Graph-theory induced gravity and strongly-degenerate fermions in a self-consistent Einstein universe

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We study UV-finite theory of induced gravity. We use scalar fields, Dirac fields and vector fields as matter fields whose one-loop effects induce the gravitational action. To obtain the mass spectrum which satisfies the UV-finiteness condition, we use a graph-based construction of mass matrices. The existence of a self-consistent static solution for an Einstein universe is shown in the presence of degenerate fermions.

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I. INTRODUCTION

The quantum nature of gravity is not yet cleared in spite of endeavor of many researchers. An old idea on this issue is that gravity emerges as quantum effects of matter fields [1]. Originally, in such an induced gravity scenario, the Newton constant is naturally obtained from the one-loop calculation with a cutoff of the Planck scale. In this case, the induced cosmological constant becomes a huge amount if no special choice of the matter-field content is considered.

We will consider a calculable model for induced gravity in the present paper. For this purpose, we first fix the choice of matter species to cancel the UV divergences. Next we should consider the mass spectra of the fields, which affect the finite contribution to the induced Newton constant and the cosmological constant.

To obtain the suitable mass spectra, we use the method of dimensional deconstruction [2] and its generalization [3]. In the generalization of the deconstruction model based on a graph, the eigenvalues of the graph Laplacian and the adjacent matrix gives the mass spectrum of the particle. Thus we can easily control the induced quantities at one-loop level in such a model [4].

We also study self-consistent static solutions for a static Einstein universe in a graph-based induced gravity. We have considered self-consistent Einstein universe at finite temperature in [5]. In the present paper, we use the calculation method with the spectral density function of the graph and search for the static solution supported by the degenerate pressure of the fermion at zero temperature.

The present paper is organized as follows. In §2, we will examine the UV-divergences in field theory with the heat kernel method. The way to construct suitable models using the knowledge of the graph structure is shown in §3. In §4, divergences in the effective gravitational action are regularized for a static Einstein space. It is shown that the technique with the density function to evaluate the effective action for an Einstein space in §5. In §6, strongly-degenerate fermions and a self-consistent solution in our model is studied. We give a summary and future prospects in the last section.
II. UV-FINITENESS CONDITION

Induced gravity has been studied by many authors [1]. In terms of the heat kernel method [6], the one-loop effective action can systematically be expressed as an integral form using Schwinger’s proper time.

The classical action for a free field can be written as a quadratic form with a differential operator on the spacetime manifolds. The operator trace (Tr) can be evaluated by the standard way to rewrite

$$\frac{1}{2} \text{Tr} \ln H = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr} \left[ e^{-tH} \right],$$

(2.1)

where $H$ is a Hessian operator which appears in the free-field action. The heat-kernel expansion can be expressed as, in four-dimensional spacetime,

$$\text{Tr} \left[ e^{-tH} \right] = \frac{1}{(4\pi t)^2} \int d^4x \sqrt{|\det g_{\mu\nu}|} \left[ \text{tr} a_0 + t \text{tr} a_1 + t^2 \text{tr} a_2 + o(t^3) \right],$$

(2.2)

where $g_{\mu\nu}$ denotes the spacetime metric and tr means the trace over the spacetime indices. The Seeley-DeWitt coefficients $a_p$ ($p = 0, 1, 2, \ldots$) depend on the background fields and the first few coefficients have been known for several types of wave operators. The one-loop effective action for the background fields is given by the collection of the contribution of various matter fields to the heat-kernel coefficients.

It is straightforward to see where the UV divergences occur, which we are interested in. The UV divergences arise from the integration in the vicinity of $t = 0$. These divergences arise from the first few terms of the heat-kernel expansion. If we manage to introduce a UV-cutoff scale $\Lambda$, the lower bound of the integration on $t$ is replaced to $1/\Lambda^2$. To seek the condition for cancellation of UV divergences from various matter fields, we need only to consider massless fields. In the present paper, minimally-coupled scalar fields, spinor fields, and vector fields are taken into consideration.

The first Seeley-DeWitt coefficient $a_0$, which is a constant value, has been found for such fields. The value for each mode is: $a_0 = 1$ for a scalar mode, $a_0 = 2$ for a spinor field, and $a_0 = 2$ for a massless vector. Then the effective Lagrangian at one-loop level includes the following cutoff-dependent term proportional to [1]

$$\frac{1}{64\pi^2}(N_0 - 2N_{1/2} + 2N_1)\Lambda^4,$$

(2.3)

where $N_0$ is the number of minimal scalar degrees of freedom, $N_{1/2}$ is the number of two-component fermion fields, and $N_1$ is the number of massless vector fields. Note that the
spinor field contributes with a negative sign for its fermionic nature. The expression (2.3) corresponds to the cosmological constant or dark energy, if we treat it as a cutoff-regularized theory.

The less divergent term comes from the coefficient \( a_1 \). The coefficient for each mode is: \( a_1 = R/6 \) for a scalar mode, \( a_1 = -R/6 \) for a spinor field, \( a_1 = -2R/3 \) for a massless vector field, where \( R \) is the scalar curvature of the spacetime. Thus the coefficient \( a_1 \) leads to the induced Einstein-Hilbert term. The effective Lagrangian at one-loop level includes the following cutoff-dependent term proportional to [1]

\[
\frac{1}{192\pi^2}(N_0 + N_{1/2} - 4N_1)\Lambda^2 R. \tag{2.4}
\]

Now we find that, to cancel the quartic and quadratic divergent terms, which diverge as \( \Lambda \to \infty \), we should choose

\[
N_0 = 2N, \quad N_{1/2} = 2N, \quad N_1 = N, \tag{2.5}
\]

where \( N = 1, 2, 3, \ldots \).

The value of the Seeley-DeWitt coefficients can be confirmed when we set a specific background-space geometry. Since the eigenvalues of the wave operators for various fields on \( S^3 \) are well known, the trace part \( \{\text{tr exp} [-Ht]\} \) for each field can be evaluated as follows and has an asymptotic form for small \( t \): [5]

\[
\sum_{\ell=0}^{\infty} (\ell + 1)^2 \exp \left[ -\frac{\ell(\ell + 2)}{a^2} t \right] = \frac{2\pi^2 a^3}{(4\pi t)^{3/2}} \left( 1 + \frac{1}{a^2} t + \cdots \right), \text{ for a scalar mode}, \tag{2.6}
\]

\[
\sum_{\ell=1}^{\infty} 2\ell(\ell + 1) \exp \left[ -\frac{(\ell + 1/2)^2}{a^2} t \right] = \frac{2(2\pi^2 a^3)}{(4\pi t)^{3/2}} \left( 1 - \frac{1}{2a^2} t + \cdots \right), \text{ for a spinor field}, \tag{2.7}
\]

\[
\sum_{\ell=2}^{\infty} 2(\ell^2 - 1) \exp \left[ -\frac{\ell^2}{a^2} t \right] = \frac{2(2\pi^2 a^3)}{(4\pi t)^{3/2}} \left( 1 - \frac{2}{a^2} t + \cdots \right), \text{ for a massless vector}, \tag{2.8}
\]

where \( a \) is the radius of \( S^3 \). Because we know that the volume of \( S^3 \) is \( 2\pi^2 a^3 \), the scalar curvature of \( S^3 \) is \( 6/a^2 \) and the second-order (Euclidean) time-derivative contribution gives a factor \( (4\pi t)^{-1/2} \), the values of \( a_0 \) and \( a_1 \) for these fields mentioned above can be verified by (2.6-2.8).

If the massless matter content satisfies the condition (2.5), there is no quartic nor quadratic divergence and also no induced gravitational action because of absence of mass scales. Thus we should consider masses of the fields to yield the finite contribution of
quantum effects. Nonetheless, for cancellation of UV divergences, the condition (2.5) is still necessary.

The algorithm to include the masses is very easy in the Schwinger time integration. We only attach the following to the integrand for each field

$$\sum_{i=1}^{N_s} e^{-(m^2_s)_i} = N_s - t \sum_{i=1}^{N_s} (m^2_s)_i + t^2 \frac{1}{2} \sum_{i=1}^{N_s} (m^4_s)_i + \cdots \equiv N_s - t \operatorname{Tr} M^2_s + t^2 \frac{1}{2} \operatorname{Tr} M^4_s + \cdots , \quad (2.9)$$

where $M^2_s$ is the mass-squared matrix for spin-$s$ field.

In addition we need some interpretations in this trick. For massive spinor fields, we replace $N'_{1/2}$ spinor fields to massive $N'_{1/2}/2$ Dirac fields. For massive vector fields, we replace $N'_1$ massless vector fields as transverse modes and $N'_1$ scalar modes as longitudinal modes to $N'_1$ massive vector fields. Now we find the additional quadratic divergence is proportional to

$$\operatorname{Tr} M^2_S - 4 \operatorname{Tr} M^2_D + 3 \operatorname{Tr} M^2_V , \quad (2.10)$$

where $M^2_S$ is the mass-squared matrix of $N'_0$ massive scalar fields, $M^2_D$ is that of $N'_{1/2}/2$ massive Dirac fields, and $M^2_V$ is that of $N'_1$ massive vector fields.

Finally, the condition for cancellation of the quartic and quadratic divergences is concluded as follows. The matter content is: $2N - N'_0 - N'_1$ massless scalar fields, $2N - N'_{1/2}$ massless Weyl spinor fields, $N - N'_1$ massless vector fields, $N'_0$ massive scalar fields, $N'_{1/2}/2$ massive Dirac fields, and $N'_1$ massive vector fields. Moreover, massive fields must have mass matrices which satisfy $\operatorname{Tr} M^2_S - 4 \operatorname{Tr} M^2_D + 3 \operatorname{Tr} M^2_V = 0$.

### III. GRAPH-BASED CONSTRUCTION OF A SPECIFIC MASS MATRIX

In this section we construct the field theory with suitable mass matrices which satisfy the UV-finite condition expressed in the previous section.

Now we remember the concept of dimensional deconstruction [2], which is equivalent to considering a higher-dimensional theory with discretized extra dimensions at a low-energy scale. A moose diagram is used to describe this theory, and is no more than a graph. The $N$-sided polygon is identified as an example of simple graphs, a cycle graph $C_N$.

A graph $G$ consists of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$, where an edge is a pair of distinct vertices of $G$. The degree of a vertex $v$, denoted by $\deg(v)$, is the number of edges incident with $v$. If all the degrees of vertices of a graph are equal, we call such a graph as a regular graph.
We can consider the orientation of an edge. The graph with directed edges is dubbed as a directed graph. An oriented edge $e = [u, v]$ connects the origin $u = o(e)$ and the terminus $v = t(e)$.

Spectral graph theory is the mathematical study of a graph by investigating various properties on eigenvalues, and eigenvectors of matrices associated with it [7]. Now we introduce various matrices that are naturally associated with a graph [3, 7] for later use.

The incidence matrix $E(G)$ is defined as

$$
(E)_{ve} = \begin{cases} 
1 & \text{if } v = o(e) \\
-1 & \text{if } v = t(e) \\
0 & \text{otherwise}
\end{cases} \quad (3.1)
$$

The adjacency matrix $A(G)$ is defined as

$$
(A)_{vv'} = \begin{cases} 
1 & \text{if } v \text{ is adjacent to } v' \\
0 & \text{otherwise}
\end{cases} \quad (3.2)
$$

The degree matrix $D(G)$ is defined as

$$
(D)_{vv'} = \begin{cases} 
\deg(v) & \text{if } v = v' \\
0 & \text{otherwise}
\end{cases} \quad (3.3)
$$

Note that $\text{Tr } A = 0$ and $\text{Tr } A^2 = \text{Tr } D$, and for a regular graph, $D$ is proportional to the identity matrix.

The graph Laplacian (or combinatorial Laplacian) $\Delta(G)$ is defined as

$$
(\Delta)_{vv'} = (D - A)_{vv'} = \begin{cases} 
\deg(v) & \text{if } v = v' \\
-1 & \text{if } v \text{ is adjacent to } v' \\
0 & \text{otherwise}
\end{cases} \quad (3.4)
$$

The most important observation is

$$
\Delta = EE^T, \quad (3.5)
$$

where $E^T$ is the transposed matrix of $E$. The Laplacian matrix is symmetric, so its eigenvalues are non-negative. Note also that $\text{Tr } \Delta = \text{Tr } D$ and $\text{Tr } \Delta^2 = \text{Tr } D^2 + \text{Tr } D$.

The simplest model of vector fields has been studied by Hill and Leibovich [8]. The generalized model associated with a general graph is written down as [3]

$$
\mathcal{L}_V = -\frac{1}{4} \sum_{v \in V} F^v_{\mu\nu} F^v_{\mu\nu} - \sum_{e \in E} (\mathcal{D}_\mu U_e)^t (\mathcal{D}^\mu U_e), \quad (3.6)
$$
where the covariant derivative is

\[ \mathcal{D}^\mu U_e \equiv (\partial^\mu + iA^\mu_{t(e)} - iA^\mu_{o(e)})U_e, \]  

(3.7)

with \(|U_e| = f\), \(f\) is a constant with the dimension of mass. The vector fields \(A^\mu_v\) are assigned at vertices of \(G\) and the scalar fields \(U_e\) are assigned at edges of \(G\) in this model.

Similarly, any kind of fields can be associated with a graph and their mass-squared matrix can be written using the graph Laplacian. For scalar fields, we assign a scalar field \(\phi_v\) to each vertex \(v\) of \(G\). A difference can be defined on each edge \(e\) as

\[ d\phi_e \equiv \phi_{t(e)} - \phi_{o(e)} = -\sum_{v \in \mathcal{V}} E^T_{ev} \phi_v. \]  

(3.8)

Thus a mass term for scalar fields can be constructed as

\[ f^2 \sum_{e \in \mathcal{E}} d\phi_e d\phi_e = f^2 \sum_{e \in \mathcal{E}} \sum_{v, v' \in \mathcal{V}} \phi_{v'} E^T_{ev} E^T_{e'v} \phi_v = f^2 \sum_{v, v' \in \mathcal{V}} \phi_v \Delta_{vv'} \phi_{v'}. \]  

(3.9)

For spinor fields, the mass term can be expressed using the incidence matrix \(E\). For example, the Lagrangian density of fermion fields can be written as [3]

\[ -\sum_{v \in \mathcal{V}} \bar{\psi}_{Rv} \mathcal{D} \psi_{Rv} - \sum_{e \in \mathcal{E}} \bar{\psi}_{Le} \mathcal{D} \psi_{Le} - f \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} [(\bar{\psi}_{Le}(E^T)_{ev}\psi_{Rv} + h.c.)], \]  

(3.10)

where the subscripts \(L\) and \(R\) denote left-handed and right-handed fermions, respectively. Namely, the left-handed fermions are assigned to the edges while the right-handed ones are assigned to the vertices. The mass-squared matrix for \(\psi_{Rv}\) is expressed as \(f^2 E E^T = f^2 \Delta\) while that for \(\psi_{Le}\) is \(f^2 E^T E \equiv f^2 \tilde{\Delta}\). The matrices \(\Delta\) and \(\tilde{\Delta}\) have the same spectrum up to zero modes. Thus the mass spectrum of fermions governed by the Lagrangian (3.10) is also given by the eigenvalues of the graph Laplacian (3.5). For details, see Ref. [3].

With the knowledge in spectral graph theory [7], we can find that the UV divergent terms are concerned with the graph Laplacian. Therefore, the UV divergences can be controlled by using the graph Laplacian and we can construct the models of UV-finite induced gravity from spectral graph theory.

A prescription is as follows. First we prepare three graphs, \(G_S\), \(G_D\) and \(G_V\). All these graphs have \(N\) vertices. We can construct Lagrangians whose mass-squared matrices satisfy

\[ \text{Tr} M^2_S = \text{Tr} M^2_D = \text{Tr} M^2_V, \quad \text{Tr} M^4_S = \text{Tr} M^4_D = \text{Tr} M^4_V, \]  

(3.11)
by choosing graphs as $D(G_S) = D(G_D) = D(G_V)$ [4]. Then we find that the induced vacuum energy at one-loop level is [4]

$$V_0 = -\frac{1}{(4\pi)^2} \int_0^\infty \frac{dt}{t^3} \text{Tr} \left[ e^{-M_2^2t} - 4e^{-M_D^2t} + 3e^{-M_V^2t} \right],$$

(3.12)

and the inverse of the Newton constant is given by [4]

$$\frac{1}{16\pi G} = -\frac{1}{6(4\pi)^2} \int_0^\infty \frac{dt}{t^2} \text{Tr} \left[ e^{-M_2^2t} + 2e^{-M_D^2t} - 3e^{-M_V^2t} \right].$$

(3.13)

In the flat-space limit, the one-loop vacuum energy has been calculated for field theory associated with the cycle graph $C_n$ [4]. The degree matrix of a cycle graph $C_n$ is an $n \times n$ diagonal matrix $\text{diag.}(2, 2, \ldots, 2)$. We select a type of non-simply-connected graphs $G\{n_i\} = C_{n_1} \cup C_{n_2} \cup \cdots = \bigcup_{n_i=N} C_{n_i}$, which has $N$ vertices. The degree matrix of $G\{n_i\}$ is an $N \times N$ diagonal matrix $\text{diag.}(2, 2, \ldots, 2)$. Therefore, if the mass-squared matrix $M^2$ is proportional to the graph Laplacian of $G\{n_i\}$, $\text{Tr} M^2$ and $\text{Tr} (M^2)^2$ are independent of the choice of the set $\{n_i\}$, as long as $\sum_i n_i = N$ is fixed. We can choose different sets $\{n_i\}$ for scalar, Dirac, and vector field model in order to obtain non-zero value for the Newton and cosmological constants [4].

IV. EVALUATION OF THE EFFECTIVE ACTION IN $S^3$ WITH ZETA FUNCTIONS

We will consider a model for the static universe with spatial topology $S^3$ with the radius $a$, in later sections. The self-consistent induced gravity model at finite temperature has been studied in Ref. [5]. We will study degenerate fermions at zero temperature and the self-consistent universe later in the present paper.

In this section, we evaluate the one-loop vacuum energy for the spacetime $R \times S^3$. To this end, we use (2.6-2.8) in the Schwinger integral form of the effective action. Here we first integrate over the proper-time $t$, but then we slightly shifted the power of $t$ in the integrand. For example, an expression which appears in the effective action is rewritten as

$$\int_0^\infty \frac{dt}{t^{3/2-s}} \sum_{\ell=0}^\infty (\ell + 1)^2 \exp \left[ -\frac{\ell(\ell + 2)}{a^2} t - m^2 t \right]$$

$$= \frac{\Gamma(s - 1/2)}{a^{1-2s}} \sum_{\ell=0}^\infty (\ell + 1)^2 \frac{(\ell + 1)^2}{[\ell(\ell + 2) + m^2 a^2]^{s-1/2}}.$$  

(4.1)
We then look for where divergences occur. We follow an analogous method used in Ref. [9], to separate a convergent summation from others. Now we convert it to

\[
\Sigma_S(m^2 a^2) \equiv \sum_{\ell=0}^{\infty} \frac{(\ell + 1)^2}{\ell(\ell + 2) + m^2 a^2} = -\sum_{\ell=1}^{\infty} \frac{\ell^2}{\ell^2 + m^2 a^2 - 1}^{s-1/2}
\]

\[
= \sum_{\ell=1}^{\infty} \left[ \frac{\ell^2}{(\ell^2 + m^2 a^2 - 1)^{s-1/2}} - \frac{1}{\ell^{2s-3}} \left( 1 + \frac{(1/2-s)(m^2 a^2 - 1)}{\ell^2} + \frac{(s^2 - 1/4)(m^2 a^2 - 1)^2}{2\ell^4} \right) \right]
\]

\[+ \zeta_R(2s - 3) + (1/2 - s)(m^2 a^2 - 1)\zeta_R(2s - 1) + \frac{(s^2 - 1/4)(m^2 a^2 - 1)^2}{2} \zeta_R(2s + 1), \quad (4.2)\]

where \(\zeta_R(z)\) is the Riemann’s zeta function. Similarly we find

\[
\Sigma_D(m^2 a^2) \equiv 4 \sum_{\ell=1}^{\infty} \frac{\ell(\ell + 1)}{[(\ell + 1/2)^2 + m^2 a^2]^{s-1/2}} = 4 \sum_{\ell=0}^{\infty} \frac{(\ell + 1/2)^2 - 1/4}{[(\ell + 1/2)^2 + m^2 a^2]^{s-1/2}}
\]

\[
= 4 \sum_{\ell=0}^{\infty} \left[ \frac{\ell(\ell + 1/2)^2 - 1/4}{[(\ell + 1/2)^2 + m^2 a^2]^{s-1/2}} - \frac{(\ell + 1/2)^2 - 1/4}{(\ell + 1/2)^2s-1} \left( 1 + \frac{(1/2-s)m^2 a^2}{(\ell + 1/2)^2} + \frac{(s^2 - 1/4)m^4 a^4}{2(\ell + 1/2)^4} \right) \right]
\]

\[+ 4 \left\{ (2^{2s-3} - 1)\zeta_R(2s - 3) + \left[ \left( \frac{1}{2} - s \right) m^2 a^2 - \frac{1}{4} \right] (2^{2s-3} - 1)\zeta_R(2s - 1)
\]

\[+ \left[ \frac{(s^2 - 1/4)m^4 a^4}{2} - \frac{(1/2-s)m^2 a^2}{4} \right] (2^{2s+1} - 1)\zeta_R(2s + 1)
\]

\[\left\}- \frac{(s^2 - 1/4)m^4 a^4}{2} (2^{2s+3} - 1)\zeta_R(2s + 3) \right\}, \quad (4.3)\]

and also

\[
\Sigma_V(m^2 a^2) \equiv 2 \sum_{\ell=2}^{\infty} \frac{\ell^2 - 1}{[\ell^2 + m^2 a^2]^{s-1/2}} = 2 \sum_{\ell=1}^{\infty} \frac{\ell^2 - 1}{[\ell^2 + m^2 a^2]^{s-1/2}}
\]

\[
= 2 \sum_{\ell=1}^{\infty} \left[ \frac{\ell^2 - 1}{[\ell^2 + m^2 a^2]^{s-1/2}} - \frac{\ell^2 - 1}{\ell^{2s-1}} \left( 1 + \frac{(1/2-s)m^2 a^2}{\ell^2} + \frac{(s^2 - 1/4)m^4 a^4}{2\ell^4} \right) \right]
\]

\[+ 2 \left\{ \zeta_R(2s - 3) + \left[ \left( \frac{1}{2} - s \right) m^2 a^2 - \frac{1}{4} \right] \zeta_R(2s - 1)
\]

\[+ \left[ \frac{(s^2 - 1/4)m^4 a^4}{2} - \frac{(1/2-s)m^2 a^2}{2} \right] \zeta_R(2s + 1) - \frac{(s^2 - 1/4)m^4 a^4}{2} \zeta_R(2s + 3) \right\}. \quad (4.4)\]

Since \(\zeta_R(-3) = \frac{1}{120}\) and \(\zeta_R(-1) = -\frac{1}{12}\) are finite, only divergent part for \(s \to 0\) in each \(\Sigma\) is the term including \(\zeta_R(2s + 1)\). The divergent parts are

\[
2\Sigma_S^{\text{div}}(m^2 a^2) = \left( s^2 - \frac{1}{4} \right) (m^2 a^2 - 1)^2 \zeta_R(2s + 1), \quad (4.5)
\]

\[
\Sigma_D^{\text{div}}(m^2 a^2) = \left[ \left( 2s^2 - \frac{1}{2} \right) m^4 a^4 - \left( \frac{1}{2} - s \right) m^2 a^2 \right] (2^{2s+1} - 1)\zeta_R(2s + 1), \quad (4.6)
\]
\[
\Sigma_{V}^{\text{div}}(m^2a^2) = \left[ (s^2 - \frac{1}{4}) m^4a^4 - (1 - 2s) m^2a^2 \right] \zeta_R(2s + 1). \tag{4.7}
\]

In the graph-based model reviewed in the previous section, we can set \( \text{Tr} M_S^4 = \text{Tr} M_D^4 = \text{Tr} M_{S'}^4 \) as well as \( \text{Tr} M_S^2 = \text{Tr} M_D^2 = \text{Tr} M_{V'}^2 \). Thus the divergence in the induced action is proportional to

\[
\lim_{s \to 0} \sum_i [2 \Sigma_{S}^{\text{div}}(m_i^2a^2) - \Sigma_{D}^{\text{div}}(m_i^2a^2) + \Sigma_{V}^{\text{div}}(m_i^2a^2)] = N \lim_{s \to 0} \left( -\frac{1}{4} + O(s) \right) \zeta_R(2s + 1). \tag{4.8}
\]

This residual divergence is only in \( \Sigma_S \) and independent of mass, in other words, it appears even in the case with massless (minimal) scalar fields. Elizalde [10] argued that this divergence should be dealt by ‘principal part prescription’. In the prescription, the pole term in the Riemann’s zeta function is discarded. This minimal subtraction yields

\[
\zeta_R(2s + 1) = \frac{1}{2s} + \gamma + O(s) \to \gamma,
\]

where \( \gamma \) is the Euler-Mascheroni constant (\( \gamma \approx 0.577216 \)).

Apart from the divergence, the divergent terms up to \( m^4 \) have been canceled. Corresponding to the analysis by using integral form à la Schwinger, we find that divergences including mass parameter can be cancelled in our graph-based models.

We now redefine the finite part of summations as

\[
\Sigma'_{S}(m^2a^2) = \sum_{\ell=1}^{\infty} \ell^2 \left[ \sqrt{\ell^2 + m^2a^2} - \ell \left( 1 + \frac{m^2a^2 - 1}{2\ell^2} - \frac{(m^2a^2 - 1)^2}{8\ell^4} \right) \right] \\
+ \zeta_R(-3) + \frac{m^2a^2 - 1}{2} \zeta_R(-1) - \frac{1}{8} \gamma,
\]

\[
\Sigma'_{D}(m^2a^2) = 4 \sum_{\ell=0}^{\infty} \left[ (\ell + 1/2)^2 - 1/4 \right] \frac{\sqrt{(\ell + 1/2)^2 + m^2a^2}}{\sqrt{(\ell + 1/2)^2 + m^2a^2}} \\
- (\ell + 1/2) \left( 1 + \frac{m^2a^2}{2(\ell + 1/2)^2} - \frac{m^4a^4}{8(\ell + 1/2)^4} \right) \\
+ 4 \left\{ -\frac{7}{8} \zeta_R(-3) \frac{1}{8} \right\} m^2a^2 - \frac{1}{8} \zeta_R(-1) + \frac{7m^4a^4}{8} \zeta_R(3),
\]

and

\[
\Sigma'_{V}(m^2a^2) = 2 \sum_{\ell=1}^{\infty} (\ell^2 - 1) \left[ \sqrt{\ell^2 + m^2a^2} - \ell \left( 1 + \frac{m^2a^2}{2\ell^2} - \frac{m^4a^4}{8\ell^4} \right) \right] \\
+ 2 \left\{ \zeta_R(-3) + \frac{1}{2} m^2a^2 - 1 \zeta_R(-1) + \frac{m^4a^4}{8} \zeta_R(3) \right\}. \tag{4.12}
\]

Then we find the effective action in the form,

\[
\frac{1}{2a} \sum_i \left[ \Sigma'_{S}(m_0^2, a^2) - \Sigma'_{D}(m_{1/2}^2, a^2) + \Sigma'_{V}(m_1^2, a^2) \right]. \tag{4.13}
\]
V. USE OF SPECTRAL DENSITY FUNCTION OF A GRAPH

In this section, we introduce the spectral density function of a graph [11]. The use of the spectral density makes the analysis of the Casimir energy very easy. In the present paper, we consider only regular graphs. Remembering that the graph Laplacian is expressed as \( \Delta = D - A \), we need only to consider the spectral density function for the adjacency matrix \( A \) in the case with a regular graph.

We start with the case for a cycle graph \( C_N \), for example. The spectrum of the eigenvalues for the adjacency matrix of \( C_N \) is

\[
\lambda_k = 2 \cos \frac{2\pi k}{N}, \quad (k = 0, 1, \ldots, N - 1)
\]

and thus the eigenvalues for \( \Delta \) are \( \Lambda_k = 2 - 2 \cos \frac{2\pi k}{N} = 4 \sin^2 \frac{\pi k}{N} \). It has been shown [11] that, since

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(\lambda_k) = \int_0^1 f(2 \cos \pi t) dt = \int_{-2}^2 f(x) \frac{dx}{\sqrt{4 - x^2}},
\]

the spectral density in the large \( N \) limit can be employed as

\[
\lim_{N \to \infty} \int_{-\infty}^{+\infty} f(x) \rho_N(x) dx = \frac{1}{\pi} \int_{-2}^2 f(x) \frac{dx}{\sqrt{4 - x^2}}.
\]

Namely, the summation about the discrete eigenvalues becomes an integration over the continuous variable \( x \) with the spectral density function \( \rho_\infty(x) \),

\[
\rho_\infty(x) = \begin{cases} 
\frac{1}{\pi \sqrt{4 - x^2}} & \text{for } -2 < x < 2 \\
0 & \text{otherwise}
\end{cases}
\]

for cycle graphs, in the large \( N \) limit. Incidentally, the precise spectral density function for \( C_N \) with a finite \( N \) is known as

\[
\rho(x) = \begin{cases} 
\frac{1}{\pi \sqrt{4 - x^2}}[1 + 2 \sum_{k=1}^{\infty} T_{kN}(x/2)] & \text{for } -2 < x < 2 \\
0 & \text{otherwise}
\end{cases}
\]

where \( T_n(z) \) denotes the Chebyshev polynomial.

The spectral density function is known for other several graphs. The trace formula for regular graph \( G \) of degree \( q + 1 \) on \( N \) vertices is [12]

\[
\frac{1}{N} \sum_{i=1}^{N} e^{i\lambda_i} = \frac{q + 1}{2\pi} \int_{-2\sqrt{q}}^{2\sqrt{q}} e^{xt} \frac{\sqrt{4q - x^2}}{(q + 1)^2 - x^2} dx + \frac{1}{N} \sum_{g} \sum_{k=1}^{\infty} \frac{\ell(g)}{2k_t(g)^2} I_{kt(g)}(2\sqrt{qt}),
\]
where \( g \) runs over the set of all oriented primitive closed geodesics in \( G \), and \( \ell(g) \) is the length of \( g \), while \( I_n(z) \) is the modified Bessel function of the first kind. Then

\[
\rho_\infty(x) = \begin{cases} 
\frac{q+1}{2\pi} \frac{\sqrt{4q-x^2}}{(q+1)^2-x^2} & \text{for } -2\sqrt{q} < x < 2\sqrt{q} \\
0 & \text{otherwise}
\end{cases}
\]

for \((q+1)\)-regular graphs. (5.7)

In the present paper, we will concentrate ourselves on the case with the graph \( G_{\{n_i\}} = C_{n_1} \cup C_{n_2} \cup \cdots = \bigcup_{i=n_i=N} C_{n_i} \). Clearly enough, one find that the spectral density function \( \rho_\infty \) is independent of the choice of \( \{n_i\} \).

This fact implies that the finite contributions for the Newton and cosmological constant come from the \( \rho_N - \rho_\infty \) if the summation is evaluated as the integration over the continuum variables. Therefore the Casimir energy behaves as \( 1/a^4 \times 2\pi^2 a^3 \) and the similar contribution which dominates if \( a \) is small are substantially calculated only by using \( \rho_\infty \) and that is independent of values for the Newton and cosmological constant in the flat-space limit. This universal conclusion may be interesting if we try to extend the present approach to the case with general graphs.

Turning to the present analysis, we assume that the mass-squared matrix is given by \( f^2 \Delta(G) \), where \( f \) is a unique mass scale in the model. For large \( N \), the effective action, where the Casimir energy is dominant, becomes

\[
\Omega_0(fa) \equiv \frac{1}{2a} \int_{-2}^{2} \left[ \Sigma_S(f^2 a^2 (2-x)) - \Sigma_D(f^2 a^2 (2-x)) + \Sigma_V(f^2 a^2 (2-x)) \right] \frac{N}{\pi \sqrt{4-x^2}} dx.
\]

(5.8)

In the next section, using this result, we study a self-consistent cosmological solution for an Einstein universe in the graph-based induced gravity model.

VI. DEGENERATE FERMIONS AND A SELF-CONSISTENT UNIVERSE

We consider a model for the static universe with spatial topology \( S^3 \) with the radius \( a \). The static homogeneous, closed space is often called an Einstein universe. The self-consistent induced gravity model at finite temperature has been studied in Ref. [5]. In the present paper, we study the self-consistent cold universe at zero temperature and we will consider degenerate fermions. Although the cold universe seems to have less relevance to the actual universe than the hot case, it can be a possible phase between quantum cosmology and classical cosmology.
In the static spacetime, it is known that the effective action can be interpreted as the total free energy of the quantum fields at finite temperature \([13]\). Similarly, we consider the thermodynamic potential for the case with a finite chemical potential.

The thermodynamic potential of a system of strongly-degenerate fermionic fields at zero temperature can be computed as \([14]\)

\[
\Omega_D = -\frac{2\pi^2 a^3}{12\pi^2} \sum_i \theta(\mu - m_i) \left[ \mu \sqrt{\mu^2 - m_i^2} \left( \mu^2 - \frac{5}{2} m_i^2 \right) + \frac{3}{2} m_i^4 \ln \left( \frac{\mu}{m_i} + \sqrt{\frac{\mu^2}{m_i^2} - 1} \right) \right], \quad (6.1)
\]

where \(\mu\) is the chemical potential and \(\theta(y)\) is the step function, \(\theta(y) = 1\) for \(y \geq 0\) and \(\theta(y) = 0\) for \(y < 0\).

For the case with the model associated with the graph which consists of a set of \(C_n\), \(\Omega_D\) in the large \(N\) (the total number of vertices) limit can be reduced to

\[
\Omega_D = -\frac{2\pi^2 a^3}{12\pi^2} \int_{-2}^{2} \theta(\mu - m(x)) \times \left[ \mu \sqrt{\mu^2 - m^2(x)} \left( \mu^2 - \frac{5}{2} m^2(x) \right) + \frac{3}{2} m^4(x) \ln \left( \frac{\mu}{m(x)} + \sqrt{\frac{\mu^2}{m^2(x)} - 1} \right) \right] \times \frac{N}{\pi \sqrt{4 - x^2}} dx, \quad (6.2)
\]

with \(m^2(x) \equiv f^2(2 - x)\).

It is known that the fastest way to obtain self-consistent equations is by using the total free energy in the finite-temperature case \([15]\). Similarly, we consider the total thermodynamic potential \(\Omega\) as the sum of the contribution of quantum effects \(\Omega_0\) derived in the previous section and that of degenerate Dirac fields \(\Omega_D\). The energy of the system is given by

\[
E = \Omega + \mu N = \frac{\partial (\mu^{-1} \Omega)}{\partial (\mu^{-1})}, \quad (6.3)
\]

where

\[
N = -\frac{\partial \Omega}{\partial \mu}, \quad (6.4)
\]

is the fermion number, which suffers no correction from \(\Omega_0\). The pressure \(P\) is obtained by

\[
P \times (2\pi^2 a^3) = -\frac{1}{3} \frac{\partial \Omega}{\partial a}, \quad (6.5)
\]

as in the finite-temperature case.

The self-consistent equations can be derived as

\[
\frac{\partial (\mu^{-1} \Omega)}{\partial (\mu^{-1})} = 0, \quad (6.6)
\]
and

$$\frac{\partial (\mu^{-1}\Omega)}{\partial a} = 0,$$  \hspace{1cm} (6.7)

where the first equation corresponds to the 00-component of the Einstein equation with one-loop corrections and the second corresponds to the diagonal component in a spatial direction. Thus the extremal point of $\mu^{-1}\Omega(fa, f/\mu)$ provides a solution to the self-consistent equation.

In FIG. 1, we show the contour plots for $\Omega/\mu$ obtained by numerical calculations, whose extremum provides a self-consistent solution. The horizontal axis indicates the scale factor $a$, while the vertical one $1/\mu$, in the unit of $f$.

Since the Casimir energy is dominant for small $a$, the solution can be found at the maximum of $\mu^{-1}\Omega$, corresponding to the Casimir regime defined in Ref. [15]. The stability is not expected, for the extremum of the potential is actually the maximum point.

**VII. SUMMARY AND OUTLOOK**

In the present paper, we have examined ultra-violet divergences of a one-loop calculable model for induced gravity. We have found that finite values for the Newton and cosmological
constant can be realized if the mass-squared matrices for scalar, spinor, and vector fields satisfy a few conditions.

It has been found that the model which has the suitable mass matrices can be obtained by the graph-based construction. In this paper, we focused on a type of the regular graph such as $G = C_{n_1} \cup C_{n_2} \cup \cdots$.

To evaluate the effective action for an Einstein universe, we need the knowledge of graph spectrum. We have introduced the spectral density function of the graph and found that it is useful to calculate the Casimir-energy dominant case, for small $a$ and large $N$.

The spectral density is also convenient to evaluate the thermodynamical potential of strongly-degenerate fermions. We have studied self-consistent Einstein universe at zero temperature with degenerate fermions in our model. We found that the Casimir regime can been seen.

In the present analysis, we have constructed models using cycle graphs, but we are also interested in the model of general graphs. As future works, trace formula for a regular graph [12] will be useful.

The universal behavior of the effective action for large $N$ and small $a$ under the condition of the fixed type of the associated graph, is interesting. If the construction of the model with dynamical selection of graphs is possible, say, utilizing the Higgs-like mechanism assigned at edges or vertices, it can be imagined that many large-scale universe with different Newton and cosmological constants would develop once from a single state with a large Casimir energy. Anyway, we should investigate some variation of the present model.

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