Time delay and time advance in resonance theory

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Abstract

We propose a theory of the resonance–antiresonance scattering process which differs considerably from the classical one (the Breit–Wigner theory), which is commonly used in the phenomenological analysis. Here both resonances and antiresonances are described in terms of poles of the scattering amplitude: the resonances by poles in the first quadrant while the antiresonances by poles in the fourth quadrant of the complex angular momentum plane. The latter poles are produced by non–local potentials, which derive from the Pauli exchange forces acting among the nucleons or the quarks composing the colliding particles.

1 Introduction

The crucial assumption in the Breit–Wigner theory of resonances is the causality condition. In its simplest form, it can be formulated as follows: the outgoing wave cannot appear before the incoming wave has reached the scatterer. If one assumes that the interaction is spherically symmetric, linear, vanishing for $R > a$ ($a$ being the finite radius of the scatterer), then it is sufficient to apply this causality condition at the surface of the scatterer ($R = a$) in order to guarantee causal propagation in the whole outside region. Then, according to Eisenbud [1], one can estimate the time delay (advance) that the incident wave packet undergoes in the scattering process, by evaluating the derivative

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of the phase–shift with respect to the energy. To be more precise, we rapidly summarize the Wigner–Eisenbud analysis in the simplest case of the $s$–wave phase–shift $\delta_0(E)$.

Using standard notations, we represent incoming and outgoing wave–packets as [2]

$$\psi_{\text{inc}}(r, t) = \int_0^{+\infty} A(E) e^{-i(kr+E_t)} dE,$$  

$$\psi_{\text{out}}(r, t) = -\int_0^{+\infty} S(E) A(E) e^{i(kr-E_t)} dE = -\int_0^{+\infty} A(E) e^{i(kr+2\delta_0-E_t)} dE,$$

where $S(E) = \exp[2i\delta_0(E)]$. Let us suppose that $A(E)$ corresponds to a narrow energy spectrum, centered upon some energy $E_0$, so that it takes appreciable values only for

$$|E - E_0| \lesssim \Delta E \quad (\Delta E \ll E_0).$$

For large $|t|$ the integrands in Eqs. (1) are rapidly oscillating functions. Then the integrals in Eqs. (1) can be evaluated by means of the stationary phase method. Writing $A(E) = |A(E)| e^{i\alpha(E)}$, the stationary phase point in (1a) can be recovered from the relation

$$\frac{d\alpha(E)}{dE} - r \frac{dk}{dE} - t = 0.$$ 

Similarly, from (1b) we have:

$$\frac{d\alpha(E)}{dE} + r \frac{dk}{dE} + 2\frac{d\delta_0(E)}{dE} - t = 0.$$ 

From (3) and (4) we can evaluate the time delay (or advance) between the center of the incoming wave packet, which moves inward, and the center of the outgoing wave packet, which moves outward; we have:

$$\Delta t = 2 \left( \frac{d\delta_0(E)}{dE} \right)_{E=E_0}. $$

If we take for $\delta_0(E)$ the phase–shift due to the scattering by an impenetrable sphere: i.e., $\delta_0(k) = -ka$ ($a =$ radius of the sphere), and adopt appropriate
units which allow us to write $E = k^2/2$, then from (5) we obtain:

$$\Delta t = -\frac{2a}{k}.\quad (6)$$

It corresponds to a path difference $2a$ (from the surface to the center and back of a sphere of radius $a$), between a wave reflected at the surface of the sphere and a wave passing through its center. Correspondingly, an outgoing signal can appear at a time earlier than it would have been possible in absence of the scatterer.

Returning to formula (5), Wigner [3] has given a physical interpretation of this result in terms of causality. The incident wave packet can be captured by the scatterer and retained for an arbitrarily long time, so that there is no upper bound for the time delay. Conversely, causality does not allow an arbitrarily large negative delay (time advance). Classically, the maximum time advance allowed is that given by formula (6). Additional corrective terms of the type $1/k$, which are of the order of a wavelength, would arise from the wave nature of matter [2].

If we transfer the results of this analysis to the representation of amplitudes and cross-sections then we must take into account two types of contributions: a pole singularity (resonance contribution) and the so-called hard sphere scattering contribution. It turns out that the experimental phase-shift $\delta_0(E)$ can be reproduced by patching up two pieces: $\delta_0^{(\text{res})}(E)$ and $\delta_0^{(\text{pot})}(E)$ (see formulae (16) in Section 2). More specifically:

(i) **Resonance contribution**: $S^{(\text{res})}(E)|_{E \in U} = e^{2i\delta_0^{(\text{res})}(E)}$, where $U$ is a neighborhood of the resonance energy. This term represents the approximation of the scattering function $S(E)$, which is appropriate in a domain close to the resonance energy. The function $S(E)$, regarded as a function of $E$, presents a two-sheeted Riemann surface, corresponding to $k = (2E)^{1/2}$. In the first sheet $S(E)$ is regular, except for possible poles on the negative real axis, which correspond to the bound states energies. On the second (unphysical) sheet, $S(E)$ presents complex poles in the neighborhood of the resonances, which always arise in complex conjugate pairs (corresponding to $k$ and $-k^*$). One can regard the poles in the fourth quadrant of the complex $k$-plane as associated with decaying states, and the poles in the third quadrant as corresponding to capture states [2]. The square of the imaginary part of the location of these poles is related to the width of the resonance, and it is inversely proportional to its lifetime, then to the time delay.

(ii) **Hard-sphere scattering contribution**: The term $S^{(\text{pot})}(k) = e^{2i\delta_0^{(\text{pot})}(k)} = e^{-2ika}$ is related to the so-called potential scattering, which is responsible, according to formula (6), for the time advance (see also formulae (12)–(16) in
The above discussion is mainly qualitative, and other definitions of the time delay have been proposed; the interested reader is referred to Ref. [2] for an exhaustive review of these theories. Particularly significant is that proposed by Smith [4], and extended by Goldberger and Watson [5] (see also Ref. [6]). In this theory, instead of formula (5), one obtains the following one:

\[ \langle \Delta t \rangle = \left\langle 2 \frac{d\delta_0(E)}{dE} \right\rangle_{\text{in}}, \]  

(7)

where the right hand side denotes the expectation value of \( 2 \frac{d\delta_0(E)}{dE} \) in the initial state (incoming wave packet); this expectation value gives the average time delay due to the interaction. In particular, if the energy spectrum of the initial state is centered around \( E = E_0 \), and is sufficiently narrow, then one recovers formula (5) again. We can thus say that all these proposals can be viewed as variations on a single theme, the causality principle remaining the milestone. Therefore, hereafter we shall refer, for simplicity, to the Breit–Wigner theory only.

We now want to raise two issues:

(i) Can the time advance be evaluated in terms of hard–core scattering in those collisions between composite particles, when Pauli exchange forces enter the game?

(ii) Time delay and time advance are not described in a symmetrical way: the former is evaluated from the pole singularities of the scattering amplitude; the latter by the scattering from an impenetrable sphere. Then a clear–cut separation between these two terms is missing, and their interference cannot be easily controlled.

Hereafter we present a theory whose peculiar character consists in evaluating both time delay and time advance by means of scattering amplitude singularities (poles), which lie, respectively, in the first and fourth quadrant of the complex angular momentum (CAM) plane. In this type of approach to scattering phenomena, the resonances are grouped in families and lie on trajectories of the angular momentum, regarded as a function of the energy. Then, one has to face the problem of connecting the resonances belonging to an ordered family, the order being given by the value of the angular momentum: the point is that between two resonances of the same family there is the so–called echo of the resonance [7], or antiresonance, which is responsible for the time advance. Then a feasible description of a family of resonances should account also for the related antiresonances. Although the term antiresonance can be misleading since it could evoke the concept of antiparticle, and this is not the case indeed, we shall speak frequently of antiresonance in order to emphasize
its deep connection with the resonance and, in what follows, the terms echo of the resonance and antiresonance will be used interchangeably.

To be more specific, let us consider the collision between two clusters (like $\alpha-\alpha$ scattering) or two composite particles (like $\pi^+ - p$ scattering); when beyond a resonance we observe an antiresonance, the latter can be due to the Pauli exchange forces which arise whenever the interacting particles penetrate each other and the fermionic character of the components (i.e., nucleons or quarks) emerges. At this point, however, we strongly remark that there does not exist a one-to-one correspondence between time advance, compositeness of the interacting particles and Pauli exchange forces, as we shall explain with more details and examples in a remark in Section 3. Coming back to the Pauli exchange forces, we note that it is precisely the Pauli antisymmetrization which leads us to introduce non-local potentials \[8\], which depend on the angular momentum. Then performing for this enlarged class of potentials the analytic continuation of the partial scattering amplitudes from integer values of the angular momentum $\ell$ to complex-valued angular momenta $\lambda \in \mathbb{C}$, a peculiar feature, which is absent in the case of local Yukawian potentials, comes out: the scattering amplitude can have pole singularities both in the first and in the fourth quadrant of the CAM plane. The poles located in the first quadrant (i.e., $\text{Re} \lambda \geq 0$, $\text{Im} \lambda > 0$) correspond to resonances (i.e., unstable states) and the time delay is related to $\text{Im} \lambda > 0$; instead, the poles lying in the fourth quadrant ($\text{Re} \lambda \geq 0$, $\text{Im} \lambda < 0$) cannot be related in any way to unstable state: they correspond to the echoes of the resonances (i.e., the antiresonances). In this type of approach both time delay and time advance are described in terms of scattering amplitude singularities (poles), which, however, act at different values of energy: when the resonance pole is dominant, the pole corresponding to the echo can be neglected, and vice versa. Therefore, in the present theory we can answer the previous questions in the following sense:

(i') When clusters of particles or composite particles penetrate each other in the collisions producing antiresonances, and these latter are due to Pauli exchange forces, then the echoes of the resonances can be described by the use of pole singularities associated with non-local, angular momentum dependent potentials, which are generated by the Pauli exchange forces.

(ii') Time delay and time advance are treated symmetrically since both are described by the use of scattering amplitude poles which act at different values of the energy: in the range of energy where a resonance pole is dominant the effect of the corresponding antiresonance pole is negligible, and vice versa.

The literature on the concepts of resonance, time delay, and on the methods for extracting the resonance parameters from the experimental data is enormous, and it is growing up in recent years (see, for instance, Refs. \[9,10,11,12,13,14,15\] and the references therein). Among these many works, we limit ourselves to quote a quite recent approach which seems promising and consists in extract-
ing the resonance parameters from the energy distribution of the time delay [16,17].

The paper is organized as follows. In Section 2 we present some phenomenological examples which show the difficulties connected with the evaluation of the time delay when the standard approach is used. In Section 3 the non–local potentials are introduced, and we show how the echoes of the resonances can be explained by introducing the Pauli exchange forces which generate non–locality. In Section 4 the representation of resonances and antiresonances in the CAM plane is treated, and, finally, in Section 5 the theory is tested on the $\alpha–\alpha$ and $\pi^+–p$ elastic scattering.

2 Time delay and time advance in the resonance–antiresonance theory

Let $f_\ell = (e^{2i\delta_\ell} - 1)/(2i)$ denote the partial wave amplitude of angular momentum $\ell$, $\delta_\ell$ being the phase–shift, and assume that the condition of elastic unitarity holds true (see the remark below). If the partial wave $f_\ell$ is represented as a vector in the complex plane, causality requires that, as the energy increases, the vector will trace a circle in counterclockwise sense. The vector describes a semi–circle of radius $\frac{1}{2}$ and center $\frac{i}{2}$ as $\delta_\ell$ varies from 0 to $\frac{\pi}{2}$; when $\delta_\ell = \frac{\pi}{2}$, $f_\ell = i$. If, for increasing energy, $\delta_\ell$ tends to $\pi$, then this circle will close. In this case the width $\Gamma$ of the resonance can be determined as the difference in energy at the opposite ends of the diameter parallel to the real axis. It is clear that this method can work only if $\delta_\ell$ increases, for increasing energy, at least up to $\delta_\ell = \frac{3}{4}\pi$; in fact, to this value of $\delta_\ell$ there corresponds the partial wave $f_\ell = -\frac{1}{2} + \frac{i}{2}$, and the distance in energy between the points $(-\frac{1}{2} + \frac{i}{2})$ and $(\frac{1}{2} + \frac{i}{2})$ can be determined. If $\delta_\ell$ does not reach $\frac{3}{4}\pi$, the method cannot be applied. This is, for instance, the case of the phase–shift $\delta_\ell(\ell=2)$ generated in the $\alpha–\alpha$ elastic scattering (see Fig. 1). Let us focus on this example and plot in Fig. 2 the vector representing the partial wave $f_\ell(\ell=2)$ as a function of $E$. This vector, which describes a circular arc with radius $\frac{1}{2}$ and center $\frac{i}{2}$, moves in counterclockwise sense up to the energy $E \simeq 4.9$ MeV; correspondingly, $\delta_\ell(\ell=2)$ reaches its maximum value, $\delta_\ell(\ell=2) \simeq 2.01$. After, for increasing energy, $\delta_\ell(\ell=2)$ decreases, passing downward through $\frac{\pi}{2}$; consequently the plot of $f_\ell(\ell=2)$ describes the same circular arc, but now covered in clockwise sense. At $E \simeq 12.4$ MeV we have $\delta_\ell(\ell=2) = \frac{\pi}{2}$, and $f_\ell=2 = i$: we have the antiresonance. For $E > 12.4$ MeV, $\delta_\ell(\ell=2)$ keeps decreasing, approaching zero; accordingly, the vector representing $f_\ell(\ell=2)$ moves in clockwise sense toward the origin. From this phenomenological example it can manifestly be seen that the above method cannot be used to determine the width $\Gamma$, in this specific case.
Fig. 1. α–α elastic scattering. Experimental phase–shifts (dots) for the partial wave ℓ = 2 and corresponding fits vs. the center of mass energy E. The experimental data are taken from Refs. [39,40,41,42,43]. The solid line indicates the phase–shift computed by using the symmetrized form of formula (44), which takes into account both resonance and antiresonance terms. The dashed line shows the phase–shift computed by using only the resonance term (see formula (41)). The numerical values of the fitting parameters are (see Section 5): \( I = 0.76 \, \text{(MeV)}^{-1} \), \( \alpha_0 = 1.6 \), \( b_1 = 1.06 \times 10^{-1} \, \text{(MeV)}^{-1/2} \), \( a_1 = 1.03 \, \text{(MeV)}^{-1/4} \), \( g_0 = 0.72 \), \( g_1 = -7.5 \times 10^{-3} \, \text{(MeV)}^{-1} \), \( g_2 = 2.0 \times 10^{-5} \, \text{(MeV)}^{-2} \), \( E^* = 4.1 \, \text{MeV} \).

Remark. We want to stress the fact that our analysis is strongly focused on the pure elastic scattering region. This choice is motivated by the consideration that in order to understand more clearly the role of the antiresonances, it is convenient to isolate the echoes of the resonances from other effects related to inelasticity.

Let us now write the partial wave amplitude as \( f_\ell = e^{i\delta_\ell} \sin \delta_\ell = \frac{1}{\cot \delta_{\ell-1}} \), and expand \( \cot \delta_\ell(E) \) in Taylor’s series in the neighborhood of the resonance, whose center of mass energy is denoted by \( E_r \). For energies close to the resonance \( \delta_\ell \simeq \frac{\pi}{2} \), and \( \cot \delta_\ell(E) \simeq 0 \) (\( E \) denotes the total energy of the two particle state in the center of mass system). Then we have:

\[
\cot \delta_\ell(E) \simeq \cot \delta_\ell(E_r) + (E - E_r) \left[ \frac{d}{dE} \cot \delta_\ell(E) \right]_{E=E_r} \simeq -\frac{2}{\Gamma} (E - E_r), \tag{8}
\]

where we have defined \( \frac{2}{\Gamma} = \left( \frac{d\delta_\ell(E)}{dE} \right)_{E=E_r} \). Additional terms in the series can
Fig. 2. α–α elastic scattering. Argand plot for the partial wave $f_{\ell=2}$ as a function of the energy.

be neglected as far as $|E - E_r| \simeq E_r$, i.e., when the width of the resonance is small compared to its energy. Then we have:

$$f_\ell(E) = \frac{1}{\cot \delta_\ell(E) - i} = \frac{\frac{\Gamma_2}{2}}{(E_r - E) - i\Gamma/2}. \quad (9)$$

Let us now recall the expression of the total cross-section for elastic scattering:

$$\sigma_{\text{el}} = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1) \sin^2 \delta_\ell = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1) \left| \frac{e^{2i\delta_\ell} - \frac{1}{2}i}{2i} \right|^2. \quad (10)$$

Assuming that near the resonance all the phase-shifts $\delta_\ell$ are zero but one, from (9) and (10) we obtain:

$$\sigma_{\text{el}} = \frac{4\pi}{k^2} (2\ell + 1) \frac{\Gamma^2/4}{(E - E_r)^2 + \Gamma^2/4}. \quad (11)$$

which is the well-known Breit–Wigner formula for a resonant cross-section. The resonance curve $\sigma_{\text{el}}(E)$ is symmetric around $E = E_r$, and the width $\Gamma$ is defined such that the cross-section at $|E - E_r| = \frac{\Gamma}{2}$ is half of its maximum value.

But, in Fig. 3, where the plot of the total α–α elastic scattering cross-section
Fig. 3. $\alpha$–$\alpha$ elastic scattering. Comparison between the total cross-section $\sigma(E)$ (see Eq. (35)) computed by accounting for both the resonance and antiresonance terms (solid line) and that computed by using only the resonance term (dashed line).

against the center of mass system energy is shown, we clearly see that the two resonance peaks, which correspond to $\ell = 2$ and $\ell = 4$, are clearly asymmetric around their resonance energies $E_r^{(\ell=2)} = 3.23$ MeV and $E_r^{(\ell=4)} = 12.6$ MeV. These asymmetries are produced precisely by the antiresonances, which correspond to the downward passage of $\delta_{(\ell=2)}$ and $\delta_{(\ell=4)}$ through $\frac{\pi}{2}$. Indeed, the antiresonances do not produce sharp peaks but an asymmetric fall in the elastic total cross-section. A similar effect appears also in Fig. 4, which refers to the $\Delta(\frac{3}{2}^+, \frac{3}{2}^-)$ resonance in the $\pi^+\mathrm{p}$ elastic scattering.

In order to remedy this defect in the Breit–Wigner theory, to the resonant partial wave $f_\ell$ in (9) it is added the contribution produced by the so–called potential scattering, whose amplitude is the same as the one for the scattering from an impenetrable sphere: i.e., hard–core scattering. Thus, instead of formula (11), the data are fitted by using the following formula [18]:

$$\sigma_{el} = \frac{4\pi}{k^2} (2\ell + 1) \left| \frac{\Gamma/2}{(E_r - E) - i\Gamma/2} + f_\ell^{\text{pot}} \right|^2,$$

where $f_\ell^{\text{pot}}$ denotes the partial wave amplitude for the scattering from an
Fig. 4. $\pi^+\text{-}p$ elastic scattering: total cross–section as a function of the laboratory beam momentum. The experimental data (dots) are taken from Ref. [38]. The solid line indicates the total cross–section computed by taking into account the contributions of both the resonance and antiresonance poles generating $\delta_\ell^{(+)}$ (formula (51)). The dashed line shows the total cross–section computed by accounting only for the resonance poles generating $\delta_\ell^{(+)}$. The fitting parameters are (see (52)): $a_0 = 6.95 \times 10^{-1}, a_1 = 9.0 \times 10^{-7} \text{ (MeV)}^{-2}, b_1 = 1.0 \times 10^{-4} \text{ (MeV)}^{-1}, b_2 = 1.4 \times 10^{-7} \text{ (MeV)}^{-2}, c_0 = -0.5, c_1 = 5.0 \times 10^{-7} \text{ (MeV)}^{-2}, g_0 = 2.0 \times 10^{-6} \text{ (MeV)}^{-2}, g_1 = 5.0 \times 10^{-12} \text{ (MeV)}^{-4}$.

impenetrable sphere, and whose corresponding phase–shifts are given by [19]:

$$\delta_\ell^{\text{pot}} = \tan^{-1} \frac{J_{\ell+1/2}(ka)}{N_{\ell+1/2}(ka)},$$

(13)

where $a$ is the radius of the sphere, while $J_{\ell+1/2}(\cdot)$ and $N_{\ell+1/2}(\cdot)$ denote the Bessel and Neumann functions of index $(\ell + \frac{1}{2})$, respectively. The asymptotic formulae representing $J_{\ell+1/2}(ka)$ and $N_{\ell+1/2}(ka)$ for large values of $k$ read [20]:

$$J_{\ell+1/2}(ka) = \left( \frac{2}{\pi ka} \right)^{1/2} \cos \left[ ka - \left( \ell + \frac{1}{2} \right) \frac{\pi}{2} - \frac{\pi}{4} \right] + O \left( (ka)^{-3/2} \right),$$

(14a)

$$N_{\ell+1/2}(ka) = \left( \frac{2}{\pi ka} \right)^{1/2} \sin \left[ ka - \left( \ell + \frac{1}{2} \right) \frac{\pi}{2} - \frac{\pi}{4} \right] + O \left( (ka)^{-3/2} \right).$$

(14b)

Eqs. (13) and (14) yield the following asymptotic formula representing $\delta_\ell^{\text{pot}}(k)$
for large values of $k$:

$$\delta_{\ell}^{\text{pot}}(k) = -ka + \frac{\pi}{2} \ell + O \left( (ka)^{-3/2} \right). \tag{15}$$

From (15) it follows that for those values of $k$ such that $ka = (\ell + 1)\frac{\pi}{2}$, then $\delta_{\ell}^{\text{pot}}(k) \simeq -\frac{\pi}{2} \pmod{\pi}$, and we have an antiresonance.

In the classical theory presented so far the phase–shift $\delta_\ell(E)$ is then fitted by adding two contributions:

$$\delta_\ell^{\text{(res)}} = \tan^{-1} \left( \frac{\Gamma/2}{E_r - E} \right), \tag{16a}$$

$$\delta_\ell^{\text{(pot)}} = -ka + \frac{\pi}{2} \ell. \tag{16b}$$

the approximation (16a) being feasible for sufficiently narrow resonance, and formula (16b) holding for sufficiently high values of $ka$. However, these last two constraints are not sufficient to justify formula (12). In fact, the step from formula (10) to (12) is admissible only if in the energy range in which the resonance term is dominant, i.e., $E \simeq E_r$, the contribution due to the potential scattering is negligible, and also vice versa, when the potential scattering effect is dominant the resonance term has to be negligible.

Now, let us come back to formulae (8)–(11). They represent what could be called the orthodox Breit–Wigner formalism. If rigidly applied, these formulae would describe appropriately the phase–shifts only near the resonance energy. The threshold is not correctly described and neither is the high energy part. A correct description of the threshold can be achieved by using a phase–space corrected width rather than a constant one. This amounts to introduce an $\ell$ dependent threshold factor to guarantee that the cross–section and the phase–shifts go smoothly to zero at threshold, instead of approaching finite constants. Now, if the threshold is correctly described, nevertheless it can happen that on the high energy side the description is worse: the width keeps growing with energy. Then the high energy behavior can be phenomenologically corrected by introducing appropriate form factors, which account also for the fact that the interacting particles are not pointlike. These corrections are widely applied in the phenomenological analysis [10,21,22,23].

We shall follow another procedure: we still introduce the antiresonances; however, the echoes will not be described by $\delta_\ell^{\text{(pot)}}$ (as in formula (16b)) but by the contribution to $\delta_\ell$ due to a pole singularity. More precisely, and this is the relevant point in the theory we present, both contributions to $\delta_\ell$, due to the resonance and the antiresonance, are described in terms of pole singularities, lying, respectively, in the first and fourth quadrant of the CAM plane, but
acting at different values of the energy. In the neighborhood of a resonance the resonance pole is dominant and the effect of the antiresonance singularity can be neglected; conversely, at energies close to the antiresonance the corresponding pole produces a large contribution while the resonance pole is negligible.

3 Antiresonances and non–local potentials

Let us first consider the scattering from a local central potential with finite first and second moments. If we also assume that the potential does not have an s–wave bound state at zero energy, then the standard Levinson’s equality for the phase–shift \( \delta_\ell(k) \) reads as follows [24]:

\[
\delta_\ell(0) - \delta_\ell(\infty) = N_\ell \pi, \tag{17}
\]

where \( N_\ell \) is the number of \( \ell \)–wave bound states. In the case of non–local potentials, equality (17) must be modified in order to take into account also the spurious bound states, that is, bound states with positive energy [25]. For the sake of simplicity, and without loss of generality, we neglect this peculiar feature of the non–local potentials, and assume that the potential being considered does not present bound states embedded in the continuum. Therefore we maintain equality (17), which, rewritten in terms of the lifetime \( \tau_\ell(E) = 2 \frac{d\delta_\ell(E)}{dE} \), reads\(^1\):

\[
\frac{1}{2\pi} \int_0^{+\infty} \tau_\ell(E) \, dE + N_\ell = 0. \tag{18}
\]

Taking this formula for granted, we have a one–to–one correspondence between \( \tau_\ell(E) > 0 \) (time delay) and \( \tau_\ell(E) < 0 \) (time advance). In fact, suppose that \( N_\ell \) is a non–negative integer number, which depends only upon the discrete spectrum, then, in the continuum, to a positive time delay due to a resonance it should correspond a negative time delay, i.e., a time advance, due to an antiresonance, in a way such that the sum rule (18) be satisfied (see also Ref. [26]). But equalities (17) and (18) are not free from ambiguities: in general the value of \( \delta_\ell(\infty) \) is unknown and, moreover, although \( \delta_\ell(0) = 0 \) mod \( \pi \), no

\(^1\) In the proof of Levinson’s theorem one does not go from formula (17) to equality (18) but vice versa from (18) to (17). Indeed, formula (17) follows from the equality \( N_\ell = -\frac{1}{\pi} \mathrm{Im} \int_0^{+\infty} \frac{f_\ell(k)}{f_\ell(-k)} \, dk = -\frac{1}{\pi} \int_0^{+\infty} \left( \frac{d}{dk} \delta_\ell(k) \right) \, dk \), which coincides with formula (18). The functions \( f_\ell(k) \) are the so–called Jost functions [24], and are related to the phase–shifts \( \delta_\ell \) as follows: \( e^{2i\delta_\ell(k)} = \frac{f_\ell(k)}{f_\ell(-k)} \). Therefore formulae (17) and (18) hold true for the same class of potentials.
hints of which multiple of \( \pi \) we are referring to is available. In spite of these ambiguities the sum rule (18) suggests that, in the pure elastic scattering and in a given partial wave, resonances and antiresonances can be related in a one-to-one fashion. In the following we shall present a theory in which resonances and antiresonances are treated symmetrically.

With this program in mind, we start from the continuity equation, which reads:

\[
\frac{\partial w}{\partial t} + \nabla \cdot j = 2(\text{Im} V_{\text{eff}})w,
\]

(19)

where \( w = \chi^*\chi \) (\( \chi \) being the wavefunction of the system), \( j \) is the current density: \( j = i\{\chi \nabla \chi^* - \chi^* \nabla \chi\} \), \( V_{\text{eff}} \) is the sum of the potential and of the centrifugal barrier. If we consider a class of local potentials which admits a continuation of the partial waves \( f_\ell \) from integer values \( \ell \) to complex values \( \lambda \) of the angular momentum, then \( \text{Im} V_{\text{eff}} \) can acquire values different from zero, even if the potential is a real-valued function. In fact, from the centrifugal barrier we have a term of the form \( \text{Im} \left( \frac{\lambda(\lambda+1)}{2\mu R^2} \right) \), which is different from zero if \( \text{Im} \lambda \neq 0 \). When we consider a resonance state, then there is a continuous drain of probability from the region of interaction, in view of the leakage of the flux of particles due to the tunnelling across the centrifugal barrier; the probability of finding the scattered particle inside a given sphere decreases with time. The only possibility of keeping up with this loss of probability, as required by the continuity equation (19), is to introduce a source somewhere: the source must be proportional to \( \text{Im} \lambda \) in order to compensate the effect of tunnelling across the centrifugal barrier. The condition for the source to be emitting is just \( \text{Im} \lambda > 0 \) (see Ref. [24]). Later on we shall relate this term to an amplitude pole singularity located in the first quadrant of the CAM plane.

**Remark:** In the resonance interaction region the probability of finding the scattered particles inside a given sphere decreases with time. This decreasing behavior appears in the relation between the imaginary part of \( E \), \( \text{Im} E = -\frac{\Gamma}{2} \), and the resonance mean life \( \tau \); \( \tau \) is large if the probability leakage rate is small [24]. Since the probability of finding the scattered particles somewhere in space is time independent, the loss of probability must be compensated by a source, which is precisely provided by the centrifugal barrier. However, this loss of probability should not be confused with the connection between the time delay and the statistical density of states in scattering. In fact, as can be shown by the Beth–Uhlenbeck formula, the density of states increases with the interaction [27].

Now, the heart of the matter is how to describe the antiresonances. In view of the symmetry between resonances and antiresonances, as suggested by the
Levinson’s sum rule, we should expect a symmetric mechanism leading to an interpretation of the antiresonances in terms of sinks, instead of sources. But, if the class of potentials under consideration is restricted to the local ones, then the pole singularities of the amplitude lie in the first quadrant of the CAM plane (i.e., Im \( \lambda > 0 \)), and again we necessarily have emitting sources [24]. The only chance is enlarging the class of the admitted potentials, including the non–local ones, which depend upon the angular momentum. Now, the following question naturally arises: can this enlarged class of potentials really describe the physical process leading to antiresonances? With this in mind, let us return to analyze the behavior of \( \delta_\ell (\ell \text{ being fixed}) \); keep in mind, as a typical example, \( \delta_{\ell=2} \) in the \( \alpha-\alpha \) elastic scattering (see Fig. 1). First it crosses the value \( \delta_\ell = \frac{\pi}{2} \) with positive derivative, i.e., \( \frac{d\delta_\ell(E)}{dE} > 0 \), and, accordingly, we observe a resonance peak in the cross–section; next, at higher energy, but fixed \( \ell \simeq kR \) (\( k \) is the momentum and \( R \) is the interparticle distance), \( \delta_\ell \) will cross \( \frac{\pi}{2} \) downward (i.e., with negative derivative \( \frac{d\delta_\ell(E)}{dE} < 0 \)), and an antiresonance is produced. Therefore we observe antiresonances at a higher value of the momentum \( k \); it follows that the interparticle distance \( R \) decreases, and the composite structure of the interacting clusters comes out. In the specific example of the \( \alpha-\alpha \) elastic collision, we can say that, at the energy close to the antiresonances, the \( \alpha \)–particles can no longer be described simply as bosons, but the fermionic character of the nucleons emerges. From the Pauli’s principle, and the corresponding antisymmetrization, it derives a repulsive force, which explains the phase–shift decrease.

When the two clusters penetrate each other, Pauli exchange forces enter the game. The nucleon–nucleon interaction which accounts for the exchange, and which is used in the antisymmetrization process, is generally represented by a potential of Gaussian form: \( V_{p,q} \propto V_0 \exp(K|r_p - r_q|^2)(C(1 + P_{pq}) \), \( P_{pq} \) being the operator that exchanges the space coordinates of the \( p^{th} \) and the \( q^{th} \) nucleons, whose locations are represented by the vectors \( r_p \) and \( r_q \), respectively; \( K \) and \( C \) are constants. Then, various procedures in use, like the resonating group, the complex–generator coordinates and the cluster coordinate methods, lead to describe the interaction between clusters by means of an integro–differential equation of the following form:

\[
[-\Delta + V_d] \chi(R) + g \int_{\mathbb{R}^3} V(R, R') \chi(R') \, dR' = E \chi(R),
\]

where \( h = 2\mu = 1 \) (\( \mu \) is the reduced mass of the clusters), \( g \) is a real–valued coupling constant, \( E \), in the case of the scattering process, denotes the scattering relative kinetic energy of the two clusters in the center of mass system, \( \Delta \) is the relative motion kinetic energy operator, and, finally, \( V_d \) is the potential due to the direct forces. Now, we assume that \( V(R, R') \) is a real and symmetric function: \( V(R, R') = V^*(R, R') = V(R', R) \), and, moreover, that
both the nucleon–nucleon potentials and the wavefunctions are rotationally invariant. Then, \( V(\mathbf{R}, \mathbf{R'}) \) depends only on the lengths of the vectors \( \mathbf{R} \) and \( \mathbf{R'} \), and on the angle \( \gamma \) between them, or equivalently on the dimension of the triangle \((0, \mathbf{R}, \mathbf{R'})\), but not on its orientation. Hence, \( V(\mathbf{R}, \mathbf{R'}) \) can be formally expanded as follows:

\[
V(\mathbf{R}, \mathbf{R'}) = \frac{1}{4\pi R R'} \sum_{s=0}^{\infty} (2s + 1) V_s(R, R') P_s(\cos \gamma),
\]

(21)

where \( \cos \gamma = \frac{\mathbf{R} \cdot \mathbf{R'}}{R R'} \), \( R = |\mathbf{R}| \), \( P_s(\cdot) \) are the Legendre polynomials, and the Fourier–Legendre coefficients are given by:

\[
V_s(R, R') = \frac{4\pi R R'}{2\ell + 1} \int_{-1}^{1} V(R, R'; \cos \gamma) P_s(\cos \gamma) d(\cos \gamma).
\]

(22)

Next, we expand the relative motion wavefunction \( \chi(\mathbf{R}) \) in the form:

\[
\chi(\mathbf{R}) = \frac{1}{R} \sum_{\ell=0}^{\infty} \chi_\ell(R) P_\ell(\cos \theta),
\]

(23)

where \( \ell \) is now the relative angular momentum between the clusters. Since \( \gamma \) is the angle between the vectors \( \mathbf{R} \) and \( \mathbf{R'} \), whose directions are determined by the angles \((\theta, \phi)\) and \((\theta', \phi')\), respectively, we have: \( \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \). Then, by using the following addition formula for the Legendre polynomials:

\[
\int_{0}^{2\pi} \int_{0}^{\pi} P_s(\cos \gamma) P_\ell(\cos \theta') \sin \theta' d\theta' d\phi' = \frac{4\pi}{2\ell + 1} P_\ell(\cos \theta) \delta_{s\ell},
\]

(24)

from (20), (21), (23) and (24) we obtain:

\[
\chi''_\ell(R) + k^2 \chi_\ell(R) - \frac{\ell(\ell + 1)}{R^2} \chi_\ell = g \int_{0}^{R_R} V_\ell(R, R') \chi_\ell(R') dR',
\]

(25)

where \( k^2 = E \) (instead of \( k^2 = 2E \) as in formula (6)), in agreement with the position \( 2\mu = 1 \) (see Eq. (20)), and the local potential is now supposed to be included into the non–local one.

In order to illustrate how a sink, instead of a source, can be obtained by introducing a non–local potential, let us consider the following very simple model. Suppose that \( V(\mathbf{R}, \mathbf{R'}) \) can be factorized as follows: \( V(\mathbf{R}, \mathbf{R'}) = \)
\[ V_\ast(R)\delta(R - R')v(\cos \gamma), \] (\( \delta \) being the Dirac distribution). Then, from (22) the partial potentials \( V_s(R, R') \) become:

\[
V_s(R, R') = 4\pi RR'V_\ast(R)\delta(R - R') \int_{-1}^{1} v(\cos \gamma)P_s(\cos \gamma) d(\cos \gamma). \tag{26}
\]

Now, we suppose that \( v(\cos \gamma) = \frac{C}{c - \cos \gamma} \) (\( C, c \) constants, \( c > 1 \)); then, from (26) we have (see Ref. [29, p. 316, formula (17)]):

\[
V_s(R, R') = 8\pi CRR'V_\ast(R)\delta(R - R')Q_s(c), \tag{27}
\]

where \( Q_s(c) \) denotes the second kind Legendre function; now, inserting Eq. (27) into (25), and assuming suitable conditions of continuity for \( V_\ast(R) \), the integral in (25) reduces to \( V_l(R)\chi_l(R) = 8\pi CR^2V_\ast(R)Q_l(c)\chi_l(R) \). Next, \( Q_l(c) \) can be regarded as the restriction of the function \( Q(\lambda; c) \) (\( \lambda \in \mathbb{C} \)) to the values \( \lambda = 0, 1, 2, \ldots \). Moreover, \( Q(\lambda; c) \) is holomorphic in the half–plane \( \text{Re} \lambda > -1 \), and tends to zero uniformly in any fixed half–plane \( \text{Re} \lambda \geq \delta > 0 \), as \( |\lambda| \to \infty \). Therefore, in view of the Carlson theorem [28], it represents the unique Carlsonian interpolation of the sequence \( \{Q_l(c)\}_{l=0}^{\infty} \) i.e., the unique analytic continuation from integer physical angular momentum \( l \) to complex–valued angular momentum \( \lambda \). We have thus realized the unique analytic interpolation of the partial potentials \( V_l(R) \), given by \( V(\lambda, R; c) = 8\pi C_0 R^2V_\ast(R)Q(\lambda; c) \) (the constant \( C_0 \) including the coupling constant \( g \)).

Next, returning to (19), and using once again the standard arguments which lead to the continuity equation [30], we obtain:

\[
\frac{\partial w}{\partial t} + \nabla \cdot j = 2w(\text{Im} V_{\text{eff}}) = 2w \left[ \frac{\text{Im}[\lambda(\lambda + 1)]}{R^2} + \text{Im} V(\lambda, R; c) \right]. \tag{28}
\]

We may thus have contributions to \( \text{Im} V_{\text{eff}} \) which derive not only from the complexification of the centrifugal barrier but also from the potential itself. In other words, even if the potential is a real–valued function, nevertheless it can generate (positive or negative) contributions to \( \text{Im} V_{\text{eff}} \) in view of its angular momentum dependence.

One can then generalize\(^2\) the model illustrated above and find the conditions to impose on the partial potentials \( V_l(R, R') \) in order to obtain a unique Carlsonian interpolation \( V(\lambda; R, R') \). Accordingly, one can obtain a generalization of Eq. (28) which presents, in any case, a contribution to \( \text{Im} V_{\text{eff}} \) depending on \( V(\lambda; R, R') \) (\( \lambda \in \mathbb{C} \)).

\(^2\) We plan to present this mathematical study elsewhere.
In the case of local Yukawian potentials it can be proved [24] that the locations of the scattering amplitude singularities (poles) are necessarily restricted to the first quadrant: i.e., \( \text{Im } \lambda \geq 0 \). This is in agreement with the connection between resonances and sources associated to the centrifugal barrier, in the sense illustrated above. Conversely, in the case of non–local potentials the scattering amplitude singularities can lie in the first and in the fourth quadrant; further, in connection with the antiresonances, repulsive forces become active and the probability of finding the scattered particles within a given sphere increases with time, since an outgoing signal can appear at a time earlier than it would have been possible in the absence of the scatterer. Using arguments similar to those presented above for the analysis of the resonances, we can argue that the only way of keeping up with this increase of probability, as required by the continuity equation, is to introduce a sink: i.e., a term proportional to \( \text{Im } \lambda < 0 \). As we shall see in the next section, this corresponds to a pole singularity of the scattering amplitude in the fourth quadrant.

**Remarks.**

(i) It must be neatly distinguished the difference between the analytic interpolation of the partial potentials \( V_\ell \) and that of the partial waves \( a_\ell \) (which will be introduced in the next subsection). The Carlsonian continuation of the partial potentials, which we have illustrated above, generates a non–local potential holomorphic in \( \text{Re } \lambda > -\frac{1}{2} \) (recall, for instance, the properties of the second kind Legendre function \( Q(\lambda; c) \)). Instead, the partial waves \( a_\ell \) are the restriction to integers of a function \( a(\lambda; E) \) \( (\lambda \in \mathbb{C}, E \text{ fixed}) \) which, in the case of local potentials of the Yukawian class, is meromorphic in the half–plane \( \text{Re } \lambda > -\frac{1}{2} \) and holomorphic for \( \text{Re } \lambda > L - \frac{1}{2} \) \( (L \geq 0) \). In the case of non–local potentials the geometry of the analyticity domain of \( a(\lambda, E) \) changes; in particular, it can be proved that \( a(\lambda, E) \) \( (E \text{ fixed}) \) is meromorphic in an angular sector \( \Lambda \) contained in the half–plane \( \text{Re } \lambda > -\frac{1}{2} \). Resonances and antiresonances are precisely related to the singularities of \( a(\lambda, E) \) lying in the angular sector \( \Lambda \) and placed in the first and in the fourth quadrant, respectively.

(ii) In Ref. [8] we have studied the interaction between clusters of particles tied up by harmonic oscillators. The spectra associated with these potentials present degeneracies, which can be removed by the action of forces depending upon the angular momentum, like non–local potentials. Once these degeneracies have been removed, the spectra proper of the rotational bands come out. Indeed, when the two clusters penetrate each other, the fermionic character of the nucleons composing the clusters emerges and, accordingly, exchange forces of the Pauli type arise. A typical example is the \( \alpha–\alpha \) interaction dis-

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\(^3\) A more precise specification of the domain \( \Lambda \) requires a detailed mathematical analysis that we plan to publish in a mathematical physics journal.
cussed above, or also the $\pi^+-p$ collision where, instead of the nucleons, one should take into account the fermionic character of the quarks. In this connection it should be pointed out that, within the limits of the present analysis, meson–exchange forces or similar do not play here any role.

Finally, it is worth recalling that the minimization of functionals, in the sense of the Ritz variational calculus, containing a Hamiltonian composed by harmonic oscillators and exchange forces gives rise to an integro–differential equation of the form (20): i.e., a Schrödinger equation for non–local potentials.

(iii) In order to avoid any misleading interpretation of our analysis, we want strongly remark:

(a) the decreasing phase–shifts, like those present in the antiresonances, are not at all a proof of the non–elementarity of the interacting particles. Repulsion can occur, indeed, also for elementary systems. Therefore, a one–to–one correspondence between antiresonances, time advance, and compositeness of the interacting particles does not hold true.

(b) There exist echoes of resonances which can be explained and fitted by the hard–core scattering, instead of by non–local potentials. This is the case of orbiting resonances in molecular scattering (for details see Ref. [31]).

(c) In conclusion, the present model, which uses pole singularities in the fourth quadrant of the CAM plane for describing antiresonances and evaluating time advance, can be appropriate if and only if Pauli exchange forces, deriving from antisymmetrization, come in play.

4 Complex angular momentum representation of resonances and antiresonances

Two kinds of solutions to Eq. (25) must be distinguished: the scattering solutions $\chi^{(s)}(k, R)$, and the bound–state solutions $\chi^{(b)}(R)$:

(i) The scattering solutions satisfy the following conditions:

$$\chi^{(s)}(k, R) = kRj_\ell(kR) + \Phi_\ell(k, R), \quad (29a)$$

$$\Phi_\ell(k, 0) = 0, \quad \lim_{R \to +\infty} \left[ \frac{d}{dR} \Phi_\ell(k, R) - i k \Phi_\ell(k, R) \right] = 0, \quad (29b)$$

where $j_\ell(kR)$ are the spherical Bessel functions, and the functions $d\Phi_\ell/dR$ are supposed to be Lipschitz–continuous.

(ii) The bound–state solutions $\chi^{(b)}(R)$ satisfy the conditions:

$$\int_0^{+\infty} \left| \chi^{(b)}_\ell(R) \right|^2 dR < \infty, \quad \chi^{(b)}_\ell(0) = 0. \quad (30)$$
As in the case of local potentials, the asymptotic behavior of the scattering solution, for large values of $R$, can be compared with the asymptotic behavior of the free radial function $j_\ell(kR)$, and, correspondingly, the phase-shifts $\delta_\ell(k)$ and the scattering amplitudes $f_\ell(k) = e^{i\delta_\ell(k)} \sin \delta_\ell(k)$ can be defined. Now, as in the standard collision theory, we can introduce the total scattering amplitude, which, in view of the rotational invariance of the total Hamiltonian, can be formally expanded in terms of Legendre polynomials as follows:

$$f(E, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell(E) P_\ell(\cos \theta),$$

(31)

where $E$ is the center of mass energy, $\theta$ is the center of mass scattering angle, $\ell$ is the relative angular momentum of the colliding clusters, and the partial scattering amplitudes $a_\ell(E)$ are given by:

$$a_\ell(E) = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{f_\ell}{k} \quad (k^2 = E, 2\mu = h = 1).$$

(32)

If we restrict, with suitable assumptions, the class of the admitted non-local potentials, then we can perform a Watson-type resummation of expansion (31) which yield the following representation:

$$f(E, \theta) = \frac{i}{2} \int_C \frac{(2\lambda + 1)a(\lambda, E)P_\lambda(\cos \theta)}{\sin \pi \lambda} d\lambda + \sum_{n=1}^{N} g_n(E)P_{\lambda_n}(\cos \theta),$$

(33)

where the path $C$ (see Fig. 5) does not necessarily has to run parallel to the imaginary axis, as in the case of high energy physics applications. The terms $\lambda_n(E)$ give the location of the poles of the scattering amplitude which belong to the angular sector $\Lambda$ (see Fig. 5) and lying either in the first or in the fourth quadrant of the CAM plane; $g_n(E)$ denotes the residue of $(2\ell + 1)a_\ell(E)$ at the poles. First, we consider the poles $\lambda_n = \alpha_n(E) + i\beta_n(E)$ lying in the first quadrant and located within the angular sector $\Lambda$. Now, suppose that, at a certain energy and for a specific value $n_0$ of $n$, $\alpha_{n_0}$ crosses an integer, while $\beta_{n_0} \ll 1$: we have a pole dominance. We can thus try, in the neighborhood of a sharp and isolated resonance, the following approximation for the amplitude:

$$f(E, \theta) \simeq g(E) \frac{P_\lambda(\cos \theta)}{\sin \pi \lambda(E)},$$

(34)

where we have dropped, for simplicity, the subscript $n_0$. The Legendre function $P_\lambda(\cos \theta)$ presents a logarithmic singularity at $\theta = 0$; therefore approximation (34) breaks down forward, where it is needed to take into account also the contribution of the background integral, in order to make the amplitude
$f(E, \theta)$ finite and regular. Conversely, approximation (34) is satisfactory at backward angles. Further, let us note that the divergence of the right-hand side of formula (34) is of logarithmic type, then the total cross-section, evaluated by the following integral:

$$\sigma_{\text{tot}} = \frac{2\pi |g(E)|^2}{|\sin \pi \lambda(E)|^2} \int_0^{\pi} |P_{\lambda}(-\cos \theta)|^2 \sin \theta \, d\theta, \quad (35)$$

converges.

Let us now project the amplitude (34) on the $\ell$th partial wave by means of the following formula:

$$\frac{1}{2} \int_{-1}^{1} P_{\ell}(z) P_{\lambda}(-z) \, dz = \frac{\sin \pi \lambda}{\pi(\lambda - \ell)(\lambda + \ell + 1)} \quad (\ell = 0, 1, 2, \ldots; \lambda \in \mathbb{C}). \quad (36)$$

We obtain:

$$a_{\ell} = \frac{e^{2i\delta_{\ell}} - 1}{2ik} = \frac{g}{\pi (\alpha_r + i\beta_r - \ell)(\alpha_r + i\beta_r + \ell + 1)}, \quad (37)$$

where we write $\lambda = \alpha_r + i\beta_r$ to emphasize that we are now referring to resonances. Next, when the elastic unitarity condition can be applied, we get the following relationship among $g$, $\alpha_r$ and $\beta_r$:

$$g = -\frac{\pi}{k} \beta_r (2\alpha_r + 1), \quad (38)$$
which, finally, yields:

\[ \delta_\ell = \sin^{-1} \frac{\beta_r(2\alpha_r + 1)}{\left\{((\ell - \alpha_r)^2 + \beta_r^2)[(\ell + \alpha_r + 1)^2 + \beta_r^2]\right\}^{1/2}}. \] (39)

If the colliding particles are identical, the scattering amplitude must be symmetrized (or antisymmetrized). Therefore, instead of approximation (34), we must write

\[ f(E, \theta) \simeq \frac{g(E)}{\sin \pi \lambda(E)} \left[ \frac{P_\lambda(\cos \theta) \pm P_\lambda(-\cos \theta)}{2} \right] \quad (0 < \theta < \pi), \] (40)

and, consequently:

\[ \delta_\ell = \sin^{-1} \left\{ \frac{1 \pm (-1)^\ell}{2} \frac{\beta_r(2\alpha_r + 1)}{\left\{((\ell - \alpha_r)^2 + \beta_r^2)[(\ell + \alpha_r + 1)^2 + \beta_r^2]\right\}^{1/2}} \right\}. \] (41)

Expanding in Taylor’s series the term \( \lambda(E) = \alpha_r(E) + i\beta_r(E) \) in a neighborhood of the resonance energy, we can derive an estimate of the resonance width \( \Gamma_r(E) \):

\[ \Gamma_r(E) = \frac{2\beta_r}{\left( \frac{d\alpha_r}{dE} \right)^2 + \left( \frac{d\beta_r}{dE} \right)^2}. \] (42)

However, it must be stressed that \( \Gamma_r(E) \), computed at the resonance energy \( E_r \) (corresponding to \( \delta_\ell(E) = \pi \)), must not be identified with the width \( \Gamma \) of the observed cross-section resonance peak.

Proceeding exactly as in the case of the resonances, we then describe the antiresonances by using poles in the fourth quadrant of the CAM plane: \( \lambda(E) = \alpha_a(E) - i\beta_a(E) \) (\( \beta_a > 0 \)). In the neighborhood of an antiresonance we have:

\[ \delta_\ell = \sin^{-1} \left\{ \frac{1 \pm (-1)^\ell}{2} \frac{-\beta_a(2\alpha_a + 1)}{\left\{((\ell - \alpha_a)^2 + \beta_a^2)[(\ell + \alpha_a + 1)^2 + \beta_a^2]\right\}^{1/2}} \right\}. \] (43)

Adding the contribution of the poles lying in the first quadrant with those lying in the fourth one, we have:

\[ \delta_\ell = \sin^{-1} \left\{ \frac{1 \pm (-1)^\ell}{2} \frac{\beta_r(2\alpha_r + 1)}{\left\{((\ell - \alpha_r)^2 + \beta_r^2)[(\ell + \alpha_r + 1)^2 + \beta_r^2]\right\}^{1/2}} \right\}. \]
\[ + \sin^{-1} \left\{ \frac{1 \pm (-1)^\ell}{2} \left[ -\beta_a (2\alpha_a + 1) \right] \frac{1}{\sqrt{[(\ell - \alpha_a)^2 + \beta_a^2][(\ell + \alpha_a + 1)^2 + \beta_a^2]}} \right\}. \] (44)

At this point the main differences between the classical Breit–Wigner theory and the present one become clear.

(i) In the Breit–Wigner theory resonances and antiresonances are described by means of different mathematical tools: pole singularities for the resonances, and hard–core potential scattering for the antiresonances. The upper bound on the time advance is rigidly fixed by the radius of the hard–core.

(ii) The resonances are not grouped in families.

Instead, the peculiar features of the present theory are:

(i') Both resonances and antiresonances are described by pole singularities (respectively, in the first and in the fourth quadrant of the CAM plane), which act at different values of energy. Their effects can be separated in a rather neat way.

(ii') Resonances and antiresonances are grouped in families, and, accordingly, the fitting parameters are constrained by the evolution of the dynamical system.

Finally, we must note that unfortunately in both theories a precise definition and evaluation of the width \( \Gamma \) of the resonances turns out to be difficult. In the Breit–Wigner theory one introduces a parameter \( \text{ad hoc} \), like the radius of the hard–core, whose value can change with the energy. More generally, the width \( \Gamma \) is often regarded as a function of the energy, and various models are in use (see refs. [10,11,22]), as we have already explained in Section 2. In the present theory, an estimate of the antiresonance width, analogous to that given in formula (42), is not available. Then, in practice, one is naturally led to use statistical methods for evaluating the distortion of the bell–shaped symmetry of the cross–section resonance peak. This statistical analysis will be outlined at the end of the next section when the specific examples of \( \alpha-\alpha \) and \( \pi^+–p \) elastic scattering will be considered.

5 Phenomenological examples

The theory presented above can be tested on the \( \alpha-\alpha \) elastic scattering. In what follows we have chosen to fit the phase–shifts, instead of the differential cross–section, so that the action of the Coulomb interaction can be easily subtracted (see Ref. [32] for more details). Since the \( \alpha \)–particles are bosons, we use the symmetrized form of formula (44). For what concerns the functions \( \alpha_r(E), \beta_r(E), \alpha_a(E) \) and \( \beta_a(E) \), we adopt the following parametrization:
Fig. 6. $\alpha-\alpha$ elastic scattering. Experimental phase-shifts for the partial waves $\ell = 2$, $\ell = 4$, and corresponding fits accounting for both the resonance and antiresonance terms (solid lines). For the numerical details see the legend of Fig. 1.

\[
\begin{align*}
\alpha_r(E)[\alpha_r(E) + 1] &= 2IE + \alpha_0, \\
\beta_r(E) &= b_1E^{1/2}, \\
\alpha_a(E) &= a_1E^{1/4}, \\
\beta_a(E) &= g_0(1 - e^{-E/E^*}) + g_1E + g_2E^2,
\end{align*}
\]

The first equality (45a) is the quantum mechanical equation of the rotator: indeed, in first approximation, the system of two $\alpha$–particles can be viewed as a rotator whose moment of inertia is given by $I = \mu R^2$, $\mu$ being the reduced mass and $R$ the interparticle distance. Formula (45b) states for $\beta_r(E)$ a growth which is fast for low energy, but slower for higher energy, in agreement with the theory concerning the evolution of the resonances into surface waves for sufficiently high energy [33,34]. For what concerns $\beta_a$, the role of the exponential term is just to make, at low energy, a smooth, though rapid, transition of $\beta_a$ from zero to $g_0$. Unfortunately, a model which prescribes the growth properties of $\alpha_a(E)$ and $\beta_a(E)$ is, at present, missing; it would require a more refined theory able to describe the evolution toward semiclassical and classical phenomena. The fits of the phase–shifts are shown in Fig. 6 (see also Fig. 1), and the values of the fitting parameters are given in the figure legends.

Next, by using the formula:

\[
\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_{\ell},
\]
the total cross-section can be fitted, the fit being shown in Fig. 3. The resulting fits are quite satisfactory. Let us recall that formula (46) holds true only in the energy region where the scattering is purely elastic. In the case of $\alpha-\alpha$ collision this condition is satisfied up to the energy of 17.25 MeV [35]. In Figs. 1, 3 and 6 the bulk of the data lie within this region. We have added some more experimental data beyond this energy to illustrate the general trend of cross-section and phase-shifts at higher energy. Indeed, beyond $E = 17.25$ MeV formula (46) can only be regarded as a drastic approximation of the total cross-section, obtained by setting the inelasticity parameter $\eta_\ell = 1$, which amounts to identify the total cross-section with the elastic one.

Let us now consider the $\pi^+ - p$ elastic scattering. It presents a family of resonances whose $J^P$ values are: $\frac{3}{2}^+, \frac{7}{2}^+, \frac{11}{2}^+, \frac{15}{2}^+, \frac{19}{2}^+$. Furthermore, we also observe in the total cross-section a resonance with $J^P = \frac{1}{2}^-$. In this analysis we must take into account the spin of the proton, and accordingly write the spin–non–flip and the spin–flip amplitudes, which read, respectively:

\[
f(k, \theta) = \frac{1}{2i k} \sum_{\ell=0}^\infty \left[ (\ell + 1)(S^{(+)}_\ell - 1) + \ell(S^{(-)}_\ell - 1) \right] P_\ell(\cos \theta), \quad (47a)\]

\[
g(k, \theta) = \frac{1}{2k} \sum_{\ell=0}^\infty (S^{(+)}_\ell - S^{(-)}_\ell) P^{(1)}_\ell(\cos \theta), \quad (47b)\]

where $P^{(1)}_\ell(\cos \theta)$ is the associated Legendre function, and

\[
S^{(+)}_\ell = e^{2i \delta^{(+)}_\ell}, \quad (48a)\]

\[
S^{(-)}_\ell = e^{2i \delta^{(-)}_\ell}, \quad (48b)\]

$\delta^{(\pm)}_\ell$ being the phase–shift associated with the partial wave whose total angular momentum $J$ is equal to $\ell \pm \frac{1}{2}$. The differential cross-section is given by

\[
\frac{d\sigma}{d\Omega} = |f|^2 + |g|^2, \quad (49)\]

if the proton target is unpolarized, and if the Coulomb scattering is neglected, as it is admissible at sufficiently high energy. Next, integrating over the angles and taking into account the orthogonality of the spherical harmonics, the following expression for the total cross-section is obtained:

\[
\sigma_{\text{tot}} = \frac{2\pi}{k^2} \sum_{J,\ell} (2J + 1) \sin^2 \delta_{\ell,J}, \quad (50)\]

where $J = \ell \pm \frac{1}{2}$, $\delta_{\ell,J} = \delta_{\ell,\ell \pm 1/2}$, $\delta_{\ell,\ell+1/2} \equiv \delta^{(+)}_\ell$, $\delta_{\ell,\ell-1/2} \equiv \delta^{(-)}_\ell$. 

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Let us now focus on the resonance $\Delta \left( \frac{3}{2}, \frac{3}{2} \right)$, which is the first member of a family of even parity resonances. It is generated by the interaction of a $\pi^+$ meson with a proton: the relative angular momentum $\pi^+ - p$ is $\ell = 1$, and the intrinsic angular momentum of the quarks in the proton is $L = 0$. Then, taking into account the spin of the proton we have: $\delta^{(+)}_{\ell} = \delta_{\ell, \ell+1/2} = \delta_{1,3/2}$.

The successive members of this sequence of resonances correspond to an even rotational band of the proton states, in which the intrinsic angular momentum of the quarks in the proton is $L = 2^+, 4^+, 6^+, 8^+$ [36]. Then, the corresponding phase–shifts are $\delta^{(+)}_{\ell} = \delta_{\ell, \ell+1/2}$, $\ell$ odd. Accordingly, the antisymmetrized form of formula (44) must be used:

$$\delta^{(+)}_{\ell} = \sin^{-1} \left\{ \frac{1 - (-1)^{\ell}}{2} \frac{\beta_r (2\alpha_r + 1)}{\left\{ [(\ell - \alpha_r)^2 + \beta_r^2] (\ell + \alpha_r + 1)^2 + \beta_r^2 \right\}^{1/2}} \right\} + \sin^{-1} \left\{ \frac{1 - (-1)^{\ell}}{2} \frac{-\beta_a (2\alpha_a + 1)}{\left\{ [(\ell - \alpha_a)^2 + \beta_a^2] (\ell + \alpha_a + 1)^2 + \beta_a^2 \right\}^{1/2}} \right\}. \quad (51)$$

Analogously, the resonance $J^P = \frac{1}{2}^-$ is generated by the $\pi^+ - p$ interaction, in which the intrinsic angular momentum of the quarks in the proton is $L = 1$, and the angular momentum of the $\pi^+ - p$ system is $\ell = 1$ since the colliding pion is a $S$–wave; then for the phase–shift we have $\delta^{(-)}_{\ell} = \delta_{\ell, \ell-1/2} = \delta_{1,1/2}$. This resonance is the first member of a sequence corresponding to an odd rotational band of the proton states, in which the intrinsic angular momentum of the quarks in the proton is $L = 3^-, 5^-, \ldots$. However, only the first member of this sequence gives an observable effect to the total cross–section. Let us note that this resonance lies on a trajectory $\lambda(E)$ of the angular momentum which is different from that of the even parity family, and then it requires an additional pole for being described. However, in the following fits we will not account for this resonance since it lies outside the purely elastic region of interaction.

The functions $\alpha_r, \beta_r, \alpha_a$ and $\beta_a$ are parametrized as follows:

$$\alpha_r = a_0 + a_1 (E^2 - E_0^2), \quad (52a)$$
$$\beta_r = b_1 (E^2 - E_0^2)^{1/2} + b_2 (E^2 - E_0^2), \quad (52b)$$
$$\alpha_a = c_0 + c_1 (E^2 - E_0^2), \quad (52c)$$
$$\beta_a = g_0 (E^2 - E_0^2) + g_1 (E^2 - E_0^2)^2, \quad (52d)$$

where $E$ is the energy in the center of mass frame, and $E_0$ is the rest mass of the $\pi^+ - p$ system. Let us note that formula (52a) differs considerably from the corresponding formula (45a). In fact, the latter refers to the non–relativistic quantum rotator, while the former is in agreement with the phenomenological simple relationship between the total spin and the squared mass, which, using
Table 1

Table 1: \( \pi^+ - p \) elastic scattering. In the present work the resonance mass is defined as the energy of the upward \( \delta^+ \) crossing of the corresponding phase–shift \( \delta_i(E) \). The purely resonant width \( \Gamma_r = \Gamma_r(E)|_{E=E_r} \) (see Eq. (42)) indicates the width of the resonance peak computed without the antiresonance contribution, while the total width \( \Gamma \) stands for the width of the resonance peak accounting also for the antiresonance term. \( \Delta S_{\text{Res}} \) indicates the relative increase of skewness of the resonance peak when the antiresonance contribution is added to the pure resonant term; here \( S \) is evaluated by means of \( S_{\text{stat}} \) (see text).

| Name       | \( J^P \) | Mass [MeV] | Mass [MeV] | \( \Gamma_r \) [MeV] | \( \Gamma \) [MeV] | \( \Delta S_{\text{Res}} \) |
|------------|-----------|------------|------------|-----------------------|-------------------|-----------------------------|
| \( \Delta(1232) \) | \( \frac{3}{2}^+ \) | 1232.8     | 1230 – 1234 | 93                   | 115               | 115 – 125 4.1               |
| \( \Delta(1950) \) | \( \frac{7}{2}^+ \) | 1951       | 1940 – 1960 | 293                  | 290 – 350        | —                           |
| \( \Delta(2420) \) | \( \frac{11}{2}^+ \) | 2463       | 2300 – 2500 | 397                  | 300 – 500        | —                           |

standard notations, reads: \( J \equiv \alpha(m^2) = \alpha_0 + \alpha'm^2 \) (\( \alpha_0, \alpha' \) constants). Let us moreover observe that from our fit (see the legend of Fig. 4) we obtain \( a_1 = 0.9/(\text{GeV})^2 \), close to the phenomenological slope which is approximately \( \alpha' \sim 1/(\text{GeV})^2 \) [37].

Substituting the values \( \delta_{\ell}^{(+)} \) in formula (50) we can fit the total cross–section (the data are taken from Ref. [38]), the result being shown in Fig. 4 (see the legend for numerical details). It is very satisfactory, and shows with clear evidence the effect of the antiresonance corresponding to the resonance \( \Delta \left( \frac{3}{2}^+, \frac{3}{2}^+ \right) \). It emerges clearly the composite structure of the interacting particles: compare, indeed, the dashed line with the solid line in Fig. 4. The fit shown in Fig. 4 extends up to \( p \sim 0.8 \text{ GeV}/c \). At higher impulse the elastic unitarity condition is largely violated, and the fitting formula should be modified accordingly. The numerical values of the parameters in Eqs. (52) have been obtained by considering, in addition to the \( \Delta \left( \frac{3}{2}^+, \frac{3}{2}^+ \right) \), also the second member of the resonance family, i.e., the \( \frac{7}{2}^+ \) resonance. Although the latter lies outside the elastic interaction range of energy, its mass and width can be recovered from the purely elastic cross–section (see Ref. [38]) and then used to constrain the parameters in the fit of the even parity family of resonances. The numerical results of this procedure have been summarized in Table 1, where also the data concerning the \( \frac{11}{2}^+ \) resonance are shown.

With regard to the \( \Delta \left( \frac{3}{2}^+, \frac{3}{2}^+ \right) \) resonance, a simple semiclassical argument can support the interpretation of the distortion effect of the bell–shaped resonance peak in terms of the composite structure of the colliding particles. If we denote by \( R \) the distance between the pion and the proton, supposed at rest, then the impulse of the projectile, i.e., the pion, in the laboratory frame is \( p_{\pi}^{\text{LAB}} \sim \sqrt{2\hbar}/R \), since \( \ell = 1 \). Now, setting \( R \) as the distance at which the two particles get in contact (or, in other words, setting \( R \) as the interaction radius), which is of the order of the proton radius [23], then the corresponding \( p_{\pi}^{\text{LAB}} \) gives an
Fig. 7. $\pi^+\!-\!p$ elastic scattering: $\Delta(\frac{3}{2},\frac{3}{2})$ resonance. Comparison between the total cross-section computed with (solid line) and without (dashed line) the antiresonance term. $E_T$ indicates the energy at which the antiresonance effect is expected to set in (see text). For better comparison, the dashed curve has been slightly translated in order to have the peaks of the two curves coincident.

The above analysis suggests a procedure for evaluating the width $\Gamma$ of the resonance $\Delta(\frac{3}{2},\frac{3}{2})$. We note, first of all, that the effect of the antiresonance is a small perturbation to the pure resonance; this means that the reference baseline of the pure resonance distribution and that of the observed cross-section (comprising both resonance and antiresonance) coincide within a good approximation. Then, from the plot of the cross-section generated by only the pure resonant term we can recover the reference baseline of this almost symmetric bell-shaped distribution by equating its second central moment to the value of $\Gamma_r$ evaluated by means of formula (42). Next, keeping fixed this baseline, we evaluate the second central moment of the distribution which fits the experimental cross-section peak (i.e., accounting also for the antiresonant term). We can take as an estimate of the total width $\Gamma$ the value of this second moment. As a measure of the degree of asymmetry of the resonance...
peak, which can be ascribed to the composite structure of the interacting particles, we take the statistical skewness $S_{\text{stat}}$ of the distribution, defined as $S_{\text{stat}} = \mu_3 / \mu_2^{3/2}$, where $\mu_2$ and $\mu_3$ are the second and third central moments of the distribution (see Table 1).

If the asymmetry of the bell–shaped resonance peak is very large, as in the case of the $\alpha$–$\alpha$ elastic scattering (see Fig. 3), and the antiresonance effect cannot be regarded as a small perturbation to the pure resonance, then the baseline of the pure resonance peak and that of the observed experimental cross–section differ significantly. The statistical method illustrated above can no longer be applied, and we are forced to follow a more pragmatic attitude. Since the symmetry due to the antiresonance effect sets in just after the resonance maximum, we can regard $\Gamma$ as the half–width at half–maximum of the resonance peak, like in the Breit–Wigner theory. From the plot of the pure resonance cross–section we obtain a bell–shaped distribution whose full–width at half–maximum agrees with the value of $\Gamma_r$ evaluated by formula (42). Then we can give an estimate of the total width $\Gamma$ by evaluating the full–width at half–maximum of the curve fitting the experimental cross–section (including both resonance and antiresonance terms). In this case the asymmetry of the resonance peak can be estimated by means of a \textit{phenomenological skewness} $S_{\text{phen}}$, defined as follows: first we compute the difference between the two half–maximum semi–widths, measured with respect to the energy of resonance $E_r$; then we define the degree of asymmetry $S_{\text{phen}}$ as the ratio between this value and the full–width $\Gamma$.

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