INTEGER KNAPSACKS:
AVERAGE BEHAVIOR OF THE FRObenius Numbers

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Abstract. Given a primitive integer vector \( a \in \mathbb{Z}^N_{>0} \), the largest integer \( b \) such that the knapsack polytope \( P = \{ x \in \mathbb{R}^N_{\geq 0} : \langle a, x \rangle = b \} \) contains no integer point is called the Frobenius number of \( a \). We show that the asymptotic growth of the Frobenius number in average is significantly slower than the growth of the maximum Frobenius number. More precisely, we prove that it does not essentially exceed \( \|a\|_{\infty}^{1+1/(N-1)} \), where \( \| \cdot \|_{\infty} \) denotes the maximum norm.

1. Introduction and statement of results

For a positive integral vector \( a = (a_1, a_2, \ldots, a_N) \in \mathbb{Z}^N_{>0} \) with \( \gcd(a) = \gcd(a_1, a_2, \ldots, a_N) = 1 \) and a positive integer \( b \) the knapsack polytope \( P = P(a, b) \) is defined as

\[
P = \{ x \in \mathbb{R}^N_{\geq 0} : \langle a, x \rangle = b \},
\]

where \( \langle \cdot , \cdot \rangle \) denotes the inner product. The integer programming feasibility problem:

\[
(1.1) \quad \text{Does the polytope } P \text{ contain an integer vector?}
\]
is called the integer knapsack problem and is well-known to be NP-complete (cf., e.g., Karp [19]).

Given the input vector \( a \in \mathbb{Z}^N \), the largest integral value \( b \) such that the instance of \( (1.1) \) is infeasible is called the Frobenius number of \( a \), denoted by \( g_N = g_N(a) \). The Frobenius number plays an important role in the analysis of integer programming algorithms (see, e.g., Aardal and Lenstra [1], Hansen and Ryan [15], and Lee, Onn and Weismantel [20]) and, vice versa, integer programming algorithms are known to be an effective tool for computing the Frobenius number (see Beihoffer et al [8]). The general problem of finding \( g_N \) has been traditionally referred to as the Frobenius problem. There is a rich literature on the various aspects of this question. For an impressive list of references see Ramirez Alfonsin [22].

Computing \( g_N \) when \( N \) is not fixed is an NP-hard problem (Ramirez Alfonsin [21]). For any fixed \( N \) the Frobenius number \( g_N \) can be found in polynomial time by a sophisticated algorithm due to Kannan [17]. One should mention here that, due to its complexity, Kannan’s algorithm has apparently never been implemented.
From the viewpoint of analysis of integer programming algorithms, upper bounds on the Frobenius number $g_N(a)$ in terms of the input vector $a$ are of primary interest. Known results include classical upper bounds by Erdős and Graham [11]

$$g_N \leq 2a_N\left\lfloor\frac{a_1}{N}\right\rfloor - a_1,$$

by Selmer [26]

$$g_N \leq 2a_{N-1}\left\lfloor\frac{a_N}{N}\right\rfloor - a_N,$$

by Vitek [27]

$$g_N \leq \left\lfloor\frac{(a_2 - 1)(a_N - 2)}{2}\right\rfloor - 1$$

and by many other authors, as well as more recent results by Beck, Diaz, and Robins [6]

$$g_N \leq \frac{1}{2}\left(\sqrt{a_1a_2a_3(a_1 + a_2 + a_3)} - a_1 - a_2 - a_3\right),$$

(assuming in (1.2)–(1.5) $a_1 \leq a_2 \leq \ldots \leq a_N$) and by Fukshansky and Robins [12], who produced an upper bound in terms of the covering radius of a lattice related to the integers $a_1, \ldots, a_N$.

In the most interesting case $a_i \sim a_j$, $i, j = 1, \ldots, N$, all known upper bounds are of order $||a||^2_\infty$, where $||\cdot||_\infty$ denotes the maximum norm. This is especially transparent in the case of the results (1.2)–(1.5). For $N = 3$ Beck and Zacks [7] conjectured that, except of a special family of input vectors, the Frobenius number does not exceed $C(a_1a_2a_3)^\alpha$ with absolute constants $C$ and $\alpha < 2/3$. This conjecture has been disproved by Schlage-Puchta [23]. As a special case, the latter result implies that, roughly speaking, cutting off special families of input vectors cannot make the order of upper bounds smaller than $||a||^2_\infty$. In general, one can show that the quantity $||a||^2_\infty$ plays a role of a limit for estimating the Frobenius number $g_N$ from above.

The next natural and important question is to derive a good upper estimate for the Frobenius number of a “typical” input vector $a$. This problem appears to be hard, and to the best of our knowledge it has firstly been systematically investigated by V. I. Arnold, see, e.g., [3, 4, 5]. In particular, he conjectured that $g_N(a)$ grows like $T^{1+1/(N-1)}$ for a “typical” $a$ of 1-norm $T$. Recently, Bourgain and Sinai [9] proved a statement in the spirit of that conjecture, which says, roughly speaking, that

$$\text{Prob}_{\infty,\alpha}\left(g_N(a)/T^{1+1/(N-1)} \geq D\right) \leq \epsilon(D),$$

where $\text{Prob}_{\infty,\alpha}(\cdot)$ is meant with respect to the uniform distribution among all points in the set

$$G_{\infty,\alpha}(N, T) = \{a \in \mathbb{Z}_{>0}^N : \gcd(a) = 1, ||a||_\infty \leq T, a_i > \alpha T, 1 \leq i \leq N\},$$
where $0 < \alpha < 1$ is a fixed number. The number $\epsilon(D)$ does not depend on $T$ and tends to zero as $D$ approaches infinity. Our main result below also implies that \([1, 6]\) (see Corollary \([1, 1]\)) holds for the more general and natural case $\alpha = 0$.

In order to state our main theorem, we have to fix some further notation. Put $\Sigma(a) = \sum_{i=1}^{N} a_i$ and $\Pi(a) = (\prod_{i=1}^{N} a_i)^{1/(N-1)}$. Theorem 2.5 of Kannan \([17]\) indicates that, from the geometric viewpoint, it is more convenient to study the quantity

$$f_N(a) = g_N(a) + \Sigma(a).$$

Clearly, $f_N = f_N(a)$ is the largest integer which is not a positive integer combination of $a_1, \ldots, a_N$. In this paper we study the asymptotic behavior of the ratio $f_N(a)/s(a)$ with

$$s(a) = \frac{\sum_{i=1}^{N} ||a[i]||a_i}{||a||^{1-1/(N-1)}},$$

where $a[i] = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$ and $|| \cdot ||$ denotes the Euclidean norm. The geometric meaning of the normalization $s(a)^{-1}$ is explained in Section \([2]\). With these notation let

$$G(N, T) = \{ a \in \mathbb{Z}_N^N : \text{gcd}(a) = 1, ||a|| \leq T \},$$

and let $\text{Prob}_{N,T}(\cdot)$ be the uniform probability distribution on $G(N, T)$. The main result of the paper is

**Theorem 1.1.** For $N \geq 3$ the inequality

$$\text{Prob}_{N,T}(f_N(a)/s(a) > t) \ll_N t^{-2}$$

holds. Here $\ll_N$ denotes the Vinogradov symbol with the constant depending on $N$ only.

In terms of $g_N$, $T$ and $\text{Prob}_{\infty,0}(\cdot)$ we obtain the following corollary.

**Corollary 1.1.** For $N \geq 3$ the inequality

$$\text{Prob}_{\infty,0}(g_N(a)/T^{1+1/(N-1)} > t) \ll_N t^{-2}$$

holds.

Beihoffer et al \([8]\) performed extensive computations which lead to a counterexample that $\Pi(a)$ is not a good predictor for the average value of $f_N(a)$. Indeed, they conjectured that the average value of $f_N(a)/\Pi(a)$ is asymptotically equal to a small constant. An analogous conjecture for $N = 3$ was proposed in Davison \([10]\).

One should remark here that $\Pi(a)$ is essentially a lower bound for $f_N$. The main result of Aliev and Gruber \([2]\) states that the inhomogeneous minimum $\mu_0 = \mu_0(S_{N-1})$ of the standard simplex

$$S_{N-1} = \{ (x_1, \ldots, x_{N-1}) \in \mathbb{R}_{\geq 0}^{N-1} : \sum_{i=1}^{N-1} x_i \leq 1 \}$$

where $0 < \alpha < 1$ is a fixed number. The number $\epsilon(D)$ does not depend on $T$ and tends to zero as $D$ approaches infinity. Our main result below also implies that \([1, 6]\) (see Corollary \([1, 1]\)) holds for the more general and natural case $\alpha = 0$.

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$$S_{N-1} = \{ (x_1, \ldots, x_{N-1}) \in \mathbb{R}_{\geq 0}^{N-1} : \sum_{i=1}^{N-1} x_i \leq 1 \}$$
is a sharp lower bound for the ratio $f_N(a)/\Pi(a)$.

The next theorem answers a question similar to the conjecture of Beihoffer et al with respect to a different normalization of $f_N$.

**Theorem 1.2.** For $N \geq 3$ we have $$\sup_T \frac{\sum_{a \in G(N,T)} f_N(a)/s(a)}{\#G(N,T)} \ll_N 1.$$ 

Observe that for all $a$ we have $s(a) \ll_N \|a\|_{1/(N-1)}^{1+1/(N-1)}$. This implies the following result.

**Corollary 1.2.** For $N \geq 3$ we have

$$\sup_T \frac{\sum_{a \in G(N,T)} f_N(a)/\|a\|_{1/(N-1)}^{1+1/(N-1)}}{\#G(N,T)} \ll_N 1. \quad (1.7)$$

Obviously, the maximum norm $\|a\|_{\infty}$ in (1.7) can be replaced by any other norm. Moreover, applying arguments similar to the one given in the proof of Corollary 1.1 one can also replace the Euclidean norm in the definition of $G(N,T)$ by any other norm, which, for example, for the maximum norm leads to the set $G_{\infty,0}(N,T)$.

Corollary 1.2 says that the asymptotic growth of the Frobenius number in average is significantly slower than the growth of the maximum Frobenius number as $\|a\|$ tends to infinity. Moreover, perhaps surprisingly, the average Frobenius number, as $N \to \infty$, does not essentially exceed $\|a\|_{\infty}$.

The next result shows that the ratio $f_N(a)/s(a)$ is unbounded along any given “direction” $\alpha \in \mathbb{R}^N$, so that Theorem 1.2 is not straightforward.

**Theorem 1.3.** For any $\epsilon > 0$, $M > 0$ and for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, 1)$, $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{N-1} \leq 1$, there exists an integer vector $a = (a_1, a_2, \ldots, a_N)$ with $0 < a_1 < a_2 < \ldots < a_N$ and $\gcd(a) = 1$ such that

$$\|\alpha - \frac{1}{a_N} a\|_{\infty} < \epsilon \quad (1.8)$$

and

$$\frac{f_N(a)}{s(a)} > M. \quad (1.9)$$

The paper is organized as follows. In Section 2 we combine Kannan’s formula for $f_N(a)$ with Jarnik’s inequalities in order to reformulate the problem via Minkowski’s successive minima. Section 3 is devoted to Schmidt’s results on the distribution of sublattices of $\mathbb{Z}^N$ on which our work heavily relies. For the proof of Theorem 1.3 we need a density lemma which will be presented in Section 4. In the subsequent sections we give the proofs of our main results.
2. Frobenius number and lattices

Following the geometric approach developed in Kannan [17] and Kannan and Lovasz [18], we will make use of tools from the geometry of numbers.

By lattice we understand a discrete submodule \( L \) of a finite–dimensional Euclidean space. Recall that a family of sets in \( \mathbb{R}^{N-1} \) is a covering if their union equals \( \mathbb{R}^{N-1} \). Given a set \( S \) and a lattice \( L \), we say that \( L \) is a covering lattice for \( S \) if the family \( \{ S + l : l \in L \} \) is a covering. The inhomogeneous minimum of the set \( S \) with respect to the lattice \( L \) is the quantity

\[
\mu(S, L) = \inf\{ \sigma > 0 : L \text{ is a covering lattice of } \sigma S \}
\]

and the quantity

\[
\mu_0(S) = \inf\{ \mu(S, L) : \det L = 1 \}
\]

is called the (absolute) inhomogeneous minimum of \( S \). If \( S \) is bounded and has inner points, then \( \mu_0(S) \) does not vanish and is finite (see Gruber and Lekkerkerker [15], Chapter 3). The quantity \( \mu_0(S) \) is closely related to the, perhaps better known, covering constant \( \Gamma(S) \) of the set \( S \), where

\[
\Gamma(S) = \sup\{ \det(L) : L \text{ is a covering lattice of } S \}
\]

Indeed, by Gruber and Lekkerkerker [15, p. 230] we have

\[
\mu_0(S) = \Gamma(S) - \frac{1}{(N-1)}.
\]

Depending on the vector \( a \in \mathbb{Z}^N > 0 \) we define the following \( S_a \) and lattice \( L_a \) by

\[
S_a = \left\{ x \in \mathbb{R}^{N-1}_{\geq 0} : \sum_{i=1}^{N-1} a_i x_i \leq 1 \right\},
\]

\[
L_a = \left\{ x \in \mathbb{Z}^{N-1} : \sum_{i=1}^{N-1} a_i x_i \equiv 0 \mod a_N \right\}.
\]

Kannan [17, Theorem 2.5] proved that

\[
(2.1) \quad f_N(a) = \mu(S_a, L_a),
\]

which provides a starting point for geometric investigations of the Frobenius number. To this end we define the hyperplane lattice \( \Lambda_a(t) \) in \( \mathbb{R}^N \) as

\[
\Lambda_a(t) = \{ x \in \mathbb{Z}^N : \langle a, x \rangle = t \}.
\]

Let \( V_a(t) = \text{aff} \Lambda_a(t) \) and \( S_a(t) \) be the \((N-1)\)-dimensional simplex \( V_a(t) \cap \mathbb{R}^N_{\geq 0} \). For convenience we will also use the notation \( V_a = V_a(0) \) and \( \Lambda_a = \Lambda_a(0) \).

Furthermore, let \( \pi(\cdot) \) denote the orthogonal projection onto coordinate hyperplane corresponding to the variables \( x_1, \ldots, x_{N-1} \). Then clearly \( S_a = \pi(S_a(1)), L_a = \pi(\Lambda_a(0)) \) and, since inhomogeneous minima are independent with respect to regular affine transformations, we can write (2.1) as

\[
(2.2) \quad f_N(a) = \mu(S_a(1), \Lambda_a(1)).
\]

Here and through the rest of the paper we consider \( V_a(t) \) as a usual \((N-1)\)-dimensional Euclidean space.
By a standard calculation (see, e.g., Fukshansky and Robins [12 (19)]) the inradius of the simplex $S_{a}(t)$ is given by

$$r_{a}(t) = \frac{t||a||}{\sum_{i=1}^{N}||a[i]||a[i]}.$$

Denoting by $B_{r}^{N}$ the ball of radius $r$ in $\mathbb{R}^{N}$ we have by (2.2)

$$f_{N}(a) \leq \mu(B_{r_{a}(1)}^{N} \cap V_{a}, \Lambda_{a}).$$

Observe that $\mu(S, tL) = t\mu(S, L)$ and $\mu(tS, L) = t^{-1}\mu(S, L)$. Thus

$$f_{N}(a) \leq \frac{||a||^{1/(N-1)}}{r_{a}(1)}\mu(B_{1}^{N} \cap V_{a}, \Gamma_{a}),$$

where the lattice $\Gamma_{a} = ||a||^{-1/(N-1)}\Lambda_{a}$ has determinant 1. In order to estimate $\mu(B_{1}^{N} \cap V_{a}, \Gamma_{a})$ we need Minkowski’s successive minima, which for a $o$-symmetric convex set $K$ and a lattice $\Lambda$ defined by (see [13, pp. 375])

$$\lambda_{i}(K, \Lambda) = \inf\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i\}, \quad 1 \leq i \leq \dim \Lambda.$$

Let $\lambda_{i} = \lambda_{i}(B_{1}^{N} \cap V_{a}, \Gamma_{a})$ be the $i$-th successive minimum of the ball $B_{1}^{N} \cap V_{a}$ with respect to the lattice $\Gamma_{a}$.

By Jarnik’s inequalities (see, e.g., Gruber and Lekkerkerer [15, pp. 99]), we have

$$\frac{1}{2}\lambda_{N-1} \leq \mu(B_{1}^{N} \cap V_{a}, \Gamma_{a}) \leq \frac{N-1}{2}\lambda_{N-1}.$$

Thus, for a fixed dimension $N$ the inhomogeneous minimum is essentially equal to the last successive minimum. By (2.3) and the right–hand side of (2.4) we obtain the inequality

$$\frac{2f_{N}(a)}{(N-1)s(a)} = \frac{2r_{a}(1)}{||a||^{1/(N-1)}}f_{N}(a) \leq \lambda_{N-1}.$$

The latter expression explains the geometric meaning of the quantity $s(a)^{-1}$. This is the normalized radius of a ball inscribed into the simplex $S_{a}(1)$.

### 3. Distribution of sublattices of $\mathbb{Z}^{n}$

In this section we will recall several results due to W. Schmidt [25] on the distribution of integer lattices. Two lattices $L, L'$ are similar if there is a linear bijection $\phi : L \to L'$ such that for some fixed $c > 0$ we have $||\phi(x)|| = c||x||$. Let $\tilde{O}_{n}$ be the group of matrices $K = (k_{1}, \ldots, k_{n}) \in GL_{n}(\mathbb{R})$ whose columns $k_{1}, \ldots, k_{n}$ have $||k_{1}|| = \cdots = ||k_{n}|| \neq 0$ and inner products $\langle k_{i}, k_{j} \rangle = 0$ for $i \neq j$. When $X = (x_{1}, \ldots, x_{n}) \in GL_{n}(\mathbb{R})$, we may uniquely write the matrix $X$ in the form

$$X = KZ,$$
where $K \in \tilde{O}_n$ and

$$Z = \begin{pmatrix}
1 & x_{12} & \cdots & x_{1n} \\
0 & y_2 & \cdots & x_{2n} \\
\vdots \\
0 & 0 & \cdots & y_n
\end{pmatrix}$$

(3.2)

with $y_2, \ldots, y_n > 0$. The matrices $Z$ as in (3.2) form the generalized upper half-plane $H = \mathcal{H}_n$. For $Z \in \mathcal{H}$ and $M \in GL_n(\mathbb{R})$, we may write $ZM$ in the form (3.1), that is we uniquely have $ZM = KZ_M$ with $K \in \tilde{O}_n$ and $Z_M \in \mathcal{H}$. Thus $GL_n(\mathbb{R})$ acts on $\mathcal{H}$; to $M$ corresponds the map $Z \mapsto Z_M$. In particular, $GL_n(\mathbb{Z})$, as a subgroup of $GL_n(\mathbb{R})$, acts on $\mathcal{H}$. We will denote by $\mathcal{F}$ a fundamental domain for the action of $GL_n(\mathbb{Z})$ on $\mathcal{H}$. We will also write $\mu$ for the $GL_n(\mathbb{R})$ invariant measure on $\mathcal{H}$ with $\mu(\mathcal{F}) = 1$.

Suppose now that $1 < n \leq m$. There is a map (see p. 38 of Schmidt [25] for detail) from lattices of rank $n$ in $\mathbb{R}^m$ onto the set $\mathcal{H}/GL_n(\mathbb{Z})$ of orbits of $GL_n(\mathbb{Z})$ in $\mathcal{H}$. The lattices $L$, $L'$ are similar precisely if they have the same image in $\mathcal{H}/GL_n(\mathbb{Z})$, hence the same image in $\mathcal{F}$. Similarity classes of lattices are parametrized by the elements of a fundamental domain $\mathcal{F}$.

A subset $\mathcal{D} \subset \mathcal{H}$ is called lean if $\mathcal{D}$ is contained in some fundamental domain $\mathcal{F}$. For $a > 0$, $b > 0$, let $\mathcal{H}(a,b)$ consists of $Z \in \mathcal{H}$ (in the form (3.2)) with

$$y_{i+1} \geq ay_i, \quad 1 \leq i < n, \quad |x_{ij}| \leq by_i, \quad 1 \leq i < j \leq n.$$ 

Here we assume $y_1 = 1$.

Recall that the Frobenius number $g_N(a)$ is well-defined only for integer vectors $a = (a_1, a_2, \ldots, a_N)$ with $\gcd(a_1, a_2, \ldots, a_N) = 1$. The vectors $a$ with this property are called primitive. More generally, a lattice $L \subset \mathbb{Z}^m$ is primitive if $L = \text{span}_\mathbb{Z}(L) \cap \mathbb{Z}^m$. Clearly, there is one-to-one correspondence between primitive vectors $b \in \mathbb{Z}^n$ and the primitive $(n-1)$-dimensional sublattices $\Lambda_b$. Note also that $\det \Lambda_b = |b|$. Let $P(\mathcal{D},T)$, where $\mathcal{D}$ is lean, be the number of primitive lattices $L \subset \mathbb{Z}^m$ with similarity class in $\mathcal{D}$ and determinant $\leq T$.

**Theorem 3.1** (Schmidt [25, Theorem 2]). Suppose $1 < n < m$ and let $\mathcal{D} \subset \mathcal{H}(a,b)$ be lean and Jordan-measurable. Then, as $T \to \infty$,

$$P(\mathcal{D},T) \sim c_2(m,n)\mu(\mathcal{D})T^m$$

(3.3)

with

$$c_2(m,n) = \frac{1}{m} \binom{m}{n} \frac{V_{m-n+1} \cdots V_m}{V_1V_2 \cdots V_n} \cdot \frac{\zeta(2) \cdots \zeta(n)}{\zeta(m-n+1) \cdots \zeta(m)}.$$ 

Here $V_l$ is the volume of the unit ball in $\mathbb{R}^l$ and $\zeta(\cdot)$ is the Riemann zeta–function.

Thus, roughly speaking, the proportion of primitive lattices with similarity class in $\mathcal{D}$ is $\mu(\mathcal{D})$. 
Given a vector $u = (u_1, u_2, \ldots, u_n)$ with $u_i \geq 1$ ($1 \leq i < n$), the lattices $L$ with
\[
\lambda_i + 1 (B^n \cap \text{span}_R(L), L) \geq u_i
\]
form a set of similarity classes, which will be denoted by $D(u)$.

**Theorem 3.2** (Schmidt [25, Theorem 5 (i)]). The set $D(u)$ may be realized as a lean, Jordan–measurable subset of $H$. We have
\[
\mu(D(u)) \ll_m,n \prod_{i=1}^{n-1} u_i^{-i(n-i)}.
\]

Here $\ll_m,n$ denotes the Vinogradov symbol with the constant depending on $m$ and $n$ only.

**Remark 3.1.** Note that the condition $D(u) \subset H(a, b)$ of Theorem 3.1 is also satisfied for some constants $a = a(n)$ and $b = b(n)$. We refer the reader [25, p. 58] for further detail.

4. A Density Lemma

**Lemma 4.1.** Let $L$ be a lattice with basis $b_1, \ldots, b_{N-1}$, $b_i \in \mathbb{Q}^N$, $1 \leq i \leq N-1$, and let $\alpha = (\alpha_1, \ldots, \alpha_{N-1}, 1) \in \mathbb{Q}^N$ be a vector orthogonal to $L$. Then there exists a sequence $a(t) = (a_1(t), \ldots, a_{N-1}(t), a_N(t)) \in \mathbb{Z}^N$, $t = 1, 2, \ldots$, such that $\gcd(a(t)) = 1$ and the following properties hold:

(i) The lattice $\Lambda_{a(t)}$ has a basis $b_1(t), \ldots, b_{N-1}(t)$ with
\[
\frac{b_{ij}(t)}{d t} = b_{ij} + O \left( \frac{1}{t} \right), \quad (1 \leq i, j \leq N),
\]
where $d \in \mathbb{N}$ is such that $d b_{ij} \in \mathbb{Z}$, $1 \leq i \leq N-1$ and $1 \leq j \leq N$.

(ii) The last component of the vector $a(t)$ satisfies
\[
a_N(t) = \det(\pi(L)) d^{N-1} t^{N-1} + O(t^{N-2}).
\]

(iii) The sequence $(a_N(t))^{-1} a(t)$ converges to $\alpha$. Indeed,
\[
\frac{a_i(t)}{a_N(t)} = \alpha_i + O \left( \frac{1}{t} \right), 1 \leq i \leq N-1.
\]

The result is a modified version of Theorem 1.2 of Aliev and Gruber [2], but in order to keep the paper self-contained as much as possible we give a short proof here.

**Proof.** Let us consider the matrices
\[
B = \begin{pmatrix}
    b_{11} & b_{12} & \ldots & b_{1N-1} & b_{1N} \\
    b_{21} & b_{22} & \ldots & b_{2N-1} & b_{2N} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    b_{N-11} & b_{N-12} & \ldots & b_{N-1N-1} & b_{N-1N}
\end{pmatrix}
\]
and

\[ M = M(t, t_1, \ldots, t_{N-1}) \]

\[
= \begin{pmatrix}
  db_{11}t + t_1 & db_{12}t & \cdots & db_{1N-1}t & db_{1N}t \\
  db_{21}t & db_{22}t + t_2 & \cdots & db_{2N-1}t & db_{2N}t \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  db_{N-11}t & db_{N-12}t & \cdots & db_{N-1N-1}t + t_{N-1} & db_{N-1N}t
\end{pmatrix},
\]

where \( t, t_1, \ldots, t_{N-1} \) are variables.

Denote by \( M_i = M_i(t, t_1, \ldots, t_{N-1}) \) and \( B_i \) the minors obtained by omitting the \( i \)th column in \( M \) or in \( B \), respectively. Note that

\[
|B_N| = |\det(b_{ij})| = \det(\pi(L)), \quad \alpha_i = \frac{|B_i|}{|B_N|}, \quad \text{and}
\]

\[
M_i = d^{N-1} B_i t^{N-1} + \text{polynomials in } t \text{ of degree less than } N - 1.
\]

Following the proof of Theorem 2 in Schinzel \[24\] we also observe that \( M_1, \ldots, M_N \) have no non-constant common factor. By \[24\], Theorem 1] with \( m = 1, F = 1 \), and \( F_{1\nu} = \mu(t, t_1, \ldots, t_{N-1}), 1 \leq \nu \leq N \), there exist integers \( t_1^*, \ldots, t_{N-1}^* \) and an infinite arithmetic progression \( P \) such that for \( at + b \in P \)

\(
\gcd(M_1(at + b, t_1^*, \ldots, t_{N-1}^*), \ldots, M_N(at + b, t_1^*, \ldots, t_{N-1}^*)) = 1.
\)

For \( t = 1, 2, \ldots \) we set

\[
a(t) = (M_1(at + b, t_1^*, \ldots, t_{N-1}^*), \ldots, (-1)^{N-1} M_N(at + b, t_1^*, \ldots, t_{N-1}^*)).
\]

Then the basis \( b_1(t), \ldots, b_{N-1}(t) \) for \( L_{a(t)} \) satisfying the statement of Lemma \[4.1\] is given by the rows of the matrix \( M(t, t_1^*, \ldots, t_{N-1}^*) \). The properties \[4.1\] of minors \( M_i \), \( B_i \) imply the properties \[4.1\] of the sequence \( a(t) \).

\[
\square
\]

5. Proof of Theorem \[1.1\]

We consider the sequence of discrete random variables \( X_T : G(N, T) \to \mathbb{R}_{\geq 0} \) defined as

\[
X_T(a) = \frac{f_N(a)}{s(a)}.
\]

Recall that the cumulative distribution function (CDF) \( F_T \) of \( X_T \) is defined for \( t \in \mathbb{R}_{\geq 0} \) as

\[
F_T(t) = \text{Prob}_{N, T}(X_T \leq t).
\]

For a real number \( u \geq 1 \), let \( v_i(u) = (u_1, u_2, \ldots, u_{N-2}) \) be the vector with \( u_i = u \) and \( u_j = 1 \) for all \( j \neq i \). Define the set \( D(u) \) of similarity classes as (cf. Section 3)

\[
D(u) = \bigcup_{i=1}^{N-2} D(v_i(u)).
\]
By (3.4) the measure of this set satisfies
\[ \mu(D(u)) \ll N \frac{1}{u^{N-2}}. \]

Let \( Y_T : G(N, T) \to \mathbb{R}_{>0} \) be the sequence of random variables defined as
\[ Y_T(a) = \sup\{ v \in \mathbb{R}_{>0} : \Lambda_a \in D(c_1 v^{2/(N-2)}) \}, \]
where the constant \( c_1 = c_1(N) \) is given by
\[ c_1 = V_{N-1}^{(N-1)(N-2)}/(N-1)^{2/(N-2)}. \]

Since the set \( D(1) \) contains all similarity classes we have for all \( a \in G(N, T) \)
\[ Y_T(a) \geq c_1^{-(N-2)/2} . \]

Let now \( \Gamma \subset \mathbb{R}^N \) be a lattice of rank \( N-1 \) and determinant 1, and let \( \lambda_i := \lambda_i(B_1^{N-1} \cap \text{span}_\mathbb{R}(L), L), 1 \leq i \leq N-1. \) We need the following simple observation

**Lemma 5.1.** Let \( \lambda_{N-1} > \lambda > 0. \) Then there exists an index \( i \in \{1, \ldots, N-2\} \) with
\[ \frac{\lambda_{i+1}}{\lambda_i} > c_2(N) \lambda^{2/(N-2)}, \]
where \( c_2(N) = 2^{-\frac{N-2}{2}} V_{N-1}^{(N-1)(N-2)}. \)

**Proof.** Suppose the opposite, i.e.,
\[ \frac{\lambda_{i+1}}{\lambda_i} \leq c_2(N) \lambda^{2/(N-2)}, \]
for all \( 1 \leq i \leq N-2. \) Then, \( \lambda_{N-1} \leq (c_2(N) \lambda^{2/(N-2)})^{N-1-i} \lambda_i, \) and by Minkowski’s second fundamental theorem (cf., e.g., [13, pp. 376])
\[ \lambda_1 \lambda_2 \cdots \lambda_{N-1} \leq \frac{2^{N-1}}{V_{N-1}}, \]
we get the contradiction
\[ \lambda_{N-1} \leq (c_2(N) \lambda^{2/(N-2)})^{\frac{(N-2)}{2}} \frac{2}{V_{N-1}^{1/(N-1)}} = \lambda. \]
\[ \square \]

We remark, that (5.3) can be slightly improved by applying Minkowski’s second theorem for balls. However we do not go further in this direction.

Let now \( F_T \) be the CDF of the random variable \( Y_T. \)

**Lemma 5.2.** For any \( T \geq 1 \) and \( t \geq 0 \) we have
\[ F_T(t) \leq \bar{F}_T(t). \]
Proof. Let $\Gamma = \Gamma_a$. By (2.5), we have
\[
\frac{f_N(a)}{s(a)} \leq \frac{(N - 1)}{2} \lambda_{N-1}.
\]
Hence, if for some $t$ holds
\[
X_T(a) = \frac{f_N(a)}{s(a)} > t
\]
then clearly $\lambda_{N-1} > \frac{2t}{(N-1)}$. By Lemma 5.1 applied with $\lambda = \frac{2t}{(N-1)}$, we get
\[
\frac{\lambda_i + 1}{\lambda_i} > c_1(N)t^{2/(N-2)}.
\]
Consequently, the lattice $\Gamma_a$ belongs to a similarity class in $\mathcal{D}(c_1t^{2/(N-2)})$, so that $Y_T(a) > t$. Therefore,
\[
\text{Prob}_N,T(X_T \leq t) = 1 - \frac{\# \{ a \in G(N, T) : f_N(a)/s(a) > t \}}{\# G(N, T)}
\]
\[
\geq 1 - \frac{\# \{ a \in G(N, T) : Y_T(a) > t \}}{\# G(N, T)} = \text{Prob}(Y_T \leq t).
\]

By Schmidt [25, Theorem 2], the number of primitive integer vectors $a \in \mathbb{Z}^N$ with $\|a\| \leq T$ and which lie on coordinate hyperplanes is essentially equal to $T^{N-1}$, so that the proportion of such vectors tends to zero as $T \to \infty$. Thus by Lemma 5.2 and Theorem 3.1 we finally obtain:
\[
\text{Prob}_N,T(f_N(a)/s(a) > t) \leq 1 - F_T(t) \leq 1 - \tilde{F}_T(t)
\]
\[
= \frac{\# \{ a \in G(N, T) : Y_T(a) > t \}}{\# G(N, T)}
\]
\[
\ll N \mu(\mathcal{D}(c_1t^{2/(N-2)})) \ll N t^{-2}.
\]
This proves the theorem.

6. Proof of Corollary

Observe that for all $a$ holds $f_N(a) > g_N(a)$. Therefore, it is enough to prove the inequality
\[
\text{Prob}_{\infty, 0}(f_N(a)/T^{1+1/(N-1)} > t) \ll_N t^{-2}.
\]
By [25, Theorem 2], we have $\# G(N, T/\sqrt{N}) \gg_N \# G(N, T)$ and thus
\[
\text{Prob}_{\infty, 0}(f_N(a)/T^{1+1/(N-1)} > t) \ll_N \frac{\# \{ a \in G(N, T) : f_N(a)/T^{1+1/(N-1)} > t \}}{\# G(N, T/\sqrt{N})}
\]
\[
\ll_N \text{Prob}_{N, T}(f_N(a)/T^{1+1/(N-1)} > t).
\]
Noting that $s(a) \ll_N T^{1+1/(N-1)}$ for $a \in G(N, T)$, we get
\[
\text{Prob}_{N, T}(f_N(a)/T^{1+1/(N-1)} > t) \leq \text{Prob}_{N, T}(f_N(a)/s(a) > \delta_N t)
\]
\[
\ll_N \text{Prob}_{N, T}(f_N(a)/s(a) > \delta_N t).
\]
with some positive constant $\delta_N$ which depends on $N$ only. Finally, by Theorem 1.1 we obtain the desired inequality:

$$\operatorname{Prob}_{\infty,0} (f_N(a)/T^{1+1/(N-1)} > t) \ll_N \operatorname{Prob}_{N,T} (f_N(a)/s(a) > \delta_N t) \ll_N t^{-2}.$$ 

7. Proof of Theorem 1.2

We will keep the notation from the proof of Theorem 1.1. Let also $E(\cdot)$ denote the mathematical expectation. Since for any nonnegative real-valued random variable $X$

$$E(X) = \int_0^\infty (1 - F_X(t)) dt,$$

Lemma 5.2 implies that

$$\sup_T E(X_T) \leq \sup_T E(Y_T)$$

Next, by Theorem 3.1 we also have

$$1 - \tilde{F}_T(t) = \frac{\# \{a \in G(N,T) : Y_T(a) > t \}}{\# G(N,T)} \ll_N \mu(D(c_1 t^{1/2})) \ll_N t^{-2}.$$ 

Thus by (7.1), (7.2) and observation (5.2), we obtain

$$\sup_T E(X_T) \ll_N \int_{c_1(N-1)/2}^\infty t^{-2} dt \ll_N 1,$$

which proves the theorem.

8. Proof of Theorem 1.3

The proof is based on Lemma 4.1 and the following continuity property of the inhomogeneous minima which follows from a more general result of Gruber [14, Satz 1]. We say that a sequence $S_t$ of star bodies in $\mathbb{R}^{N-1}$ converges to a star body $S$ if the sequence of distance functions of $S_t$ converges uniformly on the unit ball in $\mathbb{R}^{N-1}$ to the distance function of $S$. For the notions of star bodies, distance functions and convergence of a sequence of lattices to a given lattice we refer the reader to Gruber–Lekkerkerker [15].

Lemma 8.1 (Gruber [14, Satz 1]). Let $S_t$ be a sequence of star bodies in $\mathbb{R}^{N-1}$ which converges to a bounded star body $S$ and let $L_t$ be a sequence of lattices in $\mathbb{R}^{N-1}$ convergent to a lattice $L$. Then

$$\lim_{t \to \infty} \mu(S_t, L_t) = \mu(S, L).$$

For the proof of Theorem 1.3 we may assume that $\alpha \in \mathbb{Q}^N$ and

$$0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{N-1} < 1.$$ 

The simplex

$$S_\alpha(1) = \{(x_1, \ldots, x_N) \in \mathbb{R}_{\geq 0}^N : \sum_{i=1}^{N-1} \alpha_i x_i + x_N = 1\}$$
contains a ball of radius

\[ r_\alpha(1) = \frac{||\alpha||}{\sum_{i=1}^{N} ||\alpha[i]||\alpha_i}. \]

Let now \( R_\alpha \) be the radius of a ball containing \( S_\alpha(1) \), and let \( c(\alpha) = r_\alpha(1)/R_\alpha \). Recall that \( V_\alpha \) denotes the \((N-1)\)-dimensional subspace of \( \mathbb{R}^N \) orthogonal to the vector \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, 1) \). For any \( M > 0 \) one can choose a lattice \( L_M \subset V_\alpha \) of determinant 1 with

\[ \mu(B_1^{N-1} \cap V_\alpha, L_M) > \frac{4M}{c(\alpha)}. \] (8.2)

Since the inhomogeneous minima are independent of translation and since rational lattices are dense in the space of all lattices, by Lemma 8.1, we may assume that \( L_M \subset \mathbb{Q}^N \). Applying Lemma 4.1 to the lattice \( L_M \), we get a sequence \( a(t) \), where by (8.1),

\[ 0 < a_1(t) < a_2(t) < \ldots < a_N(t) \]

for sufficiently large \( t \).

Observe that (4.3) implies (1.8) with \( a_i = a_i(t), i = 1, \ldots, N, \) and \( t \) large enough. Next we show that, for sufficiently large \( t \), inequality (1.9) also holds. To this end we define the lattice \( \Gamma_t \) by

\[ \Gamma_t = ||a(t)||^{-1/(N-1)}\Lambda a(t). \]

By (4.11) and (4.2), the sequence of lattices \( L_t = \pi(\Gamma_t) \) converges to the lattice \( L = \pi(L_M) \). Now put \( \alpha(t) = (a_1(t)/a_N(t), \ldots, a_{N-1}(t)/a_N(t), 1) \).

The simplex \( S_\alpha(t) \) has the form

\[ S_\alpha(t) = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N_{\geq 0} : \sum_{i=1}^{N-1} \frac{a_i(t)}{a_N(t)} x_i + x_N = 1 \right\}. \]

The point \( p = (1/(2(N-1)), \ldots, 1/(2(N-1))) \) is an inner point of the simplex \( S = \pi(S_\alpha(1)) \) and thus of all the simplicies \( S_t = \pi(S_\alpha(t)) \) for sufficiently large \( t \). By (4.3) and Lemma 8.1, the sequence \( \mu(S_t - p, L_t) \) converges to \( \mu(S - p, L) \). Here we consider the sequence \( \mu(S_t - p, L_t) \) instead of \( \mu(S_t, L_t) \) because the distance functions of the family of star bodies in Lemma 8.1 need to converge on the unit ball. Now, since the inhomogeneous minima are independent of translation, the sequence \( \mu(S_t, L_t) \) converges to \( \mu(S, L) \). This clearly implies that the sequence \( \mu(S_\alpha(t), \Gamma_t) \) converges to \( \mu(S_\alpha(1), L_M) \).

Consequently, for all sufficiently large \( t \) we have
\[ f_N(a(t)) = \mu((a_N(t))^{-1} S_{\alpha(t)}, ||a(t)||^{1/(N-1)} \Gamma_t) \]
\[ = ||a(t)||^{1/(N-1)} a_N(t) \mu(S_{\alpha(t)}, \Gamma_t) \]
\[ > \frac{1}{2} ||a(t)||^{1/(N-1)} a_N(t) \mu(B_N^{N-1}, L_M) \]
\[ = \frac{c(\alpha)}{2r_{\alpha}(1)} ||a(t)||^{1/(N-1)} a_N(t) \mu(B_1^{N-1}, L_M) \]
\[ > 2M \frac{||a(t)||^{1/(N-1)} a_N(t)}{r_{\alpha}(1)} \]
\[ > M \frac{||a(t)||^{1/(N-1)} a_N(t)}{r_{\alpha}(1)} = M S(a(t)) \]  

The theorem is proved.

9. ACKNOWLEDGEMENT

The authors wish to thank Professor Anatoly Zhigljavsky for valuable comments and discussions.

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