From the fuzzy disc to edge currents
in Chern-Simons Theory

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Abstract

We present a brief review of the fuzzy disc, the finite algebra approximating functions on a disc, which we have introduced earlier. We also present a comparison with recent papers of Balachandran, Gupta and Kürkçıoğlu, and of Pinzul and Stern, aimed at the discussion of edge states of a Chern-Simons theory.

1 Introduction

To be working at almost the same idea of another group of people is a fairly common occurrence. But the fact that the organizer of a conference to celebrate the sixtieth birthday of a colleague, and the person to be so honoured, discover this fact at the conference qualifies it as an unusual fact. Also unusual is the near triple coincidence of another talk presenting a similar model, somehow dual to the other. This is what happened at the conference whose proceedings are collected in this volume. Since our talk was based on ref. [1] which was posted shortly after the conference, at the same time as the work of Balachandran, Gupta and Kürkçıoğlu[2], rather than writing a faithful account of the talk, we present a comparison of the common elements of these two papers, and also discuss the similarities with the work of Pinzul and Stern[3] which was also presented at the conference.

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We will discuss fuzzy approximations to spaces with boundaries. The aim is the same as studies on the lattice: the substitution of a continuous theory with a model with a finite number of degrees of freedom, a matrix model. In the limit in which matrices become large the approximation improves. The main advantage of the fuzzy approximation is that the basic symmetries of the original space are preserved. The archetype of these is the fuzzy sphere. Along the same lines other fuzzy spaces have been built. The aim of the talk was to discuss the fuzzy approximation to the disc, which is presented in more detail in [1]. A fuzzy disc (presented in section 3) is a subalgebra of the noncommutative plane, which we present is section 2. We also show that a slight modification of the construction of the fuzzy disc can lead to a strip, as noted in ref. [2]. We also briefly discuss the model where, instead of a disc, a plane with a point missing at the origin is constructed. In section 4 we introduce derivations on the fuzzy disc, which enable the construction of Chern-Simons actions and show the presence of the edge states.

2 The Noncommutative Plane

The principle behind all fuzzy spaces is to consider a noncommutative geometry, that is a noncommutative algebra generalizing the algebra of functions on a topological space, and then a finite dimensional representation, or a finite dimensional subalgebra, of this noncommutative algebra.

For the construction at hand consider functions on $\mathbb{R}^2$, with coordinates $x$ and $y$ or $z = z + iy$ and $\bar{z} = x - iy$. Then quantize the plane associating operators to functions so to have $[\hat{x}, \hat{y}] = i\frac{\theta}{2}$. The quantized versions of $z$ and $\bar{z}$ are the usual annihilation and creation operators, with

$$[a, a^\dagger] = \theta. \quad (2.1)$$

The parameter $\theta$ has the dimension of a square length. It does not have necessarily a physical meaning, like the distance between sites in a lattice approximation.

The particular quantization we choose is a map $\Omega_\theta$ which to the function $\varphi(z, \bar{z})$ associates the operator $\hat{\varphi}$ as follows. Consider the Taylor expansion:

$$\varphi(\bar{z}, z) = \varphi_{Tay}^{mn} \bar{z}^m z^n, \quad (2.2)$$

to this function associate the operator

$$\Omega_\theta(\varphi) := \hat{\varphi} = \varphi_{Tay}^{mn} a^\dagger_m a^n. \quad (2.3)$$

The inverse map is constructed defining the coherent states $a |z\rangle = z |z\rangle$, then

$$\Omega_\theta^{-1}(\hat{\varphi}) = \varphi(\bar{z}, z) = \langle z | \hat{\varphi} | z \rangle. \quad (2.4)$$

There is another useful basis on which it is possible to represent the operators. Consider the number operator

$$N = a^\dagger a, \quad (2.5)$$
and its eigenvectors which we indicate\(^2\) by \(|n\rangle\): \(N|n\rangle = n\theta|n\rangle\). We can then express the operators with a density matrix notation:

\[
\hat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn} |m\rangle \langle n| .
\]

(2.6)

The elements of the density matrix basis have a very simple multiplication rule:

\[
|m\rangle \langle n| p \rangle \langle q| = \delta_{np} |m\rangle \langle q|. 
\]

(2.7)

The connection between the expansions (2.3) and (2.6) is given by:

\[
a = \sum_{n=0}^{\infty} \sqrt{(n+1) \theta} |n\rangle \langle n+1| ; \quad a^\dagger = \sum_{n=0}^{\infty} \sqrt{(n+1) \theta} |n+1\rangle \langle n| .
\]

(2.8)

Applying (2.4) to the operator \(\hat{\varphi}\) in the number basis we obtain for the function \(\varphi\) a new expression, in terms of new coefficients:

\[
\varphi(\bar{z}, z) = e^{-|z|^2} \sum_{m,n=0}^{\infty} \varphi_{mn} \frac{\bar{z}^m z^n}{\sqrt{n!m!\theta^{m+n}}} .
\]

(2.9)

The maps \(\Omega_\theta\) and \(\Omega^{-1}_\theta\) yield a procedure of going back and forth from functions to operators. Moreover, the product of operators being noncommutative, a noncommutative \(^\ast\) product between functions is implicitly defined as

\[
(\varphi \ast \varphi')(\bar{z}, z) = \Omega^{-1}_\theta (\Omega_\theta(\varphi) \Omega_\theta(\varphi')) .
\]

(2.10)

This product (which is a variation of the Moyal-Grönewold product) was first introduced by Voros [8]. We indicate the algebra of functions on the plane with this product as \(A_\theta\). In the density matrix basis, because of (2.7), the product (2.10) simplifies to an infinite row by column matrix multiplication:

\[
(\varphi \ast \varphi')_{mn} = \sum_{k=0}^{\infty} \varphi_{mk} \varphi'_{kn} .
\]

(2.11)

When \(\theta \to 0\), the \(^\ast\) product goes to the ordinary commutative product. It is easy to see that also \(\int d^2z \varphi(\bar{z}, z) = \pi \theta \text{Tr} \Phi = \pi \theta \sum_{n=0}^{\infty} \varphi_{nn}\), where we have introduced the matrix \(\Phi\) with components \(\varphi_{nn}\).

### 3 The fuzzy Disc

We now define subalgebras (with respect to the \(^\ast\) product) of finite \(N \times N\) matrices. They are the functions whose expansion (2.9) terminates when either \(n\) or \(m\) is larger than a

\(^3\)Since coherent states corresponding to the natural numerers never appear we hope there will be no confusion between coherent states and eigenvectors of \(N\).
given integer $N$. They can be obtained easily from the full algebra of functions via a projection:

$$P^N_\theta = \sum_{n=0}^{N} \langle z| n \rangle \langle n| z \rangle = \sum_{n=0}^{N} \frac{r^{2n}}{n!} e^{-\frac{r^2}{N \theta}},$$

(3.1)

where $z = re^{i\phi}$. In the limit $N \to \infty$ and $\theta \to 0$ with

$$R^2 \equiv N\theta$$

(3.2)

fixed, the sum converges to 1 if $r < R$, and converges to 0 otherwise. It has cylindrical symmetry. For $N$ finite the function vanishes exponentially for $r$ larger than $R$ (see figure 1) and approximates well the characteristic function of the disc. The function $P^N_\theta$ is a projector of the algebra of functions on the plane with the $*$ product:

$$P^N_\theta * P^N_\theta = P^N_\theta,$$

(3.3)

and the subalgebra $\mathcal{A}^N_\theta$ is defined as

$$\mathcal{A}^N_\theta = P^N_\theta * \mathcal{A}_\theta * P^N_\theta.$$

(3.4)

Cutting at a finite $N$ the expansion provides an infrared cutoff. The cutoff is fuzzy in the sense that functions in the subalgebra are still defined outside the cutoff, but are exponentially damped. Moreover if one tries to localize a function on a scale smaller than $\sqrt{\theta}$ then there is the appearance of large values of the function on the boundary, a compact geometry version of the ultraviolet–infrared mixing as discussed in ref. [1]. In figure 2 we show the effect of projecting the function $\cos(\alpha r)$, that is we plot $P^N_\theta * \cos(\alpha r) * P^N_\theta$ for various values of $\alpha$. We see that for large values of $\alpha$ the projection is just a cutoff, while for large $\alpha$ we see the presence of a large peak on the boundary of the disc. We call fuzzy disc the space corresponding to the algebra $\mathcal{A}^N_\theta$, which is isomorphic to the algebra of $N + 1 \times N + 1$ matrices. It is important to note that what makes it a disc is the way to take the correlated limit of $\theta$ and $N$ keeping the dimensionful quantity $R$ fixed.

Figure 1: The function $P^N_\theta$ for $N = 10^2$. 
Figure 2: Profile of the cylindrically function symmetric function $P^N_\theta \ast \cos(\alpha r) \ast P^N_\theta$ for the choice, $N = 10^2$ and for the values $\alpha = R, 4R, 30R$ compared with the unprojected function. Both functions are plotted, although inside the disc they are often indistinguishable.

The procedure we just outlined can be in fact opportunely doctored to obtain subalgebras which in the limit converge to functions corresponding to portions of the plane with different shapes. This can easily be obtained with just a redefinition of $z$ as

$$z = \alpha_N x + i\beta_N y$$

with $\bar{z}$ defined accordingly, and $\alpha_N$ and $\beta_N$ appropriate functions of $N$. In general the projector has support on an ellipse with semiaxes $\sqrt{N\theta}\alpha_N$ and $\sqrt{N\theta}\beta_N$. This shape itself can change with $N$ and give a different geometrical figure. The disc discussed earlier is for $\alpha_N = \beta_N = 1$ and $\theta = 1/N$. A Noncommutative disc would have $\alpha = \beta = 1/\sqrt{N}$ with $\theta$ fixed, while a noncommutative strip[2] is described by $\alpha = 1/\sqrt{N\theta}$ and $\beta = 1$, which give an elongated ellipse which in the limit becomes a strip. The choice $\alpha_N = \beta_N = N$ gives the plane, and it must be noted that of course those choices are not unique.

If we obtained the fuzzy disc subalgebra with the projector $P^N_\theta$, we could have equally well considered the projector $1 - P^N_\theta$. In this case we have operators of the kind:

$$\hat{\phi} = \sum_{m,n=N}^\infty \varphi_{mn} |m\rangle \langle n| .$$

This describes the plane with missing a central portion, which with an appropriate limit can be reduced to a point, and has been introduced by Pinzul and Stern[3] in a different way and well before refs. [1] and [2]. The algebra can still be generated by two generators $a_N$ and $a_N^\dagger$ defined as

$$a_N = \sum_{n=N}^\infty \sqrt{(n + 1)\theta} |n\rangle \langle n + 1| ; \quad a_N^\dagger = \sum_{n=N}^\infty \sqrt{(n + 1)\theta} |n + 1\rangle \langle n| .$$

but now the commutation relation (2.1) does not hold anymore, the commutator is not a constant, but it is a diagonal matrix, and hence it corresponds to a radially symmetric function.
4 Fuzzy Chern-Simons and Edge States

We want to discuss now the possibility to identify in our matrix approximation on the disc the edge states which play a crucial role in Chern-Simons theories. In order to do this we need a matrix equivalent of the derivatives. We define

\[ \partial_z \Phi = \frac{1}{\theta} [a^\dagger, \Phi] ; \quad \partial_{\bar{z}} \Phi = \frac{1}{\theta} [a, \Phi] . \]

(4.1)

Note that in the above expression, \( a \) and \( a^\dagger \) are still infinite matrices, and therefore if \( \Phi \) is an \( N \times N \) matrix, \( \partial \Phi \) is of rank \( N+1 \times N+1 \), this is crucial for the identification of the matrix algebra with a disc, and means that the derivatives live in an extended algebra, the extension is the range of the projector

\[ E_{N+1} = \langle N+1 | \langle N+1 | , \]

(4.2)

and heuristically means that if functions vanish on the boundary of a disc, their derivatives do not necessarily do so. They are edge states. In figure 3 edge states for \( N = 10 \) and \( N = 100 \) are shown.

The definition of a matrix model corresponding to a Chern-Simons theory is straightforward:

\[ S_{cs} = \int dx_0 \text{Tr} \varepsilon_{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma + \frac{2}{3} A_\mu A_\nu A_\sigma \]

(4.3)

where \( \mu\nu\sigma = 0, z, \bar{z} \), the \( A \)'s are elements of the matrix algebra, and time has not been quantized nor fuzzyfied. A canonical analysis of this model has been performed in [2] to which we refer for details. We limit ourselves to note that the gauge invariance of the fields:

\[ A_\mu \longrightarrow A_\mu + (\partial_\mu \Lambda + i[A_\mu, A]) \]

(4.4)

forces the \( A_z \) and \( A_{\bar{z}} \) (but not \( A_0 \)) to live on the range of \( P_\mu + E_{N+1} \), that is to contain edge states. In fact these edge states are the only degrees of freedom of the theory both in the continuum and the matrix cases.

Figure 3: The edge states \( \langle z | E_N | z \rangle \) for \( N = 10 \) and \( N = 100 \).
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