Abstract

In this work we construct a general class of exactly solvable non-relativistic bi-dimensional quantum systems with position-dependent masses (PDM). These systems are isospectral to a given system with constant mass. The case of a charged particle with a PDM interacting with an external magnetic field is included in the present investigation. We apply the approach in order to construct the SU(2) coherent states in some examples which are isospectral to the two-dimensional anisotropic harmonic oscillator, and discuss the impact of the introduction of special non-homogeneous external magnetic fields.

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1 Introduction

The interest in the problem of position-dependent mass (PDM) systems has been growing in the last years, both from the non-relativistic and the relativistic point of view [1]-[31]. In fact, there are many applications to different problems of physics like quantum dots [32, 33], compositional graded crystals [34], quantum liquids [35], metal clusters [36], neutron stars [37], among others. In particular, recently the Wigner function for some classes of position-dependent Schroedinger equations were constructed and analyzed in the one-dimensional case [4]. The approach used in that work is capable to generate a class of position-dependent mass systems, which is isospectral to a given exactly solvable potential with constant mass. In this work, we will extend that approach to higher dimensions. Moreover, as far as we know, the majority part of the works dedicated to the research of PDM systems [1], deals with one-dimensional systems. Although, there are physical systems like those where a magnetic field [38] is present, which leads naturally to the need of a two-dimensional analysis. Finally, to our knowledge, no work in this subject has discussed the case of PDM in the presence of magnetic fields. In this work, we intend to partially fill this gap.

The method will be applied to the case of the anisotropic two-dimensional harmonic oscillator as well as in the case of those systems under the influence of some non-homogenous external magnetic fields. The $SU(2)$ coherent states will be constructed for a number of systems which we choose to illustrate our results.

In order to get the exact solutions for the two-dimensional Schroedinger equation we trace two routes. We begin by performing a general spatial variables change in the case of systems which are not under the effect of magnetic fields. Then, in the case of magnetic interaction, we begin by performing a time-dependent variable transformation followed by the spatial transformation. In the first case, we consider three examples of real variable transformations, which are respectively: the so-called polynomial one, the one using elliptic cylindrical coordinates and, finally, the bipolar coordinates transformation. Once we have the exact solutions for the eigenstates of the Schroedinger equation, we proceed with the construction of the $SU(2)$ coherent states [1, 39, 40, 41]. These last allow us to acquire a notion of the respective classical behavior of those position-dependent massive particles, similarly to what happens with the Wigner functions in one spatial dimension [4].
This work is organized as follows: In the section 2 we present the approach we are going to use. Then, in the section 3 we apply it to some special mass dependencies, constructing classes of two-dimensional PDM systems which are isospectral to the anisotropic bi-dimensional harmonic oscillator. In the section 4, the case of magnetic field is taken into account. Finally we trace our final comments in the section 5.

2 Class of isospectral two-dimensional position-dependent mass quantum systems

In this section we present the approach which is capable to generate a class of models with position-dependent masses from a constant one, and which can also include the interaction with a magnetic field. In this case the two-dimensional Schroedinger equation is given by

\[-\frac{\hbar^2}{2m_0} \nabla^2 \psi + \frac{\hbar}{i} \left( \nabla \cdot \vec{A} + 2 \vec{A} \cdot \nabla \right) \psi + \left( V(x,y) + \frac{e^2}{2m_0} \vec{A}^2 \right) \psi = E \psi.\] (1)

where the constant $m_0$ is the mass of the particle in the original system, $x$ and $y$ are the corresponding spatial coordinates, $p_x$ and $p_y$ are the respective momenta. Furthermore, we will work with magnetic fields in the Coulomb gauge, where $\nabla \cdot \vec{A} = 0$. Now, performing a general variable transformation in the spatial coordinates as

\[x = f(u,v), \ y = g(u,v),\] (2)

one gets

\[\partial_u \equiv \frac{\partial}{\partial u} = f_u \partial_x + g_u \partial_y, \ \partial_v \equiv \frac{\partial}{\partial v} = f_v \partial_x + g_v \partial_y,\] (3)

which can be written as

\[\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = R^{-1} \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix},\] (4)

where $R = \begin{pmatrix} f_u & g_u \\ f_v & g_v \end{pmatrix}$ is the Jacobian matrix, and its Jacobian is $J = f_u g_v - g_u f_v$, from which it can be written that

\[\partial_x = \frac{\partial}{\partial x} = \frac{g_v}{J} \frac{\partial}{\partial u} - \frac{g_u}{J} \frac{\partial}{\partial v}, \ \partial_y = \frac{\partial}{\partial y} = -\frac{f_v}{J} \frac{\partial}{\partial u} + \frac{f_u}{J} \frac{\partial}{\partial v}.\] (5)
After straightforward calculations, one can obtain the following expression for the Laplace operator in the transformed coordinates,

$$\nabla^2_{xy} \equiv \partial^2_x + \partial^2_y = \frac{1}{J^2} \left[ (g_v^2 + f_v^2) \partial^2_u + (g_u^2 + f_u^2) \partial^2_v - 2(g_u g_v + f_u f_v) \partial^2_{uv} \right] + \frac{1}{J} \left[ g_v \partial_u \left( \frac{g_v}{J} \right) - g_u \partial_v \left( \frac{g_u}{J} \right) + f_v \partial_u \left( \frac{f_v}{J} \right) - f_u \partial_v \left( \frac{f_u}{J} \right) \right] \partial_u + \frac{1}{J} \left[ g_u \partial_v \left( \frac{g_u}{J} \right) - g_v \partial_u \left( \frac{g_v}{J} \right) + f_u \partial_v \left( \frac{f_u}{J} \right) - f_v \partial_u \left( \frac{f_v}{J} \right) \right] \partial_v. \quad (6)$$

In order to achieve a system where we have a new Schroedinger-type equation with a position dependent mass one must require that the crossed derivative term $\partial^2_{uv}$, must vanish, leading to the restriction

$$g_u g_v + f_u f_v = 0. \quad (7)$$

Afterwards, we should also impose that

$$g_v^2 + f_v^2 = g_u^2 + f_u^2, \quad (8)$$

to guarantee that the mass term is the same in the both terms $\partial^2_u$ and $\partial^2_v$. Note that the equation (7) once solved leads to

$$f_u = -\frac{g_u g_v}{f_v}, \quad (9)$$

provided that $f_v \neq 0$. By substituting (9) in (8), one gets

$$g_v^2 + f_v^2 = g_u^2 + g_u^2 \left( \frac{g_v}{f_v} \right)^2 = g_u^2 \left( 1 + \frac{g_v^2}{f_v^2} \right) = g_u^2 \left( \frac{f_v^2 + g_v^2}{f_v^2} \right). \quad (10)$$

Thus, from (10) we conclude that

$$f_v^2 = g_u^2, \quad f_v = \pm g_u. \quad (11)$$

Substituting (11) in (9), one obtains

$$f_u = -\frac{g_u g_v}{\pm g_u} = \mp g_v. \quad (12)$$
In fact, in all the cases considered here, the terms linear in the derivatives in the transformed Laplace operator (6) vanish. Moreover, the transformation will keep \( \nabla \cdot \vec{A} = 0 \) for all the cases considered. Thus the Schrödinger equation in those transformed variables is written as

\[
\left[ \frac{\hbar^2}{2 M(u,v)} \left( \partial_u^2 + \partial_v^2 \right) + V_{\text{eff}}(u,v) - \frac{i \hbar}{M(u,v)} \left( \vec{A} \cdot \nabla \right) \right] \psi(u,v) = E \psi(u,v),
\]

where

\[
M(u,v) \equiv m_0 \frac{J^2}{g_u^2 + f_u^2}, \quad V_{\text{eff}}(u,v) \equiv V(f,g) + \frac{e^2}{2 M(u,v)} \vec{A}^2(f,g),
\]

with \( \vec{A} \equiv M(u,v) \vec{A}(f(u,v),g(u,v)) \).

As an illustrative example, we will first deal with two-dimensional position-dependent systems which are isospectral to the anisotropic harmonic oscillator and, then, with the ones which are isospectral to the case of a isotropic harmonic oscillator under the influence of homogeneous magnetic fields.

### 3 2D PDM systems isospectral to the anisotropic oscillator

As advertised, the first example which we will present here is the one of the anisotropic oscillator with constant mass governed by the equation

\[
-\frac{\hbar^2}{2 m_0} \nabla^2 \psi + \frac{1}{2} m_0 \left( \omega_1^2 x^2 + \omega_2^2 y^2 \right) \psi = E \psi.
\]

In this case the eigenfunctions can be straightforwardly obtained, and are given by

\[
\psi_{nm}(u,v) = \frac{1}{\sqrt{2^{m+n+1} \pi n! m! X Y}} H_m \left( \frac{\sqrt{2} f(u,v)}{X} \right) H_n \left( \frac{\sqrt{2} g(u,v)}{Y} \right) \times \exp \left[ - \left( \frac{f(u,v)}{X} \right)^2 - \left( \frac{g(u,v)}{Y} \right)^2 \right],
\]

where \( X = \sqrt{2\hbar/(m_0 \omega_1)}, Y = \sqrt{2\hbar/(m_0 \omega_2)}, \omega_1 \equiv q \) and \( \omega_2 \equiv p \).
At this point we could make a study of the eigenfunctions and eigenvalues for the PDM systems which are isospectral to this one. However, we prefer to construct the so-called $SU(2)$ coherent states, as defined in [39]-[41]. Those states present the interesting feature of having their highest probability density over a trajectory which corresponds to the classical one when $\hbar \to 0$. Furthermore, as the eigenstates, they are stationary wave-functions in contrast with the usual coherent states. As we are going to see, they will lead us to the conclusion that the isospectral PDM states will present a behavior which is very similar to the one of their “parent” anisotropic harmonic oscillator. Those coherent states can be written by using the definition introduced by Chen et al. [39]-[41]

$$\Phi(u, v, \tau) = \frac{1}{(1 + |\tau|^2)^{\frac{L}{2}}} \sum_{K=0}^{L} \left( \begin{array}{c} L \\ K \end{array} \right)^{1/2} \tau^K \psi_{nm}(f(u, v), g(u, v)), \quad (17)$$

where the quantum numbers are defined as $n = p K, m = q(L - K)$, with $K = 0, 1, 2, ..., L, p$ and $q$ being integer numbers. The complex parameter is such that $\tau = Ae^{i\phi}$, where $\phi = \frac{\pi}{2}$, written in terms of polar coordinates, is used in order to make the connection with the classical trajectory.

From now on, we will devote this section to develop explicit examples. As our first example of transformation functions, we deal with the second degree polynomials like

$$f(u, v) = \frac{1}{2}a_1u^2 + \frac{1}{2}b_1v^2 + c_1uv + d_1, \quad g(u, v) = \frac{1}{2}a_2u^2 + \frac{1}{2}b_2v^2 + c_2uv + d_2. \quad (18)$$

After imposing the restrictions (7) and (8), we are led to

$$f(u, v) = \pm \frac{1}{2}c_2u^2 \pm \frac{1}{2}c_1v^2 + c_1uv + d_1, \quad g(u, v) = \pm \frac{1}{2}c_1u^2 \pm \frac{1}{2}c_2v^2 + c_2uv + d_2, \quad (19)$$

where one can see that only four arbitrary constants are left, and one have that

$$f_u^2 + g_u^2 = f_v^2 + g_v^2 = (c_1^2 + c_2^2)(u^2 + v^2), \quad (20)$$

and the Laplace operator becomes
\[ \nabla_{uv}^2 = \frac{1}{f^2} \left[ (g_u^2 + f_v^2) \partial_u^2 + (g_v^2 + f_u^2) \partial_v^2 \right] = \frac{1}{(c_1^2 + c_2^2)(u^2 + v^2)}(\partial_u^2 + \partial_v^2), \]  

(21)

from which one conclude that the position-dependent mass is given by

\[ M(u, v) = m_0(c_1^2 + c_2^2)(u^2 + v^2). \]  

(22)

In this case, the effective anisotropic potential looks like

\[ V(u, v) = \frac{m_0}{2} \left[ \left( \frac{c_2}{2}(u^2 - v^2) + c_1 u v + d_1 \right)^2 \omega_1^2 + \left( \frac{c_1}{2}(v^2 - u^2) + c_2 u v + d_2 \right)^2 \omega_2^2 \right], \]  

(23)

and the wavefunctions are written as

\[ \psi_{nm}(u, v) = \frac{1}{\sqrt{2^{(m+n-1)}m!n!P_1 P_2}} \left[ \frac{\sqrt{2} \left( \frac{c_2}{2}(u^2 - v^2) + c_1 u v + d_1 \right)}{P_1} \right] \times H_m \left[ \frac{\sqrt{2} \left( \frac{c_1}{2}(v^2 - u^2) + c_2 u v + d_2 \right)}{P_2} \right] \times \]  

\[ \exp \left[ - \left( \frac{\left( \frac{c_2}{2}(u^2 - v^2) + c_1 u v + d_1 \right)}{P_1} \right)^2 \right] \times \]  

\[ \exp \left[ - \left( \frac{\left( \frac{c_1}{2}(v^2 - u^2) + c_2 u v + d_2 \right)}{P_2} \right)^2 \right]. \]  

(24)

(25)

In the Figures 1 and 2 (we use \( A = \hbar = m_0 = 1 \) and \( L = 20 \), when generating the plots presented throughout this work), one can see the behavior of the system in some typical situations. At this point it is interesting to note that the transformation functions can be chosen as the ones related to the parabolic cylindrical coordinates in a plane, where \( f = u v \) and \( g = (u^2 - v^2) / 2 \), which generate the following Laplace operator

\[ \nabla_{uv}^2 = - \frac{1}{(u^2 + v^2)}(\partial_u^2 + \partial_v^2). \]  

(26)

This make us remember that one could work with a general class of solutions as the ones coming from the orthogonal coordinate systems, where the metric
is diagonal. As examples we will consider the case of the elliptic cylindrical coordinates on a plane where

\[ f(u, v) = a \sinh(u) \sin(v), \quad g(u, v) = a \cosh(u) \cos(v). \quad (27) \]

In this case, the Laplace operator is written as

\[ \nabla^2 uv = -\frac{2}{a^2[\cos(2v) - \cosh(2u)]} (\partial_u^2 + \partial_v^2), \quad (28) \]

with the mass \( M(u, v) = m_0 \frac{a^2[\cos(2v) - \cosh(2u)]}{2} \). The corresponding effective potential is now given by

\[ V(u, v) = \frac{a^2m_0}{2} \left[ \sinh(u^2) \sin(v)^2 \omega_1^2 + \cosh(u^2) \cos(v)^2 \omega_2^2 \right] \quad (29) \]

The wavefunctions are written in this case as

\[ \psi_{nm}(u, v) = \frac{1}{\sqrt{2^{m+n-1}m!n!P_1P_2}} \frac{\sqrt{2}(a \sinh(u) \sin(v))}{P_1} H_m \left[ \frac{\sqrt{2}(a \cosh(u) \cos(v))}{P_2} \right] \exp \left[ -\left( \frac{a \sinh(u) \sin(v)}{P_1} \right)^2 \right] \frac{\sqrt{2}(a \cosh(u) \cos(v))}{P_2} \right] \exp \left[ -\left( \frac{a \cosh(u) \cos(v)}{P_2} \right)^2 \right] \quad (30) \]

whose typical behavior appears in the Figure 3.

Now, in the bispherical coordinates on a plane, the transformations are

\[ f(u, v) = \frac{a \sinh(u)}{\cosh(u) - \cos(v)}, \quad g(u, v) = \frac{a \sin(v)}{\cosh(u) - \cos(v)}, \quad (31) \]

and the corresponding Laplace operator looks like

\[ \nabla^2 uv = \frac{[\cos(v) - \cosh(u)]^2}{a^2} (\partial_u^2 + \partial_v^2), \quad (32) \]

and the position-dependent mass will be given by \( M(u, v) = m_0 \frac{a^2}{[\cos(v) - \cosh(u)]^2} \). In this last example of anisotropic system, the effective potential happens to be

\[ V(u, v) = \frac{a^2m_0}{2} \left[ \left( \frac{a \sinh(u)}{\cosh(u) - \cos(v)} \right)^2 \omega_1^2 + \left( \frac{a \sin(v)}{\cosh(u) - \cos(v)} \right)^2 \omega_2^2 \right], \quad (33) \]
and the wavefunctions are correspondingly given by

\[ \psi_{nm}(u, v) = \frac{1}{\sqrt{2^{m+n-1}m!n!P_1P_2}} H_m \left[ \frac{\sqrt{2}}{P_1} \left( \frac{a \sinh(u)}{\cosh(u) - \cos(v)} \right) \right] \times \\
H_n \left[ \frac{\sqrt{2}}{P_2} \left( \frac{a \sin(v)}{\cosh(u) - \cos(v)} \right) \right] \exp \left\{ - \left[ \frac{1}{P_1} \left( \frac{a \sinh(u)}{\cosh(u) - \cos(v)} \right) \right]^2 \right\} \]

whose typical behavior appears in the Figure 4.

4 The case of interaction with a magnetic field

In this section, we first construct the \( SU(2) \) coherent state for the case of a particle with constant mass in the presence of a homogeneous magnetic field which, as far as we know, was not yet obtained in the literature. Then, we proceed by using that result in order to achieve the \( SU(2) \) coherent states for the PDM particles under the presence of some non-homogeneous magnetic fields. All the results will be obtained in the Coulomb gauge. Starting with the Hamiltonian

\[ H = \frac{1}{2m_0} \vec{p}^2 - \frac{e}{2m_0} \left( \vec{A} \cdot \vec{p} + \vec{p} \cdot \vec{A} \right) + \frac{e^2}{2m_0} \vec{A}^2 + \frac{1}{2} m_0 \omega^2 \left( x^2 + y^2 \right), \quad (35) \]

where \( e \) is the electrical charge and \( \vec{A}(x, y) \) is the vector potential.

Here we choose to work with a uniform magnetic field along the direction \( z \), which is written as \( \vec{B} = B_0 \hat{z} \). In the so-called symmetrical gauge it is written as

\[ \vec{A} = \frac{B_0}{2} (-y \hat{i} + x \hat{j}), \quad (36) \]

Thus, beginning with the classical Hamiltonian defined in (35) and quantizing it, one arrives at the following Schroedinger equation

\[ -\frac{\hbar^2}{2m_0} \nabla^2 \psi + i \frac{\hbar}{2m_0} B_0 \left( x \partial_y - y \partial_x \right) \psi + \frac{1}{2} m_0 \omega^2 \left( x^2 + y^2 \right) \psi = E \psi. \quad (37) \]
with $\Omega \equiv \omega^2_0 + \frac{e^2 B_0^2}{8 m_0}$.

In order to deal with the case of a constant mass in the presence of a homogeneous magnetic field, one can follow two alternative routes. One way is to write the system in polar coordinates and the another is keeping the cartesian coordinates and performing some convenient time-dependent transformations. This last is the one we will follow here [38]. For this, we can start with the time-dependent Schroedinger equation

$$-\frac{\hbar^2}{2 m_0} \nabla^2 \sigma + i \frac{\hbar B_0 e}{2 m_0} (x \partial y - y \partial x) \sigma + \frac{1}{m_0} U_{eff} \sigma = i \hbar \frac{\partial \sigma}{\partial T}, \quad (38)$$

such that $\sigma = e^{-\frac{i}{\hbar} E_T \chi(x, y)}$. Then, we perform the time-dependent rotation

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha(t)) & \sin(\alpha(t)) \\ -\sin(\alpha(t)) & \cos(\alpha(t)) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (39)$$

which, after some manipulations [38], and choosing $\alpha$ to be given by

$$\alpha(T) = -\frac{e B_0}{2 m_0} T + c, \quad (40)$$

where $c$ is an arbitrary integration constant. One can finish with an effective isotropic two-dimensional harmonic oscillator such that

$$-\frac{\hbar^2}{2 m_0} \nabla^2 \chi + U_{eff} \chi = E \chi, \quad (42)$$

This allow us to map the original problem into one where the differential equation is given by

$$-\frac{\hbar^2}{2 M_0} \nabla^2 \chi + U_{eff} \chi = E \chi,$$

where $\chi = \chi(X_1, Y_1)$. Finally, using the usual variables separation procedure, one can arrive at the expression

$$\chi_{nm}(X_1, Y_1) = \phi(X_1) \psi(Y_1) = \phi[\cos(\alpha) x + \sin(\alpha) y] \psi[-\sin(\alpha) x + \cos(\alpha) y] = \phi X_1 \psi Y_1.$$
Finally, by using the above wave-functions, the $SU(2)$ coherent states can be easily written. Their general aspect happens to be the same of the isotropic harmonic oscillator as presented in the left plot of Figure 2.

Now, we can study the cases of PDM particles in the presence of magnetic fields. Following the same lines developed in the previous section, we make the variables transformation $x = f(u, v)$ and $y = g(u, v)$ in the Schrödinger equation and, since we already know the corresponding transformation in the Laplace operator, we only need to verify the impact of the transformation over the linear differential operator coming from $\vec{A}.\vec{p}$, which comes to be

$$x\partial_y - y\partial_x = \frac{1}{J}\{ [ f(u, v) f_u + g(u, v) g_u ] \partial_v - [ f(u, v) f_v + g(u, v) g_v ] \partial_u \}. \quad (44)$$

Then, we finish with the transformed Schrödinger equation in the spatial variables $u$ and $v$ appearing as

$$-\frac{\hbar^2}{2M(u, v)} \nabla^2 \psi + \frac{i\hbar B_0}{2M(u, v)} \left[ R(u, v) \frac{\partial \psi}{\partial u} + S(u, v) \frac{\partial \psi}{\partial v} \right] +$$

$$+ \left\{ \frac{1}{2} m_0 \Omega^2 \left[ f(u, v)^2 + g(u, v)^2 \right] \right\} \psi = E \psi. \quad (45)$$

The magnetic field will be then given by

$$\vec{B} = \nabla \times \vec{A} = \left( \frac{\partial}{\partial u} \frac{\partial}{\partial v} \frac{\partial}{\partial w} \right) = \left( \frac{\partial S}{\partial v} - \frac{\partial R}{\partial u} \right) \frac{k}{R}. \quad (46)$$
Again, the first example to be analyzed is the one defined in Equation (19). In this case we have that

\[ R(u, v) = -\frac{1}{2} \left( c_1^2 + c_2^2 \right) (v^3 + u^2 v) - (c_1 d_1 + c_2 d_2) u - (c_1 d_2 - c_2 d_1) v, \]

\[ S(u, v) = \frac{1}{2} \left( c_1^2 + c_2^2 \right) (u^3 + u v^2) + (c_2 d_1 - c_1 d_2) u + (c_1 d_1 + c_2 d_2) v. \] (47)

Here, we will have a particle with mass \( M(u, v) = m_0 (c_1^2 + c_2^2)(u^2 + v^2) \) moving in the presence of an axially symmetric field with a quadratically growing intensity,

\[ \vec{B} = B_0 (c_1^2 + c_2^2)(u^2 + v^2) \hat{k}. \] (48)

Choosing the parameters \( m_0 = d_1 = d_2 = c_1 = 0 \) and \( c_2 = 1 \), we restrict ourselves to the case of parabolic cylinder coordinates where

\[ M(u, v) = u^2 + v^2, \ R(u, v) = -\frac{1}{2} v (u^2 + v^2), \ S(u, v) = \frac{1}{2} u (u^2 + v^2), \] (49)

and the potential governing the system is

\[ V(u, v) = \frac{m_0 \omega^2}{2} \left[ \frac{1}{4} (u^2 - v^2)^2 + (u v)^2 \right] + -e^2 B_0 \sqrt{u^2 + v^2}, \] (50)

and the wavefunctions are

\[ \psi_{nm}(u, v) = \frac{1}{R \sqrt{2^{m+n-1} \pi n! m!}} H_m \left( \frac{\sqrt{2} \left[ \frac{\cos(\alpha)}{2} (u^2 - v^2) + \sin(\alpha) u v \right]}{R} \right) \]

\[ \times H_n \left( \frac{\sqrt{2} \left[ -\frac{\sin(\alpha)}{2} (u^2 - v^2) + \cos(\alpha) u v \right]}{R} \right) \]

\[ \times \exp \left[ - \left( \frac{u^2 - v^2}{2R} \right)^2 - \left( \frac{u v}{R} \right)^2 \right], \] (51)

which, after plotting the probability density of the corresponding SU(2) coherent state, presents the profile which is very similar to the one appearing in the Figure 1.
Now, in the case of elliptical cylindrical coordinates one have

$$R(v) = a^2 \cos(v) \sin(v), \quad S(u) = a^2 \cosh(u) \sinh(u). \quad (52)$$

and the magnetic field is given by

$$\vec{B} = a^2 [\cosh(2u) - \cos(2v)] \frac{\vec{k}}{k}. \quad (53)$$

The potential under which the charged particle is moving is

$$V(u, v) = m_0 a^2 \frac{\omega^2}{2} \left[ \sinh(u^2) \sin(v^2) + \cosh(u^2) \cos(v^2) \right] +$$

$$-e^2 B_0 \frac{\sqrt{\cosh(4u) - \cos(4v)}}{2 \left[ \cos(2v) - \cosh(2u) \right]}, \quad (54)$$

and the wavefunctions can be straightforwardly obtained through direct substitution of the transformation functions in the expression of the wavefunctions (43). After that, one can plot the probability density, which present the same general behavior as appearing in the Figure 3. There, one can see a sequence of regions where the highest probability density is over a closed curve, which reflects the periodicity of the transformation functions.

In the third and last example, we will deal with the bipolar coordinates. In this case the Laplace operator is written as $\nabla^2_{uv} = \left[ \frac{\cos(v)}{a^2} \right]^2 (\partial_u^2 + \partial_v^2)$, and we get

$$R(u, v) = a^2 \frac{\cosh(u) \sin(v)}{\cos(v) - \cosh(u)}, \quad S(u, v) = -a^2 \frac{\cos(v) \sinh(u)}{\cos(v) - \cosh(u)}. \quad (55)$$

the corresponding magnetic field looks like

$$\vec{B} = \frac{2 a^2}{\left[ \cos(v) - \cosh(u) \right]^2} \frac{\vec{k}}{k}, \quad (56)$$

and, in this last example, the potential is written as

$$V(u, v) = \frac{a^2 m_0 \omega^2}{2} \frac{(\sinh(u)^2 + \sin(v)^2)}{(\cosh(u) - \cos(v))^2} - \frac{e^2 B_0}{4 m_0} \sqrt{\cosh(u)^2 - \cos(v)^2}. \quad (57)$$

Once more, after constructing the wave-function for the $SU(2)$ coherent state, one can see that we arrive at a Figure which is very similar to the case (a) of the Figure 4 below.
5 Final remarks

In this work we explored a method for generating exactly solvable position-dependent mass particle in the present of some potentials. This approach was recently introduced in the case of one-dimensional systems \cite{4}. Here we see that the extension for higher dimensions is not trivial and allows one to analyze complex and interesting new systems. Moreover, we conduce our study by computing the so called $SU(2)$ coherent states, which are specially interesting wave-packets which present their maximum of probability over the classical trajectory \cite{1, 39, 40, 41}. As particular examples of variable transformation, we used a polynomial example as well as the cases of the cylindrical elliptical and the bipolar coordinates.

In the case of a charged particle under the effect of pure homogeneous external magnetic field, despite the fact that the we are dealing with time-dependent wave-function solutions, it was verified that the corresponding $SU(2)$ coherent states do not present dependence in the time variable. In that case, the maximum of the probability density is over a circle. Then, performing the change in the spatial variables, we got those states for the case of some particular spatially dependent masses and non-homogenous magnetic fields. In the case where the transformation used was that of polynomial form, it was observed that when the zero degree coefficient of the polynomial used in the transformation was non-zero, the maximum of probability splits into two ones which become more and more distant from each other when that parameter increases. Furthermore, in the case of the other two transformations used here, the probability density was naturally periodic as a direct consequence of the periodicity of the transformation.

Acknowledgements: The authors thanks to CNPq and CAPES for partial financial support. J. A. O. thanks the DFQ of UNESP, Campus de Guaratinguetá, where this work was carried out. This work was partially done during a visit (ASD) within the Associate Scheme of the Abdus Salam ICTP.

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Figure 1: Probability density for the case of polynomial solutions. Here the parameters used are \( a = c_1 = c_2 = 1 \) and: a) \( d_2 = 4 \) and c) \( d_2 = 7 \).

Figure 2: Probability density for the case of polynomial solutions. Here the parameters used are: \( d_1 = d_2 = c_1 = 0 \), \( c_2 = 1 \) and the frequencies are the following: a) \( \omega_1 = 1 \) and \( \omega_2 = 1 \) and c) \( \omega_1 = 2 \) and \( \omega_2 = 3 \).
Figure 3: Probability density for the case of elliptical cylindrical coordinates. In this case the frequencies were: a) $\omega_1 = 1$ and $\omega_2 = 1$. 
Figure 4: *Probability density for the case of bipolar coordinates. The used frequencies were: a) $\omega_1 = 1$ and $\omega_2 = 1$, b) $\omega_1 = 1$ and $\omega_2 = 2$ and c) $\omega_1 = 2$ and $\omega_2 = 3$.***