KÄHLER INFORMATION MANIFOLDS OF SIGNAL PROCESSING FILTERS IN WEIGHTED HARDY SPACES

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ABSTRACT. We generalize Kähler information manifolds of complex-valued signal processing filters by introducing weighted Hardy spaces and smooth transformations of transfer functions. We prove that the Riemannian geometry of a linear filter induced from weighted Hardy norms for the smooth transformations of its transfer function is a Kähler manifold. Additionally, the Kähler potential of the linear system geometry corresponds to the square of the weighted Hardy norms of its composite transfer functions. Based on properties of Kähler manifolds, geometric objects on the manifolds of the linear systems in weighted Hardy spaces are computed in much simpler ways. Moreover, Kähler information manifolds of signal filters in weighted Hardy spaces incorporate various well-known information manifolds under the unified framework. We also cover several examples from time series models of which metric tensor, Levi-Civita connection, and Kähler potentials are represented with polylogarithms of poles and zeros from the transfer functions with weight vectors in exponential forms.

1. INTRODUCTION

Information geometry has been thriving as an interdisciplinary study across various research areas. In particular, the differential geometric approach also has achieved theoretical enhancements and practical improvements in time series analysis and signal processing. Since the information geometry of time series models such as autoregressive and moving average (ARMA) processes and fractionally integrated ARMA (ARFIMA) processes were revealed [1,19,18], information geometry and its applications to time series models and signal processing have been understood in the framework of symplectic geometry and Kähler geometry [6,3,4,22,5,9]. Additionally, there are practical applications of information geometry such as finding improved Bayesian predictive priors [11,21,20,9,8,14]. In particular, these shrinkage priors of the time series models are easily built on the Kähler extension of the information geometry for time series and signal filters [9,8,14].

Information geometry and its Kählerian interpretation of time series analysis and signal processing are closely related to Hardy spaces in complex analysis. One of the basic assumptions on information geometry of linear systems is that the logarithmic power spectrum is a function in the Hardy space [1,2]. It is also noteworthy that Kähler structures for the information geometry of linear systems need a similar assumption on logarithmic transfer functions [9], i.e., the logarithmic transfer functions of the Kählerian signal filters satisfy finite Hardy norms. In Kählerian linear system geometry [9], the Kähler potential, encoding the geometric information of a linear system, is derived as the squared Hardy norm of the logarithmic transfer function. Since the logarithmic transfer function is expressed with the series expansion of complex cepstrum coefficients, the Hardy norm of the logarithmic transfer
function is the complex cepstrum norm of the linear system. Similarly, the Hardy norm of the logarithmic power spectrum can be understood as the power cepstrum norm of the linear system.

The Kählerian information geometry mentioned above has focused on $\alpha$-divergence-induced geometry of signal processing models, and the connection between $\alpha$-divergence and Kählerian information geometry is relatively well-understood [9]. The metric tensor of the Kählerian information geometry can be derived not only from the $\alpha$-divergence but also from the Kähler potential that is the square of the unweighted cepstrum norm of the logarithmic transfer function [9].

However, the information geometry based on weighted Hardy norms has not been paid attention. Additionally, none of works on information geometry has discussed the Kählerization process of other information manifolds. For example, although the mutual information between future and past in ARMA processes, where Martin [12] calculated the metric tensor of the mutual information, also can be interpreted as a weighted complex cepstrum norm, the Kähler extension of the mutual information geometry still remains unknown.

This paper extends the scope of Kählerian information geometry to signal filters in weighted Hardy spaces. We prove that the geometry of linear systems, where composite functions of smooth transformations and the transfer function are in weighted Hardy spaces, is Kähler geometry with an explicit Hermitian structure. On the information manifold of a linear system in weighted Hardy spaces, the Kähler potential is the squared weighted Hardy norms for the smooth transformations of the linear system transfer functions. For the logarithmic function used as a transformation function, the Kähler potential is represented with the squared weighted complex cepstrum norm.

These findings indicate the two-fold generalization of the Kählerian information geometry [9]: The generalization from the logarithmic transfer function to generic smooth transformations $\phi$ of the transfer function and the extension from the unweighted Hardy space to weighted Hardy spaces with arbitrary weight vectors $\omega$. The $(\phi, \omega)$-generalization of the Kählerian information geometry integrates various information manifolds into the unified framework. For the linear systems in the weighted Hardy spaces, a family of weight vectors generates not only the Kählerian information geometry [9] but also the geometry induced from the mutual information between past and future [12].

The structure of this paper is as follows. In next section, basic definitions of various function spaces in complex analysis and generalize these function spaces to weighted Hardy spaces. After then, the complex cepstrum in signal processing and its connection to weighted Hardy norms are reviewed. In addition, properties of Kähler manifolds are briefly covered. These concepts will be used as theoretical building blocks for further discussions.

2. Theoretical Backgrounds

In this section, we visit definitions of various function spaces in complex analysis and generalize these function spaces to weighted Hardy spaces. After then, the complex cepstrum in signal processing and its connection to weighted Hardy norms are reviewed. In addition, properties of Kähler manifolds are briefly covered. These concepts will be used as theoretical building blocks for further discussions.
2.1. Weighted Hardy spaces. A complex function $f(z)$ is given by the following Fourier series expansion for $-\pi \leq w < \pi$:

$$f(e^{jw}) = \sum_{s=-\infty}^{\infty} f_s e^{-jsw}$$

where $f_s$ is the $s$-th Fourier coefficient of the series expansion. By using Z-transformation ($e^{jw} \rightarrow z$), the function $f(z)$ in a complex domain is represented with

$$f(z) = \sum_{s=-\infty}^{\infty} f_s z^{-s}$$

where $f_s$ is the $s$-th Fourier coefficient of $f$. In the opposite direction, we can transform a polynomial in $z$ to a discrete Fourier series by using inverse Z-transformation ($z \rightarrow e^{jw}$).

Various function spaces are defined for functions on complex domains $\Omega$. The first example is the Lebesgue space ($L^p$-space) for $1 \leq p < \infty$ that is the Banach space of functions with finite $L^p$-norms:

$$L^p = \{ f : ||f||_{L^p} < \infty \}$$

where the $L^p$-norm of a complex-valued function $f$ on $\Omega$ is defined as

$$||f||_{L^p} = \left( \int_\Omega |f(z)|^p d\mu \right)^{1/p}.$$ 

For infinite $p$, $||f||_{L^\infty} = \text{ess sup} f$ where ess sup is the essential supremum.

In particular, the $p = 2$ case is more intriguing than other $p$ values. The $L^2$-space is the function space with finite $L^2$-norms:

$$L^2 = \{ f : ||f||_{L^2} < \infty \}.$$ 

The $L^2$-norm of a Fourier-transformed (or Z-transformed) function $f$ is given by

$$||f||_{L^2} = \left( \sum_{s=-\infty}^{\infty} |f_s|^2 \right)^{1/2}$$

where $f_s$ is the $s$-th Fourier (or Z-transformed) coefficient of $f$. The $L^2$-norm is also obtained from the inner product of two functions $f$ and $g$:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_\Omega f(z) \overline{g(z)} d\mu$$

such that $||f||^2_{L^2} = \langle f, f \rangle$.

Let us concentrate on analytic functions of which the domains are the unit disk $D$. In this case, the $H^p$-space is the Hardy space that is the function space of finite $H^p$-norms defined as

$$||f||_{H^p} = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

where $1 \leq p < \infty$. For infinite $p$, $||f||_{H^\infty} = \sup_{|z| < 1} |f(z)|$.

For $f$ in $H^p$-spaces, it is possible to define the radial limit of the functions:

$$\tilde{f} = \lim_{r \rightarrow 1} f(re^{i\theta})$$

This limit exists almost everywhere in $\Omega$.
for almost every $\theta$. Additionally, the relation between $H^p$-norms and $L^p$-norms is given by

$$\|f\|_{L^p} = \|f\|_{H^p}.$$

Similar to $L^p$-spaces, our interest in this paper is the $p=2$ case among various $H^p$-norms. For Fourier-transformed (or Z-transformed) functions $f$, the $H^2$-space is the function space of

$$H^2 = \{f : \|f\|_{H^2} < \infty\}$$

such that the $H^2$-norm of a Fourier-transformed (or Z-transformed) function is defined as

$$\|f\|_{H^2} = \left(\sum_{s=0}^{\infty} |f_s|^2\right)^{1/2}$$

where $f_s$ is the $s$-th Fourier (or Z-transformed) coefficient of $f$.

Similar to the $L^2$-space, the $H^2$-space is also equipped with the inner product. The inner product between two functions $f$ and $g$ in the Hardy spaces is given by

$$\langle f, g \rangle = \frac{1}{2\pi i} \oint_T f(z)g(\bar{z})d\mu(z)$$

where $\mathbb{T}$ is the unit circle.

Weighted Hardy spaces are defined as the generalization of the Hardy space such that weight vectors are introduced into the definition of the Hardy norm [14]. Given a positive sequence $\omega = (\omega_0, \omega_1, \cdots, \omega_s, \cdots)$ such that $\omega_s > 0$ for all non-negative integers $s$, the weighted Hardy norm ($H^2_\omega$-norm) is represented with

$$\|f\|_{H^2_\omega} = \|f\|_\omega = \left(\sum_{s=0}^{\infty} \omega_s |f_s|^2\right)^{1/2}$$

where $f_s$ is the $s$-th Fourier (or Z-transformed) coefficient of $f$.

The weighted Hardy space $H^2_\omega$ is defined as a function space with a finite weighted Hardy norm of a given weight vector $\omega$:

$$H^2_\omega = \{f : \|f\|_\omega < \infty\}.$$

The reproducing kernel in weighted Hardy spaces is expressed with a given weight sequence $\omega$ [15]:

$$k_u(v) = \sum_{s=0}^{\infty} \frac{\bar{u}^s v^s}{\omega_s}$$

where $u, v$ are in the unit disk $\mathbb{D}$.

It is obvious that the unweighted Hardy space is a special case of weighted Hardy spaces with the weight vector of the unit sequence, i.e., $\omega = (1, 1, \cdots)$. The reproducing kernel in the unweighted Hardy space with the weight vector of the unit sequence $\omega = (1, 1, \cdots)$ is given by

$$k_u(v) = \sum_{s=0}^{\infty} \bar{u}^s v^s = \frac{1}{1 - \bar{u}v},$$

and this reproducing kernel is also known as the Szegő kernel.

Besides the unweighted Hardy space, we can consider the weighted Hardy spaces with $\omega_s = |\rho|^{2s}$. This weight corresponds to the exponentiation in Z-transformation and means the scaling of $z^{-1} \to \rho z^{-1}$ in $z$-domain. This exponentiation also changes the region of convergence by a factor of $\rho$. The unweighted Hardy norm of a signal
filter with the exponentiation is equivalent to the weighted Hardy norm of the signal filter. The reproducing kernel of the weighted Hardy space with the weight vector of the unit sequence $\omega_s = |\rho|^{2s}$ is represented by

$$k_u(v) = \sum_{s=0}^{\infty} \frac{\bar{u}^s v^s}{|\rho|^{2s}} = \frac{1}{1 - \bar{u}v/|\rho|^2},$$

and it is straightforward to check that Eq. (3) converges to Eq. (2) in the limit of $|\rho| \to 1$.

Other function spaces are also understood as weighted Hardy spaces. One of such examples is the Sobolev space. The Sobolev space $W^{m,p}$ for an integer $m$ and $1 \leq p \leq \infty$ is defined as

$$W^{m,p} = \{ f : f \in L^p, f^{(l)} \in L^p \text{ for } l \leq m \}$$

where $f^{(l)}$ is the $l$-th derivative of $f$. This space is the function space of finite Sobolev norms.

Similar to the Hardy space, analytic functions on the unit disk with $p = 2$ are our main interest. In this case, the Sobolev norm of a Fourier-transformed function $f$ is expressed with

$$\| f \|_{W^{m,2}} = \left( \sum_{s=0}^{\infty} (1 + s^2 + s^4 + \cdots + s^{2m}) |f_s|^2 \right)^{1/2}$$

where $f_s$ is the $s$-th Fourier coefficient. It is obvious that the Sobolev space $W^{m,2}$ is the weighted Hardy space of $\omega_s = 1 + s^2 + s^4 + \cdots + s^{2m}$.

The Dirichlet space is another example of weighted Hardy spaces. The Dirichlet space $D$ is defined as the function space of the Dirichlet semi-norm given by

$$\| f \|_{D,s} = \left( \frac{1}{\pi} \iint_{\Omega} |f'(z)|^2 d\mu \right)^{1/2}$$

where $\Omega$ is the unit disk $\mathbb{D}$ in this paper. By the definition of the Dirichlet semi-norm given above, the semi-norm of a Fourier-transformed function $f$ is represented with the Fourier coefficients in the form of

$$\| f \|_{D,s} = \left( \sum_{s=1}^{\infty} s |f_s|^2 \right)^{1/2},$$

and it is straightforward to confirm that the weight vector of the Dirichlet space is given by $\omega_s = s$ for a positive integer $s$.

By plugging the weight vector of $\omega_s = s$ into Eq. (1), the reproducing kernel in the Dirichlet semi-norm space is given by

$$k_u(v) = \log (1 - \bar{u}v)$$

where $u$ and $v$ are complex numbers in the unit disk $\mathbb{D}$.

Since the semi-norm is independent of the 0-th Fourier coefficient, all constant functions belong to the identical zero norm. Several ways to extend the semi-norm to the Dirichlet norm are as follows. The simplest solution for the Dirichlet norm is including the 0-th Fourier coefficient $f_0$ into the semi-norm:

$$\| f \|_D = \left( |f_0|^2 + \| f \|_{D,s}^2 \right)^{1/2}$$.
Another solution is adding the Hardy norm to the Dirichlet semi-norm. With including the Hardy norm, the Dirichlet norm is given by

\[ \| f \|_D = \left( \| f \|_H^2 + \| f \|_D^2 \right)^{1/2} = \left( \sum_{s=0}^{\infty} (1 + s) |f_s|^2 \right)^{1/2}, \]

and the reproducing kernel in the Dirichlet space is found as

\[ k_u(v) = \frac{1}{\bar{uv}} \log (1 - \bar{uv}) \]

where \( u, v \in \mathbb{D} \setminus \{0\} \).

The Bergman space is also a weighted Hardy space. The Bergman space \( \mathcal{A} \) is the function space of a finite Bergman norm defined as

\[ \| f \|_{\mathcal{A}} = \left( \frac{1}{\pi} \int_{\Omega} |f(z)|^2 d\mu \right)^{1/2} \]

where \( \Omega \) is given by the unit disk \( \mathbb{D} \) in this paper. It is noteworthy that the Bergman norm uses \( f(z) \) instead of \( f'(z) \) in the Dirichlet semi-norm. The Bergman norm is represented with the Fourier coefficients \( f_s \) in terms of

\[ \| f \|_{\mathcal{A}} = \left( \sum_{s=0}^{\infty} |f_s|^2 \right)^{1/2} \]

such that the weight sequence is \( \omega_s = (1 + s)^{-1} \) for non-negative \( s \). From Eq. (1) and the weight sequence, the Bergman kernel is straightforwardly obtained as

\[ k_u(v) = \frac{1}{(1 - \bar{uv})^2} \]

where \( u \) and \( v \) are complex numbers inside the unit disk \( \mathbb{D} \).

An interesting property of the Bergman kernel is that the Bergman metric is emergent from the Bergman kernel:

\[ g_{uv} = \partial_u \partial_v \log k(u) = \frac{2}{(1 - \bar{uv})^2} \]

where un-barred and barred indices are holomorphic and anti-holomorphic coordinates, respectively.

Similar to the Dirichlet space and the Sobolev space, function spaces of derivative-related semi-norms can be introduced. The generalized differentiation semi-norm is given by

\[ \| f \|_{\mathcal{D}_m, s} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{d}{d\theta} f(e^{i\theta}) \right|^2 d\theta \right)^{1/2} \]

where the fractional derivative is defined as

\[ \left( \frac{d}{d\theta} \right)^m e^{is\theta} = (is)^m e^{is\theta} \]

for a real number \( m \) and a real number \( s \). With Z-transformation, the generalized differentiation semi-norm is represented with

\[ \| f \|_{\mathcal{D}_m, s} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{z}{dz} f(z) \right|^2 \frac{dz}{z} \right)^{1/2} \]
where the fractional derivative is defined as
\[
\left( \frac{d}{dz} \right)^m z^s = \frac{s^m}{z^s}
\]
for a real number \( m \) and a real number \( s \). Although a negative real number \( m \) actually corresponds to integration, we keep the convention for the integration as the generalized differentiation with a negative order in order to pursue consistency in notation.

The generalized differentiation semi-norm is represented with the Fourier (or Z-transformed) coefficients:
\[
\|f\|_{\tilde{D}^m, s} = \left( \sum_{s=0}^{\infty} s^m |f_s|^2 \right)^{1/2},
\]
and this semi-norm is regarded the weighted Hardy norm with the weight vector of \( \omega_s = s^m \). It is straightforward to obtain the reproducing kernel in the generalized differentiation semi-norm function space as
\[
k_{\mu}^{(m)}(v) = L_{\mu}(\bar{v}v)
\]
where \( u, v \) are complex numbers inside the unit disk \( \mathbb{D} \) and \( L_{\mu} \) is the polylogarithm of order \( m \) such that
\[
L_{\mu}(z) = \sum_{s=1}^{\infty} \frac{z^s}{s^m}
\]
for \( |z| < 1 \). The polylogarithm is extended to \( |z| = 1 \) by analytic continuation.

The generalized differentiation semi-norm is cross-connected to various function spaces. First of all, the \( m = 0 \) case corresponds to the unweighted Hardy space. When \( m = 1 \), the generalized differentiation semi-norm is identical to the Dirichlet semi-norm. In addition, the Sobolev norm is also obtained from the sum of the generalized differentiation semi-norms:
\[
\|f\|_{W^m, 2} = \left( \sum_{l=0}^{m} \|f\|_{\tilde{D}^l, s}^2 \right)^{1/2}.
\]
Similarly, various weighted Hardy norms are constructed from positive linear combinations of the generalized differentiation semi-norms with different \( m \) values.

As a summary, the weighted Hardy norms introduced in this subsection are represented with
\[
\|f\|_\omega = \left( \sum_{s=0}^{\infty} \omega_s |f_s|^2 \right)^{1/2}
\]
where \( \omega_s \) is the weight sequence of a given function space. For example, the weight sequences \( \omega \) of the well-known function spaces covered in this section are followings:
\[
\omega_s = \begin{cases} 
1 & \text{for unweighted Hardy space } H^2 \\
1 + s^2 + s^4 + \cdots + s^{2m} & \text{for Sobolev space } W^{m, 2} \\
s & \text{for Dirichlet space } \mathcal{D} \\
\frac{1}{1+s} & \text{for Bergman space } \mathcal{A} \\
\frac{s^m}{s^m} & \text{for differentiation semi-norm space } \tilde{D}^m
\end{cases}
\]
for a non-negative integer \( s \).
2.2. Cepstrum and weighted Hardy norms. In signal processing, the transfer function of a signal processing filter encodes crucial information on the signal filter because it describes how the system transforms input signals to output signals. The Fourier-transformed transfer function in the frequency domain is represented with

\[ h(w; \xi) = \sum_{s=-\infty}^{\infty} h_s(\xi) e^{-jsw} \]

where \( h_s \) is the \( s \)-th Fourier coefficient of the filter transfer function, and \( \xi \) is the vector of signal filter parameters.

Using the convention of \( z^{-s} \) in Z-transformation, the transfer function of a causal filter is in the unilateral form that has terms only with non-negative \( s \). The transfer function of a causal filter is expressed with the following unilateral form:

\[ h(z; \xi) = \sum_{s=0}^{\infty} h_s(\xi) z^{-s}, \]

and the transfer function is analytic outside the unit disk \( D \). The various function norms defined in the previous subsection can be applied to transfer functions in \( z^{-s} \) after the inverse projection of \( z \rightarrow z^{-1} \). In another way, the transfer functions are expressed in \( z^s \) convention and the function norms are computed.

The Hardy spaces and the Hardy norms are helpful for defining the characteristics of linear systems in signal processing. For example, the stationarity condition on linear systems is represented with the unweighted Hardy norm of the transfer functions:

\[ \sum_{s=0}^{\infty} |h_s|^2 = \| h(z; \xi) \|_{H^2}^2 < \infty, \]

and this condition means that the transfer functions of stationary filters are functions in the unweighted Hardy space.

Additionally, the concepts of the Hardy spaces and the Hardy norms can be applied to parameter estimation processes of linear systems. The \( H^p \)-norms are adopted for defining the distance between two transfer functions in order to find the optimal system parameters \( \xi^* \) under constraints:

\[ \xi^* = \arg\min_{\xi \in C} \| h(z; \xi) - \hat{h} \|_{H^p} \]

where \( \hat{h} \) is the target transfer function and \( C \) is the parameter set satisfying the constraints. One of the applications is \( H^\infty \)-optimization in control theory [10].

As generalization, we introduce a composite function \( f \) of a smooth transformation \( \phi \) and a transfer function \( h \) such that \( f = \phi \circ h : \mathbb{D} \rightarrow \mathbb{C} \) and \( f \) is analytic in the unit disk \( \mathbb{D} \). The weighted Hardy norm of a linear system described by \( f \) is given by

\[ \mathcal{I}_\omega = \| \phi \circ h \|_\omega = \left( \sum_{s=0}^{\infty} \omega_s |f_s(\xi)|^2 \right)^{1/2} \]

where \( f_s \) is the \( s \)-th Fourier coefficient of \( f = \phi \circ h \).

Similarly, the distance between two linear systems \( M_1 \) and \( M_2 \), of which the transfer functions are \( h_1(z; \xi') \) and \( h_2(z; \xi) \), respectively, is found as

\[ \mathcal{I}_\omega(M_1, M_2) = \| \phi \circ h_1 - \phi \circ h_2 \|_\omega = \left( \sum_{s=0}^{\infty} \omega_s |f_{1,s}(\xi') - f_{2,s}(\xi)|^2 \right)^{1/2} \]
where $f_i,s$ is the $s$-th Fourier coefficient of $f_i$.

It is noteworthy that the weighted Hardy norm-based distance of Eq. (4) is a distance measure in metric spaces. First of all, if two linear systems are identical, the weighted Hardy distance between the two systems is zero. Additionally, it is symmetric under exchange of $M_1$ and $M_2$ in Eq. (4). Furthermore, the triangle inequality is also satisfied by this distance measure.

Among various candidate functions to $\phi$, the most intriguing transformation, which is also applicable to signal processing, is the logarithmic function. In this case, the logarithmic transfer function corresponds to the complex cepstrum of a linear system [15]. By using the complex cepstrum and Fourier/Z-transformations, the logarithmic transfer function of a linear system can be represented with a power series of $z$:

\[
\log h(z; \xi) = \sum_{s=0}^{\infty} c_s(\xi) z^{-s}
\]

where $c_s$ is the $s$-th complex cepstrum coefficient of the transfer function. The $s$-th complex cepstrum coefficient $c_s$ in Eq. (5) is found directly from the following line integration:

\[
c_s(\xi) = \frac{1}{2\pi i} \oint_{\mathbb{T}} \log h(z, \xi) z^s \frac{dz}{z}
\]

where $\mathbb{T}$ is the unit circle. The derivation of Eq. (6) is straightforward by considering the inner product in Hardy spaces with the basis of $\{1, z^{-1}, z^{-2}, \ldots, z^{-s}, \ldots\}$: $c_s(\xi) = \langle \log h(z, \xi), z^{-s} \rangle$.

Various function norms of logarithmic transfer functions can be understood in the framework of weighted Hardy spaces. For example, the complex cepstrum norm in signal processing is the unweighted Hardy norm of a logarithmic transfer function. As an extension, weighted complex cepstrum norms correspond to weighted Hardy norms of logarithmic transfer functions. Based on the homomorphic structure of cepstrum [17], the weighted complex cepstrum distance between two linear systems $M_1$ and $M_2$, of which the transfer functions are $h_1$ and $h_2$, respectively, is represented with

\[
I_\omega(M_1, M_2) = \| \log h_1 - \log h_2 \|_\omega = \left( \sum_{s=0}^{\infty} \omega_s |c_{1,s}(\xi') - c_{2,s}(\xi)|^2 \right)^{1/2}
\]

where $c_{i,s}$ is the $s$-th complex cepstrum coefficient of a linear system $M_i$. Additionally, the weighted complex cepstrum distance is also a distance measure in metric spaces. In particular, when $h_2 = 1$, the weighted complex cepstrum norm of a transfer function is the weighted complex cepstrum distance between the output signal transformed by the filter and the input signal. In this case, the weighted Hardy norm of the logarithmic transfer function describes how different the output and the input are by using the distance measure.

### 2.3. Kähler geometry

In this subsection, we briefly visit fundamentals of Kähler manifolds for further derivations. More detailed introduction to Kähler manifolds can be found in references [13, 7].

By definition, Kähler manifolds are the Hermitian manifolds with closed Kähler forms. Simply speaking, a given manifold of dimension $n$ is a Kähler manifold if...
and only if metric tensor components of the manifold fulfill the following conditions:

\begin{align}
  \frac{\partial_i g_{j\bar{k}}}{\partial_j g_{i\bar{k}}} &= \frac{\partial_j g_{i\bar{k}}}{\partial_i g_{j\bar{k}}}, \\
  \frac{\partial_i g_{j\bar{k}}}{\partial_j g_{i\bar{k}}} &= \frac{\partial_j g_{i\bar{k}}}{\partial_i g_{j\bar{k}}} \tag{8} \\
  \frac{\partial_i g_{j\bar{k}}}{\partial_j g_{i\bar{k}}} &= \frac{\partial_j g_{i\bar{k}}}{\partial_i g_{j\bar{k}}} \tag{9}
\end{align}

where barred and unbarred indices are the holomorphic and anti-holomorphic co-ordinates of the Kähler manifold, and \(i, j, k = 1, \cdots, n\). Eq. (8) is the condition for Hermitian manifolds and Eq. (9) is for a closed Kähler form.

Kähler manifolds take several computational advantages over non-Kähler manifolds. First, non-trivial components of metric tensor and Levi-Civita connection are calculated from the Kähler potential:

\begin{align}
  g_{ij} &= (g_{i\bar{j}})^* = \partial_i \partial_{\bar{j}} K, \\
  \Gamma_{ij, \bar{k}} &= (\Gamma_{i\bar{j}, k})^* = \partial_i \partial_j \partial_{\bar{k}} K \tag{10} \tag{11}
\end{align}

where \(K\) is the Kähler potential. Eq. (10) and Eq. (11) indicate significant benefits in geometric calculation on Kähler manifolds. If the Kähler potential is given, the components of the metric tensor and Levi-Civita connection are easily obtained from taking partial derivatives with respect to only relevant coordinates. In non-Kähler geometry, these two geometric objects require more lengthy and tedious computation steps with partial derivatives and summations across all the coordinates of the manifold. Additionally, the Ricci tensor is also straightforwardly computed in a simpler way similar to Eq. (10) and Eq. (11). The Ricci tensor of Kähler manifolds is given by

\begin{align}
  R_{ij} &= -\partial_i \partial_{\bar{j}} \log G \tag{12}
\end{align}

where \(G\) is the determinant of the metric tensor Eq. (10). The Ricci tensor of non-Kähler manifolds needs much more complicated calculation procedures such as additional calculation steps for Riemann curvature tensor than Eq. (12). Moreover, the Laplace-Beltrami operator in Kähler geometry is in a much lighter form of

\begin{align}
  \Delta &= 2g^{i\bar{j}} \partial_i \partial_{\bar{j}} \\
  \Delta &= \frac{1}{G} \partial_i (\sqrt{G} g^{i\bar{j}} \partial_{\bar{j}})
\end{align}

where \(g^{i\bar{j}}\) is the component of the inverse metric tensor. It is obvious that this Laplace-Beltrami operation in Kähler geometry is much easier to calculate than that of non-Kähler manifolds in the following form:

\begin{align}
  \Delta &= \frac{1}{G} \partial_i (\sqrt{G} g^{i\bar{j}} \partial_{\bar{j}})
\end{align}

where \(G\) is the determinant of the metric tensor and \(g^{i\bar{j}}\) is the component of the inverse metric tensor.

3. Geometry of linear systems in weighted Hardy spaces

In this section, we derive the geometry of a linear system in weighted Hardy spaces. Additionally, we also prove that the induced manifolds from weighted Hardy norms for various transformations of transfer functions are Kähler manifolds. It is also shown that the Kähler potential of the signal filter geometry is given as the square of the weighted Hardy norms for composite functions of smooth transformations and transfer functions.

Let us denote a composite function \(f\) of a smooth transformation \(\phi\) and the transfer function of a linear system \(h(z; \xi)\), i.e., \(f = \phi \circ h\). Since metric tensor
components on a Riemannian manifold of dimension $n$ are extracted from an infinitesimal length, the infinitesimal weighted Hardy norm of the linear system geometry with a given weight sequence $\omega$ is straightforwardly obtained from Eq. (4):

$$\delta I^2 = \sum_{s=0}^{\infty} \omega_s |\delta f_s(\xi)|^2$$

where $\delta f_s$ is expanded in terms of the infinitesimal displacements $\delta \xi^i$ along the $i$-th coordinate $\xi^i$ of the $n$-dimensional manifold.

Metric tensor components of a Riemannian manifold are decoded from the quadratic terms of the infinitesimal displacements $\delta \xi^i$ in the length element Eq. (13). In the weighted Hardy norm-induced geometry of a linear system, the metric tensor components are given by

$$g_{ij}(\xi, \bar{\xi}; \omega) = \left( g_{\bar{i}\bar{j}}(\xi, \bar{\xi}; \omega) \right)^* = \sum_{s=0}^{\infty} \omega_s \partial_i f_s(\xi) \partial_j \bar{f}_s(\bar{\xi})$$

where $i, j = 1, \cdots, n$. By Eq. (14), it is straightforward to check that the geometry from a given weighted Hardy norm is the Hermitian manifold.

The Levi-Civita connection $\Gamma$ of the manifold is represented with

$$\Gamma_{\nu\rho,\mu} = \frac{1}{2} \left( \partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\nu\rho} \right)$$

where $\mu, \nu, \rho$ run through holomorphic and anti-holomorphic coordinates, i.e., $\mu, \nu, \rho = 1, \cdots, n, 1, \cdots, \bar{n}$. From the metric tensor expressions of Eq. (14) and Eq. (15), the non-trivial components of the Levi-Civita connection are found as

$$\Gamma_{ij,k}(\xi, \bar{\xi}; \omega) = \left( \Gamma_{\bar{i}\bar{j},\bar{k}}(\xi, \bar{\xi}; \omega) \right)^* = \frac{1}{2} \left( \partial_i g_{\bar{j}k} + \partial_j g_{ik} \right) = \sum_{s=0}^{\infty} \omega_s \partial_i \partial_j f_s(\xi) \partial_k \bar{f}_s(\bar{\xi})$$

where $i, j, k = 1, \cdots, n$. All other components of the Levi-Civita connection are vanishing.

By using the metric tensor components of Eq. (14) and Eq. (15), and the Levi-Civita connection components of Eq. (16) on a given linear system manifold where smooth transformations of a transfer function are in weighted Hardy spaces, the correspondence between Kähler manifolds and linear system manifolds is found in the form of the following theorem.

**Theorem 1.** The geometry of a linear system where a composite function of a smooth transformation and the transfer function is in weighted Hardy spaces is a Kähler manifold.

**Proof.** Eq. (14) shows that the geometry induced from a weighted Hardy norms is a Hermitian manifold of Eq. (8). It is also straightforward to verify that metric tensor components of Eq. (15) satisfy the closed Kähler two-form condition of Eq. (9).

Since the metric tensor components of the geometry fulfill Eq. (8) and Eq. (9), the information manifold of a linear system is the Hermitian manifold with a closed Kähler two-form. By the definition of Kähler manifolds, the geometry of linear systems in weighted Hardy spaces is a Kähler manifold. □
Since metric tensor components and Levi-Civita connection components in Kähler geometry are derived directly from the Kähler potential, finding the Kähler potential of the linear system geometry is our next main question. The following theorem provides the answer.

**Theorem 2.** The Kähler potential for the information manifold of a linear system in a weighted Hardy space is the square of a weighted Hardy norm for a composite function of a smooth transformation and the transfer function of the system.

**Proof.** From Eq. (15), the metric tensor of Kähler geometry is expressed with Fourier transformation (or Z-transformation) coefficients and a weight sequence:

\[ g_{ij}(\xi, \bar{\xi}; \omega) = \sum_{s=0}^{\infty} \omega_s \partial_i f_s(\xi) \partial_j \bar{f}_s(\bar{\xi}). \]

By using the product rule of derivative, the equation above is rewritten as

\[ g_{ij}(\xi, \bar{\xi}; \omega) = \partial_i \partial_j \left( \sum_{s=0}^{\infty} \omega_s |f_s(\xi)|^2 \right). \]

According to Eq. (10) stating the relation between the metric tensor and the Kähler potential, the Kähler potential \( K \) is given by

\[ (17) \quad K = \sum_{s=0}^{\infty} \omega_s |f_s(\xi)|^2 = \|f(z; \xi)\|^2. \]

up to purely holomorphic and purely anti-holomorphic functions. Since the metric tensor is independent of the purely holomorphic and purely anti-holomorphic terms, these functions are auxiliary in the definition of the Kähler potential. Eq. (17) indicates that the Kähler potential is the square of a weighted Hardy norm of a smooth transformation of the transfer function.

We also obtain the identical Kähler potential from the Levi-Civita connection. By applying the product rule of derivative to Eq. (16), the Levi-Civita connection components are expressed in the following form:

\[ \Gamma_{ij,k}(\xi, \bar{\xi}; \omega) = \sum_{s=0}^{\infty} \omega_s \partial_i \partial_j f_s \partial_k \bar{f}_s = \partial_i \partial_j \partial_k K. \]

It is obvious that the Kähler potential found above is identical to Eq. (17). Starting from \( \Gamma_{ij,k} \), it is also possible to obtain the same results for the Kähler potential. \( \square \)

By Theorem 2, it is concluded that the square of a weighted Hardy norm for a composite function of a smooth transformation and the transfer function is the Kähler potential to which the metric tensor and the Levi-Civita connection are derived by using partial derivatives. As mentioned in the previous section, this is an advantage of the Kähler manifolds that the geometric objects are obtained from the Kähler potential by simply taking holomorphic and anti-holomorphic derivatives. Using the metric tensor, the Ricci tensor in the induced geometry is also obtained from Eq. (12).

Theorem 1 and Theorem 2 can be applied to linear systems with the stationarity condition of a signal filter.
Corollary 1. The geometry of a linear system with a finite weighted stationarity condition is a Kähler manifold. The Kähler potential of the linear system geometry induced from the weighted stationarity condition is the square of the weighted Hardy norm of the transfer function of a linear system.

Proof. If \( f(z;\xi) = h(z;\xi) \), i.e., \( \phi(t) = t \), the weighted Hardy norm of \( f \) is identical to the weighted Hardy norm of the transfer function, i.e., the weighted stationarity condition of the linear system. By Theorem 1, the geometry induced from the weighted stationarity condition is a Kähler manifold.

Plugging \( f(z;\xi) = h(z;\xi) \) to Eq. (17), the Kähler potential of the geometry induced from the weighted Hardy norms of the transfer function of the linear system is given by

\[
K = \| h(z;\xi) \|_\omega^2 = \sum_{s=0}^{\infty} \omega_s | h_s(\xi) |^2
\]

where \( h_s \) is the \( s \)-th Fourier (Z-transformed) coefficient of the linear system. \( \square \)

Considering that the complex cepstrum norm is a special case of weighted Hardy norms, Theorem 1 and Theorem 2 can be applied to linear systems with weighted complex cepstrum norms.

Corollary 2. The geometry of a linear system with a finite weighted complex cepstrum norm is a Kähler manifold. The Kähler potential of the complex cepstrum geometry of the linear system is the square of the weighted complex cepstrum norm of a linear system. In other words, the Kähler potential of the complex cepstrum geometry is the square of the weighted Hardy norm of the logarithmic transfer function.

Proof. Similar to Corollary 1, this is also a special case of Theorem 1 and Theorem 2. If \( f(z;\xi) = \log |h(z;\xi)| \), i.e., \( \phi(t) = \log |t| \), the weighted Hardy norm of \( f \) is identical to the weighted complex cepstrum norm of the linear system. According to Theorem 1, the geometry induced from the weighted complex cepstrum norm is a Kähler manifold.

Plugging \( f(z;\xi) = \log |h(z;\xi)| \) to Eq. (17), the Kähler potential of the geometry induced from the weighted complex cepstrum norm is given by

\[
K = \| \log |h(z;\xi)| \|_\omega^2 = \sum_{s=0}^{\infty} \omega_s | c_s(\xi) |^2
\]

where \( c_s \) is the \( s \)-th complex cepstrum coefficient of the linear system. \( \square \)

Similarly, we can consider the weighted Hardy norm of the power cepstrum of the linear system.

Corollary 3. The geometry of a linear system with a finite weighted power cepstrum norm is a Kähler manifold. The Kähler potential of the power cepstrum geometry of the linear system is the square of the weighted power cepstrum norm of a linear system. In other words, the Kähler potential of the power cepstrum geometry is the square of the weighted Hardy norm of the logarithmic spectral density function.

Proof. Similar to Corollary 1 and Corollary 2, this is also a special case of Theorem 1 and Theorem 2. If \( f(z;\xi) = \log |h(z;\xi)|^2 \), i.e., \( \phi(t) = \log |t|^2 \), the weighted Hardy
norm is same with the weighted power cepstrum norm of the linear system. By adopting Theorem 1, the weighted power cepstrum norm-induced geometry is the Kähler manifold.

Plugging \( f(z; \xi) = \log |h(z; \xi)|^2 \) to Eq. (17), the Kähler potential of the induced geometry from the weighted power cepstrum norm is found as

\[
\mathcal{K} = \| \log |h(z; \xi)|^2 \|_{\omega}^2 = \sum_{s=0}^{\infty} \omega_s |p_s(\xi)|^2
\]

where \( p_s \) is the \( s \)-th power cepstrum coefficient of the linear system.

The theorems and corollaries proven above are the two-fold generalization of the corresponding theorems and corollaries for the Kählerian information geometry of a linear system in the unweighted Hardy space [9]. First of all, not being limited to the logarithmic function in Corollary 2 and [9], the introduction of more generic transformation functions \( \phi \) also induces the Kähler structures. When the logarithmic function is used, the information manifold of a linear system in weighted Hardy space includes the Kählerian information geometry given in [9]. The geometry for the unweighted power cepstrum norm of a spectral density function (unweighted power cepstrum) is found in Amari [1] and Amari and Nagaoka [2]. Second, it is extended from unweighted Hardy spaces to weighted Hardy spaces. By using more general weight vectors \( \omega \) not confined to the unit weight sequence, it is still able to generate the Kähler information manifolds for signal processing filters.

Various well-known information manifolds are considered as special cases of the \((\phi, \omega)\)-generalization of Kählerian information geometry of a linear system. Obviously, when \( \omega = (1, 1, \cdots) \) is used, we obtain the unweighted complex cepstrum geometry that is the Kählerian information geometry of a signal filter in the literature [9]. These information manifolds in weighted Hardy spaces are not limited to Kählerian information manifolds from the unweighted complex cepstrum [9] or the unweighted power cepstrum [2], the geometry of the weighted stationarity filters, and mutual information geometry [12].

4. Example: ARMA and ARFIMA Models

In this section, we apply the theoretical framework developed in the previous section to time series models such as ARMA models and ARFIMA models in weighted Hardy spaces.

The transfer function of the ARFIMA\((p, d, q)\) model with model parameters \( \xi = (\xi(-1), \xi(0), \cdots, \xi(p+q)) = (\sigma, d, \lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q) \) is given by

\[
h(z; \xi) = \frac{\sigma^2 (1 - \mu_1 z^{-1})(1 - \mu_2 z^{-1}) \cdots (1 - \mu_q z^{-1})}{2\pi (1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) \cdots (1 - \lambda_p z^{-1})} (1 - z^{-1})^d
\]

where \( \lambda_i (|\lambda_i| < 1) \) is a pole from the AR part of the transfer function, and \( \mu_i (|\mu_i| < 1) \) is a zero from the MA part of the transfer function. It is straightforward that the ARMA transfer function is acquired by setting up \( d = 0 \) in Eq. (19).

Among various smooth transformation functions of a transfer function, let us concentrate on the logarithmic function because of its extensive applications in signal processing and time series analysis.
Plugging Eq. (19) into the logarithmic function, the logarithmic transfer function of the ARFIMA system is obtained as

$$\log h(z; \xi) = \sum_{i=1}^{p+q} \gamma_i \log (1 - \xi^{(i)} z^{-1}) + d \log (1 - z^{-1}) + \log \frac{\sigma^2}{2\pi}$$

where \( \gamma_i = -1 \) if \( \xi^{(i)} (1 \leq i \leq p) \) is a pole of the transfer function, and \( \gamma_i = 1 \) if \( \xi^{(i)} (p + 1 \leq i \leq p + q) \) is a root of the transfer function. It is obvious that \( d = 0 \) implies the transfer function of ARMA models.

Considering the constant \( \sigma \)-submanifold, the \( s \)-th complex cepstrum coefficient \( c_s \) of the ARFIMA model is derived from the definition of complex cepstrum in Eq. (6):

$$c_s = \begin{cases} \frac{d + \sum_{i=1}^{p+q} \gamma_i (\xi^{(i)})^s}{s} & (s \neq 0) \\ 1 & (s = 0) \end{cases}$$

and it is noticeable that the complex cepstrum coefficients of the ARMA model are obtained by setting up \( d = 0 \) in the cepstrum expression of Eq. (20).

4.1. Arbitrary \( \omega_s \). With the complex cepstrum of Eq. (20) and a weight vector \( \omega \), the Kähler potential is easily found from Eq. (18):

$$K(\omega) = \sum_{s=1}^{\infty} \omega_s \left| \frac{d + \sum_{i=1}^{p+q} \gamma_i (\xi^{(i)})^s}{s} \right|^2 + \omega_0.$$ 

It is noteworthy that the convergence of the Kähler potential on the Kähler-ARFIMA manifold is not guaranteed always and depends on the weight vector \( \omega_s \) and filter parameters \( \xi \) such as the difference parameter, poles, and roots. However, since we consider only smooth transformation functions of signal filter transfer functions in weighted Hardy spaces, the convergence is obviously expected and we will comment otherwise.

Plugging the Kähler potential of Eq. (21) into Eq. (10), the metric tensor of the Kähler-ARFIMA manifolds is given by

$$g_{uv} = \left( \sum_{s=1}^{\infty} \omega_s / s^2 \right)^2 \left( \gamma_i \sum_{s=1}^{\infty} \omega_s (\xi^{(i)})^{s-1} / s \right) \left( \gamma_j \sum_{s=1}^{\infty} \omega_s (\xi^{(j)})^{s-1} / s \right)$$

where \( u, v \) run from 0 to \( p + q \), and \( i, j \) run from 1 to \( p + q \).

Similar to the metric tensor, the Levi-Civita connection is also found from Eq. (11) and Eq. (21) as

$$\Gamma_{ij,0} = -\gamma_i \sum_{s=1}^{\infty} \omega_s \left( 1 - \frac{1}{s} \right) (\xi^{(i)})^{s-2} \delta_{ij},$$

$$\Gamma_{ij,k} = \gamma_j \gamma_k \sum_{s=1}^{\infty} \omega_s (s-1)(\xi^{(j)})^{s-2}(\xi^{(k)})^{s-1} \delta_{ij}$$

where \( i, j, k \) run from 1 to \( p + q \).

Since the ARMA models are submodels of the ARFIMA model with \( d = 0 \), the information manifolds of the ARMA models are also submanifolds of the ARFIMA models. Based on this fact, metric tensor components of the Kähler-ARMA geometry are obtained from those of the Kähler-ARFIMA geometry. From Eq. (22), the
metric tensor of the Kähler-ARMA manifolds is found as

\[
g_{i\bar{j}} = \gamma_i \gamma_{\bar{j}} \sum_{s=1}^{\infty} \omega_s (\xi^i \xi^j)^{s-1}
\]

where \( i, j \) run from 1 to \( p+q \). It is also possible to derive the metric tensor directly from the Kähler potential by using Eq. (10).

Non-trivial Levi-Civita connection components are derived not only from Eq. (24) but also from Eq. (11):

\[
\Gamma_{ij\bar{k}} = \gamma_j \gamma_k \sum_{s=1}^{\infty} \omega_s (s-1)(\xi^j)^{s-2}(\xi^{\bar{k}})^{s-1}\delta_{ij}
\]

where \( i, j, k \) run from 1 to \( p+q \).

4.2. \( \omega_s = s^m \). For simplification, let us confine the weight sequence to \( \omega_s = s^m \) for a real number \( m \). As mentioned earlier, this weight sequence is related to the differentiation semi-norm spaces that are a building block for other weighted Hardy spaces.

According to Theorem 1 and Corollary 2 geometry induced from finite weighted complex cepstrum norms is Kähler geometry. By using Theorem 2 and Corollary 2 not only applying the complex cepstrum coefficients of Eq. (20) to Eq. (18) but also plugging \( \omega_s = s^m \) to Eq. (21) provide the Kähler potential of the Kähler-ARFIMA geometry given by

\[
K^{(m)} = \sum_{s=1}^{\infty} s^m \left| d + \sum_{i=1}^{p+q} \frac{\gamma_i (\xi^i)^s}{s} \right|^2 + \delta_{m,0}
\]

where \( \delta_{m,0} \) is the Kronecker delta.

It is noteworthy that the \( |d|^2 \)-term in the Kähler potential can be divergent when \( m \geq 1 \). One way of making finite complex cepstrum norms is to limit the range of \( m \) to \( m < 1 \). In this range of \( m \), the weighted complex cepstrum norm of the ARFIMA model with \( \omega_s = s^m \) becomes finite. When weighted complex cepstrum norms are convergent, the Kähler potential of the Kähler-ARFIMA geometry is decomposed into the following terms:

\[
K^{(m)} = \sum_{i,j=1}^{n} \gamma_i \gamma_j Li_{2-m}(\xi^i \bar{\xi}^j) + \sum_{i=1}^{n} \gamma_i (dLi_{2-m}(\xi^i) + \bar{d}Li_{2-m}(\xi^i))
\]

\[
+ |d|^2 Li_{2-m}(1) + \delta_{m,0}
\]

where \( Li_t \) is the polylogarithm of order \( t \).

Another way of avoiding divergent complex cepstrum norms for arbitrary \( m \) is to regularizing the divergent term in the norm by setting \( d = 0 \). Since the divergent term is dependent only on the difference parameter \( d \), plugging \( d = 0 \) not only removes the divergent term but also corresponds to the model reduction from the ARFIMA models to the ARMA models where the Kähler potential is always finite for any \( m \) values. The Kähler potential of the ARMA geometry is given in the following form of

\[
K^{(m)} = \sum_{i,j=1}^{n} \gamma_i \gamma_j Li_{2-m}(\xi^i \bar{\xi}^j) + \delta_{m,0}
\]
where $L_i$ is the polylogarithm of order $t$. It is obvious that the polylogarithm functions in the Kähler potential are all finite for the poles and the zeros of the ARMA transfer function in the unit disk $\mathbb{D}$.

By using Eq. (10), the metric tensor of the Kähler-ARFIMA geometry with the weight sequence of $\omega_s = s^m$ is obtained from the Kähler potential of Eq. (28). For $m \leq 1$, the metric tensor of the Kähler-ARFIMA manifolds is given by

$$g^{(m)}_{uv} = \begin{pmatrix} L_{i2-m}(1) & \gamma_i L_{i1-m}(\xi(i))/\xi(i) \\ \gamma_i L_{i1-m}(\xi(i))/\xi(i) & \gamma_i^2 L_{i0-m}(\xi(i)\bar{\xi}(i))/\xi(i)^2 \end{pmatrix}$$

where $u, v$ run from 0 to $q$, and $i$ runs from 1 to $p + q$.

For $m > 1$, the metric tensor of the Kähler-ARFIMA manifolds is finite for arbitrary $m$. Additionally, it is also possible to obtain the metric tensor from the submanifold of the Kähler-ARFIMA geometry, Eq. (30). The metric tensor of the Kähler-ARMA geometry is represented with Eq. (32):

$$\Gamma_{ij,k} = \gamma_j \gamma_k L_{i-m-1}(\xi(j)\bar{\xi}(k))/\xi(j)^2\bar{\xi}(k)$$

where $i, j, k$ run from 1 to $p + q$. The components also are derived from Eq. (11), partial derivatives to the Kähler potential.

Similar to the Kähler-ARFIMA geometry, the metric tensor of the Kähler-ARMA manifold is derived from partial derivatives on its Kähler potential of Eq. (29). Opposite to the Kähler-ARFIMA geometry, the Kähler potential of the Kähler-ARMA geometry is finite for arbitrary $m$. Additionally, it is also possible to obtain the metric tensor from the submanifold of the Kähler-ARFIMA geometry, Eq. (30). The metric tensor of the Kähler-ARMA geometry is represented with

$$\hat{g}^{(m)}_{ij} = \gamma_j \gamma_i L_{i-m}(\xi(i)\bar{\xi}(j))/\xi(i)^2\bar{\xi}(j)$$

where $i, j$ run from 1 to $p + q$. It is straightforward to verify that the Kähler-ARMA manifolds are the submanifolds of the Kähler-ARFIMA manifolds.

In the Kähler-ARMA geometry, non-trivial components of the Levi-Civita connection are represented with Eq. (34):

$$\Gamma_{ij,k} = \gamma_j \gamma_k L_{i-m-1}(\xi(j)\bar{\xi}(k))/\xi(j)^2\bar{\xi}(k)$$

where $i, j, k$ run from 1 to $p + q$. The components also are derived from Eq. (11), partial derivatives to the Kähler potential.

4.2.1. $m = 0$. The weight vector for $m = 0$ is the unit sequence of $\omega_s = 1$ for all non-negative integers $s$. With the unit weight sequence, the weighted complex cepstrum norm is given by

$$\mathcal{C}^{(0)} = \left( \sum_{s=0}^{\infty} |c_s|^2 \right)^{1/2},$$

and this norm is exactly the unweighted complex cepstrum norm. Since the Kähler potential of the geometry with $m = 0$ is the square of the unweighted complex cepstrum norm by Theorem 2, the Kähler potential of the geometry is represented with

$$\mathcal{K}^{(0)} = \sum_{s=1}^{\infty} \frac{d + (\mu_1^s + \cdots + \mu_p^s) - (\lambda_1^s + \cdots + \lambda_p^s)}{s} + 1.$$
It is straightforward to check that the Kähler potential of Eq. (35) is identical up to the last constant term on the right-hand side to the Kähler potential of the Kähler-ARFIMA geometry in Choi and Mullhaupt [8] where the ARFIMA model in the literature was scaled up to the constant gain. Considering the $\sigma$-submanifolds, it is straightforward to show that the Kähler potential of Eq. (35) is identical to the Kähler potential in the literature. When $d = 0$ in Eq. (35), the Kähler potential is same up to the gain term with the Kähler potential of the ARMA geometry in Choi and Mullhaupt [9] where the model is also $\sigma$-constant.

The metric tensor of the geometry is obtained by either plugging $m = 0$ to Eq. (30) or taking partial derivatives to the Kähler potential of Eq. (35). For $m = 0$, the metric tensor of the Kähler-ARFIMA geometry is found as

$$g_{uv}^{(0)} = \begin{pmatrix}
\frac{\pi^2}{6} & \frac{1}{\lambda_j} \log (1 - \lambda_j) & -\frac{1}{\mu_j} \log (1 - \mu_j) \\
-\frac{1}{\mu_i} \log (1 - \mu_i) & 1 & -\frac{1}{1 - \mu_i, \lambda_j} \\
-\frac{1}{1 - \mu_i, \lambda_j} & -\frac{1}{1 - \lambda_i, \mu_j} & \frac{1}{1 - \lambda_i, \mu_j} \end{pmatrix}$$

where $u, v$ run for $(\xi^{(0)} = d, \xi^{(1)} = \lambda_1, \cdots, \xi^{(p)} = \lambda_p, \xi^{(p+1)} = \mu_1, \cdots, \xi^{(p+q)} = \mu_q)$, the index for $\lambda_i$ runs from 1 to $p$, and the index for $\mu_i$ runs from 1 to $q$. As described above, the metric tensor is matched to the metric tensor for the Kähler information geometry of the ARFIMA models [8].

Non-vanishing components of the Levi-Civita connection on the Kähler-ARFIMA manifolds are also derived not only from Eq. (31) and Eq. (32) but also the Kähler potential of Eq. (35):

$$\Gamma_{ij,0} = -\gamma_j \frac{\xi^{(i)} - \xi^{(i)}}{(\xi^{(i)})^2} \delta_{ij},$$

$$\Gamma_{ij,k} = \gamma_j \gamma_k \frac{\xi^{(k)}}{(1 - \xi^{(j)} \xi^{(k)})^2} \delta_{ij}$$

where $i, j, k$ run from 1 to $p + q$ and $\delta_{ij}$ is the Kronecker delta. These connection components were not calculated previously in the Kähler-ARFIMA geometry [8]. However, $\Gamma_{ij,k}$ components in this Kähler-ARFIMA geometry are also matched with those in the Kähler information geometry of the ARMA models [9].

Similar to the Kähler-ARFIMA geometry, the $m = 0$ geometry of the Kähler-ARMA models is derived from the identical procedure. The metric tensor of the Kähler-ARMA manifolds is given by

$$g_{uv}^{(0)} = \begin{pmatrix}
\frac{1}{1 - \lambda_i, \lambda_j} & -\frac{1}{1 - \lambda_i, \mu_j} \\
-\frac{1}{1 - \lambda_i, \mu_j} & \frac{1}{1 - \mu_i, \lambda_j} \end{pmatrix},$$

and this metric tensor is also consistent with the metric tensor for the Kählerian information geometry of the ARMA models [9].

Non-vanishing Levi-Civita connection components in the Kähler-ARMA geometry are $\Gamma_{ij,k}$ in the Kähler-ARFIMA geometry:

$$\Gamma_{ij,k} = \gamma_j \gamma_k \frac{\xi^{(k)}}{(1 - \xi^{(j)} \xi^{(k)})^2} \delta_{ij}.$$
where \( i, j, k \) run from 1 to \( p+q \) and \( \delta_{ij} \) is the Kronecker delta. Since the geometry in the literature [9] is the constant-gain submanifold of the full Kähler-ARMA manifold, the non-trivial components of the Levi-Civita connection on the \( \sigma \)-submanifold are only \( \Gamma_{i,j,k} \), identical to \( \Gamma_{i,j,k} \) of the full Kähler-ARMA geometry in this paper.

4.2.2. \( m = 1 \). When \( m = 1 \), the weighted complex cepstrum norm with the weight sequence \( \omega_s = s \) for all positive integers \( s \) is given by

\[
\mathcal{I}_\omega^{(1)} = \left( \sum_{s=1}^{\infty} s |c_s|^2 \right)^{1/2},
\]

and it is also known as the Hilbert-Schmidt norm of the Hankel matrix in complex cepstrum coefficients. Additionally, the norm is related to mutual information between past and future in ARMA model studied by [12, 15].

Since the Kähler potential of the geometry is the square of the weighted complex cepstrum norm by Corollary 2, the Kähler potential of the Kähler-ARFIMA geometry is provided by Eq. (27):

\[
K^{(1)} = \frac{\sum_{s=1}^{\infty} |d + (\mu^1_s + \cdots + \mu^s_p) - (\lambda^1_s + \cdots + \lambda^s_p)|^2}{s}.
\]

It is noteworthy that there is no \( \sigma \)-dependence in the expression of the Kähler potential. However, this Kähler potential of the Kähler-ARFIMA models with the weight sequence of \( \omega_s = s \) is not guaranteed as finite because \( \sum_{s=1}^{\infty} |d|^2/s \) is divergent as mentioned above.

With \( m = 1 \), the Kähler potential of the ARFIMA geometry from the weighted complex cepstrum norm is convergent if and only if \( d = 0 \), i.e., the ARFIMA model is reduced to the ARMA model. In this case, the Kähler potential of the Kähler-ARMA geometry from the weighted complex cepstrum norms is found from Eq. (29):

\[
K^{(1)} = \sum_{i,j=1}^{n} \gamma_i \gamma_j L_{i1}(\xi^i \bar{\xi}^j) = \log \left( \frac{\prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \lambda_i \bar{\mu}_j) \prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \lambda_i \bar{\lambda}_j) \prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \mu_i \bar{\lambda}_j) \prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \mu_i \bar{\mu}_j)}{\prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \lambda_i \mu_j) \prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \lambda_i \lambda_j) \prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \mu_i \lambda_j) \prod_{i=1}^{p} \prod_{j=1}^{q} (1 - \mu_i \mu_j)} \right),
\]

where \( L_{i1}(z) = -\log (1 - z) \). Considering that the Kähler potential is the square of the weighted complex cepstrum norm by Corollary 2, the Kähler potential of the ARMA model for \( m = 1 \) is the mutual information between past and future in the ARMA processes studied by Martin [12]. It is also interesting that the parameters in the weighted complex cepstrum norm are only the roots and the poles of the transfer function in the ARMA model. The fact that the length measure is independent with the gain parameter \( \sigma \) is also consistent with the \( \sigma \)-quotient model mentioned in Martin [12].

From the Kähler potential found above, the metric tensor of the Kähler-ARMA geometry with \( m = 1 \) is given by

\[
g_{uv}^{(1)} = \begin{pmatrix}
\frac{1}{(1-\lambda_1 \lambda_p)^2} & \frac{1}{(1-\mu_1 \lambda_p)^2} \\
\frac{1}{(1-\lambda_1 \mu_p)^2} & \frac{1}{(1-\mu_1 \mu_p)^2}
\end{pmatrix},
\]

where \( u, v \) run for \( \xi^{(1)} = \lambda_1, \ldots, \xi^{(p+1)} = \lambda_p, \xi^{(p+2)} = \mu_1, \ldots, \xi^{(p+q)} = \mu_q \), the index for \( \lambda_i \) runs from 1 to \( p \), and the index for \( \mu_i \) runs from 1 to \( q \). The metric tensor is identical to the metric tensor from mutual information between past and future of ARMA models in Martin [12]. If only the pure AR part or the pure
MA part of the metric tensor are considered, the metric tensor of the submanifold is considered as the Poincaré metric of polydisc. Additionally, this metric is the Bergman metric defined above.

It is also straightforward to calculate non-vanishing components of the Levi-Civita connection on the Kähler-ARMA manifolds induced from the mutual information between past and future in the ARMA model. By using Eq. (24), the connection components are given by

\[ \Gamma_{ij,k} = \gamma_j \gamma_k \frac{2\tilde{\xi}^{(k)}}{(1 - \xi^{(j)}\tilde{\xi}^{(k)})^{3}} \delta_{ij}, \]

and all other components are vanishing.

4.3. \( \omega_s = |\rho|^{2s} \). By plugging \( \omega_s = |\rho|^{2s} \) into Eq. (21), the Kähler potential of the Kähler-ARFIMA manifold with the exponentiation factor is represented with

\[ K = \sum_{s=1}^{\infty} |\rho|^{2s} \left( d + \sum_{i=1}^{p+q} \gamma_i (\xi^{(i)})^s \right)^2 + 1, \]

and it is easy to check that in the limit of \( |\rho| \to 1 \), Eq. (41) is converged to the unweighted case, Eq. (35).

By either plugging the weight vector to Eq. (22) or taking partial derivatives of Eq. (41), the metric tensor of the Kähler-ARFIMA manifold with the exponentiation is given in the following form of

\[ g_{uv} = \begin{pmatrix} \sum_{s=1}^{\infty} |\rho|^{2s}/s^2 & \sum_{s=1}^{\infty} \frac{2s}{s} (\xi^{(j)})^{s-1}/s \\ \sum_{s=1}^{\infty} \frac{2s}{s} (\xi^{(i)})^{s-1}/s & \sum_{s=1}^{\infty} \frac{2s}{s} (\xi^{(i)}\tilde{\xi}^{(j)})^{s-1} \end{pmatrix}, \]

and it is also straightforward to check that Eq. (42) is reduced to the unweighted case, Eq. (36) as \( |\rho| \to 1 \).

Similarly, the Levi-Civita connection of the Kähler-ARFIMA manifold with the exponentiation factor is also found as

\[ \Gamma_{ij,k} = \gamma_j \gamma_k \sum_{s=1}^{\infty} |\rho|^{2s} (s-1)(\xi^{(j)}\tilde{\xi}^{(k)})^{s-1} \delta_{ij}, \]

where \( i, j, k \) run from 1 to \( p + q \). Both Eq. (43) and Eq. (44) converge to Eq. (37) and Eq. (38) in the limit of \( |\rho| \to 1 \), respectively.

As mentioned above, the ARMA model is a special case of the ARFIMA model with \( d = 0 \). The information manifolds of the ARMA models are indeed submanifolds of the ARFIMA geometry. Based on this fact, the metric tensor components of the Kähler-ARMA manifolds are easily acquired from those of the Kähler-ARFIMA manifolds. By taking the components of the metric tensor from Eq. (42), the metric tensor of the Kähler-ARMA manifolds is found as

\[ g_{ij} = \gamma_i \gamma_j \sum_{s=1}^{\infty} |\rho|^{2s} (\xi^{(i)}\tilde{\xi}^{(j)})^{s-1} \]

where \( i, j \) run from 1 to \( p + q \). It is also possible to gain the metric tensor directly from the Kähler potential by using Eq. (10). Similar to the ARFIMA case, Eq. (45) is converging to the unweighted case, Eq. (39) as \( |\rho| \to 1 \).
Non-trivial Levi-Civita connection components of the Kähler-ARMA geometry are derived not only from Eq. (24) but also from Eq. (11). Additionally, those can be obtained from Eq. (44):

\[ \Gamma_{ij, k} = \gamma_{ij} \gamma_{k} \sum_{s=1}^{\infty} |\rho|^{2s}(s-1)(\xi(j))^s-2(\bar{\xi}(k))^s-1 \delta_{ij} \]

where \( i, j, k \) run from 1 to \( p + q \). By taking \( |\rho| \to 1 \), the Levi-Civita connection also converge to the unweighted case, Eq. (40).

5. Conclusion

In this paper, we study the geometric properties of a linear system in weighted Hardy spaces. It is proven that the information manifold of signal filters induced from weighted Hardy norms for composite functions of smooth transformations and the transfer function is a Kähler manifold. The weighted Hardy norms of a transformed transfer function are related to the Kähler potential of the signal filter geometry. With the characteristics of Kähler manifolds, various geometric objects such as the metric tensor, the Levi-Civita connection, and the Ricci tensor of the Kähler information manifold are also straightforwardly derived from the Kähler potential.

The concepts of the weighted Hardy spaces with weight sequences \( \omega \) and smooth transformations \( \phi \) of the transfer functions are considered as the \( (\phi, \omega) \)-generalization of the Kähler information geometry in unweighted Hardy spaces found in [9]. This approach is helpful for finding underlying information manifolds of signal processing filters in various weighted Hardy spaces by leveraging the integrated framework.

For example, as a special case of weighted Hardy norms, the weighted complex cepstrum norms of a linear system also provide the information manifolds equipped with Kähler structures. The Kähler potential of the complex cepstrum geometry is the square of the weighted complex cepstrum norm. The unweighted case corresponds to the Kählerian information geometry found in [9]. Similarly, Kähler information manifolds of the weighted stationarity condition and the weighted power cepstrum of signal filters [11, 2] are also obtained from the same framework. Additionally, for \( \omega_s = |\rho|^{2s} \), the weighted Hardy norms of signal filters are equivalent to the unweighted Hardy norms of signal filters with the exponentiation of \( \rho \), i.e., information geometry of signal filters with the exponentiation of \( \rho \) can be derived from the concept of weighted Hardy norms.

ARMA and ARFIMA models with arbitrary weight vectors are studied as applications of the Kähler geometric approach using weighted Hardy norms. Several interesting information manifolds of ARMA and ARFIMA models are emergent. These information manifolds for ARMA and ARFIMA models also include the Kählerian information manifolds of the unweighted complex cepstrum norm [9], the geometry related to the mutual information between past and future [12], the information geometry for signal filters with the exponentiation. These information manifolds also can be understood under the unified framework of Kähler manifolds and weighted Hardy spaces.

Acknowledgment

We thank Xiang Shi and Andrew P. Mullhaupt for discussions on mutual information between past and future in ARMA models [12].
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