Neon2: Finding Local Minima via First-Order Oracles
(version 1)∗

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Abstract

We propose a reduction for non-convex optimization that can (1) turn a stationary-point
finding algorithm into a local-minimum finding one, and (2) replace the Hessian-vector prod-
cut computations with only gradient computations. It works both in the stochastic and the
deterministic settings, without hurting the algorithm’s performance.

As applications, our reduction turns Natasha2 into a first-order method without hurting
its performance. It also converts SGD, GD, SCSG, and SVRG into local-minimum finding
algorithms outperforming some best known results.

1 Introduction

Nonconvex optimization has become increasing popular due its ability to capture modern machine
learning tasks in large scale. Most notably, training deep neural networks corresponds to minimizing
a function \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \) over \( x \in \mathbb{R}^d \) that is non-convex, where each training sample
\( i \) corresponds to one loss function \( f_i(\cdot) \) in the summation. This average structure allows one

to perform stochastic gradient descent (SGD) which uses a random \( \nabla f_i(x) \) —corresponding to
computing backpropagation once— to approximate \( \nabla f(x) \) and performs descent updates.

Motivated by such large-scale machine learning applications, we wish to design faster first-order
non-convex optimization methods that outperform the performance of gradient descent, both in
the online and offline settings. In this paper, we say an algorithm is online if its complexity is
independent of \( n \) (so \( n \) can be infinite), and offline otherwise. In recently years, researchers across
different communities have gathered together to tackle this challenging question. By far, known
theoretical approaches mostly fall into one of the following two categories.

First-order methods for stationary points. In analyzing first-order methods, we denote
by gradient complexity \( T \) the number of computations of \( \nabla f_i(x) \). To achieve an \( \varepsilon \)-approximate
stationary point —namely, a point \( x \) with \( \| \nabla f(x) \| \leq \varepsilon \) — it is a folklore that gradient descent
(GD) is offline and needs \( T \propto O(\frac{n}{\varepsilon^2}) \), while stochastic gradient decent (SGD) is online and needs
\( T \propto O(\frac{1}{\varepsilon^2}) \). In recent years, the offline complexity has been improved to \( T \propto O(\frac{n^{2/3}}{\varepsilon^{2/3}}) \) by the

∗The result of this paper was briefly discussed at a Berkeley Simons workshop on Oct 6 and internally presented
at Microsoft on Oct 30. We started to prepare this manuscript on Nov 11, after being informed of the independent
and similar work of Xu and Yang [28]. Their result appeared on arXiv on Nov 3. To respect the fact that their work
appeared online before us, we have adopted their algorithm name Neon and called our new algorithm Neon2. We
encourage readers citing this work to also cite [28].
SVRG method [3, 23], and the online complexity has been improved to $T \propto \frac{1}{\varepsilon^{10/3}}$ by the SCSG method [18]. Both of them rely on the so-called variance-reduction technique, originally discovered for convex problems [11, 16, 24, 26].

These algorithms SVRG and SCSG are only capable of finding stationary points, which may not necessarily be approximate local minima and are arguably bad solutions for neural-network training [9, 10, 14]. Therefore,

\begin{center}
\textit{can we turn stationary-point finding algorithms into local-minimum finding ones?}
\end{center}

**Hessian-vector methods for local minima.** It is common knowledge that using information about the Hessian, one can find $\varepsilon$-approximate local minima —namely, a point $x$ with $\|\nabla f(x)\| \leq \varepsilon$ and also $\nabla^2 f(x) \succeq -\varepsilon I$.

In 2006, Nesterov and Polyak [20] showed that one can find an $\varepsilon$-approximate in $O(\frac{1}{\varepsilon^2})$ iterations, but each iteration requires an offline computation as heavy as inverting the matrix $\nabla^2 f(x)$.

To fix this issue, researchers propose to study the so-called “Hessian-free” methods that, in addition to gradient computations, also compute Hessian-vector products. That is, instead of using the full matrix $\nabla^2 f_i(x)$ or $\nabla^2 f(x)$, these methods also compute $\nabla^2 f_i(x) \cdot v$ for indices $i$ and vectors $v$.

For Hessian-free methods, we denote by gradient complexity $T$ the number of computations of $\nabla f_i(x)$ plus that of $\nabla^2 f_i(x) \cdot v$. The hope of using Hessian-vector products is to improve the complexity $T$ as a function of $\varepsilon$.

Such improvement was first shown possible independently by [1, 7] for the offline setting, with complexity $T \propto \left( \frac{n}{\varepsilon^2} + \frac{n^{3/4}}{\varepsilon^{1/2}} \right)$ so is better than that of gradient descent. In the online setting, the first improvement was by Natasha2 which gives complexity $T \propto \left( \frac{1}{\varepsilon^2} \right)$ [2].

Unfortunately, it is argued by some researchers that Hessian-vector products are not general enough and may not be as simple to implement as evaluating gradients [8]. Therefore,

\begin{center}
\textit{can we turn Hessian-free methods into first-order ones, without hurting their performance?}
\end{center}

### 1.1 From Hessian-Vector Products to First-Order Methods

Recall by definition of derivative we have $\nabla^2 f_i(x) \cdot v = \lim_{q \to 0} \{ \frac{\nabla f_i(x+qv) - \nabla f_i(x)}{q} \}$.

Given any Hessian-free method, at least at a high level, can we replace every occurrence of $\nabla^2 f_i(x) \cdot v$ with $w = \nabla f_i(x+qv) - \nabla f_i(x)$ for some small $q > 0$?

Note the error introduced in this approximation is $\|\nabla^2 f_i(x) \cdot v - w\| \propto q \|v\|^2$. Therefore, as long as the original algorithm is sufficiently stable to adversarial noise, and as long as $q$ is small enough, this can convert Hessian-free algorithms into first-order ones.

In this paper, we demonstrate this idea by converting negative-curvature-search (NC-search) subroutines into first-order processes. NC-search is a key subroutine used in state-of-the-art Hessian-free methods that have rigorous proofs (see [1, 2, 7]). It solves the following simple task:

\begin{center}
\textit{negative-curvature search (NC-search)}
\end{center}

\begin{center}
\textit{given point $x_0$, decide if $\nabla^2 f(x_0) \succeq -\delta I$ or find a unit vector $v$ such that $v^T \nabla^2 f(x_0) v \leq -\frac{\delta}{2}$.}
\end{center}

**Online Setting.** In the online setting, NC-search can be solved by Oja’s algorithm [21] which

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1. We say $A \succeq -\delta I$ if all the eigenvalues of $A$ are no smaller than $-\delta$. In this high-level introduction, we focus only on the case when $\delta = \varepsilon^{1/6}$ for some constant $C$.

2. Hessian-free methods are useful because when $f_i(\cdot)$ is explicitly given, computing its gradient is in the same complexity as computing its Hessian-vector product [22, 25], using backpropagation.
costs $\widetilde{O}(1/\delta^2)$ computations of Hessian-vector products (first proved in [6] and applied to NC-search in [2]).

In this paper, we propose a method $\text{Neon}^\text{online}$ which solves the NC-search problem via only stochastic first-order updates. That is, starting from $x_1 = x_0 + \xi$ where $\xi$ is some random perturbation, we keep updating $x_{t+1} = x_t - \eta(\nabla f_i(x_t) - \nabla f_i(x_0))$. In the end, the vector $x_T - x_0$ gives us enough information about the negative curvature.

**Theorem 1** (informal). Our $\text{Neon}^\text{online}$ algorithm solves NC-search using $\widetilde{O}(1/\delta^2)$ stochastic gradients, without Hessian-vector product computations.

This complexity $\widetilde{O}(1/\delta^2)$ matches that of Oja’s algorithm, and is information-theoretically optimal (up to log factors), see the lower bound in [6].

We emphasize that the independent work $\text{Neon}$ by Xu and Yang [28] is actually the first recorded theoretical result that proposed this approach. However, $\text{Neon}$ needs $\widetilde{O}(1/\delta^3)$ stochastic gradients, because it uses full gradient descent to find NC (on a sub-sampled objective) inspired by [15] and the power method; instead, $\text{Neon}^\text{online}$ uses stochastic gradients and is based on our prior work on Oja’s algorithm [6].

By plugging $\text{Neon}^\text{online}$ into Natasha2 [2], we achieve the following corollary (see Figure 1(c)):

**Theorem 2** (informal). $\text{Neon}^\text{online}$ turns Natasha2 into a stochastic first-order method, without hurting its performance. That is, it finds an $(\epsilon, \delta)$-approximate local minimum in $T = \widetilde{O}(\frac{1}{\epsilon^5\delta^4} + \frac{1}{\epsilon^4\delta^5} + \frac{1}{\delta^6})$ stochastic gradient computations, without Hessian-vector product computations.

(We say $x$ is an approximate local minimum if $\|\nabla f(x)\| \leq \epsilon$ and $\nabla^2 f(x) \succeq -\delta I$.)

**Offline Setting.** There are a number of ways to solve the NC-search problem in the offline setting using Hessian-vector products. Most notably, power method uses $\widetilde{O}(n/\delta)$ computations of Hessian-vector products, Lanscoz method [17] uses $\widetilde{O}(n/\sqrt{\delta})$ computations, and shift-and-invert [12] on top of SVRG [26] (that we call SI+SVRG) uses $\widetilde{O}(n + n^{3/4}/\sqrt{\delta})$ computations.

In this paper, we convert Lanscoz’s method and SI+SVRG into first-order ones:

**Theorem 3** (informal). Our $\text{Neon}^\text{det}$ algorithm solves NC-search using $\widetilde{O}(1/\sqrt{\delta})$ full gradients (or equivalently $\widetilde{O}(n/\sqrt{\delta})$ stochastic gradients), and our $\text{Neon}^\text{svrg}$ solves NC-search using $\widetilde{O}(n + n^{3/4}/\sqrt{\delta})$ stochastic gradients.

We emphasize that, although analyzed in the online setting only, the work $\text{Neon}$ by Xu and Yang [28] also applies to the offline setting, and seems to be the first result to solve NC-search using first-order gradients with a theoretical proof. However, $\text{Neon}$ uses $\widetilde{O}(1/\delta)$ full gradients instead of $\widetilde{O}(1/\sqrt{\delta})$. Their approach is inspired by [15], but our $\text{Neon}^\text{det}$ is based on Chebyshev approximation theory (see textbook [27]) and its recent stability analysis [5].

By putting $\text{Neon}^\text{det}$ and $\text{Neon}^\text{svrg}$ into the CDHS method of Carmon et al. [7], we have

**Theorem 4** (informal). $\text{Neon}^\text{det}$ turns CDHS into a first-order method without hurting its performance: it finds an $(\epsilon, \delta)$-approximate local minimum in $\widetilde{O}(\frac{n}{\epsilon^5\delta^4} + \frac{n}{\delta^5})$ full gradient computations. $\text{Neon}^\text{svrg}$ turns CDHS into a first-order method without hurting its performance: it finds an $(\epsilon, \delta)$-approximate local minimum in $T = \widetilde{O}(\frac{n}{\epsilon^5\delta^4} + \frac{n}{\delta^5} + \frac{n^{3/4}}{\epsilon^3\delta^5} + \frac{n^{3/4}}{\delta^6})$ stochastic gradient computations.

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3We note that the original paper of CDHS only proved such complexity results (although requiring Hessian-vector products) for the special case of $\delta \geq \epsilon^{1/2}$. In such a case, it requires either $\widetilde{O}(\frac{1}{\epsilon^5\delta^4})$ full gradient computations or $\widetilde{O}(\frac{n}{\epsilon^5\delta^4} + \frac{n^{3/4}}{\epsilon^3\delta^5})$ stochastic gradient computations.
Figure 1: Neon vs Neon2 for finding $(\varepsilon, \delta)$-approximate local minima. We emphasize that Neon2 and Neon are based on the same high-level idea, but Neon is arguably the first-recorded result to turn stationary-point finding algorithms (such as SGD, SCSG) into local-minimum finding ones, with theoretical proofs.

One should perhaps compare Neon2$^{\text{det}}$ to the interesting work “convex until guilty” by Carmon et al. [8]. Their method finds $\varepsilon$-approximate stationary points using $\tilde{O}(1/\varepsilon^{1.75})$ full gradients, and is arguably the first first-order method achieving a convergence rate better than $1/\varepsilon^2$ of GD. Unfortunately, it is unclear if their method guarantees local minima. In comparison, Neon2$^{\text{det}}$ on CDHS achieves the same complexity but guarantees its output to be an approximate local minimum.

Remark 1.1. All the cited works in this sub-section requires the objective to have (1) Lipschitz-continuous Hessian (a.k.a. second-order smoothness) and (2) Lipschitz-continuous gradient (a.k.a. Lipschitz smoothness). One can argue that (1) and (2) are both necessary for finding approximate local minima, but if only finding approximate stationary points, then only (2) is necessary. We shall formally discuss our assumptions in Section 2.

1.2 From Stationary Points to Local Minima

Given any (first-order) algorithm that finds only stationary points (such as GD, SGD, or SCSG [18]), we can hope for using the NC-search routine to identify whether or not its output $x$ satisfies $\nabla^2 f(x) \succeq -\delta I$. If so, then automatically $x$ becomes an $(\varepsilon, \delta)$-approximate local minima so we can terminate. If not, then we can go in its negative curvature direction to further decrease the objective.

In the independent work of Xu and Yang [28], they proposed to apply their Neon method for NC-search, and thus turned SGD and SCSG into first-order methods finding approximate local minima. In this paper, we use Neon2 instead. We show the following theorem:

**Theorem 5** (informal). To find an $(\varepsilon, \delta)$-approximate local minima,

(a) Neon2+SGD needs $T = \tilde{O}(\frac{1}{\varepsilon^4} + \frac{1}{\varepsilon^2 \delta^4} + \frac{1}{\delta^8})$ stochastic gradients;

(b) Neon2+SCSG needs $T = \tilde{O}(\frac{1}{\varepsilon^{12/7}} + \frac{1}{\varepsilon^2 \delta^4} + \frac{1}{\delta^8})$ stochastic gradients; and

(c) Neon2+GD needs $T = \tilde{O}(\frac{n}{\varepsilon^2} + \frac{n}{\varepsilon^2 \delta^4} + \frac{n}{\delta^8})$ (so $\tilde{O}(\frac{1}{\varepsilon^2} + \frac{1}{\delta^8})$ full gradients).

(d) Neon2+SVRG needs $T = \tilde{O}(\frac{n^{2/4}}{\varepsilon^2} + \frac{n}{\varepsilon^2 \delta^4} + \frac{n^{3/4}}{\delta^8})$ stochastic gradients.

We make several comments as follows.

(a) We compare Neon2+SGD to Ge et al. [13], where the authors showed SGD plus perturbation needs $T = \tilde{O}(\text{poly}(d)/\varepsilon^4)$ stochastic gradients to find $(\varepsilon, \varepsilon^{1/4})$-approximate local minima. This is the perhaps first time that a theoretical guarantee for finding local minima is given using first-order oracles.
To some extent, Theorem 5a is superior because we have (1) removed the poly(d) factor\(^4\) (2) achieved \(T = \tilde{O}(1/\varepsilon^4)\) as long as \(\delta \geq \varepsilon^{2/3}\), and (3) a much simpler analysis.

We also remark that, if using Neon instead of Neon2, one achieves a slightly worse complexity \(T = \tilde{O}(\frac{1}{\varepsilon^4} + \frac{1}{\delta^6})\), see Figure 1(a) for a comparison.\(^5\)

(b) Neon2+SCSG turns SCSG into a local-minimum finding algorithm. Again, if using Neon instead of Neon2, one gets a slightly worse complexity \(T = \tilde{O}(\frac{1}{\varepsilon^4} + \frac{1}{\delta^6} + \frac{1}{\varepsilon^7})\), see Figure 1(b).

(c) We compare Neon2+GD to Jin et al. [15], where the authors showed GD plus perturbation needs \(\tilde{O}(1/\varepsilon^2)\) full gradients to find \((\varepsilon, \varepsilon^2/7)-\text{approximate local minima}\). This is perhaps the first time that one can convert a stationary-point finding method (namely GD) into a local minimum-finding one, without hurting its performance.

To some extent, Theorem 5c is better because we use \(\tilde{O}(1/\varepsilon^2)\) full gradients as long as \(\delta \geq \varepsilon^{4/7}\).

(d) Our result for Neon2+SVRG does not seem to be recorded anywhere, even if Hessian-vector product computations are allowed.

Limitation. We note that there is limitation of using Neon2 (or Neon) to turn an algorithm finding stationary points to that finding local minima. Namely, given any algorithm \(\mathcal{A}\), if the gradient complexity for \(\mathcal{A}\) to find an \(\varepsilon\)-approximate stationary point is \(T\), then after this conversion, it finds \((\varepsilon, \delta)\)-approximate local minima in a gradient complexity that is at least \(T\). This is because the new algorithm, after combining Neon2 and \(\mathcal{A}\), tries to alternatively find stationary points (using \(\mathcal{A}\)) and escape from saddle points (using Neon2). Therefore, it must pay at least complexity \(T\). In contrast, methods such as Natasha2 swing by saddle points instead of go to saddle points and then escape. This has enabled it to achieve a smaller complexity \(T = O(\varepsilon^{-3.25})\) for \(\delta \geq \varepsilon^{1/4}\).

2 Preliminaries

Throughout this paper, we denote by \(\| \cdot \|\) the Euclidean norm. We use \(i \in_R [n]\) to denote that \(i\) is generated from \([n] = \{1, 2, \ldots, n\}\) uniformly at random. We denote by \(\mathbb{I}[\text{event}]\) the indicator function of probabilistic events.

We denote by \(\|A\|_2\) the spectral norm of matrix \(A\). For symmetric matrices \(A\) and \(B\), we write \(A \succeq B\) to indicate that \(A - B\) is positive semidefinite (PSD). Therefore, \(A \succeq -\sigma I\) if and only if all eigenvalues of \(A\) are no less than \(-\sigma\). We denote by \(\lambda_{\min}(A)\) and \(\lambda_{\max}(A)\) the minimum and maximum eigenvalue of a symmetric matrix \(A\).

Recall some definitions on smoothness (for other equivalent definitions, see textbook [19])

Definition 2.1. For a function \(f : \mathbb{R}^d \to \mathbb{R}\),

- \(f\) is L-Lipschitz smooth (or L-smooth for short) if
  \[ \forall x, y \in \mathbb{R}^d, \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|. \]

- \(f\) is second-order \(L_2\)-Lipschitz smooth (or \(L_2\)-second-order smooth for short) if
  \[ \forall x, y \in \mathbb{R}^d, \|\nabla^2 f(x) - \nabla^2 f(y)\|_2 \leq L_2\|x - y\|. \]

The following fact says the variance of a random variable decreases by a factor \(m\) if we choose \(m\) independent copies and average them. It is trivial to prove, see for instance [18].

\(^4\)We are aware that the original authors of [13] have a different proof to remove its poly(d) factor, but have not found it online at this moment.

\(^5\)Their complexity might be improvable to \(\tilde{O}(\frac{1}{\varepsilon^4} + \frac{1}{\delta^6})\) with a slight change of the algorithm, but not beyond.
algorithm & gradient complexity $T$ & Hessian-vector products & variance bound & Lip. smooth & $2^{nd}$-order smooth
\hline
stationary & SGD (folklore) & $O\left(\frac{1}{\varepsilon^4}\right)$ & no & needed & needed & no
\hline
local minima & perturbed SGD & $O\left(\frac{\text{poly}(d)}{\varepsilon^2}\right)$ (only for $\delta \geq \varepsilon^{1/4}$) & no & needed & needed & needed
\hline
local minima & Neon+SGD & $O\left(\frac{1}{\varepsilon^3} + \frac{1}{\delta^3}\right)$ & no & needed & needed & needed
\hline
local minima & Neon2+SGD & $O\left(\frac{1}{\varepsilon^3} + \frac{1}{\delta^3} + \frac{1}{\delta^6}\right)$ & no & needed & needed & needed
\hline
stationary & SCSG & $O\left(\frac{1}{\varepsilon^3}\right)$ & no & needed & needed & no
\hline
local minima & Neon+SCSG & $O\left(\frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^6}\right)$ & no & needed & needed & needed
\hline
local minima & Neon2+SCSG & $O\left(\frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^6}\right)$ & no & needed & needed & needed
\hline
local minima & Natasha2 & $O\left(\frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^6}\right)$ & needed & needed & needed & needed
\hline
local minima & Neon+Natasha2 & $O\left(\frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^6}\right)$ & no & needed & needed & needed
\hline
local minima & Neon2+Natasha2 & $O\left(\frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^{1/3}} + \frac{1}{\varepsilon^6}\right)$ & no & needed & needed & needed
\hline
stationary & GD (folklore) & $O\left(\frac{n}{\varepsilon^2}\right)$ & no & no & needed & no
\hline
local minima & perturbed GD & $O\left(\frac{n}{\varepsilon^2}\right)$ (only for $\delta \geq \varepsilon^{1/2}$) & no & no & needed & needed
\hline
local minima & Neon2+GD & $O\left(\frac{n}{\varepsilon^2} + \frac{n}{\delta^3}\right)$ & no & no & needed & needed
\hline
stationary & SVRG & $O\left(\frac{n^{2/3}}{\varepsilon^2} + n\right)$ & no & no & needed & no
\hline
local minima & Neon2+SVRG & $O\left(\frac{n^{2/3}}{\varepsilon^2} + \frac{n}{\delta^3} + \frac{n^{3/4}}{\varepsilon^{3/4}} + \frac{n^{3/4}}{\delta^{3/4}}\right)$ & no & no & needed & needed
\hline
stationary & “guilty” & $O\left(\frac{n}{\varepsilon^2}\right)$ & no & no & needed & needed
\hline
local minima & FastCubic & $O\left(\frac{n}{\varepsilon^2} + \frac{n^{3/4}}{\varepsilon^{3/4}}\right)$ (only for $\delta \geq \varepsilon^{1/2}$) & needed & no & needed & needed
\hline
local minima & CDHS & $O\left(\frac{n}{\varepsilon^2} + \frac{n}{\delta^3} + \frac{n^{3/4}}{\varepsilon^{3/4}} + \frac{n^{3/4}}{\delta^{3/4}}\right)$ & needed & no & needed & needed
\hline
local minima & Neon2+CDHS & $O\left(\frac{n}{\varepsilon^2} + \frac{n}{\delta^3} + \frac{n^{3/4}}{\varepsilon^{3/4}} + \frac{n^{3/4}}{\delta^{3/4}}\right)$ & no & no & needed & needed
\hline
\hline
Table 1: Complexity for finding $\|\nabla f(x)\| \leq \varepsilon$ and $\nabla^2 f(x) \succeq -\delta I$. Following tradition, in these complexity bounds, we assume variance and smoothness parameters as constants, and only show the dependency on $n, d, \varepsilon$.

Remark 1. Variance bounds is needed for online methods.
Remark 2. Lipschitz smoothness is needed for finding approximate stationary points.
Remark 3. Second-order Lipschitz smoothness is needed for finding approximate local minima.

Fact 2.2. If $v_1, \ldots, v_n \in \mathbb{R}^d$ satisfy $\sum_{i=1}^n v_i = 0$, and $S$ is a non-empty, uniform random subset of $[n]$. Then
$$
E \left[ \left\| \frac{1}{|S|} \sum_{i \in S} v_i \right\|^2 \right] = \frac{n-|S|}{(n-1)|S|} \cdot \frac{1}{n} \sum_{i \in [n]} \|v_i\|^2 \leq \frac{|S| \leq n}{|S|} \cdot \frac{1}{n} \sum_{i \in [n]} \|v_i\|^2 .
$$

2.1 Problem and Assumptions

Throughout the paper we study the following minimization problem
$$
\min_{x \in \mathbb{R}^d} \left\{ f(x) \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n f_i(x) \right\}
$$

Algorithm 1 Neon2online($f, x_0, \delta, p$)

Input: Function $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, vector $x_0$, negative curvature $\delta > 0$, confidence $p \in (0, 1]$.

1: for $j = 1, 2, \ldots \Theta(\log 1/p)$ do
2: \hspace{1em} $v_j \leftarrow \text{Neon2online}_\text{weak}(f, x_0, \delta, p)$;
3: \hspace{1em} if $v_j \neq \perp$ then
4: \hspace{2em} $m \leftarrow \Theta\left(\frac{L^2 \log 1/p}{\delta^2}\right)$, $v' \leftarrow \Theta\left(\frac{\delta}{L^2}\right)v$.
5: \hspace{2em} Draw $i_1, \ldots, i_m \in R\left[\mathbb{R}\right]$.
6: \hspace{2em} $z_j = \frac{1}{m\|v'\|_2^2} \sum_{j=1}^{m} (v')^T (\nabla f_{i_j}(x_0 + v') - \nabla f_{i_j}(x_0))$
7: \hspace{2em} if $z_j \leq -3\delta/4$ return $v = v_j$
8: \hspace{1em} end if
9: \hspace{1em} end for
10: return $v = \perp$.

Algorithm 2 Neon2online_\text{weak}(f, x_0, \delta, p)

1: $\eta \leftarrow \frac{\delta c_0^2 L^4 \log(d/p)}{C_2 \log\left(\frac{d}{\delta^2}\right)}$, $T \leftarrow \frac{C_2 \log(d/p)}{\eta \delta}$, \hspace{1em} for sufficiently large constant $C_0$
2: $\xi \leftarrow$ Gaussian random vector with norm $\sigma$.
3: $x_1 \leftarrow x_0 + \xi$.
4: for $t \leftarrow 1$ to $T$ do
5: \hspace{1em} $x_{t+1} \leftarrow x_t - \eta (\nabla f_i(x_t) - \nabla f_i(x_0))$ where $i \in R\left[\mathbb{R}\right]$.
6: \hspace{1em} if $\|x_{t+1} - x_0\|_2 \geq r$ then return $v = \frac{x_{t+1} - x_0}{\|x_{t+1} - x_0\|_2}$ \hspace{1em} $r \overset{\text{def}}{=} (d/p)^C \sigma$
7: \hspace{1em} end if
8: \hspace{1em} end for
9: return $v = \perp$;

where both $f(\cdot)$ and each $f_i(\cdot)$ can be nonconvex. We wish to find $(\varepsilon, \delta)$-local minima which are points $x$ satisfying

$$\|\nabla f(x)\| \leq \varepsilon \quad \text{and} \quad \nabla^2 f(x) \succeq -\delta I.$$ 

We need the following three assumptions

- Each $f_i(x)$ is $L$-Lipschitz smooth.
- Each $f_i(x)$ is second-order $L_2$-Lipschitz smooth.
  (In fact, the gradient complexity of Neon2 in this paper only depends polynomially on the second-order smoothness of $f(x)$ (rather than $f_i(x)$), and the time complexity depends logarithmically on the second-order smoothness of $f_i(x)$). To make notations simple, we decide to simply assume each $f_i(x)$ is $L_2$-second-order smooth.)
- Stochastic gradients have bounded variance: \forall $x \in \mathbb{R}^d$: $\mathbb{E}_{i \in R\left[\mathbb{R}\right]} \|\nabla f(x) - \nabla f_i(x)\|^2 \leq \mathbb{V}$.
  (This assumption is needed only for online algorithms.)

3 Neon2 in the Online Setting

We propose Neon2online formally in Algorithm 1.

It repeatedly invokes Neon2online_\text{weak} in Algorithm 2, whose goal is to solve the NC-search problem with confidence $2/3$ only; then Neon2online_\text{weak} invokes Neon2online repeatedly for $\log(1/p)$ times to boost the confidence to $1 - p$. We prove the following theorem:
**Theorem 1** (Neon2online). Let \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \) where each \( f_i \) is \( L \)-smooth and \( L^2 \)-second-order smooth. For every point \( x_0 \in \mathbb{R}^d \), every \( \delta > 0 \), every \( p \in (0, 1] \), the output
\[
v = \text{Neon2}_{\text{online}}(f, x_0, \delta, p)
\]
satisfies that, with probability at least \( 1 - p \):
1. If \( v = \perp \), then \( \nabla^2 f(x_0) \succeq -\delta I \).
2. If \( v \neq \perp \), then \( \|v\|_2 = 1 \) and \( v^\top \nabla^2 f(x_0) v \leq -\delta_2^2 \).

Moreover, the total number of stochastic gradient evaluations \( O\left(\frac{\log(d/p) L^2}{\delta^2} \right) \).

The proof of **Theorem 1** immediately follows from Lemma 3.1 and Lemma 3.2 below.

**Lemma 3.1** (Neon2weak). In the same setting as **Theorem 1** the output \( v = \text{Neon2}_{\text{weak}}(f, x_0, \delta, p) \) satisfies if \( \lambda_{\min}(\nabla^2 f(x_0)) \leq -\delta \), then with probability at least \( 2/3 \), \( v \neq \perp \) and \( v^\top \nabla^2 f(x_0) v \leq -\frac{51}{100} \delta \).

**Proof sketch of Lemma 3.1** We explain why \( \text{Neon2}_{\text{weak}} \) works as follows. Starting from a randomly perturbed point \( x_1 = x_0 + \xi \), it keeps updating \( x_{t+1} \leftarrow x_t - \eta (\nabla f_i(x_t) - \nabla f_i(x_0)) \) for some random index \( i \in [n] \), and stops either when \( T \) iterations are reached, or when \( \|x_{t+1} - x_0\|_2 > r \). Therefore, we have \( \|x_t - x_0\|_2 \leq r \) throughout the iterations, and thus can approximate \( \nabla^2 f(x_0)(x_t - x_0) \) using \( \nabla f_i(x_t) - \nabla f_i(x_0) \), up to error \( O(r^2) \). This is a small term based on our choice of \( r \).

Ignoring the error term, our updates look like \( x_{t+1} - x_0 = (I - \eta \nabla^2 f_i(x_0))(x_t - x_0) \). This is exactly the same as Oja’s algorithm [21] which is known to approximately compute the minimum eigenvector of \( \nabla^2 f(x_0) = \frac{1}{n} \sum_{i=1}^{n} f_i(x_0) \). Using the recent optimal convergence analysis of Oja’s algorithm [6], one can conclude that after \( T_1 = \Theta\left(\frac{\log L}{\eta^2}\right) \) iterations, where \( \lambda = \max\{0, -\lambda_{\min}(\nabla^2 f(x_0))\} \), then we not only have that \( \|x_{t+1} - x_0\|_2 \) is blown up, but also it aligns well with the minimum eigenvector of \( \nabla^2 f(x_0) \). In other words, if \( \lambda \geq \delta \), then the algorithm must stop before \( T \).

Finally, one has to carefully argue that the error does not blow up in this iterative process. We defer the proof details to Appendix A.2

Our Lemma 3.2 below tells us we can verify if the output \( v \) of \( \text{Neon2}_{\text{weak}} \) is indeed correct (up to additive \( \frac{\delta}{4} \)), so we can boost the success probability to \( 1 - p \).

**Lemma 3.2** (verification). In the same setting as **Theorem 1** let vectors \( x, v \in \mathbb{R}^d \). If \( i_1, \ldots, i_m \in R [n] \) and define
\[
z = \frac{1}{m} \sum_{j=1}^{m} v^\top (\nabla f_{i_j}(x + v) - \nabla f_{i_j}(x))
\]
Then, if \( \|v\| \leq \frac{\delta}{8L^2} \) and \( m = \Theta\left(\frac{L^2 \log 1/p}{\delta^2}\right) \), with probability at least \( 1 - p \),
\[
\left| \frac{z}{\|v\|_2} - \frac{v^\top \nabla^2 f(x)v}{\|v\|_2} \right| \leq \frac{\delta}{4}.
\]

4 Neon2 in the Deterministic Setting

We propose \( \text{Neon2}_{\text{det}} \) formally in Algorithm 3 and prove the following theorem:
Algorithm 3 $\text{Neon2}^\text{det}(f, x_0, \delta, p)$

**Input:** A function $f$, vector $x_0$, negative curvature target $\delta > 0$, failure probability $p \in (0, 1)$.

1. $T \leftarrow \frac{C_1^2 \log(d/p)^{\log T}}{\sqrt{\delta}}$ \quad $\diamond$ for sufficiently large constant $C_1$.
2. $\xi \leftarrow$ Gaussian random vector with norm $\sigma$;
3. $x_1 \leftarrow x_0 + \xi$, \quad $y_1 \leftarrow \xi, y_0 \leftarrow 0$
4. for $t \leftarrow 1$ to $T$ do
5. \quad $y_{t+1} = 2M(y_t) - y_{t-1}$; \quad $\diamond$ $M(y) \overset{\text{def}}{=} -\frac{1}{L} (\nabla f(x_0) + y) - \nabla f(x_0)) + (1 - \frac{4\delta}{d}) y$
6. \quad $x_{t+1} = x_0 + y_{t+1} - M(y_t)$.
7. \quad if $\|x_{t+1} - x_0\|_2 \geq r$ then return $\frac{x_{t+1} - x_0}{\|x_{t+1} - x_0\|_2}$.
8. end for
9. return $\perp$.

**Theorem 3** ($\text{Neon2}^\text{det}$). Let $f(x)$ be a function that is $L$-smooth and $L_2$-second-order smooth. For every point $x_0 \in \mathbb{R}^d$, every $\delta > 0$, every $p \in (0, 1]$, the output $v = \text{Neon2}^\text{det}(f, x_0, \delta, p)$ satisfies that, with probability at least $1 - p$:

1. If $v = \perp$, then $\nabla^2 f(x_0) \rho \preceq -\delta I$.
2. If $v \neq \perp$, then $\|v\|_2 = 1$ and $v^T \nabla^2 f(x_0) v \leq -\delta$.

Moreover, the total number full gradient evaluations is $O\left(\frac{\log^2(d/p) \sqrt{T}}{\sqrt{\delta}}\right)$.

**Proof sketch of Theorem 3** We explain the high-level intuition of $\text{Neon2}^\text{det}$ and the proof of Theorem 3 as follows. Define $M = -\frac{1}{L} \nabla^2 f(x_0) + (1 - \frac{4\delta}{d^2}) I$. We immediately notice that

- all eigenvalues of $\nabla^2 f(x_0)$ in $[-\frac{3\delta}{4}, L]$ are mapped to the eigenvalues of $M$ in $[-1, 1]$, and
- any eigenvalue of $\nabla^2 f(x_0)$ smaller than $-\delta$ is mapped to eigenvalue of $M$ greater than $1 + \frac{\delta}{4L}$.

Therefore, as long as $T \geq \tilde{\Omega}(\frac{L}{\delta})$, if we compute $x_{T+1} = x_0 + M^T \xi$ for some random vector $\xi$, by the theory of power method, $x_{T+1} - x_0$ must be a negative-curvature direction of $\nabla^2 f(x_0)$ with value $\leq \frac{1}{2} \delta$. There are two issues with this approach.

The first issue is that, the degree $T$ of this matrix polynomial $M^T$ can be reduced to $T = \tilde{\Omega}(\frac{\sqrt{L}}{\sqrt{\delta}})$ if the so-called Chebyshev polynomial is used.

**Claim 4.1.** Let $T_t(x)$ be the $t$-th Chebyshev polynomial of the first kind, defined as $[27]$:

$s_t(x) \overset{\text{def}}{=} 1,
\quad s_{t-1}(x) \overset{\text{def}}{=} 2x \cdot s_t(x) - s_{t-1}(x)$

then $T_t(x)$ satisfies: $T_t(x) = \frac{1}{2} \left( x + \sqrt{x^2 - 1} \right)^t, \left( x + \sqrt{x^2 - 1} \right)^t$ if $x \in [-1, 1]$;

\[ \frac{1}{2} \left( x + \sqrt{x^2 - 1} \right)^t, \left( x + \sqrt{x^2 - 1} \right)^t \] if $x > 1$.

Since $T_t(x)$ stays between $[-1, 1]$ when $x \in [-1, 1]$, and grows to $\approx (1 + \sqrt{x^2 - 1})^t$ for $x \geq 1$, we can use $T_t(M)$ in replacement of $M^T$. Then, any eigenvalue of $M$ that is above $1 + \frac{\delta}{4L}$ shall grow in a speed like $(1 + \sqrt{\delta/L})^T$, so it suffices to choose $T \geq \tilde{\Omega}(\frac{\sqrt{L}}{\sqrt{\delta}})$. This is quadratically faster than applying the power method, so in $\text{Neon2}^\text{det}$ we wish to compute $x_{t+1} \approx x_0 + T_t(M) \xi$.

The second issue is that, since we cannot compute Hessian-vector products, we have to use the
gradient difference to approximate it; that is, we can only use \( M(y) \) to approximate \( M(y) \) where

\[
M(y) \overset{\text{def}}{=} -\frac{1}{L} \left( \nabla f(x_0 + y) - \nabla f(x_0) \right) + \left( 1 - \frac{3\delta}{4L} \right) y .
\]

How does error propagate if we compute \( T_t(M) \xi \) by replacing \( M \) with \( M \)? Note that this is a very non-trivial question, because the coefficients of the polynomial \( T_t(x) \) is as large as \( 2^{O(t)} \).

It turns out, the way that error propagates depends on how the Chebyshev polynomial is calculated. If the so-called backward recurrence formula is used, namely,

\[
y_0 = 0, \quad y_1 = \xi, \quad y_t = 2M(y_{t-1}) - y_{t-2}
\]

and setting \( x_{T+1} = x_0 + y_{T+1} - T_t(M) \xi \), then this \( x_{T+1} \) is sufficiently close to the exact value \( x_0 + T_t(M) \xi \). This is known as the stability theory of computing Chebyshev polynomials, and is proved in our prior work [5].

We defer all the proof details to Appendix B.2 \( \Box \)

5 Neon2 in the SVRG Setting

Recall that the shift-and-invert (SI) approach [12] on top of the SVRG method [26] solves the minimum eigenvector problem as follows. Given any matrix \( A = \nabla^2 f(x_0) \) and suppose its eigenvalues are \( \lambda_1 \leq \cdots \leq \lambda_d \). Then, if \( \lambda > -\lambda_1 \), we can define positive semidefinite matrix \( B = (\lambda I + A)^{-1} \), and then apply power method to find an (approximate) maximum eigenvector of \( B \), which necessarily is an (approximate) minimum eigenvector of \( A \).

The SI approach specifies a binary search routine to determine the shifting constant \( \lambda \), and ensures that \( B = (\lambda I + A)^{-1} \) is always “well conditioned,” meaning that it suffices to apply power method on \( B \) for logarithmic number of iterations. In other words, the task of computing the minimum eigenvector of \( A \) reduces to computing matrix-vector products \( By \) for poly-logarithmic number of times. Moreover, the stability of SI was shown in a number of papers, including [12] and [4]. This means, it suffices for us to compute \( By \) approximately.

However, how to compute \( By \) for an arbitrary vector \( y \). It turns out, this is equivalent to minimizing a convex quadratic function that is of a finite sum form

\[
g(z) \overset{\text{def}}{=} \frac{1}{2} z^\top (\lambda I + A) z + y^\top z = \frac{1}{2y} \sum_{i=1}^n z^\top (\lambda I + \nabla^2 f_i(x_0)) z + y^\top z .
\]

Therefore, one can apply the a variant of the SVRG method (arguably first discovered by Shalev-Shwartz [26]) to solve this task. In each iteration, SVRG needs to evaluate a stochastic gradient \( (\lambda I + \nabla^2 f_i(x_0)) z + y \) at some point \( z \) for some random \( i \in [n] \). Instead of evaluating it exactly (which require a Hessian-vector product), we use \( \nabla f_i(x_0 + z) - \nabla f_i(x_0) \) to approximate \( \nabla^2 f_i(x_0) \cdot z \).

Of course, one needs to show also that the SVRG method is stable to noise. Using similar techniques as the previous two sections, one can show that the error term is proportional to \( O(\|z\|_2^2) \), and thus as long as we bound the norm of \( z \) is bounded (just like we did in the previous two sections), this should not affect the performance of the algorithm. We decide to ignore the detailed theoretical proof of this result, because it will complicate this paper.

**Theorem 3** \([\text{Neon2 svrg}] \). Let \( f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \) where each \( f_i \) is \( L \)-smooth and \( L_2 \)-second-order smooth. For every point \( x_0 \in \mathbb{R}^d \), every \( \delta > 0 \), every \( p \in (0, 1] \), the output \( v = \text{Neon2 svrg}(f, x_0, \delta, p) \) satisfies that, with probability at least \( 1 - p \):

1. If \( v = \perp \), then \( \nabla^2 f(x_0) \succeq -\delta I \).
2. If \( v \neq \perp \), then \( \|v\|_2 = 1 \) and \( v^\top \nabla^2 f(x_0) v \leq -\delta \).
Moreover, the total number stochastic gradient evaluations is \( \tilde{O}(n + \frac{n^{3/4} \sqrt{L}}{\sqrt{\delta}}) \).

6 Applications of Neon2

We show how Neon2 online can be applied to existing algorithms such as SGD, GD, SCSG, SVRG, Natasha2, CDHS. Unfortunately, we are unaware of a generic statement for applying Neon2 online to any algorithm. Therefore, we have to prove them individually.\(^6\)

Throughout this section, we assume that some starting vector \( x_0 \in \mathbb{R}^d \) and upper bound \( \Delta_f \) is given to the algorithm, and it satisfies \( f(x_0) - \min_x \{ f(x) \} \leq \Delta_f \). This is only for the purpose of proving theoretical bounds. In practice, because \( \Delta_f \) only appears in specifying the number of iterations, can just run enough number of iterations and then halt the algorithm, without the necessity of knowing \( \Delta_f \).

6.1 Auxiliary Claims

Claim 6.1. For any \( x \), using \( O((\frac{n}{\epsilon^2} + 1) \log \frac{1}{p}) \) stochastic gradients, we can decide

\[
\text{with probability } 1 - p: \quad \text{either } \|\nabla f(x)\| \geq \frac{\epsilon}{2} \quad \text{or} \quad \|\nabla f(x)\| \leq \epsilon .
\]

Proof. Suppose we generate \( m = O(\log \frac{1}{p}) \) random uniform subsets \( S_1, \ldots, S_m \) of \( [n] \), each of cardinality \( B = \max \{ \frac{32d}{\epsilon^4}, 1 \} \). Then, denoting by \( v_j = \frac{1}{B} \sum_{i \in S_j} \nabla f_i(x) \), we have according to Fact 2.2 that

\[
E(S_j) \left[ \|v_j - \nabla f(x)\|^2 \right] \leq \frac{\epsilon^2}{B} = \frac{\epsilon^2}{32d}.
\]

In other words, with probability at least 1/2 over the randomness of \( S_j \), we have \( \|v_j\| - \|\nabla f(x)\| \leq \frac{\epsilon}{4} \). Since \( m = O(\log \frac{1}{p}) \), we have with probability at least 1 - \( p \), it satisfies that at least \( m/2 + 1 \) of the vectors \( v_j \) satisfy \( \|v_j\| - \|\nabla f(x)\| \leq \frac{\epsilon}{4} \). Now, if we select \( v^* = v_j \) where \( j \in [m] \) is the index that gives the median value of \( \|v_j\| \), then it satisfies \( \|v_j\| - \|\nabla f(x)\| \leq \frac{\epsilon}{4} \). Finally, we can check if \( \|v_j\| \leq \frac{3\epsilon}{4} \). If so, then we conclude that \( \|\nabla f(x)\| \leq \epsilon \), and if not, we conclude that \( \|\nabla f(x)\| \geq \frac{\epsilon}{2} \).

Claim 6.2. If \( v \) is a unit vector and \( v^T \nabla^2 f(y)v \leq -\frac{\delta}{2} \), suppose we choose \( y' = y + \frac{\delta}{L_2} v \) where the sign is random, then \( f(y) - E[f(y')] \geq \frac{\delta^3}{12L_2^2} \).

Proof. Letting \( \eta = \frac{\delta}{L_2} \), then by the second-order smoothness,

\[
f(y) - E[f(y')] \geq E \left[ \langle \nabla f(y), y - y' \rangle - \frac{1}{2} (y - y')^T \nabla^2 f(y) (y - y') - \frac{L_2}{6} \|y - y'\|^3 \right]
\]

\[
= -\frac{\eta^2}{2} v^T \nabla^2 f(y) v - \frac{L_2 \eta^3}{6} \|v\|^3 \geq \frac{\eta^2 \delta}{4} - \frac{L_2 \eta^3}{6} = \frac{\delta^3}{12L_2^2} . \quad \square
\]

6.2 Neon2 on SGD and GD

To apply Neon2 to turn SGD into an algorithm finding approximate local minima, we propose the following process Neon2+SGD (see Algorithm 4). In each iteration \( t \), we first apply SGD with mini-batch size \( O(\frac{1}{T}) \) (see Line 4). Then, if SGD finds a point with small gradient, we apply Neon2 online to decide if it has a negative curvature, if so, then we move in the direction of the negative curvature (see Line 10). We have the following theorem:

\(^6\)This is because stationary-point finding algorithms have somewhat different guarantees. For instance, in mini-batch SGD we have \( f(x_t) - E[f(x_{t+1})] \geq \Omega(\|\nabla f(x_t)\|^2) \) but in SCSG we have \( f(x_t) - E[f(x_{t+1})] \geq E[\Omega(\|\nabla f(x_{t+1})\|^2)] \).
Algorithm 4 Neon2+SGD$(f, x_0, p, \varepsilon, \delta)$

**Input:** function $f(\cdot)$, starting vector $x_0$, confidence $p \in (0, 1)$, $\varepsilon > 0$ and $\delta > 0$.

1. $K \leftarrow O\left(\frac{L^2\Delta_f}{\varepsilon^2} + \frac{L\Delta_f}{\varepsilon^2}\right)$; \quad $\Delta_f$ is any upper bound on $f(x_0) - \min_x \{f(x)\}$
2. for $t \leftarrow 0$ to $K - 1$ do
3. $S \leftarrow$ a uniform random subset of $[n]$ with cardinality $|S| = B = \max\{\frac{8V}{\varepsilon^2}, 1\}$;
4. $x_{t+1/2} \leftarrow x_t - \frac{1}{L n} \sum_{i \in S} \nabla f_i(x_t)$;
5. if $\|\nabla f(x_t)\| \geq \frac{\varepsilon}{2}$ then \quad $\|\nabla f(x_t)\|$ using $O(\varepsilon^{-2} V \log(K/p))$ stochastic gradients
6. $x_{t+1} \leftarrow x_{t+1/2}$; \quad necessarily $\|\nabla f(x_t)\| \leq \varepsilon$
7. else $v \leftarrow \text{Neon2}_\text{online}(x_t, \delta, \frac{p}{2K})$; \quad necessarily $\nabla^2 f(x_t) \succeq -\delta I$
8. if $v = \bot$ then return $x_t$; \quad necessarily $v^\top \nabla^2 f(x_t)v \leq -\delta/2$
9. else $x_{t+1} \leftarrow x_t \pm \frac{\delta}{L^2} v$;
10. end if \quad necessarily $\nabla^2 f(x_t) \succeq -\delta I$
11. end for \quad necessarily $v^\top \nabla^2 f(x_t)v \leq -\delta/2$
12. end for
13. will not reach this line (with probability $\geq 1 - p$).

Algorithm 5 Neon2+GD$(f, x_0, p, \varepsilon, \delta)$

**Input:** function $f(\cdot)$, starting vector $x_0$, confidence $p \in (0, 1)$, $\varepsilon > 0$ and $\delta > 0$.

1. $K \leftarrow O\left(\frac{L^2\Delta_f}{\varepsilon^2} + \frac{L\Delta_f}{\varepsilon^2}\right)$; \quad $\Delta_f$ is any upper bound on $f(x_0) - \min_x \{f(x)\}$
2. for $t \leftarrow 0$ to $K - 1$ do
3. $x_{t+1/2} \leftarrow x_t - \frac{1}{L} \nabla f(x_t)$;
4. if $\|\nabla f(x_t)\| \geq \frac{\varepsilon}{2}$ then
5. $x_{t+1} \leftarrow x_{t+1/2}$; \quad necessarily $\nabla^2 f(x_t) \succeq -\delta I$
6. else $v \leftarrow \text{Neon2}_\text{det}(x_t, \delta, \frac{p}{2K})$;
8. if $v = \bot$ then return $x_t$; \quad necessarily $v^\top \nabla^2 f(x_t)v \leq -\delta/2$
9. else $x_{t+1} \leftarrow x_t \pm \frac{\delta}{L^2} v$;
10. end if
11. end for
12. will not reach this line (with probability $\geq 1 - p$).

**Theorem 5a.** With probability at least $1 - p$, Neon2+SGD outputs an $(\varepsilon, \delta)$-approximate local minimum in gradient complexity $T = \tilde{O}\left(\frac{1}{\varepsilon^2} + 1\left(\frac{L^2\Delta_f}{\varepsilon^2} + \frac{L\Delta_f}{\varepsilon^3}\right) + \frac{L^2}{\delta^2} \frac{L^2\Delta_f}{\varepsilon^4}\right)$.

**Corollary 6.3.** Treating $\Delta_f, V, L, L_2$ as constants, we have $T = \tilde{O}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^2\delta^2} + \frac{1}{\delta^3}\right)$.

One can similarly (and more easily) give an algorithm Neon2+GD which is the same as Neon2+SGD except that the mini-batch SGD is replaced with a full gradient descent, and the use of Neon2$\text{online}$ is replaced with Neon2$\text{det}$. We have the following theorem:

**Theorem 5c.** With probability at least $1 - p$, Neon2+GD outputs an $(\varepsilon, \delta)$-approximate local minimum in gradient complexity $\tilde{O}\left(\frac{L\Delta_f}{\varepsilon^2} + \frac{L^{1/2}\Delta_f}{\delta^2} + \frac{L^2}{\delta^2} \frac{L^2\Delta_f}{\varepsilon^4}\right)$ full gradient computations.

We only prove Theorem 5a in Appendix C and the proof of Theorem 5c is only simpler.
6.3 Neon2 on SCSG and SVRG

Background. We first recall the main idea of the SVRG method for non-convex optimization [3, 23]. It is an offline method but is what SCSG is built on. SVRG divides iterations into epochs, each of length $n$. It maintains a snapshot point $\bar{x}$ for each epoch, and computes the full gradient $\nabla f(\bar{x})$ only for snapshots. Then, in each iteration $t$ at point $x_t$, SVRG defines gradient estimator $\tilde{\nabla} f(x_t) \overset{\text{def}}{=} \nabla f_i(x_t) - \nabla f_i(\bar{x}) + \nabla f(\bar{x})$ which satisfies $\mathbb{E}_t[\tilde{\nabla} f(x_t)] = \nabla f(x_t)$, and performs update $x_{t+1} \leftarrow x_t - \alpha \tilde{\nabla} f(x_t)$ for learning rate $\alpha$.

The SCSG method of Lei et al. [18] proposed a simple fix to turn SVRG into an online method. They changed the epoch length of SVRG from $n$ to $B \approx 1/\varepsilon^2$, and then replaced the computation of $\nabla f(\bar{x})$ with $\frac{1}{|S|} \sum_{i \in S} \nabla f_i(\bar{x})$ where $S$ is a random subset of $[n]$ with cardinality $|S| = B$. To make this approach even more general, they also analyzed SCSG in the mini-batch setting, with mini-batch size $b \in \{1, 2, \ldots, B\}$. Their Theorem 3.1 [18] says that,

Lemma 6.4 ([18]). There exist constant $C > 1$ such that, if we run SCSG for an epoch of size $B$ (so using $O(B)$ stochastic gradients) with mini-batch $b \in \{1, 2, \ldots, B\}$ starting from a point $x_t$ and moving to $x_t^+$, then

$$\mathbb{E}[\|\nabla f(x_t^+)\|^2] \leq C \cdot L(b/B)^{1/3} (f(x_t) - \mathbb{E}[f(x_t^+)]) + \frac{6\gamma}{B}.$$

Our Approach. In principle, one can apply the same idea of Neon2 on SCSG to turn it into an algorithm finding approximate local minima. Unfortunately, this is not quite possible because the left hand side of Lemma 6.4 is on $\mathbb{E}[\|\nabla f(x_t^+)\|^2]$, as opposed to $\|\nabla f(x_t)\|^2$ in SGD (see (C.1)). This means, instead of testing whether $x_t$ is a good local minimum (as we did in Neon2+SGD), this time we need to test whether $x_t^+$ is a good local minimum. This creates some extra difficulty so we need a different proof.

Remark 6.5. As for the parameters of SCSG, we simply use $B = \max\{1, \frac{4\gamma V}{\varepsilon^2}\}$. However, choosing mini-batch size $b = 1$ does not necessarily give the best complexity, so a tradeoff $b = \Theta(\frac{(\varepsilon^2 + V)\varepsilon^4 L_2^2}{3^3 \delta^2})$ is needed. (A similar tradeoff was also discovered by the authors of Neon2 [28].) Note that this quantity $b$ may be larger than $B$, and if this happens, SCSG becomes essentially equivalent to one iteration of SGD with mini-batch size $b$. Instead of analyzing this boundary case $b > B$ separately, we decide to simply run Neon2+SGD whenever $b > B$ happens, to simplify our proof.

We show the following theorem (proved in Appendix C).

**Theorem 5b.** With probability at least $2/3$, Neon2+SCSG outputs an $(\varepsilon, \delta)$-approximate local minimum in gradient complexity $T = \widetilde{O}\left(\frac{\Delta_f}{\varepsilon^2} + \frac{L_2}{\delta^2}\right) \cdot \left(\frac{V}{\varepsilon^2} + \frac{L_2^2}{\delta^2}\right) + \frac{L_2}{\varepsilon} + \frac{L_2}{\delta}\right)$.

(To provide the simplest proof, we have shown Theorem 5b only with probability $2/3$. One can for instance boost the confidence to $1 - p$ by running $\log \frac{1}{p}$ copies of Neon2+SCSG.)

**Corollary 6.6.** Treating $\Delta_f, V, L, L_2$ as constants, we have $T = \widetilde{O}\left(\frac{\Delta_f}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{1}{\delta}\right)$.

---

7That is, they reduced the epoch length to $\frac{B}{2}$, and replaced $\nabla f_i(x_t) - \nabla f_i(\bar{x})$ with $\frac{1}{|S|} \sum_{i \in S'} (\nabla f_i(x_t) - \nabla f_i(\bar{x}))$ for some $S'$ that is a random subset of $[n]$ with cardinality $|S'| = b$.

8We remark that Lei et al. [18] only showed that an epoch runs in an expectation of $O(B)$ stochastic gradients. We assume it is exact here to simplify proofs. One can for instance stop SCSG after $O(B \log \frac{1}{p})$ stochastic gradient computations, and then Lemma 6.4 will succeed with probability $\geq 1 - p$. 

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Algorithm 6 Neon2+SCSG($f, x_0, \varepsilon, \delta$)

**Input:** function $f(\cdot)$, starting vector $x_0$, $\varepsilon > 0$ and $\delta > 0$.

1: $B \leftarrow \max\{1, \frac{48V}{\varepsilon^2}\};$ 
   $b \leftarrow \max\{1, \Theta\left(\frac{(\varepsilon^2 + V)^4 L_k^6}{\delta^4 L_4^3}\right)\};$
2: if $b > B$ then return Neon2+SGD($f, x_0, 2/3, \varepsilon, \delta$); ◇ for cleaner analysis purpose, see Remark 6.5
3: $K \leftarrow \Theta\left(\frac{Lb^{1/3}V}{\delta^{4/3}V}\right);$ ◇ $\Delta_f$ is any upper bound on $f(x_0) - \min_x \{f(x)\}$

4: for $t \leftarrow 0$ to $K - 1$ do
5: \hspace{0.5cm} $x_{t+1/2} \leftarrow$ apply SCSG on $x_t$ for one epoch of size $B = \max\{\Theta(V/\varepsilon^2), 1\};$
6: \hspace{0.5cm} if $\|\nabla f(x_{t+1/2})\| \geq \frac{\varepsilon}{2}$ then ◇ estimate $\|\nabla f(x_t)\|$ using $O(\varepsilon^{-2} V \log K)$ stochastic gradients
7: \hspace{0.5cm} $x_{t+1} \leftarrow x_{t+1/2};$
8: \hspace{0.5cm} else ◇ necessarily $\|\nabla f(x_{t+1/2})\| \leq \varepsilon$
9: \hspace{1cm} $v \leftarrow$ Neon2$^{\text{online}}$($f, x_{t+1/2}, \delta, 1/20K$);
10: \hspace{1cm} if $v = \perp$ then return $x_{t+1/2};$ ◇ necessarily $\nabla^2 f(x_{t+1/2}) \succeq -\delta I$
11: \hspace{1cm} else $x_{t+1} \leftarrow x_{t+1/2} \pm \frac{\delta}{L_2} v;$ ◇ necessarily $v^\top \nabla^2 f(x_{t+1/2})v \leq -\delta/2$
12: end if
13: end for
14: will not reach this line (with probability $\geq 2/3$).

As for SVRG, it is an offline method and its one-epoch lemma looks like\(^9\)

$$E[\|\nabla f(x_t^+\|^2] \leq C \cdot Ln^{1/3} \left(f(x_t) - E[f(x_t^+\right)] .$$

If one replaces the use of Lemma 6.4 with this new inequality, and replace the use of Neon2$^{\text{online}}$ with neon2$^{\text{svrg}}$, then we get the following theorem:

**Theorem 5d** With probability at least 2/3, Neon2+SVRG outputs an $(\varepsilon, \delta)$-approximate local minimum in gradient complexity $T = \tilde{O}\left((\frac{L_2^2\Delta_f L_k}{\varepsilon^2 n^{1/3}} + \frac{L_2^2\Delta_f}{\delta^4})(n + \frac{n^{3/4} L_2^4}{\sqrt{\delta}})\right)$.

For a clean presentation of this paper, we ignore the pseudocode and proof because they are only simpler than neon2+SCSG.

### 6.4 Neon2 on Natasha2 and CDHS

The recent results Carmon et al. \cite{CDHS} (that we refer to CDHS) and Natasha2 \cite{Natasha2} are both Hessian-free methods where the only Hessian-vector product computations come from the exact NC-search process we study in this paper. Therefore, by replacing their NC-search with Neon2, we can directly turn them into first-order methods without the necessity of computing Hessian-vector products.

We state the following two theorems where the proofs are exactly the same as the papers \cite{CDHS} and \cite{Natasha2}. We directly state them by assuming $\Delta_f, V, L, L_2$ are constants, to simplify our notions.

**Theorem 2.** One can replace Oja’s algorithm with Neon2$^{\text{online}}$ in Natasha2 without hurting its performance, turning it into a first-order stochastic method.

Treating $\Delta_f, V, L, L_2$ as constants, Natasha2 finds an $(\varepsilon, \delta)$-approximate local minimum in $T = \tilde{O}\left(\frac{1}{\varepsilon^2 \Delta_f^2} + \frac{L_2^4}{\delta^4} + \frac{1}{\delta^2}\right)$ stochastic gradient computations.

**Theorem 4.** One can replace Lanczos method with Neon2$^{\text{det}}$ or Neon2$^{\text{svrg}}$ in CDHS without hurting its performance, turning it into a first-order method.

\(^9\)There are at least three different variants of SVRG \cite{3, 18, 23}. We have adopted the lemma of \cite{18} for simplicity.
Treating $\Delta f, L, L_2$ as constants, CDHS finds an $(\varepsilon, \delta)$-approximate local minimum in either $\tilde{O}(\frac{1}{\varepsilon^{1.75}} + \ldots)$.

For each $s \in [T]$, we consider the random process defined as

$t \geq s: y_{t+1} = (1 - a_t)y_t, \quad y_s = b$

Consider random events

Lemma A.2. Proof.

We know that

$\mathbb{E}[a_t | \mathcal{F}_1, \ldots, \mathcal{F}_{t-1}] \leq -\lambda$.

Then, we have for every $p \in (0, 1)$: $\Pr \left[ x_T \geq T \varepsilon^{s} + 2\rho \sqrt{T \log \frac{T}{\rho}} \right] \leq p$.

Proof. We know that

$x_T \leq (1 - a_T)x_{T-1} + b$

\[\leq (1 - a_T)((1 - a_{T-1})x_{T-2} + b) + b = (1 - a_T)(1 - a_{T-1})x_{T-2} + (1 - a_T)b + b \leq \ldots \leq \sum_{s=2}^{T} \prod_{t=s}^{T}(1 - a_s)b\]

For each $s \in [T]$, we consider the random process defined as

$t \geq s: y_{t+1} = (1 - a_t)y_t, \quad y_s = b$
Therefore
\[ \log y_{t+1} = \log(1 - a_t) + \log y_t \]
For \( \log(1 - a_t) \in [2\rho, \rho] \) and \( \mathbb{E}[\log(1 - a_t) \mid \mathcal{F}_1, \cdots, \mathcal{F}_{t-1}] \leq \lambda \). Thus, we can apply Azuma-Hoeffding inequality on \( \log y_t \) to conclude that
\[
\Pr \left[ y_T \geq b e^{\lambda T + 2\rho \sqrt{T \log \frac{2}{p}}} \right] \leq \frac{1}{2}.
\]
Taking union bound over \( s \) we complete the proof. \( \square \)

### A.2 Proof of Lemma 3.1

**Proof of Lemma 3.1.** Let \( i_t \in [n] \) be the random index \( i \) chosen when computing \( x_{t+1} \) from \( x_t \) in Line 5 of \texttt{Neon2\textsuperscript{online weak}}. We will write the update rule of \( x_t \) in terms of the Hessian before we stop. By Lemma A.1, we know that for every \( t \geq 1 \),
\[
\| \nabla f_i(x_t) - \nabla f_i(x_0) - \nabla^2 f_i(x_0)(x_t - x_0) \|_2 \leq L_2 \| x_t - x_0 \|_2.
\]
Therefore, there exists error vector \( \xi_t \in \mathbb{R}^d \) with \( \| \xi_t \|_2 \leq L_2 \| x_t - x_0 \|_2 \) such that
\[
(x_{t+1} - x_0) = (x_t - x_0) - \eta \nabla^2 f_i(x_0)(x_t - x_0) + \eta \xi_t.
\]
For notational simplicity, let us denote by
\[
z_t \overset{\text{def}}{=} x_t - x_0, \quad A_t \overset{\text{def}}{=} B_t + R_t \quad \text{where} \quad B_t \overset{\text{def}}{=} \nabla^2 f_i(x_0), \quad R_t \overset{\text{def}}{=} -\frac{\xi_t z_t^\top}{\|z_t\|_2^2}
\]
then it satisfies
\[
z_{t+1} = z_t - \eta B_t z_t + \eta \xi_t = (I - \eta A_t) z_t.
\]
We have \( \|R_t\|_2 \leq L_2 \|z_t\|_2 \leq L_2 \cdot r \). By the \( L \)-smoothness of \( f_i \), we know \( \|B_t\|_2 \leq L \) and thus \( \|A_t\|_2 \leq \|B_t\|_2 + \|R_t\|_2 \leq \|B_t\|_2 + 2L r \leq 2L \).

Now, define \( \Phi_t \overset{\text{def}}{=} z_{t+1} z_{t+1}^\top = (I - \eta A_t) \cdots (I - \eta A_1) \xi_t z_t^\top (I - \eta A_1) \cdots (I - \eta A_t) \) and \( w_t \overset{\text{def}}{=} \frac{z_t}{\|z_t\|_2} \) from \( (\text{Tr}(\Phi_{t-1}))^{1/2} \). Then, before we stop, we have:
\[
\text{Tr}(\Phi_t) = \text{Tr}(\Phi_{t-1}) \left( 1 - 2\eta w_t^\top A_t w_t + \eta^2 w_t^\top A_t^2 w_t \right)
\]
\[
\leq \text{Tr}(\Phi_{t-1}) \left( 1 - 2\eta w_t^\top A_t w_t + 4\eta^2 L^2 \right)
\]
\[
\leq \text{Tr}(\Phi_{t-1}) \left( 1 - 2\eta w_t^\top B_t w_t + 2\eta \|R_t\|_2 + 4\eta^2 L^2 \right)
\]
\[
\leq \text{Tr}(\Phi_{t-1}) \left( 1 - 2\eta w_t^\top B_t w_t + 8\eta^2 L^2 \right).
\]
Above, \( \Omega \) is because our choice of parameter satisfies \( r \leq \frac{L^2}{8\eta} \). Therefore,
\[
\log \left( \text{Tr}(\Phi_t) \right) \leq \log \left( \text{Tr}(\Phi_{t-1}) \right) + \log \left( 1 - 2\eta w_t^\top B_t w_t + 8\eta^2 L^2 \right).
\]
Letting \( \lambda = -\lambda_{\text{min}}(\nabla^2 f_i(x_0)) = -\lambda_{\text{min}}(\mathbb{E}[B_t | B_t], B_t) \), since the randomness of \( B_t \) is independent of \( w_t \), we know that \( w_t^\top B_t w_t \in [-L, L] \) and for every \( w_t \), it satisfies \( \mathbb{E}_B \left[ w_t^\top B_t w_t \mid w_t \right] \geq -\lambda \). Which (by concavity of \( \log \) also implies that \( \mathbb{E}[\log(1 - 2\eta w_t^\top B_t w_t + 8\eta^2 L^2)] \leq 2\eta \lambda \) and \( \log(1 - 2\eta w_t^\top B_t w_t + 8\eta^2 L^2) \in [-2(2\eta L + 8\eta^2 L^2), 2\eta L + 8\eta^2 L^2] \) \in [-6\eta L, 12\eta L].

Hence, applying Azuma-Hoeffding inequality on \( \log(\Phi_t) \) we have
\[
\Pr \left[ \log(\Phi_t) - \log(\Phi_0) \geq 2\eta \lambda + 16\eta L \sqrt{t \log \frac{1}{p}} \right] \leq p.
\]
In other words, with probability at least $1 - p$, Neon2\textsuperscript{online} will not terminate until $t \geq T_0$, where $T_0$ is given by the equation (recall $\Phi_0 = \|z_1\|^2 = \sigma^2$):

$$2\eta \lambda T_0 + 16\eta L \sqrt{T_0 \log \frac{1}{p}} = \log \left( \frac{r^2}{\sigma^2} \right).$$

Next, we turn to accuracy. Let “true” vector $v_{t+1} \equiv (I - \eta B_t) \cdots (I - \eta B_1)\xi$ and we have

$$z_{t+1} - v_{t+1} = \prod_{s=1}^t (I - \eta A_s)\xi - \prod_{s=1}^t (I - \eta B_s)\xi = (I - \eta B_t)(z_t - v_t) - \eta R_z z_t.$$

Thus, if we call $u_t \equiv z_t - v_t$ with $u_1 = 0$, then, before the algorithm stops, we have:

$$\|u_{t+1} - (I - \eta B_t)u_t\|_2 \leq \eta \|R_z z_t\|_2 \leq \eta L 2^{-r^2}.$$

Using Young’s inequality $\|a + b\|^2 \leq (1 + \beta)\|a\|^2 + \left(\frac{1}{\beta} + 1\right)\|b\|^2$ for every $\beta > 0$, we have:

$$\|u_{t+1}\|^2 \leq (1 + \eta^2) \|(I - \eta B_t)u_t\|^2 + 8L^2 2^{-4} \leq \|u_t\|^2 \left(1 - \eta \frac{\|B_t^u u_t\|}{\|u_t\|_2} + 4\eta^2 L^2\right) + 8L^2 2^{-4}.$$

Above, $\|\|$ assumes without loss of generality that $L \geq 1$ (as otherwise we can re-scale the problem).

Therefore, applying martingale concentration Lemma A.2, we know

$$\text{Pr} \left[ \|u_t\|_2 \geq 16L 2^{-r^2} \epsilon \eta \lambda t + 8L \sqrt{t \log \frac{1}{p}} \right] \leq p.$$

Now we can apply the recent of Oja’s algorithm — [6, Theorem 4]. By our choice of parameter $\eta$, we have: with probability at least $99/100$ the following holds:

1. Norm growth: $\|v_t\|_2 \geq c(\eta \lambda - 2\eta^2 L^2)^t \sigma / d$.
2. Negative curvature: $\frac{u_t^T \nabla^2 f(x) u_t}{\|u_t\|^2} \leq -(1 - 2\eta)\lambda + O \left(\frac{\log(d)}{\eta t}\right)$.

Then let us consider the case: $\lambda \geq \delta$, let us consider a fixed $T_1$ defined as

$$T_1 \equiv \frac{\log \frac{2r}{\sigma}}{\eta \lambda - 2\eta^2 L^2} = \frac{C_0 \left(\log d/p + \log(2d)\right)}{\eta \lambda - 2\eta^2 L^2} < T.$$

At this point, by the “norm growth” property, we know that w.p. $\geq 99/100$, $\|v_{T_1}\|_2 \geq 2r$ and by our choice of parameters, we know that

$$e^{\eta \lambda T_1 + 2\eta L \sqrt{T_1 \log \frac{1}{p}}} \leq \left(\frac{r}{\sigma}\right)^2.$$

which implies that with probability at least $98/100$,

$$\frac{\|u_{T_1}\|_2}{\|v_{T_1}\|_2} \leq \frac{16dL 2^{-r^2} T_1 e^{\eta \lambda T_1 + 8\eta L \sqrt{T_1 \log \frac{T_1}{p}}} e^{(\eta \lambda - 2\eta^2 L^2)T_1 \sigma}}{\sigma} \leq \frac{16dL 2^{-r^2} T_1 e^{8\eta L \sqrt{T_1 \log \frac{T_1}{p}} + 2\eta^2 L^2 T_1}}{\sigma} \leq \frac{16dL 2^{-r^2} T_1 e^{16 \log \frac{T_1}{p}}}{\sigma^2} \leq \frac{16dL 2^{-r^2} T_1}{\sigma p} \leq \frac{\delta}{100L} \leq \frac{1}{100}.$$

Here, $\odot$ uses our choice of parameters so $\eta^2 L^2 T_1 \log \frac{T_1}{p} \leq \frac{L^2 \delta}{\lambda} \cdot \left(\log \frac{2r}{\sigma} \log \frac{T_1}{p}\right) \leq \log \frac{T_1}{p}$. $\odot$
is due to \( \frac{\nu^2}{\sigma} \leq \frac{\delta^2 p}{3200d^2L^2 \log \frac{d}{p}} \leq \frac{\delta p}{1600dT_1L^2} \). Thus, with probability at least 98/100, \( \|z_{T_1}\|_2 = \|v_{T_1} + u_{T_1}\|_2 \geq r \). This means \( \text{Neon2}_{\text{weak}} \) must terminate within \( T_1 \leq T \) iterations.

Moreover, recall with probability at least 1 - \( p \), \( \text{Neon2}_{\text{weak}} \) will not terminate before iteration \( T_0 \). Thus, at the point of \( t \in [T_0, T_1] \) of termination, we have with probability at least 98/100,
\[
\frac{v_t^T A v_t}{\|v_t\|_2^2} \leq -(1 - 2\eta)\lambda + O\left( \frac{\log(d)}{\eta} \right) \leq -(1 - 2\eta)\lambda + O\left( \frac{\log(d)}{\eta T_0} \right) \leq \frac{15}{16} \delta .
\]

Moreover, we also know that \( \|u_t + v_t\|_2 = \|z_t\|_2 \geq r \), therefore,
\[
\frac{\|u_t\|_2}{\|u_t + v_t\|_2} \leq \frac{16L^2r^2T_1 e^{\eta T_1 + 2\eta L_2} \sqrt{T_1 \log \frac{1}{p}}}{r} .
\]

By definition of \( T_1 \), we know that \( r = e^{(\eta - 2\eta L_2 T_1)} \sigma (2d) \), so we can show that \( \frac{\|u_t\|_2}{\|v_t\|_2} \leq \frac{1}{100} \) as before. Together, we have:
\[
\frac{z_t^T A z_t}{\|z_t\|_2^2} = \frac{v_t^T A v_t}{\|z_t\|_2^2} \cdot \frac{z_t^T A z_t}{\|v_t\|_2^2} \leq \frac{3}{4} \frac{v_t^T A v_t + 4L^2 \|z_t - v_t\|_2 \|v_t\|_2}{\|v_t\|_2^2}
\leq \frac{3}{4} \left( \frac{v_t^T A v_t}{\|v_t\|_2^2} + \frac{4L^2 \|u_t\|_2}{\|v_t\|_2} \right) \leq \frac{3}{4} \left( \frac{v_t^T A v_t}{\|v_t\|_2^2} + \frac{1}{4} \delta \right) \leq -\frac{51}{100} \delta .
\]

Putting everything together we complete the proof.

\[\square\]

A.3 Proof of Lemma 3.2

**Proof of Lemma 3.2** By Lemma A.1, we know for every \( i \in [n] \),
\[
\|v^T (\nabla f_i(x + v) - \nabla f_i(x) - \nabla^2 f_i(x) v)\|_2 \leq L_2 \|v\|_2^3 .
\]

Letting \( z_j = v^T (\nabla f_j(x + v) - \nabla f_j(x)) \), we know that \( z_1, \cdots, z_m \) are i.i.d. random variables with \( |z_j| \leq L \|v\|_2 + L_2 \|v\|_2^3 \). By Chernoff bound, we know that
\[
\Pr \left[ |z - \mathbb{E}[z]| \geq 2 \left( L \|v\|_2^2 + L_2 \|v\|_2^3 \right) \sqrt{\frac{1}{m} \log \frac{1}{p}} \right] \leq p .
\]

Since we also have \( |\mathbb{E}[z] - v^T \nabla^2 f(x) v| \leq L_2 \|v\|_3^3 \) from Lemma A.1, we conclude that
\[
\Pr \left[ \left| \frac{z}{\|v\|_2^2} - \frac{v^T \nabla^2 f(x) v}{\|v\|_2^2} \right| \leq 2 (L + L_2 \|v\|_2) \sqrt{\frac{1}{m} \log \frac{1}{p}} + L_2 \|v\|_2 \right] \geq 1 - p .
\]

Plugging in our assumption on \( \|v\|_2 \) and our choice of \( m \) finishes the proof.

\[\square\]

B Missing Proofs for Section 4

B.1 Stable Computation of Chebyshev Polynomials

We recall the following result from [5, Section 6.2] regarding one way to *stably* compute Chebyshev polynomials. Suppose we want to compute
\[
\tilde{s}_N \overset{\text{def}}{=} \sum_{k=0}^N T_k(M) \tilde{c}_k \in \mathbb{R}^d \quad \text{where } M \in \mathbb{R}^{d \times d} \text{ is symmetric and each } \tilde{c}_k \text{ is in } \mathbb{R}^d . \quad (B.1)
\]
Definition B.1 (inexact backward recurrence). Let $\mathcal{M}$ be an approximate algorithm that satisfies $\|\mathcal{M}(u) - Mu\|_2 \leq \varepsilon\|u\|_2$ for every $u \in \mathbb{R}^d$. Then, define inexact backward recurrence to be

\[
\hat{b}_{N+1} \overset{\text{def}}{=} 0, \quad \hat{b}_N \overset{\text{def}}{=} c_N, \quad \text{and} \quad \forall r \in \{N-1, \ldots, 0\}: \hat{b}_r \overset{\text{def}}{=} 2\mathcal{M}(\hat{b}_{r+1}) - \hat{b}_{r+2} + c_r \in \mathbb{R}^d,
\]

and define the output as $\hat{s}_N \overset{\text{def}}{=} \hat{b}_0 - \mathcal{M}(\hat{b}_1)$.

If $\varepsilon = 0$, then $\hat{s}_N = \bar{s}_N$. The following theorem gives an error analysis [5, Theorem 6.4].

Theorem B.2 (stable Chebyshev sum). For every $N \in \mathbb{N}^*$, suppose the eigenvalues of $\mathcal{M}$ are in $[a, b]$ and suppose there are parameters $C_U \geq 1, C_T \geq 1, \rho \geq 1, C_c \geq 0$ satisfying

\[
\forall k \in \{0, 1, \ldots, N\}: \left\{ \rho^k \|c_k\| \leq C_c \quad \wedge \quad \forall x \in [a, b]: \ |T_k(x)| \leq C_T \rho^k \text{ and } |U_k(x)| \leq C_U \rho^k \right\}.
\]

Then, if the inexact backward recurrence in Def. B.1 is applied with $\varepsilon \leq \frac{1}{1NC_T}$, we have

\[
\|\hat{s}_N - \bar{s}_N\| \leq \varepsilon \cdot 2(1 + 2NC_T)NC_U C_c.
\]

B.2 Proof of Theorem 3

Proof of Theorem 3. We can without loss of generality assume $\delta \leq L$. For notation simplicity, let us denote

\[
A \overset{\text{def}}{=} \nabla^2 f(x_0), \quad \mathcal{M} \overset{\text{def}}{=} \left( -\frac{1}{L} \nabla^2 f(x_0) + \left( 1 - \frac{3\delta}{4L} \right) \mathbf{I} \right) \quad \text{and} \quad \lambda \overset{\text{def}}{=} -\lambda_{\min}(A).
\]

Then, we know that the eigenvalues of $\mathcal{M}$ lie in $[-1, 1 + \lambda^{-3\delta/4}]$.

We wish to iteratively compute $x_{t+1} \approx x_0 + T_t(\mathcal{M}) \xi$, where $T_t$ is the $t$-th Chebyshev polynomial of the first kind. However, we cannot multiply $\mathcal{M}$ to vectors (because we are not allowed to use Hessian-vector products). We define

\[
\mathcal{M}(y) \overset{\text{def}}{=} -\frac{1}{L} \left( \nabla f(x_0 + y) - \nabla f(x_0) \right) + \left( 1 - \frac{3\delta}{4L} \right) y.
\]

and shall use it to approximate $\mathcal{M}y$ and then apply backward recurrence

\[
y_0 = 0, \quad y_1 = \xi, \quad y_t = 2\mathcal{M}(y_{t-1}) - y_{t-2}.
\]

If we set $x_{t+1} = x_0 + y_{t+1} - \mathcal{M}(y_t)$, following Def. B.1, it satisfies $x_{t+1} - x_0 \approx T_t(\mathcal{M}) \xi$.

Now, letting $x_{t+1}^* \overset{\text{def}}{=} x_0 + T_t(\mathcal{M}) \xi$ be the exact solution, we wish to bound the error $\|x_{t+1} - x_{t+1}^*\|_2$.

Throughout the iterations of neon2, we have

\[
y_t = 2\mathcal{M}(y_{t-1}) - y_{t-2} = 2(x_0 - x_t + y_t) - y_{t-2} \implies y_t - y_{t-2} = 2(x_t - x_0).
\]

Since we have $\|x_t - x_0\|_2 \leq \rho r$ for each $t$ before termination, we know $\|y_t\|_2 \leq 2tr$. Using this upper bound we can approximate Hessian-vector product by gradient difference. Lemma A.1 gives us

\[
\|\mathcal{M}(y_t) - \mathcal{M}y_t\|_2 \leq \frac{L_2}{L}\|y_t\|_2^2 \leq \frac{2L_2 r t}{L}\|y_t\|_2.
\]

Now, recall from Claim 4.1 that

\[
T_t(x) \in \begin{cases} [-1, 1] & \text{if } x \in [-1, 1]; \\ \left[ \frac{1}{2} \left( x + \sqrt{x^2 - 1} \right)^t, \left( x + \sqrt{x^2 - 1} \right)^t \right] & \text{if } x > 1. \end{cases}
\]

We can apply Theorem B.2 with the eigenvalues of $\mathcal{M}$ in $[a, b] = [0, 1 + \lambda^{-3\delta/4}]$ and

\[
\rho \overset{\text{def}}{=} \max \left\{ 1 + \frac{\lambda - 3\delta/4}{L}, \sqrt{\frac{2}{L} \lambda - 3\delta/4} + \frac{(\lambda - 3\delta/4)^2}{L^2}, 1 \right\}, \quad C_c = \rho^4 \sigma, \quad C_T = C_U = 2.
\]
Theorem B.2 tells us that, for every \( t \) before termination,
\[
\|x^*_{t+1} - x_{t+1}\|^2 \leq \frac{32L^2r^3\rho^t\sigma}{L}.
\]

In order to prove Theorem 3 in the rest of the proof, it suffices for us to show that, if \( \lambda_{\min}(\nabla^2 f(x_0)) \leq -\delta \), then with probability at least \( 1 - p \), it satisfies \( v \neq \perp, \|v\|_2 = 1 \), and \( v^T \nabla^2 f(x_0)v \leq -\frac{1}{2}\delta \). In other words, we can assume \( \lambda \geq \delta \).

The value \( \lambda \geq \delta \) implies \( \rho \geq 1 + \sqrt{\delta} > 1 \), so we can let
\[
T_1 \overset{\text{def}}{=} \frac{\log \frac{4dr}{\rho \sigma}}{\log \rho} \leq T.
\]

By [Claim 4.1], we know that \( \|\mathcal{T}_1(M)\|_2 \geq \frac{1}{2}\rho T_1 = \frac{2dr}{p\sigma} \). Thus, with probability at least \( 1 - p \),
\[
\|x^*_{T_1+1} - x_0\|_2 = \|\mathcal{T}_1(M)\|_2 \geq 2r.
\]
Moreover, at iteration \( T_1 \), we have:
\[
\|x^*_{T_1+1} - x_{T_1}\|_2 \leq \frac{32L_2\rho T_1^3\rho T_1\sigma}{L} \leq \frac{32L_2\rho T_1^3\rho T_1\sigma}{L} \leq \frac{4dr}{p\sigma} \leq \frac{128dL_2T_1^3\rho^2}{p} \leq \frac{\delta}{100} \rho \leq \frac{1}{16} r.
\]

Here, \( \Theta \) uses the fact that \( r \leq \frac{\delta p}{12800dL_2T_1^3} \).

This means \( \|x_{T_1+1} - x_0\|_2 \geq r \) so the algorithm must terminate before iteration \( T_1 \leq T \).

On the other hand, since \( \|\mathcal{T}_t(M)\|_2 \leq \rho^t \), we know that the algorithm will not terminate until \( t \geq T_0 \) for
\[
T_0 \overset{\text{def}}{=} \log \frac{r}{2\sigma} \log \rho.
\]

At the time \( t \geq T_0 \) of termination, by the property of Chebyshev polynomial [Claim 4.1], we know
1. \( T_t(\rho) \geq \frac{1}{2}\rho^t \geq \frac{1}{2}T_0 \geq \frac{r}{2\sigma} = (d/p)^\Theta(1) \).
2. \( \forall x \in [-1, 1], T_t(x) \in [-1, 1] \).

Since all the eigenvalues of \( A \) that are \( \geq -3/4\delta \) are mapped to the eigenvalues of \( M \) that are in \([-1, 1] \), and the smallest eigenvalue of \( A \) is mapped to the eigenvalue \( \rho \) of \( M \). So we have, with probability at least \( 1 - p \), letting \( v_t \overset{\text{def}}{=} x^*_{t+1} - x_0 \) it satisfies
\[
v_t^T A v_t \leq \frac{1}{8} \delta.
\]

Therefore, denoting by \( z_t \overset{\text{def}}{=} x_{t+1} - x_0 \), we have
\[
\frac{z_t^T A z_t}{\|z_t\|^2} = \frac{\|v_t\|^2}{\|z_t\|^2} \cdot \frac{z_t^T A z_t}{\|v_t\|^2} \leq \frac{15}{16} \left( \frac{1}{\|v_t\|^2} + 4L \frac{\|z_t - v_t\|_2}{\|v_t\|_2} \right) \leq \frac{15}{16} \left( \frac{1}{\|v_t\|^2} + \frac{1}{25} \right) \leq -\frac{51}{100} \delta.
\]

This finishes the proof because we have shown that, with probability at least \( 1 - p \), the output \( v = \frac{z_t}{\|z_t\|} \) satisfies and \( v^T \nabla^2 f(x_0)v \leq -\frac{1}{2}\delta \). \( \square \)
C Missing Proofs for Section 6

C.1 Proof of Theorem 5a

Proof of Theorem 5a Since both estimating $\|\nabla f(x_t)\|$ in Line 5 (see Claim 6.1) and invoking Neon2 online (see Theorem 1) succeed with high probability, we can assume that they always succeed. This means whenever we output $x_t$ in an iteration, it already satisfies $\|\nabla f(x_t)\| \leq \varepsilon$ and $\nabla^2 f(x_t) \succeq -\delta I$.

Therefore, it remains to show that the algorithm must output some $x_t$ in an iteration, as well as to compute the final complexity.

Recall (from classical SGD theory) if we update $x_{t+1/2} \leftarrow x_t - \frac{\alpha}{|S|} \sum_{i \in S} \nabla f_i(x_t)$ where $\alpha > 0$ is the learning rate and $|S| = B$, then

$$f(x_t) - \mathbb{E}_S[f(x_{t+1/2})] \geq \alpha \|\nabla f(x_t)\|^2 - \frac{\alpha^2 L}{2} \mathbb{E}_S \left[ \frac{1}{|S|} \sum_{i \in S} \nabla f_i(x_t) \right]^2.$$

Above, $\Theta$ is due to the smoothness of $f(\cdot)$ and $\Omega$ is due to Fact 2.2. Now, if we choose $\alpha = \frac{1}{L}$ and $B = \max\{\frac{8L}{\varepsilon^2}, 1\}$, then we have

$$f(x_t) - \mathbb{E}_S[f(x_{t+1/2})] \geq \frac{\alpha}{2} \left(\|\nabla f(x_t)\|^2 - \frac{\varepsilon^2}{8}\right). \quad (C.1)$$

In other words, as long as Line 6 is reached, we have $f(x_t) - \mathbb{E}[f(x_{t+1})] \geq \Omega(\varepsilon^2/L)$. On the other hand, whenever Line 10 is reached, then we must have $v^\top \nabla^2 f(y_0)v \leq -\frac{\delta}{2}$. By Claim 6.2, we must have $f(x_t) - \mathbb{E}[f(x_{t+1})] \geq \Omega(\delta^3/L_2^3)$.

In sum, if we choose $K = O\left(\frac{L_2^3 \Delta f}{\delta^2} + \frac{L_2^3 \Delta f}{\varepsilon^2}\right)$, then the algorithm must terminate and return $x_t$ in one of its iterations. This ensures that Line 13 will not be reached. As for the total complexity, we note that each iteration of Neon2+SGD is dominated by $O(K) = \tilde{O}(\frac{V}{\varepsilon^2} + 1)K$, as well as $\tilde{O}(\frac{L_2^3 \Delta f}{\delta^3})$ stochastic gradient computations by Neon2 online, but the latter will not happen for more than $O\left(\frac{L_2^3 \Delta f}{\delta^3}\right)$ times. Therefore, the total gradient complexity is

$$\tilde{O}\left(\frac{V}{\varepsilon^2} + 1\right)K + \frac{L_2^3 \Delta f}{\delta^2} = \tilde{O}\left(\frac{V}{\varepsilon^2} + 1\right)\left(\frac{L_2^3 \Delta f}{\delta^2} + \frac{L_2^3 \Delta f}{\varepsilon^2}\right) + \frac{L_2^3 \Delta f}{\delta^3}.$$

\[ \square \]

C.2 Proof of Theorem 5b

Proof of Theorem 5b. We first note in the special case $b = \Theta\left(\frac{(\varepsilon^2 + V)\varepsilon^4 L_2^3}{\delta^3}\right) \geq B$, or equivalently $\delta^3 \leq O\left(\frac{L_2^3 \varepsilon^4 L_2^3}{b}\right)$. Theorem 5a gives us gradient complexity $T = \tilde{O}\left(\frac{V L_2^3 \Delta f}{\varepsilon^2} + \frac{L_2^3 \Delta f}{\delta^2} + \frac{L_2^3 \Delta f}{\delta^3}\right)$ so we are done.

Therefore, in the rest of the proof we assume $\Theta\left(\frac{(\varepsilon^2 + V)\varepsilon^4 L_2^3}{\delta^3}\right) < B$ and thus $b \leq B$ is well defined. Since both estimating $\|\nabla f(x_t)\|$ in Line 6 (see Claim 6.1) and invoking Neon2 online (see Theorem 1) succeed with high probability, we can assume that they always succeed. This means whenever we output $x_{t+1/2}$ in an iteration, it already satisfies $\|\nabla f(x_{t+1/2})\| \leq \varepsilon$ and $\nabla^2 f(x_{t+1/2}) \succeq -\delta I$.

Therefore, it remains to show that the algorithm must output some $x_t$ in an iteration, as well as to compute the final complexity.
For analysis purpose, let us assume that, whenever the algorithm reaches \( v = \bot \) (in Line 10), it does not immediately halt and instead sets \( x_{t+1} = x_{t+1/2} \). This modification ensures that random variables \( x_t \) and \( x_{t+1/2} \) are well defined for \( t = 0, 1, \ldots, K - 1 \).

Let \( N_1 \) and \( N_2 \) respectively be the number of times we reach Line 9 (so we invoke \( \text{Neon2} \) and the number of times we reach Line 11 (so we update \( x_{t+1} = x_{t+1/2} \pm \frac{\delta}{L_2} v \)). Both \( N_1 \) and \( N_2 \) are random variables and it satisfies \( N_1 \geq N_2 \). To prove that \( \text{Neon2} + \text{SCSG} \) outputs in an iteration, we need to prove \( N_1 > N_2 \) holds at least with constant probability.

Let us apply the key SCSG lemma (Lemma 6.4) for an epoch with size \( B = \max\{1, \frac{48\gamma}{\varepsilon^2} \} \) and mini-batch size \( b \geq 1 \), we have

\[
\mathbb{E}[\|\nabla f(x_{t+1/2})\|^2] \leq C \cdot L (b/B)^{1/3} (f(x_t) - \mathbb{E}[f(x_{t+1/2})]) + \frac{\varepsilon^2}{8}. \tag{C.2}
\]

Now,

- if \( \|\nabla f(x_{t+1/2})\| \geq \frac{\varepsilon}{2} \) (so Line 7 is reached), we have \( x_{t+1} = x_{t+1/2} \);
- if \( v = \bot \) holds (so Line 9 is reached), we have virtually set \( x_{t+1} = x_{t+1/2} \) for analysis purpose;
- if \( v \neq \bot \) (so Line 11 is reached), we have \( \frac{\delta^3}{12L_2^2} \leq f(x_{t+1/2}) - \mathbb{E}[f(x_{t+1})] \).

Note that the third case \( v \neq \bot \) happens for \( N_2 \) times. Therefore, combining the three cases together with (C.2), we have

\[
\frac{B^{1/3}}{CLb^{1/3}} \mathbb{E}\left[ \sum_{t=0}^{K-1} (\|\nabla f(x_{t+1/2})\|^2 - \frac{\varepsilon^2}{8}) \right] + \frac{\delta^3}{12L_2^2} \cdot \mathbb{E}[N_2] \leq \Delta_f.
\]

On one hand, since we have chosen \( K \) such that \( K \geq \Omega(\frac{L b^{1/3} \Delta f}{\varepsilon^2 b^{1/3}}) = \Omega(\frac{L b^{1/3} \Delta f}{(1+\gamma)/\varepsilon^2}) \), by Markov bound (ignoring \( \mathbb{E}[N_2] \)), with probability at least \( 5/6 \), it satisfies \( \sum_{t=0}^{K-1} \|\nabla f(x_{t+1/2})\|^2 \leq \frac{\gamma^2}{4} K \). As a consequence, at least half of the indices \( t = 0, 1, \ldots, K - 1 \) will satisfy \( \|\nabla f(x_{t+1/2})\| \leq \frac{\varepsilon}{2} \). This means we have \( N_1 \geq K/2 \).

On the other hand, we have \( \frac{\delta^3}{12L_2^2} \cdot \mathbb{E}[N_2] \leq \Delta_f + \frac{KB^{1/3} \varepsilon^2}{6CLb^{1/3}} \). Since \( K \geq \Omega(\frac{L b^{1/3} \Delta f}{\varepsilon^2 b^{1/3}}) = \Omega(\frac{L b^{1/3} \Delta f}{(1+\gamma)/\varepsilon^2}) \), we have \( \frac{\delta^3}{12L_2^2} \cdot \mathbb{E}[N_2] \leq \frac{KB^{1/3} \varepsilon^2}{4CLb^{1/3}} \). As long as \( B \leq O\left(\frac{\delta^3 L^2}{L_2^3} \right) \) or equivalently \( b \geq \Omega\left(\frac{(\varepsilon^2 + \gamma)^3 L_2^3}{\delta^3 L^2} \right) \), we have \( \mathbb{E}[N_2] < K/12 \). Therefore, with probability at least \( 5/6 \), it satisfies \( N_2 < K/2 \).

Since \( N_1 > N_2 \), this means the algorithm must terminate and output some \( x_{t+1/2} \) in an iteration, with probability at least \( 2/3 \).

Finally, the per-iteration complexity of \( \text{Neon2} + \text{SCSG} \) is dominated by \( \tilde{O}(B) \) stochastic gradient computations per iteration for both SCSG and estimating \( \|\nabla f(x_{t+1/2})\| \), as well as \( \tilde{O}(L^2/\delta^2) \) for invoking \( \text{Neon2} \). This totals to gradient complexity

\[
\tilde{O}\left(K\left(B + \frac{L^2}{\delta^2}\right)\right) = \tilde{O}\left(\frac{L b^{1/3} \Delta f}{\varepsilon^2 (4/3)^{1/3}} \left(\frac{\gamma}{\varepsilon^2} + L^2 \frac{\Delta f}{\delta^2}\right)\right)
\]

\[
= \tilde{O}\left(\frac{L b^{1/3} \Delta f}{(4/3)^{1/3} \varepsilon} \left(\frac{\gamma}{\varepsilon^2} + \frac{L^2}{\delta^2}\right) + L \Delta f \frac{L^2}{\delta^2}\right)
\]

\[
= \tilde{O}\left(\frac{L \Delta f}{(4/3)^{1/3} \varepsilon} + \frac{L^2 \Delta f}{\delta^2} \right) \left(\frac{\gamma}{\varepsilon^2} + \frac{L^2}{\delta^2}\right) + \frac{L \Delta f}{(4/3)^{1/3} \varepsilon} \left(\frac{\gamma}{\varepsilon^2} + \frac{L^2}{\delta^2}\right) \right). \quad \square
\]
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