Super-geometrodynamics in higher dimensions

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Abstract
The geometrodynamics of the four-dimensional Einstein and Einstein–Maxwell theories were first studied by Wheeler and Misner more than fifty years ago, by constructing solutions of the constraints on an initial spatial slice in a Hamiltonian formulation of the theories. More recently these considerations were extended to various four-dimensional theories with additional fields, encompassing cases that arise in supergravity and the low-energy limit of compactified string theory. In this paper we extend these considerations further, by constructing solutions of the initial value constraints in higher spacetime dimensions, for wide classes of theories that include supergravities and the low-energy limits of string and M-theory. We obtain time-symmetric initial data sets for multiple black hole spacetimes and also wormholes in higher dimensions.

Keywords: higher dimensions, supergravity, general relativity

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1. Introduction

The initial value formulation of general relativity provides a powerful tool for studying time-dependent solutions, especially in situations where one cannot solve the equations explicitly. The initial value constraints can place restrictions on the possible topologies and geometries of the Cauchy surface on which the initial conditions for the problem are defined. The approach was largely pioneered by Wheeler [1] and Misner [2–4], under the name of Geometrodynamics, with further early developments by Lindquist, and Brill [5, 6] and other authors.

The initial value formulation has subsequently been extended to larger systems of matter coupled to Einstein gravity, and among these, the theories encountered in supergravities are of particular interest. First of all, such extensions of Einstein gravity have the merit of being consistent with the usual positive energy theorems of classical general relativity. Furthermore, they are potentially of intrinsic physical interest if they arise as low energy limits of string theory or M-theory. Some results in one such theory, namely a particular four-dimensional Einstein–Maxwell–Dilaton (EMD) theory coming from string theory, were studied by Ortin [7]. In [11] a larger class of EMD theories and other string-related supergravities were investigated, with an emphasis on time-symmetric initial data sets. The focus in [11] was exclusively on four-dimensional theories. Clearly, in the context of supergravity, string theory and M-theory it is of interest to extend the investigation to dimensions greater than four, and that provides the motivation for the present paper. As in [11] we shall, for simplicity, focus on the
case of time-symmetric initial data; i.e. on the case where the metric is taken to be static on
the initial time surface, and the second fundamental form vanishes there.

The bulk of the earlier studies in four dimensions involved making an ansatz introduced
by Lichnerowicz [8], in which the spatial metric on the initial surface was taken to be a con-
formal factor times a fiducial static metric $\bar{g}_{ij}$, where $\bar{g}_{ij}$ might typically be the flat Euclidean
metric, or the metric on the 3-sphere, or the metric on $S^1 \times S^2$. The Hamiltonian constraint
now becomes an equation for the spatial Laplacian of the conformal factor, allowing rather
simple solutions if a suitable restriction of its coordinate dependence, adapted to the symme-
try of the fiducial metric, is imposed. By this means data sets that describe the initial data for
multiple black holes, or black holes in a closed universe, or wormholes in the $S^1 \times S^2$ case,
can be constructed.

In higher dimensions the possibilities for choosing fiducial metrics $\bar{g}_{ij}$ become more
extended. For example, if there are $d$ spatial dimensions one can write the metric on a round
d-sphere $S^d$ in a variety of different ways, such as in terms of foliations of $S^p \times S^q$ surfaces
with a ‘latitude’ coordinate $\mu$, where $p + q = d - 1$. If one then makes an assumption that
the conformal factor depends only on $\mu$, then depending upon how the integers $p$ and $q$ parti-
tion $d - 1$, one will obtain solutions of the initial data constraints corresponding to different
distributions of black hole centres. There are also many possibilities extending the $S^1 \times S^2$
wormhole choice that was considered when $d = 3$.

In this paper, we shall consider some of these higher-dimensional generalisations in some
detail. After setting up the notation for time-symmetric initial data in section 2, we turn in
section 3 to the case of the higher-dimensional vacuum Einstein equations. We construct solu-
tions of the initial-value constraints both for a flat Euclidean fiducial metric $\bar{g}_{ij}$, and for cases
with a spherical metric, described in a variety of different ways as described above. We also
consider one example, for $d = 4$, where the fiducial metric is taken to be the Fubini–Study
metric on the complex projective plane $\mathbb{C}P^2$. In section 4 we consider the Einstein–Maxwell
equations, extending this to the Einstein–Maxwell–Dilaton system in section 5. Another gen-
eralisation, to an Einstein–Dilaton system coupled to two electromagnetic fields, is considered
in section 6. In section 7 we give a rather general discussion of Einstein gravity coupled to $p$
dilatons and $q$ Maxwell fields. This encompasses many examples that arise in supergravities in
various dimensions. We describe in section 8 how the initial-value problem for these general
Einstein–Maxwell–Dilaton systems may be mapped into the initial-value problem for corre-
spanding purely Einstein–Dilaton systems.

In section 9, we turn to a consideration of initial data for wormhole solutions in higher
dimensions, generalising results in the literature on the $d = 3$ case. We consider wormholes
associated with using a fiducial metric on $S^1 \times S^{d-1}$. We obtain solutions for wormhole ini-
tial data in higher-dimensional pure Einstein, Einstein–Maxwell, and the various Einstein–
Maxwell–Dilatons systems mentioned above. We include in our discussion a calculation of the
masses and the charges for these wormhole configurations, and the interaction energies
between multiple wormhole throats. The paper ends with conclusions in section 10.

2. The constraints for time-symmetric initial data

In the general ADM decomposition, an $n$-dimensional metric $d\hat{s}^2$ is written in the form

$$d\hat{s}^2 = -N^2 \, dt^2 + g_{ij} \,(dx^i + N^i \, dt)(dx^j + N^j \, dt),$$

(2.1)

whose inverse is given by
\[
\left( \frac{\partial}{\partial \hat{s}} \right)^2 = -\frac{1}{N^2} \left( \frac{\partial}{\partial \hat{t}} - N i \partial_i \right)^2 + g^{ij} \partial_i \otimes \partial_j. \tag{2.2}
\]

The unit vector normal to the \( t = \text{constant} \) surfaces is given by
\[
n = n^\mu \partial_\mu = \frac{1}{N} \left( \frac{\partial}{\partial \hat{t}} - N i \partial_i \right). \tag{2.3}
\]

The Hamiltonian constraint for the \( n \)-dimensional Einstein equations \( \hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \hat{g}_{\mu\nu} = \hat{T}_{\mu\nu} \) is then given by the double projection with \( n^\mu n^\nu \):
\[
\hat{R}_{\mu\nu} n^\mu n^\nu + \frac{1}{2} \hat{R} = \hat{T}_{\mu\nu} n^\mu n^\nu. \tag{2.4}
\]

By the Gauss–Codacci equations this implies
\[
R + K^2 - K^{ij} K_{ij} = 2 \hat{T}_{\mu\nu} n^\mu n^\nu, \tag{2.5}
\]
where \( R \) is the Ricci scalar of the \((n-1)\)-dimensional spatial metric \( g_{ij} \) and \( K_{ij} \) is the second fundamental form. If we consider time-symmetric data on the initial surface, which we take to be at \( t = 0 \), then \( K_{ij} = 0 \) and \( N = 0 \) on this surface, and the Hamiltonian constraint becomes
\[
R = 2N^{-2} \hat{T}_{00}. \tag{2.6}
\]

Note that the momentum constraint will simply be \( \hat{T}_{0i} = 0 \).

Following a procedure introduced by Lichnerowicz [8], we may seek solutions to the Hamiltonian constraint by considering the case where the metric \( g_{ij} \) is conformally related to a fixed, time-independent background metric \( \bar{g}_{ij} \), with \( g_{ij} = \Phi^\alpha \bar{g}_{ij} \). It is straightforward to see that if we choose \( \alpha = 4/(d-2) \) then we shall have
\[
R = \Phi^{-\frac{4(d-1)}{d-2}} \left[ -4(d-1) \square + R \right] \Phi, \quad g_{ij} = \Phi^{\frac{4(d-1)}{d-2}} \bar{g}_{ij}, \tag{2.7}
\]
where \( R \) is the Ricci scalar of the background metric \( \bar{g}_{ij} \), and \( \square \) is the covariant Laplacian \( \nabla^i \nabla_i \) in the background metric. Here, and in what follows, we are using \( d \) to denote the number of spatial dimensions, so
\[
d = n - 1. \tag{2.8}
\]

3. Data for vacuum Einstein equations

If we consider the constraint equations for the pure vacuum Einstein equations then we shall simply have the Hamiltonian constraint \( R = 0 \) on the initial \( t = 0 \) surface which, from (2.7), will give the linear equation
\[
-\square \Phi + \frac{d-2}{4(d-1)} R \Phi = 0 \tag{3.1}
\]
for \( \Phi \). Any solution of this equation will give rise to consistent time-symmetric initial data for the vacuum Einstein equations. Because the equation is linear in \( \Phi \), one can of course superpose solutions.

In the bulk of this paper where we consider various matter couplings to gravity we shall study the simplest case where the background metric is just taken to be flat, with \( \bar{g}_{ij} = \delta_{ij} \). Before doing so, in this section we shall also make some observations about the vacuum Einstein case with more complicated curved background metrics.
3.1. Vacuum data with flat \( \bar{g}_{ij} \)

The simplest choice for the background metric \( \bar{g}_{ij} \) in (2.7) is to take it to be flat, with \( \bar{g}_{ij} = \delta_{ij} \). We then get vacuum initial data by taking \( \Phi \) to be any harmonic function in the flat metric, obeying

\[
\partial_i \partial_i \Phi = 0. \tag{3.2}
\]

We may therefore take \( \Phi \) to be of the form

\[
\Phi = 1 + \frac{1}{2} \sum_{n=1}^{N} \frac{M_n}{|x - x_n|^{d-2}}, \tag{3.3}
\]

where \( x \) denotes the \( d \)-vector \( x = (x_1, x_2, \cdots, x_{d-1}) \).

In general, the case with \( N \) centres corresponds to initial data for a system of \( N \) black holes at arbitrary locations that are instantaneously at rest on the initial time-symmetric surface. When \( N \) is greater than 1 the black holes will, of course, evolve as a time-dependent solutions describing colliding black holes, which one could in principle solve numerically, but solving explicitly would not be tractable. The \( N = 1 \) case with a single singularity, however, simply gives the initial data for the \( (d+1) \)-dimensional Schwarzschild solution. Taking the singularity, without loss of generality, to be at the origin (so \( x_1 = 0 \)), and taking \( M_1 = M \), then in terms of hyperspherical polar coordinates in the Euclidean \( d \)-space we have

\[
g_{ij} dx^i dx^j = \Phi^{\frac{d-1}{2}} \left( d\rho^2 + \rho^2 d\Omega_{d-1}^2 \right), \tag{3.4}
\]

with \( \Phi = 1 + \frac{M}{2\rho^{d-2}} \).

where \( \rho^2 = x^i x^i \) and \( d\Omega_{d-1}^2 \) is the metric on the unit \( (d-1) \)-sphere. To see how this corresponds to the initial data for the Schwarzschild solution, we observe that the metric in (3.4) can be written in the standard \( d \)-dimensional Schwarzschild form

\[
g_{ij} dx^i dx^j = \left( 1 - \frac{2M}{\rho^{d-2}} \right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2 \tag{3.5}
\]

if

\[
\rho^2 = r \left( 1 + \frac{M}{2\rho^{d-2}} \right)^{\frac{1}{d-1}} \quad \text{and} \quad \left( 1 - \frac{2M}{\rho^{d-2}} \right)^{-1} dr^2 = \left( 1 + \frac{M}{2\rho^{d-2}} \right)^{-\frac{1}{d-1}} d\rho^2. \tag{3.6}
\]

A straightforward calculation shows that indeed if \( r \) is given in terms of \( \rho \) by the first equation in (3.6), then the second equation is satisfied too. Note that the first equation can be inverted to give \( \rho \) as a function of \( r \), with the result

\[
\rho = r \left[ \frac{1}{2} \left( 1 + \sqrt{1 - \frac{2M}{\rho^{d-2}}} \right) \right]^\frac{1}{d-1}. \tag{3.7}
\]

3.2. Vacuum data with non-flat background \( \bar{g}_{ij} \)

In the case of four spacetime dimensions, the Lichnerowicz procedure has been used in a variety of applications, with the background metric \( \bar{g}_{ij} \) being taken to be either flat, or else the standard metric on the unit 3-sphere or on \( S^1 \times S^2 \). These latter cases have been used to construct initial data for black holes in closed universes, or for wormholes. In higher dimensions...
the possibilities for the choice of the background metric $\tilde{g}_{ij}$ are more diverse, and there are many cases where one can solve explicitly for $\Phi$ on the initial $t = 0$ surface. (Of course, it does not necessarily mean that the data will evolve into desirable solutions, but it does provide interesting cases for further investigation.) In what follows, we present some simple examples of curved background metrics.

### 3.2.1 Unit $d$-sphere background

To illustrate some of the possibilities in higher dimensions, let us first consider the case when the background metric $\tilde{g}_{ij}$ describes the unit $d$-sphere. There are many ways that this background metric can be written; here, we shall consider the cases

$$g_{ij} \, dx^i \, dx^j = ds^2 = d\mu^2 + \sin^2 \mu \, d\Omega_2^2 + \cos^2 \mu \, d\tilde{\Omega}_4^2, \quad p + q = d - 1, \quad (3.8)$$

where $d\Omega_2^2$ and $d\tilde{\Omega}_4^2$ are unit metrics on a $p$-sphere and $q$-sphere respectively. The `latitude' coordinate $\mu$ ranges from $0 \leq \mu \leq \frac{\pi}{2}$, except when $q = 0$ when it ranges over $0 \leq \mu \leq \pi$, and $p = 0$ when it ranges over $-\frac{\pi}{2} \leq \mu \leq \frac{\pi}{2}$.

The Ricci scalar of the unit $d$-sphere is given by $R = d \,(d - 1)$, and so the equation (3.1) for $\Phi$ is the Helmholtz equation

$$-\Box \Phi + \frac{1}{4} d \,(d - 2) \, \Phi = 0. \quad (3.9)$$

A simple ansatz for solving this explicitly in the (3.8) metrics is to assume $\Phi$ is a function only of the latitude coordinate $\mu$, and so (3.9) becomes

$$\Phi'' + \left( p \, \cot \mu - q \, \sin \mu \right) \Phi' - \frac{1}{4} \left( p + q + 1 \right) \left( p + q - 1 \right) \Phi = 0, \quad (3.10)$$

where a prime denotes a derivative with respect to $\mu$.

If we consider the simplest case where $q = 0$ and hence $d = p + 1$, the unit $d$-sphere is viewed as a foliation by $(d - 1)$-spheres. The solution to (3.10) can be written as

$$\Phi = \frac{c_1}{\left( \cos \frac{1}{2} \mu \right)^{p-1}} + \frac{c_2}{\left( \sin \frac{1}{2} \mu \right)^{p-1}}, \quad (3.11)$$

where $c_1$ and $c_2$ are arbitrary constants. The first term has a singularity at the north pole, and the second term is singular at the south pole. If we choose $c_1 = c_2 = \sqrt{M} \, 2^{-\frac{3}{2}(p-1)}$, so that

$$\Phi = \sqrt{\frac{M}{2}} \, 2^{-\frac{3}{2}(p-1)} \left[ \frac{1}{\left( \cos \frac{1}{2} \mu \right)^{p-1}} + \frac{1}{\left( \sin \frac{1}{2} \mu \right)^{p-1}} \right], \quad (3.12)$$

then after defining a new radial variable $r$ by letting

$$r^{\frac{3}{2}(p-1)} = \sqrt{\frac{M}{2}} \left[ \left( \tan \frac{1}{2} \mu \right)^{\frac{3}{2}(p-1)} + \left( \cot \frac{1}{2} \mu \right)^{\frac{3}{2}(p-1)} \right], \quad (3.13)$$

the spatial $d$-metric, given as in (2.7), becomes

$$ds^2 = \Phi^{\frac{1}{2}2\pi} \left( d\mu^2 + \sin^2 \mu \, d\Omega_2^2 \right) = \left( 1 - \frac{2M}{r^{d-2}} \right)^{-1} dr^2 + r^2 \, d\tilde{\Omega}_d^2. \quad (3.14)$$

*Note that the hyperspherical harmonics on the unit $d$-sphere obey $-\Box Y = \lambda Y$ with eigenvalues $\lambda = \ell \,(\ell + d - 1)$ and $\ell = 0, 1, 2, \ldots$ and since none of these eigenvalues coincide with the eigenvalue in (3.9) (which is in fact negative), the solutions for $\Phi$ that we are seeking will necessarily have singularities on the sphere.*
This can be recognised as the time-symmetric initial data for the \((d+1)\)-dimensional generalisation of the Schwarzschild black hole. The horizon of the black hole corresponds to the equator, \(\mu = \frac{1}{2} \pi\).

Of course since the metric on a round \(d\)-sphere is conformally related to the flat Euclidean \(d\)-metric in any dimension, we can straightforwardly relate the initial data we constructed here to the previous initial data for the Schwarzschild black hole that we constructed in section 3.1 using a flat background metric. The Euclidean and sphere metrics are related by

\[
d\rho^2 + \rho^2 d\Omega_{d-1}^2 = \Omega^2 (d\mu^2 + \sin^2 \mu \, d\Omega_{d-1}^2)
\]

where

\[
\Omega = \frac{c^2}{\cos^2 \frac{1}{2} \mu}, \quad \rho = 2c^2 \tan \frac{1}{2} \mu, \tag{3.16}
\]

where \(c\) is an arbitrary constant, and using this one can easily verify that the \(\Phi\) functions given in (3.4) and (3.12) for the flat and the spherical background metrics are related by

\[
\Phi_{\text{sphere}} = \Phi_{\text{flat}} \Omega^\frac{d-2}{d-1}. \tag{3.17}
\]

In the manner described, for example, in [9, 10], one could take a superposition of solutions for \(\Phi\) like the one in (3.12), but expressed in terms of a rotated choice of polar axes for the \(d\)-sphere. By this means, one could set time-symmetric initial data for multiple black holes.

We can also consider the solutions of the equation (3.10) for \(\Phi\) in the case where \(p\) and \(q\) are both non-zero. In this description of the \(d = p + q + 1\) sphere it is foliated by \(S^p \times S^q\) surfaces. The solutions are given in terms of hypergeometric functions by

\[
\Phi = c_1 F \left[ \frac{p + q - 1}{4}, \frac{p + q + 1}{4}, \frac{q + 1}{2}; \cos^2 \mu \right] + c_2 (\cos \mu)^{1-q} F \left[ \frac{p - q + 1}{4}, \frac{p - q + 3}{4}, \frac{3 - q}{2}; \cos^2 \mu \right]. \tag{3.18}
\]

The explicit solutions are of varying complexity depending on the choice of \(p\) and \(q\). A fairly simple example is when \(p = q = 2\). In this case, the solution to (3.10) for \(\Phi\) for this metric on the unit 5-sphere is given by

\[
\Phi = \frac{1}{\cos \mu} \left( \frac{a_1}{\sin \frac{1}{2} \mu} + \frac{a_2}{\cos \frac{1}{2} \mu} \right). \tag{3.19}
\]

This has a simple pole singularity in a 3-plane times a 2-sphere surface at \(\mu = 0\) and a simple pole singularity on another 3-plane times 2-sphere surface at \(\mu = \frac{1}{2} \pi\). (Recall here the 5-sphere is spanned by \(0 \leq \mu \leq \frac{1}{2} \pi\), with a foliation of different 2-spheres contracting onto the origin of a 3-plane at each endpoint.) The initial data described by (3.19) would correspond to taking certain continuous superpositions of elementary mass-point initial data of the kind we discussed previously.

3.2.2. \(\mathbb{C}P^2\) background. Further possibilities for background metrics that could give explicitly solvable time-symmetric initial data include taking \(\bar{g}_y\) to be a metric on a product of spheres, or else taking a metric on a complex projective space or products of these, possibly with spheres as well. As a simple example, consider the case \(d = 4\) with \(\bar{g}_y\) taken to be the Fubini–Study metric on \(\mathbb{C}P^2\). The metric
\[ ds^2 = d\mu^2 + \frac{1}{4} \sin^2 \mu \cos^2 \mu \left( d\psi + \cos \theta \, d\phi \right)^2 + \frac{1}{4} \sin^2 \mu \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

(3.20)
is Einstein with \( \bar{R}_{ij} = 6\bar{g}_{ij} \) and hence \( \bar{R} = 24 \). From (3.1) we have \(-\Box \Phi + 4 \Phi = 0\), and so if we assume \( \Phi \) depends only on \( \mu \) we have

\[ \Phi'' + (3 \cot \mu - \tan \mu) \Phi' - 4 \Phi = 0, \]

(3.21)
for which the general solution is

\[ \Phi = \frac{c_1}{\sin^2 \mu} + \frac{c_2 \log \cos \mu}{\sin^2 \mu}. \]

(3.22)
The first term exhibits the leading-order \( 1/\mu^2 \) behaviour of a point charge at the NUT at \( \mu = 0 \), while the second term has the characteristic \( \log(\frac{1}{2} \pi - \mu) \) behaviour of a charge distributed over the bolt at \( \mu = \frac{1}{2} \pi \). It would be interesting to investigate what this time-symmetric initial data describes in this, and other cases.

4. Einstein–Maxwell equations

Consider the Einstein–Maxwell system in \( n = (d + 1) \) dimensions, described by the action

\[ I = \int d^n x \sqrt{-\hat{g}} (\hat{R} - \hat{F}^2), \]

(4.1)
where \( \hat{F}^2 = \hat{g}^{\mu \rho} \hat{g}^{\nu \sigma} \hat{F}_{\mu \nu} \hat{F}_{\rho \sigma} \), for which the equations of motion are

\[ \hat{R}_{\mu \nu} - \frac{1}{2} \hat{R} \hat{g}_{\mu \nu} = 2(\hat{F}_{\mu \rho} \hat{F}_{\nu \sigma} \hat{g}^{\rho \sigma} - \frac{1}{4} \hat{F}^2 \hat{g}_{\mu \nu}), \quad \hat{\nabla}_\mu \hat{F}^{\mu \nu} = 0. \]

(4.2)
We shall consider the evolution of purely electric time-symmetric initial data, for which only the components of \( \hat{F}_{\mu \nu} \) specified by

\[ n^\mu \hat{F}_{\mu i} = -E_i \]

(4.3)
are non-zero. Projecting the Einstein equations in (4.2) with \( n^n n^\nu \) then gives the Hamiltonian constraint

\[ \hat{R} = 2\hat{E}^2, \]

(4.4)
where \( \hat{E}^2 = \hat{g}^{ij} E_i E_j \). We also have the Gauss law constraint \( \nabla_i E_i = 0 \).

Generalising a discussion of Misner and Wheeler [2] to the case of general dimensions, we may seek a solution of the constraint equations, with \( g_{ij} \) given as in (2.7) and \( \bar{g}_{ij} = \delta_{ij} \) as before, in the form

\[ \Phi = (CD)^\alpha, \quad E_i = \beta \partial_i \log \frac{C}{D}. \]

(4.5)
The aim is to choose the constants \( \alpha \) and \( \beta \) appropriately, such that the constraint equations will be satisfied if \( C \) and \( D \) are arbitrary harmonic functions in the flat background Euclidean metric \( \delta_{ij} \). Substituting \( R \) from (2.7) (with \( R = 0 \) since we are taking a flat background metric \( \bar{g}_{ij} = \delta_{ij} \)) into (4.4), together with (4.5), it is straightforward to see that if we choose

\[ \alpha = \frac{1}{2}, \quad \beta = \sqrt{\frac{d - 1}{2(d - 2)}}, \]

(4.6)
then the Hamiltonian constraint is indeed satisfied if $C$ and $D$ are arbitrary harmonic functions in the Euclidean background metric $g_{ij} = \delta_{ij}$, i.e.

\[ \partial_i \partial_i C = 0, \quad \partial_i \partial_i D = 0. \]  

(4.7)

Furthermore, the Gauss law constraint $\nabla_i E^i = 0$ is also satisfied subject to (4.7). If $C$ and $D$ are taken to be sums over elementary harmonic functions corresponding to a set of mass points then the initial data set will describe an instantaneously static set of charged black holes, whose subsequent time evolution would in general be solvable only numerically, since the black holes would attract one another. Unlike the pure Einstein case described in section 8.1, here one can obtain multi black holes that are static for all time, and which are explicitly solvable, by choosing the parameters in the harmonic functions appropriately so that the gravitational attractions are exactly balanced by the electrostatic repulsions. The corresponding solutions will be the higher-dimensional generalisations of the Papapetrou–Majumdar solutions [12, 13]. We shall not present the initial data for these explicitly here, since they will be encompassed as a special case within the Einstein–Maxwell–Dilaton theories that we shall describe in section 5.

If we take the case where $C$ and $D$ both have a single pole at the origin, then in hyperspherical polar coordinates we can take

\[ C = 1 + \frac{M - Q}{2 \rho^{d-3}}, \quad D = 1 + \frac{M + Q}{2 \rho^{d-3}}. \]  

(4.8)

The spatial metric takes the form

\[ g_{ij} dx^i dx^j = \left( CD \right)^{\frac{d-3}{2}} \left( d \rho^2 + \rho^2 d\Omega_{d-1}^2 \right), \]  

(4.9)

and if we define the area coordinate $r = \rho \left( CD \right)^{1/(d-2)}$ this becomes

\[ g_{ij} dx^i dx^j = \left( 1 - \frac{2M}{\rho^{d-2}} + \frac{Q^2}{\rho^{2(d-3)}} \right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2, \]  

(4.10)

which can be recognised as the spatial part of the $(d + 1)$-dimensional Reissner–Nordström metric.

5. Einstein–Maxwell–Dilaton system

A rather general class of theories that are relevant in string theory are encompassed by the Einstein–Maxwell–Dilaton (EMD) system in $n = d + 1$ dimensions, described by the Lagrangian

\[ L = \sqrt{-g} \left( R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} e^{a\phi} F^2 \right), \]  

(5.1)

where $a$ is an arbitrary constant. The Hamiltonian and Gauss law constraints will be

\[ R = \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi + \frac{1}{2} e^{a\phi} g^{ij} E_i E_j, \]  

\[ 0 = \nabla_i \left( e^{a\phi} g^{ij} E_j \right). \]  

(5.2)

As usual, we shall consider a flat Euclidean background metric, with $g_{ij} = \Phi^{4/(d-2)} \delta_{ij}$. Upon using (2.7) the Hamiltonian constraint becomes

\[ \frac{4(d-1)}{(d-2)} \partial_i \partial_i \Phi + \frac{1}{2} \Phi \left[ \partial_i \phi \partial_i \phi + e^{a\phi} E_i E_j \right] = 0. \]  

(5.3)
It is straightforward to verify that we can solve the Hamiltonian and Gauss law constraints by introducing three arbitrary harmonic functions \( C, D \) and \( W \) in the Euclidean \( d \)-space, in terms of which we write

\[
\Phi = (CD)^{\frac{(d-2)}{2d} \cdot \frac{1}{2}} \cdot \frac{W^2}{\pi},
\]

\[
e^{a\phi} = \left( \frac{CD}{W^2} \right)^{\frac{1}{2d}},
\]

\[
E_{i} = \frac{2}{\sqrt{\Delta}} \left( \frac{CD}{W^2} \right)^{-\frac{1}{2}} \partial_{i} \log \frac{C}{D},
\]

where we have introduced the parameter \( \Delta \), as in [14], which is related to \( a \) by the expression

\[
a^2 = \Delta - \frac{2(d - 2)}{d - 1}.
\]

If general multi-centre harmonic functions are chosen for \( C, D \) and \( W \) the initial data will describe a configuration of black holes that are instantaneously at rest. As in the previous cases we discussed, these would in general evolve into time-dependent solutions describing colliding black holes, except in special cases that we discuss below.

### 5.1. Non-extremal black hole

The solution for a static non-extremal black hole in the EMD theory with arbitrary dilaton coupling \( a \) in \( n = d + 1 \) dimensions was constructed in [15], and is given by

\[
ds^2 = -h f^{-2(d - 2)} \, dt^2 + f^2 h^{-1} \, dr^2 + r^2 f^2 \, d\Omega_{d-1}^2,
\]

\[
h = 1 - \left( \frac{r_H}{r} \right)^{d-2}, \quad f = \left( 1 + \frac{\alpha}{rd^2} \right)^{-\frac{1}{\sqrt{1 + rd^{-2}}}},
\]

\[
e^{a\phi} = \left( 1 + \frac{\alpha}{rd^2} \right)^{\frac{1}{4}}, \quad A = \frac{2}{\sqrt{\Delta}} \sqrt{1 + \frac{r_H^2}{\alpha} \left( 1 + \frac{\alpha}{rd^2} \right)^{-1}} \, dt,
\]

where \( \Delta \) is defined in (5.5).

Defining a new radial coordinate \( \rho \) by

\[
r^{d-2} = \rho^{d-2} \left( 1 + \frac{u\rho}{\rho^{d-2}} \right)^2,
\]

where \( u \) and \( v \) are constants related to the horizon radius \( r_H \) and the parameter \( \alpha \) by

\[
r_H^{d-2} = 4uv, \quad \alpha = (u - v)^2,
\]

a straightforward calculation show that the metric (5.6) becomes

\[
ds^2 = -N^2 \, dt^2 + \Phi^{\frac{1}{2d}} (d\rho^2 + \rho^2 \, d\Omega_{d-1}^2),
\]

with

\[
\Phi = \left[ \left( 1 + \frac{u^2}{\rho^{d-2}} \right) \left( 1 + \frac{v^2}{\rho^{d-2}} \right) \right]^{\frac{1}{2d}} \left( 1 + \frac{u\rho}{\rho^{d-2}} \right)^{\frac{1}{4}},
\]

\[
N = \left( 1 - \frac{u\rho}{\rho^{d-2}} \right)^2 \left[ \left( 1 + \frac{u^2}{\rho^{d-2}} \right) \left( 1 + \frac{v^2}{\rho^{d-2}} \right) \right]^{\frac{2(d-1)}{4d-2}} \left( 1 + \frac{u\rho}{\rho^{d-2}} \right)^{\frac{4(d-2)}{4d-2} - 1}.
\]

(5.10)
Comparing with (5.4), we see that the static non-extremal black hole is generated by starting from the initial data in which the harmonic functions $C$, $D$ and $W$ are taken to be

$$\begin{align*}
C &= 1 + \frac{u^2}{\rho^{d-2}}, \\
D &= 1 + \frac{v^2}{\rho^{d-2}}, \\
W &= 1 + \frac{uv}{\rho^{d-2}}.
\end{align*}$$

(5.11)

If one takes more general solutions for the initial data, with $C$, $D$ and $W$ having singularities at multiple locations, the evolution would give rise to time-dependent solutions that could be constructed only numerically. However, if one takes very specific initial data with multiple singularities, it can give rise to static solutions. This will happen in the case of initial data for the multi-centre extremal black holes, discussed below:

5.2. Multi-centre extremal black holes

The static multi-centre extremal black holes in $n = d + 1$ dimensions are given by

$$\begin{align*}
ds^2 &= -C^{(d-2)} \frac{dt^2}{\Delta} + C^{\frac{1}{d-1}} \sum_i dy_i dy_i, \\
A &= \frac{2}{\sqrt{\Delta}} C^{-\frac{1}{2}} dt, \\
e^{i\phi} &= C^{\frac{d-2}{2}},
\end{align*}$$

(5.12)

where $C$ is an arbitrary harmonic function in the $d$-dimensional Euclidean space with metric $dy_i dy_i$. Comparison with (5.4) shows that indeed the harmonic function $C$ provides the initial data for these solutions, with $D = W = 1$.

6. Einstein-two-Maxwell–Dilaton system

An extension of the Einstein–Maxwell–Dilaton system containing two Maxwell fields, with just one dilaton, is of considerable interest. The theory, which we shall refer to by the acronym E2MD, has the Lagrangian

$$\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{a\phi} F_1^2 - \frac{1}{4} e^{b\phi} F_2^2 \right).$$

(6.1)

The theory and its black hole solutions were studied extensively in $n = d + 1$ dimensions in [16]. It is convenient to parameterise the dilaton coupling constants $a$ and $b$ as

$$\begin{align*}
a^2 &= \frac{4}{N_1} - \frac{2(d-2)}{d-1}, \\
b^2 &= \frac{4}{N_2} - \frac{2(d-2)}{d-1}.
\end{align*}$$

(6.2)

It was found that while black hole solutions with both field strengths carrying charge cannot be found explicitly for general values of $a$ and $b$, they can be obtained if

$$ab = -\frac{2(d-2)}{d-1},$$

(6.3)

and we shall assume this from now on. This condition also implies

$$a N_1 + b N_2 = 0, \quad N_1 + N_2 = \frac{2(d-1)}{d-2}. $$

(6.4)

The Hamiltonian and Gauss law constraints will be

$$R = \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi + \frac{1}{2} e^{a\phi} g^{ij} E_i^1 E_j^1 + \frac{1}{2} e^{b\phi} g^{ij} E_i^2 E_j^2,$$

(6.5)
\[ 0 = \nabla_i (e^{\phi} g^{ij} E^i_j), \quad 0 = \nabla_i (e^{\phi} g^{ij} E^j_i). \]  \hspace{1cm} (6.6)

We shall again consider a flat Euclidean background metric, with \( g_{ij} = \Phi^{4/(d-2)} \delta_{ij} \). Upon using (2.7) the Hamiltonian constraint (6.5) becomes

\[ \frac{4(d-1)}{(d-2)} \partial_i \partial_j \Phi + \frac{1}{2} \Phi \left[ \partial_i \phi \partial_i \phi + e^{\phi} E^i_i E^i_i + e^{\phi} E^2 E^2 \right] = 0. \]  \hspace{1cm} (6.7)

It is now a straightforward exercise to make an appropriate ansatz for solving the Hamiltonian and Gauss law constraints in terms of harmonic functions, and then to solve for the various exponents in the ansatz in order to satisfy the constrain equations. Motivated by the form of the ansatz that was employed in four dimensions in [11], we have made an ansatz here involving four harmonic functions, \( C_1, D_1, C_2 \) and \( D_2 \), and we find we can solve the Gauss law constraints (6.6) and the Hamiltonian constraint (6.7) by writing

\[
\begin{align*}
\Phi &= (C_1 D_1)^{\frac{(d-2)N_1}{2(d-2)}} (C_2 D_2)^{\frac{(d-2)N_2}{2(d-2)}}, \\
e^{\phi} &= (C_1 D_1)^{\frac{1}{4}n_1} (C_2 D_2)^{\frac{1}{4}n_2} = \left( \frac{C_1 D_1}{C_2 D_2} \right)^{\frac{1}{4}n_1}, \\
E^1_i &= \sqrt{N_1} \left( \frac{C_1 D_1}{C_2 D_2} \right)^{\frac{(d-1)N_1}{2(d-2)}} \partial_i \log \frac{C_1}{D_1}, \\
E^2_i &= \sqrt{N_2} \left( \frac{C_2 D_2}{C_1 D_1} \right)^{\frac{(d-1)N_2}{2(d-2)}} \partial_i \log \frac{C_2}{D_2}.
\end{align*}
\]  \hspace{1cm} (6.8)

### 6.1. Static non-extremal black hole

The spherically-symmetric non-extremal black hole solutions in the E2MD theory, when \( a \) and \( b \) obey the relation (6.3), can be found in [16]. They are given by

\[
\begin{align*}
ds^2 &= -(H_1^{N_1} H_2^{N_2})^{\frac{1}{(d-1)}} f \, dr^2 + (H_1^{N_1} H_2^{N_2})^{\frac{1}{(d-1)}} (f^{-1} \, dr^2 + r^2 \, d\Omega^2_{d-2}), \\
A_1 &= \frac{\sqrt{N_1} c_1}{s_1} H_1^{-1} \, dt, \\
A_2 &= \frac{\sqrt{N_2} c_2}{s_2} H_2^{-1} \, dt, \\
\phi &= \frac{1}{2} N_1 a_1 \log H_1 + \frac{1}{2} N_2 a_2 \log H_2, \\
f &= 1 - \frac{\mu}{r^{d-3}}.
\end{align*}
\]  \hspace{1cm} (6.9)

where we are using a standard notation where \( s_i = \sinh \delta_i \) and \( c_i = \cosh \delta_i \). If we make the coordinate transformation

\[ r^{d-2} = \rho^{d-2} \left( 1 + \frac{\mu}{4 \rho^{d-2}} \right)^2, \]  \hspace{1cm} (6.10)

the metric can be cast into the standard static form (5.9), with \( \Phi \) and \( \phi \) given as in (6.8), where the harmonic functions take the specific forms

\[
\begin{align*}
C_1 &= 1 + \frac{\mu s_1^2}{4 \rho^{d-2}}, \\
D_1 &= 1 + \frac{\mu e^{-2\delta_1}}{4 \rho^{d-2}}, \\
C_2 &= 1 + \frac{\mu s_2^2}{4 \rho^{d-2}}, \\
D_2 &= 1 + \frac{\mu e^{-2\delta_2}}{4 \rho^{d-2}}.
\end{align*}
\]  \hspace{1cm} (6.11)
The metric function \( g_{00} = -N^2 \) is given by
\[
N^2 = \left( 1 - \frac{\mu^2}{16 \rho^2(d-2)} \right)^2 \left[ (C_1 D_1)^{N_1} (C_2 D_2)^{N_2} \right]^{-\frac{d-2}{2}}.
\] (6.12)

After some calculation, one can verify that the field strengths in this non-extremal solution indeed imply that \( E_1^i \) and \( E_2^i \) in the initial-value data are consistent with the expressions we found in (6.8) for the general ansatz with four independent harmonic functions. Thus we conclude that in the special case where the four harmonic functions take the particular form given in (6.11), they give rise to the initial data for the non-extremal black hole (6.9).

### 7. Gravity with \( p \) Dilatons and \( q \) Maxwell fields

#### 7.1. The theories

A general class of theories that encompasses the relevant bosonic sectors of various supergravities is provided by considering the Lagrangian
\[
L = \sqrt{-g} \left( E - \frac{1}{2} \sum_{\alpha=1}^{p} (\partial \phi_{\alpha})^2 - \frac{1}{4} \sum_{I=1}^{q} X_I^{-2} F_{(i)}^2 \right), \quad X_I = e^{-\frac{1}{2}a_I \cdot \vec{\phi}}
\] (7.1)
in \( n = d + 1 \) spacetime dimensions, where \( \vec{\phi} = (\phi_1, \phi_2, \cdots, \phi_p) \) is the \( p \)-vector of dilaton fields, and \( a_I \) is a set of \( q \) constant dilaton \( p \)-vectors that characterise the couplings of the dilatons to the \( q \) Maxwell fields \( F_{(i)} \). We can obtain multi-centre BPS black hole solutions, and spherically-symmetric static non-extremal black hole solutions, whenever the dilaton vectors obey the relations [17]
\[
\vec{a}_I \cdot \vec{a}_J = 4 \delta_{IJ} - 2(d-2) \frac{\beta}{d-1}.
\] (7.2)

We shall assume the dilaton vectors obey this relation from now on.

For a given dimension and a given number \( p \) of dilaton fields, the most general theory of the form (7.1) will correspond to the case where \( q \) is chosen to be as large as possible, subject to the set of dilaton vectors \( \vec{a}_I \) obeying (7.2). Obviously, one can always find \( p \) such \( p \)-vectors. To see this, let \( \vec{e}_I \) be an orthonormal basis in \( \mathbb{R}^p \), where the vector \( \vec{e}_I \) has an entry 1 at the \( I \)th position, with all other components zero. Thus \( \vec{e}_I \cdot \vec{e}_J = \delta_{IJ} \). If we define \( \vec{e} \equiv \sum I \vec{e}_I \) then clearly the vectors
\[
\vec{a}_I = \alpha \vec{e}_I + \beta \vec{e} \tag{7.3}
\]
will obey the relations
\[
\vec{a}_I \cdot \vec{a}_J = \alpha^2 \delta_{IJ} + 2\alpha \beta + p \beta^2. \tag{7.4}
\]
Solving for \( \alpha \) and \( \beta \) such that this reproduces (7.2), we find
\[
\alpha = 2, \quad \beta = -\frac{2}{p} \pm \frac{2}{p} \sqrt{1 - \frac{(d-2)p}{2(d-1)}}. \tag{7.5}
\]

The only further question is whether one can find a set of more than these \( p \) such vectors, that all obey (7.2). Any additional dilaton vector or vectors, over and above the \( p \) already constructed above, would necessarily have to be a linear combination of the first \( p \) dilaton vectors. Let us suppose that such an additional dilaton vector \( \vec{a}_I \) existed, over and above the \( p \) dilaton vectors \( \vec{a}_I \) with \( 1 \leq I \leq p \). Thus we must have
\[ \vec{u} = \sum_{i=1}^{p} c_i \vec{a}_i, \]  

(7.6)

where \( c_i \) are some constants. The \( \vec{a}_i \) satisfy (7.2), and we must also require

\[ \vec{u} \cdot \vec{u} = 4 - \frac{2(d-2)}{d-1}, \quad \vec{u} \cdot \vec{a}_i = -\frac{2(d-2)}{d-1}. \]  

(7.7)

Using (7.2) we easily see that these conditions imply

\[ c_i = -1, \quad p = \frac{d}{d-2}. \]  

(7.8)

and so we only have solutions with integer \( p \) if \((d, p) = (3, 3)\) or \((d, p) = (4, 2)\). Thus in four spacetime dimensions we can have a theory of the type (7.1), with three dilatons and four Maxwell fields. This corresponds to the bosonic subsector of four-dimensional STU supergravity in which the three additional axionic scalars are set to zero. In five spacetime dimensions we can have a theory of the type (7.1) with two dilatons and three Maxwell fields. This corresponds to a bosonic subsector of five-dimensional STU supergravity. In all other cases, the requirement that the dilaton vectors in the Lagrangian (7.1) obey the relations (7.2) restricts us to having only \( p \) Maxwell fields when there are \( p \) dilatonic scalar fields.

72. Ansatz for initial-value constraints

The time-symmetric initial-value constraints for the theory (7.1) are

\[ R = \frac{1}{2} g^{ij} \partial_i \vec{\phi} \cdot \partial_j \vec{\phi} + \frac{1}{2} g^{ij} \sum_{I=1}^{q} X_I^{-2} E_I^i E_I^j, \]  

(7.9)

\[ 0 = \nabla_i (g^{ij} X_I^{-2} E_I^j). \]  

(7.10)

We shall, as usual, use a flat background \( d \)-metric, and so we write

\[ g_{ij} = \Phi \delta_{ij}. \]  

(7.11)

The ansatz for the initial-value data for four-dimensional STU supergravity was discussed in [11]; it involved a total of eight arbitrary harmonic functions (two for each of the four Maxwell fields). Multi-centre BPS black holes were constructed in arbitrary dimensions for the theories described by (7.1), with dilaton vectors obeying (7.2), in [18], and these provide a useful guide for writing an ansatz for initial-value data in the general case. One can also construct spherically-symmetric non-extremal black hole solutions in all dimensions. These solutions provide further guidance for writing an ansatz for initial-value data in general dimensions. In particular, we find that in order to encompass an initial-value formulation for these non-extremal solutions, we must go beyond the natural-looking generalisation of the four-dimensional STU supergravity example that would involve \( 2q \) harmonic functions for the \( q \) Maxwell fields. Namely, we must introduce one further arbitrary harmonic function, which we shall call \( W \).

After some experimentation, we are led to consider the following ansatz for the initial data:

\[ \Phi = \Pi \frac{\phi}{\sum_i \phi^i} \, W, \quad \vec{\phi} = \frac{1}{2} \sum_{i=1}^{q} \vec{a}_i \log(C_i D_i) - \vec{a} \log W, \quad E_I^i = \frac{\Phi^2}{C_i D_i} \partial_i \log \frac{C_i}{D_i}, \]  

(7.12)

\[ \Pi \equiv \prod_{i=1}^{q} C_i D_i, \quad \gamma \equiv 1 - \frac{q(d-2)}{2(d-1)}, \quad \vec{a} \equiv \sum_{i=1}^{q} \vec{a}_i. \]
(Note that for four-dimensional STU supergravity we have \(d = 3\) and \(q = 4\), implying \(\gamma = 0\), and \(\vec{a} = 0\), so the additional harmonic function \(W\) is absent in this special case.)

One can easily verify from the definition of \(X_I\) in (7.1), and using the relations (7.2), that the ansatz for \(\vec{\phi}\) in (7.12) implies

\[
X_I^{-2} = \Phi^{-4} (C_I D_I)^2,
\]

and then it is easy to see that the Gauss law constraints (7.10) are satisfied if the functions \(C_I\) and \(D_I\) are harmonic in the flat background metric,

\[
\partial_i \partial_i C_I = 0, \quad \partial_i \partial_i D_I = 0.
\]

Using (2.7), we find, upon substituting the ansätze (7.12) into the Hamiltonian constraint (7.9) that it gives

\[
-\Pi^{-1} \partial_i \partial_i \Pi + (\partial_i \log \Pi)^2 - \frac{4\gamma (d - 1)}{d - 2} W^{-1} \partial_i \partial_i W = \frac{1}{2} \sum_{I=1}^{q} \left\{ [\partial_i \log (C_I D_I)]^2 + [\partial_i \log (C_I D_I)]^2 \right\}.
\]

After some algebra, we find that this is indeed satisfied if the functions \(W, C_I\) and \(D_I\) are harmonic, obeying \(\partial_i \partial_i W = 0\) and (7.14). Thus we have established that (7.12) indeed gives a solution of the initial-value constraints (7.9) and (7.10), where \(W, C_I\) and \(D_I\) are arbitrary harmonic functions in the Euclidean background metric.

As mentioned earlier, special cases of the theories we are considering in this section include the gravity, dilaton and Maxwell-field sectors of four-dimensional STU supergravity (with \((d, p, q) = (3, 3, 4)\)) and five-dimensional STU supergravity (with \((d, p, q) = (4, 2, 3)\)). In both of these cases the constant \(\gamma\) in (7.12) is zero, and so the harmonic function \(W\) does not arise in the initial-data ansatz. The four-dimensional STU supergravity case was discussed in [11]. Some other special cases also correspond to the gravity, dilaton and Maxwell-field sectors of supergravities. These include a six-dimensional case with \((d, p, q) = (5, 2, 2)\) and a seven-dimensional case with \((d, p, q) = (6, 2, 2)\).

7.3. Extremal multi-centre black holes

As was discussed in general in [18], these solutions are given by

\[
\text{d}s^2 = -H^{-\frac{2\gamma}{d-2}} \text{d}t^2 + H^{\frac{1}{d-1}} \text{d}x^i \text{d}x^i,
\]

\[
\vec{\phi} = \frac{1}{2} \sum_{I=1}^{q} \vec{a}_I \log H_I, \quad A^I = -H_I^{-1} \text{d}t,
\]

where the \(H_I\) are arbitrary harmonic functions in the Euclidean metric \(\text{d}x^i \text{d}x^i\). Clearly this solution matches with the initial data in (7.12), in the special case with

\[
C_I = H_I, \quad D_I = 1, \quad W = 1.
\]

7.4. Non-extremal spherically-symmetric black holes

Non-extremal spherically-symmetric black holes solutions can easily be found in the theory defined by (7.1) and (7.2), and they are given by
\[ ds^2 = -H^{-\frac{2\eta}{d-2}} f \, dr^2 + H^{\frac{2\eta}{d-2}} \left( \frac{dr^2}{f} + r^2 \, d\Omega^2_{d-1} \right), \]
\[ \tilde{\phi} = \frac{1}{2} \sum_{I=1}^{s} d_I \log H_I, \quad A^I = (1 - H^{-1}_I) \coth \delta_I \, dt, \quad H_I = 1 + \frac{2 m \sinh^2 \delta_I}{\rho^{d-2}}. \] (7.18)

Introducing a new radial variable \( \rho \) by
\[ \rho^{d-2} = \rho^{d-2} \left( 1 + \frac{m}{2 \rho^{d-2}} \right)^2, \] (7.19)
we find that the metric \( ds^2 \) in (7.18) becomes
\[ ds^2 = -N^2 \, dt^2 + \Phi^{\frac{4}{d-2}} (d\rho^2 + \rho^2 \, d\Omega^2_{d-1}), \] (7.20)
where \( \Phi \) is given in (7.12) with the harmonic functions \( C_I, D_I \) and \( W \) being given by
\[ C_I = 1 + \frac{m \coth^2 \delta_I}{2 \rho^{d-2}}, \quad D_I = 1 + \frac{m \coth^{-2} \delta_I}{2 \rho^{d-2}}, \quad W = 1 + \frac{m}{2 \rho^{d-2}}. \] (7.21)

and
\[ N^2 = \Phi^{-4} W^2 \left( 1 - \frac{m}{2 \rho^{d-2}} \right)^2. \] (7.22)

The functions \( H_I \) in (7.18) are given by \( H_I = W^{-2} C_I D_I \), and hence the potentials \( A^I \) in the non-extremal solution are simply given by
\[ A^I = \left( -\frac{1}{C_I} + \frac{1}{D_I} \right) \, dt, \] (7.23)
and the dilatonic scalars are given by the expression in (7.12). Thus we see that the non-extremal spherically-symmetric black hole solutions do indeed have initial data given by (7.12), with the harmonic functions \( C_I, D_I \) and \( W \) taking the special spherically-symmetric form (7.21).

8. Mapping Einstein–Maxwell–Dilaton to Einstein-scalar

It was observed in [7], and developed further in [11], that the time-symmetric initial data for a system of gravity coupled to Maxwell fields and dilatonic scalars can straightforwardly mapped into the time-symmetric initial data for an extended system of scalar fields coupled to gravity. Although [7, 11] discussed this specifically for four-dimensional spacetimes, the extension to arbitrary dimensions is immediate.

8.1. Mapping of Einstein–Maxwell–Dilaton data

The mapping can be illustrated by considering the EMD theories with multiple dilatons and Maxwell fields that we discussed in section 7. Making the replacement
\[ E_i^I \rightarrow X_i^{-1} \partial_i \psi_I. \] (8.1)
the Hamiltonian constraint (7.9) becomes
\[ R = \frac{1}{2} g^{ij} (\partial_i \bar{\phi} \cdot \partial_j \bar{\phi} + \partial_i \psi_I \partial_j \psi_I), \] (8.2)
which is the same as the Hamiltonian constraint for a system of free scalar fields $(\vec{\phi}, \psi_I)$ coupled to gravity. In view of (7.13), the ansatz for $E^I_j$ in (7.12) becomes simply
\[
\psi_I = \log \frac{C_I}{D_I}
\] (8.3)
for the scalar fields $\psi_I$. Furthermore, under (8.1) the Gauss law constraints (7.10) give simply
\[
\partial_i \left( C_I D_I \partial_i \psi_I \right) = 0,
\] (8.4)
and so these are indeed satisfied when $\psi_A$ is given by (8.3), since $C_I$ and $D_I$ are harmonic.

8.2. General N-scalar system coupled to gravity

If we consider a general system of $N$ scalar fields $\sigma_A$, $1 \leq A \leq N$, coupled to gravity and described by the Lagrangian
\[
\mathcal{L} = \sqrt{-\hat{g}} \left( \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \sigma_A \partial_\nu \sigma_A \right),
\] (8.5)
then the Hamiltonian constraint for time-symmetric initial data is
\[
R = \frac{1}{2} g^{ij} \sum_{A=1}^{N} \partial_i \sigma_A \partial_j \sigma_A,
\]
and so writing $g_{ij} = \Phi^{4/(d-2)}$ is usual the constraint becomes
\[
-\frac{4(d-1)}{d-2} \partial_i \partial_i \Phi = \frac{1}{2} \Phi \sum_{A=1}^{N} \partial_i \sigma_A \partial_i \sigma_A.
\] (8.6)

Following the strategy used in four dimensions in [11] we make the ansatz
\[
\Phi = \prod_{a=1}^{M} X_a^{\nu_a}, \quad \sigma_A = \sqrt{\frac{8(d-1)}{d-2}} \sum_{a=1}^{M} m_a^A \log X_a,
\] (8.7)
where $X_a$ are a set of $M$ harmonic functions, $\partial_i X_a = 0$, and the constants $n_a$ and $m_a^A$ are determined by requiring that the Hamiltonian constraint (8.6) be satisfied. One finds that this holds if
\[
n_a n_b + \sum_{A=1}^{N} m_a^A m_b^A = n_a \delta_{ab}.
\] (8.8)
As in [11], by defining the $M (N+1)$-component vectors
\[
m_a = (m_a^1, n_a) \equiv (m_a^1, m_a^2, \cdots, m_a^N, n_a)
\] (8.9)
in $\mathbb{R}^{N+1}$, (8.8) becomes
\[
m_a \cdot m_b = n_a \delta_{ab}.
\] (8.10)
Defining $Q_a = m_a / \sqrt{n_a}$ one has
\[
Q_a \cdot Q_b = \delta_{ab}.
\] (8.11)
show that there is a one-one mapping between orthonormal $M$-frames in $\mathbb{R}^{N+1}$ and solutions of the Hamiltonian constraint conditions (8.8). This means that we must have $M \leq N + 1$. 
8.3. Specialisation to the scalar theories from EMD

In the mapping from Einstein–Maxwell–Dilaton theories with \( p \) dilatons and \( q \) Maxwell fields to a purely Einstein-scalar system with \( (p + q) \) scalar fields, which we described in section 8.1, the initial-value constraints were solved in terms of the harmonic functions \( C_i, D_I \) and \( W \). We may relate this to the general scalar discussion in section 8.2 by noting that, in an obvious notation, the harmonic functions \( X_a \) and the scalar fields \( \sigma_A \) are now split as

\[
X_a = \{ C_i, D_I, W \}, \quad \sigma_A = \{ \psi_I, \phi \}.
\]

Comparing the expressions for \( \Phi \), in equation (7.12), and \( \psi_I \), in equation (8.3), with equation (8.7) we see that for these cases the vectors \( m_a \) defined in (8.9) are given, following the same notation as for \( X_a \) in (8.12), by

\[
m_{C_i} = \beta (\bar{e}_i, 1/2 \bar{a}_i, 2\beta),
\]

\[
m_{D_I} = \beta (-\bar{e}_i, 1/2 \bar{a}_i, 2\beta),
\]

\[
m_w = (\bar{0}, -\beta \bar{a}, \gamma).
\]

where \( \bar{e}_i \) is an orthonormal basis for \( \mathbb{R}^p \), \( \beta = \sqrt{d-2}/8(d-1) \), and \( \gamma \) and \( \bar{a} \) are defined in (7.12). One can easily verify that the vectors defined in (8.13) indeed satisfy (8.10).

9. Wormholes

9.1. Wormhole initial data for vacuum Einstein equations

It was observed by Misner in the case of four spacetime dimensions that if one takes the spatial background metric to be \( S^1 \times S^2 \), then this can give rise to initial data that generates wormhole spacetimes. One can analogously consider \( S^1 \times S^{d-1} \) spatial backgrounds for \((d + 1)\)-dimensional spacetimes. Taking the background metric to be

\[
d\bar{s}^2 = d\mu^2 + d\sigma^2 + \sin^2\sigma d\Omega_{d-2}^2,
\]

then from (2.7), the condition for the vanishing of the Ricci scalar for the spatial metric \( ds^2 = \Phi^{4/(d-2)} \) \( d\bar{s}^2 \) is that \( \Phi \) should satisfy

\[
-\Box \Phi + \frac{(d-2)^2}{4} \Phi = 0,
\]

since the \( S^1 \times S^{d-1} \) metric (9.1) has Ricci scalar \( R = (d-1)(d-2) \). One can easily see that a solution for \( \Phi \) is given by

\[
\Phi = e^{\frac{-\mu}{2}} \left( \cosh \mu - \cos \sigma \right)^{-\frac{-\mu}{2}},
\]

where \( c \) is any constant.

If we take the solution (9.3) itself, the metric \( ds^2 = \Phi^{4/(d-2)} \) \( d\bar{s}^2 \) is nothing but the flat Euclidean metric \( ds^2 = dx^a dx^a + dz^2 \) written in bi-hyperspherical coordinates, with the Euclidean coordinates \( x' = (x^a, z) \) given by

\[
x^a = \frac{c u^a \sin \sigma}{\cosh \mu - \cos \sigma}, \quad z = \frac{c \sinh \mu}{\cosh \mu - \cos \sigma},
\]

\[
(9.4)
\]
where the $u^a$, constrained by $u^a u^a = 1$, parameterise points on the unit $(d - 2)$-sphere whose metric is $d\Omega_{d-2}^2 = du^a du^a$.

Of course since the metric (9.1) is invariant under translations of the $\mu$ coordinate, and (9.2) is a linear equation, one can form superpositions to obtain the more general solutions

$$\Phi = \sum_n A_n \left( \cosh(\mu - \mu_n) - \cos \sigma \right)^{-\frac{d-2}{2}}. \quad (9.5)$$

The $A_n$ and $\mu_n$ are arbitrary constants. Since one would like the wormhole metric to be single-valued and hence periodic in the $\mu$ coordinate on $S^1$, it is appropriate to take $\mu_n = -2n \mu_0$ and $A_n = \frac{1}{cd^{-2}}$, with the summation in (9.5) being taken over all the integers, and with $2\mu_0$ being the period of $\mu$:

$$\Phi(\mu, \sigma) = \frac{1}{cd^{-2}} \sum_{n \in \mathbb{Z}} \left( \cosh(\mu + 2n \mu_0) - \cos \sigma \right)^{-\frac{d-2}{2}}. \quad (9.6)$$

In a natural generalisation of the case of four spacetime dimensions that was discussed in [5] and elsewhere, one can easily see that if we consider the elementary harmonic function

$$\frac{1}{|x + d_n|^{d-2}} \quad (9.7)$$

in the Euclidean space with coordinates $x = \{x^1, \ldots, x^{d-1}, z\}$, where $-d_n$ is the location of the singularity, with

$$d_n = \{0, \ldots, 0, c \coth n \mu_0\}, \quad (9.8)$$

then

$$\frac{1}{|x + d_n|^{d-2}} = \frac{(\sinh n \mu_0)^{d-2}}{cd^{-2}} \left( \cosh \mu - \cos \sigma \right)^{\frac{d-2}{2}} \left( \cosh(\mu + 2n \mu_0) - \cos \sigma \right)^{-\frac{d-2}{2}}, \quad (9.9)$$

and so the periodic conformal function $\Phi$ constructed in (9.6) can be expressed as

$$\Phi = \frac{1}{cd^{-2}} \sum_{n \in \mathbb{Z}} \left( \cosh \mu - \cos \sigma \right)^{-\frac{d-2}{2}} \hat{\Phi}, \quad (9.10)$$

where

$$\hat{\Phi} = 1 + \sum_{n \geq 1} \frac{c^{d-2}}{(\sinh n \mu_0)^{d-2}} \left[ \frac{1}{|x + d_n|^{d-2}} + \frac{1}{|x - d_n|^{d-2}} \right]. \quad (9.11)$$

Thus the initial-time spatial $d$-metric metric $ds^2 = \hat{\Phi} \frac{x^a}{|x|^d} dx^a dx^b$, where $ds^2$ is the $S^1 \times S^{d-1}$ metric (9.1), can be written as

$$ds^2 = \hat{\Phi} \frac{x^a}{|x|^{d-2}} dx^a dx^b, \quad (9.12)$$

which is the metric for a sum over infinitely-many mass points at the locations $-d_n$ and $+d_n$, with strengths $c^{d-2} (\sinh n \mu_0)^{-d-2}$, giving a description of a two-centre wormhole in $(d + 1)$ spacetime dimensions. Comparing with the form of the multi-black hole initial data discussed in section 3.1, we see that the total mass of the wormhole is given by

$$M = 4c^{d-2} \sum_{n \geq 1} \frac{1}{(\sinh n \mu_0)^{d-2}}. \quad (9.13)$$
This generalises the result for the four-dimensional spacetime wormhole considered by Misner in [3], which corresponded to the case $d = 3$.

The infinite sums in the expression (9.13) can in fact be evaluated explicitly, in terms of the $q$-polygamma function

$$\psi_q^{(l)}(z) = \frac{\partial}{\partial z} \psi_q(z), \quad \psi_q(z) = \frac{1}{\Gamma_q(z)} \frac{\partial \Gamma_q(z)}{\partial z} = -\log(1 - q) + \log q \sum_{k \geq 0} \frac{q^{k+z}}{1 - q^{k+z}},$$

where $\Gamma_q(z)$ is the $q$ generalisation of the usual gamma function $\Gamma(z)$. For example, one finds that

$$\sum_{n \geq 1} \frac{1}{\sinh n \mu_0} = \frac{i \pi - \psi_q(1) + \psi_q(1 + \frac{i \pi}{\mu_0})}{\mu_0},$$

$$\sum_{n \geq 1} \frac{1}{(\sinh n \mu_0)^2} = \frac{-2 \mu_0 + \psi_q^{(1)}(1) + \psi_q^{(1)}(1 - \frac{i \pi}{\mu_0})}{\mu_0^2},$$

where $q \equiv e^{4\mu_0}$. Interestingly, in the special case $\mu_0 = \pi$, the sum for $d = 4$ has a simple expression,

$$\sum_{n \geq 1} \frac{1}{(\sinh n \pi)^2} = \frac{1}{6} - \frac{1}{2\pi}.$$

Another example with a simple expression is when $\mu_0 = \pi$ in $d = 6$, for which one has

$$\sum_{n \geq 1} \frac{1}{(\sinh n \pi)^4} = \frac{1}{3\pi} - \frac{11}{90} + \frac{[\Gamma(\frac{1}{2})]^2}{1920\pi^3}.$$  

**9.2. Wormhole initial data for Einstein–Maxwell**

By an elementary extension of the calculation described in section 4, one can verify that writing

$$ds^2 = \Phi \tau^{\frac{d-2}{4}} d\tau^2,$$

where $ds^2$ is the metric (9.1) on $S^1 \times S^{d-1}$, the initial value constraints (4.4) and $\nabla_i E^i = 0$ and for the Einstein–Maxwell system are satisfied by again writing

$$\Phi = (CD)^\frac{1}{4}, \quad E_i = \sqrt{\frac{d-1}{2(d-2)}} \partial_i \log \frac{C}{D},$$

where now $C$ and $D$ are arbitrary solutions of the Helmholtz equation

$$-\Box C + \frac{(d-2)^2}{4} C = 0, \quad -\Box D + \frac{(d-2)^2}{4} D = 0.$$

Thus we can solve the constraint equations by taking each of $C$ and $D$ to be functions of the general form (9.5). Since we would again like to construct initial data that is periodic in the circle coordinate $\mu$, it is important that we have $\Phi(\mu, \sigma) = \Phi(\mu + 2\mu_0, \sigma)$, where $2\mu_0$ is the period of $\mu$. However, as can be seen from (9.19), the functions $C$ and $D$ can be allowed to have the more general holonomy properties
\( C(\mu + 2\mu_0, \sigma) = e^{-\lambda} C(\mu, \sigma), \quad D(\mu + 2\mu_0, \sigma) = e^{\lambda} D(\mu, \sigma), \) (9.21)

where \( \lambda \) is a constant. This is compatible also with the single-valuedness of the solution for \( E_i \) in (9.19). We can construct solutions \( C \) and \( D \) with the required holonomy by taking

\[
C(\mu, \sigma) = c^{d-2} \sum_{n \in \mathbb{Z}} e^{\alpha n} \left( \cosh(\mu + 2\mu_0) - \cos \sigma \right)^{-\frac{d-2}{2}},
\]

\[
D(\mu, \sigma) = c^{d-2} \sum_{n \in \mathbb{Z}} e^{-\alpha n} \left( \cosh(\mu + 2\mu_0) - \cos \sigma \right)^{-\frac{d-2}{2}}. \tag{9.22}
\]

These series are convergent provided that \( |\lambda| < (d - 2) |\mu_0| \).

It is again useful to re-express \( C \) and \( D \) in terms of harmonic functions \( \tilde{C} \) and \( \tilde{D} \) in the conformally-related Euclidean space. Thus from (9.9) we see that \( \tilde{C} \) and \( \tilde{D} \) defined by

\[
C = \frac{c^{d-2}}{[\cosh \mu - \cos \sigma]^{d-2}} \tilde{C}, \quad D = \frac{c^{d-2}}{[\cosh \mu - \cos \sigma]^{d-2}} \tilde{D}, \tag{9.23}
\]

are given by

\[
\tilde{C} = 1 + \sum_{n \geq 1} \frac{c^{d-2}}{(\sinh \mu \mu_0)^d \mu_0^d} \left[ \frac{e^{\mu n}}{|x + \mu_n|^{d-2}} + \frac{e^{-\mu n}}{|x - \mu_n|^{d-2}} \right],
\]

\[
\tilde{D} = 1 + \sum_{n \geq 1} \frac{c^{d-2}}{(\sinh \mu \mu_0)^d \mu_0^d} \left[ \frac{e^{-\mu n}}{|x + \mu_n|^{d-2}} + \frac{e^{\mu n}}{|x - \mu_n|^{d-2}} \right]. \tag{9.24}
\]

Defining \( \tilde{\Phi} = (\tilde{C}\tilde{D})^\frac{1}{d+1} \) we therefore have

\[
d\tilde{s}^2 = \tilde{\Phi}^{-\frac{1}{d+1}} d\tilde{s}^2 = \tilde{\Phi}^{-\frac{1}{d+1}} dx^i dx^j,
\]

and so we straightforwardly find that the total mass of the Einstein–Maxwell wormhole is given by

\[
M = 4c^{d-2} \sum_{n \geq 1} \frac{\cosh \mu n}{(\sinh \mu \mu_0)^d \mu_0^d}. \tag{9.26}
\]

This reduces to the result in [5] when \( d = 3 \), corresponding to the case of Einstein–Maxwell wormholes in four spacetime dimensions. The mass is finite provided that the condition \( |\lambda| < (d - 2) |\mu_0| \) that we mentioned previously is satisfied.

The electric charge threading each wormhole throat can be calculated from a Gaussian integral

\[
Q = \frac{1}{\omega_{d-1}} \int_S E_i n^i dS, \tag{9.27}
\]

where \( \omega_{d-1} \) is the volume of the unit \((d - 1)\) sphere, and \( n^i \) is the unit vector normal to the \((d - 1)\)-surface \(S\) enclosing the charged mass points that comprise the wormhole throat under consideration. In our case the mass points at \( \mu_n \) for \( 1 \leq n \leq \infty \) are associated with one throat, and the mass points at \( -\mu_n \) with the other. Since the spatial metric \( ds^2 \) on the initial surface is equal to \( (\tilde{C}\tilde{D})^{-\frac{1}{d+1}} dx^i dx^j \), the area element \( dS = (\tilde{C}\tilde{D})^{-\frac{1}{d+1}} d\tilde{s} \) and the unit vector \( n^i = (\tilde{C}\tilde{D})^{-\frac{1}{d+1}} \tilde{n}^i \), where \( d\tilde{s} \) and \( \tilde{n}^i \) are the corresponding quantities in the Euclidean metric \( dx^i dx^j \). Thus, from (9.19) and (9.23) we have
\[ Q = \frac{1}{\omega_{d-1}} \sqrt{\frac{d - 1}{2(d - 2)}} \int_{S} \left( \hat{D} \partial_{i} \hat{C} - \hat{C} \partial_{i} \hat{D} \right) dS^i. \]  

(9.28)

The corresponding charge for the other throat will be \(-Q\).

One way to evaluate (9.28) is to convert it, using the divergence theorem, into

\[ Q = \frac{1}{\omega_{d-1}} \sqrt{\frac{d - 1}{2(d - 2)}} \int_{V} \left( \hat{D} \partial_{i} \hat{C} - \hat{C} \partial_{i} \hat{D} \right) d^d x, \]  

(9.29)

and make use of the fact that the harmonic functions \( \hat{C} \) and \( \hat{D} \), defined in (9.24), satisfy

\[ \partial_{i} \partial_{i} \hat{C} = -(d - 2) \sum_{n \geq 1} \frac{e^{d-2} \omega_{d-1}}{(\sinh n \mu_0)^{d-2}} \left[ e^{-n \lambda} \delta^d(x + d_n) + e^{-n \lambda} \delta^d(x - d_n) \right], \]

\[ \partial_{i} \partial_{i} \hat{D} = -(d - 2) \sum_{n \geq 1} \frac{e^{d-2} \omega_{d-1}}{(\sinh n \mu_0)^{d-2}} \left[ e^{-n \lambda} \delta^d(x + d_n) + e^{n \lambda} \delta^d(x - d_n) \right]. \]  

(9.30)

To calculate the total charge for the wormhole throat corresponding to the \( x = d_n \) sequence of mass points, we should choose the volume \( V \) in integral (9.29) to enclose all these mass points, but none of those located at the points \( x = -d_n \). Thus we find

\[ Q = \sqrt{\frac{(d - 1)(d - 2)}{2}} \left[ \sum_{n \geq 1} \frac{2e^{d-2} \sinh n \lambda}{(\sinh n \mu_0)^{d-2}} + \sum_{m, n \geq 1} \frac{2e^{d-4} \sinh(m + n) \lambda}{(\sinh m \mu_0 \sinh n \mu_0)^{d-2}} \right]. \]  

(9.31)

Note that the terms that arise involving \( |d_m - d_n|^{-(d - 2)} \) cancel by antisymmetry. The \( m = n \) ‘self-energy’ terms require some care, since the denominators \( |d_m - d_n|^{d-2} \) go to zero. One way to handle this is to introduce regulators by sending \( x \to x + \epsilon \) in \( \hat{C} \), and \( x \to x - \epsilon \) in \( \hat{D} \). The terms involving \( |d_m - d_n|^{-(d - 2)} \) now become \( |d_m - d_n + 2\epsilon|^{-(d - 2)} \) and still cancel by antisymmetry, prior to sending the regulator to zero. An alternative way to evaluate the charge is to work directly with the expression (9.28) for \( Q \), and evaluate the contribution for each of the included mass points \( x = d_n \) by integrating over a small \( (d - 1) \)-sphere surrounding that point. After summing over the contributions from all the mass points, one arrives at the same result (9.31) that we obtained above. In this calculation, the analogous regularisation of the potentially-divergent ‘self-energy’ terms occurs because they cancel pairwise by antisymmetry before taking the limit in which the radius of the small spheres surrounding the mass points goes to zero.

In view of the definition (9.8) for \( d_n \), we have \( |d_m + d_n| = c (\coth m \mu_0 + \coth n \mu_0) \), and hence

\[ Q = \sqrt{\frac{(d - 1)(d - 2)}{2}} \sum_{n \geq 1} \frac{2e^{d-2} \sinh n \lambda}{(\sinh n \mu_0)^{d-2}} + \sqrt{\frac{(d - 1)(d - 2)}{2}} \sum_{m, n \geq 1} \frac{2e^{d-4} \sinh(m + n) \lambda}{(\sinh(m + n) \mu_0)^{d-2}}, \]

\[ = \sqrt{\frac{(d - 1)(d - 2)}{2}} \sum_{m \geq 0} \sum_{n \geq 1} \frac{2e^{d-2} \sinh(m + n) \lambda}{(\sinh(m + n) \mu_0)^{d-2}}. \]  

(9.32)

Defining \( p = m + n \), the double summation \( \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \) can be rewritten as \( \sum_{p \geq 1} \sum_{n=1}^{p} \) and so (9.32) becomes

\[ ^{4}\text{This agrees in the special case } d = 3 \text{ with what Lindquist would have had in his result for four-dimensional space-time, if he had not accidentally omitted the factor of } p \text{ in the numerator of his expression.} \]
\[
Q = 2e^{d-2} \sqrt{\frac{(d-1)(d-2)}{2}} \sum_{p>1} \frac{p \sinh p\lambda}{(\sinh p\mu_0)^{d-2}}.
\] (9.33)

It is interesting to note from the expressions (9.26) for the total mass \(M\) and (9.33) for the charge that
\[
Q = \frac{1}{2} \sqrt{\frac{(d-1)(d-2)}{2}} \frac{\partial M}{\partial \lambda}.
\] (9.34)

In the Einstein–Maxwell case that we are considering here (but not in systems involving dilatons also, as will be seen later) the electric field \(E_i\) on the initial-value surface can be written in terms of the gradient of a potential (see (9.19)):
\[
E = -\partial_i \psi, \quad \psi = -\sqrt{\frac{d-1}{2(d-2)}} \log \frac{C}{D}.
\] (9.35)

It follows from (9.19) that the change in \(\psi\) under a translation by a complete period in \(\mu\), i.e. \(\mu \rightarrow \mu + 2\mu_0\), is given by the constant
\[
\Delta \psi \equiv \psi(\mu + 2\mu_0, \sigma) - \psi(\mu, \sigma) = 2\lambda \sqrt{\frac{d-1}{2(d-2)}}.
\] (9.36)

and so (9.34) can be expressed as
\[
Q = \frac{1}{2} (d-1) \frac{\partial M}{\partial (\Delta \psi)}.
\] (9.37)

This is suggestive of a thermodynamic first law, of the form
\[
dM = \frac{2}{d-1} Q d(\Delta \psi).
\] (9.38)

### 9.3. Einstein–Maxwell–Dilaton wormholes

The Einstein–Maxwell–Dilaton system discussed in section 5 allows one to set up wormhole initial data also. One can straightforwardly check that by taking the background metric \(d\bar{s}^2\) to be the \(S^1 \times S^{d-1}\) metric (9.1), the Hamilton and Gauss-law constraints (5.2) are satisfied if \(\Phi, \phi\) and \(E_i\) are given by (5.4) and the functions \(C, D\) and \(W\) now obey the Helmholtz equations
\[
-\Box C + \frac{(d-2)^2}{4} C = 0, \quad -\Box D + \frac{(d-2)^2}{4} D = 0, \quad -\Box W + \frac{(d-2)^2}{4} W = 0.
\] (9.39)

As in the previous wormhole examples, we can construct solutions with appropriate periodicity or holonomy properties by taking suitable linear superpositions of elementary solutions. In this example, we see that we can ensure the necessary periodicity of the conformal factor \(\Phi\), the dilaton \(\phi\) and the electric field \(E_i\) by arranging that the solutions for \(C, D\) and \(W\) obey\(^5\)
\[
C(\mu + 2\mu_0, \sigma) = e^{-\lambda} C(\mu, \sigma), \quad D(\mu + 2\mu_0, \sigma) = e^{\lambda} D(\mu, \sigma), \quad W(\mu + 2\mu_0, \sigma) = W(\mu, \sigma).
\] (9.40)

\(^5\)Note that we do not encounter the problems that were seen in [7] for the Einstein–Maxwell–Dilaton system, where the dilaton had non-trivial monodromy and was not periodic in the \(\mu\) coordinate. This is related to the fact that our ansatz involves three harmonic functions, \(C, D\) and \(W\), thus generalising the 3-function ansatz in [11], whereas the more restrictive ansatz in [7] has only two harmonic functions.
and so we may take
\[ C(\mu, \sigma) = \frac{c^{d-2}}{\omega} \sum_{n \in \mathbb{Z}} e^{n\lambda} \left( \cosh(\mu + 2n\mu_0) - \cos \sigma \right)^{-\frac{d-2}{2}} \]
\[ D(\mu, \sigma) = \frac{c^{d-2}}{\omega} \sum_{n \in \mathbb{Z}} e^{-n\lambda} \left( \cosh(\mu + 2n\mu_0) - \cos \sigma \right)^{-\frac{d-2}{2}} \]
\[ W(\mu, \sigma) = \frac{c^{d-2}}{\omega} \sum_{n \in \mathbb{Z}} \left( \cosh(\mu + 2n\mu_0) - \cos \sigma \right)^{-\frac{d-2}{2}}. \]  

(9.41)

We again have the expressions (9.25) for the metric \( ds^2 \) in terms of the \( S^1 \times S^{d-1} \) metric \( \bar{d}s^2 \) and the Euclidean metric \( d\xi d\xi \), with \( \hat{\Phi} \) related to \( \Phi \) as in (9.10) and now
\[ \hat{\Phi} = \left( \hat{C} \hat{D} \right)^{\frac{(d-1)}{2}} \hat{W}^{\frac{1}{d-2}}, \]  

(9.42)

with
\[ \hat{C} = 1 + \sum_{n \geq 1} \frac{c^{d-2}}{(\sinh n\mu_0)^{d-2}} \left[ \frac{e^{n\lambda}}{|x + d_n|^{d-2}} + \frac{e^{-n\lambda}}{|x - d_n|^{d-2}} \right], \]
\[ \hat{D} = 1 + \sum_{n \geq 1} \frac{c^{d-2}}{(\sinh n\mu_0)^{d-2}} \left[ \frac{e^{-n\lambda}}{|x + d_n|^{d-2}} + \frac{e^{n\lambda}}{|x - d_n|^{d-2}} \right], \]  

(9.43)

\[ \hat{W} = 1 + \sum_{n \geq 1} \frac{c^{d-2}}{(\sinh n\mu_0)^{d-2}} \left[ \frac{1}{|x + d_n|^{d-2}} + \frac{1}{|x - d_n|^{d-2}} \right]. \]  

(9.44)

Comparing with the form of \( \Phi \) in section 3.1 we straightforwardly find that the total wormhole mass is given by
\[ M = \frac{16(d-2)c^{d-2}}{(d-1)\Delta} \sum_{n \geq 1} \frac{\sinh^2 n\lambda}{(\sinh n\mu_0)^{d-2}} + 4c^{d-2} \sum_{n \geq 1} \frac{1}{(\sinh n\mu_0)^{d-2}}. \]  

(9.45)

The calculation of the electric charge proceeds in a very similar fashion that for the Einstein–Maxwell case, which we discussed earlier. Now, we shall have
\[ Q = \frac{1}{\omega_{d-1}} \int e^\phi E_n n^i dS, \]
\[ = \frac{2}{\sqrt{\Delta} \omega_{d-1}} \int (\partial_i \hat{C} - \hat{C} \partial_i \hat{D}) dS', \]  

(9.46)

and hence by the same steps as for Einstein–Maxwell, we find
\[ Q = \frac{4(d-2)c^{d-2}}{\sqrt{\Delta}} \sum_{p \geq 1} \frac{p \sinh p\lambda}{(\sinh p\mu_0)^{d-2}}. \]  

(9.47)

We again have a simple relation between the mass \( M \), given by (9.45), and the charge \( Q \) given by (9.47), namely
\[ Q = \frac{(d-1)\sqrt{\Delta}}{2} \frac{\partial M}{\partial \lambda}. \]  

(9.48)
9.4. Multi-Maxwell wormholes

For the remaining examples of time-symmetric initial data involving multiple Maxwell fields, which we discussed in sections 6 and 7, we shall just briefly summarise the results for wormhole initial data.

In the case of two Maxwell fields and a single dilaton, described in section 6, we find that each of the pairs of functions \( (C_1, D_1) \) and \( (C_2, D_2) \) will now take the form given in (9.22), with independent \( \lambda \) parameters \( \lambda_1 \) and \( \lambda_2 \) allowed for the two pairs, so that

\[
C_I(\mu, \sigma) = e^{-\lambda_I} C_I(\mu + 2\mu_0, \sigma), \quad D_I(\mu, \sigma) = e^{\lambda_I} D_I(\mu + 2\mu_0, \sigma), \quad I = 1, 2.
\]

The total wormhole mass is then given by

\[
M = \frac{2(d - 2)}{(d - 1)} \sum_{n \geq 1} \frac{1}{(\sinh n\mu_0)^{d-2}} \left[ N_1 \cosh n\lambda_1 + N_2 \cosh n\lambda_2 \right].
\]

There are now two charges, one for each Maxwell field, and these are given by

\[
Q_I = 2(d - 2)e^{d-2} \sqrt{N_I} \sum_{p \geq 1} \frac{p \sinh p\lambda_I}{(\sinh p\mu_0)^{d-2}}.
\]

The charges can be expressed in terms of the mass as follows:

\[
Q_I = (d - 1) \frac{\partial M}{\partial \lambda_I}.
\]

For the case of \( p \) dilaton fields and \( q \) Maxwell fields discussed in section 7, we find that the initial-value constraints can be solved, in the \( S^1 \times S^{d-1} \) background metric, by the ansatz (7.12), where now the functions \( C_I, D_I \) and \( W \) obey the equation (9.39). Single-valuedness of the metric, dilatons and electric fields requires that we have the holonomy relations

\[
C_I(\mu + 2\mu_0, \sigma) = e^{-\lambda_I} C_I(\mu, \sigma), \quad D_I(\mu + 2\mu_0, \sigma) = e^{\lambda_I} D_I(\mu, \sigma), \quad W(\mu + 2\mu_0, \sigma) = W(\mu, \sigma).
\]

We can take the functions \( C_I, D_I \) and \( W \) to be given as in (9.41), with the different \( \lambda_I \) parameters for each pair \( (C_I, D_I) \). We find the total mass of the wormhole is given by

\[
M = \frac{4(d - 2)}{(d - 1)} \sum_{l = 1}^{q} \sum_{n \geq 1} \frac{\sinh^2 n\lambda_l}{(\sinh n\mu_0)^{d-2}} + 4e^{d-2} \sum_{n \geq 1} \frac{1}{(\sinh n\mu_0)^{d-2}}.
\]

The total charges associated with one of the two wormhole throats are given by

\[
Q_I = 2(d - 2)e^{d-2} \sum_{p \geq 1} \frac{p \sinh p\lambda_I}{(\sinh p\mu_0)^{d-2}},
\]

with the other carrying charges of equal magnitudes but opposite signs. The charges and the mass are related by

\[
Q_I = (d - 1) \frac{\partial M}{\partial \lambda_I}.
\]
9.5. Wormhole interaction energy

A manifold with \(N\) Einstein–Rosen bridges in an asymptotically flat spacetime, with each bridge leading to a different asymptotically flat spacetime, has a metric of the form [6]

\[
d s^2 = \hat{\Phi}^{-2} \, d\chi^2,
\]

where

\[
\hat{\Phi} = 1 + \sum_{i=1}^{N} \frac{\alpha_i}{r_i^d},
\]

and \(r_i = |x - x_i|\), with \(x_i\) being the location of the \(i\)th mass point. The total mass \(M\) of this system is given by

\[
M = 2 \sum_{i=1}^{N} \alpha_i.
\]

When \(x\) approaches the \(i\)th mass point, we can introduce hyperspherical polar coordinates centred on \(x_i\), using \(r_i\) as the radial coordinate, which will be very small. Thus for \(j \neq i\) we can make the approximation

\[
r_j \to r_{ij} = |x_i - x_j|,
\]

and the metric then takes the form

\[
d s^2 \to \left[ \frac{\alpha_i}{r_i^d - 2} + A_i \right] \left( dr_i^2 + r_i^2 \, d\Omega_{d-1}^2 \right) = \left( \frac{\alpha_i}{r_i^d - 2} \right) \left[ 1 + A_i r_i^{d-2} \right] \left( dr_i^2 + r_i^2 \, d\Omega_{d-1}^2 \right),
\]

where the constants \(A_i\) are given by

\[
A_i = 1 + \sum_{j \neq i} \frac{\alpha_j}{r_j^d - 2}.
\]

Now introducing the new coordinate

\[
r_i^{d-2} = \frac{\alpha_i^2}{r_i^d - 2},
\]

the line element in the corresponding limit \(r_i \to \infty\) takes the form

\[
d s^2 \to \left[ 1 + \frac{A_i \alpha_i}{r_i^{d-2}} \right] \frac{\pi}{d-2} \left( dr_i^2 + r_i^2 \, d\Omega_{d-1}^2 \right).
\]

This implies that the bare mass of the individual bridge is

\[
m_i = 2A_i \alpha_i = 2\alpha_i + 2\alpha_i \sum_{j \neq i} \frac{\alpha_j}{r_j^d - 2},
\]

and their sum,

\[
\sum_{i=1}^{N} m_i = 2 \sum_{i=1}^{N} \alpha_i + 2 \sum_{i=1}^{N} \sum_{j \neq i} \frac{\alpha_i \alpha_j}{r_j^d - 2} = M + 2 \sum_{i=1}^{N} \sum_{j \neq i} \frac{\alpha_i \alpha_j}{r_j^d - 2},
\]

is not equal to the total mass of the system. Hence, the energy of gravitational interaction is
\[ M_{\text{int}} = M - \sum_{i=1}^{N} m_i = -2 \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{\alpha_i \alpha_j}{r_{ij}^{d-2}}. \]  
\hfill (9.67)

We can now use these results to obtain the interaction energy of various wormholes.

**9.5.1. Einstein wormhole.** For the wormhole manifold in pure Einstein gravity we have

\[ \Phi = 1 + \sum_{n \geq 1} \left( \frac{e^{d-2}}{(\sinh n \mu_0)^{d-2}} \left[ \frac{1}{|x + d_n|^{d-2}} + \frac{1}{|x - d_n|^{d-2}} \right] \right). \]  
\hfill (9.68)

All image points at \( d_n \) contribute to the mass \( m_1 \) of one mouth of the wormhole, and the rest (at \(-d_n\)) to the mass \( m_2 \) of the other mouth. The mass of the \( n \)th image point is

\[ m_n = 2\alpha_n + 2\alpha_n \sum_{m \neq n} \frac{\alpha_m}{r_{mn}^{d-2}}. \]  
\hfill (9.69)

where

\[ \alpha_n = \frac{e^{d-2}}{(\sinh n \mu_0)^{d-2}}. \]  
\hfill (9.70)

Hence, the masses of the wormhole mouths are

\[ m_2 = m_1 = \sum_{n \geq 1} m_n = \frac{M}{2} + 2 \sum_{n \geq 1} \sum_{m \neq n} \frac{\alpha_n \alpha_m}{r_{mn}^{d-2}}. \]  
\hfill (9.71)

The terms where \( m \) is negative will have denominators \( r_{mn} = |d_m - d_n| \) of the form \( |d_p + d_n| = c \sinh(p + n) \mu_0 \) with \( n \) and \( p = -m \) both positive. The double sum will converge for such terms. However, when \( m \) is positive the denominators will be \( |d_m - d_n| = c \sinh(m - n) \mu_0 \) with \( m \) and \( n \) positive, and even though the terms with \( m = n \) are excluded, the double sum will diverge. As discussed in detail in [5], this problem can be resolved by subtracting out the (infinite) interaction energy between the bare masses which together make up \( m_i \). In other words, one makes a ‘mass renormalisation’ by adding the infinite (negative) term

\[ \delta m_1 = -2 \sum_{n \geq 1} \sum_{m \neq n} \frac{\alpha_n \alpha_m}{r_{mn}^{d-2}} = -2 \sum_{n \geq 1} \sum_{m \neq n} \frac{\alpha_n \alpha_m}{|d_m - d_n|^{d-2}}. \]  
\hfill (9.72)

This leads to the ‘renormalised’ mass

\[ m_2 = m_1 = \frac{M}{2} + 2 \sum_{n \geq 1} \sum_{m \neq n} \frac{\alpha_n \alpha_m}{r_{mn}^{d-2}} = \frac{M}{2} + 2 \sum_{n \geq 1} \sum_{m \neq n} \frac{e^{d-2}}{|\sinh(m + n) \mu_0|^{d-2}}, \]  
\hfill (9.73)

or, after reorganising the double summation,

\[ m_2 = m_1 = \frac{M}{2} + 2 e^{d-2} \sum_{p \geq 2} \frac{p - 1}{|\sinh(p \mu_0)|^{d-2}}. \]  
\hfill (9.74)

Now, the (finite) interaction energy between the two mouths is given by

\[ M_{\text{int}} = M - (m_1 + m_2) = -4 e^{d-2} \sum_{p \geq 2} \frac{p - 1}{|\sinh(p \mu_0)|^{d-2}}. \]  
\hfill (9.75)
9.5.2. Other wormholes. We may now apply the same procedure to the case of $q$ Maxwell and $p$ dilaton fields. For this system we have

$$\Phi = \hat{\Pi} \frac{d^{-2}}{d-1} \hat{W}^q, \quad \hat{\Pi} \equiv \prod_{l=1}^{q} \hat{C}_l \hat{D}_l, \quad \gamma \equiv 1 - \frac{q(d-2)}{2(d-1)}, \quad (9.76)$$

where

$$\hat{C}_l = 1 + \sum_{n \neq 0} \frac{e^{d-2}}{(\sinh n \mu_0)^{d-2}} \left[ \frac{e^{\lambda_1}}{|x + d_1|^{d-2}} + \frac{e^{-\lambda_1}}{|x - d_1|^{d-2}} \right] = 1 + \sum_{n \neq 0} \frac{c_{ln}}{r_n^{d-2}},$$

$$\hat{D}_l = 1 + \sum_{n \neq 0} \frac{e^{d-2}}{(\sinh n \mu_0)^{d-2}} \left[ \frac{e^{-\lambda_1}}{|x + d_1|^{d-2}} + \frac{e^{\lambda_1}}{|x - d_1|^{d-2}} \right] = 1 + \sum_{n \neq 0} \frac{d_{ln}}{r_n^{d-2}},$$

$$\hat{W} = 1 + \sum_{n \neq 0} \frac{e^{d-2}}{(\sinh n \mu_0)^{d-2}} \left[ \frac{1}{|x + d_1|^{d-2}} + \frac{1}{|x - d_1|^{d-2}} \right] = 1 + \sum_{n \neq 0} \frac{w_n}{r_n^{d-2}}. \quad (9.77)$$

The total mass of wormhole is given by

$$M = \frac{\bar{M}}{2} = \sum_{n \neq 0} \left[ \frac{d-2}{4(d-1)} \left( c_{ln} + d_{ln} \right) + \gamma w_n \right]. \quad (9.78)$$

In the limit

$$r_n \to 0, \quad r_m \to r_{nm} \quad (m \neq n), \quad (9.79)$$

we get

$$\hat{C}_l \to \left[ \frac{c_{ln}}{r_n^{d-2}} + C_n \right], \quad C_n = 1 + \sum_{m \neq n} \frac{c_{ln}}{r_m^{d-2}},$$

$$\hat{D}_l \to \left[ \frac{d_{ln}}{r_n^{d-2}} + D_n \right], \quad D_n = 1 + \sum_{m \neq n} \frac{d_{ln}}{r_m^{d-2}},$$

$$\hat{W} \to \left[ \frac{w_n}{r_n^{d-2}} + W_n \right], \quad W_n = 1 + \sum_{m \neq n} \frac{w_n}{r_m^{d-2}} \quad (9.80)$$

and metric takes the form

$$ds^2 \to \left( \frac{\alpha_n}{r_n} \right)^{\frac{2}{d-2}} \prod_{l=1}^{q} \left( 1 + \frac{c_{ln}^2}{c_{ln}^2} \right) \left( 1 + \frac{d_{ln}^2}{d_{ln}^2} \right) \left[ 1 + \frac{w_n^2}{w_n^2} \right] \frac{dr_n^2}{r_n^2} \left( 1 + \frac{d\Omega_{d-2}^2}{d\Omega_{d-2}^2} \right), \quad (9.81)$$

where

$$\alpha_n = w_n^q \prod_{l=1}^{q} (c_{ln} d_{ln}) \frac{r_n^{d-2}}{r_n^{d-2}}. \quad (9.82)$$

From 9.77 we can see that

$$c_{ln} = e^{d-2} \frac{e^{-\lambda_1}}{(\sinh n \mu_0)^{d-2}}, \quad d_{ln} = e^{d-2} \frac{e^{\lambda_1}}{(\sinh n \mu_0)^{d-2}}, \quad w_n = e^{d-2} \frac{1}{(\sinh n \mu_0)^{d-2}}, \quad (9.83)$$

where

$$\hat{\Pi} \equiv \prod_{l=1}^{q} \hat{C}_l \hat{D}_l, \quad \gamma \equiv 1 - \frac{q(d-2)}{2(d-1)}.$$
and hence
\[ (c_{l_n} d_{l_n}) = w_n^2 \quad \implies \quad \alpha_n = w_n \prod_{i=1}^{q} (c_{l_n} d_{l_n})^{\frac{d-2}{d-1}} = w_n, \]
\[ (9.84) \]
i.e.
\[ \alpha_n^2 = w_n^2 = c_{l_n} d_{l_n}. \]
\[ (9.85) \]
Now, defining a new radial coordinate
\[ r_n^{d-2} = \frac{\alpha_n^2}{r_n^{d-2}}, \]
\[ (9.86) \]
the line element in the limit \( r_n^2 \to \infty \) takes the form
\[ ds^2 \to \prod_{l=1}^{q} \left[ 1 + C_{l_n} \frac{d_{l_n}}{r_n^{d-2}} \right] \left[ 1 + D_{l_n} \frac{c_{l_n}}{r_n^{d-2}} \right]^{-\frac{2\gamma}{d-2}} (dr_n^2 + r_n^2 d\Omega_{d-1}^2). \]
\[ (9.87) \]
The mass of the \( n \)th image point in this system is
\[ m_n = d - 2 - (d - 1) \left[ C_{l_n} d_{l_n} + D_{l_n} c_{l_n} \right] + 2\gamma W_{l_n} w_{l_n}, \]
\[ (9.88) \]
or
\[ m_n = \frac{d - 2}{2(d - 1)} \left[ C_{l_n} d_{l_n} + d_{l_n} \right] + 2\gamma w_{l_n} + \frac{d - 2}{2(d - 1)} \sum_{m \neq n} \left[ c_{l_m} d_{l_m} + d_{l_m} c_{l_m} \right] \frac{r_n^{d-2}}{r_m^{d-2}} + 2\gamma \sum_{m \neq n} w_n w_m, \]
\[ (9.89) \]
Following the discussion similar to previous subsection, the ‘renormalised mass’ of the wormhole mouth is
\[ m_2 = m_1 = \frac{M}{2} + \frac{d - 2}{d - 1} \left[ C_{l_n} d_{l_n} + d_{l_n} \right] + 2\gamma w_{l_n} + \frac{d - 2}{2(d - 1)} \sum_{m \neq n} \left[ c_{l_m} d_{l_m} + d_{l_m} c_{l_m} \right] \frac{r_n^{d-2}}{r_m^{d-2}} + 2\gamma \sum_{m \neq n} w_n w_m. \]
\[ (9.90) \]
The interaction energy is then given by
\[ M_{int} = M - m_1 - m_2 = \frac{4(d - 2)}{(d - 1)} d_{l_n} \sum_{p \geq 1} \left[ (p - 1) \frac{\cosh p \lambda}{\sinh p \mu_0 d_{l_n}^{d-2}} + 2\gamma d_{l_n} \sum_{p \geq 1} \frac{p - 1}{\sinh p \mu_0 d_{l_n}^{d-2}}. \right] \]
\[ (9.91) \]
In a similar fashion, in the case of two Maxwell fields and a single dilaton, described in section 6, the interaction energy is given by
\[ M_{int} = -\frac{2(d - 2)}{(d - 10)} d_{l_n} \sum_{p \geq 1} \left[ \frac{p - 1}{\sinh p \mu_0 d_{l_n}^{d-2}} \left( N_1 \cosh p \lambda_1 + N_2 \cosh p \lambda_2 \right) \right]. \]
\[ (9.92) \]
For the Einstein–Maxwell–Dilaton system discussed in section 5, the interaction energy is
\[ M_{int} = -\frac{16(d - 2)}{(d - 1)} \Delta \sum_{n \geq 1} \left( n - 1 \right) \frac{\sinh^2 \frac{p \lambda}{d_{l_n}^{d-2}}}{\left( \sinh n \mu_0 \right)^{d-2}} - 4 d_{l_n} \sum_{n \geq 1} \frac{n - 1}{\left( \sinh n \mu_0 \right)^{d-2}}. \]
\[ (9.93) \]
For the Einstein–Maxwell case, described in section 4, the interaction energy is
\[ M_{\text{int}} = -4 e^{d-2} \sum_{n \geq 1} (n - 1) \frac{\cosh n\lambda}{(\sinh n\mu_0)^{d-2}}. \] (9.94)

10. Conclusions

The geometrodynamical approach to studying solutions of the Einstein equations and the coupled Einstein–Maxwell equations was pioneered by Wheeler, Misner and others in late 1950s and early 1960s. The idea was to look at the initial-value constraints in a Hamiltonian formulation of the equations of motion, extracting as much information as possible about the properties of the (in general time-dependent) solutions that would evolve from the initial data. For simplicity, the initial data were typically taken to be time independent, corresponding to an initial slice at a moment of time-reflection symmetry in the subsequent evolution. One can calculate some general features of the solutions that will evolve from the initial data, even though in practice the explicit solution of the evolution equations is beyond reach. The early work on geometrodynamics was all focused on the case of four-dimensional spacetimes.

More recently, wider classes of four-dimensional theories were considered, in which additional matter fields of the kind occurring in supergravity theories were included [7, 11]. In this paper, we have presented results for time-symmetric initial value data satisfying the constraint equations in higher-dimensional theories of gravity coupled to scalar and Maxwell fields. These theories encompass particular cases that correspond to higher-dimensional theories of supergravity, and thus they also have relevance for the low-energy limits of string theories or M-theory. We considered initial data both for multiple black hole evolutions and also for wormhole spacetimes. In the case of wormhole spacetimes, we studied some of the properties of the solutions in detail, including the masses and charges associated with the individual wormhole throats in a multi-wormhole spacetime, and the interaction energies between the throats.

Our focus in this paper has been the construction of consistent time-symmetric initial data for multiple black holes or wormholes in higher-dimensional theories such as those that arise in supergravities or in string theory and M-theory. In general, one does not expect to be able to solve the evolution equations for the initial-data sets explicitly, but it could nonetheless be of interest to try investigate further some of the features that might be expected to arise in such solutions.

A further point is that the solutions to the initial-value constraints that we considered all made use of an ansatz introduced first by Lichnerowicz in the case of four-dimensional spacetimes, in which the spatial metric on the initial surface is taken to be a conformal factor times a fixed fiducial metric of high symmetry, such as the Euclidean metric, or the metric on \( S^3 \) or \( S^1 \times S^2 \). When one considers higher spacetime dimensions, such a conformal factor parameterises a smaller fraction of the total space of possible spatial geometries. It might therefore be interesting to explore more general ansätze for parameterising the spatial metrics on the initial surface.

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