A New approach for the pseudo-quaternionic lorentzian evolute and involute curves

Pseudo-quaternionic lorentzian evolute and involute eğriler için yeni bir yaklaşım

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Highlights

- The curve in 4-dimensional Lorentz space, which is derived from the curves obtained with Pseudo quaternions in 3D Lorentz space, is a pseudo quaternionic curve. Evolute and involute curves are defined for the obtained pseudo quaternionic timelike Lorentz curve.

- Some characterizations have been obtained for these curves in 3-dimensional and 4-dimensional Lorentz space.

Graphical Abstract

The curve in 4-dimensional Lorentz space, which is derived from the curves obtained with Pseudo quaternions in 3D Lorentz space, is a pseudo quaternionic curve. Evolute and involute curves are defined for the obtained pseudo quaternionic timelike Lorentz curve. Some characterizations have been obtained for these curves in 3-dimensional and 4-dimensional Lorentz space.

Aim

Defining the evolute and involute curves for a timelike quaternion curve in quaternionic Lorentz space

Design & Methodology

Quaternion space is defined. Then, quaternionic curves in Lorentz space are defined and using this approach, evolute and involute curves are defined.

Originality

Evolute and involute curves are defined in Lorentz space for pseudo quaternionic timelike curves with the help of quaternion algebra. Previously, the evolute / involute curve pair was defined in Lorentz space, but it was re-represented with the help of Pseudo Quaternions.

Conclusion

Some characterizations are defined for Pseudo Quaternionic evolute / involute curves obtained in Lorentz Space. Some special cases of these curves have been studied in depth. These curves; The helix can be planar or their evolutes planar or helical curves. Although evolute/involute curves have been defined in Lorentz space before, Pseudo Quaternionic timelike curves are presented in a form that has not been used before

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.
A New Approach for the Pseudo-Quaternionic Lorentzian Evolute and Involute Curves

Research Article
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ABSTRACT

In this study, the main purpose is to obtain pseudo quaternionic Lorentzian evolute-involute curves on $L^n_4$. In fact the pseudo quaternion is a quaternion which has 3-dimension and its temporal part is ziro. So it has only vectoral part of the quaternion. Thus, if defined a pseudo-Quaternionic curve on $L^n_4$ then its quaternionic part is pseudo quaternion. What it means to do here, is to write out Lorentzian evolute-involute curves of the quaternionic time-like curves in $L^n_4$ using quaternions and to reveal some relationships between these curves.

Keywords: Quaternion space, Lorentzian space, binary operation, symmetric bilinear form.

1. INTRODUCTION

The involute-evolute curves in $\mathbb{R}^3$ are well known in elementary differential geometry. If a curve is differentiable at the each point of an open interval then a set of mutually orthogonal unit vectors can be constructed. These vectors are called tangent, normal and binormal unit vectors. The set of these vectors is \( t(s), n_1(s), n_2(s) \) and it is named Serret-Frenet Frame. The set of frame vectors and curvatures of a curve is called Serret-Frenet Apparatus of the curve. The derivative equations of this vectors are
\[
\begin{align*}
t'(s) &= k_3(s)n_3(s), \\
n'_1(s) &= -k_1(s)t(s) + k_2(s)n_2(s), \\
n'_2(s) &= k_2(s)n_3(s).
\end{align*}
\]
Here \( k_1(s) \) and \( k_2(s) \) are curvatures with respect to arc length.

Let \( \alpha \) and \( \beta \) be two curves on $E^3$ which have \((t,X), (t,Y)\) coordinate neighborhoods, respectively. Let the Serret-Frenet vectors at the point \( \alpha(s) \) of the curve \( \alpha \) are \( t(s), n_1(s), n_2(s) \) and the Serret-Frenet vectors at the point \( \beta(s) \) of the curve \( \beta \) are \( t(s), n_1(s), n_2(s) \). If \( t(s) \) is orthogonally to \( t(s) \) for each parameters, that is \( \dot{t}(s).t(s) = 0 \) then \( \beta(s) \) is called involute of \( \alpha(s) \), \( \alpha(s) \) is called evolute of \( \beta(s) \), too [1].

William R. Hamilton was discovered the quaternions. He said that the quaternion is appropriate is generalization one in which the real axis. Quaternions are used in many fields. Some of these are computer images and computer graphics. They can be also used in mechanics because quaternionic formulation of equation of motion in the theory of relativity.

Quaternions are constructed as a linear combination of a 3D vector with a real value. The real quaternions are coincide with $\mathbb{R}^4$. It was a four-dimensional vector space over the real numbers. Hacisalihoğlu has worked on motion geometry and quaternions [2]. Quaternion algebra is studied by Bharathi and Nagaraj [3, 4]. Later Bilici and Çalışkan have worked in many ways as time like and space like different curves [5, 6]. Büküç and Karacan studied on the evolute and involute curves on Lorentzian space [7]. Soycdn and Güngör also have studied this issue [8]. Kalkan et al. have studied on this issue for some special curves [9]. Altın et al. studied with constant weighted curvature in Lorentzian Space [10]. Karadağ and Karadağ studied on null generalized slant helices in Lorentzian Space[11]. Later, Karadag and Sivridag have studied many cases and gave characterizations on quaternionic curve [12,13]. In fact many famous mathematicians are working on curves of Lorentz Space. Some of those are O’Neill[14],Ozturk and Ozturk, Ilarslan, [15].Then, Almaz and Külahçi are working on Lorentzian curves [16]. Eririş and Güngör studied on quaternionic Lorentzian curve ,too[17]. Karacan and at.al studied this issue [18]. Millman and Parker studied Elements of Differential Geometry [19].

The main purpose of this study, to obtain pseudo quaternionic Lorentzian evolute-involute curves on $L^n_4$. Since it is trivial task to write out Lorentzian evolute-involute curves of the quaternionic time-like curve in $L^n_4$ using quaternions and to reveal some relationships between these curves. In this study $L^n_4$ denotes the 4-dimensional Quaternion Lorentzian Space.

2. PRELIMINARIES

Like a real quaternion, it is defined as four sequential numbers accompanied by four units, where $e_4 = 1$ is the real number. The other three units are
\[
\begin{align*}
i_1 & \times i_2 = -e_2 \times e_1, \\
i_2 & \times i_4 = e_4 \times e_2 = e_A, \\
i_1 & \times i_4 = -e_4 = -1
\end{align*}
\]
The matrix form of Lorentzian this formulae for a quaternionic Lorentzian curve on $L^4_Q$ are

\[
\begin{bmatrix}
\mathbf{t} \\
\mathbf{n}_1 \\
\mathbf{n}_2 \\
N_3
\end{bmatrix} =
\begin{bmatrix}
0 & k & 0 & 0 \\
k & 0 & r & 0 \\
0 & -r & 0 & 0 \\
0 & 0 & -(r-K) & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{T} \\
\mathbf{N}_1 \\
\mathbf{N}_2 \\
N_3
\end{bmatrix}
\]  

(9)

Definition 2.1 Let $I \subset Q$ and $\beta: I \to L^4_Q$, then

\[
\beta(s) = \sum_{i=1}^{4} \beta_i(s)e_i \quad (e_4 = 1)
\]

is called a quaternionic curve which is one quaternion variable valued $\beta$ transformation.

Especially, if $\beta(t) = 0 \beta_3$ then $\beta$ is called space quaternion curve [12]. The set of space quaternions, that is $W = \{q \in Q; q + aq = 0\}$ and $\mathbb{R}^3$ (3-dimensional real Euclidean space) are isomorphic. Thus, the Frenet formulas and curvatures of curves in $W$ can be calculated with the help of space-quaternions [13].

The Serret-Frenet formulas for a time-like quaternion curve in the quaternionic Lorentz space are also calculated for a space quaternions Lorentzian curve. Using these relations, the Serret-Frenet derivative formulas for the quaternionic Lorentzian curve are calculated [13].

In this study, by using this formulation, I will be able to calculate some relations for quaternionic Lorentzian evolute-involute curves.

Definition 2.2 The two bilinear forms on $L^4_Q$ are defined as,

\[
g(x, y) = \sum_{i=1}^{3} x_i y_i - x_4 y_4, \quad h(x, y) = \sum_{i=1}^{4} x_i y_A
\]  

(10)

for $\forall x, y \in L^4_Q$ [3].

In this define, for given $g$ and $h$ bilinear forms, like equation (3) if $b$ any symmetric bilinear form and "\( \circ \)" is a binary operation then

\[
x \circ y = [x, y] + x_4 y_4 - y_4 x_4 - b(x, y)e_4
\]  

(11)

If $b = h$ then the quaternion is called pseudo quaternionic or $b = g$ then the quaternion is called quaternionic [3, 4].

3. MATERIAL and METHOD

Definition 3.1 Let $\tilde{a}$ and $\tilde{\beta}$ be two time-like curve on $L^4_Q$ which have $(I, X)$, $(I, Y)$ coordinate neighborhoods, respectively. Let the Serret-Frenet vectors at the point of $\tilde{a}(s)$ of the curve $\tilde{a}$ is $\{T(s), N_1(s), N_2(s), N_3(s)\}$ and the Serret-Frenet vectors at the point of $\tilde{\beta}(s)$ of the
curve \( \bar{\beta} \) is \( \{T^*(s), N^1_1(s), N^2_1(s), N^3_1(s) \} \). If \( T(s) \) is \( g \)-orthogonally to \( T^*(s) \) for each parameter \( s \), that is
\[
 g(T(s), T^*(s)) = 0
\]
(12)
then \( \bar{\beta} \) is called involute of \( \bar{\alpha} \), \( \bar{\alpha} \) is called evolute of \( \bar{\beta} \), too.

Theorem 3.1 If the \( \bar{\alpha} \) pseudo quaternionic time-like curve is evolute of the \( \bar{\beta} \) pseudo quaternionic time-like curve then (res. it has involute equation of \( \alpha \)) it is
\[
Y(s) = X(s) + (c - s)t(s), \quad c = constant
\]
(13)
Proof. Let \( \bar{\alpha} \) be a pseudo quaternionic time-like curve.
Then,
\[
\bar{\alpha}(s) = t(s) \text{ and } \bar{\gamma}(t(s), t(s)) = -1.
\]
From equation of evolute can be written;
\[
Y(s) = X(s) + \sigma t(s)
\]
(14)
If the derivative of equation (14) with respect to \( s \), then it can be shown as;
\[
Y'(s) = t(s) + \alpha' t(s) + \sigma t'(s)
\]
(15)
Since \( \bar{\gamma} \) is evolute of \( \bar{\beta} \)
\[
\bar{\gamma}(Y'(s), t(s)) = 0
\]
(16)
Thus, it can be obtained;
\[
\bar{g}(t(s) + \sigma' t(s) + \alpha t'(s), t(s)) = 0
\]
(17)
\[
\bar{g}(t(s), t(s)) + \sigma' \bar{g}(t(s), t(s)) + \sigma \bar{g}(t'(s), t(s)) = 0
\]
(18)
for a regular curve is known as;
\[
\bar{g}(t'(s), t(s)) = 0, \text{ hence, it can be written as;}
(1 + \sigma')\bar{g}(t(s), t(s)) = 0,
\]
and since \( X \) is a time-like curve
\[
\bar{g}(t(s), t(s)) = -1
\]
(19)
Therefore, it can be found that;
\[
-1(1 + \sigma') = 0, \quad \sigma' = -1; \sigma = -s + c
\]
Thus, it is obtained;
\[
Y(s) = X(s) + (c - s)t(s), c = constant
\]
(20)
The proof is completed.
As a result of this theorem, it can be found that,
\[
Y(s) - X(s) = (c - s)t(s)
\]
\[
||Y(s) - X(s)||^2 = ||(c - s)t(s)||^2 = g((c - s)t(s), (c - s)t(s)) = (c - s)^2 \bar{g}(t(s), t(s)) dY(s), X(s)) = |c - s| |
\]
(21)
Hence, the distance between of corresponding points to pair of evolute and involute curves is given as
\[
l = (c - s)
\]
(22)
Now, accounting the distance between of corresponding points to pair of evolute and involute curves on quaternionic Lorentzian space:
Let \( \bar{\beta} \) be a quaternionic curve which is derived \emph{X}-curve in \( L_0^1 \). That is,
\[
\bar{\beta}(s) = x_1(s) \bar{e}_1(s) + x_2(s) \bar{e}_2(s) + x_3(s) \bar{e}_3(s) + \bar{\beta}_4(s) \bar{e}_4(s)
\]
(23)
Then, consider that \( \bar{\beta} \) is a time-like quaternionic curve parametrized as
\[
\bar{\beta}'(s) = T(s), \quad g(T(s), T(s)) = -1
\]
(24)
On the other hand, the equation of the involute of the curve \( \bar{\alpha}(s) \) is
\[
\bar{\beta}(s) = \bar{\alpha}(s) + \rho T(s)
\]
(25)
then, \( \rho \) is constant.
\[
\bar{\beta}'(s) = T(s) + \rho' T(s) + \rho T'(s)
\]
(26)
and it is known that \( \bar{\beta}'(s) \) is \( g \)-orthogonally to \( T(s) \).
That is;
\[
g(\bar{\beta}'(s), T(s)) = 0
\]
(27)
If it can use of (26) in (27), it is defined as;
\[
g(T(s) + \rho T(s) + \rho T'(s), T(s)) = 0
\]
(28)
g(T(s), T(s)) = \rho g(T'(s), T(s)) + \rho g(T(s), T(s)) = 0
\]
(29)
\[
(1 + \rho')g(T'(s), T(s)) = 0
\]
(30)
from \( \bar{\alpha}(s) \) is a time-like quaternionic Lorentzian curve, \( -(1 + \rho') = 0; \rho' = -1; \rho = c - s \).
Hence the equation of evolute of the curve \( \bar{\beta}(s) \) is find as
\[
\bar{\beta}(s) = \bar{\alpha}(s) + (c - s)T(s)
\]
(31)
So it can be written:
\[
\bar{\beta}(s) - \bar{\alpha}(s) = (c - s)T(s)
\]
(32)
Therefore it has
\[
||\bar{\beta}(s) - \bar{\alpha}(s)||^2 = g((c - s)T(s), (c - s)T(s)) = (c - s)^2 \sigma g(T'(s), T(s))
\]
(33)
That is,
\[
d(\bar{\beta}(s), \bar{\alpha}(s)) = |c - s|
\]
(34)
or
\[
d\bar{\beta} = \frac{ds}{ds} = \frac{ds}{ds} d\bar{\beta}
\]
(35)
Hence, it is defined as;
\[
T = (c - s) \frac{ds}{ds} T' = (c - s) \frac{ds}{ds} KN_1
\]
(36)
and it is known that;
\[
g(\bar{T}(s), T(s)) = 0
\]
(37)
so,
\[
(c - s) \frac{ds}{ds} g(N_1(s), T(s)) = 0
\]
(38)
It is derived as;
\[
(c - s) \frac{ds}{ds} = -c_1
\]
(39)
Then,
\[
g(\bar{\bar{T}}(s), \bar{T}(s)) = g(c_1 KN_1, c_1 KN_1)
\]
(40)
\[-1 = c_1^2 K^2 g(N_1(s), N_1(s))
\]
(41)
since \( N_1 \) space-like \( g(N_1, N_1) = 1 \)
\[-1 = c_1^2 K^2
\]
(42)
\[-1 = (c - s) \frac{ds}{ds} K
\]
(43)
and it is derived as;
\[
\frac{ds}{ds} = -\frac{1}{(c-s)K}
\]
(44)
from (40) it can be written as;
\[
\bar{T}(s) = c_1 K N_1 \Rightarrow \bar{T}(s) = \rho N_1(s)
\]
(45)
Hence,
\[
\bar{T}(s) = \frac{ds}{ds} = -\frac{1}{(c-s)K}
\]
(46)
on the other hand, it has
\[
g(\bar{T}(s), \bar{T}(s)) = g(K N_1, K N_1) = K^2 g(N_1, N_1) = K^2
\]
(47)
g(\bar{N}_1, \bar{N}_1) = 1 \text{ and } g(\bar{N}_2, \bar{N}_2) = 1 \text{ indirectly, it is obtained as;}
\[
K^2 = g(\bar{T}(s), \bar{T}(s))
\]
\[
= g((KT + k N_2) - \frac{1}{(c-s)K}, (KT + k N_2) - \frac{1}{(c-s)K})
\]
Theorem 3.2 The evolutes of a planar time-like quaternionic Lorentzian curve are helices.

Proof. Let us first consider the case where \( \alpha \) is a planar time-like quaternionic Lorentz curve then its evolute is a plane curve, too. So \( \tau^* = \eta_1 \). Hence, \( \alpha(s) = \beta(s) + \lambda \tau^*(s) \Rightarrow \beta(s) = \alpha(s) - \lambda \eta_1(s) \)
\[ \Rightarrow t^* \frac{ds}{dt} = \beta'(s) = t(s) - \lambda \eta_1(s) - \lambda \eta_1'(s) \]
\[ n_1'(s) = k(s)t(s) + r(s)n_2(s) \]
Since \( \gamma(t^*, t) = 0 \)
\[ (50) \]
then \(-1 + \lambda k_1(s) = 0 \Rightarrow \lambda = \frac{1}{k(s)} \)
So \( \beta(s) = \alpha(s) - \frac{1}{k(s)} \eta_1(s) \).

The planar evolution of \( \alpha \) is the geometric location of its curvature centers. Although \( \alpha \) is a planar curve, if \( \beta \) is a non-planar evolute of \( \alpha \) then \( \beta(s) = \alpha(s) - \lambda \tau^*(s) \)
\[ \beta'(s) = \alpha'(s) - \lambda \tau^*(s) - \lambda \tau'^*(s) \]
\[ = t^*(s) = t(s) \frac{ds}{dt} + \tau^*(s) - \lambda k_1(s) \eta_1'(s) \]
\[ (52) \]

where \( \frac{ds}{dt} = -1 \);
\( \tau^* \) time - like and \( n_1, n_2 \) space - like.
\[ t(s) \frac{ds}{dt} - \lambda k_1(s) \eta_1(s) = 0 \Rightarrow t(s) = \frac{\eta_1(s)}{\lambda} \]
\[ f = (t^*(s), t(s)\eta_1(s)) \Rightarrow f^* = \frac{df}{ds} \]
\[ = (k(s) t^*(s), t(s) \eta_1(s)) \]
\[ f'(s) = 0 \Rightarrow f^* \text{ constant and} \]
\[ 4(t^*(s), t(s)\eta_1(s)) \text{ is constant.} \]

Therefore, the velocity vector of \( \beta \) is always a constant angle with the normal of planar curve \( \alpha \) so \( \beta \) is a helix.
In this case the non-planar evolutes of a \( \alpha \)-planar curve are helical curves.

Theorem 3.3 Let \( \bar{\alpha}(s) \) be a pseudo-quaternionic Lorentzian evolute curve with \( s \)-arc parameters. If \( \bar{\beta}(s) \) is the evolute curve of \( \bar{\alpha}(s) \) then,
\[ a) \bar{\beta}(s) = \bar{\alpha}(s) + \sigma n_1(s) + \rho n_2(s) \]
\[ b) \sigma = \frac{1}{k(s)} \text{ and } a^\prime - \rho^\prime = \frac{\rho^\prime + \sigma}{\sigma} \]
\[ c) r(s) = \frac{\rho(s)}{\rho^2 + \frac{1}{k(s)^2}} \]
\[ (55) \]
\[ (56) \]
\[ (57) \]

Proof.

a) Since the vector \( \bar{\beta}(s) - \bar{\alpha}(s) \) is perpendicular to the vector \( t(s) \), it can be defined as:
\[ \bar{\beta}(s) = \bar{\alpha}(s) = \sigma n_1(s) + \rho n_2(s) \]
\[ (58) \]

b) \[ \frac{d\bar{\beta}}{ds} = \bar{\alpha}'(s) + \sigma n_1(s) + \rho n_2(s) + \rho n_2(s) = t(s) + \sigma n_1(s) + \sigma(k(s) t(s) + r(s)n_2(s)) + \rho n_2(s) + \rho(-r(s)n_1(s)) \]
\[ = (1 + \sigma k(s)) t(s) + (\sigma' - \rho r(s)n_1(s) + (\sigma' + \sigma r(s))n_2(s) \]
\[ (59) \]

Since \( \frac{d\bar{\beta}}{ds}, t(s) \) is zero, then \( 1 + \sigma k(s) = 0 \Rightarrow \sigma = \frac{1}{k(s)} \)
So,
\[ \frac{d\bar{\beta}}{ds} = (\sigma' - \rho r(s))n_1(s) + (\sigma' + \sigma r(s))n_2(s) \]
\[ (60) \]
And \[ \frac{d\bar{\beta}}{ds} = \sigma' r(s) n_1(s) + (\rho' + \sigma r(s))n_2(s) \]

Vectors in the same direction. Thus
\[ (\sigma' - \frac{\rho}{k(s)} r(s)) = \frac{\sigma'}{\sigma} + \frac{r(s)}{k(s)} = u \]
\[ (63) \]
from this last equation, it is derived as;
\[ c) \rho = \rho + \sigma r(s) = \sigma \rho' + \sigma r(s) \text{ and } \sigma = -\frac{1}{k(s)} \]
then it can be obtained
\[ r(s) = \frac{\sigma'}{\sigma + \rho^2} \]
\[ (64) \]
so,
\[ r(s) = \frac{\rho' r(s)}{\rho^2 + r(s)^2} \]
\[ (65) \]

6. CONCLUSION

First, the geometric properties of transformations using quaternionic and pseudo-quaternionic multiplications in 4-dimensional real vector space are given by Baharati and Nagaraj, [3]. Later, some results were expressed by studying quaternionic curves in different spaces (e.g. Euclidean and Lorentzian spaces) by many mathematicians,[4,8,12].

As mentioned above, although the quaternionic curve and quaternionic Lorentzian curve has been studied by many mathematicians, this article is a different study with the quaternion terminology for a quaternionic Lorentzian curve. Because the aim of this study is how to derive evolution-inclusion curves for a pseudo-quaternionic (in Lorentz space) and how to give some relations between these curves.

Therefore, this article will bring a new perspective to researchers who will work in this field.
DECLARATION OF ETHICAL STANDARDS
The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS’ CONTRIBUTIONS
Müge KARADAĞ: Completed study and wrote the manuscript.

CONFLICT OF INTEREST
There is no conflict of interest in this study.

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