APPLICATION OF GEOMETRIC MEASURE THEORY IN CONTINUUM MECHANICS:
THE CONFIGURATION SPACE, PRINCIPLE OF VIRTUAL POWER AND CAUCHY’S STRESS THEORY FOR ROUGH BODIES

Thesis submitted in partial fulfillment of the requirements for the degree of “DOCTOR OF PHILOSOPHY”

BY

LIOR FALACH

ADVISOR: PROF. REUVEN SEGEV

Submitted to the Senate of Ben-Gurion University of the Negev

May 2013

BEER-SHEVA
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Approved by the advisor

Prof. Reuevn Segev: __________________ Date: ______________

Approved by the Dean of the Kreitman School of Advanced Graduate Studies

Prof. Michal Shapira: __________________ Date: ______________

May 2013

BEER-SHEVA
This work was carried out under the supervision of
Prof. Reuevn Segev
In the Department of Mechanical Engineering
Faculty of Engineering Science
Research-Student’s Affidavit when Submitting the Doctoral Thesis for Judgment

I Lior Falach, whose signature appears below, hereby declare that
(Please mark the appropriate statements):

___ I have written this Thesis by myself, except for the help and guidance offered by my Thesis Advisors.

___ The scientific materials included in this Thesis are products of my own research, culled from the period during which I was a research student.

___ This Thesis incorporates research materials produced in cooperation with others, excluding the technical help commonly received during experimental work. Therefore, I am attaching another affidavit stating the contributions made by myself and the other participants in this research, which has been approved by them and submitted with their approval.

Date: _________________

Student’s name: ________________

Signature:______________
Acknowledgment

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“We are all meant to shine, as children do. We were born to make manifest the glory of God that is within us. It’s not just in some of us; it’s in everyone. And as we let our own light shine, we unconsciously give other people permission to do the same.” [Marianne Williamson]
This work is dedicated to my parents,
Ester and Eliyaho Falach,
My heroes.
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Abstract

It is a generally agreed upon notion that the elements of geometric measure theory should play a central role in the mathematical formulation of continuum mechanics. Homological integration theory, a branch of geometric measure theory and differential geometry, is concerned with the generalization of the notion of integration on manifolds. The generalized integral is defined by a current, and within the class of currents special attention is given to integrals over polyhedral chains, normal currents and flat chains. Each of the aforementioned currents may be viewed as an integration over a domain in space of varying degree of regularity, or, as implied by our present objective, irregularity. By applying the tools of homological integration theory, generalized bodies will be introduced. The applicability of homological integration theory to continuum mechanics has been noted in several central works as for example:

“... This result is of independent mathematical interest and is intimately related to the flat form and cochains of Whitney”[Zie83],

“... We note that our development has points of contact with some ideas of geometric measure theory, the theory of distributions, and earlier developments of the mathematical foundations of mechanics”[AO79].

This thesis further explores the applicability of homological integration theory for the mathematical formulation of continuum mechanics. The proposed framework is shown to enable the inclusion of a generalized class of bodies such that a corresponding stress theory is properly formulated and a generalized principle of virtual power is presented.

In the setting of an $n$-dimensional Euclidean space, an admissible body is initially viewed a normal $n$-current induced by a set of finite perimeter. Bodies viewed as normal $n$-currents serve as our elementary building blocks which are used in the construction of a generalized Cauchy’s flux. The configuration space of bodies in the physical space is assumed to be comprises Lipschitz embeddings, which are shown to form an open subset in the space of locally Lipschitz maps endowed with the strong Lipschitz topology. Thus, virtual velocity fields are naturally viewed as locally Lipschitz maps. A field over a body is represented by the multiplication of a sharp function and a normal current. A density transport theorem is developed which is shown to be analogous to Reynolds’ transport theorem for an implicit time dependent property.
A generalized Cauchy flux is defined as a real valued function on the Cartesian product of \((n-1)\)-currents representing material surfaces and locally Lipschitz mappings representing virtual velocities. The duality between restricted velocity fields and Cauchy fluxes is studied and a generalized version of Cauchy’s postulate implies that a Cauchy flux may be uniquely extended to an \(n\)-tuple of flat \((n-1)\)-cochains. Thus, the class of admissible bodies is extended to include flat \(n\)-chains, which may be viewed as currents induced by Lebesgue integrable sets. We note that no restriction is imposed on the measure theoretic boundary of the generalized body, yet, the flux over the boundary of this generalized body is well defined as the boundary of a generalized body is viewed as a flat \((n-1)\)-chain. A general subset of the boundary may not be a flat \((n-1)\)-chain. A generalized material surface is formally introduced as a trace, defined as the intersection of the boundary with a set of finite perimeter. A trace is shown to be a flat \((n-1)\)-chain thus, the flux across such generalized material surfaces may be calculated. Wolfe’s representation theorem for flat cochains enables the identification of stress as an \(n\)-tuple of flat \((n-1)\)-forms providing an integral representation the Cauchy flux.
CHAPTER 1

Introduction

This thesis presents a framework for the formulation of some fundamental notions of continuum mechanics. Specifically, using elements from geometric measure and integration theory, we consider, within the geometric setting of $\mathbb{R}^n$, the class of admissible bodies, configurations of bodies in space, the configuration space, virtual velocities, Reynold’s transport theorem and Cauchy’s stress theory.

Cauchy’s stress theorem is one of the central results in continuum mechanics. It asserts the existence of the stress tensor which determines the traction fields on the boundaries of the various bodies. As the traditional proof relies on locality and regularity assumptions, from both the validity and the applicability aspects, stress theory is closely associated with the proper choice of the class of bodies. Furthermore, an appropriate class of bodies should allow the formulation of the Gauss-Green theorem or a generalization thereof.

In light of these observations, formulations of the fundamentals of continuum mechanics have considered, since the middle of the 20th century, the appropriate choice of the class of bodies. In \[\textit{Nol59}\], Noll sets an axiomatic scheme for continuum mechanics in which a rigorous mathematical framework for the concepts of bodies, kinematics, forces and dynamical processes is presented. A body is defined as a compact, differentiable three-dimensional manifold with piecewise smooth boundary, the manifold is assumed to be covered by a single chart and is endowed with a measure space structure. The configurations of the body in space provide charts on the body manifold and a part of the body is defined as a compact subset of the body with piecewise smooth boundary. The existence of the stress and Cauchy’s original postulate on the dependence of the traction on the exterior normal is shown to follow from the additivity assumption on the system of forces and the principle of linear momentum. In \[\textit{TT60}, \text{p. 466}\] Truesdell and Toupin ignore the formal issue of admissible bodies and tacitly assume smoothness wherever necessary. Later on, in \[\textit{Tru66}, \text{p. 4}\], Truesdell defines a body as a differential manifold endowed with a structure of a $\sigma$-finite measure space and the $\sigma$-ring of subsets are viewed as parts of
CHAPTER 1. INTRODUCTION

the body. The common ground for these early works is in the assumption that bodies
in continuum mechanics should have a smooth structure so that the classical versions
of the notions of mathematical analysis apply.

The *material universe*, a formal structure for the class of admissible bodies, was
presented by W. Noll \[Nol62, Nol73\]. The material universe is assumed to be a
partially ordered collection of sets. The collection is further furnished with the
operations \(\wedge\) (meet) and \(\vee\) (join) (generalizing the operations of intersection and
union operations on sets) such that the material universe has the structure of a
Boolean algebra.

In \[GW67\], M.Gurtin and W. Williams present an axiomatic formulation of con-
tinuum thermodynamics in which a body is viewed as a standard region, *i.e.*, the
closure of a bounded open set in a three-dimensional Euclidean space. The body’s
boundary is composed of the union of a closed set of zero area measure and a count-
able collection of two-dimensional manifolds of class \(C^1\). The collection of subbodies
defined in \[GW67\] has less structure than Noll’s material universe and it is selected
such that the collection of subbodies will enable a proof of the existence of intrinsic
thermodynamical quantities such as the radiation density, heat flux vector and the
internal entropy density.

M. Gurtin and L. Martins introduced in \[GM75\] the notion of a Cauchy flux in
order to represent the collection of total forces applied to the collection of plane surface
elements. A Cauchy flux is defined as an additive, area bounded set function acting
on the collection of compatible surface elements of the body, and a weakly balanced
Cauchy flux is defined as a volume bounded Cauchy flux. It is shown that for each
plane surface element the density of the Cauchy flux exists almost everywhere with
respect to the Hausdorff area measure. The weak balance postulate is shown to be a
necessary condition for the linearity of the dependence of the density on the normal
as well as for the formulation of a classical balance law for the Cauchy flux.

It seems that \[BF79\] and \[Zie83\] were the first to propose that the class admissible
bodies in continuum physics should consist of sets of finite perimeter. In Ziemer’s
work, admissible bodies are defined as sets of finite perimeter and a weakly balanced
Cauchy flux is shown to be represented by a measurable vector field. The works
\[GWZ86, NV88\], which followed, further extended these studies. In \[GWZ86\], the
class of admissible bodies is defined as the class of normalized sets of finite perimeter
while in \[NV88\], admissible bodies are defined as fit regions which are bounded
regularly open sets of finite perimeter and of negligible boundary. These postulates
enabled the authors to apply a version of the Green-Gauss theorem and consider sets that do not necessarily have smooth boundaries as bodies in continuum mechanics for which balance laws may be written.

In [Sil85, Sil91], Silhavy considered bodies as sets of finite perimeter in a bounded open region of \( \mathbb{R}^n \). The author employs a weak approach in the formulation of Cauchy’s flux theorem. Silhavy’s approach gives rise to a Borel set \( N_0 \), of Lebesgue measure zero and a flux vector field \( q \), such that the action of the Cauchy flux is represented the by \( q \) for any surface whose intersections with \( N_0 \) has Hausdorff area measure zero. The analysis presented in Silhavy’s work allows for singularities in the flux vector field and presents for the first time the concept of almost every surface. In [Sil91], formal definitions of the concepts of almost every body and almost every surface are given and the choice of the class of admissible bodies is shown to be intimately related with the class of representing flux vector fields. The notions of almost every body and almost every surface are further examined in [DMM99] and it is shown that the Cauchy flux is determined by its action on a collection of rectangular planar surfaces with edges parallel to the axes of \( \mathbb{R}^n \). A similar extension of the Cauchy interaction is presented in [MM03]. In the above works it is shown that a weakly balanced Cauchy flux is represented by a divergence-measure vector field.

Vector fields of bounded variations are vector fields whose components are Radon measure and whose all partial derivatives, taken in the distributional sense, are Radon measures. Divergence-measure vector fields are viewed as a generalization of bounded variation vector fields such that one requires each component to be represented by a Radon measure and the divergence, taken in the distributional sense, is represented by a Radon measure (rather than all partial derivatives). The Gauss-Green theorem has been established for sets of finite perimeter and functions of bounded variations by Federer [Fed69, Section 4.5]. Similar results were obtained for divergence-measure vector field and sets of Lipschitz deformable boundaries in [CF99, CF01]. The generalization of these results for sets of finite perimeter were obtained in [CT05, CTZ09, Sil05], and a further generalization of the theories for sets of fractal boundaries is given in [Sil09]. In [Sil05], Cauchy’s flux theory is developed for sets of finite perimeter where it is shown that a real valued Cauchy flux is represented by a divergence measure vector field. The development is extended in [Sil06] where fluxes over parts fractal boundaries are investigated and the notion of normal trace for sets of fractal boundaries is introduced. Rough bodies, introduced by Silhavy [Sil06], are sets whose measure theoretic boundaries are fractals in the
sense that the outer normal is not defined almost everywhere with respect to the \((n-1)\)-Hausdorff measure.

In \cite{Seg86}, a weak formulation of \(p\)-grade continuum mechanics, for any integer \(p \geq 1\), is presented in the setting of differential manifolds. Configurations are viewed as \(C^p\)-embeddings of the body manifold in the physical space and forces are viewed as elements of the cotangent bundle to the infinite dimensional configuration manifold of mappings. Forces are shown to be represented by measures on the \(p\)-th jet bundle. Such a measure serves as a generalization of the \(p\)-th order stress. The representation of forces by stress measures enables a natural restriction of forces to subbodies. The consistency conditions for a such a system of \(p\)-th order forces are examined in \cite{Sd91}.

The term \textit{fractal} was coined in 1975 by Mandelbrot to indicate a highly irregular geometric object (see \cite{Man83}). Mandelbrot’s seminal work was the beginning of a very large body of research concerning the fractal properties of various physical phenomena. A variety of approaches have been suggested for the adaptation of fractal objects to branches of mechanics, e.g., \cite{Tar05a, Tar05b, Tar05d, Tar05c, WY08, WY09, ES06, OS09}.

In \cite{Rod02, RS03}, Cauchy’s flux theory is formulated using Whitney’s geometric integration theory \cite{Whi57} and new developments by Harrison \cite{Har93, Har98a, Har98b, Har99}. Bodies are viewed as \(r\)-dimensional domains of integration in an \(n\)-dimensional Euclidean space with \(r \leq n\). A body is identified as an \(r\)-chain, the limit of a sequence of polyhedral chains with respect to a norm which is induced by Cauchy’s postulates. Three types of chains are examined: flat, sharp and natural chains, such that

\[
\text{polyhedral} \subset \text{flat chains} \subset \text{sharp chains} \subset \text{natural chains}.
\]

Flat \((n-1)\)-chains may represent the fractal boundaries of bodies and sharp chains are shown to represent even less regular \((n-1)\)-dimensional objects. Fluxes of a given extensive property are postulated to be \((n-1)\)-cochains, i.e., elements of the dual to the Banach space of \((n-1)\)-chains. By the duality structure of Whitney’s theory, as one allows for less regular domains of integration (chains), the resulting fluxes (cochains) become more regular, automatically.

The present work, describes a framework where the mechanics of bodies with fractal boundaries may be studied. Unlike \cite{RS03}, in which Whitney’s geometric integration theory is applied, in this work the point of view of geometric measure theory as in \cite{Fed69} is mainly adopted. Geometric measure theory can be described
best as a generalization of differential geometry by means of measure theory with
the purpose of dealing with non-smooth maps and surfaces. Geometric measure
theory has a mechanical-like origin in Plateau’s problem which considers surfaces of
minimal area having given boundaries as models of soap bubbles. For the relation
between Geometric measure theory and the Plateau problem, see [Alm66]. Early
contributions to geometric measure theory may be attributed to such mathematicians
as L.C. Young and E. De Giorgi (see [Gio06]). The theory has taken a formal structure
in Federer & Fleming’s seminal paper [FF60]. As a reference on Geometric measure
theory one may use Federer’s [Fed69] or [KP08, GMS98, Mor08, Sim84, LY02].

The universal body is modeled as an open subset of $\mathbb{R}^n$ and bodies are modeled
as flat chains. In addition to the properties of the class of admissible bodies, special
attention is given to the study of the kinematics of such bodies in space. The appropri-
ate class of admissible configurations appears to be the set of Lipschitz embeddings.
This class enjoys two significant properties. Firstly, the set of Lipschitz embeddings
of the universal body into space is an open subset of the locally convex topological
vector space of all Lipschitz mappings of the universal body into space equipped
with the Whitney, or strong, topology. In addition, for Lipschitz mappings there is
a well defined pushforward action on flat chains, such as those representing bodies.
Therefore, the images of bodies under the pushforward action induced by a Lipschitz
embedding preserve their structure and relevant properties (e.g., the availability of a
generalized Stokes theorem).

Adopting the point of view that virtual velocities are elements of the tangent
bundle of the configuration manifold, as the configuration space is open in the space
of Lipschitz mappings, virtual velocities may be identified with Lipschitz mappings
of the universal body into space. Considering force and stress theory, it is noted
that forces which are required only to be continuous linear functionals relative to the
Lipschitz topology, as would be the analogue of [Seg86], seem to be too irregular for
the setting adopted here. In order to constitute a consistent force system which is
represented by an integrable stress fields, balance and weak balance are postulated. It
is shown further that balance and weak balance are equivalent together to continuity
relative to the flat norm of chains.

The thesis is constructed as follows. Chapters 2–5 contain a short outline of the
various notions of geometric measure theory which are used in this work. Chapter 2
reviews the notion of differential forms, currents, flat chains and cochains. Chapter 3
presents sets of finite perimeter as well as the corresponding definitions for bodies
and material surfaces as currents. In Chapter 4 we discuss some of the properties of locally Lipschitz maps. In particular, the image of a flat chain under a Lipschitz mapping is examined. In addition, Lipschitz embeddings and the properties of the set they constitute are considered. This enables the presentation of a Lipschitz type configuration space in Chapter 6. In Chapter 5 we discuss the product of locally Lipschitz maps and flat chains. This multiplication operation is used in the definition of a local virtual velocity. As an example for the use this proposed setting, Reynolds transport theorem is presented in Chapter 7. Our main theorem is presented in Chapter 8 where we prove that a system of forces obeying balance and weak balance is equivalent to a unique $n$-tuple of flat $(n - 1)$-cochains. Generalized bodies and surfaces are introduced in Chapters 9. Virtual strains, or velocity gradients, stresses and a generalized form of the principle of virtual work are presented in Chapters 10 and 11.
CHAPTER 2

Elements of Geometric Measure Theory

In this chapter, some of the fundamental concepts form the theory of currents in \( \mathbb{R}^n \) are presented. Throughout, the notation is mainly adopted from [Fed69] Chapter 4. The notion of flat forms needed for Wolfe’s representation theorem, originally presented in Whitney’s Geometric Integration Theory [Whi57] Chapter VII, is formulated in this section by the tools of Federer’s Geometric Measure Theory.

Let \( U \) be an open set in \( \mathbb{R}^n \) and \( V \) a vector space. The notation \( \mathcal{D}_m (U, V) \) is used for the vector space of smooth, compactly supported \( V \)-valued differential \( m \)-forms defined on \( U \) and \( \mathcal{D}_m (U) \) is used as an abbreviation for \( \mathcal{D}_m (U, \mathbb{R}) \). The notation \( d \phi \) is use for the exterior derivative of \( \phi \in \mathcal{D}_m (U) \), an element of \( \mathcal{D}_{m+1} (U) \). The vector space \( \mathcal{D}_m (U) \) will be endowed with a locally convex topology induced by a family of semi-norms [Fed69 p. 344] as in the theory of distributions.

A continuous linear functional \( T : \mathcal{D}_m (U) \to \mathbb{R} \) is referred to as an \( m \)-dimensional current in \( U \). The collection of all \( m \)-dimensional currents defined on \( U \) forms the vector space \( \mathcal{D}_m (U) \) which is the vector space dual to \( \mathcal{D}_m (U) \). Let \( T \in \mathcal{D}_m (U) \) with \( m \geq 1 \) then \( \partial T \), the boundary of \( T \) is the element of \( \mathcal{D}_{m-1} (U) \) defined by

\[
\partial T (\phi) = T (d \phi), \quad \text{for all } \phi \in \mathcal{D}_{m-1} (U).
\]

The support of a current \( T \in \mathcal{D}_m (U) \) is defined by

\[ \text{spt} (T) = U \setminus \{ W \mid W \text{-is open, } T(\phi) = 0 \text{ for all } \phi \in \mathcal{D}_m (U), \text{spt}(\phi) \subset W \}. \]

Generally speaking, the support of a current \( T \in \mathcal{D}_m (U) \) need not be compact. However, we note that all currents introduced in this work will be of compact support.

The exterior derivative \( d \) is a continuous linear map \( d : \mathcal{D}_m (U) \to \mathcal{D}_{m+1} (U) \). Thus, the boundary operation \( \partial : \mathcal{D}_{m+1} (U) \to \mathcal{D}_m (U) \), viewed as the adjoint operator to \( d \), is a continuous linear operator on currents.

As an example of a 0-current in \( U \), let \( L^n \), denote the \( n \)-dimensional Lebesgue measure in \( \mathbb{R}^n \). Then, the restricted measure \( L^n \upharpoonright U \) is the 0-current defined as

\[
L^n \upharpoonright U (\phi) = \int_U \phi dL^n, \quad \text{for all } \phi \in \mathcal{D}^0 (U).
\]
CHAPTER 2. ELEMENTS OF GEOMETRIC MEASURE THEORY

Given $\eta$, a Lebesgue integrable $m$-vector field defined on $U$, then, $L^n \wedge \eta$ denotes the $m$-current in $U$ defined by

\[(2.3) \quad L^n \wedge \eta(\phi) = \int_U \phi(\eta) dL^n, \quad \text{for all} \quad \phi \in \mathcal{D}^m(U).\]

The inner product in $\mathbb{R}^n$ induces an inner product in $\bigwedge^m \mathbb{R}^n$ and $|\xi|$ will denote the resulting norm of an $m$-vector $\xi$. If $\xi$ is an $m$-vector given by $\xi = \sum_i \xi^i E_i$ with $\{E_i\}$ the standard base for $\bigwedge^m \mathbb{R}^n$ such that $i = 1, \ldots, \binom{n}{m}$ it follows that

\[(2.4) \quad |\xi| = \sqrt{\langle \xi, \xi \rangle} = \sqrt{(\xi^i)^2}.\]

For an $m$-covector $\alpha$, define $\|\alpha\|$ by

\[(2.5) \quad \|\alpha\| = \sup \{\alpha(\xi) \mid |\xi| \leq 1, \ \xi \text{ is a simple } m \text{-vector}\}.\]

Dually, for an $m$-vector $\xi$, $\|\xi\|$ is defined by

\[(2.6) \quad \|\xi\| = \sup \left\{ \phi(\xi) \mid \phi \in \bigwedge^m \mathbb{R}^n, \|\phi\| \leq 1 \right\},\]

which results in

\[(2.7) \quad \|\xi\| = \inf \left\{ \sum_i \|\xi_i\| \mid \sum_i \xi_i = \xi, \ \xi_i \text{ - simple } m \text{-vector} \right\}.\]

Given $\phi \in \mathcal{D}^m(U)$, for every $x \in U$, $\phi(x)$ is an $m$-covector, and so

\[(2.8) \quad \|\phi(x)\| = \sup \{\phi(x)(\xi) \mid |\xi| \leq 1, \ \xi \text{ is a simple } m \text{-vector}\}.\]

The comass of $\phi$ is defined by

\[(2.9) \quad M(\phi) = \sup_{x \in U} \|\phi(x)\|.\]

For $T \in \mathcal{D}_m(U)$ the mass of $T$ is dually defined by

\[(2.10) \quad M(T) = \sup \{T(\phi) \mid \phi \in \mathcal{D}^m(U), \ M(\phi) \leq 1 \}.\]

An $m$-dimensional current $T$ is said to be represented by integration if there exists a Radon measure $\mu_T$ and an $m$-vector valued, $\mu_T$-measurable function, $\vec{T}$, with $|\vec{T}(x)| = 1$ for $\mu_T$-almost all $x \in U$, such that

\[(2.11) \quad T(\phi) = \int_U \phi(\vec{T}) d\mu_T, \quad \text{for all} \quad \phi \in \mathcal{D}^m(U).\]
A sufficient condition for an \( m \)-dimensional current, \( T \), to be represented by integration is that \( T \) is a current of finite mass, i.e., \( M(T) < \infty \). An \( m \)-current \( T \) is said to be \textit{locally normal} if both \( T \) and \( \partial T \) are represented by integration and is said to be a \textit{normal} current if it is locally normal and of compact support. The notion of normal currents leads to the definition

\[
N(T) = M(T) + M(\partial T),
\]

and clearly, every \( T \in \mathfrak{D}_m(U) \) such that \( N(T) < \infty \) is a normal current. The vector space of all \( m \)-dimensional normal currents in \( U \) is denoted by \( N_m(U) \). For a compact set \( K \) of \( U \), set

\[
N_{m,K}(U) = N_m(U) \cap \{ T \mid \text{spt}(T) \subset K \}.
\]

For each compact subset \( K \) of \( U \), define \( F_K \), the \( K \)-flat semi-norm on \( \mathfrak{D}_m(U) \), by

\[
F_K(\phi) = \sup_{x \in K} \left\{ ||\phi(x)||, ||d\phi(x)|| \right\}.
\]

Dually, the \( K \)-flat norm for currents \( T \in \mathfrak{D}_m(U) \) is given by

\[
F_K(T) = \sup \{ T(\phi) \mid F_K(\phi) \leq 1 \}.
\]

Note that if \( T \in \mathfrak{D}_m(U) \) such that \( F_K(T) < \infty \), then, \( \text{spt}(T) \subset K \). For a given compact subset \( K \subset U \), the set \( F_{m,K}(U) \) is defined as the \( F_K \) closure of \( N_{m,K}(U) \) in \( \mathfrak{D}_m(U) \). In addition, set

\[
F_m(U) = \bigcup_K F_{m,K}(U),
\]

where the union is taken over all compact subsets \( K \) of \( U \). An element in \( F_m(U) \) is referred to as a \textit{flat \( m \)-chain in \( U \)}.

For \( T \in \mathfrak{D}_m(U) \) with \( \text{spt}(T) \subset K \) it can be shown that \( F_K(T) \) is given by

\[
F_K(T) = \inf \{ M(T - \partial S) + M(S) \mid S \in \mathfrak{D}_{m+1}(U), \text{spt}(S) \subset K \}.
\]

By taking \( S = 0 \) we note that

\[
F_K(T) \leq M(T).
\]

In addition, any element \( T \in F_{m,K}(U) \) may be represented by \( T = R + \partial S \) where \( R \in \mathfrak{D}_m(U), S \in \mathfrak{D}_{m+1}(U) \), such that \( \text{spt}(R) \subset K, \text{spt}(S) \subset K \), and

\[
F_K(T) = M(R) + M(S).
\]
Flat chains have some desirable properties. We note that the boundary of a flat $m$-chain is a flat $(m-1)$-chain. Moreover, as Section 4 will show, the flat topology is preserved under Lipschitz maps. From a geometric point of view the notion of a flat chain may be used to describe objects of irregular geometric nature such as the Sierpinski triangle. The following representation theorem reveals the measure theoretic regularity characterization of flat $m$-chains.

**Theorem 1.** [Fed69, Section 4.1.18] Let $T$ be a flat $m$-chain in $U$ with $\text{spt}(T) \subset K$. Then, for any $\delta > 0$ and $E = \{ x \mid \text{dist}(K, x) \leq \delta \} \subset U$, the current $T$ may be represented by

\begin{equation}
T = L^n \wedge \eta + \partial (L^n \wedge \xi),
\end{equation}

such that $\eta$ is an $L^n \ll U$-summable, $m$-vector field, $\xi$ is a $L^n \ll U$-summable $(m+1)$-vector field and $\text{spt}(\eta) \cup \text{spt}(\xi) \subset E$.

A linear functional $X$ defined on $F_m(U)$ such that there exists $0 < c < \infty$ with $X(T) \leq cF_K(T)$ for any compact $K \subset U$ and $T \in F_{m,K}(U)$, is referred to as a flat $m$-cochain. The flat norm of a cochains is given by

\begin{equation}
F(X) = \sup \{ X(A) \mid A \in F_m(U), \ F_K(A) \leq 1, \ K \subset U \}.
\end{equation}

By Theorem 1, a dual representation for flat cochains is available by flat forms which we shall now introduce.

Given a differentiable mapping $u$ defined on an open set of $\mathbb{R}^n$, its derivative will be denoted by $Du$ and its partial derivative with respect to the $j$-th coordinate will be denoted by $D_j u$. For a smooth $m$-vector field $\eta$ in $U$, the divergence $\text{div}\eta$ of $\eta$ is an $(m-1)$-vector field in $U$ defined by

\begin{equation}
\text{div}\eta = \sum_{j=1}^{n} D_j \eta \wedge dx_j,
\end{equation}

where $dx_i, i = 1, \ldots, n$ denote the dual base vectors relative to the standard basis $e_j, j = 1, \ldots, n$ in $\mathbb{R}^n$ [Fed69, Section 4.1.6]. For an integrable $m$-form $\phi$ in $U$, the weak exterior derivative of $\phi$ is defined as an $(m+1)$ form in $U$ denoted by $\tilde{d}\phi$ and such that the equality

\begin{equation}
\int_U \tilde{d}\phi(\eta) dL^n = - \int_U \phi(\text{div}\eta) dL^n,
\end{equation}

holds for all compactly supported, smooth $(m+1)$-vector fields $\eta$ on $U$. The weak exterior derivative is simply the exterior derivative taken in the distributional sense.
CHAPTER 2. ELEMENTS OF GEOMETRIC MEASURE THEORY

Note that $\tilde{d}\phi$ is uniquely defined up to a set of $L^n \upharpoonright U$-measure zero, thus, for $\phi \in \mathcal{D}^m(U)$, the relation $\tilde{d}\phi = d\phi$ holds $L^n \upharpoonright U$-almost everywhere.

Differential forms whose components are Lipschitz continuous are referred to as \textit{sharp} $m$-\textit{forms} (adopting Whitney’s terminology \cite[Section V.10]{Whi57}). By Rademacher’s theorem, the exterior derivative for sharp $m$-forms exists $L^n \upharpoonright U$-almost everywhere and the existence of the weak exterior derivative follows. Sharp forms are clearly a generalization of the notion of a smooth differential form and a further generalization is given by flat forms where the Lipschitz continuity is relaxed.

**Definition 2.** An $m$-form $\phi$ in $U$ is said to be \textit{flat} if

\begin{equation}
F(\phi) = \sup_{\eta, \xi} \left\{ \int_U \left( \phi(\eta) + \tilde{d}\phi(\xi) \right) dL^n \right\} < \infty,
\end{equation}

where $\eta$ and $\xi$ are respectively $m$ and $(m + 1)$ compactly supported, $L^n \upharpoonright U$-summable vector fields such that

\begin{equation}
\int_U (\|\xi\| + \|\eta\|) dL^n = 1.
\end{equation}

It is further observed that for $\phi$, a flat $m$-form in $U$,

\begin{equation}
F(\phi) = \text{ess sup}_{x \in U} \left\{ \|\phi(x)\|, \|\tilde{d}\phi(x)\| \right\}.
\end{equation}

Alternative definitions for flat forms may be found in \cite[Section IX.7]{Whi57} and \cite[Section 5.5]{Hei05}.

**Remark 3.** For $\phi$, a flat $m$-form in $U$, and $\omega$, a flat $r$-form in $U$, $\phi \wedge \omega$ is a flat $(m + r)$-form in $U$ for which we now examine the weak exterior derivative $\tilde{d}(\phi \wedge \omega)$. Let $\eta$ be a compactly supported smooth $m$-vector field and $\xi$ a compactly supported...
smooth $r$-vector field then
\[
\int_U \tilde{d}(\phi \wedge \omega)(\eta \wedge \xi) dL^n = \int_U (\phi \wedge \omega) (\div (\eta \wedge \xi)) dL^n,
\]
\[
= \int_U (\phi \wedge \omega) ((\div \eta) \wedge \xi + (-1)^m \eta \wedge \div \xi) dL^n,
\]
(2.27)
\[
= \int_U (\phi(\div \eta) \omega(\xi) + (-1)^m \phi(\eta) \omega(\div \xi)) dL^n,
\]
\[
= \int_U (\tilde{d} \phi(\eta) \omega(\xi) + (-1)^m \phi(\eta) \tilde{d} \omega(\xi)) dL^n,
\]
\[
= \int_U (\tilde{d} \phi \wedge + (-1)^m \phi \wedge \tilde{d} \omega) (\eta \wedge \xi) dL^n.
\]
Thus,
(2.28)
\[
\tilde{d}(\phi \wedge \omega) = \tilde{d} \phi \wedge + (-1)^m \phi \wedge \tilde{d} \omega.
\]
which is a generalization of the well known analogous formula for the exterior derivative of the exterior product of smooth forms.

The representation theorem of flat cochains is traditionally referred to as Wolfe's representation theorem, [Whi57, Chapter IX], [Fed69, Section 4.1.19]. It states that any flat $m$-cochain $X$ in $U$ is represented by a flat $m$-form denoted by $D_X$ such that
(2.29)
\[
X (L^n \wedge \eta + \partial (L^n \wedge \xi)) = \int_U [D_X(\eta) + \tilde{d} D_X(\xi)] dL^n,
\]
for any $\eta$ and $\xi$, compactly supported, $L^n \subseteq U$-summable $m$ and $(m+1)$-vector fields, respectively. It is further noted that the flat norm $F(X)$ for the cochain $X$ is given by
(2.30)
\[
F(X) = \esssup_{x \in U} \left\{ \|D_X(x)\|, \|\tilde{d} D_X(x)\| \right\} \equiv F(D_X).
\]
The coboundary of a flat $m$-cochain $X$ is defined as the flat $(m+1)$-cochain $dX$ such that
(2.31)
\[
dX(\mathcal{A}) = X(\partial \mathcal{A}), \text{ for all } \mathcal{A} \in F_m(U),
\]
where it is noted that the same notation is used for the coboundary operator and the exterior derivative. The coboundary is the adjoint of the boundary operator and thus a continuous linear operator taking flat $m$-chains to flat $(m+1)$-chains. It follows from the representation theorem of flat chains that the flat $(m+1)$-cochain $dX$ is represented by the flat $(m+1)$-form $D_dX = \tilde{d} D_X$. The last equality is used as the definition of the exterior derivative of a flat form in [Whi57, Section IX.12].
Given a flat $m$-cochain $X$ in $U$ and a flat $r$-cochain $Y$ in $U$, then, $X \wedge Y$ is an $(m + r)$-cochain represented by the flat $(m + r)$-form $D_{X \wedge Y} = D_X \wedge D_Y$, and for a flat $(m + r)$-chain $T = L^n \wedge \eta + \partial (L^n \wedge \xi)$, $X \wedge Y(T)$ is defined by Equation (2.29). Moreover, Equation (2.28) implies that

$$d(X \wedge Y) = dX \wedge Y + (-1)^m X \wedge dY.$$  

For a flat $m$-cochain $X$ and a flat $r$-chain $T$, such that $m \leq r$, the interior product $X \lrcorner T$ is defined as a flat $(r - m)$-chain such that

$$X \lrcorner T(\omega) = (X \wedge \omega)(T), \quad \text{for all } \omega \in \mathcal{D}^{r-m}(U),$$

where $X \wedge \omega$ is the flat $r$-cochain represented by the flat $r$-form $D_X \wedge \omega$. 


Sets of finite perimeter, bodies and material surfaces

In this chapter we lay down the basic assumptions regarding the collection of admissible bodies. Sets of finite perimeter, or Caccioppoli sets, will play a central role in the proposed framework. We first recall some of the properties of sets of finite perimeter. Extended presentations of the subject may be found in [Fed69, Zie83, Zie89, EG92].

Let $U$ be a Borel set in an open subset of $\mathbb{R}^n$ and $B(x, r)$ be the ball centered at $x \in \mathbb{R}^n$ with radius $r$. Define the $U$-density of the point $x$ by

$$d(x, U) = \lim_{r \to 0} \frac{L^n(U \cap B(x, r))}{L^n(B(x, r))},$$

where the limit exists. For $\alpha \in [0, 1]$ set

$$U^\alpha = \{ x \in \mathbb{R}^n \mid d(x, U) = \alpha \}.$$

The measure theoretic boundary, $\Gamma(U)$, of the set $U$ is defined by

$$\Gamma(U) = \mathbb{R}^n - \left( U^1 \cup U^0 \right).$$

**Definition 4.** A Borel set $U$ in $\mathbb{R}^n$ is said to be a set of finite perimeter if $L^n(U) < \infty$ and $H^{n-1}(\Gamma(U)) < \infty$, where $H^{n-1}(\Gamma(U))$ is the $(n-1)$-Hausdorff measure of $\Gamma(U)$.

For $x \in \mathbb{R}^n$, let $\nu_U(x)$ denote a unit vector in $\mathbb{R}^n$ and define

$$B^+(x, r) = B(x, r) \cap \{ y \mid (y - x) \cdot \nu_U(x) \geq 0 \},$$

$$B^-(x, r) = B(x, r) \cap \{ y \mid (y - x) \cdot \nu_U(x) \leq 0 \}.$$

The vector $\nu(x, U)$ is said to be the measure theoretic exterior normal to $U$ at $x$ if

$$\lim_{r \to 0} \frac{L^n(B^+(x, r) \cap U)}{L^n(B^+(x, r))} = 0,$$

and

$$\lim_{r \to 0} \frac{L^n(B^-(x, r) \cap U)}{L^n(B^-(x, r))} = 1.$$
For a set of finite perimeter, the exterior normal $\nu_U(x)$ to $U$ exists $H^{n-1}$-almost everywhere in $\Gamma(U)$ thus making a generalized version of the Gauss-Green theorem applicable.

Several equivalent definitions for a set of finite perimeter may be found in the literature. In [Zie89, Section 5.4.1] a set of finite perimeter is viewed as a set $U$ whose characteristic function $\chi_U: \mathbb{R}^n \to \mathbb{R}$ defined by

$$\chi_U(x) = \begin{cases} 1, & x \in U, \\ 0, & x \notin U, \end{cases}$$

is a function of bounded variation in $\mathbb{R}^n$. Let $f$ be a real valued function defined on the open set $U$. The total variation of $f$ is defined by

$$\|\text{Var}(f)\| = \sup \left\{ \int_U f \cdot \text{div}(\phi) dL^n \mid \phi \in C_0^\infty(U, \mathbb{R}^n), \phi(x) \leq 1 \text{ for all } x \in U \right\},$$

where $C_0^\infty(U, \mathbb{R}^n)$ is used to denote the space of smooth, compactly supported $\mathbb{R}^n$-valued functions defined on $V$. A function $u \in L^1(V)$ is said to be a function of bounded variation in $V$ if each of partial derivatives $D_i u$ (taken in the distributional sense) are Radon measures with a finite total variation. Alternatively, a function $u \in L^1(V)$ is a function of bounded variation in $V$ if

$$\|U\|_{BV(U)} = \int_U |f| dL^n + \|\text{Var}(f)\| < \infty.$$ 

In this sense, the measure theoretic exterior normal is defined by

$$\nu_U(x) = \lim_{r \to \infty} -\frac{D\chi_U(B(x,r))}{\|D\chi_U\|(B(x,r))}.$$

In [Fed69, Section 4.5], a set of finite perimeter is viewed as a set $U$ such that the current $(L^n \cup U) \wedge e_1 \wedge \cdots \wedge e_n$ is an integral $n$-current in $\mathbb{R}^n$. In this work, Definition 4 is chosen for its intuitive geometric interpretation.

Let $B$ be an open set in $\mathbb{R}^n$. A body in $B$ is denoted by $\mathcal{P}$ and is postulated to be a set of finite perimeter in $B$. Strictly speaking, a set of finite perimeter is determined up to a set of $L^n$ measure zero, thus as a point set, it is not uniquely defined. Formally, each set of finite perimeter determines an equivalence class of sets. A unique representation of a body is given by the identification of the body $\mathcal{P}$ with $T_\mathcal{P}$, an $n$-current in $B$ defined as $T_\mathcal{P} = (L^n \cup \mathcal{P}) \wedge e_1 \wedge \cdots \wedge e_n$. By Equation (2.3),

$$T_\mathcal{P}(\omega) = \int_{\mathcal{P}} \omega(x) (e_1 \wedge \cdots \wedge e_n) dL^n_x, \quad \text{for all } \omega \in \mathcal{D}^n(B).$$
Using the terminology of currents represented by integration, \( \mu_{T_P} = L^n \ll P \) and \( \tilde{T}_P = e_1 \wedge \cdots \wedge e_n \) are the Radon measure and unit \( n \)-vector associated with the current \( T_P \).

Objects of dimension \((n - 1)\) for which one can compute the flux will be referred to as material surfaces. Formally, a material surface is defined as a pair \( S = (\hat{S}, v) \) where \( \hat{S} \) is a Borel subset of \( B \) such that for some body \( P \) we have \( \hat{S} \subset \Gamma(P) \) and \( v \) is the exterior normal of \( P \) such that \( v(x) = v_P(x) \) is defined \( H^{n-1} \)-almost everywhere on \( \hat{S} \). Let \( v^*(x) \) be a the covector defined by

\[
v^*(x)(u) = v(x) \cdot u, \quad \text{for all } u \in \mathbb{R}^n,
\]

and set \( \tilde{T}_S \) as the \((n - 1)\)-vector

\[
\tilde{T}_S(x) = v^*(x) \cup e_1 \wedge \cdots \wedge e_n.
\]

It is easy to show that \( \tilde{T}_S(x) \) is a unit, simple \((n - 1)\)-vector \( H^{n-1} \)-almost everywhere on \( \hat{S} \). We use \( T_S \) to denote the \((n - 1)\)-current in \( B \) induced by the material surface \( S \), such that \( \mu_{T_S} = H^{n-1} \ll \hat{S} \) and \( \tilde{T}_S(x) \) are the Radon measure and \((n - 1)\)-vector associated with \( T_S \), and

\[
T_S(\omega) = \int_{\hat{S}} \omega(x)(\tilde{T}_S(x))dH^{n-1}_x, \quad \text{for all } \omega \in D^{n-1}(B).
\]

The unit \((n - 1)\)-vector \( \tilde{T}_S(x) \) is viewed as the natural \((n - 1)\)-vector tangent to the material surface \( S \). By Equation (2.3) we may write

\[
T_S = \left( H^{n-1} \ll \hat{S} \right) \wedge \tilde{T}_S.
\]

Consider the material surface \( \partial P = (\Gamma(P), v_P) \) naturally induced by the body \( P \). One has,

\[
T_{\partial P}(\omega) = \int_{\Gamma(P)} \omega(x)(\tilde{T}_{\partial P}(x))dH^{n-1}_x,
\]

\[
= \int_{\Gamma(P)} (\omega(x) \cup e_1 \wedge \cdots \wedge e_n) \cdot v_P(x)dH^{n-1}_x,
\]

\[
= \int_P d\omega(x) (e_1 \wedge \cdots \wedge e_n) dL^n_x,
\]

\[
= T_P(d\omega),
\]

\[
= \partial T_P(\omega).
\]
where in the third line above Gauss-Green theorem [Fed69, Section 4.5.6] was used. Thus, it is noted that \( T_{\partial \mathcal{P}} = \partial T_P \) as expected, and the material surface \( S \) associated with the body \( \mathcal{P} \) may be written as

\[
T_S = (\partial T_P) \downarrow \hat{S}.
\]

Since a Radon measure is a Borel regular measure, the current \( \partial T_P \downarrow \hat{S} \) is well defined for any Borel set \( \hat{S} \) [Fed69, p. 356].

For each \( T_P \), we observe that \( M(T_P) = L^n(\mathcal{P}) \) and \( M(\partial T_P) = H^{n-1}(\Gamma(\mathcal{P})) \) correspond to the “volume” of the body and “area” of its boundary, respectively. By Equation (2.12) one has \( N(T_P) = L^n(\mathcal{P}) + H^{n-1}(\Gamma(\mathcal{P})) < \infty \), so that the current \( T_P \) is a normal \( n \)-current in \( \mathcal{B} \), in particular \( T_P \) is an integral \( n \)-current. The open set \( \mathcal{B} \) is referred to as the universal body and we define the class of admissible bodies, \( \Omega_{\mathcal{B}} \), as the collection of all bodies in the universal body \( \mathcal{B} \), i.e.,

\[
\Omega_{\mathcal{B}} = \{ T_P \mid \mathcal{P} \subset \mathcal{B}, T_P = L^n(\mathcal{P}) \in N_n(\mathcal{B}) \}.
\]

The result obtained in [GWZ86] implies that in case \( \mathcal{B} \) is assumed to be a set of finite perimeter, \( \Omega_{\mathcal{B}} \) would have the structure of a Boolean algebra and would form a material universe in the sense of Noll [NoI73]. In Section 9 a generalized class of admissible bodies will be defined for which a requirement that \( \mathcal{B} \) is a bounded set will be sufficient in order to construct a Boolean algebra structure.

The collection of all material surfaces in \( \mathcal{B} \) will be denoted by \( \partial \Omega_{\mathcal{B}} \), so that

\[
\partial \Omega_{\mathcal{B}} = \{ T_S \mid T_S = (\partial T_P) \downarrow \hat{S}, T_P \in \Omega_{\mathcal{B}} \}.
\]

By the definition of \( T_S \) it follows that \( M(T_S) = H^{n-1}(\hat{S}) \) for each \( T_S \in \partial \Omega_{\mathcal{B}} \). Thus \( T_S \) is a flat \( (n-1) \)-chain of finite mass. The material surfaces \( T_S \) and \( T_{S'} \) are said to be compatible if there exists a body \( T_P \) such that \( T_S = (\partial T_P) \downarrow \hat{S} \) and \( T_{S'} = (\partial T_P) \downarrow \hat{S}' \). The material surfaces \( T_S \) and \( T_{S'} \) are said to be disjoint if \( \text{clo}(\hat{S}) \cap \text{clo}(\hat{S}') = \varnothing \).
Lipschitz mappings and Lipschitz chains

Lipschitz mappings will model configurations of bodies in space. In this chapter we review briefly some of their relevant properties.

A map \( F : U \to V \) from an open set \( U \subset \mathbb{R}^n \) to an open set \( V \subset \mathbb{R}^m \), is said to be a (globally) Lipschitz map if there exists a number \( c < \infty \) such that \( |F(x) - F(y)| \leq c|x - y| \) for all \( x, y \in U \). The Lipschitz constant of \( F \) is defined by

\[
L_F = \sup_{x,y \in U} \frac{|F(y) - F(x)|}{|y - x|}.
\]

The map \( F : U \to V \) is said to be locally Lipschitz if for every \( x \in U \) there is some neighborhood \( U_x \subset U \) of \( x \) such that the restricted map \( F|_{U_x} \) is a Lipschitz map.

Let \( F : U \to \mathbb{R}^m \) be a locally Lipschitz map defined on the open set \( U \subset \mathbb{R}^n \), then for every \( K \), a compact subset of \( U \), the restricted map \( F|_K \) is globally Lipschitz in the sense that \( L_{F,K} \), the \( K \)-Lipschitz constant of the map \( F|_K \), given by

\[
L_{F,K} = \sup_{x,y \in K} \frac{|F(x) - F(y)|}{|x - y|},
\]

is finite.

4.1. Differential topology of Lipschitz maps

The vector space of locally Lipschitz mappings from the open set \( U \subset \mathbb{R}^n \) to the open set \( V \subset \mathbb{R}^m \) is denoted by \( \mathcal{L}(U,V) \). For a compact subset \( K \subset U \), define the semi-norm

\[
\|F\|_{\mathcal{L},K} = \max \{ \|F|_K\|_{\infty}, L_{F,K} \},
\]

on \( \mathcal{L}(U,V) \), where,

\[
\|F|_K\|_{\infty} = \sup_{x \in K} |F(x)|.
\]
The vector space $\mathcal{L}(U, V)$ is endowed with the strong Lipschitz topology (see [FN05]). It is the analogue of Whitney’s topology (strong topology) for the space of differentiable mappings between open sets (see [Hir76, p. 35]) and is defined as follows.

**Definition 5.** Given $\mathcal{F} \in \mathcal{L}(U, V)$, for some indexing set $\Lambda$, let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be an open, locally finite cover of $U \subset \mathbb{R}^n$, and $\mathcal{K} = \{K_\lambda\}_{\lambda \in \Lambda}$ a family of compact subsets in $U$ such that $K_\lambda \subset U_\lambda$ and $\delta = \{\delta_\lambda\}_{\lambda \in \Lambda}$ a family of positive numbers. A neighborhood $B^C(\mathcal{F}, \mathcal{U}, \delta, \mathcal{K})$ of $\mathcal{F}$ in the strong topology is defined as the collection of all $g \in \mathcal{L}(U, V)$ such that $\|\mathcal{F} - g\|_{\mathcal{L}, K_\lambda} < \delta_\lambda$, i.e.,

$$B^C(\mathcal{F}, \mathcal{U}, \delta, \mathcal{K}) = \left\{ g \in \mathcal{L}(U, V) \mid \|\mathcal{F} - g\|_{\mathcal{L}, K_\lambda} < \delta_\lambda, \lambda \in \Lambda \right\}. $$

A map $\varphi : U \to V$, with $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, open sets such that $m \geq n$, is said to be a bi-Lipschitz map if there are numbers $0 < c \leq d < \infty$, such that [Hei00, p. 78]

$$c \leq \left| \frac{\varphi(x) - \varphi(y)}{x - y} \right| \leq d, \quad \text{for all } x, y \in U, \ x \neq y. $$

Setting $L = \max \left\{ \frac{1}{c}, d \right\}$,

$$\frac{1}{L} \leq \left| \frac{\varphi(x) - \varphi(y)}{x - y} \right| \leq L, \quad \text{for all } x, y \in U, \ x \neq y,$$

and in such a case $\varphi$ is said to be $L$-bi-Lipschitz.

The map $\mathcal{F} : U \to V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets such that $m \geq n$, is a Lipschitz immersion if for every $x \in U$ there is a neighborhood $U_x \subset U$ of $x$ such that $\mathcal{F} |_{U_x}$ is a bi-Lipschitz map, i.e., there are $0 < c_x \leq d_x < \infty$, and

$$c_x \leq \left| \frac{\varphi(y) - \varphi(z)}{y - z} \right| \leq d_x, \quad \text{for all } y, z \in U_x, \ y \neq z.$$

**Lemma 6.** The set of Lipschitz immersions is an open subset of $\mathcal{L}(U, V)$ with respect to the strong Lipschitz topology.

**Proof.** Let $g : U \to V$ be a Lipschitz immersion and for $x \in U$, let $U_x$ be a bounded open set containing $x$ such that $g|_{U_x}$ is a bi-Lipschitz map. The collection $\{U_x\}_{x \in U}$ forms an open cover of $U$. Since $\mathbb{R}^n$ is paracompact we may extract a locally finite refinement $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ which is an open subcover of $U$. For each $U_\lambda$ select an open set $V_\lambda$ such that $\text{clo}(V_\lambda) \subset U_\lambda$ and that $\{V_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $U$. Denote the sets $\text{clo}(V_\lambda)$ by $K_\lambda$ and note that $\mathcal{K} = \{K_\lambda\}_{\lambda \in \Lambda}$ is a locally finite cover of $U$ and each $K_\lambda$ is a compact set with non empty interior. For an extended proof of existence of $\mathcal{K}$ we refer to [Mun00, Section 41].
4.1. CHAPTER 4. LIPSCHITZ MAPS AND CHAINS

Let \( x \in K_{\lambda} \) for some compact set \( K_{\lambda} \), then \( g|_{K_{\lambda}} \) is a \( L_{\lambda} \)-bi-Lipschitz map for some \( 0 < L_{\lambda} < \infty \) and let \( F \in \mathcal{L} (U, V) \) then since \( F|_{K_{\lambda}} \) is a Lipschitz map it will suffice to show that \( F|_{K_{\lambda}} \) is an injective map. For every \( z, y \in K_{\lambda} \),

\[
0 < \frac{|g(z) - g(y)|}{|z - y|} \leq \frac{|(g - F)(z) - (g - F)(y)|}{|z - y|} + \frac{|F(z) - F(y)|}{|z - y|},
\]

hence

\[
\frac{|F(z) - F(y)|}{|z - y|} \geq \frac{|g(z) - g(y)|}{|z - y|} - \frac{|(g - F)(z) - (g - F)(y)|}{|z - y|},
\]

\[
\geq \frac{|g(z) - g(y)|}{|z - y|} - \|g - F\|_{L, K_{\lambda}}.
\]

Taking the infimum over \( z, y \in K_{\lambda} \) on both sides it follows that

\[
\inf_{z, y \in K_{\lambda}} \frac{|F(z) - F(y)|}{|z - y|} \geq \inf_{z, y \in K_{\lambda}} \frac{|g(z) - g(y)|}{|z - y|} - \|g - F\|_{L, K_{\lambda}}.
\]

Setting \( \delta_{\lambda} = \frac{1}{2} \inf_{z, y \in K_{\lambda}} \frac{|g(z) - g(y)|}{|z - y|} = \frac{1}{2L_{\lambda}} \) it follows that

\[
\inf_{z, y \in K_{\lambda}} \frac{|F(z) - F(y)|}{|z - y|} \geq \frac{1}{2L_{\lambda}},
\]

hence \( F|_{K_{\lambda}} \) is an injective map. Since every \( x \in U \) is contained in some \( K_{\lambda} \) it follows that \( F \) is a Lipschitz immersion. \( \Box \)

The following theorem pertaining to the set of Lipschitz embeddings is given in [FN05] for the setting of Lipschitz manifolds and its proof is analogous to the case of differentiable mappings as in [Hir76] p. 36–38.

**Definition 7.** A Lipschitz map \( \varphi : U \to V \) is said to be a Lipschitz embedding if it is a Lipschitz immersion and a homeomorphism of \( U \) onto \( \varphi(U) \).

**Theorem 8.** The set \( \mathcal{L}_{Em}(U, V) \) is open in \( \mathcal{L}(U, V) \) with respect to the strong Lipschitz topology

**Proof.** Let \( \varphi \in \mathcal{L}_{Em}(U, V) \). Apply first the proof of Lemma 6 and obtain an open set \( B^\varphi_{0}(\varphi, U, \delta, K) \) such that every element in \( B^\varphi_{0}(\varphi, U, \delta, K) \) is a Lipschitz immersion. With \( U = \{U_{\lambda}\}_{\lambda \in \Lambda} \) and \( K = \{K_{\lambda}\}_{\lambda \in \Lambda} \) selected as in the proof of Lemma 3 recall that \( \varphi|_{U_{\lambda}} \) is an \( L_{\lambda} \)-bi-Lipschitz map. Set

\[
h_{\lambda} = \text{dist} (K_{\lambda}, U - U_{\lambda}) = \inf \{|y - z| \mid y \in K_{\lambda}, z \in U - U_{\lambda}\},
\]

and if \( \lambda \in \Lambda \) then \( \varphi|_{U_{\lambda}} \) is a bi-Lipschitz map with constants \( 0 < L_{\lambda} < \infty \) and \( \varphi|_{U_{\lambda}} \) is an injective map. Since every \( x \in U \) is contained in some \( K_{\lambda} \) it follows that \( \varphi \) is a Lipschitz immersion. \( \Box \)

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and $\delta'_\Lambda = \frac{h_\Lambda}{I'\Lambda}$. Note that for $\delta' = \{\delta'_\Lambda\}_{\Lambda \in \Lambda}$, the open set $B^0_1(\delta, U, \delta', K)$ contains the collection of elements $g \in \mathcal{L}(U, V)$ such that for any $U_\Lambda \subset U$ and $K_\Lambda \subset K$, $g(K_\Lambda) \subset \varphi(U_\Lambda)$. Since $\varphi$ is an embedding for any $\Lambda \in \Lambda$ we may find disjoint open sets $A_\Lambda, B_\Lambda$ in $V$ such that

$$\varphi(\mathcal{V}_\Lambda) \subset A_\Lambda, \quad \varphi(U - K_\Lambda) \subset B_\Lambda,$$

where $\mathcal{V}_\Lambda$ is the open set satisfying $\text{clo}(\mathcal{V}_\Lambda) = K_\Lambda$. Letting $f \in B^0_1(\delta, U, \delta', K) \cap B^0_1(\varphi, U, \delta', K)$, we now show that $f$ is injective. Let $x \in \mathcal{V}_\Lambda$ and $y \in U$. If $y \in K_\Lambda$, then $f(x) \neq f(y)$ since $f|_{K_\Lambda}$ is a Lipschitz immersion. In case $y \in U - K_\Lambda$, then $f(y) \in B_\Lambda$ while $f(x) \in A_\Lambda$, thus $f(x) \neq f(y)$ and $f$ is injective.

\[\square\]

### 4.2. Maps of currents induced by Lipschitz maps

Since our objective is to represent bodies as currents, and in particular, as flat chains, and since we wish to represent configurations as Lipschitz mappings, we exhibit in the following the basic properties of the images of currents and chains under Lipschitz mappings.

Let $T$ be a current on $U$ and for open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, let $\mathcal{F} : U \rightarrow V$ be a smooth map whose restriction to $\text{spt}(T)$ is a proper map. For any $r$-form $\omega$ on $V$, the map $\mathcal{F}$ induces a form $\mathcal{F}^\#(\omega)$, the pullback of $\omega$ by $\mathcal{F}$, defined pointwise by

\[
(\mathcal{F}^\#(\omega)(x))(v_1 \wedge \cdots \wedge v_r) = (\omega(\mathcal{F}(x)))(D\mathcal{F}(v_1) \wedge \cdots \wedge D\mathcal{F}(v_r)),
\]

for all $v_1, \ldots, v_r \in \mathbb{R}^m$. It is observed that since $\mathcal{F}$ is proper only on $\text{spt}(T)$, for a form $\omega$ with a compact support, $\text{spt}(\mathcal{F}^\#(\omega))$ need not be compact. However, for a real valued function $\xi$ defined on $U$ which is compactly supported and $\xi(x) = 1$ for all $x$ in a neighborhood of $\text{spt}(T) \cap \text{spt}(\mathcal{F}^\#(\omega))$, the smooth form $\xi\mathcal{F}^\#(\omega)$ is of compact support. Thus, the pushforward $\mathcal{F}^\#(T)$ of $T$ by $\mathcal{F}$ may be defined as the current in $V$ given by

\[
\mathcal{F}^\#(T)(\omega) = T(\xi\mathcal{F}^\#(\omega)), \quad \text{for all } \omega \in \mathcal{D}^r(V),
\]

for any $\xi$ with the properties given above [GMS98, Section 2.3]. The definition of $\mathcal{F}^\#(T)(\omega)$ is independent of $\xi$ and thus will be omitted in the following. The pushforward operation satisfies

\[
\partial \mathcal{F}^\#(T) = \mathcal{F}^\#(\partial T),
\]

\[
\text{spt } (\mathcal{F}^\#T) \subset \mathcal{F}\{\text{spt } (T)\}.
\]
By a direct calculation one obtains that

(4.13) \[ M(\mathcal F\#(T)) \leq \left( \sup_{x \in K} |D\mathcal F(x)| \right)^r M(T). \]

Applying Equation (2.12) it follows that

(4.14) \[ N(\mathcal F\#(T)) \leq N(T) \sup \left\{ \left( \sup_{x \in K} |D\mathcal F(x)| \right)^r, \left( \sup_{x \in K} |D\mathcal F(x)| \right)^{r-1} \right\}, \]

and by Equation (2.19),

(4.15) \[ F_{\mathcal F\{K\}}(\mathcal F\#(T)) \leq F_K(T) \sup \left\{ \left( \sup_{x \in K} |D\mathcal F(x)| \right)^r, \left( \sup_{x \in K} |D\mathcal F(x)| \right)^{r+1} \right\}, \]

where \( \mathcal F\{K\} \) is the image of the set \( K \) under the map \( \mathcal F \).

In case \( \mathcal F: U \to V \) is a locally Lipschitz map, the map \( \mathcal F\# \) cannot be defined as in the case of smooth maps. However, given any compact \( K \subseteq U \), for \( T \in F_{r,K}(U) \), one may define the current \( \mathcal F\#(T) \) as a weak limit.

Let \( \{\mathcal F_\tau\}, \tau \in \mathbb R^+ \), be a family of smooth approximations of \( \mathcal F \) obtained by mollifiers [Fed69, Section 4.1.2]. (It is observed that flat chains have compact supports so that it is not necessary to require that \( \mathcal F \) is proper.) Set

\[ \mathcal F\#T(\omega) = \lim_{\tau \to 0} \mathcal F_{\tau\#}T(\omega), \text{ for all } \omega \in \mathcal D'(V). \]

The sequence \( \{\mathcal F_{\tau\#}(T)\} \) is a Cauchy sequence with respect to the flat norm so that the limit is well defined and one may write

(4.16) \[ \mathcal F\#(T) = \lim_{\tau \to 0} \mathcal F_{\tau\#}(T). \]

As a result, the locally Lipschitz map \( \mathcal F: U \to V \) induces a map of flat chains

\[ \mathcal F\#: F_r(U) \to F_r(V). \]

Properties (4.11) and (4.12) hold for the map \( \mathcal F\# \) induced by a locally Lipschitz map \( \mathcal F \) and

(4.17) \[ M(\mathcal F\#(T)) \leq M(T) \left( \mathcal L_{\mathcal F,\text{spt}(T)} \right)^r. \]
It follows that for normal currents
\[ F_#(T) \in N_{r,F(K)}(V), \quad \text{for all} \quad T \in N_{r,K}(U), \]
\[ N(F_#(T)) \leq N(T) \sup \left\{ \left( L_{F,spt(T)} \right)^r, \left( L_{F,spt(T)} \right)^{r-1} \right\}, \]
and for flat chains
\[ F_#(T) \in F_{r,F(K)}(V), \quad \text{for all} \quad T \in F_{r,K}(U), \]
\[ F_{F(K)}(F_#(T)) \leq F_{K}(T) \sup \left\{ \left( L_{F,spt(T)} \right)^r, \left( L_{F,spt(T)} \right)^{r+1} \right\}. \]

See [Fed69, Section 4.1.14] and [GMS98, Section 2.3] for an extended treatment.

In Whitney’s theory, the Lipschitz image of a flat chain \( A \) is defined as follows [Whi57, Chapter X]. First, for \( P = spt(A) \) consider a full sequence of simplicial subdivision \( \{P_i\} \) such that \( P_{i+1} \) is a simplicial refinement of \( P_i \). Next, let \( \{F_i\} \) be a sequence of piecewise affine approximations of the Lipschitz map \( F \) such that \( F_i(v) = F(v) \) for all vertices \( v \) in the simplicial complex \( P_i \). The chain \( F_#(A) \) is defined as the limit in the flat norm of
\[ F_#(A) = \lim_{i \to \infty} F_i(A). \]

Although Whitney’s definition of \( F_#(A) \) differs from that of Federer, the resulting chains are equivalent.

For a locally Lipschitz map \( F : U \to V \) from an open set \( U \subset \mathbb{R}^n \) to an open set \( V \subset \mathbb{R}^m \), we consider locally Lipschitz maps, a generalization of the theorem, Stepanov’s theorem [Hei00, Theorem 3.4], may be used to prove the \( L^n \)-almost existence of \( D_F \). This does not limit the validity of Equation (4.9), as a flat form is defined only \( L^n \)-almost everywhere.

Consider a locally Lipschitz map \( F : U \to V \) from an open set \( U \subset \mathbb{R}^n \) to an open set \( V \subset \mathbb{R}^m \). For a flat \( n \)-cochain \( X \) in \( V \) and a current \( T_B \) induced by an
4.2. CHAPTER 4. LIPSCHITZ MAPS AND CHAINS

$L^n$-summable set $B$ in $U$, one has

$$\mathcal{F}^\#(X)_B = \int_B \mathcal{F}^\# D_X dL^n,$$

$$\mathcal{F}^\#(X)_B = \int_B D_X(\mathcal{F}(x))(D\mathcal{F}(x)(e_1) \wedge \cdots \wedge D\mathcal{F}(x)(e_n)) dL^n_x,$$

(4.22)

$$\mathcal{F}^\#(X)_B = \int_B D_X(\mathcal{F}(x))(e_1 \wedge \cdots \wedge e_n) J_F(x) dL^n_x,$$

$$\mathcal{F}^\#(X)_B = \int_{\mathcal{F}(B)} \sum_{x \in F^{-1}(y)} D_X(y) dH^n_y.$$

In the last equation the area formula for Lipschitz maps [GMS98, Section 2.1.2] was applied and $J_F(x)$ is the Jacobian determinant of $\mathcal{F}$ at $x$. In case $\mathcal{F} : U \to V$ is injective with $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, we have

(4.23) $$\mathcal{F}^\#(X)_B = X(F^\#T_B) = \int_{\mathcal{F}(B)} D_X(y) dL^n_y = X\left(T_{\mathcal{F}(B)}\right),$$

thus, $F^\#T_B = T_{\mathcal{F}(B)}$. In particular, for a body $\mathcal{P}$, and an injective Lipschitz map $\mathcal{F}$ we note that

(4.24) $$\mathcal{F}^\#T_P = T_{\mathcal{F}(P)}.$$

For the material surface $T_{\partial \mathcal{P}}$, Equation (4.11) gives

$$\mathcal{F}^\#(T_{\partial \mathcal{P}}) = \mathcal{F}^\#(\partial T_P) = \partial \mathcal{F}^\#(T_P) = \partial T_{\mathcal{F}(\mathcal{P})},$$

and for a material surface $T_S$ Equation (3.17) implies that

(4.25) $$\mathcal{F}^\#(T_S) = T_{\mathcal{F}(S)}.$$
The representation of fields over bodies

A real valued field over a body $\mathcal{P}$ will be represented below by the product of the current $T_P$ and a *sharp function*—a real valued locally Lipschitz mapping. (The terminology is due to Whitney [Whi57, Section V.4].) The space of sharp functions will be denoted by $\mathcal{L}_s(U)$.

A sharp function $\phi \in \mathcal{L}_s(U)$ defines a flat 0-cochain $\alpha_\phi$ on $U$ as follows. Let $\xi$ be an $L^n \perp U$-measurable function compactly supported in $U$. Then, $L^n \wedge \xi$ is a 0-current of finite mass in $U$ as defined in Equation (2.2). We set

$$\alpha_\phi(L^n \wedge \xi) = \int_U \phi(x) (\xi(x)) \, dL^n_x.$$  

(5.1)

For a compactly supported $L^n \perp U$ measurable 1-vector field $\eta$, $L^n \wedge \eta$ is a 1-current of finite mass in $U$ defined in Equation (2.3). Using the existence of the weak exterior derivative $\tilde{d}\phi$, $L^n \perp U$-almost everywhere, we set

$$\alpha_\phi(\partial (L^n \wedge \eta)) = \int_U \tilde{d}\phi (\eta(x)) \, dL^n_x,$$

(5.2)

and obtain expressions analogous to Wolfe’s representation theorem (Equation (2.29)).

Let $A \in F_0(U)$ be a flat 0-chain in $U$. Applying Theorem 1, $A$ may be expressed as $A = L^n \wedge \xi + \partial (L^n \wedge \eta)$ with $\xi$ and $\eta$ as defined above. Set

$$\alpha_\phi(A) = \alpha_\phi(L^n \wedge \xi + \partial (L^n \wedge \eta)),$$

(5.3)

so that $\alpha_\phi$ defines a continuous, linear function of flat 0-chains. Applying Equation (2.30) we obtain

$$F(\alpha_\phi) = \sup_{x \in U} \left\{ |\phi(x)|, |\tilde{d}\phi(x)| \right\}.$$  

(5.4)

For $A \in F_r(U)$ and $\phi \in \mathcal{L}_s(U)$, define the multiplication $\phi A$ by $\phi A = \alpha_\phi \cup A$ using the interior product as defined in Equation (2.33). That is,

$$\phi A(\omega) = (\alpha_\phi \cup A)(\omega) = (\alpha_\phi \wedge \omega)(A), \quad \text{for all } \omega \in \mathcal{D}^r(U),$$

(5.5)
where $\alpha_\phi \wedge \omega$ is the flat $r$-cochain represented by the flat $r$-form $\phi \wedge \omega$. Note that by Equation (5.5)

$$spt (\phi A) \subset spt (\phi) \cap spt (A).$$

For the boundary of $\phi A$ we first note that

$$\partial (\phi A) (\omega) = \phi A (d \omega) = (\alpha_\phi \wedge d \omega) A, \text{ for all } \omega \in \mathcal{D}^{r-1}(U).$$

By Equation (2.32)

$$d (\alpha_\phi \wedge \omega) = (d \alpha_\phi) \wedge \omega + \alpha_\phi \wedge d \omega,$$

so that

$$\partial (\phi A) (\omega) = (d (\alpha_\phi \wedge \omega) - (d \alpha_\phi) \wedge \omega) A,$$

$$= (\phi \partial A - d \alpha_\phi \lrcorner A) (\omega).$$

Hence we can write

$$\partial (\phi A) = \phi \partial A - d \alpha_\phi \lrcorner A.$$

**Remark 9.** The multiplication of sharp functions and chains was originally defined in [Whi57, Section VII.1] using the notion of continuous chains which are $r$-vector field approximations of $r$-chains.

**Proposition 10.** Given a sharp function $\phi$, for $A \in N_{r,K}(U)$

$$N_{r,K} (\phi A) \leq \left( \sup_{x \in K} |\phi(x)| + r \mathcal{L}_{\phi,K} \right) N_{r,K} (A),$$

and for $A \in F_{r,K}(U)$ with $r < n$ (see [Whi57, p. 208])

$$F_{r,K} (\phi A) \leq \left( \sup_{x \in K} |\phi(x)| + (r + 1) \mathcal{L}_{\phi,K} \right) F_{r,K} (A),$$

and for $r = n$

$$F_{r,K} (\phi A) \leq \left( \sup_{x \in K} |\phi(x)| \right) F_{r,K} (A).$$

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PROOF. For $A \in N_{r,K}(U)$ we have

$$M(\phi A) = \sup_{\omega \in \mathcal{D}'(U)} \frac{\left| \phi A(\omega) \right|}{M(\omega)},$$

$$= \sup_{\omega \in \mathcal{D}'(U)} \frac{\left| (\alpha_\phi \wedge \omega)(A) \right|}{M(\omega)},$$

(5.14)

$$= \sup_{\omega \in \mathcal{D}'(U)} \frac{\left| \int_U (\phi(x)\omega(x)) \left( \tilde{T}_A(x) \right) d\mu_A \right|}{M(\omega)},$$

$$\leq \sup_{\omega \in \mathcal{D}'(U)} \sup_{x \in K} \left\| (\phi(x)\omega(x)) \right\| M(A),$$

$$\leq \sup_{x \in K} \left| \phi(x) \right| M(A),$$

where in the third line we used the representation by integration of $A$ and in the fourth line the term $\sup_{x \in K} \left| \phi(x) \right|$ was extracted since $\text{spt}(A) \subset K$.

In order to examine the term $M(\partial(\phi A))$, we first apply Equation (5.10)

(5.15)

$$M(\partial(\phi A)) \leq M(\phi A) + M(d\alpha_\phi \cup A).$$

For the first term on the right-hand side we have,

(5.16)

$$M(\phi A) = \sup_{\omega \in \mathcal{D}'^{-1}(U)} \frac{\left| \alpha_\phi \wedge \omega(\partial A) \right|}{M(\omega)} \leq \left( \sup_{x \in K} \left| \phi(x) \right| \right) M(A).$$

For the second term,

$$M(d\alpha_\phi \cup A) = \sup_{\omega \in \mathcal{D}'^{-1}(U)} \frac{\left| d\alpha_\phi \wedge \omega(\tilde{T}_A) \right| d\mu_A}{M(\omega)},$$

$$\leq \sup_{\omega \in \mathcal{D}'^{-1}(U)} \sup_{x \in K} \left\| d\phi(x) \wedge \omega(x) \right\| M(A),$$

(5.17)

$$\leq \sup_{\omega \in \mathcal{D}'^{-1}(U)} \left( \frac{r}{1} \right) \sup_{x \in K} \left| \tilde{d}\phi(x) \right| \frac{M(\omega) M(A)}{M(\omega)},$$

$$= r \left( \sup_{x \in K} \left| \tilde{d}\phi(x) \right| \right) M(A),$$

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where in the third line we used the fact that for an $l$-form $\omega$ and a $k$-form $\omega'$

\begin{equation}
M (\omega \wedge \omega') \leq \left( \frac{l+k}{k} \right) M(\omega) M(\omega'),
\end{equation}

as is shown in [FF60].

One concludes that

\begin{equation}
N(\phi A) = M(\phi A) + M(\partial (\phi A)),
\end{equation}

\begin{align*}
&\leq \sup_{x \in K} |\phi(x)| M(A) + \sup_{x \in K} |\phi(x)| M(\partial A) + r \mathcal{L}_{\phi,K} M(A), \\
&\leq \left( \sup_{x \in K} |\phi(x)| + r \mathcal{L}_{\phi,K} \right) N(A).
\end{align*}

For a flat $r$-chain $A \in F_{r,K}(U)$ we use the representation given in Equation (2.19) by $A = R + \partial S$ so that $F_K(A) = M(R) + M(S)$. We first observe that

\begin{equation}
M (d \alpha_{\phi} \bowtie S) = \sup_{\omega \in \mathcal{D}^r(U)} \frac{d \alpha_{\phi} \bowtie S(\omega)}{M(\omega)},
\end{equation}

\begin{align*}
&= \sup_{\omega \in \mathcal{D}^r(U), \text{spt}(\omega) \subset K} \frac{(d \alpha_{\phi} \wedge \omega)(S)}{M(\omega)}, \\
&\leq \frac{M(S) M(d \alpha_{\phi} \wedge \omega)}{M(\omega)}, \\
&\leq \frac{M(S)}{M(\omega)} \left( r + 1 \right) M(\omega) \sup_{x \in K} |\tilde{d}\phi(x)|,
\end{align*}

and conclude that

\begin{equation}
M (d \alpha_{\phi} \bowtie S) \leq (r + 1) \mathcal{L}_{\phi,K} M(S).
\end{equation}

Estimating $F_K(\phi A)$, one has
\[
F_K(\phi A) = F_K(\phi R + \phi \partial S),
\]
\[
\leq F_K(\phi R) + F_K(\phi \partial S),
\]
\[
\leq F_K(\phi R) + F_K(d\alpha_\phi \hook S + \partial(\phi S)),
\]
\[
\leq F_K(\phi R) + F_K(d\alpha_\phi \hook S) + F_K(\partial(\phi S)),
\]
\[
\leq F_K(\phi R) + F_K(d\alpha_\phi \hook S) + F_K((\phi S)),
\]
(5.22)
\[
\leq M(\phi R) + M(d\alpha_\phi \hook S) + M(\phi S),
\]
\[
\leq \sup_{x \in K} |\phi(x)| M(R) + (r + 1) \mathcal{L}_{\phi,K} M(S) + \sup_{x \in K} |\phi(x)| M(S),
\]
\[
\leq \left\{ \sup_{x \in K} |\phi(x)| + (r + 1) \mathcal{L}_{\phi,K} \right\} (M(R) + M(S)),
\]
\[
= \left\{ \sup_{x \in K} |\phi(x)| + (r + 1) \mathcal{L}_{\phi,K} \right\} F(A),
\]
where in the third line we used Equation (5.10), in the sixth line we used Equation (2.18), and in the seventh line we used Equation (5.21). For the case \( r = n \) Equation 5.13 follows from the fact that \( S = 0 \). \( \square \)

Note that by Equation (5.11) it follows that
(5.23) \[
N(\phi A) \leq (r + 1) \|\phi\|_{L^K} N(A),
\]
and by Equation (5.12) if follows that for \( r < n \)
(5.24) \[
F_K(\phi A) \leq (r + 2) \|\phi\|_{L^K} F_K(A).
\]

The vector space of sharp functions defined on \( U \) and valued in \( \mathbb{R}^m \) is identified as the space of \( m \)-tuples of real valued sharp functions defined on \( U \) i.e. \( \mathcal{L}_s(U, \mathbb{R}^m) = [\mathcal{L}_s(U)]^m \). For \( \phi \in \mathcal{L}_s(U, \mathbb{R}^m) \) and \( A \in F_{r,K}(U) \) the flat \( r \)-chain \( \phi A \) is viewed as an element of the vector space of \( (F_{r,K}(U))^m \), i.e., an \( m \)-tuple of flat \( r \)-chains in \( U \) with \( (\phi A)_i = \phi_i A \).
CHAPTER 6

Configuration space and virtual velocities

Traditionally, a configuration of a body $\mathcal{P}$ is viewed as a mapping $\mathcal{P} \rightarrow \mathbb{R}^n$ which preserves the basic properties assigned to bodies and material surfaces. Guided by our initial definition of a body $T_P$ as a current induced by $\mathcal{P}$, a set of finite perimeter in the open set $\mathcal{B}$, a configuration of the body $\mathcal{P}$ is defined as a mapping $\kappa_P \in \mathcal{L}_{\text{Em}}(\mathcal{P}, \mathbb{R}^n)$. To distinguish it from a configuration of the universal body to be considered below, such an element, $\kappa_P$, will be referred to as a local configuration. The choice of Lipschitz type configurations is a generalization of the traditional choice of $C^1$-embeddings usually taken in continuum mechanics.

It is natural therefore to refer to $\mathcal{Q}_P = \mathcal{L}_{\text{Em}}(\mathcal{P}, \mathbb{R}^n)$ as the configuration space of the body $\mathcal{P}$. Since a body is a compact set, it follows from Theorem 8 that $\mathcal{Q}_P$ is an open subset of the Banach space $\mathcal{L}(\mathcal{P}, \mathbb{R}^n) \cong \mathcal{L}(\kappa_P\{\mathcal{P}\}, \mathbb{R}^n)$.

For $\mathcal{P}, \mathcal{P}' \in \Omega_B$ the local configurations $\kappa_P, \kappa_{P'}$ are said to be compatible if

$$\kappa_P \big|_{\mathcal{P} \cap \mathcal{P}'} = \kappa_{P'} \big|_{\mathcal{P} \cap \mathcal{P}'} .$$

Note that the intersection of two sets of finite perimeter is a set of finite perimeter, thus, the restricted map may be viewed as the configuration of the body $\mathcal{P} \cap \mathcal{P}'$.

A system of compatible configurations $\kappa$, is a collection of compatible local configurations $\kappa = \{\kappa_P \mid \mathcal{P} \in \Omega_B\}$. Clearly, a system of compatible configuration is represented by a unique element of $\mathcal{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n)$. An element $\kappa \in \mathcal{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n)$ will be referred to as a global configuration, and the global configuration space $\mathcal{Q}$ is the collection of all global configurations, i.e.,

$$\mathcal{Q} = \mathcal{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n) .$$

We will view the configuration space as a trivial infinite dimensional differentiable manifold, specifically, a trivial manifold modeled on a locally convex topological vector space as in [Mic80, Chapter 9].

It is noted, in particular, that a Lipschitz embedding is injective and the image of a set of a finite perimeter in $\mathcal{B}$ is a set of finite perimeter in $\mathbb{R}^n$. In addition, as Chapter 4 indicates, Lipschitz mappings are the natural morphism in the category of
sets of finite perimeters and in the category of flat chains. Thus, an element $\kappa \in \mathcal{Q}$ preserves the structure of bodies and material surfaces as required. That is, every $\kappa \in \mathcal{Q}$ induces a map $\kappa_\#$ of flat chains. For any $T_P \in \Omega_B$, the current $\kappa_\#(T_P)$ is an element of $N_n(\mathbb{R}^n)$, and for any $T_S \in \partial \Omega_B$, the current $\kappa_\#(T_S)$ is an $(n - 1)$-chain of finite mass in $\mathbb{R}^n$. By Equations (4.24) and (4.25) it follows that $\kappa_\#(T_P) = T_{\kappa \{P\}}$ and $\kappa_\#(T_S) = T_{\kappa \{S\}}$. Applying Equation (4.18), one obtains for every $T_S \in \partial \Omega_B$ that

$$M(\kappa_\#(T_S)) \leq M(T_S) \left( L_{\kappa,S} \right)^{n-1}. \tag{6.3}$$

By Equation (4.17), for every $T_P \in \Omega_B$,

$$N(\kappa_\#(T_P)) \leq N(T_P) \sup \left\{ (L_{\kappa,P})^n, (L_{\kappa,P})^{n-1} \right\}. \tag{6.4}$$

For a global configuration $\kappa$, let $\kappa(\Omega_B)$ denote the collection of images of bodies under the configuration $\kappa$, i.e.,

$$\kappa(\Omega_B) = \{ \kappa_\#(T_P) \mid T_P \in \Omega_B \}. \tag{6.5}$$

Similarly, the collection of surfaces at the configuration $\kappa$ is

$$\kappa(\partial \Omega_B) = \{ \kappa_\#(T_S) \mid T_S \in \partial \Omega_B \}. \tag{6.6}$$

A global virtual velocity at the configuration $\kappa$ is identified with an element of the tangent space to $\mathcal{Q}$ at $\kappa$. By Theorem 8, $\mathcal{L}(B, \mathbb{R}^n)$ is naturally isomorphic to any tangent space to $\mathcal{Q}$. Moreover, $\kappa$ induces an isomorphism $\mathcal{L}(B, \mathbb{R}^n) \cong \mathcal{L}(\kappa \{B\}, \mathbb{R}^n)$ and an Eulerian virtual velocity is viewed as an element of $\mathcal{L}(\kappa \{B\}, \mathbb{R}^n)$. In what follows, we refer to $\mathcal{L}(\kappa \{B\}, \mathbb{R}^n)$ as the space of global virtual velocities at the configuration $\kappa$ and use the abbreviated notation $W_\kappa$ for it. Naturally, an element of $W_\kappa$ may be identified with an $n$-tuple of sharp functions defined on $\kappa \{B\}$, i.e., using the Whitney topology on $\mathcal{L}(\kappa \{B\})$, $W_\kappa = [\mathcal{L}(\kappa \{B\})]^n$.

Focusing our attention to a particular body $P$, one may make use of the approach of [Seg86] and define a virtual velocity of a body $P$ at a configuration $\kappa_P \in \mathcal{Q}_P$ as an element $v_P$ in the tangent space $T_{\kappa_P} \mathcal{Q}_P$. It follows from Theorem 8 that one may make the identifications $T_{\kappa_P} \mathcal{Q}_P \cong \mathcal{L}(P, \mathbb{R}^n) \cong \mathcal{L}(\kappa_P \{P\}, \mathbb{R}^n)$.

**Theorem 11.** For every body $P$, and every $\kappa_P \in \mathcal{Q}_P$, and every $\kappa \in \mathcal{Q}$ such that $\kappa |_P = \kappa_P$, the restriction mapping

$$\rho_P : T_\kappa \mathcal{Q} \rightarrow T_{\kappa_P} \mathcal{Q}_P \tag{6.7}$$

is surjective.
PROOF. We recall that Kirszbraun’s theorem asserts that a Lipschitz mapping \( f : A \to \mathbb{R}^m \) defined on a set \( A \subset \mathbb{R}^n \) may be extended to to a Lipschitz function \( F : \mathbb{R}^n \to \mathbb{R}^m \) having the same Lipschitz constant (see [Fed69, Section 2.10.43] or [Hei00, Section 6.2]). It follows immediately that any \( v_P \in \mathcal{L}(P, \mathbb{R}^n) \) may be extended to an element \( v \in \mathcal{L}(B, \mathbb{R}^n) \). □

Anticipating the properties of systems of forces to be considered below, we wish to provide the collection of restrictions of global virtual velocities to the various bodies with a finer structure than that provided by the \( \| \cdot \|_{L,K} \)-semi-norms. In particular, when considering the restriction \( v \mid_P \) of a global virtual velocity \( v \in \mathbb{W}_\kappa \) to a body \( P \), we wish that the magnitude of the resulting object will reflect the mass of \( P \). The local virtual velocity for the body \( T_P \) at the configuration \( \kappa \) induced by the global virtual velocity \( v \in \mathbb{W}_\kappa \) is defined as the \( n \)-tuple of normal \( n \)-currents given by the products \( [v_\kappa\# (T_P)] \) such that

\[
[v_\kappa\# (T_P)]_i = v_i\kappa\# (T_P), \quad \text{for all } i = 1, \ldots, n.
\]

By Equations (5.14) and (5.11), each component \( [v_\kappa\# (T_P)]_i \) is a normal \( n \)-current such that

\[
M ([v_\kappa\# (T_P)]_i) \leq \sup_{y \in \kappa(P)} |v_i(y)| M (\kappa\# (T_P)),
\]

\[
\leq \sup_{y \in \kappa(P)} |v_i(y)| (\mathcal{L}_{\kappa,P})^n M (T_P), \quad \text{(6.9)}
\]

and

\[
N ([v_\kappa\# (T_P)]_i) \leq \left( \sup_{y \in \kappa(P)} |v_i(y)| \right) + n\mathcal{L}_{v_i,P} \left( \mathcal{L}_{\kappa,P}^n \right) N (\kappa\# (T_P)),
\]

\[
\leq \left( \sup_{y \in \kappa(P)} |v_i(y)| \right) + n\mathcal{L}_{v_i,P} \left( \mathcal{L}_{\kappa,P}^n \right) \times \sup \left\{ (\mathcal{L}_{\kappa,P})^n, (\mathcal{L}_{\kappa,P})^{n-1} \right\} N(T_P). \quad \text{(6.10)}
\]

In other words, the mapping \( \mathbb{W}_\kappa \times \Omega_B \to \mathcal{D}_m (B) \) given by \( (v, T_P) \mapsto v_\kappa\# (T_P) \) is continuous with respect to both the mass norm and the normal norm.

Similarly, the assignment of a virtual velocity \( v \in \mathbb{W}_\kappa \) to a material surface \( T_S \) induces an \( n \)-tuple of \( (n - 1) \)-chains defined by the multiplication \( v_\kappa\# (T_S) \). Each component \( [v_\kappa\# (T_S)]_i \) is a chain of finite mass and applying Equation (5.14), one
obtains

\[
M ([\nu_{x#}(T_S)])_i \leq \left( \sup_{y \in x\{S\}} |v_i(y)| \right) M (x#(T_S)),
\]

(6.11)

\[
\leq \left( \sup_{y \in x\{S\}} |v_i(y)| \right) \left( \mathcal{L}_{x,S} \right)^{n-1} M (T_S).
\]
CHAPTER 7

Density transport theorem

In this chapter we apply the general setting presented thus far and present a density transport theorem which is analogous to Reynolds transport theorem for an implicit time dependent property. Using the framework introduced in Chapter 6, a motion is defined as a mapping

\[ M : \mathbb{R} \times \mathcal{B} \to \mathbb{R}^n, \]

such that for every \( t \in \mathbb{R} \) the map \( \kappa_t : \mathcal{B} \to \mathbb{R}^n \) defined by

\[ \kappa_t(x) = M(t, x), \quad \text{for all} \quad x \in \mathcal{B}, \]

is a global configuration as presented in Chapter 6, i.e., a Lipschitz embedding \( \kappa_t \in \mathcal{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n) \).

In the spirit of Chapter 5, a general Lagrangian representation of an intensive property is assumed to be given by

\[ \psi : \mathcal{B} \to \mathbb{R}, \]

where we assume that \( \psi \) is a sharp function i.e., a real valued, bounded, locally Lipschitz function. The extensive property associated with \( \psi \) and the body \( T_P \) is defined as the multiplication \( \psi T_P \), which, by Proposition 10, is a normal \( n \)-current in \( \mathcal{B} \). For any \( \omega \in \mathcal{D}^n(\mathbb{R}^n) \)

\[ \kappa_t \# (\psi T_P) (\omega) = \int_{\mathcal{P}} \psi(x) \omega(\kappa_t(x)) \left( \bigwedge_n D\kappa_t(e_1 \wedge \cdots \wedge e_n) \right) dL^n_x, \]

\[ = \int_{\mathcal{P}} \psi(\kappa_t^{-1}(\kappa_t(x))) \omega(\kappa_t(x)) \left( \bigwedge_n D\kappa_t(e_1 \wedge \cdots \wedge e_n) \right) dL^n_x, \]

\[ = \kappa_t \# (T_P)(\psi \kappa_t \wedge \omega), \]

\[ = \psi \kappa_t \kappa_t \# (T_P)(\omega). \]

Thus,

\[ \kappa_t \# (\psi T_P) = \psi \kappa_t \kappa_t \# (T_P), \]
where

\[ \psi_{\kappa_t} = \psi \circ \kappa_t^{-1} : \kappa_t(B) \to \mathbb{R}, \]

is viewed as the Eulerian representation of the property \( \psi \).

In order to develop a density transport theorem we wish to investigate the term

\[ \frac{d}{dt}(\kappa_t#(\psi T_P))_{t=0}, \]

which will be done by applying the homotopy theory for currents and the formal definition of the derivative such that

\[ \frac{d}{dt}(\kappa_t#(\psi T_P))_{t=0} = \lim_{\varepsilon \to 0} \left[ \frac{\kappa_{\varepsilon#}(\psi T_P) - \kappa_{0#}(\psi T_P)}{\varepsilon} \right], \]

we first recall some basic properties of the homotopy theorem for currents.

Let \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \) be open sets with \( T \in \mathcal{D}_k(U) \) and \( S \in \mathcal{D}_l(V) \). Then, the Cartesian product of \( T \) and \( S \) is an element of \( \mathcal{D}_{k+l}(U \times V) \) denoted by \( T \times S \) and defined as follows. Let \( p, q \) be the projection mappings

\[ p : U \times V \to U, \quad q : U \times V \to V. \]

For \( \alpha \in \mathcal{D}^r(U) \) and \( \beta \in \mathcal{D}^{m+k-r}(V) \), we note that \( p^#(\alpha) \wedge q^#(\beta) \) is an element of \( \mathcal{D}^{m+k}(U \times V) \). Thus

\[ (T \times S)(p^#(\alpha) \wedge q^#(\beta)) = \begin{cases} T(\alpha)S(\beta), & \text{in case } r = k, \\ 0 & \text{in case } r \neq k. \end{cases} \]

For the properties of the Cartesian products of currents we refer to [Fed69, Section 4.1.8].

Let \( U \subset \mathbb{R}^n \) be an open set and let \( f \) and \( g \) be locally Lipschitz mappings of \( U \) into \( \mathbb{R}^m \). For an open set \( A \) of \( \mathbb{R} \) such that \([0, 1] \subset A\), a Lipschitz homotopy from \( f \) to \( g \) is a map

\[ h : A \times U \to \mathbb{R}^m, \]

such that

\[ h(0,x) = f(x), \quad \text{and} \quad h(1,x) = g(x), \]

for all \( x \in U \). A Lipschitz homotopy \( h \) is said to be a linear homotopy if

\[ h(\tau,x) = (1-\tau)f(x) + \tau g(x). \]
In the following, we will use the following notation

\[ h_\tau(x) = h(\tau, x), \text{ for all } x \in U, \]  

and

\[ \dot{h}_\tau : U \rightarrow \mathbb{R}^m, \quad \dot{h}_\tau(x) = Dh(\tau, x)(1, 0), \text{ for all } x \in U, \]

where in the preceding equation 0 is the zero element in \( \mathbb{R}^n \). For \( T \in \mathcal{D}_r(U) \) and a homotopy \( h \) between \( f \) and \( g \), the \( h \) deformation chain of \( T \) is defined as the current

\[ h_\#([0, 1] \times T) \in \mathcal{D}_{r+1}(\mathbb{R}^m). \]

The properties of the \( h \) deformation chain are further investigated in \[Fed69\], Section 4.1.9 where it shown that for \( r > 0 \)

\[ g_\#(T) - f_\#(T) = \partial h_\#([0, 1] \times T) + h_\#([0, 1] \times \partial T). \]

For an \( r \)-current \( T \) which is represented by integration and \( \omega \in \mathcal{D}_{r+1}(\mathbb{R}^m) \),

\[ h_\#([0, 1] \times T)(\omega) = \int_{[0,1]} \left[ \int_U \omega(h_\tau(x)) \left( \dot{h}_\tau(x) \wedge Dh_\tau(x)\bar{T}(x) \right) \right] \, d\mu \, dL_1^1. \]

We now return to Equation (7.8) and let \( h_\varepsilon : [0, 1] \times \mathcal{B} \rightarrow \mathbb{R}^n \) be the linear homotopy between \( \kappa_0 \) and \( \kappa_\varepsilon \) i.e., \( h_\varepsilon(0, x) = \kappa_0(x) \) and \( h_\varepsilon(1, x) = \kappa_\varepsilon(x) \) such that

\[ h_\varepsilon(\tau, x) = \kappa_0(x)(1 - \tau) + \kappa_\varepsilon(x)\tau. \]

Applying the homotopy formula, Equation (7.16), it follows that

\[ \kappa_\varepsilon_\#(\psi T_P) - \kappa_0_\#(\psi T_P) = \partial h_\#([0, 1] \times \psi T_P) + h_\#([0, 1] \times \partial (\psi T_P)). \]

The following results are independent of any particular homotopy chosen and a linear homotopy was selected for convenience. Since \( h_\#([0, 1] \times \psi T_P) \) is an \( (n + 1) \)-current in \( \mathbb{R}^n \), the first term on the right-hand side of Equation (7.19) vanishes. Applying Equation (5.10), we obtain

\[ \kappa_\varepsilon_\#(\psi T_P) - \kappa_0_\#(\psi T_P) = h_\#([0, 1] \times \bar{\partial}\psi \cup T_P) + h_\#([0, 1] \times \psi \partial T_P). \]

Thus,

\[ \frac{d}{dt}(\kappa_{\varepsilon t}_\#(\psi T_P))|_{t=0} = \lim_{\varepsilon \to 0} \left[ \frac{h_\#([0, 1] \times \bar{\partial}\psi \cup T_P) + h_\#([0, 1] \times \psi \partial T_P)}{\varepsilon} \right]. \]

Each of the terms is examined separately by applying the integral representation of the \( h_\varepsilon \) deformation chain as given by Equation (7.17). As \( h^\varepsilon \) is a linear homotopy, by
direct calculations

\begin{align}
D h^\tau_\epsilon(x) &= (1 - \tau) D\kappa_0(x) + \tau D\kappa_\epsilon(x), \\
\dot{h}^\tau_\epsilon(x) &= \kappa_\epsilon(x) - \kappa_0(x).
\end{align}

For \( \omega \in \mathcal{D}^n(\mathbb{R}^n) \) observe that

\begin{align}
\lim_{\epsilon \to 0} \left[ \frac{h_\epsilon^\tau ([0, 1] \times \psi \partial T_P) (\omega)}{\epsilon} \right] &= \lim_{\epsilon \to 0} \left[ \frac{\left. \int_{\Gamma(P)} \psi(x) \omega \left( h^\tau_\epsilon(x) \right) \left( \dot{h}^\tau_\epsilon(x) \land \left[ \bigwedge_{n-1} D h^\tau_\epsilon(x) \right] \partial T_P(x) \right) \right| dH_{x-1}^n dL_1}{\epsilon} \right] \\
&= \int_{[0,1]} \left[ \int_{\Gamma(P)} \psi(x) \omega \left( \kappa_0(x) \right) \left( \left[ \bigwedge_{n-1} D\kappa_0(x) \right] \partial T_P(x) \right) \right] dH_{x-1}^n dL_1.
\end{align}

Here, \( \psi(x) \), defined as

\begin{align}
\psi(x) = \lim_{\epsilon \to 0} \frac{\dot{h}^\tau_\epsilon(x)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\kappa_\epsilon(x) - \kappa_0(x)}{\epsilon},
\end{align}

is viewed as the velocity of the material point \( x \in \mathcal{B} \) at the time \( t = 0 \). In addition, set \( u(x) = D\kappa_0(x)^{-1} (\psi(x)) \). Note that the integrand is independent of \( t \), thus

\begin{align}
\lim_{\epsilon \to 0} \left[ \frac{h_\epsilon^\tau ([0, 1] \times \psi \partial T_P) (\omega)}{\epsilon} \right] &= \int_{\Gamma(P)} \psi(x) \omega \left( \kappa_0(x) \right) \left( \left[ \bigwedge_{n-1} D\kappa_0(x) \right] u(x) \land \partial T_P(x) \right) \left[ dH_{x-1}^n \right] dL_1 \\
&= \left( \kappa_0^\#(\omega) \right) (\psi u \land \partial T_P), \\
&= \kappa_0^\# (\psi u \land \partial T_P) (\omega), \\
&= \psi \kappa_0^\# (u \land \partial T_P) (\omega), \\
&= \psi \kappa_0^\# (\partial T_P) (\omega)
\end{align}

Here, \( u \land \partial T_P \) is defined as the \( n \)-current such that

\begin{align}
u \land \partial T_P(\omega) = (u \hookrightarrow \omega) (\partial T_P).
\end{align}

We use \( \bigwedge_m D\kappa_0(x) \) for the map

\begin{align}
\bigwedge_m D\kappa_0(x) : \bigwedge_m \mathbb{R}^n \to \bigwedge_m \mathbb{R}^n,
\end{align}

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defined by

\[ \bigwedge_m D_\kappa^0 (x) (v_1 \land \cdots \land v_m) = (D_\kappa^0 (x) (v_1)) \land \cdots \land (D_\kappa^0 (x) (v_m)), \quad \text{for all } v_1, \ldots, v_m \in \mathbb{R}^n. \]

For the term \( \lim_{\varepsilon \rightarrow 0} \left[ \frac{h_\varepsilon (\left[ 0,1 \right] \times \tilde{d} \psi \mathcal{J} T_P)}{\varepsilon} \right] \), we obtain

\[
\lim_{\varepsilon \rightarrow 0} \left[ \frac{h_\varepsilon (\left[ 0,1 \right] \times \tilde{d} \psi \mathcal{J} T_P) (\omega)}{\varepsilon} \right] = \lim_{\varepsilon \rightarrow 0} \left[ \frac{\int_{0,1} \left[ f_P \omega (h_\varepsilon^x (x)) \left( \bigwedge_{n-1} D h_\varepsilon^x (x) \tilde{d} \psi \mathcal{J} (e_1 \land \cdots \land e_n) \right) dL_\varepsilon^n \right] dL_1^1}{\varepsilon} \right],
\]

\[
= 1 \int_P \left[ \omega (\kappa_0^0 (x)) \left( v(x) \land \left[ \bigwedge_{n=1} D_\kappa^0 (x) \tilde{d} \psi \mathcal{J} (e_1 \land \cdots \land e_n) \right) \right) dL_n^x,
\]

\[
= \int_P \left[ \omega (\kappa_0^0 (x)) \left[ \bigwedge_n D_\kappa^0 (x) \tilde{d} \psi \left( D_\kappa^{-1}_0 (x) v(x) \right) (e_1 \land \cdots \land e_n) \right) \right] dL_n^x.
\]

The term \( \tilde{d} \psi \left( D_\kappa^{-1}_0 (x) v(x) \right) \) is identified with the time-derivative of the Eulerian field describing the property \( \psi \)

\[
(7.31) \quad \tilde{d} \psi \left( D_\kappa^{-1}_0 (x) v(x) \right) = \frac{d \psi_\kappa^t}{dt} |_{t=0}.
\]

As a result,

\[
(7.32) \quad h_\varepsilon^x ([0,1] \times d \psi \mathcal{J} T_P) (\omega) = \frac{d \psi_\kappa^t}{dt} |_{t=0} \kappa_0^0 (T_P) (\omega).
\]

We concluded that

\[
(7.33) \quad \frac{d}{dt} (\kappa_\psi (\psi T_P)) |_{t=0} = \psi_\kappa^0 \lor \kappa_0^0 (\partial T_P) + \frac{d \psi_\kappa^t}{dt} |_{t=0} \kappa_0^0 (T_P),
\]

where the first term is associated as the flux of the property \( \psi \) through the boundary of the body, and the second term is the time derivative of the property in the domain of the body \( T_P \).
CHAPTER 8

Cauchy fluxes

Alluding to the approach of Seg86 again, a force on a body \( P \) at the configuration \( \kappa P \in Q_P \) is an element in the dual to the tangent space, \( T^*_{\kappa P} Q_P \). In other words, forces on \( P \) are elements of the infinite dimensional cotangent bundle \( T^* Q_P \). For \( g_P \in T^*_{\kappa P} Q_P \), and \( v_P \in T_{\kappa P} Q_P \), the action \( g_P(v_P) \) is interpreted as the virtual power performed by the force \( g_P \) for the virtual velocity \( v_P \). It follows immediately that a force on a body \( P \) at \( \kappa P \) may be identified with a linear continuous functional on the space of Lipschitz mappings. Such functionals are quite irregular and will not be considered here.

Instead, we use in this chapter the notion of a Cauchy flux at the configuration \( \kappa \), as a real valued function operating on the Cartesian product \( \kappa (\partial \Omega_B) \times W_\kappa \). These impose stricter conditions on the force system and resulting stress fields. The conditions to be imposed still imply that for a fixed body, a force is a continuous linear functional of the virtual velocities of that body.

A Cauchy flux represents a system of surface forces operating on the material surfaces, or more precisely, their images under \( \kappa \). For a given surface and a given virtual velocity field, the value returned by the Cauchy flux mapping is interpreted as the virtual power (or virtual work) performed by the force acting on the image of the material surface under \( \kappa \) for the given virtual velocity.

**Definition 12.** A Cauchy flux at the configuration \( \kappa \) is a mapping of the form

\[
\Phi_\kappa : \kappa (\partial \Omega_B) \times W_\kappa \rightarrow \mathbb{R},
\]

such that the following hold.

**Additivity:** \( \Phi_\kappa (\cdot, v) \) is additive for disjoint compatible material surfaces, i.e., for every \( \kappa#(T_S), \kappa#(T_{S'}) \in \kappa (\partial \Omega_B) \) compatible and disjoint,

\[
\Phi_\kappa (\kappa#(T_{S\cup S'}), v) = \Phi_\kappa (\kappa#(T_S), v) + \Phi_\kappa (\kappa#(T_{S'}), v),
\]

holds for every \( v \in W_\kappa \).
**Linearity:** $\Phi_\kappa (\kappa_# (T_S), \cdot)$ is a linear function on $W_\kappa$, i.e., for all $\alpha, \beta \in \mathbb{R}$ and $v, v' \in W_\kappa$,

\begin{equation}
\Phi_\kappa (\kappa_# (T_S), \alpha v + \beta v') = \alpha \Phi_\kappa (\kappa_# (T_S), v) + \beta \Phi_\kappa (\kappa_# (T_S), v')
\end{equation}

holds for every $\kappa_# (T_S) \in \kappa (\partial \Omega_B)$.

Let $v \in W_\kappa$ and $\kappa_# (T_S) \in \kappa (\partial \Omega_B)$, then, by the linearity of the Cauchy flux,

\begin{equation}
\Phi_\kappa (\kappa_# (T_S), v) = \Phi_\kappa (\kappa_# (T_S), \sum_{i=1}^{n} v_i e_i) = \sum_{i=1}^{n} \Phi_\kappa (\kappa_# (T_S), v_i e_i).
\end{equation}

Set $\Phi_i^j (\kappa_# (T_S), u) = \Phi_\kappa (\kappa_# (T_S), u e_i)$ for all $u \in L(\kappa \{B\})$, so that $\Phi_i^j$ is naturally viewed as the $i$-th component of the Cauchy flux at the configuration $\kappa$. One has,

\begin{equation}
\Phi_\kappa (\kappa_# (T_S), v) = \sum_{i=1}^{n} \Phi_i^j (\kappa_# (T_S), v_i).
\end{equation}

**Balance:** There is a number $0 < s < \infty$ such that for all components of the Cauchy flux

\begin{equation}
\Phi_i^j (\kappa_# (T_S), v) \leq s \|v\|_{L, \partial \Omega_B} M (\kappa_# (T_S)),
\end{equation}

for all $\kappa_# (T_S) \in \kappa (\partial \Omega_B)$ and $v \in W_\kappa$.

**Weak balance:** There is a number $0 < b < \infty$ such that for all components of the Cauchy flux

\begin{equation}
\Phi_i^j (\kappa_# (\partial T_P), v) \leq b \|v\|_{L, \partial P} M (\kappa_# (T_P)),
\end{equation}

for all $\kappa_# (T_P) \in \kappa (\Omega_B)$ and $v \in W_\kappa$.

It is observed that from the balance property assumed above, for each material surface $T_S$, $\Phi_\kappa (\kappa_# (T_S), \cdot)$ is continuous.

**Remark 13.** It is noted that the term $\|v\|_{L, \partial \Omega_B}$ in the balance principle, Equation (8.6), may be replaced with $\|v\|_{\infty, \partial \Omega_B} = \sup_{x \in \partial \Omega_B} |v(x)|$. We keep the former for convenience.

**Theorem 14.** Each component of the Cauchy flux $\Phi_\kappa$ induces a unique flat $(n-1)$-cochain in $\kappa \{B\}$.

The proof of Theorem 14 will be divided into three steps.

**(Step 1)** Each component of the Cauchy flux is used to defines a linear functional on the space of polyhedral $(n-1)$-chains in $\kappa \{B\}$.

**(Step 2)** The linear functional defined in the previous step is extended to a unique flat $(n-1)$-cochain.
(Step 3) The compatibility of flat \((n - 1)\) chains with Cauchy flux is established.

**PROOF.** Step 1:

Let \(\sigma^{n-1}\) be an oriented \((n - 1)\)-simplex in \(\kappa \{B\}\). Since \(\kappa \{B\}\) is open there exists some \(n\)-simplex \(\sigma^n\) in \(\kappa \{B\}\) such that \(\sigma^{n-1} \subset \partial \sigma^n\). Since \(\kappa^{-1} \{\sigma^n\}\) is a set of finite perimeter in \(B\) it follows that \(\sigma^{n-1} \in \kappa (\partial \Omega_B)\). In other words, every oriented \((n - 1)\)-simplex in \(\kappa \{B\}\) may be viewed as an element of \(\kappa (\partial \Omega_B)\).

In what follows, we use extensions of Lipschitz mappings as implied by Kirszbraun’s theorem. First, define a real valued function \(\alpha\) of \((n - 1)\)-simplices. Let \(u : \kappa \{B\} \rightarrow \mathbb{R}\) be a locally Lipschitz function in \(\kappa \{B\}\) such that \(u(x) = 1\) for \(x \in \sigma^{n-1}\), and we set

\[
\alpha \left( \sigma^{n-1} \right) = \Phi^i_{\kappa} \left( \sigma^{n-1}, u \right).
\]

The fact that the definition is independent of the choice of \(u\) follows from condition (8.6) and will be demonstrated below where \(\alpha\) is extended to polyhedral \((n - 1)\)-chains.

Consider a polyhedral \((n - 1)\)-chain \(A = \sum_{j=1}^{J} a_j \sigma_j^{n-1}\) in \(\kappa \{B\}\) such that \(\{\sigma_j^{n-1}\}_{j=1}^{J}\) are pairwise disjoint. Define the function \(u : \bigcup_{j=1}^{J} \sigma_j^{n-1} \rightarrow \mathbb{R}\) by

\[
u(x) = a_j \quad \text{if } x \in \sigma_j^{n-1}.
\]

We now apply Kirszbraun’s theorem and obtain \(\tilde{u} : \kappa \{B\} \rightarrow \mathbb{R}\), a Lipschitz extension to \(u\) defined on \(\kappa \{B\}\). By the properties postulated for Cauchy fluxes

\[
\Phi^i_{\kappa} \left( \bigcup_{j=1}^{J} \sigma_j^{n-1}, \tilde{u} \right) = \sum_{j=1}^{J} \Phi^i_{\kappa} \left( \sigma_j^{n-1}, \tilde{u} \right) = \sum_{j=1}^{J} a_j \alpha \left( \sigma_j^{n-1} \right).
\]

The function \(\alpha\) is now extended to polyhedral \((n - 1)\)-chains in \(\kappa \{B\}\) by linearity, i.e.,

\[
\alpha \left( A \right) = \alpha \left( \sum_{j=1}^{J} a_j \sigma_j^{n-1} \right) = \sum_{j=1}^{J} a_j \alpha \left( \sigma_j^{n-1} \right).
\]

Thus, \(\alpha\) is a linear functional of polyhedral \((n - 1)\)-chains. The value of \(\alpha(A)\) is independent of any particular extension of \(u\), for given \(\tilde{u}', \tilde{u}\) any two Lipschitz extensions
of \( u \),

\[
(8.12) \quad \left| \Phi'_k \left( \bigcup_{j=1}^J \sigma_j^{n-1}, \tilde{u} \right) - \Phi'_k \left( \bigcup_{j=1}^J \sigma_j^{n-1}, \tilde{u}' \right) \right| = \left| \Phi'_k \left( \bigcup_{j=1}^J \sigma_j^{n-1}, \tilde{u} - \tilde{u}' \right) \right|, \\
\leq s \| \tilde{u} - \tilde{u}' \|_{\mathcal{L}_2 \left( \bigcup_{j=1}^J \sigma_j^{n-1}, \tilde{u} \right)} M \left( T_{\bigcup_{j=1}^J \sigma_j^{n-1}} \right), \\
= 0.
\]

Step 2:
From Equation (8.6) it follows that

\[
(8.13) \quad |\alpha(\sigma^{n-1})| \leq s M(\sigma^{n-1}), \quad \text{for all } \sigma^{n-1} \in \kappa \{B\},
\]
and by Equation (8.7),

\[
(8.14) \quad |\alpha(\partial\sigma^n)| \leq b M(\sigma^n), \quad \text{for all } \sigma^n \in \kappa \{B\}.
\]
The flat norm of \( \alpha \) is defined by

\[
(8.15) \quad F(\alpha) = \sup \{ \alpha(A) | A \text{ is a polyhedral } (n-1)\text{-chain}, \\
F_K(A) \leq 1, K \subset \kappa \{B\} \}.
\]

Using Equation (2.17) the \( K \)-flat semi-norm of \( A \) is given by

\[
(8.16) \quad F_K(A) = \inf_B \{ M(A - \partial B) + M(B) | B \text{ is a polyhedral } n\text{-chain}, \text{spt}(B) \subset K \}.
\]
The flat norm of \( \alpha \) is given by

\[
(8.17) \quad F(\alpha) = \max \left\{ \sup_{\sigma^{n-1} \in \kappa \{B\}} \frac{\alpha(\sigma^{n-1})}{M(\sigma^{n-1})}, \sup_{\sigma^n \in \kappa \{B\}} \frac{\alpha(\partial\sigma^n)}{M(\sigma^n)} \right\} \leq \max \{s, b\}.
\]
To obtain the last estimate, let \( \epsilon > 0 \) and \( B_\epsilon \) be a polyhedral \( n \)-chain such that

\[
M(A - \partial B_\epsilon) + M(B_\epsilon) \leq F_K(A) + \epsilon,
\]
so that

\[
|\alpha(A)| \leq |\alpha(A - \partial B_\epsilon)| + |\alpha(\partial B_\epsilon)|,
\]
\[
\leq M(\alpha) M(A - \partial B_\epsilon) + M(d \alpha) M(B_\epsilon),
\]
\[
\leq \sup \{ M(\alpha), M(d \alpha) \} (M(A - \partial B_\epsilon) + M(B_\epsilon)),
\]
\[
\leq \sup \{ M(\alpha), M(d \alpha) \} (F_K(A) + \epsilon).
\]
Letting $\epsilon \to 0$, it follows that
\begin{equation}
F(\alpha) \leq \sup \{ M(\alpha), M(d\alpha) \}.
\end{equation}

By Equation (2.18) it follows that $M(\alpha) \leq F(\alpha)$ and we obtain
\begin{equation}
F(\alpha) = \sup \{ M(\alpha), M(d\alpha) \}.
\end{equation}

Since the terms $M(\alpha), M(d\alpha)$ are evaluated on polyhedral chains it is sufficient to evaluate it on simplices and Equation (8.17) follows.

We also recall, [Fed69, Section 4.1.23], that polyhedral chains form a dense sub-space of the space of flat chains, specifically, for every $A \in F_{n-1,K}(\mathbb{R}^n)$, a compact subset $C \subset \kappa(\mathcal{B})$ whose interior contain $K$ and $\epsilon > 0$, there is and a polyhedral $(n-1)$-chain $A_\epsilon$ supported in $C$ such that
\begin{equation}
F_C(A - A_\epsilon) \leq \epsilon.
\end{equation}

Thus, for every flat $(n-1)$-chain $A$ we have a sequence $A_j$ such that $\lim_{i \to \infty} F_j A_j = A$. The cochain $\alpha$ is uniquely extended a flat $(n-1)$-cochain $\Psi$ such that for every $A = \lim_{j \to \infty} F_j A_j$
\begin{equation}
\Psi(A) = \lim_{j \to \infty} \alpha(A_j).
\end{equation}

The foregoing part of the theorem is analogous to [Whi57, Section V.4].

Step 3:
In order to complete the proof we need to show that for $\kappa#(T_S) \in \kappa(\partial\Omega_B)$ and $v \in L_s(\kappa\{B\})$ we obtain $\Psi(v\kappa#(T_S)) = \Phi^i_k(\kappa#(T_S), v)$. By [Fed69, Section 4.1.17] the class of flat chains of finite mass is the $M$-closure of normal currents. The chain $v\kappa#(T_S)$ is a flat $(n-1)$-chain of finite mass. Hence, the sequence of polyhedral $(n-1)$-chains $\{A_j\}_{j=1}^\infty$, converging $v\kappa#(T_S)$ in the flat norm, has a convergent subsequence $\{A_{j'}\}_{j'=1}^\infty$ such that $\{A_{j'}\}$ converges to $v\kappa#(T_S)$ in the flat norm and
\begin{equation}
M(v\kappa#(T_S)) = \lim_{j' \to \infty} M(A_{j'}).
\end{equation}

By the definition of $\alpha$ and the balance principle, Equation (8.6) the sequence $\{\alpha(A_{j'})\}_{j'=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ since $|\alpha(A_m) - \alpha(A_k)| \leq sM(A_m - A_k)$. Hence
\begin{equation}
\lim_{j' \to \infty} \alpha(A_{j'}) = \Phi^i_k(\kappa#(T_S), v).
\end{equation}
Since $\Psi$ is an extension of $\alpha$ it follows that $\Psi(\mathcal{A}_j') = \alpha(\mathcal{A}_j')$ and

$$\left| \Psi(v \kappa_# (T_S)) - \Phi_k^i (\kappa_# (T_S), v) \right| = \left| \Psi(v \kappa_# (T_S)) - \lim_{j' \to \infty} \alpha(\mathcal{A}_j') \right|,$$

$$= \left| \Psi(v \kappa_# (T_S)) - \lim_{j' \to \infty} \Psi(\mathcal{A}_j) \right|,$$

$$= \left| \Psi(v \kappa_# (T_S)) - \Psi \left( \lim_{j' \to \infty} \mathcal{A}_j \right) \right|,$$

$$\leq \max \{s, b\} \lim_{j' \to \infty} F \left( v_i \kappa_# (T_S) - \mathcal{A}_j \right) = 0,$$

(8.25)

which completes the proof.

The extension of each flat $(n-1)$-cochain from $\kappa \{B\} \subset \mathbb{R}^n$ to $\mathbb{R}^n$ is done trivially by setting its representing flat $(n-1)$-form to vanish outside $\kappa \{B\}$. We conclude that a Cauchy flux $\Phi_k$ induces a unique $n$-tuple of flat $(n-1)$-cochains in $\mathbb{R}^n$ such that

(8.26) $\Phi_k (\kappa_# (T_S), v) = \sum_{i=1}^n \Psi^i (v_i \kappa_# (T_S)),$

for all $v \in W_\kappa$ and $\kappa_# (T_S) \in \kappa (\partial \Omega_B)$. The inverse implication is provided by

THEOREM 15. An $n$-tuple $\{\Psi^i\}$ of flat $(n-1)$-cochains in $\mathbb{R}^n$ induces by Equation (8.26) a unique Cauchy flux $\Phi_k$.

PROOF. For each $v \in W_\kappa$ and $\kappa_# (T_S)$, the Cauchy flux $\Phi_k (\kappa_# (T_S), v)$ will be defined by Equation (8.26), and by the components

(8.27) $\Phi_k^i (\kappa_# (T_S), v_i) = \Psi^i (v_i \kappa_# (T_S)).$

The additivity (8.2) and linearity (8.3) properties clearly hold since $\Psi^i$ is a linear function of flat $(n-1)$-chains. For the Balance (8.6) and weak balance (8.7) properties, recall that since $\Psi^i$ is a flat $(n-1)$-cochain, there exists $C > 0$ such that for every flat $(n-1)$-chain $\mathcal{A}$ with support in $K$, we may write $|\Psi^i(\mathcal{A})| \leq CF_K (\mathcal{A})$. For the balance property

$$\left| \Phi_k^i (\kappa_# (T_S), v_i) \right| = \left| \Psi^i (v_i \kappa_# (T_S)) \right|,$$

$$\leq CF_K (\kappa_# (T_S), v_i),$$

$$\leq CM (v_i \kappa_# (T_S)),$$

$$\leq C \|v_i\|_{L^\infty} M (\kappa_# (T_S)).$$

(8.28)
For the weak balance
\[ |\Phi^i_\kappa (\kappa_# (\partial T_P), v_i) | = |\Psi^i (v_i \kappa_# (\partial T_P)) |, \]
\[ \leq CF_{\kappa(P)} (v_i \kappa_# (\partial T_P)) , \]
\[ = CF_{\kappa(P)} (\partial (v_i \kappa_# (T_P)) + dv \kappa_# (T_P)) , \]
\[ \leq C \left[ F_{\kappa(P)} (\partial (v_i \kappa_# (T_P)))) + F_{\kappa(P)} (dv \kappa_# (T_P)) \right] , \]
\[ \leq C \left[ F_{\kappa(P)} (v_i \kappa_# (T_P)) + F_{\kappa(P)} (dv \kappa_# (T_P)) \right] , \]
\[ \leq C \left[ M (v_i \kappa_# (T_P)) + M (dv \partial \kappa_# (T_P)) \right] , \]
\[ \leq C \left[ \sup_{x \in \kappa(P)} |v_i(x)| M (\kappa_# (T_P)) + n \mathcal{L}_{v,\kappa(P)} M (\kappa_# (T_P)) \right] , \]
\[ \leq C(n + 1) \|v_i\|_{\mathcal{L}_{v,\kappa(P)}} M (\kappa_# (T_P)) . \]

Thus, Theorems 14 and 15 restate the point of view presented in [RS03] that the balance and weak-balance assumptions of stress theory may be replaced by the requirement that the system of forces is given in terms of an \(n\)-tuple of flat \((n - 1)\)-cochains.
CHAPTER 9

Generalized bodies and Generalized surfaces

The representation of a Cauchy flux by an \( n \)-tuple of flat \((n-1)\)-cochains enables the generalization of the class of admissible bodies and the introduction of a larger class of material surfaces. By a generalized body we will mean a subset \( \hat{\mathcal{P}} \subset \mathcal{B} \) such that the induced current \( T_{\hat{\mathcal{P}}} \) is a flat \( n \)-chain in \( \mathcal{B} \). Note that the general structure constructed thus far holds for generalized bodies. For any configuration \( \kappa \in \mathcal{L}_{Em}(\mathcal{B}, \mathbb{R}^n) \), the current \( \kappa_#(T_{\hat{\mathcal{P}}}) \) is a flat \( n \)-chain in \( \mathbb{R}^n \), and the operations \( \Psi(\kappa_#(\partial T_{\hat{\mathcal{P}}})) \) and \( d\Psi(\kappa_#(T_{\hat{\mathcal{P}}})) \) are well defined.

**Definition 16.** A generalized body is a set \( \hat{\mathcal{P}} \subset \mathcal{B} \) such that the induced current
\[
T_{\hat{\mathcal{P}}} = \hat{\mathcal{P}} \cdot \omega \, d\mathcal{L}^n,
\]
is a flat \( n \)-chain in \( \mathcal{B} \).

By [Fed69] Section 4.1.24] the current \( T_{\hat{\mathcal{P}}} \) is a rectifiable \( n \)-current or an integral flat \( n \)-chain in \( \mathcal{B} \). Moreover, we have
\[
F(T_{\hat{\mathcal{P}}}) = M(T_{\hat{\mathcal{P}}}) = L^n(\hat{\mathcal{P}}).
\]
It is recalled ([Fed69] Section 3.2.14]) that a set \( E \) is said to be \( m \)-rectifiable if there exists a Lipschitz function mapping some bounded subset of \( \mathbb{R}^m \) onto \( E \). The above definition of generalized bodies implies that a generalized body may be characterized as an \( n \)-rectifiable set in \( \mathcal{B} \), or alternatively, as an \( L^n \)-summable set in \( \mathcal{B} \). The class of generalized admissible bodies is
\[
\hat{\Omega}_\mathcal{B} = \left\{ T_{\hat{\mathcal{P}}} \mid \hat{\mathcal{P}} \subset \mathcal{B}, T_{\hat{\mathcal{P}}} \in F_n(\mathcal{B}) \right\}.
\]
As mentioned in Chapter 3, \( \hat{\Omega}_\mathcal{B} \) will have the structure of a Boolean algebra if \( \mathcal{B} \) was postulated to be a bounded set. Since \( N_n(\mathcal{B}) \subset F_n(\mathcal{B}) \), it is clear that \( \Omega_\mathcal{B} \subset \hat{\Omega}_\mathcal{B} \). Given \( T_{\hat{\mathcal{P}}}, T_{\hat{\mathcal{P}}'} \in \hat{\Omega}_\mathcal{B} \) clearly \( T_{\hat{\mathcal{P}}} \cup T_{\hat{\mathcal{P}}'} \) is an element of \( \hat{\Omega}_\mathcal{B} \). Contrary to the previous definition
of bodies, a generalized body needs not be a set of finite perimeter. Although \( \hat{\mathcal{P}} \) is a bounded set, its measure theoretic boundary, \( \Gamma (\hat{\mathcal{P}}) \), may be unbounded in the sense that \( H^{n-1} (\Gamma (\hat{\mathcal{P}})) = \infty \). Generally speaking, the boundary of a rectifiable set may not be a rectifiable set. A classical example of such a generalized body in \( \mathbb{R}^2 \) is the Koch snowflake. In [Sil06], such a body is referred to as a rough body.

**Remark 17.** It is noted that although every generalized body \( \hat{\mathcal{P}} \) induces an integral flat \( n \)-chain, not every integral flat represents a generalized body. However, it seems plausible that a flat \( n \)-class, introduced in [Zie62], is in one to one correspondence with the class of generalized bodies. This issue will not be considered in this work.

Considering a generalized surface, we first note that for a generalized body \( T_{\hat{\mathcal{P}}} \), \( \partial T_{\hat{\mathcal{P}}} \) is a flat \( (n-1) \)-chain in \( \mathcal{B} \). In addition, the following argument ([Fle66, Lemma 2.1]) indicates that the restrictions of flat chains to general Borel subsets are not necessarily flat chains. Let \( H_{\lambda,s} \) denote the closed half space defined by the linear functional \( \lambda : \mathbb{R}^n \to \mathbb{R} \) such that

\[
H_{\lambda,s} = \{ x \in \mathbb{R}^n \mid \lambda(x) \geq s \}.
\]

For a body \( T_{\hat{\mathcal{P}}} \) and a closed half-space \( H_{\lambda,s} \) the current \( \partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} \) is defined as \( T_{\hat{\mathcal{P}}} \downarrow \gamma_{\lambda,s} \) where \( \gamma_{\lambda,s} \) is the characteristic function of the half-space \( H_{\lambda,s} \). Since \( \gamma_{\lambda,s} \) defines a flat 0-cochain, we may apply Equation (5.10) and obtain

\[
\partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s}) = \partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} + T_{\hat{\mathcal{P}}} \downarrow \partial H_{\lambda,s}.
\]

Let \( T_{\hat{\mathcal{P}}} \in F_{K,n}(\mathcal{B}) \) be a generalized body in \( \mathcal{B} \) supported in a compact subset \( K \) of \( \mathcal{B} \), so that \( \partial T_{\hat{\mathcal{P}}} \) is a flat \( (n-1) \)-chain, and consider the chain \( \partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} \). One has,

\[
F_K (\partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s}) = F_K (\partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} + \partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s}) - \partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s})) ,
\]

\[
\leq F_K (\partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} - \partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s})) + F_K (\partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s})) ,
\]

\[
\leq F_K (\partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} - \partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s})) + F_K (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s}) ,
\]

\[
\leq M (\partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} - \partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s})) + M (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s}) .
\]

Since \( T_{\hat{\mathcal{P}}} \) is a chain of finite mass, \( M (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s}) < \infty \). In addition

\[
\int_{-\infty}^{\infty} M (\partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} - \partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s})) \, ds = M (T_{\hat{\mathcal{P}}}) ,
\]

and so we can show that \( M (\partial T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s} - \partial (T_{\hat{\mathcal{P}}} \downarrow H_{\lambda,s})) < \infty \) only for \( L^1 \)-almost every \( s \in \mathbb{R} \).
CHAPTER 9. GENERALIZED BODIES AND GENERALIZED SURFACES

In order to define a generalized material surface we follow [Sil06] where the various properties of flux over fractal boundaries are investigated.

**Definition 18.** For a generalized body \( \hat{\mathcal{P}} \), the subset \( \hat{\mathcal{S}} \subset \Gamma(\hat{\mathcal{P}}) \) is said to be a *trace* if there exists a set of finite perimeter \( M \) such that \( \hat{\mathcal{S}} = \Gamma(\hat{\mathcal{P}}) \cap M \) and \( H^{n-1}(\Gamma(\hat{\mathcal{P}}) \cap \Gamma(M)) = 0 \). Each trace \( \hat{\mathcal{S}} \) is associated with a unique flat \((n-1)\)-chain \( T_{\hat{\mathcal{S}}} \) given by

\[
T_{\hat{\mathcal{S}}} = \partial T_{\hat{\mathcal{P}} \cap M} - \partial T_{M \downarrow \hat{\mathcal{P}}}.
\]

For each \( \omega \in \mathcal{D}^{n-1}(\mathcal{B}) \) we have

\[
T_{\hat{\mathcal{S}}} (\omega) = \int_{\hat{\mathcal{P}} \cap M} \omega(e_1 \wedge \cdots \wedge e_n) dL^n - \int_{\Gamma(M) \cap \hat{\mathcal{P}}} \omega(\vec{T}_{\partial M}) dH^{n-1},
\]

where \( \vec{T}_{\partial M} \) is defined as in Equation (3.13). The set \( M \), of finite perimeter, is referred to as the generator of the trace \( \hat{\mathcal{S}} \) and it is shown in [Sil06] that \( \hat{\mathcal{S}} \) depends on \( M \) only through the intersection of \( \partial T_{\hat{\mathcal{P}}} \) with \( M \).

The collection of generalized material surfaces is defined as

\[
\partial \hat{\Omega}_B = \left\{ T_{\hat{\mathcal{S}}} \mid \hat{\mathcal{S}} \text{ is a trace in } \mathcal{B} \right\}.
\]

We note that by Proposition 10, for all \( T_{\hat{\mathcal{S}}} \in \partial \hat{\Omega}_B \) and \( v \in W_\kappa \), the multiplication \( v \kappa \# (T_{\hat{\mathcal{S}}} ) \) is an \( n \)-tuple of flat \((n-1)\)-chains. Thus, by Theorem 14 the Cauchy flux is naturally extended to the Cartesian product \( W_\kappa \times \kappa \left( \partial \hat{\Omega}_B \right) \).
CHAPTER 10

Virtual strains and the principle of virtual work

For \( T_\hat{\rho} \in \partial \hat{\Omega}_B \) and \( v \in W_\kappa, \partial (v_\kappa# (T_\hat{\rho})) \) is an \( n \)-tuple of flat \((n - 1)\)-chains in \( B \), whose components are defined by

\[
[\partial (v_\kappa# (T_\hat{\rho}))]_i = \partial (v_i \kappa# (T_\hat{\rho})).
\]

Thus, \( \Psi (\partial (v_\kappa# (T_\hat{\rho}))) \) is a well defined action of an \( n \)-tuple of flat \((n - 1)\)-cochains on an \( n \)-tuple of flat \((n - 1)\) chains. Applying Equation 5.10 for each component we obtain

\[
\sum_{i=1}^{n} \Psi_i (\partial (v_i \kappa# (T_\hat{\rho}))) = \sum_{i=1}^{n} \Psi_i (v_i \kappa# (\partial T_\hat{\rho})) - \sum_{i=1}^{n} \Psi_i (d\alpha_{v_i} \downarrow \kappa# (T_\hat{\rho})).
\]

Here \( \alpha_{v_i} \) is the flat 0-chain defined in Section 5.

The terms on the right-hand side of the equation above may be interpreted as follows. The term \( \sum_{i=1}^{n} \Psi_i (v_i \kappa# (\partial T_\hat{\rho})) \) is interpreted as the virtual power performed by the surface forces for the virtual velocity \( v \) on the boundary of the body \( T_\hat{\rho} \) at the configuration \( \kappa \). Next, for \( -\Psi (\partial (v_\kappa# (T_\hat{\rho}))) = -d\Psi (v_\kappa# (T_\hat{\rho})) \), the \( n \)-tuple of flat \( n \)-cochains \(-d\Psi \) is viewed as the body force. Thus the term \( -d\Psi (v_\kappa# (T_\hat{\rho})) \) is interpreted as the virtual power performed by the body forces along the virtual velocity \( v \) on the body \( T_\hat{\rho} \) at the configuration \( \kappa \). Finally, \( \sum_{i=1}^{n} \Psi_i (d\alpha_{v_i} \downarrow \kappa# (T_\hat{\rho})) \) is interpreted as the virtual power performed by the Cauchy flux along the derivative of the virtual velocity \( v \) on the body \( T_\hat{\rho} \) at the configuration \( \kappa \). The last term is traditionally viewed as the virtual power performed by the stress which we will formally present in Chapter 7.

An internal virtual velocity is viewed as an element upon which the Cauchy flux will act. Thus, a generalized internal virtual velocity is defined as an \( n \)-tuple of flat \((n - 1)\)-chains in \( \kappa \{ B \} \). A typical internal virtual velocity will be denoted by \( \chi \) and is viewed as a velocity gradient or a linear strain-like entity. Clearly, not every internal virtual velocity is derived from an external virtual velocity. Motivated by the above physical interpretation and the classical formulation of the principle of virtual work,
we introduce the kinematic interpolation map
\begin{equation}
\epsilon : \kappa(\Omega_B) \times W_k \rightarrow [F_{n-1}(\kappa(B))]^n
\end{equation}
such that each component is given by
\begin{equation}
(\epsilon(\kappa#(T_P), v))_i = v_i \partial \kappa#(T_P) - \partial (v_i \kappa#(T_P)),
\end{equation}
and for a compact set \( K \), for which \( \kappa\{\hat{P}\} \subset K \) we note that
\begin{equation}
F_K(v_i \partial \kappa#(T_P) - \partial (v_i \kappa#(T_P))) \leq F_K(v_i \partial \kappa#(T_P)) + F_K(v_i \kappa#(T_P)),
\end{equation}
\begin{equation}
\leq \left( \sup_{x \in K} |v_i(x)| + n\Sigma_{\phi,K} \right) F_K(\partial \kappa#(T_P)) + \left( \sup_{x \in K} |v_i(x)| \right) F_K(\kappa#(T_P)),
\end{equation}
\begin{equation}
\leq (n + 2) \| v_i \|_{\Sigma,K} F_K(\kappa#(T_P)).
\end{equation}
Note that the map \( \epsilon \) is disjointly additive in the first argument and is linear in the second argument. By Equation (10.4) \( \epsilon \) is continuous with respect to the flat norm of \( \kappa#(T_P) \) and the \( K \)-Lipschitz semi-norm of \( \nu \) for any compact \( K \), such that \( \hat{P} \subset K \). An internal virtual velocity \( \chi \) is said to be compatible if there are \( \hat{P} \in \hat{\Omega}_B \) and \( v \in W_k \) such that
\begin{equation}
\chi = \epsilon(\kappa#(T_P), v).
\end{equation}
Given a compatible virtual internal velocity \( \chi = \epsilon(\kappa#(T_P), v) \) we may write,
\begin{equation}
\Psi(\epsilon(\kappa#(T_P), v)) = \sum_{i=1}^{n} \Psi_i(v_i \partial \kappa#(T_P) - \partial (v_i \kappa#(T_P))),
\end{equation}
\begin{equation}
= \sum_{i=1}^{n} \Psi_i(v_i \kappa#(\partial T_P)) - \sum_{i=1}^{n} d\Psi_i(v_i \kappa#(T_P)),
\end{equation}
\begin{equation}
= \sum_{i=1}^{n} d\kappa_{v_i} \wedge \Psi_i(\kappa#(T_P)),
\end{equation}
and obtain
\begin{equation}
\Psi(\nu \kappa#(\partial T_P)) - d\Psi(\nu \kappa#(T_P)) = \Psi(\epsilon(\kappa#(T_P), v)),
\end{equation}
for all $T_p \in \hat{\Omega}_B$ and $v \in W_k$. We view the last equation as a generalization of the principle of virtual power.
CHAPTER 11

Stress

Applying the representation theorem of flat cochains, a Cauchy flux is represented by an $n$-tuple of flat $(n - 1)$-forms in $\kappa \{ B \}$. Let $\Psi_i$ denote the flat $(n - 1)$-cochain associated with the $i$-th component of the Cauchy flux. Then, $D\Psi_i$ will be used to denote its representing flat $(n - 1)$-form. The $n$-tuple of flat $(n - 1)$-forms in $\kappa \{ B \}$ representing the Cauchy flux will be denoted by $D\Psi$ and will be referred to as the Cauchy stress.

Using the representation theorem for flat forms we obtain an integral representation of the principle of virtual power given in Equation (10.7). The virtual power performed by surface forces is represented by

\[ \sum_{i=1}^{n} \Psi_i (v_i \kappa_\# (\partial T P)) \]

\[ = \sum_{i=1}^{n} \left( \kappa_\# (d (\kappa v_i \wedge \Psi_i)) \right) (T P), \]

\[ = \sum_{i=1}^{n} \int_P \tilde{d} (v_i D\Psi_i (\kappa (x))) \left( D\kappa(x)(e_1) \wedge \cdots \wedge D\kappa(x)(e_n) \right) dL^n_x, \]

\[ = \sum_{i=1}^{n} \int_P \tilde{d} (v_i D\Psi_i (\kappa (x))) \left( e_1 \wedge \cdots \wedge e_n \right) J\kappa(x) dL^n_x. \]

Equations (4.21) and (4.9) were used in the first and second lines. As noted above, the Cauchy stress $D\Psi$ is an $n$-tuple of flat $(n - 1)$-forms, and by the definition of flat forms (Definition 2), each component of the stress is an essentially bounded, $L^n$-integrable, $(n - 1)$-form whose weak exterior derivative is an essentially bounded, $L^n$-integrable $n$-form. By applying Equation (5.10) to the integrand of Equation (11.1), we note that

\[ \tilde{d} (v_i D\Psi_i) = \tilde{d} v_i \wedge D\Psi_i + v_i \wedge \tilde{d} D\Psi_i. \]

Thus, $D\Psi_i$ and $\tilde{d} D\Psi_i$ may be changed on a set of $L^n$-measure zero without affecting the value of the Cauchy flux on $\kappa_\# (\partial T \bar{P})$ for any virtual velocity $v$. For a generalized
material surface $T_{S}$, the representation of the stress as an equivalence class of $L^n$-integrable functions, may seem to be problematic as the current $T_{S}$ is supported on a set of $L^n$-measure zero. It may appear as though one can change the Cauchy flux without changing its representing flat from. In order to resolve this issue, we note that in order to apply the integral representation of the Cauchy flux we must first apply Theorem 1 and represent the chain $T_{S}$ by Lebesgue integrable vector fields in the form $T_{S} = L^n \wedge \eta + \partial (L^n \wedge \xi)$. Thus, changing the flat form of a set of $L^n$-measure zero will not effect the Cauchy flux.

The virtual power performed by body forces is represented by

$$(11.3) \quad - \sum_{i=1}^{n} d\Psi_i \left( v_i \kappa_{\#} \left( T_{\phi} \right) \right)$$

$$= - \sum_{i=1}^{n} \kappa_{\#} \left( \alpha_{v_i} \wedge d\Psi_i \right) \left( T_{\phi} \right),$$

$$= - \sum_{i=1}^{n} \int_{\phi} \left( v_i \tilde{d}D\Psi_i \left( \kappa \left( x \right) \right) \right) \left( D\kappa \left( x \right) \left( e_1 \right) \wedge \cdots \wedge D\kappa \left( x \right) \left( e_n \right) \right) dL^n_x,$$

$$= - \sum_{i=1}^{n} \int_{\phi} \left( v_i \tilde{d}D\Psi_i \left( \kappa \left( x \right) \right) \right) \left( e_1 \wedge \cdots \wedge e_n \right) J_{\kappa} \left( x \right) dL^n_x.$$

The virtual power performed by internal forces is represented by

$$(11.4) \quad \sum_{i=1}^{n} dv_i \wedge \Psi_i \left( \kappa_{\#} \left( T_{\phi} \right) \right)$$

$$= \sum_{i=1}^{n} \kappa_{\#} \left( d\alpha_{v_i} \wedge \Psi_i \right) \left( T_{\phi} \right),$$

$$= \sum_{i=1}^{n} \int_{\phi} \left( \tilde{d}v_i \wedge D\Psi_i \left( \kappa \left( x \right) \right) \right) \left( D\kappa \left( x \right) \left( e_1 \right) \wedge \cdots \wedge D\kappa \left( x \right) \left( e_n \right) \right) dL^n_x,$$

$$= \sum_{i=1}^{n} \int_{\phi} \left( \tilde{d}v_i \wedge D\Psi_i \left( \kappa \left( x \right) \right) \right) \left( e_1 \wedge \cdots \wedge e_n \right) J_{\kappa} \left( x \right) dL^n_x.$$

For $\kappa : \mathcal{B} \to \mathbb{R}^n$, a Lipschitz map, $\kappa_{\#}\Psi$ is an $n$-tuple of flat $(n-1)$-cochains in $\mathcal{B}$. Each cochain $\kappa_{\#}\Psi_i$ is represented by a flat $(n-1)$-form $D_{\kappa_{\#}\Psi_i} = \kappa_{\#}D_{\Psi_i}$. The associated $n$-tuple of flat $(n-1)$-forms, $\kappa_{\#}D_{\Psi}$ is identified as the Piola-Kirchhoff stress

$$(11.5) \quad \left( \kappa_{\#}D_{\Psi} \left( x \right) \right)_i = J_{\kappa} \left( x \right) D_{\Psi_i} \left( \kappa \left( x \right) \right).$$
Concluding remarks and further points of research

This thesis demonstrates again that the fundamental notions of continuum mechanics may be generalized by applying the tools of geometric measure theory. Identifying bodies as currents in $\mathbb{R}^n$ led to the definition of Lipschitz embedding configurations and locally Lipschitz virtual velocities. A generalized stress theory was presented in which the stress was identified with an $n$-tuple of flat $(n - 1)$-forms. This generalized stress theory enables the inclusion of $n$-rectifiable sets into the class of admissible bodies. The class of generalized bodies, viewed as flat $n$-chains, serve as an extension to the class of sets of finite perimeter representing taken as the class of admissible bodies. The inclusion of flat $n$-chains in the class of admissible bodies implies minimal restrictions on the boundary of bodies. Thus, sets of fractal boundary, rough bodies, were shown to be admissible bodies. In addition, a density transport theorem was formulated within the proposed framework and it was shown to be analogous to Reynolds’ transport theorem.

Further research is suggested in order to investigate further applications to the theory to the generalization of some fundamental notions such as:

- The mechanics of $r$-dimensional bodies for $r < n$ (analogous to the theory of plates and shells) may be introduced by the examination of bodies represented by $r$-currents.
- A generalized Reynolds’ transport theorem which
  – applies to rough bodies,
  – includes explicit time dependent properties by representing the property by a time dependent cochain.

In addition, an extension of the theory to the general setting of differentiable manifolds devoid of any metric structure or parallelism structure would be an interesting program to pursue.
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