On uniquely 3-colorable plane graphs without prescribed adjacent faces

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Abstract
A graph $G$ is uniquely $k$-colorable if the chromatic number of $G$ is $k$ and $G$ has only one $k$-coloring up to permutation of the colors. For a plane graph $G$, two faces $f_1$ and $f_2$ of $G$ are adjacent $(i, j)$-faces if $d(f_1) = i$, $d(f_2) = j$ and $f_1$ and $f_2$ have a common edge, where $d(f)$ is the degree of a face $f$. In this paper, we prove that every uniquely 3-colorable plane graph has adjacent $(3, k)$-faces, where $k \leq 5$. The bound 5 for $k$ is best possible. Furthermore, we prove that there exist a class of uniquely 3-colorable plane graphs having neither adjacent $(3, i)$-faces nor adjacent $(3, j)$-faces, where $i, j \in \{3, 4, 5\}$ and $i \neq j$. One of our constructions implies that

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there exist an infinite family of edge-critical uniquely 3-colorable plane graphs with \( n \) vertices and \( \frac{7}{3}n - \frac{14}{3} \) edges, where \( n (\geq 11) \) is odd and \( n \equiv 2 \) (mod 3).

**Keywords**: plane graph; unique coloring; uniquely 3-colorable plane graph; construction; adjacent \((i, j)\)-faces

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1 Introduction

For a plane graph \( G \), \( V(G) \), \( E(G) \) and \( F(G) \) are the sets of vertices, edges and faces of \( G \), respectively. The degree of a vertex \( v \in V(G) \), denoted by \( d_G(v) \), is the number of neighbors of \( v \) in \( G \). The degree of a face \( f \in F(G) \), denoted by \( d_G(f) \), is the number of edges in its boundary, cut edges being counted twice. When no confusion can arise, \( d_G(v) \) and \( d_G(f) \) are simplified by \( d(v) \) and \( d(f) \), respectively. A face \( f \) is a \( k \)-face if \( d(f) = k \) and a \( k^+ \)-face if \( d(f) \geq k \). Two faces \( f_1 \) and \( f_2 \) of \( G \) are adjacent \((i, j)\)-faces if \( d(f_1) = i \), \( d(f_2) = j \) and \( f_1 \) and \( f_2 \) have at least one common edge. Two distinct paths of \( G \) are internally disjoint if they have no internal vertices in common.

A graph \( G \) is **uniquely \( k \)-colorable** if \( \chi(G) = k \) and \( G \) has only one \( k \)-coloring up to permutation of the colors, where the coloring is called a **unique \( k \)-coloring** of \( G \). In other words, all \( k \)-colorings of \( G \) induce the same partition of \( V(G) \) into \( k \) independent sets, in which an independent set is called a **color class** of \( G \). In addition, uniquely colorable graphs may be defined in terms of their chromatic polynomials, which initiated by Birkhoff [2] for planar graphs in 1912, and for general graphs by Whitney [11] in 1932. Because a graph \( G \) is uniquely \( k \)-colorable if and only if its chromatic polynomial is \( k! \). For a discussion of chromatic polynomials, see Read [10].

**Theorem 1.1.** (Harary and Cartwright [6]) Let \( G \) be a uniquely \( k \)-colorable graph. Then for any unique \( k \)-coloring of \( G \), the subgraph induced by the union of any two color classes is connected.

As a corollary of Theorem [11], it can be seen that a uniquely \( k \)-colorable graph \( G \) has at least \( (k - 1)|V(G)| - \binom{k}{2} \) edges. There are many references on uniquely colorable graphs [6, 7, 8].

Chartrand and Geller [5] in 1969 started to study uniquely colorable planar graphs. They proved that uniquely 3-colorable planar graphs with at least 4 vertices contain at least two triangles, uniquely 4-colorable planar graphs are maximal planar graphs, and uniquely 5-colorable planar graphs do not exist. Aksionov [11] in 1977 improved the lower bound for the number of triangles in a uniquely 3-colorable planar graph. He proved that a uniquely 3-colorable planar graph with at least 5 vertices contains at least 3 triangles and gave a complete description of uniquely 3-colorable planar graphs containing exactly 3 triangles.

Let \( G \) be a uniquely \( k \)-colorable graph, \( G \) is **edge-critical** if \( G - e \) is not uniquely \( k \)-colorable for any edge \( e \in E(G) \). Obviously, if a uniquely
k-colorable graph \( G \) has exactly \((k - 1)|V(G)| - \binom{k}{2}\) edges, then \( G \) is edge-critical. Mel’nikov and Steinberg \[9\] in 1977 asked to find an exact upper bound for the number of edges in an edge-critical uniquely 3-colorable planar graph with \( n \) vertices. Recently, Matsumoto \[8\] proved that an edge-critical uniquely 3-colorable planar graph has at most \( \frac{8}{3}n - \frac{17}{3} \) edges and constructed an infinite family of edge-critical uniquely 3-colorable planar graphs with \( n \) vertices and \( \frac{9}{4}n - 6 \) edges, where \( n \equiv 0 (\text{mod} 4) \).

In this paper, we mainly prove Theorem 1.2.

**Theorem 1.2.** If \( G \) is a uniquely 3-colorable plane graph, then \( G \) has adjacent \((3,k)\)-faces, where \( k \leq 5 \). The bound 5 for \( k \) is best possible.

Furthermore, by using constructions, we prove that there exist uniquely 3-colorable plane graphs having neither adjacent \((3,i)\)-faces nor adjacent \((3,j)\)-faces, where \( i,j \in \{3,4,5\} \) and \( i \neq j \). One of our constructions implies that there exist an infinite family of edge-critical uniquely 3-colorable plane graphs with \( n \) vertices and \( \frac{7}{3}n - \frac{14}{3} \) edges, where \( n(\geq 11) \) is odd and \( n \equiv 2 \ (\text{mod} 3) \).

## 2 Proof of Theorem 1.2

**Lemma 2.1.** Let \( G \) be a plane graph with 3-faces. If \( G \) has no adjacent \((3,k)\)-faces, where \( k \leq 5 \), then \( |E(G)| \geq 2|F(G)| \).

**Proof.** We prove this by using a simple charging scheme. Since \( G \) has no adjacent \((3,k)\)-faces when \( k \leq 5 \), for any edge \( e \) incident to a 3-face \( f \), each edge of \( f \) is incident to a face of degree at least 6. Let \( ch(f) = d(f) \) for any face \( f \in F(G) \) and we call \( ch(f) \) the initial charge of the face \( f \). Let initial charges in \( G \) be redistributed according to the following rule.

**Rule:** For each 3-face \( f \) of \( G \) and each edge \( e \) incident with \( f \), the \( 6^+ \)-face incident with \( e \) sends \( \frac{1}{3} \) charge to \( f \) through \( e \).

Denote by \( ch'(f) \) the charge of a face \( f \in F(G) \) after applying redistributed Rule. Then

\[
\sum_{f \in F(G)} ch'(f) = \sum_{f \in F(G)} ch(f) = \sum_{f \in F(G)} d(f) = 2|E(G)| = 1
\]

On the other hand, for any 3-face \( f \) of \( G \), since the degree of each face adjacent to \( f \) is at least 6, then by the redistributed Rule, \( ch'(f) = ch(f) + 3 \cdot \frac{1}{3} = d(f) + 1 = 4 \). For any 4-face or 5-face \( f \) of \( G \), \( ch'(f) = ch(f) = d(f) \geq 4 \). For any \( 6^+ \)-face \( f \) of \( G \), since \( f \) is incident to at most \( d(f) \) edges each of which is incident to a 3-face,
then $ch'(f) \geq ch(f) - \frac{1}{3}d(f) = \frac{2}{3}d(f) \geq 4$. Therefore, we have

$$\sum_{f \in F(G)} ch'(f) \geq \sum_{f \in F(G)} 4 = 4|F(G)|$$

(2)

By the formulae (1) and (2), we have $|E(G)| \geq 2|F(G)|$. \hfill \Box

Proof of Theorem 1.2 Suppose that the theorem is not true and let $G$ be a counterexample to the theorem. Then $G$ has at least one 3-face and no adjacent $(3, k)$-faces, where $k \leq 5$. By Lemma 2.1 $|E(G)| \geq 2|F(G)|$. Using Euler’s Formula $|V(G)| - |E(G)| + |F(G)| = 2$, we can obtain $|E(G)| \leq 2|V(G)| - 4$.

Since $G$ is uniquely 3-colorable, then by Theorem 1.1 we have $|E(G)| \geq 2|V(G)| - 3$. This is a contradiction.

Note that the graph shown in Fig. 1 is a uniquely 3-colorable plane graph having neither adjacent $(3,3)$-faces nor adjacent $(3,4)$-faces. Therefore, the bound 5 for $k$ is best possible. \hfill \Box

![Figure 1: A uniquely 3-colorable plane graph having neither adjacent (3,3)-faces nor adjacent (3,4)-faces](image)

Remark. By piecing together more copies of the plane graph in Fig. 1 one can construct an infinite class of uniquely 3-colorable plane graphs having neither adjacent $(3,3)$-faces nor adjacent $(3,4)$-faces.

3 Construction of uniquely 3-colorable plane graphs without adjacent $(3,3)$-faces or adjacent $(3,5)$-faces

There are many classes of uniquely 3-colorable plane graphs having neither adjacent $(3,4)$-faces nor adjacent $(3,5)$-faces, such as even
maximal plane graphs (maximal plane graphs in which each vertex has even degree) and maximal outerplanar graphs with at least 6 vertices.

In this section, we construct a class of uniquely 3-colorable plane graphs having neither adjacent (3, 3)-faces nor adjacent (3, 5)-faces and prove that these graphs are edge-critical.

We construct a graph $G_k$ as follows:

1. $V(G_k) = \{u, w, v_0, v_1, \ldots, v_{3k-1}\}$;
2. $E(G_k) = \{v_0v_1, v_1v_2, \ldots, v_{3k-2}v_{3k-1}, v_{3k-1}v_0\} \cup \{uv_i : i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{wv_i : i \equiv 0 \text{ or } 1 \pmod{3}\}$, where $k$ is odd and $k \geq 3$.

(See an example $G_3$ shown in Fig. 2.)

**Theorem 3.1.** For any odd $k$ with $k \geq 3$, $G_k$ is uniquely 3-colorable.

**Proof.** Let $f$ be any 3-coloring of $G_k$. Since $v_0v_1 \ldots v_{3k-1}v_0$ is a cycle of odd length and each $v_i$ is adjacent to $u$ or $w$, we have $f(u) \neq f(w)$. Without loss of generality, let $f(u) = 1$ and $f(w) = 2$. By the construction of $G_k$, we know that $v_{3j+1}$ is adjacent to both $u$ and $w$, where $j = 0, 1, \ldots, k - 1$. So $v_{3j+1}$ can only receive the color 3, namely $f(v_{3j+1}) = 3, j = 0, 1, \ldots, k - 1$. Since $v_{3j}$ is adjacent to both $w$ and $v_{3j}$ in $G_k$, we have $f(v_{3j}) = 1, j = 0, 1, \ldots, k - 1$. Similarly, we can obtain $f(v_{3j+2}) = 2, j = 0, 1, \ldots, k - 1$. Therefore, the 3-coloring $f$ is uniquely decided as shown in Fig. 2 and then $G_k$ is uniquely 3-colorable.

**Theorem 3.2.** For any odd $k$ with $k \geq 3$, $G_k$ is edge-critical.

**Proof.** To complete the proof it suffices to show that $G_k - e$ is not uniquely 3-colorable for any edge $e \in E(G_k)$. Let $f$ be a uniquely
3-coloring of $G_k$ shown in Fig. 2. Denote by $E_{ij}$ the set of edges in $G_k$ whose ends colored by $i$ and $j$, respectively, where $1 \leq i < j \leq 3$. Namely

$$E_{ij} = \{xy : xy \in E(G_k), f(x) = i, f(y) = j\}, 1 \leq i < j \leq 3.$$ 

**Observation 1.** Both the subgraphs $G_k[E_{13}]$ and $G_k[E_{23}]$ of $G_k$ induced by $E_{13}$ and $E_{23}$ are trees.

**Observation 2.** The subgraph $G_k[E_{12}]$ of $G_k$ induced by $E_{12}$ consists of $k$ internally disjoint paths $uv_{3t-1}v_{3t}w$, where $i = 1, 2, \ldots, k$.

If $e \in E_{13} \cup E_{23}$, then $G_k - e$ is not uniquely 3-colorable by Observation 1. Suppose that $e \in E_{12}$. By Observation 2, there exists a number $t \in \{1, 2, \ldots, k\}$ such that $e \in \{uv_{3t-1}, v_{3t-1}v_{3t}, v_{3t}w\}$. Moreover, $G_k - e$ contains at least one vertex of degree 2. By repeatedly deleting vertices of degree 2 in $G_k - e$ we can obtain a subgraph $G_k \setminus \{v_{3t-1}, v_{3t}\}$ of $G_k$. Now we prove that $G_k \setminus \{v_{3t-1}, v_{3t}\}$ is not uniquely 3-colorable.

It can be seen that the restriction $f_0$ of $f$ to the vertices of $G_k \setminus \{v_{3t-1}, v_{3t}\}$ is a 3-coloring of $G_k \setminus \{v_{3t-1}, v_{3t}\}$. On the other hand, $G_k \setminus \{v_{3t-1}, v_{3t}, u, w\}$ is a path, denoted by $P$. Let $f'(u) = f'(w) = 1$ and alternately color the vertices of $P$ by the other two colors. We can obtain a 3-coloring $f'$ of $G_k \setminus \{v_{3t-1}, v_{3t}\}$ which is distinct from $f_0$. Since each 3-coloring of $G_k \setminus \{v_{3t-1}, v_{3t}\}$ can be extended to a 3-coloring of $G_k - e$, we know that $G_k - e$ is not uniquely 3-colorable when $e \in E_{12}$.

Since $E(G_k) = E_{12} \cup E_{13} \cup E_{23}$, $G_k - e$ is not uniquely 3-colorable for any edge $e \in E(G_k)$. $\square$

Note that $G_k$ has $3k + 2$ vertices and $7k$ edges by the construction. From Theorem 3.2 we can obtain the following result.

**Corollary 3.1.** There exist an infinite family of edge-critical uniquely 3-colorable plane graphs with $n$ vertices and $\frac{7}{3}n - \frac{14}{3}$ edges, where $n(\geq 11)$ is odd and $n \equiv 2 \pmod{3}$.

Denote by $size(n)$ the upper bound of the number of edges of edge-critical uniquely 3-colorable planar graphs with $n$ vertices. Then by Corollary 3.1 and the result due to Matsumoto [4], we can obtain the following result.

**Corollary 3.2.** For any odd integer $n$ such that $n \equiv 2 \pmod{3}$ and $n \geq 11$, we have $\frac{7}{3}n - \frac{14}{3} \leq size(n) \leq \frac{8}{3}n - \frac{17}{3}$.

4 Concluding remarks

In this paper we obtained a structural property of uniquely 3-colorable plane graphs. We proved that every uniquely 3-colorable
plane graph has adjacent \((3, k)\)-faces, where \(k \leq 5\), and the bound 5 for \(k\) is best possible. Fig. 1 shows a uniquely 3-colorable plane graph having neither adjacent \((3, 3)\)-faces nor adjacent \((3, 4)\)-faces. But this plane graph is 2-connected. This prompts us to propose the following conjecture.

**Conjecture 4.1.** Let \(G\) be a 3-connected uniquely 3-colorable plane graph. Then \(G\) has adjacent \((3, k)\)-faces, where \(k \leq 4\).

It can be seen that the uniquely 3-colorable plane graph \(G_k\) constructed in Section 3 is 3-connected. So if Conjecture 4.1 is true, then the bound 4 for \(k\) is best possible.

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