VANISHING FOURIER COEFFICIENTS OF HECKE EIGENFORMS

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Abstract. We prove that, for fixed tame level \((N, p) = 1\), there are only finitely many Hecke eigenforms \(f\) of level \(\Gamma_1(N)\) and even weight with \(a_p(f) = 0\) which are not CM.

1. Introduction

Lehmer [Leh47] raised the question of whether \(\tau(n) = 0\) for any of the non-trivial Fourier coefficients of Ramanujan’s Delta function \(\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^2 = \sum \tau(n)q^n\). He proved that if \(\tau(n) = 0\) for some \(n\), then necessarily \(\tau(p) = 0\) for a prime \(p|n\). Lehmer’s problem remains open, as does the analogous question for any cuspidal eigenform \(f\) of level one. If one weakens the hypothesis further and assumes only that \(f\) has level \(N\) for some \(N\) prime to \(p\), then there are a number of ways in which \(a_p(f) = 0\), including the following:

1. If \(f\) is a modular form with CM arising from an imaginary quadratic field \(F/\mathbb{Q}\) in which \(p\) is inert, then \(a_p(f) = 0\).
2. If \(f\) is a weight two modular form arising from an elliptic curve \(E/\mathbb{Q}\) with good supersingular reduction at \(p\), and \(p \geq 5\), then \(a_p(f) = 0\).

In this paper, we examine a vertical analogue of Lehmer’s conjecture where \(p\) is fixed and we vary the weight. Our main theorem is as follows:

**Theorem 1.1.** Fix a prime \(p > 2\) and an integer \((N, p) = 1\). Then there are only finitely many non-CM Hecke eigenforms of tame level \(N\) and even weight with \(a_p(f) = 0\).

We shall deduce from this the following:

**Corollary 1.2.** Fix a prime \(p\). There are only finitely many eigenforms of level 1 with \(a_p(f) = 0\).

Our arguments are not effective. The existence of non-CM (modular) elliptic curves \(E/\mathbb{Q}\) which are supersingular at \(p\) shows that some exceptions must be included. The assumption \(p > 2\) and the assumptions on the weight are not intrinsic to our method, but rather reflect the absence of certain \(R = T\) theorems either when \(p = 2\) or when the residual representation \(\overline{\rho}_{f/\mathbb{Q}(\zeta_p)}\) is reducible. The weight condition can be weakened to requiring either that \(n\) is even or \(n - 1\) is not divisible by \((p + 1)/2\).

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2. THE ARGUMENT

Suppose that \( f \in S_n(\Gamma_1(N), \mathbb{Z}_p) \), and let
\[
\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\overline{\mathbb{Q}}_p)
\]
denote the corresponding Galois representation. By the main theorem of [Sch90, Sa97], the \( p \)-adic representation \( \rho_f|_{G_{\mathbb{Q}_p}} \) is crystalline, and the characteristic polynomial of crystalline Frobenius is \( x^2 - a_p(f)x + p^{n-1}\chi(p) \), where \( \chi \) is the Nebentypus character of \( f \). For irreducible 2-dimensional crystalline representations of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \), the characteristic polynomial of crystalline Frobenius is enough to determine the representation uniquely. In particular, when \( a_p(f) = 0 \), there is a very simple description of the corresponding local Galois representation which we now describe. Let \( K/\mathbb{Q}_p \) denote the unique unramified quadratic extension. By local class field theory, there is a unique character \( K^\times \to \mathbb{Q}_p^\times \to K^\times \) which sends \( p \) to 1 and \( z \in \mathcal{O}_K^\times \) to \( z \). With respect to the two embeddings of \( K \) into \( \overline{\mathbb{Q}}_p \), this gives two characters \( \varepsilon_2, \varepsilon'_2 \) from \( G_K \) to \( \mathbb{Q}_p^\times \) which are permuted by the action of \( \text{Gal}(K/\mathbb{Q}_p) \). We have the following result which follows from [Bre03, Prop 3.1.2] after noting that \( \rho_f|_{G_{\mathbb{Q}_p}} \) is an unramified twist of \( V_{n,0} \):

**Theorem 2.1** (Breuil). Suppose that \( a_p(f) = 0 \). Then
\[
\rho_f|_{G_{\mathbb{Q}_p}} = \left( \text{Ind}^{G_{\mathbb{Q}_p}}_{G_K} \varepsilon_2^{n-1} \right) \otimes \psi
\]
for some unramified character \( \psi \) with \( \psi^2 = \chi|_{G_{\mathbb{Q}_p}} \).

Note that \( \psi \) is a priori only uniquely defined up to the unramified quadratic character \( \eta_K/\mathbb{Q}_p \), but since \( \rho_f|_{G_{\mathbb{Q}_p}} \otimes \eta_K/\mathbb{Q}_p \simeq \rho_f|_{G_{\mathbb{Q}_p}} \), either choice is correct. Let us now assume the hypotheses of Theorem 1.1, and suppose there exists an infinite number of non-CM modular forms of even weight and level \( \Gamma_1(N) \) with \( a_p(f) = 0 \). For all such \( f \), it follows that \( \rho_f \) has the shape implied by Theorem 2.1. If we additionally assume that \( n \) is even, we see from this description that \( \overline{\rho}_f \) is also irreducible after restriction to \( G_{\mathbb{Q}_p} \) by the following lemma:

**Lemma 2.2.** Assume that \( a_p(f) = 0 \) and \( p > 2 \).

1. The representation \( \overline{\rho}_f|_{G_{\mathbb{Q}_p}} \) is absolutely irreducible if \( n - 1 \) is not divisible by \( (p+1) \), and in particular absolutely irreducible whenever \( n \) is even.
2. The representation \( \overline{\rho}_f|_{G_{\mathbb{Q}_p}(\zeta_p)} \) is absolutely irreducible if \( n - 1 \) is not divisible by \( (p+1)/2 \).

**Proof.** If \( \overline{\rho}_f|_{G_{\mathbb{Q}_p}} \) is reducible, then the character \( \varepsilon_2^{n-1} \) extends to \( G_{\mathbb{Q}_p} \), and in particular coincides with its Galois conjugate \( \varepsilon_2^{p(n-1)} \). This implies that \( \varepsilon_2^{(p-1)(n-1)} = 1 \).

Since \( \varepsilon_2 \) has order \( p^2 - 1 \), this forces \( (n-1) \) to be divisible by \( (p + 1) \). This proves part \( \text{ii} \). If \( \overline{\rho}_f \) is irreducible but becomes reducible over \( \mathbb{Q}_p(\zeta_p) \), then it must be induced from the ramified quadratic extension \( L/\mathbb{Q}_p \) inside \( \mathbb{Q}_p(\zeta_p) \). On the other hand, \( \overline{\rho}_f \) is also induced from the unramified extension \( K/\mathbb{Q}_p \), and so it is irreducible and induced from at least two distinct quadratic fields. This implies that the projective image is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \), and thus that the ratio of the characters \( \varepsilon_2^{n-1} \) and \( \varepsilon_2^{p(n-1)} \) is a quadratic character, or equivalently that
\[
\varepsilon_2^{(p-1)(n-1)} = 1,
\]
and hence \((n - 1)\) is divisible by \((p + 1)/2\), which proves part \(2\).  

This is a key step where we use the weight \(n\) is even; we could (at this point) assume merely that \((n - 1)\) is not divisible by \((p + 1)\). There are only finitely many irreducible modular residual representations of conductor dividing \(N\), and thus we may additionally assume that all such \(f\) have the same fixed residual representation \(\overline{\rho} = \overline{\rho}_f : G_K \to \text{GL}(2, k)\), and

\[
\overline{\rho}|_{G_{Q_p}} = \left(\text{Ind}_{G_K} G_{Q_p} \varepsilon_2^{n-1}\right) \otimes \overline{\psi}
\]

for fixed \(n\) and \(\overline{\psi}\). Finally, since there only finitely many possible Nebentypus characters of tame level \(N\), we may additionally assume that \(\psi|_{G_{Q_p}}\) is also fixed for our infinite collection of non-CM forms. After increasing \(k\) if necessary, we may assume that the eigenvalue of any element in the image of \(\overline{\rho}\) lands in \(k\), and moreover that \(\mathcal{O}_K \subseteq W(k)\). Finally, by a global twist by a finite character which is unramified at \(p\), we may assume that if \(\overline{\rho}\) is ramified at \(\psi|N\) then \(\overline{\rho}|_{\ell}\) is not scalar.

Associated to \(\overline{\rho}|_{G_{Q_p}}\) is a local universal deformation ring \(R_{\text{loc}}\) which is a complete local Noetherian \(W(k)\)-algebra. We now construct a quotient of this ring corresponding to deformations which are “induced” from \(K\). Let \(V_k\) denote the underlying representation of \(\overline{\rho}\) over \(k\). After restricting to \(G_K\), there is a canonical splitting \(V_k = U_k \oplus U'_k\) such that \(G_K\) acts on \(U_k\) and \(U'_k\) by \(\varepsilon_2^{n-1} \otimes \overline{\psi}\) and \((\varepsilon'_2)^{n-1} \otimes \overline{\psi}\) respectively.

**Definition 2.3** (Locally Induced Deformations). Let \(\mathcal{C}\) denote the category of local Artinian \(W(k)\)-algebras \((A, m)\) with a fixed identification \(A/m = k\). Let \(D(A)\) denote the deformations \(V_A\) of \(\overline{\rho}\) to \(A\) up to strict equivalence (in the sense of [Man99]), and let \(D_{\text{ind}}(A)\) denote the subset of deformations \(V_A\) which admit a splitting \(V_A = U_A \oplus U'_A\) into free \(A\)-modules of rank one such that \(U_A\) and \(U'_A\) are \(G_K\)-modules lifting \(U_k\) and \(U'_k\) respectively. Let \(D_{\text{ind}}(A, \psi)\) denote the subset of \(D_{\text{ind}}(A)\) such that the action of \(p\) on \(U_A\) viewed by local class field theory as an element of \(G_K^b\) is given by \(\psi(p)\).

Let \(\mathcal{O}(p) = 1 + m_K\) denote the units in \(K^\times\) which are 1 \(\text{mod } p\). Since \(p > 2\), this may be identified as a topological group with \((\mathbb{Z}_p)^2\) via the exponential map and an arbitrary choice of a basis for \(\mathcal{O}_K\) over \(\mathbb{Z}_p\).

**Lemma 2.4.** \(D_{\text{ind}}\) is pro-represented by a complete \(W(k)\)-algebra \(R_{\text{loc, ind}}\) which is isomorphic to the universal deformation ring of

\[
\varepsilon^{n-1} \otimes \overline{\psi} : G_K \to k^\times.
\]

In particular, \(R_{\text{loc, ind}} \simeq W(k)[[\mathcal{O}(p) \oplus \mathbb{Z}_p]]\) is smooth of relative dimension 3 over \(W(k)\). \(D_{\text{ind}, \psi}\) is pro-represented by the quotient \(R_{\text{loc, ind}, \psi} \simeq W(k)[[\mathcal{O}(p)]]\) of \(R_{\text{loc, ind}}\).

**Proof.** Since \(U_k\) and \(U'_k\) are distinct as \(G_K\)-modules, any decomposition \(V_A = U_A \oplus U'_A\) is unique up to strict equivalence. Moreover, \(\text{Gal}(K/Q_p)\) permutes these factors and hence \(V_A\) is determined by \(U_A\) as a \(G_K\)-representation. On the other hand, \(U_A\) is a deformation of \(U_k\), and any such deformation gives rise to such a \(V_A\) by taking \(V_A = \text{Ind}_{G_K} G_{Q_p}(U_A)\). Hence the two deformation functors coincide.

Since \(p > 2\), we have \(K^\times \simeq (\mathbb{Z}_p)^2 \oplus \mathbb{F}_p^\times \times \mathbb{Z}\), and so the 1-dimensional deformation ring associated to any character is isomorphic to \(W(k)[[\mathcal{O}(p) \oplus \mathbb{Z}_p]]\) where the \(\mathbb{Z}_p\) factor is generated by \(p\). The choice of \(\chi\) fixes the image of the second factor. \(\Box\)
We now consider a global deformation ring $R$ of $\overline{\rho}$ subject to the following conditions:

1. The deformations are unramified outside $Np$.
2. The deformations locally at $p$ are of the form $D^{\text{ind}, \psi}$.

The ring $R$ is clearly a quotient of the deformation ring $R^{\text{glob}}$ which represents deformations which are unramified outside $Np$ and are unrestricted at $Np$. More precisely, we have an identification of $W(k)$-algebras.

$$R = R^{\text{glob}} \otimes_{R^{\text{loc}} \otimes R^{\text{glob}}} R^{\text{loc}, \text{ind}, \psi}.$$ 

Note that $R^{\text{loc}, \text{ind}}$ is a quotient of $R^{\text{loc}}$.

The construction of $R$ guarantees that all of our eigenforms with $a_p(f) = 0$ (and fixed $\overline{\rho}$) give rise to points of $R$. We now show that $R$ is small.

**Lemma 2.5.** $R$ is finite as a module over $R^{\text{loc}, \text{ind}, \psi} = W(k)[O(p)]$.

**Proof.** Let $m_{R^{\text{loc}, \text{ind}, \psi}}$ denote the maximal ideal of $R^{\text{loc}, \text{ind}, \psi}$. Any deformation $\rho$ of $\overline{\rho}$ arising from the quotient $R/m_{R^{\text{loc}, \text{ind}, \psi}}$ will necessarily be totally split at $p$ in the sense that $Q(\ker(\rho))$ is totally split over $Q(\ker(\overline{\rho}))$ at all primes dividing $p$, and hence this ring is a quotient of the corresponding split (at $p$) global deformation ring $R^{\text{split}}$. If $R^{\text{split}}$ is finite over $W(k)$, then this quotient is also finite over $W(k)/p = k$, and the result would follow by Nakayama’s Lemma. The finiteness follows immediately from [AC14 Thm.1(2)] under the additional Taylor–Wiles hypothesis that $\overline{\rho}(G(Q_p))$ is absolutely irreducible. Indeed, that reference proves the stronger claim that $R^{\text{unr}}$ is finite over $W(k)$, and $R^{\text{split}}$ is a quotient of $R^{\text{unr}}$. It suffices to consider the remaining case when the Taylor–Wiles hypothesis might fail. Using our assumption that $n$ is even, we deduce from Lemma 2.2[2] that $n - 1$ is odd and divisible by $(p+1)/2$, so $p \equiv 1 \pmod{4}$. (This is the second and final time we use that $n$ is even.) In particular, the representation $\overline{\rho}$ is irreducible but induced from an unramified character of the totally real quadratic subfield $L \subset Q(\zeta_p)$. The finiteness of the corresponding deformation ring $R^{\text{split}}$ over $W(k)$ in this case can be deduced in a similar way to [AC14] except now using results of [SW01] rather than [The12]. Since this result is somewhat orthogonal to the main results of this paper, we relegate its proof to Theorem A.1 of the appendix.

**Remark 2.6.** An alternative approach to proving finiteness is to specialize to a height one prime ideal $p$ of $R^{\text{loc}, \text{ind}, \psi}$ corresponding to a representation of the form $\varepsilon^m \otimes (\text{Ind}_{G_K}^G G_{Q_p} \varepsilon_2^{k-1}) \otimes \psi$, where $2 \leq k \leq p - 1$. In order for the corresponding residual representation to agree with $(\text{Ind}_{G_K}^G G_{Q_p} \varepsilon_2^{n-1}) \otimes \overline{\psi}$, it suffices to chose $m$ and $k$ such that the following congruence is satisfied:

$$m(p + 1) + k - 1 \equiv (n - 1) \pmod{(p + 1)} \quad \text{or} \quad (n - 1)p \equiv (p^2 - 1).$$

We may take $k \equiv n \pmod{(p + 1)}$ unless $n \equiv 0 \pmod{(p + 1)}$, and we may take $k \equiv (p + 3 - n) \equiv 2 - n \pmod{(p + 1)}$ unless $n \equiv 2 \pmod{(p + 1)}$, and so $m$ and $k$ exist as long as $p \geq 5$. The corresponding deformation ring $R^{\text{loc}, \text{ind}, \psi}/p$ is a quotient of the crystalline local deformation ring with Hodge–Tate weights $[m, m + k - 1]$, and is thus a twist of a crystalline deformation ring of weight $[0, k - 1]$ which is in the Fontaine–Laffaille range. Since one expects to be able to prove $R = T$ theorems in this context (exploiting the fact that the corresponding local deformation rings are Cohen–Macaulay), this leads to explicit bounds on $\dim_{Q_p}(R/p)[1/p]$ in terms...
of dimensions of spaces of modular forms of weight at most $p - 1$, although in this approach one would also need to deal separately with the case when $\mathfrak{P}(\mathbb{Q}(\zeta_p))$ is reducible.

Let us now study the finite $\Lambda := W(k)[\mathcal{O}(p)]$-module $R$. If $\mathfrak{P}$ is globally induced from a quadratic field $F$, the ring $R$ admits a quotient $R_{\text{CM}}$ of Galois representations which are globally induced from $F$. Any irreducible representation $\mathfrak{P}$ of dimension 2 can be induced from only finitely many possible fields $F$. (That number is typically at most 1, but can be three precisely when the projective image of $\mathfrak{P}$ is $(\mathbb{Z}/2\mathbb{Z})^2$.)

If $\mathfrak{P}$ is not globally induced, then no $\mathfrak{Q}_p$-point of $R$ can be globally induced. The eigenforms with $a_p(f) = 0$ captured by $R$ give rise to a map from $R$ to $\mathfrak{Q}_p$ and thus a prime of $R$. Any such prime is contained inside a minimal prime of $R$, and since $R$ is Noetherian, there are only finitely many minimal prime ideals, and thus we may assume that there are infinitely many non-CM points which lie inside a fixed minimal prime $\mathfrak{P}$, or equivalently lie on a fixed irreducible component $R/\mathfrak{P}$ of $R$.

**Lemma 2.7.** Suppose that there are infinitely many non-CM modular forms giving rise to points of $R/\mathfrak{P}$. Then the support of $R/\mathfrak{P}$ is all of $\Lambda$.

**Proof.** The support of a finite module is closed, and thus it suffices to show that the support includes a Zariski dense subset of $\Lambda$. If $f$ has weight $n$, then we may explicitly write down the corresponding point of $\Lambda$. We make the explicit identification:

$$\mathcal{O}(p) \cong (1 + p)\mathbb{Z}_p \oplus (1 + p)^n\mathbb{Z}_p,$$

where $\eta = \sqrt{u}$ for any fixed non-quadratic residue $u$. We may then take $X = [1 + p] - 1$ and $Y = [(1 + p)^n] - 1$. The classical modular forms we are considering all correspond to specializations where $z \in \mathcal{O}^\times$ maps to $z^{n-1}$, or equivalently to

$$X \mapsto (1 + p)^{n-1} - 1, \quad Y \mapsto (1 + p)^{n(n-1)} - 1.$$  

(1)

It suffices to show that any infinite collection of these points are Zariski dense in $\Lambda$. The problem is that the Zariski closure is trying to be given by the equation

$$H = \eta \log(1 + X) - \log(1 + Y) = 0,$$

but this is not an element of $\Lambda \otimes \mathbb{Q}_p$ because the denominators grow without bound. (Note that we have chosen the field $k$ so that $\eta \in \mathcal{O}_k \subseteq W(k)$.) Alternatively, the Zariski closure wants to be $(1 + X)^n - (1 + Y) = 0$, although this expression (unlike $H$) is not even a function on the open unit ball.

Suppose the Zariski closure of these points is given by the vanishing set of $F(X, Y)$ in $W(k)[X, Y]$. Choose a primitive $p^m$th root of unity $\zeta_m$ for each $m$ and let $\pi_m = 1 - \zeta_m$. There is an inclusion

$$F(X, Y) \in W(k)[X, Y] \subset W(k)[\pi_m][X, Y] \subset W(k)[\pi_m][X/\pi_m, Y/\pi_m],$$

which amounts to considering the restriction of functions bounded by 1 on the open unit ball $B(1)$ to functions bounded by 1 on the open ball $B(\pi_m)$. (Here $B(r)$ denotes the open ball centered at the origin with radius $|r|$.)

**Sub-lemma 1.** Suppose that $v(\eta) = 0$ but $\eta \notin \mathbb{Z}_p$. Then

$$H_m := (1 + X)^{p^m-1} - (1 + Y)^{p^m-1}.$$  

is an element of $W(k)[\pi_m, Y][X/\pi_m] \subset W(k)[\pi_m][X/\pi_m, Y/\pi_m]$, but is not an element of $W(k)[\pi_{m+1}, Y/\pi_{m+1}] \otimes \mathbb{Q}_p$. 


Proof. It suffices to analyze the growth of the coefficient of $X^n$ as $n$ varies. The
coefficient is explicitly given by a binomial coefficient, and hence its valuation is

$$v\left(\binom{\eta p^{m-1}}{n}\right) = v\left(\prod_{i=0}^{n-1} (\eta p^{m-1} - i) - v(n!)ight)$$

Our assumptions on $\eta$ imply that the valuation of $\eta p^{m-1} - i$ is $v(i)$ when $v(i) \leq m-1$
and $m-1$ otherwise. Hence the valuation of the $X^n$ coefficient is equal to

$$\sum_{k=1}^{m-1} \left(1 + \left\lfloor \frac{n-1}{p^k} \right\rfloor \right) - \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

A lower bound for this expression is given by

$$\sum_{k=1}^{m-1} \left| \frac{n}{p^k} \right| - \sum_{k=m}^{\infty} \frac{n}{p^k} \geq - \sum_{k=m}^{\infty} \frac{n}{p^k} = - \frac{n}{p^{m-1}(p-1)},$$

whereas an upper bound when $n = p^r$ and $r \geq m - 1$ is given by

$$\sum_{k=1}^{m-1} \left(1 + \left\lfloor \frac{p^r}{p^k} \right\rfloor \right) - \sum_{k=1}^{\infty} \left\lfloor \frac{p^r}{p^k} \right\rfloor = - \sum_{k=m}^{\infty} \frac{p^r}{p^k} = - \frac{n}{(p-1)p^{m-1}\left(1 - \frac{1}{p^{r-m}}\right)}.$$

Since $v(\pi_m) = 1/(p^{m-1}(p-1))$, the lower bound implies that $H_m \in W(k)[\pi_m, Y][X/\pi_m]$. On the other hand, the upper bound shows that this cannot be improved.

Now $H_m$ also vanishes at all the weights $z \mapsto z^{n-1}$. We claim that $H_m$ is
irreducible in $W(k)[\pi_m][X/\pi_m, Y/\pi_m]$. Viewing $H_m$ as an element inside the larger
ring $W(k)[\pi_m][X/\pi_1, Y/\pi_1]$, we do have the factorization

$$H_m = \prod_{i=1}^{p^{m-1}} \left(1 + Y - (1 + X)^n \zeta_{m-1}^i\right).$$

Certainly any factorization over the smaller ring is thus promoted to a factorization
over $W(k)[\pi_m, Y][X/\pi_m]$. But if there exists a factor with $r < p^{m-1}$ terms, then
the constant term considered as a polynomial in $(Y + 1)$ will be a non-zero multiple
of $(1 + X)^r$, which does not lie in this ring by Sublemma 1, a contradiction.

Because $H_m \in W(k)[\pi_m][X/\pi_m, Y/\pi_m]$ is irreducible, we deduce that
$F(X,Y)$ must vanish at all points in $B(\pi_m)$ where $H_m$ vanishes. But then $F(X,Y)$
must vanish at the finitely many pairs $(\zeta_1 - 1, \zeta_2 - 1)$ of $p$-power roots of unity
with $v(\zeta_i - 1) > v(\pi_m)$. But repeating this with $m$ arbitrarily large implies that $F(X,Y)$
vanishes at all such pairs of $p$-power roots of unity, which is impossible because they
are Zariski dense in $\Lambda$.

We note in passing that (formally) $\lim_{m \to \infty} \frac{H_m(X,Y)}{p^{m-1}} = H(X,Y)$.

Lemma 2.8. If a component $R/\mathfrak{Q}$ has infinitely many non-CM points, then
the support of CM points on $R/\mathfrak{Q}$ lies on a proper closed subscheme of $\Lambda$.

Proof. The CM points must all lie on $R/\mathfrak{Q} \otimes_R R_{CM}^F$ for one of finitely many $F$.
Either this is all of $R/\mathfrak{Q}$, which contradicts the assumption, or, because $R/\mathfrak{Q}$
is irreducible, it has positive co-dimension. But $R$ is finite over $\Lambda$, so these quotients
(and so a finite union of such quotients) has support on a proper closed subscheme.

□
To complete the proof of Theorem 1.1, it suffices to show that $R/\mathfrak{P}$ contains a Zariski dense set of points which are CM. To this end, we use a variation of the idea of Ghate–Vatsal [GV04] to prove local indecomposability of non-CM Hida families by specializations in weight one. Namely, we consider points corresponding to maps $\mathcal{O}(p) \to \overline{\mathbb{Q}}_p$ with finite image, and such that the ratio of any such map to its $\text{Gal}(K/Q_p)$-conjugate has order greater than 60. These points are clearly Zariski dense. On the other hand, for any such specialization, the corresponding Galois representation:

$$\rho : G_{\mathbb{Q}} \to \text{GL}_2(\overline{\mathbb{Q}}_p)$$

has finite image on inertia at $p$. Thus, by [PS16, Thm.0.2] when $\rho|_{G(\mathbb{Q}(\zeta_p))}$ is irreducible and by [Sas, Thm.1] in the general case (noting that we may assume that $p \equiv 1 \mod 4$ and hence $\mathfrak{p}$ is induced from a real quadratic extension), it follows that $\rho$ is modular of weight one. By construction, the projective image of inertia has order greater than 60, which ensures that the projective image is not of exceptional type ($A_4$, $S_4$, or $A_5$). It follows that $\rho$ must be of dihedral (CM) type. But then we have produced a Zariski dense set of CM points on $R/\mathfrak{P}$, a contradiction, and we are done.

**Remark 2.9.** The fact that any infinite set of characters of the form $z \mapsto z^n$ are Zariski dense in $\Lambda = \text{Spec}(\mathbb{Z}_p[[\mathcal{O}_K^\times]])$ can be viewed as a special case of a local $p$-adic analogue of Lang’s conjecture [Lan83] (See also [Ser18]). The classical analogue of our example is the statement that any infinite set of points $(\exp(\eta x), \exp(x))$ are Zariski dense in $(\mathbb{C}^\times)^2$ whenever $\eta \notin \mathbb{Q}$.

2.10. **Proof of Corollary 1.2.** If $f$ is a form of level 1 which is CM, then the corresponding automorphic representation is induced from a character on some imaginary quadratic field $K/\mathbb{Q}$. But then the level of $f$ will be divisible by any prime dividing the discriminant of $K$, a contradiction. Hence Corollary 1.2 follows immediately for all $p > 2$. For $p = 2$, we prove directly that if $a_2(f) = 0$ then $\rho_f$ is dihedral, and so $f$ is CM, from which the result follows by the argument above. Note that for $N = 1$ and $p = 2$ the representation $\rho$ will have trivial semi-simplification, and it follows that the image of $\rho_f$ factors through the maximal pro-2 extension of 2 unramified outside 2. By [Che08, Prop 1.8], it follows that the image of $\rho_f$ is isomorphic to the image of $\rho_f$ restricted to inertia at 2. But the assumption that $a_2(f) = 0$ implies that $\rho_f$ is locally induced, which now implies it is also globally induced, and thus CM.

3. **Acknowledgments**

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Appendix A. Finiteness of unramified deformation rings

Let $\overline{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(k)$ be an absolutely irreducible odd Galois representation of the form $\text{Ind}_{G_{L}}^{G_{\mathbb{Q}}}\chi$, where $L/\mathbb{Q}$ is the quadratic field $L \subset \mathbb{Q}(\sqrt{p})$. Suppose that, up to unramified twist:

\[ \overline{\rho}|_{G_{Q_p}} = \text{Ind}_{G_K}^{G_{Q_p}}\varphi^{n-1}, \quad n - 1 = \frac{p + 1}{2}. \]

The main theorem of this section is the following.

Theorem A.1. Assume that $p \equiv 1 \mod 4$, so $L/\mathbb{Q}$ is real. Let $(N,p) = 1$, and let $R^{\text{split}}$ denote the universal deformation ring of $\overline{\rho}$ consisting of representations which are unramified outside $N$ and totally split at $p$. Then $R^{\text{split}}$ is finite over $W(k)$.

Proof. We begin with some reductions. By class field theory, we may reduce to the real field $F$. By Lemma A.2, after making a further extension, it suffices to prove the finiteness of $R^{\text{split}}$ up to unramified twist: $R^{\text{split}}/\mathbb{Q}$ factors through $R^{\text{split}}/p$, and the kernel contains the image of the maximal ideal of $\Lambda$. It remains to prove:

Lemma A.2. Let $E/\mathbb{Q}_p$ be a local field of residue characteristic prime to $p$, and consider a tamely ramified representation

\[ \overline{\rho} : G_{E} \to \text{GL}_2(k) \]

whose determinant is trivial on inertia. For any finite extension $F/E$, let $R^{\text{univ}}$ denote the unrestricted trivial determinant framed local deformation ring of $\overline{\rho}|_{G_{E}}$, and let $R^{\text{univ}}_{univ}$ denote the quotient of $R^{\text{univ}}$ where one imposes the condition that, for a generator $\tau$ of tame inertia, the image $X$ of $\rho(\tau)$ satisfies $(X - 1)^2 = 0$. Let $\rho^{\text{univ}}_{E}$ and $\rho^{\text{univ}}_{F}$ denote the corresponding universal deformations. Then there exists a finite extension $F/E$ such that $\rho^{\text{univ}}_{E}|_{G_{E}}$ factors through $\rho^{\text{univ}}_{F}$. 

\[ \overline{\rho}|_{G_{Q_p}} = \text{Ind}_{G_K}^{G_{Q_p}}\varphi^{n-1}, \quad n - 1 = \frac{p + 1}{2}. \]
Proof. The image of $\rho_{\text{univ}}^E$ is certainly tamely ramified and pro-$p$, and so factors through the pro-$p$ group $\Gamma$ with presentation $\Gamma = \langle \sigma, \tau \mid \sigma \tau \sigma^{-1} = \tau^q \rangle$ where $q = q_v$. If $X$ is the image of $\rho(\tau)$, then, since $\det(X) = 1$, we may write $\text{Tr}(X^n) = 2T_n(x)$ where $2x = \text{Tr}(X) \in R^\text{univ}_E$ and $T_n$ is the $n$th Chebyshev polynomial. Since $X$ is conjugate to $X^q$, it follows that $2T_1(x) = 2T_q(x)$. From the formal identity $2T_n\left((t + t^{-1})/2\right) = t^n + t^{-n}$, it is easy to show that $T_q(x) - T_1(x)$ is a factor of $T_{q^2-1}(x) - 1$, and thus $\text{Tr}(X^{q^2-1}) = 2$. We may thus take $F$ to be the fixed field (of the inverse image in $G_E$) of the subgroup $\langle \tau^{q^2-1}, \sigma \rangle \subset \Gamma$. □

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