INVARIANTS OF RIGID SURFACE OPERATORS

CHUANZHONG LI AND BAO SHOU

Abstract. Lusztig used the symbol invariant to describe the Springer correspondence for classical groups. Similarly, the fingerprint invariant can describe the Kazhdan-Lusztig map. Both invariants pertain to rigid semisimple operators labeled by pairs of partitions \((\lambda', \lambda'')\). It is conjectured that the symbol invariant is equivalent to the fingerprint invariant for rigid surface operators. In this study, we provide a proof of this conjecture.

We classify the maps that preserve the fingerprint invariant and demonstrate that they also preserve the symbol invariant. Conversely, we classify the maps that preserve the symbol invariant and show that they also preserve the fingerprint invariant. The constructions of the symbol and fingerprint invariants in prior works are crucial to the proof.

Additionally, we found that one condition in the definition of the fingerprint invariant is redundant for rigid surface operators. In the appendix, we describe an alternative strategy to prove the equivalence of these invariants.

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1. Introduction

Surface operators are two-dimensional defects in four-dimensional gauge theory, which are generalizations of line operators such as Wilson and ’t Hooft operators. Gukov and Witten introduced surface operators in their study of the ramified case of the Geometric Langlands Program, which is realized through $N = 4$ super Yang-Mills theories [2]. $S$-duality for surface operators is studied in [3][5][9]

$S : (G, \tau) \rightarrow (G^L, -1/n_g\tau)$

with $n_g$ be 2 for $F_4$, 3 for $G_2$, and 1 for other semisimple classical groups. And $\tau$ is a complexified gauge coupling constant [2]. We will focus on surface operators in theories with gauge groups $SO(2n)$ and $Sp(2n)$, whose Langlands dual group are $SO(2n)$ and $SO(2n+1)$, respectively. Other gauge groups are trivial or more complicated.

In [4], Gukov and Witten extended their previous study of surface operators [2]. They identified a subclass of surface operators called ‘rigid’ surface operators, which are anticipated to be related to each other under $S$-duality [4][5]. There are two types of rigid surface operators: unipotent and semisimple. The rigid semisimple surface operators are characterized by pairs of partitions $(\lambda', \lambda'')$. The partition is given by

$\lambda'^1 \lambda'^2 \cdots \lambda'^m$, $\lambda_i \in \mathbb{N}$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ as shown in Fig. 1. Unipotent rigid surface operators arise when one partition of $(\lambda', \lambda'')$ is empty.

The invariant known as the symbol is based on the Springer correspondence [1], which is a map from the rigid semisimple surface operators to the unitary representation of the Weyl group. Another invariant, termed the fingerprint related to the Kazhdan-Lusztig map for the classical groups [22][23]. It is a map from the rigid semisimple surface operators to the set of conjugacy classes of the Weyl group [21]. The rigid semisimple conjugacy classes (rigid surface operators) and the conjugacy classes of the Weyl group are represented by pairs of partitions $(\lambda' ; \lambda'')$ and pairs of partitions $[\alpha ; \beta]$, respectively. The symbol invariant $\sigma$ also consists of two partitions. Thus the symbol invariant is a map from $(\lambda' ; \lambda'')$ to $[\alpha ; \beta]$. Similarly, the fingerprint invariant is a map from $(\lambda' ; \lambda'')$ to $\sigma$.

In [5], Wyllard proposed more explicit proposals for the action of $S$-duality maps acting on rigid surface operators, utilizing symbol invariant. In [15], a construction of the symbol invariant is provided. This construction makes the symbol invariant to be calculated simply and convenient to find the $S$-duality maps of rigid surface operators. The foundational properties of the fingerprint invariant and its construction are detailed in [16]. A discrepancy in the total number of rigid surface operators between the $B_n$ and $C_n$ theories, initially noted in [4][5], was addressed using the symbol invariant. This approach allowed for the construction and classification of all problematic surface operators in the $B_n/C_n$ theories [20].

![Figure 1. Partition $\lambda'^1 \lambda'^2 \cdots \lambda'^m$ with length $l = \Sigma n_i$.](image)
It has been verified that the symbol invariant of a rigid surface operator contains the same amount of information as the fingerprint invariant, as shown in Appendix B, leading to the following conjecture [5]:

**Conjecture 1.1.** The symbol invariant is equivalent to the fingerprint invariant for rigid surface operators.

In this study, we provide proof of this conjecture, as illustrated in Fig. (2).

![Figure 2. Classification of invariants preserving maps: \(F_N, F_{N-1}\) and \(F_{C-1}\) are fingerprint preserving maps, \(S_N, S_{N-1}\) and \(S_{C-1}\) are symbol preserving maps. The four \((\lambda', \lambda'')\) are different rigid surface operators with the same invariants.](image)

and \(F_{C-1}\) are the classification of the fingerprint preserving maps, we prove that they preserve the symbol invariant. Here, map (algorithm or operation) refers to a procedure that transforms one rigid surface operator into another. Conversely, \(S_N, S_{N-1}\) and \(S_{C-1}\) are the classification of the symbol preserving maps, we demonstrate that they also preserve the fingerprint invariant. The constructions of the symbol and the fingerprint in previous works play significant roles in proving the conjecture.

The following is an outline of this article. In Section 2, we introduce basic results related to rigid partitions and the rigid surface operators as a foundation. In Section 3, we introduce the definitions of symbol invariant and fingerprint invariant in [5][15]. We also introduce the construction of symbol in [15] which is the basics of this study, as well as that of the fingerprint invariant in [16]. In Section 4, we introduce examples of maps preserving symbol and prove they preserve fingerprint, offering insights and hints for the proof of the conjecture. We first prove the conjecture for \(B_n\) theory and restrict the adjoint rows of the partition \(\lambda = \lambda' + \lambda''\) to satisfy the rigid constraint. Under this constraint, the \(C2\) condition in the definition of the fingerprint invariant is unnecessary.

In Section 5, we classify the fingerprint invariant preserving maps, which are related to the conditions in the definition of the fingerprint. We then prove that these maps also preserve the symbol invariant. Conversely, in Section 6 we classify the symbol invariant preserving maps based on previous work on the construction of the symbol invariant [12][20]. And then we prove they preserve the fingerprint invariant. In Section 7, it is amazing to find that the \(C2\) condition can be omitted for rigid surface operators. Thus, the complete proof of the conjecture for rigid surface operators in \(B_n\) theory can be directly obtained and generalized to \(C_n\) and \(D_n\) theories with minor modifications. In Section 8, we summarize the proof and mention potential applications in future research.

The paper includes two appendices. Appendix A describes another strategy to prove the conjecture based on a different classification of symbol preserving maps. Appendix B
presents the rigid surface operators in the SO(11) and Sp(10) theories, which can be used to verify the proof.

2. Partitions and Rigid Surface Operator

In this section, we provide the necessary background on surface operators as a foundation. We closely following [5] to which we refer the reader for more details.

The bosonic fields of $\mathcal{N} = 4$ super-Yang-Mills theory: a gauge field as 1-form, $A_\mu$ ($\mu = 0, 1, 2, 3$), six real scalars, $\phi_I$ ($I = 1, \ldots, 6$), which take values in the adjoint representation of the gauge group $G$. Let the surface $D$ be located at $x^0 = 0$ and $x^1 = 0$. Surface operators are supported on $D$ with a certain singularity structure of fields near the surface. Since the fields satisfy the half BPS condition, the combinations $A = A_2 dx^2 + A_3 dx^3$ and $\phi = \phi_2 dx^2 + \phi_3 dx^3$ must satisfy Hitchin’s equations

\begin{align}
F_A - \phi \wedge \phi &= 0, \\
\partial_A \phi &= 0, \\
\partial_A \ast A &= 0,
\end{align}

which means a surface operator is defined as a solution of Hitchin equations with a prescribed singularity along the surface $D$.

Let $x_2 + ix_3 = re^{i\theta}$. Then the most general possible rotation-invariant Ansatz for $A$ and $\phi$ is

\begin{align}
A &= a(r) \, d\theta, \\
\phi &= -c(r) \, d\theta + b(r) \, \frac{dr}{r},
\end{align}

Substituting this Ansatz into Hitchin’s equations (2.1) and defining $s = -\ln r$, Hitchin equations (2.1) become Nahm’s equations

\begin{align}
\frac{da}{ds} &= [b, c], \\
\frac{db}{ds} &= [c, a], \\
\frac{dc}{ds} &= [a, b].
\end{align}

The surface operators, along with the communication relations for the constants $a$, $b$ and $c$, were discussed in [2]. Another conformally invariant surface operator solution is given by

\begin{align}
a &= \frac{t_x}{s + 1/f}, \\
b &= \frac{t_x}{s + 1/f}, \\
c &= \frac{t_y}{s + 1/f},
\end{align}

with $[t_x, t_y] = \varepsilon_{xyz} t_z$. $t_x$, $t_y$ and $t_z$ span a reducible representation of $su(2)$, which also belong to the adjoint representation of the gauge group.

Alternatively, the surface operators can be characterized through the complexified conjugacy class of the monodromy

\begin{align}
U = P \exp\left( \frac{2\pi i}{s + 1/f} A \right),
\end{align}

where $A = A + i\phi$. The integration contour is a constant circle near $r = 0$. From Hitchin’s equations (2.1), it follows that $\mathcal{F} = \partial_A + A \wedge A = 0$, which means that $U$ is independent of deformations of the integration contour. According to formula (2.2), $U$ can be expressed as

\begin{align}
U = P \exp\left( \frac{2\pi}{s + 1/f} t_+ \right),
\end{align}

where $t_+ \equiv t_+ + it_y$ is nilpotent, corresponding to unipotent surface operator. When considering a semisimple element $S$, the general monodromy of a surface operator is

\begin{align}
V = SU.
\end{align}
Both the unipotent and semisimple classes lead to surface operators. The field $\Psi(r, \theta)$ associated a surface operator is required to satisfy the following constraint near the surface $D$

$$S\Psi(r, \theta)S^{-1} = \Psi(r, \theta + 2\pi),$$

which breaks the gauge group to the centralizer of the semisimple element $S$. Therefore surface operators correspond to the classification of unipotent and semisimple conjugacy classes, which have been solved by mathematicians [1].

Rigid surface operators are closed on the $S$-duality and form a subset of the surface operators constructed from conjugacy classes. A unipotent conjugacy class is called rigid [1] if its dimension is strictly smaller than that of any nearby orbit. Similarly, a semisimple conjugacy class $S$ is called rigid if the centralizer of such class is larger than that of any nearby class. In summary, rigid surface operators correspond to monodromies of the form $V = SU$, where $U$ is unipotent and rigid and $S$ is semisimple and rigid. All rigid orbits have been classified [4].

For the rigid surface operators in $B_n(SO(2n+1))$, $C_n(Sp(2n))$ and $D_n(SO(2n))$ theories, the element $t_+$ in Eq. (2.2) can be described in block-diagonal basis as follows

$$t_+ = \begin{pmatrix} t_+^{n_1} \\ \vdots \\ t_+^{n_l} \end{pmatrix},$$

where $t_+^{n_i}$ is the ‘raising’ generator of the $n_i$-dimensional irreducible representation of $su(2)$. There are restrictions on the allowed dimensions of the $su(2)$ irreps. The $t_i$'s must also belong to the adjoint representation of the gauge group. From the decomposition (2.4), unipotent (nilpotent) surface operators are classified by the restricted partitions

$$n_1 \geq n_2 \geq \cdots \geq n_l$$

with $\sum_{i=1}^l n_i = n$. Partitions correspond one-to-one correspondence with Young tableaux. For instance, the partition $3^22^11^3$ corresponds to

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Young tableaux find applications in various branches of mathematics and physics [25]. They are utilized in constructing the eigenstates of the Hamiltonian System in AGT correspondence [24].

The classification of nilpotent orbits in terms of restricted partitions is as follows [2]:

- $(B_n)$: partitions of $2n+1$, $\sum \lambda_i = 2n+1$, with all even part $\lambda_i$ appearing an even number of times.
- $(D_n)$: partitions of $2n$, $\sum \lambda_i = 2n$, with all even part $\lambda_i$ appearing an even number of times.
- $(C_n)$: partitions of $2n$, $\sum \lambda_i = 2n+1$, with all odd part $\lambda_i$ appearing an even number of times.

A partition is called rigid if it satisfies the following conditions:

**Definition 2.1. (Rigid condition)**

1. No gaps i.e. $\lambda_i - \lambda_{i+1} \leq 1$ for all $i$.
2. No odd (even) part appears exactly twice in the $B_n$ or $D_n(C_n)$ theories.

We will focus on rigid partitions, corresponding to rigid operators. The following properties are crucial to this study, which are easily derivable from definitions [1].

$^1$The rigid surface operators here correspond to strongly rigid operators in [1].
Proposition 2.1. For a rigid $B_n$ partition, the longest row always contains an odd number of boxes. The two rows following the first row are either both of odd length or both of even length, continuing this pairwise pattern. If the Young tableau has an even number of rows, the shortest row must be of even length.

Proposition 2.2. For a rigid $D_n$ partition, the longest row always contains an even number of boxes. The two rows following the first row are either both of even length or both of odd length, continuing this pairwise pattern. If the Young tableau has an even number of rows, the shortest row must be of even length.

Proposition 2.3. For a rigid $C_n$ partition, the two longest rows both contain either an even or an odd number of boxes, continuing this pairwise pattern. If the Young tableau has an odd number of rows, the shortest row must contain an even number of boxes.

Examples of partitions in the $B_n$, $D_n$, and $C_n$ theories are shown in Fig. 3:

![Partitions in the $C_n$, $B_n$, and $D_n$ theories.](image)

Figure 3. Partitions in the $C_n$, $B_n$, and $D_n$ theories.

From Fig. 3, the following observations are frequently used in this study:

Lemma 2.1. (1) There is an odd number of rows under the bottom row of a pairwise rows of the partitions in the $B_n$ and $D_n$ theories.

(2) There is an even number of boxes above the top row of a pairwise rows of the partitions in the $B_n$ and $D_n$ theories.

(3) There is an even number of rows under the bottom row of a pairwise rows of the partitions in the $C_n$ theory.

(4) There is an even number of boxes above the top row of a pairwise rows of the partitions in the $C_n$ theory.

At the end of this section, the following conventions are introduced:

- For the rigid surface operator $(\lambda', \lambda'')$, the partition $\lambda$ is always given by
  \[ \lambda = \lambda' + \lambda'' , \]
  where the addition of partitions is defined by the additions of each part
  \[ \lambda_i + \mu_i . \]

- ‘Operator’ or ‘rigid semisimple surface operator’ will refer to the rigid surface operator in the rest of the paper.

- For the rigid surface operator $(\lambda', \lambda'')$ in $B_n$ theory, the first partition $\lambda'$ is in $B_n$ theory and the second partition $\lambda''$ is in $D_n$ theory.

- An even row means the length of the row is even.

- An even pairwise rows means a pair of even rows.

- Rows of a partition are indexed from bottom to top.

- Without causing confusion, a single row will be denoted with the same notation, such as $t$ and $b$, regardless of its length, the partition it belongs to, or its position in a pair of rows.

- Without causing confusion, the image of rigid surface operator $(\lambda', \lambda'')$ will also be denoted as $(\lambda', \lambda'')$ under the symbol preserving such as the ones in Fig. 2.
3. Invariants of Rigid Surface Operator

In this section, we introduce the definitions of the symbol invariant and fingerprint invariant of rigid surface operators, along with their respective constructions [3, 15, 16].

Invariants of rigid surface operators \((\lambda'; \lambda'')\), such as the fingerprint invariant, symbol invariant, and dimension, remain unchanged under the \(S\)-duality map [1, 3]. The simplest invariant of rigid surface operators is the dimension \(d\), given by the following formulas [3, 15, 16]:

\[
B_n : \quad d = 2n^2 + n - \frac{1}{3} \sum_k (s'_k)^2 - \frac{1}{3} \sum_k (s''_k)^2 + \frac{1}{2} \sum_{k \text{ odd}} r'_k + \frac{1}{2} \sum_{k \text{ odd}} r''_k,
\]

\[
D_n : \quad d = 2n^2 - n - \frac{1}{3} \sum_k (s'_k)^2 - \frac{1}{3} \sum_k (s''_k)^2 + \frac{1}{2} \sum_{k \text{ odd}} r'_k + \frac{1}{2} \sum_{k \text{ odd}} r''_k,
\]

\[
C_n : \quad d = 2n^2 + n - \frac{1}{3} \sum_k (s'_k)^2 - \frac{1}{3} \sum_k (s''_k)^2 - \frac{1}{2} \sum_{k \text{ odd}} r'_k - \frac{1}{2} \sum_{k \text{ odd}} r''_k,
\]

where \(s'_k\) denotes the number of parts of \(\lambda'\)'s that are greater than or equal to \(k\). And \(r'_k\) denotes the number of parts of \(\lambda'\) equal to \(k\). Similarly, \(s''_k\) and \(r''_k\) correspond to the other partition \(\lambda''\) of the rigid surface operator.

Other discrete quantum numbers, such as ‘centre’ and ‘topology’, are interchanged under \(S\)-duality [1]. The fingerprint invariant and the symbol invariant are finer than the dimension invariant \(d\) and the discrete quantum numbers.

3.1. Symbol Invariant of Partitions

The symbol invariant is based on the Springer correspondence extended to rigid semisimple conjugacy classes \((\lambda'; \lambda'')\) [3]. In [16], an equivalent definition of the symbol invariant for the \(C_n\) and \(D_n\) theories was presented, ensuring consistency with the \(B_n\) theory.

Definition 3.1. [14] (Symbol invariant) The symbol of a partition \(\lambda\) is calculated as follows.

- **\(B_n\) theory:**
  1. Add \(l - k\) to the \(k\)th part of the partition.
  2. Arrange the odd parts and the even parts of the sequence \(l - k + \lambda_k\) in an increasing sequence \(2f_i + 1\) and in an increasing sequence \(2g_i\), respectively.
  3. Calculate the terms
     \[
     \alpha_i = f_i - i + 1, \quad \beta_i = g_i - i + 1.
     \]
  4. The symbol invariant is written as follows
     \[
     \sigma(\lambda) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \beta_1 & \beta_2 & \cdots \end{pmatrix}.
     \]

- **\(C_n\) theory:** There are two cases.
  1. If the length of partition is even, compute the symbol as that in the \(B_n\) case, and then append an extra \(0'\) on the left of the top row of the symbol.
  2. If the length of the partition is odd, first append an extra \(0'\) as the last part of the partition. Then compute the symbol as that in the \(B_n\) case. Finally, delete a \(0'\) which is in the first entry of the bottom row of the symbol.

- **\(D_n\) theory:** first append an extra \(0'\) as the last part of the partition, and then compute the symbol as in the \(B_n\) case. Finally, delete two \(0's\) which are in the first two entries of the bottom row of the symbol.

For the symbol \(\sigma\) of rigid semisimple operators \((\lambda'; \lambda'')\), it is obtained by adding the entries that are ‘in the same place’ of symbols of \(\lambda'\) and \(\lambda''\)

\[
\sigma(\lambda') + \sigma(\lambda'').
\]

For example,

\[
\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 & 2 & 3 \end{pmatrix}.
\]
Table 1. Contributions of rows in partitions to symbol in the $B_n$, $D_n$, and $C_n$ theories.

| Parity of length of row | Length | Location in a pairwise rows | Contribution |
|-------------------------|--------|----------------------------|--------------|
| odd                     | $2n + 1$ | top                        | $(0 \ 0 \ldots 0 \ 0 \ldots 0)_{n+1}^\top$ |
| odd                     | $2n + 1$ | bottom                     | $(0 \ 0 \ldots 1 \ 1 \ldots 1)_{n+1} \ 0 \ 0 \ldots 0$ |
| even                    | $2m$    | bottom                     | $(0 \ 0 \ldots 1 \ 1 \ldots 1)_{m} \ 0 \ 0 \ldots 0$ |
| even                    | $2m$    | top                        | $(0 \ 0 \ldots 1 \ 1 \ldots 1)_{m} \ 0 \ 0 \ldots 0$ |

Proposition 3.1. The contribution to the symbol of the bottom row of an odd pairwise rows with length $L$ is the same as that of the top row in an even pairwise rows with length $L + 1$. The contribution to the symbol of the top row of an odd pairwise rows with length $L$ is the same as that of the bottom row in an even pairwise row with length $L - 1$.

Note that Proposition 3.1 and Table 1 are independent of the specific theories. This independence implies that the contribution to the symbol of a row is invariant. In other words, given the contribution to the symbol, there are always a finite number of possible lengths and locations for a row.

Remark 3.1. The following three sets give the same symbol invariant:

1. Symbol invariant $\sigma((\lambda', \lambda''))$.
2. Rigid surface operator $(\lambda', \lambda'')$.
3. Length and location of each row in $(\lambda', \lambda'')$.

3.2. Fingerprint Invariant of Partitions

The fingerprint invariant is derived from rigid surface operators $(\lambda'; \lambda'')$ using the Kazhdan-Lusztig map. It consists of a pair of partitions $[\alpha; \beta]$ associated with the Weyl group conjugacy class. The fingerprint invariant of $(\lambda', \lambda'')$ is constructed as follows.

Definition 3.2. (Fingerprint invariant)

1. First, add the two partitions $\lambda = \lambda' + \lambda''$, then construct the partition $\mu$ as follows:

\[
\mu_i = S_p(\lambda)_i = \begin{cases} 
\lambda_i + p_\lambda(i) & \text{if } \lambda_i \text{ is odd and } \lambda_i \neq \lambda_i - p_\lambda(i), \\
\lambda_i & \text{otherwise}, 
\end{cases}
\]

where $p_\lambda(i) = (-1)^{\sum_{k=1}^{i} \lambda_k}$

2. Next, define the function $\tau$ from an even positive integer $2m$ to $\pm 1$ as follows.
   - For $B_n(D_n)$ partitions,
Parity of the height of row | Sign of \((-1)^{p_\lambda(i)}\) | Change of the end of a row
--- | --- | ---
odd | - | \(\mu_i = \lambda_i - 1\)
even | - | \(\mu_i = \lambda_i + 1\)
even | + | \(\mu_i = \lambda_i\)
odd | + | \(\mu_i = \lambda_i\)

Table 2. Changes of the end of the rows in \(\lambda\) under map \(\mu\).

\[- \tau(2m) = -1, \text{ if at least one } \mu_i \text{ such that } \mu_i = 2m \text{ and any of the following three conditions is satisfied.}\]

\((C1), \quad \mu_i \neq \lambda_i\)

\((C2), \quad \sum_{k=1}^{i} \mu_k \neq \sum_{k=1}^{i} \lambda_k\)

\((C3)_{SO}, \quad \lambda_i' \text{ is odd.}\)

\[- \text{ Otherwise, } \tau(2m) = 1.\]

- For \(C_n\) partitions, the definition is the same except that the condition \((C3)_{SO}\) is replaced by

\((C3)_{Sp}, \quad \lambda_i' \text{ is even.}\)

(3) Finally, construct a pair of partitions \([\alpha; \beta]\). For each pair of parts of \(\mu\) both equal to \(a\) with \(\tau(a) = 1\), retain one part \(a\) as a part of the partition \(\alpha\). For each part of \(\mu\) of size \(2b\) with \(\tau(2b) = -1\), retain \(b\) as a part of the partition \(\beta\).

Remark 3.2. Note that the part \(2m = \lambda_i = \lambda_i' + \lambda_i''\) corresponds to the height of the 2\textit{m}th row in \(\lambda\).

- ‘(3)\(_{SO}\) \(\lambda_i'\) is odd’ is equivalent to ‘(3)\(_{SO}\) \(\lambda_i''\) is odd’.
- ‘(3)\(_{Sp}\) \(\lambda_i'\) is even’ is equivalent to ‘(3)\(_{Sp}\) \(\lambda_i''\) is even’.
- We introduce some conventions.
  - \(\mu = \mu(\lambda)\) stands for the map \(\mu\) or the image of the map.
  - \(C_i, i = 1, 2, 3\) are the conditions in formula \((3.8)\) in Definition 3.2.
  - \(p_\lambda(i) = (-1)^{\sum_{k=1}^{i} \lambda_k}\).

According to formula \((3.7)\), \(\mu_i \neq \lambda_i\) only happens at the end of a row with one box appended or deleted. The following important lemma can be get from the definition of fingerprint invariant \([16]\) (they will be used frequently.).

Lemma 3.1. Under the map \(\mu\), the change of \(i\)th row of the partition \(\lambda\) is shown in Table 3, which depends on the parity of its height and the sign of \((-1)^{p_\lambda(i)}\).

These results are visualized in Figs. 4 and 5. The symbol ‘\(\pm\)’ indicate the sign of \((-1)^{p_\lambda(i)}\) corresponding to \(i\)th part of \(\mu\).

(1) A box is deleted only at the end of rows with odd heights and the factor \((-1)^{p_\lambda(i)} = -1\), as illustrated in Fig. 4(a) and (c).

(2) A box is appended only at the end of rows with even heights and the factor \((-1)^{p_\lambda(i)} = -1\), as illustrated in Fig. 5(a).

(3) When a box is appended to a row with even height, the row above it must have a box deleted at the end of row, as illustrated in Fig. 5(a).

(4) When a row with an even height remains unchanged, the row above it also remains unchanged, as illustrated in Fig. 3(b).

Lemma 3.1 is illustrated by the following example as shown in Fig. 6.

\(^2\)In this paper, the grey box represents an appended box and the black box represents a deleted box.
Figure 4. Image of parts $k^{th}$ ($k$ is odd) under the map $\mu$.

Figure 5. Image of parts $k^{th}$ ($k$ is even) under the map $\mu$.

Figure 6. $\mu(\lambda)$ is the partition of $\lambda = \lambda' + \lambda''$ under the map $\mu$.

Example 1. Assume the number of boxes above the $(2k - 1)th$ row is even. The $2kth$ row and $(2k - 1)th$ row have different parities, then a box is deleted at the end of the $(2k - 1)th$ row and a box is appended at the end of the $(2k - 2)th$ row. This pattern continues until the $2i$th row and $(2i - 1)th$ row have different parities.

From the above lemma, we derive the following facts:

Proposition 3.2. (1) Starting from the highest row, a box is always deleted before a box is appended, as illustrated in Fig. [6].
(2) The deletion of a box must be followed by the appending of a box, ensuring that the number of boxes is balanced, i.e.,

\[ \sum_{k=1}^{i} \mu_k = \sum_{k=1}^{i} \lambda_k \]

at the end of the even height row of \( \mu(\lambda) \), as shown in Fig. 3.

(3) Fig. 4(c) is the starting part of Fig. 5(a). Fig. 4(b) is the ending part of Fig. 5(a).

(4) From Fig. 5(b), if no change happens to the part \( k \) under the map \( \mu \), \( k \)th row of \( \lambda \) is equal to the \( k \)th row of \( \mu \).

(5) From Fig. 4 and Fig. 5, for the even and odd rows of \( \lambda \), we have

\[ \text{length}(2m\text{th row of } \lambda) \leq \text{length}(2m\text{th row of } \mu(\lambda)), \]

and

\[ \text{length}((2m+1)\text{th row of } \lambda) \geq \text{length}((2m+1)\text{th row of } \mu(\lambda)). \]

From Fig. 4 and Fig. 5, more accurate, we have

\[ |\text{length}(k\text{th row of } \lambda) - \text{length}(k\text{th row of } \mu(\lambda))| \leq 1. \]

**Remark 3.3.**

(1) The difference between \( \lambda \) and \( \mu \) is less than one box in each row, as shown in Figs. 4 and 5 (or formula (mul3)). Additionally, the difference between rows with the same contribution to the symbol is also less than one box per row, as indicated in Table 3.

(2) For the above reason and Propositions 2.1, 2.2, and 2.3, the height of a row in \( \lambda \) would not change under both invariants preserving maps.

(3) For the above reason, we can give each row of \( \lambda' \) (or \( \lambda'' \)) a name, without a confusion under symbol or fingerprint preserving maps.

Note that the fingerprint invariant is finer than the invariant \( \mu \).

**Proposition 3.3.** Under the map \( \mu \), both \( \mu(\lambda) \) and \( \tau(2m) \) are invariants of operators \( (\lambda', \lambda'') \) with the same fingerprint invariant.

**Proof.** Let \((\alpha, \beta)\) be the fingerprint invariant of operators \((\lambda', \lambda'')\). According to the calculation of the fingerprint invariant, \( \mu(\lambda) \) is constructed by doubling parts of \( \alpha \) and each part of \( \beta \) multiplied by two. Thus it is the same for operators with the same fingerprint invariant.

With different values of \( \tau(2m) \), the contribution of part \( 2m \) to the fingerprint invariant is different. Thus it is the same for operators with the same fingerprint invariant.

The following proposition is critical in the proof of the conjecture. As illustrated in Figs. 4 and 5, under the map \( \mu \), the changes of the odd rows of \( \lambda \) are determined by that of the even rows.

**Proposition 3.4.** The even rows of a partition \( \lambda \) determine the fingerprint invariant completely.

**Proof.** According to Lemma 3.3, the end of the \((2k+1)\)th row will lose one box if the end of the \((2k)\)th row gains a box. The \((2k+1)\)th row remains unchanged if the \((2k)\)th row does not change. Thus, the changes in the even rows determine the changes in the odd rows under the map \( \mu \).

According to Proposition 3.3, \( \mu(\lambda) \) is an invariant and does not change under fingerprint-preserving maps.

Moreover, the conditions 3.2 in Definition 3.2 need not be considered when we calculate the partition \( \alpha \) for the odd parts of \( \lambda \). We only need to consider them \((\tau(2m))\) of the even parts \( 2m \) of \( \lambda \).
Remark 3.4. We give more evidences to this important lemma. The values of $\tau(2m)$ are derived from the conditions $C_i$’s. The following three sets give the same fingerprint invariant:

1. Rigid surface operator $(\lambda', \lambda'')$.
2. $\mu(\lambda)$ and $C_i$’s for all even parts.
3. $\rho(\lambda)$ and $\tau(2m)$ for all even parts.

The following fact is the result of Propositions 3.2(1), (2), and (3).

**Lemma 3.2.** If a part $2m$ satisfy the condition $C_2$, the first odd part before it must satisfy the condition $C_1$.

The above lemma suggests a possible example where condition $C_2$ holds, but condition $C_1$ does not.

**Example 2.** As shown in Fig. 7, the heights of the $(j−1)$th, $(j−3)$th, and $(j−5)$th rows are even. The height difference between the $(j−1)$th row and the $(j−3)$th row, as well as between the $(j−3)$th row and the $(j−5)$th row, is $\lambda_i - \lambda_{i+1} = 2$ (Note that $\lambda = \lambda' + \lambda''$ does not have to satisfy the rigid constraint.). As shown in Fig. 7, we have $\mu_m = \lambda_m$, which does not satisfy the condition $C_1$, but it satisfy the condition $C_2$

$$\sum_{k=1}^{m} \mu_k \neq \sum_{k=1}^{m} \lambda_k.$$ 

Therefore, the part $\lambda_m = j−3$ satisfies the condition $C_2$ but does not satisfy the condition $C_1$.

Immediately, we conclude:

**Lemma 3.3.** (Rigid constraint) For the partition $\lambda$, $\lambda_i - \lambda_{i+1} \leq 1$ is called rigid constraint. Under the rigid constraint of $\lambda$, the condition $C_1$ implies $C_2$.

**Remark 3.5.**

1. Note that rigid constraint is consistent with Rigid condition 2.1 but they are different.
2. Since both rows in $\lambda'$ and $\lambda''$ have different lengths, we will not get $\lambda_i - \lambda_{i+1} > 2$ in $\lambda$.
3. Not until Section 7, will we consider the partition $\lambda = \lambda' + \lambda''$ which has parts satisfy $\lambda_i - \lambda_{i+1} = 2$.

**Figure 7.** Example satisfying only condition $C_2$: Even part $\lambda_{i+1} = j−3$ with $\lambda_i - \lambda_{i+1} = 2$.

4. Examples

In this section, we introduce examples of maps that preserve symbols and demonstrate that these maps also preserve fingerprint invariants. These examples provide insights into proving the equivalence of symbol and fingerprint invariants.

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3 In Section 7.2, we will prove that this case is impossible for rigid surface operators.
INVARIANTS OF RIGID SURFACE OPERATORS

Figure 8. Symbol preserving map $EE : (\lambda', \lambda'') \rightarrow (\lambda_T', \lambda_T'')$, swapping row $e_1$ in $\lambda'$ with row $e_3$ in $\lambda''$. Rows $e_1, e_2$ and $e_3, e_4$ are even pairwise rows. Rows $e_3, e_2$ and $e_1, e_4$ are even pairwise rows.

The first example is illustrated in Fig. 8. The map $EE$ swaps the top row $e_1$ in an even pairwise rows with the top row $e_3$ in an even pairwise rows. According to Table 1, the contribution to the symbol of each row does not change under the map, thereby preserving the symbol. Under the map $EE$, $\lambda$ and $\lambda_T$ are equal as well as their images under the map $\mu$ as shown in Fig. 9. To calculate the fingerprint, we need to calculate

Figure 9. $\mu(\lambda)$ is exactly the same with $\mu(\lambda_T)$ under the map $EE$.

$\tau(2m)$ for each even part $2m$ in $\mu(\lambda)$, which is the same for $\lambda$ and $\lambda_T$ as shown in Table 3. Hence, the map $EE$ preserve the fingerprint invariant. Note the part $2m = 4$ does not satisfy both conditions $C1$ and $C3$.

| $2m$ | $C1$ | $C3$ | $\tau(m)$ |
|------|------|------|-----------|
| 6    | N    | Y    | -1        |
| 4    | N    | Y    | -1        |
| 2    | N    | Y    | -1        |

Table 3. Values $\tau(2m)$ for even rows of the partition $\mu(\lambda)$. ‘N’: Condition be not satisfied and ‘Y’: condition be satisfied.

Figure 10. Symbol preserving map $EO : (\lambda', \lambda'') \rightarrow (\lambda^{T''}, \lambda^{T''})$. It swaps $e_2$ and $o_1$, omitting one box at the end of $o_1$ and appending one box at the end of $e_2$ with $e_1 < e_2 < o_1 < o_2$.

The second example is shown in Fig. 10, the bottom row of an even pairwise rows of $\lambda'$ swap with the top row of an odd pairwise rows of $\lambda''$. After the map, $o_1$ has omitted
a box at the end of the row and $e_2$ is appended a box at the end of row. According to Table 1, the map $EO$ preserves the symbol.

Now we prove the map $EO$ preserve fingerprint. First we calculate the fingerprint invariant on the left-hand side of the map. The image of $\lambda$ under $\mu$ is given by Fig. 11.

And the function $\tau(2m)$ for even parts $m$ in $\mu(\lambda)$ is given by Table 4.

| $2m$ | $C1$ | $C3$ | $\tau(m)$ |
|------|------|------|-----------|
| 6    | N    | Y    | -1        |
| 4    | N    | Y    | -1        |
| 2    | N    | Y    | -1        |

Table 4. $\tau(m)$ for even rows of the partition $\lambda$.

Next we calculate the fingerprint invariant of the partition $\lambda_T$ on the right hand side of the map $EO$. The image of $\lambda_T$ under $\mu$ is given by Fig. 12. Note that $\mu(\lambda_T) = \mu(\lambda)$. And the function $\tau(2m)$ of $\lambda_T$ is given in Table 5. Compared with Table 4 the part $2m = 4$ satisfies different conditions with the same $\tau(4)$.

The calculation of the above two examples provides the following insights, which will be rigorously proved in subsequent sections:

- The rigid operators $(\lambda', \lambda'')$ with the same symbol can be obtained by combinations of different rows according to Table 1.
  - (1) There are combinations of rows that do not change the lengths of the rows, as shown in the first example.
(2) There are combinations of rows that do change the lengths of the rows, as demonstrated in the second example.

- The rigid operators \((\lambda', \lambda'')\) with the same fingerprint invariant correspond to the parts that satisfy different conditions in formula (3.8).

1. The condition satisfied by the part does not change under symbol preserving maps, as in the first example.
2. The condition satisfied by the part changes under symbol preserving maps, as in the second example.

5. Fingerprint Preserving Maps Preserve the Symbol Invariant

As demonstrated in the previous section, operators with the same invariant can have different configurations. In general, operators with the same symbol or fingerprint can vary significantly, as shown in Appendix B.

To prove the fingerprint invariant is equivalent to the symbol invariant, we need to prove that if two rigid operators have the same fingerprint, they also have the same symbol and vice versa. Since operators are composed of rows combined in various ways, operators with the same invariants can differ. Therefore, we consider the contribution to invariants from each row in partitions locally.

For the fingerprint, according to Proposition 3.3, the \(\mu(\lambda)\) is an invariant, meaning each row of \(\lambda\) reduces to the same row in \(\mu(\lambda)\) under the map \(\mu\). Different reductions correspond to different conditions satisfied by rows of the partition according to formula (3.8). To prove the equivalence of the two invariants, we must show that these combinations preserve the symbol invariant.

First, we classify the fingerprint preserving maps. According to Proposition 3.4, we only need to consider the even parts \(2m\) of \(\lambda\). Then, we prove that the classes of maps correspond to different classes of conditions satisfied by the even part. According to Lemma 3.3, condition \(C_1\) implies condition \(C_2\) for the partition \(\lambda\) with Rigid constraint \(3.3: \lambda_i - \lambda_{i+1} \leq 1\). Thus, we omit condition \(C2\) for the moment until Section 7.

We will prove the conjecture for \(B_n\) theory. The proof of the conjecture of other theories will be discussed in Section 7.

5.1. Classification of Fingerprint Preserving Maps

In this subsection, we derive more results on the fingerprint invariant in preparation for the next two subsections.

The following lemma, derived from Lemma 3.1, is illustrated in Fig. 5.

**Lemma 5.1.** Under the map \(\mu\), an even part \(2m\) of the partition \(\lambda\)

1. that satisfy the condition \(C_1\) implies the \(2m\)th row has been appended a box and the \((2m + 1)\)th row has been deleted a box at the end of the row, as shown in Fig. 5(a).
2. that does not satisfy the condition \(C_1\) implies no change of the \(2m\)th row and the \((2m + 1)\)th row, as shown in Fig. 5(b).

The following lemma is derived from Definition 3.2 and Remark 3.2.

**Lemma 5.2.** If the even part \(2m = \lambda'_i + \lambda''_i\) does not satisfy the condition \(C3\), then \(\lambda'_i\) is even and \(\lambda''_i\) is also even. If the even part \(2m = \lambda'_i + \lambda''_i\) satisfies the condition \(C3\), then \(\lambda'_i\) is odd and \(\lambda''_i\) is also odd.

We have the following critical proposition.

**Proposition 5.1.** The part \(2m\) can not satisfy both conditions \(C1\) and \(C3\).
Figure 13. Rigid surface operator \((\lambda', \lambda'')\) with \(\lambda = \lambda' + \lambda''\). \(t\) is the top row of an even pairwise rows in (a). The second ‘even’; means the height of the row \(t\) is even with value \(2m\).

Proof. Assume \(t\) is the \(2m\)th row of \(\lambda\) as shown in Fig. (13)(b). Suppose it satisfies both conditions \(C_1\) and \(C_3\). Since the part \(2m\) satisfy the condition \(C_3\), \(\lambda'\) is odd and \(\lambda''\) is also odd, according to Lemma 5.2. Without loss generality, assume \(t\) is the top row of a pairwise row of \(\lambda\) according to Proposition 2.1 as illustrated in Fig. (13)(a). There is an odd number of rows of \(\lambda''\) under \(t\) in \(\lambda\), according to Proposition 2.2.

According to Propositions 2.1 and 2.2, there are an even number of boxes above \(t\) in \(\lambda\). Hence, the factor \((-1)^{\rho(\lambda)} = 1\) at the end of \(t\)th row, indicating no change of the end of \(t\) under the map \(\mu(\lambda)\), according to the third row of Table 2. According to Lemma 3.1(4), the end of the \((2m + 1)\)th row does not change under the map \(\mu\).

As shown in Fig. (5)(b), the part \(2m\) does not satisfy the condition \(C_1\), which contradicts the assumption. □

According to Definition 5.2, the following proposition is evident.

Proposition 5.2. If the part \(2m\) does not satisfy both conditions \(C_1\) and \(C_3\), then \(\tau(2m) = 1\).

Remark 5.1. The above proposition implies that the even part \(2m\) of the partition \(\lambda\) either satisfies one of the conditions \(C_1\) and \(C_3\) or does not satisfy both.

Using Lemma 5.1 and Proposition 5.1, we have

Proposition 5.3. An even \(2m\) part of \(\lambda\) is classified as follows:

1. The part \(2m\) does not satisfy both the \(C_1\) condition and \(C_3\) condition \((\tau(2m) = 1)\), corresponding to Fig. (5)(b).
2. The part \(2m\) satisfies the \(C_1\) condition \((\tau(2m) = -1)\), corresponding to Fig. (5)(a).
3. The part \(2m\) satisfies the \(C_3\) condition \((\tau(2m) = -1)\), corresponding to Fig. (5)(b).

Then, we can classify the fingerprint preserving maps as follows.

Proposition 5.4. There are two classes of fingerprint preserving maps for the even \(2m\) part of \(\lambda\):

1. each case in Proposition 5.3 is preserved.
2. Swap of cases 2 and 3 in Proposition 5.3.

Proof. For the even part \(2m\), the conditions \(\tau(2m) = 1\) and \(\tau(2m) = -1\) do not change under the fingerprint preserving map.

- When \(\tau(2m) = 1\), the part \(2m\) does not satisfy both the conditions \(C_1\) and \(C_3\).
- When \(\tau(2m) = -1\), according to Proposition 5.1 the conditions \(C_1\) and \(C_3\) would either swap under the fingerprint preserving map or the part \(2m\) satisfies the same condition under the map.
Let the first class of fingerprint preserving maps be the maps preserving the conditions in the definition of fingerprint. Let the second class of fingerprint preserving maps be the maps swapping the conditions C1 and C3 when $\tau(2m) = -1$.

The following conclusion is required.

**Proposition 5.5.** The classifications in Propositions 5.3 and 5.4 are complete.

*Proof.* This is the result of Remark 5.4 and Proposition 5.1.

---

### 5.2. Fingerprint Preserving Maps Without Change of Length of Row

For the first class fingerprint preserving maps in Proposition 5.4 we have the following proposition.

**Proposition 5.6.** If the conditions satisfied by an even part $2m$ of $\lambda$ does not change under fingerprint preserving maps, the length of the $2m$th row of $\lambda$ and its location in pairwise rows does not change under fingerprint preserving maps.

*Proof.* The location of the $2m$th row of $\lambda$ in a pairwise rows corresponds to condition C3. Condition C3 (satisfied or not) is preserved under the first class of fingerprint preserving maps. Thus, the location of the $2m$th row in pairwise rows does not change.

Note that we can only append or delete one box at the end of the $2m$th row under the map $\mu$ according to Lemma 3.1.

- Assume the part $2m$ does not satisfy both the conditions C1 and C3. Then it also does not satisfy both conditions C1 and C3 under the first class of fingerprint preserving maps, which means $\tau(2m) = 1$. The part $2m$ corresponds to Fig. 5(b), according to Proposition 5.3(1), which means
  
  $$\text{length of } 2m \text{th row of } \mu(\lambda) = \text{length of } 2m \text{th row of } \lambda$$

  If right hand side of the identity changes under the fingerprint preserving map, then the value $\alpha$ in the fingerprint $(\alpha, \beta)$ involving the the part $2m$ with $\tau(2m) = 1$ changes, which is a contradiction.

- Assume the part $2m$ satisfies the condition C1 only. According to Proposition 5.3(2), it corresponds to Fig. 5(a). If the length of $2m$th row changes under fingerprint preserving map, the length of $2m$th row must be appended one box to get the same $\mu(\lambda)$, which corresponds to Fig. 5(b). To preserve $\tau(2m) = -1$, the part $2m$ satisfies condition C3. Therefore the condition satisfied by the part $2m$ changes, which is a contradiction.

- Assume the parts $2m$ satisfy the condition C3 only. According to Proposition 5.3(3), it corresponds to Fig. 5(b). If the length of $2m$th row changes under fingerprint preserving map, the length of $2m$th row must be deleted one box, and then we append a box under the map $\mu$ to get the same $\mu(\lambda)$. To preserve $\tau(2m) = -1$, the part $2m$ satisfies condition C1, which is a contradiction.

We draw the conclusion.

Consequently, we derive the following result.

**Proposition 5.7.** The fingerprint preserving maps that do not alter the conditions in formula (3.8) maintain the symbol invariant.

*Proof.* The contribution to symbol of a row is determined by its location pairwise rows and the length of row according to Propositions 5.4 (or Table 1). Then, Propositions 5.3 and 5.7 directly lead to this result.

---

4The first class maps are different from the identity map as shown in the first example in Section 4.
5.3. Fingerprint Preserving Maps With Change of Length of Row

We now examine the changes of the rows in $\lambda$ under the second class of fingerprint-preserving maps proposed in Proposition 5.4.

**Proposition 5.8.** Assume the part $2m$ of $\lambda$ satisfies conditions $C1$ or $C3$. Under fingerprint preserving maps that swap conditions $C1$ and $C3$, the length of the $2m$th row of $\lambda$ changes.

*Proof.* Assume the part $2m$ satisfies condition $C1$ but not condition $C3$. Denote the length of the $2m$th row of $\lambda$ as $L$. Then we have

$$L = \text{length of } 2m \text{th row of } \mu(\lambda) + 1$$

according to Proposition 5.3(2).

According to Proposition 5.4 under the second class of the fingerprint preserving maps, the part $2m$ satisfies the condition $C3$ but not the condition $C1$. Let length of the $2m$th row of $\lambda$ be $L'$. Then we have

$$L' = \text{length of } 2m \text{th row of } \mu(\lambda)$$

according to Proposition 5.3(3).

The partition $\mu(\lambda)$ does not change according to Proposition 5.3. So we draw the conclusion

$$L \neq L'.$$

Conversely, we can draw the same conclusion. □

According to Lemma 5.2, the change of the condition $C3$ satisfied by the even part $2m$ in $\lambda$ implies the change of the location of $2m$th row in the pairwise rows.

**Proposition 5.9.** Assume the part $2m$ of $\lambda$ satisfies the conditions $C1$ or $C3$. Under the fingerprint preserving maps, which swap the conditions $C1$ and $C3$, the location of $2m$th row in pairwise rows changes.

*Proof.* Let the part $2m$ satisfy the condition $C1$ but not the condition $C3$. Then the height of the $2m$th row in $\lambda$ is odd. Conversely, if the part $2m$ satisfy the condition $C3$ but not the condition $C1$, the height of the $2m$th row in $\lambda$ is even.

Thus swapping the conditions $C1$ and $C3$ implies the location of $2m$th row in the pairwise rows changes according to Propositions 2.1 and 2.2 (or see Fig. 3). □

Using Propositions 5.8 and 5.9 we refine the changes in the lengths of the pairwise rows under the fingerprint preserving maps.

- For the bottom row of a pairwise rows, which is the $2m$th row in $\lambda$, we have the following result.

**Lemma 5.3.** Let $b$ be the bottom row of a pairwise rows. If it is a $2m$th row in $\lambda$, then it will be appended to a box under the map $\mu$.

*Proof.* From the assumptions, the part $2m$ does not satisfy the condition $C3$. According to Proposition 5.1 it must satisfy the condition $C1$. And then, according to Fig. 3, it will append a box at the end under the map $\mu$. □

**Proposition 5.10.** Under the fingerprint preserving map, which change the length of a row, the bottom row $b$ of a pairwise rows becomes the top row of a pairwise rows with the length increased by one box.
Proof. According to Propositions 5.9, the location of row $b$ in pairwise rows changes, making the bottom row of a pairwise rows become the top row of a pairwise rows.

According to Lemma 5.3 under the map $\mu$, the row $b$ is increased by one box. To preserve $\mu(\lambda)$, this change can only be realized if the row $b$ is appended with one box after the fingerprint preserving map because we assume its length changes and the $2m$th in $\lambda$ cannot be deleted a box as shown in Fig. 5. We prove this by contradiction. Assume $b$ decreases by one box, then under map $\mu$, we have $\mu(b - 1) \leq \text{length}(b) < \text{length}(b) + 1 = \mu(b)$, where $b - 1$ means row $b$ decreases by one box and $\mu(b)$ is the image of the row $b$ under the map $\mu$. It is a contradiction. □

• For the top row of an even pairwise rows, which is the $2m$th row in $\lambda$, we have the following result.

Lemma 5.4. Let $t$ be the top row of a pairwise rows. If it is the $2m$th row in $\lambda$, then it does not change under the map $\mu$.

Proof. From the assumptions, the part $2m$ satisfies the condition $C3$. According to Proposition 5.1, it does not satisfy the condition $C1$. And then, according to Fig. 5, it does not change at the end of the row under the map $\mu$. □

Using this lemma, we can prove the following proposition.

Proposition 5.11. Under the fingerprint preserving map, which change the length of the row, the top row of a pairwise rows become the bottom row of a pairwise rows, with the length decreased by one box.

Proof. According to Propositions 5.9, the location of row $t$ in pairwise rows changes. So the top row of a pairwise rows becomes the bottom row of a pairwise rows.

According to Lemma 5.4 under the map $\mu$, row $t$ does not change. To preserve $\mu(\lambda)$, this can only occur if row $t$ is deleted one box after the fingerprint preserving map because we assume its length changes and the $2m$th row in $\lambda$ under $\mu$ cannot be deleted a box as shown in Fig. 5. We prove this by contradiction. Assume $t$ increases by one box, then under map $\mu$, we have $\mu(t + 1) \geq \text{length}(t) + 1 > \text{length}(t) = \mu(t)$, where $t + 1$ means row $t$ increases by one box. This is a contradiction. □

The above results implies the symbol preserving rules. We have the following Proposition.

Proposition 5.12. The fingerprint preserving maps, which change the length of the row, preserve the symbol invariant.

Proof. We draw the conclusion by using Propositions 5.10 and 5.11 which are just rules in Proposition 6.1 □

The preceding proposition complements Theorem 5.7. Combining these two results, we have

Theorem 5.1. For rigid surface operators, the fingerprint preserving maps preserve the symbol invariant.
6. Symbol Preserving Maps Preserve the Fingerprint Invariant.

According to Proposition 3.1, the contribution of each row in a partition to the symbol is invariant. Hence, given a symbol, there are only a finite number of possible combinations for the rows.

In this section, we prove that different combinations of rows for a symbol invariant preserve the fingerprint invariant.

6.1. Classification of Symbol Preserving Maps

Proposition 6.1 is equivalent to the following:

**Proposition 6.1.** The contribution to the symbol of the top row of a pairwise rows is equal to that of the bottom row of a pairwise rows, with the length decreased by one box, and vice versa.

Given the contribution to symbol, if the length of a row changes, the pairwise rows which the row belongs to have different parities.

Using the above proposition, we can immediately classify the symbol preserving maps.

**Proposition 6.2.** There are two classes of symbol preserving maps.

1. No change in the length of the row and its location in a pairwise rows.
2. Change in the length of the row and its location in a pairwise rows: The top row of a pairwise rows becomes the bottom row with the length decreased by one box, and vice versa.

The second class of symbol preserving maps corresponds to Proposition 6.1. Under the symbol preserving map, if the length of a row does not change, its location in a pairwise rows would not also change according to Table 1. If the length of a row changes, its location in a pairwise rows also change. Then we have

- The first class of symbol preserving maps corresponds to the change of the length of a row.
- The second class of symbol preserving maps corresponds to without change of the length of a row.

The following conclusion is required.

**Proposition 6.3.** The classification in Propositions 6.2 is complete.

*Proof:* This is the result of Remark 3.1 and Proposition 6.1.

6.2. Symbol Preserving Maps Without Change of Length of Row

For the first class symbol preserving maps, we have

**Proposition 6.4.** Let \( r \) be the \( 2m \)th row of \( \lambda \). Under the symbol preserving maps that without change the length of row \( r \) in the operator \( (\lambda', \lambda'') \), the image of row \( r \) under \( \mu \) does not change.

*Proof:* Firstly, we prove the proposition for a row with even length. Assume \( b \) is the bottom row and \( t \) is the top row of an even pairwise rows of \( \lambda' \) (or \( \lambda'' \)). According to Proposition 2.1, the height of \( b \) is even, and that of \( t \) is odd.

- Let \( r = b \) be the \( 2m \)th row of \( \lambda \). Since the height of \( b \) is even in \( \lambda' \), part \( 2m \) does not satisfy C3 condition.
  - If it satisfies the C1 condition, according to Proposition 5.3(2), we must append a box at the end of \( b \) of \( \lambda \) under the map \( \mu \), which means

\[
\text{length of } 2m \text{th row of } \mu(\lambda) = \text{length of } 2m \text{th row of } \lambda + 1.
\]
– If it does not satisfy the C1 condition, according to Proposition 5.3(1), we have

\[
\text{length of } 2m^{\text{th}} \text{ row of } \mu(\lambda) = \text{length of } 2m^{\text{th}} \text{ row of } \lambda.
\]

- Let \( r = t \) be the 2mth row of \( \lambda \). Since the height of \( t \) is odd in \( \lambda' \), part \( 2m \) satisfies the C3 condition. Thus, the length of \( t \) in \( \lambda \) does not change under the map \( \mu \) according to Proposition 5.3(3), which means

\[
\text{length of } 2m^{\text{th}} \text{ row of } \mu(\lambda) = \text{length of } 2m^{\text{th}} \text{ row of } \lambda.
\]

From the identities (6.12), (6.13), and (6.14), the length of the image of \( r \) under \( \mu \) does not change under the symbol preserving maps without change of the length of the 2mth row.

Similarly, we can prove the proposition for a row with odd length.

\[
\square
\]

6.3. Symbol Preserving Maps With Change of Length of Row

We now examine the conditions satisfied by the even part \( 2m \) of \( \lambda \) under the second class of symbol preserving maps proposed in Proposition 6.2.

First, we present a critical lemma.

**Lemma 6.1.** Under the symbol preserving maps that change the length of the \( 2m^{\text{th}} \) row of \( \lambda \), the part \( 2m \) must satisfy either conditions C1 or C3.

**Proof.** According to Proposition 5.1, the conditions C1 and C3 cannot be satisfied simultaneously by the part \( 2m \) of \( \lambda \). Therefore, the part \( 2m \) satisfies one of the following three cases of conditions:

- C1 condition.
- C3 condition.
- Neither of C1 condition and C3 condition.

We prove that the third scenario is impossible by contradiction. An counterexample is shown by the first map \( \mu \) in Fig. 14.

![Figure 14](image_url)

**Figure 14.** The \( 2m^{\text{th}} \) row is \( b \). Row \( b \) and \( r \) exchange under the symbol preserving map, changing the lengths of them. The images of \( r \) and \( b \) under map \( \mu \) are given.

Let row \( b \) be the \( 2m^{\text{th}} \) row. Under the the symbol preserving maps that change the length of the \( 2m^{\text{th}} \) row of \( \lambda \), row \( b \) and \( r \) exchange by using the rules in Table 1. Assume row \( q \) is the shortest row of \( \lambda'' \) under row \( b \) in \( \lambda \). The parity of length of pairwise rows
cr is different from that of pairwise rows bq. The parity of length of pairwise rows cb is different from that of pairwise rows rq.

Assume 2m does not satisfy both the C1 condition and the C3 condition.

- Since part 2m does not satisfy the C1 condition, under the map µ, the end of the 2mth does not change, corresponding to Fig. (5)(b)(or Proposition 5.3(1)).
- Since it does not satisfy C3 condition, there are even number of rows of λ′ and λ″ in the first 2m rows of λ according to Lemma 5.2. Thus the number of boxes above the 2mth row b is odd. Then, under the map µ, the end of the 2mth is appended a box, corresponding to Fig.(5)(a).

It is a contradiction for the above results.

In Fig.(14), the part 2m satisfies the condition C3 under the bottom map µ. Under the symbol preserving maps that change the length of the 2mth row of λ, the part 2m satisfies the condition C1 under the top map µ. In general, we have

**Lemma 6.2.** Under the symbol preserving maps that change the length of the 2mth row of λ, conditions C1 and C3 swap with each other.

**Proof.** According to Lemma 14, the part 2m must satisfy one of the conditions C1 or C3.

Assume the part 2m satisfies the condition C1. Then, the height of the 2mth row of λ is even in λ′ (or λ″). Under the symbol preserving maps which change the length of 2mth row, the height of the 2mth row of λ′ becomes odd in λ″ according to Proposition 6.1. Thus, the part 2m satisfy the condition C3.

Conversely, assume part 2m satisfy the condition C3, then the height of the 2mth row of λ is odd in λ′. Under the symbol preserving maps which change the length of 2mth row, the height of the 2mth row of λ′ becomes even in λ″ according to Proposition 6.1, which means it does not satisfy C3. Thus, it must satisfies C1 by Lemma 14.

Using these lemmas, we can draw the following conclusion.

**Proposition 6.5.** Let r be the 2mth row of λ. Under the symbol preserving maps that change the length of row r in the operators (λ′, λ″), the fingerprint invariant is preserved, which means the images of row r under µ do not change.

**Proof.** Using Lemma 6.2, the symbol preserving maps that change the length of a row is the second class of fingerprint preserving maps. They preserve the fingerprint invariant, thus they preserve µ.

The above proposition complements Proposition 6.4. Now, we can draw the following conclusion.

**Theorem 6.1.** For rigid surface operators, the symbol preserving maps preserve the fingerprint invariant.

**Proof.** According to Propositions 6.5 and 6.4, µ(2m) does not change under the symbol preserving maps.

According to Proposition 6.3, the even parts of µ with τ = 1 do not change under the symbol preserving maps. Thus, the partition α in the fingerprint invariant (α, β), which involves the even parts of µ, does not change under the symbol preserving maps.

According to the proof of Proposition 6.4, the conditions C1 or C3 are preserved under the symbol preserving maps that do not change the lengths of rows in the operators (λ′, λ″). According to the proof of Proposition 6.5, the conditions C1 and C3 swap under the symbol preserving maps that change the lengths of rows in the operators (λ′, λ″). Thus, the even parts of µ with τ = −1 do not change under the symbol preserving maps, meaning the partition β in the fingerprint invariant (α, β) also does not change.
The partition \( \alpha \) in the fingerprint invariant involves the odd parts of \( \mu \) determined by even parts of \( \mu \) according to proposition 3.4. Combining all results, the fingerprint invariant \((\alpha, \beta)\) does not change under the symbol preserving maps. □

7. Equivalence of Symbol Invariant and Fingerprint Invariant

In preceding sections, we restricted the partition \( \lambda \) to satisfy the condition \( \lambda_i - \lambda_{i+1} \leq 1 \) for all \( i \) in \( \lambda \). Due to this restriction, condition \( C2 \) can be omitted according to Lemma 3.3. Under this constraint, the equivalence of symbol invariant and fingerprint invariant is proved according to Theorems 5.1 and 6.1.

In this section, we relax the constraints and discuss the condition \( C2 \) further. Then, we prove the conjecture in the general case. The proof the conjecture for \( C_n \) and \( D_n \) theories are discussed.

7.1. Rigid Constraint: \( \lambda_i - \lambda_{i+1} \leq 1 \)

The constraints \( \lambda_i - \lambda_{i+1} \leq 1 \) are termed the rigid constraint, which is consistent with the definition of Rigid condition 2.1. Subject to rigid constraint, combining Theorems 5.1 with Theorem 6.1, we draw the following conclusion.

**Theorem 7.1.** For rigid surface operators, the symbol invariant is equivalent to the fingerprint invariant under the constraints \( \lambda_i - \lambda_{i+1} \leq 1 \) for all \( i \) in \( \lambda \).

7.2. No Rigid Constraint

If there are two equal rows in \( \lambda \), such that \( \lambda_i - \lambda_{i+1} = 2 \) in Remark 3.5, we would consider the condition \( C2 \) in Definition 3.8 as explained in Fig.(7) of Example 2. Now, we discuss the influences of the occurrence of such equal two rows in \( \lambda \) on the proofs in Sections 5 and 6.

**Fingerprint invariant implies symbol invariant (Sections 5):**

Surprisingly, we have

**Lemma 7.1.** For rigid surface operator \((\lambda', \lambda'')\), the part \( \lambda_i = 2m \) of \( \mu(\lambda) \) with \( \tau(2m) = -1 \) and \( \lambda_i - \lambda_{i+1} = 2 \) in \( \lambda \) cannot satisfy the \( C2 \) condition.

**Proof.** The 2\( m \)th row and (2\( m \) - 1)th row of \( \lambda \) with equal length must belong to different partitions of \((\lambda', \lambda'')\). And they must be in the same position of pairwise rows according to the definition of rigid \( B_n \) surface operator: Both the heights of the 2\( m \)th row and (2\( m \) - 1)th row of \( \lambda \) should have the same parity. Thus, there is an even number of boxes above the 2\( m \)th row in \( \lambda \), which means \((-1)^{P_{\lambda}(i)} = 1\) at the end of the (2\( m \) + 1)th row of \( \lambda \) corresponding to Fig.(1)(b) or (d). At the end of the (2\( m \) + 1)th row, we have \( \sum \mu_k = \sum \lambda_k \) according to Proposition 3.2(2). Thus, part 2\( m \) corresponds to Fig.(1)(b) which means it can not satisfy the condition \( C2 \). □

This lemma means the condition \( C2 \) can be omitted in the definition of the fingerprint invariant for rigid surface operator. We analyze this problem furthermore, considering two cases for rows of equal length:

- Rows in the same positions of pairwise rows: According to the proof of Lemma 7.1, the 2\( m \)th and (2\( m \) - 1)th rows of \( \lambda \) are equal to the 2\( m \)th and (2\( m \) - 1)th rows of \( \mu \), which means the lengths and the positions in pairwise rows of these two rows

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5 Hence, no box can be deleted at the end of the \( j \)th row in the example in Fig.(7), rendering this example impossible.

6 However, the condition can not omitted for general operators according to the proof of this lemma.
are preserved under the fingerprint preserving maps. Then their contributions to symbol invariant are preserved under the fingerprint preserving maps.

- Rows in different positions of pairwise rows (According to Lemma 7.1, in fact, it is impossible): Then the heights of these equal rows in $\lambda$ are odd, which can be omitted in the discussions according to Proposition 5.4.

**Symbol invariant implies fingerprint invariant (Sections 6):** From the view of the symbol invariant, we would get a more clearer picture.

According to the arguments in Lemma 7.1, the $2^m$th row and $2^{m-1}$th row in $\lambda'$ and $\lambda''$, respectively, must be in the same position of pairwise rows with the same parity. Thus, these two equal rows do not change under symbol preserving maps according to Table 1.

Then the number of boxes of $\lambda$ above these two rows is fixed under the symbol preserving maps, which means they preserve the fingerprint invariant.

In summary,

1. Condition $C_2$ in the definition of fingerprint invariant can be omitted for rigid surface operators.
2. Two equal rows of $(\lambda', \lambda'')$ equal to the image under the map $\mu$. Thus, they do not change under fingerprint preserving maps.
3. Two equal rows of $(\lambda', \lambda'')$ have the same position in pairwise rows and the same length under symbol preserving maps. Thus, they do not change under symbol preserving maps.

No changes of Two equal rows under invariants preserving maps, which means both of these maps belong to the first class maps. And these invariants preserving maps preserve each other. The presence of these two equal rows in $\lambda$ does not affect the proofs of Theorems 5.1 and 6.1 where we have not used the condition $C_2$. Hence we prove the conjecture that

**Theorem 7.2.** The symbol invariant is equivalent to the fingerprint invariant for rigid surface operators in the $B_n$ theory.

### 7.3. $C_n(Sp(2n))$ and $D_n(SO(2n + 1))$ theories

For a rigid operator $(\lambda', \lambda'')$ in $B_n$ theory, $\lambda'$ and $\lambda''$ are elements of $B_n$ and $D_n$ theories, respectively. In this subsection, we will prove Theorem 7.2 for rigid operators in $D_n$ and $C_n$ theories.

**$D_n$ theory:**

For a rigid operator $(\lambda', \lambda'')$ in $D_n$ theory, both $\lambda'$ and $\lambda''$ are elements in $D_n$ theory. To calculate the fingerprint invariant, we compute the invariant $\mu$ of $(\lambda', \lambda'')$ using the same conditions as the operators in $B_n$ theory according to Definition 5.2. For symbol invariant, since the rules in Table 1 are independent of theories, we use Table 1 as the operators in $B_n$ theory according to Definition 3.2. Therefore, for the $D_n$ theory, we can draw the same conclusions using the same strategy as for the $B_n$ theory without modifications.

**$C_n$ theory:**

For a rigid operator $(\lambda', \lambda'')$ in $C_n$ theory, we have $\lambda'$ in $C_n$ theory and $\lambda''$ in $C_n$ theory. For the calculation of the fingerprint invariant, we compute the invariant $\mu$ of $(\lambda', \lambda'')$ using the same conditions as the operators in $B_n$ theory except that the condition

$$(3)_{SO} \quad \lambda'_i \text{ is odd}$$

changes to

$$(3)_{Sp} \quad \lambda''_i \text{ is even}$$

according to Definition 3.2. For the calculation of the symbol invariant, we use Table 1 as the operators in $B_n$ theory according to Definition 3.1. Thus, for $C_n$ theory, we can draw
the same conclusions with minor modifications by changing the condition \((3)_{SO}\) to \((3)_{SP}\)
in the arguments.

In summary, the top row or bottom row of a pairwise rows has the same parities of
height for the partitions in the rigid surface operator \((\lambda', \lambda'')\) in the \(B_n\), \(C_n\), and \(D_n\)
theory. Thus, we can prove the conjecture for all theories by using the procedures in \(B_n\)
theory with minor modification. Finally, conclude

**Theorem 7.3.** The symbol invariant is equivalent to the fingerprint invariant for the
rigid surface operators in the \(B_n\), \(C_n\), and \(D_n\) theories.

8. Conclusions

The rigid surface operators are described by the partitions pair \((\lambda', \lambda'')\) \([4]\). In this paper,
we have proven the conjecture proposed in \([5]\) that the fingerprint invariant is equivalent
to the symbol invariant for rigid surface operators, leading to the result

‘Theorem 7.3: The symbol invariant is equivalent to the fingerprint
invariant for the rigid surface operators in the \(B_n\), \(C_n\), and \(D_n\) theories.’

The examples in Section 4 provide significant insights into the proof of the equivalence of
these two invariants: Type of invariants preserving maps involve the change of the length
of row. The fingerprint preserving maps, which change the length of a row, relate to the
conditions \(C_i\) satisfied by \(2m\) part in \(\lambda\).

Proposition 3.4 and Lemma 7.1 are crucial for proving the conjecture.

- Lemma 7.1 implies that condition \(C_2\) in the definition of the fingerprint invariant
  for rigid surface operators is redundant and can be omitted. Thus, calculating
  the fingerprint invariant needs to consider only conditions \(C_1\) and \(C_3\), which
  encompass all the information about \(\lambda'\) and \(\lambda''\).
- Proposition 3.4 indicates that the even rows of the partition \(\lambda\) encode all the
  useful information of the fingerprint invariant of rigid surface operators. This
  proposition allows us to focus on the even rows of \(\lambda\) and neglect the odd ones,
  reducing the complication of the classification of the fingerprint persevering maps
  as well as the entire proof.

Section 5 presents the first part of the proof, establishing that the fingerprint invariant
implies the symbol invariant for rigid surface operators, as summarized in Table 6. This
table focuses on the changes in the \(2m\)th row of \(\lambda\). The table uses the following notations:

- \(F(N)\): Fingerprint invariant or symbol invariant.
- \(C(N)\): Change of the length of row or no change.
- \(1(-1)\): Values of \(\tau(2m) = 1(-1)\).

For example, \(FN_{1}\) denotes fingerprint invariant preserving maps with a length change and
\(\tau(2m) = 1\). The propositions in brackets provide the conclusions. The first columns list
the classifications of the fingerprint invariant preserving maps, the second column specifies
the conditions satisfied, the third column describe the change of the length of row, the
fourth column identifies the location in the pairwise rows, and the fifth column references
the theorems providing the proof. The classification of fingerprint-preserving maps is on
the left of the double vertical line, while the right side demonstrates that these maps also
preserve the symbol invariant.

In Section 6, the second part of the proof, we demonstrate that the symbol invariant
implies the fingerprint invariant, as summarized in Table 7. The left side of the double
vertical line classifies the symbol-preserving maps, while the right side shows that these
maps also preserve the fingerprint invariant. The classification of the symbol preserving
maps is given by Proposition 6.2, which is based on Proposition 6.1. Proposition 6.1 is a
concise form of Table 1, derived from the construction of the symbol invariant \([13]\).

From Tables 6 and 7, we have:
| Type  | Length(2n-th row) | Location | Conditions:$C_i$ | Theorem |
|-------|------------------|----------|-----------------|---------|
| $FN_1$ | Neither(P5.4(1)) | No change(P5.6) | No change(P5.6) | P5.7    |
| $FN_{-1}$ | $C_1$ or $C_3$ (P5.4(1)) | No change(P5.6) | No change(P5.6) | P5.7    |
| $FC_{-1}$ | $C_1 \leftrightarrow C_3$(P5.4(2)) | Change(P5.8) | $b \leftrightarrow t$(P5.9) | P5.12    |

Table 6. Fingerprint invariant implies the symbol invariant.

| Type  | Length(2n-th row) | Location | Conditions:$C_i$ | Theorem |
|-------|------------------|----------|-----------------|---------|
| $SN_1$ | No(P6.2(1)) | No(P6.2(1)) | Neither (P6.2) | P6.3    |
| $SN_{-1}$ | No(P6.2(1)) | No(P6.2(1)) | $C_1$ or $C_3$ (P6.4) | P6.3 |
| $SC_{-1}$ | Change([0,2,2]) | $b \leftrightarrow t([0,2,1])$ | $C_1 \leftrightarrow C_2$(Len([0,2])) | P6.3 |

Table 7. Symbol invariant implies the fingerprint invariant.

- Note that we focus exclusively on the even rows in accordance to Proposition 3.4.
- The classification of symbol preserving maps and fingerprint preserving map is complete, according to Propositions 6.3 and 6.3. The change of the 2m-th row relates to its location in a pairwise rows in rigid surface operator and the conditions $C_1$ and $C_3$ satisfied by part 2m in $\lambda$.
- The second table is a reverse process of the first one, demonstrating their consistency: The exchange of conditions $C_1$ and $C_3$ satisfied by part 2m relates to the change of the position in a pairwise rows of the 2m-th row in the second class maps. The conditions $C_i$ and the locations are preserved in the first class maps.

The proof of the conjecture involves considering the influence of even rows of a rigid surface operator ($\lambda', \lambda''$), which is ‘local’. In Appendix A, we describe another strategy to prove that the symbol invariant implies the fingerprint invariant. This approach is based on the global classification of symbol preserving maps, differing from the approach in Proposition 6.2. Since we cannot use Proposition 5.4, the proof becomes more complex and less feasible.

Clearly, more work can be done. Given a fingerprint invariant or a symbol invariant, there are many rigid surface operators with the same invariant as shown in Appendix A. It would be interesting to find canonical-like element among them. Since the invariants are equivalent, it would be interesting to finding a formula to calculate one invariant from the other one. Hopefully our constructions will help make further progress.

In addition, the Weyl group of a simple Lie algebra is of particular importance. Both its conjugacy classes and unitary representations are parameterized by partitions but there is no canonical isomorphism between them. The Kazhdan-Lusztig map is a map from the rigid surface operator to the set of conjugacy classes of the Weyl group (fingerprint invariant). The Springer correspondence is a map from a rigid surface operator to a unitary representation of the Weyl group (symbol invariant). For rigid surface operators, we have proved the equivalence of the two invariants. Then it would be interesting to finding a canonical isomorphism between them using the canonical elements of the two invariants for rigid surface operators. The proof should shed light on further studies.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author, [BaoShou], upon reasonable request.

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Appendix A. Classification of Maps Preserving Symbol

In this section, we give a ‘global’ classification of the symbol preserving maps, distinct from that in Section 6. There are two fundamental types of symbol-preserving maps: S type and D type. The symbol preserving maps in this section are constructed using Table 1 or Proposition 3.1. We assume the surface operators are in the $B_n$ theory.

Take odd rows from one partition to another partition:

As shown in Figs. (15) and (16), the parity of the length of row $a$ and the rows above it are all even. The parities of the lengths of the last several rows of $\lambda''$ are odd.

Assume row $a$ is the bottom row of an even pairwise rows. It is inserted into the partition $\lambda'$ as the bottom row of an even pairwise rows. To preserve the symbol, the parities of the lengths and heights of the rows above $a$ in $\lambda''$ would change under the map $T_E$, as well as that of the rows above $a$ in $\lambda'$.

![Figure 15. Row $a$ is inserted into $\lambda''$ under the map $T_E$.](image)

As shown in Fig. (16), row $a$ is inserted into the partition $\lambda''$ as the top row of an odd pairwise rows. To preserve the symbol, the parities of the lengths and heights of the rows above $a$ of $\lambda'$ change under the map $T_O$, making the rows above $a$ of the partition $\lambda''$ even.

![Figure 16. The row $a$ is inserted into $\lambda''$ under the map $T_O$.](image)

Similarly, an odd row from $\lambda'$ can be inserted into $\lambda''$, leading to two additional two cases.

![Figure 17. Insertion of the row $b$ of $\lambda'$ into $\lambda''$.](image)

To preserve the symbol, inserting a row from $\lambda'$ into $\lambda''$ changes the parities of the lengths and heights of all rows above the insertion position. Inserting another shorter row halts these changes. The rows $a$ and $b$ are of pairwise rows. We insert the row $b$ of $\lambda'$ into $\lambda''$.

\footnote{We would not consider maps suffering from severe constraints, as shown in Figs. (15) and (16).}
λ′′ as shown in Fig. (17)(a). To preserve the symbol, the parities of lengths and heights of the rows in the region I change similar to the maps TE and TO, as well as the rows above the row b of λ. Next, we insert row a into λ′ as shown in Fig. (18). The parities of lengths and heights of the rows in the region III change again, recovering the original parities of lengths and heights as in Fig. (17)(a), as well as those of the rows above a of λ′. If rows a and b are not of a pairwise rows, the same result can be obtained with minor modifications.

![Figure 18: Insertion of the row a into λ''](image)

![Figure 19: Insertion of the rows a, b, and c into λ''.](image)

Inserting three rows from λ′ into λ′′ can be decomposed into two independent fundamental maps. For a rigid operator as shown in Fig. (14)(a), we insert three rows a, b, and c into λ′. To preserve the symbol, the parities of the lengths of all the rows in the region II do not change under the map O; the parities of lengths and heights of all the rows in the region I would change according to the maps TE and TO as shown in Fig. (15) and Fig. (16). Then the map O can be decomposed into two maps O1 and O2 as shown in Fig. (20) and Fig. (21).

![Figure 20: Insertion of the rows a and b into λ''.](image)

**Take even rows from one partition to another partition:**

Next, we consider the scenario where two rows from λ′ are inserted into λ′′ and vice versa. As shown in Fig. (22), a and b are rows of λ and c and d are rows of λ′, with lengths \( L(b) > L(d) > L(a) > L(c) \). The insertion of rows a and b into λ′′ and rows c and d into λ′ simultaneously can be decomposed into two separate mappings. One mapping involves inserting row a into λ′′ and row c into λ′ simultaneously, and the other mapping involves inserting row b into λ′′ and row d into λ′ simultaneously.
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Figure 21. Insertion of the rows $c$ into $\lambda''$.

Figure 22. Partitions $\lambda'$ and $\lambda''$ with lengths $L(b) > L(d) > L(a) > L(c)$.

Figure 23. Insertions of the rows $a$ and $b$ into $\lambda''$ and insertion of the rows $c$ and $d$ into $\lambda'$ happen at the same time. Regions $I$ and $II$ are the big rectangles in the middle of $\lambda'$ and $\lambda''$, respectively.

The insertions of the rows $a$ and $b$ into $\lambda''$ and the rows $c$ and $d$ into $\lambda'$ at the same time are shown in Fig. (24). The parities of lengths and heights of the rows in the region $I$ and $II$ do not change under this map. So we can insert the row $b$ into $\lambda''$ and insert the row $c$ into $\lambda'$ at the same time as shown in Fig. (26). And then insert the row $a$ into $\lambda''$ and insert the row $d$ into $\lambda'$ at the same time. Or we can insert the row $a$ into $\lambda''$ and insert the row $d$ into $\lambda'$ at the same time.

We can discuss a more complicated case: $L(b) > L(a) > L(d) > L(c)$ in Fig. (22). To save space, we omit these proofs which are easy to understand. The map $O$ in Fig. (19) can be decomposed into maps: move the longer two rows first, then move the shortest row. However, the move of the last row is the maps shown in Fig. (15) and (16). In summary, the maps preserving symbol consist of the sequence movements of two rows. There are two fundamental maps: $S$ and $D$ types.

$S$-type maps

The $S$ type symbol preserving maps which take one row from $\lambda'$ to $\lambda''$ and take one row from $\lambda''$ to $\lambda'$. The most general model is shown in Fig. (26). We can put the first or the second row of the pairwise rows in the region $II$ to the positions 1 or 2 in the region...
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Figure 24. Insertion of the row $a$ of $\lambda'$ into $\lambda''$ and insertion of the row $c$ of $\lambda''$ into $\lambda$ at the same time.

Figure 25. Insertion of the row $b$ of $\lambda'$ into $\lambda''$ and insertion of the row $d$ of $\lambda''$ into $\lambda'$ at the same time.

Figure 26. Taking one row from $\lambda'$ to $\lambda''$ and taking one row from $\lambda''$ to $\lambda'$.

2. We can put the first or the second row of the pairwise rows in the region $l3$ to the positions 1 or 2 in the region $l4$.

The rows in the regions $I$ and $II$ must have the opposite parities: (odd, even) or (even, odd). Then the number of the $S$ type maps is

\begin{equation}
NS = 2^5.
\end{equation}

**D-type maps**

The $D$ type symbol preserving maps which take two rows from $\lambda'$ to $\lambda''$. A pair of even rows can be obtained from five different blocks $\lambda'$ as shown in Fig. (27). This pairwise rows can be inserted into a block of $\lambda''$, leading to five different blocks as shown in Fig. (28).

Considering the odd version of these processes, the number of the $D$ type maps is

\begin{equation}
ND = 5 \times 5 \times 2.
\end{equation}
Figure 27. Two rows of a block of $\lambda'$ comprise an even pairwise rows.

Figure 28. An even pairwise rows is inserted into a block of $\lambda''$. 
Summary
All other symbol preserving maps can be constructed by $S$ and $D$ types of maps. Then to prove that symbol invariant implies fingerprint invariant is to prove that these two fundamental maps preserve the fingerprint invariant. However, from formulas (A.15) and (A.16), the number of $S$ and $D$ types maps is
\[ N = NS + ND = 82, \]
which means that we should check these 82 cases to preserve the fingerprint invariant to prove the symbol invariant implies the fingerprint invariant. It is unrealistic.

In [16], the fingerprint maps are classified by decomposing into several fundamental maps. However, it is more complicated than that in Section 5, which is not conventional to prove these fundamental maps preserve the symbol invariant.
Appendix B. Rigid Surface Operators in $SO(11)$ and $Sp(10)$

The second and the third columns list pairs of partitions corresponding to the surface operators in the $B_5$ and $C_5$ theories. The other columns are the dimension, symbol invariant, and fingerprint invariant of the rigid surface operators, respectively.

| Num | $Sp(10)$ | $SO(11)$ | Dim | Symbol | Fingerprint |
|-----|----------|----------|-----|--------|-------------|
| 1   | (1$^{10};\emptyset$) | (1$^{11};\emptyset$) | 0   | (0 0 0 0 0 0 1 1 1 1) | [1$^2;\emptyset$] |
| 2   | (2$^1;\emptyset$) | (1; 1$^{10}$) | 10  | (1 1 1 1 1 0 0 0 0) | [1$^4;1$] |
| 3   | (1$^8;1^2$) | (2$^2 1^7;\emptyset$) | 16  | (0 0 0 0 0 0 1 1 1 2) | [2$^11;\emptyset$] |
| 4   | (2$^2 1^4;\emptyset$) | (1$^3;1^8$) | 24  | (1 1 1 1 1 0 0 1) | [1$^2;1^3$] |
| 5   | (2$^16;1^2$) | (1; 2$^2 1^6$) | 24  | (1 1 1 1 0 0 0 1) | [1$^2;1^3$] |
| 6   | (1$^6;1^4$) | (2$^4 1^3;\emptyset$) | 24  | (0 0 0 0 1 2 2) | [2$^2 1;\emptyset$] |
| 7   | (2$^4 1^2;\emptyset$) | (1$^7;1^4$) | 28  | (0 0 1 1 1 1 1) | [1$^1;1^4$] |
| 8   | (1$^6;2 1^2$) | (3$^2 1^4;\emptyset$) | 28  | (0 0 1 1 1 1 1) | [1$^1;1^4$] |
| 9   | (2$^1;1^4$) | (1$^5;1^6$) | 30  | (1 1 1 1 1 1) | [0;1$^5$] |
| 10  | (2$^1;2^1$) | (1; 3$^2 1^3$) | 34  | (1 2 2 0 0) | [21;2] |
| 11  | (1$^2;2^3 1^2$) | (2$^2 1^3;1^6$) | 34  | (1 1 1 0 2) | [31;1] |
| 12  | (2$^3 2 1^3;\emptyset$) | (1$^3;3^2 1^4$) | 34  | (1 1 1 0 2) | [31;1] |
| 13  | – | (2$^2 1^3;1^4$) | 36  | (0 1 1 0 2) | [3;1$^2$] |
| 14  | – | (1$^3;3^2 1^2$) | 45  | (2 2 1) | [3;2] |

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