Regularity of Weak Solutions for Nonlinear Parabolic Problem with $p(x)$-Growth

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Abstract
In this paper, we study the nonlinear parabolic problem with $p(x)$-growth conditions in the space $W^{1,\infty} L^{p(x)}(Q)$, and give a regularity theorem of weak solutions for the following equation

$$\frac{\partial u}{\partial t} + A(u) = 0$$

where $A(u) = -\text{div}(a(x, t, u, \nabla u)) + a_0(x, t, u, \nabla u)$ and $a_0(x, t, u, \nabla u)$ satisfy $p(x)$-growth conditions with respect to $u$ and $\nabla u$.

Keywords: nonlinear parabolic problem, regularity, $W^{1,\infty} L^{p(x)}(Q)$ space, $p(x)$-growth condition.
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1 Introduction
In recent years, the research of variational problems with nonstandard growth conditions is an interesting topic. $p(x)$-growth problems can be regarded as a kind of nonstandard growth problems and they appear in nonlinear elastic, electrorheological fluids and other physics phenomena. Many results have been obtained on this kind of problems, for examples [1-9].

In this paper, we will qualitatively study the properties of weak solutions. For more information about qualitative analysis, we refer to [10-11]. Let $Q$ be $\Omega \times (0, T)$ where $T > 0$ is given. In [8], the authors studied the following equation in the space $W^{1,1} L^{p(x)}_{loc}(Q) \cap C(0, T; L^2_{loc}(\Omega))$,

$$u_t - \text{div}(|Du|^p(x,t)-2Du) = 0,$$

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where \( \max\{1; \frac{2N}{N+2}\} < p_1 = \inf_{(x,t) \in \Omega} p(x,t) \leq p(x,t) \leq \sup_{(x,t) \in \Omega} p(x,t) = p_2 < \infty \),
p(x,t) is dependent on the space variable \( x \) and the time variable \( t \), and satisfies
the following Logarithmic H"older condition

\[
|p(x,t) - p(y,s)| \leq \frac{C_1}{-\ln(|x-y| + C_2|t-s|^{p_2})}
\]

for all \((x,t), (y,s) \in Q, |x-y| < \frac{1}{2}, |t-s| < \frac{1}{2}\), where \( C_1, C_2 > 0 \) are constants.
The authors proved the H"older continuity of the local weak solution with the
scale transformation method. In this paper, we will study the following more
general problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + A(u) &= 0, \quad \text{in } Q, \quad (1.1) \\
u(x,t) &= 0, \quad \text{on } \partial\Omega \times (0,T), \quad (1.2) \\
u(x,0) &= \psi(x), \quad \text{in } \Omega, \quad (1.3)
\end{align*}
\]

where \( \psi(x) \) is a given function in \( L^2(\Omega) \) and \( A: W^{1,1}_{0} L^p(x)(Q) \to W^{-1,1}_{0} L^q(x)(Q) \)
is an elliptic operator of the form
\( A(u) = -\text{div}(a(x,t,u,\nabla u)) + a_0(x,t,u,\nabla u) \) with the coefficients \( a \) and \( a_0 \) satisfying the classical Leray-Lions conditions. In [12-
[13] we have proved the existence and the local boundedness of the solutions
of (1.1)-(1.3) and have obtained \( u \in W^{1,1}_{0} L^p(x)(Q) \cap L^\infty(0,T; L^2(\Omega)) \). In
this paper we will give the regularity theorem of the weak solutions in the framework
space \( W^{1,1}_{0} L^p(x)(Q) \), which can be considered as a special case of the space
\( W^{1,1}_{0} p(x,t) Q) \).

The space \( W^{1,1}_{0} L^p(x)(Q) \) provides a suitable framework to discuss some physical
problems. In [14], the authors studied a functional with variable exponent,
\( 1 \leq p(x) \leq 2 \), which provided a model for image denoising, enhancement, and
restoration. Because in [14] the direction and speed of diffusion at each location
depended on the local behavior, \( p(x) \) only depended on the location \( x \) in the
image. Consider that the space \( W^{1,1}_{0} L^p(x)(Q) \) was introduced and discussed in
[12] and [15], we think that the space \( W^{1,1}_{0} L^p(x)(Q) \) is a reasonable framework
to discuss the \( p(x) \)-growth problem (1.1)-(1.3), where \( p(x) \) only depends on the
space variable \( x \) similar to [14].

In this paper, let \( a: Q \times R \times R^N \to R^N \) and \( a_0: Q \times R \times R^N \to R \) be
the operators such that for any \( s \in R \) and \( \xi \in R^N \), \( a(x,t,s,\xi) \) and \( a_0(x,t,s,\xi) \)
are both continuous in \((t,s,\xi)\) for a.e. \( x \in \Omega \) and measurable in \( x \) for all \((t,s,\xi) \in (0,T) \times R^N \). They also satisfy that for a.e. \((x,t) \in Q\), any \( s \in R \) and \( \xi \neq \xi^* \in R^N \):

\[
\begin{align*}
|a(x,t,s,\xi)| &\leq \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \quad (1.4) \\
|a_0(x,t,s,\xi)| &\leq \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \quad (1.5) \\
[a(x,t,s,\xi) - a(x,t,\xi^*)](\xi - \xi^*) &> 0, \quad (1.6) \\
 a(x,t,s,\xi) &\geq \beta(|\xi|^{p(x)} + |s|^{p(x)}), \quad (1.7) \\
a_0(x,t,s,\xi) s &\geq \beta(|\xi|^{p(x)} + |s|^{p(x)}), \quad (1.8)
\end{align*}
\]

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where $\alpha, \beta > 0$ are constants.

Throughout this paper, unless special statement, we always suppose that $p(x)$ is Lipschitz continuous on $\Omega$, and satisfies

$$1 < p^− = \inf_{\Omega} p(x) \leq p(x) \leq \sup_{\Omega} p^+ < \infty.$$  \hfill (1.9)

Because $p(x)$ is Lipschitz continuous, there exists a constant $C > 0$ such that

$$\rho^−(p^+ − \rho^−) \leq C, \quad \forall Q_{\rho} \subset Q,$$  \hfill (1.10)

where $Q_{\rho} = K_{\rho} \times (-p^+, 0), \quad 0 < \rho < 1, \quad K_{\rho} = \{x \in \Omega \mid \max_{1 \leq i \leq N} |x_i| < \rho\}, \quad p^+_\rho = \sup_{K_{\rho}} p(x), \quad p^−_\rho = \inf_{K_{\rho}} p(x)$.

**Definition 1.1** A function $u \in W^{1,x}L^p(x)(Q) \cap L^\infty(0, T; L^2(\Omega))$ is called a weak solution of (1.1)-(1.3) if

$$-\int_Q u \frac{\partial \varphi}{\partial t} dxdt + \int_\Omega u \varphi dx_{\mid 0}^T + \int_Q [a(x, t, u, \nabla u) \nabla \varphi + a_0(x, t, u, \nabla u) \varphi] dxdt = 0$$

for all $\varphi \in C^1(0, T; C^\infty_0(\Omega))$.

**Definition 1.2** The functions $u_n \in C(0, T; C^\infty_0(\Omega))$ are called the Galerkin solutions of (1.1)-(1.3) if

$$\int_{Q'} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau + \int_{Q'} a(x, \tau, u_n, \nabla u_n) \varphi dx d\tau + \int_{Q'} a_0(x, \tau, u_n, \nabla u_n) \varphi dx d\tau = 0$$

for all $\varphi \in C^1(0, T; C^\infty_0(\Omega))$ and $Q' = \Omega \times (0, t), t \in (0, T]$.

We will prove the following regularity theorem:

**Theorem 1** Let $p^− > 2$. If $u \in W^{1,x}L^p(x)(Q) \cap L^\infty(0, T; L^2(\Omega))$ is a local weak solution of (1.1)-(1.3), then $u$ is local Hölder continuous in $Q$.

### 2 Preliminaries

We first recall some facts on spaces $L^p(x)(\Omega), W^{m,p}(x)(\Omega), W^{m,x}L^p(x)(Q)$ and parabolic space. For the details see [15-18].

Although we assume (1.9) holds in this paper, in this section we introduce the general spaces $L^p(x)(\Omega), W^{m,p}(x)(\Omega)$ and $W^{m,x}L^p(x)(Q)$.

Denote

$$E = \{\omega : \omega \text{ is a measurable function on } \Omega\},$$

where $\Omega \subset \mathbb{R}^N$ is an open subset.
Let \( p(x) : \Omega \to [1, \infty] \) be an element in \( E \). Denote \( \Omega_\infty = \{ x \in \Omega : p(x) = \infty \} \). For \( u \in E \), we define
\[
\rho(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} \, dx + \text{ess sup}_{x \in \Omega_\infty} |u(x)|.
\]
The space \( L^{p(x)}(\Omega) \) is
\[
L^{p(x)}(\Omega) = \{ u \in E : \exists \lambda > 0, \rho(\lambda u) < \infty \}
\]
endowed with the norm
\[
\| u \|_{L^{p(x)}(\Omega)} = \inf \{ \lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1 \}.
\]
We define the conjugate function \( q(x) \) of \( p(x) \) by
\[
q(x) = \begin{cases} 
\infty, & \text{if } p(x) = 1; \\
1, & \text{if } p(x) = \infty; \\
\frac{p(x)}{p(x) - 1}, & \text{if } 1 < p(x) < \infty.
\end{cases}
\]

**Lemma 2.1 (see [18])**  
(1) The dual space of \( L^{p(x)}(\Omega) \) is \( L^{q(x)}(\Omega) \), if \( 1 \leq p(x) < \infty \).
(2) The space \( L^{p(x)}(\Omega) \) is reflexive if and only if (1.9) is satisfied.

**Lemma 2.2 (see [18])**  
If \( 1 \leq p(x) < \infty \), \( C_0^\infty(\Omega) \) is dense in the space \( L^{p(x)}(\Omega) \) and \( L^{p(x)}(\Omega) \) is separable.

**Lemma 2.3 (see [18])**  
Let \( 1 \leq p(x) \leq \infty \), for every \( u(x) \in L^{p(x)}(\Omega) \) and \( v(x) \in L^{q(x)}(\Omega) \), we have
\[
\int_{\Omega} |u(x)v(x)| \, dx \leq C \| u(x) \|_{L^{p(x)}(\Omega)} \| v(x) \|_{L^{q(x)}(\Omega)},
\]
where \( C \) is only dependent on \( p(x) \) and \( \Omega \), not dependent on \( u(x), v(x) \).

Next let \( m > 0 \) be an integer. For each \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \alpha_i \) are nonnegative integers and \( |\alpha| = \sum_{i=1}^n \alpha_i \), and denote by \( D^\alpha \) the distributional derivative of order \( \alpha \) with respect to the variable \( x \).

We now introduce the generalized Lebesgue-Sobolev space \( W^{m,p(x)}(\Omega) \) which is defined as
\[
W^{m,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq m \}.
\]
\( W^{m,p(x)}(\Omega) \) is a Banach space endowed with the norm
\[
\| u \| = \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^{p(x)}(\Omega)}.
\]
The space \( W^{0,p(x)}_0(\Omega) \) is defined as the closure of \( C_0^\infty(\Omega) \) in \( W^{0,p(x)}(\Omega) \). The dual space \( (W^{m,p(x)}_0(\Omega))^* \) is denoted by \( W^{-m,q(x)}(\Omega) \) equipped with the norm
\[
\| f \|_{W^{-m,q(x)}(\Omega)} = \inf \Sigma_{|\alpha| \leq m} \| f_\alpha \|_{L^{q(x)}(\Omega)},
\]
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where infimum is taken on all possible decompositions
\[ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(\Omega). \]

**Lemma 2.4** (see [18])

1. \( W^{m,p(x)}(\Omega) \) and \( W_0^{m,p(x)}(\Omega) \) are separable if \( 1 \leq p(x) < \infty \).

2. \( W^{m,p(x)}(\Omega) \) and \( W_0^{m,p(x)}(\Omega) \) are reflexive if (1.9) holds.

We define the space \( W^{m,x}L^{p(x)}(Q) \) as the following:
\[ W^{m,x}L^{p(x)}(Q) = \{ u \in L^{p(x)}(Q) : D^\alpha u \in L^{p(x)}(Q), |\alpha| \leq m \}. \]

\( W^{m,x}L^{p(x)}(Q) \) is a Banach space with the norm \( \| u \| = \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^{p(x)}(Q)} \), where \( p(x) \) is independent of \( t \).

The space \( W_0^{m,x}L^{p(x)}(Q) \) is defined as the closure of \( C_0^\infty(Q) \) in \( W^{m,x}L^{p(x)}(Q) \) and \( W_0^{m,x}L^{p(x)}(Q) \hookrightarrow L^{p(x)}(Q) \) is continuous embedding. Let \( M \) be the number of multiindexes \( \alpha \) which satisfies \( 0 \leq |\alpha| \leq m \), then the space \( W_0^{m,x}L^{p(x)}(Q) \) can be considered as a close subspace of the product space \( \Pi_{\alpha=1}^M L^{p(x)}(Q) \). So if \( 1 < p(x) < \infty \), \( \Pi_{\alpha=1}^M L^{p(x)}(Q) \) is reflexive and further we can get that the space \( W_0^{m,x}L^{p(x)}(Q) \) is reflexive. The dual space \( (W^{m,x}L^{p(x)}(Q))^* \) is denoted by \( W^{-m,x}L^{q(x)}(Q) \) equipped with the norm
\[ \| f \|_{W^{-m,x}L^{q(x)}(Q)} = \sup_{\| u \|_{W_0^{m,x}L^{p(x)}(Q)} \leq 1} | < f, u > | = \inf \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^{q(x)}(Q)}, \]
where infimum is taken on all possible decompositions
\[ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(Q). \]

Next, we will introduce the parabolic space and some results in [16]:

**Definition 2.5** Let \( p, r \geq 1 \). A function \( f \) defined in \( Q \) belongs to the space \( L^r(0, T; L^p(\Omega)) \), if
\[ \| f \|_{p,r,Q} = \left( \int_0^T \left( \int_\Omega |f|^p dx \right)^{\frac{r}{p}} dt \right)^{\frac{1}{r}} < \infty. \]

**Definition 2.6** Let \( p, r \geq 1 \). We define the function spaces
\[ V^{r,p}(Q) = L^\infty(0, T; L^r(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)), \]
\[ V_0^{r,p}(Q) = L^\infty(0, T; L^r(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)), \]
which are both equipped with the norm
\[ \| v \|_{V^{r,p}(Q)} = \text{ess sup}_{0 < t < T} \| v(x, t) \|_{L^r(\Omega)} + \| \nabla v \|_{L^p(Q)}. \]
Lemma 2.7  Let \( \{Y_n\} \), \( n = 0,1,2, \ldots \), be a sequence of positive numbers, satisfying the inequalities \( Y_{n+1} \leq C b Y_n^{a+q} \), where \( C, b > 1 \) and \( a > 0 \) are given numbers. If \( Y_0 \leq C^{-\frac{1}{a}} b^{-\frac{1}{q}} \), then \( \{Y_n\} \) converges to 0 as \( n \to \infty \).

Lemma 2.8 Let \( r > 1 \), there exists a constant \( C \) depending only on \( N, r \), such that for every \( v \in L^\infty(0,T;L^r(\Omega)) \cap L^r(0,T;W_0^{1,r}(\Omega)) \),

\[
\|v\|_{L^r(Q)} \leq C \|v\|_{V_r(\Omega)} + \frac{1}{r} \|v\|_{V_r^\star(\Omega)}
\]

where \( \|v\| > 0 \) means \( \{v(x,t) : |v| > 0\} \).

Lemma 2.9 Let \( v \in W^{1,1}(K_\rho(x_0)) \cap C(K_\rho(x_0)) \) for some \( \rho > 0 \) and some \( x_0 \in \mathbb{R}^N \), and let \( k \) and \( h \) be any pair of real numbers such that \( k < h \), then there exists a constant \( C \) depending only upon \( N, \) such that

\[
(h - k)|A(h)| \leq C \frac{\rho^{N+1}}{K_\rho(x_0) \cdot W_0^{1,1}(\Omega)} \int_{A(k) \setminus A(h)} |\nabla v| dx
\]

where \( A(k) = \{x \in K_\rho(x_0) : v(x) > k\}, |A(k)| = \text{meas}(A(k)). \)

Let \( u \in L^1(Q) \). For any \( 0 < h < T \), we introduce the Steklov average function

\[
u_h(x,t) = \begin{cases} \frac{1}{T-h} \int_{t-h}^t u(x,\tau) d\tau, & t \in (0, T-h], \\ 0, & t > T-h. \end{cases} \]

Lemma 2.10 Let \( u \in L^r(0,T;L^p(\Omega)) \), then as \( h \to 0 \), \( u_h \to u \) in \( L^r(0,T-\varepsilon;L^p(\Omega)) \) for every \( \varepsilon > 0 \). If \( u \in C(0,T;L^2(\Omega)) \), then as \( h \to 0 \), \( u_h \to u \) in \( L^2(\Omega) \) for every \( t \in (0,T-\varepsilon) \).

Similarly, we can get the following lemma in variable exponent space.

Lemma 2.11 If \( u \in L^{p(x)}(Q) \), then as \( h \to 0 \), \( u_h \to u \) in \( L^{p(x)}(Q) \).

Proof: Because \( p(x) \) is bounded and independent of \( t \). We only need to notice that there exist \( u_k \in C_0^1(Q) \) such that \( u_k \to u \) in \( L^{p(x)}(Q) \), and by the uniform continuity of \( u_k \), we can conclude the lemma. \( \square \)

3 Regularity of Weak Solutions

In [12-13], we have obtained that for the Galerkin solution \( u_n \in C^1(0,T;C^{\infty}(\Omega)), \) \( u_n \to u \) strongly in \( L^2(Q) \) and \( L^{p(x)}(Q) \), \( u_n \to u \) weakly in \( W_0^{1,p(x)}(Q) \), \( a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u) \) and \( a_0(x,t,u_n,\nabla u_n) \to a_0(x,t,u,\nabla u) \) weakly in \( L(q(x))(Q) \), \( u_n \to u \) a.e. in \( Q \) and \( \nabla u_n \to \nabla u \) a.e. in \( Q \).

For (1.11), integrating by parts, we can get

\[
\int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau = \int_{\Omega} u_n(x,t) \varphi(x,t) dx - \int_{Q^t} u_n \frac{\partial \varphi}{\partial \tau} dx d\tau,
\]

therefore

\[
\lim_{n \to \infty} \int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau = \int_{\Omega} u(x,t) \varphi(x,t) dx - \int_{Q^t} u \frac{\partial \varphi}{\partial \tau} dx d\tau.
\]

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As \(a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)\) weakly in \(L^{q(x)}(Q)\) and \(a_0(x,t,u_n,\nabla u_n) \to a_0(x,t,u,\nabla u)\) weakly in \(L^{q(x)}(Q)\), we have

\[
\lim_{n \to \infty} \left( \int_{Q^t} a(x,\tau,u_n,\nabla u_n) \varphi d\tau + \int_{Q^t} a_0(x,\tau,u_n,\nabla u_n) \varphi d\tau \right) = \int_{Q^t} a(x,\tau,u,\nabla u) \varphi d\tau + \int_{Q^t} a_0(x,\tau,u,\nabla u) \varphi d\tau,
\]

then (1.11) can be written as

\[
\int_{Q^t} u(x,t) \varphi(x,t) dx - \int_{Q^t} u \frac{\partial \varphi}{\partial \tau} d\tau + \int_{Q^t} a(x,\tau,u,\nabla u) \varphi d\tau
+ \int_{Q^t} a_0(x,\tau,u,\nabla u) \varphi d\tau = 0.
\]

(3.1)

In (3.1), let \(\varphi\) be independent of \(t\) and \(t = t + h\), then we get

\[
\int_{Q^t} \frac{\partial u_n(x,\tau)}{\partial \tau} \varphi d\tau + \int_{Q^t} [a(x,\tau,u,\nabla u)]_h \varphi d\tau + \int_{Q^t} [a_0(x,\tau,u,\nabla u)]_h \varphi d\tau = 0,
\]

(3.2)

where \(\varphi \in C_0^\infty(\Omega)\).

**Lemma 3.1** If \(u\) is a weak solution of (1.1)-(1.3), then \(u \in C(0,T;L^2(\Omega))\).

Proof: Because \(u_n \to u\) weakly in \(W_0^{1,2}L^p(\Omega)\), there exists a convex combination of \(u_n\), denoted by \(v_n\), such that \(v_n \to u\) strongly in \(W_0^{1,2}L^p(\Omega)\) and \(v_n(x,0) \to \psi(x)\) strongly in \(L^2(\Omega)\).

Take \(\varphi = u_n - v_m\) as the testing function in (1.11),

\[
\int_{Q^t} \frac{\partial u_n}{\partial \tau}(u_n - v_m)d\tau d\tau + \int_{Q^t} a(x,\tau,u_n,\nabla u_n)(u_n - v_m)d\tau d\tau
+ \int_{Q^t} a_0(x,\tau,u_n,\nabla u_n)(u_n - v_m)d\tau d\tau = 0,
\]

then for the sufficient large \(m\), we have

\[
\lim_{n \to \infty} \int_{Q^t} \frac{\partial u_n}{\partial \tau}(u_n - v_m)d\tau d\tau 
\leq \int_{Q^t} a(x,\tau,u,\nabla u)(u - v_m)d\tau d\tau + \int_{Q^t} a_0(x,\tau,u,\nabla u)(u - v_m)d\tau d\tau 
\leq 2(\|a\|_{L^p(\Omega)} + \|a_0\|_{L^p(\Omega)}) \|\nabla(u - v_m)\|_{L^p(\Omega)} \leq \varepsilon(m)
\]

and

\[
\lim_{n \to \infty} \int_{Q^t} \frac{\partial v_m}{\partial \tau}(v_m - u_n)d\tau d\tau \leq \varepsilon(m)
\]

where \(\varepsilon(m) \to 0\) as \(m \to 0\).

In short,

\[
\lim_{n \to \infty} \int_{Q^t} \frac{\partial (u_n - v_m)}{\partial \tau}(u_n - v_m)d\tau d\tau \leq \varepsilon(m),
\]

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i.e. 
\[ \lim_{n \to \infty} \int_{\Omega} |u_n - v_m|^2 \, dx \leq \varepsilon(m). \]
therefore, for \( k > m \), we get
\[ \max_{0 < t < T} \|v_k - v_m\| \leq \max_{0 < t < T} \left[ \lim_{n \to \infty} \left( \|v_k - u_n\| + \|u_n - v_m\| \right) \right] \leq \varepsilon(k) + \varepsilon(m), \]
namely \( \{v_n\} \) is a Cauchy sequence in \( C(0, T; L^2(\Omega)) \), so we get the result. \( \square \)

Next, we will prove the main theorem.

By [13], we know that there exists a constant \( M > 0 \), such that \( \|u\|_{L_{loc}^\infty(Q)} \leq M \). Fix a point \((x_0, t_0)\) in \( Q \), let \( \rho \in (0, 1) \) be small enough such that
\[ Q(p^{\rho^\omega - \varepsilon}, 2\rho) = K_{2\rho}(x_0) \times (t_0 - \rho^{p^\omega - \varepsilon}, t_0) \subset Q, \]
where \( K_{2\rho}(x_0) = \{ x \in \Omega | \max_{1 \leq i \leq N} |x_i - x_{0,i}| < 2\rho \} \), \( p_\rho^+= \sup_{K_{2\rho}(x_0)} p(x) \), \( p_\rho^- = \inf_{K_{2\rho}(x_0)} p(x) \).

Denote \( \mu^+ = \text{ess sup}_{Q(p^{\rho^\omega - \varepsilon} - 2\rho)} u, \mu^- = \text{ess inf}_{Q(p^{\rho^\omega - \varepsilon} - 2\rho)} u, \omega = \text{ess osc}_{Q(p^{\rho^\omega - \varepsilon} - 2\rho)} u = \mu^+ - \mu^- \).

Consider the cylinder \( Q(a(p^{\rho^\omega} - \varepsilon), \frac{1}{A}) = (\frac{\omega}{A})^{p^\omega - 2} \), where \( A > 2 \) is a constant to be determined later. We assume that
\[ (\frac{\omega}{A})^{p^\omega - 2} > \rho^\omega, \quad (3.3) \]
where \( \varepsilon \in (0, 1) \) will be determined later. This implies the inclusion
\[ Q(a(p^{\rho^\omega} - \varepsilon), \rho) \subset Q(p^{\rho^\omega - \varepsilon}, 2\rho) \]
and
\[ \text{ess osc}_{Q(a(p^{\rho^\omega} - \varepsilon), \rho)} u \leq \omega. \]

If (3.3) is not hold, \( \omega \leq A(p^{\rho^\omega} - 2) \). Take \( C = A \), then the first iterative of proposition 3.4 is hold, so the proposition 3.4 is right. therefore we also assume that (3.3) is hold in the following proof.

Let \([0, t^*) + Q(l(p^{\rho^\omega}), \rho) = \{ x \in \Omega | \max_{1 \leq i \leq N} |x_i| < \rho \} \times [t^* - l(p^{\rho^\omega}), t^*] \], \( t^* = (\frac{\omega}{A})^{p^\omega - 2} \). For \([0, t^*) + Q(l(p^{\rho^\omega}), \rho) \subset Q(a(p^{\rho^\omega} - \varepsilon), \rho), -(A(p^{\rho^\omega} - 2 - 2(p^{\rho^\omega} - 2)\rho^{p^\omega} \omega^2 - \rho^\omega < t^* < 0 \). We assume \((x_0, t_0) = (0, 0) \) and define \((u - k) = \max\{|u - k, 0\} \).

**Lemma 3.2.** There exists a number \( \sigma \in (0, 1) \) independent of \( \omega, \rho \) such that if (3.3) and
\[ |(x, t) \in [(0, t^*) + Q(l(p^{\rho^\omega}), \rho)] : u < \mu^- + \frac{\omega}{2} \leq \sigma|Q(l(p^{\rho^\omega}), \rho)| \quad (3.4) \]
hold, then \( u > \mu^- + \frac{\omega}{2} \), a.e. \((x, t) \in [(0, t^*) + Q(l(p^{\rho^\omega}), \rho)] \).
Proof: Up to a translation we may assume that \((0, t^*) = (0, 0)\). Let \(\rho_m = \frac{t}{2} + \frac{p_m^+}{2}, \ k_m = \mu^* + \frac{\xi}{4} + \frac{\xi}{2 + x^2}, \ Q_{\rho_m} = K_{\rho_m} \times (-l \rho_m^+, 0), \ m = 0, 1, 2, \ldots\). We choose smooth cutoff function \(\eta_m = \xi_1(x) \xi_2(t), \) where \(0 \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1\) and

\[
\xi_1 = 1, \ \text{if} \ x \in K_{\rho_{m+1}}; \quad \xi_1 = 0, \ \text{if} \ x \in K_{\rho_m}; \quad \text{and} \quad |\nabla \xi_1| \leq \frac{1}{\rho_m - \rho_{m+1}}.
\]

\[
\xi_2 = 1, \ \text{if} \ t \geq -l \rho_m^+; \quad \xi_2 = 0, \ \text{if} \ t \leq -l \rho_{m+1}^+; \quad \text{and} \quad 0 \leq \frac{\partial \xi_2}{\partial t} \leq \frac{1}{l(\rho_m^+ - \rho_{m+1}^+)}.
\]

Take \(\varphi = -(u_n - k_m)_-\eta_m^+\) as the testing function in (1.11), then

\[
\int_{Q_{\rho_m}} \frac{\partial u_n}{\partial \tau}[-(u_n - k_m)_-\eta_m^+]dxdt + \int_{Q_{\rho_m}} a(x, \tau, u_n, \nabla u_n)[-(\nabla (u_n - k_m)_-\eta_m^+)]dxdt
\]

\[
+ \int_{Q_{\rho_m}} a(x, \tau, u_n, \nabla u_n)[-\eta_m^+](u_n - k_m)_-\eta_m^+]dxdt = 0,
\]

where \(Q_{\rho_m} = K_{\rho_m} \times (-l \rho_m^+, t), \ t \in (-l \rho_m^+, 0)\).

First, integrating by parts,

\[
\int_{Q_{\rho_m}} \frac{\partial u_n}{\partial \tau}[-(u_n - k_m)_-\eta_m^+]dxdt
\]

\[
= \frac{1}{2} \int_{Q_{\rho_m}} \frac{\partial}{\partial \tau}[(u_n - k_m)_-\eta_m^+] dx - \frac{p_m^+}{2} \int_{Q_{\rho_m}} (u_n - k_m)_-\eta_m^{p_m^+ - 1} \frac{\partial \eta_m}{\partial \tau} dxdt
\]

\[
= \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m)_-\eta_m^+(x, t) dx - \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m)_-\eta_m^+(x, -l \rho_m^+) dx
\]

\[
- \frac{p_m^+}{2} \int_{Q_{\rho_m}} [(u_n - k_m)_-\eta_m^{p_m^+ - 1} \frac{\partial \eta_m}{\partial \tau}] dxdt.
\]

Since \(u_n \to u\) in \(L^2(Q)\) and \(u \in C(0, T; L^2(\Omega))\), \(u_n \to u\) in \(L^2(\Omega)\) for \(\forall t \in (0, T)\), therefore we can get

\[
\lim_{n \to \infty} \int_{Q_{\rho_m}} \frac{\partial u_n}{\partial \tau}[-(u_n - k_m)_-\eta_m^+]dxdt
\]

\[
= \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m)_-\eta_m^+(x, t) dx - \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m)_-\eta_m^+(x, -l \rho_m^+) dx
\]

\[
- \frac{p_m^+}{2} \int_{Q_{\rho_m}} [(u_n - k_m)_-\eta_m^{p_m^+ - 1} \frac{\partial \eta_m}{\partial \tau}] dxdt.
\]

Since \(\nabla (u_n - k_m)_- \to \nabla (u - k_m)_-\) and \(a(x, t, u_n, \nabla u_n) \to a(x, t, u, \nabla u)\) a.e. in \(Q_{\rho_m}\), by Fatou lemma,
\[
\lim_{n \to \infty} \int_{Q_n} a(x, \tau, u_n, \nabla u_n) [-\nabla (u_n - k_m) - \eta_n^p] d\tau \\
\geq \int_{Q_n} a(x, \tau, u, \nabla u) [-\nabla (u - k_m) - \eta_n^p] d\tau.
\]

By the fact that \( u_n \to u \) strongly in \( L^{p(x)}(Q) \), \( a(x, t, u_n, \nabla u_n) \to a(x, t, u, \nabla u) \) weakly and \( a_0(x, t, u_n, \nabla u_n) \to a_0(x, t, u, \nabla u) \) weakly in \( L^{q(x)}(Q) \), we have

\[
\lim_{n \to \infty} \int_{Q_n} a_0(x, \tau, u_n, \nabla u_n) [-\nabla (u_n - k_m) - \eta_n^{p^+}] d\tau \\
= \int_{Q_n} a_0(x, \tau, u, \nabla u) [-\nabla (u - k_m) - \eta_n^{p^+}] d\tau,
\]
and

\[
\lim_{n \to \infty} \int_{Q_n} a_0(x, \tau, u_n, \nabla u_n) [-\nabla (u_n - k_m) - \eta_n^{p^+}] d\tau \\
= \int_{Q_n} a_0(x, \tau, u, \nabla u) [-\nabla (u - k_m) - \eta_n^{p^+}] d\tau.
\]

Let \( I = \lim_{n \to \infty} \int_{Q_n} a(x, \tau, u_n, \nabla u_n) \nabla \varphi d\tau\) for \( a_0(x, \tau, u_n, \nabla u_n) \varphi d\tau\), so

\[
I \geq - \int_{Q_n^m} a(x, \tau, u, \nabla u) [\nabla (u - k_m) - \eta_n^{p^+}] d\tau \\
- \int_{Q_n^m} a(x, \tau, u, \nabla u) [(u - k_m) - \eta_n^{p^+}] d\tau \\
- \int_{Q_n^m} a_0(x, \tau, u, \nabla u) [(u - k_m) - \eta_n^{p^+}] d\tau.
\]

By (1.4)-(1.5),(1.7)-(1.8), \( ||u||_{L_{\text{loc}}^{\infty}(Q)} \leq M \) and \( ||(u - k_m)||_{L_{\text{loc}}^{\infty}(Q)} \leq ||(k_m - u)||_{L_{\text{loc}}^{\infty}(Q)} \leq \frac{\epsilon}{2} \), we have

\[
I \geq \beta \int_{Q_n^m} (|\nabla (u - k_m)|^{p(x)} + |u|^{p(x)}) \eta_n^{p^+} d\tau \\
- \alpha \int_{Q_n^m} (|\nabla (u - k_m)|^{p(x)-1} + |u|^{p(x)-1}) (u - k_m) - \eta_n^{p^+} |\nabla \eta_n| d\tau \\
- \alpha \int_{Q_n^m} (|\nabla (u - k_m)|^{p(x)-1} + |u|^{p(x)-1}) 1 \nabla \eta_n^{p^+} d\tau \\
\geq \frac{\beta}{2} \int_{Q_n^m} |\nabla (u - k_m)|^{p(x)} \eta_n^{p^+} d\tau - C_2^{\alpha} \eta_n^{p^+} |A_m|,
\]
where \( A_m = \{(x, t) \in Q_{\rho_m} : u(x, t) < k_m\} \), \( C = C(M, p^+) \).
So we can get the following inequality
\[
\sup_{-t\rho_m^\varepsilon < t < 0} \int_{K_{\rho_m}} (u - k_m)^{p_\varepsilon} \eta_m^p \, dx + \int_{Q_{\rho_m}} |\nabla(u - k_m) - [p(x)]^\varepsilon \eta_m^p| \, dx dt \leq C 2^{m^p} \rho^{-p_\varepsilon} |A_m|,
\]
where \( C = C(M, p^+) \).

On the other hand, we have
\[
\int_{Q_{\rho_m}} |\nabla(u - k_m) - [p(x)]^\varepsilon \eta_m^p| \, dx dt \leq \int_{Q_{\rho_m}} |\nabla(u - k_m) - [p(x)]^\varepsilon \eta_m^p| \, dx dt + \int_{Q_{\rho_m}} \chi([u - k_m]_+ > 0) |\eta_m^p| \, dx dt,
\]
then by (3.5),
\[
\sup_{-t\rho_m^\varepsilon < t < 0} \int_{K_{\rho_m}} (u - k_m)^{p_\varepsilon} \eta_m^p \, dx + \frac{1}{2} \int_{Q_{\rho_m}} |\nabla(u - k_m) - [p(x)]^\varepsilon \eta_m^p| \, dx dt \leq C 2^{m^p} \rho^{-p_\varepsilon} \frac{1}{2} |A_m|.
\]

Next, we introduce the change of time-variable \( z = l^{-1} t \) which transforms \( Q_{\rho_m} \) into \( \tilde{Q}_{\rho_m} = K_{\rho_m} \times (-\rho_m^\varepsilon, 0) \). Setting also \( v(x, t) = u(x, z), \tilde{\eta}_m(x, z) = \eta_m(x, z), |A_m| = \text{meas}\{ (x, z) \in \tilde{Q}_{\rho_m} : v(x, z) < k_m \} \), then
\[
|||v - k_m|||_{L^{p_\varepsilon}(-\rho_m^\varepsilon, 0)(\tilde{Q}_{\rho_m})} \leq C \left( \sup_{-t\rho_m^\varepsilon < t < 0} \int_{K_{\rho_m}} (v - k_m)^{p_\varepsilon} \tilde{\eta}_m^p \, dx + \int_{Q_{\rho_m}} |\nabla(v - k_m) - [p(x)]^\varepsilon \tilde{\eta}_m^p| \, dx dz \right.
\]
\[
+ \int_{\tilde{Q}_{\rho_m}} |(v - k_m) - \nabla \tilde{\eta}_m^p| \, dx dz \right) \leq C 2^{m^p} \rho^{-p_\varepsilon} |A_m|.
\]

By lemma 2.8,
\[
\frac{1}{2^{m^p} (m + 1)} \left( \frac{2}{5} \right)^{p_\varepsilon} |A_{m+1}| = |k_m - k_{m+1}|^{p_\varepsilon} |A_{m+1}|
\]
\[
\leq \|(v - k_m) - [p_\varepsilon]^\varepsilon \eta_m^{p_\varepsilon} \|_{L^{p_\varepsilon}(-\rho_m^\varepsilon, 0)(\tilde{Q}_{\rho_m+1})} \leq \| (v - k_m) - \tilde{\eta}_m^{p_\varepsilon} \|_{L^{p_\varepsilon}(-\rho_m^\varepsilon, 0)(\tilde{Q}_{\rho_m})}^{p_\varepsilon}
\]
\[
\leq \| (v - k_m) - \tilde{\eta}_m^{p_\varepsilon} \| _{V_{p_\varepsilon}^{p_\varepsilon}(-\rho_m^\varepsilon, 0)(\tilde{Q}_{\rho_m})} |\tilde{A}_m|^{p_\varepsilon}\frac{p_\varepsilon}{p_\varepsilon + N} \leq C 2^{m^p} \rho^{-p_\varepsilon} |A_m|^{1 + \frac{p_\varepsilon}{p_\varepsilon + N}}.
\]

By (3.3), when \( A > 2 \), we choose \( \varepsilon \leq p_\varepsilon - 2 \), then \( \left( \frac{2}{5} \right)^{p_\varepsilon} \leq \rho^{-p_\varepsilon} \). Next, denote \( Y_m = \frac{|A_m|}{|Q_{\rho_m}|} \), then by (1.10) we obtain
exists a positive integer $K$ we get $\Psi(u > \mu)$ therefore $\Psi(u < \mu - \omega \rho)$

Proof: Set $\rho = \rho_1 \leq \rho_2 = \rho_2^+$ just satisfies the condition of this lemma, i.e.

Let (3.3)-(3.4) hold, then for every number $\sigma_1 \in (0, 1)$, therefore when $u < k$ we know

By lemma 3.2, we know $u(x, t) < \mu^+ + \omega$ a.e. in $K_\rho^p, \rho)$, by lemma 3.2 and $u \in C[0, T; L^2(\Omega)]$, we obtain $u(x, t) > \mu^+ + \omega$ a.e. in $K_\rho^p$. 

Lemma 3.3 Let (3.3)-(3.4) hold, then for every number $\sigma_1 \in (0, 1)$, there exists a positive integer $s$ such that

$|x \in K_\rho^p : u(x, t) < \mu^- + \omega \rho \leq \sigma_1 |K_\rho^p|$, $\forall t \in (-\theta, 0)$. 

Proof: Set $\rho^* = 2^{-1} \rho$, we will consider the problem in $Q(\rho^{p^*}) = K_{\rho^*} \times (0, T)$. Let $k = \mu^- + \omega \rho \leq \sigma_1 |K_\rho^p|$, then we take

By lemma 3.2, we know $u(x, t) > \mu^- + \omega \rho \leq \sigma_1 |K_\rho^p|$, a.e. in $K_{\rho^*}$, so $u(x, t) = 0$ a.e. in $K_{\rho^*} \times (-\theta, 0)$, moreover $\Psi(u(x, t)) = 0$, a.e. $x \in K_{\rho^*}$. Since $\frac{\omega}{2} \geq H_{\rho^*} \geq (u - k)_-$, we get $\Psi(u) \leq \ln \frac{\frac{\omega}{2} + 2}{\frac{\omega}{2} - 0} = m \ln 2$ and

therefore when $u < k - \omega 2^{-(m+2)}$ and

$$|\frac{\partial \Psi(u)}{\partial u}| = \begin{cases} 1 \frac{H_{\rho^*}^-}{H_{\rho^*}^- - (u - k)_- + \omega 2^{-(m+2)}}, & u < k - \omega 2^{-(m+2)}, \\ 0, & u \geq k - \omega 2^{-(m+2)}, \end{cases}$$

$$\frac{1}{2} \leq |\frac{\partial \Psi(u)}{\partial u}| \leq \frac{2^{(m+2)}}{2}. \quad (\text{EJQTDE, 2012 No. 4, p. 12}) $$
Take \( \varphi = \frac{\partial}{\partial x} (\Psi^2(d)) \eta^\rho \big|_{d=\eta_h} \) as the testing function in (3.2), where \( \eta \) is the cutoff function independent of \( t \) and satisfies \( 0 < \eta < 1 \) in \( K_{\rho^s} \), \( \eta = 1 \) in \( K_{2^{-1}\rho^s} \), and \( |\nabla \eta| \leq 4\rho^{-1} \), then

\[
\int_{Q^t(\theta, \rho^s)} \frac{\partial}{\partial d} [\Psi^2(d)] \eta^\rho \big|_{d=\eta_h} \frac{\partial \eta_h}{\partial \tau} dxd\tau \\
+ \int_{Q^t(\theta, \rho^s)} [a(x, \tau, u, \nabla u)]_h \nabla \left[ \frac{\partial}{\partial d} [\Psi^2(d)] \eta^\rho \big|_{d=\eta_h} \right] dxd\tau \\
+ \int_{Q^t(\theta, \rho^s)} [a_0(x, \tau, u, \nabla u)]_h \frac{\partial}{\partial d} [\Psi^2(d)] \eta^\rho \big|_{d=\eta_h} dxd\tau = 0, \quad (3.6)
\]

where \( Q^t(\theta, \rho^s) = K_{\rho^s} \times (-\theta, t) \), \( t \in (-\theta, 0) \).

Integrating by parts,

\[
\int_{Q^t(\theta, \rho^s)} \frac{\partial}{\partial d} [\Psi^2(d)] \eta^\rho \big|_{d=\eta_h} \frac{\partial \eta_h}{\partial \tau} dxd\tau = \int_{K_{\rho^s}} \Psi^2(u_h(x, t)) \eta^\rho dx - \int_{K_{\rho^s}} \Psi^2(u_h(x, -\theta)) \eta^\rho dx,
\]

by \( \Psi(u_h) \leq m \ln 2, \Psi(u) \leq m \ln 2, |\Psi^2(u_h) - \Psi(u)| \leq m^{2m+3} m^{2} |u_h - u| \), and \( u_h \rightarrow u \) in \( L^2(K_{\rho^s}) \) for \( \forall t \in (-\theta, 0) \), so

\[
\int_{K_{\rho^s}} \Psi^2(u_h(x, t)) \eta^\rho dx \rightarrow \int_{K_{\rho^s}} \Psi^2(u(x, t)) \eta^\rho dx,
\]

\[
\int_{K_{\rho^s}} \Psi^2(u_h(x, -\theta)) \eta^\rho dx \rightarrow \int_{K_{\rho^s}} \Psi^2(u(x, -\theta)) \eta^\rho dx,
\]

therefore we obtain

\[
\int_{Q^t(\theta, \rho^s)} \frac{\partial}{\partial d} [\Psi^2(d)] \eta^\rho \big|_{d=\eta_h} \frac{\partial \eta_h}{\partial \tau} dxd\tau \\
- \int_{K_{\rho^s}} \Psi^2(u(x, t)) \eta^\rho dx = \int_{K_{\rho^s}} \Psi^2(u(x, -\theta)) \eta^\rho dx,
\]

Denote \( \Psi'(u) = \frac{\partial \Psi(u)}{\partial u} \big|_{d=\eta_h}. \) Since \( \frac{\partial^2}{\partial u^2} (\Psi^2(d)) \big|_{d=\eta_h} = 2(1 + \Psi(u_h)) \Psi'(u_h)^2 \), for the other parts of (3.6),

\[
I \equiv \int_{Q^t(\theta, \rho^s)} [a(x, \tau, u, \nabla u)]_h \nabla \left[ \frac{\partial}{\partial d} [\Psi^2(d)] \eta^\rho \big|_{d=\eta_h} \right] dxd\tau \\
+ \int_{Q^t(\theta, \rho^s)} [a_0(x, \tau, u, \nabla u)]_h \frac{\partial}{\partial d} [\Psi^2(d)] \eta^\rho \big|_{d=\eta_h} dxd\tau \\
= 2 \int_{Q^t(\theta, \rho^s)} [a(x, \tau, u, \nabla u)]_h \nabla u_h \nabla \left( 1 + \Psi(u_h) \right) \Psi'(u_h)^2 \eta^\rho dx d\tau \\
+ 2 \int_{Q^t(\theta, \rho^s)} [a(x, \tau, u, \nabla u)]_h \Psi'(u_h) \Psi(u_h) \nabla \eta^\rho dx d\tau \\
+ 2 \int_{Q^t(\theta, \rho^s)} [a_0(x, \tau, u, \nabla u)]_h \Psi'(u_h) \Psi(u_h) \eta^\rho dx d\tau.
\]

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Next, we consider the problem on the set \( \{(x, t) \in K_{\rho^*} \times (-\theta, 0) : u(x, t) < k - \omega 2^{-(m+2)} \} \), thus \( \frac{v_0}{2} \leq |\Psi'(u)| \leq \frac{2^{m+2}}{2} \). When \( h \to 0 \), \( u_h \to u \) and \( (u_h - k)_- \to (u - k)_- \) a.e. in \( (x, t) \in Q(\theta, \rho^*) \), so \( (1 + \Psi(u_h))\Psi'(u_h)^2 \to (1 + \Psi(u))\Psi'(u)^2 \) a.e. in \( (x, t) \in Q(\theta, \rho^*) \). Since

\[
|(1 + \Psi(u_h))\Psi'(u_h)^2 - (1 + \Psi(u))\Psi'(u)^2| \leq |2(1 + m \ln 2)(\frac{2^{m+2}}{2})^2\rho_\alpha^2|
\]

and by Lebesgue’s theorem, we get

\[
(1 + \Psi(u_h))\Psi'(u_h)^2 \to (1 + \Psi(u))\Psi'(u)^2 \text{ in } L^{p(x)}(Q^*(\theta, \rho^*))
\]

for a.e. \( t \in (-\theta, 0) \). Because \( [a(x, t, u, \nabla u)]_h \to a(x, t, u, \nabla u) \) in \( L^{p(x)}(Q^*(\theta, \rho^*)) \),

\[
\int_{Q^*(\theta, \rho^*)}a(x, \tau, u, \nabla u)_h \nabla u_h(1 + \Psi(u_h))\Psi'(u_h)^2 \eta^{p(x)} \, dx \, d\tau
\]

and

\[
\int_{Q^*(\theta, \rho^*)}a(x, \tau, u, \nabla u) \nabla u(1 + \Psi(u))\Psi'(u)^2 \eta^{p(x)} \, dx \, d\tau.
\]

are both valid.

Combining these estimates, we have

\[
limit_{h \to 0} I = 2 \int_{Q^*(\theta, \rho^*)}a(x, \tau, u, \nabla u) \nabla u(1 + \Psi(u))\Psi'(u)^2 \eta^{p(x)} \, dx \, d\tau
\]

\[
+ 2 \int_{Q^*(\theta, \rho^*)}a(x, \tau, u, \nabla u) \Psi'(u) \Psi(u) \eta^{p(x)} \, dx \, d\tau
\]

(3.8)

With (1.4)-(1.5), (1.7)-(1.8), we can get

\[
limit_{h \to 0} I \geq 2\beta \int_{Q^*(\theta, \rho^*)} |\nabla u|^{p(x)} + |u|^{p(x)}(1 + \Psi(u))\Psi'(u)^2 \eta^{p(x)} \, dx \, d\tau
\]

\[
- 2\alpha \int_{Q^*(\theta, \rho^*)} |\nabla u|^{p(x)-1} + |u|^{p(x)-1}) |\Psi'(u)| \Psi(u) \eta^{p(x)} \, dx \, d\tau
\]

(3.9)

Since \( \frac{p_\alpha^{-1} - 1}{p(x) - 1} > \frac{p_\alpha^{-1} - 1}{p(x) - 1} \), by Young’s inequality,

\[
\int_{Q^*(\theta, \rho^*)} |\nabla u|^{p(x)-1} |\Psi'(u)| \Psi(u) \eta^{p(x)-1} \, dx \, d\tau
\]

\[
\leq \varepsilon \int_{Q^*(\theta, \rho^*)} |\nabla u|^{p(x)}(\Psi'(u))^2 \Psi(u) + 1 \eta^{p(x)} \, dx \, d\tau
\]

(3.10)

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In the same way, we have

\[
\begin{align*}
&\int_{Q^t(\rho, r^\ast)} |u|^{p(x)-1} \Psi'(u) \Psi(u) \eta^{p^+} \left| \nabla \eta \right| d\tau \\
\leq &\ \varepsilon \int_{Q^t(\rho, r^\ast)} |u|^{p(x)} (\Psi'(u))^2 (\Psi(u) + 1) \eta^{p^+} d\tau \\
&+ C(\varepsilon) \int_{Q^t(\rho, r^\ast)} (\Psi'(u))^2-\rho^+ (\Psi(u)) \eta^{p^+} d\tau,
\end{align*}
\]

(3.11)

and

\[
\begin{align*}
&\int_{Q^t(\rho, r^\ast)} |u|^{p(x)-1} \Psi'(u) \Psi(u) \eta^{p^+} d\tau \\
\leq &\ \varepsilon \int_{Q^t(\rho, r^\ast)} |u|^{p(x)} (\Psi'(u))^2 (\Psi(u) + 1) \eta^{p^+} d\tau \\
&+ C(\varepsilon) \int_{Q^t(\rho, r^\ast)} (\Psi'(u))^2-\rho^+ (\Psi(u)) \eta^{p^+} d\tau.
\end{align*}
\]

Combining (3.8)-(3.11),

\[
\lim_{h \to 0} I \geq (2 - 4\alpha \rho^+ \varepsilon) \int_{Q^t(\rho, r^\ast)} (|\nabla u|^{p(x)} + |u|^{p(x)}(1 + \Psi(u)) \Psi'(u))^2 \eta^{p^+} d\tau
\]

\[
- C(\varepsilon) \int_{Q^t(\rho, r^\ast)} (\Psi'(u))^2-\rho^+ (\Psi(u)) \eta^{p^+} + |\nabla \eta|^{p(x)} d\tau.
\]

Take \(4\alpha \rho^+ \varepsilon = \beta\), then

\[
\lim_{h \to 0} I \geq \beta \int_{Q^t(\rho, r^\ast)} (|\nabla u|^{p(x)} + |u|^{p(x)}(1 + \Psi(u)) \Psi'(u))^2 \eta^{p^+} d\tau
\]

\[
- C(\beta) \int_{Q^t(\rho, r^\ast)} (\Psi'(u))^2-\rho^+ (\Psi(u)) \eta^{p^+} + |\nabla \eta|^{p(x)} d\tau.
\]

(3.12)

In view of (3.7) and (3.12),

\[
\int_{K_{\rho^\ast}} \Psi^2(u(x,t)) \eta^{p^+} dx \leq C \int_{Q^t(\rho, r^\ast)} (\Psi'(u))^2-\rho^+ (\Psi(u)) \eta^{p^+} + |\nabla \eta|^{p(x)} d\tau.
\]

By \(\Psi(u) \leq m \ln 2\), \(|\Psi'(u)|^{-1} \leq \frac{2}{\beta}\), \(|\nabla \eta| \leq \frac{4}{\beta}\), \(|\Psi'(u)| \leq \frac{2m+2}{\beta}\), we can get

\[
\int_{K_{\rho^\ast}} \Psi^2(u(x,t)) \eta^{p^+} dx \leq Cm |K_{\rho^\ast}|.
\]

(3.13)

\(\forall t \in (-\theta, 0)\), for such a set \( \{ (x, t) \in K_{2\rho^\ast} : u(x, t) < \mu^- + \frac{\eta^+}{\nabla \eta} \} \) we have

\[
\Psi^2(u) \geq \ln^2 \frac{H_k^-}{H_k^- - \frac{\eta^-}{\nabla \eta} + \frac{\eta^+}{\nabla \eta}}.
\]

Since \(-\frac{\eta^-}{\nabla \eta} + \frac{\eta^+}{\nabla \eta} < 0\), we obtain \(\ln^2 \frac{H_k^-}{H_k^- - \frac{\eta^-}{\nabla \eta} + \frac{\eta^+}{\nabla \eta}}\) is decreasing about \(H_k^-\) and \(H_k^- \leq \frac{\eta^-}{\nabla \eta}\), thus

\[
\Psi^2(u) \geq \ln^2 \frac{H_k^-}{H_k^- - \frac{\eta^-}{\nabla \eta} + \frac{\eta^+}{\nabla \eta}} \geq \ln^2 \frac{\frac{\eta^-}{\nabla \eta} - \frac{\eta^-}{\nabla \eta} + \frac{\eta^+}{\nabla \eta}}{\frac{\eta^-}{\nabla \eta} - \frac{\eta^-}{\nabla \eta} + \frac{\eta^+}{\nabla \eta}} = [(m-1) \ln 2]^2.
\]

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Because $\eta = 1$ in $K_{\varphi^+}$, by (3.13)

$$|x \in K_{\varphi^+} : u(x,t) < \mu^- + \frac{\omega}{2^{m+2}}| \leq C \frac{m}{(m-1)^2} |K_{\varphi^+}|,$$

where $C = C(M, p^+)$. To prove the lemma we have only to choose $m$ sufficiently large and $s = m + 2$.$\Box$

**Lemma 3.4** Let (3.3)-(3.4) hold, then there exist $\sigma_1 \in (0, 1)$ and an integer $s > 1$ independent of $\omega$ and $\rho$, so that $u(x,t) > \mu^- + \frac{s}{2^{m+2}}$, a.e. $(x,t) \in Q(\theta, \frac{K_{\varphi^+}}{4})$.

Proof: Let $\rho_{\sigma_1} = \frac{\omega}{2^{m+2}}, k_m = \mu^- + \frac{s}{2^{m+2}}, m = 0, 1, 2, \ldots$, and $s > 1$ is to be chosen later. By lemma 3.2, for a.e. $x \in K_{\rho_{\sigma_1}}$, we have $u(x, \theta) > \mu^- + \frac{s}{2^{m+2}} \geq k_m$, thus $(u-k_m)(x, -\theta) = 0$. Let $\eta_m(x)$ be a smooth cutoff function in $K_{\rho_{\sigma_1}}$ satisfying $\eta_m \equiv 1$ in $K_{\rho_{\sigma_1}+1}$, $|\nabla \eta_m| \leq \frac{s}{2^{m+4}}$, and $\eta_m = 0$ outside $K_{\rho_{\sigma_1}}$.

We take $\varphi = -(u_k - k_m) - \eta_m^+$, as the testing function in (1.11), by the fact that

$$\|u\|_{L^\infty(Q_{\rho_{\sigma_1}})} \leq M, \quad \|(u_k - k_m)\|_{L^\infty(Q_{\rho_{\sigma_1}})} \leq \|(k_m - u)\|_{L^\infty(Q_{\rho_{\sigma_1}})} \leq \frac{\omega}{2^{m+2}}.$$ 

similar to lemma 3.2, we have

$$\sup_{0 < t < T} \int_{Q_{\rho_{\sigma_1}}} (u_k - k_m)^2 \eta_m^+ dx + \int_{Q(\theta, \rho_{m}^+)} |\nabla (u_k - k_m)|^2 \rho_{m}^+ \eta_m^+ dx dt \leq C 2^{m+p^+} \rho^{-p^+} \int_{Q(\theta, \rho_{m}^+)} \chi(\eta_m^+ > 0) dx dt.$$ 

(3.14)

On the other hand, we have

$$\int_{K_{\rho_{\sigma_1}}^+} (u_k - k_m)^2 \eta_m^+ dx \geq \left(\frac{\omega}{2}\right)^2 \int_{K_{\rho_{\sigma_1}}^+} (u_k - k_m)^2 \eta_m^+ dx \geq \frac{\theta}{2^1} \int_{K_{\rho_{\sigma_1}}^+} (u_k - k_m)^2 \eta_m^+ dx$$

and

$$\int_{Q(\theta, \rho_{m}^+)} |\nabla (u_k - k_m)|^2 \eta_m^+ dx dt \leq \int_{Q(\theta, \rho_{m}^+)} |\nabla (u_k - k_m)|^2 \eta_m^+ dx dt + \int_{Q(\theta, \rho_{m}^+)} \chi(|u_k - k_m| > 0) \eta_m^+ dx dt,$$

where $s$ is chosen so large as to satisfy the conclusion of lemma 3.3.

Combining the above two inequalities with (3.14), we get

$$\sup_{0 < t < T} \int_{K_{\rho_{\sigma_1}}^+} (u_k - k_m)^2 \eta_m^+ dx + \frac{\rho^{-p^+} |\rho^+|^2}{\theta} \int_{Q(\theta, \rho_{m}^+)} |\nabla (u_k - k_m)|^2 \eta_m^+ dx dt \leq C 2^{m+p^+} \rho^{-p^+} \left(\frac{\rho^+}{\theta}\right)^2 \int_{Q(\theta, \rho_{m}^+)} \chi(|u_k - k_m| > 0) dx dt.$$ 

We introduce the change of variable $z = t(\rho^+)^{p^+} \theta^{-1}$, which maps $Q(\theta, \rho_{m}^+)$ into $Q_m = K_{\rho_{m}^+} \times (-\rho^+)^{p^+}, 0)$. Let $v(x,t) = u(x, t(z(\rho^+)^{p^+}), \tilde{\eta}_m(x,z) =$
\[ \eta_m(x, \theta z (\rho^*)^{-p_x}) \text{, and denote } |A_m| = \text{meas}\{ (x, z) \in Q : v(x, z) < k_m \}, \]

\[ \| (v - k_m) - \eta_m \|_{V_p^+ (Q_m)}^{p_x} \overset{\text{ess sup}}{\leq} \int_{K_{R_m}} (v - k_m) \eta_m dx + \int_{Q_m} |\nabla (v - k_m) - \eta_m|^{p_x} dx dz \]

\[ \leq C (\sup_{-(\rho^*)^+}^{+} \int_{K_{R_m}} (v - k_m) \eta_m dx + \int_{Q_m} |\nabla (v - k_m) - \eta_m|^{p_x} dx dz) \]

\[ \leq C^{2m} (\rho^{-p_x^+} |A_m|, \quad \text{(3.15)} \]

by lemma 2.6 and (3.15),

\[ \frac{1}{2^{p_x^+}} \frac{1}{m+2} (\frac{\omega}{2})^{p_x^+} |A_{m+1}| = |k_m - k_{m+1}|^{p_x^+} |A_{m+1}| \]

\[ \leq \| (v - k_m) - \eta_m \|_{L^{p_x^+} (Q_m + 1)} \leq \| (v - k_m) - \eta_m \|_{L^{p_x^+} (Q_m)} \]

\[ \leq \| (v - k_m) - \eta_m \|_{V_p^+ (Q_m)}^{p_x^+} |A_m|^{p_x^+ + N} \]

\[ \leq C^{2m} (\rho^{-p_x^+} |A_m|)^{1 + \frac{p_x^+}{p_x^{+ + N}}}. \quad \text{(3.16)} \]

We take \( A > 2^* \), by (3.3), we get \( (\frac{\omega}{2})^{p_x^+ - 2} \geq \rho^x \geq \rho^{p_x^+ - 2} \), therefore \( \frac{\omega}{2} \geq \rho \).

Thus we obtain

\[ (\frac{\omega}{2})^{p_x^+} \leq \rho^{-p_x^+}. \]

Denote \( Z_m = \frac{|A_m|}{|Q_m|} \). By (3.16) and (1.10),

\[ Z_{m+1} \leq C^{4m} (Z_m)^{1 + \frac{p_x^+}{p_x^{+ + N}}} \leq C^{4m} Z_m^{1 + \frac{p_x^+}{p_x^{+ + N}}}, \]

where \( C = C(M, p^+) \). Since

\[ Z_0 = \frac{|A_0|}{|Q_0|} = \frac{\{|(x, t) \in Q(\theta, \frac{\rho^x}{2}) : u(x, t) < \mu - \frac{\omega}{2} \}}{|Q(\theta, \frac{\rho^x}{2})|}, \]

by lemma 3.3 there exists \( s \) such that \( Z_0 < \sigma_1 \) where \( \sigma_1 \equiv C \frac{\rho^x}{2^{p_x^+}} 4^{-p_x^+ (\frac{\rho^x}{2^{p_x^+}})^2} \).

Then by lemma 2.6 it follows that \( Z_m \to 0 \) as \( m \to \infty \). So we can get

\[ u(x, t) > \mu - \frac{\omega}{2}, \quad a.e. \quad (x, t) \in Q(\theta, \frac{\rho^x}{4}). \]

**Proposition 3.1** There exist \( \sigma \in (0, 1), \nu_1 \in (0, 1) \) and \( A \gg 1 \) independent of \( \omega \) and \( \rho \), such that if for some cylinder of the type \([0, t^*) + Q(l \rho^{p_x^+}, \rho)\),

\[ |(x, t) \in [0, t^*) + Q(l \rho^{p_x^+}, \rho) : u \leq \mu - \frac{\omega}{2} \leq \sigma Q(l \rho^{p_x^+}, \rho) \],

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then either
\[ \omega \leq A_1 \rho^{p^+ - 2} \]  
(3.17)
or
\[ \text{ess osc } u \leq \nu_1 \omega. \]  
(3.18)

Proof: Assume (3.17) is violated. By lemma 3.4, we can determine a positive integer number \( s \) such that
\[ \text{ess inf } Q((\Theta^{p^+}, \Theta)) u \geq \mu^- + \frac{\omega}{2^s+1}, \]
this gives
\[ - \text{ess inf } Q((\Theta^{p^+}, \Theta)) u \leq -\mu^- - \frac{\omega}{2^s+1}, \]  
(3.19)
and further
\[ \text{ess osc } u \leq (1 - \frac{1}{2^s+1})\omega. \]

therefore the proposition follows with \( \nu_1 = (1 - \frac{1}{2^s+1}) \), since \( Q((\Theta^{p^+}, \Theta)) \subset Q((\Theta, \Theta)). \)

Next assume that the condition of proposition 3.1 is violated, i.e. for every cylinder \([0, t^*) + Q((\rho^{p^+}, \rho)] \subset Q((\rho^{p^+}, \rho)), \) where \( \frac{1}{p} = (\Theta^{p^+})^{p^+ - 2}. \)
\[ |(x, t) \in [0, t^*) + Q((\rho^{p^+}, \rho)] : u \leq \mu^- + \frac{\omega}{2} | > \sigma Q((\rho^{p^+}, \rho)]. \]

Since \( \mu^- + \frac{\omega}{2} \leq \mu^+ - \frac{\omega}{2}, \) we can get
\[ |(x, t) \in [(0, t^*) + Q((\rho^{p^+}, \rho)] : u \geq \mu^- - \frac{\omega}{2} | \leq (1 - \sigma)|Q((\rho^{p^+}, \rho)]. \]  
(3.20)

**Lemma 3.5** Let (3.20) hold, then there exists a \( t \in [t^* - lp^{p^+}, t^* - \frac{\omega}{2} lp^{p^+}] \) such that
\[ \{|x \in K_\rho : u(x, t) > \mu^+ - \frac{\omega}{2} | \leq \frac{1 - \sigma}{1 - \frac{\omega}{2}}|K_\rho|. \]

Proof: If not, for all \( t \in [t^* - lp^{p^+}, t^* - \frac{\omega}{2} lp^{p^+}], \)
\[ \{|x \in K_\rho : u(x, t > \mu^+ - \frac{\omega}{2} | \geq \frac{1 - \sigma}{1 - \frac{\omega}{2}}|K_\rho| \]
and
\[ |(x, t) \in [(0, t^*) + Q((\rho^{p^+}, \rho)] : u > \mu^- - \frac{\omega}{2} | \geq \frac{1 - \sigma}{1 - \frac{\omega}{2}}\int_{t^* - lp^{p^+}}^{t^* - \frac{\omega}{2} lp^{p^+}} dt |K_\rho| \]
\[ > (1 - \frac{\omega}{2})lp^{p^+} (1 - \sigma)(1 - \frac{\omega}{2})^{-1}|K_\rho| = (1 - \sigma)|Q((\rho^{p^+}, \rho)], \]
contradicting (3.20). \( \Box \)

**Lemma 3.6** Let (3.20) hold, then there exists a positive integer \( s > 2, \) such that
\[ \{|x \in K_\rho : u(x, t) > \mu^+ - \frac{\omega}{2 p^+} | \leq (1 - (\frac{\omega}{2 p^+})^2)|K_\rho|, \ \forall t \in [t^* - \sigma lp^{p^+}, t^*]. \]  

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Proof: Let $k = \mu^+ - \frac{\sigma}{2}$, $Q_\rho = K_{(1-\alpha)\rho} \times (\bar{t}, t^*)$. Similar to lemma 3.3, we take $\varphi = \frac{\partial}{\partial t}(|\Psi^2|)\eta^p|_{t=k}$ as the testing function in (3.2), where the cutoff function $\eta$ independent of $t$ is taken so that $\eta \equiv 1$ in the cube $K_{(1-\alpha)\rho}$, $\alpha \in (0, 1)$, and $|\nabla \eta| \leq \frac{1}{\alpha\rho}, 0 < \alpha < 1$. We take $H_k^+ = \text{ess sup}_{[0, (t^*)]} (u-k)_+$, and consider

$$
\Psi(u) = \max\{0, \ln \frac{H_k^+}{H_k^+ - (u - k)_+} + \omega^{2-(m+2)}\} = \ln^+(\frac{H_k^+}{H_k^+ - (u - k)_+} + \omega^{2-(m+2)}),
$$

then

$$
\int_{K_{(1-\alpha)\rho}} \Psi^2(u(x,t)) dx \leq \int_{K_{(1-\alpha)\rho}} \Psi^2(u(x,t)) |\eta|^p dx \leq \int_{K_{\rho}} \Psi^2(u(x,t)) dx + C \int_{\bar{t}}^{t^*} \int_{K_{\rho}} \Psi(u(x,t)) |\eta|^p \Psi(u(x,t)) |\nabla \eta|^p(x) dx dt,
$$

where $|t^* - \bar{t}| \leq \ell \rho^p$, $\ell = (\frac{m}{2})^{2-p}$, $C = C(M, p^+)$. When $u(x,t) > k + \frac{\omega}{\alpha\rho} > \mu^+ - \frac{\sigma}{2}$, $\Psi^2(u(x,t)) \neq 0$, by lemma 3.5,

$$
\int_{K_{(1-\alpha)\rho}} \Psi^2(u(x,t)) dx \leq \int_{x \in K_{(1-\alpha)\rho}, u(x,t) > \mu^+ - \frac{\sigma}{2} + \frac{\omega}{\alpha\rho}} \Psi^2(u(x,t)) dx \leq \int_{x \in K_{\rho}, u(x,t) > \mu^+ - \frac{\sigma}{2}} \Psi^2(u(x,t)) dx \leq (m \ln 2)^2 (1 - \sigma)(1 - \frac{\sigma}{2})^{-1} |K_{\rho}|, \ 
$$

so we have

$$
\int_{K_{(1-\alpha)\rho}} \Psi^2(u(x,t)) dx \leq C |m^2 (1 - \sigma)(1 - \frac{\sigma}{2})^{-1} + m^2 |K_{\rho}||. \ 
$$

\forall t \in (\bar{t}, t^*)$, in $\{x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \frac{\omega}{\alpha\rho}\}$ we can get

$$
\Psi^2(u) \geq \ln^2 \frac{H_k^+}{H_k^+ - \frac{\sigma}{2} + \omega^{2-(m+1)}} \geq \ln^2 \frac{\omega^{2-2}}{\omega^{2-(m+1)}} = (m - 1)^2 \ln^2 2, 
$$

so \forall t \in (\bar{t}, t^*)

$$
|\{x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega^{2-(m+2)}\}| \leq C [(\frac{m}{m+1})^2 (1 - \sigma)(1 - \frac{\sigma}{2})^{-1} + \frac{1}{m} |\alpha\rho|^{p^+}] |K_{\rho}|. \ 
$$

On the other hand, \forall t \in (\bar{t}, t^*)

$$
|\{x \in K_{\rho} \setminus K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega^{2-(m+1)}\}| \leq |\{x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega^{2-(m+1)}\}| + \alpha N |K_{\rho}|, \ 
$$

so \forall t \in (\bar{t}, t^*)

$$
|\{x \in K_{(1-\alpha)\rho} : u(x,t) > \mu^+ - \omega^{2-(m+1)}\}| \leq C (\frac{m}{m+1})^2 [(1 - \sigma)(1 - \frac{\sigma}{2})^{-1} + \frac{1}{m} |\alpha\rho|^{p^+} + N\alpha] |K_{\rho}|. \ 
$$

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Choose $\alpha$ so small and then $m$ so large that $C\left(\frac{m}{m-1}\right)^2 \leq (1+\sigma)(1-\frac{1}{2})$, $\frac{C}{m} \alpha^{-p^\circ} \leq \frac{s}{2} \sigma^2$ and $C \alpha N \leq \frac{s}{2} \sigma^2$. Then for such a choice of $m$ the lemma follows with $s = m + 1$. $\Box$

Since (3.20) holds for all $[(0, t^*) + Q(lp^\circ, \rho)]$, the conclusion of lemma 3.6 holds for all time levels satisfying $t \geq -(a-t)lp^\circ = -(1-(\frac{2}{3})p^\circ - 2)ap^\circ$. If the number $A$ is chosen sufficiently large such that $1-(\frac{2}{3})p^\circ - 2 > \frac{2}{3}$, we deduce the following corollary.

**Corollary 3.1** Let (3.20) hold, then for all $t \in (-\frac{2}{3}lp^\circ, 0)$,

$$|(x \in K_\rho : u(x, t) \geq \mu + \omega 2^{-s})| \leq (1 - \frac{\omega}{2})(|K_\rho|).$$

**Lemma 3.7** Let (3.20) hold, then for every $\sigma \in (0, 1)$, there exists positive integer $s^* > s$, such that

$$|(x \in K_\rho : u(x, t) \geq \mu + \omega \frac{\sigma}{2^s})| \leq \bar{\sigma}|Q(2^{-1} ap^\circ, \rho)|, \quad \forall t \in (-\frac{\sigma}{2}lp^\circ, 0).$$

Proof: Consider the problem in $Q(ap^\circ, 2\rho)$. Let $k = \mu + \frac{\omega}{2^s}$, where $\bar{s} \leq s \leq s^*$. Take $\varphi = (u_{n+1} - k)_+ \eta^p$ as the testing function in (1.9), where $\eta$ is a cutoff function that equals one on $Q(\frac{\sigma}{2} lp^\circ, \rho)$, vanishes on the parabolic boundary of $Q(ap^\circ, 2\rho)$ and such that $|\nabla \eta| \leq \frac{1}{\rho^2}$, $0 \leq \eta \leq \frac{2}{a\rho^2}$. Similar to lemma 3.2, we get

$$\int_{A_s} |\nabla u|^p dx \leq \int_{Q(\frac{\sigma}{2} lp^\circ, \rho)} |\nabla (u - k)_+| \rho^p dx + |A_s|,$$

where $C = C(p^\circ)$ and

$$A_s = \{(x, t) \in Q(\frac{\sigma}{2} lp^\circ, \rho) : u(x, t) > \mu + \frac{\omega}{2^s}\},$$

$$A_s(t) = \{x \in K_\rho : u(x, t) > \mu + \frac{\omega}{2^s}\}.$$

By corollary 3.1, $\forall t \in (-\frac{\sigma}{2} lp^\circ, 0)$,

$$|(x \in K_\rho : u(x, t) \leq \mu + \frac{\omega}{2^s})| = |K_\rho| - |A_s(t)| \geq (\frac{\omega}{2})(|K_\rho|).$$

In lemma 2.8, take $k = \mu + \frac{\omega}{2^s}$, $h = \mu + \frac{\omega}{2^s}$, then $\forall t \in [-\frac{\omega}{2} lp^\circ, 0]$, by (3.21), we get

$$\frac{\omega}{2^s+1} |A_{s+1}(t)| \leq C \rho^{N+1} \frac{\rho^N}{\sigma^2} |K_\rho| \int_{A_s(t) \setminus A_{s+1}(t)} |\nabla u| dx. \quad (3.22)$$

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Take $A > 2^s$, there exists $C = C(M,p^+,p^-)$ such that \((\bar{s})^{p^-} - \bar{s} \leq C\text{ and } (\bar{s})^{p^-} \leq \rho - \bar{s}\) hold. Integrating on \((-\alpha \rho_{p^+}, 0),\text{ from } (3.22)\) we get
\[
(\bar{s})^{p^-} \frac{\partial \rho_{p^+}}{\partial t} |A_{s+1}| \leq (\bar{s})^{p^-} \frac{\partial \rho_{p^+}}{\partial t} \int A_s \triangle u |dxdt
\]
\[
\leq (\bar{s})^{p^-} \frac{\partial \rho_{p^+}}{\partial t} \int A_s \triangle u |dxdt
\]
\[
\leq \frac{1}{\bar{s}} |Q(\frac{\rho_{p^+}}{p^+} |\rho) |p_{p^+}^{\rho_{p^+} + 1}
\]
(23.23)

If $s$ is large enough so that \((\bar{s})^{p^-} \frac{\partial \rho_{p^+}}{\partial t} < 1\), from (23.23) we get
\[
|A_{s+1}| \frac{\partial \rho_{p^+}}{\partial t} \leq C \sigma \frac{\partial \rho_{p^+}}{\partial t} \int A_s \triangle u |dxdt
\]
(23.24)

for all $\bar{s} \leq s \leq s^*$. We add them for $s = \bar{s}, s+1, \bar{s} + 2, ..., s^* - 1$, then
\[
(s^* - \bar{s})|A_s| \frac{\partial \rho_{p^+}}{\partial t} \leq C \sigma \frac{\partial \rho_{p^+}}{\partial t} \int A_s \triangle u |dxdt
\]

After taking $s^*$ so large that $C(s^* - \bar{s}) \frac{\partial \rho_{p^+}}{\partial t} \leq \sigma^2 \sigma$, we conclude the lemma.□

**Lemma 3.8** Let (3.20) hold, then there exists $\sigma \in (0,1)$ so that
\[
u(x,t) \leq \mu^+ - \frac{\omega}{2^{s+1}}, \quad \text{a.e. } Q(\alpha \frac{\rho_{p^+}}{p^+}, \frac{\rho}{2})
\]
where $\frac{1}{\alpha} = (\bar{s})^{p^-} - 2, A = 2^s$.

Proof: We will consider the problem over the boxes $Q(\bar{s}, \rho_{p^+}, \rho_m)$. Let $\rho_m = \frac{\bar{s}}{2} + \frac{\partial \rho_{p^+}}{\partial t}, k_m = \mu^+ - \frac{\bar{s}}{2} - \frac{\partial \rho_{p^+}}{\partial t}$, $\zeta_m$ is a cutoff function with $0 \leq \zeta_m \leq 1$ in $Q(\bar{s}, \rho_{p^+}, \rho_m)$, $\zeta_m \equiv 1$ in $Q(\bar{s}, \rho_{p^+}, \rho_{m-1})$, $\zeta_m \equiv 0$ on the parabolic boundary of $Q(\bar{s}, \rho_{p^+}, \rho_m)$, $|\nabla \zeta_m| \leq \frac{\partial \rho_{p^+}}{\partial t}, 0 \leq \frac{\partial \rho_{p^+}}{\partial t} \leq \frac{2}{1} (\frac{\partial \rho_{p^+}}{\partial t})^{p^+}$. Take $u_m - k_m + \zeta_m$ as the testing function in (1.11), by $\|u\|_{L_infty(Q(\bar{s}, \rho_{p^+}, \rho_m))} \leq M$ and $\|(u - k_m)\|_{L_infty(Q(\bar{s}, \rho_{p^+}, \rho_m))} \leq \|(u - k_m)\|_{L_infty(Q(\bar{s}, \rho_{p^+}, \rho_m))} \leq \frac{2}{p^+}$, similar to lemma 3.2, we obtain
\[
\sup_{\frac{\rho_m}{p^+} < 0} \int_{K_{\rho_m}} (u - k_m) \chi_{\rho_{p^+}} dx + \int_{Q(\bar{s}, \rho_{p^+}, \rho_m)} |\nabla (u - k_m) + \hat{p}(z)|^{\rho_{p^+}} dx dt
\]
\[
\leq C \rho_{p^+} \int_{Q(\bar{s}, \rho_{p^+}, \rho_m)} \chi_{[u - k_m] > 0} dx dt
\]
(23.25)

On the other hand, we have
\[
\int_{K_{\rho_m}} (u - k_m) \chi_{\rho_{p^+}} dx \leq (\frac{\omega}{2^{s+1}})^{p^-} \int_{K_{\rho_m}} (u - k_m) \chi_{\rho_{p^+}} dx
\]

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By lemma 2.8 and (3.26),

\[
\int_{Q(\tilde{\rho}_{\rho m}, \rho m)} \frac{1}{|\nabla (u - k_m)|} |p_{\rho} \zeta_m|^{p_{\rho}} dx dt \leq \int_{Q(\tilde{\rho}_{\rho m}, \rho m)} \frac{1}{|\nabla (u - k_m)|} |p_{\rho} \zeta_m|^{p_{\rho}} dx dt
\]

then by (3.25),

\[
\sup_{-\tilde{\rho}_{\rho m} < t < 0} \int_{K_{\rho m}} (u - k_m)|^{p_{\rho}} \zeta_m dx + \frac{1}{a} \int_{Q(\tilde{\rho}_{\rho m}, \rho m)} |\nabla (u - k_m)|^{p_{\rho}} \zeta_m^{p_{\rho}} dx dt
\]

Next, we introduce the change of time-variable \( z = 2t^{-1}t \) which transforms \( Q(2^{-1}a \rho m, \rho m) \) into \( Q_m = K_{\rho m} \times (-\tilde{\rho}_{\rho m}, 0) \). Setting \( v(x, t) = u(x, 2^{-1}az) \), \( \tilde{\zeta}_m(x, z) = \zeta_m(x, 2^{-1}az) \), \( |A_m| = \text{meas}\{(x, z) \in Q_m : v(x, z) > k_m\} \), then

\[
\|v(v - k_m) + \tilde{\zeta}_m\|_{L^{p_{\rho}}(Q_m)} \leq C \sup_{-\tilde{\rho}_{\rho m} < t < 0} \int_{K_{\rho m}} (v - k_m)|^{p_{\rho}} \zeta_m dx + \int_{Q_m} |\nabla (v - k_m)|^{p_{\rho}} \zeta_m^{p_{\rho}} dx dz
\]

By lemma 2.8 and (3.26),

\[
\frac{1}{2^{p_{\rho} + 1}} \frac{1}{|A_{m+1}|} = |k_m - k_{m+1}|^{p_{\rho}} |A_{m+1}|
\]

\[
\leq \|v(v - k_m) + \tilde{\zeta}_m\|_{L^{p_{\rho}}(Q_m)} \leq \|v(v - k_m) + \tilde{\zeta}_m\|_{L^{p_{\rho}}(Q_m)}
\]

\[
\leq \|v(v - k_m) + \tilde{\zeta}_m\|_{L^{p_{\rho}}(Q_m)} |A_m|^{-1}_{p_{\rho} + N}
\]

\[
\leq C2^{mp_{\rho} + 1} \rho^{-p_{\rho}} |A_m|^{-1}_{p_{\rho} + N}.
\]

Take \( A = 2^{s} \), then \( \frac{1}{2^{s}} \rho^{-p_{\rho}} \leq \rho^{-p_{\rho}} \).

Next, we obtain

\[
Z_{m+1} \leq C4^{mp_{\rho} + 1} \rho^{-p_{\rho} + N}.
\]

By lemma 2.7, when \( m \to \infty \), \( Z_m \to 0 \) where \( Z_0 \leq C - \frac{n+p_{\rho}}{p_{\rho}} 4^{-p_{\rho}} \frac{(n+p_{\rho})}{p_{\rho}} \equiv \sigma \).

Thus as \( m \to \infty \),

\[
\int_{Q_m} \chi[(v - k_m)_+ > 0] dx dz \to 0,
\]

i.e. \( u(x, t) \leq \mu^+ - \frac{2}{2^{s+1}} \) a.e. in \( Q(\tilde{\rho}_{\rho m}, \rho m) \).

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Proposition 3.2  There exist $\sigma \in (0, 1)$, $\nu_2 \in (0, 1)$ and $A_2 \gg 1$ independent of $\omega$ and $\rho$, such that if for all cylinders of the type $[(0, t^*) + Q(\rho^{p^+}, \rho)]$,

$$\|(x, t) \in [(0, t^*) + Q(\rho^{p^+}, \rho)] : u > \mu^+ - \frac{\omega}{2} \leq (1 - \sigma)|Q(\rho^{p^+}, \rho)|,$$

then either

$$\omega \leq A_2 \rho^{p^+ - 2}$$

or

$$\text{ess osc } u \leq \nu_2 \omega.$$  (3.27)

Proof: Assume (3.27) is violated. By lemma 3.8, we can determine a positive integer number $s^*$ such that

$$\text{ess inf }_{Q(\bar{\mathcal{Q}}^{p^+, \mathcal{T}})} u \leq \mu^+ + \frac{\omega}{2^{s^*+1}},$$

and further

$$\text{ess osc }_{Q(\bar{\mathcal{Q}}^{p^+, \mathcal{T}})} u \leq (1 - \frac{1}{2^{s^*+1}})\omega,$$

therefore (3.28) holds with $\nu_2 = (1 - \frac{1}{2^{s^*+1}}).$ We get the conclusion. $\square$

Combine proposition 1 and proposition 2, we can get

Proposition 3.3  There exist $\nu = \max\{\nu_1, \nu_2\}$ and $\bar{A} = \{A_1, A_2\}$, such that either $\omega \leq \bar{A} \rho^{p^+ - 2}$ or $\text{ess osc } u \leq \nu \omega$, where $\nu_1, \nu_2, A_1, A_2$ are determined by proposition 1 and proposition 2.

Next we assume $\omega_1 = \max\{\nu \omega, \bar{A} \rho^{p^+ - 2}\}$ and $\frac{1}{\omega_1} = (\frac{2}{\omega})^{p^+ - 2}. Since$

$$l(\frac{2}{\omega})^{p^+} = \frac{2}{\omega} \rho^{p^+ - 2} - 2(\frac{2}{\omega})^{p^+} \geq 2 - 3p^+ \rho^{p^+ - 2} + \frac{A}{\omega_1} \rho^{p^+ - 2} \rho^{p^+} = a_1 \rho_1^{p^+},$$

where $\rho_1 = C^{-1} \rho$ and $C = \frac{\omega^{p^+ - 2} - (\frac{2}{\omega})^{p^+}}{\rho^{p^+}}$, so $Q(a_1, \rho_1) \subset Q(l(\bar{\mathcal{Q}}^{p^+, \mathcal{T}}), \bar{\mathcal{Q}}^{p^+, \mathcal{T}}).$ Then we can get $\text{ess osc } u \leq \omega_1$ and $\frac{2}{\omega_1} \rho^{p^+ - 2} > 8(\bar{\mathcal{Q}}^{p^+, \mathcal{T}})^{p^+ - 2} \rho^p.$ So for $Q(a_1, \rho_1)$

$$Q(a_1, \rho_1)^{p^+},$$

repeating the process above, we can get the similar result, and moreover the following proposition 3.4 can be obtained:

Proposition 3.4  There exist $0 < \varepsilon_0 < 1$, $\nu \in (0, 1)$, $C = C(N, M, p^+, p^-) > 1$ and $A > 1$ satisfy $\rho_0 = \rho$, $\omega_0 = \omega$, $\rho_n = C^{-n} \rho$ and $\omega_{n+1} = \max\{\nu \omega_n, C \rho_n^{p^-}\}$. $n = 1, 2, ..., such that for all boxes $Q^{(n)} = Q(\rho_n^{p^+}, \rho_n)$, $\frac{1}{\omega_0} = (\frac{2}{\omega})^{p^+ - 2}, n = 1, 2, ...$, we have

$$Q^{(n+1)} \subset Q^{(n)}, \text{ ess osc } u \leq \omega_n.$$  

In view of proposition 3.4, we get

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Proposition 3.5 There exist \( \lambda \in (0, 1) \), \( C = C(N, M, p^+, p^-) \) and \( 0 < \hat{\rho} \leq \rho \) such that for all boxes \( Q(a\rho^{\hat{\rho}^+}, \rho) \), \( \frac{1}{\alpha} = (\frac{\rho}{\hat{\rho}})^{p^- - 2} \), we have

\[
\text{ess osc}_{Q(a\rho^{\hat{\rho}^+}, \rho)} u \leq C(\omega + \rho^{\alpha})(\frac{\hat{\rho}}{\rho})^\lambda.
\]

Proof: From the iterative construction of \( \omega_n \), it follows that \( \omega_{n+1} \leq \nu \omega_n + C \rho^{\varepsilon_0} \) and by iteration

\[
\omega_n \leq \nu^n \omega + C(\Sigma_{i=0}^{n-1} \nu^i C^{-\varepsilon_0(n-i)}) \rho^{\varepsilon_0}.
\]

We may assume without loss of generality that \( \varepsilon_0 \) is so small that \( \nu \leq C^{-\varepsilon_0} \), then \( \omega_n \leq \nu^n \omega \leq C \nu(\frac{\rho}{\varepsilon_0})^{\varepsilon_0} \). Let \( 0 < \hat{\rho} \leq \rho \) be fixed, then there exists a nonnegative integer \( n \) such that

\[
C^{-(n+1)} \rho \leq \hat{\rho} \leq C^{-n} \rho,
\]

which implies the inequalities

\[
(n + 1) \geq \ln(\frac{\hat{\rho}}{\rho})^{\frac{1}{1-\nu}},
\]

\[
\nu^n \leq \nu^{-1}(\frac{\hat{\rho}}{\rho})^\lambda, \quad \lambda_1 = \frac{\ln \nu}{\ln C},
\]

\[
C \nu(\frac{\rho}{\varepsilon_0})^{\varepsilon_0} \leq C^{1+\varepsilon_0} \ln(\frac{\hat{\rho}}{\rho})^{\frac{\varepsilon_0}{1-\nu}} \rho^{\varepsilon_0} \leq C(\varepsilon_0)^{\frac{\varepsilon_0}{1-\nu}} \rho^{\varepsilon_0}.
\]

Therefore

\[
\omega_n \leq C(\omega + \rho^{\alpha})(\frac{\hat{\rho}}{\rho})^\lambda, \quad \lambda = \min\{\lambda_1, \frac{\varepsilon_0}{2}\}.
\]

On the other hand, by (3.3) we get \( \omega > C \rho^{\varepsilon_0} \). Thus by the definition of \( \omega_n \),

\[
\omega_1 = \max\{\nu \omega, C \rho^{\varepsilon_0}\} \leq \omega \quad \text{and} \quad \omega_2 = \max\{\nu \omega_1, C(C^{-1} \rho)^{\varepsilon_0}\} \leq \omega, \ldots, \text{so} \quad \omega_n \leq \omega.
\]

Since \( Q(a\rho^{\hat{\rho}^+}, \rho) \subset Q^{(n)} \), by proposition 3.4, we obtain

\[
\text{ess osc}_{Q(a\rho^{\hat{\rho}^+}, \rho)} u \leq \omega_n, \quad \text{so we conclude proposition 3.5}. \]

By proposition 3.5, we know \( u \) is Hölder continuity in \( Q(a\rho^{\hat{\rho}^+}, \rho) \), so for every point in \( Q \) we can obtain such a cylinder as \( Q(a\rho^{\hat{\rho}^+}, \rho) \), then by limited coverage theorem, \( u \) is local Hölder continuity in \( Q \), thus we get theorem 1.

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