Asymptotics of Harish-Chandra-Itzykson-Zuber integrals and free probability theory

Toshiyuki Tanaka
Department of Systems Science, Graduate School of Informatics, Kyoto University,
36-1 Yoshida Hon-machi, Sakyo-ku, Kyoto-shi, Kyoto 606-8501, Japan
E-mail: tt@i.kyoto-u.ac.jp

Abstract. We discuss an asymptotic exponent $N^{-1} \log \mathbb{E}_A[\exp(\beta \text{tr} X^*AX/2)]$ in the limit $N \to \infty$, where $\beta = 1$ or 2, where $A$ is an $N$-by-$N$ real symmetric ($\beta = 1$) or complex Hermitian ($\beta = 2$) random matrix, and where $X$ is an $N$-by-$M$ real ($\beta = 1$) or complex ($\beta = 2$) matrix, with $M$ being kept finite while taking the limit $N \to \infty$. Relation of the problem to the so-called Harish-Chandra or Itzykson-Zuber integrals are also discussed. Assuming that the result is given in terms of limiting eigenvalues of $Q = N^{-1}X^*X$, we show that the exponent is given by a sum of integrated $R$-transforms of the limiting eigenvalue distribution of $A$. We also provide some examples for which the same result holds without the above assumption.

1. Introduction
In recent years it has been widely recognized that approaches that are based on statistical mechanics are efficient in studying several problems in non-physics research fields. Those problems to which statistical mechanics framework has been successfully applied are typically characterized as ones with large-scale systems with randomness. In various problems representation of the randomness of an underlying system takes the form of random matrices, so that one can expect that more efficient and wider range of applications will be possible by better and more thorough understanding of properties of large random matrices.

Study of random matrices is itself an interdisciplinary research area. Random matrices, or matrix-valued random variables, have been studied in mathematical statistics since 1920s [1, 2]. The work of Wigner in the 1950s initiated intensive study of random matrices in physics, and numerous results have been published since then [3, 4]. Another more recent stream of research has emerged from an algebraic approach to probability, and in particular, free probability theory, or probability theory of noncommutative random variables, that was started by Voiculescu in the mid-1980s, has provided a variety of novel mathematical notions and tools in the study of random matrices in their large-dimension limit [5]. To relate these two streams of research of random matrices is thus becoming important, because of a large amount of accumulated results in physics literature, and of a relatively smaller volume of the outcomes from the algebraic approach due to its rather short history.

In this paper, we deal with a basic and simple problem of evaluating expectation of a function of a random matrix in the large-dimensional asymptotics. The problem, as well as our results, is stated in section 2. In section 3, we briefly review some problems that are relevant to ours. Some of them were studied in the context of Lie groups (“Harish-Chandra integral”), some others...
with free probability theory as well as large-deviation theory, and yet others in the context of physics ("Itzykson-Zuber integral"). Some of these studies have provided partial answers to our problem. As far as the author’s knowledge, however, no full answer has been given to our problem.

2. Problem

In this paper we treat two problems in a single unified setting, by following the convention in the literature of random matrix research of introducing a parameter \( \beta \) which takes either 1 or 2. Let \( A = (a_{ij}) \) be an \( N \)-by-\( N \) real symmetric \( (\beta = 1) \) or complex Hermitian \( (\beta = 2) \) random matrix, and let \( \lambda_1, \ldots, \lambda_N \) be its eigenvalues. We assume that the eigenvalue distribution of \( A \) converges, as \( N \to \infty \), to a compactly supported measure, denoted by \( \rho_A \), that is,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) = \rho_A(\lambda).
\]

(1)

Let \( X = (x_{\mu}) \) be an \( N \)-by-\( M \) real \((\beta = 1)\) or complex \((\beta = 2)\) matrix. We assume that eigenvalues of the \( M \)-by-\( M \) matrix \( Q = N^{-1}X^*X \) converge to deterministic limits as \( N \to \infty \), where \( X^* \) denotes transpose of \( X \) when \( \beta = 1 \), and conjugate transpose of \( X \) when \( \beta = 2 \). We would like to study the quantity

\[
E_A \left[ \exp \left( \frac{\beta}{2} \text{tr} X^*AX \right) \right],
\]

(2)

where \( E_A(\cdots) \) denotes averaging over randomness of \( A \). In this paper we discuss the limit

\[
I(\beta)(\rho_A, X) = \lim_{N \to \infty} \frac{1}{N} \log E_A \left[ \exp \left( \frac{\beta}{2} \text{tr} X^*AX \right) \right],
\]

(3)

where \( M \) is kept finite while taking the limit \( N \to \infty \).

The main contributions of this paper are twofold: In the first contribution, we make the following assumption.

**Assumption 1** \( I(\beta)(\rho_A, X) \) is of the form

\[
I(\beta)(\rho_A, X) = \frac{\beta}{2} \sum_{\mu=1}^{M} G(q_{\mu}),
\]

(4)

where \( q_1, \ldots, q_M \) are eigenvalues of the matrix \( Q = N^{-1}X^*X \) and where \( G(\cdot) \) is an analytic function that depends on \( \rho_A \).

Under Assumption 1, we show that \( G(\cdot) \) is given by

\[
G(q) = \int_{0}^{q} R_A(z) \, dz,
\]

(5)

where \( R_A(\cdot) \) is the \( R \)-transform [6] of the limiting eigenvalue distribution \( \rho_A \) of \( A \). The concept of \( R \)-transform has been first introduced in the context of free probability theory. The case \( \beta = 1 \) has been discussed in [7]. The derivation including the case \( \beta = 2 \) is given in section 4 of this paper.

1 We expect that the analysis in this paper can straightforwardly be extended to the case where \( A \) is a quaternion self-dual Hermitian random matrix with \( \beta = 4 \), even though we have not performed the extension.
As the second contribution, in section 5 we show some examples in which

\[ I^{(\beta)}(\rho_A, X) = \frac{\beta}{2} \sum_{\mu=1}^{M} \int_0^{q_\mu} R_A(z) \, dz \]  

(6)

hold without Assumption 1. These results naturally lead us to ask what conditions we have to impose on \( A \) in order to have the result (6), which is an interesting open problem.

3. Related works

The following integral has been studied extensively in the literature:

\[ I_N^{(\beta)}(D, E) = \int \exp \left[ \frac{N\beta}{2} \text{tr}(UDU^*E) \right] dm_\beta(D, E), \]  

(7)

where \( m_\beta \) denotes the Haar measure on the orthogonal group and on the unitary group when \( \beta = 1 \) and 2, respectively, and \( D \) and \( E \) are symmetric (\( \beta = 1 \)) or Hermitian (\( \beta = 2 \)) matrices, whose eigenvalues are denoted by \( \{\lambda_1, \ldots, \lambda_N\} \) and \( \{\mu_1, \ldots, \mu_N\} \), respectively.

A classical result regarding the integral (7) is the following formula, a more general version of which is attributed to Harish-Chandra [8],

\[ I_N^{(2)}(D, E) = \text{const.} \times \frac{\det[\exp(N\lambda_i\mu_j)]}{N^{N(N-1)/2} \Delta(\lambda) \Delta(\mu)}, \]  

(8)

where

\[ \Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j). \]  

(9)

Accordingly, the integral (7) is sometimes called the Harish-Chandra integral.

In some applications, large-\( N \) asymptotics are of interest. Guionnet and Zeitouni [9, 10] have shown that, when eigenvalue distributions of \( D \) and \( E \) converge, as \( N \to \infty \), weakly towards compactly supported probability measures \( \rho_D \) and \( \rho_E \), respectively, the following limit exists for \( \beta = 1 \) and 2:

\[ I^{(\beta)}(\rho_D, \rho_E) = \lim_{N \to \infty} \frac{1}{N^2} \log I_N^{(\beta)}(D, E). \]  

(10)

They have also derived an explicit formula for the limit.

Guionnet and Maïda [11] considered a different asymptotic, in which rank \( E \) is much smaller than \( N \). In this case the limit one has to consider is not (10) but

\[ I^{(\beta)}(\rho_D, E) = \lim_{N \to \infty} \frac{1}{N} \log I_N^{(\beta)}(D, E), \]  

(11)

and they have shown, assuming rank \( E = o(N^{1/2-\varepsilon}) \) for some \( \varepsilon > 0 \), that the limit is given in terms of integrals of \( R \)-transform of \( \rho_D \), which is essentially the same form as (6). Collins and Śniady [12] have further extended Guionnet and Maïda’s result to the case where rank \( E \) is \( o(N) \). The same limit as that given by Guionnet and Maïda has been calculated previously by Itzykson and Zuber [13] for \( \beta = 2 \), whose result was extended to the case \( \beta = 1 \) by Marinari, Parisi and Ritort [14]. In physics, the integral (7) is therefore sometimes called the Itzykson-Zuber integral.

The asymptotic exponent of the Itzykson-Zuber integral (11) has found applications in various fields: Opper and Winther [15] used the result in a study of their generalization of the so-called Thouless-Anderson-Palmer mean-field theory in such a way that it is applicable to problems of statistical inference consisting of non-Ising variables. In the context of communication
and information theory, Takeda et al. [16] utilized the asymptotic result in their analysis of a class of orthogonally-spread code-division multiple-access (CDMA) channels. M"uller et al. [17, 18] discussed a problem of vector precoding in multi-antenna communications, in which the asymptotic exponent was used.

Relation of the above result to our problem becomes evident if we identify \( A = UDU^* \) and \( XX^* = NE \), that is, if the random matrix \( A \) has orthogonal \((\beta = 1)\) or unitary \((\beta = 2)\) invariance, our problem is reduced to the Itzykson-Zuber integral, so that the above-mentioned result by Guionnet and Ma"ida has solved our problem with the solution (6). What remains is therefore to ask whether or not the solution (6) is still valid if the random matrix \( A \) lacks the orthogonal or unitary invariance property, which is the main question to be addressed in this paper.

4. Analysis

In this section we show derivation of our first main result (5). The derivation is basically an extension of that given in [7] in order to cover both cases, \( \beta = 1 \) and 2. We first consider a generating function \( Z(D) \) of \( \exp[(\beta/2) \text{tr} X^*AX] \), defined by

\[
Z(D) = \int \exp \left( \frac{\beta}{2} \text{tr} X^*AX - \frac{\beta}{2} \text{tr} XD^*X^* \right) dX,
\]

(12)

where \( D \) is a \( M \times M \) real symmetric \((\beta = 1)\) or complex Hermitian \((\beta = 2)\) matrix. The integral with respect to \( X \) in (12) is convergent if all eigenvalues \( \delta_1, \ldots, \delta_M \) of the matrix \( D \) are larger than the maximum eigenvalue of \( A \).

One can diagonalize \( A \) and \( D \) by appropriately choosing orthogonal \((\beta = 1)\) or unitary \((\beta = 2)\) matrices \( U \) and \( V \), respectively, and let \( A = U^*\Lambda U \) and \( D = V^*\Delta V \), where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) and \( \Delta = \text{diag}(\delta_1, \ldots, \delta_M) \). By changing variables of integration in (12) with \( X = UXV^* \), one obtains

\[
Z(D) = \int \exp \left( \frac{\beta}{2} \text{tr} \bar{X}^*\Lambda \bar{X} - \frac{\beta}{2} \text{tr} \bar{X} \Delta \bar{X}^* \right) d\bar{X}
\]

\[
= (2\pi/\beta)^N \lambda^\beta N \prod_{i\mu}(\delta_\mu - \lambda_i)^{-\beta/2}
\]

\[
\rightarrow \exp \left[ \frac{N\beta}{2} \left( M \ln \frac{2\pi}{\beta} - \sum_{\mu=1}^M \int (\delta_\mu - \lambda) \rho_A(\lambda) d\lambda \right) \right].
\]

(13)

In the last line we have considered the large-\( N \) asymptotics. The above formula evidently shows that

\[
\lim_{N \to \infty} \frac{2}{N\beta} \ln Z(D) = \lim_{N \to \infty} \frac{2}{N\beta} \ln \mathbb{E}_A[Z(D)] = M \ln \frac{2\pi}{\beta} - \sum_{\mu=1}^M \int (\delta_\mu - \lambda) \rho_A(\lambda) d\lambda.
\]

(14)

We next evaluate \( \mathbb{E}_A[Z(D)] \) using the Assumption 1 and the result will be compared with (13). One has

\[
\mathbb{E}_A[Z(D)] \approx \int \exp \left( \frac{N\beta}{2} \text{tr} G(Q) \right) \mu(Q) dQ,
\]

(15)

where \( \mu(Q) dQ \) is the measure induced from the measure \( \exp[-(\beta/2) \text{tr} XD^*X^*] dX \) in (12) via the change of variables of integration, \( Q = N^{-1}X^*X \). Noting that

\[
Q_{\mu\nu} = N^{-1} \sum_{i=1}^N \pi_{i\mu}x_{i\nu}
\]

(16)
can be regarded as an empirical mean of the independent random quantities \( \{ x_i \mu x_i' ; i = 1, \ldots, N \} \), the measure \( \mu(Q) \) satisfies the large deviation principle [19] with a rate function \( I(Q) \). One thus obtains the following heuristic formula for large \( N \):

\[
\mathbb{E}_A[Z(D)] \approx \int \exp \left( \frac{N\beta}{2} \text{tr} G(Q) - NI(Q) + o(N) \right) dQ. \tag{17}
\]

The rate function \( I(Q) \) is given by the Legendre transform

\[
I(Q) = \sup \left[ \frac{1}{2} \text{tr} QQ - \ln \mathcal{M}(\tilde{Q}) \right], \tag{18}
\]

where \( \mathcal{M}(\tilde{Q}) \) is the moment generating function of the (unnormalized) measure

\[
p(x) = \exp \left( -\frac{\beta}{2} x^* D x \right), \tag{19}
\]

where \( x = (x_1, \ldots, x_M)^* \). The moment generating function \( \mathcal{M}(\tilde{Q}) \) is evaluated to be

\[
\mathcal{M}(\tilde{Q}) = \int \exp \left( \frac{1}{2} x^* \tilde{Q} x - \frac{\beta}{2} x^* D x \right) dx = (2\pi)^{M\beta/2} \exp \left[ -\frac{\beta}{2} \text{tr} \ln(\beta D - \tilde{Q}) \right]. \tag{20}
\]

The supremum of (18) is attained at \( \tilde{Q} = \beta(D - Q^{-1}) \), so that the rate function \( I(Q) \) is explicitly expressed in terms of \( Q \) as

\[
I(Q) = \frac{\beta}{2} \left[ -M \left( 1 + \ln \frac{2\pi}{\beta} \right) + \text{tr} DQ - \ln |Q| \right]. \tag{21}
\]

Collecting these results one obtains

\[
\mathbb{E}_A[Z(D)] \approx \int \exp \left( \frac{N\beta}{2} G(Q) \right) dQ, \tag{22}
\]

where

\[
G(Q) = M \left( 1 + \ln \frac{2\pi}{\beta} \right) - \text{tr} DQ + \text{tr} G(Q) + \ln |Q|, \tag{23}
\]

and applying the saddle-point method, one has

\[
\mathbb{E}_A[Z(D)] \approx \exp \left( \frac{N\beta}{2} g \right) \tag{24}
\]

for large \( N \), where \( g = \sup_Q G(Q) \).

So far we have considered \( D \) to be a free parameter. In the following, we seek the value of \( D \) such that \( Q = N^{-1}X^*X \), with \( X \) specified in the original problem, gives the saddle point of \( G(Q) \). We restrict ourselves to considering \( D \) that can be simultaneously diagonalized with \( Q \). Let \( q_1, \ldots, q_M \) be eigenvalues of \( Q \), and let \( \delta_1, \ldots, \delta_M \) be the respective eigenvalues of \( D \). One thus has

\[
g = M \left( 1 + \ln \frac{2\pi}{\beta} \right) - \sum_{\mu=1}^M \min_{q_\mu} [\delta_\mu q_\mu - G(q_\mu) - \ln q_\mu]. \tag{25}
\]
Comparison with (13) yields
\[
\min_{q_\mu} [\delta_{q_\mu} - G(q_\mu) - \ln q_\mu] = 1 + \int \ln(\delta_\mu - \lambda) \rho_A(\lambda) d\lambda,
\] (26)
for \(\mu = 1, \ldots, M\). Since this equality should hold for arbitrary values of \(q_\mu\), we can drop the subscript \(\mu\), so that the following identity relation should hold:
\[
\min_q [\delta q - G(q) - \ln q] = 1 + \int \ln(\delta - \lambda) \rho_A(\lambda) d\lambda.
\] (27)

One can regard (27) as a formula defining the function \(G(\cdot)\). Equation (27) is the same as that has been derived using replica method by Cherrier et al. [20], on the basis of which they have argued as follows: Since
\[
\int \ln(\delta - \lambda) \rho_A(\lambda) d\lambda
\] (28)
is concave as a function of \(\delta\) whenever \(\delta\) is larger than the maximum eigenvalue of \(A\), one can regard (27) as a Legendre transform. The inverse transform reads
\[
G(q) + \ln q = \min_\delta \left[ \delta q - \int \ln(\delta - \lambda) \rho_A(\lambda) d\lambda \right] - 1,
\] (29)
from which an explicit formula giving \(G(\cdot)\) is obtained as
\[
G(q) = \delta(q) q - \int \ln[\delta(q) - \lambda] \rho_A(\lambda) d\lambda - \ln q - 1,
\] (30)
where \(\delta(q)\) is implicitly defined by
\[
q = \int \frac{\rho_A(\lambda)}{\delta(q) - \lambda} d\lambda.
\] (31)

Equation (31) states that one should choose \(D\) such that its eigenvalues should satisfy
\[
q_\mu = \int \frac{\rho_A(\lambda)}{\delta_\mu - \lambda} d\lambda, \quad \mu = 1, \ldots, M.
\] (32)

Noting that
\[
G'(q) = \delta(q) - \frac{1}{q},
\] (33)
and that \(G(0) = 0\) holds since \(\delta(q) \approx 1/q\) for \(q \ll 1\), one obtains a more compact expression for \(G(\cdot)\):
\[
G(q) = \int_0^q \left( \delta(z) - \frac{1}{z} \right) dz.
\] (34)

Comparing the result with the definition of the \(R\)-transform \(R_A(z)\) of \(\rho_A\):
\[
C_A \left( R_A(z) + \frac{1}{z} \right) = z,
\] (35)
where
\[
C_A(z) = \int \frac{\rho_A(\lambda)}{z - \lambda} d\lambda
\] (36)
is the Cauchy transform\(^2\) of \(\rho_A\), the following identity relation is established:

\[
G(q) = \int_0^q R_A(z) \, dz.
\]

(37)

To summarize, we have shown that

\[
I^{(3)}(\rho_A, X) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_A \left[ \exp \left( \frac{\beta}{2} \text{tr} X^* AX \right) \right] = \frac{\beta}{2} \sum_{\mu=1}^M G(q_\mu),
\]

(38)

where \(q_1, \ldots, q_M\) are the eigenvalues of \(Q = N^{-1} X^* X\) and where \(G(\cdot)\) is defined by (37).

Let us consider a slightly different problem of evaluating the following asymptotic exponent, with the real symmetric or complex Hermitian matrix \(B\) being of rank \(M\):

\[
\tilde{I}^{(3)}(\rho_A, B) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_A \left[ \exp \left( \frac{\beta}{2} \text{tr} AB \right) \right].
\]

(39)

It should be noted that this problem can be regarded as an extension to the problem of evaluating the asymptotic exponent of the Itzykson-Zuber integral (11), in that the average over Haar-distributed orthogonal or unitary matrices \(U\) in (7) is not taken in (39). If the matrix \(B\) is positive definite, one can decompose it as \(B = XX^*\) with an \(N\)-by-\(M\) matrix \(X\), so that the problem of evaluating (39) is reduced to our problem of evaluating (3), provided that the \(M\) nontrivial eigenvalues of \(B\) converge to deterministic limits as \(N \to \infty\). The case where \(B\) is negative definite is also reduced to our problem by letting \(-B = XX^*\) and considering \(-A\) in place of \(A\), using relation \(R_{-A}(z) = -R_A(-z)\), with the end result being given by exactly the same form (5). Although it seems natural to expect that the solution (5) remains valid even when \(B\) is not either positive or negative definite, as of writing this paper we have not succeeded in extending the argument presented in this section to cover such cases.

5. Special cases

5.1. Wigner-type random matrices

In this and next subsections, we demonstrate with some examples that the solution (6) remains valid without the help of Assumption 1 at least in some cases. We consider in this subsection the case in which \(A\) is of Wigner type, and in the next subsection the case in which \(A\) is of sample-covariance type.

In this subsection we consider the case where \(A\) is a Wigner-type real random matrix, in which \(\{a_{ij}; i \leq j\}\) are real valued and are independent and identically distributed (i.i.d.). Averaging with each independent element of \(A\) is denoted by \(\mathbb{E}_a(\cdots)\). We further assume \(\mathbb{E}_a(a) = 0\), \(\mathbb{E}_a(a^2) = v/N\), and \(\mathbb{E}_a(a^n) = o(N^{-1})\) \((m \geq 3)\). Then, one has

\[
\mathbb{E}_A \left[ \exp \left( \frac{1}{2} \text{tr} X^* AX \right) \right] = \prod_{i=1}^N \phi_a \left( \frac{1}{2} \sum_{\mu=1}^M x_{i\mu}^2 \right) \prod_{i<j} \phi_a \left( \sum_{\mu=1}^M x_{i\mu} x_{j\mu} \right),
\]

(40)

where \(\phi_a(\omega) = \mathbb{E}_a(e^{\omega a})\) is the moment generating function of \(a\). Since the arguments of \(\phi_a\) in the above formula are all \(O(1)\), one can expand \(\phi_a(\omega)\) as

\[
\phi_a(\omega) = 1 + \frac{v}{2N} \omega^2 + o(N^{-1}) = e^{\omega^2/(2N) + o(N^{-1})},
\]

(41)

\(^2\) It is also called the Stieltjes transform. Sometimes \(-C_A(z)\) or \(C_A(-z)\) is used as the definition.
so that, by rearranging terms in the exponent, one obtains

\[
\mathbb{E}_A \left[ \exp \left( \frac{1}{2} \text{tr} X^* AX \right) \right] = \exp \left[ \frac{Nv}{4} \text{tr} Q^2 + o(N) \right].
\]  

(42)

We consequently have \( G(q) = vq^2/2 \). On the other hand, it is well known that the limiting eigenvalue distribution of \( A \) is so-called Wigner’s semicircle law, and that its \( R \)-transform is given by \( R_A(z) = vz \). We have therefore shown that the solution (6) is indeed valid in this particular case. It should be noted that the random matrix \( A \) does not in general have orthogonal invariance when the distribution of \( a \) is not Gaussian. This example therefore demonstrates that one can relax to some extent the orthogonal invariance requirement in order to have the solution (6). Note also that we have not used Assumption 1 either.

The case with complex Hermitian Wigner-type \( A \) (i.e., \( \beta = 2 \)) can be treated in the same way. Under the assumption that \( \{a_{ii}; i = 1, \ldots, N\} \) and \( \{\text{Re } a_{ij}, \text{Im } a_{ij}; i < j\} \) are i.i.d. with variance \( v/(2N) \), the limiting eigenvalue distribution of \( A \) is again Wigner, its \( R \)-transform is \( R_A(z) = vz \), and calculation similar to that above yields the result

\[
\mathbb{E}_A \left[ \exp \left( \frac{\beta}{2} \text{tr} X^* AX \right) \right] = \exp \left[ \frac{N\beta v}{4} \text{tr} Q^2 + o(N) \right],
\]  

(43)

with \( \beta = 2 \). Comparing (43) with (42), one can confirm that the formula (43) is valid for both \( \beta = 1 \) and 2.

5.2. Sample-covariance-type random matrices

We next consider the case in which \( A = \Xi^* T \Xi \), where \( \Xi \) is a \( p \)-by-\( N \) matrix with \( p = \alpha N \), \( \alpha = O(1) \), and where \( T \) is a real diagonal matrix whose eigenvalue distribution converges to a compactly supported limiting distribution \( \rho_T \). We assume that \( \{\xi_{vi}; \nu = 1, \ldots, p; i = 1, \ldots, N\} \) are i.i.d. real-valued random variables. We further assume \( \mathbb{E}_\xi(\xi) = 0 \), \( \mathbb{E}_\xi(\xi^2) = v/N \), and \( \mathbb{E}_\xi(\xi^m) = o(N^{-1}) \) \((m \geq 3)\), where, as in the previous subsection, \( \mathbb{E}_\xi(\cdots) \) denotes averaging over each independent element of \( \Xi \).

Let \( \Xi^* = (\xi_1, \ldots, \xi_p) \). Then one has

\[
\text{tr} X^* AX = \sum_{\nu=1}^p \tau_\nu |\nu_\nu|^2;
\]  

(44)

where \( \nu_\nu = X^* \xi_\nu \). For given \( X \), \( \{\nu_\nu; \nu = 1, \ldots, p\} \) are independent, and as \( N \to \infty \), \( \nu_\nu \) asymptotically follows a multivariate Gaussian distribution \( N(0, vQ) \) due to the central limit theorem. Hence,

\[
\mathbb{E}_A \left[ \exp \left( \frac{1}{2} \text{tr} X^* AX \right) \right] = \prod_{\nu=1}^p \mathbb{E}_\nu \left[ \exp \left( \frac{\tau_\nu}{2} |\nu_\nu|^2 \right) \right]
\]

\[
= \exp \left[ -\frac{N\alpha}{2} \text{tr} \int \ln(I - \tau vQ) \rho_T(\tau) \, d\tau + o(N) \right].
\]  

(45)

From this formula, one can read out the result

\[
G(q) = -\alpha \int \ln(1 - \tau vQ) \rho_T(\tau) \, d\tau,
\]  

(46)

which is consistent with the fact [21]

\[
R_{\Xi^* T \Xi}(z) = \alpha \int \frac{\tau v}{1 - \tau vz} \rho_T(\tau) \, d\tau,
\]  

(47)
and the solution (6).

For the corresponding complex Hermitian case ($\beta = 2$), assuming that $\{\Re \xi_{\nu i}, \Im \xi_{\nu i}; \nu = 1, \ldots, p; i = 1, \ldots, N\}$ are i.i.d. with mean 0, variance $\nu/(2N)$, and higher-order moments being $o(N^{-1})$, we have obtained that

$$
E_A \left[ \exp \left( \frac{\beta}{2} \text{tr} X^* AX \right) \right] = \exp \left[ -\frac{N\beta\alpha}{2} \text{tr} \int \ln(I - \tau VQ) \rho_T(\tau) d\tau + o(N) \right],
$$

(48)

which can be regarded as a general formula that is valid for both $\beta = 1$ and 2. This example provides another evidence to our claim that we do not necessarily have to require either orthogonal or unitary invariance or Assumption 1 for the solution (6) to be valid.

6. Summary
We have shown, under Assumption 1, that the asymptotic exponent (3), defined on the basis of a real symmetric or complex Hermitian random matrix $A$ which does not necessarily have orthogonal or Hermitian invariance, is given in terms of $R$-transform of the limiting eigenvalue distribution $\rho_A$ of $A$. We have further shown by a number of examples that the same result remains to hold without Assumption 1 in some specific cases in which $A$ is of Wigner-type and of sample-covariance-type. Noting that the results of Guionnet and Maida, and of Collins and Śniady are applicable only in the cases where $A$ has orthogonal or unitary invariance, our results evidently suggest that the invariance requirement imposed on $A$ in their studies can be relaxed.

Acknowledgments
Support from the Grant-in-Aid for Scientific Research on Priority Areas, Ministry of Education, Culture, Sports, Science and Technology, Japan (no. 18079010) is acknowledged.

References
[1] Anderson T W 1984 An Introduction to Multivariate Statistical Analysis 2nd ed (New York: John Wiley & Sons)
[2] Muirhead R J 1982 Aspects of Multivariate Statistical Theory (New York: John Wiley & Sons)
[3] Mehta M L 2004 Random Matrices 3rd ed (Amsterdam: Elsevier)
[4] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 Rev. Mod. Phys. 53 385–479
[5] Hiai F and Petz D 2000 The Semicircle Law, Free Random Variables and Entropy (Mathematical Surveys and Monographs vol 77) ( Providence: American Mathematical Society)
[6] Voiculescu D 1986 J. Funct. Anal. 66 323–46
[7] Tanaka T 2007 Proc. 2007 IEEE Int. Symp. Info. Theory (Nice, France) pp 1651–4
[8] Harish-Chandra 1957 Amer. J. Math. 79 87–120
[9] Guionnet A and Zeitouni O 2002 J. Funct. Anal. 188 461–515
[10] Guionnet A and Zeitouni O 2004 J. Funct. Anal. 216 230–41
[11] Guionnet A and Maida M 2005 J. Funct. Anal. 222 435–90
[12] Collins B and Śniady P 2007 Ann. Inst. Henri Poincaré B: Stat. & Prob. 43 139–46
[13] Itzykson C and Zuber J B 1980 J. Math. Phys. 21 411–21
[14] Marini E, Parisi G and Ritort F 1994 J. Phys. A: Math. Gen. 27 7647–68
[15] Opper M and Winther O 2001 Phys. Rev. E 64 056131–1–14
[16] Müller R R, Guo D and Moustakas A L 2007 Proc. 2007 IEEE Int. Symp. Info. Theory (Nice, France) pp 2316–20
[17] Dembo A and Zeitouni O 1998 Large Deviation Techniques and Applications 2nd ed (New York: Springer)
[18] Cherrier R, Dean D S and Lefèvre A 2003 Phys. Rev. E 67 046112–1–10
[19] Tulino A M and Verdú S 2004 Random Matrix Theory and Wireless Communications (Hanover: now Publishers)