INTEGRAL REPRESENTATIONS OF CLOSED OPERATORS
AS BI-CARLEMAN OPERATORS
WITH ARBITRARILY SMOOTH KERNELS

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ABSTRACT. In this paper, we characterize all closed linear operators in
a separable Hilbert space which are unitarily equivalent to an integral bi-
Carleman operator in \( L_2(\mathbb{R}) \) with bounded and arbitrarily smooth kernel
on \( \mathbb{R}^2 \). In addition, we give an explicit construction of corresponding
unitary operators. The main result is a qualitative sharpening of an earlier
result of [5].

Throughout, \( \mathcal{H} \) will denote a separable Hilbert space with the inner prod-
tuct \( \langle \cdot, \cdot \rangle_\mathcal{H} \) and the norm \( \| \cdot \|_\mathcal{H} \), \( \mathcal{C}(\mathcal{H}) \) the set of all closed linear operators
densely defined in \( \mathcal{H} \), and \( \mathbb{C} \), and \( \mathbb{N} \), and \( \mathbb{Z} \), the complex plane, the set of all
positive integers, the set of all integers, respectively. For an operator \( S \)
in \( \mathcal{C}(\mathcal{H}) \), \( S^* \) will denote the Hilbert space adjoint of \( S \).

An operator \( S : D_S \to \mathcal{H} \) of \( \mathcal{C}(\mathcal{H}) \) is said to belong to the set \( \mathcal{C}_{00}(\mathcal{H}) \)
if there exist a linear manifold \( D \) dense in \( \mathcal{H} \) and an orthonormal sequence
\( \{e_n\} \) such that

\[
\{e_n\} \subset D \subset D_S \cap D_{S^*}, \quad \lim_{n \to \infty} \|S e_n\|_\mathcal{H} = 0, \quad \lim_{n \to \infty} \|S^* e_n\|_\mathcal{H} = 0.
\]

Let \( \mathbb{R} \) be the real line \( (-\infty, +\infty) \) equipped with the Lebesgue measure,
and let \( L_2 = L_2(\mathbb{R}) \) be the Hilbert space of (equivalence classes of) mea-
surable complex–valued functions on \( \mathbb{R} \) equipped with the inner product

\[
\langle f, g \rangle = \int_{\mathbb{R}} f(s) \overline{g(s)} \, ds
\]

and the norm \( \|f\| = \langle f, f \rangle^{\frac{1}{2}} \). A linear operator \( T : D_T \to L_2 \), where the
domain \( D_T \) is a dense linear manifold in \( L_2 \), is said to be integral if there
exists a measurable function \( T \) on \( \mathbb{R}^2 \), a kernel, such that, for every \( f \in D_T \),

\[
(Tf)(s) = \int_{\mathbb{R}} T(s, t) f(t) \, dt
\]

for almost every \( s \) in \( \mathbb{R} \). A kernel \( T \) on \( \mathbb{R}^2 \) is said to be Carleman if \( T(s, \cdot) \in L_2 \) for almost every fixed \( s \) in \( \mathbb{R} \). An integral operator with a kernel \( T \) is

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called Carleman if \( T \) is a Carleman kernel, and it is called bi-Carleman if both \( T \) and \( T^* (T^*(s, t) = T(t, s)) \) are Carleman kernels. Every Carleman kernel, \( T \), induces a Carleman function \( t \) from \( \mathbb{R} \) to \( L_2 \) by \( t(s) = T(s, \cdot) \) for all \( s \) in \( \mathbb{R} \) for which \( T(s, \cdot) \in L_2 \).

The characterization of closed linear operators whose unitary orbits contain integral operators was first studied by Neumann in [11] and by now is well understood [3]. The main results related to this problem are concerned with the unitary equivalence of linear operators to Carleman or bi-Carleman operators with measurable kernels. Here is one of them [3, p. 145].

**Proposition 1.** A necessary and sufficient condition that an operator \( S \in \mathcal{C}(\mathcal{H}) \) be unitarily equivalent to a bi-Carleman operator is that \( S \) belong to \( \mathcal{C}_{00}(\mathcal{H}) \).

However, the characterization problem can also be formulated in terms of kernels that satisfy various additional conditions. For example, given any non-negative integer \( m \), we consider the following question: which operators are unitarily equivalent to a bi-Carleman operator with a kernel \( K \) satisfying the conditions:

(i) the function \( K \) and all its partial derivatives on \( \mathbb{R}^2 \) up to order \( m \) are in \( C(\mathbb{R}^2, \mathbb{C}) \),

(ii) the Carleman function \( k, k(s) = K(s, \cdot) \), and all its (strong) derivatives on \( \mathbb{R} \) up to order \( m \) are in \( C(\mathbb{R}, L_2) \),

(iii) the conjugate transpose function \( K^*, K^*(s, t) = \overline{K(t, s)} \), satisfies Condition (ii), that is, the Carleman function \( k^*, k^*(s) = \overline{K^*(s, \cdot)} \), and all its (strong) derivatives on \( \mathbb{R} \) up to order \( m \) are in \( C(\mathbb{R}, L_2) \)?

Here and throughout \( C(X, B) \), where \( B \) is a Banach space (with norm \( \|\cdot\|_B \)), denote the Banach space (with the norm \( \|f\|_{C(X, B)} = \sup_{x \in X} \|f(x)\|_B \)) of continuous \( B \)-valued functions defined on a locally compact space \( X \) and vanishing at infinity (that is, given any \( f \in C(X, B) \) and \( \varepsilon > 0 \), there exists a compact subset \( X(\varepsilon, f) \subset X \) such that \( \|f(x)\|_B < \varepsilon \) whenever \( x \not\in X(\varepsilon, f) \)).

A function \( K \) that satisfies Conditions (i), (ii) is called a \( SK^m \)-kernel [3]. In addition, a \( SK^m \)-kernel \( K \) is called a \( K^m \)-kernel ([7], [5]) if it satisfies Condition (iii).

The next result is an answer to the above question; it gives a characterization of closed linear operators representable as bi-Carleman operators with \( K^m \)-kernels (cf. Proposition [1]).

**Proposition 2 ([5], [6]).** Let \( m \) be a fixed non-negative integer, and let \( S \in \mathcal{C}_{00}(\mathcal{H}) \). Then there exists a unitary operator \( U_m : \mathcal{H} \rightarrow L_2 \) such that \( T = U_m S U_m^{-1} \) is a bi-Carleman operator having \( K^m \)-kernel.

The purpose of this paper is to restrict the conclusion of Proposition [2] to arbitrarily smooth kernels. Now we define these kernels.
Definition 1. We say that a function $K$ is a $K^\infty(SK^\infty)$-kernel if it is a $K^m(SK^m)$-kernel for each non-negative integer $m$ ([9], [10]).

The following theorem is the main result of the present paper.

Theorem. If $S \in \mathcal{C}_00(\mathcal{H})$, then there exists a unitary operator $U_\infty : \mathcal{H} \to L_2$ such that the operator $T = U_\infty SU_\infty^{-1}$ is a bi-Carleman operator having $K^\infty$-kernel.

1. Proof of Theorem

The proof has an algorithmic nature and consists of three steps. In the first step we split the operator $S \in \mathcal{C}_00(\mathcal{H})$ in order to construct auxiliary operators $J, \tilde{J}, B, \tilde{B}, Q, \tilde{Q}$. Using these operators, in the second step we describe suitable orthonormal bases $\{u_n\}$ in $L_2$ and $\{f_n\}$ in $\mathcal{H}$, and then we use the mentioned bases to construct a unitary operator from $\mathcal{H}$ to $L_2$ which sends the basis $\{f_n\}$ onto the basis $\{u_n\}$, and fulfills the role of the operator $U_\infty$ cited in the theorem. The remaining part of the proof (step 3) is a direct verification that the constructed unitary operator has the desired properties. Thus, the proof of the theorem not only establishes the unitary equivalence itself, but also indicates an explicit construction of $U_\infty$.

Step 1. Let $S \in \mathcal{C}_00(\mathcal{H})$. Assume, with no loss of generality, that the sequence $\{e_k\}_{k=1}^\infty \subset D$ in (1) satisfies the condition
\begin{equation}
\sum_k \left( \|Se_k\|_{\mathcal{H}}^\frac{1}{4} + \|S^*e_k\|_{\mathcal{H}}^\frac{3}{4} \right) \leq 1
\end{equation}
(the sum notation $\sum_k$ will always be used instead of the more detailed symbol $\sum_{k=1}^\infty$). Let $H$ denote a subspace spanned by the $e_k$’s, and let $H^\perp$ be the orthogonal complement of $H$ in $\mathcal{H}$. Since $S, S^* \in \mathcal{C}(\mathcal{H})$, we have
\begin{equation}
H \subset D_S, \quad H \subset D_{S^*}.
\end{equation}

If $E$ is the orthogonal projection onto $H$, consider the decompositions
\begin{equation}
S = (1 - E)S + ES, \quad S^* = (1 - E)S^* + ES^*.
\end{equation}
Since $E \in \mathcal{R}(\mathcal{H})$, it follows via definition of the adjoint that $(ES)^* = S^*E$, and $(ES^*)^* = SE$. Observe that the operators
\begin{equation}
J = SE, \quad \tilde{J} = S^*E
\end{equation}
are nuclear because of (2) and (3). Assume, with no loss of generality, that $\dim H^\perp = \infty$, and choose an orthonormal basis $\{e_k\}_{k=1}^\infty$ for $H^\perp$ so that
\begin{equation}
\{e_k\}_{k=1}^\infty \subset (1 - E)D \subset D_S \cap D_{S^*}.
\end{equation}
For each $f \in D_S \cap D_{S^*}$ and for each $h \in \mathcal{H}$, let
\begin{align*}
\sigma(f) &= \|Sf\|_{\mathcal{H}} + \|S^*f\|_{\mathcal{H}}, \\
\delta(h) &= \|Jh\|_{\mathcal{H}}^\frac{1}{2} + \|J^*h\|_{\mathcal{H}}^\frac{1}{2} + \|\tilde{J}h\|_{\mathcal{H}}^\frac{1}{2} + \|\tilde{J}^*h\|_{\mathcal{H}}^\frac{1}{2}.
\end{align*}
If
\[ J = \sum_{n} s_n \langle \cdot, p_n \rangle_{\mathbb{H}} q_n \text{ and } \tilde{J} = \sum_{n} \tilde{s}_n \langle \cdot, \tilde{p}_n \rangle_{\mathbb{H}} \tilde{q}_n \]
are the Schmidt decompositions for \( J \) and \( \tilde{J} \), respectively, then the closedness of both \( S \) and \( S^* \) implies that, for all \( f \in D_{S^*} \) and \( g \in D_S \),
\[
ES^* f = (ES^*)^* f = (SE)^* f = J^* f = \sum_{n} s_n \langle g, q_n \rangle_{\mathbb{H}} p_n,
\]
(8)
\[
ESg = (ES)^* g = (S^* E)^* g = \tilde{J}^* g = \sum_{n} \tilde{s}_n \langle g, \tilde{q}_n \rangle_{\mathbb{H}} \tilde{p}_n;
\]
here the \( s_n \) are the singular values of \( J \) (eigenvalues of \( (JJ^*)^{\frac{1}{2}} \)), \( \{p_n\}, \{q_n\} \) are orthonormal sets (the \( p_n \) are eigenvectors for \( J^* J \) and the \( q_n \) are eigenvectors for \( JJ^* \)).

Define
\[
B = \sum_{n} s_n^4 \langle \cdot, p_n \rangle_{\mathbb{H}} q_n, \quad \tilde{B} = \sum_{n} \tilde{s}_n^4 \langle \cdot, \tilde{p}_n \rangle_{\mathbb{H}} \tilde{q}_n,
\]
and observe that, by the Schwarz inequality,
\[
\|B f\|_{\mathbb{H}} = \left\| (J^* J)^{\frac{1}{4}} f \right\|_{\mathbb{H}} \leq \|J f\|_{\mathbb{H}},
\]
\[
\|B^* f\|_{\mathbb{H}} = \left\| (J J^*)^{\frac{1}{4}} f \right\|_{\mathbb{H}} \leq \|J^* f\|_{\mathbb{H}},
\]
\[
\|\tilde{B} f\|_{\mathbb{H}} = \left\| (\tilde{J}^* \tilde{J})^{\frac{1}{4}} f \right\|_{\mathbb{H}} \leq \|\tilde{J} f\|_{\mathbb{H}},
\]
\[
\|\tilde{B}^* f\|_{\mathbb{H}} = \left\| (J J^*)^{\frac{1}{4}} f \right\|_{\mathbb{H}} \leq \|J f\|_{\mathbb{H}}
\]
if \( \|f\| = 1 \). The operators \( B \) and \( \tilde{B} \) play only an auxiliary role in what follows.

Define operators \( Q = (1 - E) S, \tilde{Q} = (1 - E) S^* \). The property (6) guarantees that the following representations hold:
\[
Q f = \sum_{k} \langle Q f, e_k^+ \rangle_{\mathbb{H}} e_k^+ = \sum_{k} \langle f, S^* e_k^+ \rangle_{\mathbb{H}} e_k^+ \quad \text{for all } f \in D_S,
\]
(11)
\[
\tilde{Q} g = \sum_{k} \langle \tilde{Q} g, e_k^+ \rangle_{\mathbb{H}} e_k^+ = \sum_{k} \langle g, S e_k^+ \rangle_{\mathbb{H}} e_k^+ \quad \text{for all } g \in D_{S^*}.
\]

**Step 2.** This step is to construct a candidate for the desired unitary operator \( U_\infty \) in the theorem.

**Notation.** If an equivalence class \( f \in L_2 \) contains a function belonging to \( C(\mathbb{R}, \mathbb{C}) \), then we shall use \([f]\) to denote that function.

Take orthonormal bases \( \{f_n\} \) for \( \mathbb{H} \) and \( \{u_n\} \) for \( L_2 \) which satisfy the conditions:

(a) the terms of the sequence \( \{[u_n]^{(i)}\} \) of derivatives are in \( C(\mathbb{R}, \mathbb{C}) \),

for each \( i \) (here and throughout, the letter \( i \) is reserved for all non-negative integers),
Rearrange, in a completely arbitrary manner, the orthonormal set
\[ j \] basis for
\[ u \]
Lemarié-Meyer wavelets we refer to [4], [1, [12]
\[ (12) \]
\[ \sum_k H_{k,i} < \infty, \sum_k z(v_k) H_{m(k),i} < \infty, \sum_k z(e_k^i) H_{n(k),i} < \infty \]
\[ \text{with } H_{k,i} = ||[h_k]^{(i)}||_{C(R,C)} \quad (k \in \mathbb{N}), \]
\[ (13) \]
\[ \sum_k d(x_k) (G_{k,i} + 1) < \infty \quad \text{with } G_{k,i} = ||[g_k]^{(i)}||_{C(R,C)} \quad (k \in \mathbb{N}), \]
where \( \{n(k)\}_{k=1}^\infty \) and \( \{m(k)\}_{k=1}^\infty \) are subsequences of \( \mathbb{N} \) such that
\( \{m(k)\}_{k=1}^\infty = \mathbb{N} \setminus \{n(k)\}_{k=1}^\infty \), and \( \{g_k\}, \{h_k\}, \{x_k\}, \) and \( \{v_k\} \), are orthonormal sets such that
\[ \{u_n\} = \{g_k\}_{k=1}^\infty \cup \{h_k\}_{k=1}^\infty, \quad \{g_k\}_{k=1}^\infty \cap \{h_k\}_{k=1}^\infty = \emptyset, \]
\[ \text{for each } i, \]
\[ (15) \]
\[ [u_n]^{(i)} \in C(R, C) \quad (n \in \mathbb{N}), \]
\[ (16) \]
\[ ||[u_n]^{(i)}||_{C(R,C)} \leq D_n A_i \quad (n \in \mathbb{N}), \]
\[ (17) \]
\[ \sum_k D_{nk} < \infty, \]
where \( \{D_n\}_{n=1}^\infty \) and \( \{A_i\}_{i=0}^\infty \) are sequences of positive numbers, and \( \{n_k\}_{k=1}^\infty \)
is a subsequence of \( \mathbb{N} \) such that \( \mathbb{N} \setminus \{n_k\}_{k=1}^\infty \) is a countable set. Since
\( d(e_k) \to 0 \) and \( z(v_k) \to 0 \) as \( k \to \infty \), the basis \( \{u_n\} \) satisfies Conditions
(a) and (b), with \( h_k = u_{nk} \ (k \in \mathbb{N}) \) and \( \{g_k\}_{k=1}^\infty = \{u_n\} \setminus \{h_k\}_{k=1}^\infty \).
To construct an example of such basis \( \{u_{n}\} \), consider a Lemarié-Meyer wavelet,
\[ u(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(\frac{s}{2}+s)} \text{sign } b(\xi) d\xi \quad (s \in \mathbb{R}), \]
with the bell function \( b \) belonging to \( C^\infty(\mathbb{R}) \) (for construction of the Lemarié-Meyer wavelets we refer to [4], [1 § 4], Example D, p. 62]). Then \( u \) belongs to the Schwartz class \( S(\mathbb{R}) \), and hence all the derivatives
\[ [u]^{(i)} \] are in \( C(\mathbb{R}, C) \). The “mother function” \( u \) generates an orthonormal basis for \( L_2 \) by
\[ u_{jk}(s) = 2^{\frac{j}{2}} u(2^j s - k) \quad (j, k \in \mathbb{Z}). \]
Rearrange, in a completely arbitrary manner, the orthonormal set \( \{u_{jk}\}_{j,k \in \mathbb{Z}} \)
into a simple sequence, so that it becomes \( \{u_n\}_{n \in \mathbb{N}} \). Since, in view of this rearrangement, to each \( n \in \mathbb{N} \) there corresponds a unique pair of integers
\( j_n, k_n \), and conversely, we can write, for each \( i \),
\[ ||[u_n]^{(i)}||_{C(R,C)} = ||[u_{jn,kn}]^{(i)}||_{C(R,C)} \leq D_n A_i, \]
where
\[ D_n = \begin{cases} 2j_n^2 & \text{if } j_n > 0, \\ \left( \frac{1}{\sqrt{2}} \right)^{|j_n|} & \text{if } j_n \leq 0, \end{cases} \]
\[ A_i = 2^{(i+\frac{1}{2})^2} \| [u]^{(i)} \|_{C(\mathbb{R}, \mathbb{C})}. \]

Whence it follows that if \( \{n_k\}_{k=1}^\infty \subset \mathbb{N} \) is a subsequence such that \( j_{n_k} \to -\infty \) as \( k \to \infty \), then
\[ \sum_k D_{n_k} < \infty. \]

Thus, the basis \( \{u_n\} \) satisfies Conditions (15) through (17) and, consequently, Conditions (a) and (b).

Let us return to the proof of the theorem. Define a candidate for the desired unitary operator \( U_\infty : \mathcal{H} \to L_2 \) on the basis vectors as follows:
\[ U_\infty x_k = g_k, \quad U_\infty v_k = h_{m(k)}, \quad U_\infty e_k^+ = h_{n(k)} \quad \text{for all } k \in \mathbb{N}, \]
where \( \{n(k)\}_{k=1}^\infty \) and \( \{m(k)\}_{k=1}^\infty \) are just those sequences which occur in Condition (12), in the harmless assumption that, for each \( k \in \mathbb{N} \),
\[ U_\infty f_k = u_k, \quad y_k = U_\infty^{-1} h_k. \]

**Step 3.** Show that the unitary operator \( U_\infty \) defined in (18) has the desired properties, that is, that \( T = U_\infty S U_\infty^{-1} \) is a bi-Carleman operator having \( K^\infty \)-kernel. For this purpose, verify first that all the operators \( P = U_\infty Q U_\infty^{-1}, \quad \tilde{P} = U_\infty Q U_\infty^{-1}, \quad F = U_\infty J^* U_\infty^{-1}, \quad \tilde{F} = U_\infty J^* U_\infty^{-1} \) are Carleman operators having \( S K^\infty \)-kernels.

Using (11), (18), one can write
\[ P f = \sum_k \left\langle f, T^* h_{n(k)} \right\rangle h_{n(k)} \quad \text{for all } f \in D_T = U_\infty D_S, \]
\[ \tilde{P} f = \sum_k \left\langle g, T h_{n(k)} \right\rangle h_{n(k)} \quad \text{for all } g \in D_{T^*} = U_\infty D_{S^*}, \]
where
\[ T^* h_{n(k)} = \sum_n \left\langle e_k^+, S f_n \right\rangle_{\mathcal{H}} u_n, \]
\[ Th_{n(k)} = \sum_n \left\langle e_k^+, S^* f_n \right\rangle_{\mathcal{H}} u_n \quad (k \in \mathbb{N}). \]

Prove that, for any fixed \( i \), the series
\[ \sum_n \left\langle e_k^+, S f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \left\langle e_k^+, S^* f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N}) \]
converge in the norm of \( C(\mathbb{R}, \mathbb{C}) \). Indeed, all these series are pointwise dominated on \( \mathbb{R} \) by one series
\[ \sum_n \left( \|S f_n\|_{\mathcal{H}} + \|S^* f_n\|_{\mathcal{H}} \right) \left| [u_n]^{(i)}(s) \right|, \]
which converges uniformly on \( \mathbb{R} \) because its component subseries (see (18), (14))
\[
\sum_{k} \left( \| J x_k \|_H + \| J x_k \|_H \right) \left[ g_k \right]^{(i)}(s),
\]
\[
\sum_{k} \left( \| S v_k \|_H + \| S v_k \|_H \right) \left[ h_m(k) \right]^{(i)}(s),
\]
\[
\sum_{k} \left( \| S e_k^\perp \|_H + \| S e_k^\perp \|_H \right) \left[ h_n(k) \right]^{(i)}(s)
\]
are in turn dominated by the convergent series
\[
\sum_{k} d(x_k) G_{k,i}, \quad \sum_{k} z(v_k) H_{m(k),i}, \quad \sum_{k} z(e_k^\perp) H_{n(k),i},
\]
respectively (see (2), (5), (7), (13), (12)). Whence it follows that, for each \( k \in \mathbb{N} \),
\[
\left\| \left[ T^* h_{n(k)} \right]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} \leq C_i^* \quad \left\| \left[ T h_{n(k)} \right]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} \leq C_i,
\]
with constants \( C_i^* \) and \( C_i \) independent of \( k \). From (7) it follows also that
\[
\left\| T^* h_{n(k)} \right\| \leq z \left( e_k^\perp \right), \quad \left\| T h_{n(k)} \right\| \leq z \left( e_k^\perp \right) \quad (k \in \mathbb{N}),
\]
since \( U_\infty \) is unitary. Consider functions \( P, \tilde{P} : \mathbb{R}^2 \to \mathbb{C} \), and \( p, \tilde{p} : \mathbb{R} \to L_2 \), defined, for all \( s, t \in \mathbb{R} \), by
\[
P(s, t) = \sum_{k} \left[ h_{n(k)} \right](s) \left[ T^* h_{n(k)} \right](t),
\]
\[
\tilde{P}(s, t) = \sum_{k} \left[ h_{n(k)} \right](s) \left[ T h_{n(k)} \right](t),
\]
\[
p(s) = \overline{P(s, \cdot)} = \sum_{k} \overline{\left[ h_{n(k)} \right](s) T^* h_{n(k)}},
\]
\[
\tilde{p}(s) = \overline{P(s, \cdot)} = \sum_{k} \overline{\left[ h_{n(k)} \right](s) T h_{n(k)}},
\]
(24) The termwise differentiation theorem implies that, for each \( i \) and each non-negative integer \( j \),
\[
\frac{\partial^{i+j} P}{\partial s^i \partial t^j}(s, t) = \sum_{k} \left[ h_{n(k)} \right]^{(i)}(s) \left[ T^* h_{n(k)} \right]^{(j)}(t),
\]
\[
\frac{\partial^{i+j} \tilde{P}}{\partial s^i \partial t^j}(s, t) = \sum_{k} \left[ h_{n(k)} \right]^{(i)}(s) \left[ T h_{n(k)} \right]^{(j)}(t),
\]
\[
\frac{d^i p}{d s^i}(s) = \sum_{k} \left[ h_{n(k)} \right]^{(i)}(s) T^* h_{n(k)},
\]
\[
\frac{d^i \tilde{p}}{d s^i}(s) = \sum_{k} \left[ h_{n(k)} \right]^{(i)}(s) T h_{n(k)},
\]
since, by (22), (23), and (12), the displayed series converge (absolutely) in \( C(\mathbb{R}^2, \mathbb{C}) \), \( C(\mathbb{R}, L_2) \), respectively. Thus,

\[
\frac{\partial^{i+j} P}{\partial s^i \partial t^j}, \quad \frac{\partial^{i+j} \tilde{P}}{\partial s^i \partial t^j} \in C(\mathbb{R}^2, \mathbb{C}), \quad \text{and} \quad \frac{d^i P}{ds^i}, \quad \frac{d^j \tilde{P}}{ds^j} \in C(\mathbb{R}, L_2).
\]

Observe also that, by (23), (12), and (24), the series (20) (viewed, of course, as ones with terms belonging to \( C(\mathbb{R}, \mathbb{C}) \)) converge (absolutely) in \( C(\mathbb{R}, \mathbb{C}) \)-norm to the functions

\[
[P f](s) \equiv \langle f, p(s) \rangle \equiv \int_{\mathbb{R}} P(s, t) f(t) \, dt,
\]

\[
[\tilde{P} g](s) \equiv \langle g, \tilde{p}(s) \rangle \equiv \int_{\mathbb{R}} \tilde{P}(s, t) g(t) \, dt,
\]

respectively. Thus, both \( P : D_T \to L_2 \) and \( \tilde{P} : D_{T^*} \to L_2 \) are Carleman operators, and \( P \) and \( \tilde{P} \) are their \( SK^\infty \)-kernels, respectively.

Since, by (5), \( \| S e_k \|_H = \| J e_k \|_H \) and \( \| S^* e_k \|_H = \| J e_k \|_H \), for all \( k \), from (2) it follows via (10) that the operators \( B \) and \( \tilde{B} \) defined in (9) are nuclear, and hence

\[
\sum_n s_n^{1/2} < \infty, \quad \sum_n s_n^{3/2} < \infty.
\]

Then, according to (8) and (9), kernels which induce the nuclear operators \( F \) and \( \tilde{F} \) can be represented by the series

\[
\sum_n s_n^{3/2} U \infty B^* q_n(s) U \infty B \tilde{p}_n(t), \quad \sum_n s_n^{3/2} U \infty \tilde{B}^* q_n(s) U \infty \tilde{B} \tilde{p}_n(t)
\]

convergent almost everywhere in \( \mathbb{R}^2 \). The functions used in these bilinear expansions can be written as the series convergent in \( L_2 \):

\[
U \infty B p_k = \sum_n \langle p_k, B^* f_n \rangle_H u_n, \quad U \infty B^* q_k = \sum_n \langle q_k, B f_n \rangle_H u_n,
\]

\[
U \infty \tilde{B} \tilde{p}_k = \sum_n \langle \tilde{p}_k, \tilde{B}^* f_n \rangle_H u_n, \quad U \infty \tilde{B}^* \tilde{q}_k = \sum_n \langle \tilde{q}_k, \tilde{B} f_n \rangle_H u_n \quad (k \in \mathbb{N}).
\]

Show that, for any fixed \( i \), the functions \( [U \infty B p_k]^{(i)}, [U \infty B^* q_k]^{(i)}, [U \infty \tilde{B} \tilde{p}_k]^{(i)}, [U \infty \tilde{B}^* \tilde{q}_k]^{(i)} \) \((k \in \mathbb{N})\) make sense, are all in \( C(\mathbb{R}, \mathbb{C}) \), and their \( C(\mathbb{R}, \mathbb{C}) \)-norms are bounded independent of \( k \). Indeed, all the series

\[
\sum_n \langle p_k, B^* f_n \rangle_H [u_n]^{(i)}(s), \quad \sum_n \langle q_k, B f_n \rangle_H [u_n]^{(i)}(s),
\]

\[
\sum_n \langle \tilde{p}_k, \tilde{B}^* f_n \rangle_H [u_n]^{(i)}(s), \quad \sum_n \langle \tilde{q}_k, \tilde{B} f_n \rangle_H [u_n]^{(i)}(s) \quad (k \in \mathbb{N})
\]

are dominated by one series

\[
\sum_n \left( \| B^* f_n \|_H + \| B f_n \|_H + \| \tilde{B}^* f_n \|_H + \| \tilde{B} f_n \|_H \right) [u_n]^{(i)}(s).
\]
This series converges uniformly in \( \mathbb{R} \), since it consists of two uniformly convergent in \( \mathbb{R} \) subseries (see (18), (19))

\[
\sum_k \left( \| B^* x_k \|_H + \| B x_k \|_H + \| \tilde{B}^* x_k \|_H + \| \tilde{B} x_k \|_H \right) \left[ |g_k|^{(i)}(s) \right],
\]

\[
\sum_k \left( \| B^* y_k \|_H + \| B y_k \|_H + \| \tilde{B}^* y_k \|_H + \| \tilde{B} y_k \|_H \right) \left[ |h_k|^{(i)}(s) \right],
\]

which are dominated by the following convergent series

\[
\sum_k d(x_k) G_{k,i} + \sum_k 2 \left( \| B \| + \| \tilde{B} \| \right) H_{k,i},
\]

respectively (see (5), (2), (7), (10), (13), (12)). Thus, for functions \( F, \tilde{F} : \mathbb{R}^2 \to \mathbb{C}, \bar{f}, \tilde{f} : \mathbb{R} \to L_2 \), defined by

\[
F(s,t) = \sum_n \frac{1}{n!} \left[ U_\infty B^* q_n \right]^{(i)}(s) U_\infty B p_n(t),
\]

\[
\tilde{F}(s,t) = \sum_n \frac{1}{n!} \left[ U_\infty \tilde{B}^* \tilde{q}_n \right]^{(i)}(s) U_\infty \tilde{B} p_n(t),
\]

\[
f(s) = \tilde{F}(s,\cdot) = \sum_n \frac{1}{n!} \left[ U_\infty B^* q_n \right]^{(i)}(s) U_\infty B p_n(t),
\]

\[
\tilde{f}(s) = \tilde{F}(s,\cdot) = \sum_n \frac{1}{n!} \left[ U_\infty \tilde{B}^* \tilde{q}_n \right]^{(i)}(s) U_\infty \tilde{B} p_n(t),
\]

one can write, for all non-negative integers \( i, j \) and all \( s, t \in \mathbb{R} \),

\[
\frac{\partial^{i+j} F}{\partial s^i \partial t^j}(s,t) = \sum_n \frac{1}{n!} \left[ U_\infty B^* q_n \right]^{(i)}(s) U_\infty B p_n(t),
\]

\[
\frac{\partial^{i+j} \tilde{F}}{\partial s^i \partial t^j}(s,t) = \sum_n \frac{1}{n!} \left[ U_\infty \tilde{B}^* \tilde{q}_n \right]^{(i)}(s) U_\infty \tilde{B} p_n(t),
\]

\[
\frac{d^j f}{ds^i}(s) = \sum_n \frac{1}{n!} \left[ U_\infty B^* q_n \right]^{(i)}(s) U_\infty B p_n(t),
\]

\[
\frac{d^j \tilde{f}}{ds^i}(s) = \sum_n \frac{1}{n!} \left[ U_\infty \tilde{B}^* \tilde{q}_n \right]^{(i)}(s) U_\infty \tilde{B} p_n(t),
\]

where the series converge in \( C(\mathbb{R}^2, \mathbb{C}), C(\mathbb{R}, L_2) \), respectively, because of (25). This implies that \( F, \tilde{F} \) are \( SK^\infty \)-kernels of \( F \) and \( \tilde{F} \), respectively.

Using (4) and (8), we obtain decompositions

\[
T f = P f + \bar{F} f \quad (f \in D_T), \quad T^* f = \bar{P} f + F f \quad (f \in D_T^*)
\]

all of whose terms are Carleman operators. Therefore \( T \) and \( T^* \) are also Carleman, and their kernels \( K \) and \( K^* \) can be defined by

\[
K(s,t) = P(s,t) + \bar{F}(s,t), \quad K^*(s,t) = \bar{P}(s,t) + F(s,t) \quad (s, t \in \mathbb{R}),
\]

respectively, where all the terms are \( SK^\infty \)-kernels. Moreover, we have \( K(s,t) = K^*(t,s), K(t,\cdot) = K^*(t,\cdot) \) for all \( s, t \in \mathbb{R} \) (cf. Corollary IV.2.17)). Thus, the operator \( T = U_\infty SU_\infty^{-1} \) is a bi-Carleman operator,
with the kernel $K$, which is a $K^\infty$-kernel. The proof of the theorem is complete.

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