On a Class of Bounded Turning Functions Subordinate to a Leaf-Like Domain

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Abstract. In the present investigation, we obtain the coefficient inequalities, Fekete-Szegö inequality, third Hankel determinant for the function $f \in R(h)$. A similar study have been done as the function $f$ defined through convolution, for the function $f^{-1}$ and for the function $z/f(z)$.

1. Introduction

Let the class of all analytic functions with normalized by $f(0) = 0$ and $f'(0) = 1$ in the open unit disc $\Delta$ denoted by $A$. Geometrically the normalization condition satisfies as $f(0) = 0$ corresponds to the translation of the image domain and the rotation or stretching or shrinking of the image domain is given by $f'(0) = 1$. The function $f$ is in the form of Maclaurin’s series given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

Thus without loss of generality an univalent analytic function can be written as in (1.1). Let the subclass of $A$, consisting of univalent functions denoted by $S$. A function $f \in A$ is said to be of bounded turning if it satisfies the condition that

$$\text{Re}\{f'(z)\} > 0, \; \forall z \in \Delta$$

This class of functions is denoted by $R$. Let $P$ be the class of all functions of the form

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (1.2)$$
that are analytic in $\Delta$ and satisfies the conditions as $\text{Re}\{P(z)\} > 0$, $\forall \ z \in \Delta$ and $|p_n| \leq 2$. Any function in the class $\mathcal{P}$ is called as a function with positive real part. For any two functions $f$ and $g$ analytic in $\Delta$. The function $f$ is said to be subordinate to the function $g$ in $\Delta$ and it can be written as $f \prec g$, if there exists a Schwartz function $w$, which is analytic in $\Delta$ with $w(0) = 0, |w(z)| < 1 (z \in \Delta)$ such that $f(z) = g(w(z))$. Furthermore, if $g$ is univalent in $\Delta$ then the following equivalence satisfies as $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

The problem of estimating the maximum value of the coefficient functional $|a_3 - \mu a_2^2|$, where $\mu$ is a real or complex parameter for the class of univalent functions. This work is initiated by Fekete and Szeg"o[2]. This problem is called as Fekete-Szeg"o problem [2].

The $q^{th}$ Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [5] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & - & - & - & a_{n+q} \\ a_{n+1} & a_{n+2} & - & - & - & a_{n+q+1} \\ - & - & a_{n+2} & - & - & - \\ - & - & - & a_{n+3} & - & - \\ a_{n+q} & a_{n+q+1} & - & - & - & a_{n+2q-2} \end{vmatrix} \quad (1.3)$$

Several authors have been investigated this determinant. For $q = 2$ and $n = 2$ in (1.3), the second Hankel determinant of the function $f$ is obtained and given by

$$[H_2(2)] = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2| \quad (1.4)$$

For $q = 3$ & $n = 1$, in (1.3) the third Hankel determinant of the function $f$ is obtained and given by

$$[H_3(1)] = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$$H_3(1) = a_3 \left\{ a_2a_4 - a_3^2 \right\} - a_4 \left\{ a_1a_4 - a_3a_2 \right\} + a_5 \left\{ a_1a_3 - a_2^2 \right\}$$

$$|H_3(1)| \leq |a_3| |a_2a_4 - a_3^2| + |a_4| |a_4 - a_3a_2| + |a_5| |a_3 - a_2^2| \quad (\because a_1 = 1) \quad (1.6)$$

The coefficient inequalities for the function $f \in S^*(q)$ subordinate to a shell shaped region $q(z)$ have been studied by Raina and Sokol[9]. These results are further improved by Sokol and Thomas [12]. In 1976, Noon and Thomas [4] defined the $q^{th}$ Hankel determinant of $f$ for $q \geq 1$ and $n \geq 0$. The second Hankel determinant for the function $f$ belonging to the subclasses of Ma-Minda starlike and convex functions have been obtained by Lee et al.[7]. The sharp results
of the second Hankel determinant for the function \( f \) have determined by Noor [8] using the concept of bounded rotation.

A function \( f \) is said to be in the class \( M_\alpha(q) \), if it satisfies the following conditions

\[
\left\{ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) \right\} < z + \sqrt{1 + z^2} = q(z); \ z = re^{i\theta} \in \Delta
\]

with \( \alpha \geq 0 \), and \( q(0) = 1 \). Here the set \( q(\Delta) \) lies in the shell shaped region contained in the right half plane, \( M_\alpha(q) \subset \mathbb{C} \). This class have been studied by Sharma and Hari Priya [10].

Motivated by the works of Raina and Sokol[9], Sokol and Thomas[12] in the present article, we introduced a function \( h(z) \) and studied the coefficient inequalities, Fekete-Szegö inequality, second Hankel determinant for the functions in these classes. A similar study have been done for the function of \( f^{-1} \) and for the function \( \frac{z}{f(z)} \).

**Definition 1.1.** The function \( h(z) = z + \sqrt{1 + z^3} \) maps the unit disc onto a leaf-like shaped region which is analytic and univalent. It is symmetric with respect to the real axis. It is a function with positive real part with \( h(0) = h'(0) = 1 \).

![Figure 1. Leaf shaped region](image)

In Figure 1, \( h(\Delta) \) is a leaf-like shaped region bounded and symmetric with respect to the real axis from \(-1\) to \(2.2\) and from \(-0.7\) to \(0.7\) on imaginary axis.

### 2. Preliminaries

To prove our results we recall the following Lemma

**Lemma 2.1.** [1] Let \( P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \) be an analytic function in \( \Delta \) satisfying \( P(0) = 1 \) and \( \Re\{P(z)\} > 0 \), for all \( z \in \Delta \), then

\[
|c_n| \leq 2 \text{ for all } n \geq 1 \quad \text{and} \quad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.
\]
The class of all such functions with positive real part are denoted by $\mathcal{P}$.

**Lemma 2.2.** Let $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ be in $\mathcal{P}$ then for any complex number $\mu$,

$$|c_2 - \mu c_1^2| \leq \max\{1, |2\mu - 1|\}$$

The result is sharp for the functions defined by $P(z) = \frac{1+z^2}{1-z^2}$ or $P(z) = \frac{1+z}{1-z}$.

**Lemma 2.3.** [3] The power series $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ converges in the open unit disc $\Delta$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants

$$D_n = \begin{bmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{bmatrix}; \forall n \in \mathbb{N}$$

and $c_{-k} = \overline{c_k}$ are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k}z)$, where $\rho_k > 0$, $t_k$ real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$ in this case $D_n > 0$ for $n < (m-1)$ and $D_n = 0$ for $n \geq m$.

For $n = 2$

$$D_2 = \begin{bmatrix} 2 & c_1 & c_2 \\ \overline{c_1} & 2 & c_1 \end{bmatrix} = 8 + 2\Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2 \geq 0$$

which is equivalent to

$$2c_2 = c_1^2 + x(4-c_1^2), \text{ for some } x, \ |x| \leq 1. \quad (2.1)$$

For $n = 3$

$$D_3 = \begin{bmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c_1} & 2 & c_1 & c_2 \\ \overline{c_2} & \overline{c_1} & 2 & c_1 \\ \overline{c_3} & \overline{c_2} & \overline{c_1} & 2 \end{bmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4-c_1^2) + c_1(2c_2 - c_1^2) - 2(2c_2 - c_1^2)|^2 \leq 2(4-c_1^2)^2 - 2(2c_2 - c_1^2)|^2.$$ 

By simple computation, we get

$$4c_3 = c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \quad (2.2)$$

for some $z$, with $|z| \leq 1$. Due to Libera [6] we state the following:
Lemma 2.4. [6] If the function $p \in \mathcal{P}$ is given by $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then
\[ 2c_2 = c_1^2 + x(4 - c_1^2) \]
\[ 4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \]
for some $x, z$ with $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

The coefficient inequalities for the function $f \in S^*(q)$ subordinate to a shell shaped region $q(z)$ have been studied by Raina and Sokol[9]. These results are further improved by Sokol and Thomas[12]. In 1976, Noonan and Thomas [4] defined the $q^h$ Hankel determinant of $f$ for $q \geq 1$ and $n \geq 0$. The second Hankel determinant for the function $f$ belonging to the subclasses of Ma-Minda starlike and convex functions have been obtained by Lee et al.[7]. Noor [8] has obtained the sharp results of the second Hankel determinant for the function and determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the function $f$ with bounded rotation. Let $M_{\alpha}(q)$ denote the class of functions $f$ analytic in the unit disc $\Delta$, normalized by $f(0) = f'(0) = 1$ and satisfies the condition
\[ \left\{ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \frac{zf'(z)}{f(z)} \right\} \prec z + \sqrt{1 + z^2} = q(z); z = re^{i\theta} \in \Delta \]
with $\alpha \geq 0$, and the branch of the square root is chosen to be $q(0) = 1$. Here the set $q(\Delta)$ lies in the shell shaped region contained in the right half plane, $M_{\alpha}(q) \subset \mathbb{C}$. This class have been studied by Sharma and Hari Priya [10].

Motivated by the works of Raina and Sokol[9], Sokol and Thomas[12] in the present paper a function $h(z)$ is introduced. The coefficient inequalities, Fekete-Szegö inequality, second and third Hankel determinant for the function $f$ in the class $R(h)$ are studied. This work also enhanced for the function defined through convolution, for $f^{-1}$ and $\frac{z}{f(z)}$.

3. Coefficient Inequalities for the function $f \in R(h)$

Theorem 3.1. If $f \in R(s)$, then
\[ |a_2| \leq \frac{1}{2} \] (3.1)
\[ |a_3| \leq \frac{1}{3} \] (3.2)
\[ |a_4| \leq \frac{3}{4} \] (3.3)
\[ |a_5| \leq \frac{1}{15} \] (3.4)

Proof. If $f \in R(h)$ then there exists a Schwartz function $w$, with $w(0) = 0$ and $|w(z)| \leq 1$ such that
\[ f'(z) = w(z) + \sqrt[3]{1 + |w(z)|^3} \] (3.5)
Define a function $P(z)$ such that
\[ P(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots. \]

Substituting $w(z)$ from (3.6) on the right hand side of the equation (3.5), it can be reduced to
\[ \sqrt[3]{1 + \left( \frac{P(z) - 1}{P(z) + 1} \right)^3 + \frac{P(z) - 1}{P(z) + 1}} = 1 + \frac{c_1}{2} z + \left\{ \frac{c_2}{2} - \frac{c_2^2}{4} \right\} z^2 + \left\{ \frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{6} \right\} z^3 + \frac{c_4^4}{48} z^4 + \ldots. \]

Upon simplification, it can be observed that
\[ f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \ldots. \]  

Comparing the coefficients of $z, z^2, z^3$ and $z^4$, from relations (3.5), (3.7) and (3.8), we have
\[ a_2 = \frac{c_1}{4} \] (3.9)
\[ a_3 = \left\{ \frac{c_2}{6} - \frac{c_1^2}{12} \right\} \] (3.10)
\[ a_4 = \left\{ \frac{c_3}{8} - \frac{c_2 c_1}{8} + \frac{c_1^3}{24} \right\} \] (3.11)
\[ a_5 = \left\{ \frac{c_4^4}{240} \right\} \] (3.12)

Upon applying Lemma 2.2 on the right hand side of the equation (3.9), it can be seen that
\[ |a_2| \leq \frac{1}{2} \] (3.13)

Now, the equation (3.10) can be reduced to
\[ |a_3| \leq \frac{1}{6} |c_2 - \nu_1 c_1^2| \] (3.14)

where $\nu_1 = \frac{1}{2}$.

Upon applying Lemma 2.3, on the right hand side of equation (3.14), one can obtain the result as in (3.2). This result $|a_3| = \frac{1}{3}$ is sharp for the function $f_2(z)$ given as below If a function $f_2$ is such that
\[ f_2' = h(z^2) = z^2 + \sqrt[3]{1 + z^6} \]

then $f_2 \in R(h)$ and
\[ f_2 = z + \sum_{n=2}^{\infty} u_n z^n = z + \frac{1}{3} z^3 + \frac{1}{18} z^6 + \ldots. \]
\[ |u_2 u_4 - u_3^2| = \frac{1}{9} \]

Upon applying Lemmas 2.1 and 2.2 on the right hand side of the equations (3.11) and (3.12), one can obtain the result as in equations (3.3) and (3.4)
Theorem 3.2. If \( f \in R(h) \), then
\[
|a_2a_3 - a_4| \leq \frac{1}{12} \tag{3.16}
\]
The result is sharp.

Proof. From relations (3.9), (3.10) and (3.11), consider
\[
\left[ a_2a_3 - a_4 \right] = \left[ \frac{c_1}{4} \left\{ \frac{c_2}{6} - \frac{c_1^2}{12} \right\} - \frac{c_3}{8} - \frac{c_2c_1}{8} + \frac{c_1^2}{24} \right] \tag{3.17}
\]
By applying Lemma 2.3, in (3.17) after simplification, we get
\[
\left[ a_2a_3 - a_4 \right] = \left\{ \frac{c_1^3}{96} - \frac{xc_1(4-c_1^2)}{48} - \frac{x^2c_1(4-c_1^2)}{36} + \frac{(1-|x|^2)(4-c_1^2)}{16} \right\} \tag{3.18}
\]
Taking \( c_1 = c \) & \( |x| = \rho \), it can be assumed without any restriction that \( c \in [0,2] \). By applying triangular inequality, it can be seen that
\[
|a_2a_3 - a_4| \leq \left| \frac{c_1^4}{96} + \frac{\rho c(4-c_1^2)}{48} + \frac{\rho^2 c(4-c_1^2)}{36} + \frac{(1-\rho^2)(4-c_1^2)}{16} \right| = S(\rho) \tag{3.19}
\]
\[
S'(\rho) = \frac{c(4-c_1^2)}{48} + \frac{c\rho(4-c_1^2)}{18} - \frac{(4-c_1^2)}{16} > 0 \tag{3.20}
\]
\( \Rightarrow \) The function \( S(\rho) \) is increasing on the closed interval \([0,1]\). Hence \( S(\rho) \leq S(1), \forall \rho \in [0,1] \)
i.e.,
\[
S(\rho) \leq \frac{c_1^4}{96} + \frac{c(4-c_1^2)}{48} + \frac{c(4-c_1^2)}{36} = G(c) \tag{3.21}
\]
Since \( c \in [0,2] \), then it follows
\[
G(c) \leq \frac{1}{12} \tag{3.22}
\]
This completes the proof of the Theorem 3.2. One can observe that the value of the coefficient functional \( |a_2a_3 - a_4| \) is sharp for the function \( f_1(z) \) defined as below
If a function \( f_1 \) is such that
\[
f'_1(z) = h(z) = z^{\sqrt{1}} + z^3
\]
then \( f_1 \in R(h) \) and
\[
f_1 = z + \sum_{n=2}^{\infty} v_n z^n = z + \frac{1}{2} z^2 + \frac{1}{112} z^4 - \frac{1}{56} z^6 + \ldots
\]
\[
|v_2v_3 - v_4| = \frac{1}{12}
\]
4. Fekete-Szeg"{o} coefficient functional for the function $f$ in the classes $R(h)$ and $R_g(h)$

**Theorem 4.1.** If $f \in R(h)$, then

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} \max \left\{ 1, \frac{3\mu}{4} \right\}$$

(4.1)

and the result is sharp.

**Proof.** From relations (3.9) and (3.10), consider

$$a_3 - \mu a_2^2 = \frac{1}{6} (c_2 - \nu_2 \frac{c_1^2}{2})$$

(4.2)

where $\nu_2 = \left(\frac{4+3\mu}{4}\right)$. Upon applying Lemma 2.2 on the right hand side of equation (4.2), one can obtain the result as in equation (4.1). The sharpness of the result is given below

$$|a_3 - \mu a_2^2| = \begin{cases} \frac{1}{3}, & p(z) = \frac{1+z^2}{1-z^2}; \\ \frac{1}{3}, & p(z) = \frac{1+z}{1-z}. \end{cases}$$

(4.3)

This completes the proof of the Theorem 3.1.

**Theorem 4.2.** If $f \in R_g(h)$, then

$$|a_3 - \mu a_2^2| \leq \frac{1}{3 \sqrt{g_3}} \left\{ 1, \frac{3\mu g_3}{g_2^2} \right\}$$

(4.4)

and the result is sharp.

**Proof.** If $f \in R(h)$ then there exists a Schwartz function $w$, with $w(0) = 0$ and $|w(z)| \leq 1$ such that

$$(f \ast g)'(z) = w(z) + \sqrt[3]{1 + |w(z)|^3}$$

(4.5)

That is

$$(f \ast g)'(z) = 1 + 2a_2g_2z + 3a_3g_3z^2 + 4a_4g_4z^3 \ldots .$$

(4.6)

From equations (3.6), (4.5) and (4.6) and comparing the coefficients of $z$, $z^2$ and $z^3$, we get

$$a_2 = \frac{c_1}{4g_2}$$

(4.7)

$$a_3 = \frac{1}{3g_3} \left[ \frac{c_2}{2} - \left( \frac{c_1^2}{4} \right) \right]$$

(4.8)
For any complex number \( \mu \), consider
\[
a_3 - \mu a_2^2 = \frac{1}{6g_3} \left( c_2 - \nu_3 c_1^2 \right)
\]
where \( \nu_3 = \left( 1 + \frac{3\mu g_3}{g_2^2} \right) \). Taking modulus on both sides and by applying Lemma 2.2 on the right hand side of the equation (4.10), one can obtain the result as in equation (4.4). The sharpness of the result is given below
\[
| a_3 - \mu a_2^2 | = \left\{ \begin{array}{ll}
\frac{1}{3g_3}, & p(z) = \frac{1+\rho^2}{1-\rho^2}; \\
\frac{\mu}{3g_2}, & p(z) = \frac{1+\rho}{1-\rho};
\end{array} \right.
\]
(4.10)

5. Second Hankel Determinant

**Theorem 5.1.** If \( f \in R(h) \), then the second Hankel determinant is
\[
H_2(2) \leq \frac{1}{9}
\]
and the result is sharp.

**Proof.** From relations (3.9), (3.10) and (3.11), consider
\[
a_2a_4 - a_3^2 = \left( \frac{c_3c_1}{32} - \frac{c_2c_1^2}{288} - \frac{c_2^2}{36} + \frac{c_4^4}{288} \right)
\]
(5.2)
Upon applying Lemma 2.2, on the right hand side of an equation (5.2), it can be seen that
\[
| a_2a_4 - a_3^2 | = \left| \frac{c_4^4}{384} + \frac{c_1(1 - |x|^2)(4 - c_1^2)}{128} + \frac{x^2c_1^2(4 - c_1^2)}{128} + \frac{x^2(4 - c_1^2)}{144} \right|
\]
(5.3)
Choosing \( c_1 = c \), and \( |x| = \rho \), by using Lemma 2.3, it may be assumed without any restriction that \( 0 \leq x \leq 1 \). Applying the triangular inequality, it can be reduced to
\[
| a_2a_4 - a_3^2 | = \frac{c_4^4}{384} + \frac{c(1 - \rho^2)(4 - c^2)}{128} + \frac{\rho^2c^2(4 - c^2)}{128} + \frac{\rho^2(4 - c^2)}{144}
\]
\[
= F(\rho)
\]
(5.4)
Consider
\[
F'(\rho) = \frac{\rho(4 - c^2)(3c - 2)(3c - 4)}{576} > 0
\]
(5.5)
\( \Rightarrow F(\rho) \) is an increasing function of \( \rho \) on the closed interval \([0,1]\). Hence \( F(\rho) \leq F(1) \), \( \forall \rho \in [0,1] \)
i.e.,
\[
F(\rho) \leq \frac{1}{9} - \frac{7c^4}{576} - \frac{7c^2}{288} = G(c)
\]
(5.6)
Since \( c \in [0, 2] \), then it follows

\[
G(c) \leq \frac{1}{9} \quad (5.7)
\]

Hence proof of the Theorem 5.1 completes. One can observe that the value of the second Hankel determinant is sharp for the function \( f_2(z) \) defined as in (3.15)

\[ \square \]

6. Third Hankel determinant

**Theorem 6.1.** If \( f \in R(h) \) then the third Hankel determinant

\[
| H_3(1) | \leq \frac{379}{2160}
\]

**Proof.** The proof follows from equation (1.6) and Theorems 3.1, 3.2, 4.1 and 5.1

\[ \square \]

7. Coefficient inequality for the function \( \frac{z}{f(z)} \)

Let the function \( G \) be defined by

\[
G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} q_n z^n
\]

**Theorem 7.1.** Let \( h(z) = z + \sqrt{1 + z^3} \). If \( f \in R(h) \) and \( G(z) = \frac{z}{f(z)} \), then for any real number \( \mu \), we have

\[
| q_2 - \mu q_1^2 | \leq \frac{1}{3} \max \left\{ 1, \left| \frac{3 - 3\mu}{2} \right| \right\} \quad (7.1)
\]

and the result is sharp.

**Proof.** As \( f \in R(h) \) and

\[
G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} q_n z^n \quad (7.2)
\]

By a simple computation one can obtain that

\[
\frac{z}{f(z)} = 1 - a_2 z + (a_2^2 - a_3) z^2 + \ldots \quad (7.3)
\]

Upon equating the coefficient of \( z \) and \( z^2 \), from relations (7.2) and (7.3), it can be seen that

\[
q_1 = -a_2; \quad (7.4)
\]

\[
q_2 = \left( -a_3 + a_2^2 \right). \quad (7.5)
\]

From equations (3.9), (3.10), (7.4) and (7.5), it can be obtained

\[
q_1 = -\frac{c_1}{4}; \quad (7.6)
\]

\[
q_2 = -\frac{1}{6} \left( c_2 - \frac{7c_1^2}{8} \right). \quad (7.7)
\]
For any complex number $\mu$, consider

$$q_2 - \mu q_1^2 = -\frac{1}{6} \left( c_2 - \nu_4 \frac{c_1^2}{2} \right)$$

(7.8)

where $\nu_4 = \frac{7 - 3\mu}{4}$. Upon applying Lemma 2.2 on the right hand side of (7.8), one can obtain the result as in (7.1). Hence this proves the result of the Theorem. The sharpness of the result is given below

$$\left| q_2 - \mu q_1^2 \right| = \begin{cases} \frac{1}{3}, & p(z) = \frac{1+z^2}{1-z^2}; \\ \left| \frac{1-\mu}{2} \right|, & p(z) = \frac{1+z}{1-z}. \end{cases}$$

(7.9)

8. Coefficient inequalities for the function $f^{-1}$

**Theorem 8.1.** If $f \in R(h)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of $f$ with $|w| < r_0$ where $r_0$ is greater than the radius of the Koebe domain of the class $f \in R(h)$, then for any complex number $\mu$, we have

$$|d_3 - \mu d_2^2| \leq \frac{1}{3} \max \left\{ 1, \left| \frac{6 - 3\mu}{4} \right| \right\}$$

(8.1)

and the result is sharp.

**Proof.** As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$$

(8.2)

is the inverse function of $f$, it can be seen that

$$f^{-1}\{f(z)\} = f\{f^{-1}(z)\} = z$$

(8.3)

From equations (1.1) and (8.3), it can be reduced to

$$f^{-1}\{z + \sum_{n=2}^{\infty} a_n z^n\} = z$$

(8.4)

From (7.3) and (8.4), one can obtain

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2d_2 + d_3)z^3 + \ldots \ldots = z$$

(8.5)

By comparing the coefficients of $z$ and $z^2$ from relation (8.5), it can be seen that

$$d_2 = -a_2$$

(8.6)

$$d_3 = 2a_2^2 - a_3$$

(8.7)
From relations (3.8), (3.9), (8.6) and (8.7)

\[ d_2 = -\frac{c_1}{4}; \]  
\[ d_3 = -\frac{1}{6} \left( c_2 - 5\frac{c_1^2}{4} \right); \]  

For any complex number \( \mu \), consider

\[ d_3 - \mu d_2^2 = -\frac{1}{6} \left( c_2 - \nu_5 \frac{c_1^2}{4} \right) \]  

where \( \nu_5 = \frac{10 - 3\mu}{4} \). Upon applying Lemma 2.2 on the right hand side of (8.10), one can obtain the result as in (8.1). Hence this completes the proof. The result is sharp as given below

\[ \left| d_3 - \mu d_2^2 \right| = \left\{ \begin{array}{ll}
\frac{1}{3}, & p(z) = 1 + \frac{z^2}{1-z^2}; \\
\frac{1}{3} \left| \frac{z^2}{1-z} \right|, & p(z) = \frac{1+z}{1-z}.
\end{array} \right. \]

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