Explosive Contagion in Networks (Supplementary Information)

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SIS SYNERGY MODEL WITH LINEAR SYNERGY RATE.

In this section, we solve the SIS model exactly for a population of individuals having the network of contacts with the topology of random $z$-regular graph with linear dependence of transmission rate on the number of ignorant/healthy neighbours and demonstrate that this solution is analogous to that for the exponential dependence of $\sigma_z(n^h(i))$ discussed in the main text.

The equilibrium states correspond to the solutions of Eq. (6) in the main text which can be recast in the following form:

$$F_1(y) = -y(\mu - z \lambda_z (1 - y)) = 0.$$  (1)

In the synergy-free case, $\lambda_z = \alpha$ and the stable solution of Eq. (1) is $y = 0$ for $\alpha \leq \alpha_c = \mu / z$ and $y = 1 - \mu / (\alpha z)$ for $\alpha > \alpha_c$.

In the model with linear synergy, the transmission rate is given by $\lambda_z = \alpha (1 + \beta z (1 - y))$ and $F_1(y)$ is a third order polynomial in $y$ which can have from one to three real roots. The equilibrium solution, $y_{sf} = 0$ (spreader-free regime), is always present while the two other roots,

$$y_{\pm} = 1 + \frac{1}{2 \beta z} \left(1 \pm \sqrt{1 + \frac{4 \beta \mu}{\alpha}}\right),$$  (2)

are real only for $\alpha \geq \alpha^*(\beta)$, where

$$\alpha^*(\beta) = -4 \beta \mu.$$  (3)

The values of $y_{\pm}$ represent equilibrium concentrations of spreaders and thus must be in the range, $y_{\pm} \in [0, 1]$. An equilibrium concentration, $y_{eq}$, corresponding to a root of $F_1(y)$ can refer to either stable (if $F_1'(y_{eq}) < 0$) or unstable (if $F_1'(y_{eq}) > 0$) equilibrium.

In the $\alpha - \beta$ parameter space, there is a special (tricritical) point, $(\alpha_{tp}, \beta_{tp}) = (2 \mu / z, -1 / (2 z))$ (see the point labelled by TP in Fig. 1), at which all three roots of $F_1(y)$ coincide, i.e. $y_{sf} = y_- = y_+ = 0$. This point separates the regimes of explosive and continuous transitions between non-invasive (spreader-free) and invasive (endemic) epidemics. Fig. 2 shows the dependence of the equilibrium concentration of spreaders, $y_{eq}$, on $\alpha$ for fixed value of $\beta$ above (panel (a)) and below (panel (b)) the tricritical point. For fixed $\beta$ above the tricritical point, $\beta > \beta_{tp}$, and values of $\alpha$ smaller than critical value,

$$\alpha_c(\beta) = \frac{\mu}{z + \beta z^2},$$  (4)

both roots $y_{\pm}$ are outside the physical range $[0, 1]$ and the only stable equilibrium at $y_{sf} = 0$ corresponds to the spreader-free state (cf. Fig. 2(a)). For $\alpha = \alpha_c(\beta)$ (see the solid line in Fig. 1), the root $y_-$ intersects the allowed range $[0, 1]$ at a point where $y_+ = y_{sf} = 0$. With increasing value of $\alpha > \alpha_c(\beta)$, the equilibrium concentration $y_-$ continuously increases in the interval $[0, 1]$ and it corresponds to the stable equilibrium ($F_1'(y_-) < 0$) while the spreader-free equilibrium, $y_{sf} = 0$, is unstable ($F_1'(y_{sf}) > 0$) for these values of $\alpha$. This means that an increase in the inherent transmission rate at fixed $\beta > \beta_{tp}$ drives the system continuously from spreader-free ($\alpha \leq \alpha_c(\beta)$) to endemic ($\alpha > \alpha_c(\beta)$) state (the region above continuous line in Fig. 1).

For values of $\beta$ below the tricritical point, $\beta < \beta_{tp}$, the scenario is very different from that described above (see Fig. 2(b)). Indeed, if $\alpha < \alpha_*$, the only acceptable root of $F_1$ is $y_{sf}$ which corresponds to the stable spreader-free state (the region below the dashed line in Fig. 1). At $\alpha = \alpha_c(\beta)$, the roots $y_{\pm}$ become real and take values in the range $(0, 1)$, i.e. $0 < y_+ = y_- < 1$. With increasing $\alpha$ in the interval $\alpha \in (\alpha_c(\beta), \alpha_c(\beta))$ (the region between dashed and dot-dashed lines in Fig. 1) at fixed $\beta$, these two roots split in such a way that $0 < y_+ < y_- < 1$. The concentration $y_-$ corresponds to the stable equilibrium while $y_+$ to the unstable one. Overall, there are two stable equilibria describing the spreader-free state with concentration of spreaders $y_{sf} = 0$ and endemic state with concentration of spreaders equal to $y_-$. The finite gap between these two equilibrium states is a signature of discontinuous explosive transition between non-invasive and invasive epidemics. With further increase of $\alpha$ for fixed value of $\beta$, the root $y_+$ leaves the physical range $[0, 1]$ when $\alpha = \alpha_c(\beta)$ (and $y_+ = 0$), and the only stable equi-
librium at \( y_- \) corresponds to the endemic state (the region above the dot-dashed line in Fig. 1). In the bi-stable regime with \( \alpha \in (\alpha_*, \alpha_c(\beta)) \), the mean-field system, depending on initial conditions, reaches the spreader-free regime, \( y_{sf} \), or the endemic regime, \( y_+ \). The dotted lines in Fig. 2(b) indicate the explosive transitions observed by increasing \( \alpha \) from \( \alpha < \alpha^* \) (up arrow) or decreasing from \( \alpha > \alpha^* \) (down arrow). A hysteresis loop of width \( \alpha_c - \alpha^* \) becomes wider as \( \beta \) becomes more negative.

MODELS WITH REMOVAL OF SPREADERS ON \( z \)-RANDOM REGULAR GRAPHS

In this section, we derive the general solution (Eq. (15) of the main text) for the mean-field models with removal of spreaders and illustrate its properties using the SIR model with linear synergistic transmission rate as a benchmark.

From Eqs. (11)-(13) of the main text and the definition of \( \lambda_z(x) = \alpha \sigma_z(x) \), one obtains,

\[
y = -\frac{1}{\alpha z \sigma_z(x) x} \frac{dx}{dt} = \frac{1}{\mu \gamma(x)} \frac{dr}{dt} . \tag{5}
\]

Integrating the second equation in Eq. (5) over time in the interval \([0, t]\) leads to the following expression:

\[
- \int_{x_0}^{x(t)} \frac{\gamma(x)}{\sigma_z(x) x} \, dx = \frac{z \alpha}{\mu} \int_0^{r(t)} dr . \tag{6}
\]

Here, we have assumed a population which initially consists of only ignorants and spreaders, \( i.e., \), \( r(0) = 0 \), \( x(0) = x_0 \leq 1 \) and \( y(0) = 1 - x_0 \). From Eq. (6), the concentration of removed individuals over time, \( r(t) \), can be expressed as a function of the concentration of ignorants as follows:

\[
r(t) = \frac{\mu}{\alpha z} \left[ F_2(x_0) - F_2(x(t)) \right] . \tag{7}
\]

The function \( F_2(x) \) is defined in Eq. (16) of the main text.

The fixed points of the system given by Eqs. (11)-(13) in the main text correspond to states without spreaders, \( y = 0 \). In general, any finite system with an initially positive concentration of spreaders, \( y_0 > 0 \), and positive removal rate, \( \gamma(x) > 0 \), evolves towards a fixed point with \( y = 0 \), \( x = x_\infty \) and \( r_\infty = 1 - x_\infty \). The condition \( y = 0 \) points out the end of the epidemic. Examples of the evolution of \( x \) and \( r \) are shown in Figs. 3 and 4 for the
SIR model with linear synergy for several values of $x_0$, $\alpha$ and $\beta$. The value of the final concentration of ignorant, $x_\infty$ (or removed, $r_\infty = 1 - x_\infty$), depends in general on the initial concentration of ignorant, $x_0 = 1 - y_0$, the inherent transmission rate, $\alpha$, as well as on the synergistic and recovery mechanisms encoded by the functions $\sigma_z$ and $\gamma$, respectively. Such dependence can be recast from Eq. (7) in the implicit form given by Eq. (15) of the main text which we repeat here for convenience:

$$\alpha = f(x_\infty; x_0) \equiv \frac{\mu}{z(1 - x_\infty)} [F_2(x_0) - F_2(x_\infty)] . \quad (8)$$

It is clear from Eq. (8) that systems characterised by a function $f(x_\infty; x_0)$ that decreases monotonically with $x_\infty$ will exhibit continuous transitions from smaller to larger $r_\infty$ (from larger to smaller $x_\infty$) with increasing $\alpha$. Examples of this type of behaviour of $f(x_\infty; x_0)$ are shown by the continuous lines in Fig. 5 for the SIR model with linear synergy rate. In contrast, discontinuous transitions can occur when $f(x_\infty; x_0)$ is not monotonic and it increases with $x_\infty$ in some sub-interval of $(0, 1)$. In this case, Eq. (8) can have several solutions for $x_\infty$, corresponding to several fixed points (cf. dashed lines in Fig. 5). The evolution given by Eqs. (11)-(13) in the main text is such that $x$ decreases with time from $x_0$ and the system evolves towards the solution corresponding to the largest value of $x_\infty$; the rest of solutions are not accessible to the system. The trajectories of the SIR model with linear synergy shown in Fig. 4 illustrate this behaviour. In particular, the trajectory for $\alpha_e$ shows both the reachable (continuous line) and unreachable (dotted line) solutions of Eq. (8).

As mentioned in the main text, the regimes with continuous and explosive transitions are separated by a critical regime for which $f(x_\infty; x_0)$ displays an inflection point at some value of $x_\infty = x_{\text{tp}} \in (0, 1)$. This situation corresponds to the tricritical point discussed in the main text. At the inflection point,

$$\left. \frac{\partial f(x; x_0)}{\partial x} \right|_{x_{\text{tp}}} = \left. \frac{\partial^2 f(x; x_0)}{\partial x^2} \right|_{x_{\text{tp}}} = 0 . \quad (9)$$

These conditions and definition of $f(x_\infty; x_0)$ given by Eq. (8) result in Eqs. (17) and (18) given in the main text.
FIG. 5. Function $f(x; x_0)$ defined by Eq. (8) with $x_\infty$ replaced by $x$, for (a) $x_0 = 1$ and (b) $x_0 = 0.95$ corresponding to SIR epidemics with linear synergy spreading on random regular graphs with $z = 2$. In both panels, the continuous and dashed lines correspond to $\beta = 0$ and $\beta = -0.45$, respectively. The horizontal dot-dashed line in (b) illustrates the solutions (circles) of Eq. (8) for $\alpha = 2$. The system evolves towards the largest solution and reaches the final concentration of ignorants $x_\infty = x_a$.

The following relations at the tricritical point:

$$x_{tp} = -\frac{1}{2z\beta_{tp}},$$

$$x_0 = -\frac{1}{z\beta_{tp}(1 + e^{2(2z\beta_{tp}+1)})},$$

$$\alpha_{tp} = -4\beta_{tp}\mu.$$  

Fig. 6 shows the phase diagram for the SIR model with linear synergistic transmission for two initial conditions: $x_0 = 1$ (i.e. a negligible initial concentration of infecteds, $y_0$) and $x_0 = 0.95$. For $x_0 = 1$, one obtains $\beta_{tp} = -1/(2z)$ from Eq. (11) which leads to $x_{tp} = 1$ and $\alpha_{tp} = -2\mu/z$. The value of $\beta_{tp}$ decreases with $x_0$ (see Fig. 7). This implies that social phenomena starting with a relatively large initial concentration of spreaders, $y_0 = 1 - x_0$, will require larger synergistic effects of the context in order for them to be explosive. However, explosive transitions exist for any initial conditions with $x_0 > 0$ since $\beta_{tp}$ is finite for any $x_0 > 0$ (from Eq. (11), it is clear that $\beta_{tp} \to -\infty$ only for $x_0 \to 0$).


FIG. 6. Contagion phase diagram for the SIR model on random \( z \)-regular graphs. The continuous black line indicates the invasion threshold for continuous transitions observed for an initial concentration of ignorants, \( x_0 = 1 \). The solid straight line displays the locus of tricritical points given by Eq. (12). The blue and green dashed lines give the explosive invasion threshold for epidemics with \( x_0 = 1 \) and \( x_0 = 0.95 \), respectively. Numerical values along the axes correspond to random \( z \)-regular graphs with \( z = 2 \) and \( \mu = 1 \).

FIG. 7. Graphical representation of the dependence of \( \beta_{tp} \) on \( x_0 \) for the SIR model with linear synergy rate (cf. Eq. (11)).