UNIQUE CONTINUATION FOR FRACTIONAL SCHRÖDINGER OPERATORS IN THREE AND HIGHER DIMENSIONS

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Abstract. We prove the unique continuation property for the differential inequality $|(-\Delta)^{\alpha/2}u| \leq |V(x)u|$, where $0 < \alpha < n$ and $V \in L^{n/\alpha, \infty}_{\text{loc}}(\mathbb{R}^n)$, $n \geq 3$.

1. Introduction

In this note we are concerned with the unique continuation property for solutions of the differential inequality

$$|(-\Delta)^{\alpha/2}u| \leq |V(x)u|, \quad x \in \mathbb{R}^n, \quad n \geq 2,$$

where $(-\Delta)^{\alpha/2}$, $0 < \alpha < n$, is defined by means of the Fourier transform $F f (= \hat{f})$:

$$F[(-\Delta)^{\alpha/2}f](\xi) = |\xi|^\alpha \hat{f}(\xi).$$

In particular, the equation $((-\Delta)^{\alpha/2} + V(x))u = 0$ has attracted interest from quantum mechanics in the case $1 < \alpha < 2$ as well as the case $\alpha = 2$. Recently, by generalizing the Feynman path integral to the Lévy one, Laskin [5] introduced the fractional quantum mechanics in which it is conjectured that physical realizations may be limited to $1 < \alpha < 2$, where averaged quantities are finite, and the fractional Schrödinger operator $(-\Delta)^{\alpha/2} + V(x)$ plays a central role. Of course, the case $\alpha = 2$ becomes equivalent to an ordinary quantum mechanics.

The unique continuation property means that a solution of (1.1) which vanishes in an open subset of $\mathbb{R}^n$ must vanish identically. In the case of $\alpha = 2$, Jerison and Kenig [1] proved the property for $V \in L^{n/2}_{\text{loc}}, n \geq 3$. An extension to $L^{n/2, \infty}_{\text{loc}}$ was obtained by Stein [9] with small norm in the sense that

$$\sup_{a \in \mathbb{R}^n} \lim_{r \to 0} \|V\|_{L^{n/2, \infty}(B(a,r))}$$

is sufficiently small. Here, $B(a,r)$ denotes the ball of radius $r > 0$ centered at $a \in \mathbb{R}^n$. These results later turn out to be optimal in the context of $L^p$ spaces ([2,3]).

On the other hand, the results when $\alpha \neq 2$ are rather scarce. Laba [4] considered the higher orders where $\alpha/2$ are integers, and obtained the property for $V \in L^{n/\alpha}_{\text{loc}}$. Recently, there was an attempt [7] to handle the non-integer orders when $n - 1 \leq \alpha < n$, $n \geq 2$, from which it turns out that the condition $V \in L^p$, $p > n/\alpha$, is sufficient to have the property. Hence this particularly gives a unique continuation
result for the fractional Schrödinger operator in the full range $1 < \alpha < 2$ when $n = 2$. Our aim here is to fill the gap, $0 < \alpha < n - 1$, for $n \geq 3$, which allows us to have the unique continuation for the fractional Schrödinger operator when $n \geq 3$ with the full range of $\alpha$.

**Theorem 1.1.** Let $n \geq 3$ and $0 < \alpha < n$. Assume that $V \in L^{n/\alpha, \infty}_{\text{loc}}$ and $u$ is a non-zero solution of (1.1) such that

\begin{equation}
(1.2) \quad u \in L^1 \cap L^{p,q} \quad \text{and} \quad (-\Delta)^{\alpha/2} u \in L^q,
\end{equation}

where $p = 2n/(n-\alpha)$ and $q = 2n/(n+\alpha)$. Then it cannot vanish in any non-empty open subset of $\mathbb{R}^n$ if

\begin{equation}
(1.3) \quad \sup_{\alpha \in \mathbb{R}^n} \lim_{r \to 0} \|V\|_{L^{n/\alpha, \infty}(B(a, r))}
\end{equation}

is sufficiently small. Here, $L^{p,q}$ denotes the usual Lorentz space.

**Remarks.** (a) The smallness condition (1.3) is trivially satisfied for $V \in L^{n/\alpha}_{\text{loc}}$ because $L^{n/\alpha}_{\text{loc}} \subset L^{n/\alpha, \infty}_{\text{loc}}$. Hence the above theorem can be seen as a natural extension of (1.1) of the results obtained in [10] for the Schrödinger operator ($\alpha = 2$). As an immediate consequence of the theorem, the same result also holds for the stationary equation

\[ ((-\Delta)^{\alpha/2} + V(x))u = Eu, \quad E \in \mathbb{C}, \]

because $(-\Delta)^{\alpha/2} u = (E - V(x))u$ and the condition (1.3) is trivially satisfied for the constant $E$.

(b) The index $n/\alpha$ is quite natural, in view of the standard rescaling: $u_{\varepsilon}(x) = u(\varepsilon x)$ takes the equation $(-\Delta)^{\alpha/2} u = V u$ into $(-\Delta)^{\alpha/2} u_{\varepsilon} = V_{\varepsilon} u_{\varepsilon}$, where $V_{\varepsilon}(x) = \varepsilon^\alpha V(\varepsilon x)$. So, $\|V_{\varepsilon}\|_{L^{p,\infty}} = \varepsilon^{\alpha-n/p} \|V\|_{L^{p,\infty}}$. Hence the $L^{p,\infty}$ norm of $V_{\varepsilon}$ is independent of $\varepsilon$ precisely when $p = n/\alpha$.

(c) When $\alpha = n$ in (1.1), there are some unique continuation results with $V \in L^p$, $p > 1$. (See [1] and [6] for $\alpha = 2$ and $\alpha = 2m$ ($m \in \mathbb{N}$), respectively.)

2. **Proof of the theorem**

From now on, we will use the letter $C$ to denote a constant that may be different at each occurrence.

Without loss of generality, we need to prove that $u$ must vanish identically if it vanishes in a sufficiently small neighborhood of zero.

Our proof is based on the following Carleman estimate which will be shown below: If $f \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$ and $(-\Delta)^{\alpha/2} f \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$, then there is a constant $C$ depending only on $\delta_t := \min_{k \in \mathbb{Z}} |t - k|$ and $n$ such that for $t \not\in \mathbb{Z}$ with $\delta_t < n - \alpha$

\begin{equation}
(2.1) \quad \| |x|^{-t-n/p} f \|_{L^{p,q}} \leq C \| |x|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} f \|_{L^q},
\end{equation}

where $p, q$ are given as in the theorem (i.e., $1/p + 1/q = 1$ and $1/q - 1/p = \alpha/n$).

Indeed, since we are assuming that $u \in L^1 \cap L^{p,q}$ and $(-\Delta)^{\alpha/2} u \in L^q$ vanish near zero (see (1.2), (1.1)), from (2.1) (with a standard limiting argument involving a $C^\infty_0$ approximate identity), we see that

\[ \| |x|^{-t-n/p} u \|_{L^{p,q}} \leq C \| |x|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} u \|_{L^q}. \]
Hence,
\[ \|x|^{t-n/p}u\|_{L^{p,q}(B(0,r))} \leq C \|x|^{t+\alpha-n/q}Vu\|_{L^{q}(B(0,r))} + C \|x|^{t+\alpha-n/q}(-\Delta)^{\alpha/2}u\|_{L^q(\mathbb{R}^n\setminus B(0,r))}. \]

The first term on the right-hand side can be absorbed into the left-hand side as follows:
\[ C \|x|^{t+\alpha-n/q} Vu\|_{L^q(B(0,r))} \leq C \|V\|_{L^{n/\alpha,\infty}(B(0,r))} \|x|^{t+\alpha-n/q} u\|_{L^{p,q}(B(0,r))} \]
\[ \leq \frac{1}{2} \|x|^{-n/p}u\|_{L^{p,q}(B(0,r))} \]
if we choose \( r \) small enough (see [13]). Here, recall that \( \alpha - n/q = -n/p \), and \( \|x|^{-n/p}u\|_{L^{p,q}(B(0,r))} \) is finite since \( u \in L^{p,q} \) vanishes near zero. So, we get
\[ \|(r/|x|)^{t+n/p}u\|_{L^{p,q}(B(0,r))} \leq 2C \|(-\Delta)^{\alpha/2} u\|_{L^{q}(\mathbb{R}^n\setminus B(0,r))} < \infty. \]

Now, we choose a sequence \( \{t_i\} \) of values of \( t \) tending to infinity such that \( \delta_{t_i} \) is independent of \( i \in \mathbb{N} \). Then, by letting \( i \to \infty \), we see that \( u = 0 \) on \( B(0,r) \), which implies \( u \equiv 0 \) by a standard connectedness argument.

**Proof of (2.1).** We will show (2.1) using Stein’s complex interpolation, as in [9], on an analytic family of operators \( T_z \) defined by
\[ T_z g(x) = \int_{\mathbb{R}^n} K_z(x,y)g(y)|y|^{-n}dy, \]
where \( K_z(x,y) = H_z(x,y)/\Gamma(n/2-z/2) \) with
\[ H_z(x,y) = |x|^{-t}|y|^{n+t-z} c_z \left(|x-y|^{-n+z} - \sum_{j=0}^{m-1} \frac{1}{j!} \left( \frac{\partial}{\partial s} \right)^j |sx-y|^{-n+z} |s=0 \right). \]

Note that for \( f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \)
\[ T_\alpha(|x|^{-t+\alpha}(-\Delta)^{\alpha/2} f(y))(x) = |x|^{-t} f(x)/\Gamma(n/2-\alpha/2) \]
(see Lemma 2.1 in [7]).

Let \( m \) be a fixed positive integer such that \( m-1 < t < m \), and recall the following two estimates for the cases of \( \Re z = 0 \) (Lemma 2.3 in [1]) and \( n-1 < \Re z < n-\delta_t \) (Lemma 4 in [9]): There is a constant \( C \) depending only on \( \delta_t \) and \( n \) such that
\[ \|T_{i\gamma}g\|_{L^2(dx/|x|^n)} \leq Ce^{c|\gamma|} \|g\|_{L^2(dx/|x|^n)}, \quad \gamma \in \mathbb{R}, \]
and
\[ \|T_z g\|_{L^r(dx/|x|^n)} \leq Ce^{c|\gamma|} \|g\|_{L^r(dx/|x|^n)}, \quad \gamma = \Im z \in \mathbb{R}, \]
where \( n-1 < \beta = \Re z < n-\delta_t \), \( 1/s - 1/r = \beta/n \) and \( 1 < s < n/\beta \).

We first consider the case where \( n-1 < \alpha < n \). Note that we can choose \( \beta \) so that \( \alpha < \beta < n-\delta_t \), since we are assuming \( \delta_t < n-\alpha \). Hence, by Stein’s complex interpolation ([8]) between (2.3) and (2.4), we see that
\[ \|T_\alpha g\|_{L^r(dx/|x|^n)} \leq C \|g\|_{L^r(dx/|x|^n)}, \]
where \( 1/s - 1/r = \alpha/n \) and \( 1 < s < n/\alpha \). From this and (2.2), we get
\[ \|x|^{-t-n/r} f\|_{L^r} \leq C \|x|^{-t+\alpha-n/s}(-\Delta)^{\alpha/2} f\|_{L^s}. \]
with the same \( r, s \) in (2.5), since we are assuming \((-\Delta)^{\alpha/2} f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\). Note that \( 1 < q < n/\alpha \). So, we can choose \( r_j, s_j, j = 1, 2 \), such that

\[
1 < s_1 < q < s_2 < n/\alpha, \quad 1/s_j - 1/r_j = \alpha/n,
\]

and for \( t_j = t + n(1/p - 1/r_j) \)

\[
m - 1 < t_j < m, \quad \delta_t/2 \leq \delta_{t_j} \leq 3\delta_t/2.
\]

Hence we can apply (2.6) with \( t = t_j \) to obtain

\[
(2.7) \quad \|x\|^{-t-n/p} f \in L_{s_j} \leq C \|x\|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} f \|L_{s_j}.
\]

for \( j = 1, 2 \). Since \( r_1 < p < r_2 \) and \( s_1 < q < s_2 \), by real interpolation (8) between the estimates in (2.7), we see that for \( 1 \leq w \leq \infty \)

\[
\|x\|^{-t-n/p} f \in L_{p,w} \leq C \|x\|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} f \|L_{q,w}.
\]

By choosing \( w = q \), we get (2.1).

Now we turn to the remaining case where \( 0 < \alpha \leq n - 1 \). In this case, (2.5) is valid for \( 1/s - 1/r = \alpha/n \) and

\[
(2.8) \quad \frac{1}{2}(1 - \frac{\alpha}{n-1}) + \frac{\alpha}{n} < \frac{1}{2} + \frac{\alpha}{2(n-1)},
\]

because we can choose \( \beta \) so that \( n - 1 < \beta < n - \delta_t \). Since (2.8) holds for \( s \) replaced by \( q \), repeating the previous argument, one can show (2.1). We omit the details. \qed

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