Planar Duality in $SU(2)$ WZW Models

G. Pradisi$^{a,c}$, A. Sagnotti$^{a,b}$ and Ya.S. Stanev$^{a,1}$

$^a$Dipartimento di Fisica
Università di Roma “Tor Vergata”
I.N.F.N. - Sezione di Roma “Tor Vergata”
Via della Ricerca Scientifica, 1 00133 Roma ITALY

$^b$Centre de Physique Théorique
Ecole Polytechnique
91128 Palaiseau FRANCE

$^c$Centro “Vito Volterra”, Università di Roma “Tor Vergata”

Abstract

We show how to generalize the $SU(2)$ WZW models to allow for open and unoriented sectors. The construction exhibits some novel patterns of Chan-Paton charge assignments and projected spectra that reflect the underlying current algebra.

\footnote{I.N.F.N. Fellow, on Leave from Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, BG-1784 Sofia, BULGARIA.}
Introduction

In a number of previous papers [1] [2] [3] [4] the idea of associating “open descendants” to left-right symmetric models of oriented closed strings [5] was shown to determine both the bulk spectra and the Chan-Paton [6] charge sectors of new models with unoriented closed and open strings. The link between these two classes of models is a general feature of Conformal Field Theory, and indeed open descendants can be constructed [2] starting from the BPZ series [7] of minimal models. The close relationship between the minimal models and the ADE [8] models of the SU(2) current algebra [9] suggests to take a closer look at this case as well. This is particularly rewarding, since the underlying current algebra introduces a number of novel features that shed new light on the meaning of the Klein-bottle and Möbius projections, while providing a finer test of the “crosscap” constraint of ref. [4].

The $A_3$ Model and its Descendants

The simplest non-trivial $SU(2)$ model belongs to the $A$ series and corresponds to $k = 2$ [8]. It is particularly instructive, since both its torus amplitude

$$T = |\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2,$$

(1)

where $\chi_{2I+1}$ denotes the character corresponding to isospin $I$, and its $S$ matrix

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

(2)

are mapped into the Ising ones by the identification of $\chi_1$, $\chi_2$ and $\chi_3$ with the three Ising characters of identity, spin and energy. Its open descendants, however, must exhibit a different structure, since $\chi_2$ has conformal weight 3/16 while the Ising spin has conformal weight 1/16. Thus, for the Ising model the matrix

$$P = T^{1/2} S T^2 S T^{1/2}$$

(3)
that relates the real bases of characters $\hat{\chi}$ for direct and transverse Möbius channels is

$$P = \begin{pmatrix} \cos\left(\frac{\pi}{8}\right) & 0 & \sin\left(\frac{\pi}{8}\right) \\ 0 & 1 & 0 \\ \sin\left(\frac{\pi}{8}\right) & 0 & -\cos\left(\frac{\pi}{8}\right) \end{pmatrix},$$

(4)

while for the $A_3$ model the $P$ matrix is

$$P = \begin{pmatrix} \sin\left(\frac{\pi}{8}\right) & 0 & \cos\left(\frac{\pi}{8}\right) \\ 0 & 1 & 0 \\ \cos\left(\frac{\pi}{8}\right) & 0 & -\sin\left(\frac{\pi}{8}\right) \end{pmatrix}.$$

(5)

This state of affairs is reminiscent of the behavior of some models discussed in ref. [1], where off-diagonal $P$ matrices result in the appearance of “complex” Chan-Paton charges. Indeed, this model can accommodate a real charge and a pair of complex charges, since the annulus amplitude

$$A = \left(\frac{n_2}{2} + m\bar{m}\right) \chi_1 + n_2(m + \bar{m}) \chi_2 + \frac{n_2^2 + m^2 + \bar{m}^2}{2} \chi_3$$

(6)

and the Möbius amplitude

$$M = \pm \left[ \frac{n_2}{2} \hat{\chi}_1 + \frac{n_2 + m + \bar{m}}{2} \hat{\chi}_3 \right]$$

(7)

are consistent both in the direct and in the transverse channel if the Klein bottle symmetrizes all three Verma modules corresponding to $\chi_1$, $\chi_2$ and $\chi_3$.

For the sake of comparison, in the Ising model the same choice of Klein-bottle projection leads to [2]

$$A = \left(\frac{n_0^2 + n_{1/2}^2 + n_{1/16}^2}{2}\right) \chi_0 + n_{1/16}(n_0 + n_{1/2})\chi_{1/16} + \left(\frac{n_{1/16}^2}{2} + n_0 n_{1/2}\right) \chi_{1/2}$$

(8)

and

$$M = \pm \left[ \frac{n_0 + n_{1/16} + n_{1/2}}{2} \hat{\chi}_0 + \frac{n_{1/16}}{2} \hat{\chi}_{1/2} \right].$$

(9)

Since the charge assignments of eqs. (8) and (9) do not follow the pattern suggested by Cardy’s analysis [10] of the Verlinde formula [11], one may wonder whether a different Klein bottle projection could result in a model with all real charges. This actually corresponds to the only other choice of closed spectrum compatible with the positivity of the vacuum Klein-bottle channel,

$$K = \frac{1}{2}\left(\chi_1 - \chi_2 + \chi_3\right),$$

(10)
whereby all the integer-isospin states are symmetrized while all the half-odd-integer-isospin ones are antisymmetrized. Indeed, in the resulting model,

\[ A = \left( \frac{n_1^2 + n_2^2 + n_3^2}{2} \right) \chi_1 + n_2(n_1 + n_3) \chi_2 + \left( \frac{n_2^2}{2} + n_1 n_3 \right) \chi_3 \]  

(11)

and

\[ M = \pm \left[ \frac{n_1 - n_2 + n_3}{2} \hat{\chi}_1 + \frac{n_2}{2} \hat{\chi}_3 \right] \]  

(12)

are consistent choices both in the direct and in the transverse channel. The annulus amplitude now reflects the fusion rules, but the Möbius amplitude involves some alternating signs whose origin is quite interesting and, as we shall see, bears a close relationship to the underlying current algebra.

One is now encouraged to repeat the same exercise for the Ising model, since after all the two \( A_3 \) models differ in their Klein-bottle projection in a way that can be traced to \( Z_2 \), the center of \( SU(2) \), that distinguishes between integer and half-odd-integer isospin representations and manifests itself in a corresponding automorphism of the fusion algebra. A similar \( Z_2 \) symmetry is present in the Ising model as well, and indeed one can construct a new class of open descendants starting from

\[ K = \frac{1}{2} \left( \chi_0 - \frac{\chi_1}{16} + \frac{\chi_1}{2} \right) \]  

(13)

again the only other projection of the closed spectrum compatible with the positivity of the vacuum Klein-bottle channel. The resulting model involves one real charge and a pair of complex charges, and

\[ A = \left( \frac{n_{1/16}^2 + m \bar{m}}{2} \right) \chi_0 + n_{1/16} (m + \bar{m}) \chi_{1/16} + \left( \frac{n_{1/16}^2 + m^2 + \bar{m}^2}{2} \right) \chi_{1/2} \]  

(14)

and

\[ M = \pm \left[ \frac{n_{1/16}}{2} \hat{\chi}_0 + \frac{m + \bar{m} - n_{1/16}}{2} \hat{\chi}_{1/2} \right] \]  

(15)

are a consistent choice both in the direct and in the transverse channel. The models in ref. [2] may thus be extended to include (infinitely many) others with a different Klein-bottle projection and with pairs of real charges replaced by complex ones [12].
These results deserve some discussion, since the “crosscap” constraint of ref. [4] singles out the usual Klein-bottle projection for the Ising model, with all three sectors symmetrized. Whereas this would appear to exclude the model of eqs. (13), (14) and (15), in some cases the constraint may actually be relaxed to some extent. This is possible whenever the fusion algebra allows some of the two-point functions in front of a crosscap $<\phi_{h_1,\bar{h}_1} \phi_{h_2,\bar{h}_2}>_c$ to behave as $Z_2$ sections [12]. Thus, including a sign $\epsilon$ depending on the (symmetric or antisymmetric) nature of one of the two fields, say $\phi_{h_1,\bar{h}_1}$, in the projected closed spectrum, the “crosscap” constraint reads

$$\epsilon_{(1,\bar{1})} (-1)^{h_1-\bar{h}_1+h_2-\bar{h}_2} C_{12k} C_{1'2'k} \Gamma_k = \sum_p C_{1'2'p} C_{12p} \Gamma_p F_{pk}(1,2,\bar{1},\bar{2})$$

where $C_{ijk}$ are the chiral structure constants, $\Gamma_k$ are the crosscap one-point coefficients and $F_{pk}$ is the duality matrix that relates the $s$-channel conformal blocks to the $u$-channel ones.

We would like to stress that eq. (16) contains more information than the Klein-bottle amplitude, that involves only the squared one-point coefficients. In the Ising model with complex charges this generalized constraint is fulfilled by all two-point functions, with a positive sign if $\phi_{h_1,\bar{h}_1}$ is the identity or the energy, both symmetrized by the Klein-bottle projection, but with a negative sign if $\phi_{h_1,\bar{h}_1}$ is the spin. A more detailed discussion will be presented in ref. [12], where eq. (16) will be shown to determine the structure of some exotic open descendants corresponding to the $E_7$ model and to the $D_{odd}$ series.

Still, we should anticipate that in WZW models for $\epsilon = 1$ integer-isospin sectors are symmetrized and half-odd-integer ones are antisymmetrized, while for the other possible choice, $\epsilon = (-1)^{2I}$, all sectors are symmetrized. The results of ref. [13], where the duality matrices for WZW models are constructed explicitly, reveal the general occurrence of this phenomenon, since the direct Klein-bottle projection involves $SU(2)$ singlets, that originate from (anti)symmetric combinations for (half)integer spins.

Similar remarks apply to the relation between the “twist” properties of the open states in the Ising and $A_3$ WZW models just discussed. Indeed, one may wonder how the new model of eqs. (14) and (15) can be compatible with the duality of disk amplitudes. The puzzle in this respect may be stated as follows. Once the behavior of the prefactor is disposed of by a suitable prescription [2], the conformal blocks should apparently deter-
mine the duality properties of all amplitudes. Thus, for instance, it is not obvious how
the relative “twist” of energy and identity states with two $n_{1/16}$ charges, opposite in the
model of eqs. (14) and (15), may be identical in the model of eqs. (8) and (9). A
closer inspection is instructive, since it reveals how the non-trivial action of the “twist”
on complex Chan-Paton charges actually ensures the consistency of both settings.

![Figure 1. “Twist” with real charges.](image1)

![Figure 2. “Twist” with complex charges.](image2)

Referring to the four-spin amplitude of fig. 1, where dashed lines denote $n_{1/16}$ charges
and continuous lines denote $n_0$ charges, let us observe that only the identity flows in the
$s$ channel (fig. 1a), while both the identity and the energy flow in the $u$ channel. Aside
from a prefactor, the amplitude of fig. 1a is

$$A_a = Tr(\Lambda_1^T \Lambda_2 \Lambda_3^T \Lambda_4) \sqrt{1 + \sqrt{1 - x}},$$

(17)

where the limit of small values for the cross-ratio $x = \frac{z_{13}z_{24}}{z_{12}z_{34}}$ exhibits the $s$ channel, while
the transformation $x \to (1 - x)$ exposes the $u$ channel. The duality transformations of
the conformal blocks amount in this case to a familiar identity and, dropping again a
prefactor, $A_a$ becomes

$$A_a = Tr(\Lambda_1 \Lambda_1^T \Lambda_2 \Lambda_3^T) \left( \sqrt{1 + \sqrt{1 - x} + \sqrt{1 - \sqrt{1 - x}}} \right).$$

(18)

The corresponding “twisted” $u$-channel contribution of fig. 1b unfolds into an am-
plitude where the identity flows in the $t$ channel while, again, both the identity and the
energy flow in the $u$ channel, with the same sign. Indeed, following ref. [14], the “twisted” contribution may be exposed by performing the transformation $x \rightarrow x/(1 + x)$ in an amplitude obtained from eq. (18) by the interchange of the external legs 2 and 3. This operation may be regarded as the the braiding $B_1 : x \rightarrow e^{i\pi} x/(1 - x)$ followed by the reflection $x \rightarrow e^{-i\pi} x$. The “twisted” contribution is then

$$A_a^{(tw)} = Tr(A_4 A_1^T A_3 A_2^T) \left( \sqrt{1 + \sqrt{1 + x + \sqrt{1 + x - 1}}} \right),$$  \hspace{1cm} (19)

and thus for all levels the identity and energy states with a pair of $n_{1/16}$ charges have identical “twist” properties, consistently with the Möbius amplitude of eq. (9).

The four-spin amplitude for the Ising model with complex charges is displayed in fig. 2. As before, dashed lines denote $n_{1/16}$ charges, but now the remaining lines carry arrows associated to the “complex” charges $m$ and $\bar{m}$, and eq. (18) is replaced by

$$A_b = Tr(M_1^\dagger M_2 M_3^\dagger M_4) \left( \sqrt{1 + \sqrt{1 - x + \sqrt{1 - 1 - x}}} \right).$$  \hspace{1cm} (20)

However, unfolding the “twisted” diagram now inverts the relative orientation of the arrows (fig. 2b) and, consistently with the partition function, the resulting amplitude

$$A_b^{(tw)} = Tr(M_1^\dagger M_3^* M_2^T M_4) \sqrt{1 - \sqrt{1 - x}}$$  \hspace{1cm} (21)

propagates the energy, not the identity, in the $t$ channel. Therefore, the “twisted” $u$-channel contribution to $A_b$,

$$A_b^{(tw)} = Tr(M_1^\dagger M_3^* M_2^T M_4) \left( \sqrt{1 + \sqrt{1 + x - \sqrt{1 + x - 1}}} \right),$$  \hspace{1cm} (22)

now involves energy and identity with opposite signs.

Returning to the $A_3$ WZW models, we would like to relate the additional signs in the Möbius amplitude to the behavior of the $SU(2)$ current algebra blocks. We confine our attention to the $A_3$ model with real charges of eqs. (11) and (12), but similar considerations apply to the model of eqs. (6) and (7). Leaving aside a prefactor, and referring again to fig. 1a, where now dashed lines denote $n_2$ charges and continuous lines denote $n_1$ charges, the $u$-channel amplitude for four isospin-1/2 states may be written

$$A_a = Tr(A_1^T A_2 A_3^T A_4) \left( S_1(x, \xi) - S_0(x, \xi) \right).$$  \hspace{1cm} (23)
where

\begin{align}
S_0(x, \xi) &= s_0(\xi) \left[ (1 - x)^{1/2} \sqrt{1 + \sqrt{1 - x}} + \frac{1}{2} x^{1/2} \sqrt{1 - \sqrt{1 - x}} \right] - \\
&\quad s_1(\xi) x^{1/2} \sqrt{1 - \sqrt{1 - x}}, \quad (24) \\
S_1(x, \xi) &= s_0(\xi) \left[ (1 - x)^{1/2} \sqrt{1 - \sqrt{1 - x}} - \frac{1}{2} x^{1/2} \sqrt{1 + \sqrt{1 - x}} \right] + \\
&\quad s_1(\xi) x^{1/2} \sqrt{1 + \sqrt{1 - x}}, \quad (25)
\end{align}

and

\begin{align}
s_0(\xi) &= \xi \quad \text{and} \quad s_1(\xi) = 1 - \frac{1}{2} \xi, \quad (26)
\end{align}

with $x$ the cross ratio defined above and $\xi$ a corresponding cross-ratio of new auxiliary variables \cite{15} present in the chiral vertex operators of the external states.

The auxiliary variables are particularly convenient, since they lead to compact expressions for the chiral vertex operators in terms of polynomials, where the powers select the different isospin projections. Indeed, if $\varphi_I^I(z)$ is a vertex operator of isospin $I$ and third isospin projection $I_3$, defining

\begin{align}
\varphi_I(z, \zeta) = \sum_{m=-I}^{I} \zeta^{I+m} (I+m)! \varphi_I^m(z), \quad (27)
\end{align}

the $SU(2)$ invariants entering the 4-point amplitude \cite{23} are polynomials in $\zeta_{ij} = \zeta_i - \zeta_j$ or, apart form a common prefactor (in our case equal to $\zeta_{13}\zeta_{24}$), polynomials in $\xi = \frac{\zeta_{12}\zeta_{34}}{\zeta_{13}\zeta_{24}}$.

The expressions $s_I(\xi)$ in eq. \cite{26} thus correspond to fields with fixed isospin $I$ flowing in the $s$ channel, and one may deal with the “twist” properties of all components at the same time. Moreover, since the $\zeta$ variables are inert under the reflection, the transformation $x \rightarrow x/(1 + x)$ should be accompanied by $\xi \rightarrow -\xi/(1 - \xi)$. Apart from a common overall factor disposed of by a suitable prescription, this alters $s_0$ and $s_1$ according to

\begin{align}
s_0 \rightarrow -s_0 \quad \text{and} \quad s_1 \rightarrow s_1, \quad (28)
\end{align}

and the additional sign is precisely responsible for the different “twist” properties of the Ising and $A_3$ models. Similar considerations apply to the model with complex Chan-Paton charges where, as in the Ising model, unfolding the “twisted” amplitude leads to a different $t$ channel contribution. In the duality matrices for the blocks these differences
result in relative signs between the elements of the two lines. Thus, for the Ising model
the matrix $F$ of eq. (16) for $(0, 1/2)$ is
\[ F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{29} \]
while for the $A_3$ WZW model the matrix for $(S_0, S_1)$ is
\[ F = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{30} \]
These observations underlie the constructions presented in the next Sections.

**The $A$ Series**

The previous results may be extended to the whole $A$ series of modular invariants. As compared to ref. [2], these amplitudes contain fewer terms but involve a number of additional subtleties that have essentially emerged in the analysis of the $A_3$ case.

For a generic $A$ model corresponding to level $k$ the $S$ matrix is
\[ S_{ab} = \sqrt{\frac{2}{k+2}} \sin \left( \frac{\pi ab}{k+2} \right), \tag{31} \]
while the $P$ matrix is
\[ P_{ab} = \frac{2}{\sqrt{k+2}} \sin \left( \frac{\pi ab}{2(k+2)} \right) \left( E_k E_{a+b} + O_k O_{a+b} \right), \tag{32} \]
with $E$ and $O$ even and odd projectors respectively. Starting from the torus amplitude
\[ T = \sum_{a=1}^{k+1} |\chi_a|^2, \tag{33} \]
the Klein-bottle projection leading to all real charges is
\[ K = \frac{1}{2} \sum_{a=1}^{k+1} (-1)^{(a-1)} \chi_a \]
(34)
where, again, the label $a$ corresponds to $2I+1$, with $I$ the isospin. As for the $A - A$ series
of minimal models, the direct-channel annulus amplitude
\[ A = \frac{1}{2} \sum_{a,b,c} N_{ab}^c n^a n^b \chi_c, \tag{35} \]
is determined by the fusion-rule coefficients according to the ansatz of ref. \[1\]. Eqs. (34) and (35) then fix the transverse Möbius amplitude, and once the resulting projection is expressed in terms of the fusion-rule coefficients, one obtains a rather pleasing expression, namely

\[
M = \pm \frac{1}{2} \sum_{a,b} (-1)^{b-1} (-1)^{\frac{k+1}{2}} N_{bb}^a n^b \chi_a ,
\]

(36)

where \(N_{bb}^a\) lets only integer-isospin states flow in the Möbius strip. The two phase factors are very interesting, and could both be anticipated in view of the discussion of the \(A_3\) models. The first phase, \((-1)^{\frac{k+1}{2}}\), accounts for the different behavior of integer and half-odd-integer isospin states in the presence of a crosscap, and is the “square root” of a similar phase in the Klein-bottle amplitude of eq. (34). The second phase, \((-1)^{b-1}\), is even more interesting, since it distinguishes among the types of Chan-Paton charges according to the isospin of the corresponding characters. Moreover, it is properly the square of the previous one, since in flowing along the boundary of the Möbius strips the charges effect two complete turns about the crosscap [14].

On the other hand, starting from a totally symmetric closed spectrum, so that

\[
K = \frac{1}{2} \sum_{a=1}^{k+1} \chi_a ,
\]

(37)

complex charges appear. If \(k\) is even, the model contains an odd number of characters in the annulus amplitude and an odd number of charges. The charge corresponding to the middle character \(\chi_{(k+2)/2}\) stays real, while the charges corresponding to \(\chi_a\) and \(\chi_{k+2-a}\) form complex pairs. On the other hand, if \(k\) is odd all charges form \((k+1)/2\) complex pairs. In both cases, all signs disappear from the Möbius projection, and the resulting open spectrum is described by

\[
A = \frac{1}{2} \sum_{a,b,c} N_{ab}^c n^a n^b \chi_{k+2-c}
\]

(38)

and

\[
M = \pm \frac{1}{2} \sum_{a,b} N_{bb}^a n^b \chi_{k+2-a} .
\]

(39)

As usual whenever complex charges are present [1], the identifications \(n_{k+2-a} = \bar{n}_a\), implicit in eqs. (38) and (39), ensure the positivity of the annulus vacuum channel. These
conditions endow the corresponding boundaries with an orientation, in the spirit of the mechanism displayed in fig. 2.

**Other Models**

In extending the construction to the other classes of WZW models, one has two distinct options for the projection of the closed spectrum whenever fields of half-integer isospin are present, to wit in the $E_7$ and $D_{odd}$ cases. Though consistent with the crosscap constraint and with the factorization of disk amplitudes, the open sectors of these models are *not* directly based on the fusion algebra, and the independent charge sectors are fewer than one would naively expect \[12\].

The other descendants correspond to the $E_{even}$ and $D_{even}$ models, all of which have an extended symmetry. They follow the pattern dictated by the fusion rules for the characters of the extended algebra, and can all be constructed systematically once one resolves the ambiguity by carefully extending $S$ and $P$ so that in all cases

\[(S)^2 = (ST)^3 = (P)^2 = C\]  \hspace{1cm} (40)

where $C$ is the charge-conjugation matrix. This introduces occasional factors of two in the fusion rules, and thus the $D_{even}$ models with $k = 8p$ have a single set of descendants with corresponding factors of two (for $p \geq 2$) in the direct annulus and Möbius amplitudes, that reflect the occurrence of more than one three point function for some sets of fields. In the $D_{even}$ models with $k = 8p + 4$ and $p \geq 1$ this amusing new feature is accompanied by the more familiar occurrence of a complex pair of Chan-Paton charges \[1\] associated to their two mutually conjugate characters, that we shall denote $\chi_{k/2+1}$ and $\tilde{\chi}_{k/2+1}$.

In order to exhibit the additional factors of two that the extended algebra introduces in the annulus amplitude, it is sufficient to display the annulus and Möbius amplitudes of the $k = 16$ $D_{even}$ model, while keeping only the two charges corresponding to the generalized characters $\chi_c = \chi_5 + \chi_{13}$ and $\chi_d = \chi_7 + \chi_{11}$. Then, letting $\chi_a = \chi_1 + \chi_{17}$, $\chi_b = \chi_3 + \chi_{15}$, denoting by $\chi_e$ and $\tilde{\chi}_e$ the two “resolved” characters, and choosing for
definiteness an overall positive sign for the Möbius amplitude,

\[ A = \frac{n_c^2 + n_d^2}{2} \chi_a + \left( \frac{n_c^2 + n_d^2}{2} + n_c n_d \right) \left( \chi_b + \chi_e + \chi_\tilde{e} \right) + \left( \frac{n_c^2 + 2 n_d^2}{2} + n_c n_d \right) \chi_c + \left( \frac{n_c^2 + 2 n_d^2}{2} + 2 n_c n_d \right) \chi_d \]  

(41)

and

\[ M = \frac{n_c + n_d}{2} \left( \hat{\chi}_a - \hat{\chi}_b + \hat{\chi}_e + \hat{\chi}_\tilde{e} \right) + \frac{n_c + 2 n_d}{2} \hat{\chi}_c - \frac{n_c}{2} \hat{\chi}_d \]  

(42)

This example exhibits rather neatly three types of unconventional Chan-Paton multiplicities. The first presents itself in the open states described by \( \chi_c \), where factors of two occur both in the annulus and in the Möbius amplitude for the charges of type \( d \). There are thus two families of such states. The other two present themselves in the open states corresponding to \( \chi_d \), where the annulus amplitude contains factors of two both for \( n_c^2 \) and for \( n_c n_d \). Since the Möbius amplitude does not contain \( n_d \), there are two sectors of states with a pair of charges of type \( d \), described by symmetric and antisymmetric matrices respectively, as well as two sectors of states with a pair of distinct charges, of types \( c \) and \( d \). These multiple sets of states reflect the occurrence in these models of multiple three point functions, a consequence of the extended symmetry.

We would like to conclude by displaying an alternative way to present our results directly in terms of the \( S \) and \( P \) matrices, that applies to all models discussed in this paper. Denoting by \( \tilde{K} \), \( \tilde{A} \) and \( \tilde{M} \) the transverse-channel amplitudes, the expressions

\[ K = \frac{1}{2} \sum_{a,b} \chi_a \frac{(P_{1b})^2 S_{ab}^\dagger}{S_{1b}} \]  

(43)

\[ \tilde{K} = \frac{1}{2} \sum_a \chi_a \left( \frac{P_{1a}}{\sqrt{S_{1a}}} \right)^2 \]  

(44)

\[ A = \frac{1}{2} \sum_{a,b,c} \chi_a n_b n_c \left( \sum_d \frac{S_{ad}^\dagger S_{bd} S_{cd}}{S_{1d}} \right) \]  

(45)

\[ \tilde{A} = \frac{1}{2} \sum_a \chi_a \left( \sum_b \frac{S_{ab} n_b^2}{\sqrt{S_{1a}}} \right) \]  

(46)

\[ M = \pm \frac{1}{2} \sum_{a,b} \hat{\chi}_a \left( \sum_d \frac{P_{1d} S_{bd} P_{1d}^\dagger}{S_{1d}} \right) \]  

(47)

\[ \tilde{M} = \pm \frac{1}{2} \sum_{a,b} \hat{\chi}_a \left( \frac{P_{1a} S_{ab} n_b}{S_{1a}} \right) \]  

(48)
(where in the $E_{\text{even}}$ and $D_{\text{even}}$ cases $S$ and $P$ are the modular matrices of the extended algebra), describe consistent open and unoriented spectra. Moreover, they relate the crosscap coefficients that solve eq. (16) to the two basic matrices $S$ and $P$ underlying the construction, so that

$$\Gamma_a = \frac{P_{1a}}{\sqrt{S_{1a}}} ,$$  \hspace{1cm} (49)$$

and display the occurrence of the new integral-valued tensor

$$Y_{abc} = \sum_d \frac{S_{ad} P_{bd} P_{cd}}{S_{1d}} S_{1d} ,$$  \hspace{1cm} (50)$$

in the Klein-bottle and Möbius amplitudes. Amusingly, $Y_{abc}$ yields an additional representation of the fusion algebra. For the $A$ and $E_6$ models, one may also choose the Klein-bottle projection of eq. (37). This alters eqs. (43) - (48), and the corresponding crosscap coefficients are then

$$\Gamma'_a = \frac{(-1)^{a+1} P_{1,k+2-a}}{\sqrt{S_{1,k+2-a}}} .$$  \hspace{1cm} (51)$$

In conclusion, we have shown that the setting of ref. [5] can encompass the construction of open descendants for all $SU(2)$ WZW models. In particular, we have described in some detail how open descendants may be associated to the $A$, $D_{\text{even}}$ and $E_{\text{even}}$ models. The resulting open spectra follow the familiar pattern based on the fusion algebra [1], but whenever the closed spectrum contains half-odd-integer isospins a totally symmetric Klein-bottle projection leads to additional models where pairs of real Chan-Paton charges are replaced with pairs of complex ones. The other models, $E_7$ and $D_{\text{odd}}$, also admit open descendants, but of a more unconventional nature, and will be described elsewhere [12].

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