Generalized $\beta$-conformal change and special Finsler spaces

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Abstract. In this paper, we investigate the change of Finsler metrics

$$L(x, y) \rightarrow L(x, y) = f(e^{\sigma(x)} L(x, y), \beta(x, y)),$$

which we refer to as a generalized $\beta$-conformal change. Under this change, we study some special Finsler spaces, namely, quasi C-reducible, semi C-reducible, C-reducible, $C_2$-like, $S_3$-like and $S_4$-like Finsler spaces. We obtain some characterizations of the energy $\beta$-change, the Randers change and the Kropina change. We also obtain the transformation of the T-tensor under this change and study some interesting special cases. We then impose a certain condition on the generalized $\beta$-conformal change, which we call the b-condition, and investigate the geometric consequences of such a condition. Finally, we give the conditions under which a generalized $\beta$-conformal change is projective and generalize some known results in the literature.

Keywords: Generalized $\beta$-conformal change, $\beta$-conformal change, Randers change, Kropina change, projective change, special Finsler spaces, b-condition, T-tensor.

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Introduction

Throughout, $M$ is an n-dimensional $C^\infty$ differentiable manifold and $F^n = (M, L)$ a Finsler space equipped with the fundamental function $L(x, y)$ on $TM$.

Finsler geometry was first introduced by Finsler himself, to be studied by many eminent mathematicians for its theoretical importance and applications in the variational calculus, mechanics and theoretical physics. Moreover, the dependence of the fundamental function $L(x, y)$ on both the positional argument $x$ and directional argument $y$ offers the possibility to use it to describe the anisotropic properties of the physical space. For a differential one-form $\beta(x, dx) = b_i(x)dx^i$ on $M$, Randers [17], in 1941, introduced a special Finsler space with the Finsler change $\overline{L} = L + \beta$, where $L$ is Riemannian, to consider a unified field theory ($L = \sqrt{a_{ij}y^iy^j}$, $a_{ij}$ being the gravitational tensor field and $b_i(x)$ the electromagnetic potential). Masumoto [11], in 1974, studied Randers and generalized Randers changes in which $L$ is Finslerian. Kropina [8] introduced the change $\overline{L} = L^2/\beta$, where $L$ is Riemannian. This change has been studied by other authors such as Shibata [18] and Matsumoto [9]. Randers and Kropina changes are closely related to physical theories and so Finsler spaces with these metrics have been investigated by many authors, from various approaches in both the physical and mathematical aspects ([3], [11], [20], [22], [23], [24], [26]). Randers change was also applied to the theory of the electron microscope by R. S. Ingarden [6]. Moreover, there is some relation between the Kropina metric and the Lagrangian function of analytic dynamics [18]. In 1984, Shibata [19] considered the general case of any $\beta$-change, that is, $\overline{L} = f(L, \beta)$, thus generalizing many changes in Finsler geometry ([11], [18]). In this context, he studied some special Finsler spaces, such as C-reducible and $S_4$-like, under Randers change.

On the other hand, in 1976, Hashiguchi [4] studied the conformal change of a Finsler metric, namely, $\overline{L} = e^{\sigma(x)}L$. In particular, he also dealt with the special conformal transformation named C-conformal. This change has been studied by Izumi [7] among others. In 2008, Abed ([1], [2]) introduced the change $\overline{L} = e^{\sigma(x)}L + \beta$, which he called a $\beta$-conformal change, thus generalizing the conformal, Randers and generalized Randers changes. Moreover, he studied some special Finsler space under this change such as C-reducible and $S_4$-like.

In [25], the present authors introduced and investigated the more general change of Finsler metrics:

$$L(x, y) \rightarrow \overline{L} = f(e^{\sigma(x)}L(x, y), \beta(x, y)) = f(\overline{L}, \beta),$$

where $\overline{L} = e^{\sigma(x)}L$ and $f$ is a positively homogeneous function of $\overline{L}$ and $\beta$ of degree one. They obtained the difference between Cartan connection associated with $(M, L)$ and Cartan connection associated with $(M, \overline{L})$, also, they established some interesting results and computed the torsion and curvature tensors of the transformed space $(M, \overline{L})$ for the four fundamental connections in Finsler geometry. This change is referred to as a generalized $\beta$-conformal change. It is clear that this change is a generalization of all the above mentioned changes and deals simultaneously with $\beta$-change and conformal change. It combines both cases of Shibata ($\overline{L} = f(L, \beta)$) and that of Hashiguchi ($\overline{L} = e^{\sigma}L$).

In this paper, we continue our investigation of the generalized $\beta$-conformal change. Under this change, we study some special Finsler spaces, compute the transformed
T-tensor, introduce what we call b-condition and study when this change becomes projective.

The present paper is organized as follows. In section 1, we introduce the necessary material and background required for the present work. In section 2, we deal with some special Finsler spaces under a generalized $\beta$-conformal change, namely, quasi C-reducible, Semi C-reducible, C-reducible, $C_2$-like, $S_3$-like and $S_4$-like. In section 3, we compute the T-tensor of the transformed space under a generalized $\beta$-conformal change and study some interesting special cases. In section 4, we impose a certain condition on the generalized $\beta$-conformal change, which we call the b-condition, and investigate the geometric consequences of such a condition. Finally, in section 5, we give the conditions under which a generalized $\beta$-conformal change is projective and generalize some known results in the literature.

1. Notations and preliminaries

Throughout the present paper we use the terminology and notations of [25]. Let $(M, L)$ be an $n$-dimensional $C^\infty$ Finsler manifold; $L$ being the fundamental Finsler function. Let $(x^i)$ be the coordinates of any point of the base manifold $M$ and $(y^i)$ a supporting element at the same point. We use the following notations:

$\partial_i$: partial differentiation with respect to $x^i$,
$\delta_i$: partial differentiation with respect to $y^i$ (basis vector fields of the vertical bundle),
$g_{ij} := \frac{1}{2}\partial_i\partial_j L^2 = \partial_i\partial_j E$: the Finsler metric tensor; $E := \frac{1}{2}L^2$: the energy function,
$l_i := \delta_i L = g_{ij}^{\frac{\partial}{\partial y^j}}$: the normalized supporting element; $l^i := \frac{\partial}{\partial y^i}$,

$l_{ij} := \delta_i l_j$,
$h_{ij} := Ll_{ij} = g_{ij} - l_il_j$: the angular metric tensor,
$C_{ijk} := \frac{1}{2}\partial_k g_{ij} = \frac{1}{2}\partial_k\partial_j l^i$: the Cartan tensor,

$G^i$: the components of the canonical spray associated with $(M, L)$,
$N^i_j := \delta_j G^i$: the Barthel (or Cartan nonlinear) connection associated with $(M, L)$,
$\delta_i := \partial_i - N^m_i\partial_m$: the basis vector fields of the horizontal bundle,
$G^i_{jk} := \delta_h N^i_j = \delta_h\delta_j G^i$: the coefficients of Berwald connection,

$C^i_{jk} := g^{ir}C_{rjk} = \frac{1}{2}g^{ir}\partial_k g_{rj}$: the h(hv)-torsion tensor,
$\gamma^i_{jk} := \frac{1}{2}g^{ir}(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk})$: the Christoffel symbols with respect to $\delta_i$,

$\Gamma^i_{jk} := \frac{1}{2}g^{ir}(\partial_j g_{kr} + \delta_k g_{jr} - \delta_r g_{jk})$: the Christoffel symbols with respect to $\delta_i$,

$(\Gamma^i_{jk}, N^i_j, C^i_{jk})$: The Cartan connection $CT$.

For a Cartan connection $(\Gamma^i_{jk}, N^i_j, C^i_{jk})$, we define

$X^i_{ijk} := \delta_k X^i_j + X^m_j\Gamma^i_{mk} - X^i_m\Gamma^m_{jk}$: the horizontal covariant derivative of $X^i_j$,

$X^i_{jk} := \delta_k X^i_j + X^m_j C^i_{mk} - X^i_m C^m_{jk}$: the vertical covariant derivative of $X^i_j$.

Let $F^n = (M, L)$ be an $n$-dimensional Finsler space with a fundamental function $L = L(x, y)$. Consider the following change of Finsler structures, which will be called a generalized $\beta$-conformal change,

$L(x, y) \rightarrow \overline{L}(x, y) = f(e^{\sigma(x)}L(x, y), \beta(x, y)) = f(\overline{L}, \beta), \quad (1.1)$
where $f$ is a positively homogeneous function of degree one in $\widetilde{L} = e^\sigma L$ and $\beta$ and $\beta = b_i(x)dx^i$. Assume that $\mathcal{F}^n = (M, \mathcal{L})$ has the structure of a Finsler space. Entities related to $\mathcal{F}^n$ will be denoted by barred symbols.

We define

$$f_1 := \frac{\partial f}{\partial L}, \quad f_2 := \frac{\partial f}{\partial \beta}, \quad f_{12} := \frac{\partial^2 f}{\partial L \partial \beta}, \ldots \text{ etc.,}$$

where $\widetilde{L} = e^\sigma L$. We use the following notations:

$q := ff_2$, \hspace{1em} $p := ff_1/L$,

$q_0 := ff_2$, \hspace{1em} $p_0 := f_2 + q_0$,

$q_{-1} := ff_{12}/L$, \hspace{1em} $p_{-1} := q_{-1} + (pf_2/f)$,

$q_{-2} := f(e^\sigma f_{11} - (f_1/L))/L^2$, \hspace{1em} $p_{-2} := q_{-2} + (e^\sigma p^2/f^2)$.

Note that the subscript under each of the above geometric objects indicates the degree of homogeneity of that object. We also use the notations:

$m_i := b_i - (\beta/L^2)y_i \neq 0$, \hspace{1em} $p_{02} := \frac{\partial p_0}{\partial \beta}$.

**Proposition 1.1.** Under a generalized $\beta$-conformal change, we have:

(a) $\ell_i = e^\sigma f_1 l_i + f_2 b_i$,

(b) $\overline{h}_{ij} = e^\sigma p h_{ij} + q_0 m_i m_j$,

(c) $\overline{g}_{ij} = e^\sigma p g_{ij} + p_0 b_i b_j + e^\sigma p_{-1}(b_i y_j + b_j y_i) + e^\sigma p_{-2} y_i y_j$.

(d) The inverse metric $\overline{g}^{ij}$ of the metric $\overline{g}_{ij}$ is given by

$$\overline{g}^{ij} = (e^{-\sigma}/p)g^{ij} - s_0 b^i b^j - s_{-1}(y^i b^j + y^j b^i) - s_{-2} y^i y^j,$$

where

$s_0 := e^{-\sigma} f_2 q_0/(\varepsilon p L^2)$, \hspace{1em} $s_{-1} := p_{-1} f_2^2/(p \varepsilon L^2)$, \hspace{1em} $s_{-2} := p_{-1}(e^\sigma m^2 p L^2 - b^2 f^2)/(\varepsilon \beta L^2)$,

$\varepsilon := f^2(e^\sigma p + m^2 q_0)/L^2 \neq 0$, \hspace{1em} $m^2 = g^{ij} m_i m_j = m^i m_i \neq 0$, \hspace{1em} $b^j = g^{ij} b_j$.

**Remark 1.2.** The quantities $s_0$, $s_{-1}$, $s_{-2}$ satisfy:

$$\beta s_0 + L^2 s_{-1} = q/\varepsilon,$$

$$b^2 s_{-1} + \beta s_{-2} = e^\sigma p_{-1} m^2 /\varepsilon.$$

Let $C_i = C_{ijk} g^{jk}$, $C^i = C^i_{jk} g^{jk}$ and $C^2 = C^i C_i$. Then, we have

**Proposition 1.3.** Under a generalized $\beta$-conformal change, we have

(a) The Cartan tensor $\overline{C}_{ijk}$ has the form

$$\overline{C}_{ijk} = e^\sigma p C_{ijk} + V_{ijk},$$

where $V_{ijk}$ is the correction term.
(b) The \((h)hv\)-torsion tensor \(\overline{C}_{ij}^l\) has the form
\[
\overline{C}_{ij}^l = C_{ij}^l + M_{ij}^l,
\]
where
\[
V_{ijk} := \frac{e^\sigma p_{-1}}{2}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{p_{02}}{2}m_im_jm_k,
\]
\[
M_{ij}^l := \frac{1}{2}(e^{-\sigma}m^l/p - m^2(s_0b^l + s_{-1}y^l))(p_{02}m_im_j + e^\sigma p_{-1}h_{ij})
- e^\sigma(s_0b^l + s_{-1}y^l)(pC_{ij\beta} + p_{-1}m_im_j) + \frac{p_{-1}}{2p}(h_{ij}^l + h_{ij}^l),
\]
and \(h_j^l = g^ji h_{ij}, C_{ij\beta} := C_{ij} b^\beta, C_{ij\beta} := C_{ijk} b^j b^k\) and so on.

(c) \(\overline{C} = C_i - e^\sigma p s_0 C_{ij\beta} + \lambda m_i,\)
where
\[
\lambda := \frac{(n + 1)p_{-1}}{2} - \frac{3e^\sigma p_{-1}m^2 s_0}{2} + \frac{p_{02}m^2}{2(e^\sigma p + q_0m^2)}.
\]

(d) \(\overline{C} = \frac{e^{-\sigma}}{p} C^i + J^i,\)
where \(J^i := \frac{\lambda e^{-\sigma}}{p} m_i - s_0 C_{ij\beta} - (C_{ij} + \lambda m^2 - e^\sigma s_0 p C_{ij\beta})(s_0 b^i + s_{-1} y^i),\)
\(C_{ij} := C_{ij} b^i.\)

(e) \(\overline{C} = \frac{e^{-\sigma}}{p} C^2 + \Phi,\)
where \(\Phi := \lambda^2 m^2 \left((e^{-\sigma}/p) - s_0 m^2\right) + C_B((2\lambda e^{-\sigma}/p) - s_0(1 + 2\lambda m^2))
+ s_0 C_{ij\beta}(1 - 3\lambda + e^\sigma s_0 p C_{ij\beta})
+ s_0 C_{ij\beta}(2\lambda e^{-\sigma} s_0 b^2 C_{ij\beta} - \lambda s_0 m^2 b^i - e^\sigma s_0 p C_{ij\beta} - 2C^\sigma).\)

**Proposition 1.4.** Under a generalized \(\beta\)-conformal change, the \(v\)-curvature tensor of \((M, L)\) is transformed as follows:
\[
\overline{S}_{ijk} = e^\sigma p S_{ijk} + \alpha_{jk}\{H_{ik}^l h_{ij} + H_{ij}^l h_{ik} + \omega_{ik} C_{ij\beta} + \omega_{ij} C_{ik\beta}\},
\]
where
\[
H_{ij} := K_1 m_im_k + K_2 C_{ij\beta} + K_3 h_{ij},\quad \omega_{ij} := K_4 m_i m_j - \frac{1}{2} e^\sigma p^2 s_0 C_{ij\beta},
\]
\[
K_1 := \frac{e^{2\sigma} p_{-1}^2}{4p}(e^{-\sigma} - 2s_0 p m^2) + e^\sigma p_{-1}p_{02} m^2 + \frac{4(e^\sigma p + q_0 m^2)}{4(e^\sigma p + q_0 m^2)},\quad K_2 := \frac{e^\sigma p_{-1}}{2} - \frac{1}{2} e^\sigma s_0 p_{-1} m^2,
\]
\[
K_3 := \frac{e^{2\sigma} p_{-1}^2 m^2}{8(e^\sigma p + q_0 m^2)},\quad K_4 := \frac{e^\sigma p_{02}}{2(e^\sigma p + q_0 m^2)} - e^\sigma s_0 p_{-1}.
\]

**Remark 1.5.**

The tensors \(H_{ij}\) and \(\omega_{ij}\) defined above have the following properties:

1. \(H_{ij}\) and \(\omega_{ij}\) are symmetric.
2. \(H_{ij}\) and \(\omega_{ij}\) are indicatory: \(H_{ij} y^i = 0, \omega_{ij} y^i = 0.\)
Proposition 1.6. Under a generalized \( \beta \)-conformal change, the vertical Ricci tensor \( \mathcal{S}_{ik} \) and the vertical scalar curvature \( \mathcal{S} \) associated with the transformed space \( (\tilde{M}, \tilde{L}) \) are given by:

\[
\mathcal{S}_{ik} = S_{ik} + Kh_{ik} + \left( s_0 m^2 - \frac{e^{-\sigma}}{(n-3)p} \right) H_{ik} + \Psi_{ik},
\]
\[
\mathcal{S} = \frac{e^{-\sigma}}{p} S + \frac{2e^{-\sigma}}{p} K \{(n-2) - e^\sigma ps_0 m^2 \} - s_0 \Psi_{\beta \beta} + \frac{e^{-\sigma}}{p} \Psi - s_0 S_{ik} b^i b^k,
\]

where

\[
K := s_0 H_{\beta \beta} \frac{e^{-\sigma}}{p} (K_1 m^2 + K_2 C_\beta + (n-1)K_3),
\]
\[
\Psi_{ik} := \frac{e^{-\sigma}}{p} \left\{ \omega_{rk} C_{i\beta}^r + \omega_{ri} C_{k\beta}^r - (K_4 m^2 - \frac{1}{2} e^{2\sigma} s_0 p^2 C_\beta) C_{ik\beta} \right\} - s_0 \left\{ H_{\beta k} m_i + H_{i\beta} m_k \right\} + \omega_{\beta k} C_{i\beta\beta} + \omega_{i\beta} C_{k\beta\beta} - \omega_{\beta \beta} C_{ik\beta} - \omega_{ik} C_{\beta \beta \beta} + e^\sigma p s_{hijk} b^i b^j \},
\]
\[
H_{\beta \beta} := H_{ij} b^i b^j, \quad \omega_{\beta \beta} := \omega_{ij} b^i b^j, \quad \Psi := \Psi_{ij} g^{ij}, \quad \Psi_{\beta \beta} := \Psi_{ij} b^i b^j.
\]

Note that the tensor \( \Psi_{ij} \) is symmetric and indicatory.

2. Special Finsler spaces

In this section we will investigate the effect of the generalized \( \beta \)-conformal change \((\mathbf{L1})\) on some special Finsler space. Some of the results obtained in this section are generalizations of known results and some are new. For a systematic study of special Finsler spaces, we refer to \([27]\).

In what follows, let \((M, L)\) be a Finsler manifold and \((\tilde{M}, \tilde{L})\) the transformed Finsler manifold under a generalized \( \beta \)-conformal change. The geometric objects associated with \((\tilde{M}, \tilde{L})\) will be denoted by barred symbols.

Theorem 2.1. For \( n > 2 \), under a generalized \( \beta \)-conformal change, the following assertions are equivalent

(a) \( p_{-1} = 0 \).

(b) \( q = k \beta; \ k \) is a nonzero constant.

(c) \( \overline{C}_{ijk} = e^\sigma p C_{ijk} \).

(d) \( \overline{L} = (k' e^{2\sigma} L^2 + k \beta^2)^\frac{1}{2}; \ k' \) is a nonzero constant.

The special \( \beta \)-conformal change \((d)\) is referred to as an energy \( \beta \)-change \([28]\).

Proof.

(a) \( \Rightarrow (b) \): Let \( p_{-1} = 0 \), then \( \frac{f f_{12}}{L} + \frac{p f_2}{f} = 0 \) which leads to \( f f_{12} + f_1 f_2 = 0 \), hence, \( \frac{\partial}{\partial L} (f f_2) = 0 \). By integration, taking the homogeneity of \( f \) into account, we get...
\( q = k\beta \), with \( k \neq 0 \).

(b)⇒(c): Let \( q = k\beta \), then \( \frac{\partial}{\partial L}(ff_2) = ff_{12} + f_1f_2 = 0 \), which leads to \( p_{-1} = 0 \). Using \( \beta p_0 + e^\sigma L^2 p_{-1} = q \), we get \( \beta p_0 = q \). By differentiating the last identity with respect to \( \beta \), we have
\[
\beta p_{02} + p_0 = f_2^2 + f_2f_2 = p_0,
\]
which leads to \( p_{02} = 0 \). Hence, by (1.4) \( V_{ijk} = 0 \) and, consequently, \( \overline{C}_{ijk} = e^\sigma p C_{ijk} \).

(c)⇒(d): Let \( V_{ijk} = 0 \), then
\[
e^\sigma p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + p_{02}m_i m_j m_k = 0.
\]
By contraction by \( b^i \), we have
\[
e^\sigma p_{-1}(2m_j m_k + m^2 h_{jk}) + p_{02}m^2 m_j m_k = 0. \tag{2.1}
\]
Contracting (2.1) again by \( b^j \), we get \( 3e^\sigma p_{-1} = m^2 p_{02} \). Hence, (2.1) reduces to \( p_{-1}(m^2 h_{jk} - m_j m_k) = 0 \), which leads to \( p_{-1} = 0 \) or \( m^2 h_{jk} - m_j m_k = 0 \). Now, if \( m^2 h_{jk} - m_j m_k = 0 \), then, \( n = 2 \) which contradicts the hypothesis. Hence, \( p_{-1} = 0 \), and consequently, \( q = k\beta \). Then, we have the partial differential equation
\[
ff_2 = k\beta.
\]
By integration with respect to \( \beta \) and using the fact that \( f \) is homogenous of degree 1 in \( \beta \) and \( \tilde{L} \), we get
\[
f^2 = k\beta^2 + \varphi(\tilde{L}),
\]
where \( \varphi(\tilde{L}) \) is a homogenous function of degree 2 in \( \tilde{L} \), which may be written as \( \varphi(\tilde{L}) = k'\tilde{L}^2 \). Hence, \( f^2 = k\beta^2 + k'\tilde{L}^2 \) and consequently,
\[
\overline{L} = (k'\tilde{L}^2 + k\beta^2)^{\frac{1}{2}} = (k'e^{2\sigma} L^2 + k\beta^2)^{\frac{1}{2}}.
\]
(d)⇒(a): It is obvious. \( \square \)

**Corollary 2.2.** For \( n > 2 \), under a generalized \( \beta \)-conformal change, if one of the above equivalent conditions holds, then the space \((M, \overline{L})\) is Riemannian if and only if \((M, L)\) is Riemannian.

We will study the change of some special Finsler spaces under a generalized \( \beta \)-conformal change.

**Definition 2.3.** A Finsler space \((M, L)\) with dimension \( n \geq 3 \) is said to be quasi-C-reducible if the Cartan tensor \( C_{ijk} \) satisfies
\[
C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j, \tag{2.2}
\]
where \( Q_{ij} \) is a symmetric indicatory tensor.
By Proposition 1.3, assuming $\lambda \neq 0$, we have

$$
\bar{C}_{ijk} = e^\sigma p C_{ijk} + \frac{e^\sigma p - 1}{2} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{p_{02}}{2} m_i m_j m_k
$$

$$
= e^\sigma p C_{ijk} + \frac{1}{6} \mathcal{S}_{ijk} \{ (3 e^\sigma p - 1) h_{ij} + p_{02} m_i m_j) \}
$$

$$
= e^\sigma p C_{ijk} + \frac{1}{6 \lambda} \mathcal{S}_{ijk} \{ (3 e^\sigma p - 1) h_{ij} + p_{02} m_i m_j) (\bar{C}_k - C_k + e^\sigma p s_0 C_{k\beta \beta}) \}
$$

$$
= e^\sigma p C_{ijk} + \frac{1}{6 \lambda} \mathcal{S}_{ijk} \{ (3 e^\sigma p - 1) h_{ij} + p_{02} m_i m_j) \bar{C}_k \}
$$

$$
= \frac{1}{6 \lambda} \mathcal{S}_{ijk} \{ (3 e^\sigma p - 1) h_{ij} + p_{02} m_i m_j) (e^\sigma p s_0 C_{k\beta \beta} - C_k) \}.
$$

Hence, we have

**Lemma 2.4.** Under a generalized $\beta$-conformal change, the transformed Cartan tensor can be written in the form

$$
\bar{C}_{ijk} = \mathcal{S}_{ijk} \{ \bar{Q}_{ij} \bar{C}_k \} + q_{ijk},
$$

where $\bar{Q}_{ij} := \frac{1}{6 \lambda} (3 e^\sigma p - 1) h_{ij} + p_{02} m_i m_j$,

$$
q_{ijk} := \frac{1}{6 \lambda} \mathcal{S}_{ijk} \{ 2 e^\sigma \lambda p C_{ijk} + (3 e^\sigma p - 1) h_{ij} + p_{02} m_i m_j) (e^\sigma p s_0 C_{k\beta \beta} - C_k) \}.
$$

By the above lemma and taking into account that the tensor $\bar{Q}_{ij}$ is symmetric and indicatory, we get the following result.

**Theorem 2.5.** If the tensor $q_{ijk} = 0$, then the space $(M, L)$ is quasi-C-reducible.

As a corollary of the above theorem, we obtain a generalized form of Matsumoto’s result [14]:

**Corollary 2.6.** Under a generalized $\beta$-conformal change, a Reimannian space $(M, L)$ is transformed to a quasi-C-reducible space.

**Definition 2.7.** A Finsler space $(M, L)$ of dimension $n \geq 3$ is called semi-C-reducible, if the Cartan tensor $C_{ijk}$ is written in the form:

$$
C_{ijk} = \frac{r}{n+1} (h_{ij} C_k + h_{ki} C_j + h_{jk} C_i) + \frac{t}{C^2} C_i C_j C_k,
$$

(2.3)

where $r$ and $t$ are scalar functions such that $r + t = 1$.

The next result has been obtained by Matsumoto and Shibata [16] in the special case of Finsler spaces with $(\alpha, \beta)$-metric.

**Theorem 2.8.** A Riemannian space is transformed to a semi-C-reducible space, by a generalized $\beta$-conformal change.
Proof. From Proposition 1.1 and Proposition 1.3 we get

\[
\bar{C}_{ijk} = \frac{1}{2} e^\sigma p^{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{1}{2} p_{02}m_i m_j m_k \\
= \frac{p^{-1}}{2p\lambda} (\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j) + \frac{m^2(pp_{02} - 3p_{-1}q_0)}{2p\lambda(e^\sigma p + m^2q_0)\bar{C}} \bar{C}_i \bar{C}_j \bar{C}_k \\
= \frac{r}{n+1} (\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j) + \frac{t}{\bar{C}^2} \bar{C}_i \bar{C}_j \bar{C}_k,
\]

where

\[
r = \frac{p^{-1}(n+1)}{2p\lambda}, \quad t = \frac{m^2(p_{02} - 3p_{-1}q_0)}{2p\lambda(e^\sigma p + m^2q_0)}, \quad r + t = 1,
\]

which means that \((M, \bar{L})\) is semi-reducible. \(\square\)

**Definition 2.9.** A Finsler space \((M, L)\) of dimension \(n \geq 3\) is called C-reducible if the Cartan tensor \(C_{ijk}\) has the form:

\[
C_{ijk} = h_{ij}A_k + h_{ki}A_j + h_{jk}A_i, \quad A_i = \frac{C_i}{n+1}. \quad (2.4)
\]

Define the tensor

\[
K_{ijk} = C_{ijk} - (h_{ij}A_k + h_{ki}A_j + h_{jk}A_i).
\]

It is clear that \(K_{ijk}\) is symmetric and indicatory. Moreover, \(K_{ijk}\) vanishes if and only if the Finsler space \((M, L)\) is C-reducible.

**Proposition 2.10.** Under a generalized \(\beta\)-conformal change, the tensor \(\bar{K}_{ijk}\) associated with the space \((M, \bar{L})\) has the form

\[
\bar{K}_{ijk} = e^\sigma pK_{ijk} + d_{ijk},
\]

where

\[
d_{ijk} := \frac{1}{n+1} \bar{G}_{ijk}\{(n+1)(\alpha_1 h_{ij} + \alpha_2 m_i m_j)m_k + q_0 m_i m_j C_k \\
+ (s_0 p q_0 m_i m_j + e^\sigma p^2 s_0 h_{ij})C_{k\beta}\}, \quad \alpha_1 := \frac{e^\sigma p^{-1}}{2} - \frac{e^\sigma p\lambda}{n+1}, \quad \alpha_2 := \frac{p_{02}}{6} - \frac{q_0\lambda}{n+1}.
\]

Consequently, we have

**Theorem 2.11.** Under a generalized \(\beta\)-conformal change, the following assertions

(a) the space \((M, L)\) is C-reducible,

(b) the space \((M, \bar{L})\) is C-reducible

are equivalent if and only if the tensor \(d_{ijk}\) vanishes.

**Corollary 2.12.** If \(\bar{L} = e^\sigma L + \beta, L\) being Finslerian, then the tensor \(d_{ijk}\) vanishes. Consequently, \((M, \bar{L})\) is C-reducible if and only if \((M, L)\) is C-reducible.
Lemma 2.13. Under a generalized $\beta$-conformal change $\mathcal{L} = f(e^\sigma L, \beta)$, with $L$ Riemannian, the tensor $d_{ijk}$ takes the form

$$d_{ijk} = \mathcal{G}_{ijk}\{\alpha_1 h_{ij}m_k + \alpha_2 m_i m_j m_k\}.$$ 

Theorem 2.14. Under a generalized $\beta$-conformal change $\mathcal{L} = f(e^\sigma L, \beta)$, with $L$ Riemannian, the following assertions are equivalent:

(a) $\alpha_1 = 0$ and $\alpha_2 = 0$,

(b) $(M, \mathcal{L})$ is C-reducible,

(c) $(M, \mathcal{L})$ is either of Randers type or of Kropina type.

Proof.

(a) $\Rightarrow$ (b): It is obvious.

(b) $\Rightarrow$ (a): Let the space $(M, \mathcal{L})$ be C-reducible, then, by Lemma 2.13 we have:

$$d_{ijk} = \mathcal{G}_{ijk}\{\alpha_1 h_{ij}m_k + \alpha_2 m_i m_j m_k\} = 0. \quad (2.5)$$

Contracting (2.5) by $g^{ij}$, we get

$$(n + 1)\alpha_1 + 3m^2\alpha_2 = 0, \quad (2.6)$$

and contracting the same equation by $b^i b^j$, we get

$$\alpha_1 + m^2\alpha_2 = 0. \quad (2.7)$$

The last two relations lead to $(n - 2)\alpha_1 = 0$. Since $n > 2$, then $\alpha_1 = 0$ and consequently $\alpha_2 = 0$, by (2.7).

(a) $\Rightarrow$ (c): If $\alpha_1 = \alpha_2 = 0$, we have $(n + 1)p_{-1} = 2p\lambda$ and $(n + 1)p_{02} = 6q_0\lambda$. Solving the last two equations for $\lambda$, we get

$$3q_0 p_{-1} = pp_{02}.$$ 

From which we obtain the partial differential equation

$$\frac{3f_{12}f_{22}}{f_1} - f_{222} = 0.$$ 

Now, if $f_{22} = 0$, by integration with respect to $\beta$ and taking the homogeneity of $f$ into account, we get $f_2 = \varphi_1(\tilde{L})$, where $\varphi_1(\tilde{L})$ is a homogenous function of degree 0 in $\tilde{L}$. Hence, by integrating $f_2$ with respect to $\beta$, we get

$$\mathcal{L} = \varphi_1(\tilde{L})\beta + \varphi_2(\tilde{L}),$$

where $\varphi_2(\tilde{L})$ is a homogenous function of degree 1 in $\tilde{L}$. By the homogeneity properties of $\varphi_1(\tilde{L})$ and $\varphi_2(\tilde{L})$, using Euler theorem, we conclude that $\varphi_1(\tilde{L}) = c_1$ and $\varphi_2(\tilde{L}) = c_2$, where $c_1$ and $c_2$ are constants. Consequently,

$$\mathcal{L} = c_2 \tilde{L} + c_1 \beta.$$
On the other hand, if $f_{22} \neq 0$, we have

$$\frac{3f_{12}}{f_1} - \frac{f_{22}}{f_2} = 0,$$

which, by integration with respect to $\beta$, gives

$$3 \ln f_1 - \ln f_{22} = \ln \varphi_3(\tilde{L}) \Rightarrow \frac{f_1^3}{f_{22}} = \varphi_3(\tilde{L}) = c_3 \tilde{L},$$

where $\varphi_3(\tilde{L})$ is a homogenous function of degree 1 in $\tilde{L}$ and $c_3$ is nonzero constant.

Using $\tilde{L} f_{11} + \beta f_{12} = 0$ and $\tilde{L} f_{21} + \beta f_{22} = 0$, we have

$$\frac{f_{11}}{f_1^3} = \frac{c_3 \beta^2}{\tilde{L}^3},$$

from which $f_1 = \frac{\tilde{L}}{\sqrt{c_3 \beta^2 + c_4 \tilde{L}^2}}$. If $c_4 \neq 0$, then

$$f = \frac{1}{c_4} \sqrt{c_3 \beta^2 + c_4 \tilde{L}^2 + c_5 \beta},$$

and if $c_4 = 0$, then

$$f = \frac{\tilde{L}^2 + \beta^2}{c_3 \beta}.$$ 

The former may be regarded as of Randers type and the later as of Kropina type.

(c) $\Rightarrow$ (a): The result follows directly by computing $\alpha_1$ and $\alpha_2$ for Randers and Kropina spaces. 

It should be noted that Matsumoto [9] showed that C-reducible Finsler spaces with $(\alpha, \beta)$-metric are either of Randers type or of Kropina type.

Definition 2.15. A Finsler space $(M, L)$ of dimension $n \geq 2$ is said to be $C_2$-like if the Cartan tensor $C_{ijk}$ satisfies

$$C^2 C_{ijk} = C_i C_j C_k. \quad (2.8)$$

Let us define the tensor

$$\eta_{ijk} = C^2 C_{ijk} - C_i C_j C_k.$$ 

It is clear that $\eta_{ijk}$ is symmetric and indicatory. Moreover, $\eta_{ijk}$ vanishes if and only if the Finsler space is $C_2$-like space.

Proposition 2.16. Under a generalized $\beta$-conformal change, the tensor $\tilde{\eta}_{ijk}$ associated with the space $(M, \tilde{L})$ has the form

$$\tilde{\eta}_{ijk} = \eta_{ijk} + I_{ijk},$$
where
\[
I_{ijk} := (e^{-\sigma}/p)C^2V_{ijk} + \Phi(e^\sigma p C_{ijk} + V_{ijk}) - \lambda^3 m_i m_j m_k \\
- \lambda^2(m_i m_k C_i + m_i m_j C_k + m_m k C_j) - \lambda(m_k C_i C_j + m_j C_k C_i + m_i C_j C_k) \\
- e^\sigma p s_0 \{C_{\beta\beta\beta}(\lambda m_j - C_i - e^\sigma p s_0 C_{i\beta\beta}) + e^\sigma p s_0 C_{j\beta\beta}(\lambda m_k + C_k) \\
- \lambda (m_k C_j + m_j C_k - \lambda m_m k) - C_j C_k \} + C_{\beta\beta\beta}(C_i C_j - e^\sigma p s_0 C_{i\beta\beta} + \lambda m_i C_j \\
+ \lambda m_j C_i + \lambda^2 m_i m_j) + \lambda C_{\beta\beta\beta}(\lambda m_i m_k + C_i C_k + m_i C_k + m_k C_i - e^\sigma p s_0 m_i C_{i\beta\beta})].
\]

**Theorem 2.17.** Under a generalized $\beta$-conformal change, the following assertions

(a) the space $(M, L)$ is $C_2$-like,

(b) the space $(M, \overline{L})$ is $C_2$-like

are equivalent if and only if the tensor $I_{ijk}$ vanishes.

**Lemma 2.18.** Starting with a Riemannian space $(M, L)$, under a generalized $\beta$-conformal change, the tensor $I_{ijk}$ takes the form:
\[
I_{ijk} = \Phi V_{ijk} - \lambda^3 m_i m_j m_k.
\]

**Theorem 2.19.** For a $\beta$-conformal change $\overline{L} = e^\sigma L + \beta$; $L$ being Finslerian, a necessary condition for the assertions

(a) the space $(M, L)$ is $C_2$-like,

(b) the space $(M, \overline{L})$ is $C_2$-like

to be equivalent is that $C_\beta = 0$.

**Proof.** In the case of $\overline{L} = e^\sigma L + \beta$; $L$ being Finslerian, $\Phi = \lambda e^{-\sigma} L(\lambda m^2 + 2 C_\beta)/\overline{L}$, $\lambda = \frac{n+1}{2\overline{L}}$ and $V_{ijk} = e^{\sigma \overline{L}}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j)$. Now, let the above assertions be equivalent, so $I_{ijk} = 0$. Contracting (2.9) by $g^{ik}$, we have $\frac{n+1}{2\overline{L}} C_\beta = 0$ and the result follows.

If $(M, L)$ is a Riemannian space and the tensor $I_{ijk}$ vanishes, i.e., $(M, \overline{L})$ is $C_2$-like, we have
\[
\lambda^2 \frac{m^2}{e^\sigma p + q_0 m^2} (e^\sigma p_{-1} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + p_{02} m_i m_j m_k) - 2\lambda^3 m_i m_j m_k = 0,
\]
contracting by $b^i b^j$ and assuming that $\lambda \neq 0$, we get
\[
(n - 2)p_{-1} = 0.
\]

Hence, we have

**Theorem 2.20.** Starting with a Riemannian space $(M, L)$, if the transformed space $(M, \overline{L})$ is $C_2$-like, then one of the following holds:

(a) $\dim M = 2$.  

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The generalized $\beta$-conformal change is an energy $\beta$-change and the transformed space is Riemannian.

**Corollary 2.21.** Let the generalized $\beta$-conformal change be of the form $\mathcal{L} = e^\sigma L + \beta$, with $L$ Riemannian. If $(M, \mathcal{L})$ is $C_2$-like, then $\dim M = 2$.

**Corollary 2.22.** A Riemannian space of dimension $\geq 3$ cannot be transformed to a non-Riemannian $C_2$-like space.

Now, we are going to study two special Finsler spaces whose defining property depends on the v-curvature tensor $S_{lijk}$, namely, the $S_3$-like and $S_4$-like Finsler spaces.

**Definition 2.23.** A Finsler space $(M^n, L)$ with dimension $n > 3$ is said to be $S_3$-like if the v-curvature tensor $S_{lijk}$ satisfies

$$S_{lijk} = \frac{S}{(n-1)(n-2)}\{h_{ik}h_{lj} - h_{ij}h_{ik}\}, \quad (2.11)$$

where $S$ is the vertical scalar curvature.

Define the following tensor

$$\mu_{lijk} = S_{lijk} - \frac{S}{(n-1)(n-2)}\{h_{ik}h_{lj} - h_{ij}h_{ik}\}.$$ 

It is clear that the tensor $\mu_{lijk}$ vanishes if and only if the space is $S_3$-like.

**Proposition 2.24.** Under a generalized $\beta$-conformal change, the tensor $\mathcal{L}_{lijk}$ associated with the space $(M, \mathcal{L})$ has the form:

$$\mathcal{L}_{lijk} = e^\sigma p\mu_{lijk} + r_{lijk},$$

where

$$r_{lijk} = \mathcal{A}_{jk}\{H_{ik}h_{lj} + H_{ij}h_{ik} + \omega_{ik}C_{lijk} + \omega_{lj}C_{lijk} - \frac{e^{2\sigma p^2\Omega}}{(n-1)(n-2)}h_{lj}h_{ik}$$

$$- \frac{q_0}{(n-1)(n-2)}(S + e^\sigma p\Omega)(h_{ik}m_im_j + h_{lj}m_im_k)\},$$

$$\Omega := \frac{e^{-\sigma}}{p}\Psi - s_0S_{ik}b^k_jb^k - s_0\Psi_{\beta\beta} + \frac{2e^{-\sigma}}{p}K(n - 2 - e^\sigma ps_0m^2).$$

**Theorem 2.25.** Under a generalized $\beta$-conformal change, the following assertions

(a) the space $(M, L)$ is $S_3$-like,

(b) the space $(M, \mathcal{L})$ is $S_3$-like

are equivalent if and only if the tensor $r_{lijk}$ vanishes.

**Proposition 2.26.** For a $\beta$-conformal change $\mathcal{L} = e^\sigma L + \beta$, the tensor $r_{lijk}$ takes the form

$$r_{lijk} = C_{lijk}h_{il} + \frac{1}{2L}m_jm_kh_{il} + \frac{m^2}{4L}h_{jk}h_{il} - A_{\beta}h_{jk}h_{il},$$

where $A_{\beta} = \frac{1}{n-1}(C_{\beta} + \frac{n+1}{4L}m^2)$. 

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From the above proposition, we retrieve a result due to Abed [1]:

**Corollary 2.27.** In the case of a $\beta$-conformal change $\overline{L} = e^\sigma L + \beta$, the following assertions

(a) the space $(M, L)$ is $S_3$-like,
(b) the space $(M, \overline{L})$ is $S_3$-like

are equivalent if and only if
\[
C_{rjk}b^r + \frac{1}{2\overline{L}} m_jm_k + \frac{m^2}{4\overline{L}} h_{jk} = A_\beta h_{jk}.
\]

Finally, we study $S_4$-like Finsler spaces under a generalized $\beta$-conformal change.

**Definition 2.28.** A Finsler space $(M, L)$ with dimension $n > 4$ is said to be $S_4$-like if the v-curvature tensor $S_{hijk}$ satisfies
\[
S_{lijk} = A_{jk}\{h_{lj}M_{ik} + h_{ik}M_{lj}\} = 0,
\]
where $M_{ik} = \frac{1}{(n - 3)}\left\{S_{ik} - \frac{S_{hik}}{2(n - 2)}\right\}$.

Define the tensor
\[
\zeta_{lijk} = S_{lijk} - A_{jk}\{h_{lj}M_{ik} + h_{ik}M_{lj}\}.\]
It is clear that the tensor $\zeta_{hijk}$ vanishes if and only if the space is $S_4$-like.

**Proposition 2.29.** Under a generalized generalized $\beta$-conformal change, the tensor $\zeta_{lijk}$ associated with the space $(M, \overline{L})$ has the form
\[
\zeta_{lijk} = e^\sigma p\zeta_{lijk} + \varepsilon_{lijk},
\]
where
\[
\varepsilon_{lijk} = F_{lijk}\{X_{lk}Y_{ij}\} = X_{lk}Y_{ij} + X_{ij}Y_{lk} - X_{lj}Y_{ik} - X_{ik}Y_{lj}.
\]

**Theorem 2.30.** Under a generalized $\beta$-conformal change, the following assertions

(a) the space $(M, L)$ is $S_4$-like,
(b) the space $(M, \overline{L})$ is $S_4$-like
are equivalent if and only if the tensor $\varepsilon_{hijk}$ vanishes.

In the case of a $\beta$-conformal change $\mathcal{T} = e^\sigma L + \beta$, the tensor $\varepsilon_{hijk}$ vanishes and we retrieve the following result of Abed [1].

**Corollary 2.31.** For a $\beta$-conformal change $\mathcal{T} = e^\sigma L + \beta$, the space $(M, L)$ is $S_4$-like if and only if the space $(M, \mathcal{T})$ is $S_4$-like.

## 3. The T-tensor $T_{hijk}$

The T-tensor is defined by [10]

$$T_{hijk} = LC_{hij|k} + C_{hijl}k + C_{hijk}l + C_{ijkl}h,$$

It should be noted that the T-tensor has a great contribution in geometric properties of special Finsler spaces. For instance, Hashiguchi [4] has shown that a Landsberg space remains Landsberg under a conformal transformation, if and only if $T_{hijk} = 0$.

On the other hand, Matsumoto [12] has obtained interesting results for spaces with $T_{hijk} = 0$ and, further, he investigated the three-dimensional Finsler spaces with vanishing T-tensor.

In this section we compute the T-tensor under a generalized $\beta$-conformal change and consider some interesting special cases.

**Theorem 3.1.** Under a generalized $\beta$-conformal change, the transformed T-tensor takes the form:

$$T_{lijk} = e^\sigma p\mathcal{T}_{lijk} - \mathcal{T}({\frac{\beta e^\sigma p_2}{2L^2}} + 2K_3)(h_i h_j k + h_i j h_k + h_i k h_j) + (h_i j k + h_i k j + h_i l j + h_l k j + h_l j k) + (e^\sigma p f_2 - \frac{1}{2} e^\sigma T_{lijk})(C_{lijk}m_k + C_{lijk}m_l + C_{lijk}m_i + C_{lijk}m_j) - \mathcal{T}(M_{ij}C_{ik\beta} + M_{jk}C_{ik\beta} + M_{ik}C_{jk\beta} + M_{ik}C_{jik\beta} + M_{ik}C_{jik\beta}) + \mathcal{T}e^{2\sigma} s_0 p^2 (C_{ijk}\beta + C_{ijkl\beta} + C_{ijkl\beta} + C_{ijkl\beta}) + \frac{1}{2} \mathcal{T}(6K_5 + p_{022})m_i m_l m_j m_k - \frac{\mathcal{T} p_{02}}{2L} (n_{ij} m_k m_l + n_{lk} m_j m_i + n_{lk} m_i m_j) + \frac{1}{2} p_{02} (\mathcal{T}_{lijk} m_k m_l + \mathcal{T}_{lijk} m_i m_j)

where

$$\nu_{ij} := \frac{1}{2} e^\sigma p_{-1} \hat{n}_{ij} - \mathcal{T}(K_1 + \frac{3e^\sigma p_{-1}}{4p})m_i m_j - \frac{\mathcal{T}e^{\sigma} p_{-1}}{2L} n_{ij},$$

$$K_5 := e^{2\sigma} s_0 p^2 - \frac{4e^\sigma p_{-1} p_{02} + p_{02} m^2}{4(e^\sigma p + q_0 m^2)},$$

$$\hat{n}_{ij} := \mathcal{T}_{lijk} m_l + \mathcal{T}_{lijk} m_i.$$
Proof. One can show that
\[
\hat{\partial}_k \overline{C}_{ij} = e^\sigma \hat{p}_k C_{ij} + e^\sigma p_{-1}(C_{ij} m_k + C_{ik} m_l + C_{jk} m_l + C_{lk} m_i)
\]
\[
- \frac{e^\sigma p_{-1}}{2L}(h_{li} n_{jk} + h_{lj} n_{ik} + h_{lj} m_k + h_{lk} n_{ij} + h_{ik} n_{jl})
\]
\[
- \frac{\beta e^\sigma p_{-1}}{2L^2}(h_{li} h_{jk} + h_{lj} h_{ik} + h_{ik} h_{ij}) - \frac{\beta p_{02}}{2L^2}(h_{ij} m_k m_l + h_{li} m_j m_k + h_{ik} m_i m_j)
\]
\[
+ h_{ijk} m_l m_i + h_{ik} m_j m_l + h_{ij} m_i m_k + \frac{1}{2} p_{022} m_j m_k m_l
\]
\[
- \frac{p_{02}}{2L}(n_{ij} m_k m_l + n_{ik} m_i m_j)
\]
(3.1)
where \(n_{ij} := l_i m_j + l_j m_i\) and \(p_{02} := \frac{\partial}{\partial \beta} p_{02}\). Similarly,
\[
\overline{C}_{ij} \overline{C}_{lk} = e^\sigma p C_{ij} C_{lk} + \frac{1}{2} e^\sigma p_{-1}(C_{ij} m_i + C_{ik} m_j + C_{jk} m_l + C_{lk} m_i)
\]
\[
+ K_4(C_{ik} \beta m_j m_l + C_{ij} \beta m_l m_k) + (K_1 + \frac{1}{4p} e^\sigma p_{-1})(h_{ij} m_k m_l + h_{ik} m_i m_j)
\]
\[
+ 2K_3 h_{ij} h_{lk} + K_2(C_{ik} \beta h_{ij} + C_{ij} \beta h_{lk}) - K_5 m_i m_j m_k - e^2s^2 s_0 C_{ij} \beta C_{lk} \beta
\]
\[
+ e^\sigma p_{-1}(h_{ij} m_k m_l + h_{ij} m_k m_i + h_{ik} m_j m_l + h_{jk} m_i m_k),
\]
(3.2)
Using (3.1) and (3.2), we get
\[
\overline{C}_{ij} \overline{C}_{lk} = \hat{\partial}_k \overline{C}_{ij} - \overline{C}_{ik} \overline{C}_{mj} - \overline{C}_{im} \overline{C}_{lj} - \overline{C}_{ik} \overline{C}_{ml}
\]
\[
= e^\sigma p C_{ij} - \left(\frac{\beta e^\sigma p_{-1}}{2L^2} + 2K_3\right)(h_{ij} h_{jk} + h_{ij} h_{ik} + h_{ik} h_{ij}) + 3K_5 m_i m_j m_k m_l
\]
\[
+ \frac{1}{2} e^\sigma p_{-1}(C_{ijk} m_i + C_{ikj} m_l + C_{jik} m_l + C_{lj} m_i) - \frac{p_{02}}{2L}(n_{ij} m_k m_l + n_{ik} m_j m_l)
\]
\[
- \frac{e^\sigma p_{-1}}{2L}(h_{li} n_{jk} + h_{lj} n_{ik} + h_{lj} n_{ik} + h_{lk} n_{ij} + h_{ik} n_{jl})
\]
\[
- (K_1 + \frac{3\beta e^\sigma p_{-1}}{4p}) (h_{ij} m_k m_l + h_{ik} m_i m_j + h_{ij} m_k m_l + h_{ik} m_i m_j)
\]
\[
+ h_{ij} m_k m_l + h_{ijk} m_l m_k) - (M_{ij} C_{ik} \beta + M_{ij} C_{ik} \beta + M_{ik} C_{jk} \beta + M_{ik} C_{ik} \beta
\]
\[
+ M_{ik} C_{ik} \beta + M_{ik} C_{ik} \beta) + e^2s^2 s_0 (C_{ij} \beta C_{ik} \beta + C_{ij} \beta C_{ik} \beta + C_{ik} \beta C_{ik} \beta),
\]
(3.3)
where \(M_{ij} := K_2 h_{ij} + K_4 m_i m_j\).

The result follows from (3.3), Proposition 1.3 and the definition of the transformed T-tensor
\[
\mathbf{T}_{hijk} = \mathbf{L} \overline{C}_{hij} \overline{C}_{lk} + \overline{C}_{hij} \overline{C}_{lk} + \overline{C}_{hik} \overline{C}_{lj} + \overline{C}_{hjk} \overline{C}_{li} + \overline{C}_{ijk} \overline{C}_{hl}.
\]

The transformed T-tensor for some important special Finsler spaces can be deduced from the above result.

Corollary 3.2. Under a Kropina change, \(\mathbf{T} = \mathbf{L}^2 / \beta\); \(\mathbf{L}\) being Reimannian, the transformed T-tensor takes the form:
\[
\mathbf{T}_{hijk} = \frac{2L}{L^2 b^2}(h_{li} h_{jk} + h_{lj} h_{ik} + h_{lk} h_{ij}) + \frac{2L^2}{\beta L^2 b^2}(h_{ij} m_k m_l + h_{ij} m_i m_k + h_{ij} m_l m_k)
\]
\[
+ h_{jk} m_i m_l + h_{ik} m_i m_j + h_{ik} m_j m_i) + \frac{6L^3}{\beta^2 L^2 b^2} m_i m_j m_k.
\]
(3.4)
Corollary 3.3. Under a conformal change $\mathcal{L} = e^\sigma L$, the transformed $T$-tensor takes the form

$$\mathcal{T}_{ijkl} = e^{3\sigma}T_{ijkl}$$

Corollary 3.4. Under a Randers change $\mathcal{L} = L + \beta$, $L$ being Riemannian, the $T$-tensor takes the form:

$$\mathcal{T}_{ijkl} = -\frac{\Theta_1}{4L^3}(h_{li}h_{jk} + h_{lj}h_{ik} + h_{lk}h_{ij}),$$

where $\Theta_1 := L^2b^2 + \beta^2 + 2L\beta$.

The above case has been studied by Matsumoto [11].

Corollary 3.5. Under a $\beta$-conformal change $\mathcal{L} = e^\sigma L + \beta$; $L$ being Finslerian, the transformed $T$-tensor takes the form:

$$\mathcal{T}_{ijkl} = \frac{e^\sigma L^2}{L^2}T_{ijkl} - \frac{e^\sigma}{4L^3}(h_{li}h_{jk} + h_{lj}h_{ik} + h_{lk}h_{ij})$$

$$+ \frac{e^\sigma}{2L}(C_{lj}m_k + C_{ik}m_l + C_{jk}m_i + C_{ik}m_j)$$

$$- \frac{e^\sigma}{2L}(h_{ij}C_{lk}\beta + h_{il}C_{jk}\beta + h_{ik}C_{lj}\beta + h_{jk}C_{il}\beta + h_{lk}C_{ij}\beta + h_{lj}C_{ik}\beta),$$

where $\Theta := L^2b^2 + \beta^2 + 2e^\sigma L\beta$.

Corollary 3.6. Under a $\beta$-conformal change, a necessary condition for the vanishing of the transformed $T$-tensor is that

$$T = \frac{(n^2 - 1)\Theta}{4LL^2} + \frac{(n - 1)L}{\mathcal{L}}C_\beta,$$

where $T = g^{ij}g^{ik}T_{ijkl}$.

4. The $b$-condition

In this section we introduce and investigate what we call the $b$-condition. We study the effect of subjecting some special Finsler spaces to this condition. In the following we assume that we are given a generalized $\beta$-conformal change $\mathcal{L} = f(e^\sigma L, \beta)$ with $\beta = b_iy^i = b^i y_i$.

A Finsler manifold $(M, L)$ is said to satisfy the $b$-condition if

$$b^i C_{ijk} = 0.$$

Theorem 4.1. For $n > 2$, the following two assertions are equivalent:

(a) The $b$-condition is invariant under a generalized $\beta$-conformal change.
(b) The generalized $\beta$-conformal change is an energy $\beta$-change.

Proof. 
(a) $\Rightarrow$(b): Let $b^i C_{ijk} = 0$. Then, $b^i C_{ijk} = 0$ and we have, by Proposition 1.3
\[ e^\sigma p_{-1} (m^2 h_{jk} + 2m_j m_k) + p_{02} m^2 m_j m_k = 0. \]
Contracting by $b^j$, we get $3e^\sigma p_{-1} = -m^2 p_{02}$. Hence,
\[ e^\sigma p_{-1} (m^2 h_{jk} - m_j m_k) = 0, \]
contracting again by $g^{jk}$, we get
\[ (n - 2)p_{-1} = 0. \]
Since $n > 2$, then $p_{-1} = 0$ and hence the result follows from Theorem 2.1 
(b) $\Rightarrow$(a): Let the generalized $\beta$-conformal change be an energy $\beta$-change. Then, by Theorem 2.1 we obtain $C_{ijk} = e^\sigma p C_{ijk}$. Hence the result. 

Theorem 4.2. Under a generalized Randers change, if $(M, L)$ satisfies the b-condition, the generalized Randers space $(M, \overline{L})$ can not satisfy the b-condition.

Proof. Let $(M, L)$ satisfy the b-condition $b^i C_{ijk} = 0$. If $(M, \overline{L})$ satisfies the b-condition, then $b^i \overline{C}_{ijk} = 0$, and consequently,
\[ \frac{1}{2L} b^i (2L C_{ijk} + h_{ij} m_k + h_{jk} m_i + h_{ik} m_j) = 0, \]
or
\[ \frac{L}{2L} (m^2 h_{jk} + 2m_j m_k) = 0, \]
which, by contraction by $g^{jk}$, yields a contradiction: $n = -1$. 

Theorem 4.3. Consider the generalized $\beta$-change (1.1). In each of the following cases 

(a) two-dimensional Finsler space,

(b) three-dimensional Finsler space satisfying the condition $L(x, -y) = L(x, y)$,

(c) quasi-C-reducible space with $b^i b^j Q_{ij} \neq 0$,

(d) C-reducible space,

(e) The transformed space $(M, \overline{L})$ with $L$ Riemannian,

if the given Finsler space $(M, L)$ satisfies the b-condition, then it is Riemannian.

Proof. 
The proof of (a) and (b) runs on in a similar manner as given in [15] for a concurrent vector fields. 
(c) Contracting (2.2) by $b^i b^j$, we get
\[ b^i b^j Q_{ij} C_k = 0. \]
Hence, \( C_k = 0 \) for \( b^i b^j Q_{ij} \neq 0 \).

(d) Contracting (2.3) by \( b^i b^j \), we get

\[
m^2 C_k = 0.
\]

Consequently, \( C_k = 0 \).

(e) Let \((M, L)\) be a Finsler space with \((\alpha, \beta)\)-metric, then

\[
\overline{C}_{ijk} = \frac{e^\sigma p_{-1}}{2} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{p_{02}}{2} m_i m_j m_k.
\]

The condition that \( b^i \overline{C}_{ijk} = 0 \) leads to

\[
e^\sigma p_{-1} (m^2 h_{jk} + 2 m_j m_k) + p_{02} m^2 m_j m_k = 0.
\]

Contracting by \( b^j \), we get \( 3 e^\sigma p_{-1} = -m^2 p_{02} \). Hence,

\[
e^\sigma p_{-1} (m^2 h_{jk} - m_j m_k) = 0,
\]

which, by contracting by \( g^{jk} \), yields

\[(n - 2)p_{-1} = 0.\]

Thus, if \( n = 2 \), the result follows by (a) and if \( p_{-1} = 0 \), then \( p_{02} = 0 \) and hence \( \overline{C}_{ijk} = 0 \).

**Theorem 4.4.** A semi-C-reducible Finsler space satisfying the b-condition is either Riemannian or C2-like.

**Proof.** Contracting (2.3) by \( b^i b^k \), we have \( r m^2 C_i = 0 \). Since \( m^2 \neq 0 \), then either \( r = 0 \), which implies that the space is C2-like, or \( C_i = 0 \), which implies that the space is Riemannian.

**Theorem 4.5.** If an \( S_3 \)-like Finsler space \((M, L)\) satisfies the b-condition, then its vertical curvature tensor \( S_{hijk} \) vanishes.

**Proof.** Contracting (2.11) by \( b^l \), we get

\[
S(h_{ik} m_j - h_{ij} m_k) = 0. \quad (4.1)
\]

Again, contracting (4.1) by \( g^{ij} \), we have \( (n - 2)S m_k = 0 \). As \( n > 4 \) and \( m_k \neq 0 \), it follows that \( S = 0 \) and consequently, \( S_{hijk} = 0 \).

**Lemma 4.6.** If a Finsler space satisfies the b-condition, then we have \( b^i|_h = 0 \) and, consequently,

\[
C_{ijk|k} b^h = C_{ijk} b^h = 0.
\]

**Proof.** From the definition of vertical covariant derivative of Cartan connection, we have

\[
b^i|_h = \dot{\hat{o}}_h b^i + b^m C^i_{mh} = \dot{\hat{o}}_h (b_j g^{ij}) = (\dot{\hat{o}}_h b_j) g^{ij} + b_j \dot{\hat{o}}_h g^{ij} = 0
\]

and hence \( C_{ijk|k} b^h = C_{ijk} (C_{ijk} b^h)|_k = 0 \).
It is well-known that if \((M, L)\) is Riemannian, then the T-tensor vanishes. But the converse is not true in general. The next result shows that the converse is true in the case where \((M, L)\) satisfies the b-condition.

**Theorem 4.7.** A Finsler space satisfying the b-condition is Riemannian if and only if the T-tensor \(T_{hijk}\) vanishes.

**Proof.** It is clear that if the space is Riemannian then the T-tensor vanishes. On the other hand, if the T-tensor vanishes, then

\[
LC_{hijk} + C_{hijl}k + C_{hijkl} + C_{hjkl}i + C_{ijkl}h = 0.
\]

Contracting by \(b^i\), using Lemma 4.6, we have \(\beta_i C_{hijk} = 0\). Hence \(C_{hijk} = 0\). \(\square\)

Let us write
\[
T_{ij} := T_{ijhk}g^{hk} = LC_{ij} + l_4C_{ij} + l_jC_i.
\]

By contracting (4.2) by \(b^i\), making use of Lemma 4.8, we have
\[
T_{ij}b^i = (\beta/L)C_j.
\]

Hence, we have

**Corollary 4.8.** A Finsler space satisfying the b-condition is Riemannian if and only if the tensor \(T_{ij}\) vanishes.

5. Projective change and generalized \(\beta\)-conformal change

In this section we will be guided by Matsomoto [13] and Shibata [18]. For two Finsler spaces \((M, L)\) and \((M, \overline{L})\) with the same underlying manifold \(M\), if every geodesic on of \((M, L)\) is also a geodesic of \((M, \overline{L})\) and vice versa, the change \(L \rightarrow \overline{L}\) of Finsler metrics is said to be projective. A geodesic on \((M, L)\) is characterized by
\[
\frac{dy^i}{dt} + 2G^i = wy^i, \quad \frac{dx^i}{dt} = y^i,
\]
where \(w = (d^2s/dt^2)/(ds/dt)^2\) and \(G^i(x, y) = \frac{1}{2}\gamma^i_{jk}y^jy^k\) is the canonical spray of \((M, L)\). We are going to find out a condition for a generalized \(\beta\)-conformal change to be projective.

Consider the left hand side of Euler-Lagrange equations
\[
\mathcal{E}_i := \partial_i L - \frac{d}{dt}(\dot{\partial}_i L)
\]

**Proposition 5.1.** Under a generalized \(\beta\)-conformal change \(\overline{L} = f(e^\sigma L, \beta)\), the functions (5.1) are transformed according to
\[
f\mathcal{E}_i = Le^p\mathcal{E}_i + Lq_0m_im^r\mathcal{E}_r + \varphi_i,
\]
where
\[
\varphi_i := L^2e^p\sigma_i - (pLe^\sigma l_i - q_0\beta m_i)\sigma_0 + 2qF_{0i} - q_0E_{00}m_i,
\]
\[
\sigma_i := \partial_i\sigma, \quad F_{ij} := \frac{1}{2}(b_{ij} - b_{ji}), \quad E_{ij} := \frac{1}{2}(b_{ij} + b_{ji}),
\]
\[
\sigma_0 = \sigma_iy^i, \quad F_{0i} = F_{ii}y^i, \quad E_{00} = E_{ij}y^iy^j.
\]
Proof. Making use of the homogeneity of \( f \), \( \overline{E}_i \) can be computed as follows.

\[
\overline{E}_i = \partial_t f - \frac{d}{dt}(\dot{\partial}_t f)
\]

\[
= f_1(\sigma, e^\sigma L + e^\sigma \partial_t L) + f_2(N^r_b b_r + b_{jj} y^j) - \frac{d}{dt}(e^\sigma l_i f_1 l_i + f_2 b_i)
\]

\[
= f_1(\sigma, e^\sigma L + f_1 e^\sigma \partial_t L + f_2 N^r_b b_r + f_2 b_{jj} y^j - f_1 l_i e^\sigma \sigma_r y^r - f_1 e^\sigma \frac{dl_i}{dt} - e^\sigma l_i \frac{df_1}{dt})
\]

\[
- b_i \frac{df_2}{dt} - f_2(b_{ij} y^j + N^r_b b_r)
\]

\[
= f_1 e^\sigma \mathcal{E}_i + f_1 L \sigma_i e^\sigma - f_1 \sigma_0 e^\sigma l_i + 2 f_2 F_{0i} - \frac{d}{dt} m_i. \tag{5.4}
\]

Using the relation \( \frac{dy_r}{dt} = y^s \partial_s y_r \), the last term \( \frac{df_2}{dt} \) of (5.4) is given by

\[
\frac{df_2}{dt} = f_{21} \frac{d\overline{L}}{dt} + f_{22} \frac{d\beta}{dt}
\]

\[
= - \frac{\beta f_{22}}{e^\sigma} e^r \frac{d e^\sigma}{dt} + \frac{d}{dt}(E_{00} + N^r_b b_r y^r + b_r \frac{dy^r}{dt})
\]

\[
= f_{22} E_{00} - L f_{32} m_i m^r \mathcal{E}_r - \beta f_{22} \sigma_0. \tag{5.5}
\]

Now, substituting (5.5) into (5.4), we get

\[
f \overline{E}_i = L e^\sigma p \mathcal{E}_i + q_0 L m_i m^r \mathcal{E}_r + L^2 e^\sigma p \sigma_i - (p L e^\sigma l_i - q_0 \beta m_i) \sigma_0 + 2 q F_{0i} - q_0 E_{00} m_i
\]

\[
= L p e^\sigma \mathcal{E}_i + q_0 L m_i m^r \mathcal{E}_r + \varphi_i. \tag{5.6}
\]

\[\square\]

Theorem 5.2. A generalized \( \beta \)-conformal change is projective if and only if the vector \( \varphi_i \) vanishes.

Proof. Let the generalized \( \beta \)-conformal change be projective. Then, \( \mathcal{E}_i = 0 \) is equivalent to \( \overline{E}_i = 0 \) and consequently, \( \varphi_i = 0 \) by (5.6).

Conversely, if \( \varphi_i = 0 \), then (5.2) shows that \( \mathcal{E}_i = 0 \) implies \( \overline{E}_i = 0 \). On the other hand, if \( \overline{E}_i = 0 \) and \( \varphi_i = 0 \), then \( e^\sigma p \mathcal{E}_i + q_0 m_i m^r \mathcal{E}_r = 0 \). Contracting the last equation by \( m^r \), taking into account that \( e^\sigma p + m^2 q_0 \neq 0 \), we get \( \mathcal{E}_r m^r = 0 \). Consequently, \( \mathcal{E}_i = 0 \).

\[\square\]

From the above theorem, we retrieve the following two results due to Shibata \[19\] and Hashiguchi and Ichijo \[5\] respectively.

Corollary 5.3. A \( \beta \)-change is projective if and only if \( 2 q F_{0i} = q_0 E_{00} m_i \).

Corollary 5.4. A Randers change is projective if and only if \( F_{0i} = 0 \), that is, \( b_i \) is gradient.

The following two results are a generalized version of Shibata’s result \[19\] and Matsumoto’s result \[13\].

Theorem 5.5. Assume that the generalized \( \beta \)-conformal change (L.1) is projective and \( L \) is Minkowskian, then the Weyl torsion \( (\overline{W})_{ij} \) and the Douglas tensor \( \overline{D}_{ijk} \) of \( (M, \overline{L}) \) vanish. Consequently, \( (M, L) \) with \( \dim M > 2 \) is projectively flat.
Proof. The Weyl torsion tensor is given by [13]:

\[ W^h_{ij} = R^h_{ij} + \frac{1}{n+1} A_{(i,j)} \{ y^h R^h_{ij} + \delta^h_i R^h_j \}, \]

where \( R_{ij} = \hat{R}_{ijk} \), \( R_j = \frac{1}{n+1} (n \hat{R}_{0j} + \hat{R}_{j0}) \) and \( \hat{R}_{ijk} \) is the h-curvature of the Berwald connection. Since \((M, L)\) is Minkowskian, then \( \hat{R}_{ijk} = 0 \), and so \( \hat{R}_{ij} = \hat{R}_i = 0 \). Consequently, \( W^h_{ij} = 0 \). By the invariance of \( W^h_{ij} \) under a projective change, we have \( \overline{W}^h_{ij} = 0 \).

The Douglas tensor is given by [13]:

\[ D^h_{ijk} = \hat{P}^h_{ijk} + \frac{1}{n+1} (y^h \hat{P}^h_{ij} |_k + \mathcal{G}_{(i,j,k)} \{ \delta^h_{ij} \hat{P}^h_{jk} \}), \]

where \( \hat{P}_{ij} = \hat{P}^h_{ij} \), \( \hat{P}^h_{ijk} \) is the hv-curvature of the Berwald connection and \( | \) denotes the vertical covariant derivative with respect to the Berwald connection \( G^h_{ij} \). Since \((M, L)\) is Minkowskian, then \( \hat{P}^h_{ijk} = 0 \), and so \( \hat{P}_{ij} = 0 \). Consequently, \( D^h_{ijk} = 0 \). By the invariance of \( D^h_{ijk} \) under a projective change, we have \( \overline{D}^h_{ijk} = 0 \).

Finally, as \( W^h_{ij} = 0 \), \( D^h_{ijk} = 0 \) and \( \dim M > 2 \), \((M, L)\) is thus projectively flat [13].

Theorem 5.6. Assume that the generalized \( \beta \)-conformal change is projective and \( L \) is Riemannian, then the projective hv-curvature tensor \( \overline{D}^h_{ijk} \) of \((M, \overline{L})\) vanishes.

Proof. Since \((M, L)\) is Riemannian, then \( \hat{P}^h_{ijk} = 0 \), and \( \hat{P}_{ij} = 0 \). Consequently, \( D^h_{ijk} = 0 \). By the invariance of \( D^h_{ijk} \) under a projective change, we have \( \overline{D}^h_{ijk} = 0 \).

Theorem 5.7. If \( \varphi_i = 0 \), then \((M, \overline{L})\) is of scalar curvature if and only if \((M, L)\) is of scalar curvature.

Proof. According to Szabó [21], a Finsler space is of scalar curvature if and only if \( W^h_{ij} \) vanishes identically. Let \( \varphi_i = 0 \), then by Theorem 5.2 the generalized \( \beta \)-conformal change is projective. Now, let \((M, L)\) be of scalar curvature, then \( W^h_{ij} = 0 \). But \( \overline{W}^h_{ij} = W^h_{ij} \), hence, \( \overline{W}^h_{ij} = 0 \). Consequently, \((M, \overline{L})\) is of scalar curvature. Conversely, let \((M, \overline{L})\) be of scalar curvature, then \( \overline{W}^h_{ij} = 0 \) which leads to \( W^h_{ij} = 0 \), hence, \((M, L)\) is of scalar curvature.

In the Riemannian case the term “of scalar curvature” reduces to the term “of constant curvature”. Thus, we generalize Yasuda and Shimada’s result [24].

Corollary 5.8. Under a generalized \( \beta \)-conformal change, if \( \varphi_i = 0 \) and \((M, L)\) is Riemannian, then the Finsler space \((M, \overline{L})\) is of scalar curvature if and only if \((M, L)\) is of constant curvature.

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