The space of symmetric squares of hyperelliptic curves and integrable Hamiltonian polynomial systems on $\mathbb{R}^4$

V. M. Buchstaber and A. V. Mikhailov

July 27, 2018

Abstract

We construct Lie algebras of vector fields on universal bundles $E^2_{N,0}$ of symmetric squares of hyperelliptic curves of genus $g = 1, 2, \ldots$, where $g = \frac{N^2 - 1}{2}$, $N = 3, 4, \ldots$. For each of these Lie algebras, the Lie subalgebra of vertical fields has commuting generators, while the generators of the Lie subalgebra of projectable fields determines the canonical representation of the Lie subalgebra with generators $L_{2q}$, $q = -1, 0, 1, 2, \ldots$. The Lie algebra $G$ of the ring of symmetric polynomials in $x$ are independent at any point of the variety of regular orbits. At each point of the variety of irregular orbits the operation of convolution of invariants was introduced and basis vector fields on $\text{Sym}^N C$ makes it possible to substantially simplify the formulas and, most importantly, employ the remarkable infinite-dimensional Lie subalgebra $W_{-1}$ of the Witt algebra $W$ in computations. The generators of the Lie algebra $W_{-1}$ are $L_{-2}, L_0, L_2, \ldots$, and the commutation relations have the form $[L_{2q_1}, L_{2q_2}] = 2(q_2 - q_1)L_{2(q_1 + q_2)}$.

For any $N$, the Lie algebra $W_{-1}$ has a faithful canonical representation in the Lie algebra $G(\text{Sym}^N \mathbb{C})$ which maps the generator $L_{2q}$ to the Newton field $L_{2q}^0 = 2 \sum x_i^{q+1} \partial_{x_i}$. The image of this representation belongs to the Lie algebra $W_{-1}(N)$, which has the structure of a free left module over the polynomial ring $\mathbb{C}(\text{Sym}^N \mathbb{C})$. We obtain an explicit expression for the vector fields $V_i$ with symmetric matrix $V_i(x)$ in terms of the Newton fields $L_{2q}^0$ (see Corollary 7). An important role in our calculations is played by a grading of variables and operators.

Introduction

This is an extended version of our paper [1]. In this article we obtain a description of the Lie algebras $G(E^2_{N,0})$ of vector fields on the spaces of the universal bundles $E^2_{N,0}$ (see Definition 2) of the symmetric squares of the hyperelliptic curve

$$V_x = \left\{(X, Y) \in \mathbb{C}^2 : Y^2 = \prod_{i=1}^{N}(X - x_i)\right\}, \quad x = (x_1, \ldots, x_N).$$

The Lie algebra $G(E^2_{N,0})$ contains the Lie subalgebra of fields lifted from the base $\text{Sym}^N \mathbb{C}$, i.e. horizontal and projectable fields (see [2] and [3] p. 337) over the polynomial Lie algebra $G(\text{Sym}^N \mathbb{C})$ (see [4] and [5]) of derivations of the ring of symmetric polynomials in $x_1, \ldots, x_N$.

The Lie algebra $G(\text{Sym}^N \mathbb{C})$ naturally arises and plays an important role in various areas of mathematics and mathematical physics, including the isospectral deformation method [6] and the classical method of separation of variables [7]. In fundamental works (see, e.g., [4] and [5]) as coordinates on $\text{Sym}^N \mathbb{C}$ the elementary symmetric functions $e_1, \ldots, e_N$ were chosen. In [4], in terms of the action of the permutation group $S_N$ on $\mathbb{C}^N$, the operation of convolution of invariants was introduced and basis vector fields on $\text{Sym}^N \mathbb{C}$ were defined, which are independent at any point of the variety of regular orbits. At each point of the variety of irregular orbits these fields generate the stratum of the discriminant hypersurface containing the given point. Zakalyukin’s well-known construction yields basis vector fields $V_i = \sum V_{i,j}(e) \frac{\partial}{\partial e_j}, \quad e = (e_1, \ldots, e_N)$, with symmetric matrix $V_{i,j}$, which are tangent to the discriminant.

In this article we show that the use of the Newton polynomials $p_1, \ldots, p_N$ makes it possible to substantially simplify the formulas and, most importantly, employ the remarkable infinite-dimensional Lie subalgebra $W_{-1}$ of the Witt algebra $W$ in computations. The generators of the Lie algebra $W_{-1}$ are $L_{-2}, L_0, L_2, \ldots$, and the commutation relations have the form $[L_{2q_1}, L_{2q_2}] = 2(q_2 - q_1)L_{2(q_1 + q_2)}$.
In this connection, we introduce variables $y_{2m}$ and $\mathcal{N}_{2k}$ by setting $e_{m} = y_{2m}$ and $p_{k} = \mathcal{N}_{2k}$ and bear in mind that $\deg x_{i} = 2$.

Note that the Jacobi identity in the $\mathbb{C}[\mathcal{N}_{2}, \ldots, \mathcal{N}_{2N}]$-polynomial Lie algebra $W_{-1}(N)$ implies the nontrivial differential relation
\[
\sum_{m=1}^{N} m \left( \mathcal{N}_{2(k+m)} \frac{\partial \mathcal{N}_{2(q+m)}}{\partial \mathcal{N}_{2m}} - \mathcal{N}_{2(q+m)} \frac{\partial \mathcal{N}_{2(k+m)}}{\partial \mathcal{N}_{2m}} \right) = (q - k)\mathcal{N}_{2(k+q+m)}
\]
for the Newton polynomials. We show that there exists a unique representation of the Lie algebra $W_{-1}$ in the Lie algebra of horizontal vector fields $L_{-2}, L_{0}, L_{2}, \ldots$ of the Lie algebroid $\mathcal{G}(\mathcal{E}_{N,0})$ (see Theorem 7). The proof of the uniqueness of this representation uses the fact that, relative to the Lie bracket $[\cdot, \cdot]$, the Lie algebra $W_{-1}$ is generated by $L_{-2}$ and $L_{4}$ satisfying the relation $[L_{2}, [L_{2}, L_{4}]] = 12[L_{4}, [L_{2}, L_{4}]]$.

On the Lie algebroid $\mathcal{G}(\mathcal{E}_{N,0})$ there exist two commuting vertical vector fields. For each point $x$, the restrictions of these fields to $\text{Sym}^{2}(V_{X})$ are the images of obviously commuting fields on $V_{X} \times V_{X}$. In our upcoming publication we shall give an explicit description of such commuting fields on $\text{Sym}^{k}(V_{X})$ for any $k \geq 2$. In Section 4 of this article we give an explicit description of these fields in the case of interest to us, namely, for $k = 2$.

Similar operators on $\text{Sym}^{m}(\mathbb{C}^{2})$ were constructed in [9] on the basis of a construction of the spectral curve and the Poisson structure. A proof of the commutativity of operators in [9] uses a method that differs from ours.

The key results of our work are a formula for the generating series $L(t)$ (see Theorem 7) of the horizontal vector fields $\mathcal{E}_{2,2}, \ldots, \mathcal{E}_{2k}, \ldots$ that determine a representation of the Lie algebra $W_{-1}$ and a commutation formula for the vertical and horizontal vector fields (see Theorem 5).

In Section 6 we construct an $N$-dimensional algebraic variety $W(N)$ in the $(N+2)$-dimensional algebraic variety $\mathcal{E}_{N,0}$ and homeomorphisms $f_{N} : \mathbb{C}^{N+2} \setminus \{ u_{4} = 0 \} \rightarrow \mathcal{E}_{N,0} \setminus W(N)$, where $\{ u_{4} = 0 \}$ is a hyperplane in $\mathbb{C}^{N+2}$ with graded coordinates $u_{2}, u_{4}, v_{N-2}, v_{N} ; y_{2}, \ldots, y_{2(N-2)}$. In the partial case $N = 5$ when we consider a universal space of symmetric squares of hyperelliptic curves of genus 2 we obtain a new proof of a well known result obtained by Dubrovin and Novikov (see section 4 in the review paper [10]). One of the main results of the present work is an explicit construction, which uses the homeomorphisms $f_{N}$, of the polynomial Lie algebras on $\mathbb{C}^{N+2}$ that are determined by the Lie algebroids $\mathcal{G}(\mathcal{E}_{N,0})$, $N = 3, 4, \ldots$ (see Theorem 3).

In Section 11 we give explicit description of integrable Hamiltonian polynomial systems in $\mathbb{R}^{4}$, associated with $\mathcal{E}_{N,0}$. In the cases $N = 3, 4, 5, 7$ represent the systems, their Hamiltonians and solutions.

\section{The Space of Symmetric Squares of Hyperelliptic Curves}

Consider a family of plane curves
\[ V_{N,0} = \{(X, Y ; x) \in \mathbb{C}^{2} \times \mathbb{C}^{N} : \pi(X, Y ; x) = 0 \}, \]  
where
\[ \pi(X, Y ; x) = Y^{2} - \prod_{k=1}^{N} (X - x_{k}). \]

The vector $\xi_{N}(x) = e$ is the parameter set for a curve in family (1). By $V_{N}$ we denote the subfamily of curves satisfying $e_{1} = p_{1} = 0$.

In this paper we use the following grading of variables: $\deg x_{k} = 2$, $k = 1, \ldots, N$, $\deg X = 2$, and $\deg Y = N$. With respect to this grading the polynomial $\pi(X, Y ; x)$ is homogeneous of degree $2N$.

The discriminant variety of family (1) is the algebraic variety
\[ \text{Disc}(V_{N,0}) = \{ \xi_{N}(x) \in \text{Sym}^{N}(\mathbb{C}) : \Delta_{N} = 0 \}, \]  
where $\Delta_{N} = \prod_{i<j}(x_{i} - x_{j})^{2}$.

The discriminant variety $\text{Disc}(V_{N})$ of the family of curves $V_{N}$ is defined similarly. The variety $\text{Disc}(V_{N,0}) \subset \mathbb{C}^{N}$ is the image under the projection $\mathbb{C}^{N} \to \text{Sym}^{N}(\mathbb{C}) = \mathbb{C}^{N}$ of the union of the so-called mirrors, that is, the hyperplanes $\{ x_{i} = x_{j}, i \neq j \}$, and $\text{Disc}(V_{N}) \subset \mathbb{C}^{N-1}$ is the image of the intersection of the space $\mathbb{C}^{N-1} = \{ e \in \mathbb{C}^{N} : e_{1} = 0 \}$ with the union of mirrors.

For $N = 3$, the variety $\text{Disc}(V_{3}) \subset \mathbb{C}^{2}$ in the coordinates $(e_{2}, e_{3})$ is determined by the equation $\Delta_{3} = 27e_{3}^{2} - 4e_{2}^{3} = 0$, i.e., is the well-known swallowtail in $\mathbb{C}^{2}$. In the book [4], it was proposed to refer to the varieties $\text{Disc}(V_{N})$ as generalized swallowtails in $\mathbb{C}^{N-1}$. 

2
Let $\mathcal{B}_{N,0}$ and $\mathcal{B}_N$ denote the open varieties $\mathbb{C}^N \setminus \text{Disc}(V_{N,0})$ and $\mathbb{C}^{N-1} \setminus \text{Disc}(V_N)$, respectively. The curves of the families $V_{N,0}$ and $V_N$ with parameters in the spaces $\mathcal{B}_{N,0}$ and $\mathcal{B}_N$ are said to be nonsingular for obvious reasons. They have genus $[\frac{N+1}{2}]$. For example, in the cases $N = 3$ and $4$, these are elliptic curves.

We set $\hat{\mathcal{E}}_{N,0} = \{(X,Y; x) \in \mathbb{C}^2 \times \mathbb{C}^N : \pi(X,Y; x) = 0, \xi(x) \in \mathcal{B}_{N,0}\}$. The group $S_N$ acts freely on $\hat{\mathcal{E}}_{N,0}$ by permutations of the coordinates $x_1, \ldots, x_N$, and therefore a regular $N!$-sheeted covering $\hat{\mathcal{E}}_{N,0} \to \mathcal{E}_{N,0} = \hat{\mathcal{E}}_{N,0}/S_N$ is defined.

**Definition 1** The universal bundle of nonsingular hyperelliptic curves of family $[4]$ is the bundle $\mathcal{E}_{N,0} \to \mathcal{B}_{N,0} : (X,Y; \xi(x)) \mapsto \xi(x)$.

The space $\mathcal{E}_{N,0}$ is the universal space of nonsingular hyperelliptic curves of genus $[\frac{N+1}{2}]$. The fiber over a point of the base $\mathcal{B}_{N,0}$ is the curve with parameters determined by this point. The universal bundle $\mathcal{E}_N \to \mathcal{B}_N$ of nonsingular hyperelliptic curves is defined similarly.

We set $\hat{\mathcal{E}}^2_{N,0} = \{(X_1,Y_1; X_2,Y_2; x) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^N : \pi(X_k,Y_k; x) = 0, k = 1, 2; X_1 - X_2 \neq 0, \xi(x) \in \mathcal{B}_{N,0}\}$. The group $G = S_2 \times S_N$ acts freely on $\hat{\mathcal{E}}^2_{N,0}$, so that the generator of $S_2$ determines the permutation $(X_1,Y_1) \leftrightarrow (X_2,Y_2)$, and the elements of the group $S_N$ determine permutations of the coordinates of the vector $x$. Therefore, a regular covering $\hat{\mathcal{E}}^2_{N,0} = \hat{\mathcal{E}}^2_{N,0}/G \to \hat{\mathcal{E}}^2_{N,0}$ is defined.

**Definition 2** The universal bundle of symmetric squares of nonsingular hyperelliptic curves of family $[4]$ is the bundle $\mathcal{E}^2_{N,0} \to \mathcal{B}_{N,0} : ([X_1,Y_1; X_2,Y_2]; [x]) \mapsto [x]$, where $[X_1,Y_1; X_2,Y_2] = \xi_2(X_1,Y_1; X_2,Y_2)$, $[x] = \xi(x)$, and $\xi_2 : \mathbb{C}^2 \times \mathbb{C}^2 \to \text{Sym}^2(\mathbb{C}^2)$ is the canonical projection onto the symmetric square of $\mathbb{C}^2$.

The space $\mathcal{E}^2_{N,0}$ is called the universal space of symmetric squares of nonsingular hyperelliptic curves of genus $[\frac{N+1}{2}]$. The fiber over a point of the base $\mathcal{B}_{N,0}$ is the variety $(\text{Sym}^2V) \setminus (X_1 - X_2 = 0)$ with parameters determined by this point.

The universal bundle $\mathcal{E}^2_{N} \to \mathcal{B}_N$ is defined similarly.

## 2 The Lie Algebra of Newton Vector Fields

In the paper [5] the theory of polynomial Lie algebras was constructed. Important examples of such infinite-dimensional Lie algebras are the Lie algebras of vector fields on $\mathbb{C}^N$ and $\mathbb{C}^{N-1}$ tangent to the varieties $\text{Disc}(V_{N,0})$ and $\text{Disc}(V_N)$ and, therefore, the Lie algebras of vector fields on $\mathcal{B}_{N,0}$ and $\mathcal{B}_N$.

In this section we give an explicit description of the Lie algebras $\mathcal{G}_P(N)$ and $\mathcal{G}_P(N)$ of vector fields in coordinates $(p_1, \ldots, p_N)$ and $(p_2, \ldots, p_N)$ determined by the Newton polynomials. We have $\deg x_i = 2$, $i = 1, \ldots, N$. To control grading, we introduce the notation

$$\mathcal{N}_{2k} = p_k(x) = \sum_{i=1}^{N} x_i^k, \quad k = 0, 1, \ldots.$$  

For graded generators of the polynomial ring $\mathcal{P}(\text{Sym}^N(\mathbb{C}))$ we take the polynomials $\mathcal{N}_2, \ldots, \mathcal{N}_{2N}$. Then $\mathcal{P}(\text{Sym}^N(\mathbb{C})) \simeq \mathbb{C}[\mathcal{N}_2, \ldots, \mathcal{N}_{2N}]$.

**Definition 3** The gradient homogeneous polynomial vector fields

$$L^{0}_{2q} = 2 \sum_{i=1}^{N} x_i^{q+1} \partial_{x_i}, \quad q = -1, 0, 1, \ldots,$$  

on $\mathbb{C}^N$ of degree $2q$ are called the Newton derivations of the ring $\mathbb{C}[x_1, \ldots, x_N]$. 

3
Lemma 1 The operators $L^0_{2q}$, $q = -1, 0, 1, \ldots$ are derivations of the ring $\mathbb{C}[N_2, \ldots, N_{2N}]$, and they are uniquely determined by the formula
\[ L^0_{2q} N_{2k} = 2kN_{2(k+q)}, \quad k = 1, 2, \ldots. \tag{4} \]

Corollary 1 The operators $L^0_{2q}$ act on the ring $\mathbb{C}[N_2, \ldots, N_{2N}]$ as
\[ L^0_{2q} = \sum_{k=1}^{N} 2kN_{2(q+k)} \frac{\partial}{\partial N_{2k}}. \tag{5} \]

Lemma 1 and Corollary 1 are verified directly.

Let us write the equation $\prod_{i=1}^{N}(x - x_i) = 0$ in the form $x^N = \sum_{j=1}^{N} (-1)^{j+1} y_{2j} x^{N-j}$.

Lemma 2 For any $k \geq 0$,
\[ x^{N+k} = \sum_{j=1}^{N} (-1)^{j+1} y_{2k,2j} x^{N-j}, \tag{6} \]
where the $\{y_{2k,2j}, k = 1, \ldots\}$, $j = 1, \ldots, N$, are the sets of symmetric functions in $x_1, \ldots, x_N$ with generating series
\[ Y_{2j} = \sum_{k=0}^{\infty} y_{2k,2j} t^k = \frac{1}{E(t)} \sum_{j=0}^{N-j} (-1)^{j} y_{2(j+s)} s^s, \quad E(t) = \prod_{i=1}^{N} (1 - x_it). \tag{7} \]

Proof According to (6), for any $k \geq 0$, we have
\[ \sum_{j=1}^{N} (-1)^{j+1} y_{2(k+1),2j} x^{N-j} = x^{N+k+1} = x \cdot x^{N+k} \]
\[ = y_{2k,2j} \sum_{j=1}^{N} (-1)^{j+1} y_{2k,2j} x^{N-j} + \sum_{j=2}^{N} (-1)^{j+1} y_{2k,2j} x^{N-j+1}. \]

Hence $y_{2(k+1),2N} = y_{2N} y_{2k,2}$ and $y_{2(k+1),2j} = y_{2j} y_{2k,2} - y_{2k,2(j+1)}$, $j < N$. We obtain the following system of equations for the generating series:
\[ Y_{2N} = y_{2N} (1 + tY_2), \]
\[ Y_{2j} = y_{2j} (1 + tY_2) - tY_{2(j+1)}, \quad j = 1, \ldots, N-1. \]

Solving this system, we obtain (7).

Corollary 2 For any $k \geq 0$,
\[ N_{2(N+k)} = \sum_{j=1}^{N} (-1)^{j+1} y_{2k,2j} N_{2(N-j)}, \tag{8} \]
\[ L^0_{2(N+k-1)} = \sum_{j=1}^{N} (-1)^{j+1} y_{2k,2j} L^0_{2(N-j-1)}. \tag{9} \]

Let us introduce the generating series $L^0(t) = \sum_{q=-1}^{\infty} L^0_{2q} t^{q+1}$.

Corollary 3 The following relation holds:
\[ L^0(t) = \frac{1}{E(t)} \sum_{m=1}^{N} E(t; m) L^0_{2(m-2)} t^{m-1}, \tag{10} \]
where
\[ E(t; m) = \sum_{k=0}^{N-m} (-1)^k y_{2k} t^k \quad \text{and} \quad E(t) = E(t; 0). \tag{11} \]
Lemma 3 The following relation holds:

\[ [L_i^0, L_j^0] = 2(q_2 - q_1)L_{2(q_1+q_2)}^0. \] (12)

Proof The required relation follows directly from (6). □

Corollary 4 For all \( k, q \in \mathbb{N} \) and \( n = 1, \ldots, \) the polynomials \( N_{2k}, k = 0, 1, \ldots, \) are related by

\[
\sum_{m=1}^{N} m \left( N_{2(k+m)} \frac{\partial N_{2(q+m)}}{\partial N_{2m}} - N_{2(q+m)} \frac{\partial N_{2(k+m)}}{\partial N_{2m}} \right) = (q - k)N_{2(k+q+n)}.
\] (13)

Proof The required relations follow directly from (5) and (12). □

Example 1 For \( N = 3 \), the polynomials \( N_2, N_4, \) and \( N_6 \) are algebraically independent and \( N_0 = 3 \). For \( k = 0, q = 1, \) and \( n = 3 \), relation (13) gives the Euler differential equation

\[
2N_2 \frac{\partial N_8}{\partial N_2} + 4N_4 \frac{\partial N_6}{\partial N_4} + 6N_6 \frac{\partial N_8}{\partial N_6} = 8N_8.
\]

Lemma 4 For all \( q = -1, 0, 1, \ldots, \)

\[ L_i^0 \Delta(N) = 4\gamma_2(q)\Delta(N), \]

where

\[
\Delta(N) = \prod_{i<j} (x_i - x_j)^2 \quad \text{and} \quad \gamma_2(q) = \sum_{i<j} \frac{x_i^{q+1} - x_j^{q+1}}{x_i - x_j}.
\] (15)

For any \( k \geq 0, \)

\[ \gamma_2(N+k)(N) = \sum_{s=1}^{N-1} (-1)^{s+1}y_{2(k+1),2s}\gamma_2(N-s-1). \]

Proof We have

\[ L_i^0 \Delta(N) = 4\Delta(N) \sum_{k=1}^{N} x_k^{q+1} \partial_k \ln \prod_{i<j} (x_i - x_j) = 4\Delta(N) \left( \sum_{i<j} \frac{x_i^{q+1} - x_j^{q+1}}{x_i - x_j} \right). \]

The formula for \( \gamma_2(N+k)(N) \) follows from (13). □

Example 2 \( \gamma_{-2}(N) = 0, \gamma_0(N) = 2N(N-1), \) and \( \gamma_1(N) = 4(N-1)N_2. \)

Corollary 5 The vector fields \( L_i^0, q = -1, 0, 1, \ldots, \), on \( \mathbb{C}^N \) determine vector fields on \( \text{Sym}^N(\mathbb{C}) \) tangent to the algebraic variety \( \text{Disc}(V_{N,0}) \subset \text{Sym}^N(\mathbb{C}). \)
Theorem 1 The Lie algebra $G_P(N)$ of vector fields on the variety $B_{N,0}$ in the coordinates $N_2,\ldots,N_{2N}$ has the structure of a free $N$-dimensional module over the ring $\mathbb{C}[N_2,\ldots,N_{2N}]$ with generators $L_q^0$, $q = -1, 0, 1, \ldots, N - 2$. The set of generators extends to an infinite set \{L_q^0\}, where the elements $L_q^0$ for $q > N - 2$ are given by $\mathcal{B}$. The operators $L_q^0$ act on $N_{2k}$ by $\mathcal{B}$.

The structure of the Lie algebra $G_{P_1}(N)$ is determined by $\mathcal{B}$ and $\mathcal{A}$, where the $N_{2k}$ for $k > N$ are the polynomials $N_{2k}(N_2,\ldots,N_{2N})$ defined recursively by $\mathcal{A}$.

We set $L_A^0(t) = tE(t)L^0(t)$. According to Corollary 5 operators $L_{A,2(k-2)}^0$ for which

$$L_A^0(t) = \sum_{m=1}^{N} L_{A,2(m-2)}^0 t^m = \sum_{m=1}^{N} E(t;m)L_{A,2(m-2)}^0 t^m$$

are defined.

Lemma 5 The following relation holds: $L_A^0(t) = \sum_{k=1}^{N} t \prod_{j \neq k} (1 - x_j t) \frac{\partial}{\partial x_k}$.

Proof We have

$$L_A^0(t) = tE(t) \sum_{q=-1}^{\infty} L_{2q}^0 t^{q+1} = E(t) \sum_{k=1}^{N} \frac{t}{1 - x_k t} \frac{\partial}{\partial x_k} = \sum_{k=1}^{N} t \prod_{j \neq k} (1 - x_j t) \frac{\partial}{\partial x_k}. $$

\[\square\]

Example 3 $L^0_{A,2} = L_{2}^0$ and $L^0_{A,2(N-2)} = y_{2N}L_{0}^0$.

Lemma 6 The generating polynomial

$$L_A^0(t)E(s) = \sum_{m=1}^{N} (-1)^m (L_A(t)y_{2m})s^m$$

in $s$ is symmetric with respect to the permutation $t \leftrightarrow s$.

Proof We have

$$L_A^0(t)E(s) = tE(t)L^0(t)E(s) = -E(t)E(s) \sum_{i=1}^{N} \frac{t}{1 - x_i t} \frac{s}{1 - x_i s}. $$

\[\square\]

Corollary 6 The action of the operators $L_{A,2(k-2)}^0$, $k = 1,\ldots,N$, on the elementary symmetric polynomials $e_m = y_{2m}$ is given by a symmetric matrix $T_{k,m}^0 = T_{k,m}^0(y_2,\ldots,y_{2N})$, that is,

$$L_{A,2(k-2)}^0 y_{2m} = L_{A,2(m-2)}^0 y_{2k}. $$

Proof We have

$$L_A^0(t)E(s) = (\sum_{k=1}^{N} (-1)^k L_{A,2(k-2)}^0 t^k) \left( \sum_{m=1}^{N} (-1)^m y_{2m}s^m \right) = \sum_{k=1}^{N} \sum_{m=1}^{N} (-1)^{k+m} (L_{A,2(k-2)}^0 y_{2m})t^k s^m. $$

It remains to use Lemma 5.

\[\square\]

The Lie algebra $G_P(N)$ of vector fields on $B_N$ in the coordinates $N_4,\ldots,N_{2N}$ is a Lie subalgebra of $G_{P_1}(N)$. It consists of the fields that leave the ideal $J_2 = (N_2) \subset C[N_2,\ldots,N_{2N}]$ invariant.

We have $L_0^0 N_2 = 2N$ and $L_{2q}^0 N_2 = 2N_{2(q+1)}$. We set $L_0 = L_0^0$ and $L_{2q} = L_{2q}^0 - \frac{1}{N} N_{2(q+1)} L_{0}. By construction L_0 N_2 = 2N_2$ and $L_{2q} N_2 = 0$ for $q \neq 0$.  

\[\square\]
Theorem 2 The Lie algebra $G_p(N)$ of vector fields on $B_N$ in the coordinates $N_1, \ldots, N_{2N}$ has the structure of a free $(N - 1)$-dimensional module over the ring $\mathbb{C}[N_1, \ldots, N_{2N}]$ with generators $L_{2q}$, $q = 0, 1, \ldots, N - 2$. The set of generators extends to an infinite set $\{L_{2q}\}$, and the elements $L_{2q}$ for $q = N + k - 1$, $k \geq 0$, are given by (see \[14\])

\[
L_{2(N+k-1)}^0 = \sum_{j=1}^{N} (-1)^{j+1} y_{2k,2j} L_{2(N-j-1)}^0,
\]

where $y_{2k,2j} \equiv y_{2k,2j} \mod \langle N_2 \rangle$ and the $y_{2k,2j}$ are the polynomials determined by the generating series \[7\].

The structure of the Lie algebra on $G_p(N)$ is introduced directly by the condition that this is a Lie subalgebra of the Lie algebra $G_p(N)$.

Let $L_A(t) = tE(t)L(t)$, where

\[
L(t) = \sum_{q=-\infty}^{\infty} L_{2q} t^{q+1} = \sum_{q=-\infty}^{\infty} \left( L_{2q} - \frac{1}{N} N_{2(q+1)} L_{2-2}^0 \right) t^{q+1}.
\]

Lemma 7 The following relation holds:

\[
L_A(t) = L_A^0(t) + \frac{1}{N} \left( L_{-2} E(t) \right) L_{-2}^0 = \left[ L_0 - \left( 1 - \frac{4}{N} \right) y_2 L_{-2} \right] t^2 + \ldots.
\]

Let us introduce operators $L_{A,2(k-2)}^0$ such that $L_A(t) = \sum_{k=2}^{N} L_{A,2(k-2)} t$.

Corollary 7 The operators $L_{A,2(k-2)}$, $k = 2, \ldots, N$, leave the ideal $\langle N_2 \rangle$ invariant. Their action on the elementary symmetric polynomials $e_m = y_{2m}$ is determined up to the ideal $\langle N_2 \rangle$ by a symmetric matrix $T_{k,m} = T_{k,m}(y_4, \ldots, y_{2N})$.

Proof The proof of this assertion is similar to that of Corollary \[6\].

The Lie algebra $G_p(N)$ of vector fields on $B_N$ in the coordinates $y_4, \ldots, y_{2N}$ has the structure of a free $(N - 1)$-dimensional module over the ring $\mathbb{C}[y_4, \ldots, y_{2N}]$ with generators $L_{A,2(k-2)}$, $k = 2, \ldots, N$. The action of these generators on $y_{2m}$ is given by a symmetric matrix $(T_{k,m}) = (T_{k,m}(N))$.

Remark 1 For each $N$, we give an explicit construction of the fields $L_{A,2(k-2)}$ and the symmetric matrix $(T_{k,m}(N))$. The notation $L_A$ is suggested by Arnold’s monograph \[12\] (see also \[13\]). These fields will be used in Section 9 to construct the Lie algebroids $G(E^2_{k,0})$ and explicitly describe an isomorphism between the Lie algebroid $G(E^2_{k,0})$ and the algebroid constructed in \[17\] from the universal bundle of Jacobians of genus 2 curves.

3 Representations of the Witt Algebra $W_{-1}$ in Lie Algebras with the Structure of a Free $N$-Dimensional Module over the Polynomial Ring

Let us introduce the following notion.

Definition 4 We define an $N$-polynomial Lie algebra $W_{-1}(N)$ as the graded Lie algebra with

- the structure of a free left module over the graded ring $A(N) = \mathbb{C}[v_2, \ldots, v_{2N}]$, $\deg v_{2k} = 2k$;
- an infinite set of generators $L^0_{2q}$, $q = -1, 0, 1, \ldots, \deg L^0_{2q} = 2q$;
operators

The theorem is proved by a direct verification of its statements.

\[ [L^{0}_{2q_1}, L^{0}_{2q_2}] = 2(q_2 - q_1)L^{0}_{2(q_1 + q_2)}, \]
\[ [L_{2q_1}, v_{2k} L_{2q_2}] = v_{2q_1, 2k} L_{2q_2} + v_{2k}[L_{2q_1}, L_{2q_2}], \]
\[ [v_{2k}, L_{2q_1}, v_{2k_2} L_{2q_2}] = v_{2k_1} v_{2q_1, 2k_2} L_{2q_2} - v_{2k_2}[L_{2q_2}, v_{2k_1} L_{2q_1}], \]

where \( v_{2q_1, 2k} \in A(N) \) is a homogeneous polynomial \( v_{2q_1, 2k}(v_2, \ldots, v_{2N}) \) of degree \( 2(q + k) \).

Using the identity \( v_{k_1}(v_{k_2} L_{2q}) = (v_{k_1} v_{k_2}) L_{2q} \) and Leibniz’ rule, we see that the skew-symmetric operation \([\cdot, \cdot]\) on the Lie algebra \( W_{-1}(N) \) is completely determined by the set of homogeneous polynomials \( v_{2q_1, 2k} = v_{2q_1, 2k}(v_2, \ldots, v_{2N}) \).

**Theorem 3** The set of polynomials \( v_{2q_1, 2k} = v_{2q_1, 2k}(v_2, \ldots, v_{2N}) \in A(N) \) determines a skew-symmetric operation on an \( N \)-polynomial Lie algebra \( W_{-1}(N) \) if and only if the homomorphism

\[ \gamma: W_{-1}(N) \rightarrow DerA(N), \quad \gamma(L^{0}_{2q}) = \sum_{k=1}^{N} v_{2q, 2k} \frac{\partial}{\partial v_{2k}}, \]

of \( A(N) \)-modules is a homomorphism of the \( N \)-polynomial Lie algebra \( W_{-1}(N) \) to the Lie algebra of polynomial derivations of the ring \( A(N) = \mathbb{C}[v_2, \ldots, v_{2N}] \).

**Proof** The theorem is proved by a direct verification of its statements.

The Lie algebra \( W_{-1} \) with generators \( L^{0}_{2q}, q = -1, 0, 1, \ldots, \), contains the Lie subalgebra generated by the three operators \( L^{0}_{-2}, L^{0}_{0}, \) and \( L^{0}_{2} \), where \([L^{0}_{-2}, L^{0}_{2}] = 4L^{0}_{0}\). The Lie algebra \( W_{-1} \) with respect to the bracket \([\cdot, \cdot]\) is generated by only two generators, \( L^{0}_{-2} \) and \( L^{0}_{2} \).

**Example 4** \( 6L^{0}_{2} = [L^{0}_{-2}, L^{0}_{4}], 4L^{0}_{0} = [L^{0}_{-2}, L^{0}_{2}], \) and \( 2L^{0}_{4} = [L^{0}_{2}, L^{0}_{4}] \).

The generators \( L_{2q}, q \geq 1 \), are given by the recurrence relation \( 2qL^{0}_{2(q+2)} = [L^{0}_{2}, L^{0}_{2(q+1)}] \). Moreover, the operators \( L^{0}_{-2}, L^{0}_{2} \) are related by commutation relations, the first of which is

\[ [L^{0}_{2}, [L^{0}_{-2}, L^{0}_{2}], L^{0}_{4}]] = 12[L^{0}_{4}, [L^{0}_{2}, L^{0}_{4}]]. \tag{18} \]

**Corollary 8** The representations \( \gamma^{j}(L^{0}_{2q}) = \sum_{k=1}^{N} v_{2q, 2k} \frac{\partial}{\partial v_{2k}}, j = 1, 2, \) of the \( N \)-polynomial algebra \( W_{-1} \) coincide if and only if \( v_{2q, 2k}^1 = v_{2q, 2k}^2 \) for \( q = -1 \) and \( 2 \).

By construction there is an embedding of the Lie algebra \( W_{-1} \) into the Lie algebra \( W_{-1}(N) \). On the other hand, the ring homomorphism \( \varphi: A(N) \rightarrow \mathbb{C}, \varphi(v_{2k}) = 0, k = 1, \ldots, N, \) induces a projection \( W_{-1}(N) \rightarrow W_{-1} \) of Lie algebras.

**Corollary 9** The homomorphism

\[ \gamma: W_{-1}(N) \rightarrow \mathcal{G}_{P, 0}(N), \quad \gamma(L^{0}_{2q}) = L^{0}_{2q} = \sum_{k=1}^{N} 2k N_{2(q+k)} \frac{\partial}{\partial N_{2k}}, \quad \gamma(v_{2k}) = N_{2k}, \]

extends to an epimorphism of Lie algebras.

Note that the nontrivial relation \( [L_{2k}, L_{2k}] \) between Newton polynomials in \( x_{1}, \ldots, x_{N} \) ensures the fulfillment of the condition

\[ \gamma([L_{2k}, L_{2k}]) = [\gamma(L_{2k}), \gamma(L_{2k})]. \]

The kernel of the homomorphism \( \gamma \) is described by \([12]\). The restriction of the homomorphism \( \gamma \) to the Lie subalgebra \( W_{-1} \) gives a representation of the Lie algebra \( W_{-1} \) in the Lie algebra \( \mathcal{G}_{P, 0}(N) \) with the structure of a free \( N \)-dimensional \( \mathbb{C}[N_2, \ldots, N_{2N}] \)-module.
4 Commuting Vector Fields on the Symmetric Square of a Plane Curve

Consider the symmetric square of the curve \( V = \{(X, Y) \in \mathbb{C}^2 : F(X, Y) = 0\} \), where \( F(X, Y) \) are polynomials in \( X \) and \( Y \). Let \( \mathcal{D}_k = F(X_k, Y_k) \partial_{X_k} - F(X_k, Y_k) \partial_{Y_k} \), \( k = 1, 2 \). We introduce the operators

\[
\mathcal{L}^1 = \frac{1}{X_1 - X_2} (\mathcal{D}_1 - \mathcal{D}_2), \quad \mathcal{L}^2 = \frac{1}{X_1 - X_2} (X_2 \mathcal{D}_1 - X_1 \mathcal{D}_2).
\]

(19)

Lemma 9 1. The operators \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are derivations of the function ring on \( \text{Sym}^2(\mathbb{C}^2) \setminus \{X_1 - X_2 = 0\} \).

2. The operators \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) annihilate the polynomials \( F(X_1, Y_1) \) and \( F(X_2, Y_2) \).

3. \( [\mathcal{L}^1, \mathcal{L}^2] \equiv 0 \).

\[ \text{Proof} \] The operators \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are derivations of the function ring on \( (\mathbb{C}^2 \times \mathbb{C}^2) \setminus \{X_1 - X_2 = 0\} \). Statements 1 and 2 are verified directly. A standard calculation shows that

\[
[(X_1 - X_2)\mathcal{L}^1, (X_1 - X_2)\mathcal{L}^2] = -F(X_2, Y_2)Y_2D_1 - F(X_1, Y_1)Y_1D_2 + (X_1 - X_2)^2[\mathcal{L}^1, \mathcal{L}^2].
\]

On the other hand, \( [D_1 - D_2, X_2D_1 - X_1D_2] = -F(X_2, Y_2)Y_2D_1 - F(X_1, Y_1)Y_1D_2 \). The coincidence of the left-hand sides of the equations and the relation \( X_1 - X_2 \neq 0 \) imply the lemma.

□

5 Lie Algebroids on the Space of Nonsingular Hyperelliptic Curves

Consider the bundle \( f : \mathcal{E}_{N,0} \to \mathcal{B}_{N,0} \) (see Definition [1]). In Section 2 we described the Lie algebra of vector fields on \( \mathcal{B}_{N,0} \) generated by the Newton fields \( \mathcal{L}_{2k}^0 \), \( k = -1, 0, 1, \ldots, N - 2 \). In this section we construct a Lie algebroid on the space \( \mathcal{E}_{N,0} \). We set \( \pi = \pi(X, Y; x) = Y^2 - P \), where \( P = P(X; x) = \prod_{i=1}^{N}(X - x_i) \). By \( \mathcal{G}(\mathbb{C}[X, Y; x]) \) we denote the Lie algebra of derivations of the ring \( \mathbb{C}[X, Y; x] \). Let us introduce the operator \( \mathcal{L}_{N-2}^0 = 2Y \partial_X + P_X \partial_Y \in \mathcal{G}(\mathbb{C}[X, Y; x]) \). We have \( \mathcal{L}_{N-2}^0 \pi \equiv 0 \). Hence, for fixed \( x \), the operator \( \mathcal{L}_{N-2}^0 \) determines a vector field on \( \mathbb{C}^2 \) that is tangent to the curve \( V = \{(X, Y) \in \mathbb{C}^2 : \pi(X, Y; x) = 0\} \). The field \( \mathcal{L}_{N-2}^0 \) determines the vertical field of the bundle \( f : \mathcal{E}_{N,0} \to \mathcal{B}_{N,0} \).

Lemma 10 Let \( D \) be a derivation of the form \( a \partial_X + b \partial_Y \) of the ring \( \mathbb{C}[X, Y; x] \), where \( x \in \mathcal{B}_{N,0} \). Then \( D\pi = \Phi\pi \), where \( \Phi \in \mathcal{G}(\mathbb{C}[X, Y; x]) \), implies \( D = \psi\mathcal{L}_{N-2}^0 + \pi D^1 \), where \( \psi \in \mathcal{G}(\mathbb{C}[X, Y; x]) \) and \( D^1 \in \mathcal{G}(\mathbb{C}[X, Y; x]) \).

\[ \text{Proof} \] We shall carry out calculations in the ring \( K = \mathbb{C}[X, Y; x]/(\pi) \). In this ring \( Y^2 = P \), and thus \( K \) is a free \( \mathbb{C}[X; x] \)-module with generators \( 1 \) and \( Y \). We set \( a = a_1 + a_2Y \) and \( b = b_1 + b_2Y \), where \( a_l, b_l \in \mathbb{C}[X; x] \), \( l = 1, 2 \). The condition that \( D\pi = 0 \) in the ring \( K \) implies \( (a_1 + a_2Y)P_X = (b_1 + b_2Y)2Y \). Hence \( a_1P_X = 2b_2P \) and \( a_2P_X = 2b_1 \). On the other hand, the condition \( D = \psi\mathcal{L}_{N-2}^0 \), where \( \psi = \psi_1 + \psi_2Y \), implies

\[
a_1 + a_2Y = 2\psi_2P + 2\psi_1Y, \quad b_1 + b_2Y = \psi_1P_X + \psi_2P_X Y.
\]

Hence \( 2\psi_1 = a_2, 2\psi_2P = a_1, \) and \( \psi_2P_X = b_2 \). Since \( x \in \mathcal{B}_{N,0} \), it follows that the polynomials \( P(X; x) \) and \( P_X(X; x) \) are coprime, and this system has a polynomial solution \( \psi_2 = \psi_2(X; x) \).

□

Consider the following sequence of derivations of the ring \( \mathbb{C}[X, Y; x] \):

\[
L_{2k}^0 = \mathcal{L}_{2k}^0 + 2X^{k+1}\partial_X + C_{2k}Y\partial_Y, \quad \text{where} \quad C_{2k} = \sum_{i=1}^{N} \frac{X^{k+1} - x_i^{k+1}}{X - x_i}.
\]

(20)

Theorem 4 The homogeneous fields \( L_{2k}^0 \), \( k = -1, 0, 1, \ldots \), of degree \( 2k \) are uniquely determined by the condition that they are lifts of the Newton fields \( \mathcal{L}_{2k}^0 \) and generate the Lie algebra of Newton horizontal vector fields on the space of the bundle \( \mathcal{E}_{N,0} \), that is,

\[
[L_{2q_1}^0, L_{2q_2}^0] = 2(q_2 - q_1)L_{2(q_1+q_2)}^0.
\]
Proof} We set \( \widetilde{L}_0^0 = L_0^0 + 2X^{k+1}\partial_X = 2(\sum_{i=1}^{N} x_i^{k+1}\partial_x_i + X^{k+1}\partial_X) \). The operator \( \widetilde{L}_0^0 \) determines a Newton derivation of the ring \( \mathbb{C}[X; x] \). It is easy to check that \( \text{(20)} \) can be written as

\[
L_0^0 = \frac{1}{2} \widetilde{L}_0^0 (\ln P) Y \partial_Y.
\]

Hence \( L_0^0(Y^2 - P) = P(\widetilde{L}_0^0 \ln P - L_0^0 \ln P) \equiv 0 \). Thus, formula \( \text{(20)} \) determines horizontal vector fields \( L_0^0 \), \( k = -1, 0, 1, \ldots \), on \( \mathcal{E}_{N,0} \), which are lifts of the fields \( L_0^0 \) on the base \( B_{N,0} \).

Now let \( L_0^{1,2} \) and \( L_0^{2,0} \) be two homogeneous horizontal vector fields on \( \mathcal{E}_{N,0} \) that are lifts of the field \( L_0^0 \) on the base \( B_{N,0} \). Then, according to Lemma \( \text{(10)} \) \( L_0^{0,1} = L_0^1 + \psi_2k+2-N L_0^{N-2} \), where \( \psi_2k+2-N = \psi_1 + \psi_2 Y \) and \( \psi_1, \psi_2 \in \mathbb{C}[X; x] \) are homogeneous polynomials such that \( \deg \psi_1 = 2m = 2k+2-N \) and \( \deg \psi_2 = 2(k+1-N) \).

Note that the degree of the function \( \psi_2k+2-N \) cannot be negative. Hence the condition \( \psi_2k+2-N \neq 0 \) implies \( N \leq 2k + 2 \). On the other hand, according to Corollary \( \text{(9)} \), the generators of the algebra \( \mathcal{W}_1 \) are completely determined by the operators \( L_0^0 \) and \( L_1^1 \). As a result, we obtain the following conditions: \( N \leq 0 \) for \( k = -1 \), \( N \leq 4 \) for \( k = 1 \), and \( N \leq 6 \) for \( k = 2 \). In the case where \( k = 2 \) and \( N = 5 \), we obtain \( \deg \psi_1 = 1 \), which contradicts \( \deg \psi_1 = 2m = 2k+2-N \) and \( \deg \psi_2 = 2(k+1-N) \).

It remains to consider the cases \( N = 3, 4, 6 \). As shown above, in the case \( N = 6 \), the lifts of the fields \( L_0^0 \) and \( L_0^3 \) are unique, and any lift of \( L_0^0 \) must have the form \( L_0^0 + \alpha L_0^3 \), where \( \alpha \in \mathbb{C} \). A direct verification shows that the commutation relation (see \( \text{(13)} \)) in the Witt algebra holds only for \( \alpha = 0 \). Thus, in the case \( N = 6 \), the lift of the fields \( L_0^0 \) is unique. Similar arguments show that this is also true in the cases \( N = 3 \) and \( 4 \).

The commutation rule \( [L_0^0, L_0^0] = 2(q_2 - q_1)L_0^0 \) follows from the fact that \( \widetilde{L}_0^0 \) is a Newton operator and from \( \text{(21)} \). This completes the proof of the theorem.

\[ \square \]

Corollary 10 The generating function for the operators \( \text{(20)} \) has the form

\[
L^0(t) = \frac{1}{2} \widetilde{L}^0(t) \ln P) Y \partial_Y, \quad \text{where} \quad \widetilde{L}^0(t) = L^0(t) + 2 \frac{1}{1 - X t} \partial_X. \tag{22}
\]

Consider the space \( \mathbb{C}^{N+1} \) with the graded coordinates \( (X, Y; N_2, \ldots, N_{2(N-1)}) \). Using the equation \( Y^2 = P(X; x) \), we can identify the space \( \mathcal{E}_{N,0} \) with an open dense subvariety in \( \mathbb{C}^{N+1} \). The Lie algebra of vector fields on \( \mathcal{E}_{N,0} \) described above determines a polynomial Lie algebra generated by the fields \( \mathcal{L}_0^{N-2} \) and the fields \( L_0^2, L_0^3, \ldots, L_0^{2(N-2)} \).

Example 5 Case \( N = 3 \). The coordinates in \( \mathbb{C}^4 \) are \( X, Y, N_2, \) and \( N_4 \). We have

\[
\frac{1}{3} N_6 = -Y^2 + X^3 - N_2 X^2 + \frac{1}{2}(N_2^2 - N_4) X + \left( \frac{1}{2} N_2 N_4 - \frac{1}{6} N_3^2 \right).
\]

Using this formula, we obtain an explicit expression for the basis polynomial fields \( L_1^1, L_0^2, L_0^3, \) and \( L_2^0 \) in \( \mathbb{C}^4 \).

6 Coordinate Rings of Spaces of Symmetric Squares of Hyperelliptic Curves

Consider the space \( \mathbb{C}^2 \times \mathbb{C}^2 \) with coordinates \( (X_1, Y_1) \) and \( (X_2, Y_2) \) graded as above, i.e., so that \( \deg X_k = 2 \) and \( \deg Y_k = N, k = 1, 2 \), and the space \( \mathbb{C}^5 \) with graded coordinates \( u_2, u_4, v_N, v_{N+2}, \) and \( v_{2N} \). Here the subscript corresponds to the degree of variables.

Lemma 11 The algebraic homogeneous map

\[
\xi: \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^5, \quad \xi((X_1, Y_1), (X_2, Y_2)) = (u_2, u_4, v_N, v_{N+2}, v_{2N}),
\]

where \( u_2 = X_1 + X_2, u_4 = (X_1 - X_2)^2, v_N = Y_1 + Y_2, v_{N+2} = (X_1 - X_2)(Y_1 - Y_2), \) and \( v_{2N} = (Y_1 - Y_2)^2 \), makes it possible to identify the algebraic variety \( (\mathbb{C}^2 \times \mathbb{C}^2)/S_2 \) with the hypersurface in \( \mathbb{C}^5 \) determined by the equation \( u_4 v_{2N} - v_{N+2} = 0 \).
In the notation of Lemma 11 we have
\[ a(t; u_2, u_4) = \sum_{k=0}^{\infty} a_{2k}(u_2, u_4)t^k. \]
For what follows we need the homogeneous polynomials \( a_{2k}(u_2, u_4) \) of degree \( 2k \) determined by the generating series
\[
\frac{1}{(1 - X_1 t)(1 - X_2 t)} = a(t; u_2, u_4) = \sum_{k=0}^{\infty} a_{2k}(u_2, u_4)t^k = 4 \left( \frac{1}{(2 - u_2 t)^2 - u_4 t^2} \right) = 1 + u_2 t + (t^2).
\]
(23)

In the notation of Lemma 11 we have
\[
\sum_{k=0}^{\infty} (X_1^k + X_2^k)t^k = \frac{1}{1 - X_1 t} + \frac{1}{1 - X_2 t} = (2 - u_2 t)a(t; u_2, u_4).
\]
(24)

Moreover,
\[
\sum_{k=2}^{\infty} (X_1 - X_2)(X_1^{k-1} - X_2^{k-1})t^{k-2}
\]
\[
= \sum_{k=2}^{\infty} [(X_1^k + X_2^k)t^{k-2} - X_1 X_2(X_1^{k-2} + X_2^{k-2})t^{k-2}] = u_44a(t; u_2, u_4),
\]
(25)

\[
\sum_{k=2}^{\infty} (Y_1 - Y_2)(X_1^k - X_2^k)t^k = (Y_1 - Y_2) \left[ \frac{1}{1 - X_1 t} - \frac{1}{1 - X_2 t} \right] = v_{N+2}a(t; u_2, u_4).
\]
(26)

We have \( Y_j^2 = X_j^N + \sum_{k=2}^{\infty} (-1)^k y_{2k} X_j^{N-k} \). Hence
\[
(Y_1^2 + Y_2^2) = \frac{1}{2}(v_2^2 + v_{2N})
\]
\[
= (2a_{2N} - u_2 a_{2N-2}) + \sum_{k=2}^{N-1} (-1)^k y_{2k}(2a_{2(N-k)} - u_2 a_{2(N-k-1)}) + (-1)^{N-2}y_{2N}.
\]

We also have
\[
(X_1 - X_2)(Y_1^2 - Y_2^2) = v_N v_{N+2} = u_4 \left( a_{2(N-2)} + \sum_{k=2}^{N-2} (-1)^k y_{2k} a_{2(N-k-2)} \right),
\]
\[
(Y_1 - Y_2)(Y_1^2 - Y_2^2) = v_N v_{2N} = v_{N+}\left( a_{2(N-1)} + \sum_{k=2}^{N-1} (-1)^k y_{2k} a_{2(N-k-1)} \right).
\]

The graded coordinate ring \( R_0(N) \) of the space \( \mathcal{E}_{N,0}^2 \) in \((\mathbb{C}^2 \times \mathbb{C}^2 \setminus \{X_1 - X_2 = 0\}) \times B_{N,0} \) (see Section 1) has the form
\[
\mathbb{C}[X_1, Y_1; X_2, Y_2; x_1, \ldots, x_N]/J,
\]
deg \( x_j = \deg X_k = 2, \ deg Y_k = N, \ j = 1, \ldots, N, \ k = 1, 2, \)

where \( J = \langle \pi_1, \pi_2 \rangle \) is the ideal generated by the polynomials \( \pi_1 = \pi_k(X_k, Y_k; x) \). Let \( R_0^G(N) \subset R_0(N) \) denote the invariant ring of the free action of \( G \) on \( R_0(N) \). Consider the graded ring \( R(N) = R_0(N)/\langle y \rangle \), where \( y = x_1 + \cdots + x_N \). Let \( R^G(N) \subset R(N) \) denote the invariant ring of the free action of \( G \) on \( R(N) \). We shall treat the ring \( R^G(N) \) as the coordinate ring of the universal space \( \mathcal{E}_N^2 \).

**Lemma 12** The ring \( R^G(N) \) is isomorphic to the graded ring
\[
R^G_0 = \mathbb{C}[u_2, u_4, v_N, v_{N+2}, v_{2N}, y]/J^G,
\]
where $\mathbf{y} = (y_1, \ldots, y_{2N})$ and the ideal $J^G$ has Gröbner basis

$$P_{2N+4} = v_{N+2}^2 - u_4 v_{2N},$$

$$P_{2N+2} = v_N v_{N+2} - u_4 \left( a_{2N-2} + \sum_{k=2}^{N-1} (-1)^k y_{2k} a_{2(N-k-1)} \right),$$

$$P_{2N} = v_N^2 + v_{2N} - (a_{2N} - u_2 a_{2(N-1)}) - \sum_{k=2}^{N-1} (-1)^k y_{2k} (2a_{2(N-k)} - u_2 a_{2(N-k-1)}) - (-1)^N 2y_{2N},$$

$$P_{3N} = v_N v_{2N} - v_{N+2} \left( a_{2(N-1)} + \sum_{k=2}^{N-1} (-1)^k y_{2k} a_{2(N-k-1)} \right).$$

The relation $v_N P_{2N+4} - v_{N+2} P_{2N+2} + u_4 P_{3N} = 0$ holds.

**Proof** The lemma follows easily from the relations obtained above.

Let us introduce the ring $A(N) = \mathbb{C}[u_2, u_4, v_{N-2}, v_N, \tilde{y}]$, where $\tilde{y} = (y_3, \ldots, y_{2(N-2)})$.

**Lemma 13** There is a ring homomorphism $\varphi: R^G_U \rightarrow A(N)$ defined by:

$$\varphi(u_2) = u_2, \quad \varphi(u_4) = u_4, \quad \varphi(v_N) = v_N, \quad \varphi(y_{2k}) = y_{2k}, \quad k = 2, \ldots, N-2,$$

$$\varphi(v_{N+2}) = u_4 v_{N-2}, \quad \varphi(v_{2N}) = u_4 v_{2N}^2,$$

$$\varphi(y_{2(N-1)}) = (-1)^{N-1} \left[ v_{N-2} v_N - a_{2(N-1)} - \sum_{k=2}^{N-2} (-1)^k y_{2k} a_{2(N-k-1)} \right],$$

$$\varphi(y_{2N}) = (-1)^N \frac{1}{2} \left[ (v_N^2 + v_{2N}) - (2 a_{2N} - u_2 a_{2(N-1)}) \right.$$

$$\left. - \sum_{k=2}^{N-1} (-1)^k y_{2k} (2a_{2(N-k)} - u_2 a_{2(N-k-1)}) \right].$$

**Proof** A direct verification shows that the homomorphism $\varphi$ maps the ideal $J^G$ to 0.

**Corollary 11** The homomorphism $\varphi[u_{4}^{-1}]: R^G_U[u_{4}^{-1}] \rightarrow A(N)[u_{4}^{-1}]$ is an isomorphism.

**Proof** The ring homomorphism

$$\eta: A(N)[u_{4}^{-1}] \rightarrow R^G_U[u_{4}^{-1}],$$

$$\eta(u_{2k}) = u_{2k}, \quad k = 1, 2, \quad \eta(v_N) = v_N, \quad \eta(y_{2k}) = y_{2k}, \quad k = 2, \ldots, N-2,$$

$$\eta(v_{N-2}) = u_{4}^{-1} v_{N+2}$$

is inverse to the homomorphism $\varphi[u_{4}^{-1}]$.

Consider the space $\mathbb{C}^{N+1}$ with the graded coordinates $(u_2, u_4, v_N, v_{N+2}, v_{2N}; \mathbf{y})$ and the space $\mathbb{C}^{N+1}$ with the graded coordinates $(u_2, u_4, v_{N-2}, v_N, \tilde{y})$. As mentioned above, the space $E^2_N$ can be identified with the algebraic subvariety in $\mathbb{C}^{N+4}$ determined by the equations $P_{2N+k} = 0$, $k = 0, 2, 4, N$ (see Lemma 13). We set

$$b_{2N}(u_2; \tilde{y}) = \frac{1}{2N-1} \left( u_2^N + \sum_{k=2}^{N-1} (-1)^k 2^k y_{2k} u_2^{N-k-1} + (-1)^N 2^{N-2} y_{2N} \right).$$

Let $W_s(N)$, $s = 1, 2$, denote the algebraic subvarieties in $\mathbb{C}^{N+4}$ determined by the equations

for $s = 1: \quad u_4 = 0, \quad v_N = 0, \quad v_{N+2} = 0, \quad v_{2N} = b_{2N}(u_2; \tilde{y}),$

for $s = 2: \quad u_4 = 0, \quad v_{N+2} = 0, \quad v_{2N} = 0, \quad v_N^2 = b_{2N}(u_2; \tilde{y}).$

We set $W(N) = W_1(N) \cup W_2(N)$. Note that, for given $\mathbf{y}$, the intersection $W_1(N) \cap W_2(N)$ is the set of roots of the equation $b_{2N}(u_2; \tilde{y}) = 0$. Clearly, $W(N) \subset E^2_N$. 

12
Consider the bundle $E_{N}$. For the curve $C$, where $d$ determines a homomorphism $f: C \rightarrow \mathbb{C}$. Theorem 5. The mapping

$$y_{2(N-1)} = (-1)^{N-1} \left[v_{N}v_{N-2} - \left(a_{2(N-1)} + \sum_{k=2}^{N-2} (-1)^{k}y_{2k}a_{2(N-k-1)}\right)\right],$$

and

$$y_{2N} = (-1)^{N} \left[v_{N}^{2} + v_{2N} - (2a_{2N} - u_{2a_{2(k-1)}}) - \sum_{k=2}^{N-1} (-1)^{k}y_{2k}(2a_{2(N-k)} - u_{2a_{2(N-k-1)}})\right],$$

determines a homomorphism $f: \mathbb{C}^{N+1} \setminus \{u_{4} = 0\} \rightarrow \mathcal{L}_{N}^{2} \setminus W(N)$.

Proof. The required assertion follows directly from Lemmas [12] and [13] and Corollary [11].

7 Lie Algebroids on the Space of Symmetric Squares of Nonsingular Hyperelliptic Curves

Consider the bundle $\mathcal{E}_{N,0}^{2} \rightarrow \mathcal{B}_{N,0}$ (see Definition [2]). We have defined an action of the Witt algebra of Newton fields $\mathcal{L}_{2k}^{0}$ (see Definition [3]) on the base $\mathcal{B}_{N,0}$. Let us introduce the following derivations of the ring $\mathbb{C}[X_{1}, Y_{1}; X_{2}, Y_{2}; x]$:

$$L_{2k}^{0} = L_{2k}^{0} + 2(X_{1}^{k+1}\partial_{X_{1}} + X_{2}^{k+1}\partial_{X_{2}}) + C_{2k}^{1}Y_{1}\partial_{Y_{1}} + C_{2k}^{2}Y_{2}\partial_{Y_{2}},$$

where

$$C_{2k}^{j} = \sum_{i=1}^{N} \frac{X_{i}^{k+1} - x_{i}^{k+1}}{X_{i} - x_{i}}, \quad j = 1, 2.$$

We set $\tilde{L}_{2k}^{0} = L_{2k}^{0} + 2(X_{1}^{k+1}\partial_{X_{1}} + X_{2}^{k+1}\partial_{X_{2}})$. It is verified directly that

$$L_{2k}^{0} = \tilde{L}_{2k}^{0} + \frac{1}{2}L_{2k}^{0}[\ln P^{(1)}]Y_{1}\partial_{Y_{1}} + [\ln P^{(2)}]Y_{2}\partial_{Y_{2}}, \quad \text{where} \quad P^{(j)} = \prod_{i=1}^{N} (X_{j} - x_{i}).$$

Now we are ready to obtain one of the main results of the present paper.

Theorem 6. The homogeneous fields $L_{2k}^{0}$, $k = -1, 0, 1, \ldots$, of degree $2k$ (see [7]) are determined uniquely by the conditions that they are lifts of the Newton fields $\mathcal{L}_{2k}^{0}$ on the base of the bundle $\mathcal{E}_{N,0}^{2} \rightarrow \mathcal{B}_{N,0}$, generate the Lie algebra of Newton horizontal vector fields on $\mathcal{E}_{N,0}^{2}$, and determine a representation of the Lie algebra $W_{-1}$.

Proof. The proof of the theorem uses explicit expressions and Lemma [10] by analogy with the proof of Theorem [4].

Using the operators $D_{k} = 2Y_{k}\partial_{X_{k}} + P^{(k)}X_{k}\partial_{Y_{k}}$, $k = 1, 2$, we infer (see Lemma 9) that in the Lie algebroid of the bundle $\mathcal{E}_{N,0}^{2}$ there are the two commuting horizontal fields

$$\mathcal{L}_{N-4}^{*} = \frac{1}{X_{1} - X_{2}} (D_{1} - D_{2}) \quad \text{and} \quad \mathcal{L}_{N-2}^{*} = \frac{1}{X_{1} - X_{2}} (X_{2}D_{1} - X_{1}D_{2}).$$

Lemma 14. For the curve $Y^{2} = \prod_{i=1}^{N} (X - x_{i})$,

$$E(t) - (1 - tX) \sum_{m=1}^{N} E(t; m)X^{m-1}t^{m-1} = t^{N}Y^{2}.$$
We have
\[
E(t) - \sum_{m=1}^{N} E(t; m)X^{m-1}t^{m-1} + \sum_{m=1}^{N} E(t; m)X^{m}t^{m} \\
= (E(t) - E(t; 1)) + \sum_{m=1}^{N-1} (E(t; m) - E(t; m + 1))X^{m}t^{m} + E(t; N)X^{N}t^{N} \\
= (-1)^{N}y_{2N}t^{N} + \sum_{m=1}^{N-1} (-1)^{m}y_{2m}X^{m}t^{N} + X^{N}t^{N} = t^{N}Y^{2}. \tag*{□}
\]

Let \( L(t) = \sum_{k=-1}^{\infty} L_{2k}^{0}t^{k+1} \).

**Theorem 7** For the generating series \( L(t) \) of the operators \( L_{2k}^{0} \), \( k = -1, 0, 1, \ldots \) (see (29)), the relation
\[
E(t)L(t) = \sum_{m=1}^{N} E(t; m)t^{m-1}L_{2(m-2)}^{0} + A_{2}(t)L_{N-4}^{*} - A_{0}(t)L_{N-2}^{*} \tag{34}
\]
holds on the variety \( (X_{1} - X_{2}) \neq 0, Y_{1}Y_{2} \neq 0 \), where
\[
A_{2}(t) = t^{N} \left[ Y_{1}X_{1} - \frac{X_{2}}{1 - tX_{2}} \right] + t^{N} a(t; u_{2}, u_{4}) \left[ \frac{1}{2}(u_{2}v_{N} + v_{N+2}) - \frac{1}{4}(u_{2}^{2} - u_{4}) \right], \\
A_{0}(t) = t^{N} \left[ Y_{1}X_{1} - \frac{X_{2}}{1 - tX_{2}} \right] + t^{N} a(t; u_{2}, u_{4}) \left[ v_{N} - \frac{1}{2}(u_{2}v_{N} - v_{N+2}) \right].
\]

**Proof** Let \( \mathcal{L} = E(t)L(t) - \sum_{m=1}^{N} E(t; m)t^{m-1}L_{2(m-2)}^{0} \). According to Corollary 14 we have \( \mathcal{L}X_{i} = 0, i = 1, \ldots, N \). Thus, the field \( \mathcal{L} \) is vertical, and therefore \( \mathcal{L} = A_{2}(t)L_{N-4}^{*} - A_{0}(t)L_{N-2}^{*} \) for some series \( A_{2}(t) \) and \( A_{0}(t) \). On the other hand, according to Lemma 14
\[
\mathcal{L}X_{j} = \frac{1}{1 - tX_{j}} \left[ E(t) - (1 - tX_{j}) \sum_{m=1}^{N} E(t; m)X^{m-1}t^{m-1} \right] = \frac{t^{N}Y^{2}}{1 - tX_{j}}.
\]
Using (23), we obtain the system of equations
\[
\frac{1}{X_{1} - X_{2}}(A_{2}(t) - X_{2}A_{0}(t)) = \frac{t^{N}Y_{1}}{1 - tX_{1}}, \quad \frac{1}{X_{2} - X_{1}}(A_{2}(t) - X_{1}A_{0}(t)) = \frac{t^{N}Y_{2}}{1 - tX_{2}},
\]
provided that \( Y_{1}Y_{2} \neq 0 \). Solving this system completes the proof of the theorem. \( \Box \)

**Corollary 12** In the basis \( \{ L_{0}^{0}, \ldots, L_{2(N-2)}^{0}; L_{N-4}^{*}, L_{N-2}^{*} \} \) the following commutation relations hold:
\[
[L_{2p}, L_{2q}] = 2(q - p)L_{2(p+q)} - 2(q - p) \left( \sum_{m=1}^{N} \omega_{p+q,m}L_{2(m-2)} + \alpha_{p+q}L_{N-4}^{*} - \beta_{p+q}L_{N-2}^{*} \right),
\]
where \( \omega_{p+q,m}, \alpha_{p+q}, \) and \( \beta_{p+q} \) are the coefficients of \( t^{p+q} \) in the series \( E(t; m)/E(t), A_{2}(t)/E(t), \) and \( A_{0}(t)/E(t) \). All these coefficients belong to the ring \( R^{G}(N) \) (see Lemma 13).

We set \( N(t) = \sum_{i=1}^{N} 1/(1 - x_{i}t) \) and \( D_{0}(t) = N(t) - 2(1/(1 - tX_{1}) + 1/(1 - tX_{2})) = N(t) - 2a(t)(2 - u_{2}t) \), where \( a(t) = a(t; u_{2}, u_{4}) \) (see (23)).

**Lemma 15** The following relations hold:
\[
[L(t), L_{N-4}]X_{1} = \frac{t}{1 - tX_{1}}D_{0}(t)L_{N-4}^{*}X_{1}, \tag{35}
\]
\[
[L(t), L_{N-2}]X_{1} = \left( \frac{2}{1 - tX_{2}} + \frac{tX_{2}}{1 - tX_{1}}D_{0}(t) \right)L_{N-4}^{*}X_{1}. \tag{36}
\]

14
Poof Using (31) and (33), we obtain
\[ L(t)X_1 = \frac{2}{1-tX_1}, \quad L(t)Y_1 = \frac{tY_1}{1-tX_1}N(t), \]
\[ \mathcal{L}^*_N X_1 = \frac{2Y_1}{X_1 - X_2}, \quad \mathcal{L}^*_N X_1 = X_2 \mathcal{L}^*_N X_1. \]
Hence
\[ L(t)\mathcal{L}^*_N X_1 = \frac{t}{1-tX_1} \left( N(t) - 2 \frac{1}{1-tX_2} \right) \mathcal{L}^*_N X_1, \] (37)
\[ \mathcal{L}^*_N L(t)X_1 = \frac{2t}{(1-tX_1)^2} \mathcal{L}^*_N X_1. \] (38)
Relations (37) and (38) imply (35). Further, we have \( L(t)\mathcal{L}^*_N X_1 = L(t)(X_2 \mathcal{L}^*_N X_1) \). Using (37), we obtain
\[ L(t)\mathcal{L}^*_N X_1 = \left[ \frac{2}{1-tX_2} + \frac{tX_2}{1-tX_1} \left( N(t) - 2 \frac{1}{1-tX_2} \right) \right] \mathcal{L}^*_N X_1, \] (39)
\[ \mathcal{L}^*_N L(t)X_1 = \frac{2tX_2}{(1-tX_1)^2} \mathcal{L}^*_N X_1. \] (40)
Relations (39) and (40) imply (36).
□
We set
\[ A_{-2}(t) = ta(t)D_0(t), \quad A_{-4}(t) = tA_{-2}(t), \]
\[ B_0(t) = a(t) \left[ 2(1-u_2) + \frac{t^2}{4} (u_2^2 - u_4)D_0(t) \right], \]
\[ B_{-2}(t) = ta(t) \left[ 2 + (1-u_2)D_0(t) \right]. \] (42) (43)

Theorem 8 On the variety \( \{ X_1 - X_2 \neq 0, Y_1Y_2 \neq 0 \} \) the commutation formulas for the generating series \( L(t) \) of horizontal fields with the vertical fields \( \mathcal{L}^*_N \) and \( \mathcal{L}^*_N \) are
\[ [L(t), \mathcal{L}^*_N] = A_{-2}(t)\mathcal{L}^*_N - A_{-4}(t)\mathcal{L}^*_N, \] (44)
\[ [L(t), \mathcal{L}^*_N] = B_0(t)\mathcal{L}^*_N - B_{-2}(t)\mathcal{L}^*_N. \] (45)
Poof The series \([L(t), \mathcal{L}^*_N]\) and \([L(t), \mathcal{L}^*_N]\) are generating series for the vertical fields. Let us find their representation in the form of linear combinations of the fields \( \mathcal{L}^*_N \) and \( \mathcal{L}^*_N \). According to Lemma 13, the coefficients \( A_{-2}(t), A_{-4}(t), B_0(t), \) and \( B_{-2}(t) \) are solutions of the systems of equations
\[ A_{-2}(t) - X_2A_{-4}(t) = \frac{t}{1-tX_1}D_0(t), \]
\[ A_{-2}(t) - X_1A_{-4}(t) = \frac{t}{1-tX_2}D_0(t) \]
and
\[ B_0(t) - X_2B_{-2}(t) = \frac{2}{1-tX_2} + \frac{tX_2}{1-tX_1}D_0(t), \]
\[ B_0(t) - X_1B_{-2}(t) = \frac{2}{1-tX_1} + \frac{tX_1}{1-tX_2}D_0(t). \]
Solving these systems, we obtain (41)–(43).
□
Corollary 13 In the basis \( \{ L_{2,2}^0, \ldots, L_{2(N-2)}^0, \mathcal{L}^*_N \} \) the following commutation relations hold:
\[ [L_{2q}, \mathcal{L}^*_N] = a_{-2,2q+2} \mathcal{L}^*_N - a_{-4,2q+2} \mathcal{L}^*_N, \]
\[ [L_{2q}, \mathcal{L}^*_N] = b_{0,2q+2} \mathcal{L}^*_N - b_{-2,2q+2} \mathcal{L}^*_N, \]
where \( a_{-2,2q+2}, a_{-4,2q+2}, b_{0,2q+2}, \) and \( b_{-2,2q+2} \) are the coefficients of \( t^{q+1} \) in the series \( A_{-2}(t), A_{-4}(t), B_0(t), \) and \( B_{-2}(t) \). All these coefficients lie in the ring \( R^G(N) \) (see Lemma 13).
8 Polynomial Lie Algebroids Determined by the Lie Algebroid on $\mathcal{E}_{N,0}^2$

In this section we give a description of the polynomial Lie algebroid $\mathcal{G}(N)$ on $\mathbb{C}^{N+1}$, which uses the homomorphism $f: \mathbb{C}^{N+1} \setminus \{u_4 = 0\} \to \mathcal{E}_{N,0}^2 \setminus W(N)$ constructed in Section 6. As generators of the Lie algebroid $\mathcal{G}(N)$ we take the horizontal vector fields $L_{2k}$ and the vertical fields $L_{N-4}^\ast$ and $L_{N-2}^\ast$, which were constructed in Section 7. Without loss of generality, it is sufficient to consider the algebroid $\mathcal{G}$ as a module over the polynomial ring $\mathcal{A}(N) = \mathbb{C}[u_2, u_4, v_{N-2}, u_N; y]$.

**Lemma 16** The action of the operators $L_{N-4}^\ast$ and $L_{N-2}^\ast$ on the coordinate functions $u_2$ and $u_4$ has the form

- $L_{N-4}^\ast u_2 = 2u_{N-2}$, $L_{N-2}^\ast u_2 = u_{2N-2} - v_N$,
- $L_{N-4}^\ast u_4 = 4v_N$, $L_{N-2}^\ast u_4 = 2u_2v_N - 2u_4v_{N-2}$.

**Proof** The required relations are derived directly from our results obtained above. □

The action of the operators $L_{N-4}^\ast$ and $L_{N-2}^\ast$ in the coordinates $X_1, Y_1; X_2, Y_2; x$ has the form

- $L_{N-4}^\ast v_{N-2} = L_{N-4}^\ast \left( \frac{Y_1 - Y_2}{X_1 - X_2} \right) = \frac{2Y_1^2 - 2Y_2^2 + (X_1 - X_2)(P_{X_1}^{(1)} + P_{X_2}^{(2)})}{(X_1 - X_2)^3}$, \hspace{1cm} (46)
- $L_{N-4}^\ast v_N = L_{N-4}^\ast (Y_1 + Y_2) = \frac{P_{X_1}^{(1)} - P_{X_2}^{(2)}}{X_1 - X_2}$, \hspace{1cm} (47)
- $L_{N-2}^\ast v_{N-2} = L_{N-2}^\ast \left( \frac{Y_1 - Y_2}{X_1 - X_2} \right) = \frac{2(Y_2 - Y_1)(X_1Y_1 + X_2Y_2) + (X_1 - X_2)(X_2P_{X_1}^{(1)} + X_1P_{X_2}^{(2)})}{(X_1 - X_2)^3}$, \hspace{1cm} (48)
- $L_{N-2}^\ast v_N = L_{N-2}^\ast (Y_1 + Y_2) = \frac{X_2P_{X_1}^{(1)} - X_1P_{X_2}^{(2)}}{X_1 - X_2}$. \hspace{1cm} (49)

Our goal is to show that this action is polynomial in the coordinates $u_2, u_4, v_{N-2}, v_N, y$. We shall use the polynomials $a_{2k}(u_2, u_4)$ (see (23)) and the polynomials $b_{2n}(u_2, u_4)$ for which $\sum b_{2n} \cdot t^n = a^2(t)$.

**Lemma 17** The action of the operators $L_{N-4}^\ast$ and $L_{N-2}^\ast$ on the coordinate functions $v_{N-2}$ and $v_N$ has the form

- $L_{N-4}^\ast v_{N-2} = \sum_{k=0}^{N-3} (-1)^k y_{2k}b_{2N-2k-6}$, \hspace{1cm} (50)
- $L_{N-4}^\ast v_N = \sum_{k=0}^{N-1} (-1)^k (N-k)y_{2k}a_{2N-2k-4}$, \hspace{1cm} (51)
- $L_{N-2}^\ast v_{N-2} = \frac{1}{2} u_2 \sum_{k=0}^{N-3} (-1)^k y_{2k}b_{2N-2k-6} - \frac{1}{2} \sum_{k=0}^{N-1} (-1)^k (N-k)y_{2k}a_{2N-2k-4}$, \hspace{1cm} (52)
- $L_{N-2}^\ast v_N = \sum_{k=0}^{N-1} (-1)^k (N-k)y_{2k}(u_{2a_{2N-2k-4} - a_{2N-2k-2}})$. \hspace{1cm} (53)

To prove this lemma, we need the following general statement.

**Lemma 18** The formula

$$r(P) = \frac{2(P^{(2)} - P^{(1)}) + (X_1 - X_2)(P_{X_1}^{(1)} + P_{X_2}^{(2)})}{(X_1 - X_2)^3}$$

defines a linear map $r: \mathbb{C}[X; y] \to \mathbb{C}[X_1; X_2; y]$ of $\mathbb{C}[y]$-modules.

**Proof** The transform $\mathcal{E}$ is $\mathbb{C}[y]$-linear; thus, it suffices to prove that $r(X^k) \in \mathbb{C}[X_1, X_2; y]$, $k = 0, 1, \ldots$. Let us take the generating series $f(t; X) = \sum_{k=0}^{\infty} X^k t^k = (1 - tX)^{-1}$. We obtain $r(f(t; X)) = t^3a^2(t)$, where
Thus, we have \( r(1) = r(X) = r(X^2) = 0 \) and \( r(X^k) = b_{2k - 6} \) for \( k \geq 3 \), where the \( b_{2k} \) are polynomials with generating series \( \sum_{n=0}^{\infty} b_{2n} t^n = a^2(t) \).

We proceed to prove Lemma 17. Using (40), we derive (50). Relation (52) can be obtained by evaluating \( \mathcal{L}_{N-4}^* u_{N-2} \), since (38) can be rewritten as
\[
\mathcal{L}_{N-2}^* u_{N-2} = \left( \frac{Y_1 - Y_2}{X_1 - X_2} \right)^2 + \frac{(X_1 + X_2)}{2} \mathcal{L}_{N-4}^* u_{N-2} - \frac{1}{2} \left( \frac{P_1^{(1)} - P_2^{(2)}}{X_1 - X_2} \right),
\]
and applying the relation
\[
P_X = \sum_{k=0}^{N-1} (-1)^k (N-k) y_{2k} X^{N-k-1}.
\]
The expression (53) for \( \mathcal{L}_{N-4}^* u_{N} \) is obtained by using (26). Relation (55) can be rewritten as
\[
\mathcal{L}_{N-2}^* v_{N} = \frac{1}{2} (X_1 + X_2) \mathcal{L}_{N-4}^* v_{N} - \frac{1}{2} (P_0^{(1)} + P_2^{(2)}).
\]
Again applying (24), we obtain (56), which proves the lemma.

Thus, we have proved the following theorem, which is one of the main results of the present paper.

**Theorem 9** For each \( N \geq 3 \), a Lie \( \mathbb{C}[u_2, u_4, v_{N-2}, v_N; y] \)-algebra with generators \( L_{0}, \ldots, L_{2(N-2)}, \mathcal{L}_{0}, \mathcal{L}_{4}, \mathcal{L}_{N-2} \) is defined. The commutation relations between these generators are described in Corollaries [22] and [23] and their action on \( u_2, u_4, v_{N-2}, v_N, y \), in Lemmas [22] and [24].

### 9 Examples of Polynomial Lie Algebras

In this section we give an explicit description of the polynomial Lie algebras \( \mathcal{G}(N) \), \( N = 3, 4, 5 \), over the rings \( \mathbb{C}[u_2, u_4, v_1, v_3] \) for \( N = 3 \), \( \mathbb{C}[u_2, u_4, v_2, v_4; y_4] \) for \( N = 4 \), and \( \mathbb{C}[u_2, u_4, v_3; y_4, y_6] \) for \( N = 5 \) with generators \( \mathcal{L}_{0}, \ldots, \mathcal{L}_{2(N-2)}, \mathcal{L}_{0}, \mathcal{L}_{4}, \mathcal{L}_{N-2} \). Here \( L_0 \) is the Euler field and, therefore, \([L_0, L_{2k}] = 2kL_{2k}\).

**Proof** Case \( N = 3 \) We have
\[
y_4 = \frac{1}{2} (-3u_2^2 - u_4 + 4v_1v_3), \quad y_6 = \frac{1}{2} (-3u_2^2 + u_2u_4 - u_4v_1^2 + 2u_2v_1v_3 - v_3^2).
\]
The action of the generators \( L_0, L_2, \) and \( \mathcal{L}_1, \mathcal{L}_1^* \) of the free left \( \mathbb{C}[u_2, u_4, v_1, v_3] \)-module is as follows:

| \( \mathcal{L}_0 \) | \( u_2 \) | \( u_4 \) | \( v_1 \) | \( v_3 \) |
|-----------------|-------------|-------------|-------------|-------------|
| \( L_0 \)       | 2u_2        | 4u_4        | v_1         | 3v_3        |
| \( L_2 \)       | \frac{1}{2}(3u_2^2 - u_4 - 8v_1v_3) | -4u_2u_4   | \frac{1}{2}(u_2v_1^2 - 3v_3) | -\frac{3}{2}(u_1v_1 + u_2v_3) |
| \( \mathcal{L}_1 \) | 2v_1        | -2(u_2v_1 - u_2v_3) | -u_2 + v_1^2 | \frac{1}{2}(3u_2^2 - u_4 - 2v_1v_3) |
| \( \mathcal{L}_1^* \) | u_2v_1 - v_3 | -2(u_4v_1 - u_2v_3) | 1             |             |

The commutation relations are
\[
[\mathcal{L}_1, L_2] = -3u_2 \mathcal{L}_1 + \mathcal{L}_1^*, \quad [\mathcal{L}_1^*, L_2] = \frac{1}{12}(9u_4 - 9u_2^2 + 16y_4) \mathcal{L}_1^*.
\]

**Proof** Case \( N = 4 \) We have
\[
y_6 = \frac{1}{4}(2u_2^3 + 2u_2u_4 - 4v_2v_4 + 4u_2y_4), \quad y_8 = \frac{1}{16}(3u_4^4 - 2u_2u_4^2 - u_4^2 + 4u_4v_2^2 - 8u_2v_4v_2 + 4v_2^4 + 4u_2y_4^2 - 4u_4y_4).
\]
The action of the generators \( L_0, L_2, L_4, \) and \( \mathcal{L}_0, \mathcal{L}_2 \) of the free left \( \mathbb{C}[u_2, u_4, v_2, v_4, y_4] \)-module is as follows:

| \( \mathcal{L}_0 \) | \( u_2 \) | \( u_4 \) |
|-----------------|-------------|-------------|
| \( L_0 \)       | 4u_4        | 2u_2        |
| \( L_2 \)       | 6u_6        | -u_2^2 - u_4 - 2y_4 | -4u_2u_4 |
| \( L_4 \)       | 8u_8        | \frac{1}{2}(u_2^3 + 3u_2u_4 + 4u_2y_4 - 6y_6) | u_4(u_2^2 + u_4 + 4y_4) |
| \( \mathcal{L}_0 \) | 0           | 2u_2        |
| \( \mathcal{L}_2 \) | 0           | u_2v_2 - v_4 | -2(u_4v_2 - u_2v_4) |
The commutation relations are

\[
\begin{align*}
[L_2, L_4] &= y_6 L_0 - y_4 L_2 - (u_4 v_2 + u_2 v_4) L_0^* + 2 v_4 L_2^*, \\
[L_0^*, L_2] &= -2 u_2 L_0^*, \\
[L_0^*, L_4] &= (3 u_2^2 + u_4 + 2 y_4) L_0^* - 2 u_2 L_2^*, \\
[L_2^*, L_2] &= \frac{1}{2} (u_4 - u_2^2 + 2 y_4) L_0^*, \\
[L_2^*, L_4] &= \frac{1}{2} (2 u_2^3 - 2 u_2 u_4 + 3 y_0) L_0^* - \frac{1}{2} (u_2^3 - u_4) L_2^*.
\end{align*}
\]

**Proof** Case \(N = 5\) We have

\[
\begin{align*}
y_8 &= \frac{1}{10} (-5 u_2^4 - 10 u_2^4 u_4 - u_2^4 + 16 u_3 v_5 - 12 u_2^2 y_4 - 4 u_4 y_4 + 16 u_2 y_6), \\
y_{10} &= \frac{1}{16} (-2 u_2^3 + 2 u_2 u_4^2 - 4 u_4 v_3^2 + 8 u_2 v_3 v_5 \\
&\quad - 4 v_3^2 - 4 u_3^2 y_4 + 4 u_2 u_4 y_4 + 4 u_2^2 y_6 - 4 u_4 y_6).
\end{align*}
\]

The action of the generators \(L_0, L_2, L_4, L_6, \) and \(L_1^*, L_3^*\) of the free left \(\mathbb{C}[u_2, u_4, v_3, v_5, y_4, y_6]\)-module is as follows:

| \(L_0\) | \(v_2\) | \(v_4\) |
|---|---|---|
| \(L_0\) | \(2v_2\) | \(4v_4\) |
| \(L_2\) | \(-2v_4\) | \(-2(u_4 v_2 + u_2 v_4)\) |
| \(L_4\) | \(\frac{1}{2}(-u_2^2 v_2 + u_4 v_2 + 4 u_2 v_4)\) | \(2u_2 v_4 + u_2^2 v_4 + u_4 v_4 + 4 v_4 y_4\) |
| \(L_0^*\) | \(2u_2\) | \(3 u_2^2 + u_4 + 2 y_4\) |
| \(L_2^*\) | \(\frac{1}{2}(-u_2^2 - u_4 + 2 u_2^2 - 2 y_4)\) | \(u_2^3 - u_2 u_4 + y_6\) |

| \(L_0\) | \(y_4\) | \(y_6\) |
|---|---|---|
| \(L_0\) | \(4y_4\) | \(6y_6\) |
| \(L_2\) | \(6y_6\) | \(-\frac{4}{5}(3 y_4^2 - 10 y_6)\) |
| \(L_4\) | \(8 y_8\) | \(-\frac{3}{2}(4 y_4 y_6 - 25 y_{10})\) |
| \(L_6\) | \(10 y_{10}\) | \(-\frac{3}{2} y_4 y_8\) |
| \(L_2^*\) | \(0\) | \(0\) |
| \(L_4^*\) | \(0\) | \(0\) |

| \(L_0\) | \(u_2\) | \(u_4\) |
|---|---|---|
| \(L_0\) | \(2 u_2\) | \(4 u_4\) |
| \(L_2\) | \(\frac{1}{2}(-5 u_2^4 - 10 u_2^4 u_4 - 5 u_2^4 + 8 v_3 v_5 - 12 u_2^2 y_4 - 4 u_4 y_4 + 16 u_2 y_6)\) | \(-4 u_2 v_4\) |
| \(L_4\) | \(\frac{1}{20}(5 u_2^3 + 15 u_2 u_4^2 + 20 u_2 y_4 - 24 y_6)\) | \(u_4(3 u_2^2 + u_4 + 4 y_4)\) |
| \(L_6\) | \(\frac{1}{20}(-5 u_2^4 - 30 u_2^2 u_4 - 5 u_2^2 - 20 u_2^2 y_4 - 20 u_2 y_4 + 40 u_2 y_6 - 64 y_8)\) | \(-2 u_4(u_2^2 + u_2 u_4 + 2 u_2 y_4 - 2 y_6)\) |
| \(L_1^*\) | \(2 v_3\) | \(4 v_5\) |
| \(L_3^*\) | \(u_2 v_3 - v_5\) | \(-2(u_4 v_3 - u_2 v_5)\) |

| \(L_0\) | \(v_3\) | \(v_5\) |
|---|---|---|
| \(L_0\) | \(3 v_3\) | \(5 v_5\) |
| \(L_2\) | \(\frac{1}{2}(-u_2 v_3 - 3 v_5)\) | \(-\frac{5}{2}(u_4 v_3 + u_2 v_5)\) |
| \(L_4\) | \(\frac{1}{2}(10 u_2 v_5 - (u_2^2 - 3 v_4 - 4 y_4) v_3)\) | \(\frac{1}{20}(15 u_2^4 + 5 u_2^2 v_5 + 5 u_4 v_5 + 12 v_5 y_4)\) |
| \(L_6\) | \(\frac{1}{2}(3 u_2^2 v_3 - 7 u_2 u_4 v_3 - 15 u_2 v_3 - 5 u_4 v_3 + 4 u_2 v_3 y_4 - 12 v_5 y_4)\) | \(\frac{1}{2}(5 u_2^3 + u_4 + 2 y_4)\) |
| \(L_2^*\) | \(\frac{1}{2}(-u_2 v_3 + 2 v_3 - 2 u_2 y_4 + 2 y_6)\) | \(\frac{1}{2}(5 u_2^4 - u_2^4 - 4 v_3 v_5 + 6 u_2^2 y_4 - 2 u_2 y_4 - 4 v_5 y_6)\) |
| \(L_4^*\) | \(\frac{1}{2}(5 u_2^4 - u_2^4 - 4 v_3 v_5 + 6 u_2^2 y_4 - 2 u_2 y_4 - 4 v_5 y_6)\) | \(\frac{1}{2}(5 u_2^4 - u_2^4 - 4 v_3 v_5 + 6 u_2^2 y_4 - 2 u_2 y_4 - 4 v_5 y_6)\) |
The commutation relations are
\[
\begin{align*}
[L_2, L_4] &= 2L_6 - \frac{2}{3} y_4 L_2 + \frac{2}{3} y_6 L_0, \\
[L_2, L_6] &= -4 y_4 L_3^* + 2(u_4 v_3 + u_{22} v_5) L_4^* - \frac{2}{3} y_4 L_4 + \frac{2}{3} y_6 L_2 - 2 y_{10} L_0, \\
[L_4, L_6] &= (u_4 v_3 + u_{22} v_5) L_3^* - \frac{1}{2}(2 u_2 u_4 v_3 + u_4 v_5) L_1^* + 2 y_4 L_6 - \frac{2}{3} y_6 L_4 + \frac{2}{3} y_6 L_2 - 2 y_{10} L_0, \\
[L_1^*, L_2] &= -L_3^* - u_2 L_1^*, \\
[L_1^*, L_4] &= -u_2 L_3^* + \frac{1}{2}(9 u_2^2 + 3 u_4 + 4 y_4) L_1^*, \\
[L_1^*, L_6] &= \frac{1}{2}(9 u_2 + 3 u_4 + 4 y_4) L_1^* - \frac{1}{2}(5 u_2^3 + 5 u_2 u_4 + 6 u_2 y_4 - 4 y_4) L_1^* , \\
[L_2^*, L_4] &= \frac{1}{20}(-5 u_2^4 + 5 u_4 + 16 y_4) L_1^*, \\
[L_2^*, L_6] &= -\frac{1}{4} (u_2^2 - u_4 + 4 y_4) L_1^* + \frac{1}{20} (5 u_2^3 - 5 u_2 u_4 + 8 y_4) L_1^*, \\
[L_3^*, L_6] &= -\frac{1}{40} (75 u_2^4 - 50 u_2^2 u_4 - 25 u_2^3 + 60 u_2 y_4 - 60 u_4 y_4 - 128 y_8) L_1^* + \frac{1}{2} u_2 (v_2^2 - u_4) L_1^*. 
\end{align*}
\]

\[\square\]

**Theorem 10** There is an isomorphism of graded rings
\[
\varphi : \mathbb{C}[u_2, u_4, v_3, v_5; y_4, y_6] \to \mathbb{C}[x_2, x_3, x_4; z_4, z_5, z_6],
\]
which determines an isomorphism of the polynomial Lie algebra described above (for \( N = 5 \)) and the polynomial Lie algebra constructed in [17] on the basis of the theory of two-dimensional sigma-functions.

**Proof** The isomorphism \( \varphi \) and its inverse are given by
\[
\begin{align*}
u_2 &\to x_2, & x_2 &\to u_2, \\
v_3 &\to \frac{2}{3} x_3, & x_3 &\to 2v_3, \\
u_4 &\to x_2^2 + 4 z_4, & x_4 &\to 5 u_2^2 + u_4 + 2 y_4, \\
v_5 &\to \frac{1}{2} (x_2 x_4 + 2 z_5), & z_4 &\to \frac{1}{2} (u_4 - u_2^2), \\
y_4 &\to \frac{1}{2} (-6 u_2^2 + x_4 - 4 z_4), & z_5 &\to v_5 - u_2 v_3, \\
y_6 &\to \frac{1}{4} (-8 x_2 z_4 + 8 x_4^2 + 2 x_4 x_2 + x_3^2 + 2 z_6), & z_6 &\to 2(u_2 y_4 + u_2 u_4 - v_2^2 - y_6).
\end{align*}
\]

A direct verification shows that this isomorphism determines the required isomorphism of polynomial Lie algebras, which is the identity isomorphism at the generators \( L_0, L_2, L_4, L_6, L_1^*, \) and \( L_3^* \). This proves the theorem. \( \square \)

## 10 Integrable Hamiltonian polynomial systems on \( \mathbb{R}^4 \)

In the previous sections we have shown that a polynomial Lie algebra of vector fields can be canonically associated with a universal space of symmetric squares of hyperelliptic curves. Commuting vector fields \( L_{N-4}^*, L_{N-2}^* \) correspond a pair of compatible dynamical systems in variables \( u_2, u_4, v_{N-2}, v_N \) which depend on parameters \( y_4, \ldots, y_{2N-4} \)
\[
\begin{align*}
\partial_y u_2 &= L_{N-4}^* u_2, \\
\partial_y u_4 &= L_{N-4}^* u_4, \\
\partial_y v_{N-2} &= L_{N-4}^* v_{N-2}, \\
\partial_y v_N &= L_{N-4}^* v_N; \\
\partial_y u_2 &= L_{N-2}^* u_2, \\
\partial_y u_4 &= L_{N-2}^* u_4, \\
\partial_y v_{N-2} &= L_{N-2}^* v_{N-2}, \\
\partial_y v_N &= L_{N-2}^* v_N.
\end{align*}
\]

The images \( H_{2N-2} = \varphi(y_{2N-2}) \) and \( H_{2N} = \varphi(y_{2N}) \) are common first integrals of the above systems. Assuming the parameters \( y_4, \ldots, y_{2N-4} \) and variables \( u_2, u_4, v_{N-2}, v_N \) to be real we obtain two integrable polynomial dynamical system on \( \mathbb{R}^4 \). Moreover, the systems obtained are Hamiltonian with the Hamiltonians \( H_{2N-2}, H_{2N} \) and the only nonzero Poisson brackets given by
\[
\{ u_2, v_{N-2} \} = 2, \quad \{ u_4, v_N \} = 4.
\]

These Hamiltonians are in involution since the corresponding vector fields commute. Thus the systems [55] are Liouville integrable. In terms of the original coordinate \( (X_1, Y_1; X_2, Y_2) \in \mathbb{C}^4 \) Poisson brackets [55] correspond to the canonical brackets
\[
\{ X_i, Y_j \} = \delta_{ij}, \quad \{ X_i, X_j \} = \{ Y_i, Y_j \} = 0.
\]
As we shall see below, in the cases \( N = 3, 4 \) these systems can be integrated in terms of elliptic functions corresponding to the Jacobian of the curve. In the case \( N = 5, 6 \) the system can be integrated in Abelian functions. For any \( N \neq 5, 6 \) the system cannot be integrated in \( 2g = 2 \left( \frac{N}{2} \right) \) periodical Abelian functions. In the case \( N = 7 \) a general solution of the system can be expressed in terms of meromorphic functions which are 6 periodic being restricted to the \( \sigma \) divisor of the Jacobian \([12]\).

10.1 Dynamical system in the case \( N = 3 \)

In the case \( N = 3 \) the commuting vector fields \( \mathcal{L}^*_3, \mathcal{L}^*_1 \) act on \( \mathbb{R}^4 \) with coordinates \( v_1, u_2, v_3, u_4 \) and define two compatible (commuting) dynamical systems

\[
\begin{align*}
\partial_\xi v_1 &= 1, \\
\partial_\xi u_2 &= 2v_1, \\
\partial_\xi v_3 &= 3u_2, \\
\partial_\xi u_4 &= 4v_3; \\
\partial_\eta v_1 &= v_1^2 - u_2, \\
\partial_\eta u_2 &= u_2 v_1 - v_3, \\
\partial_\eta v_3 &= \frac{1}{2}(3u_2^2 - u_4 - 2v_1 v_3), \\
\partial_\eta u_4 &= 2u_2 v_3 - 2u_1 v_1;
\end{align*}
\]  

(57)

where \( \partial_\xi \) and \( \partial_\eta \) stand for \( \mathcal{L}^*_3 \) and \( \mathcal{L}^*_1 \) respectively. The above dynamical systems possess two first integrals

\[
\begin{align*}
y_4 &= v_1 v_3 - \frac{1}{4}(3u_2^2 + u_4); \\
y_6 &= \frac{1}{4}(u_2 u_4 - v_1^2 + 2u_2 v_1 v_3 - u_2^2 - u_4 v_1^2).
\end{align*}
\]  

(59, 60)

in involution (which are the Hamiltonians of these systems) with respect to the Poisson structure given by \([58]\) and thus are Liouville integrable. Moreover it is not difficult to show that a general simultaneous solution to the systems can be written in the form

\[
\begin{align*}
v_1 &= \xi + f, \\
v_2 &= \xi^2 + 2\xi f + f^2 - f_0, \\
v_3 &= \xi^3 + 3\xi^2 f + 3\xi (f^2 - f_0) + f^3 - 3f_0 f + f_{\eta}, \\
v_4 &= \xi^4 + 4\xi^3 f + 6\xi^2 (f^2 - f_0) + 4\xi (f^3 - 3f_0 f + f_{\eta}) + g.
\end{align*}
\]  

(61)

where

\[
g = f^4 - 6f_0 f^2 + 9f_0^2 + 4f_{\eta} f - 2f_{\eta \eta}.
\]

and function \( f = f(\eta) \) is a general solution to the equation

\[
f_{\eta \eta}^2 = f_0^3 + 4y_4 f_{\eta} - 4y_6
\]

as it follows from system \([58]\) and the first integral \([60]\). A general solution of this equation can be expressed in terms of the Weierstrass \( \zeta \) function

\[
f = -\zeta(\eta - \eta_0; g_2, g_3) + f_0, \quad g_2 = -4y_4, \quad g_3 = 4y_6.
\]  

(62)

Thus \([51]\) together with \([62]\) represent a simultaneous general solution to the system \([57], [58]\) which depends on four arbitrary parameters \( f_0, \eta_0, y_4, y_6 \).

10.2 Dynamical system in the case \( N = 4 \)

In the case \( N = 4 \) we consider dynamical systems corresponding the symmetric square of elliptic curve

\[
Y^2 = X^4 + y_4 X^2 - y_6 X + y_8.
\]  

(63)

The commuting vector fields \( \mathcal{L}^*_5, \mathcal{L}^*_2 \) act on \( \mathbb{R}^4 \) with coordinates \( u_2, v_2, u_4, v_4 \) and define two compatible (commuting) dynamical systems depending on the parameter \( y_4 \)

\[
\begin{align*}
\partial_\xi u_2 &= 2v_2, \\
\partial_\xi v_2 &= 2u_2, \\
\partial_\xi u_4 &= 4v_4, \\
\partial_\xi v_4 &= 3u_2^2 + u_4 + 2y_4; \\
\partial_\eta u_2 &= u_2 v_2 - v_4, \\
\partial_\eta v_2 &= \frac{1}{2}(-u_2^2 - u_4 + 2v_2^2 - 2y_4), \\
\partial_\eta u_4 &= 2u_2 v_4 - 2u_4 v_2, \\
\partial_\eta v_4 &= \frac{1}{2}(3u_2^2 - u_2 u_4 - 2v_2 v_4 + 2u_2 y_4);
\end{align*}
\]  

(64)

20
where $\partial_x$ and $\partial_y$ stand for $\mathcal{L}_0^*$ and $\mathcal{L}_2^*$ respectively. The above dynamical systems possess two common first integrals (Hamiltonians)

$$y_6 = \frac{1}{4}(2u_2^2 + 2u_2u_4 - 4v_2v_4 + 4u_2y_4),$$
$$y_8 = \frac{1}{16}(3u_2^2 - 2u_2^2u_4 - u_4^2 + 4u_4v_2^2 - 8u_2v_4v_4 + 4v_4^2 + 4u_2^2y_4 - 4u_4y_4)$$  

(65)

in involution and thus are Liouville integrable.

The first system is linear inhomogeneous with constant coefficients and its general solution is of the form

$$u_2 = \alpha_1 e^{2x} + \alpha_2 e^{-2x},$$
$$v_2 = \alpha_3 e^{2x} - \alpha_2 e^{-2x},$$
$$u_4 = \alpha_4 e^{4x} + \alpha_5 e^{-4x} - 6\alpha_1 \alpha_2 - 2y_4 + 2\beta_1 e^{2x} + 2\beta_2 e^{-2x},$$
$$v_4 = \alpha_6 e^{4x} - \alpha_2 e^{-4x} + \beta_1 e^{2x} - \beta_2 e^{-2x},$$  

(66)

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary constants. In order to satisfy the second system we should assume that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are functions of $\eta$. It follows from the second system that

$$a_1 = -\beta_1, \quad a_2 = \beta_2,$$  

(67)

$$a_3 = 2\alpha_1(4\alpha_1 \alpha_2 + y_4), \quad a_4 = -2\alpha_2(4\alpha_1 \alpha_2 + y_4).$$  

(68)

Variables $\beta_1, \beta_2$ can be eliminated

$$a_1' + 2\alpha_1(4\alpha_1 \alpha_2 + y_4) = 0,$$  

(69)

$$a_2' + 2\alpha_2(4\alpha_1 \alpha_2 + y_4) = 0.$$

Substitution (65), (66) in (65) results in two first integrals

$$y_8 = a_1' a_2' + \frac{1}{4}(4\alpha_1 \alpha_2 + y_4)^2,$$
$$y_6 = 2(a_1 a_2' - a_2 a_1')$$  

(70)

of the system. They can be viewed as a result of two integrations of the system (65) with two arbitrary constants $y_6, y_8$. Equations (70) is a system of two first order equations which can be integrated using elliptic functions. Indeed, let us introduce function $\rho = \alpha_1 a_2$. Then, it follows from equations (70) that

$$r^2 + 16r^3 + 8y_4 r^2 + (y_4^2 - 4y_8^2) r - \frac{1}{4} y_6^2 = 0.$$  

(71)

We can transform this equation to the canonical Weierstrass form by a linear change of variables

$$\rho = -4r - \frac{2}{3} y_4,$$

then (71) takes the form

$$(\rho')^2 = 4\rho^3 - 2\rho - g_3, \quad g_2 = 16y_4 + \frac{4}{3} y_4^2, \quad g_3 = \frac{32}{3} y_4 y_8 - 4y_6^2 - \frac{8}{27} y_4^3.$$  

(72)

It is well known (see for example [1]) that the modular invariants $\hat{g}_2, \hat{g}_3$ of a regular elliptic curve

$$Y^2 = a_0 X^4 + 4a_1 X^3 + 6a_2 X^2 + 4a_3 X + a_4$$

are of the form

$$\hat{g}_2 = a_0 a_4 - 4a_1 a_3 + 3(a_2)^2, \quad \hat{g}_3 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - (a_2)^3 - a_0 (a_3)^2 - (a_1)^2 a_4.$$  

In our case $a_0 = 1, a_1 = 0, a_2 = y_4/6, a_3 = -y_6/2, a_4 = y_8$ and thus

$$g_2 = 4^2 \hat{g}_2, \quad g_3 = 4^3 \hat{g}_3$$

and therefore the modules of the original curve (63) and the curve corresponding to equation (72) coincide. Solution to the above equation can be expressed in terms of the Weierstrass elliptic function

$$\rho = \wp(\eta - \eta_0; g_2, g_3),$$

and therefore

$$r = -\frac{1}{4} \wp(\eta - \eta_0; g_2, g_3) - \frac{1}{6} y_4,$$

(73)

21
where \( \eta_0 \) is a constant of integration. The second equation in the system (71) can be rewritten in the form

\[
2 \left( \ln \frac{\alpha_2}{\alpha_1} \right)_\eta = \frac{y_6}{r}
\]

and integrated

\[
\frac{\alpha_2}{\alpha_1} = C \exp \left( y_6 \int \frac{d\eta}{2r} \right)
\]

where \( C \) is an arbitrary integration constant. Thus (66) with

\[
\alpha_1 = \sqrt{C^{-1}r} \exp \left( -y_6 \int \frac{d\eta}{4r} \right), \quad \alpha_2 = \sqrt{C r} \exp \left( y_6 \int \frac{d\eta}{4r} \right)
\]

and \( \beta_1 = -\alpha_1 \eta, \beta_2 = \alpha_2 \eta \) is a general solution of two compatible systems (66) with four arbitrary constants \( \eta_0, C, y_6, y_8 \) and an arbitrary parameter \( \eta_4 \).

### 10.3 Dynamical system in the case \( N = 5 \)

In the case \( N = 5 \) the commuting vector fields \( \mathcal{L}_1^*, \mathcal{L}_2^* \) (which represent \( \partial_\xi \) and \( \partial_\eta \)) act on \( \mathbb{R}^4 \) with coordinates \( u_2, v_3, u_4, v_5 \) and define two compatible (commuting) dynamical systems

\[
\begin{align*}
\partial_\xi u_2 &= 4u_3, \\
\partial_\xi v_3 &= 5u_2^2 + u_4 + 2y_4, \\
\partial_\xi u_4 &= 8u_5, \\
\partial_\xi v_5 &= 5u_2^2 + 5u_2u_4 + 6u_4y_4 - 4y_6; \\
\partial_\eta u_2 &= 4u_5 - 4u_2u_4, \\
\partial_\eta v_3 &= -4(u_3^2 - u_2u_4 - u_2y_4 + y_6), \\
\partial_\eta u_4 &= 8(u_3u_4 - u_2u_5), \\
\partial_\eta v_5 &= u_2^2 - 5u_2^2 + 4u_4u_5 - 6u_2^2z_4 + 2u_4y_4 + 4u_2y_6;
\end{align*}
\]

(73)

with two common first integrals

\[
\begin{align*}
y_8 &= \frac{1}{16}(-5u_2^4 - 10u_2^2u_4 - u_2^2 + 16v_5v_6 - 12u_2^2y_4 - 4u_4y_4 + 16u_2y_6), \\
y_{10} &= \frac{1}{16}(-2u_2^4 + 2u_2u_4 - u_2^2 + v_5v_6 + 8u_2v_5v_6 - 4v_5^2 - 4u_2^2y_4 + 4u_2u_4y_4 + 4u_2^2y_6 - 4u_4y_6).
\end{align*}
\]

This system can be integrated in terms of Abelian functions on the Jacobian. Indeed, consider the sigma-function \( \sigma = \sigma(w; \tilde{y}) \), \( w = (\xi, \eta) \) (see [11]) associated with the curve

\[
\{(X, Y) \in \mathbb{C}^2 : Y^2 = X^5 + y_1X^3 - y_6X^2 + y_8X - y_{10}\},
\]

(75)

and define Abelian \( \varphi_{i,3j} \) functions [11]

\[
\varphi_{i,3j} = \varphi_{i,3j}(w; \tilde{y}) = -\frac{\partial^{i+j}}{\partial \xi^i \partial \eta^j} \ln \sigma.
\]

Then a general common solution for the two compatible systems can be written in the form:

\[
\begin{align*}
u_2 &= \varphi_{2,0}(w - w_0; \tilde{y}), \\
v_3 &= \frac{1}{2}\varphi_{3,0}(w - w_0; \tilde{y}), \\
u_4 &= \varphi_{2,0}^2(w - w_0; \tilde{y}) + 4\varphi_{1,3}(w - w_0; \tilde{y}), \\
v_5 &= \frac{1}{2}\varphi_{2,0}(w - w_0; \tilde{y})\varphi_{3,0}(w - w_0; \tilde{y}) + \varphi_{2,3}(w - w_0; \tilde{y}).
\end{align*}
\]

The above solution depends on four arbitrary constants \( w_0 = (w_0, \eta_0) \), \( y_8, y_{10} \). Function \( u_2 \) satisfies the classical KdV equation and represents two gap solution [11]. In variables \( X_1, Y_1, X_2, Y_2 \) (Lemma [11] systems [73] and [74] correspond to Dubrovin’s systems (2.12) and (3.9) in [15] respectively. Thus our approach can be considered as a generalisation of the Dubrovin construction to the cases when the symmetric power of the curve does not coincide with its genus. Moreover, we have shown that in the case \( g = 2 \) a change of variables defined in Lemma [11] transforms Dubrovin’s systems in polynomial Hamiltonian systems. It is important that in our approach the problem to construct real valued solutions can be easily resolved, indeed it is sufficient to choose the parameters of the curve \( y_4, \ldots, y_{10} \) to be real.
10.4 Dynamical system in the case $N = 7$

In the case $N = 7$ the commuting vector fields $L^1_3, L^5_3$ (which represent $\partial_{u_1}$ and $\partial_{\eta}$) act on $\mathbb{R}^4$ with coordinates $u_2, u_4, v_5, v_7$ and define two compatible (commuting) dynamical systems

$$
\begin{align*}
\begin{cases}
\partial_{u_2} &= 2v_5, \\
\partial_{u_4} &= 4v_7, \\
\partial_{v_5} &= \frac{1}{10} (40u_2^3y_4 - 32u_2y_6 + 8u_4y_4 + 35u_4^4 + 42u_4u_2^2 + 3u_4^2 + 16y_6), \\
\partial_{v_7} &= \frac{1}{10} (40u_2^3y_4 - 48u_2y_6 + 40u_4u_2y_4 + 48u_2y_8 - 16u_4y_6 + 21u_2^5 + 70u_4u_2^3 + 21u_2u_2^2 - 32y_10);
\end{cases}
\end{align*}
$$

These two compatible systems possess two common first integrals:

$$
\begin{align*}
y_{12} &= \frac{1}{64} (-7u^6 + 35u_4u_1u_4 - 21u_2u_2u_4 - u_4^3 + 64v_5v_7 + 12u_2y_4 + 40u_4u_2y_2 + 12u_2u_2y_2 - \\
&\quad 20u_2y_4 - 40u_2u_2u_4 - 4u_2^3y_2 + 32u_2y_6 + 32u_2u_4y_6 - 48u_2y_8 - 16u_4y_6 + 64u_2y_10), \\
y_{14} &= \frac{1}{64} (-3u^7 - 7u_2u_4 + 7u_2u_2u_4 + 3u_2u_4^3 - 16u_4u_2^2 + 32u_2v_5v_7 - 16v_4^2 + 12u_2u_2y_2 + \\
&\quad + 5u_2u_4u_2 - 9u_2u_4u_2 - u_2u_4 - 8u_2y_4 + 8u_2u_4y_4 + 12u_2y_6 - 8u_2u_4y_6 - 4u_2^3y_6 - 16u_2^3y_8 + \\
&\quad + 16u_2u_4y_8 + 16u_2^2y_10 - 16u_4y_10).
\end{align*}
$$

For the hyperelliptic curve of genus 3

$$
V_\sigma = \{(X,Y) \in C^2: Y^2 = X^7 + y_4X^5 + \ldots - y_{14}\},
$$

there defined $\sigma$ function $\sigma(w_1, w_3, w_5)$ (see [13]). It is an entire function on $\mathbb{C}^3$ with co-ordinates $w_1, w_3, w_5$. In $\mathbb{C}^3$ there is an analytic surface $W = \{(w_1, w_3, w_5) \in \mathbb{C}^3 | \sigma(w_1, w_3, w_5) = 0\}$. The surface $W$ is 6-periodic in $\mathbb{C}^3$ and thus provides us with a $\sigma$-divisor, namely $W/\Lambda \subset \text{Jac}(V)$, where $\Lambda$ is the lattice of periods.

The above system can be integrated in meromorphic functions on $\mathbb{C}^4$ which can be explicitly expressed in terms of the gradient of the sigma function $\sigma$. The solutions of the above system are meromorphic on $\mathbb{C}^4$, they are not Abelian with respect to the lattice $\Lambda$ of the curve [7], but their restrictions on $W$ are 6-periodic and therefore correctly defined on the $\sigma$ divisor.

Acknowledgements

We are grateful to V. M. Rubtsof, V. V. Sokolov and A. V. Tsiganov, for useful discussions of the results of our work.

References

[1] V. M. Buchstaber and A.V. Mikhailov. Infinite dimensional Lie algebras determined by the space of symmetric squares of hyperelliptic curves. *Functional Anal. Appl.*, 51(1):4–27, 2017.

[2] Ph. J. Higgins and K. Mackenzie. Algebraic constructions in the category of the Lie algebroids. *J. Algebra*, 129:194–230, 1990.

[3] P. W. Michor. *Topics in Differential Geometry*, volume 93 of *Graduate Studies in Math.* Amer. Math. Soc., Providence, RI, 2008.
[4] V. I. Arnold. *Singularities of Caustics and Wave Fronts*, volume 62 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, 1990.

[5] V. M. Buchstaber and D. V. Leykin. Polynomial Lie algebras. *Functional Anal. Appl.*, 36(4):267–280, 2002.

[6] A. M. Perelomov. *Integrable Systems of Classical Mechanics and Lie Algebras*. Number Bd. 1. Birkhäuser Basel, 1989.

[7] A.V. Tsyganov. *Integrable systems in the method of separation of variables*. (in Russian). Modern Mathematics. R&С Dynamics, Moscow, Izhevsk, 2005.

[8] V. I. Arnold. Wave front evolution and equivariant Morse lemma. *Comm. Pure Appl. Math.*, 29(6):557–582 [correction 30:6 (1977), 823], 1976.

[9] B. Enriquez and V. Rubtsov. Commuting families in skew fields and quantization of Beauville’s fibration. *Duke Math. J.*, 119(2):197–219, 2003.

[10] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov. Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties. *Russian Math. Surveys*, 31(1):59146, 1976.

[11] V. M. Buchstaber. Polynomial dynamical systems and the Korteweg-de Vries equation. *Proc. Steklov Inst. Math.*, 294:176–200, 2016.

[12] T. Ayano and V. M. Buchstaber. The field of meromorphic functions on sigma divisor of a hyperelliptic genus 3 curve and application. *Functional Anal. Appl.*, 51(3):4–21, 2017.

[13] E.T. Whittaker and G.N. Watson. *A Course of Modern Analysis*. 1927. CUP.

[14] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leikin. Hyperelliptic Kleinian functions and applications. In *Solitons, Geometry and Topology: On the Crossroad*, volume 179 of *Amer.Math. Soc. Trans., Ser. 2*, pages 1–33. Amer. Math. Soc., Providence, RI, 1997.

[15] B. A. Dubrovin. Periodic problems for the Korteweg-de Vries equation in the class of finite band potentials. *Functional Anal. Appl.*, 9(3):215–233, 1975.