Bi-exactness of relatively hyperbolic groups

Koichi Oyakawa

Abstract

We prove that finitely generated relatively hyperbolic groups are bi-exact if and only if all peripheral subgroups are bi-exact. This is a generalization of Ozawa’s result which claims that finitely generated relatively hyperbolic groups are bi-exact if all peripheral subgroups are amenable.

1 Introduction

Bi-exactness is an analytic property of groups defined by Ozawa in [16] (by the name of class $S$). Recall that a group $G$ is exact if there exists a compact Hausdorff space on which $G$ acts topologically amenably and bi-exact if it is exact and there exists a map $\mu: G \rightarrow \text{Prob}(G)$ such that for every $s, t \in G$, we have

$$\lim_{x \to \infty} \|\mu(sxt) - s.\mu(x)\|_1 = 0.$$ 

It is known that exactness is equivalent to Yu’s property A (cf. [13]).

The notion of bi-exactness is of fundamental importance to the study of operator algebras. Notably, Ozawa proved in [14] that the group von Neumann algebra $L(G)$ of any non-amenable bi-exact icc group $G$ is prime (i.e. it cannot be decomposed as a tensor product of two II$_1$ factors) by showing that $L(G)$ is solid. In [18], Ozawa and Popa proved a unique prime factorization theorem which states that if $G_1, \ldots, G_n$ are non-amenable bi-exact icc groups and $L(G_1)\overline{\otimes} \cdots \overline{\otimes} L(G_n)$ is decomposed as a tensor product of $m$ II$_1$ factors with $m \geq n$ i.e. $L(G_1)\overline{\otimes} \cdots \overline{\otimes} L(G_n) = N_1\overline{\otimes} \cdots \overline{\otimes} N_m$, then we have $m = n$ and after permutation of indices and unitary conjugation, each $N_i$ is isomorphic to an amplification of $L(G_i)$. Subsequently, various rigidity results were proved for bi-exact groups. In particular, Sako showed measure equivalence rigidity for direct product (cf. [20]) and Chifan and Ioana showed rigidity result in von Neumann algebra sense for amalgamated free product (cf. [3]).

It is known that the class of bi-exact groups contains amenable groups, hyperbolic groups, discrete subgroups of connected simple Lie groups of rank one, and $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ (cf. [2][16][17]). On the other hand, unlike exactness, bi-exactness is not preserved under some basic group theoretic constructions. For example, direct products and increasing
unions of bi-exact groups are not necessarily bi-exact (e.g. $F_2 \times F_2$ and $G \times F_2$, where $F_2$ is a free group of rank 2 and $G$ is an infinite countable locally finite group, are exact but not bi-exact).

The standard way of proving bi-exactness is using topologically amenable action on a special compact space called ‘boundary small at infinity’. Using this method, Ozawa proved in [15] that any finitely generated group hyperbolic relative to amenable peripheral subgroups is bi-exact. In this paper, we use a different approach and generalize Ozawa’s result by proving the following.

**Theorem 1.1.** Suppose that $G$ is a finitely generated group hyperbolic relative to a collection of subgroups $\{H_\mu\}_{\mu \in \Lambda}$ of $G$. Then, $G$ is bi-exact if and only if all subgroups $H_\mu$ are bi-exact.

The ‘Only if’ direction is obvious and the whole paper is devoted to proving the ‘if’ direction. In relation to this result, it is worth mentioning the following folklore conjecture.

**Conjecture 1.2.** Suppose that a finitely generated group $G$ is hyperbolic relative to a collection of subgroups $\{H_\mu\}_{\mu \in \Lambda}$ of $G$. If all subgroups $H_\mu$ are exact, then $G$ is bi-exact relative to $\{H_\mu\}_{\mu \in \Lambda}$.

For the definition of relative bi-exactness, the reader is referred to Definition 15.1.2 of [2]. Note that the ‘if’ direction of Theorem 1.1 is not a weak version of Conjecture 1.2. Indeed, while the assumption of Theorem 1.1 is stronger, the conclusion is also stronger.

Our proof of Theorem 1.1 is based on a characterization of bi-exactness using exactness and the existence of a proper array (cf. Proposition 2.2) and uses two technologies related to relatively hyperbolic groups. The first one is a bicombing of fine hyperbolic graphs, which was constructed by Mineyev and Yaman in [10] using the same idea as in [9]. The second one is based on the notion of separating cosets of hyperbolically embedded subgroups, which was introduced by Hull and Osin in [8] and developed further by Osin in [12]. We will construct two arrays using each of these techniques and combine them to make a proper array. It is worth noting that the Mineyev-Yaman’s bicombing of relatively hyperbolic groups alone is not sufficient to derive Conjecture 1.2 for the reason explained in Remark 3.7. Therefore, Conjecture 1.2 is still considered open.

The paper is organized as follows. In Section 2, we discuss the necessary definitions and known results about bi-exact groups and relatively hyperbolic groups. In Section 3.1, we give an outline of our proof. Section 3.2 and 3.3 discuss the constructions of arrays based on ideas of [10] and [8], respectively, and Section 3.4 provides the proof of Theorem 1.1.

**Acknowledgment.** I thank Denis Osin for introducing this topic to me, for sharing his ideas, and for many helpful discussions.
2 Preliminary

2.1 Bi-exact groups

In this section, we introduce some equivalent conditions of bi-exact groups. The definition of an array given below was suggested in [4].

**Definition 2.1.** Suppose that $G$ is a group, $K$ a Hilbert space, and $\pi: G \to U(K)$ a unitary representation. A map $r: G \to K$ is called an array on $G$ into $(K, \pi)$, if $r$ satisfies (1) and (2) below. When there exists such $r$, we say that $G$ admits an array into $(K, \pi)$.

1. $\pi_{g^{-1}}(r(g^{-1})) = -r(g)$ for all $g \in G$.
2. For every $g \in G$, we have $\sup_{h \in G} \|r(gh) - \pi_{g}(r(h))\| < \infty$.

If, in addition, $r$ satisfies (3) below, $r$ is called proper.

3. For any $N \in \mathbb{N}$, $\{g \in G \mid \|r(g)\| \leq N\}$ is finite.

Conditions (2) and (3) in Proposition 2.2 are simplified versions of Proposition 2.3 of [4] and Proposition 2.7 of [19] respectively.

**Proposition 2.2.** For any countable group $G$, the following three conditions are equivalent.

1. $G$ is bi-exact.
2. $G$ is exact and admits a proper array into the left regular representation $(\ell^{2}(G), \lambda_{G})$.
3. $G$ is exact, and there exist an orthogonal representation $\eta: G \to O(K_{R})$ to a real Hilbert space $K_{R}$ that is weakly contained in the regular representation of $G$ and a map $c: G \to K_{R}$ that is proper and satisfies

$$\sup_{k \in G} \|c(gkh) - \eta_{g}c(k)\| < \infty$$

for all $g, h \in G$.

2.2 Relatively hyperbolic groups

There are several equivalent definitions of relatively hyperbolic groups due to Gromov, Bowditch, Farb, and Osin. We adopt Farb’s definition with the Bounded Coset Penetration property replaced by fineness of coned-off Cayley graphs. [7], [5, Appendix], and [11, Appendix] contain proofs of the equivalence of these definitions. To state our definition, we first prepare some terminologies. When we consider a graph $Y$ without loops or multiple edges, the edge set $E(Y)$ is defined as a irreflexive symmetric subset of $V(Y) \times V(Y)$, where $V(Y)$ is the vertex set (see Section 2.3 for details).
Definition 2.3 (Coned-off Cayley graph). Suppose that $G$ is a finitely generated group with a finite generating set $X$ and $\{H_\lambda\}_{\lambda \in \Lambda}$ is a collection of subgroups of $G$. Let $\Gamma(G, X)$ be the Cayley graph of $G$ with respect to $X$, in which we don’t allow loops nor multiple edges. More precisely, its vertex set and edge set are defined by

$V(\Gamma(G, X)) = G, \quad E(\Gamma(G, X)) = \{(g, gs), (gs, g) \mid g \in G, s \in X \setminus \{1\}\}$. (1)

We define a new graph $\hat{\Gamma}$ whose vertex set $V(\hat{\Gamma})$ and edge set $E(\hat{\Gamma})$ are given by

$V(\hat{\Gamma}) = G \cup \bigsqcup_{\lambda \in \Lambda} G/H_\lambda,$

$E(\hat{\Gamma}) = E(\Gamma(G, X)) \cup \bigsqcup_{\lambda \in \Lambda} \{(g, xH_\lambda), (xH_\lambda, g) \mid xH_\lambda \in G/H_\lambda, g \in xH_\lambda\}.$

The graph $\hat{\Gamma}$ is called coned-off Cayley graph with respect to $X$.

We set the length of any edge of $\hat{\Gamma}$ to be 1. $G$ acts on $\hat{\Gamma}$ naturally by graph automorphism. For a vertex $v \in V(\hat{\Gamma})$, we denote its stabilizer by $G_v = \{g \in G \mid gv = v\}$.

Definition 2.4 (Hyperbolic space). A geodesic metric space $(X, d)$ is called hyperbolic if there exists $\delta \in [0, \infty)$ such that for any $x, y, z \in X$, any geodesic path $[x, y], [x, z], [z, y]$, and any point $a \in [x, y]$, there exists $b \in [x, z] \cup [z, y]$ satisfying $d(a, b) \leq \delta$.

Definition 2.5 (Circuit). A closed path $p$ in a graph is called circuit, if $p$ doesn’t have self-intersection except its initial and terminal vertices.

Definition 2.6 (Fine graph). A graph $Y$ is called fine, if for any $n \in \mathbb{N}$ and any edge $e$ of $Y$, the number of circuits of length $n$ containing $e$ is finite.

The following is the definition of relatively hyperbolic groups on which we work. The reader is referred to [5, Appendix] for the equivalence of Definition 2.7 and the other definitions.

Definition 2.7. A finitely generated group $G$ is called hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$, if $\Lambda$ is finite and for some (equivalently, any) finite generating set $X$ of $G$, the coned-off Cayley graph $\hat{\Gamma}$ is fine and hyperbolic. A member of the collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ is called a peripheral subgroup.

Some definitions (e.g. Osin’s definition in [11]) can be stated without requiring $\Lambda$ to be finite. In Osin’s definition, finiteness of $\Lambda$ follows if $G$ is finitely generated.

2.3 Mineyev-Yaman’s bicombing

In [10], Mineyev and Yaman constructed a bicombing of a fine hyperbolic graph using the same idea as in [9]. We first discuss 1-chains on a graph and a unitary representation of a group acting on a graph in general and then introduce Mineyev-Yaman’s bicombing.
In this section, we consider graphs without loops nor multiple edges, following [10] and [1]. More precisely, for a graph $Y$ with a vertex set $V(Y)$, its edge set $E(Y)$ is a subset of $V(Y) \times V(Y) \setminus \{(v, v) \mid v \in V(Y)\}$ such that $\overline{E}(Y) = E(Y)$, where we define $(u, v) = (v, u)$ for any $(u, v) \in V(Y)^2$. We call a subset $E^+(Y)$ of $E(Y)$ a set of positive edges if it satisfies $E(V) = E^+(Y) \cup \overline{E}^+(Y)$. Choosing a set of positive edges $E^+(Y)$ means choosing directions of edges. Note that when we consider a group action on a graph, we allow inversion of edges.

We begin with auxiliary notations.

For a graph $Y$, we denote by $C_1(Y)$ the set of 1-chains on $Y$ over $\mathbb{C}$. More precisely, $C_1(Y)$ is defined as follows. Consider a direct product $\mathbb{C}^{E(Y)} = \{ \sum_{e \in E(Y)} c_e e \mid c_e \in \mathbb{C} \}$ of vector spaces over $\mathbb{C}$. Note that this summation notation is just a formal sum. We define a quotient space $\ell^0(Y)$ by

$$
\ell^0(Y) = \mathbb{C}^{E(Y)} / \left\{ \sum_{e \in E(Y)} c_e e \in \mathbb{C}^{E(Y)} \mid \forall e \in E(Y) c_e = c_e \right\}.
$$

Here, we denote by $[e] = [u, v]$ the element in $\ell^0(Y)$ corresponding to $e = (u, v) \in \mathbb{C}^{E(Y)}$ where $e = (u, v) \in E(Y)$. Note that for any $e = (u, v) \in E(Y)$, we have

$$[\tilde{e}] = [v, u] = -[u, v] = -[e].$$

Fix a set of positive edges $E^+(Y)$. The map

$$\mathbb{C}^{E^+(Y)} \ni \sum_{e \in E^+(Y)} c_e e \mapsto \sum_{e \in E^+(Y)} c_e [e] \in \ell^0(Y)$$

is an isomorphism of vector spaces. The space of 1-chains is defined by

$$C_1(Y) = \left\{ \sum_{e \in E^+(Y)} c_e [e] \in \ell^0(Y) \mid \exists F \subset E^+(Y), |F| < \infty \land \forall e \not\in F c_e = 0 \right\}.$$

Note that $C_1(Y)$ is independent of the choice of $E^+(Y)$. For $a, b \in V(Y)$ and a path $q = (v_0, v_1, \ldots, v_n)$ from $a = v_0$ to $b = v_n$, we define a 1-chain

$$q = [v_0, v_1] + \cdots + [v_{n-1}, v_n] \in C_1(Y). \quad (2)$$

Here, we use the same notation $q$ to denote a path and the corresponding 1-chain by abuse of notation.

For each $p \in [1, \infty)$ (in fact, we use only cases $p = 1, 2$), define a subspace $\ell^p(Y)$ of $\ell^0(Y)$ by

$$\ell^p(Y) = \left\{ \sum_{e \in E^+(Y)} c_e [e] \in \ell^0(Y) \mid \sum_{e \in E^+(Y)} |c_e|^p < \infty \right\}.$$

and for each $\xi = \sum_{e \in E^+(Y)} c_e [e] \in \ell^p(Y)$ define a norm

$$\|\xi\|_p = \left( \sum_{e \in E^+(Y)} |c_e|^p \right)^{1/p}.$$
\((\ell^p(Y), \| \cdot \|_p)\) becomes a Banach space isomorphic to \(\ell^p(E^+(Y))\). In particular, \((\ell^2(Y), \| \cdot \|_2)\) is a Hilbert space. Note that \((\ell^p(Y), \| \cdot \|_p)\) doesn’t depend on the choice of \(E^+(Y)\). We also have \(C_1(Y) \subset \ell^p(Y)\) for any \(p \in [1, \infty)\).

**Definition 2.8.** Suppose that a group \(G\) acts on \(Y\) by graph automorphisms, then \(G\) acts on \((\ell^p(Y), \| \cdot \|_p)\) isometrically by

\[
G \times \ell^p(Y) \ni (g, \sum_{(u,v) \in E^+(Y)} c_{(u,v)}[u,v]) \mapsto \sum_{(u,v) \in E^+(Y)} c_{(u,v)}[gu, gv] \in \ell^p(Y).
\]

In particular, this action on \((\ell^2(Y), \| \cdot \|_2)\) becomes a unitary representation of \(G\) and we denote this unitary representation by \((\ell^2(Y), \pi)\).

There exists a unique well-defined linear map

\[
\partial: C_1(Y) \to C_0(Y) = \left\{ \sum_{v \in F} c_v v \mid F \subset V(Y), |F| < \infty, c_v \in \mathbb{C} \right\}
\]

such that \(\partial[u, v] = v - u\) for any \((u, v) \in E(Y)\).

**Definition 2.9.** Suppose that \(Y\) is a graph. A map \(q: \text{V}(Y)^2 \to C_1(Y)\) is called a homological bicombing, if for any \((a, b) \in \text{V}(Y)^2\), we have \(\partial q[a, b] = b - a\). In particular, if all coefficients of \(q[a, b]\) are in \(\mathbb{Q}\) for any \(a, b \in \text{V}(Y)\), \(q\) is called a \(\mathbb{Q}\)-bicombing.

**Definition 2.10** (2-vertex-connectivity). A graph \(Y\) is called 2-vertex-connected, if \(Y\) is connected and for any \(v \in \text{V}(Y), Y \setminus \{v\}\) is connected, where \(Y \setminus \{v\}\) is an induced subgraph of \(Y\) whose vertex set is \(V(Y) \setminus \{v\}\).

Theorem 2.11 is a simplified version of Theorem 47 and Proposition 46 (3) in [10].

**Theorem 2.11.** Suppose that \(Y\) is a 2-vertex-connected fine hyperbolic graph and \(G\) is a group acting on \(Y\). If the number of \(G\)-orbits of \(E(Y)\) is finite and the edge stabilizer \(G_e = G_u \cap G_v\) is finite for any \(e = (u, v) \in E(Y)\), then there exists a \(\mathbb{Q}\)-bicombing \(q\) of \(Y\) satisfying the following conditions.

1. \(q\) is \(G\)-equivariant, i.e. \(q[ga, gb] = g \cdot q[a, b]\) for any \(a, b \in \text{V}(Y)\) and any \(g \in G\).
2. \(q\) is anti-symmetric, i.e. \(q[b, a] = -q[a, b]\) for any \(a, b \in \text{V}(Y)\).
3. There exists a constant \(T \geq 0\) such that for any \(a, b, c \in \text{V}(Y)\),

\[
\|q[a, b] + q[b, c] + q[c, a]\|_1 \leq T.
\]

4. There exist constants \(M' \geq 0\) and \(N' \geq 0\) such that for any \(a, b \in \text{V}(Y)\),

\[
\|q[a, b]\|_1 \leq M'd_Y(a, b) + N',
\]

where \(d_Y\) is the graph metric of \(Y\).
Remark 2.12. By examining the explicit construction of \( q \) in Section 6 of [10], we can see that for any \( a, b \in V(Y) \), \( q[a,b] \) is a convex combination of paths from \( a \) to \( b \) (see (2)), i.e. there exist paths \( p_1, \ldots, p_n \) from \( a \) to \( b \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{Q}_{\geq 0} \) with \( \sum_{j=1}^{n} \alpha_j = 1 \) such that
\[
q[a,b] = \sum_{j=1}^{n} \alpha_j p_j.
\]

Remark 2.13. Theorem 2.11 was actually proved for what Mineyev and Yaman called a ‘hyperbolic tuple’ which was defined in their paper as a quadruple consisting of a graph, a group acting on it, a subset of the vertex set, and a set of vertex stabilizers (cf. [10, Definitions 20, 27, 29, 38]). However, the statement and the proof of Theorem 47 and Proposition 46 (3) in [10] does not require the whole quadruple, and we simplify their exposition.

Remark 2.14. Theorem 2.11 (4) is necessary only for Remark 3.7.

2.4 Separating cosets of hyperbolically embedded subgroups

In this section, we define Hull-Osin’s separating cosets and explain their properties required for our discussion in Section 3.3. The notion of separating cosets of hyperbolically embedded subgroups was first introduced by Hull and Osin in [8] and further developed by Osin in [12]. There is one subtle difference in the definition of separation cosets in [8] and in [12], which is explained in Remark 2.21, though other terminologies and related propositions are mostly the same between them. With regards to this difference, we follow definitions and notations of [12] in our discussion. In this section, suppose that \( G \) is a group, \( X \) is a subset of \( G \), and \( \{H_\lambda\}_{\lambda \in \Lambda} \) is a collection of subgroups of \( G \) such that \( X \cup (\bigcup_{\lambda \in \Lambda} H_\lambda) \) generates \( G \). Note that \( \Lambda \) and \( X \) are possibly infinite. We begin with defining some auxiliary concepts.

Let \( \mathcal{H} = \bigcup_{\lambda \in \Lambda} (H_\lambda \setminus \{1\}) \). Note that this union is disjoint as sets of labels, not as subsets of \( G \). Let \( \Gamma(G,X \cup \mathcal{H}) \) be the Cayley graph of \( G \) with respect to \( X \cup \mathcal{H} \), which allows loops and multiple edges, that is, its vertex set is \( G \) and its positive edge set is \( G \setminus (X \cup \mathcal{H}) \). We call \( \Gamma(G,X \cup \mathcal{H}) \) the relative Cayley graph.

For each \( \lambda \in \Lambda \), we consider the Cayley graph \( \Gamma(H_\lambda, H_\lambda \setminus \{1\}) \), which is a subgraph of \( \Gamma(G,X \cup \mathcal{H}) \), and define a metric \( \tilde{d}_\lambda : H_\lambda \times H_\lambda \to [0, \infty] \) as follows. We say that a path \( p \) in \( \Gamma(G,X \cup \mathcal{H}) \) is \( \lambda \)-admissible, if \( p \) doesn’t contain any edge of \( \Gamma(H_\lambda, H_\lambda \setminus \{1\}) \). Note that \( p \) can contain an edge whose label is an element of \( H_\lambda \) (e.g. the case when the initial vertex of the edge is not in \( H_\lambda \)) and also \( p \) can pass vertices of \( \Gamma(H_\lambda, H_\lambda \setminus \{1\}) \). For \( f,g \in H_\lambda \), we define \( \tilde{d}_\lambda \) to be the minimum of lengths of all \( \lambda \)-admissible paths from \( f \) to \( g \). If there is no \( \lambda \)-admissible path from \( f \) to \( g \), then we define \( \tilde{d}_\lambda(f,g) = \infty \). For convenience, we extend \( \tilde{d}_\lambda \) to \( \tilde{d}_\lambda : G \times G \to [0, \infty] \) by defining \( \tilde{d}_\lambda(f,g) = \tilde{d}_\lambda(1,f^{-1}g) \) if \( f^{-1}g \in H_\lambda \) and \( \tilde{d}_\lambda(f,g) = \infty \) otherwise.

Definition 2.15. Suppose that \( G \) is a group and \( X \) is a subset of \( G \). The collection of
subgroups \( \{H_{\lambda}\}_{\lambda \in \Lambda} \) of \( G \) is said to be hyperbolically embedded in \((G, X)\), if it satisfies the two conditions below.

1. \( X \cup (\bigcup_{\lambda \in \Lambda} H_{\lambda}) \) generates \( G \) and \( \Gamma(G, X \cup \mathcal{H}) \) is hyperbolic.

2. For any \( \lambda \in \Lambda \), \( (H_{\lambda}, \widehat{d}_{\lambda}) \) is locally finite, i.e. for any \( n \in \mathbb{N} \), \( \{g \in H_{\lambda} \mid \widehat{d}_{\lambda}(1, g) \leq n\} \) is finite.

The following is a simplified version of Proposition 4.28 of [6].

**Proposition 2.16.** Suppose that \( G \) is a finitely generated group hyperbolic relative to a collection of subgroups \( \{H_{\mu}\}_{\mu \in \Lambda} \) of \( G \). Then, for any finite generating set \( X \) of \( G \), \( \{H_{\lambda}\}_{\lambda \in \Lambda} \) is hyperbolically embedded in \((G, X)\).

In the remainder of this section, suppose that \( \{H_{\lambda}\}_{\lambda \in \Lambda} \) is hyperbolically embedded in \((G, X)\).

**Definition 2.17.** [12, Definition 4.1] Suppose that \( p \) is a path in the relative Cayley graph \( \Gamma(G, X \cup \mathcal{H}) \). A subpath \( q \) of \( p \) is called an \( H_{\lambda} \)-subpath if the labels of all edges of \( q \) are in \( H_{\lambda} \). In the case \( p \) is a closed path, \( q \) can be a subpath of any cyclic shift of \( p \). An \( H_{\lambda} \)-subpath \( q \) of a path \( p \) is called \( H_{\lambda} \)-component if \( q \) is not contained in any longer \( H_{\lambda} \)-subpath of \( p \). In the case \( p \) is a closed path, we require that \( q \) is not contained in any longer \( H_{\lambda} \)-subpath of any cyclic shift of \( p \). Further, by a component, we mean an \( H_{\lambda} \)-component for some \( H_{\lambda} \). Two \( H_{\lambda} \)-components \( q_1, q_2 \) of a path \( p \) is called connected, if all vertices of \( q_1 \) and \( q_2 \) are in the same \( H_{\lambda} \)-coset. An \( H_{\lambda} \)-component \( q \) of a path \( p \) is called isolated, if \( q \) is not connected to any other \( H_{\lambda} \)-component of \( p \).

**Remark 2.18.** Note that all vertices of an \( H_{\lambda} \)-component lie in the same \( H_{\lambda} \)-coset.

The following proposition is important, which is a particular case of Proposition 4.13 of [6].

**Proposition 2.19.** [8, Lemma 2.4] There exists a constant \( C > 0 \) such that for any geodesic \( n \)-gon \( p \) in \( \Gamma(G, X \cup \mathcal{H}) \) and any isolated \( H_{\lambda} \)-component \( a \) of \( p \), we have \( \widehat{d}_{\lambda}(a_{-}, a_{+}) \leq nC \).

In what follows, we fix any constant \( D > 0 \) with \( D \geq 3C \).

We can now define separating cosets.

**Definition 2.20.** [12, Definition 4.3] A path \( p \) in \( \Gamma(G, X \cup \mathcal{H}) \) is said to penetrate a coset \( xH_{\lambda} \) for some \( \lambda \in \Lambda \), if \( p \) decomposes as \( p_1ap_2 \), where \( p_1, p_2 \) are possibly trivial, \( a \) is an
$H_\lambda$-component, and $a_- \in xH_\lambda$. Note that if $p$ is a geodesic, $p$ penetrates any coset of $H_\lambda$ at most once. In this case, $a$ is called the component of $p$ corresponding to $xH_\lambda$ and also the vertices $a_-$ and $a_+$ are called the entrance and exit points of $p$ and are denoted by $p_{\text{in}}(xH_\lambda)$ and $p_{\text{out}}(xH_\lambda)$ respectively. If in addition, we have $d_\lambda(a_-, a_+) > D$, then we say that $p$ essentially penetrates $xH_\lambda$. For $f, g \in G$ and $\lambda \in \Lambda$, if there exists a geodesic path from $f$ to $g$ in $\Gamma(G, X \cup \mathcal{H})$ which essentially penetrates an $H_\lambda$-coset $xH_\lambda$, then $xH_\lambda$ is called an $(f, g; D)$-separating coset. We denote the set of $(f, g; D)$-separating $H_\lambda$-cosets by $S_\lambda(f, g; D)$.

Remark 2.21. In Definition 3.1 of [8], whenever $f, g \in G$ are in the same $H_\lambda$-coset $xH_\lambda$ for some $\lambda \in \Lambda$, $xH_\lambda$ is included in $S_\lambda(f, g; D)$, but in our Definition 2.20, $S_\lambda(f, g; D)$ can be empty even in this case.

The following lemma is immediate from the above definition.

Lemma 2.22. [8, Lemma 3.2] For any $f, g, h \in G$ and any $\lambda \in \Lambda$, the following holds.

(a) $S_\lambda(f, g; D) = S_\lambda(g, f; D)$.

(b) $S_\lambda(hf, hg; D) = \{hxH_\lambda \mid xH_\lambda \in S_\lambda(f, g; D)\}$.

We state some nice properties of separating cosets, all of which were proven in [8]. For $f, g \in G$, we denote by $\mathcal{G}(f, g)$ the set of all geodesic paths in $\Gamma(G, X \cup \mathcal{H})$ from $f$ to $g$.

Lemma 2.23. [8, Lemma 3.3] For any $\lambda \in \Lambda$, any $f, g \in G$, and any $(f, g; D)$-separating coset $xH_\lambda$, the following holds.

(a) Every path in $\Gamma(G, X \cup \mathcal{H})$ connecting $f$ to $g$ and composed of at most 2 geodesics penetrates $xH_\lambda$.

(b) For any $p, q \in \mathcal{G}(f, g)$, we have

$$\tilde{d}_\lambda(p_{\text{in}}(xH_\lambda), q_{\text{in}}(xH_\lambda)) \leq 3C \quad \text{and} \quad \tilde{d}_\lambda(p_{\text{out}}(xH_\lambda), q_{\text{out}}(xH_\lambda)) \leq 3C.$$  

Proof of Lemma 2.23 is the same as Lemma 3.3 of [8], though their statements are slightly different. Actually, in Lemma 2.23, we don’t need to assume $f^{-1}g \notin H_\lambda$. This difference comes from the difference of definitions of separating cosets, which was mentioned in Remark 2.21.

Corollary 2.24. [8, Corollary 3.4] For any $f, g \in G$ and any $p \in \mathcal{G}(f, g)$, we have $S_\lambda(f, g; D) \subset P_\lambda(p)$, where we define $P_\lambda(p)$ to be the set of all $H_\lambda$-cosets which $p$ penetrates. In particular, we have $|S_\lambda(f, g; D)| \leq d_{X \cup \mathcal{H}}(f, g)$, hence $S_\lambda(f, g; D)$ is finite.

The following lemma makes $S_\lambda(f, g; D)$ into a totally ordered set.
Lemma 2.25. [8, Lemma 3.5] Suppose \( f, g \in G \) and that \( p \in \mathcal{G}(f, g) \) penetrates an \( H_\lambda \)-coset \( xH_\lambda \) and decomposes as \( p = p_1a_p2 \), where \( p_1, p_2 \) are possibly trivial and \( a \) is an \( H_\lambda \)-component corresponding to \( xH_\lambda \). Then, we have \( d_{X\cup H}(f, a_p) = d_{X\cup H}(f, xH_\lambda) \).

Definition 2.26. [8, Definition 3.6] Given any \( f, g \in G \), a relation \( \tilde{\cdot} \) on the set \( S_\lambda(f, g; D) \) is defined as follows.

\[
xH_\lambda \tilde{\cdot} yH_\lambda \iff d_{X\cup H}(f, xH_\lambda) \leq d_{X\cup H}(f, yH_\lambda).
\]

Next, we define a set of pairs of entrance and exit points of a separating coset. That is, for \( f, g \in G, \lambda \in \Lambda, \) and \( xH_\lambda \in S_\lambda(f, g; D) \), we define

\[
E(f, g; xH_\lambda, D) = \{ (p_{in}(xH_\lambda), p_{out}(xH_\lambda)) \mid p \in \mathcal{G}(f, g) \}.
\]

Note that because \( xH_\lambda \) is a \((f, g; D)\)-separating coset, any geodesic from \( f \) to \( g \) penetrates \( xH_\lambda \) by Lemma 2.23 (a).

Lemma 2.27. [8, Lemma 3.8] For any \( f, g, h \in G, \lambda \in \Lambda, \) and \( xH_\lambda \in S_\lambda(f, g; D) \), the following holds.

(a) \( E(f, g; xH_\lambda, D) = \{ (v, u) \mid (u, v) \in E(f, g; xH_\lambda, D) \} \).

(b) \( E(hf, hg; hxH_\lambda, D) = \{ (hu, hv) \mid (u, v) \in E(f, g; xH_\lambda, D) \} \).

(c) \( |E(f, g; xH_\lambda, D)| < \infty \).

Lemma 2.28. For any \( f, g, h \in G \) and any \( xH_\lambda \in S_\lambda(f, g; D) \), \( xH_\lambda \) is either penetrated by all geodesics from \( f \) to \( h \) or penetrated by all geodesics from \( h \) to \( g \).

Proof. Suppose there exists a geodesic \( q \in \mathcal{G}(f, h) \) which doesn’t penetrate \( xH_\lambda \). For any geodesic \( r \in \mathcal{G}(h, g) \), we can apply Lemma 2.23 (a) to a path \( qr \) and conclude \( r \) penetrates \( xH_\lambda \).

The following lemma is crucial in constructing an array that satisfies the bounded area condition.

Lemma 2.29. [8, Lemma 3.9] For any \( f, g, h \in G \) and any \( \lambda \in \Lambda, \) \( S_\lambda(f, g; D) \) can be decomposed as \( S_\lambda(f, g; D) = S' \sqcup S'' \sqcup F \) where

(a) \( S' \subset S_\lambda(f, h; D) \setminus S_\lambda(h, g; D) \) and we have \( E(f, g; xH_\lambda, D) = E(f, h; xH_\lambda, D) \) for any \( xH_\lambda \in S' \),

(b) \( S'' \subset S_\lambda(h, g; D) \setminus S_\lambda(f, h; D) \) and we have \( E(f, g; xH_\lambda, D) = E(h, g; xH_\lambda, D) \) for any \( xH_\lambda \in S'' \),

(c) \( |F| \leq 2 \).
3 Main theorem

3.1 Overview of proof

In this section, we explain the idea of the construction of a proper array in the proof of Proposition 3.20 (see (9) (10)) by considering two particular cases.

First, suppose that $G$ is a hyperbolic group i.e., the collection of peripheral subgroups is empty. We take a symmetric finite generating set $X_0$ of $G$ with $1 \in X_0$ and define

$$X = X_0^2 = \{ gh \mid g, h \in X_0 \}.$$ 

The Cayley graph $\Gamma(G, X)$, as defined by (1), has no loops or multiple edges and is a 2-vertex-connected locally finite hyperbolic graph. Let $Y$ be the barycentric subdivision of $\Gamma(G, X)$. Note that $Y$ is fine, $G$ acts on $Y$ without inversion of edges, and all edge stabilizers are trivial. By Theorem 2.11, there exists a $G$-equivariant anti-symmetric $\mathbb{Q}$-bicombing $q$ of $Y$ and a constant $T \geq 0$ such that for any $a, b, c \in V(Y)$, we have

$$\|q[a,b] + q[b,c] + q[c,a]\|_1 \leq T.$$ 

We define a map $Q: G \to \ell^2(Y)$ by (see Definition 3.4 for details of this definition)

$$Q(g) = \bar{q}[1,g].$$ 

It is straightforward to check that $Q$ is an array into $(\ell^2(Y), \pi)$ (cf. Definition 2.8) by using Lemma 3.5 (2). Also, we have

$$d_Y(1, g) \leq \|q[1,g]\|_1 = \|Q(g)\|_2^2$$ 

(see Remark 2.12, Lemma 3.3, and Lemma 3.5 (1)). Since $Y$ is a barycentric subdivision of $\Gamma(G, X)$, we have $d_Y(1, g) = 2d_X(1, g)$ for any $g \in G$, where $d_X$ is a word metric on $G$ with respect to $X$. Hence, for any $N \in \mathbb{N}$, if $g \in G$ satisfies $\|Q(g)\|_2 \leq N$, we have

$$d_X(1, g) = \frac{1}{2}d_Y(1, g) \leq \frac{1}{2}\|Q(g)\|_2^2 \leq \frac{1}{2}N^2.$$ 

This implies that $Q$ is proper. Combined with Proposition 2.2 (3), this gives another proof of the fact that hyperbolic groups are bi-exact. Here, note that $(\ell^2(Y), \pi)$ is a direct sum of copies $(\ell^2(G), \lambda_G)$, because $G$ acts on $Y$ without inversion of edges and its action on $E(Y)$ is free. Hence, it is weakly contained by $(\ell^2(G), \lambda_G)$. We can also use Mineyev’s bicombing in [9] instead of Theorem 2.11 in this case.

Second, suppose that $G$ is a free product of bi-exact groups $H_1$ and $H_2$. In this case, $\{H_1, H_2\}$ are peripheral subgroups and we can construct a proper array on $G$ using the normal forms of elements of the free product and proper arrays on $H_1$ and $H_2$ as follows. Let $r_i: H_i \to \ell^2(H_i)$ be a proper array into the left regular representation $(\ell^2(H_i), \lambda_{H_i})$ for each $i = 1, 2$. We can assume $\|r_i(a)\|_2 \geq 1$ for any $a \in H_i \setminus \{1\}$ by changing values.
of \( r_i \) on a finite subset of \( H_i \), if necessary. Also, we regard each \( r_i \) as a map from \( H_i \) to \( \ell^2(G) \) by composing it with the embedding \( \ell^2(H_i) \hookrightarrow \ell^2(G) \). For \( g \in G \setminus \{1\} \), without loss of generality, let \( g = h_1k_1 \cdots h_nk_n \) be the normal form of \( g \), where \( h_1, \ldots, h_n \in H_1 \setminus \{1\} \) and \( k_1, \ldots, k_n \in H_2 \setminus \{1\} \). We define a map \( R_1: G \rightarrow \ell^2(G) \) by

\[
R_1(g) = \lambda_G(1)r_1(h_1) + \lambda_G(h_1k_1)r_1(h_2) + \cdots + \lambda_G(h_1k_2 \cdots h_{n-1}k_{n-1})r_1(h_n).
\]

Here, we define \( R_1(1) = 0 \). In the same way, we define a map \( R_2: G \rightarrow \ell^2(G) \) from \( r_2 \). It is not difficult to show that the map \( R: G \rightarrow \ell^2(G) \oplus \ell^2(G) \) defined by

\[
R(g) = (R_1(g), R_2(g))
\]

is a proper array into \( (\ell^2(G) \oplus \ell^2(G), \lambda_G \oplus \lambda_G) \).

The case of general relatively hyperbolic groups is a combination of the above two cases. To every finitely generated relatively hyperbolic group, we associate two arrays. The first one is constructed in Section 3.2 starting with a fine hyperbolic graph as in the case of a hyperbolic group above. The second array is a generalized version of the array in the free product case. One technical remark is that the condition \( \{r_i(a)\}_2 \geq 1 \forall a \in H_i \setminus \{1\} \) in the free product case is used to prevent the set \( \{g \in G \mid \|R(g)\| = 0\} \) from becoming infinite, but in the proof of Proposition 3.20, we don’t need this condition thanks to the first array.

### 3.2 First array

This section is basically a continuation of Section 2.3. Notation and terminology related to graphs follow those in Section 2.3. The goal of this section is to prove Proposition 3.1 by using Mineyev-Yaman’s bicombing.

**Proposition 3.1.** Suppose that \( G \) is a finitely generated group hyperbolic relative to a collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \) of \( G \). Then, there exist a finite generating set \( X \) of \( G \), a 2-vertex-connected fine hyperbolic graph \( Y \) on which \( G \) acts without inversion of edges, and an array \( Q: G \rightarrow \ell^2(Y) \) into \( (\ell^2(Y), \pi) \) that satisfy the following conditions.

1. The edge stabilizer is trivial for any edge in \( E(Y) \).
2. For any \( g \in G \), we have

\[
d_{\hat{\Gamma}}(1, g) \leq \frac{1}{2} \|Q(g)\|^2,
\]

where \( \hat{\Gamma} \) is the coned-off Cayley graph of \( G \) with respect to \( X \).

**Remark 3.2.** The unitary representation \( (\ell^2(Y), \pi) \) in Proposition 3.1 is weakly contained by the left regular representation \( (\ell^2(G), \lambda_G) \). Indeed, since \( G \) acts on \( Y \) without inversion of edges and all edge stabilizers are trivial, \( (\ell^2(Y), \pi) \) is a direct sum of copies of \( (\ell^2(G), \lambda_G) \).
We first prove a few general lemmas about graphs. Lemma 3.3 can be proven for graphs with loops and multiple edges as well exactly in the same way, but we stick to our current setting.

**Lemma 3.3.** Suppose that $Y$ is a connected graph without loops or multiple edges. If $a, b \in V(Y)$ are two vertices, $(p_j)_{j=1}^N \subset C_1(Y)$ are paths from $a$ to $b$ as 1-chains, and $(\alpha_j)_{j=1}^N \subset \mathbb{C}$ are complex numbers, then we have

$$\left| \sum_{j=1}^N \alpha_j \right| \cdot d_Y(a, b) \leq \left| \sum_{j=1}^N \alpha_j p_j \right|_1.$$

*Proof.* We have $|d_Y(o(e), a) - d_Y(t(e), a)| \leq d_Y(o(e), t(e)) \leq 1$ for any $e = (u, v) \in E(Y)$, where we define $o(e) = u$ and $t(e) = v$. Hence, by defining

$$C = \{ e \in E(Y) \mid d_Y(o(e), a) = d_Y(t(e), a) \}$$

and

$$D_n = \{ e \in E(Y) \mid d_Y(o(e), a) = n - 1 \land d_Y(t(e), a) = n \}$$

$$\cup \{ e \in E(Y) \mid d_Y(t(e), a) = n - 1 \land d_Y(o(e), a) = n \}$$

for each $n \in \mathbb{N}$, we get the decomposition of edges $E(Y) = C \cup (\bigsqcup_{n \in \mathbb{N}} D_n)$. Also, we can choose directions of edges so that they point outward from $a$, that is, there exists a set of positive edges $E^+(Y)$ such that

$$D_n \cap E^+(Y) = \{ e \in E(Y) \mid d_Y(o(e), a) = n - 1 \land d_Y(t(e), a) = n \}$$

and

$$D_n \cap E^-(Y) = \{ e \in E(Y) \mid d_Y(t(e), a) = n - 1 \land d_Y(o(e), a) = n \}$$

for all $n \in \mathbb{N}$, where $E^-(Y) = \overline{E^+(Y)}$. For simplicity, we use the notation $C^+ = C \cap E^+(Y)$, $D^+_n = D_n \cap E^+(Y)$, and $D^-_n = D_n \cap E^-(Y)$. Given a 1-chain $\xi = \sum_{e \in E^+(Y)} c_e[e] \in C_1(Y)$, we define linear maps $Pr_{C^+}, Pr_{D^+_n} : C_1(Y) \to C_1(Y) (n \in \mathbb{N})$ by

$$Pr_{C^+}(\xi) = \sum_{e \in C^+} c_e[e] \quad \text{and} \quad Pr_{D^+_n}(\xi) = \sum_{e \in D^+_n} c_e[e].$$

By $E^+(Y) = C^+ \cup (\bigsqcup_{n \in \mathbb{N}} D^+_n)$, we have for any $\xi \in C_1(Y)$,

$$\xi = Pr_{C^+}(\xi) + \sum_{n \in \mathbb{N}} Pr_{D^+_n}(\xi) \quad \text{and} \quad \|\xi\|_1 = \|Pr_{C^+}(\xi)\|_1 + \sum_{n \in \mathbb{N}} \|Pr_{D^+_n}(\xi)\|_1.$$

In particular, we have

$$\left| \sum_{j=1}^N \alpha_j p_j \right|_1 = \left| Pr_{C^+}(\sum_{j=1}^N \alpha_j p_j) \right|_1 + \sum_{n \in \mathbb{N}} \left| Pr_{D^+_n}(\sum_{j=1}^N \alpha_j p_j) \right|_1.$$
We will show that, for any \( n \in \mathbb{N} \) with \( 1 \leq n \leq d_Y(a,b) \), we have

\[
| \sum_{j=1}^N \alpha_j | \leq \left\| \Pr_{D_n^+} \left( \sum_{j=1}^N \alpha_j p_j \right) \right\|_1.
\]

By identifying \( \ell^0(Y) \) with \( \mathbb{C}^E(Y) \) as explained in Section 2.3, we define a linear map \( \psi : C_1(Y) \to \mathbb{C} \) by \( \psi(\sum_{e \in E^+(Y)} c_e [e]) = \sum_{e \in E^+(Y)} c_e \). Note that the sums are finite and \( \psi \) depends on the chosen set \( E^+(Y) \) of positive edges. For any \( \xi = \sum_{e \in E^+(Y)} c_e [e] \in C_1(Y) \), we have

\[
|\psi(\xi)| = \left| \sum_{e \in E^+(Y)} c_e \right| \leq \sum_{e \in E^+(Y)} |c_e| = \|\xi\|_1.
\]

Here, for any path \( p_j \) from \( a \) to \( b \) and any \( n \in \mathbb{N} \) with \( 1 \leq n \leq d_Y(a,b) \), we have

\[
\psi(\Pr_{D_n^+}(p_j)) = 1.
\]

Indeed, let \( p_j = (e_1, \ldots, e_{k}) \) as a sequence of edges, where \( o(e_1) = a \) and \( t(e_k) = b \), and \( p_j \cap D_n = (e_{i_1}, \ldots, e_{i_{m}}) \) as its subsequence by abuse of notation, i.e. \( p_j = \sum_{j=1}^m [e_{i_j}] \) and \( \Pr_{D_n^+}(p_j) = \sum_{j=1}^m [e_{i_j}] \) as a 1-chain. We can see that \( m \) is odd, i.e. \( m = 2l - 1 \) with \( l \in \mathbb{N} \) and also \( e_{i_1} \in D^+_n, e_{i_2} \in D^-_n, e_{i_3} \in D^+_n, e_{i_4} \in D^-_n, \ldots, e_{i_{2l-1}} \in D^+_n \). This implies

\[
\psi(\Pr_{D_n^+} p_j) = \psi([e_{i_1}]) + \psi([e_{i_2}]) + \psi([e_{i_3}]) + \psi([e_{i_4}]) + \cdots + \psi([e_{i_{2l-1}}]) = 1 + (-1) + 1 + (-1) + \cdots + 1 = 1.
\]

Hence,

\[
\left\| \Pr_{D_n^+} \left( \sum_{j=1}^N \alpha_j p_j \right) \right\|_1 \geq \left| \psi(\Pr_{D_n^+} \left( \sum_{j=1}^N \alpha_j p_j \right)) \right| = \left| \psi \left( \sum_{j=1}^N \alpha_j \Pr_{D_n^+}(p_j) \right) \right| = \left| \sum_{j=1}^N \alpha_j \psi(\Pr_{D_n^+}(p_j)) \right| = \left| \sum_{j=1}^N \alpha_j \right|.
\]

Thus, we finally get

\[
\left\| \sum_{j=1}^N \alpha_j p_j \right\|_1 \geq \left\| \Pr_{C^+} \left( \sum_{j=1}^N \alpha_j p_j \right) \right\|_1 + \sum_{n \in \mathbb{N}} \left\| \Pr_{D_n^+} \left( \sum_{j=1}^N \alpha_j p_j \right) \right\|_1 \\
\geq 0 + \sum_{n=1}^{d_Y(a,b)} \left| \sum_{j=1}^N \alpha_j \right| = \sum_{j=1}^N \alpha_j \cdot d_Y(a,b).
\]

\( \square \)
**Definition 3.4.** Suppose that $Y$ is a graph without loops or multiple edges and $E^+(Y)$ is a set of positive edges. For $\xi = \sum_{e \in E^+(Y)} c_e [e] \in \ell^1(Y)$ such that $c_e \in \mathbb{R}$ for any $e \in E^+(Y)$, we define an element $\tilde{\xi} = \sum_{e \in E^+(Y)} b_e [e] \in \ell^2(Y)$ by

$$b_e = \begin{cases} \sqrt{c_e} & \text{if } c_e \geq 0 \\ -\sqrt{|c_e|} & \text{if } c_e < 0. \end{cases}$$

Note that the above definition is independent of the choice of a set of positive edges.

**Lemma 3.5.** For any $\xi_1, \xi_2 \in \ell^1(Y)$ whose coefficients are real, the following hold.

1. $\|\tilde{\xi}_1\|_2^2 = \|\xi_1\|_1$.
2. $\|\tilde{\xi}_1 - \tilde{\xi}_2\|_2^2 \leq 2\|\xi_1 - \xi_2\|_1$.

**Proof.** (1) follows trivially from Definition 3.4. We will prove (2). For any $c_1, c_2 \in \mathbb{R}_{\geq 0}$, we have

$$|\sqrt{c_1} - \sqrt{c_2}|^2 \leq |\sqrt{c_1} - \sqrt{c_2}|(\sqrt{c_1} + \sqrt{c_2}) = |c_1 - c_2|,$$

$$|\sqrt{c_1} + \sqrt{c_2}|^2 = 2(c_1 + c_2) - (\sqrt{c_1} - \sqrt{c_2})^2 \leq 2|c_1 + c_2|.$$

Therefore, given

$$\xi_1 = \sum_{e \in E^+(Y)} c_1[e], \quad \xi_2 = \sum_{e \in E^+(Y)} c_2[e], \quad \tilde{\xi}_1 = \sum_{e \in E^+(Y)} b_1[e], \quad \tilde{\xi}_2 = \sum_{e \in E^+(Y)} b_2[e],$$

we have $|b_1[e] - b_2[e]|^2 \leq 2|c_1[e] - c_2[e]|$ for any $e \in E^+(Y)$.

**Lemma 3.6.** If $G$ is a group and $X_0$ is a generating set of $G$ such that $X_0 = X_0^{-1}$ and $1 \in X_0$, then the Cayley graph $\Gamma(G, X_0^2)$ is 2-vertex-connected, where

$$X_0^2 = \{ gh \in G \mid g, h \in X_0 \}.$$ 

**Proof.** Note that $X_0^2$ generates $G$ since $X_0 \subset X_0^2$. Since $G$ acts transitively on $V(\Gamma(G, X_0^2))$ by graph automorphisms, it’s enough to prove that $\Gamma(G, X_0^2) \setminus \{1\}$ is connected. Any two vertices $g, h \in X_0 \setminus \{1\}$ are connected in $\Gamma(G, X_0^2) \setminus \{1\}$ by an edge having label $g^{-1}h \in X_0^2$. Also, for any vertex $gh \in X_0^2 \setminus \{1\}$, if $g \neq 1$, then $gh$ is connected to a vertex $g$ in $\Gamma(G, X_0^2) \setminus \{1\}$ by an edge of label $h^{-1} \in X_0$. Hence, any two vertices adjacent to 1 are connected in $\Gamma(G, X_0^2) \setminus \{1\}$ by a path of length at most 3. This implies that $\Gamma(G, X_0^2) \setminus \{1\}$ is connected.

In the following proof, recall that our coned-off Cayley graph is a graph without loops or multiple edges (see Definition 2.3).
Proof of Proposition 3.1. We take a symmetric finite generating set $X_0$ of $G$ with $1 \in X_0$, and define $X = X_0^2$. The Cayley graph $\Gamma(G, X)$ is 2-vertex-connected by Lemma 3.6. Without loss of generality, we can assume that the subgroups $H_\lambda$ are non-trivial. The coned-off Cayley graph $\hat{\Gamma}$ with respect to $X$ is 2-vertex-connected, because $\Gamma(G, X)$ is 2-vertex-connected and $\{H_\lambda\}_{\lambda \in \Lambda}$ doesn’t contain the trivial subgroup. We denote by $Y$ the barycentric subdivision of $\hat{\Gamma}$. $G$ acts on $Y$ without inversion of edges. Since $\hat{\Gamma}$ is a 2-vertex-connected fine hyperbolic graph, so is $Y$. Here, we used the fact that $\hat{\Gamma}$ doesn’t have loops to ensure $Y$ is 2-vertex-connected. Also, since the number of $G$-orbits of $E_{\hat{\Gamma}}$ is finite and the edge stabilizer is trivial for any edge in $E(\hat{\Gamma})$, the action of $G$ on $Y$ satisfies these conditions as well. Therefore, by Theorem 2.11, there exist a $G$-equivariant anti-symmetric $Q$-bicombing $q$ of $Y$ and a constant $T \geq 0$ such that for any $a, b, c \in V(Y)$, we have

$$\|q[a, b] + q[b, c] + q[c, a]\|_1 \leq T.$$  

We define a map $Q: G \to \ell^2(Y)$ by

$$Q(g) = q[1, g].$$

For every $g \in G$, we have

$$\pi_g Q(g^{-1}) = \overline{q[g, 1]} = -\overline{q[1, g]} = -Q(g),$$

because $q$ is $G$-equivariant and anti-symmetric. Given any $g, h \in G$, Lemma 3.5 (2) implies

$$\|Q(gh) - \pi_g Q(h)\|^2 = \|q[1, gh] - q[1, g] - q[gh, g]\|^2 \leq 2\|q[1, gh] - q[gh, g]\|_1 \leq 2(\|q[1, gh]\| + \|q[gh, g]\| + \|q[1, g]\|_1 + \|q[1, g]\|_1) \leq 2(T + \|q[1, g]\|_1).$$

Hence, $Q$ is an array into $(\ell^2(Y), \pi)$. By Remark 2.12, for any $g \in G$, the 1-chain $q[1, g]$ is a convex combination of paths from 1 to $g$ i.e. there exist paths $p_1, \cdots, p_N$ from 1 to $g$ and $\alpha_1, \cdots, \alpha_N \in \mathbb{Q}_{\geq 0}$ with $\sum_{j=1}^N \alpha_j = 1$ such that

$$q[1, g] = \sum_{j=1}^N \alpha_j p_j.$$  

Hence, by Lemma 3.3 and Lemma 3.5 (1), we have

$$d_Y(1, g) = \left\| \sum_{j=1}^N \alpha_j \right\| \cdot d_Y(1, g) \leq \left\| \sum_{j=1}^N \alpha_j p_j \right\|_1 = \|q[1, g]\|_1 = \|Q(g)\|^2.$$  

Since $Y$ is a barycentric subdivision of $\hat{\Gamma}$, we have for any $g \in G$,

$$2d_{\hat{\Gamma}}(1, g) = d_Y(1, g),$$

hence

$$d_{\hat{\Gamma}}(1, g) = \frac{1}{2} d_Y(1, g) \leq \frac{1}{2} \|Q(g)\|^2.$$  

$\square$
Remark 3.7. Note that $Q$ is not even proper relative to $\{H_\lambda\}_{\lambda \in \Lambda}$. For example, if $G$ is a free product of infinite groups $H_1$ and $H_2$, then by Theorem 2.11 (4), we can show that there exists some $N'' \in \mathbb{N}$ such that $\{h_1h_2 \mid h_1 \in H_1, h_2 \in H_2\} \subset \{g \in G \mid \|Q(g)\|_2 \leq N''\}$.

3.3 Second array

This section is a continuation of Section 2.4. The goal of this section is to prove Proposition 3.8. We will construct an array on $G$ from an array on a subgroup $H_\mu$ which is a member of a hyperbolically embedded collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. The construction follows Section 4 of [8] and uses the notion of separating cosets explained in Section 2.4.

Proposition 3.8. Suppose that $G$ is a group, $X$ is a subset of $G$, and $\{H_\lambda\}_{\lambda \in \Lambda}$ is a collection of subgroups hyperbolically embedded in $(G,X)$. Then, for any $\mu \in \Lambda$ and any array $r$ on $H_\mu$ into $(\ell^2(H_\mu), \lambda_{H_\mu})$, there exists an array $R$ on $G$ into $(\ell^2(G), \lambda_G)$ and a constant $K_\mu > 0$ satisfying the following: for any $g \in G$, any separating coset $xH_\mu \in S_\mu(1,g;D)$, and any geodesic path $p$ in the relative Cayley graph $\Gamma(X \cup H)$ from 1 to $g$, we have

$$\|r(p_{\text{in}}(xH_\mu)^{-1}p_{\text{out}}(xH_\mu))\|_2 \leq \|R(g)\|_2 + K_\mu. \quad (5)$$

The proof of Proposition 3.8 is essentially the same as the proof of Theorem 4.2 of [8], but because we deal with arrays instead of quasi-cocycles, we give full details with all necessary changes to make the proof self-contained.

Suppose that $r: H_\mu \to \ell^2(H_\mu)$ is an array on $H_\mu$ into the left regular representation $(\ell^2(H_\mu), \lambda_{H_\mu})$. By the embedding $\ell^2(H_\mu) \hookrightarrow \ell^2(G)$, we can think of $r$ as a map $r: H_\mu \to \ell^2(G)$. We define a map $\tilde{r}: G \times G \to \ell^2(G)$ by

$$\tilde{r}(f,g) = \begin{cases} 
\lambda_G(f)r(f^{-1}g) & \text{if } f^{-1}g \in H_\mu \\
0 & \text{if } f^{-1}g \notin H_\mu,
\end{cases}$$

where $(\ell^2(G), \lambda_G)$ is the left regular representation of $G$.

Remark 3.9. If $f, g \in G$ are in the same coset of $H_\mu$, i.e. there exists a $H_\mu$-coset $xH_\mu$ for some $x \in G$ such that $f, g \in xH_\mu$, then the support of $\tilde{r}(f,g)$ is in $xH_\mu$.

Lemma 3.10. For any $f, g, h \in G$, the following hold.

1. $\tilde{r}(g,f) = -\tilde{r}(f,g)$.
2. $\tilde{r}(hf, hg) = \lambda_G(h)\tilde{r}(f,g)$.

Proof. (1) For any $f, g \in G$, $f^{-1}g \in H_\mu$ if and only if $g^{-1}f \in H_\mu$. If $f^{-1}g \in H_\mu$, we have

$$\tilde{r}(g,f) = \lambda_G(g)r(g^{-1}f) = \lambda_G(g)(-\lambda_{H_\mu}(g^{-1}f)r((g^{-1}f)^{-1})) = -\lambda_G(f)r(f^{-1}g) = -\tilde{r}(f,g).$$
(2) If \( f^{-1}g \in H_\mu \), we have
\[
\tilde{r}(hf, hg) = \lambda_G(hf) r((hf)^{-1}hg) = \lambda_G(h) \lambda_G(f) r(f^{-1}g) = \lambda_G(h) \tilde{r}(f, g).
\]

Lemma 3.11. For any \( g \in H_\mu \), we have
\[
\sup_{h \in H_\mu} \| \tilde{r}(g, h) - \tilde{r}(1, h) \|_2 = \sup_{h \in H_\mu} \| \tilde{r}(1, h) - \tilde{r}(1, hg) \|_2 < \infty.
\]

Proof. For any \( g \in H_\mu \), we have
\[
\sup_{h \in H_\mu} \| \tilde{r}(g, h) - \tilde{r}(1, h) \|_2 = \sup_{h \in H_\mu} \| \lambda_{H_\mu}(g) r(g^{-1}h) - r(h) \|_2
\]
\[
= \sup_{h \in H_\mu} \| r(g^{-1}h) - \lambda_{H_\mu}(g^{-1}) r(h) \|_2 < \infty,
\]
and by Lemma 3.10,
\[
\sup_{h \in H_\mu} \| \tilde{r}(1, h) - \tilde{r}(1, hg) \|_2 = \sup_{h \in H_\mu} \| \lambda_{H_\mu}(h^{-1}) (\tilde{r}(1, h) - \tilde{r}(1, hg)) \|_2
\]
\[
= \sup_{h \in H_\mu} \| \tilde{r}(h^{-1}, 1) - \tilde{r}(h^{-1}, g) \|_2
\]
\[
= \sup_{h \in H_\mu} \| - \tilde{r}(1, h^{-1}) + \tilde{r}(g, h^{-1}) \|_2
\]
\[
= \sup_{h \in H_\mu} \| \tilde{r}(g, h) - \tilde{r}(1, h) \|_2.
\]

In the following, we denote
\[
K_g = \sup_{h \in H_\mu} \| \tilde{r}(g, h) - \tilde{r}(1, h) \|_2 = \sup_{h \in H_\mu} \| \tilde{r}(1, h) - \tilde{r}(1, hg) \|_2. \tag{6}
\]

Remark 3.12. For any \( g \in H_\mu \), we have
\[
K_g = \sup_{h \in H_\mu} \| \tilde{r}(1, h) - \tilde{r}(1, hg) \|_2 = \sup_{h \in H_\mu} \| \tilde{r}(1, hg^{-1}) - \tilde{r}(1, (hg^{-1})g) \|_2
\]
\[
= \sup_{h \in H_\mu} \| \tilde{r}(1, h) - \tilde{r}(1, hg^{-1}) \|_2 = K_{g^{-1}}.
\]

Remark 3.13. By Lemma 3.10 (1), \( \tilde{r}(1, 1) = 0 \), hence for any \( g \in H_\mu \), we have
\[
\| \tilde{r}(1, g) \|_2 = \| \tilde{r}(g, 1) - \tilde{r}(1, 1) \|_2 \leq \sup_{h \in H_\mu} \| \tilde{r}(g, h) - \tilde{r}(1, h) \|_2 = K_g.
\]
Lemma 3.14. For any elements $f_1, f_2, g_1, g_2 \in G$ that are in the same coset of $H_\mu$, we have
\[
\|\hat{\tau}(f_1, g_1) - \hat{\tau}(f_2, g_2)\|_2 \leq K_{f_1^{-1}f_2} + K_{g_1^{-1}g_2}.
\]

Proof. By Lemma 3.11, we have
\[
\|\hat{\tau}(f_1, g_1) - \hat{\tau}(f_2, g_2)\|_2 \leq \|\hat{\tau}(f_1, g_1) - \hat{\tau}(f_2, g_1)\|_2 + \|\hat{\tau}(f_2, g_1) - \hat{\tau}(f_2, g_2)\|_2
\]
\[
\leq \|\hat{\tau}(1, f_1^{-1}g_1) - \hat{\tau}(f_1^{-1}f_2, f_1^{-1}g_1)\|_2 + \|\hat{\tau}(1, f_2^{-1}g_1) - \hat{\tau}(1, f_2^{-1}g_1 \cdot g_1^{-1}g_2)\|_2
\]
\[
\leq K_{f_1^{-1}f_2} + K_{g_1^{-1}g_2}.
\]

For $f, g \in G$ and $xH_\mu \in S_\mu(f, g; D)$, we define $\tilde{R}(f, g; xH_\mu) \in \ell^2(G)$ by
\[
\tilde{R}(f, g; xH_\mu) = \frac{1}{|E(f, g; xH_\mu, D)|} \sum_{(u, v) \in E(f, g; xH_\mu, D)} \hat{\tau}(u, v).
\]

Remark 3.15. This is well-defined because $E(f, g; xH_\mu, D)$ is finite by Lemma 2.27 (c). Also, the support of $\tilde{R}(f, g; xH_\mu)$ is in $xH_\mu$ by Remark 3.9.

Lemma 3.16. For any $f, g, h \in G$ and $xH_\mu \in S_\mu(f, g; D)$, the following holds.

(a) $\tilde{R}(g, f; xH_\mu) = -\tilde{R}(f, g; xH_\mu)$.

(b) $\tilde{R}(hf, hg; hxH_\mu) = \lambda_G(h)\tilde{R}(f, g; xH_\mu)$.

Proof. It follows from Lemma 2.22, Lemma 2.27, and Lemma 3.10.

For $n \in \mathbb{R}_{\geq 0}$, we define a constant by
\[
K_n = \max\{K_g \mid g \in H_\mu \wedge \hat{d}_\mu(1, g) \leq n\},
\]
where $K_g$ is defined by (6). Because $(H_\mu, \hat{d}_\mu)$ is a locally finite metric space (cf. Definition 2.15 (2)), we have $K_n < \infty$.

Lemma 3.17. For any $f, g \in G$, any $xH_\mu \in S_\mu(f, g; D)$, and any $(u, v) \in E(f, g; xH_\mu, D)$, we have
\[
\|\tilde{R}(f, g; xH_\mu) - \hat{\tau}(u, v)\|_2 \leq 2K_D.
\]

Proof. For any $(u, v), (u', v') \in E(f, g; xH_\mu, D)$, we have $\hat{d}_\mu(u, u') \leq 3C \leq D$ and $\hat{d}_\mu(v, v') \leq 3C \leq D$ by Lemma 2.23 (b). This implies, by Lemma 3.14,
\[
\|\hat{\tau}(u, v) - \hat{\tau}(u', v')\|_2 \leq K_{u^{-1}u'} + K_{v^{-1}v'} \leq K_D + K_D = 2K_D.
\]

19
Thus, for any \((u, v) \in E(f, g; xH_\mu, D)\), we have

\[
\|\tilde{R}(f, g; xH_\mu) - \tilde{r}(u, v)\|_2 = \left\| \frac{1}{|E(f, g; xH_\mu, D)|} \sum_{(u', v') \in E(f, g; xH_\mu, D)} (\tilde{r}(u', v') - \tilde{r}(u, v)) \right\|_2
\]

\[
\leq \frac{1}{|E(f, g; xH_\mu, D)|} \sum_{(u', v') \in E(f, g; xH_\mu, D)} \|\tilde{r}(u', v') - \tilde{r}(u, v)\|_2
\]

\[
\leq \frac{1}{|E(f, g; xH_\mu, D)|} \sum_{(u', v') \in E(f, g; xH_\mu, D)} 2K_D = 2K_D.
\]

Finally, we define a map \(\tilde{R}: G \times G \to \ell^2(G)\) by

\[
\tilde{R}(f, g) = \sum_{xH_\mu \in S_\mu(f, g; D)} \tilde{R}(f, g; xH_\mu).
\]

This implicitly means that if \(S_\mu(f, g; D)\) is empty, then \(\tilde{R}(f, g) = 0\).

**Lemma 3.18.** For any \(f, g, h \in G\), the following hold.

(a) \(\tilde{R}(g, f) = -\tilde{R}(f, g)\).

(b) \(\tilde{R}(hf, hg) = \lambda_G(h)\tilde{R}(f, g)\).

(c) \(\|\tilde{R}(f, g)\|^2_2 = \sum_{xH_\mu \in S_\mu(f, g; D)} \|\tilde{R}(f, g; xH_\mu)\|_2^2\).

**Proof.** (a) and (b) follow from Lemma 2.22 and Lemma 3.16. (c) follows from the fact that the support of \(\tilde{R}(f, g; xH_\mu)\) is in \(xH_\mu\) as stated in Remark 3.15.

For \(g \in G\), we define

\[
L_g = \max\{\tilde{d}_\mu(u, v) \mid (u, v) \in E(1, g; xH_\mu, D), xH_\mu \in S_\mu(1, g; D)\}.
\]

(8)

\(L_g \in \mathbb{N} \cup \{0\}\) is well-defined by Corollary 2.24 and Lemma 2.27 (c).

The proof of the following lemma is similar to Lemma 4.7 of [8].

**Lemma 3.19.** For any \(g \in G\), we have

\[
\sup_{h \in G} \|\tilde{R}(1, h) + \tilde{R}(h, g) + \tilde{R}(g, 1)\|_2 < \infty.
\]
Proof. For $g,h \in G$, suppose that $\tilde{R}(1,h) + \tilde{R}(h,g) + \tilde{R}(g,1) = \sum_{v \in G} \alpha_v v \in \ell^2(G)$. We define $\xi_{xH_\mu} = \sum_{v \in xH_\mu} \alpha_v v$ for each $H_\mu$-coset $xH_\mu$. Note that we have

$$\tilde{R}(1,h) + \tilde{R}(h,g) + \tilde{R}(g,1) = \sum_{xH_\mu \in G/H_\mu} \xi_{xH_\mu}.$$ 

Let $S_\mu(1,h;D) = S_{1,h}^1 \sqcup S_{1,h}^\mu \sqcup F_{1,h}$, $S_\mu(h,g;D) = S_{h,g}^1 \sqcup S_{h,g}^\mu \sqcup F_{h,g}$, and $S_\mu(g,1;D) = S_{g,1}^1 \sqcup S_{g,1}^\mu \sqcup F_{g,1}$ be the decomposition in Lemma 2.29.

If $xH_\mu \notin S_\mu(1,h;D) \cup S_\mu(h,g;D) \cup S_\mu(g,1;D)$, then we have $\xi_{xH_\mu} = 0$ by Remark 3.15.

If $xH_\mu \in S_{1,h}^1$, we have

$$\xi_{xH_\mu} = \tilde{R}(1,h;xH_\mu) + \tilde{R}(g,1;xH_\mu) = \tilde{R}(1,h;xH_\mu) - \tilde{R}(1,g;xH_\mu) = 0$$

since $S_{1,h}^1 \subset S_\mu(g,1;D) \setminus S_\mu(h,g;D)$ and $E(1,h;xH_\mu,D) = E(1,g;xH_\mu,D)$. We can argue similarly for $S_{1,h}^1$, $S_{h,g}^1$, $S_{h,g}^\mu$, $S_{g,1}^1$, $S_{g,1}^\mu$. Hence, we have

$$\tilde{R}(1,h) + \tilde{R}(h,g) + \tilde{R}(g,1) = \sum_{xH_\mu \in F_{1,h} \cup F_{h,g} \cup F_{g,1}} \xi_{xH_\mu}.$$ 

If $xH_\mu \in F_{1,h}$, there are three cases to consider (see Figures 1, 2). We fix geodesic paths $p \in G(1,h)$, $q \in G(h,g)$, and $r \in G(g,1)$.

![Figure 1](image1.png)  

**Figure 1:** Case 1, Case 2 a), Case 3 a)  

![Figure 2](image2.png)  

**Figure 2:** Case 2 b), Case 3 b)

**Case 1:** $xH_\mu \in S_\mu(h,g;D) \cap S_\mu(g,1;D)$. Let $a,b,c$ be the $H_\mu$-components of $p,q,r$ respectively, corresponding to $xH_\mu$, i.e. we have $p = p_1 a p_2$, $q = q_1 b q_2$, $r = r_1 c r_2$. Let $e_1, e_2, e_3$ be paths of length at most 1 connecting $a_+$ to $b_-$, $b_+$ to $c_-$, $c_+$ to $a_-$ respectively, whose labels are in $H_\mu$. We claim that $e_1$ is isolated in the geodesic triangle $e_1 q_1^{-1} p_2^{-1}$. Indeed, if $e_1$ is connected to an $H_\mu$-component $f$ of $p_2$ where we have $p_2 = s_1 f s_2$, then there exists a path $t$ of length at most 1 connecting $a_-$ to $f_+$ since $a_-$ and $f_+$ are in the same
Combining with Lemma 3.17 and Lemma 3.18 (c) (also note (3)), we obtain $d$ is a geodesic. Similarly, $H$ is defined by (8). Note that the inclusion $(c_-, c_+) \in E(g, 1; xH_\mu, D)$ follows from the inclusions $r \in G(g, 1)$ and $xH_\mu \in S_\mu(g, 1; D)$. Hence, by Lemma 3.14 (see also (7)), we have
\[\|\tilde{r}(a_-, a_+) - \tilde{r}(b_-, b_-)\|_2 \leq K^{-1}a_+ + K^{-1}a_- \leq K_6C + L_g + K3C.\]
Combining with Lemma 3.17 and Lemma 3.18 (c) (also note (3)), we obtain
\[\|\xi_{xH_\mu}\|_2 = \|	ilde{R}(1, h; xH_\mu) + \tilde{R}(h, g; xH_\mu) + \tilde{R}(g, 1; xH_\mu)\|_2 \leq \|\tilde{r}(a_-, a_+) + \tilde{r}(b_-, b_-)\|_2 + \tilde{R}(g, 1; xH_\mu)\|_2 \leq 2K_D + 2K_D + K_{6C + L_g} + K_{3C} + \tilde{R}(g, 1)\|_2 \leq 10K_{10D} + K_{6C + L_g} + \tilde{R}(g, 1)\|_2.\]

Case 2: $xH_\mu \in S_\mu(g, 1; D) \setminus S_\mu(h, g; D)$ or $xH_\mu \in S_\mu(h, g; D) \setminus S_\mu(g, 1; D)$. We can assume $xH_\mu \in S_\mu(g, 1; D) \setminus S_\mu(h, g; D)$ without loss of generality.

2 a) If $q$ penetrates $xH_\mu$, then as in Case 1, let $a, b, c$ be the $H_\mu$-components of $p, q, r$ respectively, corresponding to $xH_\mu$ and $e_1, e_2, e_3$ be paths of length at most 1 connecting $a_+$ to $b_-$, $b_+$ to $c_-$, $c_+$ to $a_-$ respectively, whose labels are in $H_\mu$. In the same way as Case 1, we have $\hat{d}_\mu(a_+, b_-) \leq 3C$, $\hat{d}_\mu(b_+, c_-) \leq 3C$, $\hat{d}_\mu(c_+, a_-) \leq 3C$. Also, by $xH_\mu \notin S_\mu(h, g; D)$, we have $\hat{d}_\mu(b_-, b_+) \leq D$, hence
\[\hat{d}_\mu(a_+, c_-) \leq \hat{d}_\mu(a_+, b_-) + \hat{d}_\mu(b_-, b_+) + \hat{d}_\mu(b_+, c_-) \leq 3C + D + 3C \leq 3D.\]

2 b) If $q$ doesn’t penetrate $xH_\mu$, let $a, c$ be the $H_\mu$-components of $p, r$ respectively, corresponding to $xH_\mu$, i.e. $p = p_1ap_2$, $r = r_1cr_2$, and let $e_2, e_3$ be paths of length at most 1 connecting $a_+$ to $c_-$, $c_+$ to $a_-$ respectively, whose labels are in $H_\mu$. Because $e_3$ is isolated in the geodesic triangle $e_3p_1^{-1}r_1^{-1}$, we have $\hat{d}_\mu(c_+, a_-) \leq 3C$. Also, because $q$ doesn’t penetrate $xH_\mu$, $e_2$ is isolated in the geodesic 4-gon $e_2r_1^{-1}q_1^{-1}p_2^{-1}$, hence we have $\hat{d}_\mu(c_+, a_-) \leq 4C \leq 3D$ by Proposition 2.19.

In both 2 a) and 2 b), we have $\hat{d}_\mu(c_+, a_-) \leq 3C$ and $\hat{d}_\mu(a_+, c_-) \leq 3D$. Thus, by Lemma
3.14 and 3.17, we have
\[
\|\xi_{xH_{\mu}}\|_2 = \|\tilde{R}(1, h; xH_{\mu}) + \tilde{R}(g, 1; xH_{\mu})\|_2 \\
\leq \|\tilde{R}(1, h; xH_{\mu}) + \tilde{R}(g, 1; xH_{\mu})\|_2 - \|\tilde{R}(a_-, a_+) + \tilde{R}(c_-, c_+)\|_2 \\
+ \|\tilde{R}(a_-, a_+) - \tilde{R}(c_-, c_+)\|_2 \\
\leq 2K_D + 2K_D + K_{3C} + K_{3D} \leq 10K_{10D}.
\]
When \(xH_{\mu} \in S_{\mu}(h, g; D) \setminus S_{\mu}(g, 1; D)\), we can show \(\|\xi_{xH_{\mu}}\|_2 \leq 10K_{10D}\) in the same way.

Case 3: \(xH_{\mu} \notin S_{\mu}(h, g; D) \cup S_{\mu}(g, 1; D)\). Note that by \(xH_{\mu} \in F_{1,h} \subset S_{\mu}(1, h; D)\) and Lemma 2.23 (a), at least one of \(q\) and \(r\) penetrate \(xH_{\mu}\).

3 a) If both \(q\) and \(r\) penetrate \(xH_{\mu}\), let \(a, b, c, e_1, e_2, e_3\) be as in Case 1. Then, we have
\[
\hat{d}_{(a_+, b_-)} \leq 3C, \hat{d}_{(a_+, b_-)} \leq 3C, \hat{d}_{(a_+, c_-)} \leq 3C \text{ in the same way as Case 1. Also, by } xH_{\mu} \notin S_{\mu}(h, g; D) \cup S_{\mu}(g, 1; D), \text{ we have } \hat{d}_{(a_+, b_-)} \leq D, \hat{d}_{(a_+, c_-)} \leq D.
\]
Hence,
\[
\hat{d}_{(a_+, a_-)} \leq \hat{d}_{(a_+, b_-)} + \hat{d}_{(b_-, b_-)} + \hat{d}_{(b_-, b_-)} + \hat{d}_{(b_-, c_-)} + \hat{d}_{(c_-, c_-)} + \hat{d}_{(c_-, a_-)} \\
\leq 3C + D + 3C + D + 3C \leq 5D.
\]
In both 3 a) and 3 b), we have \(\hat{d}_{(a_+, a_-)} \leq 5D\), hence by Lemma 3.14 and Remark 3.13, we have
\[
\|\xi_{xH_{\mu}}\|_2 = \|\tilde{R}(1, h; xH_{\mu})\|_2 \leq \|\tilde{R}(1, h; xH_{\mu}) - \tilde{R}(a_-, a_+)\|_2 + \|\tilde{R}(a_-, a_+)\|_2 \\
\leq 2K_D + K_{5D} \leq 10K_{10D}.
\]
Here, we used \(\|\tilde{R}(a_-, a_+)\|_2 = \|\tilde{R}(1, a_-^{-1}a_+)\|_2 \leq K_{a_-^{-1}a_+} \leq K_{5D}\). When only \(q\) penetrates \(xH_{\mu}\), we can show \(\|\xi_{xH_{\mu}}\|_2 \leq 10K_{10D}\) in the same way.

Summarizing Case 1, 2, 3, if \(xH_{\mu} \in F_{1,h}\), we have
\[
\|\xi_{xH_{\mu}}\|_2 \leq 10K_{10D} + K_{6C+L_g} + \|\tilde{R}(g, 1)\|_2.
\]
We can argue similarly for \(F_{h,g}\) and \(F_{g,1}\) as well. Also, by Lemma 2.29 (c), we have \(|F_{1,h}|, |F_{h,g}|, |F_{g,1}| \leq 2\). Thus, for any \(g, h \in G\) we have
\[
\|\tilde{R}(1, h) + \tilde{R}(h, g) + \tilde{R}(g, 1)\|_2 \leq \sum_{xH_{\mu} \in F_{1,h}, F_{h,g}, F_{g,1}} \|\xi_{xH_{\mu}}\|_2 \\
\leq \sum_{xH_{\mu} \in F_{1,h}, F_{h,g}, F_{g,1}} \left(10K_{10D} + K_{6C+L_g} + \|\tilde{R}(g, 1)\|_2\right) \\
\leq 6 \left(10K_{10D} + K_{6C+L_g} + \|\tilde{R}(g, 1)\|_2\right).
\]
**Proof of Proposition 3.8.** We define a map \( R : G \to \ell^2(G) \) by

\[
R(g) = \tilde{R}(1, g).
\]

By Lemma 3.18 (a) and (b), we have for any \( g \in G \),

\[
\lambda_G(g)R(g^{-1}) = \tilde{R}(g, 1) = -\tilde{R}(1, g) = -R(g).
\]

Also, by Lemma 3.19, we have for any \( g \in G \),

\[
\sup_{h \in G} \|R(gh) - \lambda_G(g)R(h)\|_2 = \sup_{h \in G} \|R(g \cdot g^{-1}h) - \lambda_G(g)R(g^{-1}h)\|_2 \\
= \sup_{h \in G} \|\tilde{R}(1, h) - \tilde{R}(g, h)\|_2 \\
\leq \sup_{h \in G} \left( \|\tilde{R}(1, h) + \tilde{R}(h, g) + \tilde{R}(g, 1)\|_2 + \|\tilde{R}(1, g)\|_2 \right) \\
= \|\tilde{R}(1, g)\|_2 + \sup_{h \in G} \|\tilde{R}(1, h) + \tilde{R}(h, g) + \tilde{R}(g, 1)\|_2 \\
< \infty.
\]

Thus, \( R \) is an array. For any \( g \in G \), any \( xH_\mu \in S_\mu(1, g; D) \), and any geodesic path \( p \) in \( \Gamma(X \cup \mathcal{H}) \) from 1 to \( g \), let \( a \) be the component of \( p \) corresponding to \( xH_\mu \). By Lemma 3.17 and Lemma 3.18 (c), we have

\[
\|r(a_{-}^{-1}a_{+})\|_2 = \|\tilde{r}(a_{-}, a_{+})\|_2 \\
\leq \|\tilde{r}(1, g; xH_\mu)\|_2 + 2K_D \\
\leq \|\tilde{r}(1, g)\|_2 + 2K_D = \|R(g)\|_2 + 2K_D,
\]

hence \( R \) satisfies (5) with a constant \( K_\mu = 2K_D \).

3.4 Proof of main theorem

**Proposition 3.20.** Suppose that \( G \) is a finitely generated group hyperbolic relative to a collection of subgroups \( \{H_\mu\}_{\mu \in \Lambda} \) of \( G \). If all subgroups \( H_\mu \) are bi-exact, then \( G \) is also bi-exact.

**Proof.** Note that \( \Lambda \) is finite by definition. Because \( H_\mu \)'s are exact, \( G \) is also exact by Corollary 3 of [15]. In the following, we will verify the condition of Proposition 2.2 (3). We take a finite generating set \( X \) of \( G \), a unitary representation \((\ell^2(Y), \pi)\), and an array \( Q \) as in Proposition 3.1. Since every \( H_\mu \) is bi-exact, there exists a proper array \( r_\mu \) on \( H_\mu \) into \((\ell^2(H_\mu), \lambda_{H_\mu})\) for each \( \mu \in \Lambda \) by Proposition 2.2 (2). By Proposition 3.8, for each \( r_\mu \), there exist an array \( R_\mu \) on \( G \) into \((\ell^2(G), \lambda_G)\) and a constant \( K_\mu \geq 0 \) satisfying (5). Here, we
used the fact that \( \{H_\mu\}_{\mu \in \Lambda} \) is hyperbolically embedded in \((G, X)\) by Proposition 2.16. We define a Hilbert space \( \mathcal{K} \) and a unitary representation \( \rho \) of \( G \) by

\[
\mathcal{K} = \ell^2(Y) \oplus \left( \bigoplus_{\mu \in \Lambda} \ell^2(G) \right), \quad \rho = \pi \oplus \left( \bigoplus_{\mu \in \Lambda} \lambda_\mu \right).
\]

(9)

Since \((\mathcal{K}, \rho)\) is a direct sum of copies of \((\ell^2(G), \lambda_\mu)\) by Remark 3.2, \((\mathcal{K}, \rho)\) is weakly contained by \((\ell^2(G), \lambda_G)\). Now, we define a map

\[
P : G \ni g \mapsto (Q(g), (R_\mu(g))_{\mu \in \Lambda}) \in \ell^2(Y) \oplus \left( \bigoplus_{\mu \in \Lambda} \ell^2(G) \right) = \mathcal{K}.
\]

(10)

Because \(Q\) and \(R_\mu\)'s are arrays, \(P\) is an array on \(G\) into \((\mathcal{K}, \rho)\). Hence, for any \(g, h, k \in G\), we have

\[
\|P(kh) - \rho_gP(g^{-1}k)\| \leq \|P(kh) - P(k)\| + \|P(k) - \rho_gP(g^{-1}k)\|
\]

\[
= \| - \rho_{kh}P((kh)^{-1}) + \rho_kP(k^{-1})\| + \|\rho_{g^{-1}}P(k) - P(g^{-1}k)\|
\]

\[
= \| - P(h^{-1}k^{-1}) + \rho_{h^{-1}}P(k^{-1})\| + \|\rho_{g^{-1}}P(k) - P(g^{-1}k)\|
\]

\[
\leq C(h^{-1}) + C(g^{-1}),
\]

where we denote for each \(s \in G\),

\[
C(s) = \sup_{teG} \|P(st) - \rho_s(P(t))\| < \infty.
\]

Hence, for any \(g, h, k \in G\), we have

\[
\sup_{k \in G} \|P(kh) - \rho_gP(k)\| = \sup_{k \in G} \|P(g(g^{-1}k)h) - \rho_gP(g^{-1}k)\| \leq C(h^{-1}) + C(g^{-1}).
\]

Finally, we will show that \(P\) is proper. Let \(N \in \mathbb{N}\) and \(g \in G\) satisfy \(\|P(g)\| \leq N\). Since \(\|Q(g)\|^2 + \sum_{\mu \in \Lambda} \|R_\mu(g)\|^2 = \|P(g)\|^2\), we get \(\|Q(g)\| \leq N\) and \(\|R_\mu(g)\| \leq N\) for any \(\mu \in \Lambda\). Because the identity map \(id : (G, d_\Gamma) \to (G, d_{X \cup \mathcal{H}})\) is bi-Lipschitz, there exists \(\alpha \in \mathbb{N}\) such that \(d_{X \cup \mathcal{H}}(1, x) \leq \alpha d_\Gamma(1, x)\) for any \(x \in G\). By (4), this implies

\[
d_{X \cup \mathcal{H}}(1, g) \leq \alpha d_\Gamma(1, g) \leq \frac{1}{2} \alpha \|Q(g)\|^2 \leq \frac{1}{2} \alpha N^2.
\]

We denote \(\alpha_N = \frac{1}{2} \alpha N^2\) for simplicity.

Let \(g = w_0h_1w_1 \cdots h_nw_n\) be the label of a geodesic path from 1 to \(g\) in \(\Gamma(G, X \cup \mathcal{H})\), where \(w_i\) is a word in the alphabet \(X \cup X^{-1}\) and \(h_i \in \bigsqcup_{\mu \in \Lambda}(H_\mu \setminus \{1\})\). We have

\[
|w_0| + 1 + |w_1| + \cdots + 1 + |w_n| = d_{X \cup \mathcal{H}}(1, g) \leq \alpha_N,
\]

where \(|w_i|\) denotes the number of letters in \(w_i\). In particular, we have \(n \leq \alpha_N\) and \(|w_i| \leq \alpha_N\) for any \(w_i\).
On the other hand, for each $h_i$, there exists $\mu \in \Lambda$ such that $h_i \in H_\mu$. For simplicity, we denote

$$x_i = w_0 h_1 \cdots w_{i-1}.$$ 

If $x_i H_\mu \notin S_\mu(1, g; D)$, then we have $\hat{d}_\mu(1, h_i) = \hat{d}_\mu(x_i, x_i h_i) \leq D$ by definition of separating cosets. If $x_i H_\mu \in S_\mu(1, g; D)$, then by (5), we have

$$\|r_\mu(h_i)\|_2 \leq \|R_\mu(g)\|_2 + K_\mu \leq N + K_\mu.$$ 

In either case, we have $h_i \in A_\mu$, where

$$A_\mu = \{ h \in H_\mu \mid \hat{d}_\mu(1, h) \leq D \} \cup \{ h \in H_\mu \mid \|r_\mu(h)\|_2 \leq N + K_\mu \}.$$ 

Note that $A_\mu$ is finite, because $\hat{d}_\mu$ is a locally finite metric on $H_\mu$ and $r_\mu$ is proper. Therefore, we have

$$\{ g \in G \mid \|P(g)\| \leq N \} \subset \left\{ w_0 h_1 w_1 \cdots h_n w_n \mid n \leq \alpha_N, \ |w_i| \leq \alpha_N, \ h_i \in \bigcup_{\mu \in \Lambda} A_\mu \right\}.$$ 

Since $X$, $A_\mu$’s, and $\Lambda$ are finite, the set on the right-hand side above is finite, hence $P$ is proper. □

**Lemma 3.21.** Subgroups of countable bi-exact groups are also bi-exact.

**Proof.** Let $G$ be a countable bi-exact group and $H$ be a subgroup of $G$. $H$ is exact, because $G$ is exact and subgroups of exact groups are exact (cf. [2]). By Proposition 2.2 (2), there exists a proper array $r: G \to \ell^2(G)$ on $G$ into $(\ell^2(G), \lambda_G)$. Note that the restriction of $\lambda_G$ to $H$, that is, $(\ell^2(G), \lambda_G|_H)$ is unitarily isomorphic to $(\bigoplus_{H \in H G, G} \ell^2(H), \lambda_H)$, hence we have $(\ell^2(G), \lambda_G|_H) < (\ell^2(H), \lambda_H)$. Also, it is straightforward to show that $r|_H: H \to \ell^2(G)$ is a proper array on $H$ into $(\ell^2(G), \lambda_G|_H)$. Thus, by Proposition 2.2 (3), $H$ is bi-exact. □

**Proof of Theorem 1.1.** It follows from Proposition 3.20 and Lemma 3.21. □

**References**

[1] B. H. Bowditch, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012), no. 3, 1250016, 66. MR 2922380

[2] N. P. Brown and N. Ozawa, *$C^*$-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008. MR 2391387

[3] I. Chifan and A. Ioana, *Amalgamated free product rigidity for group von Neumann algebras*, Adv. Math. 329 (2018), 819–850. MR 3783429

26
1. Chifan, T. Sinclair, and B. Udrea, *On the structural theory of II$_1$ factors of negatively curved groups, II: Actions by product groups*, Adv. Math. 245 (2013), 208–236. MR 3084428

2. F. Dahmani, *Les groupes relativement hyperboliques et leurs bords thèse de doctorat, université louis pasteur, strasbourg I.* (2003), https://www-fourier.ujf-grenoble.fr/~dahmani/Files/These.pdf.

3. F. Dahmani, V. Guirardel, and D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. 245 (2017), no. 1156, v+152. MR 3589159

4. G. C. Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. 10 (2010), no. 3, 1807–1856. MR 2684983

5. M. Hull and D. Osin, *Induced quasicocycles on groups with hyperbolically embedded subgroups*, Algebr. Geom. Topol. 13 (2013), no. 5, 2635–2665. MR 3116299

6. I. Mineyev, *Straightening and bounded cohomology of hyperbolic groups*, Geom. Funct. Anal. 11 (2001), no. 4, 807–839. MR 1866802

7. I. Mineyev and A. Yaman, *Relative hyperbolicity and bounded cohomology*, https://faculty.math.illinois.edu/~mineyev/math/art/rel-hyp.pdf.

8. D. V. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. 179 (2006), no. 843, vi+100. MR 2182268

9. , *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016), no. 2, 851–888. MR 3430352

10. N. Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 8, 691–695. MR 1763912

11. , *Solid von Neumann algebras*, Acta Math. 192 (2004), no. 1, 111–117. MR 2079600

12. , *Boundary amenability of relatively hyperbolic groups*, Topology Appl. 153 (2006), no. 14, 2624–2630. MR 2243738

13. , *A Kurosh-type theorem for type II$_1$ factors*, Int. Math. Res. Not. (2006), Art. ID 97560, 21. MR 2211141

14. , *An example of a solid von Neumann algebra*, Hokkaido Math. J. 38 (2009), no. 3, 557–561. MR 2548235

15. N. Ozawa and S. Popa, *Some prime factorization results for type II$_1$ factors*, Invent. Math. 156 (2004), no. 2, 223–234. MR 2052608
[19] S. Popa and S. Vaes, *Unique Cartan decomposition for II$_1$ factors arising from arbitrary actions of hyperbolic groups*, J. Reine Angew. Math. 694 (2014), 215–239. MR 3259044

[20] H. Sako, *Measure equivalence rigidity and bi-exactness of groups*, J. Funct. Anal. 257 (2009), no. 10, 3167–3202. MR 2568688

Department of Mathematics, Vanderbilt University, Nashville 37240, U.S.A.
E-mail: koichi.oyakawa@vanderbilt.edu