New iterative linear matrix inequality based procedure for $H_2$ and $H_\infty$ state feedback control of continuous-time polytopic systems

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Summary
This article provides new linear matrix inequality (LMI) sufficient conditions for a generalized robust state feedback control synthesis problem for linear continuous-time polytopic systems. This generalized problem includes the robust stability, $H_2$-norm, and $H_\infty$-norm problems as special cases. Using a novel general separation result, which separates the state feedback gain from the Lyapunov matrix but with the state feedback gain synthesized from the slack variable, then allows the formulation of LMI sufficient conditions for the generalized problem. Compared to existing parameterized LMI based conditions, where auxiliary scalar parameters are introduced in order to include the quadratic stability conditions (ie, assuming a constant Lyapunov matrix) as a special case, the proposed new conditions are true LMIs and contain as a particular case the optimal quadratic stability solution. Utilizing any initial solution derived by the quadratic or some existing methods as a starting solution, we propose an algorithm based on an iterative procedure, which is recursively feasible in each update, to compute a sequence of nonincreasing upper bounds for the $H_2$-norm and $H_\infty$-norm. In addition, if no feasible initial solution can be found for some uncertain systems using any existing methods, another algorithm is presented that offers the possibility of obtaining a robust stabilizing gain. Numerical examples from the literature demonstrate that our algorithms can provide less conservative results than existing methods, and they can also find feasible solutions where all other methods fail.

KEYWORDS
$H_2$ and $H_\infty$ norms, LMI relaxations, polytopic uncertainties, robust state feedback control, slack variables, uncertain linear systems

1 | INTRODUCTION

The design of robust $H_2$-norm and $H_\infty$-norm controllers for linear time-invariant systems with polytopic uncertainties has received considerable attention in the past decades. Many studies devoted to investigating robust stability and
state feedback controller synthesis are based on linear matrix inequalities (LMIs) as they can be solved efficiently. Several methods were presented to achieve quadratic stability for such systems using a single, parameter-independent Lyapunov function. To overcome the conservatism due to the use of a single Lyapunov function, approaches using parameter-dependent Lyapunov functions have been widely addressed more recently. However, the use of a parameter-dependent Lyapunov function leads to a nonconvex robust control design problem due to the multiplication between the Lyapunov and system dynamic matrices.

To address this nonconvexity, a significant breakthrough in robust stability analysis for continuous-time systems was made by Geromel et al, where an extended LMI condition was presented, which separates the system matrices from the Lyapunov matrix but with the system matrices multiplied by two auxiliary slack variables instead. Then, imposing the condition that these slack variables are parameter-independent allows the formulation of a convex problem with a parameter-dependent Lyapunov matrix. The corresponding stability conditions for discrete-time systems appeared in Reference 9. Based on the slack variable approach, extended synthesis LMI conditions for discrete-time systems appeared in References 10 and 11, where they showed that with a simple choice of the slack variables, their conditions include the quadratic stability conditions as a special case. The synthesis problem for continuous-time systems turned out to be much more challenging. Apkarian et al proposed dilated LMI based conditions to compute $H_2$ upper bound through the use of a slack variable. However, their conditions do not contain quadratic stability conditions as a special case. To reduce this conservatism, some recent work concentrated on parameterized LMIs involving a slack matrix variable as well as an extra scalar parameter to cope with $H_2$-norm and $H_\infty$-norm control synthesis problems. However, parameterized LMIs become linear only if this scalar parameter is fixed. The major disadvantage of these approaches is that the scalar parameter belongs to an unbounded set; this results in a large computational burden because exhaustive searches on the scalar parameter need to be performed. In analogy with the work for the discrete-time case whose corresponding scalar parameter belongs to a bounded domain, Rodrigues et al proposed new extended parameterized LMI characterizations for continuous-time systems with two scalar parameters based on a change of variables and the elimination lemma. One parameter belongs to the bounded set $(-1,1)$, and another parameter, though belonging to an unbounded set, is restricted in a bounded subset through numerical experimentation. Although the search domain is limited considerably, the results may still be conservative.

The common idea of the above-mentioned class of methods is to decouple the system matrices and the Lyapunov matrix through an application of the elimination lemma or Finsler’s lemma first, and then the corresponding infinite number of conditions are converted into $N$ (the number of the vertices of the uncertain system) parameterized LMIs based on affine parameter-dependent Lyapunov matrix. Good reviews of the main existing approaches for robust stability analysis and controller synthesis problems can be found in References 6 and 26. Another type of existing approaches working with robust stabilization for continuous-time polytopic systems was addressed in the work of Geromel and Korogui, which is to convert the original infinite number of conditions into a set of $N^2$ bilinear matrix inequality (BMI) conditions, and then transform these into parameterized LMIs by separating the feedback gain and the affine Lyapunov matrix. This work was further improved by Oliveira et al. Again, an exhaustive search procedure for a scalar parameter has to be implemented in their results in order to capture the quadratic stability conditions as a particular case.

In this article, we first provide a unified parameter-dependent BMI necessary and sufficient conditions for robust stability, $H_2$-norm, and $H_\infty$-norm state feedback control synthesis problems in Section 2. We then extend the approach of Geromel and Korogui in Section 3 to derive a finite set of BMI sufficient conditions. We also outline and extend the approach of Oliveira et al to separate the terms in the bilinear product by introducing parameterized LMIs, which are shown to include the quadratic conditions as a special case. In Section 4, we propose new sufficient conditions for the aforementioned BMI conditions in terms of LMI conditions, without any scalar parameter, by using a novel separation result. Our proposed LMI conditions contain one known solution to the BMI conditions as a particular case. Therefore, starting with a feasible solution to the BMI conditions found by an existing method (eg, quadratic stability conditions, Oliveira et al or Geromel and Korogui), the upper bounds on the $H_2$-norm and $H_\infty$-norm are then iteratively reduced based on an update procedure. In the case that no feasible solution can be found using any existing method, we modify our method in Section 5 to provide the possibility of finding a feasible solution. The key idea of this method is to first relax the robust stability conditions and then iteratively search for the robust gain for increasingly stricter stability conditions until the original stability conditions are satisfied. We illustrate the effectiveness of our two algorithms through four examples from the literature in Section 6. Finally, we summarize our conclusions in Section 7.
Notation. The notation used in this article is fairly standard. \( \mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m \times m} \) denote the sets of real numbers, \( n \)-dimensional real vectors, and \( n \times m \) real matrices, respectively. \( I_n \) and \( 0_{m \times n} \) denote the \( n \)-dimensional identity matrix and the \( m \times n \) zero matrix, respectively, with the subscripts dropped if they can be inferred from context. The notation \( A > 0 \) or \( A < 0 \) denotes matrix \( A \) is symmetric positive or negative definite, respectively. \( S^m \) and \( S^n_m \) denote, respectively, the set of symmetric and symmetric positive definite matrices of dimension \( m \times m \). If \( A \in \mathbb{R}^{m \times n} \), then \( H(A) := A + A^T \). For integer \( N \geq 1 \), we define \( \mathcal{N} := \{1, \ldots, N\} \). A congruence transformation refers to effecting a congruence \( T \) that has full column rank, on a matrix inequality \( A > 0 \) to deduce that \( T^TAT > 0 \). A Schur complement argument refers to the result

\[
\begin{bmatrix}
A & B \\
\star & C
\end{bmatrix} > 0 \iff A > 0, \quad C - B^TA^{-1}B > 0 \iff C > 0, \quad A - BC^{-1}B^T > 0,
\]

where \( \star \) refers to a term readily inferred from symmetry.

2 \ PARAMETER-DEPENDENT BMI FORMULATION FOR THE ROBUST DESIGN PROBLEM

In this section, we define the robust state feedback stabilization, \( H_2 \)-norm, and \( H_\infty \)-norm design problems and embed them in a unified generalized robust design problem (GRDP) involving a parameter-dependent Lyapunov matrix variable. We give necessary and sufficient conditions for the solution of this problem in the form of parameter-dependent BMIs. We also show that in the quadratic case, when the Lyapunov function is restricted to be parameter-independent, the BMIs reduce to LMIs.

2.1 \ Robust design problem

Consider the uncertain continuous-time linear system

\[
\begin{bmatrix}
\dot{x}(t) \\
z(t)
\end{bmatrix} = \begin{bmatrix}
A(\alpha) & B(\alpha) & B_w(\alpha) \\
C(\alpha) & D(\alpha) & D_w(\alpha)
\end{bmatrix} \begin{bmatrix}
x(t) \\
u(t) \\
w(t)
\end{bmatrix},
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^{n_u} \), \( w(t) \in \mathbb{R}^{n_w} \), and \( z(t) \in \mathbb{R}^n \) are the state, input, exogenous disturbance, and cost signals, respectively, and where the system distribution matrices of appropriate dimensions lie in an uncertainty polytope spanned by the convex combination of \( N \) given vertices

\[
\begin{bmatrix}
A(\alpha) & B(\alpha) & B_w(\alpha) \\
C(\alpha) & D(\alpha) & D_w(\alpha)
\end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix}
A_i & B_i & B_{wi} \\
C_i & D_i & D_{wi}
\end{bmatrix},
\]

where \( \alpha \) is a time-invariant parameter belonging to the unit simplex

\[
\Omega_N = \left\{ \alpha \in \mathbb{R}^N : \alpha_i \geq 0, \forall i \in \mathcal{N}; \sum_{i=1}^{N} \alpha_i = 1 \right\}.
\]

The single state feedback controller \( u(t) = Kx(t) \), where \( K \in \mathbb{R}^{n_u \times n} \), is to be designed, such that the closed-loop system

\[
\begin{bmatrix}
\dot{x}(t) \\
z(t)
\end{bmatrix} = \begin{bmatrix}
A(\alpha) + B(\alpha)K & B_w(\alpha) \\
C(\alpha) + D(\alpha)K & D_w(\alpha)
\end{bmatrix} \begin{bmatrix}
x(t) \\
w(t)
\end{bmatrix},
\]

is robustly stable (the eigenvalues of \( A(\alpha) + B(\alpha)K \) lie in the open left half plane for all \( \alpha \in \Omega_N \)) and also with an upper bound for \( H_2 \)-norm or \( H_\infty \)-norm performance.
The following result, which is a simple extension of standard well-known results in the literature, gives parameter-dependent BMI equivalent conditions for RDP.

**Lemma 1** (See the work of Geromel and Korogui,[27] Altun,[15] and Rodrigues et al[21]). Consider the closed-loop system in (1).

1. **(Stability)** System (1) is robustly stable if there exists a parameter-dependent Lyapunov matrix $P(\alpha) \in S^+_n$ such that, for all $\alpha \in \Omega_N$,
   \[
   \mathcal{H}((A(\alpha) + B(\alpha)K)P(\alpha)) < 0. \tag{2}
   \]

2. **($H_2$-norm)** System (1) with $D_w(\alpha) = 0$ is robustly stable and its $H_2$-norm is less than $\mu$ if there exist parameter-dependent matrices $P(\alpha) \in S^+_n$ and $W(\alpha) \in S^n$ such that, for all $\alpha \in \Omega_N$, $\text{Trace}(W(\alpha)) < \mu^2$,
   \[
   \begin{bmatrix}
   -P(\alpha) & * \\
   (C(\alpha) + D(\alpha)K)P(\alpha) & -W(\alpha)
   \end{bmatrix} < 0, \quad \begin{bmatrix}
   \mathcal{H}((A(\alpha) + B(\alpha)K)P(\alpha)) & * \\
   (C(\alpha) + D(\alpha)K)P(\alpha) & -I_{n_w}
   \end{bmatrix} < 0. \quad \tag{3}
   \]

3. **($H_\infty$-norm)** System (1) is robustly stable and its $H_\infty$-norm is less than $\gamma$ if there exists a parameter-dependent matrix $P(\alpha) \in S^+_n$ such that, for all $\alpha \in \Omega_N$,
   \[
   \begin{bmatrix}
   \mathcal{H}((A(\alpha) + B(\alpha)K)P(\alpha)) & * & * \\
   (C(\alpha) + D(\alpha)K)P(\alpha) & -I_{n_w} & * \\
   B_w(\alpha)^T & D_w(\alpha)^T & -\gamma^2 I_{n_w}
   \end{bmatrix} < 0. \tag{4}
   \]

### 2.2 Parameter-dependent BMI formulation for a generalized robust design problem

An inspection of the conditions (2)-(4) and $P(\alpha) > 0$ verifies that they are special cases of the following more general problem considered in this article.

**Problem 1** (GRDP). Let $F \in \mathbb{R}^{m,n}$ and for all $i \in \mathcal{N}$, let $A_i \in \mathbb{R}^{m \times n}, B_i \in \mathbb{R}^{m \times n}$, and $T_i \in S^m$ be given and, for any $\alpha \in \Omega_N$ let
   \[
   \begin{bmatrix}
   A(\alpha) & B(\alpha) & T(\alpha)
   \end{bmatrix} := \sum_{i=1}^{N} a_i \begin{bmatrix}
   A_i & B_i & T_i
   \end{bmatrix}.
   \]

Find a state feedback controller $K \in \mathbb{R}^{n_w \times n}$ and a parameter-dependent Lyapunov matrix $P(\alpha) \in S^+_n$ such that for all $\alpha \in \Omega_N$,
   \[
   T(\alpha) + \mathcal{H}((A(\alpha) + B(\alpha)K)P(\alpha)F^T) < 0. \tag{5}
   \]

**Remark 1.** Note that $T(\alpha), A(\alpha),$ and $B(\alpha)$ are in general augmented parameter-dependent system matrices, that may also depend affinely on other variables, and $F$ is a constant matrix:

- For the stability case in (2): $T(\alpha) = 0$, $A(\alpha) = A(\alpha)$, $B(\alpha) = B(\alpha)$, $F = I$.
- For the $H_2$ case
  - For the first condition in (3): $T(\alpha) = \begin{bmatrix}
  -P(\alpha) & 0 \\
  0 & -W(\alpha)
  \end{bmatrix}$, $A(\alpha) = \begin{bmatrix}
  0 & 0 \\
  C(\alpha) & 0
  \end{bmatrix}$, $B(\alpha) = \begin{bmatrix}
  0 & D(\alpha) \\
  0 & 0
  \end{bmatrix}$, $F = \begin{bmatrix}
  I & 0 \\
  0 & 0
  \end{bmatrix}$.
  - For the second condition in (3): $T(\alpha) = \begin{bmatrix}
  0 & B_w(\alpha) \\
  B_w(\alpha)^T & -I
  \end{bmatrix}$, $A(\alpha) = \begin{bmatrix}
  A(\alpha) & 0 \\
  0 & 0
  \end{bmatrix}$, $B(\alpha) = \begin{bmatrix}
  B(\alpha) & 0 \\
  0 & 0
  \end{bmatrix}$. $F = \begin{bmatrix}
  I & 0 \\
  0 & 0
  \end{bmatrix}$.
- For the $H_\infty$ case in (4): $T(\alpha) = \begin{bmatrix}
  0 & 0 & B_w(\alpha) \\
  0 & -I & D_w(\alpha) \\
  B_w(\alpha)^T & D_w(\alpha)^T & -\gamma^2 I
  \end{bmatrix}$, $A(\alpha) = \begin{bmatrix}
  A(\alpha) & 0 \\
  0 & C(\alpha)
  \end{bmatrix}$, $B(\alpha) = \begin{bmatrix}
  B(\alpha) & D(\alpha) \\
  0 & 0
  \end{bmatrix}$. $F = \begin{bmatrix}
  I & 0 \\
  0 & 0
  \end{bmatrix}$.
Remark 2.

- Note that $P(\alpha)$ in (5) is required to be a general function of $\alpha$ for (2)-(4) to be necessary and sufficient. For a practical implementation, we follow the standard practice\textsuperscript{12,18,21,26,27} and restrict the Lyapunov function to be affine in the parameters:

$$P(\alpha) = \sum_{j=1}^{N} a_j P_j, \quad P_j \in S_+^n, \quad \forall j \in \mathcal{N}. \quad (6)$$

This implies that the conditions of Lemma 1 are only sufficient and this will introduce some conservatism.

- The inequality in (5) is parameter-dependent which leads to an infinite number of conditions.
- The parameter-dependent inequality (5) is bilinear due to the product terms $KP(\alpha)$ (unless $P(\alpha)$ is independent of $\alpha$).

2.3 The quadratic GRDP

The most common, simplest, though generally conservative, approach to deal with the first issue in Remark 2, which also resolves the other issues, is to assume that the Lyapunov matrix is independent of $\alpha$ so that $P(\alpha) = P$ for all $\alpha$. In this case, Problem 1 reduces to the following simple LMI problem: Find $M \in \mathbb{R}^{n \times n}$ and $P \in S_+^n$ such that the following LMIs

$$T_i + H(A_iPF^T + B_iMF^T) < 0, \quad (7)$$

are satisfied for all $i \in \mathcal{N}$, with $K = MP^{-1}$.

3 EXTENSIONS OF CURRENT RESULTS FOR AFFINE LYAPUNOV FUNCTIONS

In this section, we first extend the approach of Geromel and Korogui\textsuperscript{27} to derive a finite set of $N^2$ BMI sufficient conditions for the parameter-dependent BMI conditions (5) under the assumption of an affine parameter-dependent Lyapunov matrix, where the bilinearity is due to the multiplication between the Lyapunov and state feedback gain matrices. We then outline and extend the work of Oliveira et al\textsuperscript{26} to transform the BMI generalized problem into a parameterized LMI problem by introducing constrained slack variables to separate the bilinear terms.

3.1 Finite set of BMI sufficient conditions for GRDP

In order to address the second issue in Remark 2, we next extend the approach of Geromel and Korogui\textsuperscript{27} and convert the infinite-dimensional conditions (5) of Problem 1 into a finite number of BMIs by introducing additional $N^2$ symmetric matrix variables.

**Lemma 2.** Let all variables be as given in Problem 1 and assume that $P(\alpha)$ has the form (6). Then there exists a feasible solution to Problem 1 if, for all $i, j \in \mathcal{N}$, there exist $P_j \in S_+^n$ and $V_{ij} \in S^m$ such that

$$\sum_{i,j=1}^{N} a_i a_j V_{ij} \leq 0 \forall \alpha \in \Omega_N, \quad (8)$$

$$T_i + H(A_iP_jF^T + B_iKP_jF^T) < V_{ij}. \quad (9)$$

**Proof.** Multiplying (9) by $a_i a_j$, for all $i, j \in \mathcal{N}$ and summing gives $T(\alpha) + H((A(\alpha) + B(\alpha)K)P(\alpha)F^T) < \sum_{i,j=1}^{N} a_i a_j V_{ij} \leq 0$. □

Since characterizing (8) is intractable, we follow the work of Geromel and Korogui\textsuperscript{27} and replace it by tractable constraints, at the expense of introducing further conservatism, to define the following problem which requires a finite number of sufficient conditions for the solution of Problem 1.
Problem 2. Let all variables be as given in Problem 1. Find $K \in \mathbb{R}^{n_x \times n}$ and for all $i, j \in \mathcal{N}$, find $P_j \in \mathbb{S}^n$ and $V_{ij} \in \mathbb{S}^m$ such that (9) and
\begin{equation}
V_{ij} + V_{ji} \geq 0, \quad 1 \leq i < j \leq N, \quad \sum_{i=1}^{N} (V_{ij} + V_{ji}) \leq 0, \quad j = 1, \ldots, N,
\end{equation}
are satisfied.

Remark 3. Note that the LMI constraints in (10) are sufficient for the nonlinear constraints in (8).27 Note also that for control applications, the parameter $T_i$ and $H(A_i P_j F^T)$ are typically linear in the system matrices and other variables, for example, $P_j$. This makes Problem 2 tractable (since the inequality (9) linear) for system analysis ($K = 0$ or $K$ is given), while for controller synthesis, (9) is bilinear due to the product terms in $KP_j$; this results in nonconvexity of Problem 2. Note finally that when $P_j = P$ and $V_{ij} = 0$ for all $i, j \in \mathcal{N}$, (9) reduces to (7).

### 3.2 Parameterized LMI sufficient conditions that include the quadratic conditions

At this stage, almost all other work in the literature uses the result of Geromel et al.8 which, by introducing two slack variables $F$ and $G$, separates $K$ and $P_j$ in the terms $K P_j$ in (9) and replaces them with the terms $K F$ and $K G$. Then to enforce linearity and at the same time ensure that the solution includes the quadratic case, one of the slack variables is restricted and a scalar parameter is introduced, such that $(F, G) \rightarrow (G, rG)$. One of the least conservative approaches to deal with robust stability synthesis design in the form of (9) is the work in Oliveira et al.26 The following result is a simple extension of lemma 9 in Oliveira et al. from the robust stabilization problem in (2) to the more general Problem 2 (which includes the robust $H_\infty$-norm and $H_\infty$-norm control problems).

Theorem 1. Let all variables be as given in Problem 1. Then Problem 2 has a feasible solution if there exist $Y \in \mathbb{R}^{n_x \times n}$, $M \in \mathbb{R}^{n_x \times n}$, and, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}^n$, $V_{ij} \in \mathbb{S}^m$, and a nonzero scalar $r \in \mathbb{R}$ such that (10) and
\begin{equation}
\begin{bmatrix}
T_i + H\left( A_i P_j F^T + \frac{1}{r} B_i M F^T \right) - V_{ij} & * \\
\left( F(r P_j - Y^T) + \frac{1}{r} B_i M \right)^T & -H(Y)
\end{bmatrix} < 0,
\end{equation}
are satisfied, in which case the state feedback gain $K$ is given by $K = M Y^{-1}$. Furthermore, if the quadratic condition (7) holds, then the condition (11) holds for a sufficiently large $r$.

Proof. Effecting the congruence transformation $\begin{bmatrix} I_m & 0 \\ \frac{1}{r} K^T B_i^T & I_n \end{bmatrix}$ on (11) and using the fact that $M = K Y$ shows that (11) is equivalent to
\begin{equation}
\begin{bmatrix}
\frac{1}{r} K^T B_i^T & 0 \\
I_n & I_m
\end{bmatrix}
\begin{bmatrix}
T_i + H\left( A_i P_j F^T + B_i K P F^T \right) - V_{ij} & * \\
\left( F(r P_j - Y^T) - \frac{1}{r} B_i K Y^T \right)^T & -H(Y)
\end{bmatrix} \begin{bmatrix} I_m & 0 \\ \frac{1}{r} K^T B_i^T & I_n \end{bmatrix} < 0.
\end{equation}
This shows that (11) $\Rightarrow$ (9), so the first part is proved. For the second, setting $P_j = P$, $Y = r P$, and $V_{ij} = 0$, (11) reduces to
\begin{equation}
\begin{bmatrix}
T_i + H\left( A_i P F^T + B_i K P F^T \right) & * \\
(B_i K P)^T & -2r P
\end{bmatrix} < 0.
\end{equation}
Using a Schur complement argument, the above condition is equivalent to
\begin{equation}
T_i + H\left( A_i P F^T + B_i K P F^T \right) + \frac{1}{2r} B_i K P K^T B_i^T < 0,
\end{equation}
which is equivalent to (7) for a sufficiently large $r$.

Remark 4. Note that if (11) is satisfied, then $Y$ is nonsingular since $H(Y) > 0$ and so $K$ can be obtained from $M = K Y$. Additionally, Theorem 1 shows that (11) contains the quadratic condition (7) as a special case. Hence, Theorem 1 can
be guaranteed to provide no more conservative results than quadratic conditions. However, (11) contains an unbounded tuning scalar parameter, thus exhaustive scalar searches of this parameter have to be implemented.

4 | NEW RESULTS FOR GRDP USING AFFINE LYAPUNOV FUNCTIONS

In this section, we propose a general separation result to provide sufficient conditions for Problem 2 by removing the associated bilinearity with the help of slack variables. Our conditions are attractive from a computational point of view since they are expressed as true LMIs. Moreover, the proposed conditions contain one known solution to Problem 2 as a particular case. By adopting this attractive property, we then present an algorithm to reduce the upper bound on the $H_2$-norm or $H_\infty$-norm via an iterative procedure if one solution to Problem 2 is known.

4.1 | A general separation result

The next theorem is a general result which allows us to separate the product of two variables of the form $EX$ in a BMI without any conservatism by replacing the term $EX$ by the bilinear terms $EY$ and $EZ$, where $Y$ and $Z$ are slack variables. It also suggests a procedure for restricting these two slack variables to allow a linear solution that captures a given feasible solution of the BMI.

**Theorem 2.** Let $T \in S^m$, $E, F \in \mathbb{R}^{m \times n}$ and $X \in S^n$. Then

$$T + \mathcal{H}(EX^T) < 0, \quad X > 0,$$

(12)

if and only if there exist $Z \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times m}$ such that

$$\begin{bmatrix}
T - \mathcal{H} \left( (E - \frac{1}{2}F)Y \right) & \ast & \ast \\
Y & -X & \ast \\
((E + \frac{1}{2}F)Z)^T & 0 & X - \mathcal{H}(Z)
\end{bmatrix} < 0. \quad (13)$$

**Proof.** Effecting the congruence

$$\begin{bmatrix}
I_n & 0 & 0 \\
(E - \frac{1}{2}F)^T & I_n & 0 \\
(E + \frac{1}{2}F)^T & 0 & I_n
\end{bmatrix}$$
on (13) shows that it is equivalent to

$$\begin{bmatrix}
T - (E - \frac{1}{2}F)X(E - \frac{1}{2}F)^T + (E + \frac{1}{2}F)X(E + \frac{1}{2}F)^T & \ast & \ast \\
Y - X(E - \frac{1}{2}F)^T & -X & \ast \\
(X - Z)(E + \frac{1}{2}F)^T & 0 & X - \mathcal{H}(Z)
\end{bmatrix} < 0. \quad (14)$$

This shows that (13) $\Rightarrow$ (12) since $\mathcal{H}(EX^T) = -(E - \frac{1}{2}F)X(E - \frac{1}{2}F)^T + (E + \frac{1}{2}F)X(E + \frac{1}{2}F)^T$. Furthermore, if (12) is satisfied, then an inspection of (14) shows that by defining

$$Y = X \left( E - \frac{1}{2}F \right)^T, \quad Z = X,$$

(15)

then (12) $\Rightarrow$ (14) which is equivalent to (13) and completes the proof. \hfill \Box

**Remark 5.** Note that, provided that $F$ is constant, then both (12) and (13) are bilinear. However, (13) is linear in $X$ and, provided that the slack variables $Y$ and $Z$ are suitably restricted, for example, $Y=WY_0$ and $Z=WZ_0$ where $W$ is a variable while $Y_0$ and $Z_0$ are constant, then (13) becomes linear by considering $M:=EW$ and $W$ as decision variables, although it will be only sufficient for (12) because of this restriction.
4.2 LMI sufficient conditions that include one known solution

The following corollary is a direct application of Theorem 2 to Problem 2, and with a choice of the slack variables \( Y \) and \( Z \) suggested by (14)-(15) that will ensure that the solution includes at least one known feasible solution to Problem 2.

**Corollary 1.** Suppose that \( K \in \mathbb{R}^{n \times n} \) and for all \( i, j \in \mathcal{N}, \tilde{T}_i \in \mathbb{S}^n, \tilde{P}_j \in \mathbb{S}_+^n \), and \( \tilde{V}_{ij} \in \mathbb{S}^n \) solve Problem 2 so that

\[
\tilde{V}_{ij} + \tilde{V}_{ji} \geq 0, \quad 1 \leq i < j \leq N; \quad \sum_{i=1}^{N} (\tilde{V}_{ij} + \tilde{V}_{ji}) \leq 0, \quad j = 1, \ldots, N; \quad \tilde{T}_i + H(A_i \tilde{P}_j F^T + B_i \tilde{K} \tilde{P}_j F^T) < \tilde{V}_{ij}.
\]

If there exist \( Y \in \mathbb{R}^{n \times n} \) and \( M \in \mathbb{R}^{n \times n} \) and, for all \( i, j \in \mathcal{N}, \) there exist \( P_j \in \mathbb{S}_+^n \) and \( V_{ij} \in \mathbb{S}^n \) such that (10) and

\[
\begin{bmatrix}
T_i + H \left(A_i P_j F^T - (B_i M - \frac{1}{2} F Y) \tilde{P}_j (B_i \tilde{K} - \frac{1}{2} Y \tilde{F})^T\right) - V_{ij} & * & * \\
Y \tilde{P}_j (B_i \tilde{K} - \frac{1}{2} F)^T & -P_j & * \\
((B_i M + \frac{1}{2} F Y) \tilde{P}_j )^T & 0 & P_j - H(Y \tilde{P}_j)
\end{bmatrix} < 0,
\]

are satisfied, then with \( K = MY^{-1} \), and for all \( i, j \in \mathcal{N}, \) \( T_i, P_j, \) and \( V_{ij} \) also solve Problem 2. Furthermore, condition (16) is satisfied by

\[
(T_i, P_j, V_{ij}, Y, M) := (\tilde{T}_i, \tilde{P}_j, \tilde{V}_{ij}, I_n, \tilde{K}).
\]

**Proof.** For the first part, inequality (9) can be rewritten as the first inequality in (12) with \( T = T_i + H(A_i P_j F^T) - V_{ij}, \) \( E = B_i K, F = F, \) and \( X = P_j, \) so that it follows from Theorem 2 that (9) is satisfied if and only if there exist matrices \( Y_{ij} \) and \( Z_{ij}, \forall i, j \in \mathcal{N} \) such that

\[
\begin{bmatrix}
T_i + H \left(A_i P_j F^T - (B_i K - \frac{1}{2} F Y) Y_{ij}\right) - V_{ij} & * & * \\
Y_{ij} & -P_j & * \\
((B_i K + \frac{1}{2} F Y) Z_{ij})^T & 0 & P_j - H(Z_{ij})
\end{bmatrix} < 0.
\]

Setting \( Y_{ij} = Y \tilde{P}_j (B_i \tilde{K} - \frac{1}{2} F)^T \) and \( Z_{ij} = Y \tilde{P}_j \) give (16). Furthermore, When \( (T_i, P_j, V_{ij}, Y, M) := (\tilde{T}_i, \tilde{P}_j, \tilde{V}_{ij}, I, \tilde{K}), \) inequality (16) becomes

\[
\begin{bmatrix}
T_i + H \left(A_i \tilde{P}_j F^T - (B_i \tilde{K} - \frac{1}{2} F) \tilde{P}_j (B_i \tilde{K} - \frac{1}{2} F)^T\right) - \tilde{V}_{ij} & * & * \\
\tilde{P}_j (B_i \tilde{K} - \frac{1}{2} F)^T & -\tilde{P}_j & * \\
((B_i \tilde{K} + \frac{1}{2} F \tilde{P}_j )^T & 0 & -\tilde{P}_j
\end{bmatrix} < 0.
\]

Using a Schur complement argument shows that the above inequality is equivalent to \( \tilde{T}_i + H(A_i \tilde{P}_j F^T) + H(B_i \tilde{K} \tilde{P}_j F^T) < \tilde{V}_{ij} \) and proves the second part.

**Remark 6.** Note that \( \tilde{K} \) and \( \tilde{P}_j \) are given by a known feasible solution to Problem 2, so (16) is a true LMI and therefore it can be efficiently implemented by an LMI solver. Corollary 1 illustrates that if a feasible solution to Problem 2 can be found, then there exist solutions to Corollary 1. Furthermore, these solutions to Corollary 1 also solve Problem 2. Note also that the slack variable \( Y \) remains unconstrained since it is not restricted to have any definiteness or symmetry property, so it provides extra degrees of freedom to search for a better solution. Hence, we conclude that the new solution to Corollary 1 is at least as good as the previous known solution to Problem 2. Note also that for achieving linearity of the terms \( KY_{ij} \) and \( KZ_{ij} \) in (17), we impose the additional constraints \( Y_{ij} = Y \tilde{P}_j (B_i \tilde{K} - \frac{1}{2} F)^T \) and \( Z_{ij} = Y \tilde{P}_j \) so that \( M := KY \) and \( Y \) are decision variables.

**Remark 7.** Suppose that (16) is satisfied. Then \( H(Y \tilde{P}_j) - P_j > 0, \) and this, together with the fact that \( \tilde{P}_j \) and \( P_j \) are positive definite implies that \( Y \) is nonsingular. Thus the feedback gain \( K \) can always be recovered from \( M = KY. \)
As illustrated in Corollary 1, there must exist a feasible solution to Corollary 1 if a feasible solution to Problem 2 is available. In an optimization problem, this solution to Problem 2 may be chosen as the optimal solution obtained by an existing method (e.g., quadratic conditions, or indeed some other appropriate methods proposed in Theorem 1). This ensures that our solution is no more conservative than the optimal solution computed by any of these existing methods with one-step computation. Furthermore, this solution may be chosen as the current solution, which defines an iterative procedure if better solutions are required. We next take advantage of this useful property to present an algorithm to iteratively compute potentially less conservative bounds on the $H_2$- or the $H_{\infty}$-norm problems by utilizing the optimal solution computed by any existing method as a starting point.

**Algorithm 1.** Given $F, T, A_i, B_i$ for all $i \in N$, and tolerance level $tol$

1. **Initial solution:** Find a solution to Problem 2 by using the quadratic method of Section 2.3, other appropriate methods available in the literature (e.g., Theorem 1) or the methods of Section 5 below. Set $\bar{\mu} = \mu$ (or $\bar{\gamma} = \gamma$), $\bar{K} = K$ and $\bar{P}_j = P_j$ for all $j \in N$.
2. **Update:** Minimize $\mu$ (or $\gamma$) over the related variables in Corollary 1. Record $K, P_j, \mu$ (or $\gamma$).
3. **Stopping condition:** If $(\bar{\mu} - \mu) / \bar{\mu} \leq tol$ (or $(\bar{\gamma} - \gamma) / \bar{\gamma} \leq tol$) stop. Else set $\bar{K} = K$ and $\bar{P}_j = P_j$, $\bar{\mu} = \mu$ (or $\bar{\gamma} = \gamma$), and go to step 2.

### 5 LMI SUFFICIENT CONDITIONS WHEN NO KNOWN FEASIBLE SOLUTION EXISTS

As mentioned in the last section, the proposed Algorithm 1 requires a feasible solution to Problem 2 provided by an existing method as a starting point; then it can iteratively update the upper bounds of $H_2$-norm or $H_{\infty}$-norm. However, as illustrated in the work of Oliveira et al. and Rodrigues et al., these existing methods may fail to compute a feasible solution for some open-loop unstable uncertain system even when the system is known to be robustly closed-loop stabilizable. Hence, we next propose a modified method via an iterative procedure, based on the results in Corollary 1, to allow the possibility of finding a feedback gain that can stabilize the system when all other methods fail.

Our approach for finding a feasible solution to (12) uses the following general result that extends an idea used in Felipe et al. The result is based on Finsler’s Lemma and shows that a perturbed version of the BMI in (12) is always feasible and easily solvable.

**Lemma 3.** Let $T \in S^m, E, F \in \mathbb{R}^{m \times n}$ and $X \in S^n_+$ be given.

1. If $\text{rank}(F) = m$, then there exists $\beta \in \mathbb{R}$ such that
   \[ T + H((E - \beta F)XF^T) < 0. \]  
   (18)
2. If $\text{rank}(F) < m$, let $F_\perp$ denotes an arbitrary matrix whose columns form a basis for the null space of $F^T$ (i.e., $F^TF_\perp = 0$). Then there exists $\beta \in \mathbb{R}$ such that (18) is satisfied if and only if $F_\perp^TF_\perp < 0$.

**Proof.**

1. Suppose that $\text{rank}(F) = m$ and let $Z := T + H(EXF^T) \in S^m$ and $Y := 2FXF^T \in S^m$ so that (18) can be written as $Z - \beta Y < 0$. Since $\text{rank}(F) = m$ then $FXF^T > 0$ since $X > 0$. Therefore, $Y > 0$ and there always exists a $\beta \in \mathbb{R}$ such that $Z - \beta Y < 0$, for example, any $\beta$ larger than the largest eigenvalue of $Y^{-1}Z$.
2. Suppose that $\text{rank}(F) < m$. Since $X > 0$, then it has a Cholesky factorization $X = RR^T$, where $R \in \mathbb{R}^{n \times n}$ is nonsingular. Now, (18) can be rewritten as $Q - \mu B^TB < 0$ where $Q := T + H(EXF^T) \in S^m$, $B := RF^T \in \mathbb{R}^{n \times m}$ and $\mu := 2\beta \in \mathbb{R}$ with $\text{rank}(B) = \text{rank}(F) < m$. It follows from Finsler’s lemma (see, e.g., Oliveira and Skelton) that (18) is satisfied if and only if $F_\perp^TF_\perp < 0$ since $R$ is nonsingular.
Thus, if we cannot find a feasible solution to (12) using any of the current approaches, then Lemma 3 implies there always exists a solution to (18) for some $\beta \in \mathbb{R}$. If $\beta \leq 0$, we are done, otherwise we proceed as follows, where we carry out the analysis for robust stability since we only need a feasible stabilizing solution for the $H_2$-norm and $H_{\infty}$-norm problems.

We next introduce a slightly modified characterization of Hurwitz stability for the continuous-time closed-loop systems.

**Lemma 4.** The closed-loop system in (1) is robustly stable if, for all $i, j \in \mathcal{N}$, there exist $K \in \mathbb{R}^{n \times n}$, $\beta \leq 0$, $P_j \in \mathbb{S}^{n}_+$, and $V_{ij} \in \mathbb{S}^m$ such that (10) and

$$H((A_i - \beta I)P_j + B_iKP_j) - V_{ij} < 0,$$

(19)

are satisfied.

**Proof.** Multiplying (19) by $a_ia_j$, for all $i, j \in \mathcal{N}$, summing and using (10) yields

$$H((A(\alpha) + B(\alpha)K - \beta I)P(\alpha)) < \sum_{i,j=1}^{N} a_ia_jV_{ij} \leq 0,$$

which shows that $(A(\alpha) + B(\alpha)K - \beta I)$ is robustly stable. It follows that the closed-loop system in (1) is robustly stable for $\beta \leq 0$. \hfill $\blacksquare$

We next relax this characterization by removing the sign requirement on $\beta$ and consider the following problem:

**Problem 3.** Let all variables be as in Lemma 4. Find

$$\min\{\beta : K \in \mathbb{R}^{n \times n}, P_j \in \mathbb{S}^{n}_+, V_0 \in \mathbb{S}^m \text{ s.t. } (10) \text{ and } H(A_iP_j) + H(B_iKP_j) - V_{ij} < 2\beta P_j \text{ are satisfied, } \forall i, j \in \mathcal{N}\}. \quad (20)$$

Problem 3 is bilinear because of the product terms $KP_j$. Note that if we set $P_j = P$ and $V_{ij} = 0$ for all $i, j \in \mathcal{N}$, then it follows that the computation of the initial solution to Problem 3 can now be formulated by the following generalized eigenvalue problem (GEVP).

**Problem 4.** Let all variables be as in Lemma 4. Find

$$\min\{\beta : M \in \mathbb{R}^{n \times n}, P \in \mathbb{S}^{n}_+, H(A_iP) + H(B_iM) < 2\beta P \text{ are satisfied, } \forall i \in \mathcal{N}, \text{ with } K = MP^{-1}\}. \quad (21)$$

We will call this the relaxed quadratic stabilization problem since it follows from Lemma 3 (since $F = I$ for Problem 4) that it is always feasible. Furthermore, it is a GEVP since $P > 0$ and is therefore easily solvable. The key idea of our method is to find a feasible initial solution to Problem 3 by solving Problem 4, which means that the closed-loop system has all its eigenvalues to the left of the line $\text{Re}(s) = \beta$ in the complex plane (in general $\beta > 0$) instead of the open left half-plane required by Lemma 4. Subsequently, we try to use the degrees of freedom in the slack variable to obtain a solution for Problem 3 with a smaller value of $\beta$ through the following corollary, until $\beta \leq 0$.

**Corollary 2.** Let $K \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}$, and for all $i, j \in \mathcal{N}$, $P_j \in \mathbb{S}^{n}_+$, and $V_{ij} \in \mathbb{S}^m$ solve Problem 3 so that

$$V_{ij} + V_{ji} \geq 0, \quad 1 \leq i < j \leq N; \quad \sum_{i=1}^{N} (V_{ij} + V_{ji}) \leq 0, \quad j = 1, \ldots, N; \quad H((A_i - \beta I)P_j + B_iKP_j) - V_{ij} < 0. \quad (22)$$

If there exist $Y \in \mathbb{R}^{n \times m}$, $M \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}$, and, for all $i, j \in \mathcal{N}$, $P_j \in \mathbb{S}^{n}_+$, and $V_{ij} \in \mathbb{S}^m$ such that (10) and

$$\left[\begin{array}{ccc} H \left((A_i - \beta I)P_j - (B_iM - \frac{1}{2}Y)P_j(B_iK - \frac{1}{2}I)^T\right) & V_{ij} & P_j \\ YP_j(B_iK - \frac{1}{2}I)^T & -P_j & * \\ (B_iM + \frac{1}{2}Y)P_j^T & 0 & P_j - H(YP_j) \end{array}\right] < 0. \quad (23)$$
Remark 10. Note that although Algorithm 2 can guarantee that the computed sequence of 
more details.

Remark 8. Note that the inequalities in (23) can be written as

\[ (P_j, V_{ij}, Y, M, \beta) := (\bar{P}_j, \bar{V}_{ij}, I_n, \bar{K}, \bar{\beta}). \]

Proof. The inequality involving \( \beta \) in (20) can be rewritten as (9) with \( T_i = 0 \), \( A_i = A_i - \beta I \), \( B_i = B_i \), and \( F = I_n \), and so the result follows from Corollary 1.

Remark 9. It is worth mentioning that the traditional methods in the literature are based on parameterized LMIs, where an additional exhaustive search procedure on scalar parameters needs to be performed. Though the conditions of Corollary 2 also contain a scalar variable \( \beta \), which are quasiconvex and therefore can be solved efficiently via CVX toolbox using the bisection algorithm (see section 4.2.5 of Boyd and Vandenberghe\(^3\) for details). Moreover, in order to reduce ill-conditioning of Lyapunov matrices, in the practical implementation for Problem 4 and the conditions of Corollary 2 (which are homogeneous in the variables), we impose the constraint \( P > I_n \) for Problem 4 and \( I_n < P_j < \zeta I_n \) and take \( \zeta \) as the cost function to be minimized for Corollary 2 in the numerical examples later. See Boyd et al\(^1\) for more details.

Remark 10. Note that although Algorithm 2 can guarantee that the computed sequence of \( \beta \) is nonincreasing, the final converged value of \( \beta \) cannot be guaranteed to be nonpositive, even if Problem 3 is known to have a feasible solution for \( \beta \leq 0 \). Hence the algorithm has two stopping outcomes: (a) a stabilizing gain has been found in Step 1
TABLE 1 The minimum upper bounds on $H_2$-norm and $H_\infty$-norm comparisons for Example 1

| Method               | $\mu$        | Scalars | Method               | $\gamma$   | Scalars |
|----------------------|-------------|---------|----------------------|------------|---------|
| $H_2$ performance    |             |         | $H_\infty$ performance|            |         |
| Quadratic$^1$        | 2.5923      | —       | Quadratic$^1$        | 1.5776     | —       |
| Apkarian et al$^{12}$| 1.5526      | —       | Shaked$^{16}$        | 1.4782     | $r = 0.01$ |
| Xie$^{14}$           | 1.7025      | $r = 0.13$ | Xie$^{18}$          | 1.2416     | $r = 0.07$ |
| Altun$^{15}$         | 1.3564      | $r = 1.8$ | Rodrigues et al$^{21}$ | 1.2414     | $\epsilon = 0.1$, $\xi = 0.54$ |
| Theorem 1            | 2.1209      | $r = 7.7$ | Theorem 1            | 1.3062     | $r = 28.5$ |
| Algorithm 1          | 1.2185      | tol = 0.1% | Algorithm 1       | 1.0418     | tol = 0.1% |

or Step 2; (b) Algorithm 2 fails to find a stabilizing gain in Step 3. In addition, since $\beta$ is only a upper bound on the maximum real part of eigenvalues of the closed-loop system ($\lambda_{\max}(A(\alpha) + B(\alpha)K)$), Felipe et al$^{29}$ remarked that the actual value of $\lambda_{\max}(A(\alpha) + B(\alpha)K)$ could be negative and therefore the closed-loop system is Hurwitz stable even if $\beta > 0$. Hence, once a state feedback gain $K$ is computed in Step 2, the actual value of $\lambda_{\max}(A(\alpha) + B(\alpha)K)$ can be verified, a stabilizing gain is found and therefore Algorithm 2 terminates if it is negative. However, checking the actual value of $\lambda_{\max}(A(\alpha) + B(\alpha)K)$ in each iteration increases the complexity of Algorithm 2; this will be left to a future work.

6 | NUMERICAL EXAMPLES

In this section, we give four examples to illustrate the efficiency of our algorithms. The benefit of Algorithm 1 for the computation of less conservative upper bounds on the $H_2$-norm and $H_\infty$-norm for state feedback control design is illustrated by Examples 1 and 2. Subsequently, Examples 3 and 4 are presented to demonstrate the effectiveness of the proposed Algorithm 2 for robust stabilization. All algorithms are implemented in Matlab 9.0.0 (R2016a) using the CVX toolbox with MOSEK as solver$^{32,33}$ and running on an Intel(R) Xeon (R) CPU E5-1650, 3.5 GHz, Windows 7 Professional.

6.1 | Example 1

The problem of controlling a satellite system from References 15, 16, and 18 is considered in this example. The state space representation is given as follows:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(t), \quad z(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u(t),$$

where $k$ and $f$ denote the uncertain parameters of the system, whose uncertainty ranges are [0.09 0.4] and [0.0038 0.04], respectively. This system can be described as in Section 2.1 using $N = 4$ vertices.

Exhaustive searches on the scalar parameters are performed for current parameterized LMI based methods and Theorem 1. The comparison of the minimum upper bound $\mu$ on the $H_2$-norm and $\gamma$ on the $H_\infty$-norm obtained by Algorithm 1 and some methods in the literature is given in Table 1.

Starting with the solution provided by Theorem 1 and setting the tolerance level to be 0.1%, it can be noted from Table 1 that Algorithm 1 provides less conservative upper bounds on the $H_2$-norm and $H_\infty$-norm compared to the ones obtained from all other methods. In particular, Altun$^{15}$ gives a $\mu$ level of 1.3564 with optimized $\lambda = 1.8$, but Algorithm 1 obtains a final converged value of 1.2185 for the $H_2$-norm performance $\mu$ with state feedback gain $K = [-16.3676 -95.9845 -8.8811 -194.6665]$. Compared with the work of Rodrigues et al$^{21}$ which yields the value of 1.2414 for the $H_\infty$-norm performance $\gamma$, an upper bound of $\gamma = 1.0418$ can be achieved using Algorithm 1 with $K = -10^4 \times [0.1195 \ 1.2424 \ 0.0274 \ 1.6604]$. 

Reference:

1. HU and AIMOUKHA
2. Quadratic
3. Apkarian et al.
4. Xie
5. Altun
6. Theorem
7. Algorithm
8. Rodrigues et al.
9. Theorem
10. Algorithm
11. Table
12. Quadratic
13. Shaked
14. Xie
15. Altun
16. Theorem
17. Algorithm
18. Rodrigues et al.
19. Theorem
20. Algorithm
21. Rodrigues et al.
Figure 1 displays the relation between the computed values of $\mu$ and $\gamma$ through Algorithm 1 and the number of iterations for different values of $r$ as the initial point. It can be seen that the bounds $\mu$ and $\gamma$ are nonincreasing with the number of iterations and both converge to nearly the same final values independently of which value of $r$ is used as the initial point. This observed quadratic speed of convergence seems to be an interesting property of our iterative procedure, although a rigorous theoretical proof of this is beyond the scope of this article.

### 6.2 Example 2

Consider the following uncertain coupled spring-mass system taken from Reference 21:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ -\frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} w(t), \quad z(t) = \begin{bmatrix} 0.5k & 0.5k & -2k & -k \end{bmatrix} x(t) - 0.1u(t).$$

Here $x(t) = [x_1(t) x_2(t) x_3(t) x_4(t)]^T$ where the states $x_1(t)$ and $x_2(t)$ are the displacements of body 1 and body 2, respectively, while their respective velocities are represented by the states $x_3(t)$ and $x_4(t)$. The parameters $m_1$ and $m_2$ denote the masses of body 1 and body 2, respectively, and whose values are given as $m_1 = m_2 = 1$. The stiffness parameter $k$ is uncertain but is known to lie in the interval $[0.5, 2]$.

It has been suggested in Rodrigues et al.21 that in order to avoid the large computational burden caused by an exhaustive search procedure on scalar parameters, the search range of scalar variables is constrained to the following values (total of 12 searches): $\epsilon \in \{10^{-1}, 10^0\}$ and $\xi \in \{-0.9, -0.54, -0.18, 0.18, 0.54, 0.9\}$. For the other parameterized LMI based methods, the scalar is limited to thirteen logarithmically spaced values (total of 13 searches): $r \in \{10^{-6}, 10^{-5}, \ldots, 10^0, 10^1, \ldots, 10^6\}$. For the scalar parameter $r$ of Theorem 1, we select $r = 1$ instead of performing searches on $r$, to obtain a feasible solution used for the starting point of Algorithm 1.

Table 2 compares the results of $\mu$ achieved using our Algorithm 1 with the other existing approaches available in the literature as well as the associated solution time.

Among the current methods, Altun15 gives the minimum upper bound on the $H_2$-norm as $\mu = 1.4855$ with the mean solution time 2.65 seconds. Applying Theorem 1 gives the initial stabilizing state feedback gain $K_0 = [-0.8560 0.2864 \quad -1.7048 \quad -0.7639]$ in 0.34 second. Algorithm 1 shows improvement on $H_2$-norm performance with respect to Altun15 after 7 iterations and gives a converged value of $\mu = 0.5044$ after 15 iterations by setting $tol = 0.5\%$; the corresponding state
feedback gain is \( K = [-5.8340, 0.1948, -29.0254, -15.4930] \). The mean times for Algorithm 1 to take 7 iterations and 15 iterations are 3.05 and 6.14 seconds, respectively.

As shown in the second column in Table 3, the \( H_\infty \)-norm upper bound provided by Rodrigues et al\(^{21} \) is 1.9745. Starting with the initial stabilizing state feedback gain \( K_0 = [-0.6036, 0.0001, -1.0105, -0.4056] \) provided by Theorem 1, our Algorithm 1 can outperform Rodrigues et al\(^{21} \) after 6 iterations and finally gives \( \gamma = 1.1153 \) after 17 iterations (with \( tol = 0.5\% \)). The final resulting feedback gain is \( K = [-4.6271, -0.7696, -17.4697, -9.3864] \). Table 3 also gives the associated numerical complexity of the existing methods and Algorithm 1 for \( H_\infty \) control. For each LMI test, in contrast to the current approaches, the greater number of scalar variables and LMI rows in Corollary 1 is due to the additional matrix variables \( V_{ij} \). Note that parameterized LMI based methods require exhaustive search procedures on the scalar parameters while Algorithm 1 requires the repeated use of Corollary 1. The last column in Table 3 gives the total solution time for all methods to evaluate their computational burden.

Algorithm 1, which implements the computation of initial solution by Theorem 1 and 17 iterations to convergence, demands a longer solution time than the methods from the previous studies. However, it is worth mentioning that the number of LMI tests for the parameterized LMI methods is limited to at most 13 since the search range on scalar parameters is substantially constrained in this example, this leads to less computational time but the obtained results are in general suboptimal. Furthermore, the second row from the bottom in Table 3 indicates that Algorithm 1 is able to provide less conservative results than the ones in the literature after only 6 iterations, which takes a shorter computational time than these parameterized LMI methods.

### Table 2

| Method        | \( \mu \) | Scalars | \( T \): solution time |
|---------------|------------|---------|------------------------|
| Quadratic\(^1\) | 10.2217    | —       | 0.19 s                 |
| Xie\(^{14} \)   | 5.2704     | \( r = 0.1 \) | 2.89 s               |
| Apkarian et al\(^{12} \) | 3.2638 | —       | 0.20 s              |
| Altun\(^{15} \)   | 1.4855     | \( r = 0.1 \) | 2.65 s               |
| Theorem 1      | 6.8438     | \( r = 1 \) | 0.34 s               |
| Algorithm 1    | 1.1511     | \( it = 7 \) | 3.05 s               |
| Algorithm 1    | 0.5044     | \( it = 15 \) | 6.14 s               |

### Table 3

| Method        | \( \gamma \) | Scalars | \( V \) | \( R \) | \( T \): total solution time |
|---------------|--------------|---------|--------|--------|-------------------------------|
| Quadratic\(^1\) | 8.5569       | —       | 15     | 15     | 0.18 s                        |
| Shaked\(^{16} \) | 7.9236       | \( r = 0.01 \) | 41     | 23     | 2.52 s                        |
| Xie\(^{18} \)   | 2.3847       | \( r = 10 \) | 41     | 31     | 2.65 s                        |
| Rodrigues et al\(^{21} \) | 1.9745 | \( e = 1, \xi = 0.18 \) | 41     | 23     | 2.33 s                        |
| Theorem 1       | 9.4149       | \( r = 1 \) | 153    | 74     | 0.28 s                        |
| Algorithm 1     | 1.6275       | \( it = 6 \) | 153    | 90     | 1.99 s                        |
| Algorithm 1     | 1.1153       | \( it = 17 \) | 153    | 90     | 5.19 s                        |

### Example 3

Consider the following example for \( n = 4, n_u = 1, \) and \( N = 2 \) presented in Geromel and Korogui\(^{27} \) and Rodrigues et al\(^{28} \). The vertex system matrices of the continuous-time polytopic system are given by

\[
A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-12 - 3d & -12 - 3d & -25 & -1
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
0 \\
-6 \\
6
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-12 + 3d & -12 + 3d & -25 & -1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
6
\end{bmatrix}
\]
TABLE 4  The maximum values of $d$ within stability domain and the associated numerical complexity comparisons ($V$: scalar variables, $R$: LMI rows, $T$: total solution time)

|        | ATB01$^{12}$ | Sha01$^{16}$ | EH04$^{13}$ | GK06$^{27}$ | OdOP11$^{26}$ | ROC18$^{21,28}$ | Algorithm 2 |
|--------|--------------|--------------|-------------|------------|-------------|---------------|--------------|
| $d_{\text{max}}$ | 5.44         | 10.37        | 13.30       | 11.62      | 12.60       | 14.29         | 19.42        |
| $V$    | 40           | 40           | 40          | 120        | 80          | 40            | 40           |
| $R$    | 24           | 16           | 24          | 84         | 52          | 16            | 68           |
| $T$    | 0.20 s       | 2.57 s       | 2.59 s      | 3.59 s     | 3.48 s      | 2.45 s        | 22.04 s      |

where $d \in \mathbb{R}$ is a free parameter. For each value of $d \geq 0$, the robust stability conditions available in the literature have been solved to yield a state feedback gain $K$ to guarantee that $A(K) := \text{co}\{A_i + B_i K : i \in \mathcal{N}\}$ is robustly stable.

The scalar searches performed by the parameterized LMI based methods described in previous works follow from the values suggested in Example 2,26,28 so that a total of 13 searches for Sha01$^{16}$ EH04$^{13}$ GK06$^{27}$ and OdOP11$^{26}$ (Lemma 9) are conducted while a total of 12 searches are conducted for ROC18$^{21,28}$. The maximum value of $d$ assuring that the existing robust stability conditions and Algorithm 2 are feasible as well as the associated numerical complexity can be seen in Table 4 below.

The first row in Table 4 demonstrates that ROC18$^{21,28}$ provides the best performance of among the existing approaches in the literature; their robust stability conditions are always feasible for all $0 \leq d \leq 14.29$. However, when $d$ is slightly larger than 14.29 all the above methods give infeasibility for finding a stabilizing state feedback gain. Next, applying the proposed Algorithm 2 to this system for $d > 14.29$, setting $it_{\text{max}} = 30$, $tol = 0.1\%$, and $\epsilon = 10^{-3}$, where $\epsilon$ is the tolerance of the bisection algorithm.31 It can be verified that Algorithm 2 can stabilize the uncertain system up to $d \leq 19.42$. For $d = 19.42$, Algorithm 2 gives a feasible state feedback gain $K = [-13.0706 - 23.9313 - 0.0693 - 1.3499]$. This controller gives the maximum real part of eigenvalues of $A(K)$ as $-0.0536$ which verifies the stability of the closed-loop system.

Regarding the associated computational burden for each LMI test, Sha01$^{16}$ and ROC18$^{21,28}$ demand a lower number of scalar variables and LMI rows followed by ATB01$^{12}$ EH04$^{13}$ OdOP11$^{26}$ Algorithm 2, and GK06$^{27}$ respectively. Moreover, the last row in Table 4 gives the solution time for each method. It can be observed that the solution time for ROC18$^{21,28}$ (total of 12 LMI tests) is 2.45 seconds. For $d = 19.42$, it takes 22.04 seconds for Algorithm 2 to find the stabilizing gain, where a total of 80 LMI tests are conducted. Moreover, exhaustive numerical experiments show that the uncertain system becomes increasingly harder to be stabilized as $d$ increases, this leads to the number of LMI tests and therefore the solution time by Algorithm 2 growing accordingly. When $d$ is smaller, for example, Algorithm 2 can stabilize the uncertain system for $d = 16.5$ in 5.79 seconds (total of 21 LMI tests) and for $d = 18.5$ in 13.82 seconds (total of 50 LMI tests), respectively. In particular, when $d = 14.29$, Algorithm 2 solves 10 LMI tests in 2.75 seconds to find a stabilizing gain, the solution time is comparable to ROC18$^{21,28}$. It is also important to mention that even for $d = 16.5$, the current parameterized LMI based methods cannot provide feasible solutions even if an exhaustive search procedure on scalar parameters is implemented. In conclusion, Algorithm 2 can be used as an alternative way to find a feasible solution for robust stabilization when all other methods fail but at the price of a potentially larger computational effort.

6.4  Example 4

In this example, we compare the performance of Algorithm 2 with the previous methods in robust stabilization for uncertain systems through statistical analysis. The database of open-loop unstable uncertain systems from Oliveira et al$^{26}$ (available for download) is used in this example. We only consider uncertain systems that are guaranteed to be robustly stabilizable by a state feedback gain, but not quadratically stabilizable. In what follows, 100 systems for each combination of the dimension $n = 2, \ldots, 5$ and $N = 2, \ldots, 5$ and $n_u = 1$ are tested. We use the values of scalar parameters suggested in Example 3 for the current methods$^{26,28}$ while Algorithm 2 sets $tol = 0.1\%$, $\epsilon = 10^{-3}$, and $it_{\text{max}} = 100$.

As can be observed from Table 5, for the overall success rate of systems stabilized by all methods for all 1600 systems, ROC18$^{21,28}$ provides the highest success rate of 56.06% among the methods from the previous works, but Algorithm 2
shows a clear improvement over ROC18\textsuperscript{21,28} with 83.88\% success rate. Additionally, compared with the robust stabilization conditions of Felipe et al.,\textsuperscript{22} which use polynomial Lyapunov matrices, and which give 81.3\% success rate, Algorithm 2 still has a higher success rate even though we only use affine Lyapunov matrices. Moreover, the last row in Table 5 gives the mean solution time per system spent by each method. Compared with ROC18\textsuperscript{21,28} which demands 2.55 seconds per system, the mean solution time per system taken by Algorithm 2 is 31.11 seconds per system. Note finally that Algorithm 2 can stabilize 87.44\% of the systems in 60.21 seconds per system if $\text{tol} = 0.01\%$, $\epsilon = 10^{-4}$, and $\text{it}_{\text{max}} = 200$. The statistical results of this example corroborate our expectation that Algorithm 2 can provide better robust stabilization performance than the methods from the previous works but at the expense of more computational effort.

| $(n,N)$ | ATB01\textsuperscript{12} | Sha01\textsuperscript{16} | EH04\textsuperscript{13} | GK06\textsuperscript{27} | OdOP11\textsuperscript{26} | ROC18\textsuperscript{21,28} | Algorithm 2 |
|-----------|-----------------|----------------|----------------|----------------|----------------|----------------|--------------|
| (2, 2)    | 64              | 9              | 100            | 4              | 13             | 100            | 68           |
| (2, 3)    | 33              | 18             | 58             | 11             | 13             | 57             | 75           |
| (2, 4)    | 2               | 16             | 50             | 6              | 8              | 52             | 79           |
| (2, 5)    | 2               | 24             | 60             | 13             | 19             | 63             | 89           |
| (3, 2)    | 44              | 11             | 82             | 6              | 11             | 81             | 75           |
| (3, 3)    | 2               | 11             | 49             | 8              | 10             | 52             | 83           |
| (3, 4)    | 0               | 19             | 38             | 10             | 13             | 39             | 80           |
| (3, 5)    | 0               | 21             | 33             | 11             | 15             | 37             | 94           |
| (4, 2)    | 32              | 17             | 75             | 9              | 12             | 74             | 81           |
| (4, 3)    | 1               | 13             | 49             | 9              | 12             | 54             | 87           |
| (4, 4)    | 0               | 21             | 41             | 5              | 5              | 42             | 84           |
| (4, 5)    | 0               | 13             | 28             | 5              | 6              | 32             | 93           |
| (5, 2)    | 18              | 17             | 77             | 13             | 15             | 78             | 78           |
| (5, 3)    | 0               | 24             | 54             | 12             | 14             | 61             | 87           |
| (5, 4)    | 0               | 17             | 39             | 10             | 11             | 45             | 96           |
| (5, 5)    | 0               | 17             | 26             | 5              | 8              | 30             | 93           |

**TABLE 5** Number of uncertain systems (among 100) stabilized and the associated solution time per system demanded

| Success rate | 12.38\% | 16.75\% | 53.69\% | 8.56\% | 11.56\% | 56.06\% | 83.88\% |
|--------------|---------|---------|---------|--------|---------|---------|---------|
| Average time per system | 0.21 s | 2.71 s | 2.73 s | 7.04 s | 6.68 s | 2.55 s | 31.11 s |

7 | CONCLUSIONS

New LMI conditions for robust $H_2$-norm and $H_\infty$-norm state feedback control synthesis of continuous-time systems with polytopic uncertainty have been presented. The design conditions for robust stabilization, $H_2$-norm, and $H_\infty$-norm state feedback controllers were first expressed in terms of a unified BMI formulation. We next provided new sufficient conditions in terms of LMIs to this BMI problem by using a new separation result. It has been shown that any known solution to the BMI problem can be included as a particular case for the proposed conditions. Therefore, the new solution computed by the proposed conditions would be at least as good as the best previous known solution. We then proposed a new algorithm based on an iterative procedure, which ensures recursive feasibility, to compute increasingly tighter upper bounds for both the $H_2$-norm and $H_\infty$-norm. In addition, we also proposed another algorithm that can potentially find a robustly stabilizing gain when all other existing methods fail. Four numerical examples from the literature were presented to compare the proposed two algorithms with some methods from previous work in terms of performance and the associated numerical complexity. The results corroborate that the proposed algorithms can provide superior performance compared with the existing methods but at the expense of an increased computational effort.

The proposed approaches are based on the general separation result of Theorem 2, which is applicable to a wider range of problems than the ones considered in this work, such as the problem of output feedback control for both continuous
and discrete-time systems. We are currently investigating the extension of our method to handle the output feedback problem and compare with the methods available in the literature, for example, two-step methods\textsuperscript{34–38} and parameterized LMI based methods.\textsuperscript{39,40} Other future research directions include modifying our approach to allow the use of polynomial parameter-dependent Lyapunov matrices and improving the convergence speed to reduce the computational effort.

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