MODULAR DATA AND
REGULARITY OF MONGE-AMPÈRE EXHAUSTIONS
AND OF KOBAYASHI DISTANCE

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ABSTRACT. Regularity properties of intrinsic objects for a large class of Stein Manifolds, namely of Monge-Ampère exhaustions and Kobayashi distance, is interpreted in terms of modular data. The results lead to a construction of an infinite dimensional family of convex domains with squared Kobayashi distance of prescribed regularity properties. A new sharp refinement of Stoll’s characterization of $\mathbb{C}^n$ is also given.

1. INTRODUCTION

A well known theorem of Stoll ([19]) characterizes the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ as the unique (up to biholomorphisms) $n$-dimensional complex manifold, which admits a $C^\infty$ strictly plurisubharmonic exhaustion $\tau : \mathbb{B}^n \rightarrow [0, 1)$ such that the function $u := \log\tau|_{\{\tau > 0\}}$ satisfies the complex homogeneous Monge-Ampère equation $(dd^c u)^n = 0$ on $\mathbb{B}^n \setminus \{\tau = 0\}$.

In this theorem, it is a crucial assumption the smoothness of $\tau$ at all points and, in particular on the minimal set $\{\tau = 0\}$. In fact, Lempert ([9]) proved that for any smooth strictly linearly convex domain $D \subset \mathbb{C}^n$ and any $p \in D$, there exists a strictly plurisubharmonic exhaustion $\tau(p) : D \rightarrow [0, 1)$ such that $u := \log\tau|_{\{\tau > 0\}}$ satisfies the Monge-Ampère equation $(dd^c u)^n = 0$ and is of class $C^\infty$ at all points of $D \setminus \{p\}$, but, in general, not at $\{p\} = \{\tau = 0\}$.

In other words, from a rigid situation (the manifold is necessarily $\mathbb{B}^n$), the loss of smoothness on the minimal set leads to a quite dispersed one, represented by the infinite dimensional space of the biholomorphic classes of the strictly linearly convex domains of $\mathbb{C}^n$. Indeed, the property that the biholomorphic classes of convex domains constitute an infinite dimensional space was precisely assessed by Lempert in [10] and Bland and Duchamp in [2, 3], where complete sets of modular data, parameterizing the biholomorphic classes of pointed strictly linearly convex domains, were determined.

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The construction of Bland and Duchamp’s modular data was later simplified and extended in \cite{17}, where we proved that they actually provide a full moduli space for a large class of complex manifolds, the so-called \textit{manifolds of circular type}. This class was originally introduced by the first author in \cite{15} and properly includes all strictly linearly convex domains and all strictly pseudoconvex complete circular domains domains in \( \mathbb{C}^n \). Such manifolds are characterized by the existence of a Monge-Ampère exhaustion, i.e. a plurisubharmonic exhaustion \( \tau \) which is smooth at all points except possibly at the minimal set \( \{ \tau = 0 \} \) and with \( u := \log \tau \) satisfying the Monge-Ampère equation (see \$2.2\) below, for the detailed definition).

The existence of such Monge-Ampère exhaustion implies the existence of a very well behaved Green pluricomplex potential and it is tied with another fundamental biholomorphic invariant, the Kobayashi distance. For instance, the exhaustion \( \tau^{(p)} : D \rightarrow [0,1) \) of a strictly linearly convex domain \( D \) is equal to the square of the hyperbolic tangent of the Kobayashi distance from the point \( p \) and its second order truncation coincides with the squared Minkowski functional of the Kobayashi indicatrix at \( p \) (see e.g. \cite{16}).

These properties carry over to all manifolds of circular type and allow to relate the regularity of Monge-Ampère exhaustions of these manifolds with the regularity of the squared Kobayashi distance.

In case of the unit ball \( \mathbb{B}^n \), the squared Kobayashi distance from a point \( p \) coincides with the squared hyperbolic distance determined by the standard Kähler metric of constant negative curvature. It is real analytic and the Kobayashi indicatrix at \( p \) is the unit ball itself (up to complex linear transformations). It is therefore natural to ask whether regularity properties of Kobayashi distance, of Kobayashi indicatrix or other intrinsic objects may be used to characterize certain classes of complex manifolds, such as the unit ball.

On the other hand since long it is known that there exist strictly convex domains, which are not biholomorphic to \( \mathbb{B}^n \) and yet with the unit ball as indicatrix at some point. This was first discovered by Sibony \cite{18} and further analyzed using various kinds of modular data in \cite{10, 2, 3, 17} (see also \cite{50, 51} later). There is indeed an infinite dimensional family of non-biholomorphic complex manifolds, all with the unit ball as Kobayashi indicatrix at some point.

In this paper we consider the problem of characterization of the manifolds of circular type (and, consequently, of circular domains and strictly linearly convex domains) admitting Monge-Ampère exhaustions and Kobayashi distances of prescribed regularity. Our main tool is the existence of normal forms for manifolds of circular type, as proved in \cite{17}. They are pairs \((\mathbb{B}^n, J)\), formed by the standard unit ball \( \mathbb{B}^n \) equipped with a non-standard integrable almost complex structure \( J \), smoothly deformable into the standard complex structure \( J_0 \) and satisfying appropriate additional conditions (see \$2.3\). In \cite{17} it is shown that any manifold of circular type is biholomorphic to a normal form \((\mathbb{B}^n, J)\) and any normal form is in turn uniquely determined by a complex tensor field \( \phi_J \) on the blow up of \( \mathbb{B}^n \) at the origin, which allows to express \( J \) as
a deformation of $J_o$. Using the properties of manifold of circular type, one can show that the deformation tensor $\phi_J$ can be uniquely decomposed as a series of the form $\phi_J = \sum_{k \geq 0} \phi^{(k)}_J$, where the terms $\phi^{(k)}_J$ are tensor fields satisfying appropriate conditions and which we refer as Bland-Duchamp invariants.

It was remarked in [17] that the vanishing of all invariants $\phi^{(k)}_J$ with $k \geq 1$ is equivalent to the fact that the considered manifold is (up to a biholomorphism) a complete circular domain and coincide with its Kobayashi indicatrix at the center. In this paper, we analyze the somehow opposite situation, namely the vanishing of an initial segment of invariants, i.e. $\phi^{(k)}_J = 0$ for all $0 \leq k \leq k_0$. Our main results may be summarized as follows:

- if $\phi^{(j)}_J = 0$ for all $0 \leq j \leq k - 1$ for some $k \geq 6$, then the Monge-Ampère exhaustion $\tau$ is of class $C_2^{(k)}$ at the center (Theorem 4.1);

- conversely, if the Monge-Ampère exhaustion $\tau$ is of class $C^{2m}$ at the center for some $m \geq 1$, then $\phi^{(j)}_J = 0$ for all $0 \leq j \leq m - 1$ (Theorem 5.5).

As a consequence, the vanishing of the first six invariants $\phi^{(0)}_J, \ldots, \phi^{(5)}_J$ implies that the Monge-Ampère exhaustion $\tau$ is at least $C^3$ at the center. From this it follows that the Kobayashi indicatrix at the center is the unit ball and gives a new precise quantitative way to see that, even if one restricts to the special class of strictly linearly convex domains in $\mathbb{C}^n$, there exists an infinite dimensional family of domains, which have the ball as Kobayashi indicatrix at a point, but are not biholomorphic to the unit ball. Indeed, we are able to show that, for any $m \geq 3$, there exists a strictly linearly convex domain $D \subset \mathbb{C}^n$ and a point $x_0 \in D$, whose associated Monge-Ampère exhaustion is of class $C^m$ but not $C^\infty$ (more precisely, not $C^{4m+2}$). None of such domains is biholomorphic to $\mathbb{B}^n$, even though one can construct examples that are arbitrarily close to $\mathbb{B}^n$.

For strictly linearly convex domains, the Monge-Ampère exhaustion $\tau^{(p)}$, with center $p$, is the square of the hyperbolic tangent of the Kobayashi distance from $p$. Hence the above result implies that for any $m \geq 3$ there exist arbitrarily small deformations of $\mathbb{B}^n$, for which the squared Kobayashi distance from a point is of class $C^m$ but not $C^{4m+2}$.

We recall that, in [5], Burns presented an alternative proof of Stoll’s characterization of $\mathbb{C}^n$ in terms of (unbounded) Monge-Ampère exhaustions, in which the original smoothness assumptions are lowered to $C^5$. Since then, it looked reasonable to expect a weakening of the regularity assumptions in Stoll’s characterization of $\mathbb{B}^n$. The above examples show that this is not possible and that in Stoll’s characterization of $\mathbb{B}^n$, the regularity assumptions cannot be relaxed.

On the other hand, our results bring also to a radical simplification of the proof of Stoll’s characterization of $\mathbb{C}^n$, in which we may further relax Burn’s hypothesis and proving that it suffices that the exhaustion is of class $C^4$. 

The structure of the paper is the following. In §2, we briefly recall definitions and basic facts on integrable almost complex structures, Monge-Ampère exhaustions, manifolds of circular type and their normal forms. In §3, we clarify some delicate points on complex manifold structures of normal forms, which are crucial for the later discussions. In §4 and §5, the main results are proved. The final two sections are dedicated to the construction of Monge-Ampère exhaustions with prescribed regularity at the center and to a new proof of Stoll’s characterization of $\mathbb{C}^n$, respectively.

**Notation.** Throughout the paper, $\mathbb{B}^n$ is the unit ball of $\mathbb{C}^n$ centered at the origin and $\Delta$ is the unit disk in $\mathbb{C}$. We also denote by $e^i_1 = (1,0,\ldots,0)$, $e^n_2 = (0,0,\ldots,1)$ the standard basis of $\mathbb{C}^n$, and by $J_o$ the standard complex structure of $\mathbb{C}^n$, i.e. the tensor field of type $(1,1)$ on $\mathbb{R}^{2n} = \mathbb{C}^n$ such that $J_o \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^i}$ and $J_o \left( \frac{\partial}{\partial y^i} \right) = -\frac{\partial}{\partial x^i}$, where $x^i = \text{Re}(z^i)$ and $y^i = \text{Im}(z^i)$.

2. Preliminaries

2.1. Integrable almost complex structures and complex manifolds.

This section is devoted to recall very standard definitions and facts on complex structures. It aims to set notations and to underline some fine points about the differentiability requirements, which are crucial in what follows.

A complex manifold $M$ of dimension $n$ is a topological manifold endowed with a complete atlas $\mathcal{A}$ of $\mathbb{C}^n$-valued homeomorphisms $\xi : U \subset M \rightarrow V \subset \mathbb{C}^n$ with holomorphic overlaps. Newlander-Nirenberg Theorem establishes a natural correspondence between complex manifold structures and formally integrable almost complex structures. We briefly review this important fact.

An almost complex structure $J$ of class $\mathcal{C}^{k+\alpha}$, $0 \leq \alpha < 1$, on a $2n$-dimensional real manifold $M$, is a family of endomorphisms

$$J_x : T_x M \rightarrow T_x M , \ x \in M , \ \text{with} \ \ J_x^2 = -\text{Id}_{T_x M} ,$$

such that, denoting $J_x = J_{j}^i \frac{\partial}{\partial x^j} \otimes dx^i \big|_x$ for a given set of coordinates $(x^1, \ldots, x^{2n})$, the components $J_{j}^i$ are real functions of class $\mathcal{C}^{k+\alpha}$. If $k \geq 1$, the Nijenhuis tensor $N_J$ is a tensor field of type $(1,2)$, whose value on any pair of smooth vector fields $X, Y$ is

$$N_J(X,Y) = [X,Y] - [JX, JY] + J[X, JY] - J[JX, Y] .$$

The almost complex structure $J$ is called formally integrable if $N_J \equiv 0$.

Given an almost complex structure, each complexified tangent space $T_x^\mathbb{C} M$, $x \in M$, splits into a direct sum

$$T_x^\mathbb{C} M = T_x^{1,0} M \oplus T_x^{0,1} M ,$$

where $T_x^{1,0} M$ and $T_x^{0,1} M = T_x^{1,0} M$ are the $(+i)$- and $(-i)$-eigenspace of $J_x$, respectively. Around any point $x_o \in M$, there always exists a collection of complex vector fields $\{X_i\}_{1 \leq i \leq n}$ (with the same regularity of $J$), which give bases for the $(+i)$-eigenspaces $T_x^{1,0} M$ at all points in which they are defined.
Definition 2.1. A set of $J$-holomorphic coordinates is a homeomorphism $F = (F^1, \ldots, F^n) : \mathcal{U} \subset M \rightarrow \mathcal{V} \subset \mathbb{C}^n$ such that
\[
\overline{X}_i (F^j) = 0 \quad \text{for any } 1 \leq i, j \leq n \tag{2.1}
\]
for any $n$-tuple of local generators $\{X_i\}_{1 \leq i \leq n}$ of the distribution $T^{1,0}M \subset T^\mathbb{C}M$ of eigenspaces $T^{1,0}_xM$.

An atlas $\mathcal{A}_J$ of smoothly overlapping systems of $J$-holomorphic coordinates makes $(M, \mathcal{A}_J)$ a complex manifold of dimension $n$. Conversely, if $(M, \mathcal{A})$ is a complex manifold, the unique tensor field $J$, which in all charts of $\mathcal{A}$ is of the form
\[
J = i \frac{\partial}{\partial z^i} \otimes dz^i - i \frac{\partial}{\partial z^i} \otimes d\bar{z}^i \tag{2.2}
\]
is a formally integrable almost complex structure, for which the charts of $\mathcal{A}$ are $J$-holomorphic.

The celebrated Newlander-Niremberg Theorem ([12]) states that a smooth almost complex structure $J$ is formally integrable if and only if the set $\mathcal{A}_J$ of $J$-holomorphic coordinates is a complex atlas and $(M, \mathcal{A}_J)$ is a complex manifold. Therefore on a $2n$-dimensional manifold $M$, there is a one-to-one correspondence between the complex manifold structures and the smooth formally integrable almost complex structures $J$. Due to this, any complex manifold can be equivalently identified as the pair $(M, \mathcal{A}_J)$ or the pair $(M, J)$.

Newlander-Niremberg Theorem follows from the following existence result, which spells out the minimal regularity assumptions required for $J$.

Theorem 2.2 ([12, 11, 13, 20, 7]). Let $J$ be a formally integrable, almost complex structures on $M$ of class $\mathcal{C}^{k+\alpha}$ for some $k + \alpha > 1$. For any $x_o \in M$, there exists a system of $J$-holomorphic coordinates $F : \mathcal{U} \rightarrow \mathbb{C}^n$ of class $\mathcal{C}^{k+1+\alpha}$ on a neighborhood $\mathcal{U}$ of $x_o$.

Finally we recall the definition of the operator $dd^c_J$, which is frequently considered in this paper. For a given complex structure $J$, for any $\mathcal{C}^2$ function $f$, one has $d^c_J f := J df = -df(J(\cdot))$ and
\[
(dd^c_J) f(X, Y) = -X(JY(f)) + Y(JX(f)) + J[X, Y](f) \tag{2.3}
\]
A local computation shows that $dd^c_J = 2i\partial\bar{\partial}$.

2.2. Domains of circular type and Monge-Ampère exhaustions.

A (bounded) manifold of circular type is a complex manifold $(M, J)$ endowed with a $\mathcal{C}^0$ exhaustion $\tau : M \rightarrow [0, 1)$ such that
i) $\{\tau = 0\}$ is a singleton, say $\{x_o\}$, and $\tau|_{M\setminus\{x_o\}}$ and the pull back of $\tau$ on the blow up $\widetilde{M}$ of $M$ at $x_o$ are both of class $\mathcal{C}^\infty$. 

ii) on $M \setminus \{x_o\} = \{ \tau \geq 0 \}$

$$
\begin{cases}
2i\partial\bar{\partial}\tau = dd^c\tau > 0 , \\
2i\partial\bar{\partial}\log\tau = dd^c\log\tau \geq 0 , \\
(dd^c\log\tau)^n \equiv 0 \text{ (Monge-Ampère Equation) ;}
\end{cases}
$$

iii) in some (hence, any) system of complex coordinates $z = (z^i)$ centered at $x_o$, the exhaustion $\tau$ has a logarithmic singularity at $x_o$, i.e.

$$
\log\tau(z) = \log\|z\| + O(1).
$$

We call $\tau$ Monge-Ampère exhaustion and $x_o$ center of the exhaustion.

Any Monge-Ampère exhaustion $\tau$ defines an associated distribution

$$
\mathcal{Z} = \{ v \in T(M \setminus \{x_o\}) : (\partial\bar{\partial}\log\tau)(v, \cdot) = 0 \} \subset T(M \setminus \{x_o\}). \quad (2.4)
$$

It is an integrable complex distribution and its integral leaves are complex curves whose closures are holomorphic disks passing through $x_o$. We call it Monge-Ampère foliation associated with $\tau$.

We call domain of circular type any pair $(D, \tau)$, formed by a relatively compact domain $D \subset (M,J)$ with smooth boundary, and an exhaustion $\tau : D \to [0,1)$, which is smooth up to the boundary and such that $((D,J),\tau)$ is a manifold of circular type.

The modeling example for such domains is the unit ball $B^n \subset \mathbb{C}^n$, equipped with the standard Monge-Ampère exhaustion $\tau_o : B^n \to [0,1)$, $\tau_o(y) = \|y\|^2$, and Monge-Ampère foliation formed by the radial disks

$$
\Delta^{(v)} = \{ \zeta v , \zeta \in \Delta \} , \quad v \in \mathbb{C}^n \setminus \{0\} . \quad (2.5)
$$

The class of domains of circular type naturally includes all strictly pseudoconvex circular domains and all strictly linearly convex domain of $\mathbb{C}^n$. Indeed, for any strictly pseudoconvex circular domain $D \subset \mathbb{C}^n$, the associated squared Minkowski function $\mu^2_D$ is a Monge-Ampère exhaustion having $x_o = 0$ as center. On the other hand, for any strictly linearly convex domain $\Omega \subset \mathbb{C}^n$ and for any choice of point $x_o \in \Omega$, the exhaustion

$$
\tau : \Omega \to [0,1) , \quad \tau(y) = (\text{tanh} \delta(y))^2 , \quad (2.6)
$$

where $\delta(y)$ denotes the Kobayashi distance between $y$ and $x_o$, is a Monge-Ampère exhaustion for $\Omega$ that has $x_o$ as center.

### 2.3. Normal forms of domains of circular type.

Consider now the domain of circular type $(B^n, J_o, \tau_o)$, given by the unit ball $B^n$, with its standard complex structure $J_o$ and standard exhaustion $\tau_o$. Let also $\mathcal{Z}$ the corresponding distribution $(2.4)$ on $B^n \setminus \{0\}$ and set

$$
\mathcal{H} = (\mathcal{Z})^\perp = \{ v \in T_xB^n , \ x \neq 0 : \langle v, \mathcal{Z}_x \rangle = 0 \} , \quad (2.7)
$$

where $\langle \cdot , \cdot \rangle$ denotes the standard Euclidean metric. They are not defined at 0 but they both admit smooth extensions at all points of the blow up
\( \tilde{p} : \tilde{B}^n \rightarrow B^n \) at 0. We call \( Z \) (standard) radial distribution and \( \mathcal{H} \) (standard) normal distribution. The distributions \( Z, \mathcal{H} \) are both \( J_0 \)-invariant. We denote \( Z^{1,0}, Z^{0,1} \subset Z^C \) and \( \mathcal{H}^{1,0}, \mathcal{H}^{0,1} \subset \mathcal{H}^C \), the complex subdistributions, given by the \((+i)\)- and \((-i)\)-eigenspaces of \( J_0 \) in \( Z^C \) and \( \mathcal{H}^C \), respectively.

**Definition 2.3.** We call \( L \)-complex structure any formally integrable almost complex structure \( \tilde{J} \) on the blow up \( \tilde{B}^n \) that satisfies the following conditions:

i) \( \tilde{J} \) leaves invariant all spaces of the distributions \( Z, \mathcal{H} \);

ii) \( \tilde{J}|_Z = J_o|_Z \);

iii) there exists a smooth homotopy \( \tilde{J}_t \) of complex structures on \( \tilde{B}^n \) that satisfy (i) and (ii), with \( \tilde{J}_{t=0} = J_o \) and \( \tilde{J}_{t=1} = \tilde{J} \).

Let \( \mathcal{A}_{\tilde{J}} \) be the atlas on \( \tilde{B}^n \) of \( \tilde{J} \)-holomorphic coordinates for an \( L \)-complex structure \( J \). The blow-down of \( (\tilde{B}^n, \mathcal{A}_{\tilde{J}}) \) at 0 is called manifold in normal form associated with \( \tilde{J} \). A manifold (of circular type) in normal form is any complex manifold determined in the above fashion.

**Remark 2.4.** As topological space, any manifold \( M \) in normal form is naturally homeomorphic to \( B^n \). Hence, it can be considered as a pair \( (B^n, \mathcal{A}_J) \), formed by \( B^n \) and an appropriate atlas \( \mathcal{A}_J \) of complex charts with holomorphic overlaps. Here, \( J \) denotes for the integrable almost complex structure defined by (2.2) in the charts of \( \mathcal{A}_J \).

The existence of a blow down complex manifold \( (B^n, \mathcal{A}_J) \) for \( (\tilde{B}^n, \mathcal{A}_{\tilde{J}}) \) follows directly from the fact that \( (\tilde{B}^n, \mathcal{A}_{\tilde{J}}) \) has a plurisubharmonic exhaustion, which is strictly plurisubharmonic outside the singular set \( \tilde{p}^{-1}(0) \) (see e.g., [17]). However, the complex charts of such blow down are in general not smoothly overlapping with the standard coordinates of \( C^n \). This means that the tensor field \( J \) (which has smooth components in the charts of \( \mathcal{A}_J \)) might have non-smooth components when it is expressed in terms of the standard coordinates of \( C^n \). In §3.1 we discuss this phenomenon in detail. Here, we just point out that \( J \) is surely smooth (in standard coordinates) at the points of \( B^n \setminus \{0\} \) and that lower regularity might occur at 0.

In [17], it is proved that any manifold in normal form is of circular type and that the following holds:

**Theorem 2.5.** Let \((M, \mathcal{A}_J)\) be a manifold of circular type with Monge-Ampère exhaustion \( \tau \) and center \( x_o \). Then, there exists a biholomorphism \( \Phi : (M, \mathcal{A}_J) \rightarrow (B^n, \mathcal{A}_{J'}) \) with a manifold in normal form \( (B^n, \mathcal{A}_{J'}) \) such that

a) \( \Phi(x_o) = 0 \) and \( \tau = \tau_o \circ \Phi \);

b) \( \Phi \) maps the leaves of the Monge-Ampère foliation of \( M \) into the straight disks through the origin of \( B^n \).

A biholomorphism \( \Phi : (M, \mathcal{A}_J) \rightarrow (B^n, \mathcal{A}_{J'}) \) satisfying (a) and (b) is called normalizing map for \((M, \mathcal{A}_J, \tau)\).
2.4. Bland and Duchamp invariants of manifolds of circular type.

Let \((\mathbb{B}^n, \mathcal{A}_J)\) be a manifold in normal form. By definition, the formally integrable almost complex structure \(J\) on \(\mathbb{B}^n \setminus \{0\}\) differs from the standard complex structure \(J_0\) only by its action on the vectors in \(\mathcal{H}\). Hence it is uniquely determined by the distribution \(\mathcal{H}_{J_0}^{0,1}\) of the \((-i)\)-eigenspaces of \(J_x\) in the complex tangent spaces \(T_x^C\mathbb{B}^n, x \in T_x\mathbb{B}^n\). In [17], it is shown that \(\mathcal{H}_{J_0}^{0,1}\) is necessarily of the form

\[
\mathcal{H}_{J_0}^{0,1}\big|_x = \{ v = w + \phi_J|_x (v), \ v \in \mathcal{H}_{J_0}^{0,1}\big|_x \} \subset \mathcal{H}_x^C, \quad x \neq 0 ,
\]

for some appropriate tensor field \(\phi_J \in (\mathcal{H}_{J_0}^{0,1})^* \otimes \mathcal{H}_{J_0}^{1,0}\), called deformation tensor of \(J\) with respect \(J_0\).

The facts that \(J\) is a formally integrable almost complex structure and that \(\tau_0 = || \cdot ||^2\) is a Monge-Ampère exhaustion for \((\mathbb{B}^n, \mathcal{A}_J)\) give strong constraints on the deformation tensor \(\phi_J\). In particular, it turns out that \(\phi_J\) is sum of a series, uniformly converging on compacta, of the form

\[
\phi_J = \sum_{k \geq 0} \phi_J^{(k)} ,
\]

with each tensor field \(\phi_J^{(k)}\) is (locally) of the form

\[
\phi_J^{(k)}([w], \zeta w) = \phi_J^k([w])\zeta^k .
\]

Here, we indicate any point \(x \in \mathbb{B}^n \setminus \{0\} = \mathbb{R}^n \setminus \{0\}\) by the corresponding pair \(([w], \zeta) \in \mathbb{CP}^{n-1} \times \Delta\), with \(w \in S^{2n-1}\), such that \(x = \zeta \cdot \cdot w\). The tensor field \(\phi_J\) is constrained by the equations and inequality

\begin{align*}
&i) \quad dd^c \tau_0(\phi_J(X), Y) + dd^c \tau_0(X, \phi_J(Y)) = 0 \text{ for any } X, Y \in \mathcal{H}_{J_0}^{0,1}, \\
&ii) \quad \partial \bar{\partial} \phi_J + \frac{1}{2} [\phi_J, \phi_J] = 0 , \\
&iii) \quad dd^c \tau_0(\phi_J(X), \phi_J(Y)) < dd^c \tau_0(\bar{X}, \bar{X}) \text{ for any } 0 \neq X \in \mathcal{H}_{J_0}^{0,1}
\end{align*}

(for the detailed definition of \(\partial \bar{\partial}\), see [17]). Conversely, any sequence of tensor fields \(\phi_J^{(k)} \in (\mathcal{H}_{J_0}^{0,1})^* \otimes \mathcal{H}_{J_0}^{1,0}, 0 \leq k < \infty\), of the form (2.10), such that (2.9) converges uniformly on compacta of \(\mathbb{B}^n\) and the sum satisfies (i) - (iii), the corresponding complex distribution (2.8) determines uniquely an almost complex structure \(J\) on \(\mathbb{B}^n\), which is an \(L\)-complex structure, hence corresponds to a manifold in normal form.

The terms \(\phi_J^{(k)}\) of the series (2.9) are called Bland and Duchamp invariants of the manifold in normal form \((\mathbb{B}^n, \mathcal{A}_J)\).

Lemma 2.6. Let \((\mathbb{B}^n, \mathcal{A}_J)\) be a manifold in normal form, with associated deformation tensor \(\phi_J\). If \(X^{1,0}, Y^{0,1}\) are complex vector fields in \(\mathcal{H}_{J_0}^{1,0}, \mathcal{H}_{J_0}^{0,1}\), respectively, then

\[
dd^c_J T_0(X^{1,0} + \phi_J(X^{1,0}), Y^{0,1} + \phi_J(Y^{0,1})) = \\
= dd^c_J_T_0(X^{1,0}, Y^{0,1}) + dd^c_J_T_0(\phi_J(X^{1,0}), \phi_J(Y^{0,1})) .
\]

\[\text{(2.11)}\]
Proof. Since \([\mathcal{H}^{1,0}, \mathcal{H}^{1,0}] \subset \mathcal{H}^{1,0}\) and \([\mathcal{H}^{0,1}, \mathcal{H}^{0,1}] \subset \mathcal{H}^{0,1}\),
\[
[X^{1,0} + \phi \left( \overline{X^{1,0}} \right), Y^{0,1} + \phi \left( Y^{0,1} \right)] = [X^{1,0}, Y^{0,1}] + \left[ \phi \left( \overline{X^{1,0}} \right), \phi \left( Y^{0,1} \right) \right] \mod \mathcal{H}^C
\]
(2.12)
Since \([X^{1,0}, Y^{0,1}] := X^{1,0} + \phi \left( \overline{X^{1,0}} \right) \in \mathcal{H}_J^1\), \(Y^{0,1} := Y^{0,1} + \phi \left( Y^{0,1} \right) \in \mathcal{H}_J^0\) and the derivatives of \(\tau_o\) along vectors in \(\mathcal{H}\) are trivial, we get
\[
\frac{dd}{dt} \tau_o(X^{1,0}, Y^{0,1}) = iX^{1,0}(Y^{0,1}(\tau_o)) - iY^{1,0}(X^{0,1}(\tau_o)) - J[X^{1,0}, Y^{0,1}] (\tau_o)
\]
(2.13)
Now, using once again that the derivatives of \(\tau_o\) along vectors in \(\mathcal{H}\) are trivial and since \(J|_{\mathcal{H}} = J_o|_{\mathcal{H}}\),
\[
\frac{dd}{dt} \tau_o(X^{1,0}, Y^{0,1}) = -J_o([X^{1,0}, Y^{0,1}]) (\tau_o) - J_o \left[ \phi \left( \overline{X^{1,0}} \right), \phi \left( Y^{0,1} \right) \right] (\tau_o) = \frac{dd}{dt} \tau_o(X^{1,0}, Y^{0,1}) + \frac{dd}{dt} \tau_o(\phi(X^{1,0}), \phi(Y^{0,1})).
\]
\[\square\]

2.5. Kobayashi indicatrices and normalizing maps.

Let \(M\) be a manifold of circular type with Monge-Ampère exhaustion \(\tau\) and center \(x_0\). We recall that the value \(\kappa(v)\) of the infinitesimal Kobayashi metric \(\kappa\) at any vector \(v \in T_{x_0} M\) always exists and can be computed by the expression \(\kappa(v) = \lim_{t \to 0} \frac{d}{dt} \sqrt{\tau(\gamma_t)} \bigg|_{t=t_0}\), where \(\gamma_t\) is any smooth curve such that \(\gamma_0 = x_0\) and \(\gamma_0 = v\). The Kobayashi indicatrix of \(M\) at \(x_0\) is the domain \(I_{x_0}\) in the tangent space \(T_{x_0} M\), defined by
\[
I_{x_0} = \{ v \in T_{x_0} M \, : \, \kappa(v) < 1 \}\.
\]
Given a basis \(B = (e_i)\) for \(T_{x_0} M\), we denote by \(\ell_B\) the \(\mathbb{C}\)-linear isomorphism \(\ell_B : \mathbb{C}^n \to T_{x_0} M\) such that \(\ell_B(v^i e_i) := v^i e_i\). The circular domain of \(\mathbb{C}^n\) corresponding to \(I_{x_0}\)
\[
I_B := \ell_B^{-1}(I_{x_0}) = \{ v \in \mathbb{C}^n \, : \, \kappa(\ell_B(v)) < 1 \}\.
\]
(2.14)
is called realization of \(I_{x_0}\) determined by the basis \(B\).

Let \(p : \tilde{I}_B \to I_B\) be the blow up of the realization \(I_B\) at \(0 \in \mathbb{C}^n\). We recall that \(\tilde{I}_B\) is naturally identifiable with the set of pairs \(([v], tv)\), formed by elements \([v] \in \mathbb{C}P^{n-1}\), with \(v \in \partial I_B\), and the points \(tv \in I_B\), \(t \in [0, 1]\). Let also \(\pi : \tilde{M} \to M_0\) be the blow up of \(M\) at \(0\) and \(x_0\).

The circular representation of the manifold of circular type \((M, \mathcal{A}_J)\) is a diffeomorphism
\[
\Psi : \tilde{I}_B \to \tilde{M},
\]
which is canonically determined by the Monge-Ampère exhaustion τ and satisfies a number of crucial properties. For the definition of Ψ and its main properties, we refer to e.g. [14, 15, 4, 17]. Here, we just remind that:

a) for any ([v], tv) ∈ \( \tilde{F}(B) \), \( v \in \partial I(B) \), the map
\[
f^v : \Delta \longrightarrow M , \quad f^v(\zeta) := (\pi \circ \Psi)([v], \zeta v)
\]
is the unique stationary disk of \( M \) satisfying the conditions \( f^v(0) = x_o \) and \( f^v_* \left( \frac{\partial}{\partial x} \right)_0 = \lambda v \) for some \( \lambda > 0 \); moreover, the corresponding lifted disk \( \tilde{f}^v : \Delta \longrightarrow \tilde{M} \), defined by \( \tilde{f}^v(\zeta) := \Psi([v], \zeta v) \) is proper, holomorphic and injective;

b) the restriction \( \Psi|_{\pi^{-1}(0)} : \mathbb{C}P^{n-1} \longrightarrow \pi^{-1}(x_o) = P(T_{x_o}M) \subset \tilde{M} \) coincides with the projective map \( \ell_B \), determined by \( \ell_B \), i.e.,
\[
\Psi|_{\pi^{-1}(0)}([v]) = \ell_B([v]) := [\ell_B(v)].
\]

Moreover by [17], Lemma 3.5, there exists a diffeomorphism \( \psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n \) from the blow up of \( \mathbb{C}^n \) at the origin into itself satisfying:

1) it is a bundle map w.r.t. the \( \mathbb{C}^* \)-bundle structure \( \tilde{\pi} : \tilde{\mathbb{C}}^n \longrightarrow \mathbb{C}P^{n-1} \);
2) it preserves the distribution \( \mathcal{H} \) of \( \mathbb{C}^n \), formed by the vectors that are normal to the straight lines of \( \mathbb{C}^n \) through the origin;
3) it induces a diffeomorphism between \( \tilde{F}(B) \) and \( \tilde{\mathbb{B}}^n \);
4) there exists a smooth homotopy between \( \psi \) and \( \text{Id}_{\mathbb{C}^n} \), for which all intermediate maps \( \psi_t \) satisfy (1) and (2) and map \( \tilde{F}(B) \) onto a blow up at the origin of a circular domain.

It turns out that the diffeomorphism \( \tilde{\Phi} := \psi \circ \Psi^{-1} : \tilde{M}^n \longrightarrow \tilde{\mathbb{B}}^n \) maps the integrable almost complex structure \( \tilde{J} \) of \( M^n \) into the almost complex structure \( J' = \tilde{\Phi}_* (\tilde{J}) \) on \( \tilde{\mathbb{B}}^n \), which is an \( L \)-complex structure, and \( (\tilde{\mathbb{B}}^n, A_{J'}) \) is the blow up of a manifold in normal form \( (\mathbb{B}^n, A_J) \). Hence, the pushed-down diffeomorphism \( \Phi : (M, A_J) \longrightarrow (\mathbb{B}^n, A_{J'}) \) is a normalizing map for \( (M, A_J, \tau) \).

As a consequence of the above construction of a normalizing map, we get:

**Lemma 2.7.** If there exists a basis \( B \) for \( T_{x_o}M \) such that \( I(B) = \mathbb{B}^n \), there also exists a normalizing map \( \Phi : (M, A_J) \longrightarrow (\mathbb{B}^n, A_{J'}) \) onto a manifold in normal form whose deformation tensor \( \phi_{J'} \) has \( \phi_{J'}(0) = 0 \).

**Proof.** If \( I(B) = \mathbb{B}^n \), as a diffeomorphism \( \psi \) satisfying (1)-(4), we may take the identity map \( \psi = \text{Id}_{\mathbb{C}^n} \). Hence, the inverse \( \tilde{\Phi} = \Psi^{-1} : \tilde{M} \longrightarrow \tilde{F}(B) = \mathbb{B}^n \), of the circular representation of \( M \) is a normalizing map. By (b) and the fact that \( \ell_B \) is a \( \mathbb{C} \)-linear isomorphism, we get that
\[
\tilde{J}|_{\mathbb{C}P^{n-1}} = \Psi_{\ast}^{-1}(J)|_{P(T_{x_o}M)} = \ell_B^{-1} \ast (J)|_{P(T_{x_o}M)} = J_o
\]
where \( J_o \) denotes the standard complex structure of \( \mathbb{C}P^{n-1} \). This implies that the deformation tensor \( \phi_{J} \) of \( J \) vanishes on \( \mathbb{C}P^{n-1} \) and \( \phi(0) = 0 \). \( \square \)
3. The Almost Complex Structures of Manifolds in Normal Form

3.1. The Tangent Bundle of a Manifold in Normal Form.

Let \((\B^n, \A_J)\) be a manifold in normal form and denote by \(\A^R_J\) the complete \(C^\infty\) atlas determined by the real coordinates, given by real and imaginary parts of the complex coordinates in \(\A_J\).

By Remark 2.3, \(\A^R_J\) is in general different from the standard atlas \(\A^R_o\) of \(\B^n\), induced by the standard real manifold structure of \(\B^n(= \C^n)\).

A priori, the “tangent vectors of \(\B^n\)” are different from the usual vectors of \(\B^n\) if we consider the non-standard atlas \(\A^R_J\) on \(\B^n\) in place of the standard one. Indeed, the tangent vectors of \((\B^n, \A^R_J)\) are equivalence classes of curves, which are of class \(C^1\) when they are expressed in some coordinates of the standard atlas. On the contrary, the tangent vectors of \((\B^n, \A^R_J)\) are equivalence classes of curves, which are of class \(C^1\) in the charts of the atlas \(\A^R_J\) and might not be of class \(C^1\) for the charts in \(\A^R_o\).

Nonetheless, as we shall see below, there is a natural identification between the tangent vectors of \((\B^n, \A^R_o)\) and \((\B^n, \A^R_J)\). Using this fact, in the next sections, we shall not make any distinction between the tangent spaces of such two differentiable manifolds.

Consider the blow-up \((\tilde{\B}^n, \tilde{\A}_J)\) of \((\B^n, \A_J)\) at the origin. By construction, \(\tilde{J}\) is a formally integrable, almost complex structure, which is smooth in the standard atlas of real coordinates \(\A^R_o := \A^R_{J_0}\) of the blow up \((\tilde{\B}^n, J_0)\) of \((\B^n, J_0)\).

Hence, by Theorem 2.2, every complex chart in \(\tilde{\A}_J\) (hence, every real chart in \(\A^R_{\tilde{J}}\)) smoothly overlaps with all charts of \(\A^R_{\tilde{J}}\). This means that the atlas \(\A^R_{\tilde{J}}\) coincides with the standard atlas \(\A^R_o\) of \(\B^n\). Note also that the blow-down maps \(\tilde{\pi} : (\tilde{\B}^n, \tilde{\A}_{\tilde{J}}) \rightarrow (\B^n, \A_o)\) and \(\pi' : (\B^n, \A_{\tilde{J}}) \rightarrow (\B^n, \A_J)\), as maps between the underlying topological manifolds \(\tilde{\B}^n\) and \(\B^n\), do coincide: \(\tilde{\pi} = \pi'\). Furthermore, the restriction \(\tilde{\pi}|_{\tilde{B}^n \setminus \tilde{\pi}^{-1}(0)} = \pi'|_{\B^n \setminus \pi^{-1}(0)}\) is a diffeomorphism (indeed, it is a biholomorphism w.r.t. the complex structures involved).

These observations show that if a chart of the atlas \(\A_J\) is defined on some open subset of \(\B^n \setminus \{0\}\), it smoothly overlaps with all charts of \(\A_o\). This implies that that curves in \(\B^n \setminus \{0\}\) are of class \(C^1\) in the charts of \(\A_J\) if and only if are of class \(C^1\) in the charts of the atlas \(\A_o\) and vice versa. Hence any tangent space \(T_x \B^n, x \neq 0,\) of the real manifold \((\B^n, \A_J)\) is actually identical with the tangent space at \(x\) of the standard ball \((\B^n, \A_o)\).

Let us now focus on the tangent spaces at \(x = 0\) of \((\B^n, \A_J)\) and \((\B^n, \A_o)\). A priori, the codomain of the push-forward \(\pi_* : T\B^n \rightarrow T\B^n\) changes if one considers \(T\B^n\) as tangent bundle of \((\B^n, \A_J)\) or as tangent bundle of \((\B^n, \A_o)\). However, in both cases, the tangent space \(\B^n\) at the origin is equal to the image \(\pi_*(T\B^n|_{\pi^{-1}(0)})\). This brings to a natural identification between the two tangent spaces, which we now explain.
Firstly we observe that if \( v, v' \in T\overline{B}^n|_{\pi^{-1}(0)} \) are such that \( \pi_*(v) = \pi_*(v') \) when the target manifold is \((\mathbb{B}^n, A_o)\) then one also has \( \pi_*(v) = \pi_*(v') \) when the target manifold is \((\mathbb{B}^n, A_J)\). This can be checked as follows. When the target manifold is \((\mathbb{B}^n, A_o)\), we have \( \pi_*(v) = \pi_*(v') \) exactly if there is a pair of smooth curves \( \tilde{\gamma}, \tilde{\gamma}' : (-\varepsilon, +\varepsilon) \to \overline{B}^n \), with

i) \( y = \tilde{\gamma}(0) \) and \( y' = \tilde{\gamma}'(0) \) are both in \( \pi^{-1}(0) \) and tangent to \( v \) and \( v' \), respectively;

ii) the curves \( \gamma = \pi \circ \tilde{\gamma} \) and \( \gamma' = \pi \circ \tilde{\gamma}' \) in the manifold \((\mathbb{B}^n, A_o)\) are tangent one to the other at 0.

Since the differentiable structures induced on \( \mathbb{B}^n \setminus \{0\} \) of \((\mathbb{B}^n, A_o)\) and \((\mathbb{B}^n, A_J)\) are equal, by a direct argument based on the continuity of tangent vectors, the curves \( \gamma = \pi \circ \tilde{\gamma} \) and \( \gamma' = \pi \circ \tilde{\gamma}' \) are tangent at \( t = 0 \) also when considered as curves of \((\mathbb{B}^n, A_J)\). This proves the claim.

By this observation, we may identify every vector \( v \) of the tangent space \( T_0\overline{B}^n \) of \((\mathbb{B}^n, A_J)\) with a unique corresponding vector \( \tilde{v} \) of the tangent space \( T_0\overline{B}^n \) of \((\mathbb{B}^n, A_J)\), provided that there exists some \( \tilde{v} \in T\overline{B}^n|_{\pi^{-1}(0)} \), which is projected onto \( v \) and \( \tilde{v} \), respectively, by the corresponding push-forwards.

**Remark 3.1.** With the above identifications between tangent bundles, the tensor field \( J \) of the manifold \((\mathbb{B}^n, A_J)\) can be considered as a (possibly non-smooth) tensor field of the standard ball \((\mathbb{B}^n, A_o)\). We claim such tensor field satisfies

\[
J|_0 = J_o|_0. \tag{3.1}
\]

Indeed, \( J|_0 \) is uniquely determined by its \((+i)\)-eigenspace in \( T_0\overline{C}\mathbb{B}^n \). On the other hand, such eigenspace is spanned by the tangent vectors of the stationary disks of \((\mathbb{B}^n, A_J)\) passing through 0. This implies (3.1) as such disks of \((\mathbb{B}^n, A_J)\) coincide with the stationary disks of \((\mathbb{B}^n, A_o)\) through 0.

### 3.2. Almost complex structures of manifolds in normal form.

Let \((\mathbb{B}^n, A_J)\) be a manifold in normal form, with formally integrable almost complex structure \( J \). Using Remark 3.1 we may consider \( J \) as a tensor field on the standard unit ball \((\mathbb{B}^n, A_o)\).

As mentioned before, the components of \( J \) in the charts of \( A_J \) are smooth, but, a priori, they are not smooth in the standard coordinates. Actually, since the charts of \( A_J^0 \) on open subsets of \( \mathbb{B}^n \setminus \{0\} \), smoothly overlap with the standard coordinates, the components of \( J \) at points of \( \mathbb{B}^n \setminus \{0\} \) are necessarily smooth also in standard coordinates. On the other end, the components of \( J \) at 0 in standard coordinates might not even be continuous.

From now on, if no indication of the atlas is given, \( \mathbb{B}^n \) is considered as endowed with charts of the standard atlas \( A_o \) and \( J \) is considered as a formally integrable, almost complex structure on \((\mathbb{B}^n, A_o)\), which is of class \( C^\infty \) at all points of \( \mathbb{B}^n \setminus \{0\} \).

By construction, the Nijenhuis tensor \( N_J \) of \( J \) vanishes identically at all points of \( \mathbb{B}^n \setminus \{0\} \), i.e. \( J|_{\mathbb{B}^n \setminus \{0\}} \) is formally integrable. Moreover, every system
of $J$-holomorphic coordinates of the (non-standard) manifold $(\mathbb{B}^n, A_J)$

$$\xi_J = (z^1_J, \ldots, z^n_J) : \mathcal{U} \to \mathbb{C}^n$$

is a system of $J$-holomorphic coordinates for the almost complex structure $J$ of the (standard) manifold $\mathbb{B}^n$, as defined in Definition 2.1.

Hence for any point $x_0 \in \mathbb{B}^n$ there exists a system of $J$-holomorphic coordinates $\xi_J = (z^i_J)$ defined on a neighborhood of $x_0$. This is not a consequence of the general Theorem 2.2 as $J$ might be not even $C^0$ at 0. It simply follows from the fact that $J$ corresponds to a manifold in normal form, namely to $\mathbb{B}^n$ endowed with a (non-standard) complex manifold structure.

**Lemma 3.2.** Let $(\mathbb{B}^n, J)$ be a manifold in normal form. Then

a) if there exists a system of $J$-holomorphic coordinates $\xi_J = (z^i_J)$ in a neighborhood $\mathcal{U}$ of 0, which is of class $C^{1+r+\alpha}$ at 0 $\in \mathbb{B}^n$, for an integer $r \geq 0$ and $\alpha \in [0,1)$, then $J$ is of class $C^{r+\alpha}$ at the origin.

b) If $J$ is of class $C^{r+\alpha}$ at 0, for an integer $r \geq 1$ and $\alpha \in (0,1)$, then there exists a system $\xi_J = (z^i_J)$ of $J$-holomorphic coordinates of class $C^{1+r+\alpha}$ in a neighborhood $\mathcal{U}$ of 0.

**Proof.** (a) Assume that $\xi_J = (z^i_J)$ is of class $C^{1+r+\alpha}$ at 0. This implies that the entries of the Jacobians

$$\begin{pmatrix}
\frac{\partial z^i_J}{\partial z^m} \\
\frac{\partial z^i_J}{\partial \bar{z}^m}
\end{pmatrix}
\bigg|_{(z^m)}$$

are $C^\infty$ at all points $(z^m) \neq 0$ and of class $C^{r+\alpha}$ at 0. Using the transformation rules of (real) tensors of type $(1,1)$ and the fact that $J$ has constant components in the $J$-holomorphic coordinates $\xi_J = (z^i_J)$, the components of $J$ in standard coordinates have the same regularity of the entries of the above Jacobians. In particular, they are of class $C^{r+\alpha}$ at 0.

(b) follows directly from Theorem 2.2. \(\square\)

4. The Vanishing of the First Bland and Duchamp Invariants

**4.1. Complex polar coordinates.**

As usual, we indicate the points of the blow up $\pi : \mathbb{B}^n \to \mathbb{B}^n$ by pairs $([v], rv)$, where $v \in S^{2n-1} \subset T_0\mathbb{B}^n$, $r \in [0,1)$ and $[v]$ is the equivalence class of $v$ in $P(T_0\mathbb{B}^n \setminus \{0\}) \simeq P(C^n \setminus \{0\}) = \mathbb{C}P^{n-1}$. Recall that $\mathbb{B}^n$ can be also considered as a bundle over $\mathbb{C}P^{n-1}$, with projection $\tilde{\pi} : \mathbb{B}^n \to \mathbb{C}P^{n-1}$ defined by $\tilde{\pi}([v], rv) := [v]$, and standard fiber $\Delta$. 


For a fixed $1 \leq i_o \leq n$, we denote by $(w^1, \ldots, w^{n-1})$ the usual affine coordinates on $\mathbb{C}P^{n-1} \setminus \{[z] : z^{i_o} = 0\}$

$$(z^1 : \ldots : z^{n-1} : z^n) \mapsto \left(w^1 := \frac{z^1}{z^{i_o}}, \ldots, w^{n-1} := \frac{z^{n-1}}{z^{i_o}}\right)$$

and we call complex polar coordinates for $\mathbb{B}^n$ the associated coordinates $\tilde{\xi}_{i_o} = (w^1, \ldots, w^{n-1}, w^n)$ on $\tilde{U}_{i_o} = \pi^{-1}(\{z^{i_o} \neq 0\}) \subset \mathbb{B}^n$ defined by

$$([v], rv) \mapsto \left(\tilde{\xi}_{i_o}([v]), \ldots, w^{n-1}([v]), w^n := re^{i\arg(z^{i_o})}\right), \quad (4.1)$$

for any $v = (z^1, \ldots, z^n) \in S^{2n-1}$ (here, $\arg(\zeta)$ denotes the real number in $[0, 2\pi)$ such $\zeta = |\zeta|e^{i\arg(\zeta)}$). We finally call complex polar coordinates for $\mathbb{B}^n$ the coordinates $\xi_{i_o}$ on the set $\mathbb{B}^n \setminus \{z^{i_o} = 0\}$, defined by

$$\xi_{i_o}(z) := (w^1(\pi^{-1}(z)), \ldots, w^{n-1}(\pi^{-1}(z)), w^n(\pi^{-1}(z))) .$$

For our purposes, it is useful to have explicit information on the Jacobians

$$\begin{bmatrix}
\frac{\partial w^{i_1}}{\partial z^j} & \cdots & \frac{\partial w^{i_m}}{\partial z^j} \\
\frac{\partial \tilde{w}^{i_1}}{\partial \tilde{z}^j} & \cdots & \frac{\partial \tilde{w}^{i_m}}{\partial \tilde{z}^j}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial z^i}{\partial w^{i_1}} & \cdots & \frac{\partial z^i}{\partial w^{i_m}} \\
\frac{\partial \tilde{z}^i}{\partial \tilde{w}^{i_1}} & \cdots & \frac{\partial \tilde{z}^i}{\partial \tilde{w}^{i_m}}
\end{bmatrix} \vline_{w^{n}(z)} \quad (4.2)$$

of changes of coordinates between standard coordinates $(z^i)$ of $\mathbb{B}^n$ and complex polar coordinates $(w^i) = (w^n, \zeta)$, $1 \leq a \leq n-1$, on $\mathbb{B}^n \setminus \{z^{i_o} = 0\}$ for some $1 \leq i_o \leq n$. We show how to compute them for the case $i_o = n$: the other cases can be handled in the same way.

Let us recall that if a point has complex polar coordinates $(w^i) = (w^a, \zeta)$, the corresponding standard coordinates are

$$z^a = \frac{\zeta w^a}{\sqrt{1 + \sum_{k=1}^{n-1} |w^b|^2}} = \mu_w w^a \zeta , \quad z^n = \frac{\zeta}{\sqrt{1 + \sum_{k=1}^{n-1} |w^b|^2}} = \mu_w \zeta ,$$

where $\mu_w := \frac{1}{\sqrt{1 + \sum_{k=1}^{n-1} |w^b|^2}}$. It follows that the second Jacobian in (4.2) is

$$\begin{pmatrix}
\left(\delta_a^b - \frac{1}{2} \mu_w^2 w^a w^b \delta_{ca}\right) w^c e^{i\theta} & w^b w^a \frac{1}{|\zeta|} & -\frac{1}{2} w^b w^c \delta_{ca} \mu_w^2 e^{i\theta} & 0 \\
-\frac{1}{\mu_w} \delta_{ca} \mu_w^2 e^{i\theta} & \mu_w \frac{1}{|\zeta|} & -\frac{1}{2} w^a \delta_{ca} \mu_w^3 e^{i\theta} & 0 \\
-\frac{1}{\mu_w} \delta_{ca} \mu_w^3 e^{-i\theta} & 0 & \left(\delta_a^b - \frac{1}{2} \mu_w^2 w^a w^b \delta_{cb}\right) w^c e^{-i\theta} & \mu_w \frac{1}{|\zeta|} \\
-\frac{1}{\mu_w} \delta_{ac} \mu_w^2 e^{-i\theta} & 0 & -\frac{1}{\mu_w} \delta_{ac} \mu_w^3 e^{-i\theta} & \mu_w \frac{1}{|\zeta|}
\end{pmatrix}$$
where \( e^{i\theta} := \frac{\zeta}{|\zeta|} \). The first Jacobian in (4.2) is clearly the inverse of this matrix. Computing it explicitly and recalling that \(|\zeta| = \|z\|\) one can see that

\[
\begin{pmatrix}
\frac{\partial w^i}{\partial \bar{z}^j} & \frac{\partial w^i}{\partial z^j}
\end{pmatrix}
= \frac{1}{\|z\|} A(z, \bar{z}) \quad \text{for all } z \in \mathbb{B}^n \setminus \{z^{i_0} = 0\},
\]

where \( A(z, \bar{z}) \) is a matrix with smooth and uniformly bounded entries.

### 4.2. The vanishing of first \( \phi^{(j)} \)'s and the regularity of \( J \).

Let \((\mathbb{B}^n, A_J)\) be a manifold in normal form with associated deformation tensor \( \phi_J = \sum_k \phi^{(k)}_J \). Denote by \( \Pi \) the tensor field of type \((1,1)\) on \( \mathbb{B}^n \setminus \{0\} \), which gives the natural projections \( \Pi_x : T_x \mathbb{B}^n = \mathbb{Z}_x + \mathbb{H}_x \rightarrow \mathbb{H}_x \) and let \( \psi_J \) be the real tensor field of type \((1,1)\) on \( \mathbb{B}^n \setminus \{0\} \), given by

\[
\psi_J(X) := \phi_J(\Pi(x)), \quad \psi_J(X) := \overline{\psi(X)}, \quad \text{for any } X \in T^{0,1}(\mathbb{B}^n \setminus \{0\}).
\]

By construction, \( \psi_J|_{\mathbb{B}^n \setminus \{0\}} = \psi|_{\mathbb{B}^n \setminus \{0\}} + \psi_J|_{\mathbb{B}^n \setminus \{0\}} \). On the other hand, by Remark 3.1, \( J|_0 = J_o|_0 \). Hence, if we set \( \psi_J|_0 := 0 \), we have the equality

\[
J = J_o + \psi_J
\]

at all points of \( \mathbb{B}^n \) and the regularity of \( J \) at 0 is equal to the regularity of \( \psi_J \) at that point. Being the 2-form \( dd^c\tau_o \) non-singular and smooth at all points, such regularity is in turn equal to the regularity at 0 of the \((0,2)\)-tensor field

\[
\widetilde{\psi}_J := dd^c\tau_o(\cdot, \psi_J(\cdot)).
\]

**Theorem 4.1.** Let \( k \geq 3 \). If all Bland and Duchamp invariants \( \phi^{(j)} \), \( 0 \leq j \leq k+1 \), are identically vanishing, then

a) the almost complex structure \( J \) is of class \( C^{1 \mathbb{Z}} \) at the origin;

b) if \( k \geq 6 \), the standard exhaustion \( \tau_o : \mathbb{B}^n \rightarrow [0,1) \) is of class \( C^{1 \mathbb{Z}} \) in any set of coordinates of the atlas \( A_J \).

**Proof.** Consider the components \( \widetilde{\psi}_{ij} \) of the tensor field (4.5) in a system of complex polar coordinates \((w^a, \zeta)\) on an open subset \( \mathcal{U} \) of \( \mathbb{B}^n \setminus \{0\} \), with \( 0 \in \partial \mathcal{U} \). By the vanishing of \( \phi^{(j)} \), \( 0 \leq j \leq k-1 \), and definition of \( \psi \), they are of the form \( \widetilde{\psi}_{ij}(w^a, \zeta, \bar{w}^a, \bar{\zeta}) = |\zeta|^k g_{ij}(w^a, \zeta, \bar{w}^a, \bar{\zeta}) \) for some smooth, uniformly bounded function \( g_{ij} \). From (4.3) and the transformation rules of the coordinate components of \((0,2)\) tensors, it follows that the components of \( \widetilde{\psi}_J \) in standard coordinates are functions of the form \( \|z\|^{k-2h_{ij}(z, \bar{z})} \), for some functions \( h_{ij}(z, \bar{z}) \) on \( \mathcal{U} \) that are uniformly bounded and with uniformly bounded derivatives. Since \( 0 \in \partial \mathcal{U} \) for any \( \mathcal{U} \) where complex polar coordinates are defined, if \( k \geq 3 \), the functions \( \widetilde{\psi}_{ij} \) are of class \( C^{1 \mathbb{Z}} \) at the origin and, consequently, the same holds for \( \widetilde{\psi} \) and \( J \). Furthermore, if \( k \geq 6 \), Theorem 2.2 applies and any system of \( J \)-holomorphic coordinates \((z_j^i) \in A_J \) is of class \( C^{1 \mathbb{Z}} \) with respect to the standard coordinates. Being smooth in standard coordinates, we get that \( \tau_o \) is of class \( C^{1 \mathbb{Z}} \) in any chart of \( A_J \). \( \square \)
5. Regularity of the Monge-Ampère exhaustion implies that the first Bland and Duchamp invariants vanish

5.1. $C^2$-regularity of the exhaustion and vanishing of $\phi_{J}^{(0)}$.

**Lemma 5.1.** Let $(M, J)$ be a manifold of circular type with Monge-Ampère exhaustion $\tau$ and center $x_o$. If $\tau : M \rightarrow [0, 1)$ has second order derivatives at $x_o$, there exists a basis $B = (e_i)$ of $T_{x_o} M$ such that (the realization of) the indicatrix at $x_o$ is $I(B) = \mathbb{B}^n$.

**Proof.** By [16], for any $v \in T_{x_o} M \setminus \{0\}$ there is a unique Kobayashi extremal disk $f^v : \Delta \rightarrow (M, J)$ with $f(0) = x_o$ and $\text{Re} \left( f^v \left( \frac{\partial}{\partial \zeta} \right) \right) \in \mathbb{R} v$. This disk is such that $\tau(f^v(\zeta)) = |\zeta|^2$ for any $\zeta \in \Delta$ and

$$
\kappa(v) = 1 \quad \text{if and only if} \quad \text{Re} \left( f^v \left( \frac{\partial}{\partial \zeta} \right) \right) = v . \quad (5.1)
$$

It $\tau$ has second order derivatives at $x_o$, the 2-form $dd_J^c \tau$ is well defined at $x_o$ and, for any $v \neq 0$,

$$
1 = \partial \overline{\partial}(|\zeta|^2) = -2i dd_J^c |(\zeta|^2)|_{\zeta=0} = -2i dd_J^c \tau|_{x_o} \left( f^v \left( \frac{\partial}{\partial \zeta} \right), f^v \left( \frac{\partial}{\partial \overline{\zeta}} \right) \right) =
\frac{1}{\kappa(v)^2} \left( -2i \, dd_J^c \tau|_{x_o} (v, v) \right) . \quad (5.2)
$$

This shows that the $J$-Hermitian tensor $-2i dd_J^c \tau|_{x_o}$ is positively defined. Consider a basis $B = (e_1, \ldots, e_n)$ for $T_{x_o} M$, which is unitary with respect to $-2i dd_J^c \tau|_{x_o}$. Let $\ell_B : C^n \rightarrow T_{x_o} M$ be the $C$-linear map with $\ell_B(e^i_o) = e^i$. By construction, $\ell_B( -2i dd_J^c \tau|_{x_o} ) = < \cdot, \cdot >$ and $I(B) = \ell_B^{-1}(I_{x_o}) = \mathbb{B}^n$. \hfill $\square$

As a direct corollary of Lemma 5.1 and Lemma 2.7 we have the following:

**Theorem 5.2.** Let $(M, J)$ be a manifold of circular type with Monge-Ampère exhaustion $\tau$ and center $x_o$. If $\tau$ is of class $C^2$ at $x_o$, the Kobayashi indicatrix at $x_o$ is linearly equivalent to $\mathbb{B}^n$ and $(M, J)$ is biholomorphic, via a normalizing map, to a manifold in normal form $(\mathbb{B}^n, J')$ with $\phi_{J'}^{(0)} = 0$.

From the results of [6] it will be clear that there is an infinite dimensional family of such manifolds, none of them biholomorphic to $(\mathbb{B}^n, J_o)$.

5.2. $C^{2k}$-regularity of the exhaustion and vanishing of $\phi_{J}^{(j)}$, $j \leq k-1$.

Let $(M, J)$ be a manifold of circular type with a Monge-Ampère exhaustion $\tau : M \rightarrow [0, 1)$ and center $x_o$. We observe that the 2-form $\omega = dd_J^c \tau$ is a Kähler form for the complex manifold $(M \setminus \{x_o\}, J)$. Furthermore,

**Proposition 5.3.** ([5], Prop.1.1). The integral leaves of the distribution $Z$ of $(M \setminus \{x_o\}, J)$ are totally geodesic for the Kähler metric determined by $\omega = dd_J^c \tau$. The induced metric on any such leaf is flat.
If \( \tau \) is of class \( C^2 \) at \( x_o \), the Kähler form \( \omega = dd^c\tau \) and the Kähler metric \( g = dw^* \omega \) extend at \( x_o \) as tensor fields of class \( C^0 \). Furthermore, by Theorem 5.2, there exists a normalizing map \( \Phi : (M, J) \to (\mathbb{B}^n, J') \) onto a manifold in normal form with \( \phi_j^{(0)} = 0 \). By Lemma 2.6, the 2-forms \( dd^c\tau_o \) and \( dd^cJ\tau_o \) agree at 0 so that \( g \) is positive definite also in \( x_o \), hence a \( C^0 \) Kähler metric.

Assume now that \( \tau \) is of class \( C^k \) for some \( k \geq 2 \) at \( x_o \). Next theorem shows that this fact, together with the abundance of totally geodesic flat complex curves through \( x_o \), has important consequences.

**Theorem 5.4.** Let \( (M, J) \) be an \( n \)-dimensional complex manifold and \( g \) a smooth Kähler metric on the complement \( M \setminus \{x_o\} \) of a singleton \( \{x_o\} \). Assume also that \( g \) has a positive definite \( C^k \)-extension at \( x_o \) for some \( k \geq 2 \) and that, for any \( 0 \neq v \in T_{x_o}M \), there exists a totally geodesic, flat complex curve \( L^v \subset M \) with \( x_o \in L^v \) and \( v, Jv \in T_{x_o}L^v \).

Then the curvature \( R \) of \( g \) and the covariant derivatives \( \nabla^r R \), \( 1 \leq r \leq k-2 \), are well-defined at \( x_o \) and they all vanish at that point.

**Proof.** Since \( g \) is of class \( C^k \), \( k \geq 2 \), in any system of coordinates around \( x_o \) the Christoffel symbols of the Levi-Civita connection are of class \( C^{k-1} \) at that point and the tensors \( R|_{x_o} \), \( \nabla^r R|_{x_o} \), \( r = k-2 \), are well-defined.

Now, consider a vector \( 0 \neq v \in T_{x_o}M \). Since the complex curve \( L^v \) is flat and totally geodesic, for any quadruple of vectors \( v_1, v_2, v_3, v_4 \) at some point \( x \neq x_o \), we have that \( R|_{x_o}(v_1, v_2, v_3, v_4) = 0 \). By continuity, we get

\[
R|_{x_o}(v, Jv, v, Jv) = 0 \quad \text{for all } v \in T_{x_o}M
\]

and this implies that \( R|_{x_o} = 0 \) (see e.g. [8], Prop. IX.7.1).

Consider now an integer \( 1 \leq r \leq k-2 \) and assume the inductive hypothesis \( R|_{x_o} = \nabla^1 R|_{x_o} = \ldots = \nabla^{r-1} R|_{x_o} = 0 \). By Ricci formulas (see e.g. [1], Cor. 1.22), \( \nabla^r R|_{x_o} \) is symmetric in its first \( r \)-arguments, i.e.,

\[
\nabla^r R|_{x_o} \in S^r(T_{x_o} L^v) \otimes (\Lambda^2 T_{x_o}^* M \otimes \Lambda^2 T_{x_o}^* M)
\]

By \( \mathbb{C} \)-linearity, we may consider \( \nabla^r R|_{x_o} \) as a complex tensor in \( \bigotimes^{r+4} T_{x_o}^* M \), and, by classical formulas of Kähler geometry (see e.g. [8], p. 166) we have that, for any \( w_i \in T_{x_o}^* M \) and \( v_j \in T_{x_o}^{1,0} M \),

\[
\nabla^r w_{1 \ldots r} R|_{x_o}(v_1, v_2, w_{r+1}, w_{r+2}) = 0 = \nabla^r w_{1 \ldots r} R|_{x_o}(w_{r+1}, w_{r+2}, v_1, v_2),
\]

\[
\nabla^r w_{1 \ldots r} R|_{x_o}(v_1, v_2, v_3, v_4) = \nabla^r w_{1 \ldots r} R|_{x_o}(v_3, v_4, v_1, v_2) = \nabla^r w_{1 \ldots r} R|_{x_o}(v_1, v_3, v_2).
\]

From these equalities, using Second Bianchi Identities, we also get

\[
\nabla^r w_{1 \ldots r} v_1 R|_{x_o}(v_2, \overline{v}_3, v_4, \overline{v}_5) = \nabla^r w_{1 \ldots r} v_2 R|_{x_o}(v_1, \overline{v}_3, v_4, \overline{v}_5),
\]

\[
\nabla^r w_{1 \ldots r} v_1 R|_{x_o}(v_2, \overline{v}_3, v_4, \overline{v}_5) = \nabla^r w_{1 \ldots r} v_2 R|_{x_o}(v_1, \overline{v}_3, v_4, \overline{v}_5) = \nabla^r w_{1 \ldots r} v_3 R|_{x_o}(v_2, v_1, v_4, \overline{v}_5) = \nabla^r w_{1 \ldots r} v_4 R|_{x_o}(v_2, v_1, v_4, \overline{v}_5).
\]

\[
= \nabla^r w_{1 \ldots r} v_1 R|_{x_o}(v_2, v_3, v_4, \overline{v}_5) = \nabla^r w_{1 \ldots r} v_2 R|_{x_o}(v_1, v_3, v_4, \overline{v}_5).
\]
Therefore, the value of $\nabla^r R|_{x_0}$ on an ordered set of $r+4$ vectors, $p$ of which are holomorphic and $q = r+4-p$ are antiholomorphic, coincides with the value of $\nabla^r R|_{x_0}$ on any other ordered set, given by a permutation of the holomorphic vectors and a permutation of the antiholomorphic ones.

Consider now the following notation: given an order set of $r+4$ holomorphic vectors $w := (w_1, \ldots, w_{r+4}) \subset T^{1,0}_{x_0}M$, for any $z = (z^i) \in \mathbb{C}^{r+4}$ we define $w_z := z^i w_i$. Let also $F(z)$ be the homogeneous complex polynomial of order $r+4$ in $r+4$ variables, defined by

$$F(z) := \nabla^r_{w_1 \ldots w_z} R|_{x_0}(w_z, \overline{w_z}, w_z, \overline{w_z}).$$

We claim that $F(z)$ is $0$ for any $z$. This is clear when $z$ is such that $w_z = 0$. When $z$ is such that $w_z \neq 0$, the vector $w_z$ is holomorphic and tangent to the flat, totally geodesic complex curve $L^v$, $v = \Re(w_z)$, so that

$$F(z) = \nabla^r_{w_1 \ldots w_z} R|_{x_0}(w_z, \overline{w_z}, w_z, \overline{w_z}) = 0.$$

On the other hand, by multilinearity and symmetry properties of $\nabla^r R|_{x_0}$, the coefficient of the monomial $(\Pi_i^r z^i)^{r+3} z^{r+4}$ in the expansion of $F(z)$ is equal to $2!(r+2)|\nabla^r_{w_1 \ldots w_z} R|_{x_0}(w_{r+1}, w_{r+3}, w_{r+2}, w_{r+4})$. Since $F(z) = 0$, this implies that $\nabla^r_{w_1 \ldots w_z} R|_{x_0}(w_{r+1}, w_{r+3}, w_{r+2}, w_{r+4}) = 0$ for any choice of holomorphic vectors $w_i \in T^{1,0}_{x_0}M$.

A similar argument involving the homogenous polynomial

$$G(z) = \nabla^r_{w_1 \ldots w_z} R|_{x_0}(w_z, \overline{w_z}, w_z, \overline{w_z})$$

shows that $\nabla^r_{w_1 \ldots w_z} R|_{x_0}(w_{r+1}, w_{r+3}, w_{r+2}, w_{r+4}) = 0$ for any choice of $w_i \in T^{1,0}_{x_0}M$. Iterating this line of arguments, one shows that $\nabla^r R|_{x_0} = 0$. By induction on $r$, the theorem follows. $\square$

This result brings directly to the next theorem.

**Theorem 5.5.** Let $(M, J)$ be a manifold of circular type with Monge-Ampère exhaustion $\tau$ and center $x_0$. If $\tau$ is of class $C^2$ at $x_0$ for some integer $k \geq 1$, then there is a normalizing map mapping $(M, J)$ into a manifold in normal form $(\mathbb{B}^n, J')$ that has $\phi^{(\ell)} = 0$ for all $0 \leq \ell \leq k-1$.

**Proof.** Since $\tau$ is of class at least $C^2$ at $x_0$, by Theorem 5.2 there exists a normalizing map $\Phi : (M, J) \rightarrow (\mathbb{B}^n, J')$ for a $J'$ with $\phi^{(0)} = 0$. The claim is proved if we show that, when $k \geq 2$, the almost complex structure $J'$ satisfies also the condition $\phi^{(\ell)} = 0$ for every $1 \leq \ell \leq k-1$. For this, we first remark that, by Theorem 5.4 the Kähler metric $g = dd^c J_\tau(\cdot, J\cdot)$ on $(\mathbb{B}^n \setminus \{0\}, J')$ admits a $C^{2k-2}$ extension at $0$ such that $\nabla^m R|_0 = 0$ for any $0 \leq m \leq 2(k-2)$. Note also that such extension necessarily coincides with the standard Euclidean metric at $0$.

Pick two vectors $v, w \in S^{2n-1} \subset T_0 \mathbb{B}^n$, with $w$ orthogonal to $v$ and $J_0 v$ with respect to the standard Euclidean metric. Note that, by means of affine translations, the vectors $w$ and $Jw$ can be identified with two elements of $T_0 S^{2n-1}$ and such that $w^{1,0} := \frac{1}{2}(w - iJ_0 w) = \frac{1}{2}(w - iJ|_0 w)$ belongs to the CR distribution $\mathcal{H}^{1,0}$ of the unit sphere $S^{2n-1}$ in $T_0 \mathbb{B}^n$. 

Consider now the unique stationary disk $f^\nu: \Delta \to \mathbb{B}^n$ of $(\mathbb{B}^n, A_J)$ with $f^\nu(0) = 0$ and $f^\nu \left( \frac{\partial}{\partial z_j} |_{f^\nu(0)} \right) = \lambda v$ for some $\lambda > 0$, and let $\sigma \subset \Delta$ be a sector around the segment $(0, 1)$. Let also $\xi = (z^j = x^j + iy^j)$ be a system of $J'$-holomorphic coordinates, defined on a neighborhood of $f^\nu(\sigma)$ and satisfying the following two conditions:

a) in such coordinates, the stationary disk $f^\nu$ is of the form $f^\nu(\zeta) = (\zeta, 0, \ldots, 0)$;
b) at every $\zeta \in \sigma$, the vectors $\frac{\partial}{\partial z_j} |_{f^\nu(\zeta)}$, $2 \leq j \leq n$, are in $\mathcal{H}^C$.

Coordinates satisfying (a) and (b) can be constructed, by modifying complex polar coordinates in an appropriate way. Moreover, since the vectors $\omega_{ij}( \zeta)$ for all $\zeta \in \sigma$ are constant and $
abla \omega_{ij}$ is of the form $E_j |_{\zeta} + \phi(E_j |_{\zeta})$ for some continuous family of $J_0$-holomorphic vectors $E_j |_{\zeta} \in \mathcal{H}^{1,0}$. With no loss of generality, we may also assume that:

c) the vectors $\frac{\partial}{\partial z_j} |_{f^\nu(\zeta)}$, $2 \leq j \leq n$, are prescribed in such a way that the vectors $E_j |_{\zeta}$ have constant components with respect to standard complex polar coordinates and $\lim_{t \to 0} E_2 |_{t} = w^{1,0}$.

Let us now denote by $\omega_0 := dd^c_J \tau_0$, $\omega := dd^c_J \tau_o$ and set

$$\omega_{ij} := \omega_0 \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right), \quad \omega_{ij} := \omega \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right) = \omega_{ij} + \delta \omega_{ij}$$

for some smooth functions $\delta \omega_{ij}$. Note that, by (2.6) and condition (c), the functions $\omega_{ij} |_{f(\zeta)}$ are constant and

$$\delta \omega_{ij} |_{f(\zeta)} = dd^c_J \tau_0 (\phi(E_2), \phi(E_j)) |_{f^\nu(\zeta)} \text{ for all } \zeta \in \sigma. \quad (5.4)$$

Let also $\Gamma_{ij}^{k}$ be the Christoffel symbols of $\nabla$ in the coordinates $(z^i)$ and denote by $\Gamma_{ij}^{k} := \omega_{ik} \Gamma^{i}_{j} = \omega (\nabla \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})$. From standard facts of Kähler geometry, we know that $\Gamma_{ij}^{k} = \frac{\partial \omega_{ik}}{\partial z^j}$. Moreover, since $f^\nu(\Delta)$ is totally geodesic and flat, we also have $\Gamma_{ij}^{k}(f^\nu(\zeta)) = \Gamma_{ij}^{k}(f^\nu(\zeta)) = 0$ for any $\zeta \in \sigma \subset \Delta$. It follows that

$$R \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^l} \right) |_{f^\nu(\zeta)} = \frac{\partial \Gamma_{12}^{2}}{\partial z^1} |_{f^\nu(\zeta)} = \frac{\partial^2 \omega_{22}}{\partial z^1 \partial z^1} |_{f^\nu(\zeta)} =$$

$$= \frac{\partial^2 \left( \delta \omega_{22} |_{f^\nu(\zeta)} \right)}{\partial \zeta \partial \zeta} = \delta \omega_{22} \left( \frac{\partial^2 (dd^c_J \tau_0(\phi(E_2), \phi(E_2)) |_{f^\nu(\zeta)})}{\partial \zeta \partial \zeta} \right) |_{\zeta}. \quad (5.5)$$

Since $R |_{0} = 0$ and $\phi^{(0)} = 0$, this implies that

$$0 = \lim_{t \to 0} \frac{\partial^2 dd^c_J \tau_0 (\phi(E_2), \phi(E_j)) |_{f^\nu(\zeta)}}{\partial \zeta \partial \zeta} \bigg|_{t} = dd^c_J \tau_0 (\phi^{(1)}(w^{1,0}), \phi^{(1)}(w^{1,0})) \quad (5.6)$$
Being \(dd^c_{J_o} \tau_o > 0\), it follows that \(\phi^{(1)}_{[v]}(w^{1,0}) = 0\) for every \(v, w\), hence \(\phi^{(1)} = 0\). Due to this, the vanishing of \(\phi^{(0)} = 0\) and \([5.4]\), we also have

\[
\lim_{t \to 0} \frac{\partial}{\partial z^1} \Gamma_{1jk}(f(t)) = \lim_{t \to 0} \frac{\partial}{\partial z^1} \frac{\partial (\delta \omega^j_k)}{\partial \xi} t = 0 \quad \text{and}
\]

\[
\lim_{t \to 0} \frac{\partial \Gamma_{1jk}}{\partial z^1} f(t) = \lim_{t \to 0} \frac{\partial (\delta \omega^j_k)}{\partial \xi \partial \xi} t = 0 \quad \text{for all } 1 \leq j, k \leq n. \quad (5.6)
\]

From \([7.2], (5.2)\) and the fact that \(\nabla R|_0 = 0\), it follows that

\[
0 = \lim_{t \to 0} \nabla_{\frac{\partial}{\partial z^1}} \nabla_{\frac{\partial}{\partial z^1}} R|_{f^v(t)} \left( \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^2} \right) = \lim_{t \to 0} \frac{\partial^4 (dd^c_{J_o} \tau_o(\phi(E_2), \phi(E_2))|_{f^v(t)})}{\partial z^1 \partial z^1} = \lim_{t \to 0} \frac{\partial^4 (dd^c_{J_o} \tau_o(\phi^{(2)}_{[v]}(w^{1,0}), \phi^{(2)}_{[v]}(w^{1,0})))}{\partial z^1 \partial z^1}.
\]

Being \(v, w\) arbitrary, this implies that \(\phi^{(2)} = 0\). As before, this implies also

\[
\lim_{t \to 0} \frac{\partial^{r+s} \Gamma_{1jk}}{(\partial z^1)^r(\partial z^1)^s} f(t) = \lim_{t \to 0} \frac{\partial (\delta \omega^j_k)}{(\partial z^1)^{r+1}(\partial z^1)^{s}} t = 0 \quad (5.8)
\]

for any \(r, s\) with \(1 \leq r + s \leq 3\) and for every indices \(1 \leq j, k \leq n\). Iterating this argument, we get by induction that, for all \(0 \leq m \leq k - 2\) and for any \(v, w,\)

\[
0 = \lim_{t \to 0} \nabla_{\frac{\partial}{\partial z^1}} \ldots \nabla_{\frac{\partial}{\partial z^1}} R|_{f^v(t)} \left( \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^2} \right) = \lim_{t \to 0} \frac{\partial^{2m} R(\partial z^1, \partial z^1, \partial z^2, \partial z^2)}{(\partial z^1)^m(\partial z^1)^m} f^v(t) = \lim_{t \to 0} \frac{\partial^{2m+2} (dd^c_{J_o} \tau_o(\phi(E_2), \phi(E_2))|_{f^v(t)})}{(\partial z^1)^{m+1}(\partial z^1)^{m+1}} t = \lim_{t \to 0} \frac{\partial^{2m+2} (dd^c_{J_o} \tau_o(\phi^{(m+1)}_{[v]}(w^{1,0}), \phi^{(m+1)}_{[v]}(w^{1,0})))}{(\partial z^1)^{m+1}(\partial z^1)^{m+1}} t.
\]

This means that \(\phi^{(4)} = 0\) for every \(1 \leq \ell \leq k - 1\).

Since \(J' = J_o\) exactly when \(\phi^{(j)} = 0\) for all \(j\)’s, the previous result gives Stoll’s Theorem as immediate corollary.
Corollary 5.6 ([19]). If \((M,J)\) is a manifold of circular type with a Monge-Ampère exhaustion \(\tau\) which is \(C^\infty\) at all points, there exists a biholomorphism \(\Phi: M \to \mathbb{B}^n\) between \((M,J)\) and the unit ball \(\mathbb{B}^n\) such that \(\tau_o \circ \Phi = \tau\).

6. Examples of convex domains with Monge-Ampère exhaustions of prescribed regularity

Let \((\mathbb{B}^n, \mathcal{A}_J)\) be a circular manifold in normal form determined by a deformation tensor \(\phi = \sum \phi^{(\ell)}\). By Theorem 4.1 and Theorem 5.5, if there is an \(m \geq 3\) such that

\[
\phi^{(j)} = 0 \quad \text{for every } 0 \leq j \leq 2m - 1 \quad \text{and} \quad \phi^{(2m)} \neq 0 ,
\]

then \(\tau_o\) is of class \(C^m\) but not of class \(C^{4m+2}\)

in any system of \(J\)-holomorphic coordinates of \((\mathbb{B}^n, \mathcal{A}_J)\).

A multitude of examples satisfying (6.1) can be constructed. Indeed, observe that if (6.1) holds, clearly \(\phi^{(0)} = 0\) and therefore the formally integrable almost complex structure \(J\) of \((\mathbb{B}^n, \mathcal{A}_J)\) is one of those considered by Bland and Duchamp in [2]. By Thm. 18.2 and 18.4 of that paper, we have (for the definition of the operator \(\bar{\partial}_b\), see [2, 17]):

Theorem 6.1. Let \(f = \sum_{k \geq 1} f^{(k)}: \mathbb{B}^n \to \mathbb{C}\) be a smooth \(\mathbb{C}\)-valued function, which is sum of a series, which is uniformly converging on compacts and with terms of the form

\[
f^{(k)}(w^a, w^b, \zeta, \bar{\zeta}) = \zeta^k \bar{f}^{(k)}(w^a, \bar{w}^b)
\]

in complex polar coordinates \((w^a, \zeta)\). Let also \(\phi_f = \sum_{k \geq 1} \phi_f^{(k)}\) be the formal series defined by

\[
\phi_f^{(0)} = 0 , \quad \phi_f^{(1)} = \bar{\partial}_b(\bar{\partial}_b f^{(1)})^2 , \\
\phi_f^{(k)} = \bar{\partial}_b \left(\partial_b f^{(k)}\right)^2 + \frac{1}{2} \sum_{\ell=1}^{k-1} h_K \left(\left[\phi^{(\ell)}, \phi^{(k-\ell)}\right]\right) , \quad 1 \leq k < \infty ,
\]

where \((\partial_b f^{(j)})^2\) is the unique vector field satisfying

\[
(\partial_b f^{(j)})^j(d\Omega, \tau_o) = 2n\partial_b f^{(j)}
\]

and \(h_K\) is the homotopy operator defined in [2, Lemma 16.4].

Then \(\phi_f = \sum_{k \geq 1} \phi_f^{(k)}\) converges uniformly on compacts and is the deformation tensor of a manifold in normal form \((\mathbb{B}^n, \mathcal{A}_J)\). Moreover, if \(f^{(j)} = 0\) for all \(1 \leq j \leq m - 1\), then \(\phi_f\) is such that \(\phi_f^{(j)} = 0\) for all \(1 \leq j \leq m - 1\).

Consider now a function \(f: \hat{\mathbb{B}}^n \to \mathbb{C}\) of the form

\[
f(w^a, \bar{w}^b, \zeta, \bar{\zeta}) = \zeta^{2m} \bar{f}(w^a, \bar{w}^b) \quad \text{for some} \quad m \geq 3
\]

in complex polar coordinates \((w^a, \zeta)\). By Theorem 6.1 \(f\) determines a manifold in normal form \((\mathbb{B}^n, \mathcal{A}_J)\) satisfying (6.1) and hence with a Monge-Ampère exhaustion of class \(C^m\) but not of class \(C^{4m+2}\) at the center.
By Remark 17.4 (iv) in [2] (see also §4.5 of [3]), if \( f \) is so that \( (\overline{b}_j f^{(j)})^\sharp \) is sufficiently small in an appropriate Folland-Stein norm, the corresponding manifold in normal form \((\mathbb{B}^n, A_J)\) is biholomorphic to a smoothly bounded, strictly linearly convex domain \( D \subset \mathbb{C}^n \), which is arbitrarily close to \( \mathbb{B}^n \), but, of course, not biholomorphic to the unit ball.

By formula (2.6), this discussion immediately gives the following:

**Corollary 6.2.** For any \( m \geq 3 \), there exists a strictly linearly convex domain \( D \subset \mathbb{C}^n \), arbitrarily close to \( \mathbb{B}^n \), with squared Kobayashi distance from a point \( x_o \in D \), which is of class \( C^m \) but not \( C^\infty \) (more precisely, not \( C^{4m+2} \)).

### 7. A refinement of Stoll’s characterization of \( \mathbb{C}^n \)

Consider a complex manifold \( M \) with an unbounded continuous exhaustion \( \tau : M \to [0, \infty) \) that is equal to 0 on a singleton \( \{x_o\} \), is of class \( C^k \) on \( M \setminus \{x_o\} \) and satisfies conditions (ii) and (iii) of the Monge-Ampère exhaustions of manifolds of circular type. Any such manifold is called *unbounded manifold of circular type of class \( C^k \)* and \( \tau \) is called *Monge-Ampère exhaustion* of \( M \). Using our setting we can now easily derive the refinement of Stoll’s characterization of \( \mathbb{C}^n \) ([5, 19]) advertised in the introduction.

**Theorem 7.1.** Let \( M \) be a complex manifold with a \( C^4 \) Monge-Ampère exhaustion \( \tau : M \to [0, +\infty) \). Then there exists a biholomorphism \( \Phi : M \to \mathbb{C}^n \) such that \( \tau(z) = \|\Phi(z)\|^2 \) for all \( z \in M \). In particular, \( \tau \) is of class \( C^\omega \).

Before the proof we need the following remarks. For any unbounded manifold of circular type \((M, \tau)\), each subset \( M(r) = \{x \in M : \tau(x) < r\} \) is endowed with the exhaustion \( \tau_r \) and can be considered as a bounded manifold of circular type, the only difference being that \( \tau_r \) is just \( C^k \) away from the center and not \( C^\infty \). Due to this, most properties of bounded manifolds of circular type extend to the unbounded ones. In particular, one can directly check that the circular representation considered in [15] can be constructed for any unbounded manifold of circular type \( M \) of class \( C^k \), \( k \geq 2 \), and determines a \( C^{k-2} \)-diffeomorphism between \( M \) and the blow up \( \widehat{\mathbb{C}}^n \) of \( \mathbb{C}^n \) at the origin. Composing such circular representation with a diffeomorphism of \( \mathbb{C}^n \) satisfying (1) - (4) of §2.5 we get a natural analogue of Theorem 2.5. To state it, we first need some piece of notation.

For any \( m \geq 1 \), let us call *unbounded \( C^m \)-manifold in normal form* any complex manifold of the form \( M = (\mathbb{C}^n, A_J) \), given by a non-standard atlas \( A_J \) of complex coordinates on \( \mathbb{C}^n \), which makes \( M \) the blow-down at 0 of a complex manifold of the form \( \widetilde{M} = (\widehat{\mathbb{C}}^n, A_J) \), where \( A_J \) is a non-standard atlas satisfying the following conditions:

1) any chart of \( A_J \) is a map of class \( C^m \) if expressed in terms of the charts of the standard differentiable manifold structure of \( \widehat{\mathbb{C}}^n \);
2) the formally integrable almost complex structure \( \tilde{J} \) is given by a tensor field of \( \mathbb{C}^n \) of class \( C^{m-1} \) in the standard manifold structure of \( \mathbb{C}^n \), and satisfies (i) - (iii) of Definition 2.3.

The function \( \tau_0 = \| \cdot \|^2 \) is called the Monge-Ampère exhaustion of \( (\mathbb{C}^n, A_J) \).

**Theorem 7.2.** For any unbounded manifold of circular type \( M = (M, A_J) \) of class \( C^k \), \( k \geq 3 \), with exhaustion \( \tau \) and center \( x_o \), there exists a biholomorphism \( \Phi : (M, A_J) \rightarrow (\mathbb{C}^n, A_J) \) onto an unbounded \( C^{k-2} \)-manifold in normal form, mapping the leaves of the Monge-Ampère foliation of \( M \) into the complex lines through the origin of \( \mathbb{C}^n \) and such that \( \tau = \tau_0 \circ \Phi \).

With this result, we are able to prove Theorem 7.1.

**Proof of Theorem 7.1.** By Theorem 7.2 we may assume that \( (M, A_J) \) is an unbounded \( C^2 \)-manifold in normal form \( (\mathbb{C}^n, A_J) \) and \( \tau = \tau_0 \). The claim is proved if we can show that the tensor field \( J \) on \( \mathbb{C}^n \) is actually equal to \( J_o \).

As for the (bounded) manifolds in normal form, the tensor field \( J \) is uniquely determined by its deformation tensor \( \phi = \phi_J \), defined in (2.8). Since \( J \) is of class \( C^1 \) on \( \mathbb{C}^n \setminus \{0\} \) (and the same is true for the corresponding tensor \( \tilde{J} \) on \( \mathbb{C}^n \)), also \( \phi \) is of class \( C^1 \) and satisfies the same conditions of the deformation tensors of bounded manifolds in normal forms. In particular, it is of the form \( \phi = \sum_{k>0} \phi^{(k)} \) for some tensor fields \( \phi^{(k)} \) of the form (2.10). Moreover, since \( \tau \) is of class \( C^4 \) at the center, the same arguments of Theorem 5.2 imply that we may assume \( \phi^{(0)} = 0 \).

Consider now some \( 0 \neq v \in \mathbb{C}^n \) and the straight line \( f^v : \mathbb{C} \rightarrow \mathbb{C}^n \), \( f^v(\zeta) := v\zeta \). Consider also a \( J_v \)-holomorphic vector \( E \in T^1_0 \mathbb{C}^n \simeq \mathbb{C}^n \) of unit length and orthogonal to \( v \) and \( Jv \). By construction, the affine translation \( E_y \) of \( E \) at any point \( y \in f^v(\mathbb{C}) \) is in \( \mathcal{H}^1_0 \) (here, we denote by \( \mathcal{H} \) and \( \mathcal{H}^1_0 \subset \mathcal{H}^C \) the analogues on \( \mathbb{C}^n \) of the distributions \( \mathcal{H} \) and \( \mathcal{H}^1_0 \) of \( \mathbb{R}^n \), defined in [27]). Hence, by property (iii) of deformation tensors,

\[
\begin{align*}
\ddc_{J_y} \tau_o(\phi(E_y), \phi(E_y)) < \ddc_{J_y} \tau_o(E_y, E_y) = \| E \| = 1 .
\end{align*}
\]

(7.1)

Secondly, fix a collection \( (e_i) \) of linearly independent local generators for \( \mathcal{H}^1_0 \), which are invariant under the flow of the real vector field \( Z = \text{Re} (z^i \frac{\partial}{\partial z^i}) \) and defined on a neighborhood of \( f^v(\mathbb{C}) \setminus \{0\} \), and denote by \( (e^j) \) the corresponding field of dual coframes in \( \mathcal{H}^{1,0} \). We recall that, if the tensor field \( \phi \) is written as linear combination of the tensor fields \( e_i \otimes \bar{\partial} \), i.e.,

\[
\phi|_{(v),\zeta v} = \phi|_{(v),\zeta v} e_i \otimes \bar{\partial} , \quad ((v),\zeta v) \in \mathbb{C}^n \setminus \pi^{-1}(0) = \mathbb{C}^n \setminus \{0\} ,
\]

the functions \( \phi|_{(v),\zeta v} \) are holomorphic in \( \zeta \) (see Prop. 4.2 (iii) in [17]).

Thirdly, let \( y_o := f^v(1) = v \) and write \( E_v = E^i e_i|_v \). By invariance under the flow of \( Z \), we have that, at any point \( y = \zeta v, \zeta \neq 0 \),

\[
e_i|_{\zeta v} = |\zeta| \cdot e_i|_v , \quad e^i|_{\zeta v} = \frac{1}{|\zeta|} \cdot e^i|_v \quad \implies \quad E_{\zeta v} = \frac{1}{|\zeta|} E^i e_i|_{\zeta v}
\]
and (7.1) becomes
\[
\sum_{i,j} \left| \phi_i^j (v,\zeta v) \right|^2 \frac{\| e_i \|}{\| \zeta v \|^2} = \sum_{i,j} \left| \phi_i^j (v,\zeta v) \right|^2 \frac{\| e_i \|}{\| \zeta v \|^2} < 1. \tag{7.2}
\]
Being \( E \) arbitrary, it follows that each map \( \zeta \rightarrow \phi_i^j (v,\zeta v) \) is holomorphic and uniformly bounded on \( \mathbb{C} \), hence constant by Liouville Theorem. This implies that \( \phi = \phi^{(0)} = 0 \), i.e. that \( J = J_o \), as we needed to show. \( \square \)

**Remark 7.3.** The proof shows that the theorem holds requiring that the exhaustion \( \tau \) is just of class \( C^2 \) at the center and \( C^4 \) elsewhere. In fact, a simple modification of the above proof shows that \((M,\mathcal{A},J)\) is mapped biholomorphically onto \( \mathbb{C}^n \) even if \( \tau \) is just \( C^0 \) at the center. In this case, \( \tau \) is mapped into the square \( \mu_D^2 \) of the Minkowski functional \( \mu_D : \mathbb{C}^n \rightarrow [0, +\infty) \) of a circular domain \( D \). This fact generalizes Thm. 4.4 (i) in [15].

Indeed a more detailed analysis of the regularity of the circular map might provide similar results under (slightly) weaker differentiability assumptions on \( \tau \) as the main argument consists of a leafwise use of Liouville Theorem.

**References**

[1] A.L. Besse, Einstein manifolds, Springer-Verlag, 1986.
[2] J. Bland and T. Duchamp, *Moduli for pointed convex domains*, Invent. Math. **104** (1991), 61–112.
[3] J. Bland and T. Duchamp, *Contact geometry and CR structures on spheres*, in "Topics in Complex Analysis", Banach Center Publ., vol. 31 (1995), 99–113.
[4] J. Bland, T. Duchamp and M. Kalka, *On the automorphism group of strictly convex domains in \( \mathbb{C}^n \)*, Contemp. Math. **49** (1986), 19–29.
[5] D. Burns, *Curvatures of Monge-Ampère foliations and parabolic manifolds*, Ann. of Math. **115** (1982), 349–373.
[6] S. I. Goldberg, Curvature and Homology, Academic Press, 1970.
[7] C. D. Hill and M. Taylor, *Integrability of Rough Almost Complex Structures*, J. Geom. Anal. **13** (1) (2003), 163–172.
[8] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. 2, *John Wiley & Sons, Inc.*, 1996.
[9] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France, **109** (1981), 427–474.
[10] L. Lempert, *Holomorphic invariants, normal forms and the moduli space of convex domains*, Ann. of Math. **128** (1988), 43–78.
[11] B. Malgrange, “Sur l’intégrabilité des structure presque-complex”, in Symposia Math. vol. II, 289–296, *Academic Press*, 1969.
[12] A. Newlander and L. Niremberg, *Complex analytic coordinates in almost complex manifold*, Ann. of Math. **65** (1957), 391–404.
[13] A. Nijenhuis and W. Woolf, *Some integration problem in almost-complex manifolds*, Ann. of Math. **77** (1963), 424–489.
[14] G. Patrizio, *Parabolic Exhaustions for Strictly Convex Domains*, Manuscripta Math. **47** (1984), 271–309.
[15] G. Patrizio, *A characterization of complex manifolds biholomorphic to a circular domain*, Math. Z. **189** (1985), 343–363.
[16] G. Patrizio, *Disques extrêmaux de Kobayashi et équation de Monge-Ampère complex*, C. R. Acad. Sci. Paris, Série I, **305** (1987), 721–724.
[17] G. Patrizio and A. Spiro, Monge-Ampère equations and moduli spaces of manifolds of circular type, Adv. Math. 223 (2010), 174–197.

[18] N. Sibony, Remarks on the Kobayashi metric, Unpublished manuscript (1979).

[19] W. Stoll, The characterization of strictly parabolic manifolds, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4) 7 (1980), 87–154.

[20] S. Webster, A new proof of the Newlander-Nirenberg theorem, Math. Zeit. 201 (1989), 303–316.