A New Proof of The Strong Subadditivity Theorem

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It is well known that the strong subadditivity theorem is hold for classical system, but it is very
difficult to prove that it is hold for quantum system. The first proof of this theorem is due to Lieb
by using the Lieb’s theorem. Here we use the conditions obtained in our previous work of matrix
analysis method to give a new proof of this famous theorem. This new proof is very elementary,
it only needs to carefully analyse the minimal value of a function. This proof also shows that the
conditions obtained in our previous work are stronger than the strong subadditivity theorem.

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I. INTRODUCTION

Entropy is an important concept not only for physics but also for information science. From the definition of
entropy, we can get some simple properties of it, such as concavity, continuity property, additivity and subadditivity
[1]. But some other properties is not so obvious, such as the strong subadditivity theorem (SSA). Among all of the
properties of entropy, the most famous one is the SSA, and it is very difficult to prove this theorem for quantum
system. The content of this theorem can be expressed as the following: two overlapping subsystem $AB$ and $BC$, the
entropy of their union ($ABC$) plus the entropy of their intersection ($B$) does not exceed the sum of the entropies of
the subsystems ($AB$ and $BC$) [2], that is

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}).$$

(1)

where $S(\rho) = -\text{Tr}(\rho \ln \rho)$. It is well known that this theorem is true for classical information theory, but to prove this
theorem is true for quantum system is very difficult. This theorem is first conjectured to be true for the quantum
system by Lanford and Robibson [3]. The first proof of this conjecture is given by Lieb et al. several years later. This
proof is based on the concave of the function $S(\rho_{12}) - S(\rho_1)$ in $\rho_{12}$ [4]. Another proof based on the same fact was
proposed by Uhlmann [5,6].

Recently, quantum information theory attracts more and more attentions for its misterious properties and its
potential applications in science and technology [7,8]. The SSA plays an important role in this new field [9] too.
The fundamental scource in quantum information is entanglement between many particles which can be viewed as
the relations between the partial particles. So distinguish whether a set of the partial particles come from a single
state ($N$-representability problem) [10,11] and further to obtain its entanglement property are important in quantum
information while SSA gives a strong constraint on the partial particles and the whole system. The convenience of
the SSA is that it has explicit physical and manipulating meaning. So it is a convenient necessary criterion for the
$N$-representability problem. Recently, we use the matrix analysis method [12] to get some necessary conditions for
the $N$-presentability problem. We find that using these conditions we can get a new proof for the SSA. Our new proof
is elementary, we need only to use the Lagrange multiplier method and carefully analyse the minimum of a function.

II. THE THEOREM AND THE PROOF

There is a density matrix $\rho_{ABC}$, where the particle $A, B$ and $C$ are in $L$–dimension, $M$–dimension and
$N$–dimension Hilbert space, respectively. Let $\{\lambda_{AB}^{(1)}, \lambda_{AB}^{(2)}, \ldots, \lambda_{AB}^{(LM)}\}, \{\lambda_{BC}^{(1)}, \lambda_{BC}^{(2)}, \ldots, \lambda_{BC}^{(MN)}\}, \{\lambda_B^{(1)}, \lambda_B^{(2)}, \ldots, \lambda_B^{(M)}\}$
and $\{\lambda_{ABC}^{(1)}, \lambda_{ABC}^{(2)}, \ldots, \lambda_{ABC}^{(LMN)}\}$ are the eigenvalues of the density matrix $\rho_{AB}, \rho_{BC}, \rho_B$ and $\rho_{ABC}$, respectively (where

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\( \rho_{AB}, \rho_{BC} \) and \( \rho_B \) are gotten by tracing the other particles from \( \rho_{ABC} \), and they are arranged in increasing order. We defined vectors \( \lambda_{AB} = \{ \lambda_{AB}^{(1)}, \lambda_{AB}^{(2)}, \ldots, \lambda_{AB}^{(LM)} \} \), \( \lambda_{BC} = \{ \lambda_{BC}^{(1)}, \lambda_{BC}^{(2)}, \ldots, \lambda_{BC}^{(MN)} \} \), \( \lambda_B = \{ \lambda_B^{(1)}, \lambda_B^{(2)}, \ldots, \lambda_B^{(M)} \} \), \( \lambda_{ABC} = \{ \lambda_{ABC}^{(1)}, \lambda_{ABC}^{(2)}, \ldots, \lambda_{ABC}^{(LMN)} \} \) and \( \lambda_{AB}^B = \{ \sum_{i=1}^{L} \lambda_{AB}^{(i)} \lambda_{BC}^{(i)}, \sum_{i=1}^{M} \lambda_{BC}^{(i)} \lambda_{AB}^{(i)} \} \), \( \lambda_B^B = \{ \sum_{i=1}^{L} \lambda_B^{(i)} \} \), \( \lambda_{AB}^B = \{ \sum_{i=1}^{L} \lambda_{AB}^{(i)} \} \), and \( \lambda_{ABC}^B = \{ \sum_{i=1}^{L} \lambda_{ABC}^{(i)} \} \). Using the matrix analysis method, we get the following two lemmas on the eigenvalues [12].

**Lemma 1.** Using the notes defined before, we can get the relations between the eigenvalues of \( \rho_{BC}, \rho_{AB}, \rho_B \) and \( \rho_{ABC} \) as

\[
\lambda_{ABC}^{AB} > \lambda_{AB} \tag{2.1}
\]

\[
\lambda_{ABC}^{BC} > \lambda_{BC} \tag{2.2}
\]

\[
\lambda_{ABC}^{B} > \lambda_{B} \tag{2.3.1}
\]

\[
\lambda_{ABC}^{B} > \lambda_{B} \tag{2.3.2}
\]

**Lemma 2.** Suppose \( \text{rank}(\rho_{ABC}) = LMN - Ls \), \( \text{rank}(\rho_{BC}) = MN - s \), \( \text{rank}(\rho_{AB}) = LM - r \) and \( \text{rank}(\rho_B) = M - t \), if \( r \) and \( s \) satisfy the condition \( Nr \leq Ls \), there will be

\[
t \leq \left[ \frac{r - 1}{L} \right] + 1, \tag{3}
\]

where \( \lfloor x \rfloor \) is the maximum integer which is smaller than \( x \).

The notation \( y \succ x \) mean that the vector \( x \) is majorized by the vector \( y \). The majorization is defined as the following. Let \( x = \{ x_1, x_2, \ldots, x_n \} \) and \( y = \{ y_1, y_2, \ldots, y_n \} \) are \( n \)-dimensional vectors and the elements are arranged in increasing order. Then the vector \( x \) is majorized by vector \( y \) [13], denoted by \( y \succ x \), if for each \( k \) (\( k = 1, 2, \ldots, n \)) the following inequality is hold

\[
\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i
\]

and the equality is hold when \( k = n \). Under these two Lemmas, we can find that the SSA is hold in the following.

**Theorem** There are four normalized vectors \( \lambda_{ABC}^{AB} = \{ \lambda_A^{ABC}, \lambda_A^{ABC}, \ldots, \lambda_{LMN}^{ABC} \} \), \( \lambda_{AB}^{AB} = \{ \lambda_1^{AB}, \lambda_2^{AB}, \ldots, \lambda_{LM}^{AB} \} \), \( \lambda_{BC}^{BC} = \{ \lambda_1^{BC}, \lambda_2^{BC}, \ldots, \lambda_{MN}^{BC} \} \) and \( \lambda_B^{BC} = \{ \lambda_1^{BC}, \lambda_2^{BC}, \ldots, \lambda_{AM}^{BC} \} \), the elements of these vectors are non-negative and arranged in increasing order and define the vectors \( \lambda_{ABC}^{AB}, \lambda_{ABC}^{BC} \) and \( \lambda_B^{BC}, \lambda_B^{AB} \), which are similar to the vectors in lemma 1. If the elements of these vectors satisfy the following conditions

1. \( \lambda_{ABC}^{AB} > \lambda_{AB} \);  
2. \( \lambda_{ABC}^{BC} > \lambda_{BC} \).
3. \( \lambda_B^{BC} > \lambda_B \) and \( \lambda_B^{AB} > \lambda_B \).
4. Suppose the vector \( \lambda = \lambda_{ABC}^{AB} \) has only \( Ls \) zero elements and \( \lambda_{BC} \) has \( s \) zero elements, and if the vector \( \lambda_{AB}^{AB} \) has \( r \) zero elements, there are at least \( \left\lfloor \frac{s}{L} \right\rfloor \) 1 elements of the vector \( \lambda_{BC}^{B} \) are zeroes. If exchange the role of the vector \( \lambda_{AB}^{AB} \) and \( \lambda_{BC}^{BC} \), the similar result must be hold also.

Thus the following inequality is hold

\[
S(\lambda) + S(\lambda_B^{B}) \leq S(\lambda_{BC}^{B}) + S(\lambda_{AB}^{AB}), \tag{4}
\]

where \( S(\lambda) = \sum_{i=1}^{LMN} (-\lambda_i \ln \lambda_i) \).

The proof of the theorem is technical. We use the Lagrange multiplier method to get the minimal value of a function under the conditions 1, 2 and 3. Because there are many possible extreme points, we need to find out the minimal one. We use some facts to find that when the function gets the minimal value, all of the nonzero elements are equal to each other. Then use the condition 4 to get the minimal value of the function.

**Proof.** At first, we define a function

\[
F = \sum_{i=1}^{LM} (-\lambda_i^{AB} \ln \lambda_i^{AB}) + \sum_{i=1}^{MN} (-\lambda_i^{BC} \ln \lambda_i^{BC}) + \sum_{i=1}^{M} (\lambda_i^{B} \ln \lambda_i^{B}) + \sum_{i=1}^{LMN} (\lambda_i \ln \lambda_i), \tag{5}
\]
If we can prove that the minimum of this function is not less than 0 under the conditions 1-4, the theorem is true. So the proof becomes to find the minimal value of a function under some conditions. Obviously, the minimal value of this function exists and is finite. Now we use the Lagrange multiplier method to deal with the conditions 1, 2, 3 and define a new function

\[ G = \sum_{i=1}^{LM} (-\lambda^i_{AB} \ln \lambda^i_{AB}) + \sum_{i=1}^{MN} (-\lambda^i_{BC} \ln \lambda^i_{BC}) + \sum_{i=1}^{M} (\lambda^i_B \ln \lambda^i_B) + \sum_{i=1}^{LMN} (\lambda_i \ln \lambda_i) \]  

(6)

\[ + \sum_{i=1}^{LM-1} \alpha^i_1 \left( \sum_{j=1}^{i} \lambda^i_j - \sum_{j=1}^{N_i} \lambda_j - x^2_{1i} \right) + \sum_{k=1}^{MN-1} \beta^k_1 \left( \sum_{j=1}^{k} \lambda^k_j - \sum_{j=1}^{L_k} \lambda_j - y^2_{1k} \right) \]

\[ + \sum_{i=1}^{M-1} \alpha^i_2 \left( \sum_{j=1}^{i} \lambda^i_j - \sum_{j=1}^{J_i} \lambda_j - x^2_{2i} \right) + \sum_{k=1}^{MN-1} \beta^k_2 \left( \sum_{j=1}^{k} \lambda^k_j - \sum_{j=1}^{L_k} \lambda_j - y^2_{2k} \right) \]

\[ + \sum_{i=0}^{LM} \gamma_i (\lambda_{i+1} - \lambda_i - x^2_i) + \sum_{i=1}^{LM} u_i (\lambda^i_{AB} - \lambda^i_{AB} - r^2_i) \]

\[ + \sum_{i=1}^{MN} v_i (\lambda^i_{BC} - \lambda^i_{BC} - s^2_i) + \sum_{i=1}^{M} w_i (\lambda^i_B - \lambda^i_B - t^2_i) \]

\[ + a_1 \left( \sum_{i=1}^{LM} \lambda^i_{AB} - 1 \right) + a_2 \left( \sum_{i=1}^{MN} \lambda^i_{BC} - 1 \right) + a_3 \left( \sum_{i=1}^{M} \lambda^i_B - 1 \right) + a_4 \left( \sum_{i=1}^{LMN} \lambda_i - 1 \right), \]

where the parameters \( \alpha^j_1, \beta^j_1, u_i, v_i \) and \( w_i, \gamma_j, a_i \) are Lagrange multipliers, \( x^2_{1i}, y^2_{1k}, x^2_{2i}, y^2_{2k}, z^2_i, r^2_i, s^2_i \) and \( t_i \) are introduced to make the inequalities to be equations. We have used the conditions that \( \lambda^i_{AB}, \lambda^i_{BC}, \lambda^i_B \) and \( \lambda_i \) are arranged in increasing order and let \( \lambda_0 = 0 \).

Then when \( G \) get the minimal value, there must be some constraints on the parameters and variables. First, we can get \( \sum_{j=1}^{i} \lambda^i_j - \sum_{j=1}^{N_i} \lambda_j = x^2_{1i}, i = 1, 2, \ldots, LM - 1 \) and the similar relations between \( \lambda^i_{AB} \) and \( \lambda^i_{BC} \) can be gotten

\[ \sum_{j=1}^{i} \lambda^i_{AB} - x^2_{1i}, i = 1, 2, \ldots, LM - 1; \]

(7)

\[ \lambda^i_{i+1} - \lambda^i_{AB} = r^2_i, i = 1, \ldots, LM - 1; \]

\[ \lambda^i_B - 1 = 0, \]

and the similar relations between the vectors of \( \lambda^i_{AB}, \lambda^i_{BC} \) and \( \lambda^i_B \).

The most important constraints are the equations between the vectors \( \lambda^i_{AB}, \lambda^i_{BC}, \lambda^i_B \) and \( \lambda_i \)

\[ - \ln \lambda^i_{AB} - 1 + \sum_{j=i}^{LM-1} \alpha^j_1 - \sum_{j=\lceil \frac{i}{M} \rceil + 1}^{M-1} \alpha^j_1 + u_{i-1} - u_i + a_1 = 0 \ (i = 1, 2, \ldots, LM). \]

(9.1)

\[ - \ln \lambda^i_{BC} - 1 + \sum_{j=i}^{MN-1} \beta^j_1 - \sum_{j=\lceil \frac{i}{M} \rceil + 1}^{M-1} \beta^j_1 + v_{i-1} - v_i + a_2 = 0 \ (i = 1, 2, \ldots, MN). \]

(9.2)

\[ \ln \lambda^i_B + 1 + \sum_{j=i}^{M-1} \alpha^j_2 + \sum_{j=\lceil \frac{i}{M} \rceil + 1}^{M-1} \beta^j_2 + w_{i-1} - w_i + a_3 = 0 \ (i = 1, 2, \ldots, M). \]

(9.3)
are nonzero, then
which is inconsistent with our suppose. So this fact is true. QED.

Since the number of the possible cases are so large, it is very difficult to get the solutions directly. We point out some useful facts to reduce the possible solutions and to find the minimal value of the function \( G \).

**Fact 1.** When the function \( G \) get the minimum, suppose that parameters \( \alpha_i^1 \) and \( \alpha_i^2 \) are the nearest nonzero parameter act on the elements of vector \( \lambda^{AB} \), if the parameters \( u_i \) and \( u_{Lj} \) are zeroes, all the parameter \( u_k \) \((i \leq k \leq Lj)\) are equal to zeroes. This fact is true for the other parameters \( v_i, w_i, \gamma_i \).

**Proof.** Without loss of generality, we only consider the parameter \( u_i \). Suppose the fact is not true, there are some parameters \( u_p \) \((i < m \leq p \leq n < Lj)\) are nonzero. For simplicity, we suppose there are no more nonzero parameters \( \alpha_i^1 \) and \( \alpha_i^2 \). Then we get the conditions from (9.1)

\[
- \ln \lambda_{m}^{AB} - 1 + \alpha_1^{1} - \alpha_1^{2} + 0 - u_m + a_1 = 0,
- \ln \lambda_{m+1}^{AB} - 1 + \alpha_1^{1} - \alpha_1^{2} + u_m - u_{m+1} + a_1 = 0,
- \ln \lambda_{n}^{AB} - 1 + \alpha_1^{1} - \alpha_1^{2} + u_{n-1} - u_n + a_1 = 0,
- \ln \lambda_{n+1}^{AB} - 1 + \alpha_1^{1} - \alpha_1^{2} + u_n - a_1 = 0,
\]

where we have used the conditions that the parameters \( u_{m-1} \) and \( u_{n+1} \) are zeroes. Since the parameters \( u_p \) \((i < m \leq p \leq n < Lj)\) are nonzero, then we get \( \lambda_{m}^{AB} = \lambda_{m+1}^{AB} = \cdots = \lambda_{n}^{AB} = \lambda_{n+1}^{AB} \). So we have the relations between these nonzero parameters

\[
-u_m = u_m - u_{m+1} = \cdots = u_{n-1} - u_n = u_n,
\]

that is, \( u_n = (n - m + 1)u_m = -u_m. \) So \( u_m = 0 \), then all of the parameters \( u_p \) \((i < m \leq p \leq n < Lj)\) are zeroes, which is inconsistent with our suppose. So this fact is true. QED.

Since the fact 1, the parameters \( u_k \) affect the result only when there are some nonzero parameter \( \alpha_i^1 \) or \( \alpha_i^2 \) make \( k = i \) or \( k = Lj \). For this situation, we have the following fact.

**Fact 2.** When the function \( G \) get the minimum, if there are a set of parameters \( u_k \) \((m \leq k \leq n)\) are nonzero and there are some nonzero parameters \( \alpha_i^1 \) make \( m \leq i \leq n \). This situation is equal to the situation where the parameters \( u_k \) \((m \leq k \leq n)\) are zeroes, but two new parameters \( \alpha_{m-1}^1 \) and \( \alpha_{n+1}^1 \) should be added, and the parameters \( \gamma_i \) \((0 \leq i \leq LMN)\) should be adjusted.

**Proof.** Without loss of generality, we suppose only the nonzero parameter \( \alpha_i^1 \) satisfy the condition \( m \leq i \leq n \). For simplicity, we suppose there is no other nonzero parameters act on the eigenvalues of \( \rho_{AB} \). Since the parameter \( \alpha_i^1 \) are nonzero, then

\[
\sum_{j=1}^{i} \lambda_{j}^{AB} - \sum_{j=1}^{N} \lambda_j = 0; \tag{12}
\]

and the parameters \( u_k \) \((m \leq k \leq n)\) are nonzero, we get

\[
\lambda_{m}^{AB} = \lambda_{m+1}^{AB} = \cdots = \lambda_{n}^{AB} = \lambda_{n+1}^{AB}. \tag{13}
\]

Since we have the condition \( \sum_{j=1}^{i-1} \lambda_{j}^{AB} \geq \sum_{j=1}^{N(i-1)+1} \lambda_j \), together with the equation (12), we get \( \lambda_{i}^{AB} \leq \sum_{j=1}^{N(i-1)+1} \lambda_j \). On the other hand, \( \lambda_{i}^{AB} \geq \sum_{j=1}^{N(i-1)+1} \lambda_j \), that is, \( \sum_{j=1}^{N(i-1)+1} \lambda_j \geq \sum_{j=1}^{N(i-1)+1} \lambda_j \). Because of the condition \( \lambda_{N(i-1)+1} \leq \lambda_{N(i-1)+2} \leq \cdots \leq \lambda_{N(i+1)} \), we get the equation \( \lambda_{N(i-1)+1} = \lambda_{N(i-1)+1} = \cdots = \lambda_{N(i+1)} \). So we get \( \lambda_{i}^{AB} = \sum_{j=1}^{N(i-1)+1} \lambda_j \) and \( \lambda_{i+1}^{AB} = \sum_{j=1}^{N(i+1)} \lambda_j \), that is

\[
\sum_{j=1}^{i-1} \lambda_{j}^{AB} - \sum_{j=1}^{N(i-1)} \lambda_j = 0, \sum_{j=1}^{i} \lambda_{j}^{AB} - \sum_{j=1}^{N(i+1)} \lambda_j = 0. \tag{14}
\]

Continue to use this method we can get
From these equations, we can find this is just as there are two nonzero parameters $\alpha_{m-1}^1$ and $\alpha_{n+1}^1$, and the nonzero parameter $\alpha_1^1$ have no effect in this case. From the constraints on $\lambda_k^{AB}$, there will be

$$- \ln \lambda_k^{AB} - 1 + \alpha_1^1 + a_1 = 0 \quad (1 \leq k \leq m - 1),$$  

$$- \ln \lambda_k^{AB} - 1 + \alpha_1^1 + u_{k-1} - u_k + a_1 = 0 \quad (m \leq k \leq i),$$  

$$- \ln \lambda_k^{AB} - 1 + u_{k-1} - u_k + a_1 = 0 \quad (i + 1 \leq k \leq n + 1),$$  

$$- \ln \lambda_k^{AB} - 1 + a_1 = 0 \quad (n + 2 \leq k \leq LM).$$

Since the elements $\lambda_k^{AB}$ ($m \leq k \leq n + 1$) are equal to each other, then $u_m = u_m - u_{m+1} = \cdots = u_{i-1} - u_i = \alpha$, $u_i - u_{i+1} = u_{i+1} - u_{i+2} = \cdots = u_n = \beta$. If we let the nonzero parameters $\alpha_{m-1}^1 = -\alpha$ and $\alpha_{n+1}^1 = \beta$, the equations are the same. Now we consider the effect of this substitution on the vector $\lambda$. For simplicity, we suppose also that there are only the nonzero parameter $\alpha_1^1$ act on the vector $\lambda$. Then the equations are

$$\ln \lambda_k + 1 - \alpha_1^1 + \gamma_{k-1} - \gamma_k + a_4 = 0 \quad (1 \leq k \leq Ni),$$  

$$\ln \lambda_k + 1 + \gamma_{k-1} - \gamma_k + a_4 = 0 \quad (Ni + 1 \leq k \leq LMN).$$

Insert the parameters $\alpha_{m-1}^1$ and $\alpha_{n+1}^1$ into the equations, we can find that

$$\ln \lambda_k + 1 - \alpha_{m-1}^1 - \alpha_{n+1}^1 + \gamma_{k-1} - \gamma_k + a_4 = 0 \quad (1 \leq k \leq Ni),$$  

$$\ln \lambda_k + 1 - \alpha_{m-1}^1 + \gamma_{k-1}' - \gamma_k' + a_4 = 0 \quad (Ni + 1 \leq k \leq N(n + 1))$$  

$$\ln \lambda_k + 1 + \gamma_{k-1}' - \gamma_k + a_4 = 0 \quad (N(n + 1) + 1 \leq k \leq LMN),$$

where $\gamma'_{k}$ ($Ni + 1 \leq k \leq LMN$) are the new parameters to make the equations are the same as the equation (17). This is just as the situation that the parameters $\alpha_{m-1}^1$ and $\alpha_{n+1}^1$ are nonzero, and the parameter $\gamma_k$ is adjusted. QED

This fact is also true for the parameters $\beta_1^1$. This fact tell us that any solution found in the former situation can be found in the later case. In the following, we always suppose we have already done this change. After making these changes there is no nonzero parameters $\alpha_1^1$ or $\beta_1^1$ make the parameter $u_i(v_j)$ nonzero and the parameters $\gamma_i$ are substituted by $\gamma_i'$.

**Fact 3.** When the function $G$ get the minium, if $i$ and $j$ are the nearest indexes to make the equations $\sum_{k=1}^i \lambda_k^{AB(BC)} = \sum_{k=1}^{N(L)j} \lambda_k$ and $\sum_{k=1}^i \lambda_k^{BC(AB)} = \sum_{k=1}^{LMN} \lambda_k$ to be hold, the elements of the vector $\lambda$ between $Ni$ and $Lj$ are equal to each other.

**Proof.** We suppose this conclusion is not true, without loss of generality, let $Lj > Ni$. Then there are some elements $k$ satisfy the following conditions $\lambda_j = \lambda_j-1 = \cdots = \lambda_p \equiv \lambda_a \equiv \lambda_{Ni} = \lambda_{Ni+1} = \cdots = \lambda_q$, for simplification, we suppose that $Lj - p \geq q - Ni$. If we define the following parameters $\Delta_l$ and $\Delta_m$ as $\sum_{k=i+1}^j \lambda_k^{AB} - \sum_{k=1}^{Ni} \lambda_k = \Delta_l$ ($1 \leq i \leq \left[\frac{Ni}{Lj - p}\right]$, $[x]$ is the maximal integer which is smaller than $x$) and $\sum_{k=1}^{Nj} \lambda_k - \sum_{k=m}^{Lj} \lambda_k^{BC} = \Delta_m$ ($\left[\frac{Ni}{Lj - p}\right] \leq m \leq j$), we can find that all of these parameters are more than zero. Then we take out the minimal number from $\Delta_l$ and $\Delta_m$, we denote it by $\Delta$, obviously it is more than zero. Now we change the element $\lambda_2$ by $\lambda_2 - \frac{\lambda_1}{Lj - p}$ and $\lambda_1$ by $\lambda_1 + \frac{\lambda_1}{q - Ni}$ where the parameter $\Delta' = (q - Ni)\Delta$. After these substitution, the new elements of the vector $\lambda'$ satisfy all of the conditions. The entropy of the vector $\lambda'$ is larger than the entropy of the vector $\lambda$ and the entropy of the other vector is invariable. So the function $G$ for the new vector is smaller than the former which is inconsistent with the suppose. QED

This fact is also true for the vector $\lambda^{BC}$. Since we have the fact 3, then we want to know how many nonzero parameters $\alpha_1^1$ and $\beta_1^1$ in the section where all of the elements are the same. We have the following fact

**Fact 4.** When the function $G$ get the minimum, there is no nonzero parameters $\alpha_1^1$ and $\beta_1^1$ in the section where all of the elements of vector $\lambda$ are the same except for the edge parameters.

**Proof.** We first point out that there are at most four nonzero parameters $\alpha_1^1$ or $\beta_1^1$ in the section where all of the elements of vector $\lambda$ are equal to each other if the conclusion is not true. If this assert is not true, there are at least five nonzero parameters act on the section where all of the elements of the vector $\lambda$ are the same. So at least three of them (such as $\alpha_i^1$ ($l = i, j, k \cdots$) or $\beta_i^1$ ($l = i, j, k \cdots$)) act on the same vector. Without loss of generality,
we suppose there are three nonzero parameters $\alpha_i^1 (i = j, k)$. Since the elements $\lambda_{Ni+1} = \lambda_{Ni+2} = \cdots = \lambda_{Nj} = \lambda_{Nj+1} = \lambda_{Nj+2} = \cdots = \lambda_{Nk} \equiv \lambda_a$. We have $\lambda_{AB}^1 = N\lambda_a$ and $\lambda_{AB}^2 = N\lambda_a$, since $\lambda_{AB}^1 \leq \lambda_{AB}^2$, then all of the elements $\lambda_{AB}^1 (i + 1 \leq l \leq k)$ are equal to each other. Because we have already done the changes in the fact 2, and use the fact 1, we find all of the parameters $\nu_k (i + 1 \leq l \leq k)$ are zeros. Further more, the parameters $\alpha_i^1$ are zeros too. So the number of the nonzero parameters is no more than four, and they divide the section where all the elements are equal into three smaller sections.

Now we only need to prove the case that less than five parameters are also zeros. If these parameters are nonzero and set on the vectors as figure 1, which makes the function $G$ get the minimum, we take some sufficient small value $\Delta$ from the elements of the first section to the third section. At the same time, $\Delta$ must be taken from the left side section of the parameters $k$ and $j$ to the right side section. Using the same method of the proof of the fact 3, if the $\Delta$ is sufficient small, all the conditions will be satisfied. From the following calculating, we can find that through this manipulation the function $G$ is smaller which is inconsistent with the minimal suppose.

Let the elements of the vectors before the manipulating are $\lambda_{Ni+1} = \lambda_{Ni+2} = \cdots = \lambda_{Nk} \equiv \lambda_a$, $\lambda_{Nj+1} = \lambda_{Nj+2} = \cdots = \lambda_{Li} \equiv \lambda_b$; $\lambda_{AB}^1 = \lambda_{AB}^2 = \cdots = \lambda_{iAB} \equiv \lambda_a^2$, $\lambda_{ijAB} = \lambda_{iAB}^2 = \cdots = \lambda_{iAB}^k \equiv \lambda_b^2$, $\lambda_{iBC}^1 = \lambda_{iBC}^2 = \cdots = \lambda_{iBC}^k \equiv \lambda_c^2$. After the manipulate, the new elements are $\lambda_a' = \lambda_a - \Delta L_{k-N}^i$, $\lambda_b' = \lambda_b + \Delta L_{k-N}^i$, $\lambda_{AB}' = \lambda_{AB} - \Delta B_{k+1}^i$, $\lambda_{BC}' = \lambda_{BC} - \Delta C_{k+1}^i$, the other elements are the same as before. Since $\Delta$ is sufficient small, we can expand the function $\ln(\lambda + \Delta)$ in the first order. Using this formula, we can calculate the difference of the function $G$ between these two vectors.

$$G' - G = -\left((Lk - N)\lambda_a \ln \lambda_a - (Lk - N)\lambda_b \ln \lambda_b + (Lk - Ni)\lambda_a^2 \ln \lambda_a^2 + (Lk - Nj)\lambda_b^2 \ln \lambda_b^2\right)$$

$$+ (j - s + 1)\lambda_{AB}^1 \ln \lambda_{AB}^1 - \left((j - s)\lambda_{AB}^2 \ln \lambda_{AB}^2 - (j - s + 1)\lambda_{AB}^2 \ln \lambda_{AB}^2\right)$$

$$- (t - j)\lambda_{AB}^1 \ln \lambda_{AB}^1 + (k - u + 1)\lambda_{AB}^1 \ln \lambda_{AB}^1 + (v - k)\lambda_{BC}^1 \ln \lambda_{BC}^1$$

$$- (k - u + 1)\lambda_{AB}^2 \ln \lambda_{AB}^2 - (v - k)\lambda_{BC}^2 \ln \lambda_{BC}^2$$

$$- \Delta \ln \frac{\lambda_a \lambda_{AB}^1 \lambda_{BC}^1}{\lambda_a \lambda_{AB}^2 \lambda_{BC}^2}$$

Since $\lambda_a = \lambda_b$ and $\lambda_{AB} < \lambda_{AB}^1 < \lambda_{BC} < \lambda_{BC}^1$, there will be $G' - G < 0$. This is inconsistent with the suppose that the function $G$ get the minimum. QED

**Fact 5:** When the function $G$ get the minimum, there are at most one $\alpha_i^1$ and one $\alpha_i^2$ are nonzero and the elements $\lambda_k = 0 (k \leq Ni$ or $k \leq Lj)$, $\lambda_{AB}^1 = 0 (k \leq i)$, $\lambda_{BC}^2 = 0 (k \leq j)$.

The proof of this fact is similar to the proof of the second part of the fact 4. If there is another nonzero parameter, we can take some small value from the left of this parameter to the right of it to make the value of the function $G$ smaller, which is inconsistent with the minimal suppose of the function $G$. This fact means that all of the nonzero elements of the vector $\lambda$ are equal to each other. If there is no nonzero parameter act on the vector $\lambda$, that is, all of the parameters $\alpha_i^1$ and $\beta_j^2$ are zeros, then all of the elements of the vector $\lambda$ are $\frac{\lambda}{LMN}$, all of the elements of the vector $\lambda_{AB}^1$ are $\frac{\lambda}{LMN}$, all of the elements of the vector $\lambda_{BC}^2$ are $\frac{\lambda}{LMN}$. Now the value of the function $G$ is zero. If there is only one parameter (such as $\alpha_i^1$) is nonzero, we have the following fact.

**Fact 6:** When the function $G$ get the minimum and there is only one parameter $\alpha_i^1$ (or $\beta_j^2$) is nonzero, then all of the nonzero elements of the vector $\lambda_{BC}^2, \lambda_{AB}^2$ and $\lambda_{AB}^1$ are equal to each other.

**Proof:** Without loss of generality, we suppose the nonzero parameter is $\alpha_i^1$. The nonzero elements of the vector $\lambda_{AB}^2$ is equal to each other. We can get this result by only using the inequality between the elements of the vector $\lambda_{AB}^1$ and $\lambda$. We focus on the other part of the fact. Since the nonzero elements of the vector $\lambda_{AB}^2$ are the same, all of the parameters $\alpha_i^2$ are zero. Now we only consider the parameters $\beta_j^2$. Suppose the nonzero parameters $\beta_1^2, \beta_2^2, \cdots, \beta_{ik}^2$ are set as the figure II. From the constraints of the elements of the vector $\lambda_{AB}^1$ and $\lambda_{BC}^2$ in equations (9)

$$- \ln \lambda_{iBC}^2 - 1 - \sum_{j=1}^{M-1} \beta_j^2 + v_{i-1} - v_i + a_2 = 0 (i = 1, 2, \cdots, MN), \quad (20.1)$$

$$\ln \lambda_i^B + 1 + \sum_{j=1}^{M-1} \beta_j^2 + w_{i-1} - w_i + a_3 = 0 (i = 1, 2, \cdots, M). \quad (20.2)$$

Then we find the elements of these vectors can be divided into several groups, in each group the elements are equal to each other, that is,
We must note that all of the parameters $w_i$ which act on the vector $\lambda^B$ are zeroes. At first, if all of the indexes $i_j$ satisfy $w_{i_j} = 0$, using the fact 1, all of the parameters are zero. The second, if there are some indexes (such as $i_j$) make the parameter $w_{i_j}$ to be nonzero. Because the elements in the same section are equal to each other for the fact 3, we get $\zeta^B_{j-i} = \zeta^B_j$. Because of $\zeta^B_{j-1,j} \geq N \zeta^B_{j-1,j}$ and $\zeta^B_j \leq N \zeta^B_{j-1,j}$, then $\zeta^B_j = N \zeta^B_{j-1,j}$. So if we let $l = \lfloor \frac{N}{2} \rfloor$ and $m = \lfloor \frac{N}{2} \rfloor$, we can get the inequality $\zeta^B_{j-1,j} \geq (p_j - m) \zeta^B_{j-1,j} + (N - p_j + m) \zeta^B_{j-1,j}$ and $\zeta^B_j \leq (q_j - l) \zeta^B_{j-1,j} + (N - q_j + l) \zeta^B_{j-1,j}$. Since $\zeta^B_{j-1,j} \geq \zeta^B_{j-1,j} \geq \zeta^B_{j-1,j}$, we can get the condition that $\zeta^B_{j-1,j} = \zeta^B_{j-1,j} = \zeta^B_{j-1,j}$. Now we can get the conclusion by using the fact 1, that all of the parameters $u_k$ ($i_j - q_j \leq k \leq i_j - p_j$) and $\beta^B_{i_j}$ are zeroes. So the second situation can be reduced to the first situation. So The constraints of the vectors $\lambda^B$ and $\lambda^{AB}$ are reduced to

\begin{align}
- \ln \lambda^B_i - 1 - \sum_{j \in \{1, 2, \ldots, MN\}} \beta^2_j + v_{i-1} - v_i + a_2 = 0 \quad (i = 1, 2, \ldots, MN),
\end{align}

\begin{align}
- \ln \lambda^B_i + 1 + \sum_{j = i}^{M-1} \beta^2_i + a_3 = 0 \quad (i = 1, 2, \ldots, M).
\end{align}

If let $\chi_{i,i} = \chi_{i-1,i}$ and $\eta_i = \eta_i (i = 0, 2, \ldots, k)$, we can get $\chi_i = \eta_i$ and $\chi_{i-1,i} = \chi_{i-1,i}$ where $\omega_i + \sigma = 1$ and $\beta = \frac{p_i}{p_i + q_i}$. From these definition, we find that all of the parameters $\chi_i$ and $\chi_{i-1,i}$ are in the section $[0, 1]$. Since we have the conditions $\sum_{i=N_{i+1}}^{N_{i+1}} \lambda^B_i = \sum_{i=N_{i+1}}^{N_{i+1}} \lambda^B_i$, then we can get the equations

\begin{align}
\frac{[N(i_j - i_{j+1}) - p_j + q_j] \chi_j + p_{j+1} \chi_{j+1} \chi_{j+1} \chi_{j+1} + q_{j+1} \chi_{j+1} \chi_{j+1} \chi_{j+1}}{M - i - 1} = \frac{(i_j - i_{j+1}) \chi_j}{M - i_{j+1}}.
\end{align}

So we can get the equations

\begin{align}
(i_j - i_{j+1}) p_{i+1} (1 - \chi_i) = (M - i_j) [p_{i+1} (1 - (\frac{\chi_{i+1}}{\chi_j})^{(\sigma_{j+1})}) + q_{j+1} (1 - (\frac{\chi_{j+1}}{\chi_{j+1}})^{\omega_{j+1}})].
\end{align}

If there is a parameter $w_{N_{i+1}} = 0$, then the $m$th equations in equations (24) has no item which is including $p_{m+1}$. According to the number of the parameters which make $w_{N_{i+1}} = 0$ ($1 \leq l \leq k$), we can divide the elements of these vectors into some sections, the last equation of this section has no item which contains $p$. We can only point out that the parameter $v_{N_{i+1}}$ must be zero where the parameter $n$ satisfy the condition $\eta_n = 0$ and $\eta_{n-1} > 0$. Or the condition 4 will not be satisfied. So we always can sum up all of the equations in the same section to get

\begin{align}
(i_1 - i_{s+1}) p_1 (1 - \chi_1^{\sigma_1}) = (M - i_1) [q_1 (1 - (\frac{1}{\chi_1})^{\omega_1}) + \sum_{i=2}^{s} (q_i + p_i) (1 - \omega_i (\frac{\chi_{i-1}}{\chi_{i-1}})^{\omega_i})].
\end{align}
where the parameter $s$ means that the parameter $v_{i+1} = 0$. We first focus on the lhs. of the equation (25), and obviously, it is non-negative. Then we consider the rhs. of this equation, there is a function $f(x) = xa^{1-x}+(1-x)a^{-x}$. The value of this function is not more than 1. Then the rhs. is non-positive. To make the equation to be hold, the two sides of the equation must be zero. That is $x_1 = 1$ and $s_i(\omega_i) = 0$ or $a = 1$. For each section, we can get the same conditions which imply that all of the nonzero elements of the vector $\lambda^B$ and $\lambda^{BC}$ are equal to each other. QED

For the case there are two nonzero parameters $\alpha_i^1$ and $\beta_j^1$, using the similar method before and notice the condition 4, we can get the same result that all of the nonzero elements are equal to each other.

For the facts proved before, we can get the conclusion that when the function $G$ get the minimum, all of the nonzero elements of the vectors $\lambda^{AB}$, $\lambda^{BC}$, $\lambda^B$ and $\lambda$ are equal to each other. Using the condition 4, we can calculate that the minimum of the function $G$ is not less than zero. This is the end of the proof of the theorem. QED

Since the Lemma 1 and Lemma 2, the theorem imply that the SSA is hold. This method can be used to prove some other entropy properties between the partial density matrix and the multipartite density matrix, Such as the inequality $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$.

III. CONCLUSION

In this paper we give a new elementary proof of the SSA which is an important property of the entropy for classical information and quantum information. The proof is dependent on the analysis of the minimal value of a function under some conditions. This proof also show that the conditions in our previous work [12] are stronger than the SSA.

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Figure caption.

Figure 1. In this figure, the parameters $\alpha_i^1, \alpha_j^1, \beta_k^1, \beta_l^1$ and $\alpha_i^2, \alpha_j^2, \beta_m^2, \beta_n^2$ are nonzero. The eigenvalues $\lambda_k(i < v \leq j)$ are equal to each other. These nonzero parameters divide the eigenvalues between $i$ and $j$ into three sections. We take sufficient small value $\Delta$ from the first section to the third section. And the same time, we must take the same value from the left section of the parameter $k$ and $j$ to the right section in the eigenvalue $\lambda^{BC}$ and $\lambda^{BC}$, respectively. The bold line means that the eigenvalues in the line are the same.
Figure 2. In this figure, the parameters $\beta^2_{i_1}, \beta^2_{i_2}, \cdots, \beta^2_{i_k}$ are zero. The bold line means that all of the eigenvalues lie in the line are equal to each other for the nonzero parameters $v_i$. 
Fig. 1

Fig. 2