Univalence criterion for meromorphic functions and Loewner chains

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

Abstract. The object of the present paper is to obtain a more general condition for univalence of meromorphic functions in the $U^*$. The significant relationships and relevance with other results are also given. A number of known univalent conditions would follow upon specializing the parameters involved in our main results.

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1. Introduction

We denote by $U_r$ the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$, by $U = U_1$ the open unit disk of the complex plane and $U^* = \mathbb{C} \setminus U$, where $\overline{U}$ is closure of $U$.

Let $A$ denote the class of all analytic functions in the open unit disk $U$ normalized by

$$f(z) = z + a_2 z^2 + \ldots \quad (z \in U)$$

and we denote by $S$ the subclass of $A$ consisting of functions which are also univalent in $U$. Closely related to $S$ is the class $\Sigma$ of all meromorphic functions in $U^*$ by

$$f(\zeta) = b\zeta + b_0 + \frac{b_1}{\zeta} + \ldots \quad (\zeta \in U^*)$$

and $\Sigma_0$ stands for all functions from $\Sigma$ with normalization $b = 1$ and $b_0 = 0$. These classes have been one of the important subjects of research in complex analysis especially, Geometric Function Theory for a long time (see, for details, [12]).

Two of the most important and known univalence criteria for analytic functions defined in $U^*$ were obtained by Becker [1] and Nehari [8]. Some extensions of these two criteria were given by Lewandowski [5], [6] and Ruscheweyh [11]. During the time, unlike there were obtained a lot of univalence criteria by Miazga and Wesolowski [7], Wesolowski [13], Kanas and Srivastava [4] and Deniz and Orhan [2].

In the present paper we consider a general univalence criterion for functions $f$ belonging to the class $\Sigma$ in terms of the Schwarz derivative defined by

$$S_f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$
2. LOEWNER CHAINS AND RELATED THEOREM

Before proving our main theorem we need a brief summary of the method of Loewner chains.

Let \( L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots\), \( a_1(t) \neq 0\), be a function defined on \( U \times [0, \infty)\), where \( a_1(t)\) is a complex-valued, locally absolutely continuous function on \([0, \infty)\). \( L(z, t)\) is called a Loewner chain if \( L(z, t)\) satisfies the following conditions:

(i) \( L(z, t)\) is analytic and univalent in \( U \) for all \( t \in [0, \infty)\)

(ii) \( L(z, t) \prec L(z, s)\) for all \( 0 \leq t \leq s < \infty\),

where the symbol "\( \prec\)" stands for subordination. If \( a_1(t) = e^t\) then we say that \( L(z, t)\) is a standard Loewner chain.

In order to prove our main results we need the following theorem due to Pommerenke [9] (also see [10]). This theorem is often used to find out univalency for an analytic function, apart from the theory of Loewner chains;

**Theorem 2.1.** Let \( L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots\) be analytic in \( U_r \) for all \( t \in [0, \infty)\). Suppose that:

(i) \( L(z, t)\) is a locally absolutely continuous function in the interval \([0, \infty)\), and locally uniformly with respect to \( U_r\).

(ii) \( a_1(t)\) is a complex valued continuous function on \([0, \infty)\) such that \( a_1(t) \neq 0\), \(|a_1(t)| \to \infty\) for \( t \to \infty\) and

\[
\left\{ \frac{L(z, t)}{a_1(t)} \right\}_{t \in [0, \infty)}
\]

forms a normal family of functions in \( U_r\).

(iii) There exists an analytic function \( p : \mathbb{U} \times [0, \infty) \to \mathbb{C} \) satisfying \( \Re p(z, t) > 0 \) for all \( z \in \mathbb{U}, t \in [0, \infty) \) and

\[
(2.1) \quad z \frac{\partial L(z, t)}{\partial z} = p(z, t)\frac{\partial L(z, t)}{\partial t}, \quad z \in U_r, \ t \in [0, \infty).
\]

Then, for each \( t \in [0, \infty)\), the function \( L(z, t)\) has an analytic and univalent extension to the whole disk \( U \) or the function \( L(z, t)\) is a Loewner chain.

The equation \( (2.1)\) is called the generalized Loewner differential equation.

3. UNIVALENCE CRITERION FOR THE FUNCTIONS BELONGING TO THE CLASS \( \Sigma \)

In this section, making use of the Theorem 2.1, we obtain an univalence criterion for meromorphic functions. The method of prove is based on Theorem 2.1 and on construction of a suitable Loewner chain.
Theorem 3.1. Let \( f, g \in \mathbb{U}^* \) be locally univalent functions in \( \mathbb{U}^* \). If there exists an analytic function \( h \) such that \( \text{Re} h(\zeta) \geq \frac{1}{2} \) and \( h(\zeta) = 1 + \frac{b_1}{\zeta} + ... \) for \( \zeta \in \mathbb{U}^* \), and for arbitrary \( \alpha \in \mathbb{C} \) we have
\[
(3.1) \quad \left| \frac{1 - h(\zeta)}{h(\zeta)} |\zeta|^2 - (|\zeta|^2 - 1) \right| \left| \frac{\zeta h'(\zeta)}{h(\zeta)} + (1 - 2\alpha) \frac{\zeta f''(\zeta)}{f'(\zeta)} + 2\alpha \frac{\zeta g''(\zeta)}{g'(\zeta)} \right| + \alpha (|\zeta|^2 - 1)^2 \frac{h(\zeta)}{\zeta}
\]
\[
\left[ \left( \alpha - \frac{1}{2} \right) \left( \frac{f''(\zeta)}{f'(\zeta)} - \frac{g''(\zeta)}{g'(\zeta)} \right)^2 + S_f(\zeta) - S_g(\zeta) \right] \leq 1
\]
for all \( \zeta \in \mathbb{U}^* \), then \( f \) is univalent in \( \mathbb{U}^* \).

Proof. Without loss of generality we can consider the functions of the form
\[
f(\zeta) = \zeta + \frac{a_1}{\zeta} + ... \quad \text{and} \quad g(\zeta) = \zeta + \frac{b_1}{\zeta} + ...
\]
since the Schwarzian derivative is invariant under Möbius transformations. Consider the functions defined by
\[
v(\zeta) = \left[ \frac{g'(\zeta)}{f'(\zeta)} \right]^\alpha = 1 + \frac{v_2}{\zeta^2} + ..., \quad \alpha \in \mathbb{C}
\]
where we choose this branch of the power \((\cdot)^\alpha\), which for \( \zeta \to \infty \) has value 1, and
\[
u(\zeta) = f(\zeta)v(\zeta) = \zeta + \frac{v_2}{\zeta} + ...
\]
The functions \( u \) and \( v \) are meromorphic in \( U^* \) since \( f \) and \( g \) do not have multiple poles and \( f' \) and \( g' \) are different from zero.

For all \( t \in [0, \infty) \) and \( \frac{1}{\zeta} = z \in U \) the function \( f : \mathbb{U}_r \times [0, \infty) \to \mathbb{C} \) defined formally by
\[
(3.4) \quad f(z, t) = \left[ \frac{u \left( \frac{e^t}{z} \right) + (e^{-t} - e^t) \frac{1}{2} h \left( \frac{e^t}{z} \right) u' \left( \frac{e^t}{z} \right)}{v \left( \frac{e^t}{z} \right) + (e^{-t} - e^t) \frac{1}{2} h \left( \frac{e^t}{z} \right) v' \left( \frac{e^t}{z} \right)} \right]^{-1} = e^t z + \Psi(e^{-pt}, z^2), \quad p = 1, 2, ...
\]
is analytic in \( U \) since \( \Psi(e^{-pt}, z^2) \) is analytic function in \( U \) for each fixed \( t \in [0, \infty) \) and \( p = 1, 2, ... \). From (3.4) we have \( a_1(t) = e^t \) and
\[
\lim_{t \to \infty} |a_1(t)| = \lim_{t \to \infty} e^t = \infty.
\]
After simple calculation we obtain, for each \( z \in U \),
\[
\lim_{t \to \infty} \frac{f(z, t)}{e^t} = \lim_{t \to \infty} \left\{ z + \Psi(e^{-(p+1)t}, z^2) \right\} = z.
\]
The limit function \( k(z) = z \) belongs to the family \( \left\{ f(z, t)/e^t : t \in [0, \infty) \right\} \); then, there exists a number \( r_0 \) \((0 < r_0 < 1)\) that in every closed disk \( \mathbb{U}_{r_0} \), there exists a constant \( K_0 > 0 \), such that
\[
\left| \frac{f(z, t)}{e^t} \right| < K_0, \quad z \in \mathbb{U}_{r_0}, \ t \in [0, \infty)
\]
uniformly in this disk, provided that $t$ is sufficiently large. Thus, by Montel’s Theorem, $\{f(z,t)/e^t\}$ forms a normal family in each disk $U_{r_0}$.

Since the function $\Psi(e^{-pt}, z^2)$ is analytic in $U$, $\Psi^{(k)}(e^{-pt}, z^2), k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is continuous on the compact set, so $\Psi^{(k)}(e^{-pt}, z^2), k \in \mathbb{N}_0$ is bounded function. Thus for all fixed $T > 0$, we can write $e^t < e^T$ and we obtain that for all fixed numbers $t \in [0, T] \subset [0, \infty)$, there exists a constant $K_1 > 0$ such that

$$\left| \frac{\partial f(z,t)}{\partial t} \right| < K_1, \quad \forall z \in U_{r_0}, t \in [0, T].$$

Therefore, the function $f(z,t)$ is locally absolutely continuous in $[0, \infty)$; locally uniformly with respect to $U_{r_0}$.

After simple calculations from (3.3) we obtain

(3.5) \[ \frac{\partial f(z,t)}{\partial z} \]

\[ = \frac{1}{e^t} \left( \frac{e^t}{z} \right)^2 \left\{ \left( 1 + (e^{-2t} - 1) \left( h \left( \frac{e^t}{z} \right) + \frac{e^t}{z} h' \left( \frac{e^t}{z} \right) \right) \right) (u'v - v'u) 
\right. \\
\left. \quad + (e^{-2t} - 1) \frac{e^t}{z} h \left( \frac{e^t}{z} \right) (u''v - v''u) + (e^{-2t} - 1)^2 \frac{e^t}{z} h^2 \left( \frac{e^t}{z} \right) (u''v' - v''u') \right\} \\
\times f^2(z,t)/ \left[ v \left( \frac{e^t}{z} \right) + (e^{-t} - e^t) \frac{1}{z} h \left( \frac{e^t}{z} \right) v' \left( \frac{e^t}{z} \right) \right]^2 \]

and

(3.6) \[ \frac{\partial f(z,t)}{\partial t} \]

\[ = \frac{e^t}{z} \left\{ \left( 1 - (e^{-2t} + 1) h \left( \frac{e^t}{z} \right) + (e^{-2t} + 1) h' \left( \frac{e^t}{z} \right) \right) (u'v - v'u) 
\right. \\
\left. \quad + (e^{-2t} - 1) \frac{e^t}{z} h \left( \frac{e^t}{z} \right) (u''v - v''u) + (e^{-2t} - 1)^2 \frac{e^t}{z} h^2 \left( \frac{e^t}{z} \right) (u''v' - v''u') \right\} \\
\times f^2(z,t)/ \left[ v \left( \frac{e^t}{z} \right) + (e^{-t} - e^t) \frac{1}{z} h \left( \frac{e^t}{z} \right) v' \left( \frac{e^t}{z} \right) \right]^2 \]

where

(3.7) \[ u'v - v'u = f^\prime \left( \frac{g'}{f'} \right)^{2\alpha}, \quad \alpha \in \mathbb{C} \]

(3.8) \[ u''v - v''u = (1 - 2\alpha)f^n \left( \frac{g'}{f'} \right)^{2\alpha} + 2\alpha g'' \left( \frac{g'}{f'} \right)^{2\alpha-1}, \quad \alpha \in \mathbb{C} \]

(3.9) \[ u''v' - v''u' = \alpha f^\prime \left( \frac{g'}{f'} \right)^{2\gamma} \{ (S_f - S_g) + \left( \alpha - \frac{1}{2} \right) \left( \frac{f''}{f'} - \frac{g''}{g'} \right) \}, \quad \alpha \in \mathbb{C} \]

and $u, v, u', v', u'', v''$ are calculated at $e^t/z$. 
Consider the function \( p : \mathbb{U}_r \times [0, \infty) \to \mathbb{C} \) for \( 0 < r < r_0 \) and \( t \in [0, \infty) \), defined by

\[
p(z, t) = z \frac{\partial f(z, t)}{\partial z} / \frac{\partial f(z, t)}{\partial t}.
\]

From (3.5) to (3.9), we can easily see that the function \( p(z, t) \) is analytic in \( \mathbb{U}_r \), \( 0 < r < r_0 \). If the function

\[
(3.10) \quad w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1} = \frac{z \frac{\partial f(z, t)}{\partial z} - \frac{\partial f(z, t)}{\partial t}}{z \frac{\partial f(z, t)}{\partial z} + \frac{\partial f(z, t)}{\partial t}}
\]

is analytic in \( \mathbb{U} \times [0, \infty) \) and \( |w(z, t)| < 1 \), for all \( z \in \mathbb{U} \) and \( t \in [0, \infty) \), then \( p(z, t) \) has an analytic extension with positive real part (\( \text{Re} p(z, t) > 0 \)) in \( \mathbb{U} \), for all \( t \in [0, \infty) \).

To show this we write (3.5) and (3.6) in the equation (3.10), then we obtain

\[
(3.11) \quad w(z, t) = 2e \left\{ \left( 1 - h \left( \frac{e}{z} \right) \right) + e^{-2t} h \left( \frac{e}{z} \right) \right\} \left( u' v - v' u \right)
\]

\[
+ \frac{e^{-2t} h \left( \frac{e}{z} \right)}{2} \left( u'' v - v'' u \right) + (e^{-2t} - 1) \frac{z e^{2t}}{2} h \left( \frac{e}{z} \right) \left( u'' v - v'' u \right)
\]

\[
= e^{2t} \left( 1 - h \left( \frac{e}{z} \right) \right) + (1 - e^{2t}) \frac{e}{z} \left( h \left( \frac{e}{z} \right) \right) \left( u'' v - v'' u \right)
\]

and from (3.7) for all \( z \in \mathbb{U} \) and \( t \in [0, \infty) \)

\[
(3.12) \quad w(z, t) = e^{2t} \left( 1 - h \left( \frac{e}{z} \right) \right) + \left( 1 - e^{2t} \right) \frac{e}{z} \left( h \left( \frac{e}{z} \right) \right) + \frac{e^{2t} \left( e^{-2t} - 1 \right) \frac{z e^{2t}}{2} h \left( \frac{e}{z} \right)}{2} = 1 - h \left( \frac{e}{z} \right) + \frac{e^{2t} \left( e^{-2t} - 1 \right) \frac{z e^{2t}}{2} h \left( \frac{e}{z} \right)}{2}.
\]

The right hand side of the equation (3.12) is equal to

\[
w(z, 0) = \frac{1 - h \left( \frac{1}{z} \right)}{h \left( \frac{1}{z} \right)}
\]

for \( t = 0 \). Thus, from hypothesis of theorem for \( \frac{1}{z} = \zeta \in \mathbb{U}^* \) we have

\[
\left| \frac{1 - h \left( \zeta \right)}{h \left( \zeta \right)} \right| \leq 1.
\]

Since \( \left| \frac{e}{z} \right| > |e| > 1 \) for all \( z \in \mathbb{U} \) and \( t > 0 \), we find that \( w(z, t) \) is an analytic function in \( \mathbb{U} \). Then putting \( \tilde{e} = \zeta \in \mathbb{U}^* \), \( \zeta = \zeta e^t \), \( |\zeta| = e^t \) for \( |z| = 1 \), from (3.12) by assumption (3.1) replacing \( \tilde{e} \) by
Therefore $|w(z, t)| < 1$ for all $z \in U$ and $t \in [0, \infty)$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $f(z, t)$ is a Loewner chain or has an analytic and univalent extension to the whole unit disk $U$, for all $t \in [0, \infty)$.

From (3.2)-(3.4) it follows in particular that

$$f(z, 0) = \frac{v(\frac{1}{z})}{u(\frac{1}{z})} = \frac{1}{f(\frac{1}{z})} \in S$$

and for $\frac{1}{z} = \zeta \in U^*$ we say that $f(\zeta)$ is univalent in $U^*$. Thus the proof is completed. \hfill \square

For $\alpha = 0$ in Theorem 3.1 we obtain following new result:

**Corollary 3.2.** Let $f, g \in \Sigma$ be locally univalent function in $U^*$. If there exists an analytic function $h$ with $\text{Re} h(\zeta) \geq \frac{1}{2}$ in $U^*$ and $h(\zeta) = 1 + \frac{h_2^2}{\zeta} + \cdots$ such that

\[
|1 - \frac{h(\zeta)}{h(\zeta)}| \zeta^2 - (|\zeta|^2 - 1) \left[ \frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta f''(\zeta)}{f'(\zeta)} \right] \leq 1
\]

for all $\zeta \in U^*$, then $f$ is univalent in $U^*$.

For $\alpha = \frac{1}{2}$ in Theorem 3.1 we obtain univalence criterion given by Miazga and Wesolowski [7].

**Corollary 3.3.** Let $f, g \in \Sigma$ be locally univalent functions in $U^*$. If there exists an analytic function $h$ with $\text{Re} h(\zeta) \geq \frac{1}{2}$ in $U^*$ and $h(\zeta) = 1 + \frac{h_2^2}{\zeta} + \cdots$ such that

\[
|1 - \frac{h(\zeta)}{h(\zeta)}| \zeta^2 - (|\zeta|^2 - 1) \left[ \frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta f''(\zeta)}{g'(\zeta)} \right] + \frac{1}{2}(|\zeta^2 - 1|^2 \zeta h(\zeta) \left[ (S_f(\zeta) - S_g(\zeta)) \right] \leq 1
\]

for all $\zeta \in U^*$, then $f$ is univalent in $U^*$.

For $h(\zeta) = 1$ and $\alpha = \frac{1}{2}$ in Theorem 3.1 we obtain sufficient condition of Epstein type [3] on the exterior of the unit disk obtained earlier by Wesolowski [13].
Corollary 3.4. Let \( f, g \in \sum \) be locally univalent functions in \( \mathbb{U}^* \). If the following inequality
\[
\left| \frac{1}{2} \left( |\zeta|^2 - 1 \right)^2 \frac{\zeta}{\zeta} \left[ (S_f(\zeta) - S_g(\zeta)) \right] - \left( |\zeta|^2 - 1 \right) \frac{\zeta g''(\zeta)}{g'(\zeta)} \right| \leq 1
\]
is satisfied for all \( \zeta \in \mathbb{U}^* \), then \( f \) is univalent in \( \mathbb{U}^* \).

For \( f(\zeta) = g(\zeta) \), \( h(\zeta) = 1 \) and \( \alpha = \frac{1}{2} \) in Theorem 3.1, we obtain well-known Becker’s univalence criterion \([1]\) in \( \mathbb{U}^* \).

Corollary 3.5. Let \( f \in \sum \) be locally univalent function in \( \mathbb{U}^* \). If the following inequality
\[
\left| (|\zeta|^2 - 1) \left| \frac{\zeta f''(\zeta)}{f'(\zeta)} \right| \right| \leq 1
\]
is satisfied for all \( \zeta \in \mathbb{U}^* \), then \( f \) is univalent in \( \mathbb{U}^* \).

For \( g(\zeta) = \zeta \), \( h(\zeta) = 1 \) and \( \alpha = \frac{1}{2} \) in Theorem 3.1, we obtain Nehari type univalence criterion \([8]\) in \( \mathbb{U}^* \).

Corollary 3.6. Let \( f \in \sum \) be locally univalent function in \( \mathbb{U}^* \). If the following inequality
\[
|S_f(\zeta)| \leq \frac{2}{(|\zeta|^2 - 1)^2}
\]
is satisfied for all \( \zeta \in \mathbb{U}^* \), then \( f \) is univalent in \( \mathbb{U}^* \).

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