NON-RATIONALITY OF THE $\mathfrak{S}_6$-SYMMETRIC QUARTIC THREEFOLDS

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ABSTRACT. We prove that the quartic hypersurfaces defined by $\sum x_i = t \sum x_i^4 - (\sum x_i^2)^2 = 0$ in $\mathbb{P}^5$ are not rational for $t \neq 0, 2, 4, 6, \frac{10}{7}$.

1. INTRODUCTION

Let $V$ be the standard representation of $\mathfrak{S}_6$ (that is, $V$ is the hyperplane $\sum x_i = 0$ in $\mathbb{C}^6$, with $\mathfrak{S}_6$ acting by permutation of the basis vectors). The quartic hypersurfaces in $\mathbb{P}(V) (\cong \mathbb{P}^4)$ invariant under $\mathfrak{S}_6$ form the pencil

$$X_t : t \sum x_i^4 - (\sum x_i^2)^2 = 0, \quad t \in \mathbb{P}^1.$$  

This pencil contains two classical quartic hypersurfaces, the Burkhardt quartic $X_2$ and the Igusa quartic $X_4$ (see for instance [H]); they are both rational.

For $t \neq 0, 2, 4, 6$ and $\frac{10}{7}$, the quartic $X_t$ has exactly 30 nodes; the set of nodes $\mathcal{N}$ is the orbit under $\mathfrak{S}_6$ of $(1, 1, \rho, \rho, \rho^2, \rho^2)$, with $\rho = e^{2\pi i/3}$ ([vdG], §4). We will prove:

**Theorem.** For $t \neq 0, 2, 4, 6, \frac{10}{7}$, $X_t$ is not rational.

The method is that of [B]: we show that the intermediate Jacobian of a desingularization of $X_t$ is 5-dimensional and that the action of $\mathfrak{S}_6$ on its tangent space at 0 is irreducible. From this one sees easily that this intermediate Jacobian cannot be a Jacobian or a product of Jacobians, hence $X_t$ is not rational by the Clemens-Griffiths criterion. We do not know whether $X_t$ is unirational.

I am indebted to A. Bondal and Y. Prokhorov for suggesting the problem, to A. Dimca for explaining to me how to compute explicitly the defect of a nodal hypersurface, and to I. Cheltsov for pointing out the rationality of $X_{\frac{10}{7}}$.

2. THE ACTION OF $\mathfrak{S}_6$ ON $T_0(JX)$

We fix $t \neq 0, 2, 4, 6, \frac{10}{7}$, and denote by $X$ the desingularization of $X_t$ obtained by blowing up the nodes. The main ingredient of the proof is the fact that the action of $\mathfrak{S}_6$ on $JX$ is non-trivial. To prove this we consider the action of $\mathfrak{S}_6$ on the tangent space $T_0(JX)$, which is by definition $H^2(X, \Omega^1_X)$.

**Lemma 1.** Let $C$ be the space of cubic forms on $\mathbb{P}(V)$ vanishing along $\mathcal{N}$. We have an isomorphism of $\mathfrak{S}_6$-modules $C \cong V \oplus H^2(X, \Omega^1_X)$.

**Proof :** The proof is essentially contained in [C]; we explain how to adapt the arguments there to our situation. Let $b : P \to \mathbb{P}(V)$ be the blowing-up of $\mathbb{P}(V)$ along $\mathcal{N}$. The threefold $X$ is the strict transform of $X_t$ in $P$. The exact sequence

$$0 \to N^1_{X/P} \to \Omega^1_{P|X} \to \Omega^1_X \to 0$$
gives rise to an exact sequence

\[ 0 \rightarrow H^2(X, \Omega^1_X) \rightarrow H^3(X, N^*_X / P) \rightarrow H^3(X, \Omega^1_{P|X}) \rightarrow 0 \]

([C], proof of Theorem 1), which is $\mathcal{G}_6$-equivariant. We will compute the two last terms.

The exact sequence

\[ 0 \rightarrow \Omega^1_P(-X) \rightarrow \Omega^1_P \rightarrow \Omega^1_{P|X} \rightarrow 0 \]

provides an isomorphism $H^3(X, \Omega^1_{P|X}) \cong H^4(P, \Omega^1_P(-X))$, and the latter space is isomorphic to $H^4(\mathbb{P}(V), \Omega^1_{\mathbb{P}(V)}(-4))$ ([C], proof of Lemma 3). By Serre duality $H^4(\mathbb{P}(V), \Omega^1_{\mathbb{P}(V)}(-4))$ is dual to $H^0(\mathbb{P}(V), T_{\mathbb{P}(V)}(-1)) \cong V$. Thus the $\mathcal{G}_6$-module $H^3(X, \Omega^1_{P|X})$ is isomorphic to $V^*$, hence also to $V$.

Similarly the exact sequence $0 \rightarrow \mathcal{O}_P(-2X) \rightarrow \mathcal{O}_P(-X) \rightarrow N^*_X / P \rightarrow 0$ and the vanishing of $H^4(P, \mathcal{O}_P(-X))$ ([C], Corollary 2) provide an isomorphism of $H^3(X, N^*_X / P)$ onto $H^4(P, \mathcal{O}_P(-2X))$, which is naturally isomorphic to the dual of $C$ ([C], proof of Proposition 2). The lemma follows.

**Lemma 2.** The dimension of $C$ is 10.

**Proof:** Recall that the defect of $X_t$ is the difference between the dimension of $C$ and its expected dimension, namely:

\[ \text{def}(X_t) := \dim C - (\dim H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(3)) - \# N). \]

Thus our assertion is equivalent to $\text{def}(X_t) = 5$.

To compute this defect we use the formula of [D-S], Theorem 1.5. Let $F = 0$ be an equation of $X_t$ in $\mathbb{P}^4$; let $R := \mathbb{C}[X_0, \ldots, X_4] / (F'_0, \ldots, F'_4)$ be the Jacobian ring of $F$, and let $R^{2m}$ be the Jacobian ring of a smooth quartic hypersurface in $\mathbb{P}^4$. The formula is

\[ \text{def}(X_t) = \dim R_7 - \dim R^{2m}_7. \]

In our case we have $\dim R^{2m}_7 = \dim R^{2m}_4 = 35 - 5 = 30$; a simple computation with Singular (for instance) gives $\dim R_7 = 35$. This implies the lemma.

**Proposition.** The $\mathcal{G}_6$-module $H^2(X, \Omega^1_X)$ is isomorphic to $V$.

**Proof:** Consider the homomorphisms $a$ and $b$ of $\mathbb{C}^6$ into $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(3))$ given by $a(e_i) = x_i^3$, $b(e_i) = x_i \sum x_j^2$. They are both $\mathcal{G}_6$-equivariant and map $V$ into $C$; the subspaces $a(V)$ and $b(V)$ of $C$ do not coincide, so we have $a(V) \cap b(V) = 0$. By Lemma 2 this implies $C = a(V) \oplus b(V)$, so $H^2(X, \Omega^1_X)$ is isomorphic to $V$ by Lemma 1.

**Remark.** Suppose $t = 2, 6$ or $\frac{10}{7}$. Then the singular locus of $X_t$ is $\mathcal{N} \cup \mathcal{N}'$, where $\mathcal{N}'$ is the $\mathcal{G}_6$-orbit of the point $(1, -1, 0, 0, 0, 0)$ for $t = 2$, $(1, -1, 1, -1, 1, -1)$ for $t = 6$, $(-5, 1, 1, 1, 1, 1)$ for $t = \frac{10}{7}$. Since $x_1^3 - x_0^3$ does not vanish on $\mathcal{N}'$, the space of cubics vanishing along $\mathcal{N} \cup \mathcal{N}'$ is strictly contained in $C$. By Lemma 1 it contains a copy of $V$, hence it is isomorphic to $V$; therefore $H^2(X, \Omega^1_X)$ and $\mathcal{J}X$ are zero in these cases. We have already mentioned that $X_2$ and $X_4$ are rational. The quartic $X_{\frac{10}{7}}$ is rational: it is the image of the anticanonical map of $\mathbb{P}^3$ blown up along 6 lines which are permuted by $\mathcal{G}_6$ (see [C-S], proof of Lemma 4.5, and the references given there). We do not know whether this is the case for $X_6$. 
3. PROOF OF THE THEOREM

To prove that $X$ is not rational, we apply the Clemens-Griffiths criterion ([C-G], Cor. 3.26): it suffices to prove that $JX$ is not a Jacobian or a product of Jacobians.

Suppose $JX \cong JC$ for some curve $C$ of genus $5$. By the Proposition $S_6$ embeds into the group of automorphisms of $JC$ preserving the principal polarization; by the Torelli theorem this group is isomorphic to $\text{Aut}(C)$ if $C$ is hyperelliptic and $\text{Aut}(C) \times \mathbb{Z}/2$ otherwise. Thus we find $\# \text{Aut}(C) \geq \tfrac{1}{2}6! = 360$. But this contradicts the Hurwitz bound $\# \text{Aut}(C) \leq 84(5 - 1) = 336$.

Now suppose that $JX$ is isomorphic to a product of Jacobians $J_1 \times \ldots \times J_p$, with $p \geq 2$. Recall that such a decomposition is unique up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components ([C-G], Cor. 3.23). Thus the group $S_6$ permutes the factors $J_i$, and therefore acts on $[1,p]$; by the Proposition this action must be transitive. But we have $p \leq \dim JX = 5$, so this is impossible. ■

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