UNIQUENESS TYPE RESULT IN DIMENSION 3.

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ABSTRACT. We give some estimates of type sup $\times$ inf on Riemannian manifold of dimension 3 for the prescribed curvature type equation. As a consequence, we derive an uniqueness type result.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we deal with the following prescribed scalar curvature type equation in dimension 3:

$$\Delta u + h(x)u = V(x)u^5, \quad u > 0.$$ (E)

Where $h, V$ are two continuous functions. In the case $8h = R_g$ the scalar curvature, we call $V$ the prescribed scalar curvature. Here, we assume $h$ a bounded function and $h_0 = ||h||_{L^\infty(M)}$.

We consider three positive real number $a, b, A$ and we suppose $V$ lipschitzian:

$$0 < a \leq V(x) \leq b < +\infty \text{ and } ||\nabla V||_{L^\infty(M)} \leq A.$$ (C)

The equation (E) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = S_n$ see for example, [2-4], [11], [15]. In this case we have a sup $\times$ inf inequality.

The corresponding equation in two dimensions on open set $\Omega$ of $\mathbb{R}^2$, is:

$$\Delta u = V(x)e^u,$$ (E')

The equation (E') was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_\Omega V, ||V||_{L^\infty(\Omega)}),$$

And, precisely, in [8], [12], [16], and [19], we have:

$$C \sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, ||V||_{L^\infty(\Omega)}),$$

and,

$$\sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, ||V||_{C^\alpha(\Omega)}),$$

where $K$ is a compact subset of $\Omega$, $C$ is a positive constant which depends on $\inf_\Omega V$, $\sup_\Omega V$, and, $\alpha \in (0, 1]$.

In the case $V \equiv 1$ and $M$ compact, the equation (E) is Yamabe equation. Yamabe has tried to solve problem but he could not, see [22]. T.Aubin and R.Schoen have proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary.

When $M$ is a compact Riemannian manifold, there exist some compactness result for equation (E) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose $M$ not diffeormorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose $M$ Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [17] proved that the product sup $\times$ inf is bounded. Here we extend the result of [6].
Our proof is an extension of Brezis-Li and Li-Zhang result in dimension 3, see [7] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [10], one of the application of this method.

Here, we give an equality of type \( \sup \times \inf \) for the equation \((E)\) with general conditions \((C)\). Note that, in our proof, we do not need a classification result for some particular elliptic PDEs on \(\mathbb{R}^3\).

In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3, 5].

There are other estimates of type \( \sup + \inf \) on complex Monge-Ampere equation on compact manifolds, see [20-21]. They consider, on compact Kahler manifold \((M, g)\), the following equation:

\[
\begin{cases}
(\omega_g + \partial \bar{\partial} \varphi)^n = e^{f-\varphi} \omega_g^n, \\
\omega_g + \partial \bar{\partial} \varphi > 0 \text{ on } M
\end{cases}
\]

And, they prove some estimates of type \( \sup_M + m \inf_M \leq C \) or \( \sup_M + m \inf_M \geq C \) under the positivity of the first Chern class of \(M\).

Here, we have,

**Theorem 1.1.** For all compact set \(K\) of \(M\) and all positive numbers \(a, b, A, h_0\) there is a positive constant \(c\), which depends only on, \(a, b, A, h_0, K, M, g\) such that:

\[\sup_K u \times \inf_M u \leq c,\]

for all \(u\) solution of \((E)\) with conditions \((C)\).

This theorem generalise Li-Zhang result, see [17] in the case \(V \equiv 1\). Here, we use Li and Zhang method in [17].

In the case \(h \equiv \epsilon \in (0, 1)\) and \(u_\epsilon\) solution of:

\[\Delta u_\epsilon + \epsilon u_\epsilon = V_\epsilon u_\epsilon^5, \quad u_\epsilon > 0. \quad (E_\epsilon)\]

We have:

**Corollary 1.2.** For all compact set \(K\) of \(M\) and all positive numbers \(a, b, A\) there is a positive constant \(c\), which depends only on, \(a, b, A, K, M, g\) such that:

\[\sup_K u_\epsilon \times \inf_M u_\epsilon \leq c,\]

for all \(u_\epsilon\) solution of \((E_\epsilon)\) with conditions \((C)\).

Now, if we assume \(M\) a compact riemannian manifold and \(0 < a \leq V_\epsilon \leq b < +\infty\) we have:

**Theorem 1.3.** (see [3]). For all positive numbers \(a, b, m\) there is a positive constant \(c\), which depends only on, \(a, b, m, M, g\) such that:

\[\epsilon \sup_M u_\epsilon \times \inf_M u_\epsilon \geq c,\]

for all \(u_\epsilon\) solution of \((E_\epsilon)\) with

\[\max_M u_\epsilon \geq m > 0.\]

As a consequence of the two previous theorems, we have:
Theorem 1.4. For all positive numbers $a, b, A$ we have:

$$\max_M u_\epsilon \to 0,$$

and (up to a subsequence),

$$\max_M u_\epsilon / \epsilon^{1/4} \to w_0 > 0,$$

and

$$\min_M u_\epsilon / \epsilon^{1/4} \to w_0 > 0.$$

Remarks:

- It is not necessary to have $u_\epsilon \equiv w_0 \epsilon^{1/4}$, because if we take a nonconstant function $V$, we can find by the variational approach a nonconstant positive solution of the subcritical equation:

$$\Delta u_\epsilon + \epsilon u_\epsilon = \mu_\epsilon V u_\epsilon^{5 - \epsilon},$$

with $\mu_\epsilon, u_\epsilon > 0$.

In this case (subcritical which tends to the critical) we also have the $\sup \times \inf$ inequalities and the uniqueness type theorem.

- In fact, we prove, up to a subsequence that $u_\epsilon / \epsilon^{1/4} \to w_0 > 0$, where $w_0$ depends on $a, b$ and $A$.

2. PROOF OF THE THEOREMS

Proof of theorem 1.1:

We want to prove that:

$$\epsilon \max_{B(0, \epsilon)} u \times \min_{B(0, 4\epsilon)} u \leq c = c(a, b, A, M, g) \quad (1)$$

We argue by contradiction and we assume that:

$$\max_{B(0, \epsilon_k)} u_k \times \min_{B(0, 4\epsilon_k)} u_k \geq k \epsilon_k^{-1} \quad (2)$$

Step 1: The blow-up analysis

The blow-up analysis gives us:

For some $x_k \in B(0, \epsilon_k)$, $u_k(x_k) = \max_{B(0, \epsilon_k)} u_k$, and, from the hypothesis,

$$u_k(x_k)^2 \epsilon_k \to +\infty.$$

By a standard selection process, we can find $x_k \in B(x_k, \epsilon_k/2)$ and $\sigma_k \in (0, \epsilon_k/4)$ satisfying,

$$u_k(x_k)^2 \sigma_k \to +\infty, \quad (3)$$

$$u_k(x_k) \geq u_k(x_k), \quad (4)$$

and,

$$u_k(x) \leq C_1 u_k(x), \text{ in } B(x_k, \sigma_k), \quad (5)$$

where $C_1$ is some universal constant.

It follows from above (2), (4)) that:

$$u_k(x_k) \times \min_{\partial B(x_k, 2\epsilon_k)} u_k \epsilon_k \geq u_k(x_k) \times \min_{B(0, 4\epsilon_k)} u_k \epsilon_k \geq k \to +\infty. \quad (6)$$

We use $\{z^1, \ldots, z^n\}$ to denote some geodesic normal coordinates centered at $x_k$ (we use the exponential map). In the geodesic normal coordinates, $g = g_{ij}(z)dz^i dz^j$,

$$g_{ij}(z) = \delta_{ij} = O(r^2), \quad g := \det(g_{ij}(z)) = 1 + O(r^2), \quad h(z) = O(1), \quad (7)$$

where $r = |z|$. Thus,

$$\Delta_g u = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u) = \Delta u + b_i \partial_i u + d_{ij} \partial_j u,$$
where
\[ b_j = O(r), \quad \partial_{ij} = O(r^2) \quad (8) \]

We have a new function:
\[ u_k(y) = M_k^{-1} k(M_k^{-2} y) \] for \( |y| \leq 3\epsilon_k M_k^2 \)

where \( M_k = u_k(0) \).

From (5), (6), we have:
\[
\begin{cases}
\Delta v_k + b_i \partial_i v_k + d_{ij} \partial_{ij} v_k - \ddot{c} v_k + v_k^5 = 0 \text{ for } |y| \leq 3\epsilon_k M_k^2 \\
v_k(0) = 1 \\
v_k(y) \leq C_1 \text{ for } |y| \leq \sigma_k M_k^2 \\
\lim_{k \to +\infty} \min_{|y|=2\epsilon_k M_k^2} (v_k(y)|y|) = +\infty \quad (9)
\end{cases}
\]

where \( C_1 \) is a universal constant and
\[
\ddot{b}_i(y) = M_k^{-2} b_i(M_k^{-2} y), \quad \ddot{d}_{ij}(y) = d_{ij}(M_k^{-2} y) \quad (10)
\]

and,
\[
\ddot{c}(y) = M_k^{-4} h(M_k^{-2} y) \quad (11)
\]

We can see that for \( |y| \leq 3\epsilon_k M_k^2 \), we have:
\[
|\ddot{b}_i(y)| \leq C M_k^{-4} |y|, \quad |\ddot{d}_{ij}(y)| \leq C M_k^{-4} |y|^2, \quad |\ddot{c}(y)| \leq C M_k^{-4} \quad (12)
\]

where \( C \) depends on \( n, M, g \).

It follows from (9), (10), (11), (12) and the elliptic estimates, that, along a subsequence, \( v_k \) converges in \( C^2 \) norm on any compact subset of \( \mathbb{R}^2 \) to a positive function \( U \) satisfying:
\[
\begin{cases}
\Delta U + U^5 = 0 \text{ in } \mathbb{R}^2 \\
U(0) = 1, \quad 0 < U \leq C_1 
\end{cases} \quad (13)
\]

**Step 2: The Kelvin transform and moving-plane method**

For \( x \in \mathbb{R}^2 \) and \( \lambda > 0 \), let,
\[ v_k^{\lambda,x}(y) := \frac{\lambda}{|y-x|} v_k \left( x + \frac{\lambda^2(y-x)}{|y-x|^2} \right) \]

denote the Kelvin transformation of \( v_k \) with respect to the ball centered at \( x \) and of radius \( \lambda \).

We want to compare for fixed \( x, v_k \) and \( v_k^{\lambda,x} \). For simplicity we assume \( x = 0 \). We have:
\[ v_k^\lambda(y) := \frac{\lambda}{|y|} v_k(y^\lambda), \text{ with } y^\lambda = \frac{\lambda y}{|y|^2} \]

For \( \lambda > 0 \), we set,
\[ \Sigma_\lambda = B(0, \epsilon_k M_k^2) - \bar{B}(0, \lambda) \]

The boundary condition, (9), become:
\[
\lim_{k \to +\infty} \min_{|y|=\epsilon_k M_k^2} (v_k(y)|y|) = \lim_{k \to +\infty} \min_{|y|=2\epsilon_k M_k^2} (v_k(y)|y|) = +\infty \quad (14)
\]

We have:
\[ \Delta v_k^\lambda + V_k^\lambda (v_k^\lambda)^5 = E_1(y) \text{ for } y \in \Sigma_\lambda \quad (15) \]

where,
\[ E_1(y) = - \left( \frac{\lambda}{|y|} \right)^5 \left( \ddot{b}_i(y^\lambda) \partial_i v_k(y^\lambda) + \ddot{d}_{ij}(y^\lambda) \partial_{ij} v_k(y^\lambda) - \ddot{c}(y^\lambda) v_k(y^\lambda) \right) \quad (16) \]

Clearly, from (10), (11), there exists \( C_2 = C_2(\lambda_1) \) such that,
Note that here, we have, for simplicity, omitted $k$. We observe that by (9), (15):
\[
\Delta w_\lambda + b_i \partial_i w_\lambda + \tilde{d}_{ij} \partial_j w_\lambda - \bar{c} w_\lambda + 5\xi^4 V_k w_\lambda = E_\lambda \text{ in } \Sigma_\lambda
\]
where $\xi$ stay between $v_k$ and $v_\lambda$, and,
\[
E_\lambda = -\bar{b}_i \partial_i v_\lambda^\lambda + \tilde{d}_{ij} \partial_j v_\lambda^\lambda + \bar{c} v_\lambda^\lambda - E_1 - (V_k - V_k^\lambda)(v_\lambda^\lambda)^5.
\]
A computations give us the following two estimates:
\[
|\partial_i v_\lambda^\lambda(y)| \leq C\lambda|y|^{-2}, \quad \text{and} \quad |\partial_{ij} v_\lambda^\lambda(y)| \leq C\lambda|y|^{-3} \quad \text{in } \Sigma_\lambda
\]

From (10), (11), (20), we have,

Lemma 2.1. For some constant $C_3 = C_3(\lambda)$
\[
|E_\lambda| \leq C_3 \lambda M_k^{-4}|y|^{-1} + C_3 \lambda^5 M_k^{-2}|y|^{-4} \quad \text{in } \Sigma_\lambda
\]

we consider the following auxiliary function:

\[
h_\lambda = -C_1 AM_k^{-2}\lambda^2 \left(1 - \frac{\lambda}{|y|}\right) + C_2 AM_k^{-2}\lambda^3 \left(1 - \left(\frac{\lambda}{|y|}\right)^2\right) - C_3 M_k^{-2}\lambda(|y| - \lambda),
\]

where $C_1, C_2$ and $C_3$ are three positive numbers.

Lemma 2.2. We have,
\[
w_\lambda + h_\lambda \geq 0, \text{ in } \Sigma_\lambda \forall 0 < \lambda \leq \lambda_1
\]

Proof of Lemma 2.2. We divide the proof into two steps.

Step 1. There exists $\lambda_{0,k} > 0$ such that (22) holds:
\[
w_\lambda + h_\lambda \geq 0, \text{ in } \Sigma_\lambda \forall 0 < \lambda \leq \lambda_{0,k}.
\]

To see this, we write:
\[
w_\lambda = v_k(y) - v_k^\lambda(y) = \frac{1}{\sqrt{|y|}} \left(\sqrt{|y|}v_k(y) - \sqrt{|y|}v_k^\lambda(y)\right).
\]

Note that $y$ and $y^\lambda$ are on the same ray starting from the origin. Let, in polar coordinates,
\[
f(r, \theta) = \sqrt{r}v_k(r, \theta).
\]

From the uniform convergence of $v_k$, there exists $r_0 > 0$ and $C > 0$ independant of $k$ such that,
\[
\frac{\partial f}{\partial r}(r, \theta) > Cr^{-1/2} \text{ for } 0 < r < r_0.
\]

Consequently, for $0 < \lambda < |y| < r_0$, we have:
\[
w_\lambda(y) + h_\lambda(y) = v_k(y) - v_k^\lambda(y) + h_\lambda(y),
\]
\[
> \frac{1}{\sqrt{r_0}} C\sqrt{r_0}^{-3/2}(|y| - |y^\lambda|) + h_\lambda(y)
\]
\[
> \left(C \frac{r_0}{r_0} - C_3 AM_k^{-2}(|y| - \lambda) \text{ since } |y| - |y^\lambda| > |y| - \lambda \right.
\]
\[
> 0.
\]

(23)
Since,
\[ |h_\lambda(y)| + v_k^\lambda(y) \leq C(k, r_0) \lambda, \quad r_0 \leq |y| \leq \epsilon_k M_k^2, \]
we can pick small \( \lambda_{0,k} \in (0, r_0) \) such that for all \( 0 < \lambda \leq \lambda_{0,k} \) we have,
\[ w_{\lambda}(y) + h_\lambda(y) \geq \min_{B(0,\epsilon_k M_k^2)} v_k - C(k, r_0) \lambda_{0,k} > 0 \quad \forall \quad r_0 \leq |y| \leq \epsilon_k M_k^2 \]
Step 1 follows from (23).

Let
\[ \bar{\lambda}^k = \sup\{0 < \lambda \leq \lambda_1, w_\mu + h_\mu \geq 0, \text{ in } \Sigma_\mu, \forall 0 < \mu \leq \lambda\} \]
Step 2. \( \bar{\lambda}^k = \lambda_1 \), (22) holds.

For this, the main estimate needed is:
\[ (\Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c} + 5\xi^4 \bar{V}_k)(w_{\lambda} + h_\lambda) \leq 0 \quad \text{in } \Sigma_{\lambda} \] (25)

Thus,
\[ \Delta h_\lambda + \bar{b}_i \partial_i h_\lambda + \bar{d}_{ij} \partial_{ij} h_\lambda + (-\bar{c} + 5\xi^4 \bar{V}_k) h_\lambda + E_\lambda \leq 0 \quad \text{in } \Sigma_{\lambda}. \] (26)
We have \( h_\lambda < 0 \), and, (12) and a computation give us,
\[ |\bar{c} h_\lambda| \leq C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^2 M_k^{-6} \leq C_3 \lambda M_k^{-4} |y|^{-1}, \]
and,
\[ |\bar{b}_i \partial_i h_\lambda| + |\bar{d}_{ij} \partial_{ij} h_\lambda| \leq C_3 \lambda M_k^{-8} |y| + C_3 \lambda^3 M_k^{-6} |y|^{-1} + C_3 \lambda^5 M_k^{-6} |y|^{-2}, \]
\[ \leq C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^5 M_k^{-2} |y|^{-4} \]
Thus,
\[ |\bar{b}_i \partial_i h_\lambda| + |\bar{d}_{ij} \partial_{ij} h_\lambda| + |\bar{c} h_\lambda| \leq C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^5 M_k^{-2} |y|^{-4} \quad \text{in } \Sigma_{\lambda} \]
Thus, by (21),
\[ \Delta h_\lambda + \bar{b}_i \partial_i h_\lambda + \bar{d}_{ij} \partial_{ij} h_\lambda + (-\bar{c} + 5\xi^4 \bar{V}_k) h_\lambda + E_\lambda \leq \Delta h_\lambda + C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^5 M_k^{-2} |y|^{-4} + |E_\lambda| \leq 0, \]
because,
\[ \Delta h_\lambda = -2C_3 \lambda M_k^{-4} |y|^{-1} - 2C_3 \lambda^5 M_k^{-2} |y|^{-4}. \]
From the boundary condition and the definition of \( v_k^\lambda \) and \( h_\lambda \), we have:
\[ |h_\lambda(y)| + v_k^\lambda(y) \leq \frac{C(\lambda_1)}{|y|}, \quad \forall \quad |y| = \epsilon_k M_k^2, \]
and, thus,
\[ w_{\lambda^k}(y) + h_\lambda(y) > 0 \quad \forall \quad |y| = \epsilon_k M_k^2, \]
We can use the maximum principal and the Hopf lemma to have:
\[ w_{\lambda^k} + h_\lambda > 0, \quad \text{in } \Sigma_{\lambda}, \]
and,
\[ \frac{\partial}{\partial \nu}(w_{\lambda^k} + h_\lambda) > 0, \quad \text{in } \Sigma_{\lambda}. \]
From (25) and above we conclude that \( \bar{\lambda}^k = \lambda_1 \) and lemma 2.2 is proved.
Given any \( \lambda > 0 \), since the sequence \( v_k \) converges to \( U \) and \( h_{\lambda k} \) converges to 0 on any compact subset of \( \mathbb{R}^2 \), we have:

\[
U(y) \geq U^\lambda(y), \quad \forall \ |y| \geq \lambda, \quad \forall \ 0 < \lambda < \lambda_1.
\]

Since \( \lambda_1 > 0 \) is arbitrary, and since we can apply the same argument to compare \( v_k \) and \( v_k^{\lambda,x} \), we have:

\[
U(y) \geq U^{\lambda,x}(y), \quad \forall \ |y - x| \geq \lambda > 0.
\]

Thus implies that \( U \) is a constant which is a contradiction.

**Proof of theorem 1.4:**

From theorem 2.1 (see [3]), we have:

\[
\max_M u_\epsilon \to 0. \quad (27)
\]

We conclude with the aid of the elliptic estimates and the classical Harnack inequality that:

\[
\max_M u_\epsilon \leq C \min_M u_\epsilon, \quad (28)
\]

where \( C \) is a positive constant independent of \( \epsilon \).

Let \( G_\epsilon \) the Green function of the operator \( \Delta + \epsilon \), we have,

\[
\int_M G_\epsilon(x, y) dV_\epsilon(y) = \frac{1}{\epsilon}, \quad \forall \ x \in M. \quad (29)
\]

We write:

\[
\inf_M u_\epsilon = u_\epsilon(x_\epsilon) = \int_M G_\epsilon(x_\epsilon, y)V_\epsilon(y)u_\epsilon^5(y) dV_\epsilon(y) \geq
\]

\[
\geq a(\inf_M u_\epsilon)^5 \int_M G_\epsilon(x_\epsilon, y) dV_\epsilon(y) = a \frac{(\inf_M u_\epsilon)^5}{\epsilon},
\]

thus,

\[
\inf_M u_\epsilon \leq C_1 \epsilon^{1/4}. \quad (30)
\]

With the similar argument, we have:

\[
\sup_M u_\epsilon \geq C_2 \epsilon^{1/4}. \quad (31)
\]

Finally, we have:

\[
C_1 \epsilon^{1/4} \leq u_\epsilon(x) \leq C_2 \epsilon^{1/4} \quad \forall \ x \in M. \quad (32)
\]

Where \( C_1 \) and \( C_2 \) are two positive constant independent of \( \epsilon \).

We set \( w_\epsilon = \frac{u_\epsilon}{\epsilon^{1/4}} \), then,

\[
\Delta w_\epsilon + \epsilon w_\epsilon = \epsilon V_\epsilon u_\epsilon^5.
\]

The theorem follow from the standard elliptic estimate, the Green function of the laplacian and the Green representation formula for the solutions of the previous equation.
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