A Theoretical Analysis of Granulometry-based Roughness Measures on Cartosat DEMs

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Abstract—The study of water bodies such as rivers is an important problem in the remote sensing community. A meaningful set of quantitative features reflecting the geophysical properties help us better understand the formation and evolution of rivers. Typically, river sub-basins are analysed using Cartosat Digital Elevation Models (DEMs), obtained at regular time epochs. One of the useful geophysical features of a river sub-basin is that of a roughness measure on DEMs. However, to the best of our knowledge, there is not much literature available on theoretical analysis of roughness measures. In this article, we revisit the roughness measure on DEM data adapted from multiscale granulometries in mathematical morphology, namely multiscale directional granulometric index (MDGI). This measure was classically used to obtain shape-size analysis in greyscale images. In earlier works, MDGIs were introduced to capture the characteristic surficial roughness of a river sub-basin along specific directions. Also, MDGIs can be efficiently computed and are known to be useful features for classification of river sub-basins. In this article, we provide a theoretical analysis of a MDGI. In particular, we characterize non-trivial sufficient conditions on the structure of DEMs under which MDGIs are invariant. These properties are illustrated with some fictitious DEMs. We also provide connections to a discrete derivative of volume of a DEM. Based on these connections, we provide intuition as to why a MDGI is considered a roughness measure. Further, we experimentally illustrate on Lower-Indus, Wardha, and Barmer river sub-basins that the proposed features capture the characteristics of the river sub-basin.

Index Terms—Digital Elevation Model, Cartosat, Granulometric Index, Mathematical Morphology.

I. INTRODUCTION

The study of geophysical properties of rivers is an important problem in the remote sensing community. A study of a river sub-basin at regular time epochs over a large span of time helps understand the evolution of the river. The evolution of river sub-basins provides information required to prioritize the rivers that need immediate attention for conservation/identify natural calamities etc. However, such a study is highly dependent on extracting meaningful geophysical features of the river sub-basins. For example, the complexity of the surficial roughness of a river sub-basin provides information as to what the dominant wind directions are, in that region.

Recall that Cartosat Digital Elevation Models (DEMs) are typically used to compute geophysical features of river sub-basins. In literature, several studies indicate that surficial roughness is an important characteristic of a river sub-basin. However, to the best of our knowledge, there is not much literature available on theoretical analysis of roughness measures. In this article, we analyse in detail, a surficial roughness measure that was proposed in [5] i.e. multiscale directional granulometric index (MDGI), a special case of a more general measure namely a multiscale granulometric index.

Multiscale granulometric index was originally proposed in [3] to obtain a shape-size analysis of objects in greyscale images. As greyscale images can be viewed as digital surfaces with the greyscale intensity at each pixel representing the height of the surface, these measures have been adapted to DEMs [12]. It was shown experimentally that such an adaptation is indeed useful from an application point-of-view i.e. to classify river sub-basins [5]. A natural question would then be to ask: Can we characterize the equivalence classes of DEMs obtained by the equivalence relation - two DEMs are equivalent if their MDGI are identical? In other words, can we find necessary and sufficient conditions on the structure of a DEM under which a MDGI is invariant? In this article, we partially answer this question and provide theoretical insights on how a MDGI varies with the structure of a DEM.

In particular, the contributions of this article are as follows:

1) We provide an alternate visualization of the definition of a MDGI proposed in [5].

2) Using the alternate visualization, we characterize non-trivial sufficient conditions on DEMs under which a MDGI is invariant. The invariance properties are intuitively explained and illustrated on fictitious DEMs.

3) We analyse the relation between a MDGI and a discrete derivative of volume of a DEM. Using this analysis, we provide an intuition as to why a MDGI is considered a roughness measure.

4) Further, a preliminary application of MDGI is shown on real data i.e. Cartosat-1 DEM data of Lower-Indus, Wardha, and Barmer river sub-basins to show that these measures capture characteristics of the sub-basin.

The rest of the article is organized as follows: In section II, we provide the definitions of basic morphological operators and multiscale granulometric index. Also, the existing literature on the usage of directional granulometric indices is briefly
II. Multiscale Granulometric Index

In this section, we recall the formal definitions of a multiscale granulometric index and briefly describe the existing literature. First, we start with the basic definitions.

A. Elementary Morphological Operators

Definition 1: Let $A \subset \mathbb{Z}^2$ be a finite set. A digital elevation model (DEM) of a river basin/sub-basin is represented as a function $f : A \rightarrow H$ where $H \subset \mathbb{Z}^+$ is a finite set.

Each point $a \in A$ represents a small physical, square area and $f(a)$ represents the discretized average height of the physical area. Also, observe that $A$ is possibly non-rectangular i.e. $A = \bigcup_{i=1}^{m_1} A_i$ where $A_i = \{(i,j) : m_{1,i} \leq j \leq m_{2,i}\}$. See Fig 1 for an illustration. This is in contrast to greyscale images where $A$ is always rectangular i.e. $m_{1,i}$ and $m_{2,i}$ are independent of $i$. The number of distinct elements in $H$ are comparable to the number of grey levels in a greyscale image. A higher cardinality of $H$ indicates a finer resolution in the elevations and is analogous to a finer spectral resolution in greyscale images.

Next, we need the definition of a structuring element. Using the notion of a structuring element, dilation and erosion, the fundamental blocks of roughness measures based on multiscale granulometric index are then defined. Note that we restrict the definition of structuring element i.e. assume that the structuring element contains its origin and is symmetric. This definition suffices for the purposes of this article.

Definition 2: A structuring element $SE \subset \mathbb{Z}^2$ is a finite set such that: (1) $(0,0) \in SE$, (2) $(i,j) \in SE \Rightarrow (-i,-j) \in SE$.

The different types of structuring elements used in this article are given by $B_1 = \{(-1,1), (0,0), (1,-1)\}$, $B_2 = \{(0,1), (0,0), (0,-1)\}$, $B_3 = \{(-1,-1), (0,0), (1,1)\}$, $B_4 = \{(-1,0), (0,0), (1,0)\}$, and $B = \{(x,y) \in \mathbb{Z}^2 : -1 \leq x, y \leq 1\}$. Fig 2 provides a pictorial representation of these structuring elements. Observe that each of the structuring elements $B_1, B_2, B_3, B_4$ are 3 units long and are effectively one-dimensional.

Recall that a greyscale dilation and a greyscale erosion are defined as follows:

Definition 3: Let $f : A \rightarrow H$ be a DEM and let $SE$ be a structuring element, then a dilation of $f$ by $SE$ is given by

$$[f \oplus SE](x,y) = \max_{(s,t) \in SE} \{f(x+s,y+t)\} \tag{1}$$

where $SE$ is a structuring element.

Definition 4: Let $f : A \rightarrow H$ be a DEM and let $SE$ be a structuring element, then an erosion of $f$ by $SE$ is given by

$$[f \ominus SE](x,y) = \min_{(s,t) \in SE} \{f(x+s,y+t)\} \tag{2}$$

where $SE$ is a structuring element.

Next, we need the definition of a morphological opening and a multiscale morphological opening.

Definition 5: Let $f : A \rightarrow H$ be a DEM and let $SE$ be a structuring element, then an opening of $f$ is given by

$$[f \circ SE](x,y) = \max \{f \ominus SE \oplus SE | (x,y)\} \tag{3}$$

Definition 6: Let $f : A \rightarrow H$ be a DEM and let $SE$ be a structuring element, then a multiscale opening of $f$ is given by

$$[f \circ nSE](x,y) \tag{4}$$

where $nSE = SE \oplus SE \oplus \cdots \oplus SE$ with the number of dilations in the telescoping expression being $n-1$.

B. Directional Multiscale Granulometric Index

Before we provide a formal definition of a multiscale granulometric index, we need to define the notion of volume of a DEM.

Definition 7: Let $f : A \rightarrow H$ be a DEM. The volume of $f$, $V(f)$ is defined as follows:

$$V(f) = \sum_{a \in A} f(a) \tag{5}$$

Intuitively, the volume of a DEM captures the physical volume of a DEM on and above the altitude chosen to be zero. For example, the volume of DEM shown in Fig 1 is 46. It is easy to see that an application of a multiscale opening results in a DEM with lower volume as $n$ increases. Also, it is easy to see that there exists $N_0 \in \mathbb{N}$ such that $V(f \circ nSE) = V(f \circ (n+1)SE)$ $\forall n \geq N_0$. Recall that the definition of multiscale granulometric index is given by

Definition 8:

$$GI_{SE}(f) = -\sum_{n=0}^{\infty} p_n \log(p_n) \tag{6}$$

where

$$p_n = \frac{V(f \circ nSE) - V(f \circ (n+1)SE)}{V(f)} \tag{7}$$
Note that the existence of $N_0 \in \mathbb{N}$ such that $V(f \circ nSE) = V(f \circ (n + 1)SE) \forall n \geq N_0$ ensures that the summation is finite. The terms inside the summation for $n \geq N_0$ have to be interpreted as zero. When the structuring element $SE$ is chosen to be one of $B_1, B_2, B_3, B_4$, the obtained multiscale granulometric index is said to be a directional multiscale granulometric index or MDGI. Intuitively, this makes sense as each of $B_1, B_2, B_3, B_4$ are linear and indicate four primary directions.

C. Existing Literature

Multiscale granulometric index was first introduced in [3] to perform a shape-size analysis of objects in greyscale images. Then it was used to analyse textures in greyscale images [6]. Later, these ideas were generalized to analyse soil section image analysis [14]. Multiscale granulometric index was theoretically analyzed from the perspective of identifying shapes and sizes of objects in greyscale images. The utility of granulometries in greyscale images led to the development of efficient algorithms for specialized classes of structuring elements [15], [4].

Very recently, these ideas were adapted to DEMs. It was experimentally shown in [5] that multiscale granulometric indices obtained using specific structuring elements retain characteristic information of the river basins. A natural question would then be to ask: Can we find necessary and sufficient conditions on the structure of a DEM under which the directional granulometric index is invariant? In the next section, we partially answer this question by identifying non-trivial sufficient conditions on DEMs such that all DEMs satisfying such conditions have the same directional granulometric index.

III. THEORETICAL ANALYSIS OF DIRECTIONAL GRANULOMETRIC INDICES

In this section, we analyse the MDGIs from a theoretical perspective. Firstly, we recall some modified definitions from graph theory to suit the purposes of subsequent analysis on a MDGI. Secondly, we provide an alternate way to view a MDGI using graphs. Then, by building on this visualization of MDGI, we provide intuition on sufficient conditions under which DEMs have the MDGI. Then, we prove the main result of this section image analysis [14]. Multiscale granulometric index was first introduced in [3].

Definition 10: Let $G = (V, E, W)$ be a node-weighted graph. Let $W : V \rightarrow H$ and $h \in H$. $G_{\geq h} = (V_{\geq h}, E_{\geq h}, W_{|V_{\geq h}})$ is said to be an upper-thresholded subgraph of $G = (V, E, W)$ at elevation $h$, where $V_{\geq h} = \{v \in V : W(v) \geq h\}$, $E_{\geq h} = \{(v_i, v_j) : (v_i, v_j) \in E \text{ and } W(v_i) \geq h, W(v_j) \geq h\}$, and $W_{|V_{\geq h}}$ denotes the restriction of the function $W$ to $V_{\geq h}$.

Intuitively, an upper-thresholded subgraph of a node-weighted graph constructed on a DEM provides an abstraction of the sub-structure of the the DEM that is above an elevation level. For example, Fig 4 illustrates the upper-thresholded subgraph at elevation 3 on the node-weighted graph given by Fig 3.

Definition 11: Let $G = (V, E, W)$ be a node-weighted graph. A subset of nodes $V_1 \subset V$ is said to be connected if for every pair of nodes $v_i, v_j \in V_1$, there exists a sequence of nodes $v_i = v_0, v_1, \cdots, v_{r-1}, v_r = v_j$ such that $\{v_i, v_{i+1}\} \in E$ for every $0 \leq i \leq r - 1$. A subset of nodes $V_1 \subset V$ is said to be maximally connected if (1) $V_1$ is connected, and (2) $V_1 \subset V_2 \subset V$ and $V_2$ is connected implies $V_2 = V_1$.

We remark that given any node-weighted graph $G = (V, E, W)$, the set $V$ can be decomposed uniquely as a disjoint union of maximally connected subsets of $V$. For example, the node-weighted graph in Fig 3 has one maximally connected subset which is the vertex set itself. Similarly, the upper-thresholded graph in Fig 4 which is also a node-weighted graph has two maximally connected subsets of the vertex set.

B. Another Interpretation of a Directional Granulometric Index

Recall from subsection II-B that a multiscale granulometric index is given by Def 8 (see Eq 6 and Eq 7). Intuitively, a
A multiscale granulometric index measures the entropy of the volume loss on the series of morphological openings with increasing sizes of the structuring element.

Assume that the structuring element \( SE \) is given by one of \( B_1, B_2, B_3, B_4 \) as defined in subsection II-A. Each of the four structuring elements are effectively one-dimensional. Hence, a directional granulometric index effectively measures volume loss on linear scans (but in different directions). Fig 5 shows an illustration of the scans obtained for SEs \( B_4 \) and \( B_3 \).

![Illustration of scans](image)

Fig. 5. One dimensional scans on the DEM in Fig 1 required to obtain a theoretical analysis of \( GI_{B_4}(f) \) and \( GI_{B_3}(f) \).

Thus, same theoretical analysis on MDGI holds for each of \( B_1, B_2, B_3, B_4 \). Let \( f : A \rightarrow H \) be a DEM. In order to understand the directional granulometric index better, we first try to analyze a one-dimensional DEM i.e. working with DEMs restricting the domain to horizontal scans. The analysis for a generic two-dimensional set \( A \) would be a straightforward extension with slightly involved notation. Mathematically, such a restriction would be equivalent to working with sets of type:

\[
A_{i_0} = \{(i_0, j) : m_1 \leq j \leq m_2 \} \subset A
\]

for a fixed \( i_0 \in \mathbb{Z} \) and \( m_1 < m_2 \in \mathbb{Z} \). In general, \( m_1, m_2 \) depend on \( i_0 \) as \( A \) is not necessarily rectangular. However, we blur this detail to work with a simplified notation.

We now analyse a MDGI obtained by multiscale openings using horizontal linear structuring elements \( \{L_n : n \in \mathbb{Z}^+ \} \), where \( L_n \) denotes a horizontal structuring element with \( n \) consecutive 1s. A similar analysis holds for \( \{nB_k : n \in \mathbb{Z}^+ \} \) because \( \{nB_k : n \in \mathbb{Z}^+ \} \subseteq \{L_n : n \in \mathbb{Z}^+ \} \) (in particular \( nB_k = L_{2n+1} \) for each \( n \in \mathbb{Z}^+ \)). We are now ready to examine \( GI_{L_n}(f|A_{i_0}) \) given by Eq 1 and Eq 2 where \( f : A \rightarrow H \) is a DEM. Let \( G^f = (A_{i_0}, E_{\text{chain}}, f|A_{i_0}) \) denote a node-weighted graph with

\[
E_{\text{chain}} = \{((i_0, j), (i_0, j+1)) : m_1 \leq j \leq m_2 - 1 \}
\]

Consider the sequence of upper-thresholded subgraphs of the node-weighted graph \( G^f = (A_{i_0}, E_{\text{chain}}, f|A_{i_0}) \) at all possible elevations i.e.

\[
\{G^f_{\geq h}\}_{\min(H) \leq h \leq \max(H)} = \{(V^f_{\geq h}, E^f_{\geq h}, f|V^f_{\geq h})\}_{\min(H) \leq h \leq \max(H)}
\]

Let \( V^f_{\geq h} = \bigcup_{r=1}^{n_{t,h}} V^f_{\geq h,r} \) denote the disjoint union of maximally connected subsets for each \( h \in [\min(H), \max(H)] \). Denote \( n_{t,h} \) as

\[
n_{t,h} = |\{V^f_{\geq h,r} : |V^f_{\geq h,r}| = t\}|
\]

Here \( n_{t,h} \) denotes the number of maximally connected subsets of \( V^f_{\geq h} \) which are exactly \( t \) units long. It is easy to see that the probabilities given by Eq 7 satisfy

\[
p_k \propto k \sum_{h=\min(H)}^{\max(H)} n_{k,h},
\]

for each \( k \in \mathbb{Z}^+ \). This is because \( L_k = kL_1 \) is \( k \) units long for each \( k \in \mathbb{Z}^+ \). An opening with \( kL_1 \) removes any maximally connected subset of length less than \( k \) units. Hence, probability \( p_k \) is proportional to the volume obtained by slices of rectangular blocks that are \( k \) units long on the DEM \( f \).

We are now ready to extend these ideas to a generic two-dimensional set \( A \). In the two-dimensional case \( n_{t,h} \) for each horizontal scan given by Eq 13 would be dependent on \( i_0 \) i.e. the choice of row, denoted by \( n_{t,h}^{(i)} \). Assuming \( f : A \rightarrow H \) is the DEM on which we wish to compute the MDGI, Eq 12 would transform to:

\[
p_k \propto \sum_{i=n_1}^{n_2} \sum_{h=\min(H)}^{\max(H)} n_{k,h}^{(i)}
\]

C. Invariances of Directional Granulometric Indices

Recall from Sec II that a multiscale granulometric index is given by Def 8 (see Eq 6 and Eq 7). We are interested to characterize sufficient conditions on the structure of a DEM such that all DEMs that satisfy those conditions have the same MDGI. Mathematically, we need to find non-trivial collections of DEMs \( F_c = \{f|GI_{B_h}(f) = c\} \) where \( c > 0 \) is a positive constant. A sufficient condition for a MDGI to be invariant is that the probabilities given by Eq 7 remain the same. On a closer look at Eq 12 it is easy to see that if \( n_{t,h} \) given by Eq 13 remains constant for each \( t, h \) then the MDGI for each such DEM is the same. We now state the result formally:

**Theorem 1:** Let \( f_1 : A \rightarrow H \) and \( f_2 : A \rightarrow H \) be distinct DEMs. If the number of maximally connected subsets \( n_{t,h}^{(i)} \) (given by Eq 10) of upper-thresholded subgraphs (given by Eq 10) for every row \( i \), every length \( t \), and every elevation \( h \) are identical for both \( f_1 \) and \( f_2 \), then \( GI_{B_h}(f_1) = GI_{B_h}(f_2) \).

The proof follows from Eq 13, Eq 6 and Eq 7. To see that the sufficient conditions imposed in Theorem 1 are non-trivial, consider \( f : A \rightarrow H \). We will now construct a ‘large’ collection of DEMs different from \( f \) whose MDGI w.r.t. \( B_4 \) is identical to that of \( f \). Let \( A = \bigcup_{i=n_1}^{n_2} A_i \) where \( A_i = \{(i, j) : m_{1,i} \leq j \leq m_{2,i} \} \). Define \( \hat{f}_i : A_i \rightarrow H \) as

\[
\hat{f}_i(i, j) = f(i, m_{2,i} - j + m_{1,i})
\]

for each \( m_{1,i} \leq j \leq m_{2,i} \). Intuitively, \( \hat{f}_i \) is the mirror-reflection of \( f|A_i \). Now, consider the collection \( \mathcal{F}_{\text{Reflection}}(f) = \{g : A \rightarrow H : g|A_i = f|A_i \ or \ \hat{f}_i \} \). It is easy to see that \( |\mathcal{F}_{\text{Reflection}}(f)| = 2^{n_2-n_1+1} \). One of the elements of \( \mathcal{F}_{\text{Reflection}}(f) \) is \( f \). Hence, we could construct \( 2^{n_2-n_1+1} - 1 \) different DEMs with the same MDGI. Fig 6 provides an illustration of this construction i.e. two DEMs different from the DEM provided by Fig 1 with same MDGI \( GI_{B_4}(\cdot) \).
provided by Eq 14. However, both these DEMs have identical
cannot be constructed from the other using the construction
For example, among the DEMs illustrated in Fig 7, one DEM
compared to the sufficient conditions provided by Theorem 1.
Further, the conditions characterized
DEMs are usually different i.e. arbitrary shaped DEMs are
Encountered in practice. Also, the conditions characterized
Fig. 7. The top row corresponds to one DEM and the second row corresponds to another DEM. These DEMs have identical GI  
and hence have the same MDGI GI  
. Further, the conditions provided by Theorem 1 are sufficient
but not necessary.

Next, we discuss another type of invariance which involves
transformations on the elevation level i.e. comparing DEMs
f  
and  
where  
and  
are finite subsets of  
. Formally, we have the following result:

**Theorem 2:** Let  
and  
be finite sets. Assume that  
and  
for some arbitrary but fixed  
. The DEMs  
and  
have identical MDGIs i.e.  
for each  
. Here  
 denotes composition of functions.

The proof of Theorem 2 follows from the fact that each of the terms on right side of Eq 13 is scaled up by the same constant and hence has no effect on the LHS of the same equation.

**D. Relation to Discrete Derivative of Volume of a DEM**

In this subsection, we relate a discrete derivative of the volume of a DEM to the MDGI. Firstly, for the sake of simplified notation, assume that we are working on horizontal slices  
 of the domain  
 of the DEM  
, i.e. subsets of the type  
 of  
. Define

The quantity  
 denotes the volume of DEM  
 on and above elevation  
 on the horizontal slice  
. In particular,

\[ \Phi_{h_0}(f|_A_i) = \sum_{h=h_0}^{\text{max}(H)} \sum_{v \in V_{2h}^{f}} W(v) \]  

(15)

where  
 denotes a one-dimensional set i.e.  
 such that  
. It is easy to see that in such a case the set of discrete derivatives given by Eq 17 would effectively be a permutation of the probabilities given by Eq 12. This means that the entropy calculated on the successive discrete derivatives of volume of a DEM is identical to the MDGI when computed on a one-dimensional uni-peak DEM.

**E. Why is a Multiscale Directional Granulometric Index considered a Roughness Measure?**

In this subsection, we provide an intuition as to why a MDGI is regarded as a roughness measure. To accomplish this, we consider a special class of DEMs given by:

\[ \mathcal{F}_{\text{UniPeak}} = \{ f : A \rightarrow H : \sum_{t} n_{t,h} = 1 \forall h \in f(A) \} \]  

(19)

where  
 denotes a one-dimensional set i.e.  
 such that  
. This means that the entropy calculated on the successive discrete derivatives of volume of a DEM is identical to the MDGI when computed on a one-dimensional uni-peak DEM.

**IV. EXPERIMENTS**

In this section, we provide empirical evidence to show that MDGIs capture the characteristic features of a river sub-basin.

**A. Study Area and Data Used**

We consider lower Indus sub-basin (fluvial), Wardha sub-basin (floodplain), and Barmer sub-basin (desert) for the experiments. Lower Indus sub-basin, one of the 14 sub-basins lies in between the geographical coordinates of 73°11′ to 76°44′ East longitudes and 34°42′ to 36°9′ North latitudes, is divided into 31 watersheds of sizes ranging between 319 sq.km and 1270 sq.km. Wardha sub-basin, one of the principal tributaries of Godavari river, is situated in between the geographical coordinates of 19°18′N and 21°58′N latitudes, and 77°20′E and 79°45′E longitudes. This sub-basin has 69 watersheds. Barmer is another sub-basin of Indus Basin situated between 69°48′ and 71°43′ East longitudes, and between 25°28′ to 27°69′ North latitudes. It is fully under Thar Desert and is divided into 38 watersheds. Cartosat DEMs of the Lower Indus sub-basins, Wardha sub-basins, and Barmer sub-basins are illustrated in Fig 8.
three river sub-basins, we construct features $X$.

C. More Observations

Identifying the differences between Wardha and Lower-Indus river sub-basins. However, these order-statistics do not help in identifying the differences between Lower-Indus, Wardha, and Barmer sub-basins. The delineations highlight distinct watersheds within each of the sub-basins. (b), (f), and (j) Watersheds classified based on normalized multiscale directional granulometric indices. The MDGIs are all scaled down within each sub-basin and are color-coded as per the ranges mentioned in the legend. The ranges are arbitrarily chosen. (c), (g), and (k) high directional granulometric indices with the colors and the texture highlighting the corresponding SE for which the MDGI is highest, and (d), (h), and (l) low directional granulometric indices with the colors and the texture highlighting the corresponding SE for which the MDGI is lowest, of 31 watersheds of the Lower Indus, 69 watersheds of Wardha and 38 watersheds of Barmer sub-basins respectively.

B. Some Preliminary Observations

We compute the MDGIs for all watersheds in each of the river sub-basins using the structuring elements $B_1, B_2, B_3, B_4$. As there is a variability in the size of each of the watersheds i.e. the domain of each DEM is of different cardinalities, we normalize these MDGIs with the multiscale granulometric index obtained by using $B$ as the structuring element. As a first observation, for each river, we check the order-statistics of the normalized MDGIs. Fig 8 shows a plot of the histograms of the maximum and minimum of the normalized MDGIs among the four primary directions across the river sub-basins. These histograms can be viewed as empirical probability distributions. The maximum (respectively minimum) among the normalized MDGIs is denoted as high directional (respectively low directional) in Figs 8 and 9. It is easy to see that the Barmer sub-basins show a different pattern in the order-statistics as illustrated in Fig 9. In particular, it is often the case that the maximum is along the direction of $B_3$ and the lowest is given by $B_2$ which is not the case with the other two river sub-basins. However, these order-statistics do not help in identifying the differences between Wardha and Lower-Indus sub-basins.

C. More Observations

To identify the differences between the watersheds of all three river sub-basins, we construct features $X[0], \ldots, X[15]$ based on the normalized MDGIs. The details of the construction are as follows:

Let $GI_B(f)/GI_B(f) = Z_i$ for $i = 1, 2, 3, 4$

$$X[(i-1)4 + (j-1)] = \begin{cases} Z_i & \text{if } Z(j) = Z_i \\ 0 & \text{otherwise} \end{cases}$$

(20)

for each $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$, where $Z(j)$ denotes the $j$th order-statistic among $Z_1, Z_2, Z_3, Z_4$ i.e. $Z(1) \leq Z(2) \leq Z(3) \leq Z(4)$ form a permutation of $Z_1, Z_2, Z_3, Z_4$. As the number of watersheds (138 altogether) is small, we do not split the data into training and test sets. Instead, we try to obtain interpretable rules that can classify the watersheds into appropriate sub-basins based on the constructed features. A simple decision tree with a depth of 2 is constructed. The depth of the decision tree is restricted so as to ensure that we do not over fit the data. Also, the reason for choosing 2 as the depth is that the minimum depth of a decision tree required to classify 3 classes is 2. We observe that such a decision tree (see Fig 10) is capable of obtaining $\approx 71\%$ accuracy. Note that a random classifier on the other hand can obtain a maximum accuracy of $69/138 = 50\%$. This is because the data is class-unbalanced with 31 Indus, 69 Wardha and 38 Barmer watersheds.

This experiment indicates that the features computed from MDGIs carry characteristic information of the sub-basins. To explore the discernibility of MDGI-based features, we built decision trees of larger depths until a maximum depth of 9. We observed that the accuracies of decision trees of depths 5, 6, and 9 are $\approx 86\%, \approx 89\%$, and $\approx 94\%$ respectively. This indicates that MDGI-based features are useful for river sub-
basin classification. However, these features cannot be used as standalone features to classify sub-basins. In order to build better classifiers, one needs to incorporate domain knowledge on the river sub-basins through some form of remotely-sensed data or otherwise.

V. CONCLUSIONS AND PERSPECTIVES

In this article, we revisit the roughness measure on DEM data adapted from multiscale granulometries in mathematical morphology, namely multiscale directional granulometric index (MDGI). In earlier works, MDGIs were introduced to capture the characteristic surficial roughness of a river sub-basin along specific directions. They are known to be useful features for classification of river sub-basins. In this article, we provided a theoretical analysis of a MDGI. In particular, we characterized non-trivial sufficient conditions on the structure of DEMs under which MDGIs are invariant. These properties are illustrated with some fictitious DEMs. We also provided connections to a discrete derivative of volume of a DEM. Based on these connections, we provided intuition as to why a MDGI is considered a roughness measure. Further, we experimentally illustrated on Lower-Indus, Wardha, and Barmer river sub-basins that the proposed features capture the characteristics of the river sub-basin.

Building on the ideas from this article, one can explore at least two directions: 1) building on the main theorem, one can investigate more sufficient conditions ultimately trying to characterize sufficient and necessary conditions on the structure of a DEM such that MDGI is invariant, 2) on the experimental side, use the features proposed in the article alongside other features on river sub-basins to build better classifiers.

ACKNOWLEDGMENT

Nagajothi Kannan would like to thank Regional Remote Sensing Centre, Indian Space Research Organisation. Sravan Danda would like to acknowledge the funding received from BPGC/RIG/2020-21/11-2020/01 (Research Initiation Grant provided by BITS-Pilani K K Birla Goa Campus) and thank APPCAIR, and Computer Science and Information Systems, BITS-Pilani Goa. Aditya Challa would like to thank Indian Institute of Science (IISc) for the Raman Post Doctoral fellowship. The work of B. S. D. Sagar was supported by the DST-ITPAR-Phase-IV project under the Grant number INT/Italy/ITPAR-IV/Telecommunication/2018.

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