Diameters of Connected Components of Friends-and-Strangers Graphs Are Not Polynomially Bounded

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Abstract

Given two graphs $X$ and $Y$ on $n$ vertices, the friends-and-strangers graph $FS(X,Y)$ has as its vertices all $n!$ bijections from $V(X)$ to $V(Y)$, with bijections $\sigma, \tau$ adjacent if and only if they differ on two elements of $V(X)$, whose mappings are adjacent in $Y$. In this work, we study the diameters of friends-and-strangers graphs, which correspond to the largest number of swaps necessary to achieve one configuration from another. We provide families of constructions $X_L$ and $Y_L$ for all integers $L \geq 1$ to show that diameters of connected components of friends-and-strangers graphs fail to be polynomially bounded in the size of $X$ and $Y$, resolving a question raised by Alon, Defant, and Kravitz in the negative. Specifically, our construction yields that there exist infinitely many values of $n$ for which there are $n$-vertex graphs $X$ and $Y$ with the diameter of a component of $FS(X,Y)$ at least $n^{(\log n)/(\log\log n)}$. We also study the diameters of components of friends-and-strangers graphs when $X$ is taken to be a path graph or a cycle graph, showing that any component of $FS(Path_n,Y)$ has diameter at most $|E(Y)|$, and $diam(FS(Cycle_n,Y))$ is $O(n^3)$ whenever $FS(Cycle_n,Y)$ is connected. We conclude the work with several conjectures that aim to generalize this latter result.

1 Introduction

Defant and Kravitz ([3]) recently introduced friends-and-strangers graphs, which are defined as follows.

Definition 1.1 ([3]). Let $X$ and $Y$ be two simple graphs, each with $n$ vertices. The friends-and-strangers graph of $X$ and $Y$, denoted $FS(X,Y)$, is a graph with vertices consisting of all bijections from $V(X)$ to $V(Y)$, with any two such bijections $\sigma, \sigma'$ adjacent in $FS(X,Y)$ if and only if there exists an edge $\{a,b\}$ in $X$ such that the following hold.

- \{$\sigma(a), \sigma(b)$\} $\in E(Y)$
- $\sigma(a) = \sigma'(b)$, $\sigma(b) = \sigma'(a)$
- $\sigma(c) = \sigma'(c)$ for all $c \in V(X) \setminus \{a,b\}$.

In other words, $\sigma$ and $\sigma'$ differ precisely on two adjacent vertices of $X$, and the corresponding mappings are adjacent in $Y$. For any such $\sigma, \sigma'$, we say that $\sigma'$ is achieved from $\sigma$ by an $(X,Y)$-friendly swap.

The friends-and-strangers graph $FS(X,Y)$ acquires its name from the following intuitive understanding of Definition 1.1. Say that $V(X)$ corresponds to $n$ positions and $V(Y)$ corresponds to $n$ people, any two of whom are friends (if adjacent) or strangers (if nonadjacent). We place the $n$ people on the $n$ positions, with this configuration $\sigma$ defining the bijection in $FS(X,Y)$. From here, we can swap any two individuals if and only if their positions are adjacent in $X$ and the people placed on them are friends (i.e. adjacent in $Y$); this yields the bijection $\sigma' \in FS(X,Y)$, for which we have $\{\sigma, \sigma'\} \in E(FS(X,Y))$.

Example 1.2. See Figure 1 for an illustration of this definition.
(a) The graph $X$.

(b) The graph $Y$.

(c) A sequence of $(X,Y)$-friendly swaps. The transpositions between adjacent configurations denote the two vertices in the graph $X$ involved in the $(X,Y)$-friendly swap. Red text corresponds to vertices in $Y$ placed upon vertices of $X$, which are labeled in black; this will be a convention throughout the rest of the work. The leftmost configuration corresponds to the bijection $\sigma \in V(\mathcal{FS}(X,Y))$ such that $\sigma(x_1) = y_1$, $\sigma(x_2) = y_5$, $\sigma(x_3) = y_3$, $\sigma(x_4) = y_4$, and $\sigma(x_5) = y_2$; the other configurations analogously correspond to vertices in $\mathcal{FS}(X,Y)$.

Figure 1: A sequence of $(X,Y)$-friendly swaps in $\mathcal{FS}(X,Y)$ for the graphs $X$ and $Y$, each on 5 vertices. Any configuration in the bottom row corresponds to a vertex of $\mathcal{FS}(X,Y)$. Two consecutive configurations here differ by an $(X,Y)$-friendly swap, so the corresponding vertices in $\mathcal{FS}(X,Y)$ are adjacent.

As noted in [3], it is frequently convenient to enumerate the vertices of the graphs $X$ and $Y$ so $V(X) = V(Y) = [n]$. Here, we can rephrase the definition as $V(\mathcal{FS}(X,Y)) = \mathcal{S}_n$, and two permutations $\sigma, \sigma' \in V(\mathcal{FS}(X,Y))$ are adjacent if and only if

1. $\sigma' = \sigma \circ (i \ j)$ for some transposition $(i \ j)$
2. $\{i, j\} \in E(X)$
3. $\{\sigma(i), \sigma(j)\} \in E(Y)$.

Example 1.3. A more concrete example of an object that friends-and-strangers graphs generalize is the famous 15-puzzle, where the numbers 1 through 15 are placed on a 4-by-4 board, with one empty space to which adjacent tiles can slide. Indeed, if we let $X$ be the 4-by-4 grid graph $\text{Grid}_{4 \times 4}$ and $Y = \text{Star}_n$, then studying the graph $\mathcal{FS}(\text{Grid}_{4 \times 4}, \text{Star}_n)$ is equivalent to studying the set of possible configurations and moves that can be performed on the 15-puzzle.

1.1 Prior Work

The article [3] introduced friends-and-strangers graphs, derived many of their basic properties, studied the connected components of the graphs $\mathcal{FS}(\text{Path}_n, Y)$ and $\mathcal{FS}(\text{Cycle}_n, Y)$, and determined necessary and sufficient conditions for $\mathcal{FS}(X,Y)$ to be connected. In [5], we extend their results, showing that $\mathcal{FS}(X,Y)$ is connected for all biconnected graphs $X$ for any $Y$ such that $\mathcal{FS}(\text{Cycle}_n, Y)$ is connected, and also initiate the study of the girth of the graph $\mathcal{FS}(X, \text{Star}_n)$, to which the study of the girth of friends-and-strangers graphs can be reduced to. We also remark that although friends-and-strangers graphs were introduced recently, many existing results in the literature can be recast into this framework. In particular, [9] studies the connected components of $\mathcal{FS}(X, \text{Star}_n)$ when $X$ is a biconnected graph.

A second paper by Defant, Kravitz, and Alon [1] asks a number of probabilistic and extremal questions concerning friends-and-strangers graphs. The recent work [2] provides asymmetric generalizations of two problems posed by [1]. Specifically, they study conditions on the minimal degrees of $X$ and $Y$ to guarantee
that \( FS(X, Y) \) is connected, and a variant of this problem for \( FS(X, Y) \) to have two connected components when \( X \) and \( Y \) are taken to be edge-subgraphs of \( K_{r,r} \), the complete bipartite graph with both partition classes having size \( r \).

### 1.2 Main Results

Perhaps the best known of the canonical graph parameters is the diameter, corresponding to the “longest shortest path” between any two vertices in a graph. In the present work, we study the diameters of connected components of friends-and-strangers graphs \( FS(X, Y) \), which in this context correspond to the largest number of swaps that is necessary to achieve one configuration from another in its vertex set. In this direction, the authors of [1, 3] posed the following question.

**Question 1.4** ([1, 3]). Does there exist an absolute constant \( C > 0 \) such that for all \( n \)-vertex graphs \( X \) and \( Y \), every connected component of \( FS(X, Y) \) has diameter at most \( n^C \)?

Our main result in this article will resolve this question in the negative, so that we have the following statement; the great majority of the present work is dedicated towards proving it.

**Theorem 1.5.** There does not exist an absolute constant \( C > 0 \) such that for all \( n \)-vertex graphs \( X \) and \( Y \), every connected component of \( FS(X, Y) \) has diameter at most \( n^C \).

Broadly, we shall accomplish this derivation with families of constructions \( \mathcal{X}_L \) and \( \mathcal{Y}_L \) for all integers \( L \geq 1 \).

The following figure shows the types of graphs that are included in the families \( \mathcal{X}_3 \) and \( \mathcal{Y}_3 \).

![Graphs](image)

Figure 2: Graphs \( X_3 \in \mathcal{X}_3 \) and \( Y_3 \in \mathcal{Y}_3 \).

We extract \( X_L \in \mathcal{X}_L \) and \( Y_L \in \mathcal{Y}_L \) on the same number of vertices, and will describe two configurations \( \{\sigma_s, \sigma_f\} \subset V(FS(X_L, Y_L)) \) in the same connected component such that for sufficiently large \( n \), the distance \( d(\sigma_s, \sigma_f) \) is at least \( n^{L-1} \). Specifically, we shall prove the following result.

**Theorem 1.6.** Take any integer \( L \geq 1 \), and \( X_L \in \mathcal{X}_L \), \( Y_L \in \mathcal{Y}_L \) on \( n > (4L)^L \) vertices. Then the diameter of the connected component of \( FS(X_L, Y_L) \) that contains \( \sigma_s \) is greater than \( n^{L-1} \).
There exist infinitely many values of \( n \) for which there are \( n \)-vertex graphs \( X \) and \( Y \) such that \( FS(X,Y) \) has a connected component with diameter at least \( n^{\log n}/(\log\log n) \).

We also study the diameters of \( FS(\text{Path}_n, Y) \) and \( FS(\text{Cycle}_n, Y) \). Here, we have the following results.

**Theorem 1.8.** Any connected component of \( FS(\text{Path}_n, Y) \) has diameter at most \( |E(Y)| \).

For cycle graphs, we have the following result. In particular, this shows that whenever \( FS(\text{Cycle}_n, Y) \) is connected, its diameter is polynomially bounded (specifically, \( O(n^3) \)).

**Theorem 1.9.** Let \( Y \) be a graph on \( n \geq 3 \) vertices, and let \( n_1, \ldots, n_r \) denote the sizes of the components of \( Y \). If \( \gcd(n_1, \ldots, n_r) = 1 \), then any component of \( FS(\text{Cycle}_n, Y) \) has diameter at most \( 2n^3 + |E(Y)| \).

## 2 Preliminaries

### 2.1 Notation

Here, we review some common families of graphs and elementary graph theory terminology that we shall refer to throughout this article.

- \([n] = \{1, 2, \ldots, n\}\).
- The vertex and edge sets of a graph \( G \) will be denoted \( V(G) \) and \( E(G) \), respectively.
- Define the disjoint union of a collection of graphs \( \{G_i\}_{i \in I} \), notated \( \bigoplus_{i \in I} G_i \), to be the graph with vertex set \( \bigsqcup_{i \in I} V(G_i) \) and edge set \( \bigsqcup_{i \in I} E(G_i) \). This readily extends to expressing a graph as the disjoint union of its connected components.
- The Cartesian product of graphs \( G_1, \ldots, G_r \), denoted \( G_1 \square \cdots \square G_r \), has vertex set \( V(G_1) \times \cdots \times V(G_r) \), with \( (v_1, \ldots, v_r) \) and \( (w_1, \ldots, w_r) \) adjacent if and only if there exists \( i \in [r] \) such that \( v_i, w_i \in E(G_i) \) and \( v_j = w_j \) for all \( j \in [r] \setminus \{i\} \).

#### 2.1.1 Common Families of Graphs

Assume that the vertex set of all graphs is given by \([n]\). We define the graphs in terms of their edge sets.

- The complete graph \( K_n \) has edge set \( E(K_n) = \{i,j\} : \{i,j\} \subseteq [n], i \neq j \} \).
- The path graph \( \text{Path}_n \) has edge set \( E(\text{Path}_n) = \{i,i+1\} : i \in [n-1] \} \).
- The cycle graph \( \text{Cycle}_n \) has edge set \( E(\text{Cycle}_n) = \{i,i+1\} : i \in [n-1] \} \cup \{\{n,1\} \} \).
- The star graph \( \text{Star}_n \) has edge set \( E(\text{Star}_n) = \{i,n\} : i \in [n-1] \} \).
- For \( i+j = n \), the complete bipartite graph \( K_{i,j} \) has edge set \( E(K_{i,j}) = \{v_1,v_2\} : v_1 \in [i], v_2 \in [i+1,n] \} \). This partitions \( V(K_{i,j}) \) into two sets so that every vertex in one set is adjacent to every vertex in the other; we shall refer to these sets as partition classes of \( V(K_{i,j}) \). A special case yields the graphs \( K_{r,r} \), when the partition classes have the same number of vertices.
3 Proofs of Main Results

We begin by repeating the question that we shall resolve.

**Question 3.1** ([1, 3]). Does there exist an absolute constant \( C > 0 \) such that for all \( n \)-vertex graphs \( X \) and \( Y \), every connected component of \( \text{FS}(X, Y) \) has diameter at most \( n^C \)?

We answer Question 3.1 in the negative, and shall devote this section to deriving the following result.

**Theorem 3.2.** There does not exist an absolute constant \( C > 0 \) such that for all \( n \)-vertex graphs \( X \) and \( Y \), every connected component of \( \text{FS}(X, Y) \) has diameter at most \( n^C \).

3.1 Families of Graphs \( \mathcal{X}_L \) and \( \mathcal{Y}_L \), Notation and Initial Configurations

To motivate our construction, we begin with the following observation, mentioned in [3]. One can understand this as the central vertex of \( \text{Star}_n \) acting as a “knob” rotating around \( \text{Cycle}_n \), and all other vertices of \( V(\text{Star}_n) \) moving cyclically around it. In particular, it takes \( n(n-1) \) such swaps in the same direction for all other vertices of \( \text{Star}_n \) to return to their original positions in the starting configuration.

**Lemma 3.3.** Every connected component of \( \text{FS} (\text{Cycle}_n, \text{Star}_n) \) is isomorphic to \( \text{Cycle}_{n(n-1)} \).

**Proof.** Let \( \sigma = \sigma(1) \cdots \sigma(n) \) be in \( S_n \), and consider the component \( C \) of \( \text{FS}(\text{Cycle}_n, \text{Star}_n) \) with \( \sigma \in V(C) \). Without loss of generality, say \( \sigma(1) = n \), the central vertex of \( \text{Star}_n \) (\( C \) must have such a permutation). Let \( [n(n-1)] = \{1, 2, \ldots, n(n-1)\} \) denote the vertex set of \( \text{Cycle}_{n(n-1)} \), and define \( \varphi : V(\text{Cycle}_{n(n-1)}) \to V(C) \) by defining \( \varphi(i) \) to be the permutation achieved by starting from \( \sigma \) and swapping \( \sigma(1) = n \) rightward \( i \) times (for example, \( \varphi(1) = \sigma(2) \sigma(1) \cdots \sigma(n) \)). It follows easily that \( \varphi \) is a graph isomorphism. \( \square \)

Observe that \( \text{diam}(\text{Cycle}_{n(n-1)}) = \left\lfloor \frac{n(n-1)}{2} \right\rfloor \geq \frac{n(n-1)}{2} > n \) whenever \( n > 3 \), so that if an absolute constant \( C \) from Question 3.1 exists, necessarily \( C > 1 \). We argue similarly for general monomials \( n^d \) for \( d \in \mathbb{N} \), \( d \geq 2 \) via other choices of \( X \) and \( Y \), so that if \( C \) exists, it is necessarily greater than all natural numbers, which is contradictory. We shall construct families of graphs \( \mathcal{X}_L \) and \( \mathcal{Y}_L \), for every integer \( L \geq 1 \), that we study to prove Theorem 3.2 in the following description, assume we have fixed some arbitrary integer \( L \geq 1 \).

The Families \( \mathcal{X}_L \)  

Graphs \( X_L \in \mathcal{X}_L \) contain \( L \times 2 \) arrays of cycle subgraphs, with adjacent cycle subgraphs intersecting in exactly one vertex. The graph \( X_L \) is said to have \( L \) layers, and we refer to layer \( \ell \in [L] \). Subgraphs and vertices corresponding to the “left column” of \( X_L \) shall be subscripted by \( a \), with those in the right column subscripted by \( b \); denote the left cycle subgraph in layer \( \ell \) as \( C_a^\ell \), and right cycle subgraph of layer \( \ell \) by \( C_b^\ell \). Corresponding to each \( C_a^\ell \) and \( C_b^\ell \) is a path subgraph of \( X_L \) extending out of it: that corresponding to \( C_a^\ell \) is denoted \( P_a^\ell \), and similarly \( P_b^\ell \) for \( C_b^\ell \). Denote the subgraph of \( X_L \) consisting of the \( \ell \)-th layer by \( X^\ell \). The subgraph consisting of \( P_a^\ell \) and \( C_a^\ell \) is denoted \( X_a^\ell \), with a similar notion for \( X_b^\ell \).

Denote \( v_a^\ell = V(P_a^\ell) \cap V(C_a^\ell) \), \( v_b^\ell = V(P_b^\ell) \cap V(C_b^\ell) \), \( v^\ell = V(C_a^\ell) \cap V(C_b^\ell) \), \( a^\ell+1_a = V(C_a^\ell) \cap V(C_a^{\ell+1}) \) (whenever well-defined for \( \ell \in [L] \)). In \( C_a^\ell \), \( m \) inner vertices lie in the paths between \( \{v_{a^\ell\ell+1}^\ell, v_{a^\ell\ell+1}^\ell\}, \{v_{a^\ell\ell+1}^\ell, v_{a^\ell\ell-1}^\ell\}, \{v^\ell, v^\ell_{a^\ell-1}^\ell\}, \) and \( \{v^\ell_{a^\ell-1}^\ell, v_b^\ell\} \), with a similar statement for \( C_b^\ell \). The exceptions are layers 1 and \( L \): set \( 2m+1 \) inner vertices in the upper path between \( \{v_a^1, v_1^1\} \) in \( C_a^1 \) and the upper path between \( \{v_b^L, v_L^1\} \) in \( C_b^L \). Set \( 2m+1 \) inner vertices in the lower path between \( \{v_a^L, v_L^1\} \) in \( C_a^L \), and in the lower path between \( \{v_b^1, v_1^1\} \) in \( C_b^1 \). Take \( \nu = 4m+3 \): observe that for every \( \ell \in [L] \), \( |V(C_a^\ell)| = |V(C_b^\ell)| = \nu + 1 \). We shall also set \( |V(P_a^\ell)| = \nu + 1 \), \( |V(P_b^\ell)| = \nu \).

For brevity, we do not formally elaborate the vertex and edge sets of the graphs \( X_L \in \mathcal{X}_L \), but remark that this is straightforward (albeit tedious) from the preceding description and shall reference the subgraphs and vertices labeled above in discussing these graphs. In particular, there exists a unique \( n \)-vertex graph \( X_L \in \mathcal{X}_L \), for values \( n = 4\nu + (L-1)(4\nu - 2) = 4\nu L - 2(L-1) \) with \( \nu = 4m+3, m \geq 3 \). Figure 3 illustrates this construction for \( L = 3 \).
Henceforth, we shall often refer to vertices \( \kappa \). There exists a unique \( n \) (again, this is straightforward), and shall refer freely to the subgraphs of \( Y \). As in \( X \), we avoid formally elaborating the vertex and edge sets of the graphs \( Y \) (again, this is straightforward), and shall refer freely to the subgraphs of \( Y \) discussed above. In particular, there exists a unique \( n \)-vertex graph \( Y \) for values \( n = 4\nu L - 2(L - 1) \) with \( \nu = 4m + 3 \), \( m \geq 3 \). Henceforth, we shall often refer to vertices \( \kappa_a \) and \( \kappa_b \) as knob vertices of \( Y \). Figure 4 illustrates \( Y \) for \( L = 3 \).

The Families \( \mathcal{Y}_L \). We construct a complementary graph \( Y_L \in \mathcal{Y}_L \) for each \( X_L \in \mathcal{X}_L \), extending the intuition behind Lemma 3.3. Specifically, we assign to each cycle subgraph \( C_a \) and \( C_b \) of \( X_L \) a corresponding “knob vertex” in \( V(Y_L) \) (denoted \( \kappa_a \) and \( \kappa_b \), respectively), and for each knob vertex, we set a collection of vertices of \( V(Y_L) \) to swap only with the knob. The construction of \( Y_L \) proceeds sequentially according to \( L \) as follows: Recall the constant \( n \) from the construction of \( X_L \): take two disjoint copies of Star, denoted \( S^I_a \) and \( S^I_b \), with central vertices \( \kappa_a \) and \( \kappa_b \), respectively, and a complete bipartite graph \( K \) with \( n \) vertices in both partition classes \( K^1 \) and \( K^2 \). Now set \( \kappa_a \) and \( \kappa_b \) adjacent to all vertices in \( V(K^1) \). If \( L = 1 \), this completes the construction of \( Y_L \) \( n = 4\nu \). If not, take one vertex each in \( K^1_a \) and \( K^1_b \), which shall correspond to \( \kappa_a^2 \) and \( \kappa_b^2 \), and central vertices of subgraphs \( S^I_a \) and \( S^I_b \) respectively (again isomorphic to Star). Also construct complete bipartite graph \( K^2 \) with \( n \) vertices in both partition classes \( K^2_a \) and \( K^2_b \), and as before, set \( \kappa_a^2 \) and \( \kappa_b^2 \) adjacent to all vertices in \( V(K^2) \). Proceed similarly (for \( 1 \leq \ell \leq L \), take two vertices of \( V(K^{\ell-1}) \) in opposite partition classes and construct \( S^{\ell}_a \), \( S^{\ell}_b \), and \( K^{\ell} \), related as before) until we exhaust all \( n = 4\nu L - 2(L - 1) \) vertices, which is exactly when this procedure completes \( L \) iterations.

As in \( X_L \), we avoid formally elaborating the vertex and edge sets of the graphs \( Y_L \in \mathcal{Y}_L \) for brevity (again, this is straightforward), and shall refer freely to the subgraphs of \( Y_L \) discussed above. In particular, there exists a unique \( n \)-vertex graph \( Y_L \in \mathcal{Y}_L \) for values \( n = 4\nu L - 2(L - 1) \) with \( \nu = 4m + 3 \), \( m \geq 3 \). Henceforth, we shall often refer to vertices \( \kappa_a \) and \( \kappa_b \) as knob vertices of \( Y_L \). Figure 4 illustrates \( Y_L \in \mathcal{Y}_L \) for \( L = 3 \).

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We shall generally reserve Greek letters for vertices in \( V(Y_L) \) to distinguish them from vertices in \( V(X_L) \). The two exceptions will be \( \nu \), which shall be reserved for the parameter associated to the size of any graph \( X_L \) defined as above, and \( \lambda \), which will be used to indicate the length of a swap sequence in \( \mathcal{FS}(X_L, Y_L) \) in forthcoming arguments.
The Starting Configuration $\sigma_s$ Take $X_L \in X_L$, $Y_L \in Y_L$ on the same number of vertices. We are going to describe a specific starting configuration $\sigma_s \in V(\mathcal{FS}(X_L, Y_L))$: we will later describe a different configuration in the same connected component as $\sigma_s$, whose distance from $\sigma_s$ is strictly greater than $n^{L-1}$ for appropriate and sufficiently large values of $n$. Take all $\nu$ vertices in $V(K^1_\nu)$ and place them onto $V(P^1_\nu) \setminus \{v^1_1\}$, and the $\nu$ vertices in $V(K^1_\nu)$ onto $V(P^1_\nu)$: if $L > 1$, we place $\kappa^2_\nu$ onto the leftmost vertex of $V(P^1_\nu)$ and $\kappa^3_\nu$ onto $v^1_1$. Now take subgraph $S^1_a$ of $Y_L$: place $\kappa^1_a$ onto the middle vertex of the upper path between $v^1_1$ and $v^1$ (which has $2m + 1$ vertices), and place all $\nu - 1$ leaves of $S^1_a$ onto the remaining $\nu - 1$ vertices of $V(C^1_a) \setminus \{v^1\}$ in some way. Similarly, take $S^1_b$: place $\kappa^1_b$ onto the middle vertex of the upper path between $v^1$ and $v^1$, and place all $\nu - 1$ leaves of $S^1_b$ onto the remaining $\nu - 1$ vertices of $V(C^1_b)$. This has filled all mappings on the subgraph $V(X^1)$ of $X_L$ by vertices of $V(K^1)$, $V(S^1_b)$, and $V(S^1)$, and thus yields $\sigma_s$ if $L = 1$.

Proceed sequentially according to the layer $\ell \in [L]$: say we placed all vertices of $V(K^\ell)$, $V(S^\ell_a)$, and $V(S^\ell_b)$ for $i \leq \ell$ onto the corresponding $V(X^i)$ of $X_L$. Place all $\nu$ vertices in $V(K^\ell+1)|_{\{v^\ell+1\}}$ onto $V(P^\ell+1) \setminus \{v^\ell+1\}$, and the $\nu$ vertices in $V(K^\ell+1)$ onto $V(P^\ell+1)$: if $L > \ell + 1$, place $\kappa^\ell+2_a$ onto the leftmost vertex of $V(P^\ell+1)$ and $\kappa^\ell+2_b$ onto $v^\ell+1$. Now take $S^\ell_a$, and place its $\nu - 1$ leaves onto the remaining $\nu - 1$ vertices in $V(C^\ell+1_a) \setminus \{v^\ell+1\}$. Similarly take $S^\ell_b$, and place its $\nu - 1$ leaves onto the remaining $\nu - 1$ vertices in $V(C^\ell+1_b)$.

An illustration of this starting configuration is depicted in Figures 3 and 4, the vertices of a particular color in Figure 4 are placed upon the corresponding colored subgraph in Figure 3 to achieve $\sigma_s \in V(\mathcal{FS}(X_L, Y_L))$.

Remark 3.4. For sake of clarity and convenience of the reader, we shall explicitly list, for any vertex in $V(Y_L)$, the other vertices in $V(Y_L)$ that it is adjacent to. In particular, returning to this remark may be useful when studying the proof of Proposition 3.7.

- For $\ell \in [L]$, any vertex $\mu \in V(S^\ell_a) \setminus \{\kappa^\ell_a\}$ is adjacent to $\kappa^\ell_a$ only. Similarly, any vertex $\mu \in V(S^\ell_b) \setminus \{\kappa^\ell_b\}$ is adjacent to $\kappa^\ell_b$ only.

- The knob vertex $\kappa^1_a$ is adjacent to all vertices in $(V(S^1_b) \setminus \{\kappa^1_b\}) \cup V(K^1)$. Similarly, $\kappa^1_b$ is adjacent to all vertices in $(V(S^1_b) \setminus \{\kappa^1_b\}) \cup V(K^1)$. 

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Figure 4: Labeled schematic diagram of the construction for $Y_3 \in \mathcal{Y}_3$. Here, the vertices of $Y_3$ that are marked with a specific color correspond to the images under the starting configuration $\sigma_s$ of the vertices within the subgraph of the same color in Figure 3.
• For \( \ell \geq 2 \), the knob vertex \( \kappa_a^{\ell} \) is adjacent to all vertices in \( V(K_b^{\ell-1}) \cup \{ \kappa_a^{\ell-1}, \kappa_b^{\ell-1} \} \cup (V(S_b^\ell) \setminus \{ \kappa_a^{\ell} \}) \cup V(K^\ell) \). Similarly, the knob vertex \( \kappa_b^{\ell} \) is adjacent to all vertices in \( V(K_b^{\ell-1}) \cup \{ \kappa_a^{\ell-1}, \kappa_b^{\ell-1} \} \cup (V(S_b^\ell) \setminus \{ \kappa_b^{\ell} \}) \cup V(K^\ell) \).

• For \( \ell \in [L] \), any \( \mu \in V(K_b^\ell) \setminus \{ \kappa_a^{\ell+1} \} \) is adjacent to all vertices in \( V(K_b^\ell) \cup \{ \kappa_a^{\ell}, \kappa_b^{\ell} \} \). Similarly, any \( \mu \in V(K_b^\ell) \setminus \{ \kappa_b^{\ell+1} \} \) is adjacent to all vertices in \( V(K_b^\ell) \cup \{ \kappa_a^{\ell}, \kappa_b^{\ell} \} \). (For \( \ell = L \), this applies for any \( \mu \in V(K_b^L) \), as \( \kappa_a^{L+1} \) is not defined. The same is said for \( \mu \in V(K_b^L) \).)

**Remark 3.5.** By construction of \( \sigma_s \in V(FS(X_L, Y_L)) \), for any \( \ell \in [L] \), we have that \( (V(S_b^\ell) \setminus \{ \kappa_a^{\ell+1} \}) \subset \sigma_s (V(C_b^\ell)) \) and \( (V(S_b^\ell) \setminus \{ \kappa_b^{\ell+1} \}) \subset \sigma_s (V(C_b^\ell)) \). As such, all leaves of a star subgraph \( S_a^\ell \) or \( S_b^\ell \) of \( Y_L \) are placed onto a corresponding cycle subgraph \( C_a^\ell \) or \( C_b^\ell \) of \( X_L \), respectively. This yields for any \( \ell \in [L] \) that \( |\sigma_s (V(C_b^\ell)) \setminus (V(S_b^\ell) \setminus \{ \kappa_b^{\ell+1} \})| = 2 \) and \( |\sigma_s (V(C_b^\ell)) \setminus (V(S_b^\ell) \setminus \{ \kappa_a^{\ell+1} \})| = 2 \). In other words, the number of vertices upon any cycle subgraph \( C_a^\ell \) or \( C_b^\ell \) of \( X_L \) which are not leaves of the corresponding star subgraph of \( Y_L \) is exactly two for the configuration \( \sigma_s \).

### 3.2 Configurations in the Same Component as \( \sigma_s \)

In what follows, we refer to graphs \((X_L, Y_L)\), and prove statements concerning the vertices of the connected component of \( FS(X_L, Y_L) \) that includes \( \sigma_s \). The graphs \((X_L, Y_L)\) will be understood to refer to any arbitrary corresponding pair of graphs with \( X_L \in X_L \) and \( Y_L \in Y_L \) on the same number of vertices for an arbitrary integer \( L \geq 1 \), and \( \sigma_s \) the starting configuration for this specific instance \((X_L, Y_L)\), referring to subgraphs of \( X_L \) and \( Y_L \) as in the preceding section. As such, the propositions resulting from this section will hold for all such graphs \( FS(X_L, Y_L) \).

We elect to refer to paths in \( FS(X_L, Y_L) \) as *swap sequences*, which are denoted by the vertices and edges in \( FS(X_L, Y_L) \) that constitute the path. Specifically, a swap sequence of length \( \lambda \) from \( \sigma_0 \) is a sequence of vertices \( V = \{ \sigma_i \}_{i=0}^{\lambda} \subseteq V(FS(X_L, Y_L)) \), where \( \sigma_0 = \sigma_s \) and \( \{ \sigma_{i-1}, \sigma_i \} \in E(FS(X_L, Y_L)) \) for all \( i \in [\lambda] \).

**Remark 3.3** observes that in the starting configuration \( \sigma_s \), the leaves of any star graph \( S_a^\ell \) or \( S_b^\ell \) lie upon the vertices of \( C_a^\ell \) and \( C_b^\ell \), respectively. In particular, for any cycle subgraph \( C_a^\ell \) in \( X_L \), exactly two vertices not given by leaves of \( S_a^\ell \) lie upon them; an analogous statement holds for cycle subgraphs of form \( C_b^\ell \). This property remains true after any sequence of swaps in \( FS(X_L, Y_L) \) which begins at \( \sigma_s \).

**Proposition 3.6.** Any configuration \( \sigma \in V(FS(X_L, Y_L)) \) in the same connected component as the starting configuration \( \sigma_s \) satisfies \( V(S_a^\ell) \setminus \{ \kappa_a^\ell \} \subset \sigma(V(C_a^\ell)) \) and \( V(S_b^\ell) \setminus \{ \kappa_b^\ell \} \subset \sigma(V(C_b^\ell)) \) for all \( \ell \in [L] \).

As in Remark 3.5, this yields that for any cycle subgraph \( C_a^\ell \) or \( C_b^\ell \) in \( X_L \) and any \( \sigma \in V(FS(X_L, Y_L)) \) that is connected to \( \sigma_s \), we have \( |\sigma(V(C_a^\ell)) \setminus (V(S_a^\ell) \setminus \{ \kappa_a^\ell \})| = 2 \) and \( |\sigma(V(C_b^\ell)) \setminus (V(S_b^\ell) \setminus \{ \kappa_b^\ell \})| = 2 \) (as \( |V(S_a^\ell) \setminus \{ \kappa_a^\ell \}| = |V(S_b^\ell) \setminus \{ \kappa_b^\ell \}| = \nu - 1 \), and all cycle subgraphs have size \( \nu + 1 \)). In other words, the number of vertices upon a cycle subgraph of \( X_L \) which are not leaves of the corresponding star subgraph of \( Y_L \) via any such configuration \( \sigma \) is exactly two.

**Proof.** Assume the proposition is not true, so there exists a swap sequence \( V = \{ \sigma_i \}_{i=0}^{\lambda} \) with \( \sigma_0 = \sigma_s \) in \( FS(X_L, Y_L) \) of shortest length \( \lambda \) containing a vertex violating Proposition 3.6. \( \sigma_{\lambda+i} \) fails to satisfy Proposition 3.6, while all \( \sigma_i \) for \( i < \lambda \) do, and \( \lambda \geq 1 \). Thus, there exists a star subgraph \( S \) (of form \( S_a^\ell \) or \( S_b^\ell \)) of \( Y_L \) and a leaf \( \mu \in V(S) \) such that \( \mu \) is upon the appropriate cycle subgraph in \( \sigma_{\lambda-1} \), but swapped off with the central vertex of \( S \) to achieve \( \sigma_{\lambda} \). In particular, \( S \) is unique, since any \((X_L, Y_L)\)-friendly swap involves at most one leaf of such a subgraph.

First consider the setting \( S = S_a^\ell \) for some \( \ell \in [L] \). The leaf \( \mu \in V(S_b^\ell) \setminus \{ \kappa_b^\ell \} \) can swap only with \( \kappa_a^{\ell+1} \) by assumption, \( \mu \in \sigma_{\lambda-1}(V(C_b^\ell)) \) and \( \mu \notin \sigma_{\lambda}(V(C_b^\ell)) \), so \( \sigma_{\lambda} \) follows from \( \sigma_{\lambda-1} \) by an \((X_L, Y_L)\)-friendly swap involving \( \mu \) and \( \kappa_a^{\ell+1} \). The vertex \( \mu \) swaps out of \( V(C_b^\ell) \) from \( \sigma_{\lambda-1} \) to \( \sigma_{\lambda} \), so necessarily \( \sigma_{\lambda-1}^{-1}(\mu) \in \{ v_a^\ell, v_b^\ell, v_a^{-\ell+1}, v_b^\ell, v_a^{\ell+1} \} \) (more precisely, the nonempty subset of these that are defined for layer \( \ell \)). Figure 5 depicts the configurations in the following two cases.
Case 1: $\sigma^{-1}_\lambda(\mu) = v_\ell^\prime$. Here, $\sigma^{-1}_\xi(\kappa_\mu^\ell) \in V(P_\ell^\mu_a) \setminus \{v_\ell^\prime\}$ is adjacent to $v_\ell^\prime$. Denote $\sigma_\xi$ to be the final term of $\mathcal{V}$ before $\sigma_\lambda$ where $\sigma^{-1}_\xi(\kappa_\mu^\ell) \notin V(P_\ell^\mu_a) \setminus \{v_\ell^\prime\}$; here, $\xi < \lambda - 1$ and $\xi$ is well-defined, since $\sigma^{-1}_\xi(\kappa_\mu^\ell) \notin V(P_\ell^\mu_a) \setminus \{v_\ell^\prime\}$ and necessarily $\lambda \geq 2$. $\sigma_{\xi+1}$ is achieved from $\sigma_\xi$ by swapping $\kappa_\mu^\ell$ into $V(P_\ell^\mu_a) \setminus \{v_\ell^\prime\}$ from $v_\ell^\prime$ ($\sigma^{-1}_\xi(\kappa_\mu^\ell) = v_\ell^\prime$), while $\sigma^{-1}_\xi(\mu) \in V(C_\ell^\mu_a) \setminus \{v_\ell^\prime\}$ by minimality of $\lambda$. Since $\mu$ can swap only with $\kappa_\mu^\ell$ and $(V(C_\ell^a) \setminus \{v_\ell^\prime\}) \cap (V(P_\ell^\mu_a) \setminus \{v_\ell^\prime\}) = \emptyset$, $\sigma^{-1}_j(\mu)$ remains fixed for $\xi \leq j \leq \lambda - 1$, and in particular $\sigma^{-1}_\lambda(\mu) \neq v_\ell^\prime$, a contradiction.

Case 2: $\sigma^{-1}_\lambda(\mu) \neq v_\ell^\prime$. Here, $\sigma^{-1}_\lambda(\mu) \in \{v_\ell^\prime, v_\ell^{-1,\ell}, v_\ell^{\ell+1}\}$ (more precisely, the nonempty subset of these defined for layer $\ell$): $\{v_\ell^\prime, v_\ell^{-1,\ell}, v_\ell^{\ell+1}\} \subset V(C_\ell^\mu_a)$ are intersection vertices between $V(C_\ell^\mu_a)$ and $V(C_{\ell+1}^\mu_a)$, and $V(C_{\ell-1}^\mu_a)$, respectively. In particular, $\sigma^{-1}_\lambda(\kappa_\mu^\ell)$ lies on the corresponding cycle ($C_\ell^\mu_a$, $C_{\ell-1}^\mu_a$, or $C_{\ell+1}^\mu_a$, depending on $\sigma^{-1}_\lambda(\mu)$) for whichever intersection vertex $\sigma^{-1}_\lambda(\mu)$ equals and is adjacent to $\sigma^{-1}_\lambda(\mu)$, as $\sigma^{-1}_\lambda(\mu) \notin V(C_\ell^\mu_a)$. The minimality of $\lambda$ yields that exactly one of the two preimages (for $\mu$ and $\kappa_\mu^\ell$) differs between $\sigma_{\lambda-2}$ and $\sigma_{\lambda-1}$ (note that $\lambda \geq 2$, since $\sigma^{-1}_\lambda(\nu_\ell^\prime) \neq \sigma^{-1}_\lambda(\kappa_\mu^\ell)$), and specifically $\sigma^{-1}_{\lambda-2}(\kappa_\mu^\ell) \neq \sigma^{-1}_{\lambda-1}(\kappa_\mu^\ell)$, since $\mu$ swaps only with $\kappa_\mu^\ell$, so $\sigma^{-1}_{\lambda-1}(\mu) = \sigma^{-1}_\lambda(\mu)$. Thus, from $\sigma_{\lambda-2}$ to $\sigma_{\lambda-1}$, $\kappa_\mu^\ell$ must have swapped onto $\sigma^{-1}_{\lambda-1}(\kappa_\mu^\ell)$, with this swap on two vertices in the corresponding adjacent cycle subgraph of $X_L$ ($C_\ell^\mu_a$, $C_{\ell-1}^\mu_a$, or $C_{\ell+1}^\mu_a$, depending on $\sigma^{-1}_{\lambda-1}(\mu)$).

Assume $\sigma^{-1}_{\lambda-2}(\mu) = \sigma^{-1}_{\lambda-1}(\mu) = v_\ell^\prime$, so $\sigma^{-1}_{\lambda-2}(\{v_\ell^\prime, \nu_\ell^\prime\}) \subset V(C_{\ell+1}^\mu_a)$, as is the vertex $\kappa_\mu^\ell$ swaps with to achieve $\sigma_{\lambda-1}$ from $\sigma_{\lambda-2}$, which is distinct from $\mu$ and not in $V(S_\ell^\mu_a) \setminus \{v_\ell^\prime\}$. However, this means that for the configuration $\sigma_{\lambda-2}$, we have $|\sigma_{\lambda-2}(V(C_\ell^\mu_a)) \setminus (V(S_\ell^\mu_a) \setminus \{v_\ell^\prime\})| \geq 3$, contradicting the minimality of $\lambda$ as $\sigma_{\lambda-2}$ breaks Proposition 3.6 ($V(S_\ell^\mu_a) \setminus \{v_\ell^\prime\}) \subset \sigma_{\lambda-2}(V(C_{\ell+1}^\mu_a))$. The argument is entirely analogous for the other possible values of $\sigma^{-1}_{\lambda-1}(\mu)$ (which can include $v_\ell^{-1,\ell}$ and $v_\ell^{\ell+1}$ depending on $\ell \in [L]$), and are argued on $C_{\ell-1}^\mu_a$ and $C_{\ell+1}^\mu_a$, respectively).

![Diagram](image3.png)

(a) Following $\sigma_\xi$, $\kappa_\mu^\ell$ does not exit $V(P_\ell^\mu_a) \setminus \{v_\ell^\prime\}$, leading to a contradiction on the placement of $\mu$ in $\sigma_{\lambda-1}$. (b) $\sigma_{\lambda-1}$ results by swapping $\kappa_\mu^\ell$ along $C_\ell^\mu_a$; Proposition 3.6 is thus contradicted by $\sigma_{\lambda-2}$.

Figure 5: Illustration of configurations in the swap sequence $\mathcal{V}$ that raise a contradiction for both cases described in the proof of Proposition 3.6. Again (and as will be the case in all future such depictions), black text denotes vertices and subgraphs in $X_L$, while red text denotes vertices in $Y_L$.

As such, we cannot have $S = S_b^\ell$ for some value of $\ell \in [L]$. The argument for the setting in which $S = S_b^\ell$ is entirely analogous (where we break into cases based on $\sigma^{-1}_{\lambda-2}(\mu) \in \{v_\ell^\prime, v_\ell^{-1,\ell}, v_\ell^{\ell+1}\}$, or the nonempty subset defined for $\ell \in [L]$), leading to a contradiction on the initial assumption that such a sequence $\mathcal{V}$ can exist.

**Proposition 3.6.** Any configuration $\sigma \in V(FS(X_L, Y_L))$ in the same connected component as the starting configuration $\sigma_s$ must satisfy the following four properties.

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1. The layer 1 knob vertices lie upon the corresponding subgraph of $X^1$, i.e. $\sigma^{-1}(\kappa^1_1) \in V(X^1_a)$ and $\sigma^{-1}(\kappa^1_1) \in V(X^1_b)$.

2. For $\ell \geq 2$, the layer $\ell$ knob vertices lie upon the subgraphs $X^{\ell-1}$ or $X^\ell$, i.e. $\sigma^{-1}(\kappa^\ell_a) \in V(X^{\ell-1}) \cup V(X^\ell)$ and $\sigma^{-1}(\kappa^\ell_b) \in V(X^{\ell-1}) \cup V(X^\ell)$.

3. For $\ell \in [L-1]$, any vertex in $V(K^\ell)$ not a layer $\ell+1$ knob lies upon $X^\ell$, i.e. $\sigma^{-1}(V(K^\ell) \setminus \{\kappa^{\ell+1}_a, \kappa^{\ell+1}_b\}) \subset V(X^\ell)$, and $\sigma^{-1}(V(K^{\ell+1})) \subset V(X^\ell)$.

4. For $\ell \in [L]$, there is at most one $\mu \in V(K^\ell)$ off $V(P_a^\ell)$, $V(P_b^\ell)$, i.e. $|\sigma^{-1}(V(K^\ell)) \setminus (V(P_a^\ell) \cup V(P_b^\ell))| \leq 1$.

One can easily confirm that the starting configuration $\sigma_1$ satisfies the four properties above.

**Proof.** As in the proof of Proposition 3.6 assume this proposition is not true, so there exists a swap sequence $\mathcal{V} = \{\sigma_i\}_{i=0}^n$ with $\sigma_0 = \sigma_1 \in FS(X_{L, L})$ of minimal length $\lambda$ containing a vertex that violates Proposition 3.7. In particular, we can assume that all terms $\sigma_i \in \mathcal{V}$ satisfy Proposition 3.6 and $\sigma_1$ fails to satisfy Proposition 3.7 while all $\sigma_i$ for $i < \lambda$ do, and $\lambda \geq 1$ from the preceding comment. Specifically, $\sigma_1$ must break at least one of the four properties of Proposition 3.7. We consider each of the possibilities for the property that the configuration $\sigma_1$ breaks, and derive a contradiction in every case to deduce that none of these four properties can be broken by $\sigma_1$, which yields a contradiction on our initial assumption. We note that there is a corresponding figure to illustrate the configurations described for each case.

**Case 1:** $\sigma_1^{-1}(\kappa^1_1) \notin V(X^1_a)$ or $\sigma_1^{-1}(\kappa^1_1) \notin V(X^1_b)$. Say $\sigma^{-1}(\kappa^1_1) \notin V(X^1_a)$. To achieve $\sigma_1$ from $\sigma_{\lambda-1}$, we must have $\sigma_{\lambda-1}^{-1}(\kappa^1_1) \in \{v^{1,2}_1, v^{1,2}_2\}$. If $\sigma_{\lambda-1}^{-1}(\kappa^1_1) = v^{1,2}_1$, $\kappa^1_2$ swaps into $V(C^2_b) \setminus \{v^{1,2}_1\}$, and $\sigma_{\lambda-1}^{-1}(\kappa^1_1) = \sigma_{\lambda-1}^{-1}(v^{1,2}_1) \in V(P^\ell_b)$, as $\sigma_{\lambda-1}$, $\sigma_1$ would otherwise contradict Proposition 3.6 on the subgraph $C^1_b$. Let $\sigma_\ell$ denote the final term of $\mathcal{V}$ before $\sigma_{\lambda-1}$ with $\sigma_{\ell}^{-1}(\kappa^1_1) \in V(C^1_b)$, so in particular $\sigma_\ell^{-1}(\kappa^1_1) = v^{1,2}_1 (\xi < \lambda - 1$ is well-defined, since $\sigma_1^{-1}(\kappa^1_1) \notin V(C^1_b))$. By minimality of $\lambda$, $\sigma_\ell(v^{1,2}_1) \in V(K^\ell)$ for $\xi + 1 \leq j \leq \lambda$. As such, from $\sigma_1$, we can swap $\kappa^1_a$ along $P^\ell_b$ onto the vertex $v^{1,2}_b$, yielding a configuration contradicting Proposition 3.6 on $C^1_b$. In particular, this argument (with the analogue for the setting where $\sigma^{-1}(\kappa^1_1) \notin V(X^1_b)$) concludes the study of the first three cases for $L = 1$.

Thus, for $L \geq 2$, $\sigma_{\lambda-1}^{-1}(\kappa^1_1) = v^{1,2}_1$, and $\kappa^1_1$ swaps with one of $\{\kappa^2_a, \kappa^2_b\}$ in achieving $\sigma_1$. Assume $\kappa^1_1$ swaps with $\kappa^2_a$ ($\kappa^2_a$ swapping with $\kappa^2_b$ is analogous), so $\kappa^1_a$ swaps onto $V(C^2_b) \setminus \{v^{1,2}_1\}$ while $\sigma_{\lambda-1}(v^{1,2}_1) = \sigma_1(v^{1,2}_b) \in (V(S^2_b) \setminus \{\kappa^2_a\}) \cup V(K^\ell)$, $\sigma_{\lambda-1}^{-1}(\kappa^1_1) = \sigma_1^{-1}(\kappa^1_1) = \sigma_1^{-1}(v^{1,2}_1) \in V(X^1)$, which follows by observing $\sigma_1^{-1}(\sigma_1(V(X^1)))$ (since $\sigma_{\lambda-1}(v^{1,2}_1) = \kappa^1_1$, we have $|\sigma_{\lambda-1}(V(X^1)) \setminus \sigma_1(V(X^1))| \leq 1$ because $\sigma_{\lambda-1}$ satisfies the conditions listed in Proposition 3.7 and $\sigma^{-1}(\kappa^1_1) \notin V(X^1_b)$). Let $\sigma_\ell$ be the final term of $\mathcal{V}$ before $\sigma_{\lambda-1}$ with $\sigma_\ell(v^{1,2}_1) \neq \sigma_{\lambda-1}(v^{1,2}_b)$ ($\xi < \lambda - 1$ is well-defined since $\sigma_\ell(v^{1,2}_1) \neq \sigma_{\lambda-1}(v^{1,2}_b)$). The swap from $\sigma_\ell$ to $\sigma_{\ell+1}$ swaps $\sigma_{\lambda-1}(v^{1,2}_b)$ onto vertex $v^{1,2}_b$ from a vertex in $C^2_b$, so that for $\xi + 1 \leq j \leq \lambda$, $\sigma_{\ell}(v^{1,2}_b)$ remains unchanged. Consider $\sigma_{\ell}(v^{1,2}_b)$, which is either $\kappa^2_a$, $\kappa^2_b$ or a vertex in $V(K^\ell)$ (since $\sigma_{\lambda-1}(v^{1,2}_1) \notin V(S^2_b) \setminus \{\kappa^2_a\}) \cup V(K^\ell)$, these are the only possibilities that avoid breaking Proposition 3.6 or minimality of $\lambda$ as $\sigma_{\ell}(v^{1,2}_b)$ swaps along both $V(C^2_b)$ and $V(C^2_b)$ in moving from $v^{1,2}_b$ to $v^{1,2}_1$. If $\sigma_{\ell}(v^{1,2}_1) \in V(K^\ell)$, then $\sigma_{\ell+1}$ is achieved from $\sigma_\ell$ by swapping two elements of $V(K^\ell)$ (by the restriction on possible values of $\sigma_{\lambda-1}(v^{1,2}_1)$), but then Property (4) is broken by $\sigma_{\ell}$.

Thus, $\sigma_1^{-1}(\kappa^1_1) \notin V(X^1_b)$, for which an entirely analogous argument also raises a contradiction under all possible settings. We conclude that Proposition 3.7(1) cannot have been broken by $\sigma_1$.

**Case 2:** For some $\ell \geq 2$, we have $\sigma_1^{-1}(\kappa^\ell_1) \notin V(X^{\ell-1}) \cup V(X^\ell)$ or $\sigma_1^{-1}(\kappa^\ell_1) \notin V(X^{\ell-1}) \cup V(X^\ell)$. This case is relevant for $L \geq 2$. Assume $\sigma_1^{-1}(\kappa^\ell_1) \notin V(X^{\ell-1}) \cup V(X^\ell)$: in achieving $\sigma_1$ from $\sigma_{\lambda-1}$, $\kappa^\ell_a$ swaps with a vertex in $\{\kappa_{a-1}^\ell, \kappa_{b-1}^\ell\} \cup V(K^\ell) \cup (V(S^\ell_b) \setminus \{\kappa^\ell_a\})$ (other vertices of $V(K^{\ell-1}_a)$ would cause $\sigma_{\lambda-1}$ to break Property (4)), and $\sigma_{\lambda-1}^{-1}(\kappa^\ell_1) \in \{v^{\ell-\ell+1}_a, v^{\ell-\ell+1}_b, v^{\ell+\ell+1}_a, v^{\ell+\ell+1}_b\}$ (precisely, the subset well-defined for $\ell$), or $\kappa^\ell_a$ enters layer $\ell - 2$ or layer $\ell + 1$.
contradicting Proposition 3.6 on \( \sigma \) achieve \( \lambda \) in particular with \( \sigma \). Say \( \sigma \) is the final term in which \( \sigma \) lies in \( V(C_1^\ell) \). In \( \sigma_{\lambda-1} \), there exist three vertices (\( \kappa^\ell_a \) and \( \mu_1, \mu_2 \)) upon \( C_2^{\ell_1} \) that are not in \( V(S^1 \setminus \{k^a\}) \), so that Proposition 3.6 is broken.

(a) \( \sigma \) is the final term in which \( \kappa^a \) lies in \( V(C_1^\ell) \). In \( \sigma_{\lambda-1} \), there exist three vertices (\( \kappa^\ell_b \)) upon \( C_1^{\ell_1} \) that are not in \( V(S^1 \setminus \{k^b\}) \), so that Proposition 3.6 is broken.

(b) From \( \sigma_{\lambda+1} \) onward, \( \sigma_{\lambda-1}(v_b^{1,2}) \) is fixed on \( v_b^{1,2} \). \( \sigma(v_b^{1,2}) \) means \( \sigma(v_b^{1,2}) \) must swap to \( v_b^{1,2} \), while \( \sigma(v_b^{1,2}) \) is in \( V(K^2) \) contradicts (4) \( (\sigma_{\lambda-1}(v_b^{1,2}) \in V(S_b^2) \cup \{k^a\} \cup V(K^2)). \)

Figure 6: Configurations in \( \mathcal{V} \) used to raise a contradiction for both subcases of Case 1.

If \( \sigma_{\lambda-1}(\kappa^a) \in \{v_a^{1,2,\ell-1}, v_b^{1,2,\ell-1}\} \) (for \( \ell \geq 3 \)), \( \kappa^\ell_b \) swaps with one of \( \{\kappa^a, \kappa^\ell_a\} \) to respect Proposition 3.6 and minimality of \( \lambda \). Say \( \sigma_{\lambda-1}(\kappa^a) = v_a^{1,2,\ell-1} \) and \( \kappa^\ell_b \) swaps into \( V(C_{\lambda-1}^{\ell-2}) \setminus \{v_a^{1,2,\ell-1}\} \) with \( \kappa^a \) to achieve \( \sigma \) (the other three possibilities are analogous). Proceeding backwards in \( \mathcal{V} \), \( \sigma_{\lambda-2} \neq \sigma_{\lambda} \) (\( \lambda \) minimal; \( \sigma_{\lambda-1} \neq \sigma_{\lambda} \), so \( \sigma_{\lambda-2} \) is well-defined) and results from \( \sigma_{\lambda-1} \) by swapping \( \kappa^\ell_b \) onto \( V(C_{\lambda-1}^{\ell-2}) \setminus \{v_a^{1,2,\ell-1}\} \) to avoid contradicting Proposition 3.6 on \( \sigma_{\lambda-2} \) or minimality of \( \lambda \) (if neither \( \kappa^\ell_a \) nor \( \kappa^\ell_b \) were swapped in achieving \( \sigma_{\lambda-2} \) from \( \sigma_{\lambda-1} \), swap \( \kappa^\ell_a \) with \( \kappa^a \) from \( \sigma_{\lambda-2} \) to contradict \( \lambda \) minimal). Any other vertex with which \( \kappa^\ell_b \) can swap to yield \( \sigma_{\lambda-2} \) from \( \sigma_{\lambda-1} \) yields a contradiction: another vertex of \( V(K^\ell) \) or \( \kappa^\ell_b \) gives \( \sigma_{\lambda-2} \) breaking Property (4), a vertex of \( V(K^\ell) \) gives \( \sigma_{\lambda-2} \) breaking Property (2) or (3), and a leaf of \( V(S_b^2) \) breaks Proposition 3.6.

Thus, \( \sigma_{\lambda-1}(\kappa^a) \in \{v_a^{1,2,\ell+1}, v_b^{1,2,\ell+1}\} \) (for \( \ell < L \)), for which \( \kappa^\ell_b \) swaps with one of \( \{\kappa^a + 1, \kappa^\ell_a + 1\} \) to respect Proposition 3.6 and minimal of \( \lambda \). Say \( \sigma_{\lambda-1}(\kappa^a) = v_a^{1,2,\ell+1} \) and \( \kappa^\ell_b \) swaps into \( V(C_{\lambda-1}^{\ell+1}) \setminus \{v_a^{1,2,\ell+1}\} \) with \( \kappa^a \) (the other three possibilities are analogous). Let \( \sigma_\xi \), with \( \xi < \lambda - 1 \), be the final term in \( \mathcal{V} \) before \( \sigma_{\lambda-1} \) with \( \sigma_\xi(\kappa^a) \in V(X') \) (\( \xi \) well-defined as \( \sigma_{\xi-1}(\kappa^a) \in V(X') \)): specifically, \( \sigma_\xi(\kappa^a) = v_a^{1,2,\ell+1} \), as \( \kappa^a \) cannot traverse a path from \( v_a^{1,2,\ell+1} \) to \( v_a^{1,2,\ell+1} \) over swaps from \( \sigma_\xi \) to \( \sigma_{\lambda-1} \) without contradicting Proposition 3.6 or minimality of \( \lambda \) (see Case 1). Thus, from \( \sigma_\xi \) to \( \sigma_{\xi+1} \), \( \kappa^\ell_a \) swaps into \( V(C_{\xi+1}^{\ell+1}) \setminus \{v_a^{1,2,\ell+1}\} \) from \( v_a^{1,2,\ell+1} \), and in particular \( \sigma_{\xi+1}(v_a^{1,2,\ell+1}) \in V(S_{\xi+1}^{\ell+2}) \cup V(K^{\ell+1}) \) to respect minimality of \( \lambda \); \( \sigma_{\xi+1}(v_a^{1,2,\ell+1}) \in V(S_{\xi+1}^{\ell+2}) \) causes \( \sigma_{\xi+1}(v_a^{1,2,\ell+1}) \) to be invariant for \( \xi + 1 \leq j \leq \lambda - 1 \), so \( \sigma_{\xi+1}(v_a^{1,2,\ell+1}) \in V(K^{\ell+1}) \). Now observe that \( \sigma_{\xi+1}(v_a^{1,2,\ell+1}) \in V(K^{\ell+1}) \) for \( \xi + 1 \leq j \leq \lambda - 1 \) for any such \( \sigma_j \), \( \sigma_j(v_a^{1,2,\ell+1}) \in V(K^{\ell+1}) \) cannot swap with either \( \{v_a^{1,2,\ell+1}, \kappa^a \} \) (since \( \sigma_{\xi-1}^{-1}(\kappa^a) \in V(X') \)) another vertex in \( V(K^{\ell+1}) \), or a vertex in \( V(S_{\xi+2}^{\ell+3}) \cup V(S_{\xi+2}^{\ell+3}) \cup V(K^{\ell+2}) \) (for the setting \( \kappa^\ell_b \) (1)) without raising a contradiction on \( \lambda \) being minimal. But this contradicts \( \sigma_{\lambda-1}(v_a^{1,2,\ell+1}) = \kappa^\ell_b \).

Therefore, if Case 2 holds, necessarily \( \sigma_{\xi-1}(\kappa^a) \notin V(X^{\ell-1}) \cup V(X^\ell) \), for which we can argue analogously to conclude that Proposition 3.7(2) cannot have been broken by \( \sigma \).

**Case 3:** There exists \( \ell \in [L] \), \( \mu \in V(K^\ell) \setminus \{\kappa^\ell_a, \kappa^\ell_b\} \) with \( \sigma_{\lambda-1}(\mu) \notin V(X^\ell) \). (This becomes \( \mu \in V(K^\ell) \) for \( \ell = L \).) This case is relevant only for \( \ell \geq 2 \). From \( \sigma_{\lambda-1} \) to \( \sigma_{\lambda} \), \( \mu \) swaps with either \( \kappa^\ell_a \) or \( \kappa^\ell_b \) (\( \mu \) swapping with another \( \mu \) in \( V(K^\ell) \) means \( \sigma_{\lambda-1} \) breaks Property (4)), and \( \sigma_{\lambda-1}(\mu) \in \{v_a^{1,2,\ell-1}, v_b^{1,2,\ell-1}, v_a^{1,2,\ell+1}, v_b^{1,2,\ell+1}\} \) (precisely, the subset well-defined for \( \ell \)). Note that \( \ell \geq 2 \), since \( \ell = 1 \) gives \( \sigma_{\lambda-1} \) breaking Property (1). If \( \sigma_{\lambda-1}(\mu) \in \{v_a^{1,2,\ell+1}, v_b^{1,2,\ell+1}\} \) (for \( \ell < L \)), \( \mu \) swapping with either \( \kappa^a \) or \( \kappa^b \) yields...
Case 4: There exists \( \ell \in [L] \) such that at least two vertices of \( V(K^\ell) \) fail to have preimages under \( \sigma_\lambda \) that are in \( V(P_a^\ell) \cup V(P_b^\ell) \), i.e. \( |\sigma_\lambda^{-1}(V(K^\ell)) \setminus (V(P_a^\ell) \cup V(P_b^\ell))| \geq 2 \). For \( \sigma_{\lambda-1} \), exactly one vertex \( \mu \in V(K^\ell) \) has \( \sigma_{\lambda-1}^{-1}(\mu) \notin V(P_a^\ell) \cup V(P_b^\ell) \), since \( |\sigma_{\lambda-1}^{-1}(V(K^\ell)) \setminus (V(P_a^\ell) \cup V(P_b^\ell))| \geq 2 \), \( \lambda \) is minimal and at most one vertex of \( V(K^\ell) \) can be swapped off \( V(P_a^\ell) \cup V(P_b^\ell) \) via an \((X_L,Y_L)\)-friendly swap from \( \sigma_{\lambda-1} \) to \( \sigma_\lambda \). Towards raising a contradiction, we first show that under this setting (i.e. a swap sequence \( \mathcal{V} \) with \( \sigma_0 = \sigma_\lambda \) and \( \sigma_i \) respecting Proposition 3.7 for \( i < \lambda \), for any configuration \( \sigma_i \in \mathcal{V} \) with \( i < \lambda \) and any \( \ell \in [L] \), the following statements must hold. Note that for any \( \ell \in [L] \), all \( \sigma_i \) with \( i < \lambda \) have

\( \sigma_{\lambda-1} \) violating Property (2). Consider \( \sigma_{\lambda-1}^{-1}(\mu) \in \{v_\mu^{\ell-1}, v_\mu^{\ell-1,\ell}\} \); assume \( \sigma_{\lambda-1}^{-1}(\mu) = v_\mu^{\ell-1,\ell} \) and \( \mu \) swaps with \( \kappa^\ell_a \) (the other three cases are analogous). Proceeding backwards in \( \mathcal{V} \), \( \sigma_{\lambda-2} \neq \sigma_\lambda \) (\( \lambda \) minimal, \( \sigma_{\lambda-1} \neq \sigma_s \)) and results from \( \sigma_{\lambda-1} \) by swapping \( \mu \) with some \( \mu' \in V(K^\ell) \) on \( (V(C^\ell_a) \setminus \{v_\mu^{\ell-1,\ell}\}) \) to avoid contradicting Proposition 3.6 on \( C^\ell_a \) (if neither \( \mu \) nor \( \kappa^\ell_a \) were swapped in achieving \( \sigma_{\lambda-2} \) from \( \sigma_{\lambda-1} \), swap \( \mu \), \( \kappa^\ell_a \) from \( \sigma_{\lambda-2} \) to contradict \( \lambda \) minimal; if \( \mu' = \kappa^\ell_a \), \( \sigma_{\lambda-2} \) breaks Property (4) via \( \sigma_{\lambda-2}^{-1}(\kappa^\ell_a), \sigma_{\lambda-2}^{-1}(\kappa^\ell_b) \notin V(P_a^\ell) \cup V(P_b^\ell) \)). But then \( \sigma_{\lambda-2} \) breaks Property (4) via \( \sigma_{\lambda-2}^{-1}(\mu), \sigma_{\lambda-2}^{-1}(\mu') \notin V(P_a^\ell) \cup V(P_b^\ell) \). Therefore, \( \sigma_\lambda \) cannot break Proposition 3.7(3).

\[ \sigma_{\lambda-2} \rightarrow \sigma_{\lambda-1} \rightarrow \sigma_\lambda \]
|σ⁻¹(V(K′)) \ (V(P_a) \cup V(P_b))| \leq 1 (by minimality of λ, so that all such σ_i respect Property (4)), and at most one statement holds nontrivially for any such term σ_i (since |σ⁻¹(V(K′)) \ (V(P_a) \cup V(P_b))| ∈ \{0, 1\}).

- If σ⁻¹(V(K′)) ⊂ V(P_a) \cup V(P_b) and σ_i(\{v_a^i, v_b^i\}) ⊂ V(K′), one of κ_a^i or κ_b^i lies upon V(P_a) \ \{v_a^i\} or V(P_b) \ \{v_b^i\} (i.e. in this setting, |σ⁻¹(\{κ_a^i, κ_b^i\}) \cap (V(P_a) \ \{v_a^i\} \cup (V(P_b) \ \{v_b^i\}))| = 1).
- If there exists μ ∈ V(K′) with σ⁻¹(μ) ∉ V(P_a) \cup V(P_b) and σ_i(μ) ∈ V(K′), one of κ_a^i or κ_b^i lies upon V(P_a) \ \{v_a^i\}. Similarly, if σ_i(v_b^i) ∈ V(K′), one of {κ_a^i, κ_b^i} lies upon V(P_b) \ \{v_b^i\}.

Figure 9: An illustration of the two statements proved for all terms σ_i \in \mathcal{V}, i < λ in Case 4. Subgraphs and vertices in the figures that are colored in red correspond to preimages of vertices in V(K′): note that by Property (4), at most two vertices of V(P_a) \cup V(P_b) do not map to V(K′) under all such σ_i. For this figure, we make the special distinction of coloring preimages of elements in {κ_a^i, κ_b^i} blue, to distinguish from vertices of V(K′).

We prove these two statements inductively for all σ_i \in \mathcal{V} with i < λ. These two statements certainly hold for σ_e for all \ell \in [L], so assume them true for σ_i for some 0 \leq i < λ - 1. We now aim to prove that σ_i+1 satisfies both statements: in what follows, assume we refer (unless stated otherwise) to some fixed, arbitrary \ell \in [L]. We break into cases based on whether or not there exists a unique μ ∈ V(K′) such that σ⁻¹(μ) ∉ V(P_a) \cup V(P_b).

Subcase 4.1: No such μ ∈ V(K′) exists for σ_i (i.e. σ⁻¹(V(K′)) ⊂ V(P_a) \cup V(P_b)). Begin by considering the subcase where σ_i(\{v_a^i, v_b^i\}) ⊂ V(K′), so that one of {κ_a^i, κ_b^i} lies upon V(P_a) \ \{v_a^i\} or V(P_b) \ \{v_b^i\} by the induction hypothesis. Here, either σ_i(v_a^i) = σ_{i+1}(v_a^i) and σ_i(v_b^i) = σ_{i+1}(v_b^i) (for which the first statement certainly holds for σ_i+1, while the second holds trivially) or one of the vertices σ_i(v_a^i) or σ_i(v_b^i) swaps to achieve σ_i+1. For the latter, say σ_i(v_b^i) is swapped (the setting with σ_i(v_a^i) swapped is analogous). If σ_i(v_b^i) is swapped within V(P_a), then both statements are easily seen to hold for σ_i+1. If σ_i(v_b^i) is swapped out of V(P_a), by the induction hypothesis and the fact that σ_i respects Proposition 3.7, σ_i(v_a^i) cannot swap out of V(P_a) unless \ell = 1 and with κ_a^1, for which the second statement is upheld by σ_i+1.

Now, if σ_i(\{v_a^i, v_b^i\}) ∉ V(K′), exactly one of σ_i(v_a^i), σ_i(v_b^i) is in V(K′) (since |V(V(P_a)) \cup V(V(P_b))| = 2ν + 1 and |V(K′)| = 2ν, and σ_i((V(P_a) \ \{v_a^i\}) \cup (V(P_b) \ \{v_b^i\})) ⊂ V(K′). Say σ_i(v_a^i) ∈ V(K′) and σ_i(v_b^i) ∉ V(K′) (the setting σ_i(v_b^i) ∈ V(K′) and σ_i(v_a^i) ∉ V(K′) is analogous). In achieving σ_i+1 from σ_i, if we swap σ_i(v_a^i) off of V(P_a), then there exists μ ∈ V(K′) with σ⁻¹(μ) ∉ V(P_a) \cup V(P_b) (namely, σ_i(v_a^i)), but σ_{i+1}(v_a^i) ∉ V(K′), σ_i(v_b^i) = σ_{i+1}(v_b^i) ∉ V(K′), so both statements trivially hold. If not, we have that [3] For the remainder of the proofs for these subcases, we do not explicitly comment on the other statement holding trivially.
Subcase 4.2: There exists $\mu \in V(K^\ell)$ with $\sigma_i^{-1}(\mu) \notin V(P_a^\ell) \cup V(P_b^\ell)$. We shall further break into cases based on the subset of $\sigma_i^{-1}(\mu)$ that lies in $V(K^\ell)$. If $\sigma_i^{\ell}(\{v_a^\ell, v_b^\ell\}) \in V(K^\ell)$, then by the induction hypothesis, both $\kappa_a^\ell$ and $\kappa_b^\ell$ lie upon one of $\{v_a^\ell, v_b^\ell\}$ and $V(P_a^\ell) \setminus \{v_b^\ell\}$: this can only occur if $\ell = 1$, since $\sigma_i$ respects Property (4). If $\sigma_i(v_a^\ell) = \sigma_{i+1}(v_a^\ell)$ and $\sigma_i(v_b^\ell) = \sigma_{i+1}(v_b^\ell)$, both statements certainly remain true. Thus, it is straightforward to observe that the swap that achieves $\sigma_{i+1}(v_a^\ell) \in V(K^\ell)$, if $\sigma_i(v_a^\ell) \neq \sigma_{i+1}(v_a^\ell)$, we can have $\sigma_i(v_b^\ell)$ swapped within $P_a^\ell$, or $\sigma_i(v_b^\ell)$ swapped off $P_b^\ell$ with $\mu$ (necessarily with $\mu$, since $\sigma_{i+1}$ respects Property (4)): in both settings, the second statement is easily seen to remain true.

In any setting within this subcase, both statements are satisfied by $\sigma_i$. 

If $\sigma_i(v_a^\ell) \in V(K^\ell)$ and $\sigma_i(v_b^\ell) \notin V(K^\ell)$, then as before, either $\sigma_i(v_a^\ell)$ is swapped within $P_a^\ell$, or $\sigma_i(v_b^\ell)$ is swapped off $P_b^\ell$ with $\mu$: in both settings, the second statement remains true. The case $\sigma_i(v_a^\ell) \in V(K^\ell)$ and $\sigma_i(v_b^\ell) \notin V(K^\ell)$ is entirely analogous.

Thus, under all settings and any $\ell \in [L]$, the statements hold for $\sigma_{i+1}$, and therefore for any $\sigma_i$, with $0 \leq i < \lambda$ by the induction. From here, it is straightforward to observe that the swap that achieves $\sigma_{i+1}$ involves either $v_a^\ell$ or $v_b^\ell$ and an adjacent vertex in $V(C_a^\ell)$ or $V(C_b^\ell)$, respectively, and swaps some $\mu' \in V(K^\ell)$ off of $v_a^\ell$ or $v_b^\ell$ with either $\kappa_a^\ell$ or $\kappa_b^\ell$ (the other possibilities for the vertex swapping with $\mu'$ are other vertices of $V(K^\ell)$ or vertices that must lie in layers $\ell + 1$ or $\ell + 2$ under $\sigma_{i+1}$ by Proposition 3.6 and minimality of $\lambda$ in the case $\mu' \in \{\kappa_a^{\ell+1}, \kappa_b^{\ell+1}\}$). Here, assume that $\sigma_{i-1}(v_a^\ell) = \mu'$ and that $\mu'$ swaps with $\kappa_i^\ell$; the other cases are analogous. Recall that for $\sigma_{i-1}$, exactly one vertex $\mu \in V(K^\ell)$ has $\sigma_{i-1}^{-1}(\mu) \notin V(P_a^\ell) \cup V(P_b^\ell)$: it follows from the second statement above that $\sigma_{i-1}^{-1}(\kappa_i^\ell) \notin V(P_a^\ell) \setminus \{v_b^\ell\}$. This contradicts the minimality of $\lambda$, as $\sigma_{i-1}$ breaks Property (1) if $\ell = 1$ (since in particular, $\sigma_{i-1}^{-1}(\kappa_i^\ell), \sigma_{i-1}^{-1}(\kappa_i^\ell) \neq v_a^\ell$) and Property (4) if $\ell \geq 2$ (since $\sigma_{i-1}^{-1}(\kappa_i^\ell), \sigma_{i-1}^{-1}(\kappa_i^\ell) \notin V(P_a^{\ell-1}) \cup V(P_b^{\ell-1})$, causing $\sigma_{i-1}$ to break (4) on $\ell - 1$).

This shows that Proposition 3.7(4) cannot have been broken by $\sigma_\lambda$. Together with the conclusions of the

![Figure 10: Deriving a contradiction for Case 4, after proving the two statements for all terms $\sigma_i \in \mathcal{V}$ with $i < \lambda$. This figure depicts the setting where $\sigma_{\lambda-1}(v_a^\ell) = \mu'$, and $\mu'$ swaps with $\kappa_i^\ell$ to achieve $\sigma_\lambda$: the second statement then yields that $\kappa_i^\ell$ lies upon $V(P_a^\ell) \setminus \{v_b^\ell\}$, as some $\mu \in V(K^\ell)$ has $\sigma_{\lambda-1}^{-1}(\mu) \notin V(P_a^\ell) \cup V(P_b^\ell)$.](image-url)
other three cases, we conclude that $\sigma_\lambda$ satisfies all properties of Proposition 3.7, which contradicts $\sigma_\lambda$ failing to satisfy at least one of the properties, completing the proof.

**Remark 3.8.** We can understand Propositions 3.6 and 3.7 as separating elements of $V(Y_L)$ so that for any configuration $\sigma \in V(\text{FS}(X_L, Y_L))$ in the same connected component as $\sigma_\lambda$, specific vertices of $Y_L$ can lie only upon specific subgraphs of $X_L$. In particular, for any $\ell \in [L]$, it follows from these two results that $\sigma(V(P_{a}^{l}) \cup V(P_{b}^{l})) = \{v_{a}^{l}, v_{b}^{l}\} \subset V(K^{\ell}) \cup \{\kappa_{a}^{l}, \kappa_{b}^{l}\}$. Furthermore, we inductively argue exactly as in Case 4 of the proof for Proposition 3.7, to conclude that for any such configuration $\sigma$ reachable from $\sigma_\lambda$ via some swap sequence (i.e., $\sigma$ in the same connected component as $\sigma_\lambda$) and $\ell \in [L]$, the following must hold.

1. If $\sigma^{-1}(V(K^{\ell})) \subset V(P_{a}^{l}) \cup V(P_{b}^{l})$ and $\{\sigma(v_{a}^{l}), \sigma(v_{b}^{l})\} \subset V(K^{\ell})$, then either $\sigma^{-1}(\kappa_{a}^{l}) \in V(P_{a}^{l}) \setminus \{v_{a}^{l}\}$ or $\sigma^{-1}(\kappa_{b}^{l}) \in V(P_{b}^{l}) \setminus \{v_{b}^{l}\}$.

2. If there exists $\mu_1 \in V(K^{\ell})$ with $\sigma^{-1}(\mu_1) \notin V(P_{a}^{l}) \cup V(P_{b}^{l})$ and $\mu_2 \in V(K^{\ell})$ with $\sigma^{-1}(\mu_2) = v_{a}^{l}$ or $\sigma^{-1}(\mu_2) = v_{b}^{l}$, then either $\sigma^{-1}(\kappa_{a}^{l}) \in V(P_{a}^{l}) \setminus \{v_{a}^{l}\}$ or $\sigma^{-1}(\kappa_{b}^{l}) \in V(P_{b}^{l}) \setminus \{v_{b}^{l}\}$.

We shall again use Property (2) in the proof of Proposition 3.11.

We are now ready to prove a third invariant of any configuration in the same connected component as $\sigma_\lambda$ in $\text{FS}(X_L, Y_L)$. Toward this, we begin by introducing the following notion of ordering for elements of $V(K^{\ell})$ in the same partition class.

**Definition 3.9.** For $\ell \in [L]$ and $\mu_1, \mu_2 \in V(K^{\ell})$ in the same partition class, say that $\mu_1$ is leftwards of $\mu_2$ on a configuration $\sigma \in V(\text{FS}(X_L, Y_L))$ in the same component as $\sigma_\lambda$ if (exactly) one of the following holds.

- $\sigma^{-1}(\{\mu_1, \mu_2\}) \subset V(P_{a}^{l})$ and $d(\sigma^{-1}(\mu_2), v_{a}^{l}) < d(\sigma^{-1}(\mu_1), v_{a}^{l})$.
- $\sigma^{-1}(\{\mu_1, \mu_2\}) \subset V(P_{b}^{l})$ and $d(\sigma^{-1}(\mu_1), v_{b}^{l}) < d(\sigma^{-1}(\mu_2), v_{b}^{l})$.
- $\sigma^{-1}(\mu_1) \in V(P_{a}^{l})$ and $\sigma^{-1}(\mu_2) \in V(P_{b}^{l})$.

![Diagram](image_url)

**Figure 11:** An illustration of the three statements constituting the definition of $\mu_1$ leftwards of $\mu_2$, for vertices $\mu_1, \mu_2 \in V(K^{\ell})$ in the same partition class.

Observe that for any such $\mu_1, \mu_2 \in V(K^{\ell})$ and $\sigma \in V(\text{FS}(X_L, Y_L))$, if $\mu_1$ is leftwards of $\mu_2$ on $\sigma$, then $\sigma^{-1}(\{\mu_1, \mu_2\}) \subset V(P_{a}^{l}) \cup V(P_{b}^{l})$, and that if $\sigma^{-1}(\{\mu_1, \mu_2\}) \subset V(P_{a}^{l}) \cup V(P_{b}^{l})$, then we must have either that $\mu_1$ is leftwards of $\mu_2$ on $\sigma$ or $\mu_2$ is leftwards of $\mu_1$ on $\sigma$. It thus follows that for any $\mu_1, \mu_2 \in V(K^{\ell})$ in the same partition class, either $\mu_1$ is leftwards of $\mu_2$ on $\sigma_\lambda$ or $\mu_2$ is leftwards of $\mu_1$ on $\sigma_\lambda$, since in particular $\sigma^{-1}(V(K^{\ell})) \subset V(P_{a}^{l}) \cup V(P_{b}^{l})$. The following shows that the leftwards relation that is established by $\sigma_\lambda$ cannot change for other configurations $\sigma \in V(\text{FS}(X_L, Y_L))$ in the same connected component as $\sigma_\lambda$.

**Proposition 3.10.** Take any $\ell \in [L]$ and corresponding vertices $\mu_1, \mu_2 \in V(K^{\ell})$ in the same partition class, with $\mu_1$ leftwards of $\mu_2$ in $\sigma_\lambda$. If $\sigma \in V(\text{FS}(X_L, Y_L))$ is any configuration in the same connected component as $\sigma_\lambda$ that satisfies $\sigma^{-1}(\{\mu_1, \mu_2\}) \subset V(P_{a}^{l}) \cup V(P_{b}^{l})$, then $\mu_1$ is leftwards of $\mu_2$ in $\sigma$. 

4Statements in this proof arguing on the minimality of $\lambda$ (i.e., on $\sigma_\lambda$ being the first term in $V$ failing to satisfy one of the properties of Proposition 3.7) are instead argued by referring directly to Proposition 3.7, as this result has now been proven. Further, we induct on any arbitrary swap sequence from $\sigma_\lambda$ to $\sigma$ in the same component.
Proof. Let \( \mathcal{V} = \{ \sigma_i \}_{i=0}^{\infty} \) with \( \sigma_0 = \sigma_s \) and \( \sigma_{\lambda} = \sigma \) denote any swap sequence in \( \text{FS}(X_L, Y_L) \) starting from \( \sigma_s \) and ending at \( \sigma \). By (4) of Proposition 3.7, any \( \sigma_i \in \mathcal{V} \) must have that \( |\sigma_i^{-1}(V(\mathcal{K}_i')) \setminus (V(P_0') \cup V(P_0'))| \leq 1 \), so in particular \( |\sigma_i^{-1}\{\mu_1, \mu_2\} \setminus (V(P_0') \cup V(P_0'))| \leq 1 \). Consider the subsequence \( \mathcal{V}' = \{ \sigma_i \}_{i=0}^{\lambda'} \), \( \lambda' \leq \lambda \) consisting of \( \sigma_0 = \sigma_s \) and all configurations \( \sigma_i \in \mathcal{V} \) where \( |\sigma_i^{-1}\{\mu_1, \mu_2\} \setminus (V(P_0') \cup V(P_0'))| = 1 \) and \( |\sigma_i^{-1}\{\mu_1, \mu_2\} \setminus (V(P_0') \cup V(P_0'))| = 0 \).

The vertex \( \mu_1 \) is leftwards of \( \mu_2 \) on \( \sigma_s = \sigma_0 \), which in particular has that \( |\sigma_i^{-1}\{\mu_1, \mu_2\} \setminus (V(P_0') \cup V(P_0'))| = 0 \). If \( \mu_1 \) is leftwards of \( \mu_2 \) on \( \sigma_{i+1} \), then \( \mu_1 \) is leftwards of \( \mu_2 \) on \( \sigma_{k} \) for all \( k \geq \lambda' \), as \( \mu_1 \) and \( \mu_2 \) remain upon their corresponding paths. It thus suffices to show that \( \mu_1 \) is leftwards of \( \mu_2 \) on \( \sigma_{i+1} \), toward which we induct on \( j \) to show that \( \mu_1 \) is leftwards of \( \mu_2 \) on \( \sigma_{i,j} \) for all \( 0 \leq j \leq \lambda' \): as the statement holds for \( j = 0 \), assume \( \mu_1 \) is leftwards of \( \mu_2 \) on \( \sigma_{i,j} \) for some \( 0 \leq j < \lambda' \), and consider \( \sigma_{i,j+1} \). Take the unique vertex \( \mu \in \{ \mu_1, \mu_2 \} \) such that \( \sigma_{i,j+1}^{-1}(\mu) \notin V(P_0') \cup V(P_0') \) (recall \( |\sigma_i^{-1}((\mu_1, \mu_2)) \setminus (V(P_0') \cup V(P_0'))| = 1 \)): to achieve \( \sigma_{i,j+1} \), \( \mu \) must swap onto \( v_k' \) or \( v_k' \), for which confirming that \( \mu_1 \) is leftwards of \( \mu_2 \) on \( \sigma_{i,j+1} \) is straightforward by breaking into cases based on which of the three statements of Definition 3.9 applies for \( \mu \) leftwards of \( \mu_2 \) on \( \sigma_{i,j} \). (Specifically, the non-\( \mu \) term in \( \{ \mu_1, \mu_2 \} \) must remain upon its corresponding path throughout all configurations \( \sigma_{k}, i_j \leq k \leq i_{j+1} \), since \( \mu_1 \) and \( \mu_2 \) are nonadjacent in \( V(Y_L) \) and thus cannot swap.)

We are now ready to show the main result (in conjunction with Proposition 3.10) we will need for the proof of lower-bounding the diameter of the connected component of \( \text{FS}(X_L, Y_L) \) that has \( \sigma_s \).

**Proposition 3.11.** Take any configuration \( \sigma \in \text{FS}(X_L, Y_L) \) in the same connected component as \( \sigma_s \), and any \( \ell \in [L-1] \). If \( \sigma^{-1}(\kappa_{\ell}^{t+1}) \notin V(P_0') \cup V(P_0') \), then \( V(K_0) \subseteq \sigma(V(P_0')) \). Similarly, if \( \sigma^{-1}(\kappa_{\ell}^{t+1}) \notin V(P_0') \cup V(P_0') \), then \( V(K_0) \subseteq \sigma(V(P_0')) \).

Proof. Take arbitrary \( \ell \in [L-1] \), and \( \sigma \in \text{FS}(X_L, Y_L) \) such that \( \sigma^{-1}(\kappa_{\ell}^{t+1}) \notin V(P_0') \cup V(P_0') \). Certainly \( V(K_0) \subseteq \sigma(V(P_0') \cup V(P_0')) \), as \( |\sigma^{-1}(V(K')) \setminus (V(P_0') \cup V(P_0'))| = 1 \) by Proposition 3.7(4). To prove that \( V(K_0) \subseteq \sigma(V(P_0') \cup V(P_0')) \), consider any swap sequence \( \mathcal{V} = \{ \sigma_i \}_{i=0}^{\infty} \) from \( \sigma_0 = \sigma_s \) to \( \sigma_{\lambda} = \sigma_s \), and consider the final term \( \sigma_{\ell} \in \mathcal{V} \) before \( \sigma \) where \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) \in V(P_0') \cup V(P_0') \), \( |\ell| < \lambda \), since \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) \in V(P_0') \cup V(P_0') \), for which we must have \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) \in \{ v_k', v_k' \} \) and \( \sigma_{\ell}^{-1}(V(K_0) \setminus (\kappa_{\ell}^{t+1})) \subseteq V(P_0') \). To observe this latter claim, note that \( \kappa_{\ell}^{t+1} \) is not adjacent to any vertex in \( V(K_0') \), so we must have \( \sigma_{\ell}^{-1}(V(K_0) \setminus (\kappa_{\ell}^{t+1})) \subseteq V(P_0') \) so that \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) \) does not contradict Proposition 3.7(4) (recall \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) \notin V(P_0') \cup V(P_0') \)). Proposition 3.10 then yields that necessarily \( \sigma_{\ell}^{-1}(V(K_0) \setminus (\kappa_{\ell}^{t+1})) \subseteq V(P_0') \) as for any \( \mu \in V(K_0) \), we must have that \( \kappa_{\ell} \) is leftwards of \( \mu \) on \( \sigma_s \) since \( \kappa_{\ell} \) is leftwards of \( \mu \) on \( \sigma_s \).

Either \( V(K_0) \subseteq \sigma_{\ell}^{-1}(V(P_0')) \), from which \( V(K_0) \subseteq \sigma(V(P_0')) \) since \( \kappa_{\ell}^{-1}(\kappa_{\ell}^{t+1}) \notin V(P_0') \cup V(P_0') \) for all \( k > \xi \) yields \( V(K_0) \subseteq \sigma_{\ell}^{-1}(V(P_0')) \) for such \( k \) due to Proposition 3.7(4), or \( V(K_0) \setminus \sigma_{\ell}^{-1}(V(P_0')) \geq 1 \). For this latter setting, break into cases based on \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) \in \{ v_k', v_k' \} \). Remark 3.8(2) and Proposition 3.7(4) yield \( |V(K_0) \setminus \sigma_{\ell}^{-1}(V(P_0'))| = 1 \) (here, note that \( |\sigma_{\ell}^{-1}(V(K_0) \setminus (\kappa_{\ell}^{t+1}))| \subseteq V(P_0') \), so denote \( \mu_b \in V(K_0) \setminus \sigma_{\ell}^{-1}(V(P_0')) \), now, to achieve \( \sigma_{\ell+1} \) from \( \sigma_{\ell} \), Remark 3.8(2) and Proposition 3.7(4) yields that \( \kappa_{\ell}^{t+1} \) must swap with \( \mu_b \) to achieve \( \sigma_{\ell+1} \) if \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) = v_k' \), while this setting is impossible for \( \sigma_{\ell}^{-1}(\kappa_{\ell}^{t+1}) = v_k' \). From here, \( V(K_0) \subseteq \sigma(V(P_0')) \) follows as in the case where \( V(K_0) \subseteq \sigma_{\ell}^{-1}(V(P_0')) \).

We note that for the setting of Proposition 3.11, Proposition 3.10 determines the relative positioning of \( V(K_0) \) on \( \sigma(V(P_0')) \), as the leftwards relations between vertices of \( \sigma(V(K_0')) \) in \( \sigma_s \) must be preserved by \( \sigma \).

### 3.3 Knob Extractions

In the construction of the families \( X_L \) and \( Y_L \), we were motivated towards graphs \( \text{FS}(X_L, Y_L) \) that have two configurations with minimum distance strictly greater than \( n^\ell \) swaps apart. The idea here is the existence of a subroutine of swaps which can be executed on each layer of \( X_L \) such that one iteration of this subroutine in layer \( \ell + 1 \) necessarily requires a number of iterations in layer \( \ell \) linear in the number of vertices \( n \) of \( X_L \) and \( Y_L \). This shall be formalized by a notion that we refer to as \( \ell \)-knob extractions.
**Definition 3.12.** Take \( \sigma, \tau \in V(\text{FS}(X_L, Y_L)) \) in the same connected component as \( \sigma_s \). For \( \ell \in [L] \), call \( \tau \) an \( \ell \)-knob extraction of \( \sigma \) if one of the following two statements holds.

- \( \sigma^{-1}(V(K_a^\ell)) \subset V(P_a^\ell) \) and \( \tau^{-1}(V(K_b^\ell)) \cap V(P_a^\ell) = \emptyset \), \( \sigma^{-1}(V(K_b^\ell)) \subset V(P_a^\ell) \) and \( \tau^{-1}(V(K_a^\ell)) \cap V(P_a^\ell) = \emptyset \).

**Proof.** For any integer \( \ell \geq 1 \) and graphs \( X_L \in \mathcal{X}_L \), \( Y_L \in \mathcal{Y}_L \) on the same number of vertices, there exists an \( L \)-knob extraction of \( \sigma_s \) in the same connected component as \( \sigma_s \) in \( \text{FS}(X_L, Y_L) \).

**Proposition 3.13.** For any integer \( L \geq 1 \) and graphs \( X_L \in \mathcal{X}_L \), \( Y_L \in \mathcal{Y}_L \) on the same number of vertices, there exists an \( L \)-knob extraction of \( \sigma_s \) in the same connected component as \( \sigma_s \) in \( \text{FS}(X_L, Y_L) \).

**Proof.** For \( \ell \in [L] \) and \( \sigma \) in the same component as \( \sigma_s \), denote \( N_a^\ell = \sigma(V(C_a^\ell)) \) \( \backslash \) \( (V(S_b^\ell) \backslash \{\kappa_a^\ell\}) \) and \( N_b^\ell = \sigma(V(C_b^\ell)) \) \( \backslash \) \( (V(S_a^\ell) \backslash \{\kappa_b^\ell\}) \): recall from Proposition 3.6 that \( |N_a^\ell| = |N_b^\ell| = 2 \). As in Lemma 3.3, if \( \kappa_a^\ell \) is in \( N_a^\ell \) and commutes with the other vertex \( \mu_a \in N_b^\ell \), we can cyclically rotate \( \sigma(V(C_a^\ell)) \) \( \backslash \) \( \{\kappa_a^\ell\} \) around \( \kappa_a^\ell \): call such a swap sequence a \( \kappa_a^\ell \)-rotation involving \( \mu_a \). More precisely, define a \( \kappa_a^\ell \)-rotation involving \( \mu_a \) to be any swap sequence \( \{\sigma_i\}_{i=0}^{\lambda-1} \) with \( \lambda \) a nonzero multiple of \( \nu + 1 \) (so \( \kappa_a^\ell \) begins and ends in the same position), where \( \sigma_i(V(C_a^\ell)) = \{\mu_a\} \cup V(S_b^\ell) \) for all \( 0 \leq i \leq \lambda \), and if we enumerate \( V(C_a^\ell) = \{v_0, v_1, \ldots, v_{\nu}\} \) such that \( v_0 = \sigma_0^{-1}(k_a^\ell) \) and \( \{v_{i-1}, v_i\} \in E(C_a^\ell) \) for all \( i \in [\nu] \), then \( \sigma_j(v_i) = \kappa_a^\ell \) for all \( i \equiv j \pmod{\nu + 1} \). Analogously define a \( \kappa_b^\ell \)-rotation involving \( \mu_b \) with respect to \( \mu_b \in N_b^\ell \) and \( C_b^\ell \).

We shall prove a stronger claim. Specifically, for any integers \( L, \eta \geq 1 \), take any pair of corresponding graphs \( (X_L, Y_L) \) with starting configuration \( \sigma_s \in V(\text{FS}(X_L, Y_L)) \). Then there exists a swap sequence \( \{\sigma_i\}_{i=0}^{\lambda-1} \subset V(\text{FS}(X_L, Y_L)) \), \( \sigma_0 = \sigma_s \) with subsequence \( \{\sigma_i\}_{i=0}^{\lambda-1} \) such that the following hold.

- For every \( j \in [\eta] \), \( \sigma_j(V(P_b^\ell)) = \sigma_{j-1}(V(P_b^\ell)) \) \( \backslash \) \( \{v_a^\ell\} \) and \( \sigma_j(V(P_a^\ell)) \backslash \{v_a^\ell\} = \sigma_{j-1}(V(P_a^\ell)) \).
- For every \( j \in [\eta] \) and \( \mu \in V(K_L) \), there exists a \( \kappa_a^\ell \)-rotation and \( \kappa_b^\ell \)-rotation involving \( \mu \) that is a contiguous subsequence of \( \{\sigma_i\}_{i=1}^{\lambda-1} \).

We perform double induction by inducting on \( L \) and showing that for any fixed value of \( L \) being considered, this stronger claim holds for any \( \eta \geq 1 \). (The proposition follows by taking \( \eta = 1 \) for \( L \), with \( \sigma_L \) easily seen to be an \( L \)-knob extraction of \( \sigma_s \); general \( \eta \) is needed for the proceeding induction.)
For \( L = 1 \), consider arbitrary \((X_1, Y_1)\). Perform a \( \kappa_{b}^{1}\)-rotation involving \( \sigma_s(v_b^1) \) to move \( \sigma_s(v_b^1) \) to \( v_1^1 \), a \( \kappa_{a}^{1}\)-rotation involving \( \sigma_s(v_a^1) \) to move \( \sigma_s(v_a^1) \) to \( v_1^1 \), and swap \( \sigma_s(v_a^1) \) through \( V(P_b^1) \), yielding a vertex \( \mu \in V(K_a^1) \) upon \( v_b^1 \). Now perform a \( \kappa_{a}^{1}\)-rotation involving \( \mu \) to move \( \mu \) to \( v_1^1 \), a \( \kappa_{b}^{1}\)-rotation involving \( \mu \) to move \( \mu \) to \( v_b^1 \), and swap up through \( V(P_b^1) \), yielding a vertex in \( V(K_b^1) \) upon \( v_b^1 \). Repeating this process until we exhaust \( V(K^1) \) yields a 1-knob extraction \( \sigma \) of \( \sigma_s \) with \( \sigma(V(P_b^1)) = V(K_b^1) = \sigma_s(V(P_b^1) \setminus \{v_a^1\}) \) and \( \sigma(V(P_a^1) \setminus \{v_a^1\}) = \sigma_s(V(P_b^1)) \). It is easy to see that we can repeat these rotations and swaps to interchange the positions of \( V(K_a^1) \) and \( V(K_b^1) \) arbitrarily many times (i.e. for any \( \eta \geq 1 \)), with a \( \kappa_{a}\)-rotation and \( \kappa_{b}\)-rotation involving \( \mu \) for every \( \mu \in V(K^1) \) executed during every such interchange.

Now assume the statement holds for \( L = m \geq 1 \), and consider \((X_{m+1}, Y_{m+1})\) with starting configuration \( \sigma_s \in V(\mathcal{FS}(X_{m+1}, Y_{m+1})) \). Let \( X_m \) correspond to the first \( m \) layers of \( X_{m+1} \) and \( Y_m = Y_{m+1} |_{\sigma_s(V(X_m))} \): this is an abuse of notation, but these subgraphs are isomorphic to the corresponding \((X_m, Y_m)\) under
their usual construction, so we can extract a swap sequence \( \{ \sigma_i \}_{i=0}^{\lambda} \subset V(\mathcal{FS}(X_m, Y_m)) \) with subsequence \( \{ \sigma_i \}_{i=0}^{2^\eta+1} \) satisfying the stronger claim by the induction hypothesis (namely, we take \( L = m, \eta = 2\nu + 1 \)). In particular, \( \sigma_s \mid_{V(X_m)} = \sigma_0 \) gives the starting configuration for \( \mathcal{FS}(X_m, Y_m) \), so \( \{ \sigma_i \}_{i=0}^{\lambda} \) can be understood to be in \( \mathcal{FS}(X_{m+1}, Y_{m+1}) \) starting from \( \sigma_s \) if we were to extend the domain of all \( \sigma_i \) to \( V(X_{m+1}) \) by setting \( \sigma_i(V(X_{m+1}) \setminus V(X_m)) = \sigma_s(V(X_{m+1}) \setminus V(X_m)) \). Observe that this notion of extending \( \{ \sigma_i \}_{i=0}^{\lambda} \subset V(\mathcal{FS}(X_m, Y_m)) \) to \( V(\mathcal{FS}(X_{m+1}, Y_{m+1})) \) can be applied to any configuration upon \( V(X_{m+1}) \setminus V(X_m) \): this will be exploited frequently in the proceeding description.

We shall now construct a swap sequence \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \subset V(\mathcal{FS}(X_{m+1}, Y_{m+1})) \) with \( \tilde{\sigma}_0 = \sigma_s \) satisfying the stronger claim for \( L = m + 1, \eta = 1 \). From \( \{ \sigma_i \}_{i=0}^{\lambda} \) with subsequence \( \{ \sigma_i \}_{i=0}^{2^\nu+1} \), take \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \), which has subsequence \( \{ \sigma_i \}_{i=0}^{2^\eta+1} \subset \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \), the \( \kappa_b^m \)-rotation involving \( \kappa_b^m \). We shall construct a swap sequence \( \mathcal{S}_1 \subset V(\mathcal{FS}(X_{m+1}, Y_{m+1})) \) in three parts (denoted by the second superscript in what follows). First, extend \( \{ \sigma_i \}_{i=0}^{2^\eta+1} \) to \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \) with \( \tilde{\sigma}_i(V(X_{m+1}) \setminus V(X_m)) = \sigma_s(V(X_{m+1}) \setminus V(X_m)) \). Second, let \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \) be such a swap sequence. Then \( \mathcal{S}_1 \) results by merging the three sequences \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \), \( \{ \sigma_i \}_{i=0}^{\lambda} \), and \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \).

Now take the subsequence \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \) of \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \) and construct \( \mathcal{S}_2 \) similarly. Here, we have the following notable differences: the extension for the terms \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \) is with respect to \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \), and for \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \), the appropriate subroutine is to add a \( \kappa_b^m \)-rotation to move \( \sigma_s(v_b^m) \) upon \( v_b^m + 1 \), swap \( \sigma_s(v_b^m) \) through \( \mathcal{V} \) to yield \( \mu \in \mathcal{V}(\kappa_b^m) \) upon \( v_b^m + 1 \), and perform a \( \kappa_b^m \)-rotation to swap \( \mu \) onto \( v_b^m + 1 \).

Similarly proceed until we exhaust all vertices in \( \mathcal{V}(\kappa_b^m) \) upon \( v_b^m + 1 \); for the final \( \mu \in \mathcal{V}(\kappa_b^m) \), during the \( \kappa_b^m \)-rotation involving \( \kappa_b^m \) with \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \), simply add a \( \kappa_b^m \)-rotation to move \( \mu \) upon \( v_b^m + 1 \). This process concludes to yield the sequences \( \mathcal{S}_1, \ldots, \mathcal{S}_{2^\nu+1} \). Merging the sequences \( \mathcal{S}_1, \ldots, \mathcal{S}_{2^\nu+1} \) yields \( \tilde{\mathcal{S}}_1 \), with \( \tilde{\mathcal{S}}_1 \) an \( (m+1) \)-knob extraction of \( \sigma_0 = \sigma_s \) with \( \tilde{\mathcal{S}}_1 \mathcal{V}(\mathcal{P}_b^m) = \mathcal{V}(\mathcal{K}_a^m) = \mathcal{V}(\mathcal{P}_b^m) \setminus \{ v_a^m \} \) and \( \tilde{\mathcal{S}}_1 \mathcal{V}(\mathcal{P}_b^m) = \mathcal{V}(\mathcal{K}_a^m) = \mathcal{V}(\mathcal{P}_b^m) \). Furthermore, the above construction yields that there exists a \( \kappa_a^m \)-rotation and \( \kappa_b^m \)-rotation for every \( \mu \in \mathcal{V}(\kappa_b^m) \), so that the swap sequence \( \{ \tilde{\sigma}_i \}_{i=0}^{\lambda} \) satisfies the stronger claim for \( \eta = 1 \).

For general \( \eta \geq 1 \), construct a swap sequence \( \{ \sigma_i \}_{i=0}^{\lambda} \subset V(\mathcal{FS}(X_m, Y_m)) \) with subsequence \( \{ \sigma_i \}_{i=0}^{2^{\nu+1}} \) (i.e. \( \eta(2\nu + 1) \) interchanges in layer \( m \) by hypothesis, and proceed as elaborated for the \( \eta = 1 \) case for each contiguous subsequence \( \{ \sigma_i \}_{i=0}^{2^{\nu+1}} \subset \{ \tau \}_{i=0}^{\lambda} \). Let \( \kappa_a^m \subset V(\mathcal{FS}(X_{m+1}, Y_{m+1})) \) with the desired properties (namely, repeating the process on each such contiguous subsequence yields one term in the corresponding subsequence \( \{ \sigma_i \}_{i=0}^{\lambda} \)).

Let \( \tau_\ell \subset V(\mathcal{FS}(X_L, Y_L)) \) be any \( L \)-knob extraction in the same connected component as \( \tau \). Recalling that \( n = |V(X_L)| = |V(Y_L)| \), note in the proof of this statement that a \( 1 \)-knob extraction needed a number of swaps at least linear in \( n \) and with quadratic number of swaps in \( n \), and to perform an \( L \)-knob extraction, we performed an \( (L-1) \)-knob extraction linearly many times in \( n \). The following two propositions show this is necessary, bounding from below the length of any swap sequence from \( \tau \) to \( \tau_\ell \).

**Proposition 3.14.** Let \( \tau, \tau_\ell \subset V(\mathcal{FS}(X_L, Y_L)) \) be in the same connected component as \( \tau \), with \( \tau \) a \( 1 \)-knob extraction of \( \tau \). Any swap sequence \( \{ \sigma_i \}_{i=0}^{\lambda} \) with \( \sigma_0 = \sigma \) and \( \sigma_\lambda = \tau \) must have \( \lambda \geq \nu \).

We remark that this lower bound can be made significantly larger (which in turn can be used to yield a stronger upper bound in Corollary 3.18), but this suffices for our main result in Theorem 3.12, as for any fixed \( L, \nu \geq \frac{4\nu}{2(L-1)} \) for all values of \( n \) on which \( X_L, Y_L \) are defined (recall that \( n \)-vertex graphs \( X_L, Y_L \in \mathcal{Y}_L \) exist for values \( n = 4\nu L - 2(L-1) \)).

**Proof.** Assume \( \sigma^{-1}(V(K_a^L)) \subset V(P_a^L) \) and \( \tau^{-1}(V(K_a^L)) \cap V(P_a^L) = \emptyset \), \( \tau^{-1}(V(K_a^L)) \subset V(P_a^L) \) (i.e. the first statement in Definition 3.12 applies), and let \( \mathcal{V} = \{ \sigma_i \}_{i=0}^{\lambda} \) with \( \sigma_0 = \sigma \), \( \sigma_\lambda = \tau \) be such a swap sequence.
Figure 15: An illustration of the subroutines involved in the construction of the swap sequence \{\tilde{\sigma}_i\}^L_{i=0} for general \(L = m + 1 > 1\) and \(\eta = 1\). Preimages of \(V(K_a^m)\) are colored blue; similarly, preimages of \(V(K_b^m)\) are colored red, while preimages of \(V(K_a^{m+1})\) are colored gold, while preimages of \(V(K_b^{m+1})\) are colored pink.

Specifically, we here depict the construction of the sequence \(S_2\), which is constructed from the subsequence \{\sigma_i\}^2_{i=i_1} of the original sequence \{\sigma_i\}^1_{i=0}. Following the main text, we initially have \(\sigma_s(v_i^{m+1})\) upon \(\nu^{m+1}\), starting from \(\sigma_0^{2.1}\). At \(\sigma_0^{2.2}\), we extend the \(\kappa_m\)-rotation involving \(\kappa_a^{m+1}\) in \{\sigma_i\}^2_{i=i_1} (this is guaranteed to exist by the induction hypothesis) so that it includes a \(\kappa_m^{m+1}\)-rotation involving \(\sigma_s(v_b^{m+1})\), which is then swapped into \(P_a^{m+1}\), and we perform a \(\kappa_a^{m+1}\)-rotation involving the resulting \(\mu\) on \(v_a^{m+1}\). From \(\sigma_0^{2.3}\) to \(\sigma_2^{2.3}\) (our notation in this figure for the final configuration in \(S_2\)), we complete the remaining part of \{\sigma_i\}^2_{i=i_1}.

Recall that \(|V(K_a^1)| = \nu\); all vertices in \(K_a^1\) exit \(V(P_a^1)\) over \(V\), and at most one exits during any \((X_L, Y_L)\)-friendly swap. The statement is proved analogously for the setting \(\sigma^{-1}(V(K_a^1)) \subset V(P_a^\ell)\) and \(\tau^{-1}(V(K_b^\ell)) \cap V(P_a^\ell) = \emptyset\), \(\tau^{-1}(V(K_a^1)) \subset V(P_a^\ell)\).

**Proposition 3.15.** For any integer \(L \geq 2\) and \(\ell \in [L-1]\), let \(\sigma, \tau \in V(FS(X_L, Y_L))\) be in the same connected component as \(\sigma_s\), with \(\tau\) an \((\ell+1)\)-knob extraction of \(\sigma\). Any swap sequence \{\sigma_i\}^L_{i=0} with \(\sigma_0 = \sigma\) and \(\sigma_\ell = \tau\) must have a subsequence \{\sigma_j\}^2_{j=0} where for \(2 \leq j \leq 2\nu - 4\), there exists \(\tilde{\sigma} \in \{\sigma_i\}^L_{i=i_{j-1}}\) with \(\tilde{\sigma}\) an \(\ell\)-knob extraction of \(\sigma_{i_{j-1}}\).

An immediate corollary of this proposition is the following. For any fixed integer \(L \geq 2\), \(X_L \in X_L\) and \(Y_L \in Y_L\) on \(n = 4\nu L - 2(L - 1)\) vertices, and \(\ell \in [L - 1]\), if \(\lambda_{(L,n)}(\ell)\) denotes the length of a shortest swap sequence from \(\sigma\) to an \(\ell\)-knob extraction \(\tau\) of \(\sigma\) for \(\sigma, \tau \in V(FS(X_L, Y_L))\) in the same component as \(\sigma_s\), then \(\lambda_{(L,n)}(\ell + 1) \geq (2\nu - 5)\lambda_{(L,n)}(\ell) \geq \nu \lambda_{(L,n)}(\ell)\) (recall that \(\nu \geq 15\)). Under this notation, the result of Proposition 3.14 can be stated as \(\lambda_{(L,n)}(1) \geq \nu\), so in particular invoking Proposition 3.15 \(L - 1\) times and Proposition 3.14 once yields \(\lambda_{(L,n)}(L) \geq \nu^L\), as demonstrated in the proof of Theorem 3.16.

**Proof.** We begin by considering the setting given by the first bullet of Definition 3.12 namely the case where \(\sigma^{-1}(V(K_a^\ell)) \subset V(P_a^{\ell+1})\) and \(\tau^{-1}(V(K_b^\ell)) \cap V(P_a^{\ell+1}) = \emptyset\), \(\tau^{-1}(V(K_a^\ell)) \subset V(P_a^{\ell+1})\), with \(\nu = \{\sigma_i\}^\ell_{i=0}\), \(\sigma_0 = \sigma\), \(\lambda = \tau\) any such swap sequence. By Proposition 3.14 and Remark 3.8(2), we observe that \(|\sigma^{-1}(V(K_a^{\ell+1})) \cap V(P_a^{\ell+1})| \leq 1\), and \(|\tau^{-1}(V(K_b^{\ell+1})) \cap V(P_a^{\ell+1})| \leq 1\), so at least \(2\nu - 2\) vertices of \(V(K_b^{\ell+1})\) have swapped to the “opposite” path of layer \(\ell + 1\) over \(V\), at least \(2\nu - 4\) of which are not \(\kappa_b^{\ell+2}\) or \(\kappa_b^\ell\) (relevant for \(\ell \neq L - 1\)). Take any \(2\nu - 4\) such vertices, and enumerate these \(\{\eta_1, \eta_2, \ldots, \eta_{2\nu - 4}\} \subset V(K_b^{\ell+1})\).
in the order they first leave their initial subgraph\(^5\) during the swap sequence \(\mathcal{V}\). Construct a subsequence \(\{\sigma_{i,j}\}_{j=0}^{2\nu-4}\) of \(\mathcal{V}\) where \(i_0 = 0\) and for \(j \in [2\nu-4]\), \(\sigma_{i,j}\) is the earliest configuration where \(\sigma_{i,j}^{-1}(\eta_j) = v^{\ell+1}\) and \(\sigma_{i,j+1}^{-1}(\eta_j)\) lies outside the initial subgraph of \(\eta_j\).

If \(\eta_j \in V(K^{\ell+1}_a), \sigma_{i,j+1}\) is achieved from \(\sigma_{i,j}\) by swapping \(\eta_j\) with \(\kappa^{\ell+1}_a\). Specifically, \(\eta_j\) cannot swap with another vertex in \(V(K^{\ell+1}_a)\) by Proposition 3.7(4). If we assume that \(\eta_j\) swaps with \(\kappa^{\ell+1}_a\) (which is the only other possibility), consider \(\sigma_{\xi}\), the final term of \(\mathcal{V}\) not after \(\sigma_{i,j}\), satisfying \(\sigma_{\xi}^{-1}(\eta_j) = \sigma_{i,j}^{-1}(\eta_j) = v^{\ell+1}\) and \(\sigma_{\xi}^{-1}(\kappa^{\ell+1}_a) = \sigma_{i,j}^{-1}(\kappa^{\ell+1}_a)\); note that \(1 \leq \xi \leq i_j\), as \(\sigma_{a}^{-1}(\eta_j) \neq \sigma_{i,j}^{-1}(\eta_j)\). Then \(\sigma_{\xi}^{-1}\) cannot be achieved from \(\sigma_{\xi}\) by swapping \(\kappa^{\ell+1}_a\) or \(\eta_j\) without raising a contradiction on one of Proposition 3.6, Proposition 3.7(4), or \(\sigma_{i,j}\) being the first configuration where \(\sigma_{i,j}^{-1}(\eta_j) = v^{\ell+1}\) and \(\sigma_{i,j+1}^{-1}(\eta_j)\) lies outside the initial subgraph of \(\eta_j\).

Similarly, if \(\eta_j \in V(K^{\ell+1}_b), \sigma_{i,j+1}\) is achieved from \(\sigma_{i,j}\) by swapping \(\eta_j\) with \(\kappa^{\ell+1}_a\). Also, for any \(j \in [2\nu-4]\), observe that \(\sigma_{i,j}^{-1}(\eta_j) \in V(P^{\ell+1}_a) \cup V(P^{\ell+1}_b)\) for all \(j' \neq j\) by Proposition 3.7(4).

\[\text{Figure 16: Two possibilities for the configuration } \sigma_{i,j} \text{ for any } j \in [2\nu-4], \text{ depending on whether the corresponding vertex } \eta_j \text{ lies in } V(K^{\ell+1}_a) \text{ or } V(K^{\ell+1}_b). \text{ In this figure, subgraphs and vertices corresponding to preimages of } V(K^{\ell}_b) \text{ are colored red, while those corresponding to preimages of } V(K^{\ell}_b) \text{ are colored blue. In particular, note that the coloring of } P_a \text{ in both cases follows from Proposition 3.11.}\]

For \(2 \leq j \leq 2\nu-4\), consider consecutive terms \(\sigma_{i,j}, \sigma_{i,j}\) in the subsequence, and assume \(\eta_j \in V(K^{\ell+1}_a)\) (\(\eta_j \in V(K^{\ell+1}_b)\) is argued similarly). By Proposition 3.11, \(\sigma_{i,j}^{-1}(V(K^{\ell}_a)) \subset V(P^{\ell}_a)\), since \(\sigma_{i,j}^{-1}(\kappa^{\ell+1}_a) \notin V(P^{\ell}_a) \cup V(P^{\ell}_b)\). If \(\eta_{j-1} \in V(K^{\ell+1}_b)\), then Proposition 3.11 similarly gives \(\sigma_{j-1,j}^{-1}(V(K^{\ell}_b)) \subset V(P^{\ell}_a)\), so \(\sigma_{i,j}\) is an \(\ell\)-knob extraction of \(\sigma_{i,j-1}\) (note that \(\sigma_{i,j}^{-1}(V(K^{\ell}_b)) \cap V(P^{\ell}_a) = \emptyset\) by Remark 3.8(2)) and thus we can take \(\sigma_{i,j}\) as the desired \(\tilde{\sigma}\). If \(\eta_{j-1} \in V(K^{\ell+1}_b)\), then \(\sigma_{j-1,j}^{-1}(V(K^{\ell}_b)) \subset V(P^{\ell}_b)\) by Proposition 3.11, and \(\sigma_{j-1,j}^{-1}(\eta_j) \in V(P^{\ell}_a)\); \(\eta_j\) must swap to \(v^{\ell+1}\) over the swap sequence \(\{\sigma_{i,j}\}_{i=j+1}^{j}\), which necessarily swaps with \(\kappa^{\ell+1}_a\) upon \(C^{\ell+1}_a\) at some point. Let \(\tilde{\sigma} \in \{\sigma_{i,j}\}_{i=j+1}^{j}\) be such a configuration where \(\eta_j\) swaps with \(\kappa^{\ell+1}_a\), for which \(\tilde{\sigma}^{-1}(V(K^{\ell}_b)) \subset V(P^{\ell}_a)\), again by Proposition 3.11. This gives \(\tilde{\sigma}\) as an \(\ell\)-knob extraction of \(\sigma_{i,j-1}\) (again, \(\tilde{\sigma}^{-1}(V(K^{\ell}_b)) \cap V(P^{\ell}_a) = \emptyset\) by Remark 3.8(2)).

The statement for \(\sigma^{-1}(V(K^{\ell+1}_a)) \subset V(P^{\ell+1}_a)\) and \(\tau^{-1}(V(K^{\ell+1}_b)) \cap V(P^{\ell+1}_a) = \emptyset\), \(\tau^{-1}(V(K^{\ell+1}_a)) \subset V(P^{\ell+1}_a)\) (i.e. the setting corresponding to the second bullet of Definition 3.12) is proved analogously.\[\square\]

\[\text{\footnotesize 5If } \eta_j \in V(K^{\ell+1}_a), \text{ then } \sigma_a(\eta_j) \in V(P^{\ell+1}_a), \text{ and we say that } X^{\ell+1}_a \text{ is the initial subgraph of } \eta_j; \text{ leaving the initial subgraph corresponds to an } (X_L, Y_L)\text{-friendly swap in which } \eta_j \text{ is initially on } v^{\ell+1}\text{ and swaps onto some vertex that is not in } V(X^{\ell+1}_b). \text{ Analogously, for } \eta_j \in V(K^{\ell+1}_b), \text{ we have that } \alpha_a(\eta_j) \in V(P^{\ell+1}_b) \text{ and we say that } X^{\ell+1}_b \text{ is the initial subgraph of } \eta_j; \text{ leaving the initial subgraph corresponds to an } (X_L, Y_L)\text{-friendly swap in which } \eta_j \text{ is initially on } v^{\ell+1}\text{ and swaps onto some vertex that is not in } V(X^{\ell+1}_b).}\]
3.4 Proof of Theorem 3.2

We are finally ready to derive the relevant lower bound on the diameter of the connected component of $FS(X_L, Y_L)$ that contains $\sigma_s$. Recall the definition of $\lambda_{(L,n)}(t)$ in the discussion following Proposition 3.15.

**Theorem 3.16.** Take any integer $L \geq 1$, and $X_L \in \mathcal{X}_L$, $Y_L \in \mathcal{Y}_L$ on $n > (4L)^L$ vertices. Then the diameter of the connected component of $FS(X_L, Y_L)$ that contains $\sigma_s$ is greater than $n^{L-1}$.

**Proof.** Let $\sigma_f \in V(FS(X_L, Y_L))$ in the same connected component of $\sigma_s$ be such that $\sigma_f$ is an $L$-knob extraction of $\sigma_s$ (such a vertex $\sigma_f$ exists by Proposition 3.15), and in particular select $X_L \in \mathcal{X}_L$, $Y_L \in \mathcal{Y}_L$ on $n > (4L)^L$ vertices. Consider $d(\sigma_s, \sigma_f)$: by Propositions 3.14 and 3.15, we have the following.

$$d(\sigma_s, \sigma_f) \geq \lambda_{(L,n)}(L) \geq \nu \lambda_{(L,n)}(L-1) \geq \cdots \geq \nu^{L-1} \lambda_{(L,n)}(1) \geq \nu^L \geq \left(\frac{n}{4L}\right)^L$$

Since $\left(\frac{n}{4L}\right)^L > n^{L-1}$ whenever $n > (4L)^L$, we have the desired statement. \qed

In particular, we have the following explicit bound on the diameter which strictly involves $n$.

**Corollary 3.17.** There exist infinitely many values of $n$ for which there are $n$-vertex graphs $X$ and $Y$ such that $FS(X, Y)$ has a connected component with diameter at least $\eta(n\log n)/(\log \log n)$.

**Proof.** Recall that $n$-vertex graphs $X_L \in \mathcal{X}_L$, $Y_L \in \mathcal{Y}_L$ are defined for all $n = 4\nu L - 2(L-1)$, with $\nu = 4m + 3$ for some integer $m \geq 3$ (i.e. all $n = (16m + 10)L + 2$ for $m \geq 3$), so for $L \geq 3$ (since here, $58L + 2 < (4L)^L$), there exist $X_L \in \mathcal{X}_L$, $Y_L \in \mathcal{Y}_L$ for $(4L)^L \leq n < (4L)^L + 16L < (4(L+1))^L + 1$, for which Theorem 3.16 yields that $FS(X_L, Y_L)$ has a connected component with diameter strictly greater than $n^{L-1}$. Now, $\frac{\log n}{\log \log n} \leq \frac{\log(4L+1)^L}{\log \log(4(L+1)^L)} = (L+1)\frac{\log(4L+1)}{\log \log(4L+1)} < L - 1$ for $L$ sufficiently large. Thus, we have that $n^{L-1} > \eta(n\log n)/(\log \log n)$ for all such values of $n$ corresponding to such values of $L$, proving the claim. \qed

Theorem 3.2 follows immediately from these results, which also yields the following statement. We leave a more thorough study of the numbers $\eta(d)$ introduced in this corollary open.

**Corollary 3.18.** Let $\eta(d)$ be the smallest $n \in \mathbb{N}$ such that there exist $n$-vertex graphs $X$, $Y$ with a connected component of $FS(X, Y)$ having diameter greater than $n^d$. Then $d < \eta(d) < (4(d+1)^d+16(d+1)$.

**Proof.** The lower bound is immediate. The upper bound follows immediately from Theorem 3.16 on $L = d+1$ and the fact that $n$-vertex graphs $X_L \in \mathcal{X}_L$, $Y_L \in \mathcal{Y}_L$ exist for $n = 4\nu L - 2(L-1)$ vertices, with $\nu = 4m + 3$ for some integer $m \geq 3$. \qed

3.5 Connected $FS(X, Y)$

The proof of Theorem 3.2 relied heavily on characterizing all vertices of $FS(X_L, Y_L)$ in the same connected component of $\sigma_s$. It is thus natural to ask Question 3.1 in the setting where $FS(X, Y)$ is assumed to be connected, which was separately raised by Defant and Kravitz in [3].

**Question 3.19 (3).** Does there exist an absolute constant $C > 0$ such that for all $n$-vertex graphs $X$ and $Y$ with $FS(X, Y)$ connected, we have $\text{diam}(FS(X, Y)) \leq n^C$?

We first make note of the following result.

**Proposition 3.20 (3).** Let $FS(X, Y)$ be the friends-and-strangers graph of $X$ and $Y$. If $X$ or $Y$ is disconnected, then $FS(X, Y)$ is also disconnected. Furthermore, if $X$ and $Y$ are connected graphs on $n \geq 3$ vertices, each with a cut vertex, then $FS(X, Y)$ is disconnected.
By Proposition 3.20 we can assume without loss of generality that $X$ is biconnected and $Y$ is connected. A negative answer to Question 3.19 would mean the existence of long paths in the connected graph $FS(X,Y)$; the following result shows that the extreme end of this is not possible. First, we need a preliminary result.

**Theorem 3.21** ([3]). Let $Y$ be a graph on $n\geq 3$ vertices. The graph $FS(Cycle_n,Y)$ is connected if and only if $Y$ is a forest consisting of trees $T_1,\ldots,T_r$ such that $\gcd(|V(T_1)|,\ldots,|V(T_r)|) = 1$.

**Proposition 3.22.** For $n\geq 4$, $FS(X,Y)$ is not isomorphic to a tree on $n!$ vertices (e.g. Path$_n$) or a graph on $n!$ vertices with one edge appended (e.g. Cycle$_n$).

**Proof.** The number of edges of $FS(X,Y)$ is $|E(X)| \cdot |E(Y)| \cdot (n-2)!$, while this is $n!-1$ and $n!$ for a tree on $n!$ vertices and a tree with one edge appended on $n!$ vertices, respectively. Also, $|E(X)| \cdot |E(Y)| \cdot (n-2)!$ is divisible by 2 while $n!-1$ is not. Assume $FS(X,Y)$ is isomorphic to a tree with an edge appended to it, so $|E(X)| \cdot |E(Y)| \cdot (n-2)! = n!$, or $|E(X)| \cdot |E(Y)| = n(n-1)$. Then $X$ and $Y$ must both be connected, so that (without loss of generality) $|E(X)| = n$ and $|E(Y)| = n-1$, so $Y$ is a tree. Due to (5) of Proposition 3.20, $X$ is biconnected, so necessarily $X = Cycle_n$. But $|E(Y)| = \binom{n}{2} - (n-1)$, contradicting Theorem 3.21, which gives $|E(Y)| \leq n-1$.

4 Fixed $X$

We now study the diameters of connected components of friends-and-strangers graphs where we fix $X$ to come from a particular family of graphs. First, we introduce some notation and the notion of flip graphs on acyclic orientations.

4.1 Distances in Flip Graphs

Broadly, a flip graph typically refers to some graph whose vertices are some combinatorial objects, with any two vertices adjacent if the corresponding vertices differ by a single “flip,” or some operation that transforms one such object into another. We shall explicitly study these structures for acyclic orientations of a graph $G$, which are assignments of directions to all edges of $G$ (called orientations) that result in no directed cycles: denote the set of all acyclic orientations of $G$ by $Acyc(G)$.

The specific operations that we shall study are flips and double-flips as described by [3]: letting $\alpha \in Acyc(G)$, a flip corresponds to either converting a source of $\alpha$ into a sink or a sink of $\alpha$ into a source by reversing the directions of all incident edges, while a double-flip corresponds to similarly converting a nonadjacent source and sink of $\alpha$ into a sink and a source, respectively. Both flips and double-flips result in new acyclic orientations in $Acyc(G)$. Say that $\alpha,\alpha' \in Acyc(G)$ are flip equivalent, denoted $\alpha \sim \alpha'$, if $\alpha'$ can be achieved from $\alpha$ by some sequence of flips. Similarly say that $\alpha,\alpha' \in Acyc(G)$ are double-flip equivalent, denoted $\alpha \approx \alpha'$, if $\alpha'$ can be achieved from $\alpha$ by some sequence of double-flips. It is straightforward to show that $\sim$ and $\approx$ are equivalence relations on $Acyc(G)$, and the corresponding equivalence classes are studied by the authors of [3], who in particular show that the equivalence classes in $Acyc(G)/\approx$ are in bijection with the connected components of $FS(Cycle_n,Y)$. In particular, we shall refer to equivalence classes of $Acyc(G)/\approx$ as toric acyclic orientations (following [3]), and the toric acyclic orientation for which $\alpha$ is a representative will be denoted $[\alpha]_\approx$. Similarly, equivalence classes of $Acyc(G)/\approx$, called double-flip equivalence classes, will be denoted $[\alpha]_\approx$ (with $\alpha$ as a representative).

We provide the appropriate definitions for what we shall henceforth refer to as flip and double-flip graphs. (Note, however, that this overloads the standard terminology concerning flip graphs and that what we refer to as flip and double-flip graphs are specific instances of the more abstract notion of flip graphs.)

**Definition 4.1.** Let $G$ be a graph. The flip graph of $G$, denoted $Flip(G)$, is a graph with vertices $V(Flip(G)) = Acyc(G)$, and $\{\alpha,\alpha'\} \in E(Flip(G))$ if and only if $\alpha$ and $\alpha'$ differ by a flip.
Observe that the connected components of $\text{Flip}(G)$ correspond exactly to toric acyclic orientations, or elements of $\text{Acyc}(G)/\sim$. Specifically, the vertices of any component consist of the acyclic orientations comprising the corresponding toric acyclic orientation. For brevity, we shall refer to the operation of flipping a source of $\alpha$ to a sink as an inflip, and the operation of flipping a sink of $\alpha$ to a source as an outflip.

Now assume we enumerate the vertices of a graph $G$ such that $V(G) = [n]$, and take $\alpha \in \text{Acyc}(G)$. Associated to $\alpha$ is a poset $([n], \leq_{\alpha})$ where $i \leq_{\alpha} j$ if and only if there exists a directed path from $i$ to $j$ in $\alpha$. Define a linear extension of $\alpha$ to be any permutation $\sigma \in S_n$ such that if $i \leq_{\alpha} j$, then $\sigma^{-1}(i) \leq \sigma^{-1}(j)$. Let $\mathcal{L}(\alpha)$ denote the set of linear extensions of the acyclic orientation $\alpha$. For arbitrary $\sigma \in S_n$, there exists a unique acyclic orientation such that $\sigma \in \mathcal{L}(\alpha)$, which we denote $\alpha_G(\sigma)$: this results from directing edge $\{i,j\} \in E(G)$ from $i$ to $j$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(j)$. As such, we shall also denote $\mathcal{L}(\alpha) = \bigsqcup_{\sigma \in [n]} \mathcal{L}(\hat{\alpha})$ and $\mathcal{L}(\alpha) = \bigsqcup_{\hat{\alpha} \in [n]} \mathcal{L}(\hat{\alpha})$.

We now provide an upper bound on the diameters of connected components of graphs $\text{Flip}(G)$, or the maximum number of flips necessary to get between two acyclic orientations in the same toric acyclic orientation; this is largely a rephrasing of known results which we shall later use in deriving Theorem 4.15. For a graph $G$ and any acyclic orientation $\alpha \in \text{Acyc}(G)$, we can partition the directed edges of any (undirected) cycle subgraph $C$ of $G$ into $|C^-| = |C^+_\alpha|$ and $|C^+_\alpha|$, which correspond to edges that are directed in one of the two directions around the cycle. The article [7] studied precisely when an acyclic orientation could be achieved from another by a sequence of outflips (namely, see their Theorem 1’). We can easily extend that result, describing exactly when an acyclic orientation $\alpha'$ can be achieved from an acyclic orientation $\alpha$ be a sequence of inflips or outflips.

Lemma 4.2 ([7]). For $\alpha, \alpha' \in \text{Acyc}(G)$, $\alpha'$ can be achieved from $\alpha$ by a sequence of inflips if and only if for every cycle subgraph $C$ of $G$, $|C^-| = |C^+\alpha'|$. Similarly, $\alpha'$ can be achieved from $\alpha$ by a sequence of outflips if and only if for every cycle subgraph $C$ of $G$, $|C^+\alpha| = |C^-|$. This result provides a way to relate sequences of inflips and outflips directly to flip equivalence.

Proposition 4.3. Take a graph $G$. Acyclic orientations $\alpha, \alpha' \in \text{Acyc}(G)$ have that $\alpha \sim \alpha'$ if and only if $\alpha'$ can be achieved from $\alpha$ by a sequence of inflips. Similarly, $\alpha \sim \alpha'$ if and only if $\alpha'$ can be achieved from $\alpha$ by a sequence of outflips.

Proof. We prove the statement for inflips, for which $\alpha'$ is achieved from $\alpha$ via a sequence of inflips implying $\alpha \sim \alpha'$ is immediate. It is straightforward to observe that for any cycle subgraph $C$ of $G$ and $\alpha, \alpha' \in \text{Acyc}(G)$, if $\alpha'$ is achieved from $\alpha$ by a flip, then $|C^-\alpha| = |C^+\alpha'|$. As such, if $\alpha \sim \alpha'$, then $|C^-\alpha| = |C^+\alpha'|$, and thus $\alpha'$ can be achieved from $\alpha$ via a sequence of inflips. The statement for outflips is entirely analogous.

For $\alpha \sim \alpha'$, our definition of flip equivalence yields that $\alpha'$ can be achieved from $\alpha$ by some finite sequence of flips. The article [8] gave an upper bound on the number of outflips necessary to get from $\alpha$ to $\alpha'$ whenever $\alpha \sim \alpha'$.

Proposition 4.4 ([8]). For a graph $G$, let $\alpha, \alpha' \in \text{Acyc}(G)$ be such that $\alpha \sim \alpha'$. Then $\alpha'$ can be achieved from $\alpha$ by no more than $\binom{n}{2}$ inflips and $\binom{n}{2}$ outflips. In particular, the diameter of any connected component of $\text{Flip}(G)$ is at most $\binom{n}{2}$.

In other words, let $[\alpha]_\sim$ be a toric acyclic orientation of a graph $G$ on $n$ vertices, with $\alpha_1, \alpha_2 \in [\alpha]_\sim$. It is possible to achieve $\alpha_2$ from $\alpha_1$ by a sequence of no more than $\binom{n}{2}$ flips.

We can also similarly introduce the notion of double-flip graphs of $G$.

Definition 4.5. Let $G$ be a graph. The double-flip graph of $G$, denoted $\text{DFlip}(G)$, is a graph with vertices $V(\text{DFlip}(G)) = \text{Acyc}(G)$, and $\{\alpha, \alpha'\} \in E(\text{DFlip}(G))$ if and only if $\alpha$ and $\alpha'$ differ by a double-flip.

We leave the following question open, which asks for an analogue of Proposition 4.4 for $\text{DFlip}(G)$. We note, however, that it is relatively straightforward to show (mirroring techniques used in the proof of Theorem 4.15) that an affirmative result for this question would also resolve Conjecture 5.1.

Question 4.6. Let $G$ be an arbitrary graph on $n$ vertices. For $\alpha, \alpha' \in V(\text{DFlip}(G))$, does there exist a constant $C > 0$ such that $d(\alpha, \alpha') = O(n^C)$?
4.2 Path Graphs

In this subsection, we fix $X = \text{Path}_n$.

**Lemma 4.7** ([3]). Let $Y$ be a graph with vertex set $[n]$, and $\alpha \in \text{Acyc}(Y)$. Take any linear extension $\sigma \in L(\alpha)$, and denote $H_\alpha$ as the connected component of $\text{FS}(\text{Path}_n, Y)$ that contains $\alpha$. Then

$$\text{FS}(\text{Path}_n, Y) = \bigoplus_{\alpha \in \text{Acyc}(Y)} H_\alpha$$

and $V(H_\alpha) = L(\alpha)$. In particular, $H_\alpha$ is independent of the choice of $\sigma$ (i.e., depends only on $\alpha$).

We also define the following notion, which generalizes inversions of a permutation with respect to any element of $\mathcal{S}_n$, not just the identity.

**Definition 4.8.** Fix some $\tau \in \mathcal{S}_n$. For some other $\sigma \in \mathcal{S}_n$, we say that $\sigma$ has $\{i, j\}$ as a $\tau$-inversion for $i, j \in [n]$, $i < j$ if either $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $\tau^{-1}(j) < \tau^{-1}(i)$, or $\sigma^{-1}(j) < \sigma^{-1}(i)$ and $\tau^{-1}(i) < \tau^{-1}(j)$. We denote the number of $\tau$-inversions that $\sigma$ has as $\text{inv}_\tau(\sigma)$.

Observe that $\sigma$ has $\{i, j\}$ as a $\tau$-inversion if the relative ordering of $i, j$ in $\tau$ is opposite that of $\sigma$. Indeed, if $\tau$ is the identity, then $\text{inv}_\tau(\sigma) = \text{inv}(\sigma)$, the number of inversions of $\sigma$. It also follows immediately that $\text{inv}_\tau(\sigma) = 0$ if and only if $\tau = \sigma$. We now have the following result.

**Proposition 4.9.** Take $\alpha \in \text{Acyc}(Y)$, and let $H_\alpha$ be the corresponding component of $\text{FS}(\text{Path}_n, Y)$. Let $\mathcal{P} = ([n], \leq_\alpha)$ be the poset on $[n]$ such that $i \leq_\alpha j$ if and only if there exists a directed path from vertex $i$ to vertex $j$ in $\alpha$. Then $\text{diam}(H_\alpha) \leq \binom{n}{2} - p$, where $p$ denotes the number of ordered pairs $(i, j)$ with $i, j \in [n], i < j$ such that $i$ and $j$ are comparable in $\mathcal{P}$.

**Proof.** Let $(i, j)$ be the endpoints of any directed path in $\alpha$ (from vertex $i$ to vertex $j$). Since $V(H_\alpha) = L(\alpha)$, any $\sigma \in V(H_\alpha)$ must satisfy $\sigma^{-1}(i) < \sigma^{-1}(j)$. For $\sigma, \tau \in V(H_\alpha)$, we show $d(\sigma, \tau) = \text{inv}_\tau(\sigma)$; by the preceding observation, $\text{inv}_\tau(\sigma) \leq \binom{n}{2} - p$. Any $(\text{Path}_n, Y)$-friendly swap reduces the number of $\tau$-inversions of $\sigma$ by at most one, so $\text{inv}_\tau(\sigma) \leq d(\sigma, \tau)$. Now perform, on $\sigma = \sigma(1)\sigma(2)\cdots \sigma(n)$, a “$\tau$-bubble sort” algorithm. Specifically, say that $\sigma(i_1)$ corresponds to $\tau(1)$, and swap $\sigma(i_1)$ down to position 1 so that we achieve $\sigma^{(1)}$ with $\sigma^{(1)}(1) = \tau(1)$. Now, say $\sigma^{(1)}(i_2) = \tau(2)$ (with $i_2 \geq 2$), and move this down to position 2 to achieve $\sigma^{(2)}$ with $\sigma^{(2)}(j) = \tau(j)$ for $j \in \{1, 2\}$; proceed similarly until we achieve $\sigma^{(n)} = \tau$. This algorithm never requires swapping a pair of elements comparable in $\mathcal{P}$, since any such pair does not comprise a $\tau$-inversion of $\sigma$. Hence, $d = \text{inv}_\tau(\sigma)$, and since $\sigma$ and $\tau$ were arbitrary, we have $\text{diam}(H_\alpha) \leq \binom{n}{2} - p$. \hfill $\square$

**Remark 4.10.** The upper bound of $\binom{n}{2} - p$ on $\text{diam}(H_\alpha)$ in Theorem 4.9 corresponds to the number of pairs $\{i, j\}$ with $i < j$ and $i, j \in [n]$ that are incomparable with respect to the poset $\mathcal{P} = ([n], \leq_\alpha)$. Also, for arbitrary $\sigma, \tau \in V(H_\alpha) = L(\alpha)$, note that all $\tau$-inversions of $\sigma$ certainly must be over pairs of incomparable elements with respect to $\mathcal{P}$, and their distance in $\text{FS}(\text{Path}_n, Y)$ is exactly $\text{inv}_\tau(\sigma)$. We apply these observations to show that the upper bound on $\text{diam}(H_\alpha)$ is not tight: toward this, assume the graph shown in Figure 17 is isomorphic to a subgraph of $\overline{Y}$.

![Figure 17](image)

Figure 17: An example showing that the upper bound of Proposition 4.9 is not tight.

For some $n \geq 6$, assume the existence of $\sigma, \tau \in H_\alpha$ with distance $\text{inv}_\tau(\sigma) = \binom{n}{2} - p$, so that all pairs of incomparable elements with respect to $\mathcal{P}$ are associated with $\tau$-inversions of $\sigma$. Any two vertices in $\{1, 2, 3\}$
are not comparable in \( P \), so the relative ordering of the vertices \( \{1, 2, 3\} \) in \( \sigma \) must be exactly opposite that of \( \tau \) (i.e. the relative ordering in \( \sigma \) is that in \( \tau \) reversed). Without loss of generality, assume \( \sigma \) has relative ordering \( 1 \ 2 \ 3 \), so \( \tau \) has \( 3 \ 2 \ 1 \). Then vertex 5 necessarily follows vertex 2 in both \( \sigma \) and \( \tau \), so \( \{2, 5\} \) is not associated with a \( \tau \)-inversion of \( \sigma \), but \( \{2, 5\} \) is an incomparable pair with respect to \( P \), a contradiction.

Certainly, for any \( \alpha \in \text{Acyc}(\overline{Y}) \), the endpoints of any edge in \( \overline{Y} \) are comparable in the poset \(((\lfloor \alpha \rfloor), \leq_{\alpha})\). This yields the following statement, as we have that \( \left( \frac{\alpha}{2} \right) - p \leq \left( \frac{\alpha}{2} \right) - |E(\overline{Y})| = |E(Y)| \).

**Theorem 4.11.** Any connected component of \( FS(\text{Path}_n, Y) \) has diameter at most \( |E(Y)| \).

### 4.3 Cycle Graphs

We now fix \( X = \text{Cycle}_n \); the present work does not resolve the general question of whether diameters of connected components of \( FS(\text{Cycle}_n, Y) \) are polynomially bounded, but answers it in the affirmative for a number of settings. We begin with the setting \( Y = K_n \), which has been studied in the context of circular permutations. In particular, see Procedure 3.6 of [6] for an algorithm that achieves the minimal number of \( (\text{Cycle}_n, K_n) \)-friendly swaps between any two permutations in \( \mathcal{S}_n \).

**Proposition 4.12 ([6][10]).** \( \text{diam}(FS(\text{Cycle}_n, K_n)) = \lfloor \frac{n^2}{4} \rfloor \).

We also have the following analogue of Lemma 4.7 due to [3].

**Lemma 4.13 ([3]).** Let \( Y \) be a graph with vertex set \([n]\), and \( \alpha \in \text{Acyc}(\overline{Y}) \). Take any linear extension \( \sigma \in \mathcal{L}(\lfloor \alpha \rfloor) \), and denote \( H_{\lfloor \alpha \rfloor} \) as the connected component of \( FS(\text{Cycle}_n, Y) \) that contains \( \alpha \).

\[
FS(\text{Cycle}_n, Y) = \bigoplus_{\lfloor \alpha \rfloor \in \text{Acyc}(\overline{Y})/\approx} H_{\lfloor \alpha \rfloor}
\]

and \( V(H_{\lfloor \alpha \rfloor}) = \mathcal{L}(\lfloor \alpha \rfloor) \). In particular, \( H_{\lfloor \alpha \rfloor} \) is independent of the choice of \( \sigma \) (i.e., depends only on \([\alpha]\)).

**Proposition 4.14.** If \( |E(Y)| \leq n - 2 \) or \( Y \) has an isolated vertex, then any connected component of \( FS(\text{Cycle}_n, Y) \) has diameter at most \( |E(Y)| \). If \( \overline{Y} \) has an isolated vertex (i.e. \( Y \) has a spanning star subgraph), then any component of \( FS(\text{Cycle}_n, Y) \) has diameter at most \( (n - 1)^2 \).

**Proof.** If \( Y \) has an isolated vertex \( v \), then \( \sigma^{-1}(v) \) remains fixed throughout any sequence of \( (\text{Cycle}_n, Y) \)-friendly swaps, so any path from \( \sigma \) to \( \tau \) in \( FS(\text{Cycle}_n, Y) \) is a path in \( FS(\text{Cycle}_n, V(\text{Cycle}_n) \setminus \{\sigma^{-1}(v)\}, \overline{Y}(V) \setminus \{v\}) \), from which the result follows from Theorem 4.11.

For the setting where \( |E(Y)| \leq n - 2 \), we show that any \( \sigma, \tau \in V(FS(\text{Cycle}_n, Y)) \) in the same connected component remain in the same component after removing an edge from \( \text{Cycle}_n \), from which the result follows from Theorem 4.11. Assume for the sake of contradiction that any path from \( \sigma \) to \( \tau \) in \( FS(\text{Cycle}_n, Y) \) involves a swap on every edge in \( E(\text{Cycle}_n) \). Take a shortest path from \( \sigma \) or \( \tau \), \( \{ \sigma_i \}_{i=0}^{\lambda} \) with \( \sigma_0 = \sigma \) and \( \sigma_{\lambda} = \tau \), for which necessarily \( \lambda \geq n \), and consider the subsequence \( \{ \sigma_i \}_{i=0}^{\lambda-1} \), which involves \( n - 1 \) \( (\text{Cycle}_n, Y) \)-friendly swaps. This must be a shortest path from \( \sigma \) to \( \sigma_{n-1} \) in \( FS(\text{Cycle}_n, Y) \), and swaps along at most \( n - 1 \) edges of \( \text{Cycle}_n \); say \( e \in E(\text{Cycle}_n) \) is not swapped along, and let \( \text{Cycle}_n^e \) be \( \text{Cycle}_n \) with \( e \) removed. Then \( \{ \sigma_i \}_{i=0}^{n-1} \) is a shortest path from \( \sigma \) to \( \tau \) in \( FS(\text{Cycle}_n^e, Y) \) with length \( n - 1 \), contradicting Theorem 4.11.

Recall from Section 4 that we refer to a source-to-sink flip as an inflip, and a sink-to-source flip as an outflip.

**Theorem 4.15.** Let \( Y \) be a graph on \( n \geq 3 \) vertices, and let \( n_1, \ldots, n_r \) denote the sizes of the components of \( \overline{Y} \). If \( \gcd(n_1, \ldots, n_r) = 1 \), then any component of \( FS(\text{Cycle}_n, Y) \) has diameter at most \( 2n^3 + |E(Y)| \).

**Proof.** Certainly \( r \geq 2 \); without loss of generality, we assume \( n_1 \leq \cdots \leq n_r \), and shall refer to the corresponding components of \( \overline{Y} \) as \( \overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_r \), respectively. Take acyclic orientations \( \alpha, \alpha' \in \text{Acyc}(\overline{Y}) \) such that \( \alpha \approx \alpha' \), and call \( \alpha_i \) the acyclic orientation induced by \( \alpha \) on \( \overline{Y}_i \) (similarly, \( \alpha'_i \)). Certainly, \( \alpha_i \approx \alpha'_i \) for all \( i \in [r] \), and by Proposition 4.4 we can achieve \( \alpha_i' \) from \( \alpha_i \) in at most \( \left( \frac{n}{2} \right) \) inflips or outflips. For any \( \alpha'_i \),
we can return to $\alpha'_r$ via a sequence of $n_r$ inflips or outflips: consider a linear extension $\sigma \in \mathcal{L}(\alpha'_r)$, labeled $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n_r)$, and perform an inflip for $\alpha'_r$ on $\sigma(1) \in V(\mathcal{Y}_r')$ so that $\sigma' = \sigma(2)\cdots\sigma(n_r)\sigma(1)$ is a linear extension of the resulting acyclic orientation $\mathcal{Y}(\alpha'_r)$. Performing $n_r$ such inflips on $\alpha'_r$ returns $\sigma$ as a linear extension of the resulting acyclic orientation, which therefore must be $\alpha'_r$; a similar process for outflips yields the analogous result.

We thus proceed as follows. Starting from $\alpha$, perform a sequence of double-flips that act as inflips on sources in $\mathcal{Y}_1$ and outflips on sinks in $\mathcal{Y}_1, \ldots, \mathcal{Y}_{r-1}$ until we have achieved $\alpha'_1, \ldots, \alpha'_r$ on $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ at least once. Specifically, begin by performing inflips on $\mathcal{Y}_r$ and outflips on $\mathcal{Y}_1$ until we either achieve $\alpha'_1$ on $\mathcal{Y}_1$ (at which point we begin outflips on $\mathcal{Y}_2$ if $r \geq 2$) or $\alpha'_r$ on $\mathcal{Y}_r$ (at which point we perform inflips on $\mathcal{Y}_r$ as described previously to return to $\alpha'_r$ every $n_r$ inflips). If we achieve $\alpha'_1, \ldots, \alpha'_{r-1}$ prior to $\alpha'_r$, then perform outflips on $\mathcal{Y}_r$ (returning to $\alpha'_1$ every $n_1$ outflips) until $\alpha'_r$ is achieved, and then inflip on $\mathcal{Y}_r$ until we retain $\alpha'_1$. Else, we achieve $\alpha'_r$ prior to $\alpha'_1, \ldots, \alpha'_{r-1}$, for which $\alpha'_r$ will be offset once we have $\alpha'_1, \ldots, \alpha'_{r-1}$. In either case, call the resulting acyclic orientation $\tilde{\alpha}$, which has $\tilde{\alpha}_i = \alpha'_i$ for all $i \in [r-1]$ while $\tilde{\alpha}_r$ differs from $\alpha'_r$ by some offset $0 \leq c < n_r$. The number of double-flips performed to achieve $\tilde{\alpha}$ from $\alpha$ is thus observed to be upper bounded by $\max\{(\binom{n}{2} + n_1, \sum_{i=1}^{r-1} \binom{n_i}{2})\} \leq \sum_{i=1}^{r} n_i^2 \leq (\sum_{i=1}^{r} n_i)^2 = n^2$.

Now, by Bezout’s Lemma (recall that $\gcd(n_1, \ldots, n_r) = 1$), there exist (without loss of generality) integers $0 \leq d_1, \ldots, d_{r-1} < n_r$ such that $n_1d_1 + \cdots + n_{r-1}d_{r-1} \equiv n_r - c \pmod{n_r}$. Thus, from $\tilde{\alpha}$, we can achieve $\alpha'$ by performing $n_r$ outflips on $\tilde{\alpha}_r = \alpha'_r$ exactly $d_i$ times for $i \in [r-1]$ (with each iteration returning to $\tilde{\alpha}_i = \alpha'_i$), while performing inflips on $\alpha_r$ as discussed to achieve $\alpha'_r$. The number of double-flips performed to achieve $\alpha'$ from $\tilde{\alpha}$ is therefore upper bounded by $\sum_{i=1}^{r-1} n_i d_i \leq \max\{d_1, \ldots, d_{r-1}\}(\sum_{i=1}^{r-1} n_i) \leq n^2$, so we have an upper bound of $2n^2$ double-flips necessary to achieve $\alpha'$ from $\alpha$.

From here, take arbitrary $\sigma, \tau \in V(\mathcal{F}(\mathcal{C}_n, Y)))$ in the same component: enumerating $V(\mathcal{C}_n) = \{(Y) = [n]\}$, we have $\sigma, \tau \in \mathcal{L}(\alpha'_1) \subset \alpha'_r$ for some $\alpha'_r \in \mathcal{A}(\mathcal{Y})/\approx$. Denote $\sigma = \alpha_{\mathcal{Y}}(\sigma)$ and $\alpha' = \alpha_{\mathcal{Y}}(\tau)$, for which we can achieve $\alpha'$ from $\sigma$ by a sequence of $\lambda \leq 2n^2$ double-flips, yielding a sequence of acyclic orientations $\mathcal{V} = \{\alpha_i\}_{i=0}^{\lambda} \subset [\alpha]'$. Starting from $\sigma$, if the first double-flip inflips $v$ and outflips $w$, swap $v$ to position 1 and $w$ to position $n$ (no more than $n-1$ $(\mathcal{C}_n, Y)$-friendly swaps are necessary), then perform a $(\mathcal{C}_n, Y)$-friendly swap interchanging $\{v, w\}$ along $\{1, n\}$: the resulting configuration has that $\sigma_{\mathcal{Y}}(\tau) = \alpha_{\mathcal{Y}}(\sigma)$, so by Lemma 1.7 $\sigma, \tau$ lie in the same component of $\mathcal{F}(\mathcal{P}_n, Y)$ (specifically, the copy of $\mathcal{P}_n$ in $\mathcal{C}_n$ excluding the edge $(1, n)$). By Theorem 4.11 we can now achieve $\tau$ from $\tilde{\sigma}$ in no more than $|E(\mathcal{Y})| \cdot (\mathcal{C}_n, Y)$-friendly swaps, so we have an upper bound $d(\sigma, \tau) \leq 2n^2 + n + |E(\mathcal{Y})| = 2n^3 + |E(\mathcal{Y})|$ total $(\mathcal{C}_n, Y)$-friendly swaps necessary to achieve $\tau$ from $\sigma$, which yields the desired statement.

It follows from Lemma 3.21 and Theorem 4.15 that if $\mathcal{F}(\mathcal{C}_n, Y)$ is connected, its diameter is polynomially bounded. Recall that we can assume $X$ biconnected when studying the setting requiring $\mathcal{F}(X, Y)$ to be connected: this shows that diameters of connected friends-and-strangers graphs are indeed polynomially bounded when $X = \mathcal{C}_n$, which is considered the simplest biconnected graph.

**Corollary 4.16.** For $n \geq 3$, if $\mathcal{F}(\mathcal{C}_n, Y)$ is connected, $\text{diam}(\mathcal{F}(\mathcal{C}_n, Y)) \leq 2n^3 + |E(\mathcal{Y})|$.

We also have the following immediate corollary concerning when $X$ is Hamiltonian.

**Corollary 4.17.** Let $X$ and $Y$ be $n$-vertex graphs such that $X$ is Hamiltonian and $\bar{Y}$ is a forest with $\gcd(n_1, \ldots, n_r) = 1$, where $n_1, \ldots, n_r$ denote the sizes of the connected components of $\bar{Y}$. Then every component of $\mathcal{F}(X, Y)$ has diameter at most $2n^3 + |E(\mathcal{Y})|$.  

5 Open Questions

Recall that Theorem 3.2 of this paper proves that diameters of connected components of friends-and-strangers graphs are not polynomially bounded. There are many other interesting questions concerning diameter to be explored that remain unresolved by this article.
5.1 Improvements on Known Results

Corollary 3.18 introduces the numbers $\eta(d) \in \mathbb{N}$, which are the smallest natural numbers such that there exist graphs $X$ and $Y$ each with $\eta(d)$ vertices such that $\text{FS}(X, Y)$ has diameter at least $n^d$: tighter bounds on these values would be interesting. It would also be interesting to have general necessary and sufficient conditions that guarantee the diameter of any component of $\text{FS}(X, Y)$ is $O(n^d)$ for different values of $d \in \mathbb{N}$.

5.2 Generalizations of Theorem 4.15

Recall that Theorem 4.15 yields that if $n_1, \ldots, n_r$ are the sizes of the components of $Y$ and $\gcd(n_1, \ldots, n_r) = 1$, then the diameter of any component of $\text{FS}($Cycle$_n,$ $Y)$ is $O(n^3)$. There are two immediate ways that we can consider generalizing this result. In one direction, we can take $Y$ to be any arbitrary graph, and aim to show that the diameter of $\text{FS}(\text{Cycle}_n,$ $Y)$ remains polynomially bounded.

**Conjecture 5.1.** For any $n$-vertex graph $Y$, any component of $\text{FS}(\text{Cycle}_n,$ $Y)$ has diameter $O(n^C)$, where $C > 0$ is some universal constant.

Section 4.1 asks for an upper bound on the diameter of $\text{DFlip}(G)$ for any arbitrary graph $G$. As was remarked there, resolving the following conjecture yields Conjecture 5.1 as a corollary.

**Conjecture 5.2.** For the double-flip graph $\text{DFlip}(G)$ of any graph $G$, the diameter of any connected component of $\text{DFlip}(G)$ is $O(n^C)$ for some universal constant $C > 0$.

The other direction in which we can generalize Theorem 4.15 is to take $X$ to be any arbitrary biconnected graph, and $Y$ such that $\text{FS}(X, Y)$ is connected. We conjecture that under this setting, diameters of friends-and-strangers graphs are indeed polynomially bounded.

**Conjecture 5.3.** Take $n$-vertex graphs $X$ and $Y$ such that $\text{FS}(X, Y)$ is connected. Then $\text{diam}(\text{FS}(X, Y))$ is $O(n^C)$ for some universal constant $C > 0$.

Albeit optimistic, we provide two principal reasons why we suspect that diameters of connected friends-and-strangers graphs are polynomially bounded.

1. The proof of the negative result for Question 3.1 relies heavily on “rigging” the configurations that lie in a particular connected component of $\text{FS}(X_L,$ $Y_L)$, which allows us to argue that two particular configurations (namely, $\sigma_s$ and $\sigma_f$) are necessarily far apart. Such a strategy is not applicable if we restrict $\text{FS}(X,$ $Y)$ to be connected.

2. Recall that we can assume without loss of generality that $X$ is biconnected under this setting. Theorem 4.15 gives a positive result for Cycle$_n$, the simplest biconnected graph. Furthermore, the constructions $X_L \in X_L$ and $Y_L \in Y_L$ rely heavily on the existence of cut vertices which hold central roles in the proofs of the intermediary propositions (namely, vertices on the paths $P_{a}^{d}, P_{b}^{d}$ for $X_L$, and the knob vertices $\kappa_{a}^{d}, \kappa_{b}^{d}$ in $Y_L$).

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