Local Existence for Nonlinear Wave Equation with Radial Data in 2 + 1 Dimensions

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1 Introduction

Let $\Box_g = \partial^2_t - g\Delta_x$. In this paper, for initial data $(u_0, u_1)$ with spherical symmetry and $s > \frac{n}{2}$, we consider the local well-posed (LWP) or local existence result for the semilinear (with $g(u) \equiv 1$, SLW) and quasilinear wave equation (with $g(0) = 1$, QLW)

\[
\begin{aligned}
\Box g(u) & = p(u)(\partial_t u)^2 + q(u)(\nabla u)^2 := N(u, \partial u) \\
& \quad \text{in } \mathbb{R} \times \mathbb{R}^2;
\end{aligned}
\]

\[
\begin{aligned}
& u(0, x) = u_0 \in H^s, \quad \partial_t u(0, x) = u_1 \in H^{s-1}
\end{aligned}
\]

on $\mathbb{R} \times \mathbb{R}^2$. We use $\partial$ to stand for space-time derivatives, i.e. $\partial = (\partial_t, \partial_x)$.

For general spatial dimensions $n$, the critical index of Sobolev space for such problem is $s_c = \frac{n}{2}$ and the counterexamples for LWP give the lower bound $\max(\frac{n}{2}, \frac{n+5}{4})$ (see [3] and [7] for example). The classical theory (see [5] for example) says that this problem is LWP in $H^s \times H^{s-1}$ for $s > \frac{n+2}{2}$, which insures that $\partial u$ is bounded.

For semilinear problems, it’s known that one can improve the needed regularity to $s > \max(\frac{n+1}{2}, \frac{n+5}{4})$ with the help of Strichartz estimates (see [3] for example), and the sharp results have been proved to be $s > \max(\frac{n}{2}, \frac{n+5}{4})$ (see [10] and references therein).

In the last ten years or so, the analysis of QLW has experienced a dramatic growth. Following partial results independently obtained by Bahouri-Chemin [2], [1] and Tataru [11], [12], [13], and further work of Klainerman-Rodnianski [6], Smith and Tataru largely completes the local theory for

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general second order quasilinear hyperbolic equations in [8]. They show that for $n \leq 5$, the problem is LWP in $H^s$ for $s > \max(\frac{n+1}{2}, \frac{n+5}{4})$. Moreover, the $L^{\text{max}(2, \frac{n+1}{n-1})} L^\infty$ Strichartz estimate holds true for the corresponding wave operator $\Box g(u)$. For the detailed historical introduction, see Section 1.2 in [8] for example.

Thus, in general, the optimal regularity for LWP for the problem in two space dimensions is $\frac{7}{4}$. Note that the counterexample which gives the lower bound $\frac{7}{4}$ is non-radial. The main purpose of this paper is to show that for the radial data, the regularity can be improved further in two space dimensions by showing that we can get an improved Strichartz estimates.

In our previous paper [4], we get the following radial improvement of Strichartz estimate for the solution of linear wave equation $\Box u = 0$,

$$\| \partial u \|_{L^2_t L^\infty_x} \lesssim \| \partial u(0) \|_{H^{s-1}}$$  \hspace{1cm} (1.2)

with $s > \frac{3}{2}$ and $n = 2$. This would naturally yield the radial LWP in $H^s$ with $s > \frac{3}{2}$ for SLW. We will give the proof in Section 2. Moreover, we get a weak stability result in a more larger class, for any space dimensions. Inspired by the result for SLW, we intend to prove a similar result in the quasilinear case, by using the method in [8].

Now we begin to state our main results.

**Theorem 1** (Radial LWP for SLW). Let $n = 2$ and $s > \frac{3}{2}$. The equation (1.1) with $g(u) \equiv 1$ is radial local well-posed in $CH^s$. Precisely, for any radial data $(u_0, u_1) \in H^s \times H^{s-1}$, there exists a unique radial local in time solution $u \in CH^s$ such that $\partial u \in CH^{s-1} \cap L^2 L^\infty$, and the solution map is Lipschitz continuous on bounded sets.

As we know, the current counterexample to radial LWP shows the lower bound $s_c = \frac{5}{2}$. And here the positive result requires the regularity $s > \frac{n+1}{2}$. Thus if we combine it with the previous positive results $s > \max(\frac{n+5}{4}, \frac{n}{2})$, we know that there is still a gap for $n = 2, 3$, and $\frac{1}{2}$ gap for $n = 4$. Thus a natural problem is:

**What is the optimal regularity $s_o$ for SLW to be radial LWP?**

We conjecture that $s_o = \frac{5}{2}$. We still can’t prove or disprove the conjecture now, instead, we utilize the energy estimate to establish the following weak stability estimate for SLW.

Note that for the equation of type $\Box u = u \nabla u$ in four space dimensions, Sterbenz [9] got a relative results of global existence with small data, based on the argument of Tataru [10]. We intend to solve the conjecture by similar method in the following work.
Theorem 2 (Weak Stability for SLW). Let $s > \frac{n}{2}$ and $s \geq 1$. Consider the semilinear wave equation

$$\begin{align*}
\begin{cases}
\partial_t^2 u - \Delta u = \sum_{|\alpha|=2} q_\alpha(u) (\partial u)^\alpha \\
u(0, x) = u_0, \quad \partial_t u(0, x) = u_1
\end{cases}
\end{align*}$$

(1.3)

then there exists at most one solution in the solution class $X = \{ u \in CH^s; \partial u \in CH^{s-1} \cap L^1 L^\infty \}$. Moreover, if $u, v$ are two solutions of above equation with initial data $(u_0, u_1)$ and $(v_0, v_1)$, then

$$\| \partial (u-v) \|_{L^\infty([0,T], L^2)} \lesssim \|(u_0-v_0, u_1-v_1)\|_{H^1 \times L^2} \exp(C(T+\|\partial(u,v)\|_{L^1 L^\infty})).$$

Theorem 3 (Local Existence for QLW). Let $n = 2$ and $s > \frac{3}{2}$. For each $R > 0$, there exist constants $T, M > 0$ so that, for any radial initial data $(u_0, u_1)$ which satisfies

$$\|(u_0, u_1)\|_{H^{s} \times H^{s-1}} \leq R,$$

there exists a radial solution $u \in CH^s$ to (1.1) on $[-T, T] \times \mathbb{R}^2$ such that

$$\| \partial u \|_{C^1 H^{s-1} \cap L^2 L^\infty} \leq M.$$

(1.4)

Moreover, for $1 \leq r \leq s+1$, each $t_0 \in [-T, T]$, and any radial data $(v_0, v_1)$, the linear equation

$$\begin{align*}
\begin{cases}
\Box g(u) v = 0, \quad (t, x) \in [-T, T] \times \mathbb{R}^2 \\
v(t_0) = v_0 \in H^r, \quad \partial_t v(t_0) = v_1 \in H^{r-1}
\end{cases}
\end{align*}$$

(1.5)

admits a radial solution $v \in C([-T, T], H^r) \cap C^1([-T, T]; H^{r-1})$, and the following estimate holds:

$$\| v \|_{L^\infty_t H^r_x} + \| \partial_t v \|_{L^\infty_t H^{r-1}_x} \leq C\|(v_0, v_1)\|_{H^r \times H^{r-1}}.$$

(1.6)

Additionally, the following estimate holds, provided $\rho < r - \frac{1}{2}$,

$$\| \langle D_x \rangle^\rho v \|_{L^2_t L^\infty_x} \leq C\|(v_0, v_1)\|_{H^r \times H^{r-1}}$$

(1.7)

and the same estimate holds true with $\langle D_x \rangle^\rho$ replaced by $\langle D_x \rangle^{\rho-1} \partial$.

As in [8], for the proof of Theorem 3, we will mainly prove the following dispersive (Strichartz) estimate.
**Theorem 4** (Dispersive Estimate). Let $\epsilon \lambda \gg 1$ and $\chi_{j,k}$ $(j, k \in \mathbb{Z})$ be the “radial” wave packet which will be introduced in Section 3.2.

\begin{equation}
\left\| \sum a_{j,k} \chi_{j,k} \right\|_{L_t^2 L_x^\infty} \lesssim \epsilon_0^{-\frac{7}{4}} (\ln \lambda)^{\frac{3}{2}} \|a_{j,k}\|_{l^2_{j,k}} \tag{1.8}
\end{equation}

We give here some notations which will be used hereafter. Let $\langle x \rangle = \sqrt{1 + x^2}$ and $H(x)$ be the usual Heaviside function ($H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ else). For a set $E$, we use $|E|$ to stand for the measure or cardinality of the set $E$ depending on the context.

This paper will be organized as follows: First, for SLW, we give the proof of radial LWP result (Theorem 1) in Section 2, moreover, we give the proof of uniqueness and weak stability result (Theorem 2) in a more larger class, for any space dimensions, by utilizing the energy estimates.

Then we turn into the proof of Theorem 3. In Section 3, we reduce Theorem 3 to Theorem 4 and in Section 4, we give the sketch of the proof of Theorem 4. We will reduce the proof to the corresponding estimates for two sets of indices $(j, k)$ for wave packets separately.

The first case is that $\chi_{j,k}$ is essentially $L^\infty$-normed. For this case, we get the overlap estimate of the “radial” wave packets in Section 5, then we have the required dispersive estimate as in Section 10 of [8].

For the remained case, it turns out that this is the case when the wave packet evolve essentially along the light cone and occurs only when $j$ is small. For this case, we give a $L^\infty$ estimate for single $\chi_{j,k}$ in Section 6 which is sufficient for the proof of the dispersive estimate, as explained in the end of Section 4.

## 2 LWP and Weak Stability for SLW

In this section, we prove the results for SLW. First, we prove the radial local well posed result Theorem 1.

**Proof of Theorem 1**. The existence of the radial solution follows from the radial Strichartz estimate. Precisely, for radial $u$ such that $\Box u = 0$, we have

$$\|\partial u\|_{L_{t,x}^s L_x^\infty} \lesssim \|\partial u(0)\|_{H^{s-1}}$$

with $s > \frac{3}{2}$. We get the solution by contraction argument as usual. Let $(u_0, u_1)$ be radial and

$$\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \leq M.$$
Define a complete domain with $C$ large enough
\[ B_T = \{ u \in C([0,T], H^s) \cap C^1 H^{s-1} \mid u \text{ radial}, \|u\|_{L^\infty H^s} + \|\partial u\|_{L^\infty H^{s-1} \cap L^2 L^\infty} \leq CM \} , \]
and for $u \in B_{\epsilon, T}$, define $\Pi(u)$ be the solution of the equation
\[ \Box \Pi(u) = N(u, \partial u) \]
with prescribed initial data $(u_0, u_1)$.

Thus for $T$ small enough, by Strichartz estimate and energy estimate,
\[ \| \partial \Pi(u) \|_{L^\infty H^{s-1} \cap L^2 L^\infty} \leq C_1 (M + \|N(u, \partial u)\|_{L^1 H^{s-1}}) \]
\[ \leq C_2 (M + T^2 \|\partial u\|_{L^\infty H^{s-1} \cap L^2 L^\infty}^2) \]
\[ \leq C_3 (M + T^2 (CM)^2) \leq \frac{CM}{4} \]
and by noting that $u(t) = u_0 + \int_0^t \partial_t u$ and $s > 1$,
\[ \| \Pi(u)(t) \|_{H^s} \leq \| \partial_x \Pi(u)(t) \|_{H^{s-1}} + \| \Pi(u)(t) \|_{L^2} \]
\[ \leq \frac{CM}{4} + M + \| \partial_t u \|_{L^1 L^2} \]
\[ \leq (1 + T) \frac{CM}{4} + M \leq \frac{CM}{2} . \]

Thus $\Pi$ is closed in the ball $B_T$, similar argument shows that $\Pi$ is a contraction map in the ball $B_T$. So we get a radial local solution $u \in CH^s \cap C^1 H^{s-1}$.

It is easy to see that the radial solution is unique and the solution map is Lipschitz continuous on $B_T$ by the previous argument.

Now we give the proof of Theorem 2.

Proof of Theorem 2. Let $s > \frac{5}{2}$,
\[ X = \{ u \in C([0,T], H^s); \partial u \in CH^{s-1} \cap L^1 L^\infty \} , \]
and $u, v$ in $X$ be two solutions of equation (1.3) with initial data $(u_0, u_1)$ and $(v_0, v_1)$, then $\omega := u - v \in CH^s$, and
\[ \Box \omega = a(u, v)\omega(\partial u)^2 + q(v)\partial(u, v)\partial \omega := \tilde{N} \]
with initial data $(\omega_0, \omega_1)$. Note that $\omega(t) = \omega(0) + \int_0^t \partial_t \omega$, then
\[ \| \omega \|_{L^\infty H^1} \lesssim \| \omega \|_{L^\infty L^2} + \| \partial \omega \|_{L^\infty L^2} \lesssim \| \omega(0) \|_{L^2} + \| \partial \omega \|_{L^\infty L^2} \]
with $T \lesssim 1$. Thus by Leibnitz rule and Sobolev multiplication law, we have
\[ \| \partial \omega \|_{L^\infty L^2} \lesssim \| \partial \omega(0) \|_{L^2} + \| \tilde{N} \|_{L^1 L^2} \]
\[ \lesssim \| \partial \omega(0) \|_{L^2} + \| \omega \|_{L^\infty H^1} (\| \partial u \|_{L^1 H^{s-1}}^2 + \| \partial(u, v) \|_{L^1 L^\infty} \| \partial \omega \|_{L^\infty L^2} \]
\[ + \| \partial(u, v) \|_{L^1 L^\infty} \| \partial \omega \|_{L^\infty L^2} \]
\[ \lesssim \| (\omega_0, \omega_1) \|_{H^1 L^2} + \| \partial \omega \|_{L^\infty L^2} \| \partial(u, v) \|_{L^1 L^\infty} . \]
So we have the following stability estimate for small enough time \( T \in (0, 1] \) (such that \( \| \partial(u, v) \|_{L^1_t L^\infty_x} \ll 1 \)),
\[
\| \partial \omega \|_{L^\infty_t L^2_x} \lesssim \| (\omega_0, \omega_1) \|_{H^1_x \times L^2_x} .
\]
Thus by an induction argument we can get the final estimate
\[
\| \partial \omega \|_{L^\infty_t([0,T],L^2_x)} \lesssim \| (\omega_0, \omega_1) \|_{H^1_x \times L^2_x} \exp(C(T + \| \partial(u, v) \|_{L^1_t L^\infty_x})).
\]

3 Local Existence for Quasilinear Wave Equation

In this section we reduce our main result Theorem 3 to the dispersive estimate.

3.1 Existence Result for Smooth Initial Data

First, we show that Theorem 3 is a consequence of the following existence result for smooth initial data.

**Proposition 1** (Local existence for smooth data). Let \( n = 2 \) and \( s > 3/2 \). For each \( R > 0 \), there exist constants \( T, M > 0 \) so that, for any smooth radial data \((u_0, u_1)\) which satisfies \( \| (u_0, u_1) \|_{H^s \times H^{s-1}} \leq R \), there exists a unique smooth solution \( u \) to (1.1) on \([-T, T] \times \mathbb{R}^2\) such that
\[
\| \partial u \|_{C^1_t H^{s-1}_x \cap L^2_t L^\infty_x} \leq M .
\]
Moreover, we have the energy estimate (1.6) and Strichartz estimate (1.7) for the solution \( v \) of the equation \( \Box g(u)v = 0 \).

In fact, for any radial initial data \((u_0, u_1) \in H^s \times H^{s-1}\) such that
\[
\| (u_0, u_1) \|_{H^s \times H^{s-1}} \leq R .
\]
Let \((u_0^k, u_1^k)\) be a sequence of smooth data converging to \((u_0, u_1)\), which also satisfy the same bound. Then the conclusion of Proposition 1 applies uniformly to the corresponding solutions \( u^k \). In particular, it follows that the sequence \( \partial u^k \) is bounded in the space \( CH^{s-1} \cap L^2 L^\infty \). Thus there exists a subsequence (also denoted by \( u^k \)) which converges weakly to some \( u \) in \( CH^s \cap C^1 H^{s-1} \). We’ll show below that it’s a solution of the equation with data \((u_0, u_1)\).
Let $\phi_j(x) = \phi(j^{-1}x)$, where $\phi$ is a smooth bump function with compact support, $\phi = 1$ on the unit ball. For any fixed large $j$, define $u^k_j = \phi_j u^k$, and thus $u^k_j$ are uniformly bounded in $CH^s \cap C^1 H^{s-1}$. Thus by compactness, there is a subsequence (also denoted by $u^k_j$) which converges to some $u_j$ in $CH^{s-1} \cap C^1 H^{s-1-}$. However, since $u^k_j = u^k_j$ in $B_j$, we have $u_j = u$ in $B_j$.

As a consequence of the fractional Leibnitz rule, the right hand side term $N(u^k, \partial u^k)$ of the equations for $u^k$ are uniformly bounded in the space $L^2 H^{s-1}$. Then (1.7) combined with Duhamel’s formula show that $\partial u^k$ is uniformly bounded in $L^2 C_\delta$. Note that $s > \frac{n}{2}$ and

$$\partial u^k_j = \phi_j(x) \partial u^k + j^{-1}(\partial \phi)(j^{-1}x) u^k,$$

thus we have $\partial u^k_j$ is also uniformly bounded in $L^2 C_\delta$. Together with the above this implies that $\partial u^k_j$ converges to $\partial u_j$ in $L^2 L^\infty$. Thus we get that $\partial u^k$ converges to $\partial u$ in $CH_\text{loc}^{s-1} \cap L^2 C_\text{loc}^\delta$.

The above information is more than sufficient to allow passage to the limit in the equation (1.1) and show that $u$ is a solution in the sense of distributions, yielding the existence part of Theorem 3. The conditions (1.4), (1.6) and (1.7) hold for $u$ since they hold uniformly for $u^k$.

### 3.2 Reduction to Dispersive Estimate

Here we show briefly how Proposition 1 follows from Theorem 4.

Let us first recall some notions in [8] which is necessary for proceeding. Let $n = 2$, and $\theta = \epsilon_0^{1/2} \lambda^{-1/2}$ with $\lambda \gg 1$ stands for the frequency and $\epsilon_0 \ll 1$ s.t. $\epsilon_0 \lambda \gg 1$, we use $\chi_{j,k,\omega}$ to denote the $L^\infty_\omega$-normalized wave packet supported in the region (with $x_\omega = x \cdot \omega$ and $x'_\omega$ be the given orthonormal coordinates)

$$T_j,k,\omega = \{(x,t) : |x_\omega - t - k \lambda^{-1}| \leq \lambda^{-1}, |x'_\omega - j(\epsilon_0 \lambda)^{-1/2}| \leq (\epsilon_0 \lambda)^{-1/2}, |t| \leq 2\}$$

(3.1)

Note that for simplicity, we write here all the quantities with respect to the flat metric, and this is sufficient for us as explained at the beginning of Section 4. Precisely,

$$\chi_{j,k,\omega}(x,t) = \lambda^{-1} T_\lambda(\delta(x_\omega - t - k \lambda^{-1}) W),$$

(3.2)

where $T_\lambda$ is the convolution with a spatially localized function $\psi_{\lambda}(x) = \lambda^n \psi(\lambda x)$, and $W = W_0((\epsilon_0 \lambda)^{1/2}(x'_\omega - j(\epsilon_0 \lambda)^{-1/2}))$. The index $\omega$, which stands for the initial orientation of the wave packet at $t = -2$, varies over a maximal collection of approximately $\theta^{-1}$ unit vectors separated by at least $\theta$. 

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If we define the “radial” wave packet 

$$\chi_{j,k} = \sum_\omega \chi_{j,k,\omega},$$

(3.3) then as in [8], Proposition 1 follows from the dispersive estimate in Theorem 4 for the superposition of radial wave packet.

We outline here how Theorem 4 yields Proposition 1, for the details of the Proposition we used, one should consult the content in [8]. Firstly, Proposition 1 is the consequence of the following result which is similar to Proposition 7.2 in [8]. Let $S_\lambda$ or $S_{<\lambda}$ be the Littlewood-Paley projector at or below the frequency $\lambda$, and $g_\lambda = S_{<\lambda}g$.

**Proposition 2.** Let $\epsilon_0 \lambda \gg 1$, Then for each $(u_0, u_1) \in H^1 \times L^2$, there exists a function $u_\lambda$ in $C^\infty([-2, 2] \times \mathbb{R}^2$ with 

$$\text{supp}\left(\hat{u}_\lambda(t, \cdot)(\xi)\right) \subset \{ \xi : \lambda/8 \leq |\xi| \leq 8\lambda \},$$

such that

$$\|\Box g_\lambda u_\lambda\|_{L_t^1 L_x^2} \lesssim \epsilon_0 (\|u_0\|_{H^1} + \|u_1\|_{L^2}),$$

(3.4) 

$$u_\lambda(-2) = S_\lambda u_0, \quad \partial_t u_\lambda(-2) = S_\lambda u_1,$$ 

(3.5) 

and such that the following Strichartz estimate holds for $r > 1/2$

$$\|S_\lambda u_\lambda\|_{L_t^r L_x^\infty} \lesssim \epsilon_0^{-\frac{3}{4}} \lambda^{r-1} (\|u_0\|_{H^1} + \|u_1\|_{L^2}).$$

(3.6)

Now we use Theorem 4 to give the proof of Proposition 2. Let $u_{j,k,\omega} = \theta^{\frac{1}{2}} \chi_{j,k,\omega}$. Then by Proposition 8.7 in [8], for any radial $(u_0, u_1) \in H^1 \times L^2$, there exists a function of form

$$u = \sum_{j,k,\omega} a_{j,k} u_{j,k,\omega},$$

such that the equality (3.5) holds. Moreover, by Proposition 8.4 in [8], we have

$$\epsilon_0^{-1} \|\Box g_\lambda S_\lambda u\|_{L_t^1 L_x^2} + \|\partial S_\lambda u\|_{L_t^\infty L_x^2} \lesssim \|a_{j,k}\|_{L^2_i L^\infty_{j,k,\omega}} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2}.$$ 

(3.7)
Now if we apply Theorem 4 to $u$ here, we get that

$$\|S_\lambda u\|_{L_t^2 L_x^\infty} = \|S_\lambda u\|_{L_t^2 L_x^\infty}$$

$$= \| \sum_{j,k,\omega} a_{j,k} \chi_{j,k,\omega} \|_{L_t^2 L_x^\infty}$$

$$\lesssim \| \sum_{j,k} a_{j,k} \chi_{j,k} \|_{L_t^2 L_x^\infty}$$

$$\lesssim \theta \times \epsilon_0^{-\frac{7}{4}} (\ln \lambda)^{\frac{3}{2}} \|a_{j,k}\|_{l^2_{j,k,\omega}}$$

This is just the required Strichartz estimate at frequency $\lambda$ (3.5). Thus we complete the proof of Proposition 2.

4 Dispersive Estimate

In this section, we reduce Theorem 4 to the proof of Proposition 3, 4 and 5 below, which deal with three sets of $(j,k)$ separately.

Based on the estimate of the Hamiltonian flow in [8], without loss of generality, we need only to give the proof of Theorem 4 for the flat metric.

In the process of the study, we find that one should deal with three cases separately. Define the following subsets of the indices $(j,k)$ in $\mathbb{Z}^2$,

$$A_1 = \{(j,k) | j^2 (\epsilon_0 \lambda)^{-1} + k^2 \lambda^{-2} \gg 1\}$$

$$A_2 = \{(j,k) | j^2 (\epsilon_0 \lambda)^{-1} + k^2 \lambda^{-2} \ll 1, |j| \gg 1\}$$

$$A_3 = \{(j,k) | j^2 (\epsilon_0 \lambda)^{-1} + k^2 \lambda^{-2} \ll 1, |j| \ll 1\}$$

We will prove Theorem 4 for $(j,k) \in A_i$ separately.

In the case of $A_1$ and $A_2$, we have $\chi_{j,k} \lesssim 1$ in principle which will be clear in Proposition 5 and hence the dispersive estimate reduced to overlap estimate of the wave packet as in Proposition 10.1 of [8].

Let $P_i = (t_i, x_i)$, and define

$$N_i(P_1, P_2) = |\{(j,k) \in A_i \mid \chi_{j,k}(P_1) \chi_{j,k}(P_2) \neq 0\}|$$

Then based on the estimate Proposition 9.2 in [8], we can get the estimate of $N_i(P_1, P_2)$ for $i = 1, 2$. 

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Proposition 3. We have
\[ N_1(P_1, P_2) \lesssim \epsilon_0^{-3/2} |t_1 - t_2|^{-1/2} \]  
(4.1)

Proposition 4. We have
\[ N_2(P_1, P_2) \lesssim \epsilon_0^{-1} |t_1 - t_2|^{-1} \]  
(4.2)

For the remained case $A_3$, the previous argument doesn’t work. Instead, we prove the $L^\infty$ estimate for the $\chi_{j,k}$ with $(j,k) \in A_3$.

Proposition 5. If $|j| \lesssim 1$, $|k| \lesssim \lambda$, 
\[ |\chi_{j,k}(t,x)| \lesssim \theta^{-1} (\lambda x)^{-1/2} (\lambda |x| + 1 - |k + \lambda t|)^{-1/2} H(\lambda |x| + 1 - |k + \lambda t|) \]  
(4.3)

If $|j| \lesssim 1$ and $\lambda \ll |k| \lesssim \theta^{-2}$, then $|\chi_{j,k}(t,x)| \lesssim \epsilon_0^{-1}$. Else, $|\chi_{j,k}(t,x)| \lesssim 1$.

Corollary 6.
\[ \chi_{j,k} \lesssim \begin{cases} \epsilon_0^{-1} & (j,k) \in A_1 \\ 1 & (j,k) \in A_2 \end{cases} \]

By the previous result, we can prove Theorem 4 directly. In fact, by Proposition 3 and Corollary 6, we have (as in Proposition 10.1 of [8])
\[ \| \sum_{(j,k) \in A_1} a_{j,k} \chi_{j,k} \|_{L^2_t L^\infty} \lesssim \epsilon_0^{-\frac{7}{4}} (\ln \lambda)^{\frac{7}{2}} \|a_{j,k}\|_{l^2_{j,k}}. \]  
(4.4)

And Proposition 4 and Corollary 6 yields
\[ \| \sum_{(j,k) \in A_2} a_{j,k} \chi_{j,k} \|_{L^2_t L^\infty} \lesssim \epsilon_0^{-\frac{1}{2}} (\ln \lambda)^{\frac{3}{2}} \|a_{j,k}\|_{l^2_{j,k}}. \]  
(4.5)

By Proposition 5, we have

Proposition 7.
\[ \| \sum_{(j,k) \in A_3} a_{j,k} \chi_{j,k} \|_{L^2_t L^\infty} \lesssim \epsilon_0^{-\frac{1}{2}} \ln \lambda \|a_{j,k}\|_{l^2_{j,k}}. \]  
(4.6)

Thus, by (4.4), (4.5), (4.6), Theorem 4 is finally reduced to the proof of Proposition 3, 4 and 5.

We give the proof of Proposition 7 now.

Proof of Proposition 7. Without loss of generality, let $j = 0$ and
\[ f_k(t,m) = \langle m \rangle^{-\frac{7}{2}} \langle m + 1 - |k + t\lambda| \rangle^{-\frac{3}{2}} H(m + 1 - |k + t\lambda|), \]
then by \[(4.3),\] (4.6) is reduced to the proof of
\[
\| \sum_{|k| \leq \lambda} a_k f_k \|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}} \ln \lambda \|a_k\|_{l^2_k}.
\] (4.7)

Since $f_k \geq 0$, we may assume $a_k \geq 0$ without loss of generality. Let
\[
f_k^{\pm 1}(t, m) = f_k(t, m) H(\pm (k + t\lambda + 2))
\]
and $f_k^0 = f_k - f_k^{1} - f_k^{-1}$.

Since for any fixed $t$, there is finite $k(|k + t\lambda| < 2)$ such that $f_k^0$ nonzero. The estimate for $f_k^0$ follows directly ($|f_k^0| \leq 1$),
\[
\| \sum_{|k| \leq \lambda} a_k f_k^0 \|_{L^2_t L^\infty_x} \lesssim \lambda^{-1} \sum_k a_k^2.
\]

Thus we need only to prove \[(4.7)\] for $f_k^1$ with $a_k \geq 0$, by symmetry.

Divide the time interval $[-2, 2]$ into $I_i = \left[\frac{i}{\lambda}, \frac{i+1}{\lambda}\right]$ with $|i| \lesssim \lambda$. Then for any $t \in I_i$,
\[
\sum_k a_k f_k^1(t, m) \lesssim \sum_{1 \leq k + i \leq m+1} a_k (m)^{-\frac{1}{2}} (m + 1 - k - i)^{-\frac{1}{2}} = R(m).
\] (4.8)

Let $m_i \lesssim \lambda$ be the point such that
\[
\| R(m) \|_{L^\infty_m} = R(m_i),
\]
then
\[
\| \sum_k a_k f_k^1(t, m) \|_{L^2_t L^\infty_x}^2 \lesssim \sum_i \left| \sum_{1 \leq k + i \leq m_i+1} a_k (m_i)^{-\frac{1}{2}} (m_i + 1 - k - i)^{-\frac{1}{2}} \right|^2 \lambda^{-1}
\]
\[
\lesssim \lambda^{-1} \sum_i (m_i)^{-1} \left( \sum_k a_k^2 \left( \sum_k (m_i + 1 - k - i)^{-1} \right) \right)
\]
\[
\lesssim \lambda^{-1} \ln \lambda \sum_{1 \leq k + i \leq m_i+1} a_k^2 (m_i + 1)^{-1}
\]
\[
\lesssim \lambda^{-1} \ln \lambda \sum_k \left( a_k^2 \sum_{i \geq 1 - k} (k + i)^{-1} \right)
\]
\[
\lesssim \lambda^{-1} (\ln \lambda)^2 \sum_k a_k^2.
\]

This is just \[(4.7)\] for $f_k^1$. \[\blacksquare\]
5 Overlap estimates

We first recall Proposition 9.2 in [8] which is essential for the proof of the overlap estimates. Let \(P_i = (t_i, x_i), t_2 > t_1\) and \(t = t_2 - t_1\). Note that

\[m = \max_\omega |(x_2 - x_1) \cdot \omega - t| = |x_2 - x_1| - t,\]

where the maximum is attained at \(\omega = \alpha := \frac{x_2 - x_1}{|x_2 - x_1|}\). We define

\[N_\lambda(P_1, P_2) = |\{(j, k, \omega) \mid \chi_{j,k,\omega}(P_1) \chi_{j,k,\omega}(P_2) \neq 0\}|.\]

Lemma 1 (Proposition 9.2 in [8]).

\[
N_\lambda(P_1, P_2) \lesssim \begin{cases} 
\theta^{-1} (\lambda m)^{-\frac{1}{2}}(\lambda t)^{-\frac{1}{2}} & -4\lambda^{-1} \leq m \leq \min(2t, c(\epsilon_0 \lambda)^{-1}t^{-1}) \\
\theta^{-1} (\lambda m)^{-1} & 2t \leq m \leq c(\epsilon_0 \lambda)^{-\frac{1}{2}} \\
0 & \text{else}
\end{cases}
\]  

(5.1)

Precisely, let \(t \geq \lambda^{-1}\) and

\[A_\lambda = \{\omega \in S^1 \mid |(x_2-x_1) \cdot \omega - (t_2-t_1)| \lesssim \lambda^{-1}, \ |x_2-x_1 - (t_2-t_1)\omega| \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}}\}\]  

(5.2)

then the estimate of \(N_\lambda(P_1, P_2)\) follows from the estimate of the area of \(A_\lambda\) by the inequality

\[N_\lambda(P_1, P_2) \lesssim \theta^{-1} |A_\lambda(P_1, P_2)|.\]

Moreover, if \(|m| \lesssim \lambda^{-1}\) and \(t \gtrsim \lambda^{-1},\)

\[A_\lambda \subset \{ |\omega - \alpha| \lesssim (\lambda t)^{-\frac{1}{2}}\}.\]  

(5.3)

If \(\lambda^{-1} \lesssim m \lesssim \min(t, (\epsilon_0 \lambda)^{-1}t^{-1}),\)

\[A_\lambda \subset \{ |\omega - \alpha| \approx m^\frac{1}{2} t^{-\frac{1}{2}}\}.\]  

(5.4)

Now we are ready to give the proof of Proposition 3 and 4.

Proof of Proposition 3. Without loss of generality, we assume \(P_i = (t_i, r_i, 0)\) with \(r_2 \geq r_1 \gg 1\) and \(t = t_2 - t_1 \geq 0\). Denote the spatial clockwise rotation of \(P_2\) with angle \(\omega\) by \(P_2^{k\omega}\). Define for \(k\theta \in [0, \pi]\)

\[m_k := m(P_1, P_2^{k\theta}) = \sqrt{(r_2 - r_1)^2 + 2r_1r_2(1 - \cos(k\theta))} - t \geq r_2 - r_1 - t.\]

First, for \(t \gg (\epsilon_0 \lambda)^{-\frac{1}{2}},\) we have \(N_\lambda(P_1, P_2^{k\theta})\) nonzero only if \(m_k \in (-4\lambda^{-1}, c(\epsilon_0 \lambda t)^{-1})\) by (5.1). Thus w.l.o.g, we may assume \(r_2 - r_1 - t \leq \)
\(c(\epsilon_0 \lambda t)^{-1}\). We consider separately the cases \(|r_2 - r_1 - t| \leq c(\epsilon_0 \lambda t)^{-1}\) and \(r_2 - r_1 - t \leq -c(\epsilon_0 \lambda t)^{-1}\).

For the first case, \(|r_2 - r_1 - t| \leq c(\epsilon_0 \lambda t)^{-1}\), we have from \(m_k \leq (\epsilon_0 \lambda t)^{-1}\)

\[
1 - \cos(k\theta) \leq \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (r_2 - r_1)^2}{2r_1 r_2} \lesssim \frac{t \cdot (\epsilon_0 \lambda t)^{-1}}{r_1 r_2} \ll (\epsilon_0 \lambda)^{-1}
\]

thus \(k\theta \ll (\epsilon_0 \lambda)^{-\frac{3}{2}}\) and \(k \ll \epsilon_0^{-1}\). So

\[
N_1(P_1, P_2) \lesssim \epsilon_0^{-1} \max_k \{N_\lambda(P_1, P_2^k) \} \lesssim \epsilon_0^{-1} \cdot \theta^{-1} |\lambda t|^{-\frac{1}{2}} \lesssim \epsilon_0^{-\frac{3}{2}} t^{-\frac{1}{2}}
\]

For the second case \(r_2 - r_1 - t \leq -c(\epsilon_0 \lambda t)^{-1}\), let \(k_1 k_2\) be the number s.t. \(m_{k_1} = c(\epsilon_0 \lambda t)^{-1}\) and \(m_{k_2} = -4\lambda^{-1}\). Then

\[
1 - \cos(k_1 \theta) = \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (r_2 - r_1)^2}{2r_1 r_2} \approx \frac{t (t + r_1 - r_2 + c(\epsilon_0 \lambda t)^{-1})}{r_1 r_2} \gtrsim \theta (\epsilon_0 \lambda r_1 r_2)^{-1}.
\]

On the other hand,

\[
1 - \cos(k_1 \theta) = \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (r_2 - r_1)^2}{2r_1 r_2} \lesssim \frac{t^2}{r_1 r_2} \ll 1
\]

and thus \(1 \gg k_1 \theta \gtrsim (\epsilon_0 \lambda r_1 r_2)^{-\frac{1}{2}}\). Note that

\[
\cos(k_2 \theta) - \cos(k_1 \theta) = \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (t - 4\lambda^{-1})^2}{2r_1 r_2} \simeq (\epsilon_0 \lambda r_1 r_2)^{-1}
\]

and

\[
\cos(k_2 \theta) - \cos(k_1 \theta) = \int_{k_1 \theta}^{k_2 \theta} \sin x dx \simeq (k_1 - k_2) k_1 \theta^2.
\]

So we have

\[
k_1 - k_2 \simeq (\epsilon_0 \lambda r_1 r_2)^{-1} \theta^{-2} k_1^{-1} \lesssim (\epsilon_0 \lambda r_1 r_2)^{-\frac{1}{2}} \theta^{-1} \ll \epsilon_0^{-1}
\]

and hence \(N_1(P_1, P_2) \lesssim \epsilon_0^{-\frac{3}{2}} t^{-\frac{1}{2}}\) as before.

Secondly, for \(t \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}}\), we have \(N_\lambda(P_1, P_2^k)\) nonzero only if \(m_k \in (-4\lambda^{-1}, c(\epsilon_0 \lambda)^{-\frac{1}{2}})\) by (5.1). Then from \(m_k \leq c(\epsilon_0 \lambda)^{-\frac{1}{2}}\),

\[
1 - \cos(k\theta) = \frac{(t + c(\epsilon_0 \lambda)^{-\frac{1}{2}})^2 - (r_2 - r_1)^2}{2r_1 r_2} \lesssim \frac{(\epsilon_0 \lambda)^{-1}}{r_1 r_2} \ll (\epsilon_0 \lambda)^{-1}
\]
So \( k \ll \theta^{-1}(\epsilon_0 \lambda)^{-\frac{1}{2}} = \epsilon_0^{-1} \), and hence \( N_1(P_1, P_2) \lesssim \epsilon_0^{-\frac{3}{2}} t^{-\frac{3}{2}} \) as before. This completes the proof of Proposition 3.

**Proof of Proposition 4:** We use the same notation as in the proof of Proposition 3. Note that since \( P_i \in A_2 \), thus we may assume \( 1 \gtrsim r_2 \geq r_1 \gtrsim (\epsilon_0 \lambda)^{-\frac{1}{2}} \) and \( t \geq 0 \). Let \( \alpha_2 = k\theta \in [0, \pi] \) and note that \( P_2^{\alpha_2} - P_1 = (t, r, 0)^\alpha \) with

\[
 r^2 = (r_2 - r_1)^2 + 2r_1r_2(1 - \cos(k\theta)) \quad (5.5)
\]

and

\[
 r \sin \alpha = r_2 \sin \alpha_2. \quad (5.6)
\]

Since

\[
 N_2(P_1, P_2) \lesssim |\{(j, k, \omega) \mid \chi_{j, k, \omega}(P_1) \neq 0\}| \lesssim \theta^{-1},
\]

we may assume \( t \gtrsim (\epsilon_0 \lambda)^{-\frac{1}{2}} \) w.l.o.g.. Since we are restricted in \( A_2 \), where \(|j| \gtrsim 1\), we can modify the definition of \( A_\lambda \) as follows,

\[
 A_\lambda = \{\omega \in S^1 \mid |\sin \omega| \gtrsim (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1}, \exists j, k \text{ s.t. } \chi_{j, k, \omega}(P_1)\chi_{j, k, \omega}(P_2) \neq 0\}
\]

Note that \( m = r - t \) depend on \( \alpha_2 \),

\[
 \partial_{\alpha_2} m = \frac{r_1r_2 \sin \alpha_2}{r} = r_1 \sin \alpha. \quad (5.7)
\]

If \( (\epsilon_0 \lambda)t^{-\frac{1}{2}} \approx (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1} \), i.e., \( r_1 \lesssim t \), then by (5.3) and (5.4), for any \( \omega \in A_\lambda \), one has \(|\omega - \alpha| \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1}\). Thus we have \(|\sin \alpha| \gtrsim (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1}\) in case of \( A_\lambda \) nonempty. So

\[
 \sum_{\alpha_2} |A_\lambda(P_1, P_2^{\alpha_2})| \lesssim (\lambda t)^{-\frac{1}{2}} \frac{c(\epsilon_0 \lambda t)^{-1} - (-4\lambda^{-1})}{\sup_{\alpha_2} |\partial_{\alpha_2} m|}
\]

\[
 \lesssim (\lambda t)^{-\frac{1}{2}} \frac{(\epsilon_0 \lambda t)^{-1}}{(\epsilon_0 \lambda)^{-\frac{1}{2}}}
\]

\[
 \lesssim (\lambda t)^{-\frac{1}{2}}
\]

and

\[
 N_2(P_1, P_2) \lesssim \theta^{-1}(\lambda t)^{-\frac{1}{2}} = (\epsilon_0 t)^{-\frac{1}{2}} \lesssim (\epsilon_0 t)^{-1}.
\]

Else if \( r_1 \gg t \), we assume \( r_2 - r_1 - t \leq c(\epsilon_0 \lambda t)^{-1} \) w.l.o.g. so \( r_2 \simeq r_1 \). Let \( k_1 k_2 \) be the number s.t. \( m_{k_1} = c(\epsilon_0 \lambda t)^{-1} \) and \( m_{k_2} = \max(-4\lambda^{-1}, r_2 - r_1 - t) \). We claim that

\[ k_1 - k_2 \lesssim (\epsilon_0 r_1)^{-1}. \quad (5.8) \]
We consider separately the cases \(|r_2 - r_1 - t| \leq c(\epsilon_0 \lambda t)^{-1}\) and \(r_2 - r_1 - t \leq -c(\epsilon_0 \lambda t)^{-1}\).

For the first case, \(|r_2 - r_1 - t| \leq c(\epsilon_0 \lambda t)^{-1}\), we have

\[
1 - \cos(k_1 \theta) \leq \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (r_2 - r_1)^2}{2r_1 r_2} \lesssim \frac{t \cdot (\epsilon_0 \lambda t)^{-1}}{r_1 r_2} \lesssim (\epsilon_0 \lambda)^{-1} r_1^{-2}.
\]

thus \(k_1 \theta \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1}\).

For the second case \(r_2 - r_1 - t \leq -c(\epsilon_0 \lambda t)^{-1}\),

\[
1 - \cos(k_1 \theta) = \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (r_2 - r_1)^2}{2r_1 r_2} \simeq \frac{t(t + r_1 - r_2 + c(\epsilon_0 \lambda t)^{-1})}{r_1 r_2} \gtrsim (\epsilon_0 \lambda)^{-1} r_1^{-2}.
\]

On the other hand,

\[
1 - \cos(k_1 \theta) = \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (r_2 - r_1)^2}{2r_1 r_2} \lesssim \frac{t^2}{r_1^2} \ll 1
\]

and thus \(1 \gg k_1 \theta \gtrsim (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1}\). Note that

\[
\cos(k_2 \theta) - \cos(k_1 \theta) = \frac{(t + c(\epsilon_0 \lambda t)^{-1})^2 - (t - 4\lambda^{-1})^2}{2r_1 r_2} \simeq (\epsilon_0 \lambda)^{-1} r_1^{-2},
\]

we have \((k_1 - k_2) \theta \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1}\). This proves the claim (5.8).

Now we are ready to estimate \(N_2(P_1, P_2)\). If \((\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1} \ll (\lambda t)^{-\frac{1}{2}}\), then

\[
N_2(P_1, P_2) \lesssim \theta^{-1} \sum_{\alpha_2} |A_\lambda(P_1, P_2^{\alpha_2})| \lesssim \theta^{-1}(k_1 - k_2)(\lambda t)^{-\frac{1}{2}} \lesssim \theta^{-2}(\lambda t)^{-1} = (\epsilon_0 t)^{-1}.
\]

Else if \((\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1} \gtrsim (\lambda t)^{-\frac{1}{2}}\), let \(m_0\) be s.t. \(m_0^{\frac{1}{2}} t^{-\frac{1}{2}} = (\epsilon_0 \lambda)^{-\frac{1}{2}} r_1^{-1}\) and \(k_0\) s.t. \(m_{k_0} = \max(m_0, r_2 - r_1 - t)\). then for \(k \in [k_2, k_0]\),

\[
\partial_{\alpha_2} m = r_1 \sin \alpha \gtrsim (\epsilon_0 \lambda)^{-\frac{1}{2}}
\]

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\[
\sum_{\alpha_2} |A_\lambda(P_1, P_2^{\alpha_2})| \lesssim \frac{m_0}{(\epsilon_0 \lambda)^{\frac{1}{2}}} (\lambda t)^{-\frac{1}{2}} \\
\lesssim (\lambda t)^{-\frac{1}{2}} \\
\lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}} \theta_1^{-1} \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}} t^{-1}.
\]

For \( k \in [k_0, k_1] \),
\[
\sum_{\alpha_2} |A_\lambda(P_1, P_2^{\alpha_2})| \lesssim (k_1 - k_0)(\lambda m_0)^{-\frac{1}{2}} (\lambda t)^{-\frac{1}{2}} \\
\lesssim \theta^{-1}(\epsilon_0 \lambda)^{-\frac{1}{2}} \theta_1^{-1} \lambda^{-1} (\lambda t)^{-\frac{1}{2}} \\
= (\theta \lambda t)^{-1} = (\epsilon_0 \lambda)^{-\frac{1}{2}} t^{-1}.
\]

Thus
\[
N_2(P_1, P_2) \lesssim \theta^{-1}(\epsilon_0 \lambda)^{-\frac{1}{2}} t^{-1} = (\epsilon_0 t)^{-1}.
\]
This completes the proof of Proposition 4.

6 \( L^\infty \) estimate for radial wave packet

Since \( \chi_{j,k,\omega}(t, x) = \chi_{j-k-t,\omega}(0, x) \), we may assume \( t = 0 \) and \( j, k \geq 0 \) w.l.o.g.

We first show that
\[
T_{j,k,0} \cap T_{j,k,\theta} = \emptyset \quad \text{for } j \gg 1 \text{ or } k \gg \theta^{-2}. \tag{6.1}
\]

If \( j = 0, k \gg 1, \) and \( T_{j,k,0} \cap T_{j,k,\theta} \neq \emptyset, \) then
\[
(k - 1)\lambda^{-1} \tan \frac{\theta}{2} \leq \frac{1}{2}(\epsilon_0 \lambda)^{-\frac{1}{2}}, \tag{6.2}
\]
i.e., \( k \lesssim \theta^{-2} \). If \( k = 0, j \gg 1, \) and \( T_{j,k,0} \cap T_{j,k,\theta} \neq \emptyset, \) then
\[
(j - 1)(\epsilon_0 \lambda)^{-\frac{1}{2}} \tan \frac{\theta}{2} \leq \frac{1}{2}(\lambda)^{-1},
\]
i.e., \( j \lesssim 1 \).

If \( j, k \geq 1, \) and \( T_{j,k,0} \cap T_{j,k,\theta} \neq \emptyset, \) let \((x, y) := ((j - 1)(\epsilon_0 \lambda)^{-\frac{1}{2}}(k + 1)\lambda^{-1}) \in T_{j,k,0}, \) then \((x, y)^\theta \in T_{j,k,0}. \) Thus
\[
\begin{cases}
y \cos \theta - x \sin \theta \geq (k - 1)\lambda^{-1} \\
x \cos \theta + y \sin \theta \leq (j + 1)(\epsilon_0 \lambda)^{-\frac{1}{2}}.
\end{cases}
\]
From the first inequality, we have
\[(j-1)(\epsilon_0 \lambda)^{-\frac{1}{2}} \sin \theta \lesssim \lambda^{-1},\]
so \(j \lesssim 1\). Then by the second inequality,
\[(k+1)\lambda^{-1} \sin \theta \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}},\]
thus \(k \lesssim \theta^{-2}\). Combined these observations, we get (6.1).

Similar argument will yield
\[T_{j,k,0} \cap T_{j,k,M \theta} = \emptyset \text{ for } j \gg 1 \text{ or } k \gg M^{-1} \theta^{-2}, \quad (6.3)\]
for \(M\) s.t. \(M \theta \ll 1\). Then we have \(|\chi_{j,k}(0,x)| \lesssim 1\) for \(|j| \gg 1 \text{ or } |k| \gg \theta^{-2}\),
and \(|\chi_{j,k}(0,x)| \lesssim \epsilon_0^{-1}\) for \(|j| \lesssim 1\) and \(\lambda \lesssim |k| \lesssim \theta^{-2}\).

It remains to consider the case \(j \lesssim 1\) and \(k \lesssim \lambda\) now. Note that the estimate for \(j \lesssim 1\) can be reduced to the counterpart for \(j = 0\), Proposition 5 follows from the following Lemma 2.

**Lemma 2.** Let \(0 \leq k \leq \lambda\), we have
\[|\chi_{0,k}(0,x)| \lesssim \theta^{-1} \langle \lambda x \rangle^{-1/2} \langle \lambda x \rangle + 1 - k)^{-1/2} H(\lambda x + 1 - k). \quad (6.4)\]

**Proof.** If \(|x| < (k-1)\lambda^{-1}\), then \(\chi_{0,k,\omega}(0,x) = 0\) for any \(\omega\) and hence \(\chi_{0,k}(0,x) = 0\). Thus for the proof of (6.4), we need only to show for \(|x| \geq (k-1)\lambda^{-1}\),
\[|\chi_{j,k}(0,x)| \lesssim \theta^{-1} \langle \lambda x \rangle^{-1/2} (\lambda x + 1 - k)^{-1/2}. \quad (6.5)\]

If \(k \lesssim 1\). For the case \(|x| \lesssim \lambda^{-1}\), we use the trivial bound \(|\chi_{0,k}(0,x)| \lesssim \theta^{-1}\). For \(|x| \gg (\epsilon_0 \lambda)^{-\frac{1}{2}}\), it is obviously that \(\chi_{0,k,\omega}(0,x) = 0\) for any \(\omega\). For the remained case \(\lambda^{-1} \ll |x| \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}}\), it’s only need to calculate the number of \(l\) s.t. \((0,x) \in T_{0,k,\omega}\), denoted by \(\tau\). Then we have
\[|x| \sin \tau \theta \lesssim \lambda^{-1}.\]
Thus \(\sin \tau \theta \ll 1\) and hence \(|x|\tau \theta \lesssim \lambda^{-1}\), i.e., \(\tau \lesssim \theta^{-1}(|x|\lambda)^{-1}\).

We consider for \(k \gg 1\) now. Let \(A(x) = \{\omega| (0,x) \in T_{0,k,\omega}\}\). Since \((0,x) \in T_{0,k,\omega}\), then
\[|x \cdot \omega - k\lambda^{-1}| \lesssim \lambda^{-1}, \ |x - k\lambda^{-1}\omega| \lesssim (\epsilon_0 \lambda)^{-\frac{1}{2}}.\]
Compare it with the definition (5.2) of $A_\lambda(P_1, P_2)$, we get that

$$A(x) \subset A_\lambda((0, x), (-k\lambda^{-1}, 0)) .$$

Thus

$$|\chi_{0,k}(0, x)| \leq \sum_{|j\theta| \leq \pi} |\chi_{0,k,j\theta}(0, x)|$$

$$\leq |\{j| j\theta \in [-\pi, \pi], \chi_{0,k,j\theta}(0, x) \neq 0\}|$$

$$\lesssim \theta^{-1}|A(x)|$$

$$\lesssim \theta^{-1}|A_\lambda(P_1, P_2)| ,$$

where $P_2 = (0, x)$ and $P_1 = (-k\lambda^{-1}, 0)$.

By the notation at the beginning of Section [5] we have $t = k\lambda^{-1} \leq 1$ and $m = |x| - k\lambda^{-1}$. Then by Lemma [1] if $t \lesssim (\epsilon_0\lambda)^{-\frac{1}{2}}$, i.e., $k \lesssim \theta^{-1}$, we have

$$A_\lambda \lesssim \begin{cases} \langle \lambda m \rangle^{-\frac{1}{2}} \langle \lambda t \rangle^{-\frac{1}{2}} = \langle \lambda |x| - k \rangle^{-\frac{1}{2}} \langle k \rangle^{-\frac{1}{2}} & m \lesssim t \\ \langle \lambda m \rangle^{-1} = \langle \lambda |x| - k \rangle^{-1} & t \lesssim m \lesssim (\epsilon_0\lambda)^{-\frac{1}{2}} . \end{cases}$$

In both cases, we have

$$A_\lambda \lesssim \langle \lambda |x| - k \rangle^{-\frac{1}{2}} \langle \lambda |x| \rangle^{-\frac{1}{2}} .$$

If $(\epsilon_0\lambda)^{-\frac{1}{2}} \ll t \leq 1$, i.e., $\theta^{-1} \ll k \leq \lambda$, we have that for $m \lesssim (\epsilon_0\lambda)^{-1} t^{-1} \ll t$ (thus $\lambda |x| \lesssim k$)

$$A_\lambda \lesssim \langle \lambda m \rangle^{-\frac{1}{2}} \langle \lambda t \rangle^{-\frac{1}{2}} = \langle \lambda |x| - k \rangle^{-\frac{1}{2}} \langle k \rangle^{-\frac{1}{2}} \lesssim \langle \lambda |x| - k \rangle^{-\frac{1}{2}} \langle \lambda |x| \rangle^{-\frac{1}{2}} .$$

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