METALLIC STRUCTURES ON THE TANGENT BUNDLE OF A P-SASAKIAN MANIFOLD

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ABSTRACT. In this article, we introduce some metallic structures on the tangent bundle of a P-Sasakian manifold by complete lift, horizontal lift and vertical lift of a P-Sasakian structure \((\phi, \eta, \xi)\) on tangent bundle. Then we investigate the integrability and parallelity of these metallic structures.

1. Introduction

The lift of geometrical objects, vector fields, forms, has an important role in differential geometry. By the method of lift, we can generalize to differentiable structure on any manifold to its tangent bundle and any other bundles on manifold \([4, 5, 7]\). In this paper we study the metallic structures on tangent bundle of a P-Sasakian Riemannian manifold. The metallic structure is a generalization of the almost product structure. A metallic structure is a polynomial structure as defined by Goldenberg et al. in \([1, 2]\). In \([3]\), introduced the notation of metallic structure on a Riemannian manifold. Suppose that \(p\) and \(q\) are two positive integers.

The positive solution of the equation \(x^2 - px - q = 0\) is called members of the metallic means family. These number showed by \(\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}\), where it is a generalization of golden proportions.

**Definition 1.1.** Let \(M\) be a manifold. A metallic structure on \(M\) is an \((1,1)\) tensor field \(J\) which satisfies the equation \(J^2 = pJ + qI\), where \(p, q\) are positive integers and \(I\) is the identity operators on the Lie algebra \(\mathcal{X}(M)\) of vector fields on \(M\). If \(g\) is a Riemannian metric on \(M\), then we say that \(g\) is \(J\)-compatible whenever

\[
\begin{align*}
g(JX, Y) &= g(X, JY), & \forall X, Y \in \mathcal{X}(M) \tag{1.1} \\
g(JX, JY) &= pg(X, JY) + qg(X, Y), & \forall X, Y \in \mathcal{X}(M). \tag{1.2}
\end{align*}
\]

In this case \((M, J, g)\) is named a metallic Riemannian manifold.

Let \(J\) be a metallic structure on \(M\). Then the Nijenhuis tensor \(N_J\) of \(J\) is a tensor field of type \((1,2)\) given by

\[
N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]
\]

for \(X, Y \in \mathcal{X}(M)\).

On the other hand, at first time, Sato in \([6]\) introduced the P-Sasakian structure on a manifolds and studied several properties of these manifolds. An \(n\)-dimensional smooth manifold \(M\) is called an almost paracontact manifold if it admits an almost
paracontact structure \((\phi, \eta, \xi)\), consisting of a \((1, 1)\) tensor field \(\phi\), a 1-form \(\eta\) and a vector field \(\xi\) which satisfy the condition

\[
\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0.
\]

Let \(g\) be a Riemannian metric compatible with \((\phi, \eta, \xi)\) i.e.

\[
g(X, Y) = g(\phi X, \phi Y) + \eta(X) \eta(Y), \quad \forall X, Y \in \mathcal{X}(M),
\]

or equivalently

\[
g(X, \phi Y) = g(\phi X, Y), \quad g(X, \xi) = \eta(X), \quad \forall X, Y \in \mathcal{X}(M),
\]

where \(\mathcal{X}(M)\) is the collection of all smooth vector field on \(M\). Then \(M\) is said to be an almost paracontact Riemannian manifold.

An almost paracontact Riemannian manifold \((M, g)\) is called a P-Sasakian manifold if it satisfies

\[
(\nabla_X \phi)(Y) = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi
\]

where \(\nabla\) is the Levi-Civita connection of the Riemannian manifold. We have

\[
\nabla_X \xi = \phi X, \quad (\nabla_X \eta)(Y) = g(\phi X, Y) = (\nabla_Y \eta)(X), \quad \forall X, Y \in \mathcal{X}(M).
\]

2. Lifts of geometric structure on tangent bundle

Let \((M, g)\) be a smooth \(n\)-dimensional Riemannian manifold and denote its tangent bundle by \(TM\). We denote by \(\pi : TM \to M\) the natural projection, where it defines the natural bundle structure of \(TM\) over \(M\) and denote by \(T_k^l(M)\) the set of all tensor fields of the type \((k, l)\) in \(M\). For any point \((x, y) \in TM\), Let \(V_y = \ker \pi_* : T_y(M) \to T_x M\) and \(VTM = \cup_{y \in TM} V_y\). Also, suppose that \(HTM\) be a complement of \(VTM\) in \(TM\), that is

\[
TTM = VTM \oplus HTM.
\]

\(VTM\) and \(HTM\) are called vertical distribution and horizontal distribution, respectively. Suppose that the space \(M\) is covered by a system of coordinate neighbourhoods \((U, \varphi) = (U, x^1, x^2, ..., x^n)\), then the corresponding induced local chart on \(TM\) is \((\pi^{-1}(U), x^1, x^2, ..., x^n, y^1, y^2, ..., y^n)\). If in any point of \(x \in M\), \(\Gamma^j_k(x)\) be the Christoffel symbols of \(g\), then the sets of vector fields \(\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}\) and \(\{\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}\}\) on \(\pi^{-1}(U)\) define local frame fields for \(VTM\) and \(HTM\), respectively, where \(\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^k \Gamma^i_{kj} \frac{\partial}{\partial y^j}\). Notice that the set \(\{\frac{\partial}{\partial y^1}, \frac{\delta}{\delta x^1}\}\) defines a local frame on \(TM\). In following from [4, 7], we recall some lifts of geometrical objects of a manifold to its tangent bundle.

2.1. Vertical lifts. Let \(f\) be a function on \(M\). Then the vertical lift of \(f\) to \(TM\) is the function \(f^v\) on \(TM\) given by \(f^v = f \circ \pi\). For any vector field \(X \in \mathcal{X}(M)\), we define a vector field \(X^v\) in \(TM\) by \(X^v(\omega) = (\omega(X))^v\), where \(\omega\) is an arbitrary 1-form in \(M\). We call \(X^v\) the vertical lift of \(X\). Notice, \(X^v \in VTM\) and for all function on \(M\) we define \(X^v(df) = X.f\). Let \(F\) be a tensor field of type \((1, r)\) or \((0, r), r \geq 1\) on \(M\). Then the vertical lift of \(F\) on \(TM\) defined by

\[
F^v_y(\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_r) = \left(F_y(\pi_*(\tilde{X}_1), \pi_*(\tilde{X}_2), ..., \pi_*(\tilde{X}_r))\right)^v
\]
2.2. Complete lifts. If \( f \) is a function in \( M \), then the complete lift of \( f \) is the function \( f^c \) on \( TM \) and defined by
\[
(2.1) \\
\quad f^c(x, y) = df(x)(y), \quad y \in T_x M, \quad x \in M.
\]

Also the complete lift of vector field \( X = X^i \frac{\partial}{\partial x^i} \) on \( M \) defined by
\[
X^c = X^i \frac{\partial}{\partial x^i} - y^j \frac{\partial X^i}{\partial y^j}.
\]

Therefore we obtain \( (\frac{\partial}{\partial x^i})^c = \frac{\partial}{\partial x^i} \) and \( X^c f^c = (X f)^c \) for any function \( f \) on \( M \). Suppose that \( \omega \) is a 1-form on \( M \). The complete lift of \( \omega \) on \( TM \) defined by \( \omega^c(X) = (\omega(X))^c \), \( \omega^c(v) = (\omega(v))^c \), \( X \in \mathcal{X}(M) \). In general case, the complete lift of a tensor field \( F \) of type \((1, r)\) or \((0, r), \) \( r \geq 1, \) on \( M \) defined by \( F^c(X_1^1, ..., X_r^r) = (F(X_1^1, ..., X_r^r))^c \) for any \( X_1, ..., X_r \in \mathcal{X}(M) \). Then the complete lift of a Riemannian metric \( g \) defined by
\[
g^c = \begin{pmatrix}
y^k \frac{\partial g_{ij}}{\partial x^k} & g_{ij} \\
g_{ij} & 0 \end{pmatrix}.
\]

From [4] and [7], we have

**Proposition 2.1.** Let \( M \) be a manifold with a Riemannian metric \( g \). For any \( X, Y \in \mathcal{X}(M), \ f \in C^\infty(M), \) and \((1, 1)\) tensor field \( F \) we have :

- \( X^c f^c = 0, \ X^c f^c = X^c f^c = (X f)^c, \)
- \( F^c(X) = (F(X))^c, \)
- \( g^c \) is semi-Riemannian metric and

\[
g^c(X^c, Y^c) = g^c(X^c, Y^v) = (g(X, Y))^c, \quad g^c(X^v, Y^c) = 0, \quad g^c(X^c, Y^v) = (g(X, Y))^c
\]

- If \( P(x) \) is a polynomial in one variable \( x \) then \( P(F^c) = (P(F))^c. \)

We define the complete lift of a linear connection \( \nabla \) to \( TM \) as the unique linear connection \( \nabla^c \) on \( TM \) as \( \nabla^c_X Y = (\nabla_X Y)^c \) for \( X, Y \in \mathcal{X}(M) \), therefore
\[
\begin{align*}
\nabla^c_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} & = \Gamma^k_{ij} \frac{\partial}{\partial x^k} + y^l \frac{\partial \Gamma^k_{ij}}{\partial y^l} \frac{\partial}{\partial y^k}, \quad \nabla^c_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \\
\nabla^c_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} & = \Gamma^k_{ij} \frac{\partial}{\partial y^k}, \quad \nabla^c_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial y^k}.
\end{align*}
\]

**Proposition 2.2 (4 7).** Let \( T \) and \( R \) be the torsion and curvature tensor of \( \nabla \), respectively. Then \( T^c \) and \( R^c \) are the torsion and curvature tensor of \( \nabla^c \), respectively, and

- \( \nabla \) is symmetric if and only if \( \nabla^c \) is symmetric.
- \( \nabla \) is flat if and only if \( \nabla^c \) is flat.

**Proposition 2.3 (4 7).** Let \( F \) be a tensor field of type \((1, r)\) or \((0, r), r \geq 1, \) on \( M \) and \( X, Y \in \mathcal{X}(M), \) we have
\[
\nabla^c_X Y^v = (\nabla_X Y)^v, \quad \nabla^c_X Y^c = (\nabla_X Y)^c, \quad \nabla^c X^v Y^c = \nabla^c X^c Y^v = (\nabla_X Y)^v, \\
\nabla^c F^v = (\nabla F)^v, \quad \nabla^c F^c = (\nabla F)^c.
\]
2.3. **Horizontal lifts.** The horizontal lift $f^h$ of a function $f$ on $M$ is given by $f^h = f^* - \nabla f$, where $\nabla f = \gamma(\nabla f)$, where for any tensor field $F$ of type $(1, r)$ or $(0, r)$, $r \geq 1$, on $M$, $\gamma F = (F_X)^v$ and $F_X(X_1, ..., F_{r-1}) = F(X_1, ..., X_{r-1}, X)$. For any vector field $X = X^i \frac{\partial}{\partial x^i}$ on $M$ there exists a unique vector $X^h \in HVM$ such that $\pi_x X^h = X$, that is if $X = X^i \frac{\partial}{\partial x^i}$ then $X^h = X^i \frac{\partial}{\partial x^i} - y^i X^j \Gamma^i_{jk} \frac{\partial}{\partial y^j}$. We call $X^h$ the horizontal lift of the vector field $X$ in the point $(x, y) \in TM$. Let $\omega$ be a 1-form on $M$. Then the horizontal lift $\omega^h$ of $\omega$ is defined by $\omega^h = \omega^c - \nabla_\omega$. Then for any $X \in \mathcal{X}(M)$ we have $\omega^h(X^h) = 0$, $\omega^h(X^v) = (\omega(X))^v$. The horizontal lift of a $(1, 1)$ tensor field $F$ on $M$ defined by $F^h(X^h) = (FX)^h$, $F^h(X^v) = (FX)^v$. The horizontal lifts of a Riemannian metric $g$ defined by $g^h = g_{ij} \theta^i \otimes \eta^j + g_{ij} \eta^i \otimes \theta^j$ where $\theta^i = dx^i$, $\eta^i = g^{ij} \Gamma^c_{jk} dx^c + dy^c$.

From [4][7], we have

**Proposition 2.4.** Let $M$ be a manifold with a Riemannian metric $g$. For any $X, Y \in \mathcal{X}(M)$, $f \in C^\infty(M)$, and $(1, 1)$ tensor field $F$ we have:

- $g^h$ is semi-Riemannian metric and $g^h(X^v, Y^h) = (g(X, Y))^v$, $g^h(X^h, Y^h) = 0$,
- If $P(x)$ is a polynomial in one variable $x$ then $P(J^h) = (P(J))^h$,
- $g^h = g^f$ if and only if $\nabla g = 0$,
- $[X^h, Y^v] = 0$, $[X^v, Y^c] = [X, Y]^c$, $[X^c, Y^c] = [X, Y]^c$,
- $[X^h, Y^h] = -\nabla Y X^v$, $[X^h, Y^v] = \frac{1}{2}(\nabla X Y^c) - \gamma R(X, Y)$ where $R$ is curvature tensor of $g$ and the vertical vector lift $\gamma F$ defined by $\gamma F(y) = (F(y))^v$.

Let $\nabla$ be a linear connection on $M$, then we define the horizontal lift of $\nabla$ to $TM$ as the unique linear connection $\nabla^h$ on $TM$ given by

$$\nabla^h_{X^h, Y^v} = \nabla^h_{X, Y}^h = 0, \quad \nabla^h_{X^h, Y^v} = (\nabla X Y)^v, \quad \nabla^h_{X^h, Y^v} = (\nabla Y X)^h,$$

for $X, Y \in \mathcal{X}(M)$. Hence $\nabla^h X^c = (\nabla X Y)^c - \gamma R(\cdot, X, Y)$ where $R(\cdot, X, Y)Z = R(Z, X, Y)$.

3. **Metallic structures on the tangent bundle of a P-Sasakian manifold**

Let $M$ be an $n$-dimensional P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$. In this part of this section we study the metallic structure induced on $TM$ by the complete lift of a P-Sasakian structure.

**Proposition 3.1.** On the tangent bundle of P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$, there exists a metallic structure given by

$$(\phi^c)^2 = (\phi^2)^c = I - \eta^c \otimes \xi^c - \eta^c \otimes \xi^c, \quad \eta^c \otimes \xi^c = 1, \quad \eta^c \otimes \xi^c = 0, \quad \phi^c \otimes \xi^c = \phi^c \otimes \xi^c = 0, \quad \eta^c \circ \phi^c = \eta^c \circ \phi^c = 0.$$

**Proof.** From the definition of the almost paracontact structure of P-Sasakian manifold, we obtain the following relations

$$(\phi^c)^2 = (\phi^2)^c = I - \eta^c \otimes \xi^c - \eta^c \otimes \xi^c,$$

$$\eta^c \otimes \xi^c = 1, \quad \eta^c \otimes \xi^c = 0, \quad \phi^c \otimes \xi^c = \phi^c \otimes \xi^c = 0, \quad \eta^c \circ \phi^c = \eta^c \circ \phi^c = 0.$$
Therefore for any $\tilde{X} \in \mathcal{X}(TM)$ we have
\[
J(\xi^v) = \frac{p}{2} \xi^v - \frac{2\sigma_{p,q} - p}{2} \xi^c, \quad J(\xi^c) = \frac{p}{2} \xi^c - \frac{2\sigma_{p,q} - p}{2} \xi^v,
\]
\[
J(\phi^c \tilde{X}) = \frac{p}{2} \phi^c \tilde{X} - \frac{2\sigma_{p,q} - p}{2} (\tilde{X} - \eta^c(\tilde{X})\xi^v - \eta^v(\tilde{X})\xi^c).
\]
Now, we obtain
\[
J(\tilde{X}) = \frac{p}{2} \tilde{X} - \frac{2\sigma_{p,q} - p}{2} (\phi^c \tilde{X} + \eta^v(\tilde{X})\xi^v + \eta^c(\tilde{X})\xi^c)
\]
and
\[
J^2(\tilde{X}) = \frac{p}{2} J(\tilde{X}) - \frac{2\sigma_{p,q} - p}{2} (J(\phi^c \tilde{X}) + \eta^v(\tilde{X})J(\xi^v) + \eta^c(\tilde{X})J(\xi^c)) = pJ(\tilde{X}) + q\tilde{X},
\]
and it complete the proof.

**Proposition 3.2.** If $M$ is a P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$ and $J$ is defined by (3.1) then we have
\[
(3.2) \quad g^c(J\tilde{X}, J\tilde{Y}) = pg^c(\tilde{X}, J\tilde{Y}) + gg^c(\tilde{X}, \tilde{Y}), \quad \forall \tilde{X}, \tilde{Y} \in \mathcal{X}(TM).
\]

**Proof.** For any $X, Y \in \mathcal{X}(M)$, we have
\[
g^c(X^v, Y^v) = 0, \quad g^c(X^v, \xi^c) = (g(X, \xi))^v = (\eta(X))^v, \quad g^c((\phi X)^v, \xi^c) = (g(\phi X, \xi))^v = (g(X, \phi)^v = 0, \quad g^c(\xi^c, \xi^c) = (g(\xi, \xi))^c = 0.
\]
Therefore
\[
g^c(JX^v, JY^v) = -\frac{(2\sigma_{p,q} - p)}{2} (\eta(X))^v(\eta Y)^v,
\]
and
\[
g^c(X^v, JY^v) = -\frac{2\sigma_{p,q} - p}{2} (\eta(X))^v(\eta Y)^v.
\]
Thus
\[
g^c(JX^v, JY^v) = pg^c(X^v, JY^v) + gg^c(X^v, Y^v).
\]
Also, using $g^c(X^v, Y^c) = (g(X, Y))^v$ and $g^c(X^c, Y^c) = (g(X, Y))^c$ we have
\[
g^c(JX^v, JY^c) = \left(\frac{p^2}{2} + q\right)(g(X, Y))^v - \frac{(2\sigma_{p,q} - p)p}{2} [(g(X, \phi Y))^v - (\eta X)^v(\eta Y)^v],
\]
and
\[
g^c(X^v, JY^c) = \frac{p}{2} (g(X, Y))^v - \frac{(2\sigma_{p,q} - p)p}{2} [(g(X, \phi Y))^v - (\eta X)^v(\eta Y)^v].
\]
Hence
\[
g^c(JX^v, JY^c) = pg^c(X^v, JY^c) + gg^c(X^v, Y^c).
\]
The other cases are similar.

**Theorem 3.3.** Let $M$ be a P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$ and $J$ be defined by (3.1) then the metallic structure $J$ is integrable.
Proof. The 1-form $\eta$ defines on $(n - 1)$-dimensional distribution $D$ follows
\begin{equation}
\forall p \in M, \quad D_p = \{ v \in T_p M : \eta(v) = 0 \},
\end{equation}
the complement of $D$ is 1-dimensional distribution spanned by $\xi$. Suppose that
\begin{equation}
N^1 = N_\phi - 2d\eta \otimes \xi, \quad N^2(X, Y) = (L_{\phi X} \eta)Y - (L_{\phi Y} \eta)X, \quad N^3 = L_\xi \phi, \quad N^4 = L_\xi \eta.
\end{equation}
By Proposition 2.4 for any $X, Y$ of $C^\infty(M)$-module of all sections of distribution $D$, we have
\begin{align*}
N_J(X^v, Y^v) &= 0, \\
N_J(X^v, Y^c) &= A \left( [N^1(X, Y)]^v + 2N^2(X, Y)\xi^c \right), \\
N_J(X^c, Y^v) &= A \left( [N^1(X, Y)]^c + 2N^2(X, Y)\xi^v \right), \\
N_J(X^v, \xi^v) &= A \left( - (N^3(X))^v + N^4(X)\xi^c \right), \\
N_J(X^v, \xi^c) &= A \left( - (N^3(X))^c + (\phi N^3(X))^v - [N^4(\phi X) - N^4(X)]^v \xi^c \right), \\
N_J(X^c, \xi^c) &= A \left( - (N^3(X))^v + N^4(X) + N^2(X, \xi)[\xi^c + (\phi N^3(X)) - N^4(X)\xi]^c \right), \\
N_J(\xi^v, \xi^c) &= N_J(\xi^c, \xi^c) = N_J(\xi^v, \xi^v) = 0,
\end{align*}
where $A = \frac{(2\sigma_{p,g} - 1)^2}{2}$. But the tensor $N^1$ of a P-Sasakian manifold vanishes. On the other hand if $N^1 = 0$ then also $N^2, N^3$ and $N^4$ vanish. Hence $N_J(\tilde{X}, \tilde{Y}) = 0$ for all $\tilde{X}, \tilde{Y} \in \mathcal{X}(T M)$, that is $J$ is integrable.
\[\Box\]

Theorem 3.4. Let $M$ be a P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$ and $J$ be defined by (3.7) then the metallic structure $J$ is never parallel with respect to $\nabla^c$.

Proof. We have
\begin{align*}
(3.4)(\nabla^c_{X^c} J)^c &= \nabla^c_{X^c} (J^c) - J(\nabla^c_{X^c} \xi^c) \\
&= -\frac{2\sigma_{p,g} - 1}{2} \left[ \nabla^c_{X^c} ((\phi \xi)^c + \eta(\xi)^c \xi^v + \eta(\xi)^c \xi^c) - (\phi \nabla_X \xi)^c \right. \\
&\left. - (\eta(\nabla_X \xi))^c \xi^c - (\eta(\nabla_X \xi))^c \xi^c \right].
\end{align*}
Using $\phi X = \nabla_X \xi$ we get
\begin{equation}
(3.5)(\nabla^c_{X^c} J)^c = -\frac{2\sigma_{p,g} - 1}{2} [(\phi X)^v - X^v] \neq 0, \quad \forall X \in D \setminus \{0\},
\end{equation}
where $D$ is a distribution and defined by (3.3).
\[\Box\]

Proposition 3.5. Let $M$ be a P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$, $\nabla \phi = 0$ and $J$ be defined by (3.1). Then fundamental 2-form $\Phi$, given by
\begin{equation}
(3.6) \quad \Phi(\tilde{X}, \tilde{Y}) = g^c(\tilde{X}, J\tilde{Y}) - \frac{p}{2} g^c(\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in \mathcal{X}(T M),
\end{equation}
is closed if and only if
\begin{equation}
(3.7) \quad g(\nabla_Y X, \phi Z) + g(\nabla_Z Y, \phi X) + g(\nabla_X Z, \phi Y) = 0, \quad \forall X, Y, Z \in \mathcal{X}(M).
\end{equation}
Proof. The coboundary formula for $d$ on a 2-form $\Phi$ is

$$3d\Phi(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{X}\Phi(\tilde{Y}, \tilde{Z}) + \tilde{Y}\Phi(\tilde{Z}, \tilde{X}) + \tilde{Z}\Phi(\tilde{X}, \tilde{Y}) - \Phi([\tilde{X}, \tilde{Y}], \tilde{Z}) - \Phi([\tilde{Z}, \tilde{X}], \tilde{Y}) - \Phi([\tilde{Y}, \tilde{Z}], \tilde{X})$$

for any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{X}(TM)$. Hence for any $X, Y, Z \in \mathcal{X}(M)$ we have

$$3d\Phi(X^c, Y^c, Z^v) = X^c \Phi^v(Y^c, JZ^v) - \frac{p}{2} X^c \Phi^v(Y^c, Z^v) + Y^c \Phi^v(X^c, JX^c) - \frac{p}{2} Y^c \Phi^v(Z^v, X^c)$$

$$-\frac{p}{2} Y^c \Phi^v(Z^v, X^c) + Z^v \Phi^v(X^c, JY^c) - \frac{p}{2} Z^v \Phi^v(X^c, Y^c)$$

$$- \Phi^v([X, Y]^c, Z^v) + \frac{p}{2} \Phi^v([X, Y]^c, Z^v) - \Phi^v([X, Z]^v, JY^c) + \frac{p}{2} \Phi^v([Y, Z]^v, JX^c) + \frac{p}{2} \Phi^v([Y, Z]^v, X^c).$$

On the other hand

$$JZ^v = \frac{p}{2} Z^v - \frac{2\sigma_{p,q} - p}{2} ((\phi(Z))^v + (\eta(Z))^v \xi^c),$$

and

$$JX^c = \frac{p}{2} X^c - \frac{2\sigma_{p,q} - p}{2} ((\phi(X))^v + (\eta(X))^v \xi^c).$$

Therefore,

$$- \frac{6}{2\sigma_{p,q} - p} d\Phi(X^c, Y^c, Z^v) = \{Xg(Y, \phi Z) + Yg(Z, \phi X) + Zg(X, \phi Y)$$

$$- \Phi^v([X, Y]^c, Z^v) + \frac{p}{2} \Phi^v([X, Y]^c, Z^v) - \Phi^v([X, Z]^v, JY^c) + \frac{p}{2} \Phi^v([Y, Z]^v, JX^c) + \frac{p}{2} \Phi^v([Y, Z]^v, X^c).$$

Since $\nabla \phi = 0$ and $g(X, \phi Y) = g(Y, \phi X)$, we get

$$Xg(Y, \phi Z) - g([X, Y], \phi Z) = g(\nabla_Y X, \phi Z) + g(\nabla_X Z, \phi Y).$$

Also, $\nabla_X \xi = \phi X$ results that

$$g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi)$$

$$= Xg(Y, \xi) - g(Y, \nabla_X \xi) - Yg(X, \xi) + g(X, \nabla_Y \xi)$$

$$= X(\eta(Y)) - g(Y, \phi X) - Y(\eta(X)) + g(X, \phi Y)$$

$$= X(\eta(Y)) - Y(\eta(X)).$$

Thus

$$- \frac{6}{2\sigma_{p,q} - p} d\Phi(X^c, Y^c, Z^v) = 2 \{g(\nabla_Y X, \phi Z) + g(\nabla_Z Y, \phi X) + g(\nabla_X Z, \phi Y)\}$$

$$+ 2 \{(\eta(Y))^c(X\eta(Z))^v + (\eta(Z))^c(Y\eta(X))^v$$

$$+ (\eta(X))^c(Z\eta(Y))^v\}$$

Now, if $X, Y, Z \in \mathcal{D}$, then $d\Phi(X^c, Y^c, Z^v) = 0$ is equivalent with (3.7), where $\mathcal{D}$ is a distribution and defined by (3.3). Also, if $X = \xi$ or $Z = \xi$, we get the same result. The other cases reducible to (3.7). □
Note 3.6. $\xi I - \left(\frac{2\sigma_{p,q}}{2} - \frac{p}{2}\right)(\phi^h + \eta^v \otimes \xi^v \pm \eta^c \otimes \xi^c)$ are also metallic structures. For these structures we can obtain the similar results as for the metallic structure (3.1).

In the following we study a metallic structure on $TM$ induced by the horizontal lift.

**Proposition 3.7.** Let $M$ be a P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$. Then there exists a metallic structure on its tangent bundle, given by

$$F = \frac{p}{2} I - \left(\frac{2\sigma_{p,q}}{2} - \frac{p}{2}\right) (\phi^h + \eta^v \otimes \xi^v + \eta^c \otimes \xi^c).$$

Proof. By definition vertical lift and horizontal lift of the almost paracontact structure of P-Sasakian manifold, we have

$$(\phi^h)^2 = (\phi^2)^h = I - \eta^h \otimes \xi^v - \eta^v \otimes \xi^h,$$

$$\eta^v(\xi^h) = \eta^h(\xi^v) = 1, \quad \eta^v(\xi^v) = \eta^h(\xi^h) = 0,$$

$$\phi^h(\xi^v) = \phi^h(\xi^h) = 0, \quad \eta^v \circ \phi^h = \eta^h \circ \phi^h = 0.$$

Therefore, for any $\tilde{X} \in \mathcal{A}(TM)$ we have

$$F(\xi^v) = \frac{p}{2} \tilde{X}^v - \frac{2\sigma_{p,q} - p}{2} \tilde{X}^h, \quad J(\xi^h) = \frac{p}{2} \tilde{X}^h - \frac{2\sigma_{p,q} - p}{2} \tilde{X}^v,$$

$$F(\phi^h \tilde{X}) = \frac{p}{2} \phi^h \tilde{X} - \frac{2\sigma_{p,q} - p}{2} (\tilde{X}^v - \eta^h(\tilde{X}) \xi^v - \eta^v(\tilde{X}) \xi^h).$$

Now, we obtain

$$F(\tilde{X}) = \frac{p}{2} \tilde{X} - \frac{2\sigma_{p,q} - p}{2} (\phi^h \tilde{X} + \eta^v(\tilde{X}) \xi^v + \eta^h(\tilde{X}) \xi^h)$$

and

$$F^2(\tilde{X}) = \frac{p}{2} F(\tilde{X}) - \frac{2\sigma_{p,q} - p}{2} (F(\phi^h \tilde{X}) + \eta^v(\tilde{X}) F(\xi^v) + \eta^h(\tilde{X}) F(\xi^h)) = p F(\tilde{X}) + q \tilde{X},$$

and it finish the proof. \qed

**Definition 3.8** (Sasakian metric). Let $(M, g)$ be a Riemannian manifold. The Sasakian metric on $TM$ defined as follows

$$G(X^v, Y^h) = 0, \quad G(X^v, Y^v) = [g(X, Y)]^v, \quad G(X^h, Y^h) = [g(X, Y)]^h,$$

for any $X, Y \in \mathcal{A}(M)$.

**Proposition 3.9.** Let $M$ be a P-Sasakian manifold with structure tensor $(\phi, \eta, \xi, g)$. If $F$ defined by (3.8) on $TM$ and $G$ be Sasakian metric, then we have

$$G(F \tilde{X}, F \tilde{Y}) = p G(\tilde{X}, F \tilde{Y}) + q G(\tilde{X}, \tilde{Y}).$$

for any $\tilde{X}, \tilde{Y} \in \mathcal{A}(TM)$.

Proof. For any $X, Y \in \mathcal{A}(M)$, we have

$$\eta^h(X^v) = \eta^v(X^h) = 0, \quad \eta^h X^h = \eta^v X^v = (\eta(X))^h, \quad \phi^h X^v = (\phi X)^v, \quad \phi^h X^h = (\phi X)^h.$$ 

Hence

$$FX^v = \frac{p}{2} X^v - \frac{2\sigma_{p,q} - p}{2} (\phi X)^v.$$ 

Now by definition of Sasakian metric we get

$$G(FX^v, FY^v) = \left\{ \left( q + \frac{p^2}{2} \right) g(X, Y) - \frac{2\sigma_{p,q} - p}{2} pg(X, \phi Y) \right\}^v.$$
and
\[ G(X^v, FY^v) = \left\{ \frac{p}{2} g(X, Y) - \frac{2\sigma_{p,q} - p}{2} g(X, \phi Y) \right\}^v. \]

Therefore
\[ G(FX^v, FY^v) = pG(X^v, FY^v) + qG(X^v, Y^v). \]

the other cases are similar.

\[ \square \]

**Definition 3.10.** Let \((M, g)\) be a Riemannian manifold and \(\nabla\) the Levi-Civita connection with respect to the Riemannian metric \(g\). Let \(D\) be a distribution and defined by (3.3). Connection \(\nabla\) is called \(D\)-flat if \(\nabla_X Y \in D\) for all \(X, Y \in D\).

**Theorem 3.11.** Let \(M\) be a \(P\)-Sasakian manifold with structure tensor \((\phi, \eta, \xi, g)\). Then the metallic structure \(F\) defined by (3.3) on \(TM\) is integrable if and only if \(\nabla\) is \(D\)-flat and
\[ (3.11) \quad R(\phi X, \phi Y) + R(X, Y) - \phi\{R(\phi X, Y) + R(X, \phi Y)\} = 0, \]
where \(R\) is curvature tensor of \(M\).

Proof. Let \(X, Y \in D\) and \(U \in TM\). We have
\[ -\frac{2}{2\sigma_{p,q} - p} N_F(X^h, Y^h) U = [N^1(X, Y)]^h U + \{\eta R(\phi X, Y)U + \eta R(X, \phi Y)U\} \xi^h \]
\[ -\{R(\phi X, \phi Y)U + R(X, Y)U - \phi R(\phi X, Y)U \]
\[ + R(X, \phi Y)U\}^v + N^2(X, Y) \xi^v. \]

Also
\[ -\frac{2}{2\sigma_{p,q} - p} N_F(X^h, Y^v) = (\nabla_{\phi X} \phi Y - \phi \nabla_{\phi X} Y - \phi \nabla_X \phi Y + \nabla_X Y)^v \]
\[ -\{\eta (\nabla_{\phi X} Y) + \eta (\nabla_X \phi Y)\} \xi^h, \]
\[ N_F(X^v, Y^v) = 0, \]
\[ -\frac{2}{2\sigma_{p,q} - p} N_F(X^h, \xi^h) U = (\nabla_{\phi X} \xi - \phi \nabla_X \xi + \phi (N^3(X))^h U + [\eta R(\phi X, \xi)U] \xi^h \]
\[ + \{N^2(X, \xi) - \eta (\nabla_X \xi)\} \xi^v + [\phi R(\phi X, \xi)U]^v, \]
\[ -\frac{2}{2\sigma_{p,q} - p} N_F(X^h, \xi^v) U = [N^3(X)]^h U + \{N^2(X, \xi) + \eta R(X, \xi)U\} \xi^h \]
\[ -\{R(\phi X, \xi)U - \phi R(X, \xi)U + \phi (\nabla_{\phi X} \xi) - \nabla_X \xi\}^v \]
\[ -N^4(X) \xi^v, \]
and
\[ -\frac{2}{2\sigma_{p,q} - p} N_F(X^v, \xi^v) = -(\nabla_{\xi} \phi X - \phi \nabla_X \xi)^v + \eta (\nabla_X Y) \xi^h. \]

Hence \(N_F = 0\) if and only if (3.11) is true and
\[ (3.12) \quad \nabla_{\phi X} \phi Y - \phi \nabla_{\phi X} Y - \phi \nabla_X \phi Y + \nabla_X Y = 0. \]

From (17), (3.12) is equivalent with \(\eta (\nabla_X Y) = 0\), for any \(X, Y \in D\), i.e. \(\nabla\) is \(D\)-flat.

\[ \square \]

**Theorem 3.12.** Let \(M\) be a \(P\)-Sasakian manifold with structure tensor \((\phi, \eta, \xi, g)\). Then the metallic structure \(F\) defined by (3.3) on \(TM\) is never parallel with respect to \(\nabla^h\).
Proof. We have
\[(\nabla_{X^h} F)\xi^h = \nabla_{X^h}(F\xi^h) - F(\nabla_{X^h}\xi^h) = -\frac{2\sigma_{p,q} - p}{2}[(\phi X)^v - (\phi^2 X)^h].\]
If \(X \in \mathcal{D} \setminus \{0\}\) then \((\nabla_{X^h} F)\xi^h \neq 0\), where \(\mathcal{D}\) is defined by (3.3).

We define the fundamental 2-form \(\Phi'\) by
\[\Phi'(\tilde{X}, \tilde{Y}) = G(\tilde{X}, \tilde{F}\tilde{Y}) - \frac{p}{2}G(\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in \mathcal{X}(TM).\]

**Proposition 3.13.** Let \(M\) be a P-Sasakian manifold with structure tensor \((\phi, \eta, \xi, g)\) and \(F\) be defined by (3.8). Then fundamental 2-form \(\Phi'\), is never closed.

Proof. Let \(X \in \mathcal{D}\) be an unit vector field i.e. \(g(X, X) = 1\), where \(\mathcal{D}\) is defined by (3.3). Then by similar proof of Proposition (3.8) we have
\[-\frac{6}{2\sigma_{p,q} - p}d\Phi'(X^h, X^v, \xi^v) = -G([\xi^v, X^h], (\phi X)^v) = -g(\nabla_X\xi, \phi X)^v = -g(X, X)^v = -1.\]

\[\square\]

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