Let $H_n(q)$ be the Iwahori-Hecke algebra of type $A_{n-1}$ over a field $K$ of characteristic 0. For each Young diagram $\lambda$, an $H_n(q)$ module $S^\lambda$, called a Specht module, was defined in [DJ]. The dimension of $S^\lambda$ does not depend on the choice of $q$, and for $q$ not a root of unity, the $S^\lambda$’s provide a complete set of irreducible $H_n(q)$-modules, up to isomorphism. If $q$ is a primitive $l$-th root of unity, a complete set of simple $H_n(q)$-modules $D^\mu$ has been constructed in [DJ], where $\mu$ runs through all Young diagrams with at most $l-1$ rows of equal length. These modules are not well understood, but their dimensions can be computed if one knows the multiplicities $d_{\lambda \mu}$ of the $D^\mu$ in a composition series of $S^\lambda$, for all possible $\mu$’s and $\lambda$’s.

A result known as Nakayama conjecture, or equivalently, in the context of quantum groups, as linkage principle, gives some information about the $d_{\lambda \mu}$’s. It says that $d_{\lambda \mu} \neq 0$ only if $\mu + \rho$ is in the orbit of $\lambda + \rho$ with respect to an affine reflection group. Here $\rho = (k-1, k-2, \ldots, 1, 0)$ for some $k \geq n$, and the reflection group is that generated by the affine reflections in the hyperplanes $y_i - y_j = ml$, $1 < i < j \leq k$ and $m \in \mathbb{Z}$. However, these orbits can become quite large and can contain many diagrams $\mu$ for which $d_{\lambda \mu} = 0$.

In this paper, we explore the use of path idempotents for the Hecke algebra at roots of unity. The path idempotents are defined via the orthogonal representation, which is not well defined at roots of unity. Nevertheless, we show that certain sums of path idempotents are still well defined at roots of unity and can be used to derive information about simple modules and decomposition numbers at roots of unity.

Our main technical result (Theorem 3.3.1) concerns a (non-unital) embedding of the ($k$-row quotient of) the Hecke algebra $H_m(x)$ defined over the field of rational functions $K(x)$ into the ($k$-row quotient of the) Hecke algebra $H_n(x)$, for certain $m$ and $n$. If $q$ is a primitive $l$-th root of unity, this gives an embedding of $KS_m$ into $H_n(q)$. From this, we obtain the following applications:

(a) We show that $S^\lambda = D^\lambda$ for certain Young diagrams, namely for all diagrams $\lambda$ with $\leq k$ rows for which each component of $\lambda + \rho$ is divisible by $l$. This gives another proof of special cases of results by Dipper and James, and of James and Mathas, for Specht modules.

(b) We obtain estimates on the decomposition numbers $d_{\lambda \mu}$ which provide rather good geometric information about the coefficients and in particular about the set of $\lambda$ for which $d_{\lambda \mu} = 0$. These estimates can be derived from the irreducibility criterion mentioned above in fairly short order. (Such a derivation may be known to experts, but as far as we know it does

We would like to thank the referee for an exceptionally careful reading of a previous version of this paper, and for pointing the way towards several improvements in the exposition.
The estimates certainly are easy consequences of deep results by Soergel on Kazhdan-Lusztig polynomials and tilting modules \([S1, S2]\).

Two algorithms for computing the \(d_{\lambda\mu}\) have recently been discovered. The first, proposed by Lascoux, Leclerc, and Thibon [LLT] and proved by Ariki [Ar], computes the decomposition numbers via the lower global crystal base \([K, Ha, MM]\) of the basic \(U_q(\hat{sl}_l)\) module \(L(\omega_0)\). The second, due to Soergel [S1, S2] (see also [A2]) computes decomposition numbers for tilting modules of \(U_q(\hat{sl}_k)\) via certain Kazhdan-Lusztig polynomials for the affine Weyl group. The proofs that the two methods give the decomposition numbers [Ar, S2] are quite deep, whereas the proofs of our estimates are elementary.

It is also possible to give a more efficient version of the LLT algorithm which is a \(q\)-version of the algorithm for our estimates. This is discussed in a separate paper [GW], where we also show by a direct combinatorial method that the polynomial coefficients of the global lower crystal base of \(L(\omega_0)\) coincide with the Kazhdan-Lusztig polynomials discussed by Soergel.

It would be interesting to get a better understanding between the connection of our principal technical result, the embedding of \(H_m(x^t)\) into \(H_n(x)\), and Lusztig’s generalized Frobenius homomorphism for quantum groups. This is discussed in greater detail in Section 6.

0. Generic Algebras

This section contains some preliminaries about finite dimensional algebras whose relations depend on a parameter. We suggest that the reader skip this section on the first reading, noting however that it contains the definitions of some terms which are used in the rest of the paper. Recall that \(K\) denotes a field of characteristic 0.

0.1. Interpolation formulas. Let \(J\) be a finite set, and let \(\{a_j : j \in J\}\) be a set of variables. We define for any \(l \in J\), and for any subset \(S \subset J\) the Lagrange functions

\[
\tilde{P}_l((a_j); y) = \prod_{j \neq \ell} \frac{(y - a_j)}{(a_l - a_j)} \quad \text{and} \quad \tilde{P}_S = \sum_{l \in S} \tilde{P}_l;
\]

these are rational functions in the variables \(\{a_j\}\), and polynomials in the variable \(y\). Obviously, \(\tilde{P}_l(a_i) = \delta_{i\ell}\). One has the following well-known results regarding singularities of these rational functions, and rational functional calculus of matrices.

**Lemma 0.1.1.** The rational function \(\tilde{P}_S((a_j); y)\) has removable singularities at \(a_i = a_j\) for \(i, j \in S\) as well as for \(i, j \notin S\).

**Proof.** It follows from the definitions that we can write \(\tilde{P}_S\) as a rational function in the variables \(\{a_j\}\) with denominator

\[
\prod_{r, s \in S, r \neq s} (a_r - a_s) \prod_{s \in S, r \notin S} (a_r - a_s).
\]

The function \(\tilde{P}_S\) is symmetric in the variables \(\{a_j : j \in S\}\), while the denominator is antisymmetric in these variables. It follows that the numerator is also divisible by \(\prod_{r, s \in S, r \neq s}(a_r - a_s)\),
so \( \tilde{P}_S \) can also be written as a rational function in the variables \( \{a_j\} \) with denominator \( \prod_{s \in S, r \not\in S}(a_r - a_s) \).

\[ \square \]

**Corollary 0.1.2.** Let \( (S_i)_{i \in I} \) be a partition of the index set \( J \). Fix pairwise distinct elements \( k_i \in K \), for \( i \in I \), and an index \( \ell \in I \). Then there exists a polynomial \( P_{S_\ell,(k_i)}(y) \in K[y] \) which is obtained from \( \tilde{P}_{S_i}((a_j); y) \) by substituting \( a_j = k_i \) when \( j \in S_i \). Moreover, \( P_{S_\ell,(k_i)}(y) - 1 \) is divisible by \( (y - k_\ell)^s \), where \( s = |S_\ell| \).

**Proof.** We obtain a well-defined polynomial \( P_{S_\ell,(k_i)} \) by Lemma 0.1.1. By construction, \( \tilde{P}_{S_i}((a_j); a_r) = 1 \) for \( a_r \in S_I \); hence \( \tilde{P}_{S_\ell}((a_j); y) - 1 \) is divisible by \( \prod_{r \in S_\ell}(y - a_r) \). The last statement follows from this. \[ \square \]

### 0.2. Eigenspaces

Fix once and for all an algebraic closure \( \overline{K(x)} \) of \( K(x) \) and an algebraic closure \( K \subseteq \overline{K(x)} \) of \( K \). Let \( B \) be an \( n \times n \) matrix with entries in \( K(x) \) and with distinct eigenvalues \( r_1, \ldots, r_N \) in \( \overline{K(x)} \), with multiplicities \( m_1, \ldots, m_N \). Let \( F = K(x, r_1, \ldots, r_N) \). Recall that the generalized eigenspace of \( r_i \) is the kernel in \( F^n \) of \( (B - r_i 1)^{m_i} \). Let \( E(r_i) \) be the projection onto this eigenspace, with kernel being the direct sum of the other generalized eigenspaces; we call \( E(r_i) \) the eigenprojection of \( B \) for the eigenvalue \( r_i \).

We remark that it does not take much extra effort to deal with the case that the matrix \( B \) has its eigenvalues in an algebraic extension field of \( K(x) \), as we do here; however, in our applications in the following sections, it will never be necessary to consider such extension fields, so the reader can safely ignore the extension fields here as well.

Let \( \{v_j, 1 \leq j \leq n\} \) be a basis of \( F^n \) consisting of generalized eigenvectors of \( B \). Partition \( J = \{i \in \mathbb{N} : 1 \leq i \leq n\} \) into subsets \( S_i \), where \( j \in S_i \) if \( v_j \) is a generalized eigenvector with eigenvalue \( r_i \). Define the polynomials \( P_{S_i,(r_i)} \) for this partition, as in Corollary 0.1.2.

**Lemma 0.2.1.** We have \( E(r_\ell) = P_{S_\ell,(r_\ell)}(B) \) for any eigenvalue \( r_\ell \) of \( B \).

**Proof.** By construction, the function \( P_{S_\ell,(r_\ell)} \) has a zero of multiplicity \( m_i \) at \( r_i \) if \( i \neq \ell \). By Corollary 0.1.2, \( P_{S_\ell,(r_\ell)} - 1 \) has a zero of order \( m_\ell \) at \( r_\ell \). Hence \( P_{S_\ell,(r_\ell)}(B)v_j = 0 \) for \( v_j \) in the generalized eigenspace of \( r_i \), \( i \neq \ell \), and \( (P_{S_\ell,(r_\ell)}(B) - 1)v_j = 0 \) for \( v_j \) in the generalized eigenspace of \( r_\ell \). \[ \square \]

**Definition 0.2.2.** Let \( q \in K \). We call a matrix \( B \) over \( K(x) \) **evaluable** at \( q \) (or just evaluable, if \( q \) is understood) if none of its entries have poles at \( x = q \); that is the entries of the matrix lie in \( K(x)_q \), the local ring of rational functions with no poles at \( q \). The result of evaluating the matrix at \( q \) is denoted by \( B(q) \).
In the following $B$ will denote an evaluable matrix. The eigenvalues $r_i$ of $B$ are algebraic integers over $K(x)_q$, and the evaluation homomorphism from $K(x)_q$ to $K$ extends to a homomorphism of $K(x)[r_1, \ldots, r_N]$ to $K$; denote the images of the $r_i$ by $r_i(q)$. Let $F = K(x, r_1, \ldots, r_N)$ and let $F_q$ denote the ring consisting of quotients of elements of $K(x)_q[r_1, \ldots, r_N]$ with denominators which are non-zero upon evaluation at $q$; we refer to $F_q$ as the ring of evaluable elements of $F$. We call a matrix over $F$ evaluable if it has entries in $F_q$.

**Lemma 0.2.3.** Suppose $B$ is an evaluable matrix over $K(x)$.

(a) The eigenvalues of $B(q)$ are given by $r_i(q)$.

(b) Suppose $B$ is invertible. Then $B^{-1}$ is evaluable if and only if $B(q)$ is invertible.

**Proof.** On the one hand, $\det(y1 - B)$ maps to $\det(y1 - B(q))$ under evaluation at $q$. On the other hand, $\det(y1 - B) = \prod(y - r_i)^{m_i}$ maps to $\prod(y - r_i(q))^{m_i}$. This shows part (a). If $B^{-1}$ is evaluable, then $B^{-1}(q)$ is an inverse for $B(q)$. Conversely, if $B(q)$ is invertible, then $r_i(q) \neq 0$ for all $i$, so $\det(B)(q) \neq 0$. Then it follows from Cramer’s rule that $B^{-1}$ is evaluable.

**Proposition 0.2.4.** Let $c \in K$ be an eigenvalue of $B(q)$, and let $E(c)$ be its eigenprojection. Then the matrix $\sum_{\{\ell : r_{\ell}(q) = c\}} E(r_{\ell})$ over $F$ is evaluable and its evaluation coincides with $E(c)$.

**Proof.** For each eigenvalue $c$ of $B(q)$, let $T_c = \cup\{S_i : r_i(q) = c\}$. Then $\{T_c\}$ is a partition of $J$, and $\sum_{\{\ell : r_{\ell}(q) = c\}} P_{S_i}(a_j) = \hat{P}_{T_{\ell};(a_j)}$. But $\hat{P}_{T_{\ell};(a_j)}$ can be written as a rational function in the variables $\{a_j\}$ with denominator $\prod_{S \in T_c, \ell \not\in T_c}(a_r - a_s)$; therefore $\hat{P}_{T_{\ell};(r_{\ell})}$ lies in $F_q[y]$, and

$$\sum_{\{\ell : r_{\ell}(q) = c\}} E(r_{\ell}) = \sum_{\{\ell : r_{\ell}(q) = c\}} P_{S_i}(r_i(B)) = P_{T_c}(r_{\ell})(B)$$

is a matrix over $F_q$. Furthermore,

$$E(c) = P_{T_{\ell};(r_{\ell}(q))}(B(q)) = P_{T_{\ell};(r_{\ell})}(B)(q) = \left(\sum_{\{\ell : r_{\ell}(q) = c\}} E(r_{\ell})\right)(q).$$

**0.3.** Evaluable elements of an algebra. We apply the results of the previous section to the following set-up: Let $A$ be a finite-dimensional $K(x)$-algebra with a basis $\{a_j : j \in J\}$, for which the structure coefficients are polynomials.

**Definition 0.3.1.** For $q \in K$, we call an element $a = \sum_j s_j(x)a_j \in A$ evaluable at $x = q$ if none of the $s_j$ have a pole at $q$. Denote by $A_q$ the set of evaluable elements of $A$. 
The set $A_q$ of evaluable elements coincides with the span of $\{a_j\}$ over $K(x)_q$, and is a $K(x)_q$-algebra.

Let $A(q)$ denote the $K$-algebra with a basis also denoted by $\{a_j : j \in J\}$ whose structure coefficients are given by evaluating the structure coefficients of $A$ at $q$. Then

$$A(q) \cong A_q \otimes_{K(x)_q} K,$$

where $K(x)_q$ acts on $K$ by $f(x)\lambda = f(q)\lambda$. The evaluation map

$$a = \sum s_j(x)a_j \mapsto a(q) = \sum s_j(q)a_j,$$

or $a \mapsto a \otimes 1$, defines a $K$-algebra homomorphism from $A_q$ onto $A(q)$.

**Lemma 0.3.2.** Let $A$ and $B$ be algebras over $K(x)$ with basis $\{a_i\}, \{a_j\}$ having polynomial structure coefficients. Let $\varphi : A \to B$ be an algebra homomorphism such that $\varphi(a_i) \in B_q$ for all $i$.

(a) $\varphi(A_q) \subseteq B_q$.

(b) There is a unique $K$-algebra homomorphism $\tilde{\varphi} : A(q) \to B(q)$ satisfying $\tilde{\varphi}(a(q)) = \varphi(a)(q)$ for $a \in A_q$.

**Proof.** Part (a) is evident. The prescription $\tilde{\varphi}(a_i) = \varphi(a_i)(q)$ determines the map $\tilde{\varphi}$ in part (b). \qed

### 0.4. Evaluative representations.

We will also apply the notion of evaluability to representations of $A$ with matrix coefficients in $K(x)$:

**Definition 0.4.1.** We say that a matrix representation $\Phi$ of $A$ is evaluative at $q$ if the matrices $\Phi(a_j)$ have coefficients in $K(x)_q$ for all $j \in J$. An $A$-module $V$ is called evaluative at $q$ if it has a $K(x)$ basis with respect to which the basis elements of $A$ act via $q$-evaluable matrices.

Observe that the class of the $A(q)$ module may depend on the choice of basis. So we always assume for an evaluative $A$-module that a basis has been fixed.

**Lemma 0.4.2.** Let $\varphi$ be an evaluative representation of $A$ in $\text{Mat}_n(K(x))$.

(a) The restriction of $\varphi$ to the evaluative elements of $A$ is a $K(x)_q$-algebra homomorphism into $\text{Mat}_n(K(x)_q)$.

(b) $\varphi$ induces a $K$-algebra homomorphism from $A(q)$ into $\text{Mat}_n(K)$, determined by $\tilde{\varphi}(a(q)) = \tilde{\varphi}(a)(q)$.

**Proof.** This is a special case of Lemma 0.3.2. \qed

Consider an evaluative representation $\varphi$ of $A$ in $\text{Mat}_n(K(x))$ and an evaluative element $a \in A$. Let $\{r_i\}$ be the spectrum in $K(x)$ of left multiplication by $a$ in $A$, and put $F = K(x, r_1, \ldots, r_N)$. The spectrum of $\varphi(a)$ is contained in $\{r_i\}$. Extend the evaluation homomorphisms to $F_q$ as above. Then the spectrum of $\varphi(a)(q)$ is contained in the set $\{r_i(q)\}$.
Lemma 0.4.3. Let \( \varphi \) be an evaluable representation of \( A \).

(a) Let \( a \) be an evaluable element of \( A \). Then for any eigenprojection \( e' \) of \( \varphi(a)(q) \) there exists an evaluable eigenprojection \( e \) of \( a \) such that its evaluation \( \varphi(e)(q) \) coincides with \( e' \).

(b) Let \( p' \) be an idempotent in \( \varphi(A)(q) \). Then there exists an evaluable idempotent \( p \in A \otimes_{K(x)} F \), for some finite algebraic extension \( F \) of \( K(x) \), such that \( \varphi(p)(q) = p' \).

(c) Let \( p' \) be an idempotent in \( A(q) \). Then there exists an evaluable idempotent \( p \in A \otimes_{K(x)} F \), for some finite algebraic extension \( F \) of \( K(x) \), such that \( p(q) = p' \).

Proof. Let \( e \) be the eigenvalue corresponding to \( e' \). Then, by Proposition 0.2.4, the eigenprojection \( e \) onto the direct sum of generalized eigenspaces corresponding to the eigenvalues \( r_i \) with \( r_i(q) = e \) acts via an evaluable matrix in the left regular representation, i.e. it is an evaluable element (in \( A \otimes_{K(x)} F \)). Hence also the matrix \( \varphi(e) \) is evalu able and \( \varphi(e)(q) \) coincides with the eigenprojection \( e' \) of \( \varphi(a)(q) \), again by Proposition 0.2.4.

To show (b), let \( a' \in A(q) \), say \( a' = \sum s_j b_j \) with \( s_j \in K \), be such that \( \varphi(a') = p' \). Let \( a = \sum s_j b_j \in A \), where \( (b_j) \) is the corresponding basis for \( A \) and the \( s_j \)'s are constant rational functions. Then \( p' \) is an eigenprojection of \( \varphi(a)(q) = p' \). Hence there exists an evaluable idempotent \( p \in A \otimes_{K(x)} F \) for an appropriate extension field \( F \), such that \( \varphi(p)(q) = p' \), by part (a).

Point (c) is the special case of (b) where the representation \( \varphi \) is the left regular representation. \( \square \)

0.5. Rank vectors. We shall henceforth assume that the algebra \( A \) is a semisimple algebra over \( K(x) \) which is a direct sum of full matrix algebras over \( K(x) \):

\[
A = \bigoplus_{\lambda \in \Lambda} A_\lambda.
\]

The examples we shall deal with, the Hecke algebras of type \( A \), satisfy this assumption.

Let \( B \) be a semisimple algebra over a field \( K \) which is isomorphic to the direct sum \( \bigoplus \lambda B_\lambda \) of full matrix algebras \( B_\lambda \). Let \( p \) be an idempotent in \( B \). The rank \( r(p) \) is the vector \( (r(p)_\lambda) \), where \( r(p)_\lambda = \text{Tr}_\lambda(p) \) with \( \text{Tr}_\lambda \) the usual trace on \( \text{End}(V_\lambda) \) and \( V_\lambda \) a simple \( B_\lambda \) module.

In particular one can apply this notion of the rank vector to idempotents in \( A \).

The algebra \( A(q) \) may no longer be semisimple. Let \( \overline{A(q)} \) be its maximum semisimple quotient, with \( \overline{A(q)} \cong \bigoplus_\mu \overline{A(q)_\mu} \) and \( \overline{A(q)_\mu} \) a full matrix algebra over \( K \) for each index \( \mu \). Let \( D_\mu \) denote the simple \( \overline{A(q)_\mu} \) module. We define the (reduced) rank vector \( r_q(p') \) of an idempotent \( p' \in A(q) \) with respect to the maximum semisimple quotient \( \overline{A(q)} \).
0.6. Evaluable modules. Recall that an $A$ module $W$ is called evaluable if $W$ has a basis with respect to which the matrices representing the basis elements of $A$ are evaluable at $x = q$. An evaluable $A$ module $W$ “restricts” to an $A(q)$ module $W_q$; as a set, $W_q$ is the $K$-linear span of the distinguished basis of $W$; thus $W = W_q \otimes_K K(x)$.

Assume that $A$ has a complete set of $q$-evaluable simple modules ($S_\lambda$).

Consider a minimal idempotent in $A(q)$ which is in the simple component $A(q)_\mu$ of $A(q)$. Such an idempotent can be lifted to an idempotent $p'_\mu$ in $A(q)$, and two such liftings differ by a nilpotent element of the radical of $A(q)$.

As $S_\lambda$ is evaluable, $p'_\mu$ also acts on $S_\lambda q$. Let $d_{\lambda \mu} = \text{Tr}_{S_\lambda q}(p'_\mu)$ be its rank in this representation; this is independent of the choice of the lifting since the difference of two liftings is nilpotent.

These numbers can be interpreted in terms of a composition series of the $A(q)$ modules $S_\lambda q$.

More generally, let $W$ be a $q$-evaluable $A$ module whose restriction to an $A(q)$ module has a composition series of the form

$$0 = V_0 \subset V_1 \subset ... \subset V_k = W_q;$$

here the $V_i$'s are $A(q)$ modules such that the factors $V_i/V_{i-1}$ are simple.

Proposition 0.6.1. Let $W$ be an evaluable module, and let $p \in A$ be an evaluable idempotent.

(a) $\text{Tr}_W(p) = \text{Tr}_{W_q}(p(q))$.
(b) Suppose that $p(q) = p'_\mu$. Then $r(p) = (d_{\lambda \mu})_\lambda$.
(c) If $p(q) = p'_\mu$, then $\text{Tr}_W(p)$ is the number of factors in the composition series (*) which are isomorphic to $D_\mu$.
(d) If $W \cong \bigoplus_\lambda m_\lambda S_\lambda$ as an $A$ module, then exactly $\sum_\lambda m_\lambda d_{\lambda \mu}$ factors in (*) are isomorphic to $D_\mu$.

Proof. As $p$ is an idempotent, it acts via a matrix on $W$ whose diagonal entries (which are rational functions) add up to a constant integer. Evaluating these functions at $q$ and summing them up produces the same integer. This proves (a). Claim (b) follows from (a) and the definition of $d_{\lambda \mu}$.

To prove (c), observe that $\text{Tr}_{W_q}(p'_\mu) = \sum_i \text{Tr}_{V_{i+1}/V_i}(p'_\mu)$. As $V_{i+1}/V_i$ is simple and $p'_\mu$ is a lifting of a minimal idempotent in $A(q)$, it follows that $\text{Tr}_{V_{i+1}/V_i}(p'_\mu)$ is either 0 or 1, depending on whether $V_{i+1}/V_i$ is a simple $A(q)_\mu$ module or not. Hence $\text{Tr}(p'_\mu)$ is equal to the number of factors isomorphic to $D_\mu$. The statement now follows from (a).

Statement (d) follows from (b), (c) and $\text{Tr}_W(p) = \sum_\lambda m_\lambda \text{Tr}_{S_\lambda}(p)$.

Corollary 0.6.2. If $p$ is an evaluable idempotent in $A$, we have $r(p) = D r_q(p(q))$, where $D = (d_{\lambda \mu})$. 

0.7. Existence of idempotents.

**Proposition 0.7.1.** Let \((\pi_\lambda)_\lambda\) be not necessarily evaluable representations of \(A\) such that for each \(\lambda\), \(\pi_\lambda\) is equivalent over \(K(x)\) to the representation on \(S^\lambda\). Let \(a\) be an evaluable element of \(A\). Then there exists an evaluable idempotent \(p\) in \(A \otimes_{K(x)} F\), for some finite algebraic extension \(F\) of \(K(x)\), with the following property: whenever \(\pi_\lambda(a)\) is an evaluable matrix, and its evaluation \(\pi_\lambda(a)(q)\) is a rank \(d_\lambda\) idempotent, then \(p\) acts as a rank \(d_\lambda\) idempotent on \(S^\lambda\).

**Proof.** Consider \(a\) acting via left multiplication on \(A\). Let \(\{r_i\}\) be the spectrum of \(a\) in \(\bar{K(x)}\), \(F = K(x, r_1, \ldots, r_N)\), and let \(F_q\) denote the ring of \(q\)-evaluable elements of \(F\). Let \(p\) be the eigenprojection of \(a\) projecting on the span of generalized eigenspaces of \(a\) belonging to the eigenvalues \(r_i\) for which \(r_i(q) = 1\). By Proposition \[1.2.4\], \(p\) is evaluable, and \(p = P(a)\), where \(P \in F_q[y]\).

Suppose that \(\pi_\lambda(a)(q)\) is a rank \(d_\lambda\) idempotent. Note that \(\pi_\lambda(p) = \pi_\lambda(P(a)) = P(\pi_\lambda(a))\) is an evaluable idempotent. Furthermore its evaluation \(\pi_\lambda(p)(q) = \pi_\lambda(P(a))(q) = P(\pi_\lambda(a))(q)\) is the eigenprojection of \(\pi_\lambda(a(q))\) corresponding to its eigenvalue 1, i.e. it is equal to \(\pi_\lambda(a)(q)\), by assumption.

Finally,

\[
\text{Tr}_{S^\lambda}(p) = \text{Tr}(\pi_\lambda(p)) = \text{Tr}(\pi_\lambda(p)(q)) = \text{Tr}(\pi_\lambda(a)(q)) = d_\lambda,
\]

where the first equation follows from the equivalence of the representations, the second equation result from Proposition \[0.6.3\] (a) – applied to the algebra generated by \(p\), the third follows from the equality \(\pi_\lambda(p)(q) = \pi_\lambda(a)(q)\) observed above, and the last from the assumption on \(\pi_\lambda(a)(q)\).

**Remark 0.7.2.** We mention again that in the applications of Lemma \[1.4.3\] and Proposition \[0.7.1\] in the following sections, we can work entirely over \(K(x)\), and need not pass to an extension field. The idempotents obtained using these results will lie in the algebra \(A\) itself rather than in some \(A \otimes_{K(x)} F\).

1. Hecke algebras

1.1. Definition of Hecke algebras. Let \(q\) be a non-zero element of a field \(F\) of characteristic 0. We denote by \(H_n = H_n(q, F)\) the Iwahori-Hecke algebra of type \(A_{n-1}\), given by generators \(T_i, i = 1, 2, \ldots, n - 1\) and relations \(T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}\), \(T_iT_j = T_jT_i\) if \(|i - j| > 1\), and \(T_i^2 = (q - 1)T_i + q\). It is well-known that \(H_n\) has a \(K\)-basis \(T_w, w \in S_n\), on which the generators act by

\[
T_iT_w = \begin{cases} T_{siw} & \text{if } \ell(siw) > \ell(w) \\ (q - 1)T_w + qT_{siw} & \text{if } \ell(siw) < \ell(w). \end{cases}
\]

In fact, for any minimal expression of \(w\) as a product of simple reflections \(s_i\), \(T_w\) is the product of the corresponding \(T_i\).
Now let $K$ be a field of characteristic 0. In the following, we shall consider the Hecke algebras $H_n(x, K(x))$ over the fields $K(x)$ with parameter $q = x$, and the Hecke algebra $H_n(q, K)$ over $K$ with parameter $q \in K^\times$. Observe that $H_n(x, K(x))$ satisfies the conditions for the algebra $A$ in Section 0; to be consistent with the notation of Section 0, we shall denote $H_n(x, K(x))$ by $H_n$ and the specialization $H_n(q, K)$ by $H_n(q)$. We sometimes refer to the algebra $H_n$ over $K(x)$ as the generic Hecke algebra.

We recall the following definitions from Section 0: An element $a = \sum_w a_w(x)T_w$ of the generic Hecke algebra is said to be evaluable at $q \in K$ if the rational functions $a_w(x)$ have no poles at $x = q$. A matrix representation $\varphi : H_n(x) \to \text{Mat}_n(K(x))$ is called evaluable at $q$ if the matrices $\varphi(T_w)$ have no poles at $x = q$. Finally, an $H_n$-module $W$ is said to be evaluable at $q$ if $W$ has a $K(x)$-basis such that the matrix representation with respect to this basis is evaluable at $q$.

1.2. Some induced modules. Recall that a composition $\lambda$ of $n$ is a finite sequence $(\lambda_1, \lambda_2, \ldots)$ of non-negative integers, not necessarily decreasing, such that $\sum_i \lambda_i = n$. A partition is a composition with weakly decreasing parts. We identify partitions with Young diagrams.

Let $\lambda$ be a composition of $n$ with at most $k$ non-zero parts. Define the Young subgroup of $S_n$, $Y_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_k} \subset S_n$. It is well-known that we can also define a length function on the left cosets of $Y_\lambda$ by $\ell(L_w) = \text{min}\{\ell(v), \ v \in wY_\lambda\}$, and that one can define a representation of $H_n(q, F)$ on the module $W^\lambda$, which has a basis labelled by the left cosets, by

$$T_i L_w = \begin{cases} L_{s_{i}w} & \text{if } \ell(L_{s_{i}w}) > \ell(L_w) \\ qL_w & \text{if } L_{s_{i}w} = L_w \\ (q-1)L_w + qL_{s_{i}w} & \text{if } \ell(L_{s_{i}w}) < \ell(L_w). \end{cases}$$

In fact, consider the subalgebra $H(\lambda) \subseteq H_n(q, F)$ generated by the $T_i$ such that $s_i \in Y_\lambda$. $H(\lambda)$ acts on $F$ by $T_i \xi = q^\ell \xi$ for all generators $T_i$ of $H(\lambda)$. Then the module $W^\lambda$ is isomorphic to $H_n(q, F) \otimes_{H(\lambda)} F$, with $L_w$ corresponding to $T_w \otimes 1$.

If $\lambda$ and $\mu$ are composition which differ only in the order of their parts, then the modules $W^\lambda$ and $W^\mu$ are isomorphic.

1.3. Orthogonal representations. Consider $H_n(q, F)$, where $q$ is not a proper root of unity; this includes the case $q = 1$ and also the case $F = K(x)$ and $q = x$.

It is well-known that $H_n(q, F)$ is semisimple, its simple modules are labelled by Young diagrams, and the dimension of the simple module labelled by $\lambda$ is equal to the number of standard tableaux of shape $\lambda$.

We recall here a specific constructin of the simple $H_n(q, F)$ modules, namely a Hecke algebra version of Young’s orthogonal representations of the symmetric group ([H], [W1]). Our conventions regarding Young diagrams and tableaux are those of [Mac]; in particular, the Young diagram $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a left justified array of boxes with $\lambda_1$ boxes in the first (top) row, $\lambda_2$ in the second, and so forth.
Let 

\[ e_i = qI_i - T_i, \]

so that \( e_i \) is an essential idempotent with \( e_i^2 = (q + 1)e_i \). Obviously, a representation of \( H_n(q, F) \) is completely determined as soon as we know the matrices for the \( e_i \)’s.

For a Young diagram \( \lambda \) of size \( n \), let \( V_\lambda \) be the vector space with basis labelled by all standard tableaux of shape \( \lambda \). For a standard tableau \( t \), and for \( 1 \leq i < j \leq n \), define the quantity \( d(t; i, j) \) to be the integer, whose absolute value is one less than the length of the hook from \( i \) to \( j \), and whose sign is negative if \( j \) is northeast (i.e. to the right or above) of \( i \), and positive if it is southwest of \( i \). We abbreviate \( d(t; i, i+1) \) by \( d(t; i) \). Moreover, define the rational functions

\[ a_d(x) = \frac{1 + x + \cdots + x^d}{1 + x + \cdots x^{d-1}}, \quad \text{for } d \in \mathbb{N} \setminus \{0\}. \]

A representation of \( H_n(q, F) \) on \( V_\lambda \) is determined by

\[ e_i t = a_d t + \sqrt{a_d a_{-d}} s_i t, \]

where \( d = d(t; i) \), \( a_d = a_d(q) \), and \( s_i t \) is the tableau obtained by interchanging \( i \) with \( i+1 \). See [W1] for more details; observe the \( e_i \) here is equal to \((1 + q)\) times the \( e_i \) in [W1]. We will denote the representation of \( H_n(q, F) \) on \( V_\lambda \) by \( \pi_\lambda \).

**Remark 1.3.1.** We adjoin to our field square roots of the quantities \( a_d a_{-d} \) for \( d \in \mathbb{N} \setminus \{0\} \). It would also be possible to work with less symmetric version of these representations which avoids the introduction of square roots, namely

\[ e_i t = a_d t + a_{-d} s_i t, \]

if \( a_d \neq 0 \) and \( e_i t = 0 \) if \( a_d = 0 \).

Note that if \( i \) and \( i + 1 \) are in the same row of \( t \), then \( e_i t = 0 \), and if \( i \) and \( i + 1 \) are in the same column, then \( e_i t = (1 + q) t \). In all other cases, both \( t \) and \( s_i t \) are standard, and the restriction of \( e_i \) to the two dimensional space spanned by \( t \) and \( s_i t \) has matrix

\[ \begin{pmatrix} a_d & \sqrt{a_d a_{-d}} \\ \sqrt{a_d a_{-d}} & a_{-d} \end{pmatrix} \]

One can show that the \( V_\lambda \)'s constitute complete set of inequivalent simple modules of the semisimple algebra \( H_n(q, F) \).

It is easy to obtain the restriction rule for these modules, i.e. the way in which \( V_\lambda \) decomposes into simple \( H_m(q, F) \)-modules for \( m < n \). One shows by induction on \( n - m \) that the multiplicity of the simple \( H_m(q, F) \) module \( V_\mu \) in \( V_\lambda \) is equal to the number of standard skew tableaux of shape \( \lambda \setminus \mu \). (A standard skew tableau is a filling of the shape \( \lambda \setminus \mu \) with the numbers \( 1, 2, \ldots \) so that the entries of each row and column are strictly increasing.)
1.4. Murphy elements. We shall need a $q$-version of some symmetric group elements due to Murphy, see [M]. The $q$-version is actually simpler than the version for the symmetric group (see, for example, [LR]). Let $\Delta_n^2 = (T_1T_2 \ldots T_{n-1})^n$ in $H_n(q, F)$, and let

$$M_i = \Delta_{i-1}^{-2} \Delta_i^2 = T_{i-1}T_{i-2} \ldots T_2 T_1^2 T_2 \ldots T_{i-1}.$$  

It is well-known that $\Delta_n^2$ is in the center of $H_n(q, F)$; it follows easily from this that the elements $M_i$, $i = 2, 3, \ldots, n$ are mutually commuting. It is also well-known that the element $\Delta_n^2$ acts by the scalar $\bar{\alpha}_\lambda = q^{n(n-1)/2} \sum_{i<j} (\lambda_i+1)\lambda_j$ on $S^\lambda$ (see e.g. [W2, Lemma 3.2.1]). We deduce from this that

$$M_it = q^{i^2+c_i(t)}-r_i(t)t; \quad (*)$$

here $r_i(t)$ is the row of $t$ in which $i$ lies and $c_i(t)$ the column. Indeed, due to the restriction rule, it suffices to show this for $i = n$. In this case, $M_n$ acts on $t$ by the scalar $\bar{\alpha}_\lambda\bar{\alpha}_\lambda^{-1}$, where $\lambda'$ is $\lambda$ without the box containing $n$. The claim follows from an easy computation.

1.5. Path idempotents. We work in the generic Hecke algebra $H_n = H_n(x, K(x))$. Since $H_n$ acts faithfully on $V = \bigoplus \lambda V_\lambda$, it follows that there exist elements $p_t \in H_n$, indexed by standard tableaux of size $n$, which are completely determined by $p_t \bar{s} = \delta_{t,s}t$. We call these elements path idempotents, because of an identification of standard tableaux with certain paths, which we will describe later. Recursive formulas for these idempotents, in terms of the generators $T_i$ have been derived in [W1, Cor 2.3]. It will be more convenient, however, to express them in terms of Murphy elements.

We have seen in section 1.4 that $t$ is an eigenvector of $M_i$ with eigenvalue $\alpha_{t,i}(x) = x^{i^2+c_i(t)}-r_i(t)$. Let $P_t,i$ be the eigenprojection of $M_i$ corresponding to the eigenvalue $\alpha_{t,i}$. (Note that since the spectrum of $M_i$ is contained in $K(x)$, it is not necessary to pass to the algebra defined over some extension field in order to define the eigenprojections.) Then we have:

**Lemma 1.5.1.** $p_t = \prod_{i=2}^{n} P_{t,i}$

**Proof.** It is easy to prove by induction on $i$ that 2 tableaux $t$ and $s$ are equal if and only if $c_i(t) - r_i(t) = c_i(s) - r_i(s)$ for $i = 1, 2, \ldots, n$. As $x$ is not a root of unity, there will be at least one $M_i$ which acts via distinct eigenvalues on the the eigenvectors $t \neq s$. The result follows because the idempotents $P_{t,i}$ are mutually commuting.

□

Obviously, the Murphy elements $M_i$ are evaluable for all $q \in K$. If $q \in K^\times$ is not a proper root of unity, then all path idempotents are evaluable at $q$. This is no longer true if $q$ is a proper root of unity.

**Definition 1.5.2.** Fix an integer $l > 1$. We will say that the tableaux $t$ and $s$ are ($l$-) equivalent if $c_i(t) - r_i(t) \equiv c_i(s) - r_i(s) \mod l$ for $i = 1, 2, \ldots, n$. The orbit $[t]$ of a standard tableau $t$ consists of the collection of all standard tableaux $s$ which are equivalent to $t$. We define the orbit path idempotent $p_{[t]} \in H_n(x, K(x))$ by $p_{[t]} = \sum_s \equiv_t p_s$. 

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Lemma 1.5.3. Let $q$ be a primitive $l$-th root of unity in $K$. The orbit path idempotent $p_{[t]}$ is evaluable at $q$.

Proof. Let $P_{[t],i} = \sum_s P_{s,i}$, where the summation goes over all tableaux $s$ for which $q^{i-1+c_i(s)-r_i(s)} = q^{i-1+c_i(t)-r_i(t)}$. By Proposition 1.2.4 this idempotent is evaluable and coincides with the eigenprojection of $M_i(q)$ for the eigenvalue $q^{n-1+c_i(t)-r_i(t)}$.

Hence also $\prod_{i=2}^n P_{[t],i}$ is evaluable. It follows from the definitions that it is an idempotent which projects on the span of tableaux $s$ for which $c_i(t) - r_i(t) \equiv c_i(s) - r_i(s) \mod l$ for $i = 1, 2, \ldots, n$. Hence it coincides with $p_{[t]}$, which therefore is evaluable.

Lemma 1.5.4. For any tableau $t$ of size $n$, the path idempotent $p_t \in H_n(x, K(x))$ is evaluable at $x = 1$, and the evaluation $p_t(1)$ is the path idempotent in $K S_n$.

Proof. We show this by induction on $n$, using the recursive formulas for the $p_t$ from [W1], Corollary 2.3. (Since the derivation is brief, we rederive the formulas here.) The case $n = 1$ is trivial, so suppose $n > 1$ and fix a tableau $t$ of size $n$. For any standard tableau $s$ of size $n$, let $s'$ be the tableau obtained by removing the box containing the entry $n$. We assume inductively that $p_{t'}$ is evaluable at 1, and that $p_{t'}(1)$ is the path idempotent in $K S_n$.

For standard tableaux $s$ of size $n$, it follows from the definition of the path idempotents that

$$p_{t'} s = 0 \quad \text{if } s' \neq t'.$$

Moreover, it follows from the definition of the orthogonal representation that

$$p_{t'} e_n p_{t'} s = a_d(s)s \quad \text{if } s' = t',$$

where $d(s) = d(s; n - 1)$. Hence

$$p_t = p_{t'} (\prod_s \frac{e_n - a_d(s)}{a_d(t) - a_d(s)}) p_{t'},$$

where the product is over all $s$ such that $s \neq t$ and $s' = t'$.

We have $a_d(s)(1) = (d(s) + 1)/d(s)$. Since $d(s) \neq d(t)$ when $s \neq t$, it follows that the product is evaluable at 1, and therefore, $p_t$ is evaluable at 1. The path idempotents in the symmetric group algebra are given by the identical recursive formulas, with the $a_d$ replaced by the rational numbers $(d + 1)/d$; so it follows from the induction hypothesis that $p_t(1)$ is the path idempotent in the symmetric group algebra for the tableau $t$. \qed

1.6. Specht modules. The $H_n(x, K(x))$-modules $V_\lambda$ are not in general evaluable at $q$ when $q$ is a proper root of unity. So-called Specht modules $S^\lambda$ have been defined in [DJ] which are equivalent to $V_\lambda$ (as $H_n(x, K(x))$ modules) and which are evaluable also at roots of unity. We briefly recall the construction and properties of these modules; see [DJ] for more details.

The construction of the modules $S^\lambda$ can be carried out for $H_{n}(q, F)$ for any $q$ and $F$. It turns out that for any $q \in K^\times$, the “restriction” of the evaluable $H_n(x, K(x))$ module $S^\lambda$ to
$H_n(q)$ coincides with module $S^\lambda$ constructed over $H_n(q)$. It is safe, therefore, to use the same notation for all of these modules.

Let $Y_\lambda = S_{\lambda_1} \times \ldots \times S_{\lambda_r} \subset S_n$. We define the symmetrization and antisymmetrization operators

$$\text{Sym}_\lambda = \sum_{w \in Y_\lambda} T_w \quad \text{and} \quad A_\lambda = \sum_{w \in Y_\lambda} (-q)^{n(n-1)/2-\ell(w)} T_w.$$ 

Let $S^\lambda$ be the submodule of the $H_n(q,F)$-module $W^\lambda$ (see 1.2) generated by $A_\lambda W^\lambda$, where $\lambda'$ is the diagram obtained from $\lambda$ by interchanging rows with columns. Hence $S^\lambda \cong H A_{\lambda'} H \text{Sym}_\lambda$, where $H$ denote $H_n(q,F)$.

**Remark 1.6.1.** Let $w_\lambda$ be the permutation which maps the standard tableau of shape $\lambda$ filled column by column (i.e. the first column is filled first, then the second etc) to the standard tableau of shape $\lambda$ obtained by filling $\lambda$ row by row. (This $w_\lambda$ is the inverse of the $w_\lambda$ in [DJ].) By [DJ, Lemma 4.1], $S^\lambda \cong H_n A_{\lambda'} T_{w_\lambda} \text{Sym}_\lambda$; the latter is the Specht module in the definition of [DJ, Section 4.1].

**Theorem 1.6.2.** (Dipper-James) Write $H_n = H_n(x,K(x))$.

(a) There exists an explicit basis of the $H_n$-module $H_n A_x H_n \text{Sym}_\lambda \subseteq H_n$ which is evaluable for all $q \in K^\times$, and such that the basis elements evaluated at $q$ remain linearly independent over $K$ for all $q \in K^\times$.

(b) The action of the generators $T_i$ on $S^\lambda \cong H_n A_x H_n \text{Sym}_\lambda$ is given by evaluable matrices with respect to this basis.

(c) The restriction $S^\lambda_\Delta$ of the Specht module constructed over $H_n$ is identical with the Specht module constructed over $H_n(q)$.

(d) The $H_n$ module $S^\lambda$ is equivalent over $K(x)$ to the $H_n$-module $V_\lambda$ (defined in 1.3). For $q$ not a proper root of unity, the $H_n(q)$ module $S^\lambda$ is equivalent over $K$ to the $H_n(q)$-module $V_\lambda$.

**Proof.** This theorem is [DJ, Theorem 5.6]. The statements (b) and (d) follow easily by carrying out construction of [DJ] over the domain $K[x]$ of polynomials in $x$. 

In the following, we use two orderings on the set of Young diagrams of a given size. The dominance order is defined by $\mu \succeq \lambda$ if $\sum_{i=1}^r \mu_i \geq \sum_{i=1}^r \lambda_i$ for all $r$. This is a partial order. The lexicographic order is defined by $\mu > \lambda$ if the first non-zero difference $\mu_i - \lambda_i$ is positive. This is a total order. One has $\mu \succeq \lambda \implies \mu \geq \lambda$.

**Theorem 1.6.3.** (Dipper-James) Let $q \in K$ be a primitive $l$-th root of unity. Consider the Hecke algebra $H_n(q)$ over $K$.

(a) Let $(\ , \ )$ be the bilinear form on the $H_n(q)$-module $W^\lambda$ with orthonormal basis $(L_w)$. Then the module $D^\lambda = S^\lambda / (S^\lambda \cap (S^\lambda)^\perp)$ is either 0 or it is simple.
Let $\lambda \geq \mu$ (in dominance order). If $\lambda$ is $l$-regular, there is exactly one factor in such a series which is isomorphic to $D^\lambda$ ([DJ, Theorem 7.6]). In particular, if $\lambda$ is $l$-regular, $D^\lambda$ appears with multiplicity 1 in any composition series of $S^\lambda$.

1.7. Decomposition numbers. Recall the notions of the rank vector of an idempotent from subsection 0.5. We say that $\lambda$ is the highest component of the idempotent $p$ if $\lambda$ is the highest diagram (in lexicographic order) for which $r(p)_{\lambda} \neq 0$. Let $q$ be a primitive $l$-th root of unity in $K$. If $p$ is an idempotent of $H_n(q)$, its $q$-rank $r_q(p)$ in the maximum semisimple quotient $H_n(q)$ of $H_n(q)$ is given by a vector $(r_q(p))_{\mu}$, where $\mu$ runs through all $l$-regular diagrams.

As in section 0.6, we define numbers $d_{\lambda \mu}$ by $d_{\lambda \mu} = r(p_{\mu})_{\lambda}$, where $p_\mu$ is an evaluable idempotent in $H_n$ whose image in $H_n(q)$ is a minimal idempotent in the component $H_n(q)_\mu$. We can now reformulate the results of Section 1.6 in the following way:

**Proposition 1.7.1.** Let $q$ be a primitive $l$-th root of unity in $K$. Let $p$ be an idempotent in $H_n$ which is evaluable at $q$, and let $\mu$ be the highest component of $p$. Then we have:

(a) The highest nonzero component of $r_q(p)$ is also $\mu$. In particular, $\mu$ has to be $l$-regular.
(b) $\text{Tr}_{S^\nu}(p(q)) = \text{Tr}_{D^\mu}(p(q))$.
(c) $d_{\lambda \mu} \leq r(p)_{\lambda}/r(p)_{\mu}$ for all $\lambda$.
(d) $d_{\lambda \mu}$ is the number of factors isomorphic to $D^\mu$ in any composition series of $S^\lambda$ as in Section 1.6.

**Proof.** By assumption, $p$ acts as 0 on $S^\nu$ for any $\nu > \mu$, and hence also its evaluation $p(q)$; in particular $p(q)$ acts as 0 on $D^\nu$. As $p$ and $p(q)$ act nonzero on $S^\mu$, $D^\mu$ has to be nonzero, and $p(q)$ has to act on it as a nonzero endomorphism, by Theorem 1.6.3(c). This shows (a).

Statement (b) follows from this and 1.6.3(c).

To prove (c), let $\overline{p(q)}$ denote the image of $p(q)$ in $H_n(q)$, and decompose $\overline{p(q)}$ as $\overline{p(q)} = \overline{p(q)}_\mu + \sum_{\nu < \mu} \overline{p(q)}_\nu$. Let $p_\mu$ be a lifting of $\overline{p(q)}_\mu$ to an evaluable idempotent in $H_n$ satisfying $p_\mu = pp_\mu p$. Then we have

$$\text{Tr}_{D^\mu}(p_\mu(q)) = \text{Tr}_{D^\mu}(p(q)) = \text{Tr}_{S^\mu}(p(q)) = r(p)_{\mu},$$

using (b). Also

$$\text{Tr}_{S^\lambda}(p_\mu) \leq \text{Tr}_{S^\lambda}(p) = r(p)_{\lambda}.$$
Therefore
\[ d_{\lambda \mu} = \frac{\text{Tr}_{S^\lambda}(p_{\mu})}{\text{Tr}_{D^\nu}(p_{\mu}(q))} \leq r(p)_{\lambda}/r(p)_{\mu}. \]

Statement (d) follows from Proposition 0.6.1 and Theorem 1.6.3. \qed

Remark 1.7.2. By the last proposition, in order to get upper bounds for the coefficients \( d_{\lambda \mu} \), it suffices to construct evaluable idempotents \( p \) in \( H_n \) for which \( r(p)_{\nu} = 0 \) for all \( \nu > \mu \). This will be our strategy in sections 3 and 4.

2. The \( k \)-row quotient.

2.1. Definition of the \( k \)-row quotient.

Definition 2.1.1. Fix natural numbers \( k \leq n \). Let \( W(n, k) = W(n, k, q, F) \) be the direct sum of all \( H_n(q, F) \)-modules \( W^\lambda \), for Young diagrams \( \lambda \) of size \( n \) with at most \( k \)-rows. The \( k \)-row quotient of the Hecke algebra \( H_n(q, F) \) is the quotient by the kernel of the module \( W(n, k) \).

We remark that the \( k \)-row quotient can also be described in terms of a representation of the Hecke algebra on tensor space which has a natural connection with quantum groups of type \( A \).

Let \( V \) be a \( k \)-dimensional vector space over \( F \), with basis \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \). Then \( V^\otimes n \) has a natural basis of the form \( \varepsilon_{i_1} \otimes \ldots \otimes \varepsilon_{i_n} \), where \( 1 \leq i_j \leq k \) for \( j = 1, 2, \ldots, n \). Define a matrix \( R \) in \( \text{End}(V \otimes V) \) by

\[
R(\varepsilon_i \otimes \varepsilon_j) = \begin{cases} 
\varepsilon_j \otimes \varepsilon_i & \text{if } i < j \\
q \varepsilon_i \otimes \varepsilon_i & \text{if } i = j \\
(q-1) \varepsilon_i \otimes \varepsilon_j + q \varepsilon_j \otimes \varepsilon_i & \text{if } i > j 
\end{cases}
\]

Using these matrices \( R \), define operators \( R_i \) in \( \text{End}(V^\otimes n) \) by

\[
R_i = 1 \otimes 1 \otimes \ldots 1 \otimes R \otimes 1 \otimes \ldots \otimes 1,
\]

which acts by the matrix \( R \) on the \( i \)-th and \((i + 1)\)st factor of \( V^\otimes n \). This is the famous representation of the Hecke algebra \( H_n(q, F) \) discovered by Jimbo [Ji]. The fact that it is indeed a representation is a consequence of the following observations.

For any composition \( \lambda \) of \( n \) with at most \( k \) non-zero parts define the vector

\[ \varepsilon^\otimes \lambda = \varepsilon_1^\otimes \lambda_1 \otimes \ldots \otimes \varepsilon_k^\otimes \lambda_k, \]

where \( \varepsilon_i^\otimes m = \varepsilon_i \otimes \ldots \otimes \varepsilon_i \), \((m \text{ times})\), and

\[ V^\otimes \lambda = \text{span} \{ w(\varepsilon^\otimes \lambda) \}, \quad w \in S_n \}, \]
where the symmetric group $S_n$ acts by permuting the factors in the tensor product. It is clear that $V^\otimes n$ is the direct sum, over all compositions $\lambda$ of $n$ with at most $k$ non-zero parts, of $V^\otimes \lambda$, and that $V^\otimes \lambda$ is invariant under the operators $R_i$. We define

$$\ell(\varepsilon_{i_1} \otimes \ldots \otimes \varepsilon_{i_n}) = \#\{(i_j, i_k) : j < k \text{ and } i_j > i_k\},$$

i.e. the number of inversions of indices. As the stabilizer of $\varepsilon^\otimes \lambda$ is equal to $Y_\lambda$ (see section 1.2), the map $\iota : L_w \mapsto w(\varepsilon^\otimes \lambda)$ induces an isomorphism between the vector spaces $W^\lambda$ and $V^\otimes \lambda$, preserving the length functions defined on the basis vectors of these two vector spaces. Using this observation, it is easy to check that $\iota$ intertwines the action of $T_i$ on $W^\lambda$, and of $R_i$ on $V^\otimes \lambda$. Thus $V^\otimes \lambda$ is an $H_n(q, F)$ module, isomorphic to $W^\lambda$.

Finally, $T_i \mapsto R_i$ defines a representation of $H_n(q, F)$ on $V^\otimes n$, and

$$V^\otimes n = \bigoplus V^\otimes \lambda \cong \bigoplus W^\lambda,$$

where the sum is over all compositions $\lambda$ of $n$ with at most $k$ non-zero parts.

Given a composition $\lambda$ of $n$ let $\mu$ be the partition whose parts are the same as those of $\lambda$, but in decreasing order. Then $W^\lambda \cong W^\mu$. Thus

$$V^\otimes n \cong \bigoplus m_\lambda W^\lambda,$$

where now the sum is over partitions of $n$ with at most $k$ parts, and the $m_\lambda$ are certain multiplicities. Thus the quotient of $H_n(q, F)$ which acts faithfully on $V^\otimes n$ is the same as the $k$-row quotient defined above.

We denote by $\Phi_k$ the matrix representation of $H_n(x, K(x))$ on $(K(x)^k)^\otimes n$ with respect to the natural basis described above. This representation is evaluable at any non-zero $q \in K$, and the “restriction” to $H_n(q)$ is again the representation on $(K^k)^\otimes n$ with respect to the natural basis.

Denote by $V(n, k)$ the direct sum of the $H_n(x, K(x))$ modules $V_\lambda$ for Young diagrams $\lambda$ of size $n$ with at most $k$ rows, and by $\Pi_k$ the matrix representation of $H_n(x, K(x))$ on $V(n, k)$ with respect to the basis of standard tableaux. Since the simple direct summands of $(K(x)^k)^\otimes n$ are those labelled by Young diagrams with no more than $k$ rows, $\Pi_k(H_n(x, K(x)))$ is isomorphic to the $k$-row quotient of $H_n(x, K(x))$.

### 2.2. Affine Weyl group, $l$-cores, and blocks.

Fix an integer $k \geq 2$, and $W$ be the affine reflection group of type $A_{k-1}^{(1)}$. It is isomorphic to the semidirect product $\mathbb{Z}^{k-1} \rtimes S_k$. Fix $l \in \mathbb{N}$. Then we have a faithful action of $W$ on $\mathbb{R}^k$, with $S_k$ acting via permutation of the coordinates, and with

$$\tau_i(y_1, \ldots, y_k) = (y_1, \ldots, y_{i-l}, y_{i+1} + l, \ldots y_k),$$

where $\tau_i$ is the $i$-th generator of the translation group $\mathbb{Z}^{k-1} \subset W$. Observe that any affine reflection is a reflection in a hyperplane given by an equation of the form $y_i - y_j = ml$ for some $i, j$ with $1 \leq i, j \leq k$ and with $m \in \mathbb{Z}$. 

Define \( \rho = (k-1,k-2, \ldots, 1,0) \). Another important action (the “dot action”) of \( W \) on \( \mathbb{R}^k \) is given by

\[
w \cdot v = w(v + \rho) - \rho.
\]

We next review the relation between the orbits of the dot action of \( W \) and the notion of the \( l \)-core of a diagram.

We recall the notions of a rim hook and \( l \)-core, for example from [JK]. For \((a,b)\) a node of a Young diagram \( \lambda \), the corresponding \emph{hook} is the portion of row \( a \) to the right of \((a,b)\) together with the portion of column \( b \) below \((a,b)\), including the cell \((a,b)\); the length of the hook is \( h_{(a,b)} = \lambda_a - a + \lambda'_b - b + 1 \). The corresponding \emph{rim hook} is the portion of the rim of \( \lambda \) between \((\lambda'_b,b)\) and \((a,\lambda_a)\); the rim hook is a connected skew shape of size \( h_{(a,b)} \). A diagram is called an \( l \)-core if it has no rim hook of length \( l \). Every Young diagram contains a unique \( l \)-core, which is obtained by removing successive rim hooks of length \( l \).

Let \( D \) be the set of points \( y \in \mathbb{R}^k \) such that \( y_1 > y_2 > y_3 > \ldots > y_k \). Let \( D^+ \) be the set of points \( y \in D \) with all components nonnegative. We obtain a map from the set of Young diagrams with \( k \) rows at the most onto \( D^+ \cap \mathbb{Z}^k \), given by \( \lambda \mapsto \lambda + \rho \), where on the right hand side we identify \( \lambda \) with the vector \((\lambda_i)\). Let \( \lambda \) be a diagram with at most \( k \) rows. The point \( y = \lambda + \rho \in D^+ \) is determined by its set of components \( \{y_i\} \). It is not difficult to see that the operation of removing a rim hook of length \( l \) from \( \lambda \) corresponds exactly to the operation of reducing one element of \( \{y_i\} \) by \( l \), see [JK], Lemma 2.7.13.

The following result must be well known.

**Lemma 2.2.1.** Let \( \lambda \) and \( \mu \) be diagrams of the same size with at most \( k \) rows. Let \( y = \lambda + \rho \), \( z = \mu + \rho \). The following are equivalent:

1. \( \lambda \) and \( \mu \) have the same \( l \)-core.
2. For \( 0 \leq r \leq l - 1 \), \( |\{i : y_i \equiv r \mod l\}| = |\{i : z_i \equiv r \mod l\}| \).
3. \( \mu \) is in the orbit of \( \lambda \) under the dot action of \( W \).

**Proof.** The equivalence of statements (1) and (2) follows from the observation made just above. Furthermore, (3) \( \Rightarrow \) (2) is evident since an affine reflection changes each of two coordinates of \( y \) by a multiple of \( l \), so preserves the cardinalities of the sets \( J_r = \{i : y_i \equiv r \mod l\} \). So it remains to show (2) \( \Rightarrow \) (3). One can define a distance between \( \lambda \) and \( \mu \) as follows: Let \( y_{i,r} \) be the elements of \( J_r \) in increasing order, and similarly let \( z_{i,r} \) be the elements of \( J'_r = \{i : z_i \equiv r \mod l\} \) in increasing order. Define \( |\lambda - \mu| = \sum_r \sum_i |y_{i,r} - z_{i,r}| \). It suffices to show that if \( \lambda \neq \mu \), then there is an element \( w \in W \) such that \( w \cdot \lambda \neq \lambda \). One has \( \sum_r \sum_i y_{i,r} = \sum_r \sum_i z_{i,r} \). Suppose that there is some \( r \) such that \( \sum_i y_{i,r} \neq \sum_i z_{i,r} \). Then one can check that there is an element \( w \in W \) and \( s \neq t \) such that \( w \cdot \lambda \) is a Young diagram, \( w \) increases one element of \( J_s \) and decreases one element of \( J_t \), and \( |w \cdot \lambda - \mu| < |\lambda - \mu| \).

Suppose one the other hand that \( \sum_i y_{i,r} = \sum_i z_{i,r} \) for all \( r \). Choose \( r \) such that \( J_r \neq J'_r \). Then one can show that there is an element \( w \in W \) such that \( w \cdot \lambda \) is a Young diagram, \( w \) increases one element of \( J_r \) and decreases another, and \( |w \cdot \lambda - \mu| < |\lambda - \mu| \). \( \square \)
The blocks of the Hecke algebras of type $A$ at a root of unity are parametrized by $l$-cores:

**Theorem 2.2.2.** ([DJ2], Theorem 4.1.3) Let $q$ be a primitive $l$-th root of unity in $K$.

(a) If $\lambda$ and $\mu$ are two $l$-regular diagrams of size $n$, then $D^\lambda$ and $D^\mu$ belong to the same block of $H_n(q)$ if and only if $\lambda$ and $\mu$ have the same $l$-core.

(b) If $D^\mu$ is a composition factor of $S^\lambda$, then $\mu \succeq \lambda$ and $\mu$ and $\lambda$ have the same $l$-core.

### 2.3. Paths and path equivalence.

If $t$ is a standard tableau, we define $t(i)$ to be the Young diagram consisting of the boxes containing the numbers $1$, $2$, ..., $i$. If $t$ has $n$ boxes and at most $k$ rows, we identify $t$ with the piecewise affine path $t : [0, n] \to \mathbb{R}^k$ which takes the values $t(i)$ at $i = 0, 1, \ldots, n$ and which is affine on $[i, i+1]$. We denote by $t_\rho$ the path $t_\rho(s) = t(s) + \rho$. Observe that $t_\rho([0, n]) \subset D^+$. Likewise, if $t$ is a standard skew tableau of shape $\lambda \setminus \nu$, we identify $t$ with a piecewise affine path from $\nu$ to $\lambda$, and we also consider the path $t_\rho(s) = t(s) + \rho$ which goes from $\nu + \rho$ to $\lambda + \rho$.

Let $t$ be a standard (skew) tableau with at most $k$ rows. Let $t_\rho(i)_a$ denote the $a$-th coordinate of $t_\rho(i)$. Recall that $c_j(t)$ and $r_j(t)$ denote the column and row of the box of $t$ which contains the number $j$. Using these notations, $c_i(t) - r_i(t) = t_\rho(i)_r - k$. Hence, the tableaux $t$ and $\tilde{t}$ are equivalent in the sense of Definition 1.5.2 if and only if

$$t_\rho(i)_r(i) \equiv \tilde{t}_\rho(i)_r(i) \mod l \text{ for all } i.$$  

This definition of $l$-equivalence obviously extends to arbitrary piecewise linear paths $p : [0, n] \to \mathbb{R}^k$ for which $p(i) \in \mathbb{Z}^k$. The notion of $l$-equivalence can be characterized by defining something like an ‘action’ of the affine reflection group on paths, as follows: Assume that $t_\rho(i)$ lies in an affine hyperplane corresponding to an affine reflection $s$ (as defined in Section 2.4). Then we define

$$s^{(i)}(t_\rho(u)) = \begin{cases} t_\rho(u) & \text{if } 0 \leq u \leq i, \\ s(t_\rho(u)) & \text{if } i < u \leq n. \end{cases}$$

Geometrically, it means we reflect the part of the path $t_\rho$ after the point $t_\rho(i)$ in the hyperplane belonging to $s$. The index $i$ is necessary, as $t$ might cross the hyperplane of $s$ more than once. Observe that $s^{(i)}(t_\rho)$ may be a path which no longer belongs to a tableau, but it remains equivalent to $t_\rho$.

**Lemma 2.3.1.** Two paths $t$ and $\tilde{t}$ are $l$-equivalent if and only if there exists a finite sequence of affine reflections $s_1$, ..., $s_r$ and integers $i_j \in [1, n]$, $j = 1$, ..., $r$ such that

$$\tilde{t_\rho} = s_1^{(i_1)}(s_2^{(i_2)}(... s_r^{(i_r)}(t_\rho) ...)).$$
Proof. Assume that \( t \) and \( \tilde{t} \) are \( l \)-equivalent and assume that \( t_\rho \) and \( \tilde{t}_\rho \) coincide from 0 to \( i \), but not at \( i+1 \). Let \( a \) and \( b \) be the rows in which the box containing \( i+1 \) is added in \( t \) resp. \( \tilde{t} \). Then \( t_\rho(i+1)_a \equiv \tilde{t}_\rho(i+1)_b \mod l \), which implies \( t_\rho(i)_a \equiv \tilde{t}_\rho(i)_b \mod l \). Hence \( t_\rho(i) \) lies on a hyperplane of the form \( y_a - y_b = ml \), for some \( m \), and the paths \( \tilde{t}_\rho \) and \( s^{(i)}(t_\rho) \) coincide up to \( i + 1 \) if we define \( s \) by

\[
s(y_1, \ldots, y_a, \ldots, y_b, \ldots, y_n) = (y_1, \ldots, y_b + ml, \ldots, y_a - ml, \ldots, y_n).
\]

Repeat this construction for the paths \( \tilde{t}_\rho \) and \( s^{(i)}(t_\rho) \) until the resulting paths coincide. \( \square \)

2.4. Some simple Specht modules. The following two lemmas present very simple sufficient criteria for a Specht module to be simple. A more general criterion is given below in Corollary 3.3.4. All of these results are special cases of results of James and Dipper ([DJ], 4.11) and James and Mathas [JM].

Lemma 2.4.1. Assume that \( \lambda \) is the highest diagram of its orbit under the dot action of the affine Weyl group \( W \). Then \( D^\lambda = S^\lambda \).

Proof. If \( \mu \unrhd \lambda \) and \( \mu \) has the same \( l \)-core as \( \lambda \), then it follows that \( \mu \) has at most \( k \) rows and furthermore \( \mu \) is in the orbit of \( \lambda \) under \( W \). By assumption, \( \mu = \lambda \). It follows that \( D^\lambda \) is the only composition factor of \( S^\lambda \). \( \square \)

Lemma 2.4.2. Let \( \lambda = (l-1)\rho \). Then \( S^\lambda = D^\lambda \) and for all diagrams \( \nu \) such that \( \nu \neq \lambda \) and \( |\nu| = |\lambda| \), \( d_{\nu\lambda} = 0 \).

Proof. This follows since \( \lambda \) is its own \( l \)-core. \( \square \)

3. Big diamond elements and their matrix coefficients

In this section we show that the \( k \)-row quotient of \( H_n(x) \) contains a (non unital) subalgebra isomorphic to the \( k \)-row quotient of \( H_m(x^l) \), when \( n = ml + (l-1)(k) \). Furthermore, when \( q \) is an \( l \)-th root of unity, then the \( k \)-row quotient of \( H_n(q) \) contains a (non unital) subalgebra isomorphic to the \( k \)-row quotient of \( KS_m \). This is our main technical result.

As a corollary of this result, we obtain the simplicity of certain Specht modules over \( H_n(q) \), in characteristic 0 (a special case of results of Dipper and James, and of James and Mathas.)
3.1. $l$-Straight tableaux and $k$-critical diagrams.

**Definition 3.1.1.** We say that a skew tableau of shape $\lambda \setminus \nu$ is $l$-straight if
\[d(t; i, i + 1) \equiv -1 \mod l\]
for $i = 1, 2, |\lambda \setminus \nu|$. 

This condition can be expressed as
\[(c_{i+1}(t) - r_{i+1}(t)) - (c_i(t) - r_i(t)) \equiv 1 \mod l\]
for all $i$. Geometrically, the condition says that the path $t$ is equivalent to a path in which only one coordinate is increased, that is to a straight path.

**Definition 3.1.2.** Fix an integer $k \geq 2$. A point $y \in D^+ \cap \mathbb{Z}^k$ is said to be $k$-critical, or just critical, if $y_i - y_j$ is divisible by $l$ for all $1 \leq i, j \leq k$; it is called a reduced $k$-critical point if it is $k$-critical and $y_k = 0$. A diagram $\lambda$ with at most $k$ rows is said to be $k$-critical if $\lambda + \rho$ is a $k$-critical point, that is if $\lambda_a - \lambda_b + b - a$ is divisible by $l$ for all $a$ and $b$. It is said to be a reduced $k$-critical diagram if in addition $\lambda_k = 0$.

The smallest $k$-critical point is $l\rho$, and the smallest $k$-critical Young diagram is $(l - 1)\rho$. The size of the smallest $k$-critical Young diagram is $n_0 = (l - 1)(k)$.

**Definition 3.1.3.** We call an $l$-straight skew tableau $t$ $k$-special if the initial shape of $t$ is $k$-critical and if the length of $t$ is a multiple of $l$.

If $t$ is a $k$-special $l$-straight skew tableau with initial shape $\mu$ and length $n$, then for all $i$, at most one coordinate of $t_\rho(i) - \mu_k = t(i) + \rho - \mu_k$ is not divisible by $l$, and that coordinate is congruent to $i \mod l$, as one can see easily by induction. Consequentially $t_\rho(ml)$ is a $k$-critical point for all $m$ with $ml \leq n$.

Let $T_l^l$ denote the set of $k$-special $l$-straight skew tableaux with initial shape $(l - 1)\rho$, and, for a $k$-critical diagram $\lambda$, let $T_\lambda^l$ denote the set of $t \in T_l^l$ which end in $\lambda$.

If $t \in T_l^l$ with length $n$, then for all $m$ such that $ml \leq n$, the diagram $t_\rho(ml)/l$ has strictly decreasing components, so $\tilde{t}(m) = t_\rho(ml)/l - \rho$ is a Young diagram with $m$ boxes. It is easy to check that the sequence of Young diagrams $\tilde{t}(m)$ defines a Young tableau of shape $t_\rho(n)/l - \rho$. This shows the following lemma.

**Lemma 3.1.4.** Let $\lambda$ be a $k$-critical diagram. There is a bijection $\Psi$ between $T_\lambda^l$ and the set of all standard Young tableaux of shape $(\lambda + \rho)/l - \rho$ defined by $\Psi(t)(m) = t_\rho(ml)/l - \rho$. $lacksquare$

3.2. Big diamond elements. Recall that $e_i = x \mathbb{1} - T_i$ in the generic Hecke algebra over $K(x)$. A big diamond element is one of the form
\[E(n) = (e_{n+l}e_{n+l+1} \ldots e_{n+2l-1})(e_{n+l-1}e_l \ldots e_{n+2l-2}) \ldots (e_{n+1}e_{n+2} \ldots e_{n+l}).\]
Here $l$ is fixed, and $n$ is arbitrary. For simplicity of notation we will often normalize $n$ to be zero when discussing big diamond elements, and write $E$ for the big diamond element. For any
finite set $A$ of natural numbers containing no two consecutive numbers, let $e_A$ be the product of the commuting elements $e_j$ for $j \in A$ and $s_A$ the product of the simple transpositions $s_j$ for $j \in A$. Let $J_i = \{l - i + 1, l - i + 3, \ldots, l + i - 1\}$ for $1 \leq i \leq l$. Thus $J_1 = \{l\}$, $J_2 = \{l - 1, l + 1\}$, etc. We define $e_J = e_{J_1} e_{J_2} \ldots e_{J_2} e_{J_1}$ and $s_J = s_{J_1} \ldots s_{J_2} s_{J_1}$. We also write $e_{J,i} = e_{J_1} e_{J_2} \ldots e_{J_2} e_{J_1}$, and $s_{J,i} = s_{J_1} s_{J_2} \ldots s_{J_2} s_{J_1}$. Then one can check that
$$E = e_{J_1} e_{J_2} \ldots e_{J_2} e_{J_1} e_{J_1} e_{J_1} \ldots e_{J_2} e_{J_1}.$$
Let $s_E = s_{J_1}^{-1} s_{J_2} s_{J_1}$; thus
$$s_E(i) = \begin{cases} i + l & \text{if } i \leq l, \\ i - l & \text{if } l < i \leq 2l. \end{cases}$$

For a pair of $k$-special $l$-straight skew tableaux $t_1$ and $t_2$ with the same initial and final shapes, one has
$$\langle Et_1, t_2 \rangle = \langle e_J e_J t_1, e_J t_2 \rangle.$$  

The bilinear form used here is the one in which standard skew tableaux are orthonormal. Our goal is to obtain a formula for such matrix coefficients.

In the following, let $t$ be a $k$-special $l$-straight skew tableau of length $2l$. Let $\mu$ denote the (k-critical) initial shape of $t$; note that the first $l$ cells of $t$ are added to $\mu$ in some row $a$ and the next $l$ cells in some row $b$ (possibly $a = b$). Write $dl = \mu_a - \mu_b + b - a$.

For any integer $d$, we write $a_{\pm d} = a_d a_{-d}$.

**Lemma 3.2.1.**  For $1 \leq i \leq l - 1$,
$$e_{J_{i+1}} e_J s_{J_{i-1}} t = a_{\pm (dl + l - i)}^{i/2} e_{J_{i+1}} s_{J_{i-1}} t.$$  

**Proof.** By definition of the orthogonal representation, one has
$$e_{J_{i+1}} e_J s_{J_{i-1}} t = \sum_{H \subseteq J_i} \alpha_H e_{J_{i+1}} s_H s_{J_{i-1}} t,$$
where $\alpha_H^2$ is a rational function. We have to show that if $H \neq J_i$, then $e_{J_{i+1}} s_H s_{J_{i-1}} t = 0$.

Observe that the skew tableau $s_{J_{i-1}} t$ has the digits $1$ through $l - i$ in row $a$, followed by the elements of $J_i$ in increasing order, while in row $b$, the skew tableau has $\{1 + r : r \in J_i\}$ in increasing order followed by $l + i + 1, \ldots, 2l - 1, 2l$.

If $H \neq J_i$, let $r$ be the least element of $J_i$ such that $r \not\in H$. Then $s_H s_{J_{i-1}} t$ has the digits $r - 1$ and $r$ in successive cells in row $a$. Since $r - 1 \in J_{i+1}$, it follows that $e_{J_{i+1}} s_H s_{J_{i-1}} t = 0$.

From the description of $s_{J_{i-1}} t$, one sees that for $r \in J_i$,
$$d(s_{J_{i-1}} t, r, r + 1) = \pm (\mu_a - \mu_b + b - a + l - i) = \pm (dl + l - i).$$
Thus
$$\langle s_{J_{i-1}} t, e_J s_{J_{i-1}} t \rangle = a_{\pm (dl + l - i)}^{i/2}. \qed$$
Lemma 3.2.2. For $r, s \geq 1$,
\[
(a_{\pm r})^{s+1}(a_{\pm (r+1)})^{s} \cdots a_{\pm (r+s)} = x^{(s+2)} a_{r-1}^{-(s+1)} \frac{1 - x^{r+s+1}}{1 - x^r}.
\]

Proof. One has
\[
a_r a_{r+1} \cdots a_{r+s} = \frac{1 - x^{r+s+1}}{1 - x^r},
\]
and therefore
\[
(a_r)(a_r a_{r+1}) \cdots (a_r a_{r+1} \cdots a_{r+s}) = (1 - x^r)^{-(s+1)}(1 - x^{r+1})(1 - x^{r+2}) \cdots (1 - x^{r+s+1}).
\]
Similarly,
\[
a_{-r} a_{-r-1} \cdots a_{-r-s} = x^{s+1} \frac{1 - x^{-r-1}}{1 - x^{-r+s}},
\]
so
\[
(a_{-r})(a_{-r} a_{-r-1}) \cdots (a_{-r} a_{-r-1} \cdots a_{-r-s}) = x^{(s+2)}(1 - x^{-r-1})^{s+1}(1 - x^{-r-1})(1 - x^{-r+1})^{-1} \cdots (1 - x^{-r+s})^{-1}.
\]
The result follows from multiplying the expressions in Equations 3.2.2 and 3.2.3.

Lemma 3.2.3.
\[
e_{J_l} e_{J_l-1} t = \left( x^{\left\lfloor \frac{d}{2} \right\rfloor} a_{dl}^{l+1} \frac{1 - x^{(d+1)l}}{1 - x^{dl+1}} \right)^{1/2} e_{J_l} s_{J_l-1} t.
\]

Proof. It follows by induction from Lemma 3.2.1 that
\[
e_{J_l} e_{J_l-1} t = \prod_{i=1}^{l-1} (a_{\pm (dl+1-i)})^{i/2} e_{J_l} s_{J_l-1} t,
\]
so the result follows from an application of Lemma 3.2.2.

Write $P_t$ for the coefficient in Lemma 3.2.3. Now suppose that $t$ and $t'$ are two $k$-special $l$-straight skew tableaux with the same initial shape satisfying conditions (1) and (2) above. Then
\[
\langle t', Et \rangle = \langle e_{J_l-1} t', e_{J_l} e_{J_l-1} t \rangle = P_t P_{t'} \langle s_{J_l-1} t', e_{J_l} s_{J_l-1} t \rangle.
\]
The skew tableau $s_{J_l-1} t$ has the elements of $J_l$ in increasing order in row $a$ and those of the complementary set $\{ r + 1 : r \in J_l \}$ in increasing order in row $b$. Similarly $s_{J_l-1} t'$ has the elements of $J_l'$ in some row $a'$ and those of the complementary set in some row $b'$. Evidently $\langle s_{J_l-1} t', e_{J_l} s_{J_l-1} t \rangle = 0$ unless $\{ a, b \} = \{ a', b' \}$, and either $t = t'$, or $t' = s_E t$. One has
Proposition 3.2.4.

\[ \langle Et, t \rangle = (P^t)^2 a_{dl} = x^{(l)} \frac{1 - x^{(d+1)}l}{1 - x^{dl}} = x^{(l)} a_d(x^l), \]

and

\[ \langle Et, sEt \rangle = P^t p^s Et a_{d/l}^{1/2} = x^{(l)} (a_d(x^t)a_{-d}(x^l))^{1/2}. \]

Note that the matrix coefficients of \( x^{-l}(l) E \) with respect to \( k \)-special \( l \)-straight tableaux are just matrix coefficients of \( e_1(x^l) \) in the orthogonal representation. (Compare section 1.3).

3.3. An embedding of the \( k \)-row quotient of \( H_m(x^l) \). Given an natural number \( m \), we consider a sequence of big diamond elements \( E_1, E_2, \ldots, E_{m-1} \) in a certain \( H_m = H_n(x, K(x)) \), as follows. Let \( n = n_0 + ml \), where \( n_0 = (l - 1)(\frac{k}{2}) \) denotes the size of the smallest \( k \)-critical Young diagram \((l - 1)\rho\).

Let \( E_i \) denote the big diamond element in \( H_n \),

\[ E_i = E(n_0 + (i - 1)l), \]

for \( 1 \leq i \leq m - 1 \). The corresponding permutation \( s_{E_i} \) affects, at most, the integers between \( n_0 + (i - 1)l \) and \( n_0 + (i + 1)l \).

Consider \( H_n \) acting in the orthogonal representation on tableaux of length \( n \); for any tableau \( t \), \( E_i t \) is a linear combination of tableaux \( s \) with \( s(j) = t(j) \) for \( j \leq n_0 + (i - 1)l \) or \( j \geq n_0 + (i + 1)l \); in particular, \( s(j) = t(j) \) for \( j \leq n_0 \).

Fix a tableau \( t \) of length \( n \) satisfying the following conditions:

(1) \( t \) passes through \((l - 1)\rho\).

(2) \( t \) is a \( k \)-special \( l \)-straight skew tableau between \((l - 1)\rho \) and its final diagram \( \lambda(t) \).

One can check that for any tableau \( \bar{t} \) which is \( l \)-equivalent to \( t \) and whose final shape \( \lambda(\bar{t}) \) has at most \( k \) rows, \( \bar{t} \) also satisfies conditions (1) and (2). Conversely, any tableau satisfying conditions (1) and (2) is \( l \)-equivalent to \( t \).

We recall several notations: The sum of all path idempotents \( p_s \), where \( s \) is equivalent to \( t \) is denoted by \( p_{[t]} \), see Definition 1.5.2 and Lemma 1.5.3. The direct sum of the \( H_n(x, K(x)) \) modules \( V_\lambda \), where \( \lambda \) ranges over Young diagrams of size \( n \) with at most \( k \) rows is denoted by \( V(n, k) \); the representation of \( H_n(x, K(x)) \) on \( V(n, k) \) is denoted by \( \Pi_k \). The representation of \( H_n(x, K(x)) \) on \((K(x)^k)^{\otimes n}\) with respect to the natural basis is denoted by \( \Phi_k \).

Note that the range of \( \Pi_k(p_{[t]}) \) is spanned by the set of standard tableaux satisfying conditions (1)-(2).

Theorem 3.3.1.

(a) The images of \( p_{[t]}, p_{[t]}E_1p_{[t]}, \ldots, p_{[t]}E_{m-1}p_{[t]} \) in the \( k \)-row quotient of \( H_n(x, K(x)) \) generate a subalgebra isomorphic to the \( k \)-row quotient of \( H_m(x^l, K(x)) \).
(b) The range of \( \Pi_k(p_{[t]}(q)) \) in \( V(n, k) \) is isomorphic to \( R \otimes V(m, k) \), where \( R \) is the \( K(x) \)-vector space with basis the set of tableaux which are \( l \)-equivalent to \( t_{(l-1)\rho} \). Furthermore, 
\[ x^{-\frac{1}{l}}p_{[t]}E_ip_{[t]} \] acts as \( \mathbb{1} \otimes e_i(x^l) \) on \( R \otimes V(m, k) \).

(c) Let \( q \in K \) be a primitive \( l \)-th root of 1. The images of \( p_{[t]}(q) \), \( p_{[t]}E_ip_{[t]}(q) \), \ldots, \( p_{[t]}E_{m-1}p_{[t]}(q) \) in the \( k \)-row quotient of \( H_n(q) \) generate a subalgebra isomorphic to the \( k \)-row quotient of \( KS_m \).

\textbf{Proof.} Consider \( H_n(x, K(x)) \) acting by the orthogonal representation \( \Pi_k \) on \( V(n, k) \). For a tableau \( s \) \( l \)-equivalent to \( t \), consider \( s_{(l-t)\rho} \otimes \Psi(s) \), where \( \Psi \) is as in Lemma 3.1.4. The map
\[ \Theta : s \mapsto s_{(l-t)\rho} \otimes \Psi(s) \]
extends to a linear isomorphism from the range of \( \Pi_k(p_{[t]}(q)) \) onto \( R \otimes V(m, k) \), according to the remarks preceding the statement of the Theorem.

The formulas of Proposition 3.2.4 show that
\[ \Theta \circ x^{-\frac{1}{l}}\Pi_k(p_{[t]}E_ip_{[t]}) \circ \Theta^{-1} = \mathbb{1} \otimes \Pi_k(e_i(x^l)) \]
acting on \( R \otimes V(m, k) \). This proves points (a) and (b), since \( \Pi_k(H_n(x, K(x))) \) is isomorphic to the \( k \)-row quotient of \( H_n(x, K(x)) \).

For part (c), consider the elements \( T_i = x^{-\frac{1}{l}}p_{[t]}E_ip_{[t]} + x^ip_{[t]} \) in \( H_n(x, K(x)) \), which are evaluable at the \( l \)-th root of unity \( q \).

It follows from part (b), and the isomorphism \( \Pi_k(H_n(x, K(x))) \cong \Phi_k(H_n(x, K(x))) \), that the assignment \( T_i \mapsto \Phi_k(T_i) \) determines a non-unital \( K(x^l) \)-algebra homomorphism
\[ \varphi : H_m(x^l, K(x^l)) \rightarrow \Phi_k(H_n(x, K(x))) \]
with the image of \( \varphi \) isomorphic to the \( k \)-row quotient of \( H_m(x^l, K(x^l)) \). Since \( \Phi_k(T_i) \) is evaluable at \( x = q \), the representation \( \varphi \) is evaluable at \( x = q \) (that is, at \( x^l = 1 \)). By Lemma 1.4.2, \( \varphi \) induces a \( K \)-algebra homomorphism
\[ \tilde{\varphi} : KS_m \rightarrow \Phi_k(H_n(q, K)) \]
satisfying \( \tilde{\varphi}(s_i) = \Phi_k(T_i)(q) \). We have to show that the image of \( \tilde{\varphi} \) is isomorphic to the \( k \)-row quotient of \( KS_m \).

Let \( s \) be a standard tableau of size \( m \), and let \( p_s \) be the corresponding path idempotent in \( H_m(x^l, K(x^l)) \). According to Lemma 1.5.4, \( p_s \) is evaluable at \( x^l = 1 \), and its evaluation \( p_s |_{x^l=1} \) is the corresponding path idempotent in \( KS_m \). We need to show that \( \tilde{\varphi}(p_s |_{x^l=1}) \) is non-zero if, and only if, \( s \) has no more than \( k \) rows.

Observe that \( \varphi(p_s) \) is non-zero if, and only if \( s \) has no more than \( k \) rows, and as \( \varphi(p_s) \) is an idempotent, its rank is the same as that of \( \varphi(p_s) |_{x=q} \). But \( \varphi(p_s) |_{x=q} = \tilde{\varphi}(p_s |_{x^l=1}) \) by Lemma 1.4.2, so the desired conclusion follows.
Lemma 3.3.2. Let $y$ be a point in $D^+ \cap \mathbb{Z}^k$, and let $0 \leq i, m \leq k$ be integers such that $i + m \leq k$. Assume that $y$ has $i + m$ coordinates which are congruent to $y_1 \mod l$, among them the first $i$ (i.e. $y_1$ up to $y_i$). Let $e^{(i)} = (0, \ldots, 1, 0, \ldots, 0)$ (with $i$ 1’s and $k - i$ 0’s), and let, for given $i \in \mathbb{N}$, $t_\rho$ be the path from $y$ to $y + e^{(i)}$ such that $t_\rho(j) = y + e^{(j)}$, for $j = 1, 2, \ldots, i$.

(a) Any path $t_\rho$ which is equivalent to $t_\rho$ ends in $y + e^{(i)}$ or a point $y + e^{(i)}$ in lexicographic order.

(b) There exist at most $(m + i)!/m!$ paths in $D^+$ which are equivalent to $t_\rho$, among which $i!$ paths end in $y + e^{(i)}$.

(c) Let $w \in W$ such that $w(t_\rho)$ is a path corresponding to a skew tableau. Then there exist at most $(m + i)!/m!$ paths starting from $w(y)$ which are equivalent to $w(t_\rho)$.

Proof. Assume, for simplicity, that $y_1 \equiv 0 \mod l$. Then $t_\rho(j) \equiv 1 \mod l$, for $j = 1, 2, \ldots, i$, and the same has to hold for any tableau $t_\rho$ which is equivalent to $t_\rho$. For such a $t_\rho$, we therefore have $m + i$ possibilities for the first box, $(m + i - 1)$ possibilities for the second one, and so on. Altogether, there are $(m + 1)(m + 2) \ldots (m + i)$ possibilities, as stated. There are $i!$ different ways of adding a box in each of the first $i$ rows. This shows (b). The proof of (c) goes similarly. (a) is clear.

Lemma 3.3.3. Let $K$ be a field of characteristic 0, and let $q$ be a primitive $l$-th root of unity in $K$. Let $\lambda$ be a $k$-critical diagram. Then there exists an evaluable idempotent $p \in H_n(x, K(x))$ such that $pS^\lambda \neq 0$, and for all diagrams $\mu$ such that $\mu \neq \lambda$ and $\mu$ has at most $k$ rows, $pS^\mu = 0$.

Proof. Let us assume first that $\lambda$ is a reduced $k$-critical diagram, i.e. that $\lambda_k = 0$. Let $t$ be a tableau of shape $\lambda$ satisfying conditions (1) and (2) above. Using the formulas for central idempotents of $KS_m$ and Theorem [3.3.1], we obtain an evaluable element $\tilde{p}$ in $p_{\{t\}}H_nP_{\{t\}}$ such that $\pi_\mu(\tilde{p})$ is evaluable at $q$ and an idempotent for all $k$-row diagrams $\mu$, and is nonzero if and only if $\mu = \lambda$. By Proposition [1.7.1], there exists an evaluable idempotent $p \in H_n$ such that for diagrams $\mu$ of at most $k$ rows, $p$ is nonzero on $S^\mu$ if and only if $\mu = \lambda$.

The general case is proved by induction on $\lambda_k$, the last component of $\lambda$. The case $\lambda_k = 0$ has already been verified. For $\lambda_k > 0$, we can assume that the claim has been verified for the diagram $\lambda - e^{(k)}$; more specifically, we can make the inductive hypothesis that there exists a tableau $t$ of shape $\lambda - e^{(k)}$ and an evaluable subidempotent $p$ of $p_{\{t\}}$ which is nonzero only on $S^\lambda$. It follows from Lemma [3.3.3] that for any extension $\tilde{t}$ of $t$ into $\lambda$, all its equivalent tableaux going through $\lambda - e^{(k)}$ also end up in $\lambda$ (as $r = m = k$). Hence $pp_{\{\tilde{t}\}}$ has all the required properties.

The following result is contained in results of James and Dipper ([DJ, 4.11] and James and Mathas [JM].
Corollary 3.3.4. Let $q$ be a primitive $l$-th root of unity in $K$. For any $k$ and for any $k$-critical diagram $\lambda$, $S^\lambda = D^\lambda$. Furthermore, $d_{\nu,\lambda} = 0$ for all $\nu \neq \lambda$ such that $\nu$ has at most $k$ rows.

Proof. Suppose one has established that $S^\lambda = D^\lambda$ for a particular $k$-critical diagram $\lambda$. By Lemma 3.3.3, there is an evaluable idempotent $p \in H_n(z)$ with the property that $p$ is non-zero on $S^\lambda$ but zero on any $S^\mu$, if $\mu \neq \lambda$ and $\mu$ has at most $k$ rows. It follows from Proposition 1.7.1 that $d_{\nu,\lambda} = 0$ for all diagrams $\nu \neq \lambda$ with at most $k$-rows.

Now we proceed by induction on the reverse lexicographic order on $k$-critical diagrams of fixed size $n$. Then $\lambda$ is clearly the highest diagram in its $W$ orbit, and hence $D^\lambda = S^\lambda$, by Lemma 2.4.1. Suppose that the claim has been verified for all $k$-critical diagrams $\lambda$ such that $|\lambda| = \lambda > \lambda$. By the Nakayama conjecture, if $\mu$ is a diagram such that $D^\mu$ is a composition factor of $S^\lambda$, then $\mu \triangleright \lambda$, so in particular $\mu$ has at most $k$ rows, and $\mu$ is a $k$-critical diagram. But then it follows from the induction hypothesis that if $\mu \neq \lambda$, then $D^\mu = S^\mu$, and $d_{\lambda,\mu} = 0$. Therefore $S^\lambda$ has no composition factors other than $D^\lambda$, so $S^\lambda = D^\lambda$. \qed

Remark 3.3.5. If one knows a priori that $S^\lambda = D^\lambda$ for all $k$-critical diagrams, then it follows at once from the Nakayama conjecture that $d_{\nu,\lambda} = 0$ for all $k$-critical diagrams $\lambda$ and for all diagrams $\nu \neq \lambda$ with at most $k$ rows.

3.4. General $l$-straight tableaux. There is a version of Proposition 3.2.4 which holds for arbitrary (i.e. not $k$-special) $l$-straight skew tableaux. The matrix coefficients are computed only modulo $q^l = 1$. The idea of the computation is similar to that of Proposition 3.2.4, but the combinatorics are more complicated. We omit the details and merely state the result. We use this result in Section 5 for our remarks on the “boundary region”.

Proposition 3.4.1. Suppose $t$ and $t'$ are $l$-straight skew tableaux of length $2l$ with the same initial and final shapes. Let $Q^{t,t'} = (t',Et)$, and let $Q^t = Q^{t,t}$. Then

(a) $Q^t(q) = C(q)\frac{(d(t;l,l+1) + 1)\prod_{i=1}^{l-1} (d(t;i+1,l+l+1) - 1)}{\prod_{i=1}^{l} d(t;i,l+l)}$, where $C(q)$ does not depend on $t$.

(b) If there is a subset $A \subseteq \{1,2,\ldots,l\}$, such that $t' = \left[ \prod_{a \in A} (a,a+l) \right] t$, then $Q^{l,t'}(q) = \sqrt{Q^t(q)Q^{l,t'}(q)}$; otherwise $Q^{l,t'}(q) = 0$. Here $(a,a+l)$ denotes the transposition which interchanges $a$ and $a + l$.

Remark 3.4.2. The condition in (b) is equivalent to the existence of a subset $L \subseteq J_l$ such that $s_{J_l}t' = s_{L \cup J_l}t$. 

4. Estimates on the decomposition numbers \( d_{\lambda\mu} \).

In this section we obtain bounds on the decomposition numbers \( d_{\lambda\mu} \) for the Hecke algebra \( H_n(K, q) \) where \( K \) is a field of characteristic 0, and \( q \) is a primitive \( l \)-th root of unity.

4.1. Critical points associated to interior points. We call a point \( y \in D^+ \) interior if \( y_i - y_{i+1} \geq l \) for \( i = 1, 2, \ldots, k - 1 \). We want to access an interior point \( y \) from a convenient \( k \)-critical point.

Let \( \rho \) be an interior point, and let \( y \) be its associated critical point. The number of its conjugates is less or equal to

\[
\prod_{i=1}^{k-1} \prod_{j=d_{i+1}}^{d_i} \frac{(m(j, i) + i)!}{m(j, i)!},
\]

where \( m(i, j) = |\{ y_s : y_s \equiv j \mod l, s > i \}|. \) Among those conjugate paths, exactly

\[
\prod_{i=1}^{k-1} (i!)^{r_i} \end \]

end in \( y \).

**Proof.** One has \( c_i - c_{i+1} = y_i - y_{i+1} - r_i = [(y_i - y_{i+1})/l] \geq l \), since \( y_i - y_{i+1} \geq l \). This shows (a). Point (b) is evident from the definitions; the last inequality results from \( r_i \leq l - 1 \).

For (c) we take the path \( t_p \) from \( c \) to \( y \),

\[
y^{(k)} = c \to y^{(k-1)} = y^{(k)} + r_{k-1} e^{(k-1)} \to \ldots
\]

(4.1.1)

\[
\to y^{(i)} = y^{(i+1)} + r_i e^{(i)} \to \ldots
\]

\[
\to y^{(1)} = y,
\]

where one goes from \( y^{(i+1)} \) to \( y^{(i)} \) by adding the \( (i r_i) \) cells column-wise, as in Lemma 3.3.2. (Note that \( y^{(i+1)} + s e^{(i)} \) has at least the first \( i \) rows congruent (mod \( l \)), for \( 0 \leq s \leq r_i \).

If \( \tilde{t}_p \) is another path of length \( \sum i r_i \) starting from \( c \) which is equivalent to \( t_p \), then by the definition of equivalence, the two paths are equivalent at each step. Therefore, it follows from Lemma 3.3.2 and induction that \( \tilde{t}_p \) ends in \( y \) or in a point which is lexicographically lower than \( y \). The estimates about the number of all equivalent paths, and the number of those which end in \( y \) follows by induction on \( |y| - |c| \), using Lemma 3.3.2(b) and (c).
4.2. **An estimate on** \( d_{\lambda\mu} \). Let \( c \) be an interior \( k \)-critical point and \( t_\rho \) a path from \( c \) to some interior point \( y \). For any diagram \( \lambda \) with at most \( k \) rows, denote by \( N(t_\rho, \lambda) \) the number of paths from \( c \) to \( \lambda + \rho \) which are equivalent to \( t_\rho \).

**Theorem 4.2.1.** Let \( \mu \) be a diagram such that \( y = \mu + \rho \) is an interior point. Then \( D^\mu \neq 0 \). Let \( c \) be the critical point associated to \( \mu \) and \( t_\rho \) the path from \( c \) to \( \mu + \rho \) constructed in Lemma 4.1.3. If \( \lambda \) is a diagram with \( \leq k \) rows, the multiplicity \( d_{\lambda\mu} \) of \( D^\mu \) in \( S^\lambda \) satisfies

\[
d_{\lambda\mu} \leq N(t_\rho, \lambda)/N(t_\rho, \mu).
\]

**Proof.** Since \( \mu + \rho \) is an interior point, it follows that \( \mu \) has no two rows of the same length; so \( \mu \) is \( l \)-regular and \( D^\mu \neq 0 \).

By Corollary 3.3.4, there exists an evaluable idempotent \( z_{(c-\rho)} \) which acts as the identity on \( S^{c-\rho} \) and as zero on any Specht module \( S^{\nu} \) belonging to a Young diagram \( \nu \) with \( k \) rows at the most such that \( |\nu| = |c-\rho| \), and \( \nu \neq c-\rho \). Let \( t(0) \) be any tableau of shape \( c-\rho \), and let \( \hat{t}_\rho \) be an extension of \( t(0) \) into \( y \) by the path \( t_\rho \) constructed in Lemma 4.1.2. Then \( z_{(c-\rho)p_\ell} \) is an evaluable idempotent in \( H_n \) (\( n = |\mu| \)).

\[
\Phi_k(z_{(c-\rho)p_\ell}) = \text{projection of } (\bigoplus_{\ell(\lambda) \leq k} V_\lambda) \text{ on the span of all those paths which are equivalent to } \hat{t}_\rho \text{ and which go through } c.
\]

Hence its evaluation acts as zero on any \( S^\lambda \) with \( \lambda > y-\rho = \mu \), but not as zero on \( S^\mu \). It follows that the rank \( r \) of \( p = z_{(c-\rho)p_\ell} \) on \( S^\mu \) is also the rank by which it acts on \( D^\mu \).

The rank \( r \) is equal to the number of paths which end in \( y \) and go through \( c \). If \( r_0 \) is the number of paths which are equivalent to \( t(0)_\rho \) and end in \( c \), one sees that \( r = r_0 N(t_\rho, \mu) \). Similarly, the rank by which \( p \) acts on \( S^\lambda \) is equal to \( r_0 N(t_\rho, \lambda) \). By Proposition 1.7.1(b)

\[
d_{\lambda\mu} \leq (r_0 N(t_\rho, \lambda))/(r_0 N(t_\rho, \mu)) = N(t_\rho, \lambda)/N(t_\rho, \mu).
\]

\( \square \)

4.3. **Reduced paths.** In the bound for \( d_{\lambda\mu} \) in 4.2, the denominator is known, namely

\[
N(t_\rho, \mu) = \prod_{i=1}^{k-1} (i!)^{r_i}.
\]

The numerator is divisible by this quantity, and it remains to describe the quotient.

We define the reduced path \( t_{\rho, \text{red}} \) of \( t_\rho \) to be the sequence of points:

\[
\begin{align*}
c, c + e^{(k-1)}, c + 2e^{(k-1)}, \ldots, & \quad c + (k-1)e^{(k-1)} = y^{(k-1)}, \\
y^{(k-1)}, y^{(k-1)} + e^{(k-2)}, \ldots, & \quad y^{(k-1)} + (k-2)e^{(k-2)} = y^{(k-2)}, \\
& \quad \ldots \\
y^{(2)}, y^{(2)} + e^{(1)}, \ldots, & \quad y^{(2)} + r_1 e^{(1)} = y.
\end{align*}
\]

(4.3.1)
That is, we divide the path $t_\rho$ into segments, the first $r_{k-1}$ of length $k - 1$, the next $r_{k-2}$ of length $k - 2$, and so forth, until the last $r_1$ segments of length 1; the reduced path is the list of endpoints of these segments. Similarly, if $\tilde{t}_\rho$ is a path equivalent to $t_\rho$, we divide this path into segments of the same lengths, and define the reduced path $\tilde{t}_\rho,\text{red}$ to be the sequence of endpoints of these segments.

We define $n(\lambda, \mu)$ to be the number of distinct reduced paths belonging to paths $\tilde{t}_\rho$ which are equivalent to $t_\rho$ and which end in $\lambda + \rho$.

**Lemma 4.3.1.** Let $\mu$ be a diagram such that $y = \mu + \rho$ is an interior point. Let $c$ be the critical point associated to $y$ and $t_\rho$ the path from $c$ to $y$ constructed in Lemma 4.1.2. Let $\lambda$ be a diagram with at most $k$ rows. Then

$$N(t_\rho, \lambda)/N(t_\rho, \mu) = n(\lambda, \mu)$$

**Proof.** It follows from Theorem 4.2.1 that there is only one reduced path which ends in $y$. It suffices to observe that if $\lambda$ is a diagram such that $N(t_\rho, \lambda) \neq 0$, then the ratio $N(t_\rho, \lambda)/n(\lambda, \mu)$ is the same as $N(t_\rho, \mu)/n(\mu, \mu) = N(t_\rho, \mu)$. \hfill $\square$

**Corollary 4.3.2.** Let $\mu$ be a diagram such that $y = \mu + \rho$ is an interior point. Let $c$ be the critical point associated to $y$ and $t_\rho$ the path from $c$ to $y$ constructed in Lemma 4.1.2. If $\lambda$ is a diagram with $\leq k$ rows, the multiplicity $d_{\lambda\mu}$ of $D^\mu$ in $S^\lambda$ satisfies

$$d_{\lambda\mu} \leq n(\lambda, \mu)$$

**Remark 4.3.3.** The reduced paths described above can also be interpreted as certain semi-standard tableaux of shape $\lambda \setminus (c - \rho)$. Namely, let $\nu$ be the conjugate diagram of $(\mu + \rho - c)$, that is,

$$\nu = (k - 1)^{r_{k-1}}(k - 2)^{r_{k-2}}\cdots 1^{r_1}.$$  

Then the reduced paths correspond 1-to-1 with semi-standard tableaux of of shape $\lambda \setminus (c - \rho)$ and weight $\nu$ (that is, with $\nu_1$ 1’s, $\nu_2$ 2’s, etc.) such that each entry of $j$ has content (column index minus row index) congruent to $j - k \mod l$.

**Remark 4.3.4.** Observe that the hyperplanes passing through $c$ generate a reflection group $W_c$ which is isomorphic to $S_k$. Consider the set $A$ of all alcoves through which $t_\rho$ and its conjugates run. Let $A \in A$. Assume that for all $w \in W_c$, $w(A)$ is an alcove in the Weyl chamber which does not touch the boundary of the Weyl chamber. Then $n(\lambda, \mu) = n(w(\lambda + \rho) - \rho, \mu)$ for all $w \in W_c$. We conjecture that a similar symmetry also holds for the $d_{\lambda\mu}$.

**Corollary 4.3.5.**

$$\sum_{\lambda, \lambda_{k+1} = 0} d_{\lambda\mu} \leq \prod_{i=1}^{k-1} \prod_{j=d_{i+1}}^{d_i-1} m(j, i) + i,$$

where $m(i, j) = |\{y_s, y_s \equiv j \mod l, s > i\}|$. 

\
Proof. The right hand side is less or equal to the number of all paths within \( k \) row diagrams which are equivalent to \( t_\rho \) divided by the number of paths equivalent to \( t_\rho \) which end in \( y = \mu + \rho \). Our estimate follows from Lemma 4.1.2(c).

5. Examples

We give a geometric description of our results for \( k = 3 \). The weights \( L_i \) (1 \( \leq \) \( i \) \( \leq \) 3) of \( sl_3 \) in the vector representation (\( L_3 = -L_1 - L_2 \)) can be represented by three coplanar unit vectors which are mutually equidistant.

Consider the map \( \mathbb{R}^3 \to \mathbb{R}^2 \), \( y \mapsto \sum y_i L_i \). Partitions \( \lambda \) of length \( \leq 3 \) map to dominant integral weights \( \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 \). The open Weyl chamber is the image of \( D^+ \), namely \( C = \{ \sum y_i L_i : y_1 > y_2 > y_3 \geq 0 \} \). A path \( t_\rho \) corresponding to a tableau or skew tableau \( t \) is mapped to a path in \( C \), each of whose segments is one of the vectors \( L_i \) and whose endpoints are integer points of \( C \), namely in the integer span of the \( L_i \). \( C \) is divided into triangular tiles by the lines \( \{ \sum y_i L_i : y_i - y_j = m \lambda \} \). The 3-critical points of \( D^+ \) map to the intersection of three such lines. The boundaries of \( C \) are formed by the lines \( \{ \sum y_i L_i : y_2 - y_3 = 0 \} \) and \( \{ \sum y_i L_i : y_1 - y_2 = 0 \} \). For any point \( y \), we define the closed positive cone of \( y \) to be \( y + \bar{C} \). The interior points of \( D^+ \cap \mathbb{Z}^3 \) map to the cone \( 2L_1 + lL_2 + \bar{C} \).

In the following statements, we no longer distinguish between points of \( D^+ \) and the corresponding points of \( C \). Let \( y \) be an interior point which does not lie on any of the affine hyperplanes (i.e. lines in our case) just described. Let \( c \) be its critical point. The following is easy to check:
(a) \((y - c)_1 < 2l\) and \((y - c)_1 \neq l\),

(b) \((y - c)_1 < l\) if and only if \(y\) is in a triangle pointing upwards, with its highest vertex equal to \(c\) (see next picture below),

(c) \((y - c)_1 > l\) if and only if \(y\) is in a triangle pointing downwards; here \(c\) is the highest vertex of the triangle above the one containing \(y\) (see second picture below).

The following two diagrams show a path from \(c\) to \(y\) together with its conjugate paths in the two cases:

Next we consider the boundary region (that is the complement of the interior region) in \(D^+\) for \(k = 3\). This can be extended to certain boundary points also for larger \(k\) (see Remark 3 below). Let \(y\) be a point in the left boundary region, i.e. such that \(y_2 - y_3 < l\). We can assume \(y_3 = 0\), and hence \(y_2 < l\). We can therefore approach \(y\) by a path as shown in the figures below, which also show the conjugate paths.
Here comes the crucial step for such paths: Consider the picture to the right of picture D. Our path orbit contains 5 paths. The corresponding projection $p_{[q]}$ can be split into the sum of 2 subidempotents as follows: It is easy to check that the endpoint $a$ of the lowest and the endpoint $b$ of the rightmost path are the alphabetically highest and secondhighest points among all the endpoints of paths in that picture. Applying Proposition 3.4.1 to the skew
tableaux containing the last $2l$ steps of our paths, we obtain an evaluable subidempotent $p$ of $p[t]$ which acts as 0 on the lowest path and as 1 on the rightmost path. It again follows from Proposition 3.4.1 that $p$ acts as a rank 3 idempotent on the space spanned by our 5 paths. Hence $p[t] - p$ can be described by picture $E$.

This is not to be taken quite literally; namely $p[t] - p$ is not the sum of two ordinary path idempotents. Nevertheless, we are essentially back to the pattern of figure A in the series, and as the path $t_\rho$ is extended into further alcoves the sequence of figures A through E repeats itself.

Remarks 1. It is possible to prove a similar pattern for the right boundary region. Essentially, we use the mirror path of the path constructed for the left boundary region. Such a path can be interpreted as obtained by tensoring by the dual representation of the fundamental representation of $sl_3$; in path language, it would mean we construct all path extensions of length 2, and let the projection $p_{[12]}$ act on these extensions. It can be shown that the same algebraic theory can be applied as for the ones studied before, e.g. by using the result in [KW] which characterizes fusion categories of type $A$.

2. For $k = 3$ all our bounds are sharp, as can be seen by comparing the results with computations using the algorithm of [LLT]. However, for $k \geq 4$, our bounds are no longer sharp.

3. Our method for dealing with boundary points for $k = 3$ can be generalized to get partial results for $k \geq 4$, but it is more efficient to use the LLT algorithm.

6. **Connection to Quantum Groups**

6.1. Let $U_q(sl_k)$ be the Drinfeld-Jimbo quantum group corresponding to the root system $A_{k-1}$. In the following we will also assume $q$ to be a primitive $l$-th root of unity with $l > k$; in this case we take for $U_q(sl_k)$ the version of the quantum group as defined by Lusztig, where $q = v^2$ (see [Lu]).

The notion of *tilting modules* of a quantum group was introduced by Andersen [A], inspired by a similar notion for algebraic groups which was defined by Donkin [Do1]. For type $A$, it can be shown that any direct summand of $V^\otimes n$ is a tilting module, where $V$ is the fundamental
$k$-dimensional $U_q(sl_k)$ module. This follows from the fact that $V$ is a tilting module, and tensor products and direct summands of tilting modules are also tilting; see [A] for details. Conversely, it is possible to characterize tilting modules for type A as direct sums of direct summands of $V^\otimes n$ (with $n$ varying).

It was already observed by Jimbo [Ji] that the image of the Hecke algebra under the representation on $V^\otimes n$, as defined in Section 2, commutes with the action of $U_q(sl_k)$, and, if $q$ is not a proper root of unity, the image of the Hecke algebra and the image of the quantum group are commutants of one another; this is a $q$-analog of the famous Schur-Weyl duality between the general linear group and the symmetric group.

It has only recently been established that the same duality holds, between the Lusztig version of the quantum group of type $A$ and the Hecke algebra, when $q$ is a root of unity [DPS]. (See [J] for the case of algebraic groups with positive characteristic). It follows that indecomposable tilting modules for $U_q(sl_k)$ all have the form $pV^\otimes n$, for some $n$ and for some primitive idempotent $p$ in the $k$-row quotient of $H_n(q)$. In fact, there exists for each dominant weight $\mu$ a unique (up to isomorphism) indecomposable tilting module $T_\mu$ with highest weight equal to $\mu$ (see [A, Cor. 2.6]). If $p$ is a primitive idempotent in the $k$-row quotient of $H_n(q)$ whose image in the maximal semisimple quotient is a minimal idempotent “belonging” to the simple module $D_\mu$, then $T_\mu \cong pV^\otimes n$. Here we have identified the Young diagram $\mu$ with a dominant weight of $sl_k$ in the usual way.

Tilting modules have filtrations by standard or Weyl modules $\Delta_\lambda$, whose characters are given by the Weyl character formula. Consequently, the character $\chi^\mu_T$ of the tilting module $T_\mu$ is a linear combination of characters of Weyl modules,

$$\chi^\mu_T = \sum n_{\lambda\mu} \chi^\lambda,$$

(6.1.1)

where $n_{\lambda\mu}$ are non-negative integers.

It is well known that the multiplicities of Weyl modules in indecomposable tilting modules coincides with the multiplicities of simple Hecke algebra modules in Specht modules:

$$n_{\lambda\mu} = d_{\lambda\mu}.$$  

(6.1.2)

This equality has been derived using a duality theory for quasi-hereditary algebras; see [Do2], Chapter 4, and also [Do1], Lemma 3.1. The equality can also be derived quite directly from Schur-Weyl duality and an evaluation argument; we give this argument in section 6.2.

In [GW], we have extended the equality 6.1.2 to certain polynomial analogues of the decomposition numbers:

$$n_{\lambda\mu}(v) = d_{\lambda\mu}(v).$$  

(6.1.3)

Here the $n_{\lambda\mu}(v)$ are affine Kazhdan-Lusztig polynomials which satisfy $n_{\lambda\mu}(1) = n_{\lambda\mu}$ ([S1], [S2]), and the $d_{\lambda\mu}(v)$ are polynomials associated to a $U_v(sl_k)$-module, which satisfy $d_{\lambda\mu}(1) = d_{\lambda\mu}$ ([LLT], [Ar]).
6.2. We provide a proof of the equality \([3.1.2]\) by a fairly elementary evaluation argument. Let \(F\) be any field of characteristic 0, and \(q\) an element of \(F^\times\). Consider the commuting actions of \(H_n(q, F)\) and of the quantum group \(U_q(sl_k(F))\) on \((F^k)^\otimes n\). (If \(q\) is a root of unity, take the Lusztig version of the quantum group.) The standard basis of the Cartan subalgebra of \(H_n(q, F)\) submodule. For any idempotent \(p \in H_n(q, F)\), \(\text{Tr}_{W(\gamma)}(p)\) is the dimension of \(W(\gamma)\) free of \(\text{Tr}(F^k)^\otimes n\), that is, the multiplicity of the weight \(\gamma\) in the

\(U_q(sl_k(F))\) module \(p(F^k)^\otimes n\).

Now consider a field \(K\) of characteristic 0, put \(V(x) = K(x)^k\), and \(V = K^k\). Fix a primitive \(\ell\)-th root of unity \(q \in K\). Consider the commuting actions of \(H_n(x, K(x))\) and \(U_x(sl_k)\) on \(V(x)^\otimes n\). For each dominant integral weight \(\gamma\), the weight space \(W(\gamma)\) free of \(V(x)^\otimes n\) is an \(H_n(x, K(x))\) module (with respect to the standard basis), and its “restriction” to \(H_n(q, K)\) is the weight space \(W(\gamma)\) free of \(V^\otimes n\). If \(p \in H_n(x, K(x))\) is an evaluable idempotent, then according to Proposition \([6.6.1]\) a), \(\text{Tr}_{W(\gamma)}(x) = \text{Tr}_{W(\gamma)}(p(q))\). That is, the multiplicities of the weight \(\gamma\) in \(pV(x)^\otimes n\) and in \(p(q)V^\otimes n\) are the same.

Note also that if \(e\) is a minimal idempotent in the semisimple algebra \(H_n(x, K(x))\) belonging to the simple module \(S^\lambda\), then \(eV(x)^\otimes n\) is a simple \(U_x(sl_k)\) with highest weight \(\lambda\) and character \(\chi^\lambda\) (by Schur-Weyl duality) and \(\text{Tr}_{W(\gamma, x)}(e) = m_{\gamma}^\lambda\) is the multiplicity of \(\gamma\) in this module.

Finally, let \(p \in H_n(x, K(x))\) be an evaluable idempotent such that \(p(q)\) is a primitive idempotent in \(H_n(q, K)\) whose image in the maximal semisimple quotient is minimal and belongs to the simple module \(D^\mu\). As observed above, \(p(q)V^\otimes n \cong T^\mu\). If \(p\) is written as an orthogonal sum of minimal idempotents in \(H_n(x, K(x))\), then \(d_{\lambda\mu}\) of these minimal idempotents satisfy \(eS^\lambda \neq 0\). Thus the multiplicity of \(\gamma\) in \(T^\mu\) is

\[(6.2.1) \quad \text{Tr}_{W(\gamma)}(p(q)) = \text{Tr}_{W(\gamma, x)}(p) = \sum_{\lambda} d_{\lambda\mu} m_{\gamma}^\lambda.\]

On the other hand, it follows from Equation \((6.1.4)\) that this multiplicity is \(\sum_{\lambda} n_{\lambda\mu} m_{\gamma}^\lambda\). From the equations \(\sum_{\lambda} n_{\lambda\mu} m_{\gamma}^\lambda = \sum_{\lambda} d_{\lambda\mu} m_{\gamma}^\lambda\) for all \(\gamma\), one has \(\sum_{\lambda} n_{\lambda\mu}\chi^\lambda = \sum_{\lambda} d_{\lambda\mu}\chi^\lambda\), and hence \(n_{\lambda\mu} = d_{\lambda\mu}\) for all \(\lambda\), by linear independence of the Schur functions.

6.3. We show the connection between our Theorem \([4.2.1]\) and results obtained by Soergel: Let \(\Pi\) be the fundamental box, i.e. the region of points \(x\) in the Weyl chamber for which \(0 < (\alpha_i, x) < l\) for any simple root \(\alpha_i\), and let \(\Delta = \cup_{w \in S_n} \overline{w(\Pi)}\). A slightly weaker version of our Theorem \([4.2.1]\) can be rephrased as follows:

**Theorem 6.3.1.** Let \(\mu + \rho \in c + \Pi\), where \(c\) is a critical point in the Weyl chamber such that also \(c + \Delta\) is contained in the closure of the Weyl chamber. Then the multiplicities \(n_{\lambda\mu}\) and \(d_{\lambda\mu}\) are nonzero only if \(\lambda + \rho \in c + \Delta\).

This result follows from Theorem 5.3 and Proposition 4.19 in [S1]. The latter also contains much more precise information about these multiplicities.
6.4. If $F$ is a field of characteristic $p$, one can find an embedding of the $k$-row quotient of $FS_n$, for appropriate $n$, into the $k$-row quotient of a much larger symmetric group, using paths in the region below $\lambda_1 - \lambda_k = p^2$. One only needs to replace the $q$-numbers by ordinary numbers (mod $p$) in all our proofs. It should be possible to extend this embedding for arbitrary $n$ also in the region above $\lambda_1 - \lambda_k = p^2$, although one certainly would have to modify parts of our proof for the general case.

It should be remarked that a similar statement as Theorem 6.3.1 does NOT hold for tilting modules of algebraic groups over fields with positive characteristic. In this case, there are no uniform bounds for the number of irreducible generic characters which appear in the character of an indecomposable tilting module.

6.5. In the setting of algebraic groups, there is an interesting connection between $Sl(k, F)$ Weyl modules $V_\lambda$ and $V_{\rho, \lambda+(\rho-1)\rho}$ involving the Frobenius homomorphism (see [Ja, Section 10] for details). One would expect a similar connection between $sl_k$-modules $V_\lambda$ and $U_q sl_k$-modules $V_{\lambda+(l-1)\rho}$ for $q$ a primitive $l$-th root of unity. More precisely, it is easy to check that we have the following isomorphism of vector spaces:

$$T_{\lambda+(l-1)\rho} \cong V_\lambda \otimes T_{(l-1)\rho}, \quad (\star)$$

where $V_\lambda$ is an irreducible $sl_k$ module with highest weight $\lambda$. This follows easily from Weyl’s dimension formula (observe that the tilting modules in $(\star)$ coincide with the Weyl module with the same highest weight). Comparison with the setting for algebraic groups (see [Ja, 10.5]) would suggest that this can be done in such a way that $V_\lambda$ is an $sl_k$-module, and $T_{(l-1)\rho}$ is a $K_l$ module, where $K_l$ is the kernel of Lusztig’s Frobenius type homomorphism (see [Lu]). It would seem that our result Theorem 3.3.1 is in some sense dual to Lusztig’s result, with ‘dual’ in the sense of Schur-Weyl duality.

Similarly, one would expect from the occurrence of a representation of the symmetric group within the Hecke algebra a tensor action $\otimes$ of $Rep(sl_k)$ on direct sums of modules of the form $T_{\lambda+(l-1)\rho}$ with certain categorical properties. Using the vector space isomorphism in $(\star)$, one would expect a decomposition

$$V_\mu \otimes T_{\lambda+(l-1)\rho} = \bigoplus \nu \ c_{\lambda\mu}^\nu T_{\nu+(l-1)\rho},$$

where $V_\mu, V_\nu$ are simple highest weight modules of $sl_k$ and $c_{\lambda\mu}^\nu$ is the multiplicity of $V_\nu$ in $V_\lambda \otimes V_\mu$.

6.6. It is comparatively straightforward to prove estimates as the ones in Theorem 6.3.1 for type $A$ also for the multiplicities $n_{\lambda\mu}$ for representations of $U_q sp_{2k}$, or for those representations of $U_q so_{2k+1}$ which appear in some tensor power of the defining $k$-dimensional representation, using a $q$-version of Brauer’s centralizer algebras.
HECKE ALGEBRAS OF TYPE A

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