NATURAL OPERATIONS IN INTERSECTION COHOMOLOGY

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Abstract. Eilenberg-MacLane spaces, that classify the singular cohomology groups of topological spaces, admit natural constructions in the framework of simplicial sets. The existence of similar spaces for the intersection cohomology groups of a stratified space is a long-standing open problem asked by M. Goresky and R. MacPherson. One feature of this work is a construction of such simplicial sets. From works of R. MacPherson, J. Lurie and others, it is now commonly accepted that the simplicial set of singular simplices associated to a topological space has to be replaced by the simplicial set of singular simplices that respect the stratification. This is encoded in the category of simplicial sets over the nerve of the poset of strata. For each perversity, we define a functor from it, with values in the category of cochain complexes over a commutative ring. This construction is based upon a simplicial blow up and the associated cohomology is the intersection cohomology as it was defined by M. Goresky and R. MacPherson in terms of hypercohomology of Deligne’s sheaves. This functor admits an adjoint and we use it to get classifying spaces for intersection cohomology. Natural intersection cohomology operations are understood in terms of intersection cohomology of these classifying spaces. As in the classical case, they form infinite loop spaces. In the last section, we examine the depth one case of stratified spaces with only one singular stratum. We observe that the classifying spaces are Joyal’s projective cones over classical Eilenberg-MacLane spaces. We establish some of their properties and conjecture that, for Goresky and MacPherson perversities, all intersection cohomology operations are induced by classical ones.

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Introduction

M. Goresky and R. MacPherson introduced intersection homology in order to extend Poincaré duality from smooth manifolds to some singular spaces, the pseudomanifolds, admitting a decomposition into manifolds of different dimensions, called strata, assembled so that each point has a conical neighbourhood. Intersection homology relies on the notion of perversity, a parameter denoted $p$ which measures the tangential degree of the component of chains along the strata. For the intersection homology with rational coefficients, one gets the same picture as for topological manifolds [18, 19]. For instance, there exists a signature, which is a bordism invariant, when one applies the theory to Witt spaces [40]. But when working over a commutative ring, subtle and important differences occur.

Let us focus on Poincaré duality as an isomorphism between cohomology and homology given by a cap product with the fundamental class. If the intersection cohomology is defined from a linear dual of the intersection chain complex, we do not recover such Poincaré duality isomorphism for a general commutative ring $R$ of coefficients, without restriction on the torsion part of the intersection homology of the links of some singular strata, [23]. We refer the reader to the monographs [1, 2, 15, 27] for a detailed account of these results and their applications.

In previous works [8, 9], we have introduced a cohomology obtained from a process of blow up of singularities at the level of simplices. We call it blown up cohomology (or TW-cohomology) and denote it $H^\ast_{p}(\cdot; R)$. This cohomology coincides with the intersection cohomology obtained from the dual chain complex if coefficients are in a field but differs in general. One of its main features is the existence of cup products of classes and of cap products with intersection homology classes. In particular, the cap product with the fundamental class of a compact oriented pseudomanifold gives a Poincaré isomorphism between the blown up cohomology and the intersection homology [9, Theorem B]. Versions for the non compact case also exist in [9, 38, 37, 15].

In their second main paper ([20]), Goresky and MacPherson define and characterize complexes of sheaves whose hypercohomology coincides with intersection homology. The prototype is named Deligne sheaf and denoted $Q_p$. In [11], we prove that the sheafification of the blown up cochains is isomorphic to the Deligne sheaf in the derived category of complexes of sheaves of the space in consideration. Thus the construction that we develop in this work also applies to cohomological operations for the hypercohomology associated to $Q_p$.

One purpose of this work is the definition of a perverse analog of Eilenberg-MacLane spaces, and prove that their blown up cohomology is isomorphic to the set of intersection cohomological operations (see Theorem C below), an exact duplicate of the topological situation. In particular, our result answers the long standing question asked by Goresky and MacPherson as Problem 11 in [2]: Is there a category of spaces, maps and homotopies, and a “classifying space” $B$ so that intersection cohomology of $X$ can be interpreted as homotopy classes of maps from $X$ to $B$? Let us also mention that the existence of Steenrod squares in intersection cohomology was established by M. Goresky ([17]) and adapt to the blown up cohomology in [5]. (The main interest of [5] lies in the proof of a conjecture made in [22], see Conjecture B below.)

Operations in singular cohomology. Let us summarize the situation. If $X$ is a topological space, we can use the simplicial set $\text{Sing} X$, formed of the singular simplices, and move the problem into the simplicial paradigm. Thus, let $\text{Sset}$ be the category of simplicial sets and $M_{\text{dg}}$ be the category of cochain complexes of $R$-modules. Cohomology of simplicial sets can be defined as the homology of the normalized cochain functor,

$$N^\ast(\cdot; R): \text{Sset}^{\text{op}} \rightarrow M_{\text{dg}}.$$
If we apply it to the simplicial set $\text{Sing} X$, we recover the singular cohomology of the topological space $X$.

Let $K \in \text{Sset}$ and $M \in \mathbf{M}_{\text{dg}}$. The functor $N^*$ admits an adjoint $\langle - \rangle$, defined by $\langle M \rangle_k = \text{Hom}_{\mathbf{M}_{\text{dg}}}(M, N^*(\Delta[k]))$. Taking $M = R(n)$ with $R(n)^k = 0$ if $k \neq n$ and $R(n)^n = R$, we obtain an Eilenberg-MacLane space $K(R, n) = \langle R(n) \rangle$, [16, Corollary III.2.7], giving an isomorphism between homotopy classes and cohomology,

$$[K, K(R, n)]_{\text{Sset}} \cong H^n(K; R).$$

The category $\text{Sset}$ is simplicially enriched and we can define a simplicial set by,

$$\text{Hom}_{\text{Sset}}^\Delta(K, \langle M \rangle)_k = \text{Hom}_{\text{Sset}}(K \times \Delta[k], \langle M \rangle).$$

For $M = R(n)$, this simplicial set is an abelian simplicial group and thus of the homotopy type of a product of Eilenberg-MacLane space ([32, Théorème 6]),

$$\text{Hom}_{\text{Sset}}^\Delta(K, K(R, n)) \simeq \prod_{k=0} K(H^{n-k}(K; R), k),$$

the determination (0.1) corresponding to the image by $\pi_0$ of (0.2). Mention also that the family of Eilenberg-MacLane spaces $K(R, n)_\bullet$ is an infinite loop space, the based loop space $\Omega K(R, n)$ being homotopy equivalent to $K(R, n - 1)$.

By definition, a cohomological operation of type $(R, n, m)$ is a natural transformation between the functors $H^n(\cdot; R)$ and $H^m(\cdot; R)$, from $\text{Sset}$ to the category of $R$-modules. We denote $\text{Nat}_R(H^n, H^m)$ the set of cohomological operations of this type. The representability theorem stated in (0.1) reveals crucial in the determination of cohomology operations since, as a direct consequence of Yoneda’s lemma, there is an isomorphism

$$\text{Nat}_R(H^n, H^m) \cong H^m(K(R, n); R),$$

between the set of operations and the cohomology of Eilenberg-MacLane spaces.

These previous notions constitute a well known material and most of them are in a talk of Jean-Pierre Serre [39] in the Cartan seminar. However, they meet also recent progress and deep results in homotopy theory. The singular set $\text{Sing} X$ enters in the framework of quasi-categories developed by A. Joyal ([25, 26]) and J. Lurie ([28, 29]), where $\text{Sing} X$ has for 0-morphisms the points of $X$, 1-morphisms the paths in $X$, 2-morphisms the homotopy of paths,... Its first truncation gives the classical Poincaré groupoid. As we develop it below, our results contain the extension of this material to stratified spaces, with a presentation taking in account higher categorical structures.

**Stratified spaces.** It is time to specify the objects of our study. The prototype comes from the notion of pseudomanifolds (Definition 3.2), that covers many notions of interest as ([20]) real analytic varieties, Whitney stratified sets, Thom-Mather stratified spaces,... A more general situation is a Hausdorff topological space together with a partition

$$X = \sqcup_{S \in \mathcal{P}} S_a$$

whose elements are non-empty, locally closed subsets of $X$, called strata. If the partition is locally finite and any closure of a stratum is a union of strata, then we say that $X$, with its partition, is a **stratified space**, see Definition 1.1. A crucial property of stratified spaces is the existence of a structure of poset on $\mathcal{P}$ for the relation $S_a \preceq t$ if $S_a \subseteq S_t$. Endowing $\mathcal{P}$ with the associated Alexandrov topology, a structure of stratified space can be encoded in a continuous map $X \to \mathcal{P}$, with some additional properties, see Definition 1.4.
To obtain a simplicial representation of a stratified space $X$, we consider the nerve $\mathcal{N}(\mathcal{P})$ of the poset $\mathcal{P}$ and define a simplicial set over $\mathcal{N}(\mathcal{P})$, $\text{Sing}^p X$, as the pullback of

$$\text{Sing} X \to \text{Sing} \mathcal{P} \leftarrow \mathcal{N}(\mathcal{P})$$

This definition coincides with those of filtered simplices in [8, Definition A.3] and [30]. In terms of quasi-categories, $\text{Sing}^p X$ meets an environment similar to that of $\text{Sing} X$. Remember that the first truncation of $\text{Sing} X$ is equivalent to the Poincaré groupoid. Here, the first truncation of $\text{Sing}^p X$ is equivalent to the category of exit paths up to stratified homotopy, introduced in an unpublished work of MacPherson. (An exit path is a path which is stratum-increasing.) This similitude between $\text{Sing} X$ and $\text{Sing}^p X$ goes farther: the category of locally constant sheaves on a connected, locally contractible, topological space is well-known to be equivalent to the category of $\pi_1 X$-sets. On a stratified space, MacPherson proves that the category of sheaves that are locally constant on each stratum (also called constructible sheaves) is equivalent to the category of set valued functors on the category of exit paths up to stratified homotopy. In [42], D. Treumann extends this result into a 2-categorical framework. Finally, this has been generalized by Lurie ([28, Theorem A.9.3]) as an equivalence between quasi-categories involving the quasi-category of simplicial set-valued functors defined on $\text{Sing}^p X$, recovering the results of MacPherson and Treumann from the first and the second truncations. Let us mention also the work ([43]) of J. Woolf who refines [42] for the homotopically stratified spaces of F. Quinn ([34]).

Thus, moving the notion of stratified space in the simplicial paradigm by considering the simplicial map $\text{Sing}^p X \to \mathcal{N}(\mathcal{P})$ defined above is an exact replica of the situation with $\text{Sing} X$. In view of Lurie’s theorem, one can ask the following questions.

**Questions.** Let $X \to \mathcal{P}$ be a stratified space, $\text{Sing}^p X \to \mathcal{N}(\mathcal{P})$ the associated simplicial set over the nerve of $\mathcal{P}$ and $\overline{\mathcal{P}}$ be a perversity of associated Deligne sheaf $Q_{\overline{\mathcal{P}}}$.

A) Can we define the intersection cohomology groups of simplicial sets over $\mathcal{P}$ so that $\mathbb{H}^*(X; Q_{\overline{\mathcal{P}}})$ is the intersection cohomology of $\text{Sing}^p X$?

B) Does there exist a simplicial set over $\mathcal{P}$, $K(R, n, \mathcal{P}, \overline{\mathcal{P}})$, so that the intersection cohomology is recovered as the homotopy classes of the simplicial set $\text{Hom}_{\text{sset}}^\Delta(-, K(R, n, Q_{\overline{\mathcal{P}}}))$?

C) If (A) holds, does the intersection cohomology of $K(R, n, \mathcal{P}, \overline{\mathcal{P}})$ correspond to cohomological operations on intersection cohomology?

**The results.** Let $\text{Top}_\mathcal{P}$ be the category of topological spaces over a poset $\mathcal{P}$ and $\text{sset}_\mathcal{P}$ be the category of simplicial sets over $\mathcal{N}(\mathcal{P})$. The blown up cochain complex, $\overline{\mathcal{N}}^p_{\overline{\mathcal{P}}}(-; R): \text{Top}_\mathcal{P} \to \text{M}_{\text{dg}}$, already introduced in [7, 10, 11], factorizes through $\text{sset}_\mathcal{P}$ as

$$\begin{align*}
\overline{\mathcal{N}}^p_{\overline{\mathcal{P}}}(-; R): \text{Top}_\mathcal{P} &\xrightarrow{\text{Sing}^p} \text{sset}_\mathcal{P} &\xrightarrow{} &\text{M}_{\text{dg}}.
\end{align*}$$

By abuse of notation, we denote also $\overline{\mathcal{N}}^p_{\overline{\mathcal{P}}}(-; R): \text{sset}_\mathcal{P} \to \text{M}_{\text{dg}}$ the functor that appears in (0.4) and by $\mathcal{H}^p_{\overline{\mathcal{P}}}(-; R)$ its homology. The first question is answered in (5.3) as follows.

**Theorem A.** Let $X \to \mathcal{P}$ be a pseudomanifold over the poset $\mathcal{P}$ and $\overline{\mathcal{P}}$ be a perversity on $\mathcal{P}$. Then, there are isomorphisms

$$\mathcal{H}^p_{\overline{\mathcal{P}}}(X; R) \cong H^*(\overline{\mathcal{N}}^p_{\overline{\mathcal{P}}}(\text{Sing}^p X); R) \cong \mathbb{H}^*(X; Q_{\overline{\mathcal{P}}}).$$

For the introduction of perverse Eilenberg-MacLane spaces, we show the existence of an adjunction

$$\begin{align*}
\text{sset}_\mathcal{P} &\xrightarrow{\overline{\mathcal{N}}^p_{\overline{\mathcal{P}}}} &\text{M}_{\text{dg}}
\end{align*}$$
We take over the classical construction by setting $K(R, N, \mathcal{P}) = \langle R(n) \rangle_{\mathcal{P}}$. The following statement (see Corollary 5.7) is the answer to the second question.

**Theorem B.** Let $X \to \mathcal{P}$ be a pseudomanifold over the poset $\mathcal{P}$ and $\mathcal{P}$ be a perversity on $\mathcal{P}$. Then, there is a homotopy equivalence

\[ \text{Hom}_{\text{Sset}}(\text{Sing}^p X, K(R, n, \mathcal{P}, \mathcal{P})) \cong \prod_{k \geq 0} K(H^{n-k}(X; \mathbb{Q}_{\mathcal{P}}), k). \]

In particular, there are isomorphisms

\[ \pi_0(\text{Hom}_{\text{Sset}}(\text{Sing}^p X, K(R, n, \mathcal{P}, \mathcal{P}))) \cong [\text{Sing}^p X, K(R, n, \mathcal{P}, \mathcal{P})]_{\text{Sset}} \cong H^n(X; \mathbb{Q}_{\mathcal{P}}). \]

We also show in Theorem 5.6 that the family $(K(R, n, \mathcal{P}, \mathcal{P}))_n$ is an infinite loop object in the category $\text{Sset}_\mathcal{P}$. Finally, the behavior of cohomological operations on the hypercohomology of Deligne’s sheaves is deduced from Proposition 5.10 as follows.

**Theorem C.** Let $\mathcal{P}$ and $\mathcal{Q}$ be perversities on a poset $\mathcal{P}$. For any couple of integers $(n, m)$, there is an isomorphism

\[ \text{Nat}_R(\mathcal{K}^m_{\mathcal{P}}, \mathcal{K}^m_{\mathcal{Q}}) \cong \mathcal{K}^m_R(K(R, n, \mathcal{P}, \mathcal{P}); R). \]

Thus, an important task is the computation of intersection cohomology of the perverse Eilenberg-MacLane spaces. For that, we begin with the perversities introduced by Goresky and MacPherson in [19, 20], that we call here GM-perversities. They have the particularity of depending only on the codimension of the strata and we may choose subspaces of $\mathbb{N}$ as posets. We investigate the simplest case of isolated singularities for which we can choose $\mathcal{P} = [1] = \{0, 1\}$. We notice that the corresponding perverse Eilenberg-MacLane spaces are Joyal’s cylinders in the sense of [25, Section 7]. More precisely, with the terminology of [25], they are projective cone over classical Eilenberg-MacLane spaces. For instance, we can consider the two constant perversities, $\mathcal{U}$ and $\mathcal{R}$, with value 0 and $\infty$ respectively. For them, we get:

- $K(R, n, [1], \mathcal{U}) = \Delta[0] \times K(R, n),$
- $K(R, n, [1], \mathcal{R}) = \Delta[1] \times K(R, n)/\Delta[0] \times K(R, n).$

Our actual knowledge of perverse Eilenberg-MacLane spaces and their comparison with the classical ones leads us to the following conjecture. Recall first the existence ([6, Proposition 3.1]) of a natural chain isomorphism, $N^*(\Delta) \to N^*_{\mathcal{P}}(\Delta)$, which induces a natural chain injection $N^*(\Delta) \to N^*_{\mathcal{P}}(\Delta)$ for any positive perversity $\mathcal{P}$. We have proved this in suitably low degrees, for any ring $R$ and any perversities $\mathcal{P}$, $\mathcal{Q}$ when $m \leq n$, cf. Propositions 6.5, 6.7, 6.8 and Theorem 6.10. If Conjecture A is true, we can state a more precise conjecture, based on computations in the rational case in [8] and from [17, 22, 5] for $F_2$.

**Conjecture A.** Let $\mathcal{P} = [n]$ and $\mathcal{P}$, $\mathcal{Q}$ be two perversities of Goresky and MacPherson. Then, all perverse cohomological operations come from the classical cohomology situation; i.e., the previous natural chain injection induces an injective map

\[ \mathcal{K}^m_R(K(R, n, \mathcal{P}, \mathcal{P}); R) \longrightarrow H^m(K(R, n); R). \]

For the poset $\mathcal{P} = [1]$, we prove this conjecture in low degrees, for any ring $R$ and any perversities $\mathcal{P}$, $\mathcal{Q}$ when $m \leq n$, cf. Propositions 6.5, 6.7, 6.8 and Theorem 6.10. If Conjecture A is true, we can state a more precise conjecture, based on computations in the rational case in [8] and from [17, 22, 5] for $F_2$.

**Conjecture B.** When $\mathcal{P} = [n]$, we have the following isomorphisms of perverse algebras.

- $\mathcal{K}^m_R(K(Q, m, [n], \mathcal{P}); Q) \cong \mathcal{K}^m_R(x)$, where $\mathcal{K}^m_R(x)$ is the free rational commutative graded perverse algebra over one generator $x$ of differential degree $m$ and perverse degree $\mathcal{P}$.
- $\mathcal{K}^m_R(K(F_2, m, [n], \mathcal{P}); F_2) \cong \mathcal{K}^m_R(x)$, where $\mathcal{K}^m_R(x)$ is the free unstable perverse algebra over one generator $x$ of differential degree $m$ and perverse degree $\mathcal{P}$.
The perverse algebra $\mathcal{A}^P(x)$ is a polynomial algebra $F_2[\{\text{Sq}^I x\}_{I \in \text{Ad}}]$ generated by admissible sequences of Steenrod squares

$$\text{Sq}^I x = \text{Sq}^{i_1} \ldots \text{Sq}^{i_k} x$$

where $i_j \geq 2i_{j+1}$ and $\sum_j i_j - 2i_{j+1} < m$. The perverse degree of $\text{Sq}^I$ is computed from the following formula (conjectured in [22] and settled in [5]): “If $a$ is of perverse degree $\overline{p}$ then $\text{Sq}^j a$ is of perverse degree $\min(2\overline{p}, \overline{p} + j)$.”

**Perspective.** From our simplicial constructions, we can introduce generalized intersection cohomology theories. Taking an infinite loop space, $(S(n))_n$ in $\text{Sset}_P$, we can define, for any stratified topological space $X \to P$, an abelian group by

$$\mathcal{G}^n(X) = \pi_0 \text{Hom}_{\text{Sset}_P}^A(\text{Sing}^P X, S(n)).$$

Given a perversity $\overline{p} : P \to \mathbb{Z}$ and an infinite loop space $(L(n))_n$ in $\text{Sset}$, can we find an infinite loop space $(L(n, \overline{p}))_n$ in $\text{Sset}_P$ which brings a generalized intersection cohomology theory? For this last part, a good understanding of the situation developed in Section 6 for $P = [1]$ will be a first step.

**Outline.** In Section 1, we present the relations between stratified spaces and the category $\text{Sset}_P$ of simplicial sets over the nerve of a poset, $N(P)$. The structure of simplicial category on $\text{Sset}_P$ is detailed in Section 2; we also introduce the simplicial category $\text{Sset}_P^+$ of restricted simplicial sets over $N(P)$ and build an adjunction between these two categories. The restricted simplicial sets over $N(P)$ are crucial objects in the blown up process that we introduce in Section 3. From this construction, we define an adjunction between $\text{Sset}_P$ and the category of cochain complexes over a commutative ring in Section 4. We extend this adjunction to homotopy classes of morphisms to prepare the construction of Eilenberg-MacLane spaces done in Section 5, where we prove the results stated in Theorems A, B and C. Finally in Section 6, we analyze the case $P = [1]$, as stated before.

Two appendices complete this work. In [8], we defined a blown up cohomology for filtered face sets. These latter are simplicial sets over the poset $N$ without degeneracies, a role similar to that of $\Delta$-sets [35] for simplicial sets. In Appendix A, we introduce the category $\text{Ffs}_P$ of filtered face sets over a poset $P$ and show that the concepts introduced in the present work are compatible with that of [8]; this allows the use of results of [8] in Section 6. Finally, Appendix B is a brief reminder on homotopy and loop spaces in simplicial categories.

As a guide for the reader, we summarize in the following diagram the connections used between the category $\text{Sset}_P$ of simplicial sets over $N(P)$ and its surroundings.
The authors would like to thank Martintxo Saralegi-Aranguren for insightful advice on this work.

1. Stratified topological spaces

Stratified topological spaces and maps offer a geometric setting for the definition of intersection cohomology and the existence of morphisms between them, having regard to the level of cochain complexes. After a brief reminder, we present them as topological spaces over the poset of their strata, endowed with the Alexandrov topology. We also recall the notion of filtered simplices which prepares the study of intersection cohomology from simplicial objects.

1.1. Stratified spaces and maps. Let us introduce the stratified spaces, corresponding to the $S$-decomposition of [21, & I.1.1].

Definition 1.1. A stratified space is a Hausdorff topological space endowed with a partition

$$X = \sqcup_{s \in P} S_s$$

whose elements are non-empty, locally closed subsets of $X$, called strata, and satisfying the following properties:

(i) the Frontier condition: for any pair of strata $S$ and $S'$ with $S \cap S' \neq \emptyset$, one has $S \subset S'$,

(ii) for any subset $J \subset P$, one has $\bigcup_{s \in J} S_s = \bigcup_{s \in J} S_s$.

A stratum is regular if it is an open subset of $X$. A stratified space is said regular if it owns regular strata.

Property (ii) is satisfied if the family of strata $(S_s)_{s \in P}$ is locally finite (as in [21, & I.1.1]) and, a fortiori, if $P$ is finite. By definition, a subset $S$ is locally closed if $S = U \cap C$ with $U$ open and $C$ closed in $X$, or, equivalently, if $S = U \cap \overline{C}$. Recall also that the frontier condition is equivalent to

$$S = \sqcup_{s \in S \cap \overline{S} \neq \emptyset} S_s.$$  

Proposition 1.2. Let $X = \sqcup_{s \in P} S_s$ be a stratified space. Then the set $P$ is a poset for the relation $s \preceq t$ if $S_s \subseteq S_t$. (We write $s \prec t$ if $s \preceq t$ and $s \neq t$.)

Proof. Let $S_s$ and $S_t$ be two strata of $X$ such that $s \preceq t$ and $t \preceq s$, we have to prove $s = t$. Let $x \in S_s$. Since $S_s$ is locally closed, there exists an open subset $U$ of $X$ such that $S_s = U \cap \overline{S_s}$. As $x \in S_s \subseteq \overline{S_s}$, we have $U \cap S_s \neq \emptyset$. On the other hand, from $S_t \subset \overline{S_s}$, we get $S_s = U \cap \overline{S_s} \supset U \cap S_t \neq \emptyset$, which implies $S_s \cap S_t \neq \emptyset$ and $S_s = S_t$ since they are members of a partition. \qed

Remark 1.3. Let $(P, \preceq)$ be a poset. We endow $P$ with the Alexandrov topology. The open sets are the subsets $U$ such that if $s \in U$ and $s \preceq t$ then $t \in U$. The corresponding closed sets are the subsets $F$ such that if $s \in F$ and $t \preceq s$ then $t \in F$. Therefore, any singleton $\{s\}$ is locally closed as the intersection $]-\infty, s] \cap [s, \infty[$. Recall also that, in $P$, any union of closed subsets is closed.

If $X$ is a stratified space, we define a surjective map $\psi_X : X \to P$ by sending a point $x \in S_s$ to $s \in P$. In particular, $\psi_X$ sends a regular stratum on a maximal element of the poset.

We encode now the requirements of Definition 1.1 as properties of the map $\psi_X$, see [28] or [41] for a similar approach.

Definition 1.4. A stratification of a topological space $X$ by a poset $P$ is an open continuous surjective map $\psi_X : X \to P$, where $P$ is equipped with the Alexandrov topology.

Proposition 1.5. A Hausdorff topological space is stratified if, and only if, it admits a stratification.
In the proof, we use the following characterization of open maps.

Lemma 1.6. [41, Lemma 3.3] A map \( f: X \to Y \) between topological spaces is open if, and only if, \( f^{-1}(B) \subset f^{-1}(B) \), for any \( B \subset Y \). In particular, \( f \) is open and continuous if, and only if, \( f^{-1}(B) = f^{-1}(B) \), for any \( B \subset Y \).

Proof of Proposition 1.5. Suppose first that \( X = \sqcup_{s \in P} S_s \) is a stratified space. The frontier condition of Definition 1.1 implies that \( P \) is a poset and that \( \psi_X^{-1}([\infty, a]) = \psi_X^{-1}(a) \). Let \( B = \sqcup_{j \in J} \{s_j\} \subset P \). We have

\[
\psi_X^{-1}(B) = \psi_X^{-1}(\bigcup_{j \in J} \{s_j\}) = \psi_X^{-1}(\bigcup_{j \in J} s_j) = \bigcup_{j \in J} \psi_X^{-1}(s_j)
\]

Thus, with Lemma 1.6, \( \psi_X \) is stratification.

Reciprocally, suppose that \( \psi_X: X \to P \) is a stratification. We obtain a decomposition \( X = \sqcup_{s \in P} \psi_X^{-1}(s) \), with \( \psi_X^{-1}(s) \neq \emptyset \) and locally closed. Since \( \psi_X \) is open and continuous, we have \( \psi_X^{-1}(s) \) for any \( s \in P \). Thus, \( \psi_X^{-1}(s) \cap \psi_X^{-1}(t) = \psi_X^{-1}(s) \cap \psi_X^{-1}(t) = \psi_X^{-1}(s) \cap \psi_X^{-1}([-\infty, t]) = \psi_X^{-1}(a \cap [\infty, t]) \). With this equality, from \( \psi_X^{-1}(s) \cap \psi_X^{-1}(t) \neq \emptyset \), we deduce \( s \preceq t \), which is the frontier condition. Let \( J \subset P \). Using Lemma 1.6, we have

\[
\bigcup_{j \in J} \psi_X^{-1}(s_j) = \bigcup_{j \in J} \psi_X^{-1}(s_j) = \psi_X^{-1}(\bigcup_{j \in J} s_j) = \psi_X^{-1}(\bigcup_{j \in J} \{s_j\}) = \bigcup_{j \in J} \psi_X^{-1}(s_j).
\]

We introduce now the morphisms between stratified spaces.

Definition 1.7. A stratified map, \( f: X = \sqcup_{s \in P_X} S_s \to Y = \sqcup_{t \in P_Y} T_t \), is a continuous map between stratified spaces such that, for each stratum \( S_s \) of \( X \), there exists a unique stratum \( T_t \) of \( Y \) with \( f(S_s) \subset T_t \). We denote \( f: P_X \to P_Y \), the map \( f(s) = t \).

Proposition 1.8. If \( f: X \to Y \) is a stratified map, the map, \( f: P_X \to P_Y \) is increasing.

Proof. Let \( s_1 \preceq s_2 \) in \( P_X \). From the continuity of \( f \), we deduce \( f(S_{s_1}) \subset f(S_{s_2}) \) and \( T_{f(s_1)} \cap T_{f(s_2)} \neq \emptyset \). The frontier condition implies \( f(s_1) \preceq f(s_2) \).

The next result is a direct consequence of Propositions 1.5 and 1.8.

Corollary 1.9. A continuous map, \( f: X = \sqcup_{s \in P_X} S_s \to Y = \sqcup_{t \in P_Y} T_t \), between stratified spaces is a stratified map if, and only if, there exists a commutative diagram of continuous maps between the associated stratifications,

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\psi_X \downarrow & & \psi_Y \\
P_X & \xrightarrow{t} & P_Y
\end{array}
\]

1.2. Filtered simplices. Let us introduce the filtered simplices to prepare the simplicial approach developed in Section 2.

Let \( \Delta \) be the simplicial category whose objects are the nonnegative integers \( [n] = \{0, \ldots, n\} \) and whose morphisms are the order preserving maps. Among them, we quote the cofaces and codegeneracies, \( d^i: [n-1] \to [n] \) and \( s^i: [n+1] \to [n] \), for \( 0 \leq i \leq n \). The standard simplicial \( n \)-simplex is defined by \( \Delta[n] = \text{Hom}_\Delta([-,[n]]): \Delta^{op} \to \text{Set} \). Its realization is the geometric \( n \)-simplex, \( |\Delta[n]| = \Delta^n \).
A simplicial set is a functor $K : \Delta^{op} \to \text{Set}$, where Set is the category of sets. The elements of $K_n = K([n])$ are called $n$-simplices of $K$. The image, by the functor $K$, of the cofaces and the codegeneracies are, respectively, the faces $d_i : K_n \to K_{n-1}$ and the degeneracies $s_i : K_n \to K_{n+1}$, for $0 \leq i \leq n$. A simplicial map is a natural transformation between simplicial sets and Sset is the associated category.

Filtered simplices are the crucial notion for the existence of a simplicial blow up associated to a stratified space. We already have expressed them in the case of filtered spaces in [7], [8], [10].

**Definition 1.10.** Let $\psi_X : X = \sqcup_{s \in S} S_s \to P$ be a stratification. A filtered simplex (over $P$) of $X$ is a continuous map, $\sigma : \Delta^m = [e_0, \ldots, e_m] \to X$, such that, for any $s \in P$, $\sigma^{-1}S_s$ is the empty set, or a face $A = [e_0, \ldots, e_a]$ or a difference of two faces $B \setminus A$ with $B = [e_0, \ldots, e_b]$ and $a < b$. We denote $\text{Sing}_P^X \psi_X$ the simplicial set of the filtered simplices of the stratification $\psi_X$.

The notion of nerve of a poset allows a simplicial presentation of filtered simplices. Recall that the nerve of a poset $P$ is the simplicial set $\text{N}(P)$, whose $n$-simplices are the chains of increasing elements of $P$, $s_0 \preceq \cdots \preceq s_n$.

**Proposition 1.11.** Let $\psi_X : X = \sqcup_{s \in S} S_s \to P$ be a stratification and $\sigma : \Delta^m = [e_0, \ldots, e_m] \to X$ be a filtered simplex. Then, the association $e_i \mapsto \psi_X(\sigma(e_i))$ defines a simplicial map

$$\Psi_X : \text{Sing}_P^X \psi_X \to \text{N}(P).$$

**Proof.** First, we prove that $\psi_X(\sigma(e_i)) = s$ and $\psi_X(\sigma(e_{i+1})) = t$ imply $s \preceq t$. We may suppose $s \neq t$. From $\sigma(e_i) \in S_s$ and $\sigma([e_i, e_{i+1}]) \subset S_t$, we deduce $\sigma(e_i) \in S_t$. We therefore have $S_s \cap S_t \neq \emptyset$. The Frontier condition implies $S_s \subset S_t$ and thus $s \preceq t$. By grouping the identical strata, an iteration of this process along the vertices of $\Delta^m = [e_0, \ldots, e_m]$ gives a decomposition,

$$\Delta^m = [e_0, \ldots, e_{q_0}, e_{q_0+1}, \ldots, e_{q_0+q_1+1}, \ldots | e_{m-q_1}, \ldots, e_m],$$

such that $\sigma(e_\alpha) \in S_i$ for $\alpha = \sum_{k<i} q_k + i + j$ with $j \in \{0, \ldots, q_i\}$, $q_{i-1} = 0$ and $s_i = \psi_X(S_i) \preceq s_{i+1} = \psi_X(S_{i+1})$ for all $i \in \{0, \ldots, q\}$. Thus, $\psi_X \circ \sigma$ is a simplex of $\text{N}(P)$, that we write

$$\psi_X \circ \sigma = \frac{s_0 \preceq \cdots \preceq s_0}{q_0 + 1} \preceq \cdots \preceq \frac{s_q \preceq \cdots \preceq s_q}{q_q + 1} = \frac{s_{[0]} \preceq \cdots \preceq s_{[q]}}{q_{[0]} + 1}.$$ 

The compatibility with faces and degeneracies is immediate and $\Psi_X$ is a simplicial map. \hfill $\square$

**Remark 1.12.** From the decomposition $\Delta^m = \Delta^{q_0} * \cdots * \Delta^{q_q}$, established in (1.2), we observe that, for any $i \in \{0, \ldots, q\}$,

$$\sigma^{-1}(S_0 \sqcup \cdots \sqcup S_i) = \Delta^{q_0} * \cdots * \Delta^{q_i}.$$ 

With this filtration, a filtered simplex, $\sigma : \Delta \to X$, is a stratified map. Thus, stratified maps preserve filtered simplices.

**Proposition 1.13.** Let $f : \psi_X \to \psi_Y$ be a stratified map. Then there is a commutative diagram in Sset,

$$
\begin{array}{ccc}
\text{Sing}_P^X \psi_X & \xrightarrow{\text{Sing}(f)} & \text{Sing}_P^Y \psi_Y \\
\Psi_X \downarrow & & \downarrow \Psi_Y \\
\text{N}(P_X) & \xrightarrow{\Psi(f)} & \text{N}(P_Y),
\end{array}
$$

where $\Psi_X$, $\Psi_Y$ are defined in Proposition 1.11 and $f$ in Definition 1.7.
2. Simplicial sets over a poset

We introduce the category $\text{Sset}_P$ of simplicial sets over a poset $P$. We endow it with a structure of simplicial category and connect it to classical simplicial sets and to topological spaces over a poset. In the construction of a simplicial blow up, the notion of regular simplices turns out to be crucial. So, we introduce the category of restricted simplicial sets over a poset, $\text{Sset}_P^+$, and describe three functors between $\text{Sset}_P$ and $\text{Sset}_P^+$.

Let $P$ be a fixed poset, of associated nerve $N(P)$. Let $\Delta[P]$ be the category of simplices of $N(P)$, whose objects are the simplicial maps $\Delta[k] \to N(P)$, and morphisms the commutative triangles of simplicial maps,

\begin{equation}
\begin{array}{ccc}
\Delta[k] & \longrightarrow & \Delta[l] \\
& \searrow & \swarrow \\
& & N(P).
\end{array}
\end{equation}

**Definition 2.1.** A simplicial set over $P$ is a presheaf on the category $\Delta[P]$. We denote $\text{Sset}_P$ the category of natural transformations between simplicial sets over $P$.

If $\Phi_K : (\Delta[P])^{op} \to \text{Set}$ is a presheaf, we define a simplicial set

$$K = \{(\sigma, x) \mid \sigma \in N(P) \text{ and } x \in \Phi_K(\sigma)\}.$$

The projection $(\sigma, x) \mapsto \sigma$ is a simplicial map $\Psi_K : K \to N(P)$. Conversely, if $\Psi_K : K \to N(P)$ is a simplicial map, we define a presheaf $\Phi_K : K \to N(P)$ as follows: to any $\sigma \in N(P)$, we associate the set $\Phi_K(\sigma)$ of the lifting simplicial maps

$$\begin{array}{ccc}
\Delta[n] & \xrightarrow{a} & N(P) \\
\downarrow & & \downarrow \\
\Psi_K & \searrow & N(P).
\end{array}$$

Therefore, an object of $\text{Sset}_P$ can be seen as a simplicial map $\Psi_K : K \to N(P)$. With this point of view, a morphism of $\text{Sset}_P$ is a commutative triangle

$$\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\Psi_K & \searrow & \Psi_L \\
& N(P). & N(P).
\end{array}$$

**Remark 2.2.** The main result of this work is a presentation of perverse cohomological operations by the use of the representability of intersection cohomology. This can be achieved by considering simplicial sets over one fixed poset. However, the category of simplicial sets over posets can easily be defined as it occurs in Proposition 1.13.

Let $P$ and $Q$ be posets of associated nerves $N(P)$ and $N(Q)$. A morphism $(f, \ell)$ between two simplicial sets over posets, $(\Psi_K, P)$ and $(\Psi_L, Q)$, is a commutative diagram of simplicial maps,

$$\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\Psi_K & \searrow & \Psi_L \\
& N(P) & N(Q).
\end{array}$$
In Proposition 4.3, we place in this more general context the existence of homomorphisms between intersection cohomology groups, induced by simplicial maps.

Appendix B contains basic recalls on simplicial categories, sometimes also called simplicially enriched category. The category $Sset$ is a closed simplicial model category ([33]) with the following classes of maps: the cofibrations are the monomorphisms, the fibrations are the Kan fibrations and the weak-equivalences are the simplicial maps whose realization is a weak equivalence in the category of topological spaces. The simplicial structure comes from $K \otimes L = K \times L$ and $(\text{Hom}_{Sset}(K,L))_n = \text{Hom}_{Sset}(K \times \Delta[n], L)$.

Remark 2.3. The category $Sset_p$ inherits a structure of simplicial category with $\Psi_K \otimes L = K \times L \overset{\text{proj}}{\longrightarrow} K \overset{\Psi_K}{\rightarrow} \mathbb{N}(P)$, for $\Psi_K \in Sset_p$ and $L \in Sset$. If $\Psi_{K_1}, \Psi_{K_2} \in Sset_p$, the simplicial set $\text{Hom}_{Sset}^\Delta(\Psi_{K_1}, \Psi_{K_2})$ is the simplicial subset of $\text{Hom}_{Sset}^\Delta(K_1, K_2)$ formed of simplicial maps which commute over $\mathbb{N}(P)$. Following Definition B.2, an object $\Psi_K$ of $Sset_p$ is s-fibrant if the simplicial set $\text{Hom}_{Sset}^\Delta(\Psi_L, \Psi_K)$ is Kan for any $\Psi_L \in Sset_p$. Moreover a simplicial map over $P$, $f : \Psi_{K_1} \rightarrow \Psi_{K_2}$, is a weak-equivalence, if it induces an isomorphism $\pi_0 \text{Hom}_{Sset}^\Delta(\Psi_L, \Psi_{K_1}) \cong \pi_0 \text{Hom}_{Sset}^\Delta(\Psi_L, \Psi_{K_2})$ for any $\Psi_L \in Sset_p$.

2.1. Connection with topological spaces over a poset. Let $P$ be a fixed poset. We consider a slight more general situation than that of stratified space, called $P$-stratification in [28, Definition A.5.1].

Definition 2.4. A topological space over $P$ is a map, $\psi_X : X \rightarrow P$, which is continuous for the Alexandrov topology on $P$. We denote Top$_P$ the category of topological spaces over $P$ with morphisms, $f \in \text{Hom}_{Top_P}(\psi_X, \psi_Y)$, the continuous maps $f : X \rightarrow Y$ such that $\psi_Y \circ f = \psi_X$.

Let $\psi_X : X \rightarrow P$ be an object of Top$_P$. We define the simplicial set $\text{Sing}^P \psi_X$ as the following pullback in Sset,

\[(2.2) \quad \text{Sing}^P \psi_X = \mathbb{N}(P) \times_{\text{Sing} P} \text{Sing} X,
\]

whose elements are pairs $(\psi_X \circ \sigma, \sigma)$ with $\sigma \in \text{Sing} X$ and $\psi_X \circ \sigma \in \mathbb{N}(P)$. This construction coincides with the simplicial set of filtered simplices, introduced in Definition 1.10.

We extend the classical adjunction between the categories Top and Sset as:

$$\text{Sset}_P \xrightarrow{\text{Sing}^P} \text{Top}_P.$$

The functor $\text{Sing}^P$ is defined in (2.2). The realization functor $|-|$ sends the object $\Psi_K : K \rightarrow \mathbb{N}(P)$ of Sset$_P$ to the composite

$$\psi_{|K|} : |K| \xrightarrow{|\Psi_K|} |\mathbb{N}(P)| \xrightarrow{\chi_P} P \in \text{Top}_P.$$

Recall that the map $\chi_P$ is a natural weak homotopy equivalence called the last vertex map, see [31]. For instance, if $P = [n]$, $\chi_P$ associates to $(t_0, \ldots, t_n) \in \Delta^n = [n][n]$ the greatest index $i \in [n]$ such that $t_i \neq 0$.

Remark 2.5. The category Top endsow a structure of closed simplicial model category with $X \otimes K = X \times |K|$ and $(\text{Hom}_{Top}(X,Y))_n = \text{Hom}_{Top}(X \times \Delta^n, Y)$. With constructions similar to those of Remark 2.3, one can equip the category Top$_P$ with a structure of simplicial category.

The adjunction between simplicial sets and topological spaces over $P$ can be enriched to a simplicial adjunction, see [13, Proposition 5.1.12] for instance.
Proposition 2.6. Let $\Psi_K : K \rightarrow \mathbb{N}(\mathcal{P})$ be an object of $\text{Sset}_\mathcal{P}$ and $\psi_X : X \rightarrow \mathcal{P}$ an object of $\text{Top}_\mathcal{P}$. Then, there is an isomorphism of simplicial sets,

$$\text{Hom}^\Delta_{\text{Top}_\mathcal{P}}(\Psi_K, \psi_X) \cong \text{Hom}^\Delta_{\text{Sset}_\mathcal{P}}(\Psi_K, \text{Sing}_p^\mathcal{P} \psi_X).$$

(2.3)

An application of $\pi_0$ to (2.3) gives the following property.

Corollary 2.7. There is an isomorphism between the homotopy classes,

$$[\Psi_K, \psi_X]_{\text{Top}_\mathcal{P}} \cong [\Psi_K, \text{Sing}_p^\mathcal{P} \psi_X]_{\text{Sset}_\mathcal{P}}.$$

Remark 2.8. Let $\psi_X : X \rightarrow \mathcal{P}$ be a stratified space with $X$ conically stratified ([(28, Definition A.5.5)]). In [28, Theorem A.6.4], Lurie proves that the associated simplicial set over $\mathcal{P}$, $\Psi_X : \text{Sing}_p^\mathcal{P} \psi_X \rightarrow \mathbb{N}(\mathcal{P})$, is a fibrant object in the Joyal closed model structure on $\text{Sset}_\mathcal{P}$. This is not true in general for stratified spaces, see [13, Example 4.13]. (Fibrant objects for the Joyal structure have the lifting property for the horns $\Lambda^i[m]$ such that $0 < i < n$, $n \geq 1$.) In this work, for having a representability theorem, we use the simplicial structure on $\text{Sset}_\mathcal{P}$ and the following result is sufficient for our purpose.

Proposition 2.9. Let $\psi_X : X \rightarrow \mathcal{P}$ be a topological space over the poset $\mathcal{P}$. The associated simplicial set over $\mathcal{P}$, $\Psi_X : \text{Sing}_p^\mathcal{P} \psi_X \rightarrow \mathbb{N}(\mathcal{P})$, is s-fibrant in $\text{Sset}_\mathcal{P}$.

Proof. Let $\Psi_K : K \rightarrow \mathbb{N}(\mathcal{P})$ be any object of $\text{Sset}_\mathcal{P}$. With Definition B.2, we have to prove that the simplicial set $\text{Hom}^\Delta_{\text{Sset}_\mathcal{P}}(\Psi_K, \psi_X)$ is a fibrant simplicial set. This is equivalent to prove the surjectivity of

$$\text{Hom}_{\text{Sset}_\mathcal{P}}(\Delta[m], \text{Hom}^\Delta_{\text{Sset}_\mathcal{P}}(\Psi_K, \psi_X)) \rightarrow \text{Hom}_{\text{Sset}_\mathcal{P}}(\Lambda^k[m], \text{Hom}^\Delta_{\text{Sset}_\mathcal{P}}(\Psi_K, \psi_X)),$$

for any $m \geq 1$ and any $0 \leq k \leq m$. This amounts (cf. Definition B.1) to the surjectivity of

$$\text{Hom}_{\text{Sset}_\mathcal{P}}(\Psi_K \times \Delta[m], \psi_X) \rightarrow \text{Hom}_{\text{Sset}_\mathcal{P}}(\Psi_K \times \Lambda^k[m], \psi_X).$$

Using the adjunction $([-], \text{Sing}_p^\mathcal{P})$ and the compatibility of the realization functor with products, the previous surjectivity is equivalent to the surjectivity of

$$\text{Hom}_{\text{Top}_\mathcal{P}}([\Psi_K] \times |\Delta[m]|, \psi_X) \rightarrow \text{Hom}_{\text{Top}_\mathcal{P}}([\Psi_K] \times |\Lambda^k[m]|, \psi_X),$$

which arises from the existence of a retraction to the canonical injection $|\Lambda^k[m]| \rightarrow |\Delta[m]|$. □

2.2. Connection with simplicial sets. There is an adjunction,

$$\begin{array}{ccc}
\text{Sset}_\mathcal{P} & \xleftarrow{\mathcal{U}} & \text{Sset},
\end{array}$$

where $\mathcal{U}$ is a forgetful functor sending $\Psi_L \in \text{Sset}_\mathcal{P}$ to $L \in \text{Sset}$, and the functor $\mathbb{N}(\mathcal{P}) \times -$ sends the simplicial set $Z$ to the projection $\mathcal{P} : \mathbb{N}(\mathcal{P}) \times Z \rightarrow \mathbb{N}(\mathcal{P})$.

Proposition 2.10. The pair of functors $(\mathcal{U}, \mathbb{N}(\mathcal{P}) \times -)$ forms a simplicial adjunction: for each $Z \in \text{Sset}$ and each $\Psi_L \in \text{Sset}_\mathcal{P}$, there is an isomorphism

$$\text{Hom}^\Delta_{\text{Sset}}(\mathcal{U}([\Psi_L]), Z) \cong \text{Hom}^\Delta_{\text{Sset}_\mathcal{P}}(\Psi_L, \mathbb{N}(\mathcal{P}) \times Z).$$

Proof. To $f \in \text{Hom}_{\text{Sset}}(\mathcal{U}([\Psi_L]), Z)$, one associates $(f, \Psi_L) \in \text{Hom}_{\text{Sset}_\mathcal{P}}(\Psi_L, \mathbb{N}(\mathcal{P}) \times Z)$, and this correspondence is clearly a bijection. The simplicial adjunction follows from the following isomorphisms between the sets of $n$-simplices,

$$\begin{array}{ll}
(\text{Hom}^\Delta_{\text{Sset}}(\mathcal{U}([\Psi_L]), Z))_n & = \text{Hom}_{\text{Sset}}(\mathcal{U}([\Psi_L]) \times \Delta[n], Z) \\
& = \text{Hom}_{\text{Sset}}(\mathcal{U}([\Psi_L] \times \Delta[n]), Z) \\
& \cong \text{Hom}_{\text{Sset}_\mathcal{P}}(\Psi_L \times \Delta[n], \mathbb{N}(\mathcal{P}) \times Z) \\
& = (\text{Hom}^\Delta_{\text{Sset}_\mathcal{P}}(\Psi_L, \mathbb{N}(\mathcal{P}) \times Z))_n.
\end{array}$$
2.3. Restricted simplicial sets over a poset.

**Definition 2.11.** Let $\mathcal{P}$ be a poset. A simplex $s_0 \preceq s_1 \preceq \cdots \preceq s_\ell$ of $\mathbb{N}(\mathcal{P})$ is regular if $s_\ell$ is a maximal element of $\mathcal{P}$. We denote $\Delta[\mathcal{P}]^+$ the full sub-category of $\Delta[\mathcal{P}]$ whose objects are the regular simplices of $\mathbb{N}(\mathcal{P})$.

**Definition 2.12.** A restricted simplicial set over $\mathcal{P}$ is a functor $\Psi_K : (\Delta[\mathcal{P}]^+)^{op} \to \text{Set}$. The category of restricted simplicial sets, with morphisms the natural transformations, is denoted $\text{Sset}_\mathcal{P}^+$. As $\Delta[\mathcal{P}]^+ \subset \Delta[\mathcal{P}]$, there is a restriction functor $\mathcal{R} : \text{Sset}_\mathcal{P} \to \text{Sset}_\mathcal{P}^+$.

We also adapt the notion of regular stratum of a stratified topological space to the simplicial paradigm. We call it non singular to make a clear distinction from the previous notion of regular simplex.

**Definition 2.13.** Let $\Psi_K \in \text{Sset}_\mathcal{P}$. The non singular part of $\Psi_K$ is the restriction of $\Psi_K$ to the simplicial subset of $K$ formed of simplices $\sigma$ such that $\Psi_K \circ \sigma$ is a maximal element of $\mathcal{P}$. If each simplex of $\Psi_K$ is non singular, we say that $\Psi_K$ is a completely regular simplicial set.

**Example 2.14.** Let $\mathcal{P} = [n]$. An object $\Psi_K$ of $\text{Sset}_\mathcal{P}$ can be described as a family of sets $K_{i_0}, \ldots, i_n$, with $i_j \in \{-1\} \cup \mathbb{N}$. The elements of $K_{i_0}, \ldots, i_n$ are the simplices $\Delta[i_0] \ast \cdots \ast \Delta[i_n]$, the value -1 corresponding to an empty subset. The objects of $\text{Sset}_\mathcal{P}^+$ correspond to the families $K_{i_0}, \ldots, i_n$ with $i_n \neq -1$. The non singular part of $\Psi_K$ is the union of $K_{-1}, \ldots, i_n$ with $i_n \neq -1$.

We now introduce two functors from $\text{Sset}_\mathcal{P}^+$ to $\text{Sset}_\mathcal{P}$,

$$\begin{array}{c}
\text{Sset}_\mathcal{P}^+ \\
\mathcal{R} \\
\downarrow \\
\text{Sset}_\mathcal{P}
\end{array}$$

**Definition 2.15.** The functor $n : \text{Sset}_\mathcal{P}^+ \to \text{Sset}_\mathcal{P}$ is defined for any $\sigma \in \mathbb{N}(\mathcal{P})$ by,

$$(n(\Psi_K))_\sigma = \begin{cases} (\Psi_K)_\sigma & \text{if } \sigma \text{ is regular,} \\ v & \text{otherwise,} \end{cases}$$

where $v$ is an additional vertex.

**Definition 2.16.** The functor $i : \text{Sset}_\mathcal{P}^+ \to \text{Sset}_\mathcal{P}$ is defined by left Kan extension. First, to any restricted simplicial set over $\mathcal{P}$, $(\Delta[i_0] \ast \cdots \ast \Delta[i_n])^+$, we associate the simplicial set over $\mathcal{P}$, $\Delta[i_0] \ast \cdots \ast \Delta[i_n]$. Then, as an object $\Psi_K$ of $\text{Sset}_\mathcal{P}^+$ is a colimit in $\text{Sset}_\mathcal{P}$,

$$\Psi_K = \lim_{(\Delta[i_0] \ast \cdots \ast \Delta[i_n])^+ \to \Psi_K} (\Delta[i_0] \ast \cdots \ast \Delta[i_n])^+,$$

we set $i(\Psi_K)$ as a colimit in $\text{Sset}_\mathcal{P}$,

$$i(\Psi_K) = \lim_{(\Delta[i_0] \ast \cdots \ast \Delta[i_n])^+ \to \Psi_K} \Delta[i_0] \ast \cdots \ast \Delta[i_n].$$

We present below examples of the compositions $i \circ \mathcal{R}$ and $n \circ \mathcal{R}$.
Example 2.17. [The composition $\iota \circ R$.] Let $P = \{1\}$. This first example starts with an ordered simplicial complex, constituted of two 2-simplices with a vertex $a$ in common. The vertex $a$ is singular and the other ones completely regular:

$$K = \Delta[0] \ast \Delta[1] \cup \Delta[0] \ast \Delta[1].$$

Below, we draw $K$, its restriction $R(K) \in \text{Sset}_P^+$ and the composite $\iota(R(K)) \in \text{Sset}_P$.

The link ([8, Definition 1.11]) of the singular vertex of $K$ is not connected, which means by definition that $K$ is not normal ([8, Definition 1.55]). In contrast, $(\iota \circ R)(K)$ is normal. One can also notice that $n(R(K)) = K$ for this example.

The composition $\iota \circ R$ corresponds (see Proposition A.4) to the process of normalization introduced in [19] and justifies the following definition.

Definition 2.18. The composite $\iota \circ R : \text{Sset}_P \to \text{Sset}_P$ is called the normalization functor.

Example 2.19. [The composition $n \circ R$.] Let $P = \{1\}$. We consider the ordered simplicial complex $K = \Delta[1] \ast \Delta[0]$, with $\Delta[1]$ singular and $\Delta[0]$ completely regular. We denote $a, b$ the vertices of $\Delta[1]$ and $c$ the vertex of $\Delta[0]$. Below, we draw $K$, its restriction $R(K) \in \text{Sset}_P^+$ and the composite $n(R(K)) \in \text{Sset}_P$.

The image $n(R(K)) \in \text{Sset}_P$ is the quotient of $\Delta[1] \ast \Delta[0]$ by the relation $\Delta[1] = v$. One can also notice that $\iota(R(K)) = K$ for this example.

The next result comes directly from the definitions of these functors.

Proposition 2.20. The functors $n$ and $\iota$ are respectively the right and the left adjoints of the restriction functor, $R$. They verify $R \circ \iota = \iota \circ n = \text{id}$.

Thus $R$ preserves limits and colimits, $\iota$ is compatible with colimits and $n$ with limits. Also, the functors $R, n, \iota$ verify, for any $\Psi_K, \Psi_{K_1}, \Psi_{K_2} \in \text{Sset}_P^+$ and $\Psi_L \in \text{Sset}_P$,

$$\text{Hom}_{\text{Sset}_P}(\Psi_L, n(\Psi_K)) \cong \text{Hom}_{\text{Sset}_P}(R(\Psi_L), \Psi_K),$$

$$\text{Hom}_{\text{Sset}_P}(\iota(\Psi_K), \Psi_L) \cong \text{Hom}_{\text{Sset}_P}(\Psi_K, R(\Psi_L)),$$

which imply, with $R \circ \iota = \iota \circ n = \text{id}$,

$$\text{Hom}_{\text{Sset}_P}(\Psi_{K_1}, \Psi_{K_2}) \cong \text{Hom}_{\text{Sset}_P}(\iota(\Psi_{K_1}), n(\Psi_{K_2}))$$

$$\cong \text{Hom}_{\text{Sset}_P}(\iota(\Psi_{K_1}), \iota(\Psi_{K_2}))$$

$$\cong \text{Hom}_{\text{Sset}_P}(n(\Psi_{K_1}), n(\Psi_{K_2})).$$
2.4. **Simplicial structures.** The structure of simplicial category on $\text{Sset}_p$ induces a structure of simplicial category on $\text{Sset}_p^+$ by

\[(2.7) \quad \text{Hom}_{\text{Sset}_p^+}^\Delta(\Psi_{K_1}, \Psi_{K_2}) = \text{Hom}_{\text{Sset}_p}(\mathfrak{i}(\Psi_{K_1}), n(\Psi_{K_2})).\]

Let $\Psi_K \in \text{Sset}_+$. The products on $\text{Sset}_p$ and $\text{Sset}_p^+$ are linked by

$$\Psi_K \otimes \Delta[n] = \mathcal{R}(\mathfrak{i}(\Psi_K) \otimes \Delta[n]).$$

One can also observe from the definition of $\mathfrak{i}$ that

\[(2.8) \quad \mathfrak{i}(\Psi_K \otimes \Delta[n]) = \mathfrak{i}(\Psi_K) \otimes \Delta[n].\]

**Lemma 2.21.** The set of $n$-simplices of the simplicial set $\text{Hom}_{\text{Sset}_p^+}^\Delta(\Psi_{K_1}, \Psi_{K_2})$ is given by

$$(\text{Hom}_{\text{Sset}_p^+}^\Delta(\Psi_{K_1}, \Psi_{K_2}))_n = \text{Hom}_{\text{Sset}_p}(\Psi_{K_1} \otimes \Delta[n], \Psi_{K_2}).$$

**Proof.** Using (2.6), (2.7) and the adjunction, we have

\[
\left(\text{Hom}_{\text{Sset}_p^+}^\Delta(\Psi_{K_1}, \Psi_{K_2})\right)_n = \left(\text{Hom}_{\text{Sset}_p}(\mathfrak{i}(\Psi_{K_1}), n(\Psi_{K_2}))\right)_n = \text{Hom}_{\text{Sset}_p}(\mathfrak{i}(\Psi_{K_1} \otimes \Delta[n], n(\Psi_{K_2})) = \text{Hom}_{\text{Sset}_p}(\mathfrak{i}(\Psi_{K_1} \otimes \Delta[n], \Psi_{K_2}) = \text{Hom}_{\text{Sset}_p}(\Psi_{K_1} \otimes \Delta[n], \Psi_{K_2}).
\]

\[\square\]

**Proposition 2.22.** The functors $(\mathcal{R}, n)$ and $(\mathfrak{i}, \mathcal{R})$ are adjoint pairs of simplicial functors; i.e., for $\Psi_L \in \text{Sset}_p$ and $\Psi_K \in \text{Sset}_p^+$, we have

$$\text{Hom}_{\text{Sset}_p}(\Psi_L, n(\Psi_K)) \cong \text{Hom}_{\text{Sset}_p^+}(\mathcal{R}(\Psi_L), \Psi_K)$$

and

$$\text{Hom}_{\text{Sset}_p^+}(\Psi_K, \mathcal{R}(\Psi_L)) \cong \text{Hom}_{\text{Sset}_p}(\mathfrak{i}(\Psi_K), \Psi_L).$$

**Proof.** Using $\mathcal{R}(\Psi_L \otimes \Delta[n]) = \mathcal{R}(\Psi_L) \otimes \Delta[n]$ and $\mathfrak{i}(\Psi_K \otimes \Delta[n]) = \mathfrak{i}(\Psi_K) \otimes \Delta[n]$, the proof is a consequence of the properties of adjunctions and Lemma 2.21. \[\square\]

3. **Blown up cochains on a poset**

In this section, we recall the notion of perversity and present the blown up cohomology associated to a poset, $\mathcal{P}$, which corresponds to a local situation.

3.1. **Perversity on a poset.** First comes the principal tool for the Goresky and MacPherson theory: the notion of perversity.

**Definition 3.1.** A perversity on a poset $\mathcal{P}$ is a map $\mathfrak{p}: \mathcal{P} \to \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$ taking the value 0 on the maximal elements. If $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ are two perversities on $\mathcal{P}$, we write $\overline{\mathfrak{p}} \leq \overline{\mathfrak{q}}$ when $\overline{\mathfrak{p}}(s) \leq \overline{\mathfrak{q}}(s)$, for each $s \in \mathcal{P}$.

Given two perversities, $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$, we define a new perversity, $\overline{\mathfrak{p}} + \overline{\mathfrak{q}}$, by $(\overline{\mathfrak{p}} + \overline{\mathfrak{q}})(s) = \overline{\mathfrak{p}}(s) + \overline{\mathfrak{q}}(s)$, with the conventions $k + (-\infty) = -\infty + k = -\infty$, for any $k \in \overline{\mathbb{Z}}$, and $\ell + (+\infty) = +\infty + \ell = +\infty$, for any $\ell \in \mathbb{Z}$. (The first convention is required in the definition of the cup product in blown up cohomology, see [7, Proposition 4.2].)
For stratified topological spaces, perversities defined on the set of strata already appeared in [30] but, historically, the first ones in [19, 20] correspond to perversities defined on the poset $\mathbb{N}^{\text{op}}$. In fact, in the case of geometrical data, the strata come with an intrinsic notion of dimension and in the two seminal papers quoted above, perversities retain only this information. Let us give an illustration with pseudomanifolds. Being the spaces with singularities satisfying a Poincaré duality (see [19, 9, 38]) they play a central role in the theory.

**Definition 3.2.** A topological pseudomanifold of dimension $n$ (or a pseudomanifold) is a Hausdorff space together with a filtration by closed subsets,

$$X_{-1} = \emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X_n = X,$$

such that, for each $i \in \{0, \ldots, n\}$, $X_i \setminus X_{i-1}$ is a topological manifold of dimension $i$ or the empty set. The subspace $X_{n-1}$ is called the singular set and each point $x \in X_i \setminus X_{i-1}$ with $i \neq n$ admits

(i) an open neighborhood $V$ of $x$ in $X$, endowed with the induced filtration,
(ii) an open neighborhood $U$ of $x$ in $X_i \setminus X_{i-1}$,
(iii) a compact pseudomanifold $L$ of dimension $n - i - 1$, whose cone $\hat{c}L$ is endowed with the conic filtration, $(\hat{c}L)_i = \hat{c}L_{i-1}$,
(iv) a homeomorphism, $\varphi: U \times \hat{c}L \to V$, such that

(a) $\varphi(u, v) = u$, for any $u \in U$, where $v$ is the apex of $\hat{c}L$,
(b) $\varphi(U \times \hat{c}L) = V \cap X_{i+1}$, for any $j \in \{0, \ldots, n-i-1\}$.

The pseudomanifold $L$ is called the link of $x$.

By taking the partition $X = \bigcup X_i \setminus X_{i-1}$, we get a stratified space ([8, Theorem G]) as in Definiton 1.1 and thus a stratification over its poset of strata. As it appears in the next sections, the study of intersection cohomology of stratified spaces does not need any notion of dimension or codimension. But, as quoted above, a feature of pseudomanifolds is the existence of a geometrical notion of dimension. Moreover, in the seminal work of Goresky and MacPherson, the perversities in use need this notion of dimension. Not only, perversities need it but they have a geometrical meaning, as they control the tangential component of the singular simplices relatively to the strata of $X$.

To keep this peculiarity of pseudomanifolds, we can give preference to the opposite poset of the natural integers instead of the poset of strata. More precisely, to any $n$-dimensional pseudomanifold $X$ we associate the continuous map $\varphi_X: X \to [n]^{\text{op}}$, sending a point $x \in X$ to the codimension of the stratum $S$ with $x \in S$. Let us observe that $\varphi_X^{-1}(k) = X_k \setminus X_{k-1}$. With these observations, let us recall how perversities appear in [19].

**Definition 3.3.** A GM-perversity is a map $\mathcal{P}: [n]^{\text{op}} \to \mathbb{Z}$ such that $\mathcal{P}(0) = \mathcal{P}(1) = \mathcal{P}(2) = 0$ and $\mathcal{P}(i) \leq \mathcal{P}(i+1) \leq \mathcal{P}(i) + 1$, for all $i \geq 2$. As particular case, we have the null perversity $\mathcal{U}$ constant with value 0 and the top perversity $\mathcal{T}$ defined by $\mathcal{T}(i) = i - 2$ if $i \geq 2$. For any perversity, $\mathcal{P}$, the perversity $\mathcal{L}\mathcal{P} := \mathcal{T} - \mathcal{P}$ is called the complementary perversity of $\mathcal{P}$.

We complete this paragraph with the transfer of perversities through a map of posets. An exhaustive topological study of these operations on perversities is done in [36]. This transfer has been also used in [10] in relation with the analysis of the topological invariance of intersection homology.

**Definition 3.4.** Let $f: P \to Q$ be a morphism of posets. If $\mathcal{T}: Q \to \mathcal{Z}$ is a perversity on $Q$, the pullback perversity $f^* \mathcal{T}$ on $P$ is defined by $f^* \mathcal{T}(s) = \mathcal{T} \circ f(s)$ if $s$ is not maximal. If $\mathcal{P}: P \to \mathcal{Z}$ is a perversity on $P$, the pushforward perversity $f_* \mathcal{P}$ on $Q$ is defined on a not maximal element $t$ by

$$f_* \mathcal{P}(t) = \inf_{f(s) = t} \mathcal{P}(s).$$
3.2. Perverse degree. Recall some notations and conventions, already used in previous works as [9, Subsection 2.1]. The cone on the simplicial set $\Delta[n] = [e_0, \ldots, e_n]$ is the simplicial set $c\Delta[n] = [e_0, \ldots, e_n, v]$. The apex $v$ is called a virtual vertex and can be considered as the cone on the empty set, $c\emptyset = [v]$. Given a simplex $F$ of $\Delta[n]$, we denote by $(F, 0)$ the same simplex viewed as a face of $c\Delta[n]$ and by $(F, 1)$ the face $cF$ of $c\Delta[n]$.

The blown up cochains associated to a perversity need the introduction of an extra degree that we detail now. To any regular simplex $\Delta[m] = \Delta[q_0] \ast \cdots \ast \Delta[q_\ell]$ of $N(P)$, we associate the prism $\acute{\Delta}[m] = c\Delta[q_0] \times \cdots \times c\Delta[q_{\ell-1}] \times \Delta[q_\ell]$, called the blow up of $\Delta[m]$. A face of the blow up $\acute{\Delta}[m]$ is a product

$$ (F, \varepsilon) = (F_0, \varepsilon_0) \times \cdots \times (F_{\ell-1}, \varepsilon_{\ell-1}) \times F_{\ell}, $$

where, following the previous conventions,

- if $\varepsilon_i = 0$ or $i = \ell$, then $F_i$ is a face of $\Delta[q_i]$,
- if $\varepsilon_i = 1$ and $F_i \neq \emptyset$, then $(F_i, 1)$ is the cone $cF_i$ on the face $F_i$ of $\Delta[q_i]$,
- if $F_i = \emptyset$, then $\varepsilon_i = 1$ and $(F_i, 1) = v_i$.

For any $i \in \{0, \ldots, \ell - 1\}$, we denote

$$ \|(F, \varepsilon)\|_i = \dim((F_{i+1}, \varepsilon_{i+1}) \times \cdots \times (F_{\ell-1}, \varepsilon_{\ell-1}) \times F_{\ell}). $$

**Definition 3.5.** Let $\Delta[m] = \Delta[q_0] \ast \cdots \ast \Delta[q_\ell] = s_0^{[m]} \prec \cdots \prec s_{\ell}^{[m]}$ be a regular simplex of $N(P)$ and $(F, \varepsilon) = (F_0, \varepsilon_0) \times \cdots \times (F_{\ell-1}, \varepsilon_{\ell-1}) \times F_\ell$ be a face of the blow up $\acute{\Delta}[m]$. The perverse degree of $(F, \varepsilon)$ along the element $s \in P$ is

$$ \|(F, \varepsilon)\|_s = \begin{cases} 
\infty & \text{if } s \not\in \{s_0, \ldots, s_\ell\} \text{ or } (s = s_i \text{ and } \varepsilon_i = 1), \\
\|(F, \varepsilon)\|_i & \text{if } s = s_i \text{ and } \varepsilon_i = 0.
\end{cases} $$

**Remark 3.6.** The perverse degree of Definition 3.5 coincides with the perverse degree associated to a weight decomposition introduced in [7, Definition 3.3].

We fix a commutative ring with unit $R$. Let $j \in \mathbb{N}$ and $N^*(\Delta[j]) = \text{Hom}(N_\ast(\mathcal{R}\Delta[j]), R)$ the dual of the Moore complex associated to $\mathcal{R}\Delta[j]$. For each simplex $F \in \Delta[j]$, we write $1_F$ the element of $N^*(\Delta[j])$ taking the value 1 on $F$ and 0 otherwise. Let $\Delta[m] = \Delta[q_0] \ast \cdots \ast \Delta[q_\ell]$, with $q_i \geq 0$ for all $i$. We first define the blown up complex on $\Delta[m]$ by

$$ \acute{\Delta}^*(\Delta[m]) = N^*(c\Delta[q_0]) \otimes \cdots \otimes N^*(c\Delta[q_{\ell-1}]) \otimes N^*(\Delta[q_\ell]). $$

The elements $1_{(F, \varepsilon)} = 1_{(F_0, \varepsilon_0)} \otimes \cdots \otimes 1_{(F_{\ell-1}, \varepsilon_{\ell-1})} \otimes 1_{F_\ell}$ form a basis of $\acute{\Delta}^*(\Delta[m])$. (By convention, we also set $\varepsilon_\ell = 0$).

We describe the maps induced by the morphisms of $N(P)^+$ between the blown up complexes. Let us begin with the regular face operators of $\Delta[m] = \Delta[q_0] \ast \cdots \ast \Delta[q_\ell]$ with $q_i \geq 0$ for all $i$. Let $\alpha: \nabla \to \Delta[m]$ be a face map with $\nabla$ and $\Delta[m]$ regular. The induced filtration on $\nabla$ gives a decomposition

$$ \nabla = \nabla_0 \ast \cdots \ast \nabla_\ell, \quad \text{with } \nabla_i = \nabla \cap \Delta[q_i], $$

in which some $\nabla_i$ can be the empty set. We get rid of these empty factors to obtain what we call the solid $\Delta[m]$-decomposition of $\nabla$,

$$ \nabla = \Delta[p_0] \ast \cdots \ast \Delta[p_k], \quad \text{with } p_i \geq 0 \text{ for } 0 \leq i \leq k. $$

More precisely, as $\nabla$ is regular, there is a strictly increasing map, $\eta: \{0, \ldots, k\} \to \{0, \ldots, \ell\}$, with $\eta(k) = \ell$, defined by

$$ \nabla_j = \begin{cases} 
\emptyset & \text{if } j \notin \text{Im}(\eta), \\
\Delta[p_j] & \text{if } j = \eta(i).
\end{cases} $$
Let \( \tilde{\mathcal{N}}^*(\nabla) = N^*(c\Delta[p_0]) \otimes \cdots \otimes N^*(c\Delta[p_k]) \) be the blown up complex of \( \nabla \) endowed with its solid \( \Delta[m]- \)decomposition. The face map \( \alpha: \nabla \to \Delta[m] \) induces a cochain map,

\[
\alpha^*: \tilde{\mathcal{N}}^*(\Delta[m]) \to \tilde{\mathcal{N}}^*(\nabla),
\]

defined as follows. Let \( 1_{(F, \varepsilon)} = 1_{(F_0, \varepsilon_{0})} \otimes \cdots \otimes 1_{(F_{k-1}, \varepsilon_{k-1})} \otimes 1_{F_k} \in \tilde{\mathcal{N}}^*(\Delta[m]) \). We say that \( F \) is \( \nabla \)-compatible if \( (F_i, \varepsilon_i) = (\emptyset, 1) \) for all \( i \notin \eta(\mathfrak{n}) \). (The \( \nabla \)-compatibility ensures that \( F = F_0 \cdots F_k \) is a face of \( \nabla \).) We have:

\[
\alpha^*(1_{(F, \varepsilon)}) = \begin{cases} 0 & \text{if } F \text{ is not } \nabla \text{-compatible,} \\ 1_{(H, \varepsilon)} & \text{otherwise,} \end{cases}
\]

where \( 1_{(H, \varepsilon)} = 1_{(H_0, \varepsilon_0)} \otimes \cdots \otimes 1_{(H_{k-1}, \varepsilon_{k-1})} \otimes 1_{H_k} \), with \( (H_i, \varepsilon_i) = (\tilde{F}_{\eta(i)}, \varepsilon_{\eta(i)}) \) for all \( i \in \{0, \ldots, k\} \).

Let us consider a degeneracy map, \( \beta: \Delta[m] \to \Delta[m + 1] \). Such map \( \beta \) being the repetition of a vertex in some \( \Delta[q_i] \), we have a chain map, \( N_\varepsilon(\Delta[q_i]) \to N_\varepsilon(\Delta[q_i + 1]) \), which goes via the other components of the tensor product gives the cochain map \( \beta^*: \tilde{\mathcal{N}}^*(\Delta[m + 1]) \to \tilde{\mathcal{N}}^*(\Delta[m]) \).

Let \( \mathfrak{P} \) be a perversity on the poset \( P \). Let \( R \) be a commutative ring and \( \text{M}_{\text{dg}} \) be the category of positively graded differential graded \( R \)-modules, with a differential of degree +1. We have defined a functor \( \tilde{\mathcal{N}}^*: \Delta[\mathfrak{P}]^+ \to \text{M}_{\text{dg}} \), sending \( \Delta[m] \in \Delta[\mathfrak{P}]^+ \) on the differential complex \( \tilde{\mathcal{N}}^*(\Delta[m]) \).

**Definition 3.7.** Let \( \Delta[m] \in \Delta[\mathfrak{P}]^+ \).

1) The **perverse degree** of \( 1_{(F, \varepsilon)} \in \tilde{\mathcal{N}}^*(\Delta[m]) \) along an element \( s \in P \) is the perverse degree of \( (F, \varepsilon) \) along \( s \). For a cochain \( \omega = \sum_b \lambda_b 1_{(F_b, \varepsilon_b)} \in \tilde{\mathcal{N}}^*(\Delta[m]) \) with \( \lambda_b \neq 0 \) for all \( b \), the **perverse degree along** \( s \) is

\[
||\omega||_s = \max_b ||(F_b, \varepsilon_b)||_s.
\]

By convention, we set \( ||0||_{s} = -\infty \). We denote \( ||\omega||: P \to \mathbb{Z} \) the map which associates \( ||\omega||_s \) to any \( s \in P \).

2) The cochain \( \omega \) is \( \mathfrak{P} \)-**allowable** if \( ||\omega|| \leq \mathfrak{P} \) and of \( \mathfrak{P} \)-**intersection** if \( \omega \) and its differential \( \delta \omega \) are \( \mathfrak{P} \)-allowable. We denote \( \tilde{\mathcal{N}}^*_\mathfrak{P}(\Delta[m]; R) \) (or \( \tilde{\mathcal{N}}^*_\mathfrak{P}(\Delta[m]) \) if there is no ambiguity) the complex of \( \mathfrak{P} \)-intersection cochains on \( \Delta[m] \) and by

\[
\tilde{\mathcal{N}}^*_\mathfrak{P}: \Delta[\mathfrak{P}]^+ \to \text{M}_{\text{dg}}
\]

the associated functor. Finally, we extend it in a functor

\[
\tilde{\mathcal{N}}^*_\mathfrak{P}: \Delta[\mathfrak{P}] \to \text{M}_{\text{dg}},
\]

by setting \( \tilde{\mathcal{N}}^*(\Delta[m]) = 0 \) if \( \Delta[m] \) is a not regular simplex.

4. **Blown up cochains of simplicial sets over a poset**

Let \( R \) be a commutative ring with unit and \( \text{M}_{\text{dg}} \) be the category of positively graded differential graded \( R \)-modules, with a differential of degree +1. Let \( P \) be a poset and \( \mathfrak{P}: P \to \overline{\mathbb{Z}} \) a perversity. We define a pair of adjoint functors

\[
\text{Sset}_P \xrightarrow{\mathcal{N}^*_\mathfrak{P}} \text{M}_{\text{dg}}
\]

between the categories of simplicial sets over \( P \) and \( \text{M}_{\text{dg}} \) and prove the existence of an extension of this adjunction to homotopy classes.
4.1. Construction of the two functors. Let $P$ be a poset and $\mathfrak{p}$ be a perversity on $P$. We undertake the presentation made by Bousfield and Gugenheim for the Sullivan theory of rational homotopy type, see [3, Chapter 8]. For any $\sigma \in \mathcal{N}(P)$ and $k \in \mathbb{N}$, we set

$$\mathcal{M}_\mathfrak{p}(\sigma, k) = \mathcal{N}_\mathfrak{p}^k(\Delta[\sigma]).$$

We observe that $\mathcal{M}_\mathfrak{p}(\bullet, \ast)$ is a simplicial differential graded module over $\mathcal{N}(P)$. Let $\Psi_L \in \text{Sset}_P$ and $M \in \text{M}_{dg}$. First, we define a functor $\mathcal{N}_\mathfrak{p}^k(-): \text{Sset}_P \to \text{M}_{dg}$ by

$$\mathcal{N}_\mathfrak{p}^k(\Psi_L) = \text{Hom}_{\text{Sset}_P}(\Psi_L, \mathcal{M}_\mathfrak{p}(\bullet, k)).$$

In particular, if $\Psi_L$ is the identity map on $\mathcal{N}(P)$, denoted $\mathcal{N}(P)$, we have

$$(4.1) \quad \mathcal{N}_\mathfrak{p}^k(\mathcal{N}(P)) = \text{Hom}_{\text{Sset}_P}(\mathcal{N}(P), \mathcal{N}_\mathfrak{p}^k(\Delta[\bullet])),$$

which associates to any $\sigma \in \mathcal{N}(P)$ an element of $\mathcal{N}_\mathfrak{p}^k(\Delta[\sigma])$.

Let $M \in \text{M}_{dg}$. The element $(M)_\mathfrak{p} \in \text{Sset}_P$ is defined by the images of the $\sigma \in \Delta[P]$ and we set

$$(M)_\mathfrak{p}(\sigma) = \text{Hom}_{\text{M}_{dg}}(M, \mathcal{M}_\mathfrak{p}(\sigma, \ast)).$$

**Proposition 4.1.** The functors $(\bullet)_\mathfrak{p}$ and $\mathcal{N}_\mathfrak{p}^k(-)$ are adjoint; i.e., for any $\Psi_L \in \text{Sset}_P$ and $M \in \text{M}_{dg}$, there are bijections

$$\text{Hom}_{\text{Sset}_P}(\Psi_L, (M)_\mathfrak{p}) \xrightarrow{\beta} \text{Hom}_{\text{M}_{dg}}(M, \mathcal{N}_\mathfrak{p}^k(\Psi_L)).$$

**Proof.** The two bijections can be written down explicitly as in [3]. Let $\sigma \in \mathcal{N}(P)$, $k \in \mathbb{N}$, $x \in L_\sigma$, $w \in M^k$. We set $\alpha(g)(x)(w) = g(w)(x) \in \mathcal{M}_\mathfrak{p}(\sigma, k)$ and $\beta(f)(w)(x) = f(x)(w)$. □

This is an adjunction of contravariant functors. If we replace $\text{M}_{dg}$ by the opposite category $(\text{M}_{dg})^{op}$, these two functors give a pair of adjoint covariant functors, the functor corresponding to $\mathcal{N}_\mathfrak{p}^k(-)$ being the left adjoint. Therefore, $\mathcal{N}_\mathfrak{p}^k(-)$ transforms inductive limits in $\text{Sset}_P$ in projective limits in $\text{M}_{dg}$. This can also be seen directly from the definition and the compatibility of the Hom-functors with limits.

Similarly, we define a functor $\mathcal{N}_\mathfrak{p}^{+!*} : \text{Sset}_P^+ \to \text{M}_{dg}$ by

$$\mathcal{N}_\mathfrak{p}^{+!*}(\Psi_K) = \text{Hom}_{\text{Sset}_P^+}(\Psi_K, \mathcal{M}_\mathfrak{p}(\bullet, k))$$

and a functor $(\bullet)_\mathfrak{p}^{+} : \text{M}_{dg} \to \text{Sset}_P^+$ by

$$(M)_\mathfrak{p}^{+}(\sigma) = \text{Hom}_{\text{M}_{dg}}(M, \mathcal{M}_\mathfrak{p}(\sigma, \ast)),$$

for each $\sigma \in \Delta[P]^+$, $\Psi_K \in \text{Sset}_P^+$, $M \in \text{M}_{dg}$. We check easily

$$(4.2) \quad (M)_\mathfrak{p} = \mathcal{N}_\mathfrak{p}^{+!*}((M)_\mathfrak{p}^{+}) \quad \text{and} \quad \mathcal{N}_\mathfrak{p}^{+!*}(\Psi_L) = (\mathcal{N}_\mathfrak{p}^{+!*} \circ \mathcal{R})(\Psi_L).$$

As above, these two functors are adjoint.

**Definition 4.2.** Let $\Psi_K \in \text{Sset}_P^+$, the homology of $\mathcal{N}_\mathfrak{p}^{+!*}(\Psi_K)$ is called the blown up intersection cohomology of $\Psi_K$, for the perversity $\mathfrak{p}$, with coefficients in $R$, and denoted $\mathcal{H}_{\mathfrak{p}}^{+!*}(\Psi_K; R)$. (If there is no ambiguity, we also denote it $\mathcal{H}_{\mathfrak{p}}^{+!*}(\Psi_K)$.)

Similarly, for $\Psi_L \in \text{Sset}_P$, the homology of $\mathcal{N}_\mathfrak{p}^k(\Psi_L)$ is called the blown up intersection cohomology of $\Psi_L$ and denoted $\mathcal{H}_{\mathfrak{p}}^k(\Psi_L; R)$. 

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Let $\Psi_L \in \text{Sset}_\mathcal{P}$ and $\Psi_K \in \text{Sset}_\mathcal{P}$. From $\tilde{N}_\mathcal{P}^*(\Psi_L; R) = \tilde{N}_\mathcal{P}^{+, *}(\mathcal{R}(\Psi_L); R)$ and $\mathcal{R} \circ i = \mathcal{R} \circ n = \text{id}$, we deduce
\begin{equation}
(4.3) \quad \mathcal{H}_\mathcal{P}^*(\Psi_L) = \mathcal{H}_\mathcal{P}^{+, *}(\mathcal{R}(\Psi_L)) \quad \text{and} \quad \mathcal{H}_\mathcal{P}^{+, *}(\Psi_K) \cong \mathcal{H}_\mathcal{P}^*(\mathcal{I}(\Psi_K)) \cong \mathcal{H}_\mathcal{P}^*(n(\Psi_K)).
\end{equation}

4.2. Simplicial maps and blown up cohomology. Simplicial maps have a nice behavior with respect to blown up cohomology. We prove it in the general context of simplicial maps between simplicial sets over possibly different posets. The following result uses the notion of pullback of a perversity, introduced in Definition 3.4.

Proposition 4.3. Let $\mathcal{P}$ and $\mathcal{Q}$ be two posets, $\mathcal{P}, \mathcal{Q}$ be two perversities defined on $\mathcal{P}$ and $\mathcal{Q}$ respectively. We consider a commutative diagram of simplicial maps,
\begin{equation}
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\Psi_K \downarrow & & \downarrow \Psi_L \\
\mathcal{N}(\mathcal{P}) & \xrightarrow{f} & \mathcal{N}(\mathcal{Q}).
\end{array}
\end{equation}

We denote by $\tilde{N}_\mathcal{P}(\Psi_L, \mathcal{Q}; R)$ and $\tilde{N}_\mathcal{Q}(\Psi_K, \mathcal{P}; R)$ the blown up cochains corresponding to the perversities $\mathcal{P}, \mathcal{Q}$ and the posets $\mathcal{Q}, \mathcal{P}$ respectively. If $\mathcal{Q} \geq f^* \mathcal{P}$, then $f$ induces a cochain map $\tilde{N}_\mathcal{Q}(\Psi_L, \mathcal{Q}; R) \to \tilde{N}_\mathcal{Q}(\Psi_K, \mathcal{P}; R)$ defined by $(f^* \omega)_\sigma = \omega_{f^* \sigma}$. Therefore, there is an induced homomorphism between the associated blown up cohomology.

Proof. The association $\omega \mapsto f^* \omega$ is compatible with the face operators, $(d_i^* f^*(\omega))_\sigma = d_i^* \omega_{f^* \sigma} = \omega_{f \circ d_i \sigma}$, and similarly with the degeneracy operators. Moreover, if $\delta$ denotes the differentials, we have $(\delta f^*(\omega))_\sigma = \delta (f^*(\omega)_\sigma) = \omega_{f \circ \partial (\delta \omega)}$. Compatibility with perversities remains to be taken into account.

Let $\sigma : \Delta[n] \to K$ and $\omega \in \tilde{N}_\mathcal{P}(\Psi_L, \mathcal{Q}; R)$. The images of the simplex $\Delta[n]$ can be written
- $(\Psi_K \circ \sigma)(\Delta[n]) = s_1 < \cdots < s_{k_1} < s_{k_1 + 1} < \cdots < s_{k_r}$,
- $(\Psi_L \circ f \circ \sigma)(\Delta[n]) = t_1 < t_2 < \cdots < t_p$,
with $f(s_{k_i + j}) = t_{i+1}$, for all $j \in \{1, \ldots, k_{i+1} - k_i\}$. Therefore, by definition, we have
\begin{equation}
\|f^*(\omega)\|_{s_i} = \|\omega_{\sigma}\|_{t_i}.
\end{equation}

We fix $i$ and $j$, and denote $s = s_{k_i + j}$ and $t = t_i$. Recall that, by hypothesis, we have $\|\omega_{\sigma}\| \leq \mathcal{P}(t)$. This inequality and the hypotheses imply,
\begin{equation}
\|f^*(\omega)\|_s \leq \|f^*(\omega)\|_{s_i} = \|\omega_{\sigma}\|_t \leq \mathcal{P}(s) = \mathcal{P}(f(s)) \leq \mathcal{P}(\mathcal{I}(t)) \leq \mathcal{P}(t),
\end{equation}
and the $\mathcal{P}$-allowability of $f^* \omega$. We have proved that $f^* : \tilde{N}_\mathcal{Q}(\Psi_L, \mathcal{Q}; R) \to \tilde{N}_\mathcal{Q}(\Psi_K, \mathcal{P}; R)$ is a cochain map. \hfill $\square$

4.3. Compatibility with the homotopy classes. Recall that $\text{M}_{\mathcal{D}_\mathcal{G}}$ has a closed model structure [24, Section 2.3] where weak-equivalences are quasi-isomorphisms and fibrations are surjective chain maps. The cofibrant objects are the cochain complexes of projective $R$-modules. The rest of this section is devoted to the development of properties contributing to the proof of the following statement, which extends the adjunction between $(\cdot)_{\mathcal{P}}$ and $\tilde{N}(\cdot)_{\mathcal{P}}$ to homotopy classes.

Theorem 4.4. Let $\Psi_L$ be an $s$-fibrant object of $\text{Sset}_\mathcal{P}$, $\mathcal{P} : \mathcal{P} \to \mathcal{Z}$ a perversity and $M \in \text{M}_{\mathcal{D}_\mathcal{G}}$. Then, $(M)_{\mathcal{P}}$ is $s$-fibrant and the adjunction induces an isomorphism between the homotopy classes,
\begin{equation}
[\Psi_L, (M)_{\mathcal{P}}]_{\text{Sset}_\mathcal{P}} \cong [M, \tilde{N}(\Psi_L)]_{\text{M}_{\mathcal{D}_\mathcal{G}}}.
\end{equation}
We keep the notation of this statement all along the rest of this section. The tensorisation \( \Psi_L \otimes - \) makes reference to the tensor product in \( \text{Sset}_p \) of \( \Psi_L \) with a simplicial set. The proof consists of a construction of an ad’hoc path object in \( \text{MDG}_s \), see Corollary 4.4.

For \( j = 0, 1 \), we denote by \( t_j : \Psi_L \cong \Psi_L \otimes \{ j \} \rightarrow \Psi_L \otimes \Delta[1] \) the canonical inclusion.

**Proposition 4.5.** The following restriction morphism is a surjection,

\[
\tilde{N}_p^*(t_0) + \tilde{N}_p^*(t_1) : \tilde{N}_p^*(\Psi_L \otimes \Delta[1]) \rightarrow \tilde{N}_p^*(\Psi_L) \oplus \tilde{N}_p^*(\Psi_L).
\]

**Proof.** For constructing a section, we use the existence of a cup product at the cochain level ([7, Proposition 4.2]). Let \( \Delta[1] = [e_0, e_1] \) and \( 1_{e_i} \in \tilde{N}^0(\Delta[1]) \) the 0-cochain taking the value 1 on \( e_i \) and 0 otherwise, for \( i = 0, 1 \). We denote by \( \pi : \Psi_L \otimes \Delta[1] \rightarrow \Psi_L \) the canonical projection and consider the morphism

\[
\Phi : \tilde{N}_p^*(\Psi_L) \oplus \tilde{N}_p^*(\Psi_L) \rightarrow \tilde{N}_p^*(\Psi_L \otimes \Delta[1]),
\]

defined by

\[
\Phi(\omega_0, \omega_1) = 1_{e_0} \sim \tilde{N}_p^*(\pi)(\omega_0) + 1_{e_1} \sim \tilde{N}_p^*(\pi)(\omega_1).
\]

We observe that \( \tilde{N}_p^*(t_0) + \tilde{N}_p^*(t_1)(\Phi(\omega_0, \omega_1)) = (\tilde{N}_p^*(\pi \circ t_0)(\omega_0), \tilde{N}_p^*(\pi \circ t_1)(\omega_1)) = (\omega_0, \omega_1) \). \( \square \)

**Proposition 4.6.** For \( j = 0, 1 \), the morphism \( t_j^* : \tilde{N}_p^*(\Psi_L) \rightarrow \tilde{N}_p^*(\Psi_L \otimes \Delta[1]) \) is a trivial fibration in \( \text{MDG}_s \) and the morphisms \( t_0 \) and \( t_1 \) induce the same map in blown up cohomology.

**Proof.** The surjectivity of \( \tilde{N}_p^*(t_j) \) comes from Proposition 4.5. Let \( \pi : \Psi_L \otimes \Delta[1] \rightarrow \Psi_L \) be the canonical projection. The equality \( \pi \circ t_j = \text{id} \) implies \( \mathcal{H}_p^*(t_j) \circ \mathcal{H}_p^*(\pi) = \text{id} \). Set \( \psi_j = t_j \circ \pi : \Psi_L \otimes \Delta[1] \rightarrow \Psi_L \otimes \Delta[1] \). (We denote also \( \psi_j \) the underlying map from \( L \otimes \Delta[1] \) to \( L \otimes \Delta[1] \).) The first part of the statement is therefore reduced to the existence of a cochain homotopy \( \tilde{G} \), between the identity map and \( \tilde{N}_p^*(\psi_j) \). Such homotopy and the previous equality imply \( \mathcal{H}_p^*(t_0) = (\mathcal{H}_p^*(\pi))^{-1} = \mathcal{H}_p^*(t_1) \).

We suppose \( j = 0 \), the case \( j = 1 \) being similar. For simplicity, we denote \( \psi = \psi_0 \). Let \( \sigma : \Delta[\bullet] \rightarrow \Psi_L \) be a simplex. The map \( \psi : \Psi_L \otimes \Delta[1] \rightarrow \Psi_L \otimes \Delta[1] \) is a collection of maps, \( \psi_\Delta[k] : \Delta[k] \rightarrow \Delta[k] \), for any simplex \( \Delta[k] \) of the product \( \Delta[\bullet] \times \Delta[1] \). Such map can be extending in, \( \psi_\Delta[k] : \Delta[k] \rightarrow \Delta[k] \), by taking the identity on the cone point. We denote by \( c\psi_\Delta[k] \) and \( \psi_\Delta[k] \) the induced cochain morphisms.

The blow up of a simplex of \( \Psi_L \otimes \Delta[1] \) is a face of the product \( \tilde{\Delta} = c\Delta[j_0] \times \cdots \times c\Delta[j_{\ell-1}] \times \Delta[j_\ell] \) where each \( \Delta[j_\ell] \) is a simplex of a product \( \Delta[\bullet] \times \Delta[1] \). By naturality, it is sufficient to define the homotopy \( \tilde{G} \) at the level of \( \Delta \). Denote by \( G : N^*(L \otimes \Delta[1]) \rightarrow N^*(L \otimes \Delta[1]) \) the homotopy between the identity map and \( N^*(\tilde{\psi}) \), i.e., we have \( \delta \tilde{G} + G \delta = N^*(\tilde{\psi}) - \text{id} \). The restriction of \( G \) to a simplex \( \Delta[j] \) of \( L \otimes \Delta[1] \) is denoted \( G_{\Delta[j]} \) and its extension to \( c\Delta[j] \) by \( cG_{\Delta[j]} \). Let us also denote \( \text{id}_{\Delta[j]} \) the identity map on \( N^*(\Delta[j]) \). For

\[
1_{(F, e)} = 1_{(F_\ell, e_\ell)} \otimes \cdots \otimes 1_{(F_0, e_0)} \otimes 1_{F_\ell} \in \tilde{N}^*(\Delta[m]),
\]

we set \( |(F, e)|_{<j} = \sum_{\ell=0}^{j-1} |(F_\ell, e_\ell)| \) and we define \( \tilde{G}(1_{(F, e)}) \) as

\[
\sum_{\ell=0}^{m} (-1)^{|(F, e)|_{<\ell}} 1_{(\psi(F_0, e_0))} \otimes \cdots \otimes 1_{((\psi(F_{\ell-1}, e_{\ell-1}))} \otimes G(1_{(F_\ell, e_\ell)}) \otimes 1_{(F_\ell, e_\ell)} \otimes \cdots \otimes 1_{F_\ell}.
\]

We prove that \( \delta \tilde{G} + \delta \tilde{G} = \tilde{N}^*(\tilde{\psi}) - \text{id} \) by induction on \( \ell \), the perverse degree being taken in account at the end of the proof. It is true for \( \ell = 0 \), by choice of \( G \). Let us suppose it is true for \( \ell = j \) and we prove it for \( \ell \). The element \( 1_{(F, e)} \) can be written as \( 1_{(F_0, e_0)} \otimes B \), where \( B \) is a tensor product on which the induction hypothesis can be applied. By definition of \( \tilde{G} \), we have

\[
\tilde{G}(1_{(F_0, e_0)} \otimes B) = G(1_{(F_0, e_0)} \otimes B) + (-1)^{|(F_0, e_0)|} 1_{(F_0, e_0)} \otimes \tilde{G}(B).
\]
A computation, using $G\delta + \delta G = N^*(\psi) - \text{id}$ and the induction, gives $\tilde{G}\delta + \delta \tilde{G} = \tilde{N}^*(\psi) - \text{id}$. Finally, by construction, the homotopy $\tilde{G}$ respects the perverse degree and is the required homotopy, $\tilde{G}: \tilde{N}_p^*(\Psi_L \otimes \Delta[1]) \to \tilde{N}_p^*(\Psi_L \otimes \Delta[1])$. \hfill \Box

**Corollary 4.7.** The two injections $i_0, i_1$, and the projection $\pi$ generate a path object in the category $\mathbf{M}_{\mathbf{dg}}$,

$$\tilde{N}_p^*(\Psi_L) \xrightarrow{\pi^*} \tilde{N}_p^*(\Psi_L \otimes \Delta[1]) \xrightarrow{\iota_0^* + \iota_1^*} \tilde{N}_p^*(\Psi_L) \oplus \tilde{N}_p^*(\Psi_L).$$

Thus if $\Phi_1 \sim \Phi_2$ in $\text{Sset}_\mathcal{P}$, then $\tilde{N}_p^*(\Phi_1)$ and $\tilde{N}_p^*(\Phi_2)$ are homotopic in $\mathbf{M}_{\mathbf{dg}}$ and two homotopic maps in $\text{Sset}_\mathcal{P}$ induce the same map in blown up cohomology. In particular, a homotopy equivalence induces an isomorphism.

**Proof.** The composition $(\iota_0^* + \iota_1^*) \circ \pi^*$ is the diagonal map $\tilde{N}_p^*(\Psi_L) \to \tilde{N}_p^*(\Psi_L) \oplus \tilde{N}_p^*(\Psi_L)$. The statement is thus a direct consequence of the definition of a path object and of Propositions 4.3, 4.5 and 4.6. \hfill \Box

**Proof of Theorem 4.4.** For any $\Psi_L \in \text{Sset}_\mathcal{P}$, the simplicial set $\text{Hom}_{\text{Sset}}^\Delta(\Psi_L, (M)_\mathcal{P})$ is a simplicial group, therefore $(M)_\mathcal{P} \in \text{Sset}_\mathcal{P}$ is s-fibrant in the sense of Definition B.2. Denote by $\mathcal{T}: \text{Hom}_{\text{Sset}}(\Psi_L, (M)_\mathcal{P}) \to \text{Hom}_{\mathbf{M}_{\mathbf{dg}}}(\Psi_L, \tilde{N}_p^*(\Psi_L))$ the bijection given by the adjunction. Let $\Phi_1, \Phi_2$ be two elements of $\text{Hom}_{\text{Sset}}(\Psi_L, (M)_\mathcal{P})$. We have to prove $\Phi_1 \sim \Phi_2$ if, and only if, $\mathcal{T}(\Phi_1) \sim \mathcal{T}(\Phi_2)$.

Suppose first $\Phi_1 \sim \Phi_2$ and recall that, for $i = 1, 2$, $\mathcal{T}(\Phi_i)$ is the composition

$$M \xrightarrow{\tilde{N}_p^*(\Phi_i)} \tilde{N}_p^*(\Psi_L).$$

Thus $\mathcal{T}(\Phi_1) \sim \mathcal{T}(\Phi_2)$ is a consequence of Corollary 4.7.

Suppose now $\mathcal{T}(\Phi_1) \sim \mathcal{T}(\Phi_2)$. Then, for $i = 1, 2$, the map $\Phi_i$ is the composition

$$\Psi_L \xrightarrow{\mathcal{T}(\Phi_i)} \tilde{N}_p^*(\Psi_L) \xrightarrow{(T(\Phi_i))_\mathcal{P}} (M)_\mathcal{P}.$$

The homotopy $\mathcal{T}(\Phi_1) \sim \mathcal{T}(\Phi_2)$ consists of a map $H: M \to \tilde{N}_p^*(\Psi_L \otimes \Delta[1]) \in \mathbf{M}_{\mathbf{dg}}$ whose projection to $\tilde{N}_p^*(\Psi_L) \oplus \tilde{N}_p^*(\Psi_L)$ is $\mathcal{T}(\Phi_1) + \mathcal{T}(\Phi_2)$. Thus, by adjunction, we get a morphism in $\text{Sset}_\mathcal{P}$,

$$\Psi_L \otimes \Delta[1] \to \tilde{N}_p^*(\Psi_L \otimes \Delta[1])_\mathcal{P} \to (M)_\mathcal{P},$$

which is a homotopy between $\Phi_1$ and $\Phi_2$. \hfill \Box

We denote by $\Lambda^k[m]$ the $k$th-horn, which is the subcomplex of $\Delta[m]$ generated by all faces except the $k$th face.

**Proposition 4.8.** Let $\Psi_L$ be an object of $\text{Sset}_\mathcal{P}$ and $\mathcal{P}$ a perversity on $\mathcal{P}$. For any $m \geq 1$ and any $k$, $0 \leq k \leq m$, the canonical inclusion $\Lambda^k[m] \hookrightarrow \Delta[m]$ induces a trivial fibration,

$$\varphi: \tilde{N}_p^*(\Psi_L \otimes \Delta[m]) \to \tilde{N}_p^*(\Psi_L \otimes \Lambda^k[m]).$$

**Proof.** Let $M \in \mathbf{M}_{\mathbf{dg}}$ be cofibrant. We have to prove the existence of a dotted arrow making commutative the following diagram in $\mathbf{M}_{\mathbf{dg}}$:

$$\begin{array}{ccc}
M & \xrightarrow{\tilde{N}_p^*(\Psi_L \otimes \Lambda^k[m])} & \\
\downarrow & \nearrow & \\
\tilde{N}_p^*(\Psi_L \otimes \Delta[m]) & . & \\
\end{array}$$
By adjunction, this is equivalent to the existence of a dotted arrow making commutative the following diagram in $\text{Sset}_P$,

$$
\begin{array}{ccc}
\Psi_L \otimes \Lambda^k[m] & \rightarrow & \langle M \rangle_eta \\
\downarrow & & \downarrow \\
\Psi_L \otimes \Delta[m]. & \rightarrow & \\
\end{array}
$$

From Definition B.1, this last property amounts the existence of a dotted arrow, $g$, making commutative the following diagram in $\text{Sset}$,

$$
\begin{array}{ccc}
\Lambda^k[m] & \rightarrow & \text{Hom}^\Delta_{\text{Sset}_P}(\Psi_L, \langle M \rangle_eta) \\
\downarrow & & \downarrow \\
\Delta[m]. & \rightarrow & \\
\end{array}
$$

The morphism $g$ exists since $\text{Hom}^\Delta_{\text{Sset}_P}(\Psi_L, \langle M \rangle_p)$ is a simplicial group, therefore a Kan simplicial set. □

5. PERVERSE EILENBERG-MACLANE SIMPLICIAL SETS

Let $R$ be a commutative ring, $P$ a poset and $\overline{P}: P \rightarrow \mathbb{Z}$ a perversity. In this section, we show that the blown up cohomology is a representable functor on $\text{Sset}_P$. If $K(R, n, P, \overline{P})$ is the simplicial set over $P$ representing $\mathcal{H}^n_R(\Psi_L; R)$, we prove that the family $(K(R, n, P, \overline{P}))_n$ is an infinite loop space in $\text{Sset}_P$. We also introduce the cohomological operations on the blown up cohomology and present some basic properties. In the case of a GM-perversity on a pseudomanifold, they are cohomological operations on the Goresky-MacPherson hypercohomology of the Deligne’s sheaves.

5.1. Representability. In the classical case, a simplicial model of the Eilenberg-MacLane space $K(R, n)$ has for set of $k$-simplices the $R$-module of cocycles of degree $n$ in $N^\ast(\Delta[k]; R)$. We follow a similar treatment replacing the category of $\Delta^k$’s by $\Delta_P$.

Denote by $R(n)$ the object of $M_{\text{dg}}$, reduced to a free $R$-module generated by one cocycle of degree $n$. Recall from (4.1), that, for any $\sigma \in \Pi(P)$, we have

$$
(R(n))(\sigma) = \text{Hom}_{M_{\text{dg}}}(R(n), \tilde{N}_P^\ast(\Delta[\sigma]; R)) = Z^n \hat{N}_P^\ast(\Delta[\sigma]; R),
$$

where $Z^n$ denotes the subspace of cocycles in degree $n$. Therefore, we have

$$
(R(n))_P = Z^n \hat{N}_P^\ast(\Pi(P); R).
$$

This definition fits the case of restricted simplicial sets over $P$ and gives

$$
(R(n))_P^+ = Z^n \hat{N}_P^\ast(\Pi(P); R).
$$

Proposition 5.1. The functor sending $\Psi_L \in \text{Sset}_P$ to $\mathcal{H}^n_R(\Psi_L; R)$ is representable; i.e., for any $s$-fibrant $\Psi_L \in \text{Sset}_P$, we have an isomorphism,

$$
\mathcal{H}^n_R(\Psi_L; R) \cong [\Psi_L, (R(n))(\overline{P})]_{\text{Sset}_P}.
$$

Similarly, the functor sending $\Psi_K \in \text{Sset}_P^+$ to $\mathcal{H}^n_R(\Psi_K; R)$ is representable; i.e., for any $s$-fibrant $\Psi_K \in \text{Sset}_P^+$, we have an isomorphism,

$$
\mathcal{H}^n_R(\Psi_K; R) \cong [\Psi_K, (R(n))(\overline{P})]_{\text{Sset}_P^+}.
Proposition 5.3. The two \( \mathcal{P} \)-perverse Eilenberg-MacLane spaces are connected by 
\[
\mathcal{K}(R, n, P, \mathcal{P}) = \mathcal{N} \mathcal{M}_{\mathcal{P}}(\mathcal{P}, n, R) = \mathcal{N} \mathcal{M}_{\mathcal{P}}(\mathcal{P}, n, R) \]
and \( \mathcal{K}(R, n, P, \mathcal{P}) = \mathcal{R} \mathcal{K}(R, n, P, \mathcal{P}) \).

Proof. Recall from (4.2), the equality \( (M)_{\mathcal{P}} = n \left( (M)_{\mathcal{P}} \right) \), for any \( M \in \mathcal{M}_{\mathcal{P}} \). This gives immediately \( \mathcal{K}(R, n, P, \mathcal{P}) = \mathcal{N} \mathcal{M}_{\mathcal{P}}(\mathcal{P}, n, R) \) from their definition. The second equality follows from \( \mathcal{R} \circ n = \text{id} \), see Proposition 2.20. \( \square \)

We now specify the \((n - 1)\)-skeleton of \( \mathcal{K}(R, n, P, \mathcal{P}) \).

Proposition 5.4. For any perversity \( \mathcal{P} \), the \((n - 1)\)-skeleta of \( \mathcal{K}(R, n, P, \mathcal{P}) \) and \( \mathcal{N}(P) \) coincide.

Proof. If \( \ell < n \), the \( \mathcal{P} \)-simplices of \( \Delta[\ell] \) are degenerate. As \( N^*() \) is the normalized complex, we have \( Z^n \mathcal{N} \mathcal{P}(\Delta[\ell]) = 0 \). Thus \( (R(n))_{\mathcal{P}}(\sigma) = \mathcal{N} \mathcal{M}_{\mathcal{P}}(R(n), \mathcal{N} \mathcal{P}(\Delta[\ell])) = \{0\} \), for any \( \sigma \) of dimension \( < n \), and \( (R(n))_{\mathcal{P}}(\ell) = \mathcal{N}(P)^\ell \) if \( \ell < n \). \( \square \)

As in the classical algebraic topology situation, the family of \( \mathcal{P} \)-perverse Eilenberg-MacLane spaces is an infinite loop space. (We refer to Subsection B.2 for a reminder on infinite loop spaces in a simplicial category.) First, we have to select a basepoint in \( \mathcal{K}(R, n, P, \mathcal{P}) = (R(n))_{\mathcal{P}} \in \mathcal{Sset}_p \).

The final object of \( \mathcal{Sset}_p \) being the identity map on \( \mathcal{N}(P) \), such a basepoint is an element of 
\[
\text{Hom}_{\mathcal{Sset}_p}(\mathcal{N}(P), (R(n))_{\mathcal{P}}) \cong \text{Hom}_{\mathcal{M}_{\mathcal{P}}}(R(n), \mathcal{N} \mathcal{P}(\mathcal{N}(P))) \cong Z^n \mathcal{N} \mathcal{P}(\mathcal{N}(P)).
\]

Therefore, we can choose the map \( \epsilon: \mathcal{N}(P) \rightarrow Z^n \mathcal{N} \mathcal{P}(\mathcal{N}(P)) \) constant on 0.

Remark 5.5. The zero cocycle is a natural basepoint. Let us notice that, depending on the combinatorics of the poset \( P \), the Kan simplicial set \( Z^n \mathcal{N} \mathcal{P}(\mathcal{N}(P)) \) is not necessarily connected. But, as a simplicial group, all its connected components are homotopy equivalent, in fact isomorphic.

Theorem 5.6. Let \( \mathcal{P} : P \rightarrow \mathbb{Z} \) be a perversity on a poset \( P \) and \( n \geq 0 \). The families of \( \mathcal{P} \)-perverse Eilenberg-MacLane \( \mathcal{Sset}_p \) and \( \mathcal{Sset}_p^+ \) simplicial sets are infinite loop spaces in the categories \( \mathcal{Sset}_p \) and \( \mathcal{Sset}_p^+ \), respectively; i.e., there are weak \( s \)-equivalences, 
\[
\mathcal{K}(R, n - 1, P, \mathcal{P}) \simeq \Omega \mathcal{K}(R, n, P, \mathcal{P}) \text{ and } \mathcal{K}(R, n - 1, P, \mathcal{P})^+ \simeq \Omega \mathcal{K}(R, n, P, \mathcal{P})^+.
\]

Proof. Let \( \Psi_L : A \rightarrow \mathcal{N}(P) \in \mathcal{Sset}_p \), pointed by \( \epsilon \). The loop space associated to \( \epsilon \) being defined as a pullback (see Definition B.5), for any \( \Psi_L \in \mathcal{Sset}_p \), we have the following pullback, 
\[
\text{Hom}_{\mathcal{Sset}_p}(\Psi_L, \Omega \Psi_A) \rightarrow \text{Hom}_{\mathcal{Sset}_p}(\Psi_L, \Psi_A^{[1]}),
\]
\[
\text{Hom}_{\mathcal{Sset}_p}(\Psi_L, \mathcal{N}(P)) \rightarrow \text{Hom}_{\mathcal{Sset}_p}(\Psi_L, \Psi_A^{[1]}).
\]
If $\Psi_A = K(R, n, P, \mathcal{P})$ and $\epsilon = 0$, we deduce from Definition B.1 that

$$\text{Hom}_{\text{Sset}}(\Psi_L, \Omega_e K(R, n, P, \mathcal{P})) = \text{Ker} (\mathcal{Z}^n \tilde{N}^n_\mathcal{P}(\Psi_L \otimes \Delta[1]) \rightarrow \mathcal{Z}^{n+1} \tilde{N}^{n+1}_\mathcal{P}(\Psi_L \otimes \partial \Delta[1]))$$

and

\begin{equation}
 [\Psi_L, \Omega_e K(R, n, P, \mathcal{P})] \cong \mathcal{H}^n(\Psi_L \otimes \Delta[1], \Psi_L \otimes \partial \Delta[1]).
\end{equation}

For the determination of this relative cohomology, we consider the Ker-Coker exact sequence applied to the following morphism of short exact sequences,

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \tilde{N}^n_\mathcal{P}(\Psi_L \otimes \Delta[1], \Psi_L \otimes \partial \Delta[1]) & \rightarrow & \tilde{N}^n_\mathcal{P}(\Psi_L \otimes \Delta[1]) & \rightarrow & \tilde{N}^{n+1}_\mathcal{P}(\Psi_L) & \rightarrow & 0 \\
0 & \rightarrow & \text{Ker} & \rightarrow & \tilde{N}^n_\mathcal{P}(\Psi_L \otimes \Delta[1]) & \rightarrow & \tilde{N}^n_\mathcal{P}(\Psi_L) & \rightarrow & 0,
\end{array}
\]

where $\nu$ is the projection on the first factor. We obtain an isomorphism between $\tilde{N}^n_\mathcal{P}(\Psi_L)$ and the cokernel of $\mu$. The acyclicity of Ker (see Proposition 4.6) gives an isomorphism

\begin{equation}
\mathcal{H}^n_\mathcal{P}(\Psi_L) \cong H^{n+1}(\tilde{N}^n_\mathcal{P}(\Psi_L \otimes \Delta[1], \Psi_L \otimes \partial \Delta[1])) = \mathcal{H}^{n+1}(\Psi_L \otimes \Delta[1], \Psi_L \otimes \partial \Delta[1]).
\end{equation}

From Proposition 5.1 and the isomorphisms (5.2), (5.1), we deduce

\[
\begin{align*}
[\Psi_L, K(R, n - 1, P, \mathcal{P})] & \cong \mathcal{H}^{n-1}_\mathcal{P}(\Psi_L) \\
& \cong \mathcal{H}^n_\mathcal{P}(\Psi_L \otimes \Delta[1], \Psi_L \otimes \partial \Delta[1])) \\
& \cong [\Psi_L, \Omega_e K(R, n, P, \mathcal{P})].
\end{align*}
\]

The result follows from the Yoneda lemma. The proof of the second assertion is similar. \(\square\)

We deduce from Theorem 5.6 an isomorphism involving classical Eilenberg-MacLane spaces and perverse ones.

**Corollary 5.7.** Let $\Psi_L \in \text{Sset}_P, \mathcal{P}: P \rightarrow \mathbb{Z}$ be a perversity and $n \geq 0$. Then the simplicial set $\text{Hom}^\Delta_{\text{Sset}_P}(\Psi_L, K(R, n, P, \mathcal{P}))$ is a product of Eilenberg-MacLane spaces,

$$\text{Hom}^\Delta_{\text{Sset}_P}(\Psi_L, K(R, n, P, \mathcal{P})) \cong \prod_j K(\mathcal{H}^n_{\mathcal{P}}(\Psi_L; R), j).$$

**Proof.** The simplicial set $Z = \text{Hom}^\Delta_{\text{Sset}_P}(\Psi_L, K(R, n, P, \mathcal{P}))$ is a simplicial abelian group therefore a product of Eilenberg-MacLane spaces. We are thus reduced to the determination of its homotopy groups. We first have

$$\pi_0(Z) = [\Psi_L, K(R, n - 1, P, \mathcal{P})]_{\text{Sset}_P} \cong \mathcal{H}^n_{\mathcal{P}}(\Psi_L; R).$$

The rest follows by induction from $K(R, n - 1, P, \mathcal{P}) \simeq \Omega_e K(R, n, P, \mathcal{P}). \square$

Let $\psi_X: X \rightarrow P$ be a topological pseudomanifold of poset of strata, $P$. For any perversity, $\mathcal{P}: P \rightarrow \mathbb{Z}$, and any commutative ring $R$, we denote $Q_{\mathcal{P}}$ the Deligne’s sheaf introduced in [20] and $H^*(X; Q_{\mathcal{P}})$ its associated hypercohomology groups, which coincide with the original groups introduced in [19]. (For general perversities as those we are using here, we refer to [14].) Let $\text{Sing}^p \psi_X$ be the simplicial set over the poset of strata, $P$, introduced in (2.2). In Proposition A.3, we prove the existence of an isomorphism,

$$\mathcal{H}_{\mathcal{P}}^* (\text{Sing}^p \psi_X; R) \cong \mathcal{H}_{\mathcal{P}}^{\Delta}(\mathcal{O}_P(\text{Sing}^p \psi_X); R).$$

In [11], we denote $\mathcal{H}_{\mathcal{P}}^{\Delta}(\mathcal{O}_P(\text{Sing}^p \psi_X); R)$ simply by $\mathcal{H}_{\mathcal{P}}^* (X; R)$ and prove in [11, Theorem A] that it is isomorphic to Deligne’s hypercohomology,

$$H^*(X; Q_{\mathcal{P}}) \cong \mathcal{H}_{\mathcal{P}}^* (X; R).$$
We thus have an isomorphism,
\[
\mathcal{H}^n_{\mathbb{P}}(\text{Sing}^p \psi_X; R) \cong H^*(X; Q_{\mathbb{P}}),
\]
and Corollary 5.7 implies the following result.

**Corollary 5.8.** Let $\psi_X : X \to \mathbb{P}$ be a topological pseudomanifold of poset of strata, $\mathbb{P}$, and $\mathbb{P} : \mathbb{P} \to \mathbb{Z}$ be a perversity. Then there is an isomorphism of simplicial sets,
\[
\text{Hom}_{\text{Sset}}(\text{Sing}^p \psi_X, K(R, n, \mathbb{P}, \mathbb{P})) \cong \prod_j K(H^*(X; Q_{\mathbb{P}}), j).
\]

Moreover, in the case of pseudomanifolds, the isomorphism (5.3) identifies the cohomological operations, developed in the next subsection and in Section 6, with operations on Deligne’s hypercohomology.

### 5.2. Perverse cohomological operations

Let us define cohomological operations on intersection cohomology, reproducing the classical case.

**Definition 5.9.** Let $\mathbb{P}, \mathbb{Q} : \mathbb{P} \to \mathbb{Z}$ be two perversities on a poset $\mathbb{P}$ and $n, m$ be two integers. A perverse cohomological operation of type $(\mathbb{P}, n, \mathbb{Q}, m)$ is a natural transformation between the functors $\mathcal{H}^n_{\mathbb{P}}(\mathbb{P}, n, \mathbb{Q}, m)$ and $\mathcal{H}^m_{\mathbb{Q}}(\mathbb{P}, m, \mathbb{P}, \mathbb{Q})$, from $\text{Sset}$ to the category of $R$-modules. We denote $\text{Nat}_R(\mathcal{H}^n_{\mathbb{P}}, \mathcal{H}^m_{\mathbb{Q}})$ the set of perverse cohomological operations of this type.

Cohomological operations on intersection cohomology are also determined by the perverse Eilenberg-MacLane spaces.

**Proposition 5.10.** Let $\mathbb{P}, \mathbb{Q} : \mathbb{P} \to \mathbb{Z}$ be two perversities on a poset $\mathbb{P}$ and $n, m$ be two integers. There is an isomorphism
\[
\text{Nat}_R(\mathcal{H}^n_{\mathbb{P}}, \mathcal{H}^m_{\mathbb{Q}}) = [K(R, n, \mathbb{P}, \mathbb{P}), K(R, m, \mathbb{P}, \mathbb{Q})]_{\text{Sset}} = \mathcal{H}^m_{\mathbb{Q}}(K(R, n, \mathbb{P}, \mathbb{P}); R).
\]

**Proof.** This is a direct consequence of the Yoneda lemma and the representability of $\mathcal{H}^n_{\mathbb{P}}(\mathbb{P}, n, \mathbb{P}, \mathbb{P})$ established in Proposition 5.1.

The following result has to be compared with the classical fact that $\mathcal{H}(R, n)$ is $(n-1)$-connected.

**Proposition 5.11.** Let $\mathbb{P}, \mathbb{Q}$ be two perversities on the poset $\mathbb{P} = \mathbb{N}^{op}$. If $0 < m < n$, we have
\[
\text{Nat}_R(\mathcal{H}^n_{\mathbb{P}}, \mathcal{H}^m_{\mathbb{Q}}) = \mathcal{H}^m_{\mathbb{Q}}(K(R, n, \mathbb{P}, \mathbb{P}); R) = 0.
\]

**Proof.** The simplicial set $\mathbb{N}(\mathbb{N}^{op})$ is a cone on an acyclic simplicial set. Therefore, its reduced intersection cohomology is 0 for any perversity and the result follows from Proposition 5.4.

**Example 5.12.** From [5, 7, 17, 22], we already know some perverse cohomological operations. Let $\sigma = s_{\ell_0} \prec \cdots \prec s_{\ell_i} \in \mathbb{N}(\mathbb{P})$ and recall that $(R(n))_{\mathbb{P}}[\sigma] = Z^n \tilde{N}_\mathbb{P}(\mathbb{P}, \mathbb{P}) = Z^n \tilde{N}_\mathbb{P}(\Delta[q_0] \ast \cdots \ast \Delta[q_i])$, where $Z$ denotes the subspace of cocycles.

1. For any commutative ring, $R$, there exists a square map, $\tilde{N}_\mathbb{P}(\mathbb{P}, \mathbb{P}) \to \tilde{N}_\mathbb{P}(\mathbb{P}, \mathbb{P})$, coming from the cup product established in [7, Proposition 4.2]. This map gives an element in $\text{Nat}_R(\mathcal{H}^n_{\mathbb{P}}, \mathcal{H}^m_{\mathbb{Q}})$.
2. Let $R = \mathbb{Z}_2$ and $\mathcal{L}(2)$ be the normalized homogeneous bar resolution of the symmetric group $\Sigma_2$. In [5, Theorem A], we establish the existence of an $\mathcal{L}(2)$-algebra structure on $\tilde{N}_\mathbb{P}^*(\sigma)$, inducing $\Sigma_2$-equivariant cochain maps, $\Phi : \mathcal{L}(2) \otimes \tilde{N}_\mathbb{P}^*(\sigma) \otimes \tilde{N}_\mathbb{P}^*(\sigma) \to \tilde{N}_\mathbb{P}^*(\sigma)$. In particular, setting $\psi(e_i \otimes x \otimes y) = x \prec y$, the square maps verify $\|x \prec |x|^{-1} x\| \leq \mathbb{P} + i$, see [5, Proof of Theorem B]. With the notation $\mathcal{L}(\mathbb{P}) = \min(\mathbb{P}, \mathbb{P} + i)$, we prove in [5] that the Steenrod perverse squares $\text{Sq}_i^* \in \text{Nat}_\mathbb{Z}_2(\mathcal{H}^n_{\mathbb{P}}, \mathcal{H}^{n+i}_{\mathbb{Q}})$, as conjectured in [17] and [22].
6. Examples of operations in blown up cohomology

In this section, we suppose that $R$ is a Dedekind domain and we focus on the case of Goresky and MacPherson perversities for singular spaces with one singular stratum. We show that the perverse Eilenberg-MacLane spaces are Joyal’s projective cone over the classical Eilenberg-MacLane spaces. We also determine them for the perversities $\infty$ and $0$, as well as some sets of perverse cohomological operations.

Depth one singular spaces are singular spaces with one regular stratum and one singular stratum. They correspond to the poset $P = [1] = \{0, 1\}$ and $\mathbb{N}(P) = \Delta[1]$. An object of $\text{Sset}_{[1]}$ can be described as a family of sets $K_{k,\ell}$ with

$$(k, \ell) \in \mathbb{N}^2 \cup (\mathbb{N} \times \{-1\}) \cup (\{-1\} \times \mathbb{N}),$$

the value $-1$ corresponding to an empty subset. The objects of $\text{Sset}_{[1]}$ such that $\ell \neq -1$; i.e., with $(k, \ell) \in \mathbb{N}^2 \cup \{-1\} \times \mathbb{N}$. A completely regular simplicial set over $[1]$ verifies $K_{k,\ell} = \emptyset$ if $k > 0$. A perversity on $P = [1]$ reduces to an element $\lambda \in \mathbb{Z}$ corresponding to its value on 0, and is denoted $\lambda$.

6.1. Perverse Eilenberg-MacLane spaces as Joyal cylinders. Simplicial sets over $[1]$ have been studied by A. Joyal, see [25, Section 7]. Let us first recall his presentation.

A simplicial map $\Psi_K : K \to \Delta[1]$ is called a simplicial cylinder. The simplicial sets $K(0)$ and $K(1)$, defined by the following pullback

$$\begin{array}{ccc}
K(0) \sqcup K(1) & \longrightarrow & K \\
\downarrow \quad & \downarrow \Psi_K \quad & \downarrow \\
\partial \Delta[1] & \longrightarrow & \Delta[1],
\end{array}$$

generate a functor $i^* : \text{Sset}_{[1]} \to \text{Sset} \times \text{Sset}$, sending $\Psi_K$ to $(K(0), K(1))$. The simplicial sets $K(1)$ and $K(0)$ are respectively called the base and the cobase of the cylinder $\Psi_K$.

The functor $i^*$ admits a left adjoint $i_* : \text{Sset} \times \text{Sset} \to \text{Sset}_{[1]}$ which sends $(A, B)$ to the composite $\Psi_{A \sqcup B} : A \sqcup B \to \partial \Delta[1] \to \Delta[1]$,

$$\text{Hom}_{\text{Sset} \times \text{Sset}}((A, B), i^*(\Psi_K)) \cong \text{Hom}_{\text{Sset}_{[1]}}(\Psi_{A \sqcup B}, \Psi_K).$$

For any pair $(A, B) \in \text{Sset} \times \text{Sset}$, we can define their join, $A \ast B$ (see [25, & 3.1]) as a simplicial set whose sets of simplices are given by

$$(A \ast B)_k = \left\{ \begin{array}{ll}
\sigma & \text{with } \sigma \in A_k, \text{ denoted } (\sigma, 0), \\
\tau & \text{with } \tau \in B_k, \text{ denoted } (0, \tau), \\
(\sigma, \tau) & \text{with } \sigma \in A_i, \tau \in B_j \text{ and } i + j + 1 = k.
\end{array} \right.$$

Then, we have two canonical maps, the injection $i_{A \sqcup B} : A \sqcup B \to A \ast B$, and the surjection $\Psi_{A \ast B} : A \ast B \to \Delta[1]$, which is the join of $A \to \Delta[0]$ and $B \to \Delta[0]$. The join construction gives a right adjoint $i_* : \text{Sset} \times \text{Sset} \to \text{Sset}_{[1]}$ sending $(A, B)$ to $\Psi_{A \ast B}$,

$$\text{Hom}_{\text{Sset} \times \text{Sset}}(i_*^{\ast}(\Psi_{\phi_K}), (A, B)) \cong \text{Hom}_{\text{Sset}_{[1]}}(\Psi_{A \ast B}, \Psi_K).$$

In particular, for each cylinder $\Psi_K$, there are canonical maps

$$\Psi_{K(0) \sqcup K(1)} \to \Psi_K \to \Psi_{K(0) \ast K(1)}.$$ 

Denote by $\mathcal{C}(A, B)$ the set of cylinders $\Psi_K$ with cobase $K(0) = A$ and base $K(1) = B$. Any $\Psi_K \in \mathcal{C}(A, B)$ is characterized by a simplicial map $\Psi_K \to \Psi_{A \ast B}$ such that $A \sqcup B \to K \to A \ast B$ is the canonical inclusion. Joyal also proves that cylinders coincide with presheaves over the
category product of the category $\Delta[A]$ of simplices of $A$ by the category $\Delta[B]$ of simplices of $B$. A cylinder whose base is $B$ and cobase is a point is called a projective cone over $B$ in [25].

If $P = [1]$, any perverse Eilenberg-MacLane space is a cylinder à la Joyal. We first determine their base and cobase. By Definition 2.13, the base is the non singular part.

**Proposition 6.1.** For any perversity, a perverse Eilenberg-MacLane space over the poset $P = [1]$ is a projective cone over the classical Eilenberg-MacLane space.

**Proof.** With the notation of (6.1), the non singular part of $K(R, n, [1], \mathfrak{P})$ is obtained from the simplices $(-1, i_1)$, with $i_1 \neq -1$, and thus

$$K(R, n, [1], \mathfrak{P})(\emptyset \ast \Delta[i_1]) = \mathbb{Z}^n(N(c\emptyset) \otimes N(\Delta[i_1]) = \mathbb{Z}^n(N(\Delta[i_1])).$$

The result follows from the simplicial definition of Eilenberg-MacLane spaces.

The cobase consists of non regular simplices in the sense of Definition 2.11. The result follows from the equality $\tilde{\Delta}[n] = 0$ for non regular simplices, see Subsection 3.2. \hfill \Box

Base and cobase do not depend on the perversity; we show now that it is the same for the $n$-skeleta of perverse Eilenberg-MacLane spaces.

**Proposition 6.2.** The $n$-skeleton of $K(R, n, [1], \mathfrak{P})$ does not depend on the perversity $\mathfrak{P}$; i.e., for any perversity $\mathfrak{P}$, we have

$$(K(R, n, [1], \mathfrak{P}))^{(n)} = (K(R, n, [1], \mathfrak{P}))^{(n)}.$$

**Proof.** By definition, an $n$-simplex of $K(R, n, [1], \mathfrak{P})$ over $\Delta[a] \ast \Delta[b]$ is a linear combination of $n$-cochains of the shape $1_{c\Delta[a] \ast \Delta[a]}$ with $a + b + 1 = n$. For dimensional reasons, these cochains must “contain” the apex of $c\Delta[a]$. Thus, by Definition 3.5, they are of perverse degree $-\infty$ and the result follows. \hfill \Box

6.2. The perversity $\mathfrak{P}$. We show that the classifying space for the infinite perversity is the final element of the Joyal cylinder $C(\Delta[0], K(R, n))$.

**Proposition 6.3.** There is a homotopy equivalence, $K(R, n, [1], \mathfrak{P}) \simeq \Delta[0] \ast K(R, n)$.

**Proof.** Let $\Psi_K \in \text{Sset}_{[1]}$. The isomorphism (6.3) with $A = \Delta[0]$ and $B = K(R, n)$ implies

$$\text{Hom}_{\text{Sset}_{[1]}}(K(1), K(R, n)) \cong \text{Hom}_{\text{Sset}_{[1]}}(\Psi_K, \Psi_{\Delta[0] \ast K(R, n)}).$$

From $\Psi_K(\Delta[1]) = K(1) \otimes \Delta[1]$, we deduce also an isomorphism between the sets of homotopy classes,

$$[K(1), K(R, n)]_{\text{Sset}} \cong [\Psi_K, \Psi_{\Delta[0] \ast K(R, n)}]_{\text{Sset}_{[1]}}.$$

Recall that $K(1)$ is the completely regular part of $\Psi_K$. Thus, together with Corollary A.5 and the representability of cohomology, we have:

$$\mathcal{H}_{\mathfrak{P}}^n(\Psi_K) \cong H^n(K(1)) \cong [K(1), K(R, n)]_{\text{Sset}} \cong [\Psi_K, \Psi_{\Delta[0] \ast K(R, n)}]_{\text{Sset}_{[1]}}.$$

From the uniqueness up to homotopy equivalence of the representing object, we deduce

$$K(R, n, [1], \mathfrak{P}) \simeq \Psi_{\Delta[0] \ast K(R, n)}.$$

\hfill \Box

**Corollary 6.4.** For any perversity $\mathfrak{P}$, we have $\mathcal{H}_{\mathfrak{P}}^n(K(R, n, [1], \mathfrak{P}); R) = R$.

**Proof.** From Propositions 6.1 and 6.3, we deduce:

$$\mathcal{H}_{\mathfrak{P}}^n(K(R, n, [1], \mathfrak{P}); R) \cong H^n(K(R, n, [1], \mathfrak{P})(1)) \cong H^n(K(R, n)) = R.$$

\hfill \Box
In Proposition 5.11, we prove the nullity of $\mathcal{H}^n_{p}(k(R, n, P, \mathcal{I}); R)$ for $m < n$. Let us study now the $n$-cohomology of the $n$-skelton of $k(R, n, [1], \mathcal{I})$.

**Proposition 6.5.** Let $\mathcal{P}$, $\mathcal{Q}$ be two perversities on $P = [1]$. The $\mathcal{P}$-blown up cohomology in degree $n$ of the $n$-th skeleton, $(k(R, n, [1], \mathcal{I}))^{(n)}$, does not depend on the perversities $\mathcal{P}$ and $\mathcal{Q}$. More precisely, we have

$$\mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}))^{(n)}; R) \cong H^n(K(R, n)^{(n)}; R).$$

**Proof.** With Proposition 6.2, we may choose $\mathcal{Q} = \infty$. We know that $K(R, n, [1], \infty) = \Delta[0] \ast K(R, n)$ and that the $(n-1)$-skeleton of $K(R, n)$ is contractible. By using the exact sequence associated to the pair $((\Delta[0] \ast K(R, n))^{(n)}, \Delta[0] \ast (K(R, n)^{(n-1)}))$, we get

$$\mathcal{H}^n_{\mathcal{P}}((\Delta[0] \ast K(R, n))^{(n)}) \cong \mathcal{H}^n_{\mathcal{P}}((\Delta[0] \ast K(R, n))^{(n)}; R).$$

The excision of the apex implies

$$\mathcal{H}^n_{\mathcal{P}}((\Delta[0] \ast K(R, n))^{(n)}) \cong H^n(K(R, n)^{(n)}; R) \cong H^n(K(R, n)^{(n-1)}).$$

We determine the set of perverse cohomology operations keeping the same cohomological degree for two perversities, $\mathcal{P}$ and $\mathcal{Q}$. The result depends on the respective situation of $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{Z}$. We first establish a lemma.

**Lemma 6.6.** Let $\mathcal{P}, \mathcal{Q} \in \mathcal{Z}$ be two perversities on $P = [1]$. If $\mathcal{Q} \leq \mathcal{P}$, then the inclusion $K(R, n, [1], \mathcal{Q}) \hookrightarrow K(R, n, [1], \mathcal{P})$ and the natural transformation $\mathcal{H}^n_{\mathcal{P}} \to \mathcal{H}^n_{\mathcal{Q}}$ induce injective maps:

$$\mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{P}); R) \hookrightarrow \mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}); R) \hookrightarrow \mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}); R) = R.$$

**Proof.** This is a consequence of Corollary 6.4 and of the commutative diagram,

$$\begin{array}{c}
\mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{P}); R) \ar[d] \ar[r] & \mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}); R) \ar[d] \\
\mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{P}); R) \ar[d] \ar[r] & \mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}); R) \ar[d] \\
\mathcal{H}^n_{\mathcal{Q}}(k(R, n, [1], \mathcal{Q}); R) = R \ar[r] & \mathcal{H}^n_{\mathcal{Q}}(k(R, n, [1], \mathcal{Q}); R),
\end{array}$$

where the isomorphisms of the right-hand column come from Proposition 6.5.

**Proposition 6.7.** Let $\mathcal{P}, \mathcal{Q} \in \mathcal{Z}$ be two perversities on $P = [1]$. If $\mathcal{Q} \leq \mathcal{P}$, we have,

$$\text{Nat}_R(\mathcal{H}^n_{\mathcal{P}}, \mathcal{H}^n_{\mathcal{Q}}) = [k(R, n, [1], \mathcal{Q}), k(R, n, [1], \mathcal{P})] = \mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}); R) = R.$$

**Proof.** In the left-hand column of (6.4) with $\mathcal{P} = \mathcal{Q}$, the unit $1 \in \mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}); R) = R$ corresponds to the homotopy class of the canonical inclusion $k(R, n, [1], \mathcal{P}) \hookrightarrow k(R, n, [1], \mathcal{P})$. This is the image of the homotopy class of the identity map on $k(R, n, [1], \mathcal{P})$ by the natural transformation $\mathcal{H}^n_{\mathcal{P}}(-) = [-, k(R, n, [1], \mathcal{P})] \to \mathcal{H}^n_{\mathcal{Q}}(-) = [-, k(R, n, [1], \mathcal{Q})]$. Thus the unit 1 is reached and $\mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{P}); R) = R$. The result follows now from Lemma 6.6.

In contrast with Proposition 6.7, we have:

**Proposition 6.8.** Let $\mathcal{P}, \mathcal{Q} \in \mathcal{Z}$ be two perversities on $P = [1]$. If $\mathcal{Q} \leq \mathcal{P} < \infty$, then, for any $n > 0$, we have

$$\text{Nat}_R(\mathcal{H}^n_{\mathcal{P}}, \mathcal{H}^n_{\mathcal{Q}}) = [k(R, n, [1], \mathcal{Q}), k(R, n, [1], \mathcal{P})] = \mathcal{H}^n_{\mathcal{P}}(k(R, n, [1], \mathcal{Q}); R) = 0.$$
Proof. Let $[\omega] \in \mathcal{H}_F^n(K(R, n, [1], \overline{\theta}); R)$. Its image in $\mathcal{H}_F^n(K(R, n, [1], \overline{\theta}); R) = R$ is an element $\lambda \in R$. Consider $X = S^{n-q} \times cS^q$ and translate $[\omega]$ and $\lambda$ in terms of maps between the cohomology groups. The commutativity of the diagram

$$\mathcal{H}_F^n(X) = R \xrightarrow{[\omega]} \mathcal{H}_F^n(X) = 0 \xrightarrow{\approx}$$

$$\mathcal{H}_F^n(X) = R \xrightarrow{\lambda} \mathcal{H}_F^n(X) = R$$

implies $\lambda = 0$. The injectivity of $\mathcal{H}_F^n(K(R, n, [1], \overline{\theta}); R) \to \mathcal{H}_F^n(K(R, n, [1], \overline{\theta}); R)$ established in Lemma 6.6 gives the conclusion. \hfill \square

6.3. The perversity $\overline{\theta}$. We determine the classifying space associated to the zero perversity.

**Proposition 6.9.** The perverse Eilenberg-MacLane space $K(R, n, [1], \overline{\theta})$ is homotopically equivalent to

$$(R \circ \mathcal{R})(\Delta[1] \times K(R, n)) = (\Delta[1] \times K(R, n))/(\Delta[0] \times K(R, n)).$$

On the subcategory of $\text{Sset}_{[1]}$ formed of normal simplicial sets over $[1]$, the classifying space of the blown up cohomology reduces to $\Delta[1] \times K(R, n)$.

**Proof.** Let $\Psi_K \in \text{Sset}_{[1]}$. From (4.3), Proposition A.5 and [8, Propositions 1.57 and 1.60], we get

$$\mathcal{H}_F^n(\Psi_K; R) \cong \mathcal{H}_F^n((R \circ \mathcal{R})(\Psi_K); R) \cong H^n(\mathcal{U}(\mathcal{R}(\Psi_K); R),$$

where $\mathcal{U}: \text{Sset}_\theta \to \text{Sset}$ is the forgetful functor. Therefore, from the representability of the cohomology in $\text{Sset}$, Proposition 2.10 and Proposition 2.22, we deduce

$$\mathcal{H}_F^n(\Psi_L) \cong \mathcal{H}_F^n(\Delta[1] \times K(R, n))_{\text{Sset}} \cong ([R(\mathcal{R}(\Psi_L), \Delta[1] \times K(R, n)]_{\text{Sset}_{[1]}}$$

$$\cong [R(\mathcal{R}(\Psi_L), (\Delta[1] \times K(R, n)))]_{\text{Sset}_{[1]}} \cong [\Psi_L, n(\mathcal{R}(\Delta[1] \times K(R, n)))]_{\text{Sset}_{[1]}}].$$

The next result is a first step in the direction of Conjecture A.

**Theorem 6.10.** For any positive perversity $\overline{\theta}$, there are isomorphisms,

$$\mathcal{H}_F^n(K(R, n, [1], \overline{\theta}); R) \cong \text{Nat}_R(\mathcal{H}_F^n, \mathcal{H}_F^k) \cong H^k(K(R, n); R).$$

Before giving the proof, we need two lemmas. For sake of simplicity, we denote

$$Q(R, n) = ((\Delta[0] \ast \Delta[0]) \times K(R, n))/(\Delta[0] \ast \emptyset) \times K(R, n)),$$

the simplicial set given by the pushout

$$\begin{array}{ccc}
(\Delta[0] \ast \emptyset) \times K(R, n) & \xrightarrow{\partial} & (\Delta[0] \ast \Delta[0]) \times K(R, n) \\
\Delta[0] \ast \emptyset & \xrightarrow{\pi} & Q(R, n).
\end{array}$$

In Proposition 6.9, we have proven that $Q(R, n)$ is homotopically equivalent to the Eilenberg-MacLane space, $K(R, n, [1], \overline{0})$. The next lemma expresses the fact that the identification of non regular simplices to a point has no influence on the blown up cohomology, due to the equality $\bar{N}^*_\overline{\theta}(-) = 0$ on these elements.

**Lemma 6.11.** The canonical surjection $\pi$ in the diagram (6.5) induces an isomorphism of cochain complexes

$$\pi^*: \bar{N}^*_\overline{\theta}(Q(R, n); R) \cong \bar{N}^*_\overline{\theta}((\Delta[0] \ast \Delta[0]) \times K(R, n); R).$$
Proof. As the functor $\tilde{N}^k_P(-)$ takes the value 0 on the non regular simplices and sends inductive limits on projective limits, the result follows from the following pullback,

$$
\begin{array}{ccc}
0 = \tilde{N}^k_P((\Delta[0] \times \emptyset) \times K(R, n)) & \xrightarrow{\pi^*} & \tilde{N}^k_P((\Delta[0] \times \Delta[0]) \times K(R, n)) \\
\uparrow & & \uparrow \\
0 = \tilde{N}^k_P(\Delta[0] \times \emptyset) & \xrightarrow{\pi^*} & \tilde{N}^k_P(\Delta(R, n)).
\end{array}
$$

The second lemma introduces contractible simplicial $R$-modules.

**Lemma 6.12.** Let $k$ be a positive integer, $\mathcal{P}$ a perversity on a poset $\mathcal{P}$ and $\Psi \in \text{Sset}_\mathcal{P}$. Then the simplicial $R$-module, defined by $n \mapsto \tilde{N}^k_P(\Psi \otimes \Delta[n])$, is contractible.

Proof. The simplicial set $\tilde{N}^k_P(\Psi \otimes \Delta[\bullet])$, being a simplicial abelian group, is Kan. Thus it is sufficient to prove that its homotopy groups $\pi_i(-)$ are trivial. For that, we use [4, Proposition 1], where two cases are considered.

- For $n = 0$, we consider $x \in \tilde{N}^k_P(\Psi \otimes \Delta[0])$. We have to prove that there exists $y \in \tilde{N}^k_P(\Psi \otimes \Delta[1])$ with $d_0y = x$ and $d_1y = 0$, where $d_0, d_1: \tilde{N}^k_P(\Psi \otimes \Delta[1]) \to \tilde{N}^k_P(\Psi \otimes \Delta[0])$ are the face operators. Let $\Delta[1] = [e_0, e_1]$ and $\alpha = e_1 \in N^0(\Delta[1])$ the cochain with values 0 on $e_0$ and 1 on $e_1$. We denote $\omega = \pi^*(\alpha) \in N^0(\Psi \otimes \Delta[1])$ its pullback by the projection $\pi: \Psi \otimes \Delta[1] \to \Delta[1]$. We set

$$
y = \omega \sim s_0x \in \tilde{N}^k_P(\Psi \otimes \Delta[1]),
$$

where $s_0: \tilde{N}^k_P(\Psi \otimes \Delta[0]) \to \tilde{N}^k_P(\Psi \otimes \Delta[1])$ is the 0-degeneracy map. With these choices, from $d_1\omega = \pi^*(d_1\alpha) = 0$, $d_0\omega = \pi^*(d_0\alpha) = 1$ and the simplicial identities, we have $d_1y = 0$ and $d_0y = x$, as expected.

- If $n > 0$, we have to prove that the homotopy groups are trivial. Let $x \in \tilde{N}^k_P(\Psi \otimes \Delta[n])$ with all face restrictions $d_1x \in \tilde{N}^k_P(\Psi \otimes \Delta[n-1])$ trivial. Set $\Delta[n] = [e_1, \ldots, e_{n+1}]$ and $\Delta[n+1] = [e_0, e_1, \ldots, e_{n+1}]$. We choose $\alpha = \sum_{i > 0} e_i \in N^*(\Delta[n+1])$ and $\omega = \pi^*(\alpha)$ where $\pi: \Psi \otimes \Delta[n+1] \to \Delta[n+1]$ is the canonical projection. We set

$$
y = \omega \sim s_0x \in \tilde{N}^k_P(\Psi \otimes \Delta[n+1]).
$$

From $d_0s_0 = \text{id}$, $d_0\omega = \pi^*(d_0\alpha)$ and $d_0\alpha = 1$, we deduce $d_0y = x$. We now have to prove

$$
d_1y = d_1\omega \sim d_1s_0(x) = 0 \text{ for } i > 0.
$$

If $i > 1$, then we have $d_is_0x = s_0d_{i-1}x = 0$, by hypothesis on $x$. We still have to consider the case $i = 1$ which corresponds to

$$
d_1y = d_1\omega \sim d_1s_0x = \pi^*(d_1\alpha) \sim x.
$$

Let us notice that the 0-cochain $d_1\alpha$ is the restriction of $\alpha$ to the face $[e_0, e_2, \ldots, e_{n+1}]$ and that $\alpha(e_0) = 0$. In the cup product $\pi^*(d_1\alpha) \sim x$, the first term is evaluated on the first vertex, thus $d_1y = 0$. We have established (6.6). □

**Proof of Theorem 6.10.** Let $Z \in \text{Sset}$. The two cochain maps, $p_1^*$ and $p_2^*$, induced by the projections, $p_1: (\Delta[0] \times \Delta[0]) \times Z \to \Delta[0] \times \Delta[0]$ and $p_2: (\Delta[0] \times \Delta[0]) \times Z \to Z$, and the cup product,
\[ \sim, \quad \text{on } \tilde{N}^*(\mathcal{E}) \text{ give a cochain map } \chi = \sim \circ (p_1^* \otimes p_2^*). \] We compose it with the canonical injection to obtain a natural transformation,

\[ \theta_{\sim} : N^*(-) \xrightarrow{\sim} \tilde{N}^*(\mathcal{E}) \otimes R N^*(-) \xrightarrow{\chi} \tilde{N}^*(\mathcal{E}) \otimes (-), \]

between the functors \( N^*(-) \) and \( G(-) = \tilde{N}^*(\mathcal{E}) \otimes (-) \), from \( \text{Sset}^{op} \) to \( \text{M}_{\text{dg}} \). The theorem is proven if we show that \( \theta_{\sim} \) is a quasi-isomorphism for any \( Z \in \text{Sset} \). For that, we use [12, Proposition 3.1.14] as “a theorem of acyclic models.”

- A first step is to establish that the two functors send inductive limits on limits and cofibrations on epimorphisms. The only point which needs a proof is that \( G \) sends cofibrations on epimorphisms. For that, we consider the cochain complex, \( D(n)^* \), defined by

\[ D(n)^k = \begin{cases} R & \text{if } k = n, n + 1, \\ 0 & \text{otherwise}, \end{cases} \]

and an isomorphism \( d : D(n)^n \to D(n)^{n+1} \) as differential. We first notice that, for any cochain complex, \( C^* \), one has \( \text{Hom}_{\text{M}_{\text{dg}}}(D(n)^*, C^*) = C^n \). Let \( \Psi_L \in \text{Sset}_P \), \( Y \in \text{Sset} \). With the notations of Lemma 6.12, there are isomorphisms,

\[
\begin{align*}
\text{Hom}_{\text{Sset}}(Y, \tilde{N}^\mathcal{E}(\Psi_L \otimes \Delta[\bullet])) & \cong \text{Hom}_{\text{Sset}}(Y, \text{Hom}_{\text{M}_{\text{dg}}}(D(n), \tilde{N}^\mathcal{E}(\Psi_L \otimes \Delta[\bullet]))) \\
& \cong \text{Hom}_{\text{Sset}}(Y, \text{Hom}_{\text{Sset}}(\Psi_L \otimes \Delta[\bullet], (D(n))^\mathcal{E})) \\
& \cong \text{Hom}_{\text{Sset}}(Y, \text{Hom}_{\text{Sset}}(\Psi_L, (D(n))^\mathcal{E})) \\
& \cong \text{Hom}_{\text{Sset}}(Y, \Psi_L \otimes Y).
\end{align*}
\]

(6.7)

(6.8)

(6.9)

where (6.7) comes from Proposition 4.1, (6.8) from Remark 2.3 and (6.9) from Definition B.1.

The last isomorphism allows the determination of the set of 0-simplices,

\[
\begin{align*}
\text{Hom}_{\text{Sset}}(Y, \tilde{N}^\mathcal{E}(\Psi_L \otimes \Delta[\bullet]))_0 & \cong \text{Hom}_{\text{Sset}}(\Psi_L \otimes Y, (D(n))^\mathcal{E}) \\
& \cong \text{Hom}_{\text{M}_{\text{dg}}}(D(n), \tilde{N}^\mathcal{E}(\Psi_L \otimes Y)) \\
& \cong \tilde{N}^\mathcal{E}(\Psi_L \otimes Y).
\end{align*}
\]

(6.10)

Let \( j : X \to Y \) be a cofibration in \( \text{Sset} \). As \( \tilde{N}^\mathcal{E}(\Psi_L \otimes \Delta[\bullet]) \) is a Kan, contractible simplicial set, any diagram as below admits a dot extension,

\[
\begin{tikzcd}
X & \tilde{N}^\mathcal{E}(\Psi_L \otimes \Delta[\bullet]) \\
& Y
\end{tikzcd}
\]

This implies the surjectivity of the map obtained by composition with \( j \),

\[ j^\mathcal{E} : \text{Hom}_{\text{Sset}}(Y, \tilde{N}^\mathcal{E}(\Psi_L \otimes \Delta[\bullet])) \to \text{Hom}_{\text{Sset}}(X, \tilde{N}^\mathcal{E}(\Psi_L \otimes \Delta[\bullet])). \]

From the surjectivity at the level of the 0-simplices and (6.10), we deduce the surjectivity of

\[ (\text{id} \otimes j)^\mathcal{E} : \tilde{N}^\mathcal{E}(\Psi_L \otimes Y) \to \tilde{N}^\mathcal{E}(\Psi_L \otimes X) \]

for any \( \Psi_L \in \text{Sset}_P \). Setting \( \Psi_L = \Delta[0] \otimes \Delta[0] \) gives the desired result for the functor \( G \).

- After this first step, we can apply [12, Proposition 3.1.14]. The proof is thus reduced to the verification that \( \theta_{\Delta[m]} \) is a quasi-isomorphism for any \( m \). More specifically, we have to verify that

\[ N^*(\Delta[m]) \to \tilde{N}^\mathcal{E}((\Delta[0] \otimes \Delta[0]) \otimes \Delta[m]) \]

is a quasi-isomorphism, which is clearly the case. (Let us recall that \( \mathcal{H}^\mathcal{E}((\Delta[0] \otimes \Delta[0]) \) is isomorphic to \( H^*(\text{pt}). \)) \( \square \)
Appendix A. Filtered face sets

In [8], we study the blown up cohomology (called TW-cohomology) by using filtered face sets, in the spirit of the $\Delta$-sets of Rourke and Sanderson ([35]). In this section, we compare the blown up cohomology thus obtained with the blown up cohomology of Section 4.

Let $P$ be a poset. Let us denote by $\Delta[P]_{\text{Face}}$ the subcategory of $\Delta[P]$ whose morphisms (2.1) come from injective maps $\Delta[k] \to \Delta[l]$.

Definition A.1. A filtered face set over $P$ is a presheaf on the category $\Delta[P]_{\text{Face}}$; i.e., a functor $\Psi_T: (\Delta[P]_{\text{Face}})^{op} \to \text{Set}$. We denote $\text{Ffs}_P$ the category of natural transformations between filtered face sets over $P$.

Let $\Psi_T \in \text{Ffs}_P$ and $\Psi_L \in \text{Sset}_P$. We define a filtered face set over $P$, $\mathcal{O}_P(\Psi_L) \in \text{Ffs}_P$, by restriction and a simplicial set over $P$, $F\Psi_T(\Psi_T) \in \text{Sset}_P$, by left Kan extension as we do in Definition 2.16. These constructions are extended in functors and the functor $F\Psi_T$ is left-adjoint to $\mathcal{O}_P$.

We take over the presentation of Subsection 4.1 for the blown up cochains on filtered face sets introduced in [8]. For any $\sigma \in \mathcal{N}(P)$ and $k \in \mathbb{N}$, we have set $\mathcal{N}_P(\sigma, k) = N^k_P(\Delta[\sigma])$ and $\tilde{N}^k_P(\Psi_L) = \text{Hom}_{\text{Sset}}(\Psi_L, \Delta^k_P(\bullet, k))$ for $\Psi_L \in \text{Sset}_P$. Let $\Psi_T \in \text{Ffs}_P$ and $M \in \text{M}_{\text{dg}}$, we set

$$\tilde{N}^{\Delta[k]}_P(\Psi_T) = \text{Hom}_{\text{Ffs}_P}(\Psi_T, \mathcal{N}_P(\bullet, k))$$

and

$$(M)\tilde{\Delta} = \lim_{\sigma \in \Delta[\mathcal{P}]_{\text{Face}}} \text{Hom}_{\text{M}_{\text{dg}}}(M, \mathcal{N}_P(\sigma, \bullet)).$$

We summarize these data in the following diagram composed of three pairs of adjoint functors:

\[
\begin{array}{ccc}
\text{Sset}_P & \xrightarrow{\mathcal{O}_P} & \text{Ffs}_P \\
\downarrow F_p & & \downarrow \tilde{N}_P \quad \tilde{N}^{\Delta}_P \\
\text{M}_{\text{dg}} & \xleftarrow{(-)\tilde{\Delta}} & \text{Ffs}_P
\end{array}
\]

Remark A.2. Let $\Psi_T \in \text{Ffs}_P$ and $\Delta[J] \in \mathcal{N}(P)$. We set

$$\mathcal{R}_P(\Psi_T)(\Delta[J]) = \text{Hom}_{\text{Ffs}_P}(\mathcal{O}_P(\Delta[J]), \Psi_T).$$

This definition extends in a functor $\mathcal{R}_P: \text{Ffs}_P \to \text{Sset}_P$, which is a right adjoint to $\mathcal{O}_P$. We do not use the functor $\mathcal{R}_P$ in this work.

Proposition A.3. Let $P$ be a poset, $\overline{P}$ a perversity on $P$, $\Psi_L \in \text{Sset}_P$ and $\Psi_T \in \text{Ffs}_P$. By denoting $\mathcal{H}_P^{\Delta}(-)$ the homology of $\tilde{N}^{\Delta}_P(-)$, there are natural isomorphisms,

$$\mathcal{H}_P^{\Delta}(-)(R) \cong \mathcal{H}_P^{\Delta}(F_P(\Psi_T); R) \quad \text{and} \quad \mathcal{H}_P^{\Delta}(\Psi_L; R) \cong \mathcal{H}_P^{\Delta}(\mathcal{O}_P(\Psi_L); R).$$

Proof. Let $\Psi_T \in \text{Ffs}_P$. The functor $F_P$ is defined as a direct limit, $F_P(\Psi_T) = \lim_{\Delta[\mathcal{P}]_{\text{Face}}} \Delta[J] \to \Psi_T \Delta[J]$, and we have a natural isomorphism at the level of the complexes, $\tilde{N}^{\Delta}_P(-) \cong \tilde{N}^{\Delta}_P(F_P(\Psi_T))$.

Let $\Psi_L \in \text{Sset}_P$. The second isomorphism is not so direct since $\tilde{N}^{\Delta}_P(\Psi_L) \neq \tilde{N}^{\Delta}_P(\mathcal{O}_P(\Psi_L))$: for instance, if $P = \{0\}$ and $L = \Delta[0]$, then $\tilde{N}^{\Delta}_P(\Delta[0])$ is the complex of normalized cochains and $\tilde{N}^{\Delta}_P(\mathcal{O}_P(\Delta[0]))$ is the complex of non normalized cochains.
Denote $S(\Psi_L) = \{ \sigma : \Delta[J] \to \Psi_L \mid \Delta[J] \in \mathbb{N}(P) \}$. The functor $O_{\Psi}$ being compatible with direct limits, we have

$$O_{\Psi}(\Psi_L) = \lim_{\sigma \in S(\Psi_L)} O_{\Psi}(\Delta[J]).$$

Applying the functor $\tilde{N}^{\Delta.*}(-)$ which sends direct limits to inverse limits, we get

$$\tilde{N}^{\Delta.*}(O_{\Psi}(\Psi_L)) = \lim_{\sigma \in \tilde{S}(\Psi_L)} \tilde{N}^{\Delta.*}(O_{\Psi}(\Delta[J])).$$

On the other hand, we have $\tilde{N}^{\Delta.*}(O_{\Psi}(\Delta[J]))$.

Thus, we are reduced to prove the existence of a natural homotopy equivalence between $\tilde{N}^{\Delta.*}(\Delta[J])$ and $\tilde{N}^{\Delta.*}(O_{\Psi}(\Delta[J]))$, for $\Delta[J] \in \mathbb{N}(P)$.

If $L \in \text{Sset}$, we denote $N^*(L)$ the normalized cochain complex and $C^*(L)$ the non normalized one. There exists a natural homotopy equivalence between them, $(f, g, H)$,

$$N^*(L) \xrightarrow{g \circ f} C^*(L), \quad f \circ g = \text{id}_{N^*(L)},$$

and $H$ a natural homotopy between $g \circ f$ and $\text{id}_{C^*(L)}$.

First, we consider the global complexes $\tilde{N}^{\Delta.*}$ and $\tilde{N}^*$ without referring to a perversity. By definition, for $\Delta[J] = \Delta[j_0] * \cdots * \Delta[j_{n-1}] * \Delta[j_n]$, we have

$$\tilde{N}^{\Delta.*}(O_{\Psi}(\Delta[J])) = \tilde{N}^*(\Delta[j_0]) \otimes \cdots \otimes \tilde{N}^*(\Delta[j_{n-1}]) \otimes C^*(\Delta[j_n]),$$

with $\tilde{N}^*(\Delta[j_k]) = \lim_{(\Delta[i] \to \Delta[j_k])} N^*(c\Delta[i])$. As the apex of $c\Delta[i]$ does not appear in the limit, we do not have $\tilde{N}^*(\Delta[j_k]) = C^*(c\Delta[j_k])$. To manage with the apex, we consider the cone $c\Delta[i]$ as the direct limit of

$$\xymatrix{
\ast \\
\Delta[i] \\
\Delta[i] \ast 1
}$$

Applying the normalized functor, we obtain $N^*(c\Delta[i])$ as the inverse limit of

$$\xymatrix{
R \\
\tilde{N}^*(\Delta[i]) \\
\tilde{N}^*(\Delta[i] \times \Delta[1]).
}$$

Using the commutation of direct and inverse limits, we can write $\tilde{N}^*(\Delta[j_k])$ as the pullback of

$$\xymatrix{
R \\
\tilde{N}^*(\Delta[j_k]) \\
\lim_{(\Delta[i] \to \Delta[j_k])} N^*(\Delta[i] \times \Delta[1]).
}$$

By applying the Eilenberg-Zilber theorem and the Alexander-Whitney map, we get a natural homotopy equivalence between $\tilde{N}^*(\Delta[j_k])$ and the pullback of

$$\xymatrix{
R \\
\tilde{N}^*(\Delta[j_k]) \\
\lim_{(\Delta[i] \to \Delta[j_k])} N^*(\Delta[i] \otimes N^*(\Delta[1]).
}$$

Let us notice that the right hand expression is the tensor product $C^*(\Delta[j_k]) \otimes N^*(\Delta[1])$. Using the natural homotopy equivalence, $(f, g, H)$, the Eilenberg-Zilber theorem and the Alexander-Whitney map, we have a natural homotopy equivalence between $\tilde{N}^{\Delta.*}(\Delta[j_k])$ and the pullback of

$$\xymatrix{
R \\
C^*(\Delta[j_k]) \\
C^*(\Delta[j_k] \times \Delta[1]).
}$$

As non normalized cochains send direct limits on inverse limits, we have a natural homotopy equivalence between $C^*(c\Delta[j_k])$ and $\tilde{N}^*(\Delta[j_k])$ and thus a natural homotopy equivalence between $N^*(\Delta[J])$ and $\tilde{N}^{\Delta.*}(O_{\Psi}(\Delta[J]))$.

Finally, as the pervers degree is the sum of the degrees of some factors of the tensor product, which are preserved all along the previous process, we get a natural homotopy equivalence between $\tilde{N}^{\Delta.*}(\Delta[J])$ and $\tilde{N}^{\Delta.*}(O_{\Psi}(\Delta[J]))$, as expected. \qed
The determination of the –intersection cohomology of a filtered space as the ordinary singular cohomology of the space needs an hypothesis of normality (see [19]). This is also true for filtered face sets and we send the reader to [8, Subsection 1.5] for more details. As the reference [8] is written for \( P = [n] \), we restrict to this case for the end of this section.

**Proposition A.4.** Let \( P = [n] \) and \( \Psi_L \in \text{Sset}_P \). The map \(( \mathcal{O} \circ \mathcal{I} \circ \mathcal{R}) (\Psi_L) \to \mathcal{O}(\Psi_L)\), coming from the adjunction \((\mathcal{I}, \mathcal{R})\), is a normalization of the filtered face set \( \mathcal{O}(\Psi_L) \).

**Proof.** We refer to [8, Definitions 1.55 and 1.59] for the definitions of normal filtered face set and normalization. Set \( T = \mathcal{O}(\Psi_L) \). By definition, the expression \( \mathcal{O}(\mathcal{R}(\Psi_L)) \) corresponds to \( T^+ \) in [8]. Recall that the functor \( \mathcal{I} \) is constructed from a left Kan extension. Therefore, any simplex in \((\mathcal{I} \circ \mathcal{R})(\Psi_L)\) which is not in \((\mathcal{R} \circ \mathcal{I} \circ \mathcal{R})(\Psi_L) = \mathcal{R}(\Psi_L)\) is a non regular face of a simplex in \( \mathcal{R}(\Psi_L) \).

This gives condition (a) of [8, Definition 1.55]. The unicity condition (b) of [8, Definition 1.55] comes from the fact that all added faces by \( \mathcal{I} \) are distinct. Finally, the adaptation of the equality \((\mathcal{R} \circ \mathcal{I} \circ \mathcal{R})(\Psi_L) = \mathcal{R}(\Psi_L)\) to the notations of [8] is \( ((\mathcal{O} \circ \mathcal{I} \circ \mathcal{R})(\Psi_L))_+ = (\mathcal{O}(\Psi_L))_+ \), which is the required property of a normalization. \( \square \)

From the Propositions A.3, A.4 and from [8, Propositions 1.54, 1.57 and 1.60], we deduce immediately the following result.

**Corollary A.5.** Let \( R \) be a Dedekind domain, \( \mathfrak{p} \) a perversity, \( P = [n] \) and \( \Psi_L \in \text{Sset}_P \). Then there exist isomorphisms, and \( \mathcal{H}_\mathfrak{p}^\text{reg} (\mathcal{H}_{\mathcal{O}} (\Psi_L); R) \cong \mathcal{H}_\mathfrak{p}^\text{reg} (\mathcal{H}_{\mathcal{O}} (\mathcal{O}(\Psi_L)); R) \cong H^* (L^{reg}; R) \).

In the statement of [8, Proposition 1.57], the ring \( R \) is required to be principal, but this hypothesis is only used for the existence of a universal coefficient formula which also exists for Dedekind domains.

### Appendix B. Simplicial category

**B.1. Definitions.** Recall from [33, Chapter II] ([16, Chapter 2] or [29, Appendix A.1]) the following definition.

**Definition B.1.** A category \( \mathcal{C} \) is a simplicial category if there is a mapping space functor \( \text{Hom}^\Delta_\mathcal{C} (\mathcal{C}^{op} \times \mathcal{C}) \to \text{Sset} \), satisfying the following properties for \( A \) and \( B \) objects in \( \mathcal{C} \), \( K \) and \( L \) in \( \text{Sset} \).

(i) \( \text{Hom}^\Delta_\mathcal{C} (A, B)_0 = \text{Hom}_\mathcal{C} (A, B) \).

(ii) The functor \( \text{Hom}^\Delta_\mathcal{C} (A, -) : \mathcal{C} \to \text{Sset} \) has a left adjoint, \( A \otimes - : \text{Sset} \to \mathcal{C} \); i.e.

\[
\text{Hom}^\Delta_\mathcal{C} (A \otimes K, B) \cong \text{Hom}_\text{Sset}^\Delta (K, \text{Hom}^\Delta_\mathcal{C} (A, B)),
\]

which is associative in the sense there is an isomorphism, \( A \otimes (K \otimes L) \cong (A \otimes K) \otimes L \), natural in \( A \in \mathcal{C} \) and \( K, L \in \text{Sset} \).

(iii) The functor \( \text{Hom}^\Delta_\mathcal{C} (-, B) : \mathcal{C}^{op} \to \text{Sset} \) has a left adjoint \( B^- : \text{Sset} \to \mathcal{C}^{op} \); i.e.,

\[
\text{Hom}_\text{Sset}^\Delta (K, \text{Hom}^\Delta_\mathcal{C} (A, B)) \cong \text{Hom}^\Delta_\mathcal{C} (A, B^K),
\]

for any \( K \in \text{Sset} \) and \( A, B \in \mathcal{C} \).

With the previous notation, for all \( n \geq 0 \), we have

\[
\text{Hom}^\Delta_\mathcal{C} (A, B)_n = \text{Hom}_\mathcal{C} (A \otimes \Delta [n], B).
\]

Let \( K \) be a simplicial set. Two elements \( x, y \in K_0 \) are strictly homotopic ([33, Section II.1]) if there exists \( z \in K_1 \) with \( d_0 z = x \) and \( d_1 z = y \). The notion of homotopy is the generated equivalence relation, denoted \( \sim \). Thus, in a simplicial category \( \mathcal{C} \), we have homotopy classes defined by \( [A, B] = \pi_0 \text{Hom}^\Delta_\mathcal{C} (A, B) \). More specifically, let \( f, g : A \to B \in \mathcal{C} \). By definition, we
have $f \sim g$ if there exists $H$ (resp. $H'$) such that the left-hand (resp. right-hand) following diagram commutes,

\[
\begin{array}{ccc}
A & \xrightarrow{f \times g} & B^\Delta[1] \\
\downarrow & \downarrow & \downarrow \\
H & \cong & B \times B,
\end{array}
\quad
\begin{array}{ccc}
A \otimes \Delta[1] & \xrightarrow{H'} & B \\
\downarrow & \downarrow & \downarrow \\
A \otimes \partial \Delta[1] & \xrightarrow{(f,g)} & B.
\end{array}
\]

**Definition B.2.** Let $\mathcal{C}$ be a simplicial category. An object $A$ of a simplicial category $\mathcal{C}$ is s-fibrant if, for any object $Z$ of $\mathcal{C}$, the simplicial set $\text{Hom}_{\Delta \mathcal{C}}(Z,A)$ is a Kan complex. A map $f: A \to B \in \mathcal{C}$ is a weak s-equivalence if for any object $Z$ of $\mathcal{C}$, there is an isomorphism, $\pi_0 \text{Hom}_{\Delta \mathcal{C}}(Z, A) \xrightarrow{\cong} \pi_0 \text{Hom}_{\Delta \mathcal{C}}(Z, B)$.

**Definition B.3.** The homotopy category, $\text{Ho-s-}\mathcal{C}$, associated to a simplicial category, $\mathcal{C}$, has

- for objects, the s-fibrants objects of $\mathcal{C}$,
- for morphisms, the connected components of morphisms of $\mathcal{C}$; i.e.,

\[
[A,B] = \pi_0 \text{Hom}_{\Delta \mathcal{C}}(A,B).
\]

Two objects of $\mathcal{C}$ are homotopically equivalent, denoted by $A \simeq_{\mathcal{C}} B$, if there exist two morphisms of $\mathcal{C}$, $f: A \to B$ and $g: B \to A$ such that $f \circ g$ is homotopic to $\text{id}_B$ and $g \circ f$ homotopic to $\text{id}_A$.

In the category $\text{Sset}$, the notions of s-fibrant objects and of weak s-equivalences coincide with those of the Kan closed model structure. In particular, $\text{Ho-s-}\mathcal{C}$ is the localisation at the weak homotopy equivalences.

**Proposition B.4.** Let $\mathcal{C}$ be a simplicial category. A weak s-equivalence between s-fibrant objects of $\mathcal{C}$ is a homotopy equivalence in $\mathcal{C}$.

**Proof.** Let $f: A \to B \in \mathcal{C}$ be a weak s-equivalence with $A$ and $B$ s-fibrant. A right inverse up to homotopy of $f$ is the map $g$ given by the surjectivity of $f_*: [Z,A] \to [Z,B]$ applied to the identity on $B$,

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \downarrow & \downarrow \\
B & \xrightarrow{\text{id}} & B
\end{array}
\]

Now, the injectivity of $f_*: [A,A] \to [A,B]$ gives $g \circ f$ homotopic to $\text{id}_A$. \hfill \Box

**B.2. Infinite loop space.** Let $\mathcal{C}$ be a complete and cocomplete simplicial category. We denote $\ast$ the final object of $\mathcal{C}$. We now introduce the notion of based loop space in $\mathcal{C}$.

**Definition B.5.** A pointed object of $\mathcal{C}$ is a couple $(X, \epsilon)$ of an object $X$ of $\mathcal{C}$ and a morphism $\epsilon: \ast \to X$. The pointed loop space $\Omega_* X$ of $(X, \epsilon)$ is the pull-back

\[
\begin{array}{ccc}
\Omega_* X & \xrightarrow{\text{eval}_0, \text{eval}_1} & X^\Delta^1 \\
\downarrow & \downarrow & \downarrow \\
\ast & \xrightarrow{(\epsilon, \epsilon)} & X^\ast \cong X \times X.
\end{array}
\]

\[\text{(B.1)}\]
Let $Z$ be an object of $\mathcal{C}$, we apply the functor $\text{Hom}_C^\Delta(Z, -)$ to the previous diagram and get the pullback,

(B.2)\[
\begin{array}{ccc}
\text{Hom}^\Delta_C(Z, \Omega, X) & \longrightarrow & \text{Hom}^\Delta_C(Z, X^{\Delta[1]}) \\
\downarrow & & \downarrow \\
\text{Hom}^\Delta_C(Z, \ast) \times \text{Hom}^\Delta_C(Z, (\text{eval}_0, \text{eval}_1)) & \longrightarrow & \text{Hom}^\Delta_C(Z, X^{\partial\Delta[1]}).
\end{array}
\]

By using Definition B.1.(iii), the right-hand vertical map is induced by the canonical inclusion $\iota: \partial\Delta[1] \hookrightarrow \Delta[1]$, up to isomorphisms,

\[
\begin{array}{ccc}
\text{Hom}^\Delta_C(Z, X^{\Delta[1]}) & \cong & \text{Hom}^\Delta_{\text{Set}}(\Delta[1], \text{Hom}^\Delta_C(Z, X)) \\
\downarrow & & \downarrow \iota^* \\
\text{Hom}^\Delta_C(Z, X^{\partial\Delta[1]}) & \cong & \text{Hom}^\Delta_{\text{Set}}(\partial\Delta[1], \text{Hom}^\Delta_C(Z, X)).
\end{array}
\]

Suppose $X$ is s-fibrant, then the simplicial set $\text{Hom}^\Delta_C(Z, X)$ is Kan and $\iota^*$ is a Kan fibration ([16, Corollary 5.3]). As the diagram (B.2) is a pull-back, with $\text{Hom}^\Delta_C(Z, \ast) \cong \Delta[0]$, we obtain that $\text{Hom}^\Delta_C(Z, \Omega, X)$ is the fiber of a Kan fibration, thus it is a Kan simplicial set. Finally, we have proven that $\Omega, X$ is s-fibrant if $X$ is too.

**Definition B.6.** An infinite loop space in $\mathcal{C}$ is a sequence of s-fibrant pointed objects of $\mathcal{C}$, $\{(B, \epsilon_i)\}_{i \in \mathbb{N}}$, such that $B_i$ is weakly s-equivalent to $\Omega, B_{i+1}$.

**Remark B.7.** Let $\{(B, \epsilon_i)\}_{i \in \mathbb{N}}$ be an infinite loop space in $\mathcal{C}$. We can define a cohomological functor, $B: C^{\text{op}} \to \text{Ab-gr}$, with values in the category of graded abelian groups, by $A \mapsto \mathcal{B}^i(A) = \pi_0\text{Hom}^\Delta_C(A, B_i)$.

**References**

[1] Markus Banagl, *Topological invariants of stratified spaces*, Springer Monographs in Mathematics, Springer, Berlin, 2008. MR 2401086 (2008k:14046)

[2] A. Borel and et al., *Intersection cohomology*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008. Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition. MR 2401086 (2008k:14046)

[3] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. 8 (1976), no. 179, ix+94. MR 0431137

[4] Henri Cartan, *Théories cohomologiques*, Invent. Math. 35 (1976), 261–271. MR 341323

[5] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré, *Steenrod squares on intersection cohomology and a conjecture of M Goresky and W Pardon*, Geometric Invariants in Algebraic Topology (Göteborg, 1994), Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry, 1998, pp. 33–110. MR 1735443 (2001j:55010)

[6] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. 8 (1976), no. 179, ix+94. MR 0425956

[7] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. 8 (1976), no. 179, ix+94. MR 0425956

[8] A. Borel and et al., *Intersection cohomology*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008. Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition. MR 2401086 (2008k:14046)

[9] A. Borel and et al., *Intersection cohomology*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008. Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition. MR 2401086 (2008k:14046)

[10] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019. MR 3931682

[11] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré, *Steenrod squares on intersection cohomology and a conjecture of M Goresky and W Pardon*, Geometric Invariants in Algebraic Topology (Göteborg, 1994), Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry, 1998, pp. 33–110. MR 1735443 (2001j:55010)

[12] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019. MR 3931682
[13] Sylvain Douteau, A simplicial approach to stratified homotopy theory, Trans. Amer. Math. Soc. 374 (2021), no. 2, 955–1006. MR 4196384 11, 12

[14] Greg Friedman, Intersection homology with general perversities, Geom. Dedicata 148 (2010), 103–135. MR 2721621 25

[15] , Singular intersection homology, New Mathematical Monographs, Cambridge University Press, 2020. 2

[16] Paul G. Goerss and John F. Jardine, Simplicial homotopy theory, Progress in Mathematics, vol. 174, Birhäuser Verlag, Basel, 1999. MR 1711612 (2001d:55012) 3, 35, 37

[17] Mark Goresky, Intersection homology operations, Comment. Math. Helv. 59 (1984), no. 3, 485–505. MR 761809 (86i:55008) 2, 5, 26

[18] Mark Goresky and Robert MacPherson, La dualité de Poincaré pour les espaces singuliers, C. R. Acad. Sci. Paris Sér. A-B 284 (1977), no. 24, A1549–A1551. MR 405332 2

[19] , Intersection homology theory, Topology 19 (1980), no. 2, 135–162. MR 572580 (82b:57010) 2, 5, 14, 16, 25, 35

[20] , Intersection homology, II, Invent. Math. 72 (1983), no. 1, 77–129. MR 696691 (84i:57012) 2, 3, 5, 16, 25

[21] , Stratified Morse theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988. MR 932724 (90d:57039) 7

[22] Mark Goresky and William Pardon, Wu numbers of singular spaces, Topology 28 (1989), no. 3, 325–367. MR 1014452 2, 5, 6, 26

[23] Mark Goresky and Paul Siegel, Linking pairings on singular spaces, Comment. Math. Helv. 58 (1983), no. 1, 96–110. MR 699009 (84i:55034) 2

[24] Mark Hovey, Intersection homological algebra, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 133–150. MR 2544388 (2010g:55009)

[25] André Joyal, The theory of quasi-categories and its applications, Quadern 45, Vol. II, Centre de Recerca Matemàtica Barcelona, 2008. 3, 5, 27, 28

[26] André Joyal and Myles Tierney, Quasi-categories vs Segal spaces., Categories in algebra, geometry and mathematical physics. Conference and workshop in honor of Ross Street’s 60th birthday, Sydney and Canberra, Australia, July 11–16/July 18–21, 2005, Providence, RI: American Mathematical Society (AMS), 2007, pp. 277–326 (English). 3

[27] Frances Kirwan and Jonathan Woolf, An introduction to intersection homology theory, second ed., Chapman & Hall/CRC, Boca Raton, FL, 2006. MR 2207421 (2006k:32061)

[28] Jacob Lurie, Higher algebra, Available at http://www.math.harvard.edu/~lurie/papers/HA.pdf. 3, 4, 7, 11, 12

[29] , Higher topos theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 3, 35

[30] Robert MacPherson, Intersection homology and perverse sheaves, Unpublished AMS Colloquium Lectures, San Francisco, 1991. 4, 16

[31] Michael C. McCord, Singular homology groups and homotopy groups of finite topological spaces, Duke Math. J. 33 (1966), 465–474. MR 0196744 (33 #4930) 11

[32] J. C. Moore, Homotopie des complexes monoïdaux, II, Séminaire Henri Cartan 7 (1954-1955), no. 2 (fr), talk:19. 3

[33] Daniel G. Quillen, Homotopical algebra, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967. MR 0223432 11, 35

[34] Frank Quinn, Homotopically stratified sets, J. Amer. Math. Soc. 1 (1988), no. 2, 441–499. MR 928266 (89g:57050) 4

[35] C. P. Rourke and B. J. Sanderson, △-sets. I. Homotopy theory, Quart. J. Math. Oxford Ser. (2) 22 (1971), 321–338. MR 0300281 (45 #9327) 6, 33

[36] Martintxu Saralegi-Aranguren, Refinement invariance of intersection (co)homologies, Homology Homotopy Appl. 23 (2021), no. 1, 311–340. MR 4170473 16

[37] Martintxu Saralegi-Aranguren and Daniel Tanré, Poincaré duality, cap product and Borel-Moore intersection homology, Q. J. Math. 71 (2020), no. 3, 943–958. MR 4142716 2

[38] , Variations on Poincaré duality for intersection homology, Enseign. Math. 65 (2020), no. 1-2, 117–154. MR 4057357 2, 16

[39] Jean-Pierre Serre, Les espaces k(π, n), Séminaire Henri Cartan 7 (1954-1955), no. 1 (fr), talk:1. 3

[40] P. H. Siegel, Witt spaces: a geometric cycle theory for KO-homology at odd primes, Amer. J. Math. 105 (1983), no. 5, 1067–1105. MR 714770 2
[41] Dai Tamaki and Hiro Lee Tanaka, *Stellar Stratifications on Classifying Spaces*, arXiv e-prints (2018), arXiv:1804.11274.

[42] David Treumann, *Exit paths and constructible stacks*, Compos. Math. **145** (2009), no. 6, 1504–1532. MR 2575092

[43] Jon Woolf, *The fundamental category of a stratified space*, J. Homotopy Relat. Struct. **4** (2009), no. 1, 359–387. MR 2591969

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