle \mathbf{U}, \rho \mathbf{U} \rangle \langle \mathbf{E}_m, \mathbf{E}_n \rangle) \), and \( \langle \mathbf{E}_m, \mathbf{E}_n \rangle = \int dxdy \ E^*(r_s) \mathbf{E}(r_s) \). This result is equivalent to equations (A8), (A9) which were derived in a quantum operator Hamiltonian approach using the Heisenberg equations of motion.
In order to describe a device with multiple waveguides and multiple acoustic modes, we can carry out a similar derivation, where each of the separate optical fields can be described by equation (G8) in each of the waveguides. In the case of multiple acoustic modes, the acoustic field in equation (G2) can be described as the sum of the acoustic eigenmodes

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{2} \sum_{m=1}^{N} \left( \frac{2}{\Omega_m} \mathbf{U}_m(\mathbf{r}_c) b_m e^{i(\mathbf{q}_m \cdot \mathbf{r} - \Omega_m t)} + \text{c.c.} \right),$$

where $\mathbf{U}_m(\mathbf{r}_c)$ is the $m^{th}$ transverse acoustic eigen-mode, and $\mathbf{q}$ is the acoustic wavevector in the axial direction.

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**References**

[1] Boyd R W 2003 *Nonlinear Optics* (New York: Academic)
[2] Dabby F W and Whinnery J R 1968 Thermal self-focusing of laser beams in lead glasses *Appl. Phys. Lett.* 13 284–6
[3] Horowitz M, Daisy R, Werner O and Fischer B 1992 Large thermal nonlinearities and spatial self-phase modulation in Sr,Ba$_{1-x}$Nb$_x$O$_3$ and BaTiO$_3$ crystals *Opt. Lett.* 17 475–7
[4] Rotschild C, Alfassi B, Cohen O and Segev M 2006 Long-range interactions between optical solitons *Nat. Phys.* 2 769
[5] Izdebskaya E Y, Shvedov V G, Jung P S and Krolinskiowski W 2018 Stable vortex soliton in nonlocal media with orientational nonlinearity *Opt. Lett.* 43 66–9
[6] Shahmoon E, Grims P, Stimming H P, Mazets I and Kurizki G 2016 Highly nonlocal optical nonlinearities in atoms trapped near a waveguide *Optica* 3 725–33
[7] Sevičiê S, Henkel N, Ates C and Pohl T 2011 Nonlocal nonlinear optics in cold Rydberg gases *Phys. Rev. Lett.* 107 153001
[8] Pollard R J, Murphy A, Hendren W R, Evans P R, Atkinson R, Wurtz G A, Zayats A V and Podolskiy V A 2009 Optical nonlocalities and additional waves in epsilon-near-zero metamaterials *Phys. Rev. Lett.* 102 127405
[9] Krasanin A V, Ginzburg P, Wurtz G A and Zayats A V 2016 Nonlocality-driven supercontinent white light generation in plasmonic nanostructures *Nat. Commun.* 7 11497
[10] Fakhri P, Rashidian Vaziri M R, Jaleh B and Partovi Shabestari N 2015 Nonlocal nonlinear optical response of graphene oxide–Au nanoparticle dispersions in different solvents *J. Opt.* 18 015502
[11] Renninger W H, Khare P, Behunin R O and Rakich P T 2018 Bulk crystalline optomechanics *Nat. Phys.* 14 601
[12] Shin H, Qiu W, Jarecki R, Cox J A, Olson R H III, Starbuck A, Wang Z and Rakich P T 2013 Tailorable stimulated Brillouin scattering in nanoscale silicon waveguides *Nat. Commun.* 4 1944
[13] Kittlaus E A, Shin H and Rakich P T 2016 Large Brillouin amplification in silicon *Nat. Photon.* 10 463–7
[14] Wiederbacher G S, Dainese P and Mayer Alegre T P 2019 Brillouin optomechanics in nanophotonic structures *APL Photonics* 4 071101
[15] Eggleton B J, Poulton C G, Rakich P T, Steel M J and Bahl G 2019 Brillouin integrated photonics *Nat. Photon.* 13 664–77
[16] Van Laer R, Kuyken B, Van Thourhout D and Baets R 2015 Interaction between light and highly confined hypersound in a silicon photonic nanowire *Nat. Photon.* 9 199
[17] Shin H, Cox J A, Jarecki R, Starbuck A, Wang Z and Rakich P T 2015 Control of coherent information via on-chip photonic-phononic emitter–receivers *Nat. Commun.* 6 6427
[18] Kittlaus E A, Khare P, Otterstrom N T, Wang Z and Rakich P T 2018 RF–photonic filters via on-chip photonic-phononic emit-receive operations *J. Light. Technol.* 36 2803–9
[19] Diamandi H H, London Y and Zadok A 2017 Opto-mechanical inter-core cross-talk in multi-core fibers *Optica* 4 289–97
[20] Diamandi H H, London Y, Bashan G, Bergman A and Zadok A 2018 Highly-coherent stimulated phonon oscillations in a multi-core optical fiber *Sci. Rep.* 8 9514
[21] Kittlaus E A, Otterstrom N T, Khare P, Behunin R O and Rakich P T 2018 Non-reciprocal interband Brillouin modulation *Nat. Photon.* 12 613–9
[22] Rakich P T, Reinke C, Camacho R, Davids P and Wang Z 2012 Giant enhancement of stimulated Brillouin scattering in the subwavelength limit *Phys. Rev. X* 2 011008
[23] Sipe J and Steel MJ 2016 A Hamiltonian treatment of stimulated Brillouin scattering in nanoscale integrated waveguides *New J. Phys.* 18 045004
[24] Rakich P T, Davids P and Wang Z 2010 Tailoring optical forces in waveguides through radiation pressure and electrostrictive forces *Opt. Express* 18 14439–53
[25] Kang M S, Nazarkin A, Brenn A and Russell PS J 2009 Tightly trapped acoustic phonons in photonic crystal fibres as highly nonlinear artificial Raman oscillators *Nat. Phys.* 5 276
[26] Qiu W, Rakich P T, Shin H, Dong H, Soljaêi M and Wang Z 2013 Stimulated Brillouin scattering in nanoscale silicon step-index waveguides: a general framework of selection rules and calculating SBS gain *Opt. Express* 21 31 402–19
[27] Khare P, Behunin R O, Renninger W H and Rakich P T 2016 Noise and dynamics in forward Brillouin interactions *Phys. Rev. A* 93 063806
[28] Wolff C, Stiller B, Eggleton B J, Steel MJ and Poulton C G 2017 Cascaded forward Brillouin scattering to all Stokes orders *New J. Phys.* 19 023021
[29] London Y, Diamandi H H, Bashan G and Zadok A 2018 Invited article: distributed analysis of nonlinear wave mixing in fiber due to forward Brillouin scattering and Kerr effects *APL Photonics* 3 110804
[30] Petrov A Y and Eich M 2004 Zero dispersion at small group velocities in photonic crystal waveguides *Appl. Phys. Lett.* 85 4866–8
[31] Zhao Y, Zhang Y N, Wu D and Wang Q 2012 Wideband slow light with large group index and low dispersion in slotted photonic crystal waveguide *J. Light. Technol.* 30 2812–7
[32] Marpaung D, Roeloffzen C, Heideman R, Leinse A, Sales S and Capmany J 2013 Integrated microwave photonics *Laser Photonics Rev.* 7 506–8
[33] Raymer M G and Mostowski J 1981 Stimulated Raman scattering: unified treatment of spontaneous initiation and spatial propagation Phys. Rev. A 24 1980
[34] Boyd R W, Rzaewski K and Narum P 1990 Noise initiation of stimulated Brillouin scattering Phys. Rev. A 42 5514
[35] Marpaung D, Yao J and Capmany J 2019 Integrated microwave photonics Nat. Photon. 13 80
[36] Gertler S, Kittlaus E A, Otterstrom N T, Kharel P and Rakich P T 2020 Microwave filtering using forward Brillouin scattering in photonic-phononic emit-receive devices J. Light. Technol. (https://doi.org/10.1109/JLT.2020.2965825)
[37] Kang M S, Brenn A and Russell P S J 2010 All-optical control of gigahertz acoustic resonances by forward stimulated interpolarization scattering in a photonic crystal fiber Phys. Rev. Lett. 105 153901
[38] Kittlaus E A, Otterstrom N T and Rakich P T 2017 On-chip inter-modal Brillouin scattering Nat. Commun. 8 15819
[39] Little B E, Chu S T, Haus H A, Foresi J and Laine J-P 1997 Microring resonator channel dropping filters J. Light. Technol. 15 998–1005
[40] Noschese S, Pasquini L and Reichel L 2013 Tridiagonal toeplitz matrices: properties and novel applications Numer. Linear Algebra Appl. 20 302–26
[41] Otterstrom N T, Behunin R O, Kittlaus E A, Wang Z and Rakich P T 2018 A silicon Brillouin laser Science 360 1113–6
[42] Royer D and Dieulesaint E 2000 Elastic Waves in Solids vol 1 (Berlin: Springer) p 20
[43] Strocchi F 1966 Complex coordinates and quantum mechanics Rev. Mod. Phys. 38 36