The cd-Index: A Survey

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Abstract. This is a survey of the cd-index of Eulerian partially ordered sets. The cd-index is an encoding of the numbers of chains, specified by ranks, in the poset. It is the most efficient such encoding, incorporating all the affine relations on the flag numbers of Eulerian posets. Eulerian posets include the face posets of regular CW spheres (in particular, of convex polytopes), intervals in the Bruhat order on Coxeter groups, and the lattices of regions of oriented matroids. The paper discusses inequalities on the cd-index, connections with other combinatorial parameters, computation, and algebraic approaches.

1. Early History

The history of the cd-index starts with the combinatorial study of convex polytopes. Over one hundred years ago Steinitz proved the characterization of the face vectors of 3-dimensional polytopes [89]. Interest in the number of faces of convex polytopes in higher dimensions grew with the development of linear programming from the 1950s on. While the affine span of the face vectors of d-dimensional polytopes is known to be given just by the Euler equation, a full characterization of the face vectors of polytopes of dimensions 4 and higher still eludes us. The major breakthrough on this question came through the definition of the Stanley–Reisner ring and the discovery of the connection between polytopes and toric varieties. As a result the face vectors of simplicial polytopes were characterized by Billera and Lee [15] and Stanley [85], following a conjecture of McMullen [69].

In the 1970s Stanley, in studying balanced complexes and ranked partially ordered sets [84], broadened the focus from counting the elements of each rank to counting chains in the poset with elements from a specified rank set. Bayer and Billera applied this perspective to convex polytopes, and initiated the study of “flag vectors” of polytopes [3]. They found the “generalized Dehn–Sommerville equations,” a complete set of equations defining the affine span of the flag vectors of d-polytopes; the dimension turns out to be a Fibonacci number. As with face vectors, a complete characterization of flag vectors of polytopes is unknown.

Shortly after the proof of the generalized Dehn–Sommerville equations, Fine (see [3]) found a way to encode the flag vectors in the most efficient way, with the cd-index. The generalized Dehn–Sommerville equations apply not just to convex polytopes, but also to Eulerian posets.
polytopes, but to all Eulerian posets. Likewise, the cd-index is defined for Eulerian posets, which include the face posets of regular CW spheres, intervals in the Bruhat order on finite Coxeter groups, and the lattices of regions of oriented matroids.

There are two main issues for research on cd-indices. One is the question of the nonnegativity of the coefficients, or, more generally, inequalities on the cd-index, for Eulerian posets or for particular subclasses. The other (related) issue is the combinatorial interpretation of the coefficients, either directly in terms of the poset or in terms of other combinatorial objects. In the last thirty-five years, much research has been carried out on these issues.

2. Basic Definitions

Definition 2.1. An Eulerian poset is a graded partially ordered set such that each (nonsingleton) interval \([x, y]\) in the poset has an equal number of elements of even and odd rank.

Definition 2.2 (from [88]). A finite regular CW complex is a finite collection of disjoint open cells \(σ\) in Euclidean space such that each \(σ\) is homeomorphic to an open ball of some dimension \(n\) and its boundary is homeomorphic to a sphere of dimension \(n - 1\), which is the union of lower dimensional cells. If the complex is homeomorphic to a sphere, it is called a regular CW sphere.

(For a precise definition that does not assume Euclidean space, see [53].)

Björner [17] showed that the face posets of CW complexes are exactly the posets with a unique minimum element \(\hat{0}\) and at least one other element, and for which the order complex of every open interval \((\hat{0}, x)\) is homeomorphic to a sphere. (The order complex of a poset is the simplicial complex whose faces are the chains of the poset.)

The set of closed cells of a regular CW sphere, ordered by inclusion, along with the empty set and an adjoined maximum element, forms an Eulerian poset. A convex polytope is a regular CW sphere. The face lattice of an \(n\)-dimensional polytope is a rank \(n + 1\) Eulerian poset. (In what follows we generally do not distinguish between a polytope and its face lattice.)

Definition 2.3. For \(P\) a graded poset of rank \(n + 1\) and \(S \subseteq [n] = \{1, 2, \ldots, n\}\), the \(S\)-flag number of \(P\), denoted \(f_S(P)\), is the number of chains \(x_1 < x_2 < \cdots < x_s\) of \(P\) for which \(\{\text{rank}(x_i) : 1 \leq i \leq s\} = S\). (By convention, \(f_\emptyset(P) = 1\).) The flag vector of \(P\) is the length \(2^n\) vector \((f_S(P))_{S \subseteq [n]} \in \mathbb{N}^{2^n}\).

When the poset is the face lattice of a polytope, the indexing set is typically shifted to represent dimensions of the faces, rather than ranks in the poset. The restriction of the flag vector to terms indexed by singleton sets is the \(f\)-vector (or face vector) of \(P\).

Sommerville [83] proved the Dehn–Sommerville equations for \(f\)-vectors of simplicial polytopes by applying Euler’s formula to each interval in the face lattice. The same method gives equations on flag vectors, known as the generalized Dehn–Sommerville equations.
Theorem 2.4 ([3]). The affine dimension of the flag vectors of rank \( n + 1 \) Eulerian posets is \( F_n - 1 \), where \( (F_n) \) is the Fibonacci sequence (with \( F_0 = F_1 = 1 \)). The affine hull of the flag vectors is given by the equations

\[
\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{SU(j)}(P) = (1 - (-1)^{k-i-1}) f_S(P),
\]

where \( i \leq k - 2, i, k \in S \cup \{0, n + 1\}, \) and \( S \cap \{i + 1, \ldots, k - 1\} = \emptyset \).

This affine space is spanned by the flag vectors of convex polytopes. Bases of polytopes were given by Bayer and Billera [3] and by Kalai [61].

In the mid-1980s, Jonathan Fine discovered a compact way to represent these equations. To see this, we first need the transformation to the flag \( h \)-vector.

Definition 2.5. Let \( P \) be a rank \( n + 1 \) Eulerian poset with flag vector \( (f_S(P))_{S \subseteq [n]} \). The flag \( h \)-vector of \( P \) is the vector \( (h_S(P))_{S \subseteq [n]} \in \mathbb{N}^{2^n} \), where

\[
h_S(P) = \sum_{T \subseteq S} (-1)^{|S\setminus T|} h_T(P).
\]

This transformation is invertible: \( f_S(P) = \sum_{T \subseteq S} h_T(P) \). The flag \( h \)-vector has algebraic meaning through the Stanley–Reisner ring of the order complex. For a convex polytope (and more generally, a balanced Cohen–Macaulay complex), the entries in the flag \( h \)-vector are nonnegative [84].

The flag \( h \)-vector can be represented by a polynomial in noncommuting variables \( a \) and \( b \). Associate with \( S \subseteq [n] \) the monomial \( u_S = u_1 u_2 \cdots u_n \), where \( u_i = a \) if \( i \notin S \) and \( u_i = b \) if \( i \in S \). Then the \( ab \)-polynomial is \( \Psi_P(a, b) = \sum_{S \subseteq [n]} h_S u_S \).

Here is an equivalent formulation for the \( ab \)-polynomial: associate to each chain \( x_1 \prec x_2 \prec \cdots \prec x_s \) of \( P \) with rank set \( S \) the monomial \( w_1 w_2 \cdots w_n \), where \( w_i = a - b \) if \( i \notin S \) and \( w_i = b \) if \( i \in S \). Then \( \Psi_P(a, b) \) is the sum of these monomials over all chains of \( P \).

Fine’s inspiration was to see that when \( P \) is a convex polytope, the \( ab \)-polynomial can be written as a polynomial with integer coefficients in the noncommuting variables \( c \) and \( d \), where \( c = a + b \) and \( d = ab + ba \).

Definition 2.6. Let \( P \) be a rank \( n + 1 \) poset. The \( cd \)-index of \( P \) is the polynomial \( \Phi_P(c, d) \) such that \( \Phi_P(a + b, ab + ba) = \Psi_P(a, b) \), if such a polynomial exists.

The \( cd \)-index of a rank \( n + 1 \) poset is considered a homogeneous polynomial (in noncommuting variables) of degree \( n \) by assigning degree 1 to \( c \) and degree 2 to \( d \). A straightforward recursion shows that the number of \( cd \)-words of total degree \( n \) is the Fibonacci number \( F_n \). It is easy to see from the definition that the \( cd \)-index of the dual of a poset (reverse the order relation) is obtained from the \( cd \)-index of the poset by reversing all the \( cd \)-words.

Theorem 2.7 ([61]). Let \( P \) be a graded poset. Then \( P \) has a \( cd \)-index if and only if the flag \( f \)-vector of \( P \) satisfies the generalized Dehn–Sommerville equations. In this case the coefficients of the \( cd \)-index are integers.

We will sometimes refer to the affine space of coefficients of the \( cd \)-words of fixed degree as the generalized Dehn–Sommerville space.

The definition of the \( cd \)-index gives a way of computing it from the flag \( h \)-vector, and hence from the flag \( f \)-vector. Here are formulas for several low ranks. (Meisinger’s
dissertation [70] has many useful tables, including flag number formulas for the cd-index up through rank 9.)

**rank 3:** \( cc + (f_1 - 2)d \)

**rank 4:** \( ccc + (f_3 - 2)cd + (f_1 - 2)dc \)

**rank 5:** \( cccc + (f_4 - 2)ccd + (f_2 - f_1)cdc + (f_1 - 2)ddc + (f_3 - 2f_3 - 2f_1 + 4)dd \)

Table 1 gives the cd-indices of some familiar polytopes.

| n-gon          | \( cc + (n-2)d \) |
|----------------|---------------------|
| tetrahedron    | \( ccc + 2cd + 2dc \) |
| cube           | \( ccc + 4cd + 6dc \) |
| octahedron     | \( ccc + 6cd + 4dc \) |
| 4-simplex      | \( cccc + 3ecd + 5edd + 3ecd + 4dd \) |
| 4-cube         | \( cccc + 6ecd + 16ecd + 14ecd + 20dd \) |

Fine believed that the coefficients of the cd-index of a convex polytope are always nonnegative. This appears as a more general conjecture in [6].

**Conjecture 2.8.** The coefficients in the cd-index of every regular CW sphere are nonnegative.

The conjecture turns out to be true. In the next section we will look at the great body of work addressing this conjecture.

Note that while Fine did not publish anything about the cd-index, his calculations involving the cd-index inspired his work on an alternative approach to flag vectors [47, 48].

Stanley [87] noted that it is sometimes useful to write the cd-index as a polynomial in \( c \) and \( e^2 \), where \( e = a - b \) and thus \( e^2 = c^2 - 2d \). Purtill [77] showed that if \( P \) is a convex polytope, then \( \Phi_P(c, d) \) can be written as a polynomial in the noncommuting variables \( c, d \), and \( -e^2 = 2d - c^2 \) with nonnegative coefficients.

Next, let us consider another important parameter for convex polytopes. The \( h \)-vector of a simplicial polytope is the result of a certain linear transformation on the \( f \)-vector. This transformation was noted by Sommerville [83], but its significance was not understood for decades. For simplicial polytopes, the \( h \)-vector has interpretations in shellings of polytopes, the Stanley–Reisner ring and the toric variety associated with the polytope. Unfortunately, the particular transformation from \( f \)-vector to \( h \)-vector does not give a meaningful vector if the polytope is not simplicial. This problem was resolved by consideration of the toric variety associated with a nonsimplicial, rational polytope, and, in particular, its intersection homology. Stanley [86] gave the definition as follows. In the case of simplicial polytopes, this \( h \)-vector specializes to the aforementioned \( h \)-vector. In the general case it is sometimes referred to as the “toric \( h \)-vector.”

The recursive definition uses the following notational conventions. For an Eulerian poset \( P \), write \( 0 \) for the unique minimal element and \( 1 \) for the unique maximal element. Denote by \( P^- \) the poset \( P \setminus \{1\} \). We use interval notation in a poset; in particular \( [0, t) = \{ s \in P : 0 \leq s \prec t \} \). The rank of an element \( t \) of \( P \) is denoted \( \rho(t) \). Finally, \( k_{-1} = 0 \).
DEFINITION 2.9. Families of polynomials \( f \) and \( g \) in a single variable \( x \) are defined by the following rules:

- \( f(\emptyset, x) = g(\emptyset, x) = 1 \)
- If \( P \) is an Eulerian poset of rank \( n + 1 \geq 1 \), and if \( f(P^-, x) = \sum_{i=0}^{n} k_i x^i \), then \( g(P^-, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (k_i - k_{i-1}) x^i \).
- If \( P \) is an Eulerian poset of rank \( n + 1 \geq 1 \), then
  \[
  f(P^-, x) = \sum_{t \in P^-} g(\hat{0}, t), x(x - 1)^{n - \rho(t)}
  \]

When \( f(P^-, x) = \sum_{i=0}^{n} k_i x^i \), let \( h_i = k_{n-i} \), and call \((h_0, h_1, \ldots, h_n)\) the (toric) \( h \)-vector of \( P \).

In the case where \( P \) is the face lattice of a rational convex \( n \)-polytope, this \( h \)-vector has many of the same properties as the \( h \)-vector of a simplicial polytope; in particular, it is nonnegative \((h_i \geq 0)\) and symmetric \((h_i = h_{n-i})\).

THEOREM 2.10. For Eulerian posets, the toric \( h \)-vector is the image of a linear transformation of the flag vector. Equivalently, it is the image of a linear transformation of the \( cd \)-index.

A recursive proof of this proposition was given in [6]. Explicit formulas for the \( cd \)-index–\( h \)-vector map were given in [4]. Hetyei [57] was able to simplify the \( cd \)-index–\( h \)-vector relation by introducing the “short toric polynomial.” The paper [4] also gave a combinatorial approach involving lattice paths, due to Fine. A method for computing the toric \( h \)-vector (as well as the \( cd \)-index) of a convex polytope by sweeping a hyperplane through the polytope was given by Lee in [67].

3. Inequalities

3.1. Nonnegativity. As mentioned before, Fine believed that the \( cd \)-index has nonnegative coefficients for all polytopes, and Bayer and Klapper conjectured nonnegativity for all regular CW spheres. Here is a sequence of results and conjectures on nonnegativity. We say the \( cd \)-index of a poset is nonnegative when all its coefficients are nonnegative, and we denote this by \( \Phi_P(c, d) \geq 0 \).

THEOREM 3.1 ([77] Purtill 1993). The \( cd \)-indices of the following polytopes are nonnegative.

- polytopes of dimension at most 5
- simple polytopes
- simplicial polytopes
- quasisimplicial polytopes (all facets are simplicial)
- quasisimple polytopes (all vertex figures are simple)

THEOREM 3.2 ([87] Stanley 1994). The \( cd \)-indices of \( S \)-shellable CW spheres are nonnegative. In particular the \( cd \)-index of every convex polytope is nonnegative.

(The class of \( S \)-shellable CW spheres includes all convex polytopes.)

An important class of posets is the class of Cohen–Macaulay posets. These are the posets whose order complexes have Cohen–Macaulay Stanley–Reisner rings. A poset is Gorenstein* if and only if it is Cohen–Macaulay and Eulerian.
Conjecture 3.3 ([87] [88]).

- The cd-index of every Gorenstein* poset is nonnegative.
- The cd-index of every Gorenstein* lattice is coefficientwise greater than or equal to the cd-index of the Boolean lattice (simplex).

Stanley [87] showed that if a Gorenstein* poset is also simplicial (all of its intervals up to an element \( x \neq 1 \) are Boolean lattice), then its cd-index is nonnegative. Moreover, he showed that the inequalities of this theorem would imply all linear inequalities satisfied by the flag f-vectors of all Gorenstein* posets, and all those satisfied by the smaller class of S-shellable CW spheres.

In the most general case, Bayer [2] determined which cd-coefficients are bounded for all Eulerian posets.

Theorem 3.4.

(1) For the following cd-words \( w \), the coefficient of \( w \) as a function of Eulerian posets has greatest lower bound 0 and has no upper bound:
   (a) \( c^i d^j \), with \( \min\{i, j\} \leq 1 \),
   (b) \( c^i d^j \cdots d^j \) (at least two d’s alternating with c’s, \( i \) and \( j \) unrestricted).

(2) The coefficient of \( c^n \) in the cd-index of every Eulerian poset is 1.

(3) For all other cd-words \( w \), the coefficient of \( w \) as a function of Eulerian posets has neither lower nor upper bound.

In particular, there are Eulerian posets with some negative cd-coefficients. Easy examples in odd rank are obtained as follows. Take two copies of the “smallest” rank \( n \) Eulerian poset, namely, the poset with two elements at each rank, all pairs of elements of different ranks comparable. Identify the top and bottom elements of the two copies. For rank \( 2k + 1 \), the cd-index is \( 2c^{2k} - (c^2 - 2d)^k \) [41].

Before Stanley’s conjecture was proved by Karu for all Gorenstein* posets, there was progress on special cases. Reading [80] used the proof of the Charney–Davis Conjecture for dimension 3 (Davis and Okun [23]) and a convolution formula to prove the nonnegativity of the coefficients of certain cd-words for Gorenstein* posets. Novik [74] proved the nonnegativity of certain cd-coefficients for odd-dimensional simplicial complexes that are Eulerian and Buchsbaum (a weakening of Cohen–Macaulay), in particular for odd-dimensional simplicial manifolds. Hetyei [56] constructed a set of polyspherical CW complexes having nonnegative cd-indices. (The face posets of these complexes are Gorenstein* posets.) Hsiao [58] constructed an analogue of distributive lattices having nonnegative cd-indices. (These are Gorenstein* posets.)

Karu pursued a proof of nonnegativity of the cd-index for complete fans using methods from algebraic geometry, and was able to extend his proof to the general Gorenstein* case.

Theorem 3.5 ([62] Karu 2006). The cd-index of every Gorenstein* poset is nonnegative.

A complete characterization of the cd-indices of Gorenstein* posets is presumably beyond reach, but Murai and Nevo [72] obtained the characterization for rank 5.
3.2. Monotonicity. In this section we consider comparisons among the cd-indices of different posets.

Billera, Ehrenborg and Readdy studied the cd-indices of zonotopes (polytopes arising as the Minkowski sum of segments) and, more generally, of the lattice of regions of oriented matroids. In analogy to our notation for nonnegativity of the cd-index, we write $\Phi_Q(c,d) \geq \Phi_P(c,d)$ to mean that the coefficient of each cd-monomial in the cd-index of $Q$ is greater than or equal to the corresponding coefficient in the cd-index of $P$.

**Theorem 3.6** ([12]). Let $Q_n$ be the $n$-dimensional cube, and $Q_n^*$ its dual poset.
- If the rank $n+1$ poset $P$ is the lattice of regions of an oriented matroid, then $\Phi_P(c,d) \geq \Phi_{Q_n^*}(c,d)$.
- If $Z$ is an $n$-dimensional zonotope, then $\Phi_Z(c,d) \geq \Phi_{Q_n}(c,d)$.

In this context the cd-index can be modified to the c-2d-index, because the coefficient of every word containing $k$ $d$s is a multiple of $2^k$.

Nyman and Swartz fixed the dimension and number of zones and found the zonotopes with minimum and maximum cd-indices.

**Theorem 3.7** ([75]). For fixed $r$ and $n$, let $\mathcal{H}_L$ be an essential hyperplane arrangement with underlying geometric lattice the rank $r$ near pencil with $n$ atoms, and let $\mathcal{H}_U$ be an essential hyperplane arrangement with underlying geometric lattice the rank $r$ truncated Boolean lattice with $n$ atoms. Let $Z_L$ and $Z_U$ be the zonotopes dual to $\mathcal{H}_L$ and $\mathcal{H}_U$. Then for any $r$-dimensional zonotope $Z$ with $n$ zones,

$$\Phi_{Z_L}(c,d) \leq \Phi_Z(c,d) \leq \Phi_{Z_U}(c,d).$$

Ehrenborg [27] gave additional inequalities for the cd-index of zonotopes.

The following result of Billera and Ehrenborg is analogous to an inequality on the toric $g$-vector of rational polytopes, conjectured by Kalai [61] and proved by Braden and MacPherson [21].

**Theorem 3.8** ([11]). Let $P$ be a polytope and $F$ a face of $P$. Let $P/F$ be the polytope whose face lattice is the interval $[F,P]$ of the face lattice of $P$. Denote the pyramid over a polytope $Q$ by $\text{Pyr}(Q)$. Then

$$\Phi_P(c,d) \geq \Phi_F(c,d) \cdot \Phi_{\text{Pyr}(P/F)}(c,d)$$

$$\Phi_P(c,d) \geq \Phi_{\text{Pyr}(F)}(c,d) \cdot \Phi_{P/F}(c,d)$$

In particular, if $F$ is a facet of $P$, then $\Phi(P) \geq \Phi(\text{Pyr}(F))$, so among all polytopes having $F$ as a facet, the one with minimum cd-index is the pyramid over $F$. Repeated application of this shows that the simplex minimizes the cd-index among polytopes.

Billera and Ehrenborg were also able to show the upper bound theorem for cd-indices of polytopes. (For the upper bound theorem for $f$-vectors, see [69].)

**Theorem 3.9** ([11]). Let $P$ be an $r$-dimensional polytope with $n$ vertices, and let $C(n,r)$ be the cyclic $r$-polytope with $n$ vertices. Then

$$\Phi_P(c,d) \leq \Phi_{C(n,r)}(c,d).$$

Ehrenborg and Karu proved a decomposition theorem for the cd-index of a Gorenstein* poset, resulting in the following inequalities.
Theorem 3.10 (\cite{35}). Let $B_n$ be the Boolean lattice of rank $n$.

- If $P$ is a Gorenstein* lattice of rank $n$, then $\Phi_P(c, d) \geq \Phi_{B_n}(c, d)$.
- If $P$ is a Gorenstein* poset, and $Q$ is a subdivision of $P$, then $\Phi_Q(c, d) \geq \Phi_P(c, d)$.

3.3. Other Inequalities. Stanley \cite{87} showed that for each cd-word $w \neq c^n$ there is a sequence of Eulerian posets whose cd-indices (normalized) tend to $w$. This can be seen as a strengthening of the fact that coefficients of cd-words have no upper bound (Theorem \ref{3.4}). Another proof of this by Bayer and Hetyei is in \cite{5}, where some extreme rays of the closed cone of flag f-vectors of Eulerian posets are given.

The nonnegativity of the cd-index can be translated into inequalities on the flag h-vector and flag f-vector. Some simpler inequalities can also be extracted. Stanley \cite{87} considered the comparison of two entries of the flag h-vector. The result depends on a function of sets that looks mysterious, but makes more sense when visualizing how a cd-word (with $d$ of degree 2) “covers” an interval of integers. For $S \subseteq [n]$, let $\omega(S) = \{i \in [n-1] : \text{exactly one of } i \text{ and } i+1 \text{ is in } S\}$. Stanley \cite{87} showed that the following theorem follows from the nonnegativity of the cd-index for all Gorenstein* posets.

**Theorem 3.11.** Let $S$ and $T$ be subsets of $[n]$. The following are equivalent.

\begin{itemize}
  \item $\omega(T) \subseteq \omega(S)$
  \item For every Gorenstein* poset $P$ of rank $n+1$, $h_T(P) \leq h_S(P)$.
\end{itemize}

In particular, the largest entries in the flag h-vector for Gorenstein* posets are $h_S$, for $S = \{0, 2, 4, \ldots\}$ and $S = \{1, 3, 5, \ldots\}$. Readdy \cite{78} showed that in the case of the crosspolytope, the maximum $h_S$ occurs only for these sets.

For the specific case of the simplex (Boolean lattice), Mahajan \cite{68} looked at inequalities among the coefficients of the cd-index. He found, for example, that for the simplex the coefficient of any cd-word of the form $udv$ is greater than or equal to the coefficient of $uccv$. Furthermore the maximum coefficient is, for $n$ even, the coefficient of $cd^j c$ with $j = (n-2)/2$ and, for $n$ odd, the coefficient of $cdcd^j c$ (and that of $dcd^j c$, which is the same) with $j = (n-5)/2$.

Ehrenborg \cite{25} gave a method for lifting any cd-inequality to give inequalities in higher ranks. For flag vectors of rational polytopes, a main source of inequalities was the nonnegativity of the g-vector (as in Definition \ref{2.9}) and a form of lifting these by convolution (Kalai \cite{61}). Stenson \cite{91} showed that the inequalities described in this section for cd-indices give flag vector inequalities that are not implied by the g-vector convolution inequalities.

Murai and Yanagawa \cite{73} defined squarefree $P$-modules, a generalization of the Stanley–Reisner ring, and used it to generalize the cd-index to a class of posets they call quasi CW posets. They were then able to prove that the coefficient of $w$ for a Gorenstein* poset is less than or equal to the product of coefficients of associated cd-words having a single $d$. As a consequence, they get upper bounds on the cd-index of Gorenstein* posets when the number $f_i$ of rank $i$ elements is fixed for all $i$.

4. Computing the cd-Index

4.1. Specific Polytopes and Posets. Certain polytopes and posets have particularly nice cd-indices, often connected to other combinatorial objects. We
Purtill’s early results on nonnegativity of the cd-index (Theorem 3.1) resulted from studying CL-shellings of polytopes \[77\]. In this study he showed that the cd-index of the simplex is the (noncommutative) André polynomial of Foata and Schützenberger \[49\]; the André polynomial is a generating function for permutations satisfying certain descent properties. Purtill also extended the notion of André permutations to signed permutations, defined signed André polynomials, and showed that the signed André polynomial is the cd-index of the crosspolytope (the dual of the cube). Note that reversing the monomials in the signed André polynomial gives the cd-index of the cube. Thus, in the case of the simplex, crosspolytope and cube, each coefficient in the cd-index can be computed by counting (signed) permutations with certain descent patterns.

Subsequently, Simion and Sundaram \[92\] defined the simsun permutations, also counted by the André polynomials. Hetyei \[54\] gave an alternative set of permutations, which he called augmented André permutations, that give the cd coefficients for the cube (and thus for the crosspolytope). Billera, Ehrenborg and Readdy \[12\] gave formulas for the cd-indices of the simplex, cube, and crosspolytope, with summations over all permutations. Ehrenborg and Readdy \[43\] applied the connection in the opposite direction, and used the ab-index of the simplex and crosspolytope to study the major index of permutations and signed permutations.

Inequalities for the cd-index of zonotopes were given in Section 3.2. Billera, Ehrenborg and Readdy \[13\] showed that \(n\)-dimensional zonotopes span the generalized Dehn–Sommerville space and that they generate as an Abelian group all integral polynomials of degree \(n\) in \(c\) and \(2d\). Bayer and Sturmfels \[7\] showed that the flag vector of an oriented matroid is determined by the underlying matroid. Billera, Ehrenborg and Readdy \[12\] gave an explicit formula for the cd-index of an oriented matroid in terms of the ab-index of its lattice of flats. In particular, this gives formulas for the cd-indices of zonotopes and of essential hyperplane arrangements. Ehrenborg, Readdy and Slone \[44\] extended this to affine and toric hyperplanes. In another direction it was extended to “oriented interval greedoids” by Saliola and Thomas \[81\].

Ehrenborg and Readdy gave recursive formulas for the cd-index of the simplex and the cube \[37\]. They also gave recursive formulas for the cd-index of the lattice of regions of the braid arrangements \(A_n\) and \(B_n\) \[38\]. Jojić \[59\] then gave the cd-index of the lattice of regions of the arrangements \(D_n\) in terms of those of \(A_n\) and \(B_n\).

Hsiao \[58\] gave a general construction of a class of Gorenstein* posets, based on a signed version of the construction of a distributive lattice from the order ideals of a general poset. For the resulting “signed Birkhoff posets” he gave a combinatorial description of the cd-index in terms of peak sets of linear extensions of the underlying poset.

Two other combinatorial computations of the cd-index of a simplex (Boolean lattice) were given by Fan and He \[45\] (based on methods from the Bruhat order (Section 5)), and by Karu \[64\] (counting certain integer-valued functions).

will generally not define the associated combinatorial objects; the interested reader can find details in the references.
4.2. Operations on Posets. Among the tools used in the study of $cd$-indices are results about the effect on the $cd$-index of various operations on posets. The methods used to develop many of these involve the coproduct, introduced by Ehrenborg and Readdy \cite{37}; we postpone discussion of that until Section 6.

The most straightforward effect on the $cd$-index occurs for the join of two posets.

**Definition 4.1.** Given graded posets $P$ and $Q$, the *join* $P*Q$ of $P$ and $Q$ is the poset on the set $(P\setminus \{1\}) \cup (Q\setminus \{0\})$ with $x \leq y$ in $P*Q$ in the following cases:

- $x \leq y$ in $P\setminus \{1\}$
- $x \leq y$ in $Q\setminus \{0\}$
- $x \in P\setminus \{1\}$ and $y \in Q\setminus \{0\}$.

**Theorem 4.2 (\cite{87}).** If $P$ and $Q$ are Eulerian posets, then so is $P*Q$, and

$$\Phi_{P*Q}(c,d) = \Phi_P(c,d)\Phi_Q(c,d)$$

The *pyramid* of a poset $P$ is the Cartesian product $\text{Pyr}(P) = P \times B_1$, where $B_1$ is the two-element chain. The *prism* of a poset $P$ is the “diamond product,” $\text{Prism}(P) = P \circ B_2 = (P \setminus \{0\}) \times (B_2 \setminus \{0\}) \cup \{0\}$, where $B_2$ is the Boolean lattice on two elements. The dual operation to the prism operation takes $P$ to $\text{Bipyr}(P)$.

(The terms come from the polytope context.) These operations produce Eulerian posets from Eulerian posets. Ehrenborg and Readdy computed the effect of these operations on the $cd$-index. They expressed this in terms of a couple of derivations on $cd$-words. We show one set of formulas; for others see \cite{37}. Define a derivation $D$ on $cd$-words by $D(c) = 2d$ and $D(d) = cd + dc$.

**Theorem 4.3 (\cite{37}).** Let $P$ be an Eulerian poset. Then

- $\Phi(\text{Pyr}(P)) = \frac{1}{2} [\Phi(P) c + c \Phi(P) + D(\Phi(P))]$
- $\Phi(\text{Prism}(P)) = \Phi(P) c + D(\Phi(P))$
- $\Phi(\text{Bipyr}(P)) = c \Phi(P) + D(\Phi(P))$

Ehrenborg and Readdy \cite{37} also described the effect on the $cd$-index of other operations on polytopes: truncation at a vertex, gluing polytopes together along a common facet (in particular, performing a stellar subdivision of a facet), and taking a Minkowski sum with a segment. They also gave a formula for the $ab$-index (flag $h$-vector) of the Cartesian product of arbitrary polytopes. Ehrenborg and Fox \cite{30} gave recurrences for the $cd$-index of the Cartesian product and free join of polytopes. Slone \cite{82} gave a lattice path interpretation of the diamond product of two $cd$-words. Fox \cite{50} extended this interpretation to all $cd$-words. Ehrenborg, Johnson, Rajagopalan and Readdy \cite{34} gave formulas for the $cd$-index of the polytope resulting from cutting off a face of a polytope and for the $cd$-index of the regular CW complex resulting from contracting a face of the polytope. S. Kim \cite{65} showed how the $cd$-index of a polytope can be expressed when a polytope is split by a hyperplane. Wells \cite{93} generalized the idea of bistellar flips to (polytopal) PL-spheres and computed the effect on the $cd$-index.

For a fixed graded poset $P$, one can form the poset $I(P)$ of all closed intervals ordered by inclusion. Jojić \cite{60} studied this poset, showed that if $P$ is Eulerian then $I(P)$ is Eulerian, and computed the $cd$-index of $I(P)$ in terms of that of $P$. Jojić
also computed the effect on the cd-index of the “E_t-construction” of Paffenholz and Ziegler [76].

Heteyi [55] introduced the Tchebyshev transform on posets. He used it to construct a sequence of Eulerian posets (one in each rank) with a very simple formula for the cd-coefficients. The cc-index (a variation of the cd-index) of this poset is equivalent to the Tchebyshev (Chebyshev) polynomial. Ehrenborg and Readdy [40] continued the study of the Tchebyshev transform on general graded posets. They showed that the Tchebyshev transform (of the first kind) preserves these poset properties: Eulerian, EL-shellable and Gorenstein*. In the Eulerian case, they computed the cd-index of T(P) in terms of that of P and showed that nonnegativity of the cd-index of P implies nonnegativity of the cd-index of T(P). They showed that a second kind of Tchebyshev transform is a Hopf algebra endomorphism on the Hopf algebra of quasisymmetric functions (see Section 6).

4.3. Shelling Components. Stanley [87] decomposed the cd-index of an n-dimensional simplex (Boolean lattice) into parts based on a shelling of the simplex, and used the parts for a formula for the cd-index of a simplicial Eulerian poset. A simplicial Eulerian poset is an Eulerian poset such that for every x ≠ 1, the interval [0, x] is a Boolean lattice. Note that the h-vector of a simplicial Eulerian poset is defined by the transformation from the f-vector (mentioned in Section 2) for simplicial polytopes.

Let σ_0, σ_1, ..., σ_n be any ordering of the facets of the n-simplex Σ^n; it is a shelling order. Let \( \Phi^n_i(c,d) \) be the contribution to the cd index of Σ^n from the faces added when \( \sigma_i \) is shelled on. (For details see [87].) We refer to these as the shelling components of the cd-index.

**Theorem 4.4.**

- For all \( i, 0 \leq i \leq n - 1 \), \( \Phi^n_i(c,d) \geq 0 \).
- If \( P \) is a simplicial Eulerian poset of rank \( n+1 \) with h-vector \( (h_0,h_1,...,h_n) \), then \( \Phi_P(c,d) = \sum_{i=0}^{n-1} h_i \Phi^n_i(c,d) \).

As a consequence, Stanley proved the nonnegativity of the cd-index for Gorenstein* simplicial posets before Karu’s proof for general Gorenstein* posets. Stanley conjectured, and Heteyi [54] proved formulas for the shelling components \( \Phi^n_i(c,d) \) in terms of André permutations and in terms of simsun permutations. Ehrenborg and Readdy [37] gave a compact recursion for these shelling components, and Ehrenborg [29] gave more recursions for them.

Ehrenborg and Heteyi [32] developed the analogous results for cubical Eulerian posets, that is, Eulerian posets whose lower intervals are isomorphic to the face lattice of a cube. Billera and Ehrenborg [11] gave a formula for the contribution of each facet in a shelling of a polytope. Lee [67] described a dual approach: the calculation of the cd-index by “sweeping” a hyperplane through the polytope, keeping track of the contribution at each vertex.

5. Bruhat Order

The original motivation for the cd-index came from the combinatorial study of convex polytopes, but Reading began the study of the cd-index for another important class of Eulerian posets: intervals in the Bruhat order of Coxeter groups. In short, for \( v \) and \( w \) elements of a Coxeter group, \( v < w \) if and only if some reduced
word representation of $v$ is a subword of a reduced word for $w$. An interval in the Bruhat order is Eulerian and shellable, and hence Gorenstein*. For example, the $cd$-index of the Bruhat order of $S_4$ is $c^5 + c^3d + 2c^2dc + 2cdc^2 + dc^3 + 2cd^2 + dcd + 2d^2c$.

For more information on Bruhat order in our context, see [18, 79].

Reading [79] gave a recursive formula for the $cd$-index of a Bruhat interval. He showed that Bruhat intervals span the generalized Dehn–Sommerville space, and gave an explicit basis.

**Theorem 5.1.** The set of $cd$-indices of Bruhat intervals of rank $r$ spans the affine span of $cd$-indices of Eulerian posets of rank $r$.

The Bruhat order of a universal Coxeter group contains intervals isomorphic to the face lattices of certain polytopes, the duals of stacked polytopes. Reading conjectured that these have maximum $cd$-indices.

**Conjecture 5.2 ([79]).** Let $(W, S)$ be a Coxeter system, and let $[u, v]$ be an interval in the Bruhat order of $W$ with $u$ of length $k$ and $v$ of length $n + k + 1$. Then the $cd$-index of $[u, v]$ is coefficientwise less than or equal to the $cd$-index of a dual stacked $n$-polytope with $n + k + 1$ facets.

In particular the $cd$-index of an interval $[1, v]$ is less than or equal to the $cd$-index of a Boolean lattice.

The $cd$-index can be found in the peak algebra of quasisymmetric functions [14](see Section 6). Billera and Brenti [10] used this to define for Bruhat intervals the complete $cd$-index, a nonhomogeneous polynomial in $c$ and $d$, whose homogeneous part of top degree is the $cd$-index. They used this to give an explicit computation of the Kazhdan–Lusztig polynomials of the Bruhat intervals for any Coxeter group. They conjectured that all coefficients of the complete $cd$-index are nonnegative for all Bruhat intervals.

Besides the top degree terms, whose nonnegativity follows from Karu’s theorem, the nonnegativity of certain coefficients in the complete $cd$-index of Bruhat intervals have been verified [20, 46, 63]. Blanco [19] used CL-labeling due to Björner and Wachs [18] to describe the computation of the complete $cd$-indices of dihedral Bruhat intervals (those isomorphic to intervals in a dihedral reflection subgroup) and Bruhat intervals in universal Coxeter groups.

Blanco [20] defined the shortest path poset in a Bruhat interval, and showed that if the poset has a unique maximal rising chain then it is a Gorenstein* poset.

Y. Kim [66] studied the uncrossing partial order of matchings on $[2n]$, which is isomorphic to a subposet of the dual Bruhat order of affine permutations. He gave a recursion for the $cd$-indices of intervals in this poset.

6. Algebras

The $h$-vector of a simplicial polytope and the flag $h$-vector of Eulerian posets have interpretations in the Stanley–Reisner ring of the polytope or of the order complex of the poset. The $cd$-index is not found naturally in this ring. It turns out that other algebras are better habitats for the $cd$-index. There is an extensive literature on these algebras, and this survey will only touch the surface. For a deeper look, the reader is directed to (in chronological order) [25, 37, 16, 8, 1, 39, 14, 10, 52, 64, 22]. See Billera [9] for a survey of some of these connections. Perhaps the beginning of the story is [25], where Ehrenborg gave a Hopf algebra homomorphism from the Hopf algebra of posets (the “reduced incidence Hopf
algebra”) to the Hopf algebra of quasisymmetric functions. The homomorphism gives some (not all) known results on flag vectors of Eulerian posets.

An important concept underlying some of the algebraic structures is the convolution of flag numbers, introduced by Kalai [61]. The entries of the flag vector, \( f_n^S \) are considered as functions from rank \( n \) graded posets to nonnegative integers. A convolution product is defined: for \( P \) a poset of rank \( n + m \),

\[
\sum_{\rho(x) = n} f_n^S([\hat{0}, x]) f_m^T([x, \hat{1}]) = f_{n+m}^{S \cup [n] \cup (T+n)}(P).
\]

Ehrenborg and Readdy [37] described a coproduct on the vector space spanned by graded posets by

\[
\Delta(P) = \sum_{0 < x < 1} [\hat{0}, x] \otimes [x, \hat{1}],
\]

and a coproduct on the noncommutative polynomial ring \( k\langle a, b \rangle \) by

\[
\Delta(u_1 u_2 \cdots u_n) = \sum_{i=1}^n u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n.
\]

(Recall that the \( ab \)-polynomial of a graded poset has as its coefficients the flag \( h \)-numbers, and these coefficients can be written in terms of the flag \( f \)-vector.)

They showed that the \( ab \)-index is a Newtonian coalgebra map between the resulting coalgebras, and showed that this map takes the subalgebra spanned by Eulerian posets to the subalgebra \( k\langle c, d \rangle \). They used this to derive the formulas for the effect of various operations on the \( cd \)-index (see Section 4.2).

Billera and Liu [16] introduced a graded algebra of flag operators on posets. (In [14] it is referred to as the “algebra of forms on Eulerian posets.”) The flag number functions \( f_n^S \) span a graded vector space over \( \mathbb{Q} \), \( A = \oplus_{n \geq 0} A_n \), with \( A_n = \{ \sum_{\mathcal{S} \subseteq [n-1]} \alpha_S f_n^\mathcal{S} : \alpha_S \in \mathbb{Q} \} \). With the convolution product, \( A \) becomes a graded algebra, and can be generated by the trivial flag operators \( f_0^I \). Billera and Liu determined the two-sided ideal of \( A \) of elements that vanish for all Eulerian posets. Write \( A_\mathcal{E} \) for the quotient of \( A \) by this ideal.

**Theorem 6.1 ([16]).**

- As graded algebras, \( A \cong \mathbb{Q}\langle y_1, y_2, \ldots \rangle \), the free graded associative algebra on generators \( y_i \) of degree \( i \). The isomorphism is determined by \( f_0^I \mapsto y_i \).
- As graded algebras, \( A_\mathcal{E} \cong \mathbb{Q}\langle y_1, y_3, y_5, \ldots \rangle \).

The \( ab \)-polynomial of an Eulerian poset is in the (noncommutative) polynomial ring \( A_\mathcal{E}\langle a, b \rangle \). Billera and Liu gave another proof of the existence of the \( cd \)-index for Eulerian posets by proving

**Theorem 6.2 ([16]).** As a polynomial with coefficients in \( A_\mathcal{E} \), the \( ab \)-polynomial of every Eulerian poset is in \( A_\mathcal{E}\langle c, d \rangle \).

The authors also described one-sided ideals of the graded algebra \( A \) representing flag vector relations on simplicial polytopes and on cubical polytopes, and gave dimension arguments from the resulting \( A \)-modules.

Next we look at the peak algebra, introduced by Stembridge [90] in the study of enriched \( P \)-partitions, and described in terms of the \( cd \)-index in [14], following [8].
For $T = \{t_1, t_2, \ldots, t_k\} \subseteq [n]$, let
\[ M_T = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1}^t x_{i_2}^{t_1-t_1} x_{i_3}^{t_2-t_1} \cdots x_{i_k}^{t_{k-1}}. \]
This is a quasisymmetric function in $x_i$ ($i \geq 1$); the sum is over all increasing $k$-tuples. For a $cd$-word $w = c^{n_1} d^{n_2} e^{n_3} \cdots c^{n_k} d^{m}$ of degree $n$, define $S_w \subseteq [n]$ by $S_w = \{n_1 + 2, n_1 + n_2 + 4, \ldots, n_1 + n_2 + \cdots + n_k + 2k\}$.

**Definition 6.3** (from [14]). Let $Q$ be the algebra of quasisymmetric functions over $Q$ in the variables $x_1, x_2, \ldots$. The **peak algebra** is the subalgebra $\Pi$ of $Q$ generated by the elements $\Theta_w = \sum_{S_w \subseteq T \cup \{T+1\}} 2^{|T|+1} M_T$ for each $cd$-word $w$.

The two algebras have natural coproducts that make them Hopf algebras.

**Theorem 6.4** ([8]). The algebra of forms on Eulerian posets and the peak algebra are dual Hopf algebras.

Following [25], Billera, Hsiao and van Willigenburg [14] considered the quasisymmetric representation of the flag $f$-vector, $F(P) = \sum_{S \subseteq [n]} f_S(P) M_S$, and showed that when written in terms of Stembridge’s basis $\{\Theta_w\}$ the coefficients are the $cd$-index of $P$ (modified by factors of 2). Then an Eulerian poset has a nonnegative $cd$-index when its peak quasisymmetric function has a nonnegative representation in terms of this basis.

Aguiar [1] took an algebraic approach to constructing the $ab$-index of a poset using a morphism from the algebra of all posets to the noncommutative polynomial algebra $k\langle a, b \rangle$, considered as infinitesimal Hopf algebras. This perspective enabled him to consider generalizations of the $ab$-index to weighted posets and the relative $ab$-index. Using a map involving the zeta and Möbius functions of a poset, he found the infinitesimal Hopf algebra corresponding to Eulerian posets, that is, the $cd$-index.

Ehrenborg and Readdy [39] gave yet another proof of the existence of the $cd$-index for Eulerian posets through the homology of their Newtonian coalgebra generated by $a$ and $b$, mentioned above.

Karu [64] gave an algebraic formulation of a conjecture of Murai and Nevo [71] and proved it for a special case. The statement depends on a representation of $cd$-words by $0/1$-vectors: $mdeg(w)$ is the $0/1$-vector obtained from $w$ by replacing each $c$ by 0 and each $d$ by 10. For $w$ a $cd$-word and $v = mdeg(w)$, write $\Phi_{P,v}$, for the coefficient of $w$ in $\Phi_{P(c,d)}$. For vectors $v \in \mathbb{Z}^n$ that are not equal to $mdeg(w)$ for all $cd$-words $w$, let $\Phi_{P,v} = 0$.

**Theorem 6.5** ([64]). Let $P$ be the poset of a Gorenstein* simplicial complex of dimension $n$. Then there exists a standard $\mathbb{Z}^n$-graded $k$-algebra $A = \oplus_v A_v$ such that $\Phi_{P,v} = \dim A_v$.

Murai’s and Nevo’s conjecture was that this theorem holds for all Gorenstein* posets.

Fine [48] suggests a successor to the $cd$-index. Like the $cd$-index, it provides a special basis for the flag vector ring, which he calls $\mathcal{R}$. The successor is a basis that, conjecturally, has this property: the structure coefficients for product and pyramid (which he calls cone) are all nonnegative integers. See also Theorem 4.3 and 30 for product and pyramid in the $cd$-index. If $\mathcal{R}$ were the representation ring of some
algebraic object $G$ that satisfies Schur’s lemma, then the existence of such a basis would be immediate.

### 7. Related Parameters

Here we mention some generalizations of the cd-index to broader settings.

Ehrenborg and Readdy [36] considered a generalization of the lattice of the cube. For various $r$, take the posets $M_r$ with $r$ minimal elements and a unique maximum element, then take the Cartesian product and add a unique minimum element. The result they called an $r$-cubical lattice, where $r$ is the vector of the various values of $r$ in the factors. They defined a generalized cd-index for these lattices, and showed that the coefficients count a class of permutations generalizing the André permutations.

Ehrenborg [26] considered a relaxation of the Eulerian condition. A poset $P$ is $k$-Eulerian if every interval of rank $k$ is Eulerian. He showed that the $ab$-index of a $k$-Eulerian poset can be written in terms of $c$, $d$ and $e^{2k+1} = (a - b)^{2k+1}$. He also related the $k$-Eulerian posets to an ideal in the Newtonian coalgebra.

Ehrenborg, Hetyei and Readdy [33] considered Eulerian level posets, infinite posets with a certain uniformity at each level and whose finite intervals are Eulerian. For these posets they extended the cd-polynomial to a cd-series.

As mentioned in Section 3.3, Murai and Yanagawa [73] considered a class of “quasi CW posets” and defined an extended cd-index for this class. Grujić and Stojadinović [52] developed an analogue of the cd-index for “building sets” (see [24]) by introducing a Hopf algebra of building sets and mimicking the known Hopf algebra construction of the cd-index.

Ehrenborg, Goresky and Readdy [31] extended the definition of the cd-index to “quasi-graded posets” with a generalization of the notion of Eulerian. These posets arise from Whitney stratified manifolds. They studied in particular the cd-indices of semisuspensions. Ehrenborg and Readdy [42] applied this to study manifold arrangements.

Murai and Nevo [71] related the cd index of a class of Eulerian posets to the $f$-vector of a simplicial complex. The relation is via the $\gamma$-vector, first introduced by Gal [51]. Here the set of $S^*$-shellable spheres is a subset of Stanley’s $S$-shellable spheres, and includes convex polytopes.

**Theorem 7.1** ([71]). Let $P$ be an $(n-1)$-dimensional $S^*$-shellable regular CW sphere, with cd-index $\Phi_P(c, d)$. Define $\delta_i$ by $\Phi_P(1, d) = \delta_0 + \delta_1 d + \cdots + \delta_{[n/2]} d^{[n/2]}$. Then there exists an $[n/2]$-colored simplicial complex $\Delta$ such that $\delta_i = f_{i-1}(\Delta)$ for $0 \leq i \leq [n/2]$.

This is a survey of the cd-index, but as mentioned in Section 2 of great interest in the study of Eulerian posets and, in particular, of convex polytopes, is the toric $h$-vector. The toric $h$-vector contains much less information than the cd-index. However, Lee was led by his study of sweeping a hyperplane through a polytope to an extension of the toric $h$-vector that is equivalent to the cd-index [67].

### 8. Conclusion

The introduction of the cd-index opened up many directions of research on Eulerian posets. There are many specific open questions, but the overriding issue is to find combinatorial interpretations for the coefficients, beyond those in special
cases mentioned here. I hope that this survey will become outdated soon, because of significant research advances.

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