Non-Abelian Chiral 2-Form and M5-Branes

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Abstract

We first review self-dual (chiral) gauge field theories by studying their Lorentz non-covariant and Lorentz covariant formulations. Then, we construct a non-Abelian self-dual two-form gauge theory in six dimensions with a spatial direction compactified on a circle. This model reduces to the Yang-Mills theory in five dimensions for a small compactified radius $R$. This model also reduces to the Lorentz-invariant Abelian self-dual two-form theory when the gauge group is Abelian. The model is expected to describe multiple 5-branes in M-theory. We will discuss its decompactified limit, covariant formulation, BRST-antifield quantization and other generalizations.

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1 Introduction

A self-dual gauge field theory (or a chiral gauge field theory) is defined as a $p$-form gauge potential whose $p + 1$ form field strength is constrained by the self-duality condition, which reduces physical degrees of freedom in a theory to the half of the case without the self-duality condition. If we consider our spacetime as the Lorentz signature, real self-dual gauge fields can exist only in $4n+2$ dimensions like in $D=2$, $D=6$ or $D=10$, which are the cases we consider in this thesis. Self-dual gauge field theories received huge attention for quite a long time because of their existence in various interesting theories. For examples: They appear in the quantum hall effect and heterotic string theory. They also show up in the exotic six-dimensional 5-brane(s) worldvolume theory in M-theory, or in NS 5-brane(s) worldvolume theory in the type IIA string theory. Self-dual fields also appear in the ten dimensional type IIB supergravity too. However, this kind of theories are quite mysterious because of the lack of a standard action principle and subtle issues of the quantization on self-dual theories. Moreover, as we will see, people face even bigger challenges when trying to describe multiple M5-branes worldvolume theory, which is the main topic of this thesis (Based on [1]).

As we know, string theories in ten dimensions can be unified by the eleven dimensional M-theory, where the basic objects are M2-branes and M5-branes (the magnetic version of M2-branes in the sense that M5-branes couple to the dual background three form C-field in 11D supergravity). One could consider that M-branes are the most fundamental objects we have. (For some recent review papers of M-theory branes, see [2]), and as a way to understand the mysterious nature of the M-theory, it is desirable to understand properties of these branes as much as possible. The description of a single M2-brane or a single M5-brane had already known for quite a long time, while the understanding of multiple M2 branes gained ground in the past few years by BLG
and ABJM theory [3] [4]. It is interesting to start thinking about the question that how to describe multiple M5-branes.

The physical degrees of freedom of the six-dimensional worldvolume theory of a M5-brane [5] consist of the so-called \( N = (2, 0) \) tensor supermultiplet, which contains two-form gauge potentials with self-dual field strengths, five scalars, and two chiral spinors, where scalars and spinors can be interpreted as Goldstone bosons and fermions associated with broken translation symmetries and supersymmetries. When the 11D M theory is compactified on a circle, it gives the 10D type IIA superstring theory. Some of p-branes in the IIA theory have simple M-theory interpretations. In particular, wrapping one dimension of the M5-brane on the compact spatial dimension gives the IIA D4-brane theory. These D4-branes must arise as the dimensional reduction of the 6D five-brane world-volume theory. This is one of the criteria to construct M5-brane theory. Throughout this thesis, we will only focus on the gauge field sector, because the gauge field part brings the most subtle issues.

Just like how the Maxwell fields (abelian gauge theory) in a single D-brane become the Yang-Mill fields (non-abelian gauge theory) when D-branes start to coincide. One expects that some kind of non-abelian 2-form gauge theory will be involved in multiple M5-branes. Although we expect that multiple M5-branes should be a non-Abelian theory, it might not be an simple non-Abelian gauge theory, because the entropy of coincident N M5-branes does not scale as \( N^2 \) like the Yang-Mills theory but rather scale as \( N^3 \) [6]. In the case of multiple M2-branes, it is also not an ordinary non-Abelian theory (the entropy of coincident N M2-branes scales as \( N^{\frac{3}{2}} \)), it has the novel gauge symmetry based on the Lie 3-algebra. It is natural to ask whether similar structure plays a crucial role in multiple M5-branes. However, although there were approaches along this line, but it is not clear how they relate to M5-branes with the existence of non-abelian chiral two-form gauge potentials [7].
Some believed that a Lagrangian formulation for self-dual gauge theories was impossible, mainly because the self-duality condition imposes first order differential equations on the gauge potentials, while an ordinary kinetic term in a standard Lagrangian (take 2-form potentials in 6D as an example)

\[ L \sim H_{\mu \nu \lambda} H^{\mu \nu \lambda} \sim \partial_{[\mu} B_{\nu \lambda]} \partial^{\mu} B^{\nu \lambda} \]  

(1)

(where \( \mu, \nu = 0, 1, 2, 3, 4, 5 \)) always leads to a 2nd order differential equation. It turns out that the trick is to avoid using some of components of the gauge potentials, for example, components \( B_{i5} \) (where \( i, j = 0, 1, 2, 3, 4 \)) will enter the action only up to surface terms. So even though we get 2nd order differential equations from varying the action, the self-duality condition appears after integrating once the equations of motion. This trick was later generalized in [8] so that for a given spacetime dimension \( D \), one can write down a Lagrangian for self-dual gauge fields for arbitrary divisions of \( D \) into two positive integers \( D' \) and \( D'' \) :

\[ D' + D'' = D \]  

(2)

We call it as the \( (D' + D'') \)-formulation of self-dual gauge theories.\(^2\) Hence the Lagrangian for a self-dual gauge field theory was first constructed without manifest Lorentz symmetry. Lorentz-covariant versions are possible only when introducing auxiliary fields [10, 11].

The gauge symmetry for a single M5-brane in the trivial background is Abelian. The first non-Abelian gauge theory for self-dual 2-form potentials was found when considering an M5-brane in a large \( C \)-field background. We call it as \( NP \) M5-brane theory, where ”\( NP \)” stands for ”Nambu-Poisson”. A Nambu-Poisson structure is used to define the non-Abelian gauge symmetry

\(^2\) Some new non-covariant Lagrangians based on further decompositions of spacetime: \( D = D_1 + D_2 + D_3 \) are given in [9]
for the 2-form potential on a M5-brane. The physical origin of this Nambu-Structure is the coupling of open membranes to the $C$-field background [13]. The NP M5-brane theory was first derived from the BLG model [3]. Its gauge field content was further explored in [14, 15]. For a brief review about this topic, see [16].

A double dimension reduction of the NP M5-brane theory is in agreement with the lowest order deformation of the noncommutative D4-brane action in large NS-NS $B$ field background [12]. Thus if this NP M5-brane theory can be properly deformed such that it agrees with the noncommutative D4-brane theory to ”all orders”, it might resemble the non-abelian structure of multiple M5-branes theory, since the multiple D4-branes theory is essentially a special case of the noncommutative D4-brane theory. However, it turns out that it is extremely hard to deform the NP M5-brane theory [17]. We conclude that it takes brand new ideas to construct multiple M5-branes theory.

In the literature, there has been various attempts to construct a non-Abelian gauge theory for 2-form gauge potentials $B_{\mu\nu}$. Taking values in a Lie algebra, the corresponding geometrical structures are called ”non-Abelian gerbes”. The immediate problem to construct such a model is that we need to define covariant derivatives $D_\mu$ to have gauge covariant structures in a theory, thus we also need a 1-form potential $A$. For example, in [18], the gauge transformations of $A$ and $B$ are defined by

$$A' = gAg^{-1} + gdg^{-1} + \Lambda, \quad (3)$$
$$B' = gBg^{-1} + [A', \Lambda]_\wedge + d\Lambda + \Lambda \wedge \Lambda, \quad (4)$$

where $g \in G$ is the gauge parameter and $\Lambda \in g$ is a 1-form. Mathematically such gauge transformations are well-defined and suitable to describe some systems such as the non-Abelian generalization of the BF model [19]. It is, however, not clear if it is relevant to describe multiple M5-branes.
Physically, the introduction of an extra field like $A$ increases the physical degrees of freedom of the system. For the M5-branes system, there is no physical degrees of freedom corresponding to any new field. Furthermore, with the addition of $A$, the field $B$ is not a genuine 2-form potential in the sense that we can gauge away $A$ by $\Lambda$, and then $B$ is not independent of its longitudinal components. The result is similar to spontaneous symmetry breaking. On the other hand, if we consider that $A$ does not have physical degrees of freedom, it means that we might have a gauge symmetry to gauge it away. But this means that the covariant derivative in a theory become just $\partial_\mu$. It will be problematic to define the non-abelian gauge transformation and the field strength of $B$ in this gauge. \(^3\)

Our goal here is to have a non-Abelian gauge symmetry which includes the Abelian theory as the special case when the Lie algebra involved is Abelian. This criterium is not matched by any existing construction in literatures. Assuming the existence of an action and gauge transformation laws, a no-go theorem \([21]\) states that it does not exist any nontrivial deformation of the Abelian 2-form gauge theory. One of their assumptions was the ”locality” for the action and the gauge transformation laws. In particular, Lorentz symmetry was not assumed.

The non-existence of a local action for multiple M5-branes was argued in another way by Witten \([22]\). The M5-branes system is known to have conformal symmetry, which implies that upon double dimension reduction, the 4+1 dimensional action should be proportional to $\int d^5x \frac{1}{R}$. On the other hand, the reduction of a 5+1 dimensional local action on a circle should give $\int d^5x \ 2\pi R$, which has the opposite dependence on $R$. As long as we assume a Lorentz-

\(^3\)One attempt to construct non-Abelian 2-form gauge theory is to define it on the loop space \([20]\). This approach introduces infinitely many more degrees of freedom in a theory.
covariant formulation for M5-branes without explicit reference to the compactification radius $R$ except through the measure of integration, this gives a strong argument against the Lagrangian formulation of multiple M5-branes.

Recently, there are proposals [23, 24] claiming that the multiple M5-branes compactified on a circle of finite radius $R$ could be described by the $U(N)$ super Yang-Mills (SYM) theory for $N$ D4-branes even before taking the small $R$ limit. This would be a duality between two theories in 5 and 6 dimensions, respectively, but it can not be viewed as an example of the holographic principle of quantum gravity, because there is no gravitational force in these theories. Their proposal, if correct, would be revolutionary, however we will point out its difficulties later.

These developments suggest that it is already a tremendous progress to have a theory for multiple M5-branes compactified on a circle of finite radius $R$, if the following two criteria are satisfied:

1. In the limit $R \to 0$, the theory should be approximated by the gauge field sector of the multiple D4-branes theory, which is the $U(N)$ Yang-Mills theory in 5 dimensions.

2. When the Lie algebra of the gauge symmetry is Abelian, the theory should reduce to $N$ copies of the Abelian self-dual 2-form gauge field theory.

In view of no-go theorems [21], the absence of 6 dimensional Lorentz symmetry due to compactification of the 5-th direction does not really necessarily make the task easier. In the following we will construct an interacting theory satisfying both criteria. The cost we have to pay to meet these criteria is a nonlocal treatment of the compactified dimension as we will see in the following sections. Such a description may seem exotic, but it might be justi-

\footnote{Notice that the 2nd criterion ensures the 6 dimensional Lorentz symmetry in the broken phase in the limit $R \to \infty$.}
fied in view of the special role played by the compactified direction in defining M-theory as the strong coupling limit of type IIA string theory.

It might be worthy of note that there are some recent works on non-Abelian two-form theory and M5-branes [26], but we will not be able to review them in this thesis. On the other hand, for a more general approach to self-dual field theories and has the advantage to deal with subtle topological issues, we refer to the work of Belov and Moore [28]. It will be interesting to relate the model in this thesis to their formulation.

We organize this thesis as follows.

In section 2, we start from the simplest example among self-dual field theories: the chiral boson in two-dimensions. We will argue that why some naive methods to construct a self-dual theory do not work. Then we discuss two approaches of the chiral boson action. We will see some universal figures in self-dual theories like an extra gauge symmetry and the modified Lorentz transformation law.

In section 3, we will first introduce the standard 2-form potential gauge theory, and as an example, we briefly discuss the BF theory in four dimensions. Then we start to consider self-dual 2-form theory for a single M5-brane, which is so-called PS (Perry and Schwarz) formulation [29]. We will also study different self-dual 2-form formulations under different spacetime decompositions.

In section 4, we study Lorentz-covariant versions of the chiral boson and the chiral 2-form with the help of introducing an auxiliary field, these are so-called PST (Pasti, Sorokin and Tonin) formulations. We will see how these covaraint actions reduce to previous non-covariant formulations by a gauge fixing condition.
In section 5, which is the body of this thesis, we formulate a non-abelian 2-form theory. And we will non-abelianize PS action to see that it gives the proper action for multiple M5-branes. In particular, we will see how this model reduces to the Yang-Mills theory in 5 dimensions for small compactified radius R and it reduces to the Lorentz-invariant theory of Abelian chiral 2-forms theory when the gauge group is Abelian.

In section 6, the BRST transformation laws of this non-abelian 2-form gauge algebra are given. A crucial point is that we will need to introduce the ghost of the ghost for this kind of reducible system. A BRST invariant gauge-fixed action is also given by utilizing the BRST-antifield method.

In section 7, we generalize the theory to construct non-Abelian gauge theory for 3-form potentials.

In section 8, we conclude this thesis by pointing out some difficulties when trying to find a manifest Lorentz-covariant formulation of the non-abelian self-dual 2-form theory by using similar ideas of introducing extra auxiliary fields.
2  A Toy Model: 2D Chiral Boson

The simplest example for a self-dual field theory is in two-dimensions (2D), often called the chiral Boson, which plays an essential ingredient in the quantum Hall effect or the construction of heterotic-type string theory that is phenomenologically interesting because of the left-right asymmetry.

In 2D, the anti-symmetric tensor is just a scalar $\phi$ with field strengths defined by:

$$F_a = \partial_a \phi$$  \hspace{1cm} (5)

where $a = 0, 1$. There is a global symmetry for the scalar

$$\delta \phi = \text{constant}$$  \hspace{1cm} (6)

The self-duality condition is defined by:

$$\partial_0 \phi - \partial_1 \phi \equiv \mathcal{F} = 0$$  \hspace{1cm} (7)

The first question is: can we have an action that gives this self-duality condition? First of all, let us learn something from making mistakes. If you want to write down an action that gives the self-duality constraint equation, the first nature approach that you might try is to introduce a Lagrange-multiplier field $\lambda$ to implement this constraint and write down the following action:

$$S = \int d^2x \left[ -\frac{1}{2} F_a F^a + \lambda \mathcal{F} \right].$$  \hspace{1cm} (8)

indeed, the self-duality condition will be obtained by the variation of the action with respect to $\lambda$. However, this method turns out to be wrong since the Lagrange multiplier term itself ends up generating an extra field equation that we do not need from the field equation of the scalar $\phi$.

A clever trick introduced by Siegel (1984) \cite{30} was to consider the "squared self-duality constraint" in an action:

$$S = \int d^2x \left[ -\frac{1}{2} F_a F^a + \lambda (\mathcal{F})^2 \right]$$  \hspace{1cm} (9)
so that the field equation of Lagrange-multiplier field \( \lambda \) still gives the self-duality condition: 
\[ \partial_0 \phi - \partial_1 \phi = 0. \]
The key difference here is that the field equation of \( \phi \) could be written as:
\[
(\partial_0 + \partial_1)F - 2\lambda(\partial_0 - \partial_1)F = 0 \quad (\equiv 0 \text{ if } F = 0)
\] (10)
which does not give an extra constraint in the sense that it vanishes identically (a redundant equation of motion) by the self-duality condition! Furthermore, this redundant field equation occurs us the existence of an additional gauge symmetry in the theory. One could indeed find a gauge transformation that leaves the action invariant:
\[
\delta \phi = F \epsilon.
\] (11)
\[
\delta \lambda = \frac{1}{2}(\partial_0 + \partial_1)\epsilon + \epsilon(\partial_0 - \partial_1)\lambda + \lambda(\partial_0 - \partial_1)\epsilon.
\] (12)
with the gauge parameter \( \epsilon \). However, although the above model shows that the Lagrange multiplier does not cause further constraints, people found Sigel’s model suffers from the anomaly problems if not properly treat when quantized [32]. Furthermore, it is not totally clear whether there are enough local gauge symmetries in Seigel-like formulations in \( D = 6 \) case or \( D = 10 \) case to completely gauge away Lagrange-multiplier fields [32]. Since the case \( D = 6 \) is of our the main topic, in the following we will introduce yet another formulation that could be generalized to higher dimensions and we will see the self-duality condition could be obtained by yet another interesting way rather than the squared constraint in the Seigel’s formulation.

The formulation of chiral boson we consider was invented by R. Floreanini and R. Jackiw (1987) [31]. The FJ action is
\[
S_{1+1} = -\frac{1}{2} \int d^2x \partial_0 \phi(\partial_0 \phi - \partial_x \phi)
\] (13)
where \( 1+1 \) means that the decomposition of 2D Minkowski space
\[
R^{1+1} = R^1 \times R^1
\] (14)
which is obviously the only possible decomposition in 2D. The salient property of this action is the existence of the following gauge transformation (assuming the surface term does not contribute):

$$\delta \phi = f(t)$$  \hspace{1cm} (15)

for an arbitrary function $f(t)$ independent of $x$. We can check, since

$$\delta S \sim \int d^2x \, \partial_x (\phi \partial_t f(t)) = \text{tot.}$$  \hspace{1cm} (16)

so that the action is indeed invariant up to a total derivative. Notice that this symmetry is different from the standard constant transformation. The equation of motion is:

$$\partial_x (\partial_x \phi - \partial_t \phi) = 0$$  \hspace{1cm} (17)

this implies

$$(\partial_x \phi - \partial_t \phi) = g(t)$$  \hspace{1cm} (18)

for an arbitrary function $g(t)$ independent of $x$. Thus we find that if we can use the gauge transformation by considering

$$f(t) = - \int^t dy g(y)$$  \hspace{1cm} (19)

we can absorb the function $g(t)$ to obtain the self-duality condition:

$$\partial_x \phi - \partial_t \phi = 0$$  \hspace{1cm} (20)

On other other hand, one notices that the FJ action is not manifestly Lorentz invariant, but an interesting property is that the action is invariant under a modified Lorentz transformation law:

$$\delta \phi = \lambda^{tx} (t \partial_x - x \partial_t) \phi + \lambda^{tx} x (\partial_t \phi - \partial_x \phi)$$  \hspace{1cm} (21)

where $\lambda^{tx}$ is the Lorentz parameter. One could see that the standard transformation law is modified by the self-duality condition.
3 Abelian Chiral 2-Form: Single M5-Brane

3.1 Anti-Symmetry 2-rank Gauge Field and BF Model

A single M5-brane is described by the Abelian chiral 2-form theory. Let us first fix our notation through a short review on the anti-symmetry 2-rank gauge field. We consider $R^{1,5}$ as the 6D Minkowski space parametrized by $x^\mu$ with $\mu = 0, 1, 2, 3, 4, 5$. The 2-form potential is denoted by $B_{\mu\nu}$, as the nature of antisymmetry there is no diagonal components. The corresponding field strength is defined by

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$$  \hfill (22)

which is invariant under the following gauge transformation:

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$  \hfill (23)

in terms of the 1-form gauge parameter $\Lambda_\mu$. It seems that $\Lambda_\mu$ has six independent degrees of freedom, however, a crucial property of this gauge symmetry is its redundancy: the gauge transformation itself has an gauge transformation under:

$$\delta \Lambda_\mu = \partial_\mu \alpha$$  \hfill (24)

in terms of the 0-form gauge parameter $\alpha$. Hence only five of six gauge parameters are independent. The action is given by

$$S = -\frac{1}{6} \int d^6 x \ H^{\mu\nu\lambda} H_{\mu\nu\lambda}$$  \hfill (25)

with the field equation:

$$\partial_\mu H^{\mu\nu\lambda} = 0$$  \hfill (26)

and the Bianchi identity is:

$$\epsilon^{\mu\nu\lambda\rho\sigma} \partial_\lambda H_{\gamma\rho\sigma} = 0$$  \hfill (27)
Notice that in four dimensions a 2-form potential theory is dual to a scalar potential theory with only one physical degree of freedom, that is, in four dimensions we have

$$L_{4D} = -\frac{1}{6} H^{\mu\nu\lambda} H_{\mu\nu\lambda} = -\frac{1}{6} \times 2 \epsilon_{\mu\nu\lambda\rho} \epsilon^{\mu\nu\lambda\sigma} \partial^\rho \phi \partial^\sigma \phi = \frac{1}{2} \partial^\rho \phi \partial^\rho \phi$$  \hspace{1cm} (28)$$

where $\phi$ represents a real scalar field. In 6D that of our interesting, 2-form potential theory has six physical degrees of freedom ($C_2^{6-2} = 6$).

It is also interesting to note that, although $B_{\mu\nu}$ only increases one index compared with one-form potential $A_{\mu}$. Many things that happen in usual theories in terms of $A_{\mu}$ do not have a straightforward generalization in terms of $B_{\mu\nu}$. For example, in the Yang-Mills theory, terms like $\epsilon^{\mu\nu\lambda\rho}[A_{\mu}, A_{\nu}]$ can not have its simple generalization to $\epsilon^{\mu\nu\lambda\rho\sigma\tau}[B_{\mu\nu}, B_{\lambda\rho}]$, because it is zero by itself. Similarly, even in the abelian case, the Chern-Simon term for $A_{\mu}$:

$$L \sim \epsilon^{\mu\nu\lambda} A_{\mu} \partial_{\nu} A_{\lambda}$$  \hspace{1cm} (29)$$
gives the field equation $F_{\mu\nu} = 0$. However, a simple generalization to 2-form potential:

$$L \sim \epsilon^{\mu\nu\lambda\rho\sigma\tau} B_{\mu\nu} \partial_{\lambda} B_{\sigma\tau}$$  \hspace{1cm} (30)$$
will give a trivial vanishing field equation, i.e. $0=0$.

A famous model in four-dimensions using two-form potentials $B_{\mu\nu}$ and is quite different from the standard action is so-called the "BF" model. Let us briefly review it. For an abelian action, it is simply given by

$$S = \int d^4x \, \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho}$$  \hspace{1cm} (31)$$
where $F_{\mu\nu}$ is the Maxwell field strength. Note that the action does not involve a spacetime metric and one can see that the action is invariant (up to surface terms) under

$$\delta A_{\mu} = \partial_{\mu} \Lambda ; \quad \delta B_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$  \hspace{1cm} (32)$$
The equation of motions obtained by varying the action with respect to $B_{\mu\nu}$ and $A_{\mu}$ are simply $F_{\mu\nu} = 0$, $H_{\mu\nu\lambda} = 0$. The solutions of field equations are equivalent to $A_{\mu} = 0 = B_{\mu\nu}$ (assuming the manifold is topological trivial). We see that there is no local degree of freedom in the theory.

Now we consider a generalization of the above action

$$S = \int d^4x \, \epsilon^{\mu\nu\lambda\rho}(B_{\mu\nu}F_{\lambda\rho} - \frac{1}{2}B_{\mu\nu}B_{\lambda\rho})$$

(33)

where the second term is called the cosmological term. The action is obvious invariant under $\delta A_{\mu} = \partial_{\mu}\lambda$, the second term, however, breaks the symmetry $\delta B_{\mu\nu} = \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu}$. But one notices that this action is invariant under a larger gauge symmetry:

$$\delta A_{\mu} = \omega_{\mu} ; \; \delta B_{\mu\nu} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu}$$

(34)

thus one can use this gauge to locally set $A_{\mu} = 0$. The equation of motions obtained by varying the action with respect to $B_{\mu\nu}$ and $A_{\mu}$ are given by

$$F_{\mu\nu} = B_{\mu\nu} ; \; H_{\mu\nu\lambda} = 0$$

(35)

where the second equation is simply a consequence of the first one by recalling the Bianchi identity. We again see that there is also no local degree of freedom of the theory, i.e. $A=0=B$.

Now we consider the non-abelian generalization. The action is given by

$$S = \int d^4x \, Tr \, \epsilon^{\mu\nu\lambda\rho}B_{\mu\nu}F_{\lambda\rho}$$

(36)

where $F_{\mu\nu} = [D_{\mu}, D_{\nu}]$ is the usual Yang-Mills field strength. The action is invariant under the following sets of gauge transformations

$$\delta A_{\mu} = 0 ; \; \delta B_{\mu\nu} = [D_{\mu}, \Lambda_{\nu}] - [D_{\nu}, \Lambda_{\mu}]$$

(37)

in terms of the 1-form gauge parameter and

$$\delta A_{\mu} = [D_{\mu}, \lambda] ; \; \delta B_{\mu\nu} = [B_{\mu\nu}, \lambda]$$

(38)
in terms of the 0-form gauge parameter and

$$\delta A_\mu = 0 ; \delta B_{\mu\nu} = [F_{\mu\nu}, \omega]$$  \hspace{1cm} (39)

by another 0-form gauge parameter. The equation of motions obtained by varying the action with respect to $B_{\mu\nu}$ and $A_\mu$ are given by

$$F_{\mu\nu} = 0 ; \varepsilon^{\mu\nu\lambda}[D_\mu, B_{\nu\lambda}] = 0$$  \hspace{1cm} (40)

which means field $A$ is a flat connection: $A$ can be set 0. Then the solution of the second equation could be written as

$$B_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu$$  \hspace{1cm} (41)

for some vector $\Phi$. Now $B_{\mu\nu}$ can be set as zero by the gauge transformation (37). We see again that there is no local degree of freedom in the theory. In sum, BF theories are topological field theories and it can be generalized to any dimension. It is not necessarily that one always use two-form potentials in BF theories. BF theories have their deeper connections to knot theory, Chern Simons theory and gravity. It is also used to construct mass generation formulation for one-form gauge potential as an alternative approach of the Higgs mechanism. But we will not address these topics further in this thesis.

Now we turn to our main topic of this thesis: self-dual 2-form theory. As mentioned, self-duality means that the original six physical degrees of freedom are reduced to three physical degrees of freedom if imposing the following self-duality condition:

$$H_{\mu\nu\lambda} = \tilde{H}_{\mu\nu\lambda} \equiv \frac{1}{6} \varepsilon_{\mu\nu\lambda\gamma\rho\sigma} H^{\gamma\rho\sigma}$$  \hspace{1cm} (42)

note that one could consider a different sign above as the anti-self duality condition, however it does not matter which sign we select in the sense that it will also reduce half of the degrees of freedom.
As we have seen in the 2D chiral boson case, a general structure of chiral field formulation is the existence of an extra gauge symmetry. In the following subsections, we will provide 6D abelian actions in various splitting decompositions that all give the self-duality condition as the consequence of field equations of 2-form potentials.

3.2 5+1 Splitting Formulation

This kind of decomposition is the simplest case where only one direction will be treated differently \[29\]. In this formulation, a two-form potential is based on the 5+1 decomposition of the spacetime

\[ R^{1,5} = R^{1,4} \times R^1 \]  

hence the gauge potentials are decomposed into two sets:

\[ B_{\mu\nu} = (B_{i5}, B_{ij}) \]  

where \( i, j = (0, 1, 2, 3, 4) \) and 5-th is the special direction. One can also select the time direction as the special one, but our selection will be convenient when it comes to the dimensional reduction to D4-brane.

Let us first denote the following notations:

\[ \mathcal{H}_{ij5} \equiv H_{ij5} - \tilde{H}_{ij5} = H_{ij5} + \frac{1}{6} \epsilon_{ijnqp} H^{npq} \]  

\[ \mathcal{H}_{ijk} \equiv H_{ijk} - \tilde{H}_{ijk} = H_{ijk} - \frac{1}{2} \epsilon_{ijklm} H^{lm5} \]  

then we also have

\[ \mathcal{H}_{ij5} = \frac{1}{6} \epsilon_{ijnpq} H^{npq} \]  

The action for the abelian chiral two-form is given by

\[ S_{5+1} = -\frac{1}{12} \int d^6x \left[ \epsilon_{ijklm} H^{klm} \mathcal{H}_{ij5} \right] \]

\[ = \frac{1}{12} \int d^6x \left[ H_{\mu\nu\lambda} H^{\mu\nu\lambda} - 3 \mathcal{H}_{ij5}^2 \right] \]
where the second expression will be useful when considering the Lorentz-covariant formulation later, but in this section we can focus on the first expression.

We notice that the manifest Lorentz invariance only maintains in five dimensions. However, just like previous 2D Chiral boson’s non-covariant formulation, this 6D non-covariant action for self-dual two-form also has a modified Lorentz transformation law defined by (in the gauge $B_{i5} = 0$ where $H_{ij5} = \partial_5 B_{ij}$ and we focus on the transformations mixed with the 5-th coordinate):

$$\delta B_{ij} = (\Lambda \cdot x) \tilde{H}_{ij5} - x_5 (\Lambda \cdot \partial) B_{ij}$$

$$= (\Lambda \cdot x) \partial_5 B_{ij} - x_5 (\Lambda \cdot \partial) B_{ij} - (\Lambda \cdot x) (H_{ij5} + \frac{1}{6} \epsilon^{ijnpq} H_{npq})$$

(49)

one observes that the standard Lorentz transformation law is modified by the self-duality condition. Let us explicitly Lorentz transformation examine this symmetry. We have

$$\delta S_{5+1} \sim \int d^6 x \delta B_{ij} \epsilon^{ijklm} \partial_k (\tilde{H}_{lm5} - \partial_5 B_{lm})$$

$$= \int d^6 x \left( (\Lambda \cdot x) \tilde{H}_{ij5} - x_5 (\Lambda \cdot \partial) B_{ij} \right) \epsilon^{ijklm} \partial_k (\tilde{H}_{lm5} - \partial_5 B_{lm})$$

(50)

thus there are four terms, we write them explicitly

(1) $$= \int d^6 x (\Lambda \cdot x) \tilde{H}_{ij5} \epsilon^{ijklm} \partial_k (\tilde{H}_{lm5})$$

$$= \int d^6 x \left[ -\frac{1}{2} \epsilon^{ijklm} \Lambda_k \tilde{H}_{ij5} (\tilde{H}_{lm5}) \right] + \text{tot.}$$

$$= \int d^6 x \left[ -\Lambda_k \tilde{H}_{ij5} H^{ijk} \right] + \text{tot}$$

$$= \int d^6 x \left[ -(\Lambda \cdot \partial) B_{ij} \tilde{H}^{ij5} \right] + \text{tot.}$$

(51)

(2) $$= \int d^6 x \left[ -(\Lambda \cdot x) \tilde{H}_{ij5} \epsilon^{ijklm} \partial_k \partial_5 B_{lm} \right]$$

$$\sim \int d^6 x (\Lambda \cdot x) \tilde{H}_{ij5} \partial_5 \tilde{H}^{ij5}$$

$$= \text{tot.}$$

(52)
Thus we find that, up to total derivatives, two terms vanish and other two terms cancel out. The action is invariant under the modified Lorentz transformation.

Next we turn to the question that how to obtain the self-duality condition from the action. We start from the equation of motion of $B_{j5}$, which is

$$\partial^i \tilde{H}_{ij5} = 0$$

but this is just an identity if we recall $\epsilon_{ijklm} \partial^i \partial^k = 0$. This result implies that components $B_{i5}$ only enter the action through surface terms (total derivatives). It means that we have an additional gauge symmetry in the theory:

$$\delta B_{i5} = \Phi_i$$

for an arbitrary one-form gauge parameter $\Phi_i$. On the other hand, the equation of motion of $B_{ij}$ is given by

$$\partial^k \mathcal{H}_{ijk} = 0$$
whose solution is written as
\[ \mathcal{H}_{ijk} = \epsilon_{ijklm} \partial^l \Psi^m \] (58)

for an arbitrary one-form gauge parameter \( \Psi_m \). Now we see that under the
gauge transformation (56) one can absorb \( \Psi_i \) by using the gauge parameter
\( \Phi_i \) to obtain the self-duality condition of 2-form potentials:
\[ \mathcal{H}_{ijk} = 0 \] (59)

this result also implies \( \mathcal{H}_{ij5} = 0 \). This is the only self-duality condition that
we need in the 5+1 splitting non-covariant formulation.

We notice that we have picked 5-th as our special direction and we used
the gauge \( B_{i5} = 0 \) above. Thus if we integrate the 5-th direction on both side
of the self-duality condition, we will get:
\[ \int dx^5 (-\frac{1}{6} \epsilon_{ijklm} H^{klm}) = B_{ij} (X^5 = 2\pi R) - B_{ij} (x^5 = 0) \] (60)
as an interesting consistency condition for the self-dual theory.

3.3 3+3 Splitting Formulation

Another formulation of chiral 2-form is based on the 3+3 decomposition of
spacetime [12]:
\[ R^{1,5} = R^{1,2} \times R^3 \] (61)

Originally it was invented by studying Bagger-Lambert-Gustavsson (BLG)
multiple M2-branes model by promoting Lie 3-algebra into the infinite-dimensional
symmetry of volume preserving diffeomorphisms of an internal 3-dimensional
space. Physically this 3+3 splitting means that the 3-dimensional worldvol-
ume of M2-branes combines with extra 3-dimensional internal space to form a
6-dimensional worldvolume of a single M5-brane that carries the chiral 2-form.
The 6D Lorentz symmetry $SO(1, 5)$ is broken to $SO(1, 2) \times SO(3)$ and the gauge potentials are decomposed into three types:

$$B_{\mu \nu} = (B_{ab}, B_{a\dot{b}}, B_{\dot{a}\dot{b}})$$

where $a = (0, 1, 2)$ and $\dot{a} = (3, 4, 5)$. Notice that in this way $B_{ab}$ or $B_{\dot{a}\dot{b}}$ has three components while $B_{a\dot{b}}$ has nine components.

The action is given by

$$S_{3+3} = -\frac{1}{12} \int d^6 x [\tilde{H}_{abc}(H^{abc} - \tilde{H}^{abc}) + 3\tilde{H}_{ab\dot{c}}(H^{ab\dot{c}} - \tilde{H}^{ab\dot{c}})]$$

$$\equiv -\frac{1}{12} \int d^6 x [\tilde{H}_{ab\dot{c}} H^{ab\dot{c}} + 3\tilde{H}_{ab\dot{c}} H^{ab\dot{c}}]$$

(63)

one can check that components $B_{ab}$ enter the action only through total derivatives. Its field equation is trivial (vanishes identically) and is given by

$$(\partial^c \tilde{H}_{abc} + 3\partial^{\dot{c}} \tilde{H}_{ab\dot{c}}) = 0$$

(64)

that means there is an additional gauge symmetry in the theory:

$$\delta B_{ab} = \Phi_{ab}$$

(65)

for some arbitrary 2-form gauge parameter $\Phi_{ab}$.

Notice that in this 3+3 formulation, we will need to show both equations at the same time

$$H^{ab\dot{c}} = 0$$

(66)

$$\tilde{H}^{ab\dot{c}} = 0$$

(67)

to justify this is the right action describing chiral 2-form. However, we only have a special gauge transformation coming from $B_{ab}$ (65), is it possible to achieve both equations under a single gauge transformation? We will see in the following that the answer is positive.
The equations of motions derived from varying the action with respect to components $B_{ab}$ and $B_{\dot{a}\dot{b}}$ are given by

\begin{align}
\partial^\dot{c} H_{\dot{a}\dot{b}\dot{c}} &= 0 \tag{68} \\
\partial_c H^{\dot{a}\dot{b}c} + \partial_{\dot{c}} H^{\dot{a}\dot{b}\dot{c}} &= 0 \tag{69}
\end{align}

the first equation implies the solution

$$H_{\dot{a}\dot{b}\dot{c}} = \frac{1}{2} \epsilon_{\dot{a}\dot{b}\dot{c}} \epsilon_{abc} \partial^\dot{a} \Phi^{bc}$$ \tag{70}

for some arbitrary tensor $\Phi^{bc}$. Thus we find that after making the gauge transformation: $\delta B_{ab} = \Phi_{ab}$, one part of self-duality conditions could be obtained:

$$H_{\dot{a}\dot{b}\dot{c}} = 0$$ \tag{71}

Using this result, we could further rewrite another field equation as a total derivative:

$$\partial^\dot{c} (H_{\dot{a}\dot{b}\dot{c}} + \frac{1}{2} \epsilon_{\dot{a}\dot{b}\dot{c}} \epsilon_{abc} \partial^a B^{bc}) = 0$$ \tag{72}

then we can solve it by considering

$$H_{\dot{a}\dot{b}\dot{c}} + \frac{1}{2} \epsilon_{\dot{a}\dot{b}\dot{c}} \epsilon_{abc} \partial^a B^{bc} = \epsilon_{\dot{a}\dot{b}\dot{c}} \Psi(x_a)$$ \tag{73}

for some function $\Psi(x_a)$ independent of coordinates $x_\dot{a}$. The trick is that we could still redefine

$$B_{ab} \rightarrow B_{ab} + \frac{1}{3} \epsilon_{abc} \Phi^c(x_a)$$ \tag{74}

where $\Phi^c(x_a)$ is selected as $\partial_c \Phi^c(x_a) = \Psi(x_a)$. It is important to notice that since $\Phi^c(x_a)$ is independent of coordinates $x_\dot{a}$, this further redefinition/ transformation of $B_{ab}$ does not spoil another self-duality condition. Thus we arrive the remaining self-duality condition

$$H_{\dot{a}\dot{b}\dot{c}} = 0$$ \tag{75}

Finally, let us conclude this section by mentioning modified Lorentz transformation laws in this 3+3 formulation. The action is manifestly invariant
under the $SO(1,2) \times SO(3)$ subgroup of the full $SO(1,5)$ Lorentz group. The Lorentz symmetries mixing $x_a$ and $x_{\dot{a}}$ are no longer manifest. The claim is that the action is still invariant under the following modified Lorentz transformation laws that are parametrized by $3 \times 3$ constant matrix $\lambda_{a\dot{a}}$:

$$
\delta B^{a\dot{a}} = \lambda_b^a B^{b\dot{a}} + \lambda_c^b (x_b \partial^{\dot{c}} - x^{\dot{c}} \partial_b) B^{a\dot{c}} + \lambda_d^c x_{\dot{d}} \mathcal{H}^{ca\dot{a}}
$$

(76)

$$
\delta B^{\dot{a}\dot{b}} = \lambda_b^{\dot{a}} B^{a\dot{b}} - \lambda_a^{\dot{b}} B^{ab} + \lambda_c^{\dot{b}} (x_b \partial^{\dot{c}} - x^{\dot{c}} \partial_b) B^{\dot{a}\dot{c}}
$$

(77)

where we have considered the gauge $B_{ab} = 0$ for the simplicity. One could see that these modified transformation laws become the standard Lorentz transformation on the mass shell.

### 3.4 4+2 Splitting Formulation

We have constructed chiral 2-form theories based on $5 + 1$ splitting and $3 + 3$ splitting. It is nature to ask for a formulation based on $4+2$ splitting, which is given in [8]. Here the six dimensional Lorentz symmetry $SO(1,5)$ is broken to $SO(1,1) \times SO(4)$. In this formulation, the chiral 2-form is based on the $4+2$ decomposition of spacetime:

$$
R^{1,5} = R^{1,3} \times R^2
$$

(78)

and the gauge potentials are decomposed into:

$$
B_{\mu\nu} = (B_{ab}, B_{a\dot{b}}, B_{\dot{a}b})
$$

(79)

where $a = (0, 1)$ and $\dot{a} = (2, 3, 4, 5)$. The action is given by

$$
S_{2+4} = -\frac{1}{8} \int d^6x \left(2 \tilde{H}_{abc} \mathcal{H}^{abc} + \tilde{H}_{a\dot{b}} \mathcal{H}^{a\dot{b}}\right).
$$

(80)

The field equation of $B_{ab}$ is again trivial (vanishes identically), and is give by

$$
\partial^{\dot{a}} \tilde{H}_{a\dot{b}} = 0,
$$

(81)

that implies the gauge transformation

$$
\delta B_{ab} = \Phi_{ab}.
$$

(82)
The field equations of $B_{a\dot{a}}$ and $B_{\dot{a}\dot{b}}$ are

\begin{align}
\partial^b H_{a\dot{a}b} &= 0, \\ \partial^c (H + H)_{\dot{a}\dot{b}c} + \partial^a H_{a\dot{a}b} &= 0.
\end{align} 

(83) (84)

The solution to the first equation (83) could be written as

$$H_{a\dot{a}b} = \epsilon_{ab} \epsilon_{\dot{a}\dot{b}d} \partial^\dot{d} \Psi^{bd}$$

(85)

for some arbitrary functions $\Psi^{bd}$. By taking the Hodge-dual of both sides, we get

$$H_{a\dot{a}b} = \partial_a \Psi_{\dot{a}b} - \partial_b \Psi_{a\dot{a}}.$$

(86)

Identifying these two results, we have

$$\partial_a \Psi_{\dot{a}b} - \partial_b \Psi_{a\dot{a}} = \epsilon_{ab} \epsilon_{\dot{a}\dot{b}d} \partial^\dot{d} \Psi^{bd},$$

(87)

Now we act $\partial^\dot{b}$ on both sides of the equivalence relation to get

$$\partial^\dot{b} \partial_b \Psi_{a\dot{a}} - \partial_a \partial^\dot{b} \Psi_{ab} = 0.$$ 

(88)

Because of the solution (85) is unchanged under the transformation

$$\Psi_{a\dot{a}} \rightarrow \Psi_{a\dot{a}} + \partial_a \Lambda_{\dot{a}}.$$ 

(89)

we choose the Lorentz gauge to fix that redundant symmetry, i.e. $\partial^\dot{a} \Psi_{a\dot{a}} = 0$. Now (88) reduces to

$$\dot{\partial}^2 \Psi_{a\dot{a}} = 0, \quad (\dot{\partial}^2 \equiv \partial^\dot{a} \partial_a).$$ 

(90)

Next we consider a proper boundary condition that $\Psi^{a\dot{a}}$ vanishes at infinities of the 4D Euclidean space with coordinates $x_{\dot{a}}$ such that it has the unique solution $\Psi_{a\dot{a}} = 0$, thus we obtain one of the self-duality conditions

$$H_{a\dot{a}b} = 0.$$ 

(91)
If we plug this result into the equation of motion of $B_{\dot{a}\dot{b}}$ (84), we find we get the solution

$$\mathcal{H}_{\dot{a}\dot{b}\dot{c}} = \frac{1}{2} \varepsilon_{ab} \varepsilon_{\dot{a}\dot{b}\dot{c}} \dot{\phi}^d \phi^{ab}$$

(92)

for some arbitrary functions $\phi^{ab}$. Now we use the gauge transformation $\delta B_{ab} = \Phi_{ab}$, this equation becomes

$$\mathcal{H}_{\dot{a}\dot{b}\dot{c}} = 0.$$

(93)

so that we obtain self-duality conditions for all components of the field strengths in this 4+2 formulation.

Let me summarize this section by providing the modified Lorentz transformation laws of this 4+2 formulation. The action is manifestly invariant under the $SO(1, 1) \times SO(4)$ subgroup of the full $SO(1, 5)$ Lorentz group, while the Lorentz symmetries mixing $x_a$ and $x_{\dot{a}}$ are not manifest. The modified Lorentz transformation laws that are parametrized by $2 \times 4$ constant matrix $\lambda_{a\dot{a}}$ are given by

$$\delta B^{a\dot{a}} = \lambda^a_b B^{b\dot{a}} + \lambda^b_c (x_b \partial^\dot{a} - x^\dot{a} \partial_b) B^{a\dot{a}} + \lambda^{\dot{a}}_c x_d \mathcal{H}^{ca\dot{a}}$$

(94)

$$\delta B^{\dot{a}b} = \lambda^{\dot{a}}_a B^{ba} - \lambda^b_c (x_b \partial^\dot{a} - x^\dot{a} \partial_b) B^{\dot{a}b}$$

(95)

where we again consider the gauge $B_{ab} = 0$ for the simplicity. One could see that the modified transformation laws become the standard Lorentz transformation on the mass shell.
4 Covariant Formulations: PST Model

We have constructed various non-manifest Lorentz invariant actions for self-dual fields. However, it is always desirable to find covariant formulations especially when one wants to consider more complicated cases when fields couple to gravity and investigate the potential gravitational anomaly among chiral fields for instance.

It turns out that we need to introduce auxiliary fields to construct covariant self-dual field theories. In fact, people have considered different number of auxiliary fields that vary from infinity to only one auxiliary field. In this thesis, we will only consider the most modern approach, that is so-called PST formulation [10].

The general lesson we will learn is that there will be an extra gauge symmetry that allows us to gauge fix the auxiliary field to the proper configuration and PST covariant formulations of self-dual fields reduce to previous non-covariant self-dual actions that we have discussed in previous sections. We will start from a covariant formulation of two-dimensional chiral boson and then we consider covariant formulation of the six-dimensional self-dual two-form theory for a single M5-brane.

4.1 Covariant action for Chiral Boson: D=2

The PST covariant action for a two-dimensional chiral boson was constructed by using a single auxiliary scalar field, let us call it "a(x)". Notice that this auxiliary field should be spacetime dependent for the sake of the Lorentz symmetry. The action is given by [10]

$$ S = \int d^2x \left[ -\frac{1}{2} F^a F_a + \frac{1}{2(\partial a)^2} (\partial^b a \mathcal{F}_b)^2 \right] $$  \hspace{1cm} (96)
where $a, b = 0, 1$. And here we will not need to distinguish time or space coordinate explicitly here. We define $F_a \equiv \partial_a \phi$ as the field strength of the boson $\phi(x)$ as before and $\mathcal{F}_a$ is defined by

$$\mathcal{F}_a = F_a - \epsilon_{ab} F^b$$

as the expression for the self-duality condition.

The action has the following two sets of gauge transformation

$$\begin{align*}
(1) \quad & \delta \phi = f(a), \quad \delta a = 0 \\
(2) \quad & \delta \phi = \frac{\psi}{(\partial a)^2} \partial^b a \mathcal{F}_b, \quad \delta a = \psi(x)
\end{align*}$$

with gauge parameters $f(a)$ and $\psi(x)$. The second gauge symmetry plays a important role which allows us to gauge fix auxiliary field to, for example, fix $a = x$. This gauge-fixing choice will reduce this PST covariant action to the previous non-covariant FJ action (13).

Now let us see how to obtain the self-duality condition from this covariant action. The equation of motion of $\phi(x)$ is

$$\epsilon^{ab} \partial_b \left[ \frac{1}{2(\partial a)^2} \partial_a a \partial_c a \mathcal{F}^c \right] = 0$$

which implies the following solution

$$\frac{1}{(\partial a)^2} \partial_a a \partial_a a \mathcal{F}^c = \partial_a \Omega$$

for some arbitrary scalar $\Omega$. Then let us project this solution with two orthogonal vectors: $\partial_a a$ and $\epsilon^{ab} \partial_b a$, to get two equations

$$\begin{align*}
\frac{1}{2} \partial_c a \mathcal{F}^c &= \partial^a a \partial_a \Omega \\
\epsilon^{ab} \partial_a a \partial_b \Omega &= 0
\end{align*}$$

we see the solution to the second equation (103) can be given by

$$\Omega = \tilde{f}(a)$$
for some arbitrary function of $a(x)$. Plug it into the first equation (102) we obtain
\[ \frac{1}{2} \partial_c a F^c = \partial^a a \partial_a \tilde{f}(a) \] (105)
On the other hand, we observe that under the gauge transformation $\delta \phi = f(a), \delta a = 0$, the left hand side of the equation (102) becomes
\[ \frac{1}{2} \partial_c a \delta F^c = \frac{1}{2} \partial^a a \partial_a f(a) \] (106)
thus we could use the freedom from parameter $f(a)$ to absorb $\tilde{f}(a)$ to obtain
\[ \frac{1}{2} \partial_c a F^c = 0 \] (107)
which implies the self-duality condition. Notice that this implies $F^1 = 0$ if one fixes $a(x)$ to $x^1$, for example, it gives $\partial_0 \phi - \partial_1 \phi = 0$, which is just all we need for a chiral boson. Fixing $a(x)$ to $x^0$ will give the same result. Notice that the equation of motion of $a$ is trivial in the sense that it is proportional to the self-duality condition, as we have used the self-duality constraint, the equation of motion of $a(x)$ will not give an extra constraint on fields.

We also notice that the action (96) has the same form as Seigel’s action (9), but the key difference is that the self-duality condition obtained from Seigel’s action is through the field equation of auxiliary field $\lambda$. While in the PST formulation, the self-duality condition is obtained from the field equation of $\phi$ with an additional gauge transformation. However, these two theories are the same in classical level in the sense that they have the same equation of motion (the self-duality condition).

### 4.2 Covariant action for Chiral 2-form: D=6

The Lorentz-covariant action for the Abelian chiral 2-form theory (or the gauge sector for a single M5-brane) is constructed also with the help of a single auxiliary scalar field. We will see again that this PST covariant model has an
extra gauge symmetry which could be used to gauge fix the auxiliary field and reduces to the previous non-covariant PS action (48).

The covariant action is given by [10]:

$$S = \frac{1}{4} \int d^6x \left( -\frac{1}{3!} H_{\mu\nu\lambda} H_{\mu\nu\lambda} + \frac{1}{2} \mathcal{H}_{\mu\nu\rho} P^\sigma_{\rho} \mathcal{H}_{\mu\nu\sigma} \right)$$

(108)

where

$$\mathcal{H}_{\mu\nu\sigma} = (H - \tilde{H})_{\mu\nu\sigma}$$

(109)

with $\tilde{H}$ represents the Hodge dual of H:

$$\tilde{H}^{\mu\nu\lambda} = -\frac{1}{6} \epsilon^{\mu\nu\lambda abc} H_{abc}$$

(110)

and we denote

$$P^\mu_{\nu} = \frac{\partial^\mu b \partial^\nu b}{(\partial b)^2}$$

(111)

in terms of an auxiliary field $b(x)$.

Since the algebra in six-dimensions will be more complicated than previous two-dimensional case, let us derive equation of motions explicitly here and in the end we will observe the crucial existence of an extra gauge symmetry.

The equation of motion of $B_{\alpha\beta}$ from the first term gives:

$$\frac{\delta \left( -\frac{1}{3!} H_{\mu\nu\lambda} H_{\mu\nu\lambda} \right)}{\delta B_{\alpha\beta}} = \partial_\gamma H^{\alpha\beta\gamma} = \partial_\gamma \mathcal{H}^{\alpha\beta\gamma}$$

$$= \partial_\gamma \left( -\frac{1}{2} \epsilon^{\alpha\beta\gamma \mu\nu\lambda} P^\sigma_{\mu} \mathcal{H}_{\sigma\nu\lambda} + 3 P_{\mu}^{(\alpha} \mathcal{H}^{\beta\gamma)\mu} \right)$$

(112)

where we have considered the identity :

$$\epsilon^{\alpha\beta\gamma \mu\nu\lambda} P^\sigma_{\mu} \mathcal{H}_{\sigma\nu\lambda} = -\frac{1}{6} \epsilon^{\alpha\beta\gamma \mu\nu\lambda} \epsilon_{\sigma\nu\lambda\kappa\eta\delta} P^\sigma_{\mu} \mathcal{H}^{\kappa\eta\delta} = \frac{2!4!}{6} \delta^{(\alpha\beta\gamma\mu]} P_{\mu}^{\alpha} \mathcal{H}^{\beta\gamma)}$$

$$= \frac{1}{6} \left( -2!3! P_{\mu}^{\alpha} \mathcal{H}^{\alpha\beta\gamma} + 2!(4! - 3!) P_{\mu}^{(\alpha} \mathcal{H}^{\beta\gamma)\mu} \right)$$

$$= -2 \mathcal{H}^{\alpha\beta\gamma} + 3! P_{\mu}^{(\alpha} \mathcal{H}^{\beta\gamma)\mu}$$

(113)
while the variation of the second term gives:

$$\frac{\delta \left( \frac{1}{2} \mathcal{H}^{\mu\nu\rho} P_\rho^\sigma \mathcal{H}_{\mu\nu\sigma} \right)}{\delta B_{\alpha\beta}} = \left( \mathcal{H}^{\mu\nu\rho} P_\rho^\sigma \delta \mathcal{H}_{\mu\nu\sigma} \right) = -\partial_\gamma \left( 3 P_\mu^\alpha \mathcal{H}^\beta_\gamma \mathcal{H}^\alpha_\mu + \frac{1}{2} \epsilon^{\alpha\beta\gamma\mu\nu\lambda} P_\mu^\sigma \mathcal{H}^\gamma_\sigma \mathcal{H}^\mu_\nu \mathcal{H}^\lambda_\nu \mathcal{H}^\nu_\sigma \right)$$

Overall we have the variation on the action with respect to $B_{\mu\nu}$:

$$- \epsilon^{\alpha\beta\gamma\mu\nu\lambda} \partial_\mu b (\partial_\gamma \bar{H}^\beta_\nu) \delta B_{\alpha\beta}$$

where

$$\bar{H}^{\mu\nu} \equiv \mathcal{H}^{\mu\nu\rho} \frac{\partial_\rho b}{(\partial b)^2}$$

Next we consider the equation of motion of $b(x)$, which only appears in the second term of the action, we have

$$\frac{\delta \left( \frac{1}{2} \mathcal{H}^{\mu\nu\rho} P_\rho^\sigma \mathcal{H}_{\mu\nu\sigma} \right)}{\delta b} = -\partial_\gamma \left[ \mathcal{H}^{\mu\nu\rho} \frac{\partial_\rho b}{(\partial b)^2} \mathcal{H}_{\mu\nu\gamma} \right] + \partial_\gamma \left[ \mathcal{H}^{\mu\nu\rho} P_\rho^\sigma \frac{\partial_\gamma b}{(\partial b)^2} \mathcal{H}_{\mu\nu\sigma} \right]$$

$$= -\partial_\gamma (\bar{H}^{\mu\nu} \mathcal{H}_{\mu\nu\gamma}) + \partial_\gamma (\bar{H}^{\mu\nu} \bar{H}_{\mu\nu} \partial_\gamma b)$$

Then we notice that the first term can be written as (by (113)):

$$-\partial_\gamma (\bar{H}^{\alpha\beta} \mathcal{H}_{\alpha\beta\gamma}) = -\partial_\gamma (\bar{H}^{\alpha\beta} \frac{-1}{2} \epsilon^{\alpha\beta\gamma\mu\nu\lambda} P_\mu^\sigma \mathcal{H}_{\sigma\nu\lambda} + 3 \bar{H}^{\alpha\beta} P_\mu^\sigma \mathcal{H}^\gamma_\sigma \mathcal{H}^\gamma_\nu \mathcal{H}^\gamma_\sigma)$$

$$= -\partial_\gamma (\bar{H}^{\alpha\beta} \frac{-1}{2} \epsilon^{\alpha\beta\gamma\mu\nu\lambda} \bar{H}_{\nu\lambda} \partial_\mu b + \bar{H}^{\alpha\beta} P_\mu^\gamma \mathcal{H}^\gamma_\alpha \mathcal{H}^\gamma_\mu)$$

where we have used the fact that

$$\bar{H}_{\mu\nu} \partial_\nu b = \mathcal{H}_{\mu\nu\lambda} \frac{\partial \lambda b \partial \nu b}{(\partial b)^2} = 0$$

Overall we have the variation of the action with respect to $b(x)$:

$$\frac{1}{2} \partial_\gamma (\bar{H}^{\alpha\beta} \epsilon^{\alpha\beta\gamma\mu\nu\lambda} \bar{H}_{\nu\lambda} \partial_\mu b) \delta b = \bar{H}^{\alpha\beta} \epsilon^{\alpha\beta\gamma\mu\nu\lambda} \left( \partial_\gamma \bar{H}^\beta_\nu \right) \partial_\mu b \delta b$$

In sum, the variation of the ”Abelian” PST action gives:

$$\delta S = \int d^6x \left( - \epsilon^{\alpha\beta\rho\mu\nu\lambda} \partial_\rho b (\partial_\mu \bar{H}^\beta_\nu) \delta B_{\alpha\beta} + \epsilon^{\alpha\beta\rho\mu\nu\lambda} \partial_\mu b (\partial_\rho \bar{H}^\beta_\nu) \delta b \right)$$
so it is obvious we have an extra symmetry under the gauge parameter \( \phi \):

\[
\delta B_{\mu\nu} = \bar{H}_{\mu\nu} \phi \quad (122)
\]

\[
\delta b = \phi \quad (123)
\]

which allows us to fix \( b(x) \) to \( x^5 \) to get the non-covariant action (48). Notice that one should avoid setting \( b = 0 \) since it brings the singularity into the theory as field \( b \) appears in the denominator.

The next question is how to obtain the self-duality condition from this covariant action. We first find the general solution of the equation of motion of \( B_{\mu\nu} \) (one can read it from the first term in (121)) could be given by letting

\[
\mathcal{H}_{\mu\nu\lambda} \partial^{\lambda} b = (\partial b)^2 \partial_{[\mu} \Phi_{\nu]} + \partial_{[\mu} b \partial_{\nu]} \Phi_{\rho} \partial^\rho b + \partial^\rho b \partial_{\rho} \Phi_{[\mu} \partial_{\nu]} b \quad (124)
\]

for some arbitrary vector \( \Phi_\mu \). Just like we have studied in previous D=2 case, we should be able to find a way to "absorb" the right hand side of the above equation in order to obtain the self-duality condition. Indeed, there is an additional gauge symmetry of the action, and the transformation is given in terms of the gauge parameter \( \Phi_\mu \)

\[
\delta B_{\mu\nu} = (\partial_{\mu} b) \Phi_{\nu} - (\partial_{\nu} b) \Phi_{\mu} \quad (125)
\]

\[
\delta b = 0 \quad (126)
\]

this symmetry will generate the same form as the right hand side in (124) when considering \( \delta (\mathcal{H}_{\mu\nu\lambda} \partial^{\lambda} b) \), so that we obtain the self-duality condition for the 2-form potential. On the other hand, one can check that the equation of motion for the auxiliary field \( b \) is trivial because it is proportional to the self-duality condition. Also notice that if we fix the auxiliary field by \( a = x^5 \), the extra gauge symmetry of \( B \) (125) reduces to \( \delta B_{i5} = \Phi_{i5} \), which is just the extra gauge symmetry used to obtain the self-duality condition in the non-covariant PS action.
5 Non-Abelain Chiral 2-form: Multiple M5-Branes

Multiple M5-branes system has been the most challenging and mysterious brane system in string/M theory. A single M5-brane is described by Abelian self-dual 2-form gauge field theory that we have discussed in previous sections, when branes start to close to each other, interaction among M5-branes will appear and we expect multiple M5-branes system should be described by a certain ”non-Abelian” self-dual 2-form gauge field theory. As we have seen in previous sections, to construct an action for self-dual fields is already a highly non-trivial task, here we will face a bigger challenging which is to formulate a non-Abelian 2-form theory.

Notice that formulating a non-Abelian 2-form gauge theory should be totally independent from formulating a self-dual field theory. Only having non-Abelian 2-form gauge theory is not sufficient to describe M5-branes, we will need to combine these two characteristics in the end.

5.1 Non-locality: The hint from No-Go theorems

Several famous no-go theorems [21] claim that such exotic system (multiple M5-branes) is impossible to build. They considered general deformations of N commuting copies of Abelian self-dual 2-form gauge theory in 6 dimensions, and they found that a consistent non-Abelian deformation is always trivial in the sense that it will be equivalent to simply a change of variables that does not really deform the gauge algebra. However, every no-go theorem has their weakness, although they did not assume the deformed gauge transformations to correspond to a particular type of symmetry algebra, they did assume that the deformation was local.

However, only introducing non-locality is not enough to see where we should go and of course we will need other ideas. We will discuss them in detail in
this chapter. We will construct a non-Abelian self-dual 2-form gauge theory in
6 dimensions with a spatial direction compactified on a circle of radius R (The
reason for the compactification will be given later). It has the following two
important properties to be justified as a correct theory for M5-branes. (1) it re-
duces to the Yang-Mills theory (multiple D4-branes) in 5 dimensions for small
radius R. (2) It is equivalent to the Lorentz-invariant theory of Abelian chiral
2-forms when we turn off the coupling. Previous no-go theorems prohibiting
non-Abelian deformations are circumvented by introducing nonlocality along
the compactified dimension.[1]

5.2 The Main Ideas

The main ideas of this formulation are two-fold. First, although there is no
rule telling us that we must introduce covariant derivatives in this kind of
non-abelian higher form gauge theory. However, it is unclear how to construct
a non-Abelian theory that has the covariant structure without introducing
covariant derivatives. Also notice that if we have a non-Abelian 2-form theory,
we expect that, after compactification it should reduce to the non-Abelian 1-
form theory, which is just the standard Yang-Mills theory, where we use the
covariant derivatives and the gauge transformation is the standard one

\[ \delta A_\mu = [D_\mu, \lambda] \]

in terms of the 0-form gauge parameter \( \lambda \). Thus we might want to define
covaraint derivatives, but there should be a 1-form potential together with
ordinary derivatives to define covaraint derivatives. So the first question is:
what is the 1-form in the theory of M5-branes?

Naively, one might try to introduce a one-form potential \( A_\mu \). However, in
the theory of M5-brane(s), there is no such field. Furthermore, introducing
extra field potentially implies more physical degrees of freedom, which we
certainly do not want. One may try to consider this \( A_\mu \) as an auxiliary field, for
example, it could be gauged away by an extra gauge symmetry. But if it is the case, the theory is the same as a theory without \( A_{\mu} \), then there is no covariant derivatives. On the other hand, Chern-Simon actions are only defined in odd dimensions, it may be possible to find another kind of topological theory of one-form potential in six dimensions to construct the theory, but it is not clear how to do it. Instead, in this formulation, we define the covariant derivatives in terms of the **zero modes of** \( B_{i5} \)

\[
D_i = \partial_i + gB_{i5}^{(0)} \tag{128}
\]

Note that the zero modes of \( B_{i5}^{(0)} \) is just the Yang-Mills one-form vector potential, thus covariant derivatives we use is the same in the standard Yang-Mills theory.

We will consider the case where the worldvolume of M5-branes is on a circle  

\[
R^{1,4} \times S^1 \tag{129}
\]

where \( S^1 \) is a circle with the radius \( R \) \(^{5} \) with a periodic coordinate

\[
x^5 \sim x^5 + 2\pi R \tag{130}
\]

For an arbitrary field \( \Phi \), we have the decomposition

\[
\Phi(x_{\mu}) = \Phi^{(o)}(x_i) + \sum_{(n)} \Phi_n(x_i)e^{in\pi x_5} \\
\equiv \Phi^{(o)} + \Phi^{(KK)} \tag{131}
\]

where we use the notation that the superscript "\( (0) \)" represents zero modes while "\( (KK) \)" represents Kaluza-Klein modes. Obviously, we have

\[
\partial_5 \Phi^{(o)} = 0 \\
\partial_5 \Phi = \partial_5 \Phi^{(KK)} \tag{132}
\]

\(^{5}\)Here we consider a space-like circle for the sake of reducing to D4-branes theory in \( R^{1,4} \).
The second key idea in this model is that we will introduce the non-locality in the theory through defining a non-local operator
\[ \partial_5^{-1} \] (133)
this operator consistently acts on KK modes in the sense that when \( \partial_5 \) acts on KK modes, it gives \( \sim \frac{n}{R} \) with nonzero \( n \), so that it could be invertible. And it satisfies
\[ \partial_5^{-1} \partial_5 \Phi = \Phi^{(KK)} \] (134)
The non-locality that we introduce could be considered as the way to circumvent previous no-go theorems claiming that it is impossible to construct a theory of M5-branes, as those theorems are all based on the assumption of the locality in the theory. We will see this non-local operator indeed plays an important role in this formulation.

### 5.3 Non-Abelian Gauge Transformations for 2-form Potentials

We define the non-Abelian generalization of gauge transformation laws of the anti-symmetry tensor field (2-form) \( B_{ij} \) as [1]
\[
\begin{align*}
\delta B_{i5} &= [D_i, \Lambda_5] - \partial_5 \Lambda_i + g[B_{i5}^{(KK)}, \Lambda_5^{(0)}] \\
\delta B_{ij} &= [D_i, \Lambda_j] - [D_j, \Lambda_i] + g[B_{ij}, \Lambda_5^{(0)}] - g[F_{ij}, \partial_5^{-1} \Lambda_5^{(KK)}]
\end{align*}
\] (135) (136)
where parameter \( g \) is the coupling constant with the mass dimension.\(^6\) and \( i, j = 0, 1, 2, 3, 4 \). The covariant derivative is in terms of zero modes \( D_i = \partial_i + g B_{i5}^{(0)} \) and the 2-form field strength \( F \) is defined via the standard way
\[ F_{ij} = g^{-1}[D_i, D_j] = \partial_i B_{j5}^{(0)} - \partial_j B_{i5}^{(0)} + g[B_{i5}^{(0)}, B_{j5}^{(0)}] \] (137)
which is just the same in the Yang-Mills theory. Notice that \( F_{ij} \) in fact is zero modes, that means \( \partial_5 F_{ij} = 0 \).

\(^6\)Note that in the M-theory, there is no adjustable parameter, we will see later that the coupling constant \( g \) turns out to be the radius \( R \) of the M-circle.
More explicitly, we can decompose the gauge transformation laws into

\[ \delta B_{i5}^{(0)} = [D_i, \Lambda_5^{(0)}] \]  
\[ \delta B_{i5}^{(KK)} = [D_i, \Lambda_5^{(KK)}] - \partial_5 \Lambda_i^{(KK)} + g[B_{i5}^{(KK)}, \Lambda_5^{(0)}] \]  
\[ \delta B_{ij}^{(0)} = [D_i, \Lambda_j^{(0)}] - [D_j, \Lambda_i^{(0)}] + g[B_{ij}^{(0)}, \Lambda_5^{(0)}] \]  
\[ \delta B_{ij}^{(KK)} = [D_i, \Lambda_j^{(KK)}] - [D_j, \Lambda_i^{(KK)}] + g[B_{ij}^{(KK)}, \Lambda_5^{(0)}] - g[F_{ij}, \partial_5^{-1}\Lambda_5^{(KK)}] \]

with all quantities $B_{i5}, B_{ij}, \Lambda_5, \Lambda_i$ take values in a Lie algebra $G$.

The algebra of gauge transformation is closed and given by

\[ [\delta, \delta'] = \delta'' \]

with

\[ \Lambda_5''^{(0)} = g[\Lambda_5^{(0)}, \Lambda_5''^{(0)}] \]  
\[ \Lambda_5''^{(KK)} = g[\Lambda_5^{(0)}, \Lambda_5''^{(KK)}] - g[\Lambda_5''^{(0)}, \Lambda_5^{(KK)}] \]  
\[ \Lambda_i'' = g[\Lambda_5^{(0)}, \Lambda_i'] - g[\Lambda_5''^{(0)}, \Lambda_i] \]

Let us give some comments on these non-Abelian gauge transformation laws.

First we notice that there is a non-local term in $\delta B_{ij}^{(KK)}$ (141). Why we need that? Recall that, consider the case in six dimensional spacetime, there should be only 5 independent gauge parameters for the 2-form potential gauge theory rather than 6 since there is a redundant symmetry for the gauge parameters. In non-Abelian case, it is defined by\footnote{Although there is no rule forcing us that we must define it in this way, however we should define it in a natural way that reduces to the abelian case when the coupling is turned off.}

\[ \delta \Lambda_i^{(KK)} = [D_i, \lambda^{(KK)}] \]  
\[ \delta \Lambda_5^{(KK)} = \partial_5 \lambda^{(KK)} \]
one can check that (141) is indeed invariant under these redundant transformations. The existence of the "gauge symmetry of gauge symmetry" is crucial for this non-Abelian gauge transformation laws to be justified as the correct deformation of the Abelian gauge transformation laws of 2-form potentials.

Now we can use the redundant symmetry to fix

$$\Lambda_5^{(KK)} = 0$$

(148)

by using the freedom of $\lambda^{(KK)}$, then the gauge transformation laws are equivalent to

$$\delta B_{i5}^{(KK)} = -\partial_5 \tilde{\Lambda}_i^{(KK)} + g[B_i^{(KK)}, \Lambda_5^{(0)}]$$

(149)

$$\delta B_{ij}^{(KK)} = [D_i, \tilde{\Lambda}_j^{(KK)}] - [D_j, \tilde{\Lambda}_i^{(KK)}] + g[B_{ij}^{(KK)}, \Lambda_5^{(0)}]$$

(150)

where

$$\tilde{\Lambda}_i^{(KK)} = \Lambda_i^{(KK)} - [D_i, \partial_5^{-1}\Lambda_5^{(KK)}]$$

(151)

Notice that the zero modes $\Lambda_5^{(0)}$ can not be gauged away since $\partial_5 \lambda^{(0)} = 0$.

We notice that in non-Abelianizing the gauge transformations of a 2-form potential, the zero mode $\Lambda_5^{(0)}$ plays a special role. We associate the special role played by $\Lambda_5^{(0)}$ to its topological nature: while $\Lambda_5^{(KK)}$ can be gauged away, the zero mode $\Lambda_5^{(0)}$ corresponds to the Wilson line degrees of freedom for the gauge transformation parameter $\Lambda$ along the circle in the $x^5$ direction.

One might ask how about the gauge parameter $\Lambda_i^{(0)}$ and its redundant symmetry? The fact is that we will not use $B_{ij}^{(0)}$ and hence neither $\Lambda_i^{(0)}$ in this formulation, we will soon see the reason of it, let us discuss this point later. Also notice that the only non-local term in the gauge transformation

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8If the zero mode $\lambda^{(0)}$ can be gauged away, then there is no gauge parameter for the Yang-Mills theory of D4-branes
laws is gauged way through the change of variables, but it does not mean that this theory is local since the non-locality is hidden in the gauge parameters we redefined.

On the other hand, we have "additional non-locality" introduced in the theory in the sense that the gauge transformation laws are defined separately for the zero modes and KK modes: For instance, one may think the term $\delta B_{i5}^{(0)}$ could be deduced form $\delta B_{i5}^{(KK)}$, but we see there are in fact different by a factor 2 through the commutator term. Also notice that we did not introduce terms like

$$[B_{i5}^{(KK)}, \Lambda_{5}^{(KK)}]$$

in particular, thoughout this formulation we find that all the commutators should involve at most one KK mode. Future works about the geometric interpretation of this formulamation might help us to better understand these interesting consequences.

Finally, we could easily observe that if G is abelian, this non-Abelian gauge transformation laws are reduced to the conventional Abelian gauge transformation of 2-form gauge potential, which is used to describe a single M5-brane.

5.4 Non-Abelian 3-form Field strengths

We define 3-form field strengths as

$$H_{ij5}^{(0)} = F_{ij} = g^{-1}[D_i, D_j]$$

$$H_{ij5}^{(KK)} = [D_i, B_{j5}^{(KK)}] - [D_j, B_{i5}^{(KK)}] + \partial_5 B_{i5}^{(KK)}$$

$$H_{ijk}^{(0)} = [D_i, B_{jk}^{(0)}] + [D_j, B_{ki}^{(0)}] + [D_k, B_{ij}^{(0)}]$$

$$H_{ijk}^{(KK)} = [D_i, B_{jk}^{(KK)}] + [D_j, B_{ki}^{(KK)}] + [D_k, B_{ij}^{(KK)}]$$

$$+ g[F_{ij}, \partial_5^{-1} B_{k5}^{(KK)}] + g[F_{jk}, \partial_5^{-1} B_{i5}^{(KK)}] + g[F_{ki}, \partial_5^{-1} B_{j5}^{(KK)}] \]$$

39
They satisfy the generalized Jacobi identities

\[
\sum_{(3)} [D_i, H_{j5}^{(0)}] = 0 \tag{157}
\]

\[
\sum_{(3)} [D_i, H_{j5}^{(KK)}] = \partial_5 H_{ijk}^{(KK)} \tag{158}
\]

\[
\sum_{(4)} [D_i, H_{jkl}^{(0)}] = \sum_{(6)} [F_{ij}, B_{kl}^{(0)}] \tag{159}
\]

\[
\sum_{(4)} [D_i, H_{jkl}^{(KK)}] = \sum_{(6)} [F_{ij}, \partial_5^{-1} H_{kl5}^{(KK)}] \tag{160}
\]

where \(\sum_{(n)}\) represents a sum over \(n\) terms that totally anti-symmetrized all the indices.

These 3-form field strengths transform as

\[
\delta H_{ij5}^{(0)} = g[H_{ij5}^{(0)}, \Lambda_{5}^{(0)}] \tag{161}
\]

\[
\delta H_{ij5}^{(KK)} = g[H_{ij5}^{(KK)}, \Lambda_{5}^{(0)}] \tag{162}
\]

\[
\delta H_{ijk}^{(0)} = g[H_{ij5}^{(0)}, \Lambda_{5}^{(0)}] + g[F_{ij}, \Lambda_{k}^{(0)}] + g[F_{jk}, \Lambda_{i}^{(0)}] + g[F_{ki}, \Lambda_{j}^{(0)}] \tag{163}
\]

\[
\delta H_{ijk}^{(KK)} = g[H_{ij5}^{(KK)}, \Lambda_{5}^{(0)}] \tag{164}
\]

a potential problem here is that the transformation of \(H_{ijk}^{(0)}\) is not covaraint \(^9\), that causes a trouble to construct an gauge invaraint action. We will deal with this problem later.

Let us consider a useful gauge

\[
B_{i5}^{(KK)} = 0 \tag{165}
\]

by using the freedom from the gauge paramter \(\Lambda_{i5}^{(KK)}\). We use \(\hat{B}_{ij}\) as our variable to denote the theory under this gauge, we have

\[
H_{ij5}^{(KK)} = \partial_5 \hat{B}_{ij}^{(KK)} \tag{166}
\]

\[
H_{ijk}^{(KK)} = [D_i, \hat{B}_{jk}^{(KK)}] + [D_j, \hat{B}_{ki}^{(KK)}] + [D_k, \hat{B}_{ij}^{(KK)}] \tag{167}
\]

\(^9\)Unless \(F_{ij} = 0\) by itself, however, it will imply the zero modes, \(B_{i5}^{(0)}\), is a pure gauge and the theory is the same as no connection.
where we can also write
\[ \hat{B}_{ij} = \partial_5^{-1} H_{ij5}^{(KK)} \] (168)
which transforms covariantly \( \delta \hat{B}_{ij} = [\hat{B}_{ij}, \Lambda_5^{(0)}] \).

### 5.5 Coupling to Antisymmetry Tensors

Apart from multiple M5-branes, let us also consider applications to this formulation. Readers who want to focus on the theory of M5-branes might simply skip this section for the first reading.

A straightforward generalization of the transformation laws for \( H \) leads to the definition of gauge transformations of a totally antisymmetrized tensor field \( \phi_{\mu_1 \cdots \mu_n} \) \( (n \leq 6) \). We define their gauge transformation laws as

\[
\delta \phi^{(0)}_{i_1 \cdots i_{n-1} 5} = g[\phi^{(0)}_{i_1 \cdots i_{n-1} 5}, \Lambda_5^{(0)}],
\]
(169)
\[
\delta \phi^{(KK)}_{i_1 \cdots i_{n-1} 5} = g[\phi^{(KK)}_{i_1 \cdots i_{n-1} 5}, \Lambda_5^{(0)}],
\]
(170)
\[
\delta \phi^{(0)}_{i_1 \cdots i_n} = g[\phi^{(0)}_{i_1 \cdots i_n}, \Lambda_5^{(0)}] + g \sum_{(n)} [\phi^{(0)}_{i_1 \cdots i_{n-1} 5}, \Lambda_i^{(0)}]
\]
(171)
\[
\delta \phi^{(KK)}_{i_1 \cdots i_n} = g[\phi^{(KK)}_{i_1 \cdots i_n}, \Lambda_5^{(KK)}],
\]
(172)

where \( \sum_{(n)} \) represents a sum of \( n \) terms that totally antisymmetrizes all indices.

We see that the gauge transformation law for the component \( \phi^{(0)}_{i_1 \cdots i_n} \) is different from all other components. It is defined to mimic the gauge transformation of \( H_{ijk}^{(0)} \). We should check whether this complication will prevent us from constructing a gauge field theory. First, products of these fields \( \phi^{(0)}_{i_1 \cdots i_n} \) will also transform in the form of not being covariant (where we consider that all indices are antisymmetrized on the products). Secondly, although \( D_i \) acts on \( \phi^{(0)}_{i_1 \cdots i_n} \) does not transform covariantly, we can define a covariant exterior derivative
for $\phi^{(0)}_{i_1 \cdots i_n}$ as

$$ (D\phi)^{(0)}_{i_1 \cdots i_{n+1}} \equiv \sum_{(n+1)} [D_{i_1}, \phi^{(0)}_{i_2 \cdots i_{n+1}}] - (-1)^n \sum_{((n+1)n/2)} [B^{(0)}_{i_1i2}, \phi^{(0)}_{i_3 \cdots i_{n+1}}], \quad (173) $$

(This expression is nontrivial only if $n \leq 5$.) This covariant exterior derivative is indeed covariant, that is,

$$ \delta (D\phi)^{(0)}_{i_1 \cdots i_{n+1}} = g [(D\phi)^{(0)}_{i_1 \cdots i_{n+1}}, \Lambda^{(0)}] + g \sum_{(n+1)} [(D\phi)^{(0)}_{i_1 \cdots i_{n+5}}, \Lambda^{(0)}], \quad (174) $$

where the exterior derivative of $\phi^{(0)}_{i_1 \cdots i_{n-15}}$ is defined by

$$ (D\phi)^{(0)}_{i_1 \cdots i_{n-5}} = g \sum_{(n)} [D_{i_1}, \phi^{(0)}_{i_2 \cdots i_{n-5}}]. \quad (175) $$

It seems possible to down covariant equations of motion using exterior derivatives and totally antisymmetrized tensors.

5.6 Non-Abelianizing the Abelian Self-Dual Theory

We have mentioned that the anomalous transformation of $H^{(0)}_{ijk}$ (163) will cause the problem to define a gauge invariant action. For example, if we define a Yang-Mills-like theory, the Lagrangian should look like

$$ \frac{1}{6} \text{Tr}(H^{(0)}_{ijk}H^{(0)}_{ijk} + 3H^{(0)}_{ijk}H^{(0)}_{ij5} + H^{(KK)}_{ijk}H^{(KK)}_{ijk} + 3H^{(KK)}_{ij5}H^{(KK)}_{ijk}). \quad (176) $$

Only the first term is not gauge invariant. It is not clear how to modify the action to make it invariant. Similarly it is hard to define the usual kinetic term for the components $\phi^{(0)}_{i_1 \cdots i_n}$ of a matter field. In the following we will see that in a Lagrangian formulation of the non-Abelian self-dual gauge theory in 6 dimensions, we do not have to use the variables $B^{(0)}_{ij}$ explicitly, so these anomalous covariant transformation laws will never be used. In fact we will simply define $H^{(0)}_{ijk}$ to be the Hodge dual of $F_{ij}$

$$ H^{(0)}_{ijk} \equiv \frac{1}{2} \epsilon_{ijklm} F^{lm} \quad (177) $$
so that its gauge transformation is the same as other components. As a result the covariant transformation laws for matter fields can be uniformly defined as

\[ \delta \Phi = g[\Phi, \Lambda_5^{(0)}] \]  \hspace{1cm} (178)

for all components of a matter field.

Recall that the Abelian action for chiral 2-form is (48) (we can obtain it from a gauge-fixed PST action (108) as mentioned before)

\[ S = \frac{1}{4} T_{M5} T_{M2}^{-2} \int d^6x \left( \frac{1}{6} \varepsilon^{ijklm} H_{ijk} \left[ H_{lm5} + \frac{1}{6} \varepsilon_{lmnpq} H^{npq} \right] \right). \]  \hspace{1cm} (179)

color{consider the compactification of the Abelian theory on a circle of radius \( R \) along \( x^5 \). All fields then are decomposed into their zero modes and KK modes, the action becomes

\[ S = S^{(0)} + S^{(KK)}, \]  \hspace{1cm} (180)

where

\[ S^{(0)} = \frac{2\pi R}{12} T_{M5} T_{M2}^{-2} \int d^5x \ H^{(0)}_{ijk} H^{(0)ijk} \]  \hspace{1cm} (181)

\[ S^{(KK)} = \frac{1}{4} T_{M5} T_{M2}^{-2} \int d^6x \ \left( \frac{1}{6} \varepsilon^{ijklm} H^{(KK)}_{ijk} \left[ H^{(KK)}_{lm5} + \frac{1}{6} \varepsilon_{lmnpq} H^{(KK)npq} \right] \right) \]  \hspace{1cm} (182)

Where the zero modes \( B^{(0)}_{ij} \) are 5 dimensional 2-form potential, one can carry out the standard procedure of electric-magnetic duality for \( S^{(0)} \) to get an action for the dual 1-form potential

\[ S^{(0)}_{dual} = \frac{2\pi R}{4} T_{M5} T_{M2}^{-2} \int d^5x \ F_{ij} F^{ij}, \]  \hspace{1cm} (183)

where \( F_{ij} = H^{(0)}_{ij5} \) is the field strength of the dual 1-form potential \( B^{(0)}_{i5} \).

Let us check that how equations of motion derived from the action \( S^{(0)}_{dual} + S^{(KK)} \) leads to configurations satisfying self-duality conditions. For the zero
modes, the equation of motion derived from the action $S_{\text{dual}}^{(0)}$ is

$$\partial^j F_{ij} = 0. \quad (184)$$

Defining a 3-form field $H$ by

$$H_{ijk}^{(0)} = \frac{1}{2} \epsilon_{ijklm} F^{lm}, \quad (185)$$

we see that, due to the equation of motion $\partial^j F_{ij} = 0$, a 2-form potential $B^{(0)}$ exists locally such that $H^{(0)} = dB^{(0)}$. Since $F$ also satisfies the Jacobi identity $dF = 0$, we also have

$$\partial^k H_{ijk}^{(0)} = 0. \quad (186)$$

Note that our definition of $H_{ijk}^{(0)}$ is identical to the self-duality condition for the zero modes

$$H_{ijk}^{(0)} = \frac{1}{2} \epsilon_{ijklm} H^{(0) lm5} \quad (187)$$

hence we see that the zero modes of the self-dual gauge field can be simply described by the 5D Maxwell action.

It is natural to non-Abelianize the equation of motion for the zero modes by

$$[D^j, F_{ij}] = 0 + \cdots \quad (188)$$

up to additional covariant terms that vanish when the Lie algebra $G$ is Abelian. In the next section we will derive the complete equation from an action principle. Let us stress again that for the non-Abelian theory, we still define $H_{ijk}^{(0)}$ simply as the Hodge dual of $F_{ij}$, hence it is not necessary to introduce the new definition of $H_{ijk}^{(0)}$ which has the unusual transformation law. The transformation of $F_{ij}$ would then imply that $H^{(0)}$, transforms simply as

$$\delta H_{ijk}^{(0)} = [H_{ijk}^{(0)}, \Lambda_5^{(0)}]. \quad (189)$$

On the other hand, for the KK modes, the equations of motion derived from varying $S^{(KK)}$ is

$$\epsilon^{ijklm} \partial_k \left( H_{lm5}^{(KK)} + \frac{1}{6} \epsilon_{lmnpq} H^{(KK) npq} \right) = 0 \quad (190)$$
This implies that
\[ \epsilon^{ijklm} \left( H^{(KK)}_{lm5} + \frac{1}{6} \epsilon_{lmnpq} H^{(KK)npq} \right) = \epsilon^{ijklm} \Phi^{(KK)}_{lm} \] (191)
for some tensor \( \Phi^{(KK)}_{lm} \) satisfying
\[ \epsilon^{ijklm} \partial_k \Phi^{(KK)}_{lm} = 0. \] (192)

We can redefine \( B^{(KK)}_{lm} \) by a shift
\[ B^{(KK)}_{lm} \rightarrow B'^{(KK)}_{lm} \equiv B^{(KK)}_{lm} + \partial_5^{-1} \Phi^{(KK)}_{lm} \] (193)
such that,
\[ H^{(KK)}_{lm5} \rightarrow H'^{(KK)}_{lm5} \equiv H^{(KK)}_{lm5} + \Phi^{(KK)}_{lm} \] (194)
\[ \epsilon^{lmnpq} H^{(KK)}_{npq} \rightarrow \epsilon^{lmnpq} H'^{(KK)}_{npq} = \epsilon^{lmnpq} \Phi^{(KK)}_{npq} \] (195)

As a result of this shift (This shift is also a gauge symmetry of this theory), we have the self-duality condition
\[ H^{(KK)}_{lm5} = -\frac{1}{6} \epsilon_{lmnpq} H^{(KK)npq}. \] (196)

Let us define the non-Abelian counterpart of the equation of motion of KK modes as
\[ \epsilon^{ijklm} \left[ D_k, \left( H^{(KK)}_{lm5} + \frac{1}{6} \epsilon_{lmnpq} H^{(KK)npq} \right) \right] = 0. \] (197)
This implies that
\[ \epsilon^{ijklm} \left( H^{(KK)}_{lm5} + \frac{1}{6} \epsilon_{lmnpq} H^{(KK)npq} \right) = \epsilon^{ijklm} \Phi^{(KK)}_{lm} \] (198)
where \( \Phi^{(KK)}_{lm} \) satisfies
\[ \epsilon^{ijklm} [D_k, \Phi^{(KK)}_{lm}] = 0. \] (199)
This again can be absorbed into a shift of \( B^{(KK)}_{lm} \)
\[ B^{(KK)}_{lm} \rightarrow B'^{(KK)}_{lm} \equiv B^{(KK)}_{lm} + \partial_5^{-1} \Phi^{(KK)}_{lm}, \] (200)
so that the self-duality condition is arrived. Notice that it is the KK modes that allows us to consider this "non-local" shift as this non-local operator only consistenly acts on KK modes. Also note that the gauge transformation parameter $\Phi^{(KK)}_{lm}$ has to transform covariantly

$$\delta \Phi^{(KK)}_{lm} = [\Phi^{(KK)}_{lm}, \Lambda^{(0)}_5],$$

because the constraint of it is covariant. It can then be checked that this extra shift commutes with the gauge transformation defined previous Section.

Our task in the next section is to give an action that would lead to the non-Abelian equations of motion (188), (197), which we have shown here that it will give the self-duality condition of 2-form potentials.

### 5.7 Action

Let us consider the following action for the non-Abelian chiral 2-form potential (the gauge sector of multiple M5-branes)

$$S = S^{(0)} + S^{(KK)},$$

where

$$S^{(0)} = \frac{2\pi R}{4} T_{M5} T_{M2}^{-2} \int d^5 x \, \text{Tr}(F_{ij} F^{ij}),$$

$$S^{(KK)} = \frac{1}{4} T_{M5} T_{M2}^{-2} \int d^6 x \, \text{Tr} \left( \frac{1}{6} \epsilon^{ijklm} H^{(KK)}_{ijk} \left[ H^{(KK)}_{lm5} + \frac{1}{6} \epsilon_{lmnpq} H^{(KK)}_{npq} \right] \right).$$

this invariant action is a straightforward generalization of the action for the Abelian theory. For small $R$, the M5-branes should be approximated by D4-branes in the type II A string theory, so $S^{(0)}$ should be identified as the Yang-Mills theory for multiple D4-branes

$$S^{(0)} = \frac{1}{4} T_{D4} T_s^{-2} \int d^5 x \, \text{Tr}(f_{ij} f^{ij}),$$
where the field strength \( f_{ij} \) for multiple D4-branes is

\[
f_{ij} \equiv [\partial_i + A_i, \partial_j + A_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j].
\]  

(206)

It is known that the gauge potential \( A \) in D4-brane theory is related to the gauge potential \( B \) in M5-brane theory via the relation

\[
A_i = 2\pi R B_{i5}^{(0)}.
\]  

(207)

Plugging in the values of the parameters involved,

\[
T_{M5} = \frac{1}{2\pi} T_{M2}, \quad T_{D4} = \frac{1}{(2\pi)^4 g_s \ell_s^5}, \quad T_s = \frac{1}{2\pi \ell_s^2}, \quad R = g_s \ell_s,
\]  

(208)

we find that the coupling constant should be given by

\[
g = 2\pi R.
\]  

(209)

This factor can also be obtained by demanding that the soliton solutions which resemble instantons in the spatial 4 dimensions have momentum equal to \( n/R \) for some integer \( n \) in the \( x^5 \) direction.

Notice that the overall factor of \( 2\pi R \) due to the integration over \( x^5 \) will be multiplied by a factor of \( 1/g^2 \) where \( g \) is the Yang-Mills coupling for the zero mode field strength \( F_{ij} \), giving an overall factor of \( 1/R \), in agreement with the requirement of conformal symmetry in 6 dimensions [22]. Normally the coupling constant of an interacting field theory is independent of whether the space is compactified. Our strategy is to define a 6 dimensional field theory as the decompactification limit of a compactified theory, and the coupling depends on the compactification radius. In some sense, the coupling constant \( g \) is not really the coupling of the decompactified theory, which is a conformal field theory without any free parameter.

Assuming that we will be able to show in future works that a well defined theory does exist in uncompactified 6 dimensional spacetime as the decompactification limit of our model, one would still wonder how such a theory can
be fully Lorentz invariant, while its definition involves the choice of a special direction. We will discuss more on this point later.

The variation of $S^{(0)}$ leads to Yang-Mills equations, which can be interpreted as the self-dual equation for the zero modes. The full equation of motion for the zero modes $B_{i5}^{(0)}$ should also include variations of $S^{(KK)}$, which modifies the Yang-Mills equation by commutators that vanish in the Abelian case. Explicitly, the equations of motion is

$$[D_j, F^{ij}] = \frac{1}{2} \int_0^{2\pi R} dx^5 \left[ \hat{B}_{jk}, \left( H^{(KK)ij} - \frac{1}{4} \epsilon^{ijklm} H^{(KK)lm} \right) \right]$$

A useful feature of $S^{(KK)}$ is that it depends on $B_{i5}^{(KK)}$ and $B_{ij}^{(KK)}$ only through $\hat{B}_{ij}^{(KK)}$. Therefore, we only need to consider the variation of $\hat{B}_{ij}^{(KK)}$. Explicitly, the equations of motion is

$$\epsilon^{ijklm} \left[ D_k, \left( H_{lm5}^{(KK)} + \frac{1}{6} \epsilon_{lmnpq} H^{(KK)npq} \right) \right] = 0$$

which leads to the equation of motion that is equivalent to the self-duality condition via a shift in $B_{lm}^{(KK)}$ as we explained in the previous section.

Notice that despite the appearance of the nonlocal operator $\partial_5^{-1}$ and the nonlocal separation of KK modes from zero modes in the 5-th direction, this action is an ordinary local action from the viewpoint of the uncompactified 5 dimensional Minkowski space.

$$S = \frac{2\pi R}{4} T_{M5} T_{M2}^{-2} \int d^5 x \, \text{Tr} \left\{ F_{ij} F^{ij} \right\}$$

$$+ \frac{1}{6} \sum_{p \in \mathbb{Z}} \epsilon^{ijklm} h_{ijk}(-p) \left[ \left( \frac{i}{R} \right) \hat{b}_{lm}(p) + \frac{1}{6} \epsilon_{lmnpq} h^{npq}(p) \right]$$

where $\hat{b}_{ij}(p)$ and $h_{ijk}(p)$ are the KK mode coefficients of $\hat{B}_{ij}$ and $H_{ijk}$ defined
by
$$\hat{B}_{ij}(\vec{x}, x^5) = \sum_{p \in \mathbb{Z}} \hat{b}_{ij}(\vec{x}, p)e^{ipx^5/R},$$  \hfill (213)$$

$$H_{ijk}(\vec{x}, x^5) = \sum_{(i,j,k)} \sum_{p \in \mathbb{Z}} \left( \partial_i \hat{b}_{jk}(\vec{x}, p) + \sum_{q \in \mathbb{Z}} [A_i(\vec{x}, q), \hat{b}_{jk}(\vec{x}, p - q)] \right) e^{ipx^5/R}$$  \hfill (214)$$

where $\vec{x} = (x^0, x^1, \ldots, x^4)$. The derivation of the equations of motion above can be viewed as a collective expression of equations of motion derived by varying each KK mode or zero mode one at a time.

It is clear from these expressions that our theory is local in the directions of $\vec{x}$, and its nonlocality is restricted to the $x^5$-direction. At this moment we can not prove or disprove the causality in the direction of $x^5$. At least causality is apparently still preserved in the uncompactified directions $\vec{x}$.

5.8 Comments

In this formulation, we have avoided commutators involving two KK modes, e.g. terms of the form $[B^{(KK)}, \Lambda^{(KK)}]$. Correspondingly, there is no term of the form $[B^{(KK)}, B^{(KK)}]$ in the equations of motion or action. In fact, all gauge interactions are mediated via zero modes. Here is our interpretation.

In the limit $R \to \infty$, the Fourier expansion of a field approaches to the Fourier transform
$$\Phi(x^5) = \sum_n \Phi_n e^{inx^5/R} \rightarrow \Phi(x^5) = \int \frac{dk_5}{2\pi} \tilde{\Phi}(k_5)e^{ik_5x^5}. \hfill (215)$$

The coefficients $\Phi_n$ approach to $\tilde{\Phi}(k_5)$ as
$$\tilde{\Phi}(k_5) = 2\pi R\Phi_n \quad (k_5 = n/R).$$  \hfill (216)$$

According to this expression, the value of a specific Fourier mode $\Phi_n$ must approach to zero in the limit $R \to \infty$. In particular, the amplitude of the zero
mode approaches to zero. While all interactions are mediated via the zero mode, this does not imply that there is no interaction in the infinite \( R \) limit, because the coupling \( g = 2\pi R \to \infty \). The product of the amplitude of the zero mode with the coupling is actually kept finite in the limit.

In the limit \( R \to \infty \), the KK modes \( B_{\mu\nu}^{(KK)} \) should be identified with the 2-form potential in uncompactified 6 dimensional spacetime. In uncompactified space, the constant part of \( B_{\mu\nu} \) is not an observable, hence physically the KK modes \( B_{\mu\nu}^{(KK)} \) do not miss any physical information a 2-form potential can carry.

Regarding the large \( R \) limit, we consider that the zero modes \( B_{\mu\nu}^{(0)} \) approach to zero but a new field \( A_i \) replacing \( 2\pi R B_{i5}^{(0)} \) survives the large \( R \) limit. The field \( A_i \) can not be viewed as part of the 2-form potential, in the sense that, due to the infinite scaling of \( B_{i5}^{(0)} \) by \( R \), it can not be combined with \( B_{\mu\nu}^{(KK)} \) in a Lorentz covariant way to form a new tensor in 6 dimensions. Rather it should be understood as the 1-form needed to define gerbes (or some similar geometrical structure) together with the 2-form potential. However this does not increase the physical degrees of freedom of the 6 dimensional theory in the sense that the number of physical degrees of freedom in the 5 dimensional field \( A_i \) is negligible compared with that of a 6 dimensional field.

The fact that gauge transformation laws do not have terms of the form \([B^{(KK)}, \Lambda^{(KK)}]\), and the fact that the equations of motion do not have terms of the form \([B^{(KK)}, B^{(KK)}]\), are both telling us that our model is linearized with respect to the 2-form potential. No self-interaction of the 2-form potential is present, and all interactions are mediated by the 1-form potential \( A_i \). As the decompactification limit \( R \to \infty \) is also the strong coupling limit \( g \to \infty \), we do not expect that the classical equations of motion could give a good approximation for the quantum theory.
On the other hand, the interpretation above allows us to understand some puzzles about the proposal of recent papers [23, 24] claiming that the 5-dimensional D4-brane theory is already sufficient to describe the 6-dimensional M5-brane system even for a finite $R$. In their proposal, the momentum $p_5$ in the 5-th (compactified) direction is represented by the “instanton” number on the 4 spatial dimensions. The first puzzle with this interpretation is that, in the phase when $U(N)$ symmetry is broken to $U(1)^N$, there is no instanton solution. But physically this corresponds to having M5-brane well separated from each other, and they should still be allowed to have nonzero $p_5$. This problem does not exist in our model. In our model $p_5$ is carried by the KK modes when the Lie algebra of the gauge symmetry is Abelian. Furthermore, the Abelian case of our model is already known to be equivalent to a 6-dimensional theory which has the full Lorentz symmetry in the large $R$ limit.

The second puzzle of their proposal is that the instanton number only gives the total value of $p_5$ of a state, and there is no way to specify the distribution of $p_5$ over different physical degrees of freedom. For example, the state with $m$ units of $p_5$ contributed from the scalar field $X^1$ and $n$ units of $p_5$ from $X^2$ cannot be distinguished from the state with the numbers $m$ and $n$ switched. A possible resolution of this puzzle is that, perhaps due to strong interactions, we can no longer distinguish different distributions of $p_5$ over different degrees of freedom. In other words, it is unphysical to specify the distribution of $p_5$. If true, this would be a mysterious phenomenon, but we can not rule out this possibility, as we know very little about how to label physical degrees of freedom in a strongly correlated system.

In our model, the instanton number of the 1-form $A_i \equiv R B_{i5}^{(0)}$ should only be interpreted as the value of $p_5$ of the field $A_i$. In other words, the so-called “zero-modes” $B_{i5}^{(0)}$ can still carry nonzero $p_5$. The 5-th momentum of the 2-
form potential is manifest as the KK mode index. The scalar fields $X^I$ and the fermions $\Psi$, when they are introduced into our model, would have their own KK modes to specify their $p_5$ contribution. There is no ambiguity in the momentum carrier for a given instanton number.

The reader may wonder whether it is redundant or over-counting for $A_i$ to be able to carry nontrivial $p_5$. After all, $A_i$ is just $B_{i5}^{(0)}$ rescaled. Has not the KK modes $B_{i5}^{(KK)}$ already taken care of the contribution of $B_{i5}$ to $p_5$? How can a field carry momentum in the $x^5$-direction if it has no fluctuation (e.g. propagating wave) in that direction? The answer is simple. It is well known in classical electromagnetism that the simultaneous presence of constant electric and magnetic fields carry momentum, because the momentum density $p_i$ is proportional to $F^{0j}F_{ij}$. In the temporal gauge, $A_0 = 0$, the conjugate momentum of $A_j$ is $\Pi^j \equiv \partial_0 A^j$ and the momentum density $p_i$ is proportional to

$$F^{0j}F_{ij} = \Pi^j(\partial_i A_j) - \Pi^j(\partial_j A_i).$$

The first term is the standard contribution of a field to momentum $p_i$. We also have $(\partial_0 \phi)(\partial_i \phi)$ for a scalar field $\phi$. But there is no analogue of the 2nd term for a scalar field. It is possible for the 2nd term to be present because $A_i$ has a Lorentz index. The zero mode of $A_i$ in the $x^i$ direction can also contribute to $p_i$ through this term. Similarly, for a 3-form field strength $H$, the momentum density of $p_5$ is proportional to $H_{0ab}H^{ab5}$ $(a, b = 1, 2, 3, 4)$, which includes the zero mode contribution

$$H_{0ab}^{(0)} = \frac{1}{6} \epsilon_{abcd5} F^{ab} F^{cd}$$

because $H_{ab5}^{(0)} = F_{ab}$. This is precisely the same expression as the instanton number density. Note that there are also contributions to $p_5$ from the KK modes $H_{0ab}^{(KK)} H^{(KK)ab5}$ in addition to the zero mode contribution.
6 BRST-Antifields Quantization on Non-Abelian 2-Form

6.1 BRST Symmetry of Non-Abelian 2-form

The quantization of self-dual fields is a long-standing subtle problem. It was argued [25] that even we have a classical action of self-dual gauge fields; we do not expect to quantize the theory from the action. That means that a self-dual p-form might need to be treated quantum mechanically in the very beginning.

One approach toward a quantized M5-brane (self-dual fields) was also developed in [25]. It suggested that one can start from a non-chiral action, which has a well-defined partition function denoted by $Z$, then consider write $Z$ as an absolute value squared of the chiral fields’ partition functions. Thus as an intermediate step toward the quantization of M5-branes, it will be useful to firstly consider $Z$, which corresponds to the partition function of non-Abelian 2-form potential without the self-duality condition. In this section, although the Lorentz covariance is still broken, we investigate the BRST transformation laws of this non-local non-Abelian 2-form gauge theory and we will give a BRST invariant gauge-fixed action which is desirable before having a well-defined partition function.

The quantization of reducible gauge theories is not that straightforward. As we know, a gauge-fixing procedure is needed to render dynamical degrees of freedom by using ghost fields, which are used to compensate for the gauge degrees of freedom. In reducible gauge theories, some so-called ”ghosts of ghosts” will be needed and the ordinary Faddeev-Popov procedure become quite complicated for this kind of reducible theories. Here we will use BRST-antifield formulation (or Batalin-Vilkovisky method [33]) to deal with this situation.
The Yang-Mills-like action for the non-Abelian 2-form is given by

$$S = S^{(0)} + S^{(KK)},$$  \hspace{1cm} (219)$$

where

$$S^{(0)} = \frac{2\pi R}{4} \int d^5x \, Tr \left[ F^{(0)ij} F^{(0)}_{ij} \right]$$ \hspace{1cm} (220)$$

$$S^{(KK)} = \int d^6x \, Tr \left[ \frac{1}{6} H^{(KK)ijk} H^{(KK)}_{ijk} + \frac{1}{2} H^{(KK)ij5} H^{(KK)}_{ij5} \right]$$ \hspace{1cm} (221)$$

where we separately define zero modes and KK modes as we did before. We consider a standard form of the action that is different from the self-dual action. At first we will need to find the existence of the BRST transformation laws of this theory where gauge parameters such as

$$\Lambda_5^{(0)}, \Lambda_5^{(KK)}, \Lambda_i^{(KK)}$$ \hspace{1cm} (222)$$

become "ghost fields" with fermionic statistics and join BRST transformation laws. In the following we use the same notations for these gauge parameters as before but we should re-interpret gauge parameters as ghost fields.

The BRST transformation laws denoted by the BRST operator $s$ for this non-local non-Abelian 2-form theory are given by

$$sB_{i5}^{(0)} = [D_i, \Lambda_5^{(0)}]$$ \hspace{1cm} (223)$$

$$sB_{i5}^{(KK)} = [D_i, \Lambda_5^{(KK)}] - \partial_5 \Lambda_i^{(KK)} + g[B_{i5}^{(KK)}, \Lambda_5^{(0)}]$$ \hspace{1cm} (224)$$

$$sB_{ij}^{(KK)} = [D_i, \Lambda_j^{(KK)}] - [D_j, \Lambda_i^{(KK)}] + g[B_{ij}, \Lambda_5^{(0)}]$$ \hspace{1cm} (225)$$

$$s\Lambda_5^{(0)} = -\frac{g}{2} [\Lambda_5^{(0)}, \Lambda_5^{(0)}]$$ \hspace{1cm} (226)$$

$$s\Lambda_5^{(KK)} = \partial_5 \alpha - g[\Lambda_5^{(0)}, \Lambda_5^{(KK)}]$$ \hspace{1cm} (227)$$

$$s\Lambda_i^{(KK)} = [D_i, \alpha] - g[\Lambda_5^{(0)}, \Lambda_i^{(KK)}]$$ \hspace{1cm} (228)$$

$$s\alpha = g[\alpha, \Lambda_5^{(0)}]$$ \hspace{1cm} (229)$$

where $\alpha$ is a commuting ghost representing the redundancy in the theory. Notice that $\alpha$ appears only as the KK mode but we ignore the KK index for the
simplicity. We will not use $\Lambda_i^{(0)}$ explicitly just like we do not use $B_i^{(0)}$ explicitly throughout the theory. The BRST operator $s$ has ghost number one and the ghost of ghost ”$\alpha$” has ghost number two which can be read from the BRST transformation laws. The role of $\alpha$ is to fix the residual degrees of freedom coming from the gauge symmetry of gauge symmetry of the theory as mentioned above.

These BRST transformation laws satisfy

$$s^2 = 0 \quad (230)$$

which is the nilpotency condition for the BRST transformation. The existence of the BRST symmetry is crucial for the further analysis on quantization of a gauge theory. Physical observables are defined as those BRST invariant (closed form) but can not be expressed as a BRST variation of something (exact form). In short, observables correspond to the elements of the BRST cohomology.

6.2 Field-Antifield Quantization on Non-Abelian 2-form

Now we consider the gauge fixing process following the field-antifield method [33], where the original configuration space will be enlarged to include extra fields such as ghost fields and ghosts for ghosts. The minimal action is given by

$$S_0 = S_0^{(0)} + S_0^{(KK)} \quad (231)$$

where zero modes’ part is given by

$$S_0^{(0)} = \frac{2\pi R}{4} \int d^5x \, Tr \left[ (F_{ij}^i F_{ij} + B_{ij}^{(0)} D_{ij} \Lambda_5^{(0)}) - \Lambda_5^{(0)} \left( \frac{g}{2} [\Lambda_5^{(0)}, \Lambda_5^{(0)}] \right) \right] \quad (232)$$
and KK modes’ part is given by

\[
S_0^{(KK)} = \int d^6x \, \text{Tr} \left\{ \left[ \frac{1}{6} H^{ijk} H_{ijk} + \frac{1}{2} H^{ij5} H_{ij5} + B^{*(KK)i5} \left( [D_i, \Lambda_5^{(KK)}] - \partial_5 \Lambda_i \right) \\
+ g[B_{i5}^{(KK)}, \Lambda_5^{(0)}] \right] + B^{*(KK)ij} \left( [D_i, \Lambda_j^{(KK)}] - [D_j, \Lambda_i^{(KK)}] + g[B_{ij}, \Lambda_5^{(0)}] \right) \\
- g[F_{ij}, \partial_5^{-1} \Lambda_5^{(KK)}] \right\] + \Lambda_5^{*(KK)} \left( \Lambda_5^{(KK)} \right) + \Lambda_5^{*(KK)i} \left( [D_i, \alpha] \\
- g[\Lambda_5^{(0)}, \Lambda_i^{(KK)}] \right) + \alpha^* \left( g[\alpha, \Lambda_5^{(0)}] \right) \right\} 
\]

(233)

where \((B_{i5}^{*(KK)}, B_{ij}^{*(KK)}, B_{i5}^{*(0)}, \Lambda_5^{*(0)}, \Lambda_5^{*(KK)}, \Lambda_i^{*(KK)}, \alpha^*)\) are antifields introduced as the sources of the BRST transformation laws of corresponding gauge fields. The ghost number’s relation between a field and its corresponding antifield is given by

\[ gh(\Phi) + gh(\Phi^*) = -1 \]  

(234)

which can be read from the fact that the action has vanishing ghost number. We also notice that the relation of (mass) dimension between a field and its corresponding antifield is \(D(\Phi) + D(\Phi^*) = 6\), thus for instance, \(D(B_{\mu\nu}) = 2\) so that \(D(B_{\mu\nu}^*) = 4\), and the coupling constant has \(D(g) = -1\).

Antifields will be eliminated by using the so-called gauge-fixing fermion \(\Psi\) via

\[
\Phi^*I = \frac{\partial \Psi}{\partial \Phi^I} 
\]

(235)

which means that the partition function constructed with a gauge-fixed effective action is given by

\[
Z = \int \mathcal{D}\Phi^I \mathcal{D}\Phi^{*I} \delta \left( \Phi^{*I} - \frac{\partial \Psi}{\partial \Phi^I} \right) e^{iS_{\text{eff}}} \]

(236)

where \(\Phi^I\) represents all fields in the theory. Since \(\Psi\) must be a functional of fields only (not antifields) and its ghost number is \(-1\), we see that it is not possible to write down an acceptable gauge fermion unless we introduce extra

56
fields. Thus we introduce so-called trivial pairs (or doublet) which defined by (A,B) with the following relation

\[ sA = B \]  \tag{237}  
\[ sB = 0 \]  \tag{238}  

notice that the transformation of B is simply the consequence of the nilpotency of A.

The zero modes’ part is an irreducible system, it turns out that it is sufficient to introduce only one doublet. For the KK modes’ part, it is a reducible system and we will introduce three pairs. The extend actions are defined as

\[ S_{ex}^{(0)} = S_0^{(0)} - \frac{2\pi R}{4} \int d^5x \, Tr \, \bar{D} E \]  \tag{239}  
\[ S_{ex}^{(KK)} = S_0^{(KK)} - \int d^6x \, Tr \left( \bar{\Lambda}^i b_i + \bar{\Lambda}^5 b_5 + \bar{\alpha}^* b - \bar{\omega}^* \pi \right) \]  \tag{240}  

where we introduce doublets

\[ (\bar{D}, E) \text{ for zero modes.} \]
\[ (\bar{\Lambda}^\mu, b_\mu), (\bar{\alpha}, \bar{\omega}, \pi) \text{ for KK modes} \]  \tag{241}  

note that fields in the same doublet have the same (mass) dimension.

As an example, let us consider the gauge fermion \( \Psi \) as

\[ \Psi^{(0)} = \frac{2\pi R}{4} \int d^5x \, Tr \, \bar{D} \left( -\frac{E}{2\eta} + \partial^i B_{ij}^{(0)} \right) \]  \tag{242}  
\[ \Psi^{(KK)} = \int d^6x \, Tr \left( \bar{\Lambda}^i (\partial^j B_{ij}^{(KK)} + \partial^5 B_{i5}^{(KK)} + \partial_i \bar{\omega}) + \bar{\Lambda}^5 (\partial^i B_{i5}^{(KK)} + \partial_5 \bar{\omega}) + \bar{\alpha} \partial^\mu \Lambda_{\mu}^{(KK)} \right) \]  \tag{243}  

where \( \eta \) is an adjustable parameter. By \( \Phi^* = \frac{\partial \Psi}{\partial \Phi} \) (recall that \( \Psi \) is a functional
of fields only), we obtain the gauge fixing actions

\[ S_{gf}^{(0)} = \frac{2\pi R}{4} \int d^5x \, Tr \left[ (F^{(0)}_{ij} F^{(0)}_{ij}) - \partial^i D \left( [D_i, \Lambda^{(0)}_5] \right) \right] + \left( \frac{E}{2\eta} - \partial^i B^{(0)}_{i5} \right) E \]  

\[ S_{gf}^{(KK)} = \int d^6x \, Tr \left\{ \frac{1}{6} H^{(KK)ijk} H^{(KK)}_{ijk} + \frac{1}{2} H^{(KK)ij5} H^{(KK)ij5} \right. 
\left. - \partial^5 \Lambda^{(KK)i} \left( [D_i, \Lambda^{(KK)}_5] - \partial_5 \Lambda^{(KK)}_i + g[B^{(KK)}_{i5}, \Lambda^{(0)}_5] \right) \right. 
\left. - \partial^i \tilde{\Lambda}^{(KK)}_i \left( [D_i, \Lambda^{(KK)}_5] - [D_j \Lambda^{(KK)}_i] + g[B_{ij}, \Lambda^{(0)}_5] - g[F_{ij}, \partial_5^{-1} \Lambda^{(KK)}_5] \right) \right. 
\left. - \partial_5 \alpha \left( \partial_5 \alpha - g[\Lambda^{(0)}_5, \Lambda^{(KK)}_i] \right) - \partial^i \tilde{\alpha} \left( [D_i, \alpha] - g[\Lambda^{(0)}_5, \Lambda^{(KK)}_i] \right) \right. 
\left. - \left( \partial^i B^{(KK)}_{ij} + \partial^5 B^{(KK)}_{i5} + \partial_i \tilde{\omega} \right) b^j - \left( \partial^i B^{(KK)}_{i5} + \partial_5 \tilde{\omega} \right) b_5 \right. 
\left. - \left( \partial^\mu \Lambda^{(KK)}_\mu \right) b - \left( \partial^i \tilde{\Lambda}_i + \partial^5 \tilde{\Lambda}_5 \right) \pi \right\} \]  

(245)

it is interesting to notice that these gauge-fixed actions are indeed invarain under BRST transformation laws

\[ s(S_{gf}^{(0)}) = 0 \]  

(246)

\[ s(S_{gf}^{(KK)}) = 0 \]  

(247)

and there is no other gauge symmetry left. After we integrate auxiliary fields out, the gauge fixing conditions can be read as the following covaraint forms:

\[ \partial^i B^{(0)}_{i5} = \frac{E}{\eta} \]  

(248)

\[ \partial^\nu B^{(KK)}_{\mu\nu} + \partial_\mu \tilde{\omega} = 0 \]  

(249)

\[ \partial^\mu \Lambda^{(KK)}_\mu = 0 \]  

(250)

\[ \partial^\mu \tilde{\Lambda}_\mu = 0 \]  

(251)

if we take the parameter \( \eta \rightarrow \infty \), then the Lorenz gauge will be imposed as a delta-function condition for zero modes. The Lorentz covariance is still broken for KK modes part. The gauge fixing process here is similar with the case when we deal with the Abelian 2-form theory.
As for the counting of the degrees of freedom (Recall that a gauge field has many components while only some of those describe physical degrees of freedom. The other degrees of freedom correspond to unphysical degrees of freedom. The ghost fields’ degrees of freedom subtract since they represent these unphysical parts), the two-form potentials give $+15$, while ghost fields $\Lambda_\mu$ and $\bar{\Lambda}_\mu$ both give $-6$, then $\omega$, $\alpha$ and $\bar{\alpha}$ all give $+1$. In total we have $15-6+6+1+1+1=6$ physical degrees of freedom, and that is the expected answer for a 2-form potential theory.

Denote $S_{eff}$ as the effective action that has already intergrated out all auxiliary fields in $S_{gf}$ above. The partition function for this non-abelian 2-form theory could be formally given by

$$Z^{(0)} = \int \mathcal{D}B_{i\overline{i}}^{(0)} \mathcal{D}\Lambda_5^{(0)} \mathcal{D}\bar{D} \delta(\partial^i B_{i\overline{i}}^{(0)} - \frac{E}{\eta}) e^{\frac{i}{\hbar} S_{eff}^{(0)}}$$  \hspace{1cm} (252)

$$Z^{(KK)} = \int \mathcal{D}B_{\mu\nu}^{(KK)} \mathcal{D}\Lambda_{\mu}^{(KK)} \mathcal{D}\bar{\Lambda}_{\mu}^{(KK)} \mathcal{D}\bar{\omega} \mathcal{D}\alpha \mathcal{D}\bar{\alpha}$$

$$\delta(\partial^\nu B_{\mu\nu}^{(KK)} + \partial_\mu \bar{\omega}) \delta(\partial^\mu \Lambda_{\mu}^{(KK)}) \delta(\partial^\mu \bar{\Lambda}_{\mu}) \delta(H - \ast H)_{ij\overline{i}j} e^{\frac{i}{\hbar} S_{eff}^{(KK)}}$$ \hspace{1cm} (253)

where the field $\bar{D}$ in zero modes’ part is sometimes referred as the anti-ghost field in the standard Yang-Mills theory.

One may wonder whether it is possible to start from a standard Yang-Mills-like action as the classical action to obtain the self-duality condition in the partition function level via enlarging the configuration space through field-antifield formulation. The final partition function should look like (for KK modes)

$$Z^{(KK)} = \int \mathcal{D}B_{\mu\nu}^{(KK)} \mathcal{D}\Lambda_{\mu}^{(KK)} \mathcal{D}\bar{\Lambda}_{\mu}^{(KK)} \mathcal{D}\bar{\omega} \mathcal{D}\alpha \mathcal{D}\bar{\alpha}$$

$$\delta(\partial^\nu B_{\mu\nu}^{(KK)} + \partial_\mu \bar{\omega}) \delta(\partial^\mu \Lambda_{\mu}^{(KK)}) \delta(\partial^\mu \bar{\Lambda}_{\mu}) \delta(H - \ast H)_{ij\overline{i}j} e^{\frac{i}{\hbar} S_{eff}^{(KK)}}$$ \hspace{1cm} (254)

where $(H - \ast H)_{ij\overline{i}j}$ represents the self-duality condition. Following the method
of field-antifield formulation, the corresponding extra/extend action should be

\[ S' = \int d^6x \ Tr \ P^{*ij}Q_{ij} \tag{255} \]

where \((P_{ij}, Q_{ij})\) forms a triviar pair. And the corresponding extra gauge fermion should be

\[ \Psi' = \int d^6x \ Tr \ (H - \ast H)_{ij5}P^{ij} \tag{256} \]

so that in the end we will have a delta function that gives the self-duality condition on the partition function after integrating out \(Q_{ij}\). However, the result introduces extra degrees of freedom that come from the kinetic terms of \(P_{ij}\), which indicates that this naive approach for a self-dual theory does not work.
7 A Generalization: Non-Abelian 3-form

In this section we construct non-Abelian gauge theory for a 3-form potential. Readers who want to focus on the theory of M5-branes might simply skip this section for the first reading.

We first study the Abelian gauge theory for a 3-form potential on the spacetime of

\[ \mathbb{R}^d \times T^2 \]  

Let the torus \( T^2 \) extend in the directions of \( x^1 \) and \( x^2 \). We can decompose a field \( \Phi \) as

\[ \Phi = \Phi^{(0)} + \Phi^{(KK)}, \]  

where the zero mode \( \Phi^{(0)} \) has no dependence on \( T^2 \)

\[ \partial_a \Phi^{(0)} = 0, \]  

and the KK mode \( \Phi^{(KK)} \) can be obtained from \( \Phi \) as

\[ \Phi^{(KK)} = \Box^{-1} \Box \Phi, \]

where

\[ \Box \equiv \partial^a \partial_a \quad (a = 1, 2). \]

The Abelian gauge transformations of a 3-form potential \( B \) are given by

\[ \delta B_{i12} = \partial_i \Lambda_{12} - \partial_1 \Lambda_{i2} + \partial_2 \Lambda_{i1}, \]

\[ \delta B_{ija} = \partial_i \Lambda_{ja} - \partial_j \Lambda_{ia} + \partial_a \Lambda_{ij}, \]

\[ \delta B_{ijk} = \partial_i \Lambda_{jk} + \partial_j \Lambda_{ki} + \partial_k \Lambda_{ij}, \]

where \( a = 1, 2 \) and \( i, j, k = 0, 3, 4, \ldots, (d + 1) \). There is redundancy in the gauge transformation parameters \( \Lambda_{ia}, \Lambda_{ij} \) so that the gauge transformation
laws are invariant under the transformation

\begin{align}
\delta \Lambda_{12} &= \partial_1 \lambda_2 - \partial_2 \lambda_1, \\
\delta \Lambda_{ia} &= \partial_i \lambda_a - \partial_a \lambda_i, \\
\delta \Lambda_{ij} &= \partial_i \lambda_j - \partial_j \lambda_i.
\end{align}

(265) (266) (267)

Apparently there is also a redundancy in using \( \lambda \) to parametrize the redundancy in \( \Lambda \). There are \((d+2)\) components in \( \lambda \), but only \((d+1)\) of them are independent. Using the redundancy of \( \Lambda \), we can “gauge away” \((d+1)\) of the gauge transformation parameters. For instance, we can set

\[ \rho_i \equiv \partial^a \Lambda_{ia} = 0, \quad \Lambda_{12} = 0, \]

(268)

and use the following gauge transformation parameters

\[ \xi_i \equiv \epsilon^{ab} \partial_a \Lambda_{ib}, \quad \Lambda_{ij}, \]

(269)

so that

\[ \Lambda_{ia}^{(KK)} = -\epsilon^{ab} \Box^{-1} \partial_b \xi_i, \]

(270)

and the gauge transformation laws become

\begin{align}
\delta B_{i12} &= -\xi_i, \\
\delta B_{ija} &= -\epsilon^{ab} \Box^{-1} \partial_b (\partial_i \xi_j - \partial_j \xi_i) + \partial_a \Lambda_{ij}, \\
\delta B_{ijk} &= \partial_i \Lambda_{jk} + \partial_j \Lambda_{ki} + \partial_k \Lambda_{ij}.
\end{align}

(271) (272) (273)

Viewing \( \Lambda_{ia} \) as \( d \) copies of 1-form potentials on \( T^2 \), the \( \xi_i \)'s are the corresponding field strengths, and so their integrals over \( T^2 \) are quantized. It implies that \( \xi_i^{(0)} \) is quantized, and so we have to set

\[ \xi_i^{(0)} = 0 \]

(274)

when we use \( \xi_i \) as infinitesimal gauge transformation parameters. In the following, we have \( \xi_i = \xi_i^{(KK)} \).
Since the original gauge transformation laws have the redundancy, one can simply carry out the replacement
\[ \xi_i \rightarrow \xi_i - \partial_i \Lambda_{12}, \]
\[ \Lambda_{ij} \rightarrow \Lambda_{ij} + \frac{\Box^{-1}}{}(\partial_i \rho_j - \partial_j \rho_i). \]
(275) (276)

On the torus \( T^2 \), the gauge transformation parameter \( \Lambda_{12}^{(0)} \) corresponds to a Wilson surface degree of freedom for the 2-form \( \Lambda \).

To construct a consistent non-Abelian gauge transformation algebra for the 3-form potential, we only need to consider transformation laws for the parameters \( \xi^{(KK)}_i, \Lambda_{ij} \) and \( \Lambda_{12}^{(0)} \). In the end we get the full gauge transformation laws through the replacement
\[ \xi^{(KK)}_i \rightarrow \xi^{(KK)}_i - [D_i, \Lambda^{(KK)}_{12}], \]
\[ \Lambda^{(KK)}_{ij} \rightarrow \Lambda^{(KK)}_{ij} + \frac{\Box^{-1}}{}([D_i, \rho^{(KK)}_j] - [D_j, \rho^{(KK)}_i]), \]
where the covariant derivative \( D_i \) should be defined as
\[ D_i = \partial_i + B_{i12}^{(0)}, \]
(277) (278) (279)
and
\[ \xi^{(KK)}_i \equiv \epsilon^{ab} \partial_a \Lambda^{(KK)}_{ib}, \quad \rho^{(KK)}_i \equiv \partial_a \Lambda^{(KK)}_{ia}. \]
(280)
Here we have scaled \( B_{i12} \) to absorb the coupling constant \( g \), which is expected to be given by the area \((2\pi)^2 R_1 R_2\) of the torus.

We define the non-Abelian gauge transformations as
\[ \delta B_{i12} = [D_i, \Lambda^{(0)}_{12}] - \xi_i + [B_{i12}^{(KK)}, \Lambda^{(0)}_{12}], \]
(281)
\[ \delta B_{ija} = -\epsilon^{ab} \frac{\Box^{-1}}{} \partial_b ([D_i, \xi_j] - [D_j, \xi_i]) + \partial_a \Lambda_{ij} + [D_i, \Lambda^{(0)}_{ja}] - [D_j, \Lambda^{(0)}_{ia}] + [B_{ija}, \Lambda^{(0)}_{12}], \]
(282)
\[ \delta B_{ijk} = [D_i, \Lambda_{jk}] + [D_j, \Lambda_{ki}] + [D_k, \Lambda_{ij}] + [B_{ijk}, \Lambda^{(0)}_{12}]. \]
(283)
The algebra of gauge transformations is closed

$$[\delta, \delta'] = \delta'',$$  \hspace{1cm} (284)

with the parameters of $\delta''$ given by

$$\Lambda_{12}^{(0)''} = [\Lambda_{12}^{(0)}, \Lambda_{12}^{(0)'},] \hspace{1cm} (285)$$

$$\xi_i'' = [\xi_i, \Lambda_{12}^{(0)'},] - [\xi'_i, \Lambda_{12}^{(0)}], \hspace{1cm} (286)$$

$$\Lambda_{ij}'' = [\Lambda_{ij}, \Lambda_{12}^{(0)'},] - [\Lambda_{ij}'', \Lambda_{12}^{(0)}]. \hspace{1cm} (287)$$

The field strengths should be defined as

$$H_{ij12}^{(0)} = [D_i, D_j], \hspace{1cm} (288)$$

$$H_{ij12}^{(KK)} = [D_i, B_{j12}^{(KK)}] - [D_j, B_{i12}^{(KK)}] + \partial_i B_{ij5}^{(KK)} - \partial_j B_{ij4}^{(KK)}, \hspace{1cm} (289)$$

$$H_{ijka} = [D_i, B_{jka}] + [D_j, B_{kia}] + [D_k, B_{ija}] - \partial_a B_{ijka}$$

$$- \epsilon^{ab} \Box^{-1} \partial_b \left( [F_{ij}, B_{kl}^{(KK)}] + [F_{jk}, B_{ik}^{(KK)}] + [F_{ki}, B_{ij}^{(KK)}] \right), \hspace{1cm} (290)$$

$$H_{ijkl} = \sum_{(4)} [D_i, B_{jkl}] - \sum_{(6)} [F_{ij}, \beta_{kl}], \hspace{1cm} (291)$$

where

$$\beta_{ij} \equiv \Box^{-1} \partial^a B_{ija}, \hspace{1cm} (292)$$

so that all the field strength components transform as

$$\delta H_{ij12} = [H_{ij12}, \Lambda_{12}^{(0)}], \hspace{1cm} (293)$$

$$\delta H_{ijka} = [H_{ijka}, \Lambda_{12}^{(0)}] + \sum_{(3)} [F_{ij}, \Lambda_{ka}^{(0)}], \hspace{1cm} (294)$$

$$\delta H_{ijkl} = [H_{ijkl}, \Lambda_{12}^{(0)}] + \sum_{(6)} [F_{ij}, \Lambda_{kl}^{(0)}]. \hspace{1cm} (295)$$

It may be possible to define a non-Abelian self-dual gauge theory for a 3-form potential in 8 dimensional Euclidean space. And the same idea can be used to define a non-Abelian gauge symmetry for $p$-form potentials on $\mathbb{R}^d \times T^{p-1}$. We leave these generalizations for future study.
8 Toward non-Abelian PST and Conclusion

8.1 Toward non-Abelian PST

The most interesting open problem is to find a manifest Lorentz-covariant formulation of the non-Abelian self-dual 2-form theory. The covariant theory should reproduce the previous 5+1 non-covariant formulation under proper conditions. That is, the question here is: Could we have a non-Abelian PST-like formulation for multiple M5-branes?

Let me conclude this thesis by stressing some difficulties when one wants to find an manifest Lorentz-covariant formulation of this non-Abelian self-dual 2-form theory by using the similar ideas people (like PST) came up: introducing auxiliary fields.

We first notice that in the case of non-Abelian chiral 2-form theory here, the non-covaraincy are in fact two-fold. In the Abelian case, the field strengths $H$ are defined covariantly. (This means that its Maxwell-like action is Lorentz covariant), we then start to consider how to have an abelain covariant self-dual action (Abelian PST action). In the non-Abelian case, the Yang-Mills-like action ($s = \int H^2$) is still not Lorentz covariant since our defination of field strengths are not covaraint. One can check that the transformed field equation of the Yang-Mills-like action of the non-abelian 2-form theory under the standard Lorentz transformation is NOT up to the field equation itself, this, in some sense, might indicate we are still in a gauge. In another word, one observes that $A_5$ is absent in the algebra, which is like the axial gauge that violates Lorentz invariance, however the symmetry should be restored in the calculation of gauge-invariant S-Matrix elements.

Assuming we have a covariant version of the non-abelian 2-form theory,
which means that we assume $H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ is Lorentz invariant (we will discuss about it more later). Is it possible to have a non-Abelian PST action? Let us introduce an "Abelian" auxiliary field $b(x)$ to construct the covariant self-dual theory\(^{10}\). Let us directly non-abelianize the Abelian PST action by

$$S = \frac{1}{4} \int d^6x \ Tr \left( -\frac{1}{3!} H^{\mu\nu\lambda} H_{\mu\nu\lambda} + \frac{1}{2} H^{\mu\nu\rho} P_{\rho}^{\alpha} H_{\mu\nu\sigma} \right)$$  \hspace{1cm} (296)$$

the variation of the action\(^{11}\) gives:

$$\delta S = \int d^6x \ Tr \left( \epsilon^{\alpha\beta\rho\mu\nu\lambda} \partial_{\mu} b[D_{\rho}, H_{\nu\lambda}](\delta \hat{B}_{\alpha\beta}) - \epsilon^{\alpha\beta\rho\mu\nu\lambda} \partial_{\mu} b \partial_{\rho} H_{\nu\lambda} \bar{H}_{\alpha\beta} (\delta b) \right)$$  \hspace{1cm} (297)$$
then the claim is that we still have the extra symmetry under the gauge parameter $\phi$:

$$\delta \hat{B}_{\mu\nu} = \bar{H}_{\mu\nu} \phi$$ \hspace{1cm} (298)$$
$$\delta b = \phi$$  \hspace{1cm} (299)$$

where $a$ is the group index. if we need to introduce other fields, we simply let $\delta(\text{Other Fields}) = 0$. To check that, we see that the ordinary derivatives part are cancelled as in the abelian PST case. The extra term is:

$$\epsilon^{\alpha\beta\rho\mu\nu\lambda} \partial_{\mu} b \ f_{bc}^{a} A_{\rho}^{b} \bar{H}_{\nu\lambda}^{c} \bar{H}_{\alpha\beta}^{a} \phi = 0$$  \hspace{1cm} (300)$$

where $f_{bc}^{a}$ is the structure constant. We see this term "vanish identically" since there are three total antisymmetry indices on $\bar{H}_{\nu\lambda}^{c}$. Also notice that the equation of motion of $b(x)$ does not give extra constraint since it vanishes up to the self-duality condition. After fixing $b = x^5$, we have

$$S = \frac{1}{4} \int d^6x \ Tr \left( -\frac{1}{2} H^{ij5} H_{ij5} - \frac{1}{3!} H^{ijk} H_{ijk} + \frac{1}{2} \bar{H}^{ij5} H_{ij5} \right)$$

$$= -\frac{1}{4!} T_{M5} T_{M2}^{-2} \int d^6x \ Tr \left( 2H_{ijk} H^{ijk} + \epsilon^{ijklm} H_{klm} \partial_{5} \hat{B}_{ij} \right)$$  \hspace{1cm} (301)$$

\(^{10}\)The reason for the "Abelian" field is that we hope the field $b(x)$ could be fixed to $x^5$ in the end, and $x^5$ should not take a matrix value.

\(^{11}\)Here we consider $H_{\mu\nu\lambda} = [D_{\mu}, \bar{B}_{\nu\lambda}] + [D_{\nu}, \bar{B}_{\lambda\mu}] + [D_{\lambda}, \bar{B}_{\mu\nu}]$ for a covariant non-abelian 2-form theory in a proper gauge condition by using $\bar{B}$. And we let the connection as $A_{\mu}$. We discuss these issue more later.
which is the previous non-covariant multiple M5-branes action that gives the self-duality condition for the two-form potential.

However, we will see the main challenge is to have a covariant version of the non-Abelian 2-form theory (without self-duality), which means that it is unclear how to define $H_{\mu \nu \lambda}$ so that $L \sim H_{\mu \nu \lambda} H^{\mu \nu \lambda}$ is Lorentz invariant.

Naturally, one might introduce another Abelian auxiliary field $a_\mu(x)$ to define the covariant non-Abelian 2-form theory and the previous non-covariant algebra (gauge transformation laws and 3-form field strengths) could be obtained by an extra gauge fixing. Let us now consider this approach.

First we will consider a five dimensional one-form potential $A(x^i)$ appears as the new field replacing $RB_{i5}^{(o)}$ in large R limit and we separately define the zero modes’ part when reducing to multiple D4-branes. Let our notation as $\mu, \nu, \lambda... = 0, 1, 2, 3, 4, 5$ and $i, j, k... = 0, 1, 2, 3, 4$ and we also use the non-local operator which only consistently acts on the KK-modes that dominate contributions in 6D (In the following, we will ignore the index of (KK) notation). We define the covariant non-Abelian gauge transformation law of 2-form potential in 6D as:

$$\delta B_{\mu \nu} = [D_\mu, \Lambda_\nu] - [D_\nu, \Lambda_\mu] + [B_{\mu \nu}, \Lambda] - [F_{\mu \nu}, (a \cdot \partial)^{-1} a^\rho \Lambda_\rho] \quad (302)$$

Where $a^\mu(x)$ is the U(1) auxiliary spacetime-dependent field that is used to covariantize this non-abelian 2-form theory (Note that $a(x)$ is not $b(x)$. The field $b(x)$ is used to covariantize a self-dual theory). Here the covariant derivative is defined in terms of the one-form potential: $D_\mu = \partial_\mu + A_\mu$. It is interesting to understand more about the physical meaning of this one-form potential such as what happens when it comes to the compactification so that the role of $A$ is mapped to the zero modes $B_{i5}^{(0)}$. Notice that if $A$ field is determined by other fields, it will directly contradict the recent result that the theory is triv-
ial when one only uses $(2,0)$ tensor multiplets [27], thus one needs to consider this A-field as the new field in the interacting 6D theory. We conjecture that these new degrees of freedom should relate to the condensation of tensionless strings in M5-branes. We will discuss this A-field more later and in particular we will see this A-field is indeed the five-dimensional field hence it will not influence the degrees of freedom in six-dimensions.

We also define $F_{\mu\nu} = [D_{\mu}, D_{\nu}]$, $\delta A_\mu(x) = [D_\mu, \lambda]$, $\delta a_\mu(x) = 0$. Note that we will not introduce any coupling constant due to the conformal nature of M5-branes. Then, again, to be justified as the deformation of Abelian 2-form theory, the gauge transformation of B-field should be invariant under the "redundent transformation"

$$\delta \Lambda_\mu = [D_\mu, \alpha]$$

we find that this symmetry exists only if

$$a^\mu A_\mu = 0$$

(304)

The consequence of this condition is that if we fix a direction for $a$, says the 5-th direction, then the component $A_5$ will decouple from the theory, which is consistent with the previous model where $A_5$ is absent.

Notice that the gauge transformation on the condition (304) gives

$$a^\mu [D_\mu, \lambda] = (a \cdot \partial) \lambda = 0$$

(305)

as a consistency constraint on $\lambda$. This tells us $\lambda$ is a five-dimensional parameter, as expected.

One can check that the gauge algebra is closed by:

$$[\delta_1, \delta_2] = \delta_3$$

(306)
with

\[ \lambda_3 = [\lambda_1, \lambda_2] \quad (307) \]
\[ \Lambda_{\mu 3} = [\Lambda_{\mu 1}, \lambda_2] - [\Lambda_{\mu 2}, \lambda_1] \quad (308) \]

thus it is indeed possible to have a covariant version of the non-Abelian 2-form gauge transformation (302). We will, however, face problems when we try to define 3-form field strengths.

If we use the form as previous non-covariant formulation to define 3-form field strengths, we should define

\[
H_{\mu \nu \lambda} = [D_{\mu}, B_{\nu \lambda}] + [D_{\nu}, B_{\lambda \mu}] + [D_{\lambda}, B_{\mu \nu}] + [F_{\mu \nu}, (a \cdot \partial)^{-1} a^\rho B_{\lambda \rho}]
+ [F_{\nu \lambda}, (a \cdot \partial)^{-1} a^\rho B_{\mu \rho}] + [F_{\lambda \mu}, (a \cdot \partial)^{-1} a^\rho B_{\nu \rho}] \quad (309)
\]

if we ask this field strength transforms covariantly, that is

\[
\delta H_{\mu \nu \lambda} = [H_{\mu \nu \lambda}, \lambda] \quad (310)
\]

we will need

\[
a^\mu F_{\mu \nu} = 0 \quad (311)
\]

which is consistent with the fact that \( A_\mu \) is a five-dimensional field after a preferential choice of the vector \( a_\mu(x) \). However, we will also need to impose constraints on gauge parameters \( \Lambda_\mu \) and \( \lambda \)

\[
(a \cdot \partial)[D_{\mu}, \lambda] = 0 \quad (312)
\]
\[
(a \cdot \partial)^{-1} a^\rho [D_{\lambda}, \Lambda_\rho] = [D_{\lambda}, (a \cdot \partial)^{-1} a^\rho \Lambda_\rho] \quad (313)
\]

the problem is that these constraints on gauge parameters are too strong and it is almost saying that the field \( a_\mu \) is constant, which means that there is no manifest Lorentz symmetry.

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Furthermore, since we do not want to put equations (304), (311) by hand that contradicts the action principle. The two constraints on fields: $a^\mu A_\mu = 0, a^\mu F_{\mu\nu} = 0$ should be solved simultaneously. One can, for instance, indeed solve them by letting:

$$a_\lambda = \left( \delta^\nu_\lambda - x^{-2} x_\lambda x^\nu \right) B_\nu$$  \hspace{1cm} (314)

$$A^a_\lambda = x_\lambda [1 - (a \cdot \partial)^{-1}(a \cdot \partial)] C^a$$  \hspace{1cm} (315)

where $a$ is group index and $x_\lambda$ is the spacetime coordinate. And we introduce field $B_\nu$ (Abelian field), and field $C^a$. But the solution means that the degrees of freedom of $A$ are too small. Moreover, we should still need to find a "universal redundant" hidden symmetry for $a(x)$ that allows us to reduce:

$$\delta B_{\mu\nu} = [D_\mu, \Lambda_\nu] - [D_\nu, \Lambda_\mu] + [B_{\mu\nu}, \lambda] - [F_{\mu\nu}, (a \cdot \partial)^{-1} a^\rho \Lambda_\rho]$$  \hspace{1cm} (316)

to

$$\delta B_{ij} = [D_i, \Lambda_j] - [D_j, \Lambda_i] + [B_{ij}, \lambda] - [F_{ij}, \partial_5^{-1} \Lambda_5]$$  \hspace{1cm} (317)

and also allows us to reduce

$$H_{\mu\nu\lambda} = [D_\mu, B_{\nu\lambda}] + [D_\nu, B_{\lambda\mu}] + [D_\lambda, B_{\mu\nu}] + [F_{\mu\nu}, (a \cdot \partial)^{-1} a^\rho B_{\lambda\rho}]$$
$$+ [F_{\nu\lambda}, (a \cdot \partial)^{-1} a^\rho B_{\mu\rho}] + [F_{\lambda\mu}, (a \cdot \partial)^{-1} a^\rho B_{\nu\rho}]$$  \hspace{1cm} (318)

to

$$H_{ijk} = [D_i, B_{jk}] + [D_j, B_{ki}] + [D_k, B_{ij}] + [F_{ij}, \partial_5^{-1} B_{k5}]$$
$$+ [F_{jk}, \partial_5^{-1} B_{i5}] + [F_{ki}, \partial_5^{-1} B_{j5}]$$  \hspace{1cm} (319)

but it is not clear how to achieve them unless there are indeed constraints allowed on gauge parameters. At present we do not know the full answer to these questions, we left it for future study.
8.2 Conclusion

In this thesis, we discuss various formulations of chiral fields in different dimensions, including non-covariant and covariant formulations. We mostly focus on the formulation of chiral two-form in six-dimensions. After reviewing the Abelian theory, we construct gauge transformation laws, equations of motion and an action for its non-Abelian generalization. The new ingredient as compared to previous approaches is the non-locality in the compact direction. We expect that this model could be used to describe multiple M5-branes.

We also comment on this model about its decompactified limit and compared it with other approaches on M5-branes system. We also consider the BRST-antifield quantization of this non-Abelian 2-form gauge theory and generalize the algebra to construct non-Abelian 3-form gauge theory. Finally we see difficulties when trying to find a manifest Lorentz-covariant theory by utilizing extra auxiliary fields.

There are some main open problems:

1. A well defined $R \to \infty$ limit of the theory with the 6D Lorentz symmetry.
2. The supersymmetric extension of the theory.
3. The quantization of the theory.

It is fair to say that the full structure of M5-branes is still highly mysterious and further investigation is certainly needed.
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