FOURIER-MUKAI TRANSFORMS AND STABLE SHEAVES
ON WEIERSTRASS ELLIPTIC SURFACES

WANMIN LIU, JASON LO, AND CRISTIAN MARTINEZ

ABSTRACT. On a Weierstraß elliptic surface $X$, we define a ‘limit’ of Bridgeland stability conditions, denoted as $Z^l$-stability, by varying the polarisation along a curve in the ample cone. We describe conditions under which a slope stable torsion-free sheaf is taken by a Fourier-Mukai transform to a $Z^l$-stable object, and describe a modification upon which a $Z^l$-semistable object is taken by the inverse Fourier-Mukai transform to a slope semistable torsion-free sheaf. We also study wall-crossing for Bridgeland stability, and show that 1-dimensional twisted Gieseker semistable sheaves are taken by a Fourier-Mukai transform to Bridgeland semistable objects.

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1. INTRODUCTION

Elliptic surfaces have been intensely studied over the years. On an elliptic surface, understanding the image of a stable sheaf under a Fourier-Mukai transform has been a major problem and considered by numerous authors in works such as [7, 6, 5, 29, 30, 31, 3], just to name a few. In this article, we give a fresh approach to this problem by interpreting the Fourier-Mukai transform of slope stability for sheaves as a ‘limit’ of Bridgeland stability.

More precisely, recall that the construction of Bridgeland stability conditions depends on the choice of a polarisation $\omega$. On a Weierstraß elliptic surface $X$, by varying the polarisation $\omega$ along a curve in the ample cone, we define a ‘limit’ of Bridgeland stability conditions, denoted as ‘$Z^l$-stability’ in the article. Our main theorem, Theorem 4.1, states that on such an $X$, if $E$ is a slope stable torsion-free sheaf of positive twisted degree or a slope stable locally free sheaf, then the Fourier-Mukai transform of $E$ is a $Z^l$-stable object; on the other
hand, if $F$ is a $Z^l$-semistable object of nonzero fiber degree, then $F$ has a modification $F'$ where the inverse Fourier-Mukai transform of $F'$ is a slope semistable torsion-free sheaf. A key feature of Theorem 4.1 is, that in addition to assuming the sheaf $E$ is torsion-free, we do not fix the Chern character of $E$. In other words, we aim to understand the action of the Fourier-Mukai transform on slope stability itself, rather than on a specific moduli space of slope stable sheaves (in the sense that one usually fixes a Chern character before studying a moduli space).

After setting up the preliminaries and introducing the cohomological Fourier-Mukai transforms in Section 2, we give the precise construction of $Z^l$-stability on a Weierstraß surface in Section 3. In Section 4 we prove our main result comparing slope stability and $Z^l$-stability (Theorem 4.1). The proof of the Harder-Narasimhan property of $Z^l$-stability is included in Section 5. At this point in the article, we begin fixing Chern characters and use the theory of $Z^l$-stability we have developed to study Fourier-Mukai transforms of stable sheaves. This comes down to studying wall-crossing for Bridgeland stability conditions, and we give two approaches of different flavours.

The first approach is contained in Section 6, where we consider walls given by Chern characters $c_1$ where $c_1$ is a positive multiple of the fiber class $f$ of the elliptic surface. When the elliptic surface has Picard rank two, we use Bogomolov inequalities to bound mini-walls on the curve along which $Z^l$-stability is defined. This shows that the moduli space of Bridgeland stability at the far end of this curve coincides with the moduli space of $Z^l$-stability. As a result, we obtain Corollary 6.14, which says that if $E$ is a 1-dimensional twisted Gieseker semistable sheaf, which has positive twisted Euler characteristic and positive fiber degree $f c_1$, then its Fourier-Mukai transform is a Bridgeland stable object with 2-dimensional support.

The second approach is contained in Sections 7 and 8. For this approach, we begin by studying the asymptotics of walls in Section 7. Then, in Section 8 we apply the computations to elliptic surfaces of Picard rank two with a strictly negative section. Combined with Arcara-Miles’ result on destabilising objects for line bundles, we obtain Proposition 8.1 which roughly says that if $E$ is a line bundle of fiber degree at least 2, then it is a Bridgeland stable object, and its inverse Fourier-Mukai transform is a slope semistable locally free sheaf.

Proposition 8.1 is similar to a result due to the second author and Zhang on some Weierstraß elliptic threefolds [20, Theorem 4.4], which says that if $L$ is a line bundle of nonzero fiber degree, then its Fourier-Mukai transform is a slope stable locally free sheaf. The argument for this threefold result, however, does not appear to reduce directly to the surface case.

The essential ideas in Sections 3 through 5 have also appeared in the second author’s preceding works on a product elliptic threefold [18] and Weierstraß elliptic threefolds over a Fano or numerically $K$-trivial base [17].

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2. Preliminaries

2.1. Our elliptic fibration. Throughout this article, unless otherwise stated, we will write $p : X \to B$ to denote an elliptic surface that is a Weierstraß fibration in the sense of [3] and [24, Definition (II.3.2)]. We do not place any restriction on the Picard rank of $X$ until the second half of the paper.

2.1.1. Elliptic surface. By an elliptic surface $p : X \to B$, we mean a flat morphism where $X$ is a smooth projective surface and $B$ is a smooth projective curve, and all the fibers are Gorenstein curves of arithmetic genus 1 [3, Definition 6.8].

2.1.2. Weierstraß elliptic surface. By a Weierstraß elliptic surface, or simply a Weierstraß surface, we will mean an elliptic surface $p : X \to B$ that is also a Weierstraß fibration in the sense of [3, Definition 6.10], which requires that all the fibers of $p$ are geometrically integral, and that $p$ admits a section $\sigma : B \to X$ whose image $\Theta = \sigma(B)$ does not meet any singular point of any fiber.

2.2. The Néron-Severi group $\text{NS}(X)$. Since our elliptic fibration $p$ is assumed to be Weierstraß, there exists a section, and hence the Picard rank of $X$ is finite by the Shioda-Tate formula [24, VII 2.4], while the Néron-Severi group $\text{NS}(X)$ is generated by the fiber class $f$ and a finite number of sections $\Theta_0 := \Theta, \Theta_1, \ldots, \Theta_r$ for some $r \geq 0$ [24, VII 2.1].

2.3. Geometry of $X$. The fundamental line bundle of $p : X \to B$ is defined to be the line bundle $L := (R^1 p_* \mathcal{O}_X)^{-1} \simeq p_* \omega_{X/B}$ [24, II 3.6]. We also set $K := c_1(p_* \omega_{X/B}) \simeq c_1(L)$ and write $e := \deg(L)$. Then $p^* K \equiv ef$ while we also know that $\deg L = -\Theta^2$ [9, Theorem 7.20]. Hence $\Theta^2 = -e$. Then by [24, Proposition (III 1.1)] and [3, (6.13)],

$$\omega_X \simeq p^*(\omega_B \otimes L) \simeq p^*(\omega_B) \otimes \omega_{X/B}.$$  \hspace{1cm} (2.3.1)

By the adjunction formula, we have $\Theta_i(\Theta + K_X) = 2g(B) - 2 = \Theta_i(\Theta_i + K_X)$ and hence $K_X \equiv (2g(B) - 2 + e)f$ and $\Theta_i^2 = \Theta^2 = -e$.

A classification of Weierstraß elliptic surfaces is contained in [24, Lemma (III 4.6)].

2.4. Notation. We collect here preliminary notions and notations that will be used throughout the article.

2.4.1. Twisted Chern character. For any divisor $B$ on a smooth projective surface $X$ and any $E \in D^b(X)$, the twisted Chern character $\text{ch}^B(E)$ is defined as

$$\text{ch}^B(E) = e^{-B} \text{ch}(E) = (1 - B + \frac{B^2}{2}) \text{ch}(E).$$

We write $\text{ch}^B(E) = \sum_{i=0}^2 \text{ch}_i^B(E)$ where

$$\text{ch}_0^B(E) = \text{ch}_0(E),$$
$$\text{ch}_1^B(E) = \text{ch}_1(E) - B \text{ch}_0(E),$$
$$\text{ch}_2^B(E) = \text{ch}_2(E) - B \text{ch}_1(E) + \frac{B^2}{2} \text{ch}_0(E).$$
We sometimes refer to the divisor $B$ involved in the twisting of the Chern character as the ‘$B$-field’. In this article, there should be no risk of confusion as to whether $B$ refers to the base of the elliptic fibration $p$ or a $B$-field.

2.4.2. Cohomology. Suppose $A$ is an abelian category and $B$ is the heart of a t-structure on $D^b(A)$. For any object $E \in D^b(A)$, we will write $H^i_B(E)$ to denote the $i$-th cohomology object of $E$ with respect to the t-structure with heart $B$. When $B = A$, i.e. when the aforementioned t-structure is the standard t-structure on $D^b(A)$, we will write $H^i(E)$ instead of $H^i_A(E)$.

Given a smooth projective variety $X$, the dimension of an object $E \in D^b(X)$ will be denoted by $\dim E$, and refers to the dimension of its support, i.e.

$$\dim E = \dim \bigcup_i \supp H^i(E).$$

For a coherent sheaf $E$, we have $\dim E = \dim \supp(E)$.

2.4.3. Torsion pairs and tilting. A torsion pair $(T, F)$ in an abelian category $A$ is a pair of full subcategories $T, F$ such that

(i) $\Hom_A(E', E'') = 0$ for all $E', E'' \in T, E'' \in F$.

(ii) Every object $E \in A$ fits in an $A$-short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

for some $E' \in T, E'' \in F$.

The decomposition of $E$ in (ii) is canonical [10, Chapter 1], and we will refer to it as the $(T, F)$-decomposition of $E$ in $A$. Whenever we have a torsion pair $(T, F)$ in an abelian category $A$, we will refer to $T$ (resp. $F$) as the torsion class (resp. torsion-free class) of the torsion pair. The extension closure in $D^b(A)$

$$A' = \langle F[1], T \rangle$$

is the heart of a t-structure on $D^b(A)$ and hence an abelian subcategory of $D^b(A)$. We call $A'$ the tilt of $A$ at the torsion pair $(T, F)$. More specifically, the category $A'$ is the heart of the t-structure $(D^{\leq 0}_{A'}, D^{\geq 0}_{A'})$ on $D^b(A)$ where

$$D^{\leq 0}_{A'} = \{ E \in D^b(A) : H^i_A(E) \in T, H^j_A(E) = 0 \forall i > 0 \},$$

$$D^{\geq 0}_{A'} = \{ E \in D^b(A) : H^{-i}_A(E) \in F, H^{-j}_A(E) = 0 \forall i < -1 \}.$$

A subcategory of $A$ will be called a torsion class (resp. torsion-free class) if it is the torsion class (resp. torsion-free class) in some torsion pair in $A$. By a lemma of Polishchuk [27, Lemma 1.1.3], if $A$ is a noetherian abelian category, then every subcategory that is closed under extension and quotient in $A$ is a torsion class in $A$.

For any subcategory $C$ of an abelian category $A$, we will set

$$C^\circ = \{ E \in A : \Hom_A(F, E) = 0 \text{ for all } F \in C \}$$

when $A$ is clear from the context. Note that whenever $A$ is noetherian and $C$ is closed under extension and quotient in $A$, the pair $(C, C^\circ)$ gives a torsion pair in $A$.

2.4.4. Torsion $n$-tuples. A torsion $n$-tuple $(C_1, C_2, \cdots, C_n)$ in an abelian category $A$ as defined in [26, Section 2.2] is a collection of full subcategories of $A$ such that

- $\Hom_A(C_i, C_j) = 0$ for any $C_i \in C_i, C_j \in C_j$ where $i < j$. 

• Every object $E$ of $\mathcal{A}$ admits a filtration in $\mathcal{A}$
  \[ 0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = E \]
where $E_i/E_{i-1} \in \mathcal{C}_i$ for each $1 \leq i \leq n$.
(See also [28, Definition 3.5].) Given a torsion $n$-tuple in $\mathcal{A}$ as above, the pair
  \[(\langle \mathcal{C}_1, \cdots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \cdots, \mathcal{C}_n \rangle)\]
is a torsion pair in $\mathcal{A}$ for any $1 \leq i \leq n - 1$.

2.4.5. Fourier-Mukai transforms. For any Weierstraß elliptic fibration $p : X \to B$ in the
sense of [3, Section 6.2], there is a pair of relative Fourier-Mukai transforms $\Phi, \hat{\Phi} : D^b(X) \sim \to
D^b(X)$ whose kernels are both sheaves on $X \times_B X$, satisfying
\[ \hat{\Phi} \Phi = \text{id}_{D^b(X)[−1]} = \Phi \hat{\Phi}. \]
In particular, the kernel of $\Phi$ is the relative Poincaré sheaf for the fibration $p$, which is a
universal sheaf for the moduli problem that parametrises degree-zero, rank-one torsion-free sheaves on the fibers of $p$. An object $E \in D^b(X)$ is said to be $Φ$-WIT, if $\Phi E$ is a coherent sheaf
sitting at degree $i$. In this case, we write $\hat{E}$ to denote a coherent sheaf satisfying $\Phi E \simeq \hat{E}[-i]
up to isomorphism. The notion of $\hat{Φ}$-WIT can similarly be defined. The identities (2.4.6)
imply that, if a coherent sheaf $E$ on $X$ is $Φ$-WIT, for $i = 0, 1$, then $\hat{E}$ is $\hat{Φ}$-WIT for $i = 0, 1$.

2.4.7. Subcategories of $\text{Coh}(X)$. Let $p : X \to B$ be an elliptic surface as in 2.1. For any integers $d \geq e$, we set
\[ \text{Coh}^{\leq d}(X) = \{ E \in \text{Coh}(X) : \dim \text{supp}(E) \leq d \} \]
\[ \text{Coh}^{d}(p)_e = \{ E \in \text{Coh}(X) : \dim \text{supp}(E) = d, \dim p(\text{supp}(E)) = e \} \]
\[ \{ \text{Coh}^{\leq 0}\}^{\uparrow} = \{ E \in \text{Coh}(X) : E|_b \in \text{Coh}^{\leq 0}(X_b) \text{ for all closed points } b \in B \} \]
where $\text{Coh}^{\leq 0}(X_b)$ is the category of coherent sheaves supported in dimension 0 on the fiber $p^{-1}(b) = X_b$, for the closed point $b \in B$. We will refer to coherent sheaves that are supported on a finite number of fibers of $p$ as fiber sheaves. Adopting the notation in [18, Section 3], we also define
\[ \text{WIT}_\infty := \text{Coh}^{\leq 0}(X) \]
\[ \text{WIT}_0 := \{ E \in \text{Coh}^1(\pi)_0 : \text{all } \mu\text{-HN factors of } E \text{ have } \infty > \mu > 0 \} \]
\[ \text{WIT}_1 := \{ E \in \text{Coh}^1(\pi)_0 : \text{all } \mu\text{-HN factors of } E \text{ have } \mu = 0 \} \]
\[ \text{WIT}_\mu := \{ E \in \text{Coh}^1(\pi)_0 : \text{all } \mu\text{-HN factors of } E \text{ have } \mu < 0 \} \]
\[ \text{WIT}_{\mu} := \text{Coh}^1(\pi)_1 \cap \{ \text{Coh}^{\leq 0}\}^{\uparrow} \]
\[ \text{WIT} = \{ E \in \text{WIT}_0, \hat{\Phi} : \dim E = 2 \} \]
\[ \text{WIT}_{\text{CH}_1} = \{ E \in \Phi(\{ \text{Coh}^{\leq 1}\} \uparrow \cap \text{Coh}^{\leq 1}(X)) : \dim E = 2 \} \]
\[ \text{WIT}_{\text{CH}_1} = \{ E \in \text{WIT}_1, \hat{\Phi} : \dim E = 2, \text{fch}_1(E) \neq 0 \}. \]
Note that the definitions of $\oplus$ and $\otimes$ depend on the Fourier-Mukai functor $\hat{\Phi}$. We will use the same notation to denote the corresponding category defined using $\hat{\Phi}$; it will always be clear from the context which Fourier-Mukai functor the definition is with respect to. The Fourier-Mukai transform $\Phi$ induces the following equivalences, as already observed in [18, Remark 3.1]:

A concatenation of more than one such diagram will mean the extension closure of the categories involved; for example, the concatenation

is the extension closure of all slope semistable fiber sheaves of slope at least zero (including sheaves supported in dimension zero, which are slope semistable fiber sheaves of slope $+\infty$).

The category $\text{Coh}^{\leq d}(X)$ for any integer $d \geq 0$, as well as $\{\text{Coh}^{\leq 0}\}$ and $W_{0, \hat{\Phi}}$ are all torsion classes in $\text{Coh}(X)$. From 2.4.3, each of these torsion classes determines a tilt of $\text{Coh}(X)$, and hence determines a t-structure on $D^b(X)$. For instance, we have the torsion pairs $(W_{0, \hat{\Phi}}, W_{1, \hat{\Phi}})$ and $(\text{Coh}^{\leq d}(X), \text{Coh}^{\geq d+1}(X))$ in $\text{Coh}(X)$.

2.4.8. Slope-like functions. Suppose $\mathcal{A}$ is an abelian category. We call a function $\mu$ on $\mathcal{A}$ a slope-like function if $\mu$ is defined by

$$
\mu(F) = \begin{cases} 
C_1(F) & \text{if } C_0(F) \neq 0 \\
C_0(F) & \text{if } C_0(F) = 0
\end{cases}
$$

where $C_0, C_1 : K(\mathcal{A}) \to \mathbb{Z}$ are a pair of group homomorphisms satisfying: (i) $C_0(F) \geq 0$ for any $F \in \mathcal{A}$; (ii) if $F \in \mathcal{A}$ satisfies $C_0(F) = 0$, then $C_1(F) \geq 0$. The additive group $\mathbb{Z}$ in the definition of a slope-like function can be replaced by any discrete additive subgroup of $\mathbb{R}$. Whenever $\mathcal{A}$ is a noetherian abelian category, every slope-like function possesses the Harder-Narasimhan property [21, Section 3.2]; we will then say an object $F \in \mathcal{A}$ is $\mu$-stable (resp. $\mu$-semistable) if, for every short exact sequence $0 \to M \to F \to N \to 0$ in $\mathcal{A}$ where $M, N \neq 0$, we have $\mu(M) < (\text{resp. } \leq) \mu(N)$.

2.4.9. Slope stability. Suppose $X$ is a smooth projective surface with a fixed ample divisor $\omega$ and a fixed divisor $B$. For any coherent sheaf $E$ on $X$, we define

$$
\mu_{\omega, B}(E) = \begin{cases} 
\omega \cdot \frac{b_1^B(E)}{ch_0^\omega(E)} & \text{if } ch_0^\omega(E) \neq 0 \\
+\infty & \text{if } ch_0^\omega(E) = 0
\end{cases}
$$

A coherent sheaf $E$ on $X$ is said to be $\mu_{\omega, B}$-stable or slope stable (resp. $\mu_{\omega, B}$-semistable or slope semistable) if, for every short exact sequence in $\text{Coh}(X)$ of the form

$$0 \to M \to E \to N \to 0$$
where \( M, N \neq 0 \), we have \( \mu_{\omega,B}(M) < (\text{resp.} \leq) \mu_{\omega,B}(N) \). Note that for any coherent sheaf \( M \) on \( X \) with \( \text{ch}_0(M) \neq 0 \), we have
\[
\mu_{\omega,B}(M) = \frac{\omega \text{ch}_1^B(M)}{\text{ch}_0(M)} = \frac{\omega \text{ch}_1(M) - \omega B \text{ch}_0(M)}{\text{ch}_0(M)} = \mu_\omega(M) - \omega B.
\]
Hence \( \mu_{\omega,B} \)-stability is equivalent to \( \mu_\omega \)-stability for coherent sheaves. When \( B = 0 \), we often write \( \mu_\omega \) for \( \mu_{\omega,B} \).

**2.4.10. Bridgeland stability conditions on surfaces.** Suppose \( X \) is a smooth projective surface. For any ample divisor \( \omega \) and another divisor \( B \) on \( X \), we can define the following subcategories of \( \text{Coh}(X) \)
\[
\mathcal{T}_{\omega,B} = \{ F \in \text{Coh}(X) : F \text{ is } \mu_{\omega,B}\text{-semistable, } \mu_{\omega,B}(F) > 0 \},
\]
\[
\mathcal{F}_{\omega,B} = \{ F \in \text{Coh}(X) : F \text{ is } \mu_{\omega,B}\text{-semistable, } \mu_{\omega,B}(F) \leq 0 \}.
\]
Since the slope function \( \mu_{\omega,B} \) has the Harder-Narasimhan property, the pair \( (\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B}) \) is a torsion pair in \( \text{Coh}(X) \). The extension closure
\[
\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle
\]
in \( D^b(X) \) is thus a tilt of the heart \( \text{Coh}(X) \), i.e. \( \mathcal{B}_{\omega,B} \) is the heart of a bounded t-structure on \( D^b(X) \) and is an abelian subcategory of \( D^b(X) \). If we set
\[
Z_{\omega,B}(F) = -\int_X e^{-i\omega \text{ch}_1^B(F)} = -\text{ch}_2^B(F) + \frac{\omega^2}{2} \text{ch}_0(F) + i\omega \text{ch}_1^B(F),
\]
then the pair \( (Z_{\omega,B}, \mathcal{B}_{\omega,B}) =: \sigma_{\omega,B} \) gives a Bridgeland stability condition on \( D^b(X) \), as shown by Arcara-Bertram in [1]. In particular, for any nonzero object \( F \in \mathcal{B}_{\omega,B} \), the complex number \( Z_{\omega,B}(F) \) lies in the upper-half complex plane (that includes the negative real axis)
\[
\mathbb{H} = \{ re^{i\phi} : r > 0, \phi \in (0, 1) \}.
\]
This allows us to define the phase \( \phi(F) \) of any nonzero object \( F \in \mathcal{B}_{\omega,B} \) using the relation
\[
Z_{\omega,B}(F) \in \mathbb{R}_{>0} e^{i\phi(F)} \quad \text{where } \phi(F) \in (0, 1].
\]
We then say an object \( F \in \mathcal{B}_{\omega,B} \) is \( Z_{\omega,B}\text{-stable} \) (resp. \( Z_{\omega,B}\text{-semistable} \)) if, for all \( \mathcal{B}_{\omega,B}\text{-short exact sequence}
\[
0 \to M \to F \to N \to 0
\]
where \( M, N \neq 0 \), we have \( \phi(M) < \phi(N) \) (resp. \( \phi(M) \leq \phi(N) \)). If \( B = 0 \), we write \( Z_\omega \) and \( \mathcal{B}_\omega \) instead of \( Z_{\omega,0} \) and \( \mathcal{B}_{\omega,0} \) respectively.

**2.5. The cohomological Fourier-Mukai transforms.** For any \( E \in D^b(X) \), if we let
\[
n = \text{ch}_0(E),
\]
\[
d = f \text{ch}_1(E),
\]
\[
c = \Theta \text{ch}_1(E),
\]
\[
s = \text{ch}_2(E),
\]
then from the cohomological Fourier-Mukai transform in [3] (6.21)] we have
\[
\text{ch}_0(\Phi E) = d,
\]
\[
\text{ch}_1(\Phi E) = -\text{ch}_1(E) + dp^* K + (d - n) \Theta + (c - \frac{1}{2}cd + s)f,
\]
\[
\text{ch}_2(\Phi E) = (-c - de + \frac{1}{2}ne)
\]
where \( \Theta^2 = -e \) and \( \bar{K} = c_1(p^*\omega_{X/B}) \) as in [2.3]. Since \( p^*\bar{K} \equiv ef \), we have \( \text{ch}_1(\Phi E)f = -n \) and \( \text{ch}_1(\Phi E).\Theta = (s - \frac{e}{2}d) + ne \). In particular, for any \( m \in \mathbb{R} \) we have
\[
\text{ch}_1(\Phi E)f = -n,
\]
(2.5.3)
\[
\text{ch}_1(\Phi E)(\Theta + mf) = s - \frac{e}{2}d + (e - m)n.
\]

On the other hand, from [3, (6.22)] we have
\[
\text{ch}_0(\Phi E) = d
\]
\[
\text{ch}_1(\Phi E) = \text{ch}_1(E) - np^*\bar{K} - (d + n)\Theta + (s + en - c - \frac{e}{2}d)f,
\]
(2.5.4)
\[
\text{ch}_2(\Phi E) = -(c + de + \frac{e}{2}n).
\]

This gives \( \text{ch}_1(\Phi E)f = -n \) and \( \text{ch}_1(\Phi E).\Theta = s + \frac{e}{2}d + ne \). In particular, for any \( m \in \mathbb{R} \) we have
\[
\text{ch}_1(\Phi E)f = -n,
\]
(2.5.5)
\[
\text{ch}_1(\Phi E)(\Theta + mf) = s + \frac{e}{2}d + (e - m)n.
\]

### 2.6. Some intersection numbers.

Here we collect some intersection numbers that will be used throughout the rest of the paper. For any \( m \in \mathbb{R} \) we have
\[
(\Theta + mf)^2 = \Theta^2 + 2m = 2m - e.
\]
Recall that for any section \( \Theta \) of the fibration \( p \), the divisor \( \Theta + mf \) on \( X \) is ample for \( m \gg 0 \) [12, Proposition 1.45]. We will often work with a polarisation of the form
\[
\omega = u(\Theta + mf) + vf
\]
for some \( u, v \in \mathbb{R} \), which gives
\[
\omega^2 = (m - \frac{e}{2})u^2 + uv.
\]
If we use the notation for \( \text{ch}(E) \) in (2.5.1) then \( (\Theta + mf)\text{ch}_1(E) = c + md \) and
\[
\omega\text{ch}_1(E) = (u(\Theta + mf) + vf)\text{ch}_1(E) = uc + (um + v)d.
\]

If we also set
\[
\omega = a(\Theta + mf) + bf,
\]
where \( a, b \in \mathbb{R} \) and fix \( B = \frac{e}{2}f \) then
\[
\omega\text{ch}_1^B(E) = \omega(\text{ch}_1(E) - \frac{e}{2}f\text{ch}_0(E)) = a(c - \frac{e}{2}n) + (am + b)d.
\]
Thus when \( \omega \) is an ample divisor on \( X \), we can write the twisted slope function \( \mu_{\omega,B} \) as
\[
\mu_{\omega,B}(E) = \frac{1}{\omega}(a(c - \frac{e}{2}n) + (am + b)d).
\]

On the other hand, when \( \omega \) is an ample divisor on \( X \), with respect to the central charge (2.4.11) and using (2.5.3) we have
\[
Z_\omega(\Phi E[1]) = \text{ch}_2(\Phi E) - \frac{\omega^2}{\bar{\omega}}\text{ch}_0(\Phi E) - i\omega\text{ch}_1(\Phi E)
\]
\[
= (-c - de + \frac{e}{2}n) - ((m - \frac{e}{2})u^2 + uv)d - i(u(s - \frac{e}{2}d + (e - m)n) - vn)
\]
(2.6.2)
\[
= (-c + \frac{e}{2}n) - ((m - \frac{e}{2})u^2 + uw + e)d + i(u(-(s - \frac{e}{2}d) + (m - e)n) + vn).
\]
2.7. Heuristics. Comparing the coefficients of the characteristic classes \((e - \frac{c}{2}n)\) and \(d\) in the expressions for \(\mu_{\omega,B}(E)\) and \(Z_\omega(\Phi E[1])\), we see that for fixed \(m, a, b > 0\), if \(v \to \infty\) along the curve
\[
\frac{am + b}{a} = (m - \frac{c}{2})u^2 + uv + e,
\]
i.e.
\[
m + \frac{b}{a} = (m - \frac{c}{2})u^2 + uv + e,
\]
then \(\overline{\omega}ch^B(E)\) is a negative scalar multiple of \(\Re Z_\omega(\Phi E[1])\), while \(\Im Z_\omega(\Phi E[1])\) is dominated by a positive scalar multiple of \(\text{ch}_0(E)\). This suggests that for \(v \gg 0\), \(\mu_{\omega,B}\)-stability for \(E\) should be an ‘approximation’ of \(Z_\omega\)-stability up to the Fourier-Mukai transform \(\Phi\), or that \(Z_\omega\)-stability is a ‘refinement’ of \(\mu_{\omega,B}\)-stability for \(E\) up to \(\Phi\). We will make this idea precise in Sections 2 through 5. The computation above also motivates us to consider the change of variables
\[
\beta = b, \quad \alpha = \frac{b}{a},
\]
so that \(\overline{\omega}\) can be written as

\[
(2.7.1) \quad \overline{\omega} = \frac{\beta}{\alpha}(\Theta + mf) + \beta f.
\]

Then \(\mu_{\omega,B}\)-stability depends only on \(\alpha\) and not \(\beta\), and we can think of \(\mu_{\omega,B}\)-stability as being approximated by \(Z_\omega\)-stability as \(v \to \infty\) along the curve

\[
(2.7.2) \quad m + \alpha = (m - \frac{c}{2})u^2 + uv + e.
\]

2.8. Decomposing \(\mu_\omega\). Suppose \(F\) is an object in \(D^b(X)\). With \(\omega\) as in (2.6.1), we can rewrite \(\mu_\omega(F)\) as

\[
\mu_\omega(F) = \frac{\omega ch_1(F)}{ch_0(F)} = \frac{u(\Theta + mf)ch_1(F)}{ch_0(F)} + v \frac{f ch_1(F)}{ch_0(F)}
\]

\[
(2.8.1) \quad = u\mu_{\Theta+mf}(F) + v\mu_f(F).
\]

Recall that the divisor \(\Theta + mf\) is ample on \(X\) for \(m \gg 0\) while \(f\) is a nef divisor on \(X\). Therefore, both \(\mu_{\Theta+mf}\) and \(\mu_f\) are ‘slope-like’ functions with the Harder-Narasimhan property (see 2.4.8).

2.9. For fixed \(\beta, \alpha > 0\), with \(\overline{\omega}\) as in (2.7.1), \(\omega\) as in (2.6.1), and \(u, v > 0\) under the constraint (2.7.2), we have the following observation that will be useful later on: with the same notation for \(\text{ch}(E)\) as in 2.5, for the \(B\)-field \(B = \frac{\beta}{2}f\) we have

\[
\overline{\omega}ch^B_1(E) = \overline{\omega}(ch_1(E) - Bch_0(E))
\]

\[
= \frac{\beta}{\alpha}((c - \frac{\beta}{2}n) + (m + \alpha)d)
\]

\[
= -\frac{\beta}{\alpha}\Re Z_\omega(\Phi E[1]).
\]

(2.9.1)

In particular, if \(F\) is a \(\widehat{\Phi}\)-WIT sheaf on \(X\) of nonzero rank with \(f ch_1(F) = 0\), then \(\widehat{F} = \Phi F[1]\) is a sheaf supported in dimension 1, implying \(\overline{\omega}ch^B_1(\widehat{F}) = \overline{\omega}ch_1(\widehat{F}) > 0\). Then

\[
\Re Z_\omega(F) = \Re Z_\omega(\Phi \widehat{F}) = -\Re Z_\omega(\Phi F[1]) = \frac{\beta}{\alpha}\Re Z_\omega(\Phi F[1]) = \frac{\beta}{\alpha}\overline{\omega}ch_1(\widehat{F}) > 0.
\]
TABLE 1. A summary of notations for ‘before’ and ‘after’ the autoequivalence $\Phi[1]$

| $E$ | $\frac{\Phi[1]}{\Phi}$ | $F = \Phi E[1]$ |
|-----|----------------|-----------------|
| $B$-field $B = \frac{\frac{\omega}{\alpha}}{2} f$ | $\omega = u(\Theta + mf) + vf$ | $\omega = tH_\lambda$ via (7.2.1) |
| $\omega = \frac{\omega}{\alpha}(\Theta + mf) + \beta f$ | $Z_\omega = Z_{\omega,0}$ as (2.4.11) | $Z_l$ as (3.5) |
| Curve (2.7.2) or (7.2.2) | Limit along curve as $v \to \infty$ or $\lambda \to 0^+$ | with asymptotic curve (7.2.3) |
| $\mu_{\omega,B}$ as (2.4.9) | $\mu_{\omega,B}$ as (2.4.9) | $\Phi(Coh(X))[1]$ |
| $\widehat{\Phi}B_l$ as (3.9) | $\widehat{\Phi}B_l$ as (3.2) | $\widehat{\Phi}(Coh(X))[1]$ |

3. CONSTRUCTING A LIMIT BRIDGELAND STABILITY

Since the Bridgeland stability condition $(B_\omega, Z_\omega)$ on $X$ depends on $\omega$, varying $\omega$ will change the stability condition accordingly (see 2.4.10). In this section, we will show that when $\omega$ is written in the form

$$\omega = u(\Theta + mf) + vf$$

and $v \to \infty$ subject to the constraint (2.7.2), we obtain a notion of stability with the Harder-Narasimhan property, which can be considered as a ‘limit Bridgeland stability’.

Due to the symmetry between $\Phi$ and $\hat{\Phi}$, all the results involving $\Phi$ and $\hat{\Phi}$ in this section and beyond still hold if we interchange $\Phi$ and $\hat{\Phi}$ (except for explicit computations involving Chern classes, since the cohomological Fourier-Mukai transforms corresponding to $\Phi$ and $\hat{\Phi}$ are different - see (2.5)).

For the rest of this article, let us fix an $m > 0$ so that $\Theta + kf$ is ample for all $k \geq m$. We will write $\omega$ in the form (2.6.1) with $u, v > 0$.

**Lemma 3.1.** Suppose $u_0 > 0$ and $F \in Coh(X)$.

(1) The following are equivalent:
   (a) There exists $v_0 > 0$ such that $F \in F_\omega$ for all $(v, u) \in (v_0, \infty) \times (0, u_0)$.
   (b) There exists $v_0 > 0$ such that, for every nonzero subsheaf $A \subseteq F$, we have $\mu_\omega(A) \leq 0$ for all $(v, u) \in (0, u_0) \times (v_0, \infty)$.
   (c) For every nonzero subsheaf $A \subseteq F$, either (i) $\mu_f(A) < 0$, or (ii) $\mu_f(A) = 0$ and also $\mu_{\Theta+mf}(A) \leq 0$.

(2) The following are equivalent:
   (a) There exists $v_0 > 0$ such that $F \in T_\omega$ for all $(v, u) \in (v_0, \infty) \times (0, u_0)$.
   (b) There exists $v_0 > 0$ such that, for every nonzero sheaf quotient $F \rightarrow A$, we have $\mu_\omega(A) > 0$ for all $(v, u) \in (0, u_0) \times (v_0, \infty)$.
   (c) For any nonzero sheaf quotient $F \rightarrow A$, either (i) $\mu_f(A) > 0$, or (ii) $\mu_f(A) = 0$ and $\mu_{\Theta+mf}(A) > 0$. 
**Proof.** The proofs for parts (1) and (2) are essentially the same as those for [18, Lemma 4.1] and [18, Lemma 4.3], respectively, if we replace the slope-like function $\mu^*$ in those proofs by $\mu_{\Theta + mf}$. ■

### 3.2. A limit of the heart $B_{\omega}$. We now define the following subcategories of $\text{Coh}(X)$:
- $\mathcal{T}^l$, the extension closure of all coherent sheaves satisfying condition (2)(c) in Lemma 3.1
- $\mathcal{F}^l$, the extension closure of all coherent sheaves satisfying condition (1)(c) in Lemma 3.1

We also define the extension closure in $D^b(X)$
$$B^l = \langle \mathcal{F}^l[1], \mathcal{T}^l \rangle.$$ Following an argument as in the proof of Lemma 3.1, it is easy to check that the categories $\mathcal{T}^l$, $\mathcal{F}^l$ can equivalently be defined as
$$\mathcal{T}^l = \{ F \in \text{Coh}(X) : F \in \mathcal{T}_{\omega} \text{ for all } v \gg 0 \text{ along } (2.7.2) \}$$
$$\mathcal{F}^l = \{ F \in \text{Coh}(X) : F \in \mathcal{F}_{\omega} \text{ for all } v \gg 0 \text{ along } (2.7.2) \}.$$

The following immediate properties are analogous to those in [18, Remark 4.4]:

(i) $\text{Coh}^{\leq 1}(X) \subset \mathcal{T}^l$ since all the torsion sheaves are contained in $\mathcal{T}_{\omega}$, for any ample divisor $\omega$.

(ii) $\mathcal{F}^l \subset \text{Coh}^{= 2}(X)$ since every object in $\mathcal{F}_{\omega}$ is a torsion-free sheaf, for any ample divisor $\omega$.

(iii) $W_{0,\tilde{\phi}} \subset \mathcal{T}^l$ by the same argument as in [18, Remark 4.4(iii)].

(iv) $fch_1(F) \geq 0$ for every $F \in B^l$. This is clear from the definition of $B^l$ and Lemma 3.1. Lemma 3.3 below shows that $B^l$ is the heart of a t-structure on $D^b(X)$, and hence an abelian category. The subcategory
$$B^l_0 := \{ F \in B^l : fch_1(F) = 0 \}$$
is then a Serre subcategory of $B^l$.

(v) $\mathcal{F}^l \subset W_{1,\tilde{\phi}}$. This follows from (iii) and Lemma 3.3 below.

**Lemma 3.3.** The pair $(\mathcal{T}^l, \mathcal{F}^l)$ forms a torsion pair in $\text{Coh}(X)$, and the category $B^l$ is the heart of a bounded t-structure on $D^b(X)$.

**Proof.** By [15, Lemma 2.5], we have
$$\begin{cases} fch_1(F) \geq 0 & \text{if } F \in W_{0,\tilde{\phi}} \\ fch_1(F) \leq 0 & \text{if } F \in W_{1,\tilde{\phi}} \end{cases}.$$ Armed with this observation, the argument in the proof of [18, Lemma 4.6] applies if we replace $\mu^*$ by $\mu_{\Theta + mf}$ in that proof. ■

**Lemma 3.4.** Fix any $\alpha > 0$. For any nonzero $F \in B^l$, we have $Z_{\omega}(F) \in \mathbb{H}$ as $v \to \infty$ along the curve (2.7.2).

**Proof.** Part of the proof of $(B_{\omega}, Z_{\omega})$ being a Bridgeland stability condition on $D^b(X)$ [11, Corollary 2.1] asserts that $Z_{\omega}(F) \in \mathbb{H}$ for any nonzero object $F \in B_{\omega}$. This lemma thus follows from the characterisations of $\mathcal{T}^l, \mathcal{F}^l$ in Lemma 3.1. ■
3.5. $Z^l$-stability, limit Bridgeland stability. We can now define a ‘limit Bridgeland stability’ as follows. By Lemma 3.4, for any nonzero object $F \in B^l$ we know that $Z_\omega(F)$ lies in the upper half plane $\mathbb{H}$ for $v \gg 0$ subject to the constraint (2.7.2), i.e.
\[ m + \alpha = (m - \frac{4}{l})u^2 + uv + e. \]

We can then define a function germ $\phi(F) : \mathbb{R} \to (0, 1]$ for $v \gg 0$ via the relation
\[ Z_\omega(F) \in \mathbb{R}_{>0} e^{i\phi(F)(v)} \quad \text{for} \quad v \gg 0. \]

Although $u$ is only an implicit function in $v$ under the constraint (2.7.2), by requiring $u > 0$ we can write $u$ as a function in $v$ for $v > 0$, in which case $O(u) = O(\frac{1}{v})$ as $v \to \infty$. As a result, as $v \to \infty$, the function $Z_\omega(F)$ is asymptotically equivalent to a Laurent polynomial in $v$ over $\mathbb{C}$, allowing us to define a notion of stability as in the case of Bayer’s polynomial stability [4]: We say $F$ is $Z^l$-stable (resp. $Z^l$-semistable) if, for every $B^l$-short exact sequence
\[ 0 \to M \to F \to N \to 0 \]
where $M, N \neq 0$, we have
\[ \phi(M) < \phi(N) \quad \text{for} \quad v \gg 0 \]
(resp. $\phi(M) \leq \phi(N)$ for $v \gg 0$). We will usually write $\phi(M) < \phi(N)$ (resp. $\phi(M) \leq \phi(N)$) to mean $\phi(M) < \phi(N)$ for $v \gg 0$ (resp. $\phi(M) \leq \phi(N)$ for $v \gg 0$).

Remark 3.6. If we make a change of variables via the ‘shear matrix’
\[ \begin{pmatrix} v' \\ u' \end{pmatrix} = \begin{pmatrix} 1 & m - \frac{4}{l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} \]
then the relation (2.7.2) can be rewritten as
\[ m + \alpha = u'v' + e \]
while $\omega$ can be rewritten as $\omega = u'(\Theta + \frac{4}{l}f) + v'f$. Then $Z_\omega(F)$ is a Laurent polynomial in $v'$, and $Z^l$-stability can equivalently be defined by letting $v' \to \infty$, in which case $Z^l$-stability is indeed a polynomial stability in the sense of Bayer. Nonetheless, we will use the coordinates $(v, u)$ instead of $(v', u')$ in the rest of this article.

3.7. Torsion triple and torsion quintuple in $B^l$. We now define the following subcategories of $\mathcal{T}^l, \mathcal{F}^l$
\[ \mathcal{T}^l_{-+} = \{ F \in \text{Coh}^{=2}(X) : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) > 0 \}, \]
\[ \mathcal{T}^l_{-0} = \{ F \in \mathcal{T}^l : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) = 0 \}, \]
\[ \mathcal{T}^l_{0-} = \{ F \in \mathcal{T}^l : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) = 0 \}, \]
\[ \mathcal{T}^l_{++} = \{ F \in \text{Coh}^{=2}(X) : F \text{ is } \mu_f\text{-semistable, } \mu_f(F) < 0 \}. \]

For the same reason as in [18, Remark 4.8(iii)], we have the inclusion $\mathcal{T}^l_{-0} \subset W_{1, \tilde{\Phi}}$. Since $W_{0, \tilde{\Phi}} \subset \mathcal{T}^l$ from (3.2)(iii), we have the torsion triple in $B^l$
\[ (\mathcal{F}^l[1], W_{0, \tilde{\Phi}}, W_{1, \tilde{\Phi}} \cap \mathcal{T}^l), \]
which is an analogue of [18, (4.12)]. Also, by considering the $\mu_f$-HN filtrations of objects in $\mathcal{F}^l$ and $\mathcal{T}^l$, we obtain the torsion quintuple in $B^l$
\[ (\mathcal{F}^l_{-0}[1], \mathcal{F}^l_{-}[1], \text{Coh}^{\leq 1}(X), \mathcal{T}^l_{-+}, \mathcal{T}^l_{-0}), \]
which is an analogue of [18, (4.13)].
3.8. The category $W_{1,\Phi} \cap T^l$. From the torsion quintuple \((3.7.2)\), we see that for every object $F \in W_{1,\Phi} \cap T^l$, the $T^{l,+}$-component must be zero, or else such a component would contribute a positive intersection number $f \mathrm{ch}_1$; this implies that $F$ has a two-step filtration $F_0 \subseteq F_1 = F$ in $\text{Coh}(X)$ where $F_0 \in W_{1,\Phi} \cap \text{Coh}^{\leq 1}(X)$ and is thus a $\Phi$-WIT fiber sheaf, while $F_1/F_0 \in T^{l,0}$. Since $f \mathrm{ch}_1$ is zero for both $F_0$ and $F_1/F_0$, the transform $\hat{\Phi}F[1]$ must be a torsion sheaf.

3.9. Transforms of torsion-free sheaves. The torsion triple \((3.7.1)\) in $B^l$ is taken by $\hat{\Phi}$ to the torsion triple

$$ (\hat{\Phi}F^l[1], W_{1,\Phi}, \hat{\Phi}(W_{1,\Phi} \cap T^l)) $$

in the abelian category $\hat{\Phi}B^l$. This implies that the heart $\hat{\Phi}B^l[1]$ is a tilt of $\text{Coh}(X)$ with respect to the torsion pair $(T, \mathcal{F})$ where

$$ T = \hat{\Phi}(W_{1,\Phi} \cap T^l)[1], $$

$$ \mathcal{F} = (\hat{\Phi}F^l[1], W_{1,\Phi}). $$

By \([3.8]\) we know $T \subseteq \text{Coh}^{\leq 1}(X)$. Consequently, for every torsion-free sheaf $E$ on $X$ we have $E \in \mathcal{F} \subset \hat{\Phi}B^l$, which implies $\Phi E[1] \in B^l$.

3.10. Phases of objects. We analyse the phases of various objects in $B^l$ with respect to $Z^l$-stability. Note that if $F \in D^b(X)$ satisfies

$$ \hat{n} = \mathrm{ch}_0(F), $$

$$ \hat{d} = f \mathrm{ch}_1(F), $$

$$ \hat{c} = \Theta \mathrm{ch}_1(F), $$

$$ \hat{s} = \mathrm{ch}_2(F) $$

(3.10.1)

then

$$ Z_\omega(F) = -\mathrm{ch}_2(F) + \frac{\omega^2}{2} \mathrm{ch}_0(F) + i \omega \mathrm{ch}_1(F) $$

$$ = -\hat{s} + ((m - \frac{\epsilon}{2})u^2 + uv) \hat{n} + i(u(\hat{c} + m\hat{d}) + v\hat{d}) $$

$$ = -\hat{s} + (\alpha + (m - \epsilon))\hat{n} + i(u(\hat{c} + m\hat{d}) + v\hat{d}) $$

under the constraint \((2.7.2)\).

Now further assume $F$ is a nonzero object of $B^l$. Consider the following scenarios:

1. $F \in \text{Coh}^{\leq 0}(X)$. Then $\mathrm{ch}_2(F) > 0$, and so $Z_\omega(F) \in \mathbb{R}_{<0}$, giving $\phi(F) = 1$.
2. $F \in \text{Coh}^{\leq 1}(X)$ and $\dim F = 1$. Then $\hat{n} = 0$. We have $\hat{d} = f \mathrm{ch}_1(F) \geq 0$ in this case.
   (2.1) If $\hat{d} > 0$, then $\phi(F) \to \frac{1}{2}$.
   (2.2) If $\hat{d} = 0$, then the effective divisor $\mathrm{ch}_1(F)$ is a positive multiple of the fiber class $f$, and so $((\Theta + mf)\mathrm{ch}_1(F) = \Theta \mathrm{ch}_1(F) = \hat{c} > 0$, i.e. $\exists Z_\omega(F) = u\hat{c} > 0$.
   (2.2.1) If $\hat{s} > 0$ then $\phi(F) \to 1$.
   (2.2.2) If $\hat{s} = 0$ then $\phi(F) = \frac{1}{2}$.
   (2.2.3) If $\hat{s} < 0$ then $\phi(F) \to 0$.
3. $F \in \text{Coh}^{=2}(X)$ and $f \mathrm{ch}_1(F) = \hat{d} > 0$. Then $\phi(F) \to \frac{1}{2}$.
4. $F \in T^{l,0}$. From the definition of $T^{l,0}$, we have $\hat{d} = f \mathrm{ch}_1(F) = 0$ while $(\Theta + mf)\mathrm{ch}_1(F) > 0$; we also know $F$ is $\Phi$-WIT from \([3.7]\). Thus $\hat{F} = \Phi F[1]$ is a sheaf of rank zero, and so $\omega \mathrm{ch}_1(\hat{F})$ must be strictly positive (if $\omega \mathrm{ch}_1(\hat{F}) = 0$, then $\hat{F}$ would be supported in dimension 0, implying $F$ itself is a fiber sheaf, a contradiction). Thus from the discussion in \([2.9]\) we know

$$ 0 < -\Re Z_\omega(\Phi \hat{F}[1]) = \Re Z_\omega(F) $$
and hence \( \phi(F) \to 0 \).

(5) \( F = A[1] \) where \( A \in \mathcal{F}^{l,0} \). Then \( f\text{ch}_1(A) = 0 \) and \( (\Theta + mf)\text{ch}_1(A) \leq 0 \). In this case, \( A \) is \( \hat{\Phi} \)-WIT by \( \text{(3.2)(v)} \). By a similar computation as in (4), we have
\[
0 < -\Re Z_\omega(\hat{\Phi}A[1]) = -\Re Z_\omega(A[1]) = -\Re Z_\omega(F)
\]
and so \( \phi(F) \to 1 \).

(6) \( F = A[1] \) where \( A \in \mathcal{F}^{l,-} \). Then \( f\text{ch}_1(A) < 0 \), i.e. \( f\text{ch}_1(F) > 0 \). Hence \( \phi(F) \to \frac{1}{2} \).

3.11. Summary. We summarise the constructions in this section in the following diagram, where a wave type arrow with a pair \((T, F)\) means that (i) such pair is a torsion pair in the source heart and (ii) the target heart is the tilt at such torsion pair, i.e. the target heart is \((\mathcal{F}^l, T)\).

\[
\begin{array}{c}
\text{Coh}(X) \xrightarrow{\hat{\Phi}[1]} \Phi(\text{Coh}(X))[1] \\
\downarrow \Phi \cong \downarrow \Phi \cong \downarrow \Phi \\
((\hat{\Phi}\mathcal{F}_{[1], W_{1,\hat{\phi}}}, \hat{\Phi}(W_{1,\hat{\phi}} \cap T^l)) \xrightarrow{(\mathcal{F}^l[1], W_{0,\hat{\phi}}, W_{1,\hat{\phi}} \cap T^l)} \Phi(\text{Coh}(X)) \xrightarrow{(T^l, \mathcal{F}^l)} \text{Coh}(X) \\
\hat{\Phi} B^l \xrightarrow{\hat{\Phi}[1]} \Phi \cong \Phi \cong \Phi \xrightarrow{\Phi[1]} B^l \\
\text{limit along curve } (2.7.2) \text{ as } v \to \infty \\
\text{or } (7.2.2) \text{ as } \lambda \to 0^+
\end{array}
\]

4. Slope stability vs limit Bridgeland stability

Given any torsion-free sheaf \( E \) on \( X \), we saw in \( \text{(3.9)} \) that \( \Phi E[1] \) lies in the heart \( B^l \). In this section, we establish a comparison between \( \mu_{\lambda} \)-stability on \( E \) and \( Z^l \)-stability on the shifted transform \( \Phi E[1] \) in the form of Theorem 4.1. This theorem is the surface analogue of \( \text{[18] Theorem 5.1} \):

**Theorem 4.1.** Let \( p : X \to B \) be a Weierstraß elliptic surface with base curve \( B \).

(A) Take \( B \)-field \( B = \frac{\mathcal{F}^l}{\mathcal{F}} \). Suppose \( E \) is a \( \mu_{\lambda} \)-stable torsion-free sheaf on \( X \).

(A1) If \( \exists \text{ch}_B^l(E) > 0 \), then \( \Phi E[1] \) is a \( Z^l \)-stable object in \( B^l \).

(A2) If \( \exists \text{ch}_B^l(E) = 0 \), then \( \Phi E[1] \) is a \( Z^l \)-semistable object in \( B^l \), and the only \( B^l \)-subobjects \( G \) of \( \Phi E[1] \) where \( \phi(G) = \phi(\Phi E[1]) \) are objects in \( \Phi(\text{Coh}^{\leq 0}(X)) \).

(A3) If \( E \) is locally free, then \( \Phi E[1] \) is a \( Z^l \)-stable object in \( B^l \).

(B) Suppose \( F \in B^l \) is a \( Z^l \)-semistable object with \( f\text{ch}_1(F) \neq 0 \), and \( F \) fits in the \( B^l \)-short exact sequence (which exists by \( \text{(3.7.1)} \))
\[
0 \to F' \to F \to F'' \to 0
\]
where \( F' \in \langle \mathcal{F}^l[1], W_{0,\hat{\phi}} \rangle \) and \( F'' \in \langle W_{1,\hat{\phi}} \cap T^l \rangle \). Then \( \hat{\Phi} F' \) is a \( \mu_{\lambda} \)-semistable torsion-free sheaf on \( X \).

Note that the objects of \( \Phi(\text{Coh}^{\leq 0}(X)) \) are precisely direct sums of semistable fiber sheaves of degree 0.

Even though the proof of Theorem 4.1 is analogous to that of \( \text{[18] Theorem 5.1(A)} \), we include most of the details for ease of reference, and also to lay out explicitly the necessary changes to the proof of \( \text{[18] Theorem 5.1} \).
Proof of Theorem 4.1(A). Let us write $F = \Phi E[1]$ throughout the proof. Since $\mathrm{rk}(E) \neq 0$, we have $\phi(F) \rightarrow \frac{1}{2}$. Take any $B'$-short exact sequence

\begin{equation}
0 \to G \to F \to F/G \to 0
\end{equation}

where $G \neq 0$. This yields a long exact sequence of sheaves

\begin{equation}
0 \to \hat{\Phi}(0)G \to E \xrightarrow{\omega} \hat{\Phi}(G) \to \hat{\Phi}(G)[2]
\end{equation}

and we see $\hat{\Phi}(G)[2] = 0$. From the torsion triple (3.7.1) in $B'$, we know $G$ fits in the exact triangle

\[ \Phi(\hat{\Phi}(G))[1] \to G \to \Phi(\hat{\Phi}(G)) \to \Phi(\hat{\Phi}(G))[2] \]

where $\Phi(\hat{\Phi}(G))[1] \in \langle F'[1], W_{\phi, \phi} \rangle$ is precisely the $\Phi$-WIT$_0$ component of $G$, and $\Phi(\hat{\Phi}(G)) \in W_{1, \hat{\Phi}} \cap T'$ the $\Phi$-WIT$_1$ component of $G$.

Suppose $\mathrm{rk}(\im \alpha) = 0$. Then $\mathrm{rk}(\hat{\Phi}(G)) = \mathrm{rk} E > 0$, and so $f \chi_1(\Phi(\hat{\Phi}(G)[1]) > 0$. Now we break into two cases:

(a) $\chi_1(\im \alpha) \neq 0$. Then $\mu_{\phi, B}(\hat{\Phi}(G)) < \mu_{\phi, B}(E)$, which implies $\phi(\Phi(\hat{\Phi}(G)[1]) < \phi(F)$.

(i) If $\dim \Phi(\hat{\Phi}(G)) = 2$: from [3.8] we know $\Phi(\hat{\Phi}(G))$ fits in a short exact sequence of sheaves

\begin{equation}
0 \to A' \to \Phi(\hat{\Phi}(G)) \to A'' \to 0
\end{equation}

where $A' \in W_{1, \hat{\Phi}} \cap \text{Coh}^{\leq 1}(X) \subset \text{Coh}(\pi)_{0}$ and $A'' \in T'_{0}$. Thus $f \chi_1(\Phi(\hat{\Phi}(G)) = 0$, and $Z_\omega(F)$ is dominated by its real part. From the computation in [2.9] we know $\Re Z_\omega(\Phi(\hat{\Phi}(G)) > 0$, and so $\phi(\Phi(\hat{\Phi}(G)) \to 0$, giving us $\phi(G) < \phi(F)$ overall.

(ii) If $\dim \Phi(\hat{\Phi}(G)) \leq 1$: then the component $A''$ in (i) vanishes, and $\Phi(\hat{\Phi}(G)) = A'$ is a $\Phi$-WIT$_1$ fiber sheaf. Then

\[ Z_\omega(\Phi(\hat{\Phi}(G)) = -\bar{s} + \bar{c}u \]

where $\bar{s} = \chi_2(A') \leq 0$ while $\bar{c} = \Theta \chi_1(A') \geq 0$.

If $\bar{s} < 0$, then again we have $\phi(G) < \phi(F)$. On the other hand, if $\bar{s} = 0$ then the order of magnitude of $Z_\omega(\Phi(\hat{\Phi}(G))$ as $v \to \infty$ is $O(\frac{1}{v})$, and so we still have $\phi(G) < \phi(F)$ overall.

(b) $\chi_1(\im \alpha) = 0$. Then $\im \alpha \in \text{Coh}^{\leq 0}(X)$, in which case $\chi_1(\Phi(\hat{\Phi}(G)) = \chi_i(E)$ for $i = 0, 1$.

From the cohomological Fourier-Mukai transform [2.5.2], it follows that $\chi_0, f \chi_1$ and $\chi_2$ of $\Phi(\hat{\Phi}(G)[1]$ and $F$ agree; from [2.6.3] we also see that all the terms of $Z_\omega(\Phi(\hat{\Phi}(G)[1]$ and $Z_\omega(F)$ agree except the terms involving $u$. As in (a)(i), we have a decomposition of $\Phi(\hat{\Phi}(G)$ of the form (4.1.3).

(i) If $\dim \Phi(\hat{\Phi}(G)) = 2$: then $A'' \neq 0$, and we have $\Re Z_\omega(A'') > 0$ by [2.9] while $\Im Z_\omega(A'')$ has order of magnitude $O(\frac{1}{v})$. On the other hand, $A'$ is a $\Phi$-WIT$_1$ fiber sheaf and so $\Re Z_\omega(A') \geq 0$ while $\Im Z_\omega(A')$ also has order of magnitude $O(\frac{1}{v})$. Overall, we have $\phi(G) < \phi(F)$.

(ii) If $\dim \Phi(\hat{\Phi}(G)) \leq 1$: then $A'' = 0$ and $\Phi(\hat{\Phi}(G)) = A'$ is a $\Phi$-WIT$_1$ fiber sheaf with $\chi_2(A') \leq 0$. With $\bar{s}, \bar{c}$ as in (a)(ii) above, we observe:

* If $\bar{s} < 0$, then $\Re Z_\omega(\Phi(\hat{\Phi}(G)) > 0$ while $\Im Z_\omega(\Phi(\hat{\Phi}(G))$ has magnitude $O(\frac{1}{v})$, giving us $\phi(G) < \phi(F)$ overall.
Thus let us assume \( \hat{\omega} \cap T \). If \( \hat{\omega} \cap T \) and so same argument as in part (a) above shows that \( \hat{\omega} \cap T \) is a fiber sheaf and \( A \) is a \( \hat{\omega} \) subobject, and from the torsion triple (3.7.1) in \( F/G \) \( \in B \). Of course, scenarios (S1) and (S2) above can be ruled out if we impose the vanishing \( \text{Hom}(\hat{\Phi}(\text{Coh}^{\leq 0}(X)), F) = 0 \), i.e. \( \text{Hom}(\hat{\Phi}(\text{Coh}^{\leq 0}(X)), F) = 0 \) for every \( Q \in \text{Coh}^{\leq 0}(X) \).

Note that for any \( Q \in \text{Coh}^{\leq 0}(X) \),

\[
\text{Hom}(\hat{\Phi}Q, F) = \text{Hom}(Q, \hat{\Phi}F[1]) = \text{Hom}(Q, E[1]) = \text{Ext}^1(Q, E).
\]

Hence \( \text{Hom}(\hat{\Phi}(\text{Coh}^{\leq 0}(X)), F) = 0 \) if and only if \( \text{Ext}^1(Q, E) = 0 \) for every \( Q \in \text{Coh}^{\leq 0}(X) \), which in turn is equivalent to \( E \) being a locally free sheaf by Lemma 4.2 below. This proves statement (A3), and completes the proof of part (A).

**Lemma 4.2.** Suppose \( E \) is a torsion-free sheaf \( E \) on a smooth projective surface \( X \). Then \( E \) is locally free if and only if \( \text{Ext}^1(T, E) = 0 \) for every \( T \in \text{Coh}^{\leq 0}(X) \).
Proof. Consider the short exact sequence of sheaves

$$0 \to E \to E^* \to Q \to 0$$

where $Q$ is necessarily a sheaf in $\text{Coh}^{\leq 0}(X)$. If $E$ is not locally free, then $Q \neq 0$ and we have $\text{Ext}^1(Q, E) \neq 0$. On the other hand, if $E$ is locally free then for any $T \in \text{Coh}^{\leq 0}(X)$ we have $\text{Ext}^1(T, E) \cong \text{Ext}^1(E, T \otimes \omega_X) \cong H^1(X, E^* \otimes T) = 0$.

Proof of Theorem 4.1(B). Let $F', F, F''$ be as in the statement of the theorem. We begin by showing that $\hat{\Phi}F'$ is a torsion-free sheaf, i.e. $\text{Hom}(\text{Coh}^{\leq 1}(X), \hat{\Phi}F') = 0$, i.e.

$$(4.2.1) \quad \text{Hom}(\hat{\Phi}\text{Coh}^{\leq 1}(X)[1], F') = 0.$$

Proceeding as in the proof of [18, Lemma 5.8], we observe

$$\hat{\Phi}\text{Coh}^{\leq 1}(X)[1] \subset \langle \{ E \in W_{\hat{\Phi}} : f\text{ch}_1(E) = 0 \}, [\text{Coh}^{\leq 0}(X)[-1]] \rangle [1]$$

$$\subset \langle \text{Coh}(X)[1], \text{Coh}^{\leq 0}(X) \rangle$$

$$\subset \langle B'[1], B' \rangle.$$

Therefore, in order to prove the vanishing (4.2.1), it suffices to show the following two things:

(i) For any $G \in W_{\hat{\Phi}}$ with $f\text{ch}_1(G) = 0$, we have $\text{Hom}(B'[1], \text{Coh}^{\leq 0}(X)[1]) = 0$.

(ii) $\text{Hom}(\langle B'[1], \text{Coh}^{\leq 0}(X) \rangle, F') = 0$.

For (i), let us consider the $(T^l, F^l)$-decomposition of $G$ in $\text{Coh}(X)$

$$0 \to G' \to G \to G'' \to 0.$$

This shows $\mathcal{H}_{B'}^0(G[1]) = G''[1]$. Since $G$ is a $\hat{\Phi}$-WIT$_1$ sheaf, so is its subsheaf $G'$; thus $G' \in W_{\hat{\Phi}} \cap T^l$, and from [3.8] we have $f\text{ch}_1(G') = 0$. Since $f\text{ch}_1(G) = 0$, we also have $f\text{ch}_1(G'') = 0$. By considering the $\mu_f$-HN filtration of $G''$, we obtain $G'' \in F^l, 0$.

For any $B'$-morphism $\alpha : G''[1] \to F'$ and with $A_1$ defined as in (4.2.2) below, we now have $\text{im} \alpha \in A_1$ and $\phi(\text{im} \alpha) \to 1$ by Lemma 4.3 below. However, this gives a composition of $B'$-injections

$$\text{im} \alpha \to F' \to F.$$

Hence $\alpha$ must be zero, or else $F$ would be destabilised, proving (i). A similar argument as above proves (ii). Hence $\hat{\Phi}F'$ is a torsion-free sheaf on $X$.

Next, we show that $\hat{\Phi}F'$ is $\mu_{\text{HIT}}$-semistable. Take any short exact sequence of coherent sheaves on $X$

$$0 \to B \to \hat{\Phi}F' \to C \to 0$$

where $B, C$ are both torsion-free sheaves. Then $\hat{\Phi}[1]$ takes this short exact sequence to a $B'$-short exact sequence

$$0 \to \hat{\Phi}B[1] \to F' \to \hat{\Phi}C[1] \to 0$$

by [3.9] The $Z^l$-semistability of $F$ gives $\phi(\hat{\Phi}B[1]) \preceq \phi(F)$, which implies $\mu_{\text{HIT}}(B) \leq \mu_{\text{HIT}}(\hat{\Phi}F)$. On the other hand, since $F''$ is precisely the $\hat{\Phi}$-WIT$_1$ component of $H^0(F)$, by Lemma 4.4 below we have $F'' \in \hat{\Phi}\text{Coh}^{\leq 0}(X)$, i.e. $\hat{\Phi}F'' \in \text{Coh}^{\leq 0}(X)[-1]$. This gives

$$\mu_{\text{HIT}}(\hat{\Phi}F') = \mu_{\text{HIT}}(\hat{\Phi}F) \geq \mu_{\text{HIT}}(B).$$

Hence $\hat{\Phi}F'$ is a $\mu_{\text{HIT}}$-semistable torsion-free sheaf. ■
Let us define
\[(4.2.2) \quad A_1 = \langle \text{Coh}^{\leq 0}(X), \mathbb{F}^{l,0}[1] \rangle.\]

**Lemma 4.3.** The category $A_1$ is closed under quotient in $B^l$, and every object in this category satisfies $\phi \to 1$.

**Proof.** The second part of the lemma follows from the computations in [3.10]. For the first part, take any $A \in A_1$ and consider any $B^l$-short exact sequence of the form
\[0 \to A' \to A \to A'' \to 0.\]
We need to show that $A'' \in A_1$. Recall that $B^{l}_{0} = \{ F \in B^l : f \text{ch}_1(F) = 0 \}$ is a Serre subcategory of $B^l$; also note that $A_1$ is contained in $B^{l}_{0}$. Hence $A''$ lies in $B^{l}_{0}$, meaning $H^{-1}(A'') \in \mathbb{F}^{l,0}[1]$. On the other hand, since $H^0(A) \in \langle \text{Coh}^{\leq 0}(X) \rangle$ from the definition of $A_1$, we also have $H^0(A'') \in \langle \text{Coh}^{\leq 0}(X) \rangle$. Thus $A'' \in A_1$, and we are done. ◼

**Lemma 4.4.** Suppose $F \in B^l$ is a $Z^l$-semistable object with $f \text{ch}_1(F) \neq 0$. Then the $\tilde{\Phi}$-WIT$_1$ component of $H^0(F)$ lies in $\Phi \text{Coh}^{\leq 0}(X)$.

**Proof.** Let $G$ denote the $\tilde{\Phi}$-WIT$_1$ component of $H^0(F)$. With respect to the torsion triple $(3.7.1)$ in $B^l$, this is precisely the $W_{1,\tilde{\Phi}} \cap T^l$ component of $F$. Hence by [3.8] $G$ has a two-step filtration $G_0 \subseteq G_1 = G$ in $\text{Coh}(X)$ such that $G_1/G_0 \in T^{l,0}$ and $G_0$ is a $\tilde{\Phi}$-WIT$_1$ fiber sheaf (and so $\text{ch}_2(G_0) \leq 0$). Now we have a composition of $B^l$-surjections
\[F \to G \to G_1/G_0\]
with $\phi(F) \to 1/2$ while $\phi(G_1/G_0) \to 0$ from [3.10](4). Since $F$ is assumed to be $Z^l$-semistable, this forces $G_1/G_0 = 0$, and so $G = G_0$.

Suppose now that $\tilde{c} = \Theta \text{ch}_1(G)$ and $\tilde{s} = \text{ch}_2(G)$. Then
\[Z_\omega(G) = -\tilde{s} + ic\tilde{c}.\]
By the $Z^l$-semistability of $F$, the fiber sheaf $G$ cannot have any quotient sheaf with $\text{ch}_2 < 0$ (such a quotient would have $\phi \to 0$ by [3.10](2.2.3), destabilising $F$). Hence $G$ is a slope semistable fiber sheaf with $\text{ch}_2 = 0$, implying $G \in \Phi \text{Coh}^{\leq 0}(X)$ [3 Proposition 6.38]. ◼

5. The Harder-Narasimhan property of limit Bridgeland stability

To establish the Harder-Narasimhan property of $Z^l$-stability, we follow the line of thought in [18] Section 6] and begin by constructing a torsion triple in $B^l$ that separates objects of distinct phases. Recall the definition (4.2.2)
\[A_1 = \langle \text{Coh}^{\leq 0}(X), \mathbb{F}^{l,0}[1] \rangle.\]

**Lemma 5.1.** The category $A_1$ is a torsion class in $B^l$.

**Proof.** We already showed in Lemma [4.3] that $A_1$ is closed under quotient in $B^l$. It remains to show that every object $F \in B^l$ is the extension of an object in $A_1^l$ by an object in $A_1$.

For any $F \in B^l$, consider the $B^l$-short exact sequence
\[0 \to G[1] \to F \to F' \to 0\]
where $G[1]$ is the $\mathbb{F}^{l,0}[1]$-component of $F$ with respect to the torsion quintuple [3.7.2], equivalently, $G$ is the $\mathbb{F}^{l,0}$-component of $H^{-1}(F)$. Note that $\text{Hom}(\mathbb{F}^{l,0}[1], F') = 0$ by construction.
Suppose $F' \notin A_{1}^{0}$. Then there exists a nonzero morphism $\beta : U \to F'$ where $U \in A_{1}$. Since $A_{1}$ is closed under quotient in $B^l$, we can replace $U$ by $\text{im} \beta$ and assume $\beta$ is a $B^l$-injection. The vanishing $\text{Hom}(F^{l,0}[1], F') = 0$ then implies $H^{-1}(U) = 0$ and so $U = H^{0}(U) \in \langle \text{Coh}^{\leq 0}(X), \mathbb{P} \rangle$.

Suppose we have an ascending chain in $B^l$
$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_m \subseteq \cdots \subseteq F'$$
where $U_i \in \langle \text{Coh}^{\leq 0}(X), \mathbb{P} \rangle$ for all $i$. This induces an ascending chain of coherent sheaves
$$\widehat{\Phi}^0 U_1 \subseteq \widehat{\Phi}^0 U_2 \subseteq \cdots \subseteq \widehat{\Phi}^0 F'.$$
Thus the $U_i$ must stabilise, i.e. there exists a maximal $B^l$-subobject $U$ of $F'$ lying in the extension closure $\langle \text{Coh}^{\leq 0}(X), \mathbb{P} \rangle$. Applying the octahedral axiom to the $B^l$-surjections $F \twoheadrightarrow F' \twoheadrightarrow F'/U$ gives the diagram

\[
\begin{array}{c}
\text{G[2]} \\
\downarrow \downarrow
\end{array} 
\begin{array}{c}
F' \\
\downarrow
\end{array} 
\begin{array}{c}
M[1] \\
\downarrow
\end{array}
\begin{array}{c}
\text{F'/U} \\
\downarrow
\end{array} 
\begin{array}{c}
\text{U[1]} \\
\uparrow
\end{array} 
\begin{array}{c}
F \\
\uparrow
\end{array}
\end{array}
\]

in which every straight line is an exact triangle, and for some $M \in B^l$. The vertical exact triangle gives $H^{-1}(M) \cong G$ and $H^{0}(M) \cong U$, and so $M \in A_1$. A similar argument as in the proof of [18, Lemma 6.1(b)] then shows that $F'/U \in A_{0}^{0}$, thus finishing the proof.

We now define
$$A_{1,1/2} := \langle A_1, F^{l,-}[1], \mathbb{P}, \mathbb{P}, \mathbb{P}, \mathbb{P}, \mathbb{P} \rangle$$
$$= \langle F^{l}[1], \mathbb{P}, \mathbb{P}, \mathbb{P}, \mathbb{P}, \mathbb{P} \rangle \cdot \mathbb{P}.$$ 

(5.1.1)

**Lemma 5.2.** $A_{1,1/2}$ is a torsion class in $B^l$.

**Proof.** For the purpose of this proof, let us write
$$\mathcal{E} = \mathbb{P}, \mathbb{P}, \mathbb{P}, \mathbb{P}, \mathbb{P}. \mathbb{P}.$$ 

(Recall that concatenation of 2 by 2 boxes of the form $\mathbb{P}$ means their extension closure.) It is easy to check that $\mathcal{E}$ is a torsion class in $\text{Coh}(X)$ and that
$$\mathcal{E} = \{ H^{0}(F) : F \in A_{1,1/2} \}.$$ 

The same argument as in [18] Lemma 6.2 then shows that every object in $B^l$ can be written as the extension of an object in $\mathcal{E}$ by an object in $A_{1,1/2}$, proving the lemma.
Now that we know $A_1, A_{1,1/2}$ are both torsion classes in $B^l$ with the inclusion $A_1 \subseteq A_{1,1/2}$, we can construct the torsion triple in $B^l$

\[ (A_1, A_{1,1/2} \cap A_0, A_{1,1/2}). \]

We have the following finiteness properties for the components of this torsion triple:

**Proposition 5.3.** The following finiteness properties hold:

1. For $A = A_1$:
   a. There is no infinite sequence of strict monomorphisms in $A$
   b. There is no infinite sequence of strict epimorphisms in $A$

2. For $A = A_{1,1/2} \cap A_0^e$:
   a. There is no infinite sequence of strict monomorphisms \((5.3.1)\) in $A$.
   b. There is no infinite sequence of strict epimorphisms \((5.3.2)\) in $A$.

3. For $A = A_{1,1/2}^0$:
   a. There is no infinite sequence of strict monomorphisms \((5.3.1)\) in $A$.
   b. There is no infinite sequence of strict epimorphisms \((5.3.2)\) in $A$.

Even though the proof of this proposition is modelled after that of [18, Proposition 5.3], we lay out the details for clarity and ease of reference. For instance, since the total space of our elliptic surface $X$ does not necessarily have Picard rank 2 as in [18], the strategy of using the positivity of certain intersection numbers needs to be adjusted carefully.

**Proof.** In proving (1)(a), (2)(a) and (3)(a), we will consider the $B^l$-short exact sequences

\[ 0 \to E_{i+1} \to E_i \to G_i \to 0. \]  

On the other hand, in proving (1)(b), (2)(b) and (3)(b), we will consider the $B^l$-short exact sequences

\[ 0 \to K_i \to E_i \to E_{i+1} \to 0. \]

Since $f \text{ch}_1 \geq 0$ on $B^l$ from §5.2(iv), we know $f \text{ch}_1(E_i)$ is a decreasing sequence of nonnegative integers when proving any of the six cases of this proposition. Therefore, by omitting a finite number of terms in the sequence $E_i$ if necessary, we can always assume that the $f \text{ch}_1(E_i)$ are constant. This also implies that $f \text{ch}_1(G_i) = 0$ and $f \text{ch}_1(K_i) = 0$ for all $i$, which in turn implies $f \text{ch}_1(H^j(G_i)) = 0$ and $f \text{ch}_1(H^j(K_i)) = 0$ for all $i, j$.

Throughout the proof, we will also fix an $m > 0$ such that $\Theta + mf$ is an ample divisor on $X$.

1. (a): For any object $A \in \mathcal{F}^{l,0}[1]$, we know $f \text{ch}_1(A) = 0$ and $(\Theta + mf)\text{ch}_1(A) = \Theta \text{ch}_1(A) \geq 0$ by the definition of $\mathcal{F}^{l,0}$. In addition, any $A \in \langle \text{Coh}^{\leq 0}(X), G \rangle$ is a fiber sheaf and satisfies $\Theta \text{ch}_1(A) \geq 0$. Thus $\text{ch}_0(A) \geq 0$ on $A_1$, and by omitting a finite number of terms if necessary, we can assume that $\Theta \text{ch}_1(E_i)$ is constant and $\Theta \text{ch}_1(G_i) = 0$ for all $i$. Similarly, we can assume that $\text{ch}_0(E_i)$ is constant and $\text{ch}_0(G_i) = 0$ for all $i$.

   That $\text{ch}_0(G_i) = 0$ implies $G_i = H^0(G_i)$, and so $G_i$ is a fiber sheaf. That $\Theta \text{ch}_1(G_i) = 0$ then implies $G_i$ must be supported in dimension 0.

   The long exact sequence of cohomology from \((5.3.3)\) now looks like
   \[ 0 \to H^{-1}(E_{i+1}) \to H^{-1}(E_i) \to 0 \to H^0(E_{i+1}) \to H^0(E_i) \to H^0(G_i) \to 0, \]
from which we see the $H^{-1}(E_i)$ stabilise. From the definition of $A_1$, we also know that $c_2(H^0(A)) \geq 0$ for any $A \in A_1$. Thus $c_2(H^0(E_i))$ eventually stabilises. This then forces $c_2(H^0(G_i)) = 0$, in which case $G_i = H^0(G_i) = 0$, i.e. the sequence $E_i$ itself stabilises.

(1)(b): from the long exact sequence of cohomology of (5.3.4), the $H^0(E_i)$ must eventually stabilise since $\text{Coh}(X)$ is noetherian. Hence we can assume the $H^0(E_i)$ are constant. The remainder of the long exact sequence reads

$$0 \to H^{-1}(K_i) \to H^{-1}(E_i) \to H^{-1}(E_{i+1}) \to H^0(K_i) \to 0.$$ 

Since $ch_0 \leq 0$ on $A_1$, the sequence $ch_0(H^{-1}(E_i))$ eventually stabilises, so we can assume that $ch_0(H^{-1}(K_i)) = 0$ for all $i$ (noting $ch_0(H^0(K_i)) = 0$), i.e. $H^{-1}(K_i) = 0$, i.e. $K_i = H^0(K_i)$ is a fiber sheaf for all $i$.

As in (1)(a), we know $\Theta ch_1 \geq 0$ on $A_1$. Hence $\Theta ch_1(E_i)$ eventually stabilises, giving $\Theta ch_1(K_i) = 0$; since $K_i$ is a fiber sheaf, this forces $K_i$ to be supported in dimension $0$. The exact sequence above then gives

$$H^{-1}(E_i) \hookrightarrow H^{-1}(E_{i+1}) \twoheadrightarrow H^{-1}(E_{i+1})^\ast$$

where $H^{-1}(E_{i+1})^\ast$ is independent of $i$ for $i \gg 0$ since $H^0(K_i) \in \text{Coh}^{<0}(X)$. Thus the $H^{-1}(E_i)$ also stabilise, and the $E_i$ themselves stabilise.

(2)(a): Recall from (5.1.1) that

$$A_{1,1/2} = (A_1, \mathcal{F}^l, \mathcal{T}, \mathcal{E}) = (\mathcal{F}^l, \mathcal{T}, \mathcal{E}).$$

Since we can assume $f ch_1(H^{-1}(G_i)) = 0$ and $f ch_1(H^0(G_i)) = 0$, we have $H^{-1}(G_i) \in \mathcal{F}^l,0$ and know that $H^0(G_i)$ cannot have any subfactors in $\mathcal{G}$ or $\mathcal{G}$ since $\beta_i$ is a strict morphism in $A$, we have $G_i \in A$ and so $\text{Hom}(\mathcal{F}^l,0[1], G_i) = 0$, i.e. $H^{-1}(G_i) = 0$. This leaves $G_i \in (\mathcal{G}, \mathcal{G}, \mathcal{G})$, which means that $G_i$ is a fiber sheaf where all the HN factors with respect to the slope function $c_2/Dch_1$ (for any ample divisor $D$ on $X$) have $c_2 \geq 0$. Again by $\text{Hom}(A_1, G_i) = 0$, we have $G_i \in \mathcal{G}$.

From the long exact sequence of cohomology of (5.3.3), we know the $H^{-1}(E_i)$ are constant and

$$0 \to H^0(E_{i+1}) \to H^0(E_i) \to H^0(G_i) \to 0$$

is exact. Applying the Fourier-Mukai functor $\widehat{\phi}$, we obtain the long exact sequence of sheaves

$$0 \to \widehat{\phi}^0(H^0(E_{i+1})) \to \widehat{\phi}^0(H^0(E_i)) \to 0 \to \widehat{\phi}^1(H^0(E_{i+1})) \to \widehat{\phi}^1(H^0(E_i)) \to \widehat{\phi}^1(H^0(G_i)) \to 0.$$ 

According to Lemma 5.4 below, $\widehat{\phi}^1(H^0(E_i)) \in \text{Coh}^{<0}(X)$ for all $i$. Hence the cohomologies $\widehat{\phi}^0(H^0(E_i)), \widehat{\phi}^1(H^0(E_i))$ both stabilise for $i \gg 0$, i.e. $H^0(E_i)$ themselves stabilise. Overall, the $E_i$ stabilise.

(2)(b): As in case (1)(b), we can assume the $H^0(E_i)$ are constant and that the $f ch_1(E_i)$ are constant. The argument for describing $G_i$ in (2)(a) applies to $K_i$ here, allowing us to conclude $H^{-1}(K_i) = 0$ and $K_i = H^0(K_i) \in \mathcal{G}$. The first half of the long exact sequence of cohomology of (5.3.4) now reads

$$0 \to H^{-1}(E_i) \to H^{-1}(E_{i+1}) \to H^0(K_i) \to 0,$$

where all the terms are $\widehat{\phi}$-WITT sheaves. The Fourier-Mukai functor $\widehat{\phi}$ then takes it to a short exact sequence of sheaves

$$0 \to \widehat{\phi}^{-1}(E_i) \to \widehat{\phi}^{-1}(E_{i+1}) \to \widehat{\phi}^0(K_i) \to 0.$$
where \( H^0(K_i) \in \text{Coh}^{\leq 0}(X) \). By Lemma 5.5 below, each \( \hat{H}^{-1}(E_i) \) is a torsion-free sheaf. Hence we have the inclusions

\[
H^{-1}(E_i) \hookrightarrow \hat{H}^{-1}(E_i) \hookrightarrow \hat{H}^{-1}(E_i+1) \]

where \( (\hat{H}^{-1}(E_i+1))^{**} \) is independent of \( i \). Thus the \( H^{-1}(E_i) \) must stabilise, and so the \( E_i \) themselves stabilise.

(3)(a): Since \( \mathcal{F}[1] \) is contained in \( \mathcal{A}_{1,1/2} \), any object \( M \in \mathcal{A}_{1,1/2} \) must have \( H^{-1}(M) = 0 \), i.e. \( M = H^0(M) \). Also, since we have the inclusion \( W_{0,\hat{\Phi}} \subset \mathcal{A}_{1,1/2} \), it follows that

\[
\mathcal{A}_{1,1/2} \cap \text{Coh}(X) \subset W_{1,\hat{\Phi}} \cap \mathcal{T}.
\]

Hence \( E_i, G_i \) lie in \( W_{1,\hat{\Phi}} \cap \mathcal{T} \) for all \( i \). Then \( ch_0(E_i) \geq 0 \) for all \( i \), and we can assume \( ch_0(E_i) \) is constant while \( ch_0(G_i) = 0 \) for all \( i \) by omitting a finite number of terms. By 3.8, we know each \( G_i \) is a \( \hat{\Phi} \)-WIT fiber sheaf. Since \( \mathcal{F} \subset \mathcal{A}_{1,1/2} \), we have \( G_i \in \mathcal{F} \). The \( \mathcal{B}^l \)-short exact sequence (5.3.3) is then taken by \( \hat{\Phi}[1] \) to a short exact sequence in \( \text{Coh}^{\leq 1}(X) \)

\[
0 \to \hat{E}_{i+1} \to \hat{E}_i \to \hat{G}_i \to 0.
\]

For any ample divisor on \( X \) of the form \( \omega' = \Theta + kf \) where \( k \) is a positive integer, we see that \( \omega' ch_1(\hat{E}_i) \) is a decreasing sequence of nonnegative integers, and so must become stationary, in which case the fiber sheaf \( \hat{G}_i \) must be supported in dimension 0. This implies, however, that \( \hat{G}_i \in \text{Coh}^{\leq 0}(X) \), forcing \( G_i = 0 \), i.e. the \( E_i \) eventually stabilise.

(3)(b): As in (3)(a), the objects \( E_i, K_i \) lie in \( W_{1,\hat{\Phi}} \cap \mathcal{T} \) for all \( i \), so (5.3.4) is a short exact sequence of sheaves. Since \( \text{Coh}(X) \) is noetherian, the \( E_i \) eventually stabilise.  

Lemma 5.4. Let \( A \in \mathcal{A}_{1,1/2} \), and let \( A_1 \) denote the \( \hat{\Phi} \)-WIT component of \( H^0(A) \). Then \( \hat{\Phi}A_1[1] \in \text{Coh}^{\leq 0}(X) \).

Proof. For objects \( M \in \mathcal{B}^l \), the property

\[
\hat{\Phi}^1 M \in \text{Coh}^{\leq 0}(X)
\]

is preserved under extension in \( \mathcal{B}^l \). Since this property is satisfied for all objects in the categories that generate \( \mathcal{A}_{1,1/2} \), it is satisfied for all objects in \( \mathcal{A}_{1,1/2} \).

Lemma 5.5. Suppose \( E \in \mathcal{A}_1^0 = \{ E \in \mathcal{B}^l : \text{Hom}(A_1, E) = 0 \} \). Then \( H^{-1}(E) \) is locally free and \( H^{-1}(E) \) is torsion-free.

Proof. consider the exact sequence

\[
0 \to H^{-1}(E) \to H^{-1}(E)^{**} \to Q \to 0
\]

where \( Q \) is some coherent sheaf supported in dimension 0; this gives a \( \mathcal{B}^l \)-short exact sequence

\[
0 \to Q \to H^{-1}(E)[1] \to H^{-1}(E)^{**}[1] \to 0.
\]

Since \( E \in \mathcal{A}_1^0 \), the term \( Q \) must be zero, i.e. \( H^{-1}(E) \) is locally free.

Recall from 3.2(v) that \( H^{-1}(E) \) is \( \hat{\Phi} \)-WIT. Also, that \( E \in \mathcal{A}_1^0 \) implies Hom(\( \mathcal{F}^l,0[1], E) = 0 \), and so \( H^{-1}(E) \in \mathcal{F}^{l,-} \).

Suppose \( H^{-1}(E) \) has a subsheaf \( T \) that lies in \( \text{Coh}^{\leq 1}(X) \). Let \( T_i \) denote the \( \Phi \)-WIT component of \( T \). The composite \( T_0 \hookrightarrow T \hookrightarrow H^{-1}(E) \) in \( \text{Coh}(X) \) is then taken by \( \Phi \) to an injection
of sheaves $\tilde{T}_0 \hookrightarrow H^{-1}(E)$. Thus $\tilde{T}_0$ is a torsion-free sheaf on $X$ and lies in $\mathcal{F}^{l,-}$ since $H^{-1}(E)$ is so. However, since $\mathrm{ch}_0(\tilde{T}_0) = 0$, we must have $f \mathrm{ch}_1(\tilde{T}_0) = 0$. This forces $\tilde{T}_0$ and hence $T_0$ itself to be zero, i.e. $T$ is a $\Phi$-WIT$_1$ fiber sheaf. The inclusion $T \hookrightarrow H^{-1}(E)$ then corresponds to an element in

$$\mathrm{Hom}(T, H^{-1}(E)) \cong \mathrm{Hom}(\tilde{T}[-1], H^{-1}(E)) \cong \mathrm{Hom}(\hat{T}, H^{-1}(E)[1])$$

where $\hat{T} = \Phi T[1]$. Note that $\hat{T}$ is a $\Phi$-WIT$_0$ fiber sheaf, and so is an object in $\mathcal{A}_1$. Since $H^{-1}(E)[1]$ is a $B^1$-subobject of $E$, which lies in $\mathcal{A}_0^0$, $H^{-1}(E)[1]$ itself lies in $\mathcal{A}_0^0$, which means the injection $T \hookrightarrow H^{-1}(E)$ must be the zero map, i.e. $T = 0$. This proves that $H^{-1}(E)$ is torsion-free.

Let us now set

$$\mathcal{A}_{1/2} := \mathcal{A}_{1,1/2} \cap \mathcal{A}_1^0,$$

$$\mathcal{A}_0 := \mathcal{A}_{1,1/2}^0,$$

so that the torsion triple (5.2.1) can be rewritten as

(5.5.1) \hspace{1cm} ($\mathcal{A}_1, \mathcal{A}_{1/2}, \mathcal{A}_0$).

The following is an analogue of [18, Lemma 6.5]:

**Lemma 5.6.** For $i = 1, 1/2, 0$ and any $F \in \mathcal{A}_i$, we have $\phi(F) \rightarrow i$.

**Proof.** The case of $i = 1$ follows from the definition of $\mathcal{A}_1$ and the computation in 3.10.

For $i = 1/2$: take any $F \in \mathcal{A}_{1/2}$. If $f \mathrm{ch}_1(F) > 0$, then clearly $\phi(F) \rightarrow 1/2$ and we are done. Let us assume $f \mathrm{ch}_1(F) = 0$ from now on. Then $f \mathrm{ch}_1(H^{-1}(F)) = 0$, meaning $H^{-1}(F) \in \mathcal{F}^{l,0}$; however, $F \in \mathcal{A}_1^0$ and so $H^{-1}(F)$ must be zero, i.e. $F = H^0(F)$.

That $F \in \mathcal{A}_{1,1/2} \cap \mathrm{Coh}(X)$ with $f \mathrm{ch}_1(F) = 0$ implies $F$ cannot have any subfactors in $\begin{array}{c} \square \\ \cdot \cdot \cdot \\ \square \end{array}$ or $\begin{array}{c} \square \\ \cdot \cdot \cdot \\ \square \end{array}$. Hence $F$ is a fiber sheaf where all the HN factors with respect to slope stability have ch$_2 \geq 0$. That $F \in \mathcal{A}_1^0$ then forces $F \in \begin{array}{c} \square \\ \cdot \cdot \cdot \\ \square \end{array}$ giving us $\phi(F) = 1/2$ by 3.10(2.2.2).

For $i = 0$: take any $F \in \mathcal{A}_0$. From (5.3.5) we know $F \in W_{1,\Phi} \cap T^l$. By 3.8 we have a two-step filtration $F_0 \subseteq F_1 = F$ in $\mathrm{Coh}(X)$ where $F_0$ is a $\Phi$-WIT$_1$ fiber sheaf while $F_1/F_0 \in T^{l,0}$. From 3.10(4) we know $\phi(F_1/F_0) \rightarrow 0$, so it suffices to show $\phi(F_0) \rightarrow 0$. Since $F \in \mathcal{A}_{1,1/2}^0$, we have $\mathrm{Hom}(\begin{array}{c} \square \\ \cdot \cdot \cdot \\ \square \end{array}, F_0) = 0$, implying $F_0 \in \begin{array}{c} \square \\ \cdot \cdot \cdot \\ \square \end{array}$. By 3.10(2.2.3) we have $\phi(F_0) \rightarrow 0$ as desired. $\blacksquare$

**Lemma 5.7.** An object $F \in \mathcal{B}^l$ is $Z^l$-semistable iff, for some $i = 1, 1/2, 0$, we have:

- $F \in \mathcal{A}_i$;
- for any strict monomorphism $0 \neq F' \hookrightarrow F$ in $\mathcal{A}_i$, we have $\phi(F') \leq \phi(F)$.

**Proof.** Given Lemma 5.6 the argument in the proof of [18, Lemma 6.6] applies. $\blacksquare$

**Theorem 5.8.** The Harder-Narasimhan property holds for $Z^l$-stability on $\mathcal{B}^l$. That is, every object $F \in \mathcal{B}^l$ admits a filtration in $\mathcal{B}^l$

$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = F$

where each $F_i/F_{i-1}$ is $Z^l$-semistable, and $\phi(F_i/F_{i-1}) > \phi(F_{i+1}/F_i)$ for each $i$.

**Proof.** Using the torsion triple (5.5.1), the finiteness properties in Proposition 5.3 along with Lemma 5.7 the argument in the proof of [18, Theorem 6.7] applies. $\blacksquare$
6. Transforms of 1-dimensional sheaves

In this section, we study the stability of the Fourier-Mukai transforms of 1-dimensional sheaves. Heuristically, we will need to impose some type of stability on our 1-dimensional sheaves to deduce the $\mathbb{Z}^l$-stability of their transforms as in Section 4. Luckily, we have another type of stability at our disposal, $\mathbb{Z}_f$-$\Phi$-semistability. Since the Bridgeland slope function for 1-dimensional sheaves becomes

$$\frac{\text{ch}_2 - \frac{\chi}{\beta}\text{ch}_1 \cdot f}{\beta \text{ch}_1 \cdot \bar{\omega}};$$

where $\bar{\omega} = \frac{1}{\alpha}(mf + \Theta) + f$, then this type of stability when tested only on 1-dimensional sheaves does not depend on $\beta$. If $\text{ch}$ is the Chern character of a 1-dimensional sheaf then by [19, Theorem 1.1] we know that the only Bridgeland semistable objects with Chern character $\text{ch}$ for $\beta \gg 0$ are 1-dimensional sheaves and moreover the condition for semistability only needs to be checked on subsheaves. The following definition is in place:

**Definition 6.1.** Consider the $\mathbb{Q}$-line bundle $L = p^*(\omega_B)/2$. We say a pure 1-dimensional sheaf $E$ in $\text{Coh}(X)$ is $L$-twisted $\bar{\omega}$-Gieseker semistable, or simply twisted Gieseker semistable, if for every subsheaf $\mathcal{E} \hookrightarrow E$ we have

$$\frac{\chi_L(A)}{\text{ch}_1(A) \cdot \bar{\omega}} \leq \frac{\chi_L(E)}{\text{ch}_1(E) \cdot \bar{\omega}},$$

where the $L$-twisted Euler characteristic is defined by

$$\chi_L(E) := \chi(E \otimes L) = \text{ch}_2(E) - \frac{e}{2}\text{ch}_1(E) \cdot f + \text{ch}_0(E)\chi(O_X)$$

for every $E \in \text{Coh}(X)$.

**Proposition 6.2.** Let $E$ be a twisted Gieseker semistable 1-dimensional sheaf with $\chi_L(E) \geq 0$ and $\text{ch}_1(E) \cdot f > 0$. Then $E$ is $\Phi$-WIT. Moreover, $\Phi(E)$ is torsion-free for $m + \alpha \gg 0$.

**Proof.** Since $W_{0,\Phi}$ is closed under extensions then by using the Jordan-Holder filtration of $E$ with respect to twisted Gieseker semistability we may assume that $E$ is twisted Gieseker stable.

Using that $(W_{0,\Phi}, W_{1,\Phi})$ is a torsion pair in $\text{Coh}(X)$ we can write a short exact sequence

$$0 \to E_0 \to E \to E_1 \to 0$$

where $E_i$ is a $\Phi$-WIT, sheaf for $i = 0, 1$.

Notice that $E_1 \in \text{Coh}^{=1}(X)$ and so $\text{ch}_1(E_1) \cdot f = 0$ by [6, Lemma 6.3]. This implies that $E_1$ is either 0 or $\text{ch}_1(E_1) = \alpha'f$ for some $\alpha' > 0$ since otherwise $E_1$ would be 0-dimensional and therefore $\Phi$-WIT.

Assume that $E_1 \neq 0$. Since $\text{ch}_1(E_1) \cdot f = 0$ then $\Phi(E_1)[1] \in \text{Coh}^{\leq 1}(X)$ is a sheaf supported on fibers and therefore

$$\text{ch}_1(\Phi(E_1)[1]) \cdot \Theta = -\text{ch}_2(E_1) \geq 0.$$  

The twisted Gieseker stability of $E$ implies that

$$\frac{\chi_L(E_0)}{\text{ch}_1(E_0) \cdot \bar{\omega}} \leq \frac{\chi_L(E)}{\text{ch}_1(E) \cdot \bar{\omega}} = \frac{\chi_L(E_0) + \text{ch}_2(E_1)}{\text{ch}_1(E) \cdot \bar{\omega}}$$

and therefore

$$\chi_L(E_0)(\text{ch}_1(E_1) \cdot \bar{\omega}) < \text{ch}_2(E_1)(\text{ch}_1(E_0) \cdot \bar{\omega}),$$

a contradiction since $\chi_L(E_0) \geq 0$. Thus $E_1 = 0$ and $E$ is $\Phi$-WIT.
Now, suppose that \( \Phi(\mathcal{E}) \) is not torsion-free and let \( T \) be its torsion subsheaf so that we have a short exact sequence
\[
0 \to T \to \Phi(\mathcal{E}) \to F \to 0
\]
in \( \text{Coh}(X) \). Applying \( \Phi[1] \) we obtain the distinguished triangle
\[
\Phi(T)[1] \to \mathcal{E} \to \Phi(F)[1] \to \Phi(T)[2].
\]
Since \( \mathcal{E} \) is a sheaf then \( T \) is \( \Phi \)-WIT\(_1\) and so 1-dimensional. Moreover, \( T \) must be supported on fibers, i.e.,
\[
\text{ch}_1(T) = af, \quad \text{for some } a > 0.
\]
Consider the morphism \( g: \Phi(T)[1] \to \mathcal{E} \). The subsheaf \( \text{Im}(g) \subseteq \mathcal{E} \) is also 1-dimensional and supported on fibers, i.e.,
\[
\text{ch}_1(\text{Im}(g)) = rf \quad \text{with} \quad r > 0.
\]
Since \( \Phi(T)[1] \) is \( \Phi \)-WIT\(_0\) then so is \( \text{Im}(g) \). A simple cohomology computation then shows that \( \Phi(\text{Im}(g)) \) is a subsheaf of \( \Phi(\mathcal{E}) \) and so must be 1-dimensional and supported on fibers, i.e.,
\[
\text{ch}_1(\Phi(\text{Im}(g))) \cdot \Theta = \text{ch}_2(\text{Im}(g)) > 0.
\]
Now, from the twisted Gieseker semistability of \( \mathcal{E} \) it follows that
\[
\frac{\chi_L(\text{Im}(g))}{\text{ch}_1(\text{Im}(g)) \cdot \overline{w}} = \frac{\alpha \text{ch}_2(\text{Im}(g))}{\beta r} \leq \frac{\chi_L(\mathcal{E})}{\text{ch}_1(\mathcal{E}) \cdot \overline{w}}.
\]
Fix \( m_0 > 0 \) such that \( m_0 f + \Theta \) is in the boundary of the nef cone, then \( r \leq \text{ch}_1(\mathcal{E}) \cdot \Theta + m_0 \text{ch}_1(\mathcal{E}) \cdot f \) and so
\[
\frac{\alpha}{\beta \text{ch}_1(\mathcal{E}) \cdot (m_0 f + \Theta)} \leq \frac{\chi_L(\mathcal{E})}{\text{ch}_1(\mathcal{E}) \cdot \overline{w}},
\]
which is impossible if
\[
\text{ch}_1(\mathcal{E}) \cdot \overline{w} > \frac{\chi_L(\mathcal{E}) \text{ch}_1(\mathcal{E}) \cdot (m_0 f + \Theta)}{\alpha}.
\]
This last inequality is equivalent to
\[
m + \alpha > \frac{\text{ch}_1(\mathcal{E}) \cdot \Theta}{\text{ch}_1(\mathcal{E}) \cdot f} (\chi_L(\mathcal{E}) - 1) + m_0 \cdot \chi_L(\mathcal{E}).
\]
\[\Box\]

**Remark 6.3.** Let \( \mathcal{E} \) be a twisted Gieseker semistable 1-dimensional sheaf with \( \chi_L(\mathcal{E}) \geq 0 \) and \( \text{ch}_1(\mathcal{E}) \cdot f > 0 \). Notice that if \( m + \alpha \gg 0 \) then the torsion-free sheaf \( \Phi(\mathcal{E}) \) is \( \mu_f \)-semistable. Indeed, if
\[
0 \to E'' \to \Phi(\mathcal{E}) \to E' \to 0
\]
is a short exact sequence in \( \text{Coh}(X) \) then \( E'' \) is \( \Phi \)-WIT\(_1\) since \( \Phi(\mathcal{E}) \) is \( \Phi \)-WIT\(_1\). Therefore, by [G, Lemma 6.2] \( \mu_f(E'') \leq 0 \).

**Proposition 6.4.** Let \( \mathcal{E} \) be a twisted Gieseker semistable 1-dimensional sheaf with \( \chi_L(\mathcal{E}) \geq 0 \) and \( \text{ch}_1(\mathcal{E}) \cdot f > 0 \), and assume that \( m + \alpha \gg 0 \). Then \( \Phi(\mathcal{E}) \in \mathcal{T}^l \).

**Proof.** Assume for the moment that \( \mathcal{E} \) is stable. Since \( \Phi(\mathcal{E}) \) is \( \mu_f \)-semistable then by Lemma 3.1 we only need to prove that for every short exact sequence
\[
0 \to E'' \to \Phi(\mathcal{E}) \to E' \to 0
\]
in \( \text{Coh}(X) \) with \( \mu_f(E'') = \mu_f(E') = 0 \) we have \( \text{ch}_1(E') \cdot \Theta > 0 \).
If \( E' \) is \( \Phi \)-WIT\(_1 \) then \( \Phi(E')[1] \) is a quotient of \( E \) and therefore
\[
\frac{\chi_L(\Phi(E')[1])}{ch_1(\Phi(E')[1])} > \frac{\chi_L(E)}{ch_1(E)} \geq 0.
\]
This implies that
\[
\chi_L(\Phi(E')[1]) = -ch_2(\Phi(E')) + \frac{e}{2}ch_1(\Phi(E')) \cdot f
\]
\[
= ch_1(E') \cdot \Theta + \frac{e}{2}ch_0(E') + \frac{e}{2}(-ch_0(E'))
\]
\[
= ch_1(E') \cdot \Theta > 0.
\]
If \( E' \) is not \( \Phi \)-WIT\(_1 \) then we know that there is a short exact sequence in Coh(\( X \))
\[
0 \to E_0 \to E' \to E_1 \to 0
\]
with \( E_i \) a \( \Phi \)-WIT\(_i \), sheaf for \( i = 0, 1 \). Thus \( ch_1(E_1) \cdot \Theta > 0 \).

From the \( \mu_f \)-semistability of \( \Phi(E) \) we know that \( E' \) is also \( \mu_f \)-semistable and so by [6, Lemma 6.2] we conclude that \( ch_1(E_0) \cdot f = ch_1(E_1) \cdot f = 0 \). Thus \( ch_0(\Phi(E_0)) = 0 \) and so \( ch_1(\Phi(E_0)) \) is effective. This implies that
\[
ch_1(\Phi(E_0)) \cdot f = -ch_0(E_0) \geq 0.
\]
Therefore \( E_0 \) is torsion and \( ch_1(E_0) \cdot \Theta \geq 0 \) implying that \( ch_1(E') \cdot \Theta > 0 \).

To conclude the proof, notice that if \( E \) is strictly semistable then \( E \) is in the extension closure of finitely many 1-dimensional stable sheaves each of which is sent via \( \Phi \) to an object in \( T^1 \). Thus \( \Phi(E) \in T^l \).

\[\Box\]

**Theorem 6.5.** Let \( E \) be a twisted Gieseker semistable 1-dimensional sheaf with \( \chi_L(E) \geq 0 \) and \( ch_1(E) \cdot f > 0 \), and assume that \( m + \alpha \gg 0 \). Then \( \Phi(E) \) is \( Z^l \)-semistable.

**Proof.** We already know by Proposition [6.4] that \( \Phi(E) \in T^l \). Suppose that there is a \( Z^l \)-destabilizing sequence in \( B^l \) for \( \Phi(E) \):
\[
(6.5.1) \quad 0 \to A \to \Phi(E) \to B \to 0.
\]
We may assume that \( B \) is \( Z^l \)-stable. Since \( \Phi(E) \) is a sheaf then by analyzing the long exact sequence of cohomology sheaves it follows that \( A \) is also a sheaf. We want to show that \( B \) is a sheaf as well. Indeed, \( B \) fits into an exact sequence in \( B^l \)
\[
0 \to H^{-1}(B)[1] \to B \to H^0(B) \to 0.
\]
Since \( \Phi(E) \in T^{l,0} \) then \( \phi(\Phi(E)) \to 0 \) along the curve (2.7.2) and so \( \phi(B) \to 0 \) as well. However, from Section 3.10 we know that
\[
\phi(H^{-1}(B)[1]) > 0 \quad \text{for} \quad v \gg 0,
\]
a contradiction to our assumption that \( B \) is \( Z^l \)-stable. Thus, \( H^{-1}(B)[1] = 0 \) and (6.5.1) is a short exact sequence of sheaves.

Now, from the triangle
\[
\Phi(A)[1] \to E \to \Phi(B)[1] \to \Phi(A)[2]
\]
we know that $A$ is $\Phi$-WIT$_1$. Moreover, we obtain the long exact sequence of sheaves
\[ 0 \longrightarrow \hat{\Phi}(B) \longrightarrow \hat{\Phi}(A)[1] \longrightarrow \mathcal{E} \longrightarrow \hat{\Phi}^1(B) \longrightarrow 0 \]
where $M = \text{Im}(g)$. Notice that since $\mathcal{E}$ is 1-dimensional then
\[ \text{ch}_0(M) = 0 = \text{ch}_0(\hat{\Phi}^1(B)). \]

From Section 3.10 we know that the $\mathcal{Z}$-destabilizing subobjects of $\Phi(\mathcal{E})$ have $\text{ch}_1(A) \cdot f \leq 0$, but since $A \in \mathcal{T}$ then $\text{ch}_1(A) \cdot f = 0$. Thus
\[ \text{ch}_0(\hat{\Phi}(B)) = \text{ch}_0(\hat{\Phi}(A)[1]) = -\text{ch}_1(A) \cdot f = 0. \]

Since $B$ is a sheaf then the torsion sheaf $\hat{\Phi}(B)$ is $\Phi$-WIT$_1$ and so by [6, Lemma 6.3]
\[ \text{ch}_1(\hat{\Phi}(B)) \cdot f = 0. \]

Therefore, $\Phi(\hat{\Phi}(B))[1]$ is a torsion sheaf and the short exact sequence of sheaves
\[ 0 \to \Phi(\hat{\Phi}(B))[1] \to B \to \Phi(\hat{\Phi}^1(B)) \to 0 \]
is exact in $\mathcal{T}$, Moreover, by Section 3.10 we know that unless $\Phi(\hat{\Phi}(B))[1] = 0$, the phase $\phi(\Phi(\hat{\Phi}(B))[1]) \to \frac{1}{2}$ because $\Phi(\hat{\Phi}(B))[1]$ is a fiber sheaf. This is a contradiction since $\phi(B) \to 0$ and $B$ is $\mathcal{Z}$-stable. Therefore, $\Phi(\hat{\Phi}(B))[1] = 0$ and
\[ 0 \to A \to \Phi(\mathcal{E}) \to B \to 0 \]
is a short exact sequence in $W_{\hat{\Phi}}$, contradicting the twisted Gieseker semistability of $\mathcal{E}$ since
\[ \frac{\text{ch}_2(\hat{\Phi}(U)[1]) - \frac{\alpha}{2} \text{ch}_1(\hat{\Phi}(U)[1]) \cdot \omega}{\text{ch}_1(\hat{\Phi}(U)[1])} = -\frac{\alpha \text{ch}_1(U) \cdot \omega}{\beta(\text{ch}_2(U) - \text{ch}_0(U))} \]
for all $U \in D^b(X)$ with $\text{ch}_1(U) \cdot f = 0$ along the curve (2.7.2). \hfill \blacksquare

### 6.6. Boundedness of Bridgeland walls via Bogomolov inequalities.

From now on we will assume that the Picard rank of $X$ is 2. Recall the following results about Bogomolov type inequalities on surfaces collected in [23, Section 6]:

**Lemma 6.7.** Let $X$ be a smooth projective surface and $\omega \in N^1(X)$ be an ample real divisor class. Then there exists a constant $C_\omega \geq 0$ such that, for every effective divisor $D \subset X$, we have
\[ C_\omega(D \cdot \omega)^2 + D^2 \geq 0. \]

**Definition 6.8.** Let $X$ be a smooth projective surface and $\omega, B \in N^1(X)$ with $\omega$ ample. For $E \in D^b(X)$ we define
\[ \Delta(E) := \text{ch}_1(E)^2 - 2\text{ch}_0(E)\text{ch}_2(E) \]
\[ \Delta_b^\omega(E) := (\text{ch}_1(E) \cdot \omega)^2 - 2\text{ch}_0(E)\text{ch}_2(E)\omega^2 \]
\[ \Delta_{\omega,b}(E) := \Delta(E) + C_\omega(\text{ch}_1(E) \cdot \omega)^2. \]
**Theorem 6.9.** Let \( X \) be a smooth projective surface and \( \omega, B \in N^1(X) \) with \( \omega \) ample. Assume that \( E \) is \( Z_{\omega,B} \)-semistable. Then

\[
\bar{\Delta}_B(E) \geq 0 \quad \text{and} \quad \Delta_{\omega,B}(E) \geq 0.
\]

**Lemma 6.10.** Let \( p : X \to B \) be a Weierstrass elliptic surface with a section \( \Theta \), and suppose \( X \) has Picard rank 2. Then the nef cone \( \text{Nef}(X) \) is the set of all non-negative linear combinations of \( e \Theta + f \) and \( f \), while the cone of effective curves \( \overline{\text{NE}}(X) \) (i.e. the Mori cone) is the set of all non-negative linear combinations of \( f \) and \( \Theta \).

**Proof.** The proof for the nef cone is exactly the same as [11, Proposition V.2.20]. On the other hand, \( \overline{\text{NE}}(X) \) is the dual cone of the nef cone \( \text{Nef}(X) \). Let \( C = Af + B \Theta \) be an effective curve on \( X \), then \( B = f \cdot C \geq 0 \) and \( A = (e \Theta + f) \cdot C \geq 0 \). \( \square \)

**Proposition 6.11.** Suppose that \( X \) is a Weierstrass surface of Picard rank 2, and let \( \omega = mf + \Theta \) be an ample class. Then every constant

\[
C \geq \frac{e}{(m-e)^2}
\]

satisfies the conditions of Lemma 6.7.

**Proof.** First, note that \( D = Af + B \Theta \) is effective if and only if \( A \geq 0 \) and \( B \geq 0 \). Clearly, it is enough to bound

\[
\frac{-D^2}{(D \cdot \omega)^2}
\]

when \( D^2 \leq 0 \). Now, \( D^2 = B(2A - eB) \leq 0 \) if and only if \( 0 \leq A \leq \frac{e}{2} B \). Since the same bound will work if we replace \( D \) by a multiple of itself then we can assume \( B = 1 \) and allow \( A \) to be a rational number. Thus,

\[
\frac{-D^2}{(D \cdot \omega)^2} = \frac{e - 2A}{(A + m - e)^2} \leq \frac{e}{(m-e)^2}.
\]

\( \square \)

**Remark 6.12.** Assume that \( \omega_0 = u_0(mf + \Theta) + v_0f \) is ample and that \( C_{\omega_0} \) satisfies the condition of Lemma 6.7 for \( \omega_0 \). Then given \( \lambda > 0 \), the constant \( \lambda^2 C_{\omega_0} \) satisfies the condition of Lemma 6.7 for \( \lambda \omega_0 \). Now, since

\[
\frac{e}{u_0^2(m-e)^2} \geq \frac{e}{u_0^2 \left( m - e + \frac{m}{u_0} \right)^2}
\]

then Proposition 6.11 implies that we can choose

\[
C_{\omega_0} = \frac{e}{u_0^2(m-e)^2}.
\]

Now, let \( E \) be a \( Z^l \)-semistable sheaf in \( T^l \) with \( c_1(E) = \lambda f \) for some \( \lambda > 0 \) and assume \( c_0(E) > 0, c_2(E) \leq 0 \). Suppose that there is a destabilizing sequence

\[
0 \to A \to E \to B \to 0
\]
in \( B_{\omega_0} \) for some \( \omega_0 = u_0(mf + \Theta) + v_0f \) along the curve (2.7.2) with \( 0 < u_0 \ll 1 \). Thus, \( A \in T_{\omega} \) and so

\[
(6.12.1) \quad 0 < \text{ch}_1(A) \cdot \omega_0 < \text{ch}_1(E) \cdot \omega_0.
\]

Along the curve (2.7.2) the volume \( \omega^2 \) equals to a constant \( 2K \). Then the wall equation translates into

\[
(6.12.2) \quad \frac{\text{ch}_1(A) \cdot \omega_0}{\text{ch}_1(E) \cdot \omega_0} = \frac{\text{ch}_2(A) - \text{ch}_0(A)K}{\text{ch}_2(E) - \text{ch}_0(E)K},
\]

and (6.12.1) becomes

\[
(6.12.3) \quad \text{ch}_2(E) - \text{ch}_0(E)K < \text{ch}_2(A) - \text{ch}_0(A)K < 0,
\]

since \( \text{ch}_2(A) - \text{ch}_0(A)K \) and \( \text{ch}_2(E) - \text{ch}_0(E)K \) have the same sign and so are negative because of our assumptions on \( \text{ch}(E) \).

If \( \text{ch}_0(A) = 0 \) then inequality (6.12.3) gives us finitely many values for \( \text{ch}_2(A) \). Otherwise, using inequality (6.12.1) and Theorem 6.9 we obtain

\[
(6.12.4) \quad \lambda^2u_0^2 - 4K\text{ch}_0(A)\text{ch}_2(A) > \Delta_{\omega_0}(A) \geq 0.
\]

Taking \( u_0 \) small enough so that \( u_0^2 < 4K \), inequality (6.12.4) produces

\[
(6.12.5) \quad \text{ch}_2(A) < \frac{\lambda^2u_0^2}{4K\text{ch}_0(A)} \leq \lambda^2
\]

since \( A \) is also a sheaf. Combining inequalities (6.12.3) and (6.12.5) we obtain

\[
(6.12.6) \quad \text{ch}_2(E) - \text{ch}_0(E)K + \text{ch}_0(A)K < \text{ch}_2(A) < \lambda^2
\]

and therefore \( \text{ch}_0(A) \), \( \text{ch}_2(A) \), and consequently \( \text{ch}_0(B) \) and \( \text{ch}_2(B) \) can take only finitely many values.

For convenience of notation, let \( S = \frac{\text{ch}_2(A) - \text{ch}_0(A)K}{\text{ch}_2(E) - \text{ch}_0(E)K} \). The wall equation becomes

\[
(\text{ch}_1(A) - S\lambda f) \cdot \omega_0 = 0,
\]

and therefore the Hodge Index Theorem gives

\[
(6.12.7) \quad \text{ch}_1(A)^2 \leq 2S\lambda\text{ch}_1(A) \cdot f.
\]

On the other hand, Theorem 6.9 and Remark 6.12 give

\[
(6.12.8) \quad -\frac{e}{u_0^2(m-e)^2}S^2\lambda^2u_0^2 \leq \Delta(A).
\]

Combining inequalities (6.12.7) and (6.12.8) we obtain

\[
(6.12.9) \quad -\frac{e}{(m-e)^2}S^2\lambda^2 + 2\text{ch}_0(A)\text{ch}_2(A) \leq \text{ch}_1(A)^2 \leq 2S\lambda\text{ch}_1(A) \cdot f.
\]

Now, if \( \text{ch}_1(A) = \eta f + \gamma \Theta \) then \( \text{ch}_1(A) \cdot f = \gamma \) and \( \text{ch}_1(A)^2 = (2\eta - e\gamma)\gamma \). We will now proceed to analyze inequality (6.12.9) in the following cases:

**Case 1:** \( \gamma < 1 \). In this case, inequality (6.12.9) produces

\[
-\frac{e}{(m-e)^2}S^2\lambda^2 + 2\text{ch}_0(A)\text{ch}_2(A) \leq \text{ch}_1(A)^2 < 2S\lambda.
\]

Thus, for every pair of values for \( \text{ch}_0(A) \) and \( \text{ch}_2(A) \) there are finitely many possibilities for \( \text{ch}_1(A)^2 \). Therefore, since \( \text{ch}_1(A)^2 = (2\eta - e\gamma)\gamma \) with \( \eta \) and \( \gamma \) integers then \( \text{ch}_1(A) \) can only take finitely many values whenever \( \text{ch}_1(A)^2 \neq 0 \).
When \( ch_1(A)^2 = 0 \) then either \( \gamma = 0 \) and inequality (6.12.1) implies \( 0 < \eta < \lambda \), or \( \eta = \gamma e/2 \) and (6.12.1) implies \( 0 < \gamma K < \lambda u_0^2 < 16\lambda K^2 \). In any case, \( ch_1(A) \) can take only finitely many values.

**Case 2:** \( \gamma \geq 1 \). In this case \( ch_1(B) \cdot f < 0 \). Let \( S' = \frac{ch_2(B) - ch_0(B)K}{ch_2(E) - ch_0(E)K} \), then applying inequalities (6.12.7) y (6.12.8) to the Bridgeland semistable object \( B \) we obtain

\[
(6.12.10) \quad -\frac{e}{(m - e)^2} S'^2 \lambda^2 + 2ch_0(B)ch_2(B) \leq ch_1(B)^2 \leq 0.
\]

As in Case 1, this implies that \( ch_1(B) \) can take only finitely many values and so does \( ch_1(A) \).

This shows that the Chern character \( ch(A) \) can take only finitely many values and so there are only finitely many walls for the Chern character \( ch(E) = (ch_0(E), \lambda f, ch_2(A)) \) for \( u_0 < 4K \) along the curve (2.7.2), i.e., walls for this Chern character are bounded along the curve (2.7.2) for \( v \gg 0 \).

The following lemma gives a relation between Bridgeland stability and \( Z^l \)-stability on elliptic surfaces:

**Lemma 6.13.** Let \( p : X \to B \) be a Weierstraß surface. Suppose \( m > 0 \) is such that \( \Theta + kf \) is ample for all \( k \geq m \), and \( \omega \) is of the form (2.6.1) subject to the constraint (2.7.2). Suppose there is an object \( F \in D^b(X) \) and some \( v_0 > 0 \) such that, for all \( v > v_0 \), the divisor \( \omega \) is ample and \( F \) lies in \( B_\omega \) and is \( Z_\omega \)-semistable. Then \( F \) lies in \( B^l \) and is \( Z^l \)-semistable.

**Proof.** This follows easily from the equivalent definitions of \( T^l \) and \( F^l \) in (3.2) (See also [18, Lemma 7.1])

**Corollary 6.14.** Let \( \mathcal{E} \) be a 1-dimensional twisted Gieseker semistable sheaf with \( \chi_L(\mathcal{E}) \geq 0 \) and \( ch_1(\mathcal{E}) \cdot f > 0 \), and assume that \( m + \alpha \gg 0 \). Then \( \Phi(\mathcal{E}) \) is \( Z_\omega \)-semistable for \( v \gg 0 \) along the curve (2.7.2).

### 7. Asymptotics for Bridgeland Walls on Weierstrass Surfaces

The boundedness results for Bridgeland mini-walls obtained in Section 6 highly depend on our choice of Chern character \( ch = (ch_0, \lambda f, ch_2) \). Indeed, the same techniques will fail if we have \( ch_1 = a\Theta \), since \( ch_1 \cdot \omega \) will grow as \( v \to \infty \) along the curve (2.7.2). In this section, we want to carefully study the asymptotic behavior of the Bridgeland mini-walls instead of studying all walls at once. Results on boundedness of mini-walls similar to those in Section [6] and [19] will then yield, that Bridgeland stability in the outer-most mini-chamber on (2.7.2) implies \( Z^l \)-stability. Combined with Theorem [4.18], this would produce examples of Bridgeland semistable objects whose (inverse) Fourier-Mukai transforms are slope semistable sheaves. In Section [8] we will give an example where this program is realised.

#### 7.1. Polarisation on Weierstrass surfaces

Let \( p : X \to B \) be a Weierstrass surface with a section \( \Theta \). We do not assume that the section \( \Theta \) is unique. For \( 0 < \lambda < 1 \), define

\[
(7.1.1) \quad \begin{cases}
H = H_\lambda := \lambda(\Theta + mf) + (1 - \lambda)f \\
H^\perp = H^\perp_\lambda := -\lambda(\Theta + mf) + (1 + (2m - e - 1)\lambda)f
\end{cases}
\]

where \( m \) is a fixed positive number as in Section [3] and \( e = -\Theta^2 \). Then \( H_\lambda.H^\perp_\lambda = 0 \). We also set \( g := H_\lambda.H_\lambda \) and \( \delta := -H^\perp_\lambda.H^\perp_\lambda \) so that

\[
g = \delta = 2\lambda \left( 1 + (m - \frac{e}{2} - 1)\lambda \right) \approx 2\lambda \text{ as } \lambda \to 0^+.
\]
Notation. We refer to Appendix A for general results and notation on Bridgeland wall-chamber structures, including the definition of a frame. We will use the notation from Appendix A throughout this section.

For fixed real numbers $0 < \lambda < 1$ and $w$, we can consider the frame $(H_\lambda, H_\lambda^\perp, w)$. Then for any real numbers $s, q$ satisfying $q > \frac{1}{2}s^2$ we can define a Bridgeland stability condition $\sigma_{s,q}$ as in (A.1.2) (note that $\sigma_{s,q}$ still depends on $\lambda, w$ even though that is suppressed in the notation). As a result, we have the subset of $\text{Stab}(X)$

\begin{equation}
\{\sigma_{s,q} : (\lambda, w, s, q) \in \mathbb{R}^4, 0 < \lambda < 1, q > \frac{s^2}{2}\}
\end{equation}

which we refer to as the “$(\lambda, w, s, q)$-space”.

7.2. Change of variables and the $(\lambda, 0, 0, q)$-plane. Recall that we have parameters $u, v \in \mathbb{R}_{>0}$ related by (2.7.2) in the definition of $Z^1$-stability. We can make the change of variables

\begin{equation}
\begin{cases}
\lambda = \frac{u}{u+v} \\
t = u + v
\end{cases}, \quad \text{or equivalently} \quad \begin{cases}
u = t\lambda \\
v = t(1 - \lambda)
\end{cases}
\end{equation}

which allows us to write $\omega$ as

$$\omega = u(\Theta + mf) + vf = tH_\lambda.$$ 

At this point, together with the notation from 7.1, the parameters $\lambda, t$ correspond to our polarisation $\omega$, while $s, w$ correspond to the $B$-field $B$ (see (A.0.2)). If we set $B = 0$, i.e. $s = w = 0$, then this forces $q = \frac{1}{2}t^2$ and restricts the $(\lambda, w, s, q)$-space in (7.1.2) to a “$(\lambda, 0, 0, q)$-plane” within $\text{Stab}(X)$.

The curve (2.7.2), written in terms of $u, v$, can now be written in terms of $\lambda, q$ as

\begin{equation}
(m - \frac{1}{2} - 1)2q \lambda^2 + 2q\lambda = m + \alpha - e.
\end{equation}

Then

$$v \to +\infty \text{ along (2.7.2)} \iff \lambda \to 0^+ \text{ along (7.2.2)}$$

and the curve (7.2.2) is asymptotic to

\begin{equation}
q = \frac{1}{2\lambda} (m + \alpha - e) \text{ as } \lambda \to 0^+.
\end{equation}

7.3. Intersection numbers. Since $\Theta, \Theta_i$ are sections, we have the intersection numbers $\Theta, f = 1, \Theta, f = 1$. Recall in 2.3 we have $\Theta_\perp^2 = \Theta^2 = -e$ and $K_X \equiv (2g(B) - 2 + e)f$. Let us denote $\theta_i = \Theta, \Theta_i$. Since both $\Theta$ and $\Theta_i$ are irreducible curves, we have $\theta_i \geq 0$. Decomposing $\Theta, f$ and $\Theta_i$ with respect to the frame $(H_\lambda, H_\lambda^\perp, w)$, we have

$$\Theta = l_\Theta H_\lambda + l_\Theta^\perp H_\lambda^\perp, \quad f = l_f H_\lambda + l_f^\perp H_\lambda^\perp, \quad \Theta_i = a_i H_\lambda + b_i H_\lambda^\perp + \Delta_i,$$

where the real coefficients $l_\Theta, l_\Theta^\perp, l_f, l_f^\perp, a_i, b_i$ and the class $\Delta_i \in \{H_\lambda, H_\lambda^\perp\}^\perp$ are given as follows:

$$l_\Theta g = \Theta H_\lambda = 1 + (m - e - 1)\lambda, \quad -l_\Theta^\perp g = \Theta H_\lambda^\perp = 1 + (m - 1)\lambda,$$

$$l_f g = f H_\lambda = \lambda, \quad -l_f^\perp g = f H_\lambda^\perp = -\lambda,$$

$$a_i g = \Theta_i H_\lambda = 1 + (m + \theta_i - 1)\lambda, \quad -b_i g = \Theta_i H_\lambda^\perp = 1 + (m - \theta_i - e - 1)\lambda.$$ 

It is clear that

$$l_\Theta + l_\Theta^\perp = -\frac{e\lambda}{g}, \quad l_f + l_f^\perp = \frac{2\lambda}{g}, \quad a_i + b_i = \frac{2\lambda}{g}(\theta_i + \frac{e}{2}),$$

$$l_\Theta - l_\Theta^\perp = \frac{1}{\lambda}, \quad l_f - l_f^\perp = 0, \quad a_i - b_i = \frac{1}{\lambda}.$$
Basic computation shows that \( a_iH_\lambda + b_iH_\lambda^\perp = \Theta + (\theta_i + e)f \). Therefore
\[
(7.3.1) \quad \Delta_i = \Theta_i - \Theta - (\theta_i + e)f.
\]
In particular, the divisor class \( \Delta \) is independent of \( \lambda \).

Note that for any numerical invariant \( \text{ch} = (\text{ch}_0, \text{ch}_1, \text{ch}_2) \) with \( \text{ch}_0 \neq 0 \), we can write \( \text{ch} = e^L \left( \text{ch}_0, 0, \text{ch}_2 - \frac{\text{ch}_1^2}{2\text{ch}_0} \right) \) with \( L = \frac{\text{ch}_1}{\text{ch}_0} \). Moreover, \( \text{ch} \) is of Bogomolov type if and only if \( (\text{ch}_0, 0, \text{ch}_2 - \frac{\text{ch}_1^2}{2\text{ch}_0}) \) is so. For numerical invariants \( \text{ch} \) of this type, the following proposition gives us the asymptotic behavior of potential walls in the \( (\lambda, 0, 0, q) \)-plane as \( \lambda \to 0^+ \).

**Proposition 7.4.** (Potential walls in \( (\lambda, 0, 0, q) \)-plane for two-dimensional objects) Let \[
\text{ch} = (x, 0, z), \quad L = a_L\Theta + b_L f + \sum_i \eta_i \Theta_i,
\]
where \( x \neq 0 \) and \( xz \leq 0 \) (i.e. \( \text{ch} \) is of Bogomolov type), \( a_L, b_L, \eta_i \in \mathbb{R} \). Then the potential wall \( W(e^L\text{ch}, e^L\text{ch}') \) with
\[
\text{ch}' = (r, k\Theta + pf + \sum_i \xi_i \Theta_i, \chi),
\]
has the following asymptotic behavior in the \( (\lambda, 0, 0, q) \)-plane as \( \lambda \to 0^+ \). Write \( \Delta_L = \sum_i \eta_i \Delta_i \) and \( \Delta' = \sum_i \xi_i \Delta_i \).

(A) Suppose \( k + \sum_i \xi_i = 0 \) and \( p - ek + \sum_i \xi_i \theta_i = 0 \).

(A1) If \( a_L + \sum_i \eta_i = 0 \) and \( b_L - ea_L + \sum_i \eta_i \theta_i = 0 \), then the potential wall in the \( (\lambda, 0, 0, q) \)-plane is the entire region given by \( q > 0 \).

(A2) If \( a_L + \sum_i \eta_i \neq 0 \) or \( b_L - ea_L + \sum_i \eta_i \theta_i \neq 0 \), then there are no potential walls in the \( (\lambda, 0, 0, q) \)-plane.

(B) Suppose \( k + \sum_i \xi_i = 0 \) and \( p - ek + \sum_i \xi_i \theta_i \neq 0 \). Let us set
\[
(7.4.1) \quad A := - \left( \frac{xy - rz}{x} + \Delta' \Delta_L + (a_L + \sum_i \eta_i) \left( p - \frac{e}{2}k + \sum_i \xi_i (\theta_i + \frac{e}{2}) \right) \right) \frac{a_L + \sum_i \eta_i}{p - ek + \sum_i \xi_i \theta_i},
\]
\[
(7.4.2) \quad B := \frac{z}{x} + \frac{\Delta_L^2}{2} - \left( b_L - ea_L + \frac{e}{2} (a_L + \sum_i \eta_i) + \sum_i \eta_i \theta_i \right) \frac{xy - rz}{x} \frac{x}{p - ek + \sum_i \xi_i \theta_i}.
\]

(B1) If \( A \neq 0 \) then the potential wall is asymptotic to
\[
q = \frac{A}{2\lambda^2}.
\]

(B2) If \( A = 0 \) and \( B \neq 0 \) then the potential wall is asymptotic to
\[
q = \frac{B}{2\lambda}.
\]

(B3) If \( A = 0 \) and \( B = 0 \), then the potential wall is bounded as \( \lambda \to 0^+ \).

(C) Suppose \( k + \sum_i \xi_i \neq 0 \).
(C1) If

\[(7.4.5) \quad D := \frac{z}{x} + \frac{\Delta_L^2}{2} - \left( \frac{x \chi - rz}{x} + \Delta' \Delta_L + \left( a_L + \sum_i \eta_i \right) \left( p - \frac{e}{2}k + \sum_i \xi_i(\theta_i + \frac{e}{2}) \right) \right) \frac{a_L + \sum_i \eta_i}{k + \sum_i \xi_i} \neq 0,\]

then the potential wall is asymptotic to

\[(7.4.6) \quad q = \frac{D}{2\lambda}.\]

(C2) If \(D = 0\), then the potential wall is bounded as \(\lambda \to 0^+\).

**Proof.** Let us break the proof into five steps. Note that we are assuming \(c_1 = 0\) which means \(y_1 = y_2 = 0\) and \(\Delta = 0\).

**Step 1.** Let us decompose the given data with respect to the frame \((H, H^\perp, w)\) and compute different kinds of intersection numbers. Here \(\text{ch} = (x, 0, z)\), and so \(y_1 = y_2 = 0\) in (A.0.3). Write

\[L = a_L \Theta + b_L f + \sum_i \eta_i \Theta_i = l_1 H + l_2 H^\perp + \Delta_L\]

with real coefficients \(l_1, l_2,\) and class \(\Delta_L \in \{H, H^\perp\}^\perp\). Then

\[l_1 = a_L l_\Theta + b_L l_f + \sum_i \eta_i a_i, \quad l_2 = a_L l_\Theta + b_L l_f + \sum_i \eta_i b_i, \quad \Delta_L = \sum_i \eta_i \Delta_i.\]

In particular, the divisor class \(\Delta_L\) is independent of \(\lambda\). We have

\[(7.4.7) \quad gl_1 = (a_L + \sum \eta_i) + \left( b_L - ea_L + \left( a_L + \sum \eta_i \right)(m - 1) + \sum \eta_i \Theta_i \right) \lambda,\]

and

\[(7.4.8) \quad l_1 + l_2 = \frac{b_l - \frac{e}{2}a_L + \sum \eta_i (\theta_i + \frac{e}{2})}{1 + (m - \frac{e}{2} - 1)\lambda}, \quad l_1 - l_2 = \frac{a_L + \sum \eta_i}{\lambda}.\]

Write

\[\text{ch}_1' = k \Theta + pf + \sum \xi_i \Theta_i = c_1 H + c_2 H^\perp + \Delta',\]

with real coefficients \(c_1, c_2\) and class \(\Delta' \in \{H, H^\perp\}^\perp\) as (A.0.4). Then

\[c_1 = kl_\Theta + pl_f + \sum \xi_i a_i, \quad c_2 = kl_\Theta + pl_f + \sum \xi_i b_i, \quad \Delta' = \sum \xi_i \Delta_i.\]

Hence the divisor class \(\Delta'\) is also independent of \(\lambda\). We obtain

\[(7.4.9) \quad gc_1 = (k + \sum \xi_i) + \left( p - ek + \left( k + \sum \xi_i \right)(m - 1) + \sum \xi_i \theta_i \right) \lambda,\]

\[(7.4.10) \quad g(c_1 + c_2) = 2\lambda \left( p - \frac{e}{2}k + \sum \xi_i (\theta_i + \frac{e}{2}) \right).\]

**Step 2.** Suppose \(k + \sum \xi_i = 0\) and \(p - ek + \sum \xi_i \theta_i = 0\). Then by (7.4.9), \(c_1 = 0\), which is independent of \(\lambda\). Now that we have \(y_1 = 0\) (by assumption) and \(c_1 = 0\), we obtain \(xc_1 - ry_1 = 0\). By footnote \([2]\) in Lemma (A.3) we see that the potential wall in the \((\lambda, 0, s, q)\)-space is given by \(s = l_1\) with \(q > \frac{\lambda}{2}\).
If \(a_L + \sum_i \eta_i = 0\) and \(b_L - ea_L + \sum_i \eta_i \theta_i = 0\), then by (7.4.7), \(l_1 = 0\) and the potential wall in the \((\lambda, 0, 0, q)\)-plane is given by \(q > 0\). This shows (A1).

If \(b_L - ea_L + \sum_i \eta_i \theta_i \neq 0\), then by (7.4.7), \(l_1 \neq 0\) and there is no potential wall in the \((\lambda, 0, 0, q)\)-plane. If \(a_L + \sum_i \eta_i \neq 0\), then by (7.4.7), \(l_1 \neq 0\) as \(\lambda \to 0^+\) and again there is no potential wall in the \((\lambda, 0, 0, q)\)-plane. This shows (A2).

**Step 3.** We do some computation by assuming that \(c_1 \neq 0\). Recall the definition of \(P(ch)\) from Lemma A.2. Note that

\[
P(ch) = (0, \frac{z}{xg}), \quad C(ch, ch') = \frac{x \chi - rz}{xgc_1}.
\]

Also, we have \(y_1 = y_2 = 0\) and \(\Delta = 0\) by assumption while \(g = \delta\) from 7.1. Thus by Lemma A.3,

\[
P(e^Lch) = \left( l_1, \frac{l_1^2 - l_2^2}{2} + \frac{z}{xg} + \frac{\Delta^2}{2g} \right),
\]

and

\[
C(e^Lch, e^Lch') = C(ch, ch') + l_1 - l_2 \frac{c_2}{c_1} + \frac{\Delta \Delta L}{gc_1}.
\]

The potential wall \(W(e^Lch, e^Lch')\) in the \((\lambda, 0, s, q)\)-space (i.e. \(w = 0\)) is given by

\[
q = \left( \frac{x \chi - rz}{xgc_1} + l_1 - l_2 \frac{c_2}{c_1} + \frac{\Delta \Delta L}{gc_1} \right) (s - l_1) + \frac{l_1^2 - l_2^2}{2} + \frac{z}{xg} + \frac{\Delta^2}{2g}.
\]

By restricting to \(s = 0\), the potential wall \(W(e^Lch, e^Lch')\) in the \((\lambda, 0, 0, q)\)-plane is given by

\[
(7.4.11) \quad q = -\left( \frac{x \chi - rz}{x} + \Delta \Delta L \right) \frac{g l_1}{gc_1} g + l_1 l_2 \left( \frac{c_1 + c_2}{c_1} \right) - \frac{1}{2} (l_1 + l_2)^2 + \frac{z}{xg} + \frac{\Delta^2}{2g}.
\]

Therefore, by (7.4.7), (7.4.8), (7.4.9) and (7.4.10), we have

\[
queline{\bigg(} \frac{x \chi - rz}{x} + \Delta \Delta L \bigg) \left( \frac{a_L + \sum_i \eta_i}{(k + \sum_i \xi_i)} + \frac{(b_L - ea_L + (a_L + \sum_i \eta_i)(m - 1) + \sum_i \eta_i \theta_i)}{(p - ek + (k + \sum_i \xi_i)(m - 1) + \sum_i \xi_i \theta_i)} \right) \lambda \cdot \frac{1}{g}
\]

\[
+ \frac{1}{4} \left( \frac{b_L - \frac{\eta_i \theta_i}{2}}{1 + (m - \frac{\eta_i}{2} - 1) \lambda} \right)^2 - \frac{(a_L + \sum_i \eta_i \lambda \cdot \frac{1}{g})}{(k + \sum_i \xi_i)} \left( p - ek + (k + \sum_i \xi_i)(m - 1) + \sum_i \xi_i \theta_i \lambda \right)
\]

\[
\cdot \left( p - ek + (k + \sum_i \xi_i)(m - 1) + \sum_i \xi_i \theta_i \lambda \right)
\]

\[
- \frac{1}{2} \left( \frac{b_L - \frac{\eta_i \theta_i}{2}}{1 + (m - \frac{\eta_i}{2} - 1) \lambda} \right)^2 + \left( \frac{z}{xg} + \frac{\Delta^2}{2g} \right) \cdot \frac{1}{g}.
\]

**Step 4.** Suppose \(k + \sum_i \xi_i = 0\) and \(p - ek + \sum_i \xi_i \theta_i \neq 0\). Then by (7.4.9), \(c_1 \neq 0\). We have

\[
(7.4.12) \quad q = \frac{A}{2 \lambda^2} + \frac{B}{2 \lambda} + C(\lambda),
\]

where \(A\) and \(B\) are given as (7.4.1) and (7.4.2) and \(C(\lambda)\) is bounded as \(\lambda \to 0^+\). The claims in case (B) then follow.
Step 5. Suppose \( k + \sum_i \xi_i \neq 0 \). Then by (7.4.9), \( c_1 \neq 0 \) as \( \lambda \to 0^+ \). We have
\[
q = \frac{D}{2\lambda} + E(\lambda),
\]
where \( D \) is given as in (7.4.5) and \( E(\lambda) \) is bounded as \( \lambda \to 0^+ \). The claims in case (C) then follow. \( \blacksquare \)

We give a parallel result of Proposition 7.4 on potential walls in \((\lambda, 0, 0, q)\)-plane for one-dimensional objects in Appendix B.

8. Transforms of line bundles of fiber degree at least 2

In this section, we combine Theorem 4.1 and the structural results on walls in Section 7 to prove the following result on sheaves:

**Proposition 8.1.** Let \( p : X \to B \) be a Weierstraß elliptic surface such that \( X \) has Picard rank two and \( e > 0 \). Let \( m > 0 \) be such that \( \Theta + m f \) is ample for all \( m' \geq m \). Then for any positive integer \( a_L > 1 \) and real number \( \alpha > 0 \) satisfying
\[
m + \alpha - e \neq \frac{q}{2} a_L (a_L - 1),
\]
the line bundle \( \mathcal{O}_X(a_L \Theta) \) is \( \sigma \)-stable for any Bridgeland stability \( \sigma \) lying on the curve (7.2.2) on the \((\lambda, 0, 0, q)\)-plane with \( \lambda > 0 \) sufficiently small. Moreover, the transform \( \Phi \mathcal{O}_X(a_L \Theta) \) is a \( \mu_\sigma \)-semistable locally free sheaf of rank \( a_L \) where \( \sigma = \Theta + (m + \alpha) f \).

**Key idea of proof.** The key idea is that there is only one wall that is of the form \( W(\text{ch}(\mathcal{O}_X(a_L \Theta)), -) \), and the condition (8.1.1) ensures that, for \( \lambda > 0 \) sufficiently small, the curve along which we define ‘limit Bridgeland stability’ (7.2.2) either lies above the wall or below the wall.

**Lemma 8.2.** For any positive integer \( n \), the line bundle \( \mathcal{O}_X(n \Theta) \) is \( \Phi\text{-WIT}_0 \), and \( \widetilde{\mathcal{O}_X(n \Theta)} = \widehat{\mathcal{O}_X(n \Theta)} \) is a locally free sheaf.

**Proof.** For every closed point \( s \in B \), the restriction \( \mathcal{O}_X(n \Theta)|_s \) is a line bundle of positive degree on the fiber \( X_s \), and hence a \( \Phi_s \)-WIT sheaf [3, Proposition 6.38]. Thus \( \mathcal{O}_X(n \Theta) \) itself is \( \Phi\text{-WIT}_0 \) by [16, Lemma 3.6], and the transform \( \widehat{\mathcal{O}_X(n \Theta)} \) is torsion-free by [15, Lemma 2.11].

To see that the transform \( \mathcal{O}_X(n \Theta) \) is locally free, take any sheaf \( T \) supported in dimension 0; then
\[
\text{Ext}^1(T, \mathcal{O}_X(n \Theta)) \cong \text{Ext}^1(\widehat{T}, \mathcal{O}_X(n \Theta)[-1]) \cong \text{Hom}(\widehat{T}, \mathcal{O}_X(n \Theta)) = 0
\]
where the last equality holds since \( \mathcal{O}_X(n \Theta) \) is torsion-free, and since \( T \) is a \( \Phi\text{-WIT}_0 \) sheaf whose transform is a fiber sheaf. Lemma 4.2 then implies that \( \mathcal{O}_X(n \Theta) \) is locally free. \( \blacksquare \)

8.3. The Weierstraß elliptic surface \( X \) is a product if and only if \( \mathbb{L} = O_B \) by [24, Lemma (III.1.4)]. Therefore, if \( e > 0 \) then the Weierstraß surface \( X \) cannot be a product.

**Lemma 8.4.** Let \( p : X \to B \) be a Weierstraß elliptic surface with a section \( \Theta \), and suppose \( e > 0 \). Then \( X \) is of Picard rank two if and only if \( \Theta \) is the unique section.

**Proof.** Suppose \( X \) has Picard rank two. Then \( \text{NS}(X) \) is generated by the class of a section \( \Theta' \) and the fiber class \( f \) [24, Theorem (VII.2.1)]. We will now prove that \( \Theta' \) and \( \Theta \) are the same.
curve, and not merely the same curve class. Suppose

\( \Theta' = a\Theta + bf \) in \( \text{NS}(X) \).

Intersecting with \( f \) on both sides of (8.4.1) gives \( a = 1 \). Squaring both sides of (8.4.1) gives

\[(\Theta')^2 = \Theta^2 + 2b.\]

Now, we have \((\Theta')^2 = \Theta^2 = -e\) by adjunction, and so \( b = 0 \), giving us \( \Theta' = \Theta \) in \( \text{NS}(X) \). Thus

\[\Theta, \Theta' = \Theta^2 = -e < 0;\]

since both \( \Theta', \Theta \) are irreducible curves, this implies \( \Theta' \) and \( \Theta \) are the same curve. Thus \( p \) has a unique section.

Conversely, if \( p \) has a unique section \( \Theta \), then the Mordell-Weil group \( \text{MW}(X) \) of \( X \) is trivial. Then by the Shioda-Tate formula [24 Corollary (VII.2.4)], the Picard rank of \( X \) must be two.

\[\boxed{\text{Lemma 8.5.} \quad \text{Let} 
\quad p : X \rightarrow B \text{ be a Weierstraß elliptic surface with a section} \ \Theta, \text{ and suppose} \ \text{X has Picard rank two. Suppose also that} \ e > 0. \ \text{Then} \ \Theta \text{ is the only irreducible negative curve on} \ X.}
\]

\text{Proof.} Suppose \( C \) is an irreducible negative curve on \( X \). Then \( C \) must be extremal in \( \text{NE}(X) \) by [12] Lemma 1.22. Lemma 6.10 then implies either \( C \equiv \Theta \) or \( C \equiv f \). Since \( e > 0 \), we have \( \Theta^2 = -e < 0 \), i.e. \( \Theta \) is a negative curve, while \( f \) is not. Hence \( C \equiv \Theta \). Then \( C, \Theta = \Theta^2 < 0 \), which in turn implies the curve \( C \) coincides with the curve \( \Theta \).

Note that, under the hypotheses of Lemma 8.5 we can also conclude that \( \Theta \) must be the unique section, which is the ‘only if’ direction of Lemma 8.4.

\[\boxed{\text{8.6. An example.} \quad \text{An example of a Weierstraß surface} \ p : X \rightarrow B \text{ such that} \ X \text{ has Picard rank two, and where} \ e > 0, \text{ is an elliptic K3 surface referred to as the Bryan-Leung K3 surface in [25 Section 2.2]. In this example, we have} \ B = \mathbb{P}^1, e = 2, \text{ and} \ p \text{ has exactly 24 singular fibers, all of which are nodal.}}\]

8.7. Suppose \( p : X \rightarrow B \) is a Weierstraß surface such that \( X \) has Picard rank two and \( e > 0 \). By Lemma 8.5 there is a unique negative curve on \( X \), and it is the unique section of \( p \) (see also Lemma 8.4). A theorem of Arcara-Miles [2 Theorem 1.1] now tells us that the only object that could destabilise a line bundle \( L \) with respect to a Bridgeland stability in (7.1.2) is \( L(-\Theta) \). Following the notation in Proposition 7.4, we have

\[(x, 0, z) = (1, 0, 0) \quad \text{and} \quad (r, k\Theta + pf, \chi) = (1, -\Theta, -\frac{e}{2})\]

so that \( k = -1 \). Suppose \( L \) is of the form \( O_X(a_L\Theta) \) with \( a_L > 1 \). By Proposition 7.4(C1), the wall \( W(\text{ch}(O_X(a_L\Theta)), \text{ch}(O_X(a_L - 1)\Theta)) \) is asymptotic to

\[(8.7.1) \quad q = \frac{1}{2\lambda^2} \frac{e}{2} a_L(a_L - 1) \quad \text{as} \ \lambda \rightarrow 0^+.\]

\text{Proof of Proposition 8.1} Let \( \sigma \) be any Bridgeland stability satisfying the stated hypothesis. From 8.7 We know that \( W(\text{ch}(O_X(a_L\Theta)), \text{ch}(O_X(a_L - 1)\Theta)) \) is the only wall in the \((\lambda, 0, 0, q)\)-plane for the numerical type of \( O_X(a_L\Theta) \). Comparing the asymptotic behaviour of (7.2.2), namely (7.2.3), with the asymptotic equation of the wall, namely (8.7.1), we see that (8.1.1) ensures \( \sigma \) lies in a chamber of Bridgeland stability whenever \( \sigma \) lies on (7.2.2) with \( \lambda \) sufficiently small. (Depending on whether the curve (7.2.2) lies above or below the unique
At first glance, the statement of Proposition 8.1 appears to be similar to that of Bridgeland-Maciocia’s result [8, Theorem 1.4] which says that on a Weierstraß threefold $X$, the fibration $p : X \to S$ is $K$-trivial and $K_S$ is numerically $K$-trivial, any line bundle of nonzero fiber degree on $X$ is taken by a Fourier-Mukai transform to a $\mu_2$-stable locally free sheaf, for any polarisation $\omega$. One quickly finds, however, that the argument in [20] does not carry over directly to the situation of Proposition 8.1. A technical reason is that the base of the fibration in Proposition 8.1 is not numerically $K$-trivial.

We note that Bridgeland-Maciocia’s approach begins with a torsion-free sheaf which restricts to a stable sheaf on the generic fiber of the elliptic fibration, while our approach begins with a limit Bridgeland stable object (which is allowed to be a complex).

**Remark 8.9.** At first glance, the statement of Proposition 8.1 appears to be similar to that of [20, Theorem 4.4], which says that on a Weierstraß threefold $p : X \to S$ where $X$ is $K$-trivial and $K_S$ is numerically $K$-trivial, any line bundle of nonzero fiber degree on $X$ is taken by a Fourier-Mukai transform to a $\mu_2$-stable locally free sheaf, for any polarisation $\omega$. One quickly finds, however, that the argument in [20] does not carry over directly to the situation of Proposition 8.1. A technical reason is that the base of the fibration in Proposition 8.1 is $\mathbb{P}^1$, which is not numerically $K$-trivial.

**8.10.** In proving Proposition 8.1 we relied on Arcara-Miles’ result that there is only one possible destabilising object for a line bundle, if the surface contains a unique negative curve. This is only one half of their theorem [2, Theorem 1.1]; the other half of their theorem states that the result holds also for surfaces with no negative curves (such as $C \times \mathbb{P}^1$ where $C$ is an elliptic curve). For such and other surfaces for which Arcara-Miles’ theorem holds, it seems plausible that an analogue of Proposition 8.1 would hold.

### Appendix A. Bridgeland wall-chamber structures

Let $X$ be a smooth projective surface. We briefly recall the wall-chamber structures in the Bridgeland stability manifold $\text{Stab}(X)$. We will consider the stability conditions $\sigma_{\omega,B}$ defined in [2, 4] Our study of wall and chamber structures consists of two steps: (i) We fix a ‘frame’ and write $\omega$ and $B$ with respect to the frame as in (A.0.2), and study potential walls; (ii) we deform the frame. Step (i) follows the work of Maciocia [22]. We give an example of step (ii) on elliptic surfaces in (7.1.1), by varying a parameter $\lambda$.

By fixing a frame, we mean that we fix a triple $(H, H^\perp, w)$ where $H$ is an ample $\mathbb{R}$-divisor on $X$, $H^\perp$ is an $\mathbb{R}$-divisor satisfying $H.H^\perp = 0$, and $w$ is a real number. The divisor $H^\perp$ is taken to be zero if the Picard number of $X$ is one. In general, the divisor $H^\perp$ is not unique even up to a scalar multiple if the Picard number of $X$ is bigger than two. We set

\[(A.0.1)\]

\[g := H.H, \quad \delta := -H^\perp.H^\perp.\]

The Hodge Index Theorem implies that $\delta \geq 0$, and $\delta = 0$ if and only if $H^\perp = 0$. 

\[\text{William H.} \]
Having fixed a frame \((H, H^\perp, w)\), we can then set
\[
\begin{aligned}
\omega := tH \\
B := sH + wH^\perp
\end{aligned}
\]  
where \(t \in \mathbb{R}_{>0}, s \in \mathbb{R}\)
and think of \(\omega, B\) as depending on \(t, s\), respectively. By varying \(w\), we then obtain a \(w\)-indexed family of \((s, t)\) half-planes in \(\text{Stab}(X)\):
\[
\Pi_{(H, H^\perp, w)} := \{\sigma_{tH, sH + wH^\perp} \mid t \in \mathbb{R}_{>0}, s \in \mathbb{R}\} \subset \text{Stab}(X).
\]
Let \(\text{ch} = (\text{ch}_0, \text{ch}_1, \text{ch}_2)\) be a fixed Chern character. We can rewrite it with respect to the frame \((H, H^\perp, w)\) as
\[
\text{ch} = (\text{ch}_0, \text{ch}_1, \text{ch}_2) = (x, y_1H + y_2H^\perp + \Delta, z)
\]
for some real coefficients \(y_1, y_2\) and \(\Delta \in \{H, H^\perp\}^\perp\). Similarly, we write the potentially destabilising Chern character with respect to the frame as
\[
\text{ch}' = (\text{ch}_0', \text{ch}_1', \text{ch}_2') = (r, c_1H + c_2H^\perp + \Delta', \chi)
\]
for some real coefficients \(c_1, c_2\) and \(\Delta' \in \{H, H^\perp\}^\perp\). For fixed \(\text{ch}, \text{ch}'\), the corresponding potential wall is defined as
\[
W(\text{ch}, \text{ch}') := \{\sigma = (B, Z) \in \text{Stab}(X) \mid \Re Z(\text{ch}) \Im Z(\text{ch}') - \Re Z(\text{ch}') \Im Z(\text{ch}) = 0\}.
\]
In the notation \(\sigma = (B, Z)\) above for a Bridgeland stability, \(B\) is a heart and \(Z\) is the central charge of the stability condition. A potential wall \(W(\text{ch}, \text{ch}')\) is a Bridgeland wall if there is a \(\sigma = (B, Z) \in W(\text{ch}, \text{ch}')\) together with \(\sigma\)-semistable objects \(G \subset F \in B\) such that \(\text{ch}(F) = \text{ch}, \text{ch}(G) = \text{ch}'\).

Fix a frame \((H, H^\perp, w)\). Following the idea of Li-Zhao [13], we define \(\sigma'_{\omega, B} = (Z'_{\omega, B}, B'_{\omega, B})\) as the right action of
\[
\begin{pmatrix}
1 & 0 \\
-\frac{s}{t} & 1
\end{pmatrix}
\]
on \(\sigma_{\omega, B}\), i.e. \(B'_{\omega, B} := B_{\omega, B}\)
and
\[
(\Re Z'_{\omega, B}, \Im Z'_{\omega, B}) := (\Re Z_{\omega, B}, \Im Z_{\omega, B}) \begin{pmatrix}
1 & 0 \\
-\frac{s}{t} & 1
\end{pmatrix}.
\]
Thus
\[
(\Re Z'_{\omega, B}, \Im Z'_{\omega, B}) := (\Re Z_{\omega, B}(F) - \frac{s}{t} \Im Z_{\omega, B}(F)) + \frac{1}{t} i \Im Z_{\omega, B}(F).
\]
By varying \(w\) again, we obtain another \(w\)-indexed family of half planes with coordinates \((s, t)\) (which is different from the \(\Pi_{(H, H^\perp, w)}\) defined earlier):
\[
\Pi'_{(H, H^\perp, w)} := \{\sigma'_{tH, sH + wH^\perp} \mid t \in \mathbb{R}_{>0}, s \in \mathbb{R}\} \subset \text{Stab}(X).
\]

Lemma A.1. Fix a frame \((H, H^\perp, w)\). The above right action identifies the potential walls \(W(\text{ch}, \text{ch}')\) in the \((s, t)\)-plane \(\Pi_{(H, H^\perp, w)}\) with the potential walls \(W(\text{ch}, \text{ch}')\) in the \((s, t)\)-plane \(\Pi'_{(H, H^\perp, w)}\).

Proof. [14] Lemma 2.6. \qed
Table 2. A summary of notations for \((s,t)\)- and \((s,q)\)-planes after fixing a frame \((H,H^\perp,w)\). Here we take \((\omega,B)\) as in (A.0.2), \(s,t \in \mathbb{R}\) with \(t > 0\) and \(q\) is given by (A.1.1). In particular, \(q > \frac{1}{2}s^2\).

| \((s,t)\)-plane \(\Pi_{(H,H^\perp,w)}\) | \((s,t)\)-plane \(\Pi'_{(H,H^\perp,w)}\) | \((s,q)\)-plane \(\Sigma_{(H,H^\perp,w)}\) |
|---------------------------------|---------------------------------|---------------------------------|
| \(\sigma_{tH,sH+wH^\perp}\) | \(\sigma'_{tH,sH+wH^\perp}\) | \(\sigma_{s,q} := \sigma'_{tH,sH+wH^\perp}\) |

Fix a frame \((H,H^\perp,w)\). We introduce \((s,q)\)-coordinates in addition to \((s,t)\)-coordinates via the change of variables

\[
q := \frac{s^2 + t^2}{2}
\]

(note that \(t > 0\)). This way, there is a bijection between the \(''(s,t)\)-plane’’

\[
\{(s,t) : s \in \mathbb{R}, t \in \mathbb{R} > 0\}
\]

and the \(''(s,q)\)-plane’’

\[
\{(s, q) : s \in \mathbb{R}, q \in \mathbb{R} > 0, q > \frac{1}{2}s^2\}.
\]

The family \(\Pi'_{(H,H^\perp,w)}\) of \((s,t)\)-planes will be referred to as the family \(\Sigma_{(H,H^\perp,w)}\) when using \((s,q)\)-coordinates.

The advantage of the \((s,q)\)-coordinate is that potential walls will be semi-lines (instead of semi-circles in the \((s,t)\)-coordinate). We will write

\[
\sigma_{s,q} := \sigma'_{tH,sH+wH^\perp};
\]

the associated central charge, given by (A.0.5), can be rewritten in \((s,q)\)-coordinates as

\[
Z_{s,q}(F) := (-ch_2(F) + ch_0(F)qg) + \left(\frac{1}{2}ch_0(F)\delta w^2 + wch_1(F).H^\perp\right) + i(ch_1(F).H - ch_0(F)gs).
\]

Lemma A.2. (Bertram’s nested wall theorem in \((s,q)\)-plane) Fix a Chern character \(ch\) of Bogomolov type, i.e. \(ch^2 - 2ch_0ch_2 \geq 0\). Fix a frame \((H,H^\perp,w)\) and use the notations above.

(A) Suppose \(x \neq 0\). Then all potential walls \(W(ch,ch')\) in the \((s,q)\)-plane \(\Sigma_{(H,H^\perp,w)}\) are given by semi-lines passing through the same point \(P(ch) := \left(\frac{y_1}{x}, \frac{1}{2} \left(\frac{y_1^2}{x^2} - F(ch)\right)\right)\)

with slopes \(C(ch,ch')\).

\[
q = C(ch,ch')(s - \frac{y_1}{x}) + \frac{1}{2} \left(\frac{y_1^2}{x^2} - F(ch)\right), \quad (q > \frac{x^2}{2}),
\]

where

\[
C(ch,ch') := \frac{x\chi - rz + w\delta(xc_2 - ry_2)}{g(xc_1 - ry_1)};
\]

\[
F(ch) := \frac{\delta}{g} \left(w - \frac{y_2}{x}\right)^2 + \frac{1}{x^2g} (y_1^2g - y_2^2\delta - 2xz) \geq 0.
\]

\footnote{We use the convention that if \(xc_1 - ry_1 = 0\), then the slope is infinite and the wall is the semi-line \(s = \frac{y_1}{x}\) with \(q > \frac{x^2}{2}\).}
In particular, \( P(\text{ch}) \) is on or below the parabola \( q = \frac{s^2}{2} \).

(B) Suppose \( x = 0 \) and \( \text{ch}_1 H > 0 \) (i.e. \( y_1 > 0 \)). If \( r = 0 \), then the potential wall is given by \( y_1 \chi = z c_1 \), and there is no potential wall in the \((s, q)\)-plane. If \( r \neq 0 \), then all potential walls \( W(\text{ch}, \text{ch}') \) in the \((s, q)\)-plane are given by semi-lines of the same slope \( C = C(\text{ch}) \), and they pass through points of the form \( P'(\text{ch}') := (\frac{c_1}{r}, \frac{1}{2} (\frac{c_1^2}{r^2} - F'(\text{ch}'))) \):

\[
q = C(\text{ch})(s - \frac{c_1}{r}) + \frac{1}{2} \left( \frac{c_1^2}{r^2} - F'(\text{ch}') \right), \quad (q > \frac{s^2}{2}),
\]

where

\[
C(\text{ch}) := \frac{z + \delta wy_2}{gy_1},
\]

\[
F'(\text{ch}') := \frac{\delta}{g} (w - c_2) + \frac{1}{r^2 g} (c_1^2 g - c_2^2 \delta - 2r \chi).
\]

Moreover, if \( \text{ch}' \) is also of Bogomolov type, then \( F'(\text{ch}') \geq 0 \) and \( P'(\text{ch}') \) is on or below the parabola \( q = \frac{s^2}{2} \).

Proof. [14] Lemma 2.8. \hfill \blacksquare

Lemma A.3. (Shift by line bundle) Fix a Chern character \( \text{ch} \) of Bogomolov type. Fix a frame \((H, H^\perp, w)\) and use the notations above. Fix an \( \mathbb{R} \)-divisor \( L \) of the form

\[ L = l_1 H + l_2 H^\perp + \Delta_L \]

with real coefficients \( l_1 \) and \( l_2 \), and \( \Delta_L \in \{H, H^\perp\} \) in \( \text{NS}_\mathbb{R}(X) \).

(A) Suppose \( x \neq 0 \). Then potential walls of the form \( W(e^L \text{ch}, e^L \text{ch}') \) in the \((s, q)\)-plane are all given by semi-lines passing through the same point

\[ P(e^L \text{ch}) = P(\text{ch}) + \left( l_1, \frac{1}{2} l_1^2 + \frac{y_1}{x} l_1 - \frac{\delta}{2g} l_2^2 + \frac{\delta}{g} (w - \frac{y_2}{x}) l_2 + \frac{1}{2g} \Delta_L^2 + \frac{\Delta \Delta_L}{xy} \right) \]

with slopes\(^2\)

\[
C(e^L \text{ch}, e^L \text{ch}') = C(\text{ch}, \text{ch}') + l_2 - l_2 \frac{\delta}{g} \frac{xc_2 - ry_2}{xy_1} + \frac{x \Delta' \Delta_L - r \Delta \Delta_L}{g(xc_1 - ry_1)}
\]

in the region \( q > \frac{s^2}{2} \).

(B) Suppose \( x = 0 \) and \( \text{ch}_1 H > 0 \). Then potential walls of the form \( W(e^L \text{ch}, e^L \text{ch}') \) in the \((s, q)\)-plane are all given by semi-lines passing through points

\[ P'(e^L \text{ch}') = P'(\text{ch}') + \left( l_1, \frac{1}{2} l_1^2 + \frac{\delta}{2g} l_2^2 + \frac{\delta}{g} (w - \frac{c_2}{r}) l_2 + \frac{1}{2g} \Delta_L^2 + \frac{\Delta' \Delta_L}{rg} \right) \]

with the same slope

\[
C(e^L \text{ch}) = C(\text{ch}) + l_2 \frac{\delta}{g} \frac{y_2}{y_1} + \frac{\Delta \Delta_L}{gy_1}
\]

in the region \( q > \frac{s^2}{2} \).

\(^2\) We use the convention that if \( xc_1 - ry_1 = 0 \), then the slope is infinite and the wall is the semi-line \( s = \frac{r}{y} + l_1 \) with \( q > \frac{s^2}{2} \).
Proof. The formula (A.3.1) follows from (A.2.2). By using formula (A.2.3), we get
\[ F(e^L \text{ch}) = F(\text{ch}) + \frac{\delta l^2}{g} - \frac{2\delta}{g}(w - \frac{\delta^2}{r})l_2 - \frac{\Delta^2}{g} - \frac{2\Delta \Delta \Delta L}{rg}. \]
Thus we obtain the formula for \( P(e^L \text{ch}) \). This shows part (A). The formula (A.3.2) follows from (A.2.5). By using formula (A.2.6), we get
\[ F'(e^L \text{ch}') = F'(\text{ch}') + \frac{\delta l^2}{g} - \frac{2\delta}{g}(w - \frac{\delta^2}{r})l_2 - \frac{\Delta^2}{g} - \frac{2\Delta' \Delta L}{rg}. \]
Thus we obtain the formula for \( P'(e^L \text{ch}') \). This shows part (B). □

Suppose we are in the situation of Lemma A.2(A). By (A.2.3),
\[ F(\text{ch}) = \frac{\delta}{g} \left( w - \frac{\delta^2}{x} \right)^2 + \frac{1}{x^2g}(\text{ch}^2_1 - 2\text{ch}_0 \text{ch}_2 - \Delta^2), \]
\[ F(e^L \text{ch}) = \frac{\delta}{g} \left( w - \frac{\delta^2}{x} - l_2 \right)^2 + \frac{1}{x^2g}(\text{ch}^2_1 - 2\text{ch}_0 \text{ch}_2 - (\Delta + x \Delta L)^2). \]
Since \( H \Delta = 0 \) by assumption, the Hodge Index Theorem implies that \( -\Delta^2 \geq 0 \), and equality holds if and only if \( \Delta = 0 \). Similarly, we have \( -(\Delta + x \Delta L)^2 \geq 0 \) with equality if and only if \( \Delta + x \Delta L = 0 \). Therefore, if \( \text{ch} \) is of Bogomolov type, then \( F(\text{ch}) \geq 0 \) and \( F(e^L \text{ch}) \geq 0 \) for all \( w \). Thus the points \( P(\text{ch}) \) and \( P(e^L \text{ch}) \) are on or below the parabola \( q = \frac{\Delta^2}{2} \). If we are in the situation of Lemma A.2(B), then a similar argument works for \( P'(\text{ch}') \) and \( P'(e^L \text{ch}') \) provided \( \text{ch}' \) is of Bogomolov type.

**APPENDIX B. POTENTIAL WALLS IN \((\lambda, 0, 0, q)\)-PLANE FOR ONE-DIMENSIONAL OBJECTS**

We give a parallel result of [7.4] for potential walls in the \((\lambda, 0, 0, q)\)-plane in the case of 1-dimensional objects. We use the notation in [7.3].

Fix \( \text{ch} \) with \( \text{ch}_0 = 0 \) and \( \text{ch}_1 H_{\lambda} > 0 \). Let \( \text{ch}' \) be a destabilizing character. So \( \text{ch}_0' \neq 0 \). We have \( \text{ch}' = e^L \left( \text{ch}_0', 0, \text{ch}_2' - \frac{\text{ch}^2_1}{2\text{ch}_0'} \right) \), and \( \text{ch} = e^L (0, \text{ch}_1, \text{ch}_2 - L \text{ch}_1) \) with \( L = \frac{\text{ch}_1}{\text{ch}_0'} \).

**Proposition B.1.** (Potential walls in \((\lambda, 0, 0, 0)\)-plane for one-dimensional objects) Let
\[ \text{ch} = (0, k\Theta + pf + \sum_i \xi_i \Theta_i, z), \quad L = a_L \Theta + b_L f + \sum_i \eta_i \Theta_i, \]
where \( \text{ch}_1 H_{\lambda} > 0 \) and \( a_L, b_L, \eta_i \in \mathbb{R} \). Then the potential wall \( W(e^L \text{ch}, e^L \text{ch}') \) with
\[ \text{ch}' = (r, 0, \chi), \]
has the following asymptotic behavior in the \((\lambda, 0, 0, q)\)-plane as \( \lambda \to 0^+ \). Write \( \Delta_L = \sum_i \eta_i \Delta_i \) and \( \Delta = \sum_i \xi_i \Delta_i \).

(A) Suppose \( k + \sum_i \xi_i = 0 \) and \( pek + \sum_i \xi_i \theta_i \neq 0 \). Set
\[
(A.1) \quad A := -\left( z + \Delta \Delta_L + (a_L + \sum \eta_i)(p - \frac{e}{2}k + \sum_i \xi_i (\theta_i + \frac{\xi_i}{2})) \right) \frac{a_L + \sum \eta_i}{pek + \sum \xi_i \theta_i};
\]
\[
(B.1.1) \quad B := \frac{\chi}{r} + \frac{\Delta_L^2}{2} - \left( b_L - e a_L + \frac{e}{2} (a_L + \sum \eta_i) + \sum \eta_i \theta_i \right) \frac{z + \Delta \Delta_L}{pek + \sum \xi_i \theta_i};
\]
(B.1.2)
(A1) If $A \neq 0$ then the potential wall is asymptotic to

\[ q = \frac{A}{2\lambda^2}. \]

(B.1.3)

(A2) If $A = 0$ and $B \neq 0$ then the potential wall is asymptotic to

\[ q = \frac{B}{2\lambda}. \]

(B.1.4)

(A3) If $A = 0$ and $B = 0$, then the potential wall is bounded as $\lambda \to 0^+$.  

(B) Suppose $k + \sum \xi_i \neq 0$. Set

\[ D := \frac{\lambda}{r} + \frac{\Delta^{2}}{2} - \left( z + \Delta \Delta_L + \left( a_L + \sum \eta_i \right) \left( p - \frac{e}{2} k + \sum \xi_i (\theta_i + \frac{e}{2}) \right) \right) \frac{a_L + \sum \eta_i}{k + \sum \xi_i}. \]

(B.1.5)

(B1) If $D \neq 0$ then the potential wall is asymptotic to

\[ q = \frac{D}{2\lambda}. \]

(B.1.6)

(B2) If $D = 0$, then the potential wall is bounded as $\lambda \to 0^+$.

Proof. The proof is similar to the proof of Proposition 7.4. Now $y_1 g = ch_1 H_\lambda > 0$. The potential wall $W(e^L ch, e^L ch')$ in the $(\lambda, 0, 0, q)$-plane is given by

\[ q = -z + (\Delta \Delta_L) g h \left( \frac{y_1 + y_2}{y_1} \right) - \frac{1}{2} \left( l_1 + l_2 \right)^2 + \frac{\lambda}{r g} + \frac{\Delta^{2}}{2g}. \]

(B.1.7)

Similar computation shows that

\[ q = - (z + \Delta \Delta_L) \left( \frac{a_L + \sum \eta_i}{g c_1 g} + \left( b_L - e a_L + (a_L + \sum \eta_i) \left( m - 1 \right) + \sum \eta_i (\theta_i + \frac{e}{2}) \right) \frac{a_L + \sum \eta_i}{k + \sum \xi_i} \right) \frac{1}{g} \]

\[ + \frac{1}{4} \left( \left( \frac{b_L - \frac{e}{2} a_L + \sum \xi_i (\theta_i + \frac{e}{2})}{1 + (m - \frac{e}{2} - 1) \lambda} \right)^2 - \left( \frac{a_L + \sum \eta_i}{\lambda^2} \right)^2 \right) \frac{2\lambda \left( p - \frac{e}{2} k + \sum \xi_i (\theta_i + \frac{e}{2}) \right)}{\left( k + \sum \xi_i \right) \left( p - e k + (k + \sum \xi_i) \left( m - 1 \right) + \sum \xi_i (\theta_i) \right)} \lambda \]

\[ - \frac{1}{2} \left( \frac{b_L - \frac{e}{2} a_L + \sum \xi_i (\theta_i + \frac{e}{2})}{1 + (m - \frac{e}{2} - 1) \lambda} \right)^2 + \left( \frac{\lambda}{r g} + \frac{\Delta^{2}}{2g} \right) \cdot \frac{1}{g}. \]

The proof follows from the asymptotic analysis of above formula as $\lambda \to 0^+$.  

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CURRENT: DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, BOX 480, 751 06 UPPSALA, SWEDEN
E-mail address: wanminliu@gmail.com

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY NORTHridge, 18111 NORDHOFF STREET, NORTHridge CA 91330, USA
E-mail address: jason.lo@csun.edu

DEPARTMENT OF MATHEMATICS, UNIVERSIDAD DE LOS ANDES, BOGOTÁ 111711, COLOMBIA
E-mail address: martinecristian@gmail.com