ON THE DISTRIBUTION OF PRIME DIVISORS IN KRULL MONOID ALGEBRAS

VICTOR FADINGER AND DANIEL WINDISCH

Abstract. In the present work, we prove that every class of the divisor class group of a Krull monoid algebra contains infinitely many prime divisors. Several attempts to this result have been made in the literature so far, unfortunately with open gaps. We present a complete proof of this fact.

1. Introduction

The investigation of class groups and the distribution of prime divisors in the classes is a central topic in ring theory and has been studied for many classes of rings, e.g., for orders in number fields. A variety of realization theorems for class groups have been achieved, for instance that every abelian group occurs as the class group of a Dedekind domain (Claborn’s Realization Theorem) respectively as the class group of a simple Dedekind domain [18].

In the present paper, we study the distribution of prime divisors in the class group of commutative monoid algebras $D[S]$ that are Krull. In our setting, a monoid $S$ is a cancellative commutative unitary semigroup. For a commutative ring $D$ with unity and a monoid $S$, we denote by $D[S]$ the monoid algebra of $D$ over $S$. It is a well-known result by Chouinard, that $D[S]$ is a Krull domain if and only if $D$ is a Krull domain and $S$ is a torsion-free Krull monoid with $S^\times$ satisfying the ACC (ascending chain condition) on cyclic subgroups, where $S^\times$ denotes the group of units of $S$ (see [5]). Moreover, we have a canonical isomorphism of divisor class groups $C_v(D) \oplus C_v(S) \cong C_v(D[S])$ via $([I], [J]) \mapsto [I|J]$ [10, Corollary 16.8].

In the setting of monoids, it is known that every abelian group is the class group of a Krull monoid and the possible ways of distributing the prime divisors in the classes are completely determined [8, Theorem 2.5.4]. Considering domains, a slightly weaker form of a realization result holds true. Although every abelian group is the class group of a Dedekind domain, it is not completely clear how prime divisors behave [8, Theorem 3.7.8].

In Section 2, we fix notations concerning monoid algebras and class groups. Section 3 contains a proof of the following result concerning the distribution of prime divisors:

Theorem. Let $D$ be a Krull domain and let $S$ be a Krull monoid such that $D[S]$ is a Krull domain. Then every divisor class of $D[S]$ contains infinitely many prime divisors.

In the special case of the polynomial ring over a Krull domain, this is already known [9, Theorem 45.5]. Moreover, Kainrath [12] proved the same for finitely generated algebras over infinite fields.

Attempts to achieve such results for Krull monoid algebras have been made by Kim [13] and Chang [4]. However, there are deficiencies in their proofs as we demonstrate in Section 4.

2. Preliminaries

We assume some familiarity with monoid algebras and ideal systems. For a domain $D$ and a monoid $S$, let $D[S]$ denote the monoid algebra of $D$ over $S$. It is well known that the monoid algebra $D[S]$ is a domain if and only if the monoid $S$ is torsion-free [10] Theorem 8.1] and in that case $S$ admits a total
order $< \frac{g}{a}$ compatible with its semigroup operation. Therefore, we can write every element $f \in D[S]$ in the form $f = \sum_{i=1}^{n} d_i X_i$ with $d_i \in D$, $s_i \in S$ and $s_1 < s_2 < \ldots < s_n$. For subsets $P \subseteq D$ and $H \subseteq S$, we denote by $P[H] = \{ \sum_{i=1}^{n} p_i X_i^{h_i} \mid n \in \mathbb{N}, p_i \in P, h_i \in H \} \subseteq D[S]$. Let $K$ be the quotient field of $D$ and let $G$ be the quotient group of $S$. For an element $f = \sum_{i=1}^{n} d_i X_i \in K[G]$ with $d_i \in K$ and $s_i \in G$ we denote by $A_f$ resp. $E_f$ the fractional ideal of $D$ resp. $S$ generated by $d_1, \ldots, d_n$ resp. $s_1, \ldots, s_n$.

Let $G$ be an abelian group. An element $g \in G$ is said to be of height $(0,0,0,\ldots)$ if the equation $px = g$ has no solution $x \in G$ for any prime number $p$. Furthermore, $g$ is said to be of type $(0,0,0,\ldots)$ if the same equation has a solution for only finitely many prime numbers. The group $G$ is said to be of type $(0,0,0,\ldots)$ if every element of $G$ is of type $(0,0,0,\ldots)$. It is well known that $G$ is of type $(0,0,0,\ldots)$ if and only if it satisfies the ACC (ascending chain condition) on cyclic subgroups. For a proof of this statement and for further equivalent conditions, see [10, §14].

Let $S$ be a monoid. By $C_t(S)$ we denote the the $t$-class group of $S$ which is the quotient group of the group of $t$-invertible fractional $t$-ideals of $S$ modulo the subgroup of non-empty principal fractional ideals of $S$. If $S$ is a Krull monoid, then $C_t(S)$ coincides with the $v$-class group $C_v(S)$ (which is isomorphic to the divisor class group of $S$). Moreover, in this case $C_t(S)$ is isomorphic to the quotient of the monoid of $v$-invertible (integral) $v$-ideals modulo the submonoid of non-empty principal (integral) ideals.

If $I$ is a $t$-invertible fractional $t$-ideal of $S$, we denote by $[I] \in C_t(S)$ its $t$-class and refer to it as the divisor class of $I$ if $S$ is a Krull monoid. Moreover, we denote by $X(S)$ the set of height-one prime ideals of $S$. In the literature, the elements of $X(S)$ are also often called prime divisors of $S$ (especially, if $S$ is a Krull monoid).

Replacing the monoid $S$ by a domain $D$, the terminology and notation concerning class groups and prime divisors above is the same. For further information, see [8].

We will make intensive use of the theory of valuations on fields and groups. If $D$ is a domain and $P$ is a prime ideal of $D$ such that the localization $DP$ is a valuation domain, then we denote by $v_P$ its valuation induced on the quotient field of $D$ and call it the $P$-adic valuation. In the case of monoids, the notation is analogous. The interested reader is referred to [2] and [10].

For the remainder of this work and if not specified otherwise, $D$ denotes a domain with quotient field $K$ and $S$ is a torsion-free monoid with quotient group $G$.

3. PROOF OF THE MAIN RESULT

To ensure the existence of prime divisor in divisor classes of $D[S]$, it is necessary to construct irreducible elements in $K[G]$. The following lemma (which is due to Matsuda [10]) is a first step into this direction that will be used in a special case. For the convenience of the reader we include the full proof.

**Lemma 3.1.** ([10 Lemma 4.1], [14 Lemma 2.2]) Let $G$ be a torsion-free abelian group and let $K$ be a field. Let $g \in G$ be an element of height $(0,0,0,\ldots)$ and $a, b \in K \setminus \{0\}$. Then $a + bX^g$ is irreducible in $K[G]$.

**Proof.** Let $f, h \in K[G]$ such that $a + bX^g = fh$. Let $H$ be the subgroup of $G$ that is generated by $g$ together with the exponents of $f$ and $h$. We first show, that the subgroup $(g)$ of $H$ generated by $g$ is a direct summand of $H$. Let $p$ be a prime number. Suppose $p^a x \in (g)$ for some positive integer $a$ and $x \in H$. Then there exists $m \in \mathbb{Z}$ such that $p^a x = mg$. Since $H$ is torsion-free, there are integers $n' \geq 0$ and $m'$ such that $p^{a'} x = m' g$ and $\gcd(p, m') = 1$. It follows that $1 = kp^{a'} + lm'$ for some $k, l \in \mathbb{Z}$. Multiplying by $g$ gives $g = kp^{a'} + lm' g = p^{a'} (kg + lx)$. Since the height of $g$ equals $(0,0,0,\ldots)$, we have $n' = 0$, hence $x = m' g \in (g)$. So we have shown that $(g)$ is a pure subgroup of the finitely generated group $H$ and is therefore a direct summand by [6, Corollary 25.3]. So we write $H = (g) \oplus (e_1) \oplus \ldots \oplus (e_n)$ for some $e_1, \ldots, e_n \in H$.

The set $\{X^g, X^{e_1}, \ldots, X^{e_n}\}$ is algebraically independent over $K$. Hence $a + bX^g$ is irreducible in the polynomial ring $K[X^g, X^{e_1}, \ldots, X^{e_n}]$ and therefore also in $K[H] = K[X^g, X^{-g}, X^{e_1}, X^{-e_1}, \ldots, X^{e_n}, X^{-e_n}]$. It follows that either $f$ or $h$ is a unit in $K[H]$ and hence in $K[G]$. \(\square\)
ON THE DISTRIBUTION OF PRIME DIVISORS IN KRULL MONOID ALGEBRAS

Case 1: Let $P$ be the Laurent polynomial ring in $f K \left[ \frac{1}{f} \right]$. Note that $P$ contains infinitely many prime divisors.

**Proof.** As noted in the introduction, we have an isomorphism $C_\nu(D) \oplus C_\nu(S) \cong C_\nu(D[S])$ via $([I],[J]) \mapsto [I]$. Clearly, $f K[G] \cap D[S]$ is a height-one prime ideal of $D[S]$ and hence non-zero divisorial. So there exist divisorial ideals $I \neq (0)$ resp. $J \neq (0)$ of $D$ resp. $S$ and $h \in K[G] \setminus \{0\}$ such that $f K[G] \cap D[S] = hI[J]$ (note that $f K[G] \cap D[S] \subseteq K[G]$). It follows that $f K[G] = hK[G]$ and hence $h = uX^af$ for some $u \in K \setminus \{0\}$ and $a \in G$. Thus, $f K[G] \cap D[S] = f uI[a + J]$. A simple computation gives $uI = A_f^{-1}$ and $a + J = E_f^{-1}$.

**Lemma 3.3.** Let $D$ be a Krull domain with quotient field $K \neq D$ and let $I$ be a non-zero divisorial ideal of $D$. There exist $a, b \in K$ and $P \in \mathfrak{X}(D)$ such that $I^{-1} = (a, b)_e$ and $v_P(\frac{1}{a}) = 1$. If $D$ is not semi-local, then there are infinitely many $P \in \mathfrak{X}(D)$ such that there exist $a, b \in K$ with $I^{-1} = (a, b)_e$ and $v_P(\frac{1}{a}) = 1$.

**Proof.** By the Approximation Theorem for Krull domains, there exist $a', b \in K$ such that $I^{-1} = (a', b)_e$. If $I^{-1}$ is a principal fractional ideal, $I$ is a principal ideal, hence we can assume without loss of generality that $b$ is its generator. Now take any $P \in \mathfrak{X}(D)$ and any element $a \in D$ with $v_P(a) = 1$. In particular, if $D$ is not semi-local, there exist infinitely many such $P$. Now assume that $I^{-1}$ is not a principal fractional ideal. Thus, there exists $P \in \mathfrak{X}(D)$ such that $v_P(a') \neq v_P(b)$. Without loss of generality, let $v_P(a') > v_P(b)$. By the Approximation Theorem for Krull domains, we may choose $a \in K$ such that

$$v_Q(a) = \begin{cases} v_P(b) + 1 & \text{if } Q = P, \\ v_Q(a') & \text{if } v_Q(a') \neq 0 \text{ or } v_Q(b) \neq 0 \text{ and } Q \neq P, \\ v_Q(a) & \text{if } v_Q(a) \geq 0 \text{ otherwise,} \end{cases}$$

for $Q \in \mathfrak{X}(D)$. Then $\min\{v_Q(a), v_Q(b)\} = \min\{v_Q(a'), v_Q(b)\}$ for all $Q \in \mathfrak{X}(D)$ and hence $I^{-1} = (a, b)_e$. Moreover $v_P(\frac{1}{a}) = 1$.

If $D$ is not semi-local, then $\mathfrak{X}(D)$ is infinite, whence there are infinitely many $P \in \mathfrak{X}(D)$ such that $v_P(a') = 0 = v_P(b)$. For each of them, we construct $a$ in the following way using the Approximation Theorem for Krull domains:

$$v_Q(a) = \begin{cases} 1 & \text{if } Q = P, \\ v_Q(a') & \text{if } v_Q(a') \neq 0 \text{ or } v_Q(b) \neq 0 \text{ and } Q \neq P, \\ v_Q(a) & \text{if } v_Q(a) \geq 0 \text{ otherwise,} \end{cases}$$

for $Q \in \mathfrak{X}(D)$. Then again $\min\{v_Q(a), v_Q(b)\} = \min\{v_Q(a'), v_Q(b)\}$ for all $Q \in \mathfrak{X}(D)$ and hence $I^{-1} = (a, b)_e$. Moreover $v_P(\frac{1}{a}) = 1$.

Kim [13] showed that every divisor class of $D[G]$ (where $G \neq \{0\}$) contains a prime divisor. We copy and modify his proof in such a way that the existence of infinitely many prime divisors in each class follows.

**Lemma 3.4.** Let $G$ be a non-zero abelian group such that $D[G]$ is a Krull domain. Then each divisor class of $D[G]$ contains infinitely many prime divisors.

**Proof.** Note that $G$ is torsion-free, because $D[G]$ is Krull.

**Case 1:** Let $G$ be finitely generated. Then there is a positive integer $n$ such that $D[G]$ is isomorphic to the Laurent polynomial ring in $n$ indeterminates over $D$. If $\mathfrak{X}(D)$ is finite, then $D$ is a semi-local principal ideal domain. Hence $D[G]$ is factorial with infinitely many non-associated prime elements. Now let $\mathfrak{X}(D)$ be infinite. Let $I[G]$ be a non-zero divisorial ideal of $D[G]$ where $I$ is a non-zero divisorial ideal of $D$. By Lemma 3.3 there exist infinitely many $P \in \mathfrak{X}(D)$ such that there exist $a, b \in K$ with $I^{-1} = (a, b)_e$ and
v_P\left(\frac{a}{b}\right) = 1. For each choice of P and corresponding a, b ∈ K with the above properties, we construct a prime divisor lying in [I[G]]; Let α ∈ G with α > 0 and set g = \frac{a}{b} + X^α. Then g is irreducible in D_P[G] by a generalized version of Eisenstein’s criterion [3, Lemma 5]. Hence g is irreducible in K[G] by localization. Moreover, we have by Lemma 3.2 that gK[G] ∩ D[G] = g(\frac{a}{b}, 1)^{-1}[G] = gb(a, b)^{-1}[G] = gbI[G] and hence gK[G] ∩ D[G] is one of infinitely many prime divisors lying in the class of I[G].

**Case 2:** Let G be non-finitely generated. Again, let I[G] be a non-zero divisorial ideal of D[G] where I is a non-zero divisorial ideal of D. By Lemma 3.3 there exist a, b ∈ K such that I^{-1} = (a, b)_v. Since G is non-finitely generated, there are infinitely many elements g ∈ G of height (0, 0, 0, . . .) . For each choice of g, the element f = a + bX^g is irreducible in K[G] by Lemma 3.1. Moreover, by Lemma 3.2 we have fK[G] ∩ D[G] = f(a, b)^{-1}[G] = fI[G] and hence fK[G] ∩ D[G] is one of infinitely many prime divisors lying in the class of I[G].

**Lemma 3.5.** Let K be a field and let S be a monoid such that K[S] is a Krull domain. Then every divisor class of K[S] contains infinitely many prime divisors.

**Proof.** If S is a Krull monoid, then it has a decomposition S ≃ S_{red} × S^\times, where S_{red} denotes the reduced monoid of S [8, Theorem 2.4.8.2]. Thus, by [10, Theorem 7.1] we have K[S] ≃ K[S_{red}][S^\times], which is a Krull group algebra over K[S_{red}]. Therefore, if S is non-reduced then S^\times is non-trivial and we can use Lemma 3.4 to show that every divisor class contains infinitely many prime divisors. So, from now on, assume that S is reduced.

Let G be the quotient group of S.

**Case 1:** First assume that S is finitely generated. Then, since S is torsion-free, it is isomorphic to an additive submonoid of the group \((\mathbb{Z}^m, +)\) for some m ∈ N [3, page 50]. Therefore G is a Z-submodule of the free Z-module \(\mathbb{Z}^m\) and hence free, say of rank n ∈ N.

**Claim A:** There exists a Z-basis B of G such that B ∩ S = ∅.

**Proof of Claim A.** If n = 0 or n = 1, this is trivial. So let n > 1. Then in particular S ≠ G, because S is reduced. Let P ⊆ S be a height-one prime ideal of S and v_P : G → P the associated P-adic valuation. Let a ∈ S with v_P(a) = 1. Let G_0 be the kernel of v_P. Then G_0 is a free Z-module of rank n − 1 and we have G ≃ ker v_P ⊕ im v_P ≃ G_0 ⊕ \mathbb{Z}. So, if a_1, . . . , a_{n−1} is a Z-basis of G_0, then a_1, . . . , a_{n−1}, α is a Z-basis of G with α ∈ S.

Now let (a_1, . . . , a_n) be a Z-basis of G with a_1 ∈ S. We define \(O = K[a_1]\) with quotient field F = K(a_1). By L = K(a_1, . . . , a_n) we denote the quotient field of K[S]. Then K[S] is a finitely generated \(O\)-algebra, F is a Hilbertian field (as a finitely generated transcendental extension of a field), the Krull dimension of \(O\) equals 1, \(O\) and K[S] are integrally closed, L/F is a purely transcendental extension and hence separable and regular, and Pic(O) = 0. Therefore by [12, Theorem 2], K[S] has infinitely many prime divisors in all classes and we are done.

**Case 2:** Now let S be non-finitely generated. We have an isomorphism of divisor class groups C_v(S) → C_v(K[S]) via \([I] → [K[I]]\) ([10, Theorem 16.7]). Therefore it suffices to prove that \([K[I]]\) contains a prime ideal of K[S] for every non-empty v-ideal I of S. So let I be a non-empty v-ideal of S and let g_1, . . . , g_n ∈ G with I^{-1} = ((g_1 + S) ∪ . . . ∪ (g_n + S))_v. Since S is a reduced non-finitely generated Krull monoid, it follows from [3, Theorem 2.7.14] that the set \(X(S)\) of height-one prime ideals of S is infinite. Thus, there exist infinitely many P ∈ X(S) such that 0 = v_P(g_1) = . . . = v_P(g_n). For each choice of P, we construct a prime divisor lying in [K[I]] (and infinitely many of them are pairwise distinct). Let a ∈ S with v_P(a) = 1. Let g = X^{g_1} + . . . + X^{g_n} + X^{\alpha}, \alpha ∈ K[G].

**Claim B:** g is irreducible in K[G].

Suppose that the claim holds true. On the one hand, I^{-1} = ((g_1 + S) ∪ . . . ∪ (g_n + S))_v = ((g_1 + S) ∪ . . . ∪ (g_n + S) ∪ (g_n + a + S))_v, because a ∈ S. On the other hand, by Lemma 3.2 we have that gK[G] ∩ K[S] = gK[((g_1 + S) ∪ . . . ∪ (g_n + S) ∪ (g_n + a + S))^{-1}] = gK[I] = gK[L]. Therefore, gK[G] ∩ K[S]
is one of infinitely many prime ideals of $K[S]$ (because $g$ is irreducible by the claim) that lies in the divisor class of $K[I].$

**Proof of Claim B.** It suffices to show that $g$ is irreducible in $K[S_P],$ because $K[G]$ is the localization of $K[S_P]$ at the set $\{X^s \mid s \in P\}$ and $K[S_P]$ is a factorial domain.

To see that $g \in K[S_P]$ is irreducible, let $h_1, h_2 \in K[S_P]$ with $g = h_1 h_2.$ Since $S_P$ is a discrete rank one valuation monoid, the map $\mathbb{N}_0 \times S_P^\times \to S_P$ via $(n, s) \mapsto s a^n$ is an isomorphism (note that $a \in S_P$ was chosen with $v_P(a) = 1$). We endow $\mathbb{N}_0 \times S_P^\times$ with a total order compatible with the monoid operation in the following way: $\mathbb{N}_0$ carries the canonical order $\preceq$. On $S_P^\times$ take any total order $\preceq$ compatible with the group operation, which is possible by [10, Corollary 3.4]. Now $\mathbb{N}_0 \times S_P^\times$ is a totally ordered monoid with lexicographic order, i.e.,

$$(m, s) \preceq (n, t) \iff (m < n \lor (m = n \land s \leq t)),$$

for $m, n \in \mathbb{N}_0$ and $s, t \in S_P^\times.$

Now we can write $h_1 = \sum_{i=1}^{n} k_i X^{(a_i, b_i)}$ and $h_2 = \sum_{j=1}^{m} l_j X^{(c_j, d_j)}$ with $k_i, l_j \in K \setminus \{0\}, \ (a_i, b_i), (c_j, d_j) \in \mathbb{N}_0 \times S_P^\times,$ and $(a_i, b_i) < (a_2, b_2) < \ldots < (a_n, b_n),$ $(c_1, d_1) < (c_2, d_2) < \ldots < (c_m, d_m).$ Also $g = X^{(0, g_1)} + \ldots + X^{(0, g_s)} + X^{(1, g_s)}$ in this notation.

It follows that $(1, g_a) = (a_n, b_n) + (c_n, d_n) = (a_n + c_n, b_n + d_n),$ thus $1 = a_n + c_n.$ Without loss of generality, we can suppose that $a_n = 0$ and $c_n = 1.$ Therefore $a_i = 0$ for all $i \in \{1, \ldots, n\}.$ Let $j \in \{1, \ldots, m\}$ be minimal such that $c_j = 1.$ Then the monomial in $h_1 h_2 = g$ with exponent $(a_1, b_1) + (1, d_j) = (1, b_j + d_j)$ has a non-zero coefficient and therefore it holds that $(a_1, b_1) + (1, d_j) = (1, b_j + d_j) = (1, g_n) = (1, b_n + d_m) = (a_n, b_n) + (c_m, d_m).$ It follows that $n = 1$ and hence $h_1 = k_1 X^{(0, h_1)} \in K[S_P]$ is a unit. This proves that $g \in K[S_P]$ is irreducible. \hfill $\square$[Proof of Claim B]

**Proof of the Theorem.** Let $K$ be the quotient field of $D$ and $G$ be the quotient group of $S.$ If $D = K$ or $S = G,$ we are done by Lemma 3.4 and Lemma 3.5. So assume that $D \neq K$ and $S \neq G.$ We have an isomorphism $C_v(D) \oplus C_v(S) \to C_v(D[S])$ via $([I], [J]) \mapsto [I, J].$ Hence it suffices to prove that every class of the form $[I, J]$ contains a prime divisor, where $I$ (resp. $J$) is a non-zero (resp. non-empty) $v$-ideal of $D$ (resp. $S$). So let $I$ be a non-zero $v$-ideal of $D$ and let $J$ be a non-empty $v$-ideal of $S.$

By Lemma 3.3 let $a, b \in K$ and $P \in \mathcal{X}(D)$ such that $I^{-1} = (a, b)_v$ and $v_P(\frac{1}{a}) = 1.$ Define $p = \frac{1}{a}. \ $ Then $p$ is a prime element of $D_P.$ Moreover, let $g_1, \ldots, g_n \in G$ such that $J^{-1} = (g_1 + S) \cup \ldots \cup (g_n + S).$ For each non-negative integer $m$ we can choose $h_1, \ldots, h_m \in J^{-1}$ such that $g_1, \ldots, g_n, h_1, \ldots, h_m$ are pairwise distinct. This works out, because $S$ is torsion-free and hence infinite. Let $h$ be the maximal element in $\{g_1, \ldots, g_n, h_1, \ldots, h_m\}$ for some fixed total order on $G$ and set $M = \{g_1, \ldots, g_n, h_1, \ldots, h_m\} \setminus \{h\}.$

Now we set $g = X^h + \sum_{a \in M} p X^a \in D_P[G].$ Then $g$ is prime in $D_P[G]$ by [4, Lemma 5]. Since $K[G]$ is the localization of $D_P[G]$ at all non-zero constants and $gD_P[G]$ is a prime ideal not containing non-zero constants, it follows that $g \in K[G]$ is prime. Moreover, we have $gK[G] \cap D[S] = g(p, 1)^{-1}(((g_1 + S) \cup \ldots \cup (g_n + S) \cup (h_1 + S) \cup \ldots \cup (h_m + S)))^{-1} = gb(a, b)^{-1}[J] = gb[I, J],$ where the first equality follows by Lemma 3.5. Hence $gK[G] \cap D[S]$ is a prime divisor lying in the class of $[I, J].$ By varying $m,$ the construction leads to non-associated prime elements $g \in K[G]$ and hence the assertion follows. \hfill $\square$

**4. Past proof attempts**

As noted in the beginning of this work, Chang [4] states the fact that in a Krull monoid algebra $D[S]$ every divisor class contains a prime divisor. Unfortunately, due to an error in Lemma 7 of his work (Krull monoids in general do not satisfy the approximation property, see [11, Theorem 26.4]), the proof of his main result collapses. Another error that occurs in Chang’s argumentation is due to a mistake in [13, Corollary 5], where Kim asserts that fractional $v$-ideals of Krull monoids $S$ such that $D[S]$ is a Krull
domain are always \( v \)-generated by two elements. A counterexample to this assertion can be found in [17, Beispiel 3.8]. In the following, we give an explicit counterexample to Chang’s Lemma 7.

**Counterexample 4.1.** This example is based on zero-sum theory. For an introduction to this topic, see [8, Chapter 5]. Let \( G_0 = \{-2g, -g, g, 2g\} \subseteq G \), where \( G \) is an abelian group and \( g \in G \) is an element of infinite order. The monoid \( \mathcal{B}(G_0) \) of zero-sum sequences over \( G_0 \) is a finitely generated reduced torsion-free Krull monoid that is not factorial. Its height-one spectrum is \( \mathcal{X}(\mathcal{B}(G_0)) = \{p_i \mid i \in G_0\} \), where \( p_i = \{ S \in \mathcal{B}(G_0) \mid i \in \text{supp}(S)\} \). Now set \( \alpha_1 = (-2g) \cdot (-2g) \cdot 2g \cdot (-g) \cdot (-g) \cdot g \cdot g \in \mathcal{B}(G_0) \) and \( \alpha_2 = \alpha_1 \cdot \alpha_1 \in \mathcal{B}(G_0) \). Thus, whenever we take \( \alpha \in \mathcal{B}(G_0) \), the element \( \alpha_2 + \alpha - \alpha_1 \) has \( p_i \)-adic valuation at least 2 for every \( i \in G_0 \), which is contradiction to Lemma 7 in [4].

**References**

[1] D.D. Anderson, E.G. Houston and M. Zafrullah, *t*-linked extensions, the \( t \)-class group, and Nagata’s theorem, J. of Pure and Appl. Algebra 86 (1993), 109–124.
[2] N. Bourbaki, *Commutative Algebra*, Springer-Verlag, 2nd printing, 1989.
[3] W. Bruns and J. Gubeladze, *Rings, Polytopes and K-Theory*, Springer, 2009.
[4] G.W. Chang, Every divisor class of Krull monoid domains contains a prime ideal, J. Algebra 336 (2011), 370–377.
[5] L. Chouinard, Krull semigroups and divisor class groups, Canad. J. Math. 33 (1981), 1459–1468.
[6] L. Fuchs, *Abelian groups*, Pergamon Press, 1960.
[7] S. Gabbelli, On divisorial ideals in polynomial rings over Mori domains, Comm. Algebra 15 (1987), 2349–2370.
[8] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.
[9] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
[10] R. Gilmer, *Commutative semigroup rings*, Chicago Lectures in Mathematics, 1984.
[11] F. Halter-Koch, *Ideal Systems: An Introduction to Multiplicative Ideal Theory*, Marcel Dekker, 1998.
[12] F. Kainrath, The distribution of prime divisors in finitely generated domains, manuscripta math. 100 (1999), 203–212.
[13] H. Kim, The distribution of prime divisors in Krull monoid domains, J. Pure and Appl. Algebra 155 (2001), 203–210.
[14] R. Matsuda, On Algebraic Properties of Infinite Group Rings, Bull. Fac. Sci. Ibaraki Univ. Math. 7 (1975), 29–37.
[15] R. Matsuda, Infinite Group Rings II, Bull. Fac. Sci. Ibaraki Univ. Math. 8 (1976), 43–66.
[16] R. Matsuda, Torsion-Free Abelian Group Rings III, Bull. Fac. Sci. Ibaraki Univ. Math. 9 (1977), 1–49.
[17] L. Skula, Divisorstheorie einer Halbgruppe, Math. Z. 114 (1970), 113–120.
[18] D. Smertniq, Every abelian group is the class group of a simple Dedekind domain, Trans. Amer. Math. Soc. 369 (2017), 2477–2491.

(Victor Fadinger) Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria
Email address: victor.fadinger@uni-graz.at

(Daniel Windisch) Institut für Analysis und Zahlentheorie, Technische Universität Graz, Kopernikusgasse 24/II, 8010 Graz, Austria
Email address: dwindisch@math.tugraz.at