A version of Waldhausen’s chromatic convergence for $TC$

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Abstract
The map $TC(S)^\wedge_p \to \text{holim} TC(L_n S)^\wedge_p$ is a weak equivalence.

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1 | INTRODUCTION

Motivated by the chromatic program in stable homotopy theory, in general, and chromatic convergence of the sphere spectrum in particular, Waldhausen [26] proposed studying the interaction of the chromatic tower with algebraic $K$-theory and made some specific conjectures about what happens for the $p$-local sphere spectrum. Waldhausen conjectured that $K(S(p))$ would be the homotopy limit of the algebraic $K$-theory spectra of $L_n S$, which would inductively be built from the algebraic $K$-theory of the monochromatic categories. The monochromatic categories are simpler than the $p$-local stable category and Waldhausen asked whether their algebraic $K$-theories would be correspondingly simpler to understand than the algebraic $K$-theory of the $p$-local sphere spectrum.

The advent of trace methods and the pioneering work of McClure and Staffeldt in calculating $THH$ of the Adams summand $\ell$ reinvigorated this program. Ausoni and Rognes [2] calculated the periodic homotopy groups of $K(\ell)$ in terms of $TC(\ell)$. As part of this work, Rognes made a series of far-reaching conjectures about the interaction of algebraic $K$-theory and chromatic localization. Most notably, he formulated a redshift conjecture about $K$-theory increasing chromatic complexity, and he formulated a higher chromatic Quillen–Lichtenbaum conjecture about
Galois descent for algebraic $K$-theory spectra. Rognes’ original formulations involved the localizations $L_n$ (in place of $L_n^f$), the $p$-complete sphere, and a $p$-complete version of Waldhausen’s chromatic convergence conjecture, $K(S^\wedge_p) \simeq \holim_n K(L_n S^\wedge_p)$.

When working with the $p$-complete sphere, there is very little difference between algebraic $K$-theory and $TC$: work of Hesselholt–Madsen [16, Thm. D] and Dundas [12] shows that the cyclotomic trace $K(S^\wedge_p) \wedge p \to TC(S^\wedge_p)$ is a weak equivalence on connective covers. As a consequence, one is led to consider the $TC$ analogues of Waldhausen’s chromatic convergence conjectures. The purpose of this paper is to prove that analogue.

**Theorem 1.1.** The natural map

$$TC(S^\wedge_p) \wedge p \to \holim_n TC(L_n(S^\wedge_p)) \wedge p$$

is a weak equivalence.

The ring spectra $L_n(S^\wedge_p)$ are non-connective, and for non-connective ring spectra, we now have various potentially distinct versions of $TC(-)^\wedge_p$. Theorem 1.1 holds using either the original constructions $TC(-)^\wedge_p$ and $TC(-; p)^\wedge_p$ of Goodwillie [14] and Bökstedt–Hsiang–Madsen [7] (which are equivalent for all ring spectra), or the constructions $TC(-)^\wedge_p$ and $TC(-; p)^\wedge_p$ of Nikolaus–Scholze [23, II.1.8].

McClure and Staffeldt [22] proved a version of Waldhausen’s chromatic convergence conjecture for the connective covers $\tau \geq 0 L_n S$ in place of $L_n S$ using a direct calculation involving the plus construction. Their argument adapts in outline to $TC$ and we have a corresponding result in this case.

**Theorem 1.2.** The natural map

$$TC(S^\wedge_p) \wedge p \to \holim_n TC(\tau \geq 0 L_n(S^\wedge_p)) \wedge p$$

is a weak equivalence.

In fact, the preceding theorem holds more generally with $S$ replaced by a connective ring spectrum with the same strong chromatic convergence properties as $S$; see Corollary 3.3 for more details.

One might hope to deduce the previous theorem directly from the McClure-Staffeldt $K$-theory result using the Dundas–Goodwillie–McCarthy pullback square. However, our current state of knowledge of $\pi_0 L_n S$ makes this infeasible, and so we give a direct proof along the lines of the McClure–Staffeldt argument instead. The McClure–Staffeldt argument for Theorem 1.2 fundamentally relies on the Chromatic Convergence Theorem of Hopkins and Ravenel [25, 7.5.7] applied to $S$ or $S^\wedge_p$, propagating it through the construction of $TC$. Whereas in the $K$-theory case considered by McClure–Staffeldt, work of Bousfield–Kan [9, III§3] on pro-spaces suffices for the propagation, the $TC$ case needs the more sophisticated theory of Isaksen [18] and Fausk-Isaksen [13] for pro-spectra and equivariant pro-spectra (see Section 3). The argument for Theorem 1.1 fits a similar outline but uses the weak equivalence $THH(L_n S) \simeq L_n S$ to avoid pro-object arguments and requires chromatic convergence at a later step: it replaces the Hopkins–Ravenel Chromatic Convergence Theorem with Barthel’s criterion for chromatic
convergence \([3, 3.8]\) applied to \(\Sigma^\infty_+ BC_{p^k}\) (see Section 4). Although not directly related to Waldhausen’s conjecture or the Rognes program, Barthel’s criterion also gives the following chromatic convergence result, which is superficially similar to Theorem 1.1 and also proved in Section 4.

**Theorem 1.3.** The natural map

\[
TC(S_p^\wedge)^\wedge_p \rightarrow \text{holim}_n(L_n TC(S_p^\wedge))^\wedge_p
\]

is a weak equivalence.

We have stated Theorems 1.1, 1.2, and 1.3 in terms of \(S_p^\wedge\), but the corresponding results with \(S^\wedge_p\) replaced everywhere by \(S\) or \(S_{(p)}\) also hold; see Theorem 2.6. Indeed, the map \(TC(R)^\wedge_p \rightarrow TC(R^\wedge_p)^\wedge_p\) is a weak equivalence quite generally; we give precise statements in Propositions 2.2, 2.4, and 2.5. In the proofs, we concentrate on the case of \(S_{(p)}\), which implies the other cases.

**Conventions**

In this paper, when a point-set category of spectra is necessary or implicit in a statement, we understand “spectrum” to mean orthogonal spectrum indexed on \(\{\mathbb{R}^n\}\); “ring spectrum” means associative ring orthogonal spectrum (or the equivalent) and “commutative ring spectrum” means commutative ring orthogonal spectrum (or the equivalent). We use \(𝕋\) to denote the circle group (the unit length complex numbers) and \(C_m < 𝕋\) its cyclic subgroup of order \(m\).

2 | **TC OF A RING SPECTRUM AND ITS \(p\)-COMPLETION**

In this section, we give a concise review of the construction of topological cyclic homology in order to introduce notation and details used in later sections. We discuss both the “classic” definition of a cyclotomic spectrum as introduced by Goodwillie [14] and Bökstedt–Hsiang–Madsen [7] as well as the recent reformulation by Nikolaus–Scholze in terms of the Tate fixed point spectrum.

In the classic case, definitions of point-set categories of cyclotomic spectra and/or \(p\)-cyclotomic spectra can be found, for example, in \([8, 1.1]\) (original source), \([20, 2.4.4]\), \([16, 2.2]\), \([17, \S 1.1]\), to name a few. The notion of the homotopy theory and therefore \(∞\)-category of classic cyclotomic spectra and \(p\)-cyclotomic spectra are clear but implicit in these formulations; the paper \([5]\) sets up a model* category of \(p\)-cyclotomic spectra for the purpose of making this explicit. A model* category is a category with subcategories of cofibrations, fibrations, and weak equivalences that satisfy the usual axioms of a model category but do not necessarily admit all finite limits and colimits, but only finite coproducts and products, pushouts over cofibrations, and pullbacks over fibrations; Quillen’s homotopy theory for model categories works verbatim for such a category. We then get an \(∞\)-category from the model* structure the same way as we would for a model category.

The model* category of cyclotomic spectra starts with the model category of genuine \(𝕋\)-equivariant orthogonal spectra localized at the \(\mathcal{F}_p\)-equivalences, meaning that a map \(X \rightarrow Y\) is a weak equivalence when it is a weak equivalence on (derived genuine categorical) \(C_{p^k}\)-fixed points

\[
X^{C_{p^k}} \rightarrow Y^{C_{p^k}}
\]
for all \(k \geq 0\). (Here, “genuine” specifies the model structure; the homotopy category of this model structure is stable with respect all finite dimensional orthogonal \(\mathbb{T}\)-representations: for \(V\) any orthogonal \(\mathbb{T}\)-action on \(\mathbb{R}^n\) for some \(n\), the endofunctor \(\Sigma^V\) is an equivalence). The \(C_p\) geometric fixed point functor \(\Phi^{C_p}\) (or \(X \mapsto X^{\Phi^{C_p}}\)) goes from \(\mathbb{T}\)-spectra to \(\mathbb{T}/C_p\)-spectra and induces a left derived functor \(L\Phi\) from genuine \(\mathbb{F}_p\)-local \(\mathbb{T}\)-spectra to genuine \(\mathbb{F}_p\)-local \(\mathbb{T}/C_p\)-spectra. The \(p\)th root isomorphism \(\rho^*_p\) gives an equivalence of categories and a Quillen equivalence of model categories \(\rho^*_p\) from genuine \(\mathbb{F}_p\)-local \(\mathbb{T}/C_p\)-spectra to genuine \(\mathbb{F}_p\)-local \(\mathbb{T}\)-spectra. The structure map of a classic \(p\)-cyclotomic spectrum is a map

\[
r : \rho^*_p X^{\Phi^{C_p}} \to X
\]
that induces a weak equivalence for the derived functor

\[
\rho^*_p X^{L\Phi^{C_p}} \to \rho^*_p X^{\Phi^{C_p}} \to X.
\]

The category of \(p\)-cyclotomic spectra is the category of structure maps; the model* structure has the fibrations and weak equivalences created on the underlying \(\mathbb{T}\)-spectra in the \(\mathbb{F}_p\)-local genuine model structure. See Sections 2–5 of [5] for more details.

Construction 5.11 and Corollary 5.13 in [5] identify the mapping spectrum in \(p\)-cyclotomic spectra \(\mathcal{F}_{cyc}(X, Y)\) (for \(p\)-cyclotomic spectra \(X, Y\)). When \(X\) is cofibrant and \(Y\) is fibrant, the underlying mapping space is the space of homotopy commuting diagrams of structure maps; this in particular identifies the \(\infty\)-category of \(p\)-cyclotomic spectra as the full subcategory of the lax equalizer (in the sense of [23, II.1.4]) from the \(C_p\) geometric fixed point to the identity functors of those objects for which the map is an isomorphism.

Barwick–Glasman [4] sets up an equivalent \(\infty\)-category of \(p\)-cyclotomic spectra in a different \(\infty\)-categorical framework using a spectral Mackey functor inspired model for equivariant stable homotopy theory. This theory contains features equivalent to those discussed in the previous paragraph (cf. the proof of Theorem 3.23 there); the remainder of this section and Section 4 adapt to the Barwick–Glasman [4] context without difficulty. We expect that the arguments in Section 3 also adapt either by choosing specific point-set models or by adapting the model theoretic arguments of Isaksen [18] and Fausk–Isaksen [13] used there.

A classic \(p\)-cyclotomic spectrum \(X\) comes with two maps \(X^{C_p^k} \to X^{C_p^{k-1}}:\)

\(F\): the canonical inclusion of fixed points, and
\(R\): the composite

\[
X^{C_p^k} \cong (\rho^*_p X^{C_p})^{C_p^{k-1}} \to (\rho^*_p X^{\Phi^{C_p}})^{C_p^{k-1}} \cong X^{C_p^{k-1}}
\]

induced by the canonical map from the fixed points to geometric fixed points and the \(p\)-cyclotomic structure map.

**Definition 2.1 (Classic \(TC\)).** For a classic \(p\)-cyclotomic spectrum \(X\) and \(k > 0\), let

\[
TC^k(X; p) = \text{hoeq}(R, F : X^{C_p^k} \rightrightarrows X^{C_p^{k-1}}),
\]

the homotopy equalizer of \(R, F\). Let

\[
TC(X; p) = \text{holim}_k TC^k(X; p).
\]
Goodwillie [14, 12.1] first proposed an integral form of \( TC \) for cyclotomic spectra. It is easier to take its fundamental property [14, 14.2] and turn it into a definition as in Dundas–Goodwillie–McCarthy [11, VI.4.3.1]: define \( TC(X) \) as the homotopy pullback

\[
\begin{array}{ccc}
TC(X) & \longrightarrow & X^{h\mathbb{T}} \\
\downarrow & & \downarrow \\
\prod_p TC(X; p)^{\wedge} & \longrightarrow & \prod_p \left( \lim_k X^{hC_p^k} \right)^{\wedge}_p
\end{array}
\]

where the map \( TC(X; p) \to X^{hC_p^k} \) is the composite

\[
TC(X; p) \longrightarrow TC^k(X; p) \longrightarrow X^{C_p^k} \longrightarrow X^{hC_p^k}.
\]

We note that the map \( X^{h\mathbb{T}} \to \lim_k X^{hC_p^k} \) becomes a weak equivalence after \( p \)-completion (see, e.g., [11, VI.3.1.1]); this immediately implies the following well-known result.

**Proposition 2.2** Goodwillie [14, 14.1.(ii)]. Let \( X \) be a classic cyclotomic spectrum. The map \( TC(X)_p^{\wedge} \to TC(X; p)_p^{\wedge} \) is a weak equivalence.

Because of this proposition, we can use \( TC(X; p) \) exclusively in the classic context and deduce results for \( TC(X)_p^{\wedge} \) from the equivalence.

Nikolaus–Scholze [23, II.1.6] redefine cyclotomic spectra and \( p \)-cyclotomic spectra in terms of Borel equivariant spectra: for a topological group \( G \), the homotopy theory or \( \infty \)-category of Borel \( G \)-spectra is represented by the (relative) category of (left) \( G \)-objects in orthogonal spectra, where we take the weak equivalences to be maps that are weak equivalences of the underlying spectra. The underlying object of a cyclotomic spectrum is a Borel \( \mathbb{T} \)-spectrum. The underlying object of a \( p \)-cyclotomic spectrum is a Borel \( C_{p^{\infty}} \)-spectrum, where \( C_{p^{\infty}} = \lim_k C_p^k \). The structure map for a \( p \)-cyclotomic spectrum \( X \) is a map of Borel \( C_{p^{\infty}} \)-spectra

\[
\varphi_p : X \longrightarrow \rho_p^*(X^{tC_p})
\]

where \( tC_p \) denotes the Tate fixed point spectrum (the fixed points of the Tate spectrum of [15, p. 3]) and \( \rho_p \) is (as above) the \( p \)th root isomorphism \( \mathbb{T} \cong \mathbb{T}/C_p \). A cyclotomic spectrum has such a structure map \( \varphi_p \) for each \( p \), which is required to be a map of Borel \( \mathbb{T} \)-spectra, and we write \( \varphi \) for the induced map into the product

\[
\varphi : X \longrightarrow \prod_p \rho_p^*(X^{tC_p}).
\]

Precisely, the \( C_p \)-Tate fixed points is a functor from Borel \( \mathbb{T} \)-spectra to Borel \( \mathbb{T}/C_p \)-spectra or from Borel \( C_{p^{\infty}} \)-spectra to Borel \( C_{p^{\infty}}/C_p \)-spectra and the composite with \( \rho_p^* \) then gives an endofunctor of Borel \( \mathbb{T} \)-spectra or Borel \( C_{p^{\infty}} \)-spectra. The \( \infty \)-category of \( p \)-cyclotomic spectra is the lax equalizer (in the sense of [23, II.1.4]) from the identity to the functor \( \rho_p^*(-)^{tC_p} \) and the \( \infty \)-category of cyclotomic spectra is the lax equalizer from the identity to the product (over primes \( p \)) of the functors \( \rho_p^*(-)^{tC_p} \) [23, II.1.6].
For a Borel $\mathbb{T}$-spectrum or Borel $C_p^{\infty}$-spectrum $X$, we have a canonical map of Borel $\mathbb{T}/C_p$-spectra or Borel $C_p^{\infty}/C_p$-spectra from the $C_p$ homotopy fixed points to the $C_p$-Tate fixed points

$$X^{hC_p} \longrightarrow X^{tC_p}$$

and a canonical isomorphism

$$X^{h\mathbb{T}} \cong (\rho_p^*(X^{hC_p}))^{h\mathbb{T}} \quad \text{or} \quad X^{hC_p^{\infty}} \cong (\rho_p^*(X^{hC_p}))^{hC_p^{\infty}}.$$

We then obtain a canonical map

$$\text{can} : X^{h\mathbb{T}} \longrightarrow \prod_p (\rho_p^*(X^{tC_p}))^{h\mathbb{T}} \quad \text{or} \quad \text{can} : X^{hC_p^{\infty}} \longrightarrow (\rho_p^*(X^{tC_p}))^{hC_p^{\infty}}.$$

In the case when $X$ is a cyclotomic spectrum or $p$-cyclotomic spectrum, we have another map with the same domain and codomain obtained by taking the homotopy fixed points of $\varphi$

$$\varphi^{h\mathbb{T}} : X^{h\mathbb{T}} \longrightarrow \prod_p (\rho_p^*(X^{tC_p}))^{h\mathbb{T}} \quad \text{or} \quad \varphi^{hC_p^{\infty}} : X^{hC_p^{\infty}} \longrightarrow (\rho_p^*(X^{tC_p}))^{hC_p^{\infty}}.$$

Following [5], Nikolaus–Scholze [23, II.1.8] define $TC(X)$ and $TC(X; p)$ as mapping spectra but show in [23, II.1.9] that they may be computed as the homotopy equalizer of $\text{can}$ and $\varphi^{h\mathbb{T}}$ or $\text{can}$ and $\varphi^{hC_p^{\infty}}$; we use this theorem as a definition.

**Definition 2.3** (Nikolaus–Scholze $TC$ [23, II.1.9]). For a Nikolaus–Scholze cyclotomic spectrum $X$, define

$$TC(X) = \text{hoeq}(\text{can}, \varphi^{h\mathbb{T}} : X^{h\mathbb{T}} \Rightarrow \prod_p (\rho_p^*(X^{tC_p}))^{h\mathbb{T}}).$$

For a Nikolaus–Scholze $p$-cyclotomic spectrum $X$, define

$$TC(X; p) = \text{hoeq}(\text{can}, \varphi^{hC_p^{\infty}} : X^{hC_p^{\infty}} \Rightarrow (\rho_p^*(X^{tC_p}))^{hC_p^{\infty}}).$$

We have a forgetful functor from cyclotomic spectra to $p$-cyclotomic spectra and projection onto the $p$ factor induces a map $TC(X) \rightarrow TC(X; p)$. The following proposition is well-known and implicit in [23].

**Proposition 2.4.** Let $X$ be a Nikolaus–Scholze cyclotomic spectrum. If $X$ is either bounded below or $p$-local, then the map $TC(X)_p^\wedge \rightarrow TC(X; p)_p^\wedge$ is a weak equivalence.

**Proof.** Let $\ell \neq p$ be prime. If $X$ is $p$-local, then multiplication by $\ell$ is an isomorphism on $X$; the canonical maps

$$(\pi_*X)/C_\ell \longrightarrow \pi_*(X_{hC_\ell}) \quad \text{and} \quad \pi_*(X^{hC_\ell}) \longrightarrow (\pi_*(X))^{C_\ell}$$

are then isomorphisms and $X^{tC_\ell}$ is trivial. If $X$ is bounded below, then by [23, I.2.9], $X^{tC_\ell}$ is $\ell$-complete, $(\rho_\ell^*(X^{tC_\ell}))^{h\mathbb{T}}$ is $\ell$-complete, and the $p$-completion of $(\rho_\ell^*(X^{tC_\ell}))^{h\mathbb{T}}$ is trivial. Thus, if $X$
is either $p$-local or bounded below, the map

$$TC(X) \rightarrow \text{hoeq}(\text{can, } \varphi_p^{\text{HT}} : X^{\text{HT}} \Rightarrow (\rho_p^{\text{IC}} X)^{\text{HT}}$$

is a $p$-equivalence. The maps

$$X^{h\mathbb{T}} \rightarrow X^{hC_p^\infty} \quad \text{and} \quad (\rho_p^{\text{IC}} X)^{h\mathbb{T}} \rightarrow (\rho_p^{\text{IC}} X)^{hC_p^\infty}$$

are also $p$-equivalences (for example, by [11, A.7.6.4]) and it follows that the map $TC(X)^{\wedge}_p \rightarrow TC(X; p)^{\wedge}_p$ is a weak equivalence.

In particular, if all cyclotomic spectra under consideration are either connective or $p$-local, as is the case in the statement of Theorem 1.1, results on $TC(\_, p)^{\wedge}_p$ automatically imply the corresponding results on $TC(\_, p)^{\wedge}_p$.

The Nikolaus–Scholze setting also has an analogue of $TC^k(X; p)$: let

$$TC^k(X; p) = \text{hoeq}(\text{can, } \varphi_p^{hC_p^{k-1}} : X^{hC_p^k} \Rightarrow (X^{IC_p})^{hC_p^{k-1}}).$$

Since $(-)^{hC_p^\infty} \cong \text{holim}(-)^{hC_p^k}$, we then have an equivalence

$$TC(X; p) \cong \text{holim}_k TC^k(X; p).$$

Nikolaus–Scholze [23, II.4.10] (and its proof) show that when $X$ is bounded below, we have a weak equivalence from classic $TC^k(X; p)$ to Nikolaus–Scholze $TC^k(X; p)$, which in the homotopy limit gives a weak equivalence from classic $TC(X; p)$ to Nikolaus–Scholze $TC(X; p)$.

Finally, for a ring spectrum $R$, we get a cyclotomic spectrum $THH(R)$, first constructed by Bökstedt [6]. Now, several equivalent constructions exist [1], [23, III]; see [10, §1] for a discussion and comparison. We caution that it is standard abuse of notation to write $TC(R)$ and $TC(R; p)$ for $TC(THH(R))$ and $TC(THH(R); p)$, and this usually causes no confusion.

We need the fact from Patchkoria–Sagave [24, 3.8] that as a Borel $\mathbb{T}$-spectrum $THH(R)$ is weakly equivalent to the (left derived) cyclic bar construction $N^{C_\mathbb{F}}(R)$, constructed out of (derived) smash powers of $R$. It is a fact inherited from the smash power that when $R$ is connective, so is $THH(R)$ and also that the natural map

$$THH(R)^{\wedge}_p \rightarrow THH(R_p^{\wedge})^\wedge_p$$

is a weak equivalence of Borel $\mathbb{T}$-spectra, and a fortiori (see [5, 5.5]) a weak equivalence of both classic and Nikolaus–Scholze cyclotomic spectra.

It is well-known to experts (see, e.g., [16, Add. 6.2] or [5, 6.10]) that the $p$-completion of $TC(X; p)$ only depends on the $p$-completion of $X$. In the classic case, this is because $p$-completion commutes with fixed points. In the Nikolaus–Scholze case, this is because $p$-completion commutes with homotopy fixed points and the natural map

$$(X^{IC_p})^{\wedge}_p \rightarrow ((X^{\wedge})^{IC_p})^{\wedge}_p$$

is a weak equivalence for any Borel $C_p$-spectrum. We summarize in the following proposition.
**Proposition 2.5.** In either the classic or Nikolaus–Scholze setting, for any $p$-cyclotomic spectrum $X$, the natural map

$$TC(X; p)_p^\wedge \to TC(X^\wedge; p)_p^\wedge$$

is a weak equivalence and for any ring spectrum $R$, the natural map

$$TC(R; p)_p^\wedge \to TC(R^\wedge; p)_p^\wedge$$

is a weak equivalence.

For the purposes of our main results, we have proved the following theorem.

**Theorem 2.6.** In either the classic or the Nikolaus–Scholze setting, all pictured natural maps

$$\begin{array}{ccc}
TC(S)_p^\wedge & \to & TC(S(p))_p^\wedge \\
\downarrow & & \downarrow \\
TC(S; p)_p^\wedge & \to & TC(S(p); p)_p^\wedge \\
\downarrow & & \downarrow \\
TC(\tau_{\geq 0}L_n(S))_p^\wedge & \to & TC(\tau_{\geq 0}L_n(S(p)))_p^\wedge \\
\downarrow & & \downarrow \\
TC(\tau_{\geq 0}L_n(S); p)_p^\wedge & \to & TC(\tau_{\geq 0}L_n(S(p); p)_p^\wedge \\
\downarrow & & \downarrow \\
TC(L_n(S))_p^\wedge & \to & TC(L_n(S(p)))_p^\wedge \\
\downarrow & & \downarrow \\
TC(L_n(S; p))_p^\wedge & \to & TC(L_n(S(p); p)_p^\wedge \\
\end{array}$$

are weak equivalences.

### 3 THE HOPKINS–RAVENEL CHROMATIC CONVERGENCE THEOREM AND $TC(S(p))$

In this section, we apply the Hopkins–Ravenel Chromatic Convergence Theorem [25, 7.5.7] to prove Theorem 1.2. Specifically, Hopkins and Ravenel prove that the canonical map

$$S(p) \to \operatorname{holim}_n L_n S$$

is a weak equivalence; indeed (it is well-known that) the argument shows that the map induces a pro-isomorphism on $\pi_q$ for each $q$. The proof of Theorem 1.2 is to propagate the pro-isomorphism on homotopy groups through the $TC$ construction. We prove the following theorem.

**Theorem 3.1.** For any fixed $k \geq 0$, the map of pro-spectra $\{TC^k(S(p); p)\} \to \{TC^k(\tau_{\geq 0}L_n S; p)\}$ induces a pro-isomorphism on $\pi_q$ pro-groups for all $q \in \mathbb{Z}$.

In fact, the corresponding result holds for any connective ring spectrum that has the same strong chromatic convergence as the sphere spectrum: the previous theorem is just a special case of the following more general theorem.
Theorem 3.2. Let $R$ be a connective associative ring spectrum with the property that the system of maps $R \to L_n R$ induces a pro-isomorphism on $\pi_q$ for all $q \geq 0$. Then for any fixed $k \geq 0$, the map of pro-spectra $\{TC^k(R; p)\} \to \{TC^k(\tau_{\geq 0} L_n R; p)\}$ induces a pro-isomorphism on $\pi_q$ pro-groups for all $q \in \mathbb{Z}$.

Taking the homotopy limit
\[
TC(R; p) = \lim_k TC^k(R; p) \Rightarrow \lim_k \lim_n TC^k(\tau_{\geq 0} L_n R; p)
\]
we then get an $R$ version of Theorem 1.2.

Corollary 3.3. If $R$ satisfies the hypotheses of Theorem 3.2, then the natural map
\[
TC(R; p) \longrightarrow \lim_n TC(\tau_{\geq 0} L_n R; p)
\]
is a weak equivalence.

Specializing to the case $R = \mathbb{S}(p)$ and using Theorem 2.6, Corollary 3.3 then implies Theorem 1.2. Because $\mathbb{S}(p)$ and $\tau_{\geq 0} L_n \mathbb{S}$ are all connective spectra, both flavors of $TC$ are equivalent, and we only need to prove the theorem for classic $TC$; we discuss only this case.

We now fix a connective ring spectrum $R$ satisfying the hypothesis of Theorem 3.2. The propagation argument uses results of Isaksen [18] and Fausk–Isaksen [13] on the homotopy theory of pro-spectra. We emphasize that this theory takes place in the point-set category of pro-objects in a point-set category of spectra, in our case, the category of orthogonal spectra indexed on $\{\mathbb{R}^n\}$. In particular, Theorem 3.2 refers to a fixed (but unspecified) point-set tower of functors $TC^k(\cdot; p)$, point-set models $\tau_{\geq 0} L_n R$, and system of point-set maps $R \to \tau_{\geq 0} L_n R$. As clearly such a setup exists and the details play no role in the arguments, we omit a discussion and precise specification. On the other hand, making certain specifications avoids some awkward circumlocutions in statements, so we consider only models that satisfy the following hypothesis.

Convention 3.4. The specific models used for the spectra $R$ and $\tau_{\geq 0} L_n R$ in this section are cofibrant in one of the standard (stable) model structure on associative ring orthogonal spectra or commutative ring orthogonal spectra.

We recall that a property of a spectrum or map of spectra is said to hold levelwise for a pro-spectrum or a level map of pro-spectra if the constituent spectra or maps of spectra have that property. More generally, the property is said to hold essentially levelwise on a pro-spectrum or (general) map of pro-spectra if the pro-spectrum or map is isomorphic in the pro-category to one where the property holds levelwise. In particular, a pro-spectrum is levelwise $m$-connected if its constituent spectra are $m$-connected and a level map of pro-spectra $X \to Y$ is a levelwise $m$-equivalence if each map in the system is an $m$-equivalence. A map of pro-spectra $X \to Y$ is an essentially levelwise $m$-equivalence if there exist pro-isomorphisms $X' \cong X$ and $Y \cong Y'$ such that the composite $X' \to Y'$ is represented by a level map that is a levelwise $m$-equivalence. The essentially levelwise $m$-equivalences plays a central role in the theory of [13, 18].
Definition 3.5. A map of pro-spectra is a weak equivalence (called $\pi_\ast$-weak equivalence in [18] and $\mathcal{H}_\ast$-weak equivalence in [13]) if it is an essentially levelwise $m$-equivalence for all $m \in \mathbb{Z}$.

The main result on pro-spectra we need is the following adaptation of [18, 8.4] or [13, 9.13].

Theorem 3.6 (Isaksen [18, 8.4]). Let $X = \{X_t\}$ and $Y = \{Y_t\}$ be levelwise $(-N)$-connected pro-spectra for some $N \in \mathbb{Z}$. A map of pro-spectra $X \to Y$ is a weak equivalence in the sense of Definition 3.5 if and only if it induces on $\pi_q$ a pro-isomorphism of pro-groups for all $q > -N$.

Although not stated explicitly, the proof of the theorem gives slightly more information: the pro-isomorphic spectra $X' \cong X$ and $Y \cong Y'$ for which the composite map $X' \to Y'$ is represented by a levelwise $m$-equivalence can be chosen so that the constituent spectra are also $(-N)$-connected. We get the following immediate corollary.

Corollary 3.7. For each $m \geq 0$, there exists levelwise connective pro-spectra $X^m = \{X^m_t\}$ and $Y^m = \{Y^m_t\}$, pro-isomorphisms $X^m \cong \{R\}$ and $Y^m \cong \{\tau_{\geq 0}L_nR\}$, and a levelwise $m$-equivalence $X^m \to Y^m$ such that the diagram

$$
\begin{array}{ccc}
X^m & \longrightarrow & Y^m \\
\cong & \downarrow & \cong \\
\{R\} & \longrightarrow & \{\tau_{\geq 0}L_nR\}
\end{array}
$$

commutes.

We now begin to apply this to get comparison results for the map $\{R\} \to \{\tau_{\geq 0}L_nR\}$. In the following proposition, $((-)^{(k)}$ denotes $k$-fold smash power.

Proposition 3.8. For any fixed $k \geq 1$, the map of pro-spectra $\{R^{(k)}\} \to \{n \mapsto (\tau_{\geq 0}L_nR)^{(k)}\}$ is a weak equivalence.

Proof. Let $X^m, Y^m$ be as in Corollary 3.7. Writing $((-)^{(k)}$ for the derived smash power (in the stable category), for each $t$ in the common indexing category for $X^m, Y^m$, the map

$$
(X^m_t)^{(k)} \longrightarrow (Y^m_t)^{(k)}
$$

is an $m$-equivalence. These maps assemble into a level map of pro-objects in the stable category that fits into the following commutative diagram of maps of pro-objects in the stable category.

$$
\begin{array}{ccc}
(X^m)^{(k)} & \longrightarrow & (Y^m)^{(k)} \\
\downarrow & & \downarrow \\
\{R^{(k)}\} & \longrightarrow & \{(\tau_{\geq 0}L_nR)^{(k)}\}
\end{array}
$$

The vertical maps are pro-isomorphisms of pro-objects in the stable category and in particular induce pro-isomorphisms on homotopy groups. The top map is a levelwise isomorphism on $\pi_q$ for $q < m$, and so the bottom map is a pro-isomorphism on $\pi_q$ for $q < m$. Since $m$ was arbitrary, the statement follows from Theorem 3.6. \qed
In the following proposition, we use $N^{cy}$ to denote the cyclic bar construction for ring spectra.

**Proposition 3.9.** The map of pro-spectra $\{N^{cy}(R)\} \to \{N^{cy}(\tau_{\geq 0}L_n R)\}$ is a weak equivalence.

**Proof.** The cyclic bar construction $N^{cy}(A)$ is the geometric realization of a simplicial spectrum with $q$th simplicial level the $(q+1)$th smash power $A^{(q+1)}$. The cofibrancy hypotheses on $R$ and $\tau_{\geq 0}L_n R$ (Convention 3.4) imply that the geometric realization is equivalent to the thickened realization obtained by gluing just using the face maps. The thickened realization $N^{cy}_\Delta(A)$ has a filtration by Hurewicz cofibrations

$$* = N^{cy}_\Delta(A)_{-1} \longrightarrow N^{cy}_\Delta(A)_0 \longrightarrow N^{cy}_\Delta(A)_1 \longrightarrow N^{cy}_\Delta(A)_2 \longrightarrow \cdots$$

with $N^{cy}_\Delta(A) = \text{colim} N^{cy}_\Delta(A)_m$ and $m$th filtration quotient $A^{(m+1)} \wedge \Delta[m]/\partial$. Because the spectra $R^{(k)}$ and $(\tau_{\geq 0}L_n R)^{(k)}$ are connective for all $k$, the inclusion of the $m$th filtration piece $N^{cy}_\Delta(A)_m \to N^{cy}_\Delta(A)$ is therefore an $m$-equivalence for these ring spectra. In particular, to prove the proposition, it suffices to prove that for each $m$, the map of pro-spectra

$$\{N^{cy}_\Delta(R)_m\} \longrightarrow \{N^{cy}_\Delta(\tau_{\geq 0}L_n R)_m\}$$

induces a pro-isomorphism on each $\pi_q$ pro-group. This certainly holds for $m = -1$ where both sides are trivial and holds by induction for all $m$ by the previous proposition. □

The cyclic bar construction $N^{cy}(A)$ comes with a natural action of the Lie group $\mathbb{T}$. We can then form the homotopy orbit spectrum $N^{cy}(A)_{hc_q} = (N^{cy}(A) \wedge EC_q^+) / C_q$. By naturality, we get maps of spectra $N^{cy}(R)_{hc_q} \to N^{cy}(\tau_{\geq 0}L_n R)_{hc_q}$ and a map of pro-spectra

$$\{N^{cy}(R)_{hc_q}\} \longrightarrow \{N^{cy}(\tau_{\geq 0}L_n R)_{hc_q}\}.$$ (3.10)

To study this, we need a version of Theorem 3.6 in the Borel equivariant context where we understand weak equivalences (and $m$-equivalences) as maps that are weak equivalences (and $m$-equivalences, resp.) of the underlying spectra. Likewise, we understand $m$-connectivity in terms of the underlying spectra. We work in the setup of [13], looking at the model structure on the category of orthogonal spectra with $\mathbb{T}$-actions where the fibrations and weak equivalences are the equivariant maps that are fibrations and weak equivalences of the underlying spectra and use the Postnikov section $t$-structure. (This is the $\mathcal{F}$-model structure of [21, IV.6.5] on orthogonal spectraindexed on $\{\mathbb{R}^n\}$ where $\mathcal{F}$ is the family containing only the trivial subgroup.) This then gives a $t$-model structure in the sense of [13, 4.1], and [13, 9.13] then gives the following result, the analogue of Theorem 3.6 in the pro-category of Borel $\mathbb{T}$-equivariant orthogonal spectra.

**Theorem 3.11.** Let $X = \{X_v\}$ and $Y = \{Y_v\}$ be pro-objects in the category of Borel $\mathbb{T}$-equivariant orthogonal spectra, and assume that they are levelwise $(-N)$-connected for some $N \in \mathbb{Z}$. A map in the pro-category $X \to Y$ is an essentially levelwise $m$-equivalence for all $m$ if and only if it induces on $\pi_q$ a pro-isomorphism of pro-groups for all $q > -N$.

The observation following Theorem 3.6 also holds:
**Addendum 3.12.** Under the hypotheses of Theorem 3.11, the pro-isomorphic objects $X' \cong X$ and $Y \cong Y'$ for which the composite map $X' \to Y'$ is represented by a levelwise $m$-equivalence can be chosen so that the constituent Borel $\mathbb{T}$-equivariant orthogonal spectra are also $(-N)$-connected.

Returning to the homotopy orbits, we can now prove the following proposition.

**Proposition 3.13.** For each $q \in \mathbb{N}$, the map of pro-spectra (3.10) is a weak equivalence.

**Proof.** By Proposition 3.9 and Theorem 3.11 (including Addendum 3.12), for each $m$ we can find pro-isomorphisms

\[
X \cong N_{cy}(R), \quad N_{cy}(\tau_{\geq 0}L_n R) \cong Y
\]

in the pro-category of orthogonal $\mathbb{T}$-spectra so that the composite $X \to Y$ is represented by a level map that is a levelwise $m$-equivalence. We then have pro-isomorphisms

\[
X_{hC_q} \cong N_{cy}(R)_{hC_q}, \quad N_{cy}(\tau_{\geq 0}L_n R)_{hC_q} \cong Y_{hC_q}
\]

and the composite map $X_{hC_q} \to Y_{hC_q}$ remains a level map and a levelwise $m$-equivalence. $\square$

For the next step, we need a model for $THH(R)$ and the tower $THH(\tau_{\geq 0}L_n R)$ with the correct genuine $\mathbb{T}$-equivariant homotopy type to construct classic $TC(-; p)$ and $TC_k(-; p)$. It is convenient to assume the models to be fibrant in the category of $p$-cyclotomic spectra of [5]: this ensures that the point-set $C_{p^{k}}$-fixed points have the correct homotopy type for all $k$. With this hypothesis, we get the following comparison result for the $C_{p^{k}}$-fixed points applied to the map of pro-$\mathbb{T}$-spectra $\{THH(R)\} \to \{THH(\tau_{\geq 0}L_n R)\}$.

**Proposition 3.14.** For each fixed $k$, the map of pro-spectra

\[
\{THH(R)^{C_{p^{k}}}\} \longrightarrow \{THH(\tau_{\geq 0}L_n R)^{C_{p^{k}}}\}
\]

is a weak equivalence.

**Proof.** For $k = 0$, the $C_1$-fixed points are the underlying spectrum and the statement follows from Proposition 3.9 using the Borel equivariant weak equivalence of $THH(A)$ and $N_{cy}(A)$ (for ring spectra $A$ which are cofibrant or are cofibrant as commutative ring spectra).

Now, let $k \geq 1$. For a $p$-cyclotomic spectrum $X$, Theorem 2.2 of [16] constructs a “fundamental cofiber sequence” in the stable category

\[
X_{hC_{p^{k}}} \longrightarrow X^{C_{p^{k}}} \longrightarrow X^{C_{p^{k-1}}},
\]

natural in maps of $p$-cyclotomic spectra. Applying $\pi_q$, we get a long exact sequence, and for the map of pro-spectra

\[
\{THH(R)\} \longrightarrow \{THH(\tau_{\geq 0}L_n R)\},
\]
we get a homomorphism of long exact sequences in the pro-category of abelian groups. The map
\[ \{\pi_q(THH(R)_hC_{p^k})\} \longrightarrow \{\pi_q(THH(\tau_{\geq 0}L_nR)_hC_{p^k})\} \]
is a pro-isomorphism by Proposition 3.13 and the map
\[ \{\pi_q(THH(R)^{C_{p^{k-1}}})\} \longrightarrow \{\pi_q(THH(\tau_{\geq 0}L_nR)^{C_{p^{k-1}}})\} \]
is a pro-isomorphism by induction, and we conclude that the map
\[ \{\pi_q(THH(R)^{C_{p^k}})\} \longrightarrow \{\pi_q(THH(\tau_{\geq 0}L_nR)^{C_{p^k}})\} \]
is a pro-isomorphism by the five lemma.

Because \( TC^k(\cdot; p) \) is defined as a homotopy equalizer of a pair of maps between the fixed points, we immediately deduce the following proposition.

**Proposition 3.15.** For each fixed \( k \), the map of pro-spectra \( \{TC^k(R)\} \rightarrow \{TC^k(\tau_{\geq 0}L_nR)\} \) is a weak equivalence.

Theorem 3.2 follows easily from the preceding proposition.

### 4 Bartel’s Chromatic Convergence Criterion and \( TC(S(p)) \)

In this section, we prove Theorems 1.1 and 1.3 using Barthel’s chromatic convergence criterion. This criterion abstracts the Hopkins–Ravenel proof of the Chromatic Convergence Theorem and states it in general terms for general \( p \)-local spectra as follows.

**Theorem 4.1** Barthel [3, 3.8]. Let \( X \) be a \( p \)-local spectrum whose \( BP \)-homology \( BP_*X \) has finite projective dimension as a graded \( BP_* \)-module. Then \( X \rightarrow holim L_nX \) is a weak equivalence.

We need only the following special cases.

**Corollary 4.2.** Let \( G = \mathbb{1} \) or \( G = C_{p^k} \) for some \( k \geq 0 \); then the map \( (\Sigma^\infty_+BG)(p) \rightarrow holim_n L_n(\Sigma^\infty_+BG) \) is a weak equivalence.

**Proof.** The Atiyah–Hirzebruch spectral sequence identifies \( BP_*(B\mathbb{1}) \) as a free graded \( BP_* \)-module and it therefore has projective dimension 0. For \( G = C_{p^k} \), the case \( k = 0 \) is the Hopkins–Ravenel Chromatic Convergence Theorem. For \( k > 0 \), the Johnson–Wilson [19, (2.11)] argument generalizes from \( C_p \) to \( C_{p^k} \) to show that \( BP_*(B\mathbb{C}_{p^k}) \) is projective dimension 1 as a graded \( BP_* \)-module.

The previous corollary immediately proves Theorem 1.3.
Proof of Theorem 1.3. The main result of [7] (in the case of the trivial group) identifies $TC(S_p; p)^\wedge \simeq TC(S; p)_p^\wedge$ as

$$S_p^\wedge \vee \text{Fib}(\text{tr} : \Sigma \Sigma_+^\infty B\mathbb{T} \longrightarrow S)_p^\wedge$$

where tr denotes the transfer. Commuting wedges, suspension, and fiber sequences with homotopy limits, the result follows from Corollary 4.2.

The remainder of the section proves the following theorem.

Theorem 4.3. The map $TC(S(p); p) \rightarrow \text{holim} TC(L_n S; p)$ is a weak equivalence.

The theorem is meant to apply to both classic $TC$ and Nikolaus–Scholze $TC$, which for the non-connective spectra $L_n S$ are not equivalent. In both cases, by Theorem 2.6 the result for $S(p)$ implies the result for $S$ and $S_p^\wedge$ after $p$-completion and in particular implies Theorem 1.1.

For both classic $TC$ and Nikolaus–Scholze $TC$, the argument starts with the observation that because $L_n$-localization is smashing, $THH(L_n S)$ is $L_n$-local. It follows that the unit map $S \rightarrow THH(L_n S)$ factors in the stable category through the localization

$$S \rightarrow \text{THH}(L_n S);$$

moreover, the map $L_n S \rightarrow THH(L_n S)$ is a weak equivalence. Since the unit map is $\mathbb{T}$-equivariant, we get the weak equivalence $L_n S \rightarrow THH(L_n S)$ in the homotopy category of Borel $\mathbb{T}$-equivariant spectra for the trivial $\mathbb{T}$-action on $L_n S$. (Forgetting to the non-equivariant stable category, the universal property of localization identifies the map $L_n S \rightarrow THH(L_n S)$ as the inclusion of the zero simplices in the cyclic bar model, but that point-set model of the map is not $\mathbb{T}$-equivariant.) Taking homotopy orbits, we obtain the following proposition.

Proposition 4.4. The map $S \rightarrow \text{THH}(L_n S)$ induces a weak equivalence

$$L_n(\Sigma_+^\infty BC_p^k) \longrightarrow \text{THH}(L_n S)_{hC_p^k}.$$

This proposition combined with Corollary 4.2 then implies the following proposition.

Proposition 4.5. The map $\text{THH}(S(p))_{hC_p^k} \rightarrow \text{holim}_n \text{THH}(L_n S)_{hC_p^k}$ is a weak equivalence.

Because homotopy fixed points commute with homotopy limits, the map

$$\text{THH}(S(p))_{hC_p^\infty} \longrightarrow \text{holim}_n \text{THH}(L_n S)_{hC_p^\infty}$$

is also a weak equivalence. Applying this observation and the previous proposition in the case $k = 1$, we see that the map

$$\text{THH}(S(p))_{C_p} \longrightarrow \text{holim}_n \text{THH}(L_n S)_{C_p}$$
is a weak equivalence. Commuting homotopy fixed points with homotopy limits again, we see that the map

\[(\text{THH}(\mathbb{S}(p))_{tC_p})^{hC_p} \longrightarrow \text{holim}_n (\text{THH}(L_n \mathbb{S})_{tC_p})^{hC_p}\]

is a weak equivalence. We can now prove Theorem 4.3.

**Proof of Theorem 4.3 for Nikolaus–Scholze TC.** Using the notation of Section 2, we have shown above that the maps

\[\text{THH}(\mathbb{S}(p))^{hC_p} \longrightarrow \text{holim}_n \text{THH}(L_n \mathbb{S})^{hC_p}\]

\[(\rho^*(\text{THH}(\mathbb{S}(p))_{tC_p}))^{hC_p} \longrightarrow \text{holim}_n (\rho^*(\text{THH}(L_n \mathbb{S})_{tC_p}))^{hC_p}\]

are weak equivalences. Since Nikolaus–Scholze TC(−; p) is the fiber of a map between these spectra, we get a weak equivalence

\[\text{TC}(\mathbb{S}(p); p) \longrightarrow \text{holim}_n \text{TC}(L_n \mathbb{S}; p).\]

**Proof of Theorem 4.3 for classic TC.** Recall that the “fundamental cofiber sequence” (Theorem 2.2 of [16])

\[X_{hC_p} \longrightarrow X_{C_p} \longrightarrow X_{C_p-1}\]

inductively relates the fixed points and homotopy orbits of a p-cyclotomic spectrum. By Proposition 4.5 and induction, we see that the map

\[\text{THH}(\mathbb{S}_p)^{C^k_p} \longrightarrow \text{holim}_n \text{THH}(L_n \mathbb{S})^{C^k_p}\]

is a weak equivalence for all k and we conclude that the map

\[\text{holim}_k \text{THH}(\mathbb{S}_p)^{C^k_p} \longrightarrow \text{holim}_{k,n} \text{THH}(L_n \mathbb{S})^{C^k_p}\]

is a weak equivalence. Since TC(−; p) is the homotopy fiber of a self-map of \(\text{holim}_k (−)^{C^k}\), we get a weak equivalence

\[\text{TC}(\mathbb{S}(p); p) \longrightarrow \text{holim}_n \text{TC}(L_n \mathbb{S}; p).\]

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