DEMOCRACY FUNCTIONS AND OPTIMAL EMBEDDINGS FOR APPROXIMATION SPACES

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Abstract. We prove optimal embeddings for nonlinear approximation spaces $A^\alpha_q$, in terms of weighted Lorentz sequence spaces, with the weights depending on the democracy functions of the basis. As applications we recover known embeddings for $N$-term wavelet approximation in $L^p$, Orlicz, and Lorentz norms. We also study the “greedy classes” $\mathcal{G}^\alpha_q$ introduced by Gribonval and Nielsen, obtaining new counterexamples which show that $\mathcal{G}^\alpha_q \neq A^\alpha_q$ for most non democratic unconditional bases.

1. Introduction

Let $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ be a quasi-Banach space with a countable unconditional basis $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$. A main question in Approximation Theory consists in finding a characterization (if possible) or at least suitable embeddings for the non-linear approximation spaces $A^\alpha_q(\mathcal{B}, \mathcal{B})$, $\alpha > 0$, $0 < q \leq \infty$, defined using the N-term error of approximation $\sigma_N(x, \mathcal{B})$ (see sections 2.2 and 2.3 for definitions). Such characterizations or inclusions are often given in terms of “smoothness classes” of the sort

$$b(\mathcal{B}; \mathcal{B}) := \left\{ x = \sum_{j=1}^\infty c_j e_j \in \mathcal{B} : \{\|c_j e_j\|_\mathcal{B}\}_{j=1}^\infty \in b \right\},$$

where $b$ is a suitable sequence space whose elements decay at infinity, such as $\ell^\tau$ or more generally the discrete Lorentz classes $\ell^{\tau,q}$.

The simplest result in this direction appears when $\mathcal{B}$ is an orthonormal basis in a Hilbert space $\mathbb{H}$, and was first proved by Stechkin when $\alpha = 1/2$ and $q = 1$ (see [31] or [8] for general $\alpha, q$).

**Theorem 1.1.** ([31] [8]). Let $\mathcal{B} = \{e_j\}_{j=1}^\infty$ be an orthonormal basis in a Hilbert space $\mathbb{H}$, and $\alpha > 0$, $0 < q \leq \infty$. Then

$$A^\alpha_q(\mathcal{B}, \mathbb{H}) = \ell^{\tau,q}(\mathcal{B}; \mathbb{H}),$$

where $\tau$ is defined by $\frac{1}{\tau} = \alpha + \frac{1}{2}$.

Many results have been published in the literature similar to Theorem 1.1 when $\mathbb{H}$ is replaced by a particular space (say, $L^p$) and the basis $\mathcal{B}$ is a particular one (for example, a wavelet basis). We refer to the survey articles [5] and [35] for detailed statements and references.

Date: November 25, 2009.

2000 Mathematics Subject Classification. 41A17, 42C40.

Key words and phrases. Non-linear approximation, greedy algorithm, democratic bases, Jackson and Bernstein inequalities, discrete Lorentz spaces, wavelets.

Research supported by Grant MTM2007-60952 of Spain. The research of M. de Natividade supported by Instituto Nacional de Bolsas de Estudos de Angola, INABE.
There are also a number of results for general pairs \((\mathcal{B}, \mathcal{B})\) (even with the weaker notion of quasi-greedy basis [13, 9, 20]). We recall two of them in the setting of unconditional bases which we consider here. For simplicity, in all the statements we assume that the basis is normalized, meaning \(\|e_j\|_\mathcal{B} = 1, \forall j \in \mathbb{N}\). The first result can be found in [21] (see also [11]).

**Theorem 1.2.** ([21] Th 1, [11] Th 6.1). Let \(\mathcal{B}\) be a quasi-Banach space and \(\mathcal{B} = \{e_j\}_{j=1}^{\infty}\) a (normalized) unconditional basis satisfying the following property: there exists \(p \in (0, \infty)\) and a constant \(C > 0\) such that

\[
\frac{1}{C}\|\Gamma\|^{1/p} \leq \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathcal{B}} \leq C\|\Gamma\|^{1/p} \tag{1.1k}
\]

for all finite \(\Gamma \subset \mathbb{N}\). Then, for \(\alpha > 0\) and \(0 < q \leq \infty\) we have

\[
A^\alpha_q(\mathcal{B}, \mathcal{B}) = \ell^{q,q}(\mathcal{B}; \mathcal{B})
\]

when \(\tau\) is defined by \(\frac{1}{\tau} = \alpha + \frac{1}{p}\).

Condition \((1.1)\) is sometimes referred as \(\mathcal{B}\) having the \(p\)-Temlyakov property [20], or as \(\mathcal{B}\) being a \(p\)-space [16, 11]. For instance, wavelet bases in \(L^p\) satisfy this property [33]. The second result we quote is proved in [13] (see also [21]).

**Theorem 1.3.** ([13, Th 3.1]). Let \(\mathcal{B}\) be a Banach space and \(\mathcal{B} = \{e_j\}_{j=1}^{\infty}\) a (normalized) unconditional basis with the following property: there exist \(1 \leq p \leq q \leq \infty\) and constants \(A, B > 0\) such that when \(x = \sum_{j \in \mathbb{N}} c_j e_j \in \mathcal{B}\) we have

\[
A \left\| \{c_j\} \right\|_{\ell^p,\infty} \leq \left\| x \right\|_\mathcal{B} \leq B \left\| \{c_j\} \right\|_{\ell^q,1}. \tag{1.2}
\]

Then, for \(\alpha > 0\) and \(0 < s \leq \infty\) we have

\[
\ell^{q,s}(\mathcal{B}; \mathcal{B}) \hookrightarrow A^\alpha_q(\mathcal{B}, \mathcal{B}) \hookrightarrow \ell^{q,s}(\mathcal{B}; \mathcal{B}) \tag{1.3}
\]

where \(\frac{1}{r_p} = \alpha + \frac{1}{p}\) and \(\frac{1}{r_q} = \alpha + \frac{1}{q}\). Moreover, the inclusions given in \((1.3)\) are best possible in the sense described in section 4 of [13].

Condition \((1.2)\) is referred in [13] as \((\mathcal{B}, \mathcal{B})\) having the \((p, q)\) sandwich property, and it is shown to be equivalent to

\[
A'\|\Gamma\|^{1/q} \leq \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathcal{B}} \leq B'\|\Gamma\|^{1/p} \tag{1.4k}
\]

for all \(\Gamma \subset \mathbb{N}\) finite. Observe that \((1.4)\) coincides with \((1.1)\) when \(p = q\).

The purpose of this article is to obtain optimal embeddings for \(A_q^\alpha(\mathcal{B}, \mathcal{B})\) as in \((1.3)\) when no condition such as \((1.4)\) is imposed. More precisely, we define the right and left democracy functions associated with a basis \(\mathcal{B}\) in \(\mathcal{B}\) by

\[
h_r(N; \mathcal{B}, \mathcal{B}) \equiv \sup_{|\Gamma| = N} \left\| \sum_{k \in \Gamma} \frac{e_k}{\|e_k\|_\mathcal{B}} \right\|_\mathcal{B} \quad \text{and} \quad h_l(N; \mathcal{B}, \mathcal{B}) \equiv \inf_{|\Gamma| = N} \left\| \sum_{k \in \Gamma} \frac{e_k}{\|e_k\|_\mathcal{B}} \right\|_\mathcal{B}
\]

for \(N = 1, 2, 3, \ldots\). We refer to section 5 for various examples where \(h_l(N)\) and \(h_r(N)\) are computed explicitly (modulo multiplicative constants). As usual, when \(h_l(N) \approx h_r(N)\) for all \(N \in \mathbb{N}\) we say that \(\mathcal{B}\) is a democratic basis in \(\mathcal{B}\) [23].
The embeddings will be given in terms of weighted discrete Lorentz spaces $\ell^q_\eta$, with quasi-norms defined by
\[ \| \{c_k\} \|_{\ell^q_\eta} = \left( \sum_{k=1}^{\infty} |\eta(k) c_k|^q \frac{1}{k} \right)^{\frac{1}{q}}, \]
where $\{c_k\}$ denotes the decreasing rearrangement of $\{|c_k|\}$ and the weight $\eta = \{\eta(k)\}_{k=1}^{\infty}$ is a suitable sequence increasing to infinity and satisfying the doubling property (see section 2.4 for precise definitions and references). In the special case $\eta(k) = k^{1/\tau}$ we recover the classical definition $\ell^q_\eta = \ell^{\tau,q}$.

**Theorem 1.4.** Let $\mathcal{B}$ be a quasi-Banach space and $\mathcal{B}$ an unconditional basis. Assume that $h_\tau(N)$ is doubling. Then if $\alpha > 0$ and $0 < q \leq \infty$ we have the continuous embeddings
\[ \ell^q_{k^\alpha h_\tau(k)}(\mathcal{B}; \mathcal{B}) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}; \mathcal{B}) \hookrightarrow \ell^q_{k^\alpha h_\tau(k)}(\mathcal{B}; \mathcal{B}). \]  
Moreover, for fixed $\alpha$ and $q$ these inclusions are best possible in the scale of weighted discrete Lorentz spaces $\ell^q_\eta$, in the sense explained in sections 3, 4 and 6.

Observe that this theorem generalizes Theorems 1.2 and 1.3. In Theorem 1.2 we have $h_\tau(N) \approx h_\ell(N) \approx N^{1/p}$ and in Theorem 1.3 $h_\tau(N) \lesssim N^{1/p}$ and $h_\ell(N) \gtrsim N^{1/q}$.

When $\mathcal{B}$ is democratic in $\mathcal{B}$, Theorem 1.4 shows that $\mathcal{A}^\alpha_q(\mathcal{B}; \mathcal{B}) \approx \ell^q_{k^\alpha h(k)}(\mathcal{B}; \mathcal{B})$ with $h(k) = h_\tau(k) \approx h_\ell(k)$. Compare this result with Corollary 1 in [13, §6].

Theorem 1.4 is a consequence of the results proved in sections 3 and 4. Section 3 deals with the lower embedding in (1.5) and shows the relation to Jackson type inequalities. Section 4 deals with the upper embedding of (1.5) and its relation to Bernstein type inequalities. Section 5 contains various examples of democracy functions and embeddings with precise references; these are all special cases of Theorem 1.4. In section 6 we apply Theorem 1.4 to estimate the democracy functions $h_\ell$ and $h_\tau$ of the approximation space $\mathcal{A}^\alpha_q$.

Finally, the last section of the paper is dedicated to study the “greedy classes” $\mathcal{G}^\alpha_q(\mathcal{B}; \mathcal{B})$ introduced by Gribonval and Nielsen in [13], and their relations with the approximation spaces $\mathcal{A}^\alpha_q(\mathcal{B}; \mathcal{B})$. The classes $\mathcal{G}^\alpha_q$ are defined similarly to the approximation spaces but with the error of approximation $\sigma_N(x)$ replaced by the quantity $\|x - G_N(x)\|_\mathcal{B}$ (see section 2.3 for details). It is easy to see that $\mathcal{G}^\alpha_q(\mathcal{B}; \mathcal{B}) \subset \mathcal{A}^\alpha_q(\mathcal{B}; \mathcal{B})$, and when $\mathcal{B}$ is democratic, $\mathcal{G}^\alpha_q(\mathcal{B}; \mathcal{B}) = \mathcal{A}^\alpha_q(\mathcal{B}; \mathcal{B})$. One may conjecture that for unconditional bases $\mathcal{B}$ the converse is true, that is $\mathcal{G}^\alpha_q(\mathcal{B}; \mathcal{B}) \neq \mathcal{A}^\alpha_q(\mathcal{B}; \mathcal{B})$ implies $\mathcal{B}$ democratic. We do not know how to show this, but we can exhibit a fairly general class of non democratic pairs $(\mathcal{B}, \mathcal{B})$ for which $\mathcal{G}^\alpha_q(\mathcal{B}; \mathcal{B}) \neq \mathcal{A}^\alpha_q(\mathcal{B}; \mathcal{B})$ for all $\alpha > 0$ and $q \in (0, \infty]$. These include wavelet bases in the non democratic settings of $L^{p,q}$ and $L^p(\log L)^\alpha$. We also illustrate how irregular the classes $\mathcal{G}^\alpha_q(\mathcal{B}; \mathcal{B})$ can be when $\mathcal{B}$ is not democratic, showing in simple situations that they are not even linear spaces.

2. General Setting

2.1. Bases. Since we work in the setting of quasi-Banach spaces $(\mathcal{B}, \| \cdot \|_\mathcal{B})$, we shall often use the $\rho$-power triangle inequality
\[ \|x + y\|_\mathcal{B} \leq \|x\|_\mathcal{B}^\rho + \|y\|_\mathcal{B}^\rho, \]  
(2.1)
which holds for a sufficiently small \( \rho = \rho_\beta \in (0, 1] \) (and hence for all \( \mu \leq \rho_\beta \)); see [3, Lemma 3.10.1]. The case \( \rho_\beta = 1 \) gives a Banach space.

A sequence of vectors \( \mathcal{B} = \{ e_j \}_{j=1}^\infty \) is a basis of \( \mathbb{B} \) if every \( x \in \mathbb{B} \) can be uniquely represented as \( x = \sum_{j=1}^\infty c_j e_j \) for some scalars \( c_j \), with convergence in \( \| \cdot \|_\mathbb{B} \). The basis \( \mathcal{B} \) is unconditional if the series converges unconditionally, or equivalently if there is some \( K > 0 \) such that
\[
\left\| \sum_{j=1}^\infty \lambda_j c_j e_j \right\|_\mathbb{B} \leq K \left\| \sum_{j=1}^\infty c_j e_j \right\|_\mathbb{B} \tag{2.2}
\]
for every sequence of scalars \( \{ \lambda_j \}_{j=1}^\infty \) with \( |\lambda_j| \leq 1 \) (see eg [15, Chapter 5]).

For simplicity in the statements, throughout the paper we shall assume that \( \mathcal{B} \) is a normalized basis, meaning \( \| e_j \|_\mathbb{B} = 1 \) for all \( j \in \mathbb{N} \). We can also assume that the unconditionality constant in (2.2) is \( K = 1 \). To see so, one can introduce an equivalent quasi-norm in \( \mathbb{B} \)
\[
\| x \|_\mathbb{B} = \sup_{\Gamma \text{finite}, \| \lambda \|_1 \leq 1} \left\| \sum_{j \in \Gamma} \lambda_j x_j e_j \right\|_\mathbb{B}, \quad \text{if } x = \sum_{j=1}^\infty x_j e_j.
\]
Observe that with this renorming we still have \( \| e_j \|_\mathbb{B} = 1 \).

With the above assumptions, the following lattice property holds: if \( |y_k| \leq |x_k| \) for all \( k \in \mathbb{N} \) and \( x = \sum_{k=1}^\infty x_k e_k \in \mathbb{B} \), then the series \( y = \sum_{k=1}^\infty y_k e_k \) converges in \( \mathbb{B} \) and \( \| y \|_\mathbb{B} \leq \| x \|_\mathbb{B} \). Also, using (2.2) with \( K = 1 \) we see that, for every \( \Gamma \subset \mathbb{N} \) finite
\[
\left( \inf_{j \in \Gamma} |c_j| \right) \left\| \sum_{j \in \Gamma} e_j \right\|_\mathbb{B} \leq \left\| \sum_{j \in \Gamma} c_j e_j \right\|_\mathbb{B} \leq (\sup_{j \in \Gamma} |c_j|) \left\| \sum_{j \in \Gamma} e_j \right\|_\mathbb{B}. \tag{2.3}
\]

2.2. Non-Linear Approximation and Greedy Algorithm. Let \( \mathcal{B} = \{ e_j \}_{j=1}^\infty \) be a basis in \( \mathbb{B} \). Let \( \Sigma_N, N = 1, 2, 3, \ldots \), be the set of all \( y \in \mathbb{B} \) with at most \( N \) non-null coefficients in the unique basis representation. For \( x \in \mathbb{B} \), the \( N \)-term error of approximation with respect to \( \mathcal{B} \) is defined as
\[
\sigma_N(x) = \sigma_N(x; \mathcal{B}, \mathbb{B}) \equiv \inf_{y \in \Sigma_N} \| x - y \|_\mathbb{B}, \quad N = 1, 2, 3, \ldots
\]
We also set \( \Sigma_0 = \{ 0 \} \) so that \( \sigma_0(x) = \| x \|_\mathbb{B} \). Using the lattice property mentioned in §2.1 it is easy to see that for \( x = \sum_{j=1}^\infty c_j e_j \) we actually have
\[
\sigma_N(x) = \inf_{|\Gamma| = N} \left\{ \| x - \sum_{\gamma \in \Gamma} c_\gamma e_\gamma \|_\mathbb{B} \right\}. \tag{2.4}
\]
that is, only coefficients from \( x \) are relevant when computing \( \sigma_N(x) \); see eg [11, (2.6)]).

Given \( x = \sum_{j=1}^\infty c_j e_j \in \mathbb{B} \), let \( \pi \) denote any bijection of \( \mathbb{N} \) such that
\[
\| c_{\pi(j)} e_{\pi(j)} \| \geq \| c_{\pi(j+1)} e_{\pi(j+1)} \|, \quad \text{for all } j \in \mathbb{N}. \tag{2.5}
\]
Without loss of generality we may assume that the basis is normalized and then (2.5) becomes \( |c_{\pi(j)}| \geq |c_{\pi(j+1)}| \), for all \( j \in \mathbb{N} \). A greedy algorithm of step \( N \) is a correspondence assigning
\[
x = \sum_{j=1}^\infty c_j e_j \in \mathbb{B} \mapsto G_N^x(x) \equiv \sum_{j=1}^N c_{\pi(j)} e_{\pi(j)}
\]
for any $\pi$ as in (2.5). The **error of greedy approximation** at step $N$ is defined by

$$
\gamma_N(x) = \gamma_N(x; B, \mathbb{B}) \equiv \sup_{\pi} \| x - G_N^\pi(x) \|_B. \quad (2.6)
$$

Notice that $\sigma_N(x) \leq \gamma_N(x)$, but the reverse inequality may not be true in general. It is said that $B$ is a **greedy basis** in $\mathbb{B}$ when there is a constant $c \geq 1$ such that

$$
\gamma_N(x; B, \mathbb{B}) \leq c \sigma_N(x; B, \mathbb{B}), \quad \forall x \in \mathbb{B}, \; N = 1, 2, 3, \ldots
$$

A celebrated theorem of Konyagin and Temlyakov characterizes greedy bases as those which are unconditional and democratic [23].

### 2.3. Approximation Spaces and Greedy Classes

The classical non-linear approximation spaces $A_q^\alpha(B, \mathbb{B})$ are defined as follows: for $\alpha > 0$ and $0 < q < \infty$

$$
A_q^\alpha(B, \mathbb{B}) = \left\{ x \in \mathbb{B} : \| x \|_{A_q^\alpha} \equiv \| x \|_B + \left[ \sum_{n=1}^{\infty} (N^\alpha \sigma_N(x; B, \mathbb{B}))^q \frac{1}{N} \right]^{\frac{1}{q}} < \infty \right\}.
$$

When $q = \infty$ the definition takes the form:

$$
A_q^\infty(B, \mathbb{B}) = \left\{ x \in \mathbb{B} : \| x \|_{A_q^\infty} \equiv \| x \|_B + \sup_{N \geq 1} N^\alpha \sigma_N(x) < \infty \right\}.
$$

It is well known that $A_q^\alpha(B, \mathbb{B})$ are quasi-Banach spaces (see eg [29]). Also, equivalent quasi-norms can be obtained restricting to dyadic $N$'s:

$$
\| x \|_{A_q^\alpha} \approx \| x \|_B + \left[ \sum_{k=0}^{\infty} (2^k \sigma_{2^k}(x))^{q} \right]^{\frac{1}{q}}
$$

and likewise for $q = \infty$. This is a simple consequence of the monotonicity of $\sigma_N(x)$ (see eg [29] Prop 2] or [17 (2.3)])

The **greedy classes** $G_q^\alpha(B, \mathbb{B})$ are defined as before replacing the role of $\sigma_N(x)$ by the error of greedy approximation $\gamma_N(x)$ given in (2.6), that is

$$
G_q^\alpha(B, \mathbb{B}) = \left\{ x \in \mathbb{B} : \| x \|_{G_q^\alpha} \equiv \| x \|_B + \left[ \sum_{N=1}^{\infty} (N^\alpha \gamma_N(x; B, \mathbb{B}))^q \frac{1}{N} \right]^{\frac{1}{q}} < \infty \right\} \quad (2.7)
$$

(and similarly for $q = \infty$). We also have the equivalence

$$
\| x \|_{G_q^\alpha} \approx \| x \|_B + \left[ \sum_{k=0}^{\infty} (2^k \gamma_{2^k}(x))^{q} \right]^{\frac{1}{q}}, \quad (2.8)
$$

since $\gamma_N(x)$ is non-increasing by the lattice property in §2.1.

Since $\sigma_N(x) \leq \gamma_N(x)$ for all $x \in \mathbb{B}$ it is clear that

$$
G_q^\alpha(B, \mathbb{B}) \hookrightarrow A_q^\alpha(B, \mathbb{B}). \quad (2.9)
$$

When $B$ is a greedy basis in $\mathbb{B}$ it holds that $G_q^\alpha(B, \mathbb{B}) = A_q^\alpha(B, \mathbb{B})$ with equivalent quasi-norms. For non greedy bases, however, the inclusion may be strict, and the classes $G_q^\alpha$ may not even be linear spaces (see section 7.1 below).
2.4. Discrete Lorentz Spaces. Let \( \eta = \{\eta(k)\}_{k=1}^{\infty} \) be a sequence so that

(a) \( 0 < \eta(k) \leq \eta(k+1) \) for all \( k = 1, 2, \ldots \) and \( \lim_{k \to \infty} \eta(k) = \infty \).

(b) \( \eta \) is doubling, that is, \( \eta(2k) \leq C \eta(k) \) for all \( k = 1, 2, \ldots \), and some \( C > 0 \).

We shall denote the set of all such sequences by \( \mathbb{W} \). If \( \eta \in \mathbb{W} \) and \( 0 < r \leq \infty \), the weighted discrete Lorentz space \( \ell^r_\eta \) is defined as

\[
\ell^r_\eta = \left\{ s = \{s_k\}_{k=1}^{\infty} \in C_0 : \|s\|_{\ell^r_\eta} = \left[ \sum_{k=1}^{\infty} (\eta(k)s_k^*)^r \right]^{1/r} < \infty \right\}
\]

(with \( \|s\|_{\ell^\infty_\eta} = \sup_{k \in \mathbb{N}} \eta(k)s_k^* \) when \( r = \infty \)). Here \( \{s_k^*\} \) denotes the decreasing rearrangement of \( \{|s_k|\} \), that is \( s_k^* = |s_{\pi(k)}| \) where \( \pi \) is any bijection of \( \mathbb{N} \) such that \( |s_{\pi(k)}| \geq |s_{\pi(k+1)}| \) for all \( k = 1, 2, \ldots \) (since we are assuming \( \lim_{k \to \infty} s_k \) is always exist). When \( \eta \in \mathbb{W} \) the set \( \ell^r_\eta \) is a quasi-Banach space (see eg [4, §2.2]). Equivalent quasi-norms are given by

\[
\|s\|_{\ell^r_\eta} \approx \left[ \sum_{j=0}^{\infty} (\eta(k^j)s_{k^j}^*)^r \right]^{1/r}, \tag{2.10}
\]

for any fixed integer \( \kappa > 1 \). Particular examples are the classical Lorentz sequence spaces \( \ell^{p,r} \) (with \( \eta(k) = k^{1/p} \)), and the Lorentz-Zygmund spaces \( \ell^{p,r}(\log \ell)^\gamma \) (for which \( \eta(k) = k^{1/p} \log^\gamma (k+1) \); see eg [2, p. 285]).

Occasionally we will need to assume a stronger condition on the weights \( \eta \). For an increasing sequence \( \eta \) we define

\[
M_\eta(m) = \sup_{k \in \mathbb{N}} \frac{\eta(k)}{\eta(mk)}, \quad m = 1, 2, 3, \ldots.
\]

Observe that we always have \( M_\eta(m) \leq 1 \). We shall say that \( \eta \in \mathbb{W}_+ \) when \( \eta \in \mathbb{W} \) and there exists some integer \( \kappa > 1 \) for which \( M_\eta(\kappa) < 1 \). This is equivalent to say that the “lower dilation index” \( i_\eta > 0 \), where we let

\[
i_\eta \equiv \sup_{m \geq 1} \log M_\eta(m) \over -\log m.
\]

For example, \( \eta = \{k^a \log^b (k+1)\} \) has \( i_\eta = \alpha \), and hence \( \eta \in \mathbb{W}_+ \) iff \( \alpha > 0 \). In general, if \( \eta \) is obtained from a increasing function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) as \( \eta(k) = \phi(ak) \), for some fixed \( a > 0 \), then \( i_\eta > 0 \) iff \( i_\phi > 0 \), the latter denoting the standard lower dilation index of \( \phi \) (see eg [24, p. 54] for the definition).

Below we will need the following result:

**Lemma 2.1.** If \( \eta \in \mathbb{W}_+ \) then there exists a constant \( C > 0 \) such that

\[
\sum_{j=0}^{n} \eta(\kappa^j) \leq C \eta(\kappa^n), \quad \forall \ n \in \mathbb{N}, \tag{2.11}
\]

where \( \kappa > 1 \) is an integer as in the definition of \( \mathbb{W}_+ \).

**Proof.** Write \( \delta = M_\eta(\kappa) < 1 \). By definition \( M_\eta(\kappa) \geq \eta(\kappa^j)/\eta(\kappa^{j+1}) \), and therefore

\[
\eta(\kappa^j) \leq \delta \eta(\kappa^{j+1}), \quad \forall \ j = 0, 1, 2, \ldots. \tag{2.12}
\]
Iterating (2.12) we deduce that
\[ \eta(\kappa^j) \leq \delta^{n-j} \eta(\kappa^n), \]
for \( j = 0, 1, 2, \ldots, n \) and hence
\[ \sum_{j=0}^{n} \eta(\kappa^j) \leq \eta(\kappa^n) \sum_{j=0}^{n} \delta^{n-j} \leq \eta(\kappa^n) \frac{1}{1-\delta}. \]

\[ \square \]

**Remark 2.2.** If \( \eta \) is increasing and doubling, then \( \{k^\alpha \eta(k)\} \in \mathbb{W}_+ \) for all \( \alpha > 0 \). Also, if \( \eta \in \mathbb{W}_+ \) then \( \eta^r \in \mathbb{W}_+ \), for all \( r > 0 \).

We now estimate the **fundamental function** of \( \ell_r^\eta \). We shall denote the indicator sequence of \( \Gamma \subset \mathbb{N} \) by \( 1_\Gamma \), that is the sequence with entries 1 for \( j \in \Gamma \) and 0 otherwise.

**Lemma 2.3.** (a) If \( \eta \in \mathbb{W} \) then
\[ \|1_\Gamma\|_{\ell_\infty^\eta} = \eta(|\Gamma|), \quad \forall \text{ finite } \Gamma \subset \mathbb{N}. \]
(b) If \( \eta \in \mathbb{W}_+ \) and \( r \in (0, \infty) \) then
\[ \|1_\Gamma\|_{\ell_r^\eta} \approx \eta(|\Gamma|), \quad \forall \text{ finite } \Gamma \subset \mathbb{N} \]
with the constants involved independent of \( \Gamma \).

*Proof.* Part (a) is trivial since \( \eta \) is increasing. To prove (b) use (2.10) and the previous lemma. \( \square \)

Finally, as mentioned in §1, given a (normalized) basis \( \mathcal{B} \) in \( \mathbb{B} \) we shall consider the following subspaces
\[ \ell_r^\eta(\mathcal{B}, \mathbb{B}) := \left\{ x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} : \{c_j\}_{j=1}^{\infty} \in \ell_r^\eta \right\}, \]
endowed with the quasi-norm \( \|x\|_{\ell_r^\eta(\mathcal{B}, \mathbb{B})} := \|\{c_j\}\|_{\ell_r^\eta} \). These spaces are not necessarily complete, but they are when
\[ \| \sum_j c_j e_j \|_{\mathbb{B}} \leq C \|\{c_j\}\|_{\ell_r^\eta}, \quad \forall \text{ finite } \{c_j\}, \]
a property which holds in certain situations (see eg Remark 3.2). When this is the case, the space \( \ell_r^\eta(\mathcal{B}, \mathbb{B}) \) is just an isomorphic copy of \( \ell_r^\eta \) inside \( \mathbb{B} \).

### 2.5. Democracy Functions.
Following [23], a (normalized) basis \( \mathcal{B} \) in a quasi-Banach space \( \mathbb{B} \) is said to be **democratic** if there exists \( C > 0 \) such that
\[ \left\| \sum_{k \in \Gamma} \hat{e}_k \right\|_{\mathbb{B}} \leq C \left\| \sum_{k \in \Gamma'} \hat{e}_k \right\|_{\mathbb{B}}, \]
for all finite sets \( \Gamma, \Gamma' \subset \mathbb{N} \) with the same cardinality. This notion allows to characterize greedy bases as those which are both unconditional and democratic [23].

As we recall in §5, wavelet bases are well known examples of greedy bases for many function spaces, such as \( L^p \), Sobolev, or more generally, the Triebel-Lizorkin spaces. However, they are not democratic in some other instances such as \( BMO \), or the Orlicz \( L^p \) and Lorentz \( L^{p,q} \) spaces (when these are different from \( L^p \)). In fact, it is proved in [38] that the Haar basis is democratic in a rearrangement invariant space \( \mathcal{X} \) in \([0,1]\) if and only if \( \mathcal{X} = L^p \) for some \( p \in (1, \infty) \).
Thus, non-democratic bases are also common. To quantify the democracy of a (normalized) system \( \mathcal{B} = \{e_j\}_{j=1}^\infty \) in \( \mathbb{B} \) one introduces the following concepts:

\[
h_\ell(N; \mathcal{B}, \mathbb{B}) \equiv \sup_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} e_k \right\|_\mathbb{B} \quad \text{and} \quad h_\ell(N; \mathcal{B}, \mathbb{B}) \equiv \inf_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} e_k \right\|_\mathbb{B},
\]

which we shall call the right and left democracy functions of \( \mathcal{B} \) (see also \[9\] \[19\] \[12\]). We shall omit \( \mathcal{B} \) or \( \mathbb{B} \) when these are understood from the context.

Some general properties of \( h_\ell \) and \( h_r \) are proved in the next proposition.

**Proposition 2.4.** Let \( \mathcal{B} = \{e_j\}_{j=1}^\infty \) be a (normalized) unconditional basis in \( \mathbb{B} \) with the lattice property from \S 2.1. Then

(a) \( 1 \leq h_\ell(N) \leq h_r(N) \leq N^{1/\rho} \), \( \forall N = 1, 2, \ldots \), where \( \rho = \rho_\mathbb{B} \) as in (2.1).

(b) \( h_\ell(N) \) and \( h_r(N) \) are non-decreasing in \( N = 1, 2, 3 \ldots \)

(c) \( h_\ell(N) \) is doubling, that is, \( \exists c > 0 \) such that \( h_\ell(2N) \leq c h_\ell(N) \), \( \forall N \in \mathbb{N} \).

(d) There exists \( c \geq 1 \) such that \( h_\ell(N+1) \leq c h_\ell(N) \) for all \( N = 1, 2, 3 \ldots \)

**Proof.** (a) and (b) follow immediately from the lattice property of \( \mathcal{B} \) and the \( \rho \)-triangular inequality.

(c) Given \( N \in \mathbb{N} \), choose \( \Gamma \subset \mathbb{N} \) with \( |\Gamma| = 2N \) such that \( \left\| \sum_{k \in \Gamma} e_k \right\|_\mathbb{B} \geq h_\ell(2N)/2 \).

Partitioning arbitrarily \( \Gamma = \Gamma' \cup \Gamma'' \) with \( |\Gamma'| = |\Gamma''| = N \), and using the \( \rho \)-power triangle inequality, one easily obtains

\[
\frac{1}{2} h_\ell(2N) \leq \left\| \sum_{k \in \Gamma} e_k \right\|_\mathbb{B} = \left\| \sum_{k \in \Gamma'} e_k + \sum_{k \in \Gamma''} e_k \right\|_\mathbb{B} \leq 2^{1/\rho} h_\ell(N).
\]

(d) Given \( N \in \mathbb{N} \), choose \( \Gamma \subset \mathbb{N} \) with \( |\Gamma| = N \) such that \( \left\| \sum_{k \in \Gamma} e_k \right\|_\mathbb{B} \leq 2h_\ell(N) \). Let \( \Gamma' = \Gamma \cup \{k_\alpha\} \) for any \( k_\alpha \notin \Gamma \). Then

\[
h_\ell(N+1) \leq \left\| \sum_{k \in \Gamma'} e_k \right\|_\mathbb{B} \leq \left( \left\| \sum_{k \in \Gamma} e_k \right\|_\mathbb{B}^\rho + 1 \right)^{1/\rho} \leq (2^\rho [h_\ell(N)]^\rho + 1)^{1/\rho}.
\]

Thus, using (a) we obtain \( h_\ell(N+1) \leq (2^\rho + 1)^{1/\rho} h_\ell(N) \leq 2 \cdot 2^{1/\rho} h_\ell(N) \). \( \square \)

**Remark 2.5.** We do not know whether property (d) can be improved to show that \( h_\ell(N) \) is actually doubling. This seems however to be case in all the examples we have considered below (see \S 5).

### 3. Right Democracy and Jackson Type Inequalities

Our first result deals with inclusions for the greedy classes \( \mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \).

**Theorem 3.1.** Let \( \mathcal{B} = \{e_j\}_{j=1}^\infty \) be a (normalized) unconditional basis in \( \mathbb{B} \). Fix \( \alpha > 0 \) and \( q \in (0, \infty) \). Then, for any sequence \( \eta \) such that \( \{k^\alpha \eta(k)\}_{k=1}^\infty \in \mathcal{W}_+ \) the following statements are equivalent:

1. There exists \( C > 0 \) such that for all \( N = 1, 2, 3, \ldots \)

\[
\left\| \sum_{k \in \Gamma} e_k \right\|_\mathbb{B} \leq C \eta(N), \quad \forall \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N.
\]

2. Jackson type inequality for \( \ell_\infty^\alpha(\mathcal{B}, \mathbb{B}) \): \( \exists C_\alpha > 0 \) such that \( \forall N = 0, 1, 2 \ldots \)

\[
\gamma_N(x) \leq C_\alpha (N+1)^{-\alpha} \|x\|_{\ell_\infty^\alpha(\mathcal{B}, \mathbb{B})}, \quad \forall x \in \ell_\infty^\alpha(\mathcal{B}, \mathbb{B}).
\]
Combining (3.7) and (3.8) with the inclusion \( \ell^{q}(B, B) \leftarrow \mathcal{G}^{\alpha}_{q}(B, B) \).

4. \( \ell^{q}(B, B) \leftarrow \mathcal{G}^{\alpha}_{q}(B, B) \).

5. **Jackson type inequality for \( \ell^{q}(B, B) \):** \( \exists C_{\alpha, q} > 0 \) such that \( \forall N = 0, 1, 2, \ldots \)

\[
\gamma_{N}(x) \leq C_{\alpha, q}(N + 1)^{-\alpha} \| x \|_{\ell^{q}(B, B)}, \quad \forall x \in \ell^{q}(B, B).
\]  

(3.3)

**Proof.** “1 \( \Rightarrow \) 2” Let \( x = \sum_{k \in \mathbb{N}} c_{k}e_{k} \in \ell^{q}(B, B) \) and let \( \pi \) be a bijection of \( \mathbb{N} \) such that

\[
|c_{\pi(k)}| \geq |c_{\pi(k+1)}|, \quad k = 1, 2, 3, \ldots
\]

(3.4)

For fixed \( N = 0, 1, 2, \ldots \), denote \( \lambda_{j} = 2^{j}(N+1) \). Then, the \( \rho \)-power triangle inequality and (2.3) give

\[
\| x - G_{N}^{\pi}(x) \|^{\rho}_{B} = \left\| \sum_{k = 0}^{\infty} c_{\pi(k)}e_{\pi(k)} \right\|^{\rho}_{B} \leq \sum_{k = 0}^{\infty} \left\| \sum_{j < k \leq \lambda_{j+1}} c_{\pi(k)}e_{\pi(k)} \right\|^{\rho}_{B}.
\]

There are exactly \( \lambda_{j} = 2^{j}(N+1) \) elements in the interior sum, so using (3.3) we obtain

\[
\| x - G_{N}^{\pi}(x) \|^{\rho}_{B} \leq C^{\rho} \sum_{j = 0}^{\infty} \left( c_{\pi(j)} \eta(\lambda_{j}) \right)^{\rho} = C^{\rho} \sum_{j = 0}^{\infty} \left( \lambda_{j}^{\alpha} c_{\pi(j)} \eta(\lambda_{j}) \right)^{\rho} \lambda_{j}^{-\alpha \rho}
\]

\[
\leq C^{\rho} \| x \|^{\rho}_{\ell^{q}(B, B)}(N + 1)^{-\alpha \rho} \sum_{j = 0}^{\infty} 2^{-j \alpha \rho}
\]

\[
= C_{\alpha, \rho} (N + 1)^{-\alpha \rho} \| x \|^{\rho}_{\ell^{q}(B, B)}.
\]

The result follows taking the supremum over all bijections \( \pi \) satisfying (3.4).

**Remark 3.2.** The special case \( N = 0 \) in (3.2) says that

\[
\| x \|_{B} \leq C \| x \|_{\ell^{q}(B, B)},
\]

(3.5)

which in particular implies \( \ell^{q}(B, B) \leftarrow B \), for all \( q \in (0, \infty] \).

“2 \( \Rightarrow \) 3” This is immediate from the definition of \( \mathcal{G}^{\alpha}_{\infty} \) (and Remark 3.2), since

\[
\| x \|_{\mathcal{G}^{\alpha}_{\infty}(B, B)} := \| x \|_{B} + \sup_{N \geq 1} N^{\alpha} \gamma_{N}(x) \leq C_{\alpha} \| x \|_{\ell^{\infty}(B, B)}.
\]

“3 \( \Rightarrow \) 1” Let \( \Gamma \subset N \) with \( |\Gamma| = N \). Choose \( \Gamma' \) with \( |\Gamma'| = N \) and so that \( \Gamma \cap \Gamma' = \emptyset \), and consider \( x = \sum_{k \in \Gamma} e_{k} + \sum_{k \in \Gamma'} 2\varepsilon_{k} \). Then

\[
\gamma_{N}(x) = \left\| \sum_{k \in \Gamma} e_{k} \right\|_{B}.
\]

(3.6)

and therefore

\[
N^{\alpha} \left\| \sum_{k \in \Gamma} e_{k} \right\|_{B} = N^{\alpha} \gamma_{N}(x) \leq \| x \|_{\mathcal{G}^{\alpha}_{\infty}(B, B)}.
\]

(3.7)

On the other hand, call \( \omega(k) = k^{\alpha} \eta(k) \). By monotonicity, Lemma 2.3 and the doubling property of \( \omega \) we have

\[
\| x \|_{\ell^{\infty}(B, B)} \leq 2 \| 1_{\Gamma \cap \Gamma'} \|_{\ell^{\infty}} = 2 \omega(2N) \leq c \omega(N).
\]

(3.8)

Combining (3.7) and (3.8) with the inclusion \( \ell^{\infty}(B, B) \leftarrow \mathcal{G}^{\alpha}_{\infty}(B, B) \) gives (3.1).
“5 ⇒ 1” Let \( \Gamma \subset N \) with \( |\Gamma| = N \), and choose \( \Gamma' \) and \( x \) as in the proof of 3 ⇒ 1. As before call \( \omega(k) = k^{\alpha} \eta(k) \). Then Lemma \( \{2,3\} \) and the assumption \( \omega \in \mathbb{W}_+ \) give
\[
\|x\|_{\ell^p_{\omega}(B,\mathbb{B})} \leq 2\|1_{\Gamma \cup \Gamma'}\|_{\ell^p_\omega} \approx \omega(2N) \leq c \omega(N).
\]
Since we are assuming 5 we can write (recall \( \{3,4\} \))
\[
\left\| \sum_{k \in \Gamma} c_k \right\|_B = \gamma_N(x) \leq C_{\alpha,\rho}(N + 1)^{-\alpha} \|x\|_{\ell^p_{\omega}(B,\mathbb{B})} \lesssim N^{-\alpha} \omega(N) = \eta(N),
\]
which proves \( \{3,1\} \).

“1 ⇒ 4” The proof is similar to 1 ⇒ 2 with a few modifications we indicate next. Given \( x \in \ell^p_{\kappa_0 \eta(k)}(B,\mathbb{B}) \) and \( \pi \) as in \( \{3,4\} \) we write \( x = \sum_{j=-1}^\infty \sum_{2^j \leq k \leq 2^{j+1}} c_k \epsilon_k \). Then arguing as before (with \( N = 2^m \)) we obtain
\[
\left\| x - G_{2^m}^\pi(x) \right\|_B^\mu \leq \sum_{j=m}^\infty |c_{\pi(2^j)}|^\mu \left\| \sum_{2^j \leq k \leq 2^{j+1}} c_k \epsilon_k \right\|_B^\mu,
\]
where we choose now any \( \mu < \min\{q, \rho_B\} \). Taking the supremum over all \( \pi \)'s and using \( \{3,1\} \) we obtain
\[
\gamma_{2^m}(x;B,\mathbb{B})^\mu \leq C^\mu \sum_{j=m}^\infty (c_{2^j}^* \eta(2^j))^\mu.
\]
Therefore
\[
\left[ \sum_{m=0}^\infty (2^m \gamma_{2^m}(x))^q \right]^{1/q} \leq C \left[ \sum_{m=0}^\infty 2^{maq} \left( \sum_{j=0}^\infty |c_{2^{j+m}}^* \eta(2^{j+m})|^\mu \right)^q \right]^{1/q}.
\]
Since \( q/\mu > 1 \), we can use Minkowski’s inequality on the right hand side to obtain
\[
\left[ \sum_{m=0}^\infty (2^m \gamma_{2^m}(x))^q \right]^{1/q} \leq C \left[ \sum_{m=0}^\infty \left( \sum_{j=0}^\infty c_{2^{j+m}}^* |\eta(2^{j+m})|^q \right) \right]^{1/q} \mu/\mu
\]
\[
= C \left[ \sum_{m=0}^\infty 2^{-j\mu} \left( \sum_{j=0}^\infty c_{2^j}^* |\eta(2^j)|^q \right)^{1/q} \right]^{1/\mu} \leq C' \|\{c_k\}\|_{\ell^q_{\alpha,\eta(k)}}^q.
\]
This implies the desired estimate
\[
\|x\|_{\mathcal{Q}^q(B,\mathbb{B})} \lesssim \|\{c_k\}\|_{\ell^q_{\alpha,\eta(k)}}^q
\]
using the dyadic expressions for the norms in \( \{2,8\} \) and \( \{2,10\} \) (and Remark \( \{3,2\} \)).

“4 ⇒ 5” This is trivial since 4 implies \( \ell^q_{\kappa \eta(k)}(B,\mathbb{B}) \hookrightarrow \mathcal{Q}^q(B,\mathbb{B}) \hookrightarrow \mathcal{Q}^q_\infty(B,\mathbb{B}) \), and this clearly gives \( \{3,3\} \).

Remark 3.3. The equivalences 1 to 3 remain true under the weaker assumption \( \{k^{\alpha} \eta(k)\} \in \mathbb{W} \).

Remark 3.4. Observe that if any of the statements in 2 to 5 of Theorem \( \{3,1\} \) holds for one fixed \( \alpha > 0 \) and \( q \in (0, \infty) \), then the assertions remain true for all \( \alpha \) and \( q \) (as long as \( \{k^{\alpha} \eta(k)\} \in \mathbb{W}_+ \)), since the statement in 1 is independent of these parameters.
Corollary 3.5. Optimal inclusions into $\ell^q$. Let $B$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0, \infty)$. Then

$$\ell^q_{k^\alpha h_r(k)}(B, \mathbb{B}) \hookrightarrow \ell^q(\mathbb{B}, \mathbb{B}).$$

(3.9)

Moreover, if $\omega \in \mathbb{W}_+$, then $\ell^q(\mathbb{B}, \mathbb{B}) \hookrightarrow \ell^q(\mathbb{B}, \mathbb{B})$ if and only if $\omega(k) \geq k^\alpha h_r(k)$.

Proof. For $q < \infty$, the inclusion (3.9) is an application of 4 in the theorem with $q = h_r$ (after noticing that $\{k^\alpha h_r(k)\} \in \mathbb{W}_+$ by Proposition 2.4 and Remark 2.2). The second assertion is just a restatement of 1 with $q = \infty$. For $q = \infty$ use 3 instead of 4.

We now prove similar results for the approximation spaces $A^q$. Let $B = \{e_j\}_{j=1}^\infty$ be a (normalized) unconditional basis in $\mathbb{B}$. Fix $\alpha > 0$ and $q \in (0, \infty)$. Then, for any sequence $\eta \in \mathbb{W}_+$ the following are equivalent:

1. There exists $C > 0$ such that for all $N = 1, 2, 3, \ldots$

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq C \eta(N), \quad \forall \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N.$$  

(3.10)

2. $\ell^q(\mathbb{B}, \mathbb{B}) \hookrightarrow A^q(\mathbb{B}, \mathbb{B})$.

3. Jackson type inequality for $\ell^q_{k^\alpha \eta(k)}(B, \mathbb{B})$: $\exists C_{\alpha,q} > 0$ such that $\forall N = 0, 1, 2, \ldots$

$$\sigma_N(x) \leq C_{\alpha,q}(N + 1)^{-\alpha} \|x\|_{\ell^q_{k^\alpha \eta(k)}(B, \mathbb{B})}, \quad \forall x \in \ell^q_{k^\alpha \eta(k)}(B, \mathbb{B}).$$

(3.11)

Proof. $1 \Rightarrow 2$ follows directly from Theorem 3.4 and $\mathfrak{A} \hookrightarrow A^q$. Also, $2 \Rightarrow 3$ is trivial since $A^q \hookrightarrow A^\infty$, and 3 is equivalent to $\ell^q_{k^\alpha \eta(k)}(B, \mathbb{B}) \hookrightarrow A^\infty$.

We must show $3 \Rightarrow 1$. Let $\kappa > 1$ be a fixed integer as in the definition of the class $\mathbb{W}_+$ (and in particular satisfying (1.11)), and denote $1_\Delta = \sum_{k \in \Delta} e_k$ for a set $\Delta \subset \mathbb{N}$. For any $\Gamma_n \subset \mathbb{N}$ with $|\Gamma_n| = \kappa^n$, we can find a subset $\Gamma_{n-1}$ with $|\Gamma_{n-1}| = \kappa^{n-1}$ such that

$$\|1_{\Gamma_n} - 1_{\Gamma_{n-1}}\|_{\mathbb{B}} \leq 2\sigma_{n-1}(1_{\Gamma_n}).$$

Repeating this argument we choose $\Gamma_j \subset \Gamma_j$ with $|\Gamma_j| = \kappa^j$ and so that

$$\|1_{\Gamma_j} - 1_{\Gamma_{j-1}}\|_{\mathbb{B}} \leq 2\sigma_{j-1}(1_{\Gamma_j}), \quad \text{for } j = 1, 2, \ldots, n.$$  

Setting $\Gamma_{-1} = 0$, and using the $p$-power triangle inequality we see that

$$\left\| \sum_{j=0}^n 1_{\Gamma_j} \right\|_{\mathbb{B}} \leq \sum_{j=0}^n \|1_{\Gamma_j} - 1_{\Gamma_{j-1}}\|_{\mathbb{B}} \leq 2^p \sum_{j=0}^n \sigma_{j-1}(1_{\Gamma_j}).$$

Now, the hypothesis (3.11) and Lemma 2.3 give

$$\sigma_{\kappa^{j-1}}(1_{\Gamma_j}) \leq \kappa^{-ja} \|1_{\Gamma_j}\|_{\ell^q_{k^\alpha \eta(k)}}(B, \mathbb{B}) \approx \eta(\kappa^j).$$

Thus, combining these two expressions we obtain

$$\|1_{\Gamma_n}\|_{\mathbb{B}} \leq \left[ \sum_{j=0}^n \eta(\kappa^j)^p \right]^{1/p} \leq C \eta(\kappa^n),$$

(3.12)

where the last inequality follows from the assumption $\eta \in \mathbb{W}_+$ and Lemma 2.1. This shows (3.10) when $N = \kappa^n, n = 1, 2, \ldots$. The general case follows easily using the doubling property of $\eta$. 

\qed
Remark 3.7. As before, if any of the statements in 2 or 3 holds for one fixed \( \alpha > 0 \) and \( q \in (0, \infty] \), then the assertions remain true for all \( \alpha \) and \( q \), since 1 is independent of these parameters.

Remark 3.8. Observe also that \( 1 \Rightarrow 2 \Rightarrow 3 \) hold with the weaker assumption \( \{k^\alpha h_r(k)\} \in \mathcal{W}_+ \) from Theorem 3.1 (and in particular hold for \( \eta = h_r \) as stated in (1.5)). However, the stronger assumption \( \eta \in \mathcal{W}_+ \) is crucial to obtain \( 3 \Rightarrow 1 \), and cannot be removed as shown in Example 5.6 below.

Corollary 3.9. Optimality of the inclusions into \( A^\alpha_q \).

Let \( B \) be a (normalized) unconditional basis in \( B \). Fix \( \alpha > 0 \) and \( q \in (0, \infty] \). Then

\[
\ell_{k^\alpha h_r(k)}^q(B, B) \hookrightarrow A^\alpha_q(B, B). \tag{3.13}
\]

If for some \( \omega \in \mathcal{W}_+ \) we have \( \ell_{\omega}^q(B, B) \hookrightarrow A^\alpha_q(B, B) \), then necessarily \( \omega(k) \gtrsim k^\alpha \). Moreover if \( \omega(k) = k^\alpha \eta(k) \), with \( \eta \) increasing and doubling, then

(a) if \( \eta > 0 \), then necessarily \( \eta(k) \gtrsim h_r(k) \), and hence \( \ell_{k^\alpha \eta(k)}^q \hookrightarrow \ell_{k^\alpha h_r(k)}^q \).

(b) if \( \eta = 0 \), then \( \eta(k) \gtrsim h_r(k)/(\log k)^{1/\rho} \) and \( \ell_{k^\alpha \eta(k)}^q \hookrightarrow \ell_{(k^\alpha h_r(k)/(\log k)^{1/\rho})}^q \).

Proof. The inclusion (3.13) is actually a consequence of (3.9). Assertion (a) is just \( 2 \Rightarrow 3 \Rightarrow 1 \) in the theorem. For assertion (b) notice that in the last step of the proof of 3 \( \Rightarrow 1 \), the right hand inequality of (3.12) can always be replaced by

\[
\|1_{1_{\Gamma_n}}\|_B \lesssim \left[ \sum_{j=0}^{n} \eta(k^j)|\rho| \right]^{1/\rho} \lesssim \eta(k^n)n^{1/\rho}
\]

when \( \eta \) is increasing. Thus \( h_r(N) \lesssim \eta(N)(\log N)^{1/\rho} \) holds for \( N = k^n \), and by the doubling property also for all \( N \in \mathbb{N} \). Finally, if \( \ell_{\omega}^q(B, B) \hookrightarrow A^\alpha_q(B, B) \) for some general \( \omega \in \mathcal{W}_+ \), then given \( \Gamma \subset \mathbb{N} \) with \( |\Gamma| = N \) we trivially have

\[
\omega(N) \approx \|1_{1_{\Gamma}}\|_{\ell_\Gamma^q} \gtrsim \|1_{1_{\Gamma}}\|_{A^\alpha_q} \approx (N/2)^{\alpha} \sigma_{N/2}(1_{1_{\Gamma}}) \geq (N/2)^{\alpha}.
\]

Remark 3.10. Assertion (b) shows that the inclusion in (3.13) is optimal, except perhaps for a logarithmic loss. The logarithmic loss may actually happen, as there are Banach spaces \( B \) with \( h_r(N) \approx \log N \) and so that

\[
A^\alpha_q(B) = \ell_{k^\alpha}^q = \ell_{\{k^\alpha h_r(k)/(\log k)\}}^q.
\]

See Example 5.6 below.

4. LEFT DEMOCRACY AND BERNSTEIN TYPE INEQUALITIES

It is well known that upper inclusions for the approximation spaces \( A^\alpha_q \), as in (1.5), depend upon Bernstein type inequalities. In this section we show how the left democracy function of \( B \) is linked with these two properties.

We first remark that, for each \( \alpha > 0 \) and \( 0 < q \leq \infty \), the approximation classes \( A^\alpha_q \) and \( G^\alpha_q \) satisfy trivial Bernstein inequalities, namely, there exists \( C_{\alpha, q} > 0 \) such that

\[
\|x\|_{A^\alpha_q(B, B)} \leq \|x\|_{G^\alpha_q(B, B)} \leq C_{\alpha, q} N^\alpha \|x\|_B, \quad \forall \ x \in \Sigma_N, \ N = 1, 2, \ldots \tag{4.1}
\]

This follows easily from the definition of the norms and the trivial estimates \( \sigma_N(x) \leq \gamma_N(x) \leq \|x\|_B \).
We start with a preliminary result which is essentially known in the literature (see eg [29]). As usual \( \mathcal{B} = \{ e_j \}_{j=1}^\infty \) is a fixed (normalized) unconditional basis in \( \mathcal{B} \).

**Proposition 4.1.** Let \( \mathcal{E} \) be a subspace of \( \mathcal{B} \), endowed with a quasi-norm \( ||.||_\mathcal{E} \) satisfying the \( \rho \)-triangle inequality for some \( \rho = p_\mathcal{E} \). For each \( \alpha > 0 \) the following are equivalent:

1. \( \exists C_\alpha > 0 \) such that \( ||x||_\mathcal{E} \leq C_\alpha N_\alpha ||x||_\mathcal{B} \), \( \forall x \in \Sigma_N \), \( N = 1, 2, \ldots \)
2. \( \mathcal{A}_\rho^\alpha (\mathcal{B}, \mathcal{B}) \hookrightarrow \mathcal{E} \).
3. \( \mathcal{G}_\rho^\alpha (\mathcal{B}, \mathcal{B}) \hookrightarrow \mathcal{E} \).

**Proof.** “1 \( \Rightarrow \) 2”. Given \( x \in \mathcal{A}_\rho^\alpha (\mathcal{B}, \mathcal{B}) \), by the representation theorem for approximation spaces [29] one can write

\[
\left( \sum_{k=0}^\infty 2^{\kappa_\alpha ||x_k||_B^\rho} \right)^{1/\rho} \leq C ||x||_{\mathcal{A}_\rho^\alpha (\mathcal{B}, \mathcal{B})}.
\]

The hypothesis 1 and the \( p_\mathcal{E} \)-triangular inequality then give

\[
||x||_\mathcal{E}^\rho \leq \sum_{k=0}^\infty ||x_k||_B^\rho \leq C_\alpha^\rho \sum_{k=0}^\infty 2^{\kappa_\alpha ||x_k||_B^\rho} \leq C' ||x||_{\mathcal{A}_\rho^\alpha (\mathcal{B}, \mathcal{B})}.
\]

“2 \( \Rightarrow \) 3”. This follows from the trivial inclusion \( \mathcal{G}_\rho^\alpha (\mathcal{B}, \mathcal{B}) \hookrightarrow \mathcal{A}_\rho^\alpha (\mathcal{B}, \mathcal{B}) \).

“3 \( \Rightarrow \) 1”. This is immediate using (4.1). \( \square \)

**Theorem 4.2.** Let \( \mathcal{B} = \{ e_j \}_{j=1}^\infty \) be a (normalized) unconditional basis in \( \mathcal{B} \). Fix \( \alpha > 0 \) and \( q \in (0, \infty] \). Then, for any increasing and doubling sequence \( \{ \eta(k) \} \) the following statements are equivalent:

1. There exists \( C > 0 \) such that for all \( N = 1, 2, 3, \ldots \)

\[
\left| \sum_{k \in \Gamma} e_k \right|_B \geq \frac{1}{C} \eta(N), \quad \forall \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N.
\]  \( (4.2) \)

2. Bernstein type inequality for \( \ell^q_k(\eta(k))(\mathcal{B}, \mathcal{B}) \): \( \exists C_{\alpha,q} > 0 \) such that

\[
||x||_{\ell^q_k(\eta(k))(\mathcal{B}, \mathcal{B})} \leq C_{\alpha,q} N_\alpha ||x||_B, \quad \forall x \in \Sigma_N, \ N = 1, 2, 3, \ldots \]  \( (4.3) \)

3. \( \mathcal{A}_q^\alpha (\mathcal{B}, \mathcal{B}) \hookrightarrow \ell^q_k(\eta(k))(\mathcal{B}, \mathcal{B}) \).
4. \( \mathcal{G}_q^\alpha (\mathcal{B}, \mathcal{B}) \hookrightarrow \ell^q_k(\eta(k))(\mathcal{B}, \mathcal{B}) \).

**Proof.** “1 \( \Rightarrow \) 2”. Let \( x = \sum_{k \in \Gamma} c_k e_k \in \Sigma_N \). For any bijection \( \pi \) with \(|c_\pi(k)|\) decreasing, and any integer \( m \in \{ 1, \ldots, N \} \) we have

\[
|c_\pi(m)| \eta(m) \leq C |c_\pi(m)| \left| \sum_{j=1}^m c_\pi(j) e_{\pi(j)} \right|_B \leq C \left| \sum_{j=1}^m c_\pi(j) e_{\pi(j)} \right|_B \leq C ||x||_B,
\]

using (2.3) in the second inequality. This gives

\[
||x||_{\ell^q_k(\eta(k))} = \left[ \sum_{m=1}^N (m^\alpha \eta(m) c_m^q) \frac{1}{m} \right]^{1/q} \leq C ||x||_B \left[ \sum_{m=1}^N m^{\alpha q} \frac{1}{m} \right]^{1/q} \approx ||x||_B N_\alpha.
\]

“2 \( \Rightarrow \) 1”. For any \( \Gamma \subset \mathbb{N} \) with \(|\Gamma| = N\), applying (4.3) to \( 1_\Gamma = \sum_{k \in \Gamma} e_k \) we obtain

\[
||1_\Gamma||_B \geq \frac{1}{C_{\alpha,q}} N^{-\alpha} ||1_\Gamma||_{\ell^q_k(\eta(k))(\mathcal{B}, \mathcal{B})} \geq \eta(N),
\]
where in the last inequality we have used \( \|1_\Gamma\|_{\ell^q} \gtrsim \omega(N) \), when \( \omega \in \mathcal{W} \).

“2 \Rightarrow 3”. We have already proved that \( 1 \iff 2 \); since 1 does not depend on \( \alpha, q \), then 2 actually holds for all \( \tilde{\alpha} > 0 \). In particular, from Proposition 4.1 we have

\[
\mathcal{A}_\alpha^0 \hookrightarrow \mathcal{E} := \ell^q_{k^{\alpha}\eta(k)}(\mathcal{B}, \mathcal{B}) \tag{4.4}
\]

for \( \tilde{\alpha} \in (0, \frac{3\alpha}{2}) \) and some sufficiently small \( \rho > 0 \). Now, from the general theory developed in [7], the spaces \( \mathcal{A}_\alpha^0 \) satisfy a reiteration theorem for the real interpolation method, and in particular

\[
\mathcal{A}_\alpha^0 = (\mathcal{A}_{\alpha_0}^{\alpha_0}, \mathcal{A}_{\alpha_1}^{\alpha_1})_{1/2, q}, \tag{4.5}
\]

when \( \alpha = (\alpha_0 + \alpha_1)/2 \) with \( \alpha_1 > \alpha_0 > 0 \), and \( q_0, q_1, q \in (0, \infty) \). On the other hand, for the family of weighted Lorentz spaces it is known that

\[
(\ell^q_{\omega_0}, \ell^q_{\omega_1})_{\theta, q} = \ell^q_{\omega}, \quad 0 < \theta < 1, \quad 0 < q \leq \infty, \tag{4.6}
\]

when \( \omega_0, \omega_1 \in \mathcal{W}_+ \) and \( \omega = \omega_0^{1-\theta} \omega_1^\theta \) (see eg [25, Theorem 3]). Thus, for fixed \( \alpha \) and \( q \), we can choose the parameters accordingly, and use the inclusion (4.4), to obtain

\[
\mathcal{A}_\alpha^0 = (\mathcal{A}_{\alpha_0}^{\alpha_0}, \mathcal{A}_{\alpha_1}^{\alpha_1})_{1/2, q} \hookrightarrow (\ell^q_{k^{\alpha_0}\eta(k)}), \ell^q_{k^{\alpha_1}\eta(k)})_{1/2, q} = \ell^q_{k^{\alpha}\eta(k)}(\mathcal{B}, \mathcal{B}).
\]

“3 \Rightarrow 4”. This is trivial since \( \mathcal{G}_q \hookrightarrow \mathcal{A}_q^0 \).

“4 \Rightarrow 2”. This is trivial from (4.1).

\[\square\]

**Remark 4.3.** Observe that 3 \( \Rightarrow 4 \Rightarrow 2 \iff 1 \) hold with the weaker assumption \( \{k^{\alpha}\eta(k)\} \in \mathcal{W} \).

**Corollary 4.4.** Optimal inclusions of \( \mathcal{A}_q^\alpha \) into \( \ell^q_\omega \).

Let \( \mathcal{B} \) be a (normalized) unconditional basis in \( \mathcal{B} \). Fix \( \alpha > 0 \) and \( q \in (0, \infty] \).

(a) If \( h_\ell(N) \) is doubling then \( \mathcal{A}_q^\alpha(\mathcal{B}, \mathcal{B}) \hookrightarrow \ell^q_{k^{\alpha}h_\ell(k)}(\mathcal{B}, \mathcal{B}) \).

(b) If for some \( \omega \in \mathcal{W} \) we have \( \mathcal{A}_q^\alpha(\mathcal{B}, \mathcal{B}) \hookrightarrow \ell^q_{k^{\alpha}h_\ell(k)} \) then necessarily \( \omega(k) \lesssim k^{\alpha}h_\ell(k) \), and hence \( \ell^q_{k^{\alpha}h_\ell(k)} \hookrightarrow \ell^q_\omega \).

**Proof.** Part (a) is an application of 1 \( \Rightarrow 3 \) in the theorem with \( \eta = h_\ell \) (which under the doubling assumption satisfies \( \{k^{\alpha}h_\ell(k)\} \in \mathcal{W}_+ \) for all \( \alpha > 0 \)). Part (b) is just a restatement of 3 \( \Rightarrow 1 \) in the theorem, setting \( \eta(k) = \omega(k)/k^\alpha \) and taking into account Remark 4.3. \[\square\]

## 5. Examples and Applications

In this section we describe the democracy functions \( h_\ell \) and \( h_r \) in various examples which can be found in the literature. Inclusions for \( \mathcal{A}_q^\alpha(\mathcal{B}, \mathcal{B}) \) and \( \mathcal{G}_q^\alpha(\mathcal{B}, \mathcal{B}) \) will be obtained immediately from the results of sections 3 and 4. The most interesting case appears when \( \mathcal{B} \) is a wavelet basis, and \( \mathcal{B} \) a function or distribution space in \( \mathbb{R}^d \) which can be characterized by such basis (eg, the general Besov or Triebel-Lizorkin spaces, \( B^s_{p,q} \) and \( F^s_{p,q} \), and also rearrangement invariant spaces as the Orlicz and Lorentz classes, \( L^\Phi \) and \( L^q \)). Such characterizations provide a description of each \( \mathcal{B} \) as a sequence space, so for simplicity we shall work in this simpler setting, reminding in each case the original function space framework.

Let \( \mathcal{D} = \mathcal{D}(\mathbb{R}^d) \) denote the family of all dyadic cubes \( Q \) in \( \mathbb{R}^d \), ie

\[
\mathcal{D} = \{ Q_{j,k} = 2^{-j}([0,1]^d + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d \},
\]
We shall consider sequences indexed by \( \mathcal{D} \), \( s = \{ s_Q \}_{Q \in \mathcal{D}} \), endowed with quasi-norms of the following form
\[
\left\| \left( \sum_{Q \in \mathcal{D}} \left( |Q|^{-\frac{d}{2}} |s_Q| \chi_Q(\cdot) \right)^r \right)^{1/r} \right\|_{\mathcal{X}},
\]
where \( 0 < r \leq \infty \), \( \gamma \in \mathbb{R} \) and \( \mathcal{X} \) is a suitable quasi-Banach function space in \( \mathbb{R}^d \), such as the ones we consider below. The canonical basis \( \mathcal{B}_c = \{ e_Q \}_{Q \in \mathcal{D}} \) is formed by the sequences \( e_Q \) with entry 1 at \( Q \) and 0 otherwise. In each of the examples below, the greedy algorithms and democracy functions are considered with respect to the normalized basis \( \mathcal{B} = \{ e_Q / \| e_Q \|_B \} \). Similarly, when stating the corresponding results for the functional setting we shall write \( \mathcal{W} \) for the wavelet basis.

**Example 5.1.** \( \mathcal{X} = L^p(\mathbb{R}^d) \), \( 0 < p < \infty \). In this case, it is customary to consider the sequence spaces \( \ell^{a}_{p,r}(s) \), \( s \in \mathbb{R} \), \( 0 < r \leq \infty \), with quasi-norms given by
\[
\| s \|_{\ell^{a}_{p,r}} := \left\| \left( \sum_{Q \in \mathcal{D}} \left( |Q|^{-\frac{d}{2}} |s_Q| \chi_Q(\cdot) \right)^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^d)}.
\]
It was proved in [16, 11, 18] that, for all \( s \in \mathbb{R} \) and \( 0 < r \leq \infty \),
\[
h_t(N; \ell^{a}_{p,r}) \approx h_t(N; \ell^{a}_{p,r}) \approx N^{1/p}
\]
and
\[
\mathcal{A}^a_q(\ell^{a}_{p,r}) = \ell^{a,q}(\ell^{a}_{p,r}) = \left\{ s : \{ s_Q \| e_Q \|_B \}_{Q \in \mathcal{D}} \in \ell^{a,q} \right\},
\]
if \( \frac{1}{q} = \alpha + \frac{1}{r} \), as asserted in Theorem 1.2.

It is well-known that \( \ell^{a}_{p,r} \) coincides with the coefficient space under a wavelet basis \( \mathcal{W} \) of the (homogeneous) Triebel-Lizorkin space \( \dot{F}^{s}_{p,r}(\mathbb{R}^d) \), defined in terms of Littlewood-Paley theory (see eg [10, 26, 22]). In particular, under suitable decay and smoothness on the wavelet family (so that it is an unconditional basis of the involved spaces) the statement in (5.3) can be translated into
\[
\mathcal{A}^a_q(\mathcal{W}, \dot{F}^{s}_{p,r}(\mathbb{R}^d)) = \mathcal{A}^a_q(\mathcal{W}, \dot{F}^{s}_{p,r}(\mathbb{R}^d)) = \dot{B}^{s+a,d}_{q,q}(\mathbb{R}^d)
\]
when \( \frac{1}{q} = \frac{d}{q} + \frac{1}{p} \). We refer to [16, 17, 5, 11] for details and further results.

**Example 5.2.** Weighted Lebesgue spaces \( \mathcal{X} = L^p(w) \), \( 0 < p < \infty \). For weights \( w(x) \) in the Muckenhoupt class \( A_\infty(\mathbb{R}^d) \), one can define sequence spaces \( \ell^{a}_{p,r}(w) \) with the quasi-norm
\[
\| s \|_{\ell^{a}_{p,r}(w)} := \left\| \left( \sum_{Q \in \mathcal{D}} \left( |Q|^{-\frac{d}{2}} |s_Q| \chi_Q(\cdot) \right)^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^d,w)}.
\]
Similar computations as in the previous case in this more general situation will also lead to the identities in (5.2) and (5.3), with \( \ell^{a}_{p,r} \) replaced by \( \ell^{a}_{p,r}(w) \). We refer to [27, 21] for details in some special cases.

When \( \mathcal{W} \) is a (sufficiently smooth) orthonormal wavelet basis and \( w \) is a weight in the Muckenhoupt class \( A_p(\mathbb{R}^d) \), \( 1 < p < \infty \), then \( \ell^{a}_{p,2}(w) \) becomes the coefficient space of the weighted Lebesgue space \( L^p(w) \) (see eg [11]). One then obtains as special case
\[
h_t(N; \mathcal{W}, L^p(w)) \approx h_t(N; \mathcal{W}, L^p(w)) \approx N^{1/p}.
\]
Moreover, if \( \omega \in A_r(\mathbb{R}^d) \),
\[
A^\alpha_p(\mathcal{W}, L^p(w)) \approx \mathcal{G}^\alpha_q(\mathcal{W}, L^p(w)) \approx \hat{B}^\alpha_{r,q}(L^p(w)), \quad \text{if } \frac{1}{r} = \alpha + \frac{1}{p},
\]

where \( \hat{B}^\alpha_{r,q}(w) \) denotes a weighted Besov space (see [27] for details).

**Example 5.3. Orlicz spaces** \( \mathcal{X} = L^\Phi(\mathbb{R}^d) \). Following [12], we denote by \( f^\Phi \) the sequence space with quasi-norm
\[
\|s\|_{f^\Phi} := \left\| \left( \sum_{Q \in \mathcal{D}} \left| s_Q \right|^\frac{\varphi(t)}{|Q|^{1/2}} \right)^2 \right\|_{L^\Phi(\mathbb{R}^d)},
\]
where \( L^\Phi \) is an Orlicz space with non-trivial Boyd indices. If we denote by \( \varphi(t) = 1/\Phi^{-1}(1/t) \), the fundamental function of \( L^\Phi \), then it is shown in [12] that
\[
h_r(N; f^\Phi) \approx \inf_{s > 0} \frac{\varphi(Ns)}{\varphi(s)} \quad \text{and} \quad h_r(N; f^\Phi) \approx \sup_{s > 0} \frac{\varphi(Ns)}{\varphi(s)},
\]
with the two expressions being equivalent iff \( \varphi(t) = t^{1/p} \) (ie, iff \( L^\Phi = L^p \)). Thus, these are first examples of non-democratic spaces, with a wide range of possibilities for the democracy functions. The theorems in sections 3 and 4 recover the embeddings obtained in [12] for the approximation classes \( A^\alpha_q(\mathcal{W}, L^p) \) and \( \mathcal{G}^\alpha_q(\mathcal{W}, L^p) \), some of which can be expressed in terms of Besov spaces of generalized smoothness (see [12] for details).

**Example 5.4. Lorentz spaces** \( \mathcal{X} = L^{p,q}(\mathbb{R}^d), \) \( 0 < p, q < \infty \). Consider sequence spaces \( F^{p,q} \) defined by the following quasi-norms
\[
\|s\|_{F^{p,q}} := \left\| \left( \sum_{Q \in \mathcal{D}} \left| s_Q \right|^\frac{\varphi(t)}{|Q|^{1/2}} \right)^2 \right\|_{L^{p,q}(\mathbb{R}^d)}.
\]
Their democracy functions have been computed in [14], obtaining
\[
h_r(N; F^{p,q}) \approx N^{\frac{1}{\min(p,q)}} \quad \text{and} \quad h_r(N; F^{p,q}) \approx N^{\frac{1}{\min(p,q)}}.
\]
These imply corresponding inclusions for the classes \( A^\alpha_p(\mathcal{W}, L^{p,q}) \) and \( \mathcal{G}^\alpha_p(\mathcal{W}, L^{p,q}) \) in terms of discrete Lorentz spaces \( L^{r,s} \) (as described in the theorems of sections 3 and 4). The spaces \( F^{p,q} \) characterize, via wavelets, the usual Lorentz spaces \( L^{p,q}(\mathbb{R}^d) \) when \( 1 < p < \infty \) and \( 1 \leq q < \infty \) ([32]). Hence inclusions for \( A^\alpha_p(\mathcal{W}, L^{p,q}) \) and \( \mathcal{G}^\alpha_p(\mathcal{W}, L^{p,q}) \) can be obtained using standard Besov spaces.

**Example 5.5. Hyperbolic wavelets.** For \( 0 < p < \infty \), consider now the sequence space
\[
\|s\|_{F^p_{\text{hyp}}} := \left\| \left( \sum_{R} \left| s_R \right|^\frac{\chi_{\mathcal{D}}(\cdot t)}{|R|^{1/2}} \right)^2 \right\|_{L^p(\mathbb{R}^d)}.
\]
where \( R \) runs over the family of all dyadic rectangles of \( \mathbb{R}^d \), that is \( R = I_1 \times \ldots \times I_d \), with \( I_i \in \mathcal{D}(\mathbb{R}) \), \( i = 1, \ldots, d \). This gives another example of non-democratic basis. In fact, the following result is proved in [34, Proposition 11] (see also [34]):

\( a) \) If \( 0 < p \leq 2 \),
\[
h_r(N; f^p_{\text{hyp}}) \approx N^{1/p} (\log N)^{(\frac{1}{2} - \frac{1}{p})(d-1)} \quad \text{and} \quad h_r(N; f^p_{\text{hyp}}) \approx N^{1/p}.
\]
functions are determined by smooth wavelet bases appropriately normalized (see [36, 10, 16]).

The first part of (5.5) is easy to prove, and the second follows, for instance, by an expression in terms of Besov spaces of bounded mixed smoothness [19, 6].

However, this is not the best one can say for the approximation classes $A^\alpha_q(\mathcal{H}_d, L^p)$ and $\mathcal{G}^\alpha_q(\mathcal{H}_d, L^p)$ (see also [19, Thm 5.2]), some of which could possibly be expressed in terms of Besov spaces of bounded mixed smoothness [19, 6].

**Example 5.6. Bounded mean oscillation.** Let bmo denote the space of sequences $s = \{s_I\}_{I \in \mathcal{D}}$ with
\[
\|s\|_{\text{bmo}} = \sup_{I \in \mathcal{D}} \left( \frac{1}{|I|} \sum_{J \subsetneq I, J \in \mathcal{D}} |s_J|^2 |J| \right)^{1/2} < \infty .
\]
This sequence space gives the correct characterization of $BMO(\mathbb{R})$ for sufficiently smooth wavelet bases appropriately normalized (see [36, 10, 16]). Their democracy functions are determined by
\[
h_\epsilon(N; \text{bmo}) \approx 1 , \quad h_r(N; \text{bmo}) \approx (\log N)^{1/2} .
\]
The first part of (5.5) is easy to prove, and the second follows, for instance, by an argument similar to the one presented in the proof of [28, Lemma 3]. Our results of sections 5 and 4 give in this case the inclusions:
\[
\ell^q_{k^\alpha \sqrt{\log k}} \hookrightarrow \mathcal{G}^\alpha_q(\text{bmo}) \hookrightarrow A^\alpha_q(\text{bmo}) \hookrightarrow \ell^q_h = \ell^{1/\alpha, q} .
\]
However, this is not the best one can say for the approximation classes $A^\alpha_q$. A result proved in [30] (see also Proposition 11.6 in [16]) shows that one actually has
\[
A^\alpha_q(\text{bmo}) = A^\alpha_q(\ell^\infty) = \ell^{1/\alpha, q} ,
\]
for all $\alpha > 0$ and $q \in (0, \infty]$. For $0 < r < \infty$ one can define the space $bmo_r$ replacing the $2$ by $r$ in (5.4); it can then be shown that $h_r(N; bmo_r) \approx (\log N)^{1/r}$ and $A^\alpha_q(bmo_r) = \ell^{1/\alpha, q}$.

6. DEMOCRACY FUNCTIONS FOR $A^\alpha_q(\mathcal{B}, \mathbb{B})$ AND $\mathcal{G}^\alpha_q(\mathcal{B}, \mathbb{B})$

As usual, we fix a (normalized) unconditional basis $\mathcal{B} = \{e_j\}_{j=1}^\infty$ in $\mathbb{B}$. In this section we compute the democracy functions for the spaces $A^\alpha_q(\mathcal{B}, \mathbb{B})$ and $\mathcal{G}^\alpha_q(\mathcal{B}, \mathbb{B})$, in terms of the democracy functions in the ambient space $\mathbb{B}$. To distinguish among these notions we shall use, respectively, the notations
\[
h_\epsilon(N; A^\alpha_q), \quad h_\epsilon(N; \mathcal{G}^\alpha_q) \quad \text{and} \quad h_\epsilon(N; \mathbb{B}),
\]
and similarly for $h_r$ (recall the definitions in section 2.5). Since we shall use the embeddings in sections 3 and 4, observe first that
\[
h_\epsilon(N; \ell^q_h(\mathcal{B}, \mathbb{B})) \approx h_r(N; \ell^q_h(\mathcal{B}, \mathbb{B})) \approx \omega(N),
\]
for all $\omega \in \mathcal{W}_+$ and $0 < q \leq \infty$. This is immediate from the definition of the spaces $\ell^q_h(\mathcal{B}, \mathbb{B})$ and Lemma 2.3.

**Proposition 6.1.** Fix $\alpha > 0$ and $0 < q \leq \infty$. If $h_\epsilon(\cdot; \mathbb{B})$ is doubling then
Proposition 6.2. The required bound then follows from the doubling property of $\mathcal{B}$.

In particular, $\mathcal{B}$ is democratic in $\mathcal{G}_q^\alpha(\mathcal{B}; \mathbb{B})$ if and only if $\mathcal{B}$ is democratic in $\mathbb{B}$.

Proof. The inequalities “$\preceq$” in (a), and “$\succeq$” in (b) follow immediately from the embeddings

$$\ell_{k^{\alpha}h_r(k)}^q(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{G}_q^\alpha(\mathcal{B}; \mathbb{B}) \hookrightarrow \ell_{k^{\alpha}h_\ell(k)}^q(\mathcal{B}; \mathbb{B})$$

and the remark in (6.1). Thus we must show the converse inequalities. To establish (a), given $N = 1, 2, 3, \ldots$ choose $\Gamma$ with $|\Gamma| = N$ and so that $\|1_\Gamma\|_{\mathbb{B}} \leq 2h_\ell(N; \mathbb{B})$. Then, using the trivial bound in (4.1) we obtain

$$h_\ell(N; \mathcal{G}_q^\alpha) \leq \|1_\Gamma\|_{\mathcal{G}_q^\alpha} \lesssim N^{\alpha}\|1_\Gamma\|_{\mathbb{B}} \approx N^{\alpha}h_\ell(N; \mathbb{B}).$$

We now prove “$\succeq$” in (b). Given $N = 1, 2, \ldots$, choose first $\Gamma$ with $|\Gamma| = N$ and $\|1_\Gamma\|_{\mathbb{B}} \geq \frac{1}{2}h_r(N; \mathbb{B})$, and then any $\Gamma'$ disjoint with $\Gamma$ with $|\Gamma'| = N$. Then

$$h_r(2N; \mathcal{G}_q^\alpha) \geq \|1_{\Gamma \cup \Gamma'}\|_{\mathcal{G}_q^\alpha} \gtrsim N^{\alpha}\gamma_N(1_{\Gamma \cup \Gamma'}; \mathbb{B}) \gtrsim N^{\alpha}\|1_\Gamma\|_{\mathbb{B}} \approx N^{\alpha}h_r(N; \mathbb{B}).$$

The required bound then follows from the doubling property of $h_r$.

Proposition 6.2. Fix $\alpha > 0$ and $0 < q \leq \infty$, and assume that $h_\ell(\cdot; \mathbb{B})$ is doubling. Then

(a) $h_\ell(N; \mathcal{A}_q^\alpha) \approx N^{\alpha}h_\ell(N; \mathbb{B})$.

(b) $h_r(N; \mathcal{A}_q^\alpha) \lesssim N^{\alpha}h_r(N; \mathbb{B})$.

In particular, if $\mathcal{B}$ is democratic in $\mathbb{B}$ then $\mathcal{B}$ is democratic in $\mathcal{A}_q^\alpha(\mathcal{B}; \mathbb{B})$.

Proof. As before, “$\preceq$” in (a), and “$\succeq$” in (b) follow immediately from the embeddings

$$\ell_{k^{\alpha}h_r(k)}^q(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}; \mathbb{B}) \hookrightarrow \ell_{k^{\alpha}h_r(k)}^q(\mathcal{B}; \mathbb{B}).$$

The converse inequality in (a) follows from the previous proposition and the trivial inclusion $\mathcal{G}_q^\alpha \hookrightarrow \mathcal{A}_q^\alpha$.

As shown in Example 5.6, the converse to the last statement in Proposition 6.2 is not necessarily true. The space $\mathbb{B} = \text{bmo}$ is not democratic, but their approximation classes $\mathcal{A}_q^\alpha(\text{bmo}) = \ell^{1/\alpha, q}$ are democratic. Moreover, this example shows that the converse to the inequality in (b) does not necessarily hold, since

$$h_r(N; \mathcal{A}_q^\alpha(\text{bmo})) = N^{\alpha} \quad \text{but} \quad N^{\alpha}h_r(N; \text{bmo}) \approx N^{\alpha}(\log N)^{1/2}.$$

Nevertheless, we can give a sufficient condition for $h_r(N; \mathcal{A}_q^\alpha) \approx N^{\alpha}h_r(N; \mathbb{B})$, which turns out to be easily verifiable in all the other examples presented in §5.

PROPERTY (H). We say that $\mathcal{B}$ satisfies the Property (H) if for each $n = 1, 2, 3, \ldots$ there exist $\Gamma_n \subset \mathbb{N}$, with $|\Gamma_n| = 2^n$, satisfying the property

$$\|1_{\Gamma'}\|_{\mathbb{B}} \approx h_r(2^{n-1}; \mathbb{B}), \quad \forall \Gamma' \subset \Gamma_n \quad \text{with} \quad |\Gamma'| = 2^{n-1}.$$

Proposition 6.3. Assume that $\mathcal{B}$ satisfies the Property (H). Then, for all $\alpha > 0$ and $0 < q \leq \infty$

$$h_r(N; \mathcal{A}_q^\alpha) \approx N^{\alpha}h_r(N; \mathbb{B})$$
Assume that \( \ell \). Indeed, recall from [12, Thm 1.2] (see also Example 5.3) that \( \ell \) arithmetic loss for the embedding \( N \) size satisfies \( A \) Corollary 6.4. More about optimality for inclusions into \( \varphi \) by (6.3) and the doubling property of \( h \) result just proved and the doubling property of \( \Gamma \). Now, the property (H) (and the remark in (2.4)) give \( \sigma_{N/2}(1_{\Gamma_n}) = \inf \{ \|1_{\Gamma'}\|_\varphi : \Gamma' \subset \Gamma, |\Gamma'| = N/2 \} \approx h_r(N/2; B) \approx h_r(N; B). \)

Combining these two facts the proposition follows for \( N = 2^n \). For general \( N \) use the result just proved and the doubling property of \( h_r \). □

As an immediate consequence, the property (H) allows to remove the possible logarithmic loss for the embedding \( \ell^q_k(\mathcal{B}, B) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, B) \) discussed in Corollary 3.9.

**Corollary 6.4.** More about optimality for inclusions into \( \mathcal{A}^\alpha_q \).

Assume that \( (\mathcal{B}, B) \) satisfies property (H). If for some \( \alpha > 0, q \in (0, \infty] \) and \( \omega \in \mathcal{W}_+ \) we have \( \ell^q_k(\mathcal{B}, B) \hookrightarrow \mathcal{A}^\alpha_q(\mathcal{B}, B) \), then necessarily \( \omega(k) \gtrsim k^\alpha h_r(k) \), and therefore \( \ell^q_k \hookrightarrow \ell^q_k \).

The following examples show that Property (H) is often satisfied.

**Example 6.1.** Wavelet bases in Orlicz spaces \( L^\Phi(\mathbb{R}^d) \) satisfy the property (H). Indeed, recall from [12, Thm 1.2] (see also Example 5.3) that

\[
\|1_{\Gamma}\|_{L^\Phi} \approx \sup_{s > 0} \varphi(Ns)/\varphi(s). \tag{6.2}
\]

Moreover, any collection \( \Gamma \) of \( N \) pairwise disjoint dyadic cubes with the same fixed size \( a > 0 \) satisfies

\[
\|1_{\Gamma}\|_{L^\Phi} \approx \varphi(Na)/\varphi(a), \tag{6.3}
\]

(see eg [12, Lemma 3.1]). Thus, for each \( N = 2^n \), we first select \( a_n = 2^{\alpha n^d} \) so that \( h_r(2^n; L^\Phi) \approx \varphi(2^n a_n)/\varphi(a_n) \), and then we choose as \( \Gamma_n \) any collection of \( 2^n \) pairwise disjoint cubes with constant size \( a_n \). Then, any subfamily \( \Gamma' \subset \Gamma_n \) with \( |\Gamma'| = N/2 \), satisfies

\[
\|1_{\Gamma'}\|_{L^\Phi} \approx \varphi((N/2)a_n)/\varphi(a_n) \approx \varphi(Na_n)/\varphi(a_n) \approx h_r(N) \approx h_r(N/2),
\]

by (6.3) and the doubling property of \( \varphi \) and \( h_r \).

**Example 6.2.** Wavelet bases in Lorentz spaces \( L^{p,q}(\mathbb{R}^d) \), \( 1 < p, q < \infty \). These also satisfy the property (H). Indeed, it can be shown that any set \( \Gamma \) consisting of \( N \) disjoint cubes of the same size has

\[
\|1_{\Gamma}\|_{L^{p,q}} \approx N^{\frac{d}{p}},
\]

while sets \( \Delta \) consisting of \( N \) disjoint cubes all having different sizes satisfy

\[
\|1_{\Delta}\|_{L^{p,q}} \approx N^{\frac{d}{p}}.
\]

(see [14, (3.6) and (3.8)]). Since \( h_r(N) \approx N^{1/(p\wedge q)} \), we can define the \( \Gamma_n \)’s with sets of the first type when \( p \leq q \), and with sets of the second type when \( q < p \), to obtain in both cases a collection satisfying the hypotheses of property (H).
Example 6.3. The hyperbolic Haar system in $L^p(\mathbb{R}^d)$ from Example 5.5 also satisfies property (H). In this case, again, any set $\Gamma$ consisting of $N$ disjoint rectangles has
\[ \|1_\Gamma\|_{L^p(\mathbb{R}^d)} = N^{\frac{1}{p}}. \]
On the other hand, if $\Delta_n$ denotes the set of all the dyadic rectangles in the unit cube with fixed size $2^{-n}$, then
\[ \|1_{\Delta_n}\|_{L^p(\mathbb{R}^d)} \approx 2^{n/p} n^{(d-1)/2} \approx |\Delta_n|^{1/p} (\log |\Delta_n|)^{(d-1)/2} \left(\frac{1}{2} - \frac{1}{p}\right). \]
Moreover, it is not difficult to show that any $\Delta' \subset \Delta_n$ with $|\Delta'| = |\Delta_n|/2$ also satisfies (6.4) (with $\Delta_n$ replaced by $\Delta'$). Hence, combining these cases and using the description of $h_r(N)$ in Example 5.5, one easily establishes the property (H).

7. Counterexamples for the classes $\mathcal{Q}_q^\alpha(\mathcal{B}, \mathbb{B})$

7.1. Conditions for $\mathcal{Q}_q^\alpha \neq \mathcal{A}_q^\alpha$. Recall from section 2.3 that $\mathcal{Q}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$, with equality of the spaces when $\mathcal{B}$ is a democratic basis. It is known that there are some conditional non-democratic bases for which $\mathcal{Q}_q^\alpha = \mathcal{A}_q^\alpha$ (see [13, Remark 6.2]). For unconditional bases, however, one could ask whether non-democracy necessarily implies that $\mathcal{Q}_q^\alpha \neq \mathcal{A}_q^\alpha$. We do not know how to prove such a general result, but we can show that the inclusion $\mathcal{A}_q^\alpha \hookrightarrow \mathcal{Q}_q^\alpha$ must fail whenever the gap between $h_r(N)$ and $h_r(N)$ is at least logarithmic (and even less than that). More precisely, we have the following.

Proposition 7.1. Let $\mathcal{B}$ be an unconditional basis in $\mathbb{B}$ and $\alpha > 0$. Suppose that there exist integers $p_N, q_N \geq 1$, $N = 1, 2, \ldots$ such that
\[ \lim_{N \to \infty} \frac{p_N}{q_N} = \infty \quad \text{and} \quad \frac{h_r(q_N)}{h_r(p_N)} \geq \left(\frac{p_N}{q_N}\right)^\alpha. \]
Then the inclusion $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{Q}_q^\alpha(\mathcal{B}, \mathbb{B})$ does not hold for any $\tau \in (0, \infty]$.

Proof. For each $N$, choose $\Gamma_\ell, \Gamma_r \subset \mathbb{N}$ with $|\Gamma_\ell| = p_N, |\Gamma_r| = q_N$, and such that
\[ \|1_{\Gamma_\ell}\|_{\mathcal{B}} \leq 2h_r(p_N), \quad \|1_{\Gamma_r}\|_{\mathcal{B}} \geq \frac{1}{2} h_r(q_N). \]
Set $x_N = 1_{\Gamma_\ell} + 2 \cdot 1_{\Gamma_\ell - \Gamma_\ell \cap \Gamma_r}$. Since $|\Gamma_\ell - \Gamma_\ell \cap \Gamma_r| \geq p_N - q_N$, when $k \in [1, p_N - q_N]$ we have
\[ \|x_N - G_k(x_N)\|_{\mathcal{B}} \geq \|1_{\Gamma_\ell}\|_{\mathcal{B}} \geq \frac{1}{2} h_r(q_N). \]
Therefore, using $p_N - q_N > p_N/2$ (since $p_N/q_N > 2$ for $N$ large), we obtain that
\[ \|x_N\|_{\mathcal{Q}_q^\alpha(\mathcal{B}, \mathbb{B})} \geq \frac{1}{2} \left\{ \sum_{k=1}^{p_N/2} \left( k^{\alpha} h_r(q_N) \right)^\tau \right\}^{\frac{1}{\tau}} \geq h_r(q_N) p_N^\alpha. \]
(7.3)
On the other hand, we can estimate the norm of $x_N$ as follows:
\[ \|x_N\|_{\mathcal{B}} \lesssim \|1_{\Gamma_\ell}\|_{\mathcal{B}} + \|1_{\Gamma_\ell - \Gamma_\ell \cap \Gamma_r}\|_{\mathcal{B}} \leq h_r(q_N) + 2h_r(p_N) \lesssim h_r(q_N), \]
where the last inequality is true for $N$ large due to (7.1). Thus
\[ \sigma_k(x_N) \leq \|x_N\|_{\mathcal{B}} \lesssim h_r(q_N). \]
(7.4)
Next, if $k \geq q_N$, by (7.2)
\[ \sigma_k(x_N) \leq 2\|1_{\Gamma_\ell - \Gamma_\ell \cap \Gamma_r}\|_{\mathcal{B}} \leq 2\|1_{\Gamma_\ell}\|_{\mathcal{B}} \lesssim h_r(p_N). \]
(7.5)
Combining (7.4), (7.5), and (7.6) we see that
\[ \|x_N\|_{A_r^q(\mathcal{B},\mathbb{B})} \leq h_r(q_N) + \left[ \sum_{k=1}^{N_q-1} \left( k^{a} h_r(q_N) \right)^{\gamma} + \sum_{k=1}^{p_N+q_N} \left( k^{a} h_N(p_N) \right)^{\gamma} \right] \frac{1}{k} \]
\[ \leq \frac{1}{\gamma} h_r(q_N) \left[ h_r(q_N)^{\gamma} + h_N(p_N)^{\gamma} \right] \]
\[ \leq \frac{1}{\gamma} h_r(q_N) + h_r(q_N) \alpha \leq h_r(q_N) \alpha \]
(7.7)
where in the second inequality we have used the elementary fact \( \sum_{k=a}^{b} k^{\gamma-1} \lesssim b^\gamma \) if \( b \geq a \), and the third inequality is due to (7.1). Therefore, from (7.3) and (7.7) we deduce
\[ \frac{\|x_N\|_{A_r^q}}{\|x_N\|_{A_r^p}} \geq \frac{h_r(q_N)}{h_q(p_N)} \stackrel{(p_N)}{\rightarrow} \frac{\alpha}{\gamma} \]
as \( N \to \infty \). This shows the desired result. \( \square \)

**Corollary 7.2.** Let \( \mathcal{B} \) be an unconditional basis such that \( h_r(N) \lesssim N^{\beta_0} \) and \( h_r(N) \gtrsim N^{\beta_1} \), for some \( \beta_1 > \beta_0 \geq 0 \). Then, \( \mathcal{G}_q^a \neq A_r^q \) for all \( \alpha > 0 \) and all \( \tau \in (0,\infty] \).

**Proof.** Choose \( r, s \in \mathbb{N} \), such that \( \frac{a + \beta_0}{a + \beta_1} < \frac{r}{s} < 1 \). Take \( p_N = N^s \) and \( q_N = N^r \). Then, \( \lim_{N \to \infty} \frac{p_N}{q_N} = \lim_{N \to \infty} N^{s-r} = \infty \) and
\[ \frac{h_r(q_N)}{h_q(p_N)} \geq \frac{1}{\gamma} h_r(q_N) \left[ h_r(q_N)^{\gamma} + h_q(p_N)^{\gamma} \right] \]
\[ = \frac{1}{\gamma} h_r(q_N) \lesssim \frac{N^{\beta_1}}{N^{\beta_0}} \]
which proves (7.1) in this case, so that we can apply Proposition 7.1. \( \square \)

**Corollary 7.3.** Let \( \mathcal{B} \) be an unconditional basis such that for some \( \beta \geq 0 \) and \( \gamma > 0 \) we have either
\( i \) \( h_r(N) \gtrsim N^\beta (\log N)^\gamma \) and \( h_r(N) \lesssim N^\beta \), or
\( ii \) \( h_r(N) \gtrsim N^\beta \) and \( h_r(N) \lesssim N^\beta (\log N)^\gamma \).

Then, \( \mathcal{G}_q^a \neq A_r^q \) for all \( \alpha > 0 \) and all \( \tau \in (0,\infty] \).

**Proof.** i) Choose \( a, b \in \mathbb{N} \) such that \( 0 < \frac{a}{b} < \frac{2}{a + \beta} \). Let \( p_N = N^a 2^{N^b} \) and \( q_N = 2^{N^b} \).

Then, \( \lim_{N \to \infty} \frac{p_N}{q_N} = \lim_{N \to \infty} N^a = \infty \) and
\[ \frac{h_r(q_N)}{h_q(p_N)} \geq \frac{(2^{N^b})^\beta (\log 2^{N^b})^\gamma}{\alpha \beta (2^{N^b})^\beta} \approx \frac{N^{b_\gamma}}{N^{\alpha \beta}} \approx N^{b_\gamma - a_\beta} > N^{a_\alpha} = \left( \frac{p_N}{q_N} \right)^{\alpha} \]
which proves (7.1) in this case, so that we can apply Proposition 7.1 to conclude the result. The proof of ii) is similar with the same choice of \( p_N \) and \( q_N \). \( \square \)

7.2. **Non linearity of \( \mathcal{G}_q^a(\mathcal{B},\mathbb{B}) \).** We conclude by showing with simple examples that \( \mathcal{G}_q^a(\mathcal{B},\mathbb{B}) \) may not even be a linear space when the basis \( \mathcal{B} \) is not democratic.

Let \( \mathcal{B} = \ell^p \oplus \ell^q \), \( 0 < q < p < \infty \); that is, \( \mathcal{B} \) consists of pairs \((a,b) \in \ell_p \times \ell_q\), endowed with the quasi-norm \( \|a\|_{\ell_p} + \|b\|_{\ell_q} \). We consider the canonical basis in \( \mathcal{B} \).

Now, set \( \beta = \alpha + \frac{1}{p} \) and \( x = \{(k^{-\beta},0)\}_{k \in \mathbb{N}} \in \mathcal{B} \). For \( N = 1, 2, 3, \ldots \) we have
\[ \gamma^N(x) = \left( \sum_{k>N} \frac{1}{k^{\beta}} \right)^{1/p} \approx \left( \frac{1}{N^{\beta p - 1}} \right)^{1/p} = N^{-\alpha} \]
This shows that \( x \in \mathcal{G}_\infty^\alpha(\mathcal{B}, \mathcal{B}) \). Similarly, if we let \( \gamma = \alpha + \frac{1}{q} \), then \( y = \{(0, j^{-\gamma})\}_{j \in \mathbb{N}} \) belongs to \( \mathcal{G}_\infty^\alpha \). We will show, however, that \( x + y \notin \mathcal{G}_\infty^\alpha \). In fact, we will find a subsequence \( N_J \) of natural numbers so that

\[
\gamma_{N_J}(x + y) \approx \frac{1}{N_J^{\alpha\beta/\gamma}} \tag{7.8}
\]

(notice that \( \beta < \gamma \) since we chose \( q < p \)). To prove (7.8) let \( A_1 = \{1\} \) and

\[
A_j = \left\{ k \in \mathbb{N} : \frac{1}{j^{\gamma}} \leq \frac{1}{k^{\beta}} < \frac{1}{(j - 1)^{\gamma}} \right\}, \quad j = 2, 3, \ldots
\]

The number of elements in \( A_j \) is

\[
|A_j| \approx j^{\gamma/\beta} - (j - 1)^{\gamma/\beta} \approx j^{\gamma - 1}, \quad j = 1, 2, 3, \ldots \tag{7.9}
\]

For \( J = 2, 3, 4, \ldots \) let \( N_J = \sum_{j=1}^{J} |A_j| + J \). From (7.9) we obtain

\[
N_J \approx \sum_{j=1}^{J} j^{\gamma - 1} + J \approx J^\gamma + J \approx J^{\frac{\gamma}{\beta}},
\]

since \( \gamma > \beta \). Thus, \( \gamma_{N_J}(x + y) \approx \left( \sum_{k > J^{\frac{\gamma}{\beta}}} k^{-\beta p} \right)^{1/p} + \left( \sum_{j > J} j^{-\gamma q} \right)^{1/q} \approx \left( J^{\gamma/\beta - \beta p + 1} \right)^{1/p} + J^{-\gamma q + 1} \)

\[
= J^{-\alpha\gamma/\beta} + J^{-\alpha} \approx J^{-\alpha} \approx (N_J)^{-\alpha/\gamma},
\]

proving (7.8).

A simple modification of the above construction can be used to show that the set \( \mathcal{G}_s^\alpha(\mathcal{B}, \mathcal{B}) \) is not linear, for any \( \alpha > 0 \) and any \( s \in (0, \infty) \).

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