AN ANISOTROPIC WORMHOLE:
TUNNELLING IN TIME AND SPACE

Marco Cavaglià(a,c), Vittorio de Alfaro(a) and Fernando de Felice(b)

(a) Dipartimento di Fisica Teorica dell’Università di Torino
and INFN, Sezione di Torino, Italy
(b) Istituto di Fisica Matematica dell’Università di Torino
(c) Present address: ISAS, Trieste

ABSTRACT
We discuss the structure of a gravitational euclidean instanton obtained through coupling of gravity to electromagnetism. Its topology at fixed t is $S^1 \times S^2$. This euclidean solution can be interpreted as a tunnelling to a hyperbolic space (baby universe) at $t = 0$ or alternatively as a static wormhole that joins the two asymptotically flat spaces of a Reissner–Nordström type solution with $M = 0$.

Mail Address:
Dipartimento di Fisica Teorica
Via Giuria 1, I-10125 Torino
Electronic mail:VAXTO::VDA or VDA@TORINO.INFN.IT
1. Introduction.

Wormholes (WH) are classical euclidean solutions for the gravitational field coupled to matter or gauge fields, that connect two asymptotic fourdimensional manifolds; they are interpreted as tunnelling between the two asymptotic configurations. If a WH can be joined at $t = 0$ to a compact, hyperbolic universe, the euclidean solution can be interpreted as nucleating a baby universe (BU) from the asymptotic region and gives the semiclassical amplitude for creating a BU in the original space. The BU then evolves according to its equations of motion.

A large amount of attention has been devoted to the explicit solution of WH solutions. In particular, Giddings and Strominger [1] and Myers [2] have discussed WHs generated from coupling the gravitational field to an antisymmetric tensor of rank three (the axion), with topology $R \times S^3$; Halliwell and Laflamme [3] have discussed solutions in presence of a conformal massless field, and Coule and Maeda [4] have examined the case of the axion field coupled to a scalar Klein–Gordon field (in both cases with topology $R \times S^3$); Hawking [5] and Hosoya and Ogura [6] have dealt with gravity coupled to a Yang–Mills field. The magnetic monopole solution in four dimensions has been investigated by Keay and Laflamme [7]; its topology is $R \times S^1 \times S^2$.

In this paper we shall investigate a different WH solution of topology $R \times S^1 \times S^2$ generated by the electromagnetic (EM) field.

We shall first present the euclidean solution for the gravitational and the EM fields in the case of zero and non zero cosmological constant; then we discuss its analytic continuation at $t = 0$ into a BU.

A different continuation in hyperbolic space leads to an alternative interpretation: a Reissner–Nordström (RN) type static solution with $M = 0$ which is joined by the static wormhole to a second RN space. We briefly discuss the behaviour of charged particles and their crossing of the WH.

According to the usual interpretation, then, this is evidence for a quantum tunnelling: the WH yields the amplitude for transition from a RN space into a RN isometric space. We discuss in detail the transition probability for particles crossing between the two spaces.

This way of looking at the WH as a quantum bridge connecting two classical hyperbolic spaces opens the way to the interesting speculation that singularities in the classical domain of physical, hyperbolic solutions in general relativity can be avoided by euclidean solutions joining two spaces, as it happens in the R–N case that we discuss here.

2. The euclidean solution.

Let us start from the euclidean action for gravity minimally coupled to the EM
field:

$$S_E = \int_{\Omega} d^4x \sqrt{g} \left[ - \frac{M_p^2}{16\pi G} (R + 2\Lambda) + \frac{1}{4e^2} F^2 \right] + \int_{\partial\Omega} d^3x \sqrt{h} \frac{K}{8\pi G}.$$  \hfill (2.1)

Here $\Omega$ is a compact fourdimensional manifold, $M_p$ is the Planck mass, $R$ is the curvature scalar, $\Lambda$ is the cosmological constant, $F = F_{\mu\nu}F^{\mu\nu}$ the usual EM lagrangian, $K$ is the trace of the extrinsic curvature of the boundary $\partial\Omega$ of $\Omega$ and $h$ is the determinant of the induced metric over $\partial\Omega$. For $g_{\mu\nu}$ we look for a solution of the form

$$ds^2 = dt^2 + a^2(t)d\chi^2 + b^2(t)d\Omega_2^2$$ \hfill (2.2)

where $\chi$ is the coordinate of the 1-sphere, $0 \leq \chi < 2\pi$ and $d\Omega_2^2$ represents the line element of the 2-sphere. Let us first discuss the case $\Lambda = 0$. For the EM field we choose the Ansatz

$$A_\mu = A(t)\delta_{\chi\mu}.$$ \hfill (2.3)

The only non vanishing component of the EM field is of course

$$F_{t\chi} = -F_{\chi t} = \dot{A}(t).$$ \hfill (2.4)

From the equation of motion

$$\partial_\mu (\sqrt{g}F^{\mu\nu}) = 0$$ \hfill (2.5)

we obtain

$$\dot{A} = K \frac{a}{b^2}$$ \hfill (2.6)

($K$ is an integration constant). Substituting (2.6) into (2.1) one recovers after some algebra (details are given in the appendix) the scale factors of the 1- and 2-sphere. The solution is

$$ds^2 = dt^2 + \tilde{c}^2(t) d\chi^2 + b^2(t) d\Omega_2^2,$$ \hfill (2.7a)

$$A(t) = -\frac{\tilde{c}ceM_p}{2\sqrt{\pi}} \frac{1}{\sqrt{c^2 + t^2}}.$$ \hfill (2.7b)

$c$ is connected to $K$ by

$$c^2 = \frac{4\pi K^2}{e^2 M_p^2};$$

$\tilde{c}$ is an integration constant with dimension of length whose value will remain arbitrary.

For $t^2 \to \infty$, $a^2 \to \tilde{c}^2$ and $b^2 \to t^2$; these asymptotic behaviours ensure that the solution can be interpreted as a WH connecting two asymptotic flat space regions:

$$ds^2 = dt^2 + \tilde{c}^2 d\chi^2 + t^2 d\Omega_2^2.$$ \hfill (2.8)
At $t = 0$ the metric is singular. This is only due to the choice of the coordinates, that cover only half of the WH. Indeed, in the neighbourhood of $t = 0$, redefining the variable $\chi$, the line element becomes ($\bar{c} = c$)

$$ds^2 = dt^2 + t^2 d\chi^2 + c^2 d\Omega_2^2.$$  \hspace{1cm} (2.9)

In the neighbourhood of $t = 0$ our solution coincides with an euclidean Kasner universe [8]. Naturally the singularity at $t = 0$ can be eliminated going to cartesian coordinates in the $(t, \chi)$ plane. This particular case of singularity removable by a different choice of coordinates has been classified by Gibbons and Hawking [9] as a ‘bolt’ singularity. In the neighbourhood of $t = 0$ the topology is locally $R^2 \times S^2$ with $R^2$ contracting to zero as $t \to 0$. New variables can be defined such that the whole euclidean space is represented by a single chart. Let us define

$$t = c \tan \frac{\xi}{2},$$ \hspace{1cm} (2.10)

$\xi$ is defined in the interval $(-\pi, \pi)$. Introduce the new coordinates $u, v$ as

$$u = \frac{1 - \cos \frac{\xi}{2}}{\sin \frac{\xi}{2}} e^{1/\cos \frac{\xi}{2}} \cos \chi,$$ \hspace{1cm} (2.11a)

$$v = \frac{1 - \cos \frac{\xi}{2}}{\sin \frac{\xi}{2}} e^{1/\cos \frac{\xi}{2}} \sin \chi.$$ \hspace{1cm} (2.11b)

The expression of the line element shows that there is no singularity:

$$ds^2 = c^2 \left(1 \pm \frac{c}{\sqrt{t^2 + c^2}}\right)^2 e^{-2\sqrt{t^2 + c^2}/c}(du^2 + dv^2) + (c^2 + t^2)d\Omega_2^2.$$ \hspace{1cm} (2.12)

From

$$u^2 + v^2 = \frac{1 - \cos \frac{\xi}{2}}{1 + \cos \frac{\xi}{2}} e^{2/\cos \frac{\xi}{2}},$$ \hspace{1cm} (2.13a)

$$\frac{v}{u} = \tan \chi,$$ \hspace{1cm} (2.13b)

we see that the geodesic at constant $\chi$ is a segment of a line passing through the origin; geodesics at fixed $t$ are circumferences of radius

$$r = \sqrt{\frac{1 - \cos \frac{\xi}{2}}{1 + \cos \frac{\xi}{2}} e^{1/\cos \frac{\xi}{2}}}.$$ \hspace{1cm} (2.14)

In section 5, eq.s (2.7) will be the starting point of the discussion of the RN interpretation.
3. Non vanishing cosmological constant.

We start from (2.1). Using the same Ansatz (2.3) for the EM field, a solution is now given by

\[ ds^2 = \frac{b^2}{\lambda b^4 + b^2 - c^2} db^2 + \hat{c}^2 \frac{\lambda b^4 + b^2 - c^2}{b^2} d\chi^2 + b^2 d\Omega_2^2, \]

(3.1a)

\[ A(t) = -\frac{K}{b}, \]

(3.1b)

with \( \lambda = \Lambda / 3 \). This solution reduces to (2.7) for vanishing \( \lambda \) with the substitution \( b^2 = c^2 + t^2 \).

Let us separately discuss \( \lambda > 0 \) and \( \lambda < 0 \). In the first case it is easy to see that eq. (3.1a) is defined for \( b^2 > \hat{c}^2 = \frac{\sqrt{1 + 4\lambda c^2} - 1}{2\lambda} \).

(3.2)

Using the transformation \( b^2 = \hat{c}^2 + t^2 \)

(3.3)

eq (3.1a) takes the form

\[ ds^2 = \frac{t^2}{\lambda(\hat{c}^2 + t^2)^2 + \hat{c}^2 + t^2 - c^2} dt^2 + \hat{c}^2 \frac{\lambda(\hat{c}^2 + t^2)^2 + \hat{c}^2 + t^2 - c^2}{\hat{c}^2 + t^2} d\chi^2 + (\hat{c}^2 + t^2) d\Omega_2^2 \]

(3.4)

where now \(-\infty < t < +\infty\).

The asymptotic form of (3.4) for \( t^2 \to \infty \) is

\[ ds^2 = \frac{1}{\lambda t^2} dt^2 + \lambda \hat{c}^2 t^2 d\chi^2 + t^2 d\Omega_2^2. \]

(3.5)

This is not a flat euclidean space. Let us redefine the euclidean time by

\[ |t| = \exp(\sqrt{\lambda t^2}) \]

(3.6)

so that

\[ ds^2 = dt^2 + e^{2\sqrt{\lambda t^2}} (\lambda \hat{c}^2 d\chi^2 + d\Omega_2^2). \]

(3.7)

This line element defines an anisotropic universe whose scale factors expand exponentially; their ratio is fixed by the value of the cosmological constant.

For \( \lambda < 0 \) (3.1a) is defined when

\[ \hat{c}^2_a < b^2 < \hat{c}^2_b \]

(3.8)
where
\[
\tilde{c}_a^2 = \frac{1 - \sqrt{1 - 4|\lambda|c^2}}{2|\lambda|}, \\
\tilde{c}_b^2 = \frac{1 + \sqrt{1 - 4|\lambda|c^2}}{2|\lambda|},
\]
and (3.1a) can be interpreted like a tunnelling between two regions that are the hyperbolic continuations of the metric (3.1a) outside of the range (3.9).

4. The solution in hyperbolic spacetime.

Half of our instanton can be joined to a real, hyperbolic signature universe; this is the *bounce* solution of the tunnelling.

Let us first investigate hyperbolic solutions of the coupled gravity and EM field with the same symmetry as already investigated in the euclidean case. The solution is given by

\[
ds^2 = -dt^2 + \frac{t^2}{c^2 - t^2}d\chi^2 + (c^2 - t^2)d\Omega_2^2.
\]

The line element (4.1) describes an anisotropic universe that lapses $2c$ in time, born at $t = -c$ as a *spaghetti* configuration; at $t = 0$ contracts into a *pancake* and again tends to *spaghetti* for $t \to c$. The EM field is imaginary, however the field strength vanishes at $t = 0$, so the joining is possible. It follows that the universe $S^2 \times S^1$ created at $t = 0$ by tunnelling cannot propagate in time by EM alone.

The problem is then which engine could power a BU obtained by tunnelling described by half of our WH instanton solution. To this aim we can exploit an interesting hyperbolic solution of the same symmetry, driven by the axionic field. This solution has been obtained by Keay and Laflamme [7] and makes use of the axionic field whose energy density is negative in hyperbolic space.

Using the form (2.2) for the line element and introducing the axion field

\[
H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}h(t)
\]
from the equation of motion one gets

\[
h(t) = \frac{C}{ab^2},
\]
where $C$ is a constant. The solution for the line element has the form ($a$ is a constant with dimension of length)

\[
ds^2 = \frac{b^2}{\lambda b^4 + b^2 - c^2}db^2 + a^2 d\chi^2 + b^2 d\Omega_2^2
\]
and
\[ c'^2 = \frac{48\pi}{M_p^2} C^2. \]

At \( t = 0 \) and \( \lambda = 0 \) eq. (4.4) becomes
\[ ds^2 = a^2 d\chi^2 + c'^2 d\Omega_2^2. \] (4.5)

Let us define \( y^2 = t^2 - \epsilon^2 \) where \( \epsilon \) is some constant, hence using this into (2.7a) we obtain:
\[ ds^2 = \frac{y^2}{y^2 + \epsilon^2} dy^2 + c^2 \frac{y^2 + \epsilon^2}{y^2 + \epsilon^2 + c^2} d\chi^2 + (y^2 + \epsilon^2 + c^2) d\Omega_2^2. \] (4.6)

At \( y = 0 \) we have
\[ ds^2 = \bar{c}^2 \frac{\epsilon^2}{\epsilon^2 + c^2} d\chi^2 + (\epsilon^2 + c^2) d\Omega_2^2. \] (4.7)

Identifying
\[ \bar{c}^2 \frac{\epsilon^2}{\epsilon^2 + c^2} = a^2 \]
and
\[ \epsilon^2 + c^2 = c'^2 \]
we join our solution to the hyperbolic universe:
\[ ds^2 = -dt^2 + \bar{c}^2 \frac{\epsilon^2}{\epsilon^2 + c^2} d\chi^2 + (\epsilon^2 + c^2 - t^2) d\Omega_2^2. \] (4.8)

In the general case \( \lambda \neq 0 \) the euclidean line element (4.4) can be interpreted as a tunnelling to a hyperbolic universe
\[ ds^2 = -\frac{t^2}{t^2 - \lambda(c^2 - t^2)} dt^2 + \bar{c}^2 \lambda c^2 d\chi^2 + (c^2 - t^2) d\Omega_2^2. \] (4.9)

Let us now compute the amplitude for the formation of the BU in the case \( \Lambda = 0 \), starting from (2.1). Since \( R = 0 \), the Euclidean action is given by
\[ S_E = \frac{1}{4e^2} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu}. \] (4.10)

Remembering the form of the solution (2.7) we obtain for the nucleation of the BU at \( t = 0 \)
\[ S_E = \frac{4\pi^2 K^2}{e^2} \int_0^{+\infty} dt \frac{a}{b^2} \]
\[ = \pi M_p^2 \bar{c}c \\
= 2\pi^{3/2} \bar{c} \frac{K}{e\sqrt{G}}. \] (4.12)
The probability \( \Gamma \) of formation of the BU in a Planck volume in a Planck time is

\[
\Gamma = e^{-S_E} = \exp(-\pi M_p^2 \bar{c}c).
\] (4.13)

In order to have a finite probability, the order of magnitude of the constants appearing in the solution is

\[
\bar{c}c \approx \frac{1}{\pi} l_p^2.
\] (4.14)

5. The static wormhole interpretation.

The solution (2.7) can also be interpreted as an Euclidean wormhole solution joining two isometric, asymptotically flat space-times, described by a RN type of solution.

To obtain this solution, let us make a change of coordinates in (2.2) by substituting \( \chi \to iT \) and taking \( T \) with dimension of length; for clarity we shall also put \( t \equiv r \). Throughout this section we shall use geometrized units (velocity of light and gravitational constant equal to one). The solution we so obtain describes a hyperbolic space-time:

\[
ds^2 = dr^2 - \frac{r^2}{r^2 - Q^2}dT^2 + (r^2 - Q^2)d\Omega_2^2,
\] (5.1a)

\[
A(r) = -\frac{K}{\sqrt{r^2 - Q^2}},
\] (5.1b)

where \( Q \) is a constant. The solution (5.1) is defined for \( r^2 > Q^2 \). In this case, denoting \( R^2 \equiv r^2 - Q^2 \), we obtain

\[
ds^2 = -\left(1 + \frac{Q^2}{R^2}\right)dT^2 + \left(1 + \frac{Q^2}{R^2}\right)^{-1}dR^2 + R^2 d\Omega_2^2.
\] (5.2)

This resembles a RN metric with source mass \( M = 0 \) and radial coordinate \( R \) ranging now as \( \infty > R \geq 0 \). At \( R = 0 \) there is a true singularity.

In the region \(|r| < |Q|\) we have no hyperbolic solution, and the overall picture is that of two branches of the RN solution joined by the euclidean wormhole solution (2.7) that is regular in this region. Formally, the solution (5.1) can be obtained from (2.7) by \( \bar{c}\chi \to iT, \ c \to iQ \) since the only non vanishing component of the vector potential (2.3) is \( A_\chi \).

The condition \( M = 0 \) is rather peculiar since it implies that the contribution to the gravitational mass of the source from the electric charge is also zero \[10,11\], hence the charge itself must be zero. This is consistent with equation (2.5) which implies no charges and currents everywhere in the domain of definition of the solution (5.2).
The meaning of the parameter \( Q \) entering (5.2) requires some care. Since the electric field is radial in \( R \), its integral flux through a sphere containing the origin \( R = 0 \) is equal to
\[
\Phi = 4\pi Q \quad (5.3)
\]
The constant \( Q \) then measures the electric field flux density through the wormhole throat at \( R = 0 \). Since there are no physical charges in the field, the constant \( Q \) only fixes, as boundary conditions for equation (2.5), the amount of flux that we want through any given surface containing the origin, similar to what is done for the axionic field in [1]. Thus, since there is no support for charges and matter (hence \( M = 0 \), the electric field can extend smoothly beyond the wormhole throat to the asymptotic infinity of the isometric space-time with \( R < 0 \), generating in both cases an apparent charge \( Q \).

The traversability of the wormhole is at the basis of the very usefulness of the wormhole concept; in order to cross the wormhole, a classical particle must be able to reach it. Let us then study this aspect of the solution introduced above.

Consider the equation of motion for a test particle, having an electric charge per unit mass \( q \), total specific energy \( E \) and specific angular momentum \( L \) with respect to the flat infinity, that approaches \( R = 0 \). This is relevant since the particle can cross the wormhole throat only if it gets \( R = 0 \) (classically or via quantum tunnelling). We assume the motion in the equatorial plane, \( \theta = \pi/2 \).

The momenta and equations of motion are
\[
\begin{align*}
P_T &\equiv -\left( \frac{R^2 + Q^2}{R^2} \dot{T} + \frac{\beta}{R} \right) = -E \\
P_R &\equiv \frac{R^2}{R^2 + Q^2} \dot{R} \\
P_\phi &\equiv R^2 \dot{\phi} = L \\
\dot{R}^2 &\equiv \left( E - \frac{\beta}{R} \right)^2 - \left( 1 + \frac{L}{R^2} \right) \left( 1 + \frac{Q^2}{R^2} \right) \\
&\equiv (E - V_+)(E - V_-) \quad (5.4d)
\end{align*}
\]
where \( V_\pm \) are the potential barriers given by [11]
\[
V(R; \beta, Q, L)_\pm = R^{-2} [\beta R \pm (R^2 + Q^2)^{1/2} (R^2 + L)^{1/2}] \quad (5.5)
\]
with \( \beta = qQ \). We shall study analytically the graph of the function \( V_+ \). The behaviour of \( V_- \) is easily deduced from the relation
\[
V_-(\beta) = -V_+(\beta). \quad (5.6)
\]

The analysis of the potential barriers (5.5) with \( L \neq 0 \) shows that the barriers are repulsive for all values of the parameters. On the contrary when \( L = 0 \),
namely when the motion is strictly radial, there is a class of trajectories which can reach the wormhole throat \((R = 0)\). We shall discuss extensively this latter case, referring to App. B for the general situation.

The potential barrier \(V_+\) for \(L = 0\) reads, from (5.5):

\[
V_+(R; \beta, Q) = R^{-1} \left[ \beta + (R^2 + Q^2)^{1/2} \right].
\] (5.7)

When \(R \to \infty\), \(V_+\) behaves as

\[
V_+ \approx 1 + \frac{\beta}{R}
\]

while when \(R \to 0\) we have

\[
V_+ \approx \frac{\beta + |Q|}{R} \rightarrow \begin{cases} +\infty & \beta > -|Q| \\ 0 & \beta = -|Q| \\ -\infty & \beta < -|Q| \end{cases}
\] (5.8)

The conditions \(V_+ = 1\) and \(V_+ = 0\) are satisfied respectively when \(\beta = \beta_1 \equiv R - (R^2 + Q^2)^{1/2}\) and \(\beta = \beta_0 \equiv -(R^2 + Q^2)^{1/2}\). They are shown in figure 1a, while the graphs of \(V_+\) are shown in figure 1b for the cases \(\beta > -|Q|\), \(\beta = -|Q|\) and \(\beta < -|Q|\).

We may repeat the analysis for the case of \(V_-\) using the symmetry (5.6). The conclusion is that the point \(R = 0\) can be reached if and only if

\[
\beta^2 \geq Q^2.
\] (5.9)

In this case the particles may reach the euclidean wormhole and eventually emerge in the \(R < 0\) Universe [10, 12]. The transition probability \(T_{WH}\) for tunnelling by the Euclidean wormhole is proportional to \(\exp(-2S_{cl})\) where \(S_{cl}\), using (4.12) and remembering that \(a\) is positive (see (A.10)) is given by

\[
S_{cl} = \pi c \tilde{c} M_p^2 \frac{\sqrt{2} - 1}{\sqrt{2}}
\] (5.10)

The transition probability characterizes the wormhole and is independent of the particle’s properties since the latters, provided they satisfy (5.4), all reach the wormhole throat, regardless of their energy.

The radial particles which do not satisfy (5.9) or those which have a non zero angular momentum, may cross the wormhole reaching \(R = 0\) as a result of a quantum tunnelling with non zero quantum probability. Indeed, let us go back
to eq. (5.4d) and use it to establish the equation for the wave function taking into account that $P_R \rightarrow -i d/dR$ is given by eq. (5.4b):

$$-\frac{d^2}{dR^2}\Psi = \frac{1}{(R^2 + Q^2)^2} \left[ R^2(RE - \beta)^2 - (R^2 + L)(R^2 + Q^2) \right] \Psi \quad (5.11).$$

The particle can reach $R = 0$ by quantum barrier penetration; at $R = 0$ the solution is joined to the euclidean WH connecting to the second region $(R < 0$ for convenience). The overall transition probability is given by

$$T = T_0^2 T_{WH} \quad (5.12)$$

where $T_{WH}$ is due to the tunnelling by the euclidean wormhole and $T_0$ is the usual quantum transition probability of the barrier from $R = R_0$ to $R = 0$.

For the evaluation of $T_0$ in the WKB approximation the relevant quantity is

$$\alpha = \int_0^{R_0} dR \frac{1}{R^2 + Q^2} \left[ (R^2 + L)(R^2 + Q^2) - R^2(ER - \beta)^2 \right]^{1/2} \quad (5.13)$$

which is finite and has a particularly simple expression for $L = 0$.

In conclusion, in the present section we have used the euclidean solution in order to obtain a finite traversability amplitude from a RN space-time into an isometric space-time; we may call this a space-tunnelling WH. This also yields an interpretation of the $M = 0$ RN space-time.

Our interpretation of a space-tunnelling WH is in the direction of the proposal by John Wheeler [13; see also 14] of a pair of oppositely charged extreme RN black holes identified at their throats, in an electromagnetic field. In the present case the joining of two RN space-times happens through quantum tunnelling.

Acknowledgments

We are glad to thank Paolo Ciatti for interesting discussions.

Appendix A.

The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \quad (A.1)$$

Substituting eqs. (2.4) in the expression for $T_{\mu\nu}$ we may find its components in terms of the scale factors:

$$T_{tt} = \frac{1}{2e^2} \frac{K^2}{b^4},$$

$$T_{\chi\chi} = \frac{1}{2e^2} \frac{K^2}{b^4} a^2, \quad (A.2)$$

$$T_{ij} = -\frac{1}{2e^2} \frac{K^2}{b^4} g_{ij},$$
The ensuing equations for the two scale factors are then
\[
\frac{\dot{b}^2}{b^2} - \frac{1}{b^2} + 2\frac{\dot{a}\dot{b}}{ab} = \frac{c^2}{b^4}, \quad (A.3a)
\]
\[
2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} = \frac{c^2}{b^4}, \quad (A.3b)
\]
\[
\frac{\ddot{b}}{b} + \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} = -\frac{c^2}{b^4}. \quad (A.3c)
\]

In (A.3b) only \(b(t)\) appears; by the substitution \(\dot{b} = p\), (A.3b) takes the form
\[
f' + \frac{1}{b}f - \frac{1}{b} = \frac{c^2}{b^3} \quad (A.4)
\]
where \(f = p^2\) and we have used \(\ddot{b} = pp'\). Putting then \(b = e^h\) we get
\[
\frac{df}{dh} + f = 1 - c^2 e^{-2h} \quad (A.5)
\]
whose general solution is
\[
f = K_1 e^{-h} + 1 - c^2 e^{-2h}. \quad (A.6)
\]
\(K_1\) is an integration constant. In what follows we shall simply take \(K_1 = 0\). In the old variables then
\[
\dot{b}^2 = 1 - \frac{c^2}{b^2}. \quad (A.7)
\]
So finally
\[
b(t) = \sqrt{c^2 + t^2}. \quad (A.8)
\]

By substitution of (A.8) into (A.3a) the equation for the remaining scale factor is
\[
\frac{\dot{a}}{a} = \frac{c^2}{t(c^2 + t^2)} \quad (A.9)
\]
whose solution is
\[
a(t) = \pm \bar{c} \frac{t}{\sqrt{c^2 + t^2}} \quad (A.10)
\]
where \(\bar{c}\) is an integration constant.

The signs \pm refer respectively to the submanifolds with \(t > 0\) or \(t < 0\), having defined \(a(t)\) as non negative.

Let us list for completeness the equations corresponding to (A.3) in the case of hyperbolic signature:
\[
\frac{\dot{b}^2}{b^2} + \frac{1}{b^2} + 2\frac{\dot{a}\dot{b}}{ab} = -\frac{c^2}{b^4}, \quad (A.12a)
\]
\[
2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} = -\frac{c^2}{b^4}, \quad (A.12b)
\]
\[
\frac{\ddot{b}}{b} + \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} = +\frac{c^2}{b^4}. \quad (A.12c)
\]
from which it is easy to obtain the line element (4.1)

From (A.12a) we may control that $T_{tt}$ is negative:

$$T_{tt} = -\frac{1}{2e^2} \frac{K^2}{b^4}. \quad (A.13)$$

The electromagnetic field is imaginary; thus this solution does not describe a real evolution in hyperbolic space.

**Appendix B.**

We give here the details of the equations of motion in the case $L \neq 0$. The asymptotic behaviour of $V_+$ for large $R$ is

$$V \approx \frac{\beta + R}{R} \rightarrow \left\{ \begin{array}{cl} 1_+ & \beta \geq 0 \\ 1_- & \beta < 0 \end{array} \right.$$  

For $R \rightarrow 0$ we have

$$V_+ \approx \frac{|Q| \sqrt{L}}{R^2} \rightarrow +\infty$$

The condition $V_+ = 1$ is equivalent to

$$\beta = R^{-1} \left[ R^2 - (R^2 + Q^2)^{1/2}(R^2 + L)^{1/2} \right] \equiv \beta_1. \quad (B.1)$$

Clearly $\beta_1 < 0$ always and $\lim_{R \rightarrow \infty} \beta_1 = 0$, $\lim_{R \rightarrow 0} \beta_1 = -\infty$. The function $\beta_1$ is plotted in figure 2b. The condition $V_+ = 0$ is satisfied when

$$\beta = -R^{-1} (R^2 + Q^2)^{1/2}(R^2 + L)^{1/2} \equiv \beta_0. \quad (B.2)$$

Here again $\beta_0 < 0$ always; the graph of $\beta_0(R)$ is easily deduced from its limits

$$\lim_{R \rightarrow \infty} \beta_0 = \lim_{R \rightarrow 0} \beta_0 = -\infty$$

and from the locus of its critical points, namely

$$L = R^4 Q^{-2} \equiv L_c.$$  

The function $L_c$ is plotted in figure 2a in the $(L - R)$-plane. From (B.1) and (B.2) we find $\beta_1 = R + \beta_0$ hence $\beta_1 \geq \beta_0$, the equality sign holding only in the limit $R \rightarrow 0$. The value of $\beta_0$ at its maximum is given by

$$\beta_0(R; Q, L_c) \equiv \beta_{0c} = -\frac{R^2 + Q^2}{|Q|}$$

and its graph is plotted in figure 2b (dashed line). We are now in the position to draw the potential curves $V_+(R; \beta, Q, L)$ as function of $R$ for any given set
of values $(\beta, Q, L)$. They are shown in figure 2c for three different values of $\beta$, namely 1) $\beta > 0$; 2) $\beta < 0$ and $\beta > \beta_{0c}$; 3) $\beta < 0$ and $\beta < -\beta_{0c}$.

The classical motion is only allowed when the total energy $E$ of the charged test particle satisfies the condition $E \geq V$, hence, when the angular momentum $L$ is different from zero, we see by a direct inspection of figure 2 that,

i) the wormhole throat $R = 0$ cannot be reached classically since the field is repulsive to all particles, either charged or not;

ii) a sea of negative energy particles is allowed in the vicinity of the throat. This effect is a well known property of the RN solution and allows for electric field energy extraction via quantum tunnelling.
References.

[1] Giddings, S. B. and A. Strominger, 1988, *Nucl. Phys.* B 306, 890.

[2] Myers, R. C., 1988, *Phys. Rev.* D 38, 1327.

[3] Halliwell, J. J. and R. Laflamme, 1989, *Class. Quantum Grav.* 6, 1839.

[4] Coule, D. H. and K. Maeda, 1990, *Class. Quantum Grav.* 7, 955.

[5] Hawking, S. W., 1987, *Phys. Lett.* B 195, 337.

[6] Hosoya, A. and W. Ogura, 1989, *Phys. Lett.* B 225, 117.

[7] Keay, B. J. and R. Laflamme, 1989, *Phys. Rev.* D 40, 2118.

[8] Misner C. W., K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, New York 1973.

[9] Gibbons, G. W. and S. W. Hawking, 1979, *Commun. Math. Phys.* 66, 291.

[10] Cohen J.M. and R. Gautreau, 1979, *Phys. Rev D* 19, 2273.

[11] de Felice F. and C.J.S. Clarke, 1990, *Relativity on Curved Manifolds*, Cambridge University Press, Cambridge, England.

[12] Graves J.C. and D.R. Brill, 1960, *Phys. Rev.* 120, 1507.

[13] Wheeler J.A., 1962 *Geometrodynamics*, Academic, New York.

[14] Garfinkle D. and A. Strominger, 1991, *Phys. Lett. B* 256, 146.
Figure Captions

Fig. 1 - a) Plot of the functions $\beta_1$ (solid line) and $\beta_0$ (dashed line) when $L = 0$. b) Behaviour of the effective potential $V_+$ when $\beta > -|Q|$ (solid line); $\beta = -|Q|$ (dashed); $\beta < -|Q|$ (dot-dashed).

Fig. 2 - a) Plot of the function $L_c$ which is the locus of the points where the function $\beta_0$ has a maximum. b) Plots of the function $\beta_1$ (solid line) which identifies where $V_+ = 1$ and of the function $\beta_0$ (dot-dashed) which identifies where $V_+ = 0$. The locus of the maxima of $\beta_0$ is along the dashed curve, plot of the function $\beta_{0c}$; c) behaviour of the effective potential $V_+$ as function of $R$. The solid line represents the case $\beta > 0$; dashed line for $\beta < 0$ and $\beta > \beta_{0c}$; dot-dashed when $\beta < 0$ and $\beta < \beta_{0c}$.