Chiral de Rham complex on special holonomy manifolds

Joel Ekstrand\textsuperscript{a}, Reimundo Heluani\textsuperscript{b}, Johan Källén\textsuperscript{a} and Maxim Zabzine\textsuperscript{a}

\textsuperscript{a}Department of Physics and Astronomy, Uppsala university, Box 516, SE-751 20 Uppsala, Sweden

\textsuperscript{b}Department of Mathematics, University of California, Berkeley, CA 94720, USA

Abstract

Interpreting the chiral de Rham complex (CDR) as a formal Hamiltonian quantization of the supersymmetric non-linear sigma model, we suggest a setup for the study of CDR on manifolds with special holonomy. We discuss classical and partial quantum results. As a concrete example, we construct two commuting copies of the Odake algebra (an extension of the $N = 2$ superconformal algebra) on the space of global sections of CDR of a Calabi-Yau 3-fold. This is the first example of such a vertex subalgebra which is non-linearly generated by a finite number of superfields.
1 Introduction

The chiral de Rham complex (CDR) was introduced by Malikov, Schechtman and Vaintrob in [1]. CDR is a sheaf of supersymmetric vertex algebras over a smooth manifold $\mathcal{M}$. It is defined by gluing free chiral algebras on the overlaps of open sets of $\mathcal{M}$. Since the original work [1], there has been considerable progress in understanding the mathematical aspects of CDR (cf. [2, 3, 4] among others). In physics literature, CDR appeared in the context of half-twisted sigma models [5, 6] and in the context of infinite volume limits of sigma models [7, 8, 9]. The present work is the logical continuation of [10] where it was suggested to interpret CDR as a formal canonical quantization of the non-linear sigma model.

The question we would like to address is which vertex algebras can be attached to a manifold $\mathcal{M}$ within the CDR framework. It has been established that one can attach different versions of super-Virasoro algebras as global sections of CDR. The details of the construction depend on additional geometrical structures on $\mathcal{M}$. In this paper, we construct extensions of the $N = 2$ super-Virasoro algebra for Calabi-Yau 3-folds. Moreover, we present some partial results on the general possible extensions of the super-Virasoro algebra within the CDR framework.

In looking for possible vertex sub-algebras generated by global sections of CDR there are two possible ways to proceed, either trying to guess the answer, or trying to find a systematic way to produce it. If we think of CDR as a formal quantization of the non-linear sigma model, we can use the insight from classical sigma models in order to make an intelligent guess about the answer in CDR. Indeed, any possible extension of the super-Virasoro algebra will be related to the symmetries of the classical sigma model. Thus, the present result is inspired by the nearly 20 year old observation by P. Howe and G. Papadopoulos in [11, 12], where the relation between classical symmetries of non-linear sigma models and special holonomy manifolds was observed. Our intention is to reformulate their result in terms of Poisson vertex algebras and then try to quantize it within the framework of CDR.

In order to do so, we first describe a non-trivial way of associating global sections of CDR to differential forms on the target manifold $\mathcal{M}$. Locally, CDR is simply a $\beta\gamma$-$bc$ system with as many generators as the dimension of $\mathcal{M}$. This gives natural embeddings of the differential forms of $\mathcal{M}$ into CDR. In this article, we consider an embedding inspired by the sigma model, that is different than the one introduced in [1].

In our embedding, one needs the Levi-Civita connection on the Riemannian manifold $\mathcal{M}$, to write the correct expressions. This is detailed in section 5.
and Appendix C.

On a special holonomy manifold there exist non-trivial and well-known covariantly constant forms. The algebra generated by the corresponding superfields is a natural vertex algebra associated to any special-holonomy manifold. We show that this unified construction recovers all the known cases of [1] [3] [13]. Moreover, we show that for a Calabi-Yau threefold $\mathcal{M}$, the covariantly constant forms on $\mathcal{M}$ generate non-linearly a super-vertex algebra known as the Odake algebra. This is the first example of such a sub-super-vertex algebra of global sections of CDR which is non-linearly generated by a finite number of superfields.

The paper is organized as follows: In Section 2 we review the Hamiltonian formalism for $N = (1,1)$ supersymmetric non-linear sigma model. We also set up the notations for the rest of the paper. Section 3 deals with the classical symmetries of sigma models on special holonomy manifolds. This section presents the Hamiltonian treatment of the results from [11, 12], we also give a list of Poisson vertex algebras associated to the different cases of special holonomy. In Section 4 we briefly recall the formalism of SUSY vertex algebras and the definition of CDR. We stress the physical interpretation of CDR as a formal canonical quantization of the non-linear sigma model. In Section 5 we discuss how to promote the classical currents to well-defined sections of CDR. Section 6 presents our results at the quantum level. The main result is the construction of two commuting copies of the Odake algebra on a Calabi-Yau 3-fold. We also present some general remarks about other cases. In Section 7 we present a summary and discuss the main complications in further possible calculations. Many technicalities are collected in the appendices. In Appendix A we collect some useful formulas on different special holonomy manifolds. Appendix B contains a collection of properties of the quantum $\Lambda$-bracket. In Appendix C we present some explicit formulas accompanying those of Section 5. Appendix D presents the details of the calculation of the quantum Odake algebra.

2 Classical sigma model

In this section we will review some basic facts about the classical sigma model. Especially, we will see the connection between covariantly constant forms on the target space and symmetries of the sigma model. We will also show how to write the model in the Hamiltonian framework.
2.1 The sigma model in the Lagrangian formalism

Consider the N=(1,1) supersymmetric sigma model defined on $\Sigma = S^1 \times \mathbb{R}$. Its action is given by

$$S = \frac{1}{2} \int_{\Sigma} d\sigma dt d\theta^- d\theta^+ \; g_{ij}(\Phi) D_+ \Phi^i D_- \Phi^j. \quad (2.1)$$

We use N=(1,1) superfields $\Phi^i(\sigma, t, \theta^+, \theta^-)$. The circle $S^1$ is parametrized by $\sigma$, and $t$, the “time”, is the coordinate on $\mathbb{R}$. The pair $\theta^\pm$ labels the spinor coordinates. The fields $\Phi^i$ are maps from $\Sigma^2$ into a target manifold $M$, and $g_{ij}$ is the metric on $M$. The odd derivatives $D_\pm$ and the even derivatives $\partial_\pm = \partial_0 \pm \partial_1$ are defined by

$$D_\pm \equiv \frac{\partial}{\partial \theta^\pm} + \theta^\pm(\partial_0 \pm \partial_1), \quad \partial_\pm \equiv D_\pm^2 = \partial_0 \pm \partial_1, \quad (2.2)$$

where $\partial_0 \equiv \partial/\partial t$ and $\partial_1 \equiv \partial/\partial \sigma$. The equation of motion derived from this action is

$$D_- D_+ \Phi^i + \Gamma^i_{jk} D_- \Phi^j D_+ \Phi^k = 0, \quad (2.3)$$

where $\Gamma^i_{jk}$ are the components of the Levi-Civita connection. The model has N=(1,1) superconformal symmetry, with the corresponding current given by

$$T^\pm = g_{ij}(\Phi) D_\pm \Phi^i \partial_\pm \Phi^j. \quad (2.4)$$

The equation of motion gives $D_\pm T^\pm = 0$. This imply that $T^\pm$ are conserved, and also that $T^\pm = T^\pm(\sigma^\pm, \theta^\pm)$, i.e. we have left and right moving currents. We can multiply $T^\pm$ by any function $f^\pm(\sigma^\pm, \theta^\pm)$ to form $\tilde{T}^\pm = f^\pm T^\pm$. $\tilde{T}^\pm$ still satisfy $D_\pm \tilde{T}^\pm = 0$, and we therefore have infinitely many conserved currents. The components of the superfields $T^\pm$ are the Virasoro field and the Neveu-Schwarz supercurrent, respectively.

These are the only symmetries associated to a general Riemannian metric that we can find. However, as noticed in [11, 12], if $M$ admits covariantly constant forms, the sigma model has additional symmetries. The argument goes as follows: consider a form $\omega = \omega_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n}$ satisfying $\nabla \omega = 0$, where $\nabla$ is the Levi-Civita connection. Then

$$J^{(n)}_\pm = \omega_{i_1 \ldots i_n}(\Phi) D_\pm \Phi^{i_1} \ldots D_\pm \Phi^{i_n} \quad (2.5)$$

satisfies $D_\pm J^{(n)}_\pm = 0$ on-shell, i.e. with the use of (2.3). This implies that $J^{(n)}_\pm = J^{(n)}_\pm(\sigma^\pm, \theta^\pm)$, and the components of $J^{(n)}_\pm$ will be left and right moving currents. By the same argument as above, we have infinitely many conserved currents. The symmetries corresponding to the currents are

$$\delta_\pm \Phi^i = \epsilon_\pm g^{ij_1} \omega_{i_1 \ldots i_n} D_\pm \Phi^{j_2} \ldots D_\pm \Phi^{j_n}, \quad (2.6)$$
where the parameter $\epsilon_\pm$ satisfies $D_\mp \epsilon_\pm = 0$. The action functional (2.1) is invariant under (2.6) if $\omega$ is covariantly constant with respect to Levi-Civita connection.

### 2.2 The sigma model in the Hamiltonian formalism

The sigma model (2.1) can also be formulated in the Hamiltonian formalism [14, 15, 16]. We integrate out one odd $\theta$, and identify the Hamiltonian and the phase space structure. In order to do so, we introduce new odd coordinates $\theta_0$ and $\theta_1$ by

$$\theta_0 = \frac{1}{\sqrt{2}}(\theta^+ + i\theta^-), \quad \theta_1 = \frac{1}{\sqrt{2}}(\theta^+ - i\theta^-),$$

(2.7)

together with odd derivatives

$$D_0 = \frac{1}{\sqrt{2}}(D_+ - iD_-), \quad D_1 = \frac{1}{\sqrt{2}}(D_+ + iD_-),$$

(2.8)

which satisfy $D_0^2 = \partial_1$, $D_1^2 = \partial_1$ and $D_1 D_0 + D_0 D_1 = 2\partial_0$. We also introduce new superfields

$$\phi^i \equiv \Phi^i|_{\theta_0=0}, \quad S_i \equiv g_{ij}D_0\Phi^j|_{\theta_0=0},$$

(2.9)

and new derivatives

$$D_1 \equiv D_1|_{\theta_0=0}, \quad \partial \equiv \partial_1.$$  

(2.10)

After performing the $\theta_0$-integration, the action (2.1) becomes

$$S = \int dt d\sigma d\theta_1 \left( S_i \partial_0 \phi^i - \frac{1}{2} \mathcal{H} \right),$$

(2.11)

where

$$\mathcal{H} = \partial \phi^i D_1 \phi^j g_{ij} + g^{ij}S_i DS_j + S_k D_1 \phi^k S_l g^{kl} \Gamma^i_{ij}.$$

(2.12)

We see that the sigma model phase space corresponds to a cotangent bundle $T^*\mathcal{L}M$, where $\mathcal{L}M = \{S^{1|1} \to \mathcal{M}\}$ is a superloop space. It is equipped with a natural symplectic structure

$$\int d\sigma d\theta_1 \, \delta S_i \wedge \delta \phi^i.$$  

(2.13)

Thus, the space of functionals on $T^*\mathcal{L}M$ is equipped with a (super) Poisson bracket $\{\, , \}$ generated by the relation:

$$\{\phi^i(\sigma, \theta_1), S_j(\sigma', \theta'_1)\} = \delta^i_j \delta(\sigma - \sigma') \delta(\theta_1 - \theta'_1).$$

(2.14)
From (2.11), the Hamiltonian is:

\[ H = \frac{1}{2} \int d\sigma d\theta \mathcal{H}. \]

This Hamiltonian, together with the Poisson bracket (2.14), generates the same dynamics as we get from the action (2.1) and the variational principle. It is convenient to introduce new formal coordinates on \( S^{1|1} \): \( \xi = e^{i\sigma} \) and \( (i\xi)^{1/2}\theta = \theta_1 \), which imply

\[
(i\xi)^{1/2} D = D_1, \quad (i\xi)^{1/2} d\theta = d\theta_1, \quad (i\xi)^{1/2} S_i(\xi, \theta) = S_i(\sigma, \theta_1). \] (2.15)

Thus the Poisson bracket (2.14) becomes

\[
\{ \phi^i(\xi, \theta), S_j(\xi', \theta') \} = \delta^i_j \delta(\xi - \xi')\delta(\theta - \theta). \] (2.16)

From now on, we will use the variables \((\xi, \theta)\) on \( S^{1|1} \).

### 2.2.1 Poisson vertex algebras and \( \Lambda \)-brackets

Local functionals on \( T^*\mathcal{LM} \) form a Poisson superalgebra. This Poisson superalgebra can formally be described as a Poisson (super) vertex algebra. These in turn can be understood as semi-classical limits of vertex algebras. Alternatively, if we work locally on \( \mathcal{M} \), we can construct a sheaf of Poisson (super) vertex algebras and consider its set of global sections. For an introduction to Poisson vertex algebras in the context of superfields and \( \Lambda \)-brackets we refer the reader to [17]. For an extensive study of sheaves of Poisson vertex algebras and their relation to CDR see [4].

Below, we set the notational conventions used in the rest of the article. The Poisson bracket between two local functionals has the following general form

\[
\{ A(\xi, \theta), B(\xi', \theta') \} = \sum_{J \geq 0, 1} (-1)^J \partial^{J}_{\xi} D^{J}_{\xi'} \delta(\xi - \xi')\delta(\theta - \theta') C_{(j|j)}(\xi', \theta'), \] (2.17)

where \( C_{(j|j)} \) denotes the local functional multiplying the \((-1)^J \partial^{J}_{\xi} D^{J}_{\xi'} \delta(\xi - \xi')\delta(\theta - \theta')\)-term. This bracket can be encoded as

\[
\{ A_\Lambda B \} = \sum_{J \geq 0, 1} \Lambda^{j|j} C_{(j|j)}, \] (2.18)

where \( \Lambda^{j|j} = \lambda^j \chi^j \), with formal even \( \lambda \) and odd \( \chi \) satisfying \( \chi^2 = -\lambda \). The \( \Lambda \)'s encode derivatives of delta functions, and the translation between the two is

\[ \Lambda^{j|j} \rightarrow (-1)^J \partial^{J}_{\xi} D^{J}_{\xi'} \delta(\xi - \xi')\delta(\theta - \theta'). \]
For example, we write (2.14) as

$$\{ \phi^i \Lambda S_j \} = \delta^i_j .$$

### 2.2.2 Currents in phase space

Next, we derive the currents (2.4) and (2.5) in phase space coordinates. From (2.8) and (2.9) we note that

$$D_+ \Phi^i | \theta_0 = 0 = (g_{ij} S_j + D \phi^i) \frac{1}{\sqrt{2}} \equiv e^+_i , \quad (2.19)$$

$$D_- \Phi^i | \theta_0 = 0 = (i (g_{ij} S_j - D \phi^i)) \frac{1}{\sqrt{2}} \equiv i e^-_i . \quad (2.20)$$

The factor of $i$ in the definition of $e^-_i$ is introduced for later computational convenience. The set of fields $\{ e^\pm_i \}$ is a suitable basis for the symmetry currents. For $J^{(n)}_\pm$ the rewriting is straightforward. Since $T_\pm$ have a term with a time derivative, it requires the use of the equations of motion. We get

$$T_\pm = \pm (g_{ij} De^+_i e^+_j + g_{ij} \Gamma^i_{kl} De^+_i e^+_j) , \quad J^{(n)}_+ = \frac{1}{n!} \omega_{i_1 ... i_n} e^+_{i_1} ... e^+_{i_n} , \quad (2.21)$$

$$J^{(n)}_- = \frac{1}{n!} \omega_{i_1 ... i_n} e^-_{i_1} ... e^-_{i_n} .$$

It can be checked explicitly that $T_\pm$ generate the superconformal symmetries of the original theory and the currents $J^{(n)}_\pm$ generate the symmetry (2.6) in the Hamiltonian formalism.

### 3 Classical algebra extensions

We now investigate the classical algebra generated by the currents (2.21). A natural question to ask is whether the Poisson brackets between these currents can be expressed in terms of the same fields. The results in this section were obtained in [11, 12], although in a different framework.

Between the fields (2.19)-(2.20) we have the following brackets:

$$\{ e^+_i \Lambda e^+_j \} = \pm \chi g^{ij} + \frac{1}{\sqrt{2}} (g^{kj} \Gamma^i_{mk} e^m_+ - g^{ki} \Gamma^j_{mk} e^m_+) , \quad (3.1)$$

$$\{ e^-_i \Lambda e^-_j \} = \frac{1}{\sqrt{2}} (g^{kj} \Gamma^i_{mk} e^m_- - g^{ki} \Gamma^j_{mk} e^m_-) , \quad (3.2)$$

$$\{ e^+_i \Lambda f(\phi) \} = \frac{1}{\sqrt{2}} g^{ij} f_{,j} . \quad (3.3)$$
where \( f_j = \partial_j f \). If we only have \( T_\pm \), we get the classical (super) Virasoro algebra. In \( \Lambda \)-bracket notation, the Virasoro algebra is written as

\[
\{ T_\pm \Lambda T_\pm \} = (2\partial + \chi D + 3\lambda) T_\pm ,
\]

\[
\{ T_\mp \Lambda T_\pm \} = 0 .
\]

The algebra between the currents corresponding to covariantly constant forms is straightforward to compute. Using the above brackets, one can show that

\[
\{ T_\pm \Lambda T_\mp \} = (2\partial + \chi D + n\lambda) J_\pm^{(n)} ,
\]

\[
\{ T_\mp \Lambda T_\pm \} = 0 ,
\]

which shows that \( J_\pm^{(n)} \) have conformal weight \( \frac{n}{2} \) with respect to the left/right moving Virasoro field. Let us define

\[
B_+^{(n)} = \frac{1}{n!} g^{i_1 i_2 ... i_{n+1}} e_{+}^{i_1} ... e_{+}^{i_{n+1}} ,
\]

\[
B_-^{(n)} = \frac{1}{n!} g^{i_1 i_2 ... i_{n+1}} e_{-}^{i_1} ... e_{-}^{i_{n+1}} .
\]

We then find the following brackets between \( J_\pm^{(n+1)} \) and \( J_\pm^{(m+1)} \):

\[
\{ J_\pm^{(n+1)} \Lambda J_\pm^{(m+1)} \} = (-1)^n \left( \chi g_{ij} B_\pm^{(n)} B_\pm^{j(m)} + g_{ij} DB_\pm^{j(n)} B_\pm^{j(m)} + g_{ij} \Gamma_{kl}^{i} D\phi^l B_\pm^{k(n)} B_\pm^{j(m)} \right) ,
\]

\[
\{ J_\pm^{(n+1)} \Lambda J_\mp^{(m+1)} \} = 0 .
\]

### 3.1 Currents from holonomy groups

To further analyze the algebra (3.7), we need to know more about the covariantly constant forms \( \omega \) and the metric \( g \). The existence of covariantly constant forms is ultimately related to the holonomy of the Levi-Civita connection. Fortunately, the possible holonomy groups for the Levi-Civita connection have been classified and their relation to covariantly constant forms is known (see [18] for a review of the subject). To use this classification, let us assume that \( \mathcal{M} \) is simply-connected and that the metric \( g \) is irreducible (to avoid the holonomy group to be a product of two groups of lower dimension). Finally, we assume that \( \mathcal{M} \) is not locally a Riemannian symmetric space.

With these three assumptions, there are seven different possible cases for the Riemannian holonomy group and in each one of them we understand the properties of the corresponding covariantly constant forms. Below, we will
give the details of the algebra (3.7) in each of these seven cases. The relation between holonomy groups and the symmetries of non-linear sigma model has been noted in [11, 12].

In order to compute the structure of the corresponding algebras, we will need some algebraic properties of the invariant tensors on special holonomy manifolds. We collect the relevant formulas in Appendix A. In particular, we use the formulas derived in [19, 20]. Below, we give explicit definitions for these currents and compute their corresponding algebras.

### 3.1.1 Orientable Riemannian manifold, $SO(n)$

On a general $n$-dimensional orientable Riemannian manifold the holonomy group is $SO(n)$, and we have the covariantly constant totally anti-symmetric tensor $\epsilon_{i_1 \cdots i_n}$. For $n > 2$ the Poisson bracket between the corresponding currents is zero. For $n = 2$, since $SO(2) = U(1)$, and $\epsilon_{i_1 i_2}$ can be taken as the Kähler form, we get the $N=2$ supersymmetry algebra (see the next example).

### 3.1.2 Kähler manifold, $U(n)$

When the holonomy group is $U(n)$, $\dim \mathcal{M} = 2n$, the manifold is Kähler and we have a covariantly constant 2-form, the Kähler form $\omega$. Using $\omega = gI$, $I$ being the complex structure, the current is defined as

$$J^{(2)}_{\pm} = \pm \frac{1}{2} \omega_{ij} e_i^\pm e_j^\pm,$$

and we find that (3.7) reduces to

$$\{ J^{(2)}_{\pm}, J^{(2)}_{\mp} \} = -T_{\pm}.$$

We therefore get two commuting copies of the $N=2$ superconformal algebra when the target manifold is Kähler.

### 3.1.3 Calabi-Yau, $SU(n)$

When the holonomy group is $SU(n)$, $\dim \mathcal{M} = 2n$, we are on a Calabi-Yau manifold. We then have, in addition to the Kähler form, a covariantly constant holomorphic $n$-form $\Omega$, and its complex conjugate $\bar{\Omega}$ at our disposal. Let us denote the corresponding currents $X^{(n)}_{\pm}$ and $\bar{X}^{(n)}_{\pm}$:

$$X^{(n)}_+ = \frac{1}{n!} \Omega_{\alpha_1 \cdots \alpha_n} e_{+}^{\alpha_1} \cdots e_{+}^{\alpha_n}, \quad X^{(n)}_- = \frac{i^n}{n!} \bar{\Omega}_{\bar{\alpha_1} \cdots \bar{\alpha}_n} e_{-}^{\bar{\alpha}_1} \cdots e_{-}^{\bar{\alpha}_n},$$

$$\bar{X}^{(n)}_+ = \frac{1}{n!} \bar{\Omega}_{\bar{\alpha_1} \cdots \bar{\alpha}_n} e_{+}^{\alpha_1} \cdots e_{+}^{\alpha_n}, \quad \bar{X}^{(n)}_- = \frac{i^n}{n!} \bar{\Omega}_{\bar{\alpha_1} \cdots \bar{\alpha}_n} e_{-}^{\bar{\alpha}_1} \cdots e_{-}^{\bar{\alpha}_n}.$$
which are defined in addition to $J^{(2)}_{\pm}$ and $T_{\pm}$ on the Calabi-Yau manifold. We here introduced complex coordinates, with indices $i = (\alpha, \bar{\alpha})$. Choosing an hermitian metric, we find, using the formulas in Appendix A.2, that (3.7) reduces to

$$
\{ J^{(2)}_{\pm} X^{(n)}_{\pm} \} = -i \left( n \chi X^{(n)}_{\pm} + D X^{(n)}_{\pm} \right), \\
\{ J^{(2)}_{\pm} \bar{X}^{(n)}_{\pm} \} = +i \left( n \chi \bar{X}^{(n)}_{\pm} + D \bar{X}^{(n)}_{\pm} \right), \\
\{ X^{(n)}_{\pm} X^{(n)}_{\pm} \} = 0, \\
\{ \bar{X}^{(n)}_{\pm} \bar{X}^{(n)}_{\pm} \} = 0, \\
\{ X^{(n)}_{\pm} \bar{X}^{(n)}_{\pm} \} = \frac{i^{n^2+1}}{(n-1)!} \left( \frac{i}{2} (n-1) T \left( J^{(2)}_{\pm} \right)^{n-2} - \frac{1}{2} D \left( J^{(2)}_{\pm} \right)^{n-1} - \chi \left( J^{(2)}_{\pm} \right)^{n-1} \right),
$$

where $J^{(2)}_{\pm}$ is defined in (3.8). Note that (3.5) now reads

$$
\{ T^{(2)}_{\pm} X^{(n)}_{\pm} \} = (2\partial + \chi D + n\lambda) X^{(n)}_{\pm}, \\
\{ T^{(2)}_{\pm} \bar{X}^{(n)}_{\pm} \} = (2\partial + \chi D + n\lambda) \bar{X}^{(n)}_{\pm}, \\
\{ T^{(2)}_{\pm} X^{(n)}_{\pm} \} = \{ T^{(2)}_{\pm} \bar{X}^{(n)}_{\pm} \} = 0.
$$

These relations, together with the remaining relations for $J^{(2)}_{\pm}$ and $T_{\pm}$, give rise to the classical Odake algebra, a Poisson vertex algebra that one can attach to any Calabi-Yau manifold. Indeed, we have two commuting copies of this algebra, i.e. the plus-currents commute with the minus-currents.

Notice that the currents $(J^{(2)}_{\pm}, T^{(2)}_{\pm}, X^{(n)}_{\pm}, \bar{X}^{(n)}_{\pm})$ satisfy extra constraints. For example, from $\omega \wedge \Omega = \omega \wedge \bar{\Omega} = 0$ we would obtain the identities

$$
J^{(2)}_{\pm} X^{(n)}_{\pm} = 0, \quad J^{(2)}_{\pm} \bar{X}^{(n)}_{\pm} = 0,
$$

which in turn are needed to check the Jacobi identity for (3.12).

**3.1.4 Hyperkahler manifold, $Sp(n)$**

When the holonomy group is $Sp(n)$, $\dim \mathcal{M} = 4n$, the manifold $\mathcal{M}$ is hyperkahler. We have three complex structures $I_A$, $A = 1, 2, 3$, such that $I_A I_B = -\delta_{AB} + \epsilon_{ABC} I_C$. The metric $g$ is Hermitian with respect to all $I_A$ and the forms $\omega_A = g I_A$ are covariantly constant. For $\omega_A$ we denote the
corresponding currents $J^{(2)}_{\pm A}, A = 1, 2, 3$, where we use (3.8). The algebra (3.7) reduces to

$$\{ J^{(2)}_{\pm A} J^{(2)}_{\pm B} \} = \epsilon_{ABC} (D + 2\chi) J^{(2)}_{\pm C} - \delta_{AB} T_\pm ,$$

which, together with (3.4), generates the N=4 superconformal algebra.

### 3.1.5 Quaternionic Kähler manifold, $Sp(n) \cdot Sp(1)$

On a quaternionic Kähler manifold the holonomy group is $Sp(n) \cdot Sp(1)$.

Locally, we have three almost complex structures $J^A$ and three locally defined two-forms $\omega^A = gJ^A$. Defining the covariantly constant form:

$$\Sigma = \sum_{i=A}^3 \omega^A \wedge \omega^A ,$$

and denoting the corresponding currents by $\Sigma_\pm$:

$$\Sigma_\pm = \frac{1}{4!} \Sigma_{ijkl} e^i_\pm e^j_\pm e^k_\pm e^l_\pm ,$$

we obtain

$$\{ \Sigma_\pm \Lambda \Sigma_\pm \} = -4\Sigma_\pm T_\pm .$$

### 3.1.6 $G_2$-manifold

$G_2$ is an example of an exceptional holonomy group. A $G_2$-manifold $M$ is seven dimensional. On such a manifold there are two covariantly constant forms, a 3-form $\Pi$ and its Hodge dual $\Psi$. We denote the respective currents by the same letters $\Pi_\pm$ and $\Psi_\pm$,

$$\Pi_+ = \frac{1}{3!} \Pi_{ijkl} e^i_+ e^j_+ e^k_+ , \quad \Pi_- = \frac{1}{3!} \Pi_{ijkl} e^i_- e^j_- e^k_- , \quad \Psi_\pm = \frac{1}{4!} \Psi_{ijkl} e^i_\pm e^j_\pm e^k_\pm e^l_\pm .$$

Using extensively the formulas presented in Appendix A, we find that (3.7) reduces to

$$\{ \Pi_\pm \Lambda \Pi_\pm \} = -3D\Psi_\pm - 6\chi \Psi_\pm ,
\{ \Pi_\pm \Lambda \Psi_\pm \} = 3T_\pm \Pi_\pm ,
\{ \Psi_\pm \Lambda \Psi_\pm \} = 10T_\pm \Psi_\pm + 3\Pi_\pm D\Pi_\pm .$$

The right hand side in the last bracket can be written in many equivalent ways. The 4-form $\Psi$ is the Hodge dual of $\Pi$ and thus it is not independent data. At the level of currents we can derive the relation

$$2T_\pm \Psi_\pm + \Pi_\pm D\Pi_\pm = 0 ,$$
which follows from (A.14) and (3.19). Indeed, this relation would be needed if we study the Jacobi identity for the above algebra without any reference to the definition of the corresponding currents.

3.1.7 Spin(7)-manifold

Spin(7) is another example of an exceptional holonomy group. Spin(7)-manifolds are eight dimensional and they admit a covariantly constant 4-form Θ which is self-dual with respect to the Hodge involution. The corresponding currents Θ± are defined as

$$\Theta_\pm = \frac{1}{4!} \Theta_{ijkl} e^i_\pm e^j_\pm e^k_\pm e^l_\pm .$$

(3.22)

We find that (3.7) reduces to

$$\{ \Theta_\pm, \Lambda \Theta_\pm \} = 6T_\pm \Theta_\pm .$$

(3.23)

4 The Chiral de Rham complex as a formal canonical quantization of the sigma model

Above, we have considered classical algebra extensions. We now want to investigate the quantum counterpart of these algebras, in the CDR framework. Before doing so, in this section we give a short introduction to basic notions, such as vertex algebras, SUSY vertex algebras and CDR.

The latter is a sheaf of SUSY vertex algebras and it was introduced originally in [1], here we follow the treatment presented in [3]. Also, we review the interpretation of CDR as a formal canonical quantization of the sigma model. For further details the reader may consult [10].

4.1 SUSY vertex algebras

First, we introduce two formal variables (ξ, θ), where ξ is even and θ is odd. Given a vector space V, we define an End(V)-valued superfield as

$$A(\xi, \theta) = \sum_{j \in \mathbb{Z}} \xi^{-(j+1)} \theta^{j-1} A_{(j),j}, \quad A_{(j),j} \in \text{End}(V)$$

(4.1)

where for all \( v \in V, A_{(j),j} v = 0 \) for large enough \( j \).

A SUSY vertex algebra [17] consists of the data of a super vector space V, an even vector \(|0\rangle \in V \) (the vacuum vector), an odd endomorphism \( D \)
(whose square is an even endomorphism which we denote $\partial$), and a parity preserving linear map $A \mapsto Y(A, \xi, \theta)$ from $V$ to $\text{End}(V)$-valued superfields (the state-superfield correspondence). This data should satisfy the following set of axioms:

- **Vacuum axioms:**
  
  $Y(|0\rangle, \xi, \theta) = \text{Id}$ ,
  
  $Y(A, \xi, \theta)|0\rangle = A + O(\xi, \theta)$ ,
  
  $D|0\rangle = 0$ .

- **Translation invariance:**
  
  $[D, Y(A, \xi, \theta)] = (\partial_\theta - \theta \partial_\xi) Y(A, \xi, \theta)$ ,
  
  $[\partial, Y(A, \xi, \theta)] = \partial_\xi Y(A, \xi, \theta)$ .

- **Locality:**
  
  $(\xi - \xi')^n[Y(A, \xi, \theta), Y(B, \xi', \theta')] = 0$ , \quad $n \gg 0$ .

(The notation $O(\xi, \theta)$ denotes a power series in $\xi$ and $\theta$ without a constant term in $\xi$.)

Given the vacuum axioms for a SUSY vertex algebra, we will use the state-field correspondence to identify a vector $A \in V$ with its corresponding field $Y(A, \xi, \theta)$. Given a SUSY vertex algebra $V$ and a vector $A \in V$, we expand the fields

$$Y(A, \xi, \theta) = A(\xi, \theta) = \sum_{j \in \mathbb{Z}} \sum_{J=0,1} \xi^{-1-j} \theta^{1-J} A_{(j|J)} ,$$

and we call the endomorphisms $A_{(j|J)}$ the **Fourier modes** of $Y(A, \xi, \theta)$. Now, define the operations

$$[A_{\Lambda} B] = \sum_{j \geq 0} \sum_{J=0,1} \frac{\Lambda^{j|J}}{j!} A_{(j|J)} B ,$$

$$A B = A_{(-1|1)} B ,$$

where $\Lambda^{j|J} = \lambda^j \chi^J$, with $\lambda$ and $\chi$ being formal even and odd parameters, satisfying $\chi^2 = -\lambda$. The first operation is called the **$\Lambda$-bracket** and the second is called the **normal ordered product**. The $\Lambda$-bracket is a practical way
of writing an operator product expansion (OPE). (4.2) corresponds to the commutator of superfields,

\[ [A(\xi, \theta), B(\xi', \theta')] = \sum_{J=0,1}^{\infty} \sum_{j \geq 0} \left((-1)^{j} \partial_{\xi}^{J} D_{\xi}^{j} \delta(\xi - \xi') \delta(\theta - \theta') \right) (A_{(j|J)} B)(\xi', \theta') \]

we can therefore read the OPE from (4.2). The Λ’s encode derivatives of delta functions, and the translation between the two formalisms are given by

\[ \Lambda^{j|J} \leftrightarrow (-1)^{j} \partial_{\xi}^{J} D_{\xi}^{j} \delta(\xi - \xi') \delta(\theta - \theta') \]

In particular, the Λ-bracket contains the information of all the commutators of the Fourier modes of the respective fields.

The Λ-bracket comes with a very efficient calculus, which allows us to compute the bracket between composite fields, once the bracket between the constituent fields are known. The rules of this calculus are listed in Appendix B. For further details of the formalism the reader may consult [17].

It is worth emphasizing that the normal ordered product is not commutative nor associative. Neither does the Λ-bracket fulfill the Leibniz rule. The deviation from these properties of the different operations can be seen from the rules in the appendix. These deviations have natural interpretations as “quantum effects”, and the semiclassical limit of a SUSY vertex algebra leads to a SUSY Poisson vertex algebra (which is the same as a Poisson superalgebra of local functionals).

4.2 The Chiral de Rham complex (CDR)

As an example of a SUSY vertex algebra, consider the so called βγ-bc system. It consists of 2n superfields \((\phi^i, S_j), i, j = 1, 2 \ldots n\). \(\phi^i\) is an even field and \(S_i\) is an odd field. The defining Λ-brackets are given by

\[ [\phi^i \Lambda S_j] = \delta^i_j \]
\[ [\phi^i \Lambda \phi^j] = 0 \]
\[ [S_i \Lambda S_j] = 0 \]

If we expand these superfields

\[ \phi^i = \gamma^i + \theta c^i \]
\[ S_i = b_i + \theta \beta_i \]

then we recover the standard βγ and bc systems. Let \(g_a(\phi)\) be an invertable function of \(\phi\). The following is an automorphism of (4.3):

\[ \tilde{\phi}^a = g^a(\phi) \]
\[ \tilde{S}_a = \frac{\partial f_i}{\partial \phi^a}(g(\phi)) S_i \]
where $f$ is the inverse of $g$. The normal ordered product is used when defining
the new fields. Recall that it is not associative. The statement $[S_a \Lambda \hat{S}_b] = 0$
is therefore non-trivial.

Using this fact we can make the following geometrical construction. Con-
sider a smooth manifold $\mathcal{M}$, and attach to each coordinate patch $\{x^i\}_{i=1}^n$
the $\beta\gamma$-bc system $\{\phi^i, S_i\}_{i=1}^n$. We glue these systems on the intersections
using (4.5). We thus construct a sheaf $\Omega^{ch}(\mathcal{M})$ of $\beta\gamma$-bc systems. This sheaf
was first introduced in [1], and named the Chiral de Rham complex, which
we abbreviate CDR. Although the formalism developed in [1] works in the
analytic, algebraic and smooth settings, most of the mathematical literature
on CDR is dedicated to the algebraic setting. The setup relevant to us was
considered in [21] and in superfield formalism in [3]. In the present work, our
treatment of CDR closely follow [3].

4.3 Physical interpretation of CDR

In the above construction of CDR, we can introduce a parameter $\hbar$ in the
defining brackets (4.3) of the $\beta\gamma$-bc system:
\[
\begin{align*}
[\phi^i \Lambda S_j] &= \hbar \delta^i_j, \\
[\phi^i \Lambda \phi^j] &= 0, \\
[S_i \Lambda S_j] &= 0.
\end{align*}
\] (4.6)

Let us consider the bracket $\frac{1}{\hbar}[a \Lambda b]$. From the rules of appendix B, we see
that in the limit $\hbar \to 0$, the resulting bracket will fulfill the Leibniz rule, and
the normal ordered product is both commutative and associative, since the
deviations from these properties will be higher order in $\hbar$. The “quasi-classical”
limit $\hbar \to 0$ is a well-defined operation on Vertex Algebras, and in this limit
we get a Poisson Vertex Algebra, described in 2.2.1 (see [10] and references
therein).

In section 2.2 we showed that the phase space of the classical sigma model
is the superloop space $T^* \mathcal{LM}$, equipped with the Poisson bracket (2.14),
which in $\Lambda$-bracket notation can be written as
\[
\{ \phi^i \Lambda S_j \} = \delta^i_j.
\] (4.7)

This relation is invariant under changes of coordinates, when $\phi^i$ transforms
as a coordinate and $S_i$ as a one-form. We can therefore construct a global
object, which can formally be regarded as a sheaf of Poisson Vertex Algebras.

We can interpret the structure (4.6) as a formal canonical quantization
of the structure (4.7), and CDR as the formal canonical quantization of the
sigma model. We interpret the A-brackets for the $\beta\gamma$-bc system as equal-time commutators, and the definition of normal ordering of composite fields is the standard one.

For a further explanation of the above interpretation, and more precise statements, we refer to [10].

5 Constructing well-defined currents on CDR

Our goal is to extend the results of Section 3 from the classical setting to the quantum setting of CDR. Before calculating any brackets, we have to understand how to define the appropriate global sections of CDR from the classical expressions for $T_{\pm}$ and $J_{\pm}^{(n)}$. The main requirement is that these quantum fields coincide with their classical expressions when taking the limit $\hbar \to 0$.

We have seen above that we can glue the superfields $\phi^i$ and $S_i$ between patches using the normal ordered transformation rules (4.5). In this section, we will discuss whether composite operators, which are defined as normal ordered products of $\phi^i$, $S_j$ and functions thereof, can be glued together to give rise to globally defined sections of CDR. Especially, we will investigate whether the symmetry currents of the classical sigma model can be “quantized”. There are potential problems, since in the quantum setting the expressions are not associative nor commutative anymore. Indeed, we will see that in general the classical currents are not invariant under a change of coordinates in the quantum setup. However, we can by hand add terms to the currents which will be of higher order in $\hbar$. These extra terms will by themselves not be invariant under a change of coordinates, but their anomalous parts will precisely cancel with the anomalous parts coming from the classical part. This resembles of the introduction of a connection when constructing a covariant derivative in geometry. Amusingly, we will see below that the structure we need in order to “quantum covariantize” the set of currents $J_{\pm}^{(n)}$ is the Levi-Civita connection. Below, we will give a general prescription on which terms one must add to a current constructed from an $n$-form, for any $n$, in order for it to be a well-defined section of the CDR. This construction is limited to currents constructed from forms on $M$, the anti-symmetry of the indices is used extensively. Currents formed from other structures, for example the stress energy tensor formed from the metric, turns out to be much harder to handle, and in fact we can only give indirect proofs in special cases that the stress energy tensor is a well-defined section of CDR.

The inhomogeneous transformation properties of the Levi-Civita connection was used in [3] to construct global sections of CDR associated to Kähler
forms. Our construction below generalizes their result to forms of arbitrary degree.

5.1 Constructing well-defined sections from forms

In this section, we will show how to modify the currents constructed from \( n \)-forms, which classically are of the form

\[
J^{(n)}_{+c} = \frac{1}{n!} \omega_{i_1...i_n} e^i_1 \cdots e^i_n, \\
J^{(n)}_{-c} = \frac{i^n}{n!} \omega_{i_1...i_n} e^{-i}_1 \cdots e^{-i}_n.
\]

We here have introduced a subscript “c”, for “classical”.

The statements in the rest of this subsection applies equally well to both the plus-sector and the minus-sector, and we skip the \( \pm \)-signs and factors of \( i \).

In CDR, we in general have to choose in which order we multiply operators. For definiteness we choose the order of multiplication to be

\[
(\omega_{j_1...j_n}(\phi)) (e^j_1 \cdots (e^{j_{n-1}} e^{j_n}) \cdots),
\]

although it can be shown that for this operator the order of multiplication does not matter.

Let us define new coordinates \( \tilde{\phi}(\phi)^a \), and denote

\[
\tilde{f}^i_a \equiv \partial \tilde{\phi}^a / \partial \phi^i, \\
f^i_a \equiv \partial \phi^i / \partial \tilde{\phi}^a.
\]

The operator \( e^i \) transforms as a vector, and we have \( e^i = f^i_a e^a \). The \( n \)-form \( \omega \) transforms as \( \omega_{i_1...i_n} = f^{i_1}_{i_1} \cdots f^{i_n}_{i_n} \omega_{a_1...a_n} \). Because of the non-associativity of the normal order product, one can in general not simply pull out the factors of \( f \), and for \( n > 1 \), the operator \( (\omega_{j_1...j_n}(\phi)) (e^{j_1} \cdots (e^{j_{n-1}} e^{j_n}) \cdots) \) does not transform as a tensor. One has to add a non-tensorial object in order to get the right transformation properties. What we will do below is to introduce a new operator \( E^{(n)}_{i_1...i_n} \) such that \( \omega_{j_1...j_n}(\phi) E^{(n)}_{i_1...i_n} \) does transform as a tensor, and reduces to \( (\omega_{j_1...j_n}(\phi)) (e^{j_1} \cdots (e^{j_{n-1}} e^{j_n}) \cdots) \) in the limit \( \hbar \to 0 \).

5.1.1 The construction, plus-sector

Let us first define

\[
F^{j_1...j_k}_{\pm(0)} \equiv 1, \\
F^{j_1...j_k}_{+(k)} \equiv e^j_+(e^j_+ \cdots (e^{j_{k-1}} e^{j_k}) \cdots), \\
F^{j_1...j_k}_{-(k)} \equiv i e^j_-(i e^j_- \cdots (i e^{j_{k-1}} i e^{j_k}) \cdots).
\]

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From now on, we only work in the plus sector, and we suppress the + subscript. Comments about the minus sector can be found in the next subsection. We want to construct operators $E^{i_1 \ldots i_p}_{(p)}$ that can be used as a base for $p$-forms, i.e. an object that transforms as a $(p, 0)$-tensor. We are going to assume that $E^{i_1 \ldots i_p}_{(p)}$ is anti-symmetrized in its indices.

Our construction is recursive. Given $E^{i_1}_{(k-1)}$, we want to construct $E^{i_1 \ldots i_k}_{(k)}$. First, we investigate how $e^i E^{i_2 \ldots i_k}_{(k-1)}$ transforms under the assumption that $e^b_{(n[1]} E^{i_1}_{(k-1)} = 0 \quad \forall n > 0$, i.e. that the $\chi$-terms in the $\Lambda$-bracket between $e^b$ and $E^{i_1}_{(k-1)}$ only have a $\lambda^0$-part. We then have:

$$e^i E^{i_2 \ldots i_k}_{(k-1)} = (f^{a_1}_{x_1} \ldots f^{a_k}_{x_k}) \left( e^{a_1} E^{i_2 \ldots i_k}_{(k-1)} - \partial \tilde{f}^{a_1} f^b_{(0[1]} E^{i_2 \ldots a_k}_{(k-1)} \right), \quad (5.5)$$

where $\left( \langle i j \rangle \right)$ is the $\lambda^i \chi^j$-part of the $\Lambda$-bracket. The above assumption is satisfied if $E^{i_1}_{(k)}$ is of the form $E^{i_1}_{(k)} = \sum_{i=1}^{k} h_{i(\phi, \partial \phi)} F_{(i)}$, where $\{h_{i(\phi, \partial \phi)}\}$ are arbitrary functions of $\phi$ and $\partial \phi$. This claim will be justified below. Because of the in-homogenous transformation of the connection, we can cancel the in-homogeneous part of $(5.5)$ by adding a term proportional to $\Gamma \partial \phi$. The sought $(p, 0)$-tensorial object is then

$$E^{i_1}_{(0)} \equiv 1, \quad (5.6)$$
$$E^{i_1 \ldots i_k}_{(k)} \equiv e^i E^{i_2 \ldots i_k}_{(k-1)} + \Gamma^i_{ij} \partial \phi^k (e^j_{(0[1]} E^{i_2 \ldots i_k}_{(k-1)}), \quad (5.7)$$

We now show that the form $E^{i_1 \ldots i_k}_{(k)} = \sum_{i=1}^{k} h_{i(\phi, \partial \phi)} F_{(i)}$ implies that there are no higher powers of $\lambda$ in the $\chi$-terms of the $\Lambda$-bracket between $e^b$ and $E^{i_1}_{(k-1)}$. We have

$$[e^i \Lambda e^j] = h \chi g^{ij} + \mathcal{O}(\chi^0), \quad (5.8)$$

where we only keep the term proportional to $\chi$. From this it follows that

$$[e^i \Lambda F^{j_1 \ldots j_k}_{(k)}] = h \chi \sum_{n=1}^{k} (-1)^{k+1} g^{j_1 n}(e^{j_1} \ldots (\tilde{e}^{j_n} \ldots e^{j_k}) \ldots) + \mathcal{O}(\chi^0), \quad (5.9)$$

where the term $\tilde{e}^{j_n}$ is omitted. When the indices are anti-symmetrized, this is

$$[e^i \Lambda F^{j_1 \ldots j_k}_{(k)}] = h \chi k g^{j_1 j_k} F^{j_2 \ldots j_k}_{(k-1)} + \mathcal{O}(\chi^0). \quad (5.10)$$

We also have

$$[h(\phi, \partial \phi)e^i \Lambda F^{j_1 \ldots j_k}_{(k)}] = h(\phi, \partial \phi)[e^i \Lambda F^{j_1 \ldots j_k}_{(k)}] + \mathcal{O}(\chi^0), \quad (5.11)$$
$$[e^i \Lambda h(\phi, \partial \phi) F^{j_1 \ldots j_k}_{(k)}] = h(\phi, \partial \phi)[e^i \Lambda F^{j_1 \ldots j_k}_{(k)}] + \mathcal{O}(\chi^0), \quad (5.11)$$
where \( h \) is an arbitrary function of \( \phi \) and \( \partial\phi \). Here we used the fact that the integral term does not have any \( \chi \)-factor. From our formula (5.7), we see by induction that \( E_{(k)} \) is of the form \( E_{(k)} = \sum_{i=0}^{k} h_{(i)}(\phi, \partial\phi)F_{(i)} \), and due to (5.11), there are no higher powers of \( \lambda \) in the \( \chi \)-terms of the \( \Lambda \)-bracket between \( e^h \) and \( E_{(k-1)} \).

We now exemplify (5.7). For \( p = 1 \), \( E_{(1)}^{ij} \) is just \( e^i \). For \( p = 2 \) (5.7) yields

\[
E_{(2)}^{ij} = F_{(2)}^{ij} + h\Gamma_{kl}^{ij}g^{jk}\partial\phi^l.
\]

(5.12)

\( E_{(2)}^{ij} \) is antisymmetric on its own, and no anti-symmetrization is needed, since the symmetric part of \( F_{(2)} \) and \( \Gamma g\partial\phi \) cancel:

\[
e^{(i}e^{j)} = -h\Gamma_{kl}^{ij}g^{jk}\partial\phi^l = h\partial g^{ij}.
\]

(5.13)

For \( p = 3 \) and \( p = 4 \) we get

\[
E_{(3)}^{ij_1j_3} = e^{i_1}E_{(2)}^{j_1j_3} + 2h\Gamma_{kl}^{ij}g^{k_{i_2}j_2}\partial\phi^l e^{i_3} = F_{(3)}^{ij_1j_3} + 3h\Gamma_{kl}^{ij}g^{k_{i_2}j_2}\partial\phi^l e^{i_3},
\]

and

\[
E_{(4)}^{ij_1j_3j_4} = e^{i_1}E_{(3)}^{ij_1j_3j_4} + 3h\Gamma_{kl}^{ij}g^{k_{i_2}j_2}\partial\phi^l e^{i_3}e^{i_4}
+ 3h^2\Gamma_{kl}^{ij_1j_3j_4}g^{k_{i_2}j_2}\partial\phi^l_1\Gamma_{kl}^{ij_3j_4}g^{k_{i_4}j_4}\partial\phi^l_2
=F_{(4)}^{ij_1j_3j_4} + 6h\Gamma_{kl}^{ij}g^{k_{i_2}j_2}\partial\phi^l e^{i_3}e^{i_4}
+ 3h^2\Gamma_{kl}^{ij_1j_3j_4}g^{k_{i_2}j_2}\partial\phi^l_1\Gamma_{kl}^{ij_3j_4}g^{k_{i_4}j_4}\partial\phi^l_2,
\]

(5.15)

respectively.

Let us reintroduce the \( \pm \) subscripts. The explicit expression for general \( n \) is the following. Define the nested sum

\[
S_{r,s} = \sum_{k_s=0}^{r} \sum_{k_{s-2}=0}^{k_s+1} \cdots \sum_{k_1=0}^{k_3+1} k_1 \cdots k_{s-2} k_s
\]

and

\[
G_{j_1 \cdots j_n}^{(n)\pm(n-q-1)} = \Gamma_{k_1 \cdots k_q}^{j_1}g^{j_{k_1}j_{k_1}}\partial\phi^1 \cdots \Gamma_{k_{q+1} \cdots k_n}^{j_q}g^{j_{k_{q+1}}j_{k_{q+1}}}\partial\phi^q F_{(n-q-1)}^{j_{k_{q+2}} \cdots j_n}. \]

The subscript denotes how many \( e \)'s are present in the operator. Then the following transforms as a tensor

\[
E_{\pm(n)}^{j_1 \cdots j_n} = E_{+(n)}^{j_1 \cdots j_n} + \sum_{q=1,q \text{ odd}}^{n-1-(n \mod 2)} h_{n-q}^{\frac{n+1}{2}} S_{n-q} G_{\pm(n-q-1)}^{j_1 \cdots j_n}.
\]

(5.16)

This can be proved by induction, see Appendix C.

Using these operators, we can construct

\[
J_{\pm q}^{(n)} = \frac{1}{n!} \omega_{j_1 \cdots j_n} E_{\pm(n)}^{j_1 \cdots j_n},
\]

(5.17)

which is a well-defined section of CDR, and satisfies \( J_{\pm q}^{(n)} \to J_{\pm c}^{(n)} \) when \( h \to 0 \).
5.1.2 Comments about the minus-sector

Since the construction of well-defined tensors only uses the $\chi$-part of the $\Lambda$-bracket between the $e^i$'s, the construction in the minus-sector can be mapped to the construction in the plus-sector. Since we have

\[
\begin{align*}
[e^i_+ + \Lambda e^j_+ &= \hbar \chi g^{ij} + O(\chi^0) \\
F^{j_1...j_k}_{+(k)} &= e^j_+ (e^j_+ (\ldots (e^j_{k-1} e^j_k) \ldots)) \tag{5.18}
\end{align*}
\]

and

\[
\begin{align*}
[ie^i_- + ie^j_- &= \hbar \chi g^{ij} + O(\chi^0) \\
F^{j_1...j_k}_{-(k)} &= ie^j_+ (ie^j_+ (\ldots (ie^j_{k-1} ie^j_k) \ldots)) \tag{5.19}
\end{align*}
\]

we can immediately draw the conclusion that

\[
\begin{align*}
E^{-(0)} &= 1, \tag{5.20} \\
E^{i_1...i_k}_{-(k)} &= ie_{-1}^j E^{j_2...j_k}_{-(k-1)} + \Gamma^i_{kl} \partial \phi^k (ie_{-(0)}^l E^{j_2...j_k}_{-(k-1)}) \tag{5.21}
\end{align*}
\]

or, explicitly,

\[
E^{j_1...j_n}_{-(n)} = F^{j_1...j_n}_{-(n)} + \sum_{q=1, q \text{ odd}}^{n-1-(n \mod 2)} \hbar^{\frac{n+1}{2}} S_{n-q,q} G^{j_1...j_n}_{-(n-q-1)} \tag{5.22}
\]

transform as tensors. Using these operators, we can construct

\[
J^{(n)}_{-q} = \frac{1}{n!} \omega_{j_1...j_n} E^{j_1...j_n}_{-(n)} \tag{5.23}
\]

which is a well-defined section of CDR, and satisfies $J^{(n)}_{-q} \rightarrow J^{(n)}_{-c}$ when $\hbar \rightarrow 0$.

5.2 Well-defined sections constructed from other tensors

Above, we have constructed well-defined sections from given forms. We now investigate whether we can construct well-defined sections from other tensorial objects, in particular from the metric.

From equation (2.21), we see that classically, the energy momentum tensor for the sigma model is constructed using the metric. Also, we will later see that when the target space is a $G_2$ or a Spin(7) manifold, in the quantum case but in the flat space limit, the operator $g_{ij} \partial e^i e^j$ must be considered. It is
therefore an interesting question whether these operators, constructed using the metric, can be modified into well-defined sections of CDR.

After covariantizing the derivative, we are asking whether

\[
\begin{align*}
g_{ij} & \partial \varepsilon^i \varepsilon^j + g_{ij} \Gamma^i_{kl} \partial \phi^k \varepsilon^l \varepsilon^j, \\
g_{ij} & D \varepsilon^i \varepsilon^j + g_{ij} \Gamma^i_{kl} D \phi^k \varepsilon^l \varepsilon^j,
\end{align*}
\]

(5.24)

are well-defined sections of CDR. In the above expressions we must also choose an order of multiplication.

It turns out that this question is very hard to answer. We cannot use the same construction as in section 5.1. That construction relied heavily on the anti-symmetry of the tensors we contract with, allowing us to move around the \( \varepsilon^i \)'s to a certain extent. Since \([\varepsilon^i, \varepsilon^j] = \hbar \chi g^{ij} + O(\chi^0),\) the \( \chi \)-part vanishes when the indices are anti-symmetrized. This luxury we do not have when we contract with symmetric tensors.

A further complication occurs in the construction of the energy momentum tensor since the derivative \( D \) is involved. The presence of \( D \) adds \( \chi \)-terms in the relevant brackets, thus increasing the difficulties in the calculations.

Due to the above reasons we cannot say whether (5.24) are well-defined sections of CDR on a general Riemannian manifold. At present, we do not know in general how to extend them to well-defined sections of CDR. However, for Calabi-Yau and hyperkähler manifolds we know that the energy momentum tensor is well-defined, since we can generate it from well-defined sections due to supersymmetry, see the discussion in section 6.2.

6 Algebra extensions on CDR

6.1 General setup

We now would like to address the question to which extent the classical discussion about symmetry algebra extensions in section 3 can be taken over to the framework of CDR. In the last section, we found how to modify the classical currents into well-defined sections of CDR. We would now like to know whether these currents close under the \( \Lambda \)-bracket in the quantum setup. As a starting remark, let us mention that, although we can write down an explicit expression for \( J_q^{(n)} \), i.e. the quantum counterpart of the classical current formed from an \( n \)-form, it is quite complicated to do a general computation as in (3.7) in the quantum case. From now on, we drop the superscript \( (n) \) and the \( q \)-subscript from the currents \( J_q^{(n)} \).

Two cases have already been worked out. In [3, 13], the authors considered Kähler and hyperkähler manifolds. Interestingly, it was found that the \( N = 2 \)
algebra is anomalous unless we choose the metric to be Ricci-flat, hence CDR requires $\mathcal{M}$ to be a Calabi-Yau manifold. A nice feature of these two cases is that the issue of constructing a well-defined stress energy tensor by hand is circumvented, since it can be generated from the supersymmetry currents, which we know are well-defined sections. In summary: on a Calabi-Yau manifold $(T^\pm, J^\pm)$ are well-defined sections of CDR, and they generate two commuting copies of the superconformal $N=(2,2)$ algebra, with central charge $\frac{3}{2} \dim \mathcal{M}$.

In this article, we will add the four currents constructed from the two additional covariantly constant forms present on a Calabi-Yau 3-fold, namely the holomorphic volume form and its complex conjugate. We will show below that, together $(T^\pm, J^\pm, X^\pm, \bar{X}^\pm)$ generate two commuting copies of the extension of the $N=(2,2)$ algebra first investigated in [22], henceforth called the Odake algebra.

The obvious next step is of course to treat the two exceptional cases, $G_2$ and $Spin(7)$. Here we are in much worse shape, for two reasons. Firstly, we do not know if the stress energy tensor is a well-defined section on these manifolds. Secondly, the “quantum covariant” form of the currents constructed on these two manifolds are more complicated than in the Calabi-Yau case. This, together with the lack of nice choices of coordinates (a feature used extensively in the Calabi-Yau calculation), makes this calculation very complicated. If we were able to solve the second problem, the first problem would be solved in the same way as for the Calabi-Yau case, since we again in principle can generate the stress energy tensor from well-defined currents. Below we compute these algebras with the assumption of a flat metric, and observe that it matches with the algebras calculated in the original work [23].

In the case when the $G_2$-manifold $\mathcal{M}$ is the product of a Calabi-Yau 3-fold and $S^1$, we can attach CDR to each component in the product, and construct the $G_2$ currents from geometrical identities. This is much along the lines of [24]. Since we have reliably calculated the algebra on the Calabi-Yau factor, and the circle is a flat manifold, in this special case we are able to calculate the $G_2$ algebra within the CDR framework.

We have benefited from the Mathematica package Lambda [25], when performing many of these calculations.

### 6.2 $N=2$ algebra

When $\mathcal{M}$ is a Kähler manifold, we can use the Kähler form $\omega$ to construct two currents, which, according to section 5.1, need to be modified with one
Using (2.19)-(2.20), consider the following linear combinations of $J_{\pm}$:

$$J_1 = -J_+ - J_0 = I_j^i D\phi^i S_j + \hbar \Gamma_{jk}^i I_i^l \partial \phi^k,$$

$$J_2 = -J_+ + J_0 = \frac{1}{2} \left( \omega^i_{jk} S_j S_i - \omega^i_{ij} D\phi^i D\phi^j \right),$$

(6.2)

where $I^i_j$ is the complex structure, and $\omega^i_{jk} = \delta^i_k$. These global sections of CDR were studied in detail in [3, 13] and they both generate the $N = 2$ algebra. In particular, in [13] it has been shown that $J_{\pm}$ generate two commuting copies of $N = 2$ algebra if the Kähler manifold is equipped with a Ricci-flat metric $g$.

$$[T_+ \Lambda T_+] = \hbar (2\partial + \chi D + 3\lambda) T_+ + \hbar ^2 \frac{\dim M}{2} \lambda^2 \chi,$$

$$[T_+ \Lambda J_+] = \hbar (2\partial + \chi D + 2\lambda) J_+,$$

$$[J_+ \Lambda J_+] = -\hbar T_+ - \hbar ^2 \frac{\dim M}{2} \lambda^2 \chi,$$

(6.3)

where $T_+$ are defined by the following expressions

$$T_+ + T_- = D\phi^i D S_i + \partial \phi^i S_i - \hbar \partial D \log \sqrt{\det g_{ij}},$$

$$T_+ - T_- = g_{ij} D\phi^i \partial \phi^j + g^i_{sl} S_i D S_j + \Gamma^i_{kl} g^{il} D \phi^k (S_j S_i).$$

(6.4)

This calculations were performed in complex coordinates. It can be shown that $T_+ + T_-$ is well-defined, and moreover $T_+ - T_-$ arises as the brackets of well-defined sections of CDR. Therefore, we know that $T_{\pm}$ are well-defined sections. We could rewrite $T_{\pm}$ in terms of $e_{\pm}$, however the expressions are messy and depend on the particular choice of ordering. The crucial fact is that in the semi-classical limit $\hbar \to 0$ the above quantum $T_{\pm}$ will collapse to the classical expressions (2.21) for the super-Virasoro.

It is important to stress that, using the Jacobi identity for the $\Lambda$-bracket, we can avoid $T_+ - T_-$ in explicit calculations, and it is possible to prove that the “quantum” modified generators $(T_{\pm}, J_{\pm})$ generate two commuting copies of the $N = 2$ superconformal algebra with central charge $\frac{3}{2} \dim M$ if $M$ is Calabi-Yau. We will use similar tricks in the calculation of the extensions of this algebra on a Calabi-Yau 3-fold.
6.3 The Odake algebra

When \( \mathcal{M} \) is a Calabi-Yau 3-fold, we can construct four additional symmetry currents from the holomorphic volume form and its complex conjugate:

\[
X^+ = \frac{1}{3!} \Omega_{\alpha\beta\gamma} e^\alpha e^\beta e^\gamma, \quad X^- = \frac{\bar{\Omega}}{3!} e^\alpha e^\beta e^\gamma,
\]

\[
\bar{X}^+ = \frac{1}{3!} \bar{\Omega}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \bar{e}^\alpha \bar{e}^\beta \bar{e}^\gamma, \quad \bar{X}^- = \frac{\bar{\Omega}}{3!} \bar{e}^\alpha \bar{e}^\beta \bar{e}^\gamma.
\]

(6.5)

Here we have introduced complex coordinates \((\alpha, \bar{\alpha})\). Since we have a Kähler metric \(g\), the “quantum” corrections introduced in section 5.1 vanish, see (5.14). On a Calabi-Yau, we can choose coordinates in which the holomorphic volume forms are constant. This simplifies calculations considerably, although they are still quite lengthy. Below we state the main result of the present paper:

**Theorem.** On a Calabi-Yau 3-fold, we can construct well-defined sections of \(\text{CDR}(T^\pm, J^\pm, X^\pm, \bar{X}^\pm)\), which generates two commuting copies of the Odake algebra, given by (6.3) together with the following brackets:

\[
[X^\pm, \bar{X}^\pm] = -\frac{1}{2} \hbar (iT^\pm J^\pm - DJ^\pm J^\pm - \chi J^\pm J^\pm) + \hbar^2 (i\chi \partial J^\pm + \lambda T^\pm + i\lambda DJ^\pm + 2i\chi \lambda J^\pm) + \hbar^3 \chi \lambda^2,
\]

\[
[J^\pm, X^\pm] = -i\hbar (3\chi + D) X^\pm,
\]

\[
[J^\pm, \bar{X}^\pm] = +i\hbar (3\chi + D) \bar{X}^\pm,
\]

\[
[X^\pm, T^\pm] = \hbar (3\lambda + \chi D + 2\partial) X^\pm,
\]

\[
[\bar{X}^\pm, T^\pm] = \hbar (3\lambda + \chi D + 2\partial) \bar{X}^\pm,
\]

\[
[X^\pm, X^\pm] = 0,
\]

\[
[\bar{X}^\pm, \bar{X}^\pm] = 0,
\]

(6.7)

where the plus- and minus-sectors commute. The semi-classical limit \(\hbar \to 0\) of this vertex algebra is the Poisson vertex algebra given by (3.9) and (3.12) and discussed in section 3.

**Proof.** The calculation is done in Appendix D. In this calculation, no further geometrical constraints are found in order for the algebra to close. \(\square\)

As in the classical case, there are non-trivial constraints these currents satisfy. The quantum analog of (3.14) is given by

\[
J^\pm X^\pm = -i\hbar \partial X^\pm, \quad J^\pm \bar{X}^\pm = i\hbar \partial \bar{X}^\pm.
\]

(6.8)

These identities are needed to check the Jacobi conditions for (6.7).
6.4 $G_2$ and $Spin(7)$

As mentioned in the general discussion, we cannot choose coordinates in which the components of the covariantly constant forms, existing on $G_2$ and $Spin(7)$-manifolds, are constant. Moreover, we are not aware of a choice of coordinates in which the “quantum corrections” to the currents (see section 5) vanishes. This makes the computations of the symmetry algebras quite challenging. Taking the “flat space limit”, that is in practice choosing a constant metric and constant forms, we can compute a closed algebra, which will have several quantum terms compared to the ones computed in the classical setup. It is an open question if the algebras still hold in the general curved case, with the modifications of the currents in order to make them well-defined, since we are presently not able to do the calculations.

This subsection contains the $\beta\gamma$-$bc$ system realization of two commuting copies of the algebras found in [23, 24]. Moreover, we show that these algebras are quantizations of the classical algebras from section 3, thus showing that the Howe-Papadopoulos Poisson algebras [11, 12] for $G_2$ and $Spin(7)$ are the classical versions of the Shatashvili-Vafa vertex algebras [23].

6.4.1 $G_2$

Let us choose a flat metric $g_{ij}$ and constant $\Pi_{ijk}$, with $\ast \Pi_{ijkl} = \Psi_{ijkl}$. Let us define the currents

$$
\Pi_+ = \frac{1}{3!} \Pi_{ijk} e^i_+ e^j_+ e^k_+, \\
\Pi_- = \frac{\ell^3}{3!} \Pi_{ijk} e^i_- e^j_- e^k_-,
$$

(6.9)

$$
\Psi_\pm = -\frac{1}{4!} \Pi_{ijkl} e^i_\pm e^j_\pm e^k_\pm \pm \hbar \frac{1}{2} g_{ij} \partial e^i_\pm e^j_\pm.
$$

Then $\Pi_\pm$ are the primary fields with conformal weight $\frac{3}{2}$ with respect to the Virasoro fields $T_\pm = \pm g_{ij} D e^i_\pm e^j_\pm$ respectively. $\Psi_\pm$ have conformal weight 2 but are not primary:

$$
[T_\pm \Lambda \Psi_\pm] = \hbar (2\partial + 4\lambda + \chi D) \Psi_\pm + \hbar^2 \frac{1}{2} \chi \lambda T_\pm + \hbar^3 \frac{7}{12} \lambda^3.
$$

(6.10)
We compute the following brackets between the currents:

\[
\begin{align*}
[\Pi_{\pm} \Lambda \Pi_{\pm}] &= -3\hbar D\Psi_{\pm} - \hbar^2 3\partial T_{\pm} - 6\hbar \chi \Psi_{\pm} - \hbar^2 3\lambda T_{\pm} - \hbar^3 \frac{7}{2} \lambda^2 \chi , \\
[\Pi_{\pm} \Lambda \Psi_{\pm}] &= +3hT_{\pm} \Pi_{\pm} + \hbar^2 5\frac{5}{2} \chi \partial \Pi_{\pm} + \hbar^2 3\lambda D \Pi_{\pm} + \hbar^3 \frac{15}{2} \chi \lambda \Pi_{\pm} , \\
[\Psi_{\pm} \Lambda \Psi_{\pm}] &= +\hbar^3 \frac{9}{4} \partial^2 T_{\pm} - \hbar^2 \frac{9}{2} D \partial \Psi_{\pm} + 10hT_{\pm} \Psi_{\pm} + 3h\Pi_{\pm} D \Pi_{\pm} , \quad (6.11) \\
& \quad + \hbar^2 5 (\chi \partial + \lambda D + 2\lambda \chi) \Psi_{\pm} \\
& \quad + \hbar^3 \frac{9}{4} \lambda (\partial + \lambda) T_{\pm} + \hbar^4 \frac{35}{24} \lambda^3 \chi^3 .
\end{align*}
\]

We see that taking the limit \( \hbar \to 0 \) we get back the algebra computed with Poisson brackets in (3.20). Using the flat realization of the tensors \( \Psi \) and \( \Pi \) given in A.3.4, we find the following relation between currents

\[
h^2 \frac{1}{4} \partial^2 T_{\pm} - h^2 D \partial \Psi_{\pm} + 2T_{\pm} \Psi_{\pm} + \Pi_{\pm} D \Pi_{\pm} = 0 , \quad (6.12)
\]

which is the quantum version of the classical relation (3.21). With the formula (6.12), the constant part of \([\Psi \Lambda \Psi]\) can be written

\[
[\Psi_{\pm} \Lambda \Psi_{\pm}]_1 = \hbar^3 \frac{3}{4} \partial^2 T_{\pm} + \hbar^2 \frac{3}{2} D \partial \Psi_{\pm} + 4hT_{\pm} \Psi_{\pm} . \quad (6.13)
\]

Decomposing the currents as

\[
T_{\pm} = G_{\pm} + 2\theta L_{\pm} , \\
\Pi_{\pm} = \phi_{\pm} + \theta K_{\pm} , \\
\Psi_{\pm} = -X_{\pm} - \theta M_{\pm} , \quad (6.14)
\]

we find two copies of the same algebra as in [23].

We notice from the structure of the algebra (6.11) that both the Virasoro field \( T \) and the field \( \Psi \) is generated from the field \( \Pi \). Since we know from section 5.1 how to make \( \Pi \) well defined on a curved target manifold, in principle we can derive candidates for the Virasoro field \( T \) and the field \( \Psi \) on a general \( G_2 \) manifold. As mentioned above, these calculations are technically complicated, and we are not able to perform them at the present stage.

6.4.2 Spin(7)

Let us again choose a flat metric \( g_{ij} \) and a constant \( \Theta_{ijkl} \). Define

\[
\Theta_{\pm} = \frac{1}{4!} \Theta_{ijkl} e^i_{\pm} e^j_{\pm} e^k_{\pm} e^l_{\pm} \pm \frac{1}{2} \hbar g_{ij} \partial e^i_{\pm} e^j_{\pm} . \quad (6.15)
\]
We compute the following brackets:

\[
[T_\pm \Theta_\pm] = \hbar (2\partial + 4\lambda + \chi D) \Theta_\pm + \hbar^2 \frac{1}{2} \chi \lambda T_\pm + \hbar^3 \frac{2}{3} \lambda^3,
\]

\[
[\Theta_\pm \Theta_\pm] = \hbar^2 \frac{5}{2} \partial D \Theta_\pm + \hbar^3 \frac{5}{4} \partial^2 T_\pm + 6\hbar T_\pm \Theta_\pm + 8\hbar^2 (\chi \partial + \lambda D + 2\lambda \chi) \Theta_\pm + \hbar^3 \frac{15}{4} \lambda (\partial + \lambda) T_\pm + \hbar^4 \frac{8}{3} \lambda^3 \chi.
\] (6.16)

Taking the limit \( \hbar \to 0 \), we get back the classical result (3.23). Decomposing the superfields as:

\[
T_\pm = G_\pm + 2\theta L_\pm, \quad \Theta_\pm = \tilde{X}_\pm + \theta \tilde{M}_\pm,
\] (6.17)

we find two copies of the same algebra as in [23] (see also [17]).

6.4.3 \( G_2 = CY_3 \times S^1 \)

There is a special case of a \( G_2 \)-manifold which we can address without taking the flat space limit. Consider a \( G_2 \)-manifold \( \mathcal{M} \) of the type \( \mathcal{M} = CY_3 \times S^1 \), where \( CY_3 \) is a Calabi-Yau threefold and \( S^1 \) is a circle. This is an example of a compact \( G_2 \)-manifold.

We can attach to the Calabi-Yau factor the sheaf of Vertex algebras considered in 6.3. We can add to these generators a free \( \beta\gamma\mathrm{-bc} \) system, generated by \( e_\pm = \frac{1}{\sqrt{2}} (S \pm D\phi) \), where \( \phi \) is identified with the coordinate on the circle. The super-Virasoro field is given by \( T_\pm^{S^1} = \pm e_\pm D e_\pm \). On \( \mathcal{M} \) the corresponding 3-form \( \Pi \) and its Hodge dual \( \Psi \) are defined by the following expressions

\[
\Pi = Re(\Omega) + J \wedge dX^7, \quad \Psi = Im(\Omega) \wedge dX^7 + \frac{1}{2} J \wedge J,
\] (6.18)

where \( J \) is the Kähler form, \( \Omega \) is the holomorphic volume form and \( X^7 \) is the coordinate along the circle. It is therefore natural to construct new operators from the operators in the Odake algebra by

\[
\Pi_+ = X_+ + \bar{X}_+ + J_+ e_+,
\] (6.19)

where we temporarily only consider the plus-sector. Let us denote

\[
R_+ = X_+ + \bar{X}_+, \quad L_+ = -i(X_+ - \bar{X}_+),
\] (6.20)
(where $R$ is for real and $I$ for imaginary). We then have the following bracket between $\Pi_+$ and $\Pi_+$:

\[
[\Pi_+ \Lambda \Pi_+ ] = 3\hbar D \left( \frac{J_+ J_+}{2} - I_+ e_+ + \frac{1}{2} \hbar e_+ \partial e_+ \right) \\
+ 6\hbar \chi \left( \frac{J_+ J_+}{2} - I_+ e_+ + \frac{1}{2} \hbar e_+ \partial e_+ \right) \\
- \hbar^2 \frac{3}{2} \partial (T_+ + e_+ D e_+) - \hbar^2 3\chi (T_+ + e_+ D e_+) - \hbar^3 \frac{T}{2} \chi \lambda^2 .
\] (6.21)

We see that we reproduce the first part of the $G_2$-algebra, with $\Psi$ given by

\[
\Psi_+ = \frac{J_+ J_+}{2} - I_+ e_+ + \frac{1}{2} \hbar e_+ \partial e_+ .
\] (6.22)

This is what we expected from (6.18). When computing the rest of the algebra, we have to use the identities

\[
J_+ I_+ = -\hbar \partial R_+ , \quad J_+ R_+ = \hbar \partial I_+ .
\] (6.23)

which hold on any Calabi-Yau manifold and are just a rewritten form of (6.8). A similar analysis can be performed for the minus-sector. Computing the rest of the brackets, using the above identities, we generate the full $G_2$ algebra (6.11). Notice that we have proved the following proposition:

**Proposition.** There exists an embedding of the vertex algebra [23] (see (6.11)) associated to a manifold of special holonomy $G_2$ into the tensor product of the Odake vertex algebra [22] (see Section 6.3) associated to a Calabi-Yau 3-fold and a free boson-fermion system generated by one odd superfield $e$, such that $[ e \Lambda e ] = \chi$.

## 7 Discussion

In this article we studied extensions of the super-Virasoro algebra within the framework of the chiral de Rham complex. The main result of this work is the construction of two commuting copies of the Odake algebra associated to any Calabi-Yau 3-fold. This is the first example of a non-linearly generated algebra arising in this manner. Another result of this article is the systematic construction of global sections of CDR from antisymmetric tensors on $M$. Moreover, we presented the full classical and partial quantum results for general extensions of the super-Virasoro algebra. The central idea behind our
consideration is the interpretation of CDR as a formal canonical quantization of the non-linear sigma model.

The main unresolved questions are the full calculations of the algebras for $G_2$ and $\text{Spin}(7)$ manifolds. For $G_2$, we have a well defined generator in the curved case and in principle we can generate the other relevant operators through the brackets. By construction, these will be well-defined. Unfortunately, at the present moment, these calculations appear to be too complicated to carry out.

One motivation for studying manifolds with special holonomies comes from string theory. After compactification, let $\mathcal{M}$ be the internal manifold. In order to have space-time supersymmetry, $\mathcal{M}$ must admit a covariantly constant spinor. For different dimensions of $\mathcal{M}$, the constraint of minimal space-time supersymmetry leads to different choices of holonomy groups. For dimensions 6,7 and 8 the holonomy groups are $SU(3)$, $G_2$ and $\text{Spin}(7)$. In the quantum setup, the extensions of $N=(2,2)$ symmetry algebra for $d = 6$ were studied for the first time in [22], whereas the cases $d = 7$ and $d = 8$ were first investigated in [23]. A common feature for the calculations of these algebra extensions performed in the mentioned papers, is that they are performed in the large volume limit, which means that the metric is treated like a flat metric. With the construction of the CDR, and its interpretation as the canonical quantization of the sigma model, we can begin to compute the symmetry algebras in a reliable way without taking the large volume limit.

Finally, let us point out that our considerations give an embedding of the differential forms of $\mathcal{M}$ into CDR which is different from the original work [1]. Our embedding is motivated by sigma model considerations, and it would be very interesting to further study the properties of this map. We hope to come back to this issue elsewhere.

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Appendices

A Special holonomy manifolds

In this appendix, we collect the relevant relations of the invariant tensors on special holonomy manifolds.

A.1 Kähler manifolds

On a Kähler manifold we have the Kähler form $\omega = g I$, where $g$ is a metric and $I$ is a complex structure. We then have $\omega g^{-1} = -g$. In components

$$\omega_{ij} g^{jk} \omega_{kl} = -g_{il}.$$  (A.1)

A.2 Calabi-Yau manifolds

On a Calabi-Yau $n$-fold, we define the holomorphic volume form $\Omega$ and its complex conjugate $\bar{\Omega}$. The following relation holds

$$\Omega_{\alpha_1 \alpha_2 ... \alpha_n} g^{\alpha_1 \bar{\alpha}_1} \Omega_{\bar{\alpha}_1 \bar{\alpha}_2 ... \bar{\alpha}_n} = g_{\alpha_2 \bar{\alpha}_2} ... g_{\alpha_n \bar{\alpha}_n} + ... ,$$  (A.2)

where dots stand for the terms required by antisymmetrization in $\alpha_1 ... \alpha_n$ and $\bar{\alpha}_1 ... \bar{\alpha}_n$. In particular, for a Calabi-Yau 3-fold we have

$$\Omega_{\alpha_1 \alpha_2 \alpha_3} g^{\alpha_1 \bar{\alpha}_1} \Omega_{\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3} = g_{\alpha_2 \bar{\alpha}_2} g_{\alpha_3 \bar{\alpha}_3} - g_{\alpha_2 \bar{\alpha}_3} g_{\alpha_3 \bar{\alpha}_2} .$$  (A.3)

We also have the Kähler form $\omega$, and the following contraction between $\Omega$ and $\omega$:

$$\omega_{\alpha \beta} g^{\alpha \beta_1} \Omega_{\beta_1 \beta_2 ... \beta_n} = i \Omega_{\alpha \beta_2 ... \beta_n} .$$  (A.4)

A.3 $G_2$-manifolds

On a $G_2$ manifold we have couple of forms: a 3-form $\Pi$ and its dual $*\Pi \equiv \Psi$. The following formulas are collected from [19].

A.3.1 Contracting $\Pi$ with itself

$$\Pi_{ijk} \Pi_{abc} g^{ia} g^{jb} g^{kc} = 42$$  (A.5)
$$\Pi_{ijk} \Pi_{abc} g^{jb} g^{kc} = 6 g_{ia}$$  (A.6)
$$\Pi_{ijk} \Pi_{abc} g^{kc} = g_{ia} g_{jb} - g_{ib} g_{ja} - \Psi_{ijab}$$  (A.7)
A.3.2 Contracting Π and Ψ

\[ Π_{ijk} \Psi_{abcd} g^{ib} g^{jc} g^{kd} = 0 \] (A.8)
\[ Π_{ijk} \Psi_{abcd} g^{ic} g^{kd} = -4Π_{iab} \] (A.9)
\[ Π_{ijk} \Psi_{abcd} g^{ic} g^{jd} = g_{ia} Π_{jbc} + g_{ib} Π_{ajc} + g_{ic} Π_{abj} \] (A.10)
\[ - g_{aj} Π_{ibc} - g_{bj} Π_{aic} - g_{cj} Π_{abi} \]

A.3.3 Contracting Ψ with itself

\[ Ψ_{ijkl} \Psi_{abcd} g^{ia} g^{jb} g^{kc} g^{ld} = 168 \] (A.11)
\[ Ψ_{ijkl} \Psi_{abcd} g^{ic} g^{jd} = 24g_{ia} \] (A.12)
\[ Ψ_{ijkl} \Psi_{abcd} g^{ic} g^{jd} = 4g_{ia} g_{jb} - 4g_{ib} g_{ja} - 2Ψ_{ijab} \] (A.13)
\[ Ψ_{ijkl} \Psi_{abcd} g^{ic} g^{jd} = -Π_{ajk} Π_{ibe} - Π_{iak} Π_{jbc} - Π_{ija} Π_{kbc} + g_{ia} g_{jk} g_{ke} + g_{ib} g_{jc} g_{ka} + g_{ic} g_{ja} g_{kb} \]
\[ - g_{ja} g_{jk} g_{kb} - g_{ib} g_{ja} g_{kc} - g_{ic} g_{jb} g_{ka} \]
\[ - g_{ja} Ψ_{kbc} - g_{ja} Ψ_{kbe} - g_{ka} Ψ_{ijb} \]
\[ + g_{ab} Ψ_{ijkc} - g_{ac} Ψ_{ijkb} \]

A.3.4 Local representations of Π and Ψ

In a local orthonormal and flat basis, where the metric is \( \sum dx^i \otimes dx^i \), the forms Π and Ψ can be written [23] as

\[ Π = dx^1 dx^2 dx^3 dx^5 + dx^1 dx^3 dx^6 + dx^1 dx^4 dx^7 - dx^2 dx^3 dx^5 + \]
\[ dx^2 dx^4 dx^6 - dx^3 dx^4 dx^5 + dx^5 dx^6 dx^7, \] (A.15)
\[ Ψ = dx^1 dx^2 dx^3 dx^4 - dx^1 dx^2 dx^5 dx^7 + dx^1 dx^3 dx^5 dx^7 - \]
\[ dx^1 dx^4 dx^5 dx^6 + dx^2 dx^3 dx^5 dx^6 + \]
\[ dx^2 dx^5 dx^7 + dx^3 dx^4 dx^6 dx^7, \] (A.16)

where the product is understood as the wedge product.
A.4 \textit{Spin}(7)-manifolds

On a \textit{Spin}(7)-manifold we have a self dual 4-form $\Theta$. These formulas are collected from [20].

\begin{align}
\Theta_{ijkl}\Theta_{abcd}g^{ia}g^{jb}g^{kc}g^{ld} &= 336 \quad (A.17) \\
\Theta_{ijkl}\Theta_{abcd}g^{jb}g^{kc}g^{ld} &= 42g_{ia} \quad (A.18) \\
\Theta_{ijkl}\Theta_{abcd}g^{kc}g^{ld} &= 6g_{ia}g_{jb} - 6g_{ib}g_{ja} - 4\Theta_{ijab} \quad (A.19) \\
\Theta_{ijkl}\Theta_{abcd}g^{ld} &= g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} + g_{ic}g_{ja}g_{kb} - g_{ia}g_{jc}g_{kb} - g_{ib}g_{ja}g_{kc} - g_{ic}g_{jb}g_{ka} \quad (A.20)
\end{align}

B \textit{\Lambda}-bracket calculus

In this appendix we collect some properties of \textit{\Lambda}-bracket calculus. For further explanations and details, the reader may consult [17].

- Basic relations:

\begin{align}
D^2 &= \partial \quad [D, \partial] = 0 \quad [D, \lambda] = 0 \quad [\partial, \lambda] = 0 \quad (B.1) \\
\chi^2 &= -\lambda \quad [D, \chi] = 2\lambda \quad [\partial, \chi] = 0 \quad (B.2)
\end{align}

- Sesquilinearity:

\begin{align}
[a_{\Lambda} b] &= \chi[a_{\Lambda} b] \quad [a_{\Lambda} Db] = -(-1)^a (D + \chi) [a_{\Lambda} b] \quad (B.3) \\
[\partial a_{\Lambda} b] &= -\lambda[a_{\Lambda} b] \quad [a_{\Lambda} \partial b] = (\partial + \lambda) [a_{\Lambda} b] \quad (B.4)
\end{align}

- Skew-symmetry:

\begin{align}
[a_{\Lambda} b] &= (-1)^{ab} [b_{-\Lambda - \nabla a}] \quad (B.5)
\end{align}

The bracket on the right hand side is computed as follows: first compute $[b_{\Gamma} a]$, where $\Gamma = (\gamma, \eta)$, then replace $\Gamma$ by $(-\lambda - \partial, -\chi - D)$.

- Jacobi identity:

\begin{align}
[a_{\Lambda} [b_{\Gamma} c]] &= -(-1)^a [[a_{\Lambda} b]_{\Gamma + \Lambda} c] + (-1)^{(a+1)(b+1)} [b_{\Gamma} [a_{\Lambda} c]] \quad (B.6)
\end{align}

where the first bracket on the right hand side is computed as in (B.5).
• Quasi-commutativity:
\[ ab - (-1)^{ab} ba = \int_{-\gamma}^{0} [a_\Lambda b] d\Lambda \]  \hspace{1cm} (B.7)
where the integral \( \int d\Lambda \) is \( \partial_\chi \int d\lambda \).

• Quasi-associativity:
\[ (ab)c - a(bc) = \sum_{j \geq 0} a_{(-j-2)(j)} b_{(j)} c + (-1)^{ab} \sum_{j \geq 0} b_{(-j-2)(j)} a_{(j)} c \]  \hspace{1cm} (B.8)

• Quasi-Leibniz (non-commutative Wick formula):
\[ [a_\Lambda bc] = [a_\Lambda b]c + (-1)^{ab} [a_\Lambda c] + \int_{0}^{\Lambda} [ [a_\Lambda b] \Gamma c ] d\Gamma \]  \hspace{1cm} (B.9)

C  Proof that (5.16) solves (5.7)

In this section, we will show that
\[
E_{j_1 \ldots j_n}^{(0)} = 1 ,
\]
where
\[
E_{j_1 \ldots j_n}^{(k)} \equiv e^{j_1} E_{(k-1)}^{j_2 \ldots j_k} + \Gamma_{kl}^{j_1} \partial \phi^{k} (e_{(0)}^{j_1} E_{(k-1)}^{j_2 \ldots j_k}) .
\]  \hspace{1cm} (C.4)

We prove this for \( n \) even, for notational convenience. The proof for \( n \) odd is the same. We will always think of the free upper indices as anti-symmetrized, and we set \( \hbar = 1 \). We will extensively use the relations
\[ [e_i^j_\Lambda e^j_\Lambda x]_{\chi} = g^{ij} , \]  \hspace{1cm} (C.6)
\[ [e_i^j_\Lambda F_{(k)}^{j_1 \ldots j_k}]_{\chi} = k g^{j_1} F_{(k-1)}^{j_2 \ldots j_k} , \]  \hspace{1cm} (C.7)
\[ [h(\phi, \partial \phi) e_i^j_\Lambda F_{(k)}^{j_1 \ldots j_k}]_{\chi} = [e_i^j_\Lambda h(\phi, \partial \phi) F_{(k)}^{j_1 \ldots j_k}]_{\chi}
\]  \hspace{1cm} (C.8)
\[ = h(\phi, \partial \phi) [e_i^j_\Lambda F_{(k)}^{j_1 \ldots j_k}]_{\chi} , \]
where \([\Lambda]_\chi\) denotes the \(\chi\)-part of the bracket, and \(h\) is an arbitrary function of \(\phi\) and \(\partial\phi\).

**Proof.** We will use induction. That it is true for \(n = 2\) follows without difficulty. From the recursion relation we find

\[
E^{j_1j_2}_2 = F^{j_1j_2}_2 + \Gamma^{j_2}_{kl} \partial \phi^k g^{lj_2},
\]
and one easily convince oneself that we find the same result from (C.1).

Let us now assume that it is true for \(n = p - 2\), i.e.

\[
E^{j_3...j_p}_{p-2} = F^{j_3...j_p}_{(p-2)} + \sum_{q=1}^{p-3} S_{p-2-q,q} G^{j_3...j_p}_{(p-q-3)},
\]

(C.10)

Using the recursion relation we find

\[
E^{j_3...j_p}_p = e^{j_1} \left\{ e^{j_2} E^{j_3...j_p}_{(p-2)} + \Gamma^{j_2}_{kl} \partial \phi^k \left( e^{(0|1)}_l E^{j_3...j_p}_{p-2} \right) \right\}
+ \Gamma^{j_1}_{st} \partial \phi^s \left\{ e^{(0|1)}_l \left( e^{j_2} E^{j_3...j_p}_{p-2} + \Gamma^{j_2}_{kl} \partial \phi^k (e^{(0|1)}_l E^{j_3...j_p}_{p-2}) \right) \right\}. 
\]

(C.11)

Using the induction assumption (C.10), the first term gives us

\[
e^{j_1} e^{j_2} E^{j_3...j_p}_{p-2} = F^{j_1...j_p}_{(p)} + \sum_{q=1}^{p-3} S_{p-2-q,q} G^{j_1...j_p}_{(p-q-1)},
\]

(C.12)

Since we anti-symmetrize the indices, we are allowed to move around the \(e\)'s freely.

Next, we need to find \((e^{(0|1)}_l E^{j_3...j_p}_{p-2})\). Using the above relations we find that

\[
\left[ e^l \Lambda F^{j_3...j_p}_{p-2} \right]_\chi = (p - 2)g^{j_3} F^{j_4...j_p}_{p-3},
\]

\[
\left[ e^l \Lambda G^{j_3...j_p}_{(p-q-3)} \right]_\chi = (p - q - 3)g^{j_3} G^{j_4...j_p}_{(p-q-4)},
\]

where we have used that \(q\) is odd when moving the indices in the last line.

These two relations give us

\[
(e^{(0|1)}_l E^{j_3...j_p}_{p-2}) = (p - 2)g^{j_3} F^{j_4...j_p}_{p-3} + \sum_{q=1}^{p-3} (p - q - 3) S_{p-2-q,q} g^{j_3} G^{j_4...j_p}_{(p-q-4)}.
\]

(C.15)
From this we deduce

\[ \Gamma_{kj}^{jl} \partial \phi^k \left( e_{(0)1}^l E_{p-2}^{j_3 \ldots j_p} \right) = (p - 2)G_{p-3}^{j_2 \ldots j_p} + \sum_{q=1}^{p-3} (p - q - 3)S_{p-2-q,q}G_{(p-q-4)}^{j_2 \ldots j_p} \]

\[ \equiv H^{j_2 \ldots j_p} . \]  

(C.16)

We find the second term to be

\[ e^{j_1} H^{j_2 \ldots j_p} = (p - 2)G_{p-2}^{j_1 \ldots j_p} + \sum_{q=1}^{p-3} (p - q - 3)S_{p-2-q,q}G_{(p-q-3)}^{j_1 \ldots j_p} . \]  

(C.17)

For the third term we need

\[ \left[ e^t \Lambda e^{j_2} E_{p-2}^{j_3 \ldots j_p} \right] \chi = g^{j_2} E_{p-2}^{j_3 \ldots j_p} \]

\[ - e^{j_2} \left\{ (p - 2)g^{j_3} F_{(p-3)}^{j_1 \ldots j_p} + \sum_{q=1}^{p-3} (p - q - 3)S_{p-2-q,q}g^{j_2} G_{(p-q-4)}^{j_1 \ldots j_p} \right\} \]

\[ = g^{j_2} E_{p-2}^{j_3 \ldots j_p} \]

\[ + \left\{ (p - 2)g^{j_2} F_{(p-2)}^{j_1 \ldots j_p} + \sum_{q=1}^{p-3} (p - q - 3)S_{p-2-q,q}g^{j_2} G_{(p-q-3)}^{j_1 \ldots j_p} \right\} \]

\[ = \left( e_{(0)1}^t (e^{j_2} E_{p-2}^{j_3 \ldots j_p}) \right) . \]

(C.18)

From this we find the third term to be

\[ \Gamma^{j_1} \partial \phi^s \left( e_{(0)1}^t (e^{j_2} E_{p-2}^{j_3 \ldots j_p}) \right) = (p - 1)G_{p-2}^{j_1 \ldots j_p} \]

\[ + \sum_{q=1}^{p-3} (p - q - 2)S_{p-2-q,q}G_{(p-q-3)}^{j_1 \ldots j_p} . \]  

(C.19)

In the same way as for the other terms we find the fourth term to be

\[ \Gamma^{j_1} \partial \phi^s \left( e_{(0)1}^t H^{j_2 \ldots j_p} \right) = (p - 2)(p - 3)G_{p-4}^{j_1 \ldots j_p} \]

\[ + \sum_{q=1}^{p-3} (p - q - 3)(p - q - 4)S_{p-2-q,q}G_{(p-q-5)}^{j_1 \ldots j_p} . \]  

(C.20)
We now need to sum up (C.12), (C.17), (C.19) and (C.20), and we indeed get

\[ F_{\{j_1 \cdots j_p\}}^{(p)} + \sum_{q=1}^{p-1} S_{p-q,q} G_{\{j_1 \cdots j_p\}}^{(p-q-1)} . \]  

(C.21)

We see that the term \( F_{\{j_1 \cdots j_p\}}^{(p)} \) is present in (C.12).

Next, we find the coefficients of \( G_{\{j_1 \cdots j_p\}}^{(p)} \) for \( q = 1, 3, \ldots, p - 1 \). We treat \( q = 1 \) and \( q = 3 \) separately. Inspection of (C.12), (C.17), (C.19) and (C.20) gives us:

- **\( G_{p-2} \):**
  \[ S_{p-3,1} + (p - 2) + (p - 1) = S_{p-1,1} \]

- **\( G_{p-4} \):**
  \[ S_{p-5,3} + (p - 4)S_{p-3,1} + (p - 3)S_{p-3,1} + (p - 2)(p - 3) = \]
  \[ = S_{p-5,3} + (p - 4)S_{p-3,1} + (p - 3)S_{p-2,1} = \]
  \[ = S_{p-3,3} \]

The terms that we have not analyzed so far are

\[
\sum_{q=1}^{p-3} S_{p-2-q,q} G_{\{j_1 \cdots j_p\}}^{(p-q-1)} + \sum_{q=1}^{p-3} (p - q - 3)S_{p-2-q,q} G_{\{j_1 \cdots j_p\}}^{(p-q-3)} \\
+ \sum_{q=3}^{p-3} (p - q - 2)S_{p-2-q,q} G_{\{j_1 \cdots j_p\}}^{(p-q-3)} \\
+ \sum_{q=3}^{p-3} (p - q - 3)(p - q - 4)S_{p-2-q,q} G_{\{j_1 \cdots j_p\}}^{(p-q-5)}
\]

\[
= \left\{ \sum_{q=5}^{p-3} S_{p-2-q,q} + \sum_{q=5}^{p-1} (p - q - 1)S_{p-q,q-2} \right\} G_{\{j_1 \cdots j_p\}}^{(p-q-1)} \\
+ \left\{ \sum_{q=5}^{p-1} (p - q)S_{p-q,q-2} + \sum_{q=5}^{p-1} (p - q + 1)(p - q)S_{p+2-q,q-4} \right\} G_{\{j_1 \cdots j_p\}}^{(p-q-1)}
\]
where we in the second line have changed the summation indices and pulled out the common factor $G^{j_1...j_p}_{(p-q-1)}$. Ignoring this factor, we find the coefficient to be

$$
\sum_{q=5}^{p-3} S_{p-2-q,q} + \sum_{q=5}^{p-1} (p-q-1)S_{p-q,q-2}
$$

$$
+ \sum_{q=5}^{p-1} (p-q)S_{p-q,q-2} + \sum_{q=5}^{p-1} (p-q+1)(p-q)S_{p+2-q,q-4}
$$

$$
= \sum_{q=5}^{p-3} S_{p-2-q,q} + \sum_{q=5}^{p-1} (p-q-1)S_{p-q,q-2} + \sum_{q=5}^{p-1} (p-q)S_{p+1-q,q-2}
$$

$$
= \sum_{q=5}^{p-3} S_{p-2-q,q} + \sum_{q=5}^{p-1} (p-q-1)S_{p-q,q-2} + \sum_{q=5}^{p-1} (p-q)S_{p+1-q,q-2} + S_{2,p-3}
$$

$$
= \sum_{q=5}^{p-3} S_{p-q,q} + S_{1,p-1}
$$

$$
= \sum_{q=5}^{p-1} S_{p-q,q}
$$

and we are done. \qed

D Calculation of the Odake algebra

First, we focus on the plus-sector and define the generators $X_+, T_+$ and $J_+$. We then compute the Odake algebra. After this, we show that the plus- and minus-sectors commute. The calculation of the minus-sector can be done in an analogous way, and we will therefore not give the explicit calculation. In this section we will suppress the factors of $\hbar$.

We choose coordinates where the holomorphic volume form is constant, which in particular implies $\Gamma^\alpha_{\alpha\beta} = \Gamma^\alpha_{\bar{\alpha}\bar{\beta}} = g^{ij}g_{ij,k} = 0$ on the Kähler manifold. On Calabi-Yau manifold we can always choose such coordinates locally.
D.1 Setup

Let us define
\[ e^i_+ \equiv \frac{1}{\sqrt{2}} (g^{ij} S_j + D\phi^i) , \]
\[ e^i_- \equiv \frac{1}{\sqrt{2}} (g^{ij} S_j - D\phi^i) . \] (D.1)

These fields satisfy:
\[
[e^i_\pm e^j_\pm] = \pm \chi g^{ij} + \frac{1}{\sqrt{2}} \left(g^{kj} \Gamma^i_{mk} e^m_+ - g^{ki} \Gamma^j_{mk} e^m_+ \right) ,
\]
\[
[e^i_+ e^j_-] = \frac{1}{\sqrt{2}} \left(g^{kj} \Gamma^i_{mk} e^m_- - g^{ki} \Gamma^j_{mk} e^m_- \right), \] (D.2)
\[
[e^i_\pm f(\phi)] = \frac{1}{\sqrt{2}} g^{ij} f_{,j} ,
\]
\[
[e^i_\pm S_j] = \pm \frac{1}{\sqrt{2}} \chi \delta_j^i - g_{,j}^i S_k .
\]

On an Hermitian manifold, with the Hermitian connection, the brackets simplify:
\[
[e^a_\pm e^\beta_\pm] = 0 ,
\]
\[
[e^a_+ e^\beta_-] = 0 ,
\]
\[
[e^a_+ e^\beta_+] = \pm \chi g^{a\beta} + \frac{1}{\sqrt{2}} \left(g_{,\beta}^a e^\beta_+ - g_{,\gamma}^a e^\gamma_+ \right) , \] (D.3)
\[
[e^a_+ e^\beta_-] = \frac{1}{\sqrt{2}} \left(g_{,\beta}^a e^\beta_- - g_{,\beta}^a e^\beta_+ \right) .
\]

D.2 Defining the generators $X, T$ and $J$.

We construct the generators $X_\pm$ and $\bar{X}_\pm$ from the invariant volume forms:
\[
X_+ \equiv \frac{1}{3!} \Omega_{a\beta\gamma} e^a_+ e^\beta_+ e^\gamma_+ , \quad X_- \equiv \frac{-i}{3!} \Omega_{a\beta\gamma} e^a_- e^\beta_- e^\gamma_- , \] (D.4)
\[
\bar{X}_+ \equiv \frac{1}{3!} \bar{\Omega}_{\bar{a}\bar{\beta}\bar{\gamma}} e^a_+ e^\beta_+ e^\gamma_+ , \quad \bar{X}_- \equiv \frac{-i}{3!} \bar{\Omega}_{\bar{a}\bar{\beta}\bar{\gamma}} e^a_- e^\beta_- e^\gamma_- . \] (D.5)

Note that these are well-defined since we have associativity in the above expressions. This is consistent with the vanishing of the connection part in (5.14).
From the Kähler form, $\omega = gI$, we construct the currents $J_{\pm}$:

$$J_{\pm} = \pm \frac{1}{2} (\omega_{ij} e_i^e e_j^f e_\pm) + \frac{1}{2} \omega_{ij} \Gamma_{ik}^l g^{jk} \partial \phi^l . \quad (D.6)$$

In the coordinates where the volume form is constant the connection-part of (D.6) vanishes. From section 6.2 we know that $J_{\pm}$ generates an $N = 2$ superconformal algebra with central charge $c = 9$, i.e.

$$[J_{\pm}, J_{\pm}] = - T_{\pm} - 3 \chi \lambda , \quad (D.7)$$

where $T_{\pm}$ generates an $N = 1$ superconformal algebra.

### D.3 Calculation of the plus-sector

From now on we are going to focus on the plus-sector. In the rest of this subsection we only write out $\pm$ subscripts where necessary.

We now want to utilize (D.7) to find an expression for $T_{+}$. Going to complex coordinates, we can rewrite $J_{+}$ as

$$J_{+} = -i (g_{\alpha \bar{\beta}} e^\alpha_+ e_{\bar{\beta}}^+) . \quad (D.8)$$

Let us define a shorthand notation:

$$A_{\bar{\beta}} \equiv g_{\alpha \bar{\beta}} e^\alpha_+ , \quad (D.9)$$

so we have

$$A_{\beta} e_{\bar{\beta}}^+ = i J_{+} . \quad (D.10)$$

From the bracket

$$[A_{\bar{\beta}} \Lambda e^\alpha_+] = \chi \delta^\alpha_\beta e^\sigma_- - \frac{1}{\sqrt{2}} \Gamma^\alpha_\beta_\sigma e^\sigma_- \quad (D.11)$$

we calculate

$$[A_{\bar{\beta}} \Lambda A_\alpha e^\alpha_+] = - \chi A_{\bar{\beta}} - \frac{1}{\sqrt{2}} \Gamma^\alpha_\beta_\sigma A_\alpha e^\sigma_- \quad (D.12)$$

$$[e^\beta_+ \Lambda A_\alpha e^\alpha_+] = + \chi e^\beta_+ + \frac{1}{\sqrt{2}} \Gamma^\beta_\alpha_\sigma e^\sigma_- e^\alpha_+ \quad (D.13)$$

From this we get

$$[A_\alpha e^\alpha_+ \Lambda A_{\bar{\beta}} e_{\bar{\beta}}^+] = + (\chi A_{\bar{\beta}} + DA_{\bar{\beta}} - \frac{1}{\sqrt{2}} \Gamma^\alpha_\beta_\sigma A_\alpha e^\sigma_- ) e_{\bar{\beta}}^+$$

$$- A_{\bar{\beta}} (- \chi e^\beta_+ - De^\beta_+ + \frac{1}{\sqrt{2}} \Gamma^\beta_\alpha_\sigma e^\sigma_- e^\alpha_+)$$

$$+ 3 \chi \lambda$$

$$= DA_{\bar{\beta}} e^\beta_+ + A_{\bar{\beta}} De^\beta_+ - \sqrt{2}(A_{\bar{\beta}} e^\alpha_+ ) e^\beta_+ + 3 \chi \lambda$$

$$= 2DA_{\bar{\beta}} e^\beta_+ + \sqrt{2}(g_{\alpha \bar{\beta}, \sigma} e^\sigma_- e^\alpha_+ ) e^\beta_+ - D(A_{\bar{\beta}} e^\beta_+) + 3 \chi \lambda . \quad (D.14)$$
Let us again introduce some convenient notation:

\[ B_\beta \equiv \frac{1}{\sqrt{2}} g_{\alpha\beta\gamma} e_\gamma^\alpha e_+^\beta + DA_\beta . \] (D.15)

So, we have

\[ [J_+ J_+] = -2B_\beta e_+^\beta + iDJ - 3\chi \lambda . \] (D.16)

Comparing with (D.7) we get that

\[ B_\beta e_+^\beta = \frac{1}{2}(T_+ + iDJ_+) . \] (D.17)

**D.3.1** \([X_\Lambda \bar{X}]\)

We start with the most involved bracket, namely \([X_\Lambda \bar{X}]\). We try to present the calculation step by step.

**D.3.2** \([X_\Lambda \Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}} e_+^\alpha]\)

First, we want to calculate the bracket between \(e_+^\alpha\) and \(X\). We start with \([e_+^\alpha \Lambda e_+^\beta]\). The field \(e_+^\alpha\) anti-commute with \([e_+^\alpha \Lambda e_+^\beta]\):

\[ e_+^\alpha [e_+^\alpha \Lambda e_+^\beta] = -[e_+^\alpha \Lambda e_+^\beta] e_+^\alpha \] (D.18)

Also, the integral term in the Quasi-Leibniz formula (B.8) is zero. So, we have

\[ [e_+^\alpha \Lambda e_+^\beta] = 2[e_+^\alpha \Lambda e_+^\alpha] e_+^\beta , \] (D.19)

and, in general,

\[ [e_+^\alpha \Lambda e_+^{[\alpha} e_+^{\beta]} = p[e_+^\alpha \Lambda e_+^{[\alpha} \ldots e_+^{\alpha_p]} . \] (D.20)

So

\[ [e_+^\alpha \Lambda X] = \frac{1}{2} \chi \Omega_{\alpha\beta\gamma} g^{\alpha\beta\gamma} e_+ e_\gamma + \frac{1}{2} \chi \Omega_{\alpha\beta\gamma} \left( g^{\alpha\beta}_{\nu\rho} e_+^\nu - g^{\alpha\beta}_{\nu\rho} e_+^\rho \right) e_+^\rho e_+^\gamma . \] (D.21)

No integral terms or ordering problems occurred so far.

From (D.21) we find that

\[ [\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}} e_+^\alpha \Lambda X] = \frac{1}{2} \Omega_{\alpha\beta\gamma} \Omega_{\alpha\beta\gamma} \left( \chi g^{\alpha\beta} + \frac{1}{\sqrt{2}} \left( g^{\alpha\beta}_{\nu\rho} e_+^\nu - g^{\alpha\beta}_{\nu\rho} e_+^\rho \right) \right) e_+^\gamma . \] (D.22)
Next, we rewrite the $\Omega \bar{\Omega}$-terms using

$$\Omega_{\alpha \beta \gamma} g^{\alpha \bar{\alpha}} \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} = g_{\beta \bar{\beta}} g_{\gamma \bar{\gamma}} - g_{\beta \gamma} g_{\gamma \bar{\beta}}.$$  \hspace{1cm} (D.23)

The term

$$(\Omega_{\alpha \beta \gamma} g^{\alpha \nu} \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}}) e^{\nu}_{+} e^{\beta}_{+} e^{\gamma}_{+} = \partial_{\nu} (g_{\beta \bar{\beta}} g_{\gamma \bar{\gamma}} - g_{\beta \gamma} g_{\gamma \bar{\beta}}) e^{\beta}_{+} e^{\gamma}_{+}$$  \hspace{1cm} (D.24)

is zero because of the symmetries of the Kähler metric. We find

$$[\bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} X] = \chi g_{\beta \bar{\beta}} g_{\gamma \bar{\gamma}} e^{\beta}_{+} e^{\gamma}_{+} - \frac{1}{\sqrt{2}} e^{\beta}_{+} \partial_{\nu} (g_{\beta \bar{\beta}} g_{\gamma \bar{\gamma}}) e^{\beta}_{+} e^{\gamma}_{+},$$  \hspace{1cm} (D.25)

so

$$[X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+}] = (\chi + D) g_{\beta \bar{\beta}} g_{\gamma \bar{\gamma}} e^{\beta}_{+} e^{\gamma}_{+} + \frac{1}{\sqrt{2}} e^{\beta}_{+} \partial_{\nu} (g_{\beta \bar{\beta}} g_{\gamma \bar{\gamma}}) e^{\beta}_{+} e^{\gamma}_{+}.$$  \hspace{1cm} (D.26)

Using the shorthand notation (D.9) and (D.15), this can be written as

$$[X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+}] = \chi A_{\bar{\beta}} A_{\bar{\gamma}} + B_{[\bar{\beta} A_{\bar{\eta}]},$$  \hspace{1cm} (D.27)

where the anti-symmetrization is without any factor and we used that

$$[B_{\bar{\gamma} \Lambda A_{\bar{\beta}}}]_{\chi \lambda} = 0.$$  \hspace{1cm} (D.28)

The notation $[ \Lambda ]_{\chi \lambda}$ stands for the $\chi \lambda^n$-part of the bracket $[ \Lambda ], n \geq 0.$

**D.3.3 $[X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}]$**

We now want to calculate

$$[X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}] = [X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}] e^{\bar{\beta}}_{+} [X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}]
\begin{align*}
&+ \int_{0}^{\Lambda} [[[X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}]] e^{\beta}_{+}] d\Gamma \\
&= 2 [X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}]
\begin{align*}
&- \int_{-\Delta}^{0} [e^{\bar{\beta}}_{+} e^{\bar{\alpha}}_{+} X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}}] e^{\beta}_{+}] d\Gamma \\
&+ \int_{0}^{\Lambda} [[[X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}]] e^{\beta}_{+}] d\Gamma \\
&= 2 [X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}]
\begin{align*}
&+ \int_{0}^{\Lambda - \Delta} [[[X \Lambda \bar{\Omega}_{\bar{\alpha} \bar{\beta} \bar{\gamma}} e^{\bar{\alpha}}_{+} e^{\bar{\beta}}_{+}]] e^{\beta}_{+}] d\Gamma .
\end{align*}
\end{align*}$$  \hspace{1cm} (D.29)
where we have used that \([ e^\beta_+ \Gamma [ X_\Lambda \bar{\Omega}_{\alpha\beta\gamma} e^\alpha_+ ]]\) has no \(\eta\gamma^n\)-terms, with \(n > 0\).

Using (D.27), and the brackets

\[
[A_\gamma^\Lambda e^\beta_+]_{\chi\lambda\bar{\gamma}} = \delta_{\chi}^{\gamma}, \quad [B_\gamma^\Lambda e^\beta_+]_{\chi\lambda\bar{\gamma}} = 0 \quad (D.30)
\]

the integral term is calculated to

\[
\chi(\lambda - \partial)(-2A_\gamma) + (\lambda - \partial)(-2B_\gamma), \quad (D.31)
\]

and we get

\[
[X_\Lambda \bar{\Omega}_{\alpha\beta\gamma} e^\alpha_+ e^\beta_+] = 2\chi(A_\beta A_\gamma)e^\beta_+ + 2(B_{[\beta}A_{\gamma]}e^\beta_+ \\
+ 2\partial B_\gamma - 2\lambda B_\gamma \\
+ 2\chi \partial A_\gamma - 2\chi \lambda A_\gamma.
\]  \( (D.32) \)

### D.3.4 Calculation of \([ X_\Lambda \bar{X} ]\)

We are now ready to compute \([X_\Lambda \bar{X}]\). We choose to do the calculation as follows:

\[
[X_\Lambda \bar{X}] = \frac{1}{3!}([X_\Lambda \bar{\Omega}_{\alpha\beta\gamma} e^\alpha_+ e^\beta_+] e^\gamma_+ + [e^\beta_+ e^\gamma_+] [X_\Lambda \bar{\Omega}_{\alpha\beta\gamma} e^\alpha_+] \\
+ \int_0^\Gamma [[X_\Lambda \bar{\Omega}_{\alpha\beta\gamma} e^\alpha_+ e^\beta_+] \Gamma e^\gamma_+] d\Gamma). \quad (D.33)
\]

The two first terms are given by (D.27) and (D.32). The brackets relevant for the integral term are

\[
[(A_\beta A_\gamma)e^\beta_+]_{\chi\lambda\bar{\gamma}} = -2\chi A^\beta e^\beta_+ + 6\chi \lambda + \ldots, \quad (D.34)
\]

\[
[(B_{[\beta}A_{\gamma]}e^\beta_+]_{\chi\lambda\bar{\gamma}} = -2\chi B^\beta e^\beta_+ + \ldots, \quad (D.35)
\]

\[
[\partial A_\gamma e^\gamma_+] = -3\chi \lambda + \ldots, \quad (D.36)
\]

\[
[A_\gamma e^\gamma_+] = +3\chi + \ldots. \quad (D.37)
\]

The integral term in (D.33) therefore is

\[
\int_0^\Gamma [[X_\Lambda \bar{\Omega}_{\alpha\beta\gamma} e^\alpha_+ e^\beta_+] \Gamma e^\gamma_+] d\Gamma = +\chi(-4\lambda A_\beta e^\beta_+ + 12\frac{\lambda^2}{2} + (-4\lambda B_\beta e^\beta_+) + \chi(-6\frac{\lambda^2}{2}) - \chi \lambda(+6\lambda) \\
= -4\lambda B_\beta e^\beta_+ - 4\chi \lambda A_\beta e^\beta_+ - 3\chi \lambda^2.
\]  \( (D.38) \)
Using this, (D.27) and (D.32), we can write (D.33) as

\[
[X_\Lambda \tilde{X}] = \frac{1}{3} \chi((A_\beta A_{\bar{\alpha}}) e_+^\beta) e_+^{\alpha} + \frac{1}{3}((B_{\bar{\beta}} A_\alpha) e_+^\beta) e_+^{\bar{\alpha}} \\
+ \frac{1}{3}(\partial B_\alpha) e_+^{\alpha} - \frac{1}{3} \lambda B_\alpha e_+^{\bar{\alpha}} \\
+ \frac{1}{3} \chi(\partial A_\alpha) e_+^{\alpha} - \frac{1}{3} \chi \Lambda e_+^{\bar{\alpha}} \\
+ \frac{1}{6} \chi(e_+^\beta e_+^\alpha) (A_\beta A_\alpha) + \frac{1}{6} (e_+^\beta e_+^{\bar{\alpha}}) (B_{\bar{\beta}} A_\alpha) \\
- \frac{2}{3} \lambda B_\beta e_+^\beta - \frac{2}{3} \chi \Lambda e_+^\beta - \frac{1}{2} \chi \lambda^2 .
\]

We now go through the different terms occurring in this expression.

**D.3.5 \([X_\Lambda \tilde{X}]_1\)**

The constant part of \([X_\Lambda \tilde{X}]\) is

\[
[X_\Lambda \tilde{X}]_1 = \frac{1}{3} (e_+^\beta e_+^{\alpha}) (A_\alpha B_{\bar{\beta}}) \\
+ \frac{1}{3} ((B_\alpha A_{\bar{\beta}}) e_+^{\alpha} - (B_{\bar{\beta}} A_\alpha) e_+^{\alpha}) e_+^\beta \\
+ \frac{1}{3} \partial(B_{\bar{\beta}}) e_+^\beta ,
\]

where we have used (D.30). Using again (D.30), the first line of (D.40) can be written as

\[
\frac{1}{3} (e_+^\beta e_+^{\alpha}) (A_\alpha B_{\bar{\beta}}) = -\frac{1}{3} (B_{\bar{\beta}} A_\alpha) (e_+^{\alpha} e_+^\beta) + \frac{2}{3} \partial\left(e_+^\beta B_{\bar{\beta}}\right) .
\]

We can use quasi-associativity and move the parenthesis in the second line of (D.40). Once this is done, we can use anti-symmetry in \(\bar{\alpha}\) and \(\bar{\beta}\), and we get:

\[
-\frac{1}{3} ((B_\beta A_{\bar{\alpha}} - B_\alpha A_{\bar{\beta}}) e_+^{\alpha}) e_+^\beta = -\frac{2}{3} (B_{\bar{\beta}} A_\alpha) (e_+^{\alpha} e_+^\beta) - \frac{2}{3} \partial(e_+^\beta B_{\bar{\beta}}) .
\]

With these rewritings, and noticing that \(\partial(B_{\bar{\beta}}) e_+^\beta = e_+^\beta \partial(B_{\bar{\beta}})\), eq. (D.40) is

\[
[X_\Lambda \tilde{X}]_1 = -\left(B_{\bar{\beta}} A_\alpha\right) (e_+^{\alpha} e_+^\beta) \\
+ \frac{2}{3} \partial\left(e_+^\beta B_{\bar{\beta}}\right) - \frac{2}{3} \partial(e_+^\beta B_{\bar{\beta}}) + \frac{1}{3} e_+^\beta \partial(B_{\bar{\beta}}) .
\]

We now want to change \((B_{\bar{\beta}} A_\alpha)(e_+^{\alpha} e_+^\beta)\) to \((B_{\bar{\beta}} e_+^\beta)(A_\alpha e_+^{\alpha})\). Going through the various steps needed to change the order of the fields, we find that

\[
(B_{\bar{\beta}} A_\alpha)(e_+^{\alpha} e_+^\beta) = (B_{\bar{\beta}} e_+^\beta)(A_\alpha e_+^{\alpha}) + e_+^{\alpha} \partial(B_\alpha) .
\]

(D.44)
So, finally,

\[ [X_\Lambda \bar{X}]_1 = - \left( B_\beta e_+^\beta \right) (A_\alpha e_+^\alpha) \]
\[ + \frac{2}{3} \partial \left( e_+^\beta B_\beta \right) - \frac{2}{3} \partial (e_+^\beta B_\beta) + \frac{1}{3} e_+^\beta \partial B_\beta - e_+^\alpha \partial B_\alpha \]
\[ = - \left( B_\beta e_+^\beta \right) (A_\alpha e_+^\alpha) \]
\[ = - \frac{1}{2} (T + iDJ)iJ , \]  

(D.45)

where we have used (D.10) and (D.17) to write the result in terms of the generators.

D.3.6 \[ [X_\Lambda \bar{X}]_\chi \]

We read of the \( \chi \)-terms from (D.39):

\[ [X_\Lambda \bar{X}]_\chi = \frac{1}{3} ((A_\beta A_\alpha) e_+^\beta) e_+^\alpha + \frac{1}{3} (\partial A_\alpha) e_+^\alpha + \frac{1}{6} (e_+^\beta e_+^\alpha)(A_\beta A_\alpha) . \]  

(D.46)

We need to do similar rearranging of these terms as we did for the constant part. We calculate

\[ (e_+^\beta e_+^\alpha)(A_\beta A_\alpha) = -(A_\beta A_\alpha)(e_+^\alpha e_+^\beta) + 4\partial(A_\alpha e_+^\alpha) , \]  

(D.47)

\[ ((A_\beta A_\alpha)e_+^\beta)e_+^\alpha = -(A_\beta A_\alpha)(e_+^\alpha e_+^\beta) - 2A_\alpha \partial e_+^\alpha , \]  

(D.48)

\[ (A_\beta A_\alpha)(e_+^\beta e_+^\alpha) = (A_\beta e_+^\beta)(A_\alpha e_+^\alpha) + 3\partial A_\alpha e_+^\alpha + A_\alpha \partial e_+^\alpha . \]  

(D.49)

We then get

\[ [X_\Lambda \bar{X}]_\chi = -\frac{1}{2} (A_\beta e_+^\beta)(A_\alpha e_+^\alpha) - \frac{1}{2} \partial (A_\alpha e_+^\alpha) = +\frac{1}{2} JJ - \frac{1}{2} i\partial J . \]  

(D.50)

D.3.7 \[ [X_\Lambda \bar{X}]_\lambda \]

\[ [X_\Lambda \bar{X}]_\lambda = -B_\alpha e_+^\alpha = -\frac{1}{2} (T + iDJ) . \]  

(D.51)

D.3.8 \[ [X_\Lambda \bar{X}]_{\chi\lambda} \]

\[ [X_\Lambda \bar{X}]_{\chi\lambda} = -A_\alpha e_+^\alpha = -iJ . \]  

(D.52)

D.3.9 \[ [X_\Lambda \bar{X}]_{\lambda^2} \]

For dimensional reasons, we do not want any contributions to this term. And, indeed, \[ [X_\Lambda \bar{X}]_{\lambda^2} = 0. \]
D.3.10 \([ X_\Lambda \bar{X} ]_{\chi \lambda^2}\)

This term, we can simply read from (D.39):

\[
[ X_\Lambda \bar{X} ]_{\chi \lambda^2} = -\frac{1}{2}. \tag{D.53}
\]

D.3.11 Collecting terms

Collecting together the different terms we end up with

\[
[ X_\Lambda \bar{X} ] = -\frac{1}{2} (i T J - D J J - \chi J J + i \partial J \\
+ \lambda T + i \lambda D J + 2i \chi \lambda J + \chi \lambda^2 ) \tag{D.54}
\]

D.3.12 \([ X_\Lambda X ] \) and \([ \bar{X}_\Lambda \bar{X} ] \)

It follows immediately from the basic brackets that \([ X_\Lambda X ] = 0 \) since there are only holomorphic fields, \( e_+^\alpha \), present. In the same way, \([ \bar{X}_\Lambda \bar{X} ] = 0 \).

D.3.13 \([ J_\Lambda X ] \)

As before, we work in coordinates where the volume form is constant. We compute

\[
[ e_+^\gamma J ] = [ e_+^\gamma \omega_{\alpha \beta} e_+^\alpha e_+^\beta ] = ( g^{\gamma \delta} \omega_{\alpha \beta \delta} e_+^\alpha ) e_+^\beta + \omega_{\alpha \beta} e_+^\alpha \left( \chi g^{\gamma \beta} + \frac{g^{\gamma \beta} e_+^\alpha - g^{\beta \alpha} e_+^\alpha}{\sqrt{2}} \right) \tag{D.55}
\]

\[
= i \chi e_+^\gamma - i \frac{1}{\sqrt{2}} \Gamma_{\alpha \beta} e_+^\alpha e_+^\beta.
\]

The integral term (not written out) is a derivative on the volume form and therefore zero. Since the above bracket only has terms with \( e_+^\alpha \), we do not have to worry about integral terms when computing the full bracket (remember that \([ e_+^\alpha \Lambda e_+^\beta ] = [ e_+^\alpha \Lambda e_+^\beta ] = 0 \)). We get

\[
[ J_\Lambda X ] = -i (3 \chi + D) X + \frac{3i}{\sqrt{2}} \Gamma_{\mu \nu}^\alpha \Omega_{\alpha \beta \gamma} e_+^\mu e_+^\nu e_+^\beta e_+^\gamma = -i (3 \chi + D) X. \tag{D.56}
\]

In the last step we have used that the term with the connection can be rewritten as a covariant derivative on \( \Omega_{\alpha \beta \gamma} \), which is zero.
\section*{D.3.14 $[J_\Lambda \bar{X}]$}

The computation is similar as for $[J_\Lambda X]$, and, up to a sign, the answer is the same:

$$[J_\Lambda \bar{X}] = +i (3\chi + D) \bar{X} \quad (D.57)$$

\section*{D.3.15 $[T_\Lambda X]$}

We now want to calculate $[T_\Lambda X]$. In a direct approach, this bracket is harder then the previous ones to calculate. This is mainly because $T$ contains terms that are not linear in $e_\alpha^\Lambda$. Also, terms with $De_\alpha^\Lambda$ makes the calculations more cumbersome. We therefore need to take a small detour to do the calculation, using the Jacobi identity. We know that

$$[J_\Lambda J] = -T + 3\chi\lambda. \quad (D.58)$$

We then have, using (D.56):

$$[X_\Lambda T] = -[X_\Lambda [J_\Gamma J]] = -[[X_\Lambda J]_{\Gamma+\Lambda} J] = [J_\Gamma [X_\Lambda J]]$$

$$= -i[(3\chi + 2D)X_{\Gamma+\Lambda} J] - i[J_\Gamma (3\chi + 2D)X]$$

$$= -i(-3\chi + 2(\chi + \eta))[X_{\Gamma+\Lambda} J] - i(-3\chi - 2(D + \eta))[J_\Gamma X]$$

$$= (3\lambda + \chi D + 2\partial)X. \quad (D.59)$$

Since the bracket between $X$ and $J$ enters twice in the above calculation the different sign of $[X_\Lambda J]$ and $[\bar{X}_\Lambda J]$ does not matter, and we have

$$[\bar{X}_\Lambda T] = (3\lambda + \chi D + 2\partial)\bar{X}. \quad (D.60)$$

To summarize, $(X_+, \bar{X}_+, J_+, T_+)$ generates the Odake algebra.

\section*{D.4 Commuting sectors}

In [13] it is shown that $(J_+, T_+)$ and $(J_-, T_-)$ is two commuting copies of an $N = 2$ superconformal algebra. Above, the algebra of $(X_+, \bar{X}_+, J_+, T_+)$ is computed. We now show that these generators commute with $(X_-, \bar{X}_-, J_-, T_-)$.

The bracket between $X_+$ and $X_-$ is zero since only holomorphic fields enter. Also $[\bar{X}_+ X_-] = 0$. For $X_+$ with $\bar{X}_-$ we compute

$$[\Omega_{\alpha\beta\gamma} e^\alpha_+ \Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}} e^\bar{\beta}_- e^\bar{\gamma}_-] = -\frac{3}{\sqrt{2}} \Omega_{\alpha\beta\gamma} g^{\alpha\bar{\alpha}} \Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}} e^\bar{\beta}_+ e^\bar{\gamma}_- \quad (D.61)$$
due to the symmetries of the Kähler metric. In the same way \([X_\Lambda \bar{X}_+] = 0\). The bracket between \(X_+\) and \(J_-\) are zero because

\[
[J_- e_+^\alpha] \propto \Gamma^\alpha_{\mu\nu} e_+^\mu e_-^\nu,
\]

so

\[
[J_+ \Lambda X_+] \propto \Omega^\alpha_{\alpha\beta\gamma} \Gamma^\alpha_{\mu\nu} e_+^\mu e_+^\nu e_+^\gamma = 0
\]  \hspace{1cm} (D.64)

since \(\Omega\) is covariantly constant. The same argument goes for \([J_- \Lambda \bar{X}_+]\), \([J_+ \Lambda X_-]\), and \([J_- \Lambda \bar{X}_-]\). Using this, and (D.7) together with Jacobi identity, it follows that \([X_+ \Lambda T_-] = [\bar{X}_+ \Lambda T_-] = [X_- \Lambda T_+] = [\bar{X}_- \Lambda T_+] = 0\).

Thus, \((X_+, \bar{X}_+, J_+, T_+)\) commutes with \((X_-, \bar{X}_-, J_-, T_-)\).

### D.5 Comments about the computation of the minus-sector

As mentioned above, the calculation of the minus-sector is completely analogous to the computation of the plus-sector. However, due to the slight asymmetry between the two sectors in the defining brackets, (D.3), and the definition of the generators, (D.4), there will be some slight sign differences in the various steps. In the end, everything comes together to give us the same algebra in the minus-sector as well. Thus, \((X_-, \bar{X}_-, J_-, T_-)\) also gives us the Odake algebra.

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