On the notion of laminar and weakly turbulent elementary fluid flows: a simple mathematical model

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August 2006

Abstract

An elementary analytical fluid flow is composed by a geometric domain, a list of analytical constraints and by the function which depends on the physical properties, as Reynolds number, of the considered fluid. For this object, notions of laminar or weakly turbulent behavior are described using a simple mathematical model.

1 Elementary analytical fluid flows

Let $L$ a length unit and $\Omega$ a closed domain of $\mathbb{R}^n$, $n \geq 2$, formed by a small number (e.g. 2 or 3) of $L^2$ squares or $L^3$ cubes. Here we consider the plane case, but the notions are easily extensible to greater dimensions. Let $F$ a fluid entering in $\Omega$ from a single edge of a square and flowing out from another single edge. Let $s$, $0 \leq s \leq 1$, a parameter which describes the position of a single fluid particle along the inflow edge, so that $s_0L$ is the initial position of the generic particle associated to the particular value $s_0$ of the parameter. Let now $s \mapsto R(s)$ a function $R : [0, 1] \rightarrow [0, 1]$ which maps a value of the parameter $s$ to the value identifying the position reached by the particle on the outflow edge, so that this position is identified by the value $R(s)L$. If $x, y$ is a cartesian coordinates system in the plane of $\Omega$, assume that each streamline, or each particle path in lagrangian view, is described by a parametric curve $t \mapsto \Phi(t) = (x(t), y(t))$, with $0 \leq t \leq 1$. The parameter $t$ could not be, in general, the time variable of the flow. Let this parametric representation be determined by a set $A$ of analytical conditions regarding $\Phi(t)$ and $\dot{\Phi}(t)$, that is the passage of the particle in some suitable points of $\Omega$ and the velocity field direction in some other (or the same) points.

An elementary analytical flow is a particular list $F = \{\Omega, R, A\}$. We propose a
simple mathematical model, at least for a particular example of domain Ω, and show that the function \( R(s) \) can identify the physical properties of the particular fluid \( F \) flowing in the domain. For simplicity, in this work assume that the flow is steady.

2 A mathematical model

In this section let \( \Omega \) the domain formed by three \( L^2 \) squares, the first two along the \( x \)-axis of a cartesian coordinates system from 0 to \( 2L \), and the third above the second, from \( L \) to \( 2L \) \( y \)-coordinates.

![Figure 1: Domain with three unit squares.](image)

The inflow edge is the segment \([(0, 0), (0, L)] \) of the first square and the outflow edge is the segment \([(L, 2L), (2L, 2L)] \) of the third square.

Now the set \( A \) of analytical conditions is so defined:

- \( P_1 \). at the inflow edge, for \( t = 0 \) and position parameter \( s \), let \( \Phi(0) = (0, sL) \);
- \( P_2 \). at the outflow edge, for \( t = 1 \) and position parameter \( R = R(s) \), let \( \Phi(1) = (L(1 + R), 2L) \);
- \( P_3 \). for \( t = \frac{1}{2} \) a particle path intersects the diagonal line of the second square, that is the line of cartesian equation \( y = -x + 2L \); if a parameter \( p \), \( 0 \leq p \leq 1 \), describes the positions along this line, the condition is \( \Phi(\frac{1}{2}) = (L(2 - p), pL) \);
- \( D_1 \). at the inflow edge the velocity is parallel to \( x \)-axis, so that for every \( s \) \( \dot{y}(0) = 0 \);
- \( D_2 \). at the outflow edge the velocity is parallel to \( y \)-axis, so that for every \( s \) \( \dot{x}(1) = 0 \).

Note that condition \( P_3 \) is a logical connection between \( x \) and \( y \) coordinates, so
that for each component of $\Phi(t)$ the independent conditions are four: assuming, for simplicity, that the components are polynomial expressions on variable $t$, the candidates to satisfy the set $A = \{ P_1, P_2, P_3, D_1, D_2 \}$ of conditions are the cubics $x(t) = at^3 + bt^2 + ct + d$ and $y(t) = et^3 + ft^2 + gt + h$.

Let proceed with the computation of these cubics.

From derivation condition $D_1$ follows $g = 0$. From passage condition $P_1$ follows $d = 0$ and $h = sL$. From these partial results, from condition $P_3$ follows $c = -3a - 2b$. With these partial results, from condition $D_2$ follows $c = a + 2L(1 + R)$.

Now, using condition $P_2$, the computation gives $a = \frac{L}{2}(10 - 6R - 8p)$, so that the first component is

$$x(t) = L(10 - 6R - 8p)t^3 + L(-21 + 11R + 16p)t^2 + L(12 - 4R - 8p)t$$  \hspace{1cm} (1)

With a similar computation, the second component is

$$y(t) = L(4 + 6s - 8p)t^3 + L(-2 - 7s + 8p)t^2 + Ls$$  \hspace{1cm} (2)

Note that, with this model, the second component doesn’t depend on function $R(s)$.

For simplicity, let $p = s$, that is the particles paths are not disturbed until the line $y = -x + 2L$. In this case the $y$-component has the form

$$y(t) = L(4 - 2s)t^3 + L(-2 + s)t^2 + Ls$$  \hspace{1cm} (3)

3 The laminar case

In the laminar case assume that particles paths don’t intersect themselves, that is the outflow positions on the final edge of $\Omega$ are the same as inflow positions. The analytical expression of this geometrical condition is simply

$$R(s) = 1 - s$$  \hspace{1cm} (4)

The laminar elementary analytical flow $F = \{ \Omega, A, R \}$ is so completely defined; its particles paths have the form

$$\Phi(t) = \{ L(2 - s)(2t^3 - 5t^2 + 4t), L(2 - s)(2t^3 - t^2) + Ls \}$$  \hspace{1cm} (5)

Note that the expression (1) for $R(s)$ depends on the shape of the domain $\bar{\Omega}$, as expected; e.g., in the case of a rectangular domain with the outflow edge parallel to the inflow edge, the expression should be $R(s) = s$.

4 The weakly turbulent case

The weak turbulence notion (5) regards the physical situation of a flow where turbulence is not fully developed and it could be described by deterministic
mathematical law (4). For an elementary analytical flow, such mathematical description can be made by a suitable choice of the function $R(s)$, which maps the initial inflow position of a particle to its outflow position. In order to find a possible expression for $R(s)$, assume that some geometrical point of the final edge can be reached by more than one fluid particle, so that paths intersects themselves, as expected in a turbulent flow. Also, the analytical expression of $R(s)$ should be such that, in the case of suitable conditions, the laminar case (4) can be a mathematical limit for great values of viscosity (viscous limit, see (1)). A mathematical condition is that $R(s) \in [0, 1]$ for every $s$. Assume also that weak turbulence, as expected, is associate to the analytical condition $R \in C^m([0, 1])$ for some integer $m \geq 0$.

Note that a possible expression satisfying all the previous conditions is $R(s) = \frac{1}{2} [1 + \sin(\alpha)]$, where $\alpha = \alpha(s)$ could be computed using the viscous limit. Assume that this limit is defined for Reynolds number $Re$ approaching a value $c_F$, $\text{Re} \to c_F$, with $c_F$ a constant which should depend on the nature of the fluid and on the geometrical and physical properties of the flow. The most simple form for $\alpha$ is $\alpha = a \text{Re} s + b$, therefore

$$\text{Re} \to c_F \Rightarrow \alpha \to ac_F s + b$$  \hspace{1cm} (6)

Assuming $ac_F s + b$ sufficiently small, for $\text{Re} \approx c_F$ follows $\sin(\alpha) \approx ac_F s + b$, therefore it must be

$$R(s) = \frac{1}{2} [1 + ac_F s + b] = 1 - s$$  \hspace{1cm} (7)

from which $a = -\frac{2}{c_F}$ and $b = 1$. So the function describing a weak turbulent elementary flow is
\( R(s) = \frac{1}{2} \left[ 1 + \sin \left( 1 - \frac{2}{c_F} \Re s \right) \right] \) \hspace{1cm} (8)

Note that for high Reynolds number the function \( R(s) \) covers the interval \([0, 1]\) many times, so that the degree of turbulence increases.

Figure 3: Viscous limit (red) \( R(s) = \frac{1}{2} [1 + \sin (1 - 2s)] \), compared to laminar case \( R(s) = 1 - s \).

Figure 4: Weakly turbulent elementary flows: at left, particle paths for \( \frac{\Re}{c_F} = 3 \); at right, some particle paths in the case \( \frac{\Re}{c_F} = 2 \times 10^4 \).

5 Concluding remarks and further developments

The notion of elementary flow is characterized by a domain, a list of analytical constraints and by the function \( R = R(s) : [0, 1] \rightarrow [0, 1] \) which depends on the
It could be investigated the relation between particles paths (or streamlines) \( \Phi(t) \) and the solutions of Navier-Stokes equations, probably obtaining a mathematical relation between \( R \) and pressure, and an expression for the flow constant \( c_F \).

In the case of evolution, e.g. if the \( t \) parameter is proportional to time, function \( R(s) \) could be characterized using the Richardson’s Law \([2]\) on particles dispersion in turbulent flows.

Can be full developed turbulence a limit of this weak turbulence model? For very high Reynolds numbers the function \( R(s) \) expressed by \([8]\) formula has no limit for every \( s \), or has not a deterministic behavior: the vertical line \([0,1]\) is an accumulation line for its graph, therefore no prediction about the outflow position of a particle is possible. Perhaps it can be useful a convergence, for some kind of metric, of a succession \( \{R_n(s)\}_{n=1,2,3,...} \) of continuous \( R_s \) to some step function, in order to describe full turbulence flow and its phenomenon of intermittency \([3]\).

Finally, can be a real fluid flow described by some kind of combination of elementary flows? Perhaps a combination, or a succession, of elementary flows with a common function \( R(s) \) and variable analytical constraints \( \{A_n\}_{n=1,2,3,...} \) could be a useful representation of a real flow of a single fluid.

References

[1] P. Constantin, A few results and open problems regarding incompressible fluids, Notices of the American Mathematical Society, vol. 42, 6, (June 1995)

[2] G. Boffetta, I.M. Sokolov, Relative dispersion in fully developed turbulence: the Richardson’s Law and intermittency corrections, Physical Review Letters, 88, 9, (4 March 2002)

[3] Y. Li, C. Meneveau, Origin of non-Gaussian statistics in hydrodynamic turbulence, Physical Review Letters, PRL 95, 164502, (14 October 2005)

[4] J.C. Roux, C. Vidal, Comment naît la turbulence, Pour la Science, 39, (1981)

[5] D. Ruelle, F. Takens, On the nature of turbulence, Communications on Mathematical Physics, 20, 167-192, (1971)