Operator space approach to steering inequality

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Received 23 July 2014, revised 19 January 2015
Accepted for publication 11 February 2015
Published 12 March 2015

Abstract
In Junge and Palazuelos (2011 Commun. Math. Phys. \textbf{306} 695–746) and Junge \textit{et al} (2010 Commun. Math. Phys. \textbf{300} 715–39) the operator space theory was applied to study bipartite Bell inequalities. The aim of the paper is to follow this line of research and use the operator space technique to analyze the steering scenario. We obtain a bipartite steering functional with unbounded largest violation of steering inequality, as well as constructing all ingredients explicitly. It turns out that the unbounded largest violation is obtained by a non maximally entangled state. Moreover, we focus on the bipartite dichotomic case where we construct a steering functional with unbounded largest violation of steering inequality. This phenomenon is different to the Bell scenario where only the bounded largest violation can be obtained by any bipartite dichotomic Bell functional.

Keywords: steering inequality, unbounded largest violation, operator space

1. Introduction

The violation of local realism, called usually nonlocality, plays an important role in quantum information science. It was first studied in 1935 by Einstein \textit{et al} [7]. In their paradoxical paper, Einstein, Podolsky, and Rosen (EPR) argued that quantum mechanics does not provide a complete description of the elements of reality. Moreover, they predicted that quantum mechanics either develops to a complete theory or is replaced by another complete theory. In the paper, EPR wrote: ‘we believe that such (complete) a theory is possible’. Theories
compatible with EPR’s ideas are called ‘local-realistic (LR) theories’. Although Bohr rebutted shortly after EPR’s ideas, their arguments, without observational consequences, did not suggest a clear conclusion, hence the debate has then subsided.

The EPR arguments resurfaced in 1964 when Bell derived a constraint for correlation between two remote subsystems, known as Bell’s inequality which is satisfied by all LR theories. He proved that it is violated by quantum correlations of two spin 1/2 particles in a singlet state, i.e., an entangled state. States that violate some Bell inequalities form a strict subset of the set of entangled states [29]. In [30] the authors proposed an intermediate form of quantum correlations between Bell nonlocality and entanglement, by use of quantum steering. The latter concept was introduced by Schrödinger in 1935 in reply to the EPR paradox [24]. Wiseman et al reformulated this concept in a rigorous way [30] and showed, in particular, that the set of states admitting steering is a strict subset of entangled states on the one hand and a strict superset of states violating Bell inequalities on the other. Since then, quantum steering has attracted more and more attention both in theory [6, 17, 22, 30] and experiment [25, 26].

The simplest example of quantum steering is the following one, which was a basis for the famous EPR paradox [7] (in Bohm version [3]). Namely, when Alice and Bob share a pair of particles in a singlet state, Alice, by choosing one of two measurements, can create at Bob’s site one of two ensembles: one consisting of basis states \(|0\rangle\) and \(|1\rangle\) with equal probabilities, and the other consisting of complementary states \(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\) and \(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\), again with equal probabilities. It turns out that this would be impossible if Bob’s particle were in some well defined state, perhaps unknown to him—the so called ‘local hidden state’ (LHS)—and Alice merely used her knowledge about the state. Thus, the existence of the above Alice measurements proves that the shared state is entangled. Remarkably, Alice can in this way convince Bob that the shared state is entangled even if Bob does not trust her. Indeed, Bob can ask Alice to create one of the two ensembles at random, and upon receiving message from Alice telling which outcome she obtained, he can verify that indeed she created (or: ‘steered’) to the above states with the mentioned probabilities, provided many runs of the experiment are performed. More generally, bipartite states for which there exist measurements of Alice steering to ensembles, which does not come from the LHS, are called steerable (or admitting quantum steering).

The steering scenario, one can study ‘steering functionals’ which are analogs of Bell functionals. The violation of a steering inequality for such functionals provides a natural way to quantify the deviation from a LHS description. However it is not easy to compute the violation for a given steering functional analytically; one usually uses a numerical method, called the semi-definite program [22, 27]. In this paper, we will use the operator space theory, which has been widely developed after the pioneering and fundamental work of Effros–Ruan and Blecher–Paulsen [8, 20], to study the violation of steering inequality. Our work is motivated by a series works of Junge et al [11, 12, 23], where they used operator space theory to analyze Bell inequalities. We will briefly recall their work in subsection 2.2. In their work, operator space was connected to the largest violation of Bell inequality. As steering inequality is closely connected to Bell inequality, we are able to apply their strategy. According to their work, the following results are natural: we can construct a probabilistic steering functional with unbounded largest violation of steering inequality in the sense of ‘with high probability’. The non maximally entangled state plays an important role in violation of steering inequality for this functional.

Moreover, by mixing white noise with this non maximally entangled state, we can get a family of positive partial transpose (PPT) states. Actually, these states belong to a class of
PPT states, which was constructed in [5]. We prove that for any steering functional, only the bounded largest violation of steering inequality can be obtained by these kinds of PPT states. Thus even though there are PPT states violating some steering inequalities [16] (problem posed in [22]), our example provides some evidence that PPT states cannot provide unbounded violation.

However, not all properties of Bell functionals can be inherited by steering ones. It is well known that the quantum bound of any bipartite dichotomic Bell functional is bounded by the classical bound with an universal constant (Grothendieck constant) [21, 23]. As reported in a companion paper [15], in the steering scenario this is not true: there is a bipartite dichotomic steering functional with unbounded largest violation. Here we put the example of [15] into the framework of operator spaces and show how the inequality arises from a bounded but not completely bounded map from $\ell_1^n$ to $\mathbb{M}_n$.

In this paper, operator spaces and their tensor products are the main mathematical tools. For more information about operator space theory, see [8, 20]. We will use various kinds of operator spaces, such as $\ell_1^n$, $\ell_\infty^n$, $\mathcal{R}_n$, $\mathcal{C}_n$ and $\mathcal{O}H_n$. We strongly recommend readers to refer [11, 12] for these notions.

We finish this introduction by setting the following convention: throughout this paper, we will use $\geq$ and $\leq$ to denote the inequality up to an universal constant irrelevant to $n \in \mathbb{N}$, and we also use the Dirac symbol $|i\rangle\langle j|$, $i, j = 1, \ldots, n$ to denote the canonical basis of $\mathbb{M}_n$.

For $k \in \mathbb{N}$ we denote by $(e_1, \ldots, e_k)$ the canonical basis of $\ell_1^k$ and by $(e'_1, \ldots, e'_k)$ its dual basis in $\ell_\infty^k$. Then the system $(e_x \otimes e'_y)$, $x = 1, \ldots, n$, $a = 1, \ldots, m$ forms a basis of the space $\ell_\infty^n (\ell_\infty^m)$ as well as the system $(e'_x \otimes e_y)$, $x = 1, \ldots, n$, $a = 1, \ldots, m$ is a basis of $\ell_\infty^n (\ell_1^m)$.

2. Main result

2.1. Definition of a steering functional and its largest violation

We consider the following steering scenario [22]. Assume that there are two systems A (Alice) and B (Bob). Suppose Alice can choose among $n$ different measurement settings labeled by $x = 1, \ldots, n$. Each of them can result in one of $m$ outcomes, labeled by $a = 1, \ldots, m$. Suppose also that Bob has a $d$-dimensional quantum system $H_d$ at his disposal.

**Definition 2.1.** An assemblage is a set $\{\sigma^x_a: x = 1, \ldots, n, a = 1, \ldots, m\}$ of $d \times d$ Hermitian matrices satisfying the following conditions:

(i) $\sigma^x_a \geq 0$ (positivity);
(ii) $\sum_a \sigma^x_a$ is independent of $x$ and trace 1.

We say that a set $\{\sigma^x_a\}$ of positive matrices forms an incomplete assemblage if it satisfies the following condition instead of the above condition (ii)

(ii') $\text{Tr}(\sum_a \sigma^x_a) \leq 1$ for every $x$.

It turns out ([24] and later [10]) that any assemblage (respectively incomplete assemblage) has a quantum realization. It means that for any assemblage there exists a Hilbert space $H$, a density matrix $\rho \in B(H \otimes H_d)$ and a family $\{E^x_a\}_x$ of positive operators on $H$ such that $\sum_x E^x_a = 1$ (respectively $\sum_x E^x_a \leq 1$) for every $x$, and

$$\sigma^x_a = \text{Tr}_B((E^x_a \otimes 1_B)\rho).$$ (2.1)
We denote the set of all quantum assemblages (respectively incomplete quantum assemblages) by $Q$ (respectively by $Q^\text{in}$).

Next, we distinguish the classical part of the set assemblages.

**Definition 2.2.** We say that an assemblage (respectively incomplete assemblage) admits a LHS model, if there exists a finite set of indices $\Lambda$, nonnegative coefficients $\lambda_q$ such that $\sum_{q, x} \lambda_q \sigma_q a x = 1$ for every $x$, $\lambda$, and $a$.

for every $x$ and $a$. We denote the set of LHS assemblages (respectively LHS incomplete assemblages) by $L$.

It is known [30] that $L \subset Q$. Our aim is to quantify the difference between sets $Q$ and $L$, and our strategy is to analyze some functionals to see how much value they can obtain on the quantum assemblages comparing with values on LHS assemblages.

**Definition 2.3.** For given natural numbers $n$, $m$, and $d$, define a steering functional $F$ as a set $\{F_{x a} = \ldots \}$. The functional maps an assemblage $\sigma$ to a real number $\langle F, \sigma \rangle = \sum_{x} \sum_{a} \text{Tr} \left( F_{x a} \sigma_{x a} \right)$.

**Remark 2.4.** The notion of steering inequality was first introduced by Cavalcanti et al [4]. Given a steering functional, a steering inequality says that $\langle F, \sigma \rangle \leq B_C(F)$, $\sigma \in L^\text{in}$.

Now we can define

**Definition 2.5.** Given a steering functional $F = \{F_{x a} \in B(H_a): x = 1, \ldots, n, a = 1, \ldots, m\}$, we define the LHS bound of $F$ as a number $B_C(F) = \sup \left\{ \langle F, \sigma \rangle : \sigma \in L^\text{in} \right\}$.

and the quantum bound of $F$ as $B_Q(F) = \sup \left\{ \langle F, \sigma \rangle : \sigma \in Q^\text{in} \right\}$.

We define the largest quantum violation of steering inequality for $F$ as a positive number $LV(F) = \frac{B_Q(F)}{B_C(F)}$.

In what follows we will frequently call this number the largest violation for $F$.

2.2. Junge–Palazuelos approach to violation of Bell inequality

Junge et al studied the following largest violation of Bell inequality for a bipartite Bell functional $M$ by using operator space theory [11, 12]:
\[ LV(M) = \frac{B_Q(M)}{B_C(M)}, \]  

(2.8)

where \( M = \sum_{i,j=1}^{\infty} \sum_{a,b=1}^{n} M^a_{i,j} e_x \otimes e'_x \otimes e_y \otimes e'_y \in \ell^1_\ell(\ell^\infty_{\ell^\infty}) \otimes \ell^1_\ell(\ell^\infty_{\ell^\infty}), \) and

\[ B_Q(M) = \sup \left\{ \sum_{x,y,a,b} M_{i,j}^a e_x \otimes e'_x \otimes e_y \otimes e'_y \in \ell^1_\ell(\ell^\infty_{\ell^\infty}) \otimes \ell^1_\ell(\ell^\infty_{\ell^\infty}), \right\} \]

(2.9)

\[ B_C(M) = \sup \left\{ \sum_{x,y,a,b} M_{i,j}^a \sum_{\lambda} \rho(\lambda) \mathbb{P}(a|x, \lambda) \mathbb{P}(b|y, \lambda) \right\} : \]

\[ \sum_{a} \mathbb{P}(a|x, \lambda) \leq 1, \quad \sum_{b} \mathbb{P}(b|y, \lambda) \leq 1. \]

(2.10)

They linked the largest violation of Bell inequality to the ‘\textit{min versus } e\textit{’ problem of } M, i.e., they obtained following result (see [12, proposition 4, theorem 6]):

**Proposition 2.6.** Given \( M = \sum_{i,j=1}^{\infty} \sum_{a,b=1}^{n} M^a_{i,j} e_x \otimes e'_x \otimes e_y \otimes e'_y \in \ell^1_\ell(\ell^\infty_{\ell^\infty}) \otimes \ell^1_\ell(\ell^\infty_{\ell^\infty}), \) we have the following equivalence.

(i) Classical bound:

\[ B_C(M) \approx \| M \|_{\ell^1_\ell(\ell^\infty_{\ell^\infty}) \otimes \ell^1_\ell(\ell^\infty_{\ell^\infty})}. \]

(2.11)

(ii) Quantum bound:

\[ B_Q(M) \approx \| M \|_{\ell^1_\ell(\ell^\infty_{\ell^\infty}) \otimes \ell^1_\ell(\ell^\infty_{\ell^\infty})}. \]

(2.12)

Here the notion \( X \otimes Y \) denotes the injective tensor product of two Banach spaces \( X \) and \( Y \) [28]. And for two operator spaces \( E \) and \( F \), \( E \otimes_{\text{min}} F \) denotes the minimal tensor product of operator space [20]. \( \ell^1_\ell(\ell^\infty_{\ell^\infty}) \) is a Banach space in (2.15) and it is an operator space in (2.16). However, it is difficult to calculate the tensor norm of element in \( \ell^1_\ell(\ell^\infty_{\ell^\infty}) \otimes \ell^1_\ell(\ell^\infty_{\ell^\infty}) \). To overcome this obstacle, they used following fact (see [11, theorem 3.2]):

**Fact 2.7.** There exist \( \delta \in (0, \frac{1}{2}) \) and an universal constant \( C \) such that, for every \( n \), there are maps \( V: \ell^1_\ell(\ell^\infty_{\ell^\infty}) \rightarrow \ell^1_\ell(\ell^\infty_{\ell^\infty}) \) and \( V': \ell^1_\ell(\ell^\infty_{\ell^\infty}) \rightarrow \ell^1_\ell(\ell^\infty_{\ell^\infty}) \) satisfying \( \| V \| \leq C \sqrt{\log n}, \| V' \| \leq 1 \) and \( V' V = \text{id}_{\ell^1_\ell} \). Moreover, the map \( V' \) is completely bounded from \( \ell^1_\ell(\ell^\infty_{\ell^\infty}) \) to \( R_{\ell^0_\ell} \) and \( \| V' \|_{\ell^1_\ell} \leq K_L \leq \log n \). Where \( K_L \) is the constant in the little Grothendieck theorem.

Compared to \( \ell^1_\ell(\ell^\infty_{\ell^\infty}) \), the tensor norms on \( \ell^1_\ell \) are easier to calculate. Through this approach, they obtained a Bell inequality with unbounded largest violation of order \( \sqrt{n} \).
2.3. Unbounded largest violation of steering inequality

It is well known that there is a very close relation between Bell inequality and steering one [4]. So it is not surprising for us to find a steering functional with unbounded largest violation of steering inequality through their approach. Our main result concerns the case when all the numbers \(n, m, \) and \(d\) are equal. It can be stated as follows:

**Theorem 2.8.** For every \(n \in \mathbb{N}\), we can find a steering functional \(F = \{F^a_x \in B(H_a)\}: x, a = 1, \ldots, n\), such that

\[
LV(F) \gtrsim \frac{\sqrt{H}}{\sqrt{\log n}}.
\]  

(2.13)

Let \(F = \{F^a_x\}_{x,a}\) be a steering functional. We will identify \(F\) with the following element of the tensor product \(\otimes_{\infty} \mathbb{L} B H\):

\[
\sum_{x=1}^n \sum_{a=1}^m (e_x \otimes e_a' \otimes F^a_x).
\]  

(2.14)

We still use \(F\) to denote this element.

Now, in the spirit of proposition 2.6, we can link the problem of largest violation of steering functional to the ‘min versus \(\varepsilon\)’ problem for \(F\).

**Proposition 2.9.** Given \(F = \sum_{x,a} (e_x \otimes e_a') \otimes F^x_a \in \ell^2 (\ell^2_\infty) \otimes B(H_a)\), we have the following equivalence.

(i) LHS bound:

\[
B_C(F) \leq \| F \|_{\ell^2 (\ell^2_\infty) \otimes B(H_a)} \leq 16B_C(F).
\]  

(2.15)

(ii) Quantum bound:

\[
B_Q(F) \leq \| F \|_{\ell^2 (\ell^2_\infty) \otimes_{\min} B(H_a)} \leq 4B_Q(F).
\]  

(2.16)

**Proof.** For the LHS bound, we will use the duality between the injective and projective tensor product for finite dimensional Banach spaces. For a Hilbert space \(H\), we denote by \(S_p(H)\) the \(p\)th Schatten class for \(H\), where \(p\) is a number such that \(p \geq 1\). Any element \(\sigma \in \ell^2 (\ell^2_\infty) \otimes S_\infty(H_a)\) can be considered as a functional on \(\ell^2 (\ell^2_\infty) \otimes B(H_a)\). Its action on \(F\) is given by

\[
\langle F, \sigma \rangle = \sum_{x=1}^n \sum_{a=1}^m \text{Tr} (F^a_x \sigma^a_x),
\]  

(2.17)

where matrices \(\sigma^a_x \in B(H_a)\) are determined by the unique decomposition \(\sigma = \sum_{x,a} e_x' \otimes e_a \otimes \sigma^a_x\). Observe that the action of a steering functional on an assemblage given by (2.3) is a special case of the above duality. Given a Banach space \(X\), let \(B_X\) denote the unit ball of \(X\). Thus, by the duality, we have

\[
\| F \|_{\ell^2 (\ell^2_\infty) \otimes B(H_a)} = \sup \left\{ \| \langle F, \sigma \rangle \|: \sigma \in B_{\ell^2 (\ell^2_\infty) \otimes S_\infty(H_a)} \right\}.
\]  

(2.18)
Now, observe that
\[ \mathcal{L}^{\text{in}} \subset \mathcal{B}(\ell_2^n) \otimes S(H_n). \] (2.19)

The first inequality in (2.15) follows from inclusion (2.19).

For the quantum bound, we first recall the following fact from the [12]:

**Fact 2.10.** Given a set of incomplete POVMs \( \{E_x^a\}_{a=1}^m, x = 1, \ldots, n \) on \( B(H) \), the operator \( u: \ell_1^n(\ell_\infty^n) \to B(H) \) defined by \( u(e_x \otimes e_a)^* = E_x^a \) is a complete contraction.

The minimal tensor norm of \( F \) can be expressed as follows [20]:
\[ \| F \|_{\ell_1^n(\ell_\infty^n) \otimes \mathbb{B}(H_n)} = \sup_{H_n} \| (u \otimes \text{id})(F) \|_{\mathbb{B}(H_n) \otimes \mathbb{B}(H_n)}, \] (2.20)

where the sup is taken over all possible Hilbert spaces \( H \) and complete contractions \( u: \ell_1^n(\ell_\infty^n) \to B(H) \). Let \( \sigma \in \mathcal{Q} \), i.e. there is a Hilbert space \( H \), a density matrix \( \rho \in B(H) \otimes B(H) \), and incomplete POVMs \( \{E_x^a\} \) on \( B(H) \) such that \( \sigma^u_x = \text{Tr}_a((E_x^a \otimes I)\rho) \) for every \( x, a \). Let \( u \) be a map \( u: \ell_1^n(\ell_\infty^n) \to B(H) \) defined by \( u(e_x \otimes e_a)^* = E_x^a \). By the aforementioned fact \( u \) is a complete contraction. Thus, by the following claim it is enough to show the first inequality in (2.16)
\[ \langle F, \sigma \rangle = \text{Tr}(u(\otimes \text{id})(F))\rho. \] (2.21)

The second inequality in (2.15) and (2.16) can be proved by using the same argument in the proof of proposition 2.6. Here we omit the details.

The result which we will prove is

**Theorem 2.11.** For every \( n \in \mathbb{N} \), there exists an element \( F = \sum_{x,a=1}^n (e_x \otimes e_a^*) \otimes F_x^a \in \ell_1^n(\ell_\infty^n) \otimes B(H_n) \) such that
\[ \frac{\| F \|_{\text{min}}}{\| F \|_\infty} \geq \frac{\sqrt{n}}{\sqrt{\log n}}. \] (2.22)

Theorem 2.8 follows from this theorem and proposition 2.9.

**Proof.** Define a map \( W: \ell_1^{[\delta n]} \to B(H_n) \) by:
\[ W(e_k) = [1] \langle k |, k = 1, \ldots, [\delta n]. \] (2.23)

It is easy to check that \( W \) is a contraction, i.e. \( \| W \| \leq 1 \).

Combining with fact 2.7, we consider the element:
\[ F = (V \otimes W)(a) \in \ell_1^n(\ell_\infty^n) \otimes B(H_n), \] (2.24)

where \( a = \sum_{k=1}^{[\delta n]} e_k \otimes e_k \). On the one hand, we have
\[ \| F \|_{\ell_1^n(\ell_\infty^n) \otimes B(H_n)} = \| V \otimes W(a) \|_{\ell_1^n(\ell_\infty^n) \otimes B(H_n)} \leq \| V \| \| W(a) \|_{\ell_1^n(\ell_\infty^n) \otimes B(H_n)} \leq \sqrt{\log n}. \] (2.25)
On the other hand, the formula (2.20) implies
\[
\| F \|_{\ell_1^2(H_1) \otimes_{\text{max}} B(H_n)} \gtrsim \| (V' \otimes \text{id}) F \|_{\mathcal{R}_{\text{max}}(\mathcal{H}_n)} = \| (V' \otimes \text{id})(V \otimes W)(a) \|_{\mathcal{R}_{\text{max}}(\mathcal{H}_n)}
\]
\[
= \sum_{k=1}^{[\delta n]} \| e_k \otimes 1 \|_{\mathcal{R}_{\text{max}}} \otimes_{\text{min}} B(H_n) = \| \sum_{k=1}^{[\delta n]} |k\rangle \langle k| \|^2 \gtrsim \sqrt{n}.
\]
(2.26)

Combining equations (2.25) and (2.26) we get
\[
\frac{\| F \|_{\min}}{\| F \|_\infty} \gtrsim \frac{\sqrt{n}}{\sqrt{\log n}}.
\]
(2.27)

\[\square\]

For the specific F which was found in the above proof, we have following upper bound.

**Proposition 2.12.** The element \( F = (V \otimes W)(\alpha) \in \ell_1^2(\ell_\infty^n) \otimes B(H_n) \) defined in (2.24) verifies
\[
\| F \|_{\ell_1^2(H_1) \otimes_{\text{max}} B(H_n)} \leq \sqrt{n \log n}.
\]
(2.28)

**Proof.** Let us consider the map \( W: \ell_1^{[\delta n]} \rightarrow B(H_n) \) defined in (2.23). By the result of Pisier [20], we have
\[
\| W: \mathcal{O}_{[\delta n]} \rightarrow B(H_n) \|_{\ell_1^2} \leq n^\frac{1}{4} \sqrt{\log n}.
\]
(2.29)

Hence \( \| W: \mathcal{O}_{[\delta n]} \rightarrow B(H_n) \|_{\ell_1^2} \leq n^\frac{1}{4} \sqrt{\log n} \). On the other hand, we have following fact shown in [11]:
\[
\| V: \mathcal{O}_{[\delta n]} \rightarrow \ell_1^2(\ell_\infty^n) \|_{\ell_1^2} \leq n^\frac{1}{4} \sqrt{\log n}.
\]
(2.30)

Therefore, we obtain:
\[
\| F \|_{\ell_1^2(H_1) \otimes_{\text{max}} B(H_n)} = \| (V \otimes W)(\sum_k e_k \otimes e_k) \|_{\ell_1^2(H_1) \otimes_{\text{max}} B(H_n)}
\]
\[
\leq \sqrt{n \log n} \| \sum_k e_k \otimes e_k \|_{\mathcal{O}_{[\delta n]} \otimes_{\text{max}} \mathcal{O}_{[\delta n]}}
\]
\[
= \sqrt{n \log n} \| \sum_k e_k \otimes e_k \|_{\ell_1^{[\delta n]} \otimes_{\text{max}} \ell_1^{[\delta n]}} = \sqrt{n \log n}.
\]
(2.31)

\[\square\]
Since our work is an adaptation of [11] in the steering scenario, it is natural for us to provide the following two results.

Explicit form of the violation. Let \( \varepsilon_{k,a}^j \), \( x, a, k = 1, \ldots, n \) be independent Bernoulli sequences and let \( K \) be a positive constant. Then we define:

(i) Steering functional \( F_x^a \in B(H_{n+1}) \):

\[
F_x^a = \begin{cases} 
\frac{1}{n} \sum_{k=2}^{n+1} \varepsilon_{x,a}^{k-1} \left|1\right\rangle \left\langle k\right|, & x, a = 1, \ldots, n, \\
0, & a = n + 1.
\end{cases}
\]  

(ii) POVMs measurements [11] \( \{E_x^a\}_{x,a=1}^{n,n+1} \) in \( B(H_{n+1}) \) as

\[
E_x^a = \begin{cases} 
1, & a = 1, \ldots, n, \\
\frac{1}{nK} \left[ \begin{array}{ccccc} 
\varepsilon_{x,a}^1 & \cdots & \varepsilon_{x,a}^n \\
1 & \cdots & \varepsilon_{x,a}^n \\
\vdots & \vdots & \vdots \\
\varepsilon_{x,a}^n & \varepsilon_{x,a}^n & \cdots & 1
\end{array} \right], & a = 1, \ldots, n, \\
1 - \sum_{a=1}^{n} E_x^a, & a = n + 1
\end{cases}
\]  

for \( x = 1, \ldots, n \).

(iii) States: if \( (\alpha_i)_{i=1}^{n+1} \) is a decreasing and positive sequence then set

\[
|q_i\rangle = \sum_{i=1}^{n+1} \alpha_i |ii\rangle.
\]  

For (2.35), define two maps \( \tilde{V} : \ell_2^{n+1} \to \ell_1^n(\ell_\infty^{n+1}) \) and \( \tilde{W} : \ell_2^{n+1} \to B(H_{n+1}) \) as follows:

\[
\tilde{V}(e_k) = \begin{cases} 
0, & k = 1, \\
\frac{1}{n} \sum_{x=1}^{n} \sum_{a=1}^{x-1} \varepsilon_x \otimes \varepsilon_a \left|k\right\rangle \left\langle k\right|, & k = 2, \ldots, n + 1,
\end{cases}
\]  

and

\[
\tilde{W}(e_k) = \begin{cases} 
0, & k = 1, \\
\left|1\right\rangle \left\langle k\right|, & k = 2, \ldots, n + 1.
\end{cases}
\]
By [11, lemma 3.5], we get
\[ \| V \| : \ell^1 \rightarrow \ell^1 + \ell^\infty \| \leq C \sqrt{\log n} . \] (2.39)

Then by Chebyshev's inequality, with 'high probability' we can choose \{ e_i \} such that:
\[ \| V \| \leq C \sqrt{\log n} . \] Moreover, it is easy to see the map \( W : \ell^2 \rightarrow B(H_{n+1}) \) is a contraction, i.e. \( \| W \| \leq 1 \). Hence, by proposition 2.9, we have
\[
B_C(F) \leq \left\| \sum_{i=1}^{n+1} e_i \otimes e_i' \otimes F_i \right\|_{\ell^2(\ell^\infty) \otimes B(H_{n+1})} = \left\| V \otimes \bar{W} \left( \sum_{k=1}^{n+1} e_k \otimes e_k' \right) \right\|_{\ell^2(\ell^\infty) \otimes B(H_{n+1})} \leq C \sqrt{\log n} . \] (2.40)

For given \( \alpha \in (0, 1) \), let us consider \( |\phi_\alpha \rangle = \alpha |11 \rangle + \sum_{i=2}^{n+1} \sqrt{1 - \alpha^2/n} |ii \rangle \). It follows from the above result that \( B_Q(F) \geq \frac{1}{\alpha} \alpha \sqrt{1 - \alpha^2/n} \geq \sqrt{n} \). So, we have constructed explicitly a steering functional \( F \) such that \( LV(F) \geq \sqrt{n} \). Let us mention that this unbounded largest violation is obtained by a non maximally entangled state.

**Remark 2.13.** This construction is explicit but also probabilistic. It does not guarantee that a given functional will yield unbounded largest violation. It happens with high probability. Another natural result is:

**Larger steering violation by a non maximally entangled state.** Let \( \rho = |\psi_\alpha \rangle \langle \psi_\alpha | \) be the \( d \)-dimensional maximally entangled state, where \( |\psi_\alpha \rangle = \frac{1}{d} \sum_{i=1}^{d} |ii \rangle \). In [11, theorem 5.1] the authors provide an example of a Bell functional which gives Bell violations of order \( \sqrt{n \log n} \), but only bounded violations can be obtained by any maximally entangled state. It is not surprising that we have a similar conclusion in the steering scenario. The following notion is crucial [11, 13]: given two operator spaces \( E \) and \( F \), for any \( a \in E \otimes F \) we define its \( \psi - \) min norm:
\[
\| a \|_{\psi-\min} = \sup \| \langle \psi_\alpha | (u \otimes v)(a) |\psi_\alpha \rangle \| ,
\] (2.41)
where the supremum runs over all \( d \) and all complete contractions \( u : X \rightarrow B(H_d) \) and \( v : Y \rightarrow B(H_d) \).

The next lemma follows directly from proposition 2.9 (or see [12]).

**Lemma 2.14.** Given an element \( F = \sum_{i=1}^{n} \sum_{j=1}^{m} (e_i \otimes e_i') \otimes F_i \in \ell^2(\ell^m_n) \otimes B(H_d) \), we have:
\[
\sup_{\Theta_{\max} \in \mathcal{Q}_{\psi \alpha}} \left| \left( F, \Theta_{\max} \right) \right| \leq \| F \|_{\psi-\min} ,
\] (2.42)
where \( \mathcal{Q}_{\psi \alpha} = \{ (\sigma_j', E_j^m \otimes I_n) | \langle \psi_\alpha | \langle \sigma_j' | E_j^m \rangle \} : \{ E_j^m \} \}_{j=1}^{n} \) is a POVM, i.e., \( \mathcal{Q}_{\psi \alpha} \) is the set of all assemblages which are constructed by the \( d \)-dimensional maximally entangled state. Recall the following fact in [11, theorem 5.1]:
Fact 2.15. There are linear maps $S: \mathcal{R}_n \cap \mathcal{C}_n \to \ell_1^k(\ell_\infty^D)$ and $S^*:\ell_1^k(\ell_\infty^D) \to \mathcal{R}_n \cap \mathcal{C}_n$ such that

$$S^*S = \text{id}_{\ell_1^k}, \quad \|S^*\|_b \leq C, \quad \text{and} \quad \|S\|_b \leq C \sqrt{\log n},$$

(2.43)

where $k \leq 2^{D+n^2}$.

Similarly to the proof of theorem 2.11, we can construct an element: $F = (S \otimes W)(\sum_k e_k \otimes e_k) \in \ell_1^k(\ell_\infty^D) \otimes B(H_x)$ satisfying:

$$\|F\|_c \leq \sqrt{\log n} \quad \text{and} \quad \|F\|_{\min} \geq \sqrt{n}.$$ 

(2.44)

Moreover

$$\|F\|_{\psi_{\min}} = \left\| S \otimes W \left( \sum_k e_k \otimes e_k \right) \right\|_{\psi_{\min}} \leq \|S\|_c \|W\|_b \left\| \sum_k e_k \otimes e_k \right\|_{\psi_{\min}} \leq C \sqrt{\log n} \left\| \sum_k e_k \otimes e_k \right\|_{\psi_{\min}} \leq \sqrt{\log n}.$$ 

(2.45)

Now, we can represent $F$ as $F = \sum_{x, a} e_x \otimes e_x' \otimes F^a_x$, and define $\tilde{F}^a_x \in M_n$, such that the left-top $n \times n$ corner of $\tilde{F}^a_x$ is $F^a_x$ and other coefficients of $\tilde{F}^a_x$ are zeros. Equations (2.44) and (2.45) and lemma 2.14 lead to the conclusion that there exists a steering functional

$$\tilde{F} = \left\{ \tilde{F}^a_x \in B(H_{n+1}): x = 1, \ldots, 2^n, \quad a = 1, \ldots, n + 1 \right\},$$

(2.46)

such that:

(i) $LV(\tilde{F}) \geq \frac{\sqrt{n}}{\sqrt{\log n}}$

(ii) $\sup_{\Theta_{\max} \in \Theta_{\psi_{\max}}}(\tilde{F}^a_x, \Theta_{\max}) \leq \sqrt{\log n}$.

This steering functional $\tilde{F}$ has an unbounded largest violation of order $\frac{\sqrt{n}}{\sqrt{\log n}}$, but the unbounded largest violation can never be obtained by the maximally entangled state.

Remark 2.16. Since in the steering scenario we consider the assemblages instead of joint probabilities, the algebraic tensor product we consider is $\otimes_{\infty}^\ell$. But in some sense, $B(H_x)$ is more friendly compared to $\ell_1^k(\ell_\infty^D)$. That is the reason why we can easily apply Junge–Palazuelos’s approach [11] to the steering scenario.

2.4. Steering violation by partially entangled states including PPT states

Here we will consider the role of the partially entangled state in violation of steering inequalities. Recall that the $n$-dimensional pure partially entangled state is of the form $|\psi_p\rangle = \sum_i a_i |ii\rangle$, where $a_i > 0$ and $\sum_i a_i^2 = 1$. We have shown in subsection 2.4 that it is possible to obtain an unbounded largest violation of the order $\frac{\sqrt{n}}{\sqrt{\log n}}$ by means of a partially entangled state (for the steering functional given in (2.32)). For any state $\rho$ and a steering functional $F = (F^a_x)$ let us consider the quantum bound $B_{Q_p}(F)$ obtained by means of a partially entangled state $|\psi_p\rangle$, i.e.
\[ B_Q(F) = \sup \left\{ \sum_{x,a} \text{Tr} \left( F_x^a \text{Tr}_A\left( (E_x^a \otimes I)\rho\right) \right) : (E_x^a)_{a} \text{ are POVMs for any } x \right\}. \]

Given a partially entangled state \( |\psi_\alpha\rangle \) for any steering functional \( F = (F_x^a)_{x,a = 1, \ldots, n} \) we have

\[ B_{Q_{\psi_\alpha}}(F) = \sup \left\{ \sum_{x,a} \sum_{i,j} \alpha_i \alpha_j \langle \psi_\alpha | E_x^a | i \rangle \langle j | F_x^a | i \rangle | : E_x^a \text{ be POVMs} \right\} \]

\[ \leq \sum_{i,j} \alpha_i \alpha_j \sup \left\{ \sum_{x,a} \text{Tr} \left( E_x^a | i \rangle \langle j | \right) \text{Tr} \left( F_x^a | i \rangle \langle j | \right) | : E_x^a \right\} \]

\[ \leq \sum_{i,j} \alpha_i \alpha_j \left\| \sum_{x,a} E_x^a \otimes F_x^a \right\|_{\mathcal{L}(\mathcal{L}(\mathcal{E} \otimes \mathcal{M})}. \] (2.47)

The second inequality follows from the fact that: \( \text{Tr} \left( \cdot | i \rangle \langle j | \right) \in \mathcal{B}_X \), where \( \mathcal{B}_X \) denotes the unit ball in a norm space \( X \). Now, for any \( \lambda \in [0, 1] \) let us consider the following density matrix

\[ \rho_\lambda = (1 - \lambda) \frac{1}{n^2} + \lambda |\psi_\alpha\rangle \langle \psi_\alpha|. \]

By the preceding discussion, the quantum bound \( B_{Q_{\psi_\alpha}}(F) \), is bounded by

\[ \left( 1 - \lambda + \lambda \sum_{i,j} \alpha_i \alpha_j \right) \left\| \sum_{x,a} e_x \otimes e_a \otimes F_x^a \right\|_{\mathcal{L}(\mathcal{L}(\mathcal{E} \otimes \mathcal{M})}. \] (2.48)

Thus

\[ B_{Q_{\psi_\alpha}}(F) \leq \left( 1 - \lambda + \lambda \sum_{i,j} \alpha_i \alpha_j \right) B_C(F). \] (2.49)

In [22], the author presented a stronger version of the Peres conjecture: ‘PPT states can not violate steering inequalities, i.e., the assemblages obtained by measuring them always have LHS models’. The conjecture has been disproved in [16]. However, one can still ask whether PPT states can exhibit unbounded violation. In what follows we will consider two classes of PPT states, and show that they allow only bounded steering violation.

Firstly, let us consider PPT states among states \( \rho_\lambda \) for a given partially entangled state \( |\psi_\alpha\rangle \). The partial transpose of \( \rho_\lambda \) has the following eigenvalues
\[
\begin{cases}
\frac{1 - \lambda}{n^2} \pm \lambda \alpha_i \alpha_j & i \neq j, \\
\frac{1 - \lambda}{n^2} + \lambda \alpha_i^2 & i = 1, \ldots, n.
\end{cases}
\] (2.50)

Thus \( \rho \) is a PPT state if and only if \( \lambda \leq \min \{ \frac{1}{1 + n^2 \alpha_i \alpha_j} : i \neq j \} \). From equation (2.49), we have

\[
B_{Q_\rho} (F) \leq \left( 1 - \lambda + \lambda \sum_{i,j} \alpha_i \alpha_j \right) B_C (F)
\]

\[
\leq \left( 1 + \min \left\{ \frac{n - 1}{1 + n^2 \alpha_i \alpha_j} : i \neq j \right\} \right) B_C (F)
\]

\[
\leq \left( 1 + \frac{n - 1}{1 + n^2 \alpha} \frac{1 - \alpha^2}{n - 1} \right) B_C (F),
\] (2.51)

where \( \alpha = \max \{ \alpha_i \} \). It follows from \( \sum \alpha_i^2 = 1 \) that \( \alpha \geq \frac{1}{n} \). Thus \( B_{Q_\rho} (F) \leq B_C (F) \). Now we can make following conclusion: for PPT states \( \rho \),

\[ (1 - \lambda) \frac{1}{n^2} + \lambda |y_i \rangle \langle y_i|, \lambda \leq \min \{ \frac{1}{1 + n^2 \alpha_i \alpha_j} : i \neq j \}, \]

and any given steering functional \( F \), the quantum bound obtained by \( \rho \) is bounded by the LHS bound up to a universal constant.

In [5], Chruściński and Kossakowski introduced another class of PPT states. It is invariant under the maximal commutative subgroup of \( \mathbb{U}(n) \), and includes the previous isotropic states \( \rho \). Briefly speaking, they considered following two classes of PPT states.

(i) Isotropic-like states, \( \rho = \sum_{i=1}^n a_i |i \rangle \langle i| + \sum_{i \neq j=1}^n c_{ij} |i \rangle \langle j| \), \( (a_i)_{i,j} \geq 0, c_{ij} \geq 0, c_{ij} - |a_{ij}|^2 \geq 0, \sum_{i=1}^n a_{ii} = 1 \);

(ii) Werner-like states, \( \rho = \sum_{i=1}^n b_i |i \rangle \langle i| + \sum_{i \neq j=1}^n c_{ij} |i \rangle \langle j| \), \( (b_i)_{i,j} \geq 0, c_{ij} \geq 0, c_{ij} - |b_{ij}|^2 \geq 0, \sum_{i=1}^n b_{ii} = 1 \).

Now we will prove that for any steering functional the largest violation obtained by means of states from the above two classes of PPT states is bounded. To see this, we will use the following proposition, which is an analog of [18, theorem 2.1].

**Proposition 2.17.** Given an \( n \)-dimensional bipartite state \( \rho \in S_1^1 \otimes S_n^1 \) for any steering functional \( (F_{\Sigma})_{\Sigma,a=1,\ldots,n} \), we have

\[
B_{Q_\rho} (F) \leq \| \rho \|_{S_1^1 \otimes S_1^1} B_C (F).
\] (2.52)

**Proof.** The proof is more or less the same as the proof in [18]. By duality and proposition 2.9, we have:
Now it remains to calculate the projective norm of Isotropic-like state and Werner-like state. It can be proved that the norms of both states are bounded by 2. For instance, for any Isotropic-like state $\rho$,

\[
\sum_{x,a} \text{Tr} \left( F^a_x \text{Tr}_A (E^a_x \otimes I \rho) \right) = \sum_{x,a} \text{Tr} \left( E^a_x \otimes F^a_x \rho \right) \leq \sum_{x,a} E^a_x \otimes F^a_x = \| \rho \|_{S^1_{S^1}, S^1_{S^1}} \\
\leq \sum_{x,a} e^a_x \otimes e^a_x \otimes F^a_x = \| \rho \|_{S^1_{S^1}, S^1_{S^1}} \\
\leq B_C (F) \| \rho \|_{S^1_{S^1}, S^1_{S^1}}. \quad (2.53)
\]

Now it remains to calculate the projective norm of Isotropic-like state and Werner-like state. It can be proved that the norms of both states are bounded by 2. For instance, for any Isotropic-like state $\rho$,

\[
\| \rho \|_2 \leq \sum_{i,j=1}^n |a_{ij}| + \sum_{i \neq j}^n c_{ij} \leq 1 + \sum_{i \neq j} |a_{ij}| \leq 1 + \sqrt{\sum_{i \neq j} |c_{ij}|^2} \leq 1 + \left( \sum_{i \neq j} c_{ij} \right)^2 \leq 2. \quad (2.54)
\]

**Remark 2.18.** The PPT states described in [5] cover many PPT entangled states known in the literature, however they do not describe bound entangled states constructed via unextendible product bases (UPB) [2]. Unfortunately, up to now we can not estimate the projective norm of PPT states which are constructed by UPB.

### 3. Dichotomic case

In [23] the authors considered the dichotomic setting for the Bell scenario (see also [21]). It is more or less a reformulation of the standard setting for the Bell scenario with two outcomes. It turns out that in the steering scenario it is no longer the case: standard and dichotomic settings are not equivalent. The details of this phenomenon are discussed in [15]. Here we describe its particular exemplification by using the operator space technique.

In the dichotomic setting for the steering scenario we assume that the measurement for Alice has only two outcomes $\pm 1$. Alice prepares two correlated particles sharing a quantum state $\rho \in B(H_A \otimes H_B)$ and sends one of them to Bob. Alice wants to convince Bob that $\rho$ is an entangled state by doing a dichotomic measurement $-1 \leq E_x \leq 1$, $x = 1, \ldots, n$. After Alice’s measurement has been done, Bob obtains the conditional states

\[
\sigma_x = \text{Tr}_A \left( \rho (E_x \otimes I) \right). \quad (3.1)
\]

If the nature is described by a LHS model, then

\[
\sigma_x = \sum_{\lambda} \rho (\lambda) E_x (\lambda) \sigma_\lambda, \quad (3.2)
\]
where \( p(\lambda) \) is a probability distribution function, \( E_\lambda(\lambda) = \pm \) is the deterministic outcome obtained by Alice if she does the measurement \( E_\lambda \), and \( \sigma_\lambda \) is a density matrix of \( B(H_B) \).

For \( n = \dim(H_B) \) we define a steering functional as a set of \( n \times n \) matrices \( F_x, x = 1, \ldots, n \). Analogously to the standard case, we can define the LHS bound of \( F \) as:

\[
B_C(F) = \sup \left\{ \sum_x \text{Tr} \left( F_x \sigma_x \right) \mid \sigma_x \text{ satisfy (3.1)} \right\},
\]

(3.3)

and the quantum bound of \( F \) as:

\[
B_Q(F) = \sup \left\{ \sum_x \text{Tr} \left( F_x \sigma_x \right) \mid \sigma_x \text{ satisfy (3.2)} \right\}.
\]

(3.4)

One can also apply the argument of proposition 2.9 to obtain

\[
B_C(F) \simeq \| F \|_{\ell^m \otimes M_n} \quad \text{and} \quad B_Q(F) \simeq \| F \|_{\ell^m \otimes \min M_n}.
\]

Remark 3.1. In [23], the authors considered the Bell scenario in a dichotomic setting: for a Bell inequality \( M = \sum_{x,y=1}^n M_{x,y} \epsilon_x \otimes \epsilon_y \in \ell^m \otimes \ell^m \), the classical bound \( B_C(M) \) is equivalent to the norm \( \| M \|_{\ell^m \otimes \ell^m} \) and the quantum bound \( B_Q(M) \simeq \| M \|_{\ell^m \otimes \min \ell^m} \). By Grothendieck’s theorem [21], we have \( \ell^m \otimes \ell^m \simeq \ell^m \otimes \min \ell^m \). So for any bipartite dichotomic Bell functional \( M \), there always exists a universal constant \( K \) (not depend on the dimension), such that \( B_Q(M) \leq KB_C(M) \). They also proved that for the tripartite case, this universal constant does not exist!

The situation in the steering scenario differs from the features of the Bell scenario described in the above remark. Since \( \ell^m \otimes \ell^m \neq \ell^m \otimes \min \ell^m \), one should expect that there is a room for a steering functional with unbounded largest violation. Such a functional has been provided in [15]. Here we restate this result as the following theorem, and provide a proof referring to operator space formalism.

Theorem 3.2. For every \( n \in \mathbb{N} \), there exists a steering functional \( (F_i)_{x=1, \ldots, n} \in \ell^m \otimes M_n \) with unbounded largest violation of order \( \sqrt{\log n} \).

Proof. It is enough to prove the following claim: for every \( n \in \mathbb{N} \), there exists an element \( F = \sum_{x=1}^n e_x \otimes F_x \in \ell^m \otimes M_n \), such that

\[
\frac{\| F \|_{\min}}{\| F \|_{\ell^m}} \gtrsim \sqrt{\log n}.
\]

(3.5)

Since we know that \( \ell^m \otimes M_n = B(\ell^m, M_n) \) isometrically and \( \ell^m \otimes \min M_n = CB(\ell^m, M_n) \) completely isometrically, it is enough to prove that there exists a map \( \phi \) from \( \ell^m \) to \( M_n \), such that \( \| \phi \| \leq 1 \) and \( \| \phi \|_{\ell^m} \gtrsim \sqrt{\log n} \). Here we will use a fact described in [14]: there is a map \( \phi: \ell^m \to M_n \), such that \( \phi \) is bounded but not completely bounded. For the reader’s convenience, we just rewrite their proof. Let \( \sigma_i, i = x, y, z \) be Pauli matrices. Let
\[ A_1 = \sigma_1 \otimes 1 \otimes \ldots \otimes 1, \]
\[ A_2 = \sigma_2 \otimes \sigma_3 \otimes 1 \otimes \ldots \otimes 1, \]
\[ \vdots \]
\[ A_k = \sigma_1 \otimes \ldots \otimes \sigma_k \otimes \sigma_k \otimes 1 \otimes \ldots \otimes 1, \quad 3 \leq k \leq n. \tag{3.6} \]

It is easy to check that these \( A_i \in \mathcal{M}_2 \), \( i = 1, \ldots, n \) satisfies \( A_i = A_i^* \), \( A_iA_j + A_jA_i = 2\delta_{ij}1_{2^n} \).

For any \( A = \sum a_j A_j \in \mathcal{M}_{2^n} \), since \( A^* A + A A^* = 2 \sum |a_j|^2 1_{2^n} \), then \( \|A\| \leq \sqrt{2 \sum |a_j|^2} \). Now we define the map \( \varphi: \ell_\infty^n \rightarrow \mathcal{M}_{2^n} \) as \( \varphi(e_i) = \frac{1}{\sqrt{n}} A_i, \quad i = 1, \ldots, n \). Then

\[
\left\| \varphi \left( \sum a_i e_i \right) \right\| = \left\| \frac{1}{\sqrt{n}} \sum a_i A_i \right\| \leq \sqrt{\frac{2}{n} \left( \sum |a_j|^2 \right)} \leq \sqrt{2} \sup |a_j|. \tag{3.7}
\]

Thus \( \|\varphi\| \leq 1 \). On the other hand, we let \( \theta = \sum A_i \otimes e_i \in \mathcal{M}_{2^n} \otimes \ell_\infty^n \). Note that \( \|\theta\| = \sup_i \|A_i\| = 1 \), and by using the fact that \([14] \): there is a unit vector \( z \in \mathbb{C}^{2^n} \otimes \mathbb{C}^{2^n} \) such that \( (A_i \otimes A_i)(z) = z \) for any \( i \). Then

\[
\|\varphi\|_{cb} \geq \|1_{\mathcal{M}_{2^n}} \otimes \varphi(\theta)\| = \left\| \frac{1}{\sqrt{n}} \sum A_i \otimes A_i \right\| \geq \frac{1}{\sqrt{n}} \left( \sum (A_i \otimes A_i)(z), z \right) = \frac{1}{\sqrt{n}} \left( \sum \langle z, z \rangle \right) = \sqrt{n}. \tag{3.8}
\]

Since for every natural number \( n \geq 2 \), there exists a natural number \( m \), such that \( n \geq 2^m \), consider the diagram

\[
\ell_\infty^n \xrightarrow{\omega_1} \ell_{\infty}^m \xrightarrow{\varphi} \mathcal{M}_{2^n} \xrightarrow{\omega_2} \mathcal{M}_m, \tag{3.9}
\]

where \( \omega_1 \) projects \( \ell_\infty^n \) onto the first \( m \) coordinates, and \( \omega_2 \) embeds \( \mathcal{M}_{2^n} \) into the top \( 2^m \times 2^m \) corner of \( \mathcal{M}_m \). Set \( \Phi = \omega_2 \circ \varphi \circ \omega_1: \ell_{\infty}^m \rightarrow \mathcal{M}_m \), then

\[
\|\Phi\| = \|\varphi\| \leq 1; \quad \text{and} \quad \|\Phi\|_{cb} = \|\varphi\|_{cb} \geq \sqrt{m} \geq \sqrt{\log n}. \tag{3.10}
\]

Thus we can find a map \( \Phi: \ell_\infty^n \rightarrow \mathcal{M}_m \), such that \( \frac{\|\Phi\|_{cb}}{\|\Phi\|} \geq \sqrt{\log n} \). If \( F = \sum_{i=1}^n e_i \otimes \varphi(e_i) \), then \( F \) satisfies the statement of the theorem.

**Remark 3.3.** This result can also be traced back to the work of Paulsen [19] (or see [20, section 3.3]). For any norm space \( E \), he defined a constant

\[
\alpha(E) = \|\text{id}: \text{min}(E) \mapsto \text{max}(E)\|_{cb}, \tag{3.11}
\]

where \( \text{min}(E) \) (\( \text{max}(E) \)) is the minimal (maximal) admissible operator space structure of \( E \). It can be proved [19] that this constant is equal to \( \sup \{ \|T\|_{cb} \|T\| \leq 1, \quad T: \text{min}(E) \mapsto B(H) \} \). \( H \) is arbitrary. For \( E = \ell_\infty^n \), due to Loebl [14] and Haagerup’s [9] work, we have \( \alpha(\ell_\infty^n) \geq \sqrt{\frac{2}{n}} \) [19].

From this theorem, in the dichotomic case, the unbounded largest violation derives from some bounded but not completely bounded map. Now we will discuss for what kind of
steering functional $F_x$ we can always get a bounded largest violation. For any positive steering functional $(F_x)_{x=1, \ldots, n}$, i.e., $F_x \geq 0$ for every $x$, the LHS bound:

$$B_C(F) = \sup \left\{ \sum_x \operatorname{Tr} \left( F_x \sum_{\lambda} p(\lambda) E_x(\lambda) \sigma_x \right) \right\}$$

$$= \sup_{\lambda} \left\| \sum_x E_x(\lambda) F_x \right\|_{\mathcal{M}_n} \leq \left\| \sum_x F_x \right\|_{\mathcal{M}_n}.$$  \hspace{1cm} \text{(3.12)}

On the other hand,

$$B_C(F) \approx \left\| \sum_x e_x \otimes F_x \right\|_{\ell_1^\infty \otimes \mathcal{M}_n}$$

$$= \sup \left\{ \left\| \sum_x f(\epsilon_x) g(F_x) \right\| : f \in \mathcal{B}_{\ell_1^\infty}, g \in \mathcal{B}_{\ell_1^\infty} \right\} \geq \left\| \sum_x F_x \right\|_{\mathcal{M}_n}.$$  \hspace{1cm} \text{(3.13)}

Thus $B_C(F) \approx \left\| \sum_x F_x \right\|_{\mathcal{M}_n}$.

For the quantum bound, $B_{Q}(F) \approx \left\| \sum_x e_x \otimes F_x \right\|_{\ell^\infty_1 \otimes \mathcal{M}_n}$. It is known that [12, 20]

$$\left\| \sum_x e_x \otimes F_x \right\|_{\ell^\infty_1 \otimes \mathcal{M}_n} = \sup \left\{ \left\| \sum_x U_x \otimes F_x \right\|_{\mathcal{M}_n \otimes \mathcal{M}_n} : U_x \in \mathcal{M}_n, \ U_x U_x^* = U_x^* U_x = 1 \right\}$$

$$= \inf \left\{ \left\| \sum b_x^* b_x \right\|_{1} : \ \left\| \sum c_x^* c_x \right\|_{1} : \ F_x = b_x c_x \right\}.$$  \hspace{1cm} \text{(3.14)}

If $F_x$ is a positive matrix for every $x = 1, \ldots, n$, then by lemma 2 of [12], we know

$$\left\| \sum_x e_x \otimes F_x \right\|_{\ell^\infty_1 \otimes \mathcal{M}_n} = \left\| \sum_x F_x \right\|_{\mathcal{M}_n}.$$  \hspace{1cm} \text{(3.15)}

We end this section with the following remark.

**Remark 3.4.** If $F_x \geq 0$, then $\left\| \sum_x e_x \otimes F_x \right\|_{\ell^\infty_1 \otimes \mathcal{M}_n} \approx \left\| \sum_x e_x \otimes F_x \right\|_{\ell^\infty_1 \otimes \mathcal{M}_n}$. In other words, the quantum bound of positive dichotomic steering functional is always bounded by its LHS bound.

### 4. Conclusion

In this paper, we have used the operator space approach to study violation of steering inequality. Before, the approach had been successfully used in the Bell scenario [11, 12, 23]. In both cases, operator space was connected to the largest violation of the corresponding functional. The main difference is the algebraic tensor product considered in each case. In the Bell scenario, the algebraic tensor product is $\ell^m_n(\ell^m_k \otimes \ell^m_k)$, while in the steering scenario, it is $\ell^1_1(\ell^m_k \otimes \mathcal{B}(H_k))$. Since our work is an extension of applying the Junge–Pálazuelos approach to the steering scenario, we can easily construct a probabilistic steering functional with unbounded largest violation. And for this functional, a non maximally
entangled state will give a larger violation. However, not all properties of steering functionals can be recovered from the Bell scenario. We have shown in [15] a phenomenon characteristic only for the steering scenario, i.e., there is a bipartite dichotomic steering functional with unbounded largest violation. In this paper, we have studied this phenomenon in the framework of operator space theory.

Acknowledgments

We would like to thank Professor W A Majewski for valuable remarks and fruitful discussion. The work is supported by Foundation for Polish Science TEAM project co-financed by the EU European Regional Development Fund, Polish Ministry of Science and Higher Education Grant no. IdP2011 000361, ERC AdG grant QOLAPS and EC grant RAQUEL and a NCBiR-CHIST-ERA Project QUASAR. Part of this work was done at the National Quantum Information Center of Gdańsk. Part of this work was done when the authors attended the program Mathematical Challenges in Quantum Information at the Isaac Newton Institute for Mathematical Sciences, University of Cambridge.

References

[1] Bell J S 1964 On the Einstein–Podolsky–Rosen paradox Physics 1 195
[2] Bennett C H et al 1999 Unextendible product bases and bound entanglement Phys. Rev. Lett. 82 5385
[3] Bohm D 1951 Quantum Theory (New York: Courier Dover Publications)
[4] Cavalcanti E G, Jones S J, Wiseman H M and Reid M D 2009 Experimental criteria for steering and the Einstein–Podolsky–Rosen paradox Phys. Rev. A 80 032112
[5] Chrusciński D and Kossakowski A 2006 Class of positive partial transposition states Phys. Rev. A 74 022308
[6] Chen J L, Ye X J, Wu C F, Su H Y, Cabello A, Kwek L C and Oh C H 2013 All-versus-nothing proof of Einstein–Podolsky–Rosen steering Sci. Rep. 3 2143
[7] Einstein A, Podolsky B and Rosen N 1935 Can quantum mechanical description of physical reality be considered complete? Phys. Rev. 47 777
[8] Effros E and Ruan Z J 2000 Operator Spaces (Oxford: Oxford University Press)
[9] Haagerup U 1983 Injectivity and decompositions of completely bounded maps (Lecture Notes in Mathematics vol 1132) (Berlin: Springer) pp 170–222
[10] Hughston L P, Jozsa R and Wootters W K 1993 A complete classification of quantum ensembles having a given density matrix Phys. Lett. A 183 14–18
[11] Junge M and Palazuelos C 2011 Large violation of Bell inequalities with low entanglement Commun. Math. Phys. 306 695–746
[12] Junge M, Palazuelos C, Pérez-García D, Villanueva I and Wolf M M 2010 Unbounded violations of bipartite Bell inequalities via operator space theory Commun. Math. Phys. 300 715–39
[13] Junge M and Pisier G 1995 Bilnear forms on exact operator spaces and $\otimes_{\mathcal{B}}$ Geom. Functional Anal. 5 329–63
[14] Loebl R I 1976 A Hahn decomposition for linear maps Pac. J. Math. 65 119–33
[15] Marciniak M, Yin Z, Rutkowski A, Horodecki M and Horodecki R 2014 Unbounded violation of steering inequalities for binary output arXiv:1411.5994
[16] Moroder T, Gittsovich O, Huber M and Gühne O 2014 Steering bound entangled states: a counterexample to the stronger Peres conjecture Phys. Rev. Lett. 113 050404
[17] Navascués M and Pérez-García D 2012 Quantum steering and spacelike separation Phys. Rev. Lett. 109 160405
[18] Palazuelos C 2014 On the largest Bell violation attainable by a quantum state J. Funct. Anal. 267 1959–85
[19] Paulsen V 1992 Representation of function algebras, abstract operator spaces and Banach space geometry J. Funct. Anal. 109 113–29
[20] Pisier G 2003 Introduction to Operator Space Theory (Cambridge: Cambridge University Press)
[21] Pisier G 2012 Grothendieck’s theorem, past and present Bull. Am. Math. Soc. 49 237–323
[22] Pusey M F 2013 Negativity and steering: a stronger Peres conjecture Phys. Rev. A 88 032313
[23] Pérez-García D, Wolf M M, Palazuelos C, Villanueva I and Junge M 2008 Unbounded violation of tripartite Bell inequalities Commun. Math. Phys. 279 455–86
[24] Schrödinger E 1936 Probability relations between separated systems Proc. Camb. Phil. Soc. 32 446
[25] Saunders D J, Jones S J, Wiseman H M and Pryde G J 2010 Experimental EPR-steering using Bell-local states Nat. Phys. 6 845
[26] Smith D H et al 2012 Conclusive quantum steering with superconducting transition edge sensors Nat. Commun. 3 625
[27] Skrzypczyk P, Navascués M and Cavalcanti D 2014 Quantifying Einstein–Podolsky–Rosen steering Phys. Rev. Lett. 112 180404
[28] Takesaki M 1979 Theory of Operator Algebras I (New York: Springer)
[29] Werner R F 1989 Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model Phys. Rev. A 40 4277
[30] Wiseman H M, Jones S J and Doherty A C 2007 Steering, entanglement, nonlocality, and the EPR paradox Phys. Rev. Lett. 98 140402