MEASURING PROCESSES AND REPEATABILITY HYPOTHESIS

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Abstract

Srinivas [Commun. Math. Phys. 71 (1980), 131–158] proposed a postulate in quantum mechanics that extends the von Neumann-Lüders collapse postulate to observables with continuous spectrum. His collapse postulate does not determine a unique state change, but depends on a particular choice of an invariant mean. To clear the physical significance of employing different invariant means, we construct different measuring processes of the same observable satisfying the Srinivas collapse postulate corresponding to any given invariant means. Our construction extends the von Neumann type measuring process with the meter being the position observable to the one with the apparatus prepared in a non-normal state. It is shown that the given invariant mean corresponds to the momentum distribution of the apparatus in the initial state, which is determined as a non-normal state, called a Dirac state, such that the momentum distribution is the given invariant mean and that the position distribution is the Dirac measure.

1 Introduction

The problem of extending the von Neumann-Lüders collapse postulate [4,5] to observables with continuous spectrum is one of the major problems of the quantum theory of measurement. Recently, Srinivas [11] posed a set of postulates which gave an answer to this problem. However, it does not seem to be a complete solution. The following two problems remain.

(1) The Srinivas collapse postulate is not consistent with the \( \sigma \)-additivity of probability distributions and it requires ad hoc treatment of calculus of probability and expectation. How can we improve his set of postulates in order to retain the consistency with the \( \sigma \)-additivity of probability?
(2) His collapse postulate depends on a particular choice of an invariant mean. What is the physical significance of employing different invariant means? Can we characterize the various different ways of measuring the same observable [11; p. 149]?

The purpose of this paper is to resolve the second question by constructing different measuring processes of the same observable satisfying the Srinivas collapse postulate corresponding to the given invariant means. In our construction, the pointer position of the apparatus is the position observable and the given invariant mean corresponds to the momentum distribution at the initial state of the apparatus. Thus the choice of the invariant mean characterizes the state preparation of the apparatus.

For the general theory of quantum measurements of continuous observables, we shall refer to Davies [1], Holevo [3] and Ozawa [6–10]. The entire discussion including the solution of the first question above will be published elsewhere.

2 Formulation of the problem

In this paper, we shall deal with quantum systems with finite degrees of freedom. In the conventional formulation, the states of a system are represented by density operators on a separable Hilbert space \( \mathcal{H} \) and the observables are represented by self-adjoint operators on \( \mathcal{H} \). In this formulation, however, as shown in [7; Theorem 6.6], we cannot construct measuring processes satisfying the repeatability hypothesis, which follows from the Srinivas postulates; hence some generalization of the framework of quantum mechanics is necessary. We adopt the formulation that the states of a system are represented by norm one positive linear functionals on the algebra \( \mathcal{L}(\mathcal{H}) \) of bounded operators on \( \mathcal{H} \); states corresponding to density operators will be called normal states. For any state \( \sigma \) and compatible observables \( X, Y \) we shall denote by \( \Pr\{X \in dx, Y \in dy|\sigma\} \) the joint distribution of the outcomes of the simultaneous measurement of \( X \) and \( Y \). Our basic assumption is that \( \Pr\{X \in dx, Y \in dy|\sigma\} \) is a \( \sigma \)-additive probability distribution on \( \mathbb{R}^2 \) uniquely determined by the relation

\[
\int_{\mathbb{R}^2} f(x, y) \Pr\{X \in dx, Y \in dy|\sigma\} = \langle f(X, Y), \sigma \rangle, \quad (2.1)
\]

for all \( f \in C(\overline{\mathbb{R}}^2) \), where \( \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} \) and \( C(\overline{\mathbb{R}}^2) \) stands for the space of continuous functions on \( \overline{\mathbb{R}}^2 \). If \( \sigma \) is a normal state, Eq. (2.1) is reduced to the usual statistical formula. Apart from classical probability theory, we can consider another type of joint distributions in quantum mechanics. Let \( \langle X, Y \rangle \) be an ordered pair of any observables. We shall denote
by $\Pr\{X \in dx; Y \in dy\|\sigma\}$ the joint distribution of the outcomes of the successive measurement of $X$ and $Y$, performed in this order, in the initial state $\sigma$. Let $\eta$ be a fixed invariant mean on the space $CB(R)$ of continuous bounded functions on $R$. Let $X$ be an observable. Denote by $\mathcal{E}^X_\eta$ the norm one projection from $\mathcal{L}(\mathcal{H})$ onto $\{X(B); B \in \mathcal{B}(R)\}'$ such that

$$\Tr[\mathcal{E}^X_\eta[A] \rho] = \eta_u \Tr[e^{iuX} A e^{-iuX} \rho], \quad (2.2)$$

for all normal state $\rho$ and $A \in \mathcal{L}(\mathcal{H})$, where $\mathcal{B}(R)$ stands for the Borel $\sigma$-field of $R$, $X(B)$ stands for the spectral projection of $X$ for $B \in \mathcal{B}(R)$, and $'$ stands for the operation making the commutant in $\mathcal{L}(\mathcal{H})$. Then by a slight modification, the Srinivas collapse postulate asserts the following relation for the successive measurement of $X$ and any bounded observable $Y$:

$$\int_R y \Pr\{X \in B; Y \in dy\|\rho\} = \Tr[X(B) \mathcal{E}^X_\eta[Y] \rho], \quad (2.3)$$

for all normal state $\rho$ and $B \in \mathcal{B}(R)$. Obviously, this relation implies the following generalized Born statistical formula [11]: If $X$ and $Y$ are compatible then

$$\Pr\{X \in B; Y \in C\|\rho\} = \Tr\{X(B) Y(C) \rho\}, \quad (2.4)$$

for all normal state $\rho$ and $B, C \in \mathcal{B}(R)$. Our purpose is to construct a measuring process of $X$ which satisfies the Srinivas collapse postulate Eq.(2.3).

Throughout this paper, we shall fix an invariant mean $\eta$ which is, by a technical reason, a topological invariant mean on $CB(R)$ (cf. [2; p.24]).

3 Dirac state

In this section, we shall consider a quantum system with a single degree of freedom. Denote by $Q$ the position observable and by $P$ the momentum observable. A state $\delta$ on $\mathcal{L}(L^2(R))$ is called an $\eta$-Dirac state if it satisfies the following conditions (D1)–(D2):

(D1) For each $f \in CB(R)$, $\langle f(Q), \delta \rangle = f(0)$.

(D2) For each $f \in CB(R)$, $\langle f(P), \delta \rangle = \eta(f)$.

**Lemma 3.1** For any $f \in CB(R)$, $\mathcal{E}^Q_\eta(f(P)) = \eta(f)1$.  

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Proof. Let $\xi$ be a unit vector in $L^2(\mathbb{R})$. Let $g(p) = |\xi(-p)|^2$. Then $g$ is a density function on $\mathbb{R}$. For any $f \in CB(\mathbb{R})$, we have

$$
\langle \xi|\mathcal{E}_\eta^Q(f(P))|\xi \rangle = \eta(\xi|e^{iuQ}f(P)e^{-iuQ}|\xi)
= \eta(\xi|f(P+u1)|\xi)
= \eta \int_{\mathbb{R}} f(p+u)|\xi(p)|^2 dp
= \eta(f* g)(u) = \eta(f),
$$

where $f* g$ stands for the convolution of $f$ and $g$. It follows that $\mathcal{E}_\eta^Q(f(P)) = \eta(f)1$. QED

**Theorem 3.2** For every topologically invariant mean $\eta$, there exists an $\eta$-Dirac state.

Proof. Let $\phi$ be a state on $\{Q(B); B \in \mathcal{B}(\mathbb{R})\}'$ such that $\langle f(Q), \phi \rangle = f(0)$ for all $f \in CB(\mathbb{R})$ and $\delta$ a state on $\mathcal{L}(L^2(\mathbb{R}))$ such that $\langle A, \delta \rangle = \langle \mathcal{E}_\eta^Q(A), \phi \rangle$ for all $A \in \mathcal{L}(L^2(\mathbb{R}))$. Then by Lemma 3.1, $\delta$ is obviously an $\eta$-Dirac state. QED

### 4 Canonical measuring processes

Let $X$ be an observable of a quantum system $I$ described by a Hilbert space $\mathcal{H}$. We consider the following measuring process of $X$ by an apparatus system $II$. The apparatus system $II$ is a system with a single degree of freedom described by the Hilbert space $K = L^2(\mathbb{R})$. Thus the composite system $I+II$ is described by the Hilbert space $\mathcal{H} \otimes K$, which will be identified with the Hilbert space $L^2(\mathbb{R}; \mathcal{H})$ of all norm square integrable $\mathcal{H}$-valued functions on $\mathbb{R}$ by the Schrödinger representation of $K$. The pointer position of the apparatus system is the position observable $Q$. The interaction between the measured system $I$ and the apparatus system $II$ is given by the following Hamiltonian:

$$H_{int} = \lambda(X \otimes P), \quad (4.1)$$

where $P$ is the momentum of the apparatus. The strength $\lambda$ of the interaction is assumed to be sufficiently large that other terms in the Hamiltonian can be ignored. Hence the Schrödinger equation will be ($h = 2\pi$)

$$\frac{\partial}{\partial t}\Psi_t(q) = -\lambda \left( X \otimes \frac{\partial}{\partial t} \right) \Psi_t(q), \quad (4.2)$$

in the q-representation, where $\Psi_t \in \mathcal{H} \otimes K$. The measurement is carried out by the interaction during a finite time interval from $t = 0$ to $t = 1/\lambda$. 

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The outcome of this measurement is obtained by the measurement of $Q$ at time $t = 1/\lambda$. The statistics of this measurement depends on the initially prepared state $\sigma$ of the apparatus. According to [7;Theorem 6.6], if $\sigma$ is a normal state then this measurement cannot satisfy Eq. (2.3). Now we assume that the initial state of the apparatus is an $\eta$-Dirac state $\delta$ and we shall call this measuring process as a canonical measuring process of $X$ with preparation $\delta$.

In order to obtain the solution of Eq. (4.2), assume the initial condition

$$\Psi_0 = \psi \otimes \alpha,$$  \hspace{1cm} (4.3)

where $\psi \in \mathcal{H}$ and $\alpha \in \mathcal{K}$. The solution of the Schrödinger equation is given by

$$\Psi_t = e^{-it\lambda(X\otimes P)} \psi \otimes \alpha,$$  \hspace{1cm} (4.4)

and hence for any $\psi \in \mathcal{H}$ and $\beta \in \mathcal{K}$, we have

$$\int_{\mathbb{R}} \langle \phi \otimes \beta(q)|\Psi_t(q) \rangle dq = \int_{\mathbb{R}} e^{-it\lambda xP} \langle \phi \otimes \beta|X(dx) \otimes P(dp)|\psi \otimes \alpha \rangle \approx \int_{\mathbb{R}} \langle \beta|e^{-it\lambda xP}|\alpha \rangle \langle \phi|X(dx)|\psi \rangle$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \beta(q)^* \alpha(q - t\lambda x) dq \langle \phi|X(dx)|\psi \rangle$$

$$= \int_{\mathbb{R}} \alpha(q1 - t\lambda x) \langle \beta(q)|X(dx)|\psi \rangle dq$$

It follows that

$$\Psi_t(q) = \alpha(q1 - t\lambda X) \psi.$$  \hspace{1cm} (4.5)

For $t = 1/\lambda$, we have

$$\Psi_{1/\lambda}(q) = \alpha(q1 - X) \psi.$$  \hspace{1cm} (4.6)

**Theorem 4.1** For any $f \in L^\infty(\mathbb{R})$, we have

$$U_t^*(1 \otimes f(Q))U_t = f(t\lambda(X \otimes 1) + 1 \otimes Q),$$  \hspace{1cm} (4.7)

where $U_t = e^{-it\lambda(X\otimes P)}$. 


Lemma 5.1

Let \( X \) be an observable of the system \( I \). Then for any \( f \in CB(R^2) \),
\[
\mathcal{E}_\delta[f(X \otimes 1, 1 \otimes Q)] = f(X, 0).
\]
Proof. Let $\psi \in \mathcal{H}$. For any $\alpha \in \mathcal{K}$, we have

$$\langle \alpha | \mathcal{E}_{\psi} \cdot [f(X \otimes 1, 1 \otimes Q)] | \alpha \rangle$$

$$= \langle \psi \otimes \alpha | f(X \otimes 1, 1 \otimes Q) | \psi \otimes \alpha \rangle$$

$$= \int_R \int_R f(x, q) \langle \psi | X(dx) | \psi \rangle \langle \alpha | Q(dq) | \alpha \rangle$$

$$= \langle \alpha | F(Q) | \alpha \rangle,$$

where $F(q) = \int_R f(x, q) \langle \psi | X(dx) | \psi \rangle$. Thus $\mathcal{E}_{\psi} \cdot [f(X \otimes 1, 1 \otimes Q)] = F(Q)$. It is easy to see that $F \in CB(\mathbb{R})$ and hence $\langle F(Q), \delta \rangle = F(0)$ by (D1). We see that

$$\langle F(Q), \delta \rangle = \langle \mathcal{E}_{\psi} \cdot [f(X \otimes 1, 1 \otimes Q)], \delta \rangle$$

$$= \langle \psi | \mathcal{E}_{\delta} \cdot [f(X \otimes 1, 1 \otimes Q)] | \psi \rangle,$$

and

$$F(0) = \langle \psi | f(X, 0) | \psi \rangle.$$

It follows that $\mathcal{E}_{\delta} \cdot [f(X \otimes 1, 1 \otimes Q)] = f(X, 0)$. QED

Theorem 5.2 For any $f \in CB(\mathbb{R})$, we have

$$\mathcal{E}_{\delta} \cdot [U^* (1 \otimes f(Q)) U] = f(X).$$

Proof. From Theorem 4.1, $U^* (1 \otimes f(Q)) U = f(X \otimes 1 + 1 \otimes Q)$ and hence the assertion follows from applying Lemma 5.1 to $g \in CB(\mathbb{R}^2)$ such that $g(x, y) = f(x + y)$. QED

Theorem 5.3 For any $Y \in \mathcal{L}(\mathcal{H})$, we have

$$\mathcal{E}_{\delta} \cdot [U^* (Y \otimes 1)] = \mathcal{E}_{\eta}^X (Y).$$

(5.3)

Proof. Let $\psi \in \mathcal{H}$. For any $\alpha \in \mathcal{K}$, we have

$$\langle \alpha | \mathcal{E}_{\psi} \cdot [U^* (Y \otimes 1)] | \alpha \rangle$$

$$= \langle \psi \otimes \alpha | U^* (Y \otimes 1) U | \psi \otimes \alpha \rangle$$

$$= \int_R \int_R \langle \alpha(p) | e^{ipX} Y e^{-ipX} | \alpha(p) \rangle dp$$

$$= \int_R \langle \psi | e^{ipX} Y e^{-ipX} | \psi \rangle \langle \alpha | E^p(dp) | \alpha \rangle$$

$$= \langle \alpha | F(P) | \alpha \rangle,$$
where $F(p) = \langle \psi | e^{ipX}Y e^{-ipX} |\psi \rangle$. Consequently, $E_{\omega} \langle [U^* (Y \otimes 1) U] \rangle = F(P)$. Since $F \in CB(R)$, we have from (D2), $\langle F(P), \delta \rangle = \eta(F)$. We see that

$$\langle F(P), \delta \rangle = \langle E|\psi\rangle\langle \psi|E^*[U^*(Y \otimes I)U]|\psi\rangle,$$

and

$$\eta(F) = \eta_{\delta} \langle \psi | e^{ipX}Y e^{-ipX} |\psi \rangle = \langle \psi | E^X_{\eta}(Y)|\psi \rangle.$$

Thus, $E_{\delta}[U^*(Y \otimes 1)U] = E^X_{\eta}(Y)$. QED

**Theorem 5.4** Let $Y \in L(H)$ and $f \in CB(R)$. Then we have

$$E_{\delta}[U^*(Y \otimes f(Q))U] = f(X)E^X_{\eta}[Y]. \quad (5.4)$$

**Proof.** By the Stinespring theorem [12], there is a Hilbert space $W$, an isometry $V : H \otimes K \rightarrow W$ and a $^*$-representation $\pi : L(H) \otimes L(K) \rightarrow L(W)$ such that $E_{\delta}[U^*AU] = V^*\pi(A)V$ for all $A \in L(H) \otimes L(K)$. By Theorem 5.2, $V^*\pi(1 \otimes f(Q))V = f(X)$. Thus by easy computations,

$$(\pi(1 \otimes f(Q))V - Vf(X))^{*}(\pi(1 \otimes f(Q))V - Vf(X)) = 0.$$  

It follows that $\pi(1 \otimes f(Q))V = Vf(X)$, and hence from Theorem 5.3, we have

$$E_{\delta}[U^*(Y \otimes f(Q))U] = V^*\pi(Y \otimes f(Q))V = V^*\pi(Y \otimes 1)Vf(X)$$

$$= E^X_{\eta}[Y]f(X) = f(X)E^X_{\eta}[Y].$$

QED

Now we can prove that the canonical measuring process of $X$ with preparation $\delta$, where $\delta$ is an $\eta$-Dirac state, satisfies the Srinivas collapse postulate for the given invariant mean $\eta$.

**Theorem 5.5** For any bounded observable $Y \in L(H)$ and $B \in B(R)$, we have

$$\int_{R} y \Pr\{X \in B; Y \in dy\|\rho\} = Tr[X(B)E^X_{\eta}[Y]\rho]$$

for all normal state $\rho$.  

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Proof. Denote by \( C_0(\mathbb{R}) \) the space of continuous functions on \( \mathbb{R} \) vanishing at infinity. Let \( Y \) be a bounded observable and \( \rho \) a normal state. From Eqs. (5.1) and (5.2) and from Theorem 5.4, for any \( f, g \in C_0(\mathbb{R}) \) we have

\[
\int_{\mathbb{R}} f(x)g(y) \Pr\{X \in dx; Y \in dy\|\rho\} = \text{Tr}[f(X)\mathcal{E}_\eta^X[g(Y)]\rho].
\]

By the bounded convergence theorem and the normality of the state \( \rho \), the set of all Borel functions \( f \) satisfying the above equality is closed under bounded pointwise convergence and contains \( C_0(\mathbb{R}) \). Thus the equality holds for all bounded Borel functions \( f \). Since \( Y \) is bounded, there is a function \( h \in C_0(\mathbb{R}) \) such that \( h(y) = y \) on the spectrum of \( Y \). Let \( f = \chi_B \) and \( g = h \). We have \( f(X) = X(B) \) and \( f(Y) = Y \) so that we obtain the desired equality. \( QED \)

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