WAVE EQUATIONS WITH MOVING POTENTIALS

GONG CHEN

Abstract. In this paper, we study the endpoint reversed Strichartz estimates along general time-like trajectories for wave equations in $\mathbb{R}^3$. We also discuss some applications of the reversed Strichartz estimates and the structure of wave operators to the wave equation with one potential. These techniques are useful to analyze the stability problem of traveling solitons.

1. Introduction

Our starting point is the free wave equation ($H_0 = -\Delta$) on $\mathbb{R}^3$
\begin{equation}
\partial_{tt} u - \Delta u = 0
\end{equation}
with initial data
\begin{equation}
\begin{aligned}
u(x, 0) &= g(x), \quad u_t(x, 0) = f(x).
\end{aligned}
\end{equation}
We can write down $u$ explicitly,
\begin{equation}
u = \sin \left( t\sqrt{-\Delta} \right) f + \cos \left( t\sqrt{-\Delta} \right) g.
\end{equation}
It obeys the energy inequality,
\begin{equation}
E_F(t) = \int_{\mathbb{R}^3} |\partial_t u(t)|^2 + |\nabla u(t)|^2 \, dx \lesssim \int_{\mathbb{R}^3} |f|^2 + |\nabla g|^2 \, dx.
\end{equation}
We also have the well-known dispersive estimates for the free wave equation ($H_0 = -\Delta$) on $\mathbb{R}^3$:
\begin{align}
\left\| \sin \left( t\sqrt{-\Delta} \right) f \right\|_{L^\infty(\mathbb{R}^3)} &\lesssim \frac{1}{|t|} \left\| \nabla f \right\|_{L^1(\mathbb{R}^3)}, \\
\left\| \cos \left( t\sqrt{-\Delta} \right) g \right\|_{L^\infty(\mathbb{R}^3)} &\lesssim \frac{1}{|t|} \left\| \Delta g \right\|_{L^1(\mathbb{R}^3)}.
\end{align}
For the sake of completeness, the proofs of estimates (1.5) and (1.6) are provided in details in Appendix A. (Notice that the estimate (1.6) is slightly different from the estimates commonly used in the literature, such as Krieger-Schlag [KS] where one needs the $L^1$ norm of $D^2 g$ instead of $\Delta g$).

Strichartz estimates can be derived abstractly from these dispersive inequalities and the energy inequality. With some appropriate $(p, q, s)$, one has
\begin{equation}
\left\| u \right\|_{L^p_t L^q_x} \lesssim \left\| g \right\|_{\dot{H}^s} + \left\| f \right\|_{\dot{H}^{s-1}}.
\end{equation}

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The non-endpoint estimates for the wave equations can be found in Ginibre-Velo [GV]. Keel–Tao [KT] also obtained sharp Strichartz estimates for the free wave equation in $\mathbb{R}^n$, $n \geq 4$ and everything except the endpoint in $\mathbb{R}^3$. See Keel-Tao [KT] and Tao’s book [Tao] for more details on the subject’s background and the history.

In $\mathbb{R}^3$, there is no hope to obtain such an estimate with the $L^2_t L^\infty_x$ norm, the so-called endpoint Strichartz estimate for free wave equations, cf. Klainerman-Machedon [KM] and Machihara-Nakamura-Nakanishi-Ozawa [MNNO]. But if we reverse the order of space-time integration, one can obtain a version of reversed Strichartz estimates from the Morawetz estimate, cf. Theorem 2.3:

$$ (1.8) \left\| \sin \left( \frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}} \right) f \right\|_{L^\infty_t L^2_x} \lesssim \| f \|_{L^2(\mathbb{R}^3)}, \left\| \cos \left( \frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}} \right) g \right\|_{L^\infty_t L^2_x} \lesssim \| g \|_{H^1(\mathbb{R}^3)}. $$

These types of estimates are extended to inhomogeneous cases and perturbed Hamiltonian in Goldberg-Beceanu [BecGo]. In Section 4 we will study these estimates and their generalizations intensively.

Next, we consider a linear wave equations with a real-valued stationary potential,

$$ (1.9) H = -\Delta + V, $$

$$ (1.10) \partial_t u + Hu = \partial_t u - \Delta u + Vu = 0, $$

$$ (1.11) u(x,0) = g(x), \quad u_t(x,0) = f(x). $$

Explicitly, we have

$$ (1.12) u = \frac{\sin \left( \frac{t\sqrt{H}}{\sqrt{H}} \right)}{\sqrt{H}} f + \cos \left( \frac{t\sqrt{H}}{\sqrt{H}} \right) g. $$

For the class of short-range potentials we consider in this paper, under our hypotheses $H$ only has pure absolutely continuous spectrum on $[0, \infty)$ and a finite number of negative eigenvalues. It is very crucial to notice that if there is a negative eigenvalue $E < 0$, the associated eigenfunction responds to the wave equation propagators with a scalar factor by $\cos \left( \frac{t\sqrt{E}}{\sqrt{E}} \right)$ or $\frac{\sin \left( \frac{t\sqrt{E}}{\sqrt{E}} \right)}{E^{1/2}}$, both of which will grow exponentially since $\sqrt{E}$ is purely imaginary. Thus, Strichartz estimates for $H$ must include a projection $P_c$ onto the continuous spectrum in order to get away from this situation.

The problem of dispersive decay and Strichartz estimates for the wave equation with a potential has received much attention in recent years, see the papers by Beceanu-Goldberg [BecGo], Krieger-Schlag [KS] and the survey by Schlag [Sch] for further details and references.

The Strichartz estimates in this case are in the form:

$$ (1.13) \left\| \frac{\sin \left( \frac{t\sqrt{H}}{\sqrt{H}} \right)}{\sqrt{H}} P_c f + \cos \left( \frac{t\sqrt{H}}{\sqrt{H}} \right) P_c g \right\|_{L^p_t L^q_x} \lesssim \| g \|_{H^1} + \| f \|_{L^2}, $$

with $2 < p, \frac{1}{2} = \frac{1}{p} + \frac{2}{q}$. One also has the endpoint reversed Strichartz estimates:

$$ (1.14) \left\| \frac{\sin \left( \frac{t\sqrt{H}}{\sqrt{H}} \right)}{\sqrt{H}} P_c f + \cos \left( \frac{t\sqrt{H}}{\sqrt{H}} \right) P_c g \right\|_{L^p_t L^q_x} \lesssim \| f \|_{L^2} + \| g \|_{H^1}. $$
see Theorem 2.4.

In Section 2 and Section 4, we will systematically pass the estimates for free equations to the perturbed case via the structure formula of wave operators. This strategy also works in many other contexts provided that the free solution operators commute with certain symmetries.

For wave equations in \( \mathbb{R}^3 \), there are several difficulties. For example, the failure of the \( L^2_t L^\infty_x \) estimate and the weakness of decay power \( \frac{1}{t} \) in dispersive estimates. The reversed Strichartz estimates might circumvent these difficulties. Reversed Strichartz estimates along time-like trajectories play an important role in the analysis of wave equations of moving potentials. For example, in [GC2], we used some preliminary versions of these estimates to show Strichartz estimates for wave equations with charge transfer Hamiltonian.

There are extra difficulties when dealing with time-dependent potentials. For example, given a general time-dependent potential \( V(x,t) \), it is not clear how to introduce an analog of bound states and a spectral projection. The evolution might not satisfy group properties any more. It might also result in the growth of certain norms of the solutions, see Bourgain’s book [Bou].

The second part of this paper, we apply the endpoint reversed Strichartz estimates along trajectories to study the wave equation with one moving potential:

\[
\partial_{tt} u - \Delta u + V(x - \vec{v}(t)) = 0
\]

which appears naturally in the study of stability problems of traveling solitons.

For Schrödinger equations with moving potentials, one can find references and progress, for example in Beceanu-Soffer [BS], Rodnianski-Schlag-Soffer [RSS]. Compared with Schrödinger equations, wave equations have some natural difficulties, for example the evolution of bound states of wave equations leads to exponential growth meanwhile the evolution of bound states of Schrödinger equations are merely multiplied by oscillating factors. We also notice that Lorentz transformations are space-time rotations, therefore one can not hope to succeed by the approach used with Schrödinger equations based on Galilei transformations. The geometry becomes much more complicated in the wave equation context. A crucial step to study wave equations with moving potentials is to understand the change of the energy under Lorentz transformations. In Chen [GC2], we obtained that the energy stays comparable under Lorentz transformations. In this paper, we study this by a different approach based on local energy conservation which requires less decay of the potential. As a byproduct, we also obtain Agmon’s estimates for the decay of eigenfunctions associated to negatives eigenvalues of \( H \).

1.1. Main results.

**Definition 1.1.** A trajectory \( \vec{v}(t) \in \mathbb{R}^3 \) is said to be an admissible if \( \vec{v}(t) \) is \( C^1 \) and there exist \( 0 \leq \ell < 1 \) such \( |\vec{v}(t)| < \ell < 1 \) for \( t \in \mathbb{R} \).

Consider the solution to the free wave equation \( (H_0 = -\Delta) \),

\[
(1.16) \quad u(x, t) = \frac{\sin (t\sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos (t\sqrt{-\Delta}) g + \int^t_0 \frac{\sin ((t - s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) \, ds
\]

and let \( \vec{v}(t) \in \mathbb{R}^3 \) be an admissible trajectory. Setting

\[
(1.17) \quad u^S(x, t) := u(x + \vec{v}(t), t),
\]
we estimate
\( \sup_{x \in \mathbb{R}^3} \int |u^S(x,t)|^2 \, dt \)
in terms of the initial energy and various norms of \( F \). The idea behind these estimates is that the fundamental solution of the free wave equation is supported on the light cone. Along a time-like curve, the propagation will only meet the light cone once.

**Theorem 1.2.** Let \( \vec{v}(t) \) be an admissible trajectory. Set
\[
(1.19) \quad u(x,t) = \frac{\sin (t \sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos (t \sqrt{-\Delta}) g + \int_0^t \frac{\sin ((t-s) \sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) \, ds
\]
and
\[
(1.20) \quad u^S(x,t) := u(x + \vec{v}(t), t).
\]
Then
\[
(1.21) \quad \|u^S(x,t)\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{H^1} + \|F\|_{W^{1,1}_x L^2_t}.
\]
If \( \vec{v}(t) \) does not change the direction, then
\[
(1.22) \quad \|u^S(x,t)\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{H^1} + \|F\|_{L^1_x L^{1,1}_2}.
\]
where \( d \) is the direction of \( \vec{v}(t) \).

Let \( \vec{\mu}(t) \) be another admissible trajectory, we have the same estimates above with \( F \) replaced by
\[
(1.23) \quad F^S(x,t) := F(x + \vec{\mu}(t), t).
\]
We can extend the above estimates to wave equations with perturbed Hamiltonian,
\[
(1.24) \quad H = -\Delta + V
\]
for \( V \) decays with rate \( \langle x \rangle^{-\alpha} \) for \( \alpha > 3 \), such that \( H \) admits neither eigenfunctions nor resonances at 0. Recall that \( \psi \) is a resonance at 0 if it is a distributional solution of the equation \( H \psi = 0 \) which belongs to the space \( L^2 \left( \langle x \rangle^{-\sigma} \, dx \right) := \{ f : \langle x \rangle^{-\sigma} f \in L^2 \} \) for any \( \sigma > \frac{1}{2} \), but not for \( \sigma = \frac{1}{2} \).

**Theorem 1.3.** Let \( \vec{v}(t) \) be an admissible trajectory. Suppose
\[
(1.25) \quad H = -\Delta + V
\]
admits neither eigenfunctions nor resonances at 0. Set
\[
(1.26) \quad u(x,t) = \frac{\sin (t \sqrt{H})}{\sqrt{H}} P_c f + \cos (t \sqrt{-H}) P_c g + \int_0^t \frac{\sin ((t-s) \sqrt{H})}{\sqrt{H}} P_c F(s) \, ds
\]
and
\[
(1.27) \quad u^S(x,t) := u(x + \vec{v}(t), t),
\]
where \( P_c \) is the projection onto the continuous spectrum of \( H \).

Then
\[
(1.28) \quad \|u^S(x,t)\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{H^1} + \|F\|_{W^{1,1}_x L^2_t}.
\]
If \( \vec{v}(t) \) does not change the direction, then

\[
\|u^S(x,t)\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2} + \|g\|_{H^1} + \|F\|_{L^1_t L^{\frac{5}{2}}_x L^2_t},
\]

where \( d \) is the direction of \( \vec{v}(t) \).

Let \( \vec{\mu}(t) \) be another admissible trajectory, we have the same estimate as above with \( F \) replaced by

\[
F^S(x,t) := F(x + \vec{\mu}(t),t).
\]

We will reply on the structure formula of the wave operators by Beceanu. Although one can obtain similar results without using the structure formula, see [GC2], the goal of our exposition is to illustrate a general strategy that one can pass the estimates for the free evolution to the perturbed one via the structure formula provided there are some symmetries of the free solution operators.

As applications of the above estimates, we study both regular and reversed Strichartz estimates for scattering states to a wave equation with a moving potential. Suppose \( \vec{v}(t) \in \mathbb{R} \) is a trajectory such that there exist \( \vec{\mu} \in \mathbb{R}^3 \)

\[
|\vec{v}(t) - \vec{\mu}t| \lesssim \langle t \rangle^{-\beta}, \quad \beta > 1.
\]

Consider

\[
\partial_t u - \Delta u + V(x - \vec{v}(t)) = 0
\]

with initial data

\[
u(x,0) = g(x), \quad u_t(x,0) = f(x).
\]

An indispensable tool we need to study wave equations with moving potentials is the Lorentz transformations. Without loss of generality, we assume \( \vec{\mu} \) is along \( \vec{e}_1 \). We apply Lorentz transformation \( L \) with respect to a moving frame with speed \( |\mu| < 1 \) along the \( x_1 \) direction. Writing down the Lorentz transformation explicitly, we have

\[
\begin{align*}
t' &= \gamma (t - \mu x_1) \\
x'_1 &= \gamma (x_1 - \mu t) \\
x'_2 &= x_2 \\
x'_3 &= x_3
\end{align*}
\]

with

\[
\gamma = \frac{1}{\sqrt{1 - |\mu|^2}}.
\]

We can also write down the inverse transformation of the above:

\[
\begin{align*}
t &= \gamma (t' + vx'_1) \\
x_1 &= \gamma (x'_1 + \mu t') \\
x_2 &= x'_2 \\
x_3 &= x'_3
\end{align*}
\]

Under the Lorentz transformation \( L \), if we use the subscript \( L \) to denote a function with respect to the new coordinate \( (x',t') \), we have

\[
u_L(x_1',x_2',x_3',t') = u(\gamma (x'_1 + \mu t'), x'_2, x'_3, \gamma (t' + \mu x'_1))
\]
In order to study the equation with time-dependent potentials, we need to introduce a suitable projection. Let
\[ H = -\Delta + V \left( \sqrt{1 - |\mu|^2} x_1, x_2, x_3 \right). \]

Let \( m_1, \ldots, m_w \) be the normalized bound states of \( H \) associated to the negative eigenvalues \( -\lambda^2_1, \ldots, -\lambda^2_w \) respectively (notice that by our assumptions, 0 is not an eigenvalue). In other words, we assume
\[ H m_i = -\lambda^2_i m_i, \quad m_i \in L^2, \lambda_i > 0. \]

We denote by \( P_b \) the projections on the bound states of \( H \) and let \( P_c = Id - P_b \).

To be more explicit, we have
\[ P_b = \sum_{j=1}^{\ell} \langle \cdot, m_j \rangle m_j. \]

With Lorentz transformations \( L \) associated to the moving frame \( (x - \vec{\mu}t, t) \), we use the subscript \( L \) to denote a function under the new frame \( (x', t') \).

**Definition 1.4 (Scattering states).** Let
\[ \partial_t u - \Delta u + V (x - \vec{v}(t)) u = 0 \]
with initial data
\[ u(x, 0) = g(x), \quad u_t(x, 0) = f(x). \]

If \( u \) also satisfies
\[ \|P б u_L(t')\| \to 0, \quad t, t' \to \infty, \]
we call it a scattering state.

**Theorem 1.5 (Strichartz estimates).** Suppose \( u \) is a scattering state in the sense of Definition 1.4 which solves the equation \( \text{(1.31)} \). Then for \( p > 2 \) and \( (p, q) \) satisfying
\[ \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \]
we have
\[ \|u\|_{L^p_t([0, \infty), L^q_x)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \]

The above theorem can be extended to the inhomogeneous case, see for example [GC2].

Secondly, one has the energy estimate:

**Theorem 1.6 (Energy estimate).** Suppose \( u \) is a scattering state in the sense of Definition 1.4 which solves the equation \( \text{(1.31)} \). Then we have
\[ \sup_{t \geq 0} (\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \]

We also have the local energy decay:
Theorem 1.7 (Local energy decay). Suppose $u$ is a scattering state in the sense of Definition 1.4 which solves the equation (1.31). Then for $\forall \varepsilon > 0$, $|\vec{v}| < 1$, we have
\begin{equation}
\left\| (1 + |x - \vec{v}t|)^{\frac{3}{2} - \frac{3}{p}} (|\nabla u| + |u_t|) \right\|_{L^p_t,L^q_x} \lesssim_{\mu, \varepsilon} \| f \|_{L^2} + \| g \|_{H^1}.
\end{equation}

We also obtain the endpoint reversed Strichartz estimates for $u$.

Theorem 1.8 (Endpoint reversed Strichartz estimates). Let $\vec{h}(t)$ be an admissible trajectory. Suppose $u$ is a scattering state in the sense of Definition 1.4 which solves the equation (1.31). Then
\begin{equation}
\sup_{x \in \mathbb{R}^3} \int_0^\infty |u(x, t)|^2 \, dt \lesssim (\| f \|_{L^2} + \| g \|_{H^1})^2,
\end{equation}
and
\begin{equation}
\sup_{x \in \mathbb{R}^3} \int_0^\infty |u(x + \vec{h}(t), t)|^2 \, dt \lesssim (\| f \|_{L^2} + \| g \|_{H^1})^2.
\end{equation}

With the endpoint estimate along $(x + \vec{v}(t), t)$, one can derive the boundedness of the total energy. We denote the total energy of the system as
\begin{equation}
E_V(t) = \int |\nabla_x u|^2 + |\partial_t u|^2 + V(x - \vec{v}(t)) |u|^2 \, dx.
\end{equation}

Corollary 1.9 (Boundedness of the total energy). Suppose $u$ is a scattering state in the sense of Definition 1.4 which solves the equation (1.31). Assume
\begin{equation}
\| \nabla V \|_{L^1} < \infty,
\end{equation}
then $E_V(t)$ is bounded by the initial energy independently of $t$,
\begin{equation}
\sup_{t \geq 0} E_V(t) \lesssim \| (g, f) \|_{H^1 \times L^2}^2.
\end{equation}

Notation. “$A := B$” or “$B := A$” is the definition of $A$ by means of the expression $B$. We use the notation $(x) = (1 + |x|^2)^{\frac{3}{2}}$. The bracket $\langle \cdot, \cdot \rangle$ denotes the distributional pairing and the scalar product in the spaces $L^2, L^2 \times L^2$. For positive quantities $a$ and $b$, we write $a \lesssim b$ for $a \leq C b$ where $C$ is some prescribed constant. Also $a \simeq b$ for $a \lesssim b$ and $b \lesssim a$. Throughout, we use $\partial_t u := \frac{\partial}{\partial t} u$, $u_t := \frac{\partial}{\partial t} u$, $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial x_i}$ and occasionally, $\square := -\partial_t^2 + \Delta$.

Organization. The paper is organized as follows: In Section 2, we discuss some preliminary results for the free wave equation and the wave equation with a stationary potential. In Section 3, we analyze the change of the energy under Lorentz transformations. Agmon’s estimates are also presented as a consequence of our comparison results. In Section 4, the endpoint reversed Strichartz estimates of homogeneous and inhomogeneous forms are derived along admissible trajectories. In Section 5, we show Strichartz estimates, energy estimates, the local energy decay and the boundedness of the total energy for a scattering state to the wave equation with a moving potential. Finally, in Section 6, we confirm that a scattering state indeed scatters to a solution to the free wave equation and also obtain a version of the asymptotic completeness description of the wave equations with one moving potential. In appendices, for the sake of completeness, we show the dispersive estimates for wave equations in $\mathbb{R}^3$ based on the idea of reversed Strichartz estimates, the local energy decay of free wave equations and the global existence of solutions.
to the wave equation with a time-dependent potential. A Fourier analytic proof of the endpoint reversed Strichartz estimates is also presented.

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2. **Preliminaries**

2.1. **Strichartz estimates and the endpoint reversed Strichartz estimates.** We start with Strichartz estimates for free wave equations. Strichartz estimates can be derived abstractly from these dispersive inequalities and the energy inequality. The following theorem is standard. One can find a detailed proof in, for example, Keel-Tao [KT].

**Theorem 2.1 (Strichartz estimates).** Suppose

\[ \partial_{tt} u - \Delta u = F \]

with initial data

\[ u(x, 0) = g(x), \quad u_t(x, 0) = f(x). \]

Then for \( p, a > \frac{3}{2}, (p, q), (a, b) \) satisfying

\[ \frac{3}{2} - s = \frac{1}{p} + \frac{3}{q} \]

\[ \frac{3}{2} - s = \frac{1}{a} + \frac{3}{b} \]

we have

\[ \|u\|_{L^p_t L^q_x} \lesssim \|g\|_{H^s} + \|f\|_{H^{s-1}} + \|F\|_{L^a_t L^b_x} \]

where \( \frac{1}{a} + \frac{1}{b} = 1, \frac{1}{p} + \frac{1}{q} = 1 \).

The endpoint \((p, q) = (2, \infty)\) can be recovered for radial functions in Klainerman-Machedon [KM] for the homogeneous case and Jia-Liu-Schlag-Xu [JLSX] for the inhomogeneous case. The endpoint estimate can also be obtained when a small amount of smoothing (either in the Sobolev sense, or in relaxing the integrability) is applied to the angular variable, see Machihara-Nakamura-Nakanishi-Ozawa [MNNO].

**Theorem 2.2 (MNNO).** For any \( 1 \leq p < \infty \), suppose \( u \) solves the free wave equation

\[ \partial_{tt} u - \Delta u = 0 \]

with initial data

\[ u(x, 0) = g(x), \quad u_t(x, 0) = f(x). \]

Then

\[ \|u\|_{L^2_t L^\infty_x L^p_\omega} \leq C(p) (\|f\|_{L^2} + \|g\|_{H^1}). \]
The regular Strichartz estimates fail at the endpoint. But if one switches the order of space-time integration, it is possible to estimate the solution using the fact that the solution decays quickly away from the light cone. Therefore, we introduce reversed Strichartz estimates. Since we will only use the endpoint reversed Strichartz estimate, we will restrict our focus to that case.

**Theorem 2.3** (Endpoint reversed Strichartz estimates). Suppose
\[
\partial_t u - \Delta u = F
\]
with initial data
\[
u(x,0) = g(x), \quad u_t(x,0) = f(x).
\]
Then
\[
\|u\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2_x} + \|g\|_{\dot{H}^{1}_x} + \|F\|_{L^{3}_x L^{\frac{3}{2}}_t}.
\]
See Section 4 for the detailed proof. For the homogeneous case, one can find an alternative proof based on the Fourier transform in Appendix D.

The above results from Theorem 2.1 and Theorem 2.3 can be generalized to the wave equation with a real stationary potentials.

For the perturbed Hamiltonian,
\[
H = -\Delta + V,
\]
with \( V \lesssim \langle x \rangle^{-\alpha} \) for \( \alpha > 3 \), consider the wave equation with potential in \( \mathbb{R}^3 \):
\[
\partial_t u - \Delta u + Vu = 0
\]
with initial data
\[
u(x,0) = g(x), \quad u_t(x,0) = f(x).
\]
One can write down the solution to it explicitly:
\[
u = \frac{\sin \left( t\sqrt{H} \right)}{\sqrt{H}} f + \cos \left( t\sqrt{H} \right) g.
\]
Let \( P_b \) be the projection onto the point spectrum of \( H \), \( P_c = I - P_b \) be the projection onto the continuous spectrum of \( H \).

With the above setting, we formulate the results from [BecGo].

**Theorem 2.4** (Strichartz and reversed Strichartz estimates). Consider the perturbed Hamiltonian \( H = -\Delta + V \) in \( \mathbb{R}^3 \). Suppose \( H \) has neither eigenvalues nor resonance at zero. Then for all \( 0 \leq s \leq 1 \), \( p > \frac{3}{2} \), and \( (p,q) \) satisfying
\[
\frac{3}{2} - s = \frac{1}{p} + \frac{3}{q}
\]
we have
\[
\left\| \frac{\sin \left( t\sqrt{H} \right)}{\sqrt{H}} P_c f + \cos \left( t\sqrt{H} \right) P_c g \right\|_{L^p_x L^q_t} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}}.
\]
For the endpoint of reversed Strichartz estimates, we have
\[
\left\| \sin \left( \frac{t \sqrt{H}}{\sqrt{H}} \right) P_c f + \cos \left( \frac{t \sqrt{H}}{\sqrt{H}} \right) P_c g \right\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{H^1},
\]
\[
\left\| \int_0^t \sin \left( \frac{(t-s) \sqrt{H}}{\sqrt{H}} \right) P_c F(s) ds \right\|_{L^\infty_x L^2_t} \lesssim \|F\|_{L^2_x L^2_t}^{\frac{2}{3}} L^1_t.
\]

One can find detailed arguments and more estimates in [BecGo]. We will apply the structure of the wave operators to show the above theorem in Section 4.

2.2. Structure of wave operators and its applications. Next, we discuss the structure of wave operators. Again consider
\[
H = -\Delta + V.
\]
For wave operators, we define
\[
W^+(\cdots) = s - \lim_{t \to \infty} e^{itH} e^{it \Delta}.
\]
We know
\[
W^+ (-\Delta) = HW^+
\]
and
\[
(W^+)^* = s - \lim_{t \to \infty} e^{itH_0} e^{-itH} P_c.
\]
By Beceanu [Bec1, Bec2], we have a structure formula for $W^+$ and $(W^+)^*$.

**Theorem 2.5** ([Bec1] [Bec2]). Assume $H = -\Delta + V$ admits neither eigenfunction nor resonances at 0. Then for both $W^+$ and $(W^+)^*$, we have for $f \in L^2$,
\[
W f(x) = f(x) + \int_{S^2} \int_{R^3} g(x, y, \omega) f(S_\omega x + y) dy d\omega,
\]
for some $g(x, y, \omega)$ such that
\[
\int_{S^2} \int_{R^3} \|g(x, y, \omega)\|_{L^\infty_x} dy d\omega < \infty
\]
and where
\[
S_\omega x = x - (2x \cdot \omega) \omega.
\]
is the reflection by the plane orthogonal to $\omega$. Here $W$ is either of $W^+$ or $(W^+)^*$.

**Remark 2.6.** In [Bec1], for the potential $V$, the author only assumes that
\[
V \in B^1 \cap L^2,
\]
where
\[
B^\beta = \left\{ V \mid \sum_{k \in \mathbb{Z}} 2^{\beta k} \| \chi_{\{x \in [2^k, 2^{k+1})\}}(x) V(x) \|_{L^2} < \infty \right\}.
\]

The structure formula (2.24) in Theorem 2.5 is very powerful. One can easily pass many estimates from the free case to the perturbed case provided the solution operators of the free problem commute with certain symmetries. Here we illustrate this idea by a concrete computation based on Theorem 2.2.
Theorem 2.7. Assume $H = -\Delta + V$ admits neither eigenfunction nor resonances at 0. Setting
\begin{equation}
(u^H)^H = \sin\left(\frac{t\sqrt{H}}{\sqrt{H}}\right)P_c f + \cos\left(\frac{t\sqrt{H}}{\sqrt{H}}\right)P_c g,
\end{equation}
then for any $1 \leq p < \infty$, one has
\begin{equation}
\|u^H\|_{L^2_t L^\infty_x L^p_y} \leq C(p) (\|f\|_{L^2} + \|g\|_{H^1}).
\end{equation}

Proof. It suffices to consider
\begin{equation}
(u^H)^H = \sin\left(\frac{t\sqrt{H}}{\sqrt{H}}\right)P_c f.
\end{equation}

By construction,
\begin{equation}
\sin\left(\frac{t\sqrt{H}}{\sqrt{H}}\right)P_c = W^+ \frac{\sin\left(\frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}}\right)}{\sqrt{-\Delta}} (W^+)^*.
\end{equation}

Denoting
\begin{equation}
h = (W^+)^* P_c f,
\end{equation}
we have
\begin{equation}
\|P_c f\|_{L^2} \simeq \|h\|_{L^2}.
\end{equation}
Setting
\begin{equation}
G = \frac{\sin\left(\frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}}\right)}{\sqrt{-\Delta}} h,
\end{equation}
by Theorem 2.5 it is sufficient to consider the boundedness of
\begin{equation}
G + \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} g(x, y, \tau)G(S_\tau x + y) \ dyd\tau.
\end{equation}

Clearly, by Theorem 2.2
\begin{equation}
\|G\|_{L^2_t L^\infty_x L^p_y} \lesssim \|h\|_{L^2} \simeq \|P_c f\|_{L^2}.
\end{equation}

Next, by Minkowski’s inequality,
\begin{equation}
\left\| \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} g(x, y, \tau)G(S_\tau x + y) \ dyd\tau \right\|_{L^2_t L^\infty_x L^p_y} \lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \|g(x, y, \tau)G(S_\tau x + y)\|_{L^2_t L^\infty_x L^p_y} \ dyd\tau
\end{equation}
\begin{equation}
\|g(x, y, \tau)G(S_\tau x + y)\|_{L^2_t L^\infty_x L^p_y} \lesssim \|g(x, y, \tau)\|_{L^\infty_x} \|G(S_\tau x + y)\|_{L^2_t L^\infty_x L^p_y}.
\end{equation}

Since reflections with respect to a fixed plane and translations commute with the solution of a free wave equation, we obtain
\begin{equation}
G(S_\tau x + y) = \frac{\sin\left(\frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}}\right)}{\sqrt{-\Delta}} h(S_\tau x + y).
\end{equation}
Therefore,
\begin{equation}
\| G(S\tau x + y) \|_{L^2 L^\infty L^p} \lesssim \| h(S\tau x + y) \|_{L^2} \lesssim \| h \|_{L^2} \simeq \| P_c f \|_{L^2}.
\end{equation}
It follows
\begin{equation}
\left\| G + \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} g(x, y, \tau) G(S\tau x + y) \, dyd\tau \right\|_{L^2 L^\infty L^p} \lesssim \left( 1 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \| g(x, y, \tau) \|_{L^\infty} \, dyd\tau \right) \| P_c f \|_{L^2} \lesssim \| f \|_{L^2}.
\end{equation}
Then we conclude
\begin{equation}
\| u^H \|_{L^2 L^\infty L^p} \leq C(p, V) (\| f \|_{L^2} + \| g \|_{H^1}),
\end{equation}
as claimed. \hfill \Box

One can do similar arguments to obtain many other estimates for the perturbed wave equations, for example the local energy decay estimate, the energy estimate and many weighted estimates.

The following Christ-Kiselev Lemma is important in our derivation of Strichartz estimates.

**Lemma 2.8 (Christ-Kiselev).** Let $X, Y$ be two Banach spaces and let $T$ be a bounded linear operator from $L^\beta (\mathbb{R}^+; X)$ to $L^\gamma (\mathbb{R}^+; Y)$, such that
\begin{equation}
Tf(t) = \int_0^\infty K(t, s) f(s) \, ds.
\end{equation}
Then the operator
\begin{equation}
\tilde{T} = \int_0^t K(t, s) f(s) \, ds
\end{equation}
is bounded from $L^\beta (\mathbb{R}^+; X)$ to $L^\gamma (\mathbb{R}^+; Y)$ provided $\beta < \gamma$, and the
\begin{equation}
\| \tilde{T} \| \leq C(\beta, \gamma) \| T \|
\end{equation}
with
\begin{equation}
C(\beta, \gamma) = \left( 1 - 2^{\beta - \frac{1}{2}} \right)^{-1}.
\end{equation}

3. **Lorentz Transformations and Energy**

When we consider wave equations with moving potentials, Lorentz transformations will be important for us to reduce some estimates to stationary cases. In order to approach our problem from the viewpoint of Lorentz transformations as in [GC2], the first natural step is to understand the change of energy under Lorentz transformations.

Indeed, in [GC2], we shown that under Lorentz transformations, the energy stays comparable to that of the initial data. The method in [GC2] is based on integration by parts. Here we present an alternative approach based on the local energy conservation which is more natural and requires less decay of the potential. We notice that the method in [GC2] can be viewed as the differential version of the argument here.
Throughout this section, we perform Lorentz transformations with respect to a moving frame with speed $|v| < 1$, say, along the $x_1$ direction, i.e., the velocity is
\[ \vec{v} = (v, 0, 0). \]
Recall that after we apply the Lorentz transformation, for function $u$, under the new coordinates, we denote
\[ u_L(x'_1, x'_2, x'_3, t') = u(\gamma (x'_1 + vt'), x'_2, x'_3, \gamma (t' + vx'_1)). \]
Now let $u$ be a solution to some wave equation and set $t' = 0$. We notice that in order to show under Lorentz transformations, the energy stays comparable to that of the initial data, up to an absolute constant it suffices to prove
\[ \int |\nabla u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx \]
\[ \simeq \int |\nabla u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx. \]

Throughout this section, we will assume all functions are smooth and decay fast. We will obtain estimates independent of the additional smoothness assumption. It is easy to pass the estimates to general cases with a density argument.

**Remark 3.1.** One can observe that all discussions in this section hold for $\mathbb{R}^n$.

### 3.1. Energy comparison

In this section, a more general situation is analyzed. We consider wave equations with time-dependent potentials
\[ \partial_{tt} u - \Delta u + V(x, t)u = 0 \]
with
\[ |V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^2} \]
uniformly for $0 \leq |\mu| \leq 1$. These in particular apply to wave equations with moving potentials with speed strictly less than 1. For example, if the potential is of the form
\[ V(x, t) = V(x - vt) \]
with
\[ |V(x)| \lesssim \frac{1}{\langle x \rangle^2} \]
then it is transparent that
\[ |V(x, \mu x_1)| = |V(x - \nu \mu x_1)| \lesssim \frac{1}{\langle x \rangle^2}. \]
Suppose
\[ \partial_{tt} u - \Delta u + V(x, t)u = 0, \]
then it is clear that
\[ 0 = u_t (\Box u - V(t)u) \]
\[ = -\partial_t \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) + \text{div} (\nabla uu_t) - V(x, t)uu_t. \]
Lemma 3.2. Let $|v| < 1$. Suppose

$$
\partial_t u - \Delta u + V(x,t)u = 0
$$

and

$$
|V(x,\mu x_1)| \lesssim \frac{1}{(x)^2}
$$

uniformly for $0 \leq |\mu| < 1$. Then for arbitrary $R > 0$, there exists some constant $M(v) > 1$ depending on $v$ such that

$$
\int_{|x| > M(v)R} |\nabla_x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (x_1, x_2, x_3, vx_1)|^2 \, dx \lesssim \int_{|x| > R} |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx,
$$

where the implicit constant depends on $v, V$.

Proof. Denote

$$
L^U_+ (u, \mu, R) = \int_{|x| > M(\mu)R, x_1 > 0} |V(x, \mu x_1)| |u(x_1, x_2, x_3, \mu x_1)|^2 \, dx.
$$

$$
T^U_+ (u, \mu, R) = \int_{|x| > M(\mu)R, x_1 > 0} |V(x, \mu x_1)| |u_t(x_1, x_2, x_3, \mu x_1)|^2 \, dx.
$$

$$
E^U_+ (u, \mu, R) = \int_{|x| > M(\mu)R, x_1 > 0} |\nabla_x u (x_1, x_2, x_3, \mu x_1)|^2 + |\partial_t u (x_1, x_2, x_3, \mu x_1)|^2 \, dx,
$$

where

$$
M(\mu) = \frac{1}{1 - |\mu|}.
$$

One observes that if

$$
|V(x, \mu x_1)| \lesssim \frac{1}{(x)^2}
$$

uniformly with respect to $0 \leq |\mu| < 1$, by Hardy’s inequality,

$$
L^U_+ (u, \mu, R) + T^U_+ (u, \mu, R) \lesssim E^U_+ (u, \mu, R).
$$

With these notations, we need to show

$$
E^U_+ (u, v, R) \lesssim E^U_+ (u, 0, R).
$$

For fixed $R > 0$, we construct two regions as follows:

$L^+_R$ with equation

$$
x_1^2 + x_2^2 + x_3^2 \leq (R + t)^2, \quad 0 \leq t < \infty, \quad x_1 \geq 0
$$

and $L^-_R$ with equation

$$
x_1^2 + x_2^2 + x_3^2 \leq (R - t)^2, \quad -\infty < t \leq 0, \quad x_1 \leq 0.
$$

Denote the region bounded by $((0, \infty) \times \mathbb{R}^2 \times [0, \infty)) \setminus L^+_R$ and the plane $(x_1, x_2, x_3, vx_1)$ by $Y^+$ and use $Y^-$ to denote the region bounded by $((\infty, 0] \times \mathbb{R}^2 \times (-\infty, 0]) \setminus L^-_R$ and the plane $(x_1, x_2, x_3, vx_1)$.
By symmetry, it suffices to analyze $Y^+$. We apply the space-time divergence theorem to
\begin{equation}
\left( \nabla u_{tt}, - \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \right)
\end{equation}
in region $Y^+$. We denote the top of $Y^+$ by $Y^+_T$, the bottom as $Y^+_B$ and the lateral boundary as $Y^+_L$. We should notice $Y^+_B$ actually is $\{ x_1 > 0 \} \cap (\mathbb{R}^3 \setminus B_R(0))$.

The unit outward-pointing normal vector on the plane $(x_1, x_2, x_3, v_1)$ is
\begin{equation}
\frac{1}{\sqrt{v^2 + 1}} (-v, 0, 0, 1).
\end{equation}
The outward-pointing normal vector on the bottom of $Y^+$ is
\begin{equation}
(0, 0, 0, -1).
\end{equation}
From (3.10), one obtains
\begin{equation}
\frac{1}{\sqrt{v^2 + 1}} \int_{Y^+_T} \left[ v \partial_{x_1} u_{tt} - \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \right] dx + \int_{Y^+_B} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx
\end{equation}
(3.26)

\begin{equation}
= \frac{1}{2\sqrt{2}} \int_{Y^+_L} |\nabla u - n_L(x)u_{tt}|^2 d\sigma + \int_{Y^+} V(x, t) u_{tt} dx dt,
\end{equation}
where $n_L(x)$ is a vector of norm 1.

Note that
\begin{equation}
\int_{Y^+} |V(x, t) u_{tt}| dx dt \lesssim \int_{Y^+} |V(x, t)| \left( |u|^2 + |u_t|^2 \right) dx dt.
\end{equation}
Hence we can conclude
\begin{equation}
\frac{1}{\sqrt{v^2 + 1}} \int_{Y^+_T} \left[ -v \partial_{x_1} u_{tt} + \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \right] dx
\end{equation}
(3.28)
\begin{equation}
\lesssim \int_{Y^+_B} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx + \int_{Y^+_L} |V(x, t)| \left( |u|^2 + |u_t|^2 \right) dx dt.
\end{equation}
Consider the integral
\begin{equation}
\int_{Y^+} |V(x, t)| \left( |u|^2 + |u_t|^2 \right) dx dt,
\end{equation}
by a change of variable and Fubini’s Theorem, it follows
\begin{equation}
\int_{Y^+} |V(x, t)| \left( |u|^2 + |u_t|^2 \right) dt = \int_0^v \left( L^U_+ (u, \mu, R) + T^U_+ (u, \mu, R) \right) d\mu
\end{equation}
(3.30)
\begin{equation}
\lesssim \int_0^v E^U_+ (u, \mu, R) d\mu.
\end{equation}
Note that with $|v| < 1$ and the AM–GM inequality, we obtain
\begin{equation}
(1 - |v|) \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \leq -v \partial_{x_1} u_{tt} + \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right),
\end{equation}
(3.31)
which implies
\[
\frac{(1 - |v|)}{\sqrt{v^2 + 1}} \int_{Y^+_1} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \, dx \lesssim \int_{Y^+_0} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \, dx + \int_0^{u_+} E_u^U (u, \mu, R) \, d\mu
\]
(3.32)

With our notations, we have
\[
E_+^U (u, v, R) \lesssim \frac{\sqrt{v^2 + 1}}{|1 - |v||} \left( E_+^U (u, 0, R) + \int_0^{v} E_+^U (u, \mu, R) \, d\mu \right)
\]
(3.33)

By Grönwall’s inequality with respect to $v$, one obtains
\[
E_+^U (u, v, R) \lesssim E_+^U (u, 0, R)
\]
(3.34)
provided $|v| < 1.$

By construction, we have
\[
\int_{|x| > M(v)R, x_1 > 0} |\nabla_x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (x_1, x_2, x_3, vx_1)|^2 \, dx
\]
(3.35)
\[
\lesssim \int_{|x| > R, x_1 > 0} |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx
\]
A similar argument for $Y^-$ gives
\[
\int_{|x| > M(v)R, x_1 \leq 0} |\nabla_x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (x_1, x_2, x_3, vx_1)|^2 \, dx
\]
(3.36)
\[
\lesssim \int_{|x| > R, x_1 \leq 0} |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx.
\]
Hence, we get
\[
\int_{|x| > M(v)R} |\nabla_x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (x_1, x_2, x_3, vx_1)|^2 \, dx
\]
(3.37)
\[
\lesssim \int_{|x| > R} |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx
\]
as claimed.

\[\square\]

Lemma 3.3. Let $|v| < 1$. Suppose
\[
\partial_t u - \Delta u + V(x, t) u = 0
\]
(3.38)
and
\[
|V(x, \mu x_1)| \lesssim \frac{1}{(|x|)^2}
\]
(3.39)
uniformly for $0 \leq |\mu| < 1.$ Then for arbitrary $R > 0$, there exists some constant $M_1(v) < 1$ depending $v$ such that
\[
\int_{|x| > R} |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx
\]
(3.40)
\[
\lesssim \int_{|x| > M_1(v)R} |\nabla_x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (x_1, x_2, x_3, vx_1)|^2 \, dx,
\]
where the implicit constant depends on $v, V.$
Proof. Denote

\[ L^L_+ (u, \mu, R) = \int_{|x| > M_1(\mu)R, x_1 > 0} |V(x, \mu x_1)| \left| u(x_1, x_2, x_3, \mu x_1) \right|^2 \, dx. \]  

(3.41)

\[ T^L_+ (u, \mu, R) = \int_{|x| > M_1(\mu)R, x_1 > 0} |V(x, \mu x_1)| \left| u_t(x_1, x_2, x_3, \mu x_1) \right|^2 \, dx. \]  

(3.42)

\[ E^L_+ (u, \mu, R) = \int_{|x| > M_1(\mu)R, x_1 > 0} |\nabla_x u(x_1, x_2, x_3, \mu x_1)|^2 + |\partial_t u(x_1, x_2, x_3, \mu x_1)|^2 \, dx, \]  

where

\[ M_1(\mu) = \frac{1}{1 + |\mu|}. \]  

(3.43)

One observes that if

\[ |V(x, \mu x_1)| \lesssim \frac{1}{(x_1^2)}, \]  

(3.44)

uniformly with respect to \( 0 \leq |\mu| < 1 \), by Hardy’s inequality,

\[ L^L_+ (u, \mu, R) + T^L_+ (u, \mu, R) \lesssim E^L_+ (u, \mu, R). \]  

(3.45)

With these notations, we need to show

\[ E^L_+ (u, 0, R) \lesssim E^L_+ (u, v, R). \]  

(3.46)

For fixed \( R > 0 \), we construct two regions as follows:

- \( C^+_R \) with equation

\[ x_1^2 + x_2^2 + x_3^2 \leq (R - t)^2, \quad 0 \leq t \leq R, \quad x_1 \geq 0 \]  

(3.47)

and \( C^-_R \) with equation

\[ x_1^2 + x_2^2 + x_3^2 \leq (R + t)^2, \quad -R \leq t \leq 0, \quad x_1 \leq 0. \]  

(3.48)

Denote the region bounded by \( \{(0, \infty) \times \mathbb{R}^2 \times [0, \infty)\} \setminus \{C^+_R \} \) and the plane \( (x_1, x_2, x_3, v x_1) \) by \( K^+ \) and use \( K^- \) to denote the region \( (\infty, 0) \times \mathbb{R}^2 \times (\infty, 0) \setminus \{C^-_R \} \) and the plane \( (x_1, x_2, x_3, v x_1) \).

By symmetry, it suffices to analyze \( K^+ \). We again apply the space-time divergence theorem to

\[ \left( \nabla u u_t, -\left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \right) \]  

in region \( K^+ \) as above. We denote the top of \( K^+ \) by \( K^+_T \), the bottom as \( K^+_B \) and the lateral boundary as \( K^+_L \). The unit outward-pointing normal vector on the plane \( (x_1, x_2, x_3, v x_1) \) is

\[ \frac{1}{\sqrt{v^2 + 1}} (v, 0, 0, 1). \]  

(3.49)

The outward-pointing normal vector on the bottom of \( K^+ \) is

\[ (0, 0, 0, -1). \]  

(3.50)
One obtains from (3.10),
\[
\frac{1}{\sqrt{v^2 + 1}} \int_{K_T^+} \left[ v \partial_{x_1} uu_t - \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \right] dx + \int_{K_T^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx 
\]
(3.53) \leq -\frac{1}{2\sqrt{2}} \int_{K_L^+} |\nabla u - n(x)u_t|^2 \ d\sigma + \int_{K^+} |V(x,t)| \left( |u|^2 + |u_t|^2 \right) dx dt,

where \( n(x) \) is a vector of norm 1.

Note that
\[
\int_{K^+} |V(t)uu| \ dx dt \lesssim \int_{K^+} |V(x,t)| \left( |u|^2 + |u_t|^2 \right) dx dt.
\]

Hence we can conclude
\[
\int_{K_L^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx \lesssim \frac{1}{\sqrt{v^2 + 1}} \int_{K_T^+} \left[ -v \partial_{x_1} uu_t + \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \right] dx 
\]
(3.55) + \int_{K^+} |V(x,t)| \left( |u|^2 + |u_t|^2 \right) dx dt.

Again, consider the integral
\[
\int_{K^+} |V(x,t)| \left( |u|^2 + |u_t|^2 \right) dx dt,
\]
by a change of variable and Fubini’s Theorem, it follows
\[
\int_{K^+} |V(x,t)| \left( |u|^2 + |u_t|^2 \right) dx dt = \int_0^\nu \left( L^L_+ (u, \mu, R) + T^L_+ (u, \mu, R) \right) d\mu
\]
(3.57) \lesssim \int_0^\nu E^L_+ (u, \mu, R) d\mu.

Hence we can conclude
\[
\int_{K_L^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx \lesssim \frac{1 + |v|}{\sqrt{v^2 + 1}} \int_{K_T^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx 
\]
(3.58) + \int_0^\nu E^L_+ (u, \mu, R) d\mu.

In other words,
\[
E^L_+ (u, 0, R) \lesssim \frac{1 + |v|}{\sqrt{v^2 + 1}} \left( E^L_+ (u, v, R) + \int_0^\nu E^L_+ (u, \mu, R) d\mu \right).
\]
(3.59)

With Grönwall’s inequality again, it implies
\[
E^L_+ (u, 0, R) \lesssim E^L_+ (u, v, R)
\]
(3.60)

Therefore,
\[
\int_{K_L^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx \lesssim \int_{K_T^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx.
\]
(3.61)
By construction and a similar argument applied to $K^-$, we obtain precisely

\begin{equation}
\int_{|x|>R} |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx \lesssim \int_{|x|>M(v)R} |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx,
\end{equation}

as claimed. \hfill \Box

**Theorem 3.4.** Let $|v| < 1$. Suppose

\begin{equation}
\partial_t u - \Delta u + V(x, t)u = 0
\end{equation}

and

\begin{equation}
|V(x, \mu x_1)| \lesssim \frac{1}{|x|^2}
\end{equation}

for $0 \leq |\mu| < 1$. Then

\begin{equation}
\int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx \simeq \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx,
\end{equation}

where the implicit constant depends on $v$ and $V$.

**Proof.** We first show

\begin{equation}
\int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx \lesssim \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx.
\end{equation}

This follows from Lemma 3.2 which implies for $R > 0$,

\begin{equation}
\int_{|x|>M(v)R} |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx \lesssim \int_{|x|>R} |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx
\end{equation}

with an implicit constant independent of $R$. By the monotone convergence theorem, letting $R \to 0$, we get

\begin{equation}
\int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx \lesssim \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx.
\end{equation}
Next, we establish the converse inequality. By Lemma 3.3 for $R > 0$,

\[
\int_{|x| > R} |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx
\]

(3.69) \lesssim \int_{|x| > M_1(v)R} |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx

(3.70)

with an implicit constant independent of $R$. Letting $R \to 0$, from the dominated convergence theorem, it follows that

\[
\int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx \sim \int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx.
\]

(3.71)

Hence, we conclude

\[
\int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx
\]

(3.72)

\[\simeq \int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx.\]

The theorem is proved.

3.2. Agmon’s estimates via wave equations. As a by product of Theorem 3.4, we show Agmon’s estimates [Agmon] for the decay of eigenfunctions associated with negative eigenvalues of

\[
H = -\Delta + V.
\]

Again, we restrict our attention to the class of potentials satisfying the assumption

\[
|V(x)| \leq C_V (1 + x^2)^{-1}, \forall x \in \mathbb{R}^3.
\]

(3.74)

As in Remark 3.1, all arguments and discussions are valid for $x \in \mathbb{R}^n$.

**Theorem 3.5 (Agmon).** Let $V$ satisfy the assumption (3.74). Suppose $\phi \in W^{2,2}$

\[
-\Delta \phi + V\phi = E\phi, \, E < 0.
\]

(3.75)

Then $\forall \alpha \in [0, 2\sqrt{-E})$

\[
\int_{\mathbb{R}^3} e^{\alpha|x|} |\phi(x)|^2 \, dx \simeq \int_{\mathbb{R}^3} |\phi(x)|^2 \, dx,
\]

(3.76)

with implicit constants depending on $\alpha$, $V$.

Furthermore, if $V \in W^k(\mathbb{R}^3)$ where $k > \frac{3}{2}$ and for $0 \leq i \leq k$,

\[
|\nabla^i V(x)| \leq C_{V,i} (1 + x^2)^{-1}
\]

(3.77)

then

\[
|\phi(x)| \lesssim e^{-\frac{\alpha}{2}|x|}.
\]

(3.78)
Proof. It suffices to show $\forall \alpha \in [0,2\sqrt{-E})$

$$\int_{\mathbb{R}^3} e^{\alpha|x_j|} |\phi(x)|^2 \, dx \simeq \int_{\mathbb{R}^3} |\phi(x)|^2 \, dx, \, \forall j = 1, 2, 3.$$ 

Without loss of generality, we pick $j = 1$. With Theorem 3.4, we know if $u_{tt} + Hu = 0$, then with $|v| < 1$

$$\int |\nabla x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (\gamma x_1, x_2, x_3, vx_1)|^2 \, dx \simeq \int |\nabla x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx.$$ 

(3.79)

We can rewrite the above result using half-wave operator $e^{it\sqrt{H}}$, for $f \in L^2$ then

$$\int (|e^{iv\sqrt{-E}}f|^2)_{x=x_1} \, dx \simeq \int |f|^2 \, dx.$$ 

(3.80)

We pick $f = \phi$ satisfying

$$-\Delta \phi + V \phi = E\phi, \, E < 0,$$

then

$$\int e^{-v\sqrt{-E}} |\phi|^2 \, dx = \int |e^{-v\sqrt{-E}}\phi|^2 \, dx \simeq \int |\phi|^2 \, dx.$$ 

(3.81)

With $v$ replaced by $-v$, we obtain

$$\int e^{v\sqrt{-E}} |\phi|^2 \, dx = \int |e^{v\sqrt{-E}}\phi|^2 \, dx \simeq \int |\phi|^2 \, dx.$$ 

(3.82)

Therefore,

$$\int e^{2v\sqrt{-E}|x_j|} |\phi|^2 \, dx \simeq \int |\phi|^2 \, dx.$$ 

(3.83)

Fixed an $\alpha \in [0,2\sqrt{-E})$, we can find $|v| \in [0,1)$ such that $\alpha = |2v\sqrt{-E}|$, then it follows that

$$\int e^{\alpha|x_j|} |\phi|^2 \, dx \simeq \int |\phi|^2 \, dx.$$ 

(3.84)

Therefore the estimate (3.76) is proved.

Next we move to (3.78). Since

$$-\Delta \phi + V \phi = E\phi, \, E < 0,$$

then

$$\int |\nabla \phi|^2 \, dx + \int V |\phi|^2 \, dx = E \int |\phi|^2 \, dx.$$ 

(3.85)

$$\int |\nabla \phi|^2 \, dx \leq \|V\|_{L^\infty} \int |\phi|^2 \, dx.$$ 

(3.86)

Differentiating the equation, for any multi-index $\beta$

$$-\Delta (\partial^\beta \phi) + \partial^\beta (V \phi) = E\partial^\beta \phi$$

we can conclude

$$\int |\nabla (\partial^\beta \phi)|^2 \, dx \leq \int \partial^\beta (V \phi) \partial^\beta \phi \, dx.$$ 

(3.87)
By induction, we obtain
\begin{equation}
(3.91) \quad \int |\nabla (\partial^3 \phi)|^2 \, dx \leq \|V\|_{W^{1,\infty}} \int |\phi|^2 \, dx.
\end{equation}

Let $\psi$ be a smooth bump-cutoff function such that $\psi = 1$ in $B_1(0)$ and $\psi = 0$ in $\mathbb{R}^3 \setminus B_2(0)$. We localize our estimate,

\begin{equation}
(3.92) \quad \int (\Delta \phi(x) + V \phi(x)) \bar{\phi}(x) \psi^2(x-y) \, dx = E \int |\phi(x)|^2 \psi^2(x-y) \, dx.
\end{equation}

Integrating by parts, we know

\begin{equation}
(3.93) \quad \int (\Delta \phi(x) + V \phi(x)) \bar{\phi}(x) \psi^2(x-y) \, dx = \int V |\phi(x)|^2 \psi^2(x-y) \, dx + \int |\nabla \phi(x)|^2 \psi^2(x-y) \, dx + 2 \int \nabla \phi(x) \bar{\phi}(x) \psi(x-y) \nabla \psi(x-y) \, dx.
\end{equation}

Therefore, by the Cauchy-Schwarz inequality,

\begin{equation}
(3.94) \quad \int |\nabla \phi(x)|^2 \psi^2(x-y) \, dx \lesssim E \int |\phi(x)|^2 \psi^2(x-y) \, dx + \int V |\phi(x)|^2 \psi^2(x-y) \, dx + 2 \int |\phi(x)\nabla \psi(x-y)|^2 \, dx.
\end{equation}

It follows,

\begin{equation}
(3.95) \quad \sup_{y \in \mathbb{R}^3} \int_{|x-y| \leq 1} |\nabla \phi(x)|^2 \, dx \lesssim (\|V\|_{L^\infty} + 1 + |E|) \int_{|x-y| \leq 1} |\phi(x)|^2 \, dx.
\end{equation}

Inductively as above, we have

\begin{equation}
(3.96) \quad \sup_{y \in \mathbb{R}^3} \int_{|x-y| \leq 1} |\nabla (\partial^3 \phi)|^2 \, dx \lesssim (\|V\|_{W^{1,\infty}} + 1 + |E|) \int_{|x-y| \leq 1} |\phi(x)|^2 \, dx.
\end{equation}

Finally by Sobolev’s embedding theorem,

\begin{equation}
(3.97) \quad \sup_{y \in \mathbb{R}^3} \sup_{|x-y| \leq 1} |\phi(x)|^2 \lesssim \sum_{\beta \leq k} \sup_{y \in \mathbb{R}^3} \int_{|x-y| \leq 1} |(\partial^\beta \phi)|^2 \, dx
\lesssim \int_{|x-y| \leq 2} |\phi(x)|^2 \, dx
\lesssim e^{-\alpha|y|} \int_{|x-y| \leq 2} e^{\alpha|x|} |\phi(x)|^2 \, dx
\lesssim e^{-\alpha|y|} \int |\phi(x)|^2 \, dx
\lesssim e^{-\alpha|y|}.
\end{equation}

Hence,

\begin{equation}
(3.98) \quad \sup_{y \in \mathbb{R}^3} |\phi(y)| \lesssim e^{-\frac{\alpha}{2}|y|}
\end{equation}
4. ENDPOINT STRICHARZT ESTIMATES

In [GC2], we analyzed the endpoint reversed Strichartz estimates along slanted lines for both homogeneous and inhomogeneous cases. Intuitively, the reversed Strichartz estimates along slanted lines are based on the fact that the fundamental solutions of the wave equation in $\mathbb{R}^3$ is supported on the light cone. For fixed $x$, the propagation will only meet the light cone once. Here we further note that for a general smooth trajectory with velocity strictly less than 1, it will also only intersect the light cone once. In this section, we will study the reversed Strichartz estimates along general trajectories in several different settings.

Recall that a trajectory $\vec{v}(t) \in \mathbb{R}^3$ is called an admissible trajectory if $\vec{v}(t)$ is $C^1$ and there exist $0 \leq \ell < 1$ such $|\vec{v}(t)| < \ell < 1$ for $t \in \mathbb{R}$.

4.1. Free wave equations.

**Theorem 4.1.** Let $\vec{v}(t)$ be an admissible trajectory. Set

$$u(x,t) = \frac{\sin (t\sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos (t\sqrt{-\Delta}) g + \int_0^t \frac{\sin ((t - s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) \, ds$$

and

$$u^S(x,t) := u(x + \vec{v}(t), t).$$

First of all, for the standard case, one has

$$\|u\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L^3_d L^2_t}.$$

Consider the estimates along the trajectory $\vec{v}(t)$, one has

$$\|u^S(x,t)\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}^{1,1}_d L^2_t}.$$

If $\vec{v}(t)$ does not change the direction, then

$$\|u^S(x,t)\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L^1_d L^2_t}.$$

where $d$ is the direction of $\vec{v}(t)$.

Let $\vec{\mu}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by

$$F^S(x,t) := F(x + \vec{\mu}(t), t).$$

**Proof.** For the first term,

$$u_1(x,t) = \frac{\sin (t\sqrt{-\Delta})}{\sqrt{-\Delta}} f = \frac{1}{4\pi t} \int_{|x-y|=t} f(y) \sigma (dy).$$

So in polar coordinates,

$$\|u_1^S(x,t)\|_{L^2_t[0,\infty)}^2 \lesssim \int_0^\infty \left( \int_{\mathbb{S}} f(x + \vec{v}(r) + r\omega) r \, d\omega \right)^2 \, dr.$$
\[
\int_0^\infty \left( \int_S f(x + \vec{v}(r) + r\omega) r \, dr \right)^2 \, d\omega = \int_0^\infty \left( \int_S f(x + (\vec{v}(r)/r + \omega) r) \, dr \right)^2 \, d\omega \\
\lesssim \left( \int_0^\infty \int_S f(x + r\eta)^2 r^2 \, d\eta \, dr \right) \left( \int_{S^2} d\eta \right)
\]

(4.9)

Let
\[
\eta = \vec{v}(r)/r + \omega.
\]
Since \(|v'(t)| < \lambda < 1\), the Jacobian of this change of variable is bounded from below uniformly. Therefore,
\[
\int_0^\infty \left( \int_S f(x + (\vec{v}(r)/r + \omega) r) \, d\omega \right)^2 \, dr \approx \left( \int_0^\infty \int_S f(x + r\eta)^2 r^2 \, d\eta \, dr \right) \left( \int_{S^2} d\eta \right)
\]
\[
\lesssim \|f\|_{L^2}^2.
\]

(4.11)

A similar argument holds for
\[
u_2(x, t) = \cos \left( t\sqrt{-\Delta} \right) g.
\]
Therefore
\[
\|u_1^S\|_{L^\infty_t L^2_x} + \|u_2^S\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]
In particular,
\[
\|u_1\|_{L^\infty_t L^2_x} + \|u_2\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]

(4.14)
as claimed.

Next, we consider the inhomogenous case,
\[
D(x, t) = \int_0^t \frac{\sin \left( (t-s)\sqrt{-\Delta} \right)}{\sqrt{-\Delta}} F(s) \, ds.
\]

(4.15)

For the standard case, we consider
\[
\left\| \int_0^t \frac{\sin \left( (t-s)\sqrt{-\Delta} \right)}{\sqrt{-\Delta}} F(s) \, ds \right\|_{L^2_t} = \left\| \int_0^t \int_{|x-y|=t-s} \frac{1}{|x-y|} F(y, s) \, dy \, ds \right\|_{L^2_t}
\]
\[
= \left\| \int_{|x-y| \leq t} \frac{1}{|x-y|} F(y, t - |x-y|) \, dy \right\|_{L^2_t}
\]
\[
\lesssim \int \frac{1}{|x-y|} \|F(y, t - |x-y|)\|_{L^2_t} \, dy
\]
\[
\lesssim \sup_{x \in \mathbb{R}^3} \int \frac{1}{|x-y|} \|F(y, t)\|_{L^2_t} \, dy
\]
\[
\lesssim \|F\|_{L^{\frac{3}{2}}_t L^2_x}.
\]

Therefore, indeed,
\[
\|D\|_{L^\infty_t L^2_x} \lesssim \|F\|_{L^{\frac{3}{2}}_t L^2_x}.
\]

(4.16)
Now we consider the estimate along an admissible trajectory $\vec{v}(t) \in \mathbb{R}^3$.

We first notice that from the above discussion or the argument in Appendix D,

\begin{equation}
T := e^{it\sqrt{-\Delta}} \sqrt{-\Delta}
\end{equation}

is a bounded operator from $L^2_x$ to $L^\infty_x L^2_t$. Also the operator $T^S$:

\begin{equation}
T^S f := (T f)^S = \left( e^{it\sqrt{-\Delta}} \right)^S
\end{equation}

is a bounded operator from $L^2_x$ to $L^\infty_x L^2_t$.

Writing down the inhomogeneous evolution explicitly, one has

\begin{equation}
\left\| \int_0^t \sin \left( (t-s)\sqrt{-\Delta} \right) F(s) \, ds \right\|_{L^2_t} \lesssim \sup_{x \in \mathbb{R}^3} \left\| \int_{|x-y| \leq t} \frac{1}{|x-y|} F(y, t - |x-y|) \, dy \right\|_{L^2_t}
\end{equation}

Therefore,

\begin{equation}
\sup_{x \in \mathbb{R}^3} \left\| \int_{|x-y| \leq t} \frac{1}{|x-y|} F(y, t - |x-y|) \, dy \right\|_{L^2_t} \lesssim \sup_{x \in \mathbb{R}^3} \left\| \int_{|x-y| \leq t} \frac{1}{|x-y|} |F(y, t - |x-y|)| \, dy \right\|_{L^2_t}
\end{equation}

(4.20)

Hence we know

\begin{equation}
\sup_{x \in \mathbb{R}^3} \left\| D^S (x + v(t), t) \right\|_{L^2_t} \lesssim \sup_{x \in \mathbb{R}^3} \left\| \Re \left( T^S T^* \sqrt{-\Delta} |F| \right) \right\|_{L^2_t}
\end{equation}

(4.21)

\begin{equation}
\left\| F \right\|_{W^{1,1}_2 \cap L^2_t}.
\end{equation}

If the trajectory does not change the direction, we can obtain an estimate which does not require $\sqrt{-\Delta} F$ by a similar argument to the estimates along slanted lines in [GC2]. Without loss of generality, we assume the direction of the trajectory is along $x_1$. Then

\begin{equation}
D^S (x, t) = \int_0^t \int_{|x+\vec{v}(t)-y| = t-s} \frac{F(y,s)}{|x+\vec{v}(t)-y|} \sigma(dy) \, ds
\end{equation}

(4.22)
and
\[
\|D^S(x, \cdot)\|_{L^2_t} = \left\| \int_0^t \int_{[x + \bar{v}(t) - y] = t - s} \frac{F(y, s)}{|x + \bar{v}(t) - y|} \sigma(dy) ds \right\|_{L^2_t}
\]
(4.23)
\[
= \left\| \int_{|y| \leq t} \frac{F(x + \bar{v}(t) - y, t - |y|)}{|y|} dy \right\|_{L^2_t}
\]
\[
\leq \left\| \int_{\mathbb{R}^3} \frac{|F(x - y, t - |y + \bar{v}(t)|)|}{|y + \bar{v}(t)|} dy \right\|_{L^2_t}
\]
\[
\leq \left\| \int_{\mathbb{R}^3} \frac{|F(x - y, t - |y + \bar{v}(t)|)|}{\sqrt{y_2^2 + y_3^2}} dy \right\|_{L^2_t},
\]
where in the third line, we used a change of variable and for the last inequality and reduce the norm of $y$ to the norm of the component of $y$ orthogonal to the direction of the motion.

Finally,
\[
\left\| \int_{\mathbb{R}^3} \frac{F(x - y, t - |y + \bar{v}(t)|)}{\sqrt{y_2^2 + y_3^2}} dy \right\|_{L^2_t} \leq \int_{\mathbb{R}^3} \frac{\|F(x - y, t - |y + \bar{v}(t)|)\|_{L^2_t}}{\sqrt{y_2^2 + y_3^2}} dy
\]
(4.24)

For fixed $y$, if we apply a change of variable of $t$ here, the Jacobian is bounded by $1 - |v'|$ and $1 + |v'|$, so
\[
\int_{\mathbb{R}^3} \frac{\|F(x - y, t - |y + vt|)\|_{L^2_t}}{\sqrt{y_2^2 + y_3^2}} dy \lesssim \int_{\mathbb{R}^3} \frac{\|F(x - y, \cdot)\|_{L^2_t}}{\sqrt{y_2^2 + y_3^2}} dy
\]
(4.25)
where $\hat{v}_1$ denotes the subspace orthogonal to $x_1$ (more generally, the subspace orthogonal to the direction of the motion). Here $L^{2,1}$ is the Lorentz norm and the last inequality follows from Hölder’s inequality of Lorentz spaces. Therefore,
\[
\|D^S\|_{L^\infty_x L^2_t} \lesssim \|F\|_{L^1_{x_1} L^{2,1}_{\hat{v}_1} L^1_t}.
\]
(4.26)
as claimed.

Finally, we consider the estimate with the source term $F$ along an admissible trajectory. This follows from a duality or the same argument as in [GC2]. So we conclude that
\[
\|D^S\|_{L^\infty_x L^2_t} \lesssim \|F^{SP}\|_{\hat{W}^{1,1}_t L^2_t},
\]
(4.27)
and
\[
\|D^S\|_{L^\infty_x L^2_t} \lesssim \|F^{SP}\|_{L^1_{x_1} L^{2,1}_{\hat{v}_1} L^2_t}
\]
(4.28)
provided $\bar{v}(t)$ moves along $x_1$.

The theorem is proved. \(\square\)

**Remark 4.2.** We notice that from Sobolev’s embedding,
\[
\hat{W}^{1,1}_x \hookrightarrow L^{2,1}.
\]
Therefore indeed, the estimates along general curves requires slightly more regularity than the standard cases.
To end this subsection, we consider a truncated version which appears naturally in bootstrap arguments, for example in [GC2].

**Corollary 4.3.** Let \( \vec{v}(t) \) be an admissible trajectory. For fixed \( A > 0 \), setting

\[
D_A = \int_0^{t-A} \sin \left( (t-s)\sqrt{-\Delta} \right) F(s) \, ds,
\]

we have

\[
\|D_A^S\|_{L^\infty_v L^2_x[A,\infty]} \lesssim \frac{1}{A} \|F\|_{L^1_v L^2_x},
\]

and

\[
\|D_A\|_{L^2_t} \lesssim \frac{1}{A} \|F^{S'}\|_{L^1_v L^2_x},
\]

where

\[
F^{S'} := F(x + \vec{h}(t), t)
\]

with \( \vec{h}(t) \) is an admissible trajectory.

**Proof.** By Kirchhoff’s formula,

\[
\|D_A^S\|_{L^2_t} \lesssim \left\| \int_{A \leq |y|} \frac{|F(x + \vec{v}(t) - y, t - |y|)|}{|y|} \, dy \right\|_{L^2_t}
\]

\[
\lesssim \frac{1}{A} \|F\|_{L^1_v L^2_x}.
\]

By duality or the same arguments as in [GC2], we can also obtain,

\[
\|D_A^S\|_{L^2_t} \lesssim \frac{1}{A} \|F^{S'}\|_{L^1_v L^2_x},
\]

as claimed \( \square \)

**4.2. Perturbed wave equations.** Finally, we extend all of our estimates to the perturbed Hamiltonian. In [GC2], we relied on Duhamel expansion of the perturbed evolution, the estimates along trajectories for free ones and the standard estimates for the perturbed ones. Here we present an alternative approach based on the structure formula of the wave operators as in Section [2].

We only present the standard cases and other estimates can be obtained similarly.

**Theorem 4.4.** Let \( \vec{v}(t) \) be an admissible trajectory. Suppose

\[
H = -\Delta + V
\]

admits neither eigenfunctions nor resonances at 0. Set

\[
u(x, t) = \frac{\sin \left( t\sqrt{|H|} \right) P_c f}{\sqrt{|H|}} + \cos \left( t\sqrt{-H} \right) P_c g + \int_0^t \frac{\sin \left( (t-s)\sqrt{|H|} \right) }{\sqrt{|H|}} P_c F(s) \, ds
\]

and

\[
u^S(x, t) := u(x + \vec{v}(t), t),
\]

where \( P_c \) is the projection onto the continuous spectrum of \( H \).

First of all, for the standard endpoint reversed Strichartz estimates, we have
(4.38) \[ \left\| \frac{\sin \left( t\sqrt{H} \right)}{\sqrt{H}} P_c f + \cos \left( t\sqrt{H} \right) P_c g \right\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \]

(4.39) \[ \left\| \int_0^t \frac{\sin \left( (t-s)\sqrt{H} \right)}{\sqrt{H}} P_c F(s) \, ds \right\|_{L^\infty_x L^2_t} \lesssim \|F\|_{L^2_x L^{\frac{2}{3},1}_t}. \]

Consider the estimates along the trajectory \( \vec{v}(t) \), one has

(4.40) \[ \|u^\vec{v}(x, t)\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{W^{1,1}_x L^2_t}. \]

If \( \vec{v}(t) \) does not change the direction, then

(4.41) \[ \|u^S(x, t)\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L^1_x L^{\frac{3}{4},1}_t L^2_t}, \]

where \( d \) is the direction of \( \vec{v}(t) \).

Let \( \vec{\mu}(t) \) be another admissible trajectory, we have the same estimate above with \( F \) replaced by

\[ F^{S'}(x, t) := F(x + \vec{\mu}(t), t). \]

Proof. It suffices to consider

(4.42) \[ \sin \left( t\sqrt{H} \right) P_c f. \]

By construction,

(4.43) \[ \frac{\sin \left( t\sqrt{H} \right)}{\sqrt{H}} P_c = W^+ \frac{\sin \left( t\sqrt{-\Delta} \right)}{\sqrt{-\Delta}} (W^+)^*. \]

(4.44) \[ \frac{\sin \left( t\sqrt{H} \right)}{\sqrt{H}} P_c f = W^+ \frac{\sin \left( t\sqrt{-\Delta} \right)}{\sqrt{-\Delta}} (W^+)^* P_c f. \]

Denoting

(4.45) \[ h = (W^+)^* P_c f, \]

we have

(4.46) \[ \|P_c f\|_{L^2} \simeq \|h\|_{L^2}. \]

Setting

(4.47) \[ G = \frac{\sin \left( t\sqrt{-\Delta} \right)}{\sqrt{-\Delta}} h, \]

by Theorem 2.5 it is sufficient to consider the boundedness of

(4.48) \[ G + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x, y, \tau) G(S_{\tau} x + y) \, dy \, d\tau. \]

Clearly, by the endpoint reversed Strichartz estimate for the free case,

(4.49) \[ \|G\|_{L^\infty_x L^2_t} \lesssim \|h\|_{L^2} \simeq \|P_c f\|_{L^2}. \]
Next, by Minkowski’s inequality,

$$
\left\| \int_{S^2} \int_{\mathbb{R}^3} g(x, y, \tau) G(S_{\tau} x + y) \, dy \, d\tau \right\|_{L^\infty_x L^2_t} \lesssim \int_{S^2} \int_{\mathbb{R}^3} \|g(x, y, \tau) G(S_{\tau} x + y)\|_{L^\infty_x L^2_t} \, dy \, d\tau
$$

(4.50)

$$
\|g(x, y, \tau) G(S_{\tau} x + y)\|_{L^\infty_x L^2_t} \lesssim \|g(x, y, \tau)\|_{L^\infty_x} \|G(S_{\tau} x + y)\|_{L^\infty_x L^2_t}.
$$

(4.51)

Since reflections with respect to a fixed plane and translations commute with the solution of a free wave equation, we obtain

$$
G(S_{\tau} x + y) = \sin \left( t \sqrt{-\Delta} \right) h(S_{\tau} x + y).
$$

(4.52)

Therefore,

$$
\|G(S_{\tau} x + y)\|_{L^\infty_x L^2_t} \lesssim \|h(S_{\tau} x + y)\|_{L^2_x} \lesssim \|h\|_{L^2_x} \simeq \|P_c f\|_{L^2_x}.
$$

(4.53)

It follows

$$
\left\| G + \int_{S^2} \int_{\mathbb{R}^3} g(x, y, \tau) G(S_{\tau} x + y) \, dy \, d\tau \right\|_{L^\infty_x L^2_t} \lesssim \left( 1 + \int_{S^2} \int_{\mathbb{R}^3} \|g(x, y, \tau)\|_{L^\infty_x} \, dy \, d\tau \right) \|P_c f\|_{L^2_x} \lesssim \|f\|_{L^2_x}.
$$

(4.54)

Then we conclude

$$
\left\| \frac{\sin \left( t \sqrt{H} \right)}{\sqrt{H}} P_c f \right\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2_x},
$$

(4.55)

as claimed. □

4.3. Wave equations with moving potentials. Finally in this section, we consider the wave equation

$$
\partial_{tt} u - \Delta u + V(x - \vec{\mu} t) u = 0
$$

(4.56)

$$
u(x, 0) = g(x), \quad u_t(x, 0) = f(x).
$$

(4.57)

Again without of loss of generality, we assume \( \vec{\mu} \) is along \( \vec{e}_1 \) and \( \vec{\mu} < 1 \). Recall that associated to this model, we define

$$
H = -\Delta + V \left( \sqrt{1 - \vec{\mu}^2} x_1, x_2, x_3 \right).
$$

(4.58)

Let \( m_1, \ldots, m_w \) be the normalized bound states of \( H \) associated to the negative eigenvalues \( -\lambda^2_1, \ldots, -\lambda^2_w \), respectively (notice that by our assumptions, 0 is not an eigenvalue). We denote by \( P_b \) the projections on the the bound states of \( H \), respectively, and let \( P_c = Id - P_b \).

Performing a Lorentz transformation \( L \) with respect to the moving frame \( (x - \vec{\mu} t, t) \), we have

$$
\partial_{tt'} u_L + Hu_L = 0,
$$

(4.59)

$$
u_L(x', 0) = \tilde{g}(x'), \quad (u_L)_t(x', 0) = \tilde{f}(x').
$$

(4.60)
and

\[(4.61) \quad \| f \|_{L^2} + \| g \|_{H^1} \simeq \| \tilde{f} \|_{L^2} + \| \tilde{g} \|_{H^1}.\]

We can write

\[(4.62) \quad u_L(x', t') = \sum_{i=1}^{w} a_i(t') m_i(x') + r_L(x', t'),\]

such that

\[(4.63) \quad P_c r_L = r_L.\]

Return to our original coordinate, we have a decomposition for \(u\) that

\[(4.64) \quad u(x, t) = \sum_{i=1}^{w} a_i(\gamma(t - vx_1)) (m_i)_\mu(x, t) + r(x, t)\]

where

\[(4.65) \quad (m_i)_\mu(x, t) = m_i(\gamma(x_1 - \mu t), x_2, x_3).\]

**Theorem 4.5.** Let \(\bar{v}(t) \in \mathbb{R}^3\) be an admissible trajectory. With the notations from above, we have

\[(4.66) \quad \| r^S \|_{L_x^\infty L_t^2} \lesssim \| f \|_{L^2} + \| g \|_{H^1},\]

in particular,

\[(4.67) \quad \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\langle x - \tilde{h}(t) \rangle^\alpha} r^2(x, t) \, dx \, dt \lesssim \| f \|_{L^2} + \| g \|_{H^1}.\]

**Proof.** Notice that if \(\bar{v}(t)\) is an admissible trajectory in our original frame \((x, t)\), then if we perform a Lorentz transformation \(L(\bar{\mu})\), in the new frame, the trajectory \(v(t)\) can be written as \(\bar{v}(t')\) with \(\| \bar{v}'(t') \| < \phi(\lambda, \bar{\mu}) < 1\). In other words, in the new coordinate, the trajectory is still admissible. Then for fixed \(x \in \mathbb{R}^3\),

\[(4.68) \quad \int |r^S(x, t)|^2 \, dt \lesssim \sup_{x' \in \mathbb{R}^3} \int |r^S_L(x', t')|^2 \, dt',\]

where

\[(4.69) \quad r^S_L(x', t') = r_L(x' + \tilde{v}'(t'), t').\]

By construction and Theorem 3.4

\[(4.70) \quad \sup_{x' \in \mathbb{R}^3} \int |r^S_L(x', t')|^2 \, dt' \lesssim \left( \| \tilde{f} \|_{L^2} + \| \tilde{g} \|_{H^1} \right)^2 \simeq (\| f \|_{L^2} + \| g \|_{H^1})^2\]

and hence

\[(4.71) \quad \| r^S \|_{L_x^\infty L_t^2} \lesssim \| f \|_{L^2} + \| g \|_{H^1}.\]

The theorem is proved. \(\square\)
5. Strichartz Estimates and Energy Estimates

In this section, we establish Strichartz estimates and energy estimates for scattering states to the wave equation

\begin{align}
\partial_{tt} u - \Delta u + V(x - \vec{v}(t)) u &= 0, \\
u(x,0) &= g(x), \ u_t(x,0) = f(x)
\end{align}

with

\begin{align}
|\vec{v}(t) - \vec{\mu}t| \lesssim \langle t \rangle^{-\beta}, \ \beta > 1, \ |\vec{\mu}| < 1.
\end{align}

To simplify the problem, we assume

\begin{align}
H = -\Delta + V\left(\sqrt{1 - |\vec{\mu}|^2} x_1, x_2, x_3\right)
\end{align}

only has one bound state \( m \) such that

\begin{align}
H m = -\lambda^2 m, \ \lambda > 0.
\end{align}

One can observe that our arguments work for the general case.

We start with reversed Strichartz estimates.

\textbf{Theorem 5.1.} Let \( \vec{h}(t) \) be an admissible trajectory and \( u \) be a scattering in the sense of Definition 1.4. Then for

\begin{align}
\tag{5.5}
u^S(x,t) = u(x + \vec{h}(t), t)
\end{align}

one has

\begin{align}
\|u^S(x,t)\|_{L^\infty L^2_t[0,\infty)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\end{align}

In particular, it implies for \( \alpha > 3 \),

\begin{align}
\tag{5.6}
\int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\langle x - \vec{h}(t) \rangle} u^2(x,t) \, dx \, dt \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\end{align}

\textbf{Proof.} First of all, we need to understand the evolution of bound states. Writing the equation as

\begin{align}
\partial_{tt} u - \Delta u + V(x - \vec{\mu}t) u = [V(x - \vec{\mu}t) - V(x + \vec{v}(t))] u.
\end{align}

Recall that we assume \( \vec{\mu} \) is along \( x_1 \). Suppose \( u(x,t) \) is a scattering state. As in (4.64), we decompose the evolution as following,

\begin{align}
\tag{5.9}
u(x,t) = a(\gamma(t - \mu x_1)) \, m_{\mu}(x,t) + r(x,t)
\end{align}

where

\begin{align}
\tag{5.10}
m_{\mu}(x,t) = m(\gamma(x_1 - \mu t), x_2, x_3)
\end{align}

and

\begin{align}
\tag{5.11}
P_c(H) r_L = r_L.
\end{align}

Performing the Lorentz transformation \( L \) with respect to the moving frame \( (x - \vec{\mu}t, t) \), we have

\begin{align}
\tag{5.12}
u_L(x', t') = a(t') \, m(x') + r_L(x', t'),
\end{align}

and

\begin{align}
\tag{5.13}
\partial_{tt'} u_L + H u_L = - M(x', t') u_L
\end{align}
where
\begin{equation}
M(x', t') = -\left[ V(x - \vec{\mu}t) - V(x + \vec{v}(t)) \right]_L.
\end{equation}

When \( u \) is a scattering state in the sense Definition 1.3, the scattering condition forces \( a(t) \) to go 0.

Plugging the evolution (5.12) into the equation (5.13) and taking inner product with \( m \), we get
\begin{equation}
\ddot{a}(t') - \lambda^2 a(t') + a(t') \langle Mm, m \rangle + \langle Mr_L, m \rangle = 0
\end{equation}

Notice that
\begin{equation}
|M(x', t')| \lesssim \frac{1}{\langle \gamma (t' + \mu x') \rangle^\beta}.
\end{equation}

One can write
\begin{equation}
\ddot{a}(t') - \lambda^2 a(t') + a(t') c(t') + h(t') = 0,
\end{equation}
\begin{equation}
c(t') := \langle Mm, m \rangle
\end{equation}
and
\begin{equation}
h(t') := \langle Mr_L, m \rangle.
\end{equation}

Since \( w \) is exponentially localized by Agmon’s estimate, we know
\begin{equation}
|c(t')| \lesssim e^{-b|t'|}, \quad b > 0.
\end{equation}

The existence of the solution to the ODE (5.17) is clear. We study the long-time behavior of the solution. Write the equation as
\begin{equation}
\ddot{a}(t') - \lambda^2 a(t') = -[a(t')c(t') + h(t')],
\end{equation}
and denote
\begin{equation}
N(t') := -[a(t')c(t') + h(t')].
\end{equation}

Then
\begin{equation}
a(t') = \frac{e^{\lambda t'}}{2} \left[ a(0) + \frac{1}{\lambda} \dot{a}(0) + \frac{1}{\lambda} \int_0^{t'} e^{-\lambda s} N(s) \, ds \right] + R(t')
\end{equation}
where
\begin{equation}
|R(t')| \lesssim e^{-c|t'|},
\end{equation}
for some positive constant \( c > 0 \). Therefore, the stability condition forces
\begin{equation}
a(0) + \frac{1}{\lambda} \dot{a}(0) + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda s} N(s) \, ds = 0.
\end{equation}

Then under the stability condition (5.25),
\begin{equation}
a(t') = e^{-\lambda t'} \left[ a(0) + \frac{1}{2\lambda} \int_0^{\infty} e^{-\lambda s} N(s) \, ds \right] + \frac{1}{2\lambda} \int_0^{\infty} e^{-\lambda |t-s|} N(s) \, ds.
\end{equation}
By Young’s inequality, to estimate all \( L^p \) norms of \( a(t') \), it suffices to estimate the \( L^1 \) norm of \( h(t') \), see [GC2].

By Cauchy-Schwarz and Theorem 4.7,
\[
\int_0^\infty |\langle MrL, m \rangle| dt \lesssim \|rL\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]

Therefore,
\[
(5.27) \quad \|a(t)\|_{L^p([0, \infty))} \lesssim_p \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]

Given \( \tilde{h}(t) \) an admissible trajectory, set
\[
(5.28) \quad B(x, t) = a(\gamma(t - \mu x_1)) m_\mu(x, t),
\]
\[
(5.29) \quad B^S(x, t) = B(x + \tilde{h}(t), t).
\]

By Agmon’s estimate, see Theorem 3.5, and the \( L^1 \) norm estimate for \( h(t) \), we have
\[
(5.30) \quad \|B^S(x, t)\|_{L^\infty_x L^2_t([0, \infty))} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]

By Theorem 4.5, we also know
\[
(5.31) \quad \|r^S(x, t)\|_{L^\infty_x L^2_t([0, \infty))} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]

Therefore, one has
\[
(5.32) \quad \|u^S(x, t)\|_{L^\infty_x L^2_t([0, \infty))} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]

We notice that this in particular implies
\[
(5.33) \quad \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{|x - \tilde{h}(t)|^\alpha} u^2(x, t) \, dx \, dt \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]

The theorem is proved. \( \Box \)

Next, we show Strichartz estimates following [RS, LSch, GC2]. In the following, we use the short-hand notation
\[
L^p_t L^q_x \colon= L^p_t ([0, \infty)), \ L^q_x.
\]

**Theorem 5.2.** Suppose \( u \) is a scattering state in the sense of Definition of 1.4 which solves
\[
(5.35) \quad \partial_t u - \Delta u + V(x - \tilde{v}(t)) u = 0
\]

with initial data
\[
(5.36) \quad u(x, 0) = g(x), \ u_t(x, 0) = f(x).
\]

Then for \( p > 2 \), and \((p, q)\) satisfying
\[
(5.37) \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}
\]
we have
\[
(5.38) \quad \|u\|_{L^p_t L^q_x} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.
\]
Therefore, it reduces to show
\[ \| Af \|_{L^2} \simeq \| f \|_{H^1}, \quad \forall f \in C^\infty (\mathbb{R}^3). \] (5.39)
For real-valued \( u = (u_1, u_2) \in \mathcal{H} = H^1 (\mathbb{R}^3) \times L^2 (\mathbb{R}^3) \), we write
\[ U := Au_1 + iu_2. \] (5.40)
From (5.39), we know
\[ \| U \|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \] (5.41)
We also notice that \( u \) solves (5.35) if and only if
\[ U := Au + i\partial_t u \] satisfies
\[ i\partial_t U = AU + V (x - \vec{v}(t)) u, \] (5.42)
\[ U(0) = Ag + if \in L^2 (\mathbb{R}^3). \] (5.43)
By Duhamel’s formula,
\[ U(t) = e^{itA}U(0) - i \int_0^t e^{-i(t-s)A} V (\cdot - \vec{v}(s)) u(s) \, ds. \] (5.45)
Let \( P := A^{-1} \mathbb{R} \), then from Strichartz estimates for the free evolution,
\[ \| Pe^{itA}U(0) \|_{L^p_t L^q_x} \lesssim \| U(0) \|_{L^2}. \] (5.46)
Writing \( V = V_1 V_2 \) and with the Christ-Kiselev lemma, Lemma 2.8, it suffices to bound
\[ \left\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 (\cdot - \vec{v}(s)) u(s) \, ds \right\|_{L^p_t L^q_x}. \] (5.47)
We only need to analyze
\[ \left\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 (\cdot - \vec{v}(s)) u(s) \, ds \right\|_{L^p_t L^q_x} \lesssim \left\| \tilde{K} \right\|_{L^p_t \to L^p_t} \| V_2 (x - \vec{v}(s)) u \|_{L^q_x} \]
where
\[ (\tilde{K} F)(t) := P \int_0^\infty e^{-i(t-s)A} V_1 (\cdot - \vec{v}(s)) F(s) \, ds. \] (5.48)
To show \( \left\| \tilde{K} \right\|_{L^p_t \to L^p_t} \) is bounded, we test it against \( F \in L^2_t, \) clearly,
\[ \left\| \tilde{K} F \right\|_{L^p_t L^q_x} \leq \left\| Pe^{-itA} \right\|_{L^2 \to L^p_t L^q_x} \left\| \int_0^\infty e^{iA V_1 (\cdot - \vec{v}(s))} F(s) \, ds \right\|_{L^q_x}. \] (5.49)
The first factor on the right-hand side of (5.49) is bounded by Strichartz estimates for the free evolution. Consider the second factor, by duality, it is sufficient to show
\[ \| V_1 (\cdot - \vec{v}(t)) e^{-itA} \phi \|_{L^2_t} \lesssim \| \phi \|_{L^2}, \quad \forall \phi \in L^2 (\mathbb{R}^3). \] (5.50)
By our assumption,
\[ |\vec{v}(t) - \vec{m}| \lesssim |t|^{-\beta}, \quad \beta > 1, \quad |\vec{m}| < 1. \]
Therefore, it reduces to show
\[ \left\| (1 + |x - \vec{m}|) e^{-itA} \phi \right\|_{L^2_t} \lesssim \| \phi \|_{L^2}, \quad \forall \phi \in L^2 (\mathbb{R}^3). \] (5.51)
Notice that this is a consequence of that the energy of the free wave equation stays comparable under Lorentz transformations, Theorem 3.4. To show estimate (5.51), one can apply the Lorentz transformation \( L \). In the new frame \((x', t')\), then we can use the standard local energy decay for free wave equations, estimate (6.97) in Appendix B. Finally after applying an inverse transformation back to the original frame, we obtain (5.51).

From estimate (5.51), one does have
\[
\| V_1 (\cdot - v(t)) e^{-itA} \|_{L^2_{t,x}} \lesssim \| \phi \|_{L^2}.
\]
Therefore, indeed,
\[
\| K \|_{L^2_{t,x} \to L^p_t L^q_x} \leq C.
\]
Hence
\[
\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 (\cdot - \vec{v}(s)) u(s) \, ds \|_{L^p_t L^q_x} \lesssim \| V_2 (x - \vec{v}(s)) \|_{L^2_t} \| u \|_{L^2_{t,x}}.
\]
By our estimate (5.7),
\[
\| V_2 (x - \vec{v}(s)) \|_{L^2_t} \lesssim \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\langle x - \vec{v}(t) \rangle^a} |u(x, t)|^2 \, dx \, dt \right)^{\frac{1}{2}} \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1}.
\]
Therefore,
\[
\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 (\cdot - \vec{v}(s)) u(s) \, ds \|_{L^p_t L^q_x} \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1}.
\]
Hence one can conclude
\[
\| u \|_{L^p_t L^q_x} \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1},
\]
as we claimed. \( \square \)

The energy estimates can be established in a similar manner.

**Theorem 5.3.** Suppose \( u \) is a scattering state in the sense of Definition of 1.4 which solves
\[
\partial_t u - \Delta u + V (x - \vec{v}(t)) u = 0
\]
with initial data
\[
u(x, 0) = g(x), \quad u_t(x, 0) = f(x).
\]
Then we have
\[
\sup_{t \in \mathbb{R}} (\| \nabla u(t) \|_{L^2} + \| u_t(t) \|_{L^2}) \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1}.
\]
**Proof.** Again, we set \( A = \sqrt{-\Delta} \) and notice that
\[
\| Af \|_{L^2} \simeq \| f \|_{\dot{H}^1}, \quad \forall f \in C^\infty (\mathbb{R}^3).
\]
For real-valued \( u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1 (\mathbb{R}^3) \times L^2 (\mathbb{R}^3) \), we write
\[
U := Au_1 + iu_2.
\]
and we know
\[
\| U \|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}.
\]
We also notice that $u$ solves the original equation if and only if
\begin{equation}
U := Au + i\partial_t u
\end{equation}
satisfies
\begin{align}
i\partial_t U &= AU + V (x - \bar{v}(t)) u, \\
U(0) &= Ag + if \in L^2 (\mathbb{R}^3).
\end{align}
By Duhamel’s formula,
\begin{equation}
U(t) = e^{itA}U(0) - i \int_0^t e^{-i(t-s)A} (V (\cdot - \bar{v}(s)) u(s)) \, ds.
\end{equation}
From the energy estimate for the free evolution,
\begin{equation}
\sup_{t \in \mathbb{R}} \|e^{itA}U(0)\|_{L^2_x} \lesssim \|U(0)\|_{L^2}.
\end{equation}
Writing $V = V_1 V_2$, it suffices to bound
\begin{equation}
\sup_{t \in \mathbb{R}} \left\| \int_0^\infty e^{-i(t-s)A} V_1 V_2 (\cdot - \bar{v}(s)) u(s) \, ds \right\|_{L^2_x}.
\end{equation}
This is can be bounded in a same manner as Theorem 5.2.
It is clear that
\begin{equation}
\left\| \int_0^\infty e^{-i(t-s)A} V_1 V_2 (\cdot - \bar{v}(s)) u(s) \, ds \right\|_{L^\infty_t L^2_x} \leq \left\| \widetilde{K} \right\|_{L^2_x L^2_2 \to L^\infty_t L^2_x} \|V_2 (x - \bar{v}(t)) u\|_{L^2_t L^2_x},
\end{equation}
where
\begin{equation}
(\widetilde{K} F) (t) := \int_0^\infty e^{-i(t-s)A} V_1 (\cdot - \bar{v}(s)) F(s) \, ds.
\end{equation}
We need to estimate
\begin{equation}
\left\| \widetilde{K} \right\|_{L^2_x L^2_2 \to L^\infty_t L^2_x}.
\end{equation}
Testing against $F \in L^2_t L^2_x$, clearly,
\begin{equation}
\left\| \widetilde{K} F \right\|_{L^\infty_t L^2_x} \leq \left\| e^{-itA} \right\|_{L^2 \to L^\infty_t L^2_x} \left\| \int_0^\infty e^{isA} V_1 (\cdot - \bar{v}(s)) F(s) \, ds \right\|_{L^2}.
\end{equation}
The first factors on the right-hand side of (5.73) is bounded by the energy estimates for the free evolution. Consider the second factor, by duality, it suffices to show
\begin{equation}
\|V_1 (x - \bar{v}(t)) e^{-itA} \phi\|_{L^2_t L^2_x} \lesssim \|\phi\|_{L^2} , \forall \phi \in L^2 (\mathbb{R}^3).
\end{equation}
From our discussions Theorem 5.2, we know
\begin{equation}
\|V_1 (x - \bar{v}(t)) e^{-itA} \phi\|_{L^2_t L^2_x} \lesssim \|\phi\|_{L^2}.
\end{equation}
Hence
\begin{equation}
\left\| \int_0^\infty e^{isA} V_1 (\cdot - \bar{v}(s)) F(s) \, ds \right\|_{L^2} \lesssim \|F\|_{L^2_t L^2_x}.
\end{equation}
Therefore, indeed, we have
\begin{equation}
\left\| \widetilde{K} \right\|_{L^2_x L^2_2 \to L^\infty_t L^2_x} \leq C.
\end{equation}
and
\[
\sup_{t \in \mathbb{R}} \left\| \int_0^\infty e^{-it-s} A V_1 V_2 \left( \cdot - \vec{v}(s) \right) u(s) \, ds \right\|_{L_x^2} \lesssim \| V_2 \left( \cdot - \vec{v}(s) \right) u \|_{L_{t,x}^2}.
\]

By the weighted estimate (5.7),
\[
\| V_2 \left( \cdot - \vec{v}(s) \right) u \|_{L_x^2 L_t^2} \lesssim \left( \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\langle x - \vec{v}(t) \rangle^\alpha} |u(x, t)|^2 \, dx \, dt \right)^{\frac{1}{2}} \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1}.
\]

It implies
\[
\sup_{t \geq 0} \left\| \int_0^\infty e^{-it-s} A V_1 V_2 \left( \cdot - \vec{v}(s) \right) u(s) \, ds \right\|_{L_x^2} \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1}.
\]

Therefore, by estimates (5.68) and (5.80), we have
\[
\sup_{t \geq 0} (\| \nabla u(t) \|_{L^2} + \| u_t(t) \|_{L^2}) \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1},
\]
as claimed. \(\square\)

Similarly, one can also obtain the local energy decay estimate:

**Theorem 5.4.** Suppose \( u \) is a scattering state in the sense of Definition 1.4 which solves
\[
\partial_{tt} u - \Delta u + V(x - \vec{v}(t)) u = 0
\]
with initial data
\[
u(x, 0) = g(x), \quad u_t(x, 0) = f(x).
\]

Then for all \( \epsilon > 0 \), \( |\vec{v}| < 1 \), we have
\[
\left\| \left( 1 + |x - \vec{v}(t)| \right)^{-\frac{1}{2} - \epsilon} (|\nabla u| + |u_t|) \right\|_{L_{t,x}^2} \lesssim \| f \|_{L^2} + \| g \|_{\dot{H}^1}.
\]

**Proof.** The proof is the same as above with the energy estimate for the free wave equation replaced by the local energy decay estimate of the free wave equation.
\[
\left\| \left( 1 + |x - \vec{v}(t)| \right)^{-\frac{1}{2} - \epsilon} e^{it\sqrt{-\Delta}} f \right\|_{L_{t,x}^2} \lesssim \| f \|_{L_x^2}.
\]
The claim follows easily. \(\square\)

To finish this section, we show one important application of Theorem 5.1.

We denote
\[
E_{V}(t) = \int_{\mathbb{R}^3} \left| \nabla_x u \right|^2 + |\partial_t u|^2 + V(x - \vec{v}(t)) |u|^2 \, dx.
\]

**Corollary 5.5.** Suppose \( u \) is a scattering state in the sense of Definition 1.4 which solves
\[
\partial_{tt} u - \Delta u + V(x - \vec{v}(t)) u = 0
\]
with initial data
\[
u(x, 0) = g(x), \quad u_t(x, 0) = f(x).
\]

Assume
\[
\| \nabla V \|_{L^1} < \infty,
\]
then \( E_V(t) \) is bounded by the initial energy independently of \( t \),
\[
\sup_t E_V(t) \lesssim \|(g, f)\|_{H^1 \times L^2}^2.
\]

**Proof.** We might assume \( u \) is smooth. Taking the time derivative of \( E_V(t) \) and by the fact that \( u \) solves equation, we obtain
\[
\partial_t E_V(t) = \int_{\mathbb{R}^3} \partial_t V(x-v(t)) |u(x,t)|^2 \, dx = -v'(t) \int_{\mathbb{R}^3} \partial_y V(y) |u^S(y,t)|^2 \, dy.
\]
by a simple change of variable.

Note that
\[
\int_0^\infty |\partial_t E_V(t)| \, dt \lesssim \int_0^\infty \int_{\mathbb{R}^3} |\partial_y V(y)| |u^S(y)|^2 \, dydt,
\]
\[
\lesssim \|\partial_y V\|_{L^1_t} \|u^S\|_{L^\infty_t L^2_x}^2
\]
\[
\lesssim \|(g, f)\|_{H^1 \times L^2}^2
\]
where in the last inequality, we applied Theorem \ref{thm:5.1}.

Therefore, for arbitrary \( t \in \mathbb{R}^+ \), we have
\[
E_V(t) - E_V(0) \leq \int_0^\infty |\partial_t E_V(t)| \, dt \lesssim \|(g, f)\|_{H^1 \times L^2}^2
\]
which implies
\[
\sup_t E_V(t) \lesssim \|(g, f)\|_{H^1 \times L^2}^2.
\]
We are done. \( \square \)

With endpoint Strichartz estimates along smooth trajectories, we can also derive inhomogeneous Strichartz estimates. One can find a detailed argument in [GC2].

6. **Scattering and Asymptotic Completeness**

In this section, we show some applications of the results in this paper. We will study the long-time behaviors for a scattering state in the sense of Definition \ref{def:1.4}.

Following the notations from above section, we will still use the short-hand notation
\[
L^p_t L^q_x := L^p_t ([0, \infty), L^q_x).
\]

We reformulate the wave equation as a Hamiltonian system,
\[
U' = JE'(U)
\]
where \( J \) is a skew symmetric matrix and \( E'(U) \) is the Frechet derivative of the conserved quantity. Setting
\[
U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix},
\]
\[
J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
H_F := \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix},
\]
we can rewrite the free wave equation as
\begin{equation}
\dot{U}_0 - JHF U_0 = 0,
\end{equation}
\begin{equation}
U_0[0] = \begin{pmatrix} g_0 \\ f_0 \end{pmatrix}
\end{equation}
The solution of the free wave equation is given by
\begin{equation}
U_0 = e^{tJHF} U_0[0].
\end{equation}

**Theorem 6.1.** Suppose $u$ is a scattering state in the sense of Definition 1.4 which solves
\begin{equation}
\partial_t u - \Delta u + V(x - \vec{v}(t))u = 0
\end{equation}
with initial data
\begin{equation}
(0) u(x,0) = g(x), u_t(x,0) = f(x).
\end{equation}
Write
\begin{equation}
U = (u, u_t)^t \in C^0 \left([0, \infty); \dot{H}^1 \right) \times C^0 \left([0, \infty); L^2 \right),
\end{equation}
with initial data $U[0] = (g, f)^t \in \dot{H}^1 \times L^2$. Then there exist free data
\begin{equation}
U_0[0] = (g_0, f_0)^t \in \dot{H}^1 \times L^2
\end{equation}
such that
\begin{equation}
\| U[t] - e^{tJHF} U_0[0] \|_{\dot{H}^1 \times L^2} \to 0
\end{equation}
as $t \to \infty$.

**Proof.** We will still use the formulation in Theorem (5.38). We set $A = \sqrt{-\Delta}$ and notice that
\begin{equation}
\| Af \|_{L^2} \simeq \| f \|_{\dot{H}^1}, \quad \forall f \in C^\infty (\mathbb{R}^3).
\end{equation}
For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1 (\mathbb{R}^3) \times L^2 (\mathbb{R}^3)$, we write
\begin{equation}
U := Au_1 + iu_2.
\end{equation}
We know
\begin{equation}
\| U \|_{L^2} \simeq \| (u_1, u_2) \|_{\mathcal{H}}.
\end{equation}
We also notice that $u$ solves (6.9) if and only if
\begin{equation}
U := Au + i\partial_t u
\end{equation}
satisfies
\begin{equation}
i\partial_t U = AU + V (x - \vec{v}(t)) u, \end{equation}
\begin{equation}
U(0) = Ag + if \in L^2 (\mathbb{R}^3).
\end{equation}
By Duhamel's formula, for fixed $T$
\begin{equation}
U(T) = e^{iT A} U(0) - i \int_0^T e^{-i(T-s) A} (V (\cdot - \vec{v}(s)) u(s)) \ ds.
\end{equation}
Applying the free evolution backwards, we obtain
\begin{equation}
e^{-iT A} U(T) = U(0) - i \int_0^T e^{is A} (V (\cdot - \vec{v}(s)) u(s)) \ ds.
\end{equation}
Letting $T$ go to $\infty$, we define

$$U_0(0) := U(0) - i \int_0^\infty e^{isA} (V (\cdot - \vec{v}(s)) u(s)) \, ds$$

(6.21)

By construction, we just need to show $U_0[0]$ is well-defined in $L^2$, then automatically,

$$\|U(t) - e^{itA}U_0(0)\|_{L^2} \to 0.$$  

(6.22)

It suffices to show

$$\int_0^\infty e^{isA} (V (\cdot - \vec{v}(s)) u(s)) \, ds \in L^2.$$  

(6.23)

Then following the argument as in the proof of Theorem 5.2 we write $V = V_1 V_2$.

We consider

$$\int_0^\infty e^{isA} V_1 V_2 (\cdot - \vec{v}(s)) u(s) \, ds \leq \|K\|_{L^2_{t,x} \to L^2_{t,x}} \|V_2 (\cdot - \vec{v}(s)) u\|_{L^2_{t,x}},$$

(6.24)

where

$$(KF)(t) := \int_0^\infty e^{isA} V_1 (\cdot - \vec{v}(s)) F(s) \, ds.$$  

(6.25)

By the same argument in the proof of Theorem 5.3 one has

$$\|K\|_{L^2_{t,x} \to L^2_{t,x}} \leq C.$$  

(6.26)

Therefore

$$\int_0^\infty e^{isA} V_1 V_2 (\cdot - \vec{v}(s)) u(s) \, ds \leq \|V_2 (x - \vec{v}(t)) u\|_{L^2_{t,x}}.$$  

(6.27)

By estimate (5.7),

$$\|V_2 (x - \vec{v}(t)) u\|_{L^2_{t,x}} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{(x - \vec{v}(t))^{2\alpha}} |u(x,t)|^2 \, dx \, dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{H^1}.$$  

(6.28)

We conclude

$$\int_0^\infty e^{isA} (V (\cdot - \vec{v}(s)) u(s)) \, ds \in L^2$$

with

$$\int_0^\infty e^{isA} (V (\cdot - \vec{v}(s)) u(s)) \, ds \lesssim \|f\|_{L^2} + \|g\|_{H^1}.$$  

(6.29)

So

$$U_0(0) := U(0) - i \int_0^\infty e^{isA} (V (\cdot - \vec{v}(s)) u(s)) \, ds$$

(6.30)

is well-defined in $L^2$ and

$$\|U(t) - e^{itA}U_0(0)\|_{L^2} \to 0.$$  

(6.31)

Define

$$(g_0, f_0) := (A^{-1} \Re U_0(0), \Im U_0(0)).$$  

(6.32)

By construction, notice that

$$U[t] = (A^{-1} \Re U(t), \Im U(t))$$

(6.33)
(6.34) \[ \| U[t] - e^{\mathcal{L}Ht} U_0[0] \|_{\dot{H}^2 \times L^2} \to 0. \]

We are done. \hfill \Box

To finish this section, we show the asymptotic completeness for the wave equation with the potential moving along a straight line.

Consider the wave equation

(6.35) \[ \partial_{tt} u - \Delta u + V(x - \vec{u}t)u = 0 \]

with initial data

(6.36) \[ u(x, 0) = g(x), \quad u_t(x, 0) = f(x). \]

Without loss of generality, we still assume that \( \vec{u} \) is along \( \vec{e}_1 \).

Let \( m_1, \ldots, m_w \) be the normalized bound states of

(6.37) \[ H = -\Delta + V(\sqrt{1 - |\mu|}, x_2, x_3) \]

associated with eigenvalues \( -\lambda_1^2, \ldots, -\lambda_w^2 \) respectively with \( \lambda_i > 0, \ i = 1, \ldots, w \).

Setting

(6.38) \[ A_H = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}, \]

then the point spectrum of \( A_H \) is

(6.39) \[ \sigma_p = \bigcup_{i=1}^w \{ \pm \lambda_i \} \]

and the continuous spectrum is

(6.40) \[ \sigma_c = i(-\infty, \infty). \]

Setting

(6.41) \[ E_i^\pm = \begin{pmatrix} m_i \\ \pm \lambda_i m_i \end{pmatrix}, \ i = 1, \ldots, w, \]

we know \( E_i^\pm \) are eigenvectors of \( A_H \) with (6.4) with eigenvalues \( \pm \lambda_i \). One can define the associated Riesz projection

(6.42) \[ P_{i,\pm} (H) := \langle \cdot, JE_i^{\pm} \rangle E_i^{\pm} \]

onto \( E_i^{\pm} \). One can check

(6.43) \[ P_{i,\pm} (H) \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = (\pm \lambda_i u(t) + \partial_t u(t), m_i). \]

From the standard asymptotic completeness results, if we write

(6.44) \[ \dot{U} = A_H U \]

where as (6.3),

(6.45) \[ U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \]

and

(6.46) \[ U[0] = \begin{pmatrix} g \\ f \end{pmatrix} \]
then one can decompose the evolution as

\[ U(t) = \sum_{i=1}^{w} \langle U[0], JE_{i,\mp} \rangle e^{\pm \lambda_i t} E_i^\pm + e^{tH_F} U_0[0] + R(t) \]

where \( e^{tH_F} U_0[0] \) is the free evolution with initial data \( U_0[0] \) and

\[ \|R(t)\|_{\dot{H}^1 \times L^2} \to 0, \ t \to \infty. \]

With notations above, we can obtain a similar decomposition as (6.47) when the potential is moving.

**Corollary 6.2.** Suppose \( H \) admits no eigenfunction nor resonances at zero. Let \( u \) solve

\[ \partial_t u - \Delta u + V(x - \mu t)u = 0. \]

Write

\[ U = (u, u_t)^t \in C^0 \left([0, \infty); \dot{H}^1 \right) \times C^0 \left([0, \infty); L^2 \right), \]

with initial data \( U[0] = (g, f)^t \in \dot{H}^1 \times L^2 \). Then there exist free data

\[ U_0[0] = (g_0, f_0)^t \in \dot{H}^1 \times L^2 \]

such that with \( \gamma = \frac{1}{\sqrt{1-|\mu|^2}} \)

\[ U(t) = \sum_{i=1}^{w} a_{i,\pm} e^{\pm \lambda_i \gamma(t - \mu x_1)} E_{i,\mu}^\pm (x, t) + e^{tH_F} U_0[0] + R(t) \]

where

\[ E_{i,\mu}^\pm (x, t) = E_i^\pm (\gamma (x_1 - \mu t), x_2, x_3) \]

and

\[ \|R(t)\|_{\dot{H}^1 \times L^2} \to 0, \ t \to \infty. \]

**Proof.** Applying a Lorentz transformation such that under the new frame \((x', t')\), \( V \) is stationary, by the standard asymptotic completeness decomposition, one can write

\[ U_L(x', t') = \sum_{i=1}^{w} \langle U_L[0], JE_{i,\mp} \rangle e^{\pm \lambda_i t'} E_i^\pm (x', t') + \mathcal{R}_L(x', t'), \]

where again, we used subscript \( L \) to denote the function under the new frame. Clearly, by the above decomposition (6.53),

\[ P_b(H) \mathcal{R}_L(x', t') = 0. \]

Then in the original frame,

\[ U(t) = \sum_{i=1}^{w} a_{i,\pm} e^{\pm \lambda_i \gamma(t - \mu x_1)} E_{i,\mu}^\pm (x, t) + \mathcal{R}(x, t). \]

where

\[ a_{i,\pm} = \langle U_L[0], JE_{i,\mp} \rangle. \]

By construction, \( \mathcal{R}(x, t) \) satisfies the conditions in Theorem 6.1. Hence

\[ \mathcal{R}(x, t) = e^{tH_F} U_0[0] + R(t) \]
where \( e^{tH_F}U_0[0] \) is the free evolution with initial data \( U_0[0] \) and
\[
\| R(t)\|_{H^1 \times L^2} \to 0, \ t \to \infty.
\]
Therefore, finally, we can write
\[
U(t) = \sum_{i=1}^{w} a_{i,\pm} e^{\pm \lambda_i (t - \mu x_1)} E_{i,\mu}^\pm (x, t) + e^{tH_F}U_0[0] + R(t)
\]
with
\[
E_{i,\mu}^\pm (x, t) = E_i^\pm (\gamma (x_1 - \mu t), x_2, x_3)
\]
and
\[
\| R(t)\|_{H^1 \times L^2} \to 0, \ t \to \infty.
\]
The theorem is proved. □

Remark 6.3. As a final remark, we point out that there is no hope to establish an elegant asymptotic completeness if the potential is not moving along a straight line. If there is a perturbation from that case, the interaction among bound states becomes complicated. Basically, the mechanism is that if the evolution of one bound state is activated, say the bound state with the highest energy, then it will not only cause exponential growth with highest rate for itself but also make the evolution of other bound states grow exponentially. Meanwhile, if we have a scattering state, the evolution of bound states is controllable.

Appendix A

For the sake of completeness, in this appendix, we provide the proof of dispersive estimates for the free wave equation in \( \mathbb{R}^3 \) based on the idea of reversed Strichartz estimates.

Consider
\[
u_{tt} - \Delta u = 0 = \Box u
\]
with initial data
\[
u(0) = g, \ \nu_t(0) = f.
\]
One can write down \( u \) explicitly,
\[
u = \frac{\sin (t\sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos (t\sqrt{-\Delta}) g.
\]

Theorem 6.4. In \( \mathbb{R}^3 \), suppose \( f \in L^2, \nabla f \in L^1 \) and \( g \in L^2, \Delta g \in L^1 \). Then one has the following estimates:

\[
\left\| \frac{\sin (t\sqrt{-\Delta})}{\sqrt{-\Delta}} f \right\|_{L^\infty} \lesssim \frac{1}{|t|} \left\| \nabla f \right\|_{L^1_+},
\]
\[
\left\| \cos (t\sqrt{-\Delta}) g \right\|_{L^\infty} \lesssim \frac{1}{|t|} \left\| \Delta g \right\|_{L^1}.
\]

Remark. Note that the second estimate is slightly different from the estimates commonly used in the literature. For example, in Krieger-Schlag [KS] one needs the \( L^1 \) norm of \( D^2 g \) instead of \( \Delta g \).
Proof. First of all, we consider
\[
\sin \left( \frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}} \right) f.
\]
In \(\mathbb{R}^3\), one has
\[
\sin \left( \frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}} \right) f = \frac{1}{4\pi t} \int_{|x-y|=t} f(y) \, dy.
\]
Without loss of generality, we assume \(t \geq 0\).
Multiplying \(t\) and integrating, we obtain
\[
\int_0^\infty \left| t \sin \left( \frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}} \right) f \right| dt \lesssim \int_0^\infty \int_{S^2} |f(x + r\omega)| r^2 d\omega dr \lesssim \|f\|_{L^1_x}.
\]
Therefore,
\[
\left\| t \sin \left( \frac{t\sqrt{-\Delta}}{\sqrt{-\Delta}} \right) f \right\|_{L^\infty_t L^1_x} \lesssim \|f\|_{L^1_x}.
\]
Notice that, from the above estimate, we also have
\[
\left\| \int_0^\infty \sin \left( s\sqrt{-\Delta} \right) f ds \right\|_{L^\infty_x} \lesssim \frac{1}{|t|} \|\Delta f\|_{L^1_x}.
\]
Repeating \(f\) with \(\Delta f\), it implies that
\[
\left\| \int_t^\infty \sqrt{-\Delta} \sin \left( s\sqrt{-\Delta} \right) f ds \right\|_{L^\infty_x} \lesssim \frac{1}{|t|} \|\Delta f\|_{L^1_x}.
\]
On the other hand,
\[
\int_0^\infty \left| t \cos \left( s\sqrt{-\Delta} \right) f \right| ds \lesssim \int_0^\infty \int_{S^2} |f(x + r\omega)\Delta f(x + r\omega)| r d\omega dr
\]
where in the last inequality, we applied integration by parts in \(r\) in the first term of the RHS of the first line.
Therefore,
\[
\left\| t \cos \left( t\sqrt{-\Delta} \right) f \right\|_{L^\infty_x L^1_t} \lesssim \|\nabla f\|_{L^1_t}.
\]
Hence
\[
\left\| \int_t^\infty \cos \left( s\sqrt{-\Delta} \right) f ds \right\|_{L^\infty_x} \lesssim \frac{1}{|t|} \|\nabla f\|_{L^1_t}.
\]
Finally, we check
\[
\frac{\sin \left( t\sqrt{-\Delta} \right) f}{\sqrt{-\Delta}} = \int_t^\infty \cos \left( s\sqrt{-\Delta} \right) f ds,
\]
and
\[
\cos \left( t\sqrt{-\Delta} \right) g = \int_t^\infty \sqrt{-\Delta} \sin \left( s\sqrt{-\Delta} \right) g ds.
\]
Let \( f, g, h \) be any test functions. Define

\[
A g = \cos \left( t \sqrt{-\Delta} \right) g - \int_t^{\infty} \sqrt{-\Delta} \sin \left( s \sqrt{-\Delta} \right) g \, ds
\]

and

\[
B f = \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f - \int_t^{\infty} \cos \left( s \sqrt{-\Delta} \right) f \, ds.
\]

It is easy to check that \( A, B \) are independent of \( t \).

For \( A \), one observes that

\[
\left\langle \cos \left( t \sqrt{-\Delta} \right) g, h \right\rangle \to 0
\]

and

\[
\left\| \int_t^{\infty} \frac{\sin \left( s \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f \, ds \right\|_{L^\infty_x} \lesssim \frac{1}{|t|} \| f \|_{L^1_x}.
\]

Therefore,

\[
\left\langle A g, h \right\rangle \to 0, \quad t \to \infty.
\]

Since \( A \) is independent of \( t \), one concludes that

\[
\left\langle A g, h \right\rangle = 0
\]

for any pair of test functions and hence

\[
A = 0.
\]

Similarly, we get

\[
B = 0.
\]

Therefore by our calculations above, we can obtain the dispersive estimates for the free wave equation,

\[
\left\| \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f \right\|_{L^\infty_x} = \left\| \int_t^{\infty} \cos \left( s \sqrt{-\Delta} \right) f \, ds \right\|_{L^\infty_x} \lesssim \frac{1}{|t|} \| \nabla f \|_{L^1_x},
\]

and

\[
\left\| \cos \left( t \sqrt{-\Delta} \right) g \right\|_{L^\infty_x} = \left\| \int_t^{\infty} \sqrt{-\Delta} \sin \left( s \sqrt{-\Delta} \right) g \, ds \right\|_{L^\infty_x} \lesssim \frac{1}{|t|} \| \Delta g \|_{L^1_x}.
\]

The theorem is proved. \( \square \)
Appendix B

We derive the local energy decay estimate for the free wave equation by the Fourier method.

Recall the coarea formula: for a a real-valued Lipschitz function $u$ and a $L^1$ function $g$ then

$$\int_{\mathbb{R}^n} g(x) |\nabla u(x)| \, dx = \int_{\mathbb{R}} \int_{\{u(x)=t\}} g(x) \, d\sigma(x) \, dt,$$

where $\sigma$ is the surface measure.

**Lemma 6.5.** For $F \in C_0^\infty$, $\phi$ smooth and non-degenerate, i.e. $|\nabla \phi(x)| \neq 0$, one has

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{i\lambda \phi(x)} F(x) \, dx d\lambda = (2\pi)^n \int_{\{\phi=0\}} \frac{F(x)}{|\nabla \phi(x)|} \, d\sigma(x).$$

**Proof.** From (6.87),

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{i\lambda \phi(x)} F(x) \, dx d\lambda = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda y} \int_{\{\phi=y\}} \frac{F(x)}{|\nabla \phi(x)|} \, d\sigma(x) \, dy d\lambda.$$

Denote $\int_{\{\phi=y\}} \frac{F(x)}{|\nabla \phi(x)|} \, d\sigma(x) = g(y)$, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i\lambda \phi(x)} F(x) \, dx d\lambda = \int_{\mathbb{R}} e^{i\lambda y} g(y) \, dy d\lambda = (2\pi)^\frac{n}{2} \int_{\mathbb{R}} \hat{g}(\lambda) \, d\lambda = (2\pi)^n g(0)$$

$$= \int_{\{\phi=0\}} \frac{F(x)}{|\nabla \phi(x)|} \, d\sigma(x).$$

We are done. \qed

It suffices to consider the half wave evolution,

$$e^{i \sqrt{-\Delta} t} f.$$

**Theorem 6.6** (Local energy decay). Let $\chi \geq 0$ be a smooth cut-off function such that $\hat{\chi}$ has compact support. Then

$$\|\chi(x) e^{i \sqrt{-\Delta} t} f\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_{t,x}}.$$

**Proof.** Consider

$$\int_{\mathbb{R}^n} \left| e^{i \sqrt{-\Delta} t} f \right|^2(x) \chi(x) \, dx \, dt = \int_{\mathbb{R}} \left\langle e^{i \sqrt{-\Delta} t} f, \chi(x) e^{i \sqrt{-\Delta} t} f \right\rangle_{L^2_{t,x}} \, dt$$

$$= \int_{\mathbb{R}} \left\langle e^{i t |\xi|} \hat{f}(\xi), e^{i t |\eta|} \hat{\chi}(\eta) \hat{f}(\eta) \right\rangle_{L^2_{\eta,\xi}} \, dt$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i t (|\xi|-|\eta|)} \hat{\chi}(\xi - \eta) \hat{f}(\xi) \hat{f}(\eta) \, d\eta d\xi \, dt.$$
Applying Lemma 6.5 with \( \phi(\xi, \eta) = |\xi| - |\eta| \), the surface \{ \phi = 0 \} becomes \{ |\xi| = |\eta| \} and \| \nabla \phi \| = \sqrt{2}$. It follows that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^n} |e^{it\sqrt{-\Delta}} f|^2 (x) \chi(x) \, dx \, dt \simeq \int_{|\xi| = |\eta|} |\hat{\chi}(\xi - \eta)| \left[ |\hat{f}(\xi)|^2 + |\hat{f}(\eta)|^2 \right] \, d\sigma
\]

\[
\leq \int_{|\xi| = |\eta|} |\hat{f}(\xi)|^2 \int_{|\xi| = |\eta|} |\hat{\chi}(\xi - \eta)| \, d\sigma \, d\xi
\]

\[
\leq \sup_{\xi} |K(\xi)| \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi
\]

\[
\leq \int_{\mathbb{R}^n} |f(x)|^2 \, dx.
\]

It reduces to show that

\[
K(\xi) = \int_{|\xi| = |\eta|} |\hat{\chi}(\xi - \eta)| \, d\sigma
\]

is bounded uniformly in \( \xi \). Since \( \hat{\chi}(\xi) \) decays fast, we have

\[
|\hat{\chi}(\xi)| \lesssim \langle \xi \rangle^{-N}
\]

and

\[
|\hat{\chi}(\xi)| \lesssim |\xi|^{1-\varepsilon-n},
\]

where as usual, \( \langle \xi \rangle = \left( 1 + |\xi|^2 \right)^{1/2} \).

Note that

\[
K(\xi) = \int_{|\xi| = |\eta|} |\hat{\chi}(\xi - \eta)| \, d\sigma
\]

\[
= \int_{|\xi - \xi| = |\xi|} |\hat{\chi}(\xi)| \, d\sigma
\]

\[
\lesssim \int_{|\xi - \xi| = |\xi|, |\xi| < 1} |\hat{\chi}(\xi)| \, d\sigma
\]

\[
+ \int_{|\xi - \xi| = |\xi|, |\xi| > 1} |\hat{\chi}(\xi)| \, d\sigma
\]

\[
\lesssim C(n)
\]

which is uniformly bounded in \( \xi \) and only depends on \( n \).

Therefore, we can conclude

\[
\| \chi(x) e^{it\sqrt{-\Delta}} f \|_{L^2_{t,x}} \lesssim \|f\|_{L^2_x}.
\]

We are done.

With dyadic decomposition and weights, one has a global version of the above result:

**Corollary 6.7.** \( \forall \varepsilon > 0 \), one has

\[
\left\| (1 + |x|)^{-\frac{1}{2} - \varepsilon} e^{it\sqrt{-\Delta}} f \right\|_{L^2_{t,x}} \lesssim \varepsilon \|f\|_{L^2_x}.
\]
Proof. Let $\chi(x)$ from Theorem 6.6 be a smooth version of $1_{B_1(0)}$, the indicator function of the unit ball. It follows that

$$\| \chi(2^{-j}x) e^{it\sqrt{-\Delta}} f \|_{L^2_{t,x}} = 2^{\frac{j}{2}} 2^j \| \chi(x) \left( e^{it\sqrt{-\Delta}} f \right) (2^j x) \|_{L^2_{t,x}} = 2^{\frac{j}{2}} 2^j \| \chi(x) \left( e^{it\sqrt{-\Delta}} f (2^j \cdot) \right) \|_{L^2_{t,x}} \lesssim 2^{\frac{j}{2}} 2^j \| f \|_{L^2_x} \lesssim 2^j \| f \|_{L^2_x}.$$  

(6.98)

Notice that

$$\left(1 + |x|\right)^{-\frac{1}{2} - \epsilon} \approx \sum_{j \geq 0} 2^{-j(\frac{1}{2} + \epsilon)} \chi(2^{-j}x)$$

then with our computations above, we can conclude that

$$\left\| \sum_{j \geq 0} 2^{-j(\frac{1}{2} + \epsilon)} \chi(2^{-j}x) e^{it\sqrt{-\Delta}} f \right\|_{L^2_{t,x}} \lesssim \epsilon \| f \|_{L^2_x},$$

(6.100)

and hence

$$\left\| \left(1 + |x|\right)^{-\frac{1}{2} - \epsilon} e^{it\sqrt{-\Delta}} f \right\|_{L^2_{t,x}} \lesssim \epsilon \| f \|_{L^2_x}.$$  

(6.101)

The corollary is proved. \qed

APPENDIX C

In this appendix, we discuss the global existence of solutions to the wave equation with time-dependent potentials. Lorentz transformations are important tools in our analysis. Lorentz transformations are rotations of space-time, therefore, a priori, one needs to show the global existence of solutions to wave equations with time-dependent potentials.

**Theorem 6.8.** Assume $V(x, t) \in L^\infty_{t,x}$. Then for each $(g, f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, there is a unique solution $(u, u_t) \in C([0, T], H^1(\mathbb{R}^3)) \times C([0, T], L^2(\mathbb{R}^3))$ to

$$\partial_{tt} u - \Delta u + V(x, t) u = 0$$

(6.102)

with initial data

$$u(x, 0) = g, \quad \partial_t u(x, 0) = f.$$  

(6.103)

**Proof.** By Duhamel’s formula, we might write the solution as

$$u = \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f + \cos \left( t \sqrt{-\Delta} \right) g + \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) u(s) \, ds.$$  

(6.104)

Starting from the local existence, we try to construct the solution in

$$X = C([0, T], H^1(\mathbb{R}^3)) \times C([0, T], L^2(\mathbb{R}^3))$$

(6.105)
with $T \leq 1$. One can view $u$ as the fixed-point of the map

\begin{equation}
S(h)(t) = \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f + \cos \left( t \sqrt{-\Delta} \right) g + \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) h(s) \, ds.
\end{equation}

Let

\begin{equation}
R = 2 \left\| \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f + \cos \left( t \sqrt{-\Delta} \right) g \right\|_X.
\end{equation}

We will show when $T$ is small enough, $S$ will be a contraction map in $B_X(0, R)$.

Clearly,

\begin{equation}
\|S(h)(t)\|_X \leq \left\| \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f + \cos \left( t \sqrt{-\Delta} \right) g \right\|_X + \left\| \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) h(s) \, ds \right\|_X.
\end{equation}

By direct calculations,

\begin{equation}
\left\| \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) h(s) \, ds \right\|_{L^2_x} \leq T^2 \|V(\cdot, t) h(t)\|_{L^2_x},
\end{equation}

\begin{equation}
\left\| \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) h(s) \, ds \right\|_{H^1_x} \leq T \|V(\cdot, t) h(t)\|_{L^2_x},
\end{equation}

and

\begin{equation}
\left\| \partial_t \left( \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) h(s) \, ds \right) \right\|_{L^2_x} \leq T \|V(\cdot, t) h(t)\|_{L^2_x}.
\end{equation}

Therefore, we can pick $T \|V\|_{L^\infty_{t,x}} < \frac{1}{10}$, we have

\begin{equation}
\|S(h)(t)\|_X \leq \left\| \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f + \cos \left( t \sqrt{-\Delta} \right) g \right\|_X + \frac{1}{2} \|h\|_X.
\end{equation}

Hence, $S$ maps $B_X(0, R)$ into itself.

Next we show $S$ is a contraction. The calculations are straightforward.

\begin{equation}
\|S(h_1 - h_2)(t)\|_X \leq \left\| \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) (h_1(s) - h_2(s)) \, ds \right\|_X.
\end{equation}

The the same arguments as above give

\begin{equation}
\|S(h_1 - h_2)(t)\|_X \leq \frac{1}{2} \|(h_1 - h_2)(t)\|_X.
\end{equation}

Therefore, by fixed point theorem, there is $u \in X$ such that

\begin{equation}
u = S(u),
\end{equation}

in other words, there exist $u \in C \left( [0, T], H^1(\mathbb{R}^3) \right) \times C \left( [0, T], L^2(\mathbb{R}^3) \right)$ such that

\begin{equation}
u = \frac{\sin \left( t \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} f + \cos \left( t \sqrt{-\Delta} \right) g + \int_0^t \frac{\sin \left( (t-s) \sqrt{-\Delta} \right)}{\sqrt{-\Delta}} V(\cdot, s) u(s) \, ds.
\end{equation}
We notice that the choice of $T$ is independent of the size of the initial data. Then we can repeat the above argument with $(u(T), \partial_t u(T))$ as initial condition to construct the solution from $T$ to $2T$. Iterating this process, one can easily construct the solution $(u, u_t) \in C(\mathbb{R}, H^1(\mathbb{R}^3)) \times C(\mathbb{R}, L^2(\mathbb{R}^3))$.

Finally, we notice the uniqueness of the solution follows from Grönwall’s inequality. Suppose one has two solutions $u_1$ and $u_2$ to our equation with the same data, then

\[(6.117) \quad \|u_1 - u_2\|_{H^1 \times L^2} (t) \leq \int_0^t (t-s) \|u_1 - u_2\| (s) \, ds.\]

Applying Grönwall’s inequality over $[0, T]$, we obtain

\[(6.118) \quad \|u_1 - u_2\|_X = 0,\]

which means $u_1 \equiv u_2$ on $[0, T]$. Then by the same iteration argument as above, we can conclude that in $C(\mathbb{R}, H^1(\mathbb{R}^3)) \times C(\mathbb{R}, L^2(\mathbb{R}^3))$

\[(6.119) \quad u_1 \equiv u_2.\]

Therefore, one obtains the uniqueness.

The theorem is proved. □

In our setting, $V(x, t) = V(x - \vec{v}(t))$ satisfies the assumption of Theorem 6.8, therefore we have the global existence and uniqueness.

**Corollary 6.9.** For each $(g, f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, there is a unique global solution $(u, u_t) \in C(\mathbb{R}, H^1(\mathbb{R}^3)) \times C(\mathbb{R}, L^2(\mathbb{R}^3))$ to the wave equation

\[(6.120) \quad \partial_{tt} u - \Delta u + V(x - \vec{v}(t)) u = 0\]

with initial data

\[(6.121) \quad u(x, 0) = g, \partial_t u(x, 0) = f.\]

**Remark.** The above theorem also applies to the charge transfer model in [GC2]:

\[
\partial_{tt} u - \Delta u + \sum_{i=1}^{m} \sum_{j=1}^{m} V_{ij} (x - \vec{v}_j(t)) u = 0.
\]

**Appendix D**

In this appendix, we present an alternative approach to the homogeneous end-point reversed Strichartz estimates based on the Fourier transformation.

We only consider \(\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f = \frac{1}{2} e^{it\sqrt{-\Delta}} f - \frac{1}{2} e^{-it\sqrt{-\Delta}} f\). We can further reduce to consider

\[(6.122) \quad e^{it\sqrt{-\Delta}} f\]

With Fourier transform and polar coordinates $\xi = \lambda \omega$, we have

\[
\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} f = \int_0^\infty \int_{\mathbb{S}^2} e^{2\pi i \lambda \cdot x} e^{2\pi i \lambda \cdot \omega} (\lambda) \frac{d\omega d\lambda}{\lambda} - 1 = \int_\mathbb{R} e^{2\pi i \lambda \cdot x} \chi_{[0, \infty)}(\lambda) \int_{\mathbb{S}^2} e^{2\pi i \lambda \cdot \omega} \hat{f}(\omega) d\omega \, d\lambda
\]

\[(6.123) \quad = \int_{\mathbb{R}} e^{2\pi i \lambda x} G(x, \lambda) d\lambda\]
where
\begin{equation}
G(x, \lambda) = \chi_{[0, \infty)}(\lambda) \int_{\mathbb{S}^2} e^{2\pi i \lambda(\omega \cdot x)} \lambda \hat{f}(\lambda \omega) \, d\omega.
\end{equation}

By Plancherel’s Theorem, we know for fixed \( x \),
\begin{equation}
\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^2_t L^2_x} = \| G(x, \lambda) \|_{L^2_x}.
\end{equation}

\begin{equation}
G^2(x, \lambda) = \left( \chi_{[0, \infty)}(\lambda) \int_{\mathbb{S}^2} e^{2\pi i \lambda(\omega \cdot x)} \lambda \hat{f}(\lambda \omega) \, d\omega \right)^2
\end{equation}

\begin{equation}
\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^2_t L^2_x}^2 \lesssim \chi_{[0, \infty)}(\lambda) \int_{\mathbb{S}^2} \lambda^2 \left| \hat{f}(\lambda \omega) \right|^2 \, d\omega \, d\lambda
\end{equation}

\begin{equation}
\lesssim \int_0^\infty \int_{\mathbb{S}^2} \lambda^2 \left| \hat{f}(\lambda \omega) \right|^2 \, d\omega \, d\lambda
\end{equation}

\begin{equation}
\lesssim \int |\hat{f}(\xi)|^2 \, d\xi
\end{equation}

\begin{equation}
= \int |f(x)|^2 \, dx.
\end{equation}

Therefore,
\begin{equation}
\left\| \sin \left( t \sqrt{-\Delta} \right) f \right\|_{L^\infty_t L^2_x} \lesssim \| f \|_{L^2}.
\end{equation}

as desired.

Remark. The two dimension version was obtained in [Oh] and is mentioned in [B]:
\begin{equation}
\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^\infty_t L^2_x} \lesssim \| f \|_{B^1_{2,1}}.
\end{equation}

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URL: http://www.math.uchicago.edu/~gc/

E-mail address: gc@math.uchicago.edu

Department of Mathematics, The University of Chicago, 5734 South University Avenue, Chicago, IL 60615, U.S.A