Optimal relations for the allocation and effectiveness of the heat exchangers (hot and cold side) of an irreversible regenerative Stirling cycle

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Abstract: A stationary irreversible regenerative Stirling cycle model, with two sources of irreversibilities (finite rate of heat transfers and heat leak), is analyzed. The aim of this work is to obtain the criterion of partial optimization presented in [3]. Since with this criterion, the optimal relations for the allocation and effectiveness of the heat exchangers of Carnot-like power plant are obtained; when two design rules are applied, alternatively: internal thermal conductance fixed, or areas fixed. These optimal relations are the same for maximum specific power and efficiency. As an instance, after the substitution of these optimal values in the specific power and efficiency, the maximum specific power and efficiency are obtained. Then the maximum efficiency and specific power are compared, and it is found that the maximum efficiency is greater than maximum specific power for both design rules.

1. Introduction
In later years, the efficient use of energy has been a topic that has taken the interest of engineers and scientists. Due to that, the use of the thermal energy that is not used in the different engines has been one of the most explored subjects. Several proposals, strategies and approximations have been made considering the use of the Stirling engine, including the generation of mechanic or electrical power [1]. This engine can produce mechanical power from a heat flux in constant quantity from a working fluid; because it works with an external heat source. It is also important to mention that its thermal efficiency is higher than that from internal combustion engines. The optimization of this type of engine is one of the most important topics of development and investigation, including design parameters like not ideal regenerators [2]. We can find more technological works about Stirling engine presented in [2].

On the other hand, works about the optimization of thermal engines present optimal parameters that are not affected by the law of heat transfer corresponding to the distribution in the inventory of heat transfer on Carnot-like cycles [3]. The model presented in this work does not have temperature changes in the heat exchanger, like other studies [4] but it does have irreversibilities due to the finite heat transfer between the external heat sources corresponding to the high and low temperature; also includes a regenerator as in [5]. There is heat leak between the reservoirs [6].

2. Optimal allocation and effectiveness of the heat exchangers of hot and cold side
The model of the Stirling cycle consists of two isothermal and two isobaric processes, heat leak and finite rate heat transfer between the working fluid and external heat sources, and the inclusion of a regenerator (Figure 1).

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From the first law of thermodynamics, the power is the algebraic addition of all the heat inside the system. The first heat rate is the discard of residual heat $Q_{12}$ that is not used, during the isothermal process. It is sent to the reservoir of low temperature $T_l$.

This can be modeled by the heat $Q_2$ that is rejected to the exchanger of the cold side. On the other hand, the heat rate given by $Q_{34}$ is the heat supplied by the exchanger in the hot side of the engine to the working fluid. It also can be modeled by the heat transfer $Q_1$ received by the exchanger of the hot side.

![Figure 1. Regenerative Stirling cycle with external irreversibilities, regenerator and leak heat.](image)

Then, the rejected and received heat are:

$$Q_L = \dot{Q}_2 + \dot{Q}_1 = \beta(T_2 - T_L) + \dot{Q}_l$$

and

$$Q_H = \dot{Q}_l + \dot{Q}_1 = \alpha(T_H - T_l) + Q_l$$

where $\alpha$ and $\beta$ are the thermal conductances:

$$\alpha = UA_H \text{ or } U_H A_H; \quad \beta = UA_L \text{ or } U_L A_L$$

where $A_I (I = L, H)$ is the heat transfer area and $U$ or $U_I (I = L, H)$ are the global coefficients of heat transfer per unit area. As it is known, the heat in the isochoric process is:

$$Q_{23} = \dot{Q}_2 + \dot{m} c_v (T_a - T_b)$$

and

$$Q_{41} = \dot{m} c_v (T_b - T_a)$$

Therefore, the power in the system is:

$$P = \alpha(T_H - T_1) - \beta(T_2 - T_L) + \dot{m} c_v (T_a - T_b) + \dot{m} c_v (T_b - T_a)$$

Now, the heat transfer in the isochoric processes can be simplified [6]:

$$P = \alpha(T_H - T_1) - \beta(T_2 - T_L)$$

To get this expression it is considered that there is no internal irreversibility. The entropy balance is [6]:

$$\frac{Q_2}{T_b} - \frac{Q_1}{T_a} = 0$$
From the equations (5) and (6) and considering \( x = \frac{T_b}{T_a} \), the power can be written as:

\[
P = \dot{Q}_r(1 - x)
\]

Now, from this equation and the second equation of (1) the power is:

\[
P = \alpha (T_a - T_s) - (1-x)
\]

From the entropy balance, we have: \( x = \frac{\dot{Q}_r}{\dot{Q}_1} \). To have a value for \( T_a \) a new expression is needed. If we use equations (1), we have: \( x = \frac{\beta (T_b-T_L)}{\alpha (T_H-T_a)} = \frac{T_b}{T_a} \). Then, \( T_a = \frac{\alpha \xi T_H + \beta T_L}{\xi (\alpha + \beta)} \). Replacing this last equation in equation (8), the specific power is:

\[
p = \frac{p}{\xi T_H} = \frac{(1-x)}{\frac{\alpha}{\xi} + \frac{\beta}{\xi}}
\]

where \( \mu = \frac{T_L}{T_H} \). We notice that the equation (9) is similar the one that appears in (3) for the irreversible Carnot-like cycle. On the other hand, the efficiency is given by \( \eta = \frac{\rho}{\xi \dot{Q}_r} = \frac{\rho}{\dot{Q}_1 + \dot{Q}_{\text{leak}}} \). Here, \( \dot{Q}_r \) is the heat received by the regenerator in the process 41 and given in the process 23, and \( \dot{Q}_{\text{leak}} \) is the heat leak in the engine. In (5) it is assumed that the heat in the regenerator can be expressed with the thermal conductivity per unit of heat exchange area with the arithmetic average difference of temperature: \( \dot{Q}_r = \frac{\alpha}{2} (T_a - T_b) \), where \( \omega \) is a thermal conductance. And the heat leakage is given by: \( \dot{Q}_{\text{leak}} = k (T_H - T_L) \); also, \( k \) is a thermal conductance. From (5) and the equations for \( \dot{Q}_r \) and \( \dot{Q}_{\text{leak}} \) the the efficiency can be written as: \( \eta = \frac{\rho}{\xi - \frac{\alpha}{\xi} \omega (1-x) + k T_H (1-\mu)} \). Now, the heat transfer rate in the process 34 can be represented by the heat transferred by the hot side exchanger (second equation of (1)). Thus, the next expression is:

\[
\eta = \frac{\frac{\rho}{\xi - \frac{\alpha}{\xi} \omega (1-x) + k T_H (1-\mu)}}{\Omega (1-x) + k (1-\mu)}
\]

where \( \Omega = \frac{\alpha \xi \omega}{2} \); \( \rho = \frac{1}{m R \mu (\lambda - \alpha)} \).

The next step is to apply alternatively two design rules: fixed thermal conductance or fixed thermal transfer area, and calculate each maximum specific power. Thus: \( \alpha + \beta = \gamma \) or \( A_H + A_L = \gamma \); where \( \gamma = U A \) is the total thermal conductance. This is applied to the distribution of the exchangers, hot and cold, with the same value of global heat transfer \( U \) per unit of area in both extremes of the cycle, or, if the area \( A \) is fixed, then we will have two global heat transfer coefficients \( U_L \) and \( U_H \). Therefore, each exchanger, hot and cold, has also different effectiveness. Parametrizing these equations, the following relations are obtained:

\[
\phi_1 = \frac{\alpha}{U A}; \quad 1 - \phi_1 = \frac{\beta}{U A} \quad \text{or} \quad \phi_2 = \frac{\alpha}{\xi U A}; \quad 1 - \phi_2 = \frac{\beta}{\xi U A}
\]

Using equations (11) in equation (9), alternatively, and optimizing the power with respect to the variables \( \phi_1 \) or \( \phi_2 \), we obtain the same optimal relations for thermal conductance or heat transfer area (3):

\[
\phi_{\text{mp}C} = \frac{1}{2} \quad \text{or} \quad \phi_{\text{mp}A} = \frac{\sqrt{R_U}}{1 + \sqrt{R_U}}
\]

with \( R_U = \frac{U_L}{U_H} \). Now, let \( \phi \neq x \), then the Criterion 1 of (3) is obtained because of:
\begin{equation}
\frac{\partial p}{\partial \phi} \phi_{mp} = 0 \text{ and } \frac{\partial^2 p}{\partial \phi^2} \phi_{mp} < 0 \Rightarrow \frac{\partial p}{\partial \phi} \phi_{mp} = 0 \text{ and } \frac{\partial^2 \eta}{\partial \phi^2} \phi_{mp} < 0 . \text{ Therefore, we have the following criterion of partial optimization:}
\end{equation}

**Criterion 18.** Let \( \eta \) be the efficiency, \( x = \frac{T_b}{T_a} \) and \( \phi \neq x \) is a variable that corresponds to the allocation of different effectiveness of the heat exchangers for the Stirling model presented. Then \( \phi_{mp} \) is the point in which the specific power \( p \) achieves a maximum value if and only if \( \phi_{mp} \) is the point in which the function \( \eta \) achieves a maximum value. That is \( \phi_{mp} = \phi_{mc} \), this optimization is valid for any law of heat transfer that satisfies the equations (11).

Thus, we have obtained: \( \phi_{mp} = \phi_{mc} \) independently of the transfer heat law used. Now, we can substitute the optimal values (12) in the expressions of the specific power and the efficiency (equations (9) and (10)), and find the maximum power and efficiency with respect to \( x \).

### 3. Specific power and maximum efficiency

Including the equations (12) in the equation (9): \( p_{mpC} = \frac{U}{4} \left( 1 - \frac{\mu}{x} \right) (1 - x) \) or \( p_A = \frac{R_U(1 - \mu)(1 - x)}{(1 + \sqrt{R})^2} . \)

In optimizing with respect to \( x \) the maximum specific power is obtained:

\begin{equation}
p_{mpC} = \frac{U_A}{4} (1 - \sqrt{\mu})^2 \text{ or } p_A = \frac{R_U(1 - \sqrt{\mu})^2}{(1 + \sqrt{R})^2} . \tag{13}
\end{equation}

with \( \alpha = \beta = \frac{U_A}{2} \) and \( x_{mp} = \sqrt{\mu} \) or with \( R_U = \frac{U_A}{U} \) and \( x_{mp} = \sqrt{\mu} \). When replacing alternatively equations (12) and equation (13) in the efficiency (equation (10)), the efficiencies to maximum specific power are obtained:

\begin{equation}
\eta_{mpC} = \frac{(1 - \sqrt{\mu})^2}{1 + 4 \Omega_C(1 - \sqrt{\mu}) + 4 K_C(1 - \sqrt{\mu}) - \sqrt{\mu}} \text{ and } \eta_{mpA} = \frac{(1 - \sqrt{\mu})^2}{1 - \sqrt{\mu} + 4 \Omega_A(1 - \sqrt{\mu}) + 4 K_A(1 - \sqrt{\mu})} . \tag{14}
\end{equation}

with \( \Omega_C = \frac{\rho \alpha}{2} ; K_C = \frac{\rho}{\alpha} ; \alpha = \frac{U_A}{2} \), or with \( \Omega_A = \frac{\rho \alpha}{\lambda} ; K_A = \frac{\rho}{\lambda \alpha} ; \lambda = \frac{R_U}{(1 + \sqrt{R})^2} \) and where \( \rho = \frac{1}{m \nu} \) with \( \nu \) as mass flux and \( \lambda = \frac{V_2}{V_1} \) the volume relation. Similarly, the replacement of the equations (12) in the efficiency (equation (10)), can be optimized to each design rule, alternatively, to obtain the maximum efficiency with respect to \( x \), in both cases:

\begin{equation}
\eta_{maxC} = \frac{(1 - \sqrt{\mu})(1 - x_{meC})}{(1 - \sqrt{\mu})(1 - x_{meC}) + 4(\Omega_C(1 - x_{meC}) + K_C(1 - \mu))} \text{ or } \eta_{maxA} = \frac{(1 - \sqrt{\mu})(1 - x_{meA})}{(1 - \sqrt{\mu})(1 - x_{meA}) + 4(\Omega_A(1 - x_{meA}) + K_A(1 - \mu))} . \tag{15}
\end{equation}

where the values of maximum efficiency reached with each design rule are:

\begin{equation}
x_{meC} = \frac{\mu(1 - 4 \Omega_C + 2 \sqrt{\mu} (1 - 4 K_C (1 + \Omega_C (1 - \mu))))}{1 - 4 \Omega_C + 4 K_C(1 - \mu)} \text{ or } x_{meA} = \frac{\mu(1 - 4 \Omega_A + 2 \sqrt{\mu} (1 - 4 K_A (1 + \Omega_A (1 - \mu))))}{1 - 4 \Omega_A + 4 K_A(1 - \mu)} . \tag{16}
\end{equation}

To analyze the behaviour of the four different calculated efficiencies and compare them we consider the numerical values: \( \omega = 0.01 \frac{kW}{K} \); \( \alpha = 0.01 \frac{kW}{k} ; R_{air} = 0.267 \frac{kJ}{kg K} ; \lambda = 2 \). Then, \( \rho = 5.2593 \frac{kW}{k} \), \( \nu = 1 \frac{kg}{s} \); \( K_C = 0.01 ; \Omega_C = 0.037 ; K_A = 0.0582 ; \Omega_A = 0.0185 \). Also, of (7): \( \Lambda = 0.17157 \). The Figures 2-4 show the behaviour of the maximum efficiencies and the efficiencies to maximum power.
4. Conclusions
In this work a Stirling regenerative cycle with some irreversibilities into the heat exchangers which exchange a finite amount of heat is analyzed. There is also, a heat leak between them. The optimal relations, for the corresponding models, in the inventory of heat transfer applied alternatively, are obtained. After the replacement of these optimum values in the power and efficiency, the maximum specific power and its corresponding efficiency, and the maximum efficiency are obtained. In the Figures 2-4 the behaviour of the efficiencies is presented. It is noticed that $\eta_{\text{maxC}} \leq \eta_{\text{maxA}}$. Moreover, it is found that the efficiencies have to follow the inequality: $\eta_{\text{mpt}} \leq \eta \leq \eta_{\text{max}} (\ell=C, A)$. The results obtained in this work can be easily adapted to Ericsson cycles. Finally, the Criterion 1 of [1] is fulfilled for the model of regenerative Stirling cycle analyzed in this work. The optimal relations which, can be the same for a variety of objective functions (algebraic combinations of the power and the efficiency) and independent of the transfer heat law used. Further work is underway.

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