Abstract—In this paper, we consider a discrete-time stochastic control problem with uncertain initial and target states. We first discuss the connection between optimal transport and stochastic control problems of this form. Next, we formulate a linear-quadratic regulator problem where the initial and terminal states are distributed according to specified probability densities. A closed-form solution for the optimal transport map in the case of linear-time varying systems is derived, along with an algorithm for computing the optimal map. Two numerical examples pertaining to swarm deployment demonstrate the practical applicability of the model, and performance of the numerical method.

I. INTRODUCTION

The problem of steering the states of a linear system from an initial distribution to a terminal distribution has attracted much interest in recent years [1]–[3]. Applications of such controllers include the density control of swarms [4], [5] and networked dynamical systems [6], as well as opinion dynamics [7].

Optimal transport (OT) is a mathematical framework for deriving mass-preserving maps between specified distributions that minimize a cost of transport. The optimal transport cost, contextually called the Wasserstein metric, provides a metric on the space of probability distributions. This has been employed in a variety of fields, such as economics [8], machine learning [9], [10], computer vision [11], and image processing [12]. The Wasserstein metric also allows one to tractably compute worst-case distributions in optimization problems [13], which have been applied in areas such as state estimation [14], and machine learning [15]. The computation of the Wasserstein metric has also attracted much attention, in particular techniques allowing for computational speedup such as entropic regularization [16], [17].

The connection of optimal transport to continuous-time control began with the reformulation of optimal transport as a PDE-based fluid dynamics optimization problem [18]. In this approach, a velocity field is computed that minimizes the average kinetic energy of a fluid moving from one density to another. Equivalently, this approach can be thought of as a single-integrator particle moving from an initial state with uncertainty described by an initial distribution, to a final state with an uncertainty described by a final distribution. The cases of general linear time-varying (LTV) systems, and general LTV systems driven by noise (so-called Schrödinger bridges) were developed by [19].

The latter paper [19] employs a Lagrangian-based cost function, where the static quadratic cost is replaced with a time-varying cost with dynamical constraints. Such techniques were developed in [20], which dealt with optimal transport with nonholonomic constraints. In a similar problem configuration, the existence and uniqueness of transport maps were determined for linear–quadratic costs by [21]. Other works include distributed optimal transport for swarms of single-integrators [22], [23], Perron-Frobenius operator methods for computing optimal transport over nonlinear systems [24], and covariance control [25]–[27].

While much attention has been paid to optimal transport of dynamical systems in continuous-time, there has been a marked lack of works discussing the implementation of such controllers in discrete time, which is a gap in the literature that needs to be addressed before optimal transport techniques can be implemented on digital controllers. One contribution of this paper is to provide a rigorous analysis of the optimal transport problem for linear-quadratic regulation of LTV systems in discrete time.

In the present work, we discuss the theory and implementation of optimal transport for discrete-time linear-quadratic regulation for LTV systems. Our contributions are as follows. We formalize a previously-developed method of applying optimal transport methods to control by converting a class of optimal control problems to an optimal transport problem where the cost function is the optimal cost-to-go from an initial state to a terminal state. This formalism is then applied to derive the closed-form solution of the discrete-time LQR problem with state-density constraints. This problem is solved numerically, and the solution is then implemented on an example involving swarm deployment.

The paper is organized as follows. We outline preliminaries on optimal transport in §II. Our problem statement is outlined in §III, where we discuss formulating optimal transport problems for control systems in terms of value functions. Our results concerning optimal transport for LQR and its numerical computation are in §IV. We present numerical examples and an application to swarm deployment in §V and conclude the paper in §VI.
II. PRELIMINARIES

This paper follows the notation outlined in [28]. In this section, we summarize three seminal forms of the optimal transport problem. One may consult the excellent texts by Villani for a more in-depth discussion of the theory [29], [30].

Consider two probability spaces \((\mathcal{X}, \mu_0)\) and \((\mathcal{Y}, \mu_1)\). A transport map \(T: \mathcal{X} \to \mathcal{Y}\) is said to transport \(\mu_0\) to \(\mu_1\) if \(T\#\mu_0 = \mu_1\). This constraint can be interpreted as a sort of ‘conservation of mass’ under transport. The Monge optimal transport problem seeks to find an optimal transport map \(T\) that minimizes some cost of transport \(c(x, T(x))\); that is,

\[
\inf_T \int_{\mathcal{X}} c(x, T(x))d\mu_0(x) \\
\text{s.t. } T\#\mu_0 = \mu_1.
\]

(OT1)

In general, if one of the measures \(\mu_0, \mu_1\) has infinite second moment, then the cost of \((\text{OT1})\) may be infinite. Furthermore, the pushforward constraint of \((\text{OT1})\) makes this problem computationally intractable. Kantorovich formulated a relaxation of \((\text{OT1})\) that obtains the same minimizer under quadratic costs, i.e., \(c(x, y) = \frac{1}{2}\|x - y\|_2^2\). The problem considers the set of joint probability distributions \(\pi(x, y)\) on \(\mathcal{X} \times \mathcal{Y}\) whose marginals are the initial and target measures, \(\pi(A, \mathcal{Y}) = \mu_0(A), \pi(\mathcal{X}, B) = \mu_1(B)\), for all Borel sets \(A \subseteq \mathcal{X}\) and \(B \subseteq \mathcal{Y}\). With some abuse of notation, to denote variables of operators (e.g., optimization, integration) we may write the above as \(\pi(x, \cdot) = \mu_0(x), \pi(\cdot, y) = \mu_1(y)\). With this notation, the Kantorovich optimal transport is then formulated as:

\[
\inf_{\pi} \int_{\mathcal{X} \times \mathcal{Y}} c(x, y)d\pi(x, y) \\
\text{s.t. } \pi(x, \cdot) = \mu_0(x), \pi(\cdot, y) = \mu_1(y)\]  \hspace{1cm} (OT2)

For the case of quadratic costs, \((\text{OT2})\) obtains the same minimum as \((\text{OT1})\), and the optimal coupling satisfies \(\pi^* = (\text{Id} \times T^*)\#\mu_0\), where \(T^*(x)\) is the optimal map from \((\text{OT1})\).

One final formulation of optimal transport we describe here is given by Brenier and Benamou in the form of an optimal control problem in a fluid dynamics setting [18].

Given initial and terminal densities \(\rho_0, \rho_1\), one seeks to find a smooth, time-dependent velocity field \(v(x, t)\) taking \(\rho_0\) to \(\rho_1\) in unit time, while satisfying the continuity equation of an Eulerian fluid. The velocity field minimizes the average kinetic energy of the fluid. The problem is explicitly defined as,

\[
\sup_{\rho, v} \int_0^1 \int_{\mathbb{R}^n} \|v(x, t)\|_2^2 \rho(x, t)dxdt \\
\text{s.t. } \partial_t \rho(x, t) + \nabla \cdot (\rho(x, t)v(x, t)) = 0 \hspace{1cm} (\text{OT3})
\rho(x, 0) = \rho_0(x), \rho(x, 1) = \rho_1(x).
\]

In Lagrangian coordinates \(X(x, t)\) with \(X(x, 0) := x\), \(\partial_t X(x, t) = v(X(x, t), t)\), the solution to \((\text{OT3})\) is given by a linear interpolation with the optimal map, \(X(x, t) = x + t(T(x) - x) := T_t(x)\), and so the densities at time \(t\) satisfy \(\rho(x, t) := \rho_1(x) = (T_t)\#\rho_0(x)\).

In the following discussion, along with the problem statement, we provide an overview of how these optimal transport methods can be applied to control systems, along with the specific contribution of this paper.

III. STOCHASTIC OPTIMAL CONTROL WITH STATE-DENSITY CONSTRAINTS

In this section, we consider an optimal transport approach to the discrete-time linear-quadratic regulator. We present a formal discretization of the continuous-time controllers in [19], extended to the general case of LQR control.

We consider systems with a state \(z_k \in \mathbb{R}^n\) of the form

\[
z_{k+1} = A_k z_k + B_k u_k \\
z_0 \sim \rho_0(z)dz,
\]

(1)

where the initial condition \(z_0\) has some uncertainty described by a probability density \(\rho_0(x)\) and \(u_k \in \mathbb{R}^m\) is the control. Our goal is to translate the system \((1)\) to a terminal state \(z_{T_f} \sim \rho_1\) over a time horizon \(0 \leq k \leq T_f\), where \(\rho_1\) captures some desired uncertainty in the terminal state. The control should satisfy some optimality principle under an appropriate cost, and so an optimization problem with dynamics \((1)\) is,

\[
\min_{u, z} \mathbb{E} \left[ \sum_{k=0}^{T_f-1} c(z_k, u_k) \right] \\
\text{s.t. } z_{k+1} = A_k z_k + B_k u_k \\
z_0 \sim \rho_0(z)dz, \hspace{0.1cm} z_{T_f} \sim \rho_1(z)dz,
\]

(2)

where the expectation is with respect to a joint distribution \(\pi(z_0, z_{T_f})\), as defined in \((\text{OT2})\). The remark below formalizes a solution technique for problems of the form \((2)\) which was used by [19] to solve continuous-time optimal control problems with control costs.

Remark 1: A general method to solve problems of the form \((2)\) is to first solve the deterministic problem

\[
\min_u \sum_{k=0}^{T_f-1} c(z_k, u_k) \\
\text{s.t. } z_{k+1} = A_k z_k + B_k u_k \\
z_0 = x, \hspace{0.1cm} z_{T_f} = y
\]

(3)

to determine a formula \(C(x, y)\) for the optimal cost-to-go from \(x\) to \(y\). Thus, \((2)\) can be re-written as

\[
\min_{x, y} \int_{\mathbb{R}^n \times \mathbb{R}^n} C(x, y)dx dy \\
\text{s.t. } \pi(x, \cdot) = \rho_0(x)dx, \pi(\cdot, y) = \rho_1(y)dy
\]

(4)

where the marginal constraints on \(\pi\) encode the relevant distributions on the initial state \(x\) and terminal state \(y\). Problem \((4)\) is clearly a Kantorovich optimal transport problem of the form \((\text{OT2})\), where the cost function is now the deterministic value function encoding the cost-to-go from \(x\) to \(y\), and the optimal coupling \(\pi^*\) encodes a mapping between initial and terminal conditions \(x\) and \(y\).

The solution to \((4)\), under appropriate assumptions on the cost \(C(x, y)\), yields a coupling of the form \(\pi^*(x, y) = (\text{Id} \times T^*)\#\mu_0(x)\), where \(y = T^*(x)\). Finally, we note that \(\{u^*_k(x, T(x))\}_{k=1}^{T_f}\) solves \((2)\).

When \(c(z_k, u_k) = (z_k - y)^T Q_k (z_k - y) + u_k^T R_k u_k\) for \(y \sim \rho_1(z)dz\), we have the following LQR problem with stochastic initial and terminal constraints,

\[
\min_u \mathbb{E} \left[ \sum_{k=0}^{T_f-1} \{\|z_k - y\|_Q^2 + \|u_k\|^2_{R_k}\} \right] \\
\text{s.t. } z_{k+1} = A_k z_k + B_k u_k \\
z_0 \sim \rho_0(z)dz, \hspace{0.1cm} z_{T_f} = y \sim \rho_1(z)dz.
\]

(4)

We solve this problem in the following section.
IV. DERIVATION OF THE OPTIMAL MAP

Our main contribution in this section is the solution to Problem (4) outlined in [34]. For the simplified case with only control costs, the reader is referred to [28].

A. Discrete-Time Optimal Transport: Linear-Quadratic Case

In this subsection, we consider the more general case of a linear-quadratic cost function. The problem is formulated as

\[
\begin{align*}
\min_{\mathbf{x}} & \quad \mathbb{E} \left[ \sum_{t=0}^{T} \left\{ ||z_{k} - y||_{Q_{k}}^2 + ||u_{k}||_{R_{k}}^2 \right\} \right] \\
\text{s.t.} & \quad z_{k+1} = A_{z}z_{k} + B_{z}u_{k} \\
& \quad z_{0} = \tilde{\rho}_{0}(z)dz, \quad z_{T} = y = \tilde{\rho}_{1}(z)dz.
\end{align*}
\]

(P3)

We proceed using the methodology outlined in Remark 1 by considering the solution to the deterministic problem,

\[
\begin{align*}
\min_{\mathbf{x}} & \quad \sum_{t=0}^{T} \left\{ ||u_{k}||_{R_{k}}^2 + ||z_{k} - y||_{Q_{k}}^2 \right\} \\
\text{s.t.} & \quad z_{k+1} = A_{z}z_{k} + B_{z}u_{k} \\
& \quad z_{0} = x, \quad z_{T} = y.
\end{align*}
\]

(P4)

We summarize the cost-to-go of (P4) in the following lemma.

**Lemma 1:** The optimal cost-to-go of (P4) is quadratic in \( x, y \), in that

\[
C^{*}(x, y) = x^{T}Qx + y^{T}Qy + 2x^{T}Qxy + 2x^{T}Q_{xy}y,
\]

where \( Q_{xy} \) is invertible. The optimal control of (P4) is given by

\[
\begin{align*}
U^{*} &= K^{*}(y - \Phi(t_{f}, 0)x) \\
&\quad - \Gamma_{U_{t}}P^{-1}A_{U_{t}}^{T}Q_{xy}(\Omega x - (1_{t_{f}} \otimes I_{n})y),
\end{align*}
\]

(K*)

where the matrices comprising the optimal cost and control are given by

\[
\begin{align*}
Q_{x} &= K_{x}^{T}\bar{Q}K_{x} + K_{x}^{T}\bar{R}K_{x} \\
Q_{y} &= K_{y}^{T}\bar{Q}K_{y} + K_{y}^{T}\bar{R}K_{y} \\
Q_{xy} &= \gamma_{x}^{T}K_{x}^{T}\bar{Q}_{xy}K_{y} + \gamma_{y}^{T}K_{y}^{T}\bar{R}_{xy}K_{x} \\
K_{1} &= (I - A_{U_{t}}P^{-1}A_{U_{t}}^{T}\bar{Q})A_{x} - A_{U_{t}}P^{-1}\Gamma_{U_{t}}^{T}\bar{R}x \\
K_{2} &= (I - A_{U_{t}}P^{-1}A_{U_{t}}^{T}\bar{Q})A_{y} - A_{U_{t}}P^{-1}\Gamma_{U_{t}}^{T}\bar{R}y \\
K_{3} &= (I - A_{U_{t}}P^{-1}\Gamma_{U_{t}}^{T}\bar{R})\Gamma_{x} - A_{U_{t}}P^{-1}A_{U_{t}}^{T}\bar{Q}_{xy}A_{x} \\
K_{4} &= (I - A_{U_{t}}P^{-1}\Gamma_{U_{t}}^{T}\bar{R})\Gamma_{y} - A_{U_{t}}P^{-1}A_{U_{t}}^{T}\bar{Q}_{xy}A_{y} \\
P &= A_{U_{t}}^{T}\bar{Q}A_{U_{t}} + A_{U_{t}}^{T}\bar{R} \Gamma_{U_{t}} \\
A_{x} &= \Omega + \Psi \Gamma_{x} \\
A_{y} &= \Phi \Gamma_{y} - 1_{t_{f}} \otimes I_{n}, \\
\Gamma_{x} &= \left[ \begin{array}{c} S_{1}^{0} \Phi(t_{f}, 0) \\
\vdots \\
S_{t_{f} - 1}^{t_{f} - 1} \Phi(t_{f}, 0) \end{array} \right], \\
\Gamma_{y} &= \left[ \begin{array}{c} 0 \\
\vdots \\
0 \end{array} \right], \\
\Gamma_{U_{t}} &= \left[ \begin{array}{c} I_{(t_{f} - m)_{0}} \\
\vdots \\
I_{(t_{f} - m)_{0}} \end{array} \right],
\end{align*}
\]

(Q* and K*)

and the matrices defined by the dynamics are given by

\[
\begin{align*}
\Psi &= \left[ \begin{array}{c} \tilde{\Upsilon}(0) \\
\vdots \\
\tilde{\Upsilon}(t_{f} - 1) \end{array} \right], \\
\Omega &= \left[ \begin{array}{c} \Phi(1, 0) \\
\vdots \\
\Phi(t_{f}, 0) \end{array} \right], \\
\Upsilon(l, 0) &= [\Phi(l, 1)B_{0} \Phi(l, 2)B_{1} \cdots B_{l - 1}] \\
\Upsilon(l) := \left[ \begin{array}{c} \Upsilon(l, 0) \\
0 \\
\vdots \\
0 \end{array} \right] \in \mathbb{R}^{n \times m_{l}}.
\end{align*}
\]

An example in §V shows that the pseudoinverse in \( \Gamma_{x}, \Gamma_{y} \), and \( \Gamma_{U_{t}} \) is well-behaved, even in pathological cases. For brevity, the proof of Lemma 1 can be found in [28], which follows a similar analysis as in the simpler case in [31]. We now state the main theorem solving Problem (P3).

**Theorem 1:** Consider the setting of Problem (P3), and the Kantorovich optimal transport problem,

\[
\begin{align*}
\min_{\pi} & \quad \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} C^{*}(x, y)d\pi(x, y) \\
\text{s.t.} & \quad x \sim \rho_{0}(x)dx, \quad y \sim \rho_{1}(y)dy,
\end{align*}
\]

(OT)

where \( C^{*}(x, y) \) is given by (5). Then, the optimal coupling \( \pi^{*} \) of (5) is given by \( \pi^{*}(x, y) = (1d \times T^{*})\#\rho_{0}(x), \) where \( T_{#}\rho_{0} = \rho_{1} \). Furthermore, the optimizing control input in Problem (P3) is given by

\[
\begin{align*}
U &= K^{*}(T^{*}(x) - \Phi(t_{f}, 0)x) \\
&\quad - \Gamma_{U_{t}}P^{-1}A_{U_{t}}^{T}\tilde{Q}(\Omega x - (1_{t_{f}} \otimes I_{n})T^{*}(x)),
\end{align*}
\]

(U*)

where the relevant matrices are defined in Lemma 1.

**Remark 2:** The solution to Problem (OT) yields a Monge map \( T^{*} \) that transports \( x \sim \rho_{0}(x)dx \) to \( y := T^{*}(x) \sim \rho_{1}(y)dy \), minimizing the expected cost-to-go from \( x \) to \( y \). Another interpretation of this map is that it pairs initial and terminal states \( (x, y) \) in such a manner that it minimizes the LQR cost averaged over the distribution of initial states. We exploit this interpretation in §V where we discuss an application to swarm deployment.

Next, we examine the numerical computation of \( T^{*}(x) \).

B. Numerical Computation of the Monge Map

In general, the Monge map \( T(x) \) is difficult to compute numerically [32], [33]. In fact, (OT) was devised by Brenier and Benamou precisely to numerically compute \( T(x) \), and fast methods for OT are being actively researched [34]. In one dimension, a classical result (used in [19]) determines the Monge map in terms of the cumulative distribution functions of the initial and terminal densities as

\[
\int_{-\infty}^{x} \rho_{0}(x)dx = \int_{-\infty}^{T(x)} \rho_{1}(y)dy.
\]

(OT)

This can readily be solved with a bisection algorithm.

For systems with \( n > 1 \) states, the situation is more complicated. For example, in the single-integrator system \( x_{k+1} = x_{k} + u_{k} \), the one-stepwise Monge map exists explicitly when the initial and terminal distributions are Gaussian. Suppose \( \rho_{0}, \rho_{1} \) are, \( \rho_{0}(x) \sim N(m_{0}, \Sigma_{0}), \rho_{1}(x) \sim N(m_{1}, \Sigma_{1}) \). Then, the optimal Monge map is a shift and scaling [35], \( T(x) = A(x - m_{0}) + m_{1}, \) with \( A = \sum_{0}^{1/2} \left( \Sigma_{0}^{-1/2} \otimes \Sigma_{1}^{-1/2} \right)^{1/2} \Sigma_{0}^{-1/2} \).

For general distributions, we outline a discretization-based method for computing \( \pi^{*} \) from (OT), and then generating
the image of \( T^*(x) \) from this approximate \( \pi^* \). Suppose we discretize \( \mathcal{X} = \mathcal{Y} := \mathbb{R}^n \) into cells \( \{X_i\}_{i=1}^{n_x} \), \( \{Y_j\}_{j=1}^{n_y} \), and then define probability mass vectors \( \rho_0 \in \mathbb{R}^{n_x} \), \( \rho_1 \in \mathbb{R}^{n_y} \) representing \( \rho_0 \), \( \rho_1 \), as

\[
\rho_i = \int_{X_i} \rho_0(x) dx, \quad \rho_j = \int_{Y_j} \rho_1(y) dy.
\]

The cost in (OT2) can be written over this discrete space as

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} C(x, y)d\pi(x, y) \rightarrow \sum_{i,j} C(x_i, y_j)\pi(x_i, y_j),
\]

where \( x_i, y_j \) are representative coordinates of the cell, say their centroids. The marginal constraints on \( \pi \) are

\[
\pi 1_{n_y} = \rho_0, \quad \pi^T 1_{n_x} = \rho_1.
\]

Letting \( C(x_i, y_j) := C_{ij} \), and \( \pi(x_i, y_j) := \pi_{ij} \), we arrive at the linear program

\[
\min_{\pi} \sum_{i,j} C_{ij} \pi_{ij}
\]

\[
\text{s.t. } \pi 1_{n_y} = \rho_0, \quad \pi^T 1_{n_x} = \rho_1
\]

\[
\pi_{ij} \geq 0, \forall i, j.
\]

To recover a discrete image of the map \( T \), one has to numerically ‘un-do’ the pushforward operation that \( \pi \) represents. This is done by the element-wise division of \( \pi(x_i, y_j) \) by \( \rho_0(x_i) \) as

\[
\tau(x_i, y_j) := \frac{\pi(x_i, y_j)}{\rho_0(x_i)}.
\]

Note that this definition requires that \( \rho_0 \) must be strictly positive over the discrete domain; alternately if \( \rho_0(x_i) = 0 \), then the corresponding row of \( \pi \) must also be 0 from the constraints in (10). In this case, we can define \( \tau(x_i, y_j) \) arbitrarily, since if there is no mass to move from \( x_i \), it is irrelevant where that mass should move to. Note that Problem (10) suffers from the ‘curse of dimensionality’ due to the discretization of \( \mathbb{R}^n \). Fast approximations of optimal transport are an ongoing area of research, and one may expect that Problem (10), or approximations of it, may soon be computationally tractable for large state-spaces [16, 36, 37].

The graph of \( T \) over \( \{x_i\}_{i=1}^k \) can then be determined by applying the map \( \tau \) to the domain \( \{x_i\}_{i=1}^k \). Suppose \( \{X_1, \ldots, X_n\} \) are appropriately-vectorized coordinates in each of the \( n \) directions of the discretized domain in \( \mathbb{R}^n \). Then, the matrix \( \tau \) generates the image of \( T \) as follows:

\[
\tau[X_1, \ldots, X_n] = [T_1, \ldots, T_n],
\]

where \( T_i \) is the vectorized map over the domain in the \( i \)th direction of \( \mathbb{R}^n \).
consider $n$ agents spaced at constant intervals in the cube $[−1,1]^2$, as depicted in the middle-left subfigure of Fig. 5. Clearly, this is an approximation of a uniform distribution. Our target distribution is the logo of the Swiss Federal Institute of Technology, Zürich, discretized over a $35 \times 35$ pixel domain. A target application could be a swarm of UAVs providing a background performance act during a university event.

Using LTV discrete single-integrator dynamics

$$\begin{align*}
A_k &= Q_k = R_k = I_2, \quad 0 \leq k \leq 10, \\
B_k &= I_2, \quad 0 \leq k \leq 5, \quad B_k = 0_{2 \times 2}, \quad 5 < k \leq 10, \quad (11)
\end{align*}$$

we compute the optimal map using $[10]$ with the cost matrix $[5]$ from Lemma $[1]$ depicted in Figure 6. The simulation was again produced over a time horizon of $0 \leq k \leq 10$. This time, we plot the explicit mapping between points in a grid and their target states as generated by the map $T^*$, as shown in the bottom-right of Figure 6.

The dynamics (11) are controllable, in the sense that the controllability Gramian $W_c(t_f, 0) = \sum_{t=0}^{t_f-1} \Psi(t_f, k + 1)B_kB_k^T\Phi(t_f, k + 1)T$ is positive-definite, however the matrix $S_2$ in (7) is $0_{1 \times 2}$. Since $0_{4 \times 2} = 0_{2 \times 4}$, by (12) in [28], this simply means that the control is zero for $6 \leq k \leq 10$. As the system is controllable, it is steered to the final position by timestep $k = 5$, as evident in Figure 4.

VI. CONCLUSION

In this paper, we studied the discrete-time linear-quadratic regulator with uncertainties in the initial state, and how optimal transport can be used to guide the system to a final state with an uncertainty specified by a target probability density. We derived the form of the optimal control from the optimal transport map, and discussed numerical implementations of this map. Finally, we provided numerical examples with an application to swarm deployment.

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