Abstract. A hyperbolic polynomial (HP) is a real univariate polynomial with all roots real. By Descartes’ rule of signs, an HP with all coefficients nonvanishing has exactly \( c \) positive and exactly \( p \) negative roots counted with multiplicity, where \( c \) and \( p \) are the numbers of sign changes and sign preservations in the sequence of its coefficients. For \( c = 1 \) and \( 2 \), we discuss the question: When the moduli of all the roots of an HP are arranged in the increasing order on the real half-line, at which positions can be the moduli of its positive roots depending on the positions of the sign changes in the sequence of coefficients?

1. Introduction

We consider the classical Descartes’ rule of signs in the context of hyperbolic polynomials (HPs), i.e., real polynomials in one real variable with all roots real. We limit our study to the case when the polynomials are monic and all coefficients are nonvanishing. In this case, Descartes’ rule of signs implies that a degree \( d \) HP has exactly \( c \) positive and exactly \( p \) negative roots (counted with multiplicity), where \( c \) and \( p \) are the numbers of sign changes and sign preservations in the sequence of coefficients of the polynomial (hence \( c + p = d \)). We remind that for real, but not necessarily hyperbolic polynomials, Descartes' rule of signs says only that the number \( pos \) of positive roots is not larger than \( c \) and the difference \( c - pos \) is an even number (from which for real polynomials with all coefficients nonvanishing one can deduce that \( neg \leq p \) and \( p - neg \in 2\mathbb{N} \cup 0 \), where \( neg \) stands for the number of negative roots). A sign pattern (SP) is a finite sequence.

Mathematics Subject Classification: 26C10, 30C15.
Key words and phrases: real polynomial in one variable, hyperbolic polynomial, sign pattern, Descartes’ rule of signs.
of “+” and/or “−”-signs beginning with a +. If an HP is denoted by $P := x^d + \sum_{j=0}^{d-1} a_j x^j$, then we say that $P$ defines (or realizes) the SP (of length $d + 1$) $(+, \operatorname{sgn}(a_{d-1}), \operatorname{sgn}(a_{d-2}), \ldots, \operatorname{sgn}(a_0))$. It is true that:

1. for every SP of length $d + 1$, there exists an HP defining the given SP, see Remark 7;
2. the all-pluses SP of length $d + 1$ (hence with $c = 0$) is realizable by any monic HP having $d$ negative roots.

Descartes’ rule of signs does not impose any inequalities between the moduli of the positive and the negative roots of $P$. In the present paper, we consider, for $c = 1$ and $c = 2$, the question:

**Question 1.** When the moduli of all the roots of an HP are arranged in the increasing order on the real half-line, at which positions can the moduli of the positive roots be depending on the positions of the sign changes in the sequence of coefficients? In particular, at which positions can they be in the generic case when there are no equalities between moduli of roots?

To make formulations easier, we fix the following notation:

**Notation 1.**
1. For $c = 1$, we denote by $\Sigma_{m,n}$ the SP consisting of $m$ pluses followed by $n$ minuses, where $1 \leq m, n \leq d$, $m + n = d + 1$. For $c = 2$, we denote by $\Sigma_{m,n,q}$ the SP consisting of $m$ pluses followed by $n$ minuses followed by $q$ pluses, where $1 \leq m, n, q \leq d - 1$, $m + n + q = d + 1$.
2. For $c = 1$, we denote by $0 < \alpha$ the modulus of the positive root, and by $0 < \gamma_1 \leq \cdots \leq \gamma_{d-1}$ the moduli of the negative roots of a degree $d$ HP. For $c = 2$, we denote by $0 < \beta \leq \alpha$ the moduli of its positive and by $0 < \gamma_1 \leq \cdots \leq \gamma_{d-2}$ the moduli of its negative roots. We set $\gamma := (\gamma_1, \ldots, \gamma_{d-c})$.
3. By $e_k(\gamma)$ we denote the $k$-th elementary symmetric function of the quantities $\gamma_1, \ldots, $, e.g., $e_k(\gamma) := \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq d-c} \gamma_{j_1} \gamma_{j_2} \cdots \gamma_{j_k}$, and by $e_k(\hat{\gamma}_i)$ we denote this symmetric function of the quantities $\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{d-c}$.
4. For $c = 2$, we denote by $m^*, n^*$ and $q^*$ the numbers of moduli of negative roots of an HP defining this sign pattern which are respectively larger than $\alpha$, belonging to the interval $(\beta, \alpha)$, and smaller than $\beta$. In the absence of an equality $\gamma_j = \alpha$ or $\gamma_j = \beta$, one has $m^* + n^* + q^* = d - 2$. For $c = 1$, $m^*$ (resp. $n^*$) stands for the number of moduli of negative roots which are larger (resp. smaller) than $\alpha$. In the absence of an equality $\gamma_j = \alpha$, one has $m^* + n^* = d - 1$.

For $c = 1$ and $2$, Question 1 can be formulated as follows:
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Question 2. For a given degree \( d \), what can be the values of \( m^* \) depending on \( m \) (if \( c = 1 \)) or those of \( m^* \) and \( n^* \) depending on \( m \) and \( n \) (if \( c = 2 \))? Especially, what can these values be in the generic case when all moduli of roots are distinct?

The answer to this question is not trivial. Thus the SP \( \Sigma_{d,1} \) is realizable only by HPs with \( m^* = d - 1 \) (hence \( n^* = 0 \)), see Theorem 1. In the cases of the SPs \( \Sigma_{1,n,1} \) and \( \Sigma_{m,1,q} \), one has respectively \( m^* = m - 1, n^* = 0, q^* = q - 1 \) (see Theorem 5), and \( m^* = q^* = 0, n^* = d - 2 \) (see Theorem 2). In other situations there are several possibilities for these values, see Examples 3, 4 and 5 or Theorems 1, 3 and 4.

Remarks 1. (1) Replacing \( P(x) \) by \((-1)^dP(-x)\) means exchanging \( c \) with \( p \) and changing the signs of all roots of \( P \). Therefore, when asking the question how the moduli of the positive and negative roots of \( P \) can be ordered on the real positive half-line, it suffices to consider the cases with \( c \leq \lfloor d/2 \rfloor \). In particular, to obtain the answer to this question for \( d \leq 5 \), it is sufficient to study the cases with \( c = 1 \) and \( c = 2 \).

(2) Replacing \( P \) by its reverted polynomial \( P^R(x) := x^dP(1/x) \) means changing all roots of \( P \) by their reciprocals and reading backward the SP defined by \( P \). In particular, the SP \( \Sigma_{m,n} \) becomes \( \Sigma_{n,m} \) and the SP \( \Sigma_{m,n,q} \) becomes \( \Sigma_{q,n,m} \). In order to have again a monic polynomial, one could replace the polynomial \( P^R(x) \) by \( P^R(x)/a_0 \).

(3) For real, but not necessarily hyperbolic degree \( d \) polynomials, one can ask the question:

Question 3. Given an SP with \( c \) sign changes and \( p \) sign preservations, for which pairs of nonzero integers \((pos, neg)\) satisfying the conditions \( pos \leq c, neg \leq p \) and \( c - pos \in 2\mathbb{N} \cup \{0\} \) do there exist such polynomials defining the given SP and having exactly \( pos \) positive and \( neg \) negative roots, all distinct?

It seems that the question has been explicitly formulated for the first time in [2]. The answer to it is not trivial and the exhaustive one is known for \( d \leq 8 \), see [7], [1], [5], [8] and [9]. The proof of the realizability of certain cases is often done by means of a concatenation lemma, see Lemma 2 in Section 7.

(4) A tropical analog of Descartes’ rule of signs is proposed in [6]. Different aspects of metric inequalities involving moduli of roots of polynomials are addressed in [3] and [4]. Various problems concerning HPs are presented in [10].

The paper is structured as follows. In Section 2, we consider the case \( c = 1 \), i.e., the case of \( \Sigma_{m,n} \), see Theorem 1 and Corollary 1, which provide the exhaustive answer to Question 2 in the generic case. The sections after Section 2 concern...
the situation when \( c = 2 \). In Section 3, we consider the case \( c = 2, m = q = 1, n = d - 1 \), i.e., the case of \( \Sigma_{1,n,1} \), see Theorem 2. In Section 4, we consider the case \( c = 2, q = 1 \), i.e., the case of \( \Sigma_{m,n,1} \), see Theorems 3 and 4. In Section 5, we consider the case \( n = 1 \), i.e., the one of \( \Sigma_{m,1,q} \), \( m + q = d \), see Theorem 5. In Section 6, we give examples of SPs and HPs realizing these SPs with given strict inequalities between the quantities \( \alpha, \beta \) and \( \gamma_j \). In Section 7, we formulate a concatenation lemma (Lemma 2) which plays a key role in the construction of HPs realizing given SPs. With the help of this lemma, we explain how for \( c = 2, n \geq 2 \), one can prove the realizability of certain cases. We also summarize the realizability results of the present paper for HPs of degrees from 2 to 5, with \( c = 2 \). Finally, in Subsection 7.3, when discussing the possible further research in this domain, we prove sufficient conditions of realizability for the case \( m > n, q > n \).

2. The case \( c = 1 \)

**Theorem 1.** (1) Consider the SP \( \Sigma_{m,n} \), where \( 1 \leq n \leq m \). This SP is realizable by and only by polynomials with \( n^* \leq 2n - 2 \). In particular, for \( n = 1 \), one has \( m^* = d - 1, n^* = 0 \).

(2) All cases described in Remark 1 are realizable.

**Remark 1.** We describe here some cases with \( n = 1 \) which are realizable. Most of them are not generic. These are the cases with exactly \( s \) quantities \( \gamma_j \) which are equal to \( \alpha \), exactly \( r = n^* \) that are smaller than \( \alpha \), where \( s + r \leq 2n - 2 \), and exactly \( d - 1 - s - r = m^* \) quantities \( \gamma_j \) which are larger than \( \alpha \). As for the quantities \( \gamma_j \) which are smaller than \( \alpha \), one can realize all possible cases of equalities and/or inequalities among them. When there are \( < 2n - 2 \) quantities \( \gamma_j \) smaller than \( \alpha \), the quantities \( \gamma_j \) larger than \( \alpha \) are presumed distinct. (However, some more cases are realizable as well, see Remark 4. Nothing is claimed about the cases which remain outside the reach of Remark 4.) When there are exactly \( 2n - 2 \) quantities \( \gamma_j \) smaller than \( \alpha \), then among the quantities \( \gamma_j \) larger than \( \alpha \) one can have all possible equalities and/or inequalities.

**Corollary 1.** The SP \( \Sigma_{m,n} \) with \( 1 \leq m \leq n \) is realizable by and only by polynomials with \( m^* \leq 2m - 2 \). In particular, for \( n = d \), one has \( m^* = 0, n^* = d - 1 \).

The Corollary follows from Theorem 1, see part (2) of Remarks 1.
Remark 2. By analogy with Remark 1, one can define cases with \(1 \leq m \leq n\) which are realizable. For \(d = 2\) and \(d = 3\), all generic realizable cases covered by Theorem 1 and Corollary 1 are illustrated by Examples 1 and 2.

Remark 3. Suppose that one considers the question:

Question 4. For \(d \geq 3\), given \(n \in \mathbb{N}, 1 \leq n \leq d\), what are the possible values of \(n^*\)?

Theorem 1 and Corollary 1 imply that

\[
\max(0, 2n - d - 1) \leq n^* \leq \min(2n - 2, d - 1).
\]

Indeed, from Theorem 1 and Corollary 1 one deduces the inequalities \(n^* \leq 2n - 2\) and \(n^* \geq d - 1 - (2m - 2)\), i.e., \(n^* \leq 2n - 2\) and \(n^* \geq 2n - d - 1\).

Proof of Theorem 1. Suppose that \(\gamma_j < \alpha\) for \(j = 1, \ldots, 2n - 1\). Set \(\delta_j := \gamma_j, j = 1, \ldots, 2n - 1\), \(\delta := (\delta_1, \ldots, \delta_{2n-1})\) and

\[
Q := (x - \delta) \prod_{j=1}^{2n-1} (x + \delta_j) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x + a_0.
\]

Hence \(a_n = e_n(\delta) - \alpha e_{n-1}(\delta)\). Thus

\[
a_n = \sum_{i=1}^{2n-1} \delta_i e_{n-1}(\delta_i) - \alpha \sum_{i=1}^{2n-1} e_{n-1}(\delta_i) = \sum_{i=1}^{2n-1} (\delta_i - \alpha)e_{n-1}(\delta_i) < 0.
\]

As \(a_0 = -\alpha \delta_1 \cdots \delta_{2n-1} < 0\) and as \(P\) has one positive and \(2n - 1\) negative roots, one has exactly one sign change in the sequence \(1, a_{2n-1}, \ldots, a_1, a_0, 0\), so \(a_j < 0\) for \(j \leq n\).

Set \(a_{n+1} := 0\). The last \(n + 1\) coefficients of the polynomial \((x + \gamma_{2n})Q\) equal \(a_{j-1} + \gamma_{2n}a_j < 0\). In the same way, the last \(n + 1\) coefficients of each of the polynomials \((\prod_{s=2n}^{k}(x + \gamma_s))Q\), \(2n \leq k \leq d\), are negative, which for \(k = d\) leads to a contradiction with the definition of \(\Sigma_{m,n}\).

To prove realizability of all cases mentioned in Remark 1, we observe first that for \(R := (x + 1)^{2n-1}(x - 1) = x^{2n} + g_{2n-1}x^{2n-1} + \cdots + g_1x + g_0\), one has \(g_n = 0, g_j > 0\) for \(j > n\) and \(g_j < 0\) for \(j < n\). Consider, for \(\varepsilon > 0\) small enough, the polynomial

\[
\tilde{R} := (x + 1 + \varepsilon u)^{2n-1-s-r}(x + 1)^s(x + 1 - \varepsilon w)^r(x - 1)
\]

\[= x^{2n} + h_{2n-1}x^{2n-1} + \cdots + h_1x + h_0,
\]
where \( u > 0 \) and \( w > 0 \); we set \( \alpha := 1 \). One has

\[
h_n = (C_{2n-2}^{n-1} - C_{2n-2}^n)((2n - 1 - s - r)u - rw)\varepsilon + o(\varepsilon),
\]

with \( C_{2n-2}^{n-1} - C_{2n-2}^n \neq 0 \) and \( 2n - 1 - s - r \neq 0 \); therefore one can choose \( u \) and \( w \) such that \( h_n > 0 \) and \( h_{n-1} < 0 \). After this, one perturbs the quantities \( \gamma_i \) which are smaller than \( \alpha \) to obtain any possible case of equalities and/or inequalities among them by keeping the conditions \( h_n > 0 \) and \( h_{n-1} < 0 \). Then one sets

\[
K := (1 + \eta x)^{d-2n}R = x^d + \kappa_{d-1}x^{d-1} + \cdots + \kappa_1x + \kappa_0,
\]

where \( \eta > 0 \) is so small that \( \kappa_n > 0 \) and \( \kappa_{n-1} < 0 \). The polynomial \( K \) has a \((d - 2n)\)-fold root \(-1/\eta\) whose modulus is larger than \( \alpha \).

In the case when there are exactly \( 2n - 2 \) quantities \( \gamma_j \) smaller than \( \alpha \), one can perturb the \((d - 2n)\)-fold root \(-1/\eta\) to obtain any possible case of equalities and inequalities among the \( d - 2n \) quantities \( \gamma_j \) which are larger than \( \alpha \). When there are less than \( 2n - 2 \) quantities \( \gamma_j \) smaller than \( \alpha \), not all quantities \( \gamma_j \) larger than \( \alpha \) can be obtained by perturbing \(-1/\eta\). In this case, one can make them all distinct by perturbing \(-1/\eta\) and \(-1 - \varepsilon u\) into \( d - 2n \) and \( 2n - 1 - s - r \) distinct roots, respectively. \( \square \)

Remark 4. We call multiplicity vector a vector whose components are the multiplicities of the roots of a HP of a given degree; the roots are listed in the increasing order. Denote by \( \vec{\nu} := (\mu_1, \mu_2, \ldots, \mu_k) \) the multiplicity vector of a degree \( d - 1 - s - r \) HP. Hence \( \mu_1 + \cdots + \mu_k = d - 1 - s - r \). Suppose that \( \vec{\nu} \) satisfies the following condition:

**Condition 1.** There exists an index \( \nu \) such that \( \mu_1 + \cdots + \mu_\nu = d - 2n \), hence \( \mu_{\nu+1} + \cdots + \mu_k = 2n - 1 - s - r \).

The vector \( \vec{\nu} \) can be viewed as the multiplicity vector of the roots of a polynomial which is obtained by perturbing the product \((x + 1 + \varepsilon u)^{2n-1-s-r}(1 + \eta x)^{d-2n}\). When \( \vec{\nu} \) satisfies Condition 1, the roots of \((x + 1 + \varepsilon u)^{2n-1-s-r}\) and the ones of \((1 + \eta x)^{d-2n}\) can be perturbed independently. Thus when there are less than \( 2n - 2 \) quantities \( \gamma_j \) smaller than \( \alpha \), and when \( \vec{\nu} \) satisfies Condition 1, one can realize the case of equalities and inequalities among the roots of the HP defined by the vector \( \vec{\nu} \) by perturbing separately the roots \(-1/\eta\) and \(-1 - \varepsilon u\). It remains to observe that for \( \eta \) small enough, the root \(-1/\eta\) is smaller than the root \(-1 - \varepsilon u\).
3. The case of \( \Sigma_{1,n,1} \)

In the present Section, we consider SPs of the form \( \Sigma_{1,n,1} \), i.e., with \( c = 2, m = q = 1 \) and \( n = d - 1 \).

**Theorem 2.** For \( d \geq 4 \), the SP \( \Sigma_{1,d-1,1} \) is realizable by and only by HPs with \( n^* = d - 2 \), \( m^* = q^* = 0 \).

**Remark 5.** For \( d = 2 \), no quantity \( \gamma_j \) is defined, see Example 1. For \( d = 3 \), all possible cases of strict inequalities between the quantities \( \alpha, \beta \) and \( \gamma_1 \) are realizable, see the HPs \( P_1, P_2 \) and \( P_3 \) in Example 2, so Theorem 2 fails for \( d = 3 \).

**Proof.** Consider a polynomial \( Q := x^d + a_{d-1}x^{d-1} + \cdots + a_0 \) realizing the SP \( \Sigma_{1,d-1,1} \). Hence \( a_{d-1} < 0 \) and \( a_1 < 0 \), i.e.,

\[-\alpha - \beta + \sum_{j=1}^{d-2} \gamma_j = a_{d-1} < 0 \]

and

\[
\left( \frac{\alpha \beta}{\prod_{j=1}^{d-2} (-\gamma_j)} \right) \left( \frac{1}{\alpha} + \frac{1}{\beta} - \sum_{j=1}^{d-2} \frac{1}{\gamma_j} \right) = (-1)^{d-1}a_1,
\]

which implies

\[
\alpha + \beta > \sum_{j=1}^{d-2} \gamma_j \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{\beta} > \sum_{j=1}^{d-2} \frac{1}{\gamma_j}. \tag{1}
\]

If, for at least two indices \( j \), one has \( \gamma_j \geq \alpha \) (resp. \( \gamma_j \leq \beta \)), then the first (resp. the second) of conditions (1) fails. The same holds true if there exist two indices \( j_1 \) and \( j_2 \) for which one has \( \gamma_{j_1} \geq \alpha \geq \gamma_{j_2} \geq \beta \) (resp. \( \alpha \geq \gamma_{j_1} \geq \beta \geq \gamma_{j_2} \)). Thus for \( d \geq 5 \), the only possibility for conditions (1) to hold true is to have \( \beta < \gamma_j < \alpha \) for \( j = 1, \ldots, d - 2 \).

For \( d = 4 \), one has either \( \alpha > \gamma_2 \geq \gamma_1 > \beta > 0 \) or \( \gamma_2 \geq \alpha \geq \beta \geq \gamma_1 > 0 \) (\(*\)). So to prove the theorem, one has to refute the possibility (\(*\)). One can notice that it is impossible to have \( \gamma_2 = \alpha \) or \( \beta = \gamma_1 \), in which case at least one of conditions (1) fails. Therefore, one has \( \gamma_2 - \gamma_1 > \alpha - \beta \) (\( **\)).

Suppose that inequalities (\(*\)) and (1) hold true. Then one can continuously decrease \( \alpha \) until for the first time at least one of the three equalities holds true:

\[
\alpha = \beta \quad \text{or} \quad \alpha + \beta = \gamma_1 + \gamma_2 \quad \text{or} \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.
\]
If this is \( \alpha = \beta \), then \( 2\beta \geq \gamma_1 + \gamma_2 \) and \( 2/\beta \geq (\gamma_1 + \gamma_2)/(\gamma_1\gamma_2) \), that is
\[
4 \geq (\gamma_1 + \gamma_2)^2/(\gamma_1\gamma_2),
\]
which leads to \( (\gamma_1 - \gamma_2)^2 \leq 0 \). This is possible only if \( \alpha = \beta = \gamma_1 = \gamma_2 \), which is a contradiction. If the equality is \( \alpha + \beta = \gamma_1 + \gamma_2 \), then
\[
1/\alpha + 1/\beta = (\gamma_1 + \gamma_2)/(\alpha\beta) \geq (\gamma_1 + \gamma_2)/(\gamma_1\gamma_2),
\]
hence \( \alpha \beta \leq \gamma_1 \gamma_2 \). Set \( s := (\alpha + \beta)/2 \). Then
\[
\alpha = s + u, \quad \beta = s - u, \quad \gamma_1 = s + v, \quad \gamma_2 = s - v, \quad 0 < u < v < s.
\]
(The inequality \( u < v \) follows from \((*)\) and \((***)\).) This implies the contradiction \( \gamma_1 \gamma_2 = s^2 - v^2 < s^2 - u^2 = \alpha \beta \). Finally, if \( 1/\alpha + 1/\beta = 1/\gamma_1 + 1/\gamma_2 = 2t \), then
\[
1/\alpha = t - r, \quad 1/\beta = t + r, \quad 1/\gamma_2 = t - w, \quad 1/\gamma_1 = t + w, \quad 0 < r < w < t,
\]
hence \( \alpha \beta = 1/(t^2 - r^2) < 1/(t^2 - w^2) = \gamma_1 \gamma_2 \). However, one must have
\[
\alpha + \beta = 2t/(t^2 - r^2) > 2t/(t^2 - w^2) = \gamma_1 + \gamma_2,
\]
which is a contradiction. \( \square \)

4. The case \( q = 1 \)

Now we consider SPs of the form \( \Sigma_{m,n,1} \), i.e., with \( c = 2 \) and \( q = 1 \).

**Theorem 3.** (1) For \( d \geq 4 \), an HP defining an SP \( \Sigma_{m,n,1} \) satisfies one of the two conditions:

(i) its root of the smallest modulus is positive;

(ii) one has \( \gamma_1 \leq \beta \leq \alpha < \gamma_2 \leq \cdots \leq \gamma_{d-2} \).

(2) If condition (ii) is satisfied, then \( n = 2 \) or \( n = 3 \).

(3) For \( n = 3 \) (resp. for \( n = 2 \)), and for any \( d \geq 5 \), there exist polynomials with roots satisfying condition (ii) for all possible choices of equalities or strict inequalities (resp. condition (ii) with all inequalities strict).

**Remark 6.** For \( d = 4 \) and \( n = 3 \), one deals with the SP \( \Sigma_{1,3,1} \); this case is covered by Theorem 2. For \( d = 4 \) and \( n = 2 \), see the polynomials in part (2) of Example 3; they correspond to all generic cases allowed by Theorem 3. For the case \( n = q = 1 \), see Section 5.
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PROOF. We denote an HP defining an SP $\Sigma_{m,n,1}$ by $T := x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$. Recall that

$$1/\alpha + 1/\beta - 1/\gamma_1 - \cdots - 1/\gamma_{d-2} = -a_1/a_0 > 0 \quad (2)$$

(to see this, it suffices to consider the polynomial $T_R(x) := x^dT(1/x) = a_0x^d + a_1x^{d-1} + \cdots + 1$ whose roots are the reciprocals of the roots of $T(x)$). Hence at most one of the quantities $1/\gamma_j$ can be $\geq 1/\beta$ (so this is $1/\gamma_1$ and $\gamma_1 \leq \beta$), otherwise inequality (2) fails. If there exists exactly one such quantity, then for $j > 1$, one has $\gamma_j > \alpha$. This proves part (1).

Part (2). Suppose that condition (ii) is satisfied. Consider the polynomial $T_R$ defined above. We denote by $1/\gamma$ the $(d-3)$-tuple $(1/\gamma_2, \ldots, 1/\gamma_{d-2})$ and by $e_j$ the quantity $e_j(1/\gamma)$. One has

$$a_4/a_0 = e_4 + (1/(\alpha\beta))e_2 + (1/\gamma_1)e_3 + (1/(\alpha\beta\gamma_1))e_1 - (1/\alpha + 1/\beta)e_3 - (1/\alpha + 1/\beta)(1/\gamma_1)e_2. \quad (3)$$

The following inequality holds true:

$$e_2 = \left( (e_1)^2 - \sum_{j=2}^{d-2} (1/\gamma_j)^2 \right)/2 < (e_1)^2/2. \quad (4)$$

The inequalities (2) and $1/\beta \leq 1/\gamma_1$ imply $1/\alpha > e_1$. Thus (see (4)) $e_2 < (e_1)^2/2 < e_1/(2\alpha)$, which implies

$$(1/\alpha + 1/\beta)(1/\gamma_1)e_2 < (2/(\beta\gamma_1))e_2 < (1/(\alpha\beta\gamma_1))e_1. \quad (5)$$

The inequality

$$(1/\beta)e_3 \leq (1/\gamma_1)e_3 \quad (6)$$

results from $1/\beta \leq 1/\gamma_1$, and the inequality

$$(1/\alpha)e_3 \leq (1/(\alpha\beta))e_2 \quad (7)$$

follows from $e_3 < e_2e_1 < (1/\alpha)e_2 \leq (1/\beta)e_2$. Summing up inequalities (5), (6) and (7), one obtains $a_4/a_0 > 0$ (see (3)), hence $a_4 > 0$ and $n \leq 3$.

It remains to exclude the case $n = 1$. Suppose that the polynomial $T$ defines the SP $\Sigma_{d-1,1,1}$. Without loss of generality, we assume that $\gamma_1 = 1$ (this can be obtained by a linear change of the variable $x$). If $a_0 > 0$, $a_1 < 0$ and $a_2 > 0$, then
1/α + 1/β - 1 - e₁ = -a₁/a₀ > 0

and

-1/α - 1/β + e₁ + 1/(αβ) - (1/α + 1/β)e₁ + e₂ = a₂/a₀ > 0.

Set \( \Delta := 1/α + 1/β - 1 \). Hence \( \Delta > e₁ > 0 \) and

\[-1/α - 1/β + 1/(αβ) > \Delta e₁ - e₂ > (e₁)^2 - e₂ > 0,\]

which by \( α ≥ β ≥ γ₁ = 1 \) is impossible.

Part (3). For \( n = 3 \), consider the polynomials \( Yₙ := (x + s)^{(x - 1)^{2}(x + 1)} \) (hence \( d = s + 3 \), so \( s ≥ 2 \)). By Descartes’ rule of signs, there are exactly two sign changes in the sequence of coefficients of the polynomial \( Yₙ \). The last 5 coefficients of \( Yₙ \) are the same as the ones of the polynomial

\[Wₙ := (Cₙ^4s^{s-4}x^4 + Cₙ^{3}s^{s-3}x^3 + Cₙ^{2}s^{s-2}x² + Cₙ^{1}s^{s-1}x + s')(x - 1)^{2}(x + 1),\]

where if \( s - j < 0 \), then the term \( Cₙ^{j}s^{s-j}x^j \) is missing. These coefficients equal

\[Wₙ,₀ = s', \quad Wₙ,₁ = 0, \quad Wₙ,₂ = -(1/2)(3s + 1)s^{s-1}, \quad Wₙ,₃ = -(1/3)(s - 1)(s + 1)s^{s-2},\]

and

\[Wₙ,₄ = (1/8)(s + 1)(3s² + 3s - 2)s^{s-3}.\]

For \( s ≥ 2 \), one has \( Wₙ,₂ < 0, Wₙ,₃ < 0 \) and \( Wₙ,₄ > 0 \). By an infinitesimal shift of the \( s \)-fold root at \(-s\), one obtains the condition \( Wₙ,₁ < 0 \). This is possible to do, because the coefficient of \( x \) in the polynomial \((x + s + \varepsilon)^{(x - 1)^{2}(x + 1)}\) equals \(-s^{s-1} + o(\varepsilon)\). After this, if one wants to have strict inequalities instead of some of the equalities in condition (ii), one can use infinitesimal shiftings followed by splittings of roots. This amounts to replacing factors \((x + h)^{ℓ}\) by products \((x + h + \eta₁)^{ℓ₁} \cdots (x + h + \eta_ν)^{ℓ_ν}\), where \( ηᵢ \) are small real quantities and \( ℓ₁ + \cdots + ℓ_ν = ℓ \).

For \( n = 2 \), consider the polynomial \( P₁ \) of part (1) of Example 2. For \( ε > 0 \) small enough, the polynomial \((1 + εx)^{d-3}P₁\) defines the SP \( Σ_{d-2,2,1} \) and has a \((d - 3)\)-fold root at \(-1/ε\) and simple roots at \(-1, 1, 5, 16\). One can then perturb the root at \(-1/ε\) to make all the roots of \((1 + εx)^{d-3}P₁\) distinct. □

In the following theorem, we consider polynomials defining the SP \( Σ_{m,n,1} \) with \( m + n = d \) and satisfying the condition \( β < γ₁ \).
Theorem 4. (1) If \( m \leq n \), then there are \( \leq 2m - 1 \) quantities \( \gamma_j \) which are \( \geq \alpha \) (that is, for \( m < n - 1 \), one has \( \gamma_{d-2m-1} < \alpha \)).

(2) If \( m \leq n \), then all cases when there are exactly \( s \leq 2m - 2 \) quantities \( \gamma_j \) not less than \( \alpha \) are realizable by HPs.

(3) If \( n < m \), then there are \( \leq 2n - 1 \) quantities \( \gamma_j \) which are \( \leq \alpha \) (that is, one has \( \gamma_{2n} > \alpha \)).

(4) If \( n < m \), then all cases when there are exactly \( s \leq 2n - 2 \) quantities \( \gamma_j \) not larger than \( \alpha \) are realizable by HPs.

Proof. Part (1). Suppose that an HP \( P \) realizes the SP \( \Sigma_{m,n,1} \). Hence its derivative \( P' \) is hyperbolic and realizes the SP \( \Sigma_{m,n} \). Denote by \( \alpha' \) and \( \gamma'_1 \leq \cdots \leq \gamma'_{d-2} \) the moduli of the positive and negative roots of \( P' \). By Corollary 1, at most \( 2m - 2 \) of the quantities \( \gamma'_j \) are \( \geq \alpha' \), i.e., the inequality \( \gamma'_{d-2m} < \alpha' \) holds true (this inequality is meaningful only for \( m < n \)). For \( j \geq 2 \), one has \( \gamma'_{j-1} \leq \gamma'_j \leq \gamma_j \), so \( \gamma_{d-2m-1} \leq \gamma'_{d-2m} < \alpha' \) (the left inequality is meaningful only for \( m < n - 1 \)).

On the other hand, \( \alpha' \leq \alpha \), which proves part (1) of the theorem.

Part (2). Denote by \( Q \) a degree \( d - 1 \) HP defining the SP \( \Sigma_{m,n} \) and realizing the case \( \gamma_{d-s-2} < \alpha \leq \gamma_{d-s-1} \); this case is defined without reference to \( \beta \). The case is realizable, see Corollary 1 and Remark 2. Set \( P := (x - \varepsilon)Q \), where \( \varepsilon > 0 \) is small enough, so \( P \) defines the SP \( \Sigma_{m,n,1} \). (This statement is in fact a particular case of Lemma 2.) Hence the root \( \varepsilon \) of the polynomial \( P \) is its root of smallest modulus, and the polynomial \( P \) realizes the case

\[
\beta < \gamma_1 \leq \cdots \leq \gamma_{d-s-2} < \alpha \leq \gamma_{d-s-1} < \cdots < \gamma_{d-2}.
\]

Part (3). Suppose that an HP \( P \) realizes the SP \( \Sigma_{m,n,1} \). Hence the reverted polynomial \( P^R := x^dP(1/x) \) is hyperbolic and realizes the SP \( \Sigma_{1,n,m} \), and the polynomial \( U := dP^R - x(P^R)' \) realizes the SP \( \Sigma_{n,m} \). Denote by \( \alpha^u \) and \( \gamma_1^u \leq \cdots \leq \gamma_{d-2}^u \) the moduli of the positive and negative roots of \( U \). Hence \( \alpha^u \leq \alpha^r \) (the superscript \( r \) indicates moduli of roots of \( P^R \)), and by Corollary 1, \( \gamma_{d-2n}^u < \alpha^u \).

The zeros of the polynomials \( P^R \) and \( U \) interlace, so \( \gamma_{d-1}^r \leq \gamma_{d-1}^u \leq \gamma_{d-1}^u \). Thus \( \gamma_{d-2n-1}^r \leq \gamma_{d-2n}^u < \alpha^u \leq \alpha^r \). The roots of \( P^R \) are the reciprocals of the roots of \( P \). Hence \( \gamma_j^r = 1/\gamma_{d-1-j} \) and \( \alpha^r = 1/\alpha \), therefore the inequality \( \gamma_{d-2n-1}^r < \alpha^r \) is equivalent to \( 1/\gamma_{2n} < 1/\alpha \), i.e., to \( \gamma_{2n} > \alpha \).

The proof of part (4) is analogous to the proof of part (2) – one first finds a degree \( d - 1 \) polynomial \( Q \) defining the SP \( \Sigma_{m,n} \) and realizing the case \( \gamma_s \leq \alpha < \gamma_{s+1} \), and then constructs the polynomial \( P = (x - \varepsilon)Q \) which realizes the case

\[
\beta < \gamma_1 \leq \cdots \leq \gamma_s \leq \alpha < \gamma_{s+1} \leq \cdots < \gamma_{d-2}.
\]
5. The case \( n = 1 \)

We consider here SPs of the form \( \Sigma_{m,1,q} \) (hence \( c = 2, n = 1 \) and \( d = m + q \)).

**Theorem 5.** The SP \( \Sigma_{m,1,q} \) is realizable by and only by polynomials satisfying the condition

\[
\gamma_1 \leq \cdots \leq \gamma_{q-1} < \beta < \alpha < \gamma_q \leq \cdots \leq \gamma_{m+q-2},
\]

that is, with \( m^* = m - 1, n^* = 0 \) and \( q^* = q - 1 \).

**Remarks 2.**

(1) Unlike \( n = 1 \), for \( n = 2 \), it is not true that there is a unique possibility for \( m^* \) and \( q^* \), see Examples 3, 4 and 5. It would be interesting to know whether for \( c = n = 2 \), there is an upper bound (over all \( m \geq 1 \) and \( q \geq 1 \)) for the possible values of the quantity \( n^* \).

(2) The statement of part (1) of Theorem 1 for \( m = d, n = 1 \) (resp. the second sentence of Corollary 1) could be considered as an extension of the statement of Theorem 5 to the case \( m = d, n = 1, q = 0 \) (resp. \( m = 0, n = 1, q = d \)).

**Proof.** (1) We need the following lemma:

**Lemma 1.** There exists no polynomial realizing the SP \( \Sigma_{m,1,q} \) and satisfying the condition \( \gamma_\nu = \alpha \) or \( \gamma_\nu = \beta \) for some \( \nu \) (1 ≤ \( \nu \) ≤ \( m + q - 2 \)).

**Proof.** Suppose that such a polynomial \( P := \sum_{j=0}^{d} a_j x^j \) exists. Then

\[
P(\pm \gamma_\nu) = 0, \quad \text{which implies} \quad \sum_{k=0}^{\lfloor d/2 \rfloor} a_{2k} (\gamma_\nu)^{2k} = \sum_{k=0}^{\lfloor (d-1)/2 \rfloor} a_{2k+1} (\gamma_\nu)^{2k+1} = 0.
\]

This is impossible, because \( \gamma_\nu > 0 \) and exactly one of the coefficients \( a_j \) is negative while the rest are positive. \( \square \)

(2) For \( m = n = q = 1 \), any HP of degree 2 has just two positive roots and there is nothing to prove. For \( m = n = 1 \) and \( q = 2 \), one has

\[
\alpha + \beta - \gamma_1 > 0 \quad \text{and} \quad \alpha \beta - (\alpha + \beta) \gamma_1 > 0.
\]

If \( \gamma_1 \geq \alpha \) (resp. if \( \gamma_1 \geq \beta \)), then this leads to the contradiction \( \alpha \beta/(\alpha + \beta) > \gamma_1 > \alpha \), i.e., \( \beta/(\alpha + \beta) > 1 \) (resp. \( \alpha \beta/(\alpha + \beta) > \gamma_1 > \beta \), i.e., \( \beta/(\alpha + \beta) > 1 \)).

Hence \( \gamma_1 < \beta < \alpha \).

(3) We use induction on \( q \) for \( m \) fixed. We do this first for \( m = 1 \) the induction base being the case \( m = n = 1, q = 2 \), see part (2) of this proof; however, the induction step is performed in the same way for any \( m \geq 1 \) fixed.

Suppose that an HP \( P \) realizes the SP \( \Sigma_{m,1,q} \) with \( q > 1 \). Then its derivative \( P' \) is a degree \( d - 1 \) HP which realizes the SP \( \Sigma_{m,1,q-1} \). Consider the family of
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polynomials $P_r := r x P' + (1 - r) P$, $r \in [0, 1]$. For $r < 1$, every polynomial of this family defines the SP $\Sigma_{m,1,q}$. Every polynomial of this family is hyperbolic. By Descartes’ rule of signs, every polynomial $P_r$ has exactly two positive roots and $d - 2$ or $d - 3$ negative ones (for $r \in [0, 1)$ and $r = 1$ respectively; for $r = 1$, one of its roots equals 0). For $r = 1$, by inductive assumption, the moduli of the roots satisfy the inequalities

$$0 = \gamma_1 < \gamma_2 \leq \cdots \leq \gamma_{q-1} < \beta < \alpha \leq \cdots \leq \gamma_{m+q-2}.$$  

The roots of $P_r$ depend continuously on $r$, and for no value of $r \in [0, 1]$ one does have an equality of the form $\gamma_\nu = \alpha$ or $\gamma_\nu = \beta$, see Lemma 1. Hence for $r \in [0, 1)$, inequalities (8) hold true. From our reasoning follows the proof of Theorem 5 for $m = 1$.

(4) If an HP $P$ realizes the SP $\Sigma_{1,1,q}$ (hence $d = q + 1$), then the HP $P^R := x^d P(1/x)$ realizes the SP $\Sigma_{q,1,1}$ and $(P^R)'$ realizes the SP $\Sigma_{q,1}$. Hence the moduli $\alpha', \gamma'_1, \ldots, \gamma'_{d-2}$ of the roots of the polynomial $(P^R)'$ satisfy the conditions

$$\alpha' < \gamma'_1 \leq \cdots \leq \gamma'_{d-2},$$

see Theorem 1. Consider the family of polynomials $(P^R)_r := r x (P^R)' + (1 - r) P^R$, $r \in [0, 1]$. For $r \neq 1$ and close to 1, the moduli of the roots of the polynomial $(P^R)_r$ satisfy the inequalities

$$\beta < \alpha < \gamma_1 \leq \cdots \leq \gamma_{d-2}.$$  

One can apply Lemma 1 to conclude (by analogy with the reasoning about the family $P_r$ in part (3) of this proof) that for $r \in [0, 1)$, the above sequence of inequalities holds true, and hence Theorem 5 holds true for any SP $\Sigma_{q,1,1}$, which we for convenience denote by $\Sigma_{m,1,1}$.

(5) Now one proves Theorem 5 by induction on $q$ for each $m$ fixed by applying the reasoning developed in part (3) of this proof. The induction base are the cases $\Sigma_{m,1,1}$, see part (4) of this proof. □

6. Examples

Example 1. (1) For $d = 1$, there are two possible SPs, namely $(+, +)$ and $(+, -) = \Sigma_{1,1,1}$, realizable respectively by $x + 1$ (with $\gamma_1 = 1$) and $x - 1$ (with $\alpha = 1$).
(2) For \( d = 2 \), one has the SPs \((+, +, +), (+, +, -) = \Sigma_{2,1}, (+, -, +) = \Sigma_{1,1}\) and \((+, -, -) = \Sigma_{1,2}\). They are realizable by the HPs
\[
(x + 1)(x + 2) = x^2 + 3x + 2, \quad (x + 2)(x - 1) = x^2 + x - 2,
\]
\[
(x - 1)(x - 2) = x^2 - 3x + 2 \quad \text{and} \quad (x + 1)(x - 2) = x^2 - x - 2,
\]
with self-evident values of \( \alpha, \beta, \gamma_1 \) and \( \gamma_2 \). For any HPs realizing the SPs \( \Sigma_{2,1} \) or \( \Sigma_{1,2} \), one has \( \gamma_1 > \alpha \) or \( \gamma_1 < \alpha \), respectively.

**Example 2.** (1) For \( d = 3 \), we show SPs, HPs realizing them and inequalities between the moduli of their roots. The SP \( \Sigma_{1,2,1} \) is realizable by the HPs
\[
P_1 := (x + 1)(x - 1.5)(x - 1.6) = x^3 - 2.1x^2 - 0.7x + 2.4,
\]
\[
P_2 := (x + 1)(x - 0.5)(x - 0.6) = x^3 - 1.1x^2 - 1.2x + 0.9
\]
and
\[
P_3 := (x + 1)(x - 0.5)(x - 0.6) = x^3 - 0.1x^2 - 0.8x + 0.3,
\]
with \( \gamma_1 < \beta < \alpha \) or \( \beta < \gamma_1 < \alpha \) or \( \beta < \alpha < \gamma_1 \), respectively.

(2) The SPs \( \Sigma_{2,1,1} \) and \( \Sigma_{3,1} \) are realizable by the HPs
\[
P_4 := (x + 1)(x - 0.2)(x - 0.1) = x^3 + 0.7x^2 - 0.28x + 0.02
\]
and
\[
P_5 := (x + 1)(x + 2)(x - 0.1) = x^3 + 2.9x^2 + 1.7x - 0.2,
\]
with \( \beta < \alpha < \gamma_1 \) and \( \alpha < \gamma_1 < \gamma_2 \), respectively. Hence the SPs \( \Sigma_{1,1,2} \) and \( \Sigma_{1,3} \) are realizable by the HPs \( P_4^R \) and \( -P_5^R \), with \( \gamma_1 < \beta < \alpha \) and \( \gamma_1 < \gamma_2 < \alpha \), respectively, see part (2) of Remarks 1.

(3) The SP \( \Sigma_{2,2} \) is realizable by the HPs
\[
P_6 := (x + 1)(x + 2)(x - 0.95) = x^3 + 2.05x^2 - 0.85x - 1.9,
\]
\[
P_7 := (x + 1)(x + 2)(x - 1.5) = x^3 + 1.5x^2 - 2.5x - 3
\]
and
\[
P_8 := (x + 1)(x + 2)(x - 2.5) = x^3 + 0.5x^2 - 5.5x - 5,
\]
with \( \alpha = 0.95 < \gamma_1 = 1 < \gamma_2 = 2 \), with \( \gamma_1 = 1 < \alpha = 1.5 < \gamma_2 = 2 \) or with \( \gamma_1 = 1 < \gamma_2 = 2 < \alpha = 2.5 \), respectively.
Example 3. (1) For $d = 4$, one has
\[
Q_1 := (x - 1.2)(x - 0.8)(x + 0.97)(x + 0.98) = x^4 - 0.05x^3 - 1.9894x^2 - 0.0292x + 0.912576
\]
with $\beta = 0.8 < \gamma_1 = 0.97 < \gamma_2 = 0.98 < \alpha = 1.2$, so one realizes the SP $\Sigma_{1,3,1}$.

(2) Again for $d = 4$, one can realize the SP $\Sigma_{2,2,1}$ in different ways, with different inequalities between the quantities $\alpha$, $\beta$, $\gamma_1$ and $\gamma_2$. We list some examples here:

\[
Q_2 := (x - 4)(x - 1)(x + 2.1)(x + 3) = x^4 + 0.1x^3 - 15.2x^2 - 11.1x + 25.2,
\]
i.e., for $\beta = 1 < \gamma_1 = 2.1 < \gamma_2 = 3 < \alpha = 4$;

\[
Q_3 := (x - 0.995)(x - 0.99)(x + 1)(x + 1.001) = x^4 + 0.016x^3 - 1.985935x^2 - 0.01589995x + 0.98603505,
\]
i.e., for $\beta = 0.99 < \alpha = 0.995 < \gamma_1 = 1 < \gamma_2 = 1.001$;

\[
Q_4 := (x - 1.6)(x - 1.5)(x + 1)(x + 100) = x^4 + 97.9x^3 - 210.7x^2 - 67.6x + 240,
\]
i.e., for $\gamma_1 = 1 < \beta = 1.5 < \alpha = 1.6 < \gamma_2 = 100$;

\[
Q_5 := (x - 1)(x - 0.97)(x + 0.99)(x + 1.001) = x^4 + 0.021x^3 - 1.96128x^2 - 0.0209803x + 0.9612603,
\]
i.e., for $\beta = 0.97 < \gamma_1 = 0.99 < \alpha = 1 < \gamma_2 = 1.001$. When one replaces the latter four HPs by their reverted ones (see part (2) of Remarks 1), then one realizes the SP $\Sigma_{1,2,2}$, with $\alpha$ (resp. $\beta$ and $\gamma_1$) changed to $1/\beta$ (resp. $1/\alpha$ and $1/\gamma_3$.)

Example 4. For $d = 5$, consider the SP $\Sigma_{2,2,2}$ and some HPs defining this SP:

\[
(x - 1)(x - 1.05)(x + 1.08)(x + 1.09)(x + 1.1)
= x^5 + 1.22x^4 - 2.0893x^3 - 2.57819x^2 + 1.087824x + 1.359666,
\]

\[
(x - 1)(x - 1.05)(x + 1.02)(x + 1.09)(x + 1.1)
= x^5 + 1.16x^4 - 2.0977x^3 - 2.443760x^2 + 1.097331x + 1.284129,
\]

\[
(x - 1)(x - 1.05)(x + 1.02)(x + 1.04)(x + 1.1)
= x^5 + 1.11x^4 - 2.1012x^3 - 2.33506x^2 + 1.101036x + 1.225224,
\]
\[(x - 1)(x - 1.05)(x + 1.02)(x + 1.03)(x + 1.04)\]
\[= x^5 + 1.04x^4 - 2.1019x^3 - 2.187206x^2 + 1.1018508x + 1.1472552,\]
\[(x - 1)(x - 1.05)(x + 0.99)(x + 1.09)(x + 1.1)\]
\[= x^5 + 1.13x^4 - 2.1019x^3 - 2.376545x^2 + 1.1020845x + 1.2463605\]
and
\[(x - 1)(x - 1.05)(x + 0.99)(x + 1.04)(x + 1.1)\]
\[= x^5 + 1.08x^4 - 2.1039x^3 - 2.26927x^2 + 1.103982x + 1.189188.\]

It is easy to check that these HPs and their reverted ones realize all possible generic cases with this SP.

**Example 5.** For \(d = 7\), the HP
\[(x - 1)(x + 0.99)(x + 0.94)(x + 0.93)(x + 0.92)(x + 0.91)(x - 0.9)\]
\[= x^7 + 2.79x^6 + 0.7855x^5 - 4.244835x^4 - 3.88785176x^3\]
\[+ 0.8027291316x^2 + 2.102352335x + 0.6521052938\]
realizes the SP \(\Sigma_{3,2,3}\), with
\[\beta = 0.9 < \gamma_1 = 0.91 < \gamma_2 = 0.92 < \gamma_3 = 0.93 < \gamma_4 = 0.94 < \gamma_5 = 0.99 < \alpha = 1.\]

In this example, one has \(n^* = 5, m^* = q^* = 0\). More generally, consider the HP
\[(x^2 - 1)(x^2 - 0.9^2)(x + 0.9)^3\]
\[= x^7 + 2.7x^6 + 0.62x^5 - 4.158x^4 - 3.5883x^3 + 0.86751x^2 + 1.9683x + 0.59049\]
realizing the same SP. One can perturb its roots at \(-1\) and \(-0.9\) (the latter is 4-fold) to obtain HPs with \(n^* = 0, 1, 2, 3, 4\) or \(5\) and with all moduli of roots distinct.

### 7. Comments on the case \(c = 2\)

#### 7.1. Concatenation lemma and its applications

In the present subsection, we consider the sign pattern \(\Sigma_{m,n,q}\) with \(n > 1\) in the generic case when all moduli of roots are distinct. We remind that the quantities \(m^*, n^*\) and \(q^*\) are defined in Notation 1.
We explain how using Theorem 1 and Corollary 1 one can prove the realizability of certain cases. To this end, we recall a concatenation lemma proved in [5]. We say that a real (not necessarily hyperbolic) polynomial realizes the pair $(pos, neg)$ if it has exactly $pos$ positive and exactly $neg$ negative roots counted with multiplicity. In what follows, all real roots are assumed distinct.

**Lemma 2.** Suppose that the monic polynomials $P_1$ and $P_2$ of degrees $d_1$ and $d_2$ with SPs $\sigma_1 = (+, \hat{\sigma}_1)$ and $\sigma_2 = (+, \hat{\sigma}_2)$, respectively, realize the pairs $(pos_1, neg_1)$ and $(pos_2, neg_2)$. (Here $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the SPs obtained from $\sigma_1$ and $\sigma_2$ by deleting the initial + sign. Hence they can begin with $+$ or $-$.) Then

1. if the last position of $\hat{\sigma}_1$ is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ realizes the SP $(1, \hat{\sigma}_1, \hat{\sigma}_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$;

2. if the last position of $\hat{\sigma}_1$ is $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ realizes the SP $(1, \hat{\sigma}_1, -\hat{\sigma}_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$. (Here $-\hat{\sigma}_2$ is the SP obtained from $\hat{\sigma}_2$ by changing each $+$ to $-$ and vice versa.)

**Remark 7.** One can prove that every SP $\sigma$ of length $d + 1$, with $c$ sign changes and $p$ sign preservations, is realizable by a degree $d$ HP having $c$ distinct positive and $p$ distinct negative roots by applying Lemma 2 with $P_2$ being each time a linear polynomial $d_2 - 1$ times. If the second component of the SP $\sigma$ is $+$ (resp. is $-$), then one starts with $P_1 = x + 1$ (resp. with $P_1 = x - 1$). Suppose that one has thus constructed a degree $k$ HP $Q$, $1 \leq k \leq d - 1$, which realizes the SP $\sigma_k$ obtained from $\sigma$ by deleting the last $d - k$ components of $\sigma$. If the last two components of the SP $\sigma_{k+1}$ are distinct (resp. equal), then we apply Lemma 2 with $P_1 = Q$ and $P_2 = x - 1$ (resp. with $P_1 = Q$ and $P_2 = x + 1$). In this way, the number of sign changes (resp. of sign preservations) of $\sigma_{k+1}$ is equal to the number of positive (resp. of negative) roots of the HP which realizes it. When one successively applies Lemma 2, each next root (this is the root of $P_2(x/\varepsilon)$) has a modulus much smaller than the least of the moduli of the roots of $P_1$; this follows from the necessity to choose at each concatenation the number $\varepsilon$ sufficiently small. Therefore, the moduli of the roots of the thus constructed HP realizing the SP $\sigma$ are all distinct. Moreover, the decreasing order of the moduli of positive and negative roots on $\mathbb{R}_+$ is the same as the order of sign changes and sign preservations when the SP is read from left to right. We call this order canonical. Thus it is always possible to realize an SP by an HP with canonical order of the moduli of its positive and negative roots on $\mathbb{R}_+$. The SPs $\Sigma_{1,d}$, $\Sigma_{d,1}$, $\Sigma_{1,d-1,1}$ and $\Sigma_{m,1,q}$ have only canonical realizations, see Theorem 1, Corollary 1,
Theorem 2 and Theorem 5. For some SPs, not canonical realizations are also possible, see Examples 3, 4 and 5 or Theorems 1, 3 and 4.

Now we explain how Lemma 2 can be used to construct real polynomials defining a given SP and realizing a given pair \((\text{pos}, \text{neg})\). We are interested mainly in the case of HPs. Suppose that the polynomials \(P_1\) and \(P_2\), of degrees \(m + n^b - 1\) and \(n^d + q - 1\), define the SPs \(\Sigma_{m,n^b}\) and \(\Sigma_{n^d,q}\), where \(n^b + n^d = n + 1\). Then the polynomial \(\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)\) realizes the SP \(\Sigma_{m,n,q}\), see part (2) of Lemma 2. Suppose that:

(i) exactly \(m^*\) moduli of negative roots of \(P_1\) are larger than the modulus of its positive root \(\alpha\), and hence exactly \(m + n^b - 2 - m^*\) such moduli are smaller than \(\alpha\);

(ii) exactly \(q^*\) moduli of negative roots of \(P_2\) are smaller than the modulus of its positive root \(\beta\), and hence exactly \(n^d + q - 2 - q^*\) such moduli are larger than \(\beta\).

For \(\varepsilon > 0\) small enough, the moduli of all roots of \(P_2(x/\varepsilon)\) are smaller than the modulus of any of the roots of \(P_1\). Therefore, the polynomial \(\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)\) has exactly \(m^*\) moduli of negative roots which are larger than \(\alpha\), exactly \(m + n^b - 2 - m^* + n^d + q - 2 - q^* = m + n + q - 3 - m^* - q^* = d - 2 - m^* - q^* = n^*\) such moduli belonging to the interval \((\varepsilon\beta, \alpha)\), and exactly \(q^*\) such moduli which are smaller than \(\varepsilon\beta\). The possible values of \(m^*, n^d, n^b\) and \(q^*\) can be deduced from Theorem 1 and Corollary 1.

7.2. The case \(2 \leq d \leq 5,\ c = 2\). We summarize here what is proved above about the realizability of SPs in the generic case for \(2 \leq d \leq 5,\ c = 2\). We remind that for given \(d\), knowing the exhaustive answer to the question about the realizability of SPs with \(c\) sign changes implies knowing the one for SPs with \(d - c\) sign changes as well, see part (1) of Remarks 1. For \(c = 0\) (hence for \(c = d\)), the exhaustive answer is given by the observation (2) at the beginning of this paper. For \(c = 1\) (hence for \(c = d - 1\)) the answer is given by Theorem 1 and Corollary 1. For \(2 \leq d \leq 5\), we present here the exhaustive answer for \(c = 2\) (hence for \(c = d - 2\) as well). Thus for \(d \leq 5\), we cover all possible generic cases.

For \(d = 2\), the exhaustive answer is provided by Example 1.

For \(d = 3\), the polynomials \(P_1, P_2\) and \(P_3\) (see Example 2) show that the SP \(\Sigma_{1,2,1}\) is realizable in all three possible generic situations of inequalities between the quantities \(\alpha, \beta\) and \(\gamma_1\). The SPs \(\Sigma_{2,1,1}\) and \(\Sigma_{1,1,2}\) have only canonical realizations, see Theorem 5 and Remark 7; examples of such realizations are given by the polynomials \(P_4\) and \(P_4^R\), see Example 2.
For $d = 4$, the SPs $\Sigma_{1,1,3}$, $\Sigma_{2,1,2}$, $\Sigma_{3,1,1}$ and $\Sigma_{1,3,1}$ have only canonical realizations, see Theorems 5 and 2. The realizable generic cases for the SP $\Sigma_{2,2,1}$ are illustrated in part (2) of Example 3. The cases $\gamma_1 < \gamma_2 < \beta < \alpha$ and $\gamma_1 < \beta < \gamma_2 < \alpha$ are not realizable with this SP, see Theorem 3. The corresponding results about the SP $\Sigma_{1,2,2}$ are then deduced using part (2) of Remarks 1.

For $d = 5$, the SPs $\Sigma_{1,1,4}$, $\Sigma_{2,1,3}$, $\Sigma_{3,1,2}$, $\Sigma_{4,1,1}$ and $\Sigma_{1,4,1}$ have only canonical realizations, see Theorems 5 and 2. The SP $\Sigma_{2,2,2}$ is realizable in all generic cases, see Example 4. Consider the HP

$$(x - 0.1)(x - 1)(x + 1)^3 = x^5 + 1.9x^4 - 0.2x^3 - 2x^2 - 0.8x + 0.1. \quad (9)$$

It defines the SP $\Sigma_{2,3,1}$ and one has $\beta = 0.1$, $\alpha = \gamma_j = 1$, $j = 1, 2$ and 3. When its triple root at $-1$ splits into three simple negative roots, its coefficients depend continuously on the bifurcation, so by nearby HPs one can realize the generic cases

$$\beta < \alpha < \gamma_1 < \gamma_2 < \gamma_3,$$

$$\beta < \gamma_1 < \gamma_2 < \alpha < \gamma_3$$

and

$$\beta < \gamma_1 < \gamma_2 < \gamma_3 < \alpha.$$

The SP $\Sigma_{2,3,1}$ is realizable by the HP

$$(x + 1)(x - 1)^2(x + 2.1)^2 = x^5 + 3.2x^4 - 0.79x^3 - 7.61x^2 - 0.21x + 4.41.$$  

For $\varepsilon > 0$ small enough, this is also the case of the polynomial

$$(x + 1)(x - 1 - \varepsilon)(x - 1 - 2\varepsilon)(x + 2.1 + \varepsilon)(x + 2.1 + 2\varepsilon),$$

where

$$\gamma_1 = 1 < \beta = 1 + \varepsilon < \alpha = 1 + 2\varepsilon < \gamma_2 = 2.1 + \varepsilon < \gamma_3 = 2.1 + 2\varepsilon.$$  

According to Theorem 3, there are no other realizable generic cases with the SP $\Sigma_{2,3,1}$. Taking the reverted polynomials of the HPs realizing the SP $\Sigma_{2,3,1}$ one realizes the SP $\Sigma_{1,3,2}$ in the corresponding generic cases, see part (2) of Remarks 1.

The SP $\Sigma_{3,2,1}$ is not realizable in the generic cases with $\gamma_1 < \gamma_2 < \beta$ or with $\gamma_1 < \beta < \gamma_2 < \alpha$, see Theorem 3. It is realizable for $\gamma_1 < \beta < \alpha < \gamma_2 < \gamma_3$ by the HP

$$(x + 1)(x - 1.5)(x - 1.6)(x + 100)(x + 1000)$$

$$= x^5 + 1097.9x^4 + 97689.3x^3 - 2.107676 \times 10^5x^2 - 67360x + 2.4 \times 10^5.$$
In the generic cases $\beta < \alpha < \gamma_1 < \gamma_2 < \gamma_3$, $\beta < \gamma_1 < \alpha < \gamma_2 < \gamma_3$ and $\beta < \gamma_1 < \gamma_2 < \alpha < \gamma_3$, the SP $\Sigma_{3,2,1}$ is realizible by HPs of the form $(1+\varepsilon x)Q_3(x)$, $(1+\varepsilon x)Q_4(x)$ and $(1+\varepsilon x)Q_2(x)$, respectively, where $\varepsilon > 0$ is small enough and the HPs $Q_j$ are the ones from Example 3. Indeed, the leading coefficient in all three cases equals $1/\varepsilon$ and the other coefficients are close to the ones of $Q_j$.

The quantity $\gamma_3$ equals $1/\varepsilon$.

Proposition 1. The SP $\Sigma_{3,2,1}$ is not realizable in the generic case $\beta < \gamma_1 < \gamma_2 < \gamma_3 < \alpha$.

Before proving the proposition, let us remind that making use of part (2) of Remarks 1 and knowing the answer about the SP $\Sigma_{3,2,1}$, one obtains the answer to the question in which generic cases the SP $\Sigma_{1,2,3}$ is realizible.

Proof. We consider the polynomial $P := x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$. Denote by $R_5$ the space of the coefficients $\{a_j\}$. We consider the set $\mathbb{R}^5 \supset U := \{ \Gamma := (\alpha, \beta, \gamma_1, \gamma_2, \gamma_3) \mid \beta < \gamma_1 < \gamma_2 < \gamma_3 < \alpha \}$

and its image $V$ in $\tilde{R}$ under the Vieta mapping which sends the quintuple $\Gamma$ into the quintuple of coefficients of the polynomial $(x-\alpha)(x-\beta)(x+\gamma_1)(x+\gamma_2)(x+\gamma_3)$ (excluding the leading coefficient 1). The closure $\bar{U}$ consists of $U$ and of quintuples $\Gamma$ for which at least one of the following equalities holds true:

$$\beta = 0, \quad \beta = \gamma_1, \quad \gamma_1 = \gamma_2, \quad \gamma_2 = \gamma_3, \quad \gamma_3 = \alpha.$$  \hfill (10)

Lemma 3. There exists no HP defining the SP $\Sigma_{3,2,1}$ and satisfying at least one of the equalities (10).

The lemma is proved after the proofs of Proposition 1 and Lemma 5. Thus if some HP defined by a quintuple $\Gamma_0$ defines the SP $\Sigma_{3,2,1}$, then $\Gamma_0$ belongs to the interior of $U$. The set $U$ being contractible, one can connect $\Gamma_0$ by a $C^1$-smooth path $\mathcal{P} \subset U$ with a quintuple $\Gamma_1$ from the interior of $U$ which realizes the SP $\Sigma_{3,2,1}$. Such a quintuple $\Gamma_1$ exists, see equality (9) and the lines that follow it.

The path $\mathcal{P}$ intersects at least one of the hyperplanes $\{a_j = 0\} \subset \tilde{R}$.

Lemma 4. The Vieta mapping is a local diffeomorphism of a neighbourhood of any point of the interior of $U$ onto its image.

The lemma is proved at the end of the paper. It implies that one can modify the path $\mathcal{P}$ so that at any point of $\mathcal{P}$ at most one equality of the form $a_j = 0$ holds true. Moreover, one can parametrize $\mathcal{P}$ by $t \in [0,1]$ so that for any point satisfying the equality $a_j = 0$, there exists an open interval $(u,v) = J \subset [0,1]$ such that
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(i) \( a_j = 0 \) for \( t = (u + v)/2 \),
(ii) \( a_j \neq 0 \) for \( t \in J \setminus \{(u + v)/2\} \),
(iii) \( a_j \) has different signs for \( t \in (u, (u + v)/2) \) and \( t \in ((u + v)/2, v) \) and
(iv) for \( t \in J \), there exists a single index \( j \) with \( a_j \) satisfying properties (i)–(iii).

One cannot have \( \gamma \) for \( t \) by a quintuple from \( U \) because then for \( t \) \( \in U \) by Lemma 3 no point of the boundary of \( U \) by Lemma 5.

Consider the point \( \Gamma^* \in P \) closest to \( \Gamma_0 \) for which one has \( a_j = 0 \) for some \( j \).

Lemma 5. There exists no degree 5 HP satisfying the conditions \( 0 < \beta < \gamma_1 < \gamma_2 < \gamma_3 < \alpha \) and \( a_4 > 0 \), \( a_3 = 0 \), \( a_2 < 0 \), \( a_1 < 0 \), \( a_0 > 0 \).

The lemma (whose proof follows) finishes the proof of Proposition 1. Indeed, by Lemma 3 no point of the boundary of \( U \) realizes the SP \( \Sigma_{3,2,1} \), and we just showed that this cannot be the case of a point \( \Gamma_0 \) from the interior of \( U \) either.

Proof of Lemma 5. Suppose that such an HP exists. Recall that we denote by \( e_1 \) and \( e_2 \) the quantities \( \gamma_1 + \gamma_2 + \gamma_3 \) and \( \gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3 \). Then \( a_4 = -\alpha - \beta + e_1 > 0 \), i.e., \( 0 < \beta < \alpha - e_1 \), and

\[
a_3 = \alpha \beta - (\alpha + \beta) e_1 + e_2 = 0, \quad \text{i.e.,} \quad \beta = (\alpha e_1 - e_2)/(\alpha - e_1).
\]

But \( \alpha e_1 - e_2 = (\alpha - \gamma_1)\gamma_2 + (\alpha - \gamma_2)\gamma_3 + (\alpha - \gamma_3)\gamma_1 > 0 \), while \( \alpha - e_1 < 0 \). Hence \( \beta < 0 \), a contradiction. \( \Box \)

Proof of Lemma 3. Suppose that such an HP \( T \) exists. Then \( \beta > 0 \), otherwise \( T(0) = 0 \) and \( T \) does not define the SP \( \Sigma_{3,2,1} \). Hence \( \alpha > 0 \) and \( \gamma_j > 0 \), \( j = 1, 2 \) and 3.

Suppose that \( \beta = \gamma_1 \). Set

\[
F := (x + \gamma_2)(x + \gamma_3)(x - \alpha) = x^3 + Ax^2 + Bx + C.
\]

Then \( B = -(\gamma_2 + \gamma_3)\alpha + \gamma_2 \gamma_3 \) and

\[
T = (x^2 - \beta^2)F = x^5 + Ax^4 + (B - \beta^2)x^3 + (C - \beta^2 A)x^2 - \beta^2 Bx - \beta^2 C.
\]

The condition \( B - \beta^2 > 0 \) implies

\[
(\gamma_2 + \gamma_3)\alpha < \gamma_2 \gamma_3 - \beta^2 < \gamma_2 \gamma_3.
\]
However, \( \gamma_2 + \gamma_3 \alpha > \gamma_2 \alpha \geq \gamma_2 \gamma_3 \), which is a contradiction.

Suppose that \( \alpha = \gamma_3 \). Set 
\[
G := (x + \gamma_1)(x + \gamma_2)(x - \beta) = x^3 + A^* x^2 + B^* x + C^*.
\]
Then \( B^* = - (\gamma_1 + \gamma_2) \beta + \gamma_1 \gamma_2 \) and 
\[
T = (x^2 - \alpha^2) G = x^5 + A^* x^4 + (B^* - \alpha^2) x^3 + (C^* - \alpha^2 A^*) x^2 - \alpha^2 B^* x - \alpha^2 C^*.
\]
On the one hand, one must have \( B^* - \alpha^2 > 0 \), but on the other, 
\[
B^* - \alpha^2 = -(\gamma_1 + \gamma_2) \beta + (\gamma_1 \gamma_2 - \alpha^2) < 0,
\]
which is a contradiction. Suppose that \( \gamma_j = \gamma_{j+1} = g \), where \( j = 1 \) or \( 2 \). We set 
\[
T := (x - \alpha)(x - \beta)(x + g)^2 (x + h) = x^5 + M x^4 + N x^3 + \cdots,
\]
where \( h = \gamma_3 \) if \( j = 1 \), and \( h = \gamma_1 \) if \( j = 2 \). Then 
\[
M = -\alpha - \beta + 2g + h \quad \text{and} \quad N = \alpha \beta - 2g \alpha - 2g \beta + g^2 - h \alpha - h \beta + 2gh.
\]
Hence \( N^0 := N + \beta M = -2g \alpha + g^2 - \alpha g + 2gh - \beta^2 > 0 \). But for \( j = 1 \), one has \( g \leq h \) and 
\[
N^0 = (-2g \alpha + 2gh) + (g^2 - h \alpha) - \beta^2 < 0,
\]
while for \( j = 2 \), one has \( h \leq g \) and 
\[
N^0 = (-\alpha g + gh) + (-h \alpha + gh) + (-g \alpha + g^2) - \beta^2 < 0,
\]
which is a contradiction. \( \square \)

**Proof of Lemma 4.** Consider the Vandermonde mapping 
\[
(\beta, \gamma_1, \gamma_2, \gamma_3, \alpha) \mapsto (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5),
\]
where \( \varphi_k = \beta^{5-k} + (-\gamma_1)^{5-k} + (-\gamma_2)^{5-k} + (-\gamma_3)^{5-k} + \alpha^{5-k} \). For each point of the interior of \( U \), this mapping defines a local diffeomorphism, because its determinant is, up to a constant nonzero factor, the Vandermonde determinant \( W(\beta, \gamma_1, \gamma_2, \gamma_3, \alpha) \neq 0 \). On the other hand, the quantities \( \varphi_k \) and \( a_k \) are connected with one another by formulas of the form 
\[
(5-k)a_k = -5\varphi_k + Q_k(\varphi_4, \ldots, \varphi_{k+1}), \quad 5\varphi_k = - (5-k)a_k + Q_k^*(a_4, \ldots, a_{k+1}),
\]
where \( Q_k \) and \( Q_k^* \) are polynomials. Hence the Vieta mapping also defines a local diffeomorphism. \( \square \)
7.3. The further aims of this research. The exploration of the case \( c = 2 \) for any possible values of \( m, n \) and \( q \) can be seen as a natural continuation of this research. The exhaustive answer to Question 2, even only for some finite number of values of \( n \) (say, \( n = 2 \) and \( n = 3 \)), would be of interest. Sufficient conditions for realizability can be formulated and proved with the help of the results of the present paper:

**Theorem 6.** Suppose that \( m > n \) and \( q > n \). Then for any possible representation of the number \( n \) in the form \( n = n_1 + n_2 + 1 \), \( n_1 \in \mathbb{N} \cup \{0\} \), \( j = 1, 2 \), all generic cases (i.e., without equalities between moduli of roots) with \( n^* = n_1^* + n_2^* \), \( m^* = m + n_1 - 1 - n_1^* \) and \( q^* = q + n_2 - 1 - n_2^* \), where \( 0 \leq n_j^* \leq 2n_j \), are realizable.

**Proof.** Suppose that the polynomials \( P_j, j = 1 \) and 2, of respective degrees \( m + n_1 \) and \( q + n_2 \), realize the generic cases \( \Sigma_{m,n_1+1} \) with \( n^* = n_1^* \leq 2n_1 \) and \( \Sigma_{q,n_2+1} \) with \( n^* = n_2^* \leq 2n_2 \), respectively. We remind that by Theorem 1 all cases with \( n_1^* \leq 2n_1 \) and \( n_2^* \leq 2n_2 \) are realizable. Then the polynomial \( P_3 := x^{q+n_2}P_2(1/x) \) (of degree \( q + n_2 \)) realizes the generic case \( \Sigma_{n_2+1,q} \) with \( m^* = n_2^* \), see part (1) of Remarks 1.

Consider the polynomial \( \tilde{P} := \varepsilon^{q+n_2}P_1(x)(-P_3(x/\varepsilon)) \), see Lemma 2. For \( \varepsilon > 0 \) small enough, the moduli of all the roots of \( P_3(x/\varepsilon) \) are smaller than the smallest of the moduli of the roots of \( P_1(x) \). Denote by \( \alpha_1 \) and \( \alpha_2 \) the positive roots of \( P_1(x) \) and \( P_3(x/\varepsilon) \) respectively. Hence there are exactly \( m + n_1 - 1 - n_1^* \) negative roots of \( P_1(x) \) whose moduli are larger than \( \alpha_1 \), exactly \( n_1^* \) negative roots of \( P_1(x) \) and \( n_2^* \) negative roots of \( P_3(x/\varepsilon) \) whose moduli belong to the interval \((\varepsilon\alpha_2, \alpha_1)\), and exactly \( q + n_2 - 1 - n_2^* \) negative roots of \( P_3(x/\varepsilon) \) with moduli smaller than \( \varepsilon\alpha_2 \). This proves the theorem. \( \square \)

Acknowledgements. The author is grateful to the anonymous referee for his/her useful remarks on the text.

References

[1] A. Albouy and Y. Fu, Some remarks about Descartes’ rule of signs, *Elem. Math.* 69 (2014), 186–194.

[2] B. Anderson, J. Jackson and M. Sitharam, Descartes rule of signs revisited, *Amer. Math. Monthly* 105 (1998), 447–451.

[3] M. Avendaño, R. Kogan, M. Nisse and J. M. Rojas, Metric estimates and membership complexity for Archimedean amoebae and tropical hypersurfaces, *J. Complexity* 46 (2018), 45–65.
[4] J. Forsgård, On the multivariate Fujiwara bound for exponential sums, arXiv:1612.03738.
[5] J. Forsgård, V. P. Kostov and B. Shapiro, Could René Descartes have known this?, Exp. Math. 24 (2015), 438–448.
[6] J. Forsgård, D. Novikov and B. Shapiro, A tropical analog of Descartes’ rule of signs, Int. Math. Res. Not. IMRN 12 (2017), 3726–3750.
[7] D. J. Grabiner, Descartes rule of signs: another construction, Amer. Math. Monthly 106 (1999), 854–856.
[8] V. P. Kostov, On realizability of sign patterns by real polynomials, Czechoslovak Math. J. 68 (143) (2018), 853–874.
[9] V. P. Kostov, Polynomials, sign patterns and Descartes’ rule of signs, Math. Bohem. 144 (2019), 39–67.
[10] V. P. Kostov, Topics on Hyperbolic Polynomials in One Variable, Société Mathématique de France, Paris, 2011.

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(Received April 25, 2019; revised July 17, 2019)