Simple method for quick estimation of aquifer hydrogeological parameters

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Abstract. Development of simple and accurate methods to determine the aquifer hydrogeological parameters was of importance for groundwater resources assessment and management. Aiming at the present issue of estimating aquifer parameters based on some data of the unsteady pumping test, a fitting function of Theis well function was proposed using fitting optimization method and then a unitary linear regression equation was established. The aquifer parameters could be obtained by solving coefficients of the regression equation. The application of the proposed method was illustrated, using two published data sets. By the error statistics and analysis on the pumping drawdown, it showed that the method proposed in this paper yielded quick and accurate estimates of the aquifer parameters. The proposed method could reliably identify the aquifer parameters from long distance observed drawdowns and early drawdowns. It was hoped that the proposed method in this paper would be helpful for practicing hydrogeologists and hydrologists.

1. Introduction
In many watersheds, groundwater is one of the major sources of water supply for industrial, municipal and agricultural users [1]. However, in many parts of the world, groundwater resources have been overexploited [2]. They have been utilized at excessive rates that lead to various harmful environmental side effects that seriously hinder the development of national economy [3]. It is important for water managers and engineers to assess and calculate groundwater quantity accurately and reliably.

Aquifer hydro-geological parameters are important in groundwater resources assessment and in groundwater dynamic prediction. They are usually obtained from the data of unsteady pumping test, which is the main way to determine the aquifer parameters such as transmissivity and storage coefficient [4-6]. The first study to determine the aquifer parameters was analytically derived by Theis [7]. Later, a lot of methods based on the Theis equation were proposed to estimate the hydro-geological parameters by many scientists [8-11].

At present, in view of the non-steady flow pumping test, the Theis curve method, the straight-line method and the intelligent optimization algorithm are widely used to estimate aquifer parameters.
However, these methods have some deficiencies in practice. The main problem of the Theis curve method is that the optional is larger when the coincident points of the measured curve and the theoretical curve are less. Because drawing line and reading data may have artificial arbitrariness, the straight-line method [12] will have a certain influence on the precision of the calculation results and it only can be used for the variable \( u \) is small [13]. With the development of computer technology, the intelligent optimization algorithm [14-16] is widely applied to determine the aquifer parameters. However, the calculation process of the intelligent optimization algorithm is complex and its convergence property cannot be always fully guaranteed.

This paper is organized as follows. First, a fitting function of Theis well function is proposed using fitting optimization method when the independent variable \( u \) is in the range of 0.01 to 1.1. Second, a unary linear regression equation is established to estimate the aquifer parameters (transmissivity and storage coefficient). Then two examples are performed and their results are presented to demonstrate the advantages of the proposed method. Finally, conclusions are drawn.

2. Fitting function of the Theis well function

The well-known Theis solution, which describes the observation drawdown under the condition of homogenous, isotropy and infinitely extent aquifer, is

\[
s = -\frac{Q}{4\pi T} W(u)
\]  

(1)

where \( s \) is groundwater drawdown observed in the well, [L]; \( Q \) is constant discharge rate, [L²T⁻¹]; \( T \) is transmissivity coefficient of aquifer, [L²T⁻¹]; \( W(u) \) is dimensionless Theis well function which can be written as

\[
W(u) = \int_{u}^{\infty} \frac{1}{y} \exp(-y)dy
\]  

(2)

in which \( u \) can be written as

\[
u = \frac{r^2 S}{4 T t} = \frac{r^2}{4 a t}
\]  

(3)

where \( r \) is the radial distance from pumping well, [L]; \( S \) is the storage coefficient of aquifer, dimensionless; \( t \) is time, [T]; \( a \) is the pressure conduction coefficient of aquifer, [L²T⁻¹].

The scatter diagram of Theis well function is shown in figure 1. Because the well function can’t be integrated directly, we need to find out an approximate function to simplify the calculation. Equation (2) can also be written as

\[
W(u) = -0.5772 - \ln u + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n \cdot n!}
\]  

(4)

where after the first two items are alternating series. According to the nature of the alternating series, when \( u \) is small (\( u < 0.01 \)), that is to say, for small radial distance, or large time, equation (4) reduces to

\[
W(u) \approx -0.5772 - \ln u
\]  

(5)

The relative error involved in adopting equation (5) instead of equation (4) to compute the observation well drawdown by equation (1) is less than 2.03%.
Figure 1. The scatter diagram of the Theis well function.

However, if \( u \) is large (\( u > 0.01 \)), equation (5) cannot be used to approximately instead of Theis well function. In this paper, as the independent variable \( u \) is within the scope of 0.01 to 1.1, through fitting and optimizing for the Theis well function with the residual sum of squares are minimum, the approximate function of Theis well function is given as

\[
W(u) \approx \begin{cases} 
11.61282u^{-0.06619373} - 11.69514 & 0.01 \leq u \leq 0.26 \\
0.6463112u^{-0.602161} - 0.4250883 & 0.26 \leq u \leq 1.1 
\end{cases}
\]  

(6)

Figure 2. The fitting curve of Theis well function.

As \( u \) is within the scope of 0.01 to 1.1, the values of equations (4) and (6) are given in table 1 along with the relative error between Theis well function and the fitting function. As shown in this table, the biggest relative error is less than 1.33%. The fitting curve of Theis well function is shown in figure 2.
It is clear that equation (6) has a high alternative precision and it can satisfy the actual engineering calculation accuracy requirements.

**Table 1.** Comparison results of equations (4) and (6).

| u   | The value of equation (4) | The value of equation (6) | Relative error | u   | The value of equation (4) | The value of equation (6) | Relative error |
|-----|---------------------------|---------------------------|----------------|-----|---------------------------|---------------------------|----------------|
| 0.01| 4.0379                    | 4.056475085               | 0.46%          | 0.27| 0.9849                    | 0.997045456              | 1.23%          |
| 0.012| 3.8573                    | 3.867518545               | 0.26%          | 0.28| 0.9573                    | 0.966232585             | 0.93%          |
| 0.014| 3.7054                    | 3.709527591               | 0.11%          | 0.29| 0.9309                    | 0.937134133             | 0.67%          |
| …   | …                         | …                         | …              | …  | …                         | …                         | …              |
| 0.24| 1.0726                    | 1.068183148               | 0.41%          | 0.89| 0.2647                    | 0.268217654             | 1.33%          |
| 0.25| 1.0443                    | 1.033741162               | 1.01%          | 1   | 0.2194                    | 0.221222956             | 0.83%          |
| 0.26| 1.0139                    | 1.000737744               | 1.30%          | 1.1 | 0.186                     | 0.185165148             | 0.45%          |

3. Proposed method

In this paper, the fitting function of the Theis well function is used to simply estimate the aquifer parameters. Equation (6) can be transformed into

\[ W(u) = Au^C + B \]  

(7)

where \( A, B \) and \( C \) are constant value. The substitution of equations (3) and (7) into equation (1) gives

\[ s = \frac{Q}{4\pi T}(A\left(\frac{r^2}{4at}\right)^C + B) = \frac{AQ}{4\pi T}\left(\frac{r^2}{4a}\right)^C\left(\frac{1}{t^C}\right) + \frac{BQ}{4\pi T} \]  

(8)

by defining variables as

\[ k = \frac{AQ}{4\pi T}\left(\frac{r^2}{4a}\right)^C \]  

(9)

\[ b = \frac{BQ}{4\pi T} \]  

(10)

\[ x = \frac{1}{t^C} \]  

(11)

Equation (8) can be rewritten succinctly as

\[ s = kx + b \]  

(12)

It is clear that equation (12) is a linear equation that its independent variable is \( x \) and dependent variable is \( s \). The aquifer parameters can be estimated by calculating the coefficients of equation (12). Hence, according to the pumping test data \((s_i, x_i)(i = 1, 2, \cdots, n)\), using the least square method, \( k \) and \( b \) can be estimated by

\[
\begin{align}
\hat{k} &= \frac{\sum_{i=1}^{n} (x_i - \bar{x})(s_i - \bar{s})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \\
\hat{b} &= \bar{s} - \hat{k}\bar{x}
\end{align}
\]  

(13)
where \( \bar{x} \) and \( \bar{s} \) are average value that can be expressed as

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{14}
\]

\[
\bar{s} = \frac{1}{n} \sum_{i=1}^{n} s_i \tag{15}
\]

Finally, the aquifer parameters can be obtained by substituting equation (13) into equations (9), (10) and (11), leading to

\[
T = \frac{BQ}{4 \pi b} \tag{16}
\]

\[
a = \frac{r^2}{4} \left( \frac{Ab}{Bk} \right)^{\frac{1}{2}} \tag{17}
\]

\[
S = \frac{T}{a} \tag{18}
\]

4. Error and reliability criteria
To evaluate the reliability of the calculated aquifer parameters or the goodness-of-fit of the calculated drawdown to the observed drawdown, two different criteria, the mean error (ME) and the standard error of estimate (SEE) criteria, are used to computed the errors between the predicted and the observed drawdown. The ME is given by

\[
\text{ME} = \frac{1}{n} \sum_{j=1}^{n} (s - s_e)_j \tag{19}
\]

The SEE is given by

\[
\text{SEE} = \left( \frac{1}{n-2} \sum_{j=1}^{n} (s - s_e)_j^2 \right)^{0.5} \tag{20}
\]

where \( n \) is total drawdown number (nondimensional), \( s \) is observed drawdown [L], \( s_e \) is estimated drawdown [L]. ME gives the normality of distribution about the deviation from the zero mean, and SEE shows the presence of outliers in the data. Low values of these criteria indicate that the calculated aquifer parameters are reliable.

5. Application
In this section, two sets of pumping test data are chosen, one for the radial distance from pumping well is long and the other for the pumping test time is short. The first data are taken from Xue [17] and the second data are from WPRS [18].

5.1. Example 1. Radial distance from pumping well is long
In this example, a complete well discharges a confined aquifer with a pumping rate of 22.60 m³/h. The distance between pumping well and observation well is 117.85 m. The observation data of groundwater drawdowns and its corresponding time is listed in table 2.
Table 2. The drawdowns and time in pumping test.

| Time (h) | Drawdown (m) |
|---------|-------------|
| 8       | 0.66        |
| 10      | 0.76        |
| 12      | 0.88        |
| 15      | 1.00        |
| 20      | 1.15        |
| 25      | 1.22        |
| 30      | 1.32        |
| 35      | 1.39        |
| 40      | 1.45        |
| 50      | 1.51        |
| 60      | 1.67        |

The following steps are necessary for an effective application of the presented methodology in the previous sections:

- Calculate the variable $x$ values from equation (11) for each pumping time. As $C =$ -0.06619373, for each pumping time, $x = (1.1476, 1.1646, 1.1788, 1.1963, 1.2193, 1.2375, 1.2525, 1.2653, 1.2766, 1.2956, 1.3113). As $C =$ -0.6023161, for each pumping time, $x = (3.499, 4.0024, 4.4669, 5.1095, 6.0762, 6.9503, 7.757, 8.5117, 9.2246, 10.5516, 11.7763).

- Calculate the regression coefficients $k$ and $b$ by the use of the SPSS 19.0 software. When $u$ is within the scope of 0.26 to 1.1, the calculated $b$ is equal to 0.355886, whereas $B$ is equal to -0.4250883 in this condition, from equation (16) we can know that if $b$ and $B$ are opposite, the transmissivity of aquifer is negative, which is impossible. Hence, we can come to a conclusion that $u$ is within the scope of 0.01 to 0.26 during the pumping test. The output of linear regression is shown in table 3, from which we can know that $k = 5.961839$ and $b = -6.159595$.

| Coefficients | Standard error | t Stat | P-value | Lower 95% | Upper 95% |
|--------------|----------------|--------|---------|-----------|-----------|
| Intercept    | -6.159595185   | 0.204048| -30.187 | -6.62118  | -5.69801  |
| X Variable 1 | 5.961839207    | 0.165556| 36.011  | 5.587325  | 6.336353  |

- Take the significance level $\alpha = 0.05$ and the degree of freedom $n - 2 = 9$, then check the critical value distribution table we conclude that $|t| = 36.01099 > t_{\alpha/2}(9) = 2.262$, which has gone through the t test. Moreover, from table 3 we know that the significance probability value is $4.86 \times 10^{-11}$, which is far less than the value of $\alpha$. Hence we can conclude that under the condition of the significance level $\alpha = 0.05$ the regression is significant.

- Calculate the aquifer transmissivity value from equation (16), which yields, for the data at hand, $T = 3.414964$ m$^2$/h.

- Calculate the pressure conduction coefficient value from equation (17), which yields $a = 2359.64$ m$^2$/h.

- Calculate the storage coefficient value from equation (18). Hence, $S = 1.447 \times 10^{-3}$.

The calculation results of aquifer parameters by the approach proposed in this paper and the commonly applied method that Theis curve were listed in table 4. From table 4, it is known that the results of aquifer parameters by the approach proposed in this paper are very close to that by the Theis.
curve method. Hence, it can be said that the approach proposed in this paper is reliable for estimating aquifer parameters from unsteady pumping test data. In addition, the prediction errors are represented in Table 4. According to these errors, the values obtained from the proposed method are better than the Theis curve method.

**Table 4.** The comparison of the method in this paper and the method of Theis curve.

| Method          | $T$ (m³/h) | $S$  | $ME \times 10^{-3}$ | $SEE \times 10^{-3}$ |
|-----------------|------------|------|---------------------|----------------------|
| Proposed approach | 3.415       | 0.00145 | 0.004               | 28.582               |
| Theis curve     | 3.341       | 0.00154 | 15.351              | 33.727               |

![Figure 3. Observed drawdowns and simulated drawdowns by the proposed method.](image1)

![Figure 4. Observed drawdowns and simulated drawdowns by Theis curve method.](image2)
Figure 5. Relative error of simulated drawdowns by the proposed method and Theis curve method.

Taking the solution of aquifer parameters into equation (1), we can get the values of simulated drawdowns. Figures 3 and 4 show the drawdowns simulated by the proposed method and the Theis curve method, respectively. As shown in these figures, both methods are capable of accurately estimating aquifer parameters. Figure 5 shows the relative error of simulated drawdowns between the proposed method and the Theis curve method. It is observed from figure 5 that the accuracy of the proposed method to determine the aquifer parameters is higher than Theis curve method in this example.

5.2. Example 2. Pumping test time is short
In this example, a complete well discharges a confined aquifer with a pumping rate of 4.612 m$^3$/min. The distance between pumping well and observation well is 30.48 m. The observation data of groundwater drawdowns and its corresponding time is listed in table 5.

Table 5. Pumping test data.

| Time (min) | Drawdown (m) |
|------------|--------------|
| 20         | 0.125        |
| 25         | 0.152        |
| 30         | 0.168        |
| 40         | 0.201        |
| 50         | 0.223        |
| 60         | 0.244        |
| 70         | 0.262        |
| 80         | 0.280        |
| 90         | 0.293        |
| 100        | 0.305        |
| 120        | 0.326        |
| 180        | 0.378        |
| 240        | 0.411        |
| 300        | 0.442        |
| 360        | 0.463        |
| 420        | 0.485        |
Table 6. The comparison of the method in this paper and the method of Theis curve.

| Method            | $T$ (m²/min) | $S$ | ME×10³ | SEE×10³ |
|-------------------|---------------|-----|--------|---------|
| Proposed approach | 2.893         | 0.0662 | 5.023  | 6.521   |
| Theis curve       | 2.787         | 0.0670 | 9.572  | 11.205  |

Figure 6. Observed drawdowns and simulated drawdowns by the proposed method.

Figure 7. Observed drawdowns and simulated drawdowns by Theis curve method.

It is known that this example is an early pumping test which is another situation that $u$ is large. The application of the proposed method and the Theis curve method is given in table 6. Using the proposed method with the pumping test data results in $T$ and $S$ values of $2.893$ m²/min and $0.0662$, the values of $T$ and $S$ obtained by the Theis curve method are $2.787$ m²/min and $0.0670$ [18]. The
values of ME and SEE in table 6 show that the proposed method has a higher precision than the Theis curve method. Figures 6 and 7 show the drawdowns simulated by the proposed method and the Theis curve method, respectively. Almost all the estimated drawdowns obtained by the proposed method and Theis curve method match the observed drawdowns over the pumping test as shown in figures 6 and 7.

Thus, the proposed method is computationally simple and yields quick and acceptable estimates of aquifer parameters. It has a wider range of application compared with the Cooper-Jacob method. The intelligent optimization algorithm, such as ANN method, GA method and chaotic sequence optimization algorithm (CSOA) which can be used to determine hydro-geological parameters are complicated and can not be operated easily. Theis curve method is a commonly used method to estimate aquifer parameters, but its accuracy is not high. The proposed method can be widely used to the unsteady pumping data.

6. Conclusion
Theis well function is the basis for determination of aquifer hydro-geological parameters. The method of optimum fitting is adopted to approximately estimate the value of Theis well function. Through calculating and contrasting, we know that the maximum relative error between the fitting function and Theis well function is less than 1.33% when the independent variable \( u \) is in the range of 0.01 to 1.1. It has been demonstrated that the fitting function which creates the basic conditions for further simply estimating aquifer parameters has a high alternative precision and can satisfy the actual engineering calculation precision requirement.

According to the fitting well function, the aquifer parameters can be estimated by solving the regression coefficients of the unary linear equation. While \( u \) is large \((u > 0.01)\), Theis curve method is the commonly used method to estimate aquifer parameters. But it has a lot of disadvantages, for example, its calculation process is tedious, its accuracy is not high, and the result of anthropogenic influence is great. As to the comparison between the proposed method and the Theis curve method, two examples of actual time-drawdown data show that the proposed method performs better than the Theis curve method. Observed drawdowns of long distances from pumping well and early pumping test in a confined aquifer can be analyzed for accurately estimate aquifer parameters using the proposed method. The proposed method would be helpful to the field practitioners and engineers.

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