ON RACK COHOMOLOGY

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Abstract. We prove that the lower bounds for Betti numbers of the rack, quandle and degeneracy cohomology given in [CJKS] are in fact equalities. We compute as well the Betti numbers of the twisted cohomology introduced in [CES]. We also give a group-theoretical interpretation of the second cohomology group for racks.

1. Introduction

A rack is a pair $(X, \triangleright)$ where $X$ is a set and $\triangleright : X \times X \to X$ is a binary operation such that:

1. The map $\phi_x : X \to X$, $\phi_x(y) = x \triangleright y$, is a bijection for all $x \in X$, and
2. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ $\forall x, y, z \in X$.

It is easy to show that $(X, \triangleright)$ is a rack if and only if the map $R : X^2 \to X^2$ given by $R(x, y) = (x, x \triangleright y)$ is an invertible solution of the quantum Yang-Baxter equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$.

Racks have been studied by knot theorists in order to construct invariants of knots and links and their higher dimensional analogs (see [CS] and references therein). A basic example of a rack is a group with the operation $x \triangleright y = xyx^{-1}$ (or, more generally, a conjugation invariant subset of a group).

Several years ago, Fenn, Rourke and Sanderson [FRS] proposed a cohomology theory of racks. Namely, for each rack $X$ and an abelian group $A$, they defined cohomology groups $H^n(X, A)$. This cohomology is useful for knot theory and also, as was recently found, for the theory of pointed Hopf algebras [G]. There have been a number of results about this cohomology [LN, M, CJKS], in particular it was shown in [CJKS] that for a finite rack $X$ and a field $k$ of characteristic zero, the Betti numbers $\dim H^n(X, k)$ are bounded below by $|X/\sim|^n$, where $\sim$ is the equivalence relation on $X$ generated by the relation $z \triangleright y \sim y \forall y, z \in X$. The equality was anticipated in [CJKS], and proved in a number of cases [LN, M], but not in general.

The main result of this paper implies that the Betti numbers of a finite rack are always equal to $|X/\sim|^n$. The proof is based on a group-theoretical approach to racks, originating from the works [LYZ], [S] on set-theoretical solutions of the quantum Yang-Baxter equation. Namely, we use the structure group $G_X$ and the reduced structure group $G^0_X$ of a rack $X$ considered in [LYZ, S].

We also give a group-theoretic interpretation of the second cohomology group $H^2(X, A)$, which is used in the theory of Hopf algebras. Namely, we show that this group is isomorphic to the group cohomology $H^1(G_X, \text{Fun}(X, A))$, where $\text{Fun}(X, A)$ is the group of functions from $X$ to $A$. This is a relatively explicit description, since it is shown by Soloviev [S] that for a finite rack $X$, the group $G_X$ is a central extension of the finite group $G^0_X$ by a finitely generated abelian group. Thus the cohomology of $G_X$ can be studied using the Hochschild-Serre sequence.
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2. Definitions and notation

Definition 2.1. The structure group of a rack $X$ is the group $G_X$ with generators being the elements of $X$ and relations $x \cdot y = (x \rhd y) \cdot x \quad \forall x, y \in X$. \footnote{This group appears already in the work of Joyce [J], who pointed out that the functor $X \to G_X$ is adjoint to the functor assigning to a group the underlying rack (with the conjugation operation). Thus the group $G_X$ can be viewed as the “enveloping group” of $X$.}

The group $G_X$ acts on $X$ from the left by $\rhd$. Consider the quotient $G_X^0$ of $G_X$ by the kernel of this action, i.e. the group of transformations of $X$ generated by $\rhd$. This group is called the reduced structure group of $X$.

Remark 2.2. The groups $G_X, G_X^0$ were studied by Soloviev [S] (we note that in his work, racks are called “derived solutions”). In particular, he showed that the category of racks is equivalent to the category of quadruples $(G, X, \rho, \pi)$, where $G$ is a group, $X$ a set, $\rho : G \times X \to X$ a left action, and $\pi : X \to G$ an equivariant mapping (where $G$ acts on itself by conjugation), such that $\pi(X)$ generates $G$ and the $G$-action on $X$ is faithful. Namely, the quadruple corresponding to $X$ is simply $(G_X^0, X, \rho, \pi)$, where $\rho$ and $\pi$ are obvious.

Now let us define rack cohomology. Let $X$ be a rack. Let $G_X$ be its structure group. Let $M$ be a right $G_X$-module. We define a cochain complex $(C^\bullet(X, M), d)$, where $C^n(X, M) = \text{Fun}(X^n, M)$, $n \geq 0$, with differential

$$
d f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} \left( f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) - f(x_1, \ldots, x_{i-1}, x_i \rhd x_{i+1}, \ldots, x_{i} \rhd x_{n+1}) \right) \cdot x_i
$$

(Here $X^0$ is a set of one element, and $\text{Fun}(Y, Z)$ is the set of functions from $Y$ to $Z$ for any sets $Y, Z$).

Definition 2.3. The cohomology of $C^\bullet(X, M)$ is called the rack cohomology of $X$ with coefficients in $M$.

This includes the ordinary rack cohomology with coefficients in an abelian group $A$, introduced in [FRS] (this corresponds to taking $M = A$ with the trivial action of $G_X$), as well as the twisted rack cohomology introduced in [CES] (in this case one needs to take a $\mathbb{Z}[T, T^{-1}]$ module $M$, and define a right action of $G_X$ on it by $v x = T v$, $x \in X$).

Remark 2.4. One can also define the dual notion of rack homology. As usual, it is completely analogous to cohomology, so we will not consider it.

Remark 2.5. In [AG] there is a more general definition of cohomology, with coefficients in objects of a wider category than that of $G_X$-modules. When restricted to $G_X$-modules, the definition there takes as differential the map $d'$, defined by

$$
d' f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} \left( f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})(x_1 \rhd (x_2 \rhd (\cdots x_i)))^{-1}
\right.
\left. - f(x_1, \ldots, x_{i-1}, x_i \rhd x_{i+1}, \ldots, x_i \rhd x_{n+1}) \right)
$$
This complex is isomorphic to the one we consider here, by means of the map

$$T : (C^\bullet(X, M), d) \to (C^\bullet(X, M), d'),$$

defined by $$(T f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)(x_1 \cdots x_n)^{-1}.$$

3. The structure of rack cohomology

Let $M$ be a right $G_X$-module. Then $C^n(X, M) = \text{Fun}(X^n, M)$ is also a right $G_X$-module, with the action defined on the generators by

$$(f \cdot y)(x_1, \ldots, x_n) = f(y \triangleright x_1, \ldots, y \triangleright x_n).$$

**Lemma 3.1.** (1) The coboundary operator $d : C^n(X, M) \to C^{n+1}(X, M)$ is a map of $G_X$-modules. In particular, there is a natural right action of $G_X$ on the groups of cocycles $Z^n(X, M)$, coboundaries $B^n(X, M)$, and cohomology $H^n(X, M)$.

(2) $H^n(X, M)$ is a trivial $G_X$-module.

**Proof.**

(1) Straightforward.

(2) Let $f \in Z^n(X, M)$ and consider $f_y \in C^{n-1}(X, M)$, defined by the formula

$$f_y(x_2, \ldots, x_n) = f(y, x_2, \ldots, x_n).$$

Notice that

$$d(f_y)(x_1, \ldots, x_n) = (f - f \cdot y)(x_1, \ldots, x_n) - (df)(y, x_1, \ldots, x_n) = (f - f \cdot y)(x_1, \ldots, x_n).$$

Then $f \cdot y = f$ in $H^n(X, M)$.

\[ \square \]

**Remark 3.3.** The action $f \cdot y$ and the assignments $f \mapsto f_y$, as well as (3.2), appear in [LN].

By Lemma 3.1 we can consider the subcomplex $C^\bullet_{\text{inv}}(X, M) = C^\bullet(X, M)^{G_X}$. We define the invariant rack cohomology $H^\bullet_{\text{inv}}(X, M) = H^\bullet(C^\bullet_{\text{inv}}(X, M))$. Clearly, we have a natural map

$$\xi : H^\bullet_{\text{inv}}(X, M) \to H^\bullet(X, M),$$

induced by the inclusion of complexes.

**Remark 3.4.** If $f \in Z^n_{\text{inv}}(X, M)$, by the proof of Lemma 3.1 part 2, it is clear that $f_y \in Z^{n-1}(X, M) \forall y \in X$.

For $M, N$ right $G_X$-modules, consider the natural multiplication map

$$C^a(X, M) \times C^b(X, N) \to C^{a+b}(X, M \otimes N).$$

This map will be denoted by $f, g \to f \otimes g$.

**Lemma 3.5.** Suppose that $A$ is a trivial $G_X$-module. Then for any $f \in C^i(X, A)$, $g \in C^j_{\text{inv}}(X, N)$, one has

$$d(f \otimes g) = df \otimes g + (-1)^i f \otimes dg.$$
Lemma 3.5 shows that if $f \in Z^i(X, A)$ and $g \in Z^j_{\text{inv}}(X, N)$ then $f \otimes g \in Z^{i+j}(X, A \otimes N)$. Furthermore, by the same Lemma, the cohomology class of $f \otimes g$ depends only of the cohomology classes of $f$ and $g$. Thus, we have a product

$$H^*(X, A) \times H^*_{\text{inv}}(X, N) \to H^*(X, A \otimes N).$$

In particular, if $R$ is a (unital) ring with the trivial $G_X$-action, then $H^*_{\text{inv}}(X, R)$ is a graded algebra, and for any left $R$-module $M$ with a compatible $G_X$-action, $H^*_{\text{inv}}(X, M)$ is a graded left $H^*_{\text{inv}}(X, R)$-module.

4. Cohomology of finite racks

In this section we will assume that $X$ is a finite rack.

Let $M$ be a right $G_X$-module, such that the kernel $K$ of the action of $G_X$ on $M$ has finite index. Let $L$ be the intersection of $K$ with the kernel $\Gamma$ of the action of $G_X$ on $X$, and let $G = G_X/L$ (notice that $G$ is finite). Assume that the multiplication by $|G|$ is an isomorphism $M \to M$.

**Lemma 4.1.** Under these conditions the map $\xi : H^*_{\text{inv}}(X, M) \to H^*(X, M)$ is an isomorphism.

**Proof.** The complex $C^*(X, M)$ is a complex of $G$-modules. On each term of this complex we have a projector given by $P = \frac{1}{|G|} \sum_{g \in G} g$, which projects to $G_X$-invariants. This projector commutes with the differential, so the complex $C^*(X, M)$ is representable as a direct sum of complexes:

$$C^*(X, M) = C^*_{\text{inv}}(X, M) \oplus C^*(X, M)(1 - P).$$

By Lemma 3.1, the second summand is acyclic: indeed, any cohomology class in it satisfies $cP = 0$, while the lemma says that $cP = c$, hence $c = 0$. This implies the desired statement. \qed

In particular, for any ring $R$ with trivial $G_X$-action, such that $N = |G^0_X|$ is invertible in $R$ (for example, $R = \mathbb{Z}[1/N]$ or $R = \mathbb{Q}$), the cohomology $H^*(X, R)$ is an algebra, and if $M$ is an $R$-module with a compatible $G_X$ action then $H^*(X, M)$ is a left module over this algebra.

Let $\text{Orb}(X) = X/G_X$ be the set of $G_X$-orbits on $X$, and $m = |\text{Orb}(X)|$. The main result in this section is

**Theorem 4.2.** Under the conditions of Lemma 4.1, we have

$$H^*(X, R) \simeq T^*_R(H^1(X, R)) \simeq T^*_R(\text{Fun}(\text{Orb}(X), R)) \simeq \text{Fun}(\text{Orb}(X)^*, R)$$

as an algebra (where $T^*_R(B)$ denotes the tensor algebra of an $R$-bimodule $B$), and if $M$ is an $R$-module with a compatible $G_X$ action then

$$H^*(X, M) \simeq T^*_R(H^1(X, R)) \otimes_R M^{G_X} \simeq T^*_R(\text{Fun}(\text{Orb}(X), R)) \otimes_R M^{G_X} \simeq \text{Fun}(\text{Orb}(X)^*, M^{G_X})$$

as a left module over the algebra $H^*(X, R)$.

Before proving the theorem, we will derive a corollary.

**Corollary 4.3.** The Betti numbers of $X$ are $\dim H^i(X, \mathbb{Q}) = m^i$. Furthermore, the only primes which can appear in the torsion of $H^*(X, \mathbb{Z})$ are those dividing $N$. 

Proof. The first assertion is clear taking $R = \mathbb{Q}$. For the second one, take $R = \mathbb{Z}[\frac{1}{n}]$ (or $R = \mathbb{Z}/p$, $p \nmid N$) and apply the universal coefficient theorem.

Remark 4.4. This, together with the lower bounds for the Betti numbers of the quandle and degeneracy cohomology in [CJKS] and the splitting result of [LN], implies that those lower bounds are in fact equalities.

Proof. (of Theorem 4.2). Since $M^{G_X} = H^0(X, M)$, for any $M$ we have an obvious multiplication mapping $\mu : T^*(H^1(X, R)) \otimes_R M^{G_X} \to H^*(X, M)$, which is compatible with the algebra and module structures. Thus, all we have to show is that $\mu$ is an isomorphism.

Let us first show that $\mu$ is injective. This is in fact the lower bound of [CJKS], but we will give a different proof. The proof is by induction in degree. The base of induction is clear. Assume the statement is known in degrees $< n$, and $c \in \text{Fun}(\text{Orb}(X)^n, M^{G_X})$ is such that $\mu(c) = 0$. This means that the pullback $f : X^n \to M$ of the function $c$ is a coboundary: $f = dg$. Because $f$ is invariant (under the diagonal action of $G_X$), and $C = C^{inv} \oplus C^*(1 - P)$, we can assume that $g$ is invariant. This means that for any $y \in X$, we have $(dg)_y = d(g)_y$ (we recall that $g_y(x_1, \ldots, x_i) := g(y, x_1, \ldots, x_i)$). Thus, $f_y = dg_y$. But $f_y$ is a pullback of a function $c_y \in \text{Fun}(\text{Orb}(X)^{n-1}, M^{G_X})$, so by the induction assumption $c_y = 0$. Hence $c = 0$.

Now let us prove that $\mu$ is surjective. This for it suffices to show that $H^n(X, M) \subset H^1(X, R)H^{n-1}(X, M)$. Let $c \in H^n(X, M)$. By Lemma 4.1, the element $c$ can be represented by an invariant cycle, $f \in Z^{inv}_n(X, M)$. By Remark 3.4, $f_y \in Z^{n-1}(X, M)$ for all $y \in X$. For each $y \in X$, decompose $f_y$ as $f_y = (f_y^+ + (f_y^-)$, where

$$(f_y^+ = f_y \cdot P \in Z^{n-1}_n(X, M) \text{ and } (f_y^- = f_y \cdot (1 - P) \in Z^{n-1}_n(X, M).$$

These functions give rise to unique functions $f^+, f^- \in C^n(X, M)$ such that $(f^\pm)_y = (f_y^\pm)_y \forall y \in X$. Moreover, it is clear that $f = f^+ + f^-$. Since $(f^+_y)_y \in Z^{n-1}_n(X, M) \forall y$, it is easy to see that $f^+ \in Z^n(X, M)$. Thus, also $f^- \in Z^n(X, M)$. Let us see now that $f^\pm$ are invariant: for any $h \in C^n(X, M), g \in G_X$, we have the equality $h \cdot g = (h \cdot g)_g^{-1}$. Thus, $\mu$ is surjective.

Now let $M$ be a semisimple finite dimensional $G_X$-module over a field $k$ of characteristic zero (but we do not require the image of $G_X$ to be finite). In this case, we have

Theorem 4.5. Lemma 4.1 and Theorem 4.2 are true for such $M$.

Proof. By a Chevalley’s theorem [C], the representations $C^n(X, M) = \text{Fun}(X, k)^{\otimes n} \otimes M$ are semisimple (as tensor products of semisimple representations). Therefore, there exists an invariant projector $P : C^* \to (C^*)^{G_X}$. The rest of the proof is the same as in the previous case. 

\[\square\]
Recall \([S]\) that \(G_X\) is a central extension of the finite group \(G_X^0\) with kernel being the finitely generated abelian group \(\Gamma\).

**Corollary 4.6.** If \(M\) is a finite dimensional \(\mathbb{Q}[G_X]\)-module and \(M(1)\) the generalized eigenspace for the trivial character of \(\Gamma\), then \(H^\bullet(X,M) = H^\bullet(X,M(1))\).

**Proof.** Write \(M = \oplus \chi M(\chi)\), where \(\chi\) runs over the characters of \(\Gamma\). We have \(H^\bullet(X,M) = \oplus \chi H^\bullet(X,M(\chi))\). Now, we prove by induction on the dimension of \(M(\chi)\) that if \(\chi\) is non-trivial then \(H^\bullet(X,M(\chi)) = 0\). If \(\dim M(\chi) = 0\), the cohomology clearly vanishes. Suppose now that \(\dim M(\chi) = n > 0\) and for smaller dimensions the statement is known. Let \(M_0\) be a simple submodule of \(M(\chi)\). We have then the short exact sequence of complexes

\[0 \to C^\bullet(X,M_0) \to C^\bullet(X,M(\chi)) \to C^\bullet(X,M(\chi)/M_0) \to 0.\]

The first complex is acyclic by Theorem 4.5, the third one is acyclic by the induction assumption, so by the long exact sequence in cohomology, the complex in the middle is also acyclic. The induction step and the corollary are proved. \(\square\)

**Corollary 4.7.** Let \(M\) be a finite dimensional \(\mathbb{Q}[T^\pm 1]\)-module. Then the twisted rack cohomology \(H^1_T(X,M)\) equals the twisted rack cohomology \(H^1_T(X,M(1))\), where \(M(1)\) is the generalized eigenspace of \(T\) in \(M\) with eigenvalue 1. \(\square\)

To compute the Betti numbers of twisted cohomology, the only lacking case is that in which the elements of the rack \(X\) act on \(M\) by a Jordan block with 1 on the diagonal.

**Proposition 4.8.** Let \(M\) be a \(\mathbb{Q}G_X\)-module with basis \(\{v_1,\ldots,v_k\}\) on which the elements of \(X\) act by \(v_i \mapsto v_{i-1} + v_i\) (\(v_0 := 0\)). Then \(\dim H^n(X,M) = m^n\), where \(m = |\text{Orb}(X)|\).

Before proving the Proposition we state two easy lemmas:

**Lemma 4.9.** Let \((C^\bullet,d)\) be a complex and suppose that \(C^\bullet = C^\bullet_1 \oplus C^\bullet_2\) and that the differential \(d\) has the form \(\begin{pmatrix} d_1 & \alpha \\ 0 & d_2 \end{pmatrix}\) for this decomposition. Then \(\alpha\) induces a map \(\alpha^n_n : H^{n-1}(C^\bullet_2) \to H^n(C^\bullet_1)\). Consider then the short exact sequence of complexes

\[0 \to C^\bullet_1 \to C^\bullet \to C^\bullet_2 \to 0\]

and let \(\beta^n : H^{n-1}(C^\bullet_2) \to H^n(C^\bullet_1)\) be the connecting homomorphism. Then \(\beta^n = \alpha^n_n\).

**Proof.** Since \(d^2 = 0\), we have \(d_1\alpha = -\alpha d_2\), whence it induces a map in cohomology. The second assertion follows in a straightforward way from the definition of the connecting homomorphism. \(\square\)

**Lemma 4.10.** Let \(C^\bullet = C^\bullet_1 \oplus C^\bullet_2\) be as in Lemma 4.9. Suppose that \((C^\bullet_2,d_2')\) is a complex and that \(f : C^\bullet_2' \to C^\bullet_2\) is a quasi-isomorphism. Then \((\text{id} \oplus f) : C^\bullet_1 \oplus C^\bullet_2' \to C^\bullet\) is a quasi-isomorphism, where the first complex has differential given by \(\begin{pmatrix} d_1 & \alpha f \\ 0 & d_2' \end{pmatrix}\).

**Proof.** This follows easily from the 5-lemma. \(\square\)
Proof of Proposition 4.8. The proof is by induction on \( k \). If \( k = 1 \) the assertion is Corollary 4.3. Assume that the result is true for dimensions \( < k \). Let us decompose \( C^* = C^*(X, M_1) \oplus C^*(X, M_2) \), where \( M_1 \) is generated by \( v_1, \ldots, v_{k-1} \) and \( M_2 \) is generated by \( v_k \). Notice that the differential \( d \) in \( C^* \) can be written as
\[
\begin{pmatrix}
 d_1 & \alpha \\
 0 & d_2
\end{pmatrix},
\]
where \( d_i : C^*(X, M_i) \rightarrow C^*(X, M_i) \) are the differentials of the same complex we are considering for \( M \) of dimension \( k - 1 \) and 1 respectively.

Let us take \( C_2^* = T^*(\text{Fun}(\text{Orb}(X), \mathbb{Q})) \). By Theorem 4.2, the inclusion \( i : C_2^* \rightarrow C_1^* \) is a quasi-isomorphism, and thus by Lemma 4.10 we can work with \( C^*(X, M_1) \oplus T^*(\text{Fun}(\text{Orb}(X), \mathbb{Q})) \). We consider the long exact sequence
\[
(4.11) \quad H^{n-1}(C_2^*) \xrightarrow{\beta^n} H^n(C_1^*) \xrightarrow{i^n} H^n(C_1^* \oplus C_2^*) \xrightarrow{p^n} H^n(C_2^*) \xrightarrow{\delta^{a+n+1}} H^{n+1}(C_2^*) \rightarrow
\]
Let \( \bar{\alpha} = \alpha|_{C_2^*} \) and consider the induced map in cohomology \( \bar{\alpha}_* \), i.e.,
\[
\bar{\alpha}^n : H^{n-1}(C_2^*) = T^{n-1}(\text{Fun}(\text{Orb}(X), \mathbb{Q})) \rightarrow H^n(C_1^*) = H^n(X, M_1).
\]
By Lemma 4.9, \( \beta^n = \bar{\alpha}^n \). We claim that \( \text{rk } \bar{\alpha}_* = \text{rk } \alpha \). To see this, it suffices to prove that \( \text{Im } \alpha^n \cap B^n(C_1^*) = 0 \). Suppose that \( \alpha^n(f) \in B^n(C_1^*) \), then it has the form \( \sum b_i v_i \), where \( b_i \in C^n(X, \mathbb{Q}) \). Furthermore, it is clear that \( b_{k-1} \in B^n(X, \mathbb{Q}) \). On the other hand, if \( \pi : X \rightarrow \text{Orb}(X) \) is the canonical projection, we have
\[
\alpha^n(f)(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i f(\pi(x_1), \ldots, \pi(x_{i-1}), \pi(x_{i+1}), \ldots, \pi(x_n))v_{k-1},
\]
which shows that \( b_{k-1} \in T^n(\text{Fun}(\text{Orb}(X), \mathbb{Q})) \). But it is shown in the injectivity part of the proof of Theorem 4.2 that \( T^n(\text{Fun}(\text{Orb}(X), \mathbb{Q})) \cap B^n(X, \mathbb{Q}) = 0 \), and the claim is proved.

Then, \( \text{rk } \beta^n = \text{rk } \bar{\alpha} \). But the latter is not difficult to compute: if we consider the complex \( (D^\bullet, \hat{d}) \), where \( D^n = \text{Fun}((\text{Orb}(X))^n, \mathbb{Q}) \) and \( \hat{d} \) is given by
\[
\hat{d}(f)(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n),
\]
then it is clear that \( \bar{\alpha}^n \) and \( \hat{d}^n \) have the same rank. Furthermore, it is well known that \( D^\bullet \) is acyclic (it gives the reduced cohomology of a simplex of dimension \( m - 1 \)). It is easy then to compute the rank of \( \hat{d} \); we have \( \text{rk } \hat{d}^n = m^{n-1} - m^{n-2} + m^{n-3} - \cdots \pm 1 \). We add this computation to the long exact sequence \( (4.11) \) and we are done: we have \( \text{rk } \beta^n = m^{n-1} - m^{n-2} + \cdots \pm 1 \), and since by the inductive assumption \( \dim H^n(C_1^*) = m^n \), then \( \text{rk } i^n = m^n - m^{n-1} + \cdots \pm 1 \). Also, we have \( \text{rk } \beta^{n+1} = m^n - m^{n-1} + \cdots \pm 1 \) and since \( \dim H^n(C_2^*) = m^n \), we get \( \text{rk } p^n = m^{n-1} - m^{n-2} + \cdots \pm 1 \). Thus, \( \dim H^n(C^*) = \text{rk } i^n + \text{rk } p^n = m^n \), proving the inductive step.

Since for \( M \) as above we have \( \dim M^{G_X} = 1 \), we have proved:

**Corollary 4.12.** Let \( M \) be a right \( \mathbb{Q}G_X \)-module on which all the elements of \( X \) act by the same operator. Then \( \dim H^n(X, M) = m^n \times \dim M^{G_X} \).

**Remark 4.13.** It is interesting to study the graded algebra \( H^*_\text{inv}(X, k) \), where \( k \) is a field of characteristic \( p \) dividing \( |G_X^0| \), to which Theorem 4.2 does not apply. One may ask the following questions about this ring:

- Is it finitely generated?
- What is its Poincaré series? Is it a rational function?
5. A relation with group cohomology

In this section, for any rack $X$, we want to give a group theoretical interpretation of the group $H^2(X, A)$ (where $A$ is a trivial $G_X$-module). This group is useful in the theory of pointed Hopf algebras [G].

We start with the following obvious, but useful proposition.

**Proposition 5.1.** Let $A$ be a trivial $G_X$-module. Then one has a natural isomorphism of complexes $J : C^n(X, A) \rightarrow C^{n-1}(X, \text{Fun}(X, A))$, $n \geq 1$, where we consider the action of $G_X$ on $\text{Fun}(X, A)$ given by $(hy)(x) = h(y \triangleright x)$. It is given by $(Jf)(x_1, \ldots, x_{n-1})(x_n) = f(x_1, \ldots, x_n)$. In particular, it induces an isomorphism $H^n(X, A) \rightarrow H^{n-1}(X, \text{Fun}(X, A))$. \hfill \square

**Remark 5.2.** We note that this proposition becomes false if the action of $G_X$ on $A$ is not trivial.

Now we give the main result of this section. Let $M$ be a right $G_X$-module.

**Proposition 5.3.** $H^1(X, M) \simeq H^1(G_X, M)$.

Propositions 5.1 and 5.3 imply

**Corollary 5.4.** If $A$ is a trivial $G_X$-module, then $H^2(X, A) \simeq H^1(G_X, \text{Fun}(X, A))$. \hfill \square

**Proof.** (of Proposition 5.3) Let $C^*(G, M)$ be the standard complex of a group $G$ with coefficient in a right $G$-module $M$. Let $\eta : C^1(G, M) \rightarrow C^1(X, M)$ be the homomorphism induced by the natural map $X \rightarrow G_X$. It is easy to show that this homomorphism maps cocycles to cocycles and coboundaries to coboundaries. Thus, it induces a homomorphism $\eta : H^1(G, M) \rightarrow H^1(X, M)$. Thus, our job is to show that any $f \in Z^1(X, M)$ lifts uniquely to a $1$-cocycle on $G_X$.

To do this, recall that a map $\pi : G_X \rightarrow M$ is a $1$-cocycle iff the map $\hat{\pi} : G_X \rightarrow G_X \rtimes M$ given by $g \rightarrow (g, \pi(g))$ is a homomorphism. On the other hand, we have a map $\xi_f : X \rightarrow G_X \rtimes M$ given by $\xi_f(x) = (x, f(x))$. So we need to show that $\xi_f$ extends to a homomorphism $G_X \rightarrow G_X \rtimes M$. But the group $G_X$ is generated by $X$ with relations $xy = (x \triangleright y)x$. Thus, we only need to check that $\xi_f(x)$ satisfy the same relations. But it is easy to check that this is exactly the condition that $df = 0$. We are done. \hfill \square

Another, more conceptual, proof runs as follows: let $N$ be a right $X$-module (i.e, a right $G_X$-module) and consider on $X \times N$ the following structure:

$$(x, n) \triangleright (y, m) = (x \triangleright y, n(1 - (x \triangleright y)^{-1}) + mx^{-1}).$$

It is easy to verify that this is a rack structure on the product; we shall denote it by $(X \rtimes N, \triangleright)$ (it is actually the same structure as in [AG] for the left $X$-module $N$ with $x \cdot n = nx^{-1}$). We have then, with a straightforward proof,

**Lemma 5.5.** Let $\omega : X \rightarrow N$ and define $\hat{\omega} : X \rightarrow X \rtimes N$ by $\hat{\omega}(x) = (x, \omega(x)x^{-1})$. Then $\hat{\omega}$ is a rack homomorphism if and only if $\omega \in Z^1(X, N)$. \hfill \square

Take $\alpha : X \rtimes N \rightarrow G_X \rtimes N$, $\alpha(x, n) = (x, nx)$. One can check that in the square

$$
\begin{array}{ccc}
X & \xrightarrow{\hat{\omega}} & X \rtimes N \\
\downarrow & & \downarrow \alpha \\
G_X & \xrightarrow{\hat{\pi}} & G_X \rtimes N
\end{array}
$$
each of \( \omega, \pi \) determines uniquely the other in such a way that the diagram is commutative. \( \square \)

**Remark 5.6.** Corollary 5.4 holds also when \( A \) is nonabelian. In this case \( H^2(X, A) \) is the quotient of the set \( Z^2(X, A) = \{ f : X \times X \to A \mid f(x \triangleright y, x \triangleright z)f(x, z) = f(x, y \triangleright z)f(y, z) \} \) by the equivalence relation \( f \sim f' \) if there is a \( \gamma : X \to A \) such that \( f'(x, y) = \gamma(x \triangleright y)f(x, y)\gamma(y)^{-1} \). The proof is the same as in the abelian case.

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