GENERIC TWISTED $T$-ADIC EXPONENTIAL SUMS OF POLYNOMIALS

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Abstract. The twisted $T$-adic exponential sum associated to a polynomial in one variable is studied. An explicit arithmetic polygon is proved to be the generic Newton polygon of the $C$-function of the twisted $T$-adic exponential sum. It gives the generic Newton polygon of the $L$-functions of twisted $p$-power order exponential sums.

1. Introduction

Let $W$ be a Witt ring scheme of Witt vectors, $\mathbb{F}_q$ the field of characteristic $p$ with $q$ elements, $\mathbb{Z}_q = W(\mathbb{F}_q)$, and $\mathbb{Q}_q = \mathbb{Z}_q[\frac{1}{p}]$.

Let $\Delta \supseteq \{0\}$ be an integral convex polytope in $\mathbb{R}^n$, and $I$ the set of vertices of $\Delta$ different from the origin. Let

$$f(x) = \sum_{v \in \Delta} (a_v x^v, 0, 0, \ldots) \in W(\mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$$

with $\prod_{v \in I} a_v \neq 0$,

where $x^v = x_1^{v_1} \cdots x_n^{v_n}$ if $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$.

Let $T$ be a variable. Let $\mu_{q-1}$ be the group of $q-1$-th roots of unity in $\mathbb{Z}_q$ and $\chi = \omega^{-u}$ with $u \in \mathbb{Z}^n/(q-1)$ a fixed multiplicative character of $(\mathbb{F}_q^\times)^n$ into $\mu_{q-1}$, where $\omega : x \to \hat{x}$ is the Teichmüller character.

Definition 1.1. The sum

$$S_{f,u}(k,T) = \sum_{x \in (\mathbb{F}_q^k)^n} \chi(\text{Norm}_{\mathbb{F}_q^k/\mathbb{F}_q}(x))(1 + T)^{\text{Tr}_{\mathbb{Q}_p/k}(\hat{f}(\hat{x}))} \in \mathbb{Z}_q[[T]]$$

is called a twisted $T$-adic exponential sum. And the function

$$L_{f,u}(s,T) = \exp(\sum_{k=1}^{\infty} S_{f,u}(k,T) \frac{s^k}{k}) \in 1 + s\mathbb{Z}_q[[T]][[s]]$$

is called an $L$-function of a twisted $T$-adic exponential sum.

Definition 1.2. The function

$$C_{f,u}(s,T) = \exp(\sum_{k=1}^{\infty} -(q^k - 1)^{-1} S_{f,u}(k,T) \frac{s^k}{k}),$$

is called a $C$-function of a twisted $T$-adic exponential sums.
The L-function and C-function determine each other:

\[ L_{f,u}(s, T) = \prod_{i=0}^{n} C_{f,u}(q^i s, T)^{(-1)^{n-i+1} \binom{n}{i}}, \]

and

\[ C_{f,u}(s, T) = \prod_{j=0}^{\infty} L_{f,u}(q^j s, T)^{(-1)^{n-1-j} \binom{n-j+1}{j}}. \]

By the last identity, one sees that

\[ C_{f,u}(s, T) \in 1 + s\mathbb{Z}_q[[T]][[s]]. \]

The \( T \)-adic exponential sums were first introduced by Liu-Wan [11]. We view \( L_{f,u}(s, T) \) and \( C_{f,u}(s, T) \) as power series in the single variable \( s \) with coefficients in the \( T \)-adic complete field \( \mathbb{Q}_q((T)) \). The C-function \( C_{f,u}(s, T) \) was shown to be \( T \)-adic entire in \( s \) by Liu-Wan [11] for \( u = 0 \) and Liu [10] for all \( u \).

Let \( \zeta_{p^m} \) be a primitive \( p^m \)-th root of unity, and \( \pi_m = \zeta_{p^m} - 1 \). Then \( L_{f,u}(s, \pi_m) \) is the L-function of the exponential sums \( S_{f,u}(k, \pi_m) \) studied by Adolphson-Sperber [1–4] for \( m = 1 \) and by Liu-Wei [12] and Liu [9] for \( m \geq 1 \).

Let \( C(\Delta) \) be the cone generated by \( \Delta \). There is a degree function \( \deg \) on \( C(\Delta) \) which is \( \mathbb{R}_{\geq 0} \)-linear and takes values 1 on each co-dimension 1 face not containing 0. For \( a \not\in C(\Delta) \), we define \( \deg(a) = +\infty \). Write \( C_u(\Delta) = C(\Delta) \cap (u + (q - 1)\mathbb{Z}^n) \) and \( M_u(\Delta) = \frac{1}{q-1} C_u(\Delta) \). Let \( b \) be the least positive integer such that \( p^b u \equiv u \pmod{q - 1} \). Order elements of \( \bigcup_{i=0}^{b-1} M_{p^i u}(\Delta) \) so that \( \deg(x_1) \leq \deg(x_2) \leq \cdots \).

**Definition 1.3.** The infinite \( u \)-twisted Hodge polygon \( H_{\Delta,u}^\infty \) of \( \Delta \) is the convex function on \([0, +\infty)\) with initial point 0 which is linear between consecutive integers and whose slopes are

\[
\frac{\deg(x_{b i + 1}) + \deg(x_{b i + 2}) + \cdots + \deg(x_{b i + 1})}{b}, \quad i = 0, 1, \ldots.
\]

Write NP for the short of Newton polygon. Liu [10] proved the following Hodge bound for the C-function \( C_{f,u}(s, T) \).

**Theorem 1.4.** We have

\[ T \)-adic NP of \( C_{f,u}(s, T) \) \( \geq \text{ord}_p(q)(p - 1)H_{\Delta,u}^\infty. \]

In the rest of this paper, we assume that \( \Delta = [0, d] \) and \( a = \log_p q \).

Fix \( 0 \leq u \leq q - 2 \). Write \( u = u_0 + u_1 p + \cdots + u_{a-1} p^{a-1} \) with \( 0 \leq u_i \leq p - 1 \). Then we have

\[ \frac{u}{q - 1} = -(u_0 + u_1 p + \cdots), \quad u_i = u_{b i}. \]

Write \( p^i u = q_i (q - 1) + s_i \) for \( i \in \mathbb{N} \) with \( 0 \leq s_i < q - 1 \), then \( s_{a-l} = u_l + u_{l+1} p + \cdots + u_{a+l-1} p^{a-1} \) for \( 0 \leq l \leq a - 1 \) and \( s_i = s_{b i} \).
Lemma 1.5. The infinite $u$-twisted Hodge polygon $H_{[0,d],u}^\infty$ is the convex function on $[0, +\infty]$ with initial point 0 which is linear between consecutive integers and whose slopes are

$$
\frac{u_0 + u_1 + \cdots + u_{b-1}}{bd(p-1)} + \frac{l}{d}, \quad l = 0, 1, \cdots.
$$

Proof. One observes that $\frac{u_i}{q-1} + l$ with $0 \leq i \leq b - 1$ is just a permutation of $x_{bl+1}, x_{bl+2}, \cdots, x_{bl+1}$. The lemma follows. \qed

Definition 1.6. For any $i, n \in \mathbb{N}$, we define

$$
\delta_{\in}^{(i)}(n) = \begin{cases} 
1, & p^l + u_{b-i} \equiv n \pmod{d} \text{ for some } l < d\{\frac{n}{d}\}; \\
0, & \text{otherwise},
\end{cases}
$$

where $\{\cdot\}$ is the fractional part of a real number. We also define $\delta_{\in}^{(i)}(0) = 0$.

Definition 1.7. The arithmetic polygon $P_{[0,d],u}$ is the convex function on $[0, +\infty]$ with initial point 0 which is linear between consecutive integers and whose slopes are

$$
\omega(n) = \frac{1}{b} \sum_{i=0}^{b-1} \left( \frac{(p-1)n + u_{b-i}}{d} - \delta_{\in}^{(i)}(n) \right), \quad n \in \mathbb{N},
$$

where $[\cdot]$ is the least integer equal or greater than a real number.

Write

$$
\{x\}' = 1 + x - [x] = \begin{cases} 
\{x\}, & \text{if } \{x\} \neq 0, \\
1, & \text{if } \{x\} = 0,
\end{cases}
$$

where $\{\cdot\}$ is the fractional part of a real number. Define

$$
\varepsilon(u) = \min\{d\{\frac{u_i}{d}\}' | 1 \leq i \leq b\}.
$$

In this paper, we shall prove the following theorems.

Theorem 1.8. We have

$$
P_{[0,d],u} \geq (p-1)H_{[0,d],u}^\infty.
$$

Moreover, they coincide at the point $d$.

Theorem 1.9. Let $f(x) = \sum_{i=0}^{d} (a_ix^i, 0, 0, \cdots)$, and $p > 4d - \varepsilon(u)$. Then

$$
T - \text{adic NP of } C_{f,u}(s, T) \geq \text{ord}_p(q)P_{[0,d],u}.
$$

Theorem 1.10. Let $f(x) = \sum_{i=0}^{d} (a_ix^i, 0, 0, \cdots)$, and $p > 4d - \varepsilon(u)$. Then there is a non-zero polynomial $H_u(y) \in \mathbb{F}_q[y_i | i = 0, 1, \cdots, d]$ such that

$$
T - \text{adic NP of } C_{f,u}(s, T) = \text{ord}_p(q)P_{[0,d],u}
$$

if and only if $H_u((a_i)_{i=0,1,\cdots,d}) \neq 0$. 
Theorem 1.11. Let \( f(x) = \sum_{i=0}^{d} (a_i x^i, 0, 0, \cdots), p > 4d - \varepsilon(u) \) and \( m \geq 1 \). Then

\[
\pi_m - \text{adic NP of } C_{f,u}(s, \pi_m) \geq \text{ord}_p(q) P_{0, d, u}
\]

with equality holding if and only if \( H_u((a_i)_{i=0,1,\cdots,d}) \neq 0 \).

By a result of W. Li [7], we see if \( p \mid d \), \( L_{f,u}(s, \pi_m) \) is a polynomial of degree \( p^{m-1} \). From the above theorem, we shall deduce the following.

Theorem 1.12. Let \( f(x) = \sum_{i=0}^{d} (a_i x^i, 0, 0, \cdots), p > 4d - \varepsilon(u) \) and \( m \geq 1 \). Then

\[
\pi_m - \text{adic NP of } L_{f,u}(s, \pi_m) \geq \text{ord}_p(q) P_{0, d, u} \text{ on } [0, p^{m-1}d]
\]

with equality holding if and only if \( H_u((a_i)_{i=0,1,\cdots,d}) \neq 0 \).

Note that \( \varepsilon(0) = d \) and \( P_{0, d, 0} \) is just the arithmetic polygon \( p_{0,d} \) defined in Liu-Liu-Niu [13]. So the above results are generalizations of the corresponding results in Liu-Liu-Niu [13]. Initiated by a conjecture of Wan [15], the asymptotic behavior of generic Newton polygon of \( L_{f,u}(s, \pi_m) \) with \( m = 1 \) was studied by Zhu [16–18], Blache-Férrard [5], and Blache-Férrard-Zhu [6].

2. Arithmetic estimate

In this section, let \( R_i \) be a finite subset of \( \{1, 2, \cdots, a\} \times (\frac{s_i}{q-1} + \mathbb{N}) \) with cardinality \( a n \) for each \( 1 \leq i \leq b \), \( \tau \) a permutation of \( R = \bigcup_{i=1}^{b} R_i \).

Write \( i(l) = i \) if \( l \in R_i \). We shall estimate

\[
\sum_{i=1}^{b} \sum_{l \in R_i} \left| \frac{p\phi_i(l) - \phi_i(\tau(l))}{d} \right| \tau(l) + u_{b-i},
\]

where \( \phi_i \) is the projection \( \{1, 2, \cdots, a\} \times (\frac{s_i}{q-1} + \mathbb{N}) \rightarrow \mathbb{N} \) such that

\[
\phi_i(\cdot, \frac{s_i}{q-1} + l) = l.
\]

Define

\[
\delta^{(i)}_<(l) = \begin{cases} 1, & \text{if } \left\{ \frac{l}{d} \right\} < \left\{ \frac{pl + ub_{b-i}}{d} \right\}, \\ 0, & \text{otherwise}. \end{cases}
\]

Write \( A_n = \{0, 1, \cdots, n-1\} \) and \( \mu = \{\frac{n-1}{d}\} \).

Lemma 2.1. For any \( 1 \leq i \leq b \), we have

\[
\sum_{i=0}^{n-1} (\delta^{(i)}_<(l) - \delta^{(i)}_< (l)) = \#\{l \in A_n | \left\{ \frac{l}{d} \right\} \leq \mu < \left\{ \frac{pl + ub_{b-i}}{d} \right\} \} - \#\{l \in A_n | \left\{ \frac{l}{d} \right\} > \mu \geq \left\{ \frac{pl + ub_{b-i}}{d} \right\} \}.
\]
Proof. Note that both $\delta_\Phi^{(i)}$ and $\delta_\Psi^{(i)}$ have a period $d$ and initial value 0, so we may assume that $n \leq d$. The case $n = 1$ is trivial. We are going to show this for $n \geq 2$. Write

$$\delta_\Phi^{(i)} = \begin{cases} 1, & pl + u_{b-i} \equiv 0 \pmod{d} \text{ for some } m \leq l \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Hence by definition, we have

$$\sum_{l=0}^{n-1} \delta_\Phi^{(i)}(l) = \sharp\{1 \leq l \leq n-1\{\frac{l}{d}\} < \{\frac{pl + u_{b-i}}{d}\}\} + \delta_\Phi^{(i)}_{1,n-1}.$$

We also have

$$\sum_{l=0}^{n-1} \delta_\Psi^{(i)}(l) = \sharp\{1 \leq l \leq n-2\{l+1\} \leq d\{\frac{pl + u_{b-i}}{d}\} \leq n-1\} + 1_{\frac{1}{d} \leq \frac{u_{b-i}}{d} \leq \frac{n-1}{d}}.$$ 

Therefore

$$\sum_{l=0}^{n-1} (\delta_\Phi^{(i)}(l) - \delta_\Psi^{(i)}(l)) = \sharp\{1 \leq l \leq n-1\{\frac{pl + u_{b-i}}{d}\} > \mu\} + \delta_\Psi^{(i)}_{1,n-1} - 1_{\frac{1}{d} \leq \frac{u_{b-i}}{d} \leq \frac{n-1}{d}}.$$

Note that

$$\sharp\{l \in A_n|\{\frac{l}{d}\}' < \{\frac{pl + u_{b-i}}{d}\}'\} = \sharp\{1 \leq l \leq n-1\{\frac{pl + u_{b-i}}{d}\} > \mu\} + \delta_\Psi^{(i)}_{1,n-1}$$

and

$$\sharp\{l \in A_n|\{\frac{l}{d}\}' \geq \{\frac{pl + u_{b-i}}{d}\}'\} = 1_{\frac{1}{d} \leq \frac{u_{b-i}}{d} \leq \frac{n-1}{d}},$$

the lemma follows. \qed

Lemma 2.2. Let $A, B, C$ be sets with $A$ finite, and $\tau$ a permutation of $A$. Then

$$\#\{a \in A \mid \tau(a) \in B, a \in C\} \geq \#\{a \in A \mid a \in B, a \in C\} - \#\{a \in A \mid a \notin B, a \notin C\}.$$ 

Proof. The reader may refer [13] and we omit the proof here. \qed

For $l \in R_i$, define

$$\delta_{\Phi,\tau}^{(i)}(l) = \begin{cases} 1, & \text{if } \{\frac{\phi_{\tau,\tau}(l)}{d}\}' < \{\frac{pl + u_{b-i}}{d}\}'; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3. We have

$$\sum_{i=1}^b \sum_{l \in R_i} \delta_{\Phi,\tau}^{(i)}(l) \geq \sum_{i=1}^b \sum_{l=0}^{n-1} (\delta_\Phi^{(i)} - \delta_\Psi^{(i)})(l) - 2 \sum_{i=1}^b \sharp\{l \in R_i|\phi_i(l) > n-1\}.$$
Proof. By definition, we have

\[
\sum_{i=1}^{b} \sum_{l \in R_i} \delta_{<, \tau}^{(i)}(l) = \# \{ l \in R | \left\{ \frac{\phi_i(\tau(l))}{d}(\tau(l)) \right\} < \frac{p\phi_i(l) + u_{b-i}(l)}{d}\} \\
\geq \# \{ l \in R | \left\{ \frac{\phi_i(\tau(l))}{d}(\tau(l)) \right\} < \frac{p\phi_i(l) + u_{b-i}(l)}{d}\} \\
\sum_{i=1}^{b} \sum_{l \in R_i} \delta_{<, \tau}^{(i)}(l) \geq \# \{ l \in R | \left\{ \frac{\phi_i(\tau(l))}{d}(\tau(l)) \right\} \leq \frac{p\phi_i(l) + u_{b-i}(l)}{d}\} \\
-\# \{ l \in R | \left\{ \frac{\phi_i(\tau(l))}{d}(\tau(l)) \right\} > \mu \geq \{ \frac{p\phi_i(l) + u_{b-i}(l)}{d}\} \\
\sum_{i=1}^{b} \sum_{l \in R_i} \delta_{<, \tau}^{(i)}(l) \geq \# \{ l \in R | \left\{ \frac{\phi_i(\tau(l))}{d}(\tau(l)) \right\} \leq \frac{p\phi_i(l) + u_{b-i}(l)}{d}\} \cdot \sum_{i=1}^{b} \sum_{l \in R_i} \delta_{<, \tau}^{(i)}(l) \\
\leq \# \{ \phi_i(l) \in A_n | \{ \frac{\phi_i(\tau(l))}{d}(\tau(l)) \} \leq \mu \geq \{ \frac{p\phi_i(l) + u_{b-i}(l)}{d}\} \} + \# \{ l \in R_i, \phi_i(l) > n - 1 \} \\
= a\# \{ l \in A_n | \{ \frac{\phi_i(\tau(l))}{d}(\tau(l)) \} \} + \# \{ l \in R_i, \phi_i(l) > n - 1 \}.
\]

By \# \{ \phi_i(l) \in A_n, l \notin R_i \} = \# \{ l \in R_i, \phi_i(l) \notin A_n \}, the theorem follows from Lemma 2.1. \[\square\]

Theorem 2.4. If \( p > 4d - \varepsilon(u) \), then we have

\[
\sum_{i=1}^{b} \sum_{l \in R_i} \left\{ \frac{p\phi_i(l) - \phi_i(\tau(l))(\tau(l)) + u_{b-i}}{d} \right\} \geq abP_{[0, d], u}(n) + \# \{ l \in R, \phi_i(l) > n-1 \}.
\]

Proof. By the definition of \( \delta_{<, \tau}^{(i)}(l) \), we see

\[
\left\{ \frac{p\phi_i(l) - \phi_i(\tau(l))(\tau(l)) + u_{b-i}}{d} \right\} = \left\{ \frac{p\phi_i(l) + u_{b-i}}{d} \right\} - \left\{ \frac{\phi_i(\tau(l))(\tau(l))}{d} \right\} + \delta_{<, \tau}^{(i)}(l).
\]

Since \( p > 4d - \varepsilon(u) \), we have

\[
\sum_{l \in R_i} \left( \left\{ \frac{p\phi_i(l) + u_{b-i}}{d} \right\} - \left\{ \frac{\phi_i(l)}{d} \right\} \right)
\]
\[= \left( \sum_{\phi_i(l) \in A_n} + \sum_{l \in R_i} \sum_{l \not\in R_i} \right) \left( \frac{p\phi_i(l) + u_{b,i}}{d} - \frac{\phi_i(l)}{d} \right)\]

\[\geq a \sum_{l=0}^{n-1} \left( \frac{pl + u_{b,i}}{d} - \frac{l}{d} \right) + 3^\ast \{l \in R_i | \phi_i(l) > n-1 \}.
\]

The theorem is proved by Theorem 2.3.

3. Twisted \(T\)-adic Dwork’s trace formula

In this section we review the twisted \(T\)-adic analogy of Dwork’s theory on exponential sums.

Let \(E(x) = \exp(\sum_{n=0}^{\infty} \frac{x^n}{n}) = \sum_{n=0}^{\infty} \lambda_n x^n\) be the Artin-Hasse series. Define a new \(T\)-adic uniformizer \(\pi\) of the \(T\)-adic local ring \(\mathbb{Q}_p[[T]]\) by the formula \(E(\pi) = 1 + T\).

Recall \(M_u = M_u([0,d]) = \{v \in \mathbb{Q}_{\geq 0} | v \equiv \frac{w}{q^i} \pmod{1}\}\). Write

\[L_u = \{ \sum_{v \in M_u} b_v \pi^v x^v : b_v \in \mathbb{Q}_q[[\pi^{q^i-1}]] \} \]

and

\[B_u = \{ \sum_{v \in M_u} b_v \pi^v x^v : b_v \in \mathbb{Q}_q[[\pi^{q^i-1}]] ; \text{ord}_\pi b_v \to \infty, v \to \infty \} \]

Define \(\psi_p : L_u \to L_{p^{-1}u}\) by \(\psi_p(\sum_{v \in M_u} b_v x^v) = \sum_{v \in M_{p^{-1}u}} b_{pv} x^v\). Then the map \(\psi_p \circ E_f\) sends \(L_u\) to \(B_{p^{-1}u}\), where \(E_f(x) = \prod_{\alpha \neq 0} E(\pi \alpha x^i) \in L_0\).

The Galois group \(\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)\) can act on \(B_u\) by fixing \(\pi^{q^i-1}\) and \(x\). Let \(\sigma \in \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)\) be the Frobenius element such that \(\sigma(\zeta) = \zeta^q\) if \(\zeta\) is a \((q-1)\)-th root of unity. The operator \(\Psi = \sigma^{-1} \circ \psi_p \circ E_f(x)\) sends \(B_u\) to \(B_{p^{-1}u}\), hence \(\Psi\) operators on \(B = \bigoplus_{i=0}^{b-1} B_{p^iu}\). We call it Dwork’s \(T\)-adic semi-linear operator because it is semi-linear over \(\mathbb{Z}_q[[\pi^{q^i-1}]]\).

Note that \(\Psi^a = \psi_p^a \circ \prod_{i=0}^{a-1} E_f (x^{q^i})\). It follows that \(\Psi^a\) operates on \(B_u\) and is linear over \(\mathbb{Z}_p[[\pi^{q^i-1}]]\). Moreover, one can show that \(\Psi\) is completely continuous in sense of Serre [14], so \(\det(1 - \Psi^a s | B_u/\mathbb{Z}_q[[\pi^{q^i-1}]]\) and \(\det(1 - \Psi s | B/\mathbb{Z}_p[[\pi^{q^i-1}]]\) are well defined.

Now we state the twisted \(T\)-adic Dwork’s trace formula [9].

Theorem 3.1. We have

\[C_{f,u}(s,T) = \det(1 - \Psi^a s | B_u/\mathbb{Z}_q[[\pi^{q^i-1}]]).\]
4. The twisted Dwork semi-linear operator

In this section, we shall study the twisted Dwork semi-linear operator $\Psi$. Recall that $\Psi = \sigma^{-1} \circ \psi_p \circ E_f$, $E_f(x) = \prod_{i=0}^d E(\pi \hat{a}_i x^i) = \sum_{n=0}^\infty \gamma_n x^n$, where

$$\gamma_n = \sum_{\pi} \prod_{i=0}^d \lambda_{n_i} \hat{a}_i^{n_i}.$$ 

We see $B = \bigoplus_{i=1}^b B_{p^i u}$ has a basis represented by $\{x^{\frac{a}{q-1}+j}\}_{1 \leq i \leq b, j \in \mathbb{N}}$ over $\mathbb{Z}_q[[\pi^{1/(q-1)}]]$. We have

$$\Psi(x^{\frac{a}{q-1}+j}) = \sigma^{-1} \circ \psi_p \left( \sum_{l=0}^\infty \gamma_l (x^{\frac{a}{q-1}+l} + i) \right) = \sum_{l=0}^\infty \gamma_{pl+ub_i-j} x^{l+\frac{a_i}{q-1}}.$$ 

For $1 \leq k, i \leq b$ and $l, j \in \mathbb{N}$, define

$$\gamma_l(\frac{x^k}{q-1}+l, \frac{x_i}{q-1}+j) = \begin{cases} \gamma_{pl+ub_i-j}, & k = i - 1; \\ 0, & otherwise. \end{cases}$$

Let $\xi_1, \ldots, \xi_a$ be a normal basis of $\mathbb{Q}_q$ over $\mathbb{Q}_p$ and write

$$\xi^\sigma \gamma_{\frac{x^k}{q-1}+l, \frac{x_i}{q-1}+j} = \sum_{w=1}^a \gamma_{v(x^w-1)+(v,x_i-j)} \xi^w.$$ 

It is easy to see $\gamma_{(w, \frac{x^k}{q-1}+l), (v, \frac{x_i}{q-1}+j)} = 0$ for any $w$ and $v$ if $k \neq i - 1$. Define the $i$-th submatrix by

$$\Gamma^{(i)} = (\gamma_{(w, \frac{x^k-1}{q-1}+l), (v, \frac{x_i}{q-1}+j)})_{1 \leq w, v \leq a, i, j \in \mathbb{N}};$$

then the matrix of the operator $\Psi$ on $B$ over $\mathbb{Z}_p[[\pi^{1/(q-1)}]]$ with respect to the basis $\{\xi x^{\frac{a}{q-1}+j}\}_{1 \leq i \leq b, 1 \leq v \leq a, j \in \mathbb{N}}$ is

$$\Gamma = \begin{pmatrix} \Gamma^{(1)} & 0 & \ldots & 0 \\ 0 & \Gamma^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Gamma^{(b-1)} \\ \Gamma^{(b)} & 0 & \ldots & 0 \end{pmatrix}.$$

Hence by a result of Li-Zhu [8], we have

$$\det(1 - \Psi s|B / \mathbb{Z}_p[[\pi^{1/(q-1)}]]) = \det(1 - \Gamma s) = \sum_{n=0}^\infty (-1)^{bn} C_n s^{bn},$$

with $C_n = \sum \det(A)$ where $A$ runs over all principle $n \times n$ submatrix of $\Gamma$.

For every principle submatrix $A$ of $\Gamma$, write $A^{(i)} = A \cap \Gamma^{(i)}$ as the submatrix of $\Gamma^{(i)}$. For a principle $bn \times bn$ submatrix $A$ of $\Gamma$, by linear algebra, if one of $A^{(i)}$ is not $n \times n$ submatrix of $\Gamma^{(i)}$ then at least one row or column
of $A$ are 0 since $A$ is principle. Let $A_n$ be the set of all $bn \times bn$ principle submatrix $A$ of $\Gamma$ with $A^{(i)}$ all $n \times n$ submatrix of $\Gamma^{(i)}$ for each $1 \leq i \leq b$. Then we have

$$C_{bn} = \sum_{A \in A_n} \det(A) = \sum_{A \in A_n} (-1)^{n^2(b-1)} \prod_{i=1}^{b} \det(A^{(i)}).$$

Let $O(\pi^\alpha)$ denotes any element of $\pi$-adic order $\geq \alpha$.

**Lemma 4.1.** We have

$$\gamma_n = \pi\left[\frac{n}{d}\right] \sum_{\sum_{i=1}^{d} in_i = n} \prod_{i=1}^{d} \lambda_{n_i} a_i^{n_i} + O(\pi\left[\frac{n}{d}\right]+1).$$

**Proof.** This follows from the fact that $\sum_{i=0}^{d} n_i \geq \left[\frac{n}{d}\right]$ if $\sum_{i=0}^{d} in_i = n$. \qed

**Corollary 4.2.** For any $1 \leq i \leq b$ and $1 \leq w, v \leq a$, we have

$$\gamma(w, \frac{1}{q^i} + t)(v, \frac{1}{q^j} + j) = O(\pi\left[\frac{pl + ub - i - j}{a}\right]).$$

**Proof.** By $\xi^\sigma \gamma\left(\frac{1}{q^i} + t, \frac{1}{q^j} + j\right) = \xi^\sigma \gamma\left(\frac{1}{q^i} + t + u, \frac{1}{q^j} + j\right) = \sum_{w=1}^{a} \gamma(w, \frac{1}{q^j} + t)(v, \frac{1}{q^j} + j)\xi_w,$

this follows from Lemma 4.1. \qed

**Theorem 4.3.** If $p > 4d - \varepsilon(u)$, then we have

$$\text{ord}_\pi(C_{abn}) \geq abP_{[0,d],u}(n).$$

In particular,

$$C_{abn} = \pm \text{Norm}\left(\prod_{i=1}^{b} \det((\gamma_{pl - j + ub - i})_{l,j \in A_n})\right) + O(\pi^{abP_{[0,d],u}(n) + \frac{1}{\pi^{d(q-1)}}}),$$

where $\text{Norm}$ is the norm map from $\mathbb{Q}_q(\pi^{\frac{1}{d(q-1)}})$ to $\mathbb{Q}_p(\pi^{\frac{1}{d(q-1)}})$.

**Proof.** Let $A \in A_n$, $R_i$ the set of rows of $A^{(i)}$ as the submatrix of $\Gamma^{(i)}$, $\tau$ a permutation of $R = \bigcup_i R_i$. By the above corollary and Theorem 2.4, we have

$$\text{ord}_\pi(C_{abn}) \geq \sum_{i=1}^{b} \sum_{l \in R_i} \left[\frac{p\phi_i(l) + ub - i - \phi_i(\tau(l))(\tau(l))}{d}\right] \geq abP_{[0,d],u}(n).$$
Moreover the strict inequality holds if there exist $1 \leq i \leq b$ such that $R_i \neq \phi_i^{-1}(A_n)$, hence
\[ C_{abn} = \prod_{i=1}^{b} \det(\gamma_{(u, v, s_i^q - 1 + l)}(v, s_i^q - 1 + j))_{1 \leq w, v \leq a, j \in A_n} + O(\pi^{abP[0, d]}u(n) + \frac{1}{n^{d(q-1)}}). \]

Therefore the theorem follows from
\[ \det((\gamma_{(u, v, s_i^q - 1 + l)}(v, s_i^q - 1 + j))_{1 \leq w, v \leq a, j \in A_n}) = \pm \text{Norm}(\det(\gamma_{(u, v, s_i^q - 1 + l)}(v, s_i^q - 1 + j))_{1 \leq w, v \leq a, j \in A_n}) \]

\[ \square \]

5. HASSE POLYNOMIAL

In this section, $1 \leq n \leq d - 1$. We shall study $\det((\gamma_{pl-j+u_b-i})_{0 \leq i, j \leq n-1})$.

**Definition 5.1.** Let $S_n$ be the set of permutations of $A_n = \{0, 1, \cdots , n-1\}$. For $1 \leq i \leq b$, we define
\[ S_{n,i} = \{ \tau \in S_n | \frac{\tau(l)}{d} \geq \frac{pl + u_{b-i} - \tau(l)}{d} - \lceil \frac{pl + u_{b-i} - (n-1)}{d} \rceil \} . \]

**Lemma 5.2.** Let $p > 4d - \varepsilon(u)$ and $\tau \in S_n$. Then we have
\[ \sum_{i=1}^{b} \sum_{l=0}^{n-1} \lceil \frac{pl + u_{b-i} - \tau(l)}{d} \rceil \geq bP[0, d, u(n)], \]
with equality holding if and only if $\tau \in S_{n,i}$ for each $1 \leq i \leq b$.

**Proof.** Because
\[ \lceil \frac{pl + u_{b-i} - \tau(l)}{d} \rceil \geq \lceil \frac{pl + u_{b-i} - (n-1)}{d} \rceil \]
with equality holds if and only if
\[ \frac{\tau(l)}{d} \geq \frac{pl + u_{b-i}}{d} - \lceil \frac{pl + u_{b-i} - (n-1)}{d} \rceil , \]
it suffices to show for each $1 \leq i \leq b$,
\[ \sum_{l=0}^{n-1} \lceil \frac{(p-l)l + u_{b-i}}{d} \rceil - \delta^{(i)}_\varepsilon(l) = \sum_{l=0}^{n-1} \lceil \frac{pl + u_{b-i} - (n-1)}{d} \rceil . \]

It is trivial for $n = 1$. For $n \geq 2$, it need to show
\[ \sum_{l=0}^{n-1} (\delta^{(i)}_{\varepsilon}(l) - \delta^{(i)}_{\varepsilon}(l)) = \sharp \{ 0 \leq l \leq n-1 | \{ \frac{n-1}{d} \}' < \{ \frac{pl + u_{b-i}}{d} \}' \} - 1. \]

By Lemma 2.1 in case of $d | u_{b-i}$, the lemma follows from
\[ \delta^{(i)}_{[1, n-1]} = \frac{1}{d} \frac{u_{b-i}}{d} \leq \frac{n-1}{d} = 0, \]
and
\[ \sharp \{ 0 \leq l \leq n-1 | \{ \frac{n-1}{d} \}' < \{ \frac{pl + u_{b-i}}{d} \}' \} \]
= 1 + \#\{1 \leq l \leq n - 1\{\frac{n-1}{d}\} < \{\frac{pl}{d} + u_{b_i}\}\}. \\
In case of \(d \nmid u_{b_i}\), we have \\
\#\{0 \leq l \leq n - 1\{\frac{n-1}{d}\}' < \{\frac{pl}{d} + u_{b_i}\}'\} \\
= \#\{1 \leq l \leq n - 1\{\frac{n-1}{d}\} < \{\frac{pl}{d} + u_{b_i}\}\} + \delta_{1,n-1}^{(i)} + 1 \frac{n-1}{d} < \{\frac{u_{b_i}}{d}\}. \\
Hence it follows from the equation \(1 \frac{1}{d} \leq \frac{u_{b_i}}{d} \leq \frac{n-1}{d} + 1 \frac{n-1}{d} < \{\frac{u_{b_i}}{d}\} = 1. \)

**Definition 5.3.** Define \(H_{n,u}(y) = \prod_{i=1}^{b} H_{n,u}^{(i)}(y)\), where \\
\[H_{n,u}^{(i)}(y) = \sum_{\tau \in S_{n,i}} \text{sgn}(\tau) \prod_{l \in A_n} \sum_{j=1}^{d} \lambda_{n_j} y_j^{n_{j}}.\]

**Theorem 5.4.** Let \(p > 4d - \varepsilon(u)\), then we have \\
\[\prod_{i=1}^{b} \det((\gamma_{pl+j+u_{b_i}})_{l,j \in A_n}) = H_{n,u}(\hat{\alpha}_1, \ldots, \hat{\alpha}_d) \pi^{bp_{[0,d],u}(n)} + O(\pi^{bp_{[0,d],u}(n)+\frac{1}{d(q-1)}}).\]

**Proof.** We have \\
\[\det(\gamma_{pl+j+u_{b_i}})_{0 \leq l \leq n-1} = \sum_{\tau \in S_n} \text{sgn}(\tau) \prod_{l=0}^{n-1} \gamma_{pl+\tau(l)+u_{b_i}}.\]
The theorem now follows from Lemma \[1.1\] and \[5.2\].

**Definition 5.5.** The reduction of \(H_{n,u}\) modulo \(p\) is denoted as \(\overline{H}_{n,u}\), and is called the \(u\)-twisted Hasse polynomial of \([0,d]\) at \(n\). The \(u\)-twisted Hasse polynomial \(H_u\) of \([0,d]\) is defined by \(H_u = \prod_{n=1}^{d-1} \overline{H}_{n,u}\).

**Theorem 5.6.** The \(u\)-twisted Hasse polynomial \(H_u\) of \([0,d]\) is nonzero.

**Proof.** As a polynomial of \(y_d\), the leading terms of \(H_{n,u}^{(i)}(y)\) appear in \\
\[\sum_{\tau \in S_{n,i}} \text{sgn}(\tau) \prod_{l=0}^{n-1} \lambda_{\frac{pl+u_{b_i}+\tau(l)}{d}} y_d^{\frac{pl+u_{b_i}+\tau(l)}{d}} \delta_{\tau(l)} y_{\frac{pl+u_{b_i}+\tau(l)}{d}},\]
where \(\delta_{\tau(l)}\) is 0 or 1 depending on whether \(d|(pl + u_{b_i} - \tau(l))\) or not.

We show among the leading terms of \(H_{n,u}^{(i)}(y)\), there is exactly one minimal monomial.

For \(0 \leq l \leq n - 1\), write \(pl + u_{b_i} = q_d + r_l\), \(0 \leq r_l \leq d - 1\). Then \(\tau \in S_{n,i}\) is equivalent to \\
\[\frac{\tau(l)}{d} \geq \frac{r_l}{d} - \left\lfloor \frac{r_l - (n-1)}{d} \right\rfloor.\]
Assume we have $r_0 < r_1 < \cdots < r_m \leq n - 1 < r_{m+1} < \cdots < r_{n-1}$. Hence $\tau \in S_{n,i}$ if and only if $\tau(l_j) \geq r_{l_j}$ for $0 \leq j \leq m$.

Since

$$\sum_{l=0}^{n-1} \frac{pl + u_{b-l} - \tau(l) \cdot d}{d} = \sum_{l=0}^{n-1} q_l + \sum_{j=0}^{m} \frac{r_{l_j} - \tau(l_j) \cdot d}{d} \leq \sum_{l=0}^{n-1} q_l,$$

with equality holding if and only if $r_{l_j} = \tau(l_j)$ for all $0 \leq j \leq m$. Therefore the leading terms appear for such $\tau$ that $r_{l_j} = \tau(l_j)$ for all $0 \leq j \leq m$.

For $m + 1 \leq j \leq n - 1$, we have $\delta_\tau(l_j) = 1$ and $\{\frac{pl_j + u_{b-l_j} - \tau(l_j) \cdot d}{d}\} = \frac{r_{l_j} - \tau(l_j) \cdot d}{d}$. Hence among the leading terms, the minimal monomial appears exactly when $\tau(l_j) = r_{l_j}$ for all $0 \leq j \leq m$ and

$$\tau(l_{m+1}) = \max\{A_n - \{r_0, \cdots, r_m\}\},$$

$$\tau(l_{m+2}) = \max\{A_n - \{r_0, \cdots, r_m, \tau(l_{m+1})\}\},$$

$$\cdots$$

$$\tau(l_{n-1}) = A_n - \{r_0, \cdots, r_m, \tau(l_{m+1}), \cdots, \tau(l_{n-2})\}.$$

Now the theorem follows from $\lambda_j = \frac{1}{j} \in \mathbb{Z}_p^{\times}$ for $0 \leq j \leq p - 1$. \hfill \square

6. Proof of the main theorems

In this section we prove the main theorems of this paper.

Firstly we prove Theorem 1.8, which says that

$$P_{[0,d,u]} \geq (p - 1)H_{[0,d],u}$$

with equality holding at the point $d$.

**Proof of Theorem 1.8.** It need only to show this for $n \leq d$. By the equation

$$\sum_{l=0}^{n-1} \left[ \left( \frac{(p - 1)l + u_i}{d} \right) - \delta(l) \right] = \sum_{l=0}^{n-1} \left[ \frac{pl + u_i - (n - 1)}{d} \right]$$

that used in the proof of Lemma 5.2 it suffices to show

$$\sum_{l=0}^{n-1} \left[ \frac{pl + u_i - (n - 1)}{d} \right] \geq \sum_{l=0}^{n-1} \left( \frac{(p - 1)l + u_i}{d} \right).$$

The case $n = 1$ is trivial. For $n > 1$, since

$$\sum_{l=0}^{n-1} \left[ \frac{pl + u_i - (n - 1)}{d} \right] = \sum_{l=0}^{n-1} \left( \frac{pl + u_i}{d} \right) - \left( \frac{pl + u_i}{d} \right)' + 1 \left( \frac{n - 1}{d} \right) \left( \frac{pl + u_i}{d} \right)' \right),$$

it suffices to show

$$\sum_{l=0}^{n-1} \left( \frac{pl + u_i}{d} \right)' - 1 \left( \frac{n - 1}{d} \right) \left( \frac{pl + u_i}{d} \right)' \leq \sum_{l=0}^{n-1} \frac{l}{d}.$$
Hence the T-adic Newton polygon of det(1 − Ψ) is the lower convex closure of the points $(n, \text{ord}_T(C_{an}))$, $n = 0, 1, \cdots$.

Proof. By Lemma 6.1, we see the T-adic Newton polygon of the power series det(1 − Ψ) is the lower convex closure of the points $(n, \text{ord}_T(C_{an}))$, $n = 0, 1, \cdots$. It is clear that $(n, \text{ord}_T(C_n))$ is not a vertex of that polygon if $a \nmid n$. So that Newton polygon is the lower convex closure of the points $(an, \text{ord}_T(C_{an}))$, $n = 0, 1, \cdots$.

Hence the T-adic Newton polygon of det(1 − Ψ) is the convex closure of the points $(n, \text{ord}_T(C_{an}))$, $n = 0, 1, \cdots$. 

Now the inequality follows from
\[ \sum_{l=0}^{n-1} \left( \frac{pl + ui}{d} \right)' - 1 \left( \frac{pl + ui}{d} \right) < \left( \frac{pl + ui}{d} \right)' \]
Moreover the equalities above hold when $n = d$, the theorem is proved. □

**Lemma 6.1.** The T-adic Newton polygon of $\det(1 - \Psi s^a | B/\mathbb{Z}_p[[\pi^{1/\alpha}]]])$ coincides with that of $\det(1 - \Psi s | B/\mathbb{Z}_q[[\pi^{1/\alpha}]]])$.

**Proof.** The lemma follows from the following:
\[ \prod_{\zeta^a = 1} \det(1 - \Psi s | B/\mathbb{Z}_p[[\pi^{1/\alpha}]]]) = \det(1 - \Psi s^a | B/\mathbb{Z}_p[[\pi^{1/\alpha}]]]) \]
\[ = \text{Norm}(\det(1 - \Psi s^a | B/\mathbb{Z}_q[[\pi^{1/\alpha}]]])) \]
where Norm is the norm map from $\mathbb{Q}_q[[\pi^{1/\alpha}]]$ to $\mathbb{Q}_p[[\pi^{1/\alpha}]]$. □

**Lemma 6.2.** The T-adic Newton polygon of $C_{f,u}(s, T)^b$ coincides with that of $\det(1 - \Psi s | B/\mathbb{Z}_q[[\pi^{1/\alpha}]]])$.

**Proof.** Let $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ act on $\mathbb{Z}_q[[T]][[s]]$ by fixing $s$ and $T$, then we have $C_{pu,f}(s, T) = C_{f,u}(s, T)^\sigma$. Therefore the lemma follows from the following
\[ \prod_{i=0}^{b-1} C_{f,u}(s, T)^{\sigma^i} = \prod_{i=0}^{b-1} C_{p^i u, f}(s, T) \]
\[ = \prod_{i=0}^{b-1} \det(1 - \Psi s | B_{p^i u} / \mathbb{Z}_q[[\pi^{1/\alpha}]]]) = \det(1 - \Psi s | B/\mathbb{Z}_q[[\pi^{1/\alpha}]]]) \]

□

**Theorem 6.3.** The T-adic Newton polygon of $C_{f,u}(s, T)$ is the lower convex closure of the points
\[ (n, \frac{1}{b} \text{ord}_T(C_{abn})), \ n = 0, 1, \cdots \]
By Lemma 6.2, the $T$-adic Newton polygon of $C_{f,u}(s,T)^b$ is the lower convex closure of the points

$$(n, \ord_T(C_{an})), \ n = 0, 1, \cdots ,$$

hence the closure of the points

$$(bn, \ord_T(C_{am})), \ n = 0, 1, \cdots .$$

It follows that the $T$-adic Newton polygon of $C_{f,u}(s,T)$ is the convex closure of the points

$$(n, \frac{1}{b}\ord_T(C_{am})), \ n = 0, 1, \cdots .$$

The theorem is proved.

We now prove Theorem 1.9 which says that if $p > 4d - \varepsilon(u)$,

$$T - \text{adic NP of } C_{f,u}(s,T) \geq \ord_p(q)P_{[0,d],u}.$$ 

Proof of Theorem 1.9 The theorem follows from Theorem 4.3 and the last theorem.

Theorem 6.4. Let $A(s,T)$ be a $T$-adic entrie series in $s$ with unitary constant term. If $0 \neq |t|_p < 1$, then

$$t - \text{adic NP of } A(s,t) \geq T - \text{adic NP of } A(s,T),$$

where NP is the short for Newton polygon. Moreover, the equality holds for one $t$ if and only if it holds for all $t$.

Proof. The reader may refer [13] and we omit the proof here.

Theorem 6.5. Let $f(x) = \sum_{i=0}^{d} (a_i x^i, 0, 0, \cdots)$, and $p > 4d - \varepsilon(u)$. If the equality

$$\pi_m - \text{adic NP of } C_{f,u}(s,\pi_m) = \ord_p(q)P_{[0,d],u}$$

holds for one $m \geq 1$, then it holds for all $m \geq 1$, and we have

$$T - \text{adic NP of } C_{f,u}(s,T) = \ord_p(q)P_{[0,d],u}.$$ 

Proof. This follows from Theorem 1.9 and the last theorem.

Theorem 6.6. Let $f(x) = \sum_{i=0}^{d} (a_i x^i, 0, 0, \cdots)$. Then

$$\pi_m - \text{adic NP of } C_{f,u}(s,\pi_m) = \ord_p(q)P_{[0,d],u}$$

if and only if

$$\pi_m - \text{adic NP of } L_{f,u}(s,\pi_m) = \ord_p(q)P_{[0,d],u} \text{ on } [0, p^{m-1}d].$$
Proof. Assume that \( L_{f,u}(s, \pi_m) = \prod_{i=1}^{p^m-1} (1 - \beta_is) \). Then
\[
C_{f,u}(s, \pi_m) = \prod_{j=0}^{\infty} L_{f,u}(q^js, \pi_m) = \prod_{j=0}^{\infty} \prod_{i=1}^{p^m-1} (1 - \beta_i q^js).
\]
Therefore the slopes of the \( q \)-adic Newton polygon of \( C_{f,u}(s, \pi_m) \) are the numbers
\[
j + \text{ord}_q(\beta_i), \ 1 \leq i \leq p^{m-1}\text{Vol}(\triangle), \ j = 0, 1, \cdots.
\]
Since
\[
w(i + m^{m-1}d) = p^m - p^{m-1} + w(i),
\]
then the slopes of \( P_{[0,d],u} \) are the numbers
\[
j(p^m - p^{m-1}) + w(i), \ 1 \leq i \leq p^{m-1}d, \ j = 0, 1, \cdots.
\]
It follows that
\[
\pi_m - \text{adic NP of} \ C_{f,u}(s, \pi_m) = \text{ord}_p(q)P_{[0,d],u}
\]
if and only if
\[
\pi_m - \text{adic NP of} \ L_{f,u}(s, \pi_m) = \text{ord}_p(q)P_{[0,d],u} \text{ on } [0,p^m-1d].
\]

We now prove Theorems 1.10, 1.11 and 1.12. By the above theorems, it suffices to prove the following.

**Theorem 6.7.** Let \( f(x) = \sum_{i=0}^d (a_ix^i, 0, 0, \cdots) \), and \( p > 4d - \varepsilon(u) \). Then
\[
\pi_1 - \text{adic NP of} \ L_{f,u}(s, \pi_1) = \text{ord}_p(q)P_{[0,d],u} \text{ on } [0,d]
\]
if and only if \( H((a_i)_{0 \leq i \leq d}) \neq 0 \).

**Proof.** By a result of Liu [10], the \( q \)-adic Newton polygon of \( L_{f,u}(s, \pi_1) \) coincides with \( H_{\infty,0,d,u}^\infty \) at the point \( d \). By Theorem 1.13, \( P_{[0,d],u} \) coincides with \( (p-1)H_{[0,d],u}^\infty \) at the point \( d \). It follows that the \( \pi_1 \)-adic Newton polygon of \( L_{f,u}(s, \pi_1) \) coincides with \( \text{ord}_p(q)P_{[0,d],u} \) at the point \( d \). Therefore it suffices to show that
\[
\pi_1 - \text{adic NP of} \ L_{f,u}(s, \pi_1) = \text{ord}_p(q)P_{[0,d],u} \text{ on } [0,d-1]
\]
if and only if \( H((a_i)_{0 \leq i \leq d}) \neq 0 \).

From the identity
\[
C_{f,u}(s, \pi_1) = \prod_{j=0}^{\infty} L_{f,u}(q^js, \pi_1),
\]
and the fact the \( q \)-adic orders of the reciprocal roots of \( L_{f,u}(s, \pi_1) \) are no greater than 1, we infer that
\[
\pi_1 - \text{adic NP of} \ L_{f,u}(s, \pi_1) = \pi_1 - \text{adic NP of} \ C_{f,u}(s, \pi_1) \text{ on } [0,d-1].
\]
Therefore it suffices to show that
\[
\pi_1 - \text{adic NP of } C_{f,u}(s, \pi_1) = \text{ord}_p(q)P_{[0,d],u} \text{ on } [0, d - 1]
\]
if and only if \( H((a_i)_{0 \leq i \leq d}) \neq 0 \). The theorem now follows from the \( T \)-adic Dwork trace formula and Theorems 4.3 and 5.4. \( \square \)

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