The $\Phi^3_3$ measure via Girsanov’s theorem

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Abstract

We construct the $\Phi^3_3$ measure on a periodic three dimensional box as an absolutely continuous perturbation of a random shift of the Gaussian free field. The shifted measure is constructed via Girsanov’s theorem and the relevant filtration is the one generated by a scale parameter. As a byproduct we give a self-contained proof that the $\Phi^3_3$ measure is singular wrt. the Gaussian free field.

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1 Introduction

The $\Phi^3_3$ measure on the three dimensional torus $\Lambda = \mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is the probability measure $\nu$ on distributions $\mathscr{D}'(\Lambda)$ corresponding to the formal functional integral

$$\nu(d\varphi) = \left\{ \frac{1}{Z} \exp\left[ -\lambda \int_\Lambda (\varphi^4 - \infty \varphi^2) dx \right] \mu(d\varphi) \right\}$$

(1)

where $\mu$ is the law of the Gaussian free field with covariance $(1 - \Delta)^{-1}$ on $\Lambda$, $Z$ a normalization constant and $\lambda$ a coupling constant. The $\infty$ appearing in this expression reminds us that many things are wrong with this recipe. The key difficulty can be traced to the fact that the measure we are looking for is not absolutely continuous wrt. the reference measure $\mu$. This fact seems part of the folklore even if we could not find a rigorous proof for it in the available literature apart from a work of Albeverio and Liang [1] which however refers to the Euclidean fields at time zero. The singularity of the $\Phi^3_3$ measure is indeed a major technical difficulties in a rigorous study. Obtaining a complete construction of this formal object (both in finite and infinite volume) has been one of the main achievements of the constructive quantum field theory program [10, 8, 23, 9, 19, 5, 6].

In recent years the rigorous study of the $\Phi^3_3$ model has been pursued from the point of view of stochastic quantization. In the original formulation of Parisi–Wu [22], stochastic quantization is a way to introduce additional degrees of freedom (in particular a dependence on a fictitious time) in order to obtain an equation whose solutions describe a measure of interest, in this case the $\Phi^3_3$ measure on $\Lambda$ as in (1) or its counterpart in the full space. Rigorous analysis of stochastic quantization for simpler models like $\Phi^4_2$ (the two-dimensional analog of eq. (1)) started with the work [17]. It has been only with the fundamental work of Hairer on regularity structures [16]
that the three dimensional model could be successfully attacked, see also \cite{7,13}. This new perspective on this and related problems led to a series of new results on the global space-time control of the stochastic dynamics \cite{21,13,2,20} and to a novel proof of the constructions of non-Gaussian Euclidean quantum field theories in three dimensions \cite{12}.

A conceptual advantage of stochastic quantization is that it is insensitive to questions of absolute continuity wrt. to a reference measure. This, on the other hand, is the main difficulty of the Gibbisan point of view as expressed in eq. (1). In order to explore further the tradeoffs of different approaches we have recently developed a variational method \cite{4} for the construction and description of \( \Phi_3^4 \) with which we were able to provide an explicit formula for the Laplace transform of \( \Phi_3^4 \) in terms of a stochastic control problem. In this control problem the controlled process represent the scale-by-scale evolution of the interacting random field.

The present paper is the occasion to explore further this point of view by constructing a novel measure via a random shift of the Gaussian free field and proving that the \( \Phi_3^4 \) process can be constructed as an absolutely continuous perturbation thereof. Without entering into technical details now let us give the broad outline of this construction. We consider a Brownian martingale \((W_t)_{t \geq 0}\) with values in \( \mathcal{F}'(\Lambda) \) and such that \( W_t \) is a regularization of the Gaussian free field \( \mu \) at scale \( t \). Let us denote \( \mathbb{P} \) its law, \( \mathbb{E} \) the corresponding expectation. In particular \( W_t \to W_\infty \) as \( t \to \infty \) and \( W_\infty \) has law \( \mu \). We can identify the \( \Phi_3^4 \) measure \( \nu \) as the weak limit \( \nu^T \to \nu \) as \( T \to \infty \) of the family of probability measures \((\nu^T)_{T \geq 0}\) defined as

\[
\nu^T(A) = \mathbb{P}^T(W_T \in A),
\]

where \( \mathbb{P}^T \) is the measure on paths \((W_t)_{t \geq 0}\) with density

\[
\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{1}{Z_T} e^{-V_T(W_T)},
\]

and

\[
V_T(\varphi) := \lambda \int_{\Lambda} (\varphi(x)^4 - a_T \varphi(x)^2 + b_T) dx,
\]

is a quartic polynomial in the field \( \varphi \) with \((a_T, b_T)\) a family of (suitably diverging) renormalization constants. The presence of the scale parameter \( t \in \mathbb{R}_+ \) allows to introduce a filtration and a family of measures \( Q^v \) defined as the Girsanov transformation

\[
\frac{dQ^v}{d\mathbb{P}} \bigg|_{\mathcal{F}_T} = \exp \left( L^v_T - \frac{1}{2} \langle L^v \rangle_T \right), \quad L^v_t = \int_0^t \langle v_s, dW_s \rangle_{L^2(\Lambda)}
\]

(2)

where \( \langle (L^v)_{t \geq 0} \rangle \) is the quadratic variation of the (scalar) local martingale \((L^v_t)_{t \geq 0}\) and \((v_t)_{t \geq 0}\) is an adapted process with values in \( L^2(\Lambda) \). Let

\[
D_T := \frac{1}{Z_T} e^{-V_T(W_T)} \left( \frac{dQ^v}{d\mathbb{P}} \right)^{-1},
\]

be the density of \( \mathbb{P}^T \) wrt. \( Q^v \). We will show that it is possible to choose \( v \) in such a way that the family \((D_T)_{T \geq 0}\) is bounded in all \( L^p(Q^v) \) and that \( D_T \to D_\infty \) in \( L^1 \). With particular choice of \( v \) we call \( Q^v \) the drift measure: it is the central object of this paper. By Girsanov’s theorem the canonical process \((W_t)_{t \geq 0}\) satisfies the equation

\[
dW_t = v_t dt + d\tilde{W}_t, \quad t \geq 0,
\]

where \((\tilde{W}_t)_{t \geq 0}\) is a Gaussian martingale under \( Q^v \) (and has law equal to that of \((W_t)_{t \geq 0}\) under \( \mathbb{P} \), that is a regularized Gaussian free field). We will show also that the drift \( v_t \) can be written as
a (polynomial) function of \((\tilde{W}_s)_{s \in [0,t]}\), that is \(v_t = V_t((\tilde{W}_s)_{s \in [0,t]})\). Therefore we have an explicit description of the process \((W_t)_{t \geq 0}\) under the drift measure \(Q^v\) as the unique solution of the path-dependent SDE

\[
dW_t = V_t((\tilde{W}_s)_{s \in [0,t]})\,dt + d\tilde{W}_t, \quad t \geq 0.
\]

(3)

Let us note that this formula expresses the “interacting” random field \((W_t)_{t \geq 0}\) as a function of the “free” field \((\tilde{W}_t)_{t \geq 0}\). In this respect is a formula with very similar technical merits as the stochastic quantization approach.

Intuitively this new measure \(Q^v\), is half way between the variational description in \([4]\) and the (formal) Gibbsian description of eq. (1). It constitutes a measure which is relatively explicit, easy to construct and analyze and which can be used as reference measure for \(\Phi_3^4\), very much like the Gaussian free field can be used as reference measure for \(\Phi_3^4\) \([11]\).

As an application we provide a self-contained proof of the singularity of the \(\Phi_3^4\) measure \(\nu\) wrt. the Gaussian free field \(\mu\). As we already remarked the singularity of \(\Phi_3^4\) seems to belongs to the folklore and we were not able to trace any written proof of that. However M. Hairer, during a conference at Imperial College in 2019 showed us a unpublished proof of him of singularity using the stochastic quantization equation. Our proof and his are very similar and we do not claim any essential novelty in this respect. Albeit the proof is quite straightforward we wrote down all the details in order to provide a reference for this fact. The main point of the present paper remains that of describing the drift measure as a novel object in the context of \(\Phi_3^4\) and similar measures.

Our proof of singularity, in particular this also shows that the drift measure \(Q^v\) is singular wrt. \(\mathbb{P}\). The intuitive reason is that the drift \((V_t)_{t \geq 0}\) in the SDE (3) is not regular enough (as \(t \to \infty\)) to be along Cameron–Martin directions for the law \(\mathbb{P}\) of the process \((W_t)_{t \geq 0}\) and therefore the Girsanov transform (2) gives a singular measure when extended all the way to \(T = +\infty\).

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**Notations.** Let us fix some notations and objects.

- For \(a \in \mathbb{R}^d\) we let \(\langle a \rangle := (1 + |a|^2)^{1/2}\).
- The constant \(\varepsilon > 0\) represents a small positive number which can be different from line to line.
- Denote with \(\mathcal{S}(\Lambda)\) the space of Schwartz functions on \(\Lambda\) and with \(\mathcal{S}'(\Lambda)\) the dual space of tempered distributions. The notation \(\hat{f}\) or \(\mathcal{F}f\) stands for the Fourier transform of \(f\) and we will write \(g(D)\) to denote the Fourier multiplier operator with symbol \(g: \mathbb{R}^n \to \mathbb{R}\), i.e. \(\mathcal{F}(g(D)f) = g\mathcal{F}f\).
- \(B^\alpha_{p,q} = B^\alpha_{p,q}(\Lambda)\) denotes the Besov spaces of regularity \(\alpha\) and integrability indices \(p,q\) as usual. \(C^\alpha = C^\alpha(\Lambda)\) is the Hölder–Besov space \(B^\alpha_{\infty,\infty}\), \(W^{\alpha,p} = W^{\alpha,p}(\Lambda)\) denote the standard fractional Sobolev spaces defined by the norm \(\|f\|_{W^{\alpha,p}} := \|\langle D \rangle^\alpha f\|_{L^p}\) and \(H^\alpha = \mathcal{H}^\alpha = \mathcal{H}^\alpha(\mathbb{R}^d)\) is the usual Fourier restriction norm (a.k.a. Wiener measure) where \(\mathcal{H}^\alpha\) is the Fourier transform of \(C^\alpha(\mathbb{R}^d)\) restricted to \(\mathbb{R}^d\).
concern the Boué–Dupuis formula and certain estimates which will be important also in our analysis of absolute continuity.

2 The setting

The setting of this paper is the same as that of our variational study [4]. In this section we will briefly recall it and also state some results from that paper which will be needed below. They concern the Boué–Dupuis formula and certain estimates which will be important also in our analysis of absolute continuity.

Let \( \Omega := C\left(\mathbb{R}_+; \mathcal{C}^{-3/2-\varepsilon}(\Lambda)\right) \) and \( \mathcal{F} \) be the Borel \( \sigma \)-algebra of \( \Omega \). On \( (\Omega, \mathcal{F}) \) consider the probability measure \( \mathbb{P} \) which makes the canonical process \( (X_t)_{t \geq 0} \) a cylindrical Brownian motion on \( L^2(\Lambda) \) and let \( (\mathcal{F}_t)_{t \geq 0} \) the associated filtration. In the following \( \mathbb{E} \) without any qualifiers will denote expectations wrt. \( \mathbb{P} \) and \( \mathbb{E}_\mathbb{Q} \) will denote expectations wrt. some other measure \( \mathbb{Q} \).

On the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) there exists a collection \((B^n_t)_{n \in \mathbb{Z}^d}\) of complex (2-dimensional) Brownian motions, such that \( B^n_t = B^{\cdot, n}_t, B^{\cdot, n}_t \) independent for \( m \neq \pm n \) and \( X_t = \sum_{n \in \mathbb{Z}^d} e^{i(n, \cdot)} B^n_t \).

Fix some \( \rho \in C^\infty_0(\mathbb{R}_+, \mathbb{R}_+^d) \) such that \( \rho = 1 \) on \( B(0, 9/10) \) and \( \text{supp} \rho \subset B(0, 1) \). For \( x \in \mathbb{R}^3 \) let \( \rho_t(x) := \rho(x/t) \) with \( (x) := (1 + |x|^2)^{1/2} \) and \( \sigma_t(x) := \left( \frac{\partial}{\partial t} \rho_t^2(x) \right)^{1/2} = -2(\langle x \rangle/t)\rho(\langle x \rangle/t)\rho'(\langle x \rangle/t))^{1/2}/t^{1/2} \).

Let \( J_s = \sigma_s(D)\langle D \rangle^{-1} \) and consider the process \( (W_t)_{t \geq 0} \) defined by

\[
W_t := \int_0^t J_s dX_s = \sum_{n \in \mathbb{Z}^d} e^{i(n, \cdot)} \int_0^t \frac{\sigma_s(n)}{\langle n \rangle} dB^n_s, \quad t \geq 0. \tag{4}
\]

It is a centered Gaussian process with covariance

\[
\mathbb{E}[\langle W_t, \varphi \rangle \langle W_s, \psi \rangle] = \sum_{n \in \mathbb{Z}^d} \frac{\rho_{\min(s,t)}^2(n)}{\langle n \rangle^2} \varphi(n)\overline{\psi(n)},
\]

for any \( \varphi, \psi \in \mathcal{S}(\Lambda) \) and \( t, s \geq 0 \), by Fubini theorem and Itô isometry. By dominated convergence \( \lim_{t \to \infty} \mathbb{E}[\langle W_t, \varphi \rangle \langle W_t, \psi \rangle] = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{-2} \varphi(n)\overline{\psi(n)} \) for any \( \varphi, \psi \in L^2(\Lambda) \). For any finite “time” \( T \) the random field \( W_T \) on \( \Lambda \) has a bounded spectral support and the stopped process \( W^T_t = W_{t\wedge T} \) for any fixed \( T > 0 \), is in \( C(\mathbb{R}_+, C^\infty(\Lambda)) \). Furthermore \( (W^T_t)_t \) only depends on a finite subset of the Brownian motions \((B^n_t)_{n \in \mathbb{Z}^d}\). We write \( g(D) \) for the Fourier multiplier operator with symbol \( g \).

Observe that \( J_t \) has the property, that for some function \( f \in B^{p, p}_\alpha \) or \( f \in W^{s, p} \) with \( p \in [1, \infty) \) and \( s \in \mathbb{R} \), for any \( \alpha \in \mathbb{R} \)

\[
\| J_t f \|_{B^{p, p}_{\alpha+1-\alpha}} \lesssim \langle t \rangle^{-\alpha-1/2} \| f \|_{B^{p, p}_\alpha}.
\]

We will denote by \([W^n_t], n = 1, 2, 3\), the \( n \)-th Wick-power of the Gaussian random variable \( W_t \) (under \( \mathbb{P} \)) and introduce the convenient notations \( W^n_t := 12[W^n_t], W^n_2 := 4[W^n_2^n] \). Furthermore we will write \([\langle D \rangle^{-1/2} W_t]^n], n \in \mathbb{N} \) for the \( n \)-th Wick-power of \( \langle D \rangle^{-1/2} W_t \). It exists for any \( 0 < t < \infty \) and any \( n \geq 1 \) since it is easy to see that \( \langle D \rangle^{-1/2} W_t \) has a covariance with a diagonal behavior which can be controlled logarithmically uniformly in \( t \). These Wick powers converge as \( T \to \infty \) in spaces of distributions with regularities given in the following table:
Table 1: Regularities of the various stochastic objects, the domain of the time variable is understood to be $[0, \infty]$, $C(0, \infty) = C([0, \infty]; \mathcal{C}^\alpha)$ and $L^2(\mathbb{C}^\alpha) = L^2(\mathbb{R}_+; \mathcal{C}^\alpha)$. Estimates in these norms hold a.s. and in $L^p(\mathbb{P})$ for all $p \geq 1$ (see [4]).

We denote by $H$ the space of $(\mathcal{F}_t)_{t \geq 0}$-progressively measurable processes which are $\mathbb{P}$-almost surely in $H := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element $v$ of $H$ is a drift. Below we will need also drifts belonging to $H^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$ where $H^\alpha(\Lambda)$ is the Sobolev space of regularity $\alpha$. We denote the corresponding space with $H^\alpha_0$. For any $v \in H$ define the measure $Q^v$ on $\Omega$ by

$$\frac{dQ^v}{d\mathbb{P}} = \exp \left[ \int_0^\infty v_s dX_s - \frac{1}{2} \int_0^\infty \|v_s\|^2 ds \right].$$

Denote with $H^\alpha_0 \subset H$ the set of drifts $v \in H$ for which $Q^v(\Omega) = 1$, and set $W^v := W - I(v)$, where

$$I_t(v) = \int_0^t J_s v_s ds.$$

Below will need also the following operators. For all $t \geq 0$ let $\theta_t : \mathbb{R}^3 \to [0, 1]$ be a smooth function such that

$$\begin{align*}
\theta_t(\xi)\sigma_s(\xi) &= 0 \text{ for } s \geq t, \\
\theta_t(\xi) &= 1 \text{ for } |\xi| \leq t/2 \text{ provided that } t \geq T_0
\end{align*}$$

(5)

for some $T_0 > 0$. For example one can fix smooth functions $\tilde{\theta}, \eta : \mathbb{R}^3 \to \mathbb{R}_+$ such that $\tilde{\theta}(\xi) = 1$ if $|\xi| \leq 1/2$ and $\tilde{\theta}(\xi) = 0$ if $|\xi| \geq 2/3$, $\eta(\xi) = 1$ if $|\xi| \leq 1$ and $\eta(\xi) = 0$ if $|\xi| \geq 2$. Then let $\theta_t(\xi) := \theta(\xi/t)$ and define

$$\theta_t(\xi) = (1 - \eta(\xi))\tilde{\theta}_t(\xi) + \zeta(t)\eta(\xi)\tilde{\theta}_t(\xi)$$

where $\zeta(t) : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function such that $\zeta(t) = 0$ for $t \leq 10$ and $\zeta(t) = 1$ for $t \geq 3$. Then eq (5) will hold with $T_0 = 3$. We will let

$$f^\theta := \theta(D)f$$

(6)

for any $f \in \mathcal{F}'(\Lambda)$.

Our aim here to study the measures $\mu_T$ defined on $\mathcal{C}^{-1/2-\varepsilon}$ as

$$\frac{d\mu_T}{d\mathbb{P}} = e^{-V_T(W_T)}$$

for $\varphi \in C^\infty(\Lambda)$

$$V_T(\varphi) := \lambda \int_\Lambda (\varphi^4 - a_T \varphi^2 + b_T)dx,$$

(7)

with suitable $a_T, b_T \to \infty$. For convenience the measure $\mu_T$ is not normalized and, wrt. to the notations in the introduction we have

$$\frac{d\mathbb{P}^T}{d\mu_T} = \frac{1}{\mu_T(\Omega)}.$$

With these notations we can recall the following results of [4].
Theorem 1 For any $a_T, b_T \in \mathbb{R}$, and $f : \mathcal{C}^{-1/2-\varepsilon}(\Lambda) \to \mathbb{R}$ with linear growth, recall (7) and let $V_T^f(\varphi) := f(\varphi) + V_T(\varphi)$. Then the formula
\[
\int_{\mathcal{C}^{-1/2-\varepsilon}(\Lambda)} e^{-V_T^f(\varphi)} \, d\mu(\varphi) = -\log \mathbb{E}[e^{-V_T^f(W_T)}] = \inf_{u \in \mathcal{H}_u} \mathbb{E} \left[ V_T^f(W_T + I_T(u)) + \frac{1}{2} \int_0^T \|u_t\|_{L^2(\Lambda)}^2 \right]
\]
holds for any finite $T$.

This is a consequence of the more general Boué–Dupuis formula which can be stated as follows.

Theorem 2 (BD formula) Assume $F : C([0, T], C^\infty(\Lambda)) \to \mathbb{R}$, be Borel measurable and such that there exist $p, q \in (1, \infty)$, with $1/p + 1/q = 1$, $\mathbb{E}[\|F(W)\|^p] < \infty$ and $\mathbb{E}[e^{-F(W)}]^q < \infty$ (where we can regard $W$ as an element of $C([0, T], C^\infty(\Lambda))$ by restricting to $[0, T]$). Then
\[
-\log \mathbb{E}[e^{-F(W)}] = \inf_{u \in \mathcal{H}_u} \mathbb{E} \left[ F(W + I(u)) + \frac{1}{2} \int_0^T \|u_t\|_{L^2(\Lambda)}^2 \right].
\]

We will use several times below the Boué–Dupuis formula (9) in order to control exponential integrability of various functional. By a suitable choice of renormalization and a change of variables in the control problem (8) we were able in [4] to control the functional in Theorem 1 uniformly up to infinity:

Theorem 3 There exist a sequence $(a_T, b_T)_T$ with $a_T, b_T \to \infty$ as $T \to \infty$, such that
\[
\mathbb{E} \left[ V_T^f(W_T + I_T(u)) + \frac{1}{2} \int_0^T \|u_t\|_{L^2(\Lambda)}^2 \right] = \mathbb{E} \left[ \Psi_T^f(W, I_T(u)) + \lambda \int (I_T(u))^4 + \frac{1}{2} \|I_T(u)\|_{H_T}^2 \right]
\]
where
\[
i_T^u(u) := u_t + \lambda \mathbbm{1}_{t < T} \mathbb{W}_t^{(3)} + \lambda \mathbbm{1}_{t \in [T, \infty)} \mathbb{W}_t - I_T(u)\]
and the functional $\Psi_T^f$ satisfies the following bound
\[
|\Psi_T^f(W, I_T(u))| \leq Q_T(W) + \frac{1}{4} (\|I_T(u)\|_{L^4}^4 + \|I_T(u)\|_{H_T}^2)
\]
where $Q_T(W)$ is a function of $W$ independent of $u$ and such that $\sup_T \mathbb{E}[|Q_T(W)|] < \infty$.

Recall that $I_T^u(u) = \theta(D) I_T(u)$ by (6).

3 Construction of the drift measure

We start now to implement the strategy discussed in the introduction: construct a shifted measure sufficiently similar to $\Phi_3^4$. Intuitively the $\Phi_3^4$ measure should give rise to a canonical process which is a shift of the Gaussian free field with a drift of the form given by eq. (10). Indeed this drift $u$ should be the optimal drift in the variational formula.

A small twist is given by the fact that the relevant Gaussian free field entering these considerations is not the process $W = W(X)$ but that obtained from the shifted canonical process $X^u_t = X_t - \int_0^t u_s ds$ which we denote by
\[
W^u := W(X^u) = W - I(u).
\]
Moreover for technical reasons we have to modify the drift in large scales and add some coercive term which will allow later to prove some useful estimates. We define the functional

$$\Xi_s(W, u) := -\lambda J_s \mathbb{W}_s^3 - \lambda \mathbb{1}_{\{s \geq T\}} J_s(\mathbb{W}_s^{2} > J_s(u)) - J_s(D)^{-1/2}(\|D^{-1/2}W_s^n\|), \quad s \geq 0, \quad (11)$$

where $T > 0$ is a constant which will be fixed later and where we understand all the Wick renormalizations to be given functions of $W$. We look now for the solution $u$ of the equation

$$u = \Xi(Wu, u) = \Xi(W - I(u), u). \quad (12)$$

Expanding the Wick polynomials appearing in $\Xi(W - I(u), u)$ we obtain the equation

$$u_s = \Xi(W - I(u), u) = -\lambda J_s [\mathbb{W}_s^3 - \mathbb{W}_s^2 I_s(u) + 12W_s(I_s(u))^2 - 4(I_s(u))^3]$$

$$- \lambda \mathbb{1}_{\{s \geq T\}} J_s((\mathbb{W}_s^2 - 24W_s I_s(u) + 12(I_s(u))^2)) \lambda J_s(u)$$

$$- \sum_{n=0}^{\infty} \binom{n}{1} J_s(D)^{-1/2} \mathbb{W}_s^{-1/2} (\|D^{-1/2}W_s^n\|) (\|D^{-1/2}I_s(u)\|)^{n-1}$$

for all $s \geq 0$. This is an integral equation for $t \mapsto u_t$ with smooth coefficients depending smoothly on $W$ and can be solved via standard methods. Since the coefficients are of polynomial growth we must expect explosion in finite time, so we have to be careful. Note that for any finite time the process $(u_s)_{s \geq 0}$ has bounded spectral support. As a consequence we can solve the equation in $L^2$ and as long as $\int_0^T \|u\|_{L^2}^2 \, ds$ is finite we can see from the equation that $\text{sup}_{s \leq t} \|u_s\|_{L^2}^2$ is finite. Therefore by the existence of local solutions we have that, for all $N \geq 0$, the stopping time

$$\tau_N := \inf \left\{ t \geq 0 \mid \int_0^t \|u\|_{L^2}^2 \, ds \geq N \right\},$$

is strictly positive $\mathbb{P}$-almost surely and $u$ exists up to the (explosion) time $T_{\exp} := \sup_{N \in \mathbb{N}} \tau_N$. Moreover, by construction, the process $u_t^N := \mathbb{1}_{\{t \leq \tau_N\}} u_t$ satisfies Novikov’s condition, so it is in $\mathbb{H}_c$ and by Girsanov transformation we can define the probability measure on $C(\mathbb{R}_+, \mathbb{C}^{1/2-\varepsilon}(\Lambda))$ given by

$$dQ^u_{\tau_N} := e^{\int_0^{\tau_N} u_t^N \, dX_t - \frac{1}{2} \int_0^{\tau_N} \|u_t^N\|_{L^2(\Lambda)}^2 \, ds} \, d\mathbb{P},$$

and under which $X^u_{\tau_N} = X_t - \int_0^t u_s^N \, ds$ is a cylindrical Brownian motion. In particular $(W_t^u)^{\tau_N}_{t \geq 0}$, given by $\int_0^t J_s \, dX_s^u$ has under $Q^u_{\tau_N}$ the same law as $(W_t)_{t \geq 0}$ has under $\mathbb{P}$. Moreover we have that $W_s^{u^N} = W_s^u$ for $0 \leq s \leq \tau_N$ and that $u$ satisfies the equation

$$u_s = -\lambda J_s \mathbb{W}_s^3 - \lambda \mathbb{1}_{\{s \geq T\}} J_s(\mathbb{W}_s^{2} > J_s(u)) - J_s(D)^{-1/2}(\|D^{-1/2}W_s^n\|), \quad s \in [0, \tau_N], \quad (14)$$

where we introduced the notations $\mathbb{W}_s^{\|u\|^3} := 4\mathbb{W}_s^3$ and $\mathbb{W}_s^{u^2} := 12\mathbb{W}_s^2$.

Note that here the Wick powers are still taken to be given functions of $W$, i.e we are still taking the Wick ordering with respect to the law of $W$ under $\mathbb{P}$ (or the law of $W^u$ under $Q^u$).

If we think to the terms containing $W^u$ as given (that is, we ignore their dependence on $u$), eq. $(14)$ is a linear integral equation in $u$ which can be estimated via Gronwall-type arguments. In order to do so, let us denote by $U : H \mapsto \hat{u}$ the solution map of the equation

$$\hat{u} = \Xi(H, \hat{u}). \quad (15)$$

This last equation is linear and therefore has nice global solutions (let’s say in $C(\mathbb{R}_+, L^2)$) and by uniqueness and eq. $(14)$ we have $u_t = U_t(W^u)$ for $t \in [0, T_{\exp})$. From this perspective the
residual dependence on $u$ will not play any role since under the shifted measure the law of the process $W^u$ does not depend on $u$. By standard paraproduct estimates we have
\[
\|I_t(u)\|_{L^\infty} \lesssim \tilde{H}_t + \int_0^t \mathbb{1}_{\{s \geq T\}} \|J_s^2(W_s^u)^2 \geq I_s^2(u)\|_{L^\infty} ds \\
\lesssim \tilde{H}_t + T^{\varepsilon}\int_0^t (s)^{-3/2} \|W_s^u, \|_{\mathbb{E} \rightarrow \mathbb{E}} - 1 \cdot \|I_s^2(u)\|_{L^\infty} ds,
\]
where we have used the presence of the cutoff $\mathbb{1}_{\{s \geq T\}}$ to introduce the small factor $T^{-\varepsilon}$ and we have employed the notation
\[
\tilde{H}_t = \int_0^t \frac{1}{(s)^{1/2}} \|J_s W_s^u, \|_{\mathbb{E} \rightarrow \mathbb{E}} - 1/3 ds + \int_0^t \frac{1}{(s)^{3/2}} \|\|\|D^{-1/2} W_s^u\|\|_{H^{-1/2}} ds.
\]
Therefore, by Gronwall’s lemma
\[
\sup_{t \leq \tau_N} \|I_t(u)\|_{L^\infty} \lesssim \tilde{H}_{\tau_N} \exp \left( CT^{-\varepsilon} \int_0^{\tau_N} \|W_s^u, \|_{\mathbb{E} \rightarrow \mathbb{E}} - 1 \frac{ds}{(s)^{1+\varepsilon}} \right).
\] (16)
Under $Q^u$, the terms in $\tilde{H}_{\tau_N}$ are all in the $L^p$ spaces by hypercontractivity and moreover for any $p \geq 1$ one can choose $T$ large enough to that also the exponential term is in $L^p$. Using eq. (14) it is then not difficult to show that $\mathbb{E}_{Q_{N_1}} \|u_{N_2}\|_{H^{-1/2}}^p < \infty$ for any $p > 1$ (again provided we take $T$ large enough depending on $p$) as long as $N_1 > N_2$. By the spectral properties of $J$ and the equation for $u$, the process $t \mapsto \mathbb{1}_{\{t \leq T\}} u_t$ is spectrally supported in a ball of radius $T$, so we get in particular that
\[
\mathbb{E}_{Q_{N_1}} \left[ \int_0^{\tau_N \wedge T} \|u_t\|^2_{L^2} ds \right] \lesssim T^{1+\varepsilon},
\]
uniformly for any choice of $N_1 \geq N_2 \geq 0$.

**Lemma 1** The family $(Q^u)^N$ weakly converges to a limit $Q^u$ on $C(\mathbb{R}_+, \mathbb{E}^{-3/2-\varepsilon})$. Under $Q^u$ it holds $T_{exp} = \infty$ almost surely and $\text{Law}_{Q^u}(X^u) = \text{Law}_{\mathbb{P}_\mathbb{F}}(X)$. Moreover for any finite $T$
\[
\frac{dQ^u}{dP}_{\mathbb{F}_T} = \exp \left( \int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|^2_{L^2} ds \right).
\]

**Proof** Consider the filtration $(\mathbb{F}_N = \mathbb{F}_{\tau_N})$ and observe that $(Q^u, \mathbb{F}_N)$ is a consistent family of inner regular probability distributions and therefore there exists a unique extension $Q^u$ to $\mathbb{F}_\infty = \vee_N \mathbb{F}_N$. Next observe that $\{T_{exp} < \infty\} = \bigcup_{T \in \mathbb{N}} \{T_{exp} < T\} \subset \bigcup_{T \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \{\tau_N < T\}$ and that for any $N, T < \infty$, we have
\[
\mathbb{E}_{Q^u} \left[ \int_0^{\tau_N \wedge T} \|u_s\|^2_{L^2} ds \right] = \mathbb{E}_{Q^u} \left[ \int_0^{\tau_N \wedge T} \|u_s\|^2_{L^2} ds \right] \lesssim T^{1+\varepsilon}.
\]
On the event $\{\tau_N \leq T\}$ we have
\[
\int_0^{\tau_N \wedge T} \|u_s\|^2_{L^2} ds = N,
\]
and therefore we also have $Q^u(\{\tau_N \leq T\}) \lesssim C T^{1+\varepsilon} N^{-1}$ which in turn implies $Q^u(T_{exp} < T) = 0$. This proves that $T_{exp} = +\infty$ under $Q^u$, almost surely. As a consequence we can extend $Q^u$ to all of $\mathbb{F} = \vee_T \mathbb{F}_T$ since for any $A \in \mathbb{F}_T$ we can set
\[
Q^u(A) = Q^u(A \cap \{T_{exp} = +\infty\}) = \lim_{N} Q^u(A \cap \{T_{exp} = +\infty, \tau_N \geq T\}) = \lim_{N} Q^u(A \cap \{\tau_N \geq T\}).
\]
If \( A \in \mathcal{F}_T \) then
\[
\mathbb{E}_{Q^u}[1_A(X^u)] = \lim_{N \to \infty} \mathbb{E}_{Q^u}[1_{A \cap \{T \leq TN\}}(X^u)] = \lim_{N \to \infty} \mathbb{E}_{Q^u}[1_{A \cap \{T \leq TN\}}(X^{uN})]
\]
and
\[
\lim_{N \to \infty} \mathbb{E}_{\mathbb{P}}[1_{\{T > TN\}}(X)] = \lim_{N \to \infty} \mathbb{E}_{Q^u}[1_{\{T > TN\}}(X^{uN})] = \lim_{N \to \infty} \mathbb{E}_{Q^u}[1_{\{T > TN\}}(X^u)] \to 0.
\]
This establishes that \( \text{Law}_{Q^u}(X^u) = \text{Law}_{\mathbb{P}}(X) \). On the other hand if \( A \in \mathcal{F}_T \) we have, using the martingale property of the Girsanov density,
\[
\mathbb{E}_{Q^u}[1_A] = \lim_{N \to \infty} \mathbb{E}_{Q^u}[1_{A \cap \{T \leq TN\}}] = \lim_{N \to \infty} \mathbb{E}_{Q^u}[1_{A \cap \{T \leq TN\}}]
\]
\[
= \lim_{N \to \infty} \mathbb{E} \left[ 1_{A \cap \{T \leq TN\}} e^{\int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 ds} \right]
\]
\[
= \lim_{N \to \infty} \mathbb{E} \left[ 1_{A \cap \{T \leq TN\}} e^{\int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 ds} \right].
\]
And also
\[
\lim_{N \to \infty} \mathbb{E} \left[ 1_{A \cap \{T > TN\}} e^{\int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 ds} \right] \leq \lim_{N \to \infty} \mathbb{E} \left[ 1_{\{T > TN\}} e^{\int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 ds} \right]
\]
\[
= \lim_{N \to \infty} \mathbb{E} \left[ 1_{\{T > TN\}} e^{\int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 ds} \right] = \lim_{N \to \infty} \mathbb{E}_{Q^u}[1_{\{T > TN\}}] = 0.
\]
As a consequence
\[
\mathbb{E}_{Q^u}[1_A] = \mathbb{E} \left[ 1_A e^{\int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 ds} \right]
\]
and therefore
\[
\frac{dQ^u|_{\mathcal{F}_T}}{d\mathbb{P}|_{\mathcal{F}_T}} = e^{\int_0^T u_s dX_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 ds},
\]
as claimed. \( \square \)

The following lemma will also be useful in the sequel and it is a consequence of the above discussion:

**Lemma 2** For any \( p > 1 \) there exists a suitable choice of \( \bar{T} \) such that
\[
\mathbb{E}_{Q^u}[\sup_{t \geq 0} \|I_t(u)\|^p_{L^\infty}] < \infty.
\]

**Proof** This follows from the bound (10), after choosing \( \bar{T} \) large enough. \( \square \)

### 3.1 Proof of absolute continuity

In this section we prove that the measure \( \mu_T \) is absolutely continuous with respect to the measure \( Q^u \) we constructed in Lemma 1. First recall that the measures \( \mu_T \) defined on \( \Omega \) as
\[
\frac{d\mu_T}{d\mathbb{P}} = e^{-V_T(W_T)}
\]
can be described, using Lemma 2, as a perturbation of \( Q^u \) with density \( D_T \) given by
\[
D_T := \frac{d\mu_T}{dQ^u} = \frac{d\mu_T}{d\mathbb{P}} \frac{d\mathbb{P}}{dQ^u} = e^{-V_T(W_T) - \int_0^T u_s dX_s + \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 dt},
\]
for at least on \( \mathcal{F}_T \).
Lemma 3 There exists a $p > 1$, such that for any $K > 0$,

$$
\sup_T \mathbb{E}_{Q^u} \left[ |D_T|^p \mathbb{1}_{\{\|W_\infty\|_{\mathcal{F}^{-1/2-\varepsilon}} \leq K \}} \right] < \infty.
$$

In particular, the family $(D_T)_T$ is uniformly integrable under $Q^u$.

The proof of this claim is given in Section 3.2 below.

Corollary 1 The family of measures $(\mu_T)_{T \geq 0}$ is sequentially compact w.r.t. strong convergence on $(\Omega, \mathcal{F})$. Furthermore any accumulation point is absolutely continuous with respect to $Q^u$.

Proof We choose a subsequence (not relabeled) such that $D_T \to D_\infty$ weakly in $L^1(Q^u)$, for some $D_\infty \in L^1(Q^u)$. It always exists by uniform integrability. We now claim that for any $A \in \mathcal{F}$

$$
\lim_{T \to \infty} \mu_T(A) = \int_A D_\infty dQ^u.
$$

It is enough to check this for $A \in \mathcal{F}_S$ for any $S \in \mathbb{R}_+$ since these generate $\mathcal{F}$. But there we have for $T \geq S$,

$$
\mu_T(A) = \int_A D_T dQ^u \to \int_A D_\infty dQ^u
$$

by weak $L^1$ convergence.

Recall that the $\Phi^u_3$ measure can be defined as a weak limit of the measures $\tilde{\mu}_T$ on $\mathcal{F}^{-1/2-\varepsilon}$ given by

$$
\int f(\varphi) \tilde{\mu}_T(\mathrm{d}\varphi) = \int f(\varphi) e^{-V_T(\varphi)} \theta_T(\mathrm{d}\varphi) = \mathbb{E}_\varphi[f(W_T) e^{-V_T(W_T)}]
$$

where $\theta_T$ is the gaussian measure with covariance $\rho(D)(D)^{-2}$. From this together with the above considerations we see that any accumulation point $\tilde{\mu}_\infty$ of $\tilde{\mu}_T$ satisfies

$$
\tilde{\mu}_\infty(A) = \mathbb{E}_{Q^u}[\mathbb{1}_A(W_\infty)D_\infty]
$$

(17)

for some $D_\infty \in L^1(Q^u)$.

3.2 $L^p$ bounds

Now we will prove local $L^p$-bounds on the density $D_T$. In the sequel we will denote $\tilde{W} = W^u$, with $u$ satisfying (13), namely $u = U(\tilde{W})$. Before we proceed let us study how the functional $U(\tilde{W})$ behaves under shifts of $\tilde{W}$, since later we will want to apply the Boué–Dupuis formula and this kind of behavior will be crucial. Let $w \in L^2([0, \infty) \times \Lambda)$ and denote

$$
u_w := U(\tilde{W} + I(w)) \quad \text{and} \quad h_w := U(\tilde{W} + I(w)) + w = u_w + w.
$$

The process $h_w$ satisfies

$$
h_w - w = u_w = \Xi(\tilde{W} + I(w), u_w).
$$

More explicitly, for all $s \geq 0$ we have

$$
h_w^s - w_s = -4\lambda J_s(\tilde{W}_s^2) - 12\lambda J_s(\tilde{W}_s^2) I_s(w) - 12 \lambda J_s \tilde{W}_s(\tilde{I}_s(w))^2 - 4\lambda J_s(I_s(w))^3
$$

$$
-12\lambda \mathbb{1}_{(s \geq T)} J_s(\tilde{W}_s^2) - 24 \lambda \mathbb{1}_{(s \geq T)} (J_s(\tilde{W}_s I_s(w) > I_s(w))))
$$

$$
-12 \lambda \mathbb{1}_{(s \geq T)} J_s((I_s(w))^2 > I_s(w))
$$

$$
- \sum_{i=0}^n \binom{n}{i} J_s((D)^{-1/2} \tilde{W}_s)^i)) \times (D)^{-1/2} I_s(w))^n.
$$
Decomposing\[
\hat{W}_s^2I_s(w) = \hat{W}_s^2 \succ \theta_s I_s(w) + \hat{W}_s^2 \succ (1 - \theta_s) I_s(w) + \hat{W}_s^2 \odot I_s(w) + \hat{W}_s^2 \prec I_s(w),
\]
we can write\[
u^w = U(W + I(w)) = -4\lambda J_s(\hat{W}_s^2 \succ I_s^2(h^w)) + r_s^w, \tag{18}
\]
with\[
r_s^w = -12\lambda J_s(\hat{W}_s^2 \succ I_s^2(h^w)) - 12\lambda J_s(\hat{W}_s^2 \odot I_s(w) - 12\lambda J_s(\hat{W}_s^2 \prec I_s(w))
\]
\[= 24\lambda \mathbb{1}_{s \geq T}(I_s(w) - \theta_s I_s^2(u^w)) - \sum_{i=0}^{n} \binom{n}{i} J_s(D)^{-1/2} \left[ \left( (D)^{-1/2} W_n^2 \right)^i ( (D)^{-1/2} I_s(w) )^{n-i} \right]. \tag{19}\]

The first two terms in \([IS]\) will be used for renormalization while the remainder \(r_s^w\) contains terms of higher regularity which will have to be estimated in the sequel.

**Proof of Lemma**\([IS]\)

Observe that\[
\mathbb{1}\{\|W_n\|_{\infty, \frac{1}{2} - \epsilon < K}\} \lesssim_{K,n} \exp\left(-\|W_n\|_{\frac{1}{2} - \epsilon}^n\right) = \exp\left(-\|\hat{W}_n + I_n(U(\hat{W}))\|_{\frac{1}{2} - \epsilon}^n\right)
\]
and\[
|DT|^p = e^{-p\left[\int_T -1_{\bar{T}}(W + I(U(\bar{W}))) + \int_T^W U(\bar{W})dX + \int_T^W U_i(\bar{W})dX_i + \theta \int_T^W U_i(\bar{W})dX_i \right]^2 dt}.
\]

Combining these two facts we have\[
\mathbb{E}_{Q^n}\left[|DT|^p \mathbb{1}\{\|W\|_{\frac{1}{2} - \epsilon} < K\}\right] \lesssim \mathbb{E}_{Q^n}\left[\exp\left(-p\left(V_T(W_T + I_T(U(W))) + T_0^U U_i(W)dX_i + \theta \int_T U_i(W)dX_i + \theta \int_T U_i(W)dX_i \right)^2 dt\right)\right]
\]
\[\lesssim \mathbb{E}\left[\exp\left(-p\left(V_T(W_T + I_T(U(W))) + T_0^U U_i(W)dX_i + \theta \int_T U_i(W)dX_i + \theta \int_T U_i(W)dX_i \right)^2 dt\right)\right].
\]

The Boué–Dupuis formula\([IS]\) provides the variational bound\[
-\log \mathbb{E}_{Q^n}\left[|DT|^p \mathbb{1}\{\|W\|_{\frac{1}{2} - \epsilon} < K\}\right] \gtrsim \inf_{w \in \mathcal{R}_n} \mathbb{E}\left[p\left(V_T(W_T + I_T(h^w)) + T_0^U U_i(h^w)dX_i + \theta \int_T U_i(h^w)dX_i + \theta \int_T U_i(h^w)dX_i \right)^2 dt\right] + \mathbb{E}\left[p\left(V_T(W_T + I_T(h^w)) + T_0^U U_i(h^w)dX_i + \theta \int_T U_i(h^w)dX_i + \theta \int_T U_i(h^w)dX_i \right)^2 dt\right]
\]
where we have set \(h^w = w + U(W + I(w))\) as above. Recall now that from Theorem\([IS]\) there exists a constant \(C\), independent of \(T\), such that for each \(h^w\),\[
\mathbb{E}\left[p\left(V_T(W_T + I_T(h^w)) + \frac{1}{2} \int_0^T \|h^w\|^2 dt\right)\right] \geq -C + \frac{1}{4} \mathbb{E}_{\mathbb{P}}\left[\lambda I_T(h^w)\right]^4 + \frac{1}{2} \mathbb{E}_{\mathbb{P}}\left[\lambda I_T(h^w)\right]^2
\]
where\[
I_T^2(h^w) = h^w_t + \lambda \mathbb{1}_{s \leq T} \mathbb{W}_t^3 + \lambda \mathbb{1}_{s \leq T} J_t(\mathbb{W}_t^2 - I_T^2(h^w))
\]
Using eq.\([IS]\) we compute\[
\mathbb{1}_{s \leq T} I_t^2(h^w) = \mathbb{1}_{s \leq T} h_t^w + \lambda \mathbb{1}_{s \leq T} \mathbb{W}_t^3 + \lambda \mathbb{1}_{s \leq T} J_t(\mathbb{W}_t^2 - I_T^2(h^w))
\]
\[= \mathbb{1}_{s \leq T}(u^w_t + w_t) + \lambda \mathbb{1}_{s \leq T} \mathbb{W}_t^3 + \lambda \mathbb{1}_{s \leq T} J_t(\mathbb{W}_t^2 - I_T^2(h^w))
\]
\[= \mathbb{1}_{s \leq T}(r^w_t + w_t).\]
At this point we need a lower bound for

$$\mathbb{E}\left[ \frac{1}{t} \left( \lambda \| I_t(h)^n \|_{L^4}^4 + \int_0^T \| r_t^w + w_t \|_{L^2}^2 dt \right) + \frac{1}{2} \int_0^T \| w_t \|_{L^2}^2 dt + \| W_\infty + I_\infty(h) \|_{Q_{-1/2-\varepsilon}}^n \right] - C. $$

Given that we need to take $p > 1$, this expression presents a difficulty in the fact that the term $\int_0^T \| w_t \|_{L^2}^2 dt$ appears with a negative coefficient. Note that this term cannot easily be controlled via $\int_0^T \| r_t^w + w_t \|_{L^2}^2 dt$ since the contribution $r_t^w$, see eq. (19), contains factors which are homogeneous in $w$ of order up to $3$. This is the reason we had to localize the estimate, introduce the “good” term $\| W_\infty + I_\infty(h) \|_{Q_{-1/2-\varepsilon}}$, and introduce the term $J_s(D)^{-1/2}(\| \langle D \rangle^{-1/2} W_s \|^n)$ in (11) which will help us to control the growth of $r_t^w$. Indeed in Lemma 4 below, a Gronwall argument will allow us to show that $\int_0^T \| w_t \|_{L^2}^2 dt$ can be bounded by a combination of the other “good” terms as

$$\mathbb{E}\left[ \int_0^T \| w_t \|_{L^2}^2 dt \right] \lesssim \mathbb{E}\left[ \| I_T^2(h) \|_{L^4}^2 + \| P_T^2(h) \|_{Q_{-1/2-\varepsilon}}^n \right] + \int_0^T \| w_t + r_t^w \|_{L^2}^2 dt + 1].$$

This implies that for $1 < p \ll 2$,

$$- \log \mathbb{E}_{Q_u} \left[ |D_T|^p 1_{\{\| W \|_{Q_{-1/2-\varepsilon}} < K \}} \right] \geq \inf_{w \in \mathbb{R}_u} \mathbb{E} \left\{ \frac{1}{4} \| I_T^2(h)^n \|_{L^4}^4 + \int_0^T \| I_T^2(h)^n \|_{L^6}^6 dt \right\}$$

$$+ (1 - p)C \left[ \| P_T^2(h)^n \|_{L^4}^4 + \| P_T^2(h)^n \|_{Q_{-1/2-\varepsilon}}^n \right] + \| W_\infty + I_\infty(h)^n \|_{Q_{-1/2-\varepsilon}}^n - C$$

which gives the claim. Note that here we used the bound

$$\mathbb{E}\| I_\infty(h)^n \|_{Q_{-1/2-\varepsilon}}^n \lesssim \mathbb{E}\| W_\infty \|_{Q_{-1/2-\varepsilon}}^n + \mathbb{E}\| W_\infty + I_\infty(h)^n \|_{Q_{-1/2-\varepsilon}}^n$$

$$\lesssim C + \mathbb{E}\| W_\infty + I_\infty(h)^n \|_{Q_{-1/2-\varepsilon}}^n$$

as well as the fact that $\| P_T^2(h)^n \|_{Q_{-1/2-\varepsilon}} \lesssim \| I_\infty(h)^n \|_{Q_{-1/2-\varepsilon}}$ to conclude. \hfill \Box

The following lemmas complete the proof.

Lemma 4 For $n \in \mathbb{N}$ odd and large enough

$$\mathbb{E}\int_0^T \| w_s \|_{L^2}^2 ds \lesssim \mathbb{E}\int_0^T \| w_s + r_s^w \|_{L^2}^2 ds + \mathbb{E}\| P_T^2(h)^n \|_{Q_{-1/2-\varepsilon}}^n + \| P_T^2(h)^n \|_{L^4}^4 + 1.$$

Proof Let us introduce the notation

$$\text{Aux}_n(W, w) := \sum_{i=0}^{n} \binom{n}{i} J_s(D)^{-1/2}(\| D^{-1/2} W_s \|^n)(\langle D \rangle^{-1/2} I_s(w))^{n-i}).$$

Write $r_s^w = \tilde{r}_s^w + \text{Aux}(W, w)$ and observe that

$$w_s^2 = 2(w_s + r_s^w)^2 - 4w_s r_s^w - 2(r_s^w)^2 - w_s^2$$

$$= 2(w_s + r_s^w)^2 - 4w_s \tilde{r}_s^w - 2(r_s^w)^2 - w_s^2 - 4 \text{Aux}(W, w)$$

We can apply Itô formula to obtain

$$\int_0^T \int_\Lambda \text{Aux}_n(W, w) w_s ds = \overline{\text{Aux}(W, w)} + \text{martingale}$$

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where
\[
\tilde{\text{Aux}}_T(W, w) := \sum_{i=0}^{n} \frac{1}{n+1-i} \binom{n}{i} \int_{\Lambda} \|((D)^{-1/2}W_T)^i((D)^{-1/2}I_T(w))^{n+1-i}\|.
\]

By Lemma 5 below, we have constants \(c, C\) and a random variable \(Q_T(W)\) such that
\[
\sup_T \mathbb{E}[|Q_T(W)|] < \infty
\]
and
\[
c \int_0^T \|w_s\|_2^2 ds + c\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1} - Q_T(W) \leq \int_0^T \|w_s\|_2^2 ds + \tilde{\text{Aux}}_T(W, w)
\[
\leq C\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1} + C \int_0^T \|w_s\|_2^2 ds + Q_T(W).
\]

Integrating over space we obtain
\[
\frac{1}{dt} \mathbb{E}\left(\int_0^t \|w_s\|_2^2 ds + \tilde{\text{Aux}}_t(W, w)\right) \leq 2\mathbb{E}[\|w_t + r^w_t\|_2^2] - 4\mathbb{E}[w_t r^w_t - \|w_s\|_2^2]
\[
\leq 2\mathbb{E}[\|w_t + r^w_t\|_2^2] + 4\mathbb{E}[r^w_t \|w_s\|_2^2].
\]

Now by Lemma 6 below
\[
\langle t \rangle^{1+\varepsilon} \|\tilde{r}^w_t\|_2^2 \lesssim \int_0^t \|w_s\|_2^2 ds + \|I_t(w)\|_{W^{-1/2,n+1}}^{n+1} + \|P_t^R(h^w)\|_{L^1}^{n+1} + \|P_t^R(h^w)\|_{L^4}^{n+1} + Q_t(W)
\]
\[
\lesssim Q_t(W) + \int_0^t \|w_s\|_2^2 ds + \tilde{\text{Aux}}_t(W, w) + \|P_t^R(h^w)\|_{L^1}^{n+1} + \|P_t^R(h^w)\|_{L^4}^{n+1}.
\]

Gathering things together we have
\[
\frac{1}{\langle t \rangle^{1+\varepsilon}} \left(\int_0^t \|w_s\|_2^2 ds + \tilde{\text{Aux}}_t(W, w)\right)
\]
\[
\lesssim \frac{1}{\langle t \rangle^{1+\varepsilon}} \left(\int_0^t \|w_s\|_2^2 ds + \tilde{\text{Aux}}_t(W, w)\right) + \frac{1}{\langle t \rangle^{1+\varepsilon}} \left(\|P_t^R(h^w)\|_{L^1}^{n+1} + \|P_t^R(h^w)\|_{L^4}^{n+1}\right).
\]

Gronwall’s lemma allows then to conclude
\[
-1 + \mathbb{E}\left(\int_0^t \|w_s\|_2^2 ds\right) \lesssim \mathbb{E}\left(\int_0^t \|w_s\|_2^2 ds + \tilde{\text{Aux}}_t(W, w)\right)
\]
\[
\lesssim \mathbb{E}\left(\int_0^t \|w_s + r^w_s\|_2^2 ds + \|P_t^R(h^w)\|_{L^1}^{n+1} + \|P_t^R(h^w)\|_{L^4}^{n+1}\right) + 1.
\]

\[
\square
\]

**Lemma 5** There exists constants \(c, C\) and a random variable \(Q_T(W)\) such that
\[
\sup_T \mathbb{E}[|Q_T(W)|] < \infty,
\]
and
\[
-\frac{Q_T(W)}{2} + c \int_0^T \|w_s\|_2^2 ds + c\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1}
\]
\[
\leq \int_0^T \|w_s\|_2^2 ds + \tilde{\text{Aux}}_T(W, w)
\]
\[
\leq C\|I_T(w)\|_{W^{-1/2,n+1}}^{n+1} + C \int_0^T \|w_s\|_2^2 ds + Q_T(W)
\]

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Proof We recall that
\[ \overline{\text{Aux}}_T(W, w) = \sum_{i=0}^{n-1} \sum_{n+1-i}^{n} \int \left[ (D)^{-1/2} W_T^i (D)^{-1/2} I_T(w)^{n+1-i} \right] \]
\[ = \sum_{i=1}^{n} \sum_{n+1-i}^{n} \int \left[ (D)^{-1/2} W_T^i (D)^{-1/2} I_T(w)^{n+1-i} \right] \]
\[ + \frac{1}{n+1} \| I_T(w) \|_{W^{-1/2, n+1}} \]
and since \( \sup_{T<\infty} \mathbb{E} \left[ \| (D)^{-1/2} W_T^n \|_{L^p_{\theta, \epsilon}} \right] < \infty \) for any \( p < \infty \) and any \( \epsilon > 0 \) it is enough to bound \( \| (D)^{-1/2} I_T(w) \|_{B_{2,1}^1} \) for some \( q > 1 \) by the terms \( \| I_T(w) \|^n_{W^{-1/2, n+1}} \) and \( \| I_T(w) \|^2_{H_1} \) \( \leq \int_0^T \| w_s \|^2_{L^2} ds \). By interpolation we can estimate, for \( i \geq 1 \),
\[ \| (D)^{-1/2} I_T(w) \|_{B_{2,1}^1} \lesssim \| (D)^{-1/2} I_T(w) \|_{B_{2,1}^n} + C \]
\[ \lesssim \| I_T(w) \|_{W^{-1/2, n+1}} \| I_T(w) \|_{H_1} + C \]
Choosing \( q = n/ \left( n - \frac{1}{n-1} \right) > 1 \), we have
\[ \left( \| I_T(w) \|_{W^{-1/2, n+1}} \| I_T(w) \|_{H_1} \right)^q = \| I_T(w) \|^n_{W^{-1/2, n+1}} \| I_T(w) \|^{(n-1)/n} \|_{H_1}. \]
Now for \( n \) large enough \( \frac{n}{n-1} \leq \frac{2}{n+1} \) and using Young’s inequality we can estimate
\[ \| I_T(w) \|^n_{W^{-1/2, n+1}} \| I_T(w) \|^{(n-1)/n} \|_{H_1} \lesssim \| I_T(w) \|^n_{W^{-1/2, n+1}} \left( \| I_T(w) \|^2_{H_1} + 1 \right) \]
\[ \lesssim \| I_T(w) \|_{W^{-1/2, n+1}} + \| I_T(w) \|^2_{H_1} + 1 \]
Lemma 6 Let
\[ \tilde{r}_s^w = -12 \lambda J_s(W_s^2) \tilde{r}_s^w \geq (1 - \theta_s) I_s(w) + 12 \lambda J_s([W_s^2] \tilde{r}_s^w) + 12 \lambda J_s(W_s^2 \tilde{r}_s^w) \lesssim I_s(w) \]
\[ -12 \lambda J_s W_s (I_s(w))^2 - 4 \lambda J_s (I_s(w))^3 - 24 \lambda J_s (W_s I_s(w) \tilde{r}_s^w) + \lambda \tilde{r}_s^w \]
Setting \( h^w = u + w \), there exists a random variable \( Q_t(W) \) such that \( \sup_t \mathbb{E} |Q_t(W)| < \infty \) and
\[ (t)^{1+\epsilon} \| \tilde{r}_s^w \|^2 \leq \int_0^t \| w_s \|^2_{L^2} ds + \| I_t(w) \|_{W^{-1/2, n+1}} + \| I_t(h^w) \|_{L^2} + Q_t(W). \]
Proof Note that
\[ \| 1_{[s \leq T]} J_s(W_s^2 \tilde{r}_s^w \tilde{r}_s^w) \|^2_{L^2} \lesssim T \]
\[ \lesssim T \varepsilon \] for \( \| W_s^2 \|_{L^2} \) and \( \| I_t(h^w) \|_{L^2} + \| I_t(w) \|_{L^2} \). Moreover \( h^w = u^w + w \) implies
\[ \| I_t(h^w) \|_{L^2} \lesssim \| I_t(w) \|_{L^2} + \| I_t(u^w) \|_{L^2} \]
and \( \| I_t(u^w) \|_{L^2} \lesssim \| I_t(h^w) \|_{L^2} + \| I_t(w) \|_{L^2} \). From Lemma 6 we get
\[ \| I_t(w) \|_{L^2} \lesssim C + \int_0^t \| w_s \|^2_{L^2} ds + \| I_t(w) \|_{W^{-1/2, n+1}}. \]
The estimation for the other terms is easy but technical and postponed until Section 5. \( \square \)
4 Singularity of $\Phi_3^4$ w.r.t. the free field

The goal of this section is to prove that the $\Phi_3^4$ measure is singular with respect to the Gaussian free field. For this we have to find a set $S \subseteq C^{-1/2-\varepsilon}(\Lambda)$ such that $P(W_\infty \in S) = 1$ and $Q^u(W_\infty \in S) = 0$. Together with (17), this will imply singularity. We claim that setting

$$S := \left\{ f \in C^{-1/2-\varepsilon}(\Lambda) : \frac{1}{T_{n}^{1/2+\varepsilon}} \int_\Lambda \| (\theta_{T_n} f)^4 \| \to 0 \right\}$$

for some suitable subsequence $T_n$, does the job. Here

$$\| (\theta_{T} f)^4 \| = \| (\theta_{T} f)^4 \| - 6E[(\theta_{T} W_\infty(0))^2](\theta_{T} f)^2 + 3E[(\theta_{T} W_\infty(0))^2]^2$$

denotes the Wick ordering with respect to the Gaussian free field. Let us prove first that indeed $P(W_\infty \in S) = 1$ for some $T_n$. For later use we note that

$$\mathbb{W}_t^{\theta_{T} - 3} = 4(\theta_{T} W_t)^3 - 12E[(\theta_{T} W_t(0))^2](\theta_{T} W_t)$$

and

$$\mathbb{W}_t^{\theta_{T} - 2} = 12((\theta_{T} W_t)^2 - E[(\theta_{T} W_t(0))^2]).$$

Lemma 7 For any $\delta > 0$

$$\lim_{T \to \infty} E \left[ \left( \frac{1}{T(1+\delta)^2} \int_\Lambda \| (\theta_{T} W_\infty)^4 \| \right)^2 \right] = 0.$$

Proof Wick products corresponds to iterated Itô integrals. Introducing the notation

$$dw_{t}^{\theta_{T}} = \theta_{T} J_t dX_t,$$

we can verify by Itô formula that

$$\int_\Lambda \| \theta_{T} W_\infty^4 \| = \int_0^\infty \int_\Lambda \mathbb{W}_t^{\theta_{T} - 3} dw_{t}^{\theta_{T}} = \int_0^\infty \theta_{T} J_t \mathbb{W}_t^{\theta_{T} - 3} dX_t.$$

Since $\theta_{T} J_t = 0$ for $t \geq T$, Itô isometry gives

$$E \left[ \int_0^\infty \theta_{T} J_t \mathbb{W}_t^{\theta_{T} - 3} dX_t \right]^2 = \int_0^T \int_\Lambda (\theta_{T} J_t \mathbb{W}_t^{\theta_{T} - 3})^2 dt.$$

Then, again by Itô formula the expectation on the r.h.s. can be estimated as

$$E \left[ \int_\Lambda (\mathbb{W}_t^{\theta_{T} - 3})^2 \right] = 4E \left[ \sum_{k_1, k_2, k_3} \int_0^t \int_0^{t_1} \int_0^{t_2} dw_{t}^{\theta_{T}}(k_1) dw_{t}^{\theta_{T}}(k_2) dw_{t}^{\theta_{T}}(k_3) \right]^2$$

$$= 24E \left[ \sum_{k_1, k_2, k_3} \int_0^t \int_0^{t_1} \int_0^{t_2} \frac{\theta_{T}^2(k_1) \theta_{T}^2(k_2) \theta_{T}^2(k_3)}{(k_1)^2 (k_2)^2 (k_3)^2} ds_1 ds_2 ds_3 \right]$$

$$\leq 24E \left[ \sum_{k_1, k_2, k_3} \int_0^t \int_0^{t_1} \int_0^{t_2} \frac{\sigma_{t_1}^2(k_1) \sigma_{t_2}^2(k_2) \sigma_{t_3}^2(k_3)}{(k_1)^2 (k_2)^2 (k_3)^2} ds_1 ds_2 ds_3 \right]$$

$$\lesssim t^3.$$
Now recall that $\|J_t f\|_{L^2(\Lambda)} \lesssim (t)^{-3/2}\|f\|_{L^2(\Lambda)}$ to conclude:

$$E \left[ \frac{1}{T^{1+\delta}} \int_0^T \int_\Lambda (\theta_t J_t \mathbb{W}^{\theta_t,3}_t)^2 \, dt \right] \leq \frac{1}{T^{1+\delta}} \int_0^T \frac{1}{\ell^3} E[\|\theta_t \mathbb{W}^{\theta_t,3}_t\|_{L^2(\Lambda)}^2] \, dt \to 0$$

\[\square\]

The lemma implies that $\frac{1}{T^{1+\delta}} \int_\Lambda [(\theta_t W_\infty)^4] \to 0$ in $L^2(\mathbb{P})$. So there exists a subsequence $T_n$ such that $\frac{1}{T_n^{1+\delta}} \int_\Lambda [(\theta_{T_n} W_\infty)^4] \to 0$ almost surely.

The next step of the proof is to check that $Q^u(W_\infty \in S) = 0$. More concretely we will show that for a subsequence of $T_n$ (not relabeled)

$$\frac{1}{T_n^{1+\delta}} \int_\Lambda [(\theta_{T_n} W_\infty)^4] \to -\infty,$$

$Q^u$ almost surely. Observe that

$$\int_\Lambda [(\theta_{T} W_\infty)^4] = \int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t) \, dX_t$$

$$= \int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t dX_t) + \int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t u_t) \, dt$$

$$= \int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t dX_t) - \lambda \int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t) J_t (\mathbb{W}^{u,3}_t)_t \, dt$$

$$- \lambda \int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t) J_t (\mathbb{W}^{u,3}_t)_t \to \mathbb{P}(u) \, dt$$

We expect the term

$$\int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t) J_t (\mathbb{W}^{u,3}_t)_t \, dt$$

to go to infinity faster than $T^{1-\delta}$, $Q^u$-almost surely. To actually prove it, we start by a computation in average.

**Lemma 8** It holds

$$\lim_{T \to \infty} \frac{1}{T^{1-\delta}} E \left[ \int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t) J_t (\mathbb{W}^{u,3}_t)_t \, dt \right] = \infty.$$

**Proof** Recall that $d\mathbb{W}^{\theta_t}_t = \theta_t J_t \, dX_t$. With a slight abuse of notation we can write

$$\int_0^\infty \int_\Lambda (\theta_{T} J_t \mathbb{W}^{\theta_{T},3}_t) J_t (\mathbb{W}^{u,3}_t)_t \, dt$$

$$= 16 \int_0^\infty \sum_k \frac{\theta_T(k) \sigma^2_k(k)}{\langle k \rangle^2} \left( \sum_{k_1 + k_2 + k_3 = k} \int_0^t \int_0^{s_1}dw_{s_1} \int_0^{s_2}dw_{s_2} \int_0^{s_3}dw_{s_3} \delta_{s_1} \delta_{s_2} \delta_{s_3} \right) \, dt$$

and by Ito isometry

$$E \left[ \sum_{k_1 + k_2 + k_3 = k} \int_0^t \int_0^{s_1}dw_{s_1} \int_0^{s_2}dw_{s_2} \int_0^{s_3}dw_{s_3} \right] = 6 \sum_{k_1 + k_2 + k_3 = k} \int_0^t \int_0^{s_1}dw_{s_1} \int_0^{s_2}dw_{s_2} \int_0^{s_3}dw_{s_3} \delta_{s_1} \delta_{s_2} \delta_{s_3}$$

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For $T$ large enough and since $\sigma^2$ and $\theta$ are positive, we have
\[
\int_0^\infty \sum_{k} \theta_T(k) \frac{\sigma^2(k)}{\langle k \rangle^2} \sum_{k_1+k_2+k_3=k} \int_0^t \int_0^{t_1} \theta_T(k_1) \sigma^2_{\langle k \rangle^2} \theta_T(k_2) \sigma^2_{\langle k \rangle^2} \theta_T(k_3) \sigma^2_{\langle k \rangle^2} \frac{ds_1 ds_2 ds_3 dt}{\langle k \rangle^2}
\geq \int_{T/8}^{T/2} \sum_k \frac{\sigma^2(k)}{\langle k \rangle^2} \sum_{k_1+k_2+k_3=k} \int_0^{T/8} \int_0^{t_1} \sigma^2_{\langle k \rangle^2} \theta_T(k_1) \sigma^2_{\langle k \rangle^2} \theta_T(k_2) \sigma^2_{\langle k \rangle^2} \theta_T(k_3) \sigma^2_{\langle k \rangle^2} \frac{ds_1 ds_2 ds_3 dt}{\langle k \rangle^2}
\introduce the notation $Z^3_+ = \{ n \in Z^3 : n = (n_1, n_2, n_3) \text{ with } n_i \geq 0 \}$. After restricting the sum to $(Z^3_+)^3$ we get the bound
\[
\geq \frac{1}{T^2} \sum_{k \in Z^3_+} (\rho_T(k) - \rho_{T/8}(k)) \sum_{k_1+k_2+k_3=k} \int_{T/32}^{T/32} \int_{3T/32}^{3T/32} \int_{3T/32}^{3T/32} \sigma^2_{\langle k \rangle^2} \theta_T(k_1) \sigma^2_{\langle k \rangle^2} \theta_T(k_2) \sigma^2_{\langle k \rangle^2} \theta_T(k_3) \sigma^2_{\langle k \rangle^2} \frac{ds_1 ds_2 ds_3 dt}{\langle k \rangle^2}.
\]
Now, for large enough $T$ if $k_1+k_2+k_3 = k$ and $\langle k \rangle \leq T/8$ then $\langle k \rangle \leq T/2 \times 0.9$. Furthermore if $T$ large enough and $k_1, k_2, k_3 \in Z^3_+$ and $k_1+k_2+k_3 = k$, while $\langle k \rangle > (3T/32)\times 0.9$ (recall that if $\langle k \rangle < (3T/32)\times 0.9$ and $s > 3T/32$ then $\sigma_s(k_1) = 0$) we have $\langle k \rangle \geq T/8$. So for any $k$ for which the integral is nonzero we have $\rho_T(k) - \rho_{T/8}(k) = 1$ (recall that $\rho = 1$ on $B(0,9/10)$ and $\rho = 0$ outside of $B(0,1)$). This implies
\[
\frac{1}{T^2} \sum_{k \in Z^3_+} (\rho_T(k) - \rho_{T/8}(k)) \sum_{k_1+k_2+k_3=k} \int_{T/32}^{T/32} \int_{3T/32}^{3T/32} \int_{3T/32}^{3T/32} \sigma^2_{\langle k \rangle^2} \theta_T(k_1) \sigma^2_{\langle k \rangle^2} \theta_T(k_2) \sigma^2_{\langle k \rangle^2} \theta_T(k_3) \sigma^2_{\langle k \rangle^2} \frac{ds_1 ds_2 ds_3 dt}{\langle k \rangle^2}.
\]
Next we upgrade this bound to almost sure divergence.

**Lemma 9** There exists a $\delta_0 > 0$ such that for any $\delta_0 \geq \delta > 0$, there exists a sequence $(T_n)_n$ such that $\mathbb{P}$ – almost surely
\[
\frac{1}{T_n^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_{T_n} J_l W_l^{\theta_{T_n},3}) J_l W_l^2 dt \to \infty.
\]

**Proof** Define
\[
G_T := \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_{T} J_l W_l^{\theta_{T},3}) J_l W_l^2 dt + \sup_{t<\infty} \|W_l\|_{\ell^{-1/2}-\varepsilon}^K.
\]
We will show that $e^{-G_T} \to 0$ in $L^1(\mathbb{P})$, which implies that there exists a subsequence $T_n$ such that $e^{-G_{T_n}} \to 0$ almost surely. From this our statement follows. By the Bouch–Dupuis formula
\[
- \log \mathbb{E}[e^{-G_T}] = \inf_{v \in \mathbb{M}_a} \mathbb{E} \left[ \frac{1}{T^{1-\delta}} 16 \int_0^\infty \int_\Lambda (\theta_{T} J_l \theta_{T}((W_l + I_l(v))^3)) J_l ((W_l + I_l(v))^3) dt + \right.
\]
\[
+ \sup_{t<\infty} \|W_l + I_l(v)\|_{\ell^{-1/2}-\varepsilon}^K + \frac{1}{2} \int_0^\infty \|v_t\|_{L^2}^2 dt \right].
\]
Our aim now to prove that the last three terms are bounded below uniformly as

\[ |E_t| \lesssim \| \theta_{t} \|^2 \| \theta_{t} \|^3 \int_0^T A_i B_i^3 dt \]

\[ + \sup_{t<\infty} \| W_t + I_t(v) \|_{H^1}^2 + \frac{1}{2} \int_0^\infty \| v_t \|_{L^2}^2 dt \]

\[ \geq \inf_{v \in H^1} \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \int_0^\infty \int_0^T \sum_{i,j \in \{0,1,2,3\}} \frac{1}{T^{1-\delta}} \int_0^T A_i B_i^3 dt \right] \]

where where have used that \( \theta_{t} J_t = 0 \) for \( t \geq T \) and introduced the notations, for \( 0 \leq i \leq 3 \),

\[ A_i^3 := 4 \binom{3}{i} J_t \theta_{t} \left( \| (\theta_{t} W_t)^{3-i} \| \| (\theta_{t} I_t(v))^i \| \right), \]

and

\[ B_i^3 := 4 \binom{3}{i} J_t \left( \| W_t^{3-i} \| \right) \left( \| I_t(v) \| \right). \]

Our aim now to prove that the last three terms are bounded below uniformly as \( T \to \infty \) (while we already know that the first one diverges). For \( i \in \{1, 2, 3\} \)

\[ \| A_i \|^1_{L^2} + \| B_i \|^1_{L^2} \lesssim \langle t \rangle^{-1-\delta} \left( \| I_t(u) \|^K_{H^1} + \| I_t(u) \|^2_{H^1} + Q_t(W) \right) \]

by Lemmas 13 and 15. Here \( Q_t(W) \) is a random variable only depending on \( W \) such that \( \sup_t \mathbb{E}[\| Q_t(W) \|^p] < \infty \) for any \( p < \infty \). Then

\[ \frac{1}{T^{1-\delta}} \sum_{i,j \in \{0,1,2,3\}} \int_0^T \int_0^T A_i B_i^3 dt \]

\[ \leq \frac{1}{T^{1-\delta}} \sum_{i,j \in \{0,1,2,3\}} \int_0^T \| A_i \|^1_{L^2} + \| B_i \|^1_{L^2} dt + \frac{1}{T^{1-\delta}} \sum_{i \in \{1,2,3\}} \int_0^T \| A_i \|^2_{L^2} \| B_i \|^2_{L^2} dt + \frac{1}{T^{1-\delta}} \sum_{i \in \{1,2,3\}} \int_0^T \| A_i \|^2_{L^2} \| B_i \|^2_{L^2} dt. \]

Now for the first term we obtain

\[ \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \sum_{i,j \in \{0,1,2,3\}} \int_0^T \int_0^T (\oint_{t}^{t} \| A_i \|^1_{L^2} \| B_i \|^1_{L^2} dt \right] \]

\[ = \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \sum_{i,j \in \{0,1,2,3\}} \int_0^T (\oint_{t}^{t} \| I_t(u) \|^K_{H^1} + \| I_t(u) \|^2_{H^1} + Q_t(W) \) dt \right] \]

\[ = \frac{C}{T^{1-\delta}} \mathbb{E} \left[ \sup_t \left( \| I_t(u) \|^K_{H^1} + \| I_t(u) \|^2_{H^1} + Q_t(W) \right) \right] + \frac{C}{T^{1-\delta}}. \]

For the second term we use that \( \| A_i \|^1_{L^2} \leq Q_t(W) \) so

\[ \frac{1}{T^{1-\delta}} \mathbb{E} \left[ \int_0^T \| A_i \|^1_{L^2} \| B_i \|^1_{L^2} dt \right] \]

\[ \leq \frac{1}{T^{1-\delta}} \mathbb{E} \left[ \int_0^T (\oint_{t}^{t} \| A_i \|^1_{L^2} \| B_i \|^1_{L^2} dt + \int_0^T (\oint_{t}^{t} \| B_i \|^1_{L^2} dt \right] \]

\[ \leq \frac{1}{T^{1-\delta}} \mathbb{E} \left[ \int_0^T (\oint_{t}^{t} \| A_i \|^1_{L^2} \| B_i \|^1_{L^2} dt \right] + \frac{1}{T^{1-\delta}} \mathbb{E} \left[ \int_0^T (\oint_{t}^{t} \| I_t(u) \|^K_{H^1} + \| I_t(u) \|^2_{H^1} + Q_t(W) \) dt \right] \]

\[ \leq \frac{C}{T^{1/2-\delta}} \mathbb{E} \left[ \sup_t \left( \| I_t(u) \|^K_{H^1} + \| I_t(u) \|^2_{H^1} + Q_t(W) \right) \right] + \frac{C}{T^{1/2-\delta}}. \]
Since \( \sup_t ||I_t(u)||_{H^1}^2 \leq \int_0^\infty ||v_t||_{L^2}^2 dt \) in total we obtain for \( T \) large enough. The third term is estimated analogously.

\[
- \log \mathbb{E}[e^{-G_T}] \\
\geq \inf_{v \in H_n} \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^3 dt + \left( \frac{1}{2} - \frac{C}{T^{1/2-2\delta}} \right) \sup_{t < \infty} ||I_t(v)||_{\mathcal{F}^{-1/2-\varepsilon}}^2 \right] \\
- C \sup_{t < \infty} ||W_t||_{\mathcal{F}^{-1/2-\varepsilon}} + \left( \frac{1}{2} - \frac{C}{T^{1/2-2\delta}} \right) \int_0^\infty ||v_t||_{L^2}^2 dt - \frac{C}{T^{1/2-2\delta}} \\
\geq \mathbb{E} \left[ \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^3 dt \right] - C \rightarrow \infty
\]

Next we prove an estimate which will help with the proof of the main theorem.

**Lemma 10** We have

\[
\sup_T \mathbb{E}_{\mathbb{Q}^u} \left[ \int_0^\infty \int_\Lambda \frac{1}{t^{1+\delta}} (\theta_T J_t \mathbb{W}_t^{\theta_T,3})^2 dt \right] < \infty.
\]

Furthermore, there exists a (deterministic) subsequence \((T_n)_n\) such that

\[
\frac{1}{T_n^{1/2+\delta}} \left| \int_0^\infty \int_\Lambda \theta_{T_n} J_t \mathbb{W}_t^{\theta_T,3} dX_t^u \right| \to 0
\]

\( \mathbb{Q}^u \) almost surely.

**Proof** Recall that under \( \mathbb{Q}^u \) we have \( W_t = W_t^u + I_t(u) \) where \( u \) is defined above by (13) and \( \text{Law}_{\mathbb{Q}^u}(W^u) = \text{Law}_{\mathbb{P}}(W) \). With this in mind we compute

\[
\int_0^T \int_\Lambda \frac{1}{t^{1+\delta}} (\theta_T J_t \mathbb{W}_t^{\theta_T,3})^2 dt = \sum_{i,j} \int_0^T \int_\Lambda \frac{1}{t^{1+\delta}} A_i^t A_j^t dt,
\]

where, as above,

\[
A_i^t = 4 \binom{3}{i} J_t \theta_T ((\theta_T W_t^u)^{3-i})(\theta_T I_t(u))^i).
\]

By Lemmas [13] and [15] we have that \( \mathbb{E}_{\mathbb{Q}^u}[||A_i^t||_{L^2}^2] \leq C \) so the Cauchy–Schwarz inequality gives the result.

**Theorem 4** There exists a sequence \((T_n)_n\) such that, \( \mathbb{Q}^u \) almost surely,

\[
\frac{1}{T_n^{1-\delta}} \int_\Lambda [(\theta_T W_{T_n})^4] \to -\infty.
\]

**Proof** We have

\[
\int_\Lambda [(\theta_T W_{T_n})^4] = \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} dX_t.
\]

Now since \( dX_t = dX_t^u + u_t dt \) we have

\[
\frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} dX_t \\
= \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} dX_t^u + \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} u_t dt \\
= \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} dX_t^u - \frac{1}{T} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \mathbb{W}_t^{u,3} dt \\
- \frac{1}{T} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t (\mathbb{W}_t^{u,2} - I_t^u(u)) dt \\
- \frac{1}{T} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t (D)^{-1/2} ((D)^{-1/2} W_t^n) dt.
\]

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The first term goes to 0 $\mathbb{Q}^u$-almost surely by Lemma 10. To analyze the second term we estimate

$$\frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,3}_t J_t(\mathbb{W}^{u,2}_i) \mathbb{L}^\ast(u) dt = \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,3}_t J_t(\mathbb{W}^{u,2}_i) \mathbb{L}^\ast(u) dt \leq \frac{1}{T^{1-\delta}} \int_0^T \|\theta_T J_i \mathbb{W}^{\theta,i,3}_t\|_{L^2} \|J_t(\mathbb{W}^{u,2}_i)\|_{L^2} dt \leq \frac{1}{T^{1-\delta}} \int_0^T t^{-1/2} \|\theta_T J_i \mathbb{W}^{\theta,i,3}_t\|_{L^2} \|\mathbb{W}^{u,2}_i\|_{L^2} \|I_t(u)\|_{L^2} dt \leq T^{-1/2-2\delta} \left( \int_T \|\theta_T J_i \mathbb{W}^{\theta,i,3}_t\|_{L^2} dt \right)^{1/2} \times T^{-1/2+2\delta} \left( \int_T t^{1-\delta} \|\mathbb{W}^{u,2}_i\|_{L^2} \|I_t(u)\|_{L^2} \right)^{1/2}$$

(20)

By the computation from Lemma 11 we have

$$\mathbb{E}_{\mathbb{Q}^u} \left[ T^{-1/2-2\delta} \left( \int_T \|\theta_T J_i \mathbb{W}^{\theta,i,3}_t\|_{L^2} dt \right)^{1/2} \right] \to 0,$$

and $\sup_t \mathbb{E}_{\mathbb{Q}^u}[(\|\mathbb{W}^{u,2}_i\|_{L^2} \|I_t(u)\|_{L^2}^2) < \infty$, so (20) converges to 0 in $L^1(\mathbb{Q}^u)$. For the fourth term we proceed in the same way:

$$\int_0^\infty \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,3}_t J_t(D)^{-1/2} \mathbb{W}^i_t dt = \int_0^T \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,3}_t J_t(D)^{-1/2} \mathbb{W}^i_t dt \leq \int_0^T \|\theta_T J_i \mathbb{W}^{\theta,i,3}_t\|_{L^2} \|J_t(D)^{-1/2} \mathbb{W}^i_t\|_{L^2} dt \leq \int_0^T (\|\theta_T J_i \mathbb{W}^{\theta,i,3}_t\|_{L^2} t^{1-2\delta} \|J_t(D)^{-1/2} \mathbb{W}^i_t\|_{L^2} dt \leq \left( \int_0^T t^{-2(1-\delta)} \|\theta_T J_i \mathbb{W}^{\theta,i,3}_t\|_{L^2} dt \right)^{1/2} \left( \int_0^T t^{-2(1-\delta)} \|J_t(D)^{-1/2} \mathbb{W}^i_t\|_{L^2}^2 dt \right)^{1/2}$$

which is bounded in expectation uniformly in $T$, so the fourth term goes to 0 in $L^1(\mathbb{Q}^u)$ as well. It remains to analyze the second term. Again introducing the notation

$$A_i^j = 4 \binom{3}{i} J_i \theta_T(\mathbb{W}^{u,3}_i)^j (\theta_T I_t(u)^i),$$

$$\mathbb{W}^{\theta,i,u,3}_t = 4[(\theta_T \mathbb{W}^{u,3}_t)],$$

we have

$$\frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,3}_t J_t \mathbb{W}^{u,3}_t dt = \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,3}_t \mathbb{W}^{u,3}_t dt + \sum_{1 \leq i \leq 3} \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda A_i^j J_i \mathbb{W}^{u,3}_t dt.$$

Now observe that

$$\frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,u,3}_t J_i \mathbb{W}^{u,3}_t dt \mathbb{Q}^u \sim \mathbb{Q}^u \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda \theta_T J_i \mathbb{W}^{\theta,i,3}_t J_i \mathbb{W}^{u,3}_t dt,$$

so the lim sup of this is $\infty$ almost surely. To estimate the sum we again observe that for $i \geq 3$

$$\mathbb{E}_{\mathbb{Q}^u}(\|A_i^j\|_{L^2}^2) \lesssim \langle t \rangle^{-1-\delta}$$

and by Young's inequality

$$\int_0^T \int_\Lambda A_i^j J_i \mathbb{W}^{\theta,i,3}_t dt \leq \int_0^T \int_\Lambda A_i^j \mathbb{L}^\ast J_i \mathbb{W}^{u,3}_t dt \leq \int_0^T \int_\Lambda (t)^{1/3} \|A_i^j\|_{L^2} \langle t \rangle^{-1/3} \|J_i \mathbb{W}^{u,3}_t\|_{L^2} dt \leq \int_0^T \int_\Lambda (t)^{2/3} \|A_i^j\|_{L^2}^2 + \int_0^T \int_\Lambda (t)^{-2/3} \|J_i \mathbb{W}^{u,3}_t\|_{L^2}^2 dt.$$
Taking expectation we obtain
\[
\frac{1}{T^{1-\delta}} \mathbb{E} \left[ \int_0^T \int_\Lambda A_t^i J_t \mathbb{W}_t^{u,3} dt \right] 
\leq \frac{1}{T^{1-\delta}} \mathbb{E} \left[ \int_0^T \int_\Lambda (t)^{2/3} \|A_t^i\|_{L^2}^2 \right] + \frac{1}{T^{1-\delta}} \mathbb{E} \left[ \int_0^T \int_\Lambda (t)^{-2/3} \|J_t \mathbb{W}_t^{u,3}\|_{L^2}^2 dt \right] 
\approx \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda (t)^{-1/3+\delta} dt + \frac{1}{T^{1-\delta}} \int_0^T \int_\Lambda (t)^{-2/3} dt \to 0
\]
We have deduced that
\[
\frac{1}{T^{1-\delta}} \int_\Lambda [(\theta_T W_\infty)^3] = -\frac{1}{T^{1-\delta}} \int_\Lambda \int_0^T \theta_T J_t \mathbb{W}_t^{\theta_{T,u,3}} J_t \mathbb{W}_t^{u,3} dt + R_T,
\]
where $R_T \to 0$ in $L^1(\mathbb{Q}^u)$. We can conclude by selecting a subsequence $(T_n)_n$ such that
\[
\frac{1}{T_n^{1-\delta}} \int_0^{T_n} \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_{T,u,3}} J_t \mathbb{W}_t^{u,3} dt \to \infty
\]
$\mathbb{Q}^u$-almost surely and $R_{T_n} \to 0$, $\mathbb{Q}^u$-almost surely. □

## 5 Some analytic estimates

We collect in this final section various technical estimates needed to complete the proof of Lemma 6.

**Proposition 1** Let $1 < p < \infty$ and $p_1, p_2, p'_1, p'_2 > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} = \frac{1}{p'}$. Then for every $s, \alpha > 0$
\[
\|\langle D \rangle^s (f g)\|_{L^p} \lesssim \|\langle D \rangle^s f\|_{L^{p_1}} \|\langle D \rangle^{-\alpha} g\|_{L^{p_1}} + \|\langle D \rangle^{s+\alpha} g\|_{L^{p'_1}} \|\langle D \rangle^{-\alpha} f\|_{L^{p'_2}}.
\]
**Proof** See [15]. □

**Lemma 11** There exists $\varepsilon > 0, n \in \mathbb{N}$ such that for any $\delta > 0$ there exists $C_{\delta} < \infty$ for which the following inequality holds for any $\phi \in H^1(\Lambda)$
\[
\|\phi\|_{L^4}^{4+\varepsilon} \lesssim C\|\phi\|_{W^{-1/2,n+1}}^{n+1} + \delta \|\phi\|_{H^1}^2 + C_{\delta}.
\]
**Proof**
\[
\int \phi^4 dx \quad \lesssim \quad \|\langle D \rangle^{-1/2} \phi\|_{L^8} \|\langle D \rangle^{1/2} \phi\|_{L^8/7}^3 
\lesssim \quad \|\langle D \rangle^{-1/2} \phi\|_{L^8} \|\langle D \rangle^{1/2} \phi\|_{L^{8/5}}^2 \|\phi\|_{L^4}^2 
\lesssim \quad \|\langle D \rangle^{-1/2} \phi\|_{L^8} \|\phi\|_{H^1}^{1/2} \|\phi\|_{L^4}^{5/2}
\]
So
\[
\|\phi\|_{L^4}^{21/20} \quad \lesssim \quad \|\langle D \rangle^{-1/2} \phi\|_{L^8}^{21/20} \|\phi\|_{H^1}^{21/40} \|\phi\|_{L^4}^{104/40}
\]
and applying Young’s inequality with the exponents $(32, 32/9, 32/22)$, we obtain
\[
\|\langle D \rangle^{-1/2} \phi\|_{L^8}^{21/20} \|\phi\|_{H^1}^{21/40} \|\phi\|_{L^4}^{104/40} \quad \lesssim \quad C_{\delta} \|\langle D \rangle^{-1/2} \phi\|_{L^8}^{168/5} + \delta \|\phi\|_{H^1}^{16/9} + \delta \|\phi\|_{L^4}^{208/55} 
\lesssim \quad \|\langle D \rangle^{-1/2} \phi\|_{L^8}^{33/5} + \delta \|\phi\|_{H^1}^2 + \delta \|\phi\|_{L^4}^{4} \|\phi\|_{L^4}^{21/20} + C_{\delta}
\]
and subtracting $\delta(\|\phi\|_{L^4}^{4})^{21/20}$ on both sides of the inequality gives the result. □
Lemma 12 The following estimates hold with \( \varepsilon > 0 \) small enough

\[
\|J_t(\|W_t^2\|) \times (1 - \theta_t)I_t(w)\|_{L^2}^2 \lesssim \frac{1}{(t)^{1+\varepsilon}} \left( \int_0^t \|w_s\|^2 ds + \|I_t(w)\|_{W^{1-2/\varepsilon, +1}} + \|W_t^2\|_{C^{1-\varepsilon}} \right)
\]

\[
\|J_t(\|W_t^2\| \circ I_t(w))\|_{L^2}^2 \lesssim \frac{1}{(t)^{1+\varepsilon}} \left( \int_0^t \|w_s\|^2 ds + \|I_t(w)\|_{W^{1-2/\varepsilon, +1}} + \|W_t^2\|_{C^{1-\varepsilon}} \right)
\]

\[
\|J_tW_t^2 \times I_t(w)\|_{L^2}^2 \lesssim \frac{1}{(t)^{1+\varepsilon}} \left( \int_0^t \|w_s\|^2 ds + \|I_t(w)\|_{W^{1-2/\varepsilon, +1}} + \|W_t^2\|_{C^{1-\varepsilon}} \right)
\]

**Proof** We observe that since \( W_t^2 \) is spectrally supported in a ball or radius \( \sim t \)

\[
\|W_t^2\|_{C^{1-\varepsilon}} \lesssim (t)^{2\varepsilon} \|W_t^2\|_{C^{1-\varepsilon}}.
\]

For the first estimate we know that \( (1 - \theta_t)I_t(w) \) is supported in an annulus of radius \( \sim t \), so \( \|(1 - \theta_t)I_t(w)\|_{L^2} \lesssim (t)^{-1-\varepsilon} \|I_t(w)\|_{H^{1-\varepsilon}} \) and furthermore by interpolation \( \|I_t(w)\|_{H^{1-\varepsilon}} \lesssim \|I_t(w)\|_{H^{1-\varepsilon}} \leq \|I_t(w)\|_{H^{1+\varepsilon}} \|I_t(w)\|_{H^{1-\varepsilon}} \)

For the second term in addition observe that the function \( (t)^{1/2}J_t \) is uniformly bounded Fourier multiplier reguarlizing by 1, and putting everything together, by paraproduct estimates by 1 and putting everything together, by paraproduct estimates by 1 and putting everything together,

\[
\|J_t(\|W_t^2\| \circ I_t(w))\|_{L^2}^2 \lesssim (t)^{-1} \left( t \right)^{2\varepsilon} \left( t \right)^{-2/\varepsilon} \|W_t^2\|_{C^{1-\varepsilon}} \|I_t(w)\|_{H^{1-\varepsilon}}^2
\]

\[
\lesssim (t)^{-1} \left( t \right)^{2\varepsilon} \left( t \right)^{-2/\varepsilon} \|W_t^2\|_{C^{1-\varepsilon}} \|I_t(w)\|_{H^{1-\varepsilon}}^2
\]

\[
\lesssim (t)^{-3/2} \left( \|W_t^2\|_{C^{1-\varepsilon}} + \|I_t(w)\|_{H^{1-\varepsilon}} + \|I_t(w)\|_{L^4}^4 \right)
\]

\[
\lesssim (t)^{-3/2} \left( \int_0^t \|w_s\|^2 ds + \|I_t(w)\|_{W^{1-2/\varepsilon, +1}} + \|W_t^2\|_{C^{1-\varepsilon}} \right)
\]

For the third estimate again applying paraproduct estimates and the properties of \( J \),

\[
\|J_t(\|W_t^2\| \times I_t(w))\|_{L^2}^2 \lesssim (t)^{-3} \left( \|W_t^2\|_{C^{1-\varepsilon}} \|I_t(w)\|_{H^{1-\varepsilon}} \right)
\]

Now, the claim follows from interpolation and Young’s inequality

\[
\|W_t^2\|_{C^{1-\varepsilon}} \|I_t(w)\|_{H^{1-\varepsilon}} \lesssim \|W_t^2\|_{C^{1-\varepsilon}} \|I_t(w)\|_{H^{1-\varepsilon}} \|I_t(w)\|_{L^4}^2
\]

\[
\lesssim \|W_t^2\|_{C^{1-\varepsilon}} \|I_t(w)\|_{H^{1-\varepsilon}} + \|I_t(w)\|_{H^1} + \|I_t(w)\|_{L^4}^2
\]

\[
\lesssim \left( \int_0^t \|w_s\|^2 ds + \|I_t(w)\|_{W^{1-2/\varepsilon, +1}} + \|W_t^2\|_{C^{1-\varepsilon}} \right)
\]

\[
\Box
\]

Lemma 13 Let \( f \in C([0, \infty], C^{1-2/\varepsilon}) \) and \( g \in C([0, \infty], H^1) \) such that \( f_t, g_t \) have spectral support in a ball of radius proportional to \( t \). There exists \( n \in \mathbb{N} \) such that the following estimates hold:

\[
\|J_t(f_tg_t^n)\|_{L^2}^2 \lesssim (t)^{-3/2} \|f_t\|_{C^{1-2/\varepsilon}} \|g_t\|_{L^4}^4,
\]

\[
\|J_t(f_tg_t^n)\|_{L^2}^2 \lesssim (t)^{-3/2} \left( \|f_t\|_{C^{1-2/\varepsilon}} + \|g_t\|_{H^1}^2 + \|g_t\|_{W^{1-2/\varepsilon}}^2 \right),
\]

and

\[
\|J_t(g_t^3)\|_{L^2}^2 \lesssim (t)^{-3/2} \left( \|g_t\|_{H^1}^2 + \|g_t\|_{W^{1-2/\varepsilon}}^2 \right).
\]
Proof By the spectral properties of $J_t$,\[
\|J_t(f_t g_t^2)\|^2_{L^2} \lesssim \langle t \rangle^{-3} \|f_t\|^2_{L^\infty} \|g_t\|^4_{L^4} \lesssim \langle t \rangle^{-3/2} \|f_t\|^2_{\dot{C}^{-1/2-\delta}} \|g_t\|^4_{L^4}.
\]
Applying Young’s inequality with exponents $\left(\frac{n}{2}, \frac{n/2}{n/2-1}\right)$ with $n$ such that $\frac{2n}{(n/2-1)} \leq 4+\varepsilon$ where $\varepsilon$ is chosen as in Lemma 11 we have\[
\langle t \rangle^{-3/2} \|f_t\|^2_{\dot{C}^{-1/2-\delta}} \|g_t\|^4_{L^4} \lesssim \langle t \rangle^{-3/2} \left(\|f_t\|^n_{\dot{C}^{-1/2-\delta}} + \|g_t\|^n_{L^4}\right) \lesssim \langle t \rangle^{-3/2} \left(\|f_t\|^n_{\dot{C}^{-1/2-\delta}} + \|g_t\|^n_{W^{-1/2,n}} + \|g_t\|^2_{H^1}\right).
\]
Now the second estimate follows from choosing $n$ large enough (depending on $\delta$) and using Besov embedding after taking $f = g$.

Lemma 14 The following estimates hold\[
\langle t \rangle^{1+\varepsilon} \|J_s(W_t I_t(w) - I_t^2(u))\|^2_{L^2} \lesssim \|I_t(w)\|^4_{L^4} + \|I_t^2(u)\|^4_{L^4} + \|W_t\|^n_{\dot{C}^{-1/2-\delta}},

\langle t \rangle^{1+\varepsilon} \|J_s((I_t(w))^2 - I_t^2(u))\|^2_{L^2} \lesssim \|I_t(w)\|^4_{L^4} + \|I_t^2(u)\|^4_{L^4}.
\]
Proof For the first estimate we again use the spectral properties of $W, I, J$ and obtain by paraproduct estimate\[
\|J_s(W_t I_t(w) - I_t^2(u))\|^2_{L^2} \lesssim \langle t \rangle^{-3} \|W_t\|^2_{L^\infty} \|I_t(w)\|^2_{L^4} \|I_t^2(u)\|^2_{L^4}
\lesssim \langle t \rangle^{-3} \langle t \rangle^{1+\varepsilon} \|W_t\|^2_{\dot{C}^{-1/2-\delta}} \|I_t(w)\|^2_{L^4} \|I_t^2(u)\|^2_{L^4}
\]
and the claim follows by Young’s inequality. For the second\[
\|J_s((I_t(w))^2 - I_t^2(u))\|^2_{L^2} \lesssim \langle t \rangle^{2-2\varepsilon} \langle (I_t(w))^2_{L^4} \|I_t^2(u)\|^2_{\dot{C}^{-1/2-\delta}},
\]
and the claim follows again by Young’s inequality.

Lemma 15 Let $f_t \in C([0, \infty], \dot{C}^{-1/2-\delta})$ and $g_t \in C([0, \infty], H^1)$ such that $f_t, g_t$ have spectral support in a ball of radius proportional to $t$. Then the following estimates hold\[
\|(J_t(f_t g_t))^2\|^2_{L^2} \lesssim \langle t \rangle^{-1+2\delta} \|f_t\|^2_{\dot{C}^{-1/2-\delta}} \|g_t\|^2_{L^2}

\|(J_t(f_t g_t))^2\|^2_{L^2} \lesssim \langle t \rangle^{-1+2\delta} \langle f_t\|^2_{\dot{C}^{-1/2-\delta}} + \|g_t\|^2_{H^{-1}}.
\]
Proof\[
\|(J_t(f_t g_t))^2\|^2_{L^2} \lesssim \langle t \rangle^{-3} \|f_t\|^2_{L^\infty} \|g_t\|^2_{L^2} \lesssim \langle t \rangle^{-1+2\delta} \|f_t\|^2_{\dot{C}^{-1/2-\delta}} \|g_t\|^2_{L^2}.
\]
This proves the first estimate. For the second we continue\[
\langle t \rangle^{-1+2\delta} \|f_t\|^2_{\dot{C}^{-1/2-\delta}} \|g_t\|^2_{L^2} \lesssim \langle t \rangle^{-1+2\delta} \|f_t\|^2_{\dot{C}^{-1/2-\delta}} \|g_t\|^2_{H^{-1}} 
\lesssim \langle t \rangle^{-1+2\delta} \langle f_t\|^2_{\dot{C}^{-1/2-\delta}} + \|g_t\|^2_{H^{-1}}.
\]

Lemma 16 It holds\[
\int_0^T \int_L (J_t(W_t^2 - I_t^2(w)))^2 \lesssim T^{3\delta} \sup_t \|W_t^2\|^2_{\dot{C}^{-1/2-\delta}} (\sup_t \|I_t(w)\|^2_{L^2}),
\]
and\[
\int_0^T \int_L (J_t(W_t^2 - I_t^2(w)))^2 \lesssim T^{3\delta} \left(\sup_t \|I_t(w)\|^4_{H^{-1}} + \int_0^T \|w_t\|^2_{L^2} dt + \sup_t \|W_t^2\|^8_{\dot{C}^{-1/2-\delta}}\right).
\]
Proof This follows in the same fashion as Lemma 15.
A Besov spaces and paraproducts

In this section we will recall some well known results about Besov spaces, embeddings, Fourier multipliers and paraproducts. The reader can find full details and proofs in [3,14].

First recall the definition of Littlewood–Paley blocks. Let $\chi, \varphi$ be smooth radial functions $\mathbb{R}^d \to \mathbb{R}$ such that

- $\text{supp } \chi \subseteq B(0, R)$, $\text{supp } \varphi \subseteq B(0, 2R) \setminus B(0, R)$;
- $0 \leq \chi, \varphi \leq 1$, $\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1$ for any $\xi \in \mathbb{R}^d$;
- $\text{supp } \varphi(2^{-j} \cdot) \cap \text{supp } \varphi(2^{-i} \cdot) = \emptyset$ if $|i - j| > 1$.

Introduce the notations $\varphi_{-1} = \chi$, $\varphi_j = \varphi(2^{-j} \cdot)$ for $j \geq 0$. For any $f \in \mathcal{S}'(\Lambda)$ we define the operators $\Delta_j f := \mathcal{F}^{-1}_x(\varphi_j(\xi) \hat{f}(\xi))$, $j \geq -1$.

**Definition 1** Let $s \in \mathbb{R}, p, q \in [1, \infty]$. For a Schwarz distribution $f \in \mathcal{S}'(\Lambda)$ define the norm

$$
\|f\|_{B^s_{p,q}} := \|\langle \Delta_j f \|_{L^p}\rangle_{j\geq-1}\|_{L^q}.
$$

Then the space $B^s_{p,q}$ is the closure of Schwarz distributions under this norm. We denote $C^\alpha = B^{\alpha,\infty}_{\infty,\infty}$ the Besov–Hölder space and $H^\alpha = B^{\alpha,\infty}_{2,2}$ the Sobolev spaces.

**Proposition 2** Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Then $B^s_{p_1,q_1}$ is continuously embedded in $B^s_{p_2,q_2}$.

**Proposition 3** For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, any $p, q \in [1, \infty]$ the Besov space $B^{s_1}_{p,q}$ is compactly embedded into $B^{s_2}_{p,q}$.

**Definition 2** Let $f, g \in \mathcal{S}(\Lambda)$. We define the paraproducts

$$
\langle f \rangle g := \sum_{j < i - 1} \Delta_j f \Delta_j g, \quad \text{and} \quad f \langle g \rangle := \sum_{j > i + 1} \Delta_i f \Delta_j g = g \langle f \rangle.
$$

Moreover we introduce the resonant product

$$
\langle f \rangle g := \sum_{|i - j| \leq 1} \Delta_i f \Delta_j g.
$$

Then $fg = f \langle g \rangle + f \langle f \rangle + f \langle g \rangle$.

**Proposition 4** Let $\alpha, \beta \in \mathbb{R}$. For $f, g \in \mathcal{S}(\Lambda)$ we have the estimates

$$
\|f \langle g \rangle\|_{H^{\beta - \delta}} \lesssim \|f\|_{\mathcal{E}^\beta}\|g\|_{L^2}, \quad \|f \langle g \rangle\|_{H^{\beta - \alpha}} \lesssim \|f\|_{\mathcal{E}^\beta}\|g\|_{H^\alpha},
$$

$$
\|f \langle g \rangle\|_{L^\infty} \lesssim \|f\|_{\mathcal{E}^\beta}\|g\|_{L^\infty}, \quad \|f \langle g \rangle\|_{H^{\alpha}} \lesssim \|f\|_{\mathcal{E}^\beta}\|g\|_{H^{\alpha}}.
$$

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$. Then

$$
\|f \langle g \rangle\|_{H^{\alpha + \beta}} \lesssim \|f\|_{\mathcal{E}^\beta}\|g\|_{H^{\alpha}}, \quad \|f \langle g \rangle\|_{\mathcal{E}^{\alpha + \beta}} \lesssim \|f\|_{\mathcal{E}^\beta}\|g\|_{\mathcal{E}^{\alpha}}.
$$

By density the paraproduct and resonant product also extend to bilinear operators on the respective spaces.
Proposition 5 Let $\alpha \in (0, 1) \beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$, $\alpha + \beta + \gamma > 0$. Then for $f, g, h \in \mathcal{S}$, and for any $\delta > 0$,

$$
\| (f \circ g) \circ h - (f \circ h) \|_{H^{\alpha + \beta + \gamma - \delta}} \lesssim \| f \|_{\mathcal{E}^\gamma} \| h \|_{\mathcal{E}^\delta} \| g \|_{H^\alpha},
$$

and

$$
\| (f \circ g) \circ h - (f \circ h) \|_{H^{\alpha + \beta + \gamma}} \lesssim \| f \|_{\mathcal{E}^\gamma} \| h \|_{\mathcal{E}^\delta} \| g \|_{H^\alpha}.
$$

Proposition 6 Assume $f \in \mathcal{C}^\alpha, g \in H^\beta, h \in H^\gamma$ and $\alpha + \beta + \gamma = 0$. Then

$$
\int_{\mathbb{T}^d} [(f \circ g)h - (f \circ h)g] \lesssim \| f \|_{\mathcal{E}^\alpha} \| g \|_{H^\beta} \| h \|_{H^\gamma}.
$$

Remark 1 Proposition 6 is not proven in the above references but is quite easy and the reader can fill out a proof.

Definition 3 A smooth function $\eta : \mathbb{R}^d \to \mathbb{R}$ is said to be an $S^m$-multiplier if for every multi-index $\alpha$ there exists a constant $C_\alpha$ such that

$$
\left| \frac{\partial^\alpha}{\partial \xi^\alpha} f(\xi) \right| \lesssim (1 + |\xi|)^{m-|\alpha|}, \quad \forall \xi \in \mathbb{R}^d. \tag{21}
$$

We say that a family $(\eta_t)_{t \geq 0}$ is a uniform $S^m$-multiplier if (21) is satisfied for every $\eta_t$ with $C_\alpha$ independent of $t \geq 0$.

Proposition 7 Let $\eta$ be an $S^m$-multiplier, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $f \in B^s_{p,q}(\mathbb{T}^d)$, then

$$
\| \eta(D)f \|_{B^{s-m}_{p,q}} \lesssim \| f \|_{B^s_{p,q}}.
$$

Furthermore the constant depends only on $s, p, q, d$ and the constants $C_\alpha$ in (21).

Proposition 8 Assume $m \leq 0$, $\alpha \in (0, 1), \beta \in \mathbb{R}$. Let $\eta$ be an $S^m$-multiplier, $f \in \mathcal{C}^\beta$, $g \in H^\alpha$. Then for any $\delta > 0$,

$$
\| \eta(D)(f \circ g) - (\eta(D)f \circ g) \|_{H^{\alpha + \beta + m - \delta}} \lesssim \| f \|_{\mathcal{E}^\delta} \| g \|_{H^\alpha}.
$$

Again the constant depends only on $\alpha, \beta, \delta$ and the constants in (21).

Proposition 9 Let $\delta > 0$. We have for any $q_1, q_2 \in [1, \infty], q_1 < q_2$

$$
\| f \|_{B^q_{p,\delta}} \leq \| f \|_{B^{q_1}_{p,\delta}} \leq \| f \|_{B^{q_2}_{p,\delta}}.
$$

Furthermore, if we denote by $W^{s,p}, s \in \mathbb{R}, p \in [1, \infty]$ the fractional Sobolev spaces defined by the norm $\| f \|_{W^{s,p}} := \| (\eta(D))^s f \|_{L^p}$, then, for any $q \in [1, \infty]$,

$$
\| f \|_{B^s_{p,q}} \leq \| f \|_{W^{s,p}} \leq \| f \|_{B^{s+2\delta}_{p,q}}.
$$

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