A simple proof of the Jarzynski equality?

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1 Introduction

In this short communication, I give a very simple derivation of the Jarzynski equality [1]

\[ \Delta F = -\beta^{-1} \ln \langle e^{-\beta W} \rangle, \] (1)

which allows to compute the free energy difference \( \Delta F = F_B - F_A \) of a body, which is driven between two equilibrium states \( A \) and \( B \) by an external (time-dependent) force \( \lambda \), from the probability distribution function \( P(W) \) of the work done on the system

\[ W = \int_{t_A}^{t_B} \frac{\partial H_t(x, \lambda)}{\partial t} \, dt = \int_{t_A}^{t_B} \frac{\partial H_t(x, \lambda)}{\partial \lambda} \dot{\lambda} \, dt, \] (2)

regardless of the nature of the transformation (reversible or irreversible) between the states \( A \) and \( B \). Here \( H_t(x, \lambda) \) denotes the full (time-dependent) Hamiltonian of the system, which is supposed to be in thermal contact with a heat bath of temperature \( T = 1/k_B \beta \), the whole ensemble being isolated, \( x = (p, q) = (p_1, \ldots, p_n, q^1, \ldots, q^n) \) denotes the phase-space point of the system (which is supposed to have \( 2n \) degrees of freedom), and \( \langle f(W) \rangle = \int f(W) P(W) \, dW \) denotes an average over the ensemble of realizations of the work \( W \) done on the system (for any “smooth” function \( f \)).

There are two parts in this derivation. The first is mathematical and will be discussed later. The result is the following: let \( E(x) \) be the (time-independent) energy of a body in thermal equilibrium with a heat bath of temperature \( T \). If its energy can be written in the additive manner \( E(x) = E_0(x) + V(x) \), which can always be achieved by making certain very general assumptions, then one can compute the free energy \( F \) of the body (and all the relevant thermodynamic quantities) from the only knowledge of the equilibrium Gibbsian distribution \( \rho_0(x) = Z_0^{-1} e^{-\beta E_0(x)} \), where \( Z_0 = \int e^{-\beta E_0(x)} \, dx \) is the partition function corresponding to the “unperturbed” energy \( E_0(x) \) of the body (and to its “unperturbed” free energy \( F_0 = -\beta^{-1} \ln Z_0 \)).
2 Thermodynamic perturbation theory

The second part of the argument, which relates the perturbation energy \( V \) to the thermodynamical work \( W \) done on the system, first in the time-independent case, and then in the time-dependent case, is physical.
Let us first assume that the energy of the body (which, again, is in thermal contact with a heat bath of temperature $T$, the whole ensemble being isolated) is time-independent, and that it can be written in the additive manner

$$E(x) = E_0(x) + V(x), \quad (3)$$

where “everything” can be calculated from the “unperturbed” energy $E_0(x)$ of the body [the partition function $Z_0 = \int e^{-\beta E_0(x)} \, dx$, the free energy $F_0 = -\beta^{-1} \ln Z_0$, etc.], but not from its “full” energy $E(x)$. Therefore, the fundamental problem is to devise a method to compute, even approximatively, the partition function $Z$, the free energy $F$, etc., corresponding to the “full” energy $E(x)$ of the system. Indeed, we will show that if Eq. (3) holds, one can compute exactly $Z$ and $F$, without anymore assumptions than Eq. (3), which relies on the separability of the Hamiltonian of the system, and on thermal equilibrium.

From these very few assumptions, the algebra is very simple. Let us first compute the partition function $Z = \int e^{-\beta E(x)} \, dx$ of the system. From Eq. (3), one gets

$$Z = Z_0 \int \rho_0(x) e^{-\beta V(x)} \, dx = Z_0 \langle e^{-\beta V} \rangle_0, \quad (4)$$

where $\rho_0(x) = Z_0^{-1} e^{-\beta E_0(x)}$ is the “unperturbed” equilibrium Gibbsian distribution of the system. Therefore, the free energy of the system is given by

$$F = -\beta^{-1} \ln Z = F_0 - \beta^{-1} \ln \langle e^{-\beta V} \rangle_0. \quad (5)$$

This result was derived up to the second order [i.e. only with the first two cumulants of $V$ in Eq. (3)] by Peierls and Bogoliubov, who gave an upper bound for the free energy $F$ of the system [indeed, Eq. (6) indicates that the “full” free energy $F$ is merely the cumulants generating function of the perturbation energy $V$]. Our result is much stronger, and looks amazingly similar to the Jarzynski equality.

### 3 Dynamical interpretation

The preceeding result holds for a time-independent and equilibrium system. However, provided that the system is initially (say, at time $t = 0$) at equilibrium, these results are also true if the energy of the system depend on time through a time-dependent perturbation energy, which can be due to an external force $\lambda$ which is switched on at times $t > 0$. Therefore, if the energy of the system reads

$$E_t(x, \lambda) = E_0(x) + V_t(x, \lambda), \quad (6)$$

where the time-dependent perturbation energy $V_t$ is only non-zero for times $t > 0$, then one trivially has

$$F_t = F_0 - \beta^{-1} \ln \langle e^{-\beta V_t} \rangle_0 \quad \text{for all} \quad t > 0, \quad (7)$$
which has exactly the same meaning as Eqs. (4)-(5).

In fact, the perturbation energy \( V_t \) can be identified to the work \( W_t \) done on the system on the time interval \((0, t)\) in a very simple way [one simply has to put \( t_A = 0 \) and \( t_B = t \) in Eq. (2), but here \( B \) does not necessarily denote an equilibrium state]. Since the energy \( E_t \) of the system depends explicitly on time only through the external force \( \lambda(t) \), one has by applying the chain rule

\[
\frac{dE_t(x, \lambda)}{dt} = \frac{\partial E_t(x, \lambda)}{\partial t} = \frac{\partial E(x, \lambda)}{\partial \lambda} \frac{d\lambda}{dt}.
\]

Therefore, by Eq. (2) one has \( W_t = E_t - E_0 = V_t \), and one finally gets

\[
F_t = F_0 - \beta^{-1} \ln \langle e^{-\beta W_t} \rangle_0,
\]

which resembles amazingly the Jarzynski equality (1).

4 Discussion and conclusion

The formula (8) obtained above is not really the Jarzynski equality (1), since the Jarzynski equality computes the free energy difference of the system from the work distribution \( P(W) \), which takes into account all the (in principle) nonequilibrium fluctuations of the work \( W \) done on the system during the transformation between the (equilibrium) states \( A \) and \( B \). In our derivation, this is obviously not the case, since the average \( \langle \cdot \rangle \) is performed from the equilibrium Gibbsian distribution \( \rho_0(x) = Z_0^{-1} e^{-\beta E_0(x)} \).

However, this slight detail perhaps throws some light on the debate started by Cohen and coworkers (4), and perhaps explains why all the experiments which has been performed in order to test the Jarzynski equality are successful, even in very defavorable (irreversible) cases (and more strikingly, when the state \( B \) is not an equilibrium one) (5).

To conclude, we should add that among the hypotheses needed to the result (8), the ergodicity of the system is of course required. However, even unrealistic, this is a rather weak statement, because we only need ergodicity at times \( t \leq 0 \), that is to say without external (time-dependent) driving force \( \lambda \).

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