UPPER BOUNDS FOR THE DIMENSION OF TORI ACTING ON GKM MANIFOLDS

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Abstract. The aim of this paper is to give an upper bound for the dimension of a torus $T$ which acts on a GKM manifold $M$ effectively. In order to do that, we introduce a free abelian group of finite rank, denoted by $\mathcal{A}(\Gamma, \alpha, \nabla)$, from an (abstract) $(m,n)$-type GKM graph $(\Gamma, \alpha, \nabla)$. Here, an $(m,n)$-type GKM graph is the GKM graph induced from a $2m$-dimensional GKM manifold $M^{2m}$ with an effective $n$-dimensional torus action, say $(M^{2m}, T^n)$. Then it is shown that $\mathcal{A}(\Gamma, \alpha, \nabla)$ has rank $\ell(n)$ if and only if there exists an $(m,\ell)$-type GKM graph $(\tilde{\Gamma}, \tilde{\alpha}, \tilde{\nabla})$ which is an extension of $(\Gamma, \alpha, \nabla)$. Using this necessary and sufficient condition, we prove that the rank of $\mathcal{A}(\Gamma, \alpha, \nabla)$ for the GKM graph of $(M^{2m}, T^n)$ gives an upper bound for the dimension of a torus which can act on $M^{2m}$ effectively. As an application, we compute the rank of $\mathcal{A}(\Gamma, \alpha, \nabla)$ of the complex Grassmannian of 2-planes $G_2(C^{n+2})$ with some effective $T^{n+1}$-action, and prove that the $T^{n+1}$-action on $G_2(C^{n+2})$ is the maximal effective torus action.

1. Introduction

GKM manifolds (or more general spaces) are -roughly- spaces with torus action whose 0- and 1-dimensional orbits have the structure of a graph. This class of spaces was first appeared in the work of Goresky-Kottwitz-MacPherson [8] as a class of algebraic varieties (GKM stands for their initials). Motivated by their works, Guillemin-Zara [11] introduce a combinatorial counterpart of GKM manifold, called a(n) (abstract) GKM graph, and give some dictionary between the (symplectic) geometry (and topology) of GKM manifolds and the combinatorics of GKM graphs. This leads us to the study of geometric and topological properties of GKM manifolds using combinatorial properties of GKM graphs (see e.g. [6, 7, 9, 10, 15, 17, 19, 20] etc). In this paper, we introduce a new invariant of GKM graphs and provide a partial answer to the extension problem of torus actions on GKM manifolds.

To state our problem and main results more precisely, we briefly recall the setting of this paper and background of the problem. Let $M^{2m}$ be a $2m$-dimensional, compact, connected, almost complex manifold with effective $T^n$-action, which preserves the almost complex structure, where $T^n$ is the $n$-dimensional torus. We denote such manifold as $(M^{2m}, T^n)$, or $M^{2m}$, $(M, T)$ (if its torus action or dimensions of manifold and torus are obviously known from the context). We call $(M^{2m}, T^n)$ a GKM manifold if it satisfies the following properties (see Section 4 for details):

(1) the set of fixed points is not empty, i.e., $M^T \neq \emptyset$;
(2) the closure of a connected component of the set of one-dimensional orbits is equivariantly diffeomorphic to the 2-dimensional sphere, called an invariant 2-sphere.

It easily follows from this setting that $M^T$ is always isolated, i.e., 0-dimensional. Therefore, regarding fixed points as vertices and invariant 2-spheres as edges, this condition is equivalent to that the one-skeleton of $(M^{2m}, T^n)$ has the structure of a graph, where a one-skeleton of $(M^{2m}, T^n)$ is the orbit space of the set of 0- and 1-dimensional orbits. Note that there are several definitions of GKM manifolds. This is because the spaces with such torus actions, i.e., their one-skeleton is a graph, are widely appeared in the spaces with torus actions, and such spaces have been studied from several different points of views (homotopically, topologically, algebraically or geometrically). In this paper, we study the GKM manifolds defined by Guillemin-Zara in the paper [11]. For example, in our setting, the following manifolds are GKM manifolds: non-singular
complete toric varieties (also called toric manifolds) and homogeneous manifolds $G/H$ (where $G$ is a compact connected Lie group and $H$ is its closed subgroup with the same maximal torus) with torus invariant almost complex structures such as $S^6 = G_2/SU(3)$, flag manifolds and complex Grassmannians, etc.

Because GKM manifolds are even-dimensional and their effective torus actions have isolated fixed points, the differentiable slice theorem tells us that the following inequality holds for every GKM manifold $(M, T)$:

$$\dim T \leq \frac{1}{2} \dim M.$$  

If the equality $\dim T = \frac{1}{2} \dim M$ holds, such a GKM manifold is also known as a torus manifold (with invariant almost complex structure); famous examples are toric manifolds. Namely, the torus action on a torus manifold is (automatically) the maximal torus action. In this case, the torus invariant almost complex structures such as $G_2$/$SU(3)$, flag manifolds and complex Grassmannians etc.

Problem 1.1. When does a GKM manifold $(M^{2m}, T^n)$ extend to a GKM manifold $(M^{2m}, T^\ell)$? Here, $T^n \subset T^\ell$ and $n < \ell \leq m$.

To give an answer to this problem, we introduce a free abelian group with finite rank $\mathcal{A}(\Gamma, \alpha, \nabla)$, called a group of axial functions, for the GKM graph $(\Gamma, \alpha, \nabla)$ in Section 2. Here, a GKM graph is -roughly- the following triple (see Section 2 for details): an $m$-valent graph $\Gamma$; a function $\alpha : E(\Gamma) \to H^2(BT^n) \simeq \mathbb{Z}^n$, called an axial function; and a collection $\nabla$ of some bijective maps between out-going edges on adjacent vertices, called a connection. We call such GKM graph an $(m, n)$-type GKM graph in this paper. Note that a GKM manifold $(M^{2m}, T^n)$ defines an $(m, n)$-type GKM graph (see Section 1). The main theorem of this paper can be stated as follows (see Sections 2 and 3 for details):

**Theorem 1.2.** Let $(\Gamma, \alpha, \nabla)$ be an abstract $(m, n)$-type GKM graph. Then, the following two statements are equivalent:

1. $\text{rk } \mathcal{A}(\Gamma, \alpha, \nabla) \geq \ell$ for some $n \leq \ell \leq m$;
2. there is an $(m, \ell)$-type GKM graph $(\Gamma, \tilde{\alpha}, \nabla)$ which is an extension of $(\Gamma, \alpha, \nabla)$.

Because of the relation between GKM graphs and GKM manifolds, Theorem 1.2 implies that the maximal dimension of torus which can act on a GKM manifold $M$ is bounded from above by the rank of the group of axial functions of the GKM graph induced from $M$. Namely, we obtain the main result of this paper as follows (see Section 4 for details):

**Corollary 1.3.** Let $(M^{2m}, T^n)$ be a GKM manifold and $(\Gamma_M, \alpha_M, \nabla_M)$ be its $(m, n)$-type GKM graph. Assume that $\text{rk } \mathcal{A}(\Gamma_M, \alpha_M, \nabla_M) = \ell$. Then, the $T^n$-action on $M^{2m}$ does not extend to any $T^{\ell+1}$-action.

Furthermore, if $\text{rk } \mathcal{A}(\Gamma_M, \alpha_M, \nabla_M) = n$, then the $T^n$-action on $M^{2m}$ is the maximal torus action.

Recall that Shunji Takuma also obtains a partial answer to Problem 1.1 by introducing an obstruction class for the extension of an $(m, n)$-type GKM graph to an $(m, n + 1)$-type GKM graph in his note [19]. In contrast with Takuma’s obstruction class, our invariant is defined as the rank of a free abelian (or equivalently as a dimension of a vector space), and Theorem 1.2 may be regarded as the generalization of his result in [19].

Problem 1.1 is reminiscent of the computation of the torus degree of symmetry of a manifold $X$ (see [12]), i.e., the maximal dimension of a torus which can act on $X$ effectively (note that we do not assume the existence of fixed points or invariant almost complex structures on $X$). A
torus degree of symmetry has been studied for many classes of manifolds (see e.g. [13, 21, 23]). Corollary 1.3 may be regarded as to give an upper bound of the torus degree of symmetry of an almost complex structure of a GKM manifold. As an application of Corollary 1.3 in the final section (Section 5), we compute the torus degree of such symmetry for the complex Grassmannian of 2-planes, denoted as
\[ G_2(C^{n+2}) \simeq GL(n+2, \mathbb{C})/GL(2, \mathbb{C}) \times GL(n, \mathbb{C}) \simeq U(n+2)/U(2) \times U(n). \]

Namely, we compute \( \text{rk} \, \mathcal{A}(M, \alpha, \nabla) \) for \( M = G_2(C^{n+2}) \) with some effective \( T^{n+1} \)-action and prove the following fact:

**Proposition 1.4.** The standard effective \( T^{n+1} \)-action on \( G_2(C^{n+2}) \) is the maximal effective torus action which preserves the invariant almost complex structure.

Note that there is the natural \( T^{n+2} \)-action on \( G_2(C^{n+2}) \) which is induced from the maximal torus in \( U(n+2) \). However, this action is not effective.

The organization of this paper is as follows. In Section 2, we recall an abstract GKM graph \( (\Gamma, \alpha, \nabla) \), and introduce its group of axial functions \( \mathcal{A}(\Gamma, \alpha, \nabla) \). In Section 3 the main theorem, Theorem 1.2, is proved. In Section 4, in order to apply our results to geometry, we recall the definition of a GKM graph obtained from a GKM manifold, and show Corollary 1.3. We also prove the \( T^2 \)-action on \( S^6 = G_2/SU(3) \) is the maximal torus action in this section. In Section 5, we show the GKM graph obtained from the effective \( T^{n+1} \)-action on \( G_2(C^{n+2}) \), and compute its group of axial functions; this shows Proposition 1.4.

2. GKM Graph and its Group of Axial Functions

In this section, we first recall the basic facts about GKM graphs \( (\Gamma, \alpha, \nabla) \) (see [11]) and introduce the extension of axial functions of GKM graphs precisely. Then, a finite rank free abelian group \( \mathcal{A}(\Gamma, \alpha, \nabla) \), called a group of axial functions, is defined.

2.1. GKM Graph and its Extension. We first prepare some notation to define a GKM graph. Let \( \Gamma = (V(\Gamma), E(\Gamma)) \) be an abstract graph comprising a set \( V(\Gamma) \) of vertices and a set \( E(\Gamma) \) of oriented edges. For the given orientation on \( e \in E(\Gamma) \), we denote its initial vertex by \( i(e) \) and its terminal vertex by \( t(e) \). In this paper, we assume that there are no loops in \( E(\Gamma) \), i.e., \( i(e) \neq t(e) \) for any \( e \in E(\Gamma) \), and \( \Gamma \) is connected. The symbol \( \bar{e} \in E(\Gamma) \) represents the edge \( e \) with its orientation reversed, i.e., \( i(e) = t(\bar{e}) \) and \( t(e) = i(\bar{e}) \). The subset \( E_p(\Gamma) \subset E(\Gamma) \) is the set of out-going edges from \( p \in V(\Gamma) \); more precisely,
\[ E_p(\Gamma) = \{ e \in E(\Gamma) \mid i(e) = p \}. \]

A finite connected graph \( \Gamma \) is called an \( m \)-valent graph if \( |E_p(\Gamma)| = m \) for all \( p \in V(\Gamma) \), where the symbol \( |X| \) represents the cardinality of a finite set \( X \).

Let \( \Gamma \) be an \( m \)-valent graph. We next define a label \( \alpha : E(\Gamma) \to H^2(BT) \) on \( \Gamma \). Recall that \( BT^n \) (often denoted by \( BT \)) is a classifying space of an \( n \)-dimensional torus \( T \), and its cohomology ring (over \( \mathbb{Z} \)-coefficient) is isomorphic to the polynomial ring
\[ H^*(BT) \simeq \mathbb{Z}[a_1, \ldots, a_n], \]
with \( a_i \) is a variable with \( \text{deg} \, a_i = 2 \) for \( i = 1, \ldots, n \). So its degree 2 part \( H^2(BT) \) is isomorphic to \( \mathbb{Z}^n \). Put a label by a function \( \alpha : E(\Gamma) \to H^2(BT) \) on edges of \( \Gamma \). Set
\[ \alpha(p) = \{ \alpha(e) \mid e \in E_p(\Gamma) \} \subset H^2(BT). \]

An axial function on \( \Gamma \) is the function \( \alpha : E(\Gamma) \to H^2(BT^n) \) for \( n \leq m \) which satisfies the following three conditions:

1. \( \alpha(e) = -\alpha(\bar{e}) \);
2. for each \( p \in V(\Gamma) \), the set \( \alpha(p) \) is pairwise linearly independent, i.e., each pair of elements in \( \alpha(p) \) is linearly independent in \( H^2(BT) \);
3. for all \( e \in E(\Gamma) \), there exists a bijective map \( \nabla_e : E_{i(e)}(\Gamma) \to E_{t(e)}(\Gamma) \) such that
   \( \nabla_{i(e)} = \nabla_{t(e)}^{-1} \),
   \( \nabla_e(e) = \bar{e} \), and
(3) for each $e' \in E_{(e)}(\Gamma)$, there exists an integer $c_e(e')$ such that
\begin{equation}
\alpha(\nabla_c(e')) - \alpha(e') = c_e(e')\alpha(e) \in H^2(BT).
\end{equation}

The collection $\nabla = \{\nabla_e | e \in E(\Gamma)\}$ is called a connection on the labelled graph $(\Gamma, A)$; we denote the labelled graph with connection as $(\Gamma, A, \nabla)$, and the equation (2.1) is called a congruence relation. We call the integer $c_e(e')$ in the congruence relation a congruence coefficient of $e'$ on $e$.

The conditions as above are called an axiom of axial function. In addition, in this paper, we also assume the followings:

(4): for each $p \in V(\Gamma)$, the set $\alpha(p)$ spans $H^2(BT)$.

The axial function which satisfies (4) is called an effective axial function.

**Definition 2.1** (GKM graph [11]). If an $m$-valent graph $\Gamma$ is labeled by an axial function $\alpha : E(\Gamma) \rightarrow H^2(BT^n)$ for some $n \leq m$, then such labeled graph is said to be an (abstract) GKM graph, and denoted as $(\Gamma, \alpha, \nabla)$ (or $(\Gamma, \alpha)$ if the connection $\nabla$ is obviously determined). If such $\alpha$ is effective, $(\Gamma, \alpha, \nabla)$ is said to be an $(m, n)$-type GKM graph.

Figure 1 shows examples of GKM graphs.

![Figure 1. Examples of (2,2)-type, (3,2)-type and (3,3)-type GKM graphs from left, where $a, b$ (resp. $c$) are generators of $H^*(BT^2)$ (resp. $H^*(BT^3)$). For example, in the left $(2, 2)$-GKM graph, the axial function is defined by $\alpha(pq) = a, \alpha(qr) = -a + b$, etc.](image)

We note the following lemma proved in [11, Proposition 2.1.3].

**Lemma 2.2.** Let $(\Gamma, \alpha, \nabla)$ be a GKM graph. If $\alpha(p)$ is three-independent for every $p \in V(\Gamma)$, the connection $\nabla$ is uniquely determined.

Here, in Lemma 2.2 the set of vectors $\alpha(p)$ in $H^2(BT)$ is called a three-independent if every triple $\{\alpha(e_1), \alpha(e_2), \alpha(e_3)\} \subset \alpha(p)$ is linearly independent (e.g., the right $(3,3)$-type GKM graph in Figure 1). So, in this case, we may denote $(\Gamma, \alpha, \nabla)$ as $(\Gamma, \alpha)$ without connection $\nabla$.

We also note the following lemma:

**Lemma 2.3.** For all $e \in E(\Gamma)$, $c_e(e) = -2$.

**Proof.** By the axiom (1), (3)-(2) and (3)-(3) of axial function, it is straightforward. \qed

We close this section by defining an extension. Let $(\Gamma, \alpha, \nabla)$ be an $(m,n)$-type GKM graph. An $(m,\ell)$-type GKM graph $(\Gamma, \tilde{\alpha}, \tilde{\nabla})$ (for $n < \ell \leq m$) is said to be an extension of $(\Gamma, A, \nabla)$ if $\Gamma = \Gamma$, $\nabla = \tilde{\nabla}$ and there exists a projection $\pi : H^2(BT^\ell) \rightarrow H^2(BT^n)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
E(\Gamma) & \xrightarrow{\alpha} & H^2(BT^n) \\
\tilde{\alpha} \downarrow & & \downarrow \pi \\
H^2(BT^\ell) & & \\
\end{array}
\]

Figure 2 shows an example of extensions.
2.2. The invariant function. Let \((\Gamma, \alpha, \nabla)\) be an \((m,n)\)-type GKM graph for \(n \leq m\). We shall define an invariant function \(c(\Gamma, \alpha, \nabla) : E(\Gamma) \to \mathbb{Z}^m\) under extensions. To define it, we first fix an order of out-going edges on each vertex \(p\), i.e., set
\[
E_p(\Gamma) = \{e_1, p, \ldots, e_{m,p}\}.
\]
Then, we can define the free \(\mathbb{Z}\)-module with rank \(m\) on each \(p\), say \(\mathbb{Z}E_p(\Gamma)\), by regarding \(\{e_1, p, \ldots, e_{m,p}\}\) as the formal generator of \(\mathbb{Z}E_p(\Gamma)\). Namely,
\[
\mathbb{Z}E_p(\Gamma) := \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{m,p} \cong \mathbb{Z}^m.
\]
Because an order on each \(E_p(\Gamma)\) is fixed, the connection \(\nabla_e : E_{i(e)}(\Gamma) \to E_{i(e)}(\Gamma)\) induces the permutation on \(\{1, \ldots, m\}\). So the connection \(\nabla_e\) defines the isomorphism
\[
N_e : \mathbb{Z}E_{i(e)}(\Gamma) \to \mathbb{Z}E_{i(e)}(\Gamma) \in GL(m; \mathbb{Z})
\]
by the permutation \((m \times m)\)-square matrix. Take an edge \(e \in E(\Gamma)\). Recall that the congruence coefficient \(c_e(e')\) which is defined by \([2.1]\) is an integer attached on every edge \(e' \in E_{i(e)}(\Gamma)\) for the fixed edge \(e \in E(\Gamma)\). Therefore, the \(m\)-tuple of congruence coefficients on \(e\) defines the element in \(\mathbb{Z}E_{i(e)}(\Gamma)\) by
\[
c_e = (c_e(e_{1,i(e)}), \ldots, c_e(e_{m,i(e)})) \in \mathbb{Z}e_{1,i(e)} \oplus \cdots \oplus \mathbb{Z}e_{m,i(e)}.
\]
Thus we may define the function
\[
c(\Gamma, \alpha, \nabla) : E(\Gamma) \to \mathbb{Z}^m \quad \text{by} \quad c(\Gamma, \alpha, \nabla)(e) = c_e.
\]
Because of the following proposition (see also \([19]\)), we call this function \(c(\Gamma, \alpha, \nabla)\) an invariant function of extensions of \((\Gamma, \alpha, \nabla)\):

**Proposition 2.4.** For any extensions \((\Gamma, \tilde{\alpha}, \nabla)\) of \((\Gamma, \alpha, \nabla)\), the equation \(c(\Gamma, \alpha, \nabla) = c(\Gamma, \tilde{\alpha}, \nabla)\) holds.

**Proof.** Let \((\Gamma, \tilde{\alpha}, \nabla)\) be an \((m,\ell)\)-type GKM graph for some \(\ell > n\), and \(\tilde{c}_e(e')\) be its congruence coefficient of \(e'\) on \(e\). Fix an order of out-going edges on each vertex \(p\). By definition of the function \(c(\Gamma, \alpha, \nabla)\), it is enough to prove that the equation \(c_e(e') = \tilde{c}_e(e')\) for all \(e \in E(\Gamma)\) and \(e' \in E_{i(e)}(\Gamma)\).

By definition, there is a projection \(\pi : H^2(B\ell) \to H^2(Bm)\) such that \(\pi \circ \tilde{\alpha} = \alpha\). Together with the congruence relations \([2.1]\), we have
\[
\pi(\tilde{\alpha}(\nabla_e(e'))) = \alpha(\nabla_e(e')) = \alpha(e') + c_e(e')\alpha(e)
\]
and
\[
\pi(\tilde{\alpha}(\nabla_e(e'))) = \pi(\tilde{\alpha}(e') + \tilde{c}_e(e')\tilde{\alpha}(e))
\]
\[
= \pi(\tilde{\alpha}(e')) + \tilde{c}_e(e')\pi(\tilde{\alpha}(e))
\]
\[
= \alpha(e') + \tilde{c}_e(e')\alpha(e).
\]
Comparing these equations, we establish the statement. \(\square\)

The following lemma tells us that the \(c(\Gamma, \alpha, \nabla)(\tilde{\tau})\) is automatically determined by \(c(\Gamma, \alpha, \nabla)(e)\) and \(N_e\) defined in \([2.2]\):

**Lemma 2.5.** For any \(e \in E(\Gamma)\), the equation \(N_e(c(\Gamma, \alpha, \nabla)(\tilde{\tau})) = c(\Gamma, \alpha, \nabla)(\tau)\) holds.
Proof. Let \( \sigma \) be the permutation on \( \{1, \ldots, m\} \) induced from \( N_e \) (i.e., from \( \nabla_e \)). Then, \( \nabla_e(\sigma_{e,j},i(e)) = e_{j,i(\sigma_e)} \) and \( \nabla_{\sigma_e}(e_{j,i(e)}) = e_{\sigma_e,j,i(e)} \). Therefore, by definitions of \( N_e \) and \( c_{(\Gamma,\alpha,\nabla)} \), it is enough to show the following equality:

\[
c_e(\sigma_{e,j},i(e)) = c_{\sigma_e}(e_{\sigma_e,j,i(e)})
\]

for all \( j = 1, \ldots, m \). Because of the congruence relations \( 2.1 \) on \( e \) and \( \sigma_e \), we have

\[
\alpha(\nabla_e(\sigma_{e,j},i(e))) - \alpha(\sigma_{e,j},i(e)) = c_e(\sigma_{e,j},i(e)) \alpha(e) = \alpha(e_{j,i(\sigma_e)}) - \alpha(\sigma_{e,j},i(e))
\]

and

\[
\alpha(\nabla_{\sigma_e}(e_{j,i(e)})) - \alpha(e_{j,i(e)}) = c_{\sigma_e}(e_{\sigma_e,j,i(e)}) \alpha(\sigma_e) = \alpha(e_{\sigma_e,j,i(e)}) - \alpha(e_{j,i(e)}).
\]

By these equations and \( \alpha(e) = -\alpha(\sigma_e) \), we establish the statement. \( \square \)

Example 2.6. Figure 3 shows an example of the invariant function \( c_{(\Gamma,\alpha,\nabla)} \) induced from the (3,2)-GKM graph (\( \Gamma, \alpha \)). In this case, we put the order of \( E_p(\Gamma) \) as in Figure 3 by

\[
e_{1,p} = e_1, \quad e_{2,p} = e_2, \quad e_{3,p} = e_3,
\]

and the order of \( E_{q}(\Gamma) \) similarly by

\[
e_{1,q} = e_1, \quad e_{2,q} = e_2, \quad e_{3,q} = e_3.
\]

Then, the connection \( \nabla_{e_i} \) for \( i = 1, 2, 3 \) is determined uniquely by \( e_i \mapsto \sigma_i \) and \( e_j \mapsto \sigma_k \), where \( \{i, j, k\} = \{1, 2, 3\} \) in Figure 3. For example, for the edge \( e_1 \), by definition of \( c_{(\Gamma,\alpha,\nabla)} \) and Lemma 2.8, we have

\[
c_{(\Gamma,\alpha,\nabla)}(e_1) = (-2, 1, 1).
\]

With the similar computation (and using Lemma 2.5), we obtain \( c_{(\Gamma,\alpha,\nabla)} \) as the right graph in Figure 3.

Figure 3. The left one is the (3,2)-GKM graph \( (\Gamma, \alpha) \) and the right one is its invariant function \( c_{(\Gamma,\alpha,\nabla)} : E(\Gamma) \rightarrow \mathbb{Z}^3 \).

2.3. A group of axial functions \( \mathcal{A}(\Gamma,\alpha,\nabla) \) of \((\Gamma,\alpha,\nabla)\). Let \((\Gamma,\alpha,\nabla)\) be an \((m,n)\)-type GKM graph. In this subsection, we introduce a finitely generated free abelian group \( \mathcal{A}(\Gamma,\alpha,\nabla) \), called a group of axial functions. Before defining \( \mathcal{A}(\Gamma,\alpha,\nabla) \), we prepare some notation. Let \( f \in \mathbb{Z}E_p(\Gamma) \). The symbol \( f_e(\in \mathbb{Z}) \) represents the integer of the coefficient in \( f \) corresponding to the edge \( e \in E_p(\Gamma) \). For example, if we put the order \( E_p(\Gamma) = \{e_1, \ldots, e_m\} \) and \( f = (x_1, \ldots, x_m) \in \mathbb{Z}^m \) with respect to this order, then \( f_{e_j} = x_j \).

Definition 2.7 (Group of axial functions). A \( \mathbb{Z} \)-module \( \mathcal{A}(\Gamma,\alpha,\nabla) \) is defined by the submodule of

\[
\{f : V(\Gamma) \rightarrow \mathbb{Z}^m\} = \bigoplus_{p \in V(\Gamma)} \mathbb{Z}E_p(\Gamma) \simeq \bigoplus_{p \in V(\Gamma)} \mathbb{Z}^m
\]

which satisfies that the following relations for all \( e \in E(\Gamma) \):

\[
N_e(f(p)) - f(q) = f(q)\mathcal{A}(\Gamma,\alpha,\nabla)(\sigma_e)
\]

where \( i(e) = p, t(e) = q \) and \( f(q)\sigma_e \in \mathbb{Z} \) is the integer defined just before. This module \( \mathcal{A}(\Gamma,\alpha,\nabla) \) is said to be a group of axial functions of \((\Gamma,\alpha,\nabla)\) (also see Remark 2.9).
Remark 2.8. Because the set \( \{ f : V(\Gamma) \to \mathbb{Z}^n \} \) is a finitely generated free \( \mathbb{Z} \)-module with rank \( m | V(\Gamma) | \) i.e., free abelian group with finite rank, its submodule \( \mathcal{A}(\Gamma, \alpha, \nabla) \) is so too. It is also easy to check that two groups of axial functions defined by different orders on \( E_p(\Gamma) \) (for each \( p \in V(\Gamma) \)) are isomorphic.

Remark 2.9. We give a brief remark for our group \( \mathcal{A}(\Gamma, \alpha, \nabla) \) from the point of view of the GKM theory. Let \( M \) be a GKM manifold and \( (\Gamma_M, \alpha_M, \nabla_M) \) be its induced GKM graph (see Section 2.1). Recall the famous GKM description of equivariant cohomology of \( M \) (see e.g. \cite{8,9,11}). The GKM description of equivariant cohomology of \( M \) can be defined by taking the global sections of a sheaf of \( (\Gamma_M, \alpha_M, \nabla_M) \) in the sense of Braden-MacPherson \cite{2} (also see \cite{5}); such sheaf is called a structure sheaf of \( (\Gamma_M, \alpha_M, \nabla_M) \) in \cite{2} (or sheaf of rings in \cite{24}). More precisely, the ring defined by \( (\Gamma_M, \alpha_M, \nabla_M) \) is often denoted as \( H^*(\Gamma_M, \alpha_M) \) and it is isomorphic to the torus equivariant cohomology \( H^*_T(M; \mathbb{Q}) \) of \( M \) with the rational coefficient, if \( M \) satisfies the condition called an equivariantly formal (see \cite{3}). On the other hand, a group of axial functions \( \mathcal{A}(\Gamma_M, \alpha_M, \nabla_M) \) can be defined by taking a (little bit modified) global sections of another sheaf on \( \Gamma \), which is induced from \( c(\Gamma_M, \alpha_M, \nabla_M) \). We omit the details about this process because it is apart from the purpose of this paper. Note that we do not need the connection \( \nabla_M \) to define \( H^*(\Gamma_M, \alpha_M) \); however, for the case of \( \mathcal{A}(\Gamma_M, \alpha_M, \nabla_M) \), the connection is essential (see Remark 2.5).

The following corollary follows immediately from Definition 2.7 and Proposition 2.4.

**Corollary 2.10.** Let \( (\Gamma, \alpha, \nabla) \) be a GKM graph and \( (\Gamma, \alpha, \nabla) \) be its extension. Then, two groups of axial functions are isomorphic, i.e., \( \mathcal{A}(\Gamma, \alpha, \nabla) \simeq \mathcal{A}(\Gamma, \alpha, \nabla). \)

We note the following property of Eq. (2.3) which will be useful to compute \( \mathcal{A}(\Gamma, \alpha, \nabla). \)

**Lemma 2.11.** Let \( f : V(\Gamma) \to \mathbb{Z}^m \) be any function. If Eq. (2.3) holds for some edge \( e \in E(\Gamma) \), then \( f(\bar{e}) = -f(q) \) and Eq. (2.3) also holds for the edge \( e \), where \( p = i(e) \) and \( q = t(e). \)

**Proof.** It follows from \( \nabla_e(e) = \overline{e} \) that \( N_e(f(p)) = f(p). \) Therefore, together with Lemma 2.8, we have

\[
N_e(f(p)) - f(q) = f(p) - f(q) = -2f(q).
\]

So \( f(p) = -f(q). \) Note that \( N_{\overline{e}} = N_e^{-1} \) because \( \nabla \overline{e} = \nabla e^{-1} \). Thus, evaluating Eq. (2.8) by \( N_{\overline{e}} \) and using Lemm 2.8, we have

\[
f(p) - N_{\overline{e}}(f(q)) = -f(p) - N_{\overline{e}}(c(\Gamma_M, \alpha_M)(\overline{e})) = -f(p) - c(\Gamma_M, \alpha_M)(e).
\]

This establishes the statement. \( \square \)

**Example 2.12.** Before proving the main theorem, let us compute the \( \mathcal{A}(\Gamma, \alpha, \nabla) \) of \( (\Gamma, \alpha, \nabla) \) in Example 3. By definition of \( \mathcal{A}(\Gamma, \alpha, \nabla) \), we first have

\[
\mathcal{A}(\Gamma, \alpha, \nabla) = \{ f : p, q \in \mathbb{Z}^3 \mid N_e(f(p)) - f(q) = f(q)c(\Gamma_M, \alpha_M)(\overline{e}) \}
\]

Put \( f(p) = (x, y, z) \in \mathbb{Z}E_p(\Gamma) \) and \( f(q) = (x', y', z') \in \mathbb{Z}E_q(\Gamma). \) Then, by Lemma 2.11 \( x' = -x, \ y' = -y, \ z' = -z. \) Therefore, for example for the case when \( i = 1 \), the relation of \( \mathcal{A}(\Gamma_M, \alpha_M, \nabla) \) says that

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
- \begin{pmatrix}
-x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
-2 \\
1 \\
1
\end{pmatrix}
\]

Hence, we also have the relation \( x + y + z = 0. \) Similarly, computing for the other edges \( e_2, e_3 \) (Lemma 3.2 proved later may be also useful), we get

\[
\mathcal{A}(\Gamma, \alpha, \nabla) = \{ (f(p), f(q)) = ((x, y, z), (-x, -y, -z)) \mid x + y + z = 0 \} \simeq \mathbb{Z}^2.
\]
3. Main theorem

In this section, we prove the following main theorem.

**Theorem 3.1.** Let \((\Gamma, \alpha, \nabla)\) be an \((m,n)\)-type GKM graph. Then the following two statements are equivalent:

1. there is an \((m,\ell)\)-type GKM graph which is an extension of \((\Gamma, \alpha, \nabla)\) for some \(\ell \geq n\);
2. \(\ell \leq \text{rk} \, A(\Gamma, \alpha, \nabla) \leq m\).

In particular, if \(\text{rk} \, A(\Gamma, \alpha, \nabla) = k\), then there is an extended \((m,k)\)-type GKM graph \((\widetilde{\Gamma}, \widetilde{\alpha}, \nabla)\) which is the maximal among extensions.

3.1. **Proof of** \((1) \Rightarrow (2)\). We first prove \((1) \Rightarrow (2)\) in Theorem 3.1. The following lemma is the key lemma:

**Lemma 3.2.** Let \((\Gamma, \alpha, \nabla)\) be an \((m,n)\)-type GKM graph. Then, the rank of \(A(\Gamma, \alpha, \nabla)\) satisfies the following inequality:

\[ n \leq \text{rk} \, A(\Gamma, \alpha, \nabla) \leq m. \]

**Proof.** We first prove the inequality \(\text{rk} \, A(\Gamma, \alpha, \nabla) \leq m\). By definition, \(f \in A(\Gamma, \alpha, \nabla) \subseteq \bigoplus_{p\in V(\Gamma)} Z E_p(\Gamma)\). Under the same notations in E.q. (2.1), we put

\[ f(p) = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \quad f(q) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad c_{(\Gamma, \alpha, \nabla)}(\nabla) = \begin{pmatrix} k_{1,q} \\ \vdots \\ k_{m,q} \end{pmatrix} \]

where \(x_j, y_j\) are valuables and \(k_{j,q} \in \mathbb{Z}\) for all \(j = 1, \ldots, m\). Put \(f(q)_{\sigma} = y_j\). Then, by E.q. (2.1), for all \(i = 1, \ldots, m\) the following equation holds:

\[ x_{\sigma(i)} - y_i = y_j k_{i,q}, \]

where \(\sigma\) is the permutation on \(\{1, \ldots, m\}\) induced from \(\nabla e\). This implies that once we choose the value \(f(p) \in \mathbb{Z}^m\) for a vertex \(p\) which connects with \(q\) by an edge, then the value \(f(q) \in \mathbb{Z}^m\) is automatically and uniquely determined. Because \(\Gamma\) is a connected graph, iterating this argument on each edge, we can determine the value \(f(r)\) uniquely for all \(r \in V(\Gamma)\) if we choose a value of \(f(p)\). This implies that the restriction map

\[ \rho_p : A(\Gamma, \alpha, \nabla) \rightarrow \mathbb{Z}^m \]

is the injective homomorphism for any vertex \(p \in V(\Gamma)\), which proves \(\text{rk} \, A(\Gamma, \alpha, \nabla) \leq m\).

We next prove the rest inequality \(n \leq \text{rk} \, A(\Gamma, \alpha, \nabla)\). Because \((\Gamma, \alpha, \nabla)\) is an \((m,n)\)-type GKM graph, taking a linear basis of \(H^2(BT^n)\) as \(\{e_1,n, \ldots, e_m,n\}\), its axial function can be written as \(\alpha : E(\Gamma) \rightarrow H^2(BT^n) = \mathbb{Z} e_1, \ldots, e_m, n \simeq \mathbb{Z}^n\). Put \(\pi_i : H^2(BT^n) \rightarrow \mathbb{Z} e_i\) be the projection onto the \(i\)th coordinate of \(H^2(BT^n)\) with respect to this basis. Define

\[ \alpha_i : E(\Gamma) \xrightarrow{\alpha} H^2(BT^n) \xrightarrow{\pi_i} \mathbb{Z} e_i. \]

Recall that we choose an order on \(E_p(\Gamma) = \{e_{1,p}, \ldots, e_{m,p}\}\) for each \(p \in V(\Gamma)\). Put

\[ \alpha_{i}(e_{j,p}) = k_{j,p}^{(i)} a_i \]

for some \(k_{j,p}^{(i)} \in \mathbb{Z}\). Then, the map \(f_i : V(\Gamma) \rightarrow \mathbb{Z}^m\) is defined by

\[ f_i(p) = \begin{pmatrix} k_{1,p}^{(i)} \\ \vdots \\ k_{m,p}^{(i)} \end{pmatrix} \in \mathbb{Z} E_p(\Gamma) \simeq \mathbb{Z}^m, \]

for each \(p \in V(\Gamma)\). We claim that \(f_i \in A(\Gamma, \alpha, \nabla)\) and \(\{f_1, \ldots, f_n\}\) spans the rank \(n\) submodule in \(A(\Gamma, \alpha, \nabla)\). Let \(p = i(e), \ q = t(e) \in V(\Gamma)\) for some \(e \in E(\Gamma)\). In order to prove \(f_i \in A(\Gamma, \alpha, \nabla)\), by definition, it is enough to show the equation

\[ N_e(f_i(p)) - f_i(q) = f_i(q)_{\nabla_{(\Gamma, \alpha, \nabla)}(\nabla)}. \]

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Now we have

$$\alpha(\nabla_e(e_{j,p})) = \alpha(e_{\sigma(j), q}) = \alpha(e_{j,p}) + c_e(e_{j,p})\alpha(e).$$

Taking $\pi_i$ on these equations, we obtain

$$\alpha_i(\nabla_e(e_{j,p})) = k_{\sigma(j),q}^{(i)}a_i = k_{j,p}^{(i)}a_i + c_e(e_{j,p})k_e^{(i)}a_i$$

for all $j = 1, \ldots, m$, where $k_e^{(i)}a_i = \alpha_i(e) = f_i(p)c_i$. Therefore, we have

$$\begin{pmatrix}
  k_{\sigma(1),q}^{(i)} \\
  \vdots \\
  k_{\sigma(m),q}^{(i)}
\end{pmatrix} = \begin{pmatrix}
  k_{1,p}^{(i)} \\
  \vdots \\
  k_{m,p}^{(i)}
\end{pmatrix} + \begin{pmatrix}
  c_e(e_{1,p}) \\
  \vdots \\
  c_e(e_{m,p})
\end{pmatrix}$$

Because the permutation matrix $N_\pi$ is induced from $\nabla_e$, we have the following equation from this equation:

$$N_\pi(f_i(q)) = f_i(p) + f_i(p)c_{e(\pi,\nabla)}(e).$$

Thus, it follows from $N_\pi = N_\pi^{-1}$ and Lemma 2.5 that

$$N_\pi(f_i(p)) - f_i(q) = -f_i(p)c_{e(\pi,\nabla)}(\pi).$$

Now, by definition of $\alpha_i$, we have that $f_i(p)c_i = -f_i(q)c_{i}$ Hence, we obtain E.q. (5.2) and establish $f_i \in A(\Gamma, \alpha, \nabla)$ for all $i = 1, \ldots, n$. Now, by definition of the effective axial function, the collection $\{\alpha(e_{p,1}), \ldots, \alpha(e_{p,m})\}$ spans $H^2(BT^n) \simeq \mathbb{Z}^n$ for each $p \in V(\Gamma)$. This implies that the collection $\{f_1(p), \ldots, f_m(p)\}$ spans some $n$-dimensional subspace in $\mathbb{Z}E_p'(\Gamma)(\simeq \mathbb{Z}^m)$ for all $p \in V(\Gamma)$. Because the restriction map (5.1) is injective, this also implies that $\{f_1, \ldots, f_m\}$ spans some $n$-dimensional submodule in $A(\Gamma, \alpha, \nabla)$. This establishes that $n \leq rk A(\Gamma, \alpha, \nabla)$.

Assume that there is an $(m, \ell)$-type GKM graph $(\Gamma, \alpha, \nabla)$ which is an extension of $(\Gamma, \alpha, \nabla)$ for some $n \leq \ell \leq m$. Then, it easily follows from Corollary (2.10) and Lemma (3.2) that

$$\ell \leq rk A(\Gamma, \alpha, \nabla) = rk A(\Gamma, \alpha, \nabla) \leq m.$$ 

This establishes the statement (1) $\Rightarrow$ (2) in Theorem (3.1).

3.2. Proof of (2) $\Rightarrow$ (1). We next prove (2) $\Rightarrow$ (1) in Theorem (3.1) for an $(m,n)$-type GKM graph $(\Gamma, \alpha, \nabla)$. Assume that

$$\ell \leq rank A(\Gamma, \alpha, \nabla)(\leq m)$$

for some $\ell \geq n$. We shall prove that there exists an $(m,\ell)$-type GKM graph $(\Gamma, \alpha, \nabla)$ of its extension.

Let $f \in A(\Gamma, \alpha, \nabla)$. Put the order of $E_p(\Gamma)$ as $\{e_{1,p}, \ldots, e_{m,p}\}$ for $p \in V(\Gamma)$ and

$$f(p) = \begin{pmatrix}
  k_{1,p} \\
  \vdots \\
  k_{m,p}
\end{pmatrix}$$

with respect to this order. Then, we may define $\alpha_f^a : E(\Gamma) \to \mathbb{Z}a$ for every $a \in H^2(BT^n)$ by

$$\alpha_f^a(e_{j,p}) = k_{j,p}a.$$ 

We call this label $\alpha_f^a$ on edges an $a$-labeling with (coefficients of) $f \in A(\Gamma, \alpha, \nabla)$. The following lemma, which is the key lemma to prove (2) $\Rightarrow$ (1), tells us that the axial function $\alpha$ can be recovered from $A(\Gamma, \alpha, \nabla)$ by using $\alpha_f^a$.

**Lemma 3.3.** Let $(\Gamma, \alpha, \nabla)$ be an $(m,n)$-type GKM graph. Then, there exist $f_i \in A(\Gamma, \alpha, \nabla)$ for $i = 1, \ldots, n$ such that $\{f_1, \ldots, f_n\}$ spans an $n$-dimensional subspace of $A(\Gamma, \alpha, \nabla)$ and for the fixed basis $a_1, \ldots, a_n$ of $H^2(BT^n)$ the axial function can be split into

$$\alpha_1 + \cdots + \alpha_n = \alpha : E(\Gamma) \to H^2(BT^n) = \bigoplus_{i=1}^n \mathbb{Z}a_i,$$

where $\alpha_i := \alpha_{f_i}^{a_i}$ is the $a_i$-labeling with $f_i$. 

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Lemma 3.4. The above \((\Gamma, \tilde{\alpha}, \nabla)\) is an \((m, \ell)\)-type GKM graph and an extension of \((m, n)\)-type GKM graph \((\Gamma, \alpha, \nabla)\).

Proof. We shall find elements \(f_1, \ldots, f_n\) in \(A(\Gamma, \alpha, \nabla)\) which satisfy the conditions in the statement (this will be the converse argument of the proof of \(n \leq \text{rank } A(\Gamma, \alpha, \nabla)\) in Lemma 3.2). For \(E_p(\Gamma) = \{e_1, p, \ldots, e_m, p\}\) on each \(p \in V(\Gamma)\), we put the axial function

\[
\alpha(e_{j, p}) = \sum_{i=1}^{n} k_{j, p}^{(i)} a_i
\]

for \(k_{j, p}^{(i)} \in \mathbb{Z}\), where \(i \in \{1, \ldots, n\}\) and \(j \in \{1, \ldots, m\}\). For fixed \(i\), by picking up the coefficients on \(a_i\) around \(p \in V(\Gamma)\), we can define the function \(f_i : V(\Gamma) \to \mathbb{Z}^m\) as

\[
f_i(p) = \begin{pmatrix} k_{1, p}^{(i)} \\ \vdots \\ k_{m, p}^{(i)} \end{pmatrix}.
\]

We first claim that these functions \(f_1, \ldots, f_n\) are elements in \(A(\Gamma, \alpha, \nabla)\). For \(e \in E(\Gamma)\), put \(p = i(e), q = t(e)\) and the permutation on \(\{1, \ldots, m\}\) induced from \(\nabla_e\) as \(\sigma\). Then, by the axiom (3) of the axial function, the following equation holds:

\[
\alpha(e_{\sigma(j), p}) - \alpha(e_{j, q}) = c_{\sigma}(e_{j, q})\alpha(\sigma).
\]

Comparing the \(\alpha_i\) coefficients on both sides and using \(3.3\), we have

\[
k_{\sigma(j), p}^{(i)} - k_{j, q}^{(i)} = c_{\sigma}(e_{j, q})k_{h, q}^{(i)}
\]

where here we put \(\sigma = e_{h, q} (h \in \{1, \ldots, m\}\); namely, \(k_{h, q}^{(i)} = f_i(q)\) by using the notation of \(3.3\).

Because this equation \(3.6\) holds for all \(j = 1, \ldots, m\), we have the following equation

\[
N_\sigma(f_i(p)) - f_i(q) = f_i(q)\alpha(e_{(\Gamma, \alpha, \nabla)}(\sigma))
\]

for all \(i = 1, \ldots, n\), where \(N_\sigma : \mathbb{Z}E_p(\Gamma) \to \mathbb{Z}E_q(\Gamma)\) is an isomorphism induced from \(\nabla_e\) (see Section 2). This is nothing but the equation \(2.3\). So \(f_1, \ldots, f_n\) are elements in \(A(\Gamma, \alpha, \nabla)\).

We next claim that these functions satisfy the statement of this lemma. By the axiom of the axial function \((2)\), i.e., \(\alpha(p) = \{\alpha(e_{1, p}), \ldots, \alpha(e_{m, p})\}\) spans \(H^2(BT^n)\), we see that \(\{f_1(p), \ldots, f_n(p)\}\) spans an \(n\)-dimensional subspace in \(\mathbb{Z}^m\) for every \(p \in V(\Gamma)\). Because of \(3.4\), if \(p\) and \(q\) are connected by an edge, the value of \(f_i(q)\) is uniquely determined by the value of \(f_i(p)\); therefore, the values of \(f_i(r)\) for all \(r \in V(\Gamma)\) are uniquely determined by the value of \(f_i(p)\) because \(\Gamma\) is connected. This establishes that \(\{f_1, \ldots, f_n\} \subset A(\Gamma, \alpha, \nabla)\) spans an \(n\)-dimensional subspace. Let \(\alpha_i\) be the \(\alpha_i\)-labeling with coefficient \(f_i\) for \(i = 1, \ldots, n\). Then, by definition,

\[
\alpha_i(e_{j, p}) = k_{j, p}^{(i)} a_i.
\]

Comparing \(\alpha\) in \(3.3\), we have \(\alpha_1 \oplus \cdots \oplus \alpha_n = \alpha\). This establishes the statement. \(\Box\)

Because \(\ell \leq \text{rk } A(\Gamma, \alpha, \nabla)\), there are independent elements (as free \(\mathbb{Z}\)-module)

\[
f_1, \ldots, f_\ell \in A(\Gamma, \alpha, \nabla).
\]

Moreover, because of \(\ell \geq n\), we may choose the first part \(f_1, \ldots, f_n\) as the basis induced from \((\Gamma, \alpha, \nabla)\) in the proof of Lemma 3.3. Fix the basis of \(H^2(BT^\ell)\) as \(a_1, \ldots, a_\ell\), where the first \(n\) elements \(a_1, \ldots, a_n\) are the basis of \(H^2(BT^n)\) (see Lemma 3.3). Let \(\alpha_i\) be the \(a_i\)-labeling with \(f_i\), i.e., \(\alpha_i = \alpha_{f_i}^{a_i}\), for \(i = 1, \ldots, \ell\). Then, we can define the function as

\[
\tilde{\alpha} := \oplus_{i=1}^{\ell} \alpha_i : E(\Gamma) \to H^2(BT^\ell).
\]

The following lemma says that \((\Gamma, \tilde{\alpha}, \nabla)\) is a GKM graph extending \((\Gamma, \alpha, \nabla)\).

Lemma 3.4. The above \((\Gamma, \tilde{\alpha}, \nabla)\) is an \((m, \ell)\)-type GKM graph and an extension of \((m, n)\)-type GKM graph \((\Gamma, \alpha, \nabla)\).
Proof. By definitions of $f_i$ and $\alpha_i$ (see before the statement of this lemma) and using Lemma 2.11, we have

$$\alpha = \oplus_{i=1}^{\ell} \alpha_i,$$

for $i = 1, \ldots, n$. Therefore, it is enough to prove that $\tilde{\alpha}$ satisfies the axiom of GKM graph under the same connection $\nabla$ of $(\Gamma, \alpha, \nabla)$.

We first claim that the axiom (1) of GKM graph holds for $\tilde{\alpha}$, i.e., $\tilde{\alpha}(e) = -\tilde{\alpha}(e')$. To do this, by definition of $\tilde{\alpha}$, it is enough to show that $\alpha_i(e) = -\alpha_i(e')$ for all $i = 1, \ldots, \ell$. Let $e = e_{j,p} \in E_p(\Gamma)$ and $e' = e_{\sigma(j),q} \in E_q(\Gamma)$ for $j = 1, \ldots, m$, where $i(e) = p$, $i(e') = q$ and $\sigma$ is the permutation on $\{1, \ldots, m\}$ induced from $\nabla_e : E_q(\Gamma) \rightarrow E_p(\Gamma)$. Then, by definition of $\alpha_i$, we have that

$$\alpha_i(e) = \alpha_i(e_{j,p}) = k_{j,p}^{(i)} a_i,$$

and

$$\alpha_i(e') = \alpha_i(e_{\sigma(j),q}) = k_{\sigma(j),q}^{(i)} a_i,$$

where $f_i(p) = k_{j,p}^{(i)}$, $f_i(q) = k_{\sigma(j),q}^{(i)} \in \mathbb{Z}$. Because $f_i \in \mathcal{A}(\Gamma, \alpha, \nabla)$, it follows from Lemma 2.11 that $k_{j,p}^{(i)} = -k_{\sigma(j),q}^{(i)}$. This establish the axiom (1) of GKM graph.

We next check the condition (4) of effectiveness, i.e., $\tilde{\alpha}(p) = \{\tilde{\alpha}(e) \mid e \in E_p(\Gamma)\}$ spans $H^2(\text{BT}_\ell^r)$ for all $p \in V(\Gamma)$. Recall that for $E_p(\Gamma) := \{e_1, \ldots, e_{m,p}\}$,

$$\tilde{\alpha}(e_{j,p}) = \oplus_{i=1}^{\ell} \alpha_i(e_{j,p}) = \oplus_{i=1}^{\ell} k_{j,p}^{(i)} a_i,$$

where the integer $k_{j,p}^{(i)}$ is the $i$th coefficient of $f_i(p) \in \mathbb{Z}^m$ (see (3.4)). Now $\{f_1, \ldots, f_\ell\}$ spans an $\ell$-dimensional subspace of $\mathcal{A}(\Gamma, \alpha, \nabla)$. Because the restriction map defined in (3.3) is injective (by the similar arguments in the proof of Lemma 2.4), we have that the subset $\{f_1(p), \ldots, f_\ell(p)\} \subset \mathbb{Z}^m$ also spans a subgroup which is isomorphic to $\mathbb{Z}^\ell$ for all $p \in V(\Gamma)$. This shows that the $(m \times \ell)$-matrix $(k_{j,p}^{(i)})_{i,j}$ has rank $\ell (\leq m)$ and some minor (of $\ell \times \ell$)-smaller square matrix in $(k_{j,p}^{(i)})_{i,j}$ with determinant $\pm 1$, for all $p \in V(\Gamma)$. Therefore, there are $\ell$ elements in $\tilde{\alpha}(p) = \{\alpha(e_{1,p}), \ldots, \alpha(e_{m,p})\}$ which generate $H^2(\text{BT}_\ell^r)$. This establishes the condition (4).

We also check the axiom (2), i.e., $\tilde{\alpha}(p)$ is pairwise linearly independent for all $p \in V(\Gamma)$. Because $\alpha$ is an axilary function, $\alpha(p)$ is pairwise linearly independent for all $p \in V(\Gamma)$, i.e., $\alpha(e)$ and $\alpha(e')$ are linearly independent for all pairs $e, e' \in E_p(\Gamma)$. Moreover, we may write

$$\tilde{\alpha}(e) = \oplus_{i=1}^{\ell} \alpha_i(e) = \oplus_{i=1}^{n} \alpha_i(e) \oplus (\oplus_{i=n+1}^{\ell} \alpha_i(e)) = \alpha(e) \oplus (\oplus_{i=n+1}^{\ell} \alpha_i(e)),$$

and

$$\tilde{\alpha}(e') = \oplus_{i=1}^{\ell} \alpha_i(e') = \oplus_{i=1}^{n} \alpha_i(e') \oplus (\oplus_{i=n+1}^{\ell} \alpha_i(e')) = \alpha(e') \oplus (\oplus_{i=n+1}^{\ell} \alpha_i(e')).$$

Here, by definition of $\alpha_i$, the element $\alpha_i(e)$ (resp. $\alpha_i(e')$), for $i = n+1, \ldots, \ell$, is independent with $\alpha(e)$ (resp. $\alpha(e')$). Hence, we have that $\tilde{\alpha}(e)$ and $\tilde{\alpha}(e')$ are also pairwise linearly independent. This establishes the axiom (2).

Finally, we claim the axiom (3), i.e., $\tilde{\alpha}$ satisfies the following congruence relation: for each $e' \in E_{l(e)}(\Gamma)$

$$\tilde{\alpha}(\nabla_e(e')) = \tilde{\alpha}(e') + c_e(e')\tilde{\alpha}(e),$$

where $c_e(e')$ is the integer which satisfies that

$$\alpha(\nabla_e(e')) = \alpha(e') + c_e(e')\alpha(e).$$

Because $\tilde{\alpha} = \oplus_{i=1}^{\ell} \alpha_i$, it is enough to prove that $\alpha_i$ satisfies the congruence relation:

$$\alpha_i(\nabla_e(e')) = \alpha_i(e') + c_e(e')\alpha_i(e).$$

Set $e = e_{j,p}$ and $e' = e_{h,p}$ for some $j, h = 1, \ldots, m$, and $\nabla_e(e') = e_{\sigma(h),q}$. By definition, $\alpha_i(e_{j,p}) = k_{j,p}^{(i)} a_i$. Therefore, it is enough to check the following relation:

$$k_{\sigma(h),q}^{(i)} = k_{h,p}^{(i)} + c_e(e')k_{j,p}^{(i)}.$$

Using $f_i \in \mathcal{A}(\Gamma, \alpha, \nabla)$, Lemma 2.5, Lemma 2.11, we have

$$N_e(f_i(p)) - f_i(q) = -f_i(p) N_e(c_{(r,\alpha,\nabla)}(e))$$

(3.7)
for all $i = 1, \ldots, n$, and

$$f_i(p) = \begin{pmatrix} k_{1,p}^{(i)} \\ \vdots \\ k_{m,p}^{(i)} \end{pmatrix}.$$

Since $e = e_{j,p}$, we have $f_i(p)_e = k_{j,p}^{(i)}$. Therefore, if we put the permutation on $\{1, \ldots, m\}$ induced from $N_e$ as $\sigma$ then E.q. \ref{eq:extension} implies that

$$\begin{pmatrix} k_{\sigma(1),p}^{(i)} \\ \vdots \\ k_{\sigma(m),p}^{(i)} \end{pmatrix} - \begin{pmatrix} k_{1,q}^{(i)} \\ \vdots \\ k_{m,q}^{(i)} \end{pmatrix} = -k_{j,p}^{(i)} \begin{pmatrix} e_{\sigma(1),p} \\ \vdots \\ e_{\sigma(m),p} \end{pmatrix}.$$

This shows that

$$k_{\sigma(h),p}^{(i)} - k_{h,q}^{(i)} = -k_{j,p}^{(i)}e_{\sigma(h),p}.$$

Since $\sigma$ is the permutation, this equation establishes the equation \ref{eq:equation}.

Consequently, $\alpha$ is an extended axial function of $\alpha$. \hfill $\square$

This establishes (2) $\Rightarrow$ (1) in Theorem\ref{thm:main}. Together with Section\ref{sec:applications} we obtain Theorem\ref{thm:main}.

Remark 3.5. Lemma\ref{lem:extension} tells us that from an element of $A(\Gamma, \alpha, \nabla)$ we can construct an extension of $(\Gamma, \alpha, \nabla)$. In fact, by the similar arguments, we see that every extension of $(\Gamma, \alpha, \nabla)$ corresponds to an element of $A(\Gamma, \alpha, \nabla)$. Furthermore, it is not so difficult to show that every axial function on $\Gamma$ whose connection is $\nabla$ can be constructed by an element of $A(\Gamma, \alpha, \nabla)$. This is the reason why we call $A(\Gamma, \alpha, \nabla)$ a group of axial functions.

As a corollary, we have the following fact.

Corollary 3.6. Let $(\Gamma, \alpha, \nabla)$ be an $(m,n)$-type GKM graph. If one of the following cases hold, then there are no extensions of $(\Gamma, \alpha, \nabla)$:

1. $m = n$;
2. $A(\Gamma, \alpha, \nabla) = n$.

Example 3.7. By Corollary\ref{cor:extension} and the computation in Example\ref{ex:example} the GKM graphs in Figure\ref{fig:example} have no extensions, i.e., they are the maximal GKM graphs.

4. Applications to geometry

Guillemin-Zara studies the GKM graph as a combinatorial counterpart of some nice manifold with torus actions, called a GKM manifold. Using a notion of GKM graph, they build a bridge between the geometry of (in particular, symplectic) manifolds with torus actions and the combinatorics of GKM graphs. In this section, we give a new application of GKM graphs to study the geometry of manifolds with torus actions. More precisely, we apply our main result to study the maximal torus actions of GKM manifolds. There are several versions of definitions of GKM manifolds (see e.g.\ref{def:GKM}). In this paper, we apply our results to the GKM manifold in the original sense (introduced in\ref{def:GKM}).

4.1. GKM manifold and its GKM graph. We first briefly recall the relation between GKM manifolds and GKM graphs (see\ref{def:GKM} for details). In this paper, $(M, G)$ (resp. $(M, G, \varphi)$) represents a $G$-action on $M$ (resp. defined by $\varphi : G \times M \to M$). Let $(M, T)$ be any $T$-manifold $M$, where $T$ is a torus. Put $M_1 \subset M$ by the set of elements $x \in M$ such that the orbit $T(x) = \{x\}$ (a fixed point) or $T(x) \simeq S^1$, i.e.,

$$M_1 = \{x \in M \mid \dim T(x) \leq 1\}.$$

The set $M_1$ is called a one-skeleton of $(M, T)$.

Let $M$ be a $2m$-dimensional, compact, connected smooth manifold with an effective $n$-dimensional torus $T^n$-action, where $1 \leq n \leq m$. We call $M$ a GKM manifold or an $(m,n)$-type GKM manifold if the following three conditions hold:

- $M$ is a compact, connected, smooth manifold with a smooth action of $T^n$.
- $M$ is a GKM manifold.
- $M$ is a one-skeleton of $M$. 

In this paper, we apply our results to the GKM manifold in the original sense (introduced in\ref{def:GKM}).
(1) $M^T \neq \emptyset$;
(2) the manifold $M$ has a $T$-invariant almost complex structure;
(3) the one-skeleton of $M$ has the structure of an abstract (connected) graph $\Gamma_M$ such that its vertices $V(\Gamma_M)$ are the fixed points and its edges $E(\Gamma_M)$ are embedded 2-spheres connecting two fixed points.

Remark 4.1. The third condition implies that the orbit space of the one-skeleton is one-dimensional. Therefore, by definition of the $(m, n)$-type GKM manifold $M$, if $\dim T(\cong n) = 1$ then $M$ is equivariantly diffeomorphic to $\mathbb{CP}^1$ with a non-trivial $S^1$-action. So, in this paper, we often assume $2 \leq n \leq m$ for an $(m, n)$-type GKM manifold.

By using the differentiable slice theorem, it is easy to check that $\Gamma_M$ is an $m$-valent graph. An axial function $\alpha_M : E(\Gamma_M) \to H^2(BT)$ can be defined as the following way. Now the cohomology $H^2(BT) \cong \mathbb{Z}^n$ may be regarded as the set of all complex one-dimensional representations, or equivalently the set of all homomorphisms from $T^n$ to $S^1$, say $\text{Hom}(T^n, S^1)$. Because there is a $T$-invariant complex structure on $M$, its tangent space $T_pM$ is a complex $T$-representation space, called a tangential representation of $M$ on $p \in M^T$. Therefore, $T_pM$ decomposes into irreducible complex representation spaces:

$$T_pM = \bigoplus_{i=1}^n V(a_i)$$

where $a_i \in \text{Hom}(T^n, S^1)$ and $V(a_i)$ represents the complex one-dimensional representation space with the weight $a_i \in \text{Hom}(T, S^1)$. Since the $T$-action is effective, $\{a_1, \ldots, a_m\}$ spans $\text{Hom}(T^n, S^1) \cong \mathbb{Z}^n$. Moreover, as is well-known, the third condition of the definition of GKM manifold is equivalent to that $\{a_1, \ldots, a_m\}$ is pairwise linearly independent and the one-skeleton of $(M, T)$ is connected. Then, each $V(a_i)$ may be regarded as the tangential representation of some $T$-invariant 2-sphere on $p \in M^T$, say $S^2_{i,p} \subset M$. Recall that $E(\Gamma_M)$ is the set of $T$-invariant 2-spheres; therefore, there is the corresponding edge $e_{i,p} \in E_p(\Gamma_M)$ with $S^2_{i,p}$. This defines the function such that $e_{i,p} \mapsto a_i$, and we denote this function as $\alpha_M : E(\Gamma_M) \to H^2(BT)$. We next check that the function $\alpha_M$ satisfies the axiom of the axial function. The 1st axiom, i.e., $\alpha_M(e) = -\alpha_M(\bar{e})$, can be easily checked from the fact that the invariant 2-sphere is isomorphic to the standard $S^1$-action on $\mathbb{CP}^1(\cong S^2)$. The 2nd axiom and the 4th condition, i.e., the subset $\{\alpha_M(e) \mid E_p(\Gamma_M)\}$ in $H^2(BT)$ for each $p \in V(\Gamma_M)$ satisfies the pairwise linearly independent and spans $H^2(BT)$, have already checked in the arguments as above. In order to check the 3rd axiom, we need to define the connection on the labeled graph $(\Gamma_M, \alpha_M)$. Denote an invariant two sphere which connecting two fixed points $p, q \in M^T$ as $S^2(pq) \cong \mathbb{CP}^1$ (this might not be unique). Then, by using the restricted $T^n$-invariant almost complex structure on the restricted tangent bundle $\tau_M|S^2(pq)$, the restricted tangent bundle $\tau_M|S^2(pq)$ splits into the following $T^n$-invariant line bundles:

$$L_1 \oplus \cdots \oplus L_n,$$

where $L_i$ is a complex $T^n$-equivariant line bundle over $S^2(pq)$ for all $i = 1, \ldots, n$. Because each $L_i$ is $T^n$-equivariant, we may write the restrictions onto fixed points as $L_i|_p \cong V(a_{i,p})$ and $L_i|_q \cong V(a_{i,q})$ for some $\alpha_M(e_{i,p}) = a_{i,p}$ and $\alpha_M(e_{i,q}) = a_{i,q}$, for each $i = 1, \ldots, n$. This defines the map $e_{i,p} \mapsto e_{i,q}$ for all $i = 1, \ldots, n$. Therefore, there is the bijection $\nabla_{pq} : E_p(\Gamma_M) \to E_q(\Gamma_M)$. The collection of the bijections on each $e \in E(\Gamma_M)$ define the collection $\nabla_M = \{\nabla_e \mid e \in E(\Gamma_M)\}$. It is easy to check that this satisfies $\nabla_e(\bar{e}) = \bar{\tau}$ and $\nabla_{\tau} = \nabla_\tau^{-1}$. Now it is well-known that for every complex line bundle $L$ over $\mathbb{CP}^1$ there exists an $S^1$ representation space $\mathbb{C}_r$ by $r \in \text{Hom}(S^1, \mathbb{C}_1) \cong \mathbb{Z}$ such that

$$L \equiv L_\rho := S^3 \times_{S^1} \mathbb{C}_r,$$

where $S^1$ acts on $S^3 \subset \mathbb{CP}^2$ diagonally and $\mathbb{C}_r$ via $r$. Therefore, it easily follows from (4.1) that $\nabla_M$ satisfies the congruence relation $\nabla_\rho$. Hence, $\nabla_M$ is the connection on $(\Gamma_M, \alpha_M)$, and $\alpha_M$ is the axial function on $\Gamma_M$.

Consequently, the GKM manifold $M$ defines the GKM graph $(\Gamma_M, \alpha_M, \nabla_M)$ by using its one-skeleton and the tangential representations. In this paper, such GKM graph $(\Gamma_M, \alpha_M)$ (or $(\Gamma_M, \alpha_M, \nabla_M)$) is called an induced GKM graph from $M$.  

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Example 4.2. In Figure 1, the left GKM graph is the GKM graph induced from the standard $T^2$-action on $\mathbb{C}P^2$ and the right one is that induced from $T^3$-action on $\mathbb{C}P^3$. The middle GKM graph is induced from the $T^2$-action on $S^6 = G_2/SU(3)$, where $G_2$ is the exceptional Lie group, see [1] Section 5.2).

4.2. Extensions of torus actions. The definition of an extension of GKM graphs in Section 2.1 is motivated by an extension of a torus action on GKM manifold. We explain it more precisely in this section.

Let $(M, T^n, \varphi)$ be a manifold with an effective $n$-dimensional torus action $\varphi : T^n \times M \to M$ (not necessarily a GKM manifold). If there exists an effective $\ell$-dimensional torus action $(M, T^\ell, \varphi')$ (for $n < \ell$) and an injective homomorphism $\iota : T^n \to T^\ell$ such that the following diagram commutes:

$$
\begin{array}{ccc}
T^\ell \times M & \xrightarrow{\iota \times \text{id}} & T^n \times M \\
\varphi' & \Downarrow & \varphi \\
M & \xrightarrow{\ell} & M
\end{array}
$$

then $(M, T^\ell, \varphi')$ is called an extension of $(M, T^n, \varphi)$. We prove the following fact:

**Proposition 4.3.** If $(M^{2m}, T^\ell)$ is an extension of an $(m, n)$-type GKM manifold $(M^{2n}, T^n)$ (for $n < \ell \leq m$) and the $T^\ell$-action preserves the almost complex structure of $M$, then $(M^{2m}, T^\ell)$ is an $(m, \ell)$-type GKM manifold.

Furthermore, the induced $(m, \ell)$-type GKM graph $(\Gamma_M, \bar{\alpha}_M, \bar{\nabla}_M)$ from $(M, T^\ell)$ is an extension of the induced $(m, n)$-type GKM graph $(\Gamma_M, \alpha_M, \nabla_M)$ from $(M, T^n)$.

**Proof.** Because the $T^\ell$-action preserves the almost complex structure of $M$, it is enough to check that its one-skeleton has the structure of a graph. We note that for all $p \in M$ two orbits of $p$ of these actions satisfy $T^n(p) \subset T^\ell(p)$, because the $T^\ell$-action is an extension of $T^n$-action.

We first claim that $M^{T^n} = M^{T^\ell}$. Because $T^n(p) \subset T^\ell(p)$ for all $p \in M$, we have that $M^{T^n} \supset M^{T^\ell}$. Assume that there exists a fixed point $p \in M^{T^n}$ such that $T^n(p) \neq \{p\}$. As is well-known, there is a decomposition $T_pM = T^\ell_p(p) \oplus N_pT^\ell(p)$, where $T^\ell_p(p)$ is the tangent space and $N_pT^\ell(p)$ is the normal space of $T^\ell(p)$ on $p$. By using the differentiable slice theorem, the isotropy subgroup $T^\ell_p$ (of the $T^\ell$-action on $p$) acts on $T^\ell_p(p)$ trivially. This show that the $T^n$ (of $T^n_p$) also acts on $T^n_p(p)$ trivially. However, by the definition of GKM manifolds, for the restricted action $(T^n_pM, T^n)$, there is another decomposition $T_pM = \bigoplus_{i=1}^m V(\alpha_i)$ such that each representation $\alpha_i : T^n \to S^1$ is non-trivial. This contradicts to that $T^n$ acts on $(\{0\} \neq) T^n_pT^\ell(p) \subset T_pM$ trivially. Hence, $M^{T^n} = M^{T^\ell}$.

Take $p \in M$ such that $T^n(p) \simeq S^1$. Because we assume that the one-skeleton of $(M, T^n)$ has the structure of a connected graph, we have that $p$ is an element in an invariant 2-sphere $S^2$ of $(M, T^n)$. Because $M^{T^n} = M^{T^\ell}$, by considering the tangential representation around fixed points on this $T^n$-invariant $S^2(\equiv p)$, there exists a representation $\rho : T^\ell \to S^1$ which may be regarded as the extension of the $T^n$-action on $S^2$. Therefore, every $T^n$-invariant $S^2$ is also a $T^\ell$-invariant $S^2$, i.e., if $T^n(p) \simeq S^1$ then $T^\ell(p) \simeq S^1$. Together with $T^n(p) \subset T^\ell(p)$ for all $p \in M$, this implies that two one-skeletons of $(M^{2m}, T^n)$ and $(M^{2m}, T^\ell)$ are the same.

We next prove the final statement. By the arguments as above, we have $\bar{\Gamma}_M = \Gamma_M$. Moreover, because the extended $T^\ell$-action preserves the $T^n$-invariant almost complex structure, the splitting $\nu = \bigoplus_{i=1}^m L_i$ of the normal bundle $\nu$ of such $S^2$ by $T^n$-action is preserved by the extended $T^\ell$-action. This implies that two connections on induced GKM graphs $\nabla_M$ from the $T^n$-action and $\bar{\nabla}_M$ from the extended $T^\ell$-action are the same. Finally, put the induced homomorphism from the inclusion $\iota : T^n \to T^\ell$ as $\tau : H^2(BT^n) \to H^2(BT^\ell)$. Then, by considering the tangential representations (of both $T^n$ and $T^\ell$-actions) around fixed points, it is easy to check that there is the following
4.3. Maximal torus action on $S^6 = G_2/SU(3)$. As we mentioned in Example 4.2, the $(3,2)$-type GKM graph in Figure 4 is the induced GKM graph of the GKM manifold $(G_2/SU(3), T^2)$, where $T^2$ acts on $G_2$ as its maximal torus subgroup (e.g. see [3]). We also note that $G_2/SU(3) \simeq S^6$ (diffeomorphic). Therefore, by using Corollary 1.3 and Example 2.12, the following well-known fact can be proved (see also [3]):

**Corollary 4.4.** The $T^2$-action on $G_2/SU(3) \simeq S^6$ is the maximal torus action. In other words, there are no extended $T^2$-actions on $S^6$ of this $T^2$-action, which preserves the almost complex structure induced from the homogeneous space $G_2/SU(3)$.

**Remark 4.5.** Note that there is the $T^3$-action on $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$ defined by the standard $T^3$-action on $\mathbb{C}^3$ (see e.g. [15]). However, from Corollary 4.4, this action is not the extended action of the $T^2$-action on $S^6 = G_2/SU(3)$.

5. Maximal torus action on the complex Grassmannian $G_2(\mathbb{C}^{n+2})$

The (complex) Grassmannian (of 2-planes in $\mathbb{C}^{n+2}$), denoted by $G_2(\mathbb{C}^{n+2})$, is defined by the set of all complex 2-dimensional vector spaces in $\mathbb{C}^{n+2}$. Namely,

\begin{equation}
G_2(\mathbb{C}^{n+2}) := \{ V \subset \mathbb{C}^{n+2} \mid \dim V = 2 \}.
\end{equation}

The Grassmannian $G_2(\mathbb{C}^{n+2})$ has the natural transitive $SU(n+2)$-action which is induced from the standard $SU(n+2)$-action on $\mathbb{C}^{n+2}$. Since its isotropy group is $S(U(2) \times U(n))$, $G_2(\mathbb{C}^{n+2})$ is diffeomorphic to the homogeneous space $SU(n+2)/S(U(2) \times U(n))$ (also see [13]). In particular, this shows that

\[ \dim G_2(\mathbb{C}^{n+2}) = \dim SU(n+2)/S(U(2) \times U(n)) = 4n. \]

Since a maximal torus of $SU(n+2)$ is isomorphic to $T^{n+1}$, there is a restricted $T^{n+1}$-action on $G_2(\mathbb{C}^{n+2})$ and its one-skeleton has the structure of a graph (see [2]). We denote this action as $(SU(n+2)/S(U(2) \times U(n)), T^{n+1})$. Note that the action $(SU(n+2)/S(U(2) \times U(n)), T^{n+1})$ is not effective because there is the non-trivial center in $SU(n+2)$ (isomorphic to $\mathbb{Z}/(n+2)\mathbb{Z}$); therefore, the GKM graph obtained from this action does not satisfy the condition (4) in Section 4; i.e., the axial function is not an effective axial function. So, in this paper, we define the $T^{n+1}$-action on $G_2(\mathbb{C}^{n+2})$ by the induced action from the standard $T^{n+1}$-action on the first $(n+1)$-coordinates in $\mathbb{C}^{n+2}$ (see (5.1)). We denote this action as $(G_2(\mathbb{C}^{n+2}), T^{n+1})$. It is easy to check that $(G_2(\mathbb{C}^{n+2}), T^{n+1})$ is effective and preserves the complex structure of $G_2(\mathbb{C}^{n+2})$ induced from that of $\mathbb{C}^{n+2}$. For example, when $n = 1$, $(G_2(\mathbb{C}^{3}), T^2)$ is equivariantly diffeomorphic to the complex projective space $\mathbb{C}P^2$ with the standard $T^2$-action, i.e., the toric manifold. Note that, for $n \geq 2$, $(G_2(\mathbb{C}^{n+2}), T^{n+1})$ is not a toric manifold. We also have that $(G_2(\mathbb{C}^{n+2}), T^{n+1})$ is essentially isomorphic to $(SU(n+2)/S(U(2) \times U(n)), T^{n+1})$, i.e., the induced effective action from $(SU(n+2)/S(U(2) \times U(n)), T^{n+1})$ is $T^{n+1}$-equivariantly diffeomorphic to $(G_2(\mathbb{C}^{n+2}), T^{n+1})$ up to automorphism of $T^{n+1}$ (see [13] Definition 2.6) for details). This implies that $(G_2(\mathbb{C}^{n+2}), T^{n+1})$ is a GKM manifold defined in Section 4.1 and its GKM graph satisfies the effectiveness condition (4).

**Remark 5.1.** GKM graphs obtained from the non-effective torus actions for flag manifolds are studied by Tymoczko in [20] or Fukukawa-Ishida-Masuda in [8] etc.

In the next subsection, we compute the GKM graph of $(G_2(\mathbb{C}^{n+2}), T^{n+1})$. 

\[ \begin{array}{ccc}
\alpha_M & \sim_M & H^2(BT^2) \\
\downarrow & & \downarrow \\
E(\Gamma_M) & \overset{\alpha_M}{\longrightarrow} & H^2(BT^n) \\
\end{array} \]

This establishes the final statement. \( \square \)
5.1. The GKM graph of \((G_2(\mathbb{C}^{n+2}), T^{n+1})\). In this section, we put \(M_n = G_2(\mathbb{C}^{n+2})\) for simplicity. Let \((\Gamma_n, \alpha_n, \nabla_n)\) be the induced GKM graph from \((M_n, T^{n+1})\). Note that \(\Gamma_n = (V(\Gamma_n), E(\Gamma_n))\) is a 2n-valent graph, because the real dimension of \(M_n\) is \(4n\), where \(n \geq 1\).

We first consider the fixed points of \((M_n, T^{n+1})\). By definition, the Grassmannian \(M_n\) may be identified with the following set:

\[
\{ [v_1, v_2] \mid v_1, v_2 \text{ are linearly independent in } \mathbb{C}^{n+2} \},
\]

where the symbol \([v_1, v_2]\) represents the equivalence class such that \([v_1, v_2]\) is identified with \([w_1, w_2]\) if two pairs of vectors \([v_1, v_2]\) and \([w_1, w_2]\) span the same 2-dimensional complex vector space in \(\mathbb{C}^{n+2}\). Under this identification, the element \(t \in T^{n+1}\) acts on \([v_1, v_2] \in M_n\) by

\[
t \cdot [v_1, v_2] \mapsto [tv_1, tv_2],
\]

where \(t \in T^{n+1}\) acts on \(v \in \mathbb{C}^{n+2}\) by the standard coordinatewise multiplication on the first \((n+1)\)-coordinates. Then, the fixed points can be denoted by

\[
M^T_n := \{ [e_i, e_j] \mid i \neq j, \ i, j = 1, \ldots, n + 2 \},
\]

where \(e_1, \ldots, e_{n+2}\) are the standard basis in \(\mathbb{C}^{n+2}\). By identifying the element \([e_i, e_j] \in M^T_n\) as the subset \([i, j]\) in \([n + 2] := \{1, 2, \ldots, n + 2\}\), we may regard the set of vertices \(V(\Gamma_n)\) as

\[
V(\Gamma_n) = \{ [i, j] \subset [n + 2] \mid i \neq j \}.
\]

We also have that

\[
|V(\Gamma_n)| = \binom{n + 2}{2} = \frac{(n + 2)(n + 1)}{2}.
\]

We next consider the invariant 2-spheres in \((M_n, T^{n+1})\). Fix \([i, j] \subset [n + 2]\). Now the following subsets are \(T^{n+1}\)-invariant sets in \(M_n\), which contain \([e_i, e_j]\):

\[
S^{i, k}_{i, j} = \{ [e_i, v_{jk}] \in M_n \mid v_{jk} = a_j e_j + a_k e_k, \ (a_j, a_k) \in \mathbb{C}^2 \setminus \{0\} \},
\]

\[
S^{j, k}_{i, j} = \{ [v_{ik}, e_j] \in M_n \mid v_{ik} = a_i e_i + a_k e_k, \ (a_i, a_k) \in \mathbb{C}^2 \setminus \{0\} \},
\]

for all \(k \in [n + 2] \setminus \{i, j\}\).

Because \([e_i, v_{jk}]\) and \([e_i, \lambda v_{jk}]\) (for all \(\lambda \in \mathbb{C}^*\)) are the same element in \(M_n\), \(S^{i, k}_{i, j}\) is diffeomorphic to \(\mathbb{CP}^1\). Similarly, \(S^{j, k}_{i, j}\) is also diffeomorphic to \(\mathbb{CP}^1\). Moreover, \([e_i, e_j], [e_i, e_k] \in S^{i, k}_{i, j}\) and \([e_i, e_j], [e_k, e_j] \in S^{j, k}_{i, j}\). This shows that if \(\{i, j\} \cap \{k, l\} \neq \emptyset\) then the fixed points \([e_i, e_j]\) and \([e_k, e_l]\) are on the same invariant 2-sphere. Namely, the pair of two distinct elements \(\{i, j\}\) and \(\{k, l\}\) such that \(\{i, j\} \cap \{k, l\} \neq \emptyset\) are an element in \(E(\Gamma_n)\). We call the edge corresponding to \(S^{i, k}_{i, j}\) (resp. \(S^{j, k}_{i, j}\)) as \(E^{i, k}_{i, j} \in E(\Gamma)\) (resp. \(E^{j, k}_{i, j} \in E(\Gamma_n)\)). Note that for all \(k \in [n + 2] \setminus \{i, j\}\), \(E^{i, k}_{i, j}\) and \(E^{j, k}_{i, j}\) are out-going edges from \([i, j]\). Since \(\dim M_n = 4n\), the number of out-going edges from \([i, j]\) is \(2n\). Hence, the set of all out-going edges from \([i, j]\) can be denoted by

\[
E_{(i, j)}(\Gamma_n) = \{ E^{i, k}_{i, j}, E^{j, k}_{i, j} \mid k \in [n + 2] \setminus \{i, j\} \}.
\]

Figure 4 shows the one-skeleton induced from \(G_2(\mathbb{C}^4)\).

Next we consider the tangential representations around fixed points. To do that, we use the following notations:

- the symbol \(E(\eta)\) represents the total space of the fibre bundle \(\eta\) over \(M_n\);
- the symbol \(\eta_p\) is the restriction of \(\eta\) onto \(p \in M_n\);

Recall the structure of the tangent bundle \(\tau\) of \(M_n\). Let \(\epsilon^{n+2}_C\) be the trivial bundle \(E(\epsilon^{n+2}_C) = M_n \times \mathbb{C}^{n+2} \to M_n\). Then, the tautological vector bundle \(\gamma\) over \(M_n\) is defined as follows:

\[
E(\gamma) = \{ (V, x) \in M_n \times \mathbb{C}^{n+2} \mid x \in V \} \to M_n,
\]

where the projection of the bundle is just the projection onto the 1st factor. Note that \(\gamma\) is a complex 2-dimensional vector bundle over \(M_n\) and the diagonal \(T^{n+1}\)-action on \(M_n \times \mathbb{C}^{n+2}\) induces the \(T^{n+1}\)-action on \(E(\gamma)\); thus we may regard \(\gamma\) as the \(T^{n+1}\)-equivariant vector bundle. Let \(\gamma_\perp\) be the normal bundle of \(\gamma\) in \(\epsilon^{n+2}_C\) (we define the inner product on \(\mathbb{C}^{n+2}\) as the standard Hermitian
Figure 4. The vertices and edges of the one-skeleton of the Grassmannian $G_2(\mathbb{C}^4)$.

inner product. Since $\gamma$ is a complex 2-dimensional vector bundle, $\gamma^\perp$ is a complex $n$-dimensional vector bundle. Moreover, since the $T^{n+1}$-action on $\mathbb{C}^{n+2}$ preserves the standard Hermitian inner product, the diagonal $T^{n+1}$-action on $M_n \times \mathbb{C}^{n+2}$ induces the $T^{n+1}$-action on $\gamma^\perp$.

Similar to the case of real Grassmannian (see [18, Section 5 or proof of Theorem 14.10]), the tangent bundle $\tau$ of $M_n$ is isomorphic to the complex 2n-dimensional vector bundle $\text{Hom}(\gamma, \gamma^\perp)$. Therefore, the tangent space around $[e_i, e_j] \in M_n^T$ may be regarded as

$$\tau_{[e_i, e_j]} \cong \text{Hom}(\gamma_{[e_i, e_j]}, \gamma^\perp_{[e_i, e_j]}).$$

Because the total space of $\tau_{[e_i, e_j]}$ is $V_{ij} := \{A_i e_i + A_j e_j \mid (A_i, A_j) \in \mathbb{C}^2\}$, its normal space in $\mathbb{C}^{n+2}$ consists of

$$V_{ij}^\perp = \{ \sum_{k \in [n+2] \setminus \{i, j\}} B_k e_k \mid B_k \in \mathbb{C} \}.$$

Therefore, $\varphi \in \text{Hom}(V_{ij}, V_{ij}^\perp)$ can be denoted as

$$\varphi(A_i e_i + A_j e_j) = \sum_{k \in [n+2] \setminus \{i, j\}} f_k(A_i, A_j)e_k$$

for some linear map $f_k : \mathbb{C}^2 \rightarrow \mathbb{C}$, i.e., $f_k(A_i, A_j) = A_i \ell_{ik} + A_j \ell_{jk}$ for some $(\ell_{ik}, \ell_{jk}) \in \mathbb{C}^2$ (we may identify $f_k$ as $(\ell_{ik}, \ell_{jk})$). Then, we may regard $\varphi = (f_k)_{k \in [n+2] \setminus \{i, j\}} \in M(2, n; \mathbb{C})$ as the complex $(2 \times n)$-matrix. Now the $T^{n+1}$-actions on $\gamma$ and $\gamma^\perp$ induce the $T^{n+1}$-action on $\text{Hom}(\gamma, \gamma^\perp)$ as follows: for $\varphi \in \text{Hom}(\gamma_x, \gamma^\perp_x)$ $(x \in M_n)$ and $t \in T^{n+1}$,

$$t \cdot \varphi = t \circ \varphi \circ t^{-1} : \gamma_x \xrightarrow{t} \gamma_x \xrightarrow{\varphi} \gamma^\perp_x \xrightarrow{t^{-1}} \gamma^\perp_{tx}.$$

Therefore, on $x = [e_i, e_j]$, we have $t \cdot f_k = (t_i^{-1}t_k \ell_{ik} t_j^{-1}t_k \ell_{jk})$ for $f_k = (\ell_{ik}, \ell_{jk})$, $\varphi = (f_k) \in M(2, n; \mathbb{C})$ and $t = (t_1, \ldots, t_{n+1}, 1) \in T^{n+2}$, i.e., $t_{n+2} = 1$. Hence, on the fixed point $[e_i, e_j] \in M_n^T$, we have the tangential representation as follows:

$$\text{Hom}(\gamma_{[e_i, e_j]}, \gamma^\perp_{[e_i, e_j]}) \simeq \bigoplus_{k \in [n+2] \setminus \{i, j\}} V(-a_i + a_k) \oplus V(-a_j + a_k),$$

where $a_1, \ldots, a_{n+1}$ are the dual basis of the dual of Lie algebra $t^*$ of $T^{n+1}$ and we put $a_{n+2} = 0$. It is easy to check that the factor $V(-a_i + a_k)$ (resp. $V(-a_j + a_k)$) in (5.2) may be regarded as the tangent space on $[e_i, e_j]$ of the invariant 2-sphere $S_{i,j}^{t,k}$ (resp. $S_{i,j}^{j,k}$). Therefore, the axial function $\alpha_n : E(\Gamma_n) \rightarrow H^2(BT^{n+1}) \simeq t^*_{\mathbb{C}}$ is defined as follows:

$$\alpha_n(E^{i,k}_{i,j}) = -a_j + a_k, \quad \alpha_n(E^{j,k}_{i,j}) = -a_i + a_k.$$

By the definition of edges, the orientation reverse edge edge satisfies $E^{i,k}_{i,j} = E^{i,j}_{i,k}$ (resp. $E^{j,k}_{i,j} = E^{i,j}_{j,k}$). Therefore, by the definition of the axial function the following equation holds:

$$\alpha_n(E^{i,j}_{i,k}) = -a_k + a_j = -\alpha_n(E^{i,k}_{i,j}) \quad (\text{resp.} \quad \alpha_n(E^{j,k}_{i,j}) = -a_k + a_i = -\alpha_n(E^{j,k}_{i,j})).$$
We finally compute a connection on \((\Gamma_n, \alpha_n)\). Put the connection on the edge \(E_{i,j}^{i,k}\) as \((\nabla_n)^{i,k}_{i,j} = (\nabla_n)_{i,j}^{i,k}\). Namely,

\[
(\nabla_n)^{i,k}_{i,j} : E_{i,j}(\Gamma_n) \to E_{i,k}(\Gamma_n),
\]

where

\[
E_{i,j}(\Gamma_n) := \{ E_{i,j}^{l,i} : l \in [n + 2] \setminus \{i, j\} \},
\]

\[
E_{i,k}(\Gamma_n) := \{ E_{i,k}^{l,i} : l \in [n + 2] \setminus \{i, k\} \}.
\]

Note that the set of the weights \(\{-a_i + a_k, -a_j + a_k \mid k \in [n + 2] \setminus \{i, j\}\}\) are 3-independent for all \(\{i, j\} \subset [n + 2]\) (see (5.3)). Therefore, it follows from Lemma 2.2 that the connection \(\nabla_n\) on \((\Gamma_n, \alpha_n)\) is unique. This implies that the bijection \((\nabla_n)^{i,k}_{i,j}\) which satisfies the congruence relation (5.3) is unique. Hence, by computing (2.1) on \((\Gamma_n, \alpha_n)\), the connection must be defined as follows:

\[
(\nabla_n)^{i,k}_{i,j}(E_{i,j}^{i,k}) = E_{i,k}^{i,k},
\]

\[
(\nabla_n)^{i,k}_{i,j}(E_{i,j}^{l,i}) = E_{i,k}^{l,i} \quad \text{for } l \in [n + 2] \setminus \{i, j, k\},
\]

\[
(\nabla_n)^{i,k}_{i,j}(E_{i,j}^{k,l}) = E_{i,k}^{k,l} \quad \text{for } l \in [n + 2] \setminus \{i, j, k\},
\]

\[
(\nabla_n)^{i,k}_{i,j}(E_{i,j}^{l,k}) = E_{i,k}^{l,k}.
\]

In fact, we have that

\[
\alpha_n(E_{i,j}^{i,l}) - \alpha_n(E_{i,j}^{i,k}) = c_{i,j}^{i,k}(E_{i,j}^{i,l})\alpha_n(E_{i,j}^{i,k})
\]

\[-a_i + a_j = (a_i + a_j)
\]

and

\[
\alpha_n(E_{i,j}^{k,l}) - \alpha_n(E_{i,j}^{j,k}) = c_{i,j}^{i,k}(E_{i,j}^{k,l})\alpha_n(E_{i,j}^{j,k})
\]

\[-a_i + a_j = (a_i + a_j)
\]

for \(l \in [n + 2] \setminus \{i, j, k\}\) and

\[
\alpha_n(E_{i,j}^{j,k}) - \alpha_n(E_{i,j}^{i,k}) = c_{i,j}^{i,k}(E_{i,j}^{j,k})\alpha_n(E_{i,j}^{i,k})
\]

\[-a_i + a_j = (a_i + a_j)
\]

for some integers (congruence coefficients) \(c_{i,j}^{i,k}(E_{i,j}^{i,l}), c_{i,j}^{i,k}(E_{i,j}^{j,l}), c_{i,j}^{i,k}(E_{i,j}^{k,l}), c_{i,j}^{i,k}(E_{i,j}^{l,k})\). Therefore, together with Lemma 2.3, the congruence coefficients are

\[
c_{i,j}^{i,k}(E_{i,j}^{i,l}) = -2,
\]

\[
c_{i,j}^{i,k}(E_{i,j}^{l,i}) = -1 \quad \text{for } l \in [n + 2] \setminus \{i, j, k\},
\]

\[
c_{i,j}^{i,k}(E_{i,j}^{k,l}) = 0 \quad \text{for } l \in [n + 2] \setminus \{i, j, k\},
\]

\[
c_{i,j}^{i,k}(E_{i,j}^{l,k}) = -1.
\]

In summary we have that

**Proposition 5.2.** Let \(\Gamma_n = (V(\Gamma_n), E(\Gamma_n))\) be the abstract graph defined by

- the set of vertices \(V(\Gamma_n)\) consists of all \(\{i, j\}\) in \([n + 2]\) for \(i \neq j\);
- the set of edges \(E(\Gamma_n)\) consists of all pairs of distinct vertices \(\{i, j\}, \{k, l\}\) such that \(\{i, j\} \cap \{k, l\} \neq \emptyset\).

Define its axial function as \(\alpha_n : E(\Gamma_n) \to H^2(BT^{n+1})\) in (5.3). Then, the connection \(\nabla_n\) is uniquely determined (as above) and the triple \((\Gamma_n, \alpha_n, \nabla_n)\) is the \((2n, n + 1)\)-type GKM graph.

**Remark 5.3.** The graph in Proposition 5.2 is known as the Johnson graph \(J(n + 2, 2)\). The 1st GKM graph in Figure 1 shows the case when \(n = 1\), i.e., the Johnson graph \(J(3, 2)\), and the GKM graph in Figure 2 shows the case when \(n = 2\), i.e., the Johnson graph \(J(4, 2)\). It is known that the one-skeleton of the general Grassmannian \(G_k(C^{n+k})\) (for \(k \geq 1\)) is the Johnson graph \(J(n + k, k)\) (see 1).
5.2. The 2nd main result. Because we fix the axial function $\alpha_n$ on $\Gamma_n$ and its connection $\nabla_n$ is unique, we may write the GKM graph $(\Gamma_n, \alpha_n, \nabla_n)$ of $(M_n, T^{n+1})$ as $\Gamma_n$ for simplicity; therefore, we denote the group of axial functions $\mathcal{A}(\Gamma_n, \alpha_n, \nabla_n)$ as $\mathcal{A}(\Gamma_n)$. This final section devotes to the proof of the following theorem:

**Theorem 5.4.** The group of axial functions $\mathcal{A}(\Gamma_n)$ is isomorphic to $\mathbb{Z}^{n+1}$.

When $n = 1$, the GKM graph $\Gamma_1$ is the $(2, 2)$-type GKM graph (which is the 1st GKM graph in Figure 1). Therefore, by Theorem 3.1 we have that $\mathcal{A}(\Gamma_1) \cong \mathbb{Z}^2$. Hence, we may assume that $n \geq 2$.

To prove Theorem 5.4 we first choose an order on $E_{i,j}(\Gamma_n)$ for $i, j \in [n+2]$ as follows (see Figure 6 for $n = 2$):

- $E_{i,j} \prec E_{i,k}$ if $i < j$, where $k \in [n+2] \setminus \{i, j\}$;
- $E_{i,j} \prec E_{i,l}$ if $k < l$, where $k, l \in [n+2] \setminus \{i, j\}$.

Take $f \in \mathcal{A}(\Gamma_n)$ and put

$$f([n+1, n+2]) = \begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}$$
with respect to the order on $E_{(n+1,n+2)}(\Gamma_n)$ defined as before, i.e.,
$$E_{n+1,n+2}^{1,n+1} < \cdots < E_{n+1,n+2}^{n,n+1} < E_{n+1,n+2}^{1,n+2} < \cdots < E_{n+1,n+2}^{n,n+2}.$$ More precisely, using the notation $f(p)$, defined in Section 2.3 for $p \in V(\Gamma_n)$ and $e \in E_p(\Gamma_n)$, we define the following correspondence between edges and integers (variables):

$$E_{n+1,n+2}^{1,n+1} \rightarrow f(\{n+1,n+2\})_{E_{n+1,n+2}^{1,n+1}} = x_1;$$
$$\vdots$$
$$E_{n+1,n+2}^{n,n+1} \rightarrow f(\{n+1,n+2\})_{E_{n+1,n+2}^{n,n+1}} = x_n;$$
$$E_{n+1,n+2}^{1,n+2} \rightarrow f(\{n+1,n+2\})_{E_{n+1,n+2}^{1,n+2}} = x_{n+1};$$
$$\vdots$$
$$E_{n+1,n+2}^{n,n+2} \rightarrow f(\{n+1,n+2\})_{E_{n+1,n+2}^{n,n+2}} = x_{2n}.$$ Then, by the connectedness of $\Gamma_n$ and the definition of $A(\Gamma_n)$, the vector $f(\{i,j\})$ is denoted by the valuables $x_1,\ldots,x_{2n}$ for all $\{i,j\}$’s in $V(\Gamma_n)$. This shows that $\text{rk}A(\Gamma_n) \leq 2n$ (this is also known from Theorem 3.1 the fact that $\Gamma_n$ is a $(2n, n+1)$-type GKM graph). Therefore, in order to prove Theorem 5.4 it is enough to prove that the variables $x_{n+2},\ldots,x_{2n}$ can be denoted by the other variables $x_1,\ldots,x_{n+1}$, i.e., the following lemma holds.

**Lemma 5.5.** For $j = 0,\ldots,n - 2$ ($n \geq 2$), the following equation holds:
$$x_{2n-j} = -x_1 + x_{n-j} + x_{n+1}.$$ 

**Proof.** Recall the definition of the connection $\nabla_n$ in Section 5.4. Then, there is the triangle GKM subgraph in $\Gamma_n$ (i.e., the subgraph closed under the connection) which spanned by the vertices $\{n+1, n+2\}, \{1, n+1\}, \{1, n+2\}$ (see Figure 7). We first compute the variables corresponding edges in this triangle as in Figure 7.

![Figure 7](image-url)  

**Figure 7.** The triangle GKM subgraph with corresponding variables on edges.

We assumed $f(\{n+1,n+2\})_{E_{n+1,n+2}^{1,n+1}} = x_1$ (and $f(\{n+1,n+2\})_{E_{n+1,n+2}^{1,n+2}} = x_{n+1}$). So, by Lemma 2.1.11,

$$f(\{1,n+1\})_{E_{1,n+1}^{1,n+1}} = -x_1.$$ Moreover, the connection $(\nabla_n)_{n+1,n+2}^{1,n+1}(E_{n+1,n+2}^{1,n+2}) = E_{1,n+1}^{1,n+2}$ and the congruence coefficient $c_{n+1,n+2}^{1,n+1}(E_{1,n+1}^{1,n+2}) = -1$. Therefore, we have the following equation by the definition of $f \in A(\Gamma_n)$:

$$x_{n+1} - f(\{1,n+1\})_{E_{1,n+1}^{1,n+2}} = (-1) \times (-x_1).$$ Hence, we have $f(\{1,n+1\})_{E_{1,n+1}^{1,n+2}} = x_{n+1} - x_1$, i.e., the correspondence $E_{1,n+1}^{1,n+2} \mapsto x_{n+1} - x_1$. Together with Lemma 2.1.11 we also have the correspondence $E_{1,n+1}^{1,n+1} \mapsto x_1 - x_{n+1}$.

This establishes the variables in Figure 7.

We next consider the subgraph drawn in Figure 8 and compute the corresponding variables on edges in this subgraph as in Figure 8.
We assumed $f(\{n+1, n+2\})_{E_{n-1,n+1}^{n-1,n+2}} = x_{n-1,n+2}$ (and $f(\{n+1, n+2\})_{E_{n-1,n+1}^{n-1,n+2}} = x_{n-1,n+2}$) for $0 \leq j \leq n-2$. Because $(\nabla_n)^{1,n+1}_{n-1,n+1}(E_{n-1,n+1}^{1,n+1}) = E_{n-1,n+1}^{1,n+1}$ and $c_{n+1,n+2}(E_{n-1,n+1}^{n-1,n+2}) = 0$, we have the correspondence

$E_{1,n+1}^{1,n+1} \mapsto x_{2n-j}$.

Similarly, because $(\nabla_n)^{1,n+2}_{n+1,n+2}(E_{n-1,n+1}^{n-1,n+2}) = E_{n-1,n+1}^{1,n+1}$ and $c_{n+1,n+2}(E_{n-1,n+1}^{n-1,n+2}) = 0$, we have the correspondence

$E_{1,n+2}^{1,n+2} \mapsto x_{n-j}$.

This establishes the variables in Figure 8.

By Figure 8 and Figure 9, we have the triangle GKM subgraph with variables as in Figure 9.

In Figure 9 $(\nabla_n)^{1,n+2}_{n+1,n+1}(E_{1,n+2}^{1,n+1}) = E_{1,n+2}^{1,n+1}$ and $c_{n+1,n+2}(E_{1,n+1}^{1,n+1}) = -1$. Therefore, by definition of $f \in A(\Gamma_n)$, we have the equation

$x_{2n-j} - x_{n-j} = -1(x_1 - x_{n+1})$.

This establishes that $x_{2n-j} = -x_1 + x_{n-j} + x_{n+1}$. We obtain the statement.

Consequently, this establishes Theorem 5.4. Therefore, by Corollary 1.3, we have Proposition 1.4.
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