After introducing the notion of weak Berwald spacetime with a maximal proper acceleration, it is shown that such spacetimes are free of divergent singularities. The upper bounds in the curvature that a maximal proper acceleration implies are analyzed in two cases: for conformally flat spacetimes and for Ricci flat spacetimes.

1. Introduction

The appearance of spacetime singularities in general relativity is one of the main motivations to search for a new theory of gravity that, superseding the current paradigm, should resolve the singularities. It is also believed that new phenomenological and theoretical clues for understanding gravity will be found in regimes and domains where the gravitational interaction manifests strongly.

The strength of a classical gravitational field is characterized by the curvature tensor of a Levi-Civita connection. Hence strong gravity implies strong curvature. Curvature is also a measure of the relative separation of nearby geodesics. In particular, the Riemann curvature endomorphism is related with the relative acceleration between neighboring geodesics. Therefore, the strength of a gravitational field is related with the values of relative accelerations between geodesics.

On the other hand, it has been speculated for a long time with the idea of a maximal proper acceleration [2, 3]. For a recent review of some aspects of maximal acceleration, the reader is referred to ref. [12]. The intuitive implication of a maximal acceleration for curvature is an uniform upper bounds on the curvature(s). In this paper we confirm this insight, showing that certain type of spacetimes with metrics of maximal acceleration are free of divergent spacetime singularities, the ones that have associated a divergent curvature tensor.

The present paper is not the first time claims where maximal acceleration resolves singularities have been investigated and well-known studies of particular examples are [5] and [17]. However, our treatment is the first to show that this is the generic feature in a particular framework of spacetimes with metrics of maximal acceleration, namely, in the framework of classical higher order jet geometries [8, 9].

The structure of this paper is the following. In section 2, the notion of spacetime with a metric of maximal acceleration is reviewed from [7, 9]. In section 3, the Jacobi equation and the notion of weak Berwald spacetime is introduced in the above framework. These are examples of spacetimes of higher order jet geometry such that along the geodesics of a limit Lorentzian structure, the metric of maximal acceleration structure is geodesically equivalent to the limit Lorentzian structure. The notion of weak Berwald spacetimes allows us to implement the weak equivalence
principle in a natural way in the context of higher order jet geometries [8]. Then we discuss the general implications of the existence of a maximal proper acceleration for the Riemann curvature tensor. We show in section 4 that the spacetimes under consideration are free of divergent singularities. The meaning of the bounds on the curvature discussed in more detail in section 5. First, we investigate spacetimes where the conformal Weyl tensor is zero. Second, we consider spacetimes where the Ricci tensor is zero. In both cases, quadratic combinations of the remaining curvatures appear to be upper bounded. Few general remarks are highlighted in the discussion section, where also a very brief outlook for future developments is presented.

2. Metrics of maximal acceleration

Let us first introduce the geometric framework for this work. The spacetime manifold \( M_4 \) will be a four-dimensional smooth manifold. The metric structures that we will consider are called metrics of maximal acceleration [9]. Basically, they are metric structures that depend upon the second jet of smooth world lines \( x : I \to M \). These curves are interpreted as the world lines of test particles probing the structure of a classical spacetime, that is, a spacetime not subjected to quantum fluctuations. The metric structure determines the time that an observer will measure along its world line \( x : I \to M_4 \). If the curve \( x : I \to M_4 \) is (piecewise)-smooth causal, which means \( g(x', x') \leq 0 \), the proper time is of the form

\[
\tau[x] = \int_{t_0}^{t} \left[ -g(x', x') \right]^{\frac{1}{2}} ds,
\]

where the integration parameter will be specified later, since in general the integrands that we will consider are not homogeneous under changes of integration parameter.

If we adopt Synge’s chronodynamics postulates [18] except for Einstein’s clock hypothesis [6] and we complete such postulates with the assumption of the existence of maximal acceleration scale \( A_{\text{max}} \) embedded in the kinematical properties of the spacetime, then it is natural to adopt the following form for the integrand \( g(x', x') \) in the expression for the proper time (2.1),

\[
g(2x) = g^0(x, x', x'') + g^1(x, x', x'') \xi(x, x', x'', A_{\text{max}}^2),
\]

where \( 2x^\mu(\tau) = (x^\mu(t), x'^\mu(t), x''^\mu(t)) \) is the second jet at \( x(t) \) of the curve \( x : I \to M_4 \).

The expression (2.2) introduces a family of structures

\[
G(A_{\text{max}}) = \{g(A_{\text{max}}), A_{\text{max}} \in (0, +\infty)\}
\]

parameterized by the value of what it will be identified with the maximal proper acceleration, namely, the parameter \( A_{\text{max}} \). Thus we can consider limits when \( A_{\text{max}} \to +\infty \) in the family of metrics \( G(A_{\text{max}}) \). We require that the metric structure obtained by the limit

\[
\lim_{A_{\text{max}}^2 \to +\infty} g(2x) := g^0(x, x', x'')
\]

to be compatible with the clock hypothesis. Hence the natural form for the components of \( g \) are analytical in \( 1/A_{\text{max}}^2 \),

\[
\xi(x, x', x'', A_{\text{max}}^2) = \sum_{n=1}^{+\infty} \xi_n(x, x', x'') \left( \frac{1}{A_{\text{max}}^2} \right)^n.
\]
Compatibility with the clock hypothesis in the above mentioned limit implies,
\[ g^0(x, x', x'') = g^0(x, x'). \]

A weaker form of Synge's chronodynamics postulates implies that the metric structure \( g^0(x, x') \) is a generalized Finsler structure, but to simplify the treatment, it will be a Lorentzian metric \( \eta := g^0 \).

The Lorentzian spacetime \((M_4, \eta)\) can be thought as a limit of a spacetime endowed with a metric of maximal acceleration, associated to the proper time \((2.1)\) whose integrand is of the form
\[ (2.4) \quad g((x(\tau))(x', x')) = \left(1 + \frac{\eta(D_{x'}^x x'(\tau), D_{x'}^x x'(\tau))}{A_{\text{max}}^2 \eta(x', x')} \right) \eta(x', x') \]
and where \( D \) is the covariant derivative of the Levi-Civita connection of \( \eta \).

We specify the parameter respect to which we are taking derivatives and parameterizing curves to be the proper time parameter associated to the Lorentzian metric \( \eta \). Then the action of the metric of maximal acceleration on two arbitrary vector fields \( W, Q \) along \( x : I \to M \) is given by
\[ (2.5) \quad g(W, Q) = \left(1 + \frac{\eta(D_{x'}^x x'(t), D_{x'}^x x'(t))}{A_{\text{max}}^2 \eta(x', x')} \right) \eta(W, Q) \]

Originally, the form \((2.4)\) appeared as a covariant version \((7)\) of the metric developed by E. Caianiello and co-workers \((4)\), but it has a more natural interpretation as a higher order jet geometry \((7, 8, 9)\). From this second point of view, the metric structure of the spacetime assigns to each test particle probing the structure of the classical spacetime a line element which depends upon higher order derivatives of the test particle’s world line. Let us consider a world line \( \gamma : I \to M_4 \) with \( I = [t_0, t] \), a causal curve respect to the metric \( \eta \). Let us assume that at each point \( x(s) \in x(I) \) the spacetime is probed by a test particle with identical 2-jet than \( \gamma \). Then the proper time functional is of the form
\[ (2.6) \quad \tau[\gamma] = \int_{t_0}^t \left[ \left(1 + \frac{\eta(D_{x'}^x x'(s), D_{x'}^x x'(s))}{A_{\text{max}}^2 \eta(x', x')} \right)(-\eta(x', x')) \right]^{\frac{1}{2}} ds, \]
from where we can extract the expression \((2.4)\) and the expression \((2.5)\).

It follows that the proper acceleration respect to the limit metric \( \eta \) is bounded, in the sense that for any causal curve such that \( (g(x', x') \leq 0) \) implies the bound
\[ (2.7) \quad \eta(D_{x'}^x x', D_{x'}^x x') \leq A_{\text{max}}^2. \]
This is the reason to call the structures given by \((2.4)\) or by the form \((2.5)\) metrics of maximal acceleration. In the context of a given classical theory or model, the relation \((2.7)\) must be adopted as of universal validity, for any type of motion described from any physical frame.

The theory of metric of maximal acceleration can be given a global formulation, where the metrics are sections of a particular kind of fibre bundles over jet bundles. This should make possible to define a relevant connection theory with the corresponding notion of geodesic and geodesic deviation \((8)\). However, such a tour de force is not necessary for the purposes of this paper, as we will show in the next section.

3. Jacobi Equation and Bounds on the Lorentzian Curvature

Let us consider a regular point \( p \in M_4 \). That is, at \( p \) it is possible to choose arbitrary initial conditions for the geodesics of the Levi-Civita connection \( D \). The point \( p \) can be set in an open set \( U \subset M_4 \) such that all the points at \( U \) are regular in
the above sense. Then we can consider a *geodesic tubular neighborhood* containing \( p \), constructed as follows. Given the initial condition \((X(0), X'(0)) = (p, T(0))\) with \( T(0) \in T_p M_4 \) a causal tangent vector, then there is a geodesic \( X : I \to M_4 \) with initial conditions \((p, T(0))\). Let us consider \( V_p \in T_p M_4 \) orthogonal to \( T \) with respect to the metric \( \eta \). We construct the parallel transport \( \mathcal{P}(T(0)) \) of \( T(0) \) along the geodesic with initial conditions \((X(0), V_p)\) from the initial point \( X(0) \) to a final point \( \tilde{p} \in U \). For close enough points \( p, \tilde{p} \in U \), this construction can be done always, as consequence of elementary ordinary differential equations theory. We can consider the initial conditions \((\tilde{p}, \mathcal{P}(T(0))) \) for constructing a geodesic of the connection \( D \), extending the geodesic to the reverse direction at \( \tilde{p} \), in order that \( \tilde{p} \) is in the interior of the two geodesics with initial conditions \((\tilde{p}, \mathcal{P}(T(0))) \) and \((\tilde{p}, -\mathcal{P}(T(0))) \).

Repeating this procedure for each pair \((p, V_p)\) with \( p \) fixed and \( V_p \) orthogonal to \( T(0) \) but otherwise arbitrary, we can embed \( p \) in an open tube set \( D_p \subset M_4 \) filled by the image of a aggregate of geodesics \( x : I \to M_4 \) and such that \( D_p \) also contains the central geodesic \( X : I \to M_4 \). Moreover, by reducing enough the size of the tube, the geodesics will not cross to each other. Since the central geodesic \( X : I \to M_4 \) is causal, then the geodesics \( x : I \to M_4 \) are also causal and of the same type.

For our purposes, the above *geodesic tube* \( D_p \) containing the point \( p \) is useful because its fundamental role in the proof of the following result,

**Lemma 3.1.** If \( \tilde{p} \) is point of \( D_p \subset M_4 \), a geodesic tube of \( p \), then the maximal acceleration spacetime metric \( g \) is equivalent to \( \eta \), when probed by geodesic curves of \( \eta \).

**Proof.** For each regular point \( \tilde{p} \in M_4 \) there is a geodesic \( x : I \to M \) which pass through \( \tilde{p} \) and which image is contained in \( D_p \). When we particularize the expression \((2.4)\) for the case when \( D_x x' = 0 \) for geodesics \( x : I \to M_4 \), then \( g = \eta \) in the whole tubular neighborhood \( D_p \). \( \square \)

Assume the existence of a connection which preserves \( g \). We can say that the connection is associated to \( g \). Then the physical significance of this Lemma becomes clear if we consider the following notion,

**Definition 3.2.** A spacetime of maximal acceleration is weak Berwald if there is a connection associated to \( g \) which is also the Levi-Civita connection of the limit metric \( \eta \) along its geodesics.

This notion is related with the corresponding notion of Berwald spacetime in the theory of Finsler spacetimes. Generalized Berwald spacetimes are general structures supporting Synge’s assumptions of chronodynamics and being compatible with the weak equivalence principle.\[1\]

Let us note that the restriction to the geodesic support in Definition 3.2 is fundamental for speaking of universal free fall. If the process of probing the metric structure of the spacetime does not perturb greatly enough the spacetime geometry, as by universal free fall is required, then the geodesics of \( g \) must be the geodesics of \( \eta \), since \( \eta \) is the limit of \( g \) when the probing action does not implies back reaction. In addition, free fall motion is equivalent to select an aggregate of world lines as being the geodesics of an affine connection. Therefore, it is then natural to adopt as definition of geodesics the solutions of the system of differential equations \( D_x x' = 0 \).
This argument shows that the condition of being weak Berwald spacetime, in the framework of spacetimes with a metric of maximal acceleration, is the natural condition to accommodate the weak equivalence principle.

After the above considerations, it is direct the following

**Proposition 3.3.** A spacetime of maximal acceleration is a weak Berwald spacetime.

Several consequences of **Lemma 3.1** are of physical significance, since it allows to translate results from Lorentzian geometry to spacetimes with metrics of maximal acceleration in relevant physical situations:

**Proposition 3.4.** Let \( p \in M_4 \) be in a geodesic tube \( D_p \subset M_4 \) of \( p \). When the spacetime structure is probed by geodesic curves of \( D_p \), then the following properties hold true:

- In \( D_p \), the connection \( D \) leaves invariant \( g \).
- In \( D_p \), the Jacobi equation along the central geodesic \( X : I \rightarrow M_4 \) for \( g \) is given by the expression

\[
D_{X'}D_{X'}J + R(X', J) \cdot X' = 0,
\]

where \( R(\cdot, \cdot) \) are the curvature endomorphisms of the Levi-Civita connection of \( \eta \).

**Proof.** In order to prove that the first point holds, just note that by **Lemma 3.1** when the spacetime structure is tested by geodesics in \( D \), then \( g = \eta \) and hence \( Dg = D\eta = 0 \).

Similarly and for the same reason, in order to prove the second point, we can follow the standard derivation of Jacobi's equation in Lorentzian geometry [14]. Note that here \( J \) is a Jacobi field of the Levi-Civita connection \( D \).

**Proposition 3.4** has relevant physical consequences. Let us first emphasize that the upper bound (2.7) in the maximal acceleration should apply to the relative accelerations of the displacement vector \( J \) (a Jacobi field of \( D \)). In the case that an observer has associated the timelike geodesic \( X : I \rightarrow M_4 \), this interpretation follows from application of the principle of general covariance in the spirit of Einstein’s original theory of relativity [6]. The interpretation of the second covariant derivative \( D_{X'}D_{X'}J \) as the relative acceleration between geodesics is also valid to the case of null geodesics, although then the central geodesic \( X : I \rightarrow M_4 \) does not have associated a physical observer, but it can be seen as a limit case of a continuous series of timelike observers.

As a result of the universality of the maximal acceleration and the properties discussed in **Proposition 3.4** we have that

\[
\eta(R(X', J) \cdot X', R(X', J) \cdot X') \leq A_{\text{max}}^2.
\]

The relation (3.2) is the general form of a pointwise upper bound on the curvature of the Lorentzian metric \( \eta \). In order to understand its meaning, it is useful to consider geodesic congruences containing a given point \( p \in M_4 \). Let us consider a particular initial condition \( (X(0), X_0'(0)) \), where \( X(0) = p \in D_p \subset M_4 \) and \( X_0'(0) \in T_p M_4 \) is a causal tangent vector. For concreteness, let us assume that \( X_0'(0) \) is a time-like vector and three supplementary initial conditions \( J_{0b}(0) \) orthogonal Jacobi
fields, that without loss of generality, satisfy the relations
\[ \eta(X'_0(0), X'_0(0)) = -1, \quad \eta(J_{0b}(0), X'_0(0)) = 0, \quad \eta(J_{0b}(0), J_{0c}(0)) = \delta_{bc}, \]
\[ b, c = 1, 2, 3. \]

But for the curvature endomorphisms we have
\[ \eta(R(\dot{X}_a, J_b) \cdot X'_a, R(\dot{X}_a, J_b) \cdot X'_a) = 3 \sum_{\mu, \nu=0}^{3} \eta(R^\mu\sb{aba} E_\mu, R^\nu\sb{aba} E_\nu) \]
\[ = 3 \sum_{\mu, \nu=0}^{3} R^\mu\sb{aba} R^\nu\sb{aba} \eta(E_\mu, E_\nu) \]
\[ = 3 \sum_{\mu=0}^{3} R^\mu\sb{aba} R^\mu\sb{aba}, \]
where the index \( a \) is not summed and where \( \{E_\mu, \mu = 0, 1, 2, 3\} \) forms an orthogonal frame of vector fields along \( X : I \to M \). However, the frame \( \{E_\mu, \mu = 0, 1, 2, 3\} \) can be taken as the frame \( \{X'_a(0), J_b\} \) and this will be the case from now on.

**Proposition 3.5.** In a spacetime geometry of maximal acceleration \( (M_4, g) \), the following bounds on the Riemannian curvature of the limit Lorentzian curvature \( (M_4, \eta) \) must be satisfied:
\[ R^\mu\sb{aba} R^\mu\sb{aba} \leq A_{\text{max}}^2, \quad b, d = 0, 1, 2, 3, \quad \mu = 0, 1, 2, 3, \]
(3.3)

**Proof.** This is a direct consequence of the conditions (3.2) and the previous considerations. \( \square \)

In the relation (3.3), the repeated indexes \( a, b \) are not summed. Their meaning refers in terms of the orthonormal frame \( \{X'_a(0), J_b\} \).

4. **General Consequences of Maximal Acceleration for Singularities**

The Jacobi equation \( (3.1) \) and the relation \( (3.2) \) can be applied at the points of the spacetime that can be used as initial conditions for geodesics. Hence, singularity points of the spacetime are excluded in the above construction. The set \( S \subset M_4 \) of singularities of the spacetime is the complement of an open set (this follows by general considerations of ODE theory). Therefore, \( S \) is a close set.

Let us further assume that the topological boundary of \( M_4 \) is not empty. Then one of the following four alternatives must hold:

- The curvature is finite at the singularity points. This situation is the most complicated to treat without further assumptions. If the negation of the relation \( (3.2) \) holds, then
\[ \eta(R(X', J) \cdot X', R(X', J) \cdot X') > A_{\text{max}}^2 \]
(4.1)
at a singularity. Hence the aggregate of singularities \( S \) must be an open set. Since it is already a closed set, it must be either \( \emptyset \) or the whole \( M_4 \). Therefore, \( S \) must be the empty set.

- However, it is not fully correct to associate singularities with the condition \( (4.1) \), related with the divergence of the curvature tensor \( (3.4) \). Therefore, from now on, it is useful to consider singularities the curvature tensor actually diverges.
• The curvature tensor $R_{abcd}$ cannot be defined at a given singularity because the theoretical value diverges, but $R_{abcd}$ still can be defined at arbitrarily near points to the singularity. In this case it must be points where (3.1) holds for any value of the maximal acceleration $A_{\text{max}}$ for points near enough to the singularity. But this is a contradiction with the relation (3.2), which is applicable anywhere except at the singularity points.

• The curvature tensor $R_{abcd}$ is divergent in a whole open neighborhood of a given singularity $s \in \mathcal{S}$. Then the singularity is in the interior of $\mathcal{S}$. Let us consider any curve of singularities $\gamma_s$, contained in $\mathcal{S}$, starting at $s$ and with final point at the topological boundary of $\mathcal{S}$. At the boundary $\gamma_s \cap \partial \mathcal{S}$, we can apply the argument of the previous point, that leads to the same kind of contradictions. Thus the starting situation, namely, the singularity in the interior of $\mathcal{S}$, is impossible to hold in a spacetime with a metric of maximal acceleration as (2.5) and when the set of singularities has a boundary.

Hence we have proved the following,

**Theorem 4.1.** In a spacetime $(M_4, g)$ of maximal acceleration of weak Berwald type there are no spacetime singularities where the curvature tensor is divergent.

Note that the arguments discussed in the first three points are independent of the existence of a non-trivial topological boundary in $\mathcal{S}$. Also, let us remark that we have proved this result under the above mentioned restrictions, namely, that the spacetime is of weak Berwald type, that the singularity set has a non-trivial topological boundary and that the curvature diverges at the singularities. The second condition can be modified, for instance, assuming that singularities are isolated or that they form a regular sub-manifold of $M_4$, to obtain analogous results. The third condition possibly can be substituted by a weaker condition indicating that gravity is strong enough. The condition of being weak Berwald type is essential for these results to hold. From now on, we will assume the weak Berwald condition.

5. Implications of maximal acceleration for spacetime curvature

In order to clarify the significance of the upper bound (3.3), let us consider the *Ricci decomposition* of the Riemann curvature tensor [14, 1], that for a spacetime of dimension four is given by the expression

$$R_{abcd} = C_{abcd} - \frac{1}{2} \left\{ \eta_{ad} R_{cb} - \eta_{ac} R_{db} + \eta_{bc} R_{da} - \eta_{bd} R_{ca} \right\} - \frac{1}{6} R \left\{ \eta_{ac} \eta_{db} - \eta_{ad} \eta_{cb} \right\}.$$  

This expression relates the Riemann tensor with the Ricci tensor $R_{ab}$, the Ricci curvature $R$ and the Weyl conformal tensor $C_{abcd}$. In particular, we gave

$$R_{daba} = C_{daba} - \frac{1}{2} \left\{ \eta_{da} R_{ba} - \eta_{db} R_{aa} - \eta_{aa} R_{bd} \right\} - \frac{1}{6} \eta_{aa} \eta_{bd} R,$$

since $b \neq a$. In the derivation of this relation we have used orthonormal frames where the metric $\eta$ is diagonal and of the form $\eta = \text{diag}(-1,1,1,1)$ at a given point of the central geodesic $X(0) \in U$. 

By direct computation, using the Ricci decomposition for the geometric framework that we have discussed above, one obtains the following bounds in the curvature components with respect to the frame \( \{ X'_a, J_b \} \),

\[
\sum_d C^d_{aba} C_{daba} + C_{baba} R_{aa} - \sum_d C_{daba} R_b^d + \frac{1}{3} RC_{baba}
\]

\[
+ \frac{1}{4} \left\{ R_{aa} R_{aa} - 2 R_{aa} R_{bb} + (R_{ba})^2 + \sum_d R^d_b R_{db} \right\}
\]

\[
- \frac{1}{6} R \left\{ -R_{aa} + R_{bb} \right\} + \frac{1}{36} R^2 \leq A_{\text{max}}^2,
\]

(5.2)

where in this expression, we have imposed already that \( a \neq b \).

There are two situations where the relation (5.2) has clear physical implications:

1. **Conformally flat spacetimes.** In the first one, one makes the assumption that the conformal tensor is zero,

\[
C_{abcd} = 0.
\]

Then the relation (5.2) implies

\[
\frac{1}{4} \left\{ (R_{aa})^2 - 2 R_{aa} R_{bb} + (R_{ba})^2 + \sum_d R^d_b R_{db} \right\}
\]

\[
- \frac{1}{6} R \left\{ -R_{aa} + R_{bb} \right\} + \frac{1}{36} R^2 \leq A_{\text{max}}^2.
\]

By a re-ordering of the first line, this expression can be re-written as

\[
\frac{1}{4} \left\{ (-R_{aa} + R_{ba})^2 + (R_{ba})^2 + \sum_{d \neq b} R^d_b R_{db} \right\}
\]

\[
- \frac{1}{6} R \left\{ -R_{aa} + R_{bb} \right\} + \frac{1}{36} R^2 \leq A_{\text{max}}^2.
\]

Since

\[
\sum_{d \neq b} R^d_b R_{db} = \sum_c \sum_{d \neq b} R_{db} R_{dc} \eta^{cd} = -(R_{ba})^2 + \sum_{d \neq b.a} (R_{bd})^2,
\]

we have the expression

\[
\frac{1}{4} \left\{ (-R_{aa} + R_{ba})^2 + \sum_{d \neq b.a} (R_{bd})^2 \right\}
\]

\[
- \frac{1}{6} R \left\{ -R_{aa} + R_{bb} \right\} + \frac{1}{36} R^2 \leq A_{\text{max}}^2.
\]

From this relation, it follows the following result:

**Proposition 5.1.** For a conformally flat spacetime of maximal acceleration of weak Berwald type, we have a general bound of the form

\[
- \frac{1}{6} R \left\{ -R_{aa} + R_{bb} \right\} + \frac{1}{36} R^2 \leq A_{\text{max}}^2,
\]

(5.3)

where \( a \neq b \).

There are several consequences of **Proposition 5.1**.

**Corollary 5.2.** Assume for the Ricci scalar \( R \geq 0 \) in a spacetime of maximal acceleration of weak Berwald type which is conformally flat. If in addition the relation

\[
R_{aa} \geq R_{bb}
\]

(5.4)

holds good in the orthonormal frames considered before, then
\[ \frac{1}{36} R^2 \leq A_{\text{max}}^2. \] (5.5)

The condition \( R \geq 0 \) is illustrated by the value of the curvature of a Robertson-Walker spacetime with \( k = 0 \), where \( R \geq 0 \) is valid for dust mater source and for radiation described by perfect fluids. Therefore, the condition \( R \geq 0 \) appears as a reasonable assumption. This remark can be extrapolated to the case when \( k \neq 0 \).

We can appreciate the meaning of the condition \( R_{\alpha\alpha} \geq R_{\beta\beta} \) by considering Einstein equations without the cosmological constant. In this case, the condition (5.4) follows from the dominant energy condition in an orthonormal frame,
\[ T_{00} \geq |T_{ij}|, \quad \forall \, i, j = 1, 2, 3. \]

It is natural to think that the condition \( R_{\alpha\alpha} \geq R_{\beta\beta} \) holds at least for classical theories that quantitatively do not differ significantly from general relativity.

2. Ricci-flat spacetimes. In this case, the Ricci-flat condition \( R_{\mu\nu} = 0 \) and the general form of the bound (5.2) implies
\[ \sum_d C_{daba} C^{daba} \leq A_{\text{max}}^2. \] (5.6)

The conformal tensor is characterized by having all traces null and by having the same symmetries than the Riemann tensor. The conditions of null trace are
\[ \sum_{\mu} C^{\mu}_{\nu\rho\sigma} = 0, \quad \mu, \nu, \rho = 0, 1, 2, 3. \] (5.7)

For spacetimes of dimension four, these trace-free conditions imply 16 constrains, reducing the number of independent components of the conformal tensor to 4 functions: since the conformal tensor has the same symmetries than the Riemann tensor, in dimension four the conformal tensor has 20 components, which are subjected to the 16 trace-free conditions. On the other hand, the conditions (5.6) imply three additional constraints. Hence, the remaining four components independent of \( C_{abcd} \) are constrained by four quadratic inequalities, the result of applying coordinate changes to the relations (5.6).

**Proposition 5.3.** For a spacetime with a maximal acceleration of weak Berwald type and Ricci flat, the components of the conformal tensor respect to orthonormal frame \( \{ X(0), J_b(0) \} \) are bounded.

6. Discussion

We have shown that if the weak equivalence principle is accepted and one assumes the existence of an uniform or universal bound for the proper acceleration of test particles in the form of a weak Berwald spacetime, then there are no spacetime singularities whose curvature diverges.

The notion of spacetimes of maximal acceleration that we have considered are not necessarily originated by corrections to the classical general relativity due to a quantum origin of gravity. Indeed, we have discussed the upper bounds in the Riemann curvature in the contest of classical geometries of maximal acceleration, where the notion of higher order geometry is fundamental. Hence, our proposed resolution for the incompleteness of general relativity, namely, **Theorem 4.1**, is a classical result, in contrast with arguments involving quantum gravity [17]. The main argument in favour of a metric of maximal acceleration has been discussed elsewhere [10]. It is based upon considerations of the mathematical treatment of
radiation reaction together with an analysis of the limits of applicability of the clock hypothesis in the foundations of the theory of relativity [15]. Indeed, we have adopted the viewpoint that the existence of a maximal proper acceleration should be imprinted in the geometry of the spacetime, in a similar way as the existence of a maximal speed is imprinted in the causal structure causal structure of the spacetime.

Although the hypothesis of a maximal acceleration is fertile in consequences, there is no empirical evidence of it. However, a new approach to observe effects of maximal acceleration in laser-plasma dynamics has been recently discussed [13]. In that paper it was argued that a modification of the Lorentz force due to maximal acceleration is potentially testable in near future laser-plasma acceleration facilities.

The general result of the absence of divergent singularities in spacetimes of maximal acceleration, Theorem 4.1, applies under rather general conditions. It does not involve particular field equations or energy conditions. This shows that the assumption of a maximal proper acceleration and the validity of the weak equivalence principle and the principle of general covariance, are strong hypotheses when they are considered together. However, Theorem 4.1 provides few details about the origin and physical significance of the maximal proper acceleration.

The results of section 5 illustrate the meaning of the bound of maximal acceleration in our framework, although these results are not formulated in a covariant way, since they involve conditions satisfied in specific orthonormal frames. One is tempted to relate Proposition 5.1 and Proposition 5.3 with Weyl curvature hypothesis [16] extrapolated to frameworks of spacetimes with metrics of maximal acceleration. The Weyl curvature hypothesis states that at an initial singularity, Weyl’s curvature was zero (or very small), meanwhile the Ricci curvature was large but finite. Then we observe that maximal acceleration constrains the appearance of singular Ricci tensor for initial singularities.

A fundamental point missing in this paper is an example of complete gravitational model with maximal acceleration in the framework of higher order jet classical spacetime geometries. One can partially address this issue by considering the models of maximal acceleration of Caianiello and co-workers [4], that although based upon a mathematical formalism which is no general covariant, it could be considered a first step towards a consistent theory of spacetimes with metrics of maximal acceleration. However, a more sophisticated and rigorous line of research that all these issues raise is to develop a theory of gravity in the framework of higher order jet geometry [8].

The case when the singularities are not associated to the divergence of the curvature tensor is not directly considered in this paper. It seems to me that the methods to investigate the effect of maximal acceleration for such singularities are intrinsically different from the methods considered in this paper.

Finally, we have not considered in this paper the aspects of the thermodynamics of black holes that should be modified by the existence of a maximal acceleration. This fundamental and interesting question will be developed in future work.

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