Kolmogorov and von Mises viewpoints to the Greenburger-Horne-Zeilinger paradox

Andrei Khrennikov
Department of Mathematics, Statistics and Computer Sciences
University of Växjö, S-35195, Sweden

January 2, 2022

Abstract

We present comparative probabilistic analysis of the Greenburger-Horne-Zeilinger paradox in the frameworks of Kolmogorov’s (measure-theoretical) and von Mises’ (frequency) models of the probability theory. This analysis demonstrated that the GHZ paradox is merely a consequence of the use of Kolmogorov’s probabilistic model. By using von Mises’ frequency approach we escape the contradiction between the local realism and quantum formalism. The frequency approach implies automatically contextual interpretation of quantum formalism: different collectives induce different probability distributions. On the other hand, the formal use of Kolmogorov’s model implies the identification of such distributions with one abstract Kolmogorov measure. In the measure-theoretical approach we can escape the paradox, if we do not suppose that probability distributions corresponding to different settings of measurement devices are equivalent. We discuss the connection between equivalence/singularity dichotomy in measure theory and the existence of compatible and noncompatible observables.

1 Introduction

It is well known that violations of Bell’s inequality [1] by quantum correlations in the Einstein-Podolsky-Rosen (EPR) framework may be interpreted...
as the evidence of the impossibility to use the *local realism* in quantum theory (see, for example, [2], [3]). Such a viewpoint was strongly supported by experiments of Aspect [2] which demonstrated violations of generalized Bell’s inequality (see also [3]). Despite of the general attitude to connect violations of Bell’s inequality with such problems as *determinism and locality*, there exists sufficiently strong opposition [4]-[9] to such a conclusion. This opposition, despite of the great diversity of approaches, can be called the *probability opposition*. The general viewpoint of adherents of the probabilistic interpretation of violations of Bell’s inequality is that the derivation of this inequality is based on numerous (hidden) probabilistic assumptions. Unfortunately at the present time there are no experimental facts which can justify these probabilistic assumptions. It seems that theoretical as well as experimental investigations of the EPR paradox (in particular, Bell’s inequality) must be at least partly reoriented to the investigation of probabilistic roots of this paradox.

The viewpoint that the notion of probability plays the large role in Bell’s (and in EPR’s) considerations is not so new, [4]-[9]. The main consequence of all these probabilistic analyses is that the EPR experiment could not be described (as it was assumed by J. Bell, [1]) by the unique Kolmogorov probability distribution. In fact, these are various forms of contextual interpretation of quantum formalism): (1) De Broglie, Lochak, Nelson, De Muynck, De Baere, Marten, Stekelenborg, [4], thermodynamical approach to Bell’s problem, difference between hidden and observed probabilities; (2) Beltrametti and Cassinelli [5], quantum logic; (3) Accardi [6], quantum probabilities, no Bayes’ formula; (4) Pitowsky and Gudder [7], probability manifolds; (5) De Baere [7], fluctuating probabilities; (6) Fine and Rastal [7], no simultaneous probability distribution; (7) Muckenheim [7], negative probabilities; (8) Khrennikov [8], p-adic probabilities; fluctuating probabilities and modified Bell’s inequality [9].

However, a new strong argument in the favour of nonlocal (or nonreal) interpretation of the EPR paradox [10] was given by so called Greenberger-Horne-Zeilinger (GHZ) paradox, [11]. The GHZ scheme is based on the probability one arguments. From the first point of view all probabilistic circumstances of the GHZ scheme are so straightforward that there is no more place for probabilistic counter arguments. However, the careful probabilistic analysis demonstrates that the GHZ paradox has even deeper connection to foundations of probability theory than Bell’s inequality. Roughly speaking the root of the GHZ paradox might be in the use of the conventional
probability calculus, namely Kolmogorov’s (axiomatic) measure theoretical approach, 1933, [12].

In this paper we shall consider the GHZ paradox from the viewpoint of so-called frequency probability theory, R. von Mises, 1919 [13] (see [14] for the advanced formalism). In the opposite to Kolmogorov’s model of probability theory which is characterized by the highest degree of abstraction, von Mises’ model of probability theory is characterized by its concreteness. By R. von Mises we cannot consider a probability distributions without the relation to the concrete collective (random sequence). Von Mises’ slogan was: ”first collective and then probability distribution”. Analysis of the GHZ paradox based on von Mises’ approach demonstrated that it is rather doubtful that there exists a collective which produces the probability distribution which is formally (via Kolmogorov’s approach) used in the GHZ considerations. Hence if we use a mathematical model of probability theory which is different from Kolmogorov’s model, namely von Mises’ model, we observe no paradox in the GHZ considerations. In particular, there is no contradiction between the quantum formalism with the frequency interpretation of probability and local realism.

Of course, our probabilistic considerations could not be considered as arguments in favour of either locality or determinism. It may be that physical reality is nonlocal or even nonreal. However, Bell’s as well as GHZ’s approaches do not give definite arguments to deny locality or determinism. Both these approaches are strongly based on the use of one particular model of probability theory, Kolmogorov’s model.

Our frequency analysis clarifies the measure-theoretical roots of the GHZ paradox. In fact, this paradox can be escaped even in the measure-theoretical approach if it would not be assumed that probability distributions corresponding to different settings of measurement devices are equivalent measures. We discuss the connection between equivalence/singularity dichotomy in measure theory and the existence of compatible and incompatible observables. It seems that the splitting of physical reality to classical and quantum realities is just a consequence of the general (mathematical) property of probability measures. So it is just a property of our (mathematical) description of physical reality.

The Kolmogorov definition of a probability space is well known [12], [15]. This is a triple \((\Omega, F, \mathbf{P})\), where \(\Omega\) is an abstract set, \(F\) is a \(\sigma\)-field of subsets (events) of \(\Omega\), \(\mathbf{P}\) is the probability (normalized by 1 and \(\sigma\)-additive) measure on \(F\). On the other hand, the frequency probability theory of R. von Mises is
now days practically forgotten. So we must present an extended introduction to this approach, see section 2.

We must remark that von Mises’ theory was strongly criticized due to rather informal definition of randomness, [16]. In fact, this purely mathematical critique was one of the reasons to eliminate the frequency approach from quantum formalism. We do not relate our use of frequency theory to sophisticated mathematical problems of randomness [16]. There are two main reasons to eliminate the problem of randomness from physical considerations and justify the use of the frequency formalism. The first is von Mises’ observation that the class of place selections must be determined not by some mathematical theory, but by the concrete physical phenomenon. This viewpoint is supported by Wald’s theorem [17] by that if we fix a countable set of place selections, then there exist sufficiently many collectives with respect to this set of place selections. The second is my own observation that it seems to be that the property of randomness is not related (at least directly) to physical measurements (at least for present experiments). We are always interested only in one property of a sequence of observations: the statistical stabilization of relative frequencies \( \nu_N = \frac{n}{N} \) to some limiting quantities \( P \) (probabilities).

2 Frequency probability theory

2.1. History. The frequency probability theory was developed by R. von Mises in 1919 (see [13], [14], [9] for the details). In fact, the basis of the frequency approach was provided in the work of J. Venn, 1866, see [18]. The frequency theory was used as the motivation of Kolmogorov’s axiomatic, 1933, of the conventional probability theory (see remarks in [12]). The main advantage of the conventional theory is its abstractness. Here we work with abstract probability distributions which are not directly related to the concrete physical model. Thus results of the conventional probability theory can be used without any modification in any physical models. However, this advantage may become in some circumstances a disadvantage, because the abstractness of the formalism does not give the possibility to analyse the origin (and even the existence) of probability distributions. On the other hand, the frequency theory of probability is concrete. Here to introduce a probability distribution, we must be sure that there exists a collective (random sequence) which produces this probability distribution. The collective is more primary object than a probability distribution. The collective has more direct connection with a physical phenomenon. However, in the frequency approach
we cannot obtain results which are valid for ‘all probability distributions’. The probability distribution without a collective is nothing. Typically such a concreteness is considered as the large disadvantage of the frequency approach (comparing with the conventional measure theoretical approach). Of course, it is more attractive to prove some probabilistic statement ones and then to apply it to numerous physical models. This was one of the reasons to eliminate the frequency approach from applications in the favour of the measure-theoretical approach.

In the present paper we demonstrate that the frequency analysis of probabilistic assumptions for the derivation of Bell’s inequality can give some new sights to this problem. These sights would be impossible to obtain in the conventional abstract framework. Analysis of collectives can give more than analysis of abstract probability distributions.

2.2. Collective. Let \( \mathcal{E} \) be an ensemble of physical systems. We take elements of \( \mathcal{E} \) and form a sequence \( \pi = (\pi_1, \pi_2, ..., \pi_N, ...) \). Suppose that elements of \( \mathcal{E} \) have some properties. Suppose that these properties can be described by natural numbers, \( L = \{1, 2, ..., m\} \) (the set of ‘labels’). In principle we can consider continuous label sets, see [14]. Thus, for each \( \pi_j \in \pi \), we have a number \( \alpha_j \in L \). So \( \pi \) induces a sequence

\[
x = (\alpha_1, \alpha_2, ..., \alpha_N, ...), \quad \alpha_j \in L.
\]

For each fixed \( \alpha \in L \), we have the relative frequency \( \nu_N(\alpha) = n_N(\alpha)/N \) of the appearance of \( \alpha \) in \( (\alpha_1, \alpha_2, ..., \alpha_N) \).

R. von Mises said that \( x \) satisfies to the principle of the statistical stabilization of relative frequencies, if, for each fixed \( \alpha \in L \), \( |\nu_N(\alpha) - \nu_M(\alpha)| \to 0, N, M \to \infty \). The corresponding limit

\[
p(\alpha) = \lim_{N \to \infty} \nu_N(\alpha)
\]

is said to be a probability. This probability can be extended to the field of all subsets of \( L \):

\[
p(B) = \lim_{N \to \infty} \nu_N(\alpha \in B) = \lim_{N \to \infty} \sum_{\alpha \in B} \nu_N(\alpha) = \sum_{\alpha \in B} p(\alpha), B \subset L
\]

(the situation becomes sufficiently complex for an infinite \( L \), see Tornier [11]). We remark that \( p(L) = 1 \).

1Another reason was the problem of the rigorous mathematical definition of a collective, random sequence, see, for example, [9].

2It is not important in general either these properties are objective (properties of an object) or ‘created’ in the process of observation by an observer, see [3].
R. von Mises said that \( x \) satisfies the principle of randomness if limits (2) are invariant with respect to choices of some subsequences in \( x \). These choices of subsequences, so called place selections, have some properties, see [13], [14] or [9] (which are unimportant for our investigation). In principle the reader may forget about the principle of randomness and consider only the principle of the statistical stabilization. It seems that only this principle is important (at least at the moment) in physics in that we study behaviour of frequencies.

Sequence (1) which satisfies to two von Mises’ principles is said to be a collective; \( p \) is said to be a probability distribution of the collective \( x \). We will often use the symbols \( p(B; x) \) (and \( \nu_N(B; x), n_N(B; x) \)), \( B \subset L \), to indicate the dependence on the concrete collective \( x \).

The frequency probability formalism is not a calculus of probabilities. It is a calculus of collectives. Thus instead of operations for probabilities (as it is in the conventional probability theory), we define operations for collectives.

2.3. Operation of combining of collectives. This operation will play the crucial role in our analysis of probabilistic foundations of Bell’s arguments. Let \( x = (x_j) \) and \( y = (y_j) \) be two collectives with label sets \( L_x \) and \( L_y \), respectively. We define a new sequence \( z = (z_j) \), \( z_j = \{x_j, y_j\} \) (in general \( z \) is not a collective). Let \( a \in L_x \) and \( b \in L_y \). Among the first \( N \) elements of \( z \) there are \( n_N(a; z) \) elements with the first component equal to \( a \). As \( n_N(a; z) = n_N(a; x) \) is a number of \( x_j = a \) among the first \( N \) elements of \( x \), we obtain that \( \lim_{N \to \infty} \frac{n_N(a; x)}{N} = p(a; x) \). Among these \( n_N(a; z) \) elements, there are a number, say \( n_N(b/a; z) \) whose second component is equal to \( b \). The frequency \( \nu_N(a, b; z) \) of elements of the sequence \( z \) labeled \( (a, b) \) will then be

\[
\frac{n_N(b/a; z)}{N} = \frac{n_N(b/a; z)}{n_N(a; z)} \frac{n_N(a; z)}{N}
\]

We set \( \nu_N(b/a; z) = \frac{n_N(b/a; z)}{n_N(a; z)} \). Let us assume that, for each \( a \in L_x \), the subsequence \( y(a) \) of \( y \) which is obtained by choosing \( y_j \) such that \( x_j = a \) is an collective. Then, for each \( a \in L_x, b \in L_y \), there exists

\[
p(b/a; z) = \lim_{N \to \infty} \nu_N(b/a; z) = \lim_{N \to \infty} \nu_N(b; y(a)) = p(b; y(a)).
\]
We have $\sum_{b \in L_2} p(b/a; z) = 1$. The existence of $p(b/a; z)$ implies the existence of $p(a, b; z) = \lim_{N \to \infty} \nu_N(a, b; z)$. Moreover, we have
\[ p(a, b; z) = p(a; x) p(b/a; z) \] (5)
and $p(b/a; z) = p(a, b; z)/p(a; x)$, if $p(a; x) \neq 0$. We have
\[ \sum_{a \in L_x} \sum_{b \in L_2} p(a, b; z) = 1. \]
Thus in this case the sequence $z$ is an collective and the probability distribution $p(a; b; z)$ well defined. The collective $y$ is said to be combinable with the collective $x$. The relation of combining is a symmetric relation on the set of pairs of collectives with strictly positive probability distributions ($p > 0$).

2.4. Independent collectives. Let $x$ and $y$ be collectives. Suppose that they are combinable. The $y$ is said to be independent from $x$ if all collectives $y(a), a \in L_x$, have the same probability distribution which coincides with the probability distribution $p(b; y)$ of $y$. This implies that
\[ p(b/a; z) = \lim_{N \to \infty} \nu_N(b/a; z) = \lim_{N \to \infty} \nu_N(b; y(a)) = p(b; y). \]
Here the conditional probability $p(b/a; z)$ does not depend on $a$. Hence
\[ p(a, b; z) = p(a; x) p(b; y), a \in L_x, b \in L_y. \]

From the physical viewpoint the notion of independent collectives is more natural than the notion of independent events in the conventional probability theory. In latter the relation $p(a, b) = p(a)p(b)$ can hold just occasionally (as the result of a game with numbers, see [14] or [9], p.53).

3 Kolmorogov’s viewpoint to the GHZ scheme

From the probabilistic viewpoint the GHZ experiment can be described in the following way (in the Kolmogorov approach). Let $(\Omega, F, P)$ be a Kolmogorov probability space which describes hidden variables. For each setting $(\phi_1, \phi_2, \phi_3)$ of phase shifts we define random variables $A(\phi_1, \omega), B(\phi_2, \omega), C(\phi_3, \omega)$ corresponding to physical observables $A(\phi_1), B(\phi_2), C(\phi_3)$ (given by measurements for photons 1,2,3 respectively, in the triple (1,2,3)). Quantum formalism predicts that there exist four settings $(\phi_i^1, \phi_i^2, \phi_i^3), i = 1, 2, 3, 4$ such that
\[ A(\phi_1^i, \omega) B(\phi_2^i, \omega) C(\phi_3^i, \omega) = 1, \] (6)
\[ \omega \in \Omega_i^+ \in F, \quad P(\Omega_i^+) = 1, \quad i = 1, 2, 3; \quad (7) \]

\[ A(\phi_1^4, \omega)B(\phi_2^4, \omega)C(\phi_3^4, \omega) = -1, \quad (8) \]

\[ \omega \in \Omega_4^- \in F, \quad P(\Omega_4^-) = 1. \quad (9) \]

By using algebraic properties \((A, B, C = \pm 1)\) we obtain that

\[ \Sigma^+ = \Omega_1^+ \cap \Omega_2^+ \cap \Omega_3^+ \subset \Omega_4^+ = \Omega \setminus \Omega_4^- . \quad (10) \]

The trivial mathematical considerations in Kolmorogov’s framework imply that by (9)

\[ P(\Sigma^+) = 1. \quad (11) \]

On the other hand, by (9) and (10) we have

\[ P(\Sigma^+) = 0. \quad (12) \]

This is the GHZ paradox. The typical conclusion is that we could not use the local deterministic description.

From the Kolmorogov viewpoint it seems that all was right in the GHZ derivation.

4 Von Mises’ viewpoint to the GHZ paradox

Here we could not start with an abstract probability distribution of hidden parameters. First we have to define a collective which produces this distribution. To introduce a collective, we have to define the label set \(L\) of this collective. It is convenient to use symbol \(\Omega\) instead of \(L\) (to use formulas of the previous section). However, it is just the same symbol and nothing more. Here \(\Omega\) has the following structure: \(\Omega = \Lambda \times \Lambda_1 \times \Lambda_2 \times \Lambda_3\), where \(\Lambda\) is the set of hidden variables for a quantum system (a triple of photons), \(\Lambda_j, j = 1, 2, 3\), are sets of hidden variables for measurement devices (for \(A, B\) and \(C\), respectively).

For each setting \(\phi_1, \phi_2, \phi_3\) of phase shifts, we may consider (in the hidden variables framework) a sequence \(x_{\phi_1,\phi_2,\phi_3} = (\omega_1, \omega_2, \ldots, \omega_N, \ldots), \omega_j = (\lambda_j, \lambda_1^j, \lambda_2^j, \lambda_3^j) \in \Omega\), where \(\omega_j\) is the configuration of hidden variables for \(j\)-th quantum system \(\pi_j\) (a triple of photons) + three measurement devices at the instants of measurements \(j = 1, 2, \ldots\)

\(^4\)To simplify considerations, we assume that all sets of hidden variables are finite.
The first question is the following: *Is* $x_{\phi_1, \phi_2, \phi_3}$ *a collective?* We have no experimental reasons to suppose that micro parameters have the property of the statistical stabilization (as macro parameters). It may be that the property of the statistical stabilization on the macro level is just a consequence of the average over huge ensembles of hidden parameters. Well, suppose that $x_{\phi_1, \phi_2, \phi_3}$ is a collective. Thus the frequency probability distribution

$$P_{\phi_1, \phi_2, \phi_3}(\lambda = k, \lambda^1 = s_1, \lambda^2 = s_2, \lambda^3 = s_3) =$$

$$\lim_{N \to \infty} \frac{n_N(\lambda = k, \lambda^1 = s_1, \lambda^2 = s_2, \lambda^3 = s_3)}{N}$$

is well defined. So, for four different settings $(\phi^i_1, \phi^i_2, \phi^i_3)$ of phase shifts we have four collectives $x^i = x_{\phi^i_1, \phi^i_2, \phi^i_3}, i = 1, 2, 3, 4$, with probability distributions $P_i \equiv P_{x^i}$. By the GHZ scheme we obtain that

$$P_i(\Omega^+_i) = 1, i = 1, 2, 3, \text{ and } P_4(\Omega^+_4) = 0. \quad (13)$$

Of course, by (10) we obtain

$$P_4(\Sigma^+) = 0. \quad (14)$$

However, the first three equations in (13) do not imply that

$$P_4(\Sigma^+) = 1. \quad (15)$$

Hence there is no paradox. To obtain the paradox, we need to obtain (15). Thus there must be some special restrictions on collectives (and consequently probability distributions) which imply (13). One of such restrictions is that the probability distribution does not depend on the setting $(\phi_1, \phi_2, \phi_3)$ of phase shifts:

$$P = P_{\phi_1, \phi_2, \phi_3}. \quad (16)$$

However, such an assumption has no physical justification (compare with [4]-[9]). First of all we have to assume so called ensemble reproducibility for hidden variable $\lambda$ (see [7] and [9]): the preparation procedure for quantum systems must precisely reproduce the probability distribution of hidden

\footnote{The consideration of hidden variables for measurement apparatuses is quite natural from the physical viewpoint. In fact, it is the hidden variable representation of Bohr’s ideas.}
variables in different runs of the experiment (in particular, for different set-
tings $\phi_1, \phi_2, \phi_3$). Despite of the common opinion that such a reproducibility
is a natural property of quantum systems (preparation procedures), at the
present stage of experimental research it is impossible to test this hypothesis.
Moreover, the hypothesis of reproducibility is a form of the postulate on the
completeness of quantum mechanics. By the hypothesis on reproducibility
we have that quantum state $\psi$ uniquely determines all statistical properties
of the (ideal infinite) ensemble of quantum particles described by $\psi$.

So by accepting this hypothesis we turn back (at least indirectly) to the
original discussion of Einstein, Podolsky, Rosen and Bohr on the complete-
ness of quantum mechanics. In some sense this is the logical loop, because
one of the main aims of J.Bell and his followers was to transform the EPR
polemic on the completeness of quantum mechanics into polemic on locality
and determinism.

Remark (On the interpretation of a wave function). Of course, all our
previous considerations on the hypothesis of reproducibility and the completeness
of quantum mechanics strongly depend on the interpretation of a wave function.
In fact, we used so called statistical interpretation of quantum mechanics (see, for
example, L. Ballentine [19]): a wave function gives the description of statistical
properties of an ensemble of quantum particles. Here the statistical reproducibility
of macro properties need not be based on the statistical reproducibility of micro
properties. For an adherent of the orthodox Copenhagen interpretation (by that
the wave function provides the complete description of an individual quantum
system), there are no doubts in the validity of the hypothesis of reproducibility.

However, even if we suppose that there are no ensemble fluctuations,
there are still some doubts in the validity of (16). It is more natural to think
that different settings of apparatuses produce different distributions of micro
states of these apparatuses (compare with [4]-[9]).

5 Singularity/equivalence dichotomy and the
principle of complementarity

Of course, (16) is only a sufficient condition for obtaining the GHZ paradox.
In fact, we need only that

$$P_{\phi_1 \phi_2 \phi_3}(E) = 0 \leftrightarrow P_{\phi'_1 \phi'_2 \phi'_3}(E) = 0$$  \hspace{1cm} (17)
for any two settings $\phi_1 \phi_2 \phi_3$ and $\phi'_1 \phi'_2 \phi'_3$ of measurement devices. This condition is well known in the measure theory, namely this is the condition of equivalence of two measures: they are absolutely continuous with respect to each other. The absolute continuity implies that the transition from one setting of measurement devices to another is sufficiently smooth (in measure-theoretical sense). There exists so called Radon-Nikodim derivative:

$$\frac{dP_{\phi_1 \phi_2 \phi_3}}{dP_{\phi'_1 \phi'_2 \phi'_3}}(\omega) = f(\omega; \phi_1 \phi_2 \phi_3 / \phi'_1 \phi'_2 \phi'_3).$$

The GHZ paradox (via our frequency analysis) demonstrated that quantum measurement procedures induce probability distributions which transform nonsmoothly (in measure-theoretical sense) from one setting to another.

Measure-theoretical singularity is described by the notion of singularity: $P' \perp P''$ if there is a set $E \in F$ such that $P''(E) = 1$ and $P'(E) = 0$. Suppose that $P_i \perp P_j, i,j = 1,2,\ldots,4$, where $P_j$ are probability distributions in the GHZ scheme. Let $\Omega_j^+, j = 1,2,3$, play the role of $E$ in the definition of $P_j \perp P_4$ : $P_j(\Omega_j^+) = 1$ and $P_4(\Omega_j) = 0, j = 1,2,3$. Then $P_4(\Sigma^+) = P_4(\Omega_1^+ \cap \Omega_2^+ \cap \Omega_3^+) = 0$. Thus there is no GHZ paradox.

We remark that if the space of hidden variables has infinite dimension, then, for many classes of probability distributions (in particular, Gaussian), we have equivalence/singularity dichotomy: either equivalent or singular [15]. It may be that the split of reality into classical and quantum is just the exhibition of such a dichotomy.

6 ‘Gedanken kollektiven’ (counterfactual arguments)

We note that in the frequency approach the GHZ paradox can be obtained via counterfactual arguments (compare with [2], [20]). These arguments are represented here via the use of ‘gedanken kollektiven’. In fact, the GHZ scheme is applied to four settings $(\pi/2,0,0), (0,\pi/2,0), (0,0,\pi/2), (\pi/2, \pi/2, \pi/2)$. Let us consider a ‘gedanken kollektiv’ corresponding to the simultaneous imaginary measurement for all angles involved in the GHZ scheme: $\phi_1 = 0, \phi_2 = 0, \phi_3 = 0, \pi/2$. Such an imaginary measurement would be described by the hidden variable:

$$\tilde{\omega} = (\lambda, \lambda^1_0, \lambda^1_{\pi/2}, \lambda^2_0, \lambda^2_{\pi/2}, \lambda^3_0, \lambda^3_{\pi/2}).$$

(18)
Of course, such a measurement is forbidden by the quantum theory. We recognize this. However, we continue our frequency analysis trying to find the origin of the impossibility of such a measurement. We may image that there are two settings $\phi_1 = 0, \pi/2$ for the first photon, two settings $\phi_2 = 0, \pi/2$ for the second photon and two settings $\phi_3 = 0, \pi/2$ for the third photon in the triple. At the moment of interaction (imaginary) with photons measurement devices with these settings have hidden parameters included in (18). If we assume that the sequence of parameters $\tilde{\omega}$ corresponding to the sequence of imaginary measurements, $x = (\tilde{\omega}_j, j = 1, \ldots, \infty)$ is a collective, then we obtain the frequency probability distribution $P = P_x$ which can be used in the GHZ scheme (and induce the paradox). The origin of the nonexistence of $P$ (statistical stabilization in $x$) is that collectives corresponding to incompatible settings of measurement devices are not combinable: $x_1 = (\tilde{\omega}_1), \tilde{\omega}_1 = (\lambda, \lambda^1, \lambda^2, \lambda_0^3)$, and $x_2 = (\tilde{\omega}_2), \tilde{\omega}_2 = (\lambda, \lambda^2, \lambda^1, \lambda_0^3)$ or $x_1 = (\tilde{\omega}_1), \tilde{\omega}_1 = (\lambda, \lambda^1, \lambda^2, \lambda_0^3)$, and $x_2 = (\tilde{\omega}_2), \tilde{\omega}_2 = (\lambda, \lambda^1, \lambda^2, \lambda_0^3), \ldots$

Thus our frequency counterfactual analysis demonstrated again that the origin of the GHZ paradox is the existence of incompatible settings of measurement apparatuses (uncombinable collectives).

References

[1] J.S. Bell, Rev. Mod. Phys., 38, 447–452 (1966). J. S. Bell, *Speakable and unspeakable in quantum mechanics*. Cambridge Univ. Press (1987).

[2] J.F. Clauser , M.A. Horne, A. Shimony, R. A. Holt, Phys. Rev. Letters, 49, 1804-1806 (1969); J.F. Clauser , A. Shimony, Rep. Progr.Phys., 41 1881-1901 (1978). A. Aspect, J. Dalibard, G. Roger, Phys. Rev. Lett., 49, 1804-1807 (1982); D. Home, F. Selleri, Nuovo Cim. Rivista, 14, 2–176 (1991). H. P. Stapp, Phys. Rev., D, 3, 1303-1320 (1971); P.H. Eberhard, Il Nuovo Cimento, B, 38, N.1, 75-80(1978); Phys. Rev. Letters, 49, 1474-1477 (1982); A. Peres, Am. J. of Physics, 46, 745-750 (1978). P. H. Eberhard, Il Nuovo Cimento, B, 46, N.2, 392-419 (1978); J. Jarrett, Nos, 18, 569 (1984).

[3] B. d’Espagnat, *Veiled Reality. An anlysis of present-day quantum mechanical concepts*. Addison-Wesley(1995). A. Shimony, *Search for a naturalistic world view*. Cambridge Univ. Press (1993); T. Maudlin, *Quantum non-locality and relativity*. Blackwill.

[4] L. de Broglie, *La thermodynamique de la particule isolee*. Gauthier-Villars, Paris, 1964; G. Lochak, Found. Physics, 6, 173-184 (1976); E. Nelson, *Quantum fluctuation*. Princeton Univ. Press, 1985; W. De Muynck and W. De Baere W., Ann. Israel Phys. Soc., 12, 1-22 (1996); W. De Muynck,
W. De Baere, H. Marten, Found. of Physics, 24, 1589–1663 (1994); W. De Muynck, J.T. Stekelenborg, Annalen der Physik, 45, N.7, 222-234 (1988).

[5] E. Beltrametti, G. Cassinelli, The logic of quantum mechanics. Addison-Wesley, Reading (1981).

[6] L. Accardi, Urne e Camaleoni: Dialogo sulla realta, le leggi del caso e la teoria quantistica. Il Saggiatore, Rome (1997); Accardi L., The probabilistic roots of the quantum mechanical paradoxes. The wave–particle dualism. A tribute to Louis de Broglie on his 90th Birthday, Edited by S. Diner, D. Fargue, G. Lochak and F. Selleri. D. Reidel Publ. Company, Dordrecht, 47–55(1984);

[7] I. Pitowsky, Phys. Rev. Lett, 48, N.10, 1299-1302 (1982); Phys. Rev. D, 27, N.10, 2316-2326 (1983; S.P. Gudder, J. Math Phys., 25, 2397-2401 (1984); A. Fine, Phys. Rev. Letters, 48, 291–295 (1982); P. Rastal, Found. Phys., 13, 555 (1983). W. Muckenheim, Phys. Reports, 133, 338–401 (1986); W. De Baere, Lett. Nuovo Cimento, 39, 234-238 (1984); 25, 2397–2401 (1984).

[8] A. Yu. Khrennikov, Dokl. Akad. Nauk SSSR, ser. Matem., 322, No. 6, 1075–1079 (1992); J. Math. Phys., 32, No. 4, 932–937 (1991); Physics Letters A, 200, 119–223 (1995); Physica A, 215, 577–587 (1995); Int. J. Theor. Phys., 34, 2423–2434 (1995); J. Math. Phys., 36, No.12, 6625–6632 (1995); A.Yu. Khrennikov, p-adic valued distributions in mathematical physics. Kluwer Academic Publishers, Dordrecht (1994); A.Yu. Khrennikov, Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models. Kluwer Acad.Publ., Dordrecht, The Netherlands, 1997;

[9] A. Yu. Khrennikov, Bell and Kolmogorov: probability, reality and nonlocality. Reports of Vaxjo Univ., N. 13 (1999); A. Yu. Khrennikov, Interpretations of probability. VSP Int. Sc. Publ., Utrecht, 1999.

[10] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev., 47, 777–780 (1935).

[11] D. Greenberger, M. Horne, A. Zeilinger, Going beyond Bell’s theorem, in Bell’s theorem, quantum theory, and conceptions of the universe. Ed. M.Kafatos, Kluwer Academic, Dordrecht, 73-76 (1989).

[12] A. N. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrech Springer Verlag, Berlin (1933); reprinted: Foundations of the Probability Theory. Chelsea Publ. Comp., New York (1956).

[13] R. von Mises, Grundlagen der Wahrscheinlichkeitsrechnung, Math.Z., 5, 52–99 (1919); R. von Mises, Probability, Statistics and Truth, Macmillan, London (1957).

[14] R. von Mises, The mathematical theory of probability and statistics.
Academic, London (1964); E. Tornier, *Wahrscheinlichkeitsrechnung und allgemeine Integrationstheorie*, Univ. Press, Leipzig (1936);

[15] A. N. Shiryayev, *Probability*. Springer, New York-Berlin-Hei (1984).

[16] E. Kamke, Über neuere Begründungen der Wahrscheinlichkeitsrechnung. *Jahresbericht der Deutschen Mathematiker*, 42, 14-27 (1932); J. Ville, *Etude critique de la notion de collective*, Gauthier-Villars, Paris (1939); M. Van Lambalgen, Von Mises’ definition of random sequences reconsidered. *J. of Symbolic Logic*, 52, 3 (1987).

[17] A. Wald, Die Widerspruchsfreiheit des Kollektivbegriffs in der Wahrscheinlichkeitsrechnung. *Ergebnisse eines Math. Kolloquiums*, 8, 38-72 (1938).

[18] Venn J., *The logic of chance*. London (1866); reprint: Chelsea, New York (1962).

[19] L. E. Ballentine, Rev. Mod. Phys., 42, 358–381 (1970).

[20] I. Kvart, *A theory of Counterfactuals*. Indianapolis: Hackett(1986); D. Lewis, *Counterfactuals*. Cambridge, Mass.: Harvard Univ. Press (1973);