Existence of Solutions of a Non-Linear Eigenvalue System with a Weight

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Abstract

In this paper, we investigate the non-linear minimization problem:

\[
\inf_{u \in H^1_0(\Omega), v \in H^1_0(\Omega), \|u\|_{L^q} = 1, \|v\|_{L^q} = 1} \left[ \frac{1}{2} \int_{\Omega} a(x)|\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} b(x)|\nabla v|^2 dx - \lambda \int_{\Omega} uv dx \right]
\]

with \(q = \frac{2N}{N-2}\) and \(N \geq 4\) where \(a\) and \(b\) present a global minimum \(\alpha > 0\) at \(x_0\) with \(x_0 \in \Omega\). The main objective of this article is to show that for \(0 < \lambda < \tilde{\lambda}_1\) minimizers do exist with \(\tilde{\lambda}_1\) is the minimum between the first two eigenvalues respectively of the operators \(-\text{div}(a(x)\nabla \cdot)\) and \(-\text{div}(b(x)\nabla \cdot)\).

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1 Introduction

Given a smooth bounded domain subset \(\Omega \subset \mathbb{R}^N\), with \(N \geq 4\), throughout this paper, we are interested in the following non-linear minimization problem:

\[
Q_\lambda = \inf_{(u, v) \in (H^1_0(\Omega))^2 \setminus \{(0, 0)\}} E_\lambda(u, v),
\]

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with

\[ E_\lambda(u, v) = \frac{1}{2\|u\|_q^2} \int_\Omega a(x)|\nabla u|^2 dx + \frac{1}{2\|v\|_q^2} \int_\Omega b(x)|\nabla v|^2 dx - \frac{\lambda}{\|u\|_q\|v\|_q} \int_\Omega uv dx. \]

where \( a \) and \( b \) are positive continuous functions on \( \Omega \), \( \lambda \) is a real constant and \( q = \frac{2N}{N-2} \) is the critical exponent for the Sobolev embedding

\[ H^1_0(\Omega) \hookrightarrow L^q(\Omega). \] (2)

We notice that, in the well known paper \([5]\) Brezis and Nirenberg treated the problem \((3)\) where the weights \( a \) and \( b \) are positive constant function and they proved that the problem has at least one positive solution for \( 1 < \lambda < \lambda_1 \) when \( N \geq 4 \) and for \( \lambda^* < \lambda < \lambda_1 \) when \( N = 3 \), where \( \lambda_1 \) is the first eigenvalue of \(-\Delta\) on \( \Omega \) with zero Dirichlet boundary condition and \( \lambda^* \) is a positive constant.

In the presence of a non constant positive and bounded weight \( h \), the scalar problem has been addressed by Hadiji and Yazidi in \([14]\) where the authors showed the existence of minimizers solutions for the equation

\[ -\text{div}(h(x)\nabla u) = \lambda u + |u|^{q-2}u \quad \text{in} \ \Omega, \quad u > 0 \ \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial\Omega. \] (3)

This depends on the first eigenvalue \( \lambda_h^1 \) of \(-\text{div}(h(x)\nabla \cdot)\) in \( H^1_0(\Omega) \), the behaviour of the function \( h \) in the vicinity of its minima and the geometry of the domain \( \Omega \).

In \([9]\) Furtado and Souza generalized the problem \((3)\) when they considered non-homogeneous term \( |u|^{q-2}u \). We refer also to \([12, 13]\) for a more general weight which depends on \( u \) and \( x \).

In \([1]\) the authors considered a critical and subcritical system with critical non-linearity term \( u^{\alpha-1}v^{\beta} \), \( \alpha > 1, \beta > 1 \).

The main difficulties faced when dealing with this problem is the lack of compactness of the embedding \((2)\), using the presence of linear perturbation terms \( \lambda u \) and \( \lambda v \). Many other results in the scalar case for this kind of problems, for instance the reader can be see the references, \([12, 11, 13]\).

Geometric motivations, in particular in relation to the Yamabe problem, can be found for example in \([2, 8, 15, 16, 17]\). Note that the shape of the domain can have a strong influence on the type of results one can expect. See for example the seminal work of J.M. Coron \([3]\), or \([11]\).
In this paper, we consider the case when \( a \) and \( b \) are non constant distinct weights. We prove existence of positive solutions \((u, v)\) which depends, among others, on the behaviors of the weights \( a(\cdot) \) and \( b(\cdot) \) near their minima and the dimension of the space.

In order to state the problem and to announce our main results, we need to introduce some preliminaries.

Let us assume the existence of \( x_0, \) in \( \Omega \) such that, in a neighborhood of \( x_0, \) \( a \) and \( b \) behave like
\[
a(x) = a(x_0) + A_k|x - x_0|^k + |x - x_0|^k\theta_a(x), \quad \text{as } x \to x_0, \\
b(x) = b(x_0) + B_l|x - x_0|^l + |x - x_0|^l\theta_b(x), \quad \text{as } x \to x_0,
\]
with \( k > 0, \) \( l > 0 \) and \( A_k, B_l \) are positive constants, \( \theta_a(x) \) and \( \theta_b(x) \) tend to 0 when \( x \) tends to \( x_0. \)

Notice that the parameters \( k \) and \( l \) will play a critical role in the study of our problem. Indeed, if \( N \geq 4 \) the case \((k, l)\) with \( k > 2 \) and \( l > 2 \) is treated through a classical procedure.

For the other cases, the problem is a little delicate. We have to assume that the functions \( a \) and \( b \) satisfy the following additional conditions:
\[
kA_k \leq \frac{\tilde{a}(x)}{|x - x_0|^k} \quad \text{a.e. } x \in \Omega, \\
lB_l \leq \frac{\tilde{b}(x)}{|x - x_0|^l} \quad \text{a.e. } x \in \Omega,
\]
where
\[
\tilde{a}(x) := \nabla a(x) \cdot (x - x_0) \quad \text{and} \quad \tilde{b}(x) := \nabla b(x) \cdot (x - x_0).
\]

In order to highlight the difficulty, we consider the blow-up of \( u, v \in H^1_0(\Omega) \) around \( x = x_0, \) see \([11, 12, 13]\). Depending on the parameters \( k, l \) and \( \lambda \) different situations occur in the blow-up scale around the point where the weights are minimum. This means that one looks at the function \( w_\varepsilon \) and \( z_\varepsilon \) defined by:
\[
\forall \varepsilon > 0, \quad u(x) = \varepsilon^{-\frac{(N-2)}{2}}w_\varepsilon \left(\frac{x - x_0}{\varepsilon}\right), \\
v(x) = \varepsilon^{-\frac{(N-2)}{2}}z_\varepsilon \left(\frac{x - x_0}{\varepsilon}\right).
\]
One has \( w_\varepsilon, z_\varepsilon \in H^1_0(\Omega_\varepsilon) \) with \( \Omega_\varepsilon = \{\varepsilon^{-1}y, \ y \in \Omega\} \) and \( \|w_\varepsilon\|_{L^q(\Omega_\varepsilon)} = \|z_\varepsilon\|_{L^q(\Omega_\varepsilon)} = \|u\|_{L^q(\Omega)} = \|v\|_{L^q(\Omega)}, \) thus
\[ E_\lambda(u, v) = \frac{1}{2} \int_{\Omega} a(\varepsilon y + x_0)|\nabla w_\varepsilon(y)|^2 dy + \frac{1}{2} \int_{\Omega} b(\varepsilon y + x_0)|\nabla z_\varepsilon(y)|^2 dy - \lambda \varepsilon^2 \int_{\Omega} w_\varepsilon(y)z_\varepsilon(y)dy. \]

Consequently, the blow-up around \( x = x_0 \) gives

\[ E_\lambda(u, v) \sim \frac{1}{2} \int_{\Omega} a(x_0)|\nabla w_\varepsilon(y)|^2 dy + \frac{1}{2} \int_{\Omega} b(x_0)|\nabla z_\varepsilon(y)|^2 dy - \lambda \varepsilon^2 \int_{\Omega} w_\varepsilon(y)z_\varepsilon(y)dy + \frac{A_k}{2} \int_{\Omega} |\varepsilon y + x_0|^k|\nabla w_\varepsilon(y)|^2 dy + \frac{B_l}{2} \int_{\Omega} |\varepsilon y + x_0|^l|\nabla z_\varepsilon(y)|^2 dy - \lambda \varepsilon^2 \int_{\Omega} w_\varepsilon(y)z_\varepsilon(y)dy. \]

Then, as we can see the three last terms have different weights and the exposants \( k = 2 \) and \( l = 2 \) are critical for our problem.

Let us finally point out that the lower dimension \( N = 3 \) and the exposants \( k < 2, \) \( l < 2 \) could also be interesting for this problem, but is not yet fully understood. For dimension 3 in the scalar case we refer to [5], [7]. This paper is organized in the following way: in section 1 we state our main result. In section 2 we give a sufficient condition for the existence of minimizers. We give precise estimates of the energy in section 3. In the next section we discuss the sign of minimizers. Section 6 is concerned by some non-existence results.

## 2 Main results

Let \( x_0 \in \Omega \) such that

\[ a(x_0) = \min \{a(x), \ x \in \Omega\}. \]
\[ b(x_0) = \min \{b(x), \ x \in \Omega\}. \]

In this section, we will prove the existence of solution of problem (1). For this we define:

\[ S = \inf_{u \in H^1_0(\Omega), \|u\|_{L^q} = 1} \|\nabla u\|_2^2 \] (9)

that corresponds to the best constant for the Sobolev embedding \( H^1_0(\Omega) \) into \( L^q(\Omega) \).

In this paper, we suppose that \( a(x_0) = b(x_0) \).

Let \( \gamma_0 \) be a positive real number such that \( a(x_0) = b(x_0) = \gamma_0 \).

Now, we are able to state the main result of this paper
Theorem 1. Assume that \( a, b \in H^1(\Omega) \cap C(\Omega) \) satisfies \([1]\) and \([5]\). Let \( \lambda_a^1 \) and \( \lambda_b^1 \) be the first eigenvalues of \(-\text{div}(a(x)\nabla \cdot)\) and \(-\text{div}(b(x)\nabla \cdot)\) on \( \Omega \) with zero Dirichlet boundary condition, let denote by \( \tilde{\lambda}_1 = \min\{\lambda_a^1, \lambda_b^1\} \). We have

1. If \( N \geq 4, k > 2 \) and \( l > 2 \), then \( Q_\lambda \) is achieved for every \( \lambda \in \left[0, \lambda_a^1 \right] \cap \left[0, \lambda_b^1 \right] \).

2. If \( N \geq 5, k = 2 \) and \( l = 2 \), then there exists a constant \( \gamma(N) = \frac{(N-2)N(N+2)}{4(N-1)}(A_2 + B_2) \) such that \( Q_\lambda \) is achieved for every \( \lambda \in \left[\gamma(N), \tilde{\lambda}_1 \right] \).

3. If \( N \geq 5, k = 2 \) and \( l > 2 \), or \( k > 2 \) and \( l = 2 \), then there exists a constant \( \tilde{\gamma}(N) = \frac{N(N-2)(N+2)}{4(N-1)} A_2 \) such that \( Q_\lambda \) is achieved for every \( \lambda \in \left[\tilde{\gamma}(N), \tilde{\lambda}_1 \right] \).

We will proceed for the proof of Theorem \([4]\) as the following: First we show that \( Q_\lambda < \gamma_0 S \) and then we prove that this implies that the infimum \( Q_\lambda \) is achieved.

3 Sufficient condition for the existence of minimizers

We need first to prove the existence of \( Q_\lambda \), which is guaranteed by the following result:

Proposition 3.1. Let \( \varphi_a^1 \) and \( \varphi_b^1 \) are the first eigenfunctions of \(-\text{div}(a(x)\nabla \cdot)\) and \(-\text{div}(b(x)\nabla \cdot)\) associated to the first two eigenvalues \( \lambda_a^1 \) and \( \lambda_b^1 \). We have

(i) Assume that \( 0 < \lambda < \tilde{\lambda}_1 \), then

\[ Q_\lambda \geq 0. \]

(ii) For \( \lambda \geq \frac{\|\varphi_a^1\|_{L^q}\|\varphi_b^1\|_{L^q}}{\int_\Omega \varphi_a^1 \varphi_b^1 \, dx} |\Omega|^{1-\frac{2}{q}} \max(\lambda_a^1, \lambda_b^1), \) one has

\[ Q_\lambda \leq 0, \]

Proof (i). Let \( 0 < \lambda < \tilde{\lambda}_1(\Omega) \), and let \( u \) and \( v \) such that \( \|u\|_{L^q} = \|v\|_{L^q} = 1 \). By the definitions of \( \lambda_a^1, \lambda_b^1 \) and \( \tilde{\lambda}_1(\Omega) \) one has

\[ E_\lambda(u, v) \geq \frac{\tilde{\lambda}_1}{2} \|u\|_{L^2}^2 + \frac{\tilde{\lambda}_1}{2} \|v\|_{L^2}^2 - \lambda \int_\Omega uv \, dx. \]
By applying the Cauchy-Schwarz inequality, we find

\[ E_\lambda(u, v) \geq \tilde{\lambda}_1 \left[ \|u\|_{L^2}^2 + \|v\|_{L^2}^2 - 2\|u\|_{L^2} \|v\|_{L^2} \right]. \]

Thus

\[ E_\lambda(u, v) \geq \frac{\tilde{\lambda}_1}{2} \left( \|u\|_{L^2}^2 - \|v\|_{L^2}^2 \right)^2 \geq 0. \]

Consequently, \( Q_\lambda \geq 0. \)

(ii) We have

\[ Q_\lambda \leq E_\lambda \left( \frac{\varphi_a^a}{\|\varphi_a^a\|_{L^q}}, \frac{\varphi_b^b}{\|\varphi_b^b\|_{L^q}} \right), \]

implies that,

\[ Q_\lambda \leq \frac{1}{2\|\varphi_a^a\|_{L^q}^2} \int_\Omega a(x) |\nabla \varphi_a^a|^2 \, dx + \frac{1}{2\|\varphi_b^b\|_{L^q}^2} \int_\Omega b(x) |\nabla \varphi_b^b|^2 \, dx - \frac{\lambda}{\|\varphi_a^a\|_{L^q} \|\varphi_b^b\|_{L^q}} \int_\Omega \varphi_a^a \varphi_b^b \, dx. \]

By the definitions of \( \varphi_a^a \) and \( \varphi_b^b \), one has

\[ Q_\lambda \leq \frac{1}{2\|\varphi_a^a\|_{L^q}^2} \lambda_a^2 \int_\Omega |\varphi_a^a|^2 \, dx + \frac{1}{2\|\varphi_b^b\|_{L^q}^2} \lambda_b^2 \int_\Omega |\varphi_b^b|^2 \, dx - \frac{\lambda}{\|\varphi_a^a\|_{L^q} \|\varphi_b^b\|_{L^q}} \int_\Omega \varphi_a^a \varphi_b^b \, dx. \]

Using the embedding of \( L^q \) into \( L^2 \), there exists a positive constant \( C_1 = \frac{1}{2} - \frac{1}{q} \) such that

\[ Q_\lambda \leq \frac{1}{2\|\varphi_a^a\|_{L^q}^2} \lambda_a^2 C_1^2 \|\varphi_a^a\|_{L^q}^2 + \frac{1}{2\|\varphi_b^b\|_{L^q}^2} \lambda_b^2 C_1^2 \|\varphi_b^b\|_{L^q}^2 - \frac{\lambda}{\|\varphi_a^a\|_{L^q} \|\varphi_b^b\|_{L^q}} \int_\Omega \varphi_a^a \varphi_b^b \, dx. \]

Thus

\[ Q_\lambda \leq |\Omega|^{1 - \frac{2}{q}} \max(\lambda_a^2, \lambda_b^2) - \frac{\lambda}{\|\varphi_a^a\|_{L^q} \|\varphi_b^b\|_{L^q}} \int_\Omega \varphi_a^a \varphi_b^b \, dx. \]

As conclusion, \( Q_\lambda \leq 0. \)

This completes the proof of the Proposition. \( \blacksquare \)

**Lemma 3.1.** Let \( 0 < \lambda < \tilde{\lambda}_1 \). If \( Q_\lambda < \gamma_0 S \), then the infimum in (1) is achieved.

**Proof** Let \( \{U_n\} \subset (H^1_0(\Omega))^2 \) be a minimizing sequence for (1) that is,

\[ \|U_n\|_{L^q} = 1 \quad (\text{which means} \|u_n\|_q = 1, \ |v_n|_q = 1). \]

\[ \frac{1}{2} \int_\Omega a(x) |\nabla u_n|^2 \, dx + \frac{1}{2} \int_\Omega b(x) |\nabla v_n|^2 \, dx - \lambda \int_\Omega u_n v_n \, dx = Q_\lambda + o(1) \quad \text{as} \quad n \to \infty. \]
The sequence \( \{U_n\} \) is bounded in \((H^1_0(\Omega))^2\). Indeed, from (11), we have
\[
\frac{1}{2} \int_\Omega a(x)|\nabla u_n|^2\,dx + \frac{1}{2} \int_\Omega b(x)|\nabla v_n|^2\,dx = \lambda \int_\Omega u_n v_n\,dx + Q\lambda + o(1).
\]
Using H"older’s inequality we have
\[
\int_\Omega |u_n||v_n|\,dx \leq \|u_n\|_{L^q(\Omega)} \|v_n\|_{L^{q'}(\Omega)},
\]
then
\[
\frac{1}{2} \left[ \int_\Omega a(x)|\nabla u_n|^2\,dx + \int_\Omega b(x)|\nabla v_n|^2\,dx \right] \leq \lambda \|u_n\|_{L^\frac{q}{q-1}(\Omega)} \|v_n\|_{L^q(\Omega)} + Q\lambda + o(1).
\]
Consequently, we have
\[
\|(u_n,v_n)\|_{(H^1_0(\Omega))^2} \leq C.
\]
Then there is \( U = (u,v) \), up to a subsequence, still denoted by \( U_n = (u_n,v_n) \), such that
\[
(u_n,v_n) \rightharpoonup (u,v) \quad \text{weakly in} \quad \left( H^1_0(\Omega) \right)^2,
\]
\[
(u_n,v_n) \rightarrow (u,v) \quad \text{strongly in} \quad \left( L^2(\Omega) \right)^2,
\]
\[
(u_n,v_n) \rightarrow (u,v) \quad \text{a.e. on} \quad \Omega.
\]
with \( \|u\|_{L^q} \leq 1 \) and \( \|v\|_{L^q} \leq 1 \).
Set \( w_n = u_n - u \) and \( z_n = v_n - v \), so that
\[
(w_n,z_n) \rightharpoonup (0,0) \quad \text{weakly in} \quad \left( H^1_0(\Omega) \right)^2,
\]
\[
(w_n,z_n) \rightarrow (0,0) \quad \text{strongly in} \quad \left( L^2(\Omega) \right)^2,
\]
\[
(w_n,z_n) \rightarrow (0,0) \quad \text{a.e. on} \quad \Omega.
\]
By (10) and using the definition of \( S \) and the fact that \( \min_\Omega a(x) = \gamma_0 > 0 \) and \( \min_\Omega b(x) = \gamma_0 > 0 \), we have
\[
\frac{1}{2} \left[ \int_\Omega a(x)|\nabla u_n|^2\,dx + \int_\Omega b(x)|\nabla v_n|^2\,dx \right] \geq \frac{1}{2}\gamma_0 S.
\]
By (11), we have
\[
\frac{1}{2} \left[ \int_\Omega a(x)|\nabla u_n|^2\,dx + \int_\Omega b(x)|\nabla v_n|^2\,dx \right] - \lambda \int_\Omega u_n v_n\,dx = Q\lambda + o(1).
\]
Which implies that
\[
\frac{1}{2} \left[ \int_{\Omega} a(x)|\nabla u_n|^2 dx + \int_{\Omega} b(x)|\nabla v_n|^2 dx \right] = Q_\lambda + \lambda \int_{\Omega} u_n v_n dx + o(1).
\]

Which gives
\[
\lambda \int_{\Omega} u v dx \geq \gamma_0 S - Q_\lambda > 0
\]
and so \( u \not\equiv 0 \) and \( v \not\equiv 0 \).

Again using (11) we obtain
\[
\frac{1}{2} \left[ \int_{\Omega} a(x)|\nabla u|^2 dx + \int_{\Omega} a(x)|\nabla w_n|^2 dx + \int_{\Omega} b(x)|\nabla v|^2 dx + \int_{\Omega} b(x)|\nabla z_n|^2 dx \right]
- \lambda \int_{\Omega} u_n v_n dx = Q_\lambda + o(1).
\]

(12)

On the other hand, as \{w_n\} and \{z_n\} are bounded in \( L^q(\Omega) \) and \( w_n \longrightarrow 0 \) for a.e. \( x \) in \( \Omega \), \( z_n \longrightarrow 0 \) for a.e. \( x \) in \( \Omega \), then we can use the Lemma of Brezis-Lieb, see [4]
\[
\|u + w_n\|^q_{L^q} = \|u\|^q_{L^q} + \|w_n\|^q_{L^q} + o(1),
\]
\[
\|v + z_n\|^q_{L^q} = \|v\|^q_{L^q} + \|z_n\|^q_{L^q} + o(1).
\]

Using (10) we have
\[
1 = \|u\|^q_{L^q} + \|w_n\|^q_{L^q} + o(1),
\]
\[
1 = \|v\|^q_{L^q} + \|z_n\|^q_{L^q} + o(1).
\]

And so,
\[
1 \leq \|u\|^q_{L^q} + \|w_n\|^q_{L^q} + o(1), \quad \text{(13)}
\]
\[
1 \leq \|v\|^q_{L^q} + \|z_n\|^q_{L^q} + o(1). \quad \text{(14)}
\]

We sum the equations (13) and (14) we obtain
\[
2 \leq \|u\|^2_{L^q} + \|v\|^2_{L^q} + \frac{1}{\gamma_0 S} \left( \int_{\Omega} a(x)|\nabla w_n|^2 dx + \int_{\Omega} b(x)|\nabla z_n|^2 dx \right). \quad \text{(15)}
\]

Since \( Q_\lambda > 0 \), then
\[
Q_\lambda \leq \frac{Q_\lambda}{2} \|u\|^2_{L^q} + \frac{Q_\lambda}{2} \|v\|^2_{L^q} + \frac{Q_\lambda}{2\gamma_0 S} \left( \int_{\Omega} a(x)|\nabla w_n|^2 dx + \int_{\Omega} b(x)|\nabla z_n|^2 dx \right). \quad \text{(16)}
\]
Adding (12) and (16) we obtain
\[
\frac{1}{2} \left[ \int_\Omega a(x) |\nabla u|^2 \, dx + \int_\Omega a(x) |\nabla w_n|^2 \, dx + \int_\Omega b(x) |\nabla v|^2 \, dx + \int_\Omega b(x) |\nabla z_n|^2 \, dx \right] - \lambda \int_\Omega uv \, dx
\]
\[
\leq \frac{Q\lambda}{2} \|u\|_{L^q}^2 + \frac{Q\lambda}{2} \|v\|_{L^q}^2 + \frac{Q\lambda}{2\gamma_0 S} \left( \frac{1}{2 \gamma_0 S} - 1 \right) \left[ \int_\Omega a(x) |\nabla w_n|^2 \, dx + \int_\Omega b(x) |\nabla z_n|^2 \, dx \right].
\]
Thus
\[
\frac{1}{2} \left[ \int_\Omega a(x) |\nabla u|^2 \, dx + \int_\Omega b(x) |\nabla v|^2 \, dx \right] - \lambda \int_\Omega uv \, dx
\]
\[
\leq \frac{Q\lambda}{2} \|u\|_{L^q}^2 + \frac{Q\lambda}{2} \|v\|_{L^q}^2 + \left[ \frac{Q\lambda}{2\gamma_0 S} - \frac{1}{2} \right] \left[ \int_\Omega a(x) |\nabla w_n|^2 \, dx + \int_\Omega b(x) |\nabla z_n|^2 \, dx \right].
\]
Hence
\[
2E_\lambda(u, v) \leq Q\lambda \|u\|_{L^q}^2 + Q\lambda \|v\|_{L^q}^2 + \left[ \frac{Q\lambda}{2\gamma_0 S} - 1 \right] \left[ \int_\Omega a(x) |\nabla w_n|^2 \, dx + \int_\Omega b(x) |\nabla z_n|^2 \, dx \right] + o(1),
\]
and since \( Q\lambda < \gamma_0 S \), \( u \neq 0 \) and \( v \neq 0 \). we deduce
\[
2E_\lambda\left( \frac{u}{\|u\|_{L^q}}, \frac{v}{\|v\|_{L^q}} \right) \leq \left( \frac{1}{\|u\|_{L^q}^2} - 1 \right) \int_\Omega a(x) |\nabla u|^2 \, dx + \left( \frac{1}{\|v\|_{L^q}^2} - 1 \right) \int_\Omega b(x) |\nabla v|^2 \, dx
\]
\[
+ \lambda \int_\Omega uv \, dx - \lambda \int_\Omega \frac{uv}{\|u\|_{L^q} \|v\|_{L^q}} \, dx.
\]
Then
\[
2E_\lambda\left( \frac{u}{\|u\|_{L^q}}, \frac{v}{\|v\|_{L^q}} \right) \leq \left( \|u\|_{L^q}^2 - 1 \right) \left[ Q\lambda - \int_\Omega a(x) |\nabla u|^2 \, dx - 2\lambda \int_\Omega \frac{uv}{\|u\|_{L^q} \|v\|_{L^q}} \, dx \right]
\]
\[
+ \left( \|v\|_{L^q}^2 - 1 \right) \left[ Q\lambda - \int_\Omega b(x) |\nabla v|^2 \, dx - 2\lambda \int_\Omega \frac{uv}{\|u\|_{L^q} \|v\|_{L^q}} \, dx \right]
\]
\[
+ \lambda \int_\Omega uv \, dx - \lambda \int_\Omega \frac{uv}{\|u\|_{L^q} \|v\|_{L^q}} \, dx + 2\lambda \left( \|u\|_{L^q}^2 - 1 \right) \int_\Omega \frac{uv}{\|u\|_{L^q} \|v\|_{L^q}} \, dx + 2Q\lambda.
\]
On one hand, we have \( \|u\|_{L^q}^2 - 1 \leq 0 \) and \( \|v\|_{L^q}^2 - 1 \leq 0 \), we obtain
\[
2E_\lambda\left( \frac{u}{\|u\|_{L^q}}, \frac{v}{\|v\|_{L^q}} \right) \leq \lambda \left[ \left( \|u\|_{L^q}^2 \|v\|_{L^q} - 1 \right) + 2 \left( \|u\|_{L^q}^2 - 1 \right) + 2 \left( \|v\|_{L^q}^2 - 1 \right) \right] \int_\Omega uv \, dx + 2Q\lambda.
\]
On the other hand, we have \( \|u\|_{L^q} \|v\|_{L^q} - 1 \leq 0 \), then
\[
2E_\lambda\left( \frac{u}{\|u\|_{L^q}}, \frac{v}{\|v\|_{L^q}} \right) \leq 2Q\lambda.
\]
This means that \((u, v)\) is a minimum of \(Q_\lambda\).
4 A precise estimates of the energy

Proposition 4.1.

(a) For $N \geq 4$, and $k > 2$, $l > 2$ we have

$$Q \lambda < \gamma_0 S, \quad \text{for all } \lambda > 0.$$  

(b) For $N = 4$, and $k = 2$, $l = 2$ we have

$$Q \lambda < \gamma_0 S, \quad \text{for all } \lambda > A_2 + B_2.$$  

(c) For $N \geq 5$, and $k \geq 2$, $l \geq 2$ we have

$$Q \lambda < \gamma_0 S, \quad \text{for all } \lambda > \frac{N(N-2)(N+2)}{4(N-1)} (A_2 + B_2).$$  

(d) For $N = 4$, and $k > 2$, $l = 2$ we have

$$Q \lambda < \gamma_0 S, \quad \text{for all } \lambda > B_2.$$  

(e) For $N = 4$, and $k = 2$, $l > 2$ we have

$$Q \lambda < \gamma_0 S, \quad \text{for all } \lambda > A_2.$$  

where $A_2, B_2$ are defined by (6) and (7) and $K_3 = \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^{N-2}} dx$.

**Proof** We shall estimate the ratio $E_\lambda(u,v)$ defined in (1), with $u = u_{x_0,\varepsilon} = \zeta U_\varepsilon(x-x_0)$, where for $x \in \mathbb{R}^N$, $U_\varepsilon(x) = \frac{\varepsilon^{-\frac{N-2}{2}}}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}}$ and $\zeta \in C_0^\infty(\Omega)$ with $\zeta \geq 0$ and $\varphi \equiv 1$ on a neighborhood of $x_0$; for more details see [5] [19].

We recall from [5] that

$$\int_{\Omega} |\nabla u_{x_0,\varepsilon}(x)|^2 dx = K_1 + O(\varepsilon^{-\frac{N-2}{2}}),$$  

$$\left(\int_{\Omega} |u_{x_0,\varepsilon}(x)|^q dx\right)^{\frac{2}{q}} = K_2 + O(\varepsilon^{-\frac{N-2}{2}}),$$  

$$\int_{\Omega} |u_{x_0,\varepsilon}(x)|^2 dx = \begin{cases} K_3 \varepsilon + O(\varepsilon^{-\frac{N-2}{2}}), & \text{if } N \geq 5, \\ \frac{\omega}{2} \varepsilon |\ln \varepsilon| + o(\varepsilon |\ln \varepsilon|), & \text{if } N = 4, \end{cases}$$

$$10$$
where $K_1$ and $K_2$ are positive constants with $\frac{K_1}{K_2} = S$, $w_4$ is a the area of $S^3$ and $K_3 = \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^{N-2}} dx$.

We know by [14], the following estimations:

$$\frac{\varepsilon^{N-2}}{2} \int_\Omega a(x)|\nabla u_{x_0,\varepsilon}(x)|^2 dx \leq \begin{cases} \frac{a(x_0)K_1}{2} + O(\varepsilon^{\frac{N-2}{2}}) & \text{if } N \geq 4 \text{ and } N - 2 < k, \\ \frac{a(x_0)K_1}{2} + \frac{C_k}{2} \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } N \geq 4 \text{ and } N - 2 > k, \\ \frac{a(x_0)K_1}{2} + (N-2)^2\omega_N (A_{N-2} + M) \varepsilon^{\frac{N-2}{2}} |\ln \varepsilon| + o(\varepsilon^{\frac{N-2}{2}} |\ln \varepsilon|) & \text{if } N > 4 \text{ and } k = N - 2, \\ \frac{a(x_0)K_1}{2} + A_2 \omega_4 |\ln \varepsilon| + o(\varepsilon |\ln \varepsilon|) & \text{if } N \geq 4 \text{ and } k = 2 \end{cases}$$

(20)

and

$$\frac{\varepsilon^{N-2}}{2} \int_\Omega b(x)|\nabla u_{x_0,\varepsilon}(x)|^2 dx \leq \begin{cases} \frac{b(x_0)K_1}{2} + O(\varepsilon^{\frac{N-2}{2}}) & \text{if } N \geq 4 \text{ and } N - 2 < l, \\ \frac{b(x_0)K_1}{2} + \frac{D_k}{2} \varepsilon^{\frac{l}{2}} + o(\varepsilon^{\frac{l}{2}}) & \text{if } N \geq 4 \text{ and } N - 2 > l, \\ \frac{b(x_0)K_1}{2} + (N-2)^2\omega_N (B_{N-2} + M) \varepsilon^{\frac{N-2}{2}} |\ln \varepsilon| + o(\varepsilon^{\frac{N-2}{2}} |\ln \varepsilon|) & \text{if } N > 4 \text{ and } l = N - 2, \\ \frac{b(x_0)K_1}{2} + B_2 \omega_4 |\ln \varepsilon| + o(\varepsilon |\ln \varepsilon|) & \text{if } N \geq 4 \text{ and } l = 2 \end{cases}$$

(21)

Where $K_1 = (N - 2)^2 \int_{\mathbb{R}^N} \frac{|y|^2}{(1 + |y|^2)^N} dy$, $K_2 = \left( \int_{\mathbb{R}^N} \frac{dy}{(1 + |y|^2)^N} \right)^{\frac{N-2}{N}}$, $C_k = (N - 2)^2 A_k \int_{\mathbb{R}^N} \frac{|y|^{k+2}}{(1 + |y|^2)^N} dy$, $D_k = (N - 2)^2 B_k \int_{\mathbb{R}^N} \frac{|y|^{l+2}}{(1 + |y|^2)^N} dy$, $M$ and $M'$ are positive constants.
We claim that, as $\varepsilon \to 0$, by adding (18), (19), (20) and (21) we obtain $E(u_{x_0, \varepsilon}) \leq$

$$\begin{cases} a(x_0)K_1 + o(\varepsilon) + \frac{b(x_0)K_1 + o(\varepsilon)}{2K_2} - \lambda \frac{K_2^2 \varepsilon}{K_2} & \text{if } N \geq 5 \\
\quad + \frac{b(x_0)K_1 + D\varepsilon + o(\varepsilon)}{2K_2} & \text{if } k > 2, l > 2, \end{cases}$$

$$\begin{cases} a(x_0)K_1 + o(\varepsilon) + \frac{b(x_0)K_1 + o(\varepsilon)}{2K_2} - \lambda \frac{K_2^2 \varepsilon}{K_2} & \text{if } N \geq 5 \\
\quad + \frac{b(x_0)K_1 + D\varepsilon + o(\varepsilon)}{2K_2} + o(\varepsilon | \log \varepsilon|) & \text{if } k > 2, l > 2, \end{cases}$$

$$\begin{cases} a(x_0)K_1 + o(\varepsilon) + \frac{b(x_0)K_1 + o(\varepsilon)}{2K_2} - \lambda \frac{K_2^2 \varepsilon}{K_2} + O(1) & \text{if } N \geq 5 \\
\quad + \frac{b(x_0)K_1 + D\varepsilon + o(\varepsilon)}{2K_2} & \text{if } k > 2, l > 2, \end{cases}$$

$$\begin{cases} a(x_0)K_1 + o(\varepsilon) + \frac{b(x_0)K_1 + o(\varepsilon)}{2K_2} - \lambda \frac{K_2^2 \varepsilon}{K_2} + O(1) & \text{if } N \geq 5 \\
\quad + \frac{b(x_0)K_1 + D\varepsilon + o(\varepsilon)}{2K_2} - \lambda \frac{\omega_1}{2K_2} | \log \varepsilon | + o(\varepsilon | \log \varepsilon|) & \text{if } k > 2, l > 2, \end{cases}$$

$$\begin{cases} a(x_0)K_1 + o(\varepsilon) + \frac{b(x_0)K_1 + o(\varepsilon)}{2K_2} - \lambda \frac{K_2^2 \varepsilon}{K_2} + O(1) & \text{if } N \geq 5 \\
\quad + \frac{b(x_0)K_1 + D\varepsilon + o(\varepsilon)}{2K_2} - \lambda \frac{\omega_1}{2K_2} | \log \varepsilon | + o(\varepsilon | \log \varepsilon|) & \text{if } k > 2, l > 2, \end{cases}$$

Thus
where $\frac{C_2}{K_3} = \frac{N(N-2)(N+2)}{4(N-1)} A_2$ and $\frac{D_2}{K_3} = \frac{N(N-2)(N+2)}{4(N-1)} B_2$.

From these estimations we get the desired result.

Combining Lemma 3.1 and Proposition 4.1 we conclude that the infimum in (1) is achieved. This leads to the conclusion of Theorem 1.

Remark 4.1. If $N \geq 4$, $k < 2$ and $l < 2$ then, we can’t go strictly below $\frac{a(x_0) + b(x_0)}{2} S$.

5 The sign of the minimizers

Now, we will discuss the positivity of the solutions:

Proposition 5.1.

(i) If $Q_\lambda$ is achieved in $(u, v)$ then $u.v \geq 0$. 


(ii) There exists a solutions $u$ and $v$ of the minimization problem (1) such that $u \geq 0$ and $v \geq 0$.

Proof (i) Set $F(u,v) = \frac{1}{2} \int \nabla a(x) |\nabla u|^2 dx + \frac{1}{2} \int \nabla b(x) |\nabla v|^2 dx - \lambda \int uv dx$.

The inequality

$$F(u,v) \leq F(|u|,|v|)$$

give that

$$- \lambda \int uv dx \leq - \lambda \int |uv| dx$$

then

$$\int (uv - |uv|) dx \geq 0. \quad (22)$$

We have always $u.v \leq |uv|$ signifie that, $uv - |uv| \leq 0 \forall x \in \Omega$. Hence

$$\int (uv - |uv|) \leq 0. \quad (23)$$

Combining (22) and (23) we obtain that $uv - |uv| = 0$ and hence $|uv| = uv$.

Finally for all $x \in \Omega$, $uv \geq 0$.

(ii) From Lemma 3.1 we know there exists a minimum $(u,v)$. By Propostion 5 (i) we know that $|uv| = uv$.

On the other hand, we have

$$F(|u_n|,|v_n|) = \frac{1}{2} \int \nabla a(x) |\nabla u_n|^2 dx + \frac{1}{2} \int \nabla b(x) |\nabla v_n|^2 dx - \lambda \int |u_n v_n| dx = F(u_n,v_n).$$

Then if we have $(u_n,v_n)$ are solutions of (1) then $(|u_n|,|v_n|)$ are also minimizing solutions.

Thus, when dealing with (1), one can assume without loss of generality that $u \geq 0$ and $v \geq 0$. 

Remark 5.1. Since $u$ and $v$ are minimizers for (1) we obtain a Lagrange multipliers $\Lambda_1, \Lambda_2 \in \mathbb{R}$ such that the Euler-Lagrange equation formula associated to (1) is

$$\begin{cases} 
- \text{div}(a(x) \nabla u) - \lambda u &= \Lambda_1 u^{2^*-1} \quad \text{in} \quad \Omega \\
- \text{div}(b(x) \nabla v) - \lambda v &= \Lambda_2 v^{2^*-1} \quad \text{in} \quad \Omega \\
||u||_{2^*} = ||u||_{2^*} &= 1 \\
\quad u \geq 0, \quad v \geq 0 \quad \text{in} \quad \Omega \\
\quad u = v &= 0 \quad \text{on} \quad \partial \Omega, 
\end{cases} \quad (24)$$

such that $\frac{\Lambda_1 + \Lambda_2}{2} = Q_{\lambda} \geq 0$ according to Proposition 3.1.
6 Non-existence result

In this section, we assume that \( a, b \) verify (4) and (5) with \( k > 2, l > 2 \) and for the other cases, we obtain just a few results of non-existence modulo the additional conditions in (6) and (7).

We define
\[
\omega(a, b) := \inf_{(u, v) \in (H_0^1(\Omega))^2 \setminus \{0\}} \phi_{a, b}(u, v)
\]
where
\[
\phi_{a, b}(u, v) := \frac{1}{4} \int_{\Omega} \left( \tilde{a}(x)|\nabla u|^2 + \tilde{b}(x)|\nabla v|^2 \right) dx
\]
\[
\int_{\Omega} u(x)v(x)dx.
\]

We see that \( \omega(a, b) \in [-\infty, +\infty[. \)

The main goal of this section is the non-existence results.

**Theorem 2.** Assume that \( \lambda \leq \omega(a, b) \) and \( \Omega \) is a strictly star-shaped domain with respect to \( x_0 \). Then (11) has no minimizers.

The proof of this result follows from the following Pohozhev identity, see [18].

**Proposition 6.1.** If \( (u, v) \) is a solution of (24) then \( (u, v) \) verify the following identity:

\[
2\lambda \int_{\Omega} u(x)v(x)dx - \frac{1}{2} \int_{\Omega} \nabla b(x) \cdot (x - x_0)|\nabla v|^2 dx - \frac{1}{2} \int_{\Omega} \nabla a(x) \cdot (x - x_0)|\nabla u|^2 dx
\]
\[
= \frac{1}{2} \int_{\partial\Omega} a(x) [(x - x_0) \cdot n] |\nabla u|^2 dx + \frac{1}{2} \int_{\partial\Omega} b(x) [(x - x_0) \cdot n] |\nabla v|^2 dx.
\]

where \( n \) denotes the outward normal to \( \partial\Omega \).

**Proof** Suppose that \((u, v)\) is a solution \((24)\). We multiply the first equation in the system by \( \nabla u \cdot (x - x_0) \) and we integrate by parts we are led to

\[
\Lambda_1 \int_{\Omega} |u|^{q-2}u \nabla u(x) \cdot (x - x_0)dx = -\frac{N}{q} \Lambda_1.
\]

\[
\lambda \int_{\Omega} v(x) \nabla u(x) \cdot (x - x_0)dx = -\lambda \int_{\Omega} u(x) \nabla v(x) \cdot (x - x_0)dx - N\lambda \int_{\Omega} v(x)u(x)dx.
\]
\[
\begin{align*}
\int_{\Omega} -\text{div}(a(x)\nabla u(x)) \nabla u \cdot (x - x_0) dx &= \frac{N - 2}{2} \int_{\Omega} a(x)|\nabla u|^2 dx \\
&\quad - \frac{1}{2} \int_{\Omega} \nabla a(x) \cdot (x - x_0)|\nabla u|^2 dx \\
&\quad - \frac{1}{2} \int_{\partial\Omega} a(x) \frac{\partial u}{\partial \nu}^2 ((x - x_0) \cdot \mathbf{n}) dx,
\end{align*}
\] (27)

where \(\mathbf{n}\) denotes the outward normal to \(\partial\Omega\).

Combining (27) and (25) we get
\[
\begin{align*}
\frac{N - 2}{2} \int_{\Omega} a(x)|\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} \nabla a(x) \cdot (x - x_0)|\nabla u|^2 dx \\
&\quad - \frac{1}{2} \int_{\partial\Omega} a(x) \frac{\partial u}{\partial \nu}^2 ((x - x_0) \cdot \mathbf{n}) dx \\
&= \lambda \int_{\Omega} v(x) \nabla u(x) \cdot (x - x_0) dx - \frac{N}{q} \Lambda_1.
\end{align*}
\] (28)

Similarly, we multiply the second equation of (24) by \(\nabla v \cdot (x - x_0)\) and we integrate by parts, we get
\[
\begin{align*}
\frac{N - 2}{2} \int_{\Omega} b(x)|\nabla v|^2 dx - \frac{1}{2} \int_{\Omega} \nabla b(x) \cdot (x - x_0)|\nabla v|^2 dx \\
&\quad - \frac{1}{2} \int_{\partial\Omega} b(x) \frac{\partial v}{\partial \nu}^2 ((x - x_0) \cdot \mathbf{n}) dx \\
&= \lambda \int_{\Omega} u(x) \nabla v(x) \cdot (x - x_0) dx - \frac{N}{q} \Lambda_2.
\end{align*}
\] (29)

On the other hand, multiplying the equations in (24) by \(\frac{N - 2}{2} u\) and \(\frac{N - 2}{2} v\) respectively, and integrating and summing the obtained results, we get
\[
\frac{N - 2}{2} \int_{\Omega} a(x)|\nabla u|^2 dx = \frac{N - 2}{2} \lambda \int_{\Omega} v(x) u(x) dx + \Lambda_1 \frac{N - 2}{2}
\] (30)

and
\[
\frac{N - 2}{2} \int_{\Omega} b(x)|\nabla v|^2 dx = \frac{N - 2}{2} \lambda \int_{\Omega} u(x) v(x) dx + \Lambda_2 \frac{N - 2}{2}.
\] (31)

Combining (28), and (30) we obtain
\[
\begin{align*}
- \left(\frac{N - 2}{2}\right) \lambda \int_{\Omega} u(x) v(x) dx - \lambda \int_{\Omega} v(x) |\nabla u(x) \cdot (x - x_0)| dx \\
&= \frac{1}{2} \int_{\Omega} |\nabla u|^2 [\nabla a(x) \cdot (x - x_0)] dx + \frac{1}{2} \int_{\partial\Omega} a(x) \frac{\partial u}{\partial \nu}^2 [(x - x_0) \cdot \mathbf{n}] dx.
\end{align*}
\] (32)
On the other hand, combining (29), and (31) we get
\[
- \left( \frac{N-2}{2} \right) \lambda \int_{\Omega} u(x)v(x)dx - \lambda \int_{\Omega} u(x) [\nabla v(x) \cdot (x-x_0)] dx
= \frac{1}{2} \int_{\Omega} |\nabla v|^2 [\nabla b(x) \cdot (x-x_0)] dx + \frac{1}{2} \int_{\partial\Omega} b(x) \left| \frac{\partial v}{\partial \nu} \right|^2 [(x-x_0) \cdot n] dx. \tag{33}
\]
Adding (33) in (32) and using (26) we find
\[
2\lambda \int_{\Omega} u(x)v(x)dx - \frac{1}{2} \int_{\Omega} \nabla b(x) \cdot (x-x_0)|\nabla v|^2 dx - \frac{1}{2} \int_{\Omega} [\nabla a(x) \cdot (x-x_0)] |\nabla u|^2 dx
= \frac{1}{2} \int_{\partial\Omega} a(x) \left| \frac{\partial u}{\partial \nu} \right|^2 [(x-x_0) \cdot n] dx + \frac{1}{2} \int_{\partial\Omega} b(x) \left| \frac{\partial v}{\partial \nu} \right|^2 [(x-x_0) \cdot n] dx.
\]
Finally we get the Proposition. □

Now let us prove Theorem 2. Assume that \( \lambda \leq \omega(a,b) \) and let \( \Omega \) is a strictly starshaped domain with respect to \( x_0 \) then
\[
(x-x_0) \cdot n > 0, \quad \text{for all } x \in \partial\Omega.
\]
Suppose that \( (u,v) \) is a solution of (24). By the Proposition 6.1 we get
\[
2\lambda \int_{\Omega} u(x)v(x)dx - \frac{1}{2} \int_{\Omega} \nabla b(x) \cdot (x-x_0)|\nabla v|^2 dx - \frac{1}{2} \int_{\Omega} [\nabla a(x) \cdot (x-x_0)] |\nabla u|^2 dx > 0.
\]
It follows that
\[
\lambda > \frac{1}{4} \inf_{(u,v) \in (H_0^1(\Omega))^2 \setminus \{0\}} \frac{\int_{\Omega} [\nabla a(x) \cdot (x-x_0)|\nabla u|^2 + \nabla b(x) \cdot (x-x_0)|\nabla v|^2] dx}{\int_{\Omega} u(x)v(x)dx}
\]
which is a contradiction.

Then the problem (24) does not admit solutions. Consequently we obtain the desired result. □

6.1 Estimates of \( \omega(a,b) \)

We end this section, using the techniques in [14], we give some estimates of \( \omega(a,b) \).

Proposition 6.2.
We assume that \(a, b \in C^1(\Omega)\) and there exists \(z_0 \in \Omega\) such that \(\tilde{a}(z_0) + \tilde{b}(z_0) < 0\), then \(\omega(a, b) = -\infty\).

We assume that \(a, b \in H^1(\Omega) \cap C(\overline{\Omega})\) satisfying (4), (5) respectively and \(\tilde{a}(x) \geq 0, \tilde{b}(x) \geq 0\) a.e. \(x \in \Omega\), we have

1. If \(k > 2, l > 2\) and \(a, b \in C^1(\Omega)\), then \(\omega(a, b) = 0\).

2. If \(k = 2, l > 2\) or \(k > 2, l = 2\), satisfying (6) and (7), then

\[
\frac{N^2}{16} \min (A_2, lB_1(diam \Omega)^{l-2}) \leq \omega(a, b) \leq \frac{A_2}{2} \lambda_1(diam \Omega)^2 \tag{34}
\]

and

\[
\frac{N^2}{16} \min (kA_k(diam \Omega)^{k-2}, B_2) \leq \omega(a, b) \leq \frac{B_2}{2} \lambda_1(diam \Omega)^2 \tag{35}
\]

3. If \(0 < k \leq 2, 0 < l \leq 2\), \(a, b\) satisfy the conditions (4) and (7) respectively, then

\[
\frac{N^2}{16} \min (kA_k(diam \Omega)^{k-2}, lB_1(diam \Omega)^{l-2}) \leq \omega(a, b). \tag{36}
\]

**Proof** Let \(\varphi \in C_c(\mathbb{R}^N)\) such that \(0 \leq \varphi \leq 1\) on \(\mathbb{R}^N\), \(\varphi \equiv 1\) on \(B(0, 2r)\) and \(\varphi \equiv 0\) on \(\mathbb{R}^N \setminus B(0, 2r)\), where \(0 < r < 1\). Set \(\varphi_j(x) = \varphi(j(x - z_0))\) for \(j \in \mathbb{N}^*\), we have

\[
\omega(a, b) \leq \frac{1}{4} \int_\Omega \frac{\left(\tilde{a}(x) + \tilde{b}(x)\right) \left|\nabla \varphi_j(x)\right|^2 dx}{\int_\Omega \varphi_j^2(x) dx}
\]

\[
\leq \frac{1}{4} \int_{B(b, \frac{2r}{j})} \frac{\left(\tilde{a}(x) + \tilde{b}(x)\right) \left|\nabla \varphi_j(x)\right|^2 dx}{\int_{B(b, \frac{2r}{j})} \varphi_j^2(x) dx}.
\]

Using the change of variable \(y = j(x - z_0)\), we get

\[
\omega(a, b) \leq \frac{j^2}{4} \int_{B(0, 2r)} \frac{\left(\tilde{a}(\frac{y}{j} + z_0) + \tilde{b}(\frac{y}{j} + z_0)\right) \left|\nabla \varphi(y)\right|^2 dy}{\int_{B(0, 2r)} \varphi^2(y) dy}.
\]

Applying the Dominated Convergence Theorem, we obtain the desired result when \(j\) goes to infinity.
Now we will prove (2.4).
Using (36), (37) and since $a, b \in C^1(\Omega)$ in a neighborhood $V$ of $x_1$, we write

$$a(x) = a(x_1) + A_k|x - x_1|^k + \theta_a(x)$$  \hspace{1cm} (36)

$$b(x) = b(x_1) + B_l|x - x_1|^l + \theta_b(x),$$  \hspace{1cm} (37)

where $\theta_a(x)$ and $\theta_b(x) \in C^1(V)$ are such that

$$\lim_{x \to x_1} \frac{|\theta_a(x)|}{|x - x_1|^k} = 0 \quad \text{and} \quad \lim_{x \to x_2} \frac{|\theta_b(x)|}{|x - x_1|^l} = 0.\hspace{1cm} (38)$$

From (38), we get the existence of $r, 0 < r < 1$, such that

$$|\theta_a(x)| \leq |x - x_0|^k \quad \text{and} \quad |\theta_b(x)| \leq |x - x_0|^l, \text{ for all } x \in B(x_0, 2r) \subset \Omega.\hspace{1cm} (39)$$

Let $\varphi_j(x) = \varphi(j(x - x_0))$ define as in the proof of (1); we have

$$0 \leq \omega(a, b) \leq \frac{1}{4} \int_{\Omega} \left( \frac{\|a(x) + b(x)\|}{\|\nabla \varphi_j(x)\|^2} \right) \int_{\Omega} \varphi_j^2(x) dx$$

Using (36) and (37), we obtain

$$0 \leq \omega(a, b) \leq \frac{1}{4} \int_{B(x_0, 2r)} \left( kA_k|x - x_0|^k + lB_l|x - x_0|^l \right) \|\nabla \varphi_j(x)\|^2 dx$$

$$+ \frac{1}{4} \int_{B(x_0, 2r)} \varphi_j^2(x) dx.$$  

By a simple change of variable $y = j(x - x_0)$ and integrating by parts, we obtain

$$0 \leq \omega(a, b) \leq \frac{kA_k}{4j^{k-2}} \int_{B(0, 2r)} |y|^k \|\nabla \varphi(y)\|^2 dy$$

$$- \frac{j}{4} \int_{B(0, 2r)} \theta_a \left( \frac{y}{j} + x_0 \right) \cdot \text{div}(y|\nabla \varphi(y)|^2) dy$$

$$+ \frac{lB_l}{4j^{l-2}} \int_{B(0, 2r)} |y|^l \|\nabla \varphi(y)\|^2 dy$$

$$- \frac{j}{4} \int_{B(0, 2r)} \theta_b \left( \frac{y}{j} + x_0 \right) \cdot \text{div}(y|\nabla \varphi(y)|^2) dy.$$
Using (39), we get

\[
0 \leq \omega(a, b) \leq \frac{kA_k}{4j^{k-2}} \int_{B(0,2r)} |y|^k \nabla \varphi(y) \, dy + \frac{C}{j^{k-1}} \int_{B(0,2r)} |y|^k \, dy
\]

\[
+ \frac{lB_l}{4j^{l-2}} \int_{B(0,2r)} |y|^l |\nabla \varphi(y)|^2 \, dy + \frac{C}{j^{l-1}} \int_{B(0,2r)} |y|^l \, dy,
\]

where \(C = \max_{y \in B(0,2r)} |\text{div}(y|\nabla \varphi(y)|^2)|\).

Therefore, for \(k > 2\) and \(l > 2\), we reach that \(\omega(p,q) = 0\). This concludes the proof of (2.i).

To prove (2.ii), we start by the case \(k = 2\) and \(l > 2\). We show the left-hand inequality in (34). Since \(a\) and \(b\) satisfy (6) and (7), respectively, for all \((u,v) \in E \setminus \{0\}\), we have

\[
\phi_{a,b}(u, v) \geq \frac{1}{2} A_2 \int_{\Omega} \frac{|(x-x_0) \cdot \nabla u|^2 \, dx}{\int_{\Omega} uv \, dx} + \frac{1}{4} B_l \int_{\Omega} \frac{|x-x_0|^{l-2} |(x-x_0) \cdot \nabla v|^2 \, dx}{\int_{\Omega} uv \, dx}
\]

\[
\geq \frac{1}{2} A_2 \int_{\Omega} \frac{|(x-x_0) \cdot \nabla u|^2 \, dx}{\int_{\Omega} uv \, dx} + \frac{1}{4} B_l (\text{diam } \Omega)^{l-2} \int_{\Omega} \frac{|(x-x_0) \cdot \nabla v|^2 \, dx}{\int_{\Omega} uv \, dx}.
\]

By applying Lemma 2.1 in [14] for \(t = 0\), we find

\[
\phi_{a,b}(u, v) \geq \frac{1}{2} A_2 \left( \frac{N}{2} \right)^2 \int_{\Omega} \frac{|u|^2 \, dx}{\int_{\Omega} uv \, dx} + \frac{1}{4} B_l (\text{diam } \Omega)^{l-2} \left( \frac{N}{2} \right)^2 \int_{\Omega} \frac{|v|^2 \, dx}{\int_{\Omega} uv \, dx}.
\]

This implies that

\[
\omega(a, b) \geq \frac{N^2}{16} \min(A_2, lB_l (\text{diam } \Omega)^{l-2}).
\]

Similarly, we deduce in the case \(k > 2\) and \(l = 2\), that

\[
\omega(a, b) \geq \frac{N^2}{16} \min(kA_k (\text{diam } \Omega)^{k-2}, B_2).
\]
Now we prove the right-hand inequality in (34) and (35). Let $\psi_j(x) = \varphi_1(j(x - x_0))$ for $j \in \mathbb{N}$ large enough, where $\varphi_1$ is a positive eigenfunction corresponding to the first eigenvalue $\lambda_1$ of the operator $-\Delta$ in $H^1_0(\Omega)$.

We have

$$0 \leq \omega(a, b) \leq \frac{1}{4} \int_\Omega (\tilde{a}(x) + \tilde{b}(x)) |\psi_j(x)|^2 dx$$

Using (4) and (5), we obtain

$$0 \leq \omega(a, b) \leq \frac{4 \int_{x_0 + \frac{1}{4} \Omega} \left( 2A_1 |x - x_0|^2 + lB_1 |x - x_0|^l \right) |\nabla \psi_j(x)|^2 dx}{4 \int_{x_0 + \frac{1}{4} \Omega} \psi_j^2(x) dx}$$

$$+ \frac{\int_{x_0 + \frac{1}{4} \Omega} (\nabla \theta_a(x) \cdot (x - x_0) + \nabla \theta_b(x) \cdot (x - x_0)) |\nabla \psi_j(x)|^2 dx}{4 \int_{x_0 + \frac{1}{4} \Omega} \psi_j^2(x) dx}.$$

By a simple change of variable $y = j(x - x_0)$ and integrating by parts, we have by (38)

$$0 \leq \omega(a, b) \leq \frac{A_2}{2} \int_\Omega |y|^2 |\nabla \varphi_1(y)|^2 dy + C \int_{\Omega} |y|^k dy$$

$$+ \frac{lB_1}{4j^{-2}} \int_\Omega |y|^l |\nabla \varphi_1(y)|^2 dy + C \int_\Omega |y| dy,$$

where $C = \max_{y \in \Omega} |\text{div}(y \nabla \varphi_1(y))|^2|$. Letting $j \to \infty$ we get

$$0 \leq \omega(a, b) \leq \frac{A_2}{2} \int_\Omega |y|^2 |\nabla \varphi_1(y)|^2 dy,$$

therefore

$$0 \leq \omega(a, b) \leq \frac{A_2}{2} \lambda_1 (\text{diam} \ \Omega)^2.$$
Let us now prove (2.iii). Since $a$ and $b$ satisfy (6) and (7), respectively, for all $(u, v) \in E \setminus \{0\}$, we have

$$\phi_{a,b}(u, v) \geq \frac{k}{4} A_k \int_{\Omega} |x - x_0|^{k-2} |(x - x_0) \cdot \nabla u|^2 dx + \frac{l}{4} B_l \int_{\Omega} |x - x_0|^{l-2} |(x - x_0) \cdot \nabla v|^2 dx$$

$$\geq \frac{k}{4} A_k (\text{diam } \Omega)^{k-2} \int_{\Omega} |(x - x_0) \cdot \nabla u|^2 dx + \frac{l}{4} B_l (\text{diam } \Omega)^{l-2} \int_{\Omega} |(x - x_0) \cdot \nabla v|^2 dx.$$

By applying Lemma 2.1 in [14] for $t = 0$, we find

$$\phi_{a,b}(u, v) \geq \frac{k}{4} A_k (\text{diam } \Omega)^{k-2} \left( \frac{N}{2} \right)^2 \int_{\Omega} |u|^2 dx + \frac{l}{4} B_l (\text{diam } \Omega)^{l-2} \left( \frac{N}{2} \right)^2 \int_{\Omega} |v|^2 dx.$$

This implies that

$$\omega(a, b) \geq \frac{N^2}{16} \min\{kA_k (\text{diam } \Omega)^{k-2}, lB_l (\text{diam } \Omega)^{l-2}\}.$$
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