Complexity and neutron stars with crust and hyperon core

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Abstract. Continuing the work of K. Ch. Chatzisavvas and others [1], a measure of complexity proposed by R. López-Ruiz, H.L Mancini, and X. Calbet [2] is used to study a model of neutron star (NS) with crust and hyperon core. We employed the relativistic-mean-field approximation theory to build the Lagrangian density model of the neutron star’s core and from it, obtained the equation of state (EoS), $\epsilon(\rho)$. This $\epsilon(\rho)$ was then put into the complexity equation and the complexity-values were obtained using numerical methods. Plotted against the corresponding mass values, the results show that neutron star’s complexity-mass curves behave very similarly to an isolated-system phase complexity diagram, which was not apparent in the previous work by Chatzisavvas et al.. The reason for this is that the crust EoS put an upper limit to the complexity value and it follows that the crust itself is an ordered system of low complexity. We also show that NS have the properties of a perfect crystal although they are modeled as liquid.

1. Introduction

In the last decades, there has been a good deal of attempts at studying physical and biological systems using information theory, particularly regarding their complexity. Although a universally regarded definition of complexity is still missing, López-Ruiz, Mancini, and Calbet, have laid the satisfactory groundwork. It was first used in the field of physics itself as a trial to study atoms by Panos et al [3]. and later followed by Sahudo and Pacheco with their study of white dwarfs [4]. Finally, Chatzisavvas and others recently applied it as a tool to probe into neutron stars (NS) using a relatively simple equation of state model. Their results tell us that neutron stars are highly ordered low-complexity systems.

We continue the work done on NS based on what could be a more representative model by applying the relativistic-mean-field approximation theory to build the Lagrangian density model and then derive the EoS (equation of state). But before diving into that, let us first see what this statistical complexity equation looks like. López-Ruiz, Mancini, and Calbet defined the statistical complexity, $C$, as the product of information content of the system $H$ and disequilibrium $D$ ($C = H \cdot D$). The $H$ is an exponential “function” of information entropy, which preserves the positivity of $C$, while the disequilibrium represents the probabilistic hierarchy of the system. The physical meaning of the latter is easier to illustrate using two idealized forms of matter. When we have a perfect crystal, there is one “prevailing” state which prevents the equidistribution of states while in ideal gas, all states are equiprobably distributed. We say that the measure of distance of all states from the equiprobable is called the disequilibrium $D$ whose value is at maximum for perfect crystals and zero for ideal gas. And
since $H$ is an (exponential) function of information entropy, the reverse is true for it. Accounting for both terms, the complexity curve for an isolated system should look like the following:

![Complexity Curve](image)

**Figure 1.** The intuitive notion of the complexity phase diagram of a system.

Both ends of the complexity diagram agree with our intuition that their values should go to zero, which is to say that perfect crystals and ideal gas are both systems of minimum complexity. If our model of NS fits, then its complexity diagram should produce similar result.

In this work we began the first part by declaring the Lagrangian density of the neutron star and then derive the energy $\epsilon$ and pressure density $\rho$ to obtain the EoS, $\epsilon(\rho)$. The following calculations, starting from fitting the EoS from the $\epsilon-\rho$ relation, putting it into the TOV equation, and finally producing the complexity diagrams were done by using numerical methods. The mass-radius alongside complexity-mass diagram can be used to determine the neutron star’s theoretical maximum mass with its corresponding radius. In part 2 we describe the model used and then follow it by showing the calculation results and discussion in part 3. The last part contains our summary.

2. The model

2.1 Statistical complexity

The Shannon information entropy $S$ in position space for continuous probability distribution $\rho(r)$ is defined as,

$$ S = - \int \rho(r)ln\rho(r) \, dr, $$

and a measure of quadratic distance from the equiprobable is defined as disequilibrium $D$:

$$ D = \int \rho(r)^2dr. $$

For a system that has a finite number of accessible states, let’s say, $\{x_1, x_2, \ldots, x_N\}$, whose corresponding probabilities $\{p_1, p_2, \ldots, p_N\}$, the disequilibrium would be $D = \sum_{i=1}^{N} \left(p_i - \frac{1}{N}\right)^2$ with $\frac{1}{N}$ being the equiprobability. It would then revert back to eq. (2) when taken into its limit ($N \to \infty$). In the same vein, the information entropy is $S = - \sum_{i=1}^{N} p_i ln(p_i)$. It can be clearly seen in these forms that for a completely delocalized system with equiprobable distribution of its states ($p_i = \frac{1}{N}$), the disequilibrium $D$ is zero and the information entropy $S$ reaches its maximum value of $S = lnN$. The reverse is true for
an absolutely localized system, e.g., a perfect crystal, which has a particular “privileged” state. This would then make \( D \) goes to its maximum value and \( S \) zero since \( \ln 1 = 0 \).

We have yet to modify the information entropy and disequilibrium for our use in neutron stars. Here the probability distribution \( p_i \) can be replaced by the mass distribution \( \rho(r) \) because it relates to finding a particle in a particular location. However, mass distribution is also related to energy density \( \epsilon(r) \) since \( \rho(r) = \epsilon(r)/c^2 \). Thus, we write the information entropy and disequilibrium as:

\[
S = -b_0 \int \bar{\epsilon}(r) \ln \bar{\epsilon}(r) \, dr, \quad (3)
\]

\[
D = b_0 \int \epsilon(r)^2 \, dr. \quad (4)
\]

We have to introduce a \( b_0 \) constant since both \( S \) and \( D \) must be dimensionless and its proper value for our model is \( b_0 = 2.7 \times 10^{-1} \) km\(^3\). Finally, the dimensionless energy density \( \bar{\epsilon}(r) \) is given from the relation \( \bar{\epsilon}(r) = \epsilon(r)/\epsilon_0 \) with \( \epsilon_0 = 1 \) MeVfm\(^3\).

Taking both terms into account, we finally write the complexity equation as defined by López-Ruiz, Mancini, and Calbet:

\[
C = H \cdot D, \quad (5)
\]

where

\[
H = e^S, \quad (6)
\]

according to Sañudo et al. \[5\]. We take the exponential form simply to make \( H \) to be always positive.

### 2.2 Equation of state

According to terrestrial heavy-ion collisions experiments such as in J-PARC and GSI-FAIR, there are evidences of existing hyperons \[6\]. Based on findings such as those, the existence of hyperons in NS should be considered since the mass difference between hyperons and nucleons are roughly comparable to the nucleon Fermi energy in NS. However, this inclusion of hyperons unfortunately softens the EoS which makes the NS’s mass lower than what astrophysical observations say. We can fortunately stiffen the EoS in several ways within the relativistic-mean-field model by omitting or including the meson interactions.\[7\].

First, we start by writing the total Lagrangian density as follows:

\[
\mathcal{L} = \mathcal{L}_B^{\text{free}} + \mathcal{L}_M^{\text{free}} + \mathcal{L}_B^{\text{lin}} + \mathcal{L}_B^{\text{nonlin}} + \mathcal{L}_1^{\text{free}}, \quad (7)
\]

where the free baryons Lagrangian density is

\[
\mathcal{L}_B^{\text{free}} = \sum_{B=N,A,\Xi} \bar{\Psi}_B \left[ i \gamma^\mu \partial_\mu - M_B \right] \Psi_B. \quad (8)
\]

The second term describes the Lagrangian density for free mesons involved:

\[
\mathcal{L}_M^{\text{free}} = \frac{1}{2} \left( \partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2 \right) + \frac{1}{2} \left( \partial_\mu \sigma^* \partial^\mu \sigma^* - m_\sigma^* \sigma^*^2 \right) - \frac{1}{4} \omega_\mu \omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu - \frac{1}{2} \phi_\mu \phi^{\mu\nu} + \frac{1}{2} m_\phi^2 \phi_\mu \phi^\mu - \frac{1}{4} \rho_\mu \rho^{\mu\nu}. \quad (9)
\]

Next, the \( \mathcal{L}_B^{\text{lin}} \) describes the interaction of the baryons through mesons:

\[
\mathcal{L}_B^{\text{lin}} = \sum_{B=N,A,\Xi} \bar{\Psi}_B \left[ g_\sigma B^{\sigma} + g_{\sigma^*} B^{\sigma^*} - \gamma_\mu g_\omega B \omega^\mu - \frac{1}{2} \gamma_\mu g_\rho B \tau_B \cdot \rho^\mu - \gamma_\mu g_\phi B \phi^\mu \right] \Psi_B. \quad (10)
\]

where \( \tau_B \) are the baryons isospin matrices.
Continuing on, the nonlinear meson self-interactions Lagrangian density is
\[
\mathcal{L}_{\text{nonlin}} = -\frac{k_3 g_{\sigma N} m_\sigma^2}{6 m_N} \sigma^3 - \frac{k_4 g^2_{\pi N} m_\pi^2}{24 m_N} \sigma^4 + \frac{\zeta g^2_{\omega N}}{24} (\omega_\mu \omega^\mu)^2 + \frac{\eta_1 g_{\sigma N} m_\omega^2}{2 m_N} \sigma \omega_\mu \omega^\mu \\
+ \frac{\eta_2 g^2_{\omega N} m_\omega^2}{4 m_N} \sigma^2 \omega_\mu \omega^\mu \rho^\mu + \frac{\eta_3 g_{\omega N} m_\rho^2}{2 m_B} \sigma \rho^\mu \rho^\mu \\
+ \frac{\eta_4 g^2_{\omega N} m_\rho^2}{4 m_N} \sigma^2 \rho^\mu \rho^\mu \omega_\mu \omega^\mu \rho^\mu \rho^\mu.
\]
(11)

And finally, the Lagrangian density of the free leptons is
\[
\mathcal{L}^{\text{free}}_l = \sum_{\ell=e^{-\mu}} \bar{\psi}_\ell \left( i \gamma^\mu \partial_\mu - M_\ell \right) \psi_\ell.
\]
(12)

For eq. (10) and (11) we also define,
\[
X_{\omega H} = \begin{cases} \frac{\theta_{\omega H}}{\theta_{\omega N}}, & \text{for } \Lambda, \Sigma \text{ hyperons} \\ \frac{2}{3} \left( \frac{\theta_{\omega H}}{\theta_{\omega N}} \right), & \text{for } \Xi \text{ hyperon} \end{cases}
\]
and
\[
X_{\phi H} = \begin{cases} \frac{\theta_{\phi H}}{\theta_{\omega N}}, & \text{for } \Lambda, \Sigma \text{ hyperons} \\ \frac{1}{2} \left( \frac{\theta_{\phi H}}{\theta_{\omega N}} \right), & \text{for } \Xi \text{ hyperon} \end{cases}
\]
with \( X_{\omega H} = \frac{2}{3} \) and \( X_{\phi H} = \frac{\sqrt{2}}{3} \).

We use the already available values of the coupling constants from previous experiments and write the explicit form of the total Lagrangian density. The energy density can then be determined from the Lagrangian density using the following relation:
\[
\epsilon = \langle T_{00} \rangle = -g_{00} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \sigma} \frac{\partial \sigma}{\partial \omega_\mu} \frac{\partial \omega_\mu}{\partial \mu} \frac{\partial \mu}{\partial \sigma}.
\]
(13)

and the pressure density from
\[
p = \frac{1}{3} \langle T_{ij} \rangle = \rho_0 \frac{\partial \mu}{\partial \rho_0} - \epsilon.
\]
(14)

The summation terms of the total Lagrangian density can be written as integrals in Fermi-momentum \( k_F \) space. The energy and pressure density relation was then solved simultaneously using numerical methods by putting a range of values for the Fermi-momentum \( k_F \). Finally, a best-fit method was employed to find the \( \epsilon(p) \) relation.

### 2.3. The Tolman-Oppenheimer-Volkoff equation

The TOV equation describes the pressure of a star at a certain point on its radius,
\[
\frac{dp}{dr} = \frac{G M(r)}{r^2} \left[ 1 + \frac{p(r)}{\epsilon(r)} \right] \left[ 1 + \frac{4 \pi r^3 \rho(r)}{M(r)} \right]^{-1} \left[ 1 - \frac{2 G}{r} \right]^{-1},
\]
(15)

and the differential mass equation is,
\[
\frac{dM}{dr} = 4 \pi r^2 \epsilon(r).
\]
(16)

Inserting the \( \epsilon(p) \) into the equations and solving both equations simultaneously, yields the mass-radius diagram which tells us the theoretical maximum mass \( M \) and its radius \( R \) of the neutron star. This is done numerically by putting a range of values of central pressure density \( \rho_c \) into the \( \epsilon(\rho_c) \) relation and running the calculations from \( r = 0 \) (center of the star) until the condition \( P(r = R) \approx 0 \) (surface of the star) is satisfied. When the calculation stops, we get the \( M \) and \( R \) values for a particular \( \rho_c \) and for a range of \( \rho_c \)-values, we get a set of \( M \) and \( R \) values which could be then plotted.
3. Results and discussion

There are in total ten EoS equations employed for the calculations. They are divided into two main categories, one whose Lagrangian density model contains hyperon and the other does not as we want to see how the presence of hyperons influences the behaviour of the complexity diagram. Each of these two main categories contains five EoS characterized by the meson interactions in the Lagrangian density equation.

Table 1. Various meson self- and cross-interaction terms in the Lagrangian density model. The index ‘1’ and ‘0’ is used to indicate whether or not the corresponding term is included.

| Parameter | Self-interaction | Cross-interaction |
|-----------|------------------|-------------------|
|           | $\sigma$ | $\omega$ | $\rho$ | $\sigma-\omega$ | $\sigma-\rho$ | $\omega-\rho$ |
| BSR2      | 1     | 0     | 0     | 1    | 1    | 1     |
| BSR6      | 1     | 0     | 0     | 1    | 1    | 1     |
| BSR9      | 1     | 1     | 0     | 1    | 1    | 1     |
| GM1       | 1     | 0     | 0     | 0    | 0    | 0     |
| NL3       | 1     | 0     | 0     | 0    | 0    | 0     |

There are more parameters currently used in studies of NS using RMF approximation but we chose to limit ourselves to these five as we feel that they sufficiently represent the whole group, with an EoS using the NL3 parameter being regarded as the “stiffest” while BSR9-EoS being the “softest”. An NS with a stiff EoS would be harder to compress and its pressure increases more sharply for a corresponding rise in density. Here, we call our EoS with hyperons present as H_SU6.

3.1 Mass-radius diagram

We first start by solving the TOV equation and see what our Lagrangian density model tells us about the mass-radius diagram of a neutron star.

![Figure 2](image-url)  
(a) No Hyp  
(b) H_SU6

Figure 2. The mass-radius diagrams of neutron stars with nuclear-matter (Fig. 2a) and hyperonic cores (Fig. 2b). $M_\odot$ represents the NS mass to solar mass ratio.

The long flat end on the right side is regarded as the crust-region. Each model with its corresponding parameter (which represents which couplings or meson interactions exist) shows its own unique maximum mass value. An NS model with stronger couplings has stiffer EoS due to its baryons tend to be more strongly coupled with each other.
3.2. Statistical complexity of the neutron star model

Finally, we plugged the energy density numbers (from a range of $\rho_c$) into the complexity equation. We also show how its two components, the information entropy and disequilibrium, behave when plotted against the mass ratio.

Figure 3. The $H$, $D$, $C$ vs $M$ diagrams of neutron stars with nuclear matter and hyperonic cores.

The information entropy and disequilibrium diagrams behave like what have been shown by Chatzisavvas and others. What is interesting to note is that our complexity diagrams have turning points at the maximum complexity values very much like matter phase diagram shown earlier in Fig. 1, which suggests that given a sufficiently representative model, all isolated system would have their complexity
diagrams shaped not unlike those above. In our case, the leftmost part appears due to the presence of crust which itself is an orderly system due to decreasing complexity values. The turning point where the $C_{max}$ is could perhaps be interpreted as the transition point between crust and core, but further work has to be done.

The complexity-mass curves don’t actually terminate at the rightmost part but they have yet another turning point as shown by the following example.

![Figure 4](image1.png)

**Figure 4.** A zoomed-in view of the right tail section of the BSR9-hyperonic model’s complexity curve.

The maximum value of the mass corresponds perfectly to that derived from the TOV equation (mass-radius diagram) and it’s actually constrained by it.

Finally, using this compound diagrams below, we show how $H$, $D$, and $C$ relate to each other.

![Figure 5](image2.png)

**Figure 5.** Relation of $H$, $D$, and $C$ to $M$. 
This diagram shows that at $M_{\text{max}}$, NS has the same properties as a perfect crystal because of the low value of information entropy $H$ but the disequilibrium $D$ reaches its max. Thus we can infer that an NS is modeled as a liquid but has the properties of a perfect crystal.

### 4. Conclusion

The results show us that using RMF model of NS also produces a similar result compared to earlier work by Chatzisavvas et al., that NS in general is a system of low complexity. This RMF model also shows how the complexity raises and then drops again in the low mass region, which we assume is the crust. We think that this inclusion of crust finally shows of the complete picture of an NS.

As also has been already known, hyperons make the EoS softer which lowers the possible $M_{\text{max}}$ value. But all of these $M_{\text{max}}$ values appear to have uniformly low complexity values, which further cements the conclusion that NS is a highly ordered system of low complexity.

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