Spin-superflow turbulence in spin-1 ferromagnetic spinor Bose-Einstein condensates

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Spin-superflow turbulence (SST) in spin-1 ferromagnetic spinor Bose-Einstein condensates is theoretically and numerically studied by using the spinor Gross-Pitaevskii (GP) equations. SST is turbulence in which the disturbed spin and superfluid velocity fields are coupled. Applying the Kolmogorov-type dimensional scaling analysis to the hydrodynamic equations of spin and velocity fields, we theoretically find that the $-5/3$ and $-7/3$ power laws appear in spectra of the superflow kinetic and the spin-dependent interaction energy, respectively. Our numerical calculation of the GP equations confirms SST with the coexistence of disturbed spin and superfluid velocity field with two power laws.

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Introduction. Turbulence is a strong nonequilibrium phenomenon, exhibiting unpredictable behavior of the velocity field, that can result from the multiple degrees of freedom and nonlinearity of fluid systems [1, 2]. This situation complicates our understanding of turbulence, making it one of the unresolved problems in modern physics.

Recently, turbulence in quantum fluids has been theoretically and numerically studied in one-component atomic Bose-Einstein condensates (BECs) [3–10]. In this system, the turbulence is composed of quantized vortices with quantized circulation, which is much different from a classical fluid. Thus, the turbulence in a quantum fluid is called quantum turbulence (QT) in contrast with the classical turbulence (CT). In QT, the element of turbulence is obvious, being the quantized vortex. In contrast, in CT, the element is ambiguous because the circulation can take some continuous value. Hence, QT seems to be simpler than CT, and QT is believed to be useful for the understanding of CT.

Atomic BECs have various characteristic features, one of which is a realization of multicomponent BECs [11–13]. In this system, there exists not only a velocity field but also a (quasi) spin field. Owing to the spin degrees of freedom, various topological excitations such as monopole, skyrmion, knot, domain wall, and vortex appear [12]. Therefore, in this system, one expects novel turbulence, in which both the velocity and spin fields are much disturbed and various topological excitations are generated.

Previously, we studied turbulence in a spin-1 spinor BEC, which is a typical multicomponent BEC. In this turbulence, the spin field is disturbed, so that we call it spin turbulence (ST). Generally, in a turbulence study, it is important to investigate statistical quantities because the velocity field at a fixed time and position cannot be reproduced experimentally [1, 2]. In our previous study [14], we focused on the spectrum of the spin-dependent interaction energy corresponding to the spin correlation, finding the characteristic $-7/3$ power law. This power law is derived by a Kolmogorov-type dimensional scaling analysis, which means that this law is characteristic of the turbulence.

In this turbulence, the velocity and spin fields interact with each other, so that a coupled turbulence with two fields can be realized, lending the possibility of showing a property not seen in conventional CT and QT. However, there are no existing studies focusing on the interaction and correlation between the spin and velocity fields. Our previous study considered only the spin field.

In this Letter, we focus on the spectrum of the superflow kinetic and the spin-dependent interaction energy in a spin-1 ferromagnetic spinor BEC. The spectrum of the kinetic energy is theoretically and numerically found to show a Kolmogorov spectrum through the interaction between the spin and velocity fields. The Kolmogorov spectrum refers to the $-5/3$ power law in the kinetic energy spectrum of CT, which is also confirmed in QT. This spectrum is considered to be related to the vortex dynamics, so that it is significant to confirm it for understanding the turbulence addressed here. Furthermore, when the $-5/3$ power law is sustained, the spectrum of the spin-dependent interaction energy exhibits a $-7/3$ power law. Therefore, we obtain the coupled turbulence with the disturbed spin and superfluid velocity fields sustaining the two power laws, calling it spin-superflow turbulence (SST).

Formulation. We consider a BEC of spin-1 bosonic atoms with mass $M$ at zero temperature without trapping and magnetic fields. This system is well described by the macroscopic wave functions $\psi_m$ ($m = 1, 0, -1$) with magnetic quantum number $m$, which obey the spinor Gross-Pitaevskii (GP) equations [12, 13] given by

$$i \hbar \frac{\partial}{\partial t} \psi_m = -\frac{\hbar^2}{2M} \nabla^2 \psi_m + c_0 \rho \psi_m + c_1 \mathbf{F} \cdot \hat{F}_{mn} \psi_n. \quad (1)$$

In this Letter, Roman indices that appear twice are to
be summed over $-1$, $0$, and $1$, and Greek indices are to be summed over $x$, $y$, and $z$. The parameters $c_0$ and $c_1$ are the coefficients of the spin-independent and spin-dependent interactions, which are expressed by $4\pi\hbar^2(a_0 + 2a_2)/3M$ and $4\pi\hbar^2(a_2 - a_0)/3M$, respectively. Here, $a_0$ and $a_2$ are the s-wave scattering lengths corresponding to the total spin-0 and spin-2 channels. The total density $\rho$ and the spin density vector $F_\mu (\mu = x, y, z)$ are given by $\rho = \psi_m^* \psi_m$ and $F_\mu = \psi_m^*(\tilde{F}_\mu)_{mn} \psi_n$, where $(\tilde{F}_\mu)_{mn}$ are the spin-1 matrices.

The sign of the coefficient $c_1$ drastically changes the spin dynamics. In this Letter, we consider the ferromagnetic interaction $c_1 < 0$ because the spin hydrodynamic equations in the case for $c_1 < 0$ is simpler than that for $c_1 > 0$. This situation is described in the Supplementary Material [17].

Here, we focus of the energy spectra for the superflow kinetic and the spin-dependent interaction energy. The kinetic energy of superfluid velocity $v$ per unit mass is given by

$$E_v = \frac{1}{2N} \int d\mathbf{r} A_v(\mathbf{r})^2,$$

where $A_v = \sqrt{\rho} v$ and $N$ is a total particle number. The superfluid velocity $v_\mu$ is given by

$$v_\mu = \frac{\hbar}{2M\rho} \psi_m^* \nabla_\mu \psi_m - \psi_m^* \nabla_\mu \psi_m^*.$$

By using the Fourier series $A(\mathbf{r}) = \sum_k A(k)e^{ik\cdot\mathbf{r}}$, we define the spectrum for the kinetic energy per unit mass:

$$E_v(k) = \frac{1}{2\rho_0\Delta k} \sum_{k<|k_1|<k+\Delta k} |\tilde{A}(k_1)|^2,$$

where $\Delta k$ and $\rho_0$ are given by $2\pi/L$ and $N/L^3$, respectively, with system size $L$ and spatial dimension $n_d$. Similarly, the spectrum of the spin-dependent interaction energy is defined by

$$E_s(k) = \frac{c_1}{2M\rho_0\Delta k} \sum_{k<|k_1|<k+\Delta k} |\tilde{F}(k_1)|^2,$$

with $\tilde{F}(k) = \int F(\mathbf{r})e^{-ik\cdot\mathbf{r}} d\mathbf{r}/V$ and $V = L^3$.

Kolmogorov spectrum and spinor GP equations. We discuss the possibility for a Kolmogorov spectrum in SST with Eq. [13]. In Ref. [18], the hydrodynamic equations are derived from the spin-1 spinor GP equations. We apply the following three approximations to these hydrodynamic equations: (i) the total density is almost uniform ($\rho(t) \sim \rho_0$), (ii) the magnitude of velocity $v$ is much smaller than the density sound velocity $C_d = \sqrt{c_0\rho_0}/2M$, and (iii) the macroscopic wave functions are expressed by the fully magnetized state. Then, we obtain the equations

$$\frac{\partial}{\partial t} f_\mu \sim \frac{\hbar}{2M} \epsilon_{\mu\nu\lambda} \nabla \cdot [f_\nu (\nabla f_\lambda)],$$

$$\frac{\partial}{\partial t} v_\mu + v_\nu \nabla_\nu v_\mu \sim \frac{\hbar^2}{4M^2} \nabla_\nu \left\{ \left( \nabla_\mu f_\lambda \right) \left( \nabla_\nu f_\lambda \right) - f_\lambda \left( \nabla_\mu \nabla_\nu f_\lambda \right) \right\},$$

with $f_\mu = F_\mu/\rho$. In Eq. [7], the inertial term $v_\nu \nabla_\nu v_\mu$ is smaller than the other terms, but we retain this term for the following explanation. The detailed derivation of Eqs. [6] and [7] is described in the Supplementary Material [17].

We apply a Kolmogorov-type dimensional scaling analysis [19, 20] to Eqs. [6] and [7], obtaining the $-5/3$ power law. We consider the scale transformation $\rho \rightarrow \alpha \rho$ and $t \rightarrow \beta t$. Then, if $f_\mu$ and $v_\mu$ are transformed to $f_\mu \rightarrow \alpha^2 \beta^{-1} f_\mu$ and $v_\mu \rightarrow \alpha \beta^{-1} v_\mu$, Eqs. [6] and [7] are invariant. Thus, the velocity field satisfies $v_\mu \sim \Lambda_\nu r^{-1}$ with a nondimensional coefficient $\Lambda_\nu$. In spinor BECs, if the disturbance is not so strong, the total density of the condensate is approximately uniform because the condition $|c_0| \gg |c_1|$ is satisfied in the usual experiments. Then the spatial dependence of $A$ is almost same as that of $v$, which leads to $A_\mu \sim \Lambda_\nu r^{-1}$ with $\Lambda_\Lambda = \sqrt{\rho_0} \Lambda_\nu$. Thus, in SST, the spectrum of the superflow kinetic energy can be determined by the kinetic energy flux $\epsilon_v$ and the coefficient $\Lambda_\nu$, which, by using a Kolmogorov-type dimensional analysis, leads to

$$E_v(k) \sim \Lambda_v^{2/3} \epsilon_v^{1/3} k^{-5/3}.$$
The coefficient $\Lambda_s^{2/3}$ is nondimensional, which corresponds to the Kolmogorov constant in CT. Therefore, the spectrum of the kinetic energy of SST can obey the Kolmogorov spectrum. Applying a similar analysis to the spin field, we obtain the $-7/3$ power law given by
\[ \mathcal{E}_s(k) \sim \epsilon_s^{2/3} k^{-7/3} \]
with the spin-dependent interaction energy flux $\epsilon_s$. This was discussed in the previous study [14].

We note that this $-5/3$ power law in SST is different from that in CT, where the $-5/3$ power law is generated by the inertial term $v_v \nabla_v v_v$ in the Navier-Stokes equation. In SST, the spatial gradient of the spin vector in Eq. (7) leads to the $-5/3$ power law because, in Eq. (7), the inertial term is found to be smaller than the nonlinear spin term by the order estimation [21]. This suggests that the mechanism responsible for the $-5/3$ power law in SST should be different from that in CT.

In the following, we show numerical results to confirm these theoretical considerations.

**Numerical results.** We show our numerical results where SST is generated by the counterflow instability [14]. In this SST, the energy is injected into the initial state, so that this turbulence is the decaying turbulence. All our numerical results are generated in a two-dimensional system, where the system size is $256 \xi_\rho \times 256 \xi_\rho$, with $\xi_\rho = \hbar/\sqrt{2Mc_0\rho_0}$.

In our numerical method we introduce a phenomenological dissipation term into Eq. (13) for the following reasons. In turbulence, an energy cascade from low to high wave number is assumed, and the energy in the high-wave-number region is considered to dissipate [1, 2]. Unless this kind of dissipation takes place, energy accumulates in the high-wave-number region, which can break the power law in the spectrum. In the previous study for QT in a one-component BEC [22], when the system temperature was sufficiently lower than the BEC transition temperature, the dissipation was found to be dominant only for the higher wave number region ($k_p < k$), where $k_p$ is the wave number corresponding to the density coherence length $\xi_\rho$ of the one-component GP equation.

Thus, we introduce a similar phenomenological dissipation term into Eq. (13), but, in our calculation, the dissipation occurs only when the wave number is larger than $k_s$, which is obviously different from the previous study. Here, $k_s$ is the wave number corresponding to the spin coherence length $\xi_s = \hbar/\sqrt{2M/C_1\rho_0}$. The detailed explanation for this dissipation is described in the Supplementary Material [17].

Figures 1(a) and 1(b) show the distribution of the $x$ components of the velocity and spin fields in SST at $t/\tau = 700$ with $\tau = c_0\rho_0$. Through the counterflow instability, the $x$ components of two fields are disturbed. The $z$ components of the rotation for the two fields are shown in Figs. 1(c) and 1(d), where the rotations are also disturbed [23]. In a spin-1 spinor BEC, the superfluid velocity is related to the spin field through the Mermin-Ho relation, so that the vortical field is continuous, which is much different from the one-component BEC. Actually, as seen in Fig. 1(c), the vortical field $[\text{rot}\mathbf{v}]_z$ has a smooth spatial dependence.

The time development of the spectrum of the spin-dependent interaction and the superflow kinetic energy are shown in Fig. 2. In the early stage of the instability, these spectra have a peak corresponding to the most unstable wave number for the counterflow instability, as shown in Fig. 2(a). As time progresses, the energy is transferred from low to high wave number, as seen in

![Figure 2](https://example.com/fig2.png)

**FIG. 2:** (Color online) Time dependence of the spectrum of spin-dependent interaction (upper) and superflow kinetic (lower) energy at $t/\tau = (a) 200$, (b) 400, (c) 500, and (d) 700. The dotted lines in the upper and lower graphs are proportional to $k^{-7/3}$ and $k^{-5/3}$, respectively.
Fig. 2(b). After time $t/\tau = 500$, the spectra of the velocity and the spin-dependent interaction energy exhibit the $-5/3$ and $-7/3$ power laws in Figs. 2(c) and (d). In our calculation, the $-5/3$ power law applies in the range $500 < t/\tau < 700$. For $t/\tau > 800$, we confirm that the spectrum of the kinetic energy begins to deviate from the $-5/3$ power law; this is caused by the shortage of energy in the low-wave-number region and the energy accumulation near $k_s$.

We now consider what structure of velocity field leads to the Kolmogorov $-5/3$ power law in SST. In QT, the vortical velocity field seems to be important for the Kolmogorov spectrum. Then, to investigate the vortical flow of $\mathbf{A} = \sqrt{\rho} \mathbf{v}$ in SST, we decompose the vector $\mathbf{A}$ into incompressible $\mathbf{A}_i$ and compressible $\mathbf{A}_c$ parts [24]. The Helmholtz theorem leads to $\mathbf{A}_v = \mathbf{A}_i + \mathbf{A}_c$, where the relations $\text{div} \mathbf{A}_i = 0$ and $\text{rot} \mathbf{A}_c = 0$ are satisfied. Thus, the superflow kinetic energy is expressed by $E_v = E_i + E_c$ with $E_\alpha = \int |\mathbf{A}_\alpha|^2 dr / 2N$ $(\alpha = v, i, c, v)$. Using the Fourier series $\mathbf{A}_\alpha(r) = \sum_k \mathbf{\hat{A}}_\alpha(k)e^{ik \cdot r}$ $(\alpha = v, i, c, v)$, we can define the spectra for each kinetic energy per unit mass as

$$E_\alpha(k) = \frac{1}{2\rho_0 \Delta k} \sum_{k-|k_1|<k+\Delta k} |\mathbf{\hat{A}}_\alpha(k_1)|^2. \quad (10)$$

We calculate the time dependence of $E_\alpha$ in Fig. 3(a); one can see that the incompressible superflow kinetic energy is much larger than the compressible one. This can be caused by (i) the condition $|c_v/c_i| \gg 1$, under which the total density is hard to disturb, and (ii) the dissipation, which prevents the total density modulation from accumulating. Figure 3(b) shows the spectrum $E_c$ of the compressible kinetic energy at $t/\tau = 700$, which deviates from the $-5/3$ power law in comparison with $E_v$ in Fig. 2(d). Therefore, the vortical structure of $\mathbf{A}$ is significant for the Kolmogorov spectrum in SST, which is similar to the situation of QT.

Finally, we numerically confirm that the $-5/3$ power law in SST is different from that in CT and QT. In our theoretical consideration for the Kolmogorov spectrum, we point out that the $-5/3$ power law in CT and QT is considered to be generated by the inertial term $v_\nu \nabla_\nu v_\mu$, but, in SST, this power law originates from the nonlinear spin term of Eq. 3. To confirm this, we numerically calculate the following quantities:

$$A_\mu = \frac{1}{V} \int |v_\nu \nabla_\nu v_\mu| dr, \quad (11)$$

$$B_\mu = \frac{1}{V} \int \left| \frac{\hbar^2}{4M^2} \nabla_\nu \left\{ (\nabla_\lambda f_\lambda)(\nabla_\nu f_\lambda) - f_\lambda (\nabla_\lambda \nabla_\nu f_\lambda) \right\} \right| dr. \quad (12)$$

Figure 4 shows the time dependence of $A_x/B_x$ and $A_y/B_y$; one sees that the spatial average of the nonlinear spin term $B_\mu$ is about 20 times larger than that of the inertial term $A_\mu$ for $t/\tau > 500$. Thus we can consider the Kolmogorov spectrum in SST to be generated by the nonlinear spin term.

**Conclusion.** We theoretically and numerically studied SST in a spin-1 ferromagnetic spinor BEC at zero temperature by using the spin-1 spinor GP equations, finding that both the $-5/3$ and $-7/3$ power laws appear in the spectrum of the superflow kinetic and the spin-dependent interaction energy. First, we discussed the possibility for the Kolmogorov spectrum in SST, pointing out that this spectrum can be generated by the nonlinear spin term. Second, we showed the numerical results, whereby SST in the two-dimensional system was obtained by the counterflow instability. Our numerical results indicated that both the $-5/3$ and $-7/3$ power laws appeared. Furthermore, we estimated the magnitude of the inertial term and the nonlinear spin term of Eq. (7), numerically confirming that the Kolmogorov spectrum in SST can be generated by the latter term.

As future work, we plan to consider the relation between the power laws in SST and the topological excitations. If the system possesses many topological excitations, the properties of SST are expected to be influenced by them because they have characteristic spin and velocity structures. This direction of research connects the study of turbulence in multicomponent BECs with other research areas related to topological excitations, such as nonequilibrium physics, quantum field theory, and cosmology [25].
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I. DERIVATION OF EQUATIONS (6) AND (7) IN THE MANUSCRIPT

In this section, we derive Eqs. (6) and (7) in our manuscript by using the spin-1 spinor GP equations given by

\[ i\hbar \frac{\partial}{\partial t} \psi_m = \left( -\frac{\hbar^2}{2M} \nabla^2 - \mu \right) \psi_m + c_0 \rho \psi_m^* + c_1 F \cdot \hat{F}_m \psi_m. \]  

(13)

Throughout the discussion, Roman indices appearing twice are summed over \(-1, 0, 1\), and Greek indices are summed over \(x, y, z\). The total density \(\rho\) and the spin density vector \(F_\mu\) \((\mu = x, y, z)\) are given by \(\rho = \psi_m^* \psi_m\) and \(F_\mu = \psi_m^* (\hat{F}_\mu)_{mn} \psi_n\), where \((\hat{F}_\mu)_{mn}\) are the spin-1 matrices. The parameters \(M, c_0\), and \(c_1\) are the mass of the particle and the coefficients of the spin-independent and spin-dependent interactions.

In Sec. I. A, we introduce the hydrodynamic equations equivalent to Eq. (13), which are composed of the superfluid velocity, the spin vector, and the nematic tensor \[\hat{F}\].\] These equations are valid for an arbitrary spin state. The hydrodynamic equations for the ferromagnetic (FM) state are shown in Sec. I. B; these are derived by the relation between the spin vector and the nematic tensor being valid in the FM state. Finally, in Sec. I. C, we obtain Eqs. (6) and (7) used for the derivation of the Kolmogorov spectrum.

A. Hydrodynamic equations equivalent to the spin-1 spinor GP equations

This section describes the hydrodynamic equations equivalent to the spin-1 spinor GP equations, which is derived by Yukawa and Ueda \[\hat{F}\].\]

We introduce the spin vector and nematic tensor defined by

\[ f_\mu = \frac{1}{\rho} \psi_m^* (\hat{F}_\mu)_{mn} \psi_n, \]  

(14)

\[ n_{\mu\nu} = \frac{1}{\rho} \psi_m^* (\hat{N}_{\mu\nu})_{mn} \psi_n. \]  

(15)

with

\[ (\hat{N}_{\mu\nu})_{mn} = \frac{1}{2} [ (\hat{F}_\mu)_{mi} (\hat{F}_\nu)_{ln} + (\hat{F}_\nu)_{mi} (\hat{F}_\mu)_{ln} ] - (\hat{F}_\mu \hat{F}_\nu)_{mn}. \]  

(16)

The continuity equations for the spin vector and the nematic tensor are derived from Eq. (13). The continuity equation for the spin vector is

\[ \frac{\partial}{\partial \tau} f_\mu + \nabla \cdot \rho v_\mu = 0, \]  

(17)

where the spin current is defined by

\[ v_\mu = f_\mu - \frac{\hbar}{M} \epsilon_{\mu\lambda\nu} \left[ \frac{1}{4} f_\nu (\nabla f_\lambda) + n_{\nu\eta} (\nabla n_{\lambda\eta}) \right]. \]  

(18)

Here,

\[ v = \frac{\hbar}{2Mr} [ \psi_m^* (\nabla \psi_m) - (\nabla \psi_m^*) \psi_m ] \]  

(19)

is the superfluid velocity. Similarly, the continuity equation for the nematic tensor is given by

\[ \frac{\partial}{\partial \tau} n_{\mu\nu} + \nabla \cdot \rho v_{\mu\nu} = \frac{c_1 \rho^2}{\hbar} \left( \epsilon_{\mu\lambda\eta} f_\lambda (\nabla n_{\nu\eta}) - (\nabla f_\lambda) n_{\nu\eta} \right) + \epsilon_{\nu\lambda\eta} f_\lambda (\nabla n_{\mu\eta}) - (\nabla f_\lambda) n_{\mu\eta}. \]  

(20)

The equation for the superfluid velocity is also derived by Eq. (13), which is given by

\[ \frac{\partial}{\partial \tau} v_i + v_j \nabla_j v_i - \frac{\hbar^2}{2M^2} \nabla^2 \sqrt{\rho} \frac{\nabla^2 \sqrt{\rho}}{\rho} \]  

\[ + \frac{\hbar^2}{4M^2 \rho} \nabla_j \rho \left\{ \frac{1}{2} \left[ (\nabla_i f_\mu) (\nabla_j f_\mu) - f_\mu (\nabla_i \nabla_j f_\mu) \right] \right\} + \left[ (\nabla_i n_{\mu\nu}) (\nabla_j n_{\mu\nu}) - n_{\mu\nu} (\nabla_i \nabla_j n_{\mu\nu}) \right] \]  

\[ = c_0 (\nabla_i \rho) + c_1 f_\mu (\nabla_i \rho f_\mu). \]  

(22)
B. Hydrodynamic equations for the FM state

We assume that the system is in the FM state, where the BEC is fully magnetized. Then, the relation between the spin vector and nematic tensor is given by

\[ n_{\mu\nu} = \frac{\delta_{\mu\nu} + f_\mu f_\nu}{2}. \]  

(23)

Thus, by eliminating the nematic tensor, Eqs. (18) and (22) become

\[ v_\mu = f_\mu v - \frac{\hbar}{2M} \epsilon_{\mu\nu\lambda} f_\nu (\nabla f_\lambda), \]

(24)

\[ \frac{\partial}{\partial t} v_i + v_j \nabla_j v_i - \frac{\hbar^2}{2M^2} \nabla_i \nabla_j \sqrt{\rho} \]

\[ + \frac{\hbar^2}{4M^2 \rho} \nabla_j \rho \left\{ (\nabla_i f_\mu)(\nabla_j f_\mu) - f_\mu (\nabla_i \nabla_j f_\mu) \right\} = c_0 (\nabla_i \rho) + c_1 f_i (\nabla_i \rho f_\mu). \]

(25)

Hydrodynamic equations of this type for the FM state were discussed by several authors [2–4].

C. Hydrodynamic equations for SST

We apply the following two approximations to Eqs. [17], [24], and [25]: (i) The total density is almost uniform (ρ(t) \sim ρ_0), and (ii) the magnitude of the velocity v is much smaller than the density sound velocity C_d = \sqrt{\rho_0 \rho}/2M, where ρ_0 is defined by N/L^n and total particle number N, system size L, and spatial dimension n. The first approximation is valid if the ratio |c_0/c_1| is much larger than unity. For the second approximation, generally, the superfluid velocity may be larger than C_d inside the vortex core, the size of which is on the order of the spin coherence length ξ = h/\sqrt{2M|c_1|\rho_0} in a spin-1 spinor BEC. Thus, we expect that a velocity larger than C_d has little effect on the behavior of turbulence on the large scale (k < k_s). At present, we are interested in the large-scale property of the turbulence, so that we can neglect the velocity term. Therefore, from Eqs. [17], [24], and [25], we obtain the hydrodynamic equations (6) and (7) for the derivation of the Kolmogorov spectrum:

\[ \frac{\partial}{\partial t} f_\mu + \nabla \cdot v_\mu \sim 0, \]

(26)

\[ v_\mu \sim -\frac{\hbar}{2M} \epsilon_{\mu\nu\lambda} f_\nu (\nabla f_\lambda), \]

(27)

\[ \frac{\partial}{\partial t} v_i \sim -\frac{\hbar^2}{4M^2} \nabla_j \left\{ (\nabla_i f_\mu)(\nabla_j f_\mu) - f_\mu (\nabla_i \nabla_j f_\mu) \right\}. \]

(28)

In our numerical calculation, the time development of Eq. (28) is calculated by using the fourth-order Runge-Kutta method. Equations (13) have some conserved quantities such as total energy E_{tot} and total particle number N_{tot}, which are accurately conserved in our numerical calculation.

II. NUMERICAL METHOD

A. Spinor GP equations without dissipation

To numerically calculate Eq. (13), we use the pseudospectral method. In this method, we calculate the wave functions \hat{\psi}_m(k) in wave number space defined by \mathcal{F}[\psi_m(r)] with the Fourier transformation \mathcal{F}[\cdot] = \int \cdot e^{i \mathbf{k} \cdot \mathbf{r}} d\mathbf{r} / V and V = L^n. The spin-1 spinor GP equations in wave number space are

\[ i\frac{\partial}{\partial t} \hat{\psi}_\mu(k) = \left( \frac{\hbar^2 k^2}{2M} - \mu \right) \hat{\psi}_\mu(k) + h_m(k), \]

(29)

\[ h_m(k) = \mathcal{F}[c_0 \rho \hat{\psi}_m + c_1 \mathbf{F} \cdot \hat{\mathbf{F}} \mathbf{n} \psi_n]. \]

(30)

In our numerical calculation, the time development of Eq. (29) is calculated by using the fourth-order Runge-Kutta method. Equations (13) have some conserved quantities such as total energy E_{tot} and total particle number N_{tot}, which are accurately conserved in our numerical calculation.

As described in our manuscript, in turbulence, the energy cascade from low to high wave number is considered, and the high-wave-number modes are assumed to dissipate. However, in Eq. (13), the total energy is conserved, which means that the high-wave-number modes do not dissipate. Thus, in the spectrum of the superflow kinetic energy, energy accumulates in the high-wave-number region, which can disturb the Kolmogorov -5/3 power law. Therefore, in the following, we discuss the dissipation mechanism in QT and SST.

B. Dissipation in QT

In QT of a one-component BEC, the energy cascade can be classified into three regions: (i) the classical region (k < k_t), (ii) the quantum region (k_t < k < k_p), and (iii) the dissipation region (k \sim k_p) [3]. Here, k_t and k_p are the wave numbers corresponding to the mean intervortex...
distance $l$ and the density coherence length $\xi_\rho$. In region (i), the energy cascade is dominated by the Richardson cascade of the quantized vortices, in which vortex reconnection transfers energy from low to high wave numbers. In region (ii), nonlinear interaction of Kelvin waves transfers the energy through a Kelvin wave cascade. In region (iii), Kelvin waves with a wavelength smaller than $\xi_\rho$ change to elementary excitations. Through all three regions, the energy transferred from the low-wave-number region dissipates.

Based on this description of the energy cascade in QT, in the previous study of QT in one-component Bose-Einstein condensates a phenomenological small-scale dissipation was added to the one-component GP equation [6], which is given by

$$\left(i - \gamma(k)\right)\hbar \frac{\partial}{\partial t} \tilde{\psi}(k) = \frac{\hbar^2 k^2}{2M} \tilde{\psi}(k) - \mu(t) \tilde{\psi}(k) + p(k),$$

where the interaction coefficient and the macroscopic wave function in the wave number space for the one-component BEC are denoted by $g$ and $\psi$, respectively. The function $\gamma(k)$ is defined by $\gamma_0 \theta(k - k_s)$ with the step function $\theta$, which dissipates the energy in the high-wave-number region larger than $k_s$. Also, this dissipation reduces the particle number, so that, in the study of Ref. [6], the chemical potential is adjusted to conserve it. Thus the chemical potential $\mu(t)$ has a time dependence.

C. Dissipation in the turbulence of spinor BECs

In a spin-1 spinor BEC, there are two characteristic lengths: the density length $\xi_\rho = \hbar/\sqrt{2M_0 c_0 \rho_0}$ and the spin coherence length $\xi_s = \hbar/\sqrt{2M_1 c_1 |\rho_0|}$. In the usual experiments, $|c_0|/c_1$ is larger than unity, which leads to the condition $\xi_\rho < \xi_s$. The sizes of the spin structure such as the spin domain wall and the spin vortex is on the order of $\xi_s$. With reference to QT, we expect that the energy dissipation occurs for wave numbers greater than the wave number $k_s$ corresponding to the spin coherence length $\xi_s$.

D. Spinor GP equations with small-scale dissipation

Based on the above considerations, we add a phenomenological dissipation to Eq. (29). The spin-1 spinor GP equations with small-scale dissipation are given by

$$\left(i - \gamma(k)\right)\hbar \frac{\partial}{\partial t} \tilde{\psi}_m(k) = \frac{\hbar^2 k^2}{2M} \tilde{\psi}_m(k) - \mu \tilde{\psi}_m(k) + h_m(k),$$

where $\gamma(k)$ is defined by $\gamma_0 \theta(k - k_s)$. In the previous study, the chemical potential has a time dependence in order to conserve the total particle number. However, in our calculation, we do not adjust the chemical potential, because the total particle number hardly decreases in SST.

E. Parameters and the initial state

We show the parameters and the initial state in our numerical calculation. The system size $L \times L$ is $256\xi_\rho \times 256\xi_\rho$. The initial state is a counterflow state [7] defined by

$$\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \sqrt{\frac{\rho_0}{2}} \begin{pmatrix} \exp(i \frac{MV_k}{2\hbar} x) \\ \exp(-i \frac{MV_k}{2\hbar} x) \end{pmatrix},$$

with a relative velocity $V_R$. For the parameters, we take $\gamma_0 = 0.03$, $V_R/C_d = 24\pi \xi_\rho/L \sim 0.294$, and $|c_0|/c_1 = 20$ with a positive $c_0$. In the initial state, we add a small white noise contribution to cause the counterflow instability.

We now comment on the $V_R$ dependence of the spectrum of the superflow kinetic energy. Our calculation indicates that, under the condition of large relative velocity, the period sustaining the Kolmogorov spectrum becomes short. When the relative velocity is large, the total density is easy to modulate, which leads to the growth of the compressible velocity field. Actually, we numerically confirm the increase of the compressible kinetic energy. Therefore, we consider that this increase shortens the period sustaining the Kolmogorov spectrum.

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