Compactification of Supermembranes

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Abstract

We look at the vertical dimensional reduction of the supermembrane of M-theory to the D2-brane of Type IIA string theory. Our approach considers the soliton solutions of the two low energy field limits, $D = 11$ and $D = 10$ Type IIA supergravities, rather than the worldvolume actions. It is thus necessary to create a periodic array. The standard Kaluza-Klein procedure requires that the brane is smeared over a transverse direction, but we will keep the dependence on the compactification coordinate, seeing how the eleventh dimension comes into play when we close up on the D2-brane.

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1 Introduction

String theory has undergone a dramatic change from five separate theories, to the realization that they all are limit points in the moduli space of an underlying fundamental theory, dubbed M-theory. The five string theories are non-perturbatively equivalent, and related through various duality symmetries. M-theory is intrinsically eleven-dimensional, with $D = 11, N = 1$ supergravity as its low energy limit. It is also the strong coupling limit of Type IIA string theory[1]. Since we are unable to perform quantum mechanics in the eleven dimensional vacuum it is necessary to look at the BPS spectrum of M-theory to test and derive dualities between the various string theories in different dimensions. We can rely on such manipulations, since these states receive no quantum corrections to their masses, as long as supersymmetry remains unbroken [2].

The fundamental BPS-states of M-theory are the supermembrane [3, 4] and the superfivebrane [5], both of which appear as solutions to M-theory’s low energy approximation, $D = 11, N = 1$ supergravity. Relying on the interconnectedness of all theories the speculation is put forward in [6] that these solutions are directly related to the extended objects of Type IIA string theory [7, 8, 9, 10, 11]. The relation to the other string theories is established via dimensional reduction and U-dualities. All these extended objects, being BPS states also appear as solutions of the low energy field theories of these string theories.

The aim of this paper is to explicitly look at the relation between the supermembrane[3] and the D2-brane. There exist two approaches; the first concentrates on the world-volume action [12, 8]. The world-volume is three-dimensional giving a duality between a scalar, usually the eleventh coordi-

\footnote{From now on referred to as the M2-brane}
nate, and a vector $A_\mu$, the world-volume gauge-field of the resulting D2-brane. In this way the eleventh dimension is eliminated via a Lagrange multiplier, $A_\mu$. It is surprising that the classical supermembrane reproduces the D2-brane action, which comes from a one-loop sigma model calculation in string theory [12, 13].

The second approach looks at low energy representations of the BPS states as solutions to supergravity theories. These can be obtained mostly from the $D = 11$ version through standard Kaluza-Klein reduction. As this reduction is consistent the lower dimensional solutions also satisfy the higher dimensional field equations. However it is also possible to reduce the solutions directly [14], in two different ways. The more common one takes a $(D, p)$ solution to $(D - 1, p - 1)$ solution, by simultaneously reducing a spatial and world-volume direction. This follows closely the procedure also applicable to the world-volume approach [15, 4]. The second, more intricate reduction, takes a $(D, p)$ to a $(D, p - 1)$ solution, which is what we will be looking at here. Such reductions have been discussed previously in Refs [16, 17, 18, 19, 20, 21, 22, 11].

Looking at the movement of the solutions on a $(D, p)$ plot, we see that the former method moves the solution diagonally down, whereas the latter has a vertical movement. Hence the two procedures are called diagonal and vertical reduction respectively. E.g. diagonal reduction takes the M5-brane to the D4-brane of IIA string theory, and vertical to the NS5-brane.

Diagonal Kaluza-Klein reduction is more readily performed due to the independence of the supergravity fields on the world-volume coordinates. Such an isometry must first be created in the transverse direction for vertical reduction. This method is referred to as “construction of periodic arrays”.

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4\text{D} \text{ is the spacetime dimension, and } p \text{ the spatial extent of the } p\text{-brane}
Using the no force condition between BPS states, we create an infinite array in a direction transverse to our \( p \)-brane. The solution can then be viewed as a compactification along this particular direction, say \( \hat{z} \). The dependence on \( z \) however poses a problem for the orthodox Kaluza-Klein procedure. Therefore the original construction is modified by letting the distance between the original branes go to zero. This smears out the brane over \( \hat{z} \), removing the dependence on the transverse coordinate. Performing the reduction and conformally rescaling the metric to go back to the Einstein frame of \((D - 1)\) supergravity gives the \((D - 1, p)\) brane. When oxidizing the \((D - 1, p)\) brane to \( D \) dimensions the solution can be naively interpreted as the above smeared brane \([23]\). However, from the string point of view, it makes more sense to reinterpret the oxidation as a periodic array of string solitons, which is certainly another possibility of the oxidation \([22]\).

The limiting procedure of smearing out the brane along a transverse direction hides the fact that we are dropping the heavy modes in this construction which arise from the compactification along \( \hat{z} \). It is these that die off exponentially as the radius is shrunk, playing a role in eliminating the singularity structure encountered in most of the IIA supergravity solutions. We will see this happen explicitly for the case at hand \( \text{M2} \to \text{D2-brane} \).

The paper is organized as follows: section 2 reviews the construction of \( p \)-branes in \( D = 11 \) and \( D = 10 \) Type IIA supergravity theories and their dimensional reduction via the orthodox Kaluza-Klein method. Next we explicitly construct the stacked M2-brane and use the periodic array construction to obtain the D2-brane, comparing the discretized to the smeared brane reduction. In section 4 we discuss the ten dimensional point of view of the reduction, resolving the eleventh dimension at the horizon of the D2-brane in the next section, where we also look at the global eleven dimensional
spacetime, $\mathbb{R}^{10} \times S^1$.

2 \hspace{1em} p\text{-}branes revisited

The $p$-brane solutions we will be interested in are solutions to supergravity theories derivable from $D = 11, N = 1$ supergravity. We therefore start off from here to find the M2-brane. Next we Kaluza-Klein reduce the Lagrangian to obtain the extremal solutions of IIA supergravity, the BPS states of IIA string theory.

All brane solutions we will be considering are bosonic in nature. Therefore we can ignore the fermionic sector, needing only the fermionic transformation laws to ensure that the solutions are indeed supersymmetric. The $D = 11$ supermultiplet contains the graviton described by $e_M^A$, the gravitino, a Rarita-Schwinger vector-spinor, $\Psi_M$, and the 3-form gauge field $A_{MNP}$, with field strength $G_{MNPQ} = 4\partial_{[M}A_{NP]}$. The bosonic Lagrangian resulting is

$$\mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{48} G^{MNPQ} G_{MNPQ} \right] + \frac{1}{6 \cdot 3! 4!} \epsilon^{M_1 \ldots M_{11}} E_{M_1 \ldots M_4} E_{M_5 \ldots M_8} A_{M_9 \ldots M_{11}}. \tag{1}$$

Note that the theory has no dilaton and does not suffer from ambiguities of rescaling the metric by a conformal factor. Further, the solutions we will consider make the variation of the Chern-Simons term vanish, so that we can ignore it from now on.

One can now perform Kaluza-Klein reduction to type IIA supergravity using the ansatz for the metric

$$ds_{11}^2 = e^{2\alpha \varphi} ds_{10}^2 + e^{2\beta \varphi} (dz + A_M dx^M)^2, \tag{2}$$

where the dilaton $\varphi$, the Kaluza-Klein vector field $A$, and the ten-dimensional metric $ds_{10}^2$ are taken to be independent of the extra coordinate $z$. The gauge
field $A_{MNP}$ is split up into a 2-form and 3-form, both also independent of $z$. The two constants $\alpha, \beta$ are chosen so that the resulting lower dimensional theory is in the Einstein frame: $\alpha = -\frac{1}{12}, \beta = \frac{2}{3}$. The resulting Lagrangian has the same field content as IIA supergravity[14].

One can now proceed to look for extended brane solutions to both these theories. The ansatz for the metric and potential in either $D = 10$ or $D = 11$ is given by [7, 4]

$$ds_D^2 = e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} dy^m dy^n \delta_{mn}$$

$$A_{01...(d-1)} = e^C,$$

where $x^\mu (\mu = 0 \ldots d-1)$ are the world-volume coordinates of the $(d-1)$-brane, and $y^m (m = 1 \ldots (D-d))$ are the transverse space coordinates. The functions $A, B, C$ are all functions of the transverse radial distance $r = \sqrt{y^m y^m}$. In $D = 10$ we write the dilaton as

$$\phi = \phi(r).$$

One now imposes supersymmetry on these solutions by looking at the transformation of the fermionic sector of the theory. For the eleven-dimensional theory we have $\delta \Psi_M = \bar{D}_M \epsilon = 0$:

$$\bar{D}_M \epsilon = \left( \partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB} - \frac{1}{288} \left( \Gamma^{PQRS} M + 8 \Gamma^{PQR} \delta^S_M \right) G_{PQRS} \right) \epsilon = 0$$

[24, 4]. In (3) $\omega_M^{AB}$ is the spin connection and the Dirac matrices are defined as $\Gamma^{A_1...A_N} = \Gamma^{[A_1} \ldots \Gamma^{A_N]}$, antisymmetrized with weight one, where $M, N$ are spacetime indices and $A, B$ are tangent space indices.

Same laws exist for the type IIA supergravity theory. To find such an $\epsilon$ one splits up this spinor according to the global symmetry $(\text{Poincare})_d \times \text{SO}(D-d), \epsilon(y) = \epsilon \otimes \eta(y)$; and similarly for the $\Gamma$ matrices. Inserting the
ansätze (3), (4) into the transformation laws gives us \( A(r) \) and \( B(r) \) linearly dependent on \( C(r) \). Insertion into the equations of motion gives:

\[
\delta^{mn} \partial_m \partial_n e^{-C} = 0 \quad (7)
\]

Laplace’s equation is solved by

\[
e^{-C} = 1 + \frac{k_d}{r^{D-d-2}}, \quad r > 0. \quad (8)
\]

The ansatz also reduces \( \eta(r) \) to \( f(e^{-C})\eta_0 \), \( \eta_0 \) a constant spinor, which needs to be projected into a chiral eigenstate. The number of supersymmetries is halved, leaving us with a supersymmetric \( p \)-brane.

The final solution to the two theories considered above is now given by

\[
ds^2 = (1 + \frac{k_d}{r^{d}})^{-\frac{\tilde{d}}{2}} dx^\mu dx^{\nu} \eta_{\mu\nu} + (1 + \frac{k_d}{r^{d}})^{-\frac{\tilde{d}}{2}} dy^m dy^m \\
e^\phi = (1 + \frac{k_d}{r^{d}})^{\frac{2a}{\Delta}}, \quad (9)
\]

where \( \tilde{d} = D - d - 2 \) and \( a^2 = \Delta - \frac{2dd}{D-2}, \Delta = 4 \) in \( D = 10 \) and \( D = 11 \). For \( D = 11 \) we see that \( a = 0 \), which reflects the absence of any dilaton fields in this dimension.

The supersymmetry of these solutions implies the saturation of a Bogo- 
mol’nyi bound [4, 1], which is established by evaluating both the Noether “electric” charge \( Q \) and the mass per unit volume, \( M \), obtaining

\[
Q = \frac{1}{4\Omega_{d+1}} \int_{S^{d+1}} (\ast G_n) \geq M. \quad (10)
\]

Here \( \Omega_{d+1} \) is the volume of the sphere living at the boundary of the transverse space of the \( p \)-brane under consideration. To calculate the ADM mass we look at the first order perturbation of the metric \( g_{MN}, h_{MN} = g_{MN} - \eta_{MN} \), which falls off like \( O(\frac{1}{r^7}) \). Using cartesian coordinates, and letting \( a = 1 \ldots D - 1 \)
and $m$ run over transverse coordinates, the form of $\mathcal{M}$ is given by

\begin{equation}
\mathcal{M} = \frac{1}{4\Omega_{d+1}} \int_{S^{d+1}} d^{d+1}\Sigma^m (\partial^n h_{mn} - \partial_m h^n) \tag{11}
\end{equation}

Furthermore the mass can be related to the integration constant $k_d$ by noticing that Eq. (7) is not exactly satisfied by Eq. (8); it produces a delta function. This hints at the possibility of a source at $r = 0$, which for a $p$-brane can be coupled to the supergravity action, $I_D$ via $I_d$, where

\begin{equation}
I_d = -T_d \int d^d\xi \sqrt{-\gamma} - T_d \int_{\text{world-volume}} A_d. \tag{12}
\end{equation}

The equation of motion (11) now reads

\begin{equation}
\delta^{mn} \partial_m \partial_n e^{-C} = 2T_d \delta^{D-d}(y), \tag{13}
\end{equation}

where $d = p + 1$ is the worldvolume dimension and $\gamma$ is the worldvolume metric, giving $k_d = 2T_d/\tilde{d}\Omega_{d+1}$.

3 Vertical Dimensional Reduction of M2-brane

Having given the form of the solutions for the M2 and D2 brane in $D = 11$ and $D = 10$ dimensions respectively, we construct the periodic array necessary for the vertical dimensional reduction. This will give us an isometry in a transverse direction, which can then be eliminated via the standard Kaluza-Klein procedure.

To construct such an array along a chosen transverse axis, the $p$-brane solutions must remain static, i.e. fixed in their location. This is possible due to the no-force condition, which gives us zero interaction between the

\[5\] Throughout we have set Newton’s constant of gravity, $\kappa_{11}^2 = 1$
soliton solutions. This can be verified via Eq. (12) by inserting the ansatz for our metric, and choosing a static gauge for the coordinates parametrising the world-volume [4, 10]. We see that the Lagrangian gives a potential, which due to the chosen linear interdependence of the functions $A, B, and C$, vanishes. Mathematically this amounts to the validity of superimposing solutions to Laplace’s equation (7).

The above formulae for a single M2-brane lead to [4]:

$$\begin{align*}
 ds^2 &= H(r)^{-\frac{2}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + H(r)^{\frac{1}{3}} dy^m dy^n \\
 C_{012} &= H(r)^{-1} \\
 H(r) &= 1 - \frac{k_3}{r^6}.
\end{align*}$$

(14)

Choosing $\hat{y}_8 = \hat{z}$ as the transverse axis of compactification, the superposition of solutions gives

$$H(y) = 1 + k_3 \sum_{n \in \mathbb{Z}} \frac{1}{|y - na\hat{z}|^6},$$

(15)
a set of parallel membranes of the same orientation, with the same mass and charge; they are located periodically along the $\hat{z}$ axis with period $a = 2\pi R_{11}$. Identifying the membranes we change the topology, $\mathbb{R}^{11} \rightarrow \mathbb{R}^{10} \times S^1$.

Performing a change of variables:

$$|y - na\hat{z}|^6 = \left(\frac{(y^1)^2 + \ldots + (y^7)^2 + (y^8 - na)^2}{r^2} - \frac{(y^9)^2 + \ldots + (y^7)^2 + (y^8 - na)^2}{z - na}\right)^3,$$

$$H(\hat{r}, z) = 1 + k_3 \sum_{n \in \mathbb{Z}} \frac{1}{(\hat{r}^2 + (z - na)^2)^{\frac{3}{2}}}.$$

(16)

\footnote{Recall that this results from supersymmetry arguments. We should note that the no-force condition is not a necessary condition for the reduction, since we are constructing an infinite array. Hence, though unstable, the gravitational and electrical forces on each individual $p$-brane cancel. We can therefore also construct multi-centre solutions for non-supersymmetric cases [20].}
The explicit dependence on the compactified coordinate $z$ poses a problem for the standard Kaluza-Klein procedure. This can be ameliorated by letting $R_{11} \to 0$, creating a smeared out brane along $\hat{z}$ and replacing the sum by an integral:

$$\sum_{\alpha} \frac{k_3}{|y - y_\alpha|^6} \to \int_{-\infty}^{\infty} \frac{k_3 dz}{(\hat{r}^2 + z^2)^{3/2}} = \frac{\tilde{k}_3}{\hat{r}^5},$$

(17)

where $\tilde{k}_3 = \frac{3 \pi k_3}{8 r}$ \cite{20}, giving

$$ds^2 = \left(1 + \frac{\tilde{k}_3}{\hat{r}^5}\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{\tilde{k}_3}{\hat{r}^5}\right)^{1/3} (dz^2 + dy^m dy^\bar{m}).$$

(18)

Using Eq. (2), and the values for $\alpha$ and $\beta$ we have

$$ds^2_{11} = e^{-\frac{\phi}{6}} ds^2_{10} + e^{\frac{4\phi}{3}} dz^2,$$

(19)

giving

$$ds^2_{10} = \left(1 + \frac{\tilde{k}_3}{\hat{r}^5}\right)^{-\frac{1}{3}} dx^\mu dx^\nu \eta_{\mu\nu} \left(1 + \frac{\tilde{k}_3}{\hat{r}^5}\right)^{\frac{2}{3}} dy^m dy^\bar{m}$$

$$e^\phi = \left(1 + \frac{\tilde{k}_3}{\hat{r}^5}\right)^{1/4},$$

(20)

where the Kaluza-Klein dilaton $\varphi$ becomes the IIA dilaton field $\phi$. The solution matches with the result of the Type IIA $D=10$ supergravity equations \cite{9} \cite{14} \cite{8}.

The smearing of the brane is effectively equal to ignoring all the heavy modes along the $\hat{z}$ direction, which die off exponentially fast \cite{22}. Hence at small compactification radii, or equivalently $r \gg R_{11}$, we can see only the above solution. This solution turns out to be singular at $r = 0$, as can be seen by calculating the Ricci scalar at that point (see below). We expect that including the heavy modes will remove the singularity. This amounts to finding the explicit $z$ dependence of the prefactor: turning to
Eq. (16), writing $H(\hat{r}, z) = 1 + k_3 g(\hat{r}, z)$, with $g(\hat{r}, z) = -\frac{1}{4\pi \hat{r}} \left(-\frac{1}{2\pi \partial \hat{r}}\right)$ and $f(\hat{r}, z) = \sum_{n \in \mathbb{Z}} \frac{1}{(\hat{r}^2 + (z_n)^2)}$, we can evaluate $H(\hat{r}, z)$ by evaluating $f$. This has been done explicitly in [27, 16]:

\begin{equation}
\frac{1}{2R_{11}} \frac{\sinh(\hat{r}/R_{11})}{\cosh(\hat{r}/R_{11}) - \cos(z/R_{11})}. \tag{22}
\end{equation}

Hence

\begin{align}
H(\hat{r}, z) &= 1 + \frac{3k_3}{16R_{11}^3} \frac{\sinh(\hat{r}/R_{11})}{\cosh(\hat{r}/R_{11}) - \cos(z/R_{11})} \\
&\quad - \frac{3k_3}{16R_{11}^2} \frac{1 - \cosh(\hat{r}/R_{11}) \cos(z/R_{11})}{(\cosh(\hat{r}/R_{11}) - \cos(z/R_{11})^2)} \\
&\quad - \frac{k_3}{16R_{11}^3} \frac{\sinh(\hat{r}/R_{11}) (2 - \cos^2(z/R_{11}) - \cosh(\hat{r}/R_{11}) \cos(z/R_{11}))}{(\cosh(\hat{r}/R_{11}) - \cos(z/R_{11})^3)}.
\tag{23}
\end{align}

It is this expression that will be helpful in realizing what the eleven-dimensional character of the D2 brane is, in comparison with the picture one has of the fundamental string, a tube [13].

First, however, we will compare the limiting expression of Eq. (23) to see how it appears in ten dimensions, before we look at the eleven dimensional effects.

4 Ten Dimensional Solutions

We have three starting points for the ten dimensional picture of the D2-brane. First we can solve the Type IIA supergravity equations explicitly,

\footnote{Letting $f(z) = \phi(z) \pi \cot(\pi z)$ we can use simple complex analysis to evaluate the sum:}

\begin{equation}
\oint_{\gamma_N} f(z) = 2\pi i \left( \sum_{n=-\infty}^{\infty} \phi(n) - \text{Res}_{g \mathbb{Z}, f(z)} \right), \tag{21}
\end{equation}

where $\gamma_n$ is the square passing through $(\pm (N + \frac{1}{2}), 0)$ and $(0, \pm i (N + \frac{1}{2}))$ [28].
giving solutions of the form (9). The next approach consisted in smearing out the brane and we reobtained the same expression for the D2-brane. We are left with the third intrinsically eleven-dimensional expression (23), which has to be analysed in the appropriate limiting situations.

The limits of interest are the two extremal cases of letting the compactification radius \( R_{11} \to 0 \); and approaching the brane up close in which case \( \hat{r}, z \) become of \( O(R_{11}) \). We expect the former limit to yield the prefactor of Eq. (20), reproducing the ten dimensional solution, barring any conformal scaling.

As \( R_{11} \to 0 \), the exponential parts of (23) conspire to leave only the leading order term

\[
H(r)_{\text{multi}} = 1 + \frac{3k_3}{16R_{11}\hat{r}^5}.
\]

Using this asymptotic form of the prefactor and applying the standard Kaluza-Klein ansatz (19), we obtain the same form for the D2-brane, as in the previous two cases. The difference in these three solutions stems from the constant of integration, which represents the charge and mass of the membranes considered.

Using Eqns. (20) for the D2-brane, we can calculate the ADM mass and charge explicitly for the three cases mentioned (II):

\[
\mathcal{M} = \frac{5}{4} \hat{k}, \quad Q = \frac{5}{4} \hat{k},
\]

saturating the Bogomol'nyi inequality, indicating the supersymmetry of the solution.

Inserting the three different cases into these expressions, we find how the reduction affects our interpretation of the situation. In the solution coming directly from the IIA equations, \( k_3^{(10)} \) is completely arbitrary \( \hat{k} \). The superscripts will indicate the spacetime dimensions from which these constants arise.

\[8\]
continuum limit gave $\tilde{k}_3^{(10)} = \frac{3\pi k_3^{(11)}}{8}$, where the multiplicative factor has no physical importance since it can be absorbed into the eleven-dimensional constant of integration. Both these cases give us a continuous spectrum of masses for the D2-brane. We have to invoke the Dirac quantisation, using D2 and NS5 brane “electric-magnetic” duality, to give us discrete charges and by (25), discretising the BPS-spectrum.

However, the third reduction takes the limit $R_{11} \to 0$ after the sum has been evaluated explicitly. In this fashion we obtain an $\frac{1}{R_{11}}$ dependence in the ADM mass. Though we still have to apply Dirac quantisation to make $\tilde{k}_3 \sim \mathcal{M} \sim n, n \in \mathbb{Z}$, this $\frac{1}{R_{11}}$ dependence implies a Kaluza-Klein tower of BPS states, which would be too pathological if continuous. Further this dependence gives the correct D-brane coupling to the dilaton, $e^\phi$, unlike the usual Kaluza-Klein approach. This can be seen by following Witten’s argument [1], where we use conformal scaling and Kaluza-Klein reduction to relate the radius of the compactified dimension in the two different metrics used to measure it; the eleven dimensional metric, and the string metric.

We may now ask what the reverse procedure “Kaluza-Klein oxidation” produces when applied to Eq. (20). Since this solution starts from a continuum distribution of branes, we may expect that oxidation brings us back to it. However this need not be so. The most general resulting harmonic prefactor in the eleven-dimensional metric due to such oxidation is of the form:

$$H(\hat{r}, z) = \sum_m f_m(\hat{r}) e^{imz/R_{11}}$$

$$f_m = \frac{c_m}{\hat{r}^{d/2}} K_{d/2} \left( \frac{|m|\hat{r}}{R_{11}} \right) + \tilde{c}_m \hat{r}^{d/2} I_{d/2} \left( \frac{|m|\hat{r}}{R_{11}} \right), \quad (26)$$

where $c_m$ and $\tilde{c}_m$ are arbitrary. The heavy modes, $m \neq 0$, exponentially die of as $\hat{r} \to \infty$ and $f_0$ is the original ten dimensional prefactor[22]. There is a
slight ambiguity in what the expansion should be (if a periodic array at all), as the imprint at \( r = 0 \) is ambiguous \([6]\). The above discussion favours the periodic array.

Moving to the other limiting case of \( \hat{r}, z \sim O(R_{11}) \), the non-zero modes should effectively boost the apparent ten dimensional solution to eleven dimensions. This limit, for a small compactification radius is equivalent to letting \( \hat{r} \) become small. Ignoring the heavy modes Eq. (20) gives a Ricci scalar that blows up in the limit \( \hat{r} \to 0 \):

\[
\lim_{\hat{r} \to 0} R_{D2} = \lim_{r \to 0} \frac{75}{32} \left( 1 + \frac{k_3}{r^6} \right)^{-\frac{19}{8}} \hat{r}^{-12} k_3^\frac{2}{3} \to \infty,
\]

identifying a naked singular membrane sheet at the origin. It may be interpreted as a source to the field equations.

On the other hand, a similar calculation for the M2-brane (14) leads to the eleven-dimensional result

\[
\lim_{r \to 0} R_{M2} = \lim_{r \to 0} 6k_3^2 \left( r^6 + k_3 \right)^{-\frac{7}{3}} = 6k_3^{-\frac{1}{3}},
\]

hinting that \( r = 0 \) for the M2-brane might be a horizon \([29, 30]\), making the non-zero modes a relevant contribution to smearing out the singularity in the D2-brane. This also justifies the quantisation of such a membrane object in accordance with \([31]\), as it would counter the ability of the membrane to deform at zero energy cost. One might obtain a discrete spectrum for both the M2 and D2-brane.

Another point of view to the singularity of the D2-brane (20) is its dependence on the correct choice of metric \([3, 22]\). Eleven-dimensional supergravity does not have a dilaton introducing ambiguities in the metric due to conformal rescaling. In ten dimensional string theory, however, we have the frame of the object considered (e.g. the D2-brane); the Einstein frame
that removes the dilatonic prefactor of the Ricci term in the Lagrangian; and the dual frame, which results from performing an “electric-magnetic” duality transformation \cite{32,30}. All these frames are related by a conformal rescaling of the metric. It is these rescalings that remove the singularities in the D2-metrics, as can be checked by calculating the Ricci scalar and the proper time it takes for fundamental (dual) objects to fall into their dual counterpart. The proper time is infinite in such cases, indicating that the singularity is physically irrelevant. We are therefore always in the position of making the singularity vanish in accordance with the eleven-dimensional singularity free horizon at $r = 0$.

5 Local and Global Structure

We may now take the opposite limit from the previous section, i.e. approaching the D2-brane up close with both $\hat{r}, z \to 0$. In doing this we must be careful in which order we let the coordinate variables approach zero. There are three cases to be discerned: $\hat{r} \to 0, z = 0$; $\hat{r} = 0, z \to 0$; and a diagonal incoming trajectory $\hat{r} = \lambda z, z \to 0$. The second case might seem dubious from Eq. (23), however, it is easy to see from (16) that $\hat{r} = 0$ is allowed. In all three situations the form of the metric prefactor is of the form

$$H_{\text{multil}} \to 1 + \frac{k_3}{R^6},$$

where $R$ is the distance from $(\hat{r}, z) = (0, 0)$. Up close we reobtain the M2-brane.

To show that Eq. (28) indeed reflects the non-singularity of a single M2-brane at $r = 0$, we can construct an analytic coordinate extension past this horizon for Eqns. (14), following Refs \cite{29,30}:

$$r = k_3^{1/6}(\Phi^{-3} - 1)^{-1/6}$$

(30)
\begin{align*}
  ds^2 &= \left[ \Phi^2(-dt^2 + dx_1^2 + dx_2^2) + \frac{k_3^{1/3}}{4} \Phi^{-2}d\Phi^2 + k_3^{1/3}d\Omega_7^2 \right] \\
  &\quad + \frac{k_3^{1/3}}{4} \Phi^{-2} \left[ (1 - \Phi^3)^{-7/3} - 1 \right] d\Phi^2 \\
  &\quad + k_3^{1/2} \left[ (1 - \Phi^3)^{-1/3} - 1 \right] d\Omega_7^2 \\
  A_{\mu\nu} &= \Phi^3 \epsilon_{\mu\nu},
\end{align*}

where

\begin{align*}
  r &\to \infty \quad \Phi \to 1^- \\
  r = 0, \text{ the horizon} &\quad \Phi = 0 \\
  \text{into the horizon} &\quad \Phi < 0
\end{align*}

To see that we can pass through the horizon in these coordinates, we look at the near horizon geometry, i.e. \( \Phi \approx 0 \), which is AdS_4 \times S^7, \text{ line (B1)}. We can see this by taking AdS_4 as the quadric in \( \mathbb{R}^{3,2} \) given by

\begin{align*}
  (X^0)^2 + (X^4)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 &= \frac{k_3^{1/3}}{4} \\
  ds^2_{\text{AdS}_4} &= -d(X^0)^2 - d(X^4)^2 + d(X^1)^2 + d(X^2)^2 + d(X^3)^2,
\end{align*}

with

\begin{align*}
  (X^4 - X^3) &= \Phi \\
  X^0 &= t\Phi \\
  X^1 &= x^1\Phi \\
  X^2 &= x^2\Phi \\
  (X^4 + X^3) &= \frac{k_3^{1/3}}{4} \Phi^{-1} + (x^{12} + x^{22} - t^2)\Phi,
\end{align*}

which reproduces (B1) after insertion into (B3). \( \Phi \) is an analytic function on AdS_4 and can be continued through to negative values. Though it seems that (B1) goes bad, it is the coordinates \( \{t, x^1, x^2\} \) that fail. The metric can be brought to a healthy form at \( \Phi = 0 \) by changing the metric to the coordinates
\{X^0, X^1, X^2, X^3\}. The higher order terms also depend analytically on \(\Phi\) and can also be continued through to negative values of \(\Phi\). \(\Phi = 0\) is a horizon. Indeed there is a coordinate singularity at \(\Phi = -\infty\). Since \(r = 0, \Phi = 0\) is made up of two connected components [30], we have two horizons, and we can continue through them separately, obtaining a Carter-Penrose diagram similar to the Reissner-Nordström extreme black-hole [33].

Before going on, we first would like to introduce isotropic coordinates in the interior of a single M2-brane. The coordinate transformation (30)

\[
r = k_3^{1/6} \Phi^{1/2} (1 - \Phi^3)^{-1/6}
\]

at first sight seems to produce a non-analytic extension because of the fractional power of \(\Phi\). However, only even powers of \(r\) appear throughout the metric (14), eliminating this problem. We also do not have to worry about the complexification of \(r\) as \(\Phi\) passes through the horizon, becoming negative. For this range define \(\Phi = -\Phi_1 < 0\), such that \(r^2 = -r_1^2 = -k_3^{1/6} \Phi_1^{1/2} (1 + \Phi_1^3)^{-1/6} < 0\). By changing to these coordinates the M2-brane metric becomes

\[
 ds^2_{r^2<0} = \left(\frac{k_3}{r_1^6} - 1\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(\frac{k_3}{r_1^6} - 1\right)^{1/3} (dr_1^2 + r_1^2 d\Omega_2^2)
\]

for \(r_1 > 0\). Continuing past \(r = 0 = r_1\), we encounter a curvature singularity at \(r_1 = k_3^{1/6}\), i.e. at \(\Phi = -\infty = -\Phi_1\). This can be inferred from the blow up of the Ricci curvature scalar, which becomes infinite there.

Expanding on the above analogy with \(D = 4\) extremal black-holes, we can place several extreme Reissner-Nordström black holes in space time [34, 35, 36], resulting in a static configuration. This is what our higher-dimensional counterpart of superimposed M2-branes is. From [36] we know how the resulting space-time should look and a similar construction for black holes in \(N+1\) dimensions has been performed in [16]. We can now proceed along the
lines of [33] to see how the non-zero modes due to the presence of the other black holes affects the near-horizon geometry.

Following [33] we take Eq. (15) and concentrate on the brane located at $n = 0$. Instead of using the $(\hat{r}, z)$ split, we take the eleven dimensional isotropic radial coordinate and rewrite the sum as:

$$H = 1 + \frac{k_3}{r^6} + k_3 \sum_{n \neq 0} \frac{1}{(r^2 + (an)^2 - 2anr \cos \theta)^3};$$

(37)

where $\theta$ is the polar angle along the $y_8$-axis. We can therefore rewrite the sum in terms of $C^3_i(\cos \theta)$, ultraspherical polynomials, which form a complete set of harmonics on $S^{d-2}$:

$$\sum_{l=0}^{\infty} a_l r^l C^3_i(\cos \theta),$$

(39)

where the expansion requires $r < an$. Since we are concentrating on the M2-brane located at $n = 0$, this condition is satisfied. To continue through $r = 0$ we introduce the same variable as before:

$$r = k^{1/6} \Phi^{1/2} (1 - \Phi^3)^{-1/6}$$

$$= f(\Phi) \Phi^{1/2},$$

(40)

where $f(\Phi)$ is an analytic function of $\Phi$ at $\Phi = 0$. Hence the prefactor becomes

$$H = \Phi^{-3} + \sum_{l=0}^{\infty} a_l(\Phi) \Phi^{l/2} C^3_i(\cos \theta)$$

$$= \Phi^{-3} \left[ 1 + \sum_{l=6}^{\infty} b_l(\Phi, \Omega) \Phi^{l/2} \right],$$

(41)

\[ n = 0 \] can be chosen to label any arbitrary M2-brane, as the stacking is infinite.

10 The ultraspherical polynomials are given by the generating function

$$\frac{1}{(1 - 2tx - t^2)^{\alpha}} = \sum_{n=0}^{\infty} C^{\alpha}_n(x) t^n.$$

(38)

$\alpha = 1/2$ gives the Legendre Polynomials, for $\alpha = 0$, 1 we obtain the Tshebyscheff polynomials. In our case $\alpha = 3$. [37]
where \( a_i \) are analytic functions of \( \Phi \) and the \( b_i \) are analytic functions of \( \Phi \) and of \( S^7 \). The leading term reproduces the above discussion of \( \text{AdS}_4 \times S^7 \):

\[
d s^2 \sim 0 \quad \Phi^2 (-d t^2 + d x^{12} + d x^{22}) + k_3^{1/3} \Phi^{-2} d \Phi^2 + k_3^{1/3} d \Omega_7^2 \\
+ f(\Phi) (-d t^2 + d x^{12} + d x^{22}) + g(\Phi) d \Phi^2 + h(\Phi) d \Omega_7^2, \quad (42)
\]

where \( f(\Phi), g(\Phi), h(\Phi) \to 0 \) as they include terms of \( O(\Phi) \). The higher order terms, however, include powers of \( \Phi^{1/2} \), similarly to the single M2-brane, which seem to prevent an analytic continuation through \( r = 0 = \Phi \). Because of this apparent lack of analyticity, it seems there exists no unique extension across the horizon. One can match onto essentially any solution of the form (14) and (15) with the same total mass. Such a situation also arises in the case of dynamical multi-black holes in five dimensional de Sitter spacetime [38] and higher dimensional multi-p-branes in a static spacetime [33]. The smoothness of such solutions in a static environment was analysed in [19] removing the possibility of interpreting the finite differentiability of the metric as a result of gravitational radiation. Whether or not this lack of smoothness has a physical meaning, as the metric is always suitably differentiable in these cases \( (C^k, k \geq 2) \), was discussed for an exact solution in Ref. [33]. There it was argued that an observer could in principle keep track of the derivatives of the Riemann tensor, hence detecting when he has crossed the horizon.

However, from the physical point of view, we know that every single static black hole has a smooth analytic horizon. We would like this to hold for its dimensionally reduced counterparts and for spacetime backgrounds other than flat Minkowski spacetime. To see how this affects our current

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11 We see that \( (r, z) = (0, 0) \) is not a singularity, and one can also calculate the Ricci scalar at that membrane surface to give \( R = \frac{12}{k^{1/3}} \) indicating the possibility of passing through the horizon.
solution we may expand our series (37) in terms of the above ultraspherical polynomials. The coefficients \( a_l \) become:

\[
a_l = \sum_{n \neq 0} \frac{1}{(an)^{l+6}} \begin{cases} = 0, & l \text{ odd} \\ = \frac{2\zeta((l+6))}{a^{l+6}}, & l \text{ even} \end{cases},
\]

(43)

where \( \zeta(s) \) is the Riemann Zeta function. After inserting (40) into the expansion, we see that the power series contains only integral powers of \( \Phi \). The resulting \( f(\Phi), g(\Phi), h(\Phi) \) in Eq. (42) can be analytically continued through to negative values of \( \Phi \), hence negative values of \( r^2 \). This analytic continuation should not surprise us, as it was speculated in Ref. [38] that for multi-black hole solutions the differentiability of the horizon is increased if the masses are so arranged that the first \( n \) multiple moments vanish, which for our case of infinite “extremal black-holes” would certainly be the case. This has been verified for a low number of black-holes in Ref. [19].

As in the discussion for the single M2-brane, the power series of the prefactor contains only even powers of \( r \), so that we can proceed to formally make the change of variables \((\hat{r}, z) \rightarrow (i\hat{r}, iz)\).

\[
H(\hat{r}, z) = 1 + \frac{3k_3}{16R_{11}^{10} r^5} \frac{\sin \hat{r}/R_{11}}{\cos \hat{r}/R_{11} - \cosh z/R_{11}} - \frac{3k_3}{16R_{11}^{10} r^5} \frac{1 - \cos \hat{r}/R_{11} \cosh z/R_{11}}{(\cos \hat{r}/R_{11} - \cosh z/R_{11})^2} + \frac{k_3}{16R_{11}^{10} r^3} \frac{\sin \hat{r}/R_{11} \left(2 - \cosh^2 z/R_{11} - \cos \hat{r}/R_{11} \cosh z/R_{11}\right)}{(\cos \hat{r}/R_{11} - \cosh z/R_{11})^3}.
\]

(44)

The metric becomes:

\[
ds^2 = (-H(\hat{r}, z))^{-\frac{2}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + (-H(\hat{r}, z))^{\frac{1}{3}} \left(d\hat{r}^2 + \hat{r}^2 d\Omega_6^2 + dz^2\right).
\]

(45)

Here we have both \( \hat{r}, z > 0 \), though the origin does not correspond to a point. We can see this by looking at the \( d\Omega_6^2 \) coefficient, which is non-zero.
This is to be expected, as \((\hat{r}, z) = (0, 0)\) is the membrane horizon through which we analytically extended. We can now proceed inward with \((\hat{r}, z)\) to find the singularity that was previously located at \(\Phi = -\infty\). The location of the singularity is at \(H(\hat{r}, z) = 0\), which indeed has zeroes for suitable values of \((\hat{r}, z)\). To see that at this location we have a true singularity, we can follow the field invariant \(J = G^2 = \left(\nabla H^2\right)^2\), which diverges for \(H = 0\).

Having found the hidden singularity beyond the horizon we look at the structure of the space that the external observer cannot see, \(\Phi < 0\). We note that the \(z\) coordinate loses its periodicity when entering the horizon. However, we are not surprised by this, since the periodicity existed outside the membrane horizon separating “inside” from “outside.” Having the periodicity removed also makes the limit of \(R_{11} \to 0\) become irrelevant, as there is no radius. This seems to be in accordance with the “outside” limit becoming singular at the horizon in the lower-dimensional limit, disconnecting the “inside” from any physical observer who lives at a safe distance from the naked singularity, hence removing the physical relevance of the “inside” region. However, as stated above, this singularity is removable, depending on which metric we choose to use in ten dimensions. Since our vertical dimensional reduction keeps the membrane object as the “fundamental” element in the lower dimensional theory, this singularity seems to be quite real from the perturbative expansion point of view of the IIA theory in this particular vacuum[32].

Another way to interpret the limit of \(R_{11} \to 0\), taken outside the horizon, is to say that \(\hat{r} \gg R_{11}\). We see the eleventh dimension only becoming relevant close to the membrane. From Eq. (14) this dimension becomes irremovable inside and we are drawn to the speculation that this confinement of the eleventh dimension, might be related to the confinement of the gauge field
of a D2-brane to its worldvolume. This would be in accordance with the
duality transformations performed in Ref. [12], where the eleventh dimension
becomes the gauge field living on the world-volume.
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