Nonparametric Estimation of Surface Integrals on Density Level Sets

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Abstract

The estimation of surface integrals on density level sets is important (such as for confidence regions and bandwidth selection) in the study of nonparametric level set estimation. We consider a plug-in estimator based on kernel density estimation. By establishing a diffeomorphism between the true and estimated density level sets, we obtain the asymptotic normality of our estimator of the surface integrals when the integrand is known. We also consider the convergence rate of a plug-in bandwidth selector for density level set estimation, which involves the density derivatives as unknown integrands.

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1 Introduction

The $c$-level set of a density function $f$ on $\mathbb{R}^d$ is defined as

$$M_c = \{x \in \mathbb{R}^d : f(x) = c\}.$$ 

The set $L_c = \{x \in \mathbb{R}^d : f(x) \geq c\}$ is called the upper level set. Level set estimation finds its application in many areas such as clustering (Cadre et al. 2009), classification (Mammen and Tsybakov, 1999), and anomaly detection (Steinwart et al. 2005). It has received extensive study in the literature. See, e.g., Hartigan (1987), Polonik (1995), Tsybakov (1997), Walter (1997), Cadre (2006), Rigollet and Vert (2009), Singh et al. (2009), Rinaldo and Wasserman (2010), Steinwart (2015), and Chen et al. (2017).

We consider $d \geq 2$ in this paper. For simplicity of notation, the subscript $c$ is often omitted for $M_c$ and $L_c$. When $f$ has no flat part at the level $c$, $M$ is a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$. Given an i.i.d. sample from the density $f$, we investigate the estimation of the surface integral

$$\lambda(g) = \int_M g(x)d\mathcal{H}(x),$$

for some integrable function $g$ (which can be known or unknown), where $\mathcal{H}$ is the $(d-1)$-dimensional normalized Hausdorff measure.

The surface integral $\lambda(g)$ is an important quantity that is involved in asymptotic theory for level set estimation. For example, it appears in Cadre (2006) as the convergence limit of the G-measure of the symmetric difference between $L$ and its plug-in-type estimator under mild conditions. Using a similar measure as the risk criterion, Qiao (2018) shows that the optimal bandwidth for nonparametric level set estimation is determined by a ratio of two surface integrals in the form of $\lambda(g)$. In fact one of the motivations for this paper is to find out the convergence rate of the plug-in bandwidth selector in Qiao (2018). In a more general setting, the surface integral on a manifold (not necessarily a level set) plays an important role in the asymptotic distribution of the suprema of rescaled locally stationary Gaussian fields indexed on this manifold (see Qiao and Polonik, 2018a). An application of this extreme value theory result leads to large sample confidence regions for $M$ and $L$, for which the surface area of $M$ (a special form of $\lambda(g)$ when $g \equiv 1$) is the only unknown quantity that needs to be estimated (Qiao and Polonik, 2018b). A similar application to the asymptotic distribution for ridge estimation in Qiao and Polonik (2016) requires the estimation of surface integrals on ridges, which are low-dimensional manifolds. The quantity $\lambda(g)$ is also a key component in the concept of vertical density representation (Troutt et al. 2004). Surface integrals on (regression) level sets also appears in optimal tuning parameter selection for nearest neighbour classifiers (Hall and Kang 2005, Samworth 2012, Cannings et al. 2017). To our knowledge, the only article considering the asymptotic theory for the estimation of the surface integrals on $M$ is Cuevas et al. (2012), where the authors obtain a consistency result for the estimation of the surface...
The estimation of surface integrals on submanifolds embedded in $\mathbb{R}^d$ is a challenging question in statistics. In the literature there exists some recent work on the estimation of the surface area of the boundary of an unknown body $S \subset G$ where $G$ is a bounded set. The sampling scheme assumes a uniform distribution on $G$ and the binary labels for $S$ and $G \setminus S$ are observed with the sample. The surface area is defined as the Minkowski content, which coincides with the normalized Hausdorff measure in regular cases. In this setting the convergence rates of the estimators for surface area of $\partial S$ are derived under different shape assumptions (Cuevas et al. 2007, Pateiro-López and Rodríguez-Casal 2008, 2009, Armendáriz et al. 2009). Trillo et al. (2017) consider the same setting but use a more general notion of surface area. The consistency of the surface integral on $\partial S$ has been considered by Jiménez and Yukich (2011). Assuming the i.i.d. sampling scheme only on $S$, Arias-Castro and Rodríguez-Casal (2017) estimate the perimeter of $\partial S$ using the alpha-shape for $d = 2$ and derive the convergence rate.

Surface area has been used in the definition of “contour index”, which is the ratio between perimeter and square root of the area, and has appeared in medical imaging and remote sensing (Canzonieri and Carbone, 1998, García-Dorado et al., 1992, Salas et al., 2003). The estimation of surface area also has extensive applications in stereology (Baddeley et al., 1986, Baddeley and Jensen, 2005, Gokhale, 1990). The surface area of a level set corresponds to surface integral $\lambda(g)$ for the simplest example when $g \equiv 1$. Other examples of $\lambda(g)$ arise where $g$ can be observable temperature, humidity, or the density of some non-homogeneous material, and one is interested in the surface integrals of these quantities on an unknown level set, as explained by Jiménez and Yukich (2011) in a similar setting.

Our main contributions are summarized as follows.

(1) We derive the asymptotic normality of the plug-in kernel estimator of $\lambda(g)$ when $g$ is known. Asymptotic distributional results are few in both the fields of surface area/integral estimation and level set estimation. Armendáriz et al. (2009) obtain an asymptotic normality result for the surface area of $\partial S$, but in the typical framework of uniform distribution sampling as described above. Note that our asymptotic normality is technically different from the result in this framework, the assumptions of which eliminate the requirement for estimation of the density and even the boundary $\partial S$ to estimate the surface area. Mason and Polonik (2009) prove an asymptotic normality result for the estimation of upper level set $\mathcal{L}$, but the involved measure for integrals is not low-dimensional as what we study in this paper.

(2) We derive the convergence rate of the direct plug-in bandwidth selector for density level set estimation, which involves $\lambda(g)$ for some unknown $g$ functions. Bandwidth selection is always critical in kernel-type estimators. Qiao (2018) obtains the asymptotic optimal bandwidth for kernel estimator of density level set, which needs the estimation of surface integrals on $\mathcal{M}$ for practical plug-in bandwidth selector. Our asymptotic results complements the study of this
1.1 Set-up and structure

Let us formally provide the setting and define our estimators in the paper. Let \( X_1, \ldots, X_n \) be an i.i.d. sample generated from an unknown \( d \)-dimensional density function \( f \). The kernel density estimator of \( f \) is given by

\[
\hat{f}(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right),
\]

where \( h \) is the bandwidth and \( K \) is a \( d \)-dimensional kernel function. The plug-in estimators of \( M \) and \( L \) are given by

\[
\hat{M} = \{ x \in \mathbb{R}^d : \hat{f}(x) = c \}
\]

and

\[
\hat{L} = \{ x \in \mathbb{R}^d : \hat{f}(x) \geq c \},
\]

respectively.

For the convenience of exposition, we adopt a product kernel \( K \) with

\[
K \left( \frac{x - X_i}{h} \right) = \prod_{j=1}^{d} \tilde{K} \left( \frac{x_j - X_{ij}}{h} \right),
\]

where \( \tilde{K} \) is a univariate kernel function, \( x_j \) and \( X_{ij} \) are the elements of \( x \) and \( X_i \), respectively. The kernel function \( K \) is called a \( \nu \)th \((\nu \geq 2)\) order kernel if

\[
\int_{\mathbb{R}} |u^\nu \tilde{K}(u)|du < \infty \quad \text{and} \quad \int_{\mathbb{R}} u^l \tilde{K}(u)du = \begin{cases} 1, & \text{if } l = 0, \\ 0, & \text{if } l = 1, \ldots, \nu - 1, \\ \kappa_\nu > 0, & \text{if } l = \nu. \end{cases}
\]

The estimator \( \hat{f} \) using a high-order kernel \((\nu > 2)\) usually takes negative values in the tails, and therefore loses its interpretability for practitioners (see, e.g., Silverman, 1986, page 69, Hall and Murison, 1993). However, this is not a problem for our estimator, since we are only interested in the level set of \( f \) at a positive level, and our estimator \( \hat{M} \) does not include the negative values of \( \hat{f} \).

When \( g \) is known, our estimator for \( \lambda(g) \) is

\[
\hat{\lambda}(g) := \int_{\hat{M}} g(x)d\mathcal{H}(x).
\]

In Section 2, we show the convergence rate and asymptotic normality of \( \hat{\lambda}(g) - \lambda(g) \) under mild assumptions, as well as some useful ancillary results. Our proof relies on the construction of a diffeomorphism between \( M \) and \( \hat{M} \), in order to overcome the challenge of different integral domains in \( \lambda(g) \) and \( \hat{\lambda}(g) \). The relevant geometric concepts and the assumptions that guarantee the existence of such a diffeomorphism are given in the following subsections.
It is well-known that bandwidth selection is very important for kernel-type estimators, such as \( \hat{M} \) and \( \hat{L} \). It turns out that the bandwidth selection for level set estimation also requires estimation of the surface integral \( \lambda(g) \). Let \( \mathcal{L} \Delta \hat{\mathcal{L}} = (\mathcal{L} \setminus \hat{\mathcal{L}}) \cup (\hat{\mathcal{L}} \setminus \mathcal{L}) \), which is the symmetric difference between \( \mathcal{L} \) and \( \hat{\mathcal{L}} \). Integrals defined on \( \mathcal{L} \Delta \hat{\mathcal{L}} \) are usually used to measure the dissimilarity between \( \mathcal{L} \) and \( \hat{\mathcal{L}} \) (or between \( M \) and \( \hat{M} \)). See, e.g., Bafillo et al. (2000), Bafillo (2003), Cadre (2006), Cuevas et al. (2006), and Mason and Polonik (2009). Using \( \mathbb{E} \int_{\mathcal{L} \Delta \hat{\mathcal{L}}} |f(x) - c| dx \) as the risk function, Qiao (2018) showed that the asymptotic optimal bandwidth for \( \hat{f} \) depends on \( \lambda(g_1)/\lambda(g_2) \), where \( g_1 \) and \( g_2 \) are some algebraic functions of the first two derivatives of \( f \). Let \( \hat{g}_1 \) and \( \hat{g}_2 \) be the plug-in estimators of \( g_1 \) and \( g_2 \). The direct plug-in bandwidth selector is then \( \hat{\lambda}(\hat{g}_1)/\hat{\lambda}(\hat{g}_2) \). In Section 3, we show the convergence rate of \( \hat{\lambda}(\hat{g}_1)/\hat{\lambda}(\hat{g}_2) - \lambda(g_1)/\lambda(g_2) \). The proofs of all the results in Sections 2 and 3 are left to Section 4. The appendix contains discussions on one of the technical assumptions.

1.2 Notation and geometric concepts

For a function \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) with \( p \)th derivatives \( (p \geq 1) \), and integers \( 1 \leq i_1, \ldots, i_p, i_{p+2} \leq d \), denote \( g(i_1, \ldots, i_p)(x) = \frac{\partial^p}{\partial x_{i_1} \cdots \partial x_{i_p}} g(x) \). If \( i_1 = \cdots = i_p = i \), denote \( g(i, \ldots, i)(x) = g(i, \ldots, i)(x) \). If in addition \( g \) has \( (p+2) \)th derivatives, we denote \( g(i, \ldots, i, i_{p+1}, i_{p+2})(x) = g(i, \ldots, i, i_{p+1}, i_{p+2})(x) \). Let \( \nabla g \) and \( \nabla^2 g \) be the gradient and Hessian matrix of \( g \), respectively.

For any \( x \in \mathbb{R}^d \) and \( A \subset \mathbb{R}^d \), let \( d(x, A) = \inf_{y \in A} \|x - y\| \). The Hausdorff distance between any two sets \( A, B \subset \mathbb{R}^d \) is

\[
    d_H(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\}.
\]

The ball with center \( x \) and radius \( \epsilon \) is denoted by \( B_x(\epsilon) = \{ y \in \mathbb{R}^d : \|x - y\| \leq \epsilon \} \). For any set \( A \subset \mathbb{R}^d \) and \( \epsilon > 0 \), we denote \( A \oplus \epsilon = \bigcup_{x \in A} B_x(\epsilon) \). Let the normal projection of \( x \) onto \( A \) be \( \pi_A(x) = \{ y \in A : \|x - y\| = d(x, A) \} \). Note that \( \pi_A(x) \) may not be a single point.

We will also use the concept of reach of a manifold. For a set \( S \subset \mathbb{R}^d \), let \( \text{Up}(S) \) be the set of points \( x \in \mathbb{R}^d \) such that \( \pi_S(x) \) is unique. The reach of \( S \) is defined as

\[
    \rho(S) = \sup\{ \delta > 0 : S \oplus \delta \subset \text{Up}(S) \}.
\]

See Federer (1959). A positive reach is related to the concepts of “r-convexity” and “rolling condition”. See Cuevas et al. (2012). These are common regularity conditions for the estimation of surface area. For two manifolds \( A \) and \( B \), if the normal projections \( \pi_A : B \rightarrow A \) and \( \pi_B : A \rightarrow B \) are homeomorphisms, then \( A \) and \( B \) are called normal compatible (see Chazal et al. (2007)).
1.3 Assumptions and their discussion

We introduce the assumptions that will be used in this paper. For \( \delta > 0 \), denote \( I(\delta) = f^{-1}([c - \delta/2, c + \delta/2]) = \{ x \in \mathbb{R}^d : f(x) \in [c - \delta/2, c + \delta/2] \} \).

Assumptions:

(K1) \( K \) is a two times continuously differentiable, symmetric product kernel function of \( \nu \)th order for \( \nu \geq 2 \), with bounded support.

(F1) The density function \( f \) is uniformly continuous on \( \mathbb{R}^d \) and has bounded and continuous partial derivatives up to \( (\nu + 2) \) order on \( I(2\delta_0) \). There exist \( \delta_0 > 0 \) and \( \epsilon_0 > 0 \) such that \( \| \nabla f(x) \| > \epsilon_0 \) for all \( x \in I(2\delta_0) \).

(F2) We assume that \( M \) consists of \( N \) connected components, i.e., \( M = \bigcup_{j=1}^{N} M^j \), where \( M^j \), \( j = 1, \cdots, N \), are \( (d-1) \)-dimensional connected manifolds such that \( \inf_{1 \leq j \neq k \leq N} d_H(M^j, M^k) > 0 \). We further assume that for \( j = 1, \cdots, N \), \( M^j \) is parameterized by \( \psi_j \) on \( \Omega_j \) which is a \( (d-1) \)-dimensional Lebesgue measurable subset of \( \mathbb{R}^{d-1} \), such that the \( (d-1) \)-dimensional Lebesgue measure of \( \partial \Omega_j \) is zero, and \( \psi_j \) is injective on \( \Omega^0_j \), which is the interior of \( \Omega_j \). Denoting the Jacobian matrix of \( \psi_j \) as \( B_j \) and \( J_j = B_j^T B_j \), we require that \( B_j \) is continuous and bounded such that

\[
0 < \inf_{\theta \in \Omega_j^0} \det[J_j(\theta)] \leq \sup_{\theta \in \Omega_j^0} \det[J_j(\theta)] < \infty. \tag{1.4}
\]

(H1) The bandwidth \( h \) depends on \( n \) such that \( (\log n)^{-1} n h^{d+4} \rightarrow \infty \) as \( n \rightarrow \infty \).

Discussion of the assumptions:

1. The assumption \( \| \nabla f(x) \| > \epsilon_0 \) for \( x \in I(2\delta_0) \) in (F1) implies that the Lebesgue measure of \( M \) on \( \mathbb{R}^d \) is zero. This is a typical assumption in the literature of level set estimation (see, e.g., Tysbakov (1997), Cadre (2006), Cuevas et al. (2006), Mammen and Polonik (2013)), and guarantees that \( M \) has no flat parts and is a compact \( (d-1) \)-dimensional manifold (see Theorem 2 in Walther (1997)), which makes \( \lambda(g) \) well-defined.

2. Assumption (F2) is used in the derivation of the asymptotic normality of the surface integral estimator. An assumption similar to (F2) appears in Mason and Polonik (2009), where for \( j = 1, \cdots, N \), they take

\[
\Omega_j = \Omega := \begin{cases} [0, 2\pi), & d = 2, \\
[0, \pi]^{d-2} \times [0, 2\pi), & d > 2. \end{cases} \tag{1.5}
\]

It is known (e.g. see Lang, 1997) that the unit \( (d-1) \)-sphere \( S^{d-1} = \{ x \in \mathbb{R}^d : \| x \| = 1 \} \) can be parameterized with \( \Omega \). Mason and Polonik (2009) assume the existence of diffeomorphisms between \( M^j \) and \( S^{d-1} \) for the construction of the parameterization \( \psi_j \), for \( j = 1, \cdots, N \).
The diffeomorphisms between $M_j$ and $S^{d-1}$ can be constructed under mild conditions using Morse theory (Milnor, 1963). For $\tau \in \mathbb{R}$, let $\mathcal{M}_\tau = \bigcup_{j=1}^{N(\tau)} M_j^\tau$, where $M_j^\tau$ are the connected components of $f^{-1}(\tau)$. When $M_j^\tau$ is within a small neighborhood of a non-degenerate local maximum (or minimum), $M_j^\tau$ is diffeomorphic to $S^{d-1}$ (Lemma 2.25 in Nicolaescu, 2011). If no critical points exist between $M_j^c$ and $M_j^\tau$, then they are diffeomorphic via $\Phi_{\tau-c}(x)$, where $\Phi_t(x) = \gamma_x(t)$ is a gradient flow satisfying

$$
\frac{d\gamma_x(t)}{dt} = \frac{\nabla f(\gamma_x(t))}{\|\nabla f(\gamma_x(t))\|^2}, \quad \gamma_x(0) = x.
$$

See Theorem 2.6 in Nicolaescu (2011). This simply means $M_j^c$ is diffeomorphic to $S^{d-1}$ and the parameterization $\psi_j$ on $\Omega$ can be constructed accordingly. If there exist one or more critical points between $M_j^c$ and $M_j^\tau$, further discussion can be found in the appendix based on Morse theory.

## 2 Surface integral estimation with a known integrand

We first consider the case that the integrand $g$ is a known function. The following theorem includes the convergence rate and asymptotic normality for $\hat{\lambda}(g) - \lambda(g)$.

**Theorem 2.1** Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function with bounded continuous second order derivatives. Under assumptions (K1), (F1), (F2) and (H1), we have

(i) 

$$
\hat{\lambda}(g) - \lambda(g) = O_p \left( \frac{\log n}{nh^{d+2}} + h^{\nu} + \frac{1}{\sqrt{nh^3}} \right);
$$

(ii) if in addition $\log(n)^{-1/2}nh^{2d+1} \rightarrow \infty$ and $nh^{3+2\nu} \rightarrow \gamma \geq 0$, we have

$$
\sqrt{nh^3}[\hat{\lambda}(g) - \lambda(g)] \rightarrow_D \mathcal{N}(\sqrt{\gamma}\mu, \sigma^2),
$$

where for $N = 1$, $\mu$ is given in (4.64) and $\sigma^2$ is given in (4.55); and for $N \geq 2$, $\mu$ is given in (4.65) and $\sigma^2$ is given in (4.66).

**Remark**

1. The proof of this theorem is built upon the ancillary results below in this section. One of the main challenges is that the domains of integrals in $\lambda(g)$ and $\hat{\lambda}(g)$ are not the same, which makes the comparison difficult. Briefly speaking, our strategy is to establish a diffeomorphism between the two domains and utilize the area formula in differential geometry to convert the surface integrals into those with the same integral domains. The heuristic behind the normalizing factor $\sqrt{nh^3}$ is that, after the conversion, the source of difference arises from the integrands, which are Jacobian matrices (first derivatives), and have a standard rate of $1/\sqrt{nh^{d+2}}$. Due to the integrals on a $(d-1)$-dimensional manifold, we gain $(d-1)$ powers of $h$, which results in the normalizing factor $\sqrt{nh^3}$.
2. For part (ii), the constraints $\log(n)^{-1/2} n h^{2d+4} \to \infty$, $nh^{3+2\nu} \to \gamma \geq 0$, as well as $\log(n)^{-1/2} n h^{d+4} \to \infty$ in assumption (H1), require that the integer $\nu > d - 1$.

The following lemma shows results related to the geometric concepts introduced in Subsection 1.2 for level sets. It is similar to Lemma 1 in Chen et al. (2017), but our result is under slightly different assumptions and holds uniformly for a collection of density level sets. Also see Theorems 1 and 2 in Walther (1997) for relevant results.

**Lemma 2.1** Under assumptions (K1), (F1), and (H1), the following results hold.

(i) There exists $r_0 > 0$, such that the reach $\rho(M_\tau) > r_0$ for all $\tau \in [c - \delta_0/2, c + \delta_0/2]$.

(ii) When $n$ is sufficiently large, $\hat{M}_\tau$ and $M_\tau$ are normal compatible for all $\tau \in [c - \delta_0/2, c + \delta_0/2]$ with probability one.

For $x \in M_\tau$ with $|\tau - c| \leq \delta_0/2$, and $t \in \mathbb{R}$, let

$$\zeta_x(t) = x + \frac{\nabla f(x)}{\|\nabla f(x)\|} t.$$ 

Note that $\{\zeta_x(t) : t \in \mathbb{R}\}$ is orthogonal to the tangent space of $M$ at $x$. Furthermore, let

$$t_n(x) = \arg\min_t \left\{ |t| : \zeta_x(t) \in \hat{M}_\tau \right\},$$

and

$$P_n(x) = \zeta_x(t_n(x)).$$

Under the assumptions in Lemma 2.1, $\hat{M}_\tau$ and $M_\tau$ are normal compatible for large $n$. Hence $t_n(x)$ is uniquely defined, and $P_n$ is a homeomorphism between $M_\tau$ and $\hat{M}_\tau$. In fact, $P_n = \pi_{\hat{M}}^{-1}$. We write $P_n(M_\tau) = \hat{M}_\tau$.

The following lemma gives asymptotic results for $t_n(x)$. Recall $\mathcal{I}(\delta_0) = f^{-1}([c - \delta_0/2, c + \delta_0/2])$.

**Lemma 2.2** Suppose that assumptions (K1), (F1) and (H1) hold. For any point $x \in \mathcal{I}(\delta_0)$, we have

$$t_n(x) = \frac{f(x) - \hat{f}(x)}{\|\nabla f(x)\|} + \delta_1 n(x),$$

$$\nabla t_n(x) = \frac{\nabla f(x) - \nabla \hat{f}(x)}{\|\nabla f(x)\|} - \frac{\nabla \|\nabla f(x)\|}{\|\nabla f(x)\|^2} t_n(x) + \delta_2 n(x),$$

where

$$\sup_{x \in \mathcal{I}(\delta_0)} |\delta_1 n(x)| = O_p \left\{ \left( \frac{\log n}{nh} \right) \left( \frac{\log n}{nh^{d+2}} + h^\nu \right) \right\},$$

$$\sup_{x \in \mathcal{I}(\delta_0)} \|\delta_2 n(x)\| = O_p \left\{ \left( \frac{\log n}{nh} + h^\nu \right) \left( \frac{\log n}{nh^{d+4}} + h^\nu \right) \right\}. $$
Remark

1. Note that $|t_n(x)| = \|P_n(x) - x\|$. Then the above results link the local horizontal variation $t_n(x)$ with the local vertical variation $f(x) - \hat{f}(x)$ as well as their derivatives. Here $\|\nabla f(x)\|$ on the right-hand side of (2.4) and (2.5) can be understood as a directional derivative of $f$ in the gradient direction, i.e., $\|\nabla f(x)\| = \langle \nabla f(x), \frac{\nabla f(x)}{\|\nabla f(x)\|} \rangle$, and it reflects the asymptotic rate of change between the local vertical and horizontal variations, as $P_n(x) - x$ is parallel to the direction of $\|\nabla f(x)\|$.

Let $(\psi, \Omega)$ be one of the pairs $(\psi_j, \Omega_j)$, $j = 1, \cdots, N$ defined in assumption (F2). The following proposition establishes the asymptotic normality theory corresponding to description in the remark after Theorem 2.1. In particular, the rate $\sqrt{nh^3}$ in (2.8) has been heuristically explained and the rate $\sqrt{nh}$ in (2.9) follows a similar justification.

Proposition 2.1 Suppose that assumptions (K1), (F1) and (F2) hold. Let $w : \Omega \mapsto \mathbb{R}^d$ be a vector function with bounded continuous first derivatives. If $nh^3 \to \infty$, then

$$\sqrt{nh^3} \int_\Omega w(\theta)^T [\nabla \hat{f}(\psi(\theta)) - E \nabla \hat{f}(\psi(\theta))] d\theta \to_D \mathcal{N}(0, \sigma_1^2).$$

(2.8)

where $\sigma_1^2$ is given in (4.22).

Similarly, let $v : \Omega \mapsto \mathbb{R}$ be a bounded continuous function. If $nh \to \infty$, then

$$\sqrt{nh} \int_\Omega v(\theta)[\hat{f}(\psi(\theta)) - E \hat{f}(\psi(\theta))] d\theta \to_D \mathcal{N}(0, \sigma_2^2),$$

(2.9)

where $\sigma_2^2$ is given in (4.22).

So far we assume the integrand $g$ is known. In the next section we will consider some cases when $g$ is unknown and has to be estimated. In particular our study is in the context of convergence rate of the bandwidth selection for density level set estimation, for which $g$ is a functional of the derivatives of $f$.

3 Bandwidth selection for nonparametric level set estimation

Bandwidth selection for nonparametric estimation of density level sets is considered by Qiao (2018), where he uses the following excess risk as the criterion of bandwidth selection and shows that under regularity conditions,

$$\mathbb{E} \int_{\mathcal{L}^2 \mathcal{E}} |f(x) - c| dx = \frac{1}{2} \mathbb{E} \int_{\mathcal{M}} \frac{|\hat{f}(x) - f(x)|^2}{\|\nabla f(x)\|} d\mathcal{H}(x) \{1 + o(1)\}.$$  

(3.1)

Individual bandwidth for each dimension is used in Qiao (2018). For expositional simplicity we use a single bandwidth for all the dimensions. However, the rates obtained in this section
also carry through for the case of bandwidth vector. Following standard computation from (3.1), we have
\[
\mathbb{E} \int_{\mathbb{L} \Delta \mathbb{L}} |f(x) - c| \, dx = \frac{1}{2} \tilde{m}(h) \{1 + o(1)\},
\]
where \(\tilde{m}(h) = \frac{\kappa_2 h^{2\nu}}{(\nu!)^2} \sum_{1 \leq k, l \leq d} a_{kl}(f) + \frac{c \|f\| K_{\nu}^2}{n h^d}\) with
\[
b(f) = \lambda(\|\nabla f\|^{-1}),
\]
\[
a_{kl}(f) = \lambda \left[ f_{(k+1)} f_{(l+1)} \|\nabla f\|^{-1} \right], \quad 1 \leq k, l \leq d.
\]
The asymptotic optimal bandwidth which minimizes \(\tilde{m}(h)\) is
\[
\tilde{h}_{opt} = \left( \frac{d c b(f) \|K\|_{\nu}^2}{2 \kappa_2 \sum_{1 \leq k, l \leq d} \tilde{a}_{kl}(f)} \right)^{1/(\nu + d)}.
\]
The plug-in bandwidth selector is
\[
\hat{h}_{opt} = \left( \frac{d \hat{c} b(f) \|K\|_{\nu}^2}{2 \kappa_2 \sum_{1 \leq k, l \leq d} \hat{a}_{kl}(f)} \right)^{1/(\nu + d)},
\]
where
\[
\hat{b}(f) = \hat{\lambda}(\|\nabla \hat{f}\|^{-1}),
\]
\[
\hat{a}_{kl}(f) = \hat{\lambda} \left( \hat{f}_{(k+1)} \hat{f}_{(l+1)} \|\nabla \hat{f}\|^{-1} \right), \quad 1 \leq k, l \leq d.
\]
Note that \(\tilde{h}_{opt}\) minimizes the estimated risk function \(\tilde{m}(h) = \frac{\kappa_2 h^{2\nu}}{(\nu!)^2} \sum_{1 \leq k, l \leq d} \tilde{a}_{kl}(f) + \frac{c \|f\| K_{\nu}^2}{n h^d}\).

Three different pilot bandwidths \(h_0, h_1, h_2\) are allowed for the estimators \(\tilde{M}, \|\nabla \hat{f}\|^{-1}\) and \(\hat{f}_{(k+1)} \hat{f}_{(l+1)}\), respectively. It is suggested in Qiao (2018) that the direct plug-in bandwidths for the kernel density and its first two derivatives are used as the pilot bandwidths, which are of the orders of \(n^{-1/(d+2\nu)}\), \(n^{-1/(d+4+2\nu)}\) and \(n^{-1/(d+4+2\nu)}\), respectively (see, e.g., Wand and Jones, 1995). In the following theorem we compare the plug-in optimal bandwidth \(\hat{h}_{opt}\) with the asymptotic optimal bandwidth \(\tilde{h}_{opt}\).

**Theorem 3.1** Suppose assumptions (K1), (F1), (F2) and (H1) hold. In addition, assume that \(K\) is three times continuously differentiable on \(\mathbb{R}^d\) and \(f\) is \((\nu + 3)\) times continuously differentiable on \(I(2\delta_0)\). If \(h_0 \asymp n^{-1/(d+2\nu)}\), \(h_1 \asymp n^{-1/(d+4+2\nu)}\) and \(h_2 \asymp n^{-1/(d+4+2\nu)}\), then for \(\nu > 2\),
\[
\hat{h}_{opt} = \tilde{h}_{opt} \left\{ 1 + O_p \left( n^{-\nu/(d+4+2\nu)} \right) \right\},
\]
and
\[
\hat{m}(\hat{h}_{opt}) = \tilde{m}(\tilde{h}_{opt}) \left\{ 1 + O_p \left( n^{-\nu/(d+4+2\nu)} \right) \right\}.
\]
The remaining part of this section is denoted to some ancillary results used to prove the above theorem. When comparing \( \hat{h}_{\text{opt}} \) and \( \tilde{h}_{\text{opt}} \), we need to study the asymptotic performance of \( \hat{h}(f) - b(f) \) and \( \tilde{h}_{kl}(f) - a_{kl}(f) \), \( 1 \leq k, l \leq d \), which further depend on some specific forms of the following quantities with a generic function \( g \) and random function \( g_n \), in addition to \( \hat{\lambda}(g) - \lambda(g) \) which is studied in Theorem 2.1. Define

\[
D_{g,n}^{(0)} := \hat{\lambda}(g_n) - \lambda(g_n),
\]

\[
D_{g,n}^{(1)} := \lambda\left( g_f^2(k) \right) - \lambda\left( g_{f(k)}^2 \right),
\]

\[
D_{g,n}^{(2)} := \lambda\left( g_{f(k,k)} f(l,l) \right) - \lambda\left( g_{f(k,k)} f(l,l) \right),
\]

for integers \( k \) and \( l \) such that \( 1 \leq k, l \leq d \). Note that we omit the indices \( k \) and \( j \) for \( D_{g,n}^{(1)} \) and \( D_{g,n}^{(2)} \) for simplicity.

The following theorem gives an asymptotic result for \( D_{g,n}^{(0)} \) in (3.8).

**Proposition 3.1** Suppose that assumptions (K1), (F1), (F2), and (H1) hold. For a sequence of functions \( g_n : \mathcal{I}(\delta_0) \to \mathbb{R} \), suppose that there exist sequences \( \alpha_n \to 0 \) and \( \beta_n \to 0 \) as \( n \to \infty \), such that \( \sup_{x \in \mathcal{I}(\delta_0)} |g_n(x)| = O_p(\alpha_n) \) and \( \sup_{x \in \mathcal{I}(\delta_0)} \|\nabla g_n(x)\| = O_p(\beta_n) \). Then

\[
D_{g,n}^{(0)} = O_p \left\{ \alpha_n \left( \frac{\sqrt{\log n}}{\sqrt{n h^d}} + h^\nu \right) + \beta_n \left( \frac{\sqrt{\log n}}{\sqrt{n h^d}} + h^\nu \right) \right\}. \tag{3.11}
\]

In the next proposition we consider \( D_{g,n}^{(1)} \) in (3.9) and \( D_{g,n}^{(2)} \) in (3.10).

**Proposition 3.2** Suppose that assumptions (K1), (F1), and (H1) hold. Let \( g : \mathcal{I}(\delta_0) \to \mathbb{R} \) be a bounded continuously differentiable function. If \( n \to \infty \) and \( h \to 0 \), then

\[
D_{g,n}^{(1)} = O \left( \frac{1}{n h^d} + h^\nu \right) + O_p \left( \sqrt{\frac{1}{n h^3} + \frac{1}{n^3 h^{2d+5}} + \frac{1}{n^2 h^{d+5}}} \right), \tag{3.12}
\]

and

\[
D_{g,n}^{(2)} = O \left( \frac{1}{n h^d} + h^\nu \right) + O_p \left( \sqrt{\frac{1}{n h^5} + \frac{1}{n^3 h^{2d+9}} + \frac{1}{n^2 h^{d+9}}} \right). \tag{3.13}
\]

4 Proofs

**Proof of Lemma 2.1**
Proof. We first show the proof of assertion (i). Let \( \| f \|_{0, \text{max}}^* = \sup_{x \in \mathbb{R}^d} |f| \). For \( r = 1, 2 \) and 3, let
\[
\| f \|_{r, \text{max}}^* = \max \left\{ \| f \|_{0, \text{max}}^*, \max_{x \in I(2\delta_0)} |f(i_1, \ldots, i_p)(x)| : \max(i_1, \ldots, i_p) \leq d, 1 \leq p \leq r \right\}.
\]
Let \( b_0 = \inf_{x \in \partial I(2\delta_0)} d(x, I(\delta_0)) \), that is, the closest distance between the set \( I(\delta_0) \) and the boundary of the set \( I(2\delta_0) \). Given assumption (F1), we have \( b_0 > 0 \). This implies that for any \( \tau \) with \( |\tau - c| \leq \delta_0/2 \), and any \( x \in M_{\tau} \oplus b_0 \), we have \( \| \nabla f(x) \| \geq \epsilon_0 \). The assertion that \( M_{\tau} \) has a positive reach for \( |\tau - c| \leq \delta_0/2 \) immediately follows from Lemma 1 in Chen et al. (2017) and a positive lower bound \( r_0 \) of the reach \( \rho(M_{\tau}) \) is given by
\[
r_0 = \min \left( \frac{b_0}{2}, \| f \|_{2, \text{max}}^* \right).
\]
Since \( r_0 \) is independent of \( \tau \), we have
\[
r_0 \leq \inf_{|\tau - c| \leq \delta_0/2} \rho(M_{\tau}). \tag{4.1}
\]
Next we prove assertion (ii). By Theorem 2 in Cuevas et al. (2006), we have
\[
d_H(\hat{M}_{\tau}, M_{\tau}) = O\left( \| \hat{f} - f \|_{0, \text{max}}^* \right), \text{ a.s.}
\]
A closer examination of the proof given in Cuevas et al. (2006) reveals that the big \( O \) term above is uniform in \( \tau \in [c - \delta_0/2, c + \delta_0/2] \), i.e.
\[
\sup_{\tau \in [c - \delta_0/2, c + \delta_0/2]} d_H(\hat{M}_{\tau}, M_{\tau}) = O\left( \| \hat{f} - f \|_{0, \text{max}}^* \right), \text{ a.s.} \tag{4.2}
\]
Using Lemma 3 in Arias-Castro et al. (2016), it follows by assumptions (K1), (F1) and (H1) that for \( r = 0, 1 \) and 2,
\[
\| \hat{f} - E\hat{f} \|_{r, \text{max}}^* = O\left( \sqrt{\log n \over n h^{d+2r}} \right), \text{ a.s.} \tag{4.3}
\]
Also we follow a regular derivation procedure for kernel densities by applying a change of variables and Taylor expansion, and obtain from assumptions (K1), (F1) and (H1) that for \( r = 0, 1 \), and 2,
\[
\| E\hat{f}(x) - f(x) \|_{r, \text{max}}^* = O(h^\nu), \tag{4.4}
\]
where \( \nu \) is the order of the kernel function. Since \( \| \hat{f}(x) - f(x) \|_{r, \text{max}}^* \leq \| \hat{f}(x) - E\hat{f}(x) \|_{r, \text{max}}^* + \| E\hat{f}(x) - f(x) \|_{r, \text{max}}^* \), we get from (4.2) that almost surely
\[
\sup_{\tau \in [c - \delta_0/2, c + \delta_0/2]} d_H(\hat{M}_{\tau}, M_{\tau}) = O\left( \sqrt{\log n \over n h^{d}} \right) + O(h^\nu). \tag{4.5}
\]
Using (4.3) and (4.4), and following a similar argument for deriving (4.1), we can find \( \hat{r}_0 \) with
\[
|\hat{r}_0 - r_0| = O \left( \frac{\log n}{nh^{2+\tau}} \right) + O(h^\nu) = o(1)
\]
almost surely such that
\[
\hat{r}_0 \leq \inf_{\tau \in [c - \delta_0/2, c + \delta_0/2]} \rho(\hat{M}_\tau). \tag{4.6}
\]
By (4.1), (4.5) and (4.6), for \( n \) large enough with probability one we have
\[
\sup_{\tau \in [c - \delta_0/2, c + \delta_0/2]} d_H(\hat{M}_\tau, M_\tau) \leq (2 - \sqrt{2}) \inf_{\tau \in [c - \delta_0/2, c + \delta_0/2]} \min(\rho(\hat{M}_\tau), \rho(M_\tau))
\]
which by Theorem 1 in Chazal et al. (2007) implies that \( \hat{M}_\tau \) and \( M_\tau \) are normal compatible for all \( \tau \in [c - \delta_0/2, c + \delta_0/2] \). \( \Box \)

**Proof of Lemma 2.2**

**Proof.** We first focus on \( t_n(x) \). The fact that \( |t_n(x)| = \|P_n(x) - x\| \) and (4.5) imply
\[
\sup_{x \in I(\delta_0)} |t_n(x)| \leq \sup_{\tau \in [c - \delta_0/2, c + \delta_0/2]} d_H(\hat{M}_\tau, M_\tau) = o_p(1). \tag{4.7}
\]
Since \( \hat{f}(P_n(x)) - f(x) = 0 \), using Taylor expansion for \( \hat{f}(P_n(x)) \) we obtain
\[
0 = \hat{f} \left( x + \frac{\nabla f(x)}{\|\nabla f(x)\|} t_n(x) \right) - f(x) = \hat{f}(x) - f(x) + \frac{\nabla f(x)^T \nabla \hat{f}(x)}{\|\nabla f(x)\|} t_n(x) + e_n(x), \tag{4.8}
\]
where
\[
e_n(x) = \frac{1}{2 \|\nabla f(x)\|^2} \nabla f(x)^T \nabla^2 \hat{f} \left( x + s_1 \frac{\nabla f(x)}{\|\nabla f(x)\|} t_n(x) \right) \nabla f(x) t_n(x)^2,
\]
for some \( 0 < s_1 < 1 \). Plugging \( \nabla \hat{f}(x) = \nabla f(x) - [\nabla f(x) - \nabla \hat{f}(x)] \) into (4.8), we have
\[
t_n(x) = \frac{1}{\|\nabla f(x)\|} [f(x) - \hat{f}(x)] + \delta_{1n}(x), \tag{4.9}
\]
where
\[
\delta_{1n}(x) = \frac{1}{\|\nabla f(x)\|^2} \nabla f(x)^T [\nabla f(x) - \nabla \hat{f}(x)] t_n(x) - \frac{1}{\|\nabla f(x)\|} e_n(x).
\]
Under assumption (F1), we then have
\[
\sup_{x \in I(\delta_0)} |\delta_{1n}(x)| \
\leq \frac{1}{\epsilon_0} \sup_{x \in I(\delta_0)} \|\nabla f(x) - \nabla \hat{f}(x)\| \sup_{x \in I(\delta_0)} |t_n(x)| + \frac{1}{2\epsilon_0} \left[ \sup_{x \in I(\delta_0)} |t_n(x)| \right]^2 \sup_{x \in I(\delta_0)} \|\nabla^2 \hat{f}(x)\|_F, \tag{4.10}
\]
\]
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where $\| \cdot \|_F$ is the Frobenius norm of a matrix. Then by (4.3) and (4.4), (4.10) implies that
\[
\sup_{x \in \mathcal{I}(\delta_0)} |\delta_{1n}(x)| = o_p \left( \sup_{x \in \mathcal{I}(\delta_0)} |t_n(x)| \right),
\]
and furthermore via (4.9),
\[
\sup_{x \in \mathcal{I}(\delta_0)} |t_n(x)| \leq \frac{1}{\epsilon_0} \sup_{x \in \mathcal{I}(\delta_0)} |\hat{f}(x) - f(x)| + \sup_{x \in \mathcal{I}(\delta_0)} |\delta_{1n}(x)| = O_p \left( \sqrt{\frac{\log n}{nh^q}} + h^\nu \right).
\]
Using (4.11) again, we obtain (2.6).

Next we study $\nabla t_n(x)$. Another Taylor expansion for $\hat{f}(P_n(x))$ with fewer terms than (4.8) gives
\[
0 = \hat{f} \left( x + \frac{\nabla f(x)}{\|\nabla f(x)\|} t_n(x) \right) - f(x) = \hat{f}(x) - f(x) + t_n(x) \frac{\nabla f(x)^T r_n(x)}{\|\nabla f(x)\|}, \tag{4.11}
\]
where
\[
r_n(x) = \int_0^1 \nabla \hat{f} \left( x + s \frac{\nabla f(x)}{\|\nabla f(x)\|} t_n(x) \right) ds. \tag{4.12}
\]
Taking gradient on both sides of (4.11), we obtain
\[
0 = \nabla \hat{f}(x) - \nabla f(x) + \nabla \left( \frac{\nabla f(x)^T r_n(x)}{\|\nabla f(x)\|} \right) t_n(x) + \frac{\nabla f(x)^T r_n(x)}{\|\nabla f(x)\|} \nabla t_n(x)
\]
\[
= \nabla \hat{f}(x) - \nabla f(x) + \|\nabla f(x)\| \nabla t_n(x) + s_n(x),
\]
where
\[
s_n(x) = \nabla \left( \frac{\nabla f(x)^T r_n(x)}{\|\nabla f(x)\|} \right) t_n(x) + \frac{\nabla f(x)^T [r_n(x) - \nabla f(x)]}{\|\nabla f(x)\|} \nabla t_n(x). \tag{4.13}
\]
Therefore
\[
\nabla t_n(x) = \frac{\nabla f(x) - \nabla \hat{f}(x)}{\|\nabla f(x)\|} - \frac{s_n(x)}{\|\nabla f(x)\|}. \tag{4.14}
\]
Let
\[
u_n(x) = \nabla \left( \frac{\nabla f(x)^T r_n(x)}{\|\nabla f(x)\|} \right) - \nabla \|\nabla f(x)\| \quad \text{and} \quad \nu_n(x) = \frac{\nabla f(x)^T [r_n(x) - \nabla f(x)]}{\|\nabla f(x)\|^2}.
\]
From (4.13) we have
\[
-\frac{s_n(x)}{\|\nabla f(x)\|} = -(\|\nabla f(x)\| + u_n(x))t_n(x) + v_n(x) \nabla t_n(x).
\]
Plugging this into (4.14), we have
\[ \nabla t_n(x) = \frac{\nabla f(x) - \nabla \hat{f}(x)}{\| \nabla f(x) \|} - \frac{\nabla \| \nabla f(x) \|}{\| \nabla f(x) \|} t_n(x) + \delta_{2n}(x), \]
where
\[ \delta_{2n}(x) = -\frac{u_n(x)t_n(x)}{\| \nabla f(x) \|} + v_n(x) \nabla t_n(x). \tag{4.15} \]

Next we will show (2.7). Comparing (4.8) and (4.11), we have
\[ r_n(x) - \nabla f(x) = \nabla \hat{f}(x) - \nabla f(x) \]
\[ + \frac{1}{2} \frac{1}{\| \nabla f(x) \|^2} \nabla^2 \hat{f} \left( x + s_1 \frac{\nabla f(x)}{\| \nabla f(x) \|} t_n(x) \right) \nabla f(x) t_n(x). \]

Therefore
\[ \sup_{x \in I(\delta_0)} |v_n(x)| = O_p \left( \sqrt{\frac{\log n}{nh^{d+2}}} + h^{\nu} \right). \tag{4.16} \]

Next we consider \( u_n(x) \). Note that
\[ u_n(x) = \frac{\nabla^2 f(x)[r_n(x) - \nabla f(x)] - \nabla f(x)^T [r_n(x) - \nabla f(x)] \nabla \| \nabla f(x) \|}{\| \nabla f(x) \|^2} \]
\[ + \frac{[\nabla r_n(x) - \nabla^2 f(x)] \nabla f(x)}{\| \nabla f(x) \|^2}. \]

Using (4.12) we have
\[ \nabla r_n(x) = \int_0^1 \nabla^2 \hat{f} \left( x + s \frac{\nabla f(x)}{\| \nabla f(x) \|} t_n(x) \right) \left( I_d + s \nabla \left( \frac{\nabla f(x)}{\| \nabla f(x) \|} t_n(x) \right) \right) ds, \]
which leads to
\[ \sup_{x \in I(\delta_0)} \| \nabla r_n(x) - \nabla^2 f(x) \|_F = O_p \left( \sqrt{\frac{\log n}{nh^{d+4}}} + h^{\nu} \right). \]

This implies
\[ \sup_{x \in I(\delta_0)} \| u_n(x) \| = O_p \left( \sqrt{\frac{\log n}{nh^{d+4}}} + h^{\nu} \right). \tag{4.17} \]

Therefore by (4.15), (4.16) and (4.17) we have
\[ \sup_{x \in I(\delta_0)} \| \delta_{2n}(x) \| = O_p \left\{ \left( \sqrt{\frac{\log n}{nh^{d+4}}} + h^{\nu} \right) \left( \sqrt{\frac{\log n}{nh^{d}}} + h^{\nu} \right) \right\}. \tag{4.18} \]

This is (2.7) and our proof is completed. □
Proof of Proposition 2.1

PROOF. We first show (2.8). Let \( w(\theta) = (w_1(\theta), \cdots, w_d(\theta)) \). Note that

\[
\int_\Omega w(\theta)^T [\nabla \hat{f}(\psi(\theta))] - \mathbb{E} \nabla \hat{f}(\psi(\theta))] d\theta = \frac{1}{n} \sum_{i=1}^n (\xi_i - \mathbb{E} \xi_i),
\]

(4.19)

where \( \xi_i = \frac{1}{n} \int \sum_{j=1}^d w_j(\theta) K(j) (h^{-1}(\psi(\theta) - X_i)) d\theta, i = 1, \cdots, n \), which are i.i.d. We next compute \( \text{Var}(\xi_i) \). Note that

\[
\mathbb{E}(\xi_i^2)
\]

\[
= \frac{1}{h^{2d+2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{j=1}^d w_j(x) K(j) \left( \frac{\psi(x) - z}{h} \right) \sum_{j=1}^d w_j(y) K(j) \left( \frac{\psi(y) - z}{h} \right) f(z) dz dx
\]

\[
+ \frac{1}{h^{d+2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\{ \sum_{j=1}^d w_j(x) K(j)(u) \right\} \left\{ \sum_{j=1}^d w_j(y) K(j) \left( \frac{\psi(y) - \psi(x)}{h} + u \right) \right\} f(\psi(x) - hu) du dy dx
\]

\[
= \frac{1}{h^{d+2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\{ \sum_{j=1}^d w_j(x) K(j)(u) \right\} \left\{ \sum_{j=1}^d w_j(\psi^{-1}(s)) K(j) \left( \frac{s - \psi(x)}{h} + u \right) \right\}
\]

\[
\frac{f(\psi(x) - hu)}{\{\det [J(\psi^{-1}(s))]\}^{1/2}} du dx
\]

where in the last two equalities we apply change of variables with substitution \( u = (\psi(x) - z)/h \) and \( s = \psi(y) \), respectively, as well as the area formula (cf. Evans and Gariepy, 1992, page 117). Define \( T_{x,h} = \{h^{-1}(\psi(y) - \psi(x)) : y \in \Omega \} \) and let \( T_x(t) \) be the tangent plane of \( \mathcal{M} \) at \( \psi(x) \). If we make change of variables again with \( v = ((s - \psi(x))/h \), then \( s \in \mathcal{M} \) is equivalent to \( v \in T_{x,h} \). We then continue the above derivation and have

\[
\mathbb{E}(\xi_i^2) = \frac{1}{h^3} \int_{T_{x,h}} \int_{\mathbb{R}^d} \left\{ \sum_{j=1}^d w_j(x) K(j)(u) \right\} \left\{ \sum_{j=1}^d w_j(\psi^{-1}(\psi(x) + hv)) K(j) (v + u) \right\}
\]

\[
\frac{f(\psi(x) - hu)}{\{\det [J(\psi^{-1}(\psi(x) + hv))]\}^{1/2}} du dx
\]

Under the assumption that the kernel function \( K \) has bounded support, the integration over \( T_{x,h} \) will eventually becomes that over \( T_x \), as \( h \to 0 \). We get the following result by using this idea and a Taylor expansion, and noticing again the assumption that \( K \) has a bounded
support.

\[
\mathbb{E}(\xi_i^2) = \frac{1}{h^5} \int_{\Omega} \int_{T_x} \int_{\mathbb{R}^d} \left\{ \sum_{j=1}^d w_j(x) K_{(j)}(u) \right\} \left\{ \sum_{j=1}^d w_j(\psi^{-1}(\psi(x) + hu)) K_{(j)}(v+u) \right\} \frac{f(\psi(x) - hu)}{\det J(x)} dud\mathcal{H}(v) dx \{1 + o(1)\}
\]

\[
= \frac{1}{h^5} \int_{\Omega} \int_{T_x} \int_{\mathbb{R}^d} \left\{ \sum_{j=1}^d w_j(x) K_{(j)}(u) \right\} \left\{ \sum_{j=1}^d w_j(x) K_{(j)}(v+u) \right\} \frac{f(\psi(x))}{\det J(x)} dud\mathcal{H}(v) dx \{1 + o(1)\}
\]

(4.21)

where

\[
\sigma_1^2 = \sum_{j=1}^d \sum_{k=1}^d \int_{\Omega} w_j(x) w_k(x) \frac{f(\psi(x))}{\det J(x)} dud\mathcal{H}(v) dx.
\]

Similarly, using change of variables and Taylor expansion we get

\[
\mathbb{E}(\xi_i) = \frac{1}{h^{d+2}} \int_{\Omega} \int_{\mathbb{R}^d} \sum_{j=1}^d w_j(\theta) K_{(j)} \left( \frac{\psi(\theta) - z}{h} \right) f(z) dz d\theta
\]

\[
= \frac{1}{h^{d+2}} \int_{\Omega} \int_{\mathbb{R}^d} \sum_{j=1}^d w_j(\theta) K_{(j)}(u) f(\psi(\theta) - hu) dud\theta
\]

\[
= \frac{1}{h^{d+1}} \int_{\Omega} \int_{\mathbb{R}^d} \sum_{j=1}^d w_j(\theta) f(\psi(\theta) - hu) K_{(j)}(u) dud\theta
\]

\[
= \frac{1}{h^{d+1}} \int_{\Omega} \sum_{j=1}^d w_j(\theta) f(\psi(\theta)) d\theta \{1 + o(1)\}.
\]

(4.23)

From (4.20) and (4.23) we have \( Var(\xi_i) = \frac{1}{h^5} \sigma_1^2 \{1 + o(1)\} \) and therefore by (4.19),

\[
Var\left( \int_{\Omega} w(\theta)^T b(\psi(\theta)) d\theta \right) = \frac{Var(\xi_i)}{n} = \frac{1}{nh^3} \sigma_1^2 \{1 + o(1)\}.
\]

It only remains to show asymptotic normality of \( \frac{1}{n} \sum_{i=1}^n (\xi_i - \mathbb{E}\xi_i) \). Note that following a similar procedure of obtaining (4.21), we can show that \( \mathbb{E}(|\xi_i|^3) = O\left(\frac{1}{h^{3d+1}}h^{3d-2}\right) = O\left(\frac{1}{h^5}\right) \). Then for the third absolute central moment of \( \xi_i \), we have

\[
\mathbb{E}[|\xi_i - \mathbb{E}(\xi_i)|^3] \leq 8\mathbb{E}(|\xi_i|^3) = O\left(\frac{1}{h^5}\right).
\]
Therefore
\[
\left\{ \sum_{i=1}^{n} E[|\xi_i - E(\xi_i)|^3] \right\}^{1/3} = O \left( \frac{(n/h^5)^{1/3}}{(n/h^3)^{1/2}} \right) = O \left( \frac{1}{n^{1/6}h^{1/6}} \right) = o(1).
\]

Hence the Liapunov condition is satisfied and the asymptotic normality in (2.8) is verified.

The proof of (2.9) follows from a similar procedure. So we only show the calculation of the variance in the asymptotic normal distribution. Note that
\[
\int_{\Omega} v(\theta) \left[ \hat{f}(\psi(\theta)) - E\hat{f}(\psi(\theta)) \right] d\theta = -\frac{1}{n} \sum_{i=1}^{n} \tilde{\xi}_i - E\tilde{\xi}_i,
\]
where
\[
\tilde{\xi}_i = \frac{1}{h^2} \int_{\Omega} v(\theta) K \left( \frac{\psi(\theta) - X_i}{h} \right) d\theta.
\]

Similar to the derivation for the variance of \(\xi_i\), we have
\[
E(\tilde{\xi}_i^2)
= \frac{1}{h^2} \int_{\Omega} \int_{\mathcal{M}} \int_{\mathbb{R}^d} v(x)K(u) v(y)K \left( \frac{\psi(y) - \psi(x)}{h} + u \right) f(\psi(x) - hu) \{J(\psi^{-1}(s))\}^{1/2} dud\mathcal{H}(s) dx
= \frac{1}{h} \int_{\Omega} \int_{T_{x,h}} \int_{\mathbb{R}^d} v(x)K(u) v(\psi^{-1}(s))K \left( \frac{s - \psi(x)}{h} + u \right) \frac{f(\psi(x) - hu)}{\{J(\psi^{-1}(s))\}^{1/2}} dud\mathcal{H}(v) dx
= \frac{1}{h} \sigma_2^2 \{1 + o(1)\},
\]
where
\[
\sigma_2^2 = \int_{\Omega} \int_{T_x} \int_{\mathbb{R}^d} v(x)K(u) v(x)K(v + u) \frac{f(\psi(x))}{\{J(\psi(x))\}^{1/2}} dud\mathcal{H}(v) dx. \tag{4.24}
\]

**Proof of Theorem 2.1**
Proof. We first assume $N = 1$ in assumption (F2). For simplicity, we will drop the subscript or superscript $j$ for $\mathcal{M}, \Omega_j, \Omega^o_j, \psi_j, B_j$ and $J_j$. At the end of this proof, we will show how our derivation can be generalized to $N \geq 2$. Note that in assumption (F2), for points on the boundary of $\Omega$, the Jacobian matrix $B$ can be defined as the limit taken from the points in $\Omega^o$. This immediately implies that (1.4) can be improved by replacing $\Omega^o$ with $\Omega$.

By Lemma 2.2, $P_n(x) = x + \frac{\nabla f(x)}{\|\nabla f(x)\|}t_n(x)$ in (2.3) is a diffeomorphism between $\mathcal{M}_r$ and $\hat{\mathcal{M}}_r$ for $|\tau - c| \leq \delta_0/2$. Let $A$ be the Jacobian matrix of $P_n$. Then we can write $A(x) = I + \tilde{A}(x)$, $x \in I(\delta_0)$, where

$$\tilde{A}(x) = \nabla \left( \frac{\nabla f(x)}{\|\nabla f(x)\|} \right) t_n(x) + \frac{\nabla f(x)}{\|\nabla f(x)\|} \nabla t_n(x)^T. \quad (4.25)$$

In what follows let $\|\cdot\|_{\text{max}}$ be the max norm of a matrix, that is, for any matrix $E = [e_{ij}]_{1 \leq i,j \leq d}$, $\|E\|_{\text{max}} = \max_{1 \leq i,j \leq d} |e_{ij}|$. Then by Lemma 2.2 and using (4.3) and (4.4), we have

$$\sup_{x \in I(\delta_0)} \|\tilde{A}(x)\|_{\text{max}} = O_p \left( \sqrt{\log n/nh + h^\nu} \right). \quad (4.26)$$

Recall that $\psi$ is a parameterization function from $\Omega$ to $\mathcal{M}$, $B$ is the Jacobian matrix of $\psi$ and $J = B^T B$ is the Gram matrix. With assumption (F2), by using the area formula on manifolds, we have that

$$\lambda(g) = \int_{\Omega} g(\psi(\theta)) [\det(J(\theta))]^{1/2} d\theta. \quad (4.27)$$

By the chain rule, the Jacobian matrix of $P_n \circ \psi : \Omega \mapsto \hat{\mathcal{M}}$ is given by $A(\psi(\theta))B(\theta)$. Since $P_n \circ \psi$ is a parameterization function of $\hat{\mathcal{M}}$ from $\Omega$, another application of the area formula leads to

$$\hat{\lambda}(g) = \int_{\Omega} g(P_n(\psi(\theta))) \left\{ \det [B(\theta)^T A(\psi(\theta))^T A(\psi(\theta))B(\theta)] \right\}^{1/2} d\theta. \quad (4.28)$$

Then it follows from (4.27) and (4.28) that

$$\hat{\lambda}(g) - \lambda(g) = \int_{\Omega} \left[ g(P_n(\psi(\theta))) \left\{ \det [B(\theta)^T A(\psi(\theta))^T A(\psi(\theta))B(\theta)] \right\}^{1/2} - g(\psi(\theta)) \left\{ \det[J(\theta)] \right\}^{1/2} \right] d\theta = \mathcal{I}_n + \mathcal{II}_n + \mathcal{III}_n, \quad (4.29)$$
where

\[
I_n = \int_\Omega [g(P_n(\psi(\theta))) - g(\psi(\theta))] \{\text{det}[J(\theta)]\}^{1/2} \, d\theta, 
\]

(4.30)

\[
\text{II}_n = \int_\Omega g(\psi(\theta)) \left\{\{\text{det}(B^T(\theta)A(\psi(\theta))^T A(\psi(\theta))B(\theta))\}^{1/2} - \{\text{det}[J(\theta)]\}^{1/2}\right\} \, d\theta, 
\]

(4.31)

\[
\text{III}_n = \int_\Omega [g(P_n(\psi(\theta))) - g(\psi(\theta))] \times \left\{\{\text{det}(B^T(\theta)A(\psi(\theta))^T A(\psi(\theta))B(\theta))\}^{1/2} - \{\text{det}[J(\theta)]\}^{1/2}\right\} \, d\theta. 
\]

(4.32)

We first study \(I_n\). Since \(P_n(\psi(\theta)) - \psi(\theta) = \frac{\nabla f(\psi(\theta))}{\|\nabla f(\psi(\theta))\|} t_n(\psi(\theta))\) by definition, using Taylor expansion and Lemma 2.2 we have that,

\[
g(P_n(\psi(\theta))) - g(\psi(\theta)) = \Delta_1(\theta) + \Delta_2(\theta) + \Delta_3(\theta) + \Delta_4(\theta), 
\]

(4.33)

where

\[
\Delta_1(\theta) = \frac{f(\psi(\theta)) - \mathbb{E}\hat{f}(\psi(\theta))}{\|\nabla f(\psi(\theta))\|^2} \nabla f(\psi(\theta))^T \nabla g(\psi(\theta)), 
\]

\[
\Delta_2(\theta) = \mathbb{E}\hat{f}(\psi(\theta)) - \hat{f}(\psi(\theta)) \frac{\nabla f(\psi(\theta))^T \nabla g(\psi(\theta))}{\|\nabla f(\psi(\theta))\|^2}, 
\]

\[
\Delta_3(\theta) = \delta_{1n}(\psi(\theta)) \nabla f(\psi(\theta))^T \nabla g(\psi(\theta)), 
\]

\[
\Delta_4(\theta) = \frac{1}{2} \frac{t_n^2(\psi(\theta))}{\|\nabla f(\psi(\theta))\|^2} \nabla f(\psi(\theta))^T \nabla^2 g \left(\psi(\theta) + s \frac{\nabla f(\psi(\theta))}{\|\nabla f(\psi(\theta))\|} t_n(\psi(\theta))\right) \nabla f(\psi(\theta)), 
\]

for some \(0 < s < 1\) and \(\delta_{1n}\) is given in (2.4). Note that

\[
I_n = \int_\Omega [\Delta_1(\theta) + \Delta_2(\theta) + \Delta_3(\theta) + \Delta_4(\theta)] \{\text{det}[J(\theta)]\}^{1/2} \, d\theta, 
\]

(4.34)

and

\[
\int_\Omega \Delta_1(\theta) \{\text{det}[J(\theta)]\}^{1/2} \, d\theta = h^\nu \mu_1(1 + o(1)) = O(h^\nu), 
\]

(4.35)

where

\[
\mu_1 = \int_\Omega -\kappa_{\nu} \sum_{i=1}^d f_i(\psi(\theta)) \nabla f(\psi(\theta))^T \nabla g(\psi(\theta)) \{\text{det}[J(\theta)]\}^{1/2} \, d\theta. 
\]

(4.36)

Also it follows from Proposition 2.1 that

\[
\int_\Omega \Delta_2(\theta) \{\text{det}[J(\theta)]\}^{1/2} \, d\theta = O_p \left(\frac{1}{\sqrt{nh}}\right). 
\]

(4.37)
It follows from Lemma 2.2 that
\[
\sup_{\theta \in \Omega} |[\Delta_3(\theta) + \Delta_4(\theta)] \{\det[J(\theta)]\}^{1/2}| = O_p \left\{ \left( \frac{\log n}{nh^d} + h^{2/3} \right) \left( \frac{\log n}{nh^{d+2}} + h^{2/3} \right) \right\}, \tag{4.38}
\]
and hence
\[
\int_{\Omega} [\Delta_3(\theta) + \Delta_4(\theta)] \{\det[J(\theta)]\}^{1/2} d\theta = O_p \left\{ \left( \frac{\log n}{nh^d} + h^{2/3} \right) \left( \frac{\log n}{nh^{d+2}} + h^{2/3} \right) \right\}. \tag{4.39}
\]
From (4.35), (4.37) and (4.39), we have
\[
I_n = O_p \left\{ \frac{\log n}{nh^{d+1}} + h^{2/3} + \frac{1}{\sqrt{nh}} \right\}. \tag{4.40}
\]
Also noticing that \( \sup_{\theta \in \Omega} |\Delta_1(\theta)| = O(h^n) \) and \( \sup_{\theta \in \Omega} |\Delta_2(\theta)| = O_p \left( \frac{\log n}{nh^d} \right) \) by following standard arguments, by using (4.38) we have
\[
\sup_{\theta \in \Omega} |g(P_n(\psi(\theta))) - g(\psi(\theta))| = O_p \left( \frac{\log n}{nh^d} + h^{2/3} \right). \tag{4.41}
\]
Next we focus on \( \Pi_n \). Notice that by using \( A(x) = I_d + \tilde{A}(x) \), we have
\[
\det [B(\theta)^T A(\psi(\theta))^T A(\psi(\theta)) B(\theta)]
= \det \left\{ B(\theta)^T [I_d + \tilde{A}(\psi(\theta))]^T [I_d + \tilde{A}(\psi(\theta))] B(\theta) \right\}
= \det \left\{ J(\theta) + B(\theta)^T \tilde{A}(\psi(\theta))^T + \tilde{A}(\psi(\theta)) + \tilde{A}(\psi(\theta))^T \tilde{A}(\psi(\theta)) B(\theta) \right\}
= \det [J(\theta)] \det \{ I_d - Q(\theta) \}, \tag{4.42}
\]
where \( Q(\theta) = -[J(\theta)]^{-1} B(\theta)^T \tilde{A}(\psi(\theta))^T + \tilde{A}(\psi(\theta)) + \tilde{A}(\psi(\theta))^T \tilde{A}(\psi(\theta)) \} B(\theta) \). Denote the characteristic polynomial of \( Q(\theta) \) by \( p_Q(t) = \det(tI_d - Q(\theta)) \). Then notice that
\[
\det(I_d - Q(\theta)) = p_Q(1) = 1 - \text{tr}(Q(\theta)) + \eta_1(\theta), \quad \text{with } \eta_1(\theta) = \sum_{j=2}^{d-1} (-1)^{j-1} \text{tr}(\Lambda^j Q(\theta)), \tag{4.43}
\]
where \( \text{tr}(\Lambda^j Q(\theta)) \) is the trace of the jth external power of \( Q(\theta) \), which can be computed as the sum of all the \( j \times j \) principal minors of \( Q(\theta) \) (cf. Mirsky (1955), Theorem 7.1.2). Then using (1.26), we have
\[
\sup_{\theta \in \Omega} |\eta_1(\theta)| = O_p \left\{ \left( \frac{\log n}{nh^{d+2}} + h^{2/3} \right)^2 \right\}. \tag{4.44}
\]
Let \( D(\theta) = B(\theta)[J(\theta)]^{-1} B(\theta)^T \) and \( \eta_2(\theta) = \text{tr}[D(\theta) \tilde{A}(\psi(\theta))^T \tilde{A}(\psi(\theta))]. \) Continuing from (4.42), we have
\[
\det [B(\theta)^T A(\psi(\theta))^T A(\psi(\theta)) B(\theta)]
= \det [J(\theta)] \left\{ 1 + \text{tr} (-Q(\theta)) \right\} + \det [J(\theta)] \eta_1(\theta)
= \det [J(\theta)] \left\{ 1 + \text{tr} \left\{ D(\theta)[\tilde{A}(\psi(\theta))^T + \tilde{A}(\psi(\theta))] \right\} \right\} + \eta_1(\theta) + \eta_2(\theta)
= \det [J(\theta)] \left\{ 1 + 2\text{tr} \left\{ D(\theta) \tilde{A}(\psi(\theta))^T \right\} + \eta_1(\theta) + \eta_2(\theta) \right\}. \tag{4.44}
\]
In the last equality above we have used the fact that $D(\theta)$ is symmetric so that $\text{tr}\{D(\theta)\tilde{A}(\theta)\} = \text{tr}\{\tilde{A}(\theta)D(\theta)\} = \text{tr}\{D(\theta)\tilde{A}(\psi(\theta))^T\}$. Again by (4.26), we have

$$
sup_{\theta \in \Omega} |\eta_2(\theta)| = O_p\left(\sqrt{\frac{\log n}{n_{\delta}d+2}} + h^v\right)^2.
$$

Therefore using Taylor expansion and (4.44), we have

$$
\{ \det [B(\theta)^T \tilde{A}(\psi(\theta))^T \sigma(\psi(\theta))B(\theta)] \}^{1/2} - \{ \det [J(\theta)] \}^{1/2}
\leq \{ \det [J(\theta)] \}^{1/2} \left[ 1 + 2\text{tr} \left[ D(\theta) \tilde{A}(\psi(\theta))^T \right] + \eta_1(\theta) + \eta_2(\theta) \right]^{1/2} - 1
\leq \{ \det [J(\theta)] \}^{1/2} \text{tr} \left[ D(\theta) \tilde{A}(\psi(\theta))^T \right] + \Delta_5(\theta),
$$

where $\Delta_5(\theta)$ denotes the remainder term in the Taylor expansion. Specifically,

$$
\Delta_5(\theta) = \frac{1}{2} \{ \det [J(\theta)] \}^{1/2} (\Delta_1(\theta) + \Delta_2(\theta))
+ \sum_{i=2}^{\infty} (-1)^i \frac{(2i-3)!!}{2^i i!} \{ \det [J(\theta)] \}^{1/2} \left[ 2\text{tr} \left[ D(\theta) \tilde{A}(\psi(\theta))^T \right] + \Delta_1(\theta) + \Delta_2(\theta) \right]^i.
$$

It follows from (4.43) and (4.45) that

$$
sup_{\theta \in \Omega} |\Delta_5(\theta)| = O_p\left(\sqrt{\frac{\log n}{n_{\delta}d+2}} + h^v\right)^2.
$$

In order to further analyze (4.46), we need to decompose $\tilde{A}$ into bias and stochastic terms. By Lemma 2.2 and (4.25), for $x \in \mathcal{M}$ we can write

$$
\tilde{A}(x) = a(x)[\mathbb{E}\nabla \tilde{f}(x) - \nabla \tilde{f}(x)]^T + R(x),
$$

where $a(x) = \frac{\nabla f(x)}{||\nabla f(x)||^2}$, and with $\delta_{1n}$ and $\delta_{2n}$ given in (2.4) and (2.5), respectively,

$$
R(x) = \nabla \left( \frac{\nabla f(x)}{||\nabla f(x)||} \right) t_n(x) + \nabla \left( \frac{\nabla f(x)}{||\nabla f(x)||} \right)^T \left[ \frac{\nabla f(x) - \mathbb{E}\nabla \tilde{f}(x)}{||\nabla f(x)||} + \frac{-\nabla ||\nabla f(x)||}{||\nabla f(x)||} t_n(x) + \delta_{2n}(x) \right]^T
\leq \left\{ \nabla \left( \frac{\nabla f(x)}{||\nabla f(x)||} \right)^T \frac{\nabla f(x)(\nabla ||\nabla f(x)||)^T}{||\nabla f(x)||^2} \right\} t_n(x) + \nabla \left( \frac{\nabla f(x)}{||\nabla f(x)||} \right)^T \left[ \frac{\nabla f(x) - \mathbb{E}\nabla \tilde{f}(x)}{||\nabla f(x)||} + \delta_{2n}(x) \right]^T
\leq \left\{ \nabla \left( \frac{\nabla f(x)}{||\nabla f(x)||} \right)^T \frac{\nabla f(x)(\nabla ||\nabla f(x)||)^T}{||\nabla f(x)||^2} \right\} t_n(x) + \nabla \left( \frac{\nabla f(x)}{||\nabla f(x)||} \right)^T \left[ \frac{f(x) - \tilde{f}(x)}{||\nabla f(x)||} + \delta_{1n}(x) \right]^T
+ \nabla \left( \frac{\nabla f(x)}{||\nabla f(x)||} \right)^T \left[ \frac{\nabla f(x) - \mathbb{E}\nabla \tilde{f}(x)}{||\nabla f(x)||} + \delta_{2n}(x) \right]^T.
$$

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Let
\[ S(x) = \frac{1}{\|\nabla f(x)\|} \left\{ \nabla \left( \frac{\nabla f(x)}{\|\nabla f(x)\|} \right) - \frac{\nabla f(x)(\nabla \|\nabla f(x)\|)^T}{\|\nabla f(x)\|^2} \right\}. \]
and
\[ \tilde{R}(x) = \left[f(x) - E \hat{f}(x)\right] S(x) + \frac{\nabla f(x)}{\|\nabla f(x)\|} \left[ \nabla f(x) - E \nabla \hat{f}(x) \right]^T. \]

Then
\[ R(x) = \tilde{R}(x) + \left[E \hat{f}(x) - \hat{f}(x)\right] S(x) + \delta_{1n}(x)\|\nabla f(x)\|S(x) + \frac{\nabla f(x)}{\|\nabla f(x)\|} \delta_{2n}(x)^T. \]
(4.49)

Note that
\[ \tilde{R}(x) = U(x)h''(1 + o(1)), \]
(4.50)

where
\[ U(x) = \left\{ -\frac{1}{\nu t^n} \left[ \sum_{i=1}^{d} f_i(x)^{(i)} \right] S(x) - \frac{1}{\nu t^n} \frac{\nabla f(x)}{\|\nabla f(x)\|^2} \left[ \sum_{i=1}^{d} f_i(x)^{(i)} \right]^T \right\}. \]

Plugging (4.48) and (4.49) into (4.46), we have
\[
\{ \det \left[ B(\theta)^T A(\psi(\theta)) T A(\psi(\theta)) B(\theta) \right] \}^{1/2} - \{ \det \left[ J(\theta) \right] \}^{1/2} \\
= \{ \det \left[ J(\theta) \right] \}^{1/2} \text{tr} \left\{ D(\theta) \left[ E \nabla \hat{f}(\psi(\theta)) - \nabla \hat{f}(\psi(\theta)) \right] a(\psi(\theta))^T + R(\psi(\theta))^T \right\} + \Delta_5(\theta) \\
= \{ \det \left[ J(\theta) \right] \}^{1/2} \left\{ a(\psi(\theta))^T D(\theta) \left[ E \nabla \hat{f}(\psi(\theta)) - \nabla \hat{f}(\psi(\theta)) \right] + \text{tr} \left[ D(\theta) R(\psi(\theta))^T \right] \right\} + \Delta_5(\theta) \\
= \Delta_5(\theta) + \Delta_6(\theta) + \Delta_7(\theta) + \Delta_8(\theta) + \Delta_9(\theta),
\]
(4.51)

where
\[ \Delta_6(\theta) = \{ \det \left[ J(\theta) \right] \}^{1/2} a(\psi(\theta))^T D(\theta) \left[ E \nabla \hat{f}(\psi(\theta)) - \nabla \hat{f}(\psi(\theta)) \right], \]
\[ \Delta_7(\theta) = \{ \det \left[ J(\theta) \right] \}^{1/2} \text{tr} \left[ D(\theta) \tilde{R}(\psi(\theta))^T \right] , \]
\[ \Delta_8(\theta) = \{ \det \left[ J(\theta) \right] \}^{1/2} \text{tr} \left[ D(\theta) S(\psi(\theta))^T \right] \left[ E \hat{f}(\psi(\theta)) - \hat{f}(\psi(\theta)) \right], \]
\[ \Delta_9(\theta) = \{ \det \left[ J(\theta) \right] \}^{1/2} \text{tr} \left[ D(\theta) S(\psi(\theta))^T \right] \delta_{1n}(x)\|\nabla f(x)\| + \{ \det \left[ J(\theta) \right] \}^{1/2} \text{tr} \left[ D(\theta) \delta_{2n}(x)\|\nabla f(x)\|^2 \right]. \]

Note that from (4.31) and (4.51) we have
\[ \Pi_n = \int_{\Omega} g(\psi(\theta)) \left[ \Delta_5(\theta) + \Delta_6(\theta) + \Delta_7(\theta) + \Delta_8(\theta) + \Delta_9(\theta) \right] d\theta, \]
(4.52)
and by (4.47) we have
\[
\left| \int_{\Omega} g(\psi(\theta)) \Delta_5(\theta) d\theta \right| = O_p \left\{ \left( \sqrt{\frac{\log n}{nh^{d+2}}} + h^\nu \right)^2 \right\}. \tag{4.53}
\]

Let \( w(\theta)^T = (w_1(\theta), \ldots, w_d(\theta)) = g(\psi(\theta)) \{ \det[J(\theta)] \}^{1/2} a(\psi(\theta))^T D(\theta) \). Then by Proposition 2.1 we have that
\[
\sqrt{nh^3} \int_{\Omega} g(\psi(\theta)) \Delta_6(\theta) d\theta = -\sqrt{nh^3} \int_{\Omega} w(\theta)^T \left[ \nabla \hat{f}(\psi(\theta)) - \mathbb{E} \nabla \hat{f}(\psi(\theta)) \right] d\theta \rightarrow_D N(0, \sigma^2), \tag{4.54}
\]
where
\[
\sigma^2 = \sum_{j=1}^{d} \sum_{k=1}^{d} \int_{\Omega} w_j(x) w_k(x) \frac{f(\psi(x))}{\{ \det [J(x)] \}^{1/2}} \int_{T_x} \int_{R^d} \{ K_{(j)}(u) K_{(k)}(v + u) \} dudH(v)dx. \tag{4.55}
\]

Note that by (4.50),
\[
\int_{\Omega} g(\psi(\theta)) \Delta_7(\theta) d\theta = h^\nu \mu_{II} (1 + o(1)), \tag{4.56}
\]
where
\[
\mu_{II} = \int_{\Omega} g(\psi(\theta)) \{ \det[J(\theta)] \}^{1/2} \text{tr} \left[ D(\theta) U(\psi(\theta))^T \right] d\theta. \tag{4.57}
\]
Applying Proposition 2.1 again, we get
\[
\int_{\Omega} g(\psi(\theta)) \Delta_8(\theta) d\theta = O_p \left( \frac{1}{\sqrt{nh^3}} \right). \tag{4.58}
\]

It follows from Lemma 2.2 that
\[
\sup_{\theta \in \Omega} |\Delta_9(\theta)| = O_p \left\{ \left( \sqrt{\frac{\log n}{nh^{d+4}}} + h^\nu \right) \left( \sqrt{\frac{\log n}{nh^{d+4}}} + h^\nu \right) \right\}, \tag{4.59}
\]
and hence
\[
\int_{\Omega} g(\psi(\theta)) \Delta_9(\theta) d\theta = O_p \left\{ \left( \sqrt{\frac{\log n}{nh^{d+4}}} + h^\nu \right) \left( \sqrt{\frac{\log n}{nh^{d+4}}} + h^\nu \right) \right\}. \tag{4.60}
\]
As a result of (4.52), (4.53), (4.54), (4.56), (4.58) and (4.60), we get
\[
\Pi_n = O_p \left\{ \log n \frac{1}{nh^{d+2}} + h^\nu + \frac{1}{\sqrt{nh^3}} \right\}. \tag{4.61}
\]
Also by (4.3) and (4.4) we have
\[ \sup_{\theta \in \Omega} |\Delta_6(\theta)| = O_p\left(\sqrt{\frac{\log n}{n^d}}\right), \sup_{\theta \in \Omega} |\Delta_7(\theta)| = O(h^\nu) \]
and
\[ \sup_{\theta \in \Omega} |\Delta_8(\theta)| = O_p\left(\sqrt{\frac{\log n}{n^d + 2}}\right), \sup_{\theta \in \Omega} |\Delta_9(\theta)| = O\left(h^\nu\right) \]
and by using (4.47) and (4.59) we have
\[ \sup_{\theta \in \Omega} \left| \left\{ \det \left[ B(\theta)^T A(\psi(\theta))^T A(\psi(\theta)) B(\theta) \right] \right\}^{1/2} - \{ \det [J(\theta)] \}^{1/2} \right| = O_p\left(\sqrt{\frac{\log n}{n^d}}\right). \] (4.62)

Next we study III_n. By (4.41) and (4.62) we have
\[ |\text{III}_n| \leq \mathcal{K}_{d-1}^1(\Omega) \sup_{\theta \in \Omega} \left| g(P_n(\psi(\theta))) - g(\psi(\theta)) \right| \times \sup_{\theta \in \Omega} \left| \{ \det[B(\theta)^T A(\psi(\theta))^T A(\psi(\theta)) B(\theta)] \}^{1/2} - \{ \det [J(\theta)] \}^{1/2} \right| = O_p\left(\sqrt{\frac{\log n}{n^d + 2}} + h^\nu\right). \] (4.63)

Now we are ready to collect the above results for I_n, II_n and III_n to show the assertions (i) and (ii). Note that (2.1) follows from (4.29), (4.40), (4.61) and (4.63). When \( \log(n)^{-1/2} n h^{2d+1} \to \infty \) and \( n h^{3+2\nu} \to \gamma \geq 0 \), we have
\[ \left(\sqrt{\frac{\log n}{n^d + 2}} + h^\nu\right)^2 = o \left( \frac{1}{nh^3} \right). \]
Therefore by (4.34), (4.35), (4.37) and (4.39) for I_n, (4.52), (4.53), (4.54), (4.56), (4.58) and (4.60) for II_n, and (4.63) for III_n, we have (2.2) with
\[ \mu = \mu_I + \mu_{II}, \] (4.64)
where \( \mu_I \) and \( \mu_{II} \) are given in (4.36) and (4.57), respectively.

As indicated at the beginning of this proof, so far we take \( N = 1 \) in assumption (F2). Now we show how the above proof can be extended to the case of \( \mathcal{M} = \bigcup_{j=1}^N \mathcal{M}_j \) for \( N \geq 2 \). By using (4.33) and (4.4), for \( n \) large enough, we have \( \widehat{\mathcal{N}} = N \) with probability one, where \( \widehat{\mathcal{N}} \) is the number of connected components of \( \mathcal{M} \). Also see Theorem 3.1 in Biau et al. (2007). For \( n \) large enough let \( \widehat{\mathcal{M}}^j = P_n(\mathcal{M}_j), j = 1, \ldots, N \). It follows from Lemma 2.1 that \( \widehat{\mathcal{M}} = \bigcup_{j=1}^N \widehat{\mathcal{M}}^j \), with \( \inf_{1 \leq j \neq k \leq N} d_H(\widehat{\mathcal{M}}^j, \widehat{\mathcal{M}}^k) > 0 \) by assumption (F2). Then \( \widehat{\lambda}(g) - \lambda(g) = \sum_{j=1}^N [\widehat{\lambda}_j(g) - \lambda_j(g)] \), where
\[ \widehat{\lambda}_j(g) = \int_{\widehat{\mathcal{M}}^j} g(x) d\mathcal{H}(x) \text{ and } \lambda_j(g) = \int_{\mathcal{M}_j} g(x) d\mathcal{H}(x). \]
The argument in this proof can go through for \( \hat{\lambda}_j(g) - \lambda_j(g), \ j = 1, \cdots, N \). Immediately we obtain (2.1) for \( \hat{\lambda}(g) - \lambda(g) \). Also note that under the assumption \( \log(n)^{-1/2}nh^{d+1} \to \infty \), (4.54) is the dominant stochastic term which determines the rate in (2.2). Since the kernel \( K \) has a bounded support, when \( h \) is small enough as \( n \to \infty \), the integrals on \( \Omega_j \) for different \( j \)'s as in (4.54) are uncorrelated. This observation then leads to (2.2) with

\[
\mu = \sum_{j=1}^{N} \mu_j, \tag{4.65}
\]

\[
\sigma^2 = \sum_{j=1}^{N} \sigma_j^2, \tag{4.66}
\]

where \( \mu_j \) and \( \sigma_j^2 \) are constructed in the same fashion as (4.64) and (4.55), respectively. \( \square \)

**Proof of Proposition 3.1**

**PROOF.** Following the proof of Theorem 2.1 we can take \( N = 1 \) in assumption (F2) without loss of generality. We continue to use \( I_n, II_n \) and \( III_n \) as in (4.30), (4.31) and (4.32), where we replace \( g \) by \( g_n \).

From (4.33) by using Taylor expansion we have

\[
\sup_{\theta \in \Omega} |g_n(P_n(\psi(\theta))) - g_n(\psi(\theta))| \leq \sup_{\theta \in \Omega} |t_n(\psi(\theta))| \sup_{x \in I(\delta_0)} \| \nabla g_n(x) \| = O_p \left\{ \beta_n \left( \sqrt{\frac{\log n}{nh^d}} + h^\nu \right) \right\}. \tag{4.67}
\]

For \( I_n \), we obtain from (4.67) that

\[
I_n \leq \int_{\Omega} \left| g_n(P_n(\psi(\theta))) - g_n(\psi(\theta)) \right| \left\{ \det[J(\theta)] \right\}^{1/2} d\theta \leq \sup_{\theta \in \Omega} \left| g_n(P_n(\psi(\theta))) - g_n(\psi(\theta)) \right| \int_{\Omega} \left\{ \det[J(\theta)] \right\}^{1/2} d\theta \leq O_p \left\{ \beta_n \left( \sqrt{\frac{\log n}{nh^d}} + h^\nu \right) \right\}. \tag{4.68}
\]
For $\Pi_n$, (4.62) leads to
\[
\Pi_n \leq \int_{\Omega} |g_n(P_n(\psi(\theta)))| \left| \det(B^T(\theta)A(\psi(\theta))^T A(\psi(\theta))B(\theta)) \right|^{1/2} - \{\det[J(\theta)]\}^{1/2} \, d\theta \\
\leq \sup_{x \in I(b_0)} |g_n(x)| \sup_{\theta \in \Omega} \left| \det(B^T(\theta)A(\psi(\theta))^T A(\psi(\theta))B(\theta)) \right|^{1/2} - \{\det[J(\theta)]\}^{1/2} \mathcal{H}_{d-1}(\Omega) \\
= O_p \left\{ \alpha_n \left( \sqrt{\frac{\log n}{nh^{d+2}}} + h^\nu \right) \right\}. 
\]
(4.69)

For $\Pi_{\infty}$, by using (4.67) and (4.62) we have
\[
\Pi_{\infty} = O_p \left\{ \beta_n \left( \sqrt{\frac{\log n}{nh^{d+2}}} + h^\nu \right) \left( \sqrt{\frac{\log n}{nh^{d+2}}} + h^\nu \right) \right\}. 
\]
(4.70)

By having (4.68), (4.69) and (4.70) we conclude the proof. □

**Proof of Proposition 3.2**
The proofs of (3.12) and (3.13) are very similar, with the only difference being the number of times using integration by parts in the derivation, due to the different order of derivatives in the integrands. Hence we only present the proof for (3.13).

**Proof.** Let
\[
H_{kl}(X_i, X_j; h) = \int_{\mathcal{M}} K_{(k,k)} \left( \frac{x - X_i}{h} \right) K_{(l,l)} \left( \frac{x - X_j}{h} \right) g(x) d\mathcal{H}(x). 
\]
(4.71)

Then we can write
\[
\int_{\mathcal{M}} \widehat{f}_{(k,k)}(x) \widehat{f}_{(l,l)}(x) g(x) d\mathcal{H}(x) = \frac{1}{n^2 h^{2d+4}} (I_n + \Pi_n), 
\]
(4.72)

where
\[
I_n = \sum_{i=1}^{n} H_{kl}(X_i, X_i; h) \text{ and } \Pi_n = \sum_{i \neq j} H_{kl}(X_i, X_j; h). 
\]
(4.73)

We first consider the bias of $D_{g,n}^{(2)}$. Using the change of variables $s = (x - u)/h$ and Taylor expansion, we get
\[
E(I_n) = nE[H_{kl}(X_1, X_1; h)] = n \int_{\mathcal{M}} g(x) \int K_{(k,k)} \left( \frac{x - u}{h} \right) K_{(l,l)} \left( \frac{x - u}{h} \right) f(u) dud\mathcal{H}(x) \\
= nh^d \int_{\mathcal{M}} g(x) \int K_{(k,k)}(s) K_{(l,l)}(s) f(x - hs) ds d\mathcal{H}(x) \\
= cnh^d \int_{\mathcal{M}} g(x) \int K_{(k,k)}(s) K_{(l,l)}(s) ds d\mathcal{H}(x) + o(nh^d), 
\]
(4.74)
and

\[ E(\Pi_n) = n(n - 1) \int_M g(x) \int K_{(k,k)} \left( \frac{x - u}{h} \right) K_{(l,l)} \left( \frac{x - v}{h} \right) f(u) f(v) du dv d\mathcal{H}(x) \]

\[ = n(n - 1) h^{2d} \int_M g(x) \int K_{(k,k)}(s) K_{(l,l)}(t) f(x - hs) f(x - ht) ds dt d\mathcal{H}(x) \]

\[ = n(n - 1) h^{2d+4} \int_M g(x) \int K(s) K(t) f_{(k,k)}(x - hs) f_{(l,l)}(x - ht) ds dt d\mathcal{H}(x) \]

\[ = n(n - 1) h^{2d+4} \int_M g(x) \left[ f_{(k,k)}(x) f_{(l,l)}(x) + \frac{1}{\nu!} h^\nu \kappa_\nu \int_M f_{(l,l)}(x) \sum_{i=1}^d f_{(k,k, i\nu)}(x) \right] d\mathcal{H}(x) + o(n^2 h^{2d+4+\nu}). \quad (4.75) \]

It follows from (4.72), (4.74) and (4.75) that

\[ \mathbb{E} D_{g,n}^{(2)} = \frac{1}{n^2 h^{2d+4}} (I_n + \Pi_n) - \int_M g(x) f_{(k,k)}(x) f_{(l,l)}(x) d\mathcal{H}(x) \]

\[ = \frac{1}{n h^{d+4}} \int_M f(x) g(x) d\mathcal{H}(x) \int K_{(k,k)}(s) K_{(l,l)}(s) ds \]

\[ + \frac{1}{\nu!} h^\nu \kappa_\nu \int_M f_{(l,l)}(x) g(x) \sum_{i=1}^d f_{(k,k, i\nu)}(x) d\mathcal{H}(x) \]

\[ + \frac{1}{\nu!} h^\nu \kappa_\nu \int_M f_{(k,k)}(x) g(x) \sum_{i=1}^d f_{(l,l, i\nu)}(x) d\mathcal{H}(x) + o \left( \frac{1}{nh^{d+4}} \right) + o(h^\nu) \]

\[ = O \left( \frac{1}{nh^{d+4}} + h^\nu \right). \quad (4.76) \]

Next we consider the variance of \( D_{g,n}^{(2)} \). By (4.72) we can write

\[ \text{Var}(D_{g,n}^{(2)}) = \frac{1}{n^2 h^{2d+8}} \left[ \text{Var}(I_n) + \text{Var}(\Pi_n) + 2 \text{Cov}(I_n, \Pi_n) \right]. \quad (4.77) \]

Here \( \text{Var}(I_n) = n \text{Var} [H_{kl}(X_1, X_1; h)] \) and

\[ \text{Var}(\Pi_n) = \sum_{i \neq j} \sum_{j' \neq j} \sum_{i' \neq j'} \text{Cov} [H_{kl}(X_i, X_j; h), H_{kl}(X_{i'}, X_{j'}; h)] \]

\[ = (n^2 + o(n^2)) \text{Var} [H_{kl}(X_1, X_2; h)] + (n^2 + o(n^2)) \text{Cov} [H_{kl}(X_1, X_2; h), H_{kl}(X_2, X_1; h)] \]

\[ + (2n^3 + o(n^3)) \text{Cov} [H_{kl}(X_1, X_2; h), H_{kl}(X_2, X_3; h)] \]

\[ + (n^3 + o(n^3)) \text{Cov} [H_{kl}(X_1, X_2; h), H_{kl}(X_3, X_2; h)] \]

\[ + (n^3 + o(n^3)) \text{Cov} [H_{kl}(X_1, X_2; h), H_{kl}(X_1, X_3; h)]. \quad (4.78) \]
We first focus on $\text{Var}(I_n)$ and by the change of variables $s = (x - u)/h$ and $w = (y - x)/h$ and using Taylor expansion have

$$
E \left\{ [H_{kl}(X_1, X_1; h)]^2 \right\} = \int_{\mathcal{M}} \int_{\mathcal{M}} \int K_{(k,k)} \left( \frac{x - u}{h} \right) K_{(k,k)} \left( \frac{y - u}{h} \right) K_{(l,l)} \left( \frac{x - u}{h} \right) K_{(l,l)} \left( \frac{y - u}{h} \right) \times f(u)g(x)g(y)dud\mathcal{H}(y)d\mathcal{H}(x)
$$

$$
= h^d \int_{\mathcal{M}} \int_{\mathcal{M}} \int K_{(k,k)}(s)K_{(l,l)}(s)K_{(k,k)} \left( \frac{y - x}{h} + s \right) K_{(l,l)} \left( \frac{y - x}{h} + s \right) \times f(x - hs)g(x)g(y)dsd\mathcal{H}(y)d\mathcal{H}(x). \quad (4.79)
$$

Let $T_{x,h} = \{(y - x)/h : y \in \mathcal{M}\}$ and $T_x$ be the tangent space of $\mathcal{M}$ at $x$. Applying the change of variable $w = (y - x)/h$ in (4.79), then with the same argument as for (4.21) we have

$$
E \left\{ [H_{kl}(X_1, X_1; h)]^2 \right\} = h^{2d-1} \int_{\mathcal{M}} \int_{T_{x,h}\mathcal{M}} \int K_{(k,k)}(s)K_{(l,l)}(s)K_{(k,k)}(w + s)K_{(l,l)}(w + s) \times f(x)g(x)g(x + hw)dsd\mathcal{H}(w)d\mathcal{H}(x)
$$

$$
= h^{2d-1} \int_{\mathcal{M}} \int_{T_{x}\mathcal{M}} \int K_{(k,k)}(s)K_{(l,l)}(s)K_{(k,k)}(w + s)K_{(l,l)}(w + s) \times f(x)g(x)g(x + hw)dsd\mathcal{H}(w)d\mathcal{H}(w)(1 + o(1))
$$

$$
= ch^{2d-1} \int_{\mathcal{M}} g(x)^2 \int_{T_{x}\mathcal{M}} \int K_{(k,k)}(s)K_{(l,l)}(s)K_{(k,k)}(w + s)K_{(l,l)}(w + s)dsd\mathcal{H}(w)d\mathcal{H}(w)\{1 + o(1)\}.
$$

As a result of (4.74) and the above result we have

$$
\text{Var}(I_n) = O(nh^{2d-1}). \quad (4.80)
$$

Next we will focus on $\text{Var}(\Pi_n)$ in what follows. Note that with the change of variables $s = (x - u)/h$ and $t = (x - v)/h$,

$$
E \left\{ [H_{kl}(X_1, X_2; h)]^2 \right\} = \int_{\mathcal{M}} \int_{\mathcal{M}} \int K_{(k,k)} \left( \frac{x - u}{h} \right) K_{(l,l)} \left( \frac{x - v}{h} \right) K_{(k,k)} \left( \frac{y - u}{h} \right) K_{(l,l)} \left( \frac{y - v}{h} \right) \times f(u)f(v)g(x)g(y)dudvd\mathcal{H}(y)d\mathcal{H}(x)
$$

$$
= h^{2d} \int_{\mathcal{M}} \int_{\mathcal{M}} \int \int K_{(k,k)}(s)K_{(l,l)}(t)K_{(k,k)} \left( \frac{y - x}{h} + s \right) K_{(l,l)} \left( \frac{y - x}{h} + t \right) \times f(x - hs)f(x - ht)g(x)g(y)dstdltd\mathcal{H}(y)d\mathcal{H}(x). \quad (4.81)
$$
Applying the change of variables \( w = (y - x)/h \) in (4.81), we have

\[
E \left\{ [H_{kl}(X_1, X_2; h)]^2 \right\}
= h^{3d-1} \int_T \int_T K(k,k) (s) \left( \frac{x - u}{h} \right) K(l,l) \left( \frac{x - v}{h} \right) K(k,k) \left( \frac{y - u}{h} \right) K(l,l) \left( \frac{y - w}{h} \right)
\times f(u)f(v)f(w)g(x)g(y)dsdt\mathcal{H}(w)d\mathcal{H}(x)(1 + o(1))
\]

\[
= h^{3d-1} \int_T \int_T K(k,k) (s) \left( \frac{x - u}{h} \right) K(l,l) \left( \frac{x - v}{h} \right) K(k,k) \left( \frac{y - u}{h} \right) K(l,l) \left( \frac{y - w}{h} \right)
\times f(x)^2 g(x)^2 dsdt\mathcal{H}(w)d\mathcal{H}(x)(1 + o(1))
\]

\[
= h^{3d-1} \int_T \int_T K(k,k) (s) \left( \frac{x - u}{h} \right) K(l,l) \left( \frac{x - v}{h} \right) K(k,k) \left( \frac{y - u}{h} \right) K(l,l) \left( \frac{y - w}{h} \right)
\times dsdt\mathcal{H}(w)d\mathcal{H}(x)\{1 + o(1)\}. \tag{4.82}
\]

Similarly,

\[
E \left[ H_{kl}(X_1, X_2; h) H_{kl}(X_2, X_1; h) \right]
= c^2 h^{3d-1} \int_M g(x)^2 \int_T \int_T K(k,k) (s) \left( \frac{x - u}{h} \right) K(l,l) \left( \frac{x - v}{h} \right) K(k,k) \left( \frac{y - u}{h} \right) K(l,l) \left( \frac{y - w}{h} \right)
\times dsdt\mathcal{H}(w)d\mathcal{H}(x)\{1 + o(1)\}. \tag{4.83}
\]

Now with the change of variables \( r = (x - u)/h \) and \( s = (x - v)/h \),

\[
E \left[ H_{kl}(X_1, X_2; h) H_{kl}(X_2, X_3; h) \right]
= \int_M \int_M \int_M \int_M K(k,k) \left( \frac{x - u}{h} \right) K(l,l) \left( \frac{x - v}{h} \right) K(k,k) \left( \frac{y - u}{h} \right) K(l,l) \left( \frac{y - w}{h} \right)
\times f(u)f(v)f(w)g(x)g(y)dsdt\mathcal{H}(y)d\mathcal{H}(x)
\]

\[
= h^{3d} \int_M \int_M \int_M \int_M K(k,k) \left( \frac{x - u}{h} \right) K(l,l) \left( \frac{x - v}{h} \right) K(k,k) \left( \frac{y - u}{h} \right) K(l,l) \left( \frac{y - w}{h} \right)
\times f(x - hr)f(x - hs)f(y - ht)g(x)g(y)drdsdt\mathcal{H}(y)d\mathcal{H}(x)
\]

\[
= h^{3d+1} \int_M \int_M \int_M \int_M L(r)K(k,k)(s)K(k,k) \left( \frac{y - w}{h} \right) L(t)
\times f(k,k) \left( x - hr \right) f(x - hs) f(1,l) (y - ht) g(x) g(y) drdsdt\mathcal{H}(y)d\mathcal{H}(x)
\]

\[
= h^{3d+1} \int_M \int_M \int_M K(k,k)(s)K(k,k) \left( \frac{y - w}{h} \right)
\times f(k,k) (x) f(1,l) (y) g(x) g(y) dsd\mathcal{H}(y)d\mathcal{H}(x)\{1 + o(1)\}. \tag{4.84}
\]
Using the same argument as above, we have

\[
E[H_{kl}(X_1, X_2; h)H_{kl}(X_2, X_3; h)]
= ch^{4d+3} \int_{\mathcal{M}} \int_{T_x \mathcal{M}} \int K_{(l,l)}(s)K_{(k,k)}(s + w) \times f_{(k,k)}(x)f_{(l,l)}(x + hw)g(x)g(x + hw)d\mathcal{H}(w)d\mathcal{H}(x)(1 + o(1))
= ch^{4d+3} \int_{\mathcal{M}} f_{(k,k)}(x)f_{(l,l)}(x)g(x)^2 \int_{T_x \mathcal{M}} \int K_{(l,l)}(s)K_{(k,k)}(s + w) d\mathcal{H}(w)d\mathcal{H}(x)\{1 + o(1)\},
\]

(4.85)

Similarly

\[
E[H_{kl}(X_1, X_2; h)H_{kl}(X_3, X_2; h)]
= ch^{4d+3} \int_{\mathcal{M}} f_{(k,k)}(x)f_{(l,l)}(x)g(x) \int_{T_x \mathcal{M}} \int K_{(l,l)}(s)K_{(k,k)}(s + w) d\mathcal{H}(w)d\mathcal{H}(x)\{1 + o(1)\},
\]

(4.86)

and

\[
E[H_{kl}(X_1, X_2; h)H_{kl}(X_1, X_3; h)]
= ch^{4d+3} \int_{\mathcal{M}} f_{(l,l)}(x)g(x)^2 \int_{T_x \mathcal{M}} \int K_{(k,k)}(s)K_{(k,k)}(s + w) d\mathcal{H}(w)d\mathcal{H}(x)\{1 + o(1)\}.
\]

(4.87)

Therefore it follows from (4.78), (4.82), (4.83), (4.85), (4.86), and (4.87) that

\[
Var(\Pi_n) = O(n^2 h^{3d-1} + n^3 h^{4d+3}).
\]

(4.88)

Combining (4.77), (4.80), (4.88) and the fact that \(Cov(I_n, \Pi_n) \leq \sqrt{Var(I_n)Var(\Pi_n)}\), we get

\[
Var(D^{(2)}_{g,n}) = O \left( \frac{1}{n^2 h^{2d+g}} + \frac{1}{n^2 h^{d+g}} + \frac{1}{nh^g} \right).
\]

(4.89)

Then (3.13) is a result of (4.76) and (4.89).

**Proof of Theorem 3.1**

**Proof.** Let \(q(x) = f_{(k,k)}(x)f_{(l,l)}(x)\|\nabla f(x)\|^{-1}\) and \(\hat{q}(x) = \hat{f}_{(k,k)}(x)\hat{f}_{(l,l)}(x)\|\nabla \hat{f}(x)\|^{-1}\). With \(a_{kl}(f)\) and \(\hat{a}_{kl}(f)\) given in (3.3) and (3.5), respectively, we can write

\[
\hat{a}_{kl}(f) - a_{kl}(f) = T_1 + T_2 + T_3 + T_4,
\]

(4.90)
where

\[ T_1 = \tilde{\lambda}(\tilde{q} - q) - \lambda(q - q), \]
\[ T_2 = \tilde{\lambda}(q) - \lambda(q), \]
\[ T_3 = \lambda \left( \tilde{f}_{(k,k)} \tilde{f}_{(l,l)} \left( \| \nabla \tilde{f} \|^{-1} - \| \nabla f \|^{-1} \right) \right), \]
\[ T_4 = \lambda \left( \tilde{f}_{(k,k)} \tilde{f}_{(l,l)} - f_{(k,k)} f_{(l,l)} \right) \| \nabla f \|^{-1}. \]

We will find the convergence rates of these four quantities. Let us first focus on \( T_1 \), for which we will applying Proposition 3.1. Note that it follows from (4.3) and (4.4) that

\[ \sup_{x \in I(\delta_0)} |\tilde{f}_{(k,k)}(x)\tilde{f}_{(l,l)}(x) - f_{(k,k)}(x) f_{(l,l)}(x)| = O_p \left( \sqrt{\log n \over n h_2^{d+4} + h_2^\nu} \right), \]  

(4.91)

and

\[ \sup_{x \in I(\delta_0)} \| \nabla \tilde{f}(x) - \nabla f(x) \| = O_p \left( \sqrt{\log n \over n h_1^{d+2} + h_1^\nu} \right), \]  

(4.92)

By assumption (F1), we also have for \( j \in \{-1, 1\}, \)

\[ \sup_{x \in I(\delta_0)} \| \| \nabla \tilde{f}(x) \|^j - \| \nabla f(x) \|^j \| = O_p \left( {\log n \over n^{\nu/(d+2+2\nu)}} \right). \]  

(4.93)

Therefore,

\[ \sup_{x \in I(\delta_0)} \left| \tilde{f}_{(k,k)}(x)\tilde{f}_{(l,l)}(x) \| \nabla \tilde{f}(x) \|^{-1} - f_{(k,k)}(x) f_{(l,l)}(x) \| \nabla f(x) \|^{-1} \right| \]
\[ \leq \sup_{x \in I(\delta_0)} \left| \tilde{f}_{(k,k)}(x)\tilde{f}_{(l,l)}(x) \| \nabla \tilde{f}(x) \|^{-1} - \| \nabla f(x) \|^{-1} \right| \]
\[ + \sup_{x \in I(\delta_0)} \left| \tilde{f}_{(k,k)}(x)\tilde{f}_{(l,l)}(x) - f_{(k,k)}(x) f_{(l,l)}(x) \| \nabla f(x) \|^{-1} \right| \]
\[ = O_p \left( {\log n \over n^{\nu/(d+2+2\nu)}} \right). \]  

(4.94)

With the assumption that \( K \) is three times continuously differentiable and \( f \) is \((\nu + 3)\) times continuously differentiable on \( I(2\delta_0) \), we can extend the results in (4.3) and (4.4) to \( r = 3 \) by Lemma 3 in Arias-Castro et al. (2016). Using the same telescoping argument as above, we have

\[ \sup_{x \in I(\delta_0)} \left\| \nabla \left[ \tilde{f}_{(k,k)}(x)\tilde{f}_{(l,l)}(x) \| \nabla \tilde{f}(x) \|^{-2} \right] - \nabla \left[ f_{(k,k)}(x) f_{(l,l)}(x) \| \nabla f(x) \|^{-2} \right] \right\| \]
\[ = O_p \left( \sqrt{\log n \over n h_2^{d+6} + h_2^\nu} \right) + O_p \left( \sqrt{\log n \over n h_1^{d+4} + h_1^\nu} \right) = O_p \left( {\log n \over n^{(\nu-1)/(d+4+2\nu)}} \right). \]  

(4.95)
Applying Theorem 3.1 to $T_1$, we obtain from (4.94) and (4.95) that
\[
T_1 = O_p \left\{ \frac{\sqrt{\log n}}{n^{\nu/(d+4+2\nu)}} \left( \sqrt{\frac{\log n}{n h_0^{d+2}}} + h_0^{\nu} \right) + \frac{\sqrt{\log n}}{n^{(\nu-1)/(d+4+2\nu)}} \left( \sqrt{\frac{\log n}{n h_0^{d}}} + h_0^{\nu} \right) \right\} \\
= o_p \left( n^{-\nu/(d+4+2\nu)} \right). \tag{4.96}
\]

For $T_2$, we apply Theorem 3.1. As a result,
\[
T_2 = O_p \left( \frac{\log n}{n h_0^{d+2}} + h_0^{\nu} + \frac{1}{\sqrt{n h_0}} \right) = O_p \left( n^{-\nu/(d+2\nu)} + n^{-(d+2\nu-3)/(2(d+2\nu))} \right) \\
= o_p \left( n^{-\nu/(d+4+2\nu)} \right). \tag{4.97}
\]

Next we focus on $T_3$ and $T_4$. Note that
\[
T_3 = \lambda \left( \hat{f}_{(k,k)} \hat{f}_{(l,l)} \frac{||\nabla f||^2 - ||\tilde{f}||^2}{||\nabla f|| ||\tilde{f}|| (||\nabla f|| + ||\tilde{f}||)} \right).
\]
Let
\[
\tilde{T}_3 = \lambda \left( f_{(k,k)} f_{(l,l)} \frac{||\nabla f||^2 - ||\tilde{f}||^2}{2||\tilde{f}||^3} \right).
\]
Since $\mathcal{M}$ is a compact set, by a telescoping argument and using (4.93) and (4.94), we have
\[
|T_3 - \tilde{T}_3| = O_p \left( \frac{\sqrt{\log n}}{n^{\nu/(d+4+2\nu)}} \times \frac{\sqrt{\log n}}{n^{\nu/(d+2+2\nu)}} \right). \tag{4.98}
\]
Then applying Proposition 3.2 to $\tilde{T}_3$ and $T_4$, we have
\[
\tilde{T}_3 = O \left( \frac{1}{n h_1^{d+4} + h_1^{\nu}} \right) + O_p \left( \frac{1}{n h_1^{3} + \frac{1}{n^3 h_1^{2d+8} + \frac{1}{n^2 h_1^{d+8}}} \right) = O_p \left( n^{-\nu/(d+2+2\nu)} \right), \tag{4.99}
\]
\[
T_4 = O \left( \frac{1}{n h_2^{d+4} + h_2^{\nu}} \right) + O_p \left( \frac{1}{n h_2^{3} + \frac{1}{n^3 h_2^{2d+9} + \frac{1}{n^2 h_2^{d+9}}} \right) = O_p \left( n^{-\nu/(d+4+2\nu)} \right). \tag{4.100}
\]
It then follows from (4.90), (4.96), (4.97), (4.98), (4.99) and (4.100) that
\[
\hat{a}_{kl}(f) - a_{kl}(f) = O_p \left( n^{-\nu/(d+4+2\nu)} \right). \tag{4.101}
\]
Next we focus on $\hat{b}(f)$ and have
\[
\hat{b}(f) - b(f) = S_1 + S_2 + S_3, \tag{4.102}
\]
where
\begin{align*}
S_1 &= \lambda \left( \|\nabla \hat{f}\|^{-1} - \|\nabla f\|^{-1} \right) - \lambda \left( \|\nabla \hat{f}\|^{-1} - \|\nabla f\|^{-1} \right), \\
S_2 &= \lambda \left( \|\nabla f\|^{-1} \right) - \lambda \left( \|\nabla f\|^{-1} \right), \\
S_3 &= \lambda \left( \|\nabla \hat{f}\|^{-1} \right) - \lambda \left( \|\nabla f\|^{-1} \right).
\end{align*}

Following the similar argument for \(T_1, T_2, T_3\) and \(T_4\), we can show that
\[ \hat{b}(f) - b(f) = O_p \left( n^{-\nu/(d+2+2\nu)} \right). \]
(4.103)
The assertion in the theorem follows from (4.101) and (4.103).

5 Appendix

We continue the discussion in the remark for assumption (F2). First we provide some background of Morse theory. Suppose \(f\) is a Morse function (i.e., all of its critical points are nondegenerate). The (Morse) index of a critical point \(x\) of \(f\) is the number of negative eigenvalues of the Hessian of \(f(x)\). Let \(k\)-disk be \(D^k = \{ x \in \mathbb{R}^k : \|x\| \leq 1 \}\). The boundary of \(D^k\) is the unit \((k-1)\)-sphere, i.e., \(\partial D^k = S^{k-1}\). A \(k\)-handle of dimension \(d\) (or a handle of index \(k\)) is
\[ H_{k,d} := D^k \times D^d - \partial D^k. \]
A \(d\)-dimensional handlebody is defined as a manifold constructed by successively attaching \(m\) handles of indices \(k_1, \ldots, k_m\) to \(D^d\):
\[ D^d \cup H_{k_1,d} \cup \cdots \cup H_{k_m,d}. \]
See Chapter 2.1 of Nicolaescu (2011) for the concept of handle attachment. Since under assumption (F1) the density function \(f\) is exhaustive, meaning that the upper level set \(L_\tau = \{ x \in \mathbb{R}^d : f(x) \geq \tau \}\) is compact for all \(\tau \in \mathbb{R}\), the Morse fundamental structure theorem (Theorem 2.7 in Nicolaescu, 2011) says that if \(\mathcal{M}_\tau\) only contains one critical point \(x\) of index \(k\), then for \(\epsilon > 0\) small enough, \(L_{\tau-\epsilon}\) is diffeomorphic to \(L_{\tau+\epsilon}\) with \(H_{d-k,d}\) attached. Therefore \(L_{\tau}\) is diffeomorphic to a handlebody, whose handles are determined by the critical points of \(f\) and the indices of the critical points (see Theorem 3.4 in Matsumoto, 1997). Correspondingly, \(\mathcal{M}_\tau\) is diffeomorphic to the boundary of the handlebody. Here notice that the boundary of the handle \(H_{k,d}\) is
\[ \partial H_{k,d} = (\partial D^k \times D^d - \partial D^k) \cup (D^k \times \partial D^d - \partial D^k). \]
In assumption (F2) the parameterization for \(\mathcal{M}_j^\sharp\) with \(\psi_j\) on some \(\Omega_j \subset \mathbb{R}^{d-1}, j = 1, \ldots, N\) can be constructed through the above diffeomorphism and the parameterization for the boundary of the handlebody.

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