VERTEX CONTROL OF FLOWS IN NETWORKS

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Abstract. We study a transport equation in a network and control it in a single vertex. We describe all possible reachable states and prove a criterion of Kalman type for those vertices in which the problem is maximally controllable. The results are then applied to concrete networks to show the complexity of the problem.

1. Introduction. To motivate the problem treated in this paper we may start from the following situation. Consider a closed network of tubes (such as a system of wires or a circuit) in which material (electrons, information packets, goods) is flowing with constant speed $c_j$ on each edge $e_j$ with no friction or loss. In the nodes $v_i$ of the network the material is redistributed into the tubes according to certain weights $\omega_{ij}$ satisfying a Kirchhoff law. Simplifying the physical laws and concentrating on the structure of the network, this situation can be described by a system of linear transport equations on the edges

$$\frac{\partial}{\partial t} x_j(t, s) = c_j \frac{\partial}{\partial s} x_j(t, s)$$

and conditions in the vertices saying that

outgoing flow on edge $e_j = \omega_{ij} \sum$ incoming flows into vertex $v_i$
for every edge $e_j$ leaving from vertex $v_i$ (see Section 3 for precise definitions). The authors of [9] proposed a semigroup approach to the study of such a system. Combining graph theoretical and functional analytic tools, they proved well-posedness and described the asymptotic behavior of the solutions by the structure of the underlying graph. Based on their results we ask the following question:

*Which states (mass distributions) in the network can be approximately reached by controlling the flow (by adding or subtracting material) in a single vertex?*

Let us first look at a very simple example. Consider the network described by the directed and weighted graph presented in Figure 1 with all $c_j = 1$ and the weights $\alpha$ and $1 - \alpha$ representing the proportions of the mass leaving the vertex $v_1$ into the edges $e_1$ and $e_3$, respectively.

![Figure 1. The graph $G_1$](image)

This simple network already shows two typical phenomena that occur in general.

1. The mass distributions on the edges $e_1$ and $e_3$ will always satisfy the ratio $\frac{\alpha}{1 - \alpha}$. Therefore not every mass distribution on the edges can be attained.
2. Taking 1 into account, all other distributions can be achieved if we control in the vertices $v_2$ or $v_3$ but not by controlling in $v_1$ or $v_4$ (see Example 5.2).

In this paper we give a complete description of the maximal space of reachable states for any given network (see Lemma 4.1). Moreover, we characterize by a finite dimensional Kalman-type condition those vertices from where this maximal space can be attained (see Theorem 4.4).

It seems that the researchers in graph theory have not yet investigated the properties of these vertices. The examples in Section 5 demonstrate their interesting and complex behavior and we believe they deserve a thorough treatment (see also open problems in Section 6).

Let us remark that controlling the network in more than one vertex is obviously an easier task which can be also seen from the modified Kalman condition in Corollary 4.7.

We point out that the analogous control problem for the wave instead of the transport equation on a graph has been studied intensively. We refer to the systematic treatment in [4]. However, a deeper connection between the results in the two cases remains to be unveiled.
The paper is organized in the following way. In Section 2 we start by some basic definitions from control theory (we refer to [2] and [5] for more detailed explanation). To tackle our network control problem we use the abstract results from [6]. In Section 3 we collect terminology and results for the study of flows in networks. We use the semigroup approach as developed in [9], [10], [11], and [3]. For graph theoretical notions we refer to [1] and [7], while those needed here are all defined in [9]. We introduce our network control problem in Section 4. We apply the abstract results to vertex control in networks obtaining Theorem 4.4 as the main result. In Section 5 we explain our results for concrete graphs. In Section 6 we conclude with some open problems addressing a community of researchers from different areas.

2. Abstract boundary control. We start by recalling some notions from abstract control theory.

Abstract Framework 2.1. We consider

(i) three Banach spaces \( X, \partial X \) and \( U \) called the state, boundary and control space, respectively;
(ii) a closed, densely defined system operator \( A_m : D(A_m) \subseteq X \rightarrow X \);
(iii) a boundary operator \( Q \in \mathcal{L}(\{D(A_m)\}, \partial X) \);
(iv) a control operator \( B \in \mathcal{L}(U, \partial X) \).

The abstract boundary control system \( \Sigma_{\text{BC}}(A_m, B, Q) \) associated to the abstract Cauchy problem with boundary control on the Banach space \( X \) with boundary space \( \partial X \) and control space \( U \) is defined as

\[
\begin{aligned}
\dot{x}(t) &= A_m x(t), & t &\geq 0, \\
Q x(t) &= B u(t), & t &\geq 0, \\
x(0) &= x_0.
\end{aligned}
\tag{2}
\]

The function \( u \in L^1_{\text{loc}}(\mathbb{R}_+, U) \) and a function \( x(\cdot) = x(\cdot, x_0, u) \in C^1(\mathbb{R}_+, X) \) with \( x(t) \in D(A_m) \) for all \( t \geq 0 \) satisfying (2) is called a classical solution.

We are mainly concerned to describe all states a given system can possibly attain. Therefore we define

Definition 2.2. The approximate reachability space associated to (2) is

\[
\mathcal{R}_{\text{BC}} := \overline{\{ y \in X \mid \exists t > 0 \text{ and } u(\cdot) \in L^1([0, t], U) \text{ such that } y = x(t), \text{ where } x(\cdot) \text{ is the solution of (2) with } x(0) = 0 \}}.
\tag{3}
\]

The boundary control system \( \Sigma_{\text{BC}}(A_m, B, Q) \) is called approximately boundary controllable, if \( \mathcal{R}_{\text{BC}} = X \).

In the following Lemma we collect some results due to Greiner [8, Lem. 1.2, Lem. 1.3] we shall need to describe the space \( \mathcal{R}_{\text{BC}} \).

Lemma 2.3. Assume the following properties to hold.

(i) The restricted operator \( A \subseteq A_m \) with domain \( D(A) := \ker Q \) generates a strongly continuous semigroup \( (T(t))_{t \geq 0} \) on \( X \);
(ii) the boundary operator \( Q : D(A_m) \rightarrow \partial X \) is surjective.

Then for each \( \lambda \in \rho(A) \), \( D(A_m) = D(A) \oplus \ker(\lambda - A_m) \), the operator \( Q|_{\ker(\lambda - A_m)} \) is invertible and the inverse \( Q_\lambda := (Q|_{\ker(\lambda - A_m)})^{-1} : \partial X \rightarrow \ker(\lambda - A_m) \subseteq X \) is bounded.
For $\lambda \in \rho(A)$ we call the operator $Q_\lambda$, introduced in Lemma 2.3, the Dirichlet operator and define

$$B_\lambda := Q_\lambda B \in \mathcal{L}(U, \ker(\lambda - A_m)).$$

Recall that the spectral bound of $A$ is defined as $\omega_0(A) = \sup\{\Re \lambda : \lambda \in \sigma(A)\}$. By $\rg(C)$ we denote the range of an operator $C$. The following has been shown in [6].

**Theorem 2.4.** The approximate reachability space $\mathcal{R}^{BC}$ of $\Sigma_{BC}(A_m, B, Q)$ coincides with

(i) the smallest closed, $(T(t))_{t \geq 0}$-invariant subspace of $X$ containing $\rg(B_\mu)$ for all $\mu$ sufficiently large,

(ii) the smallest closed, $R(\mu, A)$-invariant, $\mu > \omega > \omega_0(A)$, subspace of $X$ containing $\rg(B_\mu)$ for all $\mu$ sufficiently large, and

(iii) $\bigcup_{\lambda > \omega} \rg(B_\lambda)$ for some $\omega > \omega_0(A)$.

The approximate reachability space is related to a subspace, which is independent of the specific control operator.

**Definition 2.5.** The maximal reachability space associated to (2) is

$$\mathcal{R}^{BC}_{\max} := \bigcup_{\lambda > \omega_0(A)} \ker(\lambda - A_m).$$ (4)

The system $\Sigma_{BC}(A_m, B, Q)$ is called maximally controllable if $\mathcal{R}^{BC} = \mathcal{R}^{BC}_{\max}$.

It has been shown in [6] that $\mathcal{R}^{BC} \subset \mathcal{R}^{BC}_{\max}$, so $\mathcal{R}^{BC}_{\max}$ is indeed the largest possible space of states that can be approximately reached by applying some boundary control $B$. We will see that the relevant question for controllability of our network problem is when are these two spaces equal.

3. Flows in networks. We consider a finite network modeled by a simple, directed graph. We denote by $V = \{v_1, \ldots, v_n\}$ the set of vertices and by $E = \{e_1, \ldots, e_m\}$ the set of (directed) edges of the graph. The edges are parameterized on the interval $[0, 1]$, to the contrary of their directions. The vertex $v_j(0)$ is thus called the head and the vertex $v_j(1)$ the tail of the edge $e_j \in E$. The edge $e_j$ is an incoming edge for the vertex $v_i$ if $v_i = e_j(0)$ holds, and it is called an outgoing edge for $v_i$ if $v_i = e_j(1)$ holds. We assume that in every vertex there is at least one incoming as well as at least one outgoing edge.

We will use the following graph matrices (see [9, Sect. 1]) to describe the structure of the network.

**Definition 3.1.**

(i) The outgoing incidence matrix $\Phi^- = (\phi^-_{ij})_{n \times m}$ has entries

$$\phi^-_{ij} := \begin{cases} 1, & v_i \lttext{tail of } e_j, \\ 0, & \text{else}; \end{cases}$$

(ii) The incoming incidence matrix $\Phi^+ = (\phi^+_{ij})_{n \times m}$ has entries

$$\phi^+_{ij} := \begin{cases} 1, & v_i \lttext{head of } e_j, \\ 0, & \text{else}; \end{cases}$$

(iii) The weighted outgoing incidence matrix is $\Phi^-_w = (\omega^-_{ij})_{n \times m}$, where $0 \leq \omega^-_{ij} \leq 1$ satisfy $\omega^-_{ij} = 0$ $\iff$ $\phi^-_{ij} = 0$ and $\sum_{j=1}^m \omega^-_{ij} = 1$ for all $i = 1, \ldots, n$.
(iv) The weighted adjacency matrix is defined by 
\[ A = (a_{ik})_{n \times n} := \Phi^+(\Phi_w)\top; \]

(v) The weighted adjacency matrix of the line graph is defined as 
\[ B = (b_{lj})_{m \times m} := (\Phi_w - \omega)^\top \Phi; \]

As examples for the graph matrices \( \Phi_w \) and \( A \) we refer to Example 4.2 in Section 5.

Remark 3.2. Simple computations show that
\[ \Phi^- (\Phi_w^-)^\top = I_{C^n} \]
and that both adjacency matrices \( A \) and \( B \) are column stochastic hence
\[ \|A\|_1 = 1 \quad \text{and} \quad \|B\|_1 = 1. \]

Furthermore, the relation
\[ (\Phi_w^-)^\top A = B (\Phi^-)\top \]
holds.

The mathematical model for flows in networks is as follows (see [9, Sect. 1]).
- We consider transport equations on the \( m \) edges of the graph:
\[ \frac{\partial}{\partial t} x_j(t, s) = c_j \frac{\partial}{\partial s} x_j(t, s), \]
where \( c_j > 0 \) is the velocity of the flow on the edge \( e_j \).
- The boundary conditions say that in each vertex \( v_i \) the incoming flow is distributed on the outgoing edges by fixed proportions \( \omega_{ij}^- \), called the weights of the edges \( e_j \) in vertex \( v_i \):
\[ \phi_{i1}^- x_j(t, 1) = \omega_{ij}^- \sum_{k=1}^{m} \phi_{ik}^+ x_k(t, 0). \]
Observe that by the assumption in Definition 3.1(iii) this implies that the Kirchhoff law is satisfied, i.e., the total incoming flow mass equals the total outgoing flow mass.
- We further need initial conditions on the edges:
\[ x_j(0, s) = f_j(s). \]

In the above formulas
\[ t \geq 0 \] is the time variable,
\[ s \in [0, 1] \] is the space variable on the edges,
\[ j = 1, \ldots, m \] are the indices of edges, and
\[ i = 1, \ldots, n \] are the indices of vertices.

We now rewrite this in the form of an abstract Cauchy problem
\[ \begin{cases} \dot{x}(t) = A x(t), & t \geq 0, \\ x(0) = f, \end{cases} \]
on \( X \), where
- \( X := L^1([0, 1], \mathbb{C}^m) \),
- \( A := \text{diag}(c_j \frac{d}{ds})_{j=1,\ldots,m} \) with the domain (see [9, Sect. 2])
\[ D(A) := \{ g \in W^{1,1}([0, 1], \mathbb{C}^m) \mid g(1) \in \text{rg}(\Phi_w)\top \text{ and } \Phi^- g(1) = \Phi^+ g(0) \}, \]
- \( x(t) = x(t, \cdot), \ f = (f_1, \ldots, f_m)\top \).
In the domain of \( A \), the first condition
\[
g(1) \in \text{rg}(\Phi^-)^	op
\]
means that in every vertex the total incoming flow is distributed in (given) weighted proportions to the outgoing edges. The second condition
\[
\Phi^- g(1) = \Phi^+ g(0)
\]
is the Kirchhoff’s law in each vertex.

By \([9, \text{Cor. 2.7}]\) this problem is well-posed. Furthermore, it was shown in \([3]\) that in case when all the velocities \( c_j \) coincide, we can explicitly describe the semigroup governing the problem. For this reason and in order to simplify our control problem, we assume in the following that
\[
c_j = 1, \ j = 1, \ldots, m.
\]

**Proposition 3.3.** Let (12) holds. Then the domain (11) of the operator \( A \) can be written as
\[
D(A) = \{ g \in W^{1,1}([0,1],C^m) \mid g(1) = B g(0) \}
\]
and \( (A, D(A)) \) generates the strongly continuous semigroup
\[
[T(t)g](s) = B^g(t + s - n) \quad \text{if} \quad t + s \in [n, n + 1) \quad \text{for} \quad n \in \mathbb{N},
\]
where \( B^0 := I \). Moreover, the spectral bound \( \omega_0(A) = 0 \).

**Proof.** See \([3, \text{Propositions 3.1 and 3.4}]\) and \([9, \text{Corollary 3.5}]\). \(\square\)

**Remark 3.4.** If needed one may work in the (reflexive) space \( X_p := L^p([0,1],C^m) \), \( 1 < p < \infty \), where the same proposition holds for the bounded (but not necessarily contractive) flow semigroup.

4. **Vertex control in networks.** Now we focus on maximal controllability of flows in networks by controls acting in one of the vertices only. Throughout the section we will assume that the condition (12) holds.

We start by (10) and add appropriate boundary and control operators to obtain an abstract Cauchy problem with boundary control. In our setting \( X = L^1([0,1],C^m) \) is the state space for our problem while the boundary space is \( \partial X := C^n \) corresponding to the vertices of the graph. We further need the following notations and results from \([9]\).

The *outgoing boundary operator* \( L : X \to \partial X \) is
\[
Lg := \Phi^- g(1), \quad D(L) := W^{1,1}([0,1],C^m),
\]
while the *incoming boundary operator* \( M : X \to \partial X \) is
\[
Mg := \Phi^+ g(0), \quad D(M) := W^{1,1}([0,1],C^m).
\]
Both operators are bounded and map from the Banach space \( W^{1,1}([0,1],C^m) \) to \( C^n \). Then the domain of \( A \) defined in (11) can be written as
\[
D(A) = \{ g \in W^{1,1}([0,1],C^m) \mid g(1) \in \text{rg}(\Phi^-)^	op \text{ and } (L - M)g = 0 \}.
\]
Hence, defining \( A_m = \frac{d}{dt} \) with domain
\[
D(A_m) = \{ g \in W^{1,1}([0,1],C^m) \mid g(1) \in \text{rg}(\Phi^-)^	op \}
\]
and the *boundary operator*
\[
Q := L - M \in \mathcal{L}([D(A_m)],\partial X),
\]
the abstract Cauchy problem \((10)\) obtains the form
\[
\begin{align*}
\dot{x}(t) &= A_m x(t), \quad t \geq 0, \\
Q x(t) &= 0, \quad t \geq 0, \\
x(0) &= x_0.
\end{align*}
\]

Finally, we impose control in the vertex \(v = v_i \in V\) for some fixed \(i \in \{1, \ldots, n\}\). In the following we identify this vertex with a vector \(v \in \mathbb{C}^n\)
\[
v = \begin{pmatrix}
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{pmatrix} \quad \leftarrow i^{th}.
\]

As (one dimensional) control space \(U\) and control operator \(B\) we choose
\[
U := \mathbb{C}, \quad B : U \rightarrow \text{span}\{v\} \subset \partial X = \mathbb{C}^n,
\]
where \(B\) is any (bounded) linear operator acting between the given vector spaces. With these notations we arrive at an abstract Cauchy problem with boundary control of the form \((2)\).
\[
\begin{align*}
\dot{x}(t) &= A_m x(t), \quad t \geq 0, \\
Q x(t) &= B u(t), \quad t \geq 0, \\
x(0) &= x_0.
\end{align*}
\] (15)

Now we can apply the abstract results from Section 2 to our problem. Since the eigenvectors of \(A_m\) have to satisfy the boundary conditions in the vertices (c.f. \((13)\)), it follows that in general \(\mathcal{R}_{\text{max}}^{\text{BC}}\) from \((4)\) cannot be equal to the state space \(X = L^1([0, 1], \mathbb{C}^n)\) (see section 5 for some concrete examples). Our aim is to find out when \(\mathcal{R}_{\text{max}}^{\text{BC}} = \mathcal{R}_{\text{max}}^{\text{BC}}\) can be achieved, i.e., when the system is maximally controllable. For this purpose we will give explicit descriptions of \(\mathcal{R}_{\text{max}}^{\text{BC}}\) and \(\mathcal{R}_{\text{max}}^{\text{BC}}\) in terms of the graph matrices. In the following \(\varepsilon_\lambda\) denotes the exponential function
\[
\varepsilon_\lambda(s) := e^{\lambda s} \text{ for } s \in [0, 1] \text{ and some } \lambda \in \mathbb{C}.
\]

**Lemma 4.1.** The maximal reachability space \(\mathcal{R}_{\text{max}}^{\text{BC}}\) is equal to
\[
\mathcal{R}_{\text{max}}^{\text{BC}} = \text{span} \left\{ \begin{pmatrix} a_1 g \\ \vdots \\ a_m g \end{pmatrix} \bigg| g \in L^1([0, 1], \mathbb{C}) \text{ and } \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \text{rg}(\Phi_w^{-\top}) \right\}.
\]

Briefly,
\[
\mathcal{R}_{\text{max}}^{\text{BC}} = L^1([0, 1], \mathbb{C}) \otimes \text{rg}(\Phi_w^{-\top}) = L^1([0, 1], \mathbb{C}) \otimes (\Phi_w^{-\top})\mathbb{C}^n. \quad (16)
\]

**Proof.** Using [9, p. 147] we have that
\[
\ker(\lambda - A_m) = \left\{ \begin{pmatrix} a_1 \varepsilon_\lambda \\ \vdots \\ a_m \varepsilon_\lambda \end{pmatrix} \bigg| \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \text{rg}(\Phi_w^{-\top}) \right\}. \quad (17)
\]

Observe that by the Stone-Weierstrass theorem
\[
\text{span} \bigcup_{\lambda > \omega_0(A)} \{ \varepsilon_\lambda \} = L^1([0, 1], \mathbb{C}),
\]
hence we are done. \qed

In order to describe \( \mathcal{R}^{BC} \) using Theorem 2.4(iii) we will need the form of the operators \( B_\lambda = Q_\lambda B \) for \( \lambda \) large enough. We start with

\[
Q_\lambda = (Q_{\ker(\lambda - A_m)})^{-1}
\]

from Lemma 2.3 and compute it for the boundary operator \( Q \) defined in (14).

**Lemma 4.2.** For \( \lambda > 0 = \omega_0(A) \) we have

\[
Q_\lambda = \varepsilon_\lambda (\Phi_w^{-})^\top R (e^\lambda, \lambda),
\]

where \( R (e^\lambda, \lambda) := (e^\lambda - \lambda)^{-1} \) denotes the resolvent of \( \lambda \) in \( e^\lambda \).

**Proof.** First compute

\[
(L - M)\varepsilon_\lambda (\Phi_w^{-})^\top R (e^\lambda, \lambda) = e^\lambda \Phi^{-} (\Phi_w^{-})^\top R (e^\lambda, \lambda) - \Phi^{+} (\Phi_w^{-})^\top R (e^\lambda, \lambda)
\]

\[
= e^\lambda R (e^\lambda, \lambda) - \lambda R (e^\lambda, \lambda) = I_{\mathbb{C}^n},
\]

where we used (5).

By (17), taking any \( f \in \ker(\lambda - A_m) \) we have

\[
f = \begin{pmatrix} a_1 \varepsilon_\lambda \\ \vdots \\ a_m \varepsilon_\lambda \end{pmatrix} \quad \text{for some} \quad \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = (\Phi_w^{-})^\top d, \; d \in \mathbb{C}^n.
\]

Hence, using \( \|e^{-\lambda}\lambda\|_1 < 1 \) for \( \lambda > 0 \), Definition 3.1, (5) and (6) we have

\[
\varepsilon_\lambda (\Phi_w^{-})^\top R (e^\lambda, \lambda) (L - M) f = \varepsilon_\lambda (\Phi_w^{-})^\top R (e^\lambda, \lambda) (\Phi^{-} f(1) - \Phi^{+} f(0))
\]

\[
= \varepsilon_\lambda e^{-\lambda} (\Phi_w^{-})^\top \sum_{k=0}^{\infty} e^{-\lambda k} A_k \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} - \Phi^{+} f(0)
\]

\[
= \varepsilon_\lambda e^{-\lambda} \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} - \sum_{k=0}^{\infty} e^{-\lambda k} B_k (\Phi_w^{-})^\top d - \sum_{k=0}^{\infty} e^{-\lambda k} B_{k+1} f(0)
\]

\[
= \varepsilon_\lambda \begin{pmatrix} a_1 \varepsilon_\lambda \\ \vdots \\ a_m \varepsilon_\lambda \end{pmatrix} = f.
\]

We have thus shown that

\[
\varepsilon_\lambda (\Phi_w^{-})^\top R (e^\lambda, \lambda) (L - M) = I_{\ker(\lambda - A_m)},
\]

hence we are done. \qed

Using Theorem 2.4(iii) we obtain the following.
Corollary 4.3. There exists $\omega > 0$ such that
\[
\mathcal{R}_{BC}^{\mathcal{R}} = \overline{\text{span}} \bigcup_{\lambda > \omega} \{\varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) v\} \quad (18)
\]
\[
= L^1 ([0, 1], \mathbb{C}) \otimes (\Phi_w^-)^T \left( \text{span} \{v, Av, \ldots, A^{n-1}v\} \right). \quad (19)
\]
Proof. We only have to prove the second equality. By Proposition 3.3 together with Theorem 2.4 we have
\[
T(1) (\varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) v) = \varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) v \in \mathcal{R}_{BC}.
\]
Using (6) we obtain
\[
\varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) Av \in \mathcal{R}_{BC}.
\]
Applying $T(1)$ to this vector again yields
\[
T(1) (\varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) Av) = \varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) A^2v \in \mathcal{R}_{BC}.
\]
Continuing this procedure we obtain that
\[
\varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) A^k v \in \mathcal{R}_{BC}, k = 0, 1, \ldots
\]
Since $\mathcal{R}_{BC}$ is a linear subspace, we also have that
\[
e^\lambda \cdot \varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) A^k v - \varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) A^{k+1}v
\]
\[
= \varepsilon_\lambda (\Phi_w^-)^T R (e^\lambda, A) (e^\lambda - A) A^k v
\]
\[
= \varepsilon_\lambda (\Phi_w^-)^T A^k v \in \mathcal{R}_{BC}, k = 0, 1, \ldots, n - 1.
\]
Using the Stone-Weierstrass theorem and the Neumann series expansion of $R (e^\lambda, A)$, we finally obtain the result.

Let us emphasize that the space $\mathcal{R}_{BC}^{\mathcal{R}}$ consists of all possible states in the network and is independent of the control. The space $\mathcal{R}_{BC} \subseteq \mathcal{R}_{BC}^{\mathcal{R}}$ however depends on the specific control operator, in our case on the vertex in which the control takes place. We are now able to characterize the vertices in which these two spaces coincide.

Theorem 4.4. The following assertions are equivalent for a vertex $v$.
(a) $\mathcal{R}_{BC}^{\mathcal{R}} = \mathcal{R}_{BC}^{\mathcal{R}}$, i.e., the flow is maximally controllable in the vertex $v$.
(b) $\text{span} \{v, Av, \ldots, A^{n-1}v\} = \mathbb{C}^n$.

Proof. Using (16) and (19), (a) is equivalent to
\[
\mathcal{R}_{\text{max}}^{\mathcal{R}} = L^1 ([0, 1], \mathbb{C}) \otimes (\Phi_w^-)^T \mathbb{C}^n
\]
\[
= L^1 ([0, 1], \mathbb{C}) \otimes (\Phi_w^-)^T \left( \text{span} \{v, Av, \ldots, A^{n-1}v\} \right)
\]
\[
= \mathcal{R}_{BC}.
\]
Since $(\Phi_w^-)^T$ is injective hence left invertible, this holds if and only if
\[
\text{span} \{v, Av, \ldots, A^{n-1}v\} = \mathbb{C}^n.
\]

Remark 4.5. Assertion (b) in Theorem 4.4 is a Kalman-type condition, well-known in control theory. In our situation, it guarantees that by controlling in the vertex $v$ the largest possible space of mass distributions in the network can be (approximately) reached.
In concrete situations (and for large graphs) it may be quite difficult to verify this Kalman-type criterion. In particular, it depends on the structure of the graph and on the distribution of the weights $\omega_{ij}$ (see Example 5.6 below). Therefore it is desirable to have a sufficient condition for maximal controllability which can be seen directly (and only) from the graph.

**Remark 4.6.** If there exists a vertex $v_j$ in $G$ such that the shortest (directed) path between $v_i$ and $v_j$ has length $n-1$, then the condition (b) in Theorem 4.4 is satisfied for the vertex $v_i$ (see e.g. [7, Lemma 2.5]).

This condition of the graph can be tested in linear time if the vertex is given. Unfortunately it is far from being also necessary (see Example 5.3 or 5.6).

Finally, let us mention that the proof of Theorem 4.4 can be easily adopted to the case when the control takes place in more than one vertex, obtaining the following obvious modification of the Kalman condition.

**Corollary 4.7.** Assume that we control in the vertices $v_{i_1}, \ldots, v_{i_k}$, and for the velocities (12) holds. Then the following assertions are equivalent.

(a) $R^{BC} = R^{BC}_{\text{max}}$.

(b) $\text{span} \{v_{i_1}, \mathcal{A}v_{i_1}, \ldots, \mathcal{A}^{n-1}v_{i_1}, \ldots, v_{i_k}, \mathcal{A}v_{i_k}, \ldots, \mathcal{A}^{n-1}v_{i_k}\} = \mathbb{C}^n$.

5. **Examples.** We conclude with some examples of networks showing the complexity of our problem already for small graphs.

**Example 5.1.** Starting with the basic graph $C_4$, the undirected cycle on 4 vertices, we first note that there is only one possible orientation of the edges yielding a strongly connected directed graph $G_0$, see Figure 2.

![Figure 2. The graph $G_0$](image)

By checking condition (b) of Theorem 4.4 it follows easily that our problem is maximally controllable in all the vertices of $G_0$. It is also obvious that the condition in the Remark 4.6 holds for every vertex of $G_0$.

**Example 5.2.** Now let us orient the edges of $C_4$ in a different way and add an extra edge from $v_4$ to $v_1$. We take the weights on the edges as

$\omega_{11}^1 = \alpha, \omega_{22}^1 = 1, \omega_{13}^1 = 1 - \alpha, \omega_{44}^1 = \omega_{45}^1 = 1$ for some $0 < \alpha < 1$,

and denote the network thus obtained by $G_1$, see Figure 1 (in the Introduction).
Note that the network $G_1$ is strongly connected and its incidence and adjacency matrices are

\[
(\Phi^-_w)^\top = \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 - \alpha & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\text{ and } A = \begin{pmatrix}
0 & 0 & 0 & 1 \\
\alpha & 0 & 0 & 0 \\
1 - \alpha & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}.
\]

By Lemma 4.1,

\[
\mathcal{R}_{\text{max}}^{BC} = \text{span} \left\{ \begin{pmatrix} a_1 g \\ \vdots \\ a_5 g \end{pmatrix} \middle| g \in L^1([0, 1], \mathbb{C}), a_i \in \mathbb{C}, \frac{a_3}{a_1} = \frac{1 - \alpha}{\alpha} \right\}.
\]

Verifying the Kalman-type condition in Theorem 4.4 we obtain that

\[
\mathcal{R}_{\text{max}}^{BC} = \mathcal{R}^{BC} \iff v = v_2 \text{ or } v = v_3.
\]

So, we can control the flow in the network $G_1$ only in the vertices $v_2$ or $v_3$. Also in this case, the condition from Remark 4.6 is satisfied for the vertices $v_2$ and $v_3$ – the shortest directed path between them in both directions has length 3.

**Example 5.3.** Let $G_2$ be the network obtained from $G_1$ by inserting an edge from $v_2$ to $v_3$ and taking the weights

$\omega_{11} = \alpha$, $\omega_{22} = 1 - \beta$, $\omega_{13} = 1 - \alpha$, $\omega_{34} = \omega_{45} = 1$, $\omega_{26} = \beta$ for some $0 < \alpha, \beta < 1$,

see Figure 3.

![Figure 3. The graph $G_2$](image)

Note that this is a directed version of $K_4$, the complete graph with 4 vertices or the tetrahedron graph. The appropriate adjacency matrix is

\[
A = \begin{pmatrix}
0 & 0 & 0 & 1 \\
\alpha & 0 & 0 & 0 \\
1 - \alpha & \beta & 0 & 0 \\
0 & 1 - \beta & 1 & 0
\end{pmatrix}.
\]
The maximal reachability space in this case is
\[
\mathcal{R}_{BC}^{\max} = \text{span} \left\{ \begin{pmatrix} a_1g \\ \vdots \\ a_6g \end{pmatrix} \middle| g \in L^1([0, 1], \mathbb{C}), \ a_i \in \mathbb{C}, \frac{a_3}{a_1} = \frac{1 - \alpha}{\alpha}, \frac{a_6}{a_2} = \frac{\beta}{1 - \beta} \right\}.
\]

The condition (b) in Theorem 4.4 holds for every vertex hence the problem is maximally controllable in each of the vertices. Here only the vertex \( v_3 \) has the property from the Remark 4.6.

**Example 5.4.** Let us see what happens by inserting more vertices. Take \( G_1 \) and insert a new vertex \( v_5 \) on the edge \( e_5 \), thus obtaining the network \( G_3 \) shown in Figure 4.

![Figure 4. The graph \( G_3 \)](attachment:image)

We leave it to the reader to write down the appropriate matrices and see that the problem remains maximally controllable only in the vertices \( v_2 \) or \( v_3 \). Again, the vertices \( v_2 \) and \( v_3 \) satisfy the condition from Remark 4.6.

**Example 5.5.** The situation becomes completely different by adding one more vertex to \( G_3 \). Let \( G_4 \) be the network presented in Figure 5, for some \( 0 < \alpha, \beta < 1 \).

![Figure 5. The graph \( G_4 \)](attachment:image)
Checking the Kalman-type condition for this graph we obtain
\[ R_{BC}^{\text{max}} \neq R_{BC} \text{ for all vertices } v_1, \ldots, v_6. \]
Thus we do not have control in any of the vertices! Observe that also none of the vertices has the property described in Remark 4.6.

**Example 5.6.** At the end we give an example on the impact of the weights on the edges to our problem. We add two more edges to \( G_3 \) leaving the vertex \( v_5 \) and obtain an oriented version of the graph \( W_4 \), known as the *wheel on 4 vertices*. Let \( 0 < \alpha, \beta, \gamma < 1 \) be arbitrary numbers such that \( \beta + \gamma < 1 \).

![Figure 6. The graph \( G_5 \)](image)

The graph \( G_5 \) presented in Figure 6 admits the following adjacency matrix.

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & \gamma \\
\alpha & 0 & 0 & 0 & \beta \\
1 - \alpha & 0 & 0 & 0 & 1 - \beta - \gamma \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

One can easily compute that, according to Theorem 4.4, this network is maximally controllable in the vertices \( v_2 \) and \( v_3 \), independently of the particular choice of the weights. It is not controllable in \( v_4 \) and \( v_5 \), also independently of the particular choice of the weights. However, \( G_5 \) is controllable in \( v_1 \) if and only if

\[ \alpha - \beta - \alpha \cdot \gamma \neq 0. \]

Hence, controllability in \( v_1 \) depends on the weights of the edges. Note that the condition from Remark 4.6 is independent of the weights and is not fulfilled for any of the vertices of \( G_5 \).

6. **Open problems.** The general results (Theorem 4.4 and Remark 4.6) and the above examples lead to the following open problems.

(i) A systematic investigation from the perspective of graph theory of the vertices having property Theorem 4.4(b) remains an interesting task.

(ii) The problem of vertex control in case when the velocities on the edges are different but rationally dependent (see [9]) can be reduced to the situation treated in Section 4. However, the case of rationally independent velocities is completely open.
(iii) More general transport processes in networks allowing space dependent velocities and absorption on the edges have been studied in [10]. The analogous control problem in this more realistic situation remains to be investigated.

(iv) It seems natural to ask what can be said about exact instead of approximate control (i.e. about states reachable in some fixed final time).

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REFERENCES

[1] B. Bollobás, “Modern Graph Theory,” Graduate Texts in Math., vol. 184, Springer-Verlag, New York, 1998.
[2] R. F. Curtain and H. J. Zwart, “An Introduction to Infinite-Dimensional Linear Systems Theory,” Texts Appl. Math., vol. 21, Springer-Verlag, New York, 1995.
[3] B. Dorn, Semigroups for flows in infinite networks, Semigroup Forum, 76 (2008), 341–356.
[4] R. Dáger and E. Zuazua, “Wave Propagation, Observation and Control in 1-d Flexible Multi-structures,” Mathematics & Applications, vol. 50, Springer-Verlag, Berlin, 2006.
[5] K.-J. Engel and R. Nagel, “One-Parameter Semigroups for Linear Evolution Equations,” Graduate Texts in Math., vol. 194, Springer-Verlag, New York, 2000.
[6] K.-J. Engel, M. Kramar Fijavž, B. Klöss, R. Nagel and E. Sikolya, Maximal controllability for boundary control problems, preprint.
[7] Ch. D. Godsil and G. Royle, “Algebraic Graph Theory,” Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.
[8] G. Greiner, Perturbing the boundary conditions of a generator, Houston J. Math., 13 (1987), 213–229.
[9] M. Kramar and E. Sikolya, Spectral properties and asymptotic periodicity of flows in networks, Math. Z., 249 (2005), 139–162.
[10] T. Mátrai and E. Sikolya, Asymptotic behavior of flows in networks, Forum Math., 19 (2007), 429–461.
[11] E. Sikolya, Flows in networks with dynamic ramification nodes, J. Evol. Equ., 5 (2005), 441–463.

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