INTRODUCTION: ANOMALIES FOR PEDESTRIANS

Anomalies can be viewed as a breaking of some Noether symmetry through the effects of the vacuum. In relativistic quantum field theory (QFT), such a (classical) symmetry is \textit{broken} by field quantization, cf. Refs [20, 7, 51] for recent reviews. This has important implications on such physical processes as the decay of the neutral $\pi$-meson [6], induced instanton effects [43], or underlies the postulation of the \textit{axion} in quantum chromodynamics (QCD), cf. Ref. [41].

In quantum electrodynamics (QED), Schwinger [46] demonstrated that the charge current $j$ can be retained conserved, i.e. $\langle dj \rangle = 0$, whereas the conservation of the axial current $j_5$ is broken, $\langle dj_5 \rangle \neq 0$.

In the Fujikawa approach [14], the right-hand side can obtain from considering the point-splitted current $j_5(x; \varepsilon) := \bar{\psi}(x)\gamma_5\gamma\psi(x + \varepsilon)$, where $\varepsilon$ is an infinitesimal four-vector in spacetime. Such an expression can be rendered invariant by dressing it with a path-ordered exponential

$$\bar{\psi}(x)\gamma_5\gamma\psi(x + \varepsilon) \rightarrow \bar{\psi}(x)\gamma_5\gamma\psi(x + \varepsilon)P \exp\left\{i \int_x^{x+\varepsilon} A \right\}. \quad (1)$$

The variation $\delta/\delta A$ of the current $j_5(x; \varepsilon)$ is compensated by the variation of the exponential. As the parallel transport from $x^i \rightarrow x^i + \varepsilon^i$ along the infinitesimal line element can be expanded perturbatively, it is clear that the net effect of this approach is just the standard result $\langle dj_5(x) \rangle = 2im\langle P \rangle - (1/96\pi^2)F \wedge F$ for massive fermions, where $F := dA$ is the gauge field strength. Further details of the path integral formulation were developed, e.g., in Refs. [2, 3, 50] with extension of the regularized Jacobian, as well as in the light-cone gauge [15] of the Schwinger model.

There is an intuitive physical interpretation of this result: The additional Chern-Simons (CS) term $C := A \wedge dA$ corresponds to the spin or helicity of the photon, with its spacelike part $\vec{A} \cdot \vec{B}$ known as magnetic helicity [22]. Since the axial current $j_5$ is proportional to the spin of a fermion, the deformed current $\tilde{j}_5 := j_5 + (1/96\pi^2)A \wedge dA$, includes the spin of the photon, lacking, however, gauge invariance. The chiral anomaly

\begin{footnote}
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\end{footnote}
can then be understood as the ‘conservation law’

\[ \langle d \tilde{j}_S \rangle = 0, \]

such that in QFT “...the flow of electronic spin drags some photon spin and vice versa” [53].

Anomalies were studied also in Yang-Mills type gauge models of gravity [19] with Einsteinian instanton solutions [40]. Then, the equivalence principle not only requires a coupling of gravity to the energy-momentum current of matter, but also to the spin current. Here we will focus on the intricate interaction between the chiral anomaly and the spin or helicity of the gravitational gauge field and extend it to post-Riemannian spacetimes with torsion.

**DIRAC FIELDS IN RIEMANN–CARTAN SPACETIME**

In our notation [30], a Dirac field is a bispinor–valued zero–form \( \psi \) for which \( \overline{\psi} := \psi^\dagger \gamma_0 \) denotes the Dirac adjoint. The minimal coupling to the gauge (electromagnetic) potential \( A = A_i dx^i \) is accounted for via \( D := D + iA \wedge \), where \( D\psi := d\psi + \Gamma \wedge \psi \) is the exterior covariant derivative with respect to the Riemann-Cartan (RC) connection one-form \( \Gamma_{\alpha \beta} = \Gamma_i^{\alpha \beta} dx^i \).

The Dirac Lagrangian is given by the manifestly Hermitian four–form

\[ L_D = L(\gamma, \psi, D\psi) = \frac{i}{2} \left\{ \overline{\psi} \wedge D\psi + D\overline{\psi} \wedge \psi \right\} + m \overline{\psi} \psi, \]

where \( \gamma := \gamma_0 \partial^\alpha \) is the Clifford algebra-valued coframe, see the Appendix.

The Dirac equation and its adjoint can be obtained by varying \( L_D \) independently with respect to \( \psi \) and \( \psi^\dagger \). Making use of the torsion \( \Theta := D\gamma \) and of the properties of the Hodge dual, the Dirac equation assumes the form

\[ i \gamma \wedge \left( D + \frac{i}{4} m\gamma - \frac{1}{2} T \right) \psi = 0, \]

where \( T := \frac{1}{4} Tr(\overline{\gamma} \Theta) = e_\alpha |T^\alpha| \) is the one–form of the trace (or vector) torsion. However, the covariant derivative \( D \) also contains torsion.

In order to separate out the purely Riemannian piece from torsion terms, let us decompose the Riemann–Cartan connection \( \Gamma = \Gamma^{(1)} - K \) into the Riemannian (or Christoffel) connection \( \Gamma^{(1)} \) and the contortion one–form \( K = \frac{1}{4} K^{\alpha \beta} \sigma_{\alpha \beta} \), obeying \( D\gamma = [\gamma, K] = \gamma_\alpha T^\alpha \). Accordingly, the Dirac Lagrangian (3) splits [32] into a Riemannian and a spin–contortion piece:

\[ L_D = L(\gamma, \psi, D^{(1)}\psi) - \frac{i}{2} \overline{\psi} \left( \gamma \wedge K - K \wedge \gamma \right) \psi + A \wedge j \]

\[ = L(\gamma, \psi, D^{(1)}\psi) + \frac{1}{4} \mathcal{A} \wedge jS + A \wedge j \]

\[ = L(\gamma, \psi, D^{(1)}\psi) - T^\alpha \wedge \mu_\alpha + A \wedge j. \]
The covariant derivative with respect to the Riemannian connection $\Gamma^{(\ell)}$ satisfies $D^{(\ell)} \gamma = 0$. Hence, in a RC spacetime a Dirac spinor only feels the axial torsion one–form

$$\mathcal{A} := \frac{1}{4} * Tr(\gamma \wedge D\gamma) = * (\theta^\alpha \wedge T_\alpha) = \frac{1}{2} T^{[\alpha\beta\gamma]} \eta_{\alpha\beta\gamma} = \mathcal{A} dx^i,$$

which is invariant under Weyl rescalings and chiral transformations $\gamma \rightarrow \gamma^\beta = e^{i\gamma^\beta} \gamma e^{-i\gamma^\beta}$ of the coframe, but odd under parity $P : \psi^B \rightarrow -\psi^B$, where $B = 1, 2, 3$, cf. Ref. [38].

**CLASSICAL AXIAL ANOMALY AND SPIN**

Similarly as in QED, the gravitational coupled Dirac Lagrangian $L_D = \overline{L_D} = L_D^\dagger$ is Hermitian as required, even in an anholonomic frame. Then minimal coupling prescribes us automatically with the following charge and axial currents, respectively,

$$j = \overline{\psi} \gamma \psi = j^\mu \eta_\mu, \quad j_5 := \overline{\psi} \gamma^5 \psi = \frac{1}{3} \overline{\psi} \sigma \wedge \gamma \psi = j^5_5 \eta_\mu.$$  (7)

From the Dirac equation (4) and its adjoint one can readily deduce that $d j \simeq 0$ ‘on shell’, whereas for the axial current we find the well–known “classical axial anomaly”

$$d j_5 = 2imP = 2i \overline{\psi} \gamma_5 \psi \eta$$  (8)

for massive Dirac fields [21]. The same holds in a RC spacetime. If we restore chiral symmetry in the limit $m \rightarrow 0$, this leads to classical conservation law $d j_5 = 0$ of the axial current for massless Weyl spinors, or since $d j \simeq 0$, equivalently, for the chiral current $j_\pm := \frac{1}{2} \overline{\psi} (1 \pm \gamma_5)^* \gamma \psi = \overline{\psi}_{L,R}^* \gamma \psi_{L,R}$.

As mentioned in the Introduction, the axial current has an intriguing relation to the (dynamical) spin current of the Dirac field canonically defined by the Hermitian three–form

$$\tau_{\alpha \beta} := \frac{\partial L_D}{\partial \Gamma^{\alpha \beta}} = \frac{1}{8} \overline{\psi} (\gamma^\sigma \alpha \beta + \sigma^\alpha \beta \gamma) \psi$$

$$= \frac{1}{4} \eta_{\alpha \beta \gamma} \overline{\psi} \gamma^\delta \gamma_5 \psi \eta = \tau_{\alpha \beta \gamma} \eta_\gamma = \theta_{[\alpha \wedge \beta]}.$$  (9)

Its components $\tau_{\alpha \beta \gamma} = \tau_{[\alpha \beta \gamma]}$ are totally antisymmetric. Equivalently, in Eq. (5) torsion merely couples to the two–form

$$\mu_{\alpha} = \frac{1}{4} \theta_{\alpha} \wedge * j_5,$$  (10)

commonly referred to as the spin-energy potential [18, 32]. Consequently, we obtain the remarkable result that he vector one–form

$$\mu := e_\alpha \mu^\alpha = \frac{3}{4} * j_5,$$  (11)
of the Dirac spin is dual to the axial current \( j_5 \).

There is also a relation to the axion: In Ref. [41] it is tentatively assumed that the dimensionless pseudo-scalar \( \theta \) serves as a potential for the axial torsion via \( \mathcal{A} = 2d\theta \). Then, there arises in (5) a derivative coupling of the would-be axion \( a = \theta f_a \) to two fermions via the CPT-invariant term

\[ L_a = \frac{1}{2} d\theta \wedge j_5 = \frac{1}{2f_a} da \wedge \bar{\psi}^* \gamma_5 \psi , \tag{12} \]

exactly as in the usual formulation, where the axial current \( j_5 \) is the Noether current associated with a spontaneously broken Peccei-Quinn symmetry \( U(1)_{PQ} \).

**AXIAL CURRENT IN THE EINSTEIN–CARTAN THEORY**

The Einstein–Cartan (EC) theory of a gravitationally coupled spin 1/2 Dirac field provides a dynamical understanding of the axial anomaly on a semi-classical level: The Lagrangian reads:

\[ L = \frac{i}{2\ell^2} Tr(\Omega \wedge *\sigma) + L_D = \frac{1}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + L_D , \tag{13} \]

where \( \eta^{\alpha\beta} := *((\theta^\alpha \wedge \theta^\beta) \) is dual to the unit two–form.

In EC–theory, Cartan’s algebraic relation between torsion and spin implies the following relation [37] between the axial current \( j_5 \) of the Dirac field and the translational Chern-Simons (CS) term (27), or equivalent, for the axial torsion one-form:

\[ C_{TT} \cong \frac{1}{4} j_5 , \quad \mathcal{A} = 2\ell^2 *C_{TT} = (\ell^2/2) \bar{\psi} \gamma_5 \gamma \psi . \tag{14} \]

Thus in EC-theory, the net axial current production

\[ dj_5 \cong 4dC_{TT} = \frac{2}{\ell^2} \left( T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \theta^\alpha \wedge \theta^\beta \right) \tag{15} \]

establishes a link to the NY four form [44] for massive fields.

This result, cf. [33, 37], holds on the level of first quantization. Since the Hamiltonian of the semi-classical Dirac field is not bounded from below, one has to go over to second quantization, where the vacuum expectation value \( \langle dj_5 \rangle \) of the axial current picks up anomalous terms.

Restoring chiral invariance for the Dirac fields, the limit \( m \to 0 \), implies that the NY four–form tends to zero “on shell”, i.e. \( dC_{TT} \cong (1/4) dj_5 \to 0 \). This is consistent with the fact that a Weyl spinor does not couple to torsion at all, because then the axial torsion \( \mathcal{A} \) becomes a lightlike covector, i.e. \( \mathcal{A} \wedge \mathcal{A} \eta = \mathcal{A} \wedge *\mathcal{A} \cong (\ell^2/4) *j_5 \wedge j_5 = 0 \). Here we implicitly assume that the light-cone structure of the axial covector \( *j_5 \) is not spoiled by quantum corrections, i.e. that no “Lorentz anomaly” occurs as in \( n = 4k + 2 \) dimensions [26].
CHIRAL ANOMALY IN QUANTUM FIELD THEORY

Let us recall a couple of distinguished features of the axial anomaly: Most prominent is its relation with the Atiyah–Singer index theorem \[5\]. But also from the viewpoint of perturbative QFT, the chiral anomaly has some features which signal its conceptual importance. For all topological field theories and topological effects like the anomaly, there is the remarkable fact that it does not renormalize — higher order loop corrections do not alter its one-loop value. This very fact guarantees that the anomaly can be given a topological interpretation. For the anomaly, this is the Adler–Bardeen theorem \[1\], while other topological field theories are carefully designed to have vanishing beta functions, for example. Another feature is its finiteness: in any approach, the chiral anomaly as a topological invariant is a finite quantity.

Now, to approach the anomaly in the context of spacetime with torsion, let us first switch off the Riemannian curvature and concentrate on the last but one term in the decomposed Dirac Lagrangian (5).

Then, this term can be regarded as an external axial covector \( \mathcal{A} \) coupled to the axial current \( j_5 \) of the Dirac field in an initially flat spacetime. By applying the result (11–225) of Itzykson and Zuber \[21\], we find that only the term \( d\mathcal{A} \wedge d\mathcal{A} \) arises in the axial anomaly, but not the NY type term \( d^*\mathcal{A} \sim dC_{TT} \) as was recently claimed \[10\]. After switching on the Yang-Mills field \( G \) as well as the curved spacetime of Riemannian geometry, we finally obtain for the vacuum expectation value of the axial anomaly

\[
\langle d j_5 \rangle = 2im \langle \overline{\psi} \gamma_5 \psi \rangle \eta - \frac{1}{4\pi^2} Tr(G \wedge G) - \frac{1}{96\pi^2} \left[ 2R^{(1)}_{\alpha\beta} \wedge R^{(1)}_{\alpha\beta} + d\mathcal{A} \wedge d\mathcal{A} \right]. \tag{16}
\]

This result \[25\] is based on diagrammatic techniques and the Pauli–Villars regularization scheme. In this respect, it is a typical perturbative result, and in agreement with \[16, 54, 52\] no NY term arises in the anomaly. Thus only the Weyl invariant term \( d\mathcal{A} \wedge d\mathcal{A} = -2\mathcal{F} \cdot \mathcal{F} \) for the axial torsion contributes to the axial anomaly, resembling the \( U(1) \) part \( F \wedge F = dA \wedge dA \) of the Pontrjagin term (26). Torsion terms like \( d\mathcal{A} \wedge d\mathcal{A} \) and \( d^*\mathcal{A} \wedge *(d^*\mathcal{A}) = 4\ell^2V_{NY} \wedge V_{NY} \) have been considered previously, as part of the Lagrangian, in order to make the axial torsion propagating. Due to the geometric identity (28) for the NY term \( d^*\mathcal{A} = 2\ell^2dC_{TT} = 2\ell^2V_{NY} \), the second term is really quartic in torsion and not scale invariant.

A rescaling of the tetrad has been proposed, however, one should not ignore the presence of renormalization conditions and the generation of a scale upon renormalization. Rescaling the tetrad would ultimately change the wave function renormalization \( Z \)-factor which would creep into the definition of the NY term, in sharp contrast to proper topological invariants at the quantum level, which remain unchanged under renormalization.

With no renormalization condition available for the NY term, and other methods obtaining it as zero, we can only conclude that the response function of QFT to a gauge variation (this is the anomaly) delivers no NY term. Or, saying it differently, its finite value is zero after renormalization.
Chiral anomaly in SUGRA

Simple supergravity consists in a consistent coupling of the EC to the Rarita–Schwinger spinor-valued one-form \( \Psi = \Psi_i dx^i \), cf. Refs. [49, 36] for more details.

The anomaly for the corresponding axial current \( J_5 := i\bar{\Psi} \gamma^5 \Psi \) is \( -21 \times \) the anomaly for Dirac fields, whereas for the corresponding supersymmetric Yang–Mills anomaly one finds \( 3 \times \) the Dirac result:

| Spin | Gravitational | YM anomaly |
|------|--------------|------------|
| 1/2  | 1            | 1          |
| 3/2  | -21          | 3          |

Depending on the asymptotic helicity states, there occur contributions of topological origin of the Riemannian Pontrjagin or Euler type, respectively. The role of spinors for the index theorem and in the 4D Donaldson invariants via Seiberg–Witten equation has recently been reviewed by Atiyah [5]. Six dimensional supergravity free of gauge and gravitational anomalies is studied in Ref. [12].

COMPARISON WITH THE HEAT KERNEL METHOD

In the heat kernel approach, there exists for small \( t \to +0 \) the asymptotic expansion

\[
K(t, x, \mathcal{D}^2) = (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(k-n)/2} K_k(x, \mathcal{D}^2) \tag{17}
\]

of the kernel in \( n \) dimensions, where the usual Feynman “dagger” convention \( \mathcal{A} := \gamma^\alpha \epsilon_{\alpha} A = \gamma^\alpha \epsilon_{\alpha} A = (-1)^{s+1} [\gamma \wedge A] \) for one–forms is used.

The squared Dirac operator

\[
\begin{align*}
\mathcal{D}^2 &= \frac{1}{2} \gamma^\alpha \gamma^\beta \left( \{D^\alpha, D^\beta\} + [D^\alpha, D^\beta] \right) - 2im\mathcal{D}^2 \\
&\quad - \frac{i}{4} \gamma_5 (\mathcal{D}^\alpha \mathcal{A}) + \frac{1}{2} \gamma_5 \sigma^{\alpha\beta} \mathcal{A} \gamma D^\beta + m^2 - \frac{1}{2} i m \mathcal{D}^2 \\
&\quad \approx -\square - \frac{1}{8} \sigma^{\alpha\beta} R^\alpha_{\beta \mu \nu} \sigma^{\mu \nu} \\
&\quad - \frac{i}{4} \gamma_5 (\mathcal{D}^\alpha \mathcal{A}) + \frac{1}{2} \gamma_5 \sigma^{\alpha\beta} \mathcal{A} D^\beta - \frac{1}{16} \mathcal{A} \mathcal{A} - m^2, \tag{18}
\end{align*}
\]

has been explicitly calculated in Refs. [54, 45], and the terms additional to the generally covariant Riemannian d’Alembertian operator \( \square := \partial_\mu \left( \sqrt{|g|} g^{\mu \nu} \partial_\nu \right) \) are identified [34]. Not unexpectedly, besides the familiar Riemannian curvature scalar, only the axial torsion (6) contributes to the squared Dirac operator for massive spinor fields.

The coefficients \( K_k(x, \mathcal{D}^2) \) are completely determined by the form of the second-order differential operator \( \mathcal{D}^2 \), which is positive for Euclidean signature \( \text{diag} \, o_{\alpha\beta} = 1 \).
$(-1, \cdots, -1)$. For odd $k = 1, 3, \ldots$ these coefficients are zero, while the first nontrivial terms [54], which potentially could contribute to the axial anomaly, read

$$\begin{align*}
Tr(\gamma_5 K_2) &= -d^* \mathcal{A}, \\
Tr(\gamma_5 K_4) &= \frac{1}{6} \left[ Tr \left( R^{(1)} \wedge R^{(1)} \right) - \frac{1}{4} d^* \mathcal{A} \wedge d^* \mathcal{A} + d^* \mathcal{K} \right],
\end{align*}$$

(19)

where the higher order term $d^* \mathcal{K} = d^* \bar{D} \wedge^* \bar{D}^* \mathcal{A}$ involves the covariant derivative $\bar{D} = D^{(1)} + i \mathcal{A} \gamma_5 / 4$ modified by the axial torsion.

However, there is an essential difference in the physical dimensionality of the terms $K_2$ and $K_4$. Whereas in $n = 4$ dimensions the Pontrjagin type term $K_4$ is dimensionless and thus, for $k = 4$, multiplied by $t^{(k-4)/2} = 1$, the term $K_2 \sim d^* \mathcal{A} = 2 \ell^2 dC_{TT}$ carries dimensions. Since a massive Dirac spinor has canonical dimension $[\text{length}]^{-3/2}$, it scales as $\psi \sim m^{3/2}$. Moreover, the term $t = 1 / M^2$ is related to the regulator mass $M \to \infty$ in Fujikawa method [14]. Then the second order term in the heat kernel expansion scales as $-K_2 / t = (2 \ell^2 / 4) dC_{TT} \cong (\ell^2 / 2) dD_5 = (im^{3/2} / 4) \bar{\nabla}\gamma_5 \psi \sim \ell^2 M^2 m^4 \to 0$. If we assume in the renormalization procedure, that the fundamental length $\ell$ does not scale (no running coupling constant), the second order term in the heat kernel expansion will tend to zero in the chiral limit $m \to 0$. In the case $m \neq 0$, this term diverges and the Fujikawa regulator method $M \to \infty$ cannot be applied. To rescale the coframe by $\hat{\psi}^\alpha \to \hat{\psi}^\alpha = M \psi^\alpha$ does not help, since this would change also the dimension of the Dirac field, in order to retain the physical dimension $[\hbar]$ of the Dirac action.

Thus the NY term $dC_{TT}$ does NOT contribute to the chiral anomaly in four dimensions, neither classically nor in QFT. On would surmise that in $n = 2$ dimensional models only the term $d^* \mathcal{A}$ survives in the heat kernel expansion, since it then has the correct dimensions. However, it is well-known [19] that in 2D the axial torsion $\mathcal{A}$ vanishes identically. Moreover, gravitational anomalies [26], specifically the Einstein anomaly and the Weyl anomaly, are fully determined by means of dispersion relations [8].

Let us stress the interrelation between the scale and chiral invariance: The renormalized conformal (or trace) anomaly [11]

$$\langle \hat{\psi}^\alpha \wedge \sigma_\alpha \rangle = -\frac{1}{3\pi^2} \left[ Tr(G \wedge *G) + \frac{1}{24} \left( 2R^\alpha_\beta^{(1)} \wedge R^{*\alpha_\beta}_\beta + d^* \mathcal{A} \wedge *d^* \mathcal{A} \right) \right] \tag{20}$$

for the energy-momentum current $\sigma_\alpha := \Sigma_\alpha - D_\alpha \psi$ Belinfante symmetrized via the spin energy (10) receives, in addition to the Riemannian Euler term, a kinetic contribution of the Maxwell type from the axial torsion $\mathcal{A}$. The coefficients are similar to those in Eq. (16), due to the fact that chiral and trace anomalies constitute a supermultiplet [55].

**HAMILTONIAN INTERPRETATION OF ANOMALIES**

In the canonical formulation à la Ashtekar [4], the translational NY term $dC_{TT}$ plays via

$$V_{EC}^{(\pm)} := V_{EC} \pm i dC_{TT} = \pm \frac{1}{2\ell^2} Tr \left\{ (1 \mp \gamma_5) \Omega \wedge \sigma \right\} = -\frac{1}{2\ell^2} R^\alpha_\beta \wedge \eta^\alpha_\beta + \frac{\Lambda}{\ell^2} \eta \tag{21}$$

where the fundamental length $\ell$ does not scale (no running coupling constant), the second order term in the heat kernel expansion will tend to zero in the chiral limit $m \to 0$. In the case $m \neq 0$, this term diverges and the Fujikawa regulator method $M \to \infty$ cannot be applied. To rescale the coframe by $\hat{\psi}^\alpha \to \hat{\psi}^\alpha = M \psi^\alpha$ does not help, since this would change also the dimension of the Dirac field, in order to retain the physical dimension $[\hbar]$ of the Dirac action.

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the role of the generating functional [28] for chiral, i.e. self– or antiselfdual variables \( \pm \Gamma \) in EC theory as well as in simple supergravity [33, 36].

The appearance of the Riemannian Pontrjagin term \( dC_{RR} \) in the anomaly (16) could pose problems for the canonical approach to gravity, since the anomaly does not renormalize. In the presence of gravitational instantons, which due to the necessary condition \( \Lambda \neq 0 \) could even be the dominating configurations, one gets a net production of chiral zero modes and a global symmetry is broken.

One could argue that this is a perturbative effect. In the Wilson type loop approach to gravity [9, 17, 31], the tangential complexified CS term \( C^{\{ \}}_{RR} \) is known [24] to solve the Hamiltonian constraint \( \mathcal{H}_A \Psi(\pm \Gamma) = 0 \) of gravity, where the complex Ashtekar variable \( \pm \Gamma \) is the tangential part of the self- or antiselfdual spin connection one–form. Since this solution is intrinsically non–perturbative, no anomaly should occur. In the lattice gauge approach this is indeed the case, but the problem of fermion doubling [47] appears to be another manifestation of the anomaly.

It is instructive to look at the problem from an Hamiltonian point of view since the canonical formalism of chiral gravity is closely related to the \( SU(2) \) CS gauge theory on the three–dimensional hypersurface, with \( C := \Gamma/G \) of non-equivalent gauge connections as configuration space.

Gauge anomalies are related to the global topology and have the common feature [42] that the Gauss constraint \( G^A \cong 0 \) cannot anymore be implemented on the physical states [13]. The reason is that the anomalous Ward identity

\[
\mathcal{L}_n G^A \cong n](D \tau^A), \tag{22}
\]

where \( \mathcal{L}_n := n[D + Dn] \) is the gauge–covariant Lie derivative along the normal direction, relates the time evolution of the Gauss constraint to the conservation law for the matter current \( \tau^A \) on the spacelike hypersurface [23]. Only when the individual contributions to the anomaly cancel each other, a gauge theory can be consistently quantized. In the EC formulation of the gravitationally coupled Dirac field, it is the canonical spin \( \tau^A := (1/2)\eta^{\alpha\beta}\gamma^\alpha \gamma^\beta \), which appears on the right-hand side of the Gauss constraint. Since spin is via (9,11) related to the axial current \( j_5 \), it is precisely the chiral anomaly which prevents the Gauss constraint to remain a proper constraint under time evolution. This result confronts the Ashtekar approach based on loop variables, and thus on notions of parallel transport, with the chiral anomaly [34, 35]. The teleparallelism equivalent [28] of chiral gravity, where Wilson loops are replaced by Cartan circuits [29, 31], may avoid some of these obstacles.

APPENDIX A: GRAVITATIONAL CHERN–SIMONS AND PONTRJAGIN TERMS

When the constant Dirac matrices \( \gamma_\alpha \) obeying \( \gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\sigma_{\alpha\beta} \) are saturating the index of the orthonomal coframe one–form \( \phi^\alpha = e^{\alpha}_{\cdot j} dx^j \) and its Hodge dual \( \eta^\alpha := *\phi^\alpha \),
we obtain a basis of Clifford–algebra valued exterior forms \([33, 30]\) via:

\[
\gamma := \gamma_\alpha \partial^\alpha, \quad \gamma^\ast := \gamma^\alpha \eta_\alpha. \tag{23}
\]

In terms of the Clifford algebra–valued connection \(\Gamma := \frac{1}{4} \Gamma_\alpha^\beta \sigma_{\alpha \beta} dx^\ell\), the \(SL(2,\mathbb{C})\)–covariant exterior derivative is given by \(D = d + \Gamma \wedge\), where \(\sigma_{\alpha \beta} = \frac{1}{2} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)\) are the Lorentz generators entering in the two-form \(\sigma := \frac{i}{2} \gamma \wedge \gamma = \frac{1}{2} \sigma_{\alpha \beta} \partial^\alpha \wedge \partial^\beta\).

Differentiation of these basic variables leads to the Clifford algebra–valued torsion and curvature two–forms:

\[
\Theta := D\gamma = T^\alpha \gamma_\alpha, \quad \Omega := d\Gamma + \Gamma \wedge = \frac{i}{4} R^\alpha_{\beta \gamma} \sigma_{\alpha \beta} \tag{24}
\]

in RC geometry. The Chern–Simons term for the Lorentz connection reads

\[
C_{RR} := -Tr \left( \Gamma \wedge \Omega - \frac{1}{3} \Gamma \wedge \Gamma \wedge \Gamma \right). \tag{25}
\]

The corresponding Pontrjagin topological term can be obtained by exterior differentiation

\[
dC_{RR} &= -Tr (\Omega \wedge \Omega) \\
&= \frac{1}{2} R^\{\ell\}_{\alpha \beta} \wedge R^{\{\alpha \beta} \\
&+ \frac{1}{12} d \left[ \ast \mathcal{A} \wedge R^{\{\} - \frac{1}{3} \mathcal{A} \wedge d \mathcal{A} + \frac{1}{9} \ast \mathcal{A} \wedge \ast (\mathcal{A} \wedge \ast \mathcal{A}) \right]. \tag{26}
\]

The latter contains [41], amongst others, a term proportional to the curvature scalar \(R := \ast (R^\alpha_{\beta \gamma} \eta_{\beta \alpha})\) and the axial torsion piece \(d \mathcal{A} \wedge d \mathcal{A}\) of the axial anomaly with a relative factor 9 as required by the supersymmetric path integral [27].

Since the coframe is the ‘soldered’ translational part \([39, 48]\) of the Cartan connection, a related translational CS term arises

\[
C_{TT} := \frac{1}{8 \ell^2} Tr (\gamma \wedge \Theta) = \frac{1}{2 \ell^2} \partial^\alpha \wedge T_\alpha. \tag{27}
\]

By exterior differentiation we obtain the NY four–form [44]:

\[
dC_{TT} = \frac{1}{8 \ell^2} Tr (\Theta \wedge \Theta - 4i \Omega \wedge \sigma) = \frac{1}{2 \ell^2} \left( T^\alpha \wedge T_\alpha + R^\alpha_{\beta \gamma} \wedge \partial^\alpha \wedge \partial^\beta \right). \tag{28}
\]

It is crucial to note that a fundamental length \(\ell\) necessarily occurs here for dimensional reasons. This can also be understood by a de Sitter type gauge approach, in which the \(sl(5,\mathbb{R})\)–valued connection \(\hat{\Gamma} = \Gamma + (\partial^\alpha L^4_\alpha + \partial_\beta L^4_\beta) / \ell\) is expanded into the dimensionless linear connection \(\Gamma\) plus the coframe \(\partial^\alpha = e_i^\alpha dx^i\) which carries canonical dimension \([\text{length}]\). The corresponding Pontrjagin term \(\hat{C}_{RR}\) splits via

\[
\hat{C}_{RR} = C_{RR} - 2C_{TT} \tag{29}
\]

into the linear one and the translational CS term, see the footnote 31 of Ref. [19] for details.
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