Nonlinear Resonant Wave Interaction in Vacuum

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Received September 20, 2003
PACS ref: 42.50.Pq

Abstract

The basic equations governing propagation of electromagnetic and gravitational waves in vacuum are nonlinear. As a consequence photon–photon interaction as well as photon–graviton interaction can take place without a medium. However, resonant interaction between less than four waves cannot occur in vacuum, unless the interaction takes place in a bounded region, such as a cavity or a waveguide. Recent results concerning resonant wave interaction in bounded vacuum regions are reviewed and extended.

1. Introduction

Most examples of nonlinear wave phenomena occur as a result of some nonlinear property of a medium. In particular this applies to electromagnetism since Maxwell’s equations are linear. Still, nonlinear interaction between photons in vacuum may occur as a result of scattering processes involving vacuum fluctuations, as described by quantum electrodynamics (QED). An effective theory for this can be formulated in terms of the Euler–Heisenberg Lagrangian [1,2]. It should be noted, however, that resonant interaction between less than four waves requires that the waves are parallel, in which case the nonlinear QED coupling vanishes [3,4]. Similarly, gravitons and photons couple in vacuum, as described by the Einstein–Maxwell system of equations. But the dispersion relation implies that the waves must be parallel, in case three wave interaction should be resonant. Just as for photon–photon interaction in vacuum, however, this condition means that the wave-coupling vanishes [5]. On the other hand, the situation is changed if some or all of the interacting waves are confined in bounded regions such as waveguides or cavities [5–7]. Below we will review and extend recent results concerning resonant photon–photon scattering as well as photon–graviton scattering in bounded regions.

2. Photon–Photon scattering

According to QED, the non-classical phenomenon of photon–photon scattering can take place due to the exchange of virtual electron–positron pairs. This give rise to vacuum polarization and magnetization currents, and an effective field theory can be formulated in terms of the Euler–Heisenberg Lagrangian density [1,2]

$$\mathcal{L}_{EH} = \varepsilon_0 F^2 + \xi (4F^2 + 7G^2),$$

where $\xi \equiv 20c^2\gamma^4 \hbar^3/45m_e^5\epsilon^5$, $F \equiv \frac{1}{2}(E^2 - c^2 B^2)$, $G \equiv \epsilon E \cdot B$, $\epsilon$ is the electron charge, $c$ the velocity of light, $2\pi\hbar$ the Planck constant and $m_e$ the electron mass. The latter terms in (1) represent the effects of vacuum polarization and magnetization. We note that $F = G = 0$ in the limit of parallel propagating waves. It is therefore necessary to use other wave geometries in order to obtain an effect from the QED corrections. Note that this null-result should be expected on physical grounds, since successive Lorentz boosts along the direction of propagation would decrease the amplitude of all waves without limit. However, as shown in Refs. [3,4], a resonant nonlinear interaction between three waves is possible in bounded regions, as will be considered below.

The most common approach (see e.g., Refs. [8–13]) to calculate the interaction strength is to use the general Lagrangian, apply the variational principle $\delta \int \mathcal{L}_{EH} d^3r dt = 0$ to find the influence of the QED terms in Maxwell’s equations (where the Lagrangian should be expressed in terms of the 4-potential), and proceed from there using standard techniques for weakly nonlinear waves. However, as shown in Ref. [4], the algebra is significantly reduced if one make an ansatz for the potentials corresponding to the field geometry of consideration, and then derive the evolution equations directly from the variational principle, without using Maxwell’s equations as an intermediate step.

Following the later approach we start by considering three interacting waves in a cavity with the shape of a rectangular prism. We let one of the corners lie in the origin, and we let the opposite corner have coordinates $(x_0, y_0, z_0)$. In practice we are interested in a shape where $z_0 \gg x_0, y_0$ but this assumption will not be used in the calculations. We let the large amplitude pump modes have vector potentials of the form

$$A_1 = A_1 \sin \left( \frac{\pi x}{x_0} \right) \sin \left( \frac{n_1 \pi z}{z_0} \right) \exp(-i\omega_1 t) \mathbf{\hat{\ell}} + \text{c.c.}$$

and

$$A_2 = A_2 \sin \left( \frac{\pi y}{y_0} \right) \sin \left( \frac{n_2 \pi z}{z_0} \right) \exp(-i\omega_2 t) \mathbf{\hat{x}} + \text{c.c.},$$

where c.c. denotes complex conjugate, and we chose the radiation gauge $\Phi = 0$. It is easily checked that the corresponding fields

$$B_{1z} = \left( \frac{\pi}{x_0} \right) A_1 \cos \left( \frac{\pi x}{x_0} \right) \sin \left( \frac{n_1 \pi z}{z_0} \right) \exp(-i\omega_1 t) + \text{c.c.},$$

$$B_{1x} = -\left( \frac{n_1 \pi}{z_0} \right) A_1 \sin \left( \frac{\pi x}{x_0} \right) \cos \left( \frac{n_1 \pi z}{z_0} \right) \exp(-i\omega_1 t) + \text{c.c.,}$$

$$E_{1y} = i\omega_1 A_1 \sin \left( \frac{\pi y}{y_0} \right) \sin \left( \frac{n_2 \pi z}{z_0} \right) \exp(-i\omega_1 t) + \text{c.c.,}$$

$$E_{1z} = \left( \frac{\pi}{y_0} \right) A_2 \cos \left( \frac{\pi y}{y_0} \right) \sin \left( \frac{n_2 \pi z}{z_0} \right) \exp(-i\omega_2 t) + \text{c.c.}.$$
together with $\omega_n^2 = n^2\pi^2 c^2/z_0^2 + \pi^2 c^2/x_0^2$, and

$$B_{2z} = -\frac{\pi}{\gamma_0} A_2 \cos \left( \frac{\pi}{\gamma_0} \right) \sin \left( \frac{n_2 \pi z_0}{z_0} \right) \exp(-i\omega_2 t) + \text{c.c.}, \quad (5a)$$

$$B_{2\gamma} = \left( \frac{n_2 \pi}{z_0} A_2 \sin \left( \frac{\pi}{\gamma_0} \right) \cos \left( \frac{n_2 \pi z_0}{z_0} \right) \exp(-i\omega_2 t) + \text{c.c.}, \quad (5b)$$

$$E_{1z} = i\omega_2 A_2 \sin \left( \frac{\pi}{\gamma_0} \right) \sin \left( \frac{n_2 \pi z_0}{z_0} \right) \exp(-i\omega_2 t) + \text{c.c.}, \quad (5c)$$

$$A_3 = A_1 \sin \left( \frac{\pi}{\gamma_0} \right) \sin \left( \frac{n_3 \pi z_0}{z_0} \right) \exp(-i\omega_3 t) + \text{c.c.} \quad (6)$$

where $\omega_3^2 = n_3^2\pi^2 c^2/z_0^2 + \pi^2 c^2/x_0^2$, in which case we get fields of the same form as in Eqs. (5a)–(5c). Since the QED terms are fourth order in the amplitude, the corresponding nonlinearities are cubic, implying that $\omega_3 = \pm (2\omega_0 \pm \omega_1)$ or $\omega_3 = \pm (\omega_0 \pm 2\omega_1)$ should hold for resonant interaction. We assume that the cavity dimensions are chosen such that a single eigenmode can be resonantly excited, and pick the alternative

$$\omega_3 = 2\omega_1 - \omega_2 \quad (7)$$

for definiteness. We note that when performing the variations $\delta A_3^2$, the lowest order terms proportional to $\delta A_2^2 A_1$ vanish due to the dispersion relation, and we need to include terms due to the time dependence of the amplitude of the type $A_3 \partial (\delta A_3^2)/\partial t$. For the fourth order QED corrections proportional to $\delta A_3^2$, only terms proportional to $A_1^4 A_2^2 \delta A_3^2$ survives the time integration, due to the frequency matching (7). After some algebra the corresponding evolution equation for mode 3 becomes:

$$\left( \frac{d}{dt} - \gamma \right) A_3 = \frac{\epsilon_{0}\kappa_{0}^{3}}{8i} K_{\text{rec}} A_1^2 A_2^2 \quad (8)$$

where the dimensionless coupling coefficient $K_{\text{rec}}$ is

$$K_{\text{rec}} = \frac{1}{\omega_3} \left\{ - (-, +) 2\omega_1 \omega_2 \omega_3 + 2\pi e^4 \right. \left. \times \left[ (-, +) \frac{3}{x_0^2} + \frac{n_2^2 \omega_3 n_3}{x_0^2} + (+, -) \frac{n_1^2}{x_0^2} + \frac{n_2 n_3}{x_0^2} \right] \right. \\

+ \pi^2 c^2 \left[ (-, +) \frac{5n_1^2 \omega_3 \omega_2 n_3}{z_0^2} + \frac{5n_2^2 n_3 n_3}{z_0^2} \right. \\

\left. + (+, -) \frac{2\omega_2 \omega_3}{z_0^2} + \frac{2\omega_1^2}{y_0} \right] - (-, +) \frac{7\omega_1 \omega_2 n_2 n_3}{z_0^2} \right. \\

\left. + (-, -) \frac{7\omega_1 \omega_2 \omega_1 n_2}{z_0^2} \right\} \quad (9)$$

and we have added a phenomenological linear damping term represented by $\gamma$. When deriving Eq. (9) we have assumed one of the following mode number matchings

$$2n_1 - n_2 + n_3 = 0, \quad (10)$$

$$2n_1 + n_2 - n_3 = 0, \quad (11)$$

$$2n_1 - n_2 - n_3 = 0, \quad (12)$$

in order for the QED corrections terms to survive the $z$-integration. The three different sign alternatives in (9) correspond to the mode number matching options (10)–(12) respectively. Given experimental data for possible values of the pump field strengths and the damping coefficient inside cavities, the saturation level of the excited mode can be determined from Eq. (8). The possibilities for detection of photon–photon scattering using currently available performance on microwave cavities will be discussed in the final section of the paper.

3. Gravitational interaction

Preliminaries. In vacuum, a linearized gravitational wave can be transformed into the transverse and traceless (TT) gauge. Then we have the following line-element

$$ds^2 = -c^2 dt^2 + dx^2 + [1 + h_+ (\xi)] dy^2 \\
+ [1 - h_\times (\xi)] dz^2 + 2h_\times (\xi) dy dz, \quad (13)$$

where $\xi = x - ct$, $c$ is the speed of light in vacuum, and $|h_+|, |h_\times| \ll 1$.

Neglecting terms proportional to derivatives of $h_+$ and $h_\times$ (since the gravitational frequency is assumed small), the wave equation for the magnetic field is [5]

$$\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] B = \left[ h_+ \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) + h_\times \frac{\partial^2}{\partial y \partial z} \right] B, \quad (14)$$

and similarly for the electric field.

In an isotropic dielectric medium with permittivity $\varepsilon$ different from the vacuum permeability, the Eq. (14) still holds, simply if we replace $c$ by $c/n = c/\sqrt{\varepsilon}$ in the above expressions, where $\varepsilon = \varepsilon_0 \varepsilon_i$. For the moment, we will neglect mechanical effects, i.e., effects which are associated with the varying coordinate distance of the cavity walls due to the gravitational wave.

Cavity design. The coupling of two electromagnetic modes and a gravitational wave in a cavity will depend strongly on the geometry of the electromagnetic eigenfunctions. It turns out that we can greatly magnify the coupling, as compared to a rectangular prism geometry, by varying the cross-section of the cavity, or by filling the cavity partially with a dielectric medium. The former case is of more interest from a practical point of view, since a vacuum cavity implies better detector performance, but we will here consider the later case since it can be handled analytically.

Specifically, we choose a rectangular cross-section (side lengths $x_0$ and $y_0$), and we divide the length of the cavity into three regions. Region 1 has length $l_1$ (occupying the region $-l_1 < z < 0$) and a refractive index $n_1$. Region 2 has length $l_2$ (occupying the region $0 < z < l_2$), with a refractive index $n_2$, while region 3 consists of vacuum and has length
l_3 (occupying the region l_2 < z < l_1 + l_2). We will also use l = l_1 + l_2 + l_3 for the total length. The cavity is supposed to have positive coordinates, with one of the corners coinciding with the origin. Furthermore, we require that l_2 ≪ l_1, and that the wave number in region 2 is less than in region 1. The reason for this arrangement is twofold. Firstly, we want to obtain a large coupling between the region 1. The reason for this arrangement is twofold. Secondly, we want an efficient filtering of the eigenmode with the lower frequency in region three.

The first step is to analyze the linear eigenmodes in this system. Those with the lowest frequencies are modes of the type

\[ E_j = \frac{i \omega_0}{m \pi} \tilde{B}_j \sin \left( \frac{m \pi x}{x_0} \right) \sin (k_j z + \varphi_j) e^{-i \omega t}, \] (15a)

\[ B_j = \tilde{B}_j \cos \left( \frac{m \pi x}{x_0} \right) \sin (k_j z + \varphi_j) e^{-i \omega t}, \] (15b)

\[ B_n = -\frac{k_j x_0}{m \pi} \tilde{B}_j \sin \left( \frac{m \pi x}{x_0} \right) \cos (k_j z + \varphi_j) e^{-i \omega t}, \] (15c)

in regions j = 1, 2 and 3, respectively, where the wave in region 3 is a standing wave. Furthermore, in region 3 we may also have a decaying wave

\[ E_j = \frac{i \omega_0}{m \pi} \tilde{B}_j \sin \left( \frac{m \pi x}{x_0} \right) \sinh (k_j z + \varphi_j) e^{-i \omega t}, \] (16a)

\[ B_j = \tilde{B}_j \cos \left( \frac{m \pi x}{x_0} \right) \sinh (k_j z + \varphi_j) e^{-i \omega t}, \] (16b)

\[ B_n = -\frac{k_j x_0}{m \pi} \tilde{B}_j \sin \left( \frac{m \pi x}{x_0} \right) \cosh (k_j z + \varphi_j) e^{-i \omega t}. \] (16c)

Using standard boundary conditions, it is straightforward to perform most of the eigenmode calculations analytically. Once the wavenumbers are calculated for an eigenmode, the relation between the amplitudes in the three regions is found, and thereby the mode profile. We are specifically interested in the shift from decaying to oscillatory behavior in region 3, and we denote the highest frequency which is decaying in region 3 with index a, and the wave number and decay coefficient with \( k_{1a}, k_{2a} \) and \( k_{3a} \) respectively. Similarly, the next frequency, which is oscillatory in both regions, is denoted by index b. If we have \( l \gg x_0 \) (and \( m \) the same) these two frequencies will be very close, and a gravitational wave which has a frequency equal to the difference between the electromagnetic modes causes a small coupling between these modes. An example of two such eigenmodes is shown in Fig. 1.

We define the eigenmodes to have the form \( \tilde{B}_{a,b} = \tilde{B}_{a,b}(l) \tilde{B}_{\text{norma},b}(z, y, z) \), where \( \tilde{B}_{a,b} \) is a time-dependent amplitude and the normalized eigenmodes \( \tilde{B}_{\text{norma},b} \) fulfill \( \int_V \tilde{B}_{\text{norma},b}^2 \, dV = V \). We let all electromagnetic field components be of the form \( A = A_0(r) \exp(-i \omega t) + A_* (r) \exp(-i \omega t) + c.c. \), where c.c. stands for complex conjugate, and the indices stand for the eigenmodes discussed above. The gravitational perturbation can be approximated by \( h_{y,x} = h_{y,x} \exp(-i \omega t) + c.c. \), where we neglect the spatial dependency of \( h_{y,x} \) altogether, since the gravitational wavelength is assumed to be much longer than all of the cavity dimensions. If we consider a binary system of two black holes fairly close to collapse, the gravitational frequency will not be an exact constant, but will increase slowly. During a certain interval in time, the frequency matching condition

\[ \omega_h = \omega_a + \omega_e, \] (17)

will be approximately fulfilled. Given the wave equation (14), and the above ansatz we find after integrating over the length of the cavity

\[ \frac{2 \omega_h}{c^2} \left( \frac{d}{dt} - \gamma \right) \tilde{B}_b = -h_{1a} k_{1a}^2 \tilde{B}_a I_c, \] (18)

where

\[ I_c = \frac{1}{V k_{1a}^2} \int_V \frac{d^2 B_{\text{norma}}}{d z^2} B_{\text{normb}} \, dV, \] (19)

and we have added a phenomenological linear damping term represented by \( \gamma \). Thus we note that for the given geometry, only the \( h_x \)-polarization gives a mode-coupling. (Rotating the cavity \( \pi/4 \) around the \( x \)-axis will instead give coupling to the \( h_y \)-polarization.) Furthermore, if we consider propagation in a different angle to the cavity, the result will be slightly modified. Calculations of the eigen-mode parameters show that \( I_c \) may be different from zero when \( n_{1,2} \neq 1 \), and generally \( I_c \) of the order of unity can be obtained, see Fig. 1 for an example. From Eq. (18),

![Mode profiles for the eigenmodes are shown for the parameter values](image)

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we find that the saturated value of the gravitationally excited mode is
\[ \tilde{\mathcal{B}}_{\text{sat}} = \frac{\hbar x_0 k_1^2 e^2}{2 \gamma_{\text{ph}}} I_c. \] (20)

In Fig. 1 it is shown that we can get an appreciable mode-coupling constant \( I_c \) for a cavity filled with materials with different dielectric constants, and it is of much interest whether the same can be achieved in a pure vacuum cavity. As seen by Eq. (19), the coupling is essentially determined by the wave numbers of the modes, given by \( k = (m^2 \pi^2 / x_0^2 - n^2 \alpha_0^2 / c^2)^{1/2} \). Thus by adjusting the width \( x_0 \) in a vacuum cavity, we may get the same variations in the wave numbers as when varying the index of refraction \( n \). The translation of our results to a vacuum cavity with a varying width is not completely accurate, however. Firstly, when varying \( x_0 \), the mode-dependence on \( x \) and \( z \) does not exactly factorize, in particular close to the change in width. Secondly, the contribution to the coupling \( I_c \) in each section becomes proportional to the corresponding volume, and thereby also to the cross-section. However, since most of the contribution to the integral in Eq. (19) comes from region 1, our results can still be approximately translated to the case of a vacuum cavity, by varying \( x_0 \) instead of \( n \) such as to get the same wave number as in our above example. Thus we conclude that our discussion of the sensitivity of a cavity based detector given below, can be based on a vacuum cavity rather than a “dielectric cavity”, where the former case is preferred due to the much smaller dissipation level.

4. Discussion of the detection sensitivity

For both cases considered above, as can be seen from Eqs. (8) and (20) respectively, the quality factor \( Q \) of the cavity (i.e., the damping time/period time) and the allowed field strength are the main parameters that determine the saturation level of the excited mode. For superconducting niobium cavities large surface field strengths \( (E \sim 60 \text{ MV/m}) \) and high quality factors \( Q \sim 10^{10} \) can be obtained simultaneously [14]. For the case of photon–photon scattering this typically corresponds to a few excited microwave photons in the new mode. Provided the number of excited photons beats the thermal noise level \( (N_{\text{th}} \sim kT / h \omega_0) \) a few microwave photons is enough for detection, see e.g., [15]. However, detection of such a small signal can only take place if the pump waves are filtered out. This can be done directly in the cavity using a “filtering geometry”, i.e., a cavity with a variable cross-section, where only the excited mode has a frequency above cut-off in the detection region of the cavity. Our conclusion is that detection of photon–photon scattering is possible in a microwave cavity using current technology, provided the equipment has state of the art performance, and a filtering geometry is applied.

The considerations for the gravitational wave detector are similar to the photon–photon scattering case. However, we must also take the length variations of the detector due to acoustic thermal noise into account. As a consequence, an efficient detector must then have a rather high mechanical quality factor \( Q_{\text{me}} \approx 6 \times 10^6 \) (where \( Q_{\text{me}} \) describes the relative damping of acoustic oscillations), for sensitive detection to be possible. Furthermore, we must also be aware that the gravitational wave sources of most interest—binary systems close to collapse—produces radiation with a finite frequency chirp, implying a finite time of coherent interaction, typically fractions of a second. If the above aspects are taken into account, we find that a cavity based detector of a few meters length can detect metric perturbations of the order \( h_{\text{min}} \approx 7 \times 10^{-22} \) (See Ref. [5] for a more thorough discussion of the detection level.).

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