GENERALIZED FIERZ IDENTITIES AND THE SUPERSELECTION RULE FOR GEOMETRIC MULTISPINORS *

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Abstract

The inverse problem, to reconstruct the general multivector wave function from the observable quadratic densities, is solved for 3D geometric algebra. It is found that operators which are applied to the right side of the wave function must be considered, and the standard Fierz identities do not necessarily hold except in restricted situations, corresponding to the spin-isospin superselection rule. The Greider idempotent and Hestenes quaterionic spinors are included as extreme cases of a single superselection parameter.

Key words: fierz – multivector – superselection – spinors

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1 Introduction

In a recent paper Crawford[1] explored the inverse problem of Dirac bispinor algebra, to reconstruct the wave function from the observable quadratic densities. Other authors[2,3] have presented parallel developments for multivector quantum mechanics in which column spinors are replaced by Clifford algebra aggregates. However, these expositions have only considered restricted cases (e.g. minimal ideals) for which the multivector analogies of the observable bispinor densities obey the standard Fierz[1,2,4] identities.

Previously we have proposed[5,6] a more general multivector wave function in which all the geometric degrees of freedom are used. To obtain the complete set of observable multispinor densities one needs to augment the standard sinistral[6] operators (applied to the left side of the wave function) with new dextral[6] (right-side applied) operators, and also bilateral[6] operators (multivectors applied on both sides of the wave function). It is found that if and only if the multivector wave function is restricted will the multispinor densities obey the standard Fierz identities. In this paper we propose to solve the inverse problem for the general unrestricted multivector wave function of the 8-element 3D geometric algebra \( \mathbb{C}(2) \), i.e. the Pauli algebra.

2 The Algebra of Standard Pauli Spinor Densities

In non-relativistic quantum mechanics, the electron is represented by a two-component Pauli spinor. The endomorphism algebra (module structure on spinors) is \( \mathbb{C}(2) \), i.e. two by two complex matrices. This Clifford algebra has as its basis the 8 element group generated by 3 mutually anticommuting basis vectors, \( \{ \sigma_j, \sigma_k \} = 2\delta_{jk} \) for \( j,k = 1,2,3 \), and where \( i = \sigma_1\sigma_2\sigma_3 \). As operators, their “bilinear expectation values” yield real densities which are interpreted to be the projection of the spin along the \( j \)-th spatial axis, \( S_j = \langle \psi | \sigma_j | \psi \rangle \).

The 4 densities \( \{ \rho, S_i \} \), where \( \rho = \langle \psi | \psi \rangle \), are invariant with respect to the phase parameter of the spinor. Hence they satisfy a single constraint equation which can be derived by substituting the projection operator into the square of the normalization. The magnitude of the spin is found to be constrained by the Fierz[1,2,4] identity,

\[
|S|^2 = S^k S_k = \rho^2. \tag{1}
\]

The spinor can be reconstructed in terms of a \( U(2) \) unitary rotation matrix,

\[
U(\lambda, \theta, \phi, \alpha) = \exp(ia/2) \exp(\alpha \sigma_3/2) \exp(\sigma_2 \theta/2) \exp(\sigma_1 \phi/2), \tag{2}
\]

where \( (\theta, \phi) \) are the orientation angles of the spin and \( (\lambda, \alpha) \) do not contribute in the bilinear form \( \sigma^i S_j(\theta, \phi) = \rho U \sigma_3 U^\dagger \). Choosing a starting spinor to be the “plus” eigenstate of \( \sigma_3 \), the wave function can be expressed, \( \psi(\rho, \theta, \phi, \beta) = \sqrt{\rho} U \eta \), where the net unobservable phase is \( \beta = \lambda + \alpha \).
3 The Algebra of Geometric Multispinors

We consider an unrestricted multivector wave function\(^5\) which has the same 8 degrees of freedom as the Clifford group,

\[
\psi = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (a + b\sigma_1)\frac{(1 + \sigma_3)}{2} + (c + d\sigma_1)\frac{(1 + \sigma_3)}{2}\sigma_1, \quad (3)
\]

where \{a, b, c, d\} are complex coefficients. Note each column of the matrix is a minimal left ideal of the algebra and will hence behave like a column spinor for all sinistral (left-sided) operations. Each row of the matrix is a minimal right ideal of the algebra and will behave like a row isospinor for all dextral (right-sided) operations. Hence the complete solution can be interpreted as an isospin doublet (of spinors) coupled by now-allowable dextral application of the Pauli operators.

3.1 Multivector Densities

A complete set of 16 generalized quadratic forms are defined in terms of the matrix trace (i.e. half the scalar part of the Clifford multivector)\(^9\),

\[
\rho = Tr(\psi^\dagger\psi) = |a|^2 + |b|^2 + |c|^2 + |d|^2, \quad (4a)
\]
\[
S_j = Tr(\psi^\dagger\sigma_j\psi) = Tr(\psi\psi^\dagger\sigma_j), \quad (j = 1, 2, 3), \quad (4b)
\]
\[
T_j = Tr(\psi\sigma_j\psi^\dagger) = Tr(\psi^\dagger\psi\sigma_j), \quad (j = 1, 2, 3), \quad (4c)
\]
\[
R_{jk} = Tr(\psi^\dagger\sigma_j\psi\sigma_k) = Tr(\psi\sigma_k\psi^\dagger\sigma_j), \quad (j, k = 1, 2, 3). \quad (4d)
\]

They are interpreted to be the probability, spin, isospin and bilateral densities respectively. From these we can construct the multivector densities,

\[
\psi\psi^\dagger = (\rho + \sigma_kS^k)/2, \quad (5a)
\]
\[
\psi^\dagger\psi = (\rho + \sigma_kT^k)/2, \quad (5b)
\]
\[
\psi^\dagger\sigma_j\psi = (S_j + R_{jk}\sigma^k)/2, \quad (5c)
\]
\[
\psi\sigma_k\psi^\dagger = (T_k + \sigma^jR_{jk})/2. \quad (5d)
\]

3.2 Generalized Fierz Identities

The 16 densities are all independent of the phase parameter, hence must satisfy 9 constraint equations. In general these identities have the form,

\[
Tr[(\psi^\dagger\sigma_\alpha\psi)\sigma_\beta(\psi\sigma_\gamma\psi^\dagger)\sigma_\delta] = Tr[(\psi_\beta\psi^\dagger_\alpha\psi_\gamma)(\psi\sigma_\delta\psi^\dagger_\beta)\sigma_\alpha] \quad (6a)
\]

where the indices can take on values 0 through 3, and \(\sigma_0 = 1\). The parenthesis indicate where one inserts eqs. (5abcd). It can be shown from these relations that the bilateral density eq. (4d) contains all the other densities. Further, we find that the magnitudes of the spin and isospin are equal, but that eq. (1) is no longer valid,

\[
|S|^2 = |T|^2 \leq \rho^2. \quad (6b)
\]
3.3 Interpretation of the Bilateral Density

Counting degrees of freedom, we see that there is one free internal “hidden variable” contained in $R_{jk}$ which does not affect the other densities. To gain some insight as to the nature of this parameter we consider the class of unitary (hence $\rho$ invariant) transformations that will leave the densities $\{S^j, T^j\}$ invariant, but modify the bilateral density.

The special case sinistral operator, $U(\lambda) = \exp[i\sigma_j/(2|S|)]$, will leave the spin invariant (as well as the isospin) as it corresponds to a rotation about the spin axis by angle $\lambda$. The bilateral density will be modified by this transformation, hence we should be able to parametrize $R_{jk}$ in terms of the densities $\{\rho, S^j, T^j\}$ and a bilateral phase angle $\lambda$.

4 Inverse Theorem and Superselection Rule

We assert that the projection operator for the multivector wave function has the bilateral form, $\psi = (\rho \psi + S_k \sigma^k \psi + \psi \sigma_k T^k + \sigma^j \psi \sigma^k R_{jk})/(4\rho)$.

4.1 Inverse Theorem

The multivector wave function can be reconstructed from the observable densities by applying the projection operator to an arbitrary starting solution $\eta$, and renormalizing. Hence we assert,

$$\Psi(\alpha, S^k, T^k, R^{jk}) = e^{i\alpha}(\rho \eta + S_k \sigma^k \eta + \eta \sigma_k T^k + \sigma^j \eta \sigma^k R_{jk})/N, \quad (7a)$$

where $\alpha$ is a phase factor and $\eta$ is an arbitrary starting multivector subject only to the normalized trace constraint $Tr(\eta^\dagger \eta) = 1$.

The normalization factor is most directly determined by requiring the reconstructed wave function to reproduce the probability density eq. (4a),

$$N^2 = 4[\rho + Tr(\eta^\dagger \eta \sigma^k R_{jk})] + 2[Tr(\eta^\dagger \eta \sigma^k T^k) + Tr(\eta \eta^\dagger \sigma^k S_k)], \quad (7b)$$

where identities have been used to reduce the quadratic terms to linear ones in terms of the observable densities. This construction will fail if eq. (7b) yields zero, in which case a different starting solution should be used.

4.2 Special Classes of Solutions

In order to insure a scalar norm, Hestenes[3,9] proposed a unitary or quaternionic solution which has 5 parameters,

$$\psi(\alpha, \rho, \lambda, \theta, \phi) = \sqrt{\rho/2} U(\lambda, \theta, \phi, \alpha) = \sqrt{\rho/2} e^{i\alpha/2} [r + i\sigma^j B_j], \quad (8)$$

where unitary matrix $U(\lambda, \theta, \phi, \alpha)$ is given by eq. (2). The alternate quaternionic Cayley-Klein components $\{r, B_j\}$ are all real numbers, subject to constraint $r^2 + B^2 = 1$. Only 4 parameters are however needed to describe an electron, hence Hestenes (arbitrarily?) sets the parameter $\alpha$ to zero.
This unitary class of solutions is synonymous with zero magnitude spin and isospin as defined by eqs. (4bc). The bilateral density eq. (4d) is proportional (by a factor of \( \rho \)) to the \( O(3) \) rotation matrix \( R(\lambda, \theta, \phi) \) associated with the \( U(2) \) matrix \( U(\lambda, \theta, \phi, \alpha) \). This allows Hestenes to make an alternate definition of a “spin” vector in terms of the bilateral density,

\[
S'_{ij} = R_{ij3} = \text{Tr}(\psi^\dagger \sigma_j \psi \sigma_3) = \frac{1}{2} \text{Tr}(U \sigma_3 U^\dagger \sigma_j) \rho.
\]

It is easily verified that \( R_{jk} \) is invariant with respect to the \( \lambda \) parameter of the unitary matrix, allowing Hestenes to reinterpret it as quantum phase, and dextrally applied \( i\sigma^3 \) as the quantum phase generator (replacing the usual commuting \( i \)).

In contrast, Greider\[7\] proposed an idempotent spinor which has the algebraic form of the projection operator,

\[
\psi = e^{i\alpha} \left( \frac{\rho + S_k \sigma^k}{2\sqrt{\rho}} \right) = \sqrt{\rho} U(0, \theta, \phi, \alpha) \frac{(1 + \sigma^3)}{2} U^\dagger(0, \theta, \phi, -\alpha),
\]

where the magnitude of the spin is subject to the standard Fierz constraint of eq. (1). This makes the determinant zero, hence the wave function is of the “singular class” distinctly different from the “unitary class” discussed above. There are only 4 degrees of freedom, exactly that needed to describe a single Pauli spinor (i.e. isospin is everywhere parallel to spin).

Isospin degrees of freedom can be re-introduced by applying a dextral rotation operator to eq. (10). Equivalently, consider the following factorized idempotent form,

\[
\psi = \sqrt{\rho} U(\lambda, \theta_S, \phi_S, \alpha) \frac{(1 + \sigma^3)}{2} U^\dagger(\lambda, \theta_T, \phi_T, -\alpha),
\]

\[
= e^{i\alpha} (\rho + S_k \sigma^k)(\rho + T_j \sigma^j) \left[ 4\rho(\rho^2 + S_k T^k)^{-\frac{1}{2}} \right],
\]

where the singularity constraint eq. (1) still holds. The angles \( \{\theta_S, \phi_S\} \) give the orientation of the spin, while \( \{\theta_T, \phi_T\} \) that of the isospin. Our solution has 6 degrees of freedom, exactly that which is needed to describe an isospin doublet of Pauli spinors (i.e. two particle generations in the family). The net phase \( \beta = \lambda + \alpha \) shows \( \lambda \) is indistinguishable from parameter \( \alpha \), hence \( R_{jk} = \rho S_j T_k / |\mathbf{S}|^2 \), has no \( \lambda \) dependence.

### 4.3 Superselection Parameter

Our multispinor solution is subject to a spin-isospin superselection rule\[10\]. While certain linear combinations are allowable, the superposition of “spin & isospin up” with “spin & isospin down” would yield a “forbidden” unitary class solution with zero spin and isospin. Equivalently such a state is inaccessible by any spin/isospin rotation from a “spin & isospin up” state. Mathematically this constraint manifests as requiring the determinant of our wave function to be zero.

Consider a new superselection parameter \( \delta \), defined: \( |\mathbf{S}| = \rho \cos \delta \),

\[
\Psi(\alpha, \rho, \lambda, \delta, \theta_S, \phi_S) = \sqrt{\rho} e^{i\alpha} \exp(i\sigma^k n_k \lambda/2) \left( 1 + e^{i\delta} n_k \sigma^k \right)/2,
\]

(11)
where \( n_k(\theta_S, \phi_S) = S_k/|S| \). For \( \delta = 0 \) the wave function becomes a Greider\[7\] idempotent with zero determinant, and when the spin vanishes in the limit of \( \delta = \pi/2 \), the solution is of the Hestenes\[3,9\] quaternionic form. Note the bilateral phase \( \lambda \) is independent of the ordinary imaginary phase \( \alpha \) for \( \delta > 0 \).

The remaining two isospin degrees of freedom can be reintroduced as before by a dextrad rotation operator. A complete parameterization of the general 8 degree of freedom solution can be expressed in polar form,

\[
\Psi(\rho, \alpha, \lambda, \theta_S, \phi_S, \theta_T, \phi_T) = \sqrt{\rho} \: U(\lambda, \theta_S, \phi_S, \alpha) \frac{(1 + e^{i\delta} \sigma_3)}{2} U^\dagger(\lambda, \theta_T, \phi_T, -\alpha).
\]

(12)

5 Summary

We have solved the inverse problem for the completely general eight degree of freedom wave function of 3D geometric space. Our results are more general than other treatments in that a more complete set of quadratic multispinor densities is introduced which includes sinistral, dextral and bilateral operations. The 16 densities satisfy generalized Fierz-type identities. The new bilateral density is found to contain one new independent “hidden” variable which does not affect the more familiar probability, spin and isospin densities. It is an open question as to whether this quantity can be physically measured, or is unobservable like the overall quantum phase parameter.

The standard Fierz identities (for column spinors) are found not to hold except for a restricted singular class of wave functions. This appears to be a manifestation of the spin-isospin superselection rule, and may be the critical constraint which classifies the solution as being a fermionic particle. A continuous superselection parameter is introduced for which the singular class of solutions (which includes the Greider idempotent form) is one extreme case; the Hestenes quaternionic spinor form is at the other extreme.

Extending the work to 4D Minkowski space with a 16 degree of freedom wave function we will find 136 quadratic forms, which obey 121 generalized identities. One or more new “hidden” variables will be found, and the standard Fierz identities will not be valid except for a restricted wavefunction, corresponding to the charge superselection rule of bispinors.

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