Predicting water flow in fully and partially saturated porous media: a new fractal-based permeability model

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Abstract
Predicting the permeability of porous media in saturated and partially saturated conditions is of crucial importance in many geo-engineering areas, from water resources to vadose zone hydrology or contaminant transport predictions. Many models have been proposed in the literature to estimate the permeability from properties of the porous media such as porosity, grain size or pore size. This study develops a model of the permeability for porous media saturated by one or two fluid phases with all physically based parameters using a fractal upscaling technique. The model is related to microstructural properties of porous media such as fractal dimension for pore space, fractal dimension for tortuosity, porosity, maximum radius, ratio of minimum pore radius and maximum pore radius, water saturation and irreducible water saturation. The model is favorably compared to existing and widely used models from the literature. Then, comparison with published experimental data for both unconsolidated and consolidated samples shows that the proposed model estimates the permeability from the medium properties very well.

Keywords Permeability · Water saturation · Fractal · Porous media · Porosity

Introduction
Climate change, modification of land use, and groundwater and soil contamination present society with significant challenges when faced with increasing demand for water. Understanding and predicting water flow in critical zones, especially in aquifers and soils, is of primary concern for many environmental studies and research areas (e.g., Fan et al. 2019). Permeability is one of the most crucial parameters to describe fluid flow in porous media in general (e.g., Darcy 1856; Bear 1972). Laboratory studies have shown that the permeability depends on rock properties such as porosity, cementation, pore size, pore size distribution, pore shape and pore connectivity (e.g., Rahimi 1977; Lis 2019; Ghanbarian 2020a). The permeability or relative permeability of porous media saturated by one or two phases is the key parameter that governs fluid flow in porous material and therefore plays an important role in modeling and predictions in various environmental and resources engineering. Conventionally, the permeability of porous media is determined directly in the laboratory using steady-state or unsteady-state methods. Due to the complex geometric microstructure and multiscale pore structure of porous media, much effort has been devoted to predicting permeability. The experimental approaches for the permeability determination vary from simple measurements (e.g., Rahimi 1977; Boulin et al. 2012; Sander et al. 2017) to indirect methods such as nuclear magnetic resonance measurements (e.g., Coates et al. 1991; Hidajat et al. 2002; Ioannidis et al. 2006), mercury injection capillary pressure measurement (Swanson, 1981), electrical conductivity measurement (e.g., Doussan and Ruy 2009; Jougnot et al. 2010; Revil and Cathles 1999) or spectral induced polarization measurements (e.g., Revil and Florsch 2010; Koch et al. 2012; Revil et al. 2014; Maineult et al. 2018). In the literature, permeability prediction has been proposed through theoretical models with simplified pore geometries (e.g., Kozeny 1927; Carman 1938; Bear 1972; Dullien 1992) but also advanced schemes such as effective-medium approximations (e.g., Doyen 1988; Richesson and Sahimi 2019) or critical path analysis (e.g., Katz and Thompson 1986; Hunt 2001; Daigle 2016;
Ghanbarian 2020a; Ghanbarian 2020b). Additionally, explicit numerical methods such as the finite-element, finite-difference, finite-volume, lattice Boltzmann, or pore-network modeling have been applied to predict the permeability of porous materials (among many other references: Ngo and Tamma 2001; Benzi et al. 1992; Bryant and Blunt 1992; De Vries et al. 2017).

It is shown that porous media exhibit fractal properties and their pore spaces are statistically self-similar over several length scales (e.g., Mandelbrot 1982; Katz and Thompson 1985). Theory on fractal porous media has attracted much attention in different areas (e.g., Mandelbrot 1982; Feder and Aharony 1989). Therefore, the models based on capillary tubes in combination with the fractal theory have been applied to study transport phenomena in both fully and partially saturated porous media (e.g., Li and Horne 2004; Guarracino 2007; Cai et al. 2012a, b; Liang et al. 2014; Guarracino and Jougnot 2018; Soldi et al. 2017, 2019, 2020; Thanh et al. 2018, 2019, 2020a, b) or to study hydraulic conductivity and biological clogging in bio-amended variably saturated soils (e.g., Rosenzweig et al. 2009; Samsó et al. 2016; Carles et al. 2017). The fractal theory has already been used to develop permeability models for porous materials—for example, Yu and Cheng (2002), Yu and Liu (2004) and Guarracino et al. (2014) developed a fractal permeability model for porous media under both saturated and partially saturated conditions. However, their models do not take into account the irreducible water saturation that is very important for porous media containing small pores. Moreover, their models have not been strongly validated due to only a few experimental data points used for comparison. Chen and Yao (2017) developed an improved model for the permeability estimation as an extension of Yu and Cheng (2002) and Yu and Liu (2004) by considering irreducible water saturation. The Chen and Yao (2017) model was verified by comparison with experimental data for natural sandstone samples whose pore size distribution is stated to be broader than that of samples such as glass/sand grains (e.g., Daigle 2016; Ghanbarian 2020a). Li and Horne (2004) derived a universal capillary pressure model using fractal geometry of porous media and obtained a relative permeability model using both the Purcell and the Burdine approaches, therefore obtaining a model that diverges from the fractal theory. Soldi et al. (2017) and Chen et al. (2020) proposed models to describe unsaturated flow considering the hysteresis phenomena. They assumed that porous media can be represented by a bundle of capillary tubes with a periodic pattern of pore throats and pore bodies and a fractal pore size distribution. Their models have been validated using experimental data for the relative permeability and for the hysteretic saturation curves. However, Soldi et al. (2017) did not consider the variation of the capillary length with radius in their model. Chen et al. (2020) did consider that but they only focused on the relative permeability and the water retention curve rather than the intrinsic permeability. Additionally, Soldi et al. (2017) and Chen et al. (2020) did not use much experimental data to validate their models. Similarly, Xiao et al. (2018) obtained a model for the capillary pressure and water relative permeability in unsaturated porous rocks based on the fractal distribution of pore size and tortuosity of capillaries. It is seen that the relative permeability of the water phase is a function of water saturation, porosity and the fractal dimension of the pores; however, Xiao et al. (2018) did not consider irreducible water saturation and did not have much experimental data to validate the model for the relative permeability. Recently, Meng et al. (2019) presented models for both electrical conductivity and permeability based on fractal theory by introducing the critical porosity under saturated conditions. From the obtained model, Meng et al. (2019) could explain the fact that the permeability of porous media could approach zero at a nonzero percolation porosity corresponding to a certain critical pore diameter. However, their model was validated by only two experimental data sets for the permeability as a function of porosity.

This work develops a model for the permeability of porous media containing two fluid phases in which the fractal theory and capillary tube model are utilized. The model is related to microstructural properties of porous media such as fractal dimension for pore space, fractal dimension for tortuosity, porosity, maximum radius, ratio of minimum pore radius and maximum pore radius, water saturation and irreducible water saturation. All model parameters are physically based parameters. The proposed model takes into account irreducible water saturation, the variation of the capillary length with radius. Then, the model for the saturated permeability $k_s$ and the relative permeability of the wetting phase $k_w^*$ are validated using large published experimental data sets on 111 unconsolidated and consolidated samples. The proposed model is also compared to existing and widely used models from the literature.

Model development

Flow rate at the macroscale

Porous materials can be conceptualized as a bundle of tortuous capillaries following a fractal pore-size distribution (e.g., Yu and Cheng 2002; Soldi et al. 2019; Thanh et al. 2019, 2020c). To derive analytical expressions for the permeability of porous media, this study first considers a representative elementary volume (REV) of porous media as a cube with the length of $L_0$ (see Fig. 1). The pore radius $R$ of the REV varies from a minimum value $R_{\text{min}}$ to a maximum value $R_{\text{max}}$ and conforms to the fractal scaling law. Namely, the cumulative size-distribution of pores is assumed to obey the following (e.g., Yu and Cheng 2002; Yu and Liu 2004):
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Fig. 1 A porous rock model composed of a large number of parallel capillary tubes that are either saturated by water or filled by air, depending on the capillary pressure. Note that the tortuosity of the capillaries depends on their radii.

\[ N(R) = \left( \frac{R_{\text{max}}}{R} \right)^{D_t} \]

(1)

where \( N \) is the number of capillaries (whose radius \( \geq R \)) in the REV, \( D_t \) is the fractal dimension for pore space, \( 0 < D_t < 2 \) in two-dimensional (2D) space, and \( 0 < D_t < 3 \) in three-dimensional (3D) space. Differentiating Eq. (1) with respect to \( R \), one obtains the number of pores with radii between \( R \) and \( R + dR \) as

\[ -dN = D_t R_{\text{max}}^{D_t} R^{-D_t - 1} dR \]

(2)

The negative sign in Eq. (2) implies that the number of pores decreases when the pore radius increases.

The real length of the capillary tubes \( L_\tau \) along the flow direction is generally greater than the length of the porous media \( L_0 \) (see Fig. 1). The length \( L_\tau \) is related to the pore radius \( R \) as (e.g., Yu and Cheng 2002; Yu and Liu 2004):

\[ L_\tau(R) = R^{1-D_t} L_0^{D_t} \]

(3)

where \( D_t \) is the fractal dimension for the tortuosity \( 1 \leq D_t \leq 2 \). From Eq. (3), the tortuosity \( \tau \) is determined as

\[ \tau(R) = \frac{L_\tau}{L_0} = \left( \frac{L_0}{R} \right)^{D_t - 1} \]

(4)

The fractal dimensions \( D_t \) and \( D_\tau \) can be experimentally determined by a box-counting method (e.g., Yu and Cheng 2002; Yu and Liu 2004). In this work, they are estimated from properties of porous media. Namely, the expression for \( D_t \) is given by (e.g., Yu and Cheng 2002; Yu and Liu 2004)

\[ D_t = 2 - \frac{\ln \phi}{\ln \alpha} \]

(5)

where \( \phi \) is the porosity of porous media and \( \alpha = R_{\text{min}}/R_{\text{max}} \). The expression for \( D_\tau \) is given by (e.g., Wei et al. 2015)

\[ D_\tau = (3-D_\tau) + \frac{\ln \left( \frac{D_\tau}{2F\phi^2} \right)}{\ln \phi} \]

(6)

where \( F \) is the formation factor of porous media.

The REV under partially saturated conditions was considered. The REV is assumed to be initially fully saturated and then drained when it is subjected to a pressure head \( h \) (m). For a capillary tube, the pore radius \( R_{\text{h}} \) (m) is linked to a pressure head \( h \) by (Jurin 1719)

\[ h = \frac{2\sigma \cos \beta}{\rho_w g R_{\text{h}}} \]

(7)

where \( \sigma \) (N/m) is the surface tension of the fluid, \( \beta \) is the contact angle, \( \rho_w \) (kg/m³) is the fluid density and \( g \) (m/s²) is the acceleration due to gravity. A capillary tube of porous material becomes fully desaturated under the pressure head \( h \) if its radius \( R \) is larger than \( R_{\text{h}} \) determined by Eq. (7). Hence, under the pressure head \( h \), the capillaries with radii in the range between \( R_{\text{min}} \) and \( R_{\text{h}} \) will be fully saturated.

For porous media containing small pores, the irreducible water saturation can be pretty significant since water is kept in micropores (e.g., Carsel and Parrish, 1988; Jougnot et al. 2012). This amount of water is considered in this work by setting an irreducible water radius of capillaries \( R_{\text{wirr}} \). Hence, the following is assumed: (1) for \( R_{\text{min}} \leq R \leq R_{\text{wirr}} \), the pores are filled by water that is immobile due to insufficient driving force, so it does not contribute to the water flow; (2) for \( R_{\text{wirr}} < R \leq R_{\text{h}} \), the pores are filled by mobile water, so it contributes to the water flow; (3) for \( R_{\text{h}} < R \leq R_{\text{max}} \), the pores are filled by air, so it does not contribute to the water flow (see Fig. 1). Note that film-bound water, which adheres to the pore wall because of the molecular forces acting on the hydrophilic mineral surface, is neglected in the pores with radius larger than \( R_{\text{wirr}} \). Under those assumptions, the irreducible water saturation is defined as

\[ S_{\text{wirr}} = \frac{\int_{R_{\text{min}}}^{R_{\text{wirr}}} \pi R^2 L_\tau(-dN)}{\int_{R_{\text{min}}}^{R_{\text{max}}} \pi R^2 L_\tau(-dN)} \]

(8)

Combining Eqs. (2), (3) and (8) yields the following

\[ S_{\text{wirr}} = \frac{R_{\text{wirr}}^{3-D_\tau - 1 - R_{\text{min}}^{3-D_\tau - 1}}}{R_{\text{max}}^{3-D_\tau - 1 - R_{\text{min}}^{3-D_\tau - 1}}} \]

(9)

Similarly, water saturation is defined as

\[ S_w = \frac{\int_{R_{\text{h}}}^{R_{\text{wirr}}} \pi R^2 L_\tau(-dN)}{\int_{R_{\text{min}}}^{R_{\text{max}}} \pi R^2 L_\tau(-dN)} = \frac{R_{\text{h}}^{3-D_\tau - 1 - R_{\text{min}}^{3-D_\tau - 1}}}{R_{\text{max}}^{3-D_\tau - 1 - R_{\text{min}}^{3-D_\tau - 1}}} \]

(10)
Additionally, the volume flow rate in a single pore of radius \( R \) (m) and length \( L_c \) (m) is given by Poiseuille’s law

\[
q(R) = \frac{\rho g \pi R^4 \Delta h}{8 \eta L_c} \tag{11}
\]

where \( \rho \) (kg/m³), \( \eta \) (Pa·s) are the density and viscosity of fluid, respectively, and \( \Delta h \) (m) is the pressure head drop across the REV.

The volumetric flow rate through the REV is the sum of the volumetric flow rates over all individual capillary tubes filled with water (wetting phase) and given by

\[
Q_w^{REV} = \int_{R_{wir}}^{R_h} \rho g \pi R^4 \frac{\Delta h_w}{L_c(R)} \, (-dN) \tag{12}
\]

From Eqs. (2), (3), (11) and (12), the following was obtained

\[
Q_w^{REV} = \frac{\rho g \Delta h_w}{8 \eta_w \bar{\alpha} \pi D_f^3} \left( R_h^{3+D_f} - R_w^{3+D_f} \right) \frac{1}{3 + D_c - D_f} \tag{13}
\]

From Eqs. (8) and (10) one has

\[
\bar{\alpha} = R_{\text{max}} \left[ \alpha^{3-D_c-D_f} + S_w (1-\alpha^{3-D_c-D_f}) \right] \frac{1}{3+D_c-D_f} \tag{14}
\]

and

\[
R_h = R_{\text{max}} \left[ \alpha^{3-D_c-D_f} + S_w (1-\alpha^{3-D_c-D_f}) \right] \frac{1}{3+D_c-D_f} \tag{15}
\]

And recall that \( \alpha = R_{\text{min}}/R_{\text{max}} \). From Eqs. (13), (14) and (15), one obtains

\[
Q_w^{REV} = \frac{\rho g \Delta h_w \pi D_f R_{\text{max}}^{3+D_f \bar{\alpha}}}{8 \eta_w \bar{\alpha} \pi D_f^3} \times \left\{ \left[ \alpha^{3-D_c-D_f} + S_w (1-\alpha^{3-D_c-D_f}) \right] \frac{1}{3+D_c-D_f} \right\} \tag{16}
\]

**Permeability**

The total volumetric flow rate through the REV can be expressed as (Buckingham 1907)

\[
Q_w^{REV} = \frac{\rho g \pi D_f^3}{\eta_w k_s k_w} \frac{\Delta h_w}{L_o} A_{REV} \tag{17}
\]

where \( k_s \) (m²) is the saturated permeability, \( k_w \) (no units) is the relative permeability for the wetting phase, and \( A_{REV} \) is the cross-sectional area of the REV.

\[
\phi = \frac{V_{\text{pore}}}{V_{\text{REV}}} = \frac{\pi R_{\text{max}}^2 L_c (-dN)}{L_o A_{REV}} \tag{18}
\]

Combining Eqs. (16), (17) and (18), the following is obtained

\[
k_w = \frac{R_{\text{max}}^{3D_f}}{8 L_o \phi \Delta h_w} \frac{1}{3 + D_c - D_f} \left\{ \left[ \alpha^{3-D_c-D_f} + S_w (1-\alpha^{3-D_c-D_f}) \right] \frac{1}{3+D_c-D_f} \right\} \tag{19}
\]

Using Eq. (19) and invoking \( k_w = 1 \) at \( S_w = 1 \), the following is obtained

\[
k_i = \frac{R_{\text{max}}^{3D_f}}{8 L_o \phi \Delta h_w} \frac{1}{3 + D_c - D_f} \left\{ \left[ \alpha^{3-D_c-D_f} + S_w (1-\alpha^{3-D_c-D_f}) \right] \frac{1}{3+D_c-D_f} \right\} \tag{20}
\]

Equation (20) can be written as

\[
k_i = \frac{R_{\text{max}}^{3D_f}}{8 L_o \phi \Delta h_w} \frac{1}{3 + D_c - D_f} \left\{ \left[ \alpha^{3-D_c-D_f} + S_w (1-\alpha^{3-D_c-D_f}) \right] \frac{1}{3+D_c-D_f} \right\} \tag{22}
\]

where \( \tau^{eff} \) is given by

\[
\tau^{eff} = \left( \frac{L_o}{R_{\text{max}}} \right)^{D_f-1} \tag{23}
\]

and that is defined as the effective tortuosity of the porous medium as inferred from Eq. (4) (Thanh et al. 2019, 2020c).

The length of the cubic REV is related to the cross-section area of the REV by

\[
L_o^2 = A_{REV} \tag{24}
\]

From Eqs. (18), (23) and (24), one has

\[
\tau^{eff} = \left[ 1 - \frac{\pi D_f}{\phi} \right]^{D_f-1} \tag{25}
\]

Equations (21) and (22) are the key contributions of this work. These equations show that the saturated permeability
and the relative permeability for the wetting phase are functions of microstructural parameters of porous media ($D_r, D_c, \phi, \alpha, R_{\text{max}}$), $S_w$ and $S_{\text{wirr}}$. Therefore, the proposed model provides an insight into the dependence of the saturated permeability ($k_s$) and the relative permeability ($k_r^w$) on the microstructural parameters of the porous media and it may reveal more mechanisms affecting the $k_s$ and $k_r^w$ than other models. In particular, the proposed model contains physically based parameters and that is different from some other models in literature (see Table 2) with empirical parameters such as $m$ in the RC model, $b$ which is normally taken as 180 in the KC model, $a$ and $m$ in the RGPZ model or $c$ which is normally taken as 72.2 in the CPA model.

In the case of $R_{\text{max}} >> R_{\text{min}}$ ($\alpha \rightarrow 0$), which is normally acceptable for porous rocks (see Guarracino, 2007; Soldi et al. 2019), and the negligible irreducible water saturation $S_{\text{wirr}} = 0$, Eqs. (21) and (22), respectively, become

$$k_r^w = S_w^{\frac{1-D_r-D_c}{3-D_r-D_c}}$$  \hspace{1cm} (26)

and

$$k_s = \frac{R_{\text{max}}^2 \phi (3-D_r-D_c)}{8(\tau_{\text{eff}})^2 (3 + D_c-D_r)}$$  \hspace{1cm} (27)

It is seen that Eq. (26) is similar to the power law of the Burdine-Brooks-Corey model (Brooks and Corey 1964; Burdine 1953) that is given by Ghanbarian et al. (2017a):

$$k_r^w = S_w^{\mu+1+\frac{1}{2}}$$  \hspace{1cm} (28)

where $\mu$ is the empirical tortuosity-connectivity exponent and $\lambda$ is the pore size distribution index. Obviously, the number of parameters in Eq. (26) ($S_w, D_r$ and $D_c$) is the same as that in Eq. (28) ($S_w, \mu$ and $\lambda$). Equation (27) has five parameters ($R_{\text{max}}, \tau_{\text{eff}}, \phi, D_r$ and $D_c$) which are comparable with the number of parameters in some other models in the literature as reported in Table 2 (e.g., three parameters of $d$, $m$ and $F$ in the RC model, four parameters of $d$, $\phi$, $a$ and $m$ in the RGPZ model, three parameters of $d_c$, $c$ and $F$ in the CPA model).

If one does not consider the variation of tortuosity with the capillary radius then $D_c = 1$ and Eq. (26) becomes

$$k_r^w = S_w^{\frac{4-D_c}{3-D_c}}$$  \hspace{1cm} (29)

Equation (29) is the same as that in Soldi et al. (2019).

**Results and discussion**

To predict $k_r^w$ from Eq. (21) and $k_s$ from Eq. (22), one needs to know parameters: $D_b, D_c, R_{\text{max}}, \phi, F, \alpha, S_w$ and $S_{\text{wirr}}$. Note that these model parameters are physically-based parameters. For example, the fractal dimension of the capillary size distribution $D_f$ represents the heterogeneity of the porous medium. The greater the fractal dimension, the more heterogeneous the porous media (e.g., Li and Horne 2004; Othman et al. 2010; Zainaldin et al. 2017). The fractal dimension of the tortuosity $\tau_f$ represents the extent of convolutedness of capillary pathways for fluid flow through porous media; $D_c = 1$ corresponds to straight capillary paths and a higher value of $D_c$ corresponds to a highly tortuous capillary in porous media (e.g., Yu and Cheng 2002; Yu and Liu 2004; Othman et al. 2010). The other parameters such as $R_{\text{max}}, \phi, F$ and $S_w$ and $S_{\text{wirr}}$ can be determined in the lab—for example, the porosity $\phi$ can be measured by different methods such as the mercury porosimetry, helium pycnometry, image analysis and water absorption, among other ones (e.g., Andreola et al. 2000; Nnaemeka 2010). The grain diameter $d$ can be determined by techniques such as the sieve analysis, the laser diffraction, the microscopy technique and others (e.g., Li et al. 2005; Abbireddy and Clayton 2009). The formation factor $F$ can be measured by an approach presented by Jouniaux et al. 2000 or Vinogradov et al. 2010, for example. In the context of a bundle of capillary tubes model, $R_{\text{min}}$ and $R_{\text{max}}$ correspond to the sizes of pores invaded by the nonwetting phase at the maximum and minimum values of capillary pressure. Therefore, they can be estimated by measuring the maximum capillary pressure and the minimum capillary pressure, respectively, then using the Young–Laplace equation (e.g., Ghanbarian et al. 2017b). Additionally, Daigle 2016 determined $R_{\text{min}}, R_{\text{max}}$ and therefore $\alpha$ from the micro-CT images and the nuclear magnetic resonance (NMR) measurements. He combined the micro-CT and the NMR data to provide a continuous pore size distribution in pores and therefore obtained $R_{\text{min}}, R_{\text{max}}$ and $\alpha$. Note that $S_{\text{wirr}}$ can be obtained from the soil-water retention curves that are measured by methods such as the pressure plate, tensiometers, or pressure membranes, for example (e.g., Lourenço et al. 2007; Abeykoon et al. 2017).

If the pore size distribution is unknown, the $R_{\text{max}}$ for nonconsolidated granular media can be estimated by the following (Cai et al. 2012a; Liang et al. 2014):

$$R_{\text{max}} = \frac{d}{8} \left[ \left( \frac{2\phi}{1-\phi} \right) + \left( \frac{\phi}{1-\phi} \right) + \sqrt{\frac{\pi}{4(1-\phi)^3}} \right]$$  \hspace{1cm} (30)

Based on the published work from Ghanbarian 2020a or Reviland Cathles 1999, this study uses $S_{\text{wirr}}=0$ for the intrinsic permeability to simplify parameter optimization. For the relative permeability estimation, $S_{\text{wirr}}$ is obtained through an optimization procedure. Namely, the optimization of parameters is based on data fitting and then calculating the root-mean-square error (RMSE). Model parameters are then
determined by seeking a minimum RMSE through the “fminsearch” function in MATLAB. This work uses the “fminsearch” function to optimize parameters of \( \alpha \), \( D_\tau \) and \( D_r \) for the intrinsic permeability and to optimize parameters of \( \alpha \), \( D_\tau \), \( D_r \) and \( S_{wirr} \) for the relative permeability.

**Saturated permeability**

To study the model sensitivity with model parameters such as \( \phi \), \( \alpha \), \( S_{wirr} \) and \( D_\tau \), from Eq. (22) it is possible to predict the variation of \( k_s \) with porosity as shown in Fig. 2a and with irreducible water saturation as shown in Fig. 2b. Figure 2a is obtained with \( S_{wirr} = 0 \), \( D_\tau = 1.1 \), \( R_{\text{max}} = 40 \mu\text{m} \) and three values of \( \alpha \) (0.0001, 0.001 and 0.01). Figure 2b is obtained with \( \alpha = 0.01 \), \( \phi = 0.2 \), \( R_{\text{max}} = 40 \mu\text{m} \) and three values of \( D_\tau \) (1.1, 1.15 and 1.2). Note that the input parameters for Fig. 2 are normally in the range reported in literature for porous media—for example, \( \alpha \) is commonly between 0.0001 and 0.01 (e.g., Wei et al. 2015; Thanh et al. 2020c); \( D_\tau \) is normally reported to be around 1.1 (e.g., Chen et al. 2020) and \( R_{\text{max}} \) is reported to be tens of micrometers in geological media (e.g., Hu et al. 2017). It is seen that the permeability \( k_s \) is sensitive to \( \phi \), \( \alpha \), \( S_{wirr} \) and \( D_\tau \). Namely, \( k_s \) increases with increasing porosity as expected by other models (e.g., Kozeny 1927; Revil and Cathles 1999) and with increasing \( \alpha \) as predicted by Xu and Yu (2008). Additionally, \( k_s \) decreases with an increase of \( S_{wirr} \). This is attributed to the fact that the larger \( S_{wirr} \) causes the total flow rate of the wetting phase to be smaller and therefore the permeability decreases. It is also seen that \( k_s \) decreases with an increase of \( D_\tau \). The reason is that when \( D_\tau \) increases, the flow pathways are more tortuous, causing more resistance for flow and lowers the permeability of the porous media. It should be noted that in Fig. 2, \( D_\tau \) is estimated from Eq. (5) with the knowledge of \( \phi \) and \( \alpha \).

Symbols \( d \), \( \phi \), \( F \), \( \alpha \), \( D_\tau \) and \( k_s \) stand for the grain diameter, porosity, formation factor, ratio of minimum and maximum radius, fractal dimension for pore space, fractal dimension for the tortuosity and permeability of samples, respectively. Superscripts ‘a’ measured quantities, ‘b’ estimated ones from Archie (1942), ‘c’ optimized ones by the “fminsearch” function in Matlab and ‘d’ predicted ones from Eqs. (5) and (6).

Figure 3 shows the comparison between the predicted permeability by Eq. (22) and the measured permeability for 111 uniform packs from different sources: (1) for glass beads (data from Bolève et al. 2007; Johnson et al. 1986; Chauveteau and
Table 1 Properties of the glass bead and sand packs used in this work

| Pack          | \(d^a (\mu m)\) | \(\phi^a\) (no units) | \(F^a, b\) (no units) | \(k_{\alpha s} \times 10^{-12} (m^2)\) | \(\alpha\) | \(D_T\) | \(D_T\) | Source          | Shown in       |
|---------------|-----------------|------------------------|------------------------|------------------------------------------|-----------|---------|---------|----------------|---------------|
| Glass bead    |                 |                        |                        |                                          |           |         |         |                 |               |
| 56            | 0.4             | 3.3^a                   | 2.0                    | 0.0103^c                                 | 1.803^c  | 1.05^c  | Bolève et al. | Fig. 3a       |
| 72            | 0.4             | 3.2^a                   | 3.1                    | 1.789^d                                  | 1.11^d   |         | (2007)   |                 |               |
| 93            | 0.4             | 3.4^a                   | 4.4                    | from                                     | from     |         |          |                 |               |
| 181           | 0.4             | 3.3^a                   | 27                     | Eq. (5)                                  | Eq. (6)  |         |          |                 |               |
| 256           | 0.4             | 3.4^a                   | 56                     | from                                     | from     |         |          |                 |               |
| 512           | 0.4             | 3.4^a                   | 120                    |                                          |          |         |          |                 |               |
| 3000          | 0.4             | 3.6^a                   | 14,000                 |                                          |          |         |          |                 |               |
| Glass bead    |                 |                        |                        |                                          |           |         |         |                 |               |
| 75            | 0.43            | 3.55^b                  | 5.3                    | 0.0090^c                                 | 1.900^c  | 1.14^c  | Johnson et al.| Fig. 3a       |
| 110           | 0.41            | 3.81^b                  | 8.6                    | 1.814^d                                  | 1.11^d   |         | (1986)   |                 |               |
| 500           | 0.41            | 3.81^b                  | 174.6                  | from                                     | from     |         |          |                 |               |
| Glass bead    |                 |                        |                        |                                          |           |         |         |                 |               |
| 11.5          | 0.41            | 3.81^b                  | 0.11                   | 0.0091^e                                 | 1.904^c  | 1.13^c  | Chauveteau and Zaitoun (1981) | Fig. 3a       |
| 15            | 0.41            | 3.81^b                  | 0.21                   |                                          | 1.809^d  | 1.11^d  |          |                 |               |
| 25            | 0.41            | 3.81^b                  | 0.66                   | from                                     | from     |         |          |                 |               |
| 45            | 0.41            | 3.81^b                  | 2.4                    | Eq. (5)                                  | Eq. (6)  |         |          |                 |               |
| 90            | 0.4             | 3.95^b                  | 8.4                    |                                          |          |         |          |                 |               |
| 225           | 0.4             | 3.95^b                  | 36.0                   |                                          |          |         |          |                 |               |
| 450           | 0.4             | 3.95^b                  | 137                    |                                          |          |         |          |                 |               |
| Glass bead    |                 |                        |                        |                                          |           |         |         |                 |               |
| 20            | 0.4009          | 3.90^a                  | 0.24                   | 0.0095^c                                 | 1.793^c  | 1.15^c  | Glover et al.| Fig. 3a       |
| 45            | 0.3909          | 4.02^a                  | 1.6                    |                                          | 1.797^d  | 1.12^d  | (2006)   |                 |               |
| 106           | 0.3937          | 4.05^a                  | 8.1                    |                                          | from     |         |          |                 |               |
| 250           | 0.3982          | 3.98^a                  | 50.5                   |                                          | Eq. (5)  | Eq. (6) |          |                 |               |
| 500           | 0.3812          | 4.09^a                  | 186.8                  |                                          |          |         |          |                 |               |
| 1,000         | 0.3954          | 3.91^a                  | 709.9                  |                                          |          |         |          |                 |               |
| 2,000         | 0.3856          | 4.14^a                  | 2,277.3                |                                          |          |         |          |                 |               |
| 3,350         | 0.3965          | 3.93^a                  | 7,706.9                |                                          |          |         |          |                 |               |
| Glass bead    |                 |                        |                        |                                          |           |         |         |                 |               |
| 3,000         | 0.398          | 4.21^a                  | 4,892                  | 0.0092^e                                 | 1.736^c  | 1.26^c  | Glover and Walker (2009) | Fig. 3a       |
| 4,000         | 0.385          | 4.38^a                  | 6,706                  |                                          | 1.896^d  | 1.12^d  |          |                 |               |
| 5,000         | 0.376          | 4.65^a                  | 8,584                  |                                          | from     |         |          |                 |               |
| 6,000         | 0.357          | 5.31^a                  | 8,262                  |                                          | Eq. (5)  | Eq. (6) |          |                 |               |
| 256           | 0.399          | 4.01^a                  | 41.2                   |                                          |          |         |          |                 |               |
| 512           | 0.389          | 4.36^a                  | 164                    |                                          |          |         |          |                 |               |
| 181           | 0.382          | 4.39^a                  | 18.6                   |                                          |          |         |          |                 |               |
| Glass bead    |                 |                        |                        |                                          |           |         |         |                 |               |
| 115           | 0.366          | 4.09^a                  | 8.8                    | 0.0097^e                                 | 1.753^c  | 1.11^c  | Kimura (2018) | Fig. 3b       |
| 136           | 0.364          | 4.20^a                  | 10.7                   |                                          | 1.780^d  | 1.11^d  |          |                 |               |
| 162           | 0.363          | 4.13^a                  | 18.3                   |                                          | from     |         |          |                 |               |
| 193           | 0.364          | 4.04^a                  | 26.7                   |                                          | Eq. (5)  | Eq. (6) |          |                 |               |
| 229           | 0.362          | 4.20^a                  | 33.0                   |                                          |          |         |          |                 |               |
| 273           | 0.358          | 4.17^a                  | 51.0                   |                                          |          |         |          |                 |               |
| 324           | 0.358          | 4.15^a                  | 67.4                   |                                          |          |         |          |                 |               |
| 386           | 0.356          | 4.36^a                  | 102.1                  |                                          |          |         |          |                 |               |
| 459           | 0.358          | 4.30^a                  | 134.3                  |                                          |          |         |          |                 |               |
| 545           | 0.36           | 4.06^a                  | 246.2                  |                                          |          |         |          |                 |               |
| 648           | 0.358          | 4.18^a                  | 299                    |                                          |          |         |          |                 |               |
| 771           | 0.357          | 4.29^a                  | 510.4                  |                                          |          |         |          |                 |               |
| 917           | 0.356          | 4.15^a                  | 611.9                  |                                          |          |         |          |                 |               |
| Silica sand   |                 |                        |                        |                                          |           |         |         |                 |               |
| 115           | 0.379          | 4.02^a                  | 7.0                    | 0.0066^e                                 | 1.789^c  | 1.15^c  |          |                 |               |
| 136           | 0.378          | 4.27^a                  | 10.9                   |                                          | 1.808^d  | 1.11^d  |          |                 |               |
| 162           | 0.378          | 4.21^a                  | 16.6                   |                                          | from     |         |          |                 |               |
| Pack            | \(d^a (\mu m)\) | \(\phi^a\) (no units) | \(F^a, b\) (no units) | \(k_s^a \times 10^{-12} (m^2)\) | \(\alpha\) | \(D_r\) | \(D_t\) | Source Shown in |
|----------------|------------------|------------------------|------------------------|----------------------------------|-------------|--------|--------|----------------|
| 193            | 0.378            | 4.16^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 229            | 0.38             | 4.24^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 273            | 0.38             | 4.15^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 324            | 0.38             | 4.07^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 386            | 0.38             | 4.12^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 459            | 0.381            | 4.17^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 545            | 0.383            | 4.09^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 648            | 0.385            | 4.12^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 771            | 0.388            | 4.10^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| 917            | 0.389            | 3.95^a                 |                        |                                  |             |        |        | Eq. (5) | Eq. (6) |
| Fujikawa sand  | 162              | 0.442                  | 3.75^a                 | 14.4                             | 0.0093^c     | 1.757^c| 1.20^c | –               | –            |
|                | 229              | 0.421                  | 3.83^a                 | 27.8                             | 1.814^d      | 1.12^d |        | Eq. (5) | Eq. (6) |
|                | 273              | 0.419                  | 3.79^a                 | 42.9                             | from         | from   |        | Eq. (5) | Eq. (6) |
|                | 324              | 0.416                  | 3.88^a                 | 56.5                             | Eq. (5)      | Eq. (6) |        | Eq. (5) | Eq. (6) |
|                | 386              | 0.413                  | 3.90^a                 | 81.8                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 459              | 0.414                  | 3.93^a                 | 123.8                            |             |        |        | Eq. (5) | Eq. (6) |
| Sand           | 150              | 0.45                   | 3.92^a                 | 6.7                              | 0.0092^c     | 1.612^c| 1.31^c| Biella et al. | Fig. 3c     |
|                | 300              | 0.43                   | 4.10^a                 | 49.2                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 500              | 0.40                   | 4.05^a                 | 107.7                            |             |        |        | Eq. (5) | Eq. (6) |
|                | 800              | 0.41                   | 4.29^a                 | 205.1                            |             |        |        | Eq. (5) | Eq. (6) |
|                | 1,300            | 0.40                   | 4.20^a                 | 810.2                            |             |        |        | Eq. (5) | Eq. (6) |
|                | 1,800            | 0.39                   | 4.31^a                 | 1,261.4                          |             |        |        | Eq. (5) | Eq. (6) |
|                | 2,575            | 0.37                   | 4.77^a                 | 2,563.8                          |             |        |        | Eq. (5) | Eq. (6) |
|                | 3,575            | 0.38                   | 4.88^a                 | 5,127.6                          |             |        |        | Eq. (5) | Eq. (6) |
|                | 4,500            | 0.37                   | 4.64^a                 | 5,640.4                          |             |        |        | Eq. (5) | Eq. (6) |
|                | 5,650            | 0.37                   | 4.70^a                 | 8,204.2                          |             |        |        | Eq. (5) | Eq. (6) |
|                | 7,150            | 0.37                   | 4.70^a                 | 12,306.3                         |             |        |        | Eq. (5) | Eq. (6) |
| Sand           | 192              | 0.383                  | 4.22^b                 | 21.4                             | 0.0095^c     | 1.780^c| 1.14^c| Moghadasi et al. (2004) | Fig. 3c     |
|                | 265              | 0.383                  | 4.22^b                 | 60.3                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 410              | 0.384                  | 4.20^b                 | 121                              |             |        |        | Eq. (5) | Eq. (6) |
| Quartz sand    | 180              | 0.47                   | 3.77^a                 | 17.6                             | 0.007^c      | 1.460^c| 1.41^c| Koch et al. (2012) | Fig. 3d     |
|                | 270              | 0.45                   | 3.55^a                 | 53.1                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 660              | 0.47                   | 3.25^a                 | 129                              |             |        |        | Eq. (5) | Eq. (6) |
|                | 180              | 0.48                   | 3.14^a                 | 20.8                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 240              | 0.49                   | 3.40^a                 | 33.0                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 320              | 0.49                   | 3.26^a                 | 67.5                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 500              | 0.49                   | 3.12^a                 | 171                              |             |        |        | Eq. (5) | Eq. (6) |
|                | 680              | 0.48                   | 3.10^a                 | 280                              |             |        |        | Eq. (5) | Eq. (6) |
|                | 870              | 0.49                   | 3.34^a                 | 394                              |             |        |        | Eq. (5) | Eq. (6) |
|                | 870              | 0.49                   | 3.34^a                 | 394                              |             |        |        | Eq. (5) | Eq. (6) |
|                | 180              | 0.39                   | 4.12^a                 | 11.1                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 270              | 0.39                   | 3.75^a                 | 24.0                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 660              | 0.41                   | 3.97^a                 | 75.0                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 180              | 0.40                   | 3.23^a                 | 11.7                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 240              | 0.40                   | 3.55^a                 | 19.8                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 320              | 0.42                   | 3.64^a                 | 38.1                             |             |        |        | Eq. (5) | Eq. (6) |
|                | 500              | 0.42                   | 3.52^a                 | 105.0                            |             |        |        | Eq. (5) | Eq. (6) |
Zaitoun 1981; Glover et al. 2006; Glover and Walker 2009), (2) for glass beads and silica sands (data from Kimura 2018), (3) for silica sands (data from Biella et al. 1983; Moghadasi et al. 2004), and (4) for sand grains (data from Koch et al. 2012). The sample properties ($d$, $\phi$ and $F$) as well as the measured $k_s$ are summarized in Table 1. The formation factor $F$ is not available for the samples reported by Johnson et al. (1986), Chauveteau and Zaitoun (1981), Moghadasi et al.

### Table 1 (continued)

| Pack       | $d^a$ ($\mu$m) | $\phi^a$ (no units) | $F^a, b$ (no units) | $k_s^a 10^{-12}$ (m$^2$) | $\alpha$ | $D_r$   | $D_t$   | Source                          | Shown in          |
|------------|----------------|---------------------|---------------------|---------------------------|----------|---------|---------|--------------------------------|-------------------|
| Glass bead | 1.05           | 0.411               | 3.80$^b$            | 0.00057                   | 0.009    | 1.80$^d$| 1.12$^d$| Glover and Walker (2009)       | from Dery (2010) |
|            | 2.11           | 0.398               | 3.98$^b$            | 0.00345                   |          |         |         | Eq. (5)                         |                   |
|            | 5.01           | 0.380               | 4.27$^b$            | 0.0181                    |          |         |         | Eq. (6)                         |                   |
|            | 11.2           | 0.401               | 3.94$^b$            | 0.0361                    |          |         |         |                                 |                   |
|            | 21.5           | 0.383               | 4.22$^b$            | 0.228                     |          |         |         |                                 |                   |
|            | 31             | 0.392               | 4.07$^b$            | 0.895                     |          |         |         |                                 |                   |
|            | 47.5           | 0.403               | 3.91$^b$            | 1.258                     |          |         |         |                                 |                   |
|            | 104            | 0.394               | 4.04$^b$            | 6.028                     |          |         |         |                                 |                   |
|            | 181            | 0.396               | 4.01$^b$            | 21.53                    |          |         |         |                                 |                   |
|            | 252            | 0.414               | 3.75$^b$            | 40.19                     |          |         |         |                                 |                   |
|            | 494            | 0.379               | 4.29$^b$            | 224                       |          |         |         |                                 |                   |
|            | 990            | 0.385               | 4.19$^b$            | 866.7                     |          |         |         |                                 |                   |
| Average    |                |                     |                     |                           | 0.0090   |         |         |                                 |                   |

$^a$ Measured quantities  
$^b$ Estimated from Archie (1942)  
$^c$ Optimized using the “fminsearch” function in Matlab  
$^d$ Predicted values from Eqs. (5) and (6)

Fig. 3 Comparison between measured permeability reported in literature and the one estimated by Eq. (22) in which sample properties are given in Table 1 with $S_{wirr} = 0$: a glass beads (data from Bolèvre et al. 2007; Johnson et al. 1986; Chauveteau and Zaitoun 1981; Glover et al. 2006; Glover and Walker 2009), b glass beads and silica sands (data from Kimura, 2018), c silica sands (data from Biella et al. 1983; Moghadasi et al. 2004), and d sand grains (data from Koch et al. 2012). The solid lines represent the 1:1 line.
Table 2  Some of the models for the grain-size-based permeability estimation

| Model          | Equation                                                                 | Reference                                      |
|----------------|--------------------------------------------------------------------------|-----------------------------------------------|
| RC model       | $k_s = \frac{\phi}{(\alpha D_c)^\gamma} \left[ \frac{1}{2(2-D_c)} \right]^{1+D_c}$ | Revil and Cathles (1999); Koch et al. (2012) |
| KC model       | $k_s = \frac{\phi^\gamma}{(1+\phi)^2} \left[ \frac{1}{2(2-D_c)} \right]^{1+D_c}$ | Kozeny (1927); Revil and Cathles (1999)       |
| RGPZ model     | $k_s = \frac{\phi^\gamma}{(1+\phi)^2} \left[ \frac{1}{2(2-D_c)} \right]^{1+D_c}$ | Glover et al. (2006)                          |
| XY model       | $k_s = \left( \frac{\pi D_c^{1-D_c}}{2[4(2-D_c)]^{1+D_c}} \right)^{\frac{2}{32(3+D_c-D_c)(\frac{\phi}{\alpha})^{1+D_c}}}$ | Xu and Yu (2008)                              |
| CPA model      | $k_s = \frac{d_c^2}{\tau}$                                              | Ghanbarian (2020a)                            |

(2004) and Glover and Dery (2010). Therefore, $F$ is estimated from $\phi$ by the relation $F = \phi^{-m}$ (Archie 1942) with $m = 1.5$ for spherical grains (e.g., Sen et al. 1981). Model parameters of $\alpha$, $D_c$, $D_a$, are optimized using the “fminsearch” function in MATLAB for Fig. 3 as mentioned previously and are shown in Table 1 with the superscript ‘c’. $R_{\text{max}}$ is determined from Eq. (30) with the knowledge of $d$ and $\phi$ (see columns 2 and 3 in Table 1). The average optimized value for $\alpha$ is around 0.009. That value is in good agreement with $\alpha = 0.01$ which has been effectively applied for unconsolidated samples such as sand packs or glass beads (e.g., Cai et al. 2012a; Liang et al. 2015; Thanh et al. 2018, 2019). Additionally, it is possible to also predict $D_c$ from $\phi$ and the optimized value $\alpha$ using Eq. (5) and predict $D_a$ from $D_c$, $\phi$ and $F$ using Eq. (6). The predicted values are shown in Table 1 with the superscript ‘d’. It is seen that the predicted values for $D_c$, $D_a$, are close to the optimized ones by the “fminsearch” (average difference by 4%). The comparison in Fig. 3 shows that the model predictions are in quite good agreement with experimental data Fig. 3.

Recall that $d$ is the grain diameter, $\phi$ is the porosity, $F$ is the formation factor, $m$ and $a$ are parameters taken as 1.5 and 8/3 for spherical grain samples (e.g., Sen et al. 1981; Glover et al. 2006). Note that $d_c$ and $c$ in the CPA model are the critical pore diameter and a constant coefficient equal to 72.2 (e.g., Ghanbarian 2020a).

Figure 4 shows the comparison between the predictions of various models from literature and the proposed model given by Eq. (22) for a data set of glass beads reported by Glover and Dery (2010). Sample properties are also shown in Table 1 for the glass beads of Glover and Dery (2010). Table 2 lists some of the models available for the grain-size-based permeability estimation: the RC model proposed by Revil and Cathles (1999), the KC model proposed by Kozeny (1927), the RGPZ model proposed by Glover et al. (2006), the XY model proposed by Xu and Yu (2008) based on the fractal theory and the CPA model proposed by Ghanbarian (2020a) using the critical path analysis. The common values for $m$ and $a$ in the RC model and the RGPZ model are taken to be 1.5 and 8/3 for glass beads, respectively (e.g., Sen et al. 1981; Glover et al. 2006). In the CPA model, $d_c$ is the critical pore diameter that is related to the grain diameter $d$ by $d_c = 0.42d$, and $c$ is a constant coefficient that is equal to 72.2 (e.g., Ghanbarian 2020a).

Due to the similarity between the samples of Glover and Dery (2010) and those reported in Fig. 3 (they are all made up of glass beads or sands), this study uses $\alpha = 0.009$ as an average optimized value in Table 1. Values of $D_c$, $D_a$, are predicted from Eq. (5) and Eq. (6) in the same manner as previously mentioned (see superscript ‘d’ in Table 1). The root-mean-square deviation (RMSD) values for the proposed model, KC model, RC model, RGPZ model, XY model and CPA model are calculated to be $18 \times 10^{-12}$ m$^2$, $13 \times 10^{-12}$ m$^2$, $118 \times 10^{-12}$ m$^2$, $89 \times 10^{-12}$ m$^2$, $2351 \times 10^{-12}$ m$^2$ and $85 \times 10^{-12}$ m$^2$, respectively. The representative comparison shows that the proposed model provides a remarkably good prediction with experimental data reported by Glover and
Dery (2010) and with those predicted from the other models. Note that the XY model gave a worse result compared to the others with $R_{\text{max}}$ predicted from Eq. (30). However, the prediction from the XY model could be much improved by calibrating $R_{\text{max}}$ (which is predicted from Eq. 30) using a factor of 1/3 (dividing $R_{\text{max}}$ by a factor 3), as shown by circle symbols in Fig. 4 (RMSD = $40 \times 10^{-12}$ m$^2$). It suggests that the proposed model and the XY model, which are related to $R_{\text{max}}$, could be improved by calibrating $R_{\text{max}}$.

Equation (22) is also tested in Fig. 5 for a large data set of permeability measurements on similar grain size sediments of different porosity from Chilindar (1964) using the same approach as performed for Fig. 4. The average grain diameter $d = 235 \mu$m is taken from Revil and Cathles (1999) for the fine-grained sandstone. Additionally, the KC model, RC model and RGPZ model are also used to explain experimental data reported by Chilindar (1964) and to compare with the proposed model. For the RC model and RGPZ model, $m$ is taken as 1.7 as proposed by Revil and Cathles (1999). For the proposed model, the formation factor is determined from porosity using $F = \phi^m$ with $m = 1.7$. Feng et al. (2004) and Wei et al. 2015 analyzed the best fit regression parameters to find $D_f$ using Eq. (5) for natural and artificial porous media from different studies. They found that $\alpha = 0.001$ gives the best estimate of $D_f$. Additionally, the maximum pressure ($P_{\text{max}} = 22.6$ MPa) and the minimum pressure ($P_{\text{min}} = 0.009$ MPa) are obtained from the capillary pressure measurement of Li and Horne (2006a) for a Berea core sample. Applying the Young–Laplace equation as performed in Ghanbarian et al. (2017b), one obtains $\alpha = R_{\text{min}}/R_{\text{max}} = P_{\text{max}}/P_{\text{min}} = 0.0004$, which is approximately the same order as 0.001. Therefore, $\alpha = 0.001$ is rather relevant for consolidated samples and applied in this work.

The calculated RMSD values for the proposed model, KC model, RC model and RGPZ model are $68.4 \times 10^{-14}$ m$^2$, $104 \times 10^{-14}$ m$^2$, $7.2 \times 10^{-14}$ m$^2$ and $14.8 \times 10^{-14}$ m$^2$, respectively. It is seen that the proposed model can reproduce the main trend of experimental data but less accurate than the RC model and the RGPZ model. The reason may be that Eq. (30) for determining $R_{\text{max}}$ from $d$ works quite well for unconsolidated samples that are made up of monosized spherical grains as shown in Fig. 3 or Fig. 4. However, for the consolidated samples of sandstone, the rock texture consists of mineral grains of various shapes and sizes and its pore structure is extremely complex; therefore, Eq. (30) may not be suitable. In this case, one can estimate $R_{\text{max}}$ by measuring the capillary pressure and then using the Young–Laplace equation (e.g., Ghanbarian et al. 2017b) or using the micro-CT images and

![Fig. 5](Image)

**Fig. 5** Variation of permeability with porosity for the fine-grained sandstones obtained from Chilindar (1964) (see symbols). The proposed model given by Eq. (22) with $\alpha = 0.001$, $S_{\text{wirr}} = 0$ and $F = \phi^{-2.2}$ and other ones are used for the prediction.

![Fig. 6](Image)

**Fig. 6** Variation of the relative permeability with water saturation. The symbols are experimental data from Li and Horne (2006b) for Berea sandstone. The solid and dashed lines are predicted from Eq. (21) with $D_f = 1.4$, $D_\tau = 1.05$ and $\alpha = 0.001$ and the model of Brooks and Corey (1964) with $\lambda = 1.9$, respectively.
nuclear magnetic resonance measurements (e.g., Daigle 2016). Another reason may be due to the variation of sample to sample (here $\alpha = 0.001$ is used for all samples).

Relative permeability

Figure 6 shows the variation of the relative permeability for the wetting phase with water saturation experimentally obtained from Li and Horne (2006b) for a plug of Berea sandstone (see symbols). Equation (21) is applied to predict the variation of $k_r^w$ with $S_w$ (see solid line). The irreducible water saturation $S_{wirr}$ is reported to be 0.18 (Li and Horne 2006b). Using the same approach as applied for Fig. 3, $D_t = 1.4$ and $D_r = 1.05$ are obtained. Note that $\alpha$ is taken as 0.001 for all consolidated rocks in this work as previously mentioned. Additionally, the model of Brooks and Corey (1964) $k_r^w = \left(\frac{S_w - S_{wirr}}{1 - S_{wirr}}\right)^{3+2/\lambda}$ with $\lambda = 1.9$ (best fit) is also used to explain experimental data (see dashed line). The calculated RMSD values for the proposed model and the model of Brooks and Corey are 0.0043 and 0.0041, respectively. The proposed model is in a very good agreement with experimental data and prediction from the model of Brooks and Corey (1964).

Figure 7 shows the variation of the $k_r^w$ with $S_w$ from different sources. The symbols are experimental data and the solid lines are predicted from Eq. (21). Figure 7a is obtained from data in Cerepi et al. (2017) for the Brauvilliers limestone with model parameters: $D_t = 1.1, D_r = 1.05, \alpha = 0.001, S_{wirr} = 0.28$, and for the LS2 dolostone with model parameters: $D_t = 1.3, D_r = 1.05, \alpha = 0.001, S_{wirr} = 0.37$. Figure 7b is obtained from data in Mahiya (1999) for the fired Berea core sample with model parameters: $D_t = 1.3, D_r = 1.05, \alpha = 0.001, S_{wirr} = 0.29$. Figure 7c is obtained from data in Jougnot et al. (2010) for two Callovo-Oxfordian clay-rock samples with model parameters: $D_t = 1.3, D_r = 1.05, \alpha = 0.001, S_{wirr} = 0.23$. It should be noted that all values for irreducible water saturation $S_{wirr}$ mentioned previously are taken from corresponding sources (Cerepi et al. 2017; Mahiya 1999; Jougnot et al. 2010). It is seen that the model can provide a rather good prediction of the variation of the relative permeability with water saturation.

Conclusions

A new model is proposed to predict the permeability of porous media saturated by one or two fluids based on a bundle of capillary tubes model and the fractal theory for porous media. The model is related to microstructural properties of porous media (fractal dimension for pore space, fractal dimension for tortuosity, porosity, maximum radius, ratio of minimum pore radius and maximum pore radius), water saturation and irreducible water saturation. By comparison with 111 samples of uniform glass bead and sand packs in literature, this study shows that the proposed model estimated the saturated permeability very well from sample properties. The proposed model is also compared to existing and widely used models from the literature. These results show that the proposed model is in good agreement with the others. The main advantage of the proposed model is that the input parameters are physically based parameters; therefore, it may provide an insight into the dependence of the saturated permeability $(k_s)$ and the relative permeability $(k_r^w)$ on the microstructural parameters of the porous media and it may reveal more mechanisms affecting the $k_s$ and $k_r^w$ than other models. Additionally, the model prediction for the relative permeability has been successfully
validated using experimental data for the consolidated media in literature.

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