On the dynamics of the rigid body with a fixed point: periodic orbits and integrability

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Abstract The aim of the present paper is to study the periodic orbits of a rigid body with a fixed point and quasi-spherical shape under the effect of a Newtonian force field given by different small potentials. For studying these periodic orbits, we shall use averaging theory. Moreover, we provide information on the $C^1$-integrability of these motions.

Keywords Rigid body with a fixed point · Periodic orbits · Integrability · Averaging theory

1 Introduction and statement of the main results

The dynamics of rigid bodies, in its different formulations Eulerian, Lagrangian, and Hamiltonian, has been extensively studied in the classic literature; see, for instance [8], for a classic treatment of these topics or [1, 2], and [10] for a more modern approach.

The main objectives in the study of the motion of a rigid body with a fixed point are:

1. To state the equilibria and their stabilities in rigid bodies with a fixed point.
2. To state the periodic solutions, bifurcations, and chaos of its motion.
3. To analyze the integrability and to state the first integrals for the problem.

Our main aim in this work is to study the periodic orbits of a rigid body with a fixed point under a Newtonian gravitational field using averaging theory. As a corollary, we also obtain information about $C^1$ nonintegrability.

It is well known, see for instance [3, p. 57], that the motion of a rigid body with a fixed point is described by the Hamiltonian equations associated to the Hamiltonian

$$\mathcal{H} = \frac{(G^2 - L^2)}{2} \left( \frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} \right)$$

$$+ \frac{L^2}{2C} + U(k_1, k_2, k_3) \quad (1)$$

with

$$k_1 = \left( \frac{H}{G} \right) \sqrt{1 - \left( \frac{L}{G} \right)^2} + \frac{L}{G} \sqrt{1 - \left( \frac{H}{G} \right)^2 \cos g} \sin l$$

\[ k_2 = \left( \frac{H}{G} \sqrt{1 - \left( \frac{L}{G} \right)^2} + \frac{L}{G} \sqrt{1 - \left( \frac{H}{G} \right)^2} \cos g \right) \cos l \\
- \sqrt{1 - \left( \frac{H}{G} \right)^2} \sin g \sin l, \]

\[ k_3 = \left( \frac{H}{G} \left( \frac{L}{G} \right) - \sqrt{1 - \left( \frac{L}{G} \right)^2} \sqrt{1 - \left( \frac{H}{G} \right)^2} \cos g. \]

This is a Hamiltonian in the Andoyer–Deprit canonical variables \((L, G, l, g)\) of two degree-of-freedom with the positive parameters \(A, B, C,\) and \(H.\)

As it is usual, we introduce the parameters

\[ \alpha = \frac{1}{\alpha} - \frac{1 - \alpha}{2}, \]

\[ \beta = \frac{1}{\alpha} - \frac{1}{2}. \]

The parameter \(\beta\) is known as the triaxial coefficient. Note that \(\alpha\) can take any positive value depending on the physical characteristics of the rigid body because without loss of generality we can assume that \(A \leq B \leq C,\) then \(\alpha \geq 0.\) But the triaxial coefficient \(\beta\) is bounded between zero (the oblate spheroid \(A = B\)) and one (the prolate spheroid \(B = C\)), although it is undefined in the limit case of a sphere, taking any value between zero and one depending on the direction in which we approach the limit. Therefore, \(0 \leq \beta \leq 1,\) and consequently \(0 \leq a \beta \leq \alpha.\) See [7] for more details.

In this work, we assume that \(0 < \alpha = \epsilon^k \ll 1,\) then the Hamiltonian (1) is expressed by

\[ \mathcal{H} = \frac{G^2}{2C} + \epsilon k \mathcal{P}_1 + U(k_1, k_2, k_3), \]

where

\[ \mathcal{P}_1 = \frac{1}{2C} (G^2 - L^2) (1 - \beta \cos 2l). \]

Moreover, we shall consider the following three cases:

- **Case 1:** \(U(k_1, k_2, k_3) = \epsilon V(k_1, k_2, k_3)\) and \(k = 2,\) i.e.

\[ \mathcal{H} = \frac{G^2}{2C} + \epsilon \mathcal{P}_2 + \epsilon^2 \mathcal{P}_1, \]

where \(\mathcal{P}_2 = V(k_1, k_2, k_3).\)

- **Case 2:** \(U(k_1, k_2, k_3) = \epsilon V(k_1, k_2, k_3)\) and \(k = 1,\) i.e.

\[ \mathcal{H} = \frac{G^2}{2C} + \epsilon (\mathcal{P}_1 + \mathcal{P}_2). \]

- **Case 3:** \(U(k_1, k_2, k_3) = \epsilon^2 V(k_1, k_2, k_3)\) and \(k = 1,\) i.e.

\[ \mathcal{H} = \frac{G^2}{2C} + \epsilon \mathcal{P}_1 + \epsilon^2 \mathcal{P}_2. \]

We note that \(\mathcal{P}_1\) measures the difference of the shape of the rigid body between a sphere and a triaxial ellipsoid, and \(\mathcal{P}_2\) measures the external forces acting on the rigid body. We shall assume that the perturbing functions \(\mathcal{P}_1\) are smooth in the variables \((L, l; G, g).\)

Our main results are the following three theorems, each one studying the periodic orbits of the previous three cases.

**Theorem 1** We consider the motion of the rigid body with a fixed point under the action of the Hamiltonian (2). On the energy level \(\mathcal{H} = h > 0\) if \(\epsilon \neq 0\) is sufficiently small, then for every zero \((L_0, l_0)\) of the system

\[ f_1^1(L, l) = -\frac{1}{2\pi} \sqrt{\frac{C}{2h}} \int_0^{2\pi} \frac{\partial \mathcal{P}_2}{\partial l} dg = 0, \]

\[ f_1^2(L, l) = \frac{1}{2\pi} \sqrt{\frac{C}{2h}} \int_0^{2\pi} \frac{\partial \mathcal{P}_2}{\partial L} dg = 0, \]

satisfying that

\[ \det \left( \frac{\partial (f_1^1, f_1^2)}{\partial (L, l)} \right)_{(L, l) = (L_0, l_0)} \neq 0, \]

there exists a \(2\pi\)-periodic solution \((L(g, \epsilon), l(g, \epsilon), G(g, \epsilon))\) in the variable \(g\) of the rigid body such that

\[(L(g, 0), l(g, 0), G(g, 0)) = (L_0, l_0, \sqrt{2Ch})\]

when \(\epsilon \to 0.\)

Let \(R = (a^2 - b_2^2)^2(a^2 + b_2^2) + (a^4 - 6a^2b_2^2 + b_2^4)c^2.\) An application of Theorem 1 is the following.

**Corollary 1** A spherical rigid body with Hamiltonian (2), weak potential \(\mathcal{P}_2 = ak_1 + bk_2 + ck_3\) with \(a, b,\) and \(c\) positive and \(\epsilon \neq 0\) sufficiently small has in every positive energy level at least four linear stable periodic orbits if \(R > 0,\) two linear stable periodic orbits
if \( R = 0 \), and two linear stable periodic orbits and two unstable ones if \( R < 0 \).

Note that Corollary 1 describes the motion of a nonhomogeneous sphere with center of mass at the point \((a, b, c)\) under a weak gravitational Newtonian potential.

**Theorem 2** We consider the motion of the rigid body with a fixed point under the action of the Hamiltonian (3). On the energy level \( \mathcal{H} = h > 0 \) if \( \varepsilon \neq 0 \) is sufficiently small, then for every zero \((L_0, l_0)\) of the system

\[
\begin{align*}
    f_1^1(L, l) &= -\frac{1}{2\pi} \sqrt{\frac{C}{2h}} \int_0^{2\pi} \frac{\partial (P_1 + P_2)}{\partial l} \, dg = 0, \\
    f_1^2(L, l) &= -\frac{1}{2\pi} \sqrt{\frac{C}{2h}} \int_0^{2\pi} \frac{\partial (P_1 + P_2)}{\partial L} \, dg = 0,
\end{align*}
\]

satisfying (5), there exists a \(2\pi\)-periodic solution of the form \((L(g, \varepsilon), l(g, \varepsilon), G(g, \varepsilon))\) in the variable \( g \) of the rigid body such that

\[
(L(g, 0), l(g, 0), G(g, 0)) = (L_0, l_0, \sqrt{2Ch})
\]

when \( \varepsilon \to 0 \).

An application of Theorem 2 is the following.

**Corollary 2** A quasi-spherical rigid body with Hamiltonian (3), weak potential \( P_2 = ck^3 \) with \( c > 0 \) and \( \varepsilon \neq 0 \) sufficiently small can have at least eight periodic orbits in every positive energy level.

We remark that Corollary 2 describes the motion of a nonhomogeneous quasi-spherical rigid body with center of mass at the point \((0, 0, c)\) under a weak gravitational Newtonian potential. The linear stability of the periodic orbits described in Corollary 2 can be easily studied using Theorem 5, but since there are many possibilities we do not present an explicit description here.

**Theorem 3** We consider the motion of the rigid body with a fixed point under the action of the Hamiltonian (4). On the energy level, \( \mathcal{H} = h > 0 \) if \( \varepsilon \neq 0 \) is sufficiently small and

\[
\int_0^{2\pi} \frac{\partial P_1}{\partial l} = \int_0^{2\pi} \frac{\partial P_1}{\partial L} = 0,
\]

then for every zero \((L_0, l_0)\) of the system

\[
\begin{align*}
    f_2^1(L, l) &= \frac{1}{2\pi} \int_0^{2\pi} F_1^1(L, l, g) \, dg = 0, \\
    f_2^2(L, l) &= \frac{1}{2\pi} \int_0^{2\pi} F_2^2(L, l, g) \, dg = 0,
\end{align*}
\]

where

\[
F_1^1 = \frac{1}{4h^2} \left( 2hc \left( \frac{\partial^2 P_1}{\partial l^2} \left( \int_0^{2\pi} \frac{\partial P_1}{\partial l} \, dg \right) \right) \right. \\
\left. + \frac{\partial^2 P_1}{\partial l \partial l} \left( \int_0^{2\pi} \frac{\partial P_1}{\partial l} \, dg \right) \right) \\
+ \frac{C}{(2h)^3} \left( \frac{\partial P_1}{\partial l} + \sqrt{2Ch} \frac{\partial^2 P_1}{\partial l \partial G} \right) \\
+ \frac{C}{2h} \left( \frac{\partial P_1}{\partial G} \right),
\]

\[
F_2^2 = \frac{1}{4h^2} \left( 2hc \left( \frac{\partial^2 P_1}{\partial L^2} \left( \int_0^{2\pi} \frac{\partial P_1}{\partial L} \, dg \right) \right) \right. \\
\left. + \frac{\partial^2 P_1}{\partial L \partial L} \left( \int_0^{2\pi} \frac{\partial P_1}{\partial L} \, dg \right) \right) \\
+ \frac{C}{(2h)^3} \left( \frac{\partial P_1}{\partial L} - \sqrt{2Ch} \frac{\partial^2 P_1}{\partial L \partial G} \right) \\
- \frac{C}{2h} \frac{\partial P_1}{\partial G} \left( \frac{\partial P_1}{\partial L} \right),
\]

satisfying (5), there exists a \(2\pi\)-periodic solution in the form \((L(g, \varepsilon), l(g, \varepsilon), G(g, \varepsilon))\) in the variable \( g \) of the rigid body such that

\[
(L(g, 0), l(g, 0), G(g, 0)) = (L_0, l_0, \sqrt{2Ch})
\]

when \( \varepsilon \to 0 \).

**Corollary 3** A quasi-spherical rigid body with Hamiltonian (4), weak potential \( P_2 = ck^3 \) with \( c > 0 \), energy level \( \mathcal{H} = h = 3H^2/(2C) \) and \( \varepsilon \neq 0 \) sufficiently small can have at least fourteen periodic solutions.

In the proofs of Theorems 1 and 2, we shall use the averaging theory of first order, and in the proof of Theorem 3 we shall use the averaging theory of second order. We note that the averaging method allows to find analytically periodic orbits of the rigid body
with a fixed point at any positive energy level, reducing its study to find zeros of a convenient system of two equations with two unknowns.

We shall use the periodic orbits found in the previous theorems for obtaining information about the $C^1$ nonintegrability in the sense of Liouville–Arnold of a rigid body with a fixed point.

**Theorem 4** For a rigid body with a fixed point whose motion is given by the Hamiltonian (2) (respectively (3) or (4)), we have:

1. Either the Hamiltonian system is Liouville–Arnold integrable and the gradients of two independent first integrals are linearly dependent on some points of all the periodic orbits found in Theorem 1 (respectively 2 or 3);
2. or it is not Liouville–Arnold integrable with any second first integral of class $C^1$.

Similar studies on the periodic orbits and on the integrability of the Hamiltonian systems for the Henon–Heiles Hamiltonian and the Yang–Mills Hamiltonian have been done in [5] and [6], respectively.

The rest of the paper is dedicated to the proofs of theorems and corollaries stated in this introduction. We shall prove all of them except Theorem 4 whose proof is essentially the same than the proof of Theorem 2 of [5].

2 **Proof of Theorems 1, 2, and 3**

**Proof of Theorem 1** The Hamiltonian differential equations associated to the Hamiltonian (2) are

\[
\frac{dL}{dt} = -\varepsilon \frac{\partial P_2}{\partial l} + O(\varepsilon^2), \quad \frac{dG}{dt} = -\varepsilon \frac{\partial P_2}{\partial g} + O(\varepsilon^2),
\]

\[
\frac{dl}{dt} = \varepsilon \frac{\partial P_2}{\partial L} + O(\varepsilon^2), \quad \frac{dg}{dt} = \frac{G}{C} + \varepsilon \frac{\partial P_2}{\partial G} + O(\varepsilon^2).
\]

Taking $g$ as the new independent variable the previous differential systems becomes

\[
\frac{dL}{dg} = -\varepsilon \frac{C}{G} \frac{\partial P_2}{\partial l} + O(\varepsilon^2), \quad \frac{dG}{dg} = -\varepsilon \frac{C}{G} \frac{\partial P_2}{\partial g} + O(\varepsilon^2),
\]

\[
\frac{dl}{dg} = \varepsilon \frac{C}{G} \frac{\partial P_2}{\partial L} + O(\varepsilon^2).
\]

Since we want to study the periodic orbits at the energy level $H = h$, we isolate $G$ from the energy, obtaining $G = \sqrt{2hC} + O(\varepsilon)$ because $G$ is the modulus of the angular momentum. Therefore, the equations of motion on this energy level are

\[
\frac{dL}{dg} = -\varepsilon \sqrt{\frac{C}{2h}} \frac{\partial P_2}{\partial l} + O(\varepsilon^2),
\]

\[
\frac{dl}{dg} = \varepsilon \sqrt{\frac{C}{2h}} \frac{\partial P_2}{\partial L} + O(\varepsilon^2).
\]

These differential equations are in the normal form for applying the first-order averaging theory; see Theorem 5 in the Appendix. From this theorem, the proof is over.

**Proof of Corollary 1** For proving Corollary 1 from Theorem 1, we need to compute $(f_1^1, f_1^2)$.

Indeed we have

\[
f_1 = (f_1^1, f_1^2) = \left( \frac{H\sqrt{4 - \frac{2J}{2Ch}}(b \sin l - a \cos l)}{4h}, \frac{c\sqrt{Ch}\sqrt{2 - \frac{L^2}{2Ch}} - HL\sqrt{4 - \frac{2J}{2Ch}(b \cos l + a \sin l)}}{2Ch^2\sqrt{4 - \frac{2J}{2Ch}}}, \right).
\]

Solving the system $(f_1^1, f_1^2) = (0, 0)$ we obtain the following four solutions

\[
L = \pm \frac{c\sqrt{2Ch}}{\sqrt{a^2 + b^2 + c^2}}, l = \pm \arccos \left( \frac{b}{\sqrt{a^2 + b^2}} \right).
\]

On the other hand, the determinant (5) for the four solutions is equal to

\[
\frac{(a^2 + b^2 + c^2)H^2}{8ch^3},
\]

or

\[
\frac{((a^2 - b^2)^2(a^2 + b^2) + (a^4 - 6a^2b^2 + b^4)c^2)H^2}{8(a^2 + b^2)^2Ch^3}.
\]

Analyzing the eigenvalues of the matrix, which appear in (5), the corollary follows.

**Proof of Theorem 2** The Hamiltonian differential equations associated to the Hamiltonian (3) are
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Theorem 2, we compute

Proof of Corollary 2

β(L^2 - 2Ch) \sin 2l, cH + 2hL(β \cos 2l - 1)\right) \bigg)\right).

Note that the previous system can have the solutions

p_{1,2} = \left( L = -\sqrt{2hC}, l = ± \frac{1}{2} \arccos \left( \frac{4\beta h^{3/2}}{4h^{3/2} - \sqrt{2cH}} \right) \right),

p_{3,4} = \left( L = \sqrt{2hC}, l = ± \frac{1}{2} \arccos \left( \frac{4\beta h^{3/2}}{4h^{3/2} - \sqrt{2cH}} \right) \right),

p_{5,6} = \left( L = -\frac{cH}{2h(1 + \beta)}, l = ± \frac{\pi}{2} \right),

p_{7,8} = \left( L = -\frac{cH}{2h(1 - \beta)}, l = 0, \pi \right).

We note that the first four points are defined only if the arguments of the arcsec are in (−∞, −1) ∪ (1, +∞). The last four points are always defined because β ∈ (0, 1).

It is not difficult to compute the determinant (5) for this eight points and their eigenvalues, but since there are several possibilities we do not describe all of them here.

Proof of Theorem 3 The Hamiltonian differential equations corresponding to the Hamiltonian (4) are

Proof of Corollary 2 For proving this corollary using Theorem 2, we compute f_1 = (f_1^1, f_1^2), which is equal to

\frac{dL}{dt} = -\frac{C}{G} \frac{\partial (P_1 + P_2)}{\partial l} + O(\varepsilon^2),

\frac{dG}{dt} = \frac{G}{C} + \varepsilon \frac{\partial (P_1 + P_2)}{\partial G} + O(\varepsilon^2).

These differential equations are in the normal form for applying the first-order averaging theory; see Theorem 5. From this theorem, the proof is completed.

Now taking g as the new independent variable and isolating G from the energy level H = h the previous differential system writes

\frac{dL}{dt} = \varepsilon \frac{\partial P_1}{\partial l} - \varepsilon^2 \frac{\partial P_2}{\partial l},

\frac{dG}{dt} = \varepsilon \frac{\partial P_1}{\partial g} - \varepsilon^2 \frac{\partial P_2}{\partial g},

\frac{dl}{dt} = \varepsilon \frac{\partial P_1}{\partial L} + \varepsilon^2 \frac{\partial P_2}{\partial L},

\frac{dg}{dt} = \frac{C}{G} + \varepsilon \frac{\partial P_1}{\partial G} + \varepsilon^2 \frac{\partial P_2}{\partial G}.

Now taking g as the new independent variable and isolating G from the energy level H = h the previous differential system writes

\frac{dL}{dg} = \varepsilon F_{11} + \varepsilon^2 F_{12} + O(\varepsilon^3),

\frac{dl}{dg} = \varepsilon F_{21} + \varepsilon^2 F_{22} + O(\varepsilon^3),

with

F_{11} = -\frac{C}{a_0} \frac{\partial P_1}{\partial l},\quad F_{12} = \frac{C}{a_0} \frac{\partial P_1}{\partial L},

F_{21} = \frac{1}{a_0^2} \left( a_1 \frac{\partial P_1}{\partial l} + a_0^2 \frac{\partial P_1}{\partial G} \right) - a_0 a_1 C \frac{\partial^2 P_1}{\partial l \partial G},

a_0 a_1 C \frac{\partial^2 P_1}{\partial l \partial G},\quad a_0 a_1 C \frac{\partial^2 P_1}{\partial l \partial G}.
\[ F_{22} = \frac{1}{a_0^2} \left( -a_1 C \frac{\partial P_1}{\partial L} - C^2 \frac{\partial P_1}{\partial G} \frac{\partial L}{\partial L} + a_0 C \frac{\partial P_2}{\partial L} + a_0 a_1 C \frac{\partial^2 P_1}{\partial L \partial G} \right), \]

where in the previous functions we have

\[ G = a_0 + a_1 \varepsilon + O(\varepsilon^2), \quad a_0 = \sqrt{2}C h, \quad a_1 = -\frac{CP_1}{a_0}. \]

Applying the averaging of first order described in Theorem 5, we obtain that the function \( f_1 \equiv 0 \), due to the assumption (6). So we compute the function \( f_2 = (f_2^1, f_2^2) \) corresponding to the averaging of second order, and we get the expression given in the statement of Theorem 3. \( \square \)

**Proof of Corollary 3** For proving this corollary using Theorem 3, we compute

\[ f_1 = (f_1^1, f_2^1) = \left( \frac{(L^2 - 3H^2) \beta \sin(2l)}{\sqrt{3}H}, \frac{L(c^2C^2(9H^2 - 7L^2) + 243H^6(\beta \cos(2l) - 1))}{243^{3/2} \sqrt{3}H} \right), \]

Note that the previous system can have the solutions

\[ p_{1,2,3,4} = \left( L = \pm \sqrt{3}H, \quad l = \pm \frac{1}{2} \arccos \left( \frac{81L^4 + 4c^2C^2}{81LH^4} \right) \right), \]

\[ p_{5,6,7,8} = \left( L = \pm \sqrt{3}HC^2 - 27H^4(\beta + 1), \quad l = \pm \frac{\pi}{2} \right), \]

\[ p_{9,10,11,12} = \left( L = \pm \frac{3H\sqrt{c^2C^2 - 27H^4(\beta - 1)}}{\sqrt{3}cC}, \quad l = 0, \pi \right), \]

\[ p_{13,14} = (L = 0, l = 0, \pi). \]

We note that the first four points are defined only if the arguments of the arccos are in \([-1, 1]\). The eight points from \( p_5 \) to \( p_{12} \) are defined when the corresponding radical is nonnegative. The last two points always are defined.

It is not difficult to compute the determinant (5) for this fourteen points and their eigenvalues, but since there are several possibilities we do not describe all of them here. \( \square \)

**Acknowledgements** The first and third authors were partially supported by MINECO/FEDER grant number MTM2011-22587. The second author was partially supported by MICINN/FEDER grant number MTM2008-03437, ICREA Academia, AGAUR grant number 2009SGR 410, and FP7-PEOPLE-2012-IRSES 316338 and 318999.

**Appendix**

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first- and second-order approximation for the periodic solutions of a periodic differential system; for the proof, see Theorems 11.5 and 11.6 of Verhulst [4, 11], and [9].

Consider the differential equation

\[ \dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad x(0) = x_0 \]

(7)

with \( x \in D \subset \mathbb{R}^n, t \geq 0 \). Moreover, we assume that both \( F_1(t, x) \) and \( F_2(t, x) \) are \( T \) periodic in \( t \). Separately, we consider in \( D \) the averaged differential equation

\[ \dot{y} = \varepsilon f_1(y) + \varepsilon^2 f_2(y), \quad y(0) = x_0, \]

(8)

where

\[ f_1(y) = \frac{1}{T} \int_0^T F_1(t, y) \, dt, \]

\[ f_2(y) = \frac{1}{T} \int_0^T \left[ \int_0^T \left[ D_y F_1(s, y) \int_0^s F_1(t, y) \, dt \right. \right. \]

\[ \left. + F_2(s, t) \right] \, ds. \]

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with \( T \)-periodic solutions of Eq. (8).

**Theorem 5** Consider the two initial value problems (7) and (8). Suppose:
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(i) $F_1$, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, $F_2$ and its Jacobian $\partial F_2/\partial x$ are defined, continuous, and bounded by an independent constant $\varepsilon$ in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.

(ii) $F_1$ and $F_2$ are $T$-periodic in $t$ ($T$ independent of $\varepsilon$).

(iii) $y(t)$ belongs to $\Omega$ on the interval of time $[0, 1/\varepsilon]$.

Then the following statements hold.

(a) For $t \in [1, \varepsilon]$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.

(b) If $p$ is a singular point of the averaged equation (8) and

$$\det \left( \frac{\partial (f_1 + \varepsilon f_2)}{\partial y} \right)_{y=p} \neq 0,$$

then there exists a $T$-periodic solution $\varphi(t, \varepsilon)$ of Eq. (7), which is close to $p$ such that $\varphi(0, \varepsilon) \to p$ as $\varepsilon \to 0$.

(c) The stability or instability of the limit cycle $\varphi(t, \varepsilon)$ is given by the stability or instability of the singular point $p$ of the averaged system (8). In fact, the singular point $p$ has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$.

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