GROUND STATES OF NONLINEAR SCHRÖDINGER SYSTEMS WITH PERIODIC OR NON-PERIODIC POTENTIALS

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Abstract. In this paper we study a class of weakly coupled Schrödinger system arising in several branches of sciences, such as nonlinear optics and Bose-Einstein condensates. Instead of the well known super-quadratic condition that \( \lim_{|z| \to \infty} \frac{F(x,z)}{|z|^2} = \infty \) uniformly in \( x \), we introduce a new local super-quadratic condition that allows the nonlinearity \( F \) to be super-quadratic at some \( x \in \mathbb{R}^N \) and asymptotically quadratic at other \( x \in \mathbb{R}^N \). Employing some analytical skills and using the variational method, we prove some results about the existence of ground states for the system with periodic or non-periodic potentials. In particular, any nontrivial solutions are continuous and decay to zero exponentially as \( |x| \to \infty \). Our main results extend and improve some recent ones in the literature.

1. Introduction. We consider following coupled gradient system of nonlinear Schrödinger equations:

\[
\begin{aligned}
-\Delta u + V_1(x)u &= F_u(x, u, v) \quad \text{in} \quad \mathbb{R}^N, \\
-\Delta v + V_2(x)v &= F_v(x, u, v) \quad \text{in} \quad \mathbb{R}^N, \\
u, v &\in H^1(\mathbb{R}^N),
\end{aligned}
\]  

(1.1)

where \( V_1, V_2 \in C(\mathbb{R}^N, \mathbb{R}) \), \( F: \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R} \) satisfies following basic assumptions with gradient \( \nabla F = (F_u, F_v) := F_z \),

(S0) there exist constants \( p \in (2, 2^*) \), \( C_1 > 0 \) such that

\[ |F_z(x, z)| \leq C_1(1 + |z|^{p-1}), \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2, \]

where \( 2^* := 2N/(N-2) \) if \( N \geq 3 \) and \( 2^* := +\infty \) if \( N = 1 \) or \( 2 \); 

(S1) \( F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, [0, \infty)) \) and \( |F_z(x, z)| = o(|z|) \) as \( |z| \to 0 \) uniformly in \( x \).

Coupled Schrödinger system (1.1) arises in different areas of mathematical physics and appears in many applications, such as studies of Bose-Einstein condensates in condensed matter physics [43], the propagation in birefringent optical fibers [27],

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gap solitons in photonic crystals [29] and the Hartree-Fock theory for the double condensate [16]. System (1.1) is also important for industrial applications in fiber communications systems [17] and all-optical switching devices [18]. Depending on various assumptions on the potential $V_i$, $i = 1, 2$ and the nonlinearity $F$, there are many results of existence and multiplicity solutions to problem (1.1). When $V_1 = V_2$ and $u = v$, (1.1) becomes the scalar Schrödinger equation, which has been widely investigated in the literature, see, e.g., [19, 34, 35, 40, 42]. For the case of a bounded domain, system like or similar to (1.1) has been studied by many authors, see [11, 7, 21, 25] and the references therein.

Very recently, many authors focused their attention on system (1.1) for the case of unbounded domain or the whole space case. The main difficulty of such type problem is the lack of compactness of the Sobolev embedding. By assuming symmetry property on the potential and working on the radially symmetric function space, one can recover the compactness of embedding, see, e.g., [23, 45]. Another usual way to regain the compactness is by imposing coercive assumption on the potential, see, for instance, [3, 8, 10, 35]. The concentration compactness argument is also well employed to deal with the whole space case provided that the potential and nonlinearity are periodic in the variable $x$, we refer readers to [6, 9, 26, 31] and the references therein. By using the constrained minimization method [5] and the Nehari manifold method [35, 24], and applying the bootstrap argument [9] and some skills related to ordinary differential system [23, 45], many authors studied system (1.1) for case that the potentials are nonnegative constants and obtained certain of results on the existence, regularity and uniqueness of ground state solutions. Here, a ground state(least energy) solution means a nontrivial solution $z_0 = (u, v) \in E$ with minimum energy, which can be formulated as follows,

$$\Phi(z_0) = \inf_\mathcal{M} \Phi, \quad \text{where } \mathcal{M} := \{z \in E \setminus \{(0, 0)\} : \Phi'(z) = 0\},$$

where $\Phi$ is the energy functional defined later by (2.7) and $E$ the working space given by (2.4). In [1], Ambrosetti, Cerami and Ruiz considered (1.1) for pure powers nonlinear terms and obtained bound state solutions by using linking together with the barycenter function restricted on the Nehari manifold. In recent paper [33], system (1.1) with non-autonomous and non-homogeneous nonlinearity was studied by Qin, Chen and Tang. Due to the geometrical hypotheses imposed on the potential and nonlinearity, it is not allowed to use the concentration compactness argument or work on the radially symmetric function space. To conquer the difficulties, a Nehari-Pohozaev manifold which is a combination of the usual Nehari manifold and the Pohozaev’s identity was introduced there. By applying a linking argument and minimizing the functional $\Phi$ on an appropriate subset of the manifold, bound states solutions were obtained there.

When $V_i$, $i = 1, 2$, are positive and coercive, existence of nontrivial solutions of (1.1) was established by Costa [10] under a condition which was called nonquadratic at infinity or the following classic condition introduced by Ambrosetti and Rabinowitz [2],

\[(AR) \text{ there exists a constant } \mu > 2 \text{ such that }\]

\[0 < \mu F(x, z) \leq F_z(x, z) \cdot z, \quad \forall x \in \mathbb{R}^N, \ z \in \mathbb{R}^2 \setminus \{(0, 0)\}\]

where the dot denotes the inner product in $\mathbb{R}^2$, by virtue of which the mountain-pass geometry and the Palais-Smale condition were checked there. For the periodic case, i.e.,
(V) $V_1$, $V_2 \in C(\mathbb{R}^N)$ are 1-periodic in $x_j$, $j = 1, 2, \cdots, N$, and

$$\sup [\sigma(-\Delta + V_i) \cap (-\infty, 0)] := \Lambda_i < 0 < \Lambda_i := \inf [\sigma(-\Delta + V_i) \cap (0, \infty)].$$

Chen and Ma [6] studied (1.1) for super-quadratic or asymptotically quadratic nonlinearity $F$ and obtained the existence of least energy solutions by using a generalized weak linking theorem [38]. More precisely, following super-quadratic condition (SQ) introduced by Liu-Wang [22] and technical condition (DL) introduced by Ding-Lee [13] were used there. Later, by employing the non-Nehari manifold method introduced by Tang [41], these results were improved by Qin, He and Tang [31, 32] and generalized to the asymptotically periodic case where the potentials $V_i(x)$ and the nonlinearity are allowed to be asymptotically periodic in $x$. See [26] for related results and [8] for nonexistence result.

(SQ) $\lim_{|z| \to \infty} F(x, z) = \infty$ uniformly in $x$;

(DL) $\hat{F}(x, z) := \frac{1}{2} F_z(x, z) \cdot z - F(x, z) > 0$ if $z \neq (0, 0)$, and there exist constants $r_0 > 0$, $c_1 > 0$ and $n' > \max\{1, N/2\}$ such that

$$|F_z(x, z)|^{n'} \leq c_1 \hat{F}(x, z)|z|^{n'}, \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2, \ |z| \geq r_0.$$

Since (SQ) is weaker than (AR), it is commonly used in the literature and it plays a crucial role in establishing the mountain-pass(or linking) geometry and in verifying the boundedness of the Palais-Smale(or Cerami) sequence for the energy functional $\Phi$. Indeed, it is essential to prove the existence of nontrivial solutions for periodic problem like or similar to (1.1) in all literature.

Motivated by above works, we continue to study system (1.1) and our purpose in this paper is twofold. First we introduce a local super-quadratic condition which allows the nonlinearity to be super-quadratic at some $x \in \mathbb{R}^N$ and asymptotically quadratic at other $x \in \mathbb{R}^N$. Therefore it weakens condition (SQ) properly, while it brings some new difficulties in the verification of linking geometry and boundedness of Cerami sequences for $\Phi$. Employing some new techniques and using a technical condition similar to (DL), we obtain the existence and exponential decay estimate of ground state solution for periodic system (1.1). Such an existence result, to the best of our knowledge, is up to date. Even under (SQ), the result partially extends and, in fact, complements the above mentioned results of [6, 13, 26, 31]. Secondly we consider the non-periodic case where the potentials $V_i$ and the nonlinearity $F$ are non-periodic with respect to $x$. Clearly, the concentration compactness argument is no more applicable, moreover, the potential $V_i$ considered here is allowed to be sign-changing, which is different from [1, 5, 10, 9, 24, 33]. Under the local super-quadratic condition, we obtain a continuous ground state solution of (1.1) which decays to zero exponentially, the result extends and complements related ones in [10, 8]. More precisely, we will prove Theorems 1.1 and 1.2 below by using following local super-quadratic condition instead of (SQ),

(S2) there exists a domain $\Omega \subset \mathbb{R}^N$ such that $\lim_{|z| \to \infty} \frac{F(x, z)}{|z|^2} = \infty$ a.e. $x \in \Omega$.

Before introducing our technical condition, let us define

$$\Lambda_0 := \min \{-\Lambda_1, \Lambda_1, -\Lambda_2, \Lambda_2\},$$

and $\Lambda_0 > 0$ by (V).

(S3) $\hat{F}(x, z) \geq 0$, and there exist constants $C_0 > 0$, $\delta_0 \in (0, \Lambda_0)$ and $\sigma \in (0, 1)$ such that the relation

$$\frac{|F_z(x, z)|}{|z|} \geq \frac{\sqrt{2}}{2} (\Lambda_0 - \delta_0) \Rightarrow \left(\frac{|F_z(x, z)|}{|z|^\sigma}\right)^{\kappa} \leq C_0 \hat{F}(x, z), \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$
holds for \( \kappa = \frac{2N}{2N - (1 + \sigma)(N - 2)} \) if \( N \geq 3 \), or for \( \kappa \in \left(1, \frac{2}{1 - \sigma}\right) \) if \( N = 1, 2 \).

Our first result for the periodic case reads as follows.

**Theorem 1.1.** Let \((V), (S0), (S1), (S2)\) and \((S3)\) be satisfied. If \( F(x, z) \) is 1-periodic in each of \( x_1, x_2, \cdots, x_N \), then problem \((1.1)\) has a continuous ground state solution \( z_0 \), moreover, there exist \( \tau, C > 0 \) such that

\[
|z_0(x)| \leq C e^{-\tau|x|}, \quad \forall \ x \in \mathbb{R}^N.
\]

Next, we consider the non-periodic case and make use of following assumption:

\((V')\) \( V_i \in C(\mathbb{R}^N, \mathbb{R}) \) and \( \inf_{\mathbb{R}^N} V_i(x) > -\infty \), \( i = 1, 2 \), there exists a constant \( d_0 > 0 \) such that

\[
\lim_{|y| \to +\infty} \operatorname{meas}\{ x \in \mathbb{R}^N : |x - y| \leq d_0, \ V_i(x) \leq M \} = 0, \quad \forall \ M > 0,
\]

where \( \operatorname{meas}(\cdot) \) denotes the Lebesgue measure in \( \mathbb{R}^N \).

\((S4)\) \( \tilde{F}(x, z) \geq 0 \), and there exist constants \( C_0, R_0 > 0 \) and \( \sigma \in (0, 1) \) such that the inequality

\[
\left( \frac{|F_x(x, z)|}{|z|^{\sigma}} \right)^{\kappa} \leq C_0 \tilde{F}(x, z), \quad \forall \ |z| \geq R_0
\]

holds for \( \kappa = \frac{2N}{2N - (1 + \sigma)(N - 2)} \) if \( N \geq 3 \), or for \( \kappa \in \left(1, \frac{2}{1 - \sigma}\right) \) if \( N = 1, 2 \).

\((S5)\) \( \tilde{F}(x, z) \geq 0 \), and there exist constants \( c_0, r_0, \theta > 0 \) such that the inequality

\[
|F_x(x, z)||z|^{1 - \theta} \leq c_0 \tilde{F}(x, z), \quad \forall \ |z| \geq r_0
\]

holds for \( \theta < N + p - pN/2 \) if \( N \geq 3 \), or for \( \theta < 2 \) if \( N = 1, 2 \).

Now, we are ready to state our main results for the non-periodic case.

**Theorem 1.2.** Let \((V'), (S0), (S1), (S2)\) and \((S4)\) be satisfied. Then problem \((1.1)\) has a ground state solution. Moreover \((V')\), \((S0)\) and \((S1)\) imply that any solution of \((1.1)\) is continuous and there exist \( \tau, C > 0 \) such that \((1.3)\) holds.

**Theorem 1.3.** Let \((V), (S0), (S1), (S2)\) and \((S5)\) be satisfied. Then problem \((1.1)\) has a continuous ground state solution satisfying \((1.3)\).

**Remark 1.** Condition \((V')\) has a potential well which is widely used in the literature(see [3, 8]), and implies, as a consequence of Molchanov’s result [28], that the embedding \( H := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_i(x)|u|^2dx < \infty \} \hookrightarrow L^2(\mathbb{R}^N) \) is compact, by the interpolation inequality we know that the embedding \( H \hookrightarrow L^p(\mathbb{R}^N) \) is compact for all \( p \in [2, 2^*) \), see also [4, Lemma 3.1]. Clearly, potentials \( V_i \) are allowed to be sign-changing by \((V')\), thus problem \((1.1)\) considered here is indefinite, while it is strongly indefinite under \((V)\).

**Remark 2.** Clearly, \((S1)\) yields that

\[
\frac{|F_x(x, z)|}{|z|} \geq \frac{\sqrt{2}}{2} (\Lambda_0 - \delta_0) \implies |z| \geq r \text{ for some } r > 0.
\]

Conditions \((S3)\) and \((S4)\) are similar to condition \((DL)\), indeed, they are complementary to \((DL)\) in the sense that, on the one hand, \((S4)\) together with \((S0)\) yields \((DL)\) if \( \tilde{F}(x, z) > 0 \) for \( z \neq (0, 0) \), on the other hand, when the following condition holds,

\[
\tilde{F}(x, z) \geq c|z|^\varrho, \quad \forall \ |z| \geq r_0, \text{ for some } c > 0, \quad \varrho \in (0, 2],
\]
(DL) yields (S4) with
$$\sigma = \frac{2N}{q} - N + \frac{2N}{\alpha} - 2 \in (0, 1)$$
if $N \geq 3$, or with $\kappa = (\kappa' + 1)/2$ and
$$\sigma = 1 - \frac{(\kappa' - 1)q}{\kappa' + 1} \in (0, 1)$$
if $N = 1, 2$.

Condition like or similar to (1.4) is called nonquadratic at infinity introduced by Costa and Magalhães [10, 11] with $\rho > (p - 2)\min\{1, N/2\}$, and it is widely used in the literature, see [8] with $\rho = p$ where infinitely many nontrivial solutions are obtained by Chen and Ma under some additional conditions besides $(V')$, (S0), (S1) and (SQ). Particularly, for the scalar equation of (1.1) we deduce from (S1) and (SQ) that
$$\lim_{t \to -\infty} \frac{F(x, t)}{t^2} = 2 \int_0^\infty \frac{\hat{F}(x, s)}{s^3} ds = \infty \quad \text{and} \quad \lim_{t \to -\infty} \frac{\hat{F}(x, s)}{s^{\frac{3}{2}}} ds = \infty,$$
uniformly in $x \in \mathbb{R}^N$, therefore (1.4) is a reasonable condition. However, if $\hat{F}(x, z) = 0$ for some $(x, z) \in \mathbb{R}^N \times \mathbb{R}^2$, (DL) is no longer applicable, while (S3) and (S4) are still valid in this case.

Remark 3. Condition (S5) together with (S0) yields (S4) (see Lemma 2.3), and it can be deduced from (AR) when $|F_z(x, z)| = F_z(x, z) \cdot z$ which holds naturally for scalar equation of (1.1). In this case, conditions (S2) and (S4) weaken (AR), similarly, we can certify that (AR) implies (S2) and (S3).

Before proceeding to the proof of main results, we give two nonlinear examples to illustrate the assumptions (S2), (S3) and (S4).

Remark 4. Let $N \geq 3$ and $F(x, z) = \sin^2(2\pi x_1)|z|^2 \ln|1 + |z|^2|$. Then
$$F_z(x, z) = 2 \sin^2(2\pi x_1) \left[ \ln(1 + |z|^2) + \frac{|z|^2}{1 + |z|^2} \right] z, \quad \hat{F}(x, z) = \frac{\sin^2(2\pi x_1)|z|^4}{1 + |z|^2} \geq 0.$$  

It is not difficult to verify that $F$ satisfies (S0)-(S5) with $\sigma \in (0, 1)$ and $\Omega = (1/8, 3/8) \times \mathbb{R}^{N-1}$, but neither of (AR), (SQ) and (DL).

Theorem 1.4. Let $G \subset \mathbb{R}^N$ be a closed set and $F(x, z) = |z|^{2+\alpha(x)} \left[ 1 - \frac{1}{\ln(e + |z|^2)} \right]$, where $\alpha \in C(\mathbb{R}^N, \mathbb{R})$, $\alpha(x) = 0$ for $x \in G$ and $\alpha(x) \in (0, 4/(N - 2))$ for $x \in \mathbb{R}^N \setminus G$. Then
$$F_z(x, z) = \frac{\alpha(x)}{2} |z|^2 + \alpha(x) \left[ 1 - \frac{1}{\ln(e + |z|^2)} \right] \frac{2|z|^{2+\alpha(x)}}{(e + |z|^2)(\ln(e + |z|^2))^2},$$
$$\hat{F}(x, z) = \frac{\alpha(x)}{2} |z|^2 + \alpha(x) \left[ 1 - \frac{1}{\ln(e + |z|^2)} \right] \frac{|z|^{4+\alpha(x)}}{(e + |z|^2)(\ln(e + |z|^2))^2} \geq 0.$$  

It is easy to see that $F$ satisfies (S0)-(S5) with $\sigma \in (0, 1)$ and $\Omega \subset \mathbb{R}^N \setminus G$, but neither of (AR), (SQ) and (DL). Moreover, $F(x, z)$ is asymptotically quadratic when $x \in G$ and super-quadratic when $x \in \mathbb{R}^N \setminus G$.

Let $E = E^+ \oplus E^0 \oplus E^-$ be the working space with the norm defined later by (2.9), as we shall see in Section 2, the functional corresponding to (1.1) is
$$\Phi(z) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V_1(x)|u|^2 + |\nabla v|^2.$$
To prove Theorem 1.1 and Theorem 1.2, we have to conquer two main difficulties due to the lack of the usual super-quadratic condition \( (SQ) \), which is embodied in checking the linking geometry of \( \Phi \) and verifying the boundedness of Cerami sequences. More specifically, first we need to find an \( e \in E^+ \setminus \{ (0,0) \} \) and an \( r > 0 \) such that \( \sup \Phi(\partial Q) \leq 0 \), where

\[
Q = \{ w + se : w \in E^- \oplus E^0, s \geq 0, \|w + se\| \leq r \}.
\]

To achieve the result with reduction to absurdity, it seems to be necessary to prove that

\[
\lim_{\|w + se\| \to \infty} \int_{\Omega} \frac{F(x, w + se)}{\|w + se\|^2} dx = \infty.
\]

This can be deduced from (S1), (SQ) and Fatou’s lemma if \( \Omega = \mathbb{R}^N \), since \( w + se|_{\mathbb{R}^N} \neq (0,0) \) for any \( s > 0 \) and \( w \in E^- \oplus E^0 \). However, if we replace (SQ) by (S2), the above equation becomes difficult to verify since it cannot be determined whether \( w + se|_{\Omega} \neq (0,0) \) for any \( s > 0 \) and \( w \in E^- \oplus E^0 \). Secondly, for any Cerami sequence \( \{ z_n \} \subset E \) satisfying \( z_n/\|z_n\| \to w \neq (0,0) \), it follows from (S1), (SQ) and Fatou’s lemma that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|^2} dx = \infty.
\]

Similarly, we cannot determine whether \( w|_{\Omega} \neq (0,0) \), which makes it difficult to certify the above equation when (SQ) is replaced by (S2). Therefore, some new techniques are looked forward to being introduced to surmount the above mentioned two difficulties, which is the right issue this paper intends to address. We point out that such a local super-quadratic condition can be used to study existence of nontrivial solutions for other indefinite problems, as well as multiple solutions for elliptic problems.

Remaining of this paper is organized as follows. In Section 2, we present the variational setting and give some preliminaries. In Section 3, the linking geometry for the periodic case and non-periodic case is established, respectively. Applying the generalized linking theorem \([19, 20]\), we find the Cerami sequences for \( \Phi \). Boundedness of Cerami sequences are verified in Section 4, by virtue of which we prove Theorem 1.1 and Theorem 1.2.

Throughout this paper, we use \( c_i \) and \( C_i \) \( (i = 1, 2, \ldots) \) to denote different positive constants.

2. Variational setting and preliminaries. We establish a unitary variational framework for the periodic or non-periodic elliptic system similar to or like (1.1). Let \( A_i = -\Delta + V_i \), here and in what follows \( i = 1, 2 \). Then \( A_i \) are self-adjoint in \( L^2(\mathbb{R}^N) \) with domain \( \mathcal{D}(A_i) \) (see [15, Theorem 4.26]). Let \( \{ \mathcal{E}_i(\lambda) : -\infty \leq \lambda \leq +\infty \} \) and \( |A_i| \) be the spectral family and the absolute value of \( A_i \), respectively, and \( |A_i|^{1/2} \) be the square root of \( |A_i| \). Set \( U_i = i d - \mathcal{E}_i(0) - \mathcal{E}_i(0-) \). Then \( U_i \) commutes with \( A_i \), \( |A_i| \) and \( |A_i|^{1/2} \), and \( A_i = U_i |A_i| \) is the polar decomposition of \( A_i \) (see [14, Theorem IV 3.3]). Let \( H_i = \mathcal{D}(|A_i|^{1/2}) \) and

\[
H_i^- = \mathcal{E}_i(0-) U_i, \quad H_i^0 = [\mathcal{E}_i(0) - \mathcal{E}_i(0-)] U_i, \quad H_i^+ = [i d - \mathcal{E}_i(0)] U_i.
\]
For any $u_i \in H_i$, fixing $i = 1$ or $i = 2$, it is easy to see that $u_i = u_i^- + u_i^0 + u_i^+$ with

$$u_i^- = E_i(0-)u_i \in H_i^-, \quad u_i^0 = [E_i(0) - E_i(0-)]u_i \in H_i^0, \quad u_i^+ = [id - E_i(0)]u_i \in H_i^+$$

(2.1)

and

$$A_i u_i^- = -|A_i| u_i^-, \quad A_i u_i^0 = 0, \quad A_i u_i^+ = |A_i| u_i^+, \quad \forall \ u_i = u_i^- + u_i^0 + u_i^+ \in H_i \cap D(A_i).$$

(2.2)

For fixed $i$ taking 1 or 2, define an inner product

$$\langle u, v \rangle_{H_i} = \left( |A_i|^{1/2} u_i |A_i|^{1/2} v_i \right)_{L^2} + (u^0, v^0)_{L^2}, \quad u, v \in H_i$$

(2.3)

and the corresponding norm

$$\| u \|_{H_i} = \left\| |A_i|^{1/2} u \right\|_{L^2} + \| u^0 \|_{L^2}, \quad u \in H_i,$$

where $\langle \cdot, \cdot \rangle_{L^2}$ denotes the inner product of $L^2(\mathbb{R}^N)$, $\| \cdot \|_{L^s}$ stands for the usual $L^s(\mathbb{R}^N)$ norm, $1 \leq s < \infty$. There are induced decompositions $H_i = H_i^- \oplus H_i^0 \oplus H_i^+$ which are orthogonal with respect to both $\langle \cdot, \cdot \rangle_{L^2}$ and $\langle \cdot, \cdot \rangle_{H_i}$. Then

$$\int_{\mathbb{R}^N} (|\nabla u_i|^2 + V_i(x)|u_i|^2) \, dx = \| u_i^+ \|_{H_i}^2 - \| u_i^- \|_{H_i}^2, \quad \forall \ u_i = u_i^- + u_i^0 + u_i^+ \in H_i, \ i = 1, 2.$$

If condition (V) holds, then $H_i^- \oplus H_i^+ = H_1 = H^1(\mathbb{R}^N)$ with equivalent norms, and $H_i \subset H^1(\mathbb{R}^N)$ if (V) holds. Therefore, $H_i$ embeds continuously in $L^s(\mathbb{R}^N)$ for all $2 \leq s \leq 2^\ast$.

Let $E = H_1 \times H_2$ equipped with the inner product

$$\langle z, \zeta \rangle = (u, \varphi)_{H_1} + (v, \psi)_{H_2}, \quad z = (u, v), \quad \zeta = (\varphi, \psi) \in E = H_1 \times H_2$$

(2.4)

and the corresponding norm

$$\| z \| = \left( \| u \|_{H_1}^2 + \| v \|_{H_2}^2 \right)^{1/2}, \quad z = (u, v) \in E.$$

(2.5)

(1.2) and (2.5) yield that $\| z \|^2 \geq \Lambda_0 \| z \|_{L^2}^2$ for any $z \in E$, where $\| \cdot \|_{L^s}$ stands for the usual $L^s(\mathbb{R}^N, \mathbb{R}^2)$ norm, $1 \leq s < \infty$. Moreover, by Remark 1.4 we have following lemma.

**Lemma 2.1.** Let (V) or (V’) be satisfied, then the embedding $E \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$ is continuous for $q \in [2, 2^\ast]$ and $E \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^2)$ is compact for $q \in [2, 2^\ast)$, moreover, the embedding $E \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$ is compact for $q \in [2, 2^\ast)$ if (V’) holds.

For any $\varepsilon > 0$, (S0) and (S1) yield the existence of $C_\varepsilon > 0$ such that

$$|F_\varepsilon(x, z)| \leq \varepsilon \| z \| + C_\varepsilon \| z \|^{p-1}, \quad \forall \ (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$ 

(2.6)

Under (V) or (V’), a standard argument (see [12, 46]) shows that the solutions of problem (1.1) are critical points of the functional

$$\Phi(z) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla u|^2 + V_1(x)|u|^2 + |\nabla v|^2 + V_2(x)|v|^2 \right] \, dx - \int_{\mathbb{R}^N} F(x, z) \, dx, \quad z = (u, v) \in E.$$ 

(2.7)

$\Phi$ is of class $C^1(E, \mathbb{R})$, and for $z = (u, v), \zeta = (\varphi, \psi) \in E$,

$$\langle \Phi'(z), \zeta \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V_1(x) u \varphi) \, dx + \int_{\mathbb{R}^N} (\nabla v \nabla \psi + V_2(x) v \psi) \, dx.$$
Lemma 2.2. Suppose that \((V)\) (or \((V')\)), \((S0)\) and \((S1)\) are satisfied. Then \(\Psi\) is nonnegative, weakly sequentially lower semi-continuous and \(\Psi'\) is weakly sequentially continuous.

Proof. Employing a standard argument, one can checks easily the lemma, see, for example, [19] and [46].

In following lemma, we show that \((S5)\) together with \((S0)\) yields \((S4)\), then Theorem 1.3 can be deduced from Theorem 1.2 once it is proved.

Lemma 2.3. Suppose that \((S0)\) and \((S5)\) are satisfied. Then \((S4)\) holds.
Proof. **Case 1**. \( N \geq 3 \). Let

\[
\sigma := \frac{(N + 2)\theta + p(N - 2) - 2N}{2N - (p - \theta)(N - 2)}.
\]

Since \( \theta < N + p - pN/2 \), without loss of generality, we may assume that \( \theta > [2N - p(N - 2)]/(N + 2) \), then \( \sigma \in (0, 1) \). (S0) yields the existence of a constant \( c_1 > 0 \) such that

\[
|F_z(x, z)| \leq c_1|z|^{\theta - 1}, \quad \forall \ |z| \geq r_0.
\] (2.15)

Then, setting \( \kappa = \frac{2N}{2N - (1 + \sigma)(N - 2)} \),

\[
\frac{|F_z(x, z)||z|^{\kappa - 1}}{|z|^{(1 + \sigma)\kappa}} \leq \frac{c_1^{\kappa - 1}}{|z|^{(1 + \sigma)\kappa - p(\kappa - 1)}} \leq \frac{c_2}{|z|^\theta}, \quad \forall \ |z| \geq r_0.
\]

We deduce from (S5) that

\[
\hat{F}_z(x, z) \geq \frac{1}{c_0|z|^{\sigma}}|F_z(x, z)||z|
\]

\[
\geq \frac{|F_z(x, z)||z|^{\kappa - 1}}{c_0c_2|z|^{(1 + \sigma)\kappa}}|F_z(x, z)||z|
\]

\[
= \frac{1}{c_0c_2} \left( \frac{|F_z(x, z)|}{|z|^\sigma} \right)^\kappa, \quad \forall \ |z| \geq r_0.
\]

This implies (S4).

**Case 2**. \( N = 1 \) or \( 2 \). Set

\[
\sigma := \frac{2 + (p - 3)\theta}{2(p - 1) - \theta}, \quad \kappa := \frac{2(p - 1) - \theta}{2(p - 2)}.
\]

Since \( 0 < \theta < 2 \) and \( p > 2 \), we have \( \sigma \in (0, 1) \) and \( \kappa \in (1, 2/(1 - \sigma)) \), then we deduce from (2.15) that

\[
\frac{|F_z(x, z)||z|^{\kappa - 1}}{|z|^{(1 + \sigma)\kappa}} \leq \frac{c_1^{\kappa - 1}}{|z|^{(1 + \sigma)\kappa - p(\kappa - 1)}} \leq \frac{c_3}{|z|^\theta}, \quad \forall \ |z| \geq r_0.
\]

Using (S5), we are led to

\[
\hat{F}_z(x, z) \geq \frac{1}{c_0|z|^{\sigma}}|F_z(x, z)||z|
\]

\[
\geq \frac{|F_z(x, z)||z|^{\kappa - 1}}{c_0c_3|z|^{(1 + \sigma)\kappa}}|F_z(x, z)||z|
\]

\[
= \frac{1}{c_0c_3} \left( \frac{|F_z(x, z)|}{|z|^\sigma} \right)^\kappa, \quad \forall \ |z| \geq r_0.
\]

Thus (S4) is satisfied. By cases 1)-2), the proof is completed.

3. **Linking structure for the periodic or non-periodic system.** In this section, we assume (S0) and (S1) are satisfied without mentioning. Without loss of generality, we may assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain.

**Lemma 3.1.** Let (V) or (V') be satisfied. Then there exists \( \rho > 0 \) such that

\[
\alpha := \inf_{z \in E^+, \|z\| = \rho} \Phi(z) > 0.
\]
The proof is standard, so we omit it.

We first consider the periodic case, i.e. (V) holds. Note that $E^0 = \emptyset$ in this case and $E^-$ is infinite dimensional, which is the so called strongly indefinite problem. Such type of problems have appeared extensively in the study of differential equations via critical point theory, see, for example, [12, 19, 40] and the references therein. Choose $(\nu, \nu) := e \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2) \cap C_0^\infty(\Omega, \mathbb{R}^2)$ such that

$$
\|e^+\| - \|e^-\| = \int_{\mathbb{R}^N} (|\nabla v|^2 + V_1(x)|v|^2 + |\nabla \nu|^2 + V_2(x)|\nu|^2) \, dx
$$

$$
= \int_{\Omega} (|\nabla v|^2 + V_1(x)|v|^2 + |\nabla \nu|^2 + V_2(x)|\nu|^2) \, dx \geq 1, \quad (3.1)
$$

this together with (2.10) implies that $(\nu^1_+, \nu^1_-) = e^+ \neq (0, 0)$. To obtain the linking structure of $\Phi$, we need to establish the following lemma.

**Lemma 3.2.** Let (V) and (S2) be satisfied. Then there is $r > \rho$ such that $\sup \Phi(\partial Q) \leq 0$, where

$$
Q = \{w + se^+ : \ w \in E^-, \ s \geq 0, \ \|w + se^+\| \leq r\}.
$$

**Proof.** Arguing indirectly, assume that there exists a sequence $\{z_n\} \subset E^- \oplus \mathbb{R}^+ e^+$ with $\|z_n\| \to \infty$, such that $\Phi(z_n) \geq 0$ for all $n \in \mathbb{N}$. Set $w_n = \frac{z_n}{\|z_n\|} = w^- + s_n e^+$, then $\|w_n\| = 1$. Passing to a subsequence, we may assume that $w_n \rightharpoonup w, \ (\bar{u}_n, \bar{v}_n) := w^- \mathop{\to}^* w := (\bar{u}, \bar{v})$ in $E, w_n \to w$ a.e. on $\mathbb{R}^N$ and $s_n \to s$, moreover, by Lemma 2.1 and (2.13) we have $w_n \to w^-$ in $L^q_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ for $2 \leq q < 2^*$ and

$$
0 \leq \frac{\Phi(z_n)}{\|z_n\|^2} = \frac{1}{2} s_n^2 \|e^+\|^2 - \frac{1}{2} \|w_n\|^2 - \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|^2} \, dx. \quad (3.2)
$$

If $s = 0$, (S1) and (3.2) yield that

$$
0 \leq \frac{1}{2} \|w_n\|^2 + \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|^2} \, dx \leq \frac{s_n^2}{2} \|e^+\|^2 \to 0,
$$

this implies $\|w_n\| \to 0$, we get a contradiction by $1 = \|w_n\| \to 0$. Thus $s \neq 0$.

We claim that $w|_{\Omega} = (w^- + se^+)|_{\Omega} \neq (0, 0)$. Indeed, if it is not true, suppose that

$$
(w^- + se^+)|_{\Omega} = (\bar{u} + sv^1_+, \bar{v} + sv^1_+)|_{\Omega} = (0, 0), \quad (3.3)
$$

then we deduce from (S1), (2.12) and (3.2) that

$$
0 \leq 2 \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|^2} \, dx \leq s_n^2 \|e^+\|^2 - \|w_n\|^2
$$

$$
= \int_{\mathbb{R}^N} \left( |\nabla (\bar{u}_n + s_n v^1_+)|^2 + V_1(x)|\bar{u}_n + s_n v^1_+|^2 \\
+ |\nabla (\bar{v}_n + s_n v^2_+)|^2 + V_2(x)|\bar{v}_n + s_n v^2_+|^2 \right) \, dx
$$

$$
= \left( \int_{\Omega} + \int_{\mathbb{R}^N \setminus \Omega} \right) \left( |\nabla (\bar{u}_n + s_n v^1_+)|^2 + V_1(x)|\bar{u}_n + s_n v^1_+|^2 \\
+ |\nabla (\bar{v}_n + s_n v^2_+)|^2 + V_2(x)|\bar{v}_n + s_n v^2_+|^2 \right) \, dx
$$

$$
= I_1 + I_2.
$$

Using (V), (3.3), $s_n \to s$ and $w^-_n \to w^-$ in $L^q_{loc}(\mathbb{R}^N, \mathbb{R}^2)$, we are led to

$$
I_1 = \int_{\Omega} \left( |\nabla (\bar{u}_n + s_n v^1_+)|^2 + V_1(x)|\bar{u}_n + s_n v^1_+|^2 \right)
$$
\begin{align*}
&+|\nabla (\bar{v}_n + s_n v_2^+)|^2 + V_2(x)|\bar{v}_n + s_n v_2^+|^2 \, dx \\
= \int_{\Omega} (|\nabla (\bar{u}_n - \bar{u})|^2 + V_1(x)|\bar{u}_n - \bar{u}|^2 \, dx \\
&+ |\nabla (\bar{v}_n - \bar{v})|^2 + V_2(x)|\bar{v}_n - \bar{v}|^2) \, dx + o(1) \\
= \int_{\Omega} (|\nabla \bar{u}_n|^2 - |\nabla \bar{u}|^2 + |\nabla \bar{v}_n|^2 - |\nabla \bar{v}|^2) \, dx + o(1)
\end{align*}

By (2.10), (2.12) and the fact supp \( e \subset \Omega \), we have

\begin{align*}
J_2 &= \int_{\mathbb{R}^N \setminus \Omega} (|\nabla (\bar{u}_n + s_n v_1^+)|^2 + V_1(x)|\bar{u}_n + s_n v_1^+|^2 \\
&+ |\nabla (\bar{v}_n + s_n v_2^+)|^2 + V_2(x)|\bar{v}_n + s_n v_2^+|^2) \, dx \\
&= \int_{\mathbb{R}^N \setminus \Omega} (|\nabla (\bar{u}_n - sv_1^-)|^2 + V_1(x)|\bar{u}_n - sv_1^-|^2 \\
&+ |\nabla (\bar{v}_n - sv_2^-)|^2 + V_2(x)|\bar{v}_n - sv_2^-|^2) \, dx \\
&= -\|w_n^- - se^-\|^2 - \int_{\Omega} (|\nabla (\bar{u}_n - sv_1^-)|^2 + V_1(x)|\bar{u}_n - sv_1^-|^2 \\
&+ |\nabla (\bar{v}_n - sv_2^-)|^2 + V_2(x)|\bar{v}_n - sv_2^-|^2) \, dx + o(1) \\
&= -\|w_n^- - se^-\|^2 - s^2 \int_{\Omega} (|\nabla v|^2 + V_1(x)|v|^2 \\
&+ |\nabla v|^2 + V_2(x)|v|^2) \, dx + o(1) \\
&\leq -s^2 + o(1).
\end{align*}

This is a contradiction and so \( w|_{\Omega} = (w^- + se^+) |_{\Omega} \neq (0, 0) \). Using (S1), (S2), (3.2) and Fatou’s lemma, we are led to

\begin{align*}
0 &\leq \limsup_{n \to \infty} \left[ \frac{s^2}{2} ||e^+||^2 - \frac{1}{2} ||w_n^-||^2 - \int_{\mathbb{R}^N} \frac{F(x, z_n)}{||z_n||^2} \, dx \right] \\
&\leq \lim_{n \to \infty} \frac{s^2}{2} ||e^+||^2 - \liminf_{n \to \infty} \int_{\Omega} \frac{F(x, z_n)}{|z_n|^2} |w_n|^2 \, dx \\
&\leq \frac{s^2}{2} ||e^+||^2 - \int_{\Omega} \liminf_{n \to \infty} \frac{F(x, z_n)}{|z_n|^2} |w_n|^2 \, dx = -\infty,
\end{align*}
a contradiction. This completes the proof. □

Now, we consider the non-periodic case, i.e. \((V')\) holds. Note that \(\dim(E^- \oplus E^0) := m < \infty\) in this case, denote by \(\mu_1, \mu_2, \ldots, \mu_m\) an orthogonal basis. Choose \((v, \nu) := e \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2) \cap C_0^\infty(\Omega, \mathbb{R}^2)\) such that \(e|_\Omega, \mu_1|_\Omega, \mu_2|_\Omega, \ldots, \mu_m|_\Omega\) are linearly independent and that (3.1) holds. In view of the unique continuation theorem [37], we have following result.

**Lemma 3.3.** Let \((V')\) be satisfied. If \(A_i u = -\Delta u + V_j u = 0\) and \(u|_\Omega = 0\), then \(u = 0\).

Taking advantage of the above lemma, we are able to establish following result which is crucial to demonstrate the linking geometry of \(\Phi\).

**Lemma 3.4.** Let \((V')\) and \((S2)\) be satisfied. Then there is \(r > \rho\) such that

\[
\sup_{\Phi(\partial Q)} \leq 0, \quad \text{where } Q \text{ is defined by}
\]

\[
Q = \{w + se^+ : w \in E^- \oplus E^0, s \geq 0, \|w + se^+\| \leq r\}.
\]

**Proof.** By (S1) and (2.13), we see that \(\Phi(z) \leq 0\) for \(z \in E^- \oplus E^0\). Arguing indirectly, assume that there exists a sequence \(\{z_n\} \subset E^- \oplus E^0 \oplus \mathbb{R}^+ e^+\) with \(\|z_n\| \rightarrow \infty\), such that \(\Phi(z_n) \geq 0\) for all \(n \in \mathbb{N}\). Since \(E^0 \neq \emptyset\) and \(V\) is coercive, the argument used in the proof of Lemma 3.2 is not applicable, we modify it as follows. Set \(w_n = \frac{z_n}{\|z_n\|} = w_n^0 + w_n^1 + s_n e^+\), then \(\|w_n\| = 1\). Since \(\dim(E^- \oplus E^0) := m < \infty\), up to a subsequence, we may assume that \(w_n \rightharpoonup w\) in \(E\), \(w_n^0 \rightharpoonup w^0\), \((\bar{a}_n, \bar{v}_n) := w_n^- \rightarrow w^- := (\bar{u}, \bar{v})\) in \(E\) and in \(L^q(\mathbb{R}^N, \mathbb{R}^2)\) for \(2 \leq q < 2^*\), \(w_n \rightarrow w\) a.e. on \(\mathbb{R}^N\), \(s_n \rightarrow s\) and

\[
0 \leq \frac{\Phi(z_n)}{\|z_n\|^2} = \frac{1}{2} \|e^+\|^2 - \frac{1}{2} \|w_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|^2} dx.
\]

(3.4)

To complete the proof, we consider three possible cases:

**Case 1.** \(s = 0\) and \(w^0 = (0, 0)\). We deduce from (S1) and (3.4) that

\[
0 \leq \frac{1}{2} \|w_n^-\|^2 + \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|^2} dx \leq \frac{s_n^2}{2} \|e^+\|^2 \rightarrow 0,
\]

(3.5)

which yields \(\|w^-\| \rightarrow 0\), we get a contradiction by 1 that \(\|w_n\| \rightarrow 0\).

**Case 2.** \(s = 0\) and \(w^0 := (\omega_1, \omega_2) \neq (0, 0)\). In this case, we have \(w^- = (0, 0)\) by (3.5) and \(A_i \omega_i = 0\) by (2.2) for \(i = 1, 2\). Applying Lemma 3.3, one sees that \(\omega_i|_\Omega \neq 0\) for some \(i = 1, 2\). Then we deduce from (S1), (S2), (3.4) and Fatou’s lemma that

\[
0 \leq \limsup_{n \rightarrow \infty} \left[ \frac{s_n^2}{2} \|e^+\|^2 - \frac{1}{2} \|w_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, z_n)}{\|z_n\|^2} dx \right]
\]

\[
\leq - \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, z_n)}{|z_n|^2} |w_n^- + w_n^0 + s_n e^+|^2 dx
\]

\[
\leq - \int_{\Omega} \liminf_{n \rightarrow \infty} \frac{F(x, z_n)}{|z_n|^2} |w^0|^2 dx = -\infty,
\]

which is a contradiction.

**Case 3.** \(s \neq 0\). We claim that \(w|_\Omega = (w^- + w^0 + se^+)|_\Omega \neq (0, 0)\). Indeed, if it is not true, suppose that \((w^- + w^0 + se^+)|_\Omega = (0, 0)\), then it follows from \(w^- + w^0 - s(e^- + e^0) \in E^- \oplus E^0\) that there exist constants \(a_1, a_2, \ldots, a_m\) such that \(w^- + w^0 - s(e^- + e^0) = \sum_{i=1}^m a_i \mu_i\).
then we have

\[ 0 = (w^- + w^0 + se^+)|_\Omega = \left( \sum_{i=1}^m a_i \mu_i + se \right)|_\Omega = \sum_{i=1}^m a_i \mu_i|_\Omega + se|_\Omega, \]

this implies that \( e|_\Omega, \mu_1|_\Omega, \mu_2|_\Omega, \ldots, \mu_m|_\Omega \) are linearly dependent, which contradicts with the choice of \( e \). Thus \( w|_\Omega = (w^- + w^0 + se^+)|_\Omega \neq (0, 0) \), using (S1), (S2), (3.4) and Fatou’s lemma again, we get a contradiction

\[ 0 \leq \limsup_{n \to \infty} \left[ \frac{s^2}{2} \|e^+\|^2 - \frac{1}{2} \|w^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, z_n)}{|z_n|^2} dx \right] \]
\[ \leq \lim_{n \to \infty} \frac{s^2}{2} \|e^+\|^2 - \liminf_{n \to \infty} \int_{\Omega} \frac{F(x, z_n)}{|z_n|^2} |w_n|^2 dx \]
\[ \leq \frac{s^2}{2} \|e^+\|^2 - \int \liminf_{n \to \infty} \frac{F(x, z_n)}{|z_n|^2} |w_n|^2 dx = -\infty. \]

By cases 1)-3), we complete the proof. \( \square \)

4. Boundedness of Cerami sequences. In this section, we show the boundedness of Cerami sequences for the periodic or non-periodic system (1.1) by using some new tricks. Proofs of main results will be given at the end of the section.

To find Cerami sequences for the functional \( \Phi \), we introduce following generalized linking theorem established in [19].

**Lemma 4.1.** ([19], [20, Theorem 2.1]). Let \( X \) be a Hilbert space with \( X = X^- \oplus X^+ \) and \( X^- \perp X^+ \), and let \( I \in C^1(X, \mathbb{R}) \) of the form

\[ I(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \eta(u), \quad u = u^+ + u^- \in X^+ \oplus X^- \]

Suppose that the following assumptions are satisfied:

(I1) \( \eta \in C^1(X, \mathbb{R}) \) is bounded from below and weakly sequentially lower semi-

continuous;

(I2) \( \eta' \) is weakly sequentially continuous;

(I3) there exist \( r > \rho > 0 \) and \( e \in X^+ \) with \( \|e\| = 1 \) such that

\[ \alpha := \inf \{ I(S_\rho) > \sup I(\partial Q) \}, \]

where

\[ S_\rho = \{ u \in X^+ : \|u\| = \rho \}, \quad Q = \{ se + v : v \in X^-, s \geq 0, \|se + v\| \leq \rho \}. \]

Then for some \( c \geq \alpha \), there exists a sequence \( \{u_n\} \subset X \) satisfying

\[ I(u_n) \to c, \quad \|I'(u_n)\|(1 + \|u_n\|) \to 0 \quad \text{as} \ n \to \infty. \]

Applying Lemmas 2.2, 3.1, 4.1 and Lemma 3.2 or Lemma 3.4, we obtain directly following result.

**Lemma 4.2.** Let \((S0), (S1), (S2)\) and \((V)\) or \((V')\) be satisfied. Then there exist a constant \( c \geq \alpha > 0 \) and a sequence \( \{z_n\} \subset E \) such that

\[ \Phi(z_n) \to c, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0 \quad \text{as} \ n \to \infty. \quad (4.1) \]

In the following two lemmas, we certify the boundedness of Cerami sequences obtained in Lemma 4.2 for the periodic case and non-periodic case, respectively, with the aid of the technical conditions (S3) and (S4).
Lemma 4.3. Let (V), (S0), (S1), (S2) and (S3) be satisfied. Then any sequence \( \{z_n\} \subset E \) satisfying (4.1) is bounded in \( E \).

**Proof.** Consider the case \( N \geq 3 \). By (4.1), we have

\[
C_3 \geq \Phi(z_n) - \frac{1}{2} \langle \Phi'(z_n), z_n \rangle = \int_{\mathbb{R}^N} \tilde{F}(x, z_n) dx
\]

(4.2)

for some constant \( C_3 > 0 \). To prove the boundedness of \( \{z_n\} \), arguing by contradiction, suppose that \( \|z_n\| \to \infty \). Let \( w_n = z_n / \|z_n\| \), then \( \|w_n\| = 1 \). Set

\[
\Omega_n := \left\{ x \in \mathbb{R}^N : \left| \frac{F_z(x, z_n)}{|z|} \right| \leq \frac{\sqrt{2}}{2} (\Lambda_0 - \delta_0) \right\}.
\]

(4.3)

Since \( \Lambda_0 \|z\|^2 \leq \|z\|^2 \) for any \( z \in E \), it follows from Hölder’s inequality that

\[
\int_{\Omega_n} \frac{|F_z(x, z_n)|}{|z_n|}|w_n| \left( |w_n^+| + |w_n^-| \right) dx \leq \frac{\sqrt{2}}{2} (\Lambda_0 - \delta_0) \|w_n\|_2 \left[ \int_{\mathbb{R}^N} \left( |w_n^+| + |w_n^-| \right)^2 dx \right]^{1/2}
\]

\leq (\Lambda_0 - \delta_0) \|w_n\|_2 \left( \|w_n^+\|^2 + \|w_n^-\|^2 \right)^{1/2} \leq 1 - \frac{\delta_0}{\Lambda_0}. \quad (4.4)

Using (S3), (4.2), Lemma 2.1 and Hölder inequality, we have

\[
\frac{1}{\|z_n\|^{1-\sigma}} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{|F_z(x, z_n)|}{|z_n|^{\sigma}} |w_n|^\sigma \left( |w_n^+| + |w_n^-| \right) dx \leq \frac{1}{\|z_n\|^{1-\sigma}} \left[ \int_{\mathbb{R}^N \setminus \Omega_n} \left( \frac{|F_z(x, z_n)|}{|z_n|^{\sigma}} \right)^{\frac{2^*}{2^* - 1 - \sigma}} dx \right]^{\frac{2^* - 1 - \sigma}{2^*}}
\]

\[
\|w_n\|^{2^*} \left[ \int_{\mathbb{R}^N} \left( |w_n^+| + |w_n^-| \right)^{2^*} dx \right]^{1/2^*} \leq \frac{1}{\|z_n\|^{1-\sigma}} \left( C_0 \int_{\mathbb{R}^N \setminus \Omega_n} \tilde{F}(x, z_n) dx \right)^{\frac{2^* - 1 - \sigma}{2^*}}
\]

\leq \frac{1}{\|z_n\|^{1-\sigma}} \left( C_0 C_3 \right)^{\frac{2^* - 1 - \sigma}{2^*}} C_4 = o(1), \quad (4.5)

for some constant \( C_4 > 0 \). Combining (4.4) and (4.5) and using (2.14) and (4.1), one has

\[
1 + o(1) = \frac{\|z_n\|^2 - \langle \Phi'(z_n), z_n^+ - z_n^- \rangle}{\|z_n\|^2}
\]

\[
= \frac{1}{\|z_n\|} \int_{|z_n^+| \neq 0} \frac{F_z(x, z_n) \cdot z_n^+}{|z_n^+|} |w_n^+| dx
\]

\[
- \frac{1}{\|z_n\|} \int_{|z_n^-| \neq 0} \frac{F_z(x, z_n) \cdot z_n^-}{|z_n^-|} |w_n^-| dx
\]

\[
\leq \int_{|z_n| \neq 0} \frac{|F_z(x, z_n)|}{|z_n|} |w_n| dx + \int_{|z_n| \neq 0} \frac{|F_z(x, z_n)|}{|z_n|} |w_n| dx
\]
This contradiction shows the boundedness of \( \{z_n\} \). The case \( N = 1,2 \) can be treated similarly, we omit it. The proof is completed.

**Lemma 4.4.** Let \( (V'), (S0), (S1), (S2) \) and \( (S4) \) be satisfied. Then any sequence \( \{z_n\} \subset E \) satisfying \( (4.1) \) is bounded in \( E \).

**Proof.** We only consider the case \( N \geq 3 \), since the case \( N = 1,2 \) can be dealt with similarly. Clearly, \( (4.2) \) holds also in this case. To prove the boundedness of \( \{z_n\} \), arguing by contradiction, suppose that \( \|z_n\| \to \infty \). Let \( w_n = z_n/\|z_n\| \), then \( \|w_n\| = 1 \), up to a subsequence, we may assume that \( w_n \rightharpoonup w := (u, v) \) in \( E \), \( w_n \to w \) in \( L^q(\Omega) \) by Lemma 2.1 for \( 2 \leq q < 2^* \), and \( w_n \to w \) a.e. on \( \Omega \).

There are two cases to distinguish: \( w = (0,0) \) and \( w \neq (0,0) \).

**Case 1.** \( w = (0,0) \). Since \( \dim(E^c \oplus E^0) := m < \infty \), we have \( \|w_n^+\| + \|w_n^0\| \to 0 \), \( w_n \to 0 \) in \( L^q(\Omega) \) for \( 2 \leq q < 2^* \), and \( w_n \to 0 \) a.e. on \( \Omega \). (S0) and (S1) yield

\[
\int_{0<|z_n|<R_0} \frac{|F_z(x,z_n)|}{|z_n|} \leq C_5 \|w_n\|_2 \|w_n^+\|_2 = o(1),
\]

for some constant \( C_5 > 0 \). Using (S4), \( (4.2) \) and Hölder inequality, we have

\[
\frac{1}{\|z_n\|^{1-\sigma}} \int_{|z_n| \geq R_0} \frac{|F_z(x,z_n)|}{|z_n|^\sigma} |w_n|^\sigma |w_n^+|^\sigma \leq \left( \frac{1}{\|z_n\|^{1-\sigma}} \int_{|z_n| \geq R_0} \frac{|F_z(x,z_n)|}{|z_n|^\sigma} |w_n|^\sigma \right)^{2^* - 1-\sigma} \|w_n\|_2 \|w_n^+\|_2 \leq \frac{1}{\|z_n\|^{1-\sigma}} (C_0 C_3)^{\frac{2}{2^* - 1-\sigma}} C_6 = o(1),
\]

for some constant \( C_6 > 0 \). Combining (4.7) and (4.8) and using (2.14) and (4.1), one has

\[
1 + o(1) = \frac{\|z_n\|^2 - \|z_n^0\|^2 - \|z_n^0\|^2 - \langle \Phi'(z_n), z_n^+ \rangle}{\|z_n\|^2} = \frac{1}{\|z_n\|} \int_{|z_n| \neq 0} \frac{F_z(x,z_n)}{|z_n|^\sigma} |z_n^+| |w_n^+| dx \leq \int_{|z_n| \neq 0} \frac{|F_z(x,z_n)|}{|z_n|^\sigma} |w_n| |w_n^+| dx \leq \int_{|z_n| \neq 0} \frac{|F_z(x,z_n)|}{|z_n|^\sigma} |w_n| |w_n^+| dx
\]

\[
+ \frac{1}{\|z_n\|^{1-\sigma}} \int_{|z_n| \geq R_0} \frac{|F_z(x,z_n)|}{|z_n|^\sigma} |w_n|^\sigma |w_n^+|^\sigma dx.
\]
This is a contradiction.

Case 2). $w \neq (0,0)$. Note that, for any $\zeta := (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2)$,

\[
o(1) = \left\langle \Phi'(z_n), \zeta \right\rangle = \langle z_n^+ - z_n^-, \zeta \rangle = \int_{\mathbb{R}^N} F_z(x, z_n) \cdot \zeta \, dx
\]

which implies

\[
\left\langle w_n^+ - w_n^-, \zeta \right\rangle = \frac{1}{\|z_n\|} \int_{\mathbb{R}^N} F_z(x, z_n) \cdot \zeta \, dx = o(1).
\]

Using (S0), (S1), (S4), (4.2) and Hölder’s inequality, one has

\[
\frac{1}{\|z_n\|} \int_{\mathbb{R}^N} F_z(x, z_n) \cdot \zeta \, dx = \frac{1}{\|z_n\|^{1-\sigma}} \int_{|z_n| \neq 0} \frac{|F_z(x, z_n)|}{|z_n|^\sigma} |w_n|^\sigma \, |\zeta| \, dx
\]

\[
= \frac{1}{\|z_n\|^{1-\sigma}} \left( \int_{0 < |z_n| < R_0} \frac{|F_z(x, z_n)|}{|z_n|^\sigma} |w_n|^\sigma \, |\zeta| \, dx + \int_{|z_n| \geq R_0} \frac{|F_z(x, z_n)|}{|z_n|^\sigma} |w_n|^\sigma \, |\zeta| \, dx \right)
\]

\[
\leq \frac{1}{\|z_n\|^{1-\sigma}} \left\{ C_7 \|w_n\|_2^2 \|\zeta\|_2^{2/(2-\sigma)} + \|w_n\|_2^2 \|\zeta\|_2^{2\cdot(2-\sigma)} \right\}
\]

\[
\leq \frac{C_8}{\|z_n\|^{1-\sigma}} \left[ \|\zeta\|_2^{2/(2-\sigma)} + \|\zeta\|_2^{2\cdot(2-\sigma)} \left( C_0 \int_{|z_n| \geq R_0} \tilde{F}(x, z_n) \, dx \right)^{2 - \frac{2-\sigma}{2\cdot(2-\sigma)}} \right]
\]

\[
\leq \frac{C_9}{\|z_n\|^{1-\sigma}} \left( \|\zeta\|_2^{2/(2-\sigma)} + \|\zeta\|_2^{2\cdot(2-\sigma)} \right) = o(1).
\]

from which we deduce that

\[
\left\langle w_n^+ - w_n^-, \zeta \right\rangle = o(1), \quad \forall \ \zeta := (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2),
\]

then by (2.11) and $w_n \rightharpoonup w := (u, v)$ in $E$, we get

\[
(u_1^+ - u_1^- , \varphi)_{H_1} + (v_2^+ - v_2^- , \psi)_{H_2} = 0, \quad \forall \ (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2).
\]

This together with (2.2) shows that $A_1 u = -\triangle u + V_1 u = 0$ and $A_2 v = -\triangle v + V_2 v = 0$. It follows from Lemma 3.3 that $w|_{\Omega} \neq (0,0)$. Using (4.1), (S2) and Fatou’s Lemma, we are led to a contradiction,

\[
0 = \lim_{n \to \infty} \frac{c + o(1)}{\|z_n\|^2} = \lim_{n \to \infty} \frac{\Phi(z_n)}{\|z_n\|^2}
\]

\[
= \lim_{n \to \infty} \left[ \frac{1}{2} \left( \|w_n^+\|^2 - \|w_n^-\|^2 \right) - \int_{\mathbb{R}^N} \frac{F(x, z_n)}{|z_n|^2} |w_n|^2 \, dx \right]
\]
By cases 1)-2), the boundedness of \{z_n\} is proved.

\textbf{Proof of Theorem 1.1.} Applying Lemma 4.2 and Lemma 4.3, there is a bounded sequence satisfying (4.1). By the concentration compactness argument as in [6, 31, 32], we show the existence of a ground state solution of (1.1). Under (V), (S0) and (S1), using a well known bootstrap argument as in [30, Lemma 1] and [26, Theorems 4.1] and applying Theorems C.1.1 and C.3.1 of [39], we can certify that any solution $z = (u, v) \in E$ of (1.1) is continuous and $z(x) \to (0, 0)$ as $|x| \to \infty$. Let

$$
W_1(x) = \begin{cases} 
-\frac{F_u(x, z(x))}{u(x)}, & \text{if } u(x) \neq 0, \\
0, & \text{if } u(x) = 0,
\end{cases}
W_2(x) = \begin{cases} 
-\frac{F_v(x, z(x))}{v(x)}, & \text{if } v(x) \neq 0, \\
0, & \text{if } v(x) = 0.
\end{cases}
$$

Then system (1.1) can be represented as

$$
\begin{cases} 
(-\Delta + V_1 + W_1)u = 0, \\
(-\Delta + V_2 + W_2)v = 0.
\end{cases}
$$

By showing that the multiplication operator $W_i$ is relatively compact with respect to the operator $-\Delta + V_i$, $i = 1, 2$ (see [26, proof of Theorem 4.2]), we deduce from [36, Corollary 2] that $\sigma_{ess}(-\Delta + V_i + W_i) = \sigma_{ess}(-\Delta + V_i)$ and from (V) and [36, Theorem XIII.100] that $\sigma_{ess}(-\Delta + V_i) = \sigma(-\Delta + V_i)$, where $\sigma_{ess}$ denotes the essential spectrum. In view of [39, Theorem C.3.4], we get the exponential decay of $u, v$, i.e.,

$$
|u(x)| \leq C_{10}e^{-\tau|x|}, \quad |v(x)| \leq C_{10}e^{-\tau|x|}, \quad \forall \ x \in \mathbb{R}^N.
$$

Thus (1.3) holds and the proof is completed.

\textbf{Proof of Theorem 1.2.} By Lemma 4.2 and Lemma 4.4, there is a bounded sequence \{z_n\} $\subset E$ satisfying (4.1). Up to a subsequence, we may assume that $z_n \rightharpoonup z$ in $E$. We claim that $z \neq (0, 0)$. Indeed, if it is not true, suppose that $z = (0, 0)$, then by Lemma 2.1, $z_n \to (0, 0)$ in $L^q(\mathbb{R}^N, \mathbb{R}^2)$ for $q \in [2, 2^*)$. We deduce from (S1), (2.6), (2.13), (2.14), and (4.1) that

$$
2c + o(1) = 2\Phi(z_n) = \|z_n^+\|^2 - \|z_n^-\|^2 - 2 \int_{\mathbb{R}^N} F(x, z_n)dx
\leq \|z_n^+\|^2 = \langle \Phi'(z_n), z_n^+ \rangle + \int_{\mathbb{R}^N} F_z(x, z_n) \cdot z_n^+ dx
\leq \|z_n\|_2 \|z_n^+\|_2 + C_1\|z_n\|_p \|z_n^+\|_p + o(1)
= o(1).
$$

This is a contradiction, thus $z \neq (0, 0)$. It follows from (4.1) and Lemma 2.2 that $\Phi'(z) = 0$. This shows that $z \in \mathcal{M}$ and so $\Phi(z) \geq m := \inf_{\mathcal{M}} \Phi$. By (S4), for any $z \in \mathcal{M}$, one has

$$
\Phi(z) = \Phi(z) - \frac{1}{2} \langle \Phi'(z), z \rangle = \int_{\mathbb{R}^N} F(x, z)dx \geq 0,
$$

therefore $m \geq 0$. Take a sequence \{z_n\} $\subset \mathcal{M}$ such that $\Phi(z_n) \to m$, then $\langle \Phi'(z_n), \zeta \rangle = 0$ for any $\zeta \in E$. Lemma 4.4 yields that \{z_n\} is bounded in $E$. 
Up to a subsequence, we may assume that $\varepsilon_n \to \varepsilon = \varepsilon^+ + \varepsilon^0 + \varepsilon^-$ in $E$. There are two possible cases to consider: $\varepsilon^0 = (0,0)$ and $\varepsilon^0 \neq (0,0).

**Case 1.** $\varepsilon^0 = (0,0)$. Since $\dim(E^- \oplus E^0) < \infty$, we have $\varepsilon^0_n \to (0,0)$ in $E$. It is not difficult to check that $\liminf_{n \to \infty} \|\varepsilon_n\| > 0$ in light of (2.6), Lemma 2.1 and
\[ \|\varepsilon_n\|^2 = \langle \Phi'(\varepsilon_n), \varepsilon^+_n - \varepsilon^-_n \rangle + \int_{\mathbb{R}^N} F^1(x, \varepsilon_n) \cdot (\varepsilon^+_n - \varepsilon^-_n) dx + \|\varepsilon^0_n\|^2 \]
\[ = \int_{\mathbb{R}^N} F^1(x, \varepsilon) \cdot (\varepsilon^+_n - \varepsilon^-_n) dx + o(1). \]

If $\varepsilon^+ + \varepsilon^- = (0,0)$, Lemma 2.1 yields that $\varepsilon_n \to \varepsilon^+ + \varepsilon^0 + \varepsilon^- = (0,0)$ in $L^q(\mathbb{R}^N, \mathbb{R}^2)$ for $q \in [2, 2^*)$, then the right hand side of the equation above goes to zero which implies that $\lim_{n \to \infty} \|\varepsilon_n\| = 0$, a contradiction. Therefore $\varepsilon = \varepsilon^+ + \varepsilon^- \neq (0,0)$. We deduce from $\Phi'(\varepsilon_n) = 0$, $\Phi(\varepsilon_n) \to m$, (S4) and Fatou’s lemma that $\Phi'(\varepsilon) = 0$, $\Phi(\varepsilon) \geq m$ and
\[ m = \lim_{n \to \infty} \left[ \Phi(\varepsilon_n) - \frac{1}{2} \langle \Phi'(\varepsilon_n), \varepsilon_n \rangle \right] \]
\[ = \lim_{n \to \infty} \int_{\mathbb{R}^N} \hat{F}(x, \varepsilon_n) dx \geq \int_{\mathbb{R}^N} \liminf_{n \to \infty} \hat{F}(x, \varepsilon_n) dx \]
\[ = \int_{\mathbb{R}^N} \hat{F}(x, \varepsilon) dx = \Phi(\varepsilon) - \frac{1}{2} \langle \Phi'(\varepsilon), \varepsilon \rangle \]
\[ = \Phi(\varepsilon) \]

Therefore $\Phi(\varepsilon) = m$, which yields that $\varepsilon$ is a ground state solution of problem (1.1).

**Case 2.** $\varepsilon^0 \neq (0,0)$. Then $\varepsilon \neq (0,0)$, we obtain a ground state solution of (1.1) by the same argument as above.

Cases 1)-2) yield the existence of ground state solution of (1.1). Note that (V’) implies that the point 0 is not contained in $\sigma_{ess}(-\Delta + V_i)$, using the same argument as in the proof of Theorem 1.1 and applying Theorem C.3.3 of [39], we certify the continuity and exponential decay estimate for any nontrivial solution of (1.1), cf. [30, Theorem 5] and Theorems 4.1 and 4.2 of [26] for details.

Theorem 1.3 follows directly from Theorem 1.2 and Lemma 2.3.

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