Chiral condensate in 

\[ n_f = 2 \quad \text{QCD from the} \]

Banks–Casher relation

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Based on: GPE, L. Giusti, S. Lottini, R. Sommer: “Chiral symmetry breaking in QCD Lite”, arXiv:1406.4987
Banks-Casher relate condensate $\Sigma$ to spectral density $\rho$ of Dirac operator

$$\Sigma \equiv -\frac{1}{2}\langle \bar{\psi}\psi \rangle = \pi \lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m), \quad \rho(\lambda, m) = \frac{1}{V} \sum_{k=1}^{\infty} \langle \delta(\lambda - \lambda_k) \rangle$$ (1)

Calculated on the lattice: mode number $\nu \equiv$ integrated density

The number of modes of the massive hermitian Dirac operator $D^\dagger D + m^2$, with eigenvalues $\alpha \leq \Lambda^2 + m^2$, is renormalization-group invariant

$$\nu_R(\Lambda_R, m_R) = \nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda \rho(\lambda, m)$$ (2)

To extract $\Sigma$: Define effective spectral density (removing threshold effects)

$$\tilde{\rho}_R = \frac{\pi}{2V} \frac{\nu_{2,R} - \nu_{1,R}}{\Lambda_{2,R} - \Lambda_{1,R}} \frac{V \to \infty; a, m_R, \Lambda_R \to 0}{\rightarrow} \Sigma$$ (3)

¹ L. Giusti and M. Lüscher, JHEP903(2009)13
Chiral Perturbation Theory

NLO (W)ChPT ($n_f = 2$) in the continuum$^{1,2}$ and GSM$^3$ regime$^2$

\[
\tilde{\rho}^\text{NLO}_R = \sum \left\{ 1 + \frac{m_R \Sigma}{(4\pi)^2 F^4} \left[ 3 \bar{t}_6 + 1 - \ln(2) - 3 \ln \left( \frac{\Sigma m_R}{F^2 M^2} \right) + \tilde{g}_\nu \left( \frac{\Lambda_{1,R}}{m_R}, \frac{\Lambda_{2,R}}{m_R} \right) \right] \right\}
- 32(W_0 a)^2 \frac{W'_8 m_R}{\Lambda_{1,R} \Lambda_{2,R}}
\]

with \( \tilde{g}_\nu(x_1, x_2) = \frac{f_\nu(x_1) + f_\nu(x_2)}{2} + \frac{1}{2} \frac{x_1 + x_2}{x_2 - x_1} \left[ f_\nu(x_2) - f_\nu(x_1) \right] \)

\( f_\nu(x) = \left( x - \frac{1}{x} \right) \arctan(x) - \frac{\pi}{2} x - \ln(x + x^3) \)

- No chiral logs for fixed \( \Lambda_R \); \( \tilde{g}_\nu(.) \) mild function
- 1+2 NLO LECs; \( W'_8 \) expected negative$^4$

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1. L. Giusti and M. Lüscher, JHEP903(2009)13
2. S. Necco and A. Shindler, JHEP1104(2010)31
3. “Generally small quark mass”
4. M.T. Hansen and S.R. Sharpe, PRD85(2012)14593; K. Splittorff and J. Verbaarschot, PRD85(2012)105008
### Details of the simulation

**Parameters of the simulation: \( n_f = 2 \) CLS-lattices\(^5\)**

| id | \( L/a \) | \( m_\pi \) | \( m_\pi L \) | \( a \) | \( R_{\tau_{\text{exp}}} \) | \( R_{\tau_{\text{int}}(m_\pi)} \) | \( R_{\tau_{\text{int}}(\nu)} \) | \( R_{n_{\text{it}}(\nu)} \) | \( N_{\text{cnfg}} \) |
|----|----------|---------|---------|------|----------------|----------------|----------------|----------------|-----|
| A3 | 32       | 490     | 6.0     | 0.075 | 40             | 7              | 3              | 48             | 55  |
| A4 | 380      | 4.7     |         |       | 5              |                |                | 53             | 55  |
| A5 | 330      | 4.0     |         |       | 5              |                |                | 36             | 55  |
| B6 | 48       | 280     | 5.2     | 6     | 24             |                |                | 50             |     |
| E5 | 32       | 440     | 4.7     | 0.065 | 56             | 9              | 6              | 36             | 92  |
| F6 | 48       | 310     | 5.0     | 8     | 30             |                |                | 50             |     |
| F7 | 270      | 4.3     |         |       | 7              |                |                | 27             | 50  |
| G8 | 64       | 190     | 4.1     | 8     | 24-48          |                |                | 50             |     |
| N5 | 48       | 440     | 5.2     | 0.048 | 200            | 30             | 23             | 281            | 60  |
| N6 | 340      | 4.0     |         | 100   | 10             |                |                | 128            | 60  |
| O7 | 64       | 270     | 4.2     | 100   | 15             |                |                | 76             | 50  |

- **Autocorrelation under control**, \( \tau_{\text{int}}(\nu) < n_{\text{it}}(\nu) \)
- **Finite volume effects under control**, found tiny for \( \Lambda_R \geq 20 \) MeV
- **9 values of cutoff \( \Lambda_R \) for each ensemble**, \( 20 \leq \Lambda_R \leq 120 \) MeV

\(^5\) P. Fritzsch et al., NPB865(2012)397; M. Marinkovic et al., PoS Lat(2011)232
Details of the simulation II

Stochastic evaluation of $\nu$ through VEV of projector to low modes

$$\nu = \langle \text{tr}[P_M] \rangle, \quad M = \sqrt{\Lambda^2 + m^2}$$

$$= \frac{1}{N} \sum_{k=1}^{N} (\eta_k, P_M \eta_k), \quad \eta_k \ldots \text{pseudo-fermion fields}$$

First look at numerical data

$\nu$ roughly linear in all ensembles.
Details of the simulation II

Stochastic evaluation of $\nu$ through VEV of projector to low modes

\[ \nu = \langle \text{tr}[\mathbb{P}M] \rangle, \quad M = \sqrt{\Lambda^2 + m^2} \]  
\[ = \frac{1}{N} \sum_{k=1}^{N} (\eta_k, \mathbb{P}M\eta_k), \quad \eta_k \ldots \text{pseudo-fermion fields} \]

First look on numerical data

- $\nu$ roughly linear in all ensembles.
- $\tilde{\rho}$ non-zero and flat in $\Lambda_R$ toward small $m_R$ and $a$.

\[ \tilde{\rho}_R \text{ [GeV}^3] \]
\[ m_R = 12.9 \text{ MeV} \]
\[ a = 0.048 \text{ fm} \]
Fitting strategy A: Concept

First studies\(^6\) indicated:
- Higher order effects in \(\tilde{\rho}_R\) observed.
- Functional form not well known at finite lattice spacing.

Suggest Strategy A:
- First perform continuum limit at each \((\Lambda_R, m_R)\) following Symanzik.
- Finite spectral density near the origin will suggest chiral SSB.
- Use (continuum) ChPT to remove remaining corrections.

\[
\Sigma = \lim_{\Lambda_R \to 0} \lim_{m_R \to 0} \lim_{a \to 0} \tilde{\rho}_R(\Lambda_R, m_R, a) \tag{7}
\]

A posteriori self-consistency check:
Agreement with \(M^2_\pi F^2_\pi/2\) vs. \(m\) (GMOR)?

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\(^6\) GPE et al., PoS Lat(2013)119
At each pair \((\Lambda_R, m_R)\), extrapolate \(a \to 0\).

Data agree well with linear \(a^2\)-dependence (\(\mathcal{O}(a)\)-improved theory).

Discretization effects show non-trivial \((\Lambda_R, m_R)\)-dependence:

\(\tilde{\rho}_R^{1/3}\): mild (<5%) at lightest \((\Lambda_R, m_R)\); i.g. up to \(\mathcal{O}(20\%)\)
Non-zero density at small $(\Lambda_R, m_R)$ points to chiral SSB.

Use generalized NLO ChPT to extrapolate to chiral limit\(^7\):

\[
\tilde{\rho}_R = c_0(\Lambda_R) + c_1 m_R + c_2 g(\Lambda_R, m_R)
\]

\(\Sigma = c_0(\Lambda_R) = \text{const.} \quad \text{at NLO}\) \(\quad (8)\)

\(^7\) Here and later we use the short-hand notation: \(g(\Lambda, m) = m(\bar{g}_\nu(\Lambda_1/m, \Lambda_2/m) - 3 \ln(m/\mu))\)
Fitting strategy A IV: Chiral limit

$\tilde{\rho}_R$ vs. $\Lambda_R$ in the continuum and chiral limit

- $\tilde{\rho}_R \big|_{m_R=0} = c_0(\Lambda_R)$ shows plateau at NLO ChPT.
- Identify valid range of NLO: $\Lambda_R < 80$ MeV.
- $\Sigma^{1/3} = 261(6)$ MeV in $\overline{\text{MS}}$ at 2 GeV.

$m_R = 0$ MeV
$a = 0$ fm
Fitting strategy B: Concept

Combined 3-dim fit in $(\Lambda_R, m_R, a)$

- Include all data; no interpolation required.
- Fewer fit parameters compared to Strategy A.
- Model the discretization effects:
  - Linear in $a^2$ and $m_R$.
  - Still allow for arbitrary $\Lambda_R$-dependence.

\[
\tilde{\rho}_R = c_{0,0}(\Lambda_R) + c_{0,1}(\Lambda_R) a^2 + c_{1,0} m_R + c_{1,1}(\Lambda_R) m_R a^2 + c_2 g(\Lambda_R, m_R)
\]

\[
\Sigma = c_{0,0}(\Lambda_R) = \text{const. at NLO}
\]

- Inspired by Symanzik and chiral power expansion.
- Model complies with results of Strategy A.
- Includes NLO WChPT GSM$^3$ as special case.

3 "Generally small quark mass"
Fitting strategy B II: Continuum and chiral limit

$\tilde{\rho}_R$ vs. $m_R$ and vs. $\Lambda_R$

- NLO plateau in $c_{0,0}(\Lambda_R)$ for $\Lambda_R < 80$ MeV.
- $\Sigma^{1/3} = 260(6)$ MeV in $\overline{\text{MS}}$ at 2 GeV.
- Systematic error:
  - Neglect data at coarse lattices ($a=0.075$ fm): +8 MeV.
  - Include $O(\Lambda_R^2, m_R^2)$-terms: -7 MeV.
Conclusion and check with GMOR

Ab-initio determination of the chiral condensate from Banks-Casher.

Extensive study of the spectral density:

- 3 lattice spacings: $0.048 \leq a \leq 0.076$ fm
- 4 pion masses: $190 \leq m_\pi \leq 490$ MeV
- 9 cutoffs: $20 \leq \Lambda_R \leq 120$ MeV

Separate treatment of various effects, all systematics discussed.

$\Sigma^{1/3} = 261(6)(8)$ MeV in $\overline{\text{MS}}$ at 2 GeV.

Final results agrees with GMOR-relation\(^8\).

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\(^8\) GPE, L. Giusti, S. Lottini, R. Sommer: “Chiral symmetry breaking in QCD Lite”, arXiv:1406.4987

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Chiral condensate in $n_f = 2$ QCD from the Banks–Casher relation
Data well described by linear fit passing through origin.
Strongly suggests to be actually within chiral regime.
\[ \Sigma^{1/3} = 271(8) \text{ MeV}. \]
\( \nu \neq 0.12 \)

\( \Lambda \)

\( \nu \neq 0.12 \)

\( \Lambda \neq 0.048 \text{ fm} \)

\( \Lambda \neq 0.065 \text{ fm} \)

\( \Lambda \neq 0.075 \text{ fm} \)

\( \nu \neq \nu_{\text{ref}} \)

\( m_R = 32.0 \text{ MeV} \)

\( \nu_{\text{ref}} = \nu|_{\Lambda_R=40\text{ MeV}} \)

shows flat \( a \)-dependence in dimensionful analysis.

\( \tilde{\rho}_R(\Lambda_R, m_R, a) = \tilde{\rho}_R(\Lambda_R, m_R, 0) + a^2 \Delta(\Lambda_R, m_R) \quad (11) \)

\( \Delta(\Lambda_R, m_R) = \bar{c}_{0,1}(\Lambda_R) + \bar{c}_{1,1}(\Lambda_R)m_R \quad (12) \)
|Δ\overset{\text{FV}}{\Sigma}|/\Sigma \text{ vs. } \Lambda_R \text{ for } \beta = 5.2
\[ |\Delta \tilde{\Sigma}^{FV}|/\Sigma \text{ vs. } \Lambda_R \text{ for } \beta = 5.3 \]