On Yang-Mills connections on compact Kähler surfaces

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Abstract

We extend an $L^2$-energy gap of Yang-Mills connections on principal $G$-bundles $P$ over a compact Riemannian manifold with a good Riemannian metric [6] to the case of a compact Kähler surface with a generic Kähler metric $g$, which guarantees that all ASD connections on the principal bundle $P$ over $X$ are irreducible.

1 Introduction

Let $G$ be a compact, semisimple Lie group and $P$ be a principal $G$-bundle over a closed, smooth, Riemannian manifold with Riemannian metric $g$. Suppose that $A$ is a connection on $P$ and its curvature denote by $F_A \in \Omega^2(X) \otimes g_P$. Here $\Omega^k := \Omega^k(T^*X)$ and $g_P$ is the real vector bundle associated to $P$ by the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$. We define the Yang-Mills energy function by

$$YM(A) := \int_X |F_A|^2 dvol_g,$$

the fiber metric defined through the Killing form on $g$, see [6] Section 2]. Then energy functional $YM(A)$ is gauge-invariant and thus descends to a function on the quotient space $B(P, g) := \mathcal{A}_P/\mathcal{G}_P$, of the affine space $\mathcal{A}_P$ of connections on $P$ moduli the gauge transformation. A connection $A$ is called Yang-Mills connection when it gives a critical point of the Yang-Mills functional, that is, it satisfies the Yang-Mills equation

$$d_A^* F_A = 0.$$

From the Bianchi identity $d_A F_A = 0$, a Yang-Mills connection is nothing but a connection whose curvature is harmonic with respect to the covariant exterior derivative $d_A$.

Over a 4-dimensional Riemannian manifold, $F_A$ is decomposed into its self-dual and anti-self-dual components,

$$F_A = F_A^+ + F_A^-.$$

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where $F^+_A$ denotes the projection onto the $\pm 1$ eigenspace of the Hodge star operator. A connection is called self-dual (respectively anti-self-dual) if $F_A = F^+_A$ (respectively $F_A = F^-_A$). A connection is called an instanton if it is either self-dual or anti-self-dual. On compact oriented 4-manifolds, an instanton is always an absolute minimizer of the Yang-Mills energy [21, 22]. Not all Yang-Mills connections are instantons, in [17, 20], the authors given some examples for the $SU(2)$ Yang-Mills connections on $S^4$ which are neither self-dual nor anti-self-dual.

It is a natural question whether or not there is a positive uniform gap between the energy $YM(A)$ of points $[A]$ in the stratum

$$M(P, g) := \{ [A] \in \mathcal{B}(P, g) : F^+_A = 0 \},$$

of absolute minimal of $YM(A)$ on $\mathcal{B}(P, g)$ and energies of points in the strata in $\mathcal{B}(P, g)$ of non-minimal critical points.

In [11,2], Bourguignon-Lawson proved that if $A$ is a Yang-Mills on a principal $G$-bundle over $S^4$ with its standard round metric of radius one such that $\|F^+_A\|_{L^\infty(X)} < \sqrt{3}$, then $A$ is anti-self-dual. The result was significantly improved by Min-Oo [15] and Parker [16], by replacing the preceding $L^\infty$ condition with an $L^2$-energy condition, $\|F^+_A\|_{L^2(X)} \leq \varepsilon$, where $\varepsilon = \varepsilon(g)$ is a small enough constant and by assume $X$ to be a closed, smooth, four-dimensional manifold endowed with a positive Riemannian metric $g$. In [6], Feehan extend the $L^2$-energy gap result from the case of positive Riemannian metrics [15,16] to the more general case of good Riemannian metrics. The key step in the proof of Feehan’s [6] Theorem 1 is to derive an uniform positive lower bound for the lower eigenvalue of the operator $d_A^*d_A^{+,*}$ with respect to the connection $A$, the curvature $F_A$ obeying $\|F^+_A\|_{L^2(X)} \leq \varepsilon$ for a suitable small constant $\varepsilon = \varepsilon(g)$.

In [8], the authors showed a sharp, conformally invariant improvement of these gap theorems which is nontrivial when the Yamabe invariant $Y([g])$ of $(X, g)$ is positive.

Denote by $\mathcal{A}_{YM}$ the space of Yang-Mills connections and $\mathcal{A}_{HYM}$ the space of connections whose curvature satisfies

$$\sqrt{-1} \Lambda \omega F_A = \lambda Id,$$

where $\lambda = \frac{2\pi \deg P}{\text{rank}(P) \text{vol}(X)}$. There spaces are gauge invariant with respect to the group $G_P$ of gauge transformations. In [9], the author proved that an open subset

$$W = \{ [A] : \|\sqrt{-1} \Lambda \omega F_A - \lambda Id\|_{L^2(X)} < \delta \}$$

in the orbit space $\mathcal{A}_P/G_P$ of connections with property $\mathcal{A}_{HYM}/G_P = W \cap \mathcal{A}_{YM}/G_P$ under the scalar curvature $S$ of the metric is positive.

Now if we suppose the base manifold $X$ is a Kähler surface, $P$ is a principal $SU(N)$-bundle over $X$. An ASD connection $A$ on $P$ naturally induces the Yang-Mills complex

$$\Omega^0(X, g_P) \xrightarrow{d^0 = d_A} \Omega^1(X, g_P) \xrightarrow{d^1 = d_A^*} \Omega^{2,+}(X, g_P).$$

The $i$-th cohomology group $H^i_A := \text{Ker}d^i/\text{Im}d^{i-1}$ of this complex if finite dimensional and the index $d = h^0 - h^1 + h^2 (h^i = \dim H^i_A)$ is given by $c(G)\kappa(P) - \dim G(1 - b_1 + b^+)$. Here $c(G)$ is a normalising
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constan t, \( \kappa(P) \) is a characteristic number of \( P \) obtained by evaluating a 4-dimensional characteristic class on the fundamental cycle \([X]\), \( b_1 \) is the first Betti number of \( X \) and \( b^+ \) is the rank of a maximal positive subspace for the intersection form on \( H^2(X) \). \( H^0_A \) is the Lie algebra of the stabilizer \( \Gamma_A \), the group of gauge transformation of \( P \) fixing by \( A \). From [12, Proposition 2.3] or [7, Chapter IV], the second cohomology \( H^2_A \) is \( \mathbb{R} \)-isomorphic to \( H^0_A \oplus H \), where \( H := \ker(\bar{\partial}^*_A)|_{\Omega^{0,2}(X, g_P^\ast)} \). It’s difficult to addition certain mild conditions to ensure \( H \) and \( H^0_A \) vanish at some time. The Kähler metric \( g \) often could not be good. But one can see that \( H^0_A = 0 \) is equivalent to the connection \( A \) is irreducible.

In [6], the author shown that if the close 4-manifold \( X \) admits a good metric \( g \), then the connection \( A \in B_\varepsilon(P, g) := \{[A] \in B(P, g) : \|F_A^+\|_{L^2(X)} \leq \varepsilon \} \) such that the last eigenvalue of \( d_A^+d_A^{+\ast} \) on \( L^2(\Omega^{2, +}(X, g_P)) \) has a lower positive bound, where \( \varepsilon = \varepsilon(g) \) is a suitable small positive constant. The purpose of this article is to introduce the definition of strong irreducible connection \( A \) which only guarantees that \( \lambda(A) \) has a low positive bound, See Definition 2.5.

If the Riemannian metric is good, there is a well known gluing theorem for anti-self-dual connection which due to Taubes [21]. Following the idea of Taubes’, if we suppose the connection \( A \in \mathcal{A}_P \) which obeys \( \|F_A^+\|_{L^2(X)} \leq \varepsilon \) for a suitable small positive constant and \( \lambda(A) \geq \lambda_0 > 0 \), then we could deform the connection \( A \) to an other connection \( A_\infty \) which satisfies \( \Lambda_A F_A = 0 \), see Corollary 3.5. The connection \( A_\infty \) may be not an ASD connection, but the \( (0, 2) \)-part \( F_{A_\infty}^{0,2} \) of the curvature \( F_A \) could estimated by \( \Lambda_A F_A \). Following the priori estimate in Theorem 3.1 and the vanishing Theorem 3.13 we have

**Theorem 1.1.** Let \( X \) be a compact, simply-connected, Kähler surface with a Kähler metric \( g \), \( P \) be a principal \( G \)-bundle with \( G \) being \( SU(2) \) or \( SO(3) \). Suppose that the connections \( A \in M(P, g) \) are strong irreducible ASD connections in the sense of Definition 2.5, then there is a positive constant \( \varepsilon = \varepsilon(g, P) \) with following significance. If the Yang-Mills connection \( A \) on \( P \) such that

\[
\|F_A^+\|_{L^2(X)} \leq \varepsilon,
\]

then \( A \) is anti-self-dual with respect to \( g \), i.e., \( F_A^+ = 0 \).

We may assume that any connection \( A \in M(P, g) \) is strong irreducible ASD connection in the sense of Definition 2.5 if the conditions in Theorem 2.11 are obeyed.

**Corollary 1.2.** Let \( X \) be a compact, simply-connected, Kähler surface with a Kähler metric \( g \), that is generic in the sense of Definition 2.9 \( P \) be a \( SO(3) \)-bundle over \( X \). Suppose that the second Stiefel-Whitney class, \( \omega_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \), is non-trivial, then there is a positive constant \( \varepsilon = \varepsilon(g, P) \) with following significance. If the curvature \( F_A \) of a Yang-Mills connection \( A \) on \( P \) obeying

\[
\|F_A^+\|_{L^2(X)} \leq \varepsilon,
\]

then \( A \) is anti-self-dual with respect to \( g \), i.e., \( F_A^+ = 0 \).

This paper is organised as follows. In Section 2, we first establish our notation and recall basic definitions in gauge theory over Kähler manifolds required for the remainder of this article. Following the idea
on [6], we prove that the least eigenvalue, \( \lambda(A) \), of \( d_A^* d_A \) has a positive lower bound \( \lambda_0 = \lambda_0(g, P) \) that is uniform with respect to \([A] \in \mathcal{B}(g, P)\) obeying \( \|F_A^+\|_{L^2(X)} \leq \varepsilon \), for a small enough \( \varepsilon = \varepsilon(g, P) \in (0, 1) \) and under the given sets of conditions on \( g, G \). In Section 3, using the similar way of construct of ASD connection by Taubes [21], we obtain that an approximate ASD connection \( F \) where we establish a vanishing theorem (Theorem 3.13) on the space of uniform with respect to \([g, G] \) and under the given sets of conditions on \( g, G \).

The energy functional \( \langle \Lambda_\omega F, \beta \rangle = \langle \alpha, L_\omega \beta \rangle \). Combining the curvature is harmonic, then we complete the proof of Theorem [1.1].

2 Preliminaries

2.1 Weitzenböck formula

Let \( X \) be a Kähler surface with Kähler form \( \omega \) and \( P \) be a principal \( G \)-bundle over \( X \). For any connection \( A \) on \( P \) we have the covariant exterior derivatives \( d_A : \Omega^k(X, g_P) \to \Omega^{k+1}(X, g_P) \). Like the canonical splitting the exterior derivatives \( d = \partial + \bar{\partial} \), decomposes over \( X \) into \( d_A = \partial_A + \bar{\partial}_A \). We denote also by \( \Omega^{p,q}(X, g_P^\mathbb{C}) \) the space of \( C^\infty-(p, q) \) forms on \( g_P^\mathbb{C} := g_P \otimes \mathbb{C} \). Denote by \( L_\omega \) the operator of exterior multiplication by the Kähler form \( \omega \):

\[
L_\omega \alpha = \omega \wedge \alpha, \alpha \in \Omega^{p,q}(X, g_P^\mathbb{C}),
\]

and, as usual, let \( \Lambda_\omega \) denote its pointwise adjoint, i.e.,

\[
\langle \Lambda_\omega \alpha, \beta \rangle = \langle \alpha, L_\omega \beta \rangle.
\]

Then it is well known that \( \Lambda_\omega = \#^{-1} \circ L_\omega \circ \# \). We could decompose the curvature, \( F_A \), as

\[
F_A = F_A^{2,0} + F_A^{1,1} + \frac{1}{2} \Lambda_\omega F_A \otimes \omega + F_A^{0,2},
\]

where \( F_A^{1,1} = F_A^{1,1} - \frac{1}{2} \Lambda_\omega F_A \otimes \omega \). We can write Yang-Mills functional as

\[
YM(A) = 4\|F_A^{0,2}\|^2 + \|\Lambda_\omega F_A\|^2 + \int_X tr(F_A \wedge F_A).
\]

The energy functional \( \|\Lambda_\omega F_A\|^2 \) plays an important role in the study of Hermitian-Einstein connections, see [5, 24]. If the connection \( A \) is an ASD connection, the Yang-Mills functional is minimum. We recall some identities on Yang-Mills connection over Kähler surface, see [12, Proposition 3.1] or [9, Proposition 2.1].
Proposition 2.1. Let $A$ be a Yang-Mills connection on a principal $G$-bundle $P$ over a Kähler surface $X$, then we have following identities:

1. $2\bar{\partial}_A F_A^{0,2} = \sqrt{-1} \partial_A \Lambda \omega F_A$
2. $2\bar{\partial}_A F_A^{2,0} = -\sqrt{-1} \partial_A \Lambda \omega F_A$

We define a Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\Omega^{p,q}(X, g_P^C)$ by

$$\langle \alpha, \beta \rangle_{L^2(X)} = \int_X \langle \alpha, \beta \rangle(x) dvol_g,$$

where $*$ is the $\mathbb{C}$-linearly extend Hodge operator over complex forms and $\bar{\cdot}$ is the conjugation on the bundle $g_P^C$-forms which is defined naturally. One also can see [13, Page 99] or [11]. We recall a Weitzenböck formula for Lie algebra-valued $(0, 2)$-forms, see [12, Proposition 2.3], a self-dual operator denote by $\Delta_{\partial_A} = \bar{\partial}_A \bar{\partial}_A + \partial_A \partial_A$.

Proposition 2.2. Let $X$ be a smooth Kähler surface with a Kähler metric $g$, $A$ be a connection on a principal $G$-bundle $P$ over $X$. For any $\phi \in \Omega^{0,2}(X, g_P^C)$,

$$\Delta_{\partial_A} \phi = \nabla_A^* \nabla_A \phi + [\sqrt{-1} \Lambda \omega F_A, \phi] + 2S \phi$$

where $S$ is the scalar curvature of the metric $g$.

Combining Weitzenböck formula on Proposition 2.2 with the identities on Proposition 2.1, we have an identity for Yang-Mills connection.

Proposition 2.3. Let $X$ be a smooth Kähler surface with a Kähler metric $g$, $A$ be a Yang-Mills connection on a principal $G$-bundle $P$ over $X$. Then we have

$$\nabla_A^* \nabla_A F_A^{0,2} + \frac{3}{2} [\sqrt{-1} \Lambda \omega F_A, F_A^{0,2}] + 2SF_A^{0,2} = 0$$

Furthermore, if $X$ is compact,

$$\frac{3}{4} \| \bar{\partial}_A \Lambda \omega F_A \|_{L^2(X)}^2 = \| \nabla_A F_A^{0,2} \|_{L^2(X)}^2 + \int_X 2S |F_A^{0,2}|^2 dvol_g.$$

Proof. Following Proposition 2.1, we obtain that

$$\| \bar{\partial}_A \Lambda \omega F_A \|_{L^2(X)}^2 = \langle \bar{\partial}_A \Lambda \omega F_A, \Lambda \omega F_A \rangle_{L^2(X)}$$

$$= -\langle 2\sqrt{-1} * [F_A^{0,2}, F_A^{0,2}], \Lambda \omega F_A \rangle_{L^2(X)}$$

The Weitzenböck formula for $F_A^{0,2}$ yields

$$\nabla_A^* \nabla_A F_A^{0,2} + 2SF_A^{0,2} + \frac{3}{2} [\sqrt{-1} \Lambda \omega F_A, F_A^{0,2}] = 0.$$

If $X$ is closed, taking the $L^2$-inner product of above identity with $F_A^{0,2}$ and integrating by parts, we then obtain (2.3). \qed
2.2 Irreducible connections

In this section, we first recall a definition of irreducible connection on a principal $G$-bundle $P$, where $G$ being a compact, semisimple Lie group. Given a connection $A$ on a principal $G$-bundle $P$ over $X$. We can define the stabilizer $\Gamma_A$ of $A$ in the gauge group $\mathcal{G}_P$ by

$$\Gamma_A := \{ g \in \mathcal{G}_P | g(A) = A \},$$

one also can see [5]. A connection $A$ called reducible if the connection $A$ whose stabilizer $\Gamma_A$ is larger than the centre $C(G)$ of $G$. Otherwise, the connections are irreducible, they satisfy $\Gamma_A \cong C(G)$. It’s easy to see that a connection $A$ is irreducible when it admits no nontrivial covariantly constant Lie algebra-value 0-form, i.e., $\ker d_A|_{\Omega^0(X, \mathfrak{g}_P)} = 0$. Taubes introduced the number in [23, Equation 6.3] to measure the irreducibility of $A$. We can defined the least eigenvalue $\lambda(A)$ of $d^*_A d_A$ as follow.

**Definition 2.4.** The least eigenvalue of $d^*_A d_A$ on $L^2$ section of $\Gamma(\mathfrak{g}_P)$ is

$$\lambda(A) := \inf_{v \in \Gamma(\mathfrak{g}_P) \setminus \{0\}} \frac{\|d_A v\|^2}{\|v\|^2}. \quad (2.4)$$

It is easy to see that the function $\lambda(A)$ depends only on the connection $A$. We introduce the definition of *strong* irreducible connection on a principal $G$-bundle $P$.

**Definition 2.5.** We call $A$ a smooth strong irreducible connection on $G$-bundle $P$ over a smooth $n$-dimensional, Riemannian manifold $X$, $(n \geq 2)$, if the least eigenvalue of $d_A^* d_A$ on $L^2$ section of $\Gamma(\mathfrak{g}_P)$ has a positive lower bound, i.e, there is a constant $\lambda_0 = \lambda_0(P, g) \in (0, \infty)$ such that $\lambda(A) \geq \lambda_0$.

The Sobolev norms $L^p_{k,A}$, where $1 \leq p < \infty$ and $k$ is an integer, with respect to the connections defined as:

$$\|u\|_{L^p_{k,A}(X)} := \left( \sum_{j=0}^{k} \int_X |\nabla_A^j u|^p dvol_g \right)^{1/p},$$

where $\nabla_A$ is the covariant derivative induced by the connection $A$ on $P$ and the Levi-Civita connection defined by the Riemannian metric $g$ on $T^*X$ and $\nabla_A^j := \nabla_A \circ \ldots \circ \nabla_A$ (repeated $j$ times for $j \geq 0$).

**Remark 2.6.** Let $A$ be a irreducible connection on the principal $G$-bundle over a compact manifold $X$, i.e., $\ker d_A|_{\Omega^0(X, \mathfrak{g}_P)} = 0$. Then we can assume that $\ker d_A|_{\Omega^0(X, \mathfrak{g}^C_P)} = 0$, where $\mathfrak{g}^C_P := \mathfrak{g}_P \otimes \mathbb{C}$. We denote $s$ by a section of $\Gamma(\mathfrak{g}^C_P)$, i.e., $s$ can be seen as a function over $X$ which takes value in the Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. Here $\mathfrak{g} \otimes \mathbb{C}$ is the complexification of Lie algebra $\mathfrak{g}$. See [18, Pages 11–12]. Thus in a local coordinate, there exists two $s_1, s_2 \in \mathfrak{g}$ such that $s = s_1 + is_2$. Then we have two identities, $|\nabla_A s|^2 = |\nabla_A s_1|^2 + |\nabla_A s_2|^2$ and $|s|^2 = |s_1|^2 + |s_2|^2$. Since $A$ is irreducible, $\|\nabla_A s\|^2_{L^2(X)} \geq \lambda \|s\|^2_{L^2(X)}$.

By the similar method of the proof of [6 Proposition A.3] or [5, Lemma 7.1.24], the least eigenvalue $\lambda(A)$ of $d_A^* d_A$ with respect to connection $A$ is continuous in $L^p_{loc}$-topology for $2 \leq p < 4$. Due to a result of Sedlacek [19, Theorem 4.3], we then have
Proposition 2.7. Let $G$ be a compact, semisimple Lie group and $P$ be a principal $G$-bundle over a closed, smooth, oriented, four-dimensional Riemannian manifold $X$ with a Riemannian metric $g$. If $\{A_i\}_{i \in \mathbb{N}}$ is a sequence $C^\infty$ connection on $P$ and the curvatures $\|F_{A_i}^+\|_{L^2(X)}$ converge to zero as $i \to \infty$, then there are a finite set of points, $\Sigma = \{x_1, \ldots, x_L\} \subset X$ and a subsequence $\Xi \subset \mathbb{N}$, we also denote by $\{A_i\}$, a sequence gauge transformation $\{g_i\}_{i \in \mathbb{N}}$ such that, $g_i(A_i) \to A_\infty$, a anti-self-dual connection on $P$ \mid_{X \setminus \Sigma}$ in the $L^2_\Sigma$-topology over $X \setminus \Sigma$. Furthermore, we have
\[
\lim_{i \to \infty} \lambda(A_i) = \lambda(A_\infty),
\]
where $\lambda(A)$ is as in Definition 2.4.

Theorem 2.8. Let $X$ be a closed, Riemannian 4-manifold, $P$ be a principal $G$-bundle with $G$ being compact, semisimple Lie group. Suppose that the ASD connections $A \in M(P, g)$ are strong irreducible connections in the sense of Definition 2.5 then there are positive constants $\lambda_0$ and $\varepsilon$ such that
\[
\begin{align*}
\lambda(A) &\geq \lambda_0, \forall [A] \in M(P, g), \\
\lambda(A) &\geq \frac{\lambda_0}{2}, \forall [A] \in B_\varepsilon(P, g).
\end{align*}
\]

Proof. By the definition of strong irreducible connection, there is a positive constant $\lambda_0 = \lambda_0(P, g)$ such that $\lambda(A) \geq \lambda_0$, $\forall [A] \in M(P, g)$.

Suppose that the constant $\varepsilon \in (0, 1]$ does not exist. We may then choose a sequence $\{A_i\}_{i \in \mathbb{N}}$ of connection on $P$ such that $\|F_{A_i}^+\|_{L^2(X)} \to 0$ and $\lambda(A_i) \to 0$ as $i \to \infty$. Since $\lim_{i \to \infty} \lambda(A_i) = \lambda(A_\infty)$ and $\lambda(A_\infty) \geq \lambda_0$, then it is contradict to our initial assumption regarding the sequence $\{A_i\}_{i \in \mathbb{N}}$. In particular, the preceding argument shows that the desired constant $\varepsilon$ exists.

Friedman and Morgan introduced the generic Kähler metric which ensures the connections on the compactification of moduli space of ASD connections on $E$, $\tilde{M}(P, g)$, are irreducible, See [7, Chapter IV]. The authors also given an example on a Kähler surface which admits a generic Kähler metric. Fix an algebraic surface $S$ and an ample line bundle $L$ on $S$. For every integer $c$ we have defined the moduli space $\mathcal{M}_c(S, L)$ of $L$-stable rank two holomorphic vector bundles $V$ on $S$ such that $c_1(V) = 0$, $c_2(V) = c$. Next let us determine when a Hodge metric with Kähler form $\omega$ admits reducible ASD connections. Corresponding to such a connection is an associated ASD harmonic 1-form $\alpha$, well-defined up to $\pm 1$, representing an integral cohomology class, which by the description of $\Omega^{1, -1}(X)$ is of type $(1, 1)$ and orthogonal to $\omega$. Thus Friedman-Morgan proved for an integer $c > 0$, there is an open dense subset $\mathcal{D}$ of the cone of ample divisors on $S$ such that if $g$ is a Hodge metric whose Kähler form lies in $\mathcal{D}$, $g$ is a generic metric in the sense of Definition 2.9 See [7, Chapter IV, Proposition 4.8].

Definition 2.9. Let $X$ be a compact Kähler surface, $P$ be a principal $G$-bundle over $X$ with $c_2(P) = c$. We say that a Kähler metric $g$ on $X$ is generic if for every $G$-bundle $\tilde{P}$ over $X$ with $0 < c_2(\tilde{P}) \leq c$, there are no reducible ASD connections on $\tilde{P}$. 
For a compact Kähler surface $X$ we have a moduli space of ASD connections $M(P, g)$. The Donaldson-Uhlenbeck compactification $\tilde{M}(P, g)$ of $M(P, g)$ contained in the disjoint union

$$\tilde{M}(P, g) \subset \cup (M(P_{i}, g) \times \text{Sym}^i(X)),$$

Following [5] Theorem 4.4.3], the space $\tilde{M}(P, g)$ is compact.

**Lemma 2.10.** Let $X$ be a compact, simply-connected, Kähler surface with a generic Kähler metric, $P$ be a $SO(3)$-bundle over $X$. If the second Stiefel-Whitney classes $w_2 \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ is non-trivial, then $\lambda(A) > 0$ for any $[A] \in \tilde{M}(P, g)$.

**Proof.** The only reducible anti-self-dual connection on a principal $SO(3)$-bundle over $X$ is the product connection on the product bundle $P = X \times SO(3)$ by [5] Corollary 4.3.15] and the latter possibility is excluded by our hypothesis in this case that $w_2(P) \neq 0$. Then the conclusion is a consequence of the definition of generic Kähler metric. 

Following Feehan’s idea, combining $\tilde{M}(P, g)$ is compact and $\lambda(A)$ is continuous under the Uhlenbeck topology for the connection $[A] \in \tilde{M}(P, g)$, then we have

**Theorem 2.11.** Let $X$ be a compact, simply-connected, Kähler surface with a generic Kähler metric, $P$ be a $SO(3)$-bundle over $X$. If the second Stiefel-Whitney classes $w_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ is non-trivial, then there are positive constants $\mu_0 = \mu_0(P, g)$ and $\varepsilon = \varepsilon(P, g)$ such that

$$\lambda(A) \geq \lambda_0, \forall [A] \in M(P, g),$$

$$\lambda(A) \geq \frac{\lambda_0}{2}, \forall [A] \in B_\varepsilon(P, g).$$

2.3 An estimate on Yang-Mills connections

**Lemma 2.12.** Let $X$ be a closed, smooth, four-manifold with a Riemannian metric $g$, $A$ be a connection on a principal $G$-bundle $P$ over $X$ with $G$ being a compact, semisimple Lie group. There are positive constants $\lambda = \lambda(g, P)$ and $C = C(\lambda, g, P)$ with following significance. If $\lambda(A) \geq \lambda$, where $\lambda(A)$ is as in (2.4), then for any section $v$ on $\Gamma(g_P)$,

$$\|v\|_{L^3_2(X)} \leq C\|\nabla_A^* \nabla_A v\|_{L^2(X)}. \quad (2.5)$$

**Proof.** Since $\nabla_A^* \nabla_A$ is a elliptic operator of degree 2, from a priori estimate of elliptic operator, for $p \geq 1$ and $k \geq 0$, we have

$$\|v\|_{L^p_{k+2}(X)} \leq c\|\nabla_A^* \nabla_A v\|_{L^p_k(X)} + \|v\|_{L^p(X)} \quad (2.6)$$

We take $k = 0$ and $p = 2$,

$$\|v\|_{L^3_2(X)} \leq c\|\nabla_A^* \nabla_A v\|_{L^2(X)} + \|v\|_{L^2(X)}.$$

By the definition of $\lambda(A)$, we also have

$$\|v\|_{L^2(X)} \leq \lambda^{-1}\|\nabla_A^* \nabla_A v\|_{L^2(X)}.$$

Combining the preceding inequalities yields (2.5).
We construct a priori on the Yang-Mills connection under the condition of $\Lambda_\omega F_A$ is sufficiently small in $L^2$-norm.

**Proposition 2.13.** Let $X$ be a compact Kähler surface, $A$ be a Yang-Mills connection on a principal $G$-bundle $P$ over $X$ with $G$ being a compact, semisimple Lie group, $p \in (4, \infty)$. Let $q \in (4/3, 2)$ be defined by $1/q = 1/2 + 1/p$. There are positive constants $\lambda = \lambda(g, P)$ and $\varepsilon = \varepsilon(\lambda, g, P)$ with following significance. If the curvature $F_A$ of the connection $A$ on $P$ obeying

$$\|\Lambda_\omega F_A\|_{L^2(X)} \leq \varepsilon,$$

and $\lambda(A) \geq \lambda$, where $\lambda(A)$ is as in (2.4), then

$$\|F_A^{0,2}\|_{L^p(X)} \leq C\|F_A^{0,2}\|_{L^q(X)},$$

(2.7)

where $C = C(\lambda, g, P, p)$ is a positive constant.

**Proof.** We apply the estimate (2.6) and Sobolev embedding $L^2 \hookrightarrow L^p$, then

$$\|F_A^{0,2}\|_{L^p(X)} \leq c\|F_A^{0,2}\|_{L^2(X)} \leq \|\nabla_A^* \nabla_A F_A^{0,2}\|_{L^q(X)} + \|F_A^{0,2}\|_{L^q(X)}.$$  

By using the Weizenböck formula of $F_A^{0,2}$, see equation (2.2), we then observe that

$$\|\nabla_A^* \nabla_A F_A^{0,2}\|_{L^q(X)} \leq c\|F_A^{0,2}\|_{L^q(X)} + \|\{F_A^{0,2}, \Lambda_\omega F_A\}\|_{L^q(X)}$$

$$\leq c\|F_A^{0,2}\|_{L^q(X)} + c\|\Lambda_\omega F_A\|_{L^2(X)}\|F_A^{0,2}\|_{L^p(X)}.$$  

where $c = c(g, G, p)$ is a positive constant. Combining the preceding inequalities gives

$$\|F_A^{0,2}\|_{L^p(X)} \leq c\|F_A^{0,2}\|_{L^q(X)} + c\|\Lambda_\omega F_A\|_{L^2(X)}\|F_A^{0,2}\|_{L^p(X)} + \|F_A^{0,2}\|_{L^q(X)}.$$  

where $c = c(g, G, p)$ is a positive constant. Provide $c\|\Lambda_\omega F_A\|_{L^2(X)} \leq 1/2$, rearrangement gives (2.7).  

3 Yang-Mills connection on Kähler surface

3.1 Approximate Hermitian-Yang-Mills connections

In this section we will give a general criteria under which an approximate ASD connection $A \in A_P$ could deform into an other approximate ASD connection $A_\infty$ which satisfies

$$\Lambda_\omega F_{A_\infty} = 0.$$  

(3.1)

One also can see [10, Section 4.2]. Let $A$ be a connection on a principal $G$-bundle over $X$. The equation (3.1) for a second connection $A + a$, where $a \in \Omega^1(X, g_P)$ is a bundle valued 1-form, could be rewritten to

$$\Lambda_\omega (d_A a + a \wedge a) = -\Lambda_\omega F_A.$$  

(3.2)
We seek a solution of the equation (3.2) in the form
\[ a = d_A^*(s \otimes \omega) = \sqrt{-1}(\partial_A s - \bar{\partial}_A s) \]
where \( s \in \Omega^0(X; g_P) \) is a bundle value 0-form. Then Equation (3.2) becomes the second order equation:
\[ -d_A^*d_A s + \Lambda_\omega(d_A s \wedge d_A s) = -\Lambda_\omega F_A. \quad (3.3) \]
For convenience, we define a map
\[ B(u, v) := \frac{1}{2} \Lambda_\omega [d_A u \wedge d_A v]. \]
It’s easy to check, we have the pointwise bound:
\[ |B(u, v)| \leq C |\nabla_A u| |\nabla_A v|, \]
where \( C \) is a uniform positive constant.

We would like to prove that if \( \Lambda_\omega F_A \) is small in an appropriate sense, there is a small solution \( s \) to equation (3.3).

**Theorem 3.1.** ([10, Theorem 4.8]) Let \( X \) be a compact, Kähler surface with a Kähler metric \( g \), \( P \) be a principal \( G \)-bundle over \( X \) with \( G \) being a compact, semisimple Lie group. There are positive constant \( \lambda = \lambda(g, P) \) and \( \varepsilon = \varepsilon(\lambda, g, P) \) with following significance. If the curvature \( F_A \) of a connection \( A \) on \( P \) obeying
\[ \|\Lambda_\omega F_A\|_{L^2(X)} \leq \varepsilon, \]
\[ \lambda(A) \geq \lambda, \]
where \( \lambda(A) \) is as in (2.4), then there is a section \( s \in \Gamma(g_P) \) such that the connection
\[ A_\infty := A + \sqrt{-1}(\partial_A s - \bar{\partial}_A s) \]
satisfies
(1) \( \Lambda_\omega F_{A_\infty} = 0 \)
(2) \[ \|s\|_{L^2(X)} \leq C\|\Lambda_\omega F_A\|_{L^2(X)}, \]
where \( C = C(\lambda, g) \in [1, \infty) \) is a positive constant. Furthermore, let \( p \in (2, \infty) \),
\[ \|F_{A_\infty}^{0,2} - F_A^{0,2}\|_{L^2(X)} \leq C(\|\Lambda_\omega F_A\|_{L^2(X)} + \|F_A^{0,2}\|_{L^p(X)})\|\Lambda_\omega F_A\|_{L^2(X)}, \]
for a positive constant \( C = C(\lambda, g, p) \).

Now, we begin to prove Theorem 3.1, the proof of above theorem is base on Taubes’ ideas [21] and [4]. At first, suppose \( s \) and \( f \) are sections of \( \Gamma(g_P) \) with
\[ d_A^*d_A s = f, \ i.e., \ \nabla_A^* \nabla_A s = f, \quad (3.4) \]
the first observation is
Lemma 3.2. ([10] Lemma 4.9) If $\lambda(A) \geq \lambda > 0$, then there exists a unique $C^\infty$ solution to equation (3.4). Furthermore, we have

$$
\|s\|_{L^2(X)} \leq c \|f\|_{L^2(X)},
$$

$$
\|B(s, s)\|_{L^2(X)} \leq c \|f\|^2_{L^2(X)},
$$

where $c = c(\lambda, g)$ is a positive constant.

Proof. Following the estimate on Lemma 2.12 we have

$$
\|s\|_{L^2(X)} \leq \|\nabla^* A s\|_{L^2(X)} \leq c \|f\|_{L^2(X)},
$$

for a positive constant $c = c(\lambda, g, P)$. By the Sobolev inequality in four dimension,

$$
\|B(s, s)\|_{L^2(X)} \leq C \|\nabla A s\|^2_{L^2(X)} \leq C \|\nabla A s\|^2_{L^2(X)} \leq C \|s\|^2_{L^2(X)},
$$

for a positive constant $C = C(\lambda, g, P)$. \hfill \Box

Lemma 3.3. ([10] Lemma 4.10) If $d_A^* d_A s_1 = f_1$, $d_A^* d_A s_2 = f_2$, then

$$
\|B(s_1, s_2)\|_{L^2(X)} \leq c \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}.
$$

We will prove the existence of a solution of (3.3) by the contraction mapping principle. We write $s = (d_A^* d_A)^{-1} f$ and (3.3) becomes an equation for $f$ of the form

$$
f - S(f, f) = \Lambda_{\omega} F_A,
$$

(3.5)

where $S(f, g) := B((d_A^* d_A)^{-1} f, (d_A^* d_A)^{-1} g)$. Following Lemma 3.2,

$$
\|S(f_1, f_1) - S(f_2, f_2)\|_{L^2(X)} = \|S(f_1 + f_2, f_1 - f_2)\|_{L^2(X)} \\
\leq c \|f_1 + f_2\|_{L^2(X)} \|f_1 - f_2\|_{L^2(X)}.
$$

We denote $g_k = f_k - f_{k-1}$ and $g_1 = f_1$, then

$$
g_1 = \Lambda_{\omega} F_A, \quad g_2 = S(g_1, g_1)
$$

and

$$
g_k = S\left(\sum_{i=1}^{k-1} g_i, \sum_{i=1}^{k-1} g_i\right) - S\left(\sum_{i=1}^{k-2} g_i, \sum_{i=1}^{k-2} g_i\right), \quad \forall \ k \geq 3.
$$

It is easy to show that, under the assumption of $\Lambda_{\omega} F_A$, the sequence $f_k$ defined by

$$
f_k = S(f_{k-1}, f_{k-1}) + \Lambda_{\omega} F_A,
$$

starting with $f_1 = \Lambda_{\omega} F_A$, is Cauchy with respect to $L^2$, and so converges to a limit $f$ in the completion of $\Gamma(g_P)$ under $L^2$. 

On Yang-Mills connections on compact Kähler surfaces
Proposition 3.4. ([10 Proposition 4.11]) There are positive constants \( \varepsilon \in (0, 1) \) and \( C \in (1, \infty) \) with following significance. If
\[
\| \Lambda_\omega F_A \|_{L^2(X)} \leq \varepsilon,
\]
then each \( g_k \) exists and is \( C^\infty \). Further for each \( k \geq 1 \), we have
\[
\| g_k \|_{L^2(X)} \leq C^{k-1} \| \Lambda_\omega F_A \|_{L^2(X)}^k. \tag{3.6}
\]

Proof. The proof is by induction on the integer \( k \). The induction begins with \( k = 1 \), one can see \( g_1 = \Lambda_\omega F_A \). The induction proof if completed by demonstrating that if (3.6) is satisfied for \( j < k \), then it also satisfied for \( j = k \). Indeed, since
\[
\| S(\sum_{i=1}^{k-1} g_i, \sum_{i=1}^{k-1} g_i) - S(\sum_{i=1}^{k-2} g_i, \sum_{i=1}^{k-2} g_i) \|_{L^2(X)} \leq c\| \sum_{i=1}^{k-1} g_i + \sum_{i=1}^{k-2} g_i \|_{L^2(X)}\| g_{k-1} \|_{L^2(X)},
\]
\[
\leq 2c\| g_i \|_{L^2(X)}\| g_{k-1} \|_{L^2(X)},
\]
\[
\leq \frac{2C^{k-2}\| \Lambda_\omega F_A \|_{L^2(X)}^k}{1 - C\| \Lambda_\omega F_A \|_{L^2(X)}},
\]
Now, we provide \( \varepsilon \) sufficiently small and \( C \) sufficiently large to ensure \( \| \Lambda_\omega F_A \|_{L^2(X)} \leq C^{-2}(C - 2c) \), i.e., \( \frac{2c}{1 - C\| \Lambda_\omega F_A \|_{L^2(X)}} \leq C \), hence we complete the proof of this Proposition.

Proof of Theorem 3.1. The sequence \( g_k \) is Cauchy in \( L^2 \), the limit \( f := \lim_{k \to \infty} f_k \) is a solution to (3.5). Following Lemma 3.2, we have
\[
\| s \|_{L^2(X)} \leq c\| f \|_{L^2(X)} \leq c\sum_{k=1}^\infty \| g_k \|_{L^2(X)} \leq \frac{c\| \Lambda_\omega F_A \|_{L^2(X)}}{1 - C\| \Lambda_\omega F_A \|_{L^2(X)}},
\]
for a positive constant \( c \). We provide \( \varepsilon \) and \( C \) to ensure \( C\varepsilon \leq \frac{1}{2} \), hence
\[
\| s \|_{L^2(X)} \leq 2c\| \Lambda_\omega F_A \|_{L^2(X)}.
\]
We denote \( A_\infty := A + \sqrt{-1}(\bar{\partial}_A s - \bar{\partial}_A s) \), and \( r \in (2, \infty) \) defined by \( 1/r = 1/2 - 1/p \), then
\[
\| F^{0,2}_{A_\infty} - F^{0,2}_A \|_{L^2(X)}^2 = \| -\sqrt{-1}\bar{\partial}_A \bar{\partial}_A s - \bar{\partial}_A s \|_{L^2(X)}^2
\]
\[
= \| -\sqrt{-1}[F^{0,2}_A, s] - \bar{\partial}_A s \|_{L^2(X)}^2
\]
\[
\leq 2\| \bar{\partial}_A s \|_{L^4(X)}^2 + 2\| F^{0,2}_A \|_{L^p(X)}\| s \|_{L^3(X)}
\]
\[
\leq c(\| \Lambda_\omega F_A \|_{L^2(X)} + \| F^{0,2}_A \|_{L^p(X)})\| \Lambda_\omega F_A \|_{L^2(X)},
\]
where \( c \) is a positive constant. We complete the proof of Theorem 3.1.
Corollary 3.5. Let $X$ be a compact, simply-connected, Kähler surface with a Kähler metric $g$, that is generic in the sense of Definition 2.9. $P$ be a $SO(3)$-bundle over $X$. Suppose that the second Stiefel-Whitney class, $\omega_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$, is non-trivial, then there is a positive constant $\varepsilon = \varepsilon(g, P)$ with following significance. If the curvature $F_A$ of a connection $A$ on $P$ obeying

$$\|F_A^+\|_{L^2(X)} \leq \varepsilon,$$
then there is a section $s \in \Gamma(g_P)$ such that the connection $A_\infty := A + \sqrt{-1}(\partial_A s - \bar{\partial}_A s)$ satisfies

1. $\Lambda_\omega F_{A_\infty} = 0$
2. $\|s\|_{L^2(X)} \leq C\|\Lambda_\omega F_A\|_{L^2(X)},$
where $C = C(g, P)$ is a positive constant.

3.2 Yang-Mills connection with harmonic curvature

Suppose that an integrable connection $A_{A,1}^{0,1}$ on a holomorphic bundle over a Kähler surface is Yang-Mills, then $\Lambda_\omega F_A$ is parallel, that is equivalent to $F_A^{0,2}$ being harmonic with respect to Laplacian operator $\Delta_{\partial_A}$. For a general case, we introduce the definition of a connection with harmonic curvature, See [13, p. 96].

Definition 3.6. A connection $A$ on a compact Kähler surface is said to be with a harmonic curvature if $(0, 2)$-part of curvature is harmonic, i.e., $\bar{\partial}_A F_A^{0,2} = 0$.

Lemma 3.7. Let $X$ be a compact Kähler surface, $P$ be a $G$-bundle over $X$ with $G$ being a compact, semisimple Lie group, $A$ be a Yang-Mills connection on $P$. There are positive constant $\lambda = \lambda(g, P)$ and $\varepsilon = \varepsilon(\lambda, g, P)$ with following significance. If the curvature $F_A$ of the connection $A$ obeying

$$\|F_A^+\|_{L^2(X)} \leq \varepsilon,$$
$$\lambda(A) \geq \lambda,$$
where $\lambda(A)$ is as in (2.4), then the curvature is harmonic and $\Lambda_\omega F_A = 0$.

Proof. For a suitable constant $\varepsilon$, from Theorem 3.1 there exist a connection $A_\infty$ such that

$$\|A - A_\infty\|_{L^2(X)} \leq c\|\Lambda_\omega F_A\|_{L^2(X)}$$
and $\Lambda_\omega F_{A_\infty} = 0$. We apply the Weizenböck formula (2.2) to $F_A^{0,2}$.

$$\|\bar{\partial}_{A_\infty} F_A^{0,2}\|_{L^2(X)}^2 + \|\bar{\partial}^*_{A_\infty} F_A^{0,2}\|_{L^2(X)}^2 = \|\nabla_{A_\infty} F_A^{0,2}\|_{L^2(X)}^2 + \int_X 2S|F_A^{0,2}|^2 dvol_g.$$
We observe that \( \bar{\partial}_A \Lambda \omega F_A = 0 \) and
\[
\| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)} \leq c \left\{ A - A_{\infty}, F_A^0, 2 \right\}^2_{L^2(X)} + \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)}.
\]
\[
\leq c \left\{ A - A_{\infty}, F_A^0, 2 \right\}^2_{L^2(X)} + \frac{1}{4} \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)},
\]
\[
\leq c \| A - A_{\infty} \|^2_{L^2(X)} \| F_A^0, 2 \|^2_{L^2(X)} + \frac{1}{4} \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)},
\]
\[
\leq c \| A - A_{\infty} \|^2_{L^2(X)} \| F_A^0, 2 \|^2_{L^2(X)} + \frac{1}{4} \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)},
\]
\[
\leq c \| \Lambda \omega F_A \|^2_{L^2(X)} \| F_A^0, 2 \|^2_{L^2(X)} + \frac{1}{4} \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)}.
\]

Here we use the estimates on Proposition 2.13 and Theorem 3.1 and Sobolev embedding \( L^2 \to L^4 \). Combining the preceding inequalities with integrable identity (2.3) on Proposition 2.3 gives
\[
\frac{3}{4} \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)} = \| \nabla_A F_A^0, 2 \|^2_{L^2(X)} + \int_X 2S |F_A^0, 2|^2 dvol_g
\leq \| \nabla_A F_A^0, 2 \|^2_{L^2(X)} + \int_X 2S |F_A^0, 2|^2 dvol_g + \left\{ A - A_{\infty}, F_A^0, 2 \right\}^2_{L^2(X)}
\leq c \| \Lambda \omega F_A \|^2_{L^2(X)} \| F_A^0, 2 \|^2_{L^2(X)} + \frac{1}{4} \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)}.
\]

for a positive constant \( c = c(\lambda, g) \). Thus, we have
\[
\| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)} \leq c \| \Lambda \omega F_A \|^2_{L^2(X)} \| F_A^0, 2 \|^2_{L^2(X)}. \tag{3.7}
\]

We apply Weitzenböck formula to \( \Lambda \omega F_A \), See [5] Lemma 6.1.
\[
\bar{\partial}_A^* \bar{\partial}_A \Lambda \omega F_A = \frac{1}{2} \nabla_A^* \nabla_A \Lambda \omega F_A + [\sqrt{-1} \Lambda \omega F_A, \Lambda \omega F_A],
\]
thus
\[
\| \nabla_A \Lambda \omega F_A \|^2_{L^2(X)} = 2 \| \bar{\partial}_A \Lambda \omega F_A \|^2_{L^2(X)}.
\]
Combining above identity with estimate (3.7) yields,
\[
\| \Lambda \omega F_A \|^2_{L^2(X)} \leq c \| \nabla_A \Lambda \omega F_A \|^2_{L^2(X)} \leq c \| \Lambda \omega F_A \|^2_{L^2(X)} \| F_A^0, 2 \|^2_{L^2(X)},
\]
where \( c = c(\lambda, g) \) is a positive constant. Provide \( c \| F_A^0, 2 \|^2_{L^2(X)} \leq \frac{1}{4} \), thus \( \Lambda \omega F_A \equiv 0. \)

Following the Lemma 3.7 and the eigenvalue \( \lambda(A) \) has a uniform positive lower bounded under the hypothesis of Kähler metric \( g \) is generic and the curvature \( F_A \) of the connection \([A] \) obeys \( \| F_A^+ \|^2_{L^2(X)} \leq \varepsilon \) for a small enough constant \( \varepsilon \), then we have

**Corollary 3.8.** Let \( X \) be a compact, simply-connected, Kähler surface with a Kähler metric \( g \), that is generic in the sense of Definition 2.9. \( P \) be a \( SO(3) \)-bundle over \( X \). Suppose that the second Stiefel-Whitney class, \( \omega_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \), is non-trivial, then there is a positive constant \( \varepsilon = \varepsilon(g, P) \) with following significance. If the curvature \( F_A \) of a Yang-Mills connection \( A \) on \( P \) obeying
\[
\| F_A^+ \|^2_{L^2(X)} \leq \varepsilon,
\]
then the curvature is harmonic and \( \Lambda \omega F_A = 0. \)
3.3 A vanishing theorem

Let \((X, \omega)\) be a compact Kähler surface. Given an orthonormal coframe \(\{e_0, e_1, e_2, e_3\}\) on \(X\) for which \(\omega = e^{01} + e^{23}\), where \(e^{ij} = e^i \wedge e^j\). We define \(dz^1 = e^0 + ie^1,\) \(dz^2 = e^2 + ie^3\) and \(d\bar{z}^1 = e^0 - ie^1,\) \(d\bar{z}^2 = e^2 - ie^3\), so that \(\omega = \frac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2)\).

**Proposition 3.9.** Let \(A\) be a connection on a principal \(SU(2)\) or \(SO(3)\) bundle over a compact Kähler surface. If the curvature \(F_A\) of connection \(A\) is harmonic, then \(F_A^{0,2}\) has at most rank one.

**Proof.** Since \(\bar{\partial}^* A F_A^{0,2} = 0\), we have
\[
0 = \bar{\partial}^* A \bar{\partial}^* A F_A^{0,2} = -\ast [F_A^{0,2} \wedge \ast F_A^{0,2}].
\]
(3.8)

In an orthonormal coframe, we can written \(F_A^{0,2}\) as
\[
F_A^{0,2} = (B_1 + iB_2)dz^1 \wedge d\bar{z}^2,
\]
where \(B_1, B_2\) take value in Lie algebra \(su(2)\) or \(so(3)\). Thus
\[
\ast F_A^{0,2} = (-B_1 + iB_2)dz^1 \wedge d\bar{z}^2.
\]
Following Equation (3.8), we obtain that
\[
0 = [B_1, B_2].
\]
(3.9)

Thus \(F_A^{0,2}\) has most rank one. For the details of the calculation, one also can see [14] Chapter 4. \(\square\)

We define \(\beta\) as follows, if
\[
B = B_1(e^{01} + e^{23}) + B_2(e^{02} + e^{31}) + B_3(e^{03} + e^{12}),
\]
then
\[
\beta := \frac{1}{2}(B_2 - iB_3)dz^1 \wedge dz^2, \quad \beta^* := -\frac{1}{2}(B_2 + iB_3)d\bar{z}^1 \wedge d\bar{z}^2.
\]

It follows that \(B := B_1 \omega + \beta - \beta^*\). We define a bilinear map
\[
[\cdot, \cdot] : \Omega^{2,+}(X, \mathfrak{g}_P) \otimes \Omega^{2,+}(X, \mathfrak{g}_P) \to \Omega^{2,+}(X, \mathfrak{g}_P)
\]
by \(\frac{1}{2}[\cdot, \cdot]_{\Omega^{2,+}} \otimes \cdot, \cdot \in \mathfrak{g}_P\), see [14] Section B.4. In a direct calculate, see [14] Section 7.1,
\[
-\frac{1}{4}[B, B] = [B_2, B_3](e^{01} + e^{23}) + [B_3, B_1](e^{02} + e^{31}) + [B_1, B_2](e^{03} + e^{12}).
\]

**Proposition 3.10.** Let \(G\) be a compact, semisimple Lie group, \(A\) be a connection on a principal \(G\)-bundle over a compact Kähler surface. If the curvature of the connection \(A\) satisfies \(\Lambda_\omega F_A = 0\) and \(\bar{\partial}^* F_A^{0,2} = 0\), then \(F_A^{0,2} = 0\).
Proof. We can written $F^+_A$ as $F^+_A := F^{0,2}_A + F^{2,0}_A + \frac{1}{2} A_\omega F_A \otimes \omega$. By the hypothesis of curvature and Equation (3.9), we then have $[F^+_A, F^+_A] = 0$. \hfill \Box

Before the prove of vanishing theorem 3.13, we should recall a useful lemma proved by Donaldson [5] Lemma 4.3.21.

Lemma 3.11. If $A$ is an irreducible $SU(2)$ or $SO(3)$ ASD connection on a bundle $P$ over a simply connected four-manifold $X$, then the restriction of $A$ to any non-empty open set in $X$ is also irreducible.

We recall the following simple but powerful corollary of unique continuation for ASD connections which proved in [14] Theorem 4.2.1. For the convenience of the readers, we give a proof of this theorem.

Theorem 3.12. Let $X$ be a simply-connected, oriented, smooth Riemannian four-manifold, $P$ be a principal $SU(2)$ or $SO(3)$ bundle over $X$ and $A$ be an irreducible ASD connection on $P$. If $B \in \Omega^{2,\ast}(X, g_P)$ satisfies

$$d^*_A B = 0 \text{ and } [B, B] = 0,$$

then $B = 0$.

Proof. Let $Z^c$ denote the complement of the zero set of $B$. By unique continuation of the elliptic equation $d^*_A B = 0$, $Z^c$ is either empty or dense. On $Z^c$ write $B = f \otimes \sigma$ for $\sigma \in \Omega^0(Z^c, g_P)$ with $|\sigma|^2 = 1$ and $f \in \Omega^{2,\ast}(Z^c)$. We compute

$$0 = d^*_A B = - * d_A (f \otimes \sigma) = - * (df \otimes \sigma + f \otimes d_A \sigma).$$

Taking the inner product with $\sigma$ and using the consequence of $|\sigma|^2 = 1$ that $\langle \sigma, d_A \sigma \rangle = 0$, we get $df = 0$. It follows that $f \otimes d_A \sigma = 0$. Since $f$ is nowhere zero along $Z^c$, we have $d_A \sigma = 0$ along $Z^c$. Therefore, $A$ is reducible along $Z^c$. However according to Lemma 3.11, $A$ is irreducible along $Z^c$. This is a contradiction unless $Z^c$ is empty. Therefore $Z = X$, so $B = 0$. \hfill \Box

Corollary 3.13. Let $X$ be a compact, simply-connected, Kähler surface, $P$ be a principal $G = SU(2)$ or $SO(3)$ bundle over $X$ and $A$ be an irreducible ASD connection on $P$. If $\phi \in \Omega^{0,2}(X, g_P^C)$ satisfies

$$[\phi, *\phi] = 0 \text{ and } \bar{\partial}_A^* \phi = 0,$$

then $\phi$ vanish.

Proof. By the hypothesis of $\phi$, it follows that $B := \phi - \phi^*$ satisfies $[B, B] = 0$ and $d^*_A B = 0$. Following vanishing theorem 3.12, we obtain that $B = 0$, i.e., $\phi = 0$. \hfill \Box

At first, we define a subset of $\Omega^{0,2}(X, g_P^C)$ as follow:

$$\tilde{\Omega}^{0,2}(X, g_P^C) = \{ \phi \in \Omega^{0,2}(X, g_P^C) : [\phi, *\phi] = 0 \}.$$  

(3.10)
**Definition 3.14.** The least eigenvalue of $\bar{\partial}_A \bar{\partial}_A^* v_i$ on $L^2(\tilde{\Omega}^{0,2}(X, \mathfrak{g}_P^0))$ is

$$\mu(A) := \inf_{v \in \tilde{\Omega}^{0,2}(X, \mathfrak{g}_P^0) \setminus \{0\}} \frac{\|\bar{\partial}_A^* v_i\|_2^2}{\|v\|_2^2}. \quad (3.11)$$

**Proposition 3.15.** Let $X$ be a simply-connected, compact Kähler surface with a Kähler metric $g$, $P$ be a principal $SU(2)$ or $SO(3)$ bundle over $X$. If $A$ is an irreducible anti-self-dual connection on $P$, then $\mu(A) > 0$.

**Proof.** If not, the eigenvalue $\mu(A) = 0$. We may then choose a sequence $\{v_i\}_{i \in \mathbb{N}} \subset \tilde{\Omega}^{0,2} \setminus \{0\}$ such that

$$\|\bar{\partial}_A^* v_i\|_{L^2(X)}^2 \leq \mu_i \|v_i\|_{L^2(X)}^2$$

and

$$\mu_i \to 0^+ \text{ as } i \to \infty.$$ 

Since $\left[ \frac{v}{\|v\|_{L^2}} \wedge * \frac{v}{\|v\|_{L^2}} \right] = 0$ for $v \in \tilde{\Omega}^{0,2} \setminus \{0\}$, we then noting $\|v_i\|_{L^2(X)} = 1$, $\forall i \in \mathbb{N}$. Following the Weizenböck formula, we have

$$\|\nabla_A v_i\|_{L^2(X)}^2 = -\langle S v_i, v_i \rangle_{L^2(X)} + \|\bar{\partial}_A^* v_i\|_{L^2(X)}^2.$$ 

Thus

$$\|v_i\|_{L^1}^2 \leq (C + \lambda_i) < \infty,$$

where $C$ is a positive constant only dependence on the metric. Therefore, there exist a subsequence $X \subset N$ such that $\{v_i\}_{i \in X}$ weakly convergence to $v_\infty$ in $L^2_1$, we also have $\bar{\partial}_A^* v_i$ converge weakly in $L^2$ to a limit $\bar{\partial}_A^* v_\infty = 0$. On the other hand, $L^2_1 \rightarrow L^p$, for $2 \leq p < 4$, we may choose $p = 2$, then

$$\|v_i \wedge * v_\infty\|_{L^1(X)} = \|(v_\infty - v_i) \wedge * v_\infty + v_i \wedge (v_\infty - v_i)\|_{L^1(X)}$$

$$\leq \|v_i - v_\infty\|_{L^2(X)} (\|v\|_{L^2(X)} + \|v_\infty\|_{L^2(X)}) \to 0 \text{ as } i \to \infty,$$

Hence

$$[v_\infty \wedge * v_\infty = 0], \ i.e., \ \ v_\infty \in \tilde{\Omega}^{0,2}.$$ 

Therefore the corollary 3.13 implies that $\ker \bar{\partial}_A^* |_{\tilde{\Omega}^{0,2}(X, \mathfrak{g}_P^0)} = 0$. Thus $v_\infty$ vanish. It’s contradicting to $\|v_\infty\|_{L^2(X)} = 1$. In particular, the preceding arguments shows that the $\mu(A) > 0$. 

**Lemma 3.16.** ([3] Lemma 7.2.10) There is a universal constant $C$ and for any $N \geq 2$, $R > 0$, a smooth radial function $\beta = \beta_{N,R}$ on $\mathbb{R}^4$, with

$$0 \leq \beta(x) \leq 1$$

$$\beta(x) = \begin{cases} 1 & |x| \leq R/N \\ 0 & |x| \geq R \end{cases}$$

and

$$\|\nabla \beta\|_{L^4} + \|\nabla^2 \beta\|_{L^2} < \frac{C}{\sqrt{\log N}}.$$ 

Assuming $R < R_0$, the same holds for $\beta(x - x_0)$ on any geodesic ball $B_{R}(x_0) \subset X$. 

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**On Yang-Mills connections on compact Kähler surfaces**
Following the idea in [5], we can prove that the least eigenvalue of $\partial_A^* \partial_A$ on the space $\tilde{\Omega}^{0,2}(X, g^A_F)$ with respect to the connection $A$ is continuity in the sense of $L^4_{loc}$.

**Proposition 3.17.** Let $X$ be a compact Kähler surface, $\Sigma = \{x_1, x_2, \ldots, x_L\} \subset X$ ($L \in \mathbb{N}^+$) and $0 < \rho \leq \min_{i \neq j} \text{dist}_g(x_i, x_j)$ and $U \subset X$ be the open subset give by

$$U := X \setminus \bigcup_{l=1}^L \tilde{B}_{\rho/2}(x_l).$$

Let $G$ be a compact, semisimple Lie group, $A_0$ be a $C^\infty$ connection on a principal $G$-bundle $P_0$ over $X$ obeying the curvature bounded

$$\|F_{A_0}^+\|_{L^2(X)} \leq \varepsilon$$

(3.12) where $\varepsilon \in (0, 1)$ is a sufficiently small positive constant. Let $P$ be a principal $G$-bundle over $X$ such that there is an isomorphism of principal $G$-bundles, $u : P \mid X \setminus \Sigma \cong P_0 \mid X \setminus \Sigma$, and identify $P \mid X \setminus \Sigma$ with $P_0 \mid X \setminus \Sigma$ using this isomorphism. Then there are positive constants $c = c(\rho, g) \in (0, 1], \ c \in (1, \infty)$ and $\delta \in (0, 1]$ with the following significance. Let $A$ be a $C^\infty$ connection on $P$ obeying the curvature bounded (3.12) with constant $\varepsilon$ such that

$$\|A - A_0\|_{L^4(U)} \leq \delta.$$

Then $\mu(A)$ and $\mu(A_0)$ satisfy

$$\mu(A) \leq (1 + \eta)\mu(A_0) + c((1 + \eta)(C + \delta^2) + (1 + \eta^{-1})L\rho^2\mu(A))(1 + \mu(A_0))$$

(3.13)

and

$$\mu(A_0) \leq (1 + \eta)\mu(A) + c((1 + \eta)(C + \delta^2) + (1 + \eta^{-1})L\rho^2\mu(A_0))(1 + \mu(A))$$

(3.14)

where $\eta \in (0, \infty)$ is a positive constant.

**Proof.** Assume first that $\text{supp}(v) \subset U$, write $a := A - A_0$. We then have

$$\|\tilde{\partial}_A^* v\|^2 - \|\tilde{\partial}_{A_0}^* v\|^2 \leq 2\|a\|_{L^4}^2\|v\|_{L^4}^2.$$

On the other hand, if $\text{supp}(v) \subset \bigcup_{l=1}^L \tilde{B}_{\rho/2}(x_l)$, then

$$\|v\|_{L^2}^2 \leq cL\rho^2\|v\|_{L^4}^2.$$

Let $\psi = \sum \beta_{N, \rho}(x - x_i)$ be a sum of the logarithmic cut-off functions of Lemma 3.16 and $\tilde{\psi} = 1 - \psi$. We now choose $v \in \tilde{\Omega}^{0,2}$ with $\|v\|_{L^2(X)} = 1$. At last, we observe that

$$[\tilde{\psi} v \land \ast \tilde{\psi} v] = 0, \text{ i.e., } \tilde{\psi} v \in \tilde{\Omega}^{0,2}.$$

By the definition of $\mu(A)$, we have

$$\mu(A)\|\tilde{\psi} v\|^2 \leq \|\tilde{\partial}_A^*(\tilde{\psi} v)\|^2.$$
Following the Weitzenböck formula for $v \in \Omega^{0,2}(X, g_p)$, we have

$$\|\nabla A v\|_{L^2(X)}^2 \leq C \|v\|_{L^2(X)}^2 + \|\bar{\partial}^*_A v\|_{L^2(X)}^2 + \|F_A^+\|_{L^2(X)} \|v\|_{L^4(X)}^2$$

$$\leq C \|v\|_{L^2(X)}^2 + \|\bar{\partial}^*_A v\|_{L^2(X)}^2 + C \|F_A^+\|_{L^2(X)} (\|v\|_{L^2(X)}^2 + \|\nabla A v\|_{L^2(X)}^2),$$

where $C$ is a positive constant dependence on $X, g$. Provided $C \|F_A^+\|_{L^2(X)} \leq \frac{1}{2}$, we then have a priori estimate for $v \in \Omega^{0,2}(X, g_p^C)$,

$$\|\nabla A v\|_{L^2(X)}^2 \leq C (\|v\|_{L^2(X)}^2 + \|\bar{\partial}^*_A v\|_{L^2(X)}^2).$$

Combining the above observations, we have

$$\mu(A) \|v\|_{L^2(X)}^2 \leq \mu(A) (\|\psi v\|_{L^2(X)}^2 + \|\bar{\psi} v\|_{L^2(X)}^2 + 2 \langle \psi v, \bar{\psi} v \rangle_{L^2(X)})$$

$$\leq \mu(A)(1 + \eta^{-1}) \|\psi v\|_{L^2(X)}^2 + \mu(A)(1 + \eta) \|\bar{\psi} v\|_{L^2(X)}^2,$$

where $\eta \in (0, \infty)$ is a positive constant.

For the first term on right-hand of (3.15),

$$\mu(A)(1 + \eta^{-1}) \|\psi v\|_{L^2(X)}^2 \leq c(1 + \eta^{-1}) L^2 \mu(A) \|v\|_{L^4(x)}^2,$$

for some positive constant $c = c(g)$.

For the second term on right-hand of (3.8),

$$\mu(A)(1 + \eta) \|\bar{\psi} v\|_{L^2(x)}^2 \leq c(1 + \eta) \|\bar{\psi} v\|_{L^2(x)}^2 + \|\nabla \bar{\psi} v\|_{L^2(x)}^2$$

Combining the preceding inequalities,

$$\mu(A) \leq c(1 + \eta) \|\bar{\psi} v\|_{L^2(x)}^2$$

In the space $\tilde{\Omega}^{0,2}(X, g^C_p)$, we can choose a sequence $v_\varepsilon \in \tilde{\Omega}^{0,2}$, $\varepsilon \to 0$, such that

$$\|\bar{\partial}^*_A v_\varepsilon\|_{L^2(x)}^2 \leq (\mu(A_0) + \varepsilon) \|v_\varepsilon\|_{L^2}^2 \text{ and } \|v_\varepsilon\|_{L^2}^2 = 1.$$

Therefore,

$$\mu(A) \leq c(1 + \eta) (\|\nabla \bar{\psi} v_\varepsilon\|_{L^2(x)}^2 + \|a\|_{L^2(x)}^2) + \mu(A_0) + \varepsilon \mu(A_0) + \varepsilon).$$
Let $\tilde{\xi} \to 0^+$, we then have

$$\mu(A) \leq (1 + \eta)\mu(A_0) + c(1 + \eta)\left(\|\nabla \tilde{\psi}\|_{L^2(X)}^2 + \|a\|_{L^2(U)}^2\right) + (1 + \eta^{-1})\rho^2\mu(A)(1 + \mu(A_0)).$$

Since $\|\nabla \tilde{\psi}\|_{L^2(X)}^2 \leq \frac{C'}{\log N}$ for a uniform constant $C'$, we denote $C = \frac{C'}{\log N}$, (see Lemma 3.16), we then have

$$\mu(A) \leq (1 + \eta)\mu(A_0) + c\left((1 + \eta)(C + \delta^2) + (1 + \eta^{-1})\rho^2\mu(A)\right)(1 + \mu(A_0))$$

Therefore, exchange the roles of $A$ and $A_0$ in the preceding derivation yields the inequality \eqref{3.14} for $\mu(A)$ and $\mu(A_0)$. \hfill $\square$

We then have the convergence of the least eigenvalue of $\tilde{\partial}_{A_i}\tilde{\partial}_{A_i^*}|_{\Omega^{0,2}}$ for a sequence of connections $\{A_i\}_{i \in \mathbb{N}}$ converging strongly in $L^2_{1,\text{loc}}(X \setminus \Sigma)$.

**Corollary 3.18.** Let $G$ be a compact, semisimple Lie group and $P$ be a principal $G$-bundle over a compact Kähler surface $X$ and $\{A_i\}_{i \in \mathbb{N}}$ a sequence of smooth connections on $P$ that converges strongly in $L^2_{1,\text{loc}}(X \setminus \Sigma)$, moduli a sequence $\{u_i\}_{i \in \mathbb{N}} : P_{\infty} \mid X \setminus \Sigma \cong P \mid X \setminus \Sigma$ of class $L^2_{1,\text{loc}}(X \setminus \Sigma)$ to a connection $A_{\infty}$ on a principal $G$-bundle $P_{\infty}$ over $X$. Then

$$\lim_{i \to \infty} \mu(A_i) = \mu(A_{\infty}),$$

where $\mu(A)$ is as in Definition 3.14.

**Proof.** By the Sobolev embedding $L^2_1 \hookrightarrow L^4$ and Kato inequality, we have

$$\|u_i^*(A_i) - A_{\infty}\|_{L^1(U)} \to 0$$

strongly in $L^2_{1,\infty}(U, \Omega^1 \otimes \mathfrak{g}_{P_{\infty}})$ as $i \to \infty$.

Hence from the inequalities on Proposition 3.17, we have

$$\mu(A_{\infty}) \leq (1 + \eta)\liminf_{i \to \infty} \mu(A_i) + c\left((1 + \eta)C + (1 + \eta^{-1})\rho^2\mu(A_{\infty})\right)(1 + \liminf_{i \to \infty} \mu(A_i)) \tag{3.16}$$

and

$$\limsup_{i \to \infty} \mu(A_i) \leq (1 + \eta)\mu(A_{\infty}) + c\left((1 + \eta)C + (1 + \eta^{-1})\rho^2\limsup_{i \to \infty} \mu(A_i)\right)(1 + \mu(A_{\infty})) \tag{3.17}$$

The inequalities \eqref{3.16} and \eqref{3.17} about $\liminf_{i \to \infty} \mu(A_i)$ and $\limsup_{i \to \infty} \mu(A_i)$ hold for every $\rho \in (0, \rho_0]$ and $\eta \in (0, \infty)$. It’s easy to see that $C \to 0^+$ while $\rho \to 0^+$. At first, let $\rho \to 0^+$, we then have

$$\mu(A_{\infty}) \leq (1 + \eta)\liminf_{i \to \infty} \mu(A_i) \leq (1 + \eta)\limsup_{i \to \infty} \mu(A_i) \leq (1 + \eta)^2\mu(A_{\infty}).$$

Next, let $\eta \to 0^+$, thus the conclusion follows. \hfill $\square$

We then have
**Proposition 3.19.** Let $G$ be a compact, semisimple Lie group and $P$ a principal $G$-bundle over a compact Kähler surface $X$. If $\{A_i\}_{i \in \mathbb{N}}$ is a sequence $C^\infty$ connection on $P$ and the curvatures $\|F_{A_i}\|_{L^2(X)}$ converge to zero as $i \to \infty$, then there are a finite set of points, $\Sigma = \{x_1, \ldots, x_L\} \subset X$ and a subsequence $\Xi \subset \mathbb{N}$, we also denote by $\{A_i\}$, a sequence gauge transformation $\{g_i\}_{i \in \Xi}$ such that, $g_i(A_i) \to A_\infty$, a anti-self-dual connection on $P \mid_{X \setminus \Sigma}$ in the $L^2$-topology over $X \setminus \Sigma$. Furthermore, we have

$$\lim_{i \to \infty} \mu(A_i) = \mu(A_\infty),$$

where $\mu(A)$ is as in Definition 3.14.

FollowingFeehan’s idea, we then have

**Theorem 3.20.** Let $X$ be a compact, simply-connected, Kähler surface with a Kähler metric $g$, $P$ be a principal $G$-bundle with $G$ being $SU(2)$ or $SO(3)$. Suppose that the ASD connections $A \in M(P, g)$ are strong irreducible connections in the sense of Definition 2.5. Then there are positive constants $\mu_0 = \mu_0(P, g)$ and $\epsilon = \epsilon(P, g)$ such that

$$\mu(A) \geq \mu_0, \forall [A] \in M(P, g),$$

$$\mu(A) \geq \frac{\mu_0}{2}, \forall [A] \in B_\epsilon(P, g).$$

**Proof.** Combining $\bar{M}(P, g)$ is compact, $\lambda(A)$ is continuous under the Uhlenbeck topology, $\forall [A] \in \bar{M}(P, g)$ and $A$ is a strong irreducible connection, then any connection $A \in \bar{M}(P, g)$ is strong in the sense of Definition 2.5. Following the Proposition 3.15, we then have $\mu(A) \geq \mu_0$.

Suppose that the constant $\epsilon \in (0, 1]$ does not exist. We may then choose a sequence $\{A_i\}_{i \in \mathbb{N}}$ of connection on $P$ such that $\|F_{A_i}^+\|_{L^2(X)} \to 0$ and $\mu(A_i) \to 0$ as $i \to \infty$. Since $\lim_{i \to \infty} \mu(A_i) = \mu(A_\infty)$ and $\mu(A_\infty) \geq \mu_0$, then it is contradict to our initial assumption regarding the sequence $\{A_i\}_{i \in \mathbb{N}}$. In particular, the preceding argument shows that the desired constant $\epsilon$ exists.

**Corollary 3.21.** Let $X$ be a compact, simply-connected, Kähler surface with a Kähler metric $g$, that is generic in the sense of Definition 2.9. $P$ be a $SO(3)$-bundle over $X$. If the second Stiefel-Whitney class, $\omega_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$, is non-trivial, then there are positive constants $\mu_0 = \mu_0(P, g)$ and $\epsilon = \epsilon(P, g)$ such that

$$\mu(A) \geq \mu_0, \forall [A] \in M(P, g),$$

$$\mu(A) \geq \frac{\mu_0}{2}, \forall [A] \in B_\epsilon(P, g).$$

**Proof of Theorem 1.1** For a Yang-Mills connection $A$ on $P$ with $\|F_{A}^+\|_{L^2(X)} \leq \epsilon$, where $\epsilon \in (0, 1)$ is as in the hypothesis of Corollary 3.8, then the curvature $F_A$ satisfies $\bar{\partial}_A F_A^{0,2} = 0, \Lambda_\omega F_A = 0$. Following the Definition of $\mu(A)$ and Theorem 3.20, we have

$$\frac{\mu_0}{2} \|v\|_{L^2(X)}^2 \leq \|\bar{\partial}_A v\|_{L^2(X)}^2, \forall v \in \bar{\Omega}^{0,2}(X, g_P^C).$$

where $\mu_0$ is the uniform positive lower bound in Theorem 3.20. Since $F_A^{0,2}$ is harmonic, $F_A^{0,2} = 0$ on $X$. Thus we complete this proof.
Suppose $\phi \in \Omega^{0,2}(X, g_P^C)$ takes values in a 1-dimensional subbundle of $g_P^C$, i.e., suppose that
\[ \phi = f \otimes \sigma, \]
where $f$ is a $(0, 2)$-form and where $\sigma$ is a section of $g_P^C$ with $|\sigma|^2 = 1$, following the idea of Bourguignon-Lawson [1, Proposition 3.15], we then have a useful

Lemma 3.22. Suppose that the curvature $F_A$ of the connection $A$ obeying $\Lambda_\omega F_A = 0$. The $\phi$ is harmonic with respect to Laplacian operator $\Delta_{\partial A}$ if only if $f$ is harmonic form and $\sigma$ is parallel away from the zeros of $f$.

Proof. The Weizenböck formula for any $\phi \in \Omega^{0,2}(X, g_P^C)$ yields, see (2.2)
\[ \nabla^*_A \nabla_A \phi + 2S\phi = 0. \]

A direct computation shows that
\[ \nabla^*_A \nabla_A \phi = (\nabla^* f) \otimes \sigma - \sum_j (\nabla e_j f) \otimes (\nabla_{A e_j} \sigma) + f \otimes (\nabla^*_A \nabla_A \sigma), \]
and
\[ \langle S\phi, \phi \rangle = \langle Sf, f \rangle. \]

Taking the derivative of the condition $|\sigma|^2 = 1$, we find that $\langle \nabla_A \sigma, \sigma \rangle = 0$. Consequently,
\[ \langle \nabla^*_A \nabla_A \sigma, \sigma \rangle \equiv -\langle \nabla^2_{A e_i e_i} \sigma, \sigma \rangle \equiv \sum_i \langle \nabla_{A e_i} \sigma, \nabla_{A e_i} \sigma \rangle \equiv |\nabla_A \sigma|^2. \]

We then have
\[ \langle \nabla^*_A \nabla_A \phi, \phi \rangle = \langle \nabla^* f, f \rangle + |f|^2 |\nabla_A \sigma|^2. \tag{3.18} \]

The Weitzenböck formula applied to $(0, 2)$-forms on $\Omega^{0,2}(X)$, states that
\[ (d^* d + dd^*) f = \nabla^* \nabla f + 2Sf. \]

Therefore (3.18) can be rewritten as
\[ \langle \Delta f, f \rangle + |f|^2 |\nabla_A \sigma|^2 = 0. \]

Since $\Delta \geq 0$ on $X$ we conclude that $\Delta f = 0$ and that $\nabla_A \sigma = 0$ away from the zeros of $f$. We complete the proof this lemma.

We apply the proof of Lemma 3.11 to ASD connections for group $S^1$. If $A$ is an ASD $S^1$-connection which is flat in the some ball, then in a radial gauge the connection matrix vanishes over the ball and we deduce that $A$ must be flat everywhere. This is a local argument, so applies to any closed ASD 2-form. Of course, we have just the same results for self-dual forms. We obtain then:
Corollary 3.23. ([3] Corollary 4.3.23) Suppose $\omega$ is a closed two-form on $X$ which satisfies $*\omega = \pm \omega$. Then if $\omega$ vanishes on a non-empty open set in $X$ it is identically zero.

Furthermore, if $\omega$ is a harmonic two form on $X$, then $\omega + *\omega$ or $\omega - *\omega$ is self-dual or ASD closed two-form. Then if $\omega \pm *\omega$ all vanish on a non-empty open set in $X$ it is identically zero, i.e., $\omega$ is identically zero.

Proposition 3.24. Suppose that $\omega$ is a smooth harmonic 2-form on a closed, simply-connected, four-manifold $X$. Then if $\omega$ vanishes on a non-empty open set in $X$ it is identically zero.

We then have

Theorem 3.25. Let $X$ be a compact, simply-connected, Kähler surface, $P$ be a principal $G$-bundle over $X$ with $G$ being a compact, semisimple Lie group, $A$ be an irreducible connection on $P$. If the curvature $F_A$ of the connection $A$ obeying $\Lambda_\omega F_A = 0$. Then the harmonic forms take values in a 1-dimensional subbundle of $\mathfrak{g}_P^C$ on $\Omega^{0,2}(X, \mathfrak{g}_P^C)$ with respect to Laplacian operator $\Delta_{\partial_A}$ are zero.

Proof. We set $\phi := f \otimes \sigma$ for any section $\phi$ on $\Omega^{0,2}(X, \mathfrak{g}_P^C)$. We suppose $\phi$ is harmonic with respect to $\Delta_{\partial_A}$, then $|f|\|\nabla_A \sigma\| = 0$ and $df = d^*f = 0$. We denote a closed set

$$Z := \{x \in X : f(x) = 0\} \subset X$$

by the the zero of harmonic form $f$, i.e. the zero of $\phi$. We then have $\nabla_A \sigma = 0$ along $X \setminus Z$, thus the set

$$\tilde{Z} := \{x \in X : \nabla_A \sigma \neq 0\} \subset Z.$$

Since the connection $A$ is irreducible, $\|\nabla_A \sigma\|_{L^2(X)} > 0$, (See Remark 2.6), thus the set $\tilde{Z}$ is non-empty. We could choose a point $p \in \tilde{Z}$ such that $\nabla_A \sigma \neq 0$. Then there is a constant $\rho$ such that the geodesic ball $B_\rho(p) \subset \tilde{Z} \subset Z$. Since $f$ is harmonic $(0, 2)$-form and $f(x) = 0$ for any $x \in B_\rho(p)$, $*f$ is a harmonic $(2, 0)$-form and $*f$ also vanishes on $B_\rho(p)$. Following Proposition 3.24 $f$ is identically zero. Therefore $Z = X$, so $\phi$ is identically zero.

Acknowledgment

We would like to thank Feehan for kind comments regarding his article [6]. This work was partially supported by Nature Science Foundation of China No. 11801539.

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