Proof of the Generalized Zalcman Conjecture for Initial Coefficients of Univalent Functions

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Abstract. Let $S$ denote the class of analytic and univalent (i.e., one-to-one) functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in S$, Ma proposed the generalized Zalcman conjecture that
\[ |a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad \text{for } n \geq 2, m \geq 2, \]
with equality only for the Koebe function $k(z) = z/(1-z)^2$ and its rotations. In this paper using the properties of holomorphic motion and Krushkal’s Surgery Lemma [12], we prove the generalized Zalcman conjecture when $n = 2, m = 3$ and $n = 2, m = 4$.

1. Introduction and Preliminaries

Let $H$ denote the class of analytic functions in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$. Let $A$ be the class of functions $f \in H$ such that $f(0) = 0$ and $f'(0) = 1$, and denote by $S$ the class of functions $f \in A$ which are univalent (i.e., one-to-one) in $D$. Thus $f \in S$ has the following representation
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \] 

Closely related to the class $S$ is the class $\Sigma$ of functions which are analytic and univalent in the domain $\Delta = \{z : |z| > 1\}$ exterior to $D$, except for simple pole at infinity with residue 1 and $g(z) \neq 0$ in $\Delta$. For $f \in S$ given by (1.1), let $F \in \Sigma$ be given by
\[ F(z) = \frac{1}{f\left(\frac{1}{z}\right)} = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}. \]

A simple computation in (1.2) shows that
\[ a_2 = -b_0 \]
\[ a_3 = -b_1 + b_0^2 \]
\[ a_4 = -b_2 + 2b_1 b_0 - b_0^3 \]
\[ a_5 = -b_3 + 2b_2 b_0 + b_1^2 - 3b_1 b_0^2 + b_0^4, \]
and so
\begin{align}
(a_2 a_3 - a_4) &= -b_0 b_1 + b_2 \\
(a_2 a_4 - a_5) &= -b_0 b_2 + b_1 b_0^2 + b_3 - b_1^2.
\end{align}

A set \( \Omega \subseteq \mathbb{C} \) is said to be starlike with respect to a point \( z_0 \in \Omega \) if the line segment joining \( z_0 \) to every other point \( z \in \Omega \) lies entirely in \( \Omega \). A set \( \Omega \subseteq \mathbb{C} \) is called convex if the line segment joining any two points of \( \Omega \) lies entirely in \( \Omega \). A function \( f \in \mathcal{A} \) is called starlike (convex respectively) if \( f(D) \) is starlike with respect to the origin (convex respectively). Let \( \mathcal{S}^* \) and \( \mathcal{C} \) denote the classes of starlike and convex functions in \( \mathcal{S} \) respectively. Then it is well-known that a function \( f \in \mathcal{A} \) belongs to \( \mathcal{S}^* \) if, and only if, \( \Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \) for \( z \in D \). A function \( f \in \mathcal{A} \) is called close-to-convex if there exists a real number \( \theta \) and a function \( g \in \mathcal{S}^* \) such that \( \Re \left( e^{i\theta} \frac{zf'(z)}{g(z)} \right) > 0 \) for \( z \in D \). Class of close-to-convex functions denoted by \( \mathcal{K} \). Functions in the class \( \mathcal{K} \) are known to be univalent in \( D \). Geometrically, \( f \in \mathcal{K} \) means that the complement of the image-domain \( f(D) \) is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

In the late 1960’s, Zalcman posed the conjecture that if \( f \in \mathcal{S} \) and is given by (1.1), then

\begin{equation}
|a_n^2 - a_{2n-1}| \leq (n - 1)^2 \quad \text{for } n \geq 2,
\end{equation}

with equality only for the Koebe function \( k(z) = z/(1 - z)^2 \), or its rotation. It is important to note that the Zalcman conjecture implies the celebrated Bieberbach conjecture \( |a_n| \leq n \) for \( f \in \mathcal{S} \) (see [5]), and a well-known consequence of the area theorem shows that (1.6) holds for \( n = 2 \) (see [8]). The Zalcman conjecture remains an open problem, even after de Branges’ proof of the Bieberbach conjecture [4].

For \( f \in \mathcal{S} \), Krushkal proved the Zalcman conjecture for \( n = 3 \) (see [12]), and recently for \( n = 4, 5 \) and \( 6 \) (see [13]). For a simple and elegant proof of the Zalcman conjecture for the case \( n = 3 \), see [13]. However the Zalcman conjecture for \( f \in \mathcal{S} \) is still open for \( n > 6 \). On the other hand, using complex geometry and universal Teichmüller spaces, Krushkal claimed in an unpublished work [14] to have proved the Zalcman conjecture for all \( n \geq 2 \). Personal discussions with Prof. Krushkal indicates that there is a gap in the proof of Krushkal’s unpublished work [14], and so the Zalcman conjecture remains open for the class \( \mathcal{S} \) for \( n > 6 \). In spite of these gaps, Krushkal’s work [14] contains geometric and analytic ideas which may be useful for further study on the Zalcman conjecture and other similar problems.

If \( f \in \mathcal{S} \), then the coefficients \( [f(z^2)]^{1/2} \) and \( 1/f(1/z) \) are polynomials in \( a_n \), which contains the expression of the form \( \lambda a_n^2 - a_{2n-1} \), pointed out by Pflunger [21].

The Zalcman conjecture and its other generalized form has been proved for some subclasses of \( \mathcal{S} \), such as starlike functions, typically real functions, close-to-convex
functions [5, 18, 20]. For basic properties of starlike functions, typically real functions and close-to-convex functions we refer to [5].

In 1999 Ma [19] proposed a generalized Zalcman conjecture for \( f \in S \), namely that for \( n \geq 2, m \geq 2 \),

\[
|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1),
\]

which remains an open problem till date. However Ma [19] proved this generalized Zalcman conjecture for classes \( S^* \) and \( S_R \), where \( S_R \) denotes the class of all functions in \( S \) with real coefficients.

The proofs of Theorems 4.1, 4.2 are inspired by the work of Krushkal [12]. Roughly the idea is as follows:

Any function \( f \in S \) generates a family of holomorphic motion \( f_t \) which is a family of quasiconformal mappings. Also the coefficients of the functions in this family has the same homogeneity degree as the Zalcman functional as well as the generalized Zalcman functional. Since these maps are a family of quasiconformal mappings, using the properties of extremal quasiconformal mappings we will see that for the extremal function of the generalized Zalcman functional the tangent map at \( t = 0 \) is a \(|\alpha|-\)quasiconformal deformation of the Koebe's function, \(|\alpha| \leq 1\). More precisely, we will see that any extremal function can be written as the composition of a quasiconformal mapping \( w^\sigma \) and an \(|\alpha|-\)quasiconformal deformation of the Koebe function. The goal is to show that \(|\alpha| = 1\), so that \( w^\sigma \) reduced to a rotation. Thus we will deduce that the extremal function is a rotation of a Koebe function.

Consider the Koebe function

\[
k(z) = \frac{z}{(1-z)^2}
\]

and its rotations

\[
k(z) = \frac{z}{(1-xz)^2}, \quad |x| = 1
\]

and let

\[
k_\alpha(z) = \frac{z}{(1-\alpha xz)^2}, \quad 0 \leq |\alpha| \leq 1.
\]

Before we state and prove our main results, we discuss some preliminary ideas which will be useful in our proofs.

2. QUASICONFORMAL MAPPINGS

A homeomorphism \( f \) on a domain \( D \subset \mathbb{C} \) is called \( K \)-quasiconformal if \( f_z \) and \( f_{\bar{z}} \), the partial derivatives with respect to \( z \) and \( \bar{z} \) in the sense of distribution, are locally in \( L^2 \) and satisfy

\[
|\partial_z f(z)| \leq k|\partial_{\bar{z}} f(z)|
\]

almost everywhere in \( D \), where \( k \in [0, 1] \), \( k = (K - 1)/(K + 1) \), \( f_z = \partial_z f \), \( f_{\bar{z}} = \partial_{\bar{z}} f \),

\[
\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).
\]
The function

\[ \mu(z) = \frac{\partial f}{\partial z} \text{ a.e. in } D \]

is called the complex dilatation of the map \( f \). It is Borel-measurable (since the function \( f \) is continuous) and satisfies the estimate

\[ (2.1) \quad |\mu(z)| \leq K - 1 \]

with \( K + 1 < 1 \) and \( \text{ess sup}_{z \in D} |\mu(z)| = ||\mu||_\infty \leq k. \]

**The Beltrami equation:** The equation

\[ (2.2) \quad \partial_\bar{z} f = \mu \partial_z f \]

defined on a function \( f \) in a domain \( D \), where \( \mu \) is a measurable bounded function in \( D \) with \( \text{ess sup}_{z \in D} |\mu(z)| = ||\mu||_\infty < 1 \) is called the Beltrami equation. The function \( \mu \) in the equation is called the Beltrami differential. Thus \( \mu \) belongs to the unit ball in the Banach space \( L^\infty(D) \). Consider

\[ K(f) = 1 + \frac{||\mu||_\infty}{1 - ||\mu||_\infty} \]

be the maximal dilatation of \( f \). For the fundamental properties of quasiconformal mappings we refer to [2], [15] and [16].

Consider two integral operators \( S \) and \( T \). The operator \( S \) acts on functions \( h \in \mathcal{L}^p \) for \( p > 2 \) defined by

\[ (2.3) \quad Sh(z) = -\frac{1}{\pi} \int_D \int h(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta} \right) d\xi d\eta, \quad \zeta = \xi + i\eta. \]

The operator \( T \) is defined by

\[ Th(z) = -\frac{1}{\pi} \int_D \int \frac{h(\zeta)}{(\zeta - z)^2} d\xi d\eta, \quad \zeta = \xi + i\eta. \]

The operator \( S \) is known as the Cauchy-Green integral operator, and the linear operator \( T \) is known as the Hilbert transformation. The operator \( T \) was initially defined only for functions \( C^2_0 \) \( (C_0 \text{ with compact support}) \). Functions of class \( C^2_0 \) are dense in \( \mathcal{L}^2 \), so that using the isometry property, the operator \( T \) extend to all of \( \mathcal{L}^2 \). The operator \( T \) is also correctly defined on functions \( h \in \mathcal{L}^p \) with any \( p > 1 \) and bounded in the \( \mathcal{L}^p \) norm

\[ ||Th||_{\mathcal{L}^p} \leq C_p ||h||_{\mathcal{L}^p}. \]

Moreover the bound \( C_p \) can be chosen in such a way that \( C_p \to 1 \) for \( p \to 2 \).

It is well-known [2] pg. 51 that the operator \( S \) satisfies

\[ |Sh(z)| \leq K_p ||h||_p |z|^{1-2/p} \]

with a constant \( K_p \) that depends only on \( p \).

For \( h \in \mathcal{L}^p, p > 2 \), the relations [2] Lemma 3, pg. no. 90],

\[ \partial_e(Sh) = h \]
\[ \partial_z(Sh) = Th \]
hold in the sense of distribution.

3. Holomorphic motions

Holomorphic motions were introduced by Mane, Sad and Sullivan [24] in their study of the structural stability problem for complex dynamical systems and proved the first result in the topic which is called the \( \lambda \)-lemma. There is a strong connection between holomorphic motion and quasiconformal mappings, and the \( \lambda \)-lemma illustrates this connection. After Mane, Sad and Sullivan had proved the \( \lambda \)-lemma, Sullivan and Thurston [22] proved an important result related to the extension of holomorphic motions. In the theory of holomorphic motions an important result is provided by Slodkowski’s lifting theorem, also known as the extended \( \lambda \)-lemma [23], which plays a vital role in the proof of our main result.

Basically a holomorphic motion is an isotopy of a subset \( E \) of the extended complex plane \( \mathbb{C}_\infty \) analytically parametrised by a complex variable \( t \) in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). In this paper we use a special case of holomorphic isotopies generated by analytic and univalent functions in the unit disk \( \mathbb{D} \) which represents a special case of holomorphic motions.

**Definition 3.1.** Let \( E \) be a subset of \( \mathbb{C}_\infty \) containing at least three points. A holomorphic motion of \( E \) is a function \( f : E \times \mathbb{D} \to \mathbb{C}_\infty \) such that

(i) for every fixed \( z \in E \), the function \( t \mapsto f(z, t) : \mathbb{D} \to \mathbb{C}_\infty \) is holomorphic in \( \mathbb{D} \);

(ii) for every fixed \( t \in \mathbb{D} \), the map \( f(z, t) = f_t(z) : E \to \mathbb{C}_\infty \) is injective;

(iii) \( f(z, 0) = z \) for all \( z \in E \).

The following \( \lambda \)-lemma assures that such a holomorphic motion is a holomorphic family of quasiconformal maps.

**\( \lambda \)-lemma:** If \( f : E \times \mathbb{D} \to \mathbb{C}_\infty \) is a holomorphic motion, then \( f \) has an extension \( \tilde{f} : \overline{E} \times \mathbb{D} \to \mathbb{C}_\infty \) such that

(i) \( \tilde{f} \) is a holomorphic motion of the closure \( \overline{E} \) of \( E \);

(ii) each \( \tilde{f}_t(z) = \tilde{f}(t, z) : \overline{E} \to \mathbb{C} \) is quasiconformal on the interior of \( \overline{E} \);

(iii) \( \tilde{f} \) is jointly continuous in \( (z, t) \).

The obvious question that arises is can we extend a holomorphic motion from any set to the whole sphere? The Slodkowski lifting theorem [23] solves this problem which was posed by Sullivan and Thurston.

**Extended \( \lambda \)-lemma:** Any holomorphic motion \( f : E \times \mathbb{D} \to \mathbb{C}_\infty \) can be extended to a holomorphic motion \( \tilde{f} : \mathbb{C}_\infty \times \mathbb{D} \to \mathbb{C}_\infty \), with \( \tilde{f}|_{E \times \mathbb{D}} = f \).

In view of [3, Theorem 2], the function \( \phi : \mathbb{D} \to \mathcal{M}(\mathbb{D}) \) defined by

\[
\phi(t) = \frac{\partial_2 \tilde{f}(z, t)}{\partial_2 f(z, t)}
\]
is holomorphic, where $\mathcal{M}(\mathbb{D})$ is a unit ball in $\mathcal{L}^\infty(\mathbb{D})$.

4. Main results

We now state the main results of this paper.

**Theorem 4.1.** Let $f \in \mathcal{S}$ and be given by (1.1), then

$$|a_2a_3 - a_4| \leq 2$$

with equality only for functions of the form

$$\frac{z}{(1 - e^{i\theta}z)^2}, \text{ where } \theta \text{ is real}.$$

**Theorem 4.2.** Let $f \in \mathcal{S}$ and be given by (1.1), then

$$|a_2a_4 - a_5| \leq 3$$

with equality only for functions of the form

$$\frac{z}{(1 - e^{i\theta}z)^2}, \text{ where } \theta \text{ is real}.$$

5. Proof of Theorems 4.1 and 4.2

The proofs of Theorems 4.1, 4.2 are inspired by the work of Krushkal [12]. For convenience we convert the coefficients of functions in $\mathcal{S}$ into the coefficients of functions in $\sum$.

For $f \in \mathcal{S}$, we define a holomorphic motion $f^*$ of $\mathbb{D} \cup \{\infty\}$ i.e., $f^* : \mathbb{D} \cup \{\infty\} \times \mathbb{D} \to \mathbb{C}_\infty$ by

$$f^*(z, t) = \begin{cases} f_t(z) = \frac{1}{t}f(z, t) & \text{if } z \in \mathbb{D}, \ t \in \mathbb{D} \\ \infty, & \text{if } z = \infty, \ t \in \mathbb{D}. \end{cases}$$

By the generalized $\lambda$-lemma, $f^*$ extends to a holomorphic motion say $\tilde{f}$, $\tilde{f} : \mathbb{C}_\infty \times \mathbb{D} \to \mathbb{C}_\infty$ such that $\tilde{f}|((\mathbb{D} \cup \{\infty\}) \times \mathbb{D}) = f^*$, i.e., for fixed $t \in \mathbb{D}$, $\tilde{f}_t(z)$ is quasiconformal on $\mathbb{C}_\infty$ and $\tilde{f}_t|_{\mathbb{D}} = f_t$. The Beltrami coefficient of $\tilde{f}_t$ is given by

$$\mu_{\tilde{f}_t}(z, t) = \frac{\partial \tilde{f}(z, t)}{\partial \bar{z}}.$$ 

In view of [3, Theorem 2], since the mapping $t \mapsto \mu_{\tilde{f}_t}$ is holomorphic, by Schwarz’s lemma, we have

$$||\mu_{\tilde{f}_t}(z, t)||_\infty \leq |t|.$$

Thus we have the following Taylor series expansion

$$(5.1) \quad \mu_{\tilde{f}_t}(z, t) = \mu_1(z)t + \mu_2(z)t^2 + \mu_3(z)t^3 + \cdots,$$

where

$$\mu_{\tilde{f}_t}|_{\mathbb{D}} = 0, \quad \text{since } \tilde{f}_t|_{\mathbb{D}} = f_t.$$


Similarly if \( f \in \mathcal{S} \), then
\[
F(z) = \frac{1}{f \left( \frac{1}{z} \right)} \in \Sigma, \quad F(\infty) = \infty.
\]
Therefore
\[
F \left( \frac{z}{t} \right) = \frac{1}{f \left( \frac{t}{z} \right)}, \quad F_t(z) = \frac{1}{f_t \left( \frac{1}{z} \right)} = \frac{t}{f \left( \frac{t}{z} \right)} = tF \left( \frac{z}{t} \right),
\]
and so
\[
(5.2) \quad F_t(z) = z + \sum_{n=0}^{\infty} b_n \frac{t^{n+1}}{z^n}.
\]
Now let
\[
g(z) = tF \left( \frac{z}{t} \right).
\]
Then we can define a holomorphic motion of \( \mathbb{D}^* \cup \{0\} \) say \( F^* \), where \( \mathbb{D}^* = \{ z : |z| > 1 \} \), and \( F^* : \mathbb{D}^* \cup \{0\} \times \mathbb{D} \to \mathbb{C}_\infty \) is defined by
\[
F^*(z, t) = \begin{cases} tF \left( \frac{z}{t} \right) & \text{if } z \in \mathbb{D}^*, \ t \in \mathbb{D} \\ 0, & \text{if } z = 0, \ t \in \mathbb{D}. \end{cases}
\]
Similarly we can extend \( F^* \) to a holomorphic motion of \( \mathbb{C}_\infty \) say \( \widetilde{F} \) by
\[
\widetilde{F}(z, t) = \frac{1}{f \left( \frac{1}{z}, t \right)}
\]
with Beltrami coefficient
\[
\mu_{\widetilde{F}(z, t)} = \mu_{\widetilde{F}_t(\frac{1}{z}, t)} \frac{\overline{z} \overline{t}}{\overline{z}^2} = \sum_{n=1}^{\infty} \mu_n \left( \frac{1}{z} \right) t^n \frac{\overline{z}^2}{\overline{z}^2} = \overline{\mu}_1(z) t + \overline{\mu}_2(z) t^2 + \overline{\mu}_3(z) t^3 + \cdots.
\]
Thus \( \widetilde{F}(z, t) \) satisfies
\[
\partial_{\overline{z}} \widetilde{F}(z, t) = \mu_{\widetilde{F}(z, t)} \partial_{\overline{z}} \widetilde{F}(z, t),
\]
and we know that for each fixed \( t \), \( \widetilde{F}_t(z) \) is a quasiconformal map on \( \mathbb{C}_\infty \) which is analytic on \( \mathbb{D}^* \), and so by Weyl’s lemma, \( \partial_{\overline{z}} \widetilde{F}_t(z) = 0 \) for all \( z \in \mathbb{D}^* \). Thus \( \mu_{\widetilde{F}} \) has compact support. Hence by the mapping theorem [2] pg. 54] these maps can be represented by \( \widetilde{F}_t(z) = z + S \partial_{\overline{z}} \widetilde{F}_t \), i.e.,
\[
(5.3) \quad \widetilde{F}_t(z) = z - \frac{1}{\pi} \int_{\mathbb{D}} \int \partial_{\overline{z}} \widetilde{F}_t(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta} \right) \, d\xi \, d\eta, \quad \text{where } \zeta = \xi + \eta,
\]
and where
\[
(5.4) \quad \partial_{\overline{z}} \widetilde{F}_t = \mu_{\widetilde{F}_t} + \mu_{\widetilde{F}} T \mu_{\widetilde{F}} + \mu_{\widetilde{F}} T \mu_{\widetilde{F}} T \mu_{\widetilde{F}} + \mu_{\widetilde{F}} T \mu_{\widetilde{F}} T \mu_{\widetilde{F}} T \mu_{\widetilde{F}} T \mu_{\widetilde{F}} + \cdots.
\]
In the above equation, the right hand side series converges in \( \mathcal{L}^p \), since the linear operator \( h \mapsto T(\mu h) \) on \( \mathcal{L}^p \) has normless than or equal to \( kC_p < 1 \). For \( p > 2 \)}
sufficiently close to 2.

Now for each \( n \), we will deduce the formula for \( b_n \). In view of (5.3), we have \(|z| > 1, |\zeta| < 1\) and we have \(|\zeta/z| < 1\). From (5.3) we have,

\[
\tilde{F}_t(z) = z - \frac{1}{\pi} \int_D \int \partial_{\zeta} \tilde{F}_t(\zeta) \left( \frac{z}{\zeta(z-z)} \right) d\xi d\eta
\]

which implies,

\[
\tilde{F}_t(z) = z + \frac{1}{\pi} \int_D \int \partial_{\zeta} \tilde{F}_t(\zeta) \left( \frac{1}{\zeta(1-\zeta)} \right) d\xi d\eta
\]

By comparing (5.2) and (5.3), we obtain

\[
b_n = \frac{1}{\pi} \int_D \int \partial_{\zeta} \tilde{F}_t(\zeta) \zeta^{-n-1} d\xi d\eta
\]

Now write

\[
\partial_{\zeta} \tilde{F}_t = \mu \tilde{F}_t \left( \partial_{\zeta} \tilde{F}_t - 1 \right) + \mu \tilde{F}_t.
\]

Thus in view of equations (5.6), (5.7) and [2, Lemma pg. no. 100], we have,

\[
\tilde{F}_t(z) = z - \frac{1}{\pi} \int_D \int \mu \tilde{F}_t(\zeta) \left( \frac{1}{\zeta-z} - \frac{1}{\zeta} \right) d\xi d\eta + O \left( ||\mu \tilde{F}_t||^2_{\infty} \right),
\]

\[
b_n = \frac{1}{\pi} \int_D \int \mu \tilde{F}_t(\zeta) \zeta^{-n} d\xi d\eta + O_n \left( ||\mu \tilde{F}_t||^2_{\infty} \right).
\]

Where the remainder term is approximated uniformly on any compact subset of \( \mathbb{C} \).

We next mention a lemma (cf. [12] and [10, Section 2.8]) which we will apply to the coefficients of \( \mu_{\tilde{f}} \) for the fixed \( z \).

**Lemma 5.10.** If \( h \) is an analytic self-map of \( \mathbb{D} \) fixing the origin with expansion

\[
h(z) = \sum_{n=1}^{\infty} \alpha_n z^n,
\]

then for \( n \geq 1 \),

\[
|\alpha_{n+1}| \leq 1 - |\alpha_1|^2,
\]

with equality only for \( h(z) = z \).

It therefore follows from (5.11), and Lemma 5.10, that for a fixed \( z \)

\[
|\mu_n(z)| \leq 1 - |\mu_1(z)|^2, \quad n = 2, 3, 4, \ldots
\]

for almost all \( z \in \mathbb{D} \).

We will need a much stronger version of inequality (5.11) as follows.
Proof of The Generalized Zalcman Conjecture for Initial Coefficients

(5.12) \[ ||\mu_n||_\infty \leq 1 - ||\mu_1||_\infty. \]

It is easy to see that (5.12) can be derived from (5.11) provided

(5.13) \[ |\mu_1(z)| = \text{constant a.e. on } \mathbb{D}^*. \]

To establish (5.13), Krushkal [12] proved a lemma known as the Surgery Lemma, which plays a vital role in the proof of our main result.

**Lemma 5.14** (Surgery Lemma). Given a function \( f \in S \) with \( a_2 \neq 0 \), one can construct a holomorphic motion \( f^*(z,t) \) on \( \mathbb{C}_\infty \times \mathbb{D} \) such that for any \( t \), the restriction of the fibre map \( f^*_t(z) \) to \( \mathbb{D} \) belongs to \( S \), and the family \( f^*(z,t) \) satisfies the following properties:

(i.) \( f^*(z,t) \) is also holomorphic in \( t \), and its Beltrami coefficients

(5.15) \[ \mu_{f^*_t(z)} = \mu_1^*(z) t + \mu_2^*(z) t^2 + \mu_3^*(z) t^3 + \cdots \]

satisfy

(5.16) \[ |\mu_1^*(z)| = \text{constant on } \mathbb{D}^* \]

(ii.) \( f_t^*(\infty) = \infty \) for all \( t \in \mathbb{D} \)

(iii.) \( a_n(f^*_t) = a_n(f_t) + o_n(t^n), n = 2, 3, \ldots \).

We now give the proofs of Theorems 4.1 and 4.2.

5.1. **Proof of Theorem 4.1.** If \( f \in S \) any function maximizing (4.1), then \( a_2(f) \neq 0 \). If not, let \( a_2 = 0 \), then \( b_0 = 0 \) and so \( a_2a_3 - a_4 = b_2 \). The sharp inequality

(5.17) \[ |b_2| \leq \frac{2}{3}, \]

was proved by Golusin [9]. Jenkins [11, Corollary 11] gave another proof of this, and Duren [7] has given an elementary proof. Equality in (5.17) holds only for the functions

\[ z \left( 1 + \frac{e^{i\theta}}{z^{2/3}} \right)^{2}, \quad \theta \text{ real}, \]

which map \( \{ z : |z| > 1 \} \) onto the whole \( w \)-plane except for a cut consisting of three lines, each of length \( 2^{2/3} \), originating symmetrically from the origin. Thus we obtain the following sharp inequality

\[ |b_2| \leq \frac{2}{3}, \]

and so it follows from (1.3) that \( |a_2a_3 - a_4| = |b_2| \leq 2/3 \). This contradicts our hypothesis, since for the Koebe function, \( |a_2a_3 - a_4| = 2 \), which is always greater than \( 2/3 \). Hence it is now clear that any function \( f \) maximizing (4.1) has second Taylor series coefficient \( a_2 \neq 0 \), which leads us to use the Surgery Lemma.

So now we consider the holomorphic motion

\[ \tilde{F}(z,t) = \frac{1}{f^{(1)}(\frac{1}{z},t)}. \]
with Beltrami coefficient
\[ \mu_{\tilde{F}(z,t)} = \mu_{\tilde{f}(\frac{z}{t})} \frac{z^2}{z^2} = \sum_{n=1}^{\infty} \mu_n \left( \frac{1}{z} \right) t^n \frac{z^2}{z^2} = \tilde{\mu}_1(z)t + \tilde{\mu}_2(z)t^2 + \tilde{\mu}_3(z)t^3 + \cdots \]
generated by \( \tilde{f} \). From the proof of the Surgery Lemma (see [12]), we can assume that
\[ \tilde{\mu}_1(z) = \alpha \tilde{\mu}^0(z), \]
where
\[ \tilde{\mu}^0(z) = \frac{|z|}{z}, \]
and from [12] it is clear that for small \(|t|\), we have the representation
\[ \tilde{F}_t(z) = w^\sigma \circ K_{\alpha t}(z) = z + b_0t + b_1t^2z^{-1} + \ldots, \]
where
\[ K_{\alpha t}(z) = \frac{1}{k_{\alpha t}(\frac{1}{z})} = \begin{cases} z - 2\alpha t + \alpha^2 t^2 z^{-1}, & |z| > 1, \\ z - 2\alpha t|z| + \alpha^2 t^2 z, & |z| < 1, \end{cases} \]
has the Beltrami coefficient \(-t\alpha \tilde{\mu}^0\), \( ||\sigma(z,t)||_\infty \leq |t|^2 \), and the map
\[ w^\sigma(\zeta) = \zeta + \sum_{n=0}^{\infty} b'_n \zeta^{-n} \quad (b'_n = b'_n(t)) \]
is conformal in the domain \( K_{\alpha t}(\mathbb{D}^*) \supset \{|\zeta| > 1 + O(t)\} \).

We now use the estimates of the coefficients of \( w^\sigma \) mentioned in [12], and obtain
\[ b_0' = O(t^2), \quad b_1' = O(t^3), \quad b_2' = O(t^4) \quad \text{and} \quad |b_3'| \leq \frac{(1 - |\alpha|^2)^2}{\sqrt{3}(1 - |t|^2)} |t|^4 + O(t^6), \]
which gives
\[ b_0'b_1' = O(t^5). \]

Since
\[ \tilde{F}_t(z) = z - (2\alpha t - b_0') + \frac{\alpha^2 t^2 + b_1'}{z} + \frac{2\alpha t b_1' + b_2'}{z^2} + \frac{3\alpha^2 t^2 b_1' + 4\alpha t b_2' + b_3'}{z^3} + \cdots, \]
we have
\[ b_0 = \frac{b_0' - 2\alpha t}{t}, \quad b_1 = \frac{\alpha^2 t^2 + b_1'}{t^2}, \quad b_2 = \frac{2\alpha t b_1' + b_2'}{t^3}, \quad b_3 = \frac{3\alpha^2 t^2 b_1' + 4\alpha t b_2' + b_3'}{t^3}, \]

(5.18)
and so from (1.4), we obtain

$$|a_2a_3 - a_4| = | - b_0b_1 + b_2|.$$ 

Using (5.18), after an easy computation, we obtain

$$|a_2a_3 - a_4| \leq 2|\alpha|^3,$$

where $0 < \alpha \leq 1,$ which implies that

$$|a_2a_3 - a_4| \leq 2,$$

with equality only if $|\alpha| = 1$, i.e., for the Koebe function and its rotations. This completes the proof of Theorem 4.1.

5.2. Proof of Theorem 4.2. If $f \in S$ is a function maximizing (4.2), then $a_2(f) \neq 0.$ If not, let $a_2 = 0,$ then $b_0 = 0,$ and so $a_2a_4 - a_5 = b_3 - b_1^2.$ We need to find the sharp bound for $|b_3 - b_1^2|,$ and to do this we use the method of Jenkins [11].

Jenkins [11, Corollary 13] proved that, if $g \in \Sigma,$ then for $\psi$ real and $0 \leq \sigma \leq 2,$

$$\mathfrak{R} \left\{ e^{-4i\psi} \left( b_3 + \frac{1}{2}b_2^2 - \sigma^2 e^{2i\psi}b_1 \right) \right\} \geq \begin{cases} -\frac{1}{2} - \frac{3}{16}\sigma^4 + \frac{1}{8}\sigma^4\log \left( \frac{\sigma^2}{4} \right), & 0 < \sigma \leq 2 \\ -\frac{1}{2}, & \sigma = 0. \end{cases}$$

Using the above inequality, we now find the sharp bound for $|b_3 - b_1^2|.$

Without loss in generality, choose $\psi$ real so that

$$-\mathfrak{R} \left\{ e^{-4i\psi}(b_3 - b_1^2) \right\} = |b_3 - b_1^2|,$$

and

$$-\mathfrak{R} \left\{ e^{-2i\psi}b_1 \right\} \geq 0.$$ 

Then it follows that

$$|b_3 - b_1^2| \leq \frac{1}{2} + \frac{3}{16}\sigma^4 - \frac{\sigma^4}{8} - \log \frac{\sigma^4}{4} + \frac{3}{2}\mathfrak{R}(e^{-4i\psi}b_1^2) - \sigma^2\mathfrak{R}(e^{-2i\psi}b_1), \quad 0 < \sigma \leq 2$$

$$\leq \frac{1}{2}, \quad \sigma = 0.$$ 

It is easy to see that

$$\mathfrak{R} \left\{ e^{-4i\psi}b_1^2 \right\} \leq \left(\mathfrak{R} \left\{ e^{-2i\psi}b_1 \right\} \right)^2$$

with the strict inequality unless

$$\Im \left\{ e^{-2i\psi}b_1 \right\} = 0.$$ 

Therefore

$$|b_3 - b_1^2| \leq \frac{1}{2} + \frac{3}{16}\sigma^4 - \frac{\sigma^4}{8} - \log \frac{\sigma^4}{4} + \frac{3}{2} \left(\mathfrak{R} \left\{ e^{-2i\psi}b_1 \right\} \right)^2 - \sigma^2\mathfrak{R}(e^{-2i\psi}b_1), \quad 0 < \sigma \leq 2$$

$$\leq \frac{1}{2}, \quad \sigma = 0.$$ 


If $\Re \{e^{-2i\psi b_1}\} = 0$, then taking $\sigma = 0$ we obtain

$$|b_3 - b_1^2| \leq \frac{1}{2}.$$ 

If $\Re \{e^{-2i\psi b_1}\} > 0$ then we can find $\sigma$ with $0 < \sigma \leq 2$, so that

$$\Re \{e^{-2i\psi b_1}\} = \frac{1}{4}\sigma^2 \left(1 - \log \left(\frac{\sigma^2}{4}\right)\right),$$

and using this we obtain

$$|b_3 - b_1^2| \leq \frac{1}{2} + \frac{1}{32}\sigma^4 - \frac{1}{16}\sigma^4 \log \frac{\sigma^2}{4} + \frac{3}{32}\sigma^2 \log \frac{\sigma^2}{4} = \Theta(\sigma).$$

Clearly the function $\Theta(\sigma)$ is continuous on $0 < \sigma \leq 2$, and further

$$|b_3 - b_1^2| \leq \max_{0<\sigma\leq 2} \Theta(\sigma).$$

It is easy to see that the function $\Theta$ is increasing on $[0, \sigma]$ up to the point $\sigma = 2e^{-\frac{1}{6}}$ and is decreasing after $\sigma = 2e^{-\frac{1}{6}}$. Hence the maximum of $\Theta$ occurs at this value of $\sigma$ and so

$$(5.20) \quad |b_3 - b_1^2| \leq \frac{1}{2} + e^{-\frac{2}{3}}.$$ 

Equality occurs in (5.20) only for functions of the form $h(z, 2e^{-1/6}, \psi) + k$, where $\psi$ is real, and $k$ is a constant, where $h(z, \sigma, \psi)$ is defined in [11]. Thus we have the sharp bound for $|b_3 - b_1^2|$.

Thus

$$|a_2a_4 - a_5| = |b_3 - b_1^2| \leq \frac{1}{2} + e^{-\frac{2}{3}},$$

which is a contradiction, since for the Koebe function, $|a_2a_4 - a_5| = 3$.

Thus it is clear that any function $f$ maximizing (4.2) has second Taylor series coefficient $a_2 \neq 0$. So again we can use the Surgery Lemma and proceed as before.

Using equation (1.4), we obtain

$$|a_2a_4 - a_5| = |b_3 - b_1^2| \leq \frac{1}{2} + e^{-\frac{2}{3}},$$

Using equation (5.18) in above and after an easy computation, we obtain

$$|a_2a_4 - a_5| \leq 3|\alpha|^4, \quad \text{where } 0 < \alpha \leq 1.$$ 

This implies that

$$|\lambda a_2a_4 - a_5| \leq 3,$$

with equality only if $|\alpha| = 1 \ i.e., \ for \ the \ Koebe \ function \ and \ its \ rotations. \ This \ completes \ the \ proof \ of \ Theorem \ (4.2).$

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