THE ROLE OF THE MEAN CURVATURE IN A MIXED HARDY-SOBOLEV TRACE INEQUALITY

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ABSTRACT. Let Ω be a smooth bounded domain of \( R^{N+1} \) of boundary \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \) and such that \( \partial \Omega \cap \Gamma_2 \) is a neighborhood of 0, \( h \in C^0(\partial \Omega \cap \Gamma_2) \) and \( s \in [0,1) \). We propose to study existence of positive solutions to the following Hardy-Sobolev trace problem with mixed boundaries conditions

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma_1 \\
derivative{u}{\nu} &= h(x) + \frac{u^{q(s)-1}}{d(x)^s} \quad \text{on } \Gamma_2,
\end{aligned}
\]

where \( q(s) := \frac{2(N-s)}{N-1} \) is the critical Hardy-Sobolev trace exponent and \( \nu \) is the outer unit normal of \( \partial \Omega \). In particular, we prove existence of minimizers when \( N \geq 3 \) and the mean curvature is sufficiently below the potential \( h \) at 0.

Key words: Hardy-Sobolev trace inequality, Mixed problem, Mean curvature, existence of minimizers.

1. Introduction and main result

For \( N \geq 2 \) and \( s \in [0,1] \), we consider the Hardy-Sobolev trace best constant:

\[
S_{N,s} = \inf_{u \in D} \frac{\int_{\mathbb{R}^{N+1}_+} |\nabla u|^2 \, dz}{\left( \int_{\partial \mathbb{R}^{N+1}_+} |x|^{-s} |u|^{q(s)} \, dx \right)^{2 \over q(s)}},
\]

where \( q(s) := \frac{2(N-s)}{N-1} \) is the Hardy-Sobolev trace exponent, see for instance [5] and also [3] for generalizations. Here and in the following, we denote by

\[
\mathbb{R}^{N+1}_+ = \{ z = (z_1,x) \in \mathbb{R}^{N+1} : z_1 > 0 \}
\]

with boundary \( \partial \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\} \equiv \mathbb{R}^N \). We denote and henceforth define \( D := D^{1,2}(\mathbb{R}^{N+1}_+) \) the completion of \( C_c^\infty(\mathbb{R}^{N+1}_+) \) with respect to the norm

\[
u \mapsto \left( \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2 \, dz \right)^{1/2}.
\]

Note that for \( s = 0 \) then \( q(0) = 2^\sharp \) is the critical Sobolev trace exponent while \( S_{N,0} \) coincides with the Sobolev trace constant studied by Escobar [4] and Beckner [1] with applications in
the Yamabe problem with prescribed mean curvature. Existence of cylindrical symmetric decreasing minimizers for the quotient \( S_{N,s} \) in (1.1) were obtained by Lieb [9], Theorem 5.1]. If \( s = 1 \), we recover \( S_{N,1} = 2 \frac{\Gamma^2(N + 1)}{\Gamma^2(2N + 1)} \), the relativistic Hardy constant (see e.g. [7]) which is never achieved in \( D \). In this case, it is expected that there is no influence of the curvature in comparison with the works on Hardy inequalities with singularity at the boundary or in Riemannian manifolds, see [6,12–15].

We consider a smooth domain \( \Omega \) of \( \mathbb{R}^{N+1} \), with boundary \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \) and such that \( \partial \Omega \cap \Gamma_2 \) is a smooth neighborhood of 0. Given \( h \in C^0(\Gamma_2 \cap \partial \Omega) \), we suppose the following argument of coercivity: there exists a positive constant \( C \), depending on \( \Omega \), such that

\[
\int_\Omega |\nabla u|^2 dz + \int_{\Gamma_2} h(x) u^2 dx \geq C \left( \int_\Omega |\nabla u|^2 dz + \int_\Omega u^2 dz \right) \quad \forall u \in H^1(\Omega). \tag{1.2}
\]

In [5], Lemma 2.4] authors showed the existence of a constant \( C_1(\Omega) > 0 \) such that the following inequality holds

\[
C_1(\Omega) \left( \int_{\partial \Omega} d(x)^{-s} |u|^{q(s)} dx \right)^{\frac{2}{q(s)}} \leq \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx \quad \forall u \in H^1(\Omega)
\]

where \( d(x) := \text{dist}_{\partial \Omega}(0,x) \) is the Riemannian distance on the boundary \( \partial \Omega \) of \( \Omega \). Then by (1.2) we have the existence of a positive constant \( C(\Omega) \) depending on \( \Omega \) such that

\[
C(\Omega) \left( \int_{\Gamma_2} d(x)^{-s} |u|^{q(s)} dx \right)^{\frac{2}{q(s)}} \leq \int_\Omega |\nabla u|^2 dx + \int_{\Gamma_2} h(x) u^2 dx \quad \forall u \in H^1(\Omega). \tag{1.3}
\]

Our aim in this paper is to study the existence of minimizers for the following mixed Hardy-sobolev trace quotient:

\[
\mu_s(\Omega) := \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx + \int_{\Gamma_2} h(x) u^2 dx}{\left( \int_{\Gamma_2} d(x)^{-s} |u|^{q(s)} dx \right)^{\frac{2}{q(s)}},} \tag{1.4}
\]

for \( s \in [0, 1) \). Our main result is the following

**Theorem 1.1.** Let \( N \geq 3 \), \( \Omega \) be a bounded smooth domain of \( \mathbb{R}^{N+1}_+ \) of boundary \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \) such that \( 0 \in \partial \Omega \cap \Gamma_2 \), \( s \in [0, 1) \) and \( h \in C^0(\partial \Omega \cap \Gamma_2) \). We let \( w \) to be the ground state solution of the Hardy-Sobolev trace best constant \( S_{N,s} \). We assume that the mean curvature of the boundary \( H_{\partial \Omega} \) satisfies

\[
N \frac{N - 2}{2N} + \frac{1}{N} \int_{\mathbb{R}^{N+1}_+} \frac{|\partial w}{\partial z_1}|^2 dz H_{\partial \Omega}(0) + h(0) < 0. \tag{1.5}
\]
Then \( \mu_s(\Omega) < S_{N,s} \) and \( \mu_s(\Omega) \) is achieved by a positive function \( u \in H^1(\Omega) \) satisfying

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma_1 \\
\frac{\partial u}{\partial \nu} &= h(x)u + d(x)^{-s}u^{q(s)} - 1 \quad \text{on } \Gamma_2,
\end{aligned}
\]  

(1.6)

where \( \nu \) is the unit outer normal of \( \partial \Omega \).

We mention that the study of the effect of the curvature in the Hardy-Sobolev trace inequality seems to be quite rare in the literature, see for instance the paper of the author with Fall and Minlend [5]. While the Sobolev (\( \sigma = 0 \)) inequality have been intensively studied in the last years. Our argument of proof is based on blow up analysis (see Proposition 4.1).

The paper is organized as follows. in Section 2, we recall some geometric results. In Section 3, we compare the two Hardy-Sobolev trace inequalities in order to get the existence of minimizers, see Section 4. Section 5 is devoted to the proof of our main result.

2. Preliminaries

We let \( E_i \), \( i = 2, \ldots, N + 1 \) be an orthonormal basis of \( T_0\partial \Omega \), the tangent plane of \( \partial \Omega \) at 0. We will consider the Riemannian manifold \((\partial \Omega, \tilde{g})\) where \( \tilde{g} \) is the Riemannian metric induced by \( \mathbb{R}^{N+1} \) on \( \partial \Omega \). We first introduce geodesic normal coordinates in a neighborhood (in \( \partial \Omega \)) of 0 with coordinates \( y' = (y_2, \ldots, y_{N+1}) \in \mathbb{R}^N \). We set

\[
 f(y') := \text{Exp}_{\partial \Omega}^0 \left( \sum_{i=2}^{N+1} y'_i E_i \right).
\]

It is clear that the geodesic distance \( d \) of the boundary \( \partial \Omega \) satisfies

\[
d(f(\tilde{y}')) = |\tilde{y}|. \quad (2.1)
\]

In addition the above choice of coordinates induces coordinate vector-fields on \( \partial \Omega \):

\[
 Y_i(y') = f_*(\partial_{y^i}), \quad \text{for } i = 2, \ldots, N + 1.
\]

Let \( \tilde{g}_{ij} = \langle Y_i, Y_j \rangle \), for \( i, j = 2, \ldots, N + 1 \), be the component of the metric \( \tilde{g} \). We have near the origin

\[
 \tilde{g}_{ij} = \delta_{ij} + O(|y|^2).
\]

We denote by \( N_{\partial \Omega} \) the unit normal vector field along \( \partial \Omega \) interior to \( \Omega \). Up to rotations, we will assume that \( N_{\partial \Omega}(0) = E_1 \). For any vector field \( Y \) on \( T\partial \Omega \), we define \( H(Y) = dN_{\partial \Omega}[Y] \).

The mean curvature of \( \partial \Omega \) at 0 is given by

\[
 H_{\partial \Omega}(0) = \sum_{i=2}^{N+1} \langle H(E_i), E_i \rangle.
\]

Now consider a local parametrization of a neighbourhood of 0 in \( \mathbb{R}^{N+1} \) defined as

\[
 F(y) := f(\tilde{y}) + y_1 N_{\partial \Omega}(f(\tilde{y})), \quad y = (y_1, \tilde{y}) \in B_{r_0},
\]

where \( B_{r_0} \) is a small ball centred at 0. This yields the coordinate vector-fields in \( \mathbb{R}^{N+1} \),

\[
 Y_i(y) := f_*(\partial_{y^i}) \quad i = 1, \ldots, N + 1.
\]
Let $g_{ij} = \langle Y_i, Y_j \rangle$, for $i, j = 1, \ldots, N + 1$, be the component of the flat metric $g$. We have the following (See for instance [5])

**Lemma 2.1.** For $i, j = 2, \ldots, N + 1$, Taylor expansion of the metric $g$ yields

$$g_{ij} = \delta_{ij} + 2 \langle H(E_i), E_j \rangle y^1 + O(|y|^2);$$

$$g_{i1} = 0;$$

$$g_{11} = 1.$$

We will need the following result proved by Fall-Minlend-Thiam [5, Theorem 2.1, Theorem 2.2]. Then we have

**Lemma 2.2.** Let $s \in (0, 1)$. Then, for $z = (z_1, x) \in \mathbb{R}_+^* \times \mathbb{R}^N$, $S_{N,s}$ has a positive minimizer $w \in \mathcal{D}$ that satisfies

$$\begin{cases}
\Delta w = 0 & \text{in } \mathbb{R}_{N+1}^+,
\frac{\partial w}{\partial \nu} = S_{N,s} w^{q(s)-1} |x|^{-s} & \text{on } \mathbb{R}^N,
\int_{\mathbb{R}^N} |x|^{-s} w^{q(s)} \, dx = 1
\end{cases}$$

where $\nu$ is the outer unit normal of $\partial \Omega$. Moreover we have:

(i) $w = w(z)$ only depends on $z^1$ and $|x|$, and $w$ is strictly decreasing in $|x|$.

(ii) $w(z) \leq \frac{C}{1 + |z|^{N-1}}$ for all $z \in \mathbb{R}_{N+1}^+$, for some positive constant $C$.

In [5], authors used the moving plane method to prove (i). Moreover (ii) is a direct consequence of the fact that the system (2.2) is invariant under Kelvin transformation.

3. Comparing $\mu_s(\Omega)$ and $S_{N,s}$

In this section, we construct a test function for the Hardy-Sobolev trace best constant $\mu_s(\Omega)$ in order to compare it with $S_{N,s}$. Then we recall

$$\mu_s(\Omega) = \inf_{u \in H^1(\Omega) \setminus \{0\}} J(u),$$

where the function $J$ is given by

$$J(u) = \frac{\int_{\Omega} |\nabla u|^2 \, dz + \int_{\Gamma_2} hu^2 \, dx}{\left( \int_{\Gamma_2} d(x)^{-s} |u|^{q(s)} \, dx \right)^{2/q(s)}},$$

Let $w \in \mathcal{D}$ be the positive ground state solution (positive minimizer that satisfy the corresponding Euler-Lagrange equation) given by Lemma 2.2 normalized so that

$$\int_{\partial \mathbb{R}_{N+1}^+} |x|^{-s} w^{q(s)} \, dx = 1 \quad \text{and} \quad S_{N,s} = \int_{\mathbb{R}_{N+1}^+} |\nabla w|^2 \, dz.$$
We let $\varepsilon > 0$. For $r_0 > 0$ small fix, we define

$$v_\varepsilon (F(z)) = \varepsilon \frac{i - N}{2} w \left( \frac{z}{\varepsilon} \right), \quad z = (y_1, x) \in B_{r_0}^+.$$  

Let $\eta \in C_c^\infty (F(B_{r_0}))$ such that $\eta \equiv 1$ in $F(B_{r_0/2})$ and $0 \leq \eta \leq 1$ in $\mathbb{R}^{N+1}_+$. Then we define the test function by

$$u_\varepsilon (F(z)) = \eta (F(z)) v_\varepsilon (F(z)).$$  

We have the following expansion.

**Lemma 3.1.** For all $N \geq 2$, we have

$$J (u_\varepsilon) = \int_{\mathbb{R}^{N+1}_+} |\nabla w|^2 dz + \varepsilon H_{\partial \Omega} (0) \int_{B_{r/\varepsilon}^+} z_1 |\nabla w|^2 dz - \frac{2}{N} \varepsilon H_{\partial \Omega} (0) \int_{B_{r/\varepsilon}^+} z_1 |\nabla_x w|^2 dz$$

$$+ h(0) \varepsilon \int_{\partial B_{r/\varepsilon}^+} w^2 (x) dx + O (\rho (\varepsilon))$$

where

$$\rho (\varepsilon) = \varepsilon^2 \int_{B_{r/\varepsilon}^+} |x|^2 |\nabla_x w|^2 dz + \varepsilon^3 \int_{\partial B_{r/\varepsilon}^+ \setminus \partial B_{r/2\varepsilon}^+} |x|^2 w^2 (x) dx + \varepsilon \int_{\partial B_{r/\varepsilon}^+ \setminus \partial B_{r/2\varepsilon}^+} w^2 (x) dx$$

$$+ \varepsilon^2 \int_{B_{r/\varepsilon}^+ \setminus B_{r/2\varepsilon}^+} w^2 (z) dz + \int_{\partial B_{r/\varepsilon}^+ \setminus \partial B_{r/2\varepsilon}^+} |x|^{-2} w^2 (s) dx + \varepsilon^2 \int_{\partial B_{r/\varepsilon}^+} |x|^{-2} w^2 (s) dx + \varepsilon \int_{\mathbb{R}^{N+1}_+ \setminus B_{r/2\varepsilon}^+} |\nabla w|^2 dz.$$  

**Proof.** We let

$$E (u_\varepsilon) := \int_{\Omega} |\nabla u_\varepsilon|^2 dz + \int_{\Gamma_2} h(x) u_\varepsilon^2 (x) dx.$$  

Integrating by parts, we have

$$E (u_\varepsilon) = \int_{\Omega \cap F (B_{r_0})} |\nabla v_\varepsilon|^2 dz + \int_{\Gamma_2 \cap F (B_{r_0})} hv_\varepsilon^2 dx + F_\varepsilon$$

where

$$F_\varepsilon = \int_{\Omega \cap F (B_{r_0})} (\eta^2 - 1) |\nabla v_\varepsilon|^2 dz + \int_{\Gamma_2 \cap F (B_{r_0})} h (\eta^2 - 1) v_\varepsilon^2 dx - \int_{\Omega \cap F (B_{r_0})} (\eta \Delta \eta) v_\varepsilon^2 dz.$$  

By a change of variable formula, we have

$$\int_{\Omega \cap F (B_{r_0})} |\nabla v_\varepsilon|^2 dz = \sum_{\alpha, \beta = 1}^{N+1} \int_{B_{r_0}^+} g^{\alpha \beta} (x) \left( \frac{\partial w}{\partial z^\alpha} \cdot \frac{\partial w}{\partial z^\beta} \right) (z) \sqrt{|g (x)|} dz.$$  

We deduce from Lemma 2.1 that for $i, j = 2, \ldots, N + 1$ that

$$g^{11} = 1; \quad g^{ij} (x) = \delta_{ij} - 2 \varepsilon H_{ij} z_1 + O (\varepsilon^2 |x|^2)$$

and

$$\sqrt{|g (x)|} = 1 + \varepsilon H_{\partial \Omega} (0) z_1 + O (\varepsilon^2 |x|^2).$$  

(3.5)
Therefore
\[ \int_{\Omega \cap F(B_r)} \nabla^2 u \, dz = \int_{B_r} |\nabla u|^2 + \varepsilon \int_{\partial B_r} |\nabla u|^2 + \varepsilon^2 \int_{\partial B_r}^2 w^2 (x) \, dx + O \left( \varepsilon^3 \int_{\partial B_r} |x|^2 \, dx \right). \]
Hence
\[ \int_{\Omega \cap F(B_r)} |\nabla v^\varepsilon|^2 \, dz = \int_{B_r} |\nabla w|^2 + \varepsilon H \Omega(0) \int_{B_r} z_1 |\nabla w|^2 + O \left( \varepsilon^2 \int_{B_r^+} |x|^2 \, dx \right). \]
By change of variable formula, continuity and (3.5), we get that
\[ \int_{\Omega \cap F(B_r)} |\nabla v^\varepsilon|^2 \, dz = \int_{B_r} |\nabla w|^2 + \varepsilon H \Omega(0) \int_{B_r} z_1 |\nabla w|^2 + O \left( \varepsilon^2 \int_{B_r^+} |x|^2 \, dx \right). \]
By change of variable formula and (3.5), we have
\[ F^\varepsilon = O \left( \int_{B_r^+}^2 w^2 (x) \, dx + \varepsilon \int_{\partial B_r}^2 w^2 (x) \, dx + \varepsilon^2 \int_{\partial B_r}^2 w^2 (z) \, dz \right) \]
By (3.6), (3.7) and (3.8) we obtain that
\[ E (u^\varepsilon) = \int_{B_r}^2 |\nabla w|^2 + \varepsilon H \Omega(0) \int_{B_r} z_1 |\nabla w|^2 + O \left( \varepsilon^2 \int_{B_r^+} |x|^2 \, dx \right). \]
where
\[ \rho_1 (\varepsilon) = \varepsilon^2 \int_{B_r^+} |x|^2 \, dx + O \left( \varepsilon^2 \int_{\partial B_r} |x|^2 \, dx \right). \]
We have
\[ \int_{\Gamma_2} d(x)^{-s} |u^\varepsilon|^q (s) \, dx = \int_{\Gamma_2 \cap F(B_r)} d(x)^{-s} |v^\varepsilon|^q (s) \, dx + O \left( \int_{\Gamma_2 \cap F(B_r)} d(x)^{-s} |v^\varepsilon|^q (s) \, dx \right). \]
By a change of variable formula and (3.5) we have
\[ \int_{\Gamma_2} d(x)^{-s} |u^\varepsilon|^q (s) \, dx = \int_{\partial B_r^+} |x|^{-s} w^{(s)} (0, \varepsilon x) \, dx \]
\[ \int_{\partial B_r^+} |x|^{-s} w^{(s)} (0, \varepsilon x) \, dx + O \left( \varepsilon^2 \int_{\partial B_r^+} |x|^2 \, dx \right). \]
\[ \int_{\partial B_r^+} |x|^{-s} w^{(s)} (0, \varepsilon x) \, dx + O \left( \varepsilon^2 \int_{\partial B_r^+} |x|^2 \, dx \right). \]
Using (3.9) and (3.10), we obtain that

\[
\int_{\Gamma_2} d(x)^{-s}|u_\varepsilon|^{q(s)}dx = 1 + O(\rho(\varepsilon)).
\]

Therefore by Taylor expansion, we get

\[
\left(\int_{\Gamma_2} d(x)^{-s}|u_\varepsilon|^{q(s)}dx\right)^{2/q(s)} = 1 + O(\rho(\varepsilon)).
\]

Using (3.9) and (3.10), we obtain that

\[
J(u_\varepsilon) = \int_{\mathbb{R}_+^{N+1}} |\nabla w|^2dz + \varepsilon H_{\partial\Omega}(0) \int_{B_{r/\varepsilon}^+} z_1|\nabla w|^2dz - \frac{2}{N} \varepsilon H_{\partial\Omega}(0) \int_{B_{r/\varepsilon}^+} z_1|\nabla_x w|^2dz
\]

\[
\quad - h(\varepsilon) \int_{\partial B_{r/\varepsilon}^+} w^2(x)dx + O(\rho(\varepsilon)).
\]

This ends the proof.

Lemma 3.2. Let \(\rho(\varepsilon)\) be the error term given by Proposition 3.3. Then we have

\[
\rho(\varepsilon) = o(\varepsilon) \quad \forall N \geq 3
\]

and in particular by Proposition 3.3, we have for all \(N \geq 3\) that

\[
J(u_\varepsilon) = S_{N,s} + \varepsilon H_{\partial\Omega}(0) \int_{B_{r/\varepsilon}^+} z_1|\nabla w|^2dz - \frac{2}{N} \varepsilon H_{\partial\Omega}(0) \int_{B_{r/\varepsilon}^+} z_1|\nabla_x w|^2dz + h(\varepsilon) \int_{\partial B_{r/\varepsilon}^+} w^2(x)dx + o(\varepsilon).
\]

Proof. We recall that the ground state solution \(w\) satisfies

\[
w(z) \leq \frac{C}{1 + |z|^{N-1}} \quad \text{in } \mathbb{R}_+^{N+1}.
\]

Then letting

\[
S_1(\varepsilon) := \varepsilon^3 \int_{\partial B_{r/\varepsilon}^+} |x|^2w^2(x)dx + \varepsilon \int_{\partial B_{r/\varepsilon}^+ \setminus \partial B_{r/2\varepsilon}^+} w^2(x)dx + \varepsilon^2 \int_{B_{r/\varepsilon}^+ \setminus B_{r/2\varepsilon}^+} w^2(z)dz
\]

\[
+ \int_{\partial \mathbb{R}_+^{N+1} \setminus \partial B_{r/\varepsilon}^+} |x|^{-s}w^{q(s)}dx + \varepsilon^2 \int_{\partial B_{r/\varepsilon}^+} |x|^{2-s}w^{q(s)}dx,
\]

we get by a change of variable formula that

\[
S_1(\varepsilon) = o(\varepsilon) \quad \text{for } N \geq 3.
\]

We let \(\varphi \in C^\infty_c (\mathbb{R}_+^{N+1} \setminus B_{r/2}^+)\) and we set \(\varphi_\varepsilon(z) = \varphi(\varepsilon z)\). We multiply (3.17) by \(\varphi_\varepsilon\) and we integrate by parts to get

\[
\int_{\mathbb{R}_+^{N+1} \setminus B_{r/2}^+} \varphi |\nabla w|^2dz = \frac{1}{2} \varepsilon^2 \int_{\mathbb{R}_+^{N+1} \setminus B_{r/2}^+} w^2 \Delta \varphi_\varepsilon dz + \int_{\partial \mathbb{R}_+^{N+1} \setminus \partial B_{r/2}^+} \varphi_\varepsilon |x|^{-s}w^{q(s)}dx.
\]
Then using the estimation (3.12), we obtain
\[
S_2(\varepsilon) := \int_{\mathbb{R}_+^{N+1} \setminus B_{r/2\varepsilon}} |\nabla w|^2 \, dz = O\left( \varepsilon^2 \int_{\mathbb{R}_+^{N+1} \setminus B_{r/2\varepsilon}} w^2 \, dz + \int_{\partial \mathbb{R}_+^{N+1} \setminus \partial B_{r/2\varepsilon}} |x|^{-s} w^{q(s)} \, dx \right).
\]
Then we get
\[
\int_{\mathbb{R}_+^{N+1} \setminus B_{r/2\varepsilon}} |\nabla w|^2 \, dz = O(\varepsilon) \text{ for } N \geq 3. \tag{3.14}
\]
To finish the estimation of the error term, we let \( \psi \in C_0^\infty(B_r^+) \) and we define \( \psi_\varepsilon(z) = \psi(\varepsilon z) \).

We then multiply (3.17) by \( \psi_\varepsilon w |x|^2 \) and we integrate by parts to get
\[
\int_{B_{r/\varepsilon}^+} |x|^2 \psi_\varepsilon |\nabla w|^2 \, dz = \frac{1}{2} \int_{B_{r/\varepsilon}^+} w^2 \Delta (\psi_\varepsilon |x|^2) \, dz + \int_{\partial B_{r/\varepsilon}^+} \psi_\varepsilon w^{2^*(s)} |x|^{2-s} \, dx.
\]
This implies
\[
S_3(\varepsilon) := \varepsilon^2 \int_{B_{r/\varepsilon}^+} |x|^2 |\nabla w|^2 \, dz = O\left( \varepsilon^2 \int_{B_{r/\varepsilon}^+} w^2 \, dz + \varepsilon^2 \int_{\partial B_{r/\varepsilon}^+} w^{2^*(s)} |x|^{2-s} \, dx \right).
\]
Therefore
\[
S_3(\varepsilon) = o(\varepsilon) \text{ for } N \geq 3. \tag{3.15}
\]
By (3.13), (3.14) and (3.15), we finally obtain
\[
S_1(\varepsilon) + S_2(\varepsilon) + S_3(\varepsilon) = \rho(\varepsilon) = o(\varepsilon) \text{ for } N \geq 3.
\]
This ends the proof of the Lemma. \( \square \)

**Proposition 3.3.** We assume that
\[
\left( \frac{N - 2}{2N} + \frac{1}{N} \int_{\mathbb{R}_+^{N+1}} \frac{|\partial w/\partial z_1|^2 \, dz}{z_1 |\nabla w|^2} \right) H_{\partial \Omega}(0) + h(0) < 0. \tag{3.16}
\]
Then \( \mu_s(\Omega) < S_{N,s} \).

**Proof.** We recall that the ground state solution \( w \) satisfies
\[
\begin{aligned}
\Delta w &= 0 \quad \text{in } \mathbb{R}_+^{N+1}, \\
\frac{\partial w}{\partial \nu} &= S_{N,s} w^{q(s)-1} |x|^{-s} \quad \text{on } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |x|^{-s} w^{q(s)} \, dx &= 1.
\end{aligned} \tag{3.17}
\]
We multiply (3.17) by \( z_1 \phi_\varepsilon w \) and we integrate by parts to get
\[
\int_{\mathbb{R}_+^{N+1} \setminus B_{r/\varepsilon}^+} z_1 \phi_\varepsilon |\nabla w|^2 \, dz = \frac{1}{2} \int_{\mathbb{R}_+^{N+1} \setminus B_{r/\varepsilon}^+} w^2 \Delta (\phi_\varepsilon z_1) \, dz.
\]
This implies that for all \( N \geq 3 \)
\[
\int_{\mathbb{R}^{N+1} \setminus B_{r/\varepsilon}^+} z_1 |\nabla w|^2 \, dz + \int_{\partial \mathbb{R}^{N+1} \setminus \partial B_{r/\varepsilon}^+} z_1 |\nabla w|^2 \, dz = O \left( \varepsilon^2 \int_{\mathbb{R}^{N+1} \setminus B_{r/\varepsilon}^+} w^2 \, dz + \int_{\partial \mathbb{R}^{N+1} \setminus \partial B_{r/\varepsilon}^+} w^2 \, dz \right) = o(\varepsilon).
\]
(3.18)

Moreover multiplying again (3.17) by \( z_1 w \) and integrating by parts, we get
\[
\int_{\mathbb{R}^{N+1}} z_1 |\nabla w|^2 \, dz = -\frac{1}{2} \int_{\mathbb{R}^{N+1}} \frac{\partial w^2}{\partial z_1} \, dz = -\frac{1}{2} \int_{\mathbb{R}^N} \int_0^{+\infty} \frac{\partial w^2}{\partial z_1} \, dz_1 \, dx
\]
where we can see that
\[
\int_0^{+\infty} \frac{\partial w^2}{\partial z_1} \, dz_1 = \lim_{R \to +\infty} \int_0^R \frac{\partial w^2}{\partial z_1} \, dz_1 = \lim_{R \to +\infty} w^2(R, x) - w(0, x).
\]
Since
\[
w(z) \leq \frac{C}{1 + |z|^{N-1}}
\]
we obtain
\[
\int_0^{+\infty} \frac{\partial w^2}{\partial z_1} \, dz_1 = -w(0, x).
\]
Therefore
\[
\int_{\mathbb{R}^{N+1}} z_1 |\nabla w|^2 \, dz = -\frac{1}{2} \int_{\partial \mathbb{R}^{N+1}} w^2 \, dx < +\infty \quad \forall N \geq 3. \quad (3.19)
\]
Hence by Lemma 3.2, (3.18) and (3.19), we finally obtain for all \( N \geq 3 \) that
\[
J(\eta_\varepsilon) = S_{n,s} + \varepsilon \left( \frac{N-2}{N} H_{\partial \Omega}(0) + 2h(0) \right) \int_{\mathbb{R}^{N+1}} z_1 |\nabla w|^2 \, dz + \frac{2}{N} \varepsilon H_{\partial \Omega}(0) \int_{\mathbb{R}^{N+1}} z_1 \left| \frac{\partial w}{\partial z_1} \right|^2 \, dz + o(\varepsilon).
\]
(3.20)

Since
\[
\mu_s(\Omega) \leq J(\eta_\varepsilon).
\]
we have \( \mu < S_{N,s} \) provided that
\[
\left( \frac{N-2}{N} H_{\partial \Omega}(0) + 2h(0) \right) \int_{\mathbb{R}^{N+1}} z_1 |\nabla w|^2 \, dz + \frac{2}{N} \varepsilon H_{\partial \Omega}(0) \int_{\mathbb{R}^{N+1}} z_1 \left| \frac{\partial w}{\partial z_1} \right|^2 \, dz = 0.
\]
(3.21)

That is
\[
\left( \frac{N-2}{2N} + \frac{1}{N} \int_{\mathbb{R}^{N+1}} z_1 \left| \frac{\partial w}{\partial z_1} \right|^2 \, dz \right) H_{\partial \Omega}(0) + h(0) < 0
\]
that ends the proof. \( \square \)
4. Existence of Minimizer for \( \mu_s(\Omega) \)

It is clear from Proposition 3.3 that the proof of Theorem 1.1 should be finalized by the following two results in this section. Then we have

**Proposition 4.1.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^{N+1} \) of boundary \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \) and such that \( 0 \in \partial \Omega \cap \Gamma_2 \), \( h \in C^0(\partial \Omega \cap \Gamma_2) \) and \( s \in (0, 2) \). Assume that \( \mu_s(\Omega) < S_{N,s} \). Then there exists a minimizer for \( \mu_s(\Omega) \).

**Proof.** We define \( \Phi, \Psi : H^1(\Omega) \to \mathbb{R} \) by

\[
\Phi(u) := \frac{1}{2} \left( \int_{\Omega} |\nabla u|^2dz + \int_{\Gamma_2} h(x)u^2dx \right)
\]

and

\[
\Psi(u) = \frac{1}{q(s)} \int_{\Gamma_2} d^{-s}(\sigma)|u|^{q(s)}d\sigma.
\]

By Ekland variational principle there exists a minimizing sequence \( u_n \) for the quotient \( \mu := \mu_s(\Omega) \) such that

\[
\int_{\Gamma_2} d^{-s}(\sigma)|u_n|^{q(s)}d\sigma = 1,
\]

\[
\Phi(u_n) \to \frac{1}{2}\mu_s(\Omega)
\]

and

\[
\Phi'(u_n) - \mu_s(\Omega)\Psi'(u_n) \to 0 \quad \text{in } (H^1(\Omega))',
\]

with \( (H^1(\Omega))' \) denotes the dual of \( H^1(\Omega) \). We have that

\[
\int_{\Omega} |\nabla u_n|^2dx + \int_{\Gamma_2} h(x)u_n^2dx \leq \text{Const.} \quad \forall n \geq 1.
\]

In particular, by coercivity, \( u_n \to u \) for some \( u \) in \( H^1(\Omega) \).

**Claim:** \( u \neq 0 \).

Assume by contradiction that \( u = 0 \) (that is blow up occur). By continuity, \( \|u_n\|_{H^1(\Omega)} \to 0 \) and the fact that \( s \in (0, 1] \), there exists a sequence \( r_n > 0 \) such that

\[
\int_{\Gamma_2 \cap B_{r_n}} d^{-s}(\sigma)|u_n|^{q(s)}d\sigma = \frac{1}{2}.
\]

We now show that, up to a subsequence, \( r_n \to 0 \). Indeed, by (4.1) and (4.5)

\[
\int_{\Gamma_2 \setminus B_{r_n}} d^{-s}(\sigma)|u_n|^{q(s)}d\sigma = \frac{1}{2}.
\]

Since \( q(s) < q(0) = 2^* \) for \( s > 0 \), by compactness we have

\[
r_n^s C \leq \int_{\Gamma_2 \setminus B_{r_n}} |u_n|^{q(s)}d\sigma \leq \int_{\Gamma_2} |u_n|^{q(s)}d\sigma \to 0 \quad \text{as } n \to \infty,
\]

for some positive constant \( C \).

Define \( F_n(z) = \frac{1}{r_n}F(r_nz) \) for every \( z \in B^+_{r_n} \) and put \( (g_n)_{i,j} := \langle \partial_i F_n, \partial_j F_n \rangle = g_{ij}(r_nz) \).

Clearly

\[
g_n \to g_{Euc} \quad C^1(K) \quad \text{for every compact set } K \subset \mathbb{R}^{N+1},
\]
where \( g_{Euc} \) denotes the Euclidean metric. Let 
\[
w_n(z) = \frac{N+1}{n^2} u_n(F(r_n z)) \quad \forall z \in B_{r_0}^+.
\]
Then we get 
\[
\int_{B_{r_0}^N} |\tilde{z}|^{-s} w_n q(s) d\tilde{z} = (1 + o(1)) \int_{B_{r_0}^N} |\tilde{z}|^{-s} w_n q(s) \sqrt{|g_n|} d\tilde{z}.
\]
Hence by (4.5) we have 
\[
\int_{B_{r_0}^N} |\tilde{z}|^{-s} w_n q(s) d\tilde{z} = \frac{1}{2} (1 + cr_n).
\]
Let \( \eta \in C_c^\infty(F(B_{r_0})) \), \( \eta \equiv 1 \) on \( F(B_{r_0}) \) and \( \eta \equiv 0 \) on \( \mathbb{R}^{N+1} \setminus F(B_{r_0}) \). We define 
\[
\eta_n(z) = \eta(F(r_n z)) \quad \forall z \in \mathbb{R}^{N+1}.
\]
We have that 
\[
\|\eta_n w_n\|_D \leq C \quad \forall n \in \mathbb{N},
\]
where as usual \( D = D^{1,2}(\mathbb{R}^{N+1}) \). Therefore 
\[
\eta_n w_n \to w \quad \text{in} \, D.
\]
We first show that \( w \neq 0 \). Assume by contradiction that \( w \equiv 0 \). Thus \( w_n \to 0 \) in \( L^p_{loc}(\mathbb{R}^{N+1}_+) \) and in \( L^p_{loc}(\partial \mathbb{R}^{N+1}_+) \) for every \( 1 \leq p < 2^* \). Let \( \varphi \in C_c^\infty(B_{r_0}^2) \) be a cut-off function such that \( \varphi \equiv 1 \) on \( B_{r_0}^2 \) and \( \varphi \leq 1 \) in \( \mathbb{R}^{N+1} \). Define 
\[
\varphi_n(F(y)) = \varphi(r_n^{-1} y).
\]
We multiply (1.3) by \( \varphi_n^2 u_n \) (which is bounded in \( H^1(\Omega) \)) and integrate by parts to get 
\[
\int_\Omega \nabla u_n \nabla (\varphi_n^2 u_n) dx + \int_{\Gamma_2} h(x) u_n^2 \varphi_n^2 d\sigma = \mu_s(\Omega) \int_{\Gamma_2} d^{-s}(\sigma) |\varphi_n u_n|^{q(s)-2} (\varphi_n u_n)^2 d\sigma + o(1)
\]
\[
\leq \mu_s(\Omega) \left( \int_{\Gamma_2} d^{-s}(\sigma) |\varphi_n u_n|^{q(s)} d\sigma \right)^{\frac{2}{q(s)}} + o(1),
\]
where we have used (4.1). By compactness, we can easy see that 
\[
\int_{\Gamma_2} h(x) u_n^2 \varphi_n^2 d\sigma = o(1).
\]
Then in the coordinate system and after integration by parts, (4.9) becomes 
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla (\varphi w_n)|^2 g_n \sqrt{|g_n|} d\tilde{z} \leq \mu_s(\Omega) \left( \int_{\partial \mathbb{R}^{N+1}_+} |\tilde{z}|^{-s} |\varphi w_n|^{q(s)} \sqrt{|g_n|} d\tilde{z} \right)^{\frac{2}{q(s)}} + o(1).
\]
Therefore, by (4.6), for some constant \( c > 0 \), we have 
\[
(1 - cr_n) \int_{\mathbb{R}^{N+1}_+} |\nabla (\varphi w_n)|^2 d\tilde{z} \leq \mu_s(\Omega) \left( \int_{\partial \mathbb{R}^{N+1}_+} |\tilde{z}|^{-s} |\varphi w_n|^{q(s)} d\tilde{z} \right)^{\frac{2}{q(s)}} + o(1).
\]
Hence by the Hardy-Sobolev trace inequality (1.1), we get
\[
(1 - cr_n)S_{N,s} \left( \int_{\partial R^N_{+1}} |\tilde{z}|^{-s} |\varphi w_n|^q(s) d\tilde{z} \right)^{\frac{2}{q(s)}} \leq \mu_s(\Omega) \left( \int_{\partial R^N_{+1}} |\tilde{z}|^{-s} |\varphi w_n|^q(s) d\tilde{z} \right)^{\frac{2}{q(s)}} + o(1).
\]
(4.12)

Since \( S_{N,s} > \mu \), we conclude that
\[
o(1) = \int_{\partial R^N_{+1}} |\tilde{z}|^{-s} |\varphi w_n|^q(s) d\tilde{z} = \int_{B^N_0} |\tilde{z}|^{-s} |w_n|^q(s) d\tilde{z} + o(1)
\]
because by assumption \( q(s) < 2^* \). This is clearly in contradiction with (4.7) thus \( w \neq 0 \).

Now pick \( \varphi \in C^\infty_c(\mathbb{R}^N_{+1} \setminus \{0\}) \), and put \( \varphi_n(F(y)) = \varphi(r_n^{-1} y) \) for every \( y \in B_0 \). For \( n \) sufficiently large, \( \varphi_n \in C^\infty_c(\Omega) \) and it is bounded in \( H^1(\Omega) \). We multiply (4.3) by \( \varphi_n \) and integrate by parts to get
\[
\int_{\Omega} \nabla u_n \nabla \varphi_n dx = \mu_s(\Omega) \int_{\partial \Omega} d^{-s}(\sigma)|u_n|^{r(s)-2} u_n \varphi_n d\sigma + o(1).
\]

Hence
\[
\int_{\mathbb{R}^N_{+1}} \langle \nabla w_n, \nabla \varphi \rangle g_n \sqrt{|g_n|} dz = \mu_s(\Omega) \int_{\partial \mathbb{R}^N_{+1}} |\tilde{z}|^{-s} |w_n|^q(s) - 2 w_n \varphi \sqrt{|g_n|} d\tilde{z} + o(1).
\]

Since \( \eta_n \equiv 1 \) on \( B_{r_n} \frac{1}{2r_n} \) and the support of \( \varphi \) is contained in an annulus, for \( n \) sufficiently large
\[
\int_{\mathbb{R}^N_{+1}} \langle \nabla (\eta_n w_n), \nabla \varphi \rangle g_n \sqrt{|g_n|} dz = \mu_s(\Omega) \int_{\partial \mathbb{R}^N_{+1}} |\tilde{z}|^{-s} |\eta_n w_n|^q(s) - 2 \eta_n w_n \varphi \sqrt{|g_n|} d\tilde{z} + o(1).
\]

Since also \( g_n \) converges smoothly to the Euclidean metric on the support of \( \varphi \), by passing to the limit, we infer that, for all \( \varphi \in C^\infty_c(\mathbb{R}^N_{+1} \setminus \{0\}) \)
\[
\int_{\mathbb{R}^N_{+1}} \nabla w \nabla \varphi dz = \mu_s(\Omega) \int_{\partial \mathbb{R}^N_{+1}} |\tilde{z}|^{-s} |w|^q(s) - 2 w \varphi d\tilde{z}.\]
(4.13)

Notice that \( C^\infty_c(\mathbb{R}^N_{+1} \setminus \{0\}) \) is dense in \( C^\infty_c(\mathbb{R}^N_{+1}) \) with respect to the \( H^1(\mathbb{R}^N_{+1}) \) norm when \( N \geq 2 \), see e.g. [10]. Consequently since \( w \in \mathcal{D} \), it follows that (4.13) holds for all \( \varphi \in C^\infty_c(\mathbb{R}^N_{+1}) \) by (1.1). We conclude that
\[
\begin{cases}
\Delta w = 0 & \text{in } \mathbb{R}^N_{+1}, \\
\frac{\partial w}{\partial z^1} = S_{N,s} |\tilde{z}|^{-s} |w|^{q(s)-2} w & \text{on } \partial \mathbb{R}^N_{+1}, \\
\int_{\partial \mathbb{R}^N_{+1}} |\tilde{z}|^{-s} |w|^{q(s)} \leq 1, \\
w \neq 0.
\end{cases}
\]

Multiplying this equation by \( w \) and integrating by parts, leads to \( \mu_s(\Omega) \geq S_{N,s} \) by (1.1) which is a contradiction and thus \( u = \lim u_n \neq 0 \) is a minimizer for \( \mu_s(\Omega) \).

\[ \Box \]

In the following we study the existence of minimizers for the Sobolev trace inequality.
Proposition 4.2. Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^{N+1}$ of boundary $\partial \Omega = \Gamma_1 \cup \Gamma_2$ and such that $0 \in \partial \Omega \cap \Gamma_2$ and $h \in C^0(\partial \Omega \cap \Gamma_2)$. Assume that $\mu_0(\Omega) < S_{N,0}$. Then there exists a minimizer for $\mu_0(\Omega)$.

Proof. Recall the Sobolev trace inequality, proved by Li and Zhu in [8]: there exists a positive constant $C = C(\Omega)$ such that for all $u \in H^1(\Omega)$, we have

$$S_{N,0} \left( \int_{\Gamma_2} |u|^{2^*} d\sigma \right)^{2/2^*} \leq \int_{\Omega} |\nabla u|^2 dx + C \int_{\partial \Omega} |u|^2 d\sigma. \quad (4.14)$$

Now we let $u_n$ be a minimizing sequence for $\mu$, normalized as $\|u_n\|_{L^{2^*}(\Gamma_2)} = 1$. We now show that $u_n \to u$ is not zero. Put $\theta_n := u_n - u$ so that $\theta_n \to 0$ in $H^1(\Omega)$ and $\theta_n \to 0$ in $L^2(\Omega), L^2(\partial \Omega), L^2(\Gamma_2)$. Moreover by Brezis-Lieb Lemma [2] and recalling (4.1), it holds that

$$1 - \lim_{n \to \infty} \int_{\Gamma_2} |\theta_n|^{2^*} d\sigma = \int_{\Gamma_2} |u|^{2^*} d\sigma. \quad (4.15)$$

By using (4.14), we have

$$\mu_0(\Omega) \left( \int_{\Gamma_2} |u|^{2^*} d\sigma \right)^{2/2^*} \leq \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma_2} h(\sigma) |u_n|^2 d\sigma$$

$$\leq \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Gamma_2} h(\sigma) |u_n|^2 d\sigma - \int_{\Omega} |\nabla \theta_n|^2 dx + o(1)$$

$$\leq \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Gamma_2} h(\sigma) |u_n|^2 d\sigma - S_{N,0} \left( \int_{\Gamma_2} |\theta_n|^{2^*} d\sigma \right)^{2/2^*} + o(1)$$

$$\leq \mu_0(\Omega) - S_{N,0} \left( \int_{\Gamma_2} |\theta_n|^{2^*} d\sigma \right)^{2/2^*} + o(1).$$

We take the limit as $n \to \infty$ and use (4.15) to get

$$\mu_0(\Omega) \left( \int_{\Gamma_2} |u|^{2^*} d\sigma \right)^{2/2^*} \leq \mu_0(\Omega) - S_{N,0} \left( 1 - \int_{\Gamma_2} |u|^{2^*} d\sigma \right)^{2/2^*}.$$ 

Thanks to the concavity of the function $t \mapsto t^{2^*/2}$, the above implies that

$$\int_{\Gamma_2} |u|^{2^*} d\sigma \geq 1$$

whenever $\mu < S_{N,0}$. This completes the proof. \hfill \square

5. Proof of the main Result

The proof of Theorem 1.1 is a direct consequence of Proposition 3.3, Proposition 4.1 and Proposition 4.2.

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