A CRITERION FOR THE TRIVIALITY OF THE CENTRALIZER
FOR VECTOR FIELDS AND APPLICATIONS

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Abstract. In this paper we establish a criterion for the triviality of the $C^1$-
centralizer for vector fields and flows. In particular we deduce the triviality of
the centralizer at homoclinic classes of $C^r$ vector fields ($r \geq 1$). Furthermore,
we show that the set of flows whose $C^1$-centralizer is trivial include: (i) $C^1$-
generic volume preserving flows, (ii) $C^2$-generic Hamiltonian flows on a generic
and full Lebesgue measure set of energy levels, and (iii) $C^1$-open set of non-
hyperbolic vector fields (that admit a Lorenz attractor). We also provide a
criterion for the triviality of the $C^0$-centralizer of continuous flows.

1. Introduction and statement of the main results

1.1. Introduction. In this article we study the centralizer of flows and vector
fields. The centralizer of a flow is the set of flows that commute with the original one.
In [37], Smale proposed a problem on the triviality of the centralizer in dynamical
systems, asking if ‘typical’ dynamics (meaning open and dense, or Baire generic)
would have trivial centralizers. In the case of discrete time dynamical systems
there have been substantial contributions towards both an affirmative solution to
the problem raised by Smale and also to applications that exploit the relation of the
centralizer with differentiability of conjugacies, embedding of diffeomorphisms as
time-1 maps of flows, rigidity of measurable $\mathbb{Z}^d$-actions on tori and characterization
of conjugacies for structurally stable diffeomorphisms, just to mention a few (see
e.g. [8, 11, 12, 13, 18, 27, 29, 33, 34, 39] and references therein).

The centralizer of continuous time dynamics has been less studied. Indeed, in
the continuous time setting the problem of the centralizer has been considered for
Anosov flows [14], $C^{\infty}$-Axiom A flows with the strong transversality condition [36],
and for Komuro-expansive flows, singular-hyperbolic attractors and expansive $\mathbb{R}^d$-
actions [9, 10, 25]. In most cases, the ingredients in the proofs of the previous
results involve either a strong form of hyperbolicity and/or $C^{\infty}$-smoothness of the
dynamics in order to use linearization of the dynamics in a neighborhood of a
periodic or singular point.

In order to describe a larger class of vector fields (and flows) we establish
a criterion for the triviality of the $C^1$-centralizer of vector fields based on the
denseness of hyperbolic periodic orbits (Theorem A). In particular the triviality of
the centralizer can be deduced whenever there exists a dense set of hyperbolic
periodic orbits, a condition that is satisfied by all homoclinic classes of $C^r$-vector
fields in view of Birkhoff-Smale’s theorem ($r \geq 1$) (Corollary 2). We use this

\textsuperscript{\textcopyright}2010 Mathematics Subject classification: 37C10, 37D20, 37C15
Date: March 19, 2019.
Key words and phrases. Centralizers, homoclinic classes, uniform hyperbolicity, Hamiltonian
flows, Lorenz attractors.
fact to show that $C^1$-generic volume preserving flows have a trivial centralizer (Corollary 3), obtaining the counterpart of [8] for continuous time dynamics, and that Lorenz attractors have trivial centralizer (cf. Corollary 4). In the case of Hamiltonian dynamics the relation between symmetries and elements of the centralizer of a Hamiltonian flow is more evident. Indeed, it follows from Noether’s theorem that there exists a bijection between conserved quantities, modulo constants, and the set of infinitesimal symmetries (see Subsection 1.2.3 for more details). First results on commuting Hamiltonians were obtained in [3], which described almost-periodic commuting Hamiltonian vector fields. Moreover, as the Hamiltonian is always a first integral for the Hamiltonian vector field it is natural to ask whether typical Hamiltonians have other independent first integrals. Arnold-Liouville’s integrability theorem assures that the existence of $n$ independent commuting integrals for a $C^2$-Hamiltonian $H$ implies the integrability of the Hamiltonian vector field and that the restriction of the flow to each invariant torus consists of a translation flow (see e.g. [2]). However, $C^2$-generic Hamiltonians are transitive on each connected component of generic energy levels ([6]) and, consequently, continuous first integrals are constant on the corresponding level sets. We use this fact here to prove that the centralizer of $C^2$-generic Hamiltonians on compact symplectic manifolds is trivial on generic level sets (we refer the reader to Corollary 5 for the precise statement).

The problem of the centralizer for flows and vector fields differs substantially from the discrete-time setting. For instance, the statement of the criterion for triviality established here for vector fields is false even for Anosov diffeomorphisms (see Remark 1.1). In rough terms, the first part of the strategy to the proof of Theorem A is to show that the closure of the set of hyperbolic critical elements is preserved by all flows in the $C^1$-centralizer of a flow $(X_t)_t$. The second part of the argument is to use transitivity to show that every element in the centralizer of $(X_t)_t$ is a linear (actually constant) reparametrization of $(X_t)_t$. One should mention that if one was interested to describe a Baire generic subset of dynamical systems the transitivity assumption could be removed. Indeed, as proposed by R. Thom [38], for every $1 \leq r \leq \infty$, $C^r$-generic vector fields do not admit non-trivial first integrals (see e.g. [21]).

1.2. Preliminaries. In this subsection we recall some necessary definitions and set some notation.

1.2.1. Non-wandering set, periodic orbits and hyperbolicity. Throughout, let $M$ be a compact, connected Riemannian manifold without boundary. Given a $C^1$-vector field $X \in \mathfrak{X}^1(M)$, we often denote by $(X_t)_{t \in \mathbb{R}}$ the $C^1$-flow generated by $X$ and let $\Omega(X)$ denote its non-wandering set. Given a compact $(X_t)_{t \in \mathbb{R}}$-invariant set $\Lambda \subset M$, we say that $\Lambda$ is a hyperbolic set for $(X_t)_{t \in \mathbb{R}}$ if there exists a $DX_t$-invariant splitting $T_{\Lambda}M = E^s \oplus E^0 \oplus E^u$ so that: (a) $E^0$ is one dimensional and generated by the vector field $X$, and (b) there are constants $C > 0$ and $\lambda \in (0, 1)$ so that

$$
\|DX_t(x)\|_{E_x^s} \leq C\lambda^t \quad \text{and} \quad \|(DX_t(x)\|_{E_x^u})^{-1} \leq C\lambda^t
$$

for every $x \in \Lambda$ and $t \geq 0$. The flow $(X_t)_{t \in \mathbb{R}}$ is Anosov if $\Lambda = M$ is a hyperbolic set. A point $p \in M$ is: (i) a singularity if $X_t(p) = p$ for all $t \in \mathbb{R}$, and (ii) periodic if there exists $t > 0$ such that $X_t(p) = p$ and $\pi(p) := \inf\{t > 0 : X_t(p) = p\} > 0$. We denote by $\text{Sing}((X_t)_t)$ (or $\text{Sing}(X)$) the set of singularities, by $\text{Per}((X_t)_t)$ (or $\text{Per}(X)$) the set of periodic orbits and by $\text{Crit}((X_t)_t) := \text{Sing}((X_t)_t) \cup \text{Per}((X_t)_t)$.
the set of all critical elements for the flow \((X_t)\). Every non-singular point is called regular. A flow \((X_t)\) is called Axiom A if \(\Omega(X)\) is a uniformly hyperbolic set and \(\Crit((X_t)_{t}) = \Omega(X)\). A periodic point \(p \in M\) is hyperbolic if its orbit \(\gamma_p := \{X_t(p) : t \in \mathbb{R}\}\) is a hyperbolic set. A singularity \(\sigma \in M\) is hyperbolic if \(D_X(\sigma)\) has no purely imaginary eigenvalues. Denote by \(W^s(\gamma_p)\) (resp. \(W^u(\gamma_p)\)) the usual stable (resp. unstable) manifolds of the hyperbolic periodic orbit \(\gamma_p\). The stable and unstable manifolds for a singularity \(\sigma\) are denoted similarly by \(W^s(\sigma)\) and \(W^u(\sigma)\), respectively (we refer the reader to [1, Subsection 2.1] for the classical definitions of stable and unstable manifolds). The homoclinic class associated to a hyperbolic critical element \(z\) is the compact invariant set \(H(z) = W^s(\gamma_z) \cap W^u(\gamma_z)\). Finally, we say that a flow \((X_t)\) is transitive if there exists \(x \in M\) so that \((X_t(x))_{t \in \mathbb{R}}\) is dense in \(M\). We refer to [17, 28] for more details on uniform hyperbolicity and homoclinic classes.

### 1.2.2. Commuting flows and centralizers.

Given \(r \geq 0\), let \(\mathcal{F}^r(M)\) denote the space of \(C^r\)-flows on \(M\). We say that the flows \((X_t)_{t \in \mathbb{R}}\) and \((Y_t)_{t \in \mathbb{R}}\) commute if \(Y_s \circ X_t = X_t \circ Y_s\) for all \(s, t \in \mathbb{R}\). Given a flow \((X_t)_{t \in \mathbb{R}} \in \mathcal{F}^r(M)\), the centralizer of \(X\) is the set of \(C^r\)-flows that commute with \(X\), that is,

\[
\mathcal{Z}^r((X_t)_t) = \{(Y_s)_{s \in \mathbb{R}} \in \mathcal{F}^r(M) : Y_s \circ X_t = X_t \circ Y_s, \forall s, t \in \mathbb{R}\}. \tag{1.1}
\]

It is clear from the definition that all flows obtained as smooth reparametrizations of \((X_t)_t\) belong to \(\mathcal{Z}^r((X_t)_t)\). For that reason, the centralizer of the time-one map of a flow is never a discrete subgroup of the space of diffeomorphisms. Given \(r \geq 0\), we say a flow \((X_t)_t \in \mathcal{F}^r(M)\) has quasi-trivial centralizer if for any \(Y \in \mathcal{Z}^r(X)\) there exists a continuous function \(h : M \to \mathbb{R}\) so that

(i) (orbit invariance) \(h(x) = h(X_t(x))\) for every \((t, x) \in \mathbb{R} \times M\), and

(ii) \(Y_t(x) = X_{h(x)t}(x)\) for every \((t, x) \in \mathbb{R} \times M\).

In the case that the reparametrizations \(h\) are necessarily constant then we say the centralizer is trivial. The previous notion has a dual formulation in terms of vector fields. Given \(r \geq 1\) and \(X \in \mathcal{X}^r(M)\), one can define the centralizer of the vector field \(X\) by

\[
\mathcal{Z}^r(X) = \{Y \in \mathcal{X}^r(M) : [X, Y] = L_Y X = 0\}, \tag{1.2}
\]

where \([X, Y]\) denotes the usual commutator of the vector fields \(X\) and \(Y\), and \(L_Y X\) denotes the Lie derivative of the vector field \(X\) along \(Y\). We say that the vector field \(X \in \mathcal{X}^r(M)\) has quasi-trivial centralizer if for any \(Y \in \mathcal{Z}^r(X)\) there exists a continuous \(h : M \to \mathbb{R}\) that is constant along the orbits of \((X_t)_t\) (i.e. is a first integral for the flow) and so that \(Y = h \cdot X\). As before, the centralizer is trivial if any such \(h\) is necessarily constant. Observe that any \(C^1\) reparametrization \(h : M \to \mathbb{R}\) satisfying \(X(h) = 0\) is constant along the orbits of the flow and, hence, is a first integral for the flow. If \((X_t)_t\) is transitive then any first integral is necessarily constant. For that reason, any transitive flow with a quasi-trivial centralizer has trivial centralizer. Moreover, as \(C^r\)-generic vector fields do not admit non-trivial first integrals [21], the set of \(C^1\)-flows with a quasi-trivial and non-trivial centralizer is meager.

### 1.2.3. Hamiltonian vector fields and flows.

Let \((M^{2n}, \omega)\) be a compact symplectic Riemannian manifold. Given \(r \geq 1\) and a Hamiltonian \(H \in C^{r+1}(M, \mathbb{R})\), the Hamiltonian vector field \(X_H \in \mathcal{X}^r_c(M)\) is defined by \(\omega(X_H(x), v) = DH(x)v\) for every \(x \in M\) and \(v \in T_xM\). In the case \(r \geq 2\) the Hamiltonian vector field \(X_H\) is
at least $C^1$ and we denote by $(\varphi^H_t)_t$ the Hamiltonian flow generated by $X_H$. We say that $K \in C^{r+1}(M,\mathbb{R})$ is a first integral (or an infinitesimal symmetry) for the Hamiltonian $H \in C^{r+1}(M,\mathbb{R})$ if $K(\varphi^H_t(x)) = K(x)$ for every $x \in M$ and $t \in \mathbb{R}$. A first integral $K$ for a Hamiltonian $H$ is characterized by $\{K, H\} = 0$, where the Poisson bracket $\{\cdot, \cdot\} : C^{r+1}(M,\mathbb{R}) \times C^{r+1}(M,\mathbb{R}) \to C^r(M,\mathbb{R})$, defined by

$$\{H, K\}(x) = \frac{d}{dt}H(\varphi^K_t)|_{t=0},$$

measures the derivative of $H$ along the orbits of $(\varphi^K_t)_t$. Recall $\{H, K\} = -\{K, H\}$ and that the Poisson and Lie brackets for Hamiltonians are related by $\{X, \{Y, H\}\} = \{\{X, Y\}, H\}$, see e.g. [19]). Hence, the Poisson bracket determines commutativity of Hamiltonians: $H, K \in C^r(M,\mathbb{R})$ commute if and only if the Poisson bracket $\{H, K\}$ is locally constant (see e.g. [2, Section 40]). Nevertheless, two Hamiltonian flows may commute and not preserve level sets (see Example 2.2). On the one hand, Noether’s Theorem (see e.g. [17] or [19, Theorem 18.27]) establishes a bijective correspondence between the set of vector fields $Y \in \mathfrak{X}'(M)$ such that $H(Y_t(x)) = H(x)$ and $Y^*_t \omega = \omega$ for any $t \in \mathbb{R}$ and $x \in M$ (also known as conserved quantities) and the set of first integrals for $X_H$, modulo addition by constants. On the other hand, $K$ is a first integral of $X_H$ if $\{K, H\} = 0$ (see e.g. [22, Theorem 2]).

These facts motivate the following definition. Given $r \geq 1$, a Hamiltonian $H \in C^{r+1}(M,\mathbb{R})$ with associated vector field $X_H \in \mathfrak{X}'(M)$, the Hamiltonian $C^r$-centralizer of $X_H$ is

$$Z^{r}_H(X_H) = \{X_K \in \mathfrak{X}'_r(M) : \{K, H\} = 0\}.$$  

Then, we say that the centralizer of a Hamiltonian flow $(\varphi^H_t)_t$ is: (i) quasi-trivial if for every $X_K \in Z^{r}_H(X_H)$ there exists a continuous map $h$ such that $X_H(x) = h(x)X_K(x)$ for every $x \in M$, and (ii) trivial on the connected component $E_{H,e} \subset H^{-1}(e)$ if for every $X_K \in Z^{r}_H(X_H)$ there exists $c \in \mathbb{R}$ such that $X_K(x) = cX_H(x)$ for every $x \in E_{H,e}$.

1.3. Statement of the main results. First we will state the criterion for the triviality of the centralizer of vector fields and flows.

**Theorem A.** Let $M$ be a compact Riemannian manifold and and let $(X_t)_t$ be the flow generated by $X \in \mathfrak{X}'(M)$, for some integer $r \geq 1$. Assume that $\Lambda \subset M$ is a compact, $(X_t)_t$-invariant and transitive subset. If the set of hyperbolic periodic orbits of $(X_t)_t$ is dense in $\Lambda$ then $Z^1(X | \Lambda)$ is trivial: if $Y \in Z^1(X)$ then the flow $(Y_t)_t$ preserves $\Lambda$ and there exists $c \in \mathbb{R}$ such that $Y|_\Lambda = cX|_\Lambda$.

Theorem A has a first application to non-uniformly hyperbolic flows. Assume that $(X_t)_t$ is the flow generated by $X \in \mathfrak{X}'(M)$ and let $\mu$ be a non-atomic and $(X_t)_t$-invariant probability measure. It follows from Oselechts theorem ([26]) that for $\mu$-almost every $x$ there exists a $DX_t$-invariant decomposition $T_xM = E^0 x \oplus E^1 x \oplus \cdots \oplus E^k x(x)$, called the Oselechts splitting, where $E^0 x$ is the one-dimensional subspace generated by $X(x)$ and there are well defined real numbers

$$\lambda_i(X, x) := \lim_{t \to \pm \infty} \frac{1}{t} \log \|DX_t(x) v_i\|, \quad \forall v_i \in E^1 x \setminus \{0\} \text{ and } 1 \leq i \leq k(x)$$

called the Lyapunov exponents associated to the flow $(X_t)_t$ and $x$. It is well known that if $\mu$ is ergodic then the Lyapunov exponents are almost everywhere constant and are denoted by $\lambda_i(X, \mu)$, $1 \leq i \leq k$. Such an ergodic probability measure $\mu$ is
hyperbolic if \( \lambda_i(X, \mu) \neq 0 \) for every \( 1 \leq i \leq k \). Given a \( C^2 \)-diffeomorphism \( f \), the support of any non-atomic, \( f \)-invariant, ergodic and hyperbolic probability measure is transitive and contains a dense set of periodic orbits (cf. [18, Theorem 4.1]). In the case of vector fields this result is a consequence of [20]. We refer the reader to [7] for an excellent account on non-uniform hyperbolicity. Then, Theorem A has the following immediate consequence:

**Corollary 1.** Let \( M \) be a compact Riemannian manifold and let \((X_t)\) be the flow generated by \( X \in \mathfrak{X}^r(M) \), for some integer \( r \geq 2 \). If \( \mu \) is a non-atomic \((X_t)\)-invariant, ergodic and hyperbolic probability measure then \( Z^1(X |_{\text{supp} \mu}) \) is trivial.

We note that the assumptions of Theorem A are also satisfied by homoclinic classes. Indeed, a homoclinic class \( \Lambda \) is transitive and it follows from Birkhoff-Smale’s theorem that hyperbolic periodic orbits are dense (see e.g. [17]). Thus we also get the following immediate consequence.

**Corollary 2.** Assume that \( r \geq 1 \), \( X \in \mathfrak{X}^r(M) \), \( p \in M \) is a hyperbolic critical element for \( X \) and \( \Lambda = H(p) \) is a homoclinic class for \( X \). Then \( Z^1(X |_{\Lambda}) \) is trivial.

**Remark 1.1.** Theorem A and Corollary 2 do not admit a counterpart for discrete time dynamics. Indeed, there are linear Anosov automorphisms on \( T^n \) (hence transitive, with a dense set of periodic orbits, all of them hyperbolic) that do not have trivial centralizer (see e.g. [30]). Nevertheless, any Anosov flow (e.g. suspensions of an Anosov diffeomorphisms) satisfies the hypothesis of Theorem A and, consequently, has trivial centralizer.

In order to state some consequences of Theorem A and Corollary 2 we recall that a subset \( R \) of a topological space \( E \) is called **Baire residual** if it contains a countable intersection of open and dense sets, and it is called **meager** if it is the complementary set of a Baire residual subset. It follows from Baire category theorem that if \( E \) is a Baire space then all residual subsets are dense in \( E \). Given \( r \geq 1 \), let \( \mathfrak{X}^r(M) \) denote the space of \( C^r \)-volume preserving vector fields on \( M \).

**Corollary 3.** Let \( r \geq 1 \) and let \( M \) be a compact and connected Riemannian manifold. There exists a \( C^r \)-residual subset \( \mathcal{R} \subset \mathfrak{X}^r(M) \) such that if \( X \in \mathcal{R} \), \( Y \in Z^1(X) \) and \( \Lambda \subset \text{Per}(X) \) is a transitive invariant set then there exists \( c \in \mathbb{R} \) so that \( Y = cX \) on \( \Lambda \). Moreover, \( C^1 \)-generic vector fields in \( \mathfrak{X}^1_\mu(M) \) have trivial \( C^1 \)-centralizer.

The second assertion in Corollary 3 is the counterpart of the results in [8] for \( C^1 \)-volume preserving vector fields.

Now we draw our attention to the existence of \( C^1 \)-open sets of vector fields with trivial centralizer. Kato and Morimoto [15] used the notion of topological stability to prove that all \( C^1 \)-Anosov flows have quasi-trivial centralizer (hence transitive Anosov flows have trivial centralizer). In [36], Sad used a linearization technique to prove that the centralizer is trivial for a \( C^\infty \)-open and dense set of \( C^\infty \)-Axiom A flows with the strong transversality condition. Theorem A can also be used
to exhibit new $C^1$-open sets of vector fields that have non-uniformly hyperbolic attractors on which the centralizer is trivial. These are open sets of vector fields that exhibit Lorenz attractors (we refer the reader to [1] for definitions and a large account on Lorenz attractors). More precisely we obtain the following:

**Corollary 4.** Let $U \subset \mathcal{X}^1(\mathbb{R}^3)$ be a $C^1$-open set of vector fields and let $V \subset \mathbb{R}^3$ be an open ellipsoid containing the origin such that every $X \in U$ exhibits a Lorenz attractor $\Lambda_X = \bigcap_{t \geq 0} \bar{X}_t(V)$. Then $\mathcal{Z}^1(X |_{\Lambda_X})$ is trivial for every $X \in U$: for any $Y \in \mathcal{Z}^1(X)$ the flow generated by $Y$ preserves the $(X_t)_t$-invariant set $\Lambda_X$ and there exists $c \in \mathbb{R}$ such that $Y|_{\Lambda_X} = cX|_{\Lambda_X}$.

The previous result is complementary to [9, Theorem A] where the authors proved that a $C^1$-open and $C^\infty$-dense subset of vector fields $\mathcal{X}^\infty(M^3)$ that exhibit Lorenz attractors have trivial centralizer on their topological basin of attraction.

**Remark 1.2.** The argument in the proof of Corollary 4 consists of checking that every singular-hyperbolic attractor, at least for three-dimensional flows, is a homoclinic class (see e.g. Theorem 6.8 in [1]). Hence, the triviality of the centralizer stated in Corollary 4 also holds for two other classes of Lorenz attractors. Indeed, Bautista and Rojas [4] proved that the contracting Lorenz attractors (also called Rovella attractors) are homoclinic classes, while Metzger and Morales [23] proved that the multidimensional Lorenz attractors are also homoclinic classes (we refer the reader to [4, 23] for the definitions of the attractors).

In what follows we discuss some applications of our result in the case of Hamiltonian vector fields. In this setting, the triviality of the centralizer is not immediate from the denseness of periodic orbits whose periods are isolated. Indeed, given a Hamiltonian $H \in C^2(M, \mathbb{R})$, any $K \in C^2(M, \mathbb{R})$ such that $X_K \in \mathcal{Z}_H^1(X_H)$ is a first integral for the flow $(\varphi^H_t)_t$ (cf. Subsection 1.2.3). However is not clear a priori that $K$ preserves the level sets $H^{-1}(e)$ and, even if this is the case, the first integrals $H$ and $K$ could be independent. Here we prove that $C^2$-generic Hamiltonians have trivial centralizer. More precisely:

**Corollary 5.** Let $(M^{2n}, \omega)$ be a compact symplectic Riemannian manifold. If $n \geq 2$ then there exists a residual subset $R \subset C^2(M, \mathbb{R})$ such that the following holds: for every $H \in \mathbb{R}$ there exists a residual and full Lebesgue measure subset $R_H \subset H(M)$ of energies such that if $e \in R_H$ then the centralizer of the Hamiltonian flow $(\varphi^H_t)_t$ is trivial on each connected component $\mathcal{E}_{H,e} \subset H^{-1}(e)$.

As Anosov energy levels persist by $C^1$-perturbations of the vector field and volume preserving Anosov flows are transitive, the following is an immediate consequence from Theorem A:

**Corollary 6.** Let $(M^{2n}, \omega)$ be a compact symplectic Riemannian manifold with $n \geq 2$, let $H_0 \in C^r(M, \mathbb{R})$ ($r \geq 2$) and let $e_0 \in \mathbb{R}$ be such that the Hamiltonian flow $(\varphi^H_0)_t$, restricted to a connected component $\mathcal{E}_{H_0,e_0} \subset H_0^{-1}(e_0)$ is Anosov. There is a $C^r$-open neighborhood $U$ of $H_0$ and a open neighborhood $V \subset \mathbb{R}$ of $e_0$ such that for every $(H,e) \in U \times V$ the flow $(\varphi^H_t)_t$ on has trivial centralizer on the connected component $\mathcal{E}_{H,e} \subset H^{-1}(e)$ obtained as an analytic continuation of $\mathcal{E}_{H_0,e_0}$.

One should note that in the case of surfaces one cannot avoid the presence of elliptic islands and, in particular, the triviality of the centralizer cannot be expected to hold in general (cf. Example 2.3).
We proceed by stating a result on the $C^0$-centralizer of continuous flows. Despite the fact that the proof of Theorem A makes use of some topological arguments, the $C^1$-smoothness of the vector field (and the hyperbolicity of a dense subset of periodic orbits) is used to assure that the invariant set $A$ is preserved by elements in the centralizer, and that hyperbolic periodic orbits are also preserved (cf. Lemma 3.1 and the argument following it). For general continuous flows, invariant sets may not be preserved by elements of the centralizer (cf. Example 2.3). Combining arguments used in the proof of Theorem A and in [9] we overcome this difficulty and obtained the following result:

**Theorem B.** Let $M$ be a compact and connected Riemannian manifold without boundary, and let $(X_t)_t$ be a continuous flow such that:

1. $(X_t)_t$ has at most finitely many equilibrium points,
2. $\text{Per}((X_t)_t)$ is dense in $M$,
3. $(X_t)_t$ is transitive, and
4. for every $T > 0$ the set of periodic orbits of $(X_t)_t$ with period smaller than $T$ form a subset of $M$ with finitely many disjoint closed curves.

Then $Z^0((X_t)_t)$ is trivial.

Finally, it is natural to ask whether our results extent to the setting of transitive and locally free $\mathbb{R}^d$-actions with a dense set of closed orbits. Although some open classes of $\mathbb{R}^d$-actions can be obtained as suspensions of $\mathbb{Z}^d$-actions (see e.g. [9] and references therein), to the best of our knowledge, the existence of closed orbits is itself a wide open problem in this context with a single contribution in the case of $\mathbb{R}^2$ actions on 3-manifolds [35].

This paper is organized as follows. In Section 2 we provide some examples that illustrate our main results. Section 3 is devoted to the proof of the main results. Indeed, Theorem A, its consequences (Corollaries 3, 4, 5 and 6) and Theorem B are proven in Subsection 3.1, Subsection 3.2 and Subsection 3.3, respectively.

2. Examples

In this section we give some examples that illustrate our results. In rough terms, the tubular flowbox theorem ensures that a smooth vector field is conjugated to a constant vector field in small neighborhoods of regular orbits (see the precise statement in Example 2.1 below). In particular, the following simple example of constant vector fields describes the possible local symmetries of smooth vector fields.

**Example 2.1.** Let $X \in \mathfrak{X}^r(\mathbb{R}^n)$ be the constant vector field $X(x) = (1, 0, \cdots, 0)$. If $Y \in Z^r(X)$ is written in coordinates as $Y(x) = (Y_1(x), \cdots, Y_n(x))$ and the vector fields $X$ and $Y$ commute then, in local coordinates,

$$0 = [X, Y] = \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} \left( X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) \right] \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial Y_i}{\partial x_1} \frac{\partial}{\partial x_i} = \left( \frac{\partial Y_1}{\partial x_1}, \frac{\partial Y_2}{\partial x_1}, \cdots, \frac{\partial Y_n}{\partial x_1} \right).$$

In other words, $X$ and $Y$ commute if and only if $\frac{\partial Y_i}{\partial x_1} = 0$ for all $1 \leq i \leq n$ or, equivalently, the vector field $Y$ does not depend on the $x_1$-coordinate, hence it is constant along the orbits of the constant vector field.
If $M$ is a Riemannian manifold of dimension $n$, $Z \in \mathfrak{X}'(M)$, $r \geq 1$, and $p \in M$ is a regular point for $Z$, the tubular flowbox theorem (cf. [28]) assures the existence of a neighborhood $V_p$ of $x$ in $M$ and a $C^r$-diffeomorphism $g : V_p \rightarrow h(V_p) \subset \mathbb{R}^n$ so that $g_0Z = X\big|_{h(V_p)}$. If the vector field $\tilde{Z} \in \mathfrak{X}'(M)$ commutes with $Z$ on $V_p$ then $0 = [Z, \tilde{Z}] = [g_0Z, g_0\tilde{Z}] = [X, g_0\tilde{Z}]$. Here we used that Lie brackets are invariant by induced vector fields (see e.g. [19, Corollary 8.31]). The latter implies that the vector field $g_0\tilde{Z}$ does not depend on the $x_1$-coordinate in the local coordinates of $g(V_p) \subset \mathbb{R}^n$.

Now we note that commuting Hamiltonian vector fields may not preserve level sets, which justifies the definition of the Hamiltonian centralizer.

**Example 2.2.** Consider the space $\mathbb{R}^2$ endowed with the usual symplectic form $\omega = dx \wedge dy$. Then, it is not hard to check that for any $C^{r+1}$ Hamiltonian $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ the Hamiltonian vector field $X_H \in \mathfrak{X}'(M)$ can be written as $X_H(x,y) = \left(-\frac{\partial H(y,x)}{\partial y}, \frac{\partial H(y,x)}{\partial x}\right)$, for every $(x,y) \in \mathbb{R}^2$. It is not hard to check that the Hamiltonian flows $(\varphi^H_t)_t$, $(\varphi^K_t)_t$ generated by $H, K : \mathbb{R}^2 \rightarrow \mathbb{R}$, $H(x,y) = x$ and $K(x,y) = y$, commute. However, the flow $(\varphi^K_t)_t$ does not preserve level sets of $(\varphi^H_t)_t$. In fact, the corresponding Poisson bracket is given by

$$\{H, K\} = \frac{\partial H}{\partial x} \frac{\partial K}{\partial y} - \frac{\partial H}{\partial y} \frac{\partial K}{\partial x} = 1 \neq 0.$$

The triviality of the centralizer cannot be expected in general even for vector fields on surfaces. One can produce an example on the sphere $S^2$ by a simple modification of the following vector field on the plane.

**Example 2.3.** Consider the linear vector field in $\mathbb{R}^2$ given by $X(x,y) = (-y, x)$, that generates the flow $(X_t)_{t \in \mathbb{R}}$ such that $X_t$ is the rotation of angle $t$ in $\mathbb{R}^2$. If $Z(x,y) = (ax + by, cx + dy)$ is a linear vector field in $\mathbb{R}^2$ that commutes with $X$ then

$$0 = [X, Z] = \left(\begin{array}{c}
X_1 \frac{\partial Z_1}{\partial x} - X_1 \frac{\partial X_1}{\partial x} + X_2 \frac{\partial Z_1}{\partial y} - X_2 \frac{\partial X_1}{\partial y} \\
X_1 \frac{\partial Z_2}{\partial x} - X_1 \frac{\partial X_2}{\partial x} + X_2 \frac{\partial Z_2}{\partial y} - X_2 \frac{\partial X_2}{\partial y}
\end{array}\right),$$

that is, $a = d$ and $c = -b$. Thus, the vector field $X$ commutes with the family of linear vector fields

$$Z_{a,b}(x,y) = (ax + by, -bx + ay) \quad (2.1)$$

for every $(x,y) \in \mathbb{R}^2$ $(a,b \in \mathbb{R})$. The centralizer of the linear vector field $X$ contains also nonlinear vector fields. For instance, it is not hard to check that the vector field $Z_{a,b}(x,y)$...
field \( Y \in \mathfrak{X}^\infty(\mathbb{R}^2) \) given by \( Y(x, y) = (y + x(1 - x^2 - y^2), -x + y(1 - x^2 - y^2)) \) is such that \([X, Y] = 0\). This shows that the one parameter group of rotations on \( \mathbb{R}^2 \),

which correspond to the solutions of the vector field \( X \), are rotational symmetries for the solutions of the flow generated by \( Y \in Z^1(X) \) (see Figure 2). In particular, it is clear that invariant sets may not be preserved by elements of the centralizer. For instance, any annulus \( \mathbb{D} \) centered at the origin is preserved by the flow \((X_t)_t\) while the flow generated by the vector field \( Y(x, y) = (x, y) \) commutes with \((X_t)_t\) but does not preserve \( \mathbb{D} \).

In the next example we consider the Hamiltonian centralizer of a linear center in \( \mathbb{R}^2 \).

**Example 2.4.** Consider the linear vector field \( X \in \mathfrak{X}^1(\mathbb{R}^2) \) given in the previous example. It is a Hamiltonian vector field: \( X = X_H \) for \( H \in C^2(\mathbb{R}^2, \mathbb{R}) \) given by \( H(x, y) = \tfrac{1}{2}(x^2 + y^2) \). If \( Y \in Z^1(X) \), there exists \( K \in C^2(\mathbb{R}^2, \mathbb{R}) \) such that \( Y := Y_K = (-\frac{\partial K}{\partial y}, \frac{\partial K}{\partial x}) \) and

\[
0 = \{H, K\} = \frac{\partial H}{\partial x} \frac{\partial K}{\partial y} - \frac{\partial H}{\partial y} \frac{\partial K}{\partial x} = \frac{\partial K}{\partial y} - y \frac{\partial K}{\partial x} = \bigg\langle (x, y), \left( \frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x} \right) \bigg\rangle.
\]

Hence, the vectors \((x, y)\) and \( \left( \frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x} \right) \) are orthogonal. As the vector field \( X_H(x, y) = (-y, x) \) is also orthogonal to \((x, y)\) then the vector fields \( X_H \) and \( X_K \) are collinear and, consequently, there exists \( \kappa : \mathbb{R}^+ \to \mathbb{R} \) continuous such that \( X_K(x, y) = \kappa(\|(x, y)\|) \cdot X_H(x, y) \) for all \((x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}\). In particular, since the orbits of the flow \((\varphi^K_t)_t\) are circles centered at the origin, the orbit of each point \((x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}\) by the flow \((\varphi^K_t)_t\) is either a singularity, in which case all points in a circle \( \gamma \) centered at the origin passing through \((x, y)\) are also singularities, or is periodic and the orbit coincides with the circle \( \gamma \). Moreover, \((0, 0)\) is a singularity for the vector field \( X_K \). Hence, the Hamiltonian centralizer of \( X_H \) is smaller than its centralizer in the space of all vector fields described in Example 2.3 (since there are non-volume preserving flows in the family \((2.1)\)). Nevertheless, the centralizer \( Z^1(X_H) \) is not trivial as, for instance, the Hamiltonian \( K(x, y) = \frac{1}{4}(x^2 + y^2)^2 \) satisfies \( X_K(x, y) = \kappa(\|(x, y)\|)X_H(x, y) \), where \( \kappa(z) = 2z^2 \) tends to zero as \( z \to 0^+ \), is continuous but it is not constant.
3. Proofs

3.1. Proof of Theorem A. Fix an integer $r \geq 1$ and $X \in \mathcal{X}^r(M)$, and take $Y \in \mathcal{Z}^1(X)$. Assume that the vector fields $X$ and $Y$ generate the flows $(X_t)_t$ and $(Y_s)_s$, respectively. From the commutative relation $X_t \circ Y_s = Y_s \circ X_t$ for all $t, s \in \mathbb{R}$ it follows that the diffeomorphism $Y_s$ is a conjugation between the flow $(X_t)_t$ with itself, for every $s \in \mathbb{R}$. Hence, if $p \in M$ is a periodic point of period $\pi(p) > 0$ for $(X_t)_t$ ($\gamma_p$ denotes its orbit) and $s \in \mathbb{R}$ is arbitrary then

$$X_{\pi(p)}(Y_s(p)) = Y_s(X_{\pi(p)}(p)) = Y_s(p). \quad (3.1)$$

We make use of the following simple lemma.

**Lemma 3.1.** Let $M$ be a compact Riemannian manifold and assume that the $C^1$ flow $(Y_s)_s$ on $M$ belongs to the $C^1$-centralizer of the $C^1$-flow $(X_t)_t$. Then, for every $s \in \mathbb{R}$ and every hyperbolic critical element $p \in \text{Crit}((X_t)_t)$ one has $Y_s(\gamma_p) = \gamma_p$.

**Proof.** Let $p \in \text{Per}((X_t)_t)$ be a hyperbolic periodic point of prime period $\pi(p) > 0$ and take $s \neq 0$. Differentiating $X_{\pi(p)} \circ Y_s = Y_s \circ X_{\pi(p)}$ we obtain

$$DX_{\pi(p)}(Y_s(p)) DY_s(p) = DY_s(X_{\pi(p)}(p)) DX_{\pi(p)}(p) = DY_s(p) DX_{\pi(p)}(p)$$

which shows, since $Y_s$ is a diffeomorphism, that $DX_{\pi(p)}(p)$ and $DX_{\pi(p)}(Y_s(p))$ are linearly conjugate. This proves that $(Y_s(p))_{s \in \mathbb{R}}$ is a family of hyperbolic periodic points of period $\pi(p)$ for the flow $(X_t)_t$. Since hyperbolic periodic orbits of the same period are isolated we conclude that $Y_s(\gamma_p) = \gamma_p$ for every $s \in \mathbb{R}$, which proves the lemma. \hfill \Box

The previous lemma assures that every $Y \in \mathcal{Z}^1(M)$ preserves the set of hyperbolic periodic orbits in $\Lambda$ (and so it preserves $\Lambda$, if $\Lambda$ admits a dense subset of periodic orbits).

Now, as vector fields $X$ and $Y$ are collinear on $\gamma_p$, for all periodic points $p \in \Lambda$, we use that the periodic orbits of $(X_t)_t$ are dense in $M$ to derive that the vector fields $X$ and $Y$ are collinear on $\Lambda \setminus \text{Sing}(X)$. More precisely:

**Proposition 3.2.** Let $r \geq 1$ and let $X \in \mathcal{X}^r(M)$ generate a flow $(X_t)_t$ so that $\Lambda \subset M$ is a compact, $(X_t)_t$-invariant and transitive subset with a dense subset of hyperbolic periodic orbits. For any $Y \in \mathcal{Z}^1(X)$ the flow $(Y_t)_t$ preserves $\Lambda$ and there exists a map $h : \Lambda \setminus \text{Sing}(X) \to \mathbb{R}$ such that $Y(x) = h(x) \cdot X(x)$ for all $x \in \Lambda \setminus \text{Sing}(X)$. Moreover, the following properties hold:

(a) $h$ is uniquely defined,

(b) $h$ is constant along regular orbits of $X$, and

(c) $h$ is continuous.

**Proof.** Fix $Y \in \mathcal{Z}^1(X)$ arbitrary. We know that $\Lambda$ is preserved by the flow $(Y_t)_t$. First, suppose by contradiction that exists $x_0 \in \Lambda \setminus \text{Sing}(X)$ such that the vectors $X(x_0)$ and $Y(x_0)$ are linearly independent. Consider the continuous map $r : \Lambda \setminus \text{Sing}(X) \to T_x M$ given by

$$r(x) = Y(x) - \frac{\langle Y(x), X(x) \rangle}{\langle X(x), X(x) \rangle} X(x),$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric of $M$ at $T_x M$. As $r(x_0) \neq 0$, by continuity there exists an open neighborhood $V_{x_0} \subset \Lambda$ of $x_0$ such that $r(x) \neq 0$ for all $x \in V_{x_0}$. This cannot occur because $r(x) = 0$ for all $x \in \Lambda \cap \text{Per}((X_t)_t)$.
and \( \text{Per}(X_t) \) is dense in \( \Lambda \). Therefore we conclude that \( r \) is identically zero on \( \Lambda \setminus \text{Sing}(X) \) or, in other words, the vector fields \( X \) and \( Y \) are collinear on \( \Lambda \setminus \text{Sing}(X) \). In consequence, there exists a map \( h : \Lambda \setminus \text{Sing}(X) \to \mathbb{R} \) such that \( Y(x) = h(x) \cdot X(x) \) for all \( x \in \Lambda \setminus \text{Sing}(X) \).

We proceed to prove properties (a), (b) and (c). The conclusion of (a) is immediate. Indeed, if there are \( h_1, h_2 : \Lambda \setminus \text{Sing}(X) \to \mathbb{R} \) such that \( h_1(x) X(x) = Y(x) = h_2(x) X(x) \), then \( (h_1(x) - h_2(x)) X(x) = 0 \) for all regular points \( x \in \Lambda \setminus \text{Sing}(X) \). Thus \( h_1(x) = h_2(x) \) for all \( x \in \Lambda \setminus \text{Sing}(X) \) and \( h \) is uniquely defined.

Let us prove now that \( h \) is constant along regular orbits of \( X \). Given a regular point \( p \) for \( X \), we have that
\[
Y(X_t(p)) = h(X_t(p)) X(X_t(p)) \quad \text{for every } t \in \mathbb{R}. \tag{3.2}
\]
Then, differentiating the relation \( Y_s \circ X_t = X_t \circ Y_s \) with respect to \( s \) (at \( s = 0 \)) we conclude that
\[
Y(X_t(p)) = DX_t(p) Y(p) = DX_t(p)[h(p) X(p)]
= h(p) DX_t(p) X(p) = h(p) X(X_t(p)) \tag{3.3}
\]
for every \( t \in \mathbb{R} \). Since \( X_t(p) \) is a regular point for \( X \), equations (3.2) and (3.3) show that the reparametrization \( h \) is constant along the orbits of \( X_t \). In other words, \( h(x) = h(X_t(x)) \) for all \( x \in \Lambda \setminus \text{Sing}(X) \) and \( t \in \mathbb{R} \), which proves (b).

We are left to prove the continuity of the reparametrization \( h \). First we observe that \( Y_t(x) = X_{h_t(x)}(x) \) for all \( x \in \Lambda \setminus \text{Sing}(X) \). This is a simple consequence of the uniqueness of solution of the ordinary differential equation \( u' = Y(u) \) and the fact that \( h \) is constant along the orbits of \( (X_t)_t \).

Assume by contradiction that \( h \) is not continuous at \( x \in \Lambda \setminus \text{Sing}(X) \). Then there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) convergent to \( x \) such that \( \inf_{n \in \mathbb{N}} |h(x_n) - h(x)| > 0 \). First observe that, as the vector fields \( X \) and \( Y \) are continuous on the compact \( M \), \( X(x) = \lim_{n \to \infty} X(x_n) \neq 0 \) and \( |h(x_n)| = \|Y(x_n)\|/\|X(x_n)\| \), the sequence \( (h(x_n))_n \) of real numbers is bounded. Moreover, since \( x \) is a regular point, the tubular flowbox theorem assures that there exists \( \varepsilon_x > 0 \), an open neighborhood \( V_x \subset M \) of \( x \), and a \( C^r \)-diffeomorphism \( g : V_x \to (-\varepsilon_x, \varepsilon_x) \times B(0, \varepsilon) \subset \mathbb{R} \times \mathbb{R}^{n-1} \) so that \( g_s X(y) = (1, 0, \ldots, 0) \) for every \( y \in V_x \). Choose any sequence \( (s_n)_n \) of real numbers so that \( (s_n, x_n) \to (s, x) \in (\mathbb{R} \setminus \{0\}) \times (\Lambda \setminus \text{Sing}(X)) \), that \( |h(x) \cdot s| < \frac{\varepsilon_x}{2} \) and \( |h(x_n) \cdot s_n - h(x) \cdot s| \leq \frac{\varepsilon_x}{2} \) for all \( n \in \mathbb{N} \). By construction, \( \delta_0 := \inf_{n \in \mathbb{N}} |h(x_n) \cdot s_n - h(x) \cdot s| > 0 \). Since the points \( X_{h(x_n)\cdot s_n}(x) \) and \( X_{h(x)\cdot s}(x) \) lie in the same piece of orbit in the tubular flowbox chart then there exists \( \delta_1 > 0 \) (depending on \( \delta_0 \)) so that \( d(X_{h(x_n)\cdot s_n}(x), X_{h(x)\cdot s}(x)) \geq \delta_1 > 0 \) for all \( n \in \mathbb{N} \). Therefore,
\[
\delta_1 \leq d(X_{h(x_n)\cdot s_n}(x), X_{h(x)\cdot s}(x))
\leq d(X_{h(x_n)\cdot s_n}(x), X_{h(x_n)\cdot s_n}(x_n)) + d(X_{h(x_n)\cdot s_n}(x_n), X_{h(x)\cdot s}(x))
= d(X_{h(x_n)\cdot s_n}(x), X_{h(x_n)\cdot s_n}(x_n)) + d(Y_{s_n}(x_n), Y_s(x))
\]
for all \( n \in \mathbb{N} \). The second term in the right hand side above is clearly convergent to zero as \( n \) tends to infinity by the continuous dependence on initial conditions of the flow \( (Y_s)_s \). Since \( (h(x_n))_n \) is bounded, choosing a convergent subsequence \( (h(x_{n_k}))_k \) we conclude that
\[
0 < \delta_1 \leq \lim_{k \to \infty} d(X_{h(x_{n_k})\cdot s_{n_k}}(x), X_{h(x_{n_k})\cdot s_{n_k}}(x_{n_k})) + d(Y_{s_{n_k}}(x_{n_k}), Y_s(x)) = 0
\]
leading to a contradiction. This proves item (c) and completes the proof of the proposition.

We are now in a position to complete the proof of Theorem A. The previous argument shows that there exists a continuous map \( h : \Lambda \setminus Sing(X) \to \mathbb{R} \) that is constant along orbits of \((X_t)_t\), such that \( Y(x) = h(x) \cdot X(x) \) for all \( x \in \Lambda \setminus Sing(X) \). Using that the flow \((X_t)_t\) is transitive on \( \Lambda \), there exists \( x_0 \in \Lambda \setminus Sing(X) \) such that \( \{ X_t(x_0) : t \in \mathbb{R}_+ \} = \Lambda \). As \( h \) is constant along the orbits of \((X_t)_t\) we conclude that \( h(x) = h(x_0) \) for every \( x \in \Lambda \setminus Sing(X) \). Take \( c = h(x_0) \). The later implies that the vector fields \( Y \) and \( cX \) coincide in an open and dense subset of \( \Lambda \). Thus \( Y = cX \), which proves the triviality of the centralizer on \( \Lambda \).

□

3.2. Applications. In this subsection we will consider the applications to volume preserving flows, Lorenz attractors and Hamiltonian flows.

Proof of Corollary 3. Given \( r \geq 1 \), let \( \mathcal{R} \subset \mathcal{X}^r(M) \) denote the \( C^r \)-residual subset formed by Kupka-Smale vector fields (see e.g. [28]). In particular, all critical elements of vector fields \( X \in \mathcal{R} \) are hyperbolic (hence the ones with the same period are isolated). Thus, if \( \Lambda \subset \text{Per}(X) \) is a transitive invariant set for the flow generated by \( X \in \mathcal{R} \), the argument used in the first part of the proof of Theorem A assures that all the periodic orbits in \( \Lambda \cap \text{Per}(X) \) are preserved by every \( Y \in \mathcal{Z}^1(X) \). In particular, the flow generated by \( Y \) preserves the set \( \Lambda \), and the restriction of the vector field to the set \( \Lambda \) has trivial centralizer, that is, there exists \( c \in \mathbb{R} \) so that \( Y = cX \) on \( \Lambda \). This proves the first assertion of the corollary.

In the case of volume preserving vector fields one has that \( \Omega(X) = M \) for every \( X \in \mathcal{X}_\mu^1(M) \), by Poincaré recurrence theorem. Moreover, there are \( C^1 \)-generic sets \( \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \subset \mathcal{X}_\mu^1(M) \) so that every \( X \in \mathcal{R}_1 \) generates a topologically mixing (hence transitive) flow [5], that periodic orbits of every vector field \( X \in \mathcal{R}_2 \) are dense in \( M \) [31, Theorem 11.1], and that hyperbolic periodic orbits for vector fields \( X \in \mathcal{R}_3 \) are dense in the non-wandering set (cf. [31], [32] and [24, Proposition 3.1]). These facts, together with Theorem A prove the second assertion in the corollary and complete its proof.

□

Proof of Corollary 4. Let \( \mathcal{U} \subset \mathcal{X}^1(\mathbb{R}^3) \) be a \( C^1 \)-open set of vector fields and an open ellipsoid \( V \subset \mathbb{R}^3 \) containing the origin such that every \( X \in \mathcal{U} \) exhibits a geometric Lorenz attractor \( \Lambda_X = \bigcap_{t \geq 0} X_t(V) \). For every \( X \in \mathcal{U} \) there exists a periodic point \( p_X \in \Lambda_X \) so that the Lorenz attractor \( \Lambda_X \) coincides with the homoclinic class \( H(p_X) := W^s(p_X) \cap W^u(p_X) \) (cf. [1, Proposition 3.17]). In particular, the restriction of the flow to the attractor is transitive and, by Birkhoff-Smale’s theorem, admits a dense set of hyperbolic periodic orbits. The corollary is then a direct consequence of Theorem A.

□

Proof of Corollary 5. Assume that \( n \geq 2 \). First we will verify that vector fields of \( C^2 \)-generic Hamiltonians and connected components of generic energy levels satisfy the following: the restriction of the Hamiltonian flow to such connected components is transitive and admits a dense set of hyperbolic periodic orbits. Our starting point is the following result, which assures that transitive level sets for \( C^2 \)-generic Hamiltonians are abundant from both topological and measure theoretical senses.

Theorem 3.3. [6, Theorems 2 and 3] Let \((M^{2n}, \omega)\) be a compact symplectic Riemannian manifold. There is a residual set \( \mathcal{R}_0 \) in \( C^2(M, \mathbb{R}) \) such that for any
In order to prove the corollary we are left to show that hyperbolic periodic orbits are dense in typical level sets of $C^2$-generic Hamiltonians. Although this argument is contained in [6] we include it here for completeness. Robinson [32] proved that there exists a $C^2$ residual subset $\mathcal{R}_1 \subset C^2(M, \mathbb{R})$ of Hamiltonians such that for every $H \in \mathcal{R}_1$ all closed orbits are either hyperbolic or elliptic. Note that, by Birkhoff fixed point theorem, the hyperbolic periodic points are dense on $M$ (cf. [24] Proposition 3.1, Corollary 3.2 and §6). Moreover, by Theorem 11.5 in [31], Newhouse’s result can be strengthened in a way that there exists a $C^2$-generic subset $\mathcal{R}_2 \subset C^2(M, \mathbb{R})$ such that the generic connected components of energy levels for $H \in \mathcal{R}_2$ contains a dense set of hyperbolic periodic orbits. The density of hyperbolic periodic orbits and the fact that the ones displaying the same period are isolated guarantees, as in Theorem A, that for every Hamiltonian $H \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2$, every $X_K \in \mathcal{Z}_1^H(X_H)$ and every connected component $\mathcal{E}_{H,e}$ associated to an energy $e \in \mathcal{R}_H$, there exists $h : \mathcal{E}_{H,e} \to \mathbb{R}$ continuous and invariant along regular orbits of $X_H$, such that $X_K(x) = h(x) \cdot X_H(x)$ for all $x \in \mathcal{E}_{H,e}$. Transitivity at the hypersurface $\mathcal{E}_{H,e}$ assures that the first integral $h$ is constant on $\mathcal{E}_{H,e}$, which completes the proof of Corollary 5. \hfill \Box

3.3. Proof of Theorem B. The proof explores arguments used in the proof of Theorem A and in [9]. Given the continuous flow $(X_t)_t$, denote by $\mathcal{E} = \mathcal{E}((X_t)_t)$ and $\mathcal{P} = \mathcal{P}((X_t)_t)$ the set of equilibrium and period points for $(X_t)_t$, respectively. Given $[a, b] \subset \mathbb{R}$ and $x \in M$ denote $X_{[a,b]}(x) = \{X_t(x) : t \in [a, b]\}$. Our starting point is the following lemma, which plays the role of the tubular flowbox theorem for continuous flows.

**Lemma 3.4.** Let $(X_t)_t$ be a continuous flow. For every $T > 0$ and $x \in M$ there exist open neighborhoods $U$ of the compact set $\{X_t(x) : 0 \leq t \leq T\}$ and $V \subset M$ of $x$, so that $X_{[0,T]}(y)$ is contained in $U$ for every $y \in V$.

**Proof.** This is an immediate consequence of the uniform continuity of the restriction of the continuous flow $X : \mathbb{R} \times M \to M$ to the compact set $M \times [0, T]$. \hfill \Box

Given a flow $(Y_s)_s \in \mathcal{Z}^0((X_t)_t)$, it preserves the set of equilibrium points and periodic orbits of $(X_t)_t$ of every fixed period: if $T \geq 0$ and $X_T(p) = p$ then

$$X_T(Y_s(p)) = Y_s(X_T(p)) = Y_s(p) \quad \text{for every } s \in \mathbb{R}. \quad (3.4)$$

Given $T > 0$ denote by $M_T$ the set of points in $M$ that are either equilibrium points or periodic points of period smaller or equal to $T$ for $(X_t)_t$. By assumptions (1) and (4), the set $M_T$ consists of a finite number of orbits, hence isolated. Moreover, since each map $Y_s$ is a conjugacy of the flow with itself it preserves singularities and periodic orbits of a fixed period. Hence, $Y_s(\gamma_p) = \gamma_p$ for every $p \in \text{Per}((X_t)_t) \cup \mathcal{E}$ and every $s \in \mathbb{R}$. In particular $M \setminus M_T$ is a $(X_t)_t$-invariant and dense subset of $M$. By assumption (1), the set $M_0 := M \setminus (\mathcal{E} \cup \mathcal{P})$ is $(X_t)_t$-invariant and dense. We now prove that the flow $(Y_s)_s$ preserves all orbits of $(X_t)_t$.

**Lemma 3.5.** For every $x \in M$ and every $s \in \mathbb{R}$ the point $Y_s(x)$ belongs to the orbit of $x$ by $(X_t)_t$. 

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Proof. Arguing by contradiction, assume that there are \( x_0 \in M_0 \) and \( t_0 > 0 \) so that the point \( y_0 := Y_{t_0}(x_0) \) does not belong to the orbit of \( x_0 \) by \((X_t)_t\). By uniform continuity of \( Y_{t_0} \), for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) so that \( Y_{t_0}(B(x_0, \delta)) \subset B(y_0, \varepsilon) \).

By assumption (4), equilibrium points for \((X_t)_t\) are preserved by the flow \((Y_s)_s\). Moreover, the latter condition together with the uniform continuity of the flow at compact pieces of orbits (cf. Lemma 3.4) implies that there exists an open neighborhood of the compact set

\[
\{ Y_s(x) : (x, s) \in B(x_0, \delta) \times [0, t_0] \}
\]

formed by regular points for the flow \((X_t)_t\).

We now use assumptions (2) and (4). Since periodic points are dense in \( M \) and the set of periodic orbits of a bounded period are isolated, one can pick a sequence \((x_n)\) of periodic points for \((X_t)_t\) so that \( \pi(x_n) \gg t_0 \) for every \( n \) and \( d(x_n, x_0) \to 0 \).

Since each \( x_n \) is a periodic point, and \((Y_s)_s\) preserves all periodic orbits of \((X_t)_t\), there exists \( t_n \in \mathbb{R} \) such that

\[
X_{[0, t_n]}(x_n) = Y_{[0, t_0]}(x_n) \quad \text{ (3.5)}
\]

(see Figure 3 below). In particular \( X_{t_n}(x_n) = Y_{t_0}(x_0) \in B(y_0, \varepsilon) \) for every large \( n \) and

\[
X_{t_n}(x_n) = Y_{t_0}(x_0) \to Y_{t_0}(x_0) = y_0 \quad \text{as } n \to \infty. \quad \text{ (3.6)}
\]

Assume without loss of generality that \( t_n \geq 0 \) for all \( n \geq 1 \). If the sequence \((t_n)_n\) admits a subsequence \((t_{n_k})_k\) convergent to some \( t_* \in \mathbb{R}_+ \), the continuity of the flows and their time-t maps implies

\[
d(X_{t_{n_k}}(x_{n_k}), Y_{t_0}(x_0)) \to 0 \quad \text{as } k \to \infty
\]

and, on the other hand,

\[
d(X_{t_{n_k}}(x_{n_k}), Y_{t_0}(x_0)) \to d(X_{t_*}(x_0), Y_{t_0}(x_0)) \quad \text{as } k \to \infty.
\]

This proves that \( Y_{t_0}(x_0) = X_{t_*}(x_0) \), which is a contradiction with the choice of \( x_0 \).

Alternatively, the sequence \((t_n)_n\) tends to infinity as \( n \to \infty \). Fix an arbitrary \( \ell > 0 \) and note that \( \ell < t_n \leq \pi(x_n) \) for every large \( n \geq 1 \). By continuity of the flow on compact trajectories we get

\[
X_{[0, \ell]}(x_n) \to X_{[0, \ell]}(x_0) \quad \text{as } n \to \infty. \quad \text{ (3.7)}
\]
Since the flow \((Y_s)_s\) preserves the orbits of the periodic points for \((X_t)_t\), by (3.5) there exists a unique \(s_n \geq 0\) satisfying
\[
X_{[0,\ell]}(x_n) = Y_{[0,s_n]}(x_n). \tag{3.8}
\]
Note that \(\ell < t_0\) implies \(s_n \leq t_0\). We may assume that there exists a subsequence \((x_{n_k})_k\) of periodic points so that \(s_{n_k} \rightarrow s_* := \inf_{k \geq 1} s_{n_k}\) as \(k \rightarrow \infty\). There are still two cases to consider. If \(s_* > 0\) then it follows from equalities (3.7) and (3.8) and the convergence \(Y_{[0,s_{n_k}]}(x_{n_k}) \rightarrow Y_{[0,s_*]}(x_0)\) as \(k \rightarrow \infty\) that \(X_{[0,\ell]}(x_0) = Y_{[0,s_*]}(x_0)\). In other words, the orbit of \(x_0\) by the flow \((Y_s)_s\) preserves the piece of orbit \(X_{[0,\ell]}(x_0)\).

If \(s_* = 0\) then equality (3.8) and uniform continuity of the flow ensures that
\[
X_{[0,\ell]}(x_{n_k}) = Y_{[0,s_{n_k}]}(x_{n_k}) = Y([0,s_{n_k}] \times \{x_{n_k}\}) \rightarrow Y_0(x_0) = x_0
\]
as \(k \rightarrow \infty\). Together with (3.7), this implies that \(x_0 \in E\), which contradicts the choice \(x_0 \in M_0\). This proves that the flow \((Y_s)_s\) preserves all orbits of \((X_t)_t\) and completes the proof of the lemma.

We need the following lemma, inspired by [9], which will play a role similar to Proposition 3.2.

**Lemma 3.6.** Given \(x \in M_0\) there exists a unique function \(\tau(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}\) so that:

1. \(Y_t(x) = X_{\tau(x,t)}(x)\) for every \(x \in M\) and every \(t \in \mathbb{R}\), and
2. \(\tau(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}\) is continuous for every \(x \in M_0\).

**Proof.** First, we prove that a reparametrization \(\tau(x, \cdot)\) as in the statement of the lemma does exists. As \(x \in M_0 = M \setminus (\mathcal{P} \cup \mathcal{E})\) and \(t \in \mathbb{R}\) the point \(Y_t(x)\) belongs to the orbit of \(x\) by \((X_t)_t\), hence there exists \(\tau(x, t) \in \mathbb{R}\) such that \(Y_t(x) = X_{\tau(x,t)}(x)\).

It remains to prove the continuity of \(\tau(x, \cdot)\). Assume that \(t_n \rightarrow t \in \mathbb{R}\). As \(x \not\in \mathcal{E} \cup \mathcal{P}\) then for every \(\zeta > 0\) and every large \(n \geq 1\) we have that \(Y_{t_n}(x)\) belongs to the compact piece of orbit \(\{Y_s(x) : s \in [t - \zeta, t + \zeta]\}\) containing \(x\) and
\[
X_{\tau(x,t_n)}(x) = Y_{t_n}(x) \rightarrow Y_t(x) = X_{\tau(x,t)}(x),
\]
from which we conclude that \(\tau(x, t_n) \rightarrow \tau(x, t)\). This proves the continuity of \(\tau(x, \cdot)\) as claimed. It remains to prove the uniqueness of the reparametrization.

Fix \(x \in M_0\) and assume there are \(\tau_i(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}\) \((i = 1, 2)\) satisfying (1)-(2) above. The equality \(X_\tau(x,t_1)(x) = X_{\tau_2(x,t_2)}(x)\) for every \(t \in \mathbb{R}\) ensures that
\[
X_{\tau_1(x,t) - \tau_2(x,t)}(x) = x \quad \text{for every} \quad x \in \mathbb{R}. \tag{3.9}
\]
Since \(x \in M_0\) then \(\tau_1(x,t) - \tau_2(x,t) = 0\) for all \(t \in \mathbb{R}\). This proves the uniqueness of the reparametrization and concludes the proof of the lemma.

**Lemma 3.7.** There exists a continuous map \(h : M \setminus (\mathcal{P} \cup \mathcal{E}) \rightarrow \mathbb{R}\), constant along regular orbits of \((X_t)_t\), such that \(Y_t(x) = X_{h(x,t)}(x)\) for every \(x \in M \setminus (\mathcal{E} \cup \mathcal{P})\) and every \(t \in \mathbb{R}\).

**Proof.** Let \(\tau\) be given by Lemma 3.6. We claim that there exists a function \(h : M_0 \rightarrow \mathbb{R}\) so that \(h(X_t(x)) = h(x)\) and \(\tau(x, t) = h(x)t\) for every \(x \in M_0\) and \(t \in \mathbb{R}\). Indeed, by Lemma 3.6, given \(x \in M_0\) and \(s, t \in \mathbb{R}\),
\[
X_{t + \tau(x,s)}(x) = X_t(X_{\tau(x,s)}(x)) = Y_s(X_t(x)) = Y_s(X_{h(x,t)}(x)) = X_{\tau(x,h(x,t))}(x).
\]
By uniqueness of the reparameterization, the latter proves that \( x \mapsto \tau(x, s) \) is constant along the orbits of \((X_t)_t\) for every \( s \in \mathbb{R} \). Since \( Y_s(x) \) belongs to the orbit of \( x \) by \((X_t)_t\), we conclude that

\[
X_{\tau(x,t+s)}(x) = Y_{t+s}(x) = Y_t(Y_s(x)) = X_{\tau(y_s(x),t)}(X_{\tau(x,s)}(x)) = X_{\tau(x,t)+\tau(x,s)}(x)
\]

and so \( \tau(x, t+s) = \tau(x, t) + \tau(x, s) \) for all \( x \in M_0 \) and \( s, t \in \mathbb{R} \). Since \( \tau(x, \cdot) \) is continuous (recall item (2) at Lemma 3.6) then it is linear, and there exists \( h(x) \in \mathbb{R} \) so that \( \tau(x, t) = h(x)t \) for all \( x \in M_0 \).

We are left to prove the continuity of \( h \). Assume by contradiction that \( h : M_0 \to \mathbb{R} \) is not continuous. Take \((x_n)_n \subset M_0 \) so that \((x_n)_n\) is convergent to \( x \in M_0 \) but there exists \( \delta > 0 \) and a subsequence \((x_{n_k})_k\) such that \( |h(x_{n_k}) - h(x)| \geq \delta > 0 \) for all \( k \). There are two cases to consider:

(i) **there exists a subsequence, which we denote by \((x_{n_k})_k\) for simplicity, so that \((h(x_{n_k}))_k\) converges to some \( h \neq h(x) \):**

Note that

\[
Y_t(x_{n_k}) = X_{h(x_{n_k})t}(x_{n_k}) = X_{|h(x_{n_k}) - h(x)|t}(X_{h(x)t}(x_{n_k})).
\]

Taking the limit as \( k \to \infty \) in (3.11) we conclude that

\[
Y_t(x) = X_{|h-h(x)|t}(Y_t(x)) \quad \text{for every } t \in \mathbb{R}.
\]

As \( h(x) \neq h \) the later implies that \( Y_t(x) \in \mathcal{P}(x) \) which contradicts the fact that \((Y_s)_s\) preserves the orbit of \( x \) by \((X_t)_t\).

(ii) **\( |h(x_n)| \) tends to infinity as \( n \to \infty \):**

Assume that there exists a subsequence \((x_{n_k})_k\) so that \( \lim_{k \to \infty} h(x_{n_k}) = +\infty \) (the other case is analogous) and fix \( t > 0 \). Given \( \ell > 0 \) arbitrary we have \( h(x_{n_k})t \geq h(x)t + \ell \) provided that \( k \geq 1 \) large. By continuity of the flows \((X_t)_t\) and \((Y_s)_s\) on compact pieces of orbits (Lemma 3.4) we have that

\[
X_{[0,\ell]}(X_{h(x_{n_k})t}(x_{n_k})) = X_{[0,\ell]}(Y_t(x_{n_k})) \to X_{[0,\ell]}(Y_t(x))
\]

and also \( Y_{[0,\ell]}(x_{n_k}) \to Y_{[0,\ell]}(x) \). Taking the limit as \( k \to \infty \) in (3.11) we conclude that the compact piece of orbit \( X_{[0,h(x_{n_k})-h(x)]t}(X_{h(x)t}(x_{n_k})) \) both converge to \( Y_t(x) \), hence obtaining that, on the one hand,

\[
X_{[0,\ell]}(X_{h(x)t}(x_{n_k})) \to X_{[0,\ell]}(X_{h(x)t}(x)) = Y_t(x) \quad \text{as } k \to \infty,
\]

while

\[
X_{[0,\ell]}(X_{h(x_{n_k})t}(x_{n_k})) = X_{[0,\ell]}(Y_t(x_{n_k})) \to X_{[0,\ell]}(Y_t(x)) \quad \text{as } k \to \infty.
\]

Altogether we conclude that \( Y_t(Y_t(x)) = Y_t(x) \) for every \( \ell > 0 \), which implies that \( Y_t(x) \) is a equilibrium point for \((X_t)_t\). Since the latter contradicts the fact that \((Y_s)_s\) preserves \( M_0 \) we conclude that \( h \) is continuous, completing the proof of the lemma.

We are in a position to complete the proof of Theorem B. Indeed, since \((X_t)_t\) is transitive and \( h : M_0 \to \mathbb{R} \) is constant along orbits of \((X_t)_t\) we conclude that \( h(x) = c \) for every \( x \in M_0 \). Therefore \( Y_t(x) = X_{ct}(x) \) for every \( t \in \mathbb{R} \) and every \( x \) on a dense subset of \( M \). Since the flows \((X_t)_t\) and \((Y_s)_s\) are continuous then we conclude that \( Y_t = X_{ct} \) for every \( t \in \mathbb{R} \). This proves the triviality of the \( C^0 \)-centralizer of \((X_t)_t\), as desired.
Acknowledgements: This work is part of the first author’s PhD thesis at UFBA. The second author was partially supported by CNPq-Brazil. The authors are grateful to A. Arbieto and C. Maquera for valuable comments, and indebted to the anonymous referee for the careful reading and a number of suggestions that helped to improved the manuscript.

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