Tensor product multiplicities
for crystal bases of extremal weight modules
over quantum infinite rank affine algebras
of types $B_{\infty}$, $C_{\infty}$, and $D_{\infty}$

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Abstract
Using Lakshmibai-Seshadri paths, we give a combinatorial realization of the crystal
basis of an extremal weight module of integral extremal weight over the quantized uni-
versal enveloping algebra associated to the infinite rank affine Lie algebra of type $B_{\infty}$,
$C_{\infty}$, or $D_{\infty}$. Moreover, via this realization, we obtain an explicit description (in terms
of Littlewood-Richardson coefficients) of how tensor products of these crystal bases
decompose into connected components when their extremal weights are of nonnegative
levels. These results, in types $B_{\infty}$, $C_{\infty}$, and $D_{\infty}$, extend the corresponding results due
to Kwon, in types $A_{+\infty}$ and $A_{\infty}$; our results above also include, as a special case, the
corresponding results (concerning crystal bases) due to Lecouvey, in types $B_{\infty}$, $C_{\infty}$,
and $D_{\infty}$, where the extremal weights are of level zero.

1 Introduction.
Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra over $\mathbb{C}(q)$
associated to the infinite rank affine Lie algebra $\mathfrak{g}$ of type $A_{+\infty}$, $A_{\infty}$, $B_{\infty}$, $C_{\infty}$, or $D_{\infty}$
with Cartan subalgebra $\mathfrak{h} = \bigoplus_{i \in I} \mathbb{C} h_i$ and integral weight lattice
$P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \subset \mathfrak{h}^*$, where $I$ is the (infinite) index set
for the simple roots. In [Kw2], [Kw3], Kwon studied the crystal basis $B(\lambda)$ of the extremal
weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of extremal weight $\lambda \in P$, in the cases that $\mathfrak{g}$ is of type $A_{+\infty}$
and type $A_{\infty}$; in these papers, he gave a combinatorial realization of the crystal basis $B(\lambda)$
for $\lambda \in P$ of level zero (see also [Kw1] §4.1 for the case of dominant $\lambda \in P$), by using
semistandard Young tableaux with entries in the crystal basis of the vector representation of
$U_q(\mathfrak{g})$, and furthermore described explicitly (in terms of Littlewood-Richardson coefficients) how the tensor product $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ decomposes into connected components when $\lambda, \mu \in P$ are of nonnegative levels. Also, in [Ko1], [Ko2], we obtain an explicit description of this multiplicity in terms of Littlewood-Richardson coefficients (not depending on the types $A_{\infty}, B_{\infty}, C_{\infty},$ and $D_{\infty}$); these modules will surely turn out to be extremal weight $U_q(\mathfrak{g})$-modules of extremal weight of level zero.

In this paper, we extend the results above of [Kw2], [Kw3] (in types $A_{\infty}$ and $A_{\infty}$) to the cases of type $B_{\infty}, C_{\infty},$ and $D_{\infty}$ in such a way that the results above (concerning crystal bases) of [Le] are also included. We emphasize that our approach is quite different from those in [Kw2], [Kw3], and [Le], where some double crystals (or bicrystals) play essential roles; we obtain our results entirely within the framework of Littelmann’s path model, which enable us to give a unified proof in all types $B_{\infty}, C_{\infty},$ and $D_{\infty}$. In fact, we could also have included the cases of type $A_{\infty}$ and $A_{\infty}$, but we do not dare do so, since this would make our notation more complicated and since these cases are already treated in [Kw2], [Kw3].

Let us explain our results more precisely. We set $I := \mathbb{Z}_{\geq 0}$, and $[m] := \{0,1,\ldots, m\}$ for $m \in \mathbb{Z}_{\geq 0}$. In addition, we denote by $P_+ \subset P$ the set of dominant integral weights, and by $E$ the subset of $P$ consisting of all elements of level zero. For $\lambda \in P$, let $\mathbb{B}(\lambda)$ denote the $U_q(\mathfrak{g})$-crystal consisting of all Lakshmibai-Seshadri (LS for short) paths of shape $\lambda$. Our first main result (Theorem 3.21) states that for each $\lambda \in P$, the $U_q(\mathfrak{g})$-crystal $\mathbb{B}(\lambda)$ gives a combinatorial realization of the crystal basis $\mathcal{B}(\lambda)$; in addition, the crystal graph of $\mathbb{B}(\lambda) \cong \mathcal{B}(\lambda)$ is connected. Then, by means of this realization of $\mathcal{B}(\lambda)$, we study how the tensor product $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \cong \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ decomposes into connected components when $\lambda, \mu \in P$ are of nonnegative levels, dividing the problem into four cases: the case $\lambda, \mu \in E$ (see \S 4.1); the case $\lambda \in E$, $\mu \in P_+$ (see \S 4.2); the case $\lambda \in P_+$, $\mu \in E$ (see \S 4.3); the case $\lambda, \mu \in P_+$ (see \S 4.4). It turns out that in all the cases above, each connected component of the tensor product $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is isomorphic to $\mathbb{B}(\nu)$ (or $\mathcal{B}(\nu)$) for some $\nu \in P$ of nonnegative level. In particular, in the (most difficult) case $\lambda \in P_+$, $\mu \in E$, our result (Theorem 4.15) states that for some $m \in \mathbb{Z}_{\geq 3}$, the multiplicity of each connected component $\mathbb{B}(\nu)$ in $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is equal to the (corresponding) tensor product multiplicity of finite-dimensional irreducible highest weight $U_q(\mathfrak{g}_{[m]})$-modules of highest weight $\lambda, \mu,$ and $\nu$, respectively. Here, $\mathfrak{g}_{[m]}$ denotes the “reductive” Lie subalgebra of $\mathfrak{g}$ (of type $B_{m+1}$, $C_{m+1}$, or $D_{m+1}$) corresponding to the subset $[m] \subset I = \mathbb{Z}_{\geq 0}$; note that this $m \in \mathbb{Z}_{\geq 3}$ does not depend on the connected components $\mathbb{B}(\nu)$. Moreover, by virtue of tensor product multiplicity formulas in [Ko1], [Ko2], we obtain an explicit description of this multiplicity in terms of Littlewood-Richardson coefficients (see \S 5.2).
Now, from the argument in §4.2, we also observe that for each \( \lambda \in P \) of nonnegative level, there exist \( \lambda^0 \in E \) and \( \lambda^+ \in P_+ \) such that \( B(\lambda) \cong B(\lambda^0) \otimes B(\lambda^+) \) as \( U_q(\mathfrak{g}) \)-crystals. Therefore, by combining the results in the four cases above, we finally obtain our second main result (Theorem 4.19), which yields an explicit description (in terms of Littlewood-Richardson coefficients) of the multiplicity of each connected component \( B(\nu) \) in \( B(\lambda) \otimes B(\mu) \) for general \( \lambda, \mu \in P \) of nonnegative levels.

This paper is organized as follows. In §2 we introduce basic notation for infinite rank affine Lie algebras and their quantized universal enveloping algebras. In §3 we first recall standard facts about crystal bases of extremal weight modules, and show the connectedness of (the crystal graph of) the crystal basis \( B(\lambda) \) for \( \lambda \in P \). Then, we give a combinatorial realization of \( B(\lambda) \) as the crystal \( B(\lambda) \) of all LS paths of shape \( \lambda \). In §4 we describe explicitly how the tensor product \( B(\lambda) \otimes B(\mu) \) decomposes into connected components when \( \lambda, \mu \in P \) are of nonnegative levels, deferring the proof of Proposition 4.12 (used to prove Theorem 4.15) to §5. Finally, in §5 after reviewing tensor product multiplicity formulas in [Ko1, Ko2], we show Proposition 4.12 thereby completing the proof of Theorem 4.15 (and hence Theorem 4.19).

After having finished writing up this article, we were informed by Jae-Hoon Kwon that in [Kw4], he further obtained a description of how the tensor product \( B(\lambda) \otimes B(\mu) \) decomposes into connected components for \( \lambda \in P_+ \) and \( \mu \in -P_+ \), in type \( A_\infty \).

2 Basic notation for infinite rank affine Lie algebras.

2.1 Infinite rank affine Lie algebras. Let \( \mathfrak{g} \) be the infinite rank affine Lie algebra of type \( B_\infty, C_\infty, \) or \( D_\infty \) (see [Kac, §7.11]), that is, the (symmetrizable) Kac-Moody algebra of infinite rank associated to one of the following Dynkin diagrams:

\[
B_\infty : \quad \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & \cdots & \\
\end{array}
\]

\[
C_\infty : \quad \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & \cdots & \\
\end{array}
\]

\[
D_\infty : \quad \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & \cdots & \\
\end{array}
\]
Following [Kac §7.11], we realize this Lie algebra \( \mathfrak{g} \) as a Lie subalgebra of the Lie algebra \( \mathfrak{gl}_\infty(\mathbb{C}) \) of complex matrices \( (a_{ij})_{i,j \in \mathbb{Z}} \) with finitely many nonzero entries as follows. If \( \mathfrak{g} \) is of type \( B_\infty \) (resp., \( C_\infty, D_\infty \)), then we define elements \( x_i, y_i, h_i \in \mathfrak{gl}_\infty(\mathbb{C}) \) for \( i \in I := \mathbb{Z}_{\geq 0} \) by (2.1) (resp., (2.2), (2.3)):

\[
\begin{align*}
\begin{cases}
x_0 := E_{0,1} + E_{-1,0}, & x_i := E_{i,i+1} + E_{-i,-i-1} \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \\
y_0 := 2(E_{1,0} + E_{0,-1}), & y_i := E_{i+1,i} + E_{-i,-i-1} \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \\
h_0 := 2e_0', & h_i := -e_{i-1}' + e_i' \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}.
\end{cases}
\end{align*}
\]

(2.1)

\[
\begin{align*}
\begin{cases}
x_0 := E_{0,1}, & x_i := E_{i,i+1} + E_{-i,-i+1} \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \\
y_0 := E_{1,0}, & y_i := E_{i+1,i} + E_{-i,-i-1} \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \\
h_0 := e_0', & h_i := -e_{i-1}' + e_i' \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}.
\end{cases}
\end{align*}
\]

(2.2)

\[
\begin{align*}
\begin{cases}
x_0 := E_{0,2} - E_{-1,1}, & x_i := E_{i,i+1} - E_{-i,-i+1} \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \\
y_0 := E_{2,0} - E_{1,-1}, & y_i := E_{i,i+1} - E_{-i,-i-1} \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \\
h_0 := e_0' + e_i', & h_i := -e_{i-1}' + e_i' \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}.
\end{cases}
\end{align*}
\]

(2.3)

Here, for each \( i, j \in \mathbb{Z} \), \( E_{i,j} \in \mathfrak{gl}_\infty(\mathbb{C}) \) is the matrix with a 1 in the \((i, j)\) position, and 0 elsewhere, and

\[ e_i' := \begin{cases} 
E_{-j-1,j+1} & \text{if } \mathfrak{g} \text{ is of type } B_\infty, \\
E_{-j-j} & \text{if } \mathfrak{g} \text{ is of type } C_\infty \text{ or } D_\infty,
\end{cases} \quad \text{for } j \in \mathbb{Z}_{\geq 0}.
\]

The infinite rank affine Lie algebra \( \mathfrak{g} \) is isomorphic to the Lie subalgebra of \( \mathfrak{gl}_\infty(\mathbb{C}) \) generated by \( x_i, y_i, h_i \) for \( i \in I \); the elements \( x_i, y_i \) for \( i \in I \) are the Chevalley generators, \( \mathfrak{h} := \bigoplus_{i \in I} \mathbb{C}h_i \) is the Cartan subalgebra, and \( \Pi' := \{ h_i \}_{i \in I} \) is the set of simple coroots. Let \( \Pi := \{ \alpha_i \}_{i \in I} \subset \mathfrak{h}^* := \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C}) \) be the set of simple roots for \( \mathfrak{g} \). Then we have

\[
\begin{align*}
\begin{cases}
\alpha_0 = e_0, & \alpha_i = -e_{i-1} + e_i \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \quad \text{if } \mathfrak{g} \text{ is of type } B_\infty, \\
\alpha_0 = 2e_0, & \alpha_i = -e_{i-1} + e_i \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \quad \text{if } \mathfrak{g} \text{ is of type } C_\infty, \\
\alpha_0 = e_0 + e_1, & \alpha_i = -e_{i-1} + e_i \quad \text{for} \quad i \in \mathbb{Z}_{\geq 1}, \quad \text{if } \mathfrak{g} \text{ is of type } D_\infty.
\end{cases}
\end{align*}
\]

(2.4)

Here, for each \( j \in \mathbb{Z}_{\geq 0} \), we define \( e_j \in \mathfrak{h}^* = \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C}) \) by: \( \langle e_j, e_k' \rangle = \delta_{jk} \) for \( k \in \mathbb{Z}_{\geq 0} \), where \( \langle \cdot, \cdot \rangle \) denotes the natural pairing of \( \mathfrak{h}^* \) and \( \mathfrak{h} \); note that \( \mathfrak{h} = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e_j' \).

Let \( W = \{ r_i \mid i \in I \} \subset GL(\mathfrak{h}^*) \) denote the Weyl group of \( \mathfrak{g} \), where \( r_i \) denotes the simple reflection corresponding to the simple root \( \alpha_i \) for \( i \in I \). If \( \mathfrak{g} \) is of type \( B_\infty \) (resp., \( C_\infty, D_\infty \)), then the following equation (2.5) (resp., (2.6), (2.7)) holds for \( j \in \mathbb{Z}_{\geq 0} \):

\[
r_0(e_j) = \begin{cases} 
-e_0 & \text{if } j = 0, \\
e_j & \text{otherwise},
\end{cases} \quad r_i(e_j) = \begin{cases} 
e_i & \text{if } j = i - 1, \\
e_{i-1} & \text{if } j = i, \\
e_j & \text{otherwise},
\end{cases} \quad \text{for } i \in \mathbb{Z}_{\geq 1};
\]

(2.5)
Also, for the action of $W$ on $\mathfrak{h}$, entirely similar formulas hold in all the cases above, with $e$ replaced by $e'$.

Let $\Delta := W\Pi$ be the set of roots for $\mathfrak{g}$, and $\Delta^+ := \Delta \cap \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ the set of positive roots for $\mathfrak{g}$. If $\mathfrak{g}$ is of type $B_\infty$ (resp., $C_\infty$, $D_\infty$), then the sets $\Delta$ and $\Delta^+$ are given by (2.8) (resp., (2.9), (2.10));

\[
\begin{align*}
\Delta &= \{ \pm e_j \mid 1 \leq j \} \cup \{ \pm e_j \pm e_i \mid 0 \leq j < i \}, \\
\Delta^+ &= \{ e_j \mid 1 \leq j \} \cup \{ -e_j + e_i, e_j + e_i \mid 0 \leq j < i \};
\end{align*}
\]

(2.8)

\[
\begin{align*}
\Delta &= \{ \pm 2e_j \mid 1 \leq j \} \cup \{ \pm e_j \pm e_i \mid 0 \leq j < i \}, \\
\Delta^+ &= \{ 2e_j \mid 1 \leq j \} \cup \{ -e_j + e_i, e_j + e_i \mid 0 \leq j < i \};
\end{align*}
\]

(2.9)

\[
\begin{align*}
\Delta &= \{ \pm e_j \pm e_i \mid 0 \leq j < i \}, \\
\Delta^+ &= \{ -e_j + e_i, e_j + e_i \mid 0 \leq j < i \}.
\end{align*}
\]

(2.10)

We set $E := \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{Z}e_j \subset \mathfrak{h}^*$, and then $E_\mathbb{C} := \mathbb{C} \otimes_{\mathbb{Z}} E = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e_j \subset \mathfrak{h}^*$; note that $\Delta$ is contained in $E$, and $E$ is stable under the action of the Weyl group $W$. Define a $W$-invariant, symmetric $\mathbb{C}$-bilinear form $(\cdot, \cdot)$ on $E_\mathbb{C}$ by: $(e_i, e_j) = \delta_{ij}$ for $i, j \in \mathbb{Z}_{\geq 0}$, and a $\mathbb{C}$-linear isomorphism $\psi : E_\mathbb{C} = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e_j \to \mathfrak{h} = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e'_j$ by: $\psi(e_j) = e'_j$ for $j \in \mathbb{Z}_{\geq 0}$.

Also, we define the dual root $\beta' \in \mathfrak{h}$ of a root $\beta \in \Delta$ by: $\beta' = 2\psi(\beta)/(\beta, \beta)$; observe that $\alpha_i' = h_i$ for all $i \in I$.

### 2.2 Integral weights and their levels.

For each $i \in I$, let $\Lambda_i \in \mathfrak{h}^*$ denote the $i$-th fundamental weight for $\mathfrak{g}$. We should warn the reader that these elements $\Lambda_i$, $i \in I$, are not contained in the subspace $E_\mathbb{C} = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e_j$ of $\mathfrak{h}^* = \prod_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e_j$, where the vector space $\prod_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e_j$ is thought of as a certain completion of $\bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C}e_j$. In fact, we have

\[
\begin{align*}
\Lambda_0 &= \frac{1}{2}(e_0 + e_1 + e_2 + \cdots), & \text{if } \mathfrak{g} \text{ is of type } B_\infty, \\
\Lambda_i &= e_i + e_{i+1} + e_{i+2} + \cdots & \text{for } i \in \mathbb{Z}_{\geq 1}, \\
\Lambda_i &= e_i + e_{i+1} + e_{i+2} + \cdots & \text{for } i \in I, & \text{if } \mathfrak{g} \text{ is of type } C_\infty,
\end{align*}
\]

(2.11)

(2.12)
Let $P := \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \subset \mathfrak{h}^*$ denote the integral weight lattice for $\mathfrak{g}$, and let $P_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i \subset P$ be the set of dominant integral weights for $\mathfrak{g}$.

For an integral weight $\lambda \in P$, we define $\lambda^{(j)} \in (1/2)\mathbb{Z}$ for $j \in \mathbb{Z}_{\geq 0}$ by:

$$
\lambda^{(j)} := \langle \lambda, \mathbf{e}_j \rangle, \quad \text{or equivalently,} \quad \lambda = \lambda^{(0)} \mathbf{e}_0 + \lambda^{(1)} \mathbf{e}_1 + \lambda^{(2)} \mathbf{e}_2 + \cdots;
$$

note that either $\lambda^{(j)} \in \mathbb{Z}$ for all $j \in \mathbb{Z}_{\geq 0}$, or $\lambda^{(j)} \in 1/2 + \mathbb{Z}$ for all $j \in \mathbb{Z}_{\geq 0}$. If $\lambda \in P$, then for $n$ sufficiently large, we have

$$
\lambda^{(n)} = \lambda^{(n+1)} = \lambda^{(n+2)} = \cdots,
$$

since $\lambda \in P$ is a finite sum of integer multiples of fundamental weights; in this case, we set $L_\lambda := \lambda^{(n)}$, and call it the level of the integral weight $\lambda$. Note that $E \subset P$ is identical to the set of integral weights of level zero, and that if $\lambda \in P$ is a dominant integral weight not equal to 0, then the level $L_\lambda$ of $\lambda$ is positive.

**Remark 2.1 (cf. [Ko2] statements (2)–(4) on p. 82]).** Let $\lambda \in P_+$. Then, it follows that

$$
0 \leq \langle \lambda^{(0)} \rangle \leq \lambda^{(1)} \leq \lambda^{(2)} \leq \cdots \leq \lambda^{(n-1)} \leq L_\lambda = \lambda^{(n)} = \lambda^{(n+1)} = \cdots
$$

for some $n \in \mathbb{Z}_{\geq 0}$, where for $x \in (1/2)\mathbb{Z}$, we set

$$
\langle x \rangle := \begin{cases} 
 x & \text{if } \mathfrak{g} \text{ is of type } B_\infty \text{ or } C_\infty, \\
 |x| & \text{if } \mathfrak{g} \text{ is of type } D_\infty.
\end{cases}
$$

### 2.3 Quantized universal enveloping algebras and their finite rank subalgebras.

We set $P^\vee := \bigoplus_{i \in I} \mathbb{Z} \mathbf{h}_i \subset \mathfrak{h}$, and let $U_q(\mathfrak{g}) = \langle x_i, y_i, q^{\mathbf{h}} \mid i \in I, h \in P^\vee \rangle$ denote the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{C}(q)$ with integral weight lattice $P$, and Chevalley generators $x_i, y_i, i \in I$. Also, let $U^+_q(\mathfrak{g})$ (resp., $U^-_q(\mathfrak{g})$) denote the positive (resp., negative) part of $U_q(\mathfrak{g})$, that is, the $\mathbb{C}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $x_i, i \in I$ (resp., $y_i, i \in I$). We have a $\mathbb{C}(q)$-algebra anti-automorphism $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by:

$$
\begin{align*}
(q^h)^* &= q^{-h} \quad \text{for } h \in P^\vee, \\
x_i^* &= x_i, \quad y_i^* = y_i \quad \text{for } i \in I.
\end{align*}
$$

(2.14)

Note that $U^\pm_q(\mathfrak{g})$ is stable under the $\mathbb{C}(q)$-algebra anti-automorphism $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$. 

6
For \( m, n \in \mathbb{Z}_{\geq 0} \) with \( n \geq m \), we denote the finite interval \( \{ m, m+1, \ldots, n \} \) in \( I = \mathbb{Z}_{\geq 0} \) by \([m, n]\). Also, for \( n \in \mathbb{Z}_{\geq 0} \), we write simply \([n]\) for the finite interval \([0, n] = \{ 0, 1, \ldots, n \}\) in \( I = \mathbb{Z}_{\geq 0} \). Let \( J \) be a finite interval in \( I = \mathbb{Z}_{\geq 0} \), which is of the form \([m, n]\) for some \( m, n \in \mathbb{Z}_{\geq 0} \) with \( n \geq m \). We denote by \( g_J \) the (Lie) subalgebra of \( g \) generated by \( x_i, y_i, i \in J \), and \( \mathfrak{h} \). Then, the set \( \Delta_J := \Delta \cap \bigoplus_{i \in J} \mathbb{Z} \alpha_i \) and \( \Delta_J^+ := \Delta^+ \cap \bigoplus_{i \in J} \mathbb{Z} \alpha_i \) are the sets of roots and positive roots for \( g_J \), respectively.

**Remark 2.2.** If \( g \) is of type \( B_\infty \) (resp., \( C_\infty, D_\infty \)), and \( n \in \mathbb{Z}_{\geq 3} \), then the Lie subalgebra \( g_{[n]} \) of \( g \) is a “reductive” Lie algebra of type \( B_{n+1} \) (resp., \( C_{n+1}, D_{n+1} \)). Also, if \( n \in \mathbb{Z}_{\geq 1} \), then the Lie subalgebra \( g_{[1, n]} \) of \( g \) is a “reductive” Lie algebra of type \( A_n \).

Denote by \( U_q(g_J) \) the \( \mathbb{C}(q) \)-subalgebra of \( U_q(g) \) generated by \( x_i, y_i, i \in J \), and \( q^h, h \in P^\vee \), which can be thought of as the quantized universal enveloping algebra of \( g_J \) over \( \mathbb{C}(q) \), and denote by \( U_q^+(g_J) \) (resp., \( U_q^-(g_J) \)) the positive (resp., negative) part of \( U_q(g_J) \), that is, the \( \mathbb{C}(q) \)-subalgebra of \( U_q(g_J) \) generated by \( x_i, i \in J \) (resp., \( y_i, i \in J \)). Then it is easily seen that \( U_q(g_J) \) and \( U_q^{\pm}(g_J) \) are stable under the \( \mathbb{C}(q) \)-algebra anti-automorphism \( * : U_q(g) \to U_q(g) \) given by \( (2.14) \).

Let \( W_J \) denote the (finite) subgroup of \( W \) generated by the \( r_i \) for \( i \in J \), which is the Weyl group of \( g_J \). An integral weight \( \lambda \in P \) is said to be \( J \)-dominant (resp., \( J \)-antidominant) if \( \langle \lambda, h_i \rangle \geq 0 \) (resp., \( \langle \lambda, h_i \rangle < 0 \) for all \( i \in J \). For each integral weight \( \lambda \in P \), we denote by \( \lambda_J \) the unique element of \( W_J \lambda \) that is \( J \)-dominant.

**Remark 2.3** (cf. [Ko2, statements (2)–(4) on p. 82]). Let \( n \in \mathbb{Z}_{\geq 3} \), and let \( \lambda \in P \) be an \([n]\)-dominant integral weight. Then it follows from \( (2.1) \) that

\[
0 \leq \langle \lambda^{(0)} \rangle \leq \lambda^{(1)} \leq \cdots \leq \lambda^{(n-1)} \leq \lambda^{(n)}.
\]

Now, for \( \lambda \in E = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{Z} e_J \), we set

\[
\text{Supp}(\lambda) := \{ j \in \mathbb{Z}_{\geq 0} \mid \lambda^{(j)} \neq 0 \}.
\]

**Lemma 2.4.** Let \( \lambda \in E \), and set \( p := \# \text{Supp}(\lambda) \). Let \( n \in \mathbb{Z}_{\geq 3} \) be such that \( \text{Supp}(\lambda) \subsetneq [n] \). Then, the unique \([n]\)-dominant element \( \lambda_{[n]} \) of \( W_{[n]} \lambda \) satisfies the following property:

\[
\text{Supp}(\lambda_{[n]}) = [n-p+1, n], \quad \text{and} \quad 0 < \lambda_{[n]}^{(n-p+1)} \leq \cdots \leq \lambda_{[n]}^{(n-1)} \leq \lambda_{[n]}^{(n)}.
\]

**Proof.** Since \( \text{Supp}(\lambda) \subsetneq [n] \) by assumption, and \( \lambda_{[n]} \in W_{[n]} \lambda \) by definition, it follows by using \( (2.5)–(2.7) \) that \( \text{Supp}(\lambda_{[n]}) \subsetneq [n] \). Also, we see from Remark 2.3 that

\[
0 \leq \langle \lambda_{[n]}^{(0)} \rangle \leq \lambda_{[n]}^{(1)} \leq \cdots \leq \lambda_{[n]}^{(n-1)} \leq \lambda_{[n]}^{(n)}.
\]

If \( \lambda_{[n]}^{(0)} \neq 0 \), then we have \( \text{Supp}(\lambda_{[n]}) = [n] \), which is contradiction. Therefore, we get \( \lambda_{[n]}^{(0)} = 0 \), and hence \( 0 \leq \lambda_{[n]}^{(1)} \leq \cdots \leq \lambda_{[n]}^{(n-1)} \leq \lambda_{[n]}^{(n)} \). Furthermore, since \( \# \text{Supp}(\lambda_{[n]}) = \# \text{Supp}(\lambda) = p \), we deduce that \( \text{Supp}(\lambda_{[n]}) = [n-p+1, n] \). This proves the lemma. \( \square \)
3 Path model for the crystal basis of an extremal weight module.

3.1 Extremal elements. Let $\mathcal{B}$ be a $U_q(\mathfrak{g})$-crystal (resp., $U_q(\mathfrak{g}_J)$-crystal for a finite interval $J$ in $I = \mathbb{Z}_{\geq 0}$), equipped with the maps $e_i, f_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ for $i \in I$ (resp., $i \in J$), which we call the Kashiwara operators, and the maps $\text{wt} : \mathcal{B} \to P$, $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ for $i \in I$ (resp., $i \in J$). Here, $0$ is a formal element not contained in $\mathcal{B}$. For $\nu \in P$, we denote by $\mathcal{B}_\nu$ the subset of $\mathcal{B}$ consisting of all elements of weight $\nu$.

**Definition 3.1** (cf. [Kas3, p. 389]). (1) A $U_q(\mathfrak{g})$-crystal $\mathcal{B}$ is said to be normal if $\mathcal{B}$, regarded as a $U_q(\mathfrak{g}_K)$-crystal by restriction, is isomorphic to a direct sum of the crystal bases of finite-dimensional irreducible $U_q(\mathfrak{g}_K)$-modules for every finite interval $K$ in $I = \mathbb{Z}_{\geq 0}$.

(2) Let $J$ be a finite interval in $I = \mathbb{Z}_{\geq 0}$. A $U_q(\mathfrak{g}_J)$-crystal $\mathcal{B}$ is said to be normal if $\mathcal{B}$, regarded as a $U_q(\mathfrak{g}_K)$-crystal by restriction, is isomorphic to a direct sum of the crystal bases of finite-dimensional irreducible $U_q(\mathfrak{g}_K)$-modules for every finite interval $K$ in $I = \mathbb{Z}_{\geq 0}$ contained in $J$.

If $\mathcal{B}$ is a normal $U_q(\mathfrak{g})$-crystal, then $\mathcal{B}$, regarded as a $U_q(\mathfrak{g}_J)$-crystal by restriction, is a normal $U_q(\mathfrak{g}_J)$-crystal for every finite interval $J$ in $I = \mathbb{Z}_{\geq 0}$. Also, if $\mathcal{B}$ is a normal $U_q(\mathfrak{g})$-crystal (resp., normal $U_q(\mathfrak{g}_J)$-crystal for a finite interval $J$ in $I = \mathbb{Z}_{\geq 0}$), then we have

$$
\varepsilon_i(b) = \max \{k \in \mathbb{Z}_{\geq 0} \mid e_k^i b \neq 0\} \quad \text{and} \quad \varphi_i(b) = \max \{k \in \mathbb{Z}_{\geq 0} \mid f_k^i b \neq 0\}
$$

for $b \in \mathcal{B}$ and $i \in I$ (resp., $i \in J$); in this case, we set

$$
eq \varepsilon_i^\text{max}(b) b \quad \text{and} \quad f_i^\text{max} b := f_i^{\varphi_i(b)} b
$$

for $b \in \mathcal{B}$ and $i \in I$ (resp., $i \in J$). Also, we can define an action of the Weyl group $W$ (resp., $W_J$) on $\mathcal{B}$ as follows. For each $i \in I$ (resp., $i \in J$), define $S_i : \mathcal{B} \to \mathcal{B}$ by:

$$
S_i b = \begin{cases} f_k^i b & \text{if } k := \langle \text{wt } b, h_i \rangle \geq 0, \\ e_i^{-k} b & \text{if } k := \langle \text{wt } b, h_i \rangle < 0, \end{cases} \quad \text{for } b \in \mathcal{B}. \quad (3.1)
$$

Then, these operators $S_i$, $i \in I$, give rise to a unique (well-defined) action $S : W \to \text{Bij}(\mathcal{B})$ (resp., $S : W_{[n]} \to \text{Bij}(\mathcal{B})$), $w \mapsto S_w$, of the Weyl group $W$ (resp., $W_J$) on the set $\mathcal{B}$ such that $S_i = S_i$ for all $i \in I$ (resp., $i \in J$). Here, for a set $X$, $\text{Bij}(X)$ denotes the group of all bijections from the set $X$ to itself.

**Definition 3.2.** Suppose that $\mathcal{B}$ is a normal $U_q(\mathfrak{g})$-crystal (resp., $U_q(\mathfrak{g}_J)$-crystal for a finite interval $J$ in $I = \mathbb{Z}_{\geq 0}$).

(1) An element $b \in \mathcal{B}$ is said to be extremal (resp., $J$-extremal) if for every $w \in W$ and $i \in I$ (resp., $w \in W_J$ and $i \in J$),

$$
\begin{cases} e_i S_w b = 0 & \text{if } \langle w(\text{wt } b), h_i \rangle \geq 0, \\ f_i S_w b = 0 & \text{if } \langle w(\text{wt } b), h_i \rangle \leq 0. \end{cases} \quad (3.2)
$$

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(2) An element \( b \in \mathcal{B} \) is said to be maximal (resp., \( J \)-maximal) if \( e_i b = 0 \) for all \( i \in I \) (resp., \( i \in J \)).

(3) An element \( b \in \mathcal{B} \) is said to be minimal (resp., \( J \)-minimal) if \( f_i b = 0 \) for all \( i \in I \) (resp., \( i \in J \)).

Remark 3.3. Suppose that \( \mathcal{B} \) is a normal \( U_q(\mathfrak{g}) \)-crystal.

(1) An element \( b \in \mathcal{B} \) is extremal if and only if there exists \( m \in \mathbb{Z}_{\geq 0} \) such that \( b \) is \([n]\)-extremal for all \( n \in \mathbb{Z}_{\geq m} \).

(2) Let \( n \in \mathbb{Z}_{\geq 0} \). Since the (unique) \([n]\)-maximal element of the crystal basis of the finite-dimensional irreducible \( U_q(\mathfrak{g}[n]) \)-module is \([n]\)-extremal, it follows that an \([n]\)-maximal element of \( \mathcal{B} \) is \([n]\)-extremal. Consequently, by part (1), a maximal element of \( \mathcal{B} \) is extremal.

(3) Let \( J \) be a finite interval of \( I = \mathbb{Z}_{\geq 0} \). By the same reasoning as in part (2), we see that a \( J \)-minimal element of \( \mathcal{B} \) is \( J \)-extremal.

### 3.2 Crystal bases of extremal weight modules.

In this subsection, we study some basic properties of crystal bases of extremal weight modules; note that all results in [Kas1]–[Kas4] about extremal weight modules and their crystal bases that we use in this paper remain valid in the case of infinite rank affine Lie algebras.

**Definition 3.4** (see [Kas3, Definition 8.1]). Let \( \lambda \in P \) be an integral weight.

(1) Let \( M \) be an integrable \( U_q(\mathfrak{g}) \)-module. A weight vector \( v \in M \) of weight \( \lambda \) is said to be extremal if there exists a family \( \{v_w\}_{w \in W} \) of weight vectors in \( M \) satisfying the following conditions:

(i) If \( w \) is the identity element \( e \) of \( W \), then \( v_w = v_e = v \);

(ii) for \( w \in W \) and \( i \in I \) such that \( k := \langle w \lambda, h_i \rangle \geq 0 \), we have \( x_i v_w = 0 \) and \( y_i^{(k)} v_w = v_{r_i w} \);

(iii) for \( w \in W \) and \( i \in I \) such that \( k := \langle w \lambda, h_i \rangle \leq 0 \), we have \( y_i v_w = 0 \) and \( x_i^{(k)} v_w = v_{r_i w} \).

Here, for \( i \in I \) and \( k \in \mathbb{Z}_{\geq 0} \), \( x_i^{(k)} \) and \( y_i^{(k)} \) denote the \( k \)-th \( q \)-divided powers of \( x_i \) and \( y_i \), respectively.

(2) Let \( J \) be a finite interval in \( I = \mathbb{Z}_{\geq 0} \), and let \( M \) be an integrable \( U_q(\mathfrak{g}_J) \)-module. A weight vector \( v \in M \) of weight \( \lambda \) is said to be \( J \)-extremal if there exists a family \( \{v_w\}_{w \in W_J} \) of weight vectors in \( M \) satisfying the same conditions as (i), (ii), (iii) above, with \( W \) replaced by \( W_J \), and \( I \) by \( J \).

The extremal weight \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) of extremal weight \( \lambda \) is, by definition, the integrable \( U_q(\mathfrak{g}) \)-module generated by a single element \( v_\lambda \) subject to the defining relation that the \( v_\lambda \) is an extremal vector of weight \( \lambda \) (see [Kas3, Proposition 8.2.2] and also [Kas5, §3.1]). We know from [Kas3, Proposition 8.2.2] that \( V(\lambda) \) admits the crystal basis \( (L(\lambda), \mathcal{B}(\lambda)) \), and that \( \mathcal{B}(\lambda) \) is a normal \( U_q(\mathfrak{g}) \)-crystal. If we denote by \( u_\lambda \in \mathcal{B}(\lambda) \) the element corresponding to the extremal vector \( v_\lambda \in V(\lambda) \) of weight \( \lambda \), then the element \( u_\lambda \in \mathcal{B}(\lambda) \) is extremal.
Remark 3.5 (see [Kas3, §§8.2 and 8.3]). (1) If \( \lambda \in P_+ \), then the extremal weight module \( V(\lambda) \) is isomorphic to the irreducible highest weight \( U_q(g) \)-module of highest weight \( \lambda \). Therefore, the crystal basis \( B(\lambda) \) is isomorphic, as a crystal, to the crystal basis of the irreducible highest weight \( U_q(g) \)-module of highest weight \( \lambda \).

(2) For each \( w \in W \), there exists an isomorphism \( V(\lambda) \cong V(w\lambda) \) of \( U_q(g) \)-modules between \( V(\lambda) \) and \( V(w\lambda) \). Also, for each \( w \in W \), there exists an isomorphism \( B(\lambda) \cong B(w\lambda) \) of \( U_q(g) \)-crystals between \( B(\lambda) \) and \( B(w\lambda) \).

Let \( J \) be a finite interval in \( I = \mathbb{Z}_{\geq 0} \). As in the case of \( V(\lambda) \), the extremal weight \( U_q(g)_J \)-module \( V_J(\lambda) \) of extremal weight \( \lambda \) is, by definition, the integrable \( U_q(g)_J \)-module generated by a single element \( v_\lambda \) subject to the defining relation that the \( v_\lambda \) is an \( J \)-extremal vector of weight \( \lambda \). We denote by \((L_J(\lambda), B_J(\lambda)) \) the crystal basis of \( V_J(\lambda) \).

Remark 3.6. Let \( J \) be a finite interval in \( I = \mathbb{Z}_{\geq 0} \). Then, results entirely similar to those in Remark 3.5 hold for extremal weight \( U_q(g)_J \)-modules and their crystal bases. If \( \lambda \) is \( J \)-dominant (resp., \( J \)-antidominant), then \( V_J(\lambda) \) is the finite-dimensional irreducible \( U_q(g)_J \)-module of highest weight \( \lambda \) (resp., lowest weight \( \lambda \)). Also, for \( \lambda \in P \), we have \( B_J(\lambda) \cong B_J(\lambda_J) \) as \( U_q(g)_J \)-crystals since \( \lambda_J \in W_J \lambda \). Because \( B_J(\lambda_J) \) is isomorphic, as a \( U_q(g)_J \)-crystal, to the crystal basis of the finite-dimensional irreducible \( U_q(g)_J \)-module of highest weight \( \lambda_J \), it follows that the crystal graph of \( B_J(\lambda) \) is connected, and \( \#(B_J(\lambda)_\nu) = 1 \) for all \( \nu \in W_J \lambda \).

To prove Proposition 3.7 below, we need to recall the description of the crystal basis \( B(\lambda) \) for \( \lambda \in P \) from [Kas3]. Let \((\mathcal{L}(\pm \infty), B(\pm \infty)) \) denote the crystal basis of \( U_q^+(g) \). Recall from [Kas1] Proposition 5.2.4] that the crystal lattice \( \mathcal{L}(\pm \infty) \) of \( U_q^+(g) \) is stable under the \( \mathbb{C}(q) \)-algebra anti-automorphism \( * : U_q(g) \rightarrow U_q(g) \) given by (2.14). Furthermore, we know from [Kas2] Theorem 2.1.1] that the \( \mathbb{C} \)-linear automorphism (also denoted by \( * \)) on \( \mathcal{L}(\pm \infty)/q\mathcal{L}(\pm \infty) \) induced by \( * : \mathcal{L}(\pm \infty) \rightarrow \mathcal{L}(\pm \infty) \) stabilizes the crystal basis \( B(\pm \infty) \), and gives rise to a map \( * : B(\pm \infty) \rightarrow B(\pm \infty) \). Now, we set

\[
B^\lambda := B(\infty) \otimes T_\lambda \otimes B(-\infty) \quad \text{for} \quad \lambda \in P,
\]

where \( T_\lambda := \{t_\lambda\} \) is a \( U_q(g) \)-crystal consisting of a single element \( t_\lambda \) of weight \( \lambda \in P \) (see [Kas3] Example 1.5.3, part 2]), and then set \( \tilde{B} := \bigoplus_{\lambda \in P} B^\lambda \). Also, we define a map \( * : \tilde{B} \rightarrow \tilde{B} \) by:

\[
(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-(\text{wt}b_1 + \lambda + \text{wt}b_2)} \otimes b_2^*
\]

for \( b_1 \in B(\infty), \lambda \in P, \) and \( b_2 \in B(-\infty) \) (cf. [Kas3, Corollary 4.3.3]). It follows from [Kas3] Theorem 3.1.1 and Theorem 2.1.2 (v)] that \( B^\lambda \) is normal for all \( \lambda \in P \), and hence so is \( \tilde{B} = \bigoplus_{\lambda \in P} B^\lambda \). Moreover, by [Kas3] Proposition 8.2.2, Theorem 3.1.1, and Corollary 4.3.3], the subset

\[
\{b \in B^\lambda \mid b^* \text{ is extremal}\} \subset B^\lambda
\]

(3.3)
is a $U_q(g)$-subcrystal of $B^\lambda$, and it is isomorphic, as a $U_q(g)$-crystal, to the crystal basis $B(\lambda)$; under this isomorphism, the extremal element $u_\lambda \in B(\lambda)$ corresponds to the element $u_\infty \otimes t_\lambda \otimes u_{-\infty} \in B^\lambda$, where $u_{\pm \infty}$ is the element of $B(\pm \infty)$ corresponding to the identity element $1 \in U_q^\mp(g)$. Thus, we can identify $B(\lambda)$ with the $U_q(g)$-subcrystal given by (5.3), and identify $u_\lambda \in B(\lambda)$ with $u_\infty \otimes t_\lambda \otimes u_{-\infty} \in B^\lambda$.

We have the following proposition (see also [Kw2, Proposition 3.1] in type $A_{+\infty}$ and [Kw3, Proposition 4.1] in type $A_{\infty}$).

**Proposition 3.7.** Let $\lambda \in P$ be an integral weight. Then, the crystal graph of the crystal basis $B(\lambda)$ is connected.

**Proof.** While the argument in the proof of [Kw2, Proposition 3.1] in type $A_{+\infty}$ (or, of [Kw3, Proposition 4.1] in type $A_{\infty}$) still works in the case of type $B_\infty$, $C_\infty$, or $D_\infty$, we prefer to give a different proof.

For each $n \in \mathbb{Z}_{\geq 0}$, let $(L_{[n]}(\pm \infty), B_{[n]}(\pm \infty))$ denote the crystal basis of $U_q^+(g_{[n]}) \subset U_q^+(g)$. We deduce from the definitions that $U_q^+(g_{[n]})$ is stable under the Kashiwara operators $e_i$ and $f_i$, $i \in [n]$, on $U_q^+(g)$, and that their restrictions to $U_q^+(g_{[n]})$ are exactly the Kashiwara operators $e_i$ and $f_i$, $i \in [n]$, on $U_q^+(g_{[n]})$, respectively. Therefore, the crystal lattice $L_{[n]}(\pm \infty)$ of $U_q^+(g_{[n]})$ is identical to the $A$-submodule of the crystal lattice $L(\pm \infty)$ of $U_q^+(g)$ generated by those elements of the form: $X \cdot 1 \in U_q^+(g_{[n]}) \subset U_q^+(g_{[n]})$ for some monomial $X$ in the Kashiwara operators $e_i$ and $f_i$ for $i \in [n]$, where $A := \{ f(q) \in \mathbb{C}(q) \mid f(q) \text{ is regular at } q = 0 \}$. Consequently, the crystal basis $B_{[n]}(\pm \infty)$ of $U_q^+(g_{[n]})$ is identical to the subset of $B(\pm \infty)$ consisting of all elements $b$ of the form: $b = X u_{\pm \infty}$ for some monomial $X$ in the Kashiwara operators $e_i$ and $f_i$ for $i \in [n]$.

Now, we define a subset $B_{[n]}^\lambda$ of $B^\lambda$ by:

$$B_{[n]}^\lambda = \{ b_1 \otimes t_\lambda \otimes b_2 \in B^\lambda \mid b_1 \in B_{[n]}(\infty), \ b_2 \in B_{[n]}(-\infty) \}.$$ 

Then, it is obvious from the tensor product rule for crystals that $B_{[n]}^\lambda$ is a $U_q(g_{[n]})$-crystal isomorphic to the tensor product $B_{[n]}(\infty) \otimes T_\lambda \otimes B_{[n]}(-\infty)$ of $U_q(g_{[n]})$-crystals; namely, as $U_q(g_{[n]})$-crystals,

$$B_{[n]}^\lambda \cong B_{[n]}(\infty) \otimes T_\lambda \otimes B_{[n]}(-\infty), \quad b_1 \otimes t_\lambda \otimes b_2 \mapsto b_1 \otimes t_\lambda \otimes b_2.$$  

(3.4)

Hence, by [Kas3, Theorem 3.1.1 and Theorem 2.1.2 (v), $B_{[n]}^\lambda$ is a normal $U_q(g_{[n]})$-crystal. We also remark that

$$B(\lambda) \cap B_{[n]}^\lambda \subset B(\lambda) \cap B_{[n+1]}^\lambda \quad \text{for all } n \in \mathbb{Z}_{\geq 0}, \quad B(\lambda) = \bigcup_{n \geq 0} (B(\lambda) \cap B_{[n]}^\lambda).$$

(3.5)

**Claim.** The crystal basis $B_{[n]}(\lambda)$ of the extremal weight $U_q(g_{[n]})$-module of extremal weight $\lambda$ is isomorphic, as a $U_q(g_{[n]})$-crystal, to $B(\lambda) \cap B_{[n]}^\lambda$.
Proof of Claim. We have a $\mathbb{C}(q)$-algebra anti-automorphism $\star : U_q(\mathfrak{g}[n]) \to U_q(\mathfrak{g}[n])$ defined by:
\[
\begin{aligned}
(q^h)^* &= q^{-h} \quad \text{for } h \in P^r, \\
x_i^* &= x_i, \quad y_i^* = y_i \quad \text{for } i \in [n].
\end{aligned}
\] (3.6)
As in the case of $\mathcal{B}(\pm \infty)$, this $\mathbb{C}(q)$-algebra anti-automorphism $\star : U_q(\mathfrak{g}[n]) \to U_q(\mathfrak{g}[n])$ induces a map $\star : \mathcal{B}_q(\mathfrak{g}[n]) \to \mathcal{B}_q(\mathfrak{g}[n])(\pm \infty)$. We set $\widetilde{\mathcal{B}}[n] := \bigoplus_{\lambda \in \mathcal{P}} \mathcal{B}_q^\lambda \subset \widetilde{\mathcal{B}}$, which is a normal $U_q(\mathfrak{g}[n])$-crystal, and then define $\star : \widetilde{\mathcal{B}}[n] \to \widetilde{\mathcal{B}}[n]$ by:
\[
(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-(wt(b_1 + \lambda + wt(b_2))} \otimes b_2^*
\]
for $b_1 \in \mathcal{B}_q[\lambda](\infty)$, $\lambda \in P$, and $b_2 \in \mathcal{B}_q[\lambda](-\infty)$. Then, we see from [Kas93] Proposition 8.2.2, Theorem 3.1.1, and Corollary 4.3.3], together with (3.4), that
\[
\{ b \in \mathcal{B}_q^\lambda \mid b^* \text{ is } [n]\text{-extremal} \} \subset \mathcal{B}_q^\lambda \subset \mathcal{B}_q^\lambda
\] (3.7)
is a $U_q(\mathfrak{g}[n])$-subcrystal of $\mathcal{B}_q^\lambda$ isomorphic to the crystal basis $\mathcal{B}_q[\lambda](\lambda)$. We identify $\mathcal{B}_q[\lambda](\lambda)$ with the $U_q(\mathfrak{g}[n])$-subcrystal of $\mathcal{B}_q^\lambda$ given by (3.7).

Also, observe that the $\mathbb{C}(q)$-algebra anti-automorphism $\star : U_q(\mathfrak{g}[n]) \to U_q(\mathfrak{g}[n])$ given by (3.6) is identical to the restriction to $U_q(\mathfrak{g}[n])$ of the $\mathbb{C}(q)$-algebra anti-automorphism $\star : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by (2.13). Consequently, the map $\star : \mathcal{B}_q[\lambda](\pm \infty) \to \mathcal{B}_q[\lambda](\pm \infty)$ is identical to the restriction to $\mathcal{B}_q[\lambda](\pm \infty)$ of the map $\star : \mathcal{B}(\pm \infty) \to \mathcal{B}(\pm \infty)$, and hence the map $\star : \mathcal{B}_q[\lambda] \to \mathcal{B}_q[\lambda]$ is identical to the restriction to $\mathcal{B}_q[\lambda]$ of the map $\star : \mathcal{B} \to \mathcal{B}$. Therefore, we have
\[
\mathcal{B}[\lambda] = \{ b \in \mathcal{B}_q^\lambda \mid b^* \text{ is } [n]\text{-extremal} \} \subset \mathcal{B}[\lambda] \cap \mathcal{B}_q^\lambda.
\] (3.8)
Because we know from Remark 3.6 with $J = [n]$ that the crystal graph of the $U_q(\mathfrak{g}[n])$-crystal $\mathcal{B}_q[\lambda](\lambda)$ is connected, we conclude from (3.8) that $\mathcal{B}_q[\lambda](\lambda) = \mathcal{B}[\lambda] \cap \mathcal{B}_q^\lambda$. This proves the claim.

We now complete the proof of Proposition 3.7. Take $b_1, b_2 \in \mathcal{B}(\lambda)$ arbitrarily. We show that there exists a monomial $X$ in the Kashiwara operators $e_i$ and $f_i$ for $i \in I$ such that $Xb_1 = b_2$. It follows from (3.5) that there exists $n \in \mathbb{Z}_{\geq 0}$ such that $b_1$ and $b_2$ are both contained in $\mathcal{B}(\lambda) \cap \mathcal{B}_q^\lambda$. Since $\mathcal{B}(\lambda) \cap \mathcal{B}_q^\lambda \cong \mathcal{B}_q[\lambda](\lambda)$ as $U_q(\mathfrak{g}[n])$-crystals by the claim above, we deduce from Remark 3.6 with $J = [n]$ that there exists a monomial $X$ in the Kashiwara operators $e_i$ and $f_i$ for $i \in [n]$ such that $Xb_1 = b_2$. This finishes the proof of Proposition 3.7.

\[\square\]

**Proposition 3.8.** Let $\lambda \in P$ be an integral weight. For each $w \in W$, we have $\mathcal{B}(\lambda)_{w,\lambda} = \{ S_wu_\lambda \}$.\]

**Proof.** By using the action of the Weyl group $W$ on $\mathcal{B}(\lambda)$, we are reduced to the case $w = e$. Now, we show that $\mathcal{B}(\lambda)_\lambda = \{ u_\lambda \}$. Let $b \in \mathcal{B}(\lambda)_\lambda$. Then, there exists $n \in \mathbb{Z}_{\geq 0}$ such that
Let $\lambda, \mu \in P$ be integral weights (not necessarily dominant). Then, $\mathcal{B}(\lambda) \cong \mathcal{B}(\mu)$ as $U_q(\mathfrak{g})$-crystals if and only if $\lambda \in W \mu$.

**Proof.** We know from Remark 3.5(2) that if $\lambda \in W \mu$, then $\mathcal{B}(\lambda) \cong \mathcal{B}(\mu)$ as $U_q(\mathfrak{g})$-crystals. Conversely, suppose that $\mathcal{B}(\lambda) \cong \mathcal{B}(\mu)$ as $U_q(\mathfrak{g})$-crystals. Let $b \in \mathcal{B}(\lambda)$ be the element corresponding to $u_\mu \in \mathcal{B}(\mu)$ under the isomorphism $\mathcal{B}(\lambda) \cong \mathcal{B}(\mu)$ of $U_q(\mathfrak{g})$-crystals; note that $b \in \mathcal{B}(\lambda)$ is an extremal element of weight $\mu$ since so is $u_\mu$. Take $n \in \mathbb{Z}_{\geq 0}$ such that $b \in \mathcal{B}(\lambda) \cap \mathcal{B}^\lambda_{[n]}$, and let $w \in W_{[n]}$ be such that $w \mu = \mu_{[n]}$ (recall that $\mu_{[n]}$ is the unique element of $W_{[n]} \mu$ that is $[n]$-dominant). Then, since $S_w$ is defined by using only the Kashiwara operators $e_i$ and $f_i$ for $i \in [n]$, we see that $S_w b$ is contained in $\mathcal{B}(\lambda) \cap \mathcal{B}^\lambda_{[n]}$. Furthermore, since $b \in \mathcal{B}(\lambda)$ is an extremal element of weight $\mu$, and since $\text{wt}(S_w b) = w \mu = \mu_{[n]}$ is $[n]$-dominant, we deduce from Definition 3.2(1) that $S_w b \in \mathcal{B}(\lambda) \cap \mathcal{B}^\lambda_{[n]}$ is an $[n]$-maximal element. Now we recall that, by the Claim in the proof of Proposition 3.7 and by Remark 3.6 with $J = [n]$, the $U_q(\mathfrak{g}_{[n]})$-crystal $\mathcal{B}(\lambda) \cap \mathcal{B}^\lambda_{[n]}$ is isomorphic to the crystal basis of the finite-dimensional irreducible $U_q(\mathfrak{g}_{[n]})$-module of highest weight $\lambda_{[n]}$. Therefore, we deduce that the element $S_w b$ is the (unique) $[n]$-maximal element of $\mathcal{B}(\lambda) \cap \mathcal{B}^\lambda_{[n]}$ of weight $\lambda_{[n]}$, and hence that $\mu_{[n]} = w \mu = \text{wt}(S_w b) = \lambda_{[n]} \in W_{[n]} \lambda$, which implies that $\lambda \in W \mu$. Thus we have proved the proposition. 

### 3.3 Lakshmibai-Seshadri paths and crystal structure on them

In this subsection, following [Li1] and [Li2], we review basic facts about Lakshmibai-Seshadri paths and crystal structure on them; note that all results in [Li1] and [Li2] that we use in this paper remain valid in the case of infinite rank affine Lie algebras. We take and fix an arbitrary (not necessarily dominant) integral weight $\lambda \in P$.

**Definition 3.11.** Let $\mu, \nu$ be elements of $W \lambda$. We write $\mu > \nu$ if there exist a sequence $\mu = \xi_0, \xi_1, \ldots, \xi_l = \nu$ of elements of $W \lambda$ and a sequence $\beta_1, \ldots, \beta_l \in \Delta^+$ of positive roots for $\mathfrak{g}$, with $l \geq 1$, such that $\xi_m = r_{\beta_m} (\xi_{m-1})$ and $\langle \xi_{m-1}, \beta_m \rangle \in \mathbb{Z}_{<0}$ for all $1 \leq m \leq l$, where $r_\beta \in W$ denotes the reflection with respect to a root $\beta \in \Delta$. In this case, the sequence $\xi_0, \xi_1, \ldots, \xi_l$ above is called a chain for $(\mu, \nu)$ in $W \lambda$. If $\mu > \nu$, then we define $\text{dist}(\mu, \nu)$ to be the maximum length $l$ of all possible chains for $(\mu, \nu)$ in $W \lambda$. 

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Remark 3.12. Let \( \mu, \nu \) be elements of \( W\lambda \) such that \( \mu > \nu \). If \( \mu = \xi_0, \xi_1, \ldots, \xi_l = \nu \) is a chain for \( (\mu, \nu) \) in \( W\lambda \), with corresponding positive roots \( \beta_1, \ldots, \beta_l \in \Delta^+ \), then we have

\[
\nu - \mu = \sum_{m=1}^{l} \mathbb{Z}_{\geq 0}\beta_m < \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i \setminus \{0\}.
\] (3.9)

Definition 3.13. Let \( \mu, \nu \) be elements of \( W\lambda \) such that \( \mu > \nu \), and let \( 0 < a < 1 \) be a rational number. An \( a \)-chain for \( (\mu, \nu) \) in \( W\lambda \) is, by definition, a sequence \( \mu = \xi_0 > \xi_1 > \cdots > \xi_l = \nu \) of elements of \( W\lambda \), with \( l \geq 1 \), such that dist\((\xi_{m-1}, \xi_m)\) = 1 and \( \langle \xi_{m-1}, \beta_m^\vee \rangle \in a^{-1}\mathbb{Z}_{<0} \) for all \( 1 \leq m \leq l \), where \( \beta_m \in \Delta^+ \) is the positive root for \( g \) corresponding to a chain for \( (\xi_{m-1}, \xi_m) \) in \( W\lambda \).

Definition 3.14 ([Li2, §4]). Let \( \lambda \in P \), and let \( (\nu; a) \) be a pair of a sequence \( \nu : \nu_1 > \nu_2 > \cdots > \nu_s \) of elements of \( W\lambda \) and a sequence \( a : 0 = a_0 < a_1 < \cdots < a_s = 1 \) of rational numbers, with \( s \geq 1 \). The pair \( (\nu; a) \) is called a Lakshmibai-Seshadri path (LS path for short) of shape \( \lambda \) (for \( g \)), if for every \( u = 1, 2, \ldots, s-1 \), there exists an \( a_u \)-chain for \( (\nu_u, \nu_{u+1}) \) in \( W\lambda \). We denote by \( \mathcal{B}(\lambda) \) the set of all LS paths of shape \( \lambda \).

Remark 3.15. (1) It is easily seen from the definition that \( \pi_\nu := (\nu; 0, 1) \in \mathcal{B}(\lambda) \) for all \( \nu \in W\lambda \).

(2) It is obvious from the definitions that \( \mathcal{B}(w\lambda) = \mathcal{B}(\lambda) \) for all \( w \in W \).

Let \( J \) be a finite interval in \( I = \mathbb{Z}_{\geq 0} \). We define an LS path of shape \( \lambda \) for \( g_J \) as follows. First we define a partial order \( \succ_J \) on \( W_J\lambda \), which is entirely similar to the one in Definition 3.11 with \( W \) replaced by \( W_J \), and \( \Delta^+ \) by \( \Delta_J^+ = \Delta^+ \cap \bigoplus_{i \in J} \mathbb{Z}\alpha_i \). If \( \mu, \nu \in W_J\lambda \) are such that \( \mu >_J \nu \), then we denote by dist\(_J(\cdot, \cdot)\) the maximal length of all possible chains for \( (\mu, \nu) \) in \( W_J\lambda \). Next, for a rational number \( 0 < a < 1 \), we define \( a \)-chains in \( W_J\lambda \) as in Definition 3.13 with \( W \) replaced by \( W_J \), dist\(_J(\cdot, \cdot)\) by dist\(_J(\cdot, \cdot)\), and \( \Delta^+ \) by \( \Delta_J^+ \). Finally, we define an LS path of shape \( \lambda \) for \( g_J \) in the same way as in Definition 3.14, with \( W \) replaced by \( W_J \). We denote by \( \mathcal{B}_J(\lambda) \) the set of all LS paths of shape \( \lambda \) for \( g_J \).

Lemma 3.16. Let \( J \) be a finite interval of \( I = \mathbb{Z}_{\geq 0} \). Then, the set \( \mathcal{B}_J(\lambda) \) is identical to the subset of \( \mathcal{B}(\lambda) \) consisting of all elements \( \pi = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s) \), with \( s \geq 1 \), satisfying the condition that \( \nu_u \in W_J\lambda \) for all \( 1 \leq u \leq s \).

Proof. First we show that the partial order \( \succ_J \) on \( W_J\lambda \) coincides with the restriction to \( W_J\lambda \) of the partial order \( > \) on \( W\lambda \). Let \( \mu, \nu \in W_J\lambda \). It is obvious from the definitions that if \( \mu >_J \nu \), then \( \mu > \nu \). Conversely, suppose that \( \mu > \nu \), and let \( \mu = \xi_0, \xi_1, \ldots, \xi_l = \nu \) be a chain for \( (\mu, \nu) \) in \( W\lambda \), with corresponding positive roots \( \beta_1, \ldots, \beta_l \in \Delta^+ \). Then, by (3.9), we have \( \nu - \mu = \sum_{m=1}^{l} \mathbb{Z}_{\geq 0}\beta_m \). Also, since \( \mu, \nu \in W_J\lambda \), we deduce that \( \nu - \mu \in \bigoplus_{i \in J} \mathbb{Z}\alpha_i \). Therefore, if follows that \( \beta_m \in \Delta_J^+ \cap \bigoplus_{i \in J} \mathbb{Z}\alpha_i = \Delta_J^+ \) for all \( 1 \leq m \leq l \), and hence \( \xi_m \in W_J\lambda \) for all \( 0 \leq m \leq l \). This implies that \( \mu >_J \nu \), as desired. Furthermore, we see from the
argument above that if $\mu, \nu \in W_J \lambda$ are such that $\mu > \nu$ (or equivalently, $\mu >_J \nu$), then $\text{dist}(\mu, \nu) = \text{dist}_J(\mu, \nu)$.

Now, let $\mu, \nu \in W_J \lambda$ be such that $\mu > \nu$, and let $0 \leq a \leq 1$ be a rational number. Then we deduce from what is shown above, along with the definitions of $a$-chains in $W \lambda$ and of those in $W_J \lambda$, that there exists an $a$-chain for $(\mu, \nu)$ in $W \lambda$ if and only if there exists an $a$-chain for $(\mu, \nu)$ in $W_J \lambda$. Consequently, it follows immediately from the definitions of an LS path of shape $\lambda$ for $g$ and of the one for $g_J$ that

$$
\mathcal{B}_J(\lambda) = \{(\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s) \in \mathcal{B}(\lambda) \mid s \geq 1, \text{ and } \nu_a \in W_J \lambda, 1 \leq u \leq s\}.
$$

This proves the lemma. \qed

It follows from Lemma 3.16 that

$$
\mathcal{B}_{[n]}(\lambda) \subseteq \mathcal{B}_{[n+1]}(\lambda) \quad \text{for all } n \in \mathbb{Z}_{\geq 0}, \text{ and } \mathcal{B}(\lambda) = \bigcup_{n \geq 0} \mathcal{B}_{[n]}(\lambda).
$$

(3.10)

Now, for real numbers $a, b \in \mathbb{R}$ with $a \leq b$, we set $[a, b]_\mathbb{R} := \{t \in \mathbb{R} \mid a \leq t \leq b\}$. A path is, by definition, a piecewise linear, continuous map $\pi : [0, 1]_\mathbb{R} \to \mathbb{R} \otimes_\mathbb{Z} \mathbb{P}$ from $[0, 1]_\mathbb{R}$ to $\mathbb{R} \otimes_\mathbb{Z} \mathbb{P}$ such that $\pi(0) = 0$. Let $\pi = (\underline{\nu}; \underline{a})$ be a pair of a sequence $\nu : \nu_1, \nu_2, \ldots, \nu_s$ of integral weights in $\mathbb{P}$ and a sequence $a : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers. We associate to the pair $\pi = (\underline{\nu}; \underline{a})$ the following path $\pi : [0, 1]_\mathbb{R} \to \mathbb{R} \otimes_\mathbb{Z} \mathbb{P}$:

$$
\pi(t) = \sum_{v=1}^{u-1} (a_v - a_{v-1})\nu_v + (t - a_{u-1})\nu_u \quad \text{for } a_{u-1} \leq t \leq a_u, 1 \leq u \leq s.
$$

(3.11)

It is easily seen that for pairs $\pi = (\underline{\nu}; \underline{a}) \in \mathcal{B}(\lambda)$ and $\pi' = (\underline{\nu'}; \underline{a'}) \in \mathcal{B}(\lambda)$, $\pi$ is identical to $\pi'$ (i.e., $\underline{\nu} = \underline{\nu'}$ and $\underline{a} = \underline{a'}$) if and only if $\pi(t) = \pi'(t)$ for all $t \in [0, 1]_\mathbb{R}$. Hence we can identify the set $\mathcal{B}(\lambda)$ of all LS paths of shape $\lambda$ with the set of corresponding paths via (3.11). Note that the element $\pi_\nu = (\nu; 0, 1) \in \mathcal{B}(\lambda)$ for $\nu \in W \lambda$ corresponds to the straight line path connecting $0 \in \mathbb{R} \otimes_\mathbb{Z} \mathbb{P}$ to $\nu \in \mathbb{P} \subset \mathbb{R} \otimes_\mathbb{Z} \mathbb{P}$.

Remark 3.17. Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s)$ be as above. If $a_u = u/s$ for all $0 \leq u \leq s$, then we write the corresponding path simply as: $\pi = (\nu_1, \nu_2, \ldots, \nu_s)$. Now, noting that the set $\{\langle \lambda, \beta \rangle \mid \beta \in \Delta\} \subseteq \mathbb{Z}$ is a finite set (see (2.8)–(2.10)), we define $N = N_\lambda \in \mathbb{Z}_{\geq 1}$ to be the least common multiple of all nonzero integers in $\{\langle \lambda, \beta \rangle \mid \beta \in \Delta\} \cup \{1\}$. It follows from the definition of $a$-chains that if $\pi = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s) \in \mathcal{B}(\lambda)$, then $Na_u \in \mathbb{Z}$ for all $0 \leq u \leq s$. Consequently, the path corresponding to $\pi$ is identical to the path

$$
\left(\begin{array}{c}
\nu_1, \ldots, \nu_1 \text{ (}\,\text{b}_1\text{ times)} \\
\nu_2, \ldots, \nu_2 \text{ (}\,\text{b}_2\text{ times)} \\
\nu_s, \ldots, \nu_s \text{ (}\,\text{b}_s\text{ times)}
\end{array}\right),
$$

where $b_u := N(a_u - a_{u-1})$ for $1 \leq u \leq s$. 

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We define a $U_q(\mathfrak{g})$-crystal structure on the set $\mathcal{B}(\lambda)$ of all LS paths of shape $\lambda$ as follows. First, recalling from [Li2, Lemma 4.5 a)] that $\pi(1) \in P$ for all $\pi \in \mathcal{B}(\lambda)$, we define $\text{wt}: \mathcal{B}(\lambda) \to P$ by: $\text{wt} \pi = \pi(1)$.

**Remark 3.18.** For every $\pi \in \mathcal{B}(\lambda)$, the level of $\text{wt} \pi = \pi(1)$ is equal to the level $L_\lambda$ of $\lambda$. Indeed, it follows from Remark 3.17 that $\pi(1)$ can be written as: $\pi = (\nu_1, \nu_2, \ldots, \nu_N)$ for some $\nu_1, \nu_2, \ldots, \nu_N \in W_\lambda$. Therefore, by the definition of $\text{wt}: \mathcal{B}(\lambda) \to P$, we have

$$\nu := \text{wt} \pi = \pi(1) = \sum_{M=1}^{N} \nu_M.$$  \hspace{1cm} (3.12)

If we take $n \in \mathbb{Z}_{\geq 3}$ such that $\nu_M \in W_{[n]} \lambda$ for all $1 \leq M \leq N$, and such that $\lambda^{(m)} = L_\lambda$ for all $m \geq n + 1$, then we deduce by using (2.5)–(2.7) that for all $m \geq n + 1$ and $1 \leq M \leq N$, $\nu_M^{(m)} = \lambda^{(m)} = L_\lambda$, and hence

$$\nu^{(m)} = \frac{1}{N} \sum_{M=1}^{N} \nu_M^{(m)} = \frac{1}{N} \sum_{M=1}^{N} L_\lambda = L_\lambda.$$ 

This implies that $L_\nu = L_\lambda$. In particular, if $\lambda \in E$, then $\text{wt} \pi \in E$ for all $\pi \in \mathcal{B}(\lambda)$; recall that $E \subset P$ is identical to the set of integral weights of level zero.

Next, for $\pi \in \mathcal{B}(\lambda)$ and $i \in I$, we define $e_i \pi$ as follows. Set

$$H_i^\pi(t) := \langle \pi(t), h_i \rangle \text{ for } t \in [0, 1]_\mathbb{R}, \text{ and } m_i^\pi := \min\{H_i^\pi(t) \mid t \in [0, 1]_\mathbb{R}\}.$$ 

We know from [Li2, Lemma 4.5 d)] that $m_i^\pi \in \mathbb{Z}_{\leq 0}$. If $m_i^\pi = 0$, then we set $e_i \pi := 0$. If $m_i^\pi \leq -1$, then we set

$$(e_i \pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t_0) + r_i(\pi(t) - \pi(t_0)) & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) + \alpha_i & \text{if } t_1 \leq t \leq 1,
\end{cases}$$

where

$$t_1 := \min\{t \in [0, 1]_\mathbb{R} \mid H_i^\pi(t) = m_i^\pi\},$$
$$t_0 := \max\{t \in [0, t_1]_\mathbb{R} \mid H_i^\pi(t) = m_i^\pi + 1\}.$$ 

It follows from [Li2] Corollary 2 in §4 that $e_i \pi \in \mathcal{B}(\lambda) \cup \{0\}$. Thus, we obtain a map $e_i: \mathcal{B}(\lambda) \to \mathcal{B}(\lambda) \cup \{0\}$. Similarly, for $\pi \in \mathcal{B}(\lambda)$ and $i \in I$, we define $f_i \pi$ as follows. Note that $H_i^\pi(1) - m_i^\pi \in \mathbb{Z}_{\geq 0}$. If $H_i^\pi(1) - m_i^\pi = 0$, then we set $f_i \pi := 0$. If $H_i^\pi(1) - m_i^\pi \geq 1$, then we set

$$(f_i \pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t_0) + r_i(\pi(t) - \pi(t_0)) & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) - \alpha_i & \text{if } t_1 \leq t \leq 1,
\end{cases}$$
Theorem 3.21. Let \( \lambda \in P \) be a (not necessarily dominant) integral weight. Then, the crystal basis \( B(\lambda) \) of the extremal weight \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) of extremal weight \( \lambda \) is isomorphic, as a \( U_q(\mathfrak{g}) \)-crystal, to the crystal \( B(\lambda) \) consisting of all LS paths of shape \( \lambda \).

Proof. We use the notation in the proof of Proposition 3.7. It follows from the Claim in the proof of Proposition 3.7 and (3.13) with \( J = [n] \) that for each \( n \in \mathbb{Z}_{\geq 0} \), there exists an isomorphism

\[
\Phi_n : B(\lambda) \cap B^\lambda \rightarrow B([n])(\lambda)
\]
of $U_q(\mathfrak{g}[n])$-crystals. We show that the following diagram is commutative for all $n \in \mathbb{Z}_{\geq 0}$ (see also \((3.7\) and \((3.10\)):

$$
\begin{array}{ccc}
\mathcal{B}(\lambda) \cap B^\lambda_{[n]} & \xrightarrow{c} & \mathcal{B}(\lambda) \cap B^\lambda_{[n+1]} \\
\Phi_n \downarrow & & \downarrow \Phi_{n+1} \\
\mathbb{B}_{[n]}(\lambda) & \xrightarrow{c} & \mathbb{B}_{[n+1]}(\lambda);
\end{array}
$$

(3.14)

this amounts to showing the equality $\Phi_{n+1}(b) = \Phi_n(b)$ for all $b \in \mathcal{B}(\lambda) \cap B^\lambda_{[n]}$. Now, observe that by the definition of $\mathcal{B}(\lambda) \cap B^\lambda_{[n]}$, the extremal element $u_\lambda = u_\infty \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}(\lambda)$ is contained in $\mathcal{B}(\lambda) \cap B^\lambda_{[n]}$ (and hence in $\mathcal{B}(\lambda) \cap B^\lambda_{[n+1]}$). Therefore, for each $b \in \mathcal{B}(\lambda) \cap B^\lambda_{[n]}$, there exists a monomial $X$ in the Kashiwara operators $e_i$ and $f_i$ for $i \in [n]$ such that $b = Xu_\lambda$, since the crystal graph of the $U_q(\mathfrak{g}[n])$-crystal $\mathcal{B}(\lambda) \cap B^\lambda_{[n]}$ (and hence in $\mathcal{B}(\lambda) \cap B^\lambda_{[n+1]}$). Also, we see from Lemma 3.16 that $\pi_\lambda$ is contained in $\mathbb{B}_{[n]}(\lambda)$ (and hence in $\mathbb{B}_{[n+1]}(\lambda)$), and from Remark 3.20 that $\mathbb{B}_{[n]}(\lambda)_\lambda = \mathbb{B}_{[n+1]}(\lambda)_\lambda = \{\pi_\lambda\}$, which implies that $\Phi_n(u_\lambda) = \Phi_{n+1}(u_\lambda) = \pi_\lambda$. Combining these, we have

$$
\Phi_n(b) = \Phi_n(Xu_\lambda) = X\Phi_n(u_\lambda) = X\pi_\lambda = X\Phi_{n+1}(u_\lambda) = \Phi_{n+1}(Xu_\lambda) = \Phi_{n+1}(b),
$$
as desired.

The commutative diagram (3.14) allows us to define a map $\Phi : \mathcal{B}(\lambda) \to \mathbb{B}(\lambda)$ as follows: for $b \in \mathcal{B}(\lambda)$,

$$
\Phi(b) := \Phi_n(b) \quad \text{if } b \in \mathcal{B}(\lambda) \cap B^\lambda_{[n]} \text{ for some } n \in \mathbb{Z}_{\geq 0} \quad \text{(see (3.5))}.
$$

By using the definition, we can easily verify that the map $\Phi : \mathcal{B}(\lambda) \to \mathbb{B}(\lambda)$ is indeed an isomorphism of $U_q(\mathfrak{g})$-crystals. Thus we have proved the theorem. \(\square\)

The next corollary follows immediately from Theorem 3.21 and Propositions 3.7 3.8

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**Corollary 3.22.** (1) For each $\lambda \in P$, the crystal $\mathbb{B}(\lambda)$ consisting of all LS paths of shape $\lambda$ is a normal $U_q(\mathfrak{g})$-crystal whose crystal graph is connected.

(2) For each $\lambda \in P$ and $w \in W$, we have $\mathbb{B}(\lambda)_{w,\lambda} = \{ S_w\pi_\lambda \} = \{ \pi_{w,\lambda} \}$.

(3) Let $\lambda, \mu \in P$. Then, $\mathbb{B}(\lambda) \cong \mathbb{B}(\mu)$ as $U_q(\mathfrak{g})$-crystals if and only if $\lambda \in W\mu$.

Note that the tensor product of normal $U_q(\mathfrak{g})$-crystals is also a normal $U_q(\mathfrak{g})$-crystal. Hence, by Corollary 3.22(1), the tensor product $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ for $\lambda, \mu \in P$ is a normal $U_q(\mathfrak{g})$-crystal. The following proposition plays a key role in the next section.

**Proposition 3.23.** Let $\lambda, \mu \in P$, and suppose that $\pi \otimes \eta \in \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is an extremal element of weight $\nu \in P$. If we denote by $\mathbb{B}(\pi \otimes \eta)$ the connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ containing the extremal element $\pi \otimes \eta$, then $\mathbb{B}(\pi \otimes \eta)$ is isomorphic, as a $U_q(\mathfrak{g})$-crystal, to $\mathbb{B}(\nu)$ (and hence to $\mathcal{B}(\nu)$).
Proof. For \( n \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{B}_{[n]}(\pi \otimes \eta) \) denote the subset of \( \mathcal{B}(\pi \otimes \eta) \) consisting of all elements of the form: \( X(\pi \otimes \eta) \) for some monomial \( X \) in the Kashiwara operators \( e_i \) and \( f_i \) for \( i \in [n] \); the crystal graph of the \( U_q(\mathfrak{g}_{[n]}) \)-crystal \( \mathcal{B}_{[n]}(\pi \otimes \eta) \) is clearly connected. We claim that there exists an isomorphism \( \Psi_n : \mathcal{B}_{[n]}(\pi \otimes \eta) \cong \mathcal{B}_{[n]}(\nu) \) of \( U_q(\mathfrak{g}_{[n]}) \)-crystals. Indeed, let \( w \in W_{[n]} \) be such that \( w\nu = \nu_{[n]} \). We deduce from the definitions of \( S_w \) and \( \mathcal{B}_{[n]}(\pi \otimes \eta) \) that \( S_w(\pi \otimes \eta) \) is contained in \( \mathcal{B}_{[n]}(\pi \otimes \eta) \). In addition, since \( \pi \otimes \eta \) is extremal by assumption, it is \([n]\)-extremal. Consequently, the element \( S_w(\pi \otimes \eta) \in \mathcal{B}_{[n]}(\pi \otimes \eta) \) is an \([n]\)-extremal element of weight \( w\nu = \nu_{[n]} \). Furthermore, since \( \nu_{[n]} \) is \([n]\)-dominant by definition, we see from Definition 3.2(1) that \( S_w(\pi \otimes \eta) \) is an \([n]\)-maximal element of \( \mathcal{B}_{[n]}(\pi \otimes \eta) \). Also, because \( \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \) is a normal \( U_q(\mathfrak{g}) \)-crystal, and because the crystal graph of the \( U_q(\mathfrak{g}_{[n]}) \)-crystal \( \mathcal{B}_{[n]}(\pi \otimes \eta) \) is connected, we find that \( \mathcal{B}_{[n]}(\pi \otimes \eta) \) is isomorphic, as a \( U_q(\mathfrak{g}_{[n]}) \)-crystal, to the crystal basis of a finite-dimensional irreducible \( U_q(\mathfrak{g}_{[n]}) \)-module. From these facts, along with Remark 3.20, we conclude that \( \mathcal{B}_{[n]}(\pi \otimes \eta) \cong \mathcal{B}_{[n]}(\nu_{[n]}) = \mathcal{B}_{[n]}(\nu) \) as \( U_q(\mathfrak{g}_{[n]}) \)-crystals.

Now, it is obvious from the definitions that

\[
\mathcal{B}_{[n]}(\pi \otimes \eta) \subset \mathcal{B}_{[n+1]}(\pi \otimes \eta) \quad \text{for all } n \in \mathbb{Z}_{\geq 0}, \quad \text{and} \quad \mathcal{B}(\pi \otimes \eta) = \bigcup_{n \geq 0} \mathcal{B}_{[n]}(\pi \otimes \eta). \tag{3.15}
\]

Furthermore, in exactly the same way as in the proof of Theorem 3.21, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{B}_{[n]}(\pi \otimes \eta) & \xrightarrow{\psi_n} & \mathcal{B}_{[n+1]}(\pi \otimes \eta) \\
\downarrow \mathcal{B}_{[n]}(\nu) & & \downarrow \mathcal{B}_{[n+1]}(\nu) \\
\mathcal{B}_{[n]}(\nu) & \xrightarrow{\psi_{n+1}} & \mathcal{B}_{[n+1]}(\nu).
\end{array}
\]

This commutative diagram allows us to define a map \( \Psi : \mathcal{B}(\pi \otimes \eta) \to \mathcal{B}(\nu) \) as follows: for \( b \in \mathcal{B}(\pi \otimes \eta) \),

\[
\Psi(b) := \psi_n(b) \quad \text{if } b \in \mathcal{B}_{[n]}(\pi \otimes \eta) \text{ for some } n \in \mathbb{Z}_{\geq 0}.
\]

By using the definition, we can easily verify that the map \( \Psi \) is an isomorphism of \( U_q(\mathfrak{g}) \)-crystals. Thus we have proved the proposition. \( \square \)

4 Decomposition of tensor products into connected components.

In this section, we consider the decomposition (into connected components) of the tensor product \( \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \) for \( \lambda, \mu \in P \) with \( L_\lambda, L_\mu \geq 0 \); in fact, it turns out that each connected component is isomorphic to \( \mathcal{B}(\nu) \) for some \( \nu \in P \). Our main aim is to give an explicit description of the multiplicity \( m_{\lambda, \mu}^\nu \) of a connected component \( \mathcal{B}(\nu) \) for \( \nu \in P \) in this decomposition. It should be mentioned that our results in this section can be regarded as extensions of the corresponding results in [Kw2] (in type \( A_{\infty} \)) and [Kw3] (in type \( A_\infty \)) to the cases of type \( B_\infty, C_\infty, D_\infty \); see also [Le] for the case \( \lambda, \mu \in E \) (in all types \( A_{\infty}, B_\infty, C_\infty, D_\infty \)).

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4.1 The case $\lambda, \mu \in E$. A partition is, by definition, a weakly increasing sequence $\rho = (\rho^{(0)} \geq \rho^{(1)} \geq \rho^{(2)} \geq \cdots)$ of nonnegative integers such that $\rho^{(j)} = 0$ for all but finitely many $j \in \mathbb{Z}_{\geq 0}$; we call $\rho^{(j)}$ the $j$-th part of $\rho$ for $j \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{P}$ denote the set of all partitions. For $\rho = (\rho^{(0)} \geq \rho^{(1)} \geq \rho^{(2)} \geq \cdots) \in \mathcal{P}$, we define the length of $\rho$, denoted by $l(\rho)$, to be the number of nonzero parts of $\rho$, and define the size of $\rho$, denoted by $|\rho|$, to be the sum of all parts of $\rho$, i.e., $|\rho| := \sum_{j \in \mathbb{Z}_{\geq 0}} \rho^{(j)}$. Also, for $\rho, \kappa, \omega \in \mathcal{P}$, let $LR_{\rho, \kappa}^\omega$ denote the Littlewood-Richardson coefficient for $\rho, \kappa$, and $\omega$ (see, for example, [Le, Chapter 5] or [Ko2, §2]); it is well-known that $LR_{\rho, \kappa}^\omega \neq 0$ only if $|\omega| = |\rho| + |\kappa|.$

Now, let $P_{1, \infty}$ denote the subset of $E$ consisting of all elements $\nu$ such that the sequence $(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \ldots)$ is a partition; observe that if $\nu \in P_{1, \infty}$, then $\langle \nu, h_i \rangle \leq 0$ for all $i \in \mathbb{Z}_{\geq 1}$ (see (2.4)), and hence $\nu$ is $[1, n]$-antidominant for all $n \in \mathbb{Z}_{\geq 1}$. We identify the set $P_{1, \infty}$ with the set $\mathcal{P}$ of all partitions via the correspondence: $\nu \mapsto (\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \ldots)$; under this correspondence, we have $|\nu| = \sum_{j \in \mathbb{Z}_{\neq 0}} |\nu^{(j)}|$, and $l(\nu) = \# Supp(\nu) = \max Supp(\nu) + 1$ for $\nu \in P_{1, \infty}$. Furthermore, using (2.5)–(2.7), we can easily verify that for each $\lambda \in E$, there exists a unique element $\lambda_\dagger \in W\lambda$ such that $\lambda_\dagger \in P_{1, \infty}$ ($\cong \mathcal{P}$). In fact, the corresponding partition $(\lambda^{(0)}_\dagger, \lambda^{(1)}_\dagger, \lambda^{(2)}_\dagger, \ldots) \in \mathcal{P}$ is obtained by arranging the sequence $(|\lambda^{(0)}_\dagger|, |\lambda^{(1)}_\dagger|, |\lambda^{(2)}_\dagger|, \ldots)$ of nonnegative integers in weakly increasing order. Thus, $P_{1, \infty}$ ($\cong \mathcal{P}$) is a complete set of representatives for $W$-orbits in $E$. It follows from Corollary 3.22(3) that if $\nu \neq \nu'$ for $\nu, \nu' \in P_{1, \infty}$, then $B(\nu) \not\cong B(\nu')$ as $U_q(\mathfrak{g})$-crystals.

The following is the main result of this subsection (cf. [Le, Corollary 3.2.5 and Remark following Theorem 4.1.6], and also [Kw2, Theorem 4.10] in type $A_{\infty}$, [Kw3, Proposition 4.9] in type $A_\infty$).

**Theorem 4.1.** Let $\lambda, \mu \in E$. Then, we have the following decomposition into connected components:

\[
\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) = \bigoplus_{\nu \in P_{1, \infty}} \mathbb{B}(\nu)^{\oplus m_{\lambda, \mu}^{\nu}} \text{ as } U_q(\mathfrak{g})\text{-crystals},
\]

where for each $\nu \in P_{1, \infty}$ ($\cong \mathcal{P}$), the multiplicity $m_{\lambda, \mu}^{\nu}$ is equal to the Littlewood-Richardson coefficient $LR_{\lambda_\dagger, \mu_\dagger}^{\nu}$ for the partitions $\lambda_\dagger, \mu_\dagger,$ and $\nu$.

**Remark 4.2.** Because $LR_{\lambda_\dagger, \mu_\dagger}^{\nu} \neq 0$ only if $|\nu| = |\lambda_\dagger| + |\mu_\dagger|$ as noted above, it follows that the total number of connected components of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is finite for $\lambda, \mu \in E$.

Before proving Theorem 4.1, we need the following proposition.

**Proposition 4.3.** Keep the setting of Theorem 4.1. Each connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ contains an extremal element.

**Proof.** Recall from Remark 3.18 that if $\pi \in \mathbb{B}(\lambda)$, then $wt \pi = \pi(1) \in E$. First we claim that

\[
(wt \pi, wt \pi) \leq (\lambda, \lambda) \text{ for all } \pi \in \mathbb{B}(\lambda).
\]
Indeed, define $N = N_\lambda$ as in Remark 3.17 and write $\pi \in \mathbb{B}(\lambda)$ as: $\pi = (\nu_1, \nu_2, \ldots, \nu_N)$ for some $\nu_1, \nu_2, \ldots, \nu_N \in W\lambda \subset E$; note that $\omega \pi = (1/N) \sum_{M=1}^N \nu_M$ by (3.12). Since $(\cdot, \cdot)$ is positive definite on $\bigoplus_{j \in \mathbb{Z}_{>0}} \mathbb{R}e_j$, it follows from the Cauchy-Schwarz inequality that

$$
(\omega \pi, \omega \pi) = \frac{1}{N^2} \left( \sum_{M=1}^N \nu_M, \sum_{M=1}^N \nu_M \right) \leq \frac{1}{N^2} \left( \sum_{M=1}^N (\nu_M, \nu_M) \right)^{1/2}.
$$

In addition, since $(\cdot, \cdot)$ is $W$-invariant, we have

$$
\frac{1}{N^2} \left( \sum_{M=1}^N (\nu_M, \nu_M) \right)^{1/2} = \frac{1}{N^2} \left( \sum_{M=1}^N (\lambda, \lambda) \right)^{1/2} = (\lambda, \lambda).
$$

Combining these, we obtain $(\omega \pi, \omega \pi) \leq (\lambda, \lambda)$, as desired. Similarly, we obtain

$$
(\omega \eta, \omega \eta) \leq (\mu, \mu) \quad \text{for all } \eta \in \mathbb{B}(\mu).
$$

From these inequalities, we see, again by the Cauchy-Schwarz inequality, that for every $\pi \otimes \eta \in \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$,

$$
(\omega (\pi \otimes \eta), \omega (\pi \otimes \eta)) = (\omega \pi + \omega \eta, \omega \pi + \omega \eta) \leq \{(\omega \pi, \omega \pi) + (\omega \eta, \omega \eta)\}^{1/2} \leq \{(\lambda, \lambda) + (\mu, \mu)\}^{1/2}.
$$

Now, let $\mathbb{B}$ be an arbitrary connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$. We deduce from (4.2) that the subset $\{(\omega b, \omega b) \mid b \in \mathbb{B}\}$ of $\mathbb{Z}_{>0}$ is bounded above, and hence that there exists $b_0 \in \mathbb{B}$ for which the equality

$$
(\omega b_0, \omega b_0) = \max\{(\omega b, \omega b) \mid b \in \mathbb{B}\}
$$

holds. From this equality, by arguing as in [Kas3, §9.3], we find that the element $b_0 \in \mathbb{B}$ is extremal. Thus we have proved the proposition. \qed

Combining Propositions 3.23 and 4.3, we conclude that each connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is isomorphic, as a $U_q(\mathfrak{g})$-crystal, to $\mathbb{B}(\xi)$ for some $\xi \in E$, and hence to $\mathbb{B}(\nu)$ for some $\nu \in P_{-1, \infty}^{\lambda}$ by Remark 3.15(2); recall that $P_{-1, \infty}^{\lambda}$ is a complete set of representatives for $W$-orbits in $E$. We will prove that for each $\nu \in P_{-1, \infty}^{\lambda}$, the multiplicity $m_{\lambda, \mu}^{\nu}$ in the decomposition (4.1) is equal to the Littlewood-Richardson coefficient $LR_{\lambda, \mu}^{\nu}$. Because $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) = \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ by Remark 3.15(2), we may and do assume that $\lambda = \lambda_1 \in P_{-1, \infty}$ and $\mu = \mu_1 \in P_{-1, \infty}$ for the rest of this subsection.

**Proposition 4.4.** Keep the setting above.

1. Let $\nu \in P_{-1, \infty}^{\lambda}$. If $\mathbb{B}(\nu)$ is isomorphic, as a $U_q(\mathfrak{g})$-crystal, to a connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$, then we have $|\nu| = |\lambda| + |\mu|$. 

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(2) Let \( m \in \mathbb{Z}_{\geq 1} \) be such that \( m \geq |\lambda| + |\mu| \), and let \( \nu \in P_{[1, \infty)} \). If \( \pi \otimes \eta \in B(\lambda) \otimes B(\mu) \) is an extremal element of weight \( \nu \), then \( \pi \otimes \eta \) is a \([1, m]\)-minimal element contained in \( B_{[1, m]}(\lambda) \otimes B_{[1, m]}(\mu) \).

(3) Let \( m \in \mathbb{Z}_{\geq 1} \) be such that \( m \geq |\lambda| + |\mu| \). Every \([1, m]\)-minimal element of \( B_{[1, m]}(\lambda) \otimes B_{[1, m]}(\mu) \) is extremal.

**Proof.** (1) We prove the assertion by induction on \( |\mu| = \sum_{j \in \mathbb{Z}_{\geq 0}} |\mu^{(j)}| \). If \( |\mu| = 0 \) (and hence \( \mu = 0 \)), then the assertion is obvious. Assume now that \( |\mu| = 1 \), and hence \( \mu = e_0 \). By direct computation, we can check that if \( g \) is of type \( B_\infty \) (resp., \( C_\infty, D_\infty \)), then the crystal graph of \( B(e_0) \) is given by (4.3) (resp., (4.4), (4.5); cf. [Le, §3.2]):

\[
\begin{array}{ccccccc}
& & 3 & & 2 & & 1 & & 0 & & 1 & & 2 & & 3 \\
\pi_{-e_2} & & \pi_{-e_1} & & \pi_0 & & \pi_{e_0} & & \pi_{e_1} & & \pi_{e_2} & & \\
\end{array}
\]  

Here, \( \pi_0 = (-e_0, e_0; 0, 1/2, 1) \).

\[
\begin{array}{ccccccc}
& & 3 & & 2 & & 1 & & 0 & & 1 & & 2 & & 3 \\
\pi_{-e_2} & & \pi_{-e_1} & & \pi_{e_0} & & \pi_{e_1} & & \pi_{e_2} & & \\
\end{array}
\]

Thus, we have

\[
B(e_0) = \{ \pi_{\pm e_j} \mid j \in \mathbb{Z}_{\geq 0} \} \cup \{ \pi_0 = (-e_0, e_0; 0, 1/2, 1) \} \quad \text{if} \ g \text{ is of type } B_\infty,
\]

\[
B(e_0) = \{ \pi_{\pm e_j} \mid j \in \mathbb{Z}_{\geq 0} \} \quad \text{if} \ g \text{ is of type } C_\infty \text{ or } D_\infty.
\]

By our assumption on \( \nu \in P_{[1, \infty)} \), there exists an extremal element \( \pi \otimes \eta \in B(\lambda) \otimes B(\nu) \) of weight \( \nu \). Take \( n \in \mathbb{Z}_{\geq 0} \) such that \( n > \# \text{Supp}(\lambda) \) (\( = \ell(\lambda) \) since \( \lambda \in P_{[1, \infty)} \)) and such that \( \pi \in B_{[n]}(\lambda) \) (see (5.1)). We deduce from Remark 3.20 that there exists a monomial \( X \in e_i^{\text{max}}, i \in [n] \), such that \( X \pi = \pi_{\lambda^{[n]}} \). It follows from the tensor product rule for crystals that

\[
X(\pi \otimes \eta) = (X \pi) \otimes \eta' = \pi_{\lambda^{[n]}} \otimes \eta' \quad \text{for some } \eta' \in B(e_0).
\]

Here we remark that the element \( \pi_{\lambda^{[n]}} \otimes \eta' \) is extremal, since so is \( \pi \otimes \eta \), and \( X \) is a monomial in \( e_i^{\text{max}}, i \in [n] \); also, if we set \( \xi := \text{wt}(\pi_{\lambda^{[n]}} \otimes \eta') \), then \( \xi \) is contained in \( W_{[n]} \nu \subset W \nu \).

Suppose, by contradiction, that \( g \) is of type \( B_\infty \) and \( \eta' = \pi_0 = (-e_0, e_0; 0, 1/2, 1) \). Since \( 0 \not\in \text{Supp}(\lambda^{[n]}) \) by Lemma 2.4 and the choice of \( n \), we have \( \langle \lambda^{[n]}, h_0 \rangle = 0 \), which implies that \( e_0 \pi_{\lambda^{[n]}} = f_0 \pi_{\lambda^{[n]}} = 0 \). Also, we see from the crystal graph (4.3) that neither \( e_0 \eta' \) nor \( f_0 \eta' \) is equal to \( 0 \). Therefore, by the tensor product rule for crystals, we deduce that neither...
\( e_j(\pi_{\lambda[n]} \otimes \eta') \) nor \( f_j(\pi_{\lambda[n]} \otimes \eta') \) is equal to \( 0 \), which contradicts the fact that \( \pi_{\lambda[n]} \otimes \eta' \) is extremal. Also, suppose, by contradiction, that \( \eta' = \pi_{-e_k} \) for some \( k \in \text{Supp}(\lambda[n]) \). We set \( w := r_{k+2}r_{k+3} \cdots r_nr_{n+1} \in W \) (if \( k = n \), then we set \( w := e \), the identity element of \( W \)). Since \( \langle -e_k, h_j \rangle = 0 \) for all \( k + 2 \leq j \leq n + 1 \), we deduce by the tensor product rule for crystals along with Corollary 3.22 (2) that

\[
S_w(\pi_{\lambda[n]} \otimes \eta') = S_w(\pi_{\lambda[n]} \otimes \pi_{-e_k}) = (S_w\pi_{\lambda[n]}) \otimes \pi_{-e_k} = \pi_{w\lambda[n]} \otimes \pi_{-e_k}.
\]

It is easily seen from Lemma 2.4 by using (2.4) and (2.5)–(2.7), that \( \langle w\lambda[n], h_{k+1} \rangle < 0 \), which implies that \( e_{k+1}(w\lambda[n] \otimes \pi_{-e_k}) \) is equal to \( 0 \), which contradicts the fact that \( \pi_{w\lambda[n]} \otimes \pi_{-e_k} = S_w(\pi_{\lambda[n]} \otimes \eta') \) is extremal. Thus, we conclude that \( \eta' = \pi_{e_k} \) for some \( k \in \mathbb{Z}_{\geq 0} \), or \( \eta' = \pi_{-e_k} \) for some \( k \in \mathbb{Z}_{\geq 0} \setminus \text{Supp}(\lambda[n]) \).

Now, we set \( |\xi| := \sum_{j \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_{[n]}| \) (recall that \( \xi = \text{wt}(\pi_{\lambda[n]} \otimes \eta') \)). If \( \eta' = \pi_{e_k} \) for some \( k \in \mathbb{Z}_{\geq 0} \), then since \( \lambda^{(j)}_{[n]} \geq 0 \), \( j \in \mathbb{Z}_{\geq 0} \) (see Lemma 2.4), we find that

\[
|\xi| = \sum_{j \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_{[n]}| + \delta_{jk} = \left( \sum_{j \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_{[n]}| \right) + 1.
\]

Also, if \( \eta' = \pi_{-e_k} \) for some \( k \in \mathbb{Z}_{\geq 0} \setminus \text{Supp}(\lambda[n]) \), then we find that

\[
|\xi| = \sum_{j \in \text{Supp}(\lambda[n])} |\lambda^{(j)}_{[n]}| + \sum_{j \in \mathbb{Z}_{\geq 0} \setminus \text{Supp}(\lambda[n])} |\delta_{jk}| = \left( \sum_{j \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_{[n]}| \right) + 1.
\]

Here it is easily seen by using (2.5)–(2.7) that \( \sum_{j \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_{[n]}| = |\lambda| \), since \( \lambda[n] \in W\lambda \). Similarly, we see that \( |\xi| = |\nu| \) since \( \xi \in W\nu \). Combining these equalities, we obtain \( |\nu| = |\xi| = |\lambda| + 1 = |\lambda| + |\mu| \), as desired.

Assume, therefore, that \( |\mu| > 1 \). We set \( p := \text{max Supp}(\mu) \in \mathbb{Z}_{\geq 0} \), and \( \mu' := \mu - e_p \); note that \( \mu' \in P^{[1,\infty]} \) and \( |\mu'| = |\mu| - 1 \).

**Claim.** The element \( \pi_{e_p} \otimes \pi_{\mu'} \in \mathbb{B}(e_0) \otimes \mathbb{B}(\mu') \) is an extremal element of weight \( \mu \). Therefore, the connected component \( \mathbb{B}(\pi_{e_p} \otimes \pi_{\mu'}) \) of \( \mathbb{B}(e_0) \otimes \mathbb{B}(\mu') \) containing \( \pi_{e_p} \otimes \pi_{\mu'} \) is isomorphic, as a \( U_q(\mathfrak{g}) \)-crystal, to \( \mathbb{B}(\mu) \).

**Proof of Claim.** First, we show that \( f_i(\pi_{e_p} \otimes \pi_{\mu'}) = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \). Since \( \mu' \in P^{[1,\infty]} \), we have \( \langle \mu', h_i \rangle \leq 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \), which implies that \( f_i\pi_{\mu'} = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \). If \( p = 0 \), then we see from (2.1)–(2.3) that \( \langle e_0, h_i \rangle \leq 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \), and hence \( f_i\pi_{e_0} = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \). In this case, by the tensor product rule for crystals, we obtain \( f_i(\pi_{e_0} \otimes \pi_{\mu'}) = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \). So, suppose that \( p > 0 \). We see from (2.1)–(2.3) that \( \langle e_p, h_i \rangle \leq 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \) with \( i \neq p \). Hence, by the same reasoning as above, we obtain \( f_i(\pi_{e_p} \otimes \pi_{\mu'}) = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \) with \( i \neq p \).
Now, since \( \mu' = \mu - e_p \) and \( \langle \mu, h_p \rangle \leq 0 \), with \( \langle e_p, h_p \rangle = 1 \) by (2.1)–(2.3), it follows that \( \langle \mu', h_p \rangle = \langle \mu, h_p \rangle - 1 \leq -1 \), which implies that \( \varepsilon_p(\pi_{\mu'}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_p^k \pi_{\mu'} = 0\} \geq 1 \).

Also, since \( \langle e_p, h_p \rangle = 1 \), we deduce that \( \varphi_p(\pi_{ep}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_p^k \pi_{ep} = 0\} = 1 \).

Consequently, by the tensor product rule for crystals, we obtain \( f_p(\pi_{ep} \otimes \pi_{\mu'}) = \pi_{ep} \otimes f_p \pi_{\mu'} = 0 \). Thus, we have shown that \( f_i(\pi_{ep} \otimes \pi_{\mu'}) = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \).

To prove that \( \pi_{ep} \otimes \pi_{\mu'} \) is extremal, we show that \( \pi_{ep} \otimes \pi_{\mu'} \) is \([n]\)- extremal for all \( n \in \mathbb{Z}_{\geq 1} \) (see Remark 3.3(1)). Fix \( n \in \mathbb{Z}_{\geq 1} \). Since \( \pi_{ep} \otimes \pi_{\mu'} \) is \([1, n]\)-minimal as shown above, it is \([1, n]\)-extremal (see Remark 3.3(3)). Let \( w = w_0^{1, n} \) be the longest element of \( W_{[1, n]} \), and set \( \pi \otimes \eta := S_w(\pi_{ep} \otimes \pi_{\mu'}) \in \mathbb{B}(e_0) \otimes \mathbb{B}(\mu') \); note that \( \pi \otimes \eta \) is \([1, n]\)-extremal. Since the weight of \( \pi \otimes \eta \) is equal to \( w\mu \), which is \([1, n]\)- dominant, and since \( \pi \otimes \eta \) is \([1, n]\)-extremal, it follows immediately from (3.2) that \( \pi \otimes \eta \) is a \([1, n]\)- maximal element. Now, we show that \( e_0(\pi \otimes \eta) = 0 \).

Because the operator \( S_w \) is defined by using only the Kashiwara operators \( e_i \) and \( f_i \) for \( i \in [1, n] \), the element \( \pi \in \mathbb{B}(e_0) \) must be of the form: \( \pi = X \pi_{ep} \) for some monomial \( X \) in the Kashiwara operators \( e_i \) and \( f_i \) for \( i \in [1, n] \). From this fact, we can easily verify by using (4.8)–(4.13) that \( \pi = \pi_{eq} \) for some \( q \in \mathbb{Z}_{\geq 0} \). Hence, noting that \( \langle e_q, h_0 \rangle \geq 0 \) (see (2.1)), we get \( e_0 \pi = e_0 \pi_{eq} = 0 \). Similarly, the element \( \eta \in \mathbb{B}(\mu') \) must be of the form: \( \eta = Y \pi_{\mu'} \) for some monomial \( Y \) in the Kashiwara operators \( e_i \) and \( f_i \) for \( i \in [1, n] \). Define \( N = N_{\mu'} \) as in Remark 3.17 and write \( \eta = Y \pi_{\mu'} \in \mathbb{B}(\mu') \) as: \( \eta = (\nu_1, \nu_2, \ldots, \nu_N) \) for some \( \nu_1, \nu_2, \ldots, \nu_N \in W_{\mu'} \). Then, we deduce from the definition of the Kashiwara operators \( e_i \) and \( f_i \) for \( i \in [1, n] \) that \( \nu_M \in W_{[1, n]}^{\mu'} \) for all \( 1 \leq M \leq N \). Since \( \mu' \in P_{-1, \infty} \), it is easily seen by using (2.5)–(2.7) that \( \nu_M^{(j)} \geq 0 \) for all \( j \in \mathbb{Z}_{\geq 0} \) and \( 1 \leq M \leq N \). Hence we see from (2.4) that \( \langle \nu_M, h_0 \rangle \geq 0 \) for all \( 1 \leq M \leq N \), which implies that \( e_0 \eta = 0 \). From the fact \( e_0 \pi = e_0 \eta = 0 \) just observed, and the tensor product rule for crystals, we obtain \( e_0(\pi \otimes \eta) = 0 \), as desired. Thus, we have shown that \( \pi \otimes \eta \) is \([n]\)- maximal, and hence it is \([n]\)- extremal by Remark 3.3(2). Since \( \pi \otimes \eta = S_w(\pi_{ep} \otimes \pi_{\mu'}) \) and \( w \in W_{[1, n]} \subset W_{[n]} \), we conclude that \( \pi_{ep} \otimes \pi_{\mu'} \) is \([n]\)- extremal. This proves the claim.

By the claim above, we obtain the following embedding of \( U_q(\mathfrak{g})\)- crystals:

\[
\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \hookrightarrow \mathbb{B}(\lambda) \otimes \mathbb{B}(e_0) \otimes \mathbb{B}(\mu').
\]

By our assertion for the case \( \mu = e_0 \) (which is already proved), each connected component of \( \mathbb{B}(\lambda) \otimes \mathbb{B}(e_0) \) is isomorphic, as a \( U_q(\mathfrak{g}) \)-crystal, to \( \mathbb{B}(\xi) \) for some \( \xi \in P_{-1, \infty} \) such that \( |\xi| = |\lambda| + 1 \). Furthermore, it follows from our induction hypothesis that for \( \xi \in P_{-1, \infty} \), each connected component of \( \mathbb{B}(\xi) \otimes \mathbb{B}(\mu') \) is isomorphic, as a \( U_q(\mathfrak{g}) \)-crystal, to \( \mathbb{B}(\nu) \) for some \( \nu \in P_{-1, \infty} \) such that \( |\nu| = |\xi| + |\mu'| \); here, recall that \( |\mu'| = |\mu| - 1 \). Therefore, we conclude that each connected component of \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) is isomorphic, as a \( U_q(\mathfrak{g}) \)-crystal, to \( \mathbb{B}(\nu) \) for some \( \nu \in P_{-1, \infty} \) such that \( |\nu| = |\lambda| + |\mu| \). This completes the proof of part (1).

(2) Since \( \langle \nu, h_i \rangle \leq 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \), and since \( \pi \otimes \eta \) is an extremal element of weight \( \nu \) by assumption, it follows immediately from (3.2) that \( f_i(\pi \otimes \eta) = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \).
Note that by Proposition 3.23, the connected component of $\mathbb{B}_{[1,m]}(\lambda)$ and $\eta \in \mathbb{B}_{[1,m]}(\mu)$. Define $N_\lambda$ and $N_\mu$ as in Remark 3.17 and write $\pi \in \mathbb{B}(\lambda)$ and $\eta \in \mathbb{B}(\mu)$ as:

$$\pi = (\zeta_1, \zeta_2, \ldots, \zeta_{N_\lambda}), \quad \eta = (\xi_1, \xi_2, \ldots, \xi_{N_\mu})$$

for some $\zeta_1, \zeta_2, \ldots, \zeta_{N_\lambda} \in W\lambda$ and $\xi_1, \xi_2, \ldots, \xi_{N_\mu} \in W\mu$, respectively. Then we have (see 3.12)

$$\nu = \frac{1}{N_\lambda} \sum_{M=1}^{N_\lambda} \zeta_M + \frac{1}{N_\mu} \sum_{L=1}^{N_\mu} \xi_L. \quad (4.6)$$

Hence we obtain

$$|\nu| = \sum_{j \in \mathbb{Z}_{\geq 0}} \nu^{(j)} = \sum_{j \in \mathbb{Z}_{\geq 0}} \left\{ \frac{1}{N_\lambda} \sum_{M=1}^{N_\lambda} \zeta^{(j)}_M + \frac{1}{N_\mu} \sum_{L=1}^{N_\mu} \xi^{(j)}_L \right\}$$

$$= \frac{1}{N_\lambda} \sum_{M=1}^{N_\lambda} \left\{ \sum_{j \in \mathbb{Z}_{\geq 0}} \zeta^{(j)}_M \right\} + \frac{1}{N_\mu} \sum_{L=1}^{N_\mu} \left\{ \sum_{j \in \mathbb{Z}_{\geq 0}} \xi^{(j)}_L \right\}.$$

Since $\zeta_M \in W\lambda$ for all $1 \leq M \leq N_\lambda$, and $\lambda \in P_{1,\infty}^-$, we deduce that

$$\sum_{j \in \mathbb{Z}_{\geq 0}} \zeta^{(j)}_M \leq |\lambda| = \sum_{j \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}|$$

for all $1 \leq M \leq N_\lambda$; note that

$$\sum_{j \in \mathbb{Z}_{\geq 0}} \zeta^{(j)}_M = |\lambda| \quad \text{if and only if} \quad \zeta^{(j)}_M \geq 0 \text{ for all } j \in \mathbb{Z}_{\geq 0}. \quad (4.7)$$

Similarly, since $\xi_L \in W\mu$ for all $1 \leq L \leq N_\mu$, and $\mu \in P_{1,\infty}^-$, we have

$$\sum_{j \in \mathbb{Z}_{\geq 0}} \xi^{(j)}_L \leq |\mu| = \sum_{j \in \mathbb{Z}_{\geq 0}} |\mu^{(j)}|$$

for all $1 \leq L \leq N_\mu$; note that

$$\sum_{j \in \mathbb{Z}_{\geq 0}} \xi^{(j)}_L = |\mu| \quad \text{if and only if} \quad \xi^{(j)}_L \geq 0 \text{ for all } j \in \mathbb{Z}_{\geq 0}. \quad (4.8)$$

Combining these, we infer that

$$|\nu| = \frac{1}{N_\lambda} \sum_{M=1}^{N_\lambda} \left\{ \sum_{j \in \mathbb{Z}_{\geq 0}} \zeta^{(j)}_M \right\} + \frac{1}{N_\mu} \sum_{L=1}^{N_\mu} \left\{ \sum_{j \in \mathbb{Z}_{\geq 0}} \xi^{(j)}_L \right\} \leq |\lambda| + |\mu|. \quad (4.9)$$

Note that by Proposition 3.23 the connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ containing $\pi \otimes \eta$ is isomorphic, as $U_q(\mathfrak{g})$-crystal, to $\mathbb{B}(\nu)$. Therefore, by part (1), we obtain $|\nu| = |\lambda| + |\mu|$.  

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From this fact and (4.9), we deduce that \( \sum_{j \in \mathbb{Z}_{\geq 0}} \zeta_{M}^{(j)} = |\lambda| \) for all \( 1 \leq M \leq N_{\lambda} \), and \( \sum_{j \in \mathbb{Z}_{\geq 0}} \xi_{L}^{(j)} = |\mu| \) for all \( 1 \leq L \leq N_{\mu} \), and hence that by (4.7) and (4.8),
\[
\begin{cases}
\zeta_{M}^{(j)} \geq 0 & \text{for all } j \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq M \leq N_{\lambda}, \\
\xi_{L}^{(j)} \geq 0 & \text{for all } j \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq L \leq N_{\mu}.
\end{cases}
\]

(4.10)

Since \( \ell(\nu) \leq |\nu| = |\lambda| + |\mu| \), and \( m \geq |\lambda| + |\mu| \) by assumption, we have \( \nu^{(j)} = 0 \) for all \( j \geq m + 1 \). Consequently, by (4.10) and (4.6),
\[
\begin{cases}
\zeta_{M}^{(j)} = 0 & \text{for all } j \geq m + 1 \text{ and } 1 \leq M \leq N_{\lambda}, \\
\xi_{L}^{(j)} = 0 & \text{for all } j \geq m + 1 \text{ and } 1 \leq L \leq N_{\mu}.
\end{cases}
\]

(4.11)

Now, taking (4.10) and (4.11) into consideration, we see through use of (2.5)–(2.7) that for each \( 1 \leq M \leq N_{\lambda} \), there exists \( w_{M} \in W_{[1, m]} \) such that \( w_{M} \zeta_{M} \in P_{[-1, \infty]} \). However, since \( \zeta_{M} \in W_{\lambda} \) for \( 1 \leq M \leq N_{\lambda} \) and \( \lambda = \lambda_{1} \in P_{[1, \infty]} \), it follows from the uniqueness of \( \lambda_{1} \in W_{\lambda} \) that \( w_{M} \zeta_{M} = \lambda_{1} = \lambda \) for all \( 1 \leq M \leq N_{\lambda} \). Similarly, since \( \mu = \mu_{1} \in P_{[1, \infty]} \), we see from (4.10) and (4.11) that for each \( 1 \leq L \leq N_{\mu} \), there exists \( z_{L} \in W_{[1, m]} \) such that \( z_{L} \xi_{L} = \mu \). Therefore, by Lemma 3.16, we conclude that \( \pi \in B_{[1, m]}(\lambda) \) and \( \eta \in B_{[1, m]}(\mu) \). This completes the proof of part (2).

(3) Let \( \pi \otimes \eta \in B_{[1, m]}(\lambda) \otimes B_{[1, m]}(\mu) \) be an arbitrary \([1, m]\)-minimal element. First, we show that \( f_{i}(\pi \otimes \eta) = 0 \) for all \( i \geq m + 1 \), and hence for all \( i \in \mathbb{Z}_{\geq 1} \). Define \( N = N_{\lambda} \) as in Remark 3.17 and write \( \pi \in B_{[1, m]}(\lambda) \subset B(\lambda) \) as:
\[
\pi = (\zeta_{1}, \zeta_{2}, \ldots, \zeta_{N})
\]
for some \( \zeta_{1}, \zeta_{2}, \ldots, \zeta_{N} \in W_{\lambda} \). Since \( \pi \in B_{[1, m]}(\lambda) \), it follows from Lemma 3.16 that \( \zeta_{M} \in W_{[1, m]} \lambda \) for all \( 1 \leq M \leq N \). Because \( \lambda \in P_{[1, \infty]} \) and \( \text{Supp}(\lambda) = [\ell(\lambda) - 1] \subset [m] \), we deduce through use of (2.5)–(2.7) that for each \( 1 \leq M \leq N \), \( \zeta_{M}^{(j)} \geq 0 \) for all \( j \in \mathbb{Z}_{\geq 0} \), and \( \zeta_{M}^{(j)} = 0 \) for all \( j \geq m + 1 \). Therefore, by (2.4), we obtain \( \langle \zeta_{M}, h_{i} \rangle \leq 0 \) for all \( 1 \leq M \leq N \) and \( i \geq m + 1 \), from which it easily follows that \( f_{i} \pi = 0 \) for all \( i \geq m + 1 \). An entirely similar argument shows that \( f_{i} \eta = 0 \) for all \( i \geq m + 1 \). Consequently, by the tensor product rule for crystals, we obtain \( f_{i}(\pi \otimes \eta) = 0 \) for all \( i \geq m + 1 \). Furthermore, in the course of the argument above, we also find (see (4.10)) that if we set \( \nu := \text{wt}(\pi \otimes \eta) \), then \( \nu^{(j)} \geq 0 \) for all \( j \in \mathbb{Z}_{\geq 0} \). In addition, since \( B(\lambda) \otimes B(\mu) \) is a normal \( U_{q}(\mathfrak{g}) \)-crystal, and \( f_{i}(\pi \otimes \eta) = 0 \) for all \( i \in \mathbb{Z}_{\geq 1} \), it follows (for example, from the representation theory of \( \text{sl}_{2}(\mathbb{C}) \)) that \( \langle \nu, h_{i} \rangle \leq 0 \) for all \( i \in \mathbb{Z}_{\geq 2} \). Combining these inequalities, we conclude that \( \nu \in P_{[1, \infty]} \).

To prove that \( \pi \otimes \eta \) is extremal, we show that \( \pi \otimes \eta \) is \([n]\)-extremal for all \( n \in \mathbb{Z}_{\geq m} \) (see Remark 3.3(1)). Fix \( n \in \mathbb{Z}_{\geq 2} \). Let \( w = w_{0}^{[1, n]} \) be the longest element of \( W_{[1, n]} \), and set \( \pi' \otimes \eta' := S_{w}(\pi \otimes \eta) \in B_{[1, n]}(\lambda) \otimes B_{[1, n]}(\mu) \). Since \( \pi \otimes \eta \) is \([1, n]\)-minimal from what is shown above, we see from Remark 3.3(3) that \( \pi \otimes \eta \) is \([1, n]\)-extremal, and hence so is
\( \pi' \otimes \eta' = S_w(\pi \otimes \eta) \). Because the weight of \( \pi' \otimes \eta' \) is equal to \( w\nu \), which is \([1, n]\)-dominant, and because \( \pi' \otimes \eta' \) is \([1, n]\)-extremal, it follows immediately from (3.2) that \( \pi' \otimes \eta' \) is \([1, n]\)-maximal. Furthermore, by the argument used to show the equality “\( e_0 \eta = 0' \)” in the proof of the Claim (for part (1)), we can show that \( e_0 \pi' = e_0 \eta' = 0 \), and hence \( e_0(\pi' \otimes \eta') = 0 \). Thus, we have shown that \( \pi' \otimes \eta' \) is \([n]\)-maximal, and hence it is \([n]\)-extremal by Remark 3.3(2).

This implies that \( \pi \otimes \eta \) is \([n]\)-extremal since \( \pi' \otimes \eta' = S_w(\pi \otimes \eta) \). This completes the proof of part (3).

\[ \square \]

**Proof of Theorem 4.7** Let \( \nu \in P_1^{[1, \infty]} \). If \( |\nu| \neq |\lambda| + |\mu| \), then it follows from Proposition 4.3(1) that there exists no connected component of \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) isomorphic to \( \mathbb{B}(\nu) \), and hence that \( m_{\lambda, \mu}^\nu = 0 \) in the decomposition (4.1). Also, recall that \( \mathbb{L}_{\lambda, \mu}^\nu = 0 \) if \( |\nu| \neq |\lambda| + |\mu| \). Hence we have \( m_{\lambda, \mu}^\nu = 0 = \mathbb{L}_{\lambda, \mu}^\nu \) in this case.

Assume, therefore, that \( |\nu| = |\lambda| + |\mu| \). We see from Corollary 3.22(3) and Proposition 3.8(2) that there exists a one-to-one correspondence between the set of extremal elements of weight \( \nu \) in \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) and the set of connected components of \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) that is isomorphic to \( \mathbb{B}(\nu) \). From this fact, we deduce that the multiplicity \( m_{\lambda, \mu}^\nu \) in the decomposition (4.1) is equal to the number of extremal elements of weight \( \nu \) in \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \); namely, we obtain

\[
m_{\lambda, \mu}^\nu = \{ \pi \otimes \eta \in \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \mid \pi \otimes \eta \text{ is an extremal element of weight } \nu \}.
\]

Fix \( m \in \mathbb{Z}_{\geq 1} \) such that \( m \geq |\lambda| + |\mu| \). Then we deduce from Proposition 4.3(2), (3) that the set of extremal elements of weight \( \nu \) in \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) is identical to the set of \([1, m]\)-minimal elements of weight \( \nu \) in \( \mathbb{B}[1, m](\lambda) \otimes \mathbb{B}[1, m](\mu) \). Thus,

\[
m_{\lambda, \mu}^\nu = \{ \pi \otimes \eta \in \mathbb{B}[1, m](\lambda) \otimes \mathbb{B}[1, m](\mu) \mid \pi \otimes \eta \text{ is a } [1, m]\text{-minimal element of weight } \nu \}.
\]

Recall that \( \mathfrak{g}[1, m] \) is a “reductive” Lie algebra of type \( A_m \) (see Remark 2.2), and that \( \lambda, \mu, \) and \( \nu \) are \([1, m]\)-antidominant since they are contained in \( P_1^{[1, \infty]} \). Therefore, by Remark 3.20 \( \mathbb{B}[1, m](\lambda) \) (resp., \( \mathbb{B}[1, m](\mu) \)) is isomorphic, as a \( U_q(\mathfrak{g}[1, m]) \)-crystal, to the crystal basis of the finite-dimensional irreducible \( U_q(\mathfrak{g}[1, m]) \)-module \( V[1, m](\lambda) \) (resp., \( V[1, m](\mu) \)) of lowest weight \( \lambda \) (resp., \( \mu \)). Consequently, the number of \([1, m]\)-minimal elements of weight \( \nu \) in the tensor product \( \mathbb{B}[1, m](\lambda) \otimes \mathbb{B}[1, m](\mu) \) is equal to the multiplicity \( [V[1, m](\lambda) \otimes V[1, m](\mu) : V[1, m](\nu)] \) of the finite-dimensional irreducible \( U_q(\mathfrak{g}[1, m]) \)-module \( V[1, m](\nu) \) of lowest weight \( \nu \) in the tensor product \( \mathbb{U}_q(\mathfrak{g}[1, m]) \)-module \( V[1, m](\lambda) \otimes V[1, m](\mu) \). Summarizing, we have

\[
m_{\lambda, \mu}^\nu = [V[1, m](\lambda) \otimes V[1, m](\mu) : V[1, m](\nu)].
\]

Here, noting that

\[
\begin{align*}
\ell(\lambda) & \leq |\lambda| \leq |\lambda| + |\mu| \leq m, \\
\ell(\mu) & \leq |\mu| \leq |\lambda| + |\mu| \leq m, \\
\ell(\nu) & \leq |\nu| = |\lambda| + |\mu| \leq m,
\end{align*}
\]
we can easily check that the Young diagram corresponding canonically to the highest weight of the finite-dimensional irreducible $U_q(\mathfrak{g}_{[m]})$-module $V_{[1,m]}(\lambda)$ (resp., $V_{[1,m]}(\mu)$, $V_{[1,m]}(\nu)$) is identical to the Young diagram of the partition $\lambda$ (resp., $\mu$, $\nu$) $\in P_{[1,\infty]} \cong \mathcal{P}$. Therefore, it follows immediately (see, for example, [Kw, Chapter 8, Section 3, Corollary 2]) that

$$m_{\lambda, \mu}^{\nu} = [V_{[1,m]}(\lambda) \otimes V_{[1,m]}(\mu) : V_{[1,m]}(\nu)] = LR_{\lambda, \mu}^{\nu}.$$ 

Thus, we have proved Theorem 4.1. \[\]

4.2 The case $\lambda \in E$, $\mu \in P_+$. First we prove the following proposition (cf. [Kw2, the proof of Corollary 4.11] in type $A_{+\infty}$ and [Kw3, Proposition 3.11] in type $A_{\infty}$).

**Proposition 4.5.** Let $\lambda \in E$, and $\mu \in P_+$. The crystal graph of the tensor product $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is connected.

**Proof.** Since $\mu \in P_+$, it follows from Theorem 3.21 and Remark 3.5(1) that $\mathcal{B}(\mu)$ is isomorphic, as a $U_q(\mathfrak{g})$-crystal, to the crystal basis of the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight $\mu$. Therefore, the weight $\text{wt}(\eta)$ of an element $\eta \in \mathcal{B}(\mu)$ is of the form:

$$\text{wt}(\eta) = \mu - \sum_{i \in I} m_i \alpha_i$$

for some $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$; in this case, we set

$$\text{dep}(\eta) := \sum_{i \in I} m_i \in \mathbb{Z}_{\geq 0}.$$ 

Let $\mathcal{B}(\pi_\lambda \otimes \pi_\mu)$ denote the connected component of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ containing $\pi_\lambda \otimes \pi_\mu$. We will show by induction on $\text{dep}(\eta)$ that every element $\pi \otimes \eta \in \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is contained in this connected component $\mathcal{B}(\pi_\lambda \otimes \pi_\mu)$.

First, suppose that $\text{dep}(\eta) = 0$. Then we have $\eta = \pi_\mu$. Since the crystal graph of $\mathcal{B}(\lambda)$ is connected by Corollary 3.22(1), there exists a monomial $X_1$ in the Kashiwara operators $e_i$ and $f_i$ for $i \in I$ such that $X_1 \pi = \pi_\lambda$. Note that if $f_i \pi' \neq 0$ (resp., $e_i \pi' \neq 0$) for some $\pi' \in \mathcal{B}(\lambda)$ and $i \in I$, then $f_i(\pi' \otimes \pi_\mu) = (f_i \pi') \otimes \pi_\mu$ (resp., $e_i(\pi' \otimes \pi_\mu) = (e_i \pi') \otimes \pi_\mu$) by the tensor product rule for crystals. From this, it follows that

$$X_1(\pi \otimes \pi_\mu) = (X_1 \pi) \otimes \pi_\mu = \pi_\lambda \otimes \pi_\mu,$$

as desired.

Next, suppose that $\text{dep}(\eta) > 0$; note that $e_i \eta \neq 0$ for some $i \in I$ since $\eta$ is not the (unique) maximal element $\pi_\mu$ of $\mathcal{B}(\mu)$. Now we take $n \in \mathbb{Z}_{\geq 0}$ such that

(i) $\text{Supp}(\lambda) \subseteq [n] = \{0, 1, \ldots, n\}$;

(ii) $\pi \in \mathcal{B}_{[n]}(\lambda)$ (see (3.10));

(iii) $e_i \eta \neq 0$ for some $i \in [n]$.

It follows from condition (ii) and Remark 3.20 that there exists a monomial $X_2$ in $e_i^{\text{max}}$, $i \in [n]$, such that $X_2 \pi = \pi_{\lambda_{[n]}}$; by the tensor product rule for crystals, we have

$$X_2(\pi \otimes \eta) = (X_2 \pi) \otimes \eta' = \pi_{\lambda_{[n]}} \otimes \eta'.$$
for some $\eta' \in \mathbb{B}(\mu)$. If $\eta' \neq \eta$, then we have $\text{dep}(\eta') < \text{dep}(\eta)$ since $X_2$ is a monomial in $e_{i}^{\text{max}}$, $i \in [n]$. Therefore, by our induction hypothesis, $\pi_{\lambda[n]} \otimes \eta'$ is contained in the connected component $\mathbb{B}(\pi_{\lambda} \otimes \pi_{\mu})$, and hence so is $\pi \otimes \eta$. Thus, we may assume that $\eta' = \eta$, i.e.,

$$X_2(\pi \otimes \eta) = \pi_{\lambda[n]} \otimes \eta. \quad (4.12)$$

We set $w_i := r_{i+1}r_{i+2} \cdots r_{n}r_{n+1} \in W$ for $0 \leq i \leq n$, and $w_{n+1} := e$, the identity element of $W$.

**Claim.** We have

$$\langle w_i \lambda[n], h_i \rangle \leq 0 \quad \text{for all } 0 \leq i \leq n+1.$$

**Proof of Claim.** Set $p := \# \text{Supp}(\lambda)$. It follows from Lemma 2.4 and condition (i) that

$$\text{Supp}(\lambda[n]) = \left[ n - p + 1, n \right], \quad \text{and} \quad 0 < \lambda_{[n]}^{(n-p+1)} \leq \cdots \leq \lambda_{[n]}^{(n-1)} \leq \lambda_{[n]}^{(n)}.$$

Let $0 \leq i \leq n+1$. By using (2.5)–(2.7), we deduce from (2.1)–(2.3) that

$$w_i^{-1} h_i = \begin{cases} w_i^{-1}(-e_{i-1}' + e_i') = -e_{i-1}' + e_{i+1}' & \text{if } i \geq 1, \\
0^{-1}(2e_0') = 2e_{n+1}' & \text{if } i = 0, \text{ and } \mathfrak{g} \text{ is of type } B_{\infty}, \\
0^{-1}(e_0') = e_{n+1}' & \text{if } i = 0, \text{ and } \mathfrak{g} \text{ is of type } C_{\infty}, \\
0^{-1}(e_{n+1}' + e_0') = e_0' + e_{n+1}' & \text{if } i = 0, \text{ and } \mathfrak{g} \text{ is of type } D_{\infty}. \end{cases}$$

Since $n - p + 1 > 0$ by condition (i), we have $\langle w_i \lambda[n], h_i \rangle = \langle \lambda[n], w_i^{-1} h_i \rangle \leq 0$ in all the cases above. This proves the claim. □

Let $0 \leq i \leq n$. By the tensor product rule for crystals, we have

$$e_{i+1}^{\text{max}} e_{i+2}^{\text{max}} \cdots e_n^{\text{max}} e_{n+1}^{\text{max}} X_2(\pi \otimes \eta) 
= e_{i+1}^{\text{max}} e_{i+2}^{\text{max}} \cdots e_n^{\text{max}} e_{n+1}^{\text{max}} (\pi_{\lambda[n]} \otimes \eta) \quad \text{by (4.12)}$$

$$= (e_{i+1}^{\text{max}} e_{i+2}^{\text{max}} \cdots e_n^{\text{max}} e_{n+1}^{\text{max}} \pi_{\lambda[n]}) \otimes \eta_i$$

for some $\eta_i \in \mathbb{B}(\mu)$. Using the claim above successively, we find that

$$e_{i+1}^{\text{max}} e_{i+2}^{\text{max}} \cdots e_n^{\text{max}} e_{n+1}^{\text{max}} \pi_{\lambda[n]} = \pi_{r_{i+1}r_{i+2} \cdots r_{n}r_{n+1} \lambda[n]} = \pi_{w_i \lambda[n]},$$

and hence

$$e_{i+1}^{\text{max}} e_{i+2}^{\text{max}} \cdots e_n^{\text{max}} e_{n+1}^{\text{max}} X_2(\pi \otimes \eta) = \pi_{w_i \lambda[n]} \otimes \eta_i.$$
by the claim above, and because $e_i\eta \neq 0$ by the choice of $i \in [n]$, it follows from the tensor product rule for crystals that
\[
e_i e_i^{\max} e_i^{\max} \cdots e_i^{\max} e_i^{\max} X_2(\pi \otimes \eta) = e_i (\pi_{\omega_i \lambda_{[n]}} \otimes \eta) = \pi_{\omega_i \lambda_{[n]}} \otimes (e_i \eta) \quad (\neq \mathbf{0}).
\]

Since \(\text{dep}(e_i \eta) < \text{dep}(\eta)\), the element \(\pi_{\omega_i \lambda_{[n]}} \otimes (e_i \eta)\) is contained in \(\mathcal{B}(\pi_{\lambda} \otimes \pi_{\mu})\) by our induction hypothesis. This implies that \(\pi \otimes \eta \in \mathcal{B}(\pi_{\lambda} \otimes \pi_{\mu})\), thereby completing the proof of Proposition 4.5. 

The following theorem extends [Kw2, Corollary 4.11] in type \(A_{++}\) and [Kw3, Theorem 4.6] in type \(A_{\infty}\).

**Theorem 4.6.** Let \(\lambda \in E\), and \(\mu \in P_+\). Then,
\[
\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \cong \mathcal{B}(\xi + \mu) \quad \text{as } U_q(\mathfrak{g})\text{-crystals}
\]
for some \(\xi \in W\lambda\).

**Proof.** We set \(q := \min \{ j \in \mathbb{Z}_{\geq 0} \mid \mu^{(j)} = L_\mu \}\); we see from Remark 2.1 that
\[
0 \leq \langle \mu^{(0)} \rangle \leq \langle \mu^{(1)} \rangle \leq \cdots \leq \mu^{(q-1)} \leq L_\mu = \mu^{(q)} = \mu^{(q+1)} = \cdots.
\]
If \(q = 0\), then this part is omitted.

Also, we set \(p := \#\text{Supp}(\lambda)\). Then we deduce by using (2.5)–(2.7) that there exists a unique element \(\xi \in W\lambda\) satisfying the conditions that \(\text{Supp}(\xi) = [q, q + p - 1]\), and that
\[
0 < \xi^{(q)} \leq \xi^{(q+1)} \leq \cdots \leq \xi^{(q+p-1)};
\]
note that \(\pi_\xi \in \mathcal{B}(\lambda)\) (see Remark 3.15(1), (2)). We will prove that \(\pi_\xi \otimes \pi_\mu \in \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)\) is an extremal element. If we prove this assertion, then it follows immediately from Propositions 3.28 and 4.5 that \(\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \cong \mathcal{B}(\xi + \mu)\) as \(U_q(\mathfrak{g})\)-crystals, which is what we want to prove.

First we show the following claim.

**Claim.** Let \(\beta \in \Delta\) be a root for \(\mathfrak{g}\).

1. If \(\langle \xi, \beta^\vee \rangle > 0\), then \(\langle \mu, \beta^\vee \rangle \geq 0\).
2. If \(\langle \mu, \beta^\vee \rangle > 0\), then \(\langle \xi, \beta^\vee \rangle \geq 0\).

**Proof of Claim.** The assertions are obvious in the case \(\xi = 0\) (or equivalently, \(\lambda = 0\)). We may therefore assume that \(\xi \neq 0\) (or equivalently, \(\lambda \neq 0\)); note that \(p = \text{Supp}(\lambda) > 0\).

1. If \(\beta\) is a positive root, then we have \(\langle \mu, \beta^\vee \rangle \geq 0\) since \(\mu \in P_+\). Therefore, we may assume that \(\beta\) is a negative root; note that the set \(\Delta^-\) of negative roots is as follows (see (2.8)–(2.10)):
\[
\Delta^- = \begin{cases} 
\{-e_j \mid 1 \leq j \} \cup \{e_j - e_i, -e_j - e_i \mid 0 \leq j < i \} & \text{if } \mathfrak{g} \text{ is of type } B_\infty, \\
\{-2e_j \mid 1 \leq j \} \cup \{e_j - e_i, -e_j - e_i \mid 0 \leq j < i \} & \text{if } \mathfrak{g} \text{ is of type } C_\infty, \\
\{e_j - e_i, -e_j - e_i \mid 0 \leq j < i \} & \text{if } \mathfrak{g} \text{ is of type } D_\infty.
\end{cases}
\]

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The assumption \( \langle \xi, \beta' \rangle > 0 \) now implies that \( \beta \) is of the form: \( e_j - e_i \) for some \( q \leq j \leq q+p-1 \) and \( i \geq q+p \). Since \( \mu^{(j)} = L_\mu \) for all \( j \geq q \) by the definition of \( q \), it follows that \( \langle \mu, \beta' \rangle = 0 \). This proves part (1) of the claim.

(2) The assumption \( \langle \mu, \beta' \rangle > 0 \) implies that \( \beta \) is a positive root. If \( \beta \) is a sum of positive integer multiples of \( e_j \)'s, \( j \in \mathbb{Z}_{\geq 0} \), then it is obvious that \( \langle \xi, \beta' \rangle \geq 0 \). Therefore, we may assume that \( \beta \) is of the form: \(-e_j + e_i \) for some \( 0 \leq j < i \). Since \( \mu^{(j)} = L_\mu \) for all \( j \geq q \) by the definition of \( q \), and since \( \langle \mu, \beta' \rangle > 0 \) by assumption, it follows that \( j \leq q - 1 \). Because \( \text{Supp}(\xi) = \{ q, q+1, \ldots, q+p-1 \} \) and \( 0 < \xi^{(q)} \leq \xi^{(q+1)} \leq \cdots \leq \xi^{(q+p-1)} \), we obtain \( \langle \xi, \beta' \rangle \geq 0 \), as desired. This proves part (2) of the claim.

By an argument entirely similar to the one for \([AK\) Lemma 1.6 (1)]\), we deduce, using the claim above, that \( S_w(\pi_\xi \otimes \pi_\mu) = \pi_{w_\xi} \otimes \pi_{w_\mu} \) for all \( w \in W \) (the proof proceeds by induction on the length of \( w \in W \) with the help of Corollary \([3.22(2)](\)\). From this, it easily follows that \( \pi_\xi \otimes \pi_\mu \) is extremal. This completes the proof of Theorem \([4.6]\) \(\square\)

**Remark 4.7.** Let \( \lambda \in P \) be an integral weight such that \( L_\lambda \geq 0 \). We set

\[
q := \# \{ j \in \mathbb{Z}_{\geq 0} \mid |\lambda^{(j)}| < L_\lambda \} \quad \text{and} \quad p := \# \{ j \in \mathbb{Z}_{\geq 0} \mid |\lambda^{(j)}| > L_\lambda \}.
\]

Then, we deduce by using \([2.5]–[2.7]\) that there exists a unique element \( \nu \in W\lambda \) satisfying the conditions that

\[
\begin{align*}
0 \leq \nu^{(0)} \leq \nu^{(1)} \leq \cdots \leq \nu^{(q-1)} < L_\lambda \leq \nu^{(q)} \leq \nu^{(q+1)} \leq \cdots \leq \nu^{(q+p-1)} \\
\text{If } q = 0, \text{ then this part is omitted.} \\
\nu^{(j)} = L_\lambda \quad \text{for all } j \geq p + q.
\end{align*}
\]

We set

\[
\lambda^+ := \sum_{j=0}^{q-1} \nu^{(j)} e_j + L_\lambda (e_q + e_{q+1} + \cdots) \quad \text{and} \quad \lambda^0 := \sum_{j=q}^{q+p-1} (\nu^{(j)} - L_\lambda) e_j;
\]

note that \( \lambda^+ \in P_+ \), \( \lambda^0 \in E \), and \( \nu = \lambda^+ + \lambda^0 \). Theorem \([4.6]\), along with its proof, yields an isomorphism of \( U_q(\mathfrak{g})\)-crystals:

\[
\mathbb{B}(\lambda) \cong \mathbb{B}(\nu) \cong \mathbb{B}(\lambda^0) \otimes \mathbb{B}(\lambda^+).
\]

The following proposition will be used in the proof of Theorem \([4.19]\) below (cf. \([Kw2\) Lemma 5.1] in type \( A_{\infty} \) and \([Kw3\) Proposition 3.12] in type \( A_\infty \)).

**Proposition 4.8.** Let \( \lambda_1, \lambda_2 \in E \), and \( \mu_1, \mu_2 \in P_+ \). Then,

\[
\mathbb{B}(\lambda_1) \otimes \mathbb{B}(\mu_1) \cong \mathbb{B}(\lambda_2) \otimes \mathbb{B}(\mu_2)
\]

as \( U_q(\mathfrak{g})\)-crystals if and only if \( \lambda_1 \in W\lambda_2 \) and \( \mu_1 = \mu_2 \).
Proof. The “if” part follows immediately from Corollary 3.22(3). We show the “only if” part. As in the proof of Theorem 4.6, we set \( q_m := \min \{ j \in \mathbb{Z}_{\geq 0} \mid \mu_m^{(j)} = L_{\mu_m} \} \) and \( p_m := \# \text{Supp}(\lambda_m) \) for \( m = 1, 2 \). Also, for \( m = 1, 2 \), let \( \xi_m \in W\lambda_m \) be the unique element of \( W\lambda_m \) such that
\[
0 < \xi_m^{(q_m)} \leq \xi_{m}^{(q_m+1)} \leq \cdots \leq \xi_{m}^{(q_m+p_m-1)}.
\]

Then, the proof of Theorem 4.6 shows that
\[
\mathcal{B}(\lambda_m) \otimes \mathcal{B}(\mu_m) \cong \mathcal{B}(\xi_m + \mu_m) \quad \text{as } U_q(\mathfrak{g})\text{-crystals}
\]
for \( m = 1, 2 \); observe that if we set \( \nu_m := \xi_m + \mu_m \) for \( m = 1, 2 \), then we have
\[
0 \leq \nu_m^{(0)} \leq \nu_m^{(1)} \leq \cdots \leq \nu_m^{(q_m+1)} < L_{\mu_m} < \nu_m^{(q_m+2)} \leq \cdots \leq \nu_m^{(q_m+p_m-1)},
\]
for all \( j \geq p_m + q_m \).

Because
\[
\mathcal{B}(\nu_1) \cong \mathcal{B}(\lambda_1) \otimes \mathcal{B}(\mu_1) \cong \mathcal{B}(\lambda_2) \otimes \mathcal{B}(\mu_2) \cong \mathcal{B}(\nu_2) \quad \text{as } U_q(\mathfrak{g})\text{-crystals}
\]
by assumption, we infer by Corollary 3.22(3) that \( \nu_1 \in W
\nu_2 \). However, by using (2.5)–(2.7), we see that \( \nu_2 \) is a unique element of \( W
\nu_2 \) satisfying condition (4.16), and hence that \( \nu_1 = \nu_2 \).

From this equality, it is easily seen that \( L_{\mu_1} = L_{\mu_2} \), and \( q_1 = q_2, p_1 = p_2 \). Therefore, it follows from the definitions of \( \nu_1 \) and \( \nu_2 \) that \( \mu_1 = \mu_2 \), and hence \( \xi_1 = \xi_2 \). Thus we have proved the proposition.

4.3 The case \( \lambda \in P_+, \mu \in E \). Throughout this subsection, we fix \( \lambda \in P_+ \) and \( \mu \in E \). We set \( p := \min \{ j \in \mathbb{Z}_{\geq 0} \mid \lambda^{(j)} = L_{\lambda} \} \in \mathbb{Z}_{\geq 0} \); it follows from Remark 2.1 that
\[
0 \leq \langle \lambda^{(0)} \rangle \leq \langle \lambda^{(1)} \rangle \leq \cdots \leq \langle \lambda^{(p-1)} \rangle < L_{\lambda} = \lambda^{(p)} = \lambda^{(p+1)} = \cdots.
\]

Also, we set
\[
q := \min \{ q \in \mathbb{Z}_{\geq 1} \mid \text{Supp}(\mu) \subset [q - 1] \text{ and } q \geq |\mu| \},
\]
where \( |\mu| := \sum_{j \in \mathbb{Z}_{\geq 0}} |\mu^{(j)}| \in \mathbb{Z}_{\geq 0} \); note that \( q \geq \# \text{Supp}(\mu) \). Recall from Remark 3.17 that if \( N = N_{\mu} \in \mathbb{Z}_{\geq 1} \) denotes the least common multiple of nonzero integers in \( \{ \langle \mu, \beta \rangle \mid \beta \in \Delta \} \cup \{ 1 \} \), then each LS path \( \eta \in \mathcal{B}(\mu) \) of shape \( \mu \) can be written as:
\[
\eta = (\nu_1, \nu_2, \ldots, \nu_N)
\]
for some \( \nu_1, \nu_2, \ldots, \nu_N \in W\mu \), with \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_N \).
Let \( n \in \mathbb{Z}_{\geq 0} \) be such that \( n > p + (N + 1)q \); note that \( n \geq 3 \). Let \( P^{[n]}_+(\lambda, \mu) \) denote the subset of \( P \) consisting of all \([n]\)-dominant integral weights \( \nu \in P \) satisfying the conditions that

\[
\begin{aligned}
L_\nu &= L_\lambda, \\
\nu^{(j)} &= L_\nu (= L_\lambda) \quad \text{for all } j \geq n + 1, \text{ and} \\
\# \{ 0 \leq j \leq n \mid \nu^{(j)} = L_\nu (= L_\lambda) \} &> n - p - Nq;
\end{aligned}
\]

note that \( n - p - Nq \geq q \geq |\mu| \).

Lemma 4.9. Let \( n \in \mathbb{Z}_{\geq 0} \) be such that \( n > p + (N + 1)q \). If \( \pi \otimes \eta \in \mathcal{B}_{[n]}(\lambda) \otimes \mathcal{B}_{[n]}(\mu) \) is an \([n]\)-maximal element, then \( \pi = \pi_\lambda \). Moreover, the weight \( \text{wt}(\pi \otimes \eta) \) of \( \pi \otimes \eta \) is contained in \( P^{[n]}_+(\lambda, \mu) \).

Proof. For the first assertion, observe that if \( e_i \pi \neq 0 \) for some \( i \in [n] \), then \( e_i (\pi \otimes \eta) \neq 0 \) by the tensor product rule for crystals. Now, since \( \pi \otimes \eta \) is \([n]\)-maximal by assumption, we have \( e_i \pi = 0 \) for all \( i \in [n] \), which implies that \( \pi = \pi_{\lambda_{[n]}} = \pi_\lambda \) (see Remark 3.20), as desired.

We show that \( \nu := \text{wt}(\pi \otimes \eta) = \text{wt}(\pi \otimes \eta) = \lambda + \text{wt} \eta \) is contained in the set \( P^{[n]}_+(\lambda, \mu) \). Since \( \pi \otimes \eta \) is \([n]\)-maximal and \( \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \) is a normal \( U_q(\mathfrak{g}) \)-crystal, it is clear that \( \nu = \text{wt}(\pi \otimes \eta) \) is \([n]\)-dominant. Also, we see from Remark 3.18 that the level \( L_\nu \) of \( \nu = \lambda + \text{wt} \eta \) is equal to \( L_\lambda + 0 = L_\lambda \). We write \( \eta \in \mathcal{B}_{[n]}(\mu) \) as: \( \eta = (\nu_1, \nu_2, \ldots, \nu_N) \) for some \( \nu_1, \nu_2, \ldots, \nu_N \in W_\mu \); note that \( \nu_M \in W_{[n]}(\mu) \) for all \( 1 \leq M \leq N \) by Lemma 3.16. Since \( \text{Supp}(\mu) \subset [q - 1] \subset [n] \) by our assumptions, we have \( \text{Supp}(\nu_M) \subset [n] \) for all \( 1 \leq M \leq N \). Therefore, it follows from the equation

\[
\nu = \lambda + \text{wt} \eta = \lambda + \frac{1}{N} \sum_{M=1}^{N} \nu_M
\]

that \( \nu^{(j)} = L_\lambda \) for all \( j \geq n + 1 \) (see Remark 3.18). Furthermore, since \( \# \text{Supp}(\nu_M) = \# \text{Supp}(\mu) \leq q \) for all \( 1 \leq M \leq N \), and since

\[
\text{Supp}(\text{wt} \eta) \subset \bigcup_{1 \leq M \leq N} \text{Supp}(\nu_M),
\]

we have

\[
\# \{ 0 \leq j \leq n \mid \nu^{(j)} = L_\nu (= L_\lambda) \} \geq \# \{ 0 \leq j \leq n \mid \nu^{(j)} = L_\lambda \}
\geq (n - p + 1) - \# \text{Supp}(\text{wt} \eta) \geq (n - p + 1) - \sum_{M=1}^{N} \# \text{Supp}(\nu_M)
\geq (n - p + 1) - Nq > n - p - Nq.
\]

Thus, we have shown that \( \nu = \text{wt}(\pi \otimes \eta) \in P^{[n]}_+(\lambda, \mu) \), thereby proving the lemma. \( \square \)
Let $n \in \mathbb{Z}_{\geq 0}$ be such that $n > p + (N + 1)q$. We deduce from the definition of $P_+^{[n]}(\lambda, \mu)$ that an element $\nu \in P_+^{[n]}(\lambda, \mu)$ satisfies the conditions

\begin{align*}
0 \leq (\nu(0)) \leq \cdots \leq \nu(u_0 - 1) < L_\lambda, \\
\nu(u_0) = \nu(u_0 + 1) = \cdots = \nu(u_1 - 1) = L_\lambda, \\
L_\lambda < \nu(u_1) \leq \nu(u_1 + 1) \leq \cdots \leq \nu(n), \\
\nu(j) = L_\lambda \quad \text{for all } j \geq n + 1,
\end{align*}

(4.18)

for some $0 \leq u_0 < u_1 \leq n + 1$; it follows from the definition of $P_+^{[n]}(\lambda, \mu)$ that $u_1 - u_0 > n - p - Nq > q > 0$. Here, if $u_0 = 0$ (resp., $u_1 = n + 1$), then the first condition (resp., the third condition) is dropped. For the element $\nu \in P_+^{[n]}(\lambda, \mu)$ above, we define $\xi \in P$ by:

\begin{align*}
\xi(j) = \nu(j) & \quad \text{for } 0 \leq j \leq u_0 - 1, \\
\xi(j) = \nu(j) = L_\lambda & \quad \text{for } u_0 \leq j \leq u_1 - 1, \\
\xi(u_1) = L_\lambda, \\
\xi(j) = \nu(j - 1) & \quad \text{for } u_1 + 1 \leq j \leq n + 1, \\
\xi(j) = \nu(j) = L_\lambda & \quad \text{for } j \geq n + 2,
\end{align*}

(4.19)

where $u_0$ and $u_1$ are as in (4.18). It is easily verified that $\xi \in P_+^{[n+1]}(\lambda, \mu)$.

**Lemma 4.10.** Keep the setting above. We have $\xi = \nu_{[n+1]}$. Moreover, if we define a map $\Theta_n : P_+^{[n]}(\lambda, \mu) \to P_+^{[n+1]}(\lambda, \mu)$ by $\Theta_n(\nu) := \nu_{[n+1]}$ for $\nu \in P_+^{[n]}(\lambda, \mu)$, then the map $\Theta_n$ is bijective.

**Proof.** We set $w := r_{u_1 + 1} \cdot \cdots \cdot r_{n + 1} W_{[n+1]}$, where $u_1$ is as in (4.18). Then, by using (2.9)–(2.11), we find that the $\xi$ given by (4.19) is identical to the element $w \nu \in W_{[n+1]} \nu$. Since $\xi$ is $[n+1]$-dominant by the definition (4.19) of $\xi$, we conclude that $\xi = \nu_{[n+1]}$. Thus we have shown that if $\nu \in P_+^{[n]}(\lambda, \mu)$, then $\Theta_n(\nu) = \nu_{[n+1]} \in P_+^{[n+1]}(\lambda, \mu)$. Now, the bijectivity of the map $\Theta_n$ easily follows by examining (4.18) and (4.19). This proves the lemma. \(\square\)

Let $n \in \mathbb{Z}_{\geq 0}$ be such that $n > p + (N + 1)q$. Let $\pi \otimes \eta$ be an $[n + 1]$-maximal element of $B_{[n+1]}(\lambda) \otimes B_{[n+1]}(\mu)$. Then, we have $\text{wt}(\pi \otimes \eta) \in P_+^{[n+1]}(\lambda, \mu)$ by Lemma 4.9 and hence $\text{wt}(\pi \otimes \eta) = \nu_{[n+1]}$ for some $\nu \in P_+^{[n]}(\lambda, \mu)$ by Lemma 4.10.

**Proposition 4.11.** Keep the setting above. Let $x \in W_{[n+1]}$ be such that $x \nu_{[n+1]} = \nu$. Then, $S_x(\pi \otimes \eta)$ is an $[n]$-maximal element of weight $\nu$ contained in $B_{[n]}(\lambda) \otimes B_{[n]}(\mu)$.

**Proof.** We want to prove that there exists $w \in W_{[n+1]}$ such that $S_w(\pi \otimes \eta)$ is an $[n]$-maximal element contained in $B_{[n]}(\lambda) \otimes B_{[n]}(\mu)$. First of all, we see by Lemma 4.9 that $\pi = \pi_\lambda$. We write $\eta \in B_{[n+1]}(\mu)$ as: $\eta = (\nu_1, \nu_2, \ldots, \nu_N)$ for some $\nu_1, \nu_2, \ldots, \nu_N \in W_\mu$, with $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_N$. By the same reasoning as in the proof of Lemma 4.9, we see that
\(\nu_M \in W_{[n+1]M}\) and \(\text{Supp}(\nu_M) \subset [n+1]\) for all \(1 \leq M \leq N\). Furthermore, since \(\langle \lambda, h_i \rangle = 0\) for all \(i \geq p + 1\) by (1.17), and since \(\pi \otimes \eta = \pi_\lambda \otimes \eta\) is \([n+1]\)-maximal by assumption, it follows from the tensor product rule for crystals that \(e_i \eta = 0\) for all \(p + 1 \leq i \leq n + 1\). Consequently, by the definition of the Kashiwara operators \(e_i\), \(p + 1 \leq i \leq n + 1\), we must have \(m_i^\eta = \min\{\langle \eta(t), h_i \rangle \mid t \in [0, 1]_\mathbb{R}\}\) = 0 for all \(p + 1 \leq i \leq n + 1\). Therefore, it follows that \(\langle \nu_1, h_i \rangle \geq 0\) for all \(p + 1 \leq i \leq n + 1\). From this fact, using (2.4), we deduce that

\[
\begin{cases}
\nu_1^{(p)} \leq \nu_1^{(p+1)} \leq \cdots \leq \nu_1^{(s_0)} < 0, \\
\nu_1^{(s_0+1)} = \nu_1^{(s_0+2)} = \cdots = \nu_1^{(s_1)} = 0, \\
0 < \nu_1^{(s_1+1)} \leq \nu_1^{(s_1+2)} \leq \cdots \leq \nu_1^{(n+1)}
\end{cases}
\]  

(4.20)

for some \(p - 1 \leq s_0 \leq s_1 \leq n + 1\). Here, since \(#\text{Supp}(\nu_1) = #\text{Supp}(\mu) \leq q\), the number \((s_0 - p + 1) + (n - s_1 + 1)\) of nonzero elements in (4.20) is less than or equal to \(q\), and hence we have

\[
s_1 - s_0 = (n - p + 2) - \{(s_0 - p + 1) + (n - s_1 + 1)\} \\
> (n - p + 2) - q \geq p + (N + 1)q - p + 2 - q = Nq + 2.
\]  

(4.21)

Also, by using (3.12), we obtain

\[
\text{Supp}(\omega t_\eta) \subset \bigcup_{1 \leq M \leq N} \text{Supp}(\nu_M) \subset [n+1],
\]  

(4.22)

and hence

\[
#\text{Supp}(\omega t_\eta) \leq \sum_{M=1}^{N} #\text{Supp}(\nu_M) = \sum_{M=1}^{N} #\text{Supp}(\mu) \leq \sum_{M=1}^{N} q = Nq.
\]

From this inequality and (4.21), we conclude that there exists \(s_0 + 1 < s \leq s_1\) such that \(s \notin \text{Supp}(\omega t_\eta)\). We set

\[
w_1 := r_{n+1} r_n \cdots r_{s+2} r_{s+1} \in W_{[n+1]},
\]

and \(\eta_1 := S_{w_1} \eta \in \mathbb{B}_{[n+1]}(\mu)\); note that \(s + 1 > s_0 + 2 \geq p + 1\). Since \(\langle \lambda, h_i \rangle = 0\) for all \(i \geq p + 1\), by the tensor product rule for crystals, there follows

\[
S_{w_1}(\pi \otimes \eta) = S_{w_1}(\pi_\lambda \otimes \eta) = \pi_\lambda \otimes (S_{w_1} \eta) = \pi_\lambda \otimes \eta_1.
\]

We claim that \(\eta_1 \in \mathbb{B}_{[n]}(\mu)\). Write it as: \(\eta_1 = (\xi_1, \xi_2, \ldots, \xi_N)\) for some \(\xi_1, \xi_2, \ldots, \xi_N \in W\mu\), with \(\xi_1 \geq \xi_2 \geq \cdots \geq \xi_N\). Then we have \(\xi_M^{(n+1)} \geq \xi_1^{(n+1)} \geq 0\) for all \(1 \leq M \leq N\). Indeed, it follows from the definitions of the Kashiwara operators \(e_i\) and \(f_i\) for \(i \in [n+1]\) (see also [L20 Proposition 4.7]) that \(\xi_i = z \nu_i\) for some subword \(z\) of \(w_1 = r_{n+1} r_n \cdots r_{s+2} r_{s+1}\). Since \(s_0 + 1 < s \leq s_1\), we infer from (4.20), by using (2.5)–(2.7), that \(\xi_1^{(n+1)} \geq 0\). Now, by the same reasoning as in the proof of Lemma 4.9 we see that \(\text{Supp}(\xi_M) \subset [n+1]\) for all \(1 \leq M \leq N\),
and hence \( \text{Supp}(\xi_M - \xi_1) \subseteq [n + 1] \). Also, since \( \xi_1 \geq \xi_M \), we have \( \xi_M - \xi_1 \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subseteq E \) by Remark \( 3.12 \). Combining these facts, we deduce that \( \xi_M - \xi_1 \in \sum_{i=0}^{n+1} \mathbb{Z}_{\geq 0} \alpha_i \), since \( \alpha_i = -e_{i-1} + e_i \) for \( i \geq 1 \) (see (2.4)); observe that by (2.4), \( \xi \in \sum_{i=0}^{n+1} \mathbb{Z}_{\geq 0} \alpha_i \) implies that \( \xi^{(n+1)} \geq 0 \), since \( \alpha^{(n+1)}_i = 1 > 0 \) and \( \alpha^{(n+1)}_i = 0 \) for all \( 0 \leq i \leq n \). Consequently, we conclude that \( (\xi_M - \xi_1)^{(n+1)} \geq 0 \), and hence \( \xi_M^{(n+1)} \geq \xi_1^{(n+1)} \geq 0 \), as desired. Furthermore, since \( s \notin \text{Supp}(\text{wt } \eta) \), and \( \text{Supp}(\text{wt } \eta) \subseteq [n + 1] \) by (4.22), we can easily verify by using (2.5)–(2.7) that

\[
\text{Supp}(\text{wt } \eta_1) = \text{Supp}(w_1(\text{wt } \eta)) \subseteq [n].
\]

Therefore, by taking (3.12) into consideration, we obtain \( \xi_M^{(n+1)} = 0 \) for all \( 1 \leq M \leq N \), and hence \( \text{Supp}(\xi_M) \subseteq [n] \) for all \( 1 \leq M \leq N \). In addition, since \( \xi_M \in W_\mu \) for \( 1 \leq M \leq N \), we see through use of (2.6)–(2.7) that \( \# \text{Supp}(\xi_M) = \# \text{Supp}(\mu) \leq q < n \), and hence \( \text{Supp}(\xi_M) \nsubseteq [n] \) for all \( 1 \leq M \leq N \); recall that \( \text{Supp}(\mu) \subseteq [q - 1] \nsubseteq [n] \) by the definition of \( q \). Hence we can apply Lemma 2.4 to \( \xi_M \) for \( 1 \leq M \leq N \), and also to \( \mu \). Then, noting that \( \xi_M \in W_\mu \) for \( 1 \leq M \leq N \), we find through use of (2.5)–(2.7) that the unique \( [n] \)-dominant element of \( W_{[n]} \xi_M \) is identical to the unique \( [n] \)-dominant element of \( W_{[n]} \mu \) for all \( 1 \leq M \leq N \). As a consequence, we conclude that \( \xi_M \in W_{[n]} \mu \) for all \( 1 \leq M \leq N \), which implies that \( \eta_1 \in \mathcal{B}_{[n]}(\mu) \) by Lemma 3.16. Thus we have shown that \( S_{w_1}(\pi \otimes \eta) = \pi_\lambda \otimes \eta_1 \in \mathcal{B}_{[n]}(\lambda) \otimes \mathcal{B}_{[n]}(\mu) \).

Because \( \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \) is a normal \( \mathcal{U}_q(\mathfrak{g}) \)-crystal, and because \( \pi \otimes \eta = \pi_\lambda \otimes \eta \) is \( [n+1] \)-maximal by assumption, it follows from Remark 3.3(2) that \( \pi \otimes \eta = \pi_\lambda \otimes \eta \) is \( [n+1] \)-extremal, and hence so is \( \pi_\lambda \otimes \eta_1 = S_{w_1}(\pi \otimes \eta) \) (recall that \( w_1 \in W_{[n+1]} \)). If we take \( w_2 \in W_{[n]} \) such that \( w_2(\text{wt}(\pi_\lambda \otimes \eta_1)) \) is \( [n] \)-dominant, then we see from (3.2) that the element \( S_{w_2}(\pi_\lambda \otimes \eta_1) \in \mathcal{B}_{[n]}(\lambda) \otimes \mathcal{B}_{[n]}(\mu) \) is \( [n] \)-maximal since it is \( [n] \)-extremal. Now we set \( w = w_2w_1 \in W_{[n+1]} \). Then the element \( S_{w}(\pi \otimes \eta) = S_{w_2}(\pi_\lambda \otimes \eta_1) \) is an \( [n] \)-maximal element contained in \( \mathcal{B}_{[n]}(\lambda) \otimes \mathcal{B}_{[n]}(\mu) \). This proves the assertion at the very beginning of our proof.

It remains to show that \( S_{w}(\pi \otimes \eta) \) is identical to \( S_x(\pi \otimes \eta) \). If we set \( \xi := \text{wt}(S_w(\pi \otimes \eta)) \), then \( \xi \in P_+^n(\lambda, \mu) \) by Lemma 1.9. Since \( \nu_{[n+1]} = \text{wt}(\pi \otimes \eta) = w^{-1} \xi \in W_{[n+1]} \xi \), we have \( \xi_{[n+1]} = \nu_{[n+1]} \). Therefore, by the bijectivity of the map \( \Theta_n : P_+^n(\lambda, \mu) \rightarrow P_+^{[n]+1}(\lambda, \mu) \) (see Lemma 3.10), we obtain \( \xi = \nu \), and hence \( w\nu_{[n+1]} = \nu \). Since \( \nu_{[n+1]} \) is \( [n+1] \)-dominant, and since \( x \) is elements of \( W_{[n+1]} \) such that \( w\nu_{[n+1]} = \nu = x\nu_{[n+1]} \), there exists \( u \in W_{[n+1]} \) such that \( u\nu_{[n+1]} = \nu_{[n+1]} \) and \( w = xu \); note that \( u \) is equal to a product of the \( r_i \)'s for \( i \in [n + 1] \) such that \( \langle h_{\nu_i}, r_i \rangle = 0 \). Consequently, by using (3.1), we obtain

\[
S_{w}(\pi \otimes \eta) = S_{xu}(\pi \otimes \eta) = S_x S_u(\pi \otimes \eta) = S_x(\pi \otimes \eta).
\]

This completes the proof of the proposition.

Let \( n \in \mathbb{Z}_{\geq 0} \) be such that \( n > p + (N+1)q \). Let \( \mathcal{B}_{\text{max}}^{[n]} \) denote the subset of \( \mathcal{B}_{[n]}(\lambda) \otimes \mathcal{B}_{[n]}(\mu) \) consisting of all \( [n] \)-maximal elements, and set \( \mathcal{B}_{\text{max}, \nu}^{[n]} := \mathcal{B}_{\text{max}}^{[n]} \cap (\mathcal{B}_{[n]}(\lambda) \otimes \mathcal{B}_{[n]}(\mu))_\nu \) for
\( \nu \in P^\dagger_+(\lambda, \mu) \). Then, by Lemma 4.9
\[
\mathbb{B}_\text{max}^{[n]} = \bigcup_{\nu \in P^\dagger_+(\lambda, \mu)} \mathbb{B}_\text{max}^{[n], \nu}.
\]

Also, we see by Lemmas 4.9 and 4.10 that
\[
\mathbb{B}_\text{max}^{[n+1]} = \bigcup_{\xi \in P^\dagger_+(\lambda, \mu)} \mathbb{B}_\text{max}^{[n+1], \xi} = \bigcup_{\nu \in P^\dagger_+(\lambda, \mu)} \mathbb{B}_\text{max}^{[n+1], \nu[n+1]}.
\]

Let \( \nu \in P^\dagger_+(\lambda, \mu) \), and let \( x \in W_{[n+1]} \) be such that \( x \nu_{[n+1]} = \nu \). By Proposition 4.11 we obtain an injective map \( S_x \) from \( \mathbb{B}_\text{max}^{[n+1], \nu_{[n+1]}} \) into \( \mathbb{B}_\text{max}^{[n], \nu} \):
\[
S_x : \mathbb{B}_\text{max}^{[n+1], \nu_{[n+1]}} \rightarrow \mathbb{B}_\text{max}^{[n], \nu}.
\]

**Proposition 4.12.** Keep the setting above. The map \( S_x : \mathbb{B}_\text{max}^{[n+1], \nu_{[n+1]}} \rightarrow \mathbb{B}_\text{max}^{[n], \nu} \) is bijective.

The proof of this proposition is given in [35]. In the rest of this subsection, we take and fix an arbitrary \( m \in \mathbb{Z}_{\geq 0} \) such that \( m > p + (N + 1)q \).

**Corollary 4.13.** Every element \( \pi \otimes \eta \) of \( \mathbb{B}_\text{max}^{[m]} \) is extremal.

**Proof.** We show that \( \pi \otimes \eta \) is \([n]\)-extremal for all \( n \geq m \) (see Remark 3.3(1)). We set \( \nu := \text{wt}(\pi \otimes \eta) \in P^\dagger_+([m]) \). Since \( \nu_{[m]} = \nu \) and \( \nu_{[n+1]} = (\nu_{[n]})_{[n+1]} \) for all \( n \geq m \) by the definitions, we can easily show by induction on \( n \), using Lemma 4.10 that \( \nu_{[n]} \in P^\dagger_+([m]) \) for all \( n \geq m \). For each \( n \geq m \), we take \( x_n \in W_{[n+1]} \) such that \( x_n(\nu_{[n+1]}) = x_n((\nu_{[n]})_{[n+1]}) = \nu_{[n]} \).

Since \( \nu_{[n]} \in P^\dagger_+([m]) \) as seen above, Proposition 4.12 asserts that the map
\[
S_{x_n} : \mathbb{B}_\text{max}^{[n+1], \nu_{[n+1]}} \rightarrow \mathbb{B}_\text{max}^{[n], \nu_{[n]}}
\]
is bijective.

Now, fix \( n \in \mathbb{Z}_{\geq 0} \) such that \( n \geq m \), and set \( y_n := x_m x_{m+1} \cdots x_{n-2} x_{n-1} \); note that \( y_n \in W_{[n]} \). Then, the argument above shows that the composite \( S_{y_n} = S_{x_m} S_{x_{m+1}} \cdots S_{x_{n-2}} S_{x_{n-1}} \) yields a bijective map from \( \mathbb{B}_\text{max}^{[m], \nu_{[n]}} \) onto \( \mathbb{B}_\text{max}^{[m], \nu_{[m]}} \) as follows:
\[
\mathbb{B}_\text{max}^{[m], \nu_{[n]}} \xrightarrow{S_{y_{n-1}}} \mathbb{B}_\text{max}^{[n-1], \nu_{[n-1]}} \xrightarrow{S_{y_{n-2}}} \cdots \xrightarrow{S_{y_1}} \mathbb{B}_\text{max}^{[m+1], \nu_{[m+1]}} \xrightarrow{S_{y_m}} \mathbb{B}_\text{max}^{[m], \nu_{[m]}} = \mathbb{B}_\text{max}^{[m], \nu}.
\]

Consequently, the element \( S_{y_{n-1}}(\pi \otimes \eta) \) is contained in \( \mathbb{B}_\text{max}^{[m], \nu_{[n]}} \), and hence is an \([n]\)-maximal element. Because \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) is a normal \( U_q(\mathfrak{g}) \)-crystal, it follows from Remark 3.3(2) that \( S_{y_{n-1}}(\pi \otimes \eta) \) is \([n]\)-extremal, and hence so is \( \pi \otimes \eta \). This proves the corollary. \( \square \)

**Proposition 4.14.** Each connected component of \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) contains a unique element of \( \mathbb{B}_\text{max}^{[m]} \).
Proof. Let $\pi \otimes \eta \in \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$, and take $n \in \mathbb{Z}_{\geq 0}$, with $n \geq m$, such that $\pi \otimes \eta \in \mathbb{B}[n](\lambda) \otimes \mathbb{B}[n](\mu)$ (see (3.10)). From Remark 3.20 we see that each connected component of $\mathbb{B}[n](\lambda) \otimes \mathbb{B}[n](\mu)$ is isomorphic, as a $U_q(\mathfrak{g}[n])$-crystal, to the crystal basis of a finite-dimensional irreducible $U_q(\mathfrak{g}[n])$-module. Therefore, there exists a monomial $X$ in the Kashiwara operators $e_i$ for $i \in [n]$ such that $X(\pi \otimes \eta) \in \mathbb{B}[n]$. Let $\xi \in P$ be the weight of $X(\pi \otimes \eta)$; note that $\xi \in P_+^{[n]}(\lambda, \mu)$ by Lemma 4.9. We set

$$\nu := \Theta_m^{-1} \Theta_{m+1}^{-1} \cdots \Theta_{n-2}^{-1} \Theta_{n-1}^{-1}(\xi) \in P_+^{[m]}(\lambda, \mu);$$

it is clear that $\xi = \nu_{[n]}$ by Lemma 4.10. Then, the argument in the proof of Corollary 4.13 shows that there exists $y \in W_{[n]}$ such that $S_{\nu-1}$ yields a bijective map from $\mathbb{B}_{\max, \xi} = \mathbb{B}_{\max, \nu_{[n]}}$ onto $\mathbb{B}_{\max, \nu}$. In particular, we have $S_{\nu-1}X(\pi \otimes \eta) \in \mathbb{B}_{\max, \nu} \subset \mathbb{B}_{\max}$. Also, observe that since $y \in W_{[n]}$, the element $S_{\nu-1}X(\pi \otimes \eta)$ lies in the connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ containing $\pi \otimes \eta$. Thus, we have proved that each connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ contains an element of $\mathbb{B}_{\max}^{[m]}$.

It remains to prove the uniqueness assertion. Suppose that $b_1, b_2 \in \mathbb{B}_{\max}^{[m]}$ are contained in the same connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$. We set $\nu := \text{wt } b_1 \in P_+^{[m]}(\lambda, \mu)$ and $\xi := \text{wt } b_2 \in P_+^{[m]}(\lambda, \mu)$. Since $b_1$ and $b_2$ are both extremal by Corollary 4.13, we deduce from Proposition 3.23 that the connected component containing both $b_1$ and $b_2$ is isomorphic to $\mathbb{B}(\nu)$, and also to $\mathbb{B}(\xi)$. Consequently, by Corollary 3.22 (3), we obtain $\nu \in W_{[n]}$. If we take $n \in \mathbb{Z}_{\geq 0}$, with $n \geq m$, such that $\nu \in W_{[n]}$, then we have $\nu_{[n]} = \xi_{[n]}$, which implies that $\nu = \xi$ by Lemma 4.10. Thus, $b_1$ and $b_2$ are both elements of weight $\nu$ in a connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ isomorphic to $\mathbb{B}(\nu)$. Therefore, by Corollary 3.22 (2), we conclude that $b_1 = b_2$, as desired. This proves the proposition.

The following is the main result of this subsection (cf. [Kw2 Corollary 7.3] in type $A_{+\infty}$ and [Kw3 Proposition 5.13] in type $A_\infty$).

**Theorem 4.15.** Let $\lambda \in P_+$ and $\mu \in E$. We take $m \in \mathbb{Z}_{\geq 0}$ such that $m > p + (N + 1)q$ as above. Then, we have the following decomposition into connected components:

$$\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) = \bigoplus_{\nu \in P_+^{[m]}(\lambda, \mu)} \mathbb{B}(\nu)^{\otimes m^{\nu}_{\lambda, \mu}},$$

where for each $\nu \in P_+^{[m]}(\lambda, \mu)$, the multiplicity $m^{\nu}_{\lambda, \mu}$ is equal to $\# \mathbb{B}_{\max, \nu}^{[m]}$.

**Remark 4.16.** Recall from Remark 2.2 that if $\mathfrak{g}$ is of type $B_\infty$ (resp., $C_\infty$, $D_\infty$), then $\mathfrak{g}_{[m]}$ is a “reductive” Lie algebra of type $B_{m+1}$ (resp., $C_{m+1}$, $D_{m+1}$); note that $m > p + (N + 1)q \geq 2$. Furthermore, we know from Remark 3.20 that $\mathbb{B}_{[m]}(\lambda)$ (resp., $\mathbb{B}_{[m]}(\mu)$) is isomorphic, as a $U_q(\mathfrak{g}_{[m]})$-crystal, to the crystal basis of the finite-dimensional irreducible $U_q(\mathfrak{g}_{[m]})$-module $V_{[m]}(\lambda_{[m]}) = V_{[m]}(\lambda)$ (resp., $V_{[m]}(\mu_{[m]})$) of highest weight $\lambda_{[m]} = \lambda$ (resp., $\mu_{[m]}$). Therefore, for
each $\nu \in P_+^{[m]}(\lambda, \mu)$, the number $\#\mathbb{B}_{\max, \nu}^{[m]}$ is equal to the multiplicity $[V_{[m]}(\lambda) \otimes V_{[m]}(\mu_{[m]}): V_{[m]}(\nu)]$ of the finite-dimensional irreducible $U_q(\mathfrak{g}_{[m]})$-module $V_{[m]}(\nu)$ of highest weight $\nu$ in the tensor product $U_q(\mathfrak{g}_{[m]})$-module $V_{[m]}(\lambda) \otimes V_{[m]}(\mu_{[m]})$. Thus,

$$m_{\lambda, \mu}^{\nu} = \#\mathbb{B}_{\max, \nu}^{[m]} = [V_{[m]}(\lambda) \otimes V_{[m]}(\mu_{[m]}): V_{[m]}(\nu)]$$

for each $\nu \in P_+^{[m]}(\lambda, \mu)$. In particular, the number of those elements $\nu \in P_+^{[m]}(\lambda, \mu)$ for which $m_{\lambda, \mu}^{\nu} \neq 0$ is finite, and hence the total number of connected components of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is finite; see [5.3] for an explicit description of the number $\#\mathbb{B}_{\max, \nu}^{[m]}$ in terms of Littlewood-Richardson coefficients.

**Proof of Theorem 4.15.** Let $\mathbb{B}$ be an arbitrary connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$. Then, we know from Proposition 4.14 that there exists a unique element $\pi \otimes \eta$ of $\mathbb{B}$ that is contained in $\mathbb{B}_{\max}^{[m]}$; we set $\nu := \text{wt}(\pi \otimes \eta) \in P_+^{[m]}(\lambda, \mu)$. Since the element $\pi \otimes \eta \in \mathbb{B} \cap \mathbb{B}_{\max}^{[m]}$ is extremal by Corollary 4.13, we see from Proposition 3.23 that $\mathbb{B}$ is isomorphic, as a $U_q(\mathfrak{g})$-crystal, to $\mathbb{B}(\nu)$.

Proposition 4.14 and Corollary 4.13 together with Proposition 3.23 show that for each $\nu \in P_+^{[m]}(\lambda, \mu)$, there exists an one-to-one correspondence between the set of connected components of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ isomorphic to $\mathbb{B}(\nu)$, and the subset $\mathbb{B}_{\max, \nu}^{[m]}$ of $\mathbb{B}_{\max}^{[m]}$ consisting of all elements of weight $\nu$. This implies immediately that $m_{\lambda, \mu}^{\nu} = \#\mathbb{B}_{\max, \nu}^{[m]}$. Thus we have proved the theorem. □

4.4 The case $\lambda, \mu \in P_+$. Let $\lambda, \mu \in P_+$. In this case, we see from Theorem 3.21 and Remark 3.5(1) that $\mathbb{B}(\lambda)$ (resp., $\mathbb{B}(\mu)$) is isomorphic, as a $U_q(\mathfrak{g})$-crystal, to the crystal basis of the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight $\lambda$ (resp., $\mu$). For $\eta \in \mathbb{B}(\mu)$, we say that $\lambda + \eta$ is dominant (resp., $[n]$-dominant for $n \in \mathbb{Z}_{\geq 0}$) if $\langle \lambda + \eta(t), h_i \rangle \geq 0$ for all $t \in [0, 1]_\mathbb{R}$ and $i \in I$ (resp., $i \in [n]$); remark that such an element $\eta \in \mathbb{B}(\mu)$ is said to be “$\lambda$-dominant” (resp., “$\lambda$-dominant” with respect to $\mathfrak{g}_{[n]}$) in the terminology in [11]. It is easy to verify that $\lambda + \eta$ is dominant (resp., $[n]$-dominant) if and only if $\pi_\lambda \otimes \eta \in \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is maximal (resp., $[n]$-maximal).

We have the following (see [2] Proposition 2.3.2; cf. [3] Proposition 4.10 in type $A_\infty$).

**Theorem 4.17.** Let $\lambda, \mu \in P_+$.  

(1) We have the following decomposition into connected components:

$$\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) = \bigoplus_{\eta \in \mathbb{B}(\mu), \lambda + \eta \text{ is dominant}} \mathbb{B}(\lambda + \text{wt}(\eta)).$$

(4.23)

In particular, each connected component of $\mathbb{B}(\lambda) \otimes \mathbb{B}(\mu)$ is isomorphic, as a $U_q(\mathfrak{g})$-crystal, to $\mathbb{B}(\nu)$ for some $\nu \in P_+$.  

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(2) Let \( \nu \in P_+ \). If \( L_\nu \neq L_\lambda + L_\mu \), then the multiplicity \( m^\nu_{\lambda, \mu} \) of \( \mathbb{B}(\nu) \) in the decomposition

\[
\text{(4.23)}
\]

is equal to 0. If \( L_\nu = L_\lambda + L_\mu \), then the multiplicity \( m^\nu_{\lambda, \mu} \) of \( \mathbb{B}(\nu) \) in the decomposition

\[
\text{(4.23)}
\]

is equal to the number

\[
\# \{ \eta \in \mathbb{B}_{[n]}(\mu) \mid \lambda + \eta \text{ is } [n]-\text{dominant, and } \text{wt}(\pi_\lambda \otimes \eta) = \nu \}
\]

for an arbitrary \( n \in \mathbb{Z}_{\geq 3} \) such that \( \lambda^{(j)} = L_\lambda, \mu^{(j)} = L_\mu \), and \( \nu^{(j)} = L_\nu = L_\lambda + L_\mu \) for all \( j \geq n \).

**Remark 4.18.** Recall from Remark 2.2 that if \( \mathfrak{g} \) is of type \( B_\infty \) (resp., \( C_\infty, D_\infty \)), then \( \mathfrak{g}_{[n]} \) is a “reductive” Lie algebra of type \( B_{n+1} \) (resp., \( C_{n+1}, D_{n+1} \)). Furthermore, we know from Remark 3.20 that \( \mathbb{B}_{[n]}(\lambda) \) (resp., \( \mathbb{B}_{[n]}(\mu) \)) is isomorphic, as a \( U_q(\mathfrak{g}_{[n]}) \)-crystal, to the crystal basis of the finite-dimensional irreducible \( U_q(\mathfrak{g}_{[n]}) \)-module of highest weight \( \lambda_{[n]} = \lambda \) (resp., \( \mu_{[n]} = \mu \)). Therefore, it follows from part (2) of Theorem 4.17 together with the result in [Li2, §10], that for each \( \nu = \nu_{[n]} \in P_+ \) such that \( L_\nu = L_\lambda + L_\mu \), the multiplicity \( m^\nu_{\lambda, \mu} \) in the decomposition (4.23) is equal to the tensor product multiplicity of the corresponding finite-dimensional irreducible \( U_q(\mathfrak{g}_{[n]}) \)-modules; see [5.2] for an explicit description of this tensor product multiplicity in terms of Littlewood-Richardson coefficients. In particular, we have \( m^\nu_{\lambda, \mu} < \infty \) for all \( \nu \in P_+ \). However, in the case \( \lambda, \mu \in P_+ \), the total number of connected components of \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) is infinite in general; compare with the other cases, in which the total number of connected components of \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) is finite (see Remarks 4.2 and 4.16 and Theorem 4.6). For example, let

\[
\lambda = \mu := e_1 + e_2 + e_3 + \cdots.
\]

Then, for every \( p \in \mathbb{Z}_{\geq 1} \),

\[
\xi_p := -(e_1 + e_2 + \cdots + e_{p-1}) + e_p + e_{p+1} + \cdots
\]

is an element of \( W_\mu \), and \( \pi_{\xi_p} \) is an element of \( \mathbb{B}(\mu) \) such that \( \lambda + \pi_{\xi_p} \) is dominant. Consequently, it follows from part (1) of Theorem 4.17 that \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) has infinitely many connected components.

**Proof of Theorem 4.17.** The formula (4.23) is just a restatement of the result in [Li2, §10]. While part (2) is essentially the same as [Le, Proposition 2.3.2] (see also its proof), we prefer to give another proof. Let \( \nu \in P_+ \). It is obvious from Remark 3.18 that if \( L_\nu \neq L_\lambda + L_\mu \), then \( m^\nu_{\lambda, \mu} = 0 \) since the level of the weight of an element in \( \mathbb{B}(\lambda) \otimes \mathbb{B}(\mu) \) is equal to \( L_\lambda + L_\mu \). Assume, therefore, that \( L_\nu = L_\lambda + L_\mu \). Fix \( n \in \mathbb{Z}_{\geq 3} \) such that \( \lambda^{(j)} = L_\lambda, \mu^{(j)} = L_\mu \), and \( \nu^{(j)} = L_\nu = L_\lambda + L_\mu \) for all \( j \geq n \). We claim that

\[
\big\{ \eta \in \mathbb{B}(\mu) \mid \lambda + \eta \text{ is dominant, and } \text{wt}(\pi_\lambda \otimes \eta) = \nu \big\} = \\
\big\{ \eta \in \mathbb{B}_{[n]}(\mu) \mid \lambda + \eta \text{ is } [n]-\text{dominant, and } \text{wt}(\pi_\lambda \otimes \eta) = \nu \big\};
\]

(4.25)
part (2) follows immediately from this equation and part (1). Let \( \eta \) be an element in the set on the left-hand side of (1.25). Since \( \lambda + \eta \) is dominant, it is obvious that \( \lambda + \eta \) is \([n]\)-dominant. We show that \( \eta \in \mathcal{B}_{[n]}(\mu) \). Define \( N = N_\mu \) as in Remark 3.17 and write \( \eta \in \mathcal{B}(\mu) \) as: \( \eta = (\xi_1, \xi_2, \ldots, \xi_N) \) for some \( \xi_1, \xi_2, \ldots, \xi_N \in W_\mu \). Then, we have by (3.12),

\[
\nu = \text{wt}(\pi_\lambda \otimes \eta) = \lambda + \frac{1}{N} \sum_{M=1}^{N} \xi_M.
\]

Let \( j \in \mathbb{Z}_{\geq 0} \) be such that \( j \geq n \). Then, by the choice of \( n \), we have \( \lambda^{(j)} = L_\lambda \) and \( \nu^{(j)} = L_\nu = L_\lambda + L_\mu \). Consequently, we obtain

\[
L_\lambda + L_\mu = \nu^{(j)} = \lambda^{(j)} + \frac{1}{N} \sum_{M=1}^{N} \xi_M^{(j)} = \lambda + \frac{1}{N} \sum_{M=1}^{N} \xi_M^{(j)},
\]

and hence \( L_\mu = (1/N) \sum_{M=1}^{N} \xi_M^{(j)} \). Also, by using (2.5)–(2.7), we see from Remark 2.1 that \( \xi_M^{(j)} \leq L_\mu \) for all \( 1 \leq M \leq N \). Combining these, we find that \( \xi_M^{(j)} = L_\mu \) for all \( 1 \leq M \leq N \). Now, for each \( 1 \leq M \leq N \), let \( w_M \in W_{[n]} \) be such that \( \zeta_M := w_M \xi_M \) is the unique \([n]\)-dominant element in \( W_{[n]} \xi_M \). Then we see from Remark 2.3 that

\[
0 \leq \zeta_M^{(0)} \leq \cdots \leq \zeta_M^{(n-1)} \leq \zeta_M^{(n)}.
\]

In addition, by using (2.5)–(2.7), we see that \( \zeta_M^{(n)} \leq L_\mu \) from Remark 2.1 and that \( \zeta_M^{(j)} = L_\mu \) for all \( j \geq n + 1 \) since \( w_M \in W_{[n]} \) and \( \xi_M^{(j)} = L_\mu \) for all \( j \geq n \) as shown above. Therefore, it follows through use of (2.4) that \( \zeta_M \) is dominant, and hence that \( \zeta_M \in W_\mu \) is identical to \( \mu \in P_+ \). Thus, we conclude that \( \xi_M \in W_{[n]} \mu \) for all \( 1 \leq M \leq N \), which implies that \( \eta \in \mathcal{B}_{[n]}(\mu) \) by Lemma 3.16.

Conversely, let \( \eta \) be an element in the set on the right-hand side of (1.25). We show that \( \lambda + \eta \) is dominant. Since \( \lambda + \eta \) is \([n]\)-dominant, it suffices to show that \( \langle \lambda + \eta(t), h_i \rangle \geq 0 \) for all \( t \in [0, 1]_\mathbb{R} \) and \( i \geq n + 1 \). Define \( N = N_\mu \) as in Remark 3.17 and write \( \eta \in \mathcal{B}(\mu) \) as:

\[
\eta = (\xi_1, \xi_2, \ldots, \xi_N) \text{ for some } \xi_1, \xi_2, \ldots, \xi_N \in W_\mu.
\]

Then, by the same reasoning as above, we obtain \( \xi_M^{(j)} = L_\mu \) for all \( j \geq n + 1 \) and \( 1 \leq M \leq N \). Consequently, we derive from (2.1)–(2.3) that \( \langle \xi_M, h_i \rangle = 0 \) for all \( 1 \leq M \leq N \) and \( i \geq n + 1 \), which implies that \( \langle \eta(t), h_i \rangle = 0 \) for all \( t \in [0, 1]_\mathbb{R} \) and \( i \geq n + 1 \). Hence it follows that \( \langle \lambda + \eta(t), h_i \rangle = \langle \lambda, h_i \rangle \geq 0 \) for all \( t \in [0, 1]_\mathbb{R} \) and \( i \geq n + 1 \). Thus we have proved that \( \lambda + \eta \) is dominant. This completes the proof of Theorem 4.17. \( \square \)

4.5 The general case. Finally, in this subsection, we consider the decomposition (into connected components) of the tensor product \( \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \) for general \( \lambda, \mu \in P \) such that \( L_\lambda, L_\mu \geq 0 \). Define \( \lambda^+, \mu^+ \in P_+ \) and \( \lambda^0, \mu^0 \in E \) as in Remark 4.7. Then we have

\[
\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \cong \mathcal{B}(\lambda^0) \otimes \mathcal{B}(\lambda^+) \otimes \mathcal{B}(\mu^0) \otimes \mathcal{B}(\mu^+).
\]
as $U_q(\mathfrak{g})$-crystals. Since $\lambda^+ \in P_+$ and $\mu^0 \in E$, it follows from Theorem 4.15 that
\[
\mathcal{B}(\lambda^0) \otimes \mathcal{B}(\lambda^+) \otimes \mathcal{B}(\mu^0) \otimes \mathcal{B}(\mu^+) \cong \bigoplus_{\xi \in P_{+}^{[m]}(\lambda^+, \mu^0)} \mathcal{B}(\lambda^0) \otimes \mathcal{B}(\xi) \otimes \mathcal{B}(\mu^+) \quad (4.26)
\]
as $U_q(\mathfrak{g})$-crystals, where we take $m \in \mathbb{Z}_{\geq 0}$ (sufficiently large) as in \([1.3]\) with $\lambda$ replaced by $\lambda^+$, and $\mu$ by $\mu^0$. If we define $\xi^+ \in P_+$ and $\xi^0 \in E$ for each $\xi \in P_{+}^{[m]}(\lambda^+, \mu^0)$ as in Remark 4.7, then we have
\[
\mathcal{B}(\lambda^0) \otimes \mathcal{B}(\xi) \otimes \mathcal{B}(\mu^+) \cong \mathcal{B}(\lambda^0) \otimes \mathcal{B}(\xi^0) \otimes \mathcal{B}(\mu^+) \quad (4.27)
as $U_q(\mathfrak{g})$-crystals. Since $\lambda^0$, $\nu^0 \in E$, it follows from Theorem 4.1 that
\[
\mathcal{B}(\lambda^0) \otimes \mathcal{B}(\xi^0) \cong \bigoplus_{\zeta \in P_{+}^{[1, \infty)}} \mathcal{B}(\zeta) \quad \text{as } U_q(\mathfrak{g})\text{-crystals.}
\]
Also, since $\xi^+$, $\mu^+ \in P_+$, it follows from Theorem 4.17 that
\[
\mathcal{B}(\xi^+) \otimes \mathcal{B}(\mu^+) \cong \bigoplus_{\chi \in P_+} \mathcal{B}(\chi) \quad \text{as } U_q(\mathfrak{g})\text{-crystals.}
\]
Combining these, we find that
\[
\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \cong \bigoplus_{\xi \in P_{+}^{[m]}(\lambda^+, \mu^0), \zeta \in P_{+}^{[1, \infty]}, \chi \in P_+} \mathcal{B}(\zeta) \otimes \mathcal{B}(\chi) \quad (4.28)
as $U_q(\mathfrak{g})$-crystals. Recall that $P_{+}^{[1, \infty]}$ is a complete set of representatives for $W$-orbits in $E$. Let $P/W$ denote the set of all $W$-orbits in $P$, which we regard as a subset of $P$ by taking a complete set of representatives for $W$-orbits in $P$; recall Corollary 3.22.(3). Now, using Theorem 4.6 and Remark 4.7 together with Proposition 4.8, we obtain from (4.27) the following theorem (cf. \([Kw2]\) Corollary 7.4] in type $A_{+\infty}$ and \([Kw3]\) Theorem 5.14] in type $A_{\infty}$).

**Theorem 4.19.** Let $\lambda, \mu \in P$ be integral weights of nonnegative levels; namely, $L_\lambda$, $L_\mu \geq 0$. Then, we have the following decomposition into connected components:
\[
\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) = \bigoplus_{\nu \in P/W} \mathcal{B}(\nu)^{m_{\lambda, \mu}^{\nu}}, \quad (4.28)
\]
where for each $\nu \in P/W$, the multiplicity $m_{\lambda, \mu}^{\nu}$ is given as follows:
\[
m_{\lambda, \mu}^{\nu} = \sum_{\xi \in P_{+}^{[m]}(\lambda^+, \mu^0)} m_{\lambda^+, \mu^0}^{\xi} m_{\lambda^+, \xi^0}^{\nu} m_{\xi^+, \mu^+}^{\nu}.
\]
5 Proof of Proposition 4.12.

5.1 Basic notation for Young diagrams. Recall that $\mathcal{P}$ denotes the set of all partitions; we usually identify a partition $\rho \in \mathcal{P}$ with the corresponding Young diagram, which is also denoted by $\rho$ (see, for example, [F]). For $\rho, \kappa, \omega \in \mathcal{P}$, we write $\rho \subset \kappa$ if the Young diagram of $\rho$ is contained in the Young diagram of $\kappa$; in this case, we denote by $\kappa/\rho$ the skew Young diagram obtained from the Young diagram of $\kappa$ by removing that of $\rho$, and by $|\kappa/\rho|$ the total number of boxes in this skew Young diagram, i.e., $|\kappa/\rho| = |\kappa| - |\rho|$. It is well-known that for $\rho, \kappa, \omega \in \mathcal{P}$, the Littlewood-Richardson coefficient $LR_{\rho,\kappa}^{\omega}$ is nonzero only if $\rho \subset \omega$ and $\kappa \subset \omega$. Also, the conjugate of the partition $\rho \in \mathcal{P}$ is denoted by $^t\rho$. In what follows, for $L \in \mathbb{Z}_{\geq 0}$ and $\rho \in \mathcal{P}$, let $\iota_L(\rho)$ denote the partition whose Young diagram is obtained from the Young diagram of $\rho$ by inserting one row with exactly $L$ boxes between an appropriate pair of adjacent rows of the Young diagram of $\rho$ in such a way that the resulting diagram is also a Young diagram. By convention, we set $\iota_L(\rho) = \rho$ if $L = 0$.

We fix $\ell \in \mathbb{Z}_{\geq 3}$; recall from Remark 2.2 that if $\mathfrak{g}$ is of type $B_\infty$ (resp., $C_\infty$, $D_\infty$), then $\mathfrak{g}[\ell]$ is a “reductive” Lie algebra of type $B_{\ell+1}$ (resp., $C_{\ell+1}$, $D_{\ell+1}$). Let $\rho = (\rho^{(0)} \geq \rho^{(1)} \geq \rho^{(2)} \geq \cdots) \in \mathcal{P}$ be a partition whose length $\ell(\rho)$ is less than or equal to $\ell + 1$, and let $J$ be a finite subset of $\mathbb{Z}_{\geq 1}$. We define $\rho^{j,\ell} \in \mathcal{P}$ as follows (cf. [Ko2, §8]). Write the conjugate partition $^t\rho$ of $\rho$ as: $^t\rho = (b_1^{(0)} \geq \cdots \geq b_j^{(0)} \geq \cdots \geq b_j^{(1)} \geq \cdots)$; note that the first part $b_j^{(0)}$ is equal to the length $\ell(\rho)$ of $\rho$, which is less than or equal to $\ell + 1$ by assumption. Then we set

$$b_j^{(t)} = (b_j)_{j \in \mathbb{Z}_{\geq 1}} := (b_j^{(0)} > b_j^{(1)} > \cdots > b_j^{(j-1)} > (j - 1) > \cdots), \quad (5.1)$$

and define $b_j^{(t),\ell} = (b_j^{(t),\ell})_{j \in \mathbb{Z}_{\geq 1}}$ by: for each $j \in \mathbb{Z}_{\geq 1}$,

$$b_j^{(t),\ell} = \begin{cases} b_j & \text{if } j \notin J, \\ R_\ell - b_j & \text{if } j \in J, \end{cases}$$

where $R_\ell := \begin{cases} 2\ell + 3 & \text{if } \mathfrak{g} \text{ is of type } B_\infty, \\ 2\ell + 4 & \text{if } \mathfrak{g} \text{ is of type } C_\infty, \\ 2\ell + 2 & \text{if } \mathfrak{g} \text{ is of type } D_\infty. \quad (5.2)\end{cases}$

Observe that

$$b_j^{(t),\ell} \geq \begin{cases} \ell + 2 & \text{if } \mathfrak{g} \text{ is of type } B_\infty, \\ \ell + 3 & \text{if } \mathfrak{g} \text{ is of type } C_\infty, \text{ for all } j \in J, \\ \ell + 1 & \text{if } \mathfrak{g} \text{ is of type } D_\infty, \end{cases}$$

and that $b_j^{(t),\ell} \neq b_k^{(t),\ell}$ for all $j, k \in \mathbb{Z}_{\geq 1}$ such that $j \neq k$. Let $\tau$ be the (unique) finite permutation of the set $\mathbb{Z}_{\geq 1}$ such that

$$b_{\tau(1)}^{(t),\ell} > b_{\tau(2)}^{(t),\ell} > \cdots > b_{\tau(j)}^{(t),\ell} > b_{\tau(j+1)}^{(t),\ell} > \cdots;$$

note that $\tau(j) = j$ for all $j > \max J$. Now we define $\rho^{j,\ell}$ to be the conjugate of the partition

$$(b_{\tau(1)}^{(t),\ell} \geq b_{\tau(2)}^{(t),\ell} + 1 \geq \cdots \geq b_{\tau(j)}^{(t),\ell} + (j - 1) \geq b_{\tau(j+1)}^{(t),\ell} + j \geq \cdots). \quad (5.3)$$
Also, we define
\[
sgn^J,\ell(\rho) := \begin{cases} 
\operatorname{sgn}(\tau) & \text{if } \rho \text{ is of type } B_\infty \text{ or } D_\infty, \\
\operatorname{sgn}(\tau) \times (-1)^\#J & \text{if } \rho \text{ is of type } C_\infty,
\end{cases}
\]
where \(\operatorname{sgn}(\tau)\) denotes the sign of the finite permutation \(\tau\) of the set \(\mathbb{Z}_{\geq 1}\).

**Remark 5.1.** Suppose that \(\rho\) is of type \(D_\infty\) and \(\ell(\rho) = \ell + 1\). Let \(J\) be a finite subset of \(\mathbb{Z}_{\geq 2}\), and set \(K := J \cup \{1\}\). Since \(\ell(\rho^{(0)}) = \ell(\rho) = \ell + 1\) in (5.1), it follows from (5.2) that \(b_j^{\ell,\ell} = b_j^{K,\ell}\) for all \(j \in \mathbb{Z}_{\geq 1}\). Thus, we obtain \(\rho_j^{\ell,\ell} = \rho_j^{K,\ell}\) and \(\operatorname{sgn}_J^J,\ell(\rho) = \operatorname{sgn}_{K}^K,\ell(\rho)\).

### 5.2 Tensor product multiplicity formulas in terms of Littlewood-Richardson coefficients

In this subsection, we review some tensor product multiplicity formulas from [Ko1] and [Ko2], which we use in the proof of Proposition 4.12. Fix \(\ell \in \mathbb{Z}_{\geq 3}\). For an \([\ell]\)-dominant integral weight \(\lambda \in P\), we denote \(\operatorname{ch} V_{[\ell]}(\lambda)\) by the formal character of the finite-dimensional irreducible \(U_q(\mathfrak{g}_{[\ell]})\)-module \(V_{[\ell]}(\lambda)\) of highest weight \(\lambda\). We set
\[
P_{\ell}^+: = \{ \lambda \in P \mid \lambda \text{ is } [\ell]\text{-dominant, and } \lambda^{(j)} = L_\lambda \text{ for all } j \in \mathbb{Z}_{\geq \ell + 1} \}.
\]

For each \(\lambda \in P^+_{[\ell]}\), we define a sequence \(\phi_{\ell}(\lambda)\) by:
\[
\phi_{\ell}(\lambda) = (\lambda^{(\ell)}, \ldots, \lambda^{(1)}, \langle \lambda^{(0)} \rangle, 0, 0, \ldots).
\]
Note that by Remark 2.3, we have \(\lambda^{(\ell)} \geq \cdots \geq \lambda^{(1)} \geq \langle \lambda^{(0)} \rangle \geq 0\), and hence that if \(L_\lambda \in \mathbb{Z}\), then \(\phi_{\ell}(\lambda)\) is a partition.

For the rest of this subsection, we fix \(\lambda, \mu \in P^+_{[\ell]}\). Because every weight of \(V_{[\ell]}(\lambda)\) (resp., \(V_{[\ell]}(\mu)\)) is contained in the set \(\lambda - \sum_{i \in [\ell]} \mathbb{Z}_{\geq 0} \alpha_i\) (resp., \(\mu - \sum_{i \in [\ell]} \mathbb{Z}_{\geq 0} \alpha_i\)), we deduce through use of (2.11) that every weight \(\nu \in P\) of \(V_{[\ell]}(\lambda) \otimes V_{[\ell]}(\mu)\) satisfies the condition that
\[
\nu^{(j)} = \lambda^{(j)} + \mu^{(j)} = L_\lambda + L_\mu \quad \text{for all } j \in \mathbb{Z}_{\geq \ell + 1}.
\]
In particular, we have \(L_\nu = L_\lambda + L_\mu\).

**Case of type** \(C_\infty\). Note that in this case, \(\mathfrak{g}_{[\ell]}\) is of type \(C_{\ell + 1}\), and \(L_\lambda, L_\mu \in \mathbb{Z}\) (see (2.12)), and hence \(\phi_{\ell}(\lambda), \phi_{\ell}(\mu) \in \mathcal{P}\). We have the following formula by [Ko2, Theorem 6.6 (2)]:
\[
\operatorname{ch} V_{[\ell]}(\lambda) \otimes \operatorname{ch} V_{[\ell]}(\mu) = \sum_{\nu \in P^+_{[\ell]}, L_\nu = L_\lambda + L_\mu} \sum_{J \subset \mathbb{Z}_{\geq 1}, \# J < \infty, \omega_1, \omega_2, \omega_3} \frac{\operatorname{LR}_{\omega_1, \omega_2, \omega_3}^{\phi_{\ell}(\lambda)} \operatorname{LR}_{\omega_1, \omega_2, \omega_3}^{\phi_{\ell}(\mu)} \operatorname{LR}_{\omega_1, \omega_2, \omega_3}^{\phi_{\ell}(\nu), \ell, \ell} \operatorname{sgn}_J^J,\ell(\phi_{\ell}(\nu))}{\operatorname{ch} V_{[\ell]}(\nu)}.
\]

For simplicity of notation, we set
\[
C_{\rho_1, \rho_2}^{\rho, \ell} := \sum_{J \subset \mathbb{Z}_{\geq 1}, \# J < \infty, \omega_1, \omega_2, \omega_3} \operatorname{LR}_{\omega_1, \omega_2, \omega_3}^{\rho_1} \operatorname{LR}_{\omega_2, \omega_3}^{\rho_2} \operatorname{LR}_{\omega_1, \omega_2, \omega_3}^{\rho, \ell, \ell} \operatorname{sgn}_J^J(\rho)
\]
for partitions \(\rho_1, \rho_2, \rho \in \mathcal{P}\) whose lengths are less than or equal to \(\ell + 1\).
Case of type $B_\infty$. Note that in this case, $g_{[\ell]}$ is of type $B_{\ell+1}$. Suppose first that $L_\lambda, L_\mu \in \mathbb{Z}$, and hence $\phi_\ell(\lambda), \phi_\ell(\mu) \in \mathcal{P}$. Then we have the following formula by \cite[Theorem 6.6 (1)]{Ko2} along with Definition 1.2 (1)]:

$$ch V_{[\ell]}(\lambda) \times ch V_{[\ell]}(\mu) = \sum_{\nu \in P^R_{[\ell]}, L_\nu = L_\lambda + L_\mu} \left\{ \sum_{J \subset \mathbb{Z}_{\geq 1}, \#J < \infty, \omega_1, \omega_2, \omega_3 \in \mathcal{P}} LR_{\omega_1, \omega_2}^\phi(\lambda) LR_{\omega_2, \omega_3}^\phi(\mu) LR_{\omega_3, \omega_1}^\phi(\nu) J, \ell \text{ sgn}(J, \ell)(\phi_\ell(\nu)) \right\} ch V_{[\ell]}(\nu). \tag{5.6}$$

For simplicity of notation, we set

$$B^R_{\rho_1, \rho_2, \rho} := \sum_{J \subset \mathbb{Z}_{\geq 1}, \#J < \infty, \omega_1, \omega_2, \omega_3 \in \mathcal{P}} LR_{\omega_1, \omega_2}^\rho(\mu_1) LR_{\omega_2, \omega_3}^\rho(\mu_2) LR_{\omega_3, \omega_1}^\rho(\mu_3) \text{ sgn}(J, \ell)(\rho) \tag{5.7}$$

for partitions $\rho_1, \rho_2, \rho \in \mathcal{P}$ whose lengths are less than or equal to $\ell + 1$.

Suppose next that $L_\lambda \in (1/2) + \mathbb{Z}$ and $L_\mu \in \mathbb{Z}$, and hence $\phi_\ell(\lambda) \notin \mathcal{P}$, but $\phi_\ell(\mu) \in \mathcal{P}$ since $\lambda(j) \in (1/2) + \mathbb{Z}_{\geq 0}$ for all $j \in [\ell]$. Note that $\Lambda_0 \in P_{+}^{[\ell]}$ by (2.11), and that if $\xi \in P_{+}^{[\ell]}$ is such that $L_\xi \in (1/2) + \mathbb{Z}$, then the level of $\delta := \xi - \Lambda_0 \in \mathcal{P}$ equal to $L_\xi - 1/2 \in \mathbb{Z}$; also, observe that $\phi_\ell(\delta)$ is a partition, and

$$\phi_\ell(\delta) \left( \frac{\delta(\ell)}{2}, \ldots, \frac{\delta(\ell)}{2}, \frac{\delta(0)}{2}, 0, 0, \ldots \right).$$

We have the following formula by \cite[1.2.1)]{Ko1}:

$$ch V_{[\ell]}(\lambda) \times ch V_{[\ell]}(\mu) = \sum_{\nu \in P^R_{[\ell]}, L_\nu = L_\lambda + L_\mu} \left\{ \sum_{\rho_1 \in \mathcal{P}, \rho_2 \in \phi_\ell(\mu), \phi_\ell(\rho_1)/\phi_\ell(\rho_2): \text{vertical strip}} C^\phi(\nu - \Lambda_0; \ell, \phi_\ell(\lambda - \Lambda_0), \rho_2) \right\} ch V_{[\ell]}(\nu). \tag{5.8}$$

Case of type $D_\infty$. Note that in this case, $g_{[\ell]}$ is of type $D_{\ell+1}$. For $\lambda \in P_{+}^{[\ell]}$, we set (cf. \cite[Definition 1.1)]{Ko2})

$$ch^{(+)} V_{[\ell]}(\lambda) := ch V_{[\ell]}(\lambda) + ch V_{[\ell]}(\overline{\lambda}),$$

$$ch^{(-)} V_{[\ell]}(\lambda) := ch V_{[\ell]}(\lambda) - ch V_{[\ell]}(\overline{\lambda}),$$

where $\overline{\lambda}$ is the element of $P_{+}^{[\ell]}$ defined by: $\overline{\lambda}(0) = -\lambda(0)$ and $\overline{\lambda}(j) = \lambda(j)$ for all $j \in \mathbb{Z}_{\geq 1}$. In addition, for $\lambda \in P_{+}^{[\ell]}$ such that $L_\lambda \in \mathbb{Z}$, we set (cf. \cite[Definition 1.2 (2),(3)]{Ko2})

$$\widetilde{ch} V_{[\ell]}(\lambda) := \begin{cases} ch V_{[\ell]}(\lambda) & \text{if } \lambda(0) = 0, \\ ch^{(+)} V_{[\ell]}(\lambda) & \text{if } \lambda(0) \neq 0. \end{cases}$$
Suppose first that $L_\lambda, L_\mu \in \mathbb{Z}$, and hence $\phi_\ell(\lambda), \phi_\ell(\mu) \in \mathcal{P}$. Then we have the following formula by [Ko2, Theorem 6.6 (3) along with Definition 1.2 (2), (3)] and Remark 5.1:

$$\tilde{\text{ch}}V_\ell(\lambda) \times \tilde{\text{ch}}V_\ell(\mu) =$$

$$\sum_{\nu \in \mathcal{P}_+^{[\ell]}, L_\nu = L_\lambda + L_\mu, \nu^{(0)} \geq 0} \begin{cases} c(\nu) \sum_{J \subset \mathbb{Z}_{\geq 1}, \# J < \infty, \omega_1, \omega_2, \omega_3 \in \mathcal{P}} LR^{\phi_\ell /(\omega_1, \omega_2)} LR^{\phi_\ell /(\omega_3, \omega_1)} LR^{\phi_\ell (\nu) \ell, \ell} \text{sgn}^{J, \ell}(\phi_\ell(\nu)) \end{cases} \tilde{\text{ch}}V_\ell(\nu), \quad (5.9)$$

where for $\nu \in \mathcal{P}_+^{[\ell]}$ with $L_\nu \in \mathbb{Z}$, we set

$$c(\nu) := \begin{cases} \frac{1}{2} & \text{if } \ell(\phi_\ell(\nu)) = \ell + 1, \\
1 & \text{otherwise}. \end{cases}$$

For simplicity of notation, we set

$$D^{\rho, \ell}_{\rho_1, \rho_2} := \sum_{J \subset \mathbb{Z}_{\geq 1}, \# J < \infty, \omega_1, \omega_2, \omega_3 \in \mathcal{P}} LR^{\rho_1, \omega_1} LR^{\rho_2, \omega_2} LR^{\rho_3, \omega_3} \text{sgn}^{J, \ell}(\rho) \quad (5.10)$$

for partitions $\rho_1, \rho_2, \rho \in \mathcal{P}$ whose lengths are less than or equal to $\ell + 1$. Also, if $\lambda^{(0)} > 0$, then we have the following formula by [Ko1, (1.2.8)]:

$$\text{ch}^{(-)}V_\ell(\lambda) \times \tilde{\text{ch}}V_\ell(\mu) =$$

$$\sum_{\nu \in \mathcal{P}_+^{[\ell]}, L_\nu = L_\lambda + L_\mu, \nu^{(0)} > 0} \begin{cases} (-1)^{|\omega|/\rho_2} C^{\phi_\ell(\nu) - (1^{\ell+1}) \ell, \ell}_{\phi_\ell(\lambda) - (1^{\ell+1}) \ell, \rho_2} \end{cases} \text{ch}^{(-)}V_\ell(\nu), \quad (5.11)$$

where for a partition $\rho = (\rho^{(0)} \geq \rho^{(1)} \geq \cdots) \in \mathcal{P}$ of length $\ell + 1$, we set

$$\rho - (1^{\ell+1}) := (\rho^{(0)} - 1, \rho^{(1)} - 1, \ldots, \rho^{(\ell)} - 1, 0, 0, \ldots) \in \mathcal{P}.$$

Suppose next that $L_\lambda \in 1/2 + \mathbb{Z}$, $\lambda^{(0)} > 0$, and $L_\mu \in \mathbb{Z}$, and hence $\phi_\ell(\mu) \in \mathcal{P}$, but $\phi_\ell(\lambda) \notin \mathcal{P}$ since $\lambda^{(j)} \in (1/2) + \mathbb{Z}$ for all $j \in [\ell]$. Note that $\Lambda_0 \in \mathcal{P}_+^{[\ell]}$ and $\Lambda_1 \in \mathcal{P}_+^{[\ell]}$ by (2.13), and that if $\xi \in \mathcal{P}_+^{[\ell]}$ is such that $L_\xi \in (1/2) + \mathbb{Z}$ and $\xi^{(0)} > 0$, then the level of $\delta := \xi - \Lambda_0 \in \mathcal{P}_+^{[\ell]}$ is equal to $L_\xi - 1/2 \in \mathbb{Z}$, and $\delta^{(0)} \geq 0$; also, observe that $\phi_\ell(\delta) = (\delta^{(\ell)}, \ldots, \delta^{(1)}, \delta^{(0)} = \delta^{(0)}, 0, 0, \ldots)$ is a partition, and

$$\phi_\ell(\xi) = \left( \delta^{(\ell)} + \frac{1}{2}, \ldots, \delta^{(1)} + \frac{1}{2}, \delta^{(0)} + \frac{1}{2}, 0, 0, \ldots \right).$$
We have the following formulas by \([Ko1]\) (1.2.3) and (1.2.2):

\[
\text{ch}^{(+)} V[\ell](\lambda) \times \tilde{\text{ch}} V[\ell](\mu) =
\sum_{\nu \in P^+ \cap \mathbb{N}, \mu^\prime + L \mu \leq L}{\sum_{\rho_2 \in \mathcal{P}, \rho_2 \subset \phi[\ell](\mu), \phi[\ell](\mu)/\rho_2 : \text{vertical strip}} (-1)^{|\phi[\ell](\lambda-A_\ell)|+|\rho_2|+|\phi[\ell](\nu-A_\ell)|} B^{\phi[\ell](\nu-A_\ell); \ell}_{\phi[\ell](\lambda-A_\ell), \rho_2}} \text{ch}^{(+)} V[\ell](\nu);
\]

\[
\text{ch}^{(-)} V[\ell](\lambda) \times \tilde{\text{ch}} V[\ell](\mu) =
\sum_{\nu \in P^+ \cap \mathbb{N}, \mu^\prime + L \mu \leq L}{\sum_{\rho_2 \in \mathcal{P}, \rho_2 \subset \phi[\ell](\mu), \phi[\ell](\mu)/\rho_2 : \text{vertical strip}} (-1)^{|\phi[\ell](\mu)/\rho_2|} B^{\phi[\ell](\nu-A_\ell); \ell}_{\phi[\ell](\lambda-A_\ell), \rho_2}} \text{ch}^{(-)} V[\ell](\nu).
\]

5.3 **Proof of Proposition 4.12.** The following proposition plays an essential role in the proof of Proposition 4.12.

**Proposition 5.2.** Let \(L \in \mathbb{Z}_{\geq 0}\), and \(n \in \mathbb{Z}_{\geq 3}\). Let \(\rho_1, \rho_2, \rho \in \mathcal{P}\) be partitions satisfying the following conditions:

(i) The lengths of these partitions are all less than or equal to \(n+1\).

(ii) The first part of \(\rho_1\) is equal to \(L\).

(iii) There hold the inequalities

\[
y_1 := \# \{1 \leq j \leq n+1 \mid \text{the } j\text{-th part of } \rho_1 \text{ is equal to } L \} > |\rho_2|,
\]

\[
y := \# \{1 \leq j \leq n+1 \mid \text{the } j\text{-th part of } \rho \text{ is equal to } L \} > |\rho_2|.
\]

If we set \(\kappa_1 := \iota_L(\rho_1)\) and \(\kappa := \iota_L(\rho)\), then we have

\[
X_{\kappa_1, \rho_2} = X_{\kappa_1; \rho_2}^{\kappa_1, n+1} \quad \text{for } X = B, C, D.
\]

The proof of this proposition will be given in \([5,4]\).

**Proof of Proposition 4.12.** We fix \(n \in \mathbb{Z}_{\geq 0}\) such that \(n > p + (N + 1)q\), and \(\nu \in P^+_{\ell}(\lambda, \mu)\) as in Proposition 4.12. Recall from Remark 3.20 that \(B_{\nu}[\ell](\lambda)\) (resp., \(B_{\nu}[\ell](\mu) = B_{\nu}[\ell](\mu[\nu])\)) is isomorphic, as a \(U_q(\mathfrak{g}[\nu])\)-crystal, to the crystal basis of the finite-dimensional irreducible \(U_q(\mathfrak{g}[\nu])\)-module of highest weight \(\lambda\) (resp., \(\mu[\nu]\)), and similar statements hold for \(B_{\nu}[\ell](\lambda)\) and \(B_{\nu}[\ell](\mu) = B_{\nu}[\ell](\mu[\nu])\). Therefore, the sets \(B_{\mu[\kappa], \nu}[\ell] \subset B_{\nu}[\ell](\lambda) \otimes B_{\nu}[\ell](\mu)\) and \(B_{\mu[\kappa], \nu}[\ell] \subset B_{\nu}[\ell](\lambda) \otimes B_{\nu}[\ell](\mu)\) are both finite sets. Because it is already shown that the map \(S_x : B_{\mu[\kappa], \nu}[\ell] \rightarrow B_{\mu[\kappa], \nu}[\ell]\) is injective, it suffices to show that \(#B_{\mu[\kappa], \nu}[\ell]\) is equal to the multiplicity...
of $V_n(\nu)$ (resp., $V_{n+1}(\nu_{n+1})$) in the tensor product $V_n(\lambda) \otimes V_{n+1}(\mu_{n+1})$ (resp., $V_{n+1}(\lambda) \otimes V_{n+1}(\mu_{n+1})$), and hence to the coefficient of $\text{ch} V_n(\nu)$ (resp., $\text{ch} V_{n+1}(\nu_{n+1})$) in the product $\text{ch} V_n(\lambda) \times \text{ch} V_n(\mu_{n+1})$ (resp., $\text{ch} V_{n+1}(\lambda) \times \text{ch} V_{n+1}(\mu_{n+1})$). Below we will use an explicit description of the numbers $\#B^n_{\max, \nu}$ and $\#B^{n+1}_{\max, \nu_{n+1}}$ in terms of Littlewood-Richardson coefficients obtained from the formulas in [5.2].

Since $n > p + (N + 1)q > q$ and $\text{Supp}(\mu) \subset [q - 1]$, it follows from Lemma [2.1] that $\text{Supp}(\mu_{[\ell]}) \subset [\ell]$ and $\mu_{[\ell]}(0) = 0$ for $\ell = n, n + 1$, and hence that $\mu_{[\ell]} \in P^+_{[\ell]}$ for $\ell = n, n + 1$. In addition, it is easily seen that $\phi_n(\mu_{[n]}) = \phi_{n+1}(\mu_{[n+1]}) = \mu_{[\uparrow]}$, and that $|\mu| = \sum_{\ell \in \mathbb{Z}_{\geq 0}} |\mu(\ell)|$ is equal to the sum $|\mu|$ of all parts of $\mu_{[\uparrow]}$; note that $n - p + 1 \geq (N + 1)q + 1 > |\mu| = |\mu_{[\uparrow]}|$. Also, we infer from Remark [2.1] and the choice of $n$ that

$$\ldots = \lambda^{(n+1)} = \lambda^{(n)} = \ldots = \lambda^{(p)} = L_\lambda > \lambda^{(p-1)} \geq \ldots \geq \lambda^{(1)} \geq \lambda^{(0)} \geq 0,$$

and hence that $\lambda$ is contained in both of the sets $P^n_{[n]}$ and $P^{n+1}_{[n+1]}$. Since $\nu \in P^n_{[n]}(\lambda, \mu)$ ($\subset P^n_{[n]}$), we see from (4.18) that

$$\left\{\begin{array}{l}
0 \leq \langle \nu^{(0)} \rangle \leq \nu^{(1)} \leq \cdots \leq \nu^{(u_0-1)} < L_\lambda, \\
\nu^{(u_0)} = \nu^{(u_0+1)} = \cdots = \nu^{(u_1-1)} = L_\lambda, \\
L_\lambda < \nu^{(u_1)} \leq \nu^{(u_1+1)} \leq \cdots \leq \nu^{(n)}, \\
\nu_{\uparrow} = L_\lambda \quad \text{for all } j \geq n + 1.
\end{array}\right. \tag{5.15}$$

for some $0 \leq u_0 < u_1 \leq n + 1$, with

$$u_1 - u_0 > n - p - Nq \geq |\mu| = |\mu_{[\uparrow]}|.$$

Furthermore, we deduce from Lemma [4.10] that $\nu_{[n+1]} \in P^{n+1}_{[n+1]}(\lambda, \mu)$ ($\subset P^{n+1}_{[n+1]}$) is given by:

$$\left\{\begin{array}{l}
\nu^{(j)}_{[n+1]} = \nu^{(j)} \quad \text{for } 0 \leq j \leq u_0 - 1, \\
\nu^{(j)}_{[n+1]} = L_\lambda \quad \text{for } u_0 \leq j \leq u_1 - 1, \\
\nu^{(u_1)}_{[n+1]} = L_\lambda, \\
\nu^{(j)}_{[n+1]} = \nu^{(j-1)} \quad \text{for } u_1 + 1 \leq j \leq n + 1, \\
\nu^{(j)}_{[n+1]} = L_\lambda \quad \text{for } j \geq n + 2.
\end{array}\right. \tag{5.16}$$

Case of type $C_\infty$. Since $L_\lambda \in \mathbb{Z}_{\geq 0}$, it follows that $\phi_\ell(\lambda)$, $\ell = n, n + 1$, and $\phi_n(\nu)$, $\phi_{n+1}(\nu_{n+1})$ are all partitions. We infer from [5.4] that

$$\#B^n_{\max, \nu} = C_{\phi_n(\nu); n, \mu_{[\uparrow]}}, \quad \#B^{n+1}_{\max, \nu_{n+1}} = C_{\phi_{n+1}(\nu_{n+1}); n+1, \mu_{[\uparrow]}}. \tag{5.17}$$
Here, we see from (5.14) that the first part of $\phi_n(\lambda)$ is equal to $L := L_\lambda$, and
\[
\#\{1 \leq j \leq n + 1 \mid \text{the } j\text{-th part of } \phi_n(\lambda) \text{ is equal to } L\} = n - p + 1 > |\mu_1|,
\]
\[
\#\{1 \leq j \leq n + 1 \mid \text{the } j\text{-th part of } \phi_n(\nu) \text{ is equal to } L\} = u_1 - u_0 > |\mu_1|.
\]

Also, we deduce from (5.14), and (5.15), (5.16) that $\phi_{n+1}(\lambda) = \iota_L(\phi_n(\lambda))$ and $\phi_{n+1}(\nu_{[n+1]}) = \iota_L(\phi_n(\nu))$. Therefore, by Proposition 5.2, we conclude that
\[
\mathbb{B}^{[n]}[\mu] = \mathbb{C}^{\phi_n(\nu) ; n}_{\phi_n(\lambda), \mu_1} = \mathbb{C}^{\phi_{n+1}(\nu_{[n+1]}); n+1}_{\phi_{n+1}(\lambda), \mu_1} = \#\mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}},
\]
as desired.

**Case of type $B_\infty$.** If $L_\lambda \in \mathbb{Z}_{\geq 0}$, then we can show by the same reasoning as in the case of type $C_\infty$, this time using (5.16), that
\[
\#\mathbb{B}^{[n]}[\mu] = \mathbb{B}^{\phi_n(\nu) ; n}_{\phi_n(\lambda), \mu_1} = \mathbb{B}^{\phi_{n+1}(\nu_{[n+1]}); n+1}_{\phi_{n+1}(\lambda), \mu_1} = \#\mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}},
\]

Therefore, it remains to consider the case in which the level $L_\lambda$ of $\lambda$ is contained in $1/2 + \mathbb{Z}_{\geq 0}$. In this case, we infer from (5.18) that
\[
\#\mathbb{B}^{[n]}[\mu] = \sum_{\rho_2 \in \mathcal{P}, \rho_2 \subset \mu_1, \mu_1/\rho_2 : \text{vertical strip}} C^{\phi_n(\nu - \Lambda_0); n}_{\phi_n(\lambda - \Lambda_0), \rho_2}.
\]

Here, observe that the first part of $\phi_n(\lambda - \Lambda_0)$ is equal to $L := L_\lambda - 1/2 \in \mathbb{Z}_{\geq 0}$, and
\[
\#\{1 \leq j \leq n + 1 \mid \text{the } j\text{-th part of } \phi_n(\lambda - \Lambda_0) \text{ is equal to } L\} > |\mu_1| \geq |\rho_2|,
\]
\[
\#\{1 \leq j \leq n + 1 \mid \text{the } j\text{-th part of } \phi_n(\nu - \Lambda_0) \text{ is equal to } L\} > |\mu_1| \geq |\rho_2|
\]

for all $\rho_2 \in \mathcal{P}$ such that $\rho_2 \subset \mu_1$. Also, we deduce from (5.14), and (5.15), (5.16) that $\phi_{n+1}(\lambda - \Lambda_0) = \iota_L(\phi_n(\lambda - \Lambda_0))$ and $\phi_{n+1}(\nu_{[n+1]} - \Lambda_0) = \iota_L(\phi_n(\nu - \Lambda_0))$. Therefore, by Proposition 5.2 we conclude that
\[
C^{\phi_n(\nu - \Lambda_0); n}_{\phi_n(\lambda - \Lambda_0), \rho_2} = C^{\phi_{n+1}(\nu_{[n+1]} - \Lambda_0); n+1}_{\phi_{n+1}(\lambda - \Lambda_0), \rho_2}
\]
for all $\rho_2 \in \mathcal{P}$ such that $\rho_2 \subset \mu_1$, and hence that
\[
\#\mathbb{B}^{[n]}[\mu] = \sum_{\rho_2 \in \mathcal{P}, \rho_2 \subset \mu_1, \mu_1/\rho_2 : \text{vertical strip}} C^{\phi_n(\nu - \Lambda_0); n}_{\phi_n(\lambda - \Lambda_0), \rho_2}
\]
\[
= \sum_{\rho_2 \in \mathcal{P}, \rho_2 \subset \mu_1, \mu_1/\rho_2 : \text{vertical strip}} C^{\phi_{n+1}(\nu_{[n+1]} - \Lambda_0); n+1}_{\phi_{n+1}(\lambda - \Lambda_0), \rho_2} = \#\mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}},
\]
as desired.
Case of type $D_\infty$. Note that

\[ \widetilde{\text{ch}} V_{[\ell]}(\mu_{[\ell]}) = \text{ch} V_{[\ell]}(\mu_{[\ell]}) \quad \text{for } \ell = n, n + 1 \]

since $\mu_{[\ell]} = 0$, and also that $\nu_{[n+1]} = \nu^{(0)}$ since $u_1 > 0$ in (5.15) and (5.16).

Suppose first that $L_\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda^{(0)} = 0$; we have $\widetilde{\text{ch}} V_{[\ell]}(\lambda) = \text{ch} V_{[\ell]}(\lambda)$ for $\ell = n, n + 1$ by definition. In this case, we infer from (5.9) that

\[ \# \mathbb{B}^{[n]}_{\max, \nu} = c(\nu) D_{\phi_n(\lambda), \mu_1}^{\nu} \quad \text{and} \quad \# \mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}} = c(\nu_{[n+1]}) D_{\phi_{n+1}(\lambda), \mu_1}^{\nu_{[n+1]} \nu_{[n+1]} n+1}. \quad (5.19) \]

We can show by the same reasoning as in the case of type $C_\infty$ that

\[ D_{\phi_n(\nu), \mu_1}^{\nu} = D_{\phi_{n+1}(\nu_{[n+1]} n+1), \mu_1}. \]

Also, since $\nu_{[n+1]} = \nu^{(0)}$, it is obvious from the definitions that $c(\nu_{[n+1]}) = c(\nu)$. Combining these facts with (5.19), we find that

\[ \# \mathbb{B}^{[n]}_{\max, \nu} = \# \mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}}. \]

Suppose next that $L_\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda^{(0)} \neq 0$; we have $\widetilde{\text{ch}} V_{[\ell]}(\lambda) = \text{ch}(+) V_{[\ell]}(\lambda)$ for $\ell = n, n + 1$ by definition. If $\nu_{[n+1]} = \nu^{(0)} = 0$, then we infer from (5.9) and (5.11) that

\[ \# \mathbb{B}^{[n]}_{\max, \nu} = \frac{1}{2} D_{\phi_n(\nu), \mu_1}^{\nu} \quad \text{and} \quad \# \mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}} = \frac{1}{2} D_{\phi_{n+1}(\nu_{[n+1]} n+1), \mu_1}. \quad (5.20) \]

Now, in exactly the same way as above, we can show that

\[ D_{\phi_n(\nu), \mu_1}^{\nu} = D_{\phi_{n+1}(\nu_{[n+1]} n+1), \mu_1}, \quad \text{and hence} \quad \# \mathbb{B}^{[n]}_{\max, \nu_{[n]}} = \# \mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}}. \]

If $\nu_{[n+1]} = \nu^{(0)} \neq 0$, then we infer from (5.9) and (5.11) that

\[ \# \mathbb{B}^{[n]}_{\max, \nu_{[n]}} = \frac{1}{2} \left( \frac{1}{2} D_{\phi_n(\nu), \mu_1}^{\nu} + (-1)^c \sum_{\rho_2, \omega \in P, \rho_2 \subset \omega \subset \mu_1, \mu_1 / \omega: \text{vertical strip}, \omega / \rho_2: \text{vertical strip}} (-1)^{[\omega / \rho_2]} C_{\rho_1, \rho_2}^{n, \rho_1, \rho_2} \right) \quad (5.21) \]

and

\[ \# \mathbb{B}^{[n+1]}_{\max, \nu_{[n+1]}} = \frac{1}{2} \left( \frac{1}{2} D_{\phi_{n+1}(\nu_{[n+1]} n+1), \mu_1}^{\nu_{[n+1]} n+1} + (-1)^c \sum_{\rho_2, \omega \in P, \rho_2 \subset \omega \subset \mu_1, \mu_1 / \omega: \text{vertical strip}, \omega / \rho_2: \text{vertical strip}} (-1)^{[\omega / \rho_2]} C_{\rho_1, \rho_2}^{n+1, \rho_1, \rho_2} \right), \quad (5.22) \]
where we set
\[ c := \begin{cases} 0 & \text{if } \lambda^{(0)}\nu^{(0)}_{[n+1]} = \lambda^{(0)}\nu^{(0)} > 0, \\ 1 & \text{otherwise,} \end{cases} \]
and
\[\rho_1 := \phi_n(\lambda) - (1^{n+1}), \quad \rho := \phi_n(\nu) - (1^{n+1}), \quad \kappa_1 := \phi_{n+1}(\lambda) - (1^{n+2}), \quad \kappa := \phi_{n+1}(\nu_{[n+1]}) - (1^{n+2}).\]

In exactly the same way as above, we can show that
\[ D_{\phi_n(\nu); n} \rho_1 = D_{\phi_{n+1}(\nu_{[n+1]}); n+1} \kappa_1. \]

Here, observe that the first part of \( \rho_1 \) is equal to \( L - 1 = L_\lambda - 1 \), and
\[
\begin{align*}
\# \{1 \leq j \leq n+1 \mid \text{the } j\text{-th part of } \rho_1 \text{ is equal to } L - 1 \} & > |\mu| > |\rho_2|, \\
\# \{1 \leq j \leq n+1 \mid \text{the } j\text{-th part of } \rho \text{ is equal to } L - 1 \} & > |\mu| > |\rho_2|
\end{align*}
\]
for all \( \rho_2 \in \mathcal{P} \) such that \( \rho_2 \subset \mu_\dagger \). Also, we deduce from (5.14), and (5.15), (5.16) that \( \kappa_1 = \iota_{L-1}(\rho_1) \) and \( \kappa = \iota_{L-1}(\rho) \). Therefore, by Proposition 5.2, we conclude that
\[ C_{\rho_1, \rho_2} = C_{\kappa_1, \kappa_2} \]
for all \( \rho_2 \in \mathcal{P} \) such that \( \rho_2 \subset \mu_\dagger \). Combining these facts with (5.21), (5.22), we find that
\[ \# B_{\max, \nu^{(n)}} = \# B_{\max, \nu_{[n+1]}}. \]

Finally, we consider the case in which the level \( L_\lambda \) of \( \lambda \) is contained in \( 1/2 + \mathbb{Z}_{\geq 0} \). In this case, we infer from (5.12) and (5.13) that
\[
\begin{align*}
\# B_{\max, \nu^{(n)}} & = \frac{1}{2} \sum_{\rho_2 \in \mathcal{P}, \rho_2 \subset \mu_\dagger, \mu_\dagger / \rho_2 : \text{vertical strip}} \left\{ (-1)^{|\mu_\dagger / \rho_2|} + (-1)^{|\rho_2|+|\rho_1|+|\rho|+c} \right\} B_{\rho_1, \rho_2}; \quad (5.23) \\
\end{align*}
\]
and
\[
\begin{align*}
\# B_{\max, \nu_{[n+1]}} & = \frac{1}{2} \sum_{\rho_2 \in \mathcal{P}, \rho_2 \subset \mu_\dagger, \mu_\dagger / \rho_2 : \text{vertical strip}} \left\{ (-1)^{|\mu_\dagger / \rho_2|} + (-1)^{|\rho_2|+|\kappa_1|+|\kappa|+c} \right\} B_{\kappa_1, \rho_2}; \quad (5.24) \\
\end{align*}
\]
where we set
\[ c := \begin{cases} 0 & \text{if } \lambda^{(0)}\nu^{(0)}_{[n+1]} = \lambda^{(0)}\nu^{(0)} > 0, \\ 1 & \text{otherwise}, \end{cases} \]
and

\[ \rho_1 := \phi_n(\lambda - \Lambda_0), \quad \rho := \phi_n(\nu - \Lambda_0), \]
\[ \kappa_1 := \phi_{n+1}(\lambda - \Lambda_0), \quad \kappa := \phi_{n+1}(\nu_{n+1} - \Lambda_0). \]

Here, observe that the first part of \( \rho_1 \) is equal to \( L := L_\lambda - 1/2 \), and

\[ \#\{1 \leq j \leq n + 1 \mid \text{the } j\text{-th part of } \rho_1 \text{ is equal to } L \} > |\mu_1| \geq |\rho_2|, \]
\[ \#\{1 \leq j \leq n + 1 \mid \text{the } j\text{-th part of } \rho \text{ is equal to } L \} > |\mu_1| \geq |\rho_2| \]

for all \( \rho_2 \in \mathcal{P} \) such that \( \rho_2 \subset \mu_1 \). Also, we deduce from (5.14), and (5.15), (5.16) that \( \kappa_1 = \iota_L(\rho_1) \) and \( \kappa = \iota_L(\rho) \). Therefore, by Proposition 5.2, we conclude that

\[ B_{\rho_1, \rho_2} = B_{\kappa_1, \kappa_2} \]

for all \( \rho_2 \in \mathcal{P} \) such that \( \rho_2 \subset \mu_1 \). In addition, since \( |\kappa_1| + |\kappa| = |\rho_1| + |\rho| + 2(L_\lambda - 1/2) \), we have \((-1)^{\kappa_1 + |\rho|} = (-1)^{\kappa_1 + |\kappa|}\). Combining these facts with (5.23), (5.24), we find that

\[ \#R_{\max, \nu} = \#R_{\max, \nu_{n+1}}. \]

This completes the proof of Proposition 4.12. \( \square \)

5.4 Proof of Proposition 5.2

If \( L = 0 \), then \( \rho_1 \) is the empty partition \( \emptyset \) by assumption (ii), and hence so is \( \kappa_1 = \iota_L(\rho_1) = \rho_1 \). In this case, we deduce from the definitions that

\[ X^{\rho; n}_{\emptyset, \rho_2} = \begin{cases} 1 & \text{if } \rho = \rho_2, \\ 0 & \text{otherwise}, \end{cases} \quad X^{\kappa; n+1}_{\emptyset, \rho_2} = \begin{cases} 1 & \text{if } \kappa = \rho_2, \\ 0 & \text{otherwise}. \end{cases} \]

Since \( \kappa = \iota_L(\rho) = \rho \), we obtain \( X^{\rho; n}_{\rho_1, \rho_2} = X^{\rho; n}_{\emptyset, \rho_2} = X^{\kappa; n+1}_{\emptyset, \rho_2} = X^{\kappa; n}_{\kappa_1, \rho_2} \), as desired.

Assume, therefore, that \( L > 0 \). We set

\[ \mathcal{Q}(\rho_2) := \{ (\omega_2, \omega_3) \in \mathcal{P} \times \mathcal{P} \mid LR_{\omega_2, \omega_3}^{\rho_2} \neq 0 \}, \]
\[ \mathcal{R}(\rho_1, \omega_2) := \{ \omega_1 \in \mathcal{P} \mid \omega_1 \subset \rho_1, \ |\rho_1| = |\omega_1| + |\omega_2| \} \text{ for } \omega_2 \in \mathcal{P}, \]
\[ \mathcal{R}(\kappa_1, \omega_2) := \{ \omega_1 \in \mathcal{P} \mid \omega_1 \subset \kappa_1, \ |\kappa_1| = |\omega_1| + |\omega_2| \} \text{ for } \omega_2 \in \mathcal{P}. \]

Then, from the definitions, we have

\[ X^{\rho; n}_{\rho_1, \rho_2} = \sum_{J \subset \mathbb{Z}_{\geq 1}, \# J < \infty, (\omega_2, \omega_3) \in \mathcal{Q}(\rho_2), \omega_1 \in \mathcal{R}(\rho_1, \omega_2)} LR_{\omega_1, \omega_2}^{\rho_1} LR_{\omega_2, \omega_3}^{\rho_2} LR_{\omega_3, \omega_1}^{\rho; n} \sgn^{J, n}(\rho), \tag{5.25} \]
\[ X^{\kappa; n+1}_{\kappa_1, \rho_2} = \sum_{J \subset \mathbb{Z}_{\geq 1}, \# J < \infty, (\omega_2, \omega_3) \in \mathcal{Q}(\rho_2), \omega_1 \in \mathcal{R}(\kappa_1, \omega_2)} LR_{\omega_1, \omega_2}^{\kappa_1} LR_{\omega_2, \omega_3}^{\rho_2} LR_{\omega_3, \omega_1}^{\kappa; n+1} \sgn^{J, n+1}(\kappa). \tag{5.26} \]
Claim 1. (1) Fix $(\omega_2, \omega_3) \in Q(\rho_2)$ and $\omega_1 \in R(\rho_1, \omega_2)$. If $LR^{\rho_2, n}_{\omega_3, \omega_1} \neq 0$ for a finite subset $J$ of $\mathbb{Z}_{\geq 1}$, then $J$ is contained in $[1, L] = \{1, 2, \ldots, L\}$. Also, if $J \subset [1, L]$, then the number of parts of $\rho^{j,n}$ that are equal to $L$ is greater than or equal to $y$.

(2) Fix $(\omega_2, \omega_3) \in Q(\rho_2)$ and $\omega_1 \in R(\kappa_1, \omega_2)$. If $LR^{\kappa_2, n+1}_{\omega_3, \omega_1} \neq 0$ for a finite subset $J$ of $\mathbb{Z}_{\geq 1}$, then $J$ is contained in $[1, L] = \{1, 2, \ldots, L\}$.

Proof of Claim 1. We give a proof only for part (1), since the proof of part (2) is similar. Let $u \in \mathbb{Z}_{\geq 1}$ be such that the $u$-th part of $\rho$ is equal to $L$, and such that the $(u+1)$-st part of $\rho$ is less than $L$; note that $u \geq y > |\rho_2|$ by assumption (iii). It follows that $u$ and $u-y$ are equal to the $L$-th part and the $(L+1)$-st part of the conjugate partition $^t\rho$ of $\rho$, respectively (see the figures below).

Therefore, if we define $b(^t\rho) = (b_j)_{j \in \mathbb{Z}_{\geq 1}}$ as in (5.1), then we have $b_L = u - (L-1)$, and hence

$$\# \{ j \in \mathbb{Z}_{\geq 1} \mid b_j \geq u - (L-1) \} = \# [1, L] = L.$$  

Also, if we define $b(^t\rho)^{j,n} = (b^{j,n}_j)_{j \in \mathbb{Z}_{\geq 1}}$ as in (5.2), then we have $b^{j,n}_j \geq n+1 \geq u - (L-1)$ for all $j \in J$ since $b_L = u - (L-1) \leq b_1 \leq n+1$. Consequently,

$$\{ j \in \mathbb{Z}_{\geq 1} \mid b^{j,n}_j \geq u - (L-1) \} = [1, L] \cup J.$$  

For the first assertion, suppose that $J$ is not contained in $[1, L]$. Then we get

$$\# \{ j \in \mathbb{Z}_{\geq 1} \mid b^{j,n}_j \geq u - (L-1) \} \geq L + 1.$$  

Let $\tau$ be the finite permutation of the set $\mathbb{Z}_{\geq 1}$ such that

$$b^{j,n}_{\tau(1)} > b^{j,n}_{\tau(2)} > \cdots > b^{j,n}_{\tau(j)} > b^{j,n}_{\tau(j+1)} > \cdots.$$
We deduce from (5.28) that \( b_{\tau(L+1)}^{J,n} \geq u - (L - 1) \), and hence that the \((L + 1)\)-st part \( v \) of the conjugate partition

\[
(b_{\tau(1)}^{J,n} \geq b_{\tau(2)}^{J,n} + 1 \geq \cdots \geq b_{\tau(j)}^{J,n} + (j - 1) \geq b_{\tau(j+1)}^{J,n} + j \geq \cdots)
\]

of \( \rho^{J,n} \) is greater than or equal to \( u - (L - 1) + L = u + 1 \). From this fact, we find that for each \( 1 \leq j \leq v \), the \( j \)-th row of the Young diagram of \( \rho^{J,n} \) has more than \( L + 1 \) boxes (see the figures below).

If \( \omega \not\subseteq \rho^{J,n} \), then \( LR_{\omega_3,\omega_1}^{\rho^{J,n}} = 0 \). Now, suppose that \( \omega_1 \subseteq \rho^{J,n} \). Since \( \omega_1 \in R(\rho_1, \omega_2) \), we have \( \omega_1 \subseteq \rho_1 \), which implies that each row of the Young diagram of \( \omega_1 \) has at most \( L \) boxes. Consequently, we deduce that \( |\rho^{J,n}/\omega_1| \geq v \geq u + 1 > |\rho_2| \geq |\omega_3| \), and hence \( |\rho^{J,n}| > |\omega_3| + |\omega_1| \). Then, it follows that \( LR_{\omega_3,\omega_1}^{\rho^{J,n}} = 0 \), as desired.

For the second assertion, suppose that \( J \subset [1, L] \). As above, let \( \tau \) be the finite permutation of the set \( \mathbb{Z}_{\geq 1} \) such that

\[
b_{\tau(1)}^{J,n} > b_{\tau(2)}^{J,n} > \cdots > b_{\tau(j)}^{J,n} > b_{\tau(j+1)}^{J,n} > \cdots.
\]

Since

\[
\{ j \in \mathbb{Z}_{\geq 1} \mid b_{\tau(j)}^{J,n} \geq u - (L - 1) \} = [1, L] \cup J = [1, L]
\]

by (5.27), we have \( b_{\tau(L)}^{J,n} \geq u - (L - 1) \). Also, since \( J \subset [1, L] \), we see that \( \tau(j) = j \) for all \( j \geq L + 1 \), and hence \( b_{\tau(L+1)}^{J,n} = b_{L+1}^{J,n} = b_{L+1} \). Recall that \( b_{L+1} + L \) is equal to the \((L + 1)\)-st part of \( \tau_{\rho} \), which is equal to \( u - y \). Therefore, we have

\[
\{b_{\tau(L)}^{J,n} + (L - 1)\} - \{b_{\tau(L+1)}^{J,n} + L\} \geq \{u - (L - 1) + (L - 1)\} - (b_{L+1} + L) = u - (b_{L+1} + L) = y.
\]
Because \( b_{r(L)}^J, n + (L-1) \) and \( b_{r(L+1)}^J, n + L \) are the \( L \)-th part and the \((L+1)\)-st part of \( t^J, n \), respectively, we conclude that the Young diagram of \( \rho^J, n \) has more than \( y \) rows having exactly \( L \) boxes (see the figures below).

\[
\begin{align*}
\text{This proves the claim.} & \quad \blacksquare \\
\text{By using Claim 1, we deduce from (5.25) and (5.26) that} & \\
X_{\rho_1, \rho_2}^{\rho_1, n} &= \sum_{J \subseteq \{1, \ldots, L\}, (\omega_2, \omega_3) \in \mathcal{Q}(\rho_2)} \text{LR}_{\omega_1, \omega_2}^{\rho_1} \text{LR}_{\omega_2, \omega_3}^{\rho_2} \text{LR}_{\omega_3, \omega_1}^{\rho_3, n} \text{sgn}^{J, n}(\rho), \quad (5.29) \\
X_{\kappa_1, \rho_2}^{\kappa_1, n+1} &= \sum_{J \subseteq \{1, \ldots, L\}, (\omega_2, \omega_3) \in \mathcal{Q}(\rho_2)} \text{LR}_{\omega_1, \omega_2}^{\kappa_1} \text{LR}_{\omega_2, \omega_3}^{\rho_2} \text{LR}_{\omega_3, \omega_1}^{\rho_3, n+1} \text{sgn}^{J, n+1}(\kappa). \quad (5.30)
\end{align*}
\]

Claim 2. Fix \((\omega_2, \omega_3) \in \mathcal{Q}(\rho_2)\). For every \( \omega_1 \in \mathcal{R}(\rho_1, \omega_2) \), we have \( \iota_L(\omega_1) \in \mathcal{R}(\kappa_1, \omega_2) \). Thus, \( \iota_L \) yields a map from \( \mathcal{R}(\rho_1, \omega_2) \) to \( \mathcal{R}(\kappa_1, \omega_2) \). Moreover, this map is bijective.

Proof of Claim 2. Recall that
\[
\begin{align*}
\mathcal{R}(\rho_1, \omega_2) &= \{ \omega_1 \in \mathcal{P} \mid \omega_1 \subseteq \rho_1, |\rho_1| = |\omega_1| + |\omega_2| \}, \\
\mathcal{R}(\kappa_1, \omega_2) &= \{ \omega_1 \in \mathcal{P} \mid \omega_1 \subseteq \kappa_1, |\kappa_1| = |\omega_1| + |\omega_2| \}.
\end{align*}
\]
Let \( \omega_1 \in \mathcal{R}(\rho_1, \omega_2) \). Since the first part of \( \rho_1 \) is equal to \( L \) by assumption (ii), we see that the first part of \( \omega_1 \) is less than or equal to \( L \). Therefore, the Young diagrams of \( \kappa_1 = \iota_L(\rho_1) \) (resp., \( \iota_L(\omega_1) \)) is obtained by adding one row having exactly \( L \) boxes just above the top row of the Young diagram of \( \rho_1 \) (resp., \( \omega_1 \)). Hence it is obvious that \( \iota_L(\omega_1) \in \mathcal{R}(\kappa_1, \omega_2) \).
We need to show that the map $\iota_L : \mathcal{R}(\rho_1, \omega_2) \to \mathcal{R}(\kappa_1, \omega_2)$ is bijective. Since the injectivity of the map $\iota_L$ is obvious, it remains to show the surjectivity of the map $\iota_L$. If $\delta_1 \in \mathcal{R}(\kappa_1, \omega_2)$, then there exists a part of $\delta_1$ that is equal to $L$. Indeed, since $\delta_1 \subset \kappa_1 = \iota_L(\rho_1)$, all parts of $\delta_1$ are less than or equal to $L$. Suppose, contrary to our assertion, that all parts of $\delta_1$ are less than $L$. Then the skew Young diagram $\kappa_1/\delta_1$ must contain at least $y_1 + 1$ boxes, where $y_1 + 1$ is equal to the number of rows of the Young diagram of $\kappa_1$ having exactly $L$ boxes since $\kappa_1 = \iota_L(\rho_1)$. Because $y_1 + 1$ is greater than $|\rho_2|$ by assumption (iii), and because $(\omega_2, \omega_3) \in Q(\rho_2)$, we deduce that $|\kappa_1/\delta_1| \geq y_1 + 1 > |\rho_2| \geq |\omega_2|$, and hence $|\kappa_1| > |\delta_1| + |\omega_2|$, which contradicts the assumption that $\delta_1 \in \mathcal{R}(\kappa_1, \omega_2)$. Therefore, if we let $\omega_1$ be the partition whose Young diagram is obtained from the Young diagram of $\delta_1$ by removing one row having exactly $L$ boxes, then we have $\omega_1 \in \mathcal{R}(\rho_1, \omega_2)$ and $\iota_L(\omega_1) = \delta_1$. This proves the surjectivity of the map $\iota_L$, as desired. 

**Claim 3.** For each $J \subset [1, L]$, we have $\iota_L(\rho^{J,n}) = \kappa^{J,n+1}$ and $\text{sgn}^{J,n}(\rho) = \text{sgn}^{J,n+1}(\kappa)$.

**Proof of Claim 3.** Since $\kappa = \iota_L(\rho)$, we deduce that

$$t' \kappa = t \rho + \underbrace{(1, 1, \ldots, 1, 0, 0, \ldots)}_{L \text{ times}}.$$ 

Consequently, if we define the sequences $b^{(t) \kappa} = (c_j)_{j \in \mathbb{Z}_{\geq 1}}$ and $b^{(t) \rho} = (b_j)_{j \in \mathbb{Z}_{\geq 1}}$ as in (5.1), then

$$b^{(t) \kappa} = b^{(t) \rho} + \underbrace{(1, 1, \ldots, 1, 0, 0, \ldots)}_{L \text{ times}}.$$ 

Furthermore, if we define the sequences $b^{J,n+1} = (c_j^{J,n+1})_{j \in \mathbb{Z}_{\geq 1}}$ and $b^{J,n} = (b_j^{J,n})_{j \in \mathbb{Z}_{\geq 1}}$ as in (5.2), then

$$c_j^{J,n+1} = c_j = b_j + 1 = b_j^{J,n} + 1 \quad \text{for} \ j \in [1, L] \setminus J;$$

$$c_j^{J,n+1} = R_{n+1} - c_j = R_n + 2 - (b_j + 1)$$

$$= R_n - b_j + 1 = b_j^{J,n} + 1 \quad \text{for} \ j \in J \subset [1, L];$$

$$c_j^{J,n+1} = c_j = b_j = b_j^{J,n} \quad \text{for} \ j > L.$$ 

These equations imply that

$$b^{(t) \kappa} = b^{(t) \rho} + \underbrace{(1, 1, \ldots, 1, 0, 0, \ldots)}_{L \text{ times}}.$$

Let $\tau$ be the finite permutation of the set $\mathbb{Z}_{\geq 1}$ such that

$$c_{\tau(1)}^{J,n+1} > c_{\tau(2)}^{J,n+1} > \cdots > c_{\tau(j)}^{J,n+1} > c_{\tau(j+1)}^{J,n+1} > \cdots.$$ 

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note that \( \tau(j) = j \) for all \( j \geq L + 1 \) since \( J \subset [1, L] \). It follows from (5.31) that
\[
 b^J_{r(1)} > b^J_{r(2)} > \ldots > b^J_{r(j)} > b^J_{r(j+1)} > \ldots,
\]
and hence that \( \text{sgn}^{J,n+1}(\kappa) = \text{sgn}^J,\rho). \) Also, it follows from the definitions that
\[
\kappa^{J,n+1} = \rho^J,n + \underbrace{1, 1, \ldots, 1, 0, 0, \ldots}_{L \text{ times}},
\]
which means that \( \iota_L(\rho^J,n) = \kappa^{J,n+1} \). This proves the claim.

Using Claims \( \mathcal{P} \) and \( \mathcal{K} \) we deduce from (5.30) that
\[
 X^{\kappa,n+1}_{\rho_1, \rho_2} = \sum_{J \subset [1, L], \omega_1 \in \mathcal{R}(\rho_1, \omega_2)} LR_{\omega_1, \omega_2} LR_{\omega_2, \omega_3} LR_{\omega_3, \omega_1} \text{sgn}^{J,n+1}(\kappa)
\]
\[
= \sum_{J \subset [1, L], \omega_1 \in \mathcal{R}(\rho_1, \omega_2)} LR_{\iota_L(\omega_1)} LR_{\omega_2, \omega_3} LR_{\omega_3, \iota_L(\omega_1)} \text{sgn}^J(\rho).
\]
First, observe that for each \( J \subset [1, L] \), \( \omega_1, \omega_2, \omega_3 \in \mathcal{Q}(\rho_2) \), and \( \omega_1 \in \mathcal{R}(\rho_1, \omega_2) \), there holds
\[
 LR_{\iota_L(\omega_1)} LR_{\omega_2, \omega_3} LR_{\omega_3, \iota_L(\omega_1)} = LR_{\omega_1, \omega_2}.
\]
Indeed, since \( \omega_1 \subset \rho_1 \) by the definition of \( \mathcal{R}(\rho_1, \omega_2) \), it follows from assumption (ii) that the Young diagram of \( \iota_L(\rho_1) \) (resp., \( \iota_L(\omega_1) \)) is obtained by adding one row having exactly \( L \) boxes just above the top row of the Young diagram of \( \rho_1 \) (resp., \( \omega_1 \)). This implies that the skew Young diagram \( \iota_L(\rho_1)/\iota_L(\omega_1) \) is identical to the skew Young diagram \( \rho_1/\omega_1 \). Therefore, by the Littlewood-Richardson rule (see, for example, \( \mathcal{E} \) Chapter 5, Section 2, Proposition 3 or \( \mathcal{Ko2} \) §2), we obtain (5.33). Next, we show that for each \( J \subset [1, L] \), \( \omega_1, \omega_2, \omega_3 \in \mathcal{Q}(\rho_2) \), and \( \omega_1 \in \mathcal{R}(\rho_1, \omega_2) \), there holds
\[
 LR_{\omega_1, \omega_2, \omega_3} LR_{\omega_3, \iota_L(\omega_1)} = LR_{\omega_1, \omega_2}.
\]
Suppose that \( \omega_1 \not\subset \rho^J,n \). Let \( j \in \mathbb{Z}_{\geq 1} \) be such that the \( j \)-th part of \( \omega_1 \) is greater than the \( j \)-th part of \( \rho^J,n \). Since \( \omega_1 \subset \rho_1 \) by the definition of \( \mathcal{R}(\rho_1, \omega_2) \), it follows from assumption (ii) that
\[
 L \geq \text{the } j \text{-th part of } \omega_1 > \text{the } j \text{-th part of } \rho^J,n,
\]
and hence that
\[
 \text{the } (j+1)\text{-st part of } \iota_L(\omega_1) = \text{the } j \text{-th part of } \omega_1
\]
\[
> \text{the } j \text{-th part of } \rho^J,n = \text{the } (j+1)\text{-st part of } \iota_L(\rho^J,n).
\]
This completes the proof of Proposition 5.2.

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[5] L. ω boxes between the y

In this case, there exist j0 ∈ Z such that the j0-th parts of ρ1 and ω1 are both equal to L. Indeed, it follows from Claim B(1) that the number of parts of ρ1 that are equal to 1 is greater than or equal to y. Let j1, j2 ∈ Z be such that the j1-th and j2-th parts of ρ1 are equal to 1. We may, therefore, assume that Lρ1 = 0 = LRω1. We may, therefore, assume that \( Lρ1 \neq 0 = LRω1 \). This implies that \( ρ1 \neq ω1 \). This implies that \( |ρ1| = |ω1| \).

Also, if \( Lρ1 \neq 0 = LRω1 \), then it is obvious that \( |ρ1| \neq |ω1| \).

\( \sum_{(\omega, L, \rho)} L_{(\omega, L, \rho)} \) into (5.32), we finally obtain (5.33) and (5.34). We may, therefore, assume that \( LR^{L}_{ω} ω \neq 0 = LR^{L}_{ω} \). We may, therefore, assume that \( LR^{L}_{ω} ω \neq 0 = LR^{L}_{ω} \).
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