COHEN-LENSTRA-GERTH HEURISTICS VIA AUTOMORPHISM COUNTS

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Abstract. For a finite abelian 2-group $G$, we study the frequency with which quadratic imaginary number fields $K$ have 2-part of their class group $K$ isomorphic to $G$. A philosophy enunciated by Gerth extends the Cohen-Lenstra heuristics for imaginary quadratic number fields to the case $p = 2$, by referencing both the 2-rank and the 4-rank of the group in question. A recent paper by Smith provides relative density statements about the $2^{k+1}$-rank of such a class group given its $2^1$- through $2^k$-ranks, for $k \geq 2$. We deduce from Smith’s results an explicit automorphism-count-theoretic statement of the Cohen-Lenstra-Gerth heuristics, also describing connections to “higher Rédei matrices” introduced by Kolster to study the $2^k$-ranks of the class group of $K$.

1. Introduction

The Cohen-Lenstra heuristics describe, among other things, the distribution of the $p$-class groups (i.e. the $p$-part of the class groups) of imaginary quadratic number fields for $p$ odd. One direct prediction made by these heuristics implies that among such fields, the density of those whose $p$-class groups are isomorphic to a given finite abelian $p$-group $G$ varies inversely with the number of automorphisms of $G$. This statement is more formally encoded in terms of the Cohen-Lenstra measure on the set $\mathcal{G}$ of (isomorphism classes of) finite abelian $p$-groups, obtained by assigning a preliminary weight of $\frac{1}{|\text{Aut}(G)|}$ to each $G \in \mathcal{G}$, and then uniformly normalizing so that $\mu(\mathcal{G}) = 1$. Now, consider the set $\mathcal{K}(x)$ of imaginary quadratic number fields $K$ (with discriminant $\Delta_K$ and $p$-class group $\text{Cl}_p(K)$) with $|\Delta_K| \leq x$. Then the heuristics make the following prediction:

**Conjecture 1** (Cohen-Lenstra). Let $G$ be a finite abelian $p$-group with $p$ odd. Then

\[
\lim_{x \to \infty} \frac{\# \{ K \in \mathcal{K}(x) : \text{Cl}_p(K) \cong G \}}{\# \{ K \in \mathcal{K}(x) \}} = \mu(G) = \frac{c_p}{|\text{Aut}(G)|},
\]

where

\[
c_p := \prod_{k=1}^{\infty} (1 - p^{-k}).
\]

When $p = 2$, the construction of the measure $\mu$ proceeds identically, but the direct analog of (1) fails by genus theory. Namely, since the 2-rank $\text{rk}_2(G) := \dim_{\mathbb{F}_2}(G/2G)$ of such a class group is one less than the number of distinct prime factors of the discriminant, a 2-group $G$ can only be obtained as a 2-class-group via discriminants with precisely $\text{rk}_2(G) + 1$ distinct prime factors. Thus for the analogous left-hand side of (1) with $p = 2$ and any finite abelian 2-group $G$, the
denominator grows like a constant times $x$ and the numerator has order of magnitude \((\log x)^{\frac{r_k}{2}}\) \((\text{[3]}, \text{Theorem 437})\), so we are forced to conclude that all 2-groups occur with density zero. In particular, \((\text{1})\) does not hold.

It was not until Gerth \([3]\) that a version of the Cohen-Lenstra heuristic was put forward for \(p = 2\), postulating that the Cohen-Lenstra heuristics should hold not for the group \(\text{Cl}_2(K)\) itself, whose data is tainted by genus theory, but for its group \(2 \text{Cl}_2(K)\) of squares. So successful are these extensions of the Cohen-Lenstra heuristics that one frequently encounters comments of the form “...and by Gerth, the analogous statement should hold for \(p = 2\) by replacing \(G\) with \(2G\)” And yet, for those who view the Cohen-Lenstra heuristic as a statement about densities of class groups in terms of automorphism counts, no replacement for \((\text{1})\) seems to have been clearly stated in the literature. The natural attempt, replacing \(G\) with \(2G\) in equation \((\text{1})\), fails immediately – both \(G = (1)\) and \(G = \mathbb{Z}/2\mathbb{Z}\) have \(2G = (1)\), but it is far from being the case that these two groups arise as 2-class groups with the same frequency.

The aim of the paper is to articulate the direct analogue of \((\text{1})\) for \(p = 2\). In fact, we will use the remarkable new results of Smith \([10]\) together with previous work of Gerth to prove that this analogue holds. To this end, for each natural number \(r\), the Cohen-Lenstra measure \(\mu\) on the set of finite abelian 2-groups induces by restriction and renormalization a probability measure \(\mu_r\) on the set of such groups with 2-rank equal to \(r\). The key correction to the naive analog of \((\text{1})\) discussed above is to condition the probability of \(G\) occurring against the 2- and 4-ranks of the group in question (the 4-rank being defined by \(\text{rk}_4(G) := \dim_{\mathbb{F}_2}(2G/4G)\)), leaving us with a non-zero limit on the left-hand side. Finally, for simplicity of notation, let us abbreviate \(\text{rk}_2(K) := \text{rk}_2(\text{Cl}_2(K))\) and similarly for \(\text{rk}_4(K)\).

**Theorem 2.** Let \(G\) be a finite abelian 2-group of 2-rank \(r_2\) and 4-rank \(r_4\). Then

\[
\lim_{x \to \infty} \frac{\# \{ K \in \mathcal{K}(x) : \text{rk}_2(K) = r_2, \text{rk}_4(K) = r_4 \}}{\# \{ K \in \mathcal{K}(x) : \text{rk}_2(K) = r_2, \text{rk}_4(K) = r_4, \text{Cl}_2(K) \cong G \}} = \frac{\tilde{c}_{r_4}}{|\text{Aut}(2G)|},
\]

where

\[
\tilde{c}_r := 2^{-r^2} \prod_{k=1}^{r} (1 - 2^{-k})^{-2}.
\]

It is worth noting that the limiting density in \((\text{2})\) does not depend on \(r_2\). Previous work of Gerth \([3]\) computes the limiting proportion

\[
(3) \quad d_{r_2, r_4} := \lim_{x \to \infty} \frac{\# \{ K \in \mathcal{K}(x) : \text{rk}_2(K) = r_2, \text{rk}_4(K) = r_4 \}}{\# \{ K \in \mathcal{K}(x) : \text{rk}_2(K) = r_2 \}}.
\]

Combining this with Theorem \(2\) we conclude the following corollary.

**Corollary 3.** Let \(G\) be a finite abelian 2-group of 2-rank \(r_2\) and 4-rank \(r_4\). Then

\[
\lim_{x \to \infty} \frac{\# \{ K \in \mathcal{K}(x) : \text{Cl}_2(K) \cong G \}}{\# \{ K \in \mathcal{K}(x) : \text{rk}_2(K) = r_2 \}} = d_{r_2, r_4} \cdot \mu_{r_4}(2G) = \frac{d_{r_2, r_4} \tilde{c}_{r_4}}{|\text{Aut}(2G)|},
\]

where \(d_{r_2, r_4}\) is as in \((\text{3})\).

It is worth remarking briefly on the philosophy of 2-class groups implied by the result, and it is here convenient to abuse the language of probability, considering class groups of “random” imaginary quadratic number fields. For \(p\) odd, the situation is reasonably straightforward: the probability that \(\text{Cl}_p(K)\) is isomorphic to a
given (finite abelian \( p \)-) group \( G \) is only a normalization (the \( c_p \) factor) away from being literally the reciprocal of its automorphism count. By contrast, for \( p = 2 \) Theorem 2 suggests that this is only true after taking into account the 2-rank and 4-rank of the group. For a finite abelian 2-group \( G \) with 2-rank \( r_2 \) and 4-rank \( r_4 \), we think of the probabilistic selection of a 2-class group as occurring in three independent phases.

- Genus theory dictates the 2-rank of the class groups, one might begin by choosing a random negative fundamental discriminant with \( r_2 + 1 \) distinct prime factors.
- The work of Gerth [3] computes the probability that the discriminant chosen in the previous step will correspond to a 4-rank of \( r_4 \).
- Finally, Theorem 2 computes the probability that given a discriminant giving the correct 2-rank and 4-rank will have 2-class group precisely \( G \).

Interestingly, there is an extent to which all three of these bullets can be thought of as special cases of a broader notion of genus theory. Namely, Gerth’s results follow from the classical study of the ranks of Rédei matrices, certain matrices over \( \mathbb{F}_2 \) formed by Hilbert symbols amongst primes dividing the discriminant. More generally, in Section 4 we highlight the work of Kolster [6], who described an algorithm for computing the full 2-part of the class group in terms of the ranks of so-called “higher Rédei matrices.” The reason for the discrepancy between the second and the third bullets in the list is that while the first Rédei matrices have some symmetry forced into them by quadratic reciprocity, the higher Rédei matrices appear to behave like random binary matrices of the appropriate size, as far as their ranks are concerned. This last claim has recently been proved by Smith (see Theorem 5 below), and forms a principal ingredient to the proof of Theorem 2. For related work involving Cohen-Lenstra heuristics that also includes experimental data, see [5] and [2].

**Notation.** Let \( \mathfrak{G} \) denote the set of isomorphism classes of finite abelian 2-groups, and, as mentioned earlier, let \( \mu : \mathfrak{G} \to [0, 1] \) denote the Cohen-Lenstra probability measure on \( \mathfrak{G} \), given explicitly as

\[
\mu(G) = \frac{c_2}{\# \text{Aut}(G)}, \quad c_2 := \prod_{k=1}^{\infty} (1 - 2^{-k}).
\]

The symbol \( G \) will always denote a group in \( \mathfrak{G} \), written additively. For \( G \in \mathfrak{G} \) and any \( k \in \mathbb{N} \), the 2\( k \)-ranks of \( G \) are defined by

\[
\text{rk}_{2^k}(G) := \dim_{\mathbb{F}_2}(2^{k-1}G/2^kG).
\]

We note that

\[
\text{rk}_{2^k}(2G) = \text{rk}_{2^{k+1}}(G).
\]

For any non-negative integer \( r \), we define

\[
\mathfrak{G}_r := \{ G \in \mathfrak{G} : \text{rk}_2(G) = r \}
\]

and we denote by \( \mu_r : \mathfrak{G}_r \to [0, 1] \) the probability measure induced by \( \mu \), given explicitly as

\[
\mu_r(G) = \frac{\mu(G)}{\mu(\mathfrak{G}_r)} \quad (G \in \mathfrak{G}_r).
\]

We will occasionally use “type notation” to reference 2-groups, e.g., \( (8, 8, 4, 2) \) for the group \( G = \mathbb{Z}/8 \times \mathbb{Z}/8 \times \mathbb{Z}/4 \times \mathbb{Z}/2 \). Note that the ranks are easily read off...
Proposition 4. agree, as encoded in the following proposition.

For any group $G ∈ \mathfrak{G}$ is uniquely determined by its series of ranks. We continue to let $\mathcal{K}(x)$ denote the set of imaginary quadratic number fields with discriminant satisfying $|\Delta_K| ≤ x$. For $K ∈ \mathcal{K}(x)$, we adopt the simplifying notation $rk_{2k}(K) = rk_{2k} Cl_2(K)$.

2. Reduction via Telescoping

The overarching idea is to re-write each side of Theorem 2 as a telescoping infinite product, and then check that the constituent factors of these products are pairwise equal.

To this end, for each $n$ we let $\mathfrak{G}(G; n)$ denote the subset of $\mathfrak{G}$ consisting of groups $H$ whose $2^1$ through $2^{n−1}$-ranks agree with those of $2G$, i.e.

$$\mathfrak{G}(G; n) = \{H ∈ \mathfrak{G} : rk_{2k}(H) = rk_{2k}(2G) \text{ for all } 1 ≤ k ≤ n − 1\}.$$

Then for fixed $G$ and all sufficiently large $n$ we have $\mathfrak{G}(G; n + 1) = \mathfrak{G}(G; n) = \{2G\}$ and so all but finitely many terms of the following telescoping product are 1:

$$(2R) \quad \mu_{rk_4(2G)}(2G) = \mu_{rk_4(2G)}(2G) = \frac{\mu(2G)}{\mu(\mathfrak{G}(G; 2))} = \prod_{n=2}^{\infty} \frac{\mu(\mathfrak{G}(G; n + 1))}{\mu(\mathfrak{G}(G; n))}.$$

We decompose the left-hand side of (2) similarly, picking out number fields whose class groups have the same first few ranks as a given $G$:

$$D_{G,n}(x) = \#\{K ∈ \mathcal{K}(x) : rk_{2k}(K) = rk_{2k}(G) \text{ for all } 1 ≤ k ≤ n\}.$$

Note that $D_{G,n}(x) = D_{G,n+1}(x) = \#\{K ∈ \mathcal{K}(x) : Cl_2(K) ≅ G\}$ for all sufficiently large $n$, so the telescoping product

$$(2L) \quad \lim_{x \to \infty} \frac{\#\{K ∈ \mathcal{K}(x) : Cl_2(K) ≅ G\}}{\#\{K ∈ \mathcal{K}(x) : rk_{2k}(K) = r_2, rk_{4k}(K) = r_4\}} = \prod_{n=2}^{\infty} \lim_{x \to \infty} \frac{D_{G,n+1}(x)}{D_{G,n}(x)}$$

gain contains only finitely many non-trivial factors. Comparing the product expansions for (2L) and (2R), it clearly suffices to verify that the corresponding factors agree, as encoded in the following proposition.

Proposition 4. For any group $G$ and integer $n ≥ 2$, we have

$$(4) \quad \lim_{x \to \infty} \frac{D_{G,n+1}(x)}{D_{G,n}(x)} = \frac{\mu(\mathfrak{G}(G; n + 1))}{\mu(\mathfrak{G}(G; n))}$$

The next section will detail the proof of this claim, though we will first describe how the terms on the left of (4) were recently addressed by Smith, making precise the perspective from the introduction that the ranks of higher Redei matrices behave like ranks of random matrices over $\mathbb{F}_2$. Specifically, let $M(s, t)$ denote the proportion of the binary $s \times s$ matrices in $M_{s \times s}(\mathbb{F}_2)$ whose rank is precisely $s − t$. We have the following Theorem.

Theorem 5 (Smith, [10]). For any $n ≥ 2$, we have

$$\lim_{x \to \infty} \frac{D_{G,n+1}(x)}{D_{G,n}(x)} = M(rk_{2n}(G), rk_{2n+1}(G))$$
Formulas for the values of these proportions are used frequently in enumerative combinatorics (e.g., in the study of determinantal varieties). We will make use of the formula

\[ M(s, t) = 2^{-t^2} \prod_{i=1}^{s-t} \frac{(1 - 2^{-i-t})^2}{1 - 2^{-i}}, \]

(apparently first due to Landsberg \[7\] (see, e.g., \[1\]), to establish our final proposition.

**Proposition 6.** For any \( G \in \mathcal{G} \) and any \( n \geq 2 \) we have

\[ M(\text{rk}_2^n(\mathcal{G}), \text{rk}_2^{n+1}(\mathcal{G})) = \frac{\mu(\mathcal{G}; n+1)}{\mu(\mathcal{G}; n)}. \]

It is clear that Proposition 6 and Theorem 5 prove Proposition 4; the next section of the paper will be devoted to proving Proposition 6.

From a computational perspective, combining Smith’s result with (2L), (5), and the to-be-proven Theorem 2, provides a reasonably simple combinatorial calculation of the desired Cohen-Lenstra densities, as we now illustrate.

**Example.** Let \( G = (8, 8, 4, 2) \), so \( \text{rk}_2 = 4 \), \( \text{rk}_4 = 3 \), \( \text{rk}_8 = 2 \), and \( \text{rk}_{2^k} = 0 \) for \( k \geq 4 \).

\[ M(3, 2)M(2, 0) = \frac{49}{512} \cdot \frac{3}{8} = \frac{147}{4096} \approx 0.35888. \]

On the other hand, we have have \( 2G = (4, 4, 2) \), which has \( \# \text{Aut}((4, 4, 2)) = 1536 \) automorphisms. Further, the family \( \mathcal{G}(G; 2) \) of 2-groups of 2-rank 3 can be calculated to have a total Cohen-Lenstra weight of \( \frac{8}{441} \).

\[ \mu_3(2G) = \frac{1/1536}{8/441} = \frac{147}{4096} \approx 0.35888. \]

This common value reflects the density of imaginary quadratic number fields with class group \( G \) among those with \( \text{rk}_2(K) = 4 \) and \( \text{rk}_4(K) = 3 \).

### 3. Proof of Proposition 6

Summarizing the results of the previous section, what remains is to show that, for a finite abelian 2-group \( G \) and integer \( n \geq 1 \), we have

\[ \frac{\mu(\mathcal{G}; n+1)}{\mu(\mathcal{G}; n)} = 2^{-t^2} \prod_{i=1}^{s-t} \frac{(1 - 2^{-i-t})^2}{1 - 2^{-i}}, \]

for \( s = \text{rk}_2^n(G) \) and \( t = \text{rk}_2^{n+1}(G) \). In terms of automorphism counts, the left-hand side of equation (6) is given by

\[ \frac{\mu(\mathcal{G}; n+1)}{\mu(\mathcal{G}; n)} = \frac{\sum_{G \in \mathcal{G}(G; n+1)} \# \text{Aut}(G)}{\sum_{G \in \mathcal{G}(G; n)} \# \text{Aut}(G)}. \]

The principal observation is that the groups indexing the numerator and denominator of this expression are sufficiently close to expect significant cancellations to occur in this fraction. The main tool used here is the classical formula for the automorphism count of a finite abelian group due to Ranum \[9\], and in particular we recall the version of these formulas found in \[8\].
Theorem 7 (Ranum, [8]). Let \( p \) be any prime number. If \( G \cong \prod_{i=1}^{k} (\mathbb{Z}/p^{e_i})^{a_i} \) with \( e_1 > e_2 > \cdots > 0 \) and each \( a_i > 0 \), then

\[
\# \text{Aut}(G) = \prod_{i=1}^{k} \prod_{s=1}^{a_i} (1 - p^{-s}) \prod_{1 \leq i,j \leq k} p^{\min(e_i, e_j)a_i a_j}
\]

(7)

The upcoming two corollaries connect these counts to the type of cancellation mentioned above.

Lemma 8. Suppose \( p \) is any prime and let \( G = H \oplus I \), with invariants indexed as follows:

\[
G = \prod_{i=1}^{k} (\mathbb{Z}/p^{e_i})^{a_i}, \quad H = \prod_{i=1}^{k_1} (\mathbb{Z}/p^{e_i})^{a_i}, \quad I = \prod_{i=k_1+1}^{k} (\mathbb{Z}/p^{e_i})^{a_i}
\]

each with notation as in Theorem 7, including that \( e_{k_i} > e_{k_{i+1}} \). Then

\[
\# \text{Aut}(H \oplus I) = \# \text{Aut}(H) \cdot \# \text{Aut}(I) \cdot p^{\rk_2(H) \sum_{k_1 < j \leq k} 2e_j a_j}.
\]

Proof. We view the right-hand side of (7) as consisting of the two obvious factors, and see how each behaves under direct sum. The first of the two factors is multiplicative:

\[
\prod_{i=1}^{k} \prod_{s=1}^{a_i} (1 - p^{-s}) = \prod_{i=1}^{k_1} \prod_{s=1}^{a_i} (1 - p^{-s}) \cdot \prod_{i=k_1+1}^{k} \prod_{s=1}^{a_i} (1 - p^{-s}),
\]

whereas the second factor collects an additional multiplier:

\[
\prod_{1 \leq i,j \leq k} p^{\min(e_i, e_j)a_i a_j} = \prod_{1 \leq i,j \leq k_1} p^{\min(e_i, e_j)a_i a_j} \prod_{k_1 < i,j \leq k} p^{\min(e_i, e_j)a_i a_j} \prod_{1 \leq i \leq k_1} \prod_{k_1 < j \leq k} p^{2 \min(e_i, e_j)a_i a_j}
\]

Putting this together, we conclude

\[
\# \text{Aut}(H \oplus I) = \# \text{Aut}(H) \cdot \# \text{Aut}(I) \cdot \prod_{1 \leq i \leq k_1} \prod_{k_1 < j \leq k} p^{2e_j a_j},
\]

and note that

\[
\prod_{1 \leq i \leq k_1} \prod_{k_1 < j \leq k} p^{2e_j a_j} = p^{(\sum_{k_1 < j \leq k} 2e_j a_j) - (\sum_{1 \leq i \leq k_1} a_i)} = p^{\sum_{k_1 < j \leq k} 2e_j a_j - \rk_2(H)}.
\]

Corollary 9. Suppose \( G_1 = H_1 \oplus I \) and \( G_2 = H_2 \oplus I \) are finite abelian 2-groups of the same 2-rank, with \( I \) of exponent less than the order of every generator of \( H_1 \) and \( H_2 \). Then

\[
\frac{\# \text{Aut}(G_1)}{\# \text{Aut}(G_2)} = \frac{\# \text{Aut}(H_1)}{\# \text{Aut}(H_2)}, \quad \text{and so} \quad \frac{\mu(G_1)}{\mu(G_2)} = \frac{\mu(H_1)}{\mu(H_2)}.
\]

Proof. Note that \( \rk_2(H_1) = \rk_2(H_2) \), so the extra factor arising from applying Lemma 8 is the same in each and so cancel.

Lemma 10. Let \( c \in \mathbb{N} \) and suppose \( G \) and \( H \) are 2-groups of the same 2-rank, all of whose generators have order greater than \( 2^c \). Then

\[
\frac{\# \text{Aut}(G)}{\# \text{Aut}(2^c G)} = \frac{\# \text{Aut}(H)}{\# \text{Aut}(2^c H)}
\]
Proof. Writing $G$ as in Theorem 7 and substituting into Equation 7 gives

$$\frac{\# \text{Aut}(G)}{\# \text{Aut}(2^c G)} = \prod_{1 \leq i, j \leq k} 2^{a_i a_j} = 2^{c(rk_2(G))^2},$$

showing that the ratio depends only on $c$ and the 2-rank of the group. \hfill \qed

Corollary 11. Suppose $G$ and $H$ are finite abelian 2-groups, that $rk_2(G) = rk_2(H)$, and that all generators of $G$ and $H$ have order greater than $2^c$. Then

$$\frac{\# \text{Aut}(G)}{\# \text{Aut}(2^c G)} = \frac{\# \text{Aut}(2^c G)}{\# \text{Aut}(2^c H)},$$

and so

$$\frac{\mu(G)}{\mu(H)} = \frac{\mu(2^c G)}{\mu(2^c H)}.$$

Corollaries 9 and 11 both demonstrate factors that appear in common in automorphism group counts of 2-groups with similar structure. The benefit of this is that such factors can be simultaneously dismissed from expressions of the form $\sum_{H} \frac{\mu(G)}{\mu(H)}$, as long as the summation in the denominator runs over groups all of which satisfy the hypotheses of the corresponding corollary. Before diving into the final calculation, let us return to the earlier example as an illustration.

Example. Consider equation (4) with $G = (8, 8, 4, 2)$ and $n = 3$. Then $s = 2$ and $t = 0$, and so the right-hand side gives $\frac{3}{4}$. (Or more easily, is $M(2, 0) = \frac{1}{4}$, as 6 of the 16 binary $2 \times 2$ matrices have rank 2.) On the left-hand side, $\mathcal{S}(G; 4)$ consists of finite abelian 2-groups whose 2-, 4-, 8-, and 16- ranks are respectively 4, 3, 2, and 0, so $\mathcal{S}(G; 4) = \{G\}$. Similarly, $\mathcal{S}(G; 3)$ consists of finite abelian 2-groups whose 2-, 4-, and 8-ranks are 4, 3, and 2, and so consists of all groups of the form $(2^a, 2^b, 4, 2)$ with $a \geq b \geq 3$. With the aid of Corollaries 9 and 11 (with $c = 2$), as well as Theorem 7 below, we compute that we have agreement between the two sides:

$$\frac{\mu(\mathcal{S}(G; 4))}{\mu(\mathcal{S}(G; 3))} = \frac{\# \mathcal{S}(G; 4)}{\# \mathcal{S}(G; 3)} = \frac{\# \mathcal{S}(G; 4)}{\# \mathcal{S}(G; 3)} = \frac{\mu(2^2)}{\mu(2^2)} = \frac{3}{8}.$$

To summarize, the remaining claim necessary to establish Theorem 4 is the generalization of this calculation, that two different ways of computing the proportion of groups $G$ with $2^{k+1}$-rank equal to $t$ out of those with $2^k$-rank is equal to $s'$ are indeed equal. The first is Smith's established result that the actual distribution of this statistic satisfied by class groups of imaginary quadratic number fields is given by $M(s, t)$, which we evaluate using (4). The second is the quotient of inverse automorphism count sums which identify these two families of groups. That is, it remains to show that, for any $0 \leq t \leq s$, we have

$$M(s, t) = \frac{\sum_{a_1 \geq a_2 \geq \ldots \geq a_t \geq 2^{s-t}} \mu(2^{a_1}, 2^{a_2}, \ldots, 2^{a_t}, 2^{s-t}, 2, \ldots, 2)}{\sum_{a_1 \geq a_2 \geq \ldots \geq a_t \geq 1} \mu(2^{a_1}, 2^{a_2}, \ldots, 2^{a_t}, 2^{a_{t+1}}, \ldots, 2^{a_s})}$$

Note that we can simplify the numerator of the right-hand side by taking $I = (\mathbb{Z}/2\mathbb{Z})^{s-t}$ in Lemma 8. Since $a_t \geq 2$, the lemma applies and gives

$$\mu(2^{a_1}, \ldots, 2^{a_t}, 2, \ldots, 2) = \frac{1}{2} \mu(2^{a_1}, \ldots, 2^{a_t}) \mu(2, 2, \ldots, 2) \cdot 2^{-2(t+s-t)}$$
(recall that \( c_2 := \prod_{i=1}^{\infty} (1 - 2^{-i}) \)). Similarly, applying Lemma \([10]\) with \( c = 1 \) we obtain
\[
\sum_{a_1 \geq a_2 \geq \ldots \geq a_t \geq 2} \mu(2^{a_1}, 2^{a_2}, \ldots, 2^{a_t}) = 2^{-t^2} \sum_{a_1 \geq a_2 \geq \ldots \geq a_t \geq 1} \mu(2^{a_1}, 2^{a_2}, \ldots, 2^{a_t}).
\]
This reduces the right-hand side to
\[
2^{-2t(s-t) - t^2} \sum_{a_1 \geq a_2 \geq \ldots \geq a_t \geq 2} \mu(2^{a_1}, 2^{a_2}, \ldots, 2^{a_t})
\]
where recall that \( \mathcal{G}_r \) denotes the set of finite abelian 2-group of 2-rank equal to \( r \). Finally, we invoke the \( r = 2 \)-case for these two measures:

**Theorem 12.** We have
\[
\mu(\mathcal{G}_r) = \frac{2^{-r^2} c_2}{\prod_{i=1}^{\infty} (1 - 2^{-i})^2}.
\]

In particular, we find
\[
\frac{\mu(\mathcal{G}_1)}{\mu(\mathcal{G}_s)} = 2^{s^2 - t^2} \prod_{i=t+1}^{s} (1 - 2^{-i})^2.
\]

Plugging in this and using that \( \#\text{Aut}((\mathbb{Z}/2)^{s-t}) = \prod_{i=1}^{s-t} (2^{s-t} - 2^{i-1}) \) leaves us with
\[
\sum_{a_1 \geq a_2 \geq \ldots \geq a_t \geq 2} \mu(2^{a_1}, 2^{a_2}, \ldots, 2^{a_t}) = 2^{-t^2} \prod_{i=1}^{s-t} (1 - 2^{-i})^2.
\]

In light of \([8]\), this proves \([8]\), thus concluding the proof of Proposition \([8]\).

4. Rédei matrices and \( 2^k \)-ranks of the class group

In this section, we will discuss Rédei matrices associated to an imaginary quadratic number field \( K \) and their connections to the 2-part of the class group of \( K \), expanding on our brief remarks from Section \([1]\) and following the construction in \([3]\). This connection will provide an explanation for the random matrix counts occurring in Theorem \([3]\). Furthermore, it will highlight the difference between the 4-rank case (as studied by Gerth \([3]\)) and the \( 2^k \)-rank case for \( k \geq 3 \) (as studied by Smith \([10]\)).

Fix an imaginary quadratic field \( K \), and denote by \( r \) the number of prime numbers that ramify in \( K \) (i.e. \( r = \omega(\Delta_K) \)); by genus theory, we additionally have \( r = \text{rk}_2(K) + 1 \). We define the first Redei matrix \( R_1(K) \in M_{r \times r}(\mathbb{F}_2) \) by

\[
R_1(K) := \begin{pmatrix}
    a_{1,1} & a_{1,2} & \ldots & a_{1,r} \\
    a_{2,1} & a_{2,2} & \ldots & a_{2,r} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{r,1} & a_{r,2} & \ldots & a_{r,r}
\end{pmatrix}
\]

\((-1)^{a_{i,j}} := (p_i, \Delta_K)_{p_j} \),
where \( p_1, p_2, \ldots, p_r \) are the ramified primes of \( K \) and \((\cdot, \cdot)_p\) denotes the local Hilbert symbol at \( p_j \), defined for \( a, b \in \mathbb{Q}_p^\times \) by

\[(a, b)_p := \begin{cases} 1 & \text{if } \exists (x, y, z) \in \mathbb{Q}_p^3 - \{(0, 0, 0)\} \text{ with } ax^2 + by^2 = z^2, \\ -1 & \text{otherwise.} \end{cases}\]

We recall that \((a, b)_p = 1\) if and only if \( a \in N_{\mathbb{Q}_p(\sqrt{b})^\times/\mathbb{Q}_p^\times} (\mathbb{Q}_p(\sqrt{b})^\times) \). The Hilbert reciprocity law

\[
\prod_p (a, b)_p = 1
\]

imply that

\[(10) \quad \sum_{j=1}^r a_{i,j} = 0, \quad (\forall i \in \{1, 2, \ldots, r\}).\]

and so in particular we have \( \text{rk}(R_1(K)) \leq r - 1 \). It will be more natural for us to deal with matrices of full rank, and so we consider the sub-matrix \( R'_1(K) \) obtained by removing the first row and first column from \( R_1(K) \). (One could alternatively consider the sub-matrix obtained by removing the \( s \)-th row and column of \( R_1(K) \) for any fixed \( s \) with \( 1 \leq s \leq r \).

A theorem of Rédei asserts the following relation between \( \text{rk}(R_1(K)) \) and the 4-rank of \( K \):

\[
\text{rk}_4(K) = r - 1 - \text{rk}(R_1(K)) = r - 1 - \text{rk}(R'_1(K)).
\]

4.1. **Modeling the 4-rank.** With the thought in mind to model \( \text{rk}_4(K) \) as \( r - 1 - \text{rk}(R'_1(K)) \), we are led to the question: To what extent can \( R'_1(K) \) be viewed as a random matrix in \( M_{(r-1)\times(r-1)}(\mathbb{F}_2) \)? As we shall see presently, the matrix \( R'_1(K) \) (and also \( R_1(K) \)) possesses additional symmetry, coming from quadratic reciprocity. Indeed, assume for simplicity that \( \Delta_K \) is odd (as observed in [3], for the purpose of computing \( d_{r_2,r_4} \) this is the only case needed). If we order the primes \( p_1, p_2, \ldots, p_r \) as

\[(11) \quad p_1, p_2, \ldots, p_{s+1}, p_{s+2}, \ldots, p_r, \quad p_i \equiv 3 \mod 4, 1 \leq i \leq s, \quad p_j \equiv 1 \mod 4, s < j \leq r, \]

then, by the formula

\[
(u_1p^{m_1}, u_2p^{m_2})_p = \left( \frac{-1}{p} \right)^{m_1m_2} \left( \frac{u_1}{p} \right)^{m_2} \left( \frac{u_2}{p} \right)^{m_3}
\]

(for \( m_i \in \mathbb{Z} \) and odd \( p \) with \( v_p(u_i) = 0 \)) together with quadratic reciprocity, we see using [3] that

\[(12) \quad a_{i,j} \neq a_{j,i} \quad \text{if } 1 \leq i, j \leq s \text{ and } i \neq j \]

\[a_{i,j} = a_{j,i} \quad \text{otherwise.}\]

Viewing \( R'_1(K) \) as \((a_{i,j})_{2 \leq i,j \leq r} \), the same relations apply to the coefficients of \( R'_1(K) \). Gerth’s work shows that, as \( K \in \mathcal{K}(x) \) with \( \text{rk}_3(K) = r - 1 \), the matrix \( R'_1(K) \in M_{(r-1)\times(r-1)}(\mathbb{F}_2) \) distributes randomly, modulo the condition (12) (equivalently, the matrix \( R_1(K) \in M_{r \times r}(\mathbb{F}_2) \) distributes randomly modulo the conditions (11) and (12)). This is then used to compute the density \( d_{r_2,r_4} \) in Corollary [3].
4.2. Modeling the $2^k$ rank for $k \geq 2$. Having the first Rédei matrix $R'_1(K)$ in hand, the higher Rédei matrices $\{R'_k(K)\}_{k \geq 2}$ are then constructed inductively \cite{6}: Given $R'_k(K) \in M_{(r-1) \times (r-1)}(\mathbb{F}_2)$ of rank $r_k < r - 1$ and satisfying
\begin{equation}
\text{rk}_{2k+1}(K) = r - 1 - \text{rk}(R'_k(K)),
\end{equation}
the next matrix $R'_{k+1}(K)$ is obtained as follows. An appropriate permutation of the lowest $r - 1 - r_{k-1}$ rows transforms $R'_k(K)$ into a matrix $\tilde{R}'_{k+1}(K)$ which may be written block-wise as
\begin{equation}
\tilde{R}'_{k+1}(K) = \begin{pmatrix} P_k & Q_k \\ \tilde{U}_k & T_k \end{pmatrix},
\end{equation}
where $P_k \in M_{r_k \times r_k}(\mathbb{F}_2)$, the other matrices are of appropriate shapes to fill out an $(r-1) \times (r-1)$ matrix, and the top $r_k \times (r-1)$ rectangle $(P_k, Q_k)$ has rank $r_k$.

Next, the bottom blocks $\tilde{U}_k$ and $T_k$ are replaced by new blocks $U_k$ and $T_k$ whose coefficients in $\mathbb{F}_2$ are quadratic residue symbols of solutions to certain recursively-defined norm equations. To specify these norm equations, we recall that the rows of $R'_k(K)$ are indexed by the primes dividing $\Delta_K$ and so by keeping track of the row exchanges used to construct $R'_2(K)$, the rows of $(\tilde{U}_1, \tilde{T}_1)$ correspond to some subset of these primes. It follows from properties of Hilbert symbols that for any $p$ in this subset, there exists an integer $t$ and a primitive totally positive solution $z \in \mathcal{O}_K$ to the equation
\begin{equation}
N_{K/Q}(z) = t^2 p.
\end{equation}

The construction proceeds by replacing the row of $(\tilde{U}_1, \tilde{T}_1)$ corresponding to the prime $p$ with the row $[t_1, \ldots, t_{r-1}]$, where analogously to the construction of $R_1(K)$, we set $(-1)^{t_j} = (t, \Delta_K)_p$. The analogous norm equations for higher Rédei matrices are defined recursively, solving a variant of \eqref{15}, in which we replace $p$ with the product of the ramified primes which correspond to rows demonstrating a linear dependence in $(\tilde{U}_k, \tilde{T}_k)$. We refer the reader to \cite{6} for details.

After solving all of the norm equations as above, we are left with the new $(r - 1) \times (r - 1)$ matrix
\begin{equation}
R'_{k+1}(K) = \begin{pmatrix} P_k & Q_k \\ U_k & T_k \end{pmatrix},
\end{equation}
of rank $r_{k+1} \geq r_k$, and, as shown in \cite{6}, we have $\text{rk}_{2k+2}(K) = r - 1 - r_{k+1}$. The next matrix
\begin{equation}
R'_{k+2}(K) = \begin{pmatrix} P_{k+1} & Q_{k+1} \\ U_{k+1} & T_{k+1} \end{pmatrix},
\end{equation}
is obtained by rearranging the lowest $r - 1 - r_k$ rows of $R'_{k+1}(K)$ so that the topmost rows become linearly independent over $\mathbb{F}_2$, and then replacing the bottom $r - 1 - r_k$ rows via solving norm equations as before. This process terminates as soon as $r_k = r - 1$, and the $2^{k+1}$-ranks of the class group of $K$ are recovered from the ranks of the matrices $R'_k(K)$ via \eqref{13}.

It is through this iterative process that such a stark divide arises between the $k = 1$ and $k \geq 2$ cases: Whereas $R_1(K)$ has symmetry restrictions ultimately stemming from quadratic reciprocity, such symmetry is broken in the the higher Rédei matrices by the inherently asymmetric construction via the solutions to these norm equations. Thus, whereas the first Rédei matrix is modeled by a constrained random matrix as described above, it is reasonable to expect higher Rédei matrices to
behave as completely unconstrained random matrices (and in particular, their ranks should be governed by the appropriate proportions $M(s, t)$). More precisely, appropriate row and column operations simultaneously transform the matrices $\tilde{R}'_{k+1}(K)$ in (14) and $R'_{k+1}(K)$ in (19) into $\tilde{R}''_{k+1}(K)$ and $R''_{k+1}(K)$, where

$$\tilde{R}'_{k+1}(K) := \begin{pmatrix} I_r & 0 \\ 0 & T''_k \end{pmatrix}, \quad R''_{k+1}(K) := \begin{pmatrix} I_r & 0 \\ 0 & T''_k \end{pmatrix}.$$ 

Since $\text{rk}(R'_{k+1}(K)) = \text{rk}(R''_{k+1}(K)) = r_k + \text{rk}(T''_k)$, we find that the new rank $r_{k+1}$ is determined by the rank of $T''_k$. From this perspective, Theorem 5 may be interpreted as asserting that, for $k \geq 2$, the rank of the new block matrix $T''_k$ behaves like that of a random matrix in $M((r_1-r_k)(r_1-r_k), (F_2)$. 

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