A NOTE ON A THEOREM OF MUNKRES

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Abstract. We prove that given a $C^\infty$ Riemannian manifold with boundary, any fat triangulation of the boundary can be extended to the whole manifold. We also show that this result holds to $C^1$ manifolds, and that in dimensions 2, 3 and 4 it also holds for PL manifolds. We employ the main result to prove that given any orientable $C^\infty$ Riemannian manifold with boundary admits quasimeromorphic mappings onto $\mathbb{R}^n$. In addition some generalizations are given.

1. Introduction

The existence of triangulations for $C^1$ manifolds without boundary is known since the classical work of Whithead ([31], 1940). This result was extended in 1960 by Munkres ([17]) to include $C^r$, $1 \leq r \leq \infty$, manifolds with boundary. To be more precise, he proved that any $C^r$ triangulation of the boundary can be extended to a $C^r$ triangulation of the whole manifold.

Yet earlier, in 1934-1935 (even before Whithead’s work) Cairns\(^1\) ([3], [4]) proved the existence of fat triangulations for compact $C^1$ manifolds and for manifolds with boundary having a finite number of compact boundary components. Moreover, his triangulations were fat (see definition below), something which Munkres’ method achieved only away from the boundary (see Section 2.2). Unfortunately, it seems that little interest existed during the following decades, for studying generalizations of the results above ([8] representing a notable exception). The interest in the existence of fat triangulation was rekindled by the study of quasiregular and quasimeromorphic functions, since the existence of fat triangulations is crucial in the proof of existence of quasiregular (quasimeromorphic) mappings (see [15], [29]) and in 1992 Peltonen ([18]) proved the existence of fat triangulations for $C^\infty$ Riemannian manifolds, using methods partially based upon another technique of Cairns (originally developed for triangulating manifolds of class $\geq C^2$).

In this paper we extend Munkres’ Theorem to the case of fat triangulations of manifolds (and orbifolds – see Section 5.2.) with or without boundary and we show how to apply this main result in order to prove the existence of quasimeromorphic functions. Our main result is the following Theorem:

Theorem 1.1. Let $M^n$ be an $n$-dimensional $C^\infty$ manifold with boundary. Then any fat triangulation of $\partial M^n$ can be extended to a fat triangulation of $M^n$.

\(^1\) Although far better known and widely cited, Whitehead’s work is rooted in Cairns’ studies, to whom it gives due credit in the very opening phrase: "This paper is supplementary to S.S. Cairns’ work on the triangulation ... of manifolds of class $C^1$."
where a fat triangulations is defined as follows:

**Definition 1.2.** Let $\tau \subset \mathbb{R}^n$; $0 \leq k \leq n$ be a $k$-dimensional simplex. The fatness $\varphi$ of $\tau$ is defined as being:

\[
\varphi = \varphi(\tau) = \inf_{\sigma \subset \tau, \dim \sigma = l} \frac{Vol(\sigma)}{diam^l(\sigma)}
\]

The infimum is taken over all the faces of $\tau$, $\sigma < \tau$, and $Vol_{eul}(\sigma)$ and $diam(\sigma)$ stand for the Euclidian $l$-volume and the diameter of $\sigma$ respectively. (If $\dim \sigma = 0$, then $Vol_{eul}(\sigma) = 1$, by convention.)

A simplex $\tau$ is $\varphi_0$-fat, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$. A triangulation (of a submanifold of $\mathbb{R}^n$) $T = \{\sigma_i\}_{i \in I}$ is $\varphi_0$-fat if all its simplices are $\varphi_0$-fat. A triangulation $T = \{\sigma_i\}_{i \in I}$ is fat if there exists $\varphi_0 \geq 0$ s.t. all its simplices are $\varphi_0$-fat; $\forall i \in I$.

**Remark 1.3.** There exists a constant $c(k)$ that depends solely upon the dimension $k$ of $\tau$ s.t.

\[
1/c(k) \cdot \varphi(\tau) \leq \min_{\sigma \subset \tau, \dim \sigma = l} \angle(\tau, \sigma) \leq c(k) \cdot \varphi(\tau),
\]

and

\[
\varphi(\tau) \leq \frac{Vol(\sigma)}{diam^l(\sigma)} \leq c(k) \cdot \varphi(\tau);
\]

where $\angle(\tau, \sigma)$ denotes the (internal) dihedral angle of $\sigma < \tau$. (For a formal definition, see [6], pp. 411-412, [23].)

**Remark 1.4.** The definition above is the one introduced in [6]. For different, yet equivalent definitions of fatness, see [3, 4, 18, 29].

The idea of the proof of Theorem 1.1. is first to build two fat triangulations: $T_1$ of a product neighbourhood $N$ of $\partial M^n$ in $int M^n$ and $T_2$ of $int M^n$, and then to ”mash” the two triangulations into a new triangulation $T$, while retaining their fatness.

While the mashing procedure of the two triangulations is basically that developed in the original proof of Munkres’ theorem, the triangulation of $T_1$ was modified, in order to ensure the fatness of the simplices of $T_1$. The existence of the second triangulation is assured by Peltonen’s result. Thus our main efforts are dedicated to the task of fattening the newly obtained triangulation into a new fat triangulation. The technique we employ is essentially the one developed in [6].

Once a fat triangulation of an orientable manifold $M^n$ is provided, the construction of the required quasimeromorphic mapping is canonical (see [1, 15, 17, 29]) and is based upon the so called ”Alexander Trick”, which we present here succinctly (in a nutshell): one starts by constructing a suitable triangulation of $M^n$. Since $M^n$ is orientable, an orientation consistent with the given triangulation (i.e. such that two given $n$-simplices having a $(n-1)$-dimensional face in common will have opposite orientations) can be chosen. Then one quasiconformally maps the simplices of the triangulation into $\mathbb{R}^n$ in a chess-table manner: the positively oriented ones onto the interior of the standard simplex in $\mathbb{R}^n$ and the negatively oriented ones onto $\mathbb{R}^n$. For a more direct approach in dimensions 2 and 3 see [21]. Also, for the treatment of the same problem in the context of Computational Geometry, see [7].
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its exterior. If the dilatations of the quasiconformal maps constructed above are uniformly bounded – which condition is fulfilled if the simplices of the triangulation are of uniform fatness – then the resulting map will be quasimeromorphic.

This paper is organized as follows: in Section 2 we present the main techniques we employ: Peltonen’s method of triangulating \( \text{int} \ M^n \) and the Proof of Munkres’ Theorem on the extension of the triangulation of \( \partial M^n \) to \( \text{int} \ M^n \). Section 3 is dedicated to the main task of fattening the common triangulation. In Section 4 we show how to apply the main result in the construction of a quasimeromorphic mappings from \( M^n \) to \( \mathbb{R}^n \). Finally, in Section 5 we propose some generalizations.

2. Extending \( \mathcal{T}_1 \) to \( \text{int} \ M^n \)

2.1. Peltonen’s Technique. Peltonen’s method is an extension of one due to Cairns, developed in order to triangulate \( C^2 \)-compact manifolds ([5]). It is based on the subdivision of the given manifold into a closed cell complex generated by a Dirichlet (Voronoy) type partition whose vertices are the points of a maximal set that satisfy a certain density condition. We give below a sketch of the Peltonen’s method, refering the interested reader to [18] for the full details.

The construction devised by Peltonen consists of two parts:

Part 1 This part proceeds in two steps:

Step A Build an exhaustion \( \{ E_i \} \) of \( M^n \), generated by the pair \((U_i, \eta_i)\), where:

1. \( U_i \) is the relatively compact set \( E_i \setminus E_{i-1} \) and
2. \( \eta_i \) is a number that controls the fatness of the simplices of the triangulation of \( E_i \), that will be constructed in Part 2, such that it will not differ to much on adjacent simplices, i.e.:
   (i) The sequence \((\eta_i)\geq 1\) descends to 0;
   (ii) \( 2\eta_i \geq \eta_{i-1} \).

Step B

1. Produce a maximal set \( A_i \), \( |A| \leq \aleph_0 \), s.t. \( A \cap U_i \) satisfies:
   (i) a density condition, and
   (ii) a "gluing" condition (for \( U_i, U_{i+1} \)).
2. Prove that the Dirichlet complex \( \{ \bar{\gamma}_i \} \) defined by the sets \( A_i \) is a cell complex and every cell has a finite number of faces (so it can be triangulated in a standard manner).

Part 2 Consider first the dual complex \( \Gamma \) and prove that it is a Euclidian simplicial complex with a "good" density, then project \( \Gamma \) on \( M^n \) (using the normal map). Finally, prove that the resulting complex can be triangulated by fat simplices.

Remark 2.1. In the course of Peltonen’s construction \( M^n \) is presumed to be isometrically embedded in some \( \mathbb{R}^{N_1} \), where the existence of \( N_1 \) is guaranteed by Nash’s Theorem (see [18], [25]).

2.2. The Extension of \( \mathcal{T}_1 \) to \( \text{int} \ M^n \). We first establish some notations and definitions:

Let \( K \) denote a simplicial complex, and let \( K' < K \) denote a subcomplex of \( K \).

Definition 2.2. Let \( f_i : K_i \to \mathbb{R}^n \), \( i = 1, 2 \) be s.t. \( f_i(|K_i|) \) is closed. We say that \( (K_1, f_1), (K_2, f_2) \) intersect in a subcomplex iff:

(i) \( f_i^{-1}(f_1(|K_1|) \cap f_2(|K_2|)) = |L_i| \); where \( L_i < K_i \), \( i = 1, 2 \).
and

(ii) \( f_2^{-1} \circ f_1 : L_1 \to L_2 \) is a linear isomorphism.\(^3\)

**Definition 2.3.** Let \( L < K \). \( L \) is called **full** iff \( \sigma \cap L \) either is a face of \( \sigma \) or else it is empty; \( \forall \sigma \in K \).

**Remark 2.4.** \( L \) is full \( \iff \partial \sigma \cap L \neq \partial \sigma; \forall \sigma \in K \).

If \((K_1, f_1), (K_2, f_2)\) intersect in a full subcomplex, then there exist a complex \( K \) and a homeomorphism \( f : K \to \mathbb{R}^n \) s.t. the following diagram is commutative:

\[
\begin{array}{ccc}
K_1 & \xrightarrow{f_1} & K_2 \\
\downarrow i_1 & \downarrow f & \downarrow i_2 \\
K & \to & \mathbb{R}^n \\
\end{array}
\]

Here \( i_1, i_2 \) are linear isomorphisms. The pair \((K, f)\) is unique up to isomorphism.

**Definition 2.5.** Let \((K_1, f_1), (K_2, f_2)\) and \((K, f)\) be as above. Then \((K, f)\) is called the **union** of \((K_1, f_1)\) and \((K_2, f_2)\).

**Definition 2.6.** Let \( f : K \to \mathbb{R}^n \) be a \( C^r \) map, and let \( \delta : K \to \mathbb{R}^n_+ \) be a continuous function. Then \( g : |K| \to \mathbb{R}^n \) is called a **\( \delta \)-approximation** to \( f \) iff:

(i) There exists a subdivision \( K' < K \) s.t. \( g \in C^r(K', \mathbb{R}^n) \);

(ii) \( d_{\text{eucl}}(f(x), g(x)) < \delta(x), \forall x \in |K| \);

(iii) \( d_{\text{eucl}}(df_a(x), dg_a(x)) \leq \delta(a) \cdot d_{\text{eucl}}, \forall a \in |K|, \forall x \in \overline{\text{St}}(a, K') \)

**Definition 2.7.** Let \( K < K' \), \( U = \overset{\circ}{U} \), and let \( f \in C^r(K, \mathbb{R}^n), g \in C^r(K, \mathbb{R}^n) \). \( g \) is called a **\( \delta \)-approximation** of \( f \) on \( U \) iff conditions (i) and (ii) of Definition 2.1 hold for any \( a \in U \).

**Definition 2.8.** Let \( K < K' \) and let \( f \in C^r(K, \mathbb{R}^n), g \in C^r(K', \mathbb{R}^n) \) be non-degenerate\(^4\) mappings and let \( U = \overset{\circ}{U} \subset |K| \). \( g \) is called an **\( \alpha \)-approximation** (of \( f \) on \( U \)) iff:

\[
(2.1) \quad \angle(df_a(x), dg_a(x)) \leq \alpha; \forall a \in U, \forall x \in \overline{\text{St}}(a, K'), a \neq x.
\]

We now bring Munkres’ Theorem. While we will initially apply it for \( C^\infty \) manifolds, we give the proof for the general case of \( C^r \) manifolds, \( 1 \leq r \leq \infty \). We modify the original construction so the triangulation of a certain neighbourhood of \( \partial M^n \) will be fat.

**Theorem 2.9** ([17], 10.6). Let \( M^n \) be a \( C^r \)-manifold with boundary. Then any \( C^r \)-triangulation of \( \partial M^n \) can be extended to a \( C^r \)-triangulation of \( M^n \), \( 1 \leq r \leq \infty \).

\(^3\) i.e. \( (i) \ f : |L_1| \xrightarrow{\sim} |L_2| \) and \( (ii) \ f|_\sigma \) is linear, \( \forall \sigma \in L_1 \).

\(^4\) i.e. \( \text{rank}(f|_\sigma) = \dim \sigma, \forall \sigma \in K \).
Proof. Let $f : J \to \partial M^n$ be a $\varphi_{\partial M}$-fat $C^r$ triangulation, for some $\varphi_{\partial M}$. We construct a triangulation of $|J| \times [0, 1)$ in the following way:
If $J$ is isometrically embedded in $\mathbb{R}^{N_2}$, we consider (in $\mathbb{R}^{N_2}$) the cells of type:

\[
\sigma_{1,n} = \sigma \times \left[ \frac{k}{n_0}, \frac{k + 1}{n_0} \right]; \quad k = 1, \ldots, n_0 - 1.
\]

and

\[
\sigma_{2,n} = \sigma \times \left\{ \frac{k}{n_0} \right\}; \quad \forall \sigma \in J.
\]

Let $K$ denote the resulting simplicial complex: $|K| = |J| \times [0, 1)$. The cells of the complex above may be divided in simplices without subdividing the cells of type $\sigma_{2,n}$. (See Fig. 1 for the case $N_2 = 2$.)

![Figure 1.](image)

For reasons that will become clear in the course of the proof of Theorem 3.6, we choose $n_0$ such that the fatness of any simplex $\sigma \in K$ is $\geq \varphi_0$, for some $\varphi_0$ \footnote{depends upon $\varphi_{\partial M}$ and $\varphi_{\text{int.} M}$.} and such that $\text{diam} \sigma \leq \text{diam} \tau$, $\forall \sigma \in K_0, \tau \in L_0$ \footnote{To attain this inequalities, further subdivision may be necessary – their number depending upon the respective "$\eta_n"$-s given by Peltonen’s construction.}, where $K_0, L_0$ are defined as follows:
Let $K_0$ be the subcomplex of $K$ s.t. $|K_0| = |J| \times \left[ 0, \frac{k_1}{n_0} \right]$, $k_1 = \left\lceil \frac{\text{diam}}{\varphi_0} \right\rceil$; and let $\psi : \partial M^n \times [0, 1) \to M^n$ be a product neighbourhood of $\partial M^n$ (in $M^n$). (Here $M^n$ is supposed to be embedded in $N = \max\{N_1, N_2\}$.) Then, if $g$ makes the following diagram commutative:
Then \( g \) is a \( C^r \) embedding s.t.

(i) \( g(K_0) = \overline{g(K_0)} \) (in \( M^n \))

and

(ii) \( \psi(\partial M^n \times [0, \frac{k_2}{n_0}]) \subset \text{int} g(K_0) \), \( k_2 = \lfloor \frac{4n_1}{5} \rfloor \).

Now, if \( h : L \to M^n \) is a \( C^r \) triangulation of \( \text{int} M^n \), then, by further (eventual) subdivision, we may suppose that: \( \sigma' \cap \psi(\partial M^n \times [0, \frac{k_2}{n_0}]) = \emptyset \), \( k_3 = \lfloor \frac{3n_1}{4} \rfloor \); \( \forall \sigma' \in L; \sigma' \cap \psi(\partial M^n \times \{ \frac{k_2}{n_0} \}) \neq \emptyset \).

Let \( L_0 \) be the complex given by:

\[
\begin{align*}
L_0' &= \{ \sigma \in L \mid h(\sigma) \cap (M^n \setminus \psi(\partial M^n \times [0, \frac{k_2}{n_0}])) \neq \emptyset \}; \\
L_0 &= L_0' \cap L_0'.
\end{align*}
\]

Then, by \[7\], Theorem 10.4, (see also Fig. 28) \( \exists g' : K_0' \to M^n, h' : L_0' \to M^n \);

where \( g' \) is a \( \delta \)-approximation of \( g \) and \( h \) is a \( \delta \)-approximation of \( h \), s.t.

(i) \( g'(K_0') \cap h'(L_0') \) is full

and

(ii) The union of \( (K_0', g') \) and \( (L_0', h') \) is an embedding.

Also, by applying again \[7\], Theorem 10.4, we may suppose that

(a) \( K_0' \mid J \times [0, \frac{k_3}{n_0}] \equiv K_0 \mid J \times [0, \frac{k_2}{n_0}] \)

(b) \( g_0' \mid J \times [0, \frac{k_3}{n_0}] \equiv g_0 \mid J \times [0, \frac{k_2}{n_0}] \); \( k_4 = \lfloor \frac{2n_1}{3} \rfloor \)

Then \( (K_0', g') \cup (L_0', h') \) will be the sought for triangulation, but only if the following condition also holds:

\[
g'(|K_0|) \cup h'(|L_0|) = M^n.
\]

But this condition also takes hold in our case, by virtue of a more general result about topological manifolds (see \[7\], pp. 36-38, 105).

\[\square\]

3. Fattening Triangulations

First let us establish some definitions and notations:

**Definition 3.1.** Let \( \sigma_i \subset K \), \( \dim \sigma_i = k_i \), \( i = 1, 2 \); s.t. \( \text{diam} \sigma_1 \leq \text{diam} \sigma_2 \). We say that \( \sigma_1, \sigma_2 \) are \( \delta \)-transverse iff

(i) \( \dim (\sigma_1 \cap \sigma_2) = \max(0, k_1 + k_2 - n) \);

(ii) \( 0 < \delta < \Delta(\sigma_1, \sigma_2) \);

and if \( \sigma_3 \subset \sigma_1, \sigma_4 \subset \sigma_2 \), s.t. \( \dim \sigma_3 + \dim \sigma_4 < n = \dim K \), then

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7 see [17].

8 after [17].
(iii) \( \text{dist}(\sigma_3, \sigma_4) > \delta \cdot \eta_1 \).

In this case we write: \( \sigma_1 \cap \delta \sigma_2 \).

We begin by triangulating and fattening the intersection of two individual simplices belonging to the two given triangulations, respectively. Given two closed simplices \( \sigma_1, \sigma_2 \), their intersection (if not empty) is a closed, convex polyhedral cell: \( \bar{\gamma} = \bar{\sigma}_1 \cap \bar{\sigma}_2 \). One canonically triangulates \( \bar{\gamma} \) by using the \textit{barycentric subdivision} \( \bar{\gamma}^* \) of \( \bar{\gamma} \), defined inductively upon the dimension of the cells of \( \partial \bar{\gamma} \) in the following manner: for each cell \( \beta \subset \partial \bar{\gamma} \), choose an interior point \( p_\beta \in \text{int} \beta \) and construct the join \( J(p_\beta, \beta_i), \forall \beta_i \subset \partial \beta \).

We first show that if the simplices are fat and if they intersect \( \delta \)-transversally, then one can choose the points s.t. the barycentric subdivision \( \bar{\gamma}^* \) will be composed of fat simplices. More precisely, we prove the following Proposition:

**Proposition 3.2.** Let \( \sigma_1, \sigma_2 \subset \mathbb{R}^m \), where \( m = \max(\dim \sigma_1, \dim \sigma_2) \), s.t. \( d_1 = \text{diam} \sigma_1 \leq d_2 = \text{diam} \sigma_2 \), and s.t. \( \sigma_1, \sigma_2 \) have common fatness \( \varphi_0 \).

If \( \sigma_1 \cap \delta \sigma_2 \), then there exists \( c = c(m, \varphi_0, \delta) \)

\[9\] If \( \dim \beta = 0 \) or \( \dim \beta = 1 \), then \( \beta \) is already a simplex.
(1) If \( \sigma_3 \subset \sigma_1, \sigma_4 \subset \sigma_2 \) and if \( \sigma_1 \cap \sigma_2 \neq \emptyset \), then \( \sigma_1 \cap \sigma_2 = \gamma_0 \) is an \((k_3 + k_4 - m)\)-cell, where \( k_3 = \dim \sigma_3, k_4 = \dim \sigma_4 \) and:

\[
(3.1) \quad \text{Vol}_{euc} (\gamma_0) \geq c \cdot d_1^{k_3 + k_4 - m}.
\]

(2) \( \forall \gamma_0 \) as above, \( \exists p \in \gamma_0 \), s.t.

\[
(3.2) \quad \text{dist} (p, \partial \gamma_0) > c \cdot d_1.
\]

(3) If the points employed in the construction of \( \gamma^* \) satisfy the condition (3.2) above, then each \( l \)-dimensional simplex \( \tau \in \gamma^* \) satisfies the following inequalities:

\[
(3.3) \quad \varphi_l \geq \text{Vol}_{euc} (\tau) / d_1^l \geq c \cdot d_1.
\]

**Proof.** First, consider the following remarks:

**Remark 3.3.** The following sets are compact:
\[
S_1 = \{ \sigma_1 \mid \text{diam} \sigma_1 = 1, \varphi (\sigma_1) \geq \varphi_0 \}, \quad S_2 = \{ \sigma_2 \mid \text{diam} \sigma_2 = 2(1 + \delta), \varphi (\sigma_2) \geq \varphi_0 \},
\]

\[
S (\varphi_0, \delta) \subset S_1 \cap S_2, \quad S (\varphi_0, \delta) = \{(\sigma_1, \sigma_2) \mid \exists v_0, \text{s.t.} v_0 \in \sigma_1, \forall \sigma_1 \in S_1 \cap S_2 \}.
\]

**Remark 3.4.** There exists a constant \( c(\varphi) \) s.t. \( S = S' \), where \( S = \{ \sigma_1 \cap \sigma_2 \mid \text{diam} \sigma_2 \leq \delta_2 \}, \quad S' = \{ \sigma_1 \cap \sigma_2 \mid \text{diam} \varphi (1 + \delta) d_1 \} \), i.e. the sets of all possible intersections remains unchanged under controlled dilations of one of the families of simplices.

Now, from the fact that \( \sigma_1 \cap \sigma_2 \) it follows that \( \sigma_3 \cap \sigma_4 \neq \emptyset \Leftrightarrow \bar{\sigma}_3 \cap \bar{\sigma}_4 \neq \emptyset \) (see [2], p. 436). Therefore, the function \( \text{Vol}_{euc} (\gamma_0) \) attains a positive minimum, as a positive, continuous function defined on the compact set \( \bar{\sigma}_3 \cap \bar{\sigma}_4 \), thus proving the first assertion of the proposition.

Let be \( \gamma \) be a \( q \)-dimensional cell, and let \( \beta \) be a face of \( \partial \gamma \). Then:

\[
(3.4) \quad \text{Vol}_{euc} (\beta) \leq d_1^q.
\]

Choose \( p \in \gamma \), such that \( p = \text{dist} (p, \partial \gamma) = \max \{ \text{dist} (r, \partial \gamma) \mid r \in \gamma \} \). Then, if \( \beta = \beta^j \) denotes a \( j \)-dimensional face of \( \partial \gamma \), we have that:

\[
(3.5) \quad \gamma \subset \bigcup_{\beta^j \subset \partial \gamma} N_p (\beta^j);
\]

where: \( N_p (\beta^j) = \{ r \mid \text{dist} (r, \beta^j) \leq \rho \} \). But:

\[
(3.6) \quad \text{Vol}_{euc} (\beta^j \cap \gamma) \leq c \cdot \rho^{q-j} \cdot \text{Vol}_{euc} (\beta^j),
\]

for some \( c' = c'(q) \).

Moreover, the number of faces \( \sigma_3 \cap \sigma_4 \) of \( \gamma \) is at most \( 2^{\dim \sigma_1 + \dim \sigma_2 + 2} \), where \( \sigma_1, \sigma_2 \) are as in Remark 3.4. and \( \dim \sigma_3 \leq \dim \sigma_1, \quad \dim \sigma_4 \leq \dim \sigma_2 \).

Thus (3.4) in conjunction with (3.6) imply that there exists \( c_1 = c_1 (m, \varphi_0, \delta) \), such that:

\[
(3.7) \quad c_1 d_1^q \leq \sum_{j=0}^{q-1} \rho^{q-j} d_1^j.
\]
and (3.2) follows from this last inequality.

The last inequality follows from (3.2) and (3.3) by induction.

□

Next we show that given two fat Euclidean triangulations that intersect \( \delta \)-transversally, then one can infinitesimally move any given point of one of the triangulations s.t. the resulting intersection will be \( \delta' \)-transversal, where \( \delta' \) depends only on \( \delta \), the common fatness of the given triangulations, and on the displacement length. More precisely one can show that the following results holds:

**Proposition 3.5.** Let \( K_1, K_2 \subset \mathbb{R}^n \) be \( n \)-dimensional simplicial complexes, of common fatness \( \varphi_0 \) and \( d_1 = \text{diam} \sigma_1 \leq d_2 = \text{diam} \sigma_2 \). Let \( v_0 \in K_1 \) be a 0-dimensional simplex of \( K_1 \). Consider the complex \( K_1' \) obtained by replacing \( v_0 \) by \( v_0' \in \mathbb{R}^n \) and keeping fixed the rest of the 0-dimensional vertices of \( K_1 \) fixed. Consider also \( L_2 \subseteq K_2, L_2 = \{ \sigma \in K_2 | \sigma \cap B(v_0,2d_1) \neq \emptyset \} \).

Then, if there exists \( k \) s.t. \( \tau \subseteq \partial St(v_0) \) are \( \delta \)-transversal to \( L_2 \), there exist \( \varphi_0, \delta, \varepsilon > 0, \delta^* = \delta^*(\varphi_0, \delta, \varepsilon) \) and there exists \( v_0^* \) s.t. \( \text{dist}(v_0, v_0^*) < \varepsilon \cdot d_1 \) s.t.

\[
\tau^* \cap_{\delta^*} L_2 ; \forall \tau \subseteq St(v_0^*) \setminus \partial St(v_0^*) \text{, } \dim \tau^* = k + 1.
\]

**Proof.** Let \( N(r) = |\{ \sigma \in K_1 | \sigma \subseteq B_r(v_0) \}| \). Then there exists a constant \( c_n \) s.t. \( N(r) \leq \frac{c_n}{r^n} \). It follows that the set \( St(v_0) \) is compact, since there are at most \( \frac{c_n}{\varphi_0} \) possible edge lengths, which can take values in the interval \([d_1 \varphi_0, d_1]^{10}\). Therefore if a \( D^* \) satisfying (3.8) exist, it depends only on \( \varphi_0, \delta \) and \( \varepsilon \) (and not on \( K_1, K_2 \)).

Let \( \sigma_1, \ldots, \sigma_{l_1} \) and \( \tau_1, \ldots, \tau_{l_2} \) be orderings of the simplices of \( L_2 \) and of the \( k \)-simplices of \( \partial St(v_0) \), respectively. Then, by \([6]\), Lemma 7.4, there exists \( \varepsilon_{1,1} \) and \( v_{1,1}, d(v_{1,1}, v_0) = \varepsilon_{1,1} \), such that the hyperplane \( \Pi(v_{1,1}, \tau_1) \) determined by \( v_1 \) and by \( \tau_1 \) is transversal to \( \sigma_1 \). By replacing \( \tau_1 \) by \( \tau_2 \) and \( v_0 \) by \( v_{1,1} \) we obtain \( v_{1,2} \) and \( \varepsilon_{1,2} \) s.t. \( \Pi(v_{1,2}, \tau_1) \cap \sigma_2 \). (See Fig. 3.)

Moreover, by choosing \( \varepsilon_{1,2} \) sufficiently small, one can ensure that \( \Pi(v_{1,2}, \tau_1) \cap \sigma_1 \), also. Repeating the process for \( \tau_3, \ldots, \tau_{l_2} \), one determines a point \( v_{1,l_2} \) such that \( \Pi(v_{1,l_2}, \tau_j) \cap L_2 \cap_{\delta}\cap \sigma_{l_2} \). In the same manner and by choosing at each stage an \( \varepsilon_{i,j} \) small enough, one finds points \( v_{i,j} \) s.t. \( \Pi(v_{i,j}, \tau_j) \cap L_2, i = 1, \ldots, l_1, j = 1, \ldots, l_2 \). Then \( v_{i,j} = v_{i,l_2} \) satisfies: \( \Pi(v_{i,j}, \tau_j) \cap L_2, j = 1, \ldots, l_2 \).

□

We are now prepared to prove the main result of this section namely:

**Theorem 3.6.** Let \( T_1, T_2 \) be two fat triangulations of open sets \( U_1, U_2 \subset M^n \), \( U_1 \cap U_2 \neq \emptyset \) having common fatness \( \geq \varphi_0 \), and such that \( T_1 \cap T_2 \neq \emptyset \). Then there exist fat triangulations \( \overline{T}_1, \overline{T}_2 \) and there exist open sets \( U \subset U_1 \cap U_2 \subset V \), such that

1. \( (\overline{T}_1 \cap \overline{T}_2) \cap (U \setminus V) = \overline{T}_i, i = 1, 2; \)
2. \( (\overline{T}_1 \cap \overline{T}_2) \cap U = \overline{T}; \)

where

3. \( T \) is a fat triangulation of \( U \).

\[10\text{i.e. the number of possible combinatorial structures on } St(v_0) \text{ depends only on } \varphi_0.\]
Proof. Let $K_1, K_2$ denote the underlying complexes of $T_1, T_2$, respectively. By considerations similar to those of Proposition 3.5, it follows that given $\varphi_0 > 0$, there exists $d(\varphi_0) > 0$ such that given a $k$-dimensional simplex $\sigma \subset \mathbb{R}^n$, $diam(\sigma) = d_1$ has fatness $\varphi_0$, then translating each vertex of $\sigma$ by a distance $d(\varphi_0) \cdot d_1$ renders a simplex of fatness $\geq \varphi_0/2$. Also, it follows that given $\varphi_0, \delta > 0$, exists $\delta(\varphi_0, \delta)$ satisfying the following condition: if every vertex $u \in \sigma \subset K$ is replaced by a vertex $u'$ s.t. $dist(u, u') \leq \delta(\varphi_0, \delta) \cdot d_1$, then the resulting simplex $\sigma'$ is $\cap_{\delta/2}$-transversal to $K$: for any $n$-dimensional simplicial complex $K$ of fatness $\varphi_0$ and such that $diam \sigma = d_2 \geq d_1$.

Let $v_0 \in U_1 \cap U_2$. Define the following subcomplexes of $K_1, K_2$, respectively:

- $L_2 = \{ \bar{\sigma} \subset K_2 \mid \bar{\sigma} \subset B_2(v_0), d_1 \leq dist(\bar{\sigma}, \partial B_2(v_0)) \leq d_2 \}$
- $M_2 = \{ \sigma \subset K_2 \mid \sigma \subset \bar{\tau} \subset B_2(v_0), dim \tau = n, \bar{\sigma} \cap L_2 \neq \emptyset \}$
- $L_1 = \{ \bar{\sigma} \subset K_1 \mid dist(\bar{\sigma}, L_2) \leq d_2 \}$
- $M_1 = \{ \sigma \subset K_1 \mid \sigma \subset \tau \subset B_2(v_0), dim \tau = n, \tau \cap L_1 \neq \emptyset \}$

(See Fig. 4.)

Consider an ordering $v_1, \ldots, v_p$ of the vertices of $L_1$. It follows from Proposition 3.5. that, if all the vertices of $L_1$ are moved by at most $t_0$, where

\[
(3.9) \quad t_0 = \frac{d_1}{n} \min \{ \frac{1}{2}, d(\varphi_0) \},
\]

then there exists

\[
(3.10) \quad \delta_0^* = \delta_0^*(\varphi_0, 1, \frac{t_0}{d_1}),
\]
such that

\[(3.11) \quad S^0(L_{1,0}) \cap \delta_0^* K_2,\]

where $S^0(L_{1,0})$ denotes the 0-skeleton of $L_{1,0}$.

Now define inductively

\[(3.12) \quad t_i = \frac{d_1}{n} \min \left\{ \frac{1}{2} d(\varphi_0), \delta\left(\varphi_0, \frac{\delta_0^*}{2}\right), \ldots, \delta\left(\varphi_0, \frac{\delta_{i-1}^*}{2}\right) \right\},\]

where

\[(3.13) \quad \delta_i^* = \delta_i^*(\varphi_0, \delta_{i-1}^*, \frac{t_i}{d_1}); \quad i = 1, \ldots, n - 1.\]

Then $t_0 \geq t_1 \geq \ldots \geq t_{n-1}$.

Moving each and every vertex of $L_1$ by a distance $\leq t_i$, $i = 1, \ldots, n$, renders complexes $L_{1,1}, \ldots, L_{1,n-1}$ s.t.

1. $L_{1,i} \cap (B_\epsilon(v_0) \setminus M_1) \equiv L_1,$
2. $L_{1,i}$ are $\varphi_0$-fat; $i = 0, \ldots, n - 1$.

By inductively applying Proposition 3.5, it follows that

\[(3.14) \quad S^i(L_{1,i}) \cap \delta_i^* K_2,\]
Where $S^i(L_{1,i})$ denotes the $i$-skeleton of $L_{1,i}$.

Moreover,

\[(3.15)\quad S^i(L_{1,i}) \cap \delta^*_j K_2, \forall j > i.\]

It follows that

\[(3.16)\quad L_{1,n-1} \cap \delta^*_K K_2,\]

where

\[(3.17)\quad \delta^*_K = \frac{1}{2} \min\{\delta_0^*, \ldots, \delta_i^*\}.\]

By Proposition 3.2. the barycentric subdivision of $L_{1,n-1} \cap L_2$ is fat. We extend it to a fat subdivision of $M_2$ in the following manner: given a simplex $\sigma \subseteq M_2 \setminus L_2$, subdivide $\sigma$ by constructing all the simplices with vertices $v_i$, where $v_i$ is either the vertex of a simplex $\sigma \subseteq M_2 \setminus L_2$, $\sigma \cap L_2 \neq \emptyset$, or it is a vertex of a closed simplex $\overline{\sigma}$ of the barycentric subdivision of $L_{1,n-1} \cap L_2$, such that $\overline{\sigma} \subseteq \partial L_2 \cap M_2$, $i = 1, \ldots, k_0$. The triangulation $K_2$ thus obtained is a fat extension of $K_2 \setminus M_2$.

In an analogous manner one constructs a similar fat extension $\tilde{K}_2$ of $K_2 \setminus M_2$.

Now let $T_1, T_2$ be the triangulations of $\partial M^n \times [0,1)$ and $\text{int} M^n$, respectively, given by Theorem 2.9. Then the local fat triangulation obtained in Theorem 3.6. extends globally to a fat triangulation of $T_1 \cap T_2$, by applying Lemma 10.2 and Theorem 10.4 of [17]. This concludes the

\[\text{Proof of Theorem 1.1}\]

4. The Existence of Quasimeromorphic Mappings

4.1. Quasimeromorphic Mappings.

**Definition 4.1.** Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$, and let $f : D \to \mathbb{R}^m$. $f$ is called $ACL$ (absolutely continuous on lines) iff:

\begin{enumerate}
  \item $f$ is continuous
  \item for any $n$-interval $Q = \overline{Q} = \{a_i \leq x_i \leq b_i \mid i = 1, \ldots, n\}$, $f$ is absolutely continuous on almost every line segment in $Q$, parallel to the coordinate axes.
\end{enumerate}

**Lemma 4.2** ([30], 26.4). If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is $ACL$, then $f$ admits partial derivatives almost everywhere.

The result above justifies the following Definition:

**Definition 4.3.** $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is $ACL^p$ iff its derivatives are locally $L^p$ integrable, $p \geq 1$.

**Definition 4.4.** Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$ and let $f : D \to \mathbb{R}^m$ be a continuous mapping. $f$ is called

\begin{enumerate}
  \item quasiregular iff (i) $f$ is $ACL^n$ and
  \item $\exists K \geq 1$ s.t.
\end{enumerate}

\[(4.1)\quad |f'(x)| \leq K J_f(x) \text{ a.e.}\]
where \( f'(x) \) denotes the formal derivative of \( f \) at \( x \), \( |f'(x)| = \sup_{|h| = 1} |f'(x)h| \), and where \( J_f(x) = det f'(x) \).

The smallest \( K \) that satisfies (4.1) is called the outer dilatation of \( f \).

1. quasiconformal iff \( f : D \to \mathbb{R}^n \) is a local homeomorphism.
2. quasimeromorphic iff \( f : D \to \mathbb{R}^n \setminus \{\infty\} \) is quasiregular, where the condition of quasiregularity at \( f^{-1}(\infty) \) can be checked by conjugation with auxiliary Möbius transformations.

**Remark 4.5.** One can extend the definitions above to oriented \( C^\infty \) Riemannian \( n \)-manifolds by using coordinate charts.

### 4.2. Alexander’s Trick

The technical ingredient in Alexander’s trick is the following Lemma:

**Lemma 4.6.** (\[13\], \[18\]) Let \( T \) be a fat triangulation of \( M^n \subset \mathbb{R}^n \), and let \( \tau, \sigma \in T \), \( \tau = (p_1, \ldots, p_n) \), \( \sigma = (q_1, \ldots, q_n) \); and denote \( |\tau| = \tau \cup int \tau \).

Then there exists a sense-preserving homeomorphism \( h = h_\tau : |\tau| \to \mathbb{R}^n \) s.t.

1. \( h(|\tau|) = |\sigma| \), if \( det(p_1, \ldots, p_n) > 0 \) and \( h(|\tau|) = \mathbb{R}^n \setminus |\sigma| \), if \( det(p_1, \ldots, p_n) < 0 \).
2. \( h(p_i) = q_i \), \( i = 1, \ldots, n \).
3. \( h|_{\partial|\sigma|} \) is a PL-homeomorphism.
4. \( h|_{int|\sigma|} \) is quasiconformal.

**Proof** Let \( \tau_0 = (p_{0,1}, \ldots, p_{0,n}) \) denote the equilateral \( n \)-simplex inscribed in the unit sphere \( S^{n-1} \). The radial linear stretching \( \varphi : \tau \to \mathbb{R}^n \) is onto and bi-lipschitz (see \[13\]). Moreover, by a result of Gehring and Väisälä, \( \varphi \) is also quasiconformal (see \[30\]). We can extend \( \varphi \) to \( \mathbb{R}^n \) by defining \( \varphi(\infty) = \infty \).

The Existence Theorem of quasimeromorphic mappings now follows immediately:

**Theorem 4.7.** Let \( M^n \) be a \( C^\infty \) Riemannian manifold with or without boundary. Then there exists a quasimeromorphic mapping \( f : M^n \to \mathbb{R}^n \).

**Proof** Let \( f : M^n \to \mathbb{R}^n \) be defined by: \( f|_{|\sigma|} = h_\tau \), where \( h \) is the homeomorphism constructed in the Lemma above. Then \( f \) is a local homeomorphism on the \((n-1)\)-skeleton of \( T \) too, while its branching set \( B_f \) is the \((n-2)\)-skeleton of \( T \). By its construction \( f \) is quasiregular. Moreover, given the uniform fatness of the triangulation \( T \), the dilatation of \( f \) depends only on the dimension \( n \).

### 5. Generalizations and Further Research

We succinctly present some immediate generalizations of Theorems 1.1. and 2.9.

#### 5.1. Smoothings

Theorem 1.1. was restricted to \( C^\infty \) manifolds because the triangulation \( T_2 \) of \( int M^n \) was obtained by applying Peltonen’s Theorem; so our overall argument is valid only for \( C^\infty \) manifolds. But the class of any \( n \)-manifold may be elevated up to \( C^\infty \) (see \[17\], Theorems 4.8 and 5.13), so we can apply the methods of \[18\] on the smoothed \( C^\infty \) manifold, and then project the fat triangulation.
received to the original structure. Since in the smoothing process we employed only \( \delta \)-approximations that are, by Lemma 8.7, \( \alpha \)-approximations too, we will obtain a fat triangulation, as desired. We can thus formulate the following Corollary:

**Corollary 5.1.** Let \( M^n \) be an \( n \)-dimensional \( C^r \), \( 1 \leq r \leq \infty \) manifold with boundary. Then any fat triangulation of \( \partial M^n \) can be extended to a fat triangulation of \( M^n \).

Moreover, every \( PL \) manifold of dimension \( n \leq 4 \) admits a (unique, for \( n \leq 3 \)) smoothing (see [16, 17, 28]), and every topological manifold of dimension \( n \leq 3 \) admits a \( PL \) structure (cf. [12, 28]). Therefore we can start with a \( PL \) manifold (or even just a topological one in dimensions 2 and 3) and smooth it, thus receiving

**Corollary 5.2.** Let \( M^n \) be an \( n \)-dimensional, \( n \leq 4 \) (resp. \( n \leq 3 \)), \( PL \) (resp. topological) manifold with boundary. Then any fat triangulation of \( \partial M^n \) can be extended to a fat triangulation of \( M^n \).

Using again Alexander’s Trick renders the following result:

**Corollary 5.3.** Let \( M^n \) be an \( n \)-dimensional manifold (\( n \geq 2 \)), with or without boundary. Then in the following cases there exists a quasimeromorphic mapping \( f : M^n \to \mathbb{R}^n \):

1. \( M^n \) is of class \( C^r \), \( 1 \leq r \leq \infty \), \( \forall n \geq 2 \);
2. \( M^n \) is a \( PL \) manifold and \( n \leq 4 \);
3. \( M^n \) is a topological manifold and \( n \leq 3 \).

**Remark 5.4.** It may be that during the smoothing process the dilatation will increase in an unbounded fashion, thus rendering impossible the proof of existence of fat triangulations quasimeromorphic mappings. However, the dilatation may increase only when we linearize the tangent cone at cone points. Fortunately, the nature of linearization process is such that, when the cone angles are bounded from below, then the dilatations will be bounded from above\(^1\).

### 5.2. Kleinian Groups

Since the construction of fat triangulations was motivated mainly by the study of \( G \)-automorphic quasimeromorphic mappings with respect to a Kleinian group \( G \), i.e. a discontinuous group of orientation preserving isometries of \( \mathbb{H}^n \), it is natural to employ Theorems 1.1. and 2.9. to prove the following result, that represents a generalization of a result of Tukia ([29]):

**Theorem 5.5.** Let \( G \) be a Kleinian group with torsion acting upon \( \mathbb{H}^n \), \( n \geq 3 \).\(^2\)

If the elliptic elements (i.e. torsion elements) of \( G \) have uniformly bounded orders, then there exists a non constant \( G \)-automorphic quasimeromorphic mapping \( f : \mathbb{H}^n \to \mathbb{R}^n \), i.e. such that

\[
(5.1) \quad f(g(x)) = f(x), \quad \forall x \in \mathbb{H}^n, \forall g \in G.
\]

While for full details we refer the reader to [22] and – for a different fattening method (albeit in dimension 3 only), to [21] – we bring here the following

\(^1\) For some details regarding linearizations, see [28].

\(^2\) The case \( n = 2 \) being trivial, since in this case \( \mathbb{H}^n/G \) is always a manifold.
Sketch of Proof

By Lemma 4.6, it suffices to produce a fat $G$-invariant triangulation of $\mathbb{H}^n$. The singular locus $L$ of $\mathbb{H}^n/G$ is the image, under the natural projection $\pi : \mathbb{H}^n \to \mathbb{H}^n/G$, of the union $A = \bigcup_{i \in N} A_{f_i}$ of the elliptic axes of $G$. For each elliptic axes $A_{f_i}$, it is possible to choose a collar $N_i$ and triangulate it in an $f_i$-invariant manner. Define $T_i$ as the $f_i$-invariant triangulation of $N_i$. Put $N = \bigcup_{i \in N} N_i$. Then $M_e = (\mathbb{H}^n \setminus N)/G$ is a manifold with boundary. Then $\partial M_e = \bigcup_{i \in N} \partial N_i$ has the triangulation induced by that of $N$. Since the orders of the elliptic elements are bounded from above, the induced triangulation will be fat. By Theorem 1.1, this triangulation can be extended to a fat triangulation $T$ of $\mathbb{H}^n/G$. Then $\pi^{-1}(T) \cup \bigcup_{i \in N} T_i$ will represent the desired fat $G$-invariant triangulation.

□

Remark 5.6. It seems feasible to adapt this argument for any geometric orbifold with tame singular locus (at least in dimension 3).

5.3. Lipschitz Manifolds. The existence of triangulations for Lipschitz manifolds was already stated by Cairns ([5]), yet it was never proved in full detail. Also it seems possible relax the smoothness condition even further, as to include quasisymmetric manifolds, i.e. manifolds of which local charts are given by quasisymmetric mappings, where:

**Definition 5.7.** An embedding $f : \mathbb{R}^m \to \mathbb{R}^n$ is called quasisymmetric iff there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all $x, a, b \in \mathbb{R}^m$ and for all $t \in [0, \infty)$ the following holds:

$$(5.2) \quad \text{dist}(x, a) \leq t \cdot \text{dist}(x, b) \implies \text{dist}(f(x), f(a)) \leq \eta(t) \cdot \text{dist}(f(x), f(b)).$$

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13. $A = \{A_{f_i}\}$ is a countable set, by the discreteness of $G$.
14. Some more care is needed in producing the fat triangulation of $\mathcal{N}$ if there exist $i \neq j$ s.t. $A_{f_i} \cap A_{f_j} \neq \emptyset$. 
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