MEAN DIMENSION AND AH-ALGEBRAS WITH DIAGONAL MAPS

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Abstract. Mean dimension for AH-algebras is introduced. It is shown that if a simple unital AH-algebra with diagonal maps has mean dimension zero, then it has strict comparison on positive elements. In particular, the strict order on projections is determined by traces. Moreover, a lower bound of the mean dimension is given in term of comparison radius. Using classification results, if a simple unital AH-algebra with diagonal maps has mean dimension zero, it must be an AH-algebra without dimension growth.

Two classes of AH-algebras are shown to have mean dimension zero: The class of AH-algebras with at most countably many extremal traces, and the class of AH-algebras with numbers of extreme traces which induce same state on the $K_0$-group being uniformly bounded (in particular, this class includes AH-algebras with real rank zero).

Contents

1. Introduction 1
2. Notation and preliminaries 3
3. Mean dimension for AH-algebras 5
4. A local approximation theorem 16
5. Mean dimension zero and AH-algebras with diagonal maps 19
6. Radius of comparison for AH-algebras with diagonal maps 25
7. Cuntz mean dimension for AH-algebras with generalized diagonal maps 27
8. Variation mean dimension for general AH-algebras 35

References 37

1. Introduction

Dimension growth—roughly speaking, the limit ratio of the dimension of the base space and dimension of the irreducible representation—plays a crucial role in the classification of AH-algebras. On one hand, the class of simple unital AH-algebras without dimension growth is classified by their K-theory invariant (see [3], [7], and [4]). On the other hand, AH-algebras with fast dimension growth are lack of certain regularities in their invariants (see, for example, [22], [23], [19]).

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In this paper, a definition of mean dimension for AH-algebras is introduced (under a mild condition that the eigenvalue patterns of the AH-algebras split into continuous maps). The definition is motivated by the dynamical system mean dimension introduced by Elon Lindenstrauss and Benjamin Weiss in [14], and it is the limit ratio of the dimensions of certain open covers of the base spaces (instead of the base spaces themselves) and the dimensions of the irreducible representations. As dimension growth, it also measures the size of the space in which one can maneuver a vector bundle (a projection), but in a more intrinsic way.

The small boundary property (SBP) (Definition 3.12) and small boundary refinement property (SBRP) (Definition 3.15) are also introduced, and it is shown that AH-algebras with SBP or SBRP always have mean dimension zero. This enable us to show that any AH-algebra of real rank zero has mean dimension zero (Theorem 3.24), and any AH-algebra with at most countably many extremal traces has mean dimension zero (Corollary 3.18). Unlike mean dimension and small boundary property for minimal dynamical systems, where they are equivalent (Theorem 6.2 of [13]), there are simple AH-algebras with mean dimension zero, but without small boundary property (Remark 3.19).

If one focuses on the class of AH-algebras with diagonal maps, it turns out that mean dimension is then a very useful measurement for the its regularity on the invariant. In fact, one has the following approximation theorem (Theorem 4.2): let A be a simple AH-algebra has diagonal maps, then A can be locally approximated by homogeneous C*-algebras with dimension ratio no more than its mean dimension. In particular, if the AH-algebra has mean dimension zero, it has local slow dimension growth. Using this local approximation, AH-algebras with mean dimension zero and diagonal maps are shown to have strict comparison on positive elements (Theorem 5.9), and hence the strict order on projections are determined by traces (Theorem 4.3). In particular, AH-algebra with real rank zero and diagonal maps has the strict order on projections determined by traces. Combining with a result of Huaxin Lin in [9], real rank zero AH-algebras with diagonal maps are tracially AF algebra, and therefore they are AH-algebras without dimension growth.

Also as an application of the above-mentioned approximation theorem, the Cuntz semigroups of AH-algebras with diagonal maps and mean dimension zero are calculated. Together with the recent classification results, these AH-algebras are AH-algebras without dimension growth.

Another application of the approximation theorem is to comparison radius of AH-algebras with diagonal maps: in Section 5 as one expects, the comparison radius is shown to be dominated by one half of the mean dimension.

However, this version of mean dimension only work well for AH-algebras with diagonal maps. For instance, the Villadsen algebras of the second type (see [23]) always have mean dimension zero, but they never have slow dimension growth, even locally. Thus, modified versions of mean dimension—the Cuntz mean dimension, for AH-algebras with generalized diagonal maps, and the variation mean dimension, for general AH-algebras—are introduced. These new versions of mean dimension are able to detect the regularity of the K-theoretical invariant in a boarder context, for example, AH-algebras with zero variation mean dimension always have strict comparison on positive elements. But not as the original version, these new versions of mean dimension are
more difficult to calculate. However, for the class of AH-algebras with diagonal maps, all of them agree with each other.

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2. Notation and preliminaries

Definition 2.1. Let $a$ and $b$ be positive elements in a C*-algebra $A$. The element $a$ is said to be Cuntz smaller than $a$, denoted by $a \preceq b$, if there exist sequences $(x_n)$ and $(y_n)$ in $A$ such that

$$\lim_{n \to \infty} x_n y_n = a.$$ 

The elements $a$ and $b$ are Cuntz equivalent, denoted by $a \sim b$, if $a \preceq b$ and $b \preceq a$. It is an equivalent relation.

For any positive element $a \in A$ and any $\varepsilon > 0$, denoted by $(a - \varepsilon)_+$, the element $h_\varepsilon(a)$, where $h_\varepsilon(t) = \min\{0, t - \varepsilon\}$. One then has the following fact on comparison of positive elements $a$ and $b$.

1. $a \preceq b$ if there are $\{c_n\}$ such that $c_n^* b c_n \to a$.
2. $a \preceq b$ if and only if $(a - \varepsilon)_+ \preceq b$ for all $\varepsilon > 0$.
3. $a \preceq b$ if and only if for any $\varepsilon > 0$, there exists $x$ such that $x^* bx = (a - \varepsilon)_\varepsilon$.
4. if $\|a - b\| < \varepsilon$, then $(a - \varepsilon)_+ \preceq b$.
5. $a + b \preceq a \oplus b$; if $a \perp b$, then $a + b \sim a \oplus b$.

The proofs can be found in [2] and [16].

Definition 2.2. Let $A$ be a C*-algebra, and consider matrix algebras $M_\infty(A) := \bigcup_n M_n(A)$ over $A$. For any positive elements $a \in M_n(A)$ and $b \in M_m(A)$, still denoted by $a \preceq b$ if there are $\{c_n\} \subseteq M_{m,n}(A)$ such that $c_n^* b c_n \to a$, and $a \sim b$ if $a \preceq b$ and $b \preceq a$. Then $\sim$ is an equivalent relation on $A_+$. Denoted by $[a]$ the equivalent class containing $a$. Then $W(A) := \{[a]; a \in M_\infty(A)^+\}$ is a positive ordered abelian semigroup with

$$[a] + [b] := [a \oplus b] \quad \text{and} \quad [a] \leq [b] \text{ if } a \preceq b.$$

Definition 2.3. Let $a$ be a positive element in a C*-algebra $A$, and let $\tau$ be a state of $A$. Define

$$d_\tau(a) := \lim_{\varepsilon \to 0^+} \tau(f_\varepsilon(a)) = \sup_{\varepsilon > 0} \tau(f_\varepsilon(a)) = \mu_\tau((0, \|a\|)),$$

where $f_\varepsilon(x) := \min\{x/\varepsilon, 1\}$, and $\mu_\tau$ is the measure induced by $\tau$ on $C^*(1, a) \cong C(\text{sp}(a))$.

The C*-algebra $A$ is said to have strict comparison of positive elements if for any positive elements $a$ and $b$ with

$$d_\tau(a) < d_\tau(b), \quad \forall \tau \in T(A),$$

one has that $a \preceq b$. 

Note that the dimension function \( d_\tau \) induces a positive map from \( W(A) \) to \( \text{SAff}(T(A)) \), the set of functions on \( T(A) \) which are pointwise suprema of increasing sequences of continuous, affine, and strictly positive functions on \( T(A) \), equipped with the natural order \( (\ll) \). If the C*-algebra \( A \) has strict comparison of positive element, then the order structure on \( W(A) \) is determined by its image in \( \text{SAff}(T(A)) \).

**Definition 2.4.** An AH-algebra is an inductive limit of C*-algebras

\[
pM_n(C(X))p
\]

where \( X \) is a compact Hausdorff space, and \( p \) is a projection in \( M_n(C(X)) \).

Consider an AH-algebra \( A = \lim_{i \to \infty} (A_i, \varphi_i) \), and write \( A_i = \bigoplus_{l=1}^{h_i} p_{i,l}M_{r_{i,l}}(C(X_{i,l}))p_{i,l} \), where each \( p_{i,l} \) has constant rank. Denote by \( A_{i,l} \) the direct summand \( p_{i,l}M_{r_{i,l}}(C(X_{i,l}))p_{i,l} \). For any \( i, j \), if there exist a unitary \( U \in A_j \) such that the restriction of the map \( \text{ad}(U) \circ \varphi_{i,j} \) to any direct summands \( A_{i,l} \) and \( A_{j,k} \) has the form

\[
f \mapsto \begin{pmatrix}
    f \circ \lambda_1 \\
    \vdots \\
    f \circ \lambda_n
\end{pmatrix}
\]

for some continuous maps \( \lambda_1, \ldots, \lambda_n : X_{j,k} \to X_{i,l} \), then \( A \) is called an AH-algebra with diagonal maps.

The following example is due to N. C. Phillips.

**Example 2.5 (AH-model for dynamical systems).** Let \( X \) be a compact Hausdorff space, and let \( \sigma \) be a homeomorphism on \( X \). The the AH-model of the system \((X, \sigma)\) is the following AH-algebra \( A(X, \sigma) \) with diagonal map:

\[
C(X) \xrightarrow{f} M_2(C(X)) \xrightarrow{f} M_2(C(X)) \xrightarrow{f} \cdots
\]

with the connecting map induced by

\[
f \mapsto \begin{pmatrix}
f \\
\sigma
\end{pmatrix}.
\]

The AH-algebra \( A(X, \sigma) \) is simple if and only if \((X, \sigma)\) is minimal, and the tracial simplex of \( A(X, \sigma) \) is canonically isomorphic to the simplex of the invariant probability measures on \( X \).

**Definition 2.6.** Let \( A = \lim_{i \to \infty} (A_i, \varphi_i) \) be an AH-algebra with \( A_i = \bigoplus_{l=1}^{h_i} p_{i,l}M_{r_{i,l}}(C(X_{i,l}))p_{i,l} \), where each \( p_{i,l} \) has constant rank. Then the dimension growth of \( A \) is defined by

\[
\liminf_{i \to \infty} \max_l \left\{ \frac{\dim(X_{i,l})}{\text{rank}(p_l)} \right\},
\]

where \( \dim(X_{i,l}) \) is the covering dimension of \( X \). If there is an inductive decomposition of \( A \) using homogeneous C*-algebras such that the dimension growth is zero, then \( A \) is said to have slow dimension growth. Moreover, if the dimensions of the base spaces are uniformly bounded, \( A \) is said to be an AH-algebra without dimension growth.
Remark 2.7. The class of simple unital real rank zero AH-algebra with slow dimension growth has been classified in [3], and the class of simple AH-algebra without dimension growth has been classified in [7] and [4].

Let $A$ be a unital C*-algebra and let $\tau$ be a tracial state. Recall that the restriction of $\tau$ to projections induces a positive linear functional on $K_0(A)$, and this in fact induces an affine map $\rho : T(A) \to S_u(K_0(A))$, where $S_u(K_0(A))$ is the convex of positive linear functional on $K_0(A)$ which send $[1_A]_0$ to 1. Note that $\rho$ is surjective when $A$ is exact, and $\rho$ is injective if and only if projections of $A$ is separated by traces. Recall that a C*-algebra $A$ is real rank zero if the projections span a dense subspace of $A$. It is evident that if $A$ is real rank zero, then projections of $A$ separate traces.

3. Mean dimension for AH-algebras

Mean dimension was introduced by E. Lindenstrauss and B. Weiss in [14] for dynamical systems. In this section, AH-algebras with eigenvalue maps separated by continuous maps are considered, and the mean dimension will be introduced for these AH-algebras.

Let $X$ be a topological space, and let $\alpha$ be a finite open cover of $X$. We say that a cover $\beta$ refines $\alpha$ ($\beta \succ \alpha$) if every element of $\beta$ is a subset of some element of $\alpha$. Denote the set of finite open covers of $X$ by $C(X)$.

Definition 3.1 (Definition 2.1 and Definition 2.2 of [14]). If $\alpha$ is an open cover of $X$, denote by

$$\text{ord}(\alpha) := \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1,$$

and denote by

$$D(\alpha) = \min_{\beta \succ \alpha} \text{ord}(\beta).$$

A continuous map $f : X \to Y$ is $\alpha$-compatible if it is possible to find a finite open cover of $f(X)$, $\beta$, such that $f^{-1}(\beta) \succ \alpha$.

Remark 3.2. It follows from Proposition 2.3 of [14] that a map $f : X \to Y$ is $\alpha$-compatible if for any $y \in Y$, $f^{-1}(y)$ is a subset of some $U \in \alpha$.

The following proposition is a characterization of $D(\alpha)$.

Proposition 3.3 (Proposition 2.4 of [14]). If $\alpha$ is an open cover of $X$, then

$$D(\alpha) \leq k$$

if and only if there is an $\alpha$-compatible continuous function $f : X \to K$ where $K$ has topological dimension $k$.

Recall that if $\alpha$ and $\beta$ are two open covers of $X$, then $\alpha \cup \beta$ is the open cover of $X$ with $U \cap V$ where $U \in \alpha$ and $V \in \beta$. It follows from Corollary 2.5 of [14] that

$$D(\alpha \cup \beta) \leq D(\alpha) + D(\beta).$$
Lemma 3.4. Let $\alpha$ be an open cover of $Y$, and let $f : X \to Y$ be a continuous map. Then

$$\mathcal{D}(f^{-1}(\alpha)) \leq \mathcal{D}(\alpha).$$

Proof. Let $\beta$ an open cover with $\beta \succ \alpha$ with $\text{ord}(\beta) = \mathcal{D}(\alpha)$. Consider the open cover $f^{-1}(\beta)$. It is clear that $f^{-1}(\beta) \succ f^{-1}(\alpha)$. Pick $x \in X$ such that

$$\text{ord}(f^{-1}(\beta)) = \sum_{V \in f^{-1}(\beta)} 1_V(x) - 1.$$

However, since

$$\sum_{V \in f^{-1}(\beta)} 1_V(x) = \sum_{U \in \beta} 1_U(f(x)),$$

one has

$$\mathcal{D}(f^{-1}(\alpha)) \leq \text{ord}(f^{-1}(\beta)) = \sum_{V \in f^{-1}(\beta)} 1_V(x) - 1 = \sum_{U \in \beta} 1_U(f(x)) - 1 \leq \text{ord}(\alpha),$$

as desired. \qed

Consider unital homogeneous C*-algebras

$$A_i = \bigoplus_{l=1}^{h_i} p_{i,l} M_{r_{i,l}}(C(X_{i,l})) p_{i,l},$$

where each compact Hausdorff space $X_{i,l}$ is connected. Denote by $n_{i,l}$ the rank of $p_{i,l}$, by $A_{i,l}$ the $l$th component of $A_i$. Consider a family of homomorphisms $\varphi_{i_1,i_3} : A_i \to A_j$ with $\varphi_{i_1,i_3} = \varphi_{i_2,i_3} \circ \varphi_{i_1,i_2}$ for any $i_1 \leq i_2 \leq i_3$. Denote by $m_{i,j}^{k}$ the multiplicity of the restriction of the map $\varphi_{i,j}$ to $A_{j,k}$, and by $m_{i,j}^{l,k}$ the multiplicity of the restriction of the map $\varphi_{i,j}$ to $A_{i,l}$ and $A_{j,k}$. Note that

$$m_{i,j}^{k} = \sum_{l=1}^{h_i} m_{i,j}^{l,k}.$$ 

In this paper, let us assume that the eigenvalue maps

$$\Gamma_{i,j}^{k,l} : X_{j,k} \to \prod_{l=1}^{m_{i,j}^{l,k}} X_{i,l}/\sim$$

do not take on the value $\infty$. Let $A_{j,k} = p_{j,k} M_{r_{j,k}}(C(X_{j,k})) p_{j,k}$ be always induced by continuous maps $\{\lambda_{i,j}^{l,k}(m) : 1 \leq m \leq m_{i,j}^{l,k}\}$, that is

$$\Gamma_{i,j}^{l,k}(x) = \{\lambda_{i,j}^{l,k}(1)(x), ..., \lambda_{i,j}^{l,k}(m_{i,j}^{l,k})(x)\}.$$
Consider the unital inductive limit of $A_i$:

$$A_1 \xrightarrow{\varphi_{1,2}} A_2 \xrightarrow{\varphi_{2,3}} \cdots \xrightarrow{\varphi_{i-1,i}} A_i \xrightarrow{\varphi_{i,i+1}} A = \lim_{i \to \infty} A_i.$$ 

Let $\alpha_{i,l}$ be an open cover of $X_{i,l}$. Consider the open cover

$$(\lambda_{i,j}^{l,k}(1))^{-1}(\alpha_{i,l}) \vee \cdots \vee (\lambda_{i,j}^{l,k}(m_{i,j})^{-1}(\alpha_{i,l}),$$

and denoted by $\varphi_{i,j}^{l,k}(\alpha_{i,l})$.

Let $\alpha_i$ be an open cover of $X_i$ with each member a subset of a connected component. Denote by $\alpha_{i,l}$ the cover of $X_{i,l}$ induced by $\alpha_i$. Then set

$$\varphi_{i,j}^k(\alpha_i) = \varphi_{i,j}^{l,k}(\alpha_{i,1}) \vee \cdots \vee \varphi_{i,j}^{h_{i,k}}(\alpha_{i,h_i}).$$

Consider $D(\varphi_{i,j}^k(\alpha_i))$. One then has

$$D(\varphi_{i,j}^k(\alpha_i)) = D(\varphi_{i,j}^{l,k}(\alpha_{i,1}) \vee \cdots \vee \varphi_{i,j}^{h_{i,k}}(\alpha_{i,h_i})) \leq \sum_{l=1}^{h_i} \sum_{m=1}^{m_{i,j}} D((\lambda_{i,j}^{l,k}(m))^{-1}(\alpha_{i,l})) \leq \sum_{l=1}^{h_i} \sum_{m=1}^{m_{i,j}} D(\alpha_{i,l}) \text{ by Lemma 3.4} \leq \sum_{l=1}^{h_i} \sum_{m=1}^{m_{i,j}} D(\alpha_{i,l}),$$

and hence

$$D(\varphi_{i,j}^k(\alpha_i)) \leq \sum_{l=1}^{h_i} \sum_{m=1}^{m_{i,j}} D(\alpha_{i,l})/n_{j,k} = \sum_{l=1}^{h_i} \sum_{m=1}^{m_{i,j}} D(\alpha_{i,l})/n_{j,k}.$$

Lemma 3.5. Let $a_1, \ldots, a_n, b_1, \ldots, b_n, m_1, \ldots, m_n$ be positive numbers with $b_i$ and $m_i$ nonzero. If $a_i/b_i \leq c$ for some number $c$ for any $1 \leq i \leq n$, then

$$\frac{m_1a_1 + \cdots + m_na_n}{m_1b_1 + \cdots + m_nb_n} \leq c.$$

Proof. Since $a_i/b_i \leq c$, one has that $a_i \leq cb_i$, and hence $m_ia_i \leq cm_ib_i$ for all $1 \leq i \leq n$. Therefore

$$m_1a_1 + \cdots + m_na_n \leq cm_1b_1 + \cdots + cm_nb_n = c(m_1b_1 + \cdots + m_nb_n),$$

and

$$\frac{m_1a_1 + \cdots + m_na_n}{m_1b_1 + \cdots + m_nb_n} \leq c,$$

as desired. \qed
From the lemma above and Equation (3.1), one has
\[
\frac{D(\varphi_{i,j}(\alpha_i))}{n_{j,k}} \leq \max \{ \frac{D(\alpha_l)}{n_{i,l}} ; 1 \leq l \leq h_i \}.
\]

Thus the sequence
\[
\max \{ \frac{D(\varphi_{i,j}(\alpha_i))}{n_{j,k}} ; 1 \leq k \leq h_j \}, \quad j = i, i + 1, ...
\]
is a decreasing sequence, and the limit exists.

**Definition 3.6.** Set
\[
\gamma_i(A) := \sup_{\alpha_i \in C(X_i)} \lim_{j \to \infty} \max \{ \frac{D(\varphi_{i,j}(\alpha_i))}{n_{j,k}} ; 1 \leq k \leq h_j \}.
\]

Note that \( \{ \gamma_n \} \) is an increasing sequence. The mean dimension of the inductive limit \( A \) is the limit
\[
\gamma(A) = \lim_{i \to \infty} \gamma_i.
\]

**Remark 3.7.** The mean dimension of the AH-model \( A(X, \sigma) \) is exactly the mean dimension of the dynamical system \( (X, \sigma) \).

**Remark 3.8.** Since \( D(\varphi_{i,j}(\alpha_i)) \leq \dim(X_j, k) \), it is clear that if an AH-algebra has slow dimension growth is zero, then it has mean dimension zero.

Let \( f \) be a positive element in \( A_i \). Then for any \( x \in X_{j,k} \), one has
\[
\text{Tr}(\varphi_{i,j}^k(f)(x)) = \sum_{l=1}^{h_i} \sum_{m=1}^{m_{i,j}^{l,k}} \text{Tr}(f(\lambda_{i,j}^{l,k}(m)(x)))
\]
\[
\leq \sum_{l=1}^{h_i} \sum_{m=1}^{m_{i,j}^{l,k}} \sup_{x \in X_{i,l}} \text{Tr}(f(x))
\]
\[
= \sum_{l=1}^{h_i} m_{i,j}^{l,k} \sup_{x \in X_{i,l}} \text{Tr}(f(x)).
\]

Hence
\[
\sup_{x \in X_{j,k}} \text{Tr}(\varphi_{i,j}^k(f)(x)) \leq \sum_{l=1}^{h_i} m_{i,j}^{l,k} \sup_{x \in X_{i,l}} \text{Tr}(f(x)),
\]
and
\[
\frac{1}{n_{j,k}} \sup_{x \in X_{j,k}} \text{Tr}(\varphi_{i,j}^k(f)(x)) \leq \frac{\sum_{l=1}^{h_i} m_{i,j}^{l,k} \sup_{x \in X_{i,l}} \text{Tr}(f(x))}{\sum_{l=1}^{h_i} m_{i,j}^{l,k} n_{i,l}} \leq \max_{1 \leq l \leq h_i} \frac{1}{n_{i,l}} \sup_{x \in X_{i,l}} \text{Tr}(f(x)).
\]
Thus the sequence
\[
\left\{ \max_{1 \leq k \leq h_j} \sup_{x \in X_{j,k}} \frac{1}{n_{j,k}} \text{Tr}(\varphi_{i,j}^{k}(f)(x)), \quad j = i, i + 1, \ldots \right\}
\]
is decreasing, and the limit exists.

**Definition 3.9.** Let \( f \in (A_{i,l})^+ \), and let \( E \) be a closed subset of \( X_{i,l} \). For any \( X_{j,k} \) with \( j \geq i \), the orbit capacities of \( f \) and \( E \) at \( X_{j,k} \), denoted by \( \text{ocap}_{j,k}(f) \) and \( \text{ocap}_{j,k}(E) \) respectively, are
\[
\text{ocap}_{j,k}(f) := \sup_{x \in X_{j,k}} \frac{1}{n_{j,k}} \text{Tr}(\varphi_{i,j}(f)(x)),
\]
and
\[
\text{ocap}_{j,k}(E) := \sup_{x \in X_{j,k}} \frac{\#\{1 \leq m \leq m_{i,j}^{l,k}; \lambda_{i,j}^{l,k}(m)(x) \in E\}}{n_{j,k}} \#\{1 \leq m \leq m_{i,j}^{l,k}; \lambda_{i,j}^{l,k}(m)(x) \in E\}.
\]
The orbit capacity of \( f \) and \( E \), denote by \( \text{ocap}(f) \) and \( \text{ocap}(E) \) respectively, is
\[
\text{ocap}(f) := \lim_{j \to \infty} \max_{1 \leq k \leq h_j} \text{ocap}_{j,k}(f),
\]
and
\[
\text{ocap}(E) := \lim_{j \to \infty} \max_{1 \leq k \leq h_j} \text{ocap}_{j,k}(E).
\]

**Remark 3.10.** For any closed set \( E \subseteq X_{i,l} \), one has
\[
\text{ocap}(E) \leq \inf\{\text{ocap}(f); \; f \text{ is positive in } A_l' \cap A_{i,l} \text{ and } f|_{E} = 1\}.
\]

**Remark 3.11.** For any \( f \in (A_{i,l})^+ \), one has
\[
\text{ocap}(f) = \sup_{\tau \in T(A)} \tau(\varphi_{i,\infty}(f)).
\]
Indeed, using the compactness argument, it is clear that
\[
\text{ocap}(f) \leq \sup_{\tau \in T(A)} \tau(\varphi_{i,\infty}(f)).
\]
On the other hand, let \( \tau \) be a tracial state with \( \tau(\varphi_{i,\infty}(f)) \) maximum. Let \( \{\mu_j\} \) be the sequence of probability measures with \( \mu_j \) supported on \( X_j \) induced by the restriction of \( \tau \). Then,
\[
\tau_{\mu_j}(\varphi_{i,j}(f)) = \frac{1}{n_{j,1}} \int_{X_{j,1}} \text{Tr}(\varphi_{i,j}^{1}(f))d\mu_j^{1} + \cdots + \frac{1}{n_{j,h_j}} \int_{X_{j,1}} \text{Tr}(\varphi_{i,j}^{h_j}(f))d\mu_j^{h_j}
\]
\[
\leq \frac{1}{n_{j,1}} \sup_{x \in X_{j,1}} \text{Tr}(\varphi_{i,j}^{1}(f)(x)) |\mu_j^{1}| + \cdots + \frac{1}{n_{j,h_j}} \sup_{x \in X_{j,1}} \text{Tr}(\varphi_{i,j}^{h_j}(f)(x)) |\mu_j^{h_j}|
\]
\[
\leq \max_{1 \leq k \leq h_j} \sup_{x \in X_{j,k}} \frac{1}{n_{j,k}} \text{Tr}(\varphi_{i,j}(f)(x)),
\]
and hence
\[
\sup_{\tau \in T(A)} \tau(\varphi_{i,\infty}(f)) = \tau(\varphi_{i,\infty}(f)) \leq \text{ocap}(f).
\]

**Definition 3.12.** An AH-algebra \( A \) has the small boundary property (SBP) if for any \( X_{i,l} \), any point \( x \in X_{i,l} \), any open \( U \ni x \), there is a neighbourhood \( V \subseteq U \) such that \( \text{ocap}(\partial V) = 0 \).
Remark 3.13. The small boundary property for AH-algebra is an analogue to the small boundary property for dynamical systems introduced in [14] by Lindenstrauss and Weiss.

Using Remark 3.11 and a cardinality argument, one immediately has the following lemma.

Lemma 3.14. Denote by $c$ the cardinality of the extremal tracial state of $A$. If $\aleph_0 c$ is strictly less than the continuum, then $A$ has the SBP.

Definition 3.15. An AH-algebra $A$ has small boundary refinement property (SBRP) if for any $X_{i,l}$, any finite open cover $\alpha$ of $X_{i,l}$, and any $\varepsilon > 0$, there exists $L$ such that for any $X_{j,k}$ with $j > L$, there is a refinement $\alpha' \succ \alpha$ and $\delta > 0$ (both may depend on $X_{j,k}$) such that there is a one-to-one correspondence between the elements $U$ of $\alpha$ and $U'$ of $\alpha'$ with $\overline{U} \subseteq U$, and

$$\text{ocap}_{j,k} \left( \bigcup_{U' \in \alpha'} B(\partial U', \delta) \right) < \varepsilon,$$

where $B(\partial U', \delta)$ is the closed $\delta$-neighbourhood of $\partial U'$.

Lemma 3.16. If an AH-algebra has SBP, then it has SBRP.

Proof. Using the small boundary property, there is a cover of $X_{i,l}$ by open set with small boundary that refines $\alpha$. Then, by taking union of these sets, there is a refinement of $\alpha$ satisfying Definition 3.15. □

Theorem 3.17. If a simple AH-algebra $A$ has SBRP, then $A$ has mean dimension zero.

Proof. The proof is a modification of the proof of Theorem 5.4 of [14]. Assume that $A$ has the SBRP, and let $\alpha$ be an open cover of $X_i := \bigsqcup X_{i,l}$. Fix $i$ and $l$, and consider the unit $e$ of $A_{i,l}$. Since $A$ is simple, there exists $\delta_l > 0$ such that $\tau(e) > \delta_l$ for any tracial state $\tau$ on $A$. Therefore, there exists $N'$ such that for any $j > N'$, one has that

$$\frac{n_{i,l} m_{i,j}^{l,k}}{n_{j,k}} > \delta_l, \quad \forall 1 \leq k \leq h_j.$$

Let $d$ be a compatible metric on $X_{i,l}$. For any $\varepsilon > 0$, since $A$ has SBRP, there exists $N > N'$ such that for any $X_{j,k}$ with $j > N$, there is an open cover $\alpha' \succ \alpha$ and $\delta > 0$ such that there is a one-to-one correspondence between the elements $U_s$ of $\alpha_{i,l}$ and $U'_s$ of $\alpha'_{i,l}$ with $\overline{U'_s} \subseteq U_s$, such that

$$\frac{n_{i,l}}{n_{j,k}} \# \{1 \leq m \leq m_{i,j}^{l,k}: \lambda_{i,j}^k(m)(x) \in B\left( \bigcup_{s=1}^{\lvert \alpha_{i,l} \rvert} \partial U'_s, \delta \right) \} < \varepsilon \delta_l, \quad \forall x \in X_{j,k},$$

and $B(\partial U'_s, \delta) \subseteq U_s$, where $B(\partial U'_s, \delta)$ is the closed $\delta$-neighbourhood of $\partial U'_s$.

Fix $X_{j,k}$ and let us show that $D(\varphi_{i,j}^k(\alpha))/n_{j,k}$ is bounded by $\varepsilon \lvert \alpha \rvert$.

Define

$$\varphi'_s = \begin{cases} 1 & \text{if } x \in U'_s, \\ \max(0, 1 - \delta^{-1} d(x, \partial U'_s)) & \text{otherwise}, \end{cases}$$
and define

\[ \begin{align*}
\phi_1(x) &= \phi'_1(x), \\
\phi_2(x) &= \min(\phi'_2(x), 1 - \phi'_1(x)), \\
\phi_3(x) &= \min(\phi'_3(x), 1 - \phi'_1(x) - \phi'_2(x)), \\
& \vdots
\end{align*} \]

Therefore, one has a subordinate partition of unity \( \phi_s : X_{i,l} \to [0, 1] \) such that

1. \( \sum_{s=1}^{\alpha_{i,l}} \phi_s(x) = 1, \)
2. \( \text{supp}(\phi_s) \subset U \) for some \( U \in \alpha, \)
3. \( \frac{n_{i,l}}{n_{j,k}} \#\{1 \leq m \leq m_{i,j}^{l,k}; \lambda_{i,j}^{l,k}(m)(x) \in \bigcup_{s=1}^{\alpha_{i,l}} \phi_s^{-1}(0, 1)\} < \varepsilon \delta_l, \quad \forall x \in X_{j,k}, \forall k. \)

Note that \( j > N > N', \) one has that

\[ \frac{n_{i,l} m_{i,j}^{l,k}}{n_{j,k}} > \delta_l, \quad \forall 1 \leq k \leq h_j. \]

Also note that \( n_{j,k} = \sum_{l=1}^{h_i} n_{i,l} m_{i,j}^{l,k}. \) Hence

\[ \sum_{l=1}^{h_i} m_{i,j}^{l,k} \leq n_{j,k}. \]

Set \( E = \bigcup_{s=1}^{\alpha_{i,l}} \phi_s^{-1}(0, 1), \) and choose \( j \) large enough such that for any \( 1 \leq l \leq h_i, \)

\[ \frac{n_{i,l}}{n_{j,k}} \#\{1 \leq m \leq m_{i,j}^{l,k}; \lambda_{i,j}^{l,k}(m)(x) \in E \cap X_{i,l}\} < \varepsilon \delta_l, \quad \forall x \in X_{j,k}, \forall 1 \leq k \leq h_j. \]

Thus,

\[ (3.2) \quad \frac{1}{m_{j,k}} \#\{1 \leq m \leq m_{i,j}^{l,k}; \lambda_{i,j}^{l,k}(m)(x) \in E \cap X_{i,l}\} < \varepsilon, \quad \forall x \in X_{j,k}, \forall 1 \leq k \leq h_j. \]

Define \( \Phi : X_i \to \mathbb{R}^{\alpha|} \) by

\[ x \mapsto (\phi_1(x), ..., \phi_{|\alpha|}(x)), \]

and set

\[ \Gamma_{i,j}^k : X_{j,k} \ni x \mapsto (\lambda_{i,j}^{l,k}(1)(x), ..., \lambda_{i,j}^{l,k}(m_{i,j}^{l,k})(x)) \in (X_i)^{m_{i,j}^{l,k}}. \]

Consider the map

\[ f_{i,j}^{l,k} = \Phi \circ \Gamma_{i,j}^k : X_{j,k} \to \mathbb{R}^{\alpha|n_{i,j}^{l,k}}. \]

Then the set \( f_{i,j}^{l,k}(X_{j,k}) \) is a subset of the image of finite number of \( \varepsilon |\alpha| m_{i,j}^{l,k} \) dimensional affine subspace of \( \mathbb{R}^{\alpha|n_{i,j}^{l,k}}. \) Indeed, let \( e_j^i, i = 1, ..., m_{ij}^{l,k}, j = 1, ..., |\alpha| \) be the standard base of \( \mathbb{R}^{\alpha|n_{i,j}^{l,k}}. \)

For every \( I = \{i_1, ..., i_{N'}\}, N' \leq \varepsilon m_{i,j}^{l,k}, \) and every \( \xi \in \{0, 1\}^{\alpha|n_{i,j}^{l,k}}, \) set

\[ C(I, \xi) = \text{span}\{e_j^i : i \in I, 1 \leq j \leq |\alpha|\} + \xi. \]
Then, by (3.2),

\[ f_{l,k}^{i,j}(X_{j,k}) \subseteq \bigcup_{|I|<\epsilon m_{i,j}^{l,k}} \pi(C(I,\xi)). \]

It is easy to see that \( f_{l,k}^{i,j} \) is \( \varphi_{i,j}^{l,k}(\alpha_{i,l}) \) compatible. By Proposition 3.3 one has that

\[ D(\varphi_{i,j}^{l,k}(\alpha_{i,l})) < \varepsilon |\alpha| m_{i,j}^{l,k}, \]

and hence

\[ D(\varphi_{i,j}^{k}(\alpha)) = D(\varphi_{i,j}^{1,k}(\alpha_1)) + \cdots + D(\varphi_{i,j}^{h_i,k}(\alpha_{h_i})) \]

\[ \leq \varepsilon |\alpha| (m_{i,j}^{1,k} + \cdots + m_{i,j}^{h_i,k}) \]

\[ \leq \varepsilon |\alpha| n_{j,k}. \]

Therefore, \( A \) has mean dimension zero, as desired. \( \square \)

**Corollary 3.18.** Denote by \( c \) the cardinality of the extremal tracial state of \( A \). If \( \aleph_0 c \) is strictly less than the continuum, then \( A \) has the SBP, and hence has mean dimension zero.

**Proof.** It follows from Lemma 3.14 and Theorem 3.17. \( \square \)

**Remark 3.19.** Unlike the small boundary property for dynamical systems, which is equivalent to the mean dimension zero for minimal dynamical systems (Theorem 6.2 of [13]), the small boundary property for AH-algebra is strictly stronger than mean dimension zero. The following is an example of a simple AH-algebra which has mean dimension zero but does not satisfies the small boundary property (it is one of the Goodearl algebras):

Let \((m_n)\) be a sequence of natural numbers such that there is \( c > 0 \) such that

\[ \left( \frac{m_n - 1}{m_n} \right) \left( \frac{m_n - 1}{m_n - 1} \right) \cdots \left( \frac{m_1 - 1}{m_1} \right) > c, \quad \forall n \in \mathbb{N}. \]

Set \( r_n = m_{n-1} \cdots m_1 \) (assume \( r_1 = 1 \)), and set \( A_n = M_{r_n}(C([0,1])) \). Let \( \{x_n\} \) be a dense sequence in \([0,1]\). Set the eigenvalue maps (there are \( m_n \) of them) between \( A_n \) and \( A_{n+1} \) to be

\[ \lambda_1 = id, \ldots, \lambda_{m_n-1} = id, \lambda_{m_n}(x) = x_n. \]

Then the inductive limit \( A = \lim_{\rightarrow} A_n \) is a simple AI-algebra, and hence has mean dimension zero (see Remark 3.8). However, it does not have SBP. In fact, the following stronger statement holds: for any \( z \in [0,1] \) in the spectrum of \( A_n \), one has that \( \text{ocap}(\{z\}) > 0. \)

Indeed, for any \( k > n \), there are \( (m_n - 1)(m_{n+1} - 1) \cdots (m_{k-1} - 1) \) of eigenvalue maps in total of \( m_n m_{n+1} \cdots m_{k-1} \) between \( A_n \) and \( A_k \) are identity map. Thus,

\[ \sup_{x \in X_{j,k}} r_k' \# \{1 \leq m \leq m_n m_{n+1} \cdots m_{k-1}; \lambda_m(x) = z\} \]

\[ \geq \left( \frac{m_k - 1}{m_k} \right) \left( \frac{m_{k-1} - 1}{m_{k-1}} \right) \cdots \left( \frac{m_1 - 1}{m_1} \right) \]

\[ \geq \left( \frac{m_k - 1}{m_k} \right) \left( \frac{m_{k-1} - 1}{m_{k-1}} \right) \cdots \left( \frac{m_1 - 1}{m_1} \right) > c, \]
and $\operatorname{ocap}\{\varepsilon\} > c$.

**Definition 3.20.** Let $F$ be a self-adjoint element in a homogeneous C*-algebra $A$, the variation of the trace of $F$ is
\[
\max_{l} \sup_{s,t \in X_{l}} \frac{1}{n_{l,t}} |\operatorname{Tr}(F(s)) - \operatorname{Tr}(F(t))|.
\]

An AH-algebra $A$ is said to have the property of small variation of trace (SVT) if for any self-adjoint element $F \in A$ and any $\varepsilon > 0$, there exists $N > 0$ such that for any $j > N$, the variation of the trace of $\varphi_{i,j}(F)$ is less than $\varepsilon$.

The following lemma is well known.

**Lemma 3.21.** If the projections of an AH-algebra separate traces, then it has SVT.

**Proof.** If this were not true, then there exists $\varepsilon > 0$, $F \in A$, and $j_{m} \to \infty$ as $m \to \infty$ such that for any $A_{j_{m}}$, there is a connected component $X_{j_{m},k_{m}}$ and $s_{j_{m}}, t_{j_{m}} \in X_{j_{m},k_{m}}$ such that
\[
\frac{1}{n_{j_{m},k_{m}}}|\operatorname{Tr}(\varphi_{i,j_{m}}(F)(s_{j_{m}})) - \operatorname{Tr}(\varphi_{i,j_{m}}(F)(t_{j_{m}}))| \geq \varepsilon.
\]

Therefore there are tracial states $\tau_{j_{m}}^{(0)}$ and $\tau_{j_{m}}^{(1)}$ on $A_{j_{m}}$ such that
\[
\left|\tau_{j_{m}}^{(0)}(\varphi_{i,j_{m}}(F)) - \tau_{j_{m}}^{(1)}(\varphi_{i,j_{m}}(F))\right| \geq \varepsilon.
\]

Pick $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$, and consider
\[
x \mapsto \omega(\tau_{j_{m}}^{(l)}(\varphi_{i,j_{1}}(x)), \tau_{j_{2}}^{(l)}(\varphi_{i,j_{2}}(x)), \ldots, \tau_{j_{m}}^{(l)}(\varphi_{i,j_{m}}(x)), \ldots)
\]
with $l = 0, 1$. It then induces two tracial states $\tau^{(0)}$ and $\tau^{(1)}$ on $A$, and $\tau^{(0)} \neq \tau^{(1)}$. However, for any projection $p \in A$, one has that $\tau^{(0)}(p) = \tau^{(1)}(p)$, and this contradicts to the assumption. □

**Definition 3.22.** Let $A$ be a C*-algebra, let $\tau_{1}$ and $\tau_{2}$ be two tracial states, and let $\mathcal{F}$ be a finite subset of $A$. We write
\[
\|\tau_{1} - \tau_{2}\|_{\mathcal{F}} = \max\{|\tau_{1}(f) - \tau_{2}(f)|; f \in \mathcal{F}\}.
\]

Let $\Delta$ be a subset of $T(A)$. Write $\tau \in \mathcal{F}_{\varepsilon} \Delta$ if there is $\tau' \in \Delta$ such that $\|\tau - \tau'\|_{\mathcal{F}} < \varepsilon$.

If $A$ is a homogeneous C*-algebra with spectrum $X$, for any $x \in X$, denote by $\tau_{x}$ the tracial state of $A$ induced by the Dirac measure on $\{x\}$.

The following lemma can be regarded as a generalization of Lemma 3.21.

**Lemma 3.23.** Let $A$ be a simple AH-algebra, and let $M > 0$. If $\rho^{-1}(\kappa)$ has at most $M$ extreme points for any $\kappa \in S_{u}(K_{0}(A))$, then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subseteq A$, there exists $N$ such that for any $X_{j,k}$ with $j > N$, there exists a convex $\Delta_{j,k} \subseteq T(A)$ with at most $M$ extreme points such that
\[
\varphi_{i,j}^{*}(\tau_{x}) \in \mathcal{F}_{\varepsilon} \Delta_{j,k}, \quad \forall x \in X_{j,k}.
\]
Proof. If this were not true, there exist a finite subset \( F \subseteq A \) and \( \varepsilon > 0 \) such that for any \( n \), there exists \( j_n > n \) and \( k_n \) such that for any convex \( \Delta \subseteq T(A) \) with at most \( M \) extreme points, there exists \( x_{j_n,k_n}(\Delta) \) such that

\[
\varphi^*_{i,j_n}(\tau_{x_{j_n,k_n}(\Delta)}) \notin F, \varepsilon \Delta
\]

(3.3)

Let us show that it is impossible.

Choose an arbitrary point \( y_{j_n,k_n} \in X_{j_n,k_n} \), and consider the tracial state \( \tau_{y_{j_n,k_n}} \). Extend it to a state on \( A \), and consider an accumulation point \( \tau_y \). It is a straightforward argument to show that \( \tau_y \) is in fact a tracial state on \( A \).

Consider \( \Delta_\infty := \rho^{-1}(\rho(\tau_y)) \).

It then has at most \( M \) extreme points, hence

\[
\Delta_i := \varphi^*_{i,\infty}(\Delta_\infty)
\]

has at most \( M \) extreme points.

Consider points \( x_{j_n,k_n}(\Delta_i) \) and tracial states \( \tau_{x_{j_n,k_n}(\Delta_i)} \). The same argument as above shows that there is a tracial state \( \tau_x \) on \( A \) such that \( \tau_x \) is an accumulation point of \( \{ \tau_{x_{j_n,k_n}(\Delta_i)} \} \). However, since \( x_{j_n,k_n} \) is in the same connected component \( y_{j_n,k_n} \), one has

\[
\rho(\tau_x) = \rho(\tau_y)
\]

and hence

\[
\tau_x \in \Delta_\infty.
\]

Since \( \tau_x \) is an accumulation point of \( \{ \tau_{x_{j_n,k_n}(\Delta_i)} \} \), one has that

\[
\| \varphi^*_{j_n,\infty}(\tau_x) - \tau_{x_{j_n,k_n}(\Delta_i)} \|_F < \varepsilon
\]

for sufficiently large \( n \), thus

\[
\| \varphi^*_{i,\infty}(\tau_x) - \varphi^*_{i,j_n}\tau_{x_{j_n,k_n}(\Delta_i)} \|_F < \varepsilon,
\]

which is a contradiction to (3.3) as desired. \( \square \)

**Theorem 3.24.** Let \( A \) be a simple AH-algebra. If there is \( M > 0 \) such that \( \rho^{-1}(\kappa) \) has at most \( M \) extreme points for any \( \kappa \in S(K_0(A)) \), then \( A \) has SBRP, and hence has mean dimension zero. In particular, AH-algebras with real rank zero have mean dimension zero.

**Proof.** Consider \( X_{i,l} \), and let \( d \) be a compatible metric on \( X_{i,l} \). We assert that in order to prove the theorem, it is enough to show that for any open ball \( B(x,t) \), any \( \varepsilon > 0 \), and any \( s < t \), there exists \( L > 0 \) such that for any \( X_{j,k} \) with \( j > L \), there are \( s < r_1 < r_2 < t \) such that

\[
\text{ocap}_{j,k}(\text{Ann}(x,r_1,r_2)) < \varepsilon,
\]

where \( \text{Ann}(x,r_1,r_2) \) is the closed annulus with centre \( x \) and radius \( r_1 \) and \( r_2 \).

Indeed, let \( \alpha \) be a finite open cover of \( X_{i,l} \) and denote by \( \delta \) the Lebesgue number of \( \alpha \). Let \( \{ B(x_n,s_n); n = 1, \ldots, m \} \) a finite cover of \( X_{i,l} \) with open balls such that for each \( B(x_n,s_n) \), one has that \( B(x_n,s_n) \subseteq U \) for some \( U \in \alpha \).
For each $B(x_n,s_n)$, choose $t_n > s_n$ such that $B(x_n,t_n) \subset U$ if $B(x_n,s_n) \subset U$. Choose $L_n > 0$ such that for any $X_{j,k}$ with $j > L_n$, there are $s_k < r_{k,1} < r_{k,2} < t_n$ such that

$$\text{ocap}_{j,k}(\text{Ann}(x_n,r_{n,1},r_{n,2})) < \varepsilon/m.$$  

Set $L = \max\{L_1, ..., L_m\}$, and consider for a space $X_{j,k}$ with $j > L$.

For each $U \in \alpha$, set

$$U' = \bigcup_{B(x_n,s_n) \subset U} B(x_n,r_n),$$

where $r_n = (r_n,1 + r_n,2)/2$. Set $\delta = \min\{(r_{n,2} - r_{n,1})/2; 1 \leq n \leq m\}$.

Then,

$$\bigcup_{k} U'_{k} \supseteq \bigcup_{n=1}^{m} B(x_n,r_n) \supseteq \bigcup_{n=1}^{m} B(x_n,s_n) = X_{i,l},$$

and hence $\{U'_k\}$ is an open cover of $X_{i,l}$. Moreover, one has that $U'_k \subset U_k$, and

$$B(\partial U'_k, \delta) \subset \bigcup_{n=1}^{m} \text{Ann}(x_n,r_{n,1},r_{n,2}))$$

Hence

$$\text{ocap}_{j,k}\left(\bigcup_{j=1}^{\alpha} \partial U'_j\right) \leq \sum_{j=1}^{m} \text{ocap}_{j,k}(\text{Ann}(x_j,r_{j,1},r_{j,2})) < \varepsilon.$$

This proves the assertion.

Consider a ball $B(x,t)$, any $\varepsilon > 0$ and $s < t$. Set $m = \lceil 2/\varepsilon \rceil^M$, where $\lceil 1/\varepsilon \rceil$ is the smallest integer bigger than $1/\varepsilon$. Set $d = t - s$. For each $1 \leq n \leq m$, choose a continuous function $0 \leq f_n \leq 1$ such that $f_n(y) = 0$ for any $y \notin \text{Ann}(x,s + (n - 1)d/m, s + nd/m)$ and $f_n(y) = 1$ for any $y \in \text{Ann}(x,s + (4n - 3)d/4m, s + (4n - 1)d/4m)$.

It is clear that $f_n$ are mutually orthogonal. Denote by $\mathcal{F} = \{f_1, ..., f_m\}$. By Lemma 3.23 there exists $L$ such that for any $X_{j,k}$ with $j > L$, there exists a convex $\Delta_{j,k} \subseteq T(A_i)$ with at most $M$ extreme points such that

$$\varphi_{i,j}^*(\{\tau_x\}) \in \mathcal{F}, \forall x \in X_{j,k}.$$

Denote by $\mu_1, ..., \mu_M$ the extreme points of $\Delta_{j,k}$. Then, there exists $f_n \in \mathcal{F}$ such that

$$\mu_t(f_n) < \varepsilon/2, \quad 1 \leq t \leq M,$$

and hence

$$\varphi_{i,j}^*(\{\tau_x\})(f_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall x \in X_{j,k}.$$

Set $r_1 = s + (4n - 3)d/4m$ and $r_2 = s + (4n - 1)d/4m$. Note that

$$\text{ocap}_{j,k}(\text{Ann}(x,r_1,r_2)) \leq \max_{x \in X_{i,j}} \{\varphi_{i,j}^*(\{\tau_x\})(f_n)\} < \varepsilon,$$

as desired. 

\[ \square \]

Remark 3.25. The author would like to thank Professor XiaoQiang Zhao for encouraging him to fill a gap in the proof of the theorem above.
4. A LOCAL APPROXIMATION THEOREM

Let \( X \) be a compact Hausdorff space. Since \( D(\alpha) \leq \dim(X) \) for any open cover \( \alpha \) of \( X \), one has that if an AH-algebra \( A \) has slow dimension growth, then \( A \) has mean dimension zero.

In this section, let us consider AH-algebras with diagonal maps. Recall that an connection map \( \varphi_{i,j} : A_i \to A_j \) is an diagonal map if there exists a unitary \( U \in A_j \) such that the restriction of \( \text{ad}(U) \circ \varphi_{i,j} \) to \( A_{i,l} \) and \( A_{j,k} \) has the form

\[
\varphi_{i,j} : f \mapsto \begin{pmatrix} f \circ \lambda_1 & \cdots & f \circ \lambda_n \end{pmatrix}
\]

for some continuous maps \( \lambda_1, \ldots, \lambda_n : X_{j,k} \to X_{i,l} \).

It is interesting to point out that any simple AH-algebra with diagonal maps has stable rank one, i.e., invertible elements are dense. See [5].

**Remark 4.1.** If \( A \) is an AH-algebra with diagonal maps, then one can assume that all connection map has the form above.

For any \( f \in A_i \), if there are open sets \( U_1, \ldots, U_n \) such that \( \|f(x) - f(y)\| \leq \varepsilon \) for any \( x, y \in U_i \), then for any \( x, y \in \bigcap \lambda^{-1}(U_i) \) one has that \( \|\varphi(f)(x) - \varphi(f)(y)\| \leq \varepsilon \). Therefore, for any \( f \in A_i \), any \( \varepsilon > 0 \) and any open cover \( \alpha \) of \( X_i \) satisfying \( \|f(x) - f(y)\| \leq \varepsilon \), \( \forall x, y \in U, \forall U \in \alpha \), one has

\[
\|\varphi_{i,j}(f)(x) - \varphi_{i,j}(f)(y)\| \leq \varepsilon, \quad \forall x, y \in V, \forall V \in \varphi_{i,j}(\alpha).
\]

**Theorem 4.2.** Let \( A \) be an AH-algebra with diagonal map, and denote by \( \gamma \) the mean dimension of \( A \). Then for any finite subset \( F \subseteq A \) and any \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \), there exists a unital sub-C*-algebra

\[
C \cong \bigoplus p_i M_{n_i}(C(\Omega_i)) p_i \subseteq A
\]

such that \( F \subseteq \varepsilon_1 C \), and

\[
\frac{\dim \Omega_i}{\text{rank}(p_i)} < \gamma + \varepsilon_2.
\]

**Proof.** Without loss of generality, assume that \( F \subseteq A_1 \). Choose an open cover \( \alpha \) of \( X_1 \) such that for any \( U \in \alpha \), any \( x, y \in U \), and any \( f \in F \), one has

\[
\|f(x) - f(y)\| < \varepsilon_1.
\]

Since \( A \) has mean dimension \( \gamma \), there exists \( A_j \) such that

\[
\frac{D(\alpha_{1,j}^k)}{n_{j,k}} < \gamma + \varepsilon_2
\]

for any \( k \). Without loss of generality, assume that \( j = 2 \).

On each \( X_{2,k} \), consider an open cover \( \beta \) with \( \beta \supset \alpha_{1,2}^k \) with

\[
\text{ord}(\beta) = D(\alpha_{1,2}^k).
\]
Note that for any $U \in \beta$ and any $x, y \in U$, and any $f \in F_1$,

$$\left\| \varphi_{1,2}^k(f)(x) - \varphi_{1,2}^k(f)(y) \right\| < \varepsilon_1$$

Choose a partition of unity $\phi_i : X_2 \to [0, 1]$ subordinate to $\beta$, and choose $x_i \in U_i$ for each $i$. Then, for any $x \in X_2$ and any $f \in F$, one has

$$\left\| \sum_i \phi_i(x) \varphi_{1,2}(f)(x_i) - \varphi_{1,2}(f)(x) \right\| = \left\| \sum_i \phi_i(x) \varphi_{1,2}(f)(x_i) - \sum_i \phi_i(x) \varphi_{1,2}(f)(x) \right\|
= \left\| \sum_i \phi_i(x) \varphi_{1,2}(f)(x_i) - \varphi_{1,2}(f)(x) \right\|
< \varepsilon_1.$$

Define a c.p. map $\Theta : A_2 \to A_2$ by

$$f(x) \mapsto \sum_i \phi_i(x) f(x_i),$$

and therefore, one has

$$\| \Theta(f) - f \| < \varepsilon_1, \quad \forall f \in F.$$

Consider a ord($\beta$)-dimensional simplicial complex $\Delta$ as follows: The vertices of $\Delta$ correspond to the elements of $\alpha$, the $s$-dimensional simplices correspond to all $U_1, \ldots, U_s$ with $\bigcap U_i \neq \emptyset$. The point in each simplex $\{U_1, \ldots, U_s\}$ can be parameterized as

$$\sum_{i=1}^s \lambda_i [U_i] \quad \text{with} \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_i \lambda_i = 1.$$

Define a map $\xi : X_2 \to \Delta$ by

$$x \mapsto \sum_{x \in U_i \in \beta} \phi_i(x) [U_i],$$

and it induces a $*$-homomorphism $\Xi : \bigoplus_k M_{n_{2,k}}(C(\Omega_k)) \to A_2$.

For each $f \in A_2$, define function $\tilde{f}$ on $\Delta$ as follows: On vertices, set

$$\tilde{f}([U_i]) = f(x_i)$$

and extend it linearly to the simplicial complex. Then, it is clear that

$$\Xi(\tilde{f}) = \Theta(f).$$

Therefore, the image of $\Theta$ is contained in the image of $\Xi$, and hence $F$ can be approximated by the elements in the image of $\Xi$ within $\varepsilon_1$.

Identify the image of $\varphi_{2,\infty} \circ \Xi$ in $A$ with $C = \bigoplus_k M_{n_k}(C(\Omega_k))$ with $n_k = n_{2,k}$. Then one has that $F \subseteq \varepsilon_1 C$. Note that each $\Omega_k$ is a closed subset of $\Delta_{2,k}$, and hence for each $k$,

$$\dim(\Omega_k) \leq \text{ord}(\beta) = D(\alpha_{1,2}^k).$$
Therefore, one has
\[ \frac{\dim(\Omega_k)}{n_k} \leq \frac{\mathcal{D}(\alpha_{1,2}^k)}{n_k} < \gamma + \varepsilon_2, \]
as desired. \(\square\)

If an AH-algebra with diagonal map has mean dimension zero, then, it has local slow dimension growth by Theorem 4.2. Then the following theorem states that the strict order on projections is in fact determined by their traces.

**Theorem 4.3.** Let \( A \) be an AH-algebra with diagonal map. If \( A \) has mean dimension zero, then for any projections \( p \) and \( q \) in \( A \), if \( \tau(p) > \tau(q) \) for any tracial state \( \tau \) on \( A \), \( q \) is Murray-von Neumann equivalent to a sub-projection of \( p \).

**Proof.** Since the simplex of tracial states is compact, there exists \( \delta > 0 \) such that \( \tau(p) > \tau(q) + \delta \) for any tracial state \( \tau \) on \( A \).

We assert that one can find a homogeneous C*-algebra \( C \) using Theorem 4.2 such that \( \frac{1}{4} C \subseteq C \) and for any tracial state \( \tau \) on \( C \), one has
\[ \tau(p') > \tau(q') + \delta, \]
where \( p' \) and \( q' \) are projections in \( C \) which are close to \( p \) and \( q \) within 1/2 respectively.

If this were not true, there is a sequence of \( \{C_n\} \) such that such that for any \( n \), there is a tracial state \( \tau_n \) on \( C_n \) with
\[ \tau_n(p') \leq \tau_n(q') + \delta, \]
and moreover, \( \{C_n\} \) approximates arbitrary large finite subset with arbitrary tolerance. Extend each \( \tau_n \) to a state on \( A \), and still denote it by \( \tau_n \).

Pick an accumulation point \( \tau_\infty \). It is clear that \( \tau_\infty \) is a trace on \( A \). However,
\[ \tau_\infty(p) = \lim \tau_n(p') \leq \lim \tau_n(q') + \delta = \tau_\infty(q) + \delta, \]
which contradicts to the assumption. This prove the assertion.

Therefore, by Theorem 4.2, there exists \( C \cong \bigoplus M_{n_i}(C(\Omega_i)) \subseteq A \) with
\[ \frac{\dim \Omega_i}{n_i} < \delta \]
such that \( \{p, q\} \subseteq \frac{1}{4} C \), and for any tracial state \( \tau \) on \( C \),
\[ \tau(p') > \tau(q') + \delta, \]
where \( p' \) and \( q' \) are projections in \( C \) which are close to \( p \) and \( q \) within 1/2 respectively. Note that the projections \( p \) and \( q \) are unitary equivalent to the projections \( p' \) and \( q' \) respectively. Therefore, in order to proof the theorem, it is enough to show that \( q' \) is Murray-von Neumann equivalent to a sub-projection of \( p' \).

Indeed, it follows from (4.1) that
\[ \text{rank}(p'(x)) > \text{rank}(q'(x)) + \delta n_i \quad \text{if} \quad x \in \Omega_i. \]
Since \( \dim(\Omega_i) < n_i \delta \), by Theorem (9)1.2 and Theorem (9)1.5 of [8], one has that \( q' \) is Murray-von Neumann equivalent to a sub-projection of \( p' \), as desired. \(\square\)
Corollary 4.4. Any simple unital real rank zero AH-algebra with diagonal maps (without assuming slow dimension growth) has slow dimension growth.

Proof. By Theorem 3.24 such an AH-algebra has mean dimension zero. It then follows from Theorem 4.3 that the $K_0$-group has the strict comparison, and hence it is tracially AF by Theorem 2.1 of [9]. By the classification of tracially AF algebras, it has slow dimension growth. □

5. Mean dimension zero and AH-algebras with diagonal maps

In this section, the Cuntz semigroup of AH-algebra $A$ with diagonal maps are calculated to be $V(A) \sqcup \text{SAff}(T(A))$. Together with a recent result of W. Winter, this implies that $A$ is $\mathcal{Z}$-stable, and hence $A$ is a AH-algebra without dimension growth.

The following Lemma is due to M. Rørdam. See 4.3 of [20] for a proof.

Lemma 5.1. Let $A$ be a C*-algebra, and let $\varepsilon > 0$. Let $b$ and $b'$ be positive elements with $\|b' - (b - \varepsilon/2)\| \leq \varepsilon/4$. Set $\tilde{b} = (b' - \varepsilon/4)_+$. Then one has that $\|b - \tilde{b}\| < \varepsilon$ and $(b - \varepsilon)_+ \preceq \tilde{b} \preceq (b - \varepsilon/2)_+ \preceq b$.

Definition 5.2. A C*-algebra $A$ has radius of comparison less than $r$, write $\text{rc}(A) < r$, if for any positive elements $a, b \in A$ with $d_\tau(a) + r < d_\tau(b)$, $\forall \tau \in T(A)$, one has that $a \preceq b$. The radius of comparison of $A$ is $\text{rc}(A) := \inf\{r; \text{rc}(A) < r\}$.

Definition 5.3. An ordered abelian semigroup $(W, +, \leq)$ is said to be almost unperforated if for any element $a$ and $b$, if $(n + 1)a \leq nb$ for some $n \in \mathbb{N}$, then $a \leq b$.

One then has the following lemma.

Lemma 5.4 (Corollary 4.6 of [17]). Let $A$ be a simple unital exact C*-algebra. If $W(A)$ is almost unperforated, then $A$ has strict comparison of positive elements.

The dimension function $d_\tau(\cdot)$ is lower semicontinuous, both with $\cdot$ and with $\tau$.

Lemma 5.5. For any state $\tau$ on $A$, the map $a \rightarrow d_\tau(a)$ is lower semicontinuous, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $b$ with $\|a - b\| \leq \delta$, one has $d_\tau(b) > d_\tau(a) - \varepsilon$.

Proof. There exists $\varepsilon_1$ such that $|d_\tau(a) - \tau(f_{\varepsilon_1}(a))| < \varepsilon/2$.

Choose $\delta > 0$ such that $\|f_{\varepsilon_1}(a) - f_{\varepsilon_1}(b)\| < \varepsilon/2$ whenever $\|a - b\| \leq \delta$. One then has that

$$d_\tau(b) = \sup_{\varepsilon' > 0} \tau(f_\varepsilon'(b)) \geq \tau(f_{\varepsilon_1}(b)) > \tau(f_{\varepsilon_1}(a)) - \varepsilon/2 > d_\tau(a) - \varepsilon,$$

as desired. □
Lemma 5.6. For any positive element \( a \) in \( A \), the map \( \tau \to d_\tau(a) \) is lower semicontinuous, i.e., for any \( \varepsilon > 0 \), there exists a neighbourhood \( U \ni \tau \) such that for any \( \rho \in U \), one has \( d_\rho(a) > d_\tau(a) - \varepsilon \).

Proof. There exists \( \varepsilon_1 \) such that
\[
|d_\tau(a) - \tau(f_{\varepsilon_1}(a))| < \varepsilon/2.
\]
Choose a neighbourhood \( U \) in the simplex of traces of \( A \) such that
\[
|\rho(f_{\varepsilon_1}(a)) - \tau(f_{\varepsilon_1}(a))| < \varepsilon/2, \quad \forall \rho \in U.
\]
One then has
\[
d_\rho(a) = \sup_{\varepsilon' > 0} \rho(f_{\varepsilon'}(a)) \geq \rho(f_{\varepsilon_1}(a)) > \tau(f_{\varepsilon_1}(a)) - \varepsilon/2 > d_\tau(a) - \varepsilon,
\]
as desired. \( \Box \)

Lemma 5.7. Let \( A \) be a C*-algebra satisfying the following property: there is a collection of C*-algebras \( \{ C_i \} \) such that for any finite subset \( F \subseteq A \) and any \( \varepsilon > 0 \), there is a sub-C*-algebra \( C_i \subseteq A \) such that \( F \subseteq \varepsilon C_i \).

Then, for any positive elements \( a \) and \( b \) in \( A \) with \( a \precsim b \), and any \( \varepsilon > 0 \), there exists \( C_i \) and positive elements \( a_i, b_i \in C_i \) such that
\[
\|a - a_i\| \leq \varepsilon, \quad \|b - b_i\| \leq \varepsilon, \quad \text{and} \quad (a_i - \varepsilon)_+ \precsim b_i
\]
in \( C_i \), and \( b_i \precsim b \).

Proof. Since \( a \precsim b \), there is a sequence \( \{v_i\} \) in \( A \) such that
\[
\lim_{i \to \infty} v_i^* bv_i = a.
\]
For each \( i \), choose a sub-C*-algebra \( C_i \) with \( u_i, a_i, b_i \) in \( C_i \) such that
\[
\|v_i^* bv_i - u_i^* b_i u_i\| \leq 1/2^i \quad \text{and} \quad \|a_i - a\| \leq 1/2^i.
\]
Moreover, by Lemma 5.1, \( b_i \) can be chosen so that \( b_i \precsim b \). Hence one has
\[
\lim_{i \to \infty} u_i^* b_i u_i = a
\]
and
\[
\lim_{i \to \infty} a_i = a \quad \text{and} \quad \lim_{i \to \infty} b_i = b.
\]
Moreover, \( b_i \precsim b \).

Choose \( C_i \) large enough such that \( \|a - a_i\| < \varepsilon/4 \) and \( \|b - b_i\| < \varepsilon/4 \), and
\[
\|u_i^* b_i u_i - a\| < \varepsilon/4.
\]
Hence, one has
\[
\|u_i^* b_i u_i - a_i\| < \varepsilon/2,
\]
and therefore
\[
(a_i - \varepsilon)_+ \precsim b_i
\]
in \( C_i \), and \( b_i \precsim b \), as desired. \( \Box \)
Lemma 5.8. Let $A$ be a simple $C^*$-algebra satisfying the following property: For any finite subset $F \subseteq A$ and any $\varepsilon_1, \varepsilon_2 > 0$, there is a sub-$C^*$-algebra $C$ such that $F \subseteq \varepsilon_1 C$ and $\text{rc}(C) < \varepsilon_2$, then $W(A)$ is almost unperforated, in particular, the $C^*$-algebra $A$ has the strict comparison of positive elements.

Proof. Assume that one has positive elements $a$ and $b$ in $A$ with $(n + 1)[a] \precsim n[b]$ for some $n$.

Fix $\varepsilon > 0$ for the time being. It follows from Lemma 5.7 that there exist $C_i$ and $a_i, b_i \in C_i$ such that

$$\|a_i - a\| \leq \varepsilon, \quad \|b_i - b\| \leq \varepsilon \quad \text{and} \quad b_i \precsim b,$$

and

$$\bigoplus_{n+1}(a_i - \varepsilon)_+ \precsim \bigoplus_n b_i$$

in $C_i$. Moreover, using the simplicity of $A$, the compactness of the simplex of tracial states of $A$, and the lower-semicontinuity of $d_\tau$, one can assume that there is a strictly positive number $c$ which is independent of $C_i$ such that

$$d_\tau(b_i) > c$$

for any tracial state $\tau$ on $C_i$.

Indeed, for any $\varepsilon' > 0$, consider $f_{\varepsilon'}(b_i)$. It is clear that $f_{\varepsilon'}(b_i) \to f_{\varepsilon'}(b)$. Since $A$ is simple, there exists $c > 0$ such that

$$\tau(f_{\varepsilon'}(b_i)) > c$$

for any tracial state $\tau$ on $A$. Then, there exists a sub-$C^*$-algebra $C_i$ such that

$$\tau(f_{\varepsilon'}(b_i)) > c$$

for all tracial state $\tau$ on $C_i$. If this were not true, there is a sequence of $C_i$ with dense union in $A$, positive elements $b_i \in C_i$ and a sequence of tracial state $\tau_i$ on $C_i$ such that

$$\|f_{\varepsilon'}(b_i) - f_{\varepsilon'}(b)\| \leq 1/2^i \quad \text{and} \quad \tau_i(f_{\varepsilon'}(b_i)) \leq c.$$

Extend $\tau_i$ to a state on $A$, and pick an accumulation point $\tau_\infty$, and assume that $\tau_i \to \tau_\infty$. One then has

$$\tau_i(f_{\varepsilon'}(b)) \leq c + 1/2^i,$$

and

$$\tau_\infty(f_{\varepsilon'}(b)) = \lim_{i \to \infty} \tau_i(f_{\varepsilon'}(b)) \leq c,$$

which is a contradiction. This proves (5.2). Thus,

$$d_\tau(b_i) = \sup_{\varepsilon' > 0} f_{\varepsilon'}(b_i) \geq \tau(f_{\varepsilon'}(b_i)) > c.$$ 

Note that $c$ is independent of $C_i$.

Therefore, one may assume that $C_i$ is large enough such that $\text{rc}(C_i) < c/(n + 1)$. It follows from (5.1) that

$$d_\tau((a_i - \varepsilon)_+) + \frac{1}{n + 1}d_\tau(b_i) \leq d_\tau(b_i)$$
for any tracial state $\tau$ on $C_i$. Since $d_{\tau}(b_i) > c > 0$ for all $\tau$, one has
\[ d_{\tau}((a_i - \varepsilon)_+) + \frac{c}{n + 1} \leq d_{\tau}(b_i) \]
for any tracial state $\tau$ on $C_i$.

Since $\text{rc}(C_i) < c/(n + 1)$, one has that
\[ (a_i - \varepsilon)_+ \preceq b_i. \]
Note that $\| (a - \varepsilon)_+ - (a_i - \varepsilon)_+ \| \leq 3\varepsilon$, one has that $(a - 4\varepsilon)_+ \preceq (a_i - \varepsilon)_+$, and hence
\[ (a - 4\varepsilon)_+ \preceq (a_i - \varepsilon)_+ \preceq b_i \preceq b. \]

Since $\varepsilon$ is arbitrary, one has that $a \preceq b$, as desired. \qed

**Theorem 5.9.** Let $A$ be a simple unital AH-algebra with diagonal maps. If $A$ has mean dimension zero, then it has strict comparison of positive elements.

**Proof.** It follows directly from Theorem 4.2, Lemma 5.8 and Theorem 3.15 of [20]. \qed

The next theorem is analogous to Theorem 3.4 of [21] for local approximations, and the proof is similar.

**Lemma 5.10.** Let $A$ be a simple separable unital $C^*$-algebra satisfying the following property: For any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there is a unital sub-$C^*$-algebra $C$ such that $\mathcal{F} \subseteq C$ and $C$ is a recursive sub-homogeneous $C^*$-algebra with dimension ratio less than $\varepsilon$. Then any strictly positive continuous affine function $f$ on $T(A)$ and any $\varepsilon > 0$, there exists positive elements $a \in M_n(A)$ for some $n$ such that
\[ |f(\tau) - d_{\tau}(a)| < \varepsilon \quad \forall \tau \in T(A). \]

**Proof.** Since $f$ is strictly positive, there exists $\delta > 0$ such that $f(\tau) > \delta$ for all $\tau \in T(A)$. Without loss of generality, one may assume that $\varepsilon < \delta/8$.

Let $(\mathcal{F}_i)$ be an increasing sequence of finite subsets of $A$ with dense union, and let $(C_i)$ be the sequence of corresponding sub-$C^*$-algebras with $\mathcal{F}_i$ approximates within $\varepsilon/i$ of $C_i$. Since $A$ is simple, one may assume that the dimension of the irreducible representation of $C_i$ is at least $i^2$ and the dimension ratio of $C_i$ is less that $\varepsilon/i$.

Denote by $\iota_i$ the embedding of $C_i$, and denoted by $\iota'_i : \text{Aff}(T(C_i)) \to \text{Aff}(T(A))$ the map induced by $\iota_i$. Since $A$ can be locally approximated by $\{C_i\}$, it is well known that the set $\bigcup_i \iota'_i(\text{Aff}(T(C_i)))$ is dense in $\text{Aff}(T(A))$. Therefore, for sufficiently large $k$, the set $\iota'_i(\text{Aff}(T(C_k)))$ approximately contains $f$ up to $\varepsilon/2$, and thus, one may assume that $\iota'_i(\text{Aff}(T(C_k)))$ approximately contains $f$ up to $\varepsilon/2$ for all $i$.

For each $i$, pick $f_i \in \text{Aff}(T(C_i))$ such that $\| \iota'_i(f_i) - f \| < \varepsilon/2$ . Since $f$ is strictly positive, by an asymptotic argument, one may assume further that the sub-$C^*$-algebras $\{C_i\}$ and $\{f_i\}$ are chosen in a way such that $f_i(\tau) > \delta/2$ for all $k$ and for all $\tau \in T(C_i)$.

Fix a $C_i$ with $i > \max\{8+2/\varepsilon, 8/\delta\}$. Denote by $\{X_1, ..., X_i\}$ the base spaces of $C_i$, and denote by $X = \bigsqcup_{k=1}^i X_k$. Note that each $x \in X$ induces a tracial state of $C_i$, and denote it by $\tau_x$. 


For each $1 \leq k \leq l$, consider the functions $h_k(x) = \lceil n_x f_i(\tau_x) \rceil - 2$ and $g_k(x) = \lceil n_x f_i(\tau_x) \rceil - \varepsilon n_x / 2 - 3$, where $x \in X_k$, and $n_x$ is the matrix size of $C_i$ at $x$, and $d_x$ is the dimension of the connected component of the base space $X$ containing $x$. Then, the functions $h_k$ and $g_k$ are lower semicontinuous and upper semicontinuous respectively. Note that

$$h_k(x) - g_k(x) = 2 + \frac{\varepsilon n_x}{2} \geq 4d_x.$$ 

Assert that there exists $a \in C_i$ such that

$$g_k(x) \leq \text{rank}(\pi_k(a)(x)) \leq h_k(x),$$

where $\pi_k$ is the restriction of to $X_k$.

If $l = 0$, since $f_i(\tau_x) > \delta / 2$, $d_x / n_x < \varepsilon / 2$, $n_x > i^2$, and $\varepsilon < \delta / 8$, one has that

$$g(x) > n_x \delta / 2 - 4d_x - 2 > i^2 \delta / 4 - 2.$$ 

Thus the function $g(x)$ is positive. Hence, by Proposition 2.9 of [21] (with $Y = \emptyset$), there exists a positive element $a \in M_n(C_i)$ for some $n$ such that

$$g(x) \leq \text{rank}(a(x)) \leq h(x),$$

and hence

$$|d_{\tau_x}(a) - f_i(\tau_x)| \leq \left| \frac{g(x)}{n_x} - \frac{h(x)}{n_x} \right| < \frac{\varepsilon}{2}.$$ 

Assume that the assertion is true for $l = K$. There exists $a_K \in R^{(K)}$ such that

$$g_k(x) \leq \text{rank}(\pi_k(a_K)(x)) \leq h_k(x), \quad 0 \leq k \leq K.$$ 

Consider $b = \varphi(a_K) \in M_n(C(Y_{K+1}))$. Then, for any $y \in Y_{K+1},$

$$g_{K+1}(y) \leq \text{rank}(b(y)) \leq h_{K+1}(y).$$ 

Indeed, there are points $\{y_1, ..., y_m\} \subseteq \bigsqcup_{j=1}^{K} X_j$ such that the representation $\pi_y$ of $R^{(k)}$ is unitarily equivalent to the direct sum of $\pi_{y_j}$. Therefore,

$$n_y \tau_y = \sum n_{y_j} \tau_{y_j};$$

and hence

$$\text{rank}(b(y)) = \text{rank}(a(y_j)) = \sum h_j(y_j) \leq \sum \lceil n_{y_j} f_i(\tau(y_j)) \rceil - 2 \leq \lceil n_y f_i(\tau_y) \rceil - 2 = h_{K+1}(y).$$ 

The same argument shows that $g_{K+1}(y) \leq \text{rank}(b(y))$. Therefore, by Proposition 2.9 of [21], there is a function $a' \in M_n(C(X_{K+1}))$ such that the restriction of $a'$ to $Y_{K+1}$ equals to $b$ and

$$g_{K+1}(x) \leq \text{rank}(a(x)) \leq h_{K+1}(x), \quad \forall x \in X_{K+1}.$$ 

Thus, the assertion holds for $l = K + 1$, and hence for $C_i$ with arbitrary length.
Therefore it follows from the assertion that \(|d_\tau(a) - f_\tau| < \varepsilon/2\) for all \(\tau \in T(C_i)\), and hence for all \(\tau \in T(A)\), one has

\[
|d_\tau(a) - f_\tau| \leq |d_\tau(a) - \iota'_i(f_i(\tau))| + |\iota'_i(f_i(\tau) - f_\tau)|
< |d_\tau_\circ_\iota_i(a) - f_\iota_\circ_\iota_i(\tau)| + \varepsilon/2
< \varepsilon,
\]
as desired. \(\square\)

**Corollary 5.11.** The class of simple unital AH-algebras with diagonal maps and mean dimension zero are AH-algebras without dimension growth.

**Proof.** By the Theorem 5.9 and Lemma 5.10, the Cuntz semigroup of AH-algebra with mean dimension zero is \(V(A) \sqcup \text{SAff}(T(A))\). It then follows from [24] that such algebras are \(\mathcal{Z}\)-stable. By [25], [11], [12], and [10], these C*-algebras are AH-algebras without dimension growth. \(\square\)

**Corollary 5.12.** If a unital simple AH-algebra with diagonal maps has at most countably many extremal tracial states, or if there is \(M > 0\) such that \(\rho^{-1}(\kappa)\) has at most \(M\) extreme points for any \(\kappa \in S(K_0(A))\), then it is an AH-algebra without dimension growth.

**Proof.** It follows from Corollary 5.11, Theorem 3.24, and Corollary 3.18. \(\square\)

**Theorem 5.13.** Let \(A\) be a simple unital AH-algebra with diagonal maps. If \(A\) has finite mean dimension, then \(A \otimes A\) is an AH-algebra without dimension growth.

**Proof.** Let us show that for any finite subset \(F \subseteq A \otimes A\) and any \(\varepsilon > 0\), there is a homogeneous C*-algebra \(D\) with \(\dim(\text{sp}(D))/\dim(\pi) < \varepsilon\) and \(F \subseteq \varepsilon D\). Then the theorem follows from Lemma 5.8 and Lemma 5.10.

Without loss of generality, one can assume that

\[
f_i = \sum_{j=1}^{l_i} f_{i,j}^{(1)} \otimes f_{i,j}^{(2)}
\]

for each \(f_i \in F\). Set \(l = \max_i\{l_i\}\), \(s = \max_{i,j}\{\|f_{i,j}^{(1)}\|, \|f_{i,j}^{(2)}\|\}\), and

\[
G = \{f_{i,j}^{(1)} \otimes f_{i,j}^{(2)} ; f_i \in F\}.
\]

By Theorem 4.2, there exists a homogeneous C*-algebra \(C \subseteq A\) such that \(G \subseteq \varepsilon C\) and

\[
\dim(\text{sp}(C))/\dim(\pi) < r + \varepsilon.
\]

Since \(A\) is simple, one can assume that the dimension of the irreducible representation of \(C\) is bigger than \(2(r + \varepsilon)/\varepsilon\). One then has that \(F \subseteq \varepsilon C \otimes C\). Moreover, the algebra \(D := C \otimes C\) is a homogeneous C*-algebra with

\[
\dim(\text{sp}(D))/\dim(\pi) = 2\dim(\text{sp}(A))/\dim^2(\pi) < (r + \varepsilon)^2/\dim(\pi) < \varepsilon,
\]
as desired. \(\square\)
6. Radius of comparison for AH-algebras with diagonal maps

In this section, a lower bound for the mean dimension is given in the term of radius of comparison.

**Lemma 6.1.** Let $A$ be a simple exact $C^*$-algebra satisfying the following property: There is $r \geq 0$ such that for any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon_1, \varepsilon_2 > 0$, there is a sub-$C^*$-algebra $C$ such that $\mathcal{F} \subseteq \varepsilon_1 C$ and $\text{rc}(C) < r + \varepsilon_2$, then one has that $\text{rc}(A) \leq r$.

**Proof.** Since $A$ is exact (and hence all sub-$C^*$-algebras $C_i$), any lower semicontinuous dimension function on $W(A)$ is induced by a tracial state.

Let $a$ and $b$ be positive elements in $A$ with

$$d_{\tau}(a) + r' < d_{\tau}(b), \quad \forall \tau \in \mathcal{T}(A)$$

for some $r' > r$. Let us show that $a \lesssim b$.

Since $d_{\tau}(a \oplus r'1) < d_{\tau}(b)$ for any tracial state on $A$, there is a rational number $0 < c < 1$ such that $d_{\tau}(a \oplus r'1) < cd_{\tau}(b)$ for all tracial state $\tau$ on $A$, and hence there are natural numbers $m < m'$ such that

$$m'd_{\tau}(a \oplus r'1) < md_{\tau}(b).$$

By Proposition 3.1 of [16], there exists $n \in \mathbb{N}$ an $d \in W(A)$ such that

(6.1) $$n(m'[a \oplus r'1]) + d \leq nm[b] + d$$

in $W(A)$. Applying (6.1) $k$ times, one has

$$km'n([a \oplus r'1]) + d \leq kmn[b] + d, \quad \forall k \in \mathbb{N}.$$  

Since $W(A)$ is simple, there exists $l$ such that $d \leq l[b]$, and hence

$$km'n[a \oplus r'1] \leq (kmn + l)[b] \quad \forall k \in \mathbb{N}.$$  

Choose $k$ sufficiently large so that $km'n > kmn + l$, and set $N = km'n - 1$, one has

$$(N + 1)[a \oplus r'1] \leq N[b].$$

One can then follows the same proof as that of Lemma 5.8. Fix $\varepsilon > 0$ for the time being. It follows from Lemma 5.7 that there exist $C_i$ and $a_i, b_i \in C_i$ such that

$$\|a_i - a\| \leq \varepsilon, \quad \|b_i - b\| \leq \varepsilon \quad \text{and} \quad b_i \lesssim b,$$

and

(6.2) $$\bigoplus_{N+1}((a_i - \varepsilon) \oplus r'1) \lesssim \bigoplus_{N} b_i$$

in $C_i$. Moreover, using the simplicity of $A$, the compactness of the simplex of tracial states of $A$, and the lower-semicontinuity of $d_{\tau}$, one can assume there is a strictly positive number $c$ which is independent of $C_i$ such that

(6.3) $$d_{\tau}(b_i) > c$$

for any tracial state $\tau$ on $C_i$. 


Indeed, for any $\varepsilon' > 0$, consider $f_{\varepsilon'}(b_i)$. Fix $\varepsilon'$, and it is clear that $f_{\varepsilon'}(b_i) \to f_{\varepsilon'}(b)$. Since $A$ is simple, there exists $c > 0$ such that
\[ \tau(f_{\varepsilon'}(b)) > c \]
for any tracial state $\tau$ on $A$. Then, there exists a sub-C*-algebra $C_i$ such that
\[ \tau(f_{\varepsilon'}(b)) > c \]
for all tracial state $\tau$ on $C_i$. If this were not true, there is a sequence of $C_i$ with dense union in $A$, positive elements $b_i \in C_i$ and a sequence of tracial state $\tau_i$ on $C_i$ such that
\[ \|f_{\varepsilon'}(b_i) - f_{\varepsilon'}(b)\| \leq 1/2^i \text{ and } \tau_i(f_{\varepsilon'}(b_i)) \leq c. \]
Extend $\tau_i$ to a state on $A$, and pick an accumulation point $\tau_\infty$, and assume that $\tau_i \to \tau_\infty$. One then has
\[ \tau_i(f_{\varepsilon'}(b)) \leq c + 1/2^i, \]
and
\[ \tau_\infty(f_{\varepsilon'}(b)) = \lim_{i \to \infty} \tau_i(f_{\varepsilon'}(b)) \leq c, \]
which is a contradiction. This proves (6.4).

Note that $c$ is independent of $C_i$.

Therefore, one may assume that $C_i$ is large enough such that $rc(C_i) < r + c/(N+1)$. It follows from (6.2) that
\[ d_\tau((a_i - \varepsilon)_+) + r' + \frac{1}{N+1}d_\tau(b_i) \leq d_\tau(b_i) \]
for any tracial state $\tau$ on $C_i$. By (6.3), $c_i = \inf_\tau\{d_\tau(b_i)\} \geq c$ is strictly positive, and hence one has
\[ d_\tau((a_i - \varepsilon)_+) + r' + \frac{c}{N+1} \leq d_\tau(b_i) \]
for any tracial state $\tau$ on $C_i$.

Since $rc(C_i) < r + c/(N+1)$, one has that
\[ (a_i - \varepsilon)_+ \preceq b_i. \]
Note that $\|(a - \varepsilon)_+ - (a_i - \varepsilon)_+\| \leq 3\varepsilon$, one has that $(a - 4\varepsilon)_+ \preceq (a_i - \varepsilon)_+$, and hence
\[ (a - 4\varepsilon)_+ \preceq (a_i - \varepsilon)_+ \preceq b_i \preceq b. \]
Since $\varepsilon$ is arbitrary, one has that $a \preceq b$, as desired. \qed

**Theorem 6.2.** If an AH-algebra $A$ with diagonal maps has mean dimension $\gamma$, then $rc(A) \leq \gamma/2$.

**Proof.** Since $A$ has mean dimension $\gamma$, by Theorem 4.2, for any finite subset $F \subseteq A$ and any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists a unital sub-C*-algebra
\[ C \cong \bigoplus M_{n_i}(C(\Omega_i)) \subseteq A \]
such that $\mathcal{F} \subseteq_{\varepsilon_1} C$, and

$$\frac{\dim \Omega_i}{n_i} < \gamma + \varepsilon_2.$$ 

By Corollary 5.2 of $[18]$, $rc(C) \leq \gamma/2 + \varepsilon_2$.

It then follows from Lemma 6.1 that $rc(A) \leq \gamma/2$. \hfill \Box

7. Cuntz mean dimension for AH-algebras with generalized diagonal maps

**Definition 7.1.** A branched open cover $\tilde{\alpha}$ of a compact Hausdorff space $X$ consists pairs

$$\{(U_\lambda, \kappa_\lambda); \lambda \in \Lambda\}$$

with $U_\lambda$ an open subset of $X$ and $\kappa$ a Murray-von Neumann class of projections in $C(\overline{U_\lambda}, K)$ such that $\bigcup_{\lambda \in \Lambda} U_\lambda = X$. Note $\{U_\lambda\}$ is an ordinary cover of $X$, denote it by $\alpha$.

For any $U \in \alpha$, define

$$K_U = \{\kappa_V; (V, \kappa_V) \in \tilde{\alpha}, U = V\}.$$ 

Then the Murray von Neumann equivalence relation induces a partition

$$K_U = \mathcal{E}_U(1) \sqcup \cdots \sqcup \mathcal{E}_U(n_U).$$

For any $\mathcal{E}_U(i)$, define

$$\text{rank}(\mathcal{E}_U(i)) = \text{rank}(p),$$

where $p$ is any representative of $\mathcal{E}_U(i)$.

Define the multiplicity of $U$ by

$$\text{mul}(U) := \min\{|\mathcal{E}_U(i)| \cdot \text{rank}(\mathcal{E}_U(i)); 1 \leq i \leq n_U\},$$

and define the multiplicity of $\tilde{\alpha}$ to be

$$\text{mul}(\tilde{\alpha}) := \min\{\text{mul}(U); U \in \alpha\}.$$

**Definition 7.2.** Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two branched covers. Define $\tilde{\alpha} \lor \tilde{\beta}$ to be the branched cover consisting of

$$\{(U \cap V, \kappa_{U \cap V}); U \in \alpha, V \in \beta, \kappa_{U \cap V} = \kappa_U \text{ or } \kappa_{U \cap V} = \kappa_V\}.$$ 

**Definition 7.3.** Let $\tilde{\alpha}$ be a branched cover of $X$, and let $\beta$ be any ordinary cover with $\beta \succ \alpha$. For any $W \in \beta$, consider

$$\tilde{\alpha}_W = \{(U, \kappa_U); (U, \kappa_U) \in \tilde{\alpha} \text{ and } W \subseteq U\}, \text{ and } \mathcal{K}_{W}^{\tilde{\alpha}} = \{(\kappa_U)|_W; (U, \kappa_U) \in \tilde{\alpha}_W\}.$$ 

Then

$$\text{Ind}_{\beta}^{\tilde{\alpha}} := \{(W, \kappa_W); W \in \beta, \kappa_W \in \mathcal{K}_{W}^{\tilde{\alpha}}\}$$

is a branched cover of $X$ induced by $\alpha$ based on $\beta$. 
7.4. Consider the homogeneous C*-algebra $A = p(C(X) \otimes \mathcal{K})p$, where $X$ is a compact Hausdorff space, and $p$ is a projection. Let $w$ be an element in $W(A)$ such that there exist an open set $U$ and $d \in \mathbb{N}$ such that
\begin{equation}
\tau_{\tau}(w) = \begin{cases} 
d & \text{if } x \in U \\
0 & \text{otherwise}
\end{cases},
\end{equation}
where $\tau$ is the Dirac measure concentrated on $x$.

Lemma 7.5. Let $w$ be the Cuntz semigroup element as above. Then, for any open subset $U' \subset U$ with $\overline{U'} \subset U$, there is a vector bundle $\xi$ on $\overline{U'}$ such that $w|_{U'}$ is induced by $\xi$.

Proof. Let $a(x) \in M_n(A^+)$ be a representative of $w$. Then one has that $\text{supp}(a) = \overline{U}$, and for any $x \in U$, $\text{rank}(a(x)) = d$. Then, $\text{Im}(a(x))$ is the desired vector bundle. \hfill \Box

Definition 7.6. An element $w$ in $W(A)$ is said to be type 0 if it satisfies (7.1), and $U$ is called an open support of $w$.

7.7. Let $w_1, ..., w_n$ be elements of type 0 in $W(A)$, and denoted by $U_1, ..., U_n$ the their open supports respectively. If $\{U_1, ..., U_n\}$ forms an open cover of $X$, then $w_1, ..., w_n$ induce a branched cover of $X$ in the natural way.

Definition 7.8 ([23]). A homomorphism: $\varphi : C(X) \otimes \mathcal{K} \to C(Y) \otimes \mathcal{K}$ is called generalized diagonal if there exist $k \in \mathbb{N}$ and maps $\lambda_1, ..., \lambda_k : Y \to X$ and mutually orthogonal projections $p_1, ..., p_k$ in $C(Y) \otimes \mathcal{K}$ such that $\varphi = (\text{id}_{C(X)} \otimes \vartheta) \circ (\tilde{\varphi} \otimes \text{id}_{\mathcal{K}})$ where

$$\tilde{\varphi} : C(X) \to C(Y) \otimes \mathcal{K} : f \mapsto \sum_{i=1}^{k} (f \circ \lambda_i)p_i$$

and $\vartheta : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ is an isomorphism. In this case, one say the map $\varphi$ comes from $(\lambda_i, p_i)_{i=1}^{k}$.

Remark 7.9. The homomorphisms in the form above are referred to as diagonal maps in [23], which are different from the diagonal maps considered in this paper.

7.10. Let $X_i$ be a sequence of compact Hausdorff spaces, and consider a sequence of diagonal maps $\varphi_i : C(X_i) \otimes \mathcal{K} \to C(X_{i+1}) \otimes \mathcal{K}$. Let $q_i$ be a projection in $C(X_1) \otimes \mathcal{K}$, and denote by $q_i = \varphi_{i-1} \circ \cdots \circ \varphi_1(q_1)$. Set

$$A_i = q_i(C(X_i) \otimes \mathcal{K})q_i.$$ 

The the inductive limit $\lim_{\rightarrow}(A_i, \varphi_i)$ is called a unital AH-algebra with generalized diagonal maps.

Let $\alpha$ be an open cover of $X_1$. For each $U \in \alpha$, consider any complex valued function $\phi_U$ with $\phi_U(x) \neq 0$ for all $x \in U$ and $\phi_U(x) = 0$ for all $x \notin U$. Consider $[U] = [\phi_U \cdot q_i]$ in the Cuntz semigroup of $A_i$. This element is uniquely determined by $U$.

Consider the image $[\varphi_i([U])]$. Then,

$$[\varphi_i([U])] = \sum_{j=1}^{m_i} [(\phi_U \cdot \lambda_j)p_j \otimes q_i] = \sum_{j=1}^{m_i} [(\phi \lambda_{i,j}(U))p_j \otimes q_i].$$
7.11. Let $\alpha$ be an open cover of $X_i$, and let $\varphi_i : A_i \to A_{i+1}$ be a generalized diagonal map. Assume that $X_{i+1}$ is connected and the map $\varphi_i$ is induced by $(\lambda_j, p_j)$. Then, the pair $(\lambda_j, p_j)$ induces a branched cover
\[
(\lambda, p_j)^{-1}(\alpha) := \{(\lambda_j^{-1}(U), p_j|_{\lambda_j^{-1}(U)}); U \in \alpha\}
\]
of $X_{i+1}$.
Consider the branched cover
\[
\bar{\alpha}_{i+1} := (\lambda_1, p_1)^{-1}(\alpha) \lor (\lambda_2, p_2)^{-1}(\alpha) \lor \cdots \lor (\lambda_{m_i}, p_{m_i})^{-1}(\alpha)
\]
and the ordinary cover
\[
\alpha_{i+1} := \lambda_1^{-1}(\alpha) \lor \lambda_2^{-1}(\alpha) \lor \cdots \lor \lambda_{m_i}^{-1}(\alpha).
\]
For any $\beta \succ \alpha_{i+1}$, consider the branched cover $\text{Ind}_{\beta}^{\bar{\alpha}_{i+1}}$. Set
\[
n'_{i+1}(\beta) := \text{mul}(\text{Ind}_{\beta}^{\bar{\alpha}_{i+1}}),
\]
and
\[
r'_{i+1}(\alpha) = \inf\{\frac{\text{ord}(\beta)}{n_{i+1}(\beta)}; \beta \succ \alpha_{i+1}\}.
\]
Note that $r'_{i+1}$ depends on the pair $(\lambda_j, p_j)$, but the map $\varphi_i$ may be induced by different pairs. Hence one defines
\[
r_{i+1}(\alpha) = \inf\{r'_{i+1}; (\lambda_j, p_j) \text{ induces } \varphi_i\}.
\]
If $X_{i+1}$ has more than one connected component, say, $X^{(1)}_{i+1}, \ldots, X^{(k)}_{i+1}$, then define
\[
r_{i+1}(\alpha) = \max\{r_{i+1}(\alpha) \text{ for the restriction of } \varphi_i \text{ to } X^{(l)}_{i+1}; 1 \leq l \leq k\}.
\]

**Remark 7.12.** Although it is possible to define $r_{i+1}(\alpha)$ without considering connected components of base spaces (direct sum of generalized diagonal maps is a general diagonal map by itself), it would be overkill to compare the overall order of $\beta$ on $X_{i+1}$ to the overall multiplicity of $\beta$. In other words, the order of $\beta$ might be obtained on one connected component, but the multiplicity might be achieved on another.

**Definition 7.13.** Let $A$ be an AH-algebra with generalized diagonal maps. Set
\[
\gamma_i(A) := \sup_{\alpha \in \mathcal{C}(X_i)} \sup_{j \geq i} \liminf_{\beta \succ \alpha} r_{j}(\alpha).
\]
The Cuntz mean dimension of $A$ is defined by
\[
\gamma_c(A) = \lim_{i \to \infty} \gamma_i(A).
\]

**Remark 7.14.** For an AH-algebra with generalized diagonal maps, if for each diagonal map $\varphi$, all $p_j$ are Murray-von Neumann equivalent, (in particular, if the connection map $\varphi_i$ is diagonal), then $n_{i+1}(\beta) = \text{rank}(q_{i+1})$ for any $\beta \succ \alpha_{i+1}$, and hence
\[
\gamma_i = \frac{\mathcal{D}(\alpha_{i+1})}{\text{rank}(q_{i+1})},
\]
and hence $\gamma_c(A) = \gamma(A)$.
Definition 7.15. Let $X$ and $\Delta$ be two compact Hausdorff space. Let $S$ be a unital sub-C*-algebra of $A = p(C(X) \otimes K)p$ and let $\xi : X \to \Delta$ be a continuous map. Then $S$ is said to separate $\Delta$ if

1. for any $y \in \Delta$ and any open set $U \subset \Delta$ containing $y$, there exists $f \in S \cap A'$ such that $f|_{\xi^{-1}(x)} \neq 0$ and $f|_{\xi^{-1}(U)} = 0$;
2. for any $x_1$ and $x_2$ in $X$ with $\xi(x_1) = \xi(x_2)$, the representation $\pi_{x_1}|S$ is unitarily equivalent to $\pi_{x_2}|S$.

Lemma 7.16. With the notation as above, one then has that for any subset $D \subseteq X$ with a closed image and $x \in X$ with $\xi(x) \notin \xi(D)$, there exists $f \in S \cap A'$ such that $f(x) \neq 0$ but $f|_{D} = 0$.

Proof. Set $y = \xi(x)$. Since $\xi(x) \notin \xi(D)$ and $\xi(D)$ is closed, there is an open set $U \subseteq \Delta$ containing $y$, and $f \in S \cap A'$ such that $U \cap \xi(D) = \emptyset$, $f|_{\xi^{-1}(y)} \neq 0$ and $f|_{\xi^{-1}(U)} = 0$. Note that $D \subseteq \xi^{-1}(U)$ and the lemma follows. \hfill $\square$

Lemma 7.17. Let $X$ and $\Delta$ be two compact Hausdorff space. Let $S$ be a unital sub-C*-algebra of $A = p(C(X) \otimes K)p$ and let $\xi : X \to \Delta$ be a continuous map. Suppose that $S$ separates $\Delta$. Then $S$ has a recursive subhomogeneous C*-algebra decomposition with topological dimension at most $\dim(\Delta)$.

Proof. Let us first show that $\dim(\Prim_n(S)) \leq \dim(\Delta)$ for all $1 \leq n \leq N$, where $N$ is the largest dimension of the irreducible representations of $S$.

For any $\sigma \in \Prim(S)$, there exists $x \in X$ such that $\sigma$ is a direct summand of the evaluation $\pi_x$. (Note that there might be more than one $x$ such that $\pi_x$ contains $\sigma$.) Define a map

$$F : \Prim(S) \ni \sigma \mapsto \xi(x) \in \Delta.$$ 

Assert that $F$ does not depend on the choice of $x$. In fact, let $x_1, x_2 \in X$ such that both $\pi_{x_1}|S$ and $\pi_{x_2}|S$ contain $\sigma$. If $\xi(x_1) \neq \xi(x_2)$, since $S$ separates $\Delta$, by Lemma 7.16 there exists $f \in S \cap A'$ such that $\pi_{x_1}(f) \neq 0$ but $\pi_{x_2}(f) = 0$. In particular, $\pi_{x_1}|S$ and $\pi_{x_2}|S$ cannot have a common factor, and thus $\xi(x_1) = \xi(x_2)$.

Note that any pre-image $E$ of a closed subset $D' \subseteq \Delta$ under $F$ has the following form:

$$E = \{\sigma; \sigma \text{ is a direct summand of } \pi_x|S \text{ for some } x \in D\},$$

where $D = \xi^{-1}(D')$.

Let us show that $E$ is closed. Denote by

$$I(E) := \{f \in S; \sigma(f) = 0, \forall \sigma \in E\}.$$ 

Let $\sigma'$ be an irreducible representation of $S$ such that $\sigma'(I(E)) = \{0\}$. There is $x' \in X$ such that $\sigma'$ is a direct summand of $\pi_{x'}|S$. If $x' \notin D = \xi^{-1}(D')$, then $\xi(x') \notin D' = \xi(D)$. By Lemma 7.16 there exists $f \in S \cap A'$ such that $f(x') \neq 0$ and $f|_{D} = 0$. Therefore, $f \in I(E)$, but $\sigma'(f) \neq 0$, and this contradicts the assumption. Thus $x' \in D$, $\sigma' \in E$, and $E$ is a closed set.

Hence the map $F$ is a continuous map. Let $y \in \Delta$, and consider $F^{-1}\{\{y\}\}$. Let $x \in X$ such that $\xi(x) = y$. Since $S$ separates $\Delta$, by (2) of Definition 7.15, the set $F^{-1}\{\{y\}\}$ agrees with the set of direct summands of $\pi_x|S$, and therefore $F^{-1}\{\{y\}\}$ only has finitely many elements.
For each $1 \leq n \leq N$, denote by $F_n$ the restriction of $F$ to $\text{Prim}_n(S)$. Note that $\text{Prim}_n(S)$ is locally compact, Hausdorff, and second countable, and hence is metrizable. By Theorem 1.12.7, Theorem 4.1.5, Corollary 3.1.20 of [6], one has that
\[
\dim(\text{Prim}_n(S)) = \dim(F_n(\text{Prim}_n(S))) \leq \dim(\Delta).
\]
Thus, $S$ is a unital subhomogeneous C*-algebra with $\dim(\text{Prim}_n(S)) \leq \dim(\Delta)$ for all $1 \leq n \leq N$. By Theorem 2.16 of [15], $S$ has a recursive subhomogeneous C*-algebra decomposition with topological dimension at most $\dim(\Delta)$, as desired. \hfill $\square$

**Definition 7.18.** Let $\tilde{\alpha}$ be a branched cover of $X$. A family of projections $\{p_{(U,\kappa_U)}; (U, \kappa_U) \in \tilde{\alpha}\}$ is compatible to $\tilde{\alpha}$ if
\begin{enumerate}
\item each $p_{(U,\kappa_U)}$ is a projection in $C(\overline{U}) \otimes \mathcal{K}$;
\item for each $U \in \alpha$, the projections $\{p_{(U,\kappa)}; (U, \kappa) \in \tilde{\alpha}\}$ are mutually orthogonal;
\item $[p_{(U,\kappa_U)}] = \kappa_U$.
\end{enumerate}

**Lemma 7.19.** Let $\tilde{\alpha}$ be a branched cover of $X$, and let $\{p_{(U,\kappa_U)}\}$ be a family of projections which is compatible with $\tilde{\alpha}$. Consider the homogeneous C*-algebra $A = p(C(X) \otimes \mathcal{K})p$, where $p$ is a projection in $C(X) \otimes \mathcal{K}$. Assume that for each $U \in \alpha$, one has $\sum_{(U,\kappa) \in \tilde{\alpha}} p_{(U,\kappa)} = p|_U$.

Let $\{\phi_i\}$ be a partition of identity according to $\alpha$, then, there is a sub-C*-algebra $S \subseteq A$ such that
\[
\{\phi_U \cdot p_{(U,\kappa_U)}; (U, \kappa_U) \in \tilde{\alpha}\} \subseteq S,
\]
and $S$ has a recursive subhomogeneous decomposition of topological dimension at most $\text{ord}(\alpha)$, and the dimensions of the irreducible representations of $S$ at least $\text{mul}(\tilde{\alpha})$.

**Proof.** Consider a simplicial complex $\Delta$ as the following: The vertices of $\Delta$ consists of elements of $\alpha$. The $s$-dimensional simplices correspond to all $U_1, ..., U_s$ with $\bigcap_{i=1}^s U_i \neq \emptyset$. Let $\{\phi_U; U \in \alpha\}$ be a partition of unit coordinating to $\alpha$. Define the map $\xi : X \to \Delta$ by
\[
x \mapsto \sum_{U \in \alpha} \phi_U(x)[U].
\]
Note that $\dim(\Delta) = \text{ord}(\alpha)$.

For any $U \in \alpha$, consider $\mathcal{K}_U = \mathcal{E}_U(1) \cup \cdots \cup \mathcal{E}_U(n_U)$. Write
\[
\mathcal{E}_U(k) = \{\kappa_1, ..., \kappa_m\}
\]
where $1 \leq k \leq n_U$. Since $\kappa_i, i = 1, ..., m_k$ are mutually equivalent, there exists a system of matrix units $\{(e_{U})_{i,j}; 1 \leq i, j \leq m_k\} \subseteq A|_U$ such that $[(e_{U})_{i,j}] = \kappa_i, (e_{U})_{i,i} = p_{(U,\kappa_i)}$. Note that
\[
(e_{U})_{i,j} \cdot \phi_U \in A.
\]

Consider the elements
\[
\mathcal{G}_{\tilde{\alpha}} := \{(e_{U})_{i,j} \cdot \phi_U; U \in \alpha, 1 \leq k \leq n_U, 1 \leq i, j \leq m_k\},
\]
and the sub-C*-algebra
\[
S = C^*(\mathcal{G}_{\tilde{\alpha}}) \subseteq A.
\]
Note that for any $(U, \kappa_U) \in \tilde{\alpha}$,
\[
\phi_U \cdot p_{(U,\kappa_U)} = \phi_U \cdot (e_{U})_{i,i} \in S.
\]
Then, $S$ is a subhomogeneous $C^*$-algebra. For any $x \in X$, the restriction of the representation $\pi_x$ to $S$ has a decomposition

$$\pi_x(S) = \bigoplus_i M_{n_i}(\mathbb{C}),$$

with $n_i \geq \text{mul}(\tilde{\alpha})$.

Let us show that $S$ separates $\Delta$. Note that for each $U \in \alpha$,

$$\phi_U \cdot p = \phi_U \cdot p|_U = \phi_U \cdot (\sum_{(U,\kappa U) \in \tilde{\alpha}} p_{(U,\kappa U)}) \in S,$$

and it is in the centre of $A$. Note that the map $\xi$ induces a homomorphism $\Xi : \mathbb{C}(\Delta) \to \mathbb{C}(X) \sim A'$. By considering the image of $X$ in $\Delta$, we may assume that $\Xi$ is injective. By (7.3), one has that $\phi_U \cdot p \in \Xi(\mathbb{C}(\Delta))$ for any $U \in \alpha$. If $\xi(x_1) \neq \xi(x_2)$ for some $x_1, x_2 \in X$, then there exists $\phi_U$ such that $\phi_U(x_1) \neq \phi_U(x_2)$. Therefore, the elements $\{\phi_U \cdot p\}$ separate points of the algebra $\Xi(\mathbb{C}(\Delta))$. By Stone-Weierstrass Theorem, the elements $\{\phi_U \cdot p\}$ generate $\Xi(\mathbb{C}(\Delta))$, that is, $S$ contains $\Xi(\mathbb{C}(\Delta))$. Therefore, for any $y \in \Delta$ and $V \subseteq \Delta$ an open ball containing $y$, there exists $f \in S \cap A'$ such that $f|_{\xi^{-1}(y)} = 1$ and $f|_{\xi^{-1}(V^c)} = 0$. This show the Condition (1) of Definition 7.15.

For (2) of Definition 7.15, if $\xi(x_1) = \xi(x_2)$ for some $x_1, x_2 \in X$, then, $\phi_U(x_1) = \phi_U(x_2)$ for all $U \in \alpha$, and it is clear that $\pi_{x_1}|_S$ is unitarily equivalent to $\pi_{x_2}|_S$ from the construction.

Thus, the subhomogeneous $C^*$-algebra $S$ separates $\Delta$. By Lemma 7.17, the $C^*$-algebra $S$ has a recursive subhomogeneous decomposition with topological dimension at most $\text{ord}(\alpha)$. Moreover, by (7.2), the irreducible representations of $S$ has dimension at least $\text{mul}(\tilde{\alpha})$. □

**Theorem 7.20.** Let $A$ be a simple AH-algebra with generalized diagonal maps. Then $\text{rc}(A) \leq \frac{\gamma_c(A)}{2}$.

**Proof.** Let us show that for any finite subset $F \subseteq A$ and any $\varepsilon$, there exists a sub-$C^*$-algebra $S \subseteq A$ such that $F \subseteq S$, and $\text{rc}(S) \leq \gamma/2 + \varepsilon$. Then, the theorem follows from Lemma 6.1.

Without loss of generality, one may assume that $F \subseteq A_1$. Then, there is an open cover $\alpha$ of $X_1$ such that for any $f \in F$ and any $U \in \alpha$, one has

$$\|f(x) - f(y)\| < \varepsilon \quad \forall x, y \in U.$$

Choose $j$ and an open cover $\beta$ of $X_j$ with $\beta \succ \alpha_j$ such that

$$\frac{\text{ord}(\beta)}{n_j(\beta)} < \gamma_c(A) + \varepsilon.$$

Consider the branched cover $\tilde{\beta} := \text{Ind}_{\beta}^{\tilde{\alpha}_j}$. Note that

$$\varphi_j(f) = \sum_{i=1}^k (f \circ \lambda_i)p_i, \quad \forall f \in A_1,$$
where \( \lambda_i \) are continuous maps from \( X_j \) to \( X_1 \), and \( \{p_i; 1 \leq i \leq k\} \) is a family of mutually orthogonal projections in \( C(X_j) \otimes \mathcal{K} \). Then, the family of projections

\[
\{p_i|_V; 1 \leq i \leq k, V \in \beta\}
\]

is compatible to \( \tilde{\beta} \). Indeed, for each projection \( p_i|_V \), it corresponds to \( (V, [p_i|_V]) \in \tilde{\beta} \).

Let \( \{\phi_V; V \in \beta\} \) be a partition of identity coordinating to \( \beta \), and let \( \{x_V; V \in \beta\} \) be a set of points with \( x_V \in V \). Then, for any \( f \in \mathcal{F} \), one has that for any \( x \in X_j \),

\[
\begin{align*}
(7.4) & \quad \left\| \sum_{i=1}^{k} \sum_{V \in \beta} (f \circ \lambda_i)(x_V) \phi_V(x)p_i(x) - \varphi_j(f)(x) \right\| \\
(7.5) & \quad = \left\| \sum_{i=1}^{k} \sum_{V \in \beta} (f \circ \lambda_i)(x_V) \phi_V(x)p_i(x) - \left( \sum_{i=1}^{k} (f \circ \lambda_i)(x)p_i(x) \right) \left( \sum_{V \in \beta} \phi_V(x) \right) \right\| \\
(7.6) & \quad = \left\| \sum_{i=1}^{k} \sum_{V \in \beta} (f \circ \lambda_i)(x_V) \phi_V(x)p_i(x) - \sum_{i=1}^{k} \sum_{V \in \beta} (f \circ \lambda_i)(x) \phi_V(x)p_i(x) \right\| \\
(7.7) & \quad \leq \max_{1 \leq i \leq k} \sum_{V \in \beta} \| (f \circ \lambda_i)(x_V) - (f \circ \lambda_i)(x) \phi_V(x)p_i(x) \| \leq \varepsilon,
\end{align*}
\]

and thus

\[
\varphi_j(\mathcal{F}) \subseteq \varepsilon \{\phi_V|_V; 1 \leq i \leq k, V \in \beta\}.
\]

By Lemma \( \ref{lem:sub-C*-algebra} \) there is a sub-C*-algebra \( S \subseteq A \) such that

\[
\varphi_j(\mathcal{F}) \subseteq \varepsilon S,
\]

and \( S \) has a recursive subhomogeneous C*-algebra decomposition with topological dimension at most \( \text{ord}(\beta) \), and the dimensions of the irreducible representations of \( S \) at most \( n_j = \text{mul}(\tilde{\beta}) \).

By Theorem 5.1 of \( \ref{thm:gamma} \), one has that \( \text{rc}(S) \leq \gamma_c(A)/2 + \varepsilon \). By Lemma \( \ref{lem:gamma} \) one has that \( \text{rc}(A) \leq \gamma_c(A)/2 \), as desired.

\[ \Box \]

**Corollary 7.21.** Let \( A \) be a simple AH-algebra with generalized diagonal maps. If \( A \) has Cuntz mean dimension zero, then the Cuntz semigroup of \( A \) is almost unperforated. In particular, the C*-algebra \( A \) has strict comparison of positive elements.

**Proof.** Using the same argument as that of Theorem 7.20 for any finite subset \( \mathcal{F} \) and any \( \varepsilon > 0 \), there is a sub-C*-algebra \( S \) such that \( \mathcal{F} \subseteq \varepsilon S \) and \( \text{rc}(S) < \varepsilon \). Then, the corollary follows from Lemma \( \ref{lem:gamma} \). \[ \Box \]

**Theorem 7.22.** Let \( A \) be a simple AH-algebra with generalized diagonal maps. If \( A \) has Cuntz mean dimension zero, then \( A \) is isomorphic to an AH-algebra without dimension growth.
Proof. By Corollary 7.21 and Lemma 5.10, the Cuntz semigroup of $A$ is $V(A) \sqcup \operatorname{SAff}(T(A))$. Then, the C*-algebra $A$ is $\mathcal{Z}$-stable and hence is isomorphic to an AH-algebra without dimension growth.

One has the following corollary immediately.

**Corollary 7.23.** Let $A$ be an AH-algebra with generalized diagonal maps with all projections $p_1,\ldots,p_k$ are equivalent (in particular, if all base spaces of $A_i$ are contractible). If $A$ has at most countably many extreme tracial states, or there exists $M > 0$ such that $\rho^{-1}(\kappa)$ has at most $M$ extreme points for any $\kappa \in S(K_0(A))$, then $A$ is isomorphic to an AH-algebra without dimension growth.

**Proof.** It follows from Corollary 3.18 and Theorem 3.24 that $\gamma(A) = 0$. By Remark 7.14 one has that $\gamma_c(A) = \gamma(A) = 0$. Then the AH-algebra $A$ is isomorphic to an AH-algebra without dimension growth by Theorem 7.22.

This corollary can be generalized as the following:

**Definition 7.24.** An AH-algebra with generalized diagonal maps has Property (D) if there is $\delta > 0$ such that any connection map can be induced by a pair $(\lambda_i,p_i)$ with a grouping among $\{p_i\}$

$$\\{\{p_{1,1},\ldots,p_{1,c_1}\},\ldots,\{p_{l,1},\ldots,p_{l,c_l}\}\}$$

such that that any two projection in one group are Murray-von Neumann equivalent and

$$\frac{\sum_{k=1}^{c_j} \operatorname{rank}(p_{j,k})}{\sum_i \operatorname{rank}(p_i)} \geq \delta$$

for any $j$.

**Corollary 7.25.** Let $A$ be an AH-algebra with generalized diagonal maps with Property (D). If $A$ has at most countably many extreme tracial states, or there exists $M > 0$ such that $\rho^{-1}(\kappa)$ has at most $M$ extreme points for any $\kappa \in S(K_0(A))$, then $A$ is isomorphic to an AH-algebra without dimension growth.

**Proof.** Assume that each $X_i$ has only one connected component. Other cases can be proved in a similar way. Note that $\gamma(A) = 0$. Fix $A_i$ and consider an open cover $\alpha$ of $X_i$. Denote by $\delta$ the Property (D) constant for $A$. Fix an arbitrary $\varepsilon > 0$. Since $\gamma(A) = 0$, one may assume that there is an open cover $\beta$ of $X_{i+1}$ such that $\beta \succ \varphi_{i,i+1}(\alpha)$ and $\operatorname{ord}(\beta) < \varepsilon \delta d$, where $d$ is the dimension of the irreducible representation of $A_{i+1}$.

Consider the induced branched cover by $\beta$. One then has

$$\frac{\operatorname{ord}(\beta)}{n_{i+1}(\beta)} \leq \frac{\operatorname{ord}(\beta)}{d \delta} \leq \varepsilon.$$

Thus, $\gamma_c(A) = 0$, and hence $A$ is an AH-algebra without dimension growth, as desired.
8. Variation mean dimension for general AH-algebras

For general AH-algebras, the variation mean dimension is considered in this section. Similar to the mean dimension, the class of AH-algebra with variation mean dimension zero are classifiable. However, the variation mean dimension is not well behaved.

Let $A = \lim_{i} (A_i, \varphi_i)$ be an AH-algebra. For any finite subset $F \subseteq A_i$ and any $\varepsilon > 0$, consider $\text{Cov}(F, \varepsilon)$, the collection of open covers $\alpha$ of $X_i$ satisfying that for any $U \in \alpha$ and any $x, y \in U$, one has that $\|f(x) - f(y)\| < \varepsilon$, $\forall f \in F$. Note that if $\alpha \in \text{Cov}(F, \varepsilon)$ and $\beta \supset \alpha$, then $\beta \in \text{Cov}(F, \varepsilon)$.

**Definition 8.1.** For any $F \in A_i$ and any $\varepsilon > 0$, set
\[
\mathcal{D}(F, \varepsilon) = \min\{\text{ord}(\alpha); \alpha \in \text{Cov}(F, \varepsilon)\},
\]
and set
\[
\nu_i = \sup_{F \subseteq A_i, \varepsilon > 0} \lim_{j \to \infty} \inf \max_{1 \leq k \leq h_j} \frac{\mathcal{D}(\varphi^k_{i,j}(F), \varepsilon)}{n_{j,k}}.
\]
The sequence $(\nu_i)$ is increasing, and the limit $\nu$ is the variation mean dimension of $A$.

**Lemma 8.2.** If $A$ is an AH-algebra with diagonal maps, then $\gamma(A) = \nu(A)$.

**Proof.** For any $F \subseteq A_i$, $\varepsilon > 0$, there is an open cover $\alpha$ of $X_i$ such that for any $U \in \alpha$ and any $x, y \in U$, $\|f(x) - f(y)\| \leq \varepsilon$ for any $f \in F$. Since all connection maps of $A$ are diagonal, one has $\varphi^k_{i,j}(\alpha) \in \text{Cov}(\varphi^k_{i,j}(F), \varepsilon)$, and hence
\[
\mathcal{D}(\varphi^k_{i,j}(\alpha)) \geq \mathcal{D}(\varphi^k_{i,j}(F), \varepsilon).
\]
Therefore, $\gamma(A) \geq \nu(A)$.

On the other hand, for any $\varepsilon > 0$, consider any open cover $\alpha$ of $X_i$, and denote by $d = \text{ord}(\alpha)$.

Pick a partition of unit $\{\psi_U; U \in \alpha\}$ subordinate to $\beta$, and regard them as central elements of $A_i$. Set $F = \{\psi_U; U \in \beta\} \subseteq A_i$.

Assume there is an open subset $V$ such that $\forall x, y \in V$, $\|f(x) - f(y)\| < 1/(d + 1)$, $\forall f \in F$. Since $\text{ord}(\alpha) = d$, for any $x \in X_i$, there exists $\psi_U$ such that $\psi_U(x) \geq 1/(d + 1)$ (note that $\sum \psi_U(x) = 1$).

Fix $x_0 \in V$ and $U_0 \in \beta$ such that $\psi_{U_0}(x_0) \geq 1/(d + 1)$. Since for any $x \in V$,
\[
\|\psi_U(x) - \psi_{U_0}(x_0)\| \leq 1/(d + 1),
\]
ones that $\phi_{U_0}(x) > 0$ and hence $x \in U_0$. Therefore, $V \subseteq U_0$.

Thus, for any open cover $\alpha \in \text{Cov}(F, 1/(d + 1))$, one has that $\alpha \supset \alpha$, and hence
\[
\mathcal{D}(F, 1/(d + 1)) \geq \mathcal{D}(\alpha).
\]
Consider $A_j$ with $j > i$. If there is an open subset $V \subseteq X_j$ such that $\forall x, y \in V, \forall f \in F$,
\[
\|\varphi^k_{i,j}(f)(x) - \varphi^k_{i,j}(f)(y)\| < 1/(d + 1),
\]
then
\[
\|\psi_U \circ \lambda^{i,k}_{i,j}(m)(x) - \psi_U \circ \lambda^{i,k}_{i,j}(m)(y)\| \leq 1/(d + 1)
\]
for all $U \in \alpha$, $1 \leq l \leq h_i$, and $1 \leq m \leq m_{i,j}^{l,k}$.

For any $l$ and $m$, fix $x_0 \in V$ and $U_0 \in \alpha$ such that

$$\psi_{U_0}(\lambda_{i,j}^{l,k}(m)(x_0)) \geq 1/(d + 1).$$

Hence one has

$$\lambda_{i,j}^{l,k}(m)(x) \in U_0 \quad \forall x \in V,$$

and

$$V \subseteq (\lambda_{i,j}^{l,k}(m))^{-1}(U_0).$$

Therefore

$$V \in \varphi_{i,j}^k(\alpha).$$

Thus, for any open cover $o \in \text{Cov}(\varphi_{i,j}^k(F), 1/(d + 1))$, one has that $o \succ \varphi_{i,j}^k(\alpha)$, and

$$D(\varphi_{i,j}^k(F), 1/(d + 1)) \geq D(\varphi_{i,j}^k(\alpha)).$$

Hence $\nu(A) \geq \gamma(A)$, as desired. \qed

Remark 8.3. In general, the mean dimension might not equal to the variation mean dimension. For example, consider the Villadsen’s algebras of second type ([23]). These AH-algebras have zero mean dimension zero, but its variation mean dimension is nonzero (Theorem 8.5).

Using the same argument as that of Theorem 4.2, one has the following approximation theorem:

Theorem 8.4. Let $A$ be an AH-algebra, and denote by $\nu$ the variation mean dimension of $A$. Then for any finite subset $F \subseteq A$ and any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists a unital sub-$\text{C}^*$-algebra

$$C \cong \bigoplus_p M_{n_i}(C(\Omega_i))p_i \subseteq A$$

such that $F \subseteq \varepsilon_1 C$, and

$$\frac{\text{rank}(p_i)}{n_i} < \gamma + \varepsilon_2.$$

Proof. The proof is the same as that of Theorem 4.2 with the following small modification (the same notation as those in the proof of Theorem 4.2 are used): Once the complex $\Delta$ is constructed, consider the map $\xi : X_2 \to \Delta$. Define $C$ to be

$$C := \{f \in A_2; \text{f is constant on } \xi^{-1}(y) \text{ for any } y \in \Delta \} \subseteq A_2 \subseteq A.$$

Then $C$ is the desired sub-$\text{C}^*$-algebra. \qed

Using this approximation theorem, one has the following theorems:

Theorem 8.5. If a simple AH-algebra $A$ has variation mean dimension zero, then it has strict comparison of positive elements, and hence is classifiable.

Theorem 8.6. If an AH-algebra $A$ has variation mean dimension $\nu$, then $\text{rc}(A) \leq \nu/2$.

Remark 8.7. The variation mean dimension seems depend on the inductive decomposition. Even if one replaces the connection maps between building blocks by map which are unitarily equivalent (this will not change the inductive limit algebra), the variation mean dimension might change.
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