A model for pest control using integrated approach: impact of latent and gestation delays

Fahad Al Basir · Monirul Hassan Noor

Received: 13 September 2021 / Accepted: 19 December 2021 / Published online: 4 February 2022
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Abstract In this article, we have established a mathematical model using impulsive differential equations for the dynamics of crop pest management in the presence of a pest with its predator and bio-pesticides. The pest population is divided into two subpopulations, namely, the susceptible pests and the infected pests. In this control process, bio-pesticides (generally virus) infect the susceptible pest through viral infection within the pest and make it infected so that predators can consume it quickly. We assume that pest controlling, using this integrated approach, is a delayed process and thus incorporated latent time of susceptible pest and gestation delay of predator in the model as time delay parameters. The system dynamics have been analyzed using qualitative theory: the existence of the equilibrium points and their stability properties has been derived. Hopf bifurcation of the coexisting equilibrium point is presented for both the delayed and non-delayed system. Detail numerical simulations are performed in support of analytical results and illustrate the different dynamical regimes observed in the system. We have observed that the system becomes free of infection when the latent time of the pest is large. Coexisting equilibrium exists for the lower value of latent delay, and it can change the stability properties from stable to unstable when it crosses its critical value. In contrast, gestation delay affects the stability switches of coexisting equilibrium only. The combined effect of the two delays on the system is shown numerically. Also, viral replication rate, infection rate (from virus to pest) is also significant from the pest management perspective. In summary, both the delay are essential for crop pest management, and pest control will be successful with tolerable delays.

Keywords Biopesticides · Predator · Mathematical model · Latent and gestation delay · Hopf bifurcation · Numerical stability analysis.

1 Introduction

Food scarcity gives us the challenge to improve our agricultural tools and techniques at an advanced level and consequently increase the production of crops. Chemicals affect insect populations, pests, and beneficial insects to increase the yield from a crop field. In classical biological control, the pest population is controlled from the region of its origin by the introduction of natural enemies [1]. However, chemical control acts as a hazard to the ecosystem. It makes lots of unwanted harmful effects on other related and nonrelated populations. This kind of problem can be solved if Integrated Pest Management (IPM) is adopted.
IPM is becoming more acceptable among researchers, and its growing applicability in the field by marginal farmers. In an integrated pest control process, we need to reduce the dependency on pesticides by emphasizing the improvement of the applicability of biological control agents. The role of microbial pesticides in the IPM has been recently studied for agriculture, forestry, and public health by various researchers [9]. As an integrated approach, bio-control has a significant role in pest control [14]. On the other hand, there are many animals and birds whose food is the pest population, but these creatures do not hamper the crop. Henceforth, they can be used as one of the biological control agents for the control of pest.

In [25], we can find the use of predator populations in removing pest population. Further, we can infect the susceptible pest population by inducing viral or bacterial diseases within the pest. For example, baculovirus usually grows in plants, and this virus has no direct effect on the production of crops. However, they can be involved to reduce the pest population [27]. Thus, infection to the pest population is one way to eliminate pests from the agriculture field successfully.

Prey–predator type pest control models, with a disease in the prey, (pest) populations are very helpful to describe the dynamics of pest and its natural enemy [24,26]. However, there are few theoretical modelling works on the dynamics of pests and their predators and biopesticides [2,4,5]. In [2], authors have assumed a mathematical model for integrated pest control using bilinear infection rate. Roy et al. [5] have proposed and analyzed mathematical models for biological pest control, particularly in Jatropha sp. plantation using a virus as a controlling agent. They have used a separate crop population and a logistic type growth for pest population. Later in [4], they have introduced a predator population and shown that integrated pest control is helpful if biopesticide is release in an impulse way. In their paper, Tan and Chen [7] discussed pest control biologically by introducing infected pests. Also, Wang and Song [23] used mathematical models to control a pest population by releasing infected pests. All the ecoepidemic models with susceptible prey, infected prey, and the predators can be used to discuss the nature of the susceptible pest, infected pest, and predators but consider time delay as crucial in the biological system dynamics like crop pest control.

Mathematical models using delay differential equations are more realistic as time delays occur in almost all biological systems. In the system under consideration, it is reasonable to consider that the reproduction of the predator after predating the prey is not an immediate process. It has to pass through the gestation of the predator. After predation, some amounts of energy in the biomass of prey assimilate into the predator’s energy in the form of biomass. However, this biophysiological process is not simple. Also, the exchange of prey energy to predator energy is not instantaneous. Different steps are involved in this mechanism. Moreover, there is another time lag present in the pest management system. A time lag exists between the infection of a pest using the virus (biopesticide) and the time it becomes fully infected. This time lag is known as the latent time or latent period of the disease in the pests. In our model, we consider both the gestation and latent delay.

There are few mathematical models for the dynamics of pest control considering time-delay effects. Zhang et al. [8] used a delayed stage-structured epidemic model for pest management. Jana and Kar [10] have considered gestation delay in their prey–predator type model with infection in the prey population. In [11], latent delay was considered in a pest control model using biopesticides, whereas in [12,13], mathematical models were introduced using gestation period as a delay parameter.

In the present study, we develop a mathematical model to study the dynamics of pest controlling system using biopesticides and a predator population. We consider an integrated approach (releasing biopesticides and natural enemy of pest, i.e., predator). Thus, the model is an epidemiological model for the dynamics of bio-control of a pest. Then, we find out the effects of latent and gestation delays on the system analytically and numerically. To our knowledge, this is the first new work considering both the latent delay and gestation delay in a mathematical model for pest control.

The article is arranged in the following manner. The delay model is derived in Sect. 2, including some basic mathematical properties. The existence of equilibria and their stability is performed in Sect. 3 and Sect. 4, respectively. Possible occupancy of Hopf bifurcation and its possible controls is discussed in Sect. 5. Detail numerical calculations are presented in Sect. 6 with biological discussion. Finally, we conclude the paper in Sect. 7 with a discussion.
2 The mathematical model

Here, a model is formulated considering four populations, namely the susceptible pest, infected pest, predator, and bio-pesticide. The following assumptions are taken to develop the mathematical model.

A crop plantation system is considered where pest consumes the plant. The pests are the prey, and the natural enemy of pest is the predator. Pest is controlled using bio-pesticides spraying. In this bio-controlling process, the pest is made infected as infected pests are less harmful to the plant. Also, predators are incorporated into the system. They uptake the pest population. They consume the susceptible pest at a lower rate than consuming the infected pest.

Let \( S(t) \) be the susceptible pest, \( I(t) \) be the infected pest, \( P(t) \) is the predator (natural enemy of pest) and \( V(t) \) is the bio-pesticide (Virus) at any time \( t \).

- Logistic growth is assumed for plant biomass, with net growth rate \( r \) and carrying capacity \( K \). Let \( \alpha \) and \( \beta \) are the predation rates. As predator can catch infected pest easily, we assume \( \beta > \alpha \) and \( m \leq 1 \) is the conversion efficacy. Predator-pest (prey) interaction can be studied with the Holling type II functional response. Here, \( a \) is the half saturation constant.

- There is an infection rate of healthy pest, \( \lambda \), due to the use of biopesticide (virus), and \( \kappa \) is the virus replication parameter. Finally, \( \epsilon \) is the intra-specific competition rate of a predator.

- Let \( \delta \) is the mortality rate of infected pest induced infection, and \( d \) is the natural mortality rate of a predator.

- Infection of susceptible pest by biopesticide is assumed as a delayed process. Here, \( \tau_1 > 0 \) is the time delay due to the delayed response of biopesticide. We take another delay \( \tau_2 > 0 \) due to the gestation of a predator.

Based on the above assumptions, the following mathematical model is derived:

\[
\begin{align*}
\frac{dS}{dt} &= rS \left(1 - \frac{S + I}{K}\right) - \lambda SV - \frac{\alpha SP}{\alpha + S}, \\
\frac{dI}{dt} &= \lambda e^{-\delta \tau_1} S(t - \tau_1) V(t - \tau_1) - \delta I - \beta IP, \\
\frac{dP}{dt} &= \frac{m\alpha S(t - \tau_2) P(t - \tau_2)}{a + S(t - \tau_2)} - dP - \epsilon P^2, \\
\frac{dV}{dt} &= \kappa \delta I - \mu V, \\
\end{align*}
\]

Let us assume \( \tau = \max\{\tau_1, \tau_2\} \) and denote \( C \) as the Banach space of continuous functions \( \phi : [-\tau, 0] \rightarrow \mathbb{R}^4 \), equipped with the \( \sup \)-norm,

\[
\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \{|\phi_1(\theta)|, |\phi_2(\theta)|, |\phi_3(\theta)|, |\phi_4(\theta)|\},
\]

where, \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in C([-\tau, 0], \mathbb{R}) \). For biological reasons, populations always have nonnegative values if they survive, therefore, the initial function for model (1) is taken as below:

\[
\begin{align*}
S(\theta) &= \phi_1(\theta), \quad I(\theta) = \phi_2(\theta), \quad P(\theta) = \phi_3(\theta), \quad V(\theta) = \phi_4(\theta) \\
&\text{with } \phi_i(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0, \quad i = 1, 2, 3, 4. 
\end{align*}
\]

3 Preliminary results

In this division, some fundamental properties such as positive invariance and boundedness of the solutions of the delayed system (1) are depicted.

Biologically, positivity signifies the survival of the populations. Therefore, we require the following theorem.

**Theorem 1** Each solution of the system (1) with the given initial conditions (2) is non-negative and is ultimately bounded.

**Proof** Using the standard theory of functional differential Eq. [30], it is easy to a unique solution of the system (1) exists in the region \( \mathbb{R}^4_+ \).

We rewrite the first equation of (1) as

\[
\begin{align*}
\frac{dS}{dt} - rS \left(1 - \frac{S + I}{K}\right) - \lambda SV - \frac{\alpha SP}{\alpha + S} &= 0. 
\end{align*}
\]

Let \( f(S, I, V) = r \left(1 - \frac{S + I}{K}\right) - \lambda V - \frac{\alpha P}{\alpha + S} \), then from (3), we have

\[
\begin{align*}
\frac{dS}{dt} - Sf(S, I, V) &= 0 \\
\Rightarrow \left[ \frac{dS}{dt} - Sf(S, I, V) \right] \\
\exp \left(- \int_0^t f(S(\xi), I(\xi), V(\xi)) d\xi \right) &= 0 \\
\Rightarrow \frac{d}{dt} S(\exp \left(- \int_0^t f(S(\xi), I(\xi), V(\xi)) d\xi \right) &= 0, \\
\Rightarrow S(t) = S(0) \exp \left(\int_0^t f(S(\xi), I(\xi), V(\xi)) d\xi \right). 
\end{align*}
\]
Since, \( S(0) = \phi_1(0) > 0 \), then \( S(t) > 0 \) for \( t \geq 0 \). We shall prove the nonnegativity of \( I(t) \) by contradiction.

Suppose there exists a \( t_0 > 0 \) for which \( I(t) > 0 \) for \( t < t_0 \) and \( I(t) = 0 \) for \( t = t_0 \) and \( I(t) < 0 \) for \( t \in (t_0, t_0 + \epsilon) \). Now for \( t = t_0 \), from the second equation of (1) we have

\[
I'(t_0) = \lambda e^{-\delta t} S(t_0 - \tau) V(t_0 - \tau) > 0,
\]

which is a contradiction. This shows \( I(t) > 0 \) for all \( t > 0 \). Using similar argument, the nonnegativity of the rest population can be established.

To ensure the model remains biologically plausible, both plant and vector populations have to remain bounded during their time evolution.

To prove the boundedness of \( S(t) \), \( I(t) \) and \( P(t) \), let us define

\[
G(t) = me^{-\delta t} S(t) + m I(t + \tau_1) + P(t + \tau_2)
\]

Then,

\[
\frac{dG(t)}{dt} = me^{-\delta t} \frac{dS}{dt} + m \frac{dI(t + \tau_1)}{dt} + \frac{dP(t + \tau_2)}{dt}
\]

\[
< e^{-\delta t} r S \left[ 1 - \frac{S + I}{K} \right] - \delta I(t + \tau_1) - P(t + \tau_2)
\]

\[
< e^{-\mu r} \left[ (r + m) S - \frac{r S^2}{K} \right] - q G(t),
\]

where \( q = \min \left\{ \frac{\delta}{m}, \frac{d}{m} \right\} \).

Now, \( (r + m) S - \frac{r S^2}{K} \) is a quadratic in \( x \) and its maximum value is \( \frac{K(r+m)^2}{4r} \). Also, the maximum value of \( e^{-\mu r} \) is 1. Using this, we can write

\[
\frac{dG(t)}{dt} < \frac{K(r+m)^2}{4r} - q G(t).
\]

This implies

\[
\lim_{t \to \infty} \sup G(t) \leq \frac{K(r+m)^2}{4rq}.
\]

and, hence, the \( S(t) \) and \( I(t) \) are bounded. As \( I \) is bounded, using the well-known comparison principle, from the last equation of (1), \( V(t) \) is also bounded.

3.1 Existence of equilibria

The model (1) has five equilibria namely (i) the trivial equilibrium \( E_0(0, 0, 0, 0) \); (ii) the axial equilibrium \( E_1(K, 0, 0, 0) \), (iii) biopesticide and infected pest free equilibrium \( E_2(\bar{S}, \bar{I}, \bar{P}, 0) \), where,

\[
\bar{P} = \frac{r(K - \bar{S})(\alpha + \bar{S})}{\alpha K}
\]

and \( \bar{S} \) is the positive root of

\[
P_0 S^3 + P_1 S^2 + P_2 S + P_3 = 0,
\]

with coefficients

\[
P_0 = -\epsilon r, \quad P_1 = -2\alpha \epsilon + \epsilon K, \quad P_2 = -ma^2 K + d a K - ra^2 \epsilon + 2\epsilon K a, \quad P_3 = \epsilon K a^2 + ma K a.
\]

The feasibility of the equilibrium \( E_2 \) is analyzed in Proposition 1.

(iv) the predator-free equilibrium \( E_3(\bar{S}, \bar{I}, 0, \bar{V}) \), where,

\[
\bar{S} = \frac{\mu}{\lambda e^{-\delta \tau_1} K}, \quad \bar{I} = \frac{\mu(K, e^{-\delta \tau_1} K - \mu r)}{\lambda e^{-\delta \tau_1} K (\delta K \lambda K + \mu r)}, \quad \bar{V} = \frac{\delta (K, e^{-\delta \tau_1} K - \mu r)}{\lambda e^{-\delta \tau_1} (\delta K \lambda K + \mu r)}
\]

The predator-free equilibrium \( E_3 \) of the system (1) exists if the following condition holds

\[
K \lambda K > \mu e^{\delta \tau_1}.
\]

(v) the coexistence equilibrium, \( E^*(X^*, S^*, I^*, A^*) \), given by

\[
P^* = \frac{\lambda e^{-\delta \tau_1} \delta S^* - \mu \delta}{\mu \beta}, \quad V^* = \frac{\kappa \delta I^*}{\mu}, \quad I^* = \frac{\mu \beta d + \epsilon (a + S^*)(\lambda e^{-\delta \tau_1} K \delta S^* - \mu \delta)}{\mu m \beta (a + S^*)},
\]

and \( S^* \) is the positive root of

\[
A_1 S^2 + B_1 S + C_1 = 0.
\]

\[
A_1 = -\delta^2 e K K^2 \lambda^2 e^{-\delta \tau_1} - \delta e \kappa \lambda e^{-\delta \tau_1} \mu r - \beta^2 m \mu^2 r, \quad B_1 = -a \delta^2 e K K^2 \lambda^2 e^{-\delta \tau_1} + \delta e K K \mu \mu - \alpha \beta \delta K \kappa e^{-\delta \tau_1} m \mu - a \delta e \kappa \lambda e^{-\delta \tau_1} \mu r + \delta e \mu^2 r - \beta^2 m \mu^2 r + \beta^2 K m \mu^2 r, \quad C_1 = -\beta d \delta K \kappa \lambda \mu + a \delta^2 e K K \lambda \mu + \alpha \beta \delta K m \mu^2 + \beta d \mu^2 r + a \delta e \mu^2 r + a \beta^2 K m \mu^2 r.
\]

The existence conditions of the coexisting equilibrium \( E^* \) are discussed in Proposition 2.

Proposition 1 Let

\[
\Delta = 18 P_1 P_2 P_3 P_4 - 4 P_2^2 P_3^2 + P_2 P_3^2 P_4^2 - 4 P_1 P_3^3 - 27 P_1 P_2^2 P_3^3
\]

be the discriminant of (6). Then, for \( \Delta > 0 \), the followings hold:

(i) if \( P_2 < 0 \) or \( P_3 > 0 \), then there exists a unique equilibrium \( E_2 \) (the biopesticide-infected pest-free equilibrium) of the system 1.
(ii) if \( P_2 > 0, P_3 < 0 \), then there may exist three distinct feasible \( E_2 \) of the system (1).

**Proof** Descartes’ rule of signs states that the number of real positive roots of polynomials with real coefficients is equal to the number of sign changes in those coefficients or is less than the number of sign changes by an even number. That means, if there is a single sign change in the polynomial coefficients, this theorem immediately implies the existence of a single real positive root.

In Eq. (6), we see that \( P_1 < 0 \) and \( P_4 > 0 \), thus the following the cases may be raised:

(i) if \( P_2 < 0 \) or \( P_3 > 0 \), then there is one change of sign in the coefficients of Eq. (6), thus according to Descartes’ rule of signs \( (6) \) will have one positive root. Consequently, the system (1) will have a unique feasible equilibrium \( E^* \),

(b) if \( P_2 > 0, P_3 < 0 \), there is three change of sign in the coefficients of Eq. (6), thus according to Descartes’ rule of signs the Eq. (6) may have three positive roots. The roots will be distinct when \( \Delta > 0 \). Consequently, there may be three possible feasible \( E_2 \) of the system (1).

\[ \square \]

**Proposition 2** Let

\[ \Delta_2 = B_1^2 - 4A_1C_1 \]

be the discriminant of Eq. (8). Suppose \( \Delta_2 > 0 \), then

(i) if \( B_1 < 0, C_1 < 0 \), then \( (8) \) has no feasible endemic equilibrium \( E^* \).

(ii) if \( C_1 > 0 \), then there exists a unique coexisting equilibrium \( E^* \).

(iii) if \( B_1 > 0, C_1 < 0 \), then precisely two distinct feasible coexisting equilibrium is possible.

**Proof** Noticing that \( A_1 < 0 \) and thus using the Descartes rule of sign following cases may be possible:

(i) for \( B_1 < 0, C_1 < 0 \), there is no change of sign in the coefficient of \( (8) \). Thus, there is no positive root, which means there is no feasible endemic equilibrium \( E^* \) of the system (1),

(ii) for \( B_1 > 0, C_1 < 0 \), there is two change of sign in the coefficient of \( (8) \). Thus, there may be two positive roots of \( (8) \). They will be distinct if the discriminant \( \Delta_2 > 0 \). Thus, two feasible coexisting equilibrium is possible,

(iii) for \( C_1 > 0 \), whatever the sign of \( B_1 \) may be, there is one change of sign in the coefficient of \( (8) \). Thus, there exists exactly one positive root of \( (8) \). That is, a unique coexisting equilibrium \( E^* \) for the system (1).

\[ \square \]

3.2 Characteristic equation

Characteristic equation of the Jacobian matrix is needed for the stability of equilibria. For this, we linearize the system (1) about \( E(S, I, P, V) \) as follows:

\[ \frac{dX}{dt} = FX(t) + GX(t - \tau_1) + VX(t - \tau_2). \]  

Here, \( F, G \) are \( 4 \times 4 \) matrices, given as below:

\[ F = [F_{ij}] = \begin{bmatrix} F_{11} & -\frac{rS}{K} & -\frac{aS}{a+S} & -\lambda S \\ 0 & -\delta - \beta P & -\beta I & 0 \\ 0 & 0 & -d - 2\epsilon P & 0 \\ 0 & \kappa\delta & 0 & -\mu \end{bmatrix}, \]

\[ G = [G_{ij}] = \begin{bmatrix} \lambda e^{-\delta\tau_1}I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ H = [H_{ij}] = \begin{bmatrix} \alpha \beta P & m \beta P & H_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

where \( F_{11} = r(1 - \frac{2S+I}{K} - \lambda V - \frac{aP}{a+S}) \), \( H_{33} = \frac{maS}{a+S} + m\beta I \). The characteristic equation of the delay system (1) is given by,

\[ | \xi I - F - e^{-\xi\tau_1}G - e^{-\xi\tau_2}H | = 0. \]  

This gives the following characteristic equation,

\[ \phi(\rho, \tau_1, \tau_2) = \rho^4 + l_1\rho^3 + l_2\rho^2 + l_3\rho + l_4 + e^{-\rho\tau_1}(b_1\rho^2 + b_2\rho + b_3) + e^{-\rho\tau_2}(a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4) + e^{-\rho(\tau_1+\tau_2)}(q_1\rho + q_2) = 0, \]  

Coefficients of Eq. (12) are given in Appendix A.
4 Stability of equilibria without delay

In this section, the local stability analysis of the equilibria of the system (1) without delay is studied. For the stability of any equilibrium, we need the distribution of the roots of the characteristic Eq. (12). From this, we show that the trivial-equilibrium point $E_0(0,0,0,0)$ is always unstable.

Next, we analyze the stability of virus-free equilibrium $E_2(\bar{S},0,\bar{P},0)$.

4.1 Stability of virus-free equilibrium $E_2(\bar{S},0,\bar{P},0)$

In this case, the Jacobian matrix,

$$ J = [J_{ij}]_{4 \times 4} = \begin{bmatrix} J_{11} & -\frac{\bar{S}}{K} & -\frac{\alpha \bar{S}}{a + \bar{S}} & -\lambda \bar{S} \\ 0 & -\delta - \beta \bar{P} & 0 & \lambda e^{-\delta \tau_1} \bar{S} \\ \frac{ma \bar{P} \alpha}{(a + \bar{S})^2} & m \beta \bar{P} & J_{33} & 0 \\ 0 & \kappa \delta & 0 & -\mu \end{bmatrix} $$

where, $J_{11} = r(1 - \frac{2 \bar{S}}{K}) - \frac{\alpha a \bar{P}}{(a + \bar{S})^2}$, $J_{33} = \frac{ma \bar{S}}{a + \bar{S}} - d - 2 \epsilon \bar{P}$.

The characteristic Eq. at $E_2(\bar{S},0,\bar{P},0)$ is

$$ \xi^4 + M_1 \xi^3 + M_2 \xi^2 + M_3 \xi + M_4 = 0, \quad (13) $$

with

$$ M_1 = -J_{11} - J_{22} - J_{33} - J_{44}, $$
$$ M_2 = J_{11} J_{22} - J_{13} J_{31} + J_{11} J_{33} + J_{12} J_{23} + J_{14} J_{43}, $$
$$ J_{22} J_{44} + J_{33} J_{44} - J_{42} J_{24}, $$
$$ M_3 = J_{13} J_{22} J_{31} - J_{11} J_{23} J_{33} - J_{12} J_{23} J_{44}, $$
$$ J_{13} J_{31} J_{44} + J_{11} J_{33} J_{44} - J_{23} J_{31} J_{44} + J_{11} J_{42} J_{24} + J_{12} J_{42} J_{23} + J_{13} J_{42} J_{24}, $$
$$ M_4 = -J_{13} J_{22} J_{31} J_{44} + J_{11} J_{22} J_{33} J_{44}, $$
$$ J_{13} J_{31} J_{42} J_{24} - J_{11} J_{33} J_{42} J_{24}. $$

For the stability of an equilibrium point, we use the Routh–Hurwitz criterion and Eq. (13) and get the following theorem.

**Theorem 2** The virus-free equilibrium $E_2$ is asymptotically stable if the following conditions are satisfied

$$ M_1 > 0, \quad M_2 > 0, \quad M_3 > 0, \quad M_4 > 0, \quad M_1 M_2 - M_1 M_3 - M_2^2 - M_3 M_4^2 > 0. $$

In the following subsection, we study the stability of predator-free equilibrium $E_3$.

4.2 Stability of $E_3(\bar{S}, \bar{I}, 0, \bar{V})$ without delay

We have the following theorem for the stability of $E_3$.

**Theorem 3** For $\tau_1 = 0 = \tau_2$, the predator-free equilibrium $E_3(\bar{S}, \bar{I}, 0, \bar{V})$ is stable if and only if

$$ \sigma_1 > 0, \quad \sigma_2 > 0, \quad \sigma_3 > 0, \quad \sigma_1 \sigma_2 - \sigma_3 > 0. \quad (14) $$

**Proof** For stability of an equilibrium point, the characteristic equation should have roots with negative real parts.

At $E_3$, the Jacobian matrix is,

$$ L = [l_{ij}] = \begin{bmatrix} l_{11} - \frac{\bar{S}}{K} & -\alpha \bar{S} & -\lambda \bar{S} \\ \lambda \bar{I} & -\delta - d & \beta \bar{I} & \lambda \bar{S} \\ 0 & 0 & -d & 0 \\ 0 & \kappa \delta & 0 & -\mu \end{bmatrix} $$

where, $l_{11} = r(1 - \frac{2 \bar{S}}{K}) - \frac{\alpha a \bar{P}}{(a + \bar{S})^2} - \lambda \bar{V}$.

Thus for $\tau_1 = 0 = \tau_2$, the characteristic equation at $E_3$ becomes

$$ (\xi + d) \cdot (\xi^3 + \sigma_1 \xi^2 + \sigma_2 \xi + \sigma_3) = 0 \quad (15) $$

where,

$$ \sigma_1 = -(l_{11} + l_{22} + l_{33}), $$
$$ \sigma_2 = l_{11} l_{22} + l_{11} l_{33} + l_{22} l_{33} - l_{12} l_{21} - l_{13} l_{32} - l_{23} l_{32}, $$
$$ \sigma_3 = l_{12} l_{32} l_{21} - l_{11} l_{22} l_{33} - l_{13} l_{32} l_{21} + l_{1} l_{13} l_{32} l_{23}. $$

Since one of the roots of (15) is $-d < 0$. Now for all parameter values, using Routh–Hurwitz criterion [31], we can say that the characteristic Eq. (18) has roots with negative real parts if the following conditions are satisfied

$$ \sigma_1 > 0, \quad \sigma_2 > 0, \quad \sigma_3 > 0, \quad \sigma_1 \sigma_2 - \sigma_3 > 0. \quad (16) $$

4.3 Stability of $E^*$ without delay

For stability and Hopf bifurcation of $E^*$, we have the following theorem.

**Theorem 4** The endemic $E^*$ with $\tau_1 = 0 = \tau_2$ is stable if
A_2 > 0, \ A_3 > 0, \ A_4 > 0, \ A_1 A_2 - A_3 > 0, \\
A_1 A_2 A_3 - A_2^2 - A_4 A_1^2 > 0.

It undergoes a Hopf bifurcation at \( \lambda = \lambda^* \in (0, \infty) \) if and only if
\[
A_2(\lambda^*) > 0, \quad A_3(\lambda^*) > 0, \quad A_4(\lambda^*) > 0, \\
A_1(\lambda^*) A_2(\lambda^*) - A_3(\lambda^*) > 0, \\
\psi(\lambda^*) = 0, \quad \text{and } A_1^2 A_2^2 A_3(1 - 3A_3) \\
\neq (A_2 A_1^2 - 2A_3^2)(A_3^2 - A_1^2 A_3^2).
\] (17)

where \( \psi(\lambda) \) is a continuously differentiable function \( \psi: (0, \infty) \rightarrow \mathbb{R} \) of \( \lambda \) as follows,
\[
\psi(\lambda) := A_1(\lambda) A_2(\lambda) A_3(\lambda) - A_2^2(\lambda) - A_4(\lambda) A_1^2(\lambda).
\]

Moreover, at \( \lambda = \lambda^* \), two eigenvalues \( \rho(\lambda) \) of the characteristic equation are purely imaginary, and the other two have purely imaginary parts. Primes denote differentiation concerning the parameter \( \lambda \).

Proof For \( \tau_1 = 0 = \tau_2 \), the characteristic equation at the endemic equilibrium \( E^* \) is
\[
H(\xi) = \xi^4 + A_1 \xi^3 + A_2 \xi^2 + A_3 \xi + A_4 = 0. \quad (18)
\]

For stability of an equilibrium point, the characteristic equation should have roots with negative real parts. Now, for all parameter values, using Routh–Hurwitz criterion [31], we can say that the characteristic Eq. (18) has roots with negative real parts if the following conditions are satisfied:
\[
A_1 > 0, \ A_2 > 0, \ A_3 > 0, \ A_4 > 0, \ A_1 A_2 \\
- A_3 > 0, \ A_1 A_2 A_3 - A_2^2 - A_4 A_1^2 > 0, \quad (19)
\]
where,
\[
A_1 = l_1 + a_1, \ A_2 = l_2 + b_1 + a_2, \\
A_3 = l_3 + b_2 + a_3 + q_1, \\
A_4 = l_4 + b_3 + a_4 + q_2.
\] (20)

Now, we prove the existence of Hopf bifurcation.

Using conditions (17), the characteristic Eq. (18) can be equivalently rewritten in the form
\[
\left( \xi^2 + \frac{A_3}{A_1} \right) \left( \xi^2 + A_1 \xi + \frac{A_1 A_4}{A_3} \right) = 0.
\] (21)

Two roots of this equation are given by
\[
\xi_{1,2} = \pm i \omega_0, \quad \omega_0 = \sqrt{\frac{A_3}{A_1}},
\]
while the other two roots, \( \xi_3 \) and \( \xi_4 \) satisfy the equation
\[
\xi^2 + A_1 \xi + \frac{A_1 A_4}{A_3} = 0,
\]
and from the Routh–Hurwitz criterion, they both have a negative real part.

To verify the transversality condition, we first note that \( \Phi(\lambda^*) \) is a continuous function of its argument, and hence, there exists an open interval \( \lambda \in (\lambda^* - \epsilon, \lambda^* + \epsilon) \) where \( \xi_1 \) and \( \xi_2 \) are complex conjugate roots of the characteristic equation, which can be written in the general form as
\[
\xi_{1,2}(\lambda) = \xi(\lambda) \pm i \nu(\lambda),
\]
with \( \xi_{1,2}(\lambda^*) = \pm i \nu_0 \). Substituting \( \xi_j(\lambda) = \xi(\lambda) \pm i \nu(\lambda) \) into the characteristic Eq. (18), differentiating with respect to \( \lambda \), and separating real and imaginary parts gives
\[
P(\lambda) \xi' - Q(\lambda) v'(\lambda) + R(\lambda) = 0, \\
Q(\lambda) \xi' + P(\lambda) v'(\lambda) + S(\lambda) = 0, \quad (22)
\]
where
\[
P(\lambda) = 4 \xi^3 - 12 \xi v^2 + 3 A_1 (\xi^2 - v^2) + 2 A_2 \xi + A_3, \\
Q(\lambda) = 12 \xi^2 v + 6 A_1 \xi v - 4 \xi^3 + 2 A_2 \xi, \\
R(\lambda) = A_1 \xi^3 - 3 A_1' \xi v^2 + A_2' (\xi^2 - v^2) + A_3', \\
S(\lambda) = 3 A_1^2 \xi^2 v - A_1' v^3 + 2 A_2' \xi v + A_3' \xi.
\]
Solving the (22) for \( \xi'(\lambda^*) \) and using the condition (17) yields
\[
\left[ \frac{d \text{Re}[\xi_j(\lambda)]}{d \lambda} \right]_{\lambda = \lambda^*} = \xi'(\lambda^*)
\]
\[
= - \frac{Q(\lambda^*) S(\lambda^*) + P(\lambda^*) R(\lambda^*)}{P^2(\lambda^*) + Q^2(\lambda^*)} \\
= \frac{A_1^2 A_3^2 A_3(A_1 - 3A_3) - 2(A_2 A_1^2 - 2A_3^2)(A_1 A_1^2 - A_1 A_3^2)}{A_1^2 (A_1 - 3A_3)^2 + 4(A_2 A_1^2 - 2A_3^2)^2} \\
\neq 0.
\]

Thus, the transversality condition holds, and consequently, a Hopf bifurcation occurs at \( \lambda = \lambda^* \).

Remark 1 Replacing \( \lambda \) in the above analysis by any model parameter (for example, \( \alpha, \beta \), etc.), the Hopf bifurcation of endemic equilibrium point can be investigated at the critical value of that parameter.

5 Stability analysis with delay

In this section, we have only provided the local stability of the equilibria \( E_3 \) and \( E^* \), respectively, with delay, i.e., when \( \tau_1 > 0 \) and \( \tau_2 > 0 \).
5.1 Stability of $E_3$ with delay

For the delay model, the characteristic Eq. (23) is transcendental in $\xi$ with infinitely many roots. Any steady state, $E(S, I, P, V)$, is locally stable (or unstable) if all the roots of the corresponding characteristic equation have negative real parts (or have positive real parts). Also, Hopf-bifurcating periodic solution persists if at least one purely imaginary root occurs. We shall examine the possible existence of Hopf bifurcation.

Now, if $S_3 < 0$ holds, then Eq. (27) has at least one positive root. Let $\theta$ be the least positive root of (27) for which the characteristic Eq. (23) has pair of purely imaginary roots $\pm i\sqrt{\theta}$.

From the Eq. (25), writing $\sqrt{\theta} = w$, we get the values of $\tau_1$ for which (27) has positive roots is

$$
\tau_1 = \frac{1}{w} \cos^{-1} \left[ \frac{\tilde{a}_4 w^4 - (\tilde{a}_2 \tilde{a}_4 - \tilde{a}_1 \tilde{a}_3) w^2 - \tilde{a}_3 \tilde{a}_5}{\tilde{a}_4^2 w^2 + \tilde{a}_3^2} \right] + \frac{2k\pi}{w}, \quad k = 0, 1, 2, \ldots, \tag{29}
$$

Let

$$
\tau_{10}^* = \min_{k \geq 0} \{ \tau_k^* \}, \quad w_0 = w(\tau_{10}^*).
$$

Then, we have the following proposition.

**Proposition 3** Suppose that the interior equilibrium point $E^*$ exists and is locally asymptotically stable for $\tau_1 = 0$, i.e., (19) is satisfied. Now, if either, $S_3 < 0$, then $E^*$ is asymptotically stable when $\tau < \tau_{10}^*$ and unstable when $\tau > \tau_{10}^*$, where

$$
\tau_{10} = \frac{1}{w_0} \arccos \left[ \frac{\tilde{a}_4 w_0^4 - (\tilde{a}_2 \tilde{a}_4 - \tilde{a}_1 \tilde{a}_3) w_0^2 - \tilde{a}_3 \tilde{a}_5}{\tilde{a}_4^2 w_0^2 + \tilde{a}_3^2} \right]. \tag{30}
$$

Thus, when $\tau_1 = \tau_{10}^*$, Hopf bifurcation occurs, i.e., a family of periodic solution bifurcates at $E^*$ as $\tau$ passes through the critical value $\tau_{10}^*$ provided the transversality condition, $3\omega_0^4 + 2S_1 \omega_0^2 + S_2 \neq 0$ is satisfied.

**Proof** Clearly, for the proof of the theorem we have to verify only the transversality condition.

Denoting $\xi = \xi(\tau_1)$ and differentiating (23), we obtain

$$
\left( \frac{d\xi(\tau_1)}{d\tau_1} \right)_{\tau_1 = \tau_{10}^*}^{-1} = -\frac{3\xi^2 + 2\tilde{a}_1 \xi + \tilde{a}_2}{\xi(\tilde{a}_4 \xi + \tilde{a}_3) - \frac{\tau_{10}^*}{\xi}},
$$

which leads to

$$
\frac{\xi(\tau_1)}{\xi(\tau_{10}^*)} = \frac{\tau_{10}^*}{\tau_1}
$$

$$
\frac{\xi(\tau_1)}{\xi(\tau_{10}^*)} = \frac{\tau_{10}^*}{\tau_1}
$$

Putting $\theta = w^2$, the Eq. (26) is reduced to

$$
H(\theta) = \theta^3 + S_1 \theta^2 + S_2 \theta + S_3 = 0, \tag{27}
$$

where,

$$
S_1 = \tilde{a}_1^2 - 2\tilde{a}_2, \quad S_2 = \tilde{a}_2^2 - 2\tilde{a}_1 \tilde{a}_5 - \tilde{a}_4^2, \quad S_3 = \tilde{a}_3^2 - \tilde{a}_5^2. \tag{28}
$$

\[ \square \]
Remark 2 Stability of $E_3$ for $\tau_1 = 0, \tau_2 > 0$ and for $\tau_1 > 0, \tau_2 > 0$ can be studied using the analysis in the following sub-section.

5.2 Stability of $E^*$ with delay

In this section, we check the stability switches of endemic equilibrium for with delay, i.e., when $\tau_1 > 0$ and $\tau_2 > 0$. We will study three cases, namely (i) $\tau_1 > 0, \tau_2 = 0$, (ii) $\tau_1 = 0, \tau_2 > 0$ and (iii) $\tau_1 > 0, \tau_2 > 0$.

(i) For $\tau_1 > 0, \tau_2 = 0$, the characteristic equation at the endemic equilibrium $E^*$ (derived from (12)) is

$$\phi(\xi, \tau_1, 0) = \xi^4 + C_1 \xi^3 + C_2 \xi^2 + C_3 \xi + C_4 + e^{-\xi \tau_1} [D_1 \xi^2 + D_2 \xi + D_3] = 0,$$  \hspace{1cm} (33)

where,

$$C_1 = (a_1 + l_1), \quad C_2 = (a_2 + l_2), \quad C_3 = (a_3 + l_3),$$

$$C_4 = (a_4 + l_4),$$

$$D_1 = b_1, \quad D_2 = b_2 + q_1, \quad D_3 = b_3 + q_2.$$

The terms $a_1, a_2, ..., a_4, b_1, b_2$, etc., are given in Appendix-A.

For the stability changes to occur, we have to show that there exists a pair of purely imaginary roots of the characteristic equation for a critical value of $\tau_1 > 0$. Suppose that there exists a purely imaginary root say $i \theta$ of (33).

We substitute $\xi = i \theta$ in (33) and then separating real and imaginary parts we finally obtain

$$D_1 \sin \theta \tau_1 + D_2 \cos \theta \tau_1 = -\theta^4 + C_2 \theta^2 - C_4, \quad (34)$$

$$D_1 \cos \theta \tau_1 - D_2 \sin \theta \tau_1 = C_1 \theta^3 - C_3 \theta.$$ \hspace{1cm} (35)

First, we take the square and then add the above two equations to get,

$$l^8 + \omega_1 l^6 + \omega_2 l^4 + \omega_3 l^2 + \omega_4 = 0.$$ \hspace{1cm} (36)

Again we substitute $\theta^2 = l$ in (36) and get the following equation

$$l^4 + \omega_1 l^3 + \omega_2 l^2 + \omega_3 l + \omega_4 = 0.$$ \hspace{1cm} (37)

The coefficients of (37) are

$$\omega_1 = C_1^2 - 2C_2, \quad \omega_2 = C_2^2 + 2C_4 - 2C_1 C_3,$$

$$\omega_3 = -2C_2 C_4 + C_3^2 - D_1^2, \quad \omega_4 = C_4^2 - (D_1^2 + D_2^2).$$

Let us define $H(l)$ by

$$H(l) = l^4 + \omega_1 l^3 + \omega_2 l^2 + \omega_3 l + \omega_4.$$ 

Given that $H(0) = \omega_4 < 0$ and also note that $\lim_{l \to \infty} H(l) = \infty$. Hence, there exists an $l_0 \in (0, \infty)$ so that $H(l_0) = 0$. Thus, we have the following lemma.

**Lemma 1** The Eq. (37) has at least one positive root when $\omega_4 < 0$.

Without loss of generality, we assume that the Eq. (37) has four positive roots. We denote them as $l^*_r, r = 1, 2, 3, 4$. Then, (36) also has four positive roots, $\theta_r = \sqrt{\frac{\omega_2}{\omega_1}}, r = 1, 2, 3, 4$. From Eq. (34), the values of $\tau_1$ are calculated as

$$\tau_1^n = \frac{1}{\theta_r} \cos^{-1} \left[ \frac{\theta_r^2 D_2 C_1 \theta_0^2 - C_3 + (D_1 \theta_0^2 - D_3)(\theta_0^2 - C_2 \theta_0^2 + C_4)}{(D_1 \theta_0^2 - D_3)^2 + D_2^2 \theta_0^2} \right]$$

$$+ \frac{2\pi n}{\theta_r}, \quad r = 1, 2, 3, 4 \text{ and } n = 0, 1, 2, 3, ...$$ \hspace{1cm} (38)

Thus, we establish that $\pm \hat{i} \theta_0$ is a pair of purely imaginary roots of (33). Let us write the following

$$\tau_1^* = \tau_{1l^*_r} = \min_{n \geq 0, 1 \leq r \leq 4} \{\tau_1^n\}, \quad \theta_0 = \theta_{l^*_r}, \quad l_0 = l^*_{r_0}. \quad (39)$$

Then, we can have the following results.

**Theorem 5** Suppose that the $E^*$ is stable without delay. Then, for $\omega_4 < 0$, the steady state $E^*$ is locally asymptotically stable for $\tau_1 < \tau_1^*$, and it is unstable when $\tau_1 > \tau_1^*$. Furthermore, the occurrence of Hopf bifurcation at $E^*$ is confirmed when $\tau_1 = \tau_1^*$ provided that the transversality condition $4\theta_0^2 + A_1 \theta_0^3 + A_2 \theta_0^2 + A_3 \neq 0$ is satisfied.

**Proof** First part of the theorem is immediate from the above analysis. Thus, we have to prove the last part of the theorem only.

We Differentiate (33) with respect to $\tau_1$ to obtain

$$\frac{d\tau_1}{d\xi} = \frac{4\xi^3 + 3(C_1 \xi^2 + 2C_2 \xi + C_3)}{D_2 \xi^3 + D_3 \xi^2 + D_4 \xi} e^{\xi \tau_1}$$

$$+ \frac{2 D_2 \xi^3 + D_3 \xi^2 + D_4 \xi}{D_2 \xi^3 + D_3 \xi^2 + D_4 \xi} \frac{\tau_1}{\xi^2}.$$ 

Considering Eq. (34), it is easy to show the following result,

$$Sgn \left[ \frac{d(Re \xi)}{d\tau_1} \right]_{\tau_1 = \tau_1^*} = Sgn [Re(\frac{d\xi}{d\tau_1})_{\tau_1 = \tau_1^*}],$$

\hspace{1cm} Springer
Table 1 Values of parameters used in numerical calculation for system

| Parameter | Short description | Values |
|-----------|-------------------|--------|
| $r$       | Growth rate of pest | 0.1    |
| $K$       | Maximum density of pest | 1      |
| $\lambda$ | Infection rate of susceptible pest | 0–0.2 |
| $d$       | Natural mortality rate of pest | 0.012 |
| $\delta$  | Additional mortality of infected pest | 0.02  |
| $\alpha$  | Predator catching rate of susceptible pest | 0.01  |
| $\beta$   | Predator catching rate of infected pest | 0.09  |
| $n$       | Lysis of predator due to competition | 5     |
| $m$       | Conversion factor for predator | 0.6   |
| $\mu$     | Decay rate of virus | 0.1   |
| $a$       | Half-saturation constant | 0.5   |

\[
= \text{Sgn} \left[ \frac{4\xi_1^6 + \omega_1\xi_1^4 + \omega_2\xi_1^2 + \omega_3}{D_3^2 + [-D_2\xi_2^2 + D_4]^2} \right], \quad (40)
\]

Since $D_3^2 + [-D_2\xi_2^2 + D_4]^2$ is always positive. Also, from Eq. (39) we see that $H'(l_0) \neq 0$ if $4\xi_0^6 + \omega_1\xi_0^4 + \omega_2\xi_0^2 + \omega_3 \neq 0$. Thus, it follows that

\[
\text{Sgn} \left[ \frac{d(Re \xi)}{d\tau_1} \right]_{\tau_1=\tau^*} \neq 0.
\]

(ii) The analysis for $\tau_2 > 0$, $\tau_1 = 0$ is given in Appendix B. (iii) For $\tau_1 > 0$, $\tau_2 > 0$, we provide the result without proof in the following theorem.

**Theorem 6** Suppose that the non-delayed system is asymptotically stable for $\tau_2 \in (0, \tau_2^*)$. Now, if $\omega_4 < 0$ holds, then there exists $\tau^*$ for which stability switch occurs at $E^*$ when $\tau_1$ passes the critical value $\tau^*$. Furthermore, $E^*$ will undergo a Hopf bifurcation when $\tau_1 = \tau^*$, provided that

\[
\frac{d(Re \xi)}{d\tau_1} \bigg|_{\tau_1=\tau^*} > 0.
\]

In this case, the characteristic equation is

\[
\phi(\rho, \tau_1, \tau_2) = \rho^4 + l_1\rho^3 + l_2\rho^2 + l_3\rho + l_4 + e^{-\rho\tau_1}(b_1\rho^2 + b_2\rho + b_3) + e^{-\rho\tau_2}(a_1\rho^2 + a_2\rho + a_3) + e^{-\rho(\tau_1+\tau_2)}(q_1\rho^2 + q_2\rho + q_3) = 0,
\]

(41)

Thus, comparing the results in [32] (Theorem 5 and Theorem 6 of [32]), a formal proof of the Theorem 6 can be analyzed. A similar result can be provided for $\tau_1$ when $\tau_2 \in (0, \tau_2^*)$.

6 Numerical simulations

This section provides numerical examples to analyze the results and interpretations we have made in the previous sections to gain a better understanding of how different parameters affect the dynamics. We examine the stability of the equilibrium points numerically and solve the system ((1)) to illustrate various types of behavior.

Numerical solution of the model system without delay is plotted in Fig. 1 for different value of $\lambda$ (the rate of infection of susceptible pest by biopesticides). From this figure, it is confirmed that for $\lambda = 0.012$ the system is asymptotically stable (confirmed by Theorem 4). All the system populations oscillate initially but finally converges to the endemic equilibrium $E^*(2.212, 0.8425, 22.67, 6.71)$. For the higher value of the infection rate $\lambda = 0.01388$, stable periodic orbit is observed. Hopf bifurcation diagram of coexisting equilibrium $E^*$ is plotted in Fig. 2. It indicates that system population bifurcates into the periodic solution at $E^*$ when $\lambda$ crosses its critical value $\lambda^* = 0.01388$. It is also obvious that for higher values of $\lambda$, $E^*$ is not stable, but $E_3$ exists and is unstable. Another bifurcation diagram of endemic equilibrium is plotted in Fig. 3 for $\beta$ (consumption rate of predator). From this figure, we observe that for higher values of the parameter,
coexisting equilibrium $E^*$ is stable. We conclude that the stabilizing nature of the parameter is seen.

In Fig. 4, bifurcation diagram of the system is plotted for the value of delay $\tau_1$. The system (1) has periodic orbits when $\tau_1$ lies in the range $\tau_1 \in (0, 50)$. That is endemic equilibrium $E^*$ is stable for higher values of delay $\tau_1$. It is clear from Fig. 4 that the stability switches of the endemic equilibrium $E^*$ in terms of time delay $\tau_1$ at two points, nearly at $\tau \approx 3.86$ days and $\tau \approx 41.25$ days from stable to unstable and from unstable to stable, respectively. Bifurcation plot of Fig. 5 shows that gestation delay destabilizes the system when its value is larger than $\tau^{*}_1 = 4.78$.

The region of stability of endemic equilibrium $E^*$ is presented in Fig. 6. In Fig. 6a, stability of $E^*$ is shown in $\beta - \lambda$ parameter plane. We have observed the
stabilizing role of $\beta$. It can be recognized that for lower values of both the parameters, $E^*$ is not feasible. For the higher value of $\lambda$, $E^*$ is unstable but if the value of $\beta$ crosses a threshold value, $E^*$ becomes stable. Thus, the critical value $\beta^*$ is dependent on the value of $\lambda$ and vice versa. Figure 6b shows that the effects of gestation delay $\tau_2$ are also dependent on other model parameters, for example, virus replication rate $\kappa$. It can be observed that for a larger value of $\kappa$, endemic equilibrium does not exist. In Fig. 6c, the region of stability in $\tau_1 - \tau_2$ parameter plane is shown. This figure contains the combined effect of the two delays as discussed in Figs. 4 and 5.
7 Discussion and conclusion

In this paper, a mathematical model using delay differential equations has been derived for the dynamics of integrated pest management (IPM) system using biopesticide and predatory insects as managing agents. In this system, biopesticides (generally virus) attack the susceptible pest and make it infected after a latent period. We consider this period as latent delay $\tau_1$. We have also introduced predatory insects population in the model system. Infected pest are less harmful, and also, predators can consume the infected easily than susceptible pest.

We investigated the model system from the perspective of stability and persistence. We have explained the nonnegativity and boundedness of solutions of the proposed delayed model. We have obtained five equilibrium points. Using Routh–Hurwitz criteria, we have derived their stability conditions. Numerically we have seen that the stability changes of endemic equilibrium point occurred through Hopf bifurcation. Finally, we have assumed that the infection of pests by biopesticide is a delayed process and modified the model accord-
ingly. It is seen that the dual role of latent delay. It sometimes stabilizes the, but harmful for the system as the infected pest density decreases for increasing latent delay. In that case, we have to increase the rate of biopesticides release. Sometimes it is suggested to use chemical pesticides, i.e., pest culling [5,29]. Also, we have seen the destabilizing role of gestation delay. Consequently, incorporation of additional food is recommended [24].

In conclusion, our work shows that spraying of pesticides along with the predatory insect is successful if this integrated approach is used correctly, i.e., considering the resistance in terms of latency of pest and gestation period (or maturation period of a predator). Hence, time delay should be included in the modeling of a pest management system.

The present work can be extended in many ways. It can also be interesting how the enrichment of additional food for predator can effect the system. The pest maturation delay effect can also be studied.

Data Availability The data used for this research are included within the article.

Declarations

Conflict of Interest Authors declare that they have no conflict of interest.

A Appendix

The coefficients of (12) are given below:

\begin{align*}
l_1 &= -F_{11} - F_{22} - F_{33} - F_{44} \\
l_2 &= F_{11}F_{22} + F_{11}F_{33} + F_{22}F_{33} + F_{11}F_{44} \\
&+ F_{22}F_{44} + F_{33}F_{44} \\
l_3 &= -F_{11}F_{22}F_{33} - F_{11}F_{22}F_{44} - F_{11}F_{33}F_{44} \\
&- F_{22}F_{33}F_{44} \\
l_4 &= F_{11}F_{22}F_{33}F_{44}, \\
b_1 &= -F_{12}G_{21} - F_{42}G_{24} \\
b_2 &= F_{12}F_{33}G_{21} - F_{14}F_{42}G_{21} + F_{12}F_{44}G_{21} \\
&+ F_{11}F_{42}G_{24} + F_{33}F_{42}G_{24} \\
b_3 &= F_{14}F_{33}F_{42}G_{21} - F_{12}F_{33}F_{44}G_{21} - F_{11}F_{33}F_{42}G_{24}, \\
a_1 &= -H_{33} \\
a_2 &= -F_{13}H_{31} - F_{23}H_{32} + F_{11}H_{33} + F_{22}H_{33} + F_{44}H_{33} \\
a_3 &= F_{13}F_{22}H_{31} - F_{12}F_{23}H_{31} + F_{13}F_{44}H_{31} + F_{11}F_{23}H_{32} + F_{23}F_{44}H_{32} - F_{11}F_{22}H_{33} - F_{11}F_{44}H_{33} - F_{22}F_{44}H_{33} \\
a_4 &= -F_{14}F_{23}H_{31} - F_{13}F_{22}F_{44}H_{31} + F_{11}F_{22}F_{44}H_{33} - F_{11}F_{23}F_{44}H_{32} + F_{12}F_{23}F_{44}H_{31} + F_{13}F_{42}G_{24}H_{31}, \\
n_1 &= -F_{13}G_{21}H_{32} + F_{12}G_{21}H_{33} + F_{42}G_{24}H_{33} \\
n_2 &= F_{13}F_{44}G_{21}H_{32} + F_{14}F_{42}G_{21}H_{33} - F_{12}F_{44}G_{21}H_{33} - F_{11}F_{44}G_{24}H_{33} - F_{11}F_{24}G_{24}H_{33}.
\end{align*}

B Appendix

For \( \tau_2 > 0 \) and \( \tau_1 = 0 \), the characteristic equation becomes

\[
\psi(\xi, \tau) = \xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4 + e^{-\xi\tau}[b_1\xi^3 + b_2\xi^2 + b_3\xi + b_4].
\] (42)

For \( \tau > 0 \), then (42) will have infinitely many roots. To determine the nature of the stability, the sign of the real parts of the roots of the characteristic Eq. (42) is required. A necessary condition for a stability changes of \( E^* \) is that the characteristic Eq. (42) should have purely imaginary solutions. Let \( i\xi \) be a root of Eq. (42) and from which we get,

\[
\xi^4 - a_2\xi^2 + a_4 = [\xi^2b_2 - b_4]\cos\xi\tau - [\xi b_3]\sin\xi\tau,
\]

\[
a_1\xi^3 - a_3\xi = [\xi^2b_2 - b_4]\sin\xi\tau + [\xi b_3]\cos\xi\tau.
\]

Squaring and adding above two equations,

\[
\xi^8 + a_1\xi^6 + a_2\xi^4 + a_3\xi^2 + a_4 = 0.
\] (44)

Simplifying and substituting \( \xi^2 = l \) in Eq. (44), we get the following equation

\[
l^4 + a_1l^3 + a_2l^2 + a_3l + a_4 = 0.
\] (45)

The roots of Eq. (45) have negative real parts if and only if its coefficients satisfy the Routh–Hurwitz criterion. In such case, (42) does not have purely imaginary roots. Thus, we summarize the results in the following proposition.

**Proposition 4** Suppose that the system without delay is stable. The endemic equilibrium \( E^* \) is LAS for all \( \tau_2 > 0 \) if the following conditions are satisfied:

\[
\omega_1 > 0, \quad \omega_4 > 0, \quad \omega_1\omega_2 - \omega_3 > 0, \quad (\omega_1\omega_2 - \omega_3)\omega_3 - \omega^2_1\omega_4 > 0.
\]
If \( \omega_4 < 0 \) holds, then Eq. (45) will admit at least one positive root. If \( \theta_0^* \) is the minimum positive root of (45), then \( \theta \) will be a purely imaginary root, \( \pm i\theta_0 \) corresponding to the delay \( \tau_2 \), then the endemic equilibrium \( E^* \) remains stable for \( \tau_2 < \tau_2^* \).

We can evaluate the critical value of \( \tau_2 \) for which the endemic equilibrium \( E^* \) remains stable. From Eq. (43),

\[
\tau_2^* = \frac{1}{\theta_0} \cos^{-1} \left[ \frac{b_2(-\theta_0^4 + a_2\theta_0^2 - a_4) + b_1a_1\theta_0^3}{b_1^2 + b_2^2} \right] + \frac{2\pi n}{\theta_0}, \quad n = 0, 1, 2, 3, \ldots
\]

From the above analysis, the following theorem follows.

**Theorem 7** If \( \omega_4 < 0 \) is satisfied, then the steady state \( E^* \) is LAS for \( \tau_2 < \tau_2^* \) and becomes unstable for \( \tau_2 > \tau_2^* \). Furthermore, the system will undergo a Hopf bifurcation at \( E^* \) at \( \tau_2 = \tau_2^* \) provided

\[
4\theta_0^6 + A_1\theta_0^4 + A_2\theta_0^2 + A_3 \neq 0,
\]

where,

\[
A_1 = 3a_1 - 6a_2, \quad A_2 = 2a_2 + 4a_4 - 4a_1a_3, \\
A_3 = a_3^2 - 2a_2a_4 - b_1^2.
\]

**Proof** We need to prove the last conditions only. Now, differentiating (42) with respect to \( \tau_2 \) we get:

\[
\frac{d\tau_2}{d\xi} = 4\xi^3 + 3a_1\xi^2 + 2a_2\xi + a_3 \frac{e^{\xi\tau_2}}{b_1\xi^2 + b_2} + \frac{\tau_2}{\xi}.
\]

Now, using the relation (43) one can obtain:

\[
\text{sgn} \left[ \frac{d(Re\xi)}{d\tau_2} \right]_{\tau_2 = \tau_2^*} = \text{sgn} \left[ \text{Re} \left( \frac{d\xi}{d\tau_2} \right) \right]_{\xi = i\theta_0} = \text{sgn} \left[ \frac{4\theta_0^6 + A_1\theta_0^4 + A_2\theta_0^2 + A_3}{b_1\theta_0^2 + b_2^2} \right]
\]

and the latter is positive if \( 4\theta_0^6 + A_1\theta_0^4 + A_2\theta_0^2 + A_3 \neq 0 \), i.e., the transversality condition holds and the system undergoes Hopf bifurcation at \( \tau_2 = \tau_2^* \). \( \square \)

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