A LARGE DEVIATION PRINCIPLE FOR LAST PASSAGE TIMES IN AN ASYMMETRIC BERNOULLI POTENTIAL

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Abstract. We prove a large deviation principle and give an expression for the rate function, for the last passage time in a Bernoulli environment. The model is exactly solvable and its invariant version satisfies a Burke-type property. Finally, we compute explicit limiting logarithmic moment generating functions for both the classical and the invariant models. The shape function of this model exhibits a flat edge in certain directions, and we also discuss the rate function and limiting log-moment generating functions in those directions.

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1. Introduction

1.1. Brief description of the model. We study large deviations for the last passage time in a Bernoulli environment. The original model was introduced in [48] as a simplified model of directed first passage percolation. In this model, the environment \( \eta = \{ \eta_v^{\kappa, \lambda} \}_{v \in \mathbb{Z}_+^2} \) is a...
collection of i.i.d. Bernoulli($p$) under a background measure $\mathbb{P}$ with marginals $\mathbb{P}\{\eta_v = \lambda\} = p = 1 - \mathbb{P}\{\eta_v = \kappa\}$, $\kappa > \lambda \in \mathbb{R}_+$, $v \in \mathbb{Z}_+^2$.

The set of admissible paths from $(0, 0)$ to $(m, n) \in \mathbb{Z}_+^2$ is denoted by $\Pi_{m,n}$ and it contains all paths of the form

$$\pi_{(0,0),(m,n)} = \{0 = v_0, v_1, \ldots, v_{m+n} = (m, n)\},$$

so that $v_{i+1} - v_i \in \mathcal{R} = \{e_1, e_2\}$. We say that $\mathcal{R}$ is the set of admissible steps. The random variable under consideration is the “first passage time”

$$L_{\kappa,\lambda,p}^{(0,0),(m,n)} = \inf_{\pi \in \Pi_{m,n}} \sum_{v_i \in \pi} V(T_{v_i} \eta, v_{i+1} - v_i),$$

where $T_v$ denotes the shift by $v \in \mathbb{Z}_+^2$ and $V : \Omega \times \mathcal{R} \to \mathbb{R}$ is the potential function given by

$$V(\eta, z) = \eta e_1 \mathbb{1}\{z = e_1\} + \tau_0 \mathbb{1}\{z = e_2\}.$$  

Value $\tau_0$ was constant and fixed from the beginning. The interest was to find the explicit shape function $\mu(s, t) = \lim_{n \to \infty} \frac{L_{\kappa,\lambda,p}^{(0,0),(|ns|,|nt|)}}{n}$.

The model can be mapped into a last passage directed percolation by two observations. First, because the admissible paths are directed the number of vertical increments $z = e_2 \in \mathbb{R}$ are fixed for any fixed endpoint $(m, n)$ (in fact they are $n$) and the cost for crossing them is deterministic $\tau_0$. Thus, for simplification $\tau_0$ can be set to be zero. Second, since $\lambda < \kappa$, to minimize $L_{\kappa,\lambda,p}^{(0,0),(m,n)}$ one should try and take horizontal steps $e_2 \in \mathcal{R}$ when the value of the environment at the target site is $\lambda$. Define new environment

$$\omega_v = \frac{1}{\kappa - \lambda} (\kappa - \eta_v) \sim \text{Ber}(p) \in \{0, 1\}.$$  

Then define the last passage time

$$G_{(0,0),(m,n)}^{V} = \max_{\pi_{(0,0),(m,n)} \in \Pi_{(0,0),(m,n)}} \left\{ \sum_{v_i \in \pi} V(T_{v_i} \omega, v_{i+1} - v_i) \right\}.$$  

The value of $G_{(0,0),(m,n)}^{V}$ gives the number of horizontal steps through environment $\omega_v = 1$, equivalently $\eta_v = \lambda$. Each of the remaining horizontal steps contributes $\kappa$ to $L_{\kappa,\lambda,p}^{(0,0),(m,n)}$ and therefore we have

$$L_{(0,0),(m,n)}^{V} = (\lambda - \kappa) G_{(0,0),(m,n)}^{V} + \kappa n + \tau_0 n.$$  

Therefore, for simplicity we study the last passage time $G_{(0,0),(m,n)}^{V}$ given by (1.2), in environment $\omega$ given (1.1), under potential $V$ given by

$$V(\omega, z) = \omega e_1 \mathbb{1}\{z = e_1\}.$$  

By (1.3) one can translate all results to $L_{(0,0),(m,n)}^{V}$. Now that $V$ is specified we omit it from the notation. We also omit $(0, 0)$ as the starting point, when it is implied. Therefore, the last passage time (1.2) is simply denoted by $G_{m,n}$. If the starting point is $(k, \ell)$ we write $G_{(k,\ell),(m,n)}$.
The law of large numbers for $G_{m,n}$ was first found in [48] by first obtaining invariant distributions for an embedded totally asymmetric particle system. Most recently the LLN was reproved in [6] using an invariant boundary model with sources and sinks. The same idea was utilised in the same article for the discrete version of Hammersley’s process [30], introduced in [47]. The theorem states

**Theorem 1.1** (The shape function for $G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}$ [6, 48]). Fix $p$ in $(0, 1)$ and $(s, t) \in \mathbb{R}_+^2$. Then we have the explicit law of large numbers limit

$$g_{pp}(s, t) = \lim_{N \to \infty} \frac{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}}{N} = \begin{cases} (\sqrt{ps} + \sqrt{(1-p)t})^2 - t, & t < s \frac{1-p}{p} \\ s, & t \geq s \frac{1-p}{p}. \end{cases}$$

(1.5)

This is a concave, symmetric, 1-homogeneous differentiable function which is continuous up to the boundaries of $\mathbb{R}_+^2$. Together with the shape function for the discrete Hammersley [47], are the first completely explicit shape functions for which strict concavity is not valid. In fact, the formula above indicates one flat edge, for $t > s \frac{1-p}{p}$. This simplified Bernoulli model was studied further in [28] where Tracy-Widom distributional limits were obtained for this and a generalised inhomogeneous version where the probability of success of the Bernoulli environment changes with the first coordinate of the site. Then the LLN was used for certain estimates in proving generalised properties of the shape functions of last passage percolation in [37].
Other models for which a flat edge of the shape function exists are common, and well studied. A flat edge for the contact process was observed in [18,19]. The discrete Hammersley model treated in [13,22,23,47] and the inhomogeneous model in [20] allow for an exact derivation of the limiting shape function and they also exhibit two flat edges. Large deviations for the latter were obtained in [21]. In the present article, we also study the behaviour of large deviations in directions for which the shape is flat for this classical Bernoulli model.

Historically, models of last passage percolation for which explicit invariant distributions could be obtained for the embedded particle system [1,45,47,48], satisfied a Burke-type property that led to invariant boundary corner growth models [3,6,9,10,13,29,33]. The same is true for this model, and we discuss the Burke-type property in Section 2. Boundaries can be constructed using sequences of i.i.d. Bernoulli and Geometric random variables, that play the role of Busemann functions for the last passage time model. We use the invariance granted by the boundaries to reprove Theorem 1.1 and verify the variational formula of [24].

Because of the asymmetry of the model and the dependence of $V$ on the path increment $z \in \mathcal{R}$, the general findings of [25,26] cannot be directly applied but the fact that invariant boundary models exist is a good indication that those theorems can hold more generally.

The Burke property guarantees enough analytical tractability to classify this as an exactly solvable model of the KPZ class [15]. Several well-studied models of last passage percolation and directed polymers exhibit this characteristic. Aside from the ones already mentioned above, there are also the continuum directed polymer studied in [2], the log-gamma polymer introduced in [50], the polymer in a Brownian environment with continuous-time random walk paths, discovered in [41], subsequently worked on by [38,39,51], the strict-weak gamma polymer studied in [16] and [40] and the random walk in Beta-distributed random potential [5]. The exactly solvable planar polymer models with two admissible steps were recently classified in [11]. Exactly solvable models which present environment inhomogeneity are for the corner growth model [20] and for totally asymmetric particle systems associated to growth models [8,36].

1.2. Large deviations. Large deviations rate functions for last passage times (for LPP) and partition functions (for directed polymers) have been computed in several cases when the model is exactly solvable. Below $G$ stands for a generic last passage time random variable. Define the upper (or right) and lower (or left) tale for the rate function as

$$\lim_{n \to \infty} -n^{-1} \log \mathbb{P}\{G_{n,n} \geq rn\} = J_u(r), \quad \lim_{n \to \infty} -n^{-2} \log \mathbb{P}\{G_{n,n} \leq rn\} = J_\ell(r),$$

A priori the existence of the limits is not even guaranteed, and it depends for example on the potential $V$ and the environment $\omega$ among other things. The existence of $J_u(r)$ and $J_\ell(r)$ was proved for the exponential and geometric corner growth model in $\mathbb{Z}^2$ [33]. An earlier work where the right-tail rate function is explicitly computed appeared in [46]. Existence of the rate functions was also known in the case of the Hammersley process. Its fluctuations in the large deviations regime were studied in [17], obtaining also precise results for the upper and lower exponential tails. An explicit right-tail rate function was computed in [49], using the invariant distributions for the particle system and studying deviations for the tagged particle. In the framework of particles systems, functional large deviation principle
for Totally Asymmetric Simple Exclusion Process (TASEP), which is closely connected to Exponential LPP, was obtained, for the \( n \)-speed tail in \( [32, 52] \) and for the \( n^2 \)-speed tail recently in \( [42] \).

Using the invariance structure offered by Burke’s property, a right-tail large deviation rate function with speed \( n \) for the partition function in the log-gamma polymer was proven \([27]\). Large deviations and KPZ fluctuations were computed for a random walk in a dynamic i.i.d. beta random environment in \([4]\). The idea of \([27]\) was later extended for the free energy in the O’Connell-Yor polymer by \([31]\), which is also a model with asymmetry like the one in this article. We utilise this techniques here, and we are also able to prove explicit limiting log-moment generating functions.

The approach for the existence of the right tail rate function is probabilistic in nature and utilizes super-additivity and the explicit expression is computed using probabilistic arguments. In general, the speed \( n^2 \) and the existence of lower-tail rate functions remained elusive, including for non-solvable models of last passage percolation, if one was to use only probabilistic techniques. In \([35]\) it was shown under a boundedness condition on the environment that the \( n^2 \) speed was correct, but with no existence of the rate function results. This was for first passage percolation models (FPP). FPP and LPP have the same qualitative behavior with the role of upper and lower tails reversed, an artefact of sub-additivity vs super-additivity. Existence of the \( n^2 \) speed rate function is proven in \([7]\) and the result is expected to extend for LPP with the same probabilistic approach. A variant of this result was earlier proved in \([12]\) for line-to-line first passage time.

1.3. Structure of the paper. The paper is organised as follows: In Section 2 we discuss the Burke property and the invariant version of the model.

In Section 3 we prove a full large deviation principle (LDP) for \( \lfloor Ns \rfloor, \lfloor Nt \rfloor \) at speed \( n \). General properties of the rate function are also proven, including that its Legendre dual is the limiting logarithmic moment generating function (l.m.g.f.) of \( \lfloor Ns \rfloor, \lfloor Nt \rfloor \) via Varadhan’s lemma. Existence of the full LDP is a direct consequence of the existence of a right-tail rate function. We prove an explicit variational formula for the right-tail rate function and its Legendre dual, that we ten proceed to explicitly solve and obtain a closed formula in Section 4. Finally, in Section 5 we prove an explicit expression of the limiting l.m.g.f. for the invariant boundary model.

The remaining part of the article are the Appendices, where we present variations of known results, that we needed tailored to our model. The last Appendix is a long computation needed in computing the explicit formulas. The goal was to make this article self-sufficient.

1.4. Commonly used notation. Throughout the paper, \( \mathbb{N} \) denotes the natural numbers, and \( \mathbb{Z}_+ \) the non-negative integers. Symbol \( G \) is always denoting a last passage time. As we already mentioned, the superscript \( V \) will be omitted as there is no confusion on the potential; in our case we always use \([1.4]\). Letter \( \pi \) signifies a generic admissible path.

Bold-face letters (e.g. \( v \)) indicate two-dimensional vectors (e.g. \( w = (w_1, w_2) \)). In the rare cases where we write \( v \leq w \) we mean the inequality holds coordinate-wise.
The Legendre (convex) dual of a function \( f : \mathbb{R} \rightarrow (-\infty, \infty] \) is defined as:

\[
f^* (y) = \sup_{x \in \mathbb{R}} \{ xy - f(x) \}.
\]

The statement \( f = f^{**} \) is used throughout the article without any special mention, and it is true if and only if \( f \) is convex and lower semicontinuous, which is why we pay particular attention into having the rate function lower-semicontinuous at the boundaries of their set that they are finite. Finally, in two occasions we need the infimal convolution of two generalised convex functions \( f, g \), and we write

\[
f \Box g (r) = \inf_{x \in \mathbb{R}} \{ f(x) + g(r - x) \}.
\]

The important fact is that \((f \Box g)^* = f^* + g^*\). We refer to [44] for the necessary convex analysis.

2. The Invariant Model

The boundary model is constructed defining different distribution of the weights on the two axes and they depends on a parameter \( u \in (p, 1] \). We can freely choose the value of this parameter according to its domain and different \( u \) values defines different boundary distributions. The weight at the origin remains unchanged respect to the original model and it is set to \( \omega_0 = 0 \). On the horizontal axis, for any \( k \in \mathbb{N} \) we set the weights \( \omega_{ke} \sim \text{Bernoulli}(u) \), with independent marginals

\[
\mathbb{P}\{\omega_{ke} = 1\} = u = 1 - \mathbb{P}\{\omega_{ke} = 0\}.
\]

On the vertical axis, for any \( k \in \mathbb{N} \), we set \( \omega_{ke} \sim \text{Geometric} \left( \frac{u-p}{u(1-p)} \right) \) with independent marginals

\[
\mathbb{P}\{\omega_{ke} = \ell\} = \frac{u-p}{u(1-p)} \left( \frac{p(1-u)}{u(1-p)} \right)^\ell, \quad \ell \in \mathbb{Z}_+.
\]

We do not alter the weights in the bulk \( \{\omega_w\}_{w \in \mathbb{N}^2} \). Therefore they have i.i.d. \( \text{Ber}(p) \) marginal distributions. We use the superscript \( \omega^{(u)} \) to highlight the fact that the distributions on the two axes are different than the ones in the bulk and they depend on \( u \). To sum up, for any \( i \geq 1, j \geq 1 \), the \( \omega^{(u)} \) marginals are independent with marginals

\[
\omega^{(u)}_{i,j} \sim \begin{cases} 
\text{Ber}(p), & \text{if } (i,j) \in \mathbb{N}^2, \\
\text{Ber}(u), & \text{if } i \in \mathbb{N}, j = 0, \\
\text{Geom} \left( \frac{u-p}{u(1-p)} \right), & \text{if } i = 0, j \in \mathbb{N}, \\
\delta_0, & \text{if } i = 0, j = 0.
\end{cases}
\]

If we consider any path \( \pi \) starting from 0, we observe from the previous display that the invariant model allows for the possibility to \( \pi \) to collect different weights along the two axes. In particular, if the path moves horizontally before entering the bulk, then it collects the Bernoulli(\( u \)) weights until it takes the first vertical step, and after that, it collects weight according to the potential function \( (1.4) \). If \( \pi \) moves vertically from 0 then it will collect the geometric weights on the vertical axis, and after it enters the bulk, it again collects according to \( V \). This is the only difference from the potential \( V \) of the i.i.d. model, namely while on the \( y \)-axis, the path can still collect positive weight.
Given a parameter \( u \in (p, 1] \) we define the last passage time for the invariant model from 0 to \( w \) as

\[
G_{0,w}^{(u)} = \max_{1 \leq k \leq w} \left\{ \sum_{i=1}^{k} \omega_{i}e_1 + G_{ke_1 + e_2,w} \right\}
\]

(2.4)

This formula comes from the variational equality using the above description. An explicit formula for the shape function for this model is given by the following theorem.

**Theorem 2.1.** [Law of large numbers for \( G_{[Ns],[Nt]}^{(u)} \)] For fixed parameter \( p < u \leq 1 \) and \( (s,t) \in \mathbb{R}^2_+ \) we have

\[
g_{pp}^{(u)}(s,t) = \lim_{N \to \infty} \frac{G_{[Ns],[Nt]}^{(u)}}{N} = su + t \frac{p(1-u)}{u-p}, \quad \mathbb{P} - a.s.
\]

(2.5)

It is convenient to introduce to passage times, depending on the first step of the set of paths we are optimizing over. Define

\[
G_{[Ns],[Nt]}^{(u),\text{hor}} = \max_{1 \leq k \leq [Ns]} \left\{ \sum_{i=1}^{k} \omega_{i,0} + G_{(k,1),([Ns],[Nt])} \right\}
\]

(2.6)

and

\[
G_{[Ns],[Nt]}^{(u),\text{ver}} = \max_{1 \leq \ell \leq [Nt]} \left\{ \sum_{j=1}^{\ell} \omega_{0,j} + \omega_{1,\ell} + G_{(1,\ell),([Ns],[Nt])} \right\}
\]

(2.7)

Then, by (2.4)

\[
G_{[Ns],[Nt]}^{(u)} = G_{[Ns],[Nt]}^{(u),\text{hor}} \lor G_{[Ns],[Nt]}^{(u),\text{ver}}.
\]

Passage times (2.6) and (2.7) satisfy a law of large numbers as well, given in the next

**Theorem 2.2.** Let \( s,t \geq 0, u \in (p, 1] \).

(a) The following limit exists and is given by

\[
g_{pp}^{(u),\text{hor}}(s,t) = \lim_{N \to \infty} N^{-1} G_{[Ns],[Nt]}^{(u),\text{hor}} = \begin{cases} g_{pp}^{(u)}(s,t) & \text{if } t < s \frac{(u-p)^2}{p(1-p)}, \\ g_{pp}(s,t) & \text{if } t \geq s \frac{(u-p)^2}{p(1-p)}, \end{cases}
\]

(2.9)

(b) The following limit exists and is given by

\[
g_{pp}^{(u),\text{ver}}(s,t) = \lim_{N \to \infty} N^{-1} G_{[Ns],[Nt]}^{(u),\text{ver}} = \begin{cases} g_{pp}^{(u)}(s,t) & \text{if } t > s \frac{(u-p)^2}{p(1-p)}, \\ g_{pp}(s,t) & \text{if } t \leq s \frac{(u-p)^2}{p(1-p)}. \end{cases}
\]

(2.10)
As is usual in the exactly solvable models of last passage percolation, there is the notion of a characteristic direction. In this case, for the model with boundaries for a given boundary parameter \( u \in (p, 1] \), there exists a unique direction \((m(N), n(N))\) whose scaled direction, as \( N \to \infty \), converges to the macroscopic characteristic direction

\[
N^{-1}(m_u(N), n_u(N)) \to \left(1, \frac{(u-p)^2}{p(1-p)}\right),
\]

which gives that for large enough \( N \) the endpoint \((m(N), n(N))\) is always below the critical line \( y = \frac{1-p}{p} x \) that separates the flat edge from the strictly concave part of \( g_{pp}(s,t) \) in Theorem 1.1. Here the characteristic direction already manifested in Theorem 2.2 as the cutting line between feeling the boundary effect versus entering the bulk.

We will prove Theorems 2.1 and 2.2 at the end of this section.

To simplify the notation in what follows, set \( w = (i,j) \in \mathbb{Z}_+^2 \) and define the last passage time gradients by

\[
I^{(u)}_{i+1,j} = G^{(u)}_{0,(i+1,j)} - G^{(u)}_{0,(i,j)}, \quad \text{and} \quad J^{(u)}_{i,j+1} = G^{(u)}_{0,(i,j+1)} - G^{(u)}_{0,(i,j)}.
\]

When there is no confusion we will drop the superscript \((u)\) from the \( I, J \) notation. When \( j = 0 \) we have that \( \{I^{(u)}_{i,0}\}_{i \in \mathbb{N}} \) are i.i.d. Bernoulli\((u)\) random variables since \( I^{(u)}_{i,0} = \omega_{i,0} \). When \( i = 0 \), \( \{J^{(u)}_{0,j}\}_{j \in \mathbb{N}} \) are i.i.d. Geometric\((\frac{u-p}{u(1-p)})\) random variables. There are three recursive equations satisfied in this model, which we summarize below.

**Lemma 2.3.** Let \( u \in (p, 1] \) and \((i,j) \in \mathbb{N}^2\). Then the last passage time can be recursively computed as

\[
G^{(u)}_{0,(i,j)} = \max \{G^{(u)}_{0,(i,j-1)}, G^{(u)}_{0,(i-1,j)} + \omega_{i,j} \}.
\]

Furthermore, the last passage time gradients satisfy the recursive equations

\[
I^{(u)}_{i,j} = \max\{I^{(u)}_{i,j-1} - J^{(u)}_{i-1,j}, \omega_{i,j}\},
\]

\[
J^{(u)}_{i,j} = (J^{(u)}_{i-1,j} - I^{(u)}_{i,j-1} + \omega_{i,j})^+.
\]

**Proof.** Equation (2.13) follows from the model description in the introduction. Note that the same recursive equation is actually satisfied for the model without boundaries. We indicatively prove (2.14) for the \( J \) and the other one is obtained in a similar way.

\[
J^{(u)}_{i,j} = G^{(u)}_{0,(i,j)} - G^{(u)}_{0,(i,j-1)}
\]

\[
= \max\{G^{(u)}_{0,(i,j-1)}, G^{(u)}_{0,(i-1,j)} + \omega_{i,j}\} - G^{(u)}_{0,(i,j-1)} \quad \text{by (2.13)},
\]

\[
= \max\{G^{(u)}_{0,(i,j-1)} - G^{(u)}_{0,(i-1,j)}, G^{(u)}_{0,(i,j-1)} - G^{(u)}_{0,(i,j-1)} + \omega_{i,j}\}
\]

\[
= \max\{0, G^{(u)}_{0,(i,j-1)} - G^{(u)}_{0,(i-1,j)} + G^{(u)}_{0,(i-1,j-1)} - G^{(u)}_{0,(i-1,j-1)} + \omega_{i,j}\}
\]

\[
= (J^{(u)}_{i-1,j} - I^{(u)}_{i,j-1} + \omega_{i,j})^+.
\]

\( \square \)
Using the gradients (2.14) and the environment \( \{ \omega_{i,j} \}_{(i,j) \in \mathbb{N}^2} \) we also define new random variables \( \alpha_{i,j} \) on \( \mathbb{Z}_2^+ \)
(2.15) \[ \alpha_{i-1,j-1} = \min \{ I_{i-1,j-1}^{(u)}, J_{i-1,j}^{(u)} + \omega_{i,j} \} \quad \text{for } (i,j) \in \mathbb{N}^2. \]
Since the \( I_{i,j}^{(u)} \) are Bernoulli, so are the \( \alpha_{i,j} \). The following lemma gives the joint distribution of the vector \( (I_{i,j}^{(u)}, J_{i,j}^{(u)}, \alpha_{i-1,j-1}) \). This is It is the analogue of Burke’s property that is a common aspect of many solvable models.

**Lemma 2.4** (Burke’s property). Let independent random variables be distributed by
(2.16) \[ (I_{i,j-1}^{(u)}, J_{i-1,j}^{(u)}, \omega_{i,j}) \sim \left( \text{Ber}(u), \text{Geom} \left( \frac{u-p}{u(1-p)} \right), \text{Ber}(p) \right), \]
where we assume \( u > p \). Then, for \( (i,j) \in \mathbb{N}^2 \), the vector obtained via equations (2.14), (2.15) is a vector of mutually independent marginals, with the same distributions, i.e.
(2.17) \[ (I_{i,j}^{(u)}, J_{i,j}^{(u)}, \alpha_{i-1,j-1}) \sim \left( \text{Ber}(u), \text{Geom} \left( \frac{u-p}{u(1-p)} \right), \text{Ber}(p) \right). \]

We first present a series of key technical lemmas, and we encourage the reader familiar with these techniques to proceed to the proof of Proposition 4.1.

A down-right path \( \psi \) on the lattice \( \mathbb{Z}_2^+ \) is an ordered sequence of sites \( \{ v_i \}_{i \in \mathbb{Z}} \) that satisfy
\[ v_i - v_{i-1} \in \{ e_1, -e_2 \}. \]
For a given down-right path \( \psi \), define \( \psi_i = v_i - v_{i-1} \) to be the \( i \)-th edge of the path and set
\[ L_{\psi_i} = \begin{cases} I_{\psi_i}^{(u)}, & \text{if } \psi_i = e_1 \\ J_{\psi_i-1}^{(u)}, & \text{if } \psi_i = -e_2. \end{cases} \]
Also define the interior sites \( I_\psi \) of \( \psi \) to be
\[ I_\psi = \{ w \in \mathbb{Z}_2^+ : \exists v_i \in \psi \text{ s.t. } w < v_i \text{ coordinate-wise} \}. \]
A convenient way to state Lemma 2.4 is the following.

**Corollary 2.5.** Fix a down-right path \( \psi \). Then the random variables
(2.18) \[ \{ \{ \alpha_w \}_{w \in I_\psi}, \{ L_{\psi_i} \}_{i \in \mathbb{Z}} \} \]
are mutually independent, with marginals
\[ \alpha_w \sim \text{Ber}(p), \quad L_{\psi_i} \sim \begin{cases} \text{Ber}(u), & \text{if } \psi_i = e_1 \\ \text{Geom} \left( \frac{u-p}{u(1-p)} \right), & \text{if } \psi_i = -e_2. \end{cases} \]

**Theorem 2.6** (Variational formula for the LLN of the non boundary model). Fix \( p \) in \( (0,1) \) and \( (s,t) \in \mathbb{R}_2^+ \). Then we have the explicit law of large numbers limit
\[ g_{pp}(s,t) = \inf_{p<u \leq 1} \left\{ s \mathbb{E}(I^{(u)}) + t \mathbb{E}(J^{(u)}) \right\} = \inf_{p<u \leq 1} g_{pp}^{(u)}(s,t). \]
**Remark 2.7.** To see the characteristic direction manifesting in a different way, start from
the formula in Theorem 2.6:

\[ g_{pp}(s, t) = \inf_{p < u \leq 1} \{ s \mathbb{E}(I^{(u)}) + t \mathbb{E}(J^{(u)}) \} = \inf_{p < u \leq 1} g_{pp}^{(u)}(s, t). \]

This can be immediately seen from (B.4) and the fact that \( g_{pp}^{(u)}(s, t) = sg_{pp}(1, ts^{-1}) \) for example. Without loss set \( s = 1 \). Then the \( u^* \) that minimizes the expression above is \( u^* = p + \sqrt{tp(1-p)} \) if \( t < q/p \) and 1 otherwise. Assume \( t < q/p \). Solve the expression for \( t \) we obtain

\[ t = \frac{(u^* - p)^2}{p(1 - p)}. \]

In other words, \( g_{pp}(1, t) = g_{pp}^{(u^*)}(1, t) \) and direction \((1, t)\) is characteristic according to (2.11) for the boundary model with parameter \( u^* \). Note that the range of characteristic directions only covers the directions for which \( g_{pp}^{(u)}(s, t) \) is strictly concave. The flat edge of \( g_{pp} \) corresponds to \( u^* = 1 \).

**Remark 2.8.** Along the characteristic direction the last passage time at point \( N(m, n) \) it is expected to have variance of order \( O(N^{2/3}) \) for large \( N \), while in the other directions the fluctuations of \( G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \) to have order of magnitude \( N^{1/2} \) and they are asymptotically Gaussian. Finally it is possible to prove using similar arguments as in [13] that the order of the variance in the flat edge is \( o(1) \).

From these considerations, we expect that the large deviations, for the boundary model, to be ‘unusual’ in the characteristic direction, while in the off-characteristic directions to be the typical decay of order \( e^{-N} \) for both tails. We can show that the right tail has deviations of order \( e^{-cN} \), but conditional on one of the boundaries being absent. This is essentially equation (5.12). In Lemma 3.5 we give a bound on the left tail that indicates superexponential decay when we move along direction (2.11) for the boundary model.

**Proof of Theorem 2.7.** From equations (2.12) we may write the last passage time of the invariant model as

\[ G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}^{(u)} = \sum_{j=1}^{\lfloor Ns \rfloor} I_{1,0}^{(u)} + \sum_{j=1}^{\lfloor Nt \rfloor} J_{\lfloor Ns \rfloor, j}^{(u)} \]

where the \( I, J \) variables are respectively the horizontal and vertical increments of the passage time. By the definition of the boundary model, the \( I \) variables are i.i.d. Ber(\( u \)). Scaled by \( N \), the first sum converges to \( s\mathbb{E}(I_{1,0}) \) by the law of large numbers.

By Corollary 2.5 the \( J \) variables are i.i.d. Geom(\( \frac{u-p}{u(1-p)} \)), since they belong on the downright path that goes from \((0, \lfloor Nt \rfloor)\) horizontally to \((\lfloor Ns \rfloor, \lfloor Nt \rfloor)\) and then vertically down to \((\lfloor Ns \rfloor, 0)\). At this point we cannot immediately evoke the law of large numbers as before since the whole sequence (and therefore the whole second sum) changes with \( N \). To show that this does not alter the law of large numbers limit, we first appeal to the Borel-Cantelli...
lemma via a large deviation estimate. Fix an $\varepsilon > 0$.

$$\mathbb{P}\left\{ \frac{1}{N-1} \sum_{j=1}^{\lfloor Nt \rfloor} J_{[Nk],j}^{(u)} \notin \left( \frac{p(1-u)}{u-p} - \varepsilon, \frac{p(1-u)}{u-p} + \varepsilon \right) \right\}$$

$$= \mathbb{P}\left\{ \frac{1}{N-1} \sum_{j=1}^{\lfloor Nt \rfloor} J_{[0],j}^{(u)} \notin \left( t \frac{p(1-u)}{u-p} - \varepsilon, t \frac{p(1-u)}{u-p} + \varepsilon \right) \right\}$$

$$\leq e^{-c(u,p,t,\varepsilon)N},$$

for some proper positive constant $c(u,p,t,\varepsilon)$. Then for each $\varepsilon > 0$ there exists a random $N_\varepsilon$ so that for all $N > N_\varepsilon$

$$t \frac{p(1-u)}{u-p} - \varepsilon < \frac{1}{N-1} \sum_{j=1}^{\lfloor Nt \rfloor} J_{[Nk],j}^{(u)} \leq t \frac{p(1-u)}{u-p} + \varepsilon,$$

from the Borel-Cantelli lemma. Then we have

$$su + t \frac{p(1-u)}{u-p} - \varepsilon \leq \lim_{N \to \infty} G^{(u)}(u) \frac{1}{N} \leq \frac{1}{N} \leq \lim_{N \to \infty} \frac{G^{(u)}(u)}{N} \leq su + t \frac{p(1-u)}{u-p} + \varepsilon.$$

The proof is complete when $\varepsilon$ is sent to 0.

**Proof of Theorem 2.2**. By definition (2.6) and (1.5) we have

$$g^{(u), \text{hor}}_{pp}(s,t) = \lim_{N \to \infty} N^{-1} G_{[Ns],[Nt]}^{(u), \text{hor}}$$

$$= \lim_{N \to \infty} \max_{1 \leq k \leq [Nk]} \left\{ \frac{1}{N-1} \sum_{i=1}^{k} I_{[0],i}^{(u)} + N^{-1} G_{[Nk],[Nt]}^{(u)} \right\}$$

$$= \sup_{0 \leq a \leq s} \{ au + g_{pp}(s-a,t) \}.$$

The last line follows by the same coarse graining arguments as in the proof of Theorem 2.6 (see Appendix B).

If $t < \frac{1}{p} s$

$$g_{pp}^{(u), \text{hor}}(s,t) = \sup_{0 \leq a \leq s - \frac{pt}{1-p}} \{ au + (\sqrt{p(s-a)} + \sqrt{(1-p)t})^2 - t \} \lor \sup_{s - \frac{pt}{1-p} < a \leq s} \{ a(u-1) + s \}.$$

The second supremum is attained at the boundary point $s - \frac{pt}{1-p}$ since it optimizes a decreasing function of $a$. In the first supremum, a unique minimizing point exists and it is either a boundary point or the critical point $a^*$ of the derivative of $f(a) = au + (\sqrt{p(s-a)} + \sqrt{(1-p)t})^2 - t$, given by

$$a^* = s - \frac{p(1-p)t}{(u-p)^2}.$$
If $s - \frac{p(1-p)t}{u-p} < 0$ then we have that $a^* = 0$. Otherwise, note that $a^*$ we can substitute $a^*$ into $f(a)$ and obtain

$$f(a^*) = su + \frac{p(1-u)}{u-p}t = g_{pp}^{(u)}(s,t).$$

Finally, if $t \geq \frac{1-p}{p}s$

$$g_{pp}^{(u)}(s,t) = \sup_{0 \leq a \leq s} \{au + s - a\} = s = g_{pp}(s,t).$$

The proof for $g_{pp}^{(u,\text{ver})}(s,t)$ is similar and left to the reader. 

3. I.i.d. Model: Full LDP

We first focus on the model without boundaries. Recall that the maximal path can collect Bernoulli weights only when it takes a step to the right. The full rate function is described in Theorem 3.4. As it is usually the case with models of last passage percolation, large deviations of the passage time above its mean are of different exponential scale than the deviations below its mean. With this in mind, in order to obtain a full LDP, one only needs the right-tail rate function. This is our beginning point.

Suppose that the target point is $(s,t)$, then, since the last passage time collects Bernoulli weights only through the right step, (3.1) implies that the probability

$$\mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \geq Nr\} \neq 0 \text{ if and only if } r \leq s.$$  

In the particular case where $s$ is rational, the probability above can be strictly positive for certain values of $N$, but otherwise it is 0.

**Theorem 3.1.** For $((s,t),r)$ with $0 \leq r < s < \infty$ and $t \in \mathbb{R}^+$, the following $\mathbb{R}^+$-valued limit exists:

$$-\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \geq Nr\} = J_{s,t}(r).$$

$J_{s,t}(r)$, as a function of $((s,t),r)$ is a continuous convex function on the interior of the set $A = \{(s,t),r) : s \geq r \vee 0, t \in \mathbb{R}^+, r \in \mathbb{R}^+\}$. It can be uniquely extended to a finite continuous convex function on the closure $\bar{A}$. Moreover, $J_{s,t}(r) > 0$ for $r > g_{pp}(s,t)$.

**Remark 3.2.** From this point onwards, we assume that $J_{s,t}(r)$ will be the function with domain $\bar{A}$, to have a generalised lower-semicontinuous convex function on $\mathbb{R}^3$. The only thing one needs to keep in mind is that when we are discussing boundary values, the limit representation (3.1) is no longer available to us. This does not affect the results that follow.

We present the proof of this basic result in Appendix C. The methodology utilises the super-additivity of $G$ and follows the proof steps of [46].

The continuous extension up to $\bar{A}$ makes the function $J_{s,t}(r)$ lower-semicontinuous on $\mathbb{R}^3$ where it takes the value $\infty$ outside of $\bar{A}$.

It will be useful to also know some of the boundary values of the lower semi-continuous extension. We summarise the results in the following corollary:
Figure 2. Graphical representation of the function $J_{s,t}(r)$. In both figures we used $p = 1/2$ and $t = 1$. To the left we have the lower-semicontinuous version of $J_{s,1}(r)$ as a function of $(s,r)$. You can see that it is finite for $s \leq r$.

To the right is the function $J_{2,1}(r)$.

Corollary 3.3. The lower-semicontinuous extension of $J_{s,t}(r)$ takes the following values on $\partial A$

$$J_{s,t}(r) = \begin{cases} 0, & t = 0, r \leq 0, s \in \mathbb{R}_+, \\ 0, & t \in \mathbb{R}_+, r \leq 0, s = 0, \\ sI_B^{(p)}(r/s), & t = 0, 0 \leq r \leq s, s \in \mathbb{R}_+ \\ \lim_{r \nearrow s} J_{s,t}(r), & t, s \in \mathbb{R}_+, r = s. \end{cases}$$

This follows from the fact that $J_{s,t}$ needs to be lower-semicontinuous, and it is briefly discussed after the proof of Theorem 3.1 in the Appendix. Above we defined $I_B$ to be the Cramér rate function for sums of i.i.d. $\omega_i \sim \text{Bernoulli}(p)$,

$$I_B^{(p)}(r) = \begin{cases} -\lim_{N \to \infty} N^{-1} \log \mathbb{P}\left\{ \sum_{i=1}^N \omega_i \geq Nr \right\} = r \log \frac{r}{p} + (1 - r) \log \frac{1 - r}{1 - p}, & r \in [p, 1], \\ 0, & r < p \\ \infty, & r > 1. \end{cases}$$

Finally, we show the existence of a good rate function $I_{s,t}(r)$ and list its properties; this is the content of the next theorem. We restrict $r \in [0, s]$ because $I_{s,t}(r) = \infty$ for any $r$ outside this interval.

Theorem 3.4. Let $\omega_{i,j} \sim \text{Bernoulli}(p)$ with $i,j \geq 1$ and $(s,t) \in (0, \infty)^2$. Then there exists a generalised function $I_{s,t}(r)$ so that the distributions of $N^{-1}G_{[Ns],[Nt]}$ satisfy an LDP with normalisation $N$ and rate function $I_{s,t}(r)$. To be precise, the following bounds hold for any open set $H$ and any closed set $F$ in $[0, s]$:

$$\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{ N^{-1}G_{[Ns],[Nt]} \in F \} \leq - \inf_{r \in F} I_{s,t}(r)$$
and
\[(3.6) \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{N^{-1}G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \in H\} \geq -\inf_{r \in H} I_{s,t}(r).\]

The rate function \(I_{s,t}(r)\) is defined by
\[(3.7) I_{s,t}(r) = \begin{cases} J_{s,t}(r), & r \in [g_{pp}(s,t), s], \\ \infty, & \text{otherwise.} \end{cases}\]

Rate function \(J_{s,t}(r)\) is the right-tail rate function computed in Theorem \(3.1\). In particular, on \([g_{pp}(s,t), s]\) the rate function \(I_{s,t}\) is finite, strictly increasing, continuous and convex. Moreover, the unique zero of \(I_{s,t}(r)\) is at \(r = g_{pp}(s,t)\).

In order to obtain a full large deviation principle, we must estimate the lower tail for the probabilities of the last passage time. As is usual in the solvable models of last passage percolation, the speed for the lower tail is different than \(N\). Our first lemma establishes the same fact for this model. In turn, this gives left tail bounds strong enough to imply \(I_{s,t}(r) = \infty\) for \(r < g_{pp}(s,t)\) for both boundary and i.i.d. model.

**Lemma 3.5.** There exist constants \(c > 0, C < \infty\) that depend on parameters \(s, t, p, u\), so that for all \(N \geq 1\) the following estimates hold:

(a) For \((s, t) \in (0, \infty)^2\) and \(r \in [0, g_{pp}(s,t)]\)
\[\mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \leq Nr\} \leq C e^{-cN^2}.\]

(b) For \((s, t) = \alpha\left(1, \frac{(u-p)^2}{p(1-p)}\right)\) for some \(\alpha > 0\), parallel to the characteristic direction, and \(r \in [0, g_{pp}^{(u)}(s,t)]\),
\[\mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}^{(u)} \leq Nr\} \leq C e^{-cN^2}.\]

**Proof of Lemma 3.5.** We prove (b) but similar arguments work for (a). We bound \(G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}\) from below, using the superadditivity property of the last passage times.

For this reason we consider a subset of lattice paths, arranged in a collection of i.i.d. partition function over subsets of rectangles. This block argument proof was first used in [35] and later adapted in [46] for the last passage time and in [27,31] for the log-gamma polymer and the brownian polymer model respectively.

Note that if \((s,t)\) are chosen in the characteristic direction it is immediate to see that \(g_{pp}^{(u)}(s,t) = g_{pp}(s,t)\).

We first show the result for \((s,t) \in \mathbb{Q}_+^2\). In order to highlight this distinction we assume that the target point is \((q_1, q_2) \in \mathbb{Q}_+^2\). Fix \(0 < \varepsilon < 1/4(g_{pp}^{(u)}(q_1, q_2) - r).\) Define a new scale parameter \(m \in \mathbb{N}\) large enough so that \(m(q_1 \wedge q_2) \geq 1, mq_1, mq_2 \in \mathbb{N}_+\) and
\[(3.8) \mathbb{E}[G_{mq_1, mq_2}] > m(r + 2\varepsilon).\]

We will use \(mq_1\) and \(mq_2\) to coarse-grain our environment. Let \(R_{a,b}^{k,\ell} = \{a, \ldots, a + k - 1\} \times \{b, \ldots, \ell + b - 1\}\) denote the \(k \times \ell\) rectangle with lower left corner at \((a, b)\). For \(i, \ell \geq 0\)
define pairwise disjoint \(mq_1 \times mq_2\) rectangles
\[\mathcal{R}_i^\ell = \mathcal{R}_{\ell(mq_1+i+1)+1}^{mq_1+1,mq_2+1}.\]
The rectangles \(\mathcal{R}_i^\ell\) are separated by the inter-site distance to avoid a scenario where a path goes along a common edge between two rectangles. This way, we will be able to clearly say in which one of the two rectangles the path goes through. For each \(i\) we define the diagonal union of rectangles as \(\Delta_i = \bigcup_{\ell \geq 0} \mathcal{R}_i^\ell, i \geq 0\) and in the sequence we are considering potential paths that stay in a fixed \(\Delta_i\).

Moreover, note that the last passage times \(G_{v_i,T,v_i^\ell}\) in each rectangle are all identically distributed, where \(v_i^w = (\ell+i)mq_1+1+i, \ell(mq_2+1)+1\) and \(v_i^e = ((i+1)mq_1+\ell mq_1, (1+\ell)(mq_2+1))\) are respectively the south-west and north-east corners of \(\mathcal{R}_i^\ell\).

Define \(B,M = M(B) \in \mathbb{N}\) the maximal integers which satisfy
\[
\begin{align*}
(M+1)mq_1 + Bmq_1 &\leq Nq_1 \quad \text{and} \\
1 + (B)(mq_2 + 1) &\leq Nq_2.
\end{align*}
\]
The fact that \(B\) is maximal and (3.10) imply that
\[B = \left\lfloor \frac{Nq_2}{mq_2+1} \right\rfloor - 1.
\]
Substituting (3.11) in (3.9) we obtain
\[M = \left\lfloor \frac{N}{m} - \left\lfloor \frac{Nq_2}{mq_2+1} \right\rfloor \right\rfloor \text{ and hence } \left\lfloor \frac{N}{m(mq_2+1)} \right\rfloor \leq M \leq \left\lfloor \frac{Nmq_2+2}{m(mq_2+1)} \right\rfloor.
\]
Since \(m\) is a constant and assumed much smaller than \(N\), we have that \(B = B(N) = O(N)\) and \(M = M(N) = O(N)\). Fix a diagonal \(\Delta_i\) for \(0 \leq i \leq M\) and define the union of rectangles in \(\Delta_i \cap ([0,Nq_1] \times [0,Nq_2])\) as \(\Delta_i^B = \bigcup_{0 \leq \ell \leq B} \mathcal{R}_i^\ell\).

Let \(G_{i}^\Delta\) be the last passage time of all lattice paths in \(\Delta_i^B\) from the lower left corner of \(\mathcal{R}_i^0\) to the upper right corner of \(\mathcal{R}_i^B\). \(G_{i}^\Delta\) are i.i.d, where in particular \(G_0^\Delta\) is the sum of the \(B\) last passage times of rectangle \(\mathcal{R}_0\) whose mean is controlled by (3.8). A standard large deviation estimate for an i.i.d sum gives the following bound
\[
\mathbb{P}\{G_{[Nq_1],[Nq_2]}^\Delta \leq N\} \leq \mathbb{P}\{G_i^\Delta \leq N\, \text{for } 0 \leq i \leq M\}
\]
\[
\leq \mathbb{P}\{G_0^\Delta \leq N\}^M \leq \mathbb{P}\left\{ \sum_{k=0}^{B(N)} G_k^0 \leq N\right\}^{M(N)}
\]
\[
\leq e^{-cB(N)M(N)} \leq e^{-c_1N^2}.
\]
This completes the proof for \((s,t) \in \mathbb{Q}^2_+\).

Finally we show (3.13) holds also for \(s,t \in \mathbb{R}_+\). We bound \(G_{[Nq_1],[Nq_2]}^\Delta\) using \(G_{[Nq_1],[Nq_2]}^\Delta\) for some special \((q_1,q_2) \in \mathbb{Q}^2_+\) which are close enough to \((s,t) \in \mathbb{R}_+^2\). For any \((q_1,q_2) \leq (s,t)\) we have that
\[
\mathbb{P}\{G_{[Nq_1],[Nq_2]}^\Delta \leq N\} \leq \mathbb{P}\{G_{[Nq_1],[Nq_2]}^\Delta \leq N\}, \text{ for all } r \in [0,g_{pp}^u(s,t)].
\]
Figure 3. Representation of the coarse grained $[ms] \times [mt]$ rectangles and the diagonals $\Delta_i$ in the proof of Lemma 3.5. The blue thick line is one of the possible maximal paths. For the bound needed, we are allowed to ignore the path segments outside of the coarse-grained diagonals, particularly we may ignore the correlated segments when candidate paths traverse the south and north boundary of $[0, [Ns]] \times [0, [Nt]]$. Passage times in each $\Delta_1$ are i.i.d. and smaller than the overall passage time.

For any $\delta > 0$ and find $(q_1, q_2)$ so that $\delta > g_{pp}^{(u)}(s, t) - g_{pp}^{(u)}(q_1, q_2) > 0$. This is possible by the continuity and monotonicity of $g_{pp}^{(u)}$. We choose $\delta < \frac{g_{pp}^{(u)}(s, t) - r}{2}$ and therefore

$$r < g_{pp}^{(u)}(s, t) - 2\delta < g_{pp}^{(u)}(q_1, q_2) - \delta < g_{pp}^{(u)}(s, t),$$

for some $(q_1, q_2) \in Q_+^2$. Then (3.14) is a left-tail large deviation anyway for $G^{(u)}_{[Nq_1], [Nq_2]}$ so (3.13) holds.

Proof of Theorem 3.4. This proof is a consequence of the lemmas and theorems that we have already proved. Define for $r \in \mathbb{R}$ function $I_{s,t}(r)$ by (3.7).

Then, the regularity properties proved for $J$ in Theorems 3.1 and 3.3 are also valid for $I_{s,t}$. For the upper large deviation bound (3.5) we consider two cases:

1. if $F \subseteq [0, g_{pp}(s, t)]$, then $r^* = \max\{x : x \in F\} < g_{pp}(s, t)$ and we have

$$\mathbb{P}\{N^{-1}G_{[Ns], [Nt]} \in F\} \leq \mathbb{P}\{G_{[Ns], [Nt]} \leq Nr^*\} \leq e^{-N^2}.$$
The last inequality comes from Lemma 3.5. Take logarithms on both sides, divide by \( N \), take the limit \( N \to \infty \) and finally by definition (3.7) conclude that

\[
\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in F\} = -\infty = - \inf_{r \in F} I_{s,t}(r).
\]

(2) If \( F \cap [g_{pp}(s,t), s] \neq \emptyset \) then we split into two different cases:

Case 1: \( F \not\ni g_{pp}(s,t) \). Then there exists an \( \varepsilon > 0 \) such that \((g_{pp}(s,t) - \varepsilon, g_{pp}(s,t) + \varepsilon) \subseteq F^c\). Then we bound

\[
\mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in F\} \leq \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \leq g_{pp}(s,t) - \varepsilon\}
+ \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in F \cap [g_{pp}(s,t) + \varepsilon, s]\}.
\]

By the previous calculations, we already control the first addend by \( e^{-cN^2} \) therefore we focus only on the second one which will be of an exponential order of magnitude larger and control the value of the \( \lim \). Since \( F \) and \([g_{pp}(s,t) + \varepsilon, s]\) are two closed sets there exists an \( r^* \) such that \( r^* = \min\{ r : r \in F \cap [g_{pp}(s,t) + \varepsilon, s]\} \). It follows that

\[
\mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in F\} \leq e^{-cN^2} + \mathbb{P}\{G_{[N_s],[N_t]} \geq Nr^*\}.
\]

Now take the logarithm of both sides, divide by \( N \) and take the \( \lim \)

\[
\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in F\} \leq \lim_{N \to \infty} N^{-1} \log(e^{-cN^2} + \mathbb{P}\{G_{[N_s],[N_t]} \geq Nr^*\})
= -J_{s,t}(r^*) = - \inf_{r \in F} I_{s,t}(r).
\]

The last line is obtained using (3.2), (3.7) and the fact that \( I_{s,t}(r) \) is a strictly increasing function.

Case 2: \( F \ni g_{pp}(s,t) \). In this case, \( \inf_{r \in F} I_{s,t}(r) = 0 \), therefore, inequality (3.5) is automatically satisfied.

For the lower large deviation bound (3.6), we need to consider three cases according to \( H \):

(1) If \( g_{pp}(s,t) \in H \), then \( \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in H\} \to 1 \) and (3.6) holds as an equality

(2) If \( H \subseteq [0, g_{pp}(s,t)) \), (3.6) holds because its right-hand side is \( -\infty \).

(3) The remaining case is the one where \( H \) contains an interval \((a, b) \subset (g_{pp}(s,t), s)\). Then for any \( \varepsilon > 0 \) small enough, we can find a non-trivial interval \([a + \varepsilon, b - \varepsilon] \subseteq H \) and bound

\[
N^{-1} \log \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in H\} \geq N^{-1} \log \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \in [a + \varepsilon, b - \varepsilon]\}
= N^{-1} \log \left( \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \geq N(a + \varepsilon)\} - \mathbb{P}\{N^{-1}G_{[N_s],[N_t]} \geq N(b - \varepsilon)\} \right)
\]

(3.15) \( \to -J_{s,t}(a + \varepsilon) \).

Equation (3.15) follows after taking \( \lim \) on both sides and keeping in mind that the two terms in the logarithm have different exponential orders of magnitude.
Monotonicity and convexity $J_{s,t}$ on $[g_{pp}(s,t), s]$ implies that for some constant $C$, $J_{s,t}(a + \varepsilon) \leq J_{s,t}(a) + C\varepsilon$. Then, (3.15) becomes
\[
\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{N^{-1}G_{|Ns|,|Nt|} \in H\} \geq -J_{s,t}(a) - C\varepsilon
\]

Let $\varepsilon \to 0$ in the last display. Then take $a = \inf H \cap (g_{pp}(s,t), s)$ to finish using
\[
J_{s,t}(a) = \inf_{r \in H \cap (g_{pp}(s,t), s)} I_{s,t}(r) = \inf_{r \in H} I_{s,t}(r).
\]

**Corollary 3.6.** Let $\xi \in \mathbb{R}$. Then
\[
(3.16) \lim_{N \to \infty} N^{-1} \log \mathbb{E}e^{\xi G_{0,(|Ns|,|Nt|)}} = I_{s,t}^*(\xi) = \begin{cases} J_{s,t}^*(\xi) & \text{if } \xi > 0, \\ 0 & \text{if } \xi = 0, \\ \xi g_{pp}(s,t) & \text{if } \xi < 0, \end{cases}
\]

**Proof of Corollary 3.6.** Since $G_{|Ns|,|Nt|} \leq Ns$, for any $\gamma > 1$ and $\xi \in \mathbb{R},$
\[
\sup_{N} \left( \mathbb{E}e^{\gamma \xi G_{0,(|Ns|,|Nt|)}} \right)^{1/N} < \infty.
\]
This bound together with Theorem 3.4 suffice to apply Varadhan’s theorem (e.g. page 38 in [43]) which gives
\[
\lim_{N \to \infty} N^{-1} \log \mathbb{E}e^{\xi G_{0,(|Ns|,|Nt|)}} = I_{s,t}^*(\xi) = \sup_{r \in \mathbb{R}} \{r \xi - I_{s,t}(r)\} = \sup_{r \in [g_{pp}(s,t), s]} \{r \xi - I_{s,t}(r)\}.
\]
The first equality on the second line is because $I_{s,t}(r) = \infty$ if $r \in (-\infty, g_{pp}(s,t))$ or $r > s$ and there is no difference in excluding that interval from the supremum.

Then we can compute $I_{s,t}^*$. $I_{s,t}$ is increasing for $r \in [g_{pp}(s,t), s]$, therefore if $\xi < 0$, the supremum is always attained at $r = g_{pp}(s,t)$. Instead, when $\xi \geq 0$, $I_{s,t}(\xi) = J_{s,t}^*(\xi)$ since $r$ can range over all of $\mathbb{R}$ and the last supremum will still be attained for some $r \in [g_{pp}(s,t), s]$.

4. **I.I.D. model: Right tail rate function and log moment generating function**

The main goal of this section is to prove an explicit variational formula for the rate function $J_{s,t}(r)$. That formula, while precise does not enjoy enough analytical tractability to further obtain a closed formula. However, its dual $J_{s,t}^*(\xi)$ will be explicitly computed by the end of the section. The variational characterization of $J$ requires the log-moment generating functions for Bernoulli($p$) random variables, given by
\[
(4.1) C^{(p)}_B(\xi) = \log(1 - p + pe^{\xi}), \xi \in \mathbb{R},
\]
and Geometric($p$) random variables given by
\[
(4.2) C^{(p)}_G(\xi) = \begin{cases} \log \frac{p}{1-(1-p)e^{\xi}}, & \xi < -\log(1-p) \\ \infty, & \text{otherwise}. \end{cases}
\]
Both log-moment generating functions can be seen as the Legendre duals of the rate functions for sums of i.i.d. Bernoulli \(^{(4.1)}\) and for sums of i.i.d. geometric random variables, given by

\[
J^*(s,t) = \inf_{u \in (p,1]} \left\{ s C_B^{(u)}(\xi) - t C_G^{(u-p)/(u(1-p))}(\xi) \right\}, \quad \text{if } \xi > 0, \\
0, \quad \text{if } \xi = 0, \\
\infty, \quad \text{if } \xi < 0.
\]

The two theorems that give the precise forms for \(J\) and \(J^*\) follow.

**Proposition 4.1.** Let \((s,t) \in \mathbb{R}^2_+\). Then for all \(\xi \in \mathbb{R}\), the convex dual \(J^*_s(t)(\xi)\) is given by

\[
J^*_s(t)(\xi) = \begin{cases} 
\inf_{u \in (p,1]} \left\{ s C_B^{(u)}(\xi) - t C_G^{(u-p)/(u(1-p))}(\xi) \right\}, & \text{if } \xi > 0, \\
0, & \text{if } \xi = 0, \\
\infty, & \text{if } \xi < 0. 
\end{cases}
\]

The closed form for \(J^*\) is given in the following.

**Theorem 4.2.** Fix \(p \in (0,1), \xi \geq 0\) and \((s,t) \in \mathbb{R}^2_+\). Define

\[
\Delta = \Delta_{p,s,t,\xi} = p(1-p)(e^\xi + e^{-\xi} - 2)[p(1-p)(s+t)^2(e^\xi + e^{-\xi} - 2) + 4st].
\]

Then,

\[
J^*_s(t)(\xi) = \begin{cases} 
\frac{s \log(p(1-p)(s+t)(e^\xi + e^{-\xi} - 2) + 2s + \sqrt{\Delta})}{\Delta + t \log(p(1-p)(s+t)(e^\xi + e^{-\xi} - 2) + \sqrt{\Delta})(1-p(1-e^{-\xi}))}, & \text{if } t < \frac{1-p}{p}s, \\
\frac{2s(1-p(1-e^{-\xi}))}{p(1-p)(t-s)(e^\xi + e^{-\xi} - 2) + \sqrt{\Delta}}, & \text{if } t \geq \frac{1-p}{p}s. 
\end{cases}
\]

4.1. **Exact computations for** \(J_{s,t}(r)\). We first present a series of key technical lemmas, and we encourage the reader familiar with these techniques to proceed to the proof of Proposition 4.1.

We will use the invariance property of the model with boundaries first. Consider the last passage time in the model with boundary \(G_{[N_s],[N_t]}^{(u)}\) and we iteratively apply equation \((2.12)\) to obtain

\[
G_{[N_s],[N_t]}^{(u)} - G_{0,[N_t]}^{(u)} = \sum_{i=1}^{[N_s]} I^{(u)}_{i,[N_t]}. 
\]

Focus on the left hand side. From equation \((2.4)\) and \((2.12)\) we can write the previous difference as

\[
\sum_{i=1}^{[N_s]} I^{(u)}_{i,[N_t]} = G_{[N_s],[N_t]}^{(u)} - G_{0,[N_t]}^{(u)} \\
= \max_{1 \leq k \leq [N_s]} \left\{ \sum_{i=1}^{k} I_{i,0}^{(u)} + G_{(k,1),([N_s],[N_t])} - \sum_{j=1}^{[N_t]} J_{0,j}^{(u)} \right\}
\]
\[
\max_{1 \leq k \leq \lfloor N_t \rfloor} \left\{ \sum_{j=1}^{k} J_{0,j}^{(u)} + \omega_{1,k} + G_{(1,k),([N_s],[N_t])} - \sum_{j=1}^{\lfloor N_t \rfloor} J_{0,j}^{(u)} \right\}
\]

\[
= \max_{1 \leq k \leq \lfloor N_s \rfloor} \left\{ \sum_{i=1}^{k} I_{i,0}^{(u)} - \sum_{j=1}^{\lfloor N_t \rfloor} J_{0,j}^{(u)} + G_{(k,1),([N_s],[N_t])} \right\}
\]

\[
\max_{1 \leq k \leq \lfloor N_t \rfloor} \left\{ - \sum_{j=1}^{\lfloor N_t \rfloor} J_{0,j}^{(u)} + \omega_{1,k} + G_{(1,k),([N_s],[N_t])} \right\}.
\]

To compactify notation we use a convention where the y-axis is labeled by negative indices and we define

\[(4.6) \quad \eta_k = \begin{cases} 
- \sum_{j=\lfloor -N_t \rfloor + 1}^{\lfloor k \rfloor} J_{0,j}^{(u)} & k \leq 0, \\
\sum_{i=1}^{\lfloor N_s \rfloor} I_{i,0}^{(u)} - \sum_{j=1}^{\lfloor N_t \rfloor} J_{0,j}^{(u)} & k \geq 1.
\end{cases}
\]

As such, we can say that the last passage time can be obtained on path that enters the bulk \( \mathbb{N}^2 \) at a point \( v(z) \) defined by

\[(4.7) \quad v(z) = \begin{cases} 
(1, \lfloor -z \rfloor) & z \leq -1, \\
(1, 1) & -1 < z < 1, \\
(\lfloor z \rfloor, 1) & z \geq 1,
\end{cases}
\]

and the gradient can be written as

\[
\sum_{i=1}^{\lfloor N_s \rfloor} I_{i,\lfloor N_t \rfloor}^{(u)} = \max_{\lfloor -N_t \rfloor \leq k \leq \lfloor N_s \rfloor, k \neq 0} \left\{ \eta_k + \omega_v(k) \mathbb{1}\{k < 0\} + G_v(k,([N_s],[N_t])) \right\}.
\]

Then the following inequalities are immediate:

\[(4.8) \quad \eta_k + G_v(k,([N_s],[N_t])) \leq \sum_{i=1}^{\lfloor N_s \rfloor} I_{i,\lfloor N_t \rfloor}^{(u)}
\]

\[(4.9) \quad \leq \max_{\lfloor -N_t \rfloor \leq k \leq \lfloor N_s \rfloor, k \neq 0} \left\{ \eta_k + G_v(k,([N_s],[N_t])) \right\} + 1.
\]

This inequality will be crucial for our purposes. We briefly discuss the main idea.

The second line in the last display is a sum of i.i.d. Bernoulli, so it has a known large deviation rate function. A deviation for the \( \sum I^{(u)} \) is controlled above and below by deviations for the expressions \( \eta_k + G_v(k,([N_s],[N_t])) \). \( \eta_k \) itself is either a sum of i.i.d. geometric random variables or a difference of two independent sums; in either case the large deviation rate function for \( \eta_k \) is computable, and the only unknown will be the large deviation rate function for \( G \) (albeit in a complicated expression). The subsection is devoted into following this program and solve for the rate function of \( G \).
It will be crucial to understand the function defined by

\[ H^{a,b}_{s,t}(r) = - \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{\eta_{[Na]} + G_{\nu(Nb),([Ns],[Nt])} \geq Nr\}, \]

where \( a, b \in [-t, s] \). We first argue why the limit exists. This fact will be a direct consequence of Lemma C.2 when we show that the \( \eta_{[Na]} \) and \( G_{\nu(Nb),([Ns],[Nt])} \) will have a right tail rate function.

We begin by computing the rate function for the \( \eta_k \). For real \( a \in [-t, s] \), and \( r \in \mathbb{R} \) define

\[ \kappa_a(r) = - \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{\eta_{[Na]} \geq Nr\}. \]

From (4.6) we observe that if \( k \leq 0 \) \( \eta_k \) is a sum of i.i.d. geometric distributed random variables while if \( k \geq 1 \), \( \eta_k \) is the difference of two independent sums of i.i.d. random variables.

The convex dual is

\[ \kappa^*(\xi) = \sup_{r \in \mathbb{R}} \{ \xi r - \kappa_a(r) \} \]

\[ = \begin{cases} \left( t + a \right) \left[ \log \frac{u-p}{u(1-p)} - \log \left( 1 - \frac{p(1-u)}{u(1-p)} e^{-\xi} \right) \right], & \text{for } \xi > \log \left( \frac{p(1-u)}{u(1-p)} \right), -t \leq a \leq 0, \\ t \left[ \log \frac{u-p}{u(1-p)} - \log \left( 1 - \frac{p(1-u)}{u(1-p)} e^{-\xi} \right) \right] + a \log (ue^\xi + 1 - u), & \text{for } \xi > \log \left( \frac{p(1-u)}{u(1-p)} \right), 0 < a \leq s, \\ \infty, & \text{otherwise}. \end{cases} \]

\[ = \begin{cases} \left( t + a \right) C^{\left( \frac{u-p}{u(1-p)} \right)}_G (-\xi), & \text{for } \xi > \log \left( \frac{p(1-u)}{u(1-p)} \right), -t \leq a \leq 0, \\ t C^{\left( \frac{u-p}{u(1-p)} \right)}_G (-\xi) + a C^{(u)}_G (\xi), & \text{for } \xi > \log \left( \frac{p(1-u)}{u(1-p)} \right), 0 < a \leq s, \\ \infty, & \text{otherwise}. \end{cases} \]

The first line in (4.12) follows from Cramer’s theorem when the random variables are geometric. The second line follows from Lemma C.2 when \( L_N = \sum_{i=1}^{[Na]} f^{(u)}_{i,0} \) and \( Z_N = - \sum_{j=1}^{[Nt]} f^{(u)}_{0,j} \), and the fact that the dual of an infimal convolution is the sum of the corresponding duals.

\textbf{Remark 4.3.} The condition on \( \xi \) can be stated equivalently in terms of \( u \). In fact, if \( \xi \in \mathbb{R} \) is fixed, the above inequality becomes \( u > \frac{pe^{-\xi}}{1-p+pe^{-\xi}} \). Moreover if \( \xi > 0 \), \( \frac{pe^{-\xi}}{1-p+pe^{-\xi}} < p \) and so it remains \( u \in (p, 1] \).

The rightmost zero \( m_{\kappa,a} \) of \( \kappa_a \) is the law of large numbers limit

\[ m_{\kappa,a} = \lim_{N \to \infty} N^{-1} \eta_{[Na]} = \begin{cases} -(t + a) \frac{u-p}{p(1-u)}, & -t \leq a \leq 0, \\ au - t \frac{u-p}{p(1-u)}, & 0 < a \leq s. \end{cases} \]

Note that when viewed as functions of \( a, \kappa_a, \kappa^*_a \) and \( m_{\kappa,a} \) are all continuous at \( a = 0 \).
For the rate function of \( G_{\nu(Na),([Ns],[Nt])} \), we first introduce the equivalent macroscopic version of (4.7) for \( a \in \mathbb{R} \), by

\[
N^{-1} \nu(Na) \to \tilde{\nu}(a) = \begin{cases} 
(0, -a), & -t \leq a \leq 0, \\
(a, 0), & 0 < a \leq s.
\end{cases}
\]

(4.14)

With this notation, the rate function of the last past passage time in the interior is

\[
J_{(s,t) - \tilde{\nu}(a)}(r) = -\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{\nu(Na),([Ns],[Nt])} \geq Nr\}.
\]

(4.15)

This is because \( G_{\nu(Na),([Ns],[Nt])} \) equals in distribution \( G_{(0,0),([Ns],[Nt])} - \nu(Na) \). There will be a small discrepancy between \( ([Ns],[Nt]) - \nu(Na) \) and \( N((s,t) - \tilde{\nu}(a)) \) but Lemma C.1 proves that it is negligible in the limit.

Let \( m_{\kappa,a} \) and \( m_{J,b} \) be the rightmost zeros respectively of \( \kappa_a \) (defined by (4.13)) and \( J_{(s,t) - \tilde{\nu}(b)} \) (which equals \( g_{pp}(s,t) - \tilde{\nu}(b) \)). Using Lemma C.2 for \( (a,b) \in [-t,s]^2 \), we have

\[
H^{a,b}_{s,t}(r) = \begin{cases} 
0, & r < m_{\kappa,a} + m_{J,b}, \\
\inf_{m_{\kappa,a} \leq x \leq r - m_{J,b}} \{ \kappa_a(x) + J_{(s,t) - \tilde{\nu}(b)}(r - x) \}, & m_{\kappa,a} + m_{J,b} \leq r \leq s.
\end{cases}
\]

(4.16)

The following regularity lemma follows from the continuity properties we discussed up to this point, and the details are left to the reader.

**Lemma 4.4.** Fix \( s, t \in (0, \infty) \) and fix any compact set \( K \subseteq (-\infty, s] \). Then \( H^{a,b}_{s,t}(r) \) is a uniformly continuous function of \( (b, r) \in [-t, s] \times K \), uniformly in \( a \in [-t, s] \). In symbols

\[
\lim_{\delta \to 0} \sup_{a,b,b' \in [-t, s], r, r' \in K: |b-b'| \leq \delta, |r-r'| \leq \delta} |H^{a,b}_{s,t}(r) - H^{a,b'}_{s,t}(r')| = 0.
\]

(4.17)

When \( a = b \) we simplify the notation as \( H^{a}_{s,t}(r) = H^{a,a}_{s,t}(r) \). Observe that at this point an expression involving \( J_{s,t} \) manifested on the right-hand side of (4.16). Our goal is to invert the relation and isolate \( J_{s,t} \).

The next lemma is the continuous version of the discrete inequalities (4.8), (4.9) at the level of the rate functions.

**Lemma 4.5.** Let \( s, t \in (0, \infty) \) and \( r \in [0, s] \). Then

\[
sI^{(u)}_G(r/s) = \inf_{-t \leq a \leq s} H^a_{s,t}(r).
\]

(4.18)

Proof. For any \( a \in [-t, s] \), by (4.8)

\[
-sI^{(u)}_G(r/s) = \lim_{N \to \infty} N^{-1} \log \mathbb{P}\left\{ \sum_{i=1}^{[Ns]} I^{(u)}_i [Nt] \geq Nr \right\}
\]

\[
\geq \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{ \eta_{[Na]} + G_{\nu([Na]),([Ns],[Nt])} \geq Nr \}
\]

\[
= -H^a_{s,t}(r).
\]
This is true for an arbitrary \( a \), therefore

\[
(4.19) \quad s I_B^{(u)}(r/s) \leq \inf_{-t \leq a \leq s} H^0_{s,t}(r). 
\]

To get the lower bound we use (4.9) together with a coarse graining argument.

We begin describing the partition which will be helpful when we will use (4.9). Fix a small enough \( \delta > 0 \) to partition the interval \([-t,s]\). In particular, define \(-t = a_0 < a_1 < \cdots < a_q = 0 < \cdots < a_m = s\) where \( |a_{i+1} - a_i| < \delta \). Moreover, we fix an \( \varepsilon > 0 \) and we assume that \( N \) is large enough so that \( N\varepsilon > 1 \).

When \( a_i \geq 0 \), for any \( k \in [\lfloor Na_i \rfloor, \lfloor Na_{i+1} \rfloor] \cap \mathbb{Z} \),

\[
P\{\eta_k + G_{k,(\lfloor Na_i \rfloor, \lfloor Na_{i+1} \rfloor)} \geq Nr\} \leq \mathbb{P}\{\eta_{\lfloor Na_{i+1} \rfloor} + G_{\lfloor Na_{i+1} \rfloor, (\lfloor Na_i \rfloor, \lfloor Na_{i+1} \rfloor)} \geq Nr\}.
\]

Similarly, when \( a_i < 0 \) and \( \lfloor Na_i \rfloor < k \leq \lfloor Na_{i+1} \rfloor \) the bound becomes

\[
P\{N\varepsilon \leq N r - \varepsilon \}
\]

From (4.9) we bound

\[
P\left\{ \sum_{i=1}^{\lfloor Ns \rfloor} I_{i,[Nt]} \geq Nr \right\} \leq \mathbb{P}\left\{ \max_{k \neq 0} \left\{ \eta_k + G_{k,(\lfloor Na_i \rfloor, \lfloor Na_{i+1} \rfloor)} \right\} + 1 \geq Nr \right\}
\]

\[
\leq \mathbb{P}\left\{ \max_{k \neq 0} \left\{ \eta_k + G_{k,(\lfloor Na_i \rfloor, \lfloor Na_{i+1} \rfloor)} \right\} \geq N(r - \varepsilon) \right\}.
\]

Take logarithm on both sides and divide by \( N \) and use a union bound to obtain

\[
N^{-1} \log \mathbb{P}\left\{ \sum_{i=1}^{\lfloor Ns \rfloor} I_{i,[Nt]} \geq Nr \right\} \leq N^{-1} \log m + \left\{ \max_{0 \leq i \leq q-1} \left\{ N^{-1} \log \mathbb{P}\{\eta_{\lfloor Na_i \rfloor} + G_{\lfloor Na_{i+1} \rfloor, (\lfloor Na_i \rfloor, \lfloor Na_{i+1} \rfloor)} \geq N(r - \varepsilon)\} \right\} \right\}
\]

\[
\vee \left\{ \max_{q \leq i \leq m-1} \left\{ N^{-1} \log \mathbb{P}\{\eta_{\lfloor Na_{i+1} \rfloor} + G_{\lfloor Na_i \rfloor, (\lfloor Na_{i+1} \rfloor, \lfloor Na_i \rfloor)} \geq N(r - \varepsilon)\} \right\} \right\}.
\]

Take \( N \to \infty \) to get

\[
-s I_B^{(u)}(r/s) \leq \max_{0 \leq i \leq q-1} \left\{ -H^{a_i,a_{i+1}}_{s,t}(r - \varepsilon) \right\} \vee \max_{q \leq i \leq m-1} \left\{ -H^{a_i+1,a_i}_{s,t}(r - \varepsilon) \right\} \leq \sup_{a,b \in [-t,s]; |a-b| \leq \delta} \{-H^{a,b}_{s,t}(r - \varepsilon)\}.
\]

Use Lemma 4.4 by letting \( \delta \to 0 \); this also implies \( b \to a \). Then let \( \varepsilon \to 0 \).

The following lemma is the last technical tool we need in order to finally solve (4.18) for the unknown rate function \( J \). It proves convexity and lower semi-continuity of the Legendre dual of \( J \).

**Lemma 4.6.** For a fixed \( \xi \in \mathbb{R}_+ \), the function \( J^*_s(\xi) \), as a function of \( (s,t) \), is continuous and finite on \( \mathbb{R}^2_+ \).
Proof. By definition $J_{s,t}^*(\xi) = \sup_{r \in \mathbb{R}} \{\xi r - J_{s,t}(r)\}$, but, since $J_{s,t}(r) = \infty$ for $r > s$, and $J_{s,t}(r) = 0$ for $r < g_{pp}(s,t)$, we can write for $\xi \geq 0$ that

$$J_{s,t}^*(\xi) = \sup_{r \in [g_{pp}(s,t), s]} \{\xi r - J_{s,t}(r)\}.$$ 

Then it is immediate to see that $J_{s,t}^*(\xi) \leq \xi s$, for all $(s, t) \in \mathbb{R}^2_+$. Continuity will follow once we prove that $M_{\xi}(s, t)$ is a concave finite function. Let $\lambda \in (0, 1)$ and $(s, t) = \lambda(s_1, t_1) + (1 - \lambda)(s_2, t_2)$ for some $(s_i, t_i) \in \mathbb{R}^2_+$. Recall that $J$ is convex and lower-semicontinuous in $(s,t,r)$ from Theorem 3.1. Write $r$ as the convex combination $r = \lambda r_1 + (1 - \lambda)r_2$ for some $r_1, r_2 \in \mathbb{R}$. By convexity

$$\inf_{r \in \mathbb{R}} \{J_{s,t}(r) - \xi r\}$$

$$\leq \inf_{r \in \mathbb{R}} \left\{ \inf_{(r_1, r_2) \in \mathbb{R}^2} \{\lambda(J_{s_1,t_1}(r_1) - \xi r_1) + (1 - \lambda)(J_{s_2,t_2}(r_2) - \xi r_2)\} \right\}$$

$$= \inf_{(r_1, r_2) \in \mathbb{R}^2} \{\lambda(J_{s_1,t_1}(r_1) - \xi r_1) + (1 - \lambda)(J_{s_2,t_2}(r_2) - \xi r_2)\}$$

$$= \lambda \inf_{r_1 \in \mathbb{R}} \{J_{s_1,t_1}(r_1) - \xi r_1\} + (1 - \lambda) \inf_{r_2 \in \mathbb{R}} \{J_{s_2,t_2}(r_2) - \xi r_2\}$$

$$= -\lambda J_{s_1,t_1}^*(\xi) - (1 - \lambda) J_{s_2,t_2}^*(\xi).$$

In the end we have

$$J_{s,t}^*(\xi) \geq \lambda J_{s_1,t_1}^*(\xi) + (1 - \lambda) J_{s_2,t_2}^*(\xi),$$

which is enough to prove the concavity of $J_{s,t}^*(\xi)$ in $(s, t)$. □

Figure 4. Graphical representation of the function $J_{s,t}^*(r)$. In both figures we used $p = 0.1$ and $t = 1$. To the left we have $J_{s,1}^*(\xi)$ as a function of $(s, \xi)$ and one see the directions of convexity when $s$ is fixed and $\xi$ varies, and the direction of concavity ranges when $\xi$ is fixed and $s$ varies as described in the proof of Lemma 4.6. The blue line is at $s = 1/9$ which the is characteristic point for $p = 0.1$ and $t = 1$. For smaller $s$, $J_{s,1}^*(\xi) = s\xi$. To the right is the convex continuous function $J_{10,1}^*(\xi)$. 
In order to prove Proposition 4.1, we need the following technical result. This is in the spirit of Proposition 3.10 in [31] but tailored to our particular case. For this reason we postpone the proof until Appendix A.

**Proposition 4.7.** Let \( I = (a, b) \subseteq \mathbb{R} \) with \( a, b \in \mathbb{R} \). Let the convex functions \( h, g: I \to \mathbb{R} \) be twice continuously differentiable with \( h'(u) > 0 \) and \( g'(u) < 0 \) for every \( u \in I \). Define
\[
\begin{align*}
    f_{s,t}(u) &= sh(u) + tg(u) \quad \text{with} \quad (s, t) \in \mathbb{R}^2. \\
\end{align*}
\]
Suppose that \( f_{s,t}''(u) > 0 \) for all \( (s, t) \in \mathbb{R}^2_+ \), \( \lim_{u \to a} f_{s,t}(u) = \infty \) and \( f_{s,t}(b) = c < \infty \) with \( c \in \mathbb{R} \). If \( \Lambda(s, t) \) is a continuous function in \( (s, t) \) with the property that for all \( (s, t) \in \mathbb{R}^2_+ \) and \( u \in I \) the identity
\[
\begin{align*}
    0 &= \sup_{0 \leq z \leq s} \{ \Lambda(s - z, t) - f_{s-z,t}(u) \} \vee \sup_{0 \leq z \leq t} \{ \Lambda(s, t - z) - f_{s,t-z}(u) \}
\end{align*}
\]
holds, then for every \( t < -\frac{h'(b)}{g'(b)} s \),
\[
\Lambda(s, t) = \min_{u \in I} \{ f_{s,t}(u) \}.
\]

Using this we can now find a variational expression for \( J^* \).

**Proof of Proposition 4.1.** If \( \xi < 0 \), by definition
\[
\begin{align*}
    J_{s,t}^*(\xi) &= \sup_{r \in \mathbb{R}} \{ r\xi - J_{s,t}(\xi) \} = \sup_{r < g_{pp}(s,t)} \{ r\xi - J_{s,t}(r) \} \vee \sup_{r \geq g_{pp}(s,t)} \{ r\xi - J_{s,t}(r) \}.
\end{align*}
\]
Note that the first supremum is \( +\infty \) since \( J_{s,t}(r) = 0 \) for \( r < g_{pp}(s,t) \) and \( \xi < 0 \). Therefore \( J_{s,t}^*(\xi) = \infty \) if \( \xi < 0 \).

If \( \xi \geq 0 \), equation (4.16) gives that \( H_{s,t}^a(r) \) is the infimal convolution of \( \kappa_a \) and \( J_{(s,t),-(s,t)}(a) \) since the value of the infimum does not change when we allow \( x \) to range over all of \( \mathbb{R} \). We compactify the notation by writing \( H_{s,t}^a(r) = \kappa_a \square J_{(s,t),-(s,t)}(a)(r) \). By Theorem 16.4 in [44], the addition operation is dual to the infimal convolution operation. From (4.18) of Lemma 4.5, take the Legendre dual on both sides to obtain
\[
\begin{align*}
    sC^u_B(\xi) &= \sup_{-t \leq a \leq s} \left\{ \sup_{r \in \mathbb{R}} \{ r\xi - (\kappa_a \square J_{(s,t),-(s,t)}(a))(r) \} \right\} \\
    &= \sup_{-t \leq a \leq s} \{ (\kappa_a \square J_{(s,t),-(s,t)}(\xi))^*(a) \} = \sup_{-t \leq a \leq s} \{ \kappa_a^*(\xi) + J_{s,t,-a}^*(\xi) \}. 
\end{align*}
\]
From (4.12) we can substitute the explicit expression of \( \kappa_a^*(\xi) \). Define
\[
\begin{align*}
    -\ell(\xi)(u) &= C^u_g\left(\frac{u}{u(1-p)}\right)(-\xi) = \log \frac{u - p}{u(1 - p) - p(1 - u)e^{-\xi}}, \\
    d(\xi)(u) &= C^u_B(\xi) = \log(ue^\xi + 1 - u).
\end{align*}
\]
Use this to simplify (4.21) into
\[
\begin{align*}
    sd(\xi)(u) + t\ell(\xi)(u) &= \sup_{0 \leq a \leq t} \{ a\ell(\xi)(u) + J_{s,t-a}^*(\xi) \} \vee \sup_{0 \leq a \leq s} \{ ad(\xi)(u) + J_{s-a,t}^*(\xi) \}.
\end{align*}
\]
Subtract $sd\xi(u) + tl\xi(u)$ to either side
\[
0 = \sup_{0 \leq z \leq s} \{J_{s,a,t}^*(\xi) - [(s - a)d\xi(u) + tl\xi(u)]\} \vee \sup_{0 \leq z \leq t} \{J_{s,t-a}^*(\xi) - [sd\xi(u) + (t - a)l\xi(u)]\}.
\]

Use Proposition 4.7 identifying as $I = (p, 1], \Lambda(s, t) = J_{s,t}^*(\xi)$, $h(u) = d\xi(u)$, $g(u) = l\xi(u)$ and therefore $f_{s,t}(u) = sd\xi(u) + tl\xi(u)$. The only hypothesis that is not immediately verifiable is continuity of $J^*$ in $s, t$, but that is now covered by Lemma 4.6. Therefore, if $t < \frac{1-p}{p}s$
\[
J_{s,t}^*(\xi) = \min_{u \in [p, 1]} \{sd\xi(u) + tl\xi(u)\} = \min_{u \in [p, 1]} \{sC_B^{(u)}(\xi) - tC_G^{(\frac{u-p}{u(1-p)})}(\xi)\}.
\]

For $t \geq \frac{1-p}{p}s$ we reason directly: $J_{s,t}(r) = +\infty1\{r > s\}$ and its convex dual will be $s\xi$ for $\xi > 0$. This is also the $\min_{u \in [p, 1]}\{sC_B^{(u)}(\xi) - tC_G^{(\frac{u-p}{u(1-p)})}(\xi)\}$, with the minimum obtained at $u = 1$.

4.2. Closed formula for $J_{s,t}^*(\xi)$.

Proof of Theorem 4.2: The aim of this proof is to find an analytical result for the infimum in Proposition 4.1 when $t < \frac{1-p}{p}s$. Therefore we start computing the derivatives of the two cumulant-generating function and to find the optimizing point we solve the equation
\[
0 = s\frac{\partial C_B^{(u)}(\xi)}{\partial u} - t\frac{\partial C_G^{(\frac{u-p}{u(1-p)})}(\xi)}{\partial u} = s \frac{e^\xi - 1}{1 + u(e^\xi - 1)}
\]

or equivalently, after the algebraic simplification of denominators
\[
0 = u^2s[(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})] - up[(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + 2s(1 - e^{-\xi})]
\]

The minimum is in fact attained to the solution to this equation (for further details see Appendix [D]. The minimizing point is
\[
J_{s,t}^*(\xi) = sC_B^{(u^*)}(\xi) - tsC_G^{(u^*)}(\xi).
\]

with $\Delta = p(1 - p)(e^\xi + e^{-\xi} - 2)(1 - p)p(s + t)2(e^\xi + e^{-\xi} - 2) + 4st]$. Then (4.5) follows directly by
\[
0 = \frac{p(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + 2sp(1 - e^{-\xi}) + \sqrt{\Delta}}{2s[(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})]}.
\]
5. Invariant model: Limiting log-moment generating functions

Define the last passage time’s l.m.g.f. for the boundary model

\[
\Lambda^{(u)}_{(s,t)}(\xi) = \lim_{N \to \infty} N^{-1} \log E e^{\xi G^{(u)}_{[Ns],[Nt]}}.
\]

In this section we find \(\Lambda^{(u)}_{(s,t)}(\xi)\) when \(\xi > 0\).

It will be convenient to also define the l.m.g.f. for the two passage times conditional on the first step being \(e_1\) or \(e_2\), \(G^{(u),\text{hor}}_{[Ns],[Nt]}\) and \(G^{(u),\text{hor}}_{[Ns],[Nt]}\) given by (2.6) and (2.7) respectively. The corresponding l.m.g.f. are

\[
\Lambda^{(u),\text{hor}}_{(s,t)}(\xi) = \lim_{N \to \infty} N^{-1} \log E e^{\xi G^{(u),\text{hor}}_{[Ns],[Nt]}}
\]

and

\[
\Lambda^{(u),\text{ver}}_{(s,t)}(\xi) = \lim_{N \to \infty} N^{-1} \log E e^{\xi G^{(u),\text{ver}}_{[Ns],[Nt]}}.
\]

The existence of the above limits is verified in Lemma 5.3 below, but we state it as part of the main Theorem 5.1.

The existence of the two limits above then gives rise to the formula

\[
\Lambda^{(u)}_{(s,t)}(\xi) = \Lambda^{(u),\text{hor}}_{(s,t)}(\xi) \lor \Lambda^{(u),\text{ver}}_{(s,t)}(\xi)
\]

for any \(\xi > 0\).

Thus, finding \(\Lambda^{(u)}_{(s,t)}(\xi)\) is equivalent to finding \(\Lambda^{(u),\text{hor}}_{(s,t)}(\xi), \Lambda^{(u),\text{ver}}_{(s,t)}(\xi)\), which is the content of Theorem 5.1 below.

Heuristically, one expects the creation of some critical direction for \((s, t)\) that will depend on \(\xi, p, u\); below the direction the boundary effect will be felt at the l.g.m.f. level, and otherwise the model will behave like the boundary is not present. This was also observed at the LLN level in Theorem 2.2. In fact this is the case.

For \(\xi > 0\) we define

\[
k^{(u)}(\xi) = \left( \frac{\partial C^{(u)}_B(\xi)}{\partial u} \right) / \left( \frac{\partial C^{(u)}_G(-\xi)}{\partial u} \right).
\]

The relevant conditions that create a critical line are

\[
t = k^{(u)}(\xi)s, \quad \text{and} \quad t = k^{(u)}(-\xi)s,
\]

for \(\Lambda^{(u),\text{hor}}\) and \(\Lambda^{(u),\text{ver}}\) respectively. Recall that l.m.g.f. of \(G_{[Ns],[Nt]}\) is given by Corollary 3.6 and is equal to \(J^{(u)}_{s,t}(\xi) = J^{(u)}_{s,t}(\xi)\). For uniformity of notation in the section, set \(\Lambda^{(u)}_{(s,t)}(\xi) = I^{(u)}_{s,t}(\xi)\).

**Theorem 5.1.** Let \(s, t \geq 0, u \in (p, 1)\) and \(\xi \geq 0\).

(a) The limit in (5.2) exists and is given by

\[
\Lambda^{(u),\text{hor}}_{(s,t)}(\xi) = \begin{cases} 
s C^{(u)}_B(\xi) - t C^{(u)}_G \left( \frac{p - u}{1 - p} \right)(-\xi) & \text{if } t < k^{(u)}(\xi)s, \\
\Lambda^{(u)}_{(s,t)}(\xi) & \text{if } t \geq k^{(u)}(\xi)s. 
\end{cases}
\]
(b) The limit in (5.3) exists and is given by

\[
\Lambda_{(u),\text{ver}}^t(\xi) = \begin{cases} 
  tC_G^{(u)}(\xi) - sC_B^{(u)}(\xi), & \text{if } \xi \in [0, \log p^{(1-p)/(1-u)}] \text{ and } t > k^{(u)}(-\xi)s, \\
  \Lambda_{(u),\text{ver}}^t(\xi), & \text{if } \xi \in [0, \log p^{(1-p)/(1-u)}] \text{ and } t \leq k^{(u)}(-\xi)s, \\
  \infty, & \text{if } \xi \in \left[\log p^{(1-p)/(1-u)}, \infty\right).
\end{cases}
\]

The last theorem proves the full l.m.g.f. for the boundary model. Define

\[
\ell(u)(\xi) = \frac{C_B^{(u)}(\xi) + C_B^{(u)}(-\xi)}{C_G^{(u)}(\xi) + C_G^{(u)}(-\xi)}.
\]

Then, the l.g.m.f. for the boundary last passage time is given by

**Theorem 5.2.** Let \( s, t \geq 0 \) and \( u \in (p, 1] \). Then the limit in (5.11) exists for \( \xi \geq 0 \) and is given by

\[
\Lambda_{(u)}(s,t)(\xi) = \begin{cases} 
  sc_B^{(u)}(\xi) - tc_G^{(u)}(\xi), & \text{if } \xi \in [0, \log p^{(1-p)/(1-u)}] \text{ and } t < \ell(u)(\xi)s, \\
  tc_G^{(u)}(\xi) - sc_B^{(u)}(\xi), & \text{if } \xi \in [0, \log p^{(1-p)/(1-u)}] \text{ and } t \geq \ell(u)(\xi)s, \\
  \infty, & \text{if } \xi \in \left[\log p^{(1-p)/(1-u)}, \infty\right).
\end{cases}
\]

Before the two proofs, we begin by verifying the existence of limits (5.2) and (5.3). We begin by noting that similar arguments as in Lemma 1.5 give that

\[
-\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{[N_s],[Nt]}^{(u),\text{hor}} \geq Nr\} = \inf_{a \in [0,s]} \inf_{x \in \mathbb{R}} \inf \{aI_B^{(u)}((r-x)/a) + J_{s-a,t}(x)\}.
\]

Equation (5.11) in particular verifies the existence of the limit in the left-hand side, and we denote it by

\[
-\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{[N_s],[Nt]}^{(u),\text{hor}} \geq Nr\} = J_{s,t}^{(u),\text{hor}}(r).
\]

Finally, observe that we take the Legendre transform, equation (5.11) becomes

\[
(J_{s,t}^{(u),\text{hor}})^{*}(\xi) = \sup_{a \in [0,s]} \{aC_B^{(u)}(\xi) + J_{s-a,t}^{(u)}(\xi)\}.
\]

Symmetric definitions and arguments give similar equations for \( J_{s,t}^{(u),\text{ver}} \).

**Lemma 5.3.** Let \( G_{[N_s],[Nt]}^{(u),\text{hor}} \) be the last passage time given by (2.6), and let \( (J_{s,t}^{(u),\text{hor}})^{*}(\xi) \) given by (5.13). Then for \( \xi > 0 \),

\[
-\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{e^{\xi G_{[N_s],[Nt]}^{(u),\text{hor}}}} = (J_{s,t}^{(u),\text{hor}})^{*}(\xi).
\]

Corresponding statements hold for \( G_{[N_s],[Nt]}^{(u),\text{ver}} \).
Proof. Let $\xi \geq 0$. Set
\[
\gamma = \lim_{N \to \infty} N^{-1} \log \mathbb{E}[e^{\xi G_{[Ns],[Nt]}^{(u)\text{hor}}}] \quad \text{and} \quad \bar{\gamma} = \lim_{N \to \infty} N^{-1} \log \mathbb{E}[e^{\xi G_{[Ns],[Nt]}^{(u)\text{hor}}}] .
\]
The lower bound is immediate using the exponential Chebyshev inequality
\[
N^{-1} \log \mathbb{P}\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r\} \leq -\xi r + N^{-1} \log \mathbb{E}[e^{\xi G_{[Ns],[Nt]}^{(u)\text{hor}}}] .
\]
Letting $N \to \infty$ along a suitable subsequence gives $\bar{\gamma} \geq \xi r - J_{s,t}^{(u)\text{hor}}(r)$ for all $r \in [0,s]$. Thus $\gamma \geq J_{s,t}^{(u)\text{hor}}(\xi)$. For the upper bound we first claim that for every $r > s$
\[
(5.15) \quad \lim_{N \to \infty} N^{-1} \log \mathbb{E}[e^{\xi G_{[Ns],[Nt]}^{(u)\text{hor}}} 1\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r\}] = -\infty .
\]
To see this, apply Holder’s inequality to the expectation in (5.15). For any $\alpha > 1$,
\[
N^{-1} \log \mathbb{E}[e^{\alpha \xi G_{[Ns],[Nt]}^{(u)\text{hor}}} 1\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r\}] \leq N^{-1} \log \left\{ \mathbb{E}[e^{\alpha \xi G_{[Ns],[Nt]}^{(u)\text{hor}}}]^{\frac{\alpha}{\alpha-1}} \mathbb{E}[1\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r\}]^{\frac{\alpha-1}{\alpha}} \right\} = (\alpha N)^{-1} \log \left( \mathbb{E}[e^{\alpha \xi G_{[Ns],[Nt]}^{(u)\text{hor}}}] \right) + (\alpha - 1) \alpha^{-1} N^{-1} \log \mathbb{P}\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r\} .
\]
The first term is finite since $G_{[Ns],[Nt]}^{(u)\text{hor}} \leq [Ns]$ and for the same reason the second term equals $-\infty$.
To show the upper bound in (5.14) pick a $\delta > 0$ and partition $\mathbb{R}$ with $r_i = i\delta, i \in \mathbb{Z}$:
\[
(5.16) \quad N^{-1} \log \mathbb{E}[e^{\xi G_{[Ns],[Nt]}^{(u)\text{hor}}}] \leq N^{-1} \log \left[ \sum_{i=-m}^{m} e^{N \xi r_{i+1}} \mathbb{P}\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r_i\} + e^{N \xi r_m} + \mathbb{E}[e^{\xi G_{[Ns],[Nt]}^{(u)\text{hor}}} 1\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r_m\}] \right] .
\]
By (5.15), for each $M > 0$ there exists $m = m(M)$ so that for all $N$ large enough
\[
N^{-1} \log \mathbb{E}[e^{\xi G_{[Ns],[Nt]}^{(u)\text{hor}}} 1\{G_{[Ns],[Nt]}^{(u)\text{hor}} \geq N r_m\}] < -M .
\]
Take a limit as $N \to \infty$ along any subsequence that achieves $\bar{\gamma}$ to see that (5.16) implies
\[
\bar{\gamma} \leq \max_{-m \leq i \leq m} \{ \xi r_{i+1} - J_{s,t}^{(u)\text{hor}}(r_i) \} \vee \xi r_m \vee (-M) \leq \left( \sup_{r \in [0,s]} \{ \xi r - J_{s,t}^{(u)\text{hor}}(r) \} + \xi \delta \right) \vee \xi r_m \vee (-M) .
\]
The statement of the Lemma follows by letting $\delta \to 0$, $m \to \infty$ and $M \to \infty$. 
In order to repeat the estimates for \( G^{(u),\text{ver}}_{[N_1],[N_2]} \) the equivalent statement for (5.15) is
\[
\lim_{r \to \infty} \lim_{N \to \infty} N^{-1} \log \mathbb{E}[e^{\xi G^{(u),\text{ver}}_{[N_1],[N_2]}(s,t)} \mathbb{1}\{G^{(u),\text{ver}}_{[N_1],[N_2]} \geq N r\}] = -\infty.
\]
We omit the remaining details; the interested reader can find a similar calculation in [27]. □

Proof of Theorem 5.1. The existence of limit (5.2) is verified by Lemma 5.3. Then, use in sequence equations (5.13) and (5.14) and Proposition 4.1 to write
\[
\Lambda_{(s,t)}^{(u),\text{hor}}(\xi) = \sup_{a \in [0,s]} \{a C_B^{(u)}(\xi) + J_{s-a,t}^{*}(\xi)\}
\]
\[
= \sup_{a \in [0,s]} \left\{ \inf_{\theta \in [p,1]} \{a \left(C_B^{(u)}(\xi) - C_B^{(\theta)}(\xi)\right) + s C_B^{(\theta)}(\xi) - t C_G^{(\frac{\theta-p}{1-p})(-\xi)}\} \right\}.
\]
The sup and inf can be interchanged by a minimax theorem (e.g. [34]). The function inside the supremum is linear in \( a \). Thus the supremum will be reached at one of the two boundary points according to the sign of the difference
\[
C_B^{(u)}(\xi) - C_B^{(\theta)}(\xi) \begin{cases} > 0, & \text{if } \theta \in (u,1], \\ = 0, & \text{if } \theta = u, \\ < 0, & \text{if } \theta \in (p,u). \end{cases}
\]
Therefore we have
\[
\Lambda_{(s,t)}^{(u),\text{hor}}(\xi) = \inf_{\theta \in [u,1]} \{s C_B^{(u)}(\xi) - t C_G^{(\frac{\theta-p}{1-p})(-\xi)}\} \wedge \{s C_B^{(u)}(\xi) - t C_G^{(\frac{u-p}{1-p})(-\xi)}\}
\]
(5.17)
\[
\wedge \inf_{\theta \in (p,u)} \{s C_B^{(\theta)}(\xi) - t C_G^{(\frac{\theta-p}{1-p})(-\xi)}\}.
\]
Note that, since \(- C_G^{(\frac{\theta-p}{1-p})(-\xi)}\) is increasing in \( \theta \), the first term on the right-hand side of (5.17) is always greater than the second one. So, it remains to compare the second and the third term.

Call \( \theta^* \) the minimizing point in \((p,1]\) for the expression \( s C_B^{(\theta)}(\xi) - t C_G^{(\frac{\theta-p}{1-p})(-\xi)}\) in this specific case. Then, there are two possible cases:

1. If \( \theta^* \leq u \), then
\[
\Lambda_{(s,t)}^{(u),\text{hor}}(\xi) = \inf_{\theta \in (p,u]} \{s C_B^{(\theta)}(\xi) - t C_G^{(\frac{\theta-p}{1-p})(-\xi)}\} = s C_B^{(\theta^*)}(\xi) - t C_G^{(\frac{\theta^*-p}{1-p})(-\xi)} = \Lambda_{(s,t)}(\xi).
\]
2. If \( \theta^* > u \) then
\[
\Lambda_{(s,t)}^{(u),\text{hor}}(\xi) = s C_B^{(u)}(\xi) - t C_G^{(\frac{u-p}{1-p})(-\xi)}.
\]
This concludes the proof of (5.7). For the analogous result in the vertical case, first note that we may write
\[
(5.18) \quad \Lambda_{(s,t)}(\xi) = J_{s,t}^{*}(\xi) = \inf_{u \in (p,1]} \{t C_G^{(\frac{u-p}{1-p})(\xi)} - s C_B^{(u)}(-\xi)\}.
\]
That is possible to prove either by repeating the same computation in the subseccion 4.1 but starting $G_{\lfloor N_s \rfloor : \lfloor N_t \rfloor}^{(u)} - G_{\lfloor N_s \rfloor , 0}^{(u)} = \sum_{j=1}^{\lfloor N_t \rfloor} J_j^{(u)}$, or by computing (5.18) as in the proof of Theorem 4.2 and observe that it gives the same result.

Then as in the case for the horizontal boundary only,

$$\Lambda_{(s,t),ver}^{(u)}(\xi) = \sup_{a \in [0,t]} \{ a C_G^{(u)}(\xi) + J_{s,t-a}^{(u)}(\xi) \} = \sup_{a \in [0,t]} \left\{ \inf_{\theta \in (p,1)} \{ a \left( C_G^{(u)}(\xi) - C_G^{(\theta)}(\xi) \right) + t C_G^{(\theta)}(\xi) - s C_B(\xi) \} \right\}.$$  

From this expression we see that we need to restrict $\xi \in [0, \log \frac{u(1-p)}{p(1-u)} ]$, otherwise $\Lambda_{(s,t),ver}^{(u)}(\xi)$ is not finite. Then, as before, for $\xi \in [0, \log \frac{u(1-p)}{p(1-u)} ]$

$$\Lambda_{(s,t),ver}^{(u)}(\xi) = \begin{cases} 
\inf_{\theta \in (p,1)} \{ t C_G^{(\theta)}(\xi) - s C_B(\xi) \} = \Lambda_{(s,t)}(\xi) & \text{if } t \leq k^{(u)}(\xi) \theta s, \\
\inf_{\theta \in (p,u)} \{ t C_G^{(\theta)}(\xi) - s C_B(\xi) \} = t C_G^{(\theta)}(\xi) - s C_B(\xi) & \text{if } t > k^{(u)}(\xi) \theta s.
\end{cases}$$

This concludes the proof of the theorem. \hfill \Box

**Proof of Theorem 5.2.** All the proof is based on (5.4). First note that by Proposition 4.1 and (5.18) we have that for any $u$

$$(5.19) \quad \Lambda_{(s,t)}(\xi) \leq s C_B(\xi) - t C_G^{(\frac{u-p}{u(1-p)})}(\xi) \quad \text{and} \quad \Lambda_{(s,t)}(\xi) \leq t C_G^{(\frac{u-p}{u(1-p)})}(\xi) - s C_B(\xi).$$

Therefore, if $\xi \in [\log \frac{u(1-p)}{p(1-u)} , \infty)$, $\Lambda_{(s,t)}^{(u)}(\xi) = \infty$.

If $\xi \in (0, \log \frac{u(1-p)}{p(1-u)} )$ we define three regions in the quadrant by

$$L = \{(s,t) : t < k^{(u)}(\xi) s \}, \quad M = \{(s,t) : k^{(u)}(\xi) s \leq t \leq k^{(u)}(\xi) s \}, \quad U = \mathbb{R}^2 \setminus (M \cup L).$$

$k^{(u)}(\xi)$ is defined by (5.5) and one can directly verify that $k^{(u)}(\xi) s < k^{(u)}(\xi)$, for $(s,t) \in L$, $\Lambda_{s,t}^{(u)}(\xi) = s C_B^{(u)}(\xi) - t C_G^{(\frac{u-p}{u(1-p)})}(\xi) = \Lambda_{(s,t)}^{(u),hor}(\xi)$ by (5.4), (5.19), since $\Lambda_{(s,t)}^{(u),ver}(\xi) = \Lambda_{(s,t)}(\xi)$.

For $(s,t) \in U$ the arguments are symmetric, with $\Lambda_{s,t}^{(u)}(\xi) = t C_G^{(\frac{u-p}{u(1-p)})}(\xi) - s C_B^{(u)}(\xi)$.

From (5.4), (5.19) and Theorem 5.1 we have that

$$\Lambda_{s,t}^{(u)}(\xi) = \begin{cases} 
\Lambda_{(s,t)}^{(u),ver}(\xi), & t \geq k^{(u)}(\xi) s, \\
\Lambda_{s,t}^{(u),ver}(\xi) \lor \Lambda_{s,t}^{(u),hor}(\xi), & k^{(u)}(\xi) s < t < k^{(u)}(\xi) s, \\
\Lambda_{s,t}^{(u),hor}(\xi), & t \leq k^{(u)}(\xi) s.
\end{cases}$$

By (5.4) and Theorem 5.1 $\Lambda_{s,t}^{(u)}(\xi)$ is continuous in $(s,t)$. From this and the fact that the middle branch above is linear in $(s,t)$, we conclude that the slope $\ell^{(u)}(\xi)$ of the line

$$t = \ell^{(u)}(\xi) s \iff \{(s,t) \in \mathbb{R}^2_+ : \Lambda_{(s,t)}^{(u),ver}(\xi) = \Lambda_{(s,t)}^{(u),hor}(\xi) \}$$
satisfies \( k^{(u)}(\xi) \geq \ell^{(u)}(\xi) \geq k^{(u)}(-\xi) \) and therefore
\[
\Lambda_{(s,t)}^{(u)}(\xi) = \begin{cases} 
sc_B^{(u)}(\xi) - tsc_G^{(u)}(\xi), & \text{if } k^{(u)}(\xi) \leq t \leq \ell^{(u)}(\xi), \\
tsc_G^{(u)}(\xi) - ssc_B^{(u)}(\xi), & \text{if } \ell^{(u)}(\xi) < t < k^{(u)}(\xi).
\end{cases}
\]
This gives the theorem. \( \square \)

APPENDIX A. A CONVEX ANALYSIS PROPOSITION

Proof of Proposition 4.7. Fix \((s, t) \in \mathbb{R}^2_+\) and call the line (A.2) below this line (A.2) a solution to (A.1) exists and is giving the minimizing argument. We hold \( u \) respectively, to obtain (A.3) and (A.4)
f
(A.5) \[ \Lambda(s, t - z) - f_{s,t-z}(u^{*}_{s,t-z}) \leq f_{s,t-z}(u^{*}_{s,t-z}) - f_{s,t-z}(u). \]

Since the minimizer is unique we have that \( f_{s-z,t}(u^{*}_{s-z,t}) - f_{s,z,t}(u) < 0 \) unless \( u = u^{*}_{s,z,t} \) and \( f_{s,t-z}(u^{*}_{s,t-z}) - f_{s,t-z}(u) < 0 \) unless \( u = u^{*}_{s,t-z} \). Set \( u = u^{*}_{s,t} \) and substitute it in (A.3) and (A.4)
(A.6) \[ \Lambda(s, t - z) - f_{s,t-z}(u^{*}_{s,t}) \leq f_{s,t-z}(u^{*}_{s,t}) - f_{s,t-z}(u^{*}_{s,t}). \]

Note that (4.20) implies that there exists a sequence \( z_n \to z \in [0, s] \) or \( \tilde{z}_n \to \tilde{z} \in [0, t] \) such that at least one of the following limits holds
(A.7) \[ \Lambda(s - z_n, t) - f_{s,z_n,t}(u^{*}_{s,t}) \to 0, \]
(A.8) \[ \Lambda(s, t - \tilde{z}_n) - f_{s,t-	ilde{z}_n}(u^{*}_{s,t}) \to 0. \]
If \( t < -\frac{h'(b)}{g'(b)} s \) then the point \((s, t - \tilde{z})\) is below the critical line for every \( \tilde{z} \in [0, t] \). The point \((s - z, t)\) can be above or below the critical line according to the value of \( z \). We analyse these two cases for the first supremum in (4.20). The case for the second supremum is identical to case (a) below, as for all \( \tilde{z} \), the index point stays below the critical line.

(a) If \( 0 \leq z < s + t \frac{g'(b)}{h'(b)} \), we have that both \( u^*_s, t, u^*_s, t, z \in (a, b) \). In particular

\[
(A.9) \quad h'(u^*_1, \nu) + \nu g'(u^*_1, \nu) = 0.
\]

By the implicit function theorem we can take the derivative of the previous expression respect to \( \nu \) and find

\[
\frac{du^*_1, \nu}{d\nu} = -\frac{g'(u^*_1, \nu)}{h''(u^*_1, \nu) + \nu g''(u^*_1, \nu)} > 0.
\]

This implies that for all \( z \in (0, s + t \frac{g'(b)}{h'(b)}) \) and \( \tilde{z} \in (0, t) \), \( u^*_s, t - \tilde{z} < u^*_s, \tilde{z} < u^*_s, t \).

We want to show that (A.7) is possible if only \( z_n \to 0 \) from which the result follows from continuity. The right hand side in (A.5) is negative and therefore, by continuity we can argue that the supremum will be attained at one of the boundary points. Thus, we have only to show that

\[
(A.10) \quad \lim_{z \to s + t \frac{g'(b)}{h'(b)}} f_{s-z, t}(u^*_s, z, t) - f_{s-z, t}(u^*_s, t) < 0.
\]

For any fixed \( z \in (0, s + t \frac{g'(b)}{h'(b)}) \) we have that

\[
f_{s-z, t}(u^*_s, z, t) - f_{s-z, t}(u^*_s, t) < 0.
\]

Therefore we obtain the proof if we show that the last expression is decreasing in \( z \). Take the derivative in \( z \), use (A.9), recall that \( u^*_s, t < u^*_s, z, t \) and \( h(u) \) is an increasing function by hypothesis

\[
\frac{d}{dz} \left( (s - z)h(u^*_s, z, t) + tg(u^*_s, z, t) - [(s - z)h(u^*_s, t) + tg(u^*_s, t)] \right)
\]

\[
= -h(u^*_s, z, t) + ((s - z)h'(u^*_s, z, t) + tg'(u^*_s, z, t)) \frac{du^*_s, z, t}{dz} + h(u^*_s, t)
\]

\[
= h(u^*_s, t) - h(u^*_s, z, t) < 0.
\]

(b) If \( s + t \frac{g'(b)}{h'(b)} \leq z \leq s \), we have that \( u^*_s, z, t = b \). Note that \( u^*_s, t < u^*_s, z, t = b \) in this case and therefore \( f_{s-z, t}(b) - f_{s-z, t}(u^*_s, t) < 0 \) for every \( z \in [s + t \frac{g'(b)}{h'(b)}, s] \). This implies that (A.7) can never be true for \( z \in (s + t \frac{g'(b)}{h'(b)}, s] \). But the boundary point \( z = s + t \frac{g'(b)}{h'(b)} \) is also not optimal by continuity considerations and (A.10).

Therefore, the potential maximum happens at \( z = 0 \). Similarly, this will be true for \( \tilde{z} = 0 \) and therefore \( \Lambda(s, t) = f_{s, t}(u^*_s, t) \) as required. \( \square \)
APPENDIX B. LAW OF LARGE NUMBERS AND PROOF OF BURKE’S PROPERTY

Proof of Lemma 2.4. We omit the superscripts and indices from the $I, J$ and we simply denote

$$I = \max\{I - J, \omega\}, \quad \text{and} \quad \bar{I} = (I - J + \omega)^+.$$ 

The marginal distributions of $(\bar{I}, \bar{J}, \alpha)$ can be computed directly, using equations (2.14), (2.15). For example, since $\alpha$ only takes the values 0 or 1 it suffices to compute

$$P\{\alpha = 1\} = P\{\min\{I, J + \omega\} = 1\} = P\{I = 1, J + \omega \geq 1\} = u(p + (1 - p)(1 - \frac{u - p}{u(1 - p)})) = p.$$ 

The remaining calculations are left to the reader.

The proof of independence goes by calculating the Laplace transform of the triple $(\bar{I}, \bar{J}, \alpha)$. Let $x, z \in \mathbb{R}$ and $y > \log[p(1-u)/(u(1-p))]$. Recall that $u \in (p, 1)$. Then compute, using (2.14) and (2.15), the joint Laplace transform

$$E(e^{-x\bar{I} - y\bar{J} - z\alpha}) = E[e^{-x \max\{I-J,\omega\} - y(1-I+\omega)^+ - z \min\{I, J\}}]$$

$$= puE[e^{-x(\max\{1-J,1\}) - y - z \min\{1, J+1\}}] + p(1-u)E[e^{-x-y(\max\{1-J,1\})}]$$

$$+ (1-p)uE[e^{-x(\max\{1-J,1\}) - y(\max\{1-J,1\}) - z(1\wedge J)}] + (1-p)(1-u)E[e^{-yJ}]$$

$$= pu \frac{u - p}{u(1-p)} e^{-(x+z)} \sum_{j=0}^{\infty} \left( \frac{p(1-u)}{u(1-p)} \right)^j e^{-yj}$$

$$+ p(1-u) \frac{u - p}{u(1-p)} e^{-(x+y)} \sum_{j=0}^{\infty} \left( \frac{p(1-u)}{u(1-p)} \right)^j e^{-yj}$$

$$+ (1-p)u \frac{u - p}{u(1-p)} \left( e^{-x} + \sum_{j=1}^{\infty} \left( \frac{p(1-u)}{u(1-p)} \right)^j e^{-y(j-1)-z} \right)$$

$$+ (1-p)(1-u) \sum_{j=0}^{\infty} \left( \frac{p(1-u)}{u(1-p)} \right)^j e^{-yj}$$

$$= \frac{\frac{u - p}{u(1-p)} e^{-y}}{1 - \frac{p(1-u)}{u(1-p)} e^{-y}} \left( pu e^{-(x+z)} + p(1-u) e^{-(x+y)} + (1-p)(1-u) \right)$$

$$+ (1-p)u \frac{u - p}{u(1-p)} \left[ e^{-x} \left( 1 - \frac{p(1-u)}{u(1-p)} e^{-y} \right) + e^{-z} \frac{p(1-u)}{u(1-p)} \right]$$

$$= \frac{\frac{u - p}{u(1-p)} e^{-y}}{1 - \frac{p(1-u)}{u(1-p)} e^{-y}} \left( pu e^{-(x+z)} + (1-p)(1-u) + (1-p)u e^{-x} + p(1-u) e^{-z} \right)$$

$$= E(e^{-y\bar{J}})E(e^{-x\bar{I}})E(e^{-z\alpha}) \quad \square$$
Proof of Corollary 2.5. The proof is inductive. Consider the countable set of paths $\Psi$ that connect the $y$-axis to the $x$-axis. The trivial case is when $I_{\psi_0} = \emptyset$ (i.e. $\psi_0$ is the union of the two axes, $\psi_0 \in \Psi$) and then the statement reduces to the independence of the $\omega_{i,j}$’s on the $x$ and $y$ axes which is true by the definition of the environment.

Assume that for a $\psi \in \Psi$ the statement holds. We say that a lattice vertex $v_{i_0}$ on $\psi$ $(i, j) \in \mathbb{Z}^2_+$ is a west-south corner of $\psi$ if

$$(v_{i_0-1}, v_{i_0}, v_{i_0+1}) = ((i, j + 1), (i, j), (i + 1, j)).$$

Now define a new path $\tilde{\psi}$ by replacing $v_{i_0}$ with $\tilde{v}_{i_0} = (i + 1, j + 1)$ and keep all the other points intact which means that $v_i = \tilde{v}_i$ for $i \neq i_0$. In this way we have $I_{\tilde{\psi}} = I_{\psi} \cup \{(i, j)\}$.

Going from $\psi$ to $\tilde{\psi}$ we have also a change in the set of random variables in (2.18). In fact (B.1)

$${I_{i+1,j}, J_{i+1,j}}$$

have been replaced by

(B.2) $${I_{i+1,j+1}, J_{i+1,j+1}, \alpha_{i+1,j+1}}.$$

By (2.14) and (2.15) the variables in (B.2) are determined by (B.1) and $\omega_{i+1,j+1}$. By construction $\omega_{i+1,j+1}$ is independent of (2.18) for the $\psi$ under consideration. By construction the triple $\{I_{i+1,j}, J_{i+1,j}, \omega_{i+1,j+1}\}$ are independent random variables and by the induction assumption we have they are in turn independent of the all other variables (2.18). Finally Lemma 2.4 implies that also the triple $\{I_{i+1,j+1}, J_{i+1,j+1}, \omega_{i,j}\}$ are independent random variables with the correct marginal distribution and they are independent of all the random variables of $\tilde{\psi}$. All these observations prove that also $\tilde{\psi}$ satisfies the statement of the corollary.

Note that if we start with $\psi_0$, we can build a path $\psi \in \Psi$ by flipping west-south corners finitely many times. The induction argument guarantees that class $\Psi$ satisfies the corollary.

The general statement follows also for an arbitrary down-right path $\psi$ using the independence of finite subcollections. Consider any square $\mathcal{R} = \{i \leq 0, j \leq M\}$ large enough so that the corner $(M, M)$ lies outside $\psi \cup I_{\psi}$. The $\alpha$ and $L(\psi)$ variables associated to $\psi$ that lie in $\mathcal{R}$ are a subset of the variables of the path $\tilde{\psi}$ that goes through the points $(0, M), (M, M)$ and $(M, 0)$. This path $\tilde{\psi}$ connects the axes so the first part of the proof applies to it. Thus the variables (2.18) that lie inside an arbitrarily large square are independent. $\square$

B.1. Laws of large numbers.

Proof of Theorems 2.6, 1.1. Recall the definitions of $g^{(u), \text{ver}}_{pp}(s, t)$ and $g^{(u), \text{hor}}_{pp}(s, t)$ from Theorem 2.2. In the sequence we use freely the facts that $g_{pp}(s, t)$ is 1-homogeneous and concave.

If $t < \frac{1-p}{p}s$, start from equation (2.4). Divide by $N$ to obtain the macroscopic variational formulation

$$g^{(u)}_{pp}(s, t) = g^{(u), \text{hor}}_{pp}(s, t) \vee g^{(u), \text{ver}}_{pp}(s, t)$$

$$= \sup_{0 \leq z \leq s} \{g^{(u)}_{pp}(z, 0) + g_{pp}(s - z, t)\} \vee \sup_{0 \leq \z \leq t} \{g^{(u)}_{pp}(0, \z) + g_{pp}(s, t - \z)\}$$
(B.3) \[ \sup_{0 \leq z \leq s} \{ z \mathbb{E}(I^{(u)}(s-z,t)) \} \sup_{0 \leq z \leq t} \{ z \mathbb{E}(J^{(u)}(s,t-z)) \}. \]

How we obtain equation [B.3] is not immediate to see. Since this step is technical and it is not the main goal of this proof, we postpone it until the end. Therefore assume for the moment that [B.3] holds. Subtract \( g_{pp}^{(u)}(s,t) \) to either side of [B.3]

\[ 0 = \sup_{0 \leq z \leq s} \{ g_{pp}(s-z,t) - [(s-z)u + t \frac{p(1-u)}{u-p}] \} \sup_{0 \leq z \leq t} \{ g_{pp}(s,t-z) - [(t-z) \frac{p(1-u)}{u-p} + su] \}. \]

We use Proposition 4.7 by identifying as \( I = (p,1) \), \( \Lambda(s,t) = g_{pp}(s,t) \), \( h(u) = s \), \( g(u) = \frac{p(1-u)}{u-p} \) and therefore \( f_{s,t}(u) = su + t \frac{p(1-u)}{u-p} \). Note that \( h'(u) > 0 \), \( g'(u) < 0 \) for every \( u \in (p,1) \) and in particular \( f''_{s,t}(u) > 0 \) for every \( (s,t) \in \mathbb{R}_+^2 \). Moreover \( \lim_{u \searrow p} f_{s,t}(u) = \infty \) and \( f_{s,t}(1) = s < \infty \). Therefore

(B.4) \[ g_{pp}(s,t) = \min_{u \in (p,1)} \left\{ su + t \frac{p(1-u)}{u-p} \right\} = \left( \sqrt{ps} + \sqrt{(1-p)t} \right)^2 - t, \quad \text{if } t < s \frac{1-p}{p}. \]

If \( t \geq \frac{1-p}{p} s \), We want to find an upper and a lower bound for \( G_{|[Ns],[Nt]} \). The upper bound is trivial since by model definition \( G_{|[Ns],[Nt]} \leq [Ns] \). For the lower bound, force a macroscopic distance from the critical line, i.e. assume that it is possible to find a \( \varepsilon > 0 \) so that the sequence of endpoints \( ([Ns]_t, [Nt]_t) \) satisfy

(B.5) \[ \lim_{N \to \infty} \frac{[Nt]}{[Ns]} \geq \frac{1-p}{p} + \varepsilon. \]

Then consider the following strategy: construct an approximate maximal path \( \pi \) for \( G_{|[Ns],[Nt]} \), knowing that for large \( \left| Nt \right| \geq \left( \frac{1-p}{p} + \varepsilon \right) [Ns] \). \( \pi \) starts from \((0,0)\) and moves up until it finds a weight to collect horizontally on his right. After that this procedure repeats. For each iteration of this procedure, the vertical length of this path increases by a random Geometric\((p)\) length, independently of the past. Define \( Y \sim \text{Geometric}(p) \) with range on \( 0,1,... \). By construction, we have

\[ \left\{ \sum_{i=1}^{[Ns]} Y_i > [Nt] \right\} \supseteq \{ G_{|[Ns],[Nt]} < [Ns] \}. \]

The relation on \((s,t)\) implies that the larger event above is large deviation event, and therefore by the Borel-Cantelli lemma, \( G_{|[Ns],[Nt]} = [Ns] \). Scaling by \( N \) and letting it tend to \( \infty \) completes the proof.

We finally prove [B.3]. For a lower bound, fix any \( z \in [0,s] \) and \( \bar{z} \in [0,t] \). Then if we move on the horizontal axis

\[ G_{|[Ns],[Nt]}^{(u)} \geq \sum_{i=1}^{[Nz]} I_{i,0}^{(u)} + G_{([Nz],1,([Ns],[Nt]))}. \]

Divide by \( N \). Observe that the left hand side converges a.s. to \( g_{pp}^{(u)}(s,t) \). While the first term on the right converges a.s. to \( z \mathbb{E}(I^{(u)}) \). The second on the right, converges in probability
to $g_{pp}(s - z, t)$. In particular, we can find a subsequence $N_k$ such that the convergence is almost sure for the second term. Taking limits on this subsequence, we conclude

$$g_{pp}^{(u)}(s, t) \geq z\mathbb{E}(I^u) + g_{pp}(s - z, t).$$

Since $z$ is arbitrary we can take supremum over $z$ in both sides of the inequality above. The same arguments will work if we move on the vertical axis. Thus, we obtain the lower bound for (B.3). For the upper bound, we partition the two axes. Fix $\varepsilon, \tilde{\varepsilon} > 0$ and let $\{0 = q_0, \varepsilon = q_1, 2\varepsilon = q_2, \ldots, s \mid \varepsilon \} \in (\varepsilon, s = q_M$} a partition of $(0, s)$ and $\{0 = q_0, \tilde{\varepsilon} = q_1, 2\tilde{\varepsilon} = q_2, \ldots, t \mid \tilde{\varepsilon} \} \in (\tilde{\varepsilon}, t = q_M$} a partition of $(0, t)$. The maximal path that utilises $G_{N_k, N}^{(u)}$ has to exit between $[Nk\varepsilon]$ and $[N(k + 1)\varepsilon]$ for some $k$ if it chooses to go through the $x$-axis and between $[N\tilde{k}\tilde{\varepsilon}]$ and $[N(\tilde{k} + 1)\tilde{\varepsilon}]$ for some $\tilde{k}$ if it goes through the $y$-axis. Therefore, we may write

$$G_{[Ns],[Nt]}^{(u)} \leq \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \left\{ \left[ N(k + 1)\varepsilon \right] I_{k,0}^{(u)} + G_{(\lfloor Nk\varepsilon \rfloor,1),([Ns],[Nt])} \right\} \cup \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \left\{ \left[ N(k + 1)\tilde{\varepsilon} \right] J_{k,0}^{(u)} + G_{(1,([N\tilde{k}\tilde{\varepsilon}]),([Ns],[Nt]))} \right\}.$$

Divide by $N$. The right-hand side converges in probability to the constant

$$\max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \left\{ (k + 1)\varepsilon u + g_{pp}(s - \varepsilon k, t) \right\}$$

$$= \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \left\{ k\varepsilon u + g_{pp}(s - \varepsilon k, t) \right\} + \varepsilon u$$

$$= (\max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \left\{ k\varepsilon \frac{p(1-u)}{u-p} + g_{pp}(s, t - \varepsilon k) \right\} + \varepsilon \frac{p(1-u)}{u-p})$$

$$= (\max_{q_k} \left\{ q_k u + g_{pp}(s - q_k, t) \right\} + \varepsilon u)$$

$$\leq \left( \sup_{0 \leq z \leq s} \left\{ zu + g_{pp}(s - z, t) \right\} + \varepsilon u \right)$$

$$\leq (\max_{0 \leq t \leq \tilde{\varepsilon}} \left\{ \tilde{\varepsilon} \frac{p(1-u)}{u-p} + g_{pp}(s, t - \tilde{\varepsilon}) \right\} + \varepsilon \frac{p(1-u)}{u-p})$$

The convergence becomes a.s. on a subsequence. The upper bound for (B.3) now follows by letting $\varepsilon \to 0$ and $\tilde{\varepsilon} \to 0$ in the final equation. \qed
Appendix C. Basic properties of the rate function

Proof of Theorem 3.1. First we prove the existence of limit (3.2). Take \( m, n \in \mathbb{N} \) and an error due to the floor function \( r \) and \( Nr \).

By (3.1)

\[ \mathbb{P}\{G_{[(m+n)s],[m+n]t} \geq (m+n)r\} \]

and superadditivity

\[ \geq \mathbb{P}\{G_{[ms],[mt]} + G_{[(m+n)s],[m+n)t]} \geq (m+n)r\}, \quad \text{by superadditivity} \]

By (3.1) \( \mathbb{P}\{G_{[x_{m,n}] \geq 0}\} = 1 \). Take logarithms in the last inequality; then by Fekete’s lemma the limit

\[ \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{[Ns],[Nt]} \geq Nr\} \]

exists for any \((s,t) \in \mathbb{R}^2 \setminus \{0\}\) and \( r \in [0,s]\) and in fact equals \( \sup_N N^{-1} \log \mathbb{P}\{G_{[Ns],[Nt]} \geq Nr\} \). The value of the limit is now denoted by \(-J_{s,t}(r)\).

From the superadditivity of \( G \) we can also obtain the convexity of the limit. Pick any \( \lambda \in (0,1) \) and define the triple \((s,t,r) = \lambda((s_1,t_1),r_1) + (1-\lambda)((s_2,t_2),r_2)\) with \( r_1 \in [0,s_1] \) and \( r_2 \in [0,s_2] \). Then

\[ N^{-1} \log \mathbb{P}\{G_{[Ns],[Nt]} \geq Nr\} \]

\[ \geq \lambda(\lambda N)^{-1} \log \mathbb{P}\{G_{[N\lambda s_1],[N\lambda t_1]} \geq N\lambda r_1\} \]

\[ + (1-\lambda)((1-\lambda)N)^{-1} \log \mathbb{P}\{G_{[N(1-\lambda)s_2],[N(1-\lambda)t_2]} \geq N(1-\lambda)r_2\}. \]

Multiply both sides by \(-1\) and invert the sign of the inequality to obtain for \( N \to \infty \)

\[ J_{s,t}(r) \leq \lambda J_{s_1,t_1}(r_1) + (1-\lambda)J_{s_2,t_2}(r_2). \]

From (3.1) we know that \( J \) is finite and we have just proven that it is also convex. This implies that \( J \) is continuous on \( A \) and upper semicontinuous on the whole set \( A \), from Theorems 10.1 and 10.2 in [44]. Moreover, \( J_{s,t}(r) \) on \( A \) can be uniquely extended to a continuous function on \( A \) by Theorem 10.3 in [44].

Finally, the law of large numbers for the last passage time implies \( J_{s,t}(r) = 0 \) for \( r < gp(s,t) \) and then by continuity for \( r \leq gp(s,t) \). Use the same method of proof of Proposition 3.1(b) of [14] to get the concentration inequality:

\[ \mathbb{P}\{G_{[Ns],[Nt]} - \mathbb{E}[G_{[Ns],[Nt]}] \geq N\varepsilon\} \leq 2e^{-c\varepsilon^2 n} \quad \forall n \in \mathbb{N}. \]

This holds for a given \((s,t) \in \mathbb{R}^2_+\) and \( \varepsilon > 0 \). Constant \( c > 0 \) will depend on \( s, t, \varepsilon \).

Since \( N^{-1}\mathbb{E}[G_{[Ns],[Nt]}] \to gp(s,t) \), this implies that \( J_{s,t}(r) > 0 \) for \( r > gp(s,t) \) (without excluding the value \( 0 \)). \( \square \)

Lemma C.1 (Continuity in the macroscopic directions). Let \((s,t) \in \mathbb{R}^2_{\geq 0}\) and \( u_N = (s_N,t_N) \in \mathbb{Z}_+^2\) an increasing sequence such that \( N^{-1}u_N \to (s,t) \). Then for \( r \in [0,s] \)

\[ \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{s_N,t_N} \geq Nr\} = -J_{s,t}(r). \]
Proof. Since $u_N$ and $([N s], [N t])$ are non-decreasing in $N$, for each $N$ we can find two sequences $\ell_N$ and $m_N$ such that

$$[\ell_N(s, t)] \leq u_N \leq [m_N(s, t)] \quad \text{with} \quad N - m_N, N - \ell_N = o(N).$$

Then it is immediate that

$$G_{[\ell_N s], [\ell_N t]} \leq G_{s_N, t_N} \leq G_{[m_N s], [m_N t]},$$

which gives

$$\mathbb{P}\{G_{[m_N s], [m_N t]} \geq N r\} \geq \mathbb{P}\{G_{s_N, t_N} \geq N r\} \geq \mathbb{P}\{G_{[\ell_N s], [\ell_N t]} \geq N r\}. $$

Taking the limit of both sides and by the continuity of the rate function we have

$$\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{s_N, t_N} \geq N r\} \leq \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{[m_N s], [m_N t]} \geq N r\} \leq \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{[\ell_N s], [\ell_N t]} \geq N r\}. $$

Taking the limit of both sides and by the continuity of the rate function we have

$$\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{G_{s_N, t_N} \geq N r\} \leq \lim_{N \to \infty} m_N^{-1} \log \mathbb{P}\{G_{[m_N s], [m_N t]} \geq N r\} = m_N r - (m_N - N)r \leq \lim_{N \to \infty} m_N^{-1} \log \mathbb{P}\{G_{[m_N s], [m_N t]} \geq m_N(r - \varepsilon)\}$$

for any $\varepsilon > 0$ and $N$ large enough

$$= - J_{s,t}(r - \varepsilon).$$

Then let $\varepsilon \to 0$ and invoke the continuity of $J$ for the upper bound. Same arguments are valid for the lower bound, using $\lim_{N \to \infty}$. \hfill $\square$

From Theorem 3.1 we have that $J_{s,t}(r)$ can be continuously extended to the boundary of the domain $A = \{(s, t, r) : J_{s,t}(r) < \infty\}$,

$$\partial A = \{s = 0, t \geq 0, r \leq 0\} \cup \{t = 0, s \geq r \vee 0\} \cup \{s = r, t \geq 0\}. $$

It will be convenient to understand the values of the continuation of $J_{s,t}(r)$ on $\partial A$.

For any $s, t > 0$ and $r \leq 0$, $J_{s,t}(r) = 0$. Therefore, we will have that

$$J_{s,0}(r) = J_{0,t}(r) = 0, \quad r \leq 0.$$

Now for the $r > 0$ case. Since we want $J_{s,0}(r)$ with $(s \geq r)$ continuous we define $J_{s,h}(r) = \lim_{h \to 0} J_{s,h}(r)$. An approximation using thin rectangles as in [27] gives that

$$J_{s,0}(r) = s I_B(r/s) = r \log \frac{r}{sp} + (s - r) \log \frac{1 - r/s}{1 - p}.$$ 

Recall that $I_B$ is the Cramér rate function for sums of i.i.d. $\omega_i \sim$ Bernoulli($p$). This discussion is summarised in Corollary 3.3.

**Lemma C.2** (Infimal convolutions). For each $N$ let $L_N$ and $Z_N$ be two independent random variables. Assume their rate functions

(C.4) \quad $\lambda(s) = - \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{L_N \geq N s\},$

(C.5) \quad $\phi(s) = - \lim_{N \to \infty} N^{-1} \log \mathbb{P}\{Z_N \geq N s\}$

exists and
(1) \( \lambda(s) \) is finite in \((-\infty, b)\) with \( b \in \mathbb{R} \) and \( \lambda(s) = \infty \) when \( s > b \).

(2) \( \lambda \) is continuous at all points for which is finite and lower semi-continuous on \( \mathbb{R} \).

(3) \( \phi(s) \) is finite for all \( s \in \mathbb{R} \).

(4) \( \lambda(a_{\lambda}) = \phi(a_{\phi}) = 0 \) for some \( a_{\lambda}, a_{\phi} \in \mathbb{R} \).

Then for \( r \in \mathbb{R} \)

\[
\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{L_N + Z_N \geq Nr\} = \begin{cases} 
- \inf_{a_{\lambda} \leq s \leq b \wedge (r - a_{\phi})} \{\phi(r - s) + \lambda(s)\}, & r > a_{\phi} + a_{\lambda}, \\
0, & r \leq a_{\phi} + a_{\lambda}.
\end{cases}
\]

(C.6)

Proof. First observe that the infimum in (C.6) is obtained when \( s \) satisfies \( a_{\lambda} \leq s \leq b \wedge (r - a_{\phi}) \).

The lower bound follows from the independence of the two random variables

\[
\mathbb{P}\{L_N + Z_N \geq Nr\} \geq \mathbb{P}\{Z_N \geq \mathbb{N}(r - s)\}\mathbb{P}\{L_N \geq Ns\}.
\]

To upper bound for \( r \leq a_{\lambda} + a_{\phi} \) is immediate.

We therefore only discuss the case \( r > a_{\lambda} + a_{\phi} \). Take a finite partition \( a_{\lambda} = q_0 < \cdots < q_{m-1} = b \wedge (r - a_{\phi}) < q_m = q_{m+1} \).

Use a union bound and the independence of \( L_N, Z_N \) to derive

\[
\mathbb{P}\{L_N + Z_N \geq Nr\} \leq \mathbb{P}\{L_N + Z_N \geq Nr, L_N < Nq_0\} + \sum_{i=0}^{m-1} \mathbb{P}\{L_N + Z_N \geq Nr, nq_i \leq L_N \leq Nq_{i+1}\} + P\{L_N \geq Nq_m\}
\]

\[
\leq P\{Z_N \geq \mathbb{N}(r - q_0)\} + \sum_{i=0}^{m-1} \mathbb{P}\{Z_N \geq \mathbb{N}(r - q_{i+1})\}\mathbb{P}\{L_N \geq Nq_i\} + P\{L_N \geq Nq_m\}.
\]

Now take the logarithm on both sides, divide by \( N \) and finally take \( N \to \infty \) to obtain

\[
\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{L_N + Z_N \geq Nr\} \leq - \min \left\{ \phi(r - q_0), \min_{0 \leq i \leq m-1} \{\phi(r - q_{i+1}) + \lambda(q_i)\}, \lambda(q_m) \right\}.
\]

We may simplify the last inequality as

\[
\mathbb{P}\{L_N + Z_N \geq Nr\} \leq - \min_{-1 \leq i \leq m} \{\phi(r - q_{i+1}) + \lambda(q_i)\}
\]

This is because \( \lambda(q_0) = 0 \). Also, if \( b \leq r - a_{\phi} \) then \( \lambda(q_m) = \infty \) and it can be omitted from the minimum. If \( b > r - a_{\phi} \) then \( \phi(r - q_m) = 0 \). The result then follows by the continuity of \( \lambda \) on \([a_{\lambda}, b]\) by arbitrarily refining the partition. \( \square \)
We want to find the solutions to
\begin{equation}
0 = u^2 2 s [(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})] - u p [(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + 2 s (1 - e^{-\xi})] \\
+ (e^{-\xi} - 1)p ((1 - p)(s + t) - s),
\end{equation}
when \( \xi \geq 0 \) and prove that the corresponding solution is the minimizing argument for the function
\[
f(u) = s C_B^{(u)} (\xi) - t C_G^{(\frac{u - p}{1 - p})} (-\xi).
\]
The discriminant \( \Delta \) of (D.1) is
\[
\Delta = [(1 - p)p(s + t)e^{-2\xi}(e^\xi - 1)^2 + e^{-\xi}4p(1 - p)st(e^\xi - 1)^2 \geq 0.
\]
Therefore two solutions and are given by
\[
u^*_\pm = \frac{p(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + 2sp(1 - e^{-\xi}) \pm \sqrt{\Delta}}{2s(2p - 1 + (1 - p)e^\xi - pe^{-\xi})}.
\]
We begin by checking if \( \nu^*_+ \in (p, 1) \). It is immediate to check that the \(-\sqrt{\Delta}\) solution is not larger than \( p \) when \( \xi > 0 \) and \( \nu^*_- < p \), so we focus on the plus one and \( \nu^*_+ \). In that case, the following inequalities are equivalent:
\[
\frac{p(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + 2sp(1 - e^{-\xi}) + \sqrt{\Delta}}{2s[(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})]} > p \\
\frac{p(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + 2sp(1 - e^{-\xi})(1 - p) + \sqrt{\Delta} - p2s(1 - p)(e^\xi - 1)}{2s[(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})]} > 0 \\
\frac{p(1 - p)(t - s)(e^\xi + e^{-\xi} - 2) + \sqrt{\Delta}}{2s[(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})]} > 0,
\]
which is immediately true since the numerator and denominator are always positive for \( \xi > 0 \).

The other bound
\[
\frac{p(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + 2sp(1 - e^{-\xi}) + \sqrt{\Delta}}{2s[(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})]} \leq 1 \iff \\
\frac{p(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + \sqrt{\Delta} - 2s(1 - p)(e^\xi - 1)}{2s[(1 - p)(e^\xi - 1) + p(1 - e^{-\xi})]} \leq 0.
\]
The denominator is always positive for \( \xi > 0 \), therefore the overall fraction is negative if and only if
\begin{equation}
p(1 - p)(s + t)(e^\xi + e^{-\xi} - 2) + \sqrt{\Delta} - 2s(1 - p)(e^\xi - 1) \leq 0.
\end{equation}
We will verify (D.2) when \( t < \frac{1 - p}{p}s \). When \( t \) satisfies this condition it automatically satisfies
\[
t < \left( \frac{2}{p(1 - e^{-\xi})} - 1 \right)s.
\]
When this holds, (D.2) can be equivalently written as
\[ \sqrt{\Delta} < 2s(1-p)(e^\xi - 1) - p(1-p)(s+t)(e^{2\xi} + e^{-\xi} - 2). \]
Both sides of the above inequality are positive, so by squaring both sides we reach the equivalent sequence of inequalities
\[
4(1-p)^2p^2(s+t)^2(cosh \xi - 1)^2 + 8stp(1-p)(cosh \xi - 1) \\
\leq 4p^2(1-p)^2(s+t)^2(cosh \xi - 1)^2 + 4s^2(1-p)^2(e^{2\xi} + 1 - 2e^{\xi}) \\
- 8sp(1-p)^2(s+t)(cosh \xi - 1)(e^{\xi} - 1) \\
\iff stp(e^{\xi} - p(e^{\xi} - 1)) \leq s^2(1-p)e^{\xi} - s^2p(1-p)(e^{\xi} - 1) \\
\iff t \leq \frac{(1-p)}{p}s,
\]
which is true from our hypothesis. This implies that \( u^* \in (p, 1] \) It remains to argue that \( u^*_+ \) is the minimizing point.

For \( \xi > 0 \), the derivative is positive whenever
\[
s = \frac{e^\xi - 1}{1 + u(e^\xi - 1)} - \frac{p(p-1)(e^{-\xi} - 1)}{u^2(1+p(e^{-\xi} - 1)) - up[1 + e^{-\xi} + p(e^{-\xi} - 1)] + p^2e^{-\xi}} = \frac{N(u, \xi)}{D(u, \xi)} > 0.
\]
The numerator \( N(u, \xi) \) is given by the right hand side of (D.1) and by what we discussed up to this point, for \( \xi > 0 \) and \( t < \frac{1-p}{p}s \)
\[
N(u, \xi) \begin{cases} 
\geq 0 & \text{if } u \in [u^*_+, 1], \\
< 0 & \text{if } u \in (p, u^*_+).
\end{cases}
\]
The denominator is
\[
D(u, \xi) = [1 - u + ue^\xi][u^2(1-p(1 - e^{-\xi})) - up[1 + e^{-\xi} + p(e^{-\xi} - 1)] + p^2e^{-\xi}].
\]
The first factor is always positive for this reason we study the sign of the parabola in the second factor. The coefficient of \( u^2 \) is positive, for every \( \xi > 0 \) and the factor itself has zeros \( u^*\pm \) given by \( u^*\pm = p, \ u^*_+ = \frac{pe^{-\xi}}{1-p+pe^{-\xi}} \leq e^{\xi} \). Since our range is \( u > p \), the second factor, and hence \( D(u, \xi) > 0 \). Overall,
\[
\frac{N(u, \xi)}{D(u, \xi)} \begin{cases} 
\geq 0 & \text{if } u \in [u^*_+, 1], \\
< 0 & \text{if } u \in (p, u^*_+).
\end{cases}
\]
Therefore \( u^*_+ \) is a global minimum.

\[ \quad \]

References
\[ \begin{array}{ll}
[1] & \text{David Aldous and Persi Diaconis. Hammersley’s interacting particle process and longest increasing subsequences.} \\
& \text{Probability theory and related fields, 103(2):199–213, 1995.} \\
[2] & \text{Gideon Amir, Ivan Corwin, and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1+ 1 dimensions.} \\
& \text{Communications on pure and applied mathematics, 64(4):466–537, 2011.}
\end{array} \]
[3] M. Balázs, E. Cator, and T. Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.*, 11:no. 42, 1094–1132 (electronic), 2006.

[4] Márton Balázs, Firas Rassoul-Agha, and Timo Seppäläinen. Large deviations and wandering exponent for random walk in a dynamic beta environment. *ArXiv Mathematics e-prints*, 2018.

[5] Guillaume Barraquand and Ivan Corwin. Random-walk in Beta-distributed random environment. *Probab. Theory Related Fields*, 167(3-4):1057–1116, April 2017.

[6] A-L Basdevant, N Enriquez, L Gerin, and J-B Gouéré. Discrete Hammersley’s lines with sources and sinks. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13:33–52, 2016.

[7] Riddhipratim Basu, Shirshendu Ganguly, and Allan Sly. Upper tail large deviations in first passage percolation. *arXiv preprint arXiv:1712.01255*, 2017.

[8] Alexei Borodin and Leonid Petrov. Inhomogeneous exponential jump model. *Probability Theory and Related Fields*, 172(1-2):323–385, 2017.

[9] Eric Cator and Piet Groeneboom. Hammersley’s process with sources and sinks. *Ann. Probab.*, 33(3):879–903, 2005.

[10] Eric Cator and Piet Groeneboom. Second class particles and cube root asymptotics for Hammersley’s process. *Ann. Probab.*, 34(4):1273–1295, 2006.

[11] Hans Chaumont and Christian Noack. Characterizing stationary 1+ 1 dimensional lattice polymer models. *Electronic Journal of Probability*, 23, 2018.

[12] Yunshyong Chow and Yu Zhang. Large deviations in first-passage percolation. *The Annals of Applied Probability*, 13(4):1601–1614, 2003.

[13] Federico Ciech and Nicos Georgiou. Order of the variance in the discrete hammersley process with boundaries. *arXiv preprint arXiv:1712.06479*, 2017.

[14] Francis Comets and Nobuo Yoshida. Branching random walks in space–time random environment: Survival probability, global and local growth rates. *Journal of Theoretical Probability*, 24:657–687, 2011.

[15] Ivan Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1(1):1130001, 76, 2012.

[16] Ivan Corwin, Timo Seppäläinen, and Hao Shen. The strict-weak lattice polymer. *Journal of Statistical Physics*, 160(4):1027–1053, 2015.

[17] Jean-Dominique Deuschel and Ofer Zeitouni. On increasing subsequences of iid samples. *Combinatorics, Probability and Computing*, 8(3):247–263, 1999.

[18] Richard Durrett and David Griffeath. Supercritical contact processes on Z. *The Annals of Probability*, pages 1–15, 1983.

[19] Richard Durrett and Thomas M Liggett. The shape of the limit set in Richardson’s growth model. *The Annals of Probability*, pages 186–193, 1981.

[20] Elhur Emrah. Limit shapes for inhomogeneous corner growth models with exponential and geometric weights. *Electron. Commun. Probab.*, 21(42):1–16, 2016.

[21] Elhur Emrah and Chris Janjigian. Large deviations for some corner growth models with inhomogeneity. *Markov Process. Related Fields*, 23:267–312, 2017.

[22] Nicos Georgiou. Soft edge results for longest increasing paths on the planar lattice. *Electronic Communications in Probability*, 15:1–13, 2010.

[23] Nicos Georgiou and Janosch Ortmann. Optimality regions and fluctuations for bernoulli last passage models. *Mathematical Physics, Analysis and Geometry*, 21(3):22, 2018.

[24] Nicos Georgiou, Firas Rassoul-Agha, and Timo Seppäläinen. Variational formulas and cocycle solutions for directed polymer and percolation models. *Communications in Mathematical Physics*, 346(2):741–779, 2016.

[25] Nicos Georgiou, Firas Rassoul-Agha, and Timo Seppäläinen. Geodesics and the competition interface for the corner growth model. *Probab. Theory Relat. Fields*, 169:223–255, 2017.

[26] Nicos Georgiou, Firas Rassoul-Agha, and Timo Seppäläinen. Stationary cocycles and Busemann functions for the corner growth model. *Probab. Theory Relat. Fields*, 169:177–222, 2017.

[27] Nicos Georgiou and Timo Seppäläinen. Large deviation rate functions for the partition function in a log-gamma distributed random potential. *Ann. Probab.*, 2013. To appear.
[28] Janko Gravner, Craig A Tracy, and Harold Widom. Limit theorems for height fluctuations in a class of discrete space and time growth models. *Journal of Statistical Physics*, 102(5-6):1085–1132, 2001.

[29] Piet Groeneboom. Ulam’s Problem and Hammersley’s Process. *The Annals of Probability*, 29(2):683–690, 2001.

[30] John M. Hammersley. A few seedlings of research. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Theory of Statistics*. The Regents of the University of California, 1972.

[31] Chris Janjigian. Large deviations of the free energy in the o’connell–yor polymer. *Journal of Statistical Physics*, 160(4):1054–1080, 2015.

[32] L. Jensen. *The asymmetric exclusion process in one dimension*. PhD thesis, Ph. D. dissertation, New York Univ., New York, 2000.

[33] Kurt Johansson. Shape fluctuations and random matrices. *Communications in mathematical physics*, 209(2):437–476, 2000.

[34] G. Kassay. A simple proof for König’s minimax theorem. *Acta Math. Hungar.*, 63(4):371–374, 1994.

[35] Harry Kesten. Aspects of first passage percolation. In *École d’été de probabilités de Saint Flour XIV-1984*, pages 125–264. Springer, 1986.

[36] Alisa Knizel, Leonid Petrov, and Axel Saenz. Generalizations of tasep in discrete and continuous inhomogeneous space. *arXiv preprint arXiv:1808.09855*, 2018.

[37] James B. Martin. Limiting shape for directed percolation models. *Ann. Probab.*, 32(2):2908–2937, 2004.

[38] J. Moriarty and N. O’Connell. On the free energy of a directed polymer in a Brownian environment. *Markov Process. Related Fields*, 13(2):251–266, 2007.

[39] Neil O’Connell. Directed polymers and the quantum Toda lattice. *Ann. Probab.*, 40(2):437–458, 2012.

[40] Neil O’Connell and Janosch Ortmann. Tracy-widom asymptotics for a random polymer model with gamma-distributed weights. *Electronic Journal of Probability*, 20, 2015.

[41] Neil O’Connell and Marc Yor. Brownian analogues of Burke’s theorem. *Stochastic Process. Appl.*, 96(2):285–304, 2001.

[42] Stefano Olla and Li-Cheng Tsai. Exceedingly large deviations of the totally asymmetric exclusion process. *arXiv preprint arXiv:1708.07052*, 2017.

[43] Firas Rassoul-Agha and Timo Seppäläinen. *A course on large deviations with an introduction to Gibbs measures*, volume 162. American Mathematical Soc., 2015.

[44] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[45] H. Rost. Nonequilibrium behaviour of a many particle process: density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete*, 58(1):41–53, 1981.

[46] Timo Seppäläinen. Coupling the totally asymmetric simple exclusion process with a moving interface. *Markov Process. Related Fields*, 4(4):593–628, 1998. I Brazilian School in Probability (Rio de Janeiro, 1997).

[47] Timo Seppäläinen. Increasing sequences of independent points on the planar lattice. *The Annals of Applied Probability*, 7(4):886–898, 1997.

[48] Timo Seppäläinen. Exact limiting shape for a simplified model of first-passage percolation on the plane. *Ann. Probab.*, 26(3):1232–1250, 1998.

[49] Timo Seppäläinen. Large deviations for increasing sequences on the plane. *Probability theory and related fields*, 112(2):221–244, 1998.

[50] Timo Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.*, 40(1):19–73, 2012. Corrected version available at *arXiv:0911.2446*.

[51] Timo Seppäläinen and Benedek Valkó. Bounds for scaling exponents for a 1 + 1 dimensional directed polymer in a Brownian environment. *ALEA Lat. Am. J. Probab. Math. Stat.*, 7:451–476, 2010.

[52] Srinivasa RS Varadhan. Large deviations for the asymmetric simple exclusion process. In *Stochastic Analysis on Large Scale Interacting Systems, Proceedings of Shonan/Kyoto meetings, 2002*. Math. Soc. Japan, 2004.
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