CATEGORIAL PROPERTIES OF COMPRESSED ZERO-DIVISOR GRAPHS OF FINITE COMMUTATIVE RINGS

ALEN ĐURIĆ, SARA JEVDENIĆ, AND NIK STOPAR

Abstract. We define a compressed zero-divisor graph $\Theta(K)$ of a finite commutative unital ring $K$, where the compression is performed by means of the associatedness relation. We prove that this is the best possible compression which induces a functor $\Theta$, and that this functor preserves categorial products (in both directions). We use the structure of $\Theta(K)$ to characterize important classes of finite commutative unital rings, such as local rings and principal ideal rings.

Key Words: compressed zero-divisor graph, categorial product, local ring, principal ideal ring

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1. Introduction

The aim of this paper is to study the zero-divisor graphs of finite commutative rings with special attention devoted to categorial properties. The zero-divisor graph of a commutative ring was first introduced by Beck [10], to investigate the structure of commutative rings. For a given commutative ring $K$, Beck’s zero-divisor graph $G(K)$ is a simple graph with vertex set $K$, such that two distinct vertices $a$ and $b$ are adjacent if and only if $ab = 0$. Beck was mainly interested in the chromatic number and the clique number of the graph. Later Anderson and Livingston [6] defined a simplified version $\Gamma(K)$ of Beck’s zero-divisor graph by including only nonzero zero-divisors of $K$ in the vertex set and leaving the definition of edges the same. In particular, this graph is still a simple graph, but may have far fewer vertices in general. Their motivation for this simplification was to better capture the essence of the zero-divisor structure of the ring. Several properties of $\Gamma(K)$ have been investigated, such as connectedness, diameter, girth, chromatic number, etc. [6, 2]. In addition, the isomorphism problem for such graphs has been solved for finite reduced rings [3]. Several authors have also investigated rings $K$ whose graph $\Gamma(K)$ belongs to a certain family of graphs, such as star graphs [6], complete graphs [2], complete $r$-partite graphs and planar graphs [1, 21]. Similar type of zero-divisor graphs have been considered in other algebraic structures as well, namely, semirings and semigroups [7, 8, 12, 13, 14, 15, 16].

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Although smaller, the graph $\Gamma(K)$ may still have very large set of vertices and edges. To further reduce the size of the graph, Mula [20] introduced the graph of equivalence classes of zero-divisors $\Gamma_E(K)$, which was later called compressed zero-divisor graph by Anderson and LaGrange [4]. Two elements $a$ and $b$ of a commutative unital ring $K$ are equivalent if $\text{ann}_K a = \text{ann}_K b$. The vertex set of $\Gamma_E(K)$ is the set of all equivalence classes of nonzero zero-divisors of $K$ and two distinct equivalence classes $[r]$ and $[s]$ are adjacent if and only if $rs = 0$. Compressed zero-divisor graphs were investigated in more details by Spiroff and Wickham [22], Coykendall, Sather-Wagstaff, Sheppardson and Spiroff [11] and Anderson and LaGrange [4, 5]. They considered similar graph properties that were previously considered for the zero-divisor graph. The main advantage of the compressed zero-divisor graph $\Gamma_E(K)$ over the noncompressed graph $\Gamma(K)$ is that it can be relatively small even if the ring itself is large. In particular, $\Gamma_E(K)$ can be a finite graph even if $K$ is an infinite ring and $\Gamma(K)$ an infinite graph. Nevertheless, graph $\Gamma_E(K)$ still captures the essence of the zero-divisor structure of the ring, since the elements that are identified by the above equivalence have the same neighbourhood in $\Gamma(K)$.

In this paper we will be dealing with a type of compressed zero-divisor graph of finite commutative unital rings. Our main focus will be to investigate the categorial properties of such graphs. Our compressed zero-divisor graph, denoted by $\Theta(K)$ (see Definition 3.2), is essentially a compression of Beck’s original zero-divisor graph $G(K)$, except that we allow loops in the graph. Unlike in the definition of $\Gamma_E(K)$, here, the compression is performed by means of the associatedness relation (recall that $a, b \in K$ are associated if $a = bu$ for some invertible element $u \in K$). We remark that the associatedness relation is a refinement of the relation used in the definition of $\Gamma_E(K)$, hence, the graph $\Gamma_E(K)$ can easily be obtained from $\Theta(K)$ by simply identifying the vertices of $\Theta(K)$ with the same neighbourhood, and eliminating those vertices that do not correspond to zero-divisors.

The advantage of $\Theta(K)$ over $\Gamma_E(K)$ is that it can be extended in a natural way to a functor from the category of commutative rings to the category of graphs. The main reason why $\Gamma_E(K)$ does not extend to a functor in a natural way is that the corresponding equivalence relation induced by annihilator ideals is too coarse, it compresses the zero-divisor graph too much. In fact, we show that in the class of finite unital rings the associatedness relation is the coarsest equivalence relation that still induces a functor $\Theta$ (see Propositions 3.1 and 3.3 for details). Graph $\Theta(K)$ is thus the best possible candidate for a categorial approach to compressed zero-divisor graphs of finite rings.

It turns out that functor $\Theta$ has several favourable properties that connect the ring structure of $K$ and the graph structure of $\Theta(K)$. In particular, it preserves categorial products, not only in the forward direction but, in some sense, also in the backward direction - a decomposition of graph $\Theta(K)$ induces a decomposition of ring $K$ (see Theorem 3.6). In the class of finite
commutative rings this reduces the problem to local rings. In addition, our main results, Theorems 5.3 and 5.8, show that the structure of $\Theta(K)$ can be used to characterize important families of rings within the category of finite commutative unital rings, namely, local rings and principal ideal rings.

2. Preliminaries

Throughout the paper $K$ will be a finite commutative unital ring with unity 1, unless specified otherwise. In particular, we consider the zero ring to be unital. We denote by $\sim$ the associatedness relation on the set of elements of $K$. By definition $a \sim b$ if and only if $a = bu$ for some invertible element $u \in K$. The associatedness class of an element $a \in K$, i.e. the equivalence class of $a$ with respect to $\sim$, will be denoted by $[a]$. The equivalence class with respect to any other equivalence relation $\approx$ will be denoted by $[\phantom{a}]_{\approx}$. Recall that in a finite commutative unital ring $K$ every element is either a unit or a zero-divisor. Indeed, if $a \in K$ is not a zero-divisor, then the map $x \mapsto ax$ is injective and hence surjective, which means that $a$ is invertible. The ring of integers modulo $m$ will be denoted by $\mathbb{Z}_m$.

Let $G$ be an arbitrary, possibly non-simple, graph and $v$ a vertex in $G$. The neighbourhood of $v$, i.e. the set of all vertices adjacent to $v$ (including possibly $v$), will be denoted by $N(v)$. The graphs we will be dealing with will have no multiple edges and no multiple loops. We will adopt the convention that a loop on vertex $v$ contributes 1 to the degree of $v$, denoted $\text{deg}(v)$. With this convention our graphs will satisfy $\text{deg}(v) = |N(v)|$.

3. Definition and categorial properties of $\Theta(K)$

It is easily verified that finite commutative unital rings form a category with arrows being ring homomorphisms that preserve the identity element. We will denote this category by $\text{FinCRing}$. The category of undirected graphs and graph morphisms will be denoted by $\text{Graph}$. Given a category $\mathcal{C}$, we will denote the class of objects of $\mathcal{C}$ by $\text{obj}\mathcal{C}$. For $X, Y \in \text{obj}\mathcal{C}$, the set of morphisms from $X$ to $Y$ will be denoted by $\mathcal{C}(X, Y)$.

In this paper we will take a categorial approach to zero-divisor graphs. We will focus on compressed zero-divisor graphs since these are usually much smaller then the standard zero-divisor graphs. As mentioned in the introduction, Mulay’s compressed zero-divisor graph $\Gamma_E(K)$ is not a good candidate for a categorial approach, because the formation of $\Gamma_E(K)$ does not extend to a functor $\text{FinCRing} \rightarrow \text{Graph}$ in a natural way. The problem is that graph $\Gamma_E(K)$ is compressed too much. Hence, our definition of zero-divisor graph will be different. We want to compress the zero-divisor graph as much as possible, in such a way, that it will still induce a functor. The following proposition (along with Proposition 3.3) essentially states that the associatedness relation is the best equivalence relation to do this.
Proposition 3.1. For each \( K \in \text{objFinCRing} \), let \( \approx_K \) be an equivalence relation on \( K \), such that the family \( \{\approx_K\}_{K \in \text{objFinCRing}} \) induces a well-defined functor \( F : \text{FinCRing} \to \text{Graph} \) in the following way.

(i) For \( K \in \text{objFinCRing} \), the vertices of \( F(K) \) are equivalence classes of \( \approx_K \), and there is an edge between vertices \( [a]_{\approx_K} \) and \( [b]_{\approx_K} \) if and only if \( ab = 0 \).

(ii) For \( f \in \text{FinCRing}(K, L) \), we have \( F(f) ([a]_{\approx_K}) = [f(a)]_{\approx_L} \).

Then, for every \( K \in \text{objFinCRing} \), \( a \approx_K b \) implies \( a \sim b \).

Proof. Suppose \( x \approx_K 0 \). By (i) there is an edge joining \( [1]_{\approx_K} \) and \( [0]_{\approx_K} = [x]_{\approx_K} \). Since edges have to be well-defined, we deduce \( 1 \cdot x = 0 \). This shows that \( [0]_{\approx_K} = \{0\} \) for any \( K \).

Now, suppose \( a \approx_K b \) and let \( q : K \to K/aK \) be the canonical projection. Then, by (ii)

\[
[q(b)]_{\approx_K/aK} = F(q) ([b]_{\approx_K}) = F(q) ([a]_{\approx_K}) = [q(a)]_{\approx_K/aK} = [0]_{\approx_K/aK}.
\]

Thus, the above implies \( q(b) = 0 \), hence \( b \in aK \). Similarly, \( a \in bK \), so \( aK = bK \). As remarked by Kaplansky in [19, §2], in any artinian ring and, in particular, in any finite ring, this implies \( a \sim b \). \( \square \)

The above result thus motivates us to define compressed zero-divisor graphs in the following way.

Definition 3.2. For a finite commutative unital ring \( K \), \( \Theta (K) \) is a graph whose vertices are associatedness classes (including \( [0] \) and \( [1] \)) of elements of \( K \) and vertices \([u]\) and \([v]\) (not necessarily distinct) are adjacent if and only if \( uv = 0 \).

Observe that the edges of graph \( \Theta (K) \) are well-defined. In addition, the associatedness classes form a monoid under the well-defined multiplication \( [x] \cdot [y] = [xy] \).

We remark that class \( [0] \) contains only 0 and class \( [1] \) consists of all the units of the ring. We need to keep these two classes in the graph and also allow loops because we need them in order to obtain a functor. Every other class is represented by a nonzero zero-divisor, because in a finite ring every element is either a zero-divisor or a unit.

Proposition 3.3. The mapping \( K \mapsto \Theta (K) \) extends to a functor \( \Theta : \text{FinCRing} \to \text{Graph} \).

Proof. Let \( f : K \to L \) be a unital ring homomorphism, where \( K \) and \( L \) are finite commutative unital rings. Define \( \Theta (f) : \Theta (K) \to \Theta (L) \) by \( \Theta (f) ([x]) = [f(x)] \). Observe that \( \Theta (f) \) is well-defined since \( f \) preserves units, and clearly, \( \Theta (f) \) is a graph homomorphism. In addition, \( \Theta (\text{id}_K) = \text{id}_{\Theta (K)} \) and \( \Theta (g \circ f) = \Theta (g) \circ \Theta (f) \) for all morphisms \( f : K \to L \) and \( g : L \to M \). So \( \Theta : \text{FinCRing} \to \text{Graph} \) is a functor. \( \square \)
Theorem 3.6. The reverse direction as well.

We see that

On the other hand, associatedness classes in $V(\Theta(K) \times \Theta(L))$ form the set

We see that $V(\Theta(K)) = \{\{0\}, \{1, 1 + x, 1 + x^2, 1 + x + x^2\}, \{x, x + x^2\}, \{x^2\}\}$. We see that $\Theta(f) = \Theta(id_K) = id_{\Theta(K)}$, so the equalizer of $\Theta(id_K)$ and $\Theta(f)$ is $\Theta(K) \xrightarrow{id_{\Theta(K)}} \Theta(K)$. Since $|V(\Theta(E))| \neq |V(\Theta(K))|$, we conclude that equalizer of $id_K$ and $f$ is not preserved by functor $\Theta : \text{FinCRing} \to \text{Graph}$.

The following example shows that, in a sense, the product is preserved in the reverse direction as well.

**Proposition 3.4.** The functor $\Theta : \text{FinCRing} \to \text{Graph}$ preserves finite products.

**Proof.** It is sufficient to show that functor $\Theta$ preserves binary products and final object. Let $K, L \in \text{objFinCRing}$. Since operations in $K \times L$ are defined coordinate-wise, we have that $[(x, y)] = [(x', y')]$ in $K \times L$ if and only if both $[x] = [x']$ and $[y] = [y']$. This shows that $V(\Theta(K \times L)) = V(\Theta(K)) \times V(\Theta(L))$. In addition, $[(x, y)]$ is adjacent to $[(z, w)]$ in $\Theta(K \times L)$ if and only if both $[x]$ is adjacent to $[z]$ in $\Theta(K)$ and $[y]$ is adjacent to $[w]$ in $\Theta(L)$.

Hence, $\Theta(K \times L)$ is isomorphic to the tensor product of graphs $\Theta(K)$ and $\Theta(L)$ by the map $[(x, y)] \mapsto ([x], [y])$. Clearly, $\Theta(0)$, the graph of the zero ring, is the graph with precisely one vertex and one loop.

As the following example shows, functor $\Theta : \text{FinCRing} \to \text{Graph}$ does not preserve finite limits in general, hence it has no left adjoint functor.

**Example 3.5.** Let $K = \mathbb{Z}_2[x] / (x^3)$ and let $f : K \to K$ be a unital ring homomorphism which maps $x$ to $x + x^2$. It is easily verified that the equalizer of $id_K$ and $f$ (i.e. the limit cone over the diagram $K \xrightarrow{f} K$), is $E \xrightarrow{i} K$, where $E = \{0, 1, x^2, 1 + x^2\}$ and $i$ is an inclusion. Observe that

$$V(\Theta(E)) = \{\{0\}, \{x^2\}, \{1, 1 + x^2\}\}.$$ 

On the other hand, associatedness classes in $K$ form the set

$$V(\Theta(K)) = \{\{0\}, \{1, 1 + x, 1 + x^2, 1 + x + x^2\}, \{x, x + x^2\}, \{x^2\}\}.$$ 

We see that $\Theta(f) = \Theta(id_K) = id_{\Theta(K)}$, so the equalizer of $\Theta(id_K)$ and $\Theta(f)$ is $\Theta(K) \xrightarrow{id_{\Theta(K)}} \Theta(K)$. Since $|V(\Theta(E))| \neq |V(\Theta(K))|$, we conclude that equalizer of $id_K$ and $f$ is not preserved by functor $\Theta : \text{FinCRing} \to \text{Graph}$.

The following theorem shows that, in a sense, the product is preserved in the reverse direction as well.

**Theorem 3.6.** Suppose $K, L_1, L_2 \in \text{objFinCRing}$ such that $\Theta(K) \cong \Theta(L_1) \times \Theta(L_2)$. Then $K = K_1 \times K_2$ for some subrings $K_1, K_2 \subseteq K$ with $\Theta(K_1) \cong \Theta(L_1)$ and $\Theta(K_2) \cong \Theta(L_2)$. 


Proof. If \( \Theta (L_1) \cong \Theta (0) \), then \( \Theta (L_1) \times \Theta (L_2) \cong \Theta (L_2) \) so we may take \( K_1 = 0 \) and \( K_2 = K \). We argue similarly if \( \Theta (L_2) \cong \Theta (0) \). So assume \( \Theta (L_1) \not\cong \Theta (0) \) and \( \Theta (L_2) \not\cong \Theta (0) \).

Let \( f : \Theta (L_1) \times \Theta (L_2) \rightarrow \Theta (K) \) be any isomorphism. Choose \( k_1, k_2 \in K \) such that \( f (([1], [0])) = [k_1] \) and \( f ([0], [1])) = [k_2] \), and define

\[
(1) \quad K_1 = \text{ann} (k_2) \quad \text{and} \quad K_2 = \text{ann} (k_1).
\]

Clearly, \( K_1 \) and \( K_2 \) are ideals of \( K \). If \( x \in K_1 \cap K_2 \), then

\[
[x] \in N ([k_1]) \cap N ([k_2]) = f (N ([1], [0])) \cap N ([0], [1])) = f ([0], [0])) = [0].
\]

Thus, \( K_1 \cap K_2 = 0 \).

Note that the subgraph of \( \Theta (L_1) \times \Theta (L_2) \), induced by \( N ([0], [1])) \), is isomorphic to \( \Theta (L_1) \). Hence, the subgraph \( G_1 \) of \( \Theta (K) \), induced by \( N ([k_1]) \), is also isomorphic to \( \Theta (L_1) \). Clearly, \( V (G_1) = \{ x \in V (\Theta (K)) : x \in K_1 \} \) and \( k_1 \in K_1 \). Since \( K_1 \) is an ideal, we thus have \( [k_1^2] \in V (G_1) \). Observe that \( [k_1^2] \neq [0] \), since \( [k_1] \) has no loop due to the fact that \( L_1 \neq 0 \). Suppose \( [k_1^2] \neq [k_1] \). Then \( [k_1^2] \in V (G_1) \setminus \{ [0], [k_1] \} \). Since \( G_1 \cong \Theta (L_1) \), there is only one vertex in \( G_1 \) that is adjacent to every vertex in \( G_1 \), i.e. \([0]\), and there is only one vertex in \( G_1 \) whose only neighbour in \( G_1 \) is \([0]\), i.e. \( f ([1], [0])) = [k_1] \). This implies that \( [k_1^2] \) has a neighbour in \( G_1 \) different from \([0]\), say \([a]\), where \( a \in K_1 \). Hence, \( k_1^2 a = 0 \) because \( G_1 \) is an induced subgraph of \( \Theta (K) \). This implies that \( [k_1 a] \) is adjacent to \([k_1]\) in \( \Theta (K) \), and since \( K_1 \) is an ideal, \( [k_1 a] \in V (G_1) \). Therefore, \( k_1 a = 0 \) because, by the above, \( [0]\) is the only neighbour of \([k_1]\) in \( G_1 \). Similarly, this implies that \([a]\) is adjacent to \([k_1]\), hence \( a = 0 \), a contradiction. We have thus shown that \( [k_1^2] = [k_1] \). In particular, \( k_1 = k_1^2 u_1 \) for some unit \( u_1 \in K \).

Observe that \( k_1 (1 - k_1 u_1) = 0 \), hence \( 1 - k_1 u_1 \in K_2 \) by (1). If \( 1 - k_1 u_1 = 0 \), then \( k_1 \) is a unit in \( K \), hence \( [k_1] = [1] \). But this would imply that \([0]\) is the only neighbour of \([k_1]\) in \( \Theta (K) \), which would further imply \( K_2 = 0 \). In this case, \( \Theta (L_2) \cong \Theta (0) \), a contradiction. So \( 1 - k_1 u_1 \neq 0 \).

Suppose \( [1 - k_1 u_1] \neq [k_2] \). Let \( G_2 \) be the subgraph of \( \Theta (K) \), induced by \( N ([k_1]) \). Then the same argument as above shows that \( [1 - k_1 u_1] \in V (G_2) \setminus \{ [0], [k_2] \} \) has a neighbour in \( G_2 \) different from \([0]\), say \([b]\), where \( 0 \neq b \in K_2 \). Hence,

\[
(2) \quad (1 - k_1 u_1) b = 0
\]

because \( G_2 \) is an induced subgraph of \( \Theta (K) \). Since \( k_2 k_1 = 0 \), we have \( k_2 = k_2 (1 - k_1 u_1) \). Hence, \( k_2 b = 0 \) by (2). This implies \( b \in K_1 \), so \( b \in K_1 \cap K_2 = 0 \), a contradiction. Thus, \( [1 - k_1 u_1] = [k_2] \), and consequently \( 1 = k_1 u_1 + k_2 u_2 \) for some unit \( u_2 \in K \). This shows that \( K = K_1 + K_2 \). Since we already know that \( K_1 \cap K_2 = 0 \), we conclude that \( K = K_1 \times K_2 \).

Observe that if \( x \in K_1 \) and \( x \sim y \) in \( K \), then \( y \in K_1 \) and \( x \sim y \) in \( K_1 \). Hence, \( \Theta (K_1) \cong G_1 \cong \Theta (L_1) \) and similarly \( \Theta (K_2) \cong \Theta (L_2) \). \( \square \)
4. The graph of the ring of integers modulo \( m \)

In this section we describe the graph \( \Theta(Z_m) \) since it will play an important role in the rest of the paper. We remark that graph \( \Gamma_E(Z_m) \) (see \( \S \) for definition) is obtained from \( \Theta(Z_m) \) by removing vertices \([0]\) and \([1]\) and all loops.

**Proposition 4.1.** Let \( k \) be a nonnegative integer. Up to graph isomorphism there exists a unique graph \( SG_k \) such that \( |V(SG_k)| = k + 1 \) and the degrees of vertices of \( SG_k \) are \( 1, 2, \ldots, k + 1 \). In addition, if we let \( v_i \in V(SG_k) \), \( 0 \leq i \leq k \), be the vertex with degree \( i + 1 \), then \( SG_k \) has the following properties:

(i) \( N(v_i) = \{v_{k-i}, v_{k-i+1}, \ldots, v_k\} \) for all \( 0 \leq i \leq k \),

(ii) \( N(v_0) \subset \ldots \subset N(v_{k-1}) \subset N(v_k) \).

We will call \( SG_k \) the staircase graph with index \( k \).

**Proof.** Let \( G \) be a graph with vertices \( \{v_0, v_1, v_2, \ldots, v_k\} \), where vertices \( v_i \) and \( v_j \) (not necessarily distinct) are adjacent if and only if \( i + j \geq n \). Then, clearly, \( \deg(v_i) = i + 1 \) for all \( 0 \leq i \leq k \), and graph \( G \) satisfies (i) and (ii). Thus, it remains to prove the uniqueness of \( SG_k \).

Let \( H \) be any graph with \( k + 1 \) vertices with degrees \( 1, 2, \ldots, k + 1 \). One of the vertices has to have degree \( k + 1 \), so it has to be adjacent to every vertex, including itself. We label that vertex by \( u_k \). One of the remaining vertices has to have degree 1, so it has no other neighbour besides \( u_k \). We label that vertex by \( u_0 \). One of the remaining, not yet labeled, vertices has to have degree \( k \), so it has to be adjacent to every vertex (including itself) except \( u_k \). We label it by \( u_k \). One of the remaining vertices has to have degree \( k - 1 \), so it has to be adjacent to every vertex (including itself) except \( u_0 \) and \( u_k \). We label that vertex by \( u_{k-2} \). Continuing this process, we eventually label all the vertices of \( H \), and since \( H \) has precisely \( k + 1 \) vertices, the labels we use are precisely \( u_0, u_1, \ldots, u_k \). It is clear from the labeling process that we have

\[ N(u_i) = \{u_{k-i}, u_{k-i+1}, \ldots, u_k\}, \]

hence the map \( H \to G \), defined by \( u_i \mapsto v_i \), is a graph isomorphism. This shows the uniqueness of \( SG_k \). \( \square \)

Observe that the adjacency matrix of a staircase graph, with vertices ordered by degree, resembles a staircase, hence the name.

We now describe the zero-divisor graphs of rings \( Z_m \). By slight abuse of notation we will denote the elements of \( Z_m \) simply by integers instead of cosets of integers.

**Proposition 4.2.** Let \( m = \prod_{i=1}^{n} p_i^{k_i} \) be a canonical representation of a positive integer \( m \). Then \( \Theta(Z_m) \cong \prod_{i=1}^{n} SG_{k_i} \).
Proof. Observe that \( \mathbb{Z}_m \cong \prod_{i=1}^{n} \mathbb{Z}_{p_i^{k_i}} \). It is easy to see that, for a prime \( p \) and a nonnegative integer \( k \), the graph \( \Theta(\mathbb{Z}_{p^k}) \) has vertices \( [p^0], [p^1], [p^2], \ldots, [p^k] \), and the degree of \( [p^j] \) is \( j + 1 \). Hence, \( \Theta(\mathbb{Z}_{p^k}) \cong SG_k \) by Proposition 4.1. The result now follows from Proposition 3.1. \( \square \)

Let \( G = \Theta(\mathbb{Z}_m) \) for some positive integer \( m \). By [17] Lemma 4.3, every vertex in \( \Theta(\mathbb{Z}_m) \) is represented by a uniquely determined positive divisor of \( m \). Let \( n \) denote the number of distinct prime divisors of \( m \). We will show that the structure of \( G \) determines uniquely the number \( n \) and the set of exponents in the canonical representation of \( m \). Starting from graph \( G \), with no labels on vertices, we describe how to reconstruct the labels of \( G \) (as associatedness classes) in terms of graph properties. Of course, by Proposition 4.2, the structure of \( G \) does not determine the prime factors of \( m \), so besides the graph itself, we will also need additional information on which primes are involved.

The only vertex in \( G \) adjacent to every vertex (including itself), is \([0]\). The only vertex of degree 1, is \([1]\) and it is adjacent only to \([0]\). So we can label these two vertices immediately.

Observe that a vertex \( v \in V(G) \) corresponds to some prime \( p \) if and only if \( \text{deg}(v) = 2 \), since its neighbours in this case are precisely \([0]\) and \([m/p]\). Since the structure of \( G \) does not determine the prime factors of \( m \), we have to assign the degree 2 vertices some specific distinct primes, say \( p_1, p_2, \ldots, p_n \), where \( n \) is just the number of degree 2 vertices in \( G \). So, now we have labels \([0], [1], [p_1], \ldots, [p_n]\) and we want to reconstruct the labels of all the other vertices.

For a divisor \( d \) of \( m \) we will call vertex \([m/d]\) the complement of vertex \([d]\). First we can identify the complements of \([p_1], \ldots, [p_n]\), since the complement of \([p_i]\), is the unique neighbour of \([p_i]\) different from \([0]\). We label the complement of \([p_i]\) by \([m/p_i]\), however this is not a true label yet, since we do not know yet what \( m \) is or rather what the exponent of each \( p_i \) in the factorization of \( m \) is. We determine these exponents now.

Fix some \( i \in \{1, 2, \ldots, n\} \). The classes of powers of \( p_i \) are those neighbours of \([m/p_i]\) that are not neighbours of any \([m/p_j]\), \( j \neq i \). The number of such neighbours gives us the highest power of \( p_i \) that divides \( m \), say \( p_i^{m_i} \). This, in particular, determines \( m \) so we can now truly label the complement of each \([p_i]\). We can also label the vertices that correspond to powers of \( p_i \). By the above let \( v \) be a neighbour of \([m/p_i]\) that is not a neighbour of any \([m/p_j]\), \( j \neq i \). Then the label for \( v \) is \([p_i^{\text{deg}(v) - 1}]\). This is because the neighbours of \([p_i^k]\) are precisely \([m], [m/p_i], \ldots, [m/p_i^k]\). Observe that we have not changed the label of \([p_i]\) in this step.

Next, we label the complements of powers of primes. Fix some \( i \in \{1, 2, \ldots, n\} \). The vertex \([m/p_i]\) is already labeled. For \( k \geq 2 \), the only neighbour of \([p_i^k]\) that is not a neighbour of \([p_i^{k-1}]\), must be labeled \([m/p_i^k]\).
Finally, we can label all the remaining vertices. Let \( v \) be a vertex and for each \( i \in \{1, 2, \ldots, n\} \) let \( k_i \leq m_i \) be the greatest nonnegative integer such that \( v \) is a neighbour of \( \left[ m/p_i^{k_i}\right] \). Then the label of \( v \) is \( \left[ p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}\right] \).

We remark that, although in this algorithm we label some vertices more than once, the labels are consistent.

Having a labeled zero-divisor graph of \( \mathbb{Z}_m \) it is now easy to reconstruct \( \Gamma(\mathbb{Z}_m) \), the standard non-compressed zero-divisor graph \( \Gamma(\mathbb{Z}_m) \) of \( \mathbb{Z}_m \), as defined in \([6]\). To do this we first exclude vertices \([0]\) and \([1]\) from each \( A_d \). Then we replace each remaining vertex \([Z] \). The union of all these sets is the set of vertices of \( \Gamma(d) \) in \( \Theta(\mathbb{Z}_m) \).

Having a labeled zero-divisor graph of \( \mathbb{Z}_m \) it is now easy to reconstruct \( \Gamma(\mathbb{Z}_m) \), the standard non-compressed zero-divisor graph \( \Gamma(\mathbb{Z}_m) \) of \( \mathbb{Z}_m \), as defined in \([6]\). To do this we first exclude vertices \([0]\) and \([1]\) from \( \Theta(\mathbb{Z}_m) \).

Then we replace each remaining vertex \([d], d|m, \) of \( \Theta(\mathbb{Z}_m) \) by the set

\[
A_d = \left\{ ds : s \in \left\{1, 2, \ldots, \frac{m}{d} - 1\right\}, \gcd (s, m) = 1 \right\}.
\]

The union of all these sets is the set of vertices of \( \Gamma(\mathbb{Z}_m) \). If \([d_1]\) was adjacent to \([d_2]\) in graph \( \Theta(\mathbb{Z}_m) \), then every \( x \in A_{d_1} \) is adjacent to every \( y \in A_{d_2} \) in \( \Gamma(\mathbb{Z}_m) \). In particular, if there was a loop on vertex \([d]\) in \( \Theta(\mathbb{Z}_m) \), then \( A_d \) is a clique (with no loops) in \( \Gamma(\mathbb{Z}_m) \), and if there was no loop on \([d]\) in \( \Theta(\mathbb{Z}_m) \), then \( A_d \) is an independant set in \( \Gamma(\mathbb{Z}_m) \). This “blow up” process has already been described by Spiroff and Wickham \([22] \) end of §1. However, in our situation, conveniently, the loops in \( \Theta(\mathbb{Z}_m) \) determine the edges between the vertices of \( A_d \) in \( \Gamma(\mathbb{Z}_m) \). So this blow up process could be done entirely graph-theoretically if one was to encode in \( \Theta(\mathbb{Z}_m) \) also the size of associatedness classes, that is the size of sets \( A_d \), say as weights of vertices.

5. Local rings and principal ideal rings

Recall that every finite commutative unital ring is isomorphic to a finite direct product of finite local rings (see for example \([9]\)). Hence, a finite commutative unital ring is local if and only if it is directly indecomposable.

**Corollary 5.1.** If \( \Theta (K) \cong \Theta (L) \) and \( K \) is local, then \( L \) is local as well.

**Proof.** Suppose \( L \) is not local. Then \( L = L_1 \times L_2 \) where \( L_1, L_2 \neq 0 \). Hence, \( \Theta (K) = \Theta (L) = \Theta (L_1) \times \Theta (L_2) \) by Proposition 3.4.

By Theorem 3.6 there exist subrings \( K_1, K_2 \subseteq K \) such that \( K = K_1 \times K_2 \) and \( \Theta (K_1) \cong \Theta (L_1) \) and \( \Theta (K_2) \cong \Theta (L_2) \). Since \( L_1 \neq 0 \), we have \( \Theta (K_1) \cong \Theta (L_1) \neq \Theta (0) \), hence also \( K_1 \neq 0 \). Similarly, \( K_2 \neq 0 \). This is a contradiction because \( K \) is local.

From Corollary 5.1 and the fact that \( \Theta (\mathbb{Z}_{p^n}) \cong SG_n \), we immediately obtain the following result.

**Corollary 5.2.** If \( \Theta (K) \) is isomorphic to the staircase graph \( SG_n \), then \( K \) is a local ring.

It turns out that locality of a finite commutative unital ring is a property that can be characterized by the structure of its zero-divisor graph as is shown by the next theorem. For \( a \in K \), we will adopt the convention that \( a^0 = 1 \) even when \( a = 0 \). If \( K \) is a finite local unital ring with maximal ideal
Then either the second equality also implies \( N \) for all \( k, l \in \mathbb{N} \) \( (K(N(k, a) \cup N(l, b)) = 0 \) or \( a^{k-1}b^{l-1} = 0 \). This shows that \( N(a) \cup N(b) \subseteq N(ab) \).

Next, we show by induction on \( k+l \) that \( a^k b^l \neq 0 \) for all \( k, l \geq 0 \). This holds by definition of \( m \) and \( n \). So, suppose \( k, l \geq 1 \) and assume, on the contrary, that \( a^k b^l = 0 \). Hence, either \( a^k b^l = 1 \) or \( a^k b^l = 0 \). But this is impossible by induction.

Let \( \Theta \) be a direct product of local rings, hence \( K = K_1 \times K_2 \) for some nonzero rings \( K_1 \) and \( K_2 \). If we take \( a = (1, 0) \in K \), then clearly \( [a] \notin \{[0], [1]\} \) and \( N([aa]) = N([a]) = N([a]) \cup N([a]) \).

**Corollary 5.4.** Let \( K \) be a finite commutative unital local ring which is not a field. If \( [a] \neq [1] \) has the least degree in \( \Theta(K) \), apart from [1], then \( a \in K \) is an irreducible element.

**Proof.** Suppose \( a = bc \), where \( b \) and \( c \) are not units. Observe that \( a \neq 0 \) since \( K \) is not a field. Then by Theorem 5.3, \( N([b]) \cup N([c]) \subseteq N([bc]) = N([a]) \) which implies \( |N([b])|, |N([c])| < |N([a])| \), a contradiction. □

Suppose \( \Theta(K) \cong SG_n \). Each vertex in this graph corresponds to the associatedness class of some element in \( K \). We want to find a nice set of elements that represent the vertices of \( \Theta(K) \). To this end we need the following two lemmas.
therefore a
Hence, by Lemma 5.5, $N([a_i]) \not\subseteq N([a_j])$, we have $\text{ann}(a_i) \not\subseteq \text{ann}(a_j)$, hence $\text{ann}(ya_i) \subseteq \text{ann}(ya_j)$, so $N([ya_i]) \subseteq N([ya_j])$.

(ii) Since neighbourhoods of distinct vertices of $\Theta(K) \cong SG_n$ are distinct, we must have $[ya_i] = [ya_j]$, hence $ya_i = ya_ju$ for some unit $u$. So,

$$y(a_i - a_ju) = 0.$$  
Since $N([a_i]) \not\subseteq N([a_j])$, we can choose the greatest $k$ such that $[a_k] \in N([a_j]) \setminus N([a_i])$. Then

$$a_k(a_i - a_ju) = a_k a_i \neq 0.$$  
Since $y \in m$, there exists $l > 0$ such that $[y] = [a_i]$. From (3) and (4) we conclude $N([a_i]) \not\subseteq N([a_k])$, hence $l > k$. Proposition 4.1 tells us that $N([a_j]) = \{[a_{n-1}], [a_{n-2}], \ldots, [a_n]\}$. This implies $[a_i] \in N([a_j])$, hence $[a_i] \in N([a_j])$ by the choice of $k$. Therefore, $a_k a_j = a_k a_i = 0$ and consequently $ya_j = ya_i = 0$.

Lemma 5.6. Let $K$ be a local ring with maximal ideal $m$, such that $\Theta(K) \cong SG_n$. If $x \in m$, with $x^m = 0$ and $x^{m-1} \neq 0$, then $[x^k] \neq [x^l]$ for all $0 \leq k < l \leq m$.

Proof. Suppose otherwise, that $[x^k] = [x^l]$. Then $x^k = x^l u$, where $u$ is a unit, so that $x^k (1 - x^{l-k} u) = 0$. Since $l - k \geq 1$ and $x$ is nilpotent, $1 - x^{l-k} u$ is a unit. But then $x^k = 0$, a contradiction.

We can now shows that the vertices of a zero-divisor graph which is isomorphic to a staircase graph can be labeled by powers of a single element of the ring.

Proposition 5.7. Let $K$ be a local ring with maximal ideal $m$, such that $\Theta(K) \cong SG_n$. Denote representatives of associatedness classes in such a way that $N([a_0]) \not\subseteq \ldots \not\subseteq N([a_{n-1}]) \not\subseteq N([a_n])$. Then $[a_i] = [a_i]$ for all $i \in \{0, \ldots, n\}$.

Proof. We prove the claim by induction on $i$. Clearly, $[a_0] = [1]$, so the claim is true for $i = 0$ and also for $i = 1$.

Let $i \in \{2, \ldots, n\}$. We examine the products $a_{i-1} a_1, a_{i-1} a_2, a_{i-1} a_3, \ldots, a_{i-1} a_{n-i}, a_{i-1} a_{n-i+1}$. By Proposition 4.1 we have

$$N([a_{i-1}]) = \{[a_{n-i+1}], [a_{n-i+2}], \ldots, [a_n]\},$$  
therefore $a_{i-1} a_1 \neq 0$, $a_{i-1} a_2 \neq 0$, $\ldots$, $a_{i-1} a_{n-i} \neq 0$ and $a_{i-1} a_{n-i+1} = 0$. Hence, by Lemma 5.5

$$N([a_{i-1} a_1]) \not\subseteq N([a_{i-1} a_2]) \not\subseteq \ldots \not\subseteq N([a_{i-1} a_{n-i}]) \not\subseteq N([a_{i-1} a_{n-i+1}]).$$
Since this is a subchain of the chain \( N ([a_0]) \varsubsetneq \ldots \varsubsetneq N ([a_{n-1}]) \subseteq N ([a_n]) \), we conclude that \( N ([a_{i-1}a_1]) \subseteq N ([a_i]) \). By induction, \([a_{i-1}] = [a_{i-1}^{-1}]\), which implies \([a_{i-1}a_1] = [a_{i-1}^{-1}a_1] = [a_1]\). Thus, \( N ([a_1]) \subseteq N ([a_i]) \), so there exists \( j \leq i \) such that \([a_1] = [a_j]\). If \( j < i \), then by induction \([a_1] = [a_i]\), which contradicts Lemma 5.6 unless \( a_{i-1}^{-1} = 0 \). But the latter would imply \([a_{i-1}] = [a_1^{-1}] = [0] = [a_n]\) and consequently \( i = n + 1 \) which is not the case. \( \square \)

Recall that a principal ideal ring, abbreviated PIR, is a commutative unital ring in which every ideal is principal. Being a PIR is another property that can be characterized by the structure of \( \Theta(K) \).

**Theorem 5.8.** A finite commutative unital ring \( K \) is a PIR if and only if \( \Theta(K) \) is isomorphic to a finite tensor product of staircase graphs.

**Proof.** Suppose \( \Theta(K) \cong \prod_{i=1}^{n} SG_{k_i} \cong SG_{k_1} \times \prod_{i=2}^{n} SG_{k_i} \). By Proposition 4.2, we have \( \Theta(K) \cong \Theta \left( \mathbb{Z}_{p_1} \right) \times \Theta \left( \mathbb{Z}_{p_2 \ldots p_n} \right) \) for some distinct primes \( p_1, p_2, \ldots, p_n \). Theorem 3.3 implies \( K \cong K_1 \times K'_1 \), where \( \Theta(K_1) \cong SG_{k_1} \) and \( \Theta(K'_1) \cong \prod_{i=2}^{n} SG_{k_i} \). By induction, \( K \cong \prod_{i=1}^{n} K_i \), where \( \Theta(K_i) \cong SG_{k_i} \). Since the direct product of PIR's is a PIR, it suffices to prove that each \( K_i \) is a PIR. By Corollary 5.2, \( K_i \) is a local ring. Denote its maximal ideal by \( \mathfrak{m}_i \). Then by Proposition 5.7, there exists \( x \in K_i \) such that \([x^0], [x^1], [x^2], \ldots, [x^{k_i}]\) are all of the vertices of \( \Theta(K_i) \). This clearly implies that every ideal of \( K_i \) is principal, generated by the least power of \( x \) it contains.

Conversely, suppose \( K \) is a PIR. Then by a result of Hungerford [18, Theorem 1], \( K \) is a finite direct product of homomorphic images of PID's, say \( K \cong \prod_{i=1}^{n} K_i/I_i \), where \( K_i \) is a PID (not necessarily finite) and \( I_i \triangleleft K_i \) for all \( i \in \{1, 2, \ldots, n\} \). By Proposition 5.4, it suffices to prove that each \( \Theta(K_i/I_i) \) is a tensor product of staircase graphs. If \( I_i = 0 \), then \( K_i \) is finite and every finite PID is a field. In this case, \( \Theta(K_i/I_i) \) is isomorphic to either \( SG_1 \) or \( SG_0 \). If \( I_i = K_i \), then \( \Theta(K_i/I_i) \cong SG_0 \). Now, assume \( 0 \neq I_i \neq K_i \). Then, \( I_i \) is generated by some \( a_i = u \cdot p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m} \), where \( m \geq 1 \), \( \alpha_1 \geq 1 \), \( p_j \) are prime elements and \( u \) is a unit in \( K_i \). By the Chinese Remainder Theorem, \( K_i/I_i \cong \prod_{j=1}^{m} K_i/(p_j^{\alpha_j}) \), hence it suffices to prove that \( \Theta \left( K_i/(p_j^{\alpha_j}) \right) \cong SG_{\alpha_j} \). This is easily shown upon observing that Every element in \( K_i/(p_j^{\alpha_j}) \) is a product of some power of \( p_j \) and some unit. \( \square \)

The following corollary easily follows from the proof of Theorem 5.8.

**Corollary 5.9.** Let \( K \) be a finite commutative unital ring. Then \( K \) is a local PIR if and only if \( \Theta(K) \cong SG_n \) for some nonnegative integer \( n \). In fact, \( n \) is the index of nilpotency of the maximal ideal of \( K \).

We remark that for a fixed positive integer \( n \) there exist many non-isomorphic local PIR's with \( \Theta(K) \cong SG_n \). For example, the rings \( \mathbb{Z}_{16} \),...
\[ \mathbb{Z}_2[x]/(x^4), \mathbb{Z}_4[x]/(x^2 - 2) \text{ and } \mathbb{Z}_4[x]/(x^2 - 2x - 2) \] all have the compressed zero-divisor graph \( \Theta(K) \) isomorphic to \( SG_3 \) and, in addition, they all have the residue field isomorphic to \( \mathbb{Z}_2 \). Moreover, they have the corresponding associatedness classes of the same sizes, which means that they also have the same non-compressed zero-divisor graphs \( \Gamma(K) \). All the above can be verified by hand and we leave the verification to the reader.

Corollary 5.9 shows that for a finite local PIR \( K \) the index of nilpotency of its maximal ideal can be extracted from the structure of \( \Theta(K) \). We were not able to establish whether the same holds for any finite local ring so we leave it as an open question.

**Question 5.10.** Let \( K \) be a finite local unital ring with maximal ideal \( \mathfrak{m} \).

(a) Does the graph structure of \( \Theta(K) \) determine the index of nilpotency of \( \mathfrak{m} \)?

(b) Does the graph structure of \( \Theta(K) \) determine the minimal number of generators of \( \mathfrak{m} \)?

### 6. Infinite rings

Finally, we remark that the definition of graph \( \Theta(K) \) can be extended to infinite commutative unital rings, however a verbatim extension is not the best way to do so. In view of the proof of Proposition 3.1, we believe that the right way to extend the definition is to compress the zero-divisor graph by the relation \( \approx \), defined by \( a \approx b \) if and only if \( aK = bK \), and define edges in a similar way as in the finite case. By this definition, the equivalence classes are in a bijective correspondence \( [a] \approx \leftrightarrow aK \) with the principal ideals of \( K \), and two classes are connected by an edge if and only if the product of the corresponding principal ideals is 0. Hence, we propose the following extension of Definition 3.2.

**Definition 6.1.** For an arbitrary commutative unital ring \( K \), \( \Theta(K) \) is a graph whose vertices are principal ideals of \( K \) (including 0 and \( K \)) and vertices \( I \) and \( J \) (not necessarily distinct) are adjacent if and only if \( IJ = 0 \).

As remarked by Kaplansky in [19, §2], for any artinian commutative unital ring \( K \), the equality \( aK = bK \) holds if and only if \( a \sim b \). Hence, for any artinian commutative unital ring, and in particular for any finite commutative unital ring, Definition 6.1 is equivalent to Definition 3.2.

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(A. Đurić) Faculty of Natural Sciences and Mathematics, University of Banja Luka, Mladen Stojanovića 2, 78000 Banja Luka, Bosnia and Herzegovina

E-mail address: alen.djuric@protonmail.com

(S. Jevđenić) Faculty of Natural Sciences and Mathematics, University of Banja Luka, Mladen Stojanovića 2, 78000 Banja Luka, Bosnia and Herzegovina

E-mail address: sarajevdjenic9@gmail.com

(N. Stopar) Faculty of Electrical Engineering, University of Ljubljana, Tržaška cesta 25, 1000 Ljubljana, Slovenia

E-mail address: nik.stopar@fe.uni-lj.si