ON COSET $N$-VALUED TOPOLOGICAL GROUPS
ON $S^3$ AND $\mathbb{R}P^3$

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To my son Vanya

Recall the notion of an $n$-valued topological group (see an overview [1]).

**Definition 1.** A Hausdorff path-connected topological space $X$ with a base point $e \in X$, together with an $n$-valued continuous multiplication $\mu: X \times X \rightarrow \text{Sym}^n X = X^n/S_n$ and a continuous involution $\text{inv}: X \rightarrow X$, $\text{inv}(e) = e$, is called an $n$-valued topological group, if the following conditions hold true:

1. $\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) \in \text{Sym}^n X$ for all $x, y, z \in X$;
2. $\mu(x, e) = \mu(e, x) = [x, x, \ldots, x]$ for all $x \in X$;
3. $\mu(x, \text{inv}(x)) \ni e$ if $\mu(\text{inv}(x), x) \ni e$ for all $x \in X$.

We will use the following construction of a coset $n$-valued groups (see [1]).

Let us take an arbitrary compact connected Lie group $W$ and any subgroup $G \subset \text{Aut}(W)$ of order $n$ in its automorphism group. Denote by $X$ the quotient space $W/G$ and by $\pi: W \rightarrow X$ the canonic projection. Then the space $X$ can be endowed with a natural structure of an $n$-valued topological group with the identity $e = \pi(e_W)$, the inverse map $\pi(\pi(w)) = \pi(w^{-1})$ and multiplication $\mu(\pi(a), \pi(b)) = [\pi(ag_1(b)), \pi(ag_2(b)), \ldots, \pi(ag_n(b))]$ for all $a, b \in W$, where $G = \{g_1, g_2, \ldots, g_n\}$.

In dimension 3 there are only three compact connected Lie groups: $T^3, Sp(1) \rtimes SO(3)$. The aim of the research is to describe all coset $n$-valued topological groups, arising from $Sp(1)$ and $SO(3)$.

It is known that any automorphism $\varphi: SO(3) \rightarrow SO(3)$ is an inner conjugation by an element $g \in SO(3)$, which is 1-1 correspondence to $\varphi$. Therefore the group $\text{Aut}(SO(3))$ coincides with $SO(3)$. Also the classical isomorphism $Sp(1)/\{1, -1\} \cong SO(3)$ implies that any automorphism $\psi: Sp(1) \rightarrow Sp(1)$ is an inner conjugation by an element $\pm q \in Sp(1)$. Hence the group $\text{Aut}(Sp(1))$ coincides again with $SO(3)$.

Let us use the well known classification of finite subgroups in $SO(3)$.

1. $C_n$, a cyclic group, generated by a $n$-fold rotation about a line;
2. $D_m$, $n = 2m$, a dihedral group, generated by an $n$-fold rotation about a line, and a reflection in a line (half-turn) which is orthogonal to the first line;
3. $T$, $n = 12$, the group of orientation-preserving symmetries of a regular tetrahedron;
4. $O$, $n = 24$, the group of orientation-preserving symmetries of a cube;
5. $I$, $n = 60$, the group of orientation-preserving symmetries of a regular icosahedron.

The following fact is known:

**Fact 1.** For an arbitrary smooth connected orientable manifold $M^m, m = 2, 3$ and any finite group $G$, acting on $M^m$ smoothly, effectively and orientation-preserving, the orbit space $X = M^m/G$ is a topological orientable $m$-dimensional manifold.
Therefore, if a coset topological group $X$ is derived from $Sp(1)$ or $SO(3)$, then $X$ is an orientable connected compact topological 3-manifold.

**Theorem 1.** Let $W = Sp(1)$ and $G \subset \text{Aut}(Sp(1)) = SO(3)$ — an arbitrary finite subgroup of order $n$. Then the coset $n$-valued topological group $Sp(1)/G$ is homeomorphic to $S^3$.

The case of a 2-valued coset group structure on $S^3$ was introduced by V.M. Buchstaber in 1993.

**Theorem 2.** Let $W = SO(3)$ and $G \subset SO(3)$ — an arbitrary finite subgroup of order $n$. If $n$ is even, then the coset group $SO(3)/G$ is homeomorphic to $S^3$. And if $n$ is odd, then the coset group $SO(3)/G$ is homeomorphic to $\mathbb{R}P^3$.

**Proof of theorem 1.** It is obvious that the action of the group $G$ preserves the real part of quaternions $x \in Sp(1)$, fix points $\pm 1$, and also preserves the standard metric of the sphere $S^3 = Sp(1)$. It follows that the quotient space $Sp(1)/G$ is the (unreduced) suspension over the quotient space $S^2/G$, where $S^2$ is a sphere of purely imaginary quaternions of unit length. By the fact that the action of the group $G$ is smooth and orientation-preserving and due to fact 1, the quotient space $S^2/G$ is a compact orientable surface $M^2$. As the map $S^2 \to M^2 = S^2/G$ has a nonzero degree, then the genus of a surface $M^2$ could not increase, so $M^2 = S^2$. \(\square\)

**Proof of theorem 2.** Let us take a finite subgroup $G \subset SO(3)$ of order $n$. Denote by $2G$ its preimage under the canonic epimorphism $Sp(1) \to SO(3)$. Set $G = \{g_1, g_2, \ldots, g_n\}$ and $2G = \{\pm q_1, \pm q_2, \ldots, \pm q_n\}$. The quotient space $SO(3)/G$ may be derived from the universal cover $SO(3) = Sp(1) = S^3$ by the following action of a finite group $\tilde{G} := G \times C_2$, $C_2 = \{1, -1\}$. Namely, $(g, \varepsilon)(x) := \varepsilon(q, xq^{-1})$, for all $x \in Sp(1)$ and $\varepsilon \in \{1, -1\}$. The required orbit space $Sp(1)/\tilde{G}$ could be reached into two steps. Firstly, we need to get the quotient space $Sp(1)/\tilde{G} \cong S^3$. Secondly, we need to factorize the obtained sphere $S^3$ by an involution $\tau$, which arise from antipodal involution $x \mapsto -x$ on $Sp(1)$. It is clear, that the involution $\tau$ preserves the orientation.

Classical S.Illman’s theorem \[2\] states that for any smooth manifold $M^m$ and any finite group $F$, which acts smoothly on $M^m$, there exists a triangulation of $M^m$ for which the action of $F$ is simplicial. It follows that the involution $\tau: S^3 \to S^3$ is simplicial.

F.Walhadsen’ theorem \[3\] states that any preserving orientation simplicival involution of the sphere $S^3$ is conjugated in the group of homeomorphisms to one of standard involutions: $(y_1, y_2, y_3, y_4) \mapsto (-y_1, -y_2, -y_3, -y_4)$ (no fixed points), or $(y_1, y_2, y_3, y_4) \mapsto (-y_1, -y_2, y_3, y_4)$ (there are fixed points). In the first case the orbit space $S^3/\tau$ is $\mathbb{R}P^3$. In the second case $S^3/\tau \cong S^3$.

So, to prove our theorem 2 it is sufficient to obtain a criteria for the involution $\tau$ to have fixed points. The desired existence of fixed points for $\tau$ is equivalent to the fact that for some $g_i, 1 \leq i \leq n$, the element $(g_i, -1) \in \tilde{G}$ possesses at least one fixed point $x \in Sp(1)$, i.e. a point $x \in Sp(1)$ with the condition $-q_i x q_i^{-1} = x$.

Consider the equation $q x q^{-1} = -x$ on the group $Sp(1)$ for a given $q \in Sp(1)$. Since $\text{Re}(q x q^{-1}) = \text{Re}(x)$, then a solution $x$ of our equation could be only a purely imaginary quaternion of unit length. It is known, that the epimorphism $Sp(1) \to SO(3)$ maps a quaternion $q \in Sp(1)$ onto the element $g \in SO(3)$ such that $g(x) = q x q^{-1}$, where $x$ is an arbitrary purely imaginary quaternion of unit length. As any element $g \in SO(3)$ is a rotation about a line by some angle, then the equation $g(x) = -x$ has a solution iff $g$ is a rotation by the angle $\pi$, i.e. $g^2 = e$. Since $g \in G$, then the sought condition is equivalent to the order of the group $G$ be even. \(\square\)

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