Quaternionic Construction of the $W(F_4)$ Polytopes with Their Dual Polytopes and Branching under the Subgroups $W(B_4)$ and $W(B_3) \times W(A_1)$

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Abstract

4-dimensional $F_4$ polytopes and their dual polytopes have been constructed as the orbits of the Coxeter-Weyl group $W(F_4)$ where the group elements and the vertices of the polytopes are represented by quaternions. Branchings of an arbitrary $W(F_4)$ orbit under the Coxeter groups $W(B_4)$ and $W(B_3) \times W(A_1)$ have been presented. The role of group theoretical technique and the use of quaternions have been emphasized.

1 Introduction

Exceptional Lie groups $G_2[1],F_4[2],E_6[3],E_7[4]$ and $E_8[5]$ have been proposed as models in high energy physics. In particular, the largest exceptional group $E_8$ turned out to be the unique gauge symmetry $E_8 \times E_8$ of the heterotic superstring theory [6]. It is not yet clear as to how any one of these groups will describe the symmetry of any natural phenomenon. A recent experiment by Coldea et.al [7] on neutron scattering over $CoNb_2O_6$ (cobalt niobate) forming a one-dimensional quantum chain reveals evidence of the scalar particles describable by the $E_8$ [8] symmetry. The Coxeter-Weyl groups associated with exceptional Lie groups describe the symmetries of certain polytopes. For example, the Coxeter-Weyl groups $W(E_6), W(E_7)$ and $W(E_8)$ describe the symmetries of the Gosset’s polytopes [9] in six, seven and eight dimensions respectively. The Coxeter-Weyl group $W(F_4)$ is a unique group in the sense that it describes the symmetry of a uniform polytope 24-cell in 4-dimensions.
which has no correspondence in any other dimensions. It is also interesting from the point of view that the automorphism group \( \text{Aut}(F_4) \approx W(F_4) : C_2 \) can be described by the use of quaternionic representation of the binary octahedral group \([10]\) that will be described in the next section.

In this paper we study regular and semi-regular 4D polytopes whose symmetries and vertices are described by the quaternionic representations of the \( W(F_4) \) symmetry and the discrete quaternions respectively. We also give the projections of the polytopes under the symmetries \( W(B_4) \) and \( W(B_3) \times W(A_1) \). The latter decomposition helps how to view 4D polytopes in three dimensions. The dual polytopes of the group \( W(F_4) \) have not been constructed in any mathematical literature. By following the method which we have developed for the Catalan solids \([11]\), duals of the Archimedean solids, we construct the duals of the regular and semi-regular \( F_4 \)-polytopes.

The paper is organized as follows. In Section 2 we construct the group \( \text{Aut}(F_4) \approx W(F_4) : C_2 \) in terms of the quaternionic elements of the binary octahedral group \([12]\). A review of construction of the maximal subgroups \( W(B_4) \) and \( W(B_3) \times W(A_1) \) in terms of quaternions is presented \([13]\). Section 3 deals with the regular and semi-regular \( F_4 \)-polytopes and their branchings under the maximal subgroups \( W(B_4) \) and \( W(B_3) \times W(A_1) \). In Section 4 we construct the dual polytopes of the \( F_4 \)-polytopes. Section 5 is devoted to discussion and results.

## 2 Construction of \( \text{Aut}(F_4) \approx W(F_4) : C_2 \) with quaternions

In this section we introduce the quaternionic root system of the Coxeter diagram of \( F_4 \) and give a proof that the \( \text{Aut}(F_4) \approx W(F_4) : C_2 \) can be represented as the left-right action of the quaternionic representation of the binary octahedral group. The \( F_4 \) diagram with the quaternionic simple roots is shown in Figure 1. Note that all simple roots of the Coxeter diagrams have equal norms.

Let \( q = q_0 e_0 + q_i e_i, \ (e_0 = 1; i = 1, 2, 3) \) be a real unit quaternion with its conjugate defined by \( \bar{q} = q_0 e_0 - q_i e_i \) and the norm \( qq = \bar{q}q = 1 \) where the quaternionic imaginary units satisfy the relations

\[
\epsilon_i \epsilon_i = -\delta_{ij} + \epsilon_{ijk} \epsilon_k, \ (i, j, k = 1, 2, 3). \tag{1}
\]

Here \( \delta_{ij} \) and \( \epsilon_{ijk} \) are the Kronecker and Levi-Civita symbols and summation over the repeated indices is implied. We define the scalar product by the
\[
\sqrt{2}(1-e_1-e_2-e_3) \quad \sqrt{2}e_3 \\
 r_1 \quad r_2 \quad r_3 \quad r_4
\]

Figure 1: The \( F_4 \) diagram with quaternionic simple roots

relation

\[(p, q) = \frac{1}{2}(\bar{p}q + \bar{q}p). \quad (2)\]

In general, a reflection generator \( r \) of an arbitrary Coxeter group with respect to a hyperplane represented by the vector \( \alpha \) is given by the action

\[r : \Lambda \rightarrow \Lambda - \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \alpha. \quad (3)\]

When \( \Lambda \) and \( \alpha \) are represented by quaternions the equation (3) reads

\[r : \Lambda \rightarrow -\frac{\alpha\bar{\alpha}}{(\alpha, \alpha)}, \quad (4)\]

where we define the product in (3) symbolically as \( r : \Lambda \rightarrow \left[-\frac{\alpha}{\sqrt{(\alpha, \alpha)}}, \frac{\alpha}{\sqrt{(\alpha, \alpha)}}\right]^*\Lambda \) and drop the factor \( \Lambda \). Then the quaternionic generators of the Coxeter group \( W(F_4) \) are given by

\[
\begin{align*}
r_1 &= \left[-\frac{1}{2}(1-e_1-e_2-e_3), \frac{1}{2}(1-e_1-e_2-e_3)\right]^*; \\
r_2 &= \left[-e_3, e_3\right]^*; \\
r_3 &= \left[-\frac{(e_2-e_3)}{\sqrt{2}}, \frac{(e_2-e_3)}{\sqrt{2}}\right]^*; \\
r_4 &= \left[-\frac{(e_1-e_2)}{\sqrt{2}}, \frac{(e_1-e_2)}{\sqrt{2}}\right]^*.
\end{align*}
\quad (5)
\]

The Coxeter group \( W(F_4) \) generated by these generators can be compactly written as follows [14]

\[W(F_4) = \{[p, q] \oplus [r, s] \oplus [p, q]^* \oplus [r, s]^*]\}; \quad p, q \in T, \quad r, s \in T'. \quad (6)\]
Table 1: Multiplication table of the binary octahedral group.

|    | $V_0$ | $V_+ | V_- | V_1 | V_2 | V_3 |
|----|-------|-------|-------|-------|-------|-------|
| $V_0$ | $V_0$ | $V_2$ | $V_- | V_1 | V_2 | V_3 |
| $V_+$ | $V_+ | V_- | V_0 | V_3 | V_1 | V_2 |
| $V_- | V_- | V_0 | V_+ | V_2 | V_3 | V_1 |
| $V_1$ | $V_1 | V_2 | V_3 | V_0 | V_+ | V_- |
| $V_2$ | $V_2 | V_3 | V_1 | V_- | V_0 | V_+ |
| $V_3$ | $V_3 | V_1 | V_2 | V_+ | V_- | V_0 |

Here the notation stands for $[p, q]Λ = pΛq$ where $T$ and $T'$ are the subsets of the binary octahedral group $O = T \oplus T'$ of order 48. They can further be written as the union of subsets

$$T = V_0 \oplus V_+ \oplus V_-; \quad T' = V_1 \oplus V_2 \oplus V_3$$

where the subsets are given by the quaternions

$$V_0 = \{ \pm 1, \pm e_1, \pm e_2, \pm e_3 \},$$

$$V_+ = \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \ (\text{even number of} \ (+) \text{ sign}),$$

$$V_- = V_+,$n

$$V_1 = \left\{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3) \right\},$$

$$V_2 = \left\{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1) \right\},$$

$$V_3 = \left\{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2) \right\}. \quad (8)$$

The set $V_0$ is the quaternion group and the set $T$ is the binary tetrahedral group.

The set $V_0$ is an invariant subgroup of the binary octahedral group $O = T \oplus T'$. The left or right coset decomposition of the binary octahedral group under the quaternion group can be written as $O = \sum_{i=1}^{6} g_i V_0$ with $g_i$ taking the values

$$1, \frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3), \frac{1}{\sqrt{2}}(e_3 - e_1), \frac{1}{2}(1 + e_1 + e_2 + e_3), \frac{1}{2}(1 - e_1 - e_2 - e_3). \quad (9)$$

The coset representatives in (9) form a group isomorphic to the permutation group $S_3$. This is reflected in the multiplication Table 1.
Of course, the binary octahedral group has a larger invariant subgroup $T$ which follows from the multiplication $TT \subset T$, $TT' \subset T'$, $TT \subset T'$, $TT' \subset T$. Now we can write the equation (6) in a symbolic form as

$$W(F_4) = \{ [T, T] \oplus [T', T'] \oplus [T, T']^* \oplus [T', T]^* \}. \quad (10)$$

It is a group of order $288 \times 4 = 1152$ from which one can identify many maximal subgroups of order 576 [14]. The diagram symmetry of $F_4, \alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3$ can be generated by $D = (-\frac{1}{\sqrt{2}}(e_2 + e_3), e_2)$. The extension of the Coxeter group by the generator $D$ leads to the automorphism group $\text{Aut}(F_4) \approx W(F_4) : C_2$ of order 2304 and can be compactly written as [14]

$$\text{Aut}(F_4) \approx W(F_4) : C_2 = \{ [O, O] \oplus [O, O]^* \}. \quad (11)$$

The Cartan matrix of the Coxeter group $F_4$ and its inverse are given respectively by the matrices

$$C_{F_4} = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -\sqrt{2} & 0 \\
0 & -\sqrt{2} & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix}, \quad (12)$$

$$(C_{F_4})^{-1} = \begin{bmatrix}
2 & 3 & 2\sqrt{2} & \sqrt{2} \\
3 & 6 & 4\sqrt{2} & 2\sqrt{2} \\
2\sqrt{2} & 4\sqrt{2} & 6 & 3 \\
\sqrt{2} & 2\sqrt{2} & 3 & 2
\end{bmatrix}$$

where the simple roots and the basis vectors in the dual space satisfy respectively the relations $(\alpha_i, \alpha_j) = C_{ij}$, $(\omega_i, \omega_j) = (C^{-1})_{ij}$. Here the basis vectors of the dual space are defined through the relation $(\alpha_i, \omega_j) = \delta_{ij}$. The $F_4$-polytopes can be generated by the orbit $W(F_4)\Lambda$ where the vector $\Lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4$ is defined in the dual space which can also be represented as $\Lambda = (a_1, a_2, a_3, a_4)$. The components of the vector $\Lambda$ are real numbers $a_i \geq 0$, $(i=1,2,3,4)$. The maximal Coxeter subgroups of the group $W(F_4)$ that we deal with are $W(B_4)$ and $W(B_3) \times W(A_1)$. There are two subgroups $W(B_{3L}) = \langle r_1, r_2, r_3 \rangle$ and $W(B_{3R}) = \langle r_2, r_3, r_4 \rangle$ which are not conjugates in the group $W(F_4)$ but they are conjugates in the larger group $\text{Aut}(F_4)$. We use the notation $\langle r_1, r_2, r_3, ..., r_n \rangle$ for the Coxeter group generated by reflection generators $r_i$, $(i = 1, 2, 3, ..., n)$. The octahedral group $W(B_3)$ plays an important role when an arbitrary polytope of the group
$W(F_4)$ is projected into 3D. The quaternionic representations of the groups $W(B_3) \times C_2$ are given by

\begin{align}
W(B_{3R}) \times C_2 &= [T, \pm \bar{T}] \oplus [T', \pm \bar{T}'] \oplus [T, \pm \bar{T}']^* \oplus [T', \pm \bar{T}']^* \\
W(B_{3L}) \times C_2 &= \{ [T, \pm q \bar{T}q] \oplus [T', \pm q \bar{T}'q] \oplus [T, \pm q \bar{T}q]^* \oplus [T', \pm q \bar{T}'q]^* \},
\end{align}

(13a) $q = \frac{1 + e_1}{\sqrt{2}}.$

(13b)

Here $[T, \pm \bar{T}]$ or any other pair represents the group element $[p, \pm \bar{p}], p \in T$. The Coxeter group $W(B_3)$ is isomorphic to the octahedral group $S_4 \times C_2$ of order 48. Therefore the group $W(B_3) \times C_2$ of order 96 can be embedded in the group $W(F_4)$ in 12 different ways, in each embedding, a quaternion with $\pm$ sign left invariant. In equations (13a) and (13b) the groups leave the quaternions $\pm 1$ and $\pm (\frac{1 + e_1}{\sqrt{2}})$ invariant respectively. It is clear from these arguments that while the conjugates of the group $W(B_{3R}) \times C_2$ leaving the vectors of the set $T'$ invariant the conjugates of the group $W(B_{3L}) \times C_2$ leave invariant the vectors from $T$.

## 3 The $W(F_4)$ polytopes and branching under the group $W(B_4)$

The Coxeter diagram of the group $W(B_4)$ is shown in Figure 2. The Cartan matrix and its inverse are respectively given by the matrices

\begin{center}
\begin{tikzpicture}
\begin{scope}[scale=0.8]
\draw (0,0) -- (1,0) node[above] {$1-e_1$} -- (2,0) node[above] {$e_1-e_2$} -- (3,0) node[above] {$e_2-e_3$} -- (4,0) node[above] {$\sqrt{2}e_3$};
\draw (1,0) -- (1,-0.5);
\draw (2,0) -- (2,-0.5);
\draw (3,0) -- (3,-0.5);
\draw (4,0) -- (4,-0.5);
\draw (0,0) circle (0.1);
\draw (1,0) circle (0.1);
\draw (2,0) circle (0.1);
\draw (3,0) circle (0.1);
\draw (4,0) circle (0.1);
\node at (0.5,-0.5) {$r_1$};
\node at (1.5,-0.5) {$r_2$};
\node at (2.5,-0.5) {$r_3$};
\node at (3.5,-0.5) {$r_4$};
\end{scope}
\end{tikzpicture}
\end{center}

Figure 2: The $B_4$ diagram with quaternionic simple roots
\[
C_{B_4} = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -\sqrt{2} \\
0 & 0 & -\sqrt{2} & 2
\end{bmatrix},
\]

\[
(C_{B_4})^{-1} = \begin{bmatrix}
1 & 1 & 1 & \frac{\sqrt{2}}{2} \\
1 & 2 & 2 & \sqrt{2} \\
\frac{1}{\sqrt{2}} & \sqrt{2} & \frac{3}{\sqrt{2}} & 2
\end{bmatrix}.
\]  

If we chose the quaternionic simple roots as
\[
\alpha_1 = 1 - e_1, \quad \alpha_2 = e_1 - e_2, \quad \alpha_3 = e_2 - e_3, \quad \alpha_4 = \sqrt{2}e_3
\]
the Coxeter group can be represented as [10]
\[
W(B_4) = \{ [V_0, V_0] \oplus [V_+, V_-] \oplus [V_-, V_+] \oplus [V_1, V_1] \oplus [V_2, V_2] \oplus [V_3, V_3] \\
\oplus [V_0, V_0]^* \oplus [V_+, V_-]^* \oplus [V_-, V_+]^* \\
\oplus [V_1, V_1]^* \oplus [V_2, V_2]^* \oplus [V_3, V_3]^* \}
\]

The validity of the above equation can be checked by using Table 1. The group \(W(B_4)\) can be embedded in the group \(W(F_4)\) triply symmetric way which leads to the coset decomposition
\[
W(F_4) = \sum_{i=1}^{3} W(B_4)g_i
\]
\[
g_1 = [1, 1], \quad g_2 = [\omega_0, 1], \quad g_3 = [\omega_0^2, 1],
\]
\[
\omega_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3).
\]

When the vector \(\Lambda = (a_1, a_2, a_3, a_4)\) is expressed in terms of quaternions we can find the vectors \(\Lambda_1 \equiv \Lambda, \quad \Lambda_2 \equiv \omega_0\Lambda, \quad \Lambda_3 \equiv \omega_0^2\Lambda\) and operate by \(W(B_4)\) on the left do determine the branching of the orbits of the group \(W(B_4)\) in the orbit of the group \(W(F_4)\). If all components of the vector \(\Lambda\) are different from zero then the orbit \(W(F_4)\Lambda\) represents a polytope with 1152 vertices. The branching of the orbit \(\Lambda = (a_1, a_2, a_3, a_4)\) under the orbits of the group \(W(B_4)\) can be written as follows
\[
(a_1, a_2, a_3, a_4)_{F_4} = \{ (\sqrt{2}a_1 + \sqrt{2}a_2 + a_3), a_4, a_3, a_2 \}_{B_4} \\
+ \{ a_3, a_4, (\sqrt{2}a_2 + a_3), a_1 \}_{B_4} \\
+ \{ (\sqrt{2}a_2 + a_3), a_4, a_3, (a_1 + a_2) \}_{B_4}.
\]
We use the notation $(G)A = (a_1, a_2, a_3, a_4)_G$ for the orbit of the Coxeter group where $G$ represents the Coxeter diagram. Let us give a few examples. A complete list of decomposition of the regular and semi-regular polytopes are given in the Appendix 1.

The polytope $(1, 0, 0, 0)_{F_4} = W(F_4)\omega_1 = W(F_4)\sqrt{2} = \sqrt{2}T$ represents the set of 24 quaternions $T = V_0 \oplus V_+ \oplus V_-$ up to a scale factor $\sqrt{2}$. They are the elements of the the binary tetrahedral group and represents the vertices of the regular polytope 24-cell. This is a unique polytope which has no correspondence in any other dimensions. This is perhaps due to the fact that $W(F_4)$ is associated with the exceptional Lie group $F_4$. The polytope 24-cell consists of 24 octahedral cells, every vertex of which, is shared by 6 octahedra. To illustrate this let us consider the vertex represented by the quaternion 1. The following quaternions, e.g., represent the vertices of an octahedron involving the quaternion 1:

$$1, e_1, \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3).$$

The center of this octahedron is represented, up to a scale factor, by the quaternion $\frac{1 + e_1}{\sqrt{2}} \in T'$. If we introduce the orthogonal vectors $p_0 = \frac{1 + e_1}{\sqrt{2}}, p_i = e_i p_0, i = 1, 2, 3$ then (19) can be written as $p_0 \pm p_i (i = 1, 2, 3)$ which represent an octahedron shifted in the 4th dimension by $p_0$. We tabulate in equation (20) the vertices of the six octahedra and their centers sharing the vertex 1.

| Vertices       | Centers(scaled)          |
|----------------|--------------------------|
| $1, e_1, \frac{1}{2}(1 + e_1 \pm e_2 \pm e_3)$ | $1 + e_1$               |
| $1, -e_1, \frac{1}{2}(1 - e_1 \pm e_2 \pm e_3)$ | $1 - e_1$               |
| $1, e_2, \frac{1}{2}(1 \pm e_1 + e_2 \pm e_3)$ | $1 + e_2$               |
| $1, -e_2, \frac{1}{2}(1 \pm e_1 - e_2 \pm e_3)$ | $1 - e_2$               |
| $1, e_3, \frac{1}{2}(1 \pm e_1 \pm e_2 + e_3)$ | $1 + e_3$               |
| $1, -e_3, \frac{1}{2}(1 \pm e_1 \pm e_2 - e_3)$ | $1 - e_3$.              |

When we discuss the dual polytope of the $(1, 0, 0, 0)_{F_4}$ we will see that those centers of the six octahedra form another octahedron with the vertices $1 \pm e_i, (i = 1, 2, 3)$ where the vertices belong to the set $T' = \frac{1}{\sqrt{2}}(0, 0, 0, 1)_{F_4}$ which forms another 24-cell. Actually the diagram symmetry operator $D$ exchanges the two 24-cells so that the polytope 24-cell is said to be self-dual.

Now we can use (18) to check how 24-cell decomposes under the subgroup $W(B_4)$. It is straightforward to see that

$$(1, 0, 0, 0)_{F_4} = (\sqrt{2}, 0, 0, 0)_{B_4} + (0, 0, 0, 1)_{B_4}.$$
This is nothing but writing the set $T$ in the form $T = V_0 \oplus (V_+ \oplus V_-)$. The set $V_0$ represents the vertices of a hyperoctahedron (octahedron in 4D or called a 16 cell) whose cells are tetrahedra, 8 of which, share the same vertex. The set $(V_+ \oplus V_-)$ represents the hypercube (8-cell) with 16 vertices. The hypercube consists of 8-cubes, 4 of which, meet at one vertex. The hyperoctahedron and the hypercube are the duals of each other; the centers of the 16 tetrahedra are represented by the vertices of the hypercube and vice versa.

The dual polytope 24-cell $(0, 0, 0, 1)_{F_4}$ decomposes under the subgroup $W(B_4)$ as

\[(0, 0, 0, 1)_{F_4} = (0, 1, 0, 0)_{B_4}\]

where the vertices, edges, faces and cells respectively decompose as $24 = 24, 96=96, 96=32+64, 24=16+8$.

4 The $W(F_4)$ polytopes and branching under the group $W(B_{3R})$

As we mentioned earlier there are two non-conjugate octahedral groups in the Coxeter group $W(F_4)$. Here we will give the projections of the $W(F_4)$ polytopes under the octahedral group $W(B_{3R})$. Decomposition with respect to the other group $W(B_{3L})$ can be done using a similar technique. The group $W(B_{3R})$ up to a conjugation can be represented by the quaternions as follows

\[W(B_{3R}) = \{[T, \tilde{T}] \oplus [T', \tilde{T}'] \oplus [T, \tilde{T}]^* \oplus [T', \tilde{T}']^*\}\]

which leaves the unit quaternion 1 invariant. The right coset decomposition of the Coxeter group $W(F_4)$ under the octahedral group $W(B_{3R})$ can be given as $W(F_4) = \sum_{i=1}^{24} W(B_{3R})g_i$ with $g_i \in [T, 1]$. Note that $[T, 1]$ and $[1, T]$ are two invariant subgroups of the Coxeter group $W(F_4)$. Applying the group $W(F_4)$ on the vector $\Lambda$ we obtain the orbit $W(F_4)\Lambda = \sum_{i=1}^{24} W(B_{3R})g_i\Lambda$. Defining the vectors $\Lambda_i = g_i\Lambda, \, i = 1, 2, ..., 24$ with $\Lambda_1 \equiv \Lambda$ we obtain, in general, 24 different orbits of the group $W(B_{3R})$ in the decomposition of the orbit $W(F_4)\Lambda$. Let a vector represented in the dual space of the root system of the Coxeter group $W(B_{3R})$ be given by $b_1v_1 + b_2v_2 + b_3v_3$. When the quaternionic simple roots of the group $W(B_{3R})$ are given by $\alpha_1 = \sqrt{2}e_3, \, \alpha_2 = e_2 - e_3, \, \alpha_3 = e_1 - e_2$ then basis vectors in the dual space are given by $v_1 = \frac{1}{\sqrt{2}}(e_1 + e_2 + e_3), \, v_2 = e_1 + e_2, \, v_3 = e_1$. Applying the group $W(B_{3R})$ on the quaternionic representations of the vectors $\Lambda_i = g_i\Lambda, \, i = 1, 2, ..., 24$ and expressing them in the basis $v_i, (i = 1, 2, 3)$ then we obtain the following
decomposition:

\[
(a_1, a_2, a_3, a_4)_{F_4} = \{(a_2, a_3, a_4)_{B_3} \pm \left(a_1 + \frac{3a_2}{2} + \frac{2a_3 + a_4}{\sqrt{2}}\right)\}
\]

\[
+ \{(a_2, a_3, (\sqrt{2}a_1 + \sqrt{2}a_2 + a_3 + a_4))_{B_3}
\pm \left(\frac{a_2}{2} + \frac{a_3 + a_4}{\sqrt{2}}\right)\}
\]

\[
+ \{(a_2, (a_3 + a_4), (\sqrt{2}a_1 + \sqrt{2}a_2 + a_3))_{B_3}
\pm \left(\frac{a_2}{2} + \frac{a_3}{\sqrt{2}}\right)\}
\]

\[
+ \{((a_2 + \sqrt{2}a_3), a_4, (\sqrt{2}a_1 + \sqrt{2}a_2 + a_3))_{B_3}
\pm \left(\frac{a_2}{2}\right)\}
\]

\[
+ \{((a_1 + 2a_2 + \sqrt{2}a_3), a_4, a_3)_{B_3}
\pm \left(\frac{a_1}{2}\right)\}
\]

\[
+ \{((a_1 + a_2), a_3, a_4)_{B_3}
\pm \left(\frac{1}{2}a_1 + 3a_2 + 2\sqrt{2}a_3 + \sqrt{2}a_4\right)\}
\]

\[
+ \{((a_1 + a_2), a_3, (\sqrt{2}a_2 + a_3 + a_4))_{B_3}
\pm \left(\frac{1}{2}a_1 + a_2 + 2\sqrt{2}a_3 + \sqrt{2}a_4\right)\}
\]

\[
+ \{((a_1 + a_2), (a_3 + a_4), (\sqrt{2}a_1 + \sqrt{2}a_2 + a_3))_{B_3}
\pm \left(\frac{1}{2}a_1 + a_2 + \sqrt{2}a_3\right)\}
\]

\[
+ \{\frac{1}{2}(a_1 + a_2 + \sqrt{2}a_3), a_4, (\sqrt{2}a_2 + a_3))_{B_3}
\pm \frac{1}{2}(a_1 + a_2)\}
\]

\[
+ \{((a_1, (\sqrt{2}a_2 + a_3), a_4))_{B_3}
\pm \left(\frac{a_1}{2} + a_2 + \frac{2a_3 + a_4}{\sqrt{2}}\right)\}
\]

\[
+ \{((a_1, (\sqrt{2}a_2 + a_3 + a_4), a_3)_{B_3}
\pm \left(\frac{a_1}{2} + a_2 + \frac{a_3}{\sqrt{2}}\right)\}
\]

\[
+ \{((a_1, (\sqrt{2}a_2 + a_3), (a_3 + a_4))_{B_3}
\pm \frac{1}{2}(a_1 + 2a_2 + \sqrt{2}a_3 + \sqrt{2}a_4)\}. \quad (24)
\]
The term starting with the ± represents a $W(A_1) \approx C_2$ vector. For a general $W(F_4)$ orbit the formula (24) shows the sphere $S^3$ representing the vertices of the $W(F_4)$ polytope sliced by 24 hyperplanes. Each crosssection of the sphere $S^3$ with an hyperplane orthogonal to the vector represented by the group $W(A_1) \approx C_2$ is an $S^2$ sphere representing a particular $W(B_3)$ polyhedron. Let us discuss a few cases. Consider the 24-cell represented by the orbit $\frac{1}{\sqrt{2}}(1,0,0,0)_{F_4} = T$ which can be written as

$$
\frac{1}{\sqrt{2}}(1,0,0,0)_{F_4} = \{(0,0,0)_{B_3} \pm \frac{1}{\sqrt{2}}\} + \{(0,0,1)_{B_3} \pm (0)\}
\quad + \{\frac{1}{\sqrt{2}}(1,0,0)_{B_3} \pm \frac{1}{\sqrt{2}}\}.
\quad (25)
$$

When expressed in terms of quaternionic vertices they correspond to the decomposition

$$
T = \pm 1 + (\pm e_1, \pm e_2, \pm e_3) + \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3) + \frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3). \quad (26)
$$

Either from the analysis of the equation (25) [15] or directly reading the equation (26) the points $\pm 1$ corresponds to the two opposite poles of the sphere $S^3$. The set of vertices $(\pm e_1, \pm e_2, \pm e_3)$ represent an octahedron obtained as the intersection of the hyperplane through the center of the sphere $S^3$. The last two sets of quaternions in (26) represent cubes oppositely placed with respect to the center of the sphere $S^3$. The dual 24-cell represented by the orbit $\frac{1}{\sqrt{2}}(0,0,0,1)_{F_4} = T'$ decomposes as

$$
\frac{1}{\sqrt{2}}(0,0,0,1)_{F_4} = \{\frac{1}{\sqrt{2}}(0,0,1)_{B_3} \pm \frac{1}{2}\} + \{\frac{1}{\sqrt{2}}(0,1,0)_{B_3} \pm (0)\}
\quad (27)
$$

which can also be represented by the set of quaternions

$$
T' = \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm 1 \pm e_3)\} + \\
\{\frac{1}{\sqrt{2}}(\pm e_1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)\}. \quad (28)
$$

The first set represents two octahedra oppositely placed with respect to the origin along the fourth direction, the second set represents a cuboctahedron with 12 vertices around the center of the sphere $S^3$. The decompositions of quaternionic vertices of 24-cell indicate that it can have two different decompositions under the group $W(B_3R)$. If the subgroup $W(B_{3L})$ were chosen for the projection of the 24-cell into 3D then the above decomposition of the sets represented by $T$ and $T'$ would be interchanged. Using equation (23), any interested reader can work out the projection of any $W(F_4)$ polytope into 3D with the octahedral residual symmetry.
5 Dual polytopes of the uniform polytopes of the Coxeter-Weyl group $W(F_4)$

The Catalan solids which are the duals of the Archimedean solids can be derived from the Coxeter diagrams $A_3$, $B_3$, $H_3$ with a simple technique [11]. Following the same group theoretical technique we have already constructed the dual polytopes of the $W(H_4)$ polytopes [16]. Here we apply the same technique to determine the vertices and the symmetries of the dual polytopes of the uniform $W(F_4)$ polytopes. To find the vertices of the dual polytope one should determine the vectors representing the centers of the cells at the vertex $(a_1, a_2, a_3, a_4)$. The relative magnitudes of the vectors of interest can be obtained by noting that the hyperplane formed by the vectors representing the centers of the cells are orthogonal to the vertex $(a_1, a_2, a_3, a_4)$. The dual polytopes are cell transitive while the polytopes are vertex transitive. In this section we will construct the vertices of a dual polytope and determine its cell structure. We shall also use the group theoretical technique to determine the numbers of vertices $N_0$, edges $N_1$, faces $N_2$ and the cells $N_3$ of an arbitrary polytope. All 4D polytopes satisfy the Euler’s topological formula $N_0 - N_1 + N_2 - N_3 = 0$.

5.1 Dual polytope of the 24-cell $(1, 0, 0, 0)_{F_4} = W(F_4)\omega_1$

The 24-cell has 24 vertices, 96 edges, 240 faces and 24-cells. First we discuss how these numbers follow from the group theoretical calculations. We note first that the vertex represented by the vector $\omega_1$ is left invariant under the subgroup $W(B_3 R) = \langle r_2, r_3, r_4 \rangle$ of order 48. The number of vertices then is the number of left coset representatives under the decomposition $W(F_4) = \sum g_i W(B_3 R)$ which is equal to the index $\frac{|W(F_4)|}{|W(B_3 R)|} = \frac{1152}{48} = 24$. The number of edges can be determined as follows. The difference of the vectors $\omega_1 - r_i \omega_1 = \alpha_1$ or $-\alpha_1$ represents an edge. The set $(\pm \alpha_1)$ is left invariant by the group generated by $\langle r_1, r_2, r_3 \rangle$ of order 12 so that its index in the group $W(F_4)$ equals 96. Similarly from the vector $(1, 0)$ of the subgroup $W(A_2)$ one can generate an equilateral triangle by the group $W(A_2) = \langle r_1, r_2 \rangle$ of order 6. Therefore the group leaving the triangle invariant is the subgroup generated by $\langle r_1, r_2, r_4 \rangle = W(A_2) \times W(A_1)$ of order 12 and its index in the group $W(F_4)$, 96, is the number of triangular faces. The octahedral cell is generated by the group $W(B_3 L) = \langle r_1, r_2, r_3 \rangle$ whose index, 24, is the number of octahedral cells hence the name of the polytope 24-cell.

The center of the octahedron $W(B_3 L) \omega_1$ can be represented by the vector $\omega_4$.
up to a scale factor. The number of octahedra at the vertex $\omega_1$ is given by the formula

$$\frac{|<r_2,r_3,r_4>|}{|<r_2,r_3>|} = 6$$

(29)

which is the index of the subgroup $<r_2,r_3>$ of order 8 fixing the vertex $\omega_1$ of the octahedron in the group $<r_2,r_3,r_4>$ which fixes the vertex $\omega_1$.

The vertices of the cell representing the centers of the 6 octahedra can be determined by applying $W(B_3) = <r_2,r_3,r_4>$ on the vector $\omega_4$ which leads to another octahedron. We have discussed this case in detail in Section 3.

5.2 Dual polytope of the polytope $(0,1,0,0)_{F_4} = W(F_4)\omega_2$

Applying the technique discussed in Section 5.1 we determine $N_0 = 96$ vertices, $N_1 = 288$ edges, $N_2 = 240$ faces and $N_3 = 48$ cells. The polytope has 96 triangular, 144 square faces and 48 cells which consist of 24 cuboctahedra and 24 cubes whose symmetries are respectively are the conjugate groups of the groups $W(B_3L)$ and $W(B_3R)$.

The number of cuboctahedral cells at the vertex $\omega_2$ is determined by the index $\frac{|<r_1,r_3,r_4>|}{|<r_1,r_3>|} = 3$. The group in the numerator is a subgroup of the group $W(F_4)$ fixing the vertex $\omega_2$ and the group in the denominator is the subgroup of the octahedral group $W(B_3L)$ fixing the same vertex. The group $W(B_3L)$ which generates the cuboctahedra leaves the vector $\omega_4$ invariant which can be taken as the vector representing its center. So the vectors representing the centers of the 3 cuboctahedra at the vertex $\omega_2$ can be taken as $\omega_4$, $r_3r_4\omega_4$, $(r_3r_4)^2\omega_4$.

Similarly the number of the cubes at the same vertex is 2 whose centers can be represented by the vectors $\lambda \omega_1$ and $\lambda r_1\omega_1$ where $\lambda$ is a scale factor determined from the equation $(\lambda \omega_1 - \omega_4).\omega_2 = 0$ as $\lambda = 2\sqrt{2}/3$. These five vertices form a cell which is a semi-regular bipyramid whose center is represented by the vector $\omega_2$. The bipyramid consists of six isosceles triangles of sides $\sqrt{\frac{10}{3}}$ and $\sqrt{\frac{2}{2}}$. It has the symmetry $<r_1,r_3,r_4> \approx C_2 \times D_3$ of order 12 which fixes the vector $\omega_2$. The set of vertices of the dual polytope is the union of the orbits $\frac{2\sqrt{2}}{3}(1,0,0,0)_{F_4} \oplus (0,0,0,1)_{F_4}$. They define two concentric spheres $S^3$ with the fraction of radii $\frac{R_1}{R_2} = \frac{3}{2\sqrt{2}} \approx 1.06$. Let us define the quaternionic unit vectors $p_0 = \frac{\omega_2}{|\omega_2|} = \frac{3 + e_1 + e_2 + e_3}{2\sqrt{2}}$, $p_i = e_i p_0$, ($i = 1,2,3$). Note that all the cells of the dual polytope are bipyramids and their centers are the vertices of the polytope $(0,1,0,0)_{F_4}$. After expressing five vertices of the bipyramid in terms of the new quaternionic units and omitting the common 4th component proportional to $p_0$, a scaled set of the vectors representing the vertices of the
Figure 3: Plot of the bipyramid representing the vertices in (30). (a) Side view. (b) Top view.

A plot of this bipyramid is shown in Figure 3.

5.3 Dual polytope of the polytope \((0, 0, 1, 0)_{F_4} = W(F_4)\omega_3\)

This polytope is obtained from the polytope \((0, 1, 0, 0)_{F_4} = W(F_4)\omega_2\) by exchanging \(\omega_2\) and \(\omega_3\).

5.4 Dual polytope of the polytope \((1, 1, 0, 0)_{F_4} = W(F_4)(\omega_1 + \omega_2)\)

This polytope has \(N_0 = 192\) vertices, \(N_1 = 384\) edges, \(N_2 = 240\) faces and \(N_3 = 48\) cells. The faces are of two types: 96 hexagons and 144 squares. The cells consist of 24 truncated octahedra and 24 cubes. To find the dual polytope we have to determine the vectors representing the centers of these cells. The vectors representing the centers of the truncated octahedra and the cube at the vertex \((\omega_1 + \omega_2)\) are given by

\[
\omega_1, \, r_3r_4\omega_4, \, (r_3r_4)^2\omega_1, \, \lambda\omega_1. \tag{31}
\]

The relative scale parameter is determined to be \(\lambda = \frac{3\sqrt{2}}{4}\). The cell is a pyramid consisting of an equilateral triangular base of side \(\sqrt{2}\) and three
isosceles triangular faces of sides \( \sqrt{2} \) and \( \sqrt{\frac{26}{25}} \). Defining the unit quaternion \( p_0 = \frac{\omega_1 + \omega_3}{|\omega_1 + \omega_3|} = \frac{3 + 6\sqrt{2} + 1 + e_1 + e_2}{2\sqrt{2}} \) and \( p_i = e_i p_0, \) \( (i = 1, 2, 3) \) one obtains the vertices in 3D of the cell of the dual polytope as

\[
\omega_4 \approx 4p_1 - 2p_3, \quad r_3 r_4 \omega_4 \approx -2p_2 + 4p_3,
\]

\[
r_4 r_3 \omega_4 \approx -2p_1 + 4p_2, \quad \lambda \omega_1 \approx -\frac{6}{5}(p_1 + p_2 + p_3).
\]

A plot of this cell is shown in Figure 4. It is clear that this solid has a symmetry group generated by \( < r_3, r_4 > \).

The orbit of the vertices of the dual cell consists of two concentric spheres \( S^3 \) with the ratio of the radii \( \frac{R_4}{R_1} \approx 1.18 \).

\section{5.5 Dual polytope of the polytope \( (1, 0, 1, 0) F_4 = W(F_4)(\omega_1 + \omega_3) \)}

This polytope has \( N_0 = 288 \) vertices, \( N_1 = 864 \) edges, \( N_2 = 720 \) faces, and \( N_3 = 144 \) cells. Reading from left to right the faces consist of 96 triangles, 288+144 squares, and 192 triangles. The cells consist of 24 small rhombicuboctahedra, 24 cubeoctahedra, and 96 triangular prisms. The vectors representing the centers of the cells at the vertex \( (\omega_1 + \omega_3) \) are given by

\[
\lambda \omega_4, \lambda r_4 \omega_4, \rho \omega_1, \omega_2, r_2 \omega_2.
\]

The relative scale parameters are determined to be \( \lambda = \frac{1 + \sqrt{2}}{2}, \rho = \frac{5 - \sqrt{2}}{4} \). The cell of the dual polytope consists of five vertices. Defining the unit quaternion \( p_0 = \frac{\omega_1 + \omega_3}{|\omega_1 + \omega_3|} = \frac{2 + \sqrt{3} + e_1 + e_2}{2\sqrt{3} + 2\sqrt{2}} \) and \( p_i = e_i p_0, \) \( (i = 1, 2, 3) \) one obtains the vertices in 3D of the cell of the dual polytope as
\[
\lambda \omega_4 \approx \frac{1 + 9\sqrt{2}}{7}[(1 + \sqrt{2})p_1 - p_2 - p_3],
\]
\[
\lambda r_4 \omega_4 \approx \frac{1 + 9\sqrt{2}}{7}[-p_1 + (1 + \sqrt{2})p_2 + p_3],
\]
\[
\rho \omega_1 \approx \frac{2 - 5\sqrt{2}}{2}[p_1 + p_2],
\]
\[
\omega_2 \approx [p_1 + (1 - \sqrt{2})p_2 + (1 + \sqrt{2})p_3],
\]
\[
r_2 \omega_2 \approx [(1 - \sqrt{2})p_1 + p_2 - (1 + \sqrt{2})p_3]. \tag{34}
\]

A plot of this cell is shown in Figure 5. It is clear that this solid has a symmetry of the Klein four-group generated by \(< r_2, r_4 >\). The dual polytope consists of the union of the orbits \(\rho(1, 0, 0, 0)_{F_4} \oplus (0, 1, 0, 0)_{F_4} \oplus \lambda(0, 0, 0, 1)_{F_4}\) with 144 vertices, 288 cells, 864 faces and 720 edges. The radii of the \(S^3\) spheres are respectively \(R_1 \approx 2.536, R_2 \approx 2.450, R_4 \approx 2.774\).

### 5.6 Dual polytope of the polytope \((1, 0, 0, 1)_{F_4} = W(F_4)(\omega_1 + \omega_4)\)

The polytope has \(N_0 = 144\) vertices, \(N_1 = 576\) edges, \(N_2 = 672\) faces and \(N_3 = 240\) cells. Reading from left to right the faces consist of 192 triangles, 288 squares, and 192 triangles. The cells consist of 24+24 octahedra, 96+96 triangular prisms. The vectors representing the centers of the cells at the vertex \((\omega_1 + \omega_4)\) are given by

\[
\lambda \omega_1, \lambda \omega_4, \omega_2, r_2 r_3 \omega_2, (r_2 r_3)^2 \omega_2, (r_2 r_3)^3 \omega_2, \omega_3, r_2 r_3 \omega_3, (r_2 r_3)^2 \omega_3, (r_2 r_3)^3 \omega_3. \tag{35}
\]
Figure 6: The plot of the tetragonal trapezohedron with vertices given by (36). (a) Side view: Three kites meet at one vertex, (b) Top view: Four kites meet at one vertex.

The scale factor is found to be \( \lambda = \frac{2 + \sqrt{2}}{2} \). This cell of the dual polytope is a solid called tetragonal trapezohedron with 10 vertices with the symmetry \( D_4 : C_2 \) of order 16. The dihedral group \( D_4 \) is generated by \( D_4 \approx < r_2, r_3 > \) and \( C_2 \) is the group generated by the diagram symmetry. Defining the new set of unit vectors by \( p_0 = \frac{\omega_1 + \omega_4}{|\omega_1 + \omega_4|} \) and \( p_i = e_i p_0 \), \( (i = 1, 2, 3) \) we obtain the vertices of the solid in 3D as

\[
\begin{align*}
\lambda \omega_1 &\approx -p_1, \quad \lambda \omega_4 \approx p_1 \\
\omega_2 &\approx (2\sqrt{2} - 3)p_1 + (\sqrt{2} - 1)p_2 + p_3 \\
r_2r_3\omega_2 &\approx (2\sqrt{2} - 3)p_1 + p_2 + (\sqrt{2} - 1)p_3 \\
(r_2r_3)^2\omega_2 &\approx (2\sqrt{2} - 3)p_1 - p_2 + (\sqrt{2} - 1)p_3 \\
\omega_3 &\approx (3 - 2\sqrt{2})p_1 + p_2 + (\sqrt{2} - 1)p_3 \\
r_2r_3\omega_3 &\approx (3 - 2\sqrt{2})p_1 + (\sqrt{2} - 1)p_2 + p_3 \\
(r_2r_3)^2\omega_3 &\approx (3 - 2\sqrt{2})p_1 - p_2 + (\sqrt{2} - 1)p_3 \\
(r_2r_3)^3\omega_3 &\approx (3 - 2\sqrt{2})p_1 - (\sqrt{2} - 1)p_2 + p_3.
\end{align*}
\]

(36)

A plot of this tetragonal trapezoid is shown in Figure 6.

It has eight faces and each face is a kite with sides \( \sqrt{16 - 10\sqrt{2}} \approx 1.363 \) and \( \sqrt{80 - 56\sqrt{2}} \approx 0.897 \) and the area 0.934. So the dual polytope
has 144 cells of tetragonal trapezohedra shown in Figure 6. The dual polytope is cell transitive and it has 240 vertices, 672 edges and 576 faces of kites described above. The polytope is the union of the orbits \( \lambda(1,0,0,0)_{F_4} \oplus \lambda(0,0,0,1)_{F_4} \oplus (0,1,0,0)_{F_4} \oplus (0,0,1,0)_{F_4} \). The vertices of the dual polytope is on two concentric radii of the spheres \( S^3 \) with radii \( R_1 = R_4 = 2.414 \) and \( R_2 = R_3 = 2.449 \).

5.7 Dual polytope of the polytope \((0,1,1,0)_{F_4} = W(F_4)(\omega_2 + \omega_3)\)

The polytope has \( N_0 = 288 \) vertices, \( N_1 = 576 \) edges, \( N_2 = 336 \) faces and \( N_3 = 48 \) cells. Reading from left to right the faces consist of 96 triangles, 144 octagons, and 96 triangles. The vertices of the dual polytope are on two concentric radii of the spheres \( S^3 \) with radii \( R_1 = R_4 = 2.414 \) and \( R_2 = R_3 = 2.449 \).

5.8 Dual polytope of the polytope \((1,1,1,0)_{F_4} = W(F_4)(\omega_1 + \omega_2 + \omega_3)\)

The polytope has \( N_0 = 576 \) vertices, \( N_1 = 1152 \) edges, \( N_2 = 720 \) faces and \( N_3 = 144 \) cells. Reading from left to right the faces consist of 96 hexagons, 288 squares, 144 octagons and 192 triangles. The cells consist of 24 great rhombicuboctahedra, 24 truncated cubes, and 96 triangular prisms. The
vectors representing the centers of the cells at the vertex \((\omega_1 + \omega_2 + \omega_3)\) are given by \(\omega_4, r_4\omega_4, \lambda\omega_1,\) and \(\rho\omega_2\). It is a solid with four vertices which has a symmetry generated by \(< r_4 \rangle\). The parameters are determined to be \(\lambda = \frac{3(1+3\sqrt{2})}{17}, \rho = \frac{3(1+\sqrt{2})}{9+4\sqrt{2}}\). The vertices of the dual cell is the union of the orbits \(\lambda(1,0,0,0)_{F_4} \oplus \rho(0,1,0,0)_{F_4} \oplus (0,0,0,1)_{F_4}\). The vertices lie on three \(S^3\) spheres with the radii \(R_1 : R_2 : R_4 = 1:1.21:1.41\). Defining the new set of unit vectors by \(p_0 = \frac{\omega_1 + \omega_2 + \omega_3}{|\omega_1 + \omega_2 + \omega_3|}\) and \(p_i = e_ip_0, \ (i = 1, 2, 3)\) we obtain the vertices of the cell as

\[
\begin{align*}
\lambda\omega_1 &\approx -\frac{3(8 + 7\sqrt{2})}{17}p_1 - \frac{3(8 + 7\sqrt{2})}{17}p_2 - \frac{3(6 + \sqrt{2})}{17}p_3, \\
\rho\omega_2 &\approx \frac{3}{9 + 4\sqrt{2}}[(2 + \sqrt{2})p_1 + (2 + \sqrt{2})p_2 + (4 + 3\sqrt{2})p_3], \\
\omega_4 &\approx (4 + \sqrt{2})p_1 - \sqrt{2}p_2 - (2 + \sqrt{2})p_3, \\
r_4\omega_4 &\approx -(2 + \sqrt{2})p_1 + (4 + \sqrt{2})p_2 + \sqrt{2}p_3.
\end{align*}
\]  

A plot of this solid is depicted in Figure 8.

### 5.9 Dual polytope of the polytope \((1, 1, 0, 1)_{F_4} = W(F_4)(\omega_1 + \omega_2 + \omega_4)\)

The polytope has \(N_0 = 576\) vertices, \(N_1 = 1440\) edges, \(N_2 = 1104\) faces and \(N_3 = 240\) cells. Reading from left to right the faces consist of 192 hexagons, 144+288+288 squares, and 192 triangles. The cells consist of 24 truncated octahedra, 24 small rhombicuboctahedra, 96 hexagonal prisms, and 96 triangular prisms. The vectors representing the centers of the cells at the vertex...
Figure 8: The plot of the cell of the dual polytope of the polytope \((1, 1, 1, 0)_{F_4} = W(F_4)(\omega_1 + \omega_2 + \omega_3)\)

\((\omega_1 + \omega_2 + \omega_3)\) are given by \(\omega_3, r_3 \omega_3, \lambda \omega_1, \) and \(\rho \omega_1, \eta \omega_2\). It is a solid with five vertices which has a symmetry generated by \(< r_3 >\). The parameters are determined to be \(\lambda = \frac{3+6\sqrt{2}}{2+3\sqrt{2}}, \rho = \frac{3+6\sqrt{2}}{5+\sqrt{2}}, \) and \(\eta = \frac{3+6\sqrt{2}}{9+2\sqrt{2}}\). The vertices of the dual cell is the union of the orbits \(\rho(1, 0, 0, 0)_{F_4} \oplus \eta(0, 1, 0, 0)_{F_4} \oplus (0, 0, 1, 0)_{F_4}\). Defining the new set of unit vectors by \(p_0 = \omega_1 + \omega_2 + \omega_4\) and \(p_i = e_i p_0, \ (i = 1, 2, 3)\) we obtain the vertices of the cell as

\[
\begin{align*}
\rho \omega_1 & \approx -\frac{3}{5+\sqrt{2}}[6 + 5\sqrt{2}]p_1 + (4 + \sqrt{2})p_2 + (4 + \sqrt{2})p_3, \\
\eta \omega_2 & \approx \frac{6(1+2\sqrt{2})}{4+9\sqrt{2}}[(1 - \sqrt{2})p_1 + p_2 + (1 + \sqrt{2})p_3], \\
\lambda \omega_1 & \approx \frac{12(1+2\sqrt{2})}{2+3\sqrt{2}}p_1 - \frac{6(1+2\sqrt{2})}{2+3\sqrt{2}}p_3, \\
\omega_3 & \approx (2 - \sqrt{2})p_1 + (4 + \sqrt{2})p_2 + (-2 + \sqrt{2})p_3, \\
r_3 \omega_3 & \approx (4 - \sqrt{2})p_1 - (2 + \sqrt{2})p_2 + (2 + \sqrt{2})p_3.
\end{align*}
\] (39)

A plot of this solid is depicted in Figure 9.

5.10 Dual polytope of the polytope \((1, 1, 1, 1)_{F_4} = W(F_4)(\omega_1 + \omega_2 + \omega_3 + \omega_4)\)

The polytope has \(N_0 = 1152\) vertices, \(N_1 = 2304\) edges, \(N_2 = 1392\) faces and \(N_3 = 240\) cells. Reading from left to right the faces consist of 192 hexagons, 288+288+288 squares, 144 octagons and 192 hexagons. The cells consist of 24+24 great rhombicuboctahedra and 96+96 hexagonal prisms. The vectors representing the centers of the cells at the vertex \((\omega_1 + \omega_2 + \omega_3 + \omega_4)\) are given by \(\omega_1, \omega_4, \) and \(\rho \omega_3, \rho \omega_3\) with \(\rho = \frac{3+6\sqrt{2}}{9+2\sqrt{2}}\). It is a solid with four vertices which has a diagram symmetry only. The 240 vertices of the dual cell is the union
of the orbits $(1, 0, 0, 0)_{F_4} \oplus \rho(0, 1, 0, 0)_{F_4} \oplus \rho(0, 0, 1, 0) \oplus (0, 0, 0, 1)_{F_4}$. Defining the new set of unit vectors by $p_0 = \frac{\omega_1 + \omega_2 + \omega_3 + \omega_4}{|\omega_1 + \omega_2 + \omega_3 + \omega_4|}$ and $p_i = e_ip_0$, $(i = 1, 2, 3)$ we obtain the vertices of the cell as

$$\begin{align*}
\omega_1 &\approx -(4 + \sqrt{2})p_1 - (2 + \sqrt{2})p_2 - \sqrt{2}p_3, \\
\omega_4 &\approx (4 + \sqrt{2})p_1 - \sqrt{2}p_2 - (2 + \sqrt{2})p_3, \\
\rho\omega_2 &\approx \frac{\sqrt{8} + \sqrt{2}}{9 + \sqrt{2}}[(\sqrt{2} - 2)p_1 + (\sqrt{2} - 2)p_2 + (\sqrt{2} + 4)p_3], \\
\rho\omega_3 &\approx \frac{\sqrt{8} + \sqrt{2}}{9 + \sqrt{2}}[(-\sqrt{2} + 2)p_1 + (\sqrt{2} + 4)p_2 + (\sqrt{2} - 2)p_3].
\end{align*}$$

Note that the Coxeter diagram symmetry leads to the transformation $p_0 \rightarrow p_0$, $p_1 \rightarrow -p_1$, $p_2 \leftrightarrow p_3$ which results in $\omega_1 \leftrightarrow \omega_4$ and $\rho\omega_2 \leftrightarrow \rho\omega_3$. A plot of this solid is depicted in Figure 10.

6 Conclusion

4D polytopes can be classified with respect to their symmetries represented by the Coxeter groups $W(A_4)$, $W(B_4)$, $W(H_4)$ and $W(F_4)$. Here we constructed the regular and semiregular polytopes of the Coxeter group $W(F_4)$ with their dual polytopes. We have represented the group elements in terms of quaternions. It is an interesting observation that the automorphism group of the Coxeter diagram $F_4$ can be determined in terms of the binary octahedral group as $\text{Aut}(F_4) = \{[O, O] \oplus [O, O]^*\}$. The dual polytopes of the $W(F_4)$ have been constructed for the first time in this work. We have explicitly given the decomposition of a given polytope under the maximal subgroup $W(B_4)$. The projection of an arbitrary $W(F_4)$ polytope in 3D space has been made
Figure 10: The plot of the cell of the dual polytope of the polytope 
$(1,1,1,1)_{F_4} = W(F_4)(\omega_1 + \omega_2 + \omega_3 + \omega_4)$

with the use of the octahedral group $W(B_3)$ which allows us to view the 4D polytope in 3D space. For a detail decomposition see Appendix 2. This work can be related to the quantum information theory based on 2-qubit s since we use a finite subgroup of the orthogonal group $SU(2) \times SU(2) \approx O(4)/C_2$.

Appendix 1

The decomposition of the $W(F_4)$ weights under $W(B_4)$ is given as follows:

$(0,0,0,1)_{F_4} = (0,1,0,0)_{B_4}$

$(0,0,1,0)_{F_4} = (1,0,1,0)_{B_4}$

$(0,0,1,1)_{F_4} = (1,1,1,0)_{B_4}$

$(0,1,0,0)_{F_4} = (0,0,\sqrt{2},0)_{B_4} + (\sqrt{2},0,0,1)_{B_4}$. 

$(0,1,0,1)_{F_4} = (0,1,\sqrt{2},0)_{B_4} + (\sqrt{2},1,0,1)_{B_4}$. 

$(0,1,1,0)_{F_4} = (1,0,1+\sqrt{2},0)_{B_4} + (1+\sqrt{2},0,1,1)_{B_4}$. 

$(0,1,1,1)_{F_4} = (1,1,1+\sqrt{2},0)_{B_4} + (1+\sqrt{2},1,1,1)_{B_4}$. 

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(1, 0, 0, 0)_{F_4} = (0, 0, 0, 1)_{B_4} + (\sqrt{2}, 0, 0, 0)_{B_4}.
(1, 0, 0, 1)_{F_4} = (0, 1, 0, 1)_{B_4} + (\sqrt{2}, 1, 0, 0)_{B_4}.
(1, 0, 1, 0)_{F_4} = (1, 0, 1, 1)_{B_4} + (1 + \sqrt{2}, 0, 1, 0)_{B_4}.
(1, 0, 1, 1)_{F_4} = (1, 1, 1, 1)_{B_4} + (1 + \sqrt{2}, 1, 1, 0)_{B_4}.
(1, 1, 0, 0)_{F_4} = (0, 0, \sqrt{2}, 1)_{B_4} + (\sqrt{2}, 0, 0, 2)_{B_4} + (2\sqrt{2}, 0, 0, 1)_{B_4}.
(1, 1, 0, 1)_{F_4} = (0, 1, \sqrt{2}, 1)_{B_4} + (\sqrt{2}, 1, 0, 2)_{B_4} + (2\sqrt{2}, 1, 0, 1)_{B_4}.
(1, 1, 1, 0)_{F_4} = (1, 0, 1 + \sqrt{2}, 1)_{B_4} + (1 + \sqrt{2}, 0, 1, 2)_{B_4} + (1 + 2\sqrt{2}, 0, 1, 1)_{B_4}.
(1, 1, 1, 1)_{F_4} = (1, 1, 1 + \sqrt{2}, 1)_{B_4} + (1 + \sqrt{2}, 1, 1, 2)_{B_4} + (1 + 2\sqrt{2}, 1, 1, 1)_{B_4}.

Appendix 2

The decomposition of the $W(F_4)$ weights under $W(B_3)$ is given as follows:

$(0, 0, 0, 1)_{F_4} =$

$\{(0, 0, 1)_{B_3} \pm (\frac{1}{\sqrt{2}})\} + \{(0, 1, 0)_{B_3} \pm (0)\}.$

$(0, 0, 1, 0)_{F_4} =$

$\{(0, 1, 0)_{B_3} \pm (\sqrt{2})\} + \{(0, 1, 1)_{B_3} \pm (\frac{1}{\sqrt{2}})\} + \{(\sqrt{2}, 0, 1)_{B_3} \pm (0)\}.$

$(0, 0, 1, 1)_{F_4} =$

$\{(0, 1, 1)_{B_3} \pm (\frac{3}{\sqrt{2}})\} + \{(0, 1, 2)_{B_3} \pm (\sqrt{2})\} + \{(0, 2, 1)_{B_3} \pm (\frac{1}{\sqrt{2}})\} + \{(\sqrt{2}, 1, 1)_{B_3} \pm (0)\}.$

$(0, 1, 0, 0)_{F_4} =$

$\{(0, \sqrt{2}, 0)_{B_3} \pm (1)\} + \{(1, 0, 0)_{B_3} \pm (\frac{3}{2})\} + \{(1, 0, \sqrt{2})_{B_3} \pm (\frac{1}{2})\} + \{(2, 0, 0)_{B_3} \pm (0)\}.$

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\((0, 1, 0, 1)_{F_4} =\)
\{ (0, \sqrt[2]{2}, 1)_{B_3} \pm (1 + \frac{1}{\sqrt{2}}) \} + \{ (0, 1 + \sqrt[2]{2}, 0)_{B_3} \pm (1) \} + \{ (1, 0, 1)_{B_3} \pm (\frac{3}{2} + \frac{1}{\sqrt{2}}) \} + \{ (1, 0, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{1}{2} + \frac{1}{\sqrt{2}}) \} + \{ (1, 1, \sqrt[2]{2})_{B_3} \pm (\frac{1}{2}) \} + \{ (2, 1, 0)_{B_3} \pm (0) \}.

\((0, 1, 1, 0)_{F_4} =\)
\{ (0, 1 + \sqrt[2]{2}, 0)_{B_3} \pm (1 + \sqrt[2]{2}) \} + \{ (0, 1 + \sqrt[2]{2}, 1)_{B_3} \pm (1 + \frac{1}{\sqrt{2}}) \} + \{ (1, 1, 0)_{B_3} \pm (\frac{3}{2} + \sqrt[2]{2}) \} + \{ (1, 1, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{1}{2} + \frac{1}{\sqrt{2}}) \} + \{ (1 + \sqrt[2]{2}, 0, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{1}{2}) \} + \{ (2 + \sqrt[2]{2}, 0, 1)_{B_3} \pm (0) \}.

\((0, 1, 1, 1)_{F_4} =\)
\{ (0, 1 + \sqrt[2]{2}, 1)_{B_3} \pm (1 + \frac{1}{\sqrt[2]{2}}) \} + \{ (0, 1 + \sqrt[2]{2}, 2)_{B_3} \pm (1 + \sqrt[2]{2}) \} + \{ (0, 2 + \sqrt[2]{2}, 1)_{B_3} \pm (1 + \frac{1}{\sqrt{2}}) \} + \{ (1, 1, 1)_{B_3} \pm (\frac{1}{2} + \sqrt[2]{2}) \} + \{ (1 + \sqrt[2]{2}, 0, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{3}{2} + \sqrt[2]{2}) \} + \{ (1, 2, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{1}{2} + \frac{1}{\sqrt{2}}) \} + \{ (1 + \sqrt[2]{2}, 1, 1)_{B_3} \pm (0) \}.

\((1, 0, 0, 0)_{F_4} =\)
\{ (0, 0, 0)_{B_3} \pm (1) \} + \{ (0, 0, \sqrt[2]{2})_{B_3} \pm (0) \} + \{ (1, 0, 0)_{B_3} \pm (\frac{1}{2}) \}.

\((1, 0, 0, 1)_{F_4} =\)
\{ (0, 0, 1)_{B_3} \pm (1 + \frac{1}{\sqrt{2}}) \} + \{ (0, 0, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{1}{\sqrt{2}}) \} + \{ (0, 1, \sqrt[2]{2})_{B_3} \pm (0) \} + \{ (1, 0, 1)_{B_3} \pm (\frac{1}{2} + \frac{1}{\sqrt{2}}) \} + \{ (1, 1, 0)_{B_3} \pm (\frac{1}{2}) \}.

\((1, 0, 1, 0)_{F_4} =\)
\{ (0, 1, 0)_{B_3} \pm (1 + \sqrt[2]{2}) \} + \{ (0, 1, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{1}{2} + \sqrt[2]{2}) \} + \{ (1, 1, 0)_{B_3} \pm (\frac{1}{2} + \frac{1}{\sqrt{2}}) \} + \{ (1, 1, 1)_{B_3} \pm (\frac{3}{2} + \sqrt[2]{2}) \} + \{ (1 + \sqrt[2]{2}, 0, 1 + \sqrt[2]{2})_{B_3} \pm (0) \} + \{ (1 + \sqrt[2]{2}, 0, 1)_{B_3} \pm (\frac{1}{2}) \}.

\((1, 0, 1, 1)_{F_4} =\)
\{ (0, 1, 1)_{B_3} \pm (1 + \frac{3}{\sqrt{2}}) \} + \{ (0, 1, 2 + \sqrt[2]{2})_{B_3} \pm (\sqrt[2]{2}) \} + \{ (0, 2, 1 + \sqrt[2]{2})_{B_3} \pm (\frac{1}{2} + \frac{1}{\sqrt{2}}) \} + \{ (1, 1, 1)_{B_3} \pm (\frac{1}{2} + \frac{3}{\sqrt{2}}) \} + \{ (1, 1, 2)_{B_3} \pm (\frac{1}{2} + \sqrt[2]{2}) \} + \{ (1, 2, 1)_{B_3} \pm (\frac{1}{2} + \frac{1}{\sqrt{2}}) \} + \{ (\sqrt[2]{2}, 1, 1 + \sqrt[2]{2})_{B_3} \pm (0) \} + \{ (1 + \sqrt[2]{2}, 1, 1)_{B_3} \pm (\frac{1}{2}) \}.

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\((1, 1, 0, 0)_{F_4} = \)
\[
\{(1, 0, 0)_{B_3} \pm \left(\frac{\sqrt{2}}{2}\right)\} + \{(1, 0, 2\sqrt{2})_{B_3} \pm \left(\frac{1}{2}\right)\} + \{(1, \sqrt{2}, 0)_{B_3} \pm \left(\frac{\sqrt{3}}{3}\right)\} + \{(2, 0, 0)_{B_3} \pm (2)\} + \{(2, 0, \sqrt{2})_{B_3} \pm (1)\} + \{(3, 0, 0)_{B_3} \pm \left(\frac{1}{2}\right)\}.
\]
\[(1, 1, 0, 1)_{F_4} = \)
\[
\{(1, 1, 0)_{B_3} \pm \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\} + \{(1, 0, 1 + 2\sqrt{2})_{B_3} \pm \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\} + \{(1, 1, 2\sqrt{2})_{B_3} \pm \left(\frac{1}{2}\right)\} + \{(1, \sqrt{2}, 1)_{B_3} \pm \left(\frac{\sqrt{3}}{3} + \frac{1}{\sqrt{2}}\right)\} + \{(1, 1 + \sqrt{2}, 0)_{B_3} \pm \left(\frac{\sqrt{3}}{3}\right)\} + \{(2, 0, 1)_{B_3} \pm (2 + \sqrt{2})\} + \{(2, 0, 1 + \sqrt{2})_{B_3} \pm (1 + \frac{1}{\sqrt{2}})\} + \{(2, 1, \sqrt{2})_{B_3} \pm (1)\} + \{(3, 1, 0)_{B_3} \pm \left(\frac{1}{2}\right)\}.
\]
\[(1, 1, 1, 0)_{F_4} = \)
\[
\{(1, 1, 0)_{B_3} \pm \left(\frac{\sqrt{2}}{2} + \sqrt{2}\right)\} + \{(1, 1, 1 + 2\sqrt{2})_{B_3} \pm \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\} + \{(1, 1 + \sqrt{2}, 0)_{B_3} \pm \left(\frac{\sqrt{3}}{3} + \sqrt{2}\right)\} + \{(1, 1 + \sqrt{2}, 1)_{B_3} \pm \left(\frac{\sqrt{3}}{3} + \frac{1}{\sqrt{2}}\right)\} + \{(2, 1, 0)_{B_3} \pm (2 + \sqrt{2})\} + \{(2, 1, 1 + \sqrt{2})_{B_3} \pm (1 + \frac{1}{\sqrt{2}})\} + \{(1 + \sqrt{2}, 0, 1 + 2\sqrt{2})_{B_3} \pm \left(\frac{1}{2}\right)\} + \{(2 + \sqrt{2}, 0, 1 + \sqrt{2})_{B_3} \pm (1)\} + \{(3 + \sqrt{2}, 0, 1)_{B_3} \pm \left(\frac{1}{2}\right)\}.
\]
\[(1, 1, 1, 1)_{F_4} = \)
\[
\{(1, 1, 0)_{B_3} \pm \left(\frac{\sqrt{2}}{2} + \sqrt{2}\right)\} + \{(1, 1, 2 \left(1 + \sqrt{2}\right))_{B_3} \pm \left(\frac{1}{2} + \sqrt{2}\right)\} + \{(1, 2, 1 + 2\sqrt{2})_{B_3} \pm \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\} + \{(1, 1 + \sqrt{2})_{B_3} \pm \left(\frac{3}{2} \left(1 + \sqrt{2}\right)\right)\} + \{(1, 1 + \sqrt{2}, 2)_{B_3} \pm \left(\frac{3}{2} + \sqrt{2}\right)\} + \{(1, 1 + \sqrt{2}, 1)_{B_3} \pm \left(\frac{3}{2} \left(1 + \sqrt{2}\right)\right)\} + \{(1, 2 + \sqrt{2}, 1)_{B_3} \pm \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\right)\} + \{(2, 1, 1)_{B_3} \pm \left(2 + \frac{1}{\sqrt{2}}\right)\} + \{(2, 1, 2 + \sqrt{2})_{B_3} \pm \left(1 + \sqrt{2}\right)\} + \{(2, 2, 1 + \sqrt{2})_{B_3} \pm \left(1 + \frac{1}{\sqrt{2}}\right)\} + \{(1 + \sqrt{2}, 1, 1 + 2\sqrt{2})_{B_3} \pm \left(\frac{1}{2}\right)\} + \{(2 + \sqrt{2}, 1, 1 + \sqrt{2})_{B_3} \pm (1)\} + \{(3 + \sqrt{2}, 1, 1)_{B_3} \pm \left(\frac{1}{2}\right)\}.
\]

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