Bounded Languages Meet Cellular Automata with Sparse Communication

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Cellular automata are one-dimensional arrays of interconnected interacting finite automata. We investigate one of the weakest classes, the real-time one-way cellular automata, and impose an additional restriction on their inter-cell communication by bounding the number of allowed uses of the links between cells. Moreover, we consider the devices as acceptors for bounded languages in order to explore the borderline at which non-trivial decidability problems of cellular automata classes become decidable. It is shown that even devices with drastically reduced communication, that is, each two neighboring cells may communicate only constantly often, accept bounded languages that are not semilinear. If the number of communications is at least logarithmic in the length of the input, several problems are undecidable. The same result is obtained for classes where the total number of communications during a computation is linearly bounded.

1 Introduction

Cellular automata are linear arrays of identical copies of deterministic finite automata, where the single nodes, which are called cells, are homogeneously connected to both their immediate neighbors. They work synchronously at discrete time steps. In the general case, in every time step the state of each cell is communicated to its neighbors. That is, on the one hand the state is sent regardless of whether it is really required, and on the other hand, the number of bits sent is determined by the number of states. Devices with bounded bandwidth of the inter-cell links are considered in [12, 19, 20, 23]. In [22] two-way cellular automata are considered where the number of proper state changes is bounded. There are strong relations to inter-cell communication. Roughly speaking, a cell can remember the states received from its neighbors. As long as these do not change, no communication is necessary.

Due to their temporal and structural restrictions real-time one-way cellular automata define one of the weakest classes of cellular automata. However, they are still powerful enough to accept non-context-free (even non-semilinear) languages (see, e.g., the surveys [10, 11]). Moreover, almost all of the commonly investigated decidability questions are known not to be semidecidable [14]. In order to explore the borderline at which non-trivial decidability problems become decidable, additional structural and computational restrictions have been imposed. Here, we investigate real-time one-way cellular automata where the communication is quantitatively measured by the number of uses of the links between cells. Bounds on the sum of all communications of a computation, as well as bounds on the maximal number of communications that may appear between each two cells are considered. Reducing the communication drastically, but still enough to have non-trivial devices, we obtain systems where each two neighboring cells may communicate only constantly often, and systems where the total number of communications during a computation depends linearly on the length of the input. However, it has been shown in [13] that even these restrictions do not lead to decidable properties.

An approach often investigated and widely accepted is to consider a given type of device for special purposes only, for example, for the acceptance of languages having a certain structure or form. From
this point of view it is natural to start with unary languages (e.g., [1 4 9 16 17]). For general real-time one-way cellular automata it is known that they accept only regular unary languages [18]. Since the proof is constructive, we derive that the borderline in question has been crossed. So, we generalize unary languages to bounded languages. For several devices it is known that they accept non-semilinear languages in general, but only semilinear bounded languages. Since for semilinear sets several properties are decidable [5], constructive proofs lead to decidable properties for these devices in connection with bounded languages [4 6 7 8].

2 Definitions and preliminaries

We denote the positive integers and zero \{0, 1, 2, \ldots\} by \mathbb{N}. The empty word is denoted by \lambda, the reversal of a word \(w\) by \(w^R\), and for the length of \(w\) we write \(|w|\). For the number of occurrences of a subword \(x\) in \(w\) we use the notation \(|w|_x\). We use \(\subseteq\) for inclusions and \(\subset\) for strict inclusions. A language \(L\) over some alphabet \{\(a_1, a_2, \ldots, a_k\)\} is said to be bounded, if \(L \subseteq a_1^*a_2^*\cdots a_k^*\).

A cellular automaton is a linear array of identical deterministic finite state machines, sometimes called cells. Except for the leftmost cell and rightmost cell each one is connected to both its nearest neighbors. We identify the cells by positive integers. The state transition depends on the current state of each cell and on the information which is currently sent by its neighbors. The information sent by a cell depends on its current state and is determined by so-called communication functions. The two outermost cells receive a boundary symbol on their free input lines once during the first time step from the outside world. Subsequently, these input lines are never used again. A formal definition is

**Definition 1.** A cellular automaton (CA) is a system \((S, F, A, B, \#, b_l, b_r, \delta)\), where \(S\) is the finite, non-empty set of cell states, \(F \subseteq S\) is the set of accepting states, \(A \subseteq S\) is the nonempty set of input symbols, \(B\) is the set of communication symbols, \(# \notin B\) is the boundary symbol, \(b_l, b_r: S \to B \cup \{\bot\}\) are communication functions which determine the information to be sent to the left and right neighbors, where \(\bot\) means nothing to send, and \(\delta: (B \cup \{\#, \bot\}) \times S \times (B \cup \{\#, \bot\}) \to S\) is the local transition function.

A configuration of a cellular automaton \((S, F, A, B, \#, b_l, b_r, \delta)\) at time \(t \geq 0\) is a description of its global state, which is actually a mapping \(c_t: \mathbb{N} \to S\), for \(n \geq 1\). The operation starts at time 0 in a so-called initial configuration. For a given input \(w = a_1 \cdots a_n \in A^+\) we set \(c_0(w)(i) = a_i\), for \(1 \leq i \leq n\). During the course of its computation a CA steps through a sequence of configurations, whereby successor configurations are computed according to the global transition function \(\Delta\): Let \(c_t, t \geq 0\), be a configuration. Then its successor configuration \(c_{t+1} = \Delta(c_t)\) is as follows. For \(2 \leq i \leq n - 1\),

\[
c_{t+1}(i) = \delta(b_r(c_t(i-1)), c_t(i), b_l(c_t(i+1))),
\]

and for the leftmost and rightmost cell we set

\[
c_1(1) = \delta(\#, c_0(1), b_l(c_0(2))),
\]

\[
c_{t+1}(1) = \delta(\bot, c_t(1), b_l(c_t(2))), \text{ for } t \geq 1, \text{ and}
\]

\[
c_1(n) = \delta(b_r(c_0(n-1)), c_0(n), \#),
\]

\[
c_{t+1}(n) = \delta(b_r(c_t(n-1)), c_t(n), \bot), \text{ for } t \geq 1.
\]

Thus, the global transition function \(\Delta\) is induced by \(\delta\).

An input \(w\) is accepted by a CA \(\mathcal{M}\) if at some time \(i\) during the course of its computation the leftmost cell enters an accepting state. The language accepted by \(\mathcal{M}\) is denoted by \(L(\mathcal{M})\). Let \(t: \mathbb{N} \to \mathbb{N}\),
Let \( t(n) \geq n \), be a mapping. If all \( w \in L(M) \) are accepted with at most \( t(|w|) \) time steps, then \( M \) is said to be of time complexity \( t \).

An important subclass of cellular automata are the so-called one-way cellular automata (OCA), where the flow of information is restricted to one way from right to left. For a formal definition it suffices to require that \( b_r \) maps all states to \( \bot \), and that the leftmost cell does not receive the boundary symbol during the first time step.

In the following we study the impact of communication in cellular automata. Communication is measured by the number of uses of the links between cells. It is understood that whenever a communication symbol not equal to \( \bot \) is sent, a communication takes place. Here we do not distinguish whether either or both neighboring cells use the link. More precisely, the number of communications between cell \( i \) and cell \( i+1 \) up to time step \( t \) is defined by

\[
\text{com}(i, t) = |\{ j \mid 0 \leq j < t \text{ and } (b_r(c_j(i)) \neq \bot \text{ or } b_l(c_j(i+1)) \neq \bot \}|.
\]

For computations we now distinguish the maximal number of communications between two cells and the total number of communications. Let \( c_0, c_1, \ldots, c_{t(|w|)} \) be the sequence of configurations computed on input \( w \) by some cellular automaton with time complexity \( t(n) \), that is, the computation on \( w \). Then we define

\[
\text{mcom}(w) = \max\{ \text{com}(i, t(|w|)) \mid 1 \leq i \leq |w| - 1 \} \quad \text{and} \quad \text{scom}(w) = \sum_{i=1}^{|w|-1} \text{com}(i, t(|w|)).
\]

Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be a mapping. If all \( w \in L(M) \) are accepted with computations where \( \text{mcom}(w) \leq f(|w|) \), then \( M \) is said to be max communication bounded by \( f \). Similarly, if all \( w \in L(M) \) are accepted with computations where \( \text{scom}(w) \leq f(|w|) \), then \( M \) is said to be sum communication bounded by \( f \). In general, it is not expected to obtain tight bounds on the exact number of communications but rather tight bounds up to a constant multiplicative factor. For the sake of readability we denote the class of CAs that are max communication bounded by some function \( g \in O(f) \) by MC(\( f \))-CA, where it is understood that \( f \) gives the order of magnitude. Corresponding notation is used for OCAs and sum communication bounded CAs and OCAs. (SC(\( f \))-CA and SC(\( f \))-OCA). The family of all languages which are accepted by some device \( X \) with time complexity \( t \) is denoted by \( \mathcal{L}_t(X) \). In the sequel we are particularly interested in fast computations and call the time complexity \( t(n) = n \text{ real time} \) and write \( \mathcal{L}_{rt}(X) \).

## 3 Computational capacity

It has been shown in [13] that the family MC(1)-OCA contains the non-context-free languages

\[
\{ a^n a^n_2 \cdots a^n_k \mid n \geq 1 \} \quad \text{and} \quad \{ a^n b^m c^m d^m \mid n, m \geq 1 \},
\]

as well as the languages \( \{ a^n w \mid n \geq 1 \land w \in (b^* c^k b^* \land |w|_b = n) \} \), for all constants \( k \geq 0 \). All of these languages are either semilinear or non-bounded. But in contrast to many other computational devices, for

\[
\# \longrightarrow a_1 \longrightarrow a_2 \longrightarrow a_3 \cdots \longrightarrow a_n \longrightarrow \#
\]

\[ \text{Figure 1: A two-way cellular automaton.} \]
example certain multi-head finite automata, parallel communicating finite automata, and certain parallel communicating grammar systems, MC(1)-OCAs can accept non-semilinear bounded languages.

**Example 2.** The language \( L_1 = \{ a^n b^n + \lfloor \sqrt{n} \rfloor \mid n \geq 1 \} \) belongs to the family \( \mathcal{L}_{rt}(\text{MC}(1)-\text{OCA}) \).

In [13] a CA is constructed such that its cell \( n \) enters a designated state exactly at time step \( 2n + \lfloor \sqrt{n} \rfloor \), and at most \( n \) cells are used for the computation. In fact, the CA constructed is actually an OCA. Additionally, each cell performs only a finite number of communication steps. Thus, the CA constructed is an MC(1)-OCA.

An MC(1)-OCA accepting \( L_1 \) implements the above construction on the \( a \)-cells of the input \( a^n b^n m \).

Thus, the leftmost cell enters the designated state \( q \) at time step \( 2n + \lfloor \sqrt{n} \rfloor \). Additionally, in the rightmost cell a signal \( s \) with maximum speed is sent to the left. When this signal arrives in an \( a \)-cell exactly at a time step at which the cell would enter the designated state \( q \), the cell changes to an accepting state instead. So, if \( m = n + \lfloor \sqrt{n} \rfloor \), then \( s \) arrives at time \( 2n + \lfloor \sqrt{n} \rfloor \) at the leftmost cell and the input is accepted. In all other cases the input is rejected. Clearly, the OCA constructed is an MC(1)-OCA. \( \diamond \)

**Example 3.** The language \( L_2 = \{ a^{2^n} b^n c^{2^n} + n \mid n \geq 1 \} \) belongs to the family \( \mathcal{L}_{rt}(\text{MC}(1)-\text{OCA}) \).

The rough idea of the construction is sketched as follows. We first describe a real-time MC(1)-CA accepting \( \{ b^n a^{2^n} \mid n \geq 1 \} \). Then this two-way real-time MC(1)-CA is simulated by a one-way linear-time MC(1)-OCA accepting the reversal language \( \{ a^{2^n} b^n \mid n \geq 1 \} \) in time \( 2 \cdot (n + 2^n) \). The time additionally needed is provided by adding the suffix \( c^{2^n} + n \) to the input. Finally, the correct length of the suffix is checked.

In more detail, we first consider the construction of a signal with speed \( 2^n \) given in [15]. There a CA is described whose cell \( n \) enters a designated state exactly at time step \( 2^n \), where at most \( 2^n \) cells are used for the computation. The CA constructed is in fact an MC(1)-CA. Similar to the construction in Example 2 we implement the construction of the signal \( 2^n \) on the \( b \)-cells, and the rightmost \( a \)-cell sends a signal \( s \) with maximum speed to the left. We know that the rightmost \( b \)-cell enters a designated state \( q \) exactly at time step \( 2^n \) if the input is \( b^n a^m \).

Moreover, signal \( s \) arrives at time step \( m + 1 \) in cell \( n \). If \( m = 2^n \), then cell \( n \) has entered the state \( q \) exactly one time step before, and now changes to an accepting state which is sent with maximum speed to the leftmost cell. In all other cases the input is rejected. Altogether, we derive that \( \{ b^n a^{2^n} \mid n \geq 1 \} \in \mathcal{L}_{rt}(\text{MC}(1)-\text{CA}) \).

Next, we want to accept \( L_2 \) by some real-time MC(1)-OCA. To this end, we utilize the fact that the reversal of every language accepted by some real-time CA can be accepted in twice the time by some OCA [2 [10] [21]. The essence of the proof is that each cell of the one-way device collects the states of its both neighbors to the right in an additional time step. With this information it can simulate the behavior of its immediate neighbor to the right in the two-way device. In this way, \( n \) steps of the CA can be simulated in \( 2n \) steps by the OCA on reversed input. It follows that the resulting OCA is an MC(1)-OCA if the given CA was an MC(1)-CA. In particular, we obtain that \( \{ a^{2^n} b^n \mid n \geq 1 \} \) can be accepted by some MC(1)-OCA in time \( 2 \cdot (n + 2^n) \). Thus, the simulation can be performed on input \( a^{2^n} b^n c^{2^n} + n \) in real-time.

Finally, the number of \( cs \) remains to be checked. To this end, we consider the already mentioned language \( \{ a^n b^n \mid n \geq 1 \} \) which belongs to the family \( \mathcal{L}_{rt}(\text{MC}(1)-\text{OCA}) \) [13]. Here we match the number of \( as \) and \( bs \) against the number of \( cs \) on an additional track. So, we obtain the desired MC(1)-OCA. \( \diamond \)

### 4 Decidability questions

This section is devoted to decidability problems. In fact, the results show undecidability of various questions for real-time MC(\( \log n \))-OCAs and SC(\( n \))-OCAs accepting bounded languages. First we
show that emptiness is undecidable for real-time MC(\(\log n\))-OCAs and SC(n)-OCAs accepting bounded languages by reduction from Hilbert’s tenth problem which is known to be undecidable. The problem is to decide whether a given polynomial \(p(x_1,\ldots,x_n)\) with integer coefficients has an integral root. That is, to decide whether there are integers \(\alpha_1,\ldots,\alpha_n\) such that \(p(\alpha_1,\ldots,\alpha_n) = 0\). In \[8\] Hilbert’s tenth problem was used to show that emptiness is undecidable for certain multi-counter machines. As is remarked in \[8\], it is sufficient to restrict the variables \(x_1,\ldots,x_n\) to take non-negative integers only. If \(p(x_1,\ldots,x_n)\) contains a constant summand, then we may assume that it has a negative sign. Otherwise, \(p(x_1,\ldots,x_n)\) is multiplied by \(-1\). Then, such a polynomial has the following form: 
\[
p(x_1,\ldots,x_n) = t_1(x_1,\ldots,x_n) + \cdots + t_r(x_1,\ldots,x_n),
\]
where each \(t_j(x_1,\ldots,x_n)\) (\(1 \leq j \leq r\)) is a term of the form \(s_j x_1^{i_{j1}} \cdots x_n^{i_{jn}}\) with \(s_j \in \{+1,-1\}\) and \(i_{j1},\ldots,i_{jn} \geq 0\). It should be remarked that some terms \(t_j\) may be equal. Additionally, we may assume that the summands are ordered according to their sign, i.e., there exists \(1 \leq p \leq r\) such that \(s_1 = \cdots = s_p = 1\) and \(s_{p+1} = \cdots = s_r = -1\). Moreover, constant terms occur only at the end of the sum. I.e., \(t_r = \cdots = t_{r-c+1} = -1\), if \(p\) contains \(c > 0\) constant terms.

We first look at the positive terms \(t_j\), \(1 \leq j \leq p\), and define languages \(L(t_j)\) as follows.

\[
L(t_j) = \{ b_1^{\alpha_1} \cdots b_n^{\alpha_n} c_1^{i_{j1}} \cdots c_n^{i_{jn}} d_1^{2n-\alpha_1} \cdots d_n^{2n-\alpha_n} \mid \alpha_1,\ldots,\alpha_n \geq 0 \}
\]

For the negative, non-constant terms \(t_j\) with \(p+1 \leq j \leq r\) the definition of \(L(t_j)\) is identical except for the fact that each symbol \(d_1\) is replaced by some symbol \(d_2\). For each negative, constant term \(t_j\), we define \(L(t_j) = \{ d_2^{\alpha} \} \). Since \(n\) is a constant depending on the given polynomial \(p\), we can observe that each \(L(t_j)\) is a bounded language.

**Lemma 4.** For \(1 \leq j \leq r\), the language \(L(t_j)\) belongs to the family \(\mathcal{L}_{rt}(CA)\).

**Proof:** If \(t_j\) is a constant term, then \(L(t_j)\) is a regular language and belongs to \(\mathcal{L}_{rt}(CA)\). Otherwise, \(L(t_j)\) can be accepted by a real-time CA as follows. We may assume that the input is correctly formatted, since this can be checked with some leftward signal starting in the rightmost cell.

The first task is to check for \(1 \leq m \leq n\) that the number of symbols \(c_m\) is equal to the number of symbols \(b_m\) to the power \(i_{jm}\), that is, the number of symbols \(c_m\) is equal to \(\alpha_{i_{jm}}^m\). In \[15\] for every \(k \geq 2\), a two-way CA is constructed whose leftmost cell enters a designated state \(q_m\) exactly at every time step \(x^k\), for \(x \geq 1\). The CAs do not use more than \(x\) cells.

Since \(i_{jm}\) is a constant, we can implement this construction for \(k = i_{jm}\) on the \(c_m\)-cells, for all \(1 \leq m \leq n\). Whenever the leftmost cell of the block consisting of \(c_m\)-cells, called \(c_m\)-block, enters the designated state \(q_m\), a signal is sent to the left which marks one symbol \(b_m\). Additionally, the rightmost cell of the \(c_m\)-block sends a signal \(s_m\) with maximum speed to the left. When \(s_m\) arrives in the leftmost cell of the \(c_m\)-block, it checks whether the cell would enter the state \(q_m\) at that time step. Moreover, the signal \(s_m\) checks that all symbols \(b_m\) have been marked and no signal failed to mark a \(b_m\) since there are too few of them. The latter can be observed and remembered by the cell carrying the leftmost \(b_m\).

The second task is to check that the number of symbols \(d_1\) is equal to \(2^n \cdot \alpha_{i_{jm}} \cdots \alpha_{i_{jn}}\). We next describe the construction and remark that the construction for symbols \(d_2\) is identical. The principal idea is as follows: In the rightmost cell of a \(c_m\)-block a signal \(s'_m\) is set up that moves through the block back and forth. The signal \(s'_n\) moves with maximum speed. The signal \(s'_m\) through some other \(c_m\)-block is stopped by the arrival of the signal \(s'_{m+1}\) at its leftmost cell. Whenever this happens, an auxiliary signal is sent from the rightmost \(c_m\)-cell to the left that moves the signal \(s'_m\) one cell. Clearly, by dropping \(s'_n\) the whole process gets frozen.

Now, the rightmost cell of the array emits a signal \(s_d\) to the left at initial time. When \(s_d\) arrives at the rightmost \(c_m\)-cell at the same time as signal \(s'_n\), the latter is dropped. Now, signal \(s_d\) continues to move
to the left and checks that all signals \( s_{m} \) stay at the rightmost \( c_{m} \)-cell and, additionally, that the \( c_{1} \)-block has been passed through back and forth by \( s_{1}^{t} \) exactly once, which can be remembered by \( s_{1}^{t} \). In this case, the number of \( d_{i} \) is exactly 2 \( \cdot \alpha_{1}^{t} \cdot 2 \cdot \alpha_{2}^{t} \cdot 2 \cdot \alpha_{2}^{t} = 2^{n} \cdot \alpha_{1}^{t} \cdot \alpha_{n}^{t} \). A schematic computation may be found in Figure 2.

Now, we can construct a real-time CA accepting \( \tilde{L}(t) \) by realizing both tasks on different tracks and checking the correctness of the input format. \( \square \)

![Figure 2: Schematic computation of \( a^{2}b^{2}c^{2} \cdot a^{3}b^{3}c^{3} \).](image)

We next consider the following regular languages \( R_{j} \) depending on the sign of \( t_{j} \). If \( s_{j} = 1 \), then \( R_{j} = b_{1}^{s} \cdots b_{n}^{s} c_{1}^{s} \cdots c_{n}^{s} d_{1}^{s} \). If \( s_{j} = -1 \) and \( t_{j} \) is non-constant, then \( R_{j} = b_{1}^{s} \cdots b_{n}^{s} c_{1}^{s} ... c_{n}^{s} d_{2}^{s} \). Otherwise, we set \( R_{j} = d_{2}^{s} \). Then, we define for positive terms \( t_{j} \)

\[
\tilde{L}(t_{j}) = \left\{ a_{1}^{\alpha_{1}} \cdots a_{n}^{\alpha_{n}} w_{1} \cdots w_{j-1} a_{1}^{\alpha_{1}} \cdots a_{n}^{\alpha_{n}} d_{1}^{r} \cdots d_{1}^{t} \cdots a_{n}^{\alpha_{n}} d_{1}^{r} \cdots a_{n}^{\alpha_{n}} \phi w_{j+1} \cdots w_{r} \mid \alpha_{1}, \ldots, \alpha_{n} \geq 0 \right. \\
\text{and } w_{i} \in R_{i} \text{ for } 1 \leq i \leq r \text{ with } i \neq j \}.
\]

The languages \( \tilde{L}(t_{j}) \) for negative, non-constant terms are defined analogously. For negative, constant terms we define

\[
\tilde{L}(t_{j}) = \left\{ a_{1}^{\alpha_{1}} \cdots a_{n}^{\alpha_{n}} w_{1} \cdots w_{j-1} d_{2}^{\alpha_{n}} \phi w_{j+1} \cdots w_{r} \mid \alpha_{1}, \ldots, \alpha_{n} \geq 0 \right. \\
\text{and } w_{i} \in R_{i} \text{ for } 1 \leq i \leq r \text{ with } i \neq j \}.
\]

Now, we consider the language \( \tilde{L}(p) = \bigcap_{j=1}^{r} \tilde{L}(t_{j}) \) and observe that \( \tilde{L}(t_{j}) \) and, thus, \( \tilde{L}(p) \) are still bounded languages (up to a renaming of symbols).

**Lemma 5.** The language \( \tilde{L}(p) \) belongs to the family \( \mathcal{L}_{rt}(CA) \).

**Proof:** Since \( \mathcal{L}_{rt}(CA) \) is closed under intersection, we have to show that each language \( \tilde{L}(t_{j}) \) belongs to \( \mathcal{L}_{rt}(CA) \). First, we can observe that \( \tilde{L}(t_{j}) \) is a regular language and, thus, is accepted by a real-time CA, if \( t_{j} \) is a negative, constant term. We next present the construction for \( t_{j} \) being a non-constant term.
Due to Lemma \ref{lemma:bpc} we can construct a real-time CA accepting $L(t_j)$. Then we concatenate the regular language $a_1^* \cdots a_n^* R_1 \cdots R_{j-1}$ on the left and the regular language $R_{j+1} \cdots R_r$ on the right of $L(t_j)$. Since the family $\mathcal{L}_{rt}(CA)$ is closed under marked concatenation \cite{18}, we again obtain a real-time CA language. It remains to be shown that the number of $a_i$s is equal to the number of $b_i$s, for $1 \leq i \leq n$. It is known that the language $\{ a^n b^n \mid n \geq 1 \}$ can be accepted by a real-time CA. The principal idea is to send the $b$s to the left and to match them against the $a$s. If both numbers are equal, the input is accepted and otherwise rejected. By an obvious generalization of this idea, we can check with a real-time CA that the number of $a_i$s is equal to the number of $b_i$s. Additionally, we have to take care that $b_i$s occurring in $w_1, \ldots, w_{j-1}$ are not matched against $a_i$s. To this end, the cells carrying $a_i$s are ignoring the first $j-1$ blocks of $b_i$s. Altogether, language $\tilde{L}(t_j)$ can be accepted by a real-time CA.

Finally, let $L(p) = \{ w \in \tilde{L}(p) \mid |w|_{d_1} = |w|_{d_2} \}$.

**Lemma 6.** The language $L(p)$ belongs to the family $\mathcal{L}_{rt}(CA)$.

**Proof:** Here we have to check that the input belongs to $\tilde{L}(p)$ and that the number of occurring symbols $d_1$ is equal to the number of occurring symbols $d_2$. The former task can be realized by some real-time CA due to Lemma \ref{lemma:count}. The latter task can also be realized by some real-time CA: Due to the format of words in $\tilde{L}(p)$ we know that each word can be divided into two parts. The first part contains symbols $d_1$ and no symbols $d_2$ whereas the second part contains symbols $d_2$ and no symbols $d_1$. By sending symbols $d_2$ to the left and matching them against symbols $d_1$, we can check that their number is equal.

**Lemma 7.** The language $L_1(p) = \{ wa^{|w||b^{2|w|}} \mid w \in L(p)^r \}$ belongs to the family $\mathcal{L}_{rt}(SC(n)-OCA)$.

**Proof:** Let us first give evidence that the language $\{ a^m b^{2m} \mid m \geq 1 \}$ belongs to $\mathcal{L}_{rt}(SC(n)-OCA)$. To this end, we implement a binary counter in the $a$-cells, i.e., we store the binary encoding of the currently counted value in these cells. The least significant bit is simulated in the rightmost $a$-cell. The information to be communicated to the left are carry-overs. Now, the counter is increased at every time step. Furthermore, the rightmost $b$-cell emits a signal to the left at initial time. When this signal arrives at the rightmost $a$-cell, it checks successively whether all $a$-cells passed through are in a state indicating that they have produced a carry-over before. If it arrives in an $a$-cell that is in a carry-over state for the first time, that cell enters an accepting state. An example computation is depicted in Figure \ref{fig:counter}

![Figure 3: A binary counter accepting $b^{2^4} = b^{16}$. A + denotes a carry-over and some primed state indicates that the cell has produced a carry-over at some time before. The latter is checked by signal $g$, where $G$ indicates an accepting state.](image)

Let us consider an input $a^m b^{2m}$. The rightmost $a$-cell performs $2^{m-1}$ communication steps to send the carry-overs during the counting phase. Its left neighbor performs $2^{m-2}$ communication steps and so
Thus, we obtain that all $a$-cells perform not more than $\sum_{i=1}^{m} 2^{i-1} \in O(2^m)$ communication steps. Furthermore, the $b$-cells perform only a constant number of communication steps. Altogether, $O(2^m)$ communication steps are performed in total, and an SC$(n)$-OCA can be constructed.

Next, we consider the language $\{ wa^{|w|} | w \in L(p)^R \}$. The correct number of $a$s can be checked in a similar way as for the language $\{ a^m b^m | m \geq 1 \}$. This check costs a constant number of communication steps per cell. In the proof of Lemma 6 a real-time CA is constructed which accepts $L(p)$. It is known that an OCA accepting $L(p)^R$ in twice the time can be constructed $[2] [10] [21]$. We implement this construction in order to check the $w$. By concatenating the additional symbols $a$, the OCA works still in real time. The rightmost $a$-cell additionally sends a signal which freezes the computation in the first $|w|$ cells.

Now, it is easy to construct a real-time OCA accepting $L_1(p)$. To conclude the proof we have to show that the real-time OCA constructed is a real-time SC$(n)$-OCA. The $b$-cells perform a constant number of communication steps per cell. The $a$-cells perform $O(2^m)$ communication steps to realize the counter. Additionally, a constant number of communications per cell is needed to check the length of $w$ with the number of $a$s and to send the freezing signal. Finally, each of the first $|w|$ cells can perform at most $2m$ communication steps due to the freezing signal. In total, they perform at most $2m^2$ communication steps. Altogether, we obtain that at most $O(2^m)$ communication steps are performed which shows that $L_1(p) \in \mathcal{L}_{rt}(\text{SC}(n)\text{-OCA})$.

**Lemma 8.** The language $L_2(p) = \{ wa^{|w|} b^2w^{|w|} c^{|w|} | w \in L(p)^R \}$ belongs to the family

$$\mathcal{L}_{rt}(\text{MC}(\log n)\text{-OCA}).$$

**Proof:** Let us first consider the language $\{ b^{2^m} c^m | m \geq 1 \}$. It is shown in $[10]$ that the unary language $\{ b^{2^m} | m \geq 1 \}$ can be accepted by a $(2^m + m)$-time OCA. The principal idea is to construct a moving binary counter which starts with one time step delay in the rightmost cell and moves to the left, whereby the cells passed through are counted. If necessary, the length of the counter is increased. Additionally, the cells passed through check whether all bits are 1. In this case, some number $2^m - 1$ has been counted. Taking into account the delayed start we obtain that some number $2^m$ has been counted. In this case a cell enters an accepting state. An example computation is depicted in Figure 4. Since the counter moves from right to left and all cells passed through enter an accepting or rejecting permanent state, we can observe that the number of communication steps of each cell is bounded by the length of the counter, that is, by the logarithm of the length of the input.

So, the language $\{ b^{2^m} | m \geq 1 \}$ is accepted by an MC$(\log n)$-OCA which needs more than real time. It is easy to modify the construction such that the language $\{ b^{2^{2^m}} c^m | m \geq 1 \}$ is accepted in real time.

Next, let us consider the language $\{ wa^{|w|} | w \in L(p)^R \}$. In the proof of Lemma 7 an SC$(n)$-OCA accepting this language is constructed. So, we obtain that $\{ wa^{|w|} | w \in L(p)^R \}$ belongs to $\mathcal{L}_{rt}(\text{OCA})$.

Finally, we concatenate both languages and check the same number of $a$s and $c$s by sending $c$s to the left and matching them against $a$s. Altogether, we obtain a real-time OCA accepting $L_2(p)$. To conclude the proof we have to show that the real-time OCA constructed is in fact a real-time MC$(\log n)$-OCA. The length of the input is $3m + 2^m$. Thus, we have to show that each cell does not perform more than $O(m)$ communication steps. Concerning the $b$-cells not more than $O(m)$ communication steps are performed due to the construction provided. The matching of $c$s against $a$s causes not more than $O(m)$ additional communication steps. Therefore, the number of communication steps in $b$- and $c$-cells is of order $O(m)$. The $a$-cells receive $m$ signals from the $c$-cells and send a block of $a$s to be matched against $w$. Altogether, not more than $O(m)$ communication steps are performed. Finally, due to the freezing signal the first $|w|$
Figure 4: Computation of an OCA accepting an input of the language \( \{ b^{2^m} \mid m \geq 1 \} \) in time \( 2^m + m \).

cells can perform at most \( 2m \in O(m) \) communication steps after the computation whether \( w \in L(p)^R \). Altogether, we obtain that \( L_2(p) \) belongs to \( L_{rt}(MC(\log n)-OCA) \).

Now, we are prepared to derive the undecidability results.

**Theorem 9.** Given an arbitrary real-time SC\((n)\)-OCA or MC\((\log n)\)-OCA \( M \) accepting a bounded language, it is undecidable whether \( L(M) \) is empty.

**Proof:** Due to Lemma\[7\] we can construct a real-time SC\((n)\)-OCA \( M \) accepting \( L_1(p) \). By the construction of \( L_1(p) \), it is not difficult to observe that \( M \) accepts the empty set if and only if \( 2^n \cdot p(x_1, \ldots, x_n) \) has no solution in the non-negative integers. The latter is true if and only if \( p(x_1, \ldots, x_n) \) has no solution in the non-negative integers. Since Hilbert’s tenth problem is undecidable, we obtain that the emptiness problem for real-time SC\((n)\)-OCAs is undecidable. The argumentation for MC\((\log n)\)-OCA is similar considering \( L_2(p) \) and Lemma\[8\].

By standard techniques (see, e.g., [11]) one can show the following results.

**Theorem 10.** The problems of testing finiteness, infiniteness, universality, inclusion, equivalence, regularity, and context-freedom are undecidable for arbitrary real-time SC\((n)\)-OCAs and MC\((\log n)\)-OCAs accepting bounded languages.

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