Giant Atoms in Synthetic Frequency Dimensions

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(Dated: November 11, 2021)

Giant atoms with nonlocal couplings to real-space waveguides have attracted extensive attention due to their unique interference effects. Here, we propose a feasible scheme for constructing giant atoms in synthetic frequency dimensions with, e.g., a dynamically modulated microring resonator and a tailored multi-level atom. Both analytical and numerical calculations show good agreement between this scheme and real-space two-level giant atoms. Our scheme is proven to have potential applications for manipulating the frequency of light, such as excitation confinement and release in the frequency domain. More importantly, time-reversal symmetry of the model can be readily broken by tuning the relative phase between the external fields applied on the atom, enabling implementations of chiral quantum optics in frequency dimensions.

Giant atoms, which support nonlocal couplings with the surrounding environment (waveguides), have attracted growing research interest in the past few years due to various intriguing phenomena arising from them [1]. In general, giant atoms can be achieved by coupling (artificial) atoms to propagating fields whose wavelengths are much smaller than the atomic sizes (e.g., surface acoustic waves) [2–10], or by coupling atoms to meandering waveguides at separated points [11–13]. With such nonlocal interactions, one should naturally consider phase accumulations of photons between different atom-waveguide coupling points, which lead to a series of striking phenomena that are absent for small atoms, such as frequency-dependent Lamb shifts and relaxation rates [11], decoherence-free interaction between atoms through the waveguide [12, 14–16], and oscillating bound states [13, 18], to name a few.

On the other hand, the concept of synthetic dimensions has been recently proposed and extensively explored in a variety of physical systems such as photonic structures [1–4, 6, 7, 25], cold atoms [26–28], and superconducting circuits [8–10, 30, 31]. With synthetic dimensions, it is possible to explore richer physical effects with fewer geometric dimensions. A simple way to create synthetic dimensions in photonic systems is to actively couple modes at different frequencies via dynamical modulations [1–4, 6, 7]. Moreover, one can also create synthetic dimensions based on other internal degrees of freedom of photons such as momentum [35, 36] and orbital angular momentum [37–39]. The construction of synthetic dimensions not only enables significant reduction of physical resources, but also provide possibilities for manipulating the relevant degrees of freedom [4].

In this Letter, we demonstrate how to implement giant atoms in synthetic frequency dimensions, where the one-dimensional (1D) frequency lattice acting as a discrete waveguide can be readily achieved with a dynamically modulated microring resonator [1–4]. We consider a cyclic four-level atom, where the two lower transitions couple to different sites of the frequency lattice and the two upper transitions are driven by two external fields. The atom is assumed to exhibit two two-photon resonant transitions, such that it can be effectively described as a two-level giant atom that couples nonlocally with the lattice. We reveal that such an effective model can not only support some typical effects of real-space giant atoms such as long-lived populations, but also shows the potential for manipulating the frequency of light.

As shown in Fig. 1(a), we consider a microring resonator which is modulated by an electro-optic modulator (EOM). For the finite frequency range where the group velocity dispersion can be neglected, the resonant frequencies of the ring (labeled by \( m = 0, \pm 1, \pm 2, \ldots \)) are approximately given by \( \omega_m = \omega_0 + m\Omega_s \), where \( \omega_0 \) is the central frequency of the considered range; \( \Omega_s = 2\pi v_g/L \) is the free spectral range, with \( v_g \) and \( L \) the group velocity at \( \omega_0 \) and the circumference of the ring, respectively. Due to the dynamic modulation introduced by the EOM, adjacent resonant modes are effectively coupled and thus a synthetic 1D frequency lattice is formed [1–4]. For weak modulations, the Hamiltonian of the synthetic lattice can be written as (see Sec. I of [40] for more details; \( h = 1 \) throughout this Letter)

\[
H_r = \sum_m [\omega_m a_m^{\dagger} a_m + 2J \cos (\Omega t)(a_m^{\dagger} a_{m+1} + \text{H.c.})],
\]

which in the interaction picture becomes

\[
H_r' = \sum_m J(a_m^{\dagger} a_{m+1} + \text{H.c.})
\]

if the modulation frequency \( \Omega = \Omega_s \) (we focus on this case hereafter). Here \( a_m \) (\( a_m^{\dagger} \)) is the annihilation (creation) operator of the \( m \)th resonant mode of the ring; \( J = \alpha/2\tau_R \) is the effective coupling strength between
adjacent modes, with $\alpha$ the modulation amplitude and $T_R$ the round-trip propagation time of the ring; H.c. denotes the Hermitian conjugate.

In addition, we consider a four-level atom with a ground state $|g\rangle$, two middle states $|f_1\rangle$ and $|f_2\rangle$, and an excited state $|e\rangle$. We select $|g\rangle$ as the reference so that the energies of $|f_1\rangle$, $|f_2\rangle$, and $|e\rangle$ are denoted by $\omega_{f_1}$, $\omega_{f_2}$, and $\omega_e$, respectively. The two lower transitions $|g\rangle \leftrightarrow |f_1\rangle$ and $|g\rangle \leftrightarrow |f_2\rangle$ are coupled with modes $a_0$ and $a_N$, respectively, while the upper transitions $|f_1\rangle \leftrightarrow |e\rangle$ and $|f_2\rangle \leftrightarrow |e\rangle$ are driven by two external fields of amplitudes (frequencies) $\eta_1$ and $\eta_2$ ($\omega_{d,1}$ and $\omega_{d,2}$), respectively. In the interaction picture and with the rotating-wave approximation, the total Hamiltonian of the system can be written as (see Sec. II of [40] for more details)

$$H = H' + \left( g_1 a_0^\dagger |g\rangle \langle f_1| e^{-i\Delta_{d,1} t} + g_2 a_N^\dagger |g\rangle \langle f_2| e^{-i\Delta_{d,2} t} + \eta_1 e^{i\theta} |e\rangle \langle f_1| e^{i\Delta_{d,1} t} + \eta_2 |e\rangle \langle f_2| e^{i\Delta_{d,2} t} + \text{H.c.} \right),$$

where $g_1$ ($g_2$) is the coupling strength between mode $a_0$ ($a_N$) and the transition $|g\rangle \leftrightarrow |f_1\rangle$ ($|g\rangle \leftrightarrow |f_2\rangle$), and is assumed to be real hereafter for simplicity; $\theta$ is the relative phase between the external fields; $\Delta_{d,1} = \omega_0 - \omega_{f_1}$ ($\Delta_{d,2} = \omega_N - \omega_{f_2}$) is the detuning between mode $a_0$ ($a_N$) and the transition $|g\rangle \leftrightarrow |f_1\rangle$ ($|g\rangle \leftrightarrow |f_2\rangle$); $\Delta_{d,1} = \omega_e - \omega_{f_1} - \omega_{d,1}$ ($\Delta_{d,2} = \omega_e - \omega_{f_2} - \omega_{d,2}$) is the detuning between the transition $|f_1\rangle \leftrightarrow |e\rangle$ ($|f_2\rangle \leftrightarrow |e\rangle$) and the external field $\eta_1$ ($\eta_2$). In the case of $\Delta_{d,1}(2) = \Delta_{d,2}(2)$, where two two-photon resonant transitions between $|g\rangle$ and $|e\rangle$ are constructed [41–43], Eq. (3) is equivalent to the time-independent Hamiltonian

$$H' = H'_0 - \Delta_1 |f_1\rangle \langle f_1| - \Delta_2 |f_2\rangle \langle f_2| + \left( g_1 a_0^\dagger |g\rangle \langle f_1| + g_2 a_N^\dagger |g\rangle \langle f_2| + \eta_1 e^{i\theta} |e\rangle \langle f_1| + \eta_2 |e\rangle \langle f_2| + \text{H.c.} \right).$$

Note that we only have considered resonant modes propagating along one direction of the ring. Generally speaking, counterpropagating modes should also be taken into account due to the randomness of the atomic spontaneous emission. However, chiral atom-waveguide interactions can be achieved, for example, with the spin-momentum locking effect [44–46].

Assuming the single-excitation states of the system as

$$|\psi(t)\rangle = \sum_m u_m(t) a_m^\dagger |0, g\rangle + \sum_{\beta=1,2} u_\beta(t) |0, \beta\rangle$$

and solving the Schrödinger equation, we have

$$i\dot{u}_e = \eta_1 e^{i\theta} u_f - \eta_2 u_{f_2},$$

$$i\dot{u}_f = -\Delta_1 u_f + g_1 u_0 + \eta_1 u e^{i\theta},$$

$$i\dot{u}_{f_2} = -\Delta_2 u_{f_2} + g_2 u_0 + \eta_2 u e^{i\theta},$$

$$i\dot{u}_m = J (u_{m+1} - u_{m-1}) + g_1 u_f \delta_{m,0} + g_2 u_{f_2} \delta_{m,N},$$

where $u_m$ and $u_\beta$ are the probability amplitudes of exciting the $m$th mode and the state $|\beta\rangle$, respectively.

Considering the regime of $\{\Delta_1, \Delta_2\} \gg \{g_1, g_2, \eta_1, \eta_2\}$, where $|f_1\rangle$ and $|f_2\rangle$ can be adiabatically eliminated if they are initially unpopulated [41–43], Eq. (6) can be approximately simplified to

$$i\dot{u}_e = \Delta_{e,c} u e + g_{e,c} e^{i\theta} u_0 + g_{e,2} u_{N},$$

$$i\dot{u}_m = J (u_{m+1} - u_{m-1}) + (g_{e,c} e^{i\theta} u_e + \Delta_{e,1} u_0) \delta_{m,0} + (g_{e,2} u_e + \Delta_{e,2} u_{N}) \delta_{m,N},$$

where $\Delta_{e,c} = \eta_1^2 \Delta_1 + \eta_2^2 \Delta_2$, $\Delta_{e,1} = g_{e,1}^2 \Delta_1$, $\Delta_{e,2} = g_{e,2}^2 \Delta_2$, $\Delta_{e,c} = \eta_1^2 \Delta_1 + \eta_2^2 \Delta_2$, and $\delta_{m,0}$ is the effective coupling strength between $a_0$ ($a_N$) and the transition $|g\rangle \leftrightarrow |e\rangle$ due to the corresponding two-photon transition. For simplicity, we assume $\Delta_1 = \Delta_2 = \Delta$ and $\eta_1 = \eta_2 = \eta$ hereafter. In the case of $\theta = 0$ and $\{\Delta_{e,c,1}, \Delta_{e,1,2}\} \to 0$, Eq. (7) is similar to the dynamic equation of a real-space two-level giant atom coupled to a 1D lattice at two separated sites [14], which reads

$$i\dot{w}_e = \lambda_1 w_0 + \lambda_2 w_{N},$$

$$i\dot{w}_m = J (w_{m+1} + w_{m-1}) + w_0 (\lambda_1 \delta_{m,0} + \lambda_2 \delta_{m,N}).$$

Here $w_m$ and $w_e$ are the probability amplitudes of exciting the $m$th site of the lattice and the atom, respectively; $\lambda_1$ ($\lambda_2$) is the atom-waveguide coupling

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a) Schematic illustration of the model under consideration. The ring resonator is modulated by an electro-optic modulator. A four-level atom interacts with two resonant modes of the ring and two external fields, which form two two-photon resonant transitions between $|g\rangle$ and $|e\rangle$. (b) Effective two-level giant atom after diabatic elimination. (c) Ladder-type implementation scheme of the effective giant atom.}
\end{figure}
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2(a)
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FIG. 2. Dynamic evolutions of (a) \( P_c(t) \) and \( P'_c(t) \) with \( \Delta/J = 60 \) and different values of \( N \), (b) \( P_c(t) \) with \( N = 2 \) and different values of \( \Delta \). In panel (a), we assume \( \lambda_1 = \lambda_2 = 0.1J \) for the real-space case. Other parameters are \( g_1 = g_2 = 3J \), \( \eta = 2J \), \( \theta = 0 \), and \( |\psi(t = 0)\rangle = |0,e\rangle \).

strength at the 0th (Nth) site. In view of this, our model is equivalent to a two-level giant atom in the frequency dimension, as shown in Fig. 1(b).

We first verify the validity of the effective giant atom in the familiar case of \( \theta = 0 \) by numerically solving Eq. (6) with appropriate parameters and comparing the results with those of the real-space analogue [obtained with Eq. (8)]. In practice, the frequency dimension typically extends over a limited number of modes where the group velocity dispersion can be neglected. In view of this, we consider 25 lattice sites (resonant modes) in total, which is experimentally accessible [39], and assume that \( N \) is much smaller than the lattice length to avoid undesired boundary effects. Figure 2(a) shows the dynamic evolutions of the populations \( P_c(t) = |u_e(t)|^2 \) and \( P'_c(t) = |u_e(t)|^2 \) with the initial state \( |\psi(t = 0)\rangle = |0,e\rangle \) and with different values of \( N \). Clearly, the evolutions of \( P_c(t) \) and \( P'_c(t) \) can be well fitted if \( |\Delta| \gg \{g_1, g_2, \eta_1, \eta_2\} \) and \( \lambda_1(2) = g_{e,1(2)} \). In particular, long-lived population can be observed if \( N = 2 \) and \( g_{e,1} = g_{e,2} \), similar to a decoherence-free giant atom in real space [11, 14]. Physically, this is because the two atom-waveguide coupling channels interfere destructively in this case. Moreover, we see that the decay rate of \( P_c(t) \) depends on \( N \), which embodies the phase-dependent spontaneous emissions of real-space giant atoms.

We also demonstrate in Fig. 2(b) the dependence of the dynamics on detuning \( \Delta \). The long-lived population becomes more and more ideal as \( \Delta \) increases. Physically, this is due to two reasons: on one hand, the adiabatic elimination method becomes more accurate as \( \Delta \) increases; on the other hand, a larger \( \Delta \) leads to longer coherence time \( t_c \) of the emitted photons (\( t_c \propto 1/(g_{e,1}^2 + g_{e,2}^2) \) [48]) and thus the propagation time \( \tau \) between the two coupling points (\( \tau = N/2J \) [14]) tends to be negligible compared with \( t_c \). Consequently, the non-Markovian retardation effect can be ignored and the giant atom becomes almost decoherence-free with large enough \( \Delta \) [14, 49].

In Fig. 2, we have considered the regime of \( J \gg \{g_{e,1}, g_{e,2}\} \), where the effective giant atom can be almost decoherence-free. In such a regime, however, the giant atom can hardly affect the dynamic diffusion of a single-mode excitation along the frequency lattice due to the weak atom-waveguide couplings, as shown in Fig. S1(a) [40]. In view of this, we now consider the regime where the effective coupling strengths \( g_{e,1} \) and \( g_{e,2} \) are comparable with \( J \). Nevertheless, the effective frequency shifts \( \Delta_{e,1} \) and \( \Delta_{e,2} \) can also be considerable in this case such that the excitations residing between sites \( a_0 \) and \( a_N \) are always strongly confined therein regardless of the giant-atom structure (i.e., the value of \( N \); see Sec. III of [40] for more details). Therefore, we introduce opposite frequency shifts for modes \( a_0 \) and \( a_N \) by coupling them with two extra auxiliary modes \( b_1 \) and \( b_2 \), respectively, i.e.,

\[
\begin{align*}
&i\dot{u}_m = J(u_{m+1} + u_{m-1}) + g_1(u_{f_1} + u_{b_1})\delta_{m,0} \\
&\quad + g_2(u_{f_2} + u_{b_2})\delta_{m,N}, \\
&i\dot{u}_{b_1(2)} = \Delta u_{b_1(2)} + g_{1(2)}u_0,
\end{align*}
\]

where \( u_{b_{1,2}} \) are the probability amplitudes of exciting \( b_1 \) and \( b_2 \), respectively. The dynamic evolutions of \( u_e, u_{f_1}, \) and \( u_{f_2} \) are identical with those in Eq. (6).

We plot in Figs. 3(a) and 3(b) the dynamic evolutions of the modal excitation probabilities \( P_m(t) = |u_m(t)|^2 \) in the presence of the auxiliary modes [solving Eq. (9) numerically] with \( N = 2 \) and \( N = 4 \), respectively, and demonstrate in Figs. 3(c) and 3(d) the dynamic evolutions of \( P'_m(t) = |u_m(t)|^2 \) of the real-space case for comparison (the initial state is assumed to be \( |\psi(t = 0)\rangle = a_1^\dagger|0,g\rangle \) in this case). We find that the effective giant atom exhibits essentially the same evolutions as those of the real-space analogue for both \( N = 2 \) and \( N = 4 \). Different from the case shown in Fig. S1(a) [40], where excitation confinement always can be observed (regardless of \( N \)) due to the large effective frequency shifts, here the single-mode excitation can be strongly confined only if \( N = 2 \) whereas it diffuses along the frequency lattice in a more free manner if \( N = 4 \). We point out that the confined excitation can be dynamically released by turning off the external fields. As shown in Figs. 3(c) and 3(f), the giant atom disappears effectively at \( Jt = 3 \) with the assumption \( \eta(t > 3) = 0 \), such that the excitation is released and diffuses freely along the frequency lattice.

Physically, the phenomena above can be understood from an effective “giant-small atom model” [50], where a lattice (waveguide) is coupled simultaneously with a small atom at \( m = M \) and with a giant atom at \( m = 0 \).
and $m = N$. In fact, the dynamics of our model with $|\psi(t = 0)\rangle = a_M^0|0,g\rangle$ can be effectively described by that of the giant-small atom model with an initially excited small atom located at $m = M$. When $\theta = 0$ and $0 < M < N$, the giant atom is effectively coupled to the small one with symmetric coupling strength $g_{GS} \propto (e^{ik_0M} + e^{ik_0D})$ in the Markovian limit, where $D = N - M$ and $k_0 = -\pi/2$ (see Sec. IV of [40] for more details). For $M = D = 1$ in Figs. 3(a) and 3(c), the excitation oscillation between the small atom (or say, mode $a_1$ of the ring) and the giant one is dominated by the nonzero coupling strength $g_{GS}$. Note that $M = D = 1$ is the only choice achieving nonzero $g_{GS}$ as long as $N = 2$ and $\theta = 0$ [40]. For $M = 1, D = 3$ in Figs. 3(b) and 3(d), however, the giant atom is decoupled from the small one due to $g_{GS} = 0$. In this case, the excitation should diffuse freely as in a bare lattice. However, considering the nonnegligible retardation effects and the finite effective frequency shift $\Delta_{e,e}$ in this case, the giant atom can still be weakly excited and thus the excitation diffusion is modified as shown in Figs. 3(b) and 3(d).

Up to now, we have considered the case $\theta = 0$, which simulates in the synthetic frequency dimension a conventional two-level giant-atom. Now we turn to consider the case $\theta \neq 0$, which can be readily achieved by tuning the phase difference between the external fields. In this case, the time-reversal symmetry of the Hamiltonian Eq. (4) is broken such that the spontaneous emission of the giant atom should be chiral. In particular, the giant atom exhibits almost unidirectional emission for $\theta = \pi/2$ and $N = 3$, as shown in Fig. 4(a). This phenomenon can be understood again from the giant-small atom models for $M < 0 < N$ and $0 < N < M$ (see Sec. IV of [40] for more details), where the small atom is located at the left and right sides of the giant atom, respectively. For $\theta = \pi/2$ and $M < 0 < N$ ($0 < N < M$), the effective coupling strength from the giant atom to the small one, $g_{G \rightarrow S} \propto (e^{i(k_0|M|+\pi/2)} + e^{ik_0D})$, is always zero (nonzero) in the Markovian limit, implying that the resonant modes on the left (right) side of the giant atom cannot (can) be excited. We point out that the weak excitations of the left-side modes here arise from the finite-size effect of the lattice. The chiral profile tends to become more ideal with the increase of the evolution time (and also with the increase of the lattice length to avoid boundary effects), as shown in Fig. S2 [40]. On the other hand, such a giant atom can hardly be excited by photons coming from the right side, as shown in Fig. 4(b). This is because the effective coupling strength from a right-side small atom to the giant one, $g_{S \rightarrow G} \propto (e^{i(k_0|N|+\pi/2)} + e^{ik_0D})$, is always zero in the case of $M > 3$. This thus provides the possibility of realizing cascaded interactions [51–53] in the frequency dimension, with which the excitation of the present giant atom can be transferred to another such one but not vice versa, as shown in Fig. S3 [40].

Moreover, we demonstrate that chiral diffusion of modal excitations can also be achieved based on the mechanism above. As shown in Fig. 4(e), for $N = 3$, the modal excitation profile becomes quite asymmetric with respect to the initially excited mode if $\theta = \pi/2$ and...
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|ψ(t = 0)| = a^g_0(0, g), or if θ = −π/2 and |ψ(t = 0)| = a^e_0(0, g). The two choices of parameters lead to reversed chiral profiles. This result arises from the fact that a portion of the excitation is transferred to the giant atom and reemitted in the chiral manner shown in Fig. 4(a), and then it interferes with another portion of the excitation that has diffused along the lattice. We have checked that the chiral diffusion is most obvious when J ≈ {g_e,1, g_e,2} and disappears gradually when deviating from this condition. Note that the giant atom can hardly be excited (and thus the chiral modal profile almost disappears) if one only reverses θ but keeps the initial state unchanged [see the green line with squares in Fig. 4(c) and Sec. IV of [40] for more details].

Finally, we would like to point out that the effective giant atom in the frequency dimension can also be implemented with a Ladder-type three-level atom, which is coupled to different sites of the synthetic lattice via multiple two-photon resonant transitions, as shown in Fig. 1(c). The feasibility and advantages of such a scheme are discussed in detail in Sec. V of [40] using effective Hamiltonian theory [16]. Considering the implementations of synthetic dimensions based on superconducting circuits [8, 9, 30, 31], this also shows the possibility of constructing effective giant atoms in the microwave regime.

In summary, we propose in this Letter a feasible scheme for constructing giant atoms in synthetic frequency dimensions. Compared with real-space giant atoms, our scheme has several advantages: (i) The present model shows much smaller scale for practical on-chip integration and is more resource-efficient than the real-space analogue, which often requires large numbers of optical elements. (ii) Higher-dimensional giant atoms [56] can be simulated with systems of lower geometric dimensions, which are easier to construct experimentally. In particular, the zero-dimensional model here greatly reduces the difficulty of introducing other physical effects, such as nonlinearity that may bring richer exotic phenomena to giant atoms [57, 58]. (iii) The present scheme supports some useful applications for manipulating the frequency of light, such as excitation confinement and release, as well as chiral atom-waveguide interactions in frequency dimensions. The scheme in this Letter can also be extended to include other synthetic dimensions [39], which paves a promising way for manipulating various internal degrees of freedom of photons.

We acknowledge helpful discussions with G. C. La Rocca and Peng Zhang. YL is supported by the National Natural Science Foundation of China (under Grants No. 12074030 and No. U1930402). AFK acknowledges support from the Swedish Research Council (grant number 2019-03696), and from the Knut and Alice Wallenberg Foundation through the Wallenberg Centre for Quantum Technology (WACQT). JHW acknowledges support from the National Natural Science Foundation of China (under Grant No. 12074061).

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[1] A. F. Kockum, “Quantum optics with giant atoms—the first five years,” p125-p146 in Mathematics for Industry (Springer Singapore, 2021).
[2] M. V. Gustafsson, T. Aref, A. F. Kockum, M. K. Ekström, G. Johansson, and P. Delsing, Propagating phonons coupled to an artificial atom, Science 346, 207-211 (2014).
[3] R. Manenti, A. F. Kockum, A. Patterson, T. Behrle, J. Rahamin, G. Tancredi, F. Nori, and P. J. Leek, Circuit quantum acoustodynamics with surface acoustic waves, Nat. Commun. 8, 1-6 (2017).
[4] A. Noguchi, R. Yamazaki, Y. Tabuchi, and Y. Nakamura, Qubit-Assisted Transduction for a Detection of Surface Acoustic Waves near the Quantum Limit, Phys. Rev. Lett. 119, 180505 (2017).
[5] K. J. Satzinger, Y. P. Zhong, H.-S. Chang, G. A. Pears, A. Bienfait, M.-H. Chou, A. Y. Cleland, C. R. Conner, É. Dumur, J. Grebel, I. Gutierrez, B. H. November, R. G. Povey, S. J. Whiteley, D. I. Awschalom, D. I. Schuster, and A. N. Cleland, Quantum control of surface acoustic wave phonons, Nature 563, 661-665 (2018).
[6] B. A. Moores, L. R. Sletten, J. J. Viennot, and K. W. Lehnert, Cavity Quantum Acoustic Device in the Multimode Strong Coupling Regime, Phys. Rev. Lett. 120, 227701 (2018).
[7] A. N. Bolgar, J. I. Zotova, D. D. Kirichenko, I. S. Besedin, A. V. Semenov, R. S. Shaikhkhalidorov, and O. V. Astafiev, Quantum Regime of a Two-Dimensional Phonon Cavity, Phys. Rev. Lett. 120, 223603 (2018).
[8] A. Bienfait, K. J. Satzinger, Y. P. Zhong, H.-S. Chang, M.-H. Chou, C. R. Conner, É. Dumur, J. Grebel, G. A. Pears, R. G. Povey, and A. N. Cleland, Phonon-mediated quantum state transfer and remote qubit entanglement, Science 364, 368-371 (2019).
[9] G. Andersson, B. Suri, L. Guo, T. Aref, and P. Delsing, Non-exponential decay of a giant artificial atom, Nat. Phys. 15, 1123-1127 (2019).
[10] G. Andersson, M. K. Ekström, and P. Delsing, Electromagnetically Induced Acoustic Transparency with a Superconducting Cavity, Phys. Rev. Lett. 124, 240402 (2020).
[11] A. F. Kockum, P. Delsing, and G. Johansson, Designing frequency-dependent relaxation rates and Lamb shifts for a giant artificial atom, Phys. Rev. A 90, 013837 (2014).
[12] B. Kannan, M. Ruckriegel, D. Campbell, A. F. Kockum, J. Braumüller, D. Kim, M. Kjaergaard, P. Krantz, A. Melville, B. M. Niedzieski, A. Vepsäläinen, R. Winik, J. Yoder, F. Nori, T. P. Orlando, S. Gustavsson, and W. D. Oliver, Waveguide quantum electrodynamics with superconducting artificial giant atoms, Nature 583, 775-779 (2020).
[13] A. M. Vadiraj, A. Ask, T. G. McConkey, I. Nsanzineza, C. W. Sandbo Chang, A. F. Kockum, and C. M. Wilson, Engineering the level structure of a giant artificial atom in waveguide quantum electrodynamics, Phys. Rev. A 103, 023710 (2021).
[14] A. F. Kockum, G. Johansson, and F. Nori, Decoherence-Free Interaction between Giant Atoms in Waveguide Quantum Electrodynamics, Phys. Rev. Lett. 120, 140404 (2018).
tory theory for cascaded open systems, Phys. Rev. Lett. **70**, 2273-2276 (1993).

[53] A. Metelmann and A. A. Clerk, Nonreciprocal photon transmission and amplification via reservoir engineering, Phys. Rev. X **5**, 021025 (2015).

[54] X. Q. Li, X. Z. Zhang, G. Zhang, and Z. Song, Asymmetric transmission through a flux-controlled non-Hermitian scattering center, Phys. Rev. A **91**, 032101 (2015).

[55] C. Li, L. Jin, and Z. Song, Non-Hermitian interferometer: Unidirectional amplification without distortion, Phys. Rev. A **95**, 022125 (2017).

[56] A. González-Tudela, C. S. Muñoz, and J. I. Cirac, Engineering and Harnessing Giant Atoms in High-Dimensional Baths: A Proposal for Implementation with Cold atoms, Phys. Rev. Lett. **122**, 203603 (2019).

[57] E. Lustig, S. Weimann, Y. Plotnik, Y. Lumer, M. A. Bandres, A. Szameit, and M. Segev, Photonic topological insulator in synthetic dimensions, Nature **567**, 356-360 (2019).

[58] Z. Wang, T. Jaako, P. Kirton, and P. Rabl, Super-correlated Radiance in Nonlinear Photonic Waveguides, Phys. Rev. Lett. **124**, 213601 (2020).

[59] D. F. James and J. Jerke, Effective Hamiltonian theory and its applications in quantum information, Can. J. Phys. **85**, 625-632 (2007).
I. IMPLEMENTATIONS OF SYNTHETIC FREQUENCY LATTICES

In this section, we briefly review some physical implementations of the synthetic frequency lattice considered in the main text. For a ring resonator with zero group velocity dispersion in a specific frequency range, the resonant modes of the ring can be assumed to be equally spaced in frequency [1–4]. The frequency of the mth resonant mode (we use \( m = 0, \pm 1, \pm 2, \ldots \)) to label the resonant modes and select the central mode of frequency \( \omega_0 \) as the 0th mode) is then given by

\[
\omega_m = \omega_0 + m\Omega_s, \tag{S1}
\]

where \( \Omega_s = 2\pi v_g/L \) is the free spectral range, with \( v_g \) and \( L \) the group velocity at \( \omega_0 \) and the circumference of the ring, respectively. In view of this, the electric field in the ring can be expanded as [5]

\[
E(t, r_\perp, z) = \sum_m a_m(t, z)\varepsilon_m(r_\perp)e^{-i\omega_m t}, \tag{S2}
\]

where \( z \) and \( r_\perp \) correspond to the propagation direction along the ring and the direction perpendicular to \( z \), respectively; \( \varepsilon_m(r_\perp) \) is the modal profile of the ring; \( a_m(t, z) \) is the modal amplitude of the \( m \)th mode, which satisfies the periodic boundary condition \( a_m(t, z + L) = a_m(t, z) \) due to the geometry of the ring. If the ring resonator is dynamically modulated by an electric-optic modulator (EOM) located at \( z = z_0 \), the electric field acquires a time-dependent phase whenever it passes through the EOM, i.e.,

\[
E(t^+, r_\perp, z_0) = E(t^-, r_\perp, z_0)e^{i\alpha \cos(\Omega t)}, \tag{S3}
\]

where \( \alpha \) (\( \Omega \)) is the modulation amplitude (frequency) of the EOM and \( t^\pm = t \pm 0^\pm \). Substituting Eq. (S3) into Eq. (S2) and using the Jacobi-Anger expansion, one can obtain

\[
a_m(t^+, z_0) = J_0(\alpha)a_m(t^-, z_0) + \sum_{l=1}^{\infty} i^l J_l(\alpha) [a_{m+l}(t^-, z_0)e^{-il\delta_s t} + a_{m-l}(t^-, z_0)e^{il\delta_s t}], \tag{S4}
\]

where \( \delta_s = \Omega_s - \Omega \) and \( J_l \) is the Bessel function of the \( l \)th order. For weak modulations (i.e., \( \alpha \to 0 \)), one can only keep the terms \( l = \pm 1 \) and approximately has \( J_0(\alpha) \approx 1 \) and \( J_1(\alpha) \approx \alpha/2 \). In this way, we have

\[
\frac{\partial a_m}{\partial t} = iJ(a_{m+1}e^{-i\delta_s t} + a_{m-1}e^{i\delta_s t}), \tag{S5}
\]

where \( J = \alpha/2T_R \) with \( T_R \) the round-trip propagation time of the ring. Clearly, Eq. (S5) describes similar dynamics as those of one-dimensional tight-binding models, implying that a one-dimensional synthetic frequency lattice is created. Such dynamics can also be described by an equivalent Hamiltonian [1–3] (\( \hbar = 1 \) throughout this Supplemental Material)

\[
H_r = \sum_m [\omega_m a_m^\dagger a_m + 2J \cos(\Omega t)(a_m^\dagger a_{m+1} + \text{H.c.})], \tag{S6}
\]

which in the interaction picture becomes

\[
H_r' = \sum_m J(a_m^\dagger a_{m+1}e^{-i\delta_s t} + \text{H.c.}) \tag{S7}
\]
by taking the rotating wave approximation \( \delta_s = \Omega_s - \Omega \approx 0 \).

In addition to the ring-resonator scheme discussed above, such a synthetic frequency lattice can also be achieved by dynamically modulating the refractive index of a waveguide in terms of \( n(z, t) = n_0 + \Delta n \cos(\Omega t - qz + \phi) \) \([6, 7]\), where \( n_0 \) is the static background refractive index, \( \Delta n, \Omega, q, \) and \( \phi \) are the amplitude, frequency, wave vector, and initial phase of the modulation, respectively. With the phase-matching condition \( \Omega/q = c/n_0 \), one can obtain formally the same dynamic equation as that in Eq. (S5).

Besides photonic systems, superconducting quantum circuits have also shown the potential in creating synthetic dimensions, for both microwave photons \([8, 9]\) and surface acoustic waves \([10]\). In this case, a three-level \( \Delta \)-type structure is sufficient to simulate the effective giant atom of our interest, without the need of one more intermediate state as that in Fig. 1(a) in the main text. To be specific, a single-photon transition \( |g\rangle \leftrightarrow |e\rangle \) (resonantly coupled to the 0th site of the lattice with coupling strength \( g_1 \)) and a two-photon resonant transition \( |g\rangle \leftrightarrow |f\rangle \leftrightarrow |e\rangle \) (one of the two transitions is coupled to the \( N \)th site off-resonantly with coupling strength \( g_2 \)) while the other one is driven by an external field coexist in this case \([11]\). For artificial atoms, \( \Delta \)-type structure can be readily achieved with qubits operated away from their optimal points, and \( g_1 \) can be engineered much smaller than \( g_2 \) \([12]\) (in this way, the effective coupling strength between \( |g\rangle \) and \( |e\rangle \) induced by the two-photon process can be comparable with \( g_1 \)). Moreover, the effective giant atom can also be simulated with a Ladder-type three-level structure, as shown later in Sec. V, which is even simpler to construct than a \( \Delta \)-type one. However, the \( \Delta \)-type scheme here has the advantage that only one external field is needed.

II. TOTAL HAMILTONIAN IN THE INTERACTION PICTURE

According to the main text, the total Hamiltonian of the system can be written as

\[
H = H_a + H_r + H_{\text{int}},
\]

where \( H_a = \omega_{f,1}|f_1\rangle\langle f_1| + \omega_{f,2}|f_2\rangle\langle f_2| + \omega_e|e\rangle\langle e| \) is the free Hamiltonian of the atom, with \( \omega_{f,1}, \omega_{f,2}, \) and \( \omega_e \) the frequencies of the states \( |f_1\rangle, |f_2\rangle, \) and \( |e\rangle \) with respect to the state \( |g\rangle \), respectively; \( H_r \) is the Hamiltonian of the modulated ring resonator, which is given in Eq. (S6); \( H_{\text{int}} \) is the Hamiltonian that describes the interaction of the atom with the ring resonator and the external fields, which reads

\[
H_{\text{int}} = g_1 a_0^\dagger |g\rangle\langle f_1| + g_2 a_N^\dagger |g\rangle\langle f_2| + \eta_1 e^{-i\omega_d 1 t} e^{i\theta} |e\rangle\langle f_1| + \eta_2 e^{-i\omega_d 2 t} |e\rangle\langle f_2| + \text{H.c.},
\]

where \( g_1 (g_2) \) is the coupling strength between mode \( a_0 \) (\( a_N \)) and the transition \( |g\rangle \leftrightarrow |f_1\rangle (|g\rangle \leftrightarrow |f_2\rangle) \); \( \theta \) is the relative phase between the two external fields. In the interaction picture with respect to \( H_0 = H_a + \sum_m \omega_m a_m^\dagger a_m \) and using the rotating-wave approximation, Eq. (S8) can be transformed into

\[
H' = \sum_m J(a_m^\dagger a_{m+1} + \text{H.c.}) + \left(g_1 a_0^\dagger |g\rangle\langle f_1| e^{i\Delta_1 t} + g_2 a_N^\dagger |g\rangle\langle f_2| e^{i\Delta_2 t}
\]

\[
+ \eta_1 e^{i\theta} |e\rangle\langle f_1| e^{i\Delta_1 1 t} + \eta_2 |e\rangle\langle f_2| e^{i\Delta_2 2 t} + \text{H.c.}\right)
\]

in the case of \( \delta_s = 0 \), where \( \Delta_1 = \omega_0 - \omega_{f,1} \) (\( \Delta_2 = \omega_N - \omega_{f,2} \)) is the detuning between mode \( a_0 \) (\( a_N \)) and the transition \( |g\rangle \leftrightarrow |f_1\rangle (|g\rangle \leftrightarrow |f_2\rangle) \); \( \Delta_{d,1} = \omega_e - \omega_{f,1} - \omega_{d,1} \) (\( \Delta_{d,2} = \omega_e - \omega_{f,2} - \omega_{d,2} \)) is the detuning between the transition \( |f_1\rangle \leftrightarrow |e\rangle (|f_2\rangle \leftrightarrow |e\rangle) \) and the external field \( \eta_1 (\eta_2) \). Clearly, the Hamiltonian in Eq. (S10) is exactly the time-independent one in Eq. (4) in the main text.

III. EXCITATION CONFINEMENT WITHOUT AUXILIARY MODES

As discussed in the main text, the effective giant atom can hardly affect the propagation of a single-mode excitation along the synthetic frequency lattice in the regime of \( J \gg \{g_{e,1}, g_{e,2}\} \). Indeed, as shown in Fig. S1(a), the dynamic evolution of the initial state \( |\psi(t = 0)\rangle = a_1^\dagger |0, q\rangle \) behaves as in a bare lattice due to the weak atom-waveguide couplings. In view of this, we consider the regime where the effective coupling strengths \( g_{e,1} \) and \( g_{e,2} \) are comparable with \( J \). As shown in Figs. S1(b)-S1(d), the two atom-waveguide coupling points located at \( m = 0 \) and \( m = N \) act as two cavity mirrors between which the excitation is reflected back and forth. The oscillation region is fully determined by the separation \( N \) of the two coupling points. In particular, the excitation is almost confined at the initial mode \( a_1 \) for \( N = 2 \) due to the narrow region between the two “mirrors”.
FIG. S1. Dynamic evolutions of the modal excitation probability $P_m(t)$ with (a) $J \gg \{g_{e,1}, g_{e,2}\}$ and (b)-(d) $J \sim \{g_{e,1}, g_{e,2}\}$.

In panel (a) we assume $g_1 = g_2 = 3J$, $\eta = 2J$, $\Delta = 60J$, and $N = 4$, while in panels (b)-(d) we assume $g_1 = g_2 = 40J$, $\eta = 5J$, $\Delta = 200J$, and $N = 4, 3, 2$, respectively. The initial state is assumed to be $|\psi(t=0)\rangle = a_1^\dagger |0,g\rangle$.

It is worth mentioning, however, that the confinements above are actually caused by the large effective frequency shifts of modes $a_0$ and $a_N$, i.e., $\Delta_{e,1} = g_1^2 / \Delta$ and $\Delta_{e,2} = g_2^2 / \Delta$, which serve as two $\delta$-like potentials to confine the excitations between them. In order to remove the potentials and thereby focus on the giant-atom effect, we introduce opposite frequency shifts for modes $a_0$ and $a_N$ by coupling them with two extra auxiliary modes $b_1$ and $b_2$, respectively, as shown in Eq. (9) in the main text.

**IV. EFFECTIVE GIANT-SMALL ATOM MODEL**

In this section, we provide physical interpretations for the results in Figs. 3 and 4 in the main text with the help of an effective “giant-small atom model”. Considering a one-dimensional discrete waveguide (lattice) that is coupled with a two-level giant atom (at sites $m = 0$ and $m = N$) and a two-level small one (at site $m = M$) simultaneously, the Hamiltonian of the system can be written as

$$H = \omega_0 (b^\dagger b + c^\dagger c) + \left[ J \sum_m a_m^\dagger a_{m+1} + gb^\dagger (a_0 e^{i\theta} + a_N) + \xi c^\dagger a_M + \text{H.c.} \right], \quad (S11)$$

where $\omega_0$ is the transition frequency of the two atoms; $a_m$ and $b$ ($c$) are the annihilation operator of the $m$th lattice site and the lowering operator of the giant (small) atom, respectively; $g$ ($\xi$) is the coupling strength between the lattice and the giant (small) atom; $\theta$ is the relative phase between the two coupling channels of the giant atom. By transforming the lattice operators into the Bloch representation, i.e.,

$$a_k = \frac{1}{\sqrt{2\pi}} \sum_m a_m e^{-ikm}, \quad (S12)$$

with $-\pi \leq k < \pi$ the Bloch wave vector, Eq. (S11) can be rewritten as

$$H' = \omega_0 (b^\dagger b + c^\dagger c) + \int dk \omega_k a_k^\dagger a_k + \int dk [(G_{k,1} b^\dagger + G_{k,2} c^\dagger) a_k + \text{H.c.}], \quad (S13)$$

where $\omega_k = 2J \cos k$ is the dispersion relation of the lattice; $G_{k,1} = g[\exp(i\theta) + \exp(ikN)]/\sqrt{2\pi}$ and $G_{k,2} = \xi \exp(ikM)/\sqrt{2\pi}$ are the spectral coupling functions. In the single-excitation subspace, the wave function of the system is given by

$$|\psi(t)\rangle = \int dk u_k(t) e^{-i\omega_k t} a_k^\dagger |V\rangle + \sum_{\beta=b,c} u_\beta(t) e^{-i\omega_\beta t} \beta^\dagger |V\rangle, \quad (S14)$$
where $u_k$ and $u_b$ ($u_c$) are the probability amplitudes of creating a photon with wave vector $k$ in the lattice and of exciting the giant (small) atom to the excited state, respectively; $|V\rangle$ is the vacuum state of the whole system. Solving the Schrödinger equation, we have

$$
\frac{du_k}{dt} = -i \int dk G_{k,1} u_k e^{-i(\omega_k - \omega_0)t},
$$

$$
\frac{du_b}{dt} = -i \int dk G_{k,2} u_k e^{-i(\omega_k - \omega_0)t},
$$

$$
\frac{du_c}{dt} = -i(\gamma^*_{k,1} u_b + \gamma^*_{k,2} u_c) e^{i(\omega_k - \omega_0)t},
$$

(S15)

The formal solution of $u_k$ is written as

$$
u_k(t) = -i \int_{t_0}^{t} dt' e^{i(\omega_k - \omega_0)t'} [G_{k,1}^* u_b(t') + G_{k,2}^* u_c(t')],
$$

(S16)

where $t_0 < t$ is the initial time and $u_k(t_0) = 0$ has been assumed due to the initially vacuum lattice. Substituting Eq. (16) into the dynamic equations of $u_b$ and $u_c$ in Eq. (15), we have

$$
\frac{du_b}{dt} = -\frac{1}{2\pi} \int_{t_0}^{t} dt' e^{i\omega_0(t-t')} \int_{-\infty}^{+\infty} dk \left\{ 2g^2(1 + \cos(\theta - kN)) u_b(t') + g\xi (e^{i(\omega - kM)} + e^{iM} u_c(t')) e^{-i\omega_0(t-t')} \right\},
$$

(S17)

and $\frac{du_c}{dt}$ with $D = N - M$. By changing the integration variable with $\int dk \to \int d\omega / \nu_g$, we further have

$$
\frac{du_b}{dt} = -\frac{1}{2\pi} \int_{t_0}^{t} dt' e^{i\omega_0(t-t')} \int_{0}^{+\infty} d\omega \nu_g \left\{ 4g^2(1 + \cos \theta \cos kN) u_b(t') + 2g\xi (e^{i\theta} \cos kM + \cos kD) u_c(t') \right\} e^{-i\omega_0(t-t')},
$$

(S18)

where $\nu_g = -2J \sin k$ is the group velocity of photons in the lattice. According to the Weisskopf-Wigner theory [5], the intensity of the emission spectrum is concentrated around the atomic transition frequency $\omega_0$. Moreover, $\nu_g$ varies quite slowly and thus can be treated as a constant if $\omega_0$ is far away from the cut-off frequency of the lattice (i.e., if $\omega_0 \to 0$). In this case, one has $\omega_0 \simeq \omega_0 + \nu = \omega_0 + (k - k_0) \nu_0$ with $\omega_0 = 2J \cos k_0$ and $\nu_0 = -2J \sin k_0$, which lead to

$$
\frac{du_b}{dt} = -\int_{-\infty}^{+\infty} d\nu \int_{t_0}^{t} dt' e^{-i\nu(t-t')} \left\{ \frac{\gamma_b}{2\pi} \left[ 2 + \cos \theta \left( e^{i(k_0N+\nu\frac{\nu_0}{\nu})} + e^{-i(k_0N+\nu\frac{\nu_0}{\nu})} \right) \right] u_b(t') + \gamma_b e^{i\nu k_0 M + \nu \frac{\nu_0}{\nu}} + \gamma_b e^{-i\nu k_0 M + \nu \frac{\nu_0}{\nu}} + e^{i\nu k_0 D + \nu \frac{\nu_0}{\nu}} + e^{-i\nu k_0 D + \nu \frac{\nu_0}{\nu}} \right\} u_c(t'),
$$

(S19)

$$
\frac{du_c}{dt} = -\int_{-\infty}^{+\infty} d\nu \int_{t_0}^{t} dt' e^{-i\nu(t-t')} \left\{ \gamma_c u_c(t') + \frac{\gamma_c}{2\pi} \left[ e^{i\nu k_0 M + \nu \frac{\nu_0}{\nu}} + e^{-i\nu k_0 M + \nu \frac{\nu_0}{\nu}} \right] \right\} u_b(t'),
$$

where $\gamma_b = g^2 / \nu_0$, $\gamma_c = \xi^2 / \nu_0$, and $\gamma_{bc} = g\xi / \nu_0$. With the definition of the delta function $\int d\nu e^{-i\nu t} = 2\pi \delta(t)$ and assuming $0 < M < N$ (the small atom is located between the two atom-waveguide coupling points of the giant one),
Similarly, one has

\[ \frac{du_b(t)}{dt} = -2\gamma_b(1 + \cos\theta)e^{ik_0 N}u_b(t) - \gamma_{bc}[e^{i(k_0 M + \theta)} + e^{ik_0 D}]u_c(t), \]

\[ \frac{du_c(t)}{dt} = -\gamma_c u_c(t) - \gamma_{bc}[e^{i(k_0 M - \theta)} + e^{ik_0 D}]u_b(t). \]  

If \( M < 0 < N \) (the small atom is located at the left side of the giant one), and

\[ \frac{du_b(t)}{dt} = -2\gamma_b(1 + \cos\theta)e^{ik_0 N}u_b(t) - \gamma_{bc}[e^{i(k_0 M + \theta)} + e^{ik_0 D}]u_c(t), \]

\[ \frac{du_c(t)}{dt} = -\gamma_c u_c(t) - \gamma_{bc}[e^{i(k_0 M - \theta)} + e^{ik_0 D}]u_b(t) \]  

if \( 0 < N < M \) (the small atom is located at the right side of the giant one). Note that for the case of \( \omega_0 \to 0 \) (i.e., \( \Delta_{\alpha,\pi} \to 0 \)) considered in this Letter, one has \( k_0 \approx -\pi/2 \) [14, 15].

The results in Fig. 3 in the main text can be well understood from Eq. (S21): the dynamics of the our model in Eq. (S10) with the initial state \( |\psi(t = 0)\rangle = a_M^\dagger |0, g\rangle \) can be effectively described by the giant-small atom model with
an initially excited small atom located at \( m = M \). For the case of \( \theta = 0 \) in Fig. 3, the effective coupling strength between the giant and small atoms \( g_{GS} \propto -(e^{ik_0M} + e^{ik_0D}) \) is reciprocal. When \( M = 1 \) and \( D = 1 \) [Figs. 3(a) and 3(c)], the excitation can oscillate between the two atoms (or say, between mode \( a_1 \) and the state \( | e \rangle \)) due to the nonzero effective coupling strength and thus it is confined in the frequency dimension in an oscillating manner. Note that this is the only choice to achieve nonzero effective coupling as long as \( N = 2 \) and \( \theta = 0 \), i.e., \( g_{GS} \) is always zero for other values of \( M \) and \( D \), as can be seen from Eqs. (S21)-(S23). When \( M = 1 \) and \( D = 3 \) [Figs. 3(b) and 3(d)], the giant atom decouples from the small one (mode \( a_1 \)) due to \( g_{GS} = 0 \), such that the excitation diffuses as in a bare lattice. The results in Figs. 3(b) and 3(d) deviate slightly from the bare-lattice case due to the nonnegligible non-Markovian retardation effect in the case of \( J \approx \{g_{e,1}, g_{e,2}\} \).

Equations (S22) and (S23) reveal that if \( \theta \neq n\pi \) (\( n \) is an arbitrary integer), the effective couplings between the two atoms become different for \( M < 0 < N \) and \( 0 < N < M \) (the effective coupling strength depends on which side of the giant atom the small one is located at). In particular, for \( \theta = \pi/2 \) and \( N = 3 \), the effective coupling strength from the giant atom to the small one \( g_{GS} \propto -(e^{ik_0M} - e^{ik_0D}) \) is always zero as long as \( M < 0 \), implying that the lattice sites (resonant modes) on the left side of the giant atom cannot be excited in the Markovian limit. We would like to point out that the weak excitations of the left-side modes in Fig. 4(a) in the main text arise from the finite-size effect of the lattice. As shown in Figs. S2(a)-S2(c), the chiral modal excitation profile tends to become more ideal (i.e., the weak excitations of the left-side modes tend to vanish) with the increase of the evolution time. To avoid undesired boundary effects, we also extend the length of the frequency lattice accordingly [the total number of the lattice sites is assumed to be \( m_{tot} = 25, 65, 105 \) for Figs. S2(a)-S2(c), respectively], which is experimentally available with synthetic lattices of other internal degrees of freedom such as orbital angular momentum. Note that such a chiral spontaneous emission can be reversed by tuning \( \theta \) from \( \pi/2 \) to \(-\pi/2\).

On the other hand, as mentioned in the main text, such a giant atom can hardly be excited by photons coming from its right side (or say, by an excited small atom located at its right side). This can be understood again from Eq. (S23), which shows that the effective coupling strength from a right-side small atom to the giant one \( g_{GS} \propto -(e^{ik_0M+\theta} + e^{ik_0D}) \) is always zero as long as \( \theta = \pi/2 \) and \( N = 3 \). Thus provides the possibility of realizing cascaded interactions between multiple such giant atoms, i.e., the interatomic interactions can be directional based on this mechanism. As an example, we consider here that another giant atom (labeled as “atom \( B' \)) is coupled to the frequency lattice at sites \( m = 4 \) and \( m = 7 \), while the present one (labeled as “atom \( A \)) is coupled to the lattice at \( m = 0 \) and \( m = 3 \). The two giant atoms are assumed to be identical except for the positions of the coupling points. We plot in Fig. S3 the dynamic evolutions of the atomic populations \( P_{e,A}(t) \) of atom \( A \) and \( P_{e,B}(t) \) of atom \( B \) with two initial states \( |\psi(t = 0)\rangle = |0, e,g \rangle \) and \( |\psi(t = 0)\rangle = |0, g,e \rangle \). As predicted above, in the case of \( \theta = \pi/2 \), the excitation of atom \( A \) can be partially transferred to atom \( B \) [Fig. S3(a)], whereas atom \( A \) can hardly be excited if atom \( B \) is initially excited [Fig. S3(b)]. This indicates that one can implement cascaded systems in the synthetic frequency dimension.

Finally, one can see from Eq. (S21) that the effective coupling strength from the small atom to the giant one \( g_{GS} \propto -(e^{ik_0M+\theta} + e^{ik_0D}) \) is zero if \( M = 2, D = 1 \), and \( \theta = -\pi/2 \), or if \( M = 1, D = 2 \), and \( \theta = \pi/2 \). In these cases, the giant atom cannot be excited and thus the chiral modal excitation profile almost disappears, as shown in Fig. 4(c) in the main text (see the green line with squares). On the contrary, \( g_{GS} \) reaches its maximum if \( M = 1, D = 2 \), and \( \theta = -\pi/2 \), or if \( M = 2, D = 1 \), and \( \theta = \pi/2 \). The excited giant atom exhibits reversed chiral spontaneous emission for \( \theta = \pi/2 \) and \( \theta - \pi/2 \), as discussed above, and thus yields reversed modal excitation profiles as shown in Fig. 4(c) in the main text (see the blue line with dots and the red line with circles).
V. EFFECTIVE HAMILTONIAN OF THE LADDER-TYPE SCHEME

In this section, we would like to demonstrate that the effective giant atom of our interest also can be implemented by exploiting a Ladder-type three-level atom coupled with two sites of the synthetic frequency lattice via two two-photon resonant transitions. For example, as shown in Fig. 1(c) in the main text, the lower transition \(|g \leftrightarrow |f\rangle\) of the Ladder-type atom is coupled off-resonantly to the 0th and \(N\)th lattice sites with coupling strengths (detunings) \(g_1\) and \(g_2\) (\(\Delta_1\) and \(\Delta_2\)), respectively. Meanwhile, the upper transition \(|f \leftrightarrow |e\rangle\) is driven by two external fields of amplitudes \(\eta_1\) and \(\eta_2\) in order to construct two two-photon resonant transitions. In the interaction picture with and the rotating-wave approximation, the total Hamiltonian can be written as

\[
H_{\text{tot}} = H'_L + H_L(t) \text{ with}
\]

\[
H_L(t) = g_1 a_0^\dagger |g\rangle \langle f| e^{i \Delta_1 t} + g_2 a_N^\dagger |g\rangle \langle f| e^{i \Delta_2 t} + \eta_1 e^{i \theta} |e\rangle \langle f| e^{i \Delta_1 t} + \eta_2 |e\rangle \langle f| e^{i \Delta_2 t} + \text{H.c.}
\]  

(S24)

Now we exploit effective Hamiltonian theory [16] to remove the time dependence of \(H_L(t)\). According to this theory, if the time-dependent Hamiltonian has the form of

\[
H(t) = \sum_m h_m e^{i \Delta_m t} + \text{H.c.},
\]

then the effective Hamiltonian can be given by

\[
H_{\text{eff}} = -iH(t) \int_0^t dt' H(t')
\]

\[
= \sum_{m,n} \frac{-1}{m} [\Delta_m e^{i \Delta_m t} + \Delta_n e^{i \Delta_n t} + \frac{\Delta_m}{h_n} e^{i \Delta_m t} + \frac{\Delta_n}{h_m} e^{i \Delta_n t} + \frac{\Delta_m}{h_n} e^{i \Delta_m t} + \frac{\Delta_n}{h_m} e^{i \Delta_n t}],
\]

(S26)

where we have discarded the high-frequency terms containing \(\exp(\pm i \Delta_m t)\) in Eq. (S26). For \(H_L(t)\) given in Eq. (S24), we have

\[
H_1 = g_1 a_0^\dagger |g\rangle \langle f|, \quad H_2 = g_2 a_N^\dagger |g\rangle \langle f|, \quad H_3 = \eta_1 e^{i \theta} |e\rangle \langle f|, \quad H_4 = \eta_2 |e\rangle \langle f|.
\]

(S27)

Substituting Eq. (S27) into Eq. (S26), the effective Hamiltonian can be derived as

\[
H_{\text{eff}} = \sum_{m=n} \ldots + \sum_{m \neq n} \ldots
\]

(S28)

with

\[
\sum_{m=n} \ldots = \frac{g_1^2}{\Delta_1} |n_0| |g\rangle \langle f| + (n_0 + 1) |f\rangle \langle f|
\]

\[
+ \frac{g_2^2}{\Delta_2} |n_N| |g\rangle \langle f| - (n_N + 1) |f\rangle \langle f|
\]

\[
+ \frac{\eta_1^2}{\Delta_1} |e\rangle \langle e| - |f\rangle \langle f|
\]

\[
+ \frac{\eta_2^2}{\Delta_2} |e\rangle \langle e| - |f\rangle \langle f|
\]

(S29)

and

\[
\sum_{m \neq n} \ldots = \frac{g_1 g_2}{\Delta_1} a_0^\dagger a_N^\dagger |g\rangle \langle f| e^{i \Delta_2 - \Delta_1} t - \frac{g_1 g_2}{\Delta_2} a_N^\dagger a_0^\dagger |f\rangle \langle f| e^{i \Delta_2 - \Delta_1} t
\]

\[
+ \frac{g_1 g_2}{\Delta_1} e^{i \theta} a_0^\dagger a_N^\dagger |g\rangle \langle f| - \frac{g_1 g_2}{\Delta_2} e^{i \theta} a_N^\dagger a_0^\dagger |f\rangle - \frac{\eta_1 \eta_2}{\Delta_1} |e\rangle \langle e| - \frac{\eta_1 \eta_2}{\Delta_2} |e\rangle \langle e|
\]

\[
+ \frac{\eta_1 \eta_2}{\Delta_1} |e\rangle \langle e| - \frac{\eta_1 \eta_2}{\Delta_2} |e\rangle \langle e|
\]

(S30)

where \(n_0 = a_0^\dagger a_0\) and \(n_N = a_N^\dagger a_N\) are the photon number operators of the 0th and \(N\)th lattice sites, respectively. Considering that \(\Delta_2 - \Delta_1 = N \Omega_s \gg 0\) in our model (\(\Omega_s \gg \{g_1, g_2, \eta_1, \eta_2\}\)), Eqs. (S29) and (S30) can be further simplified to

\[
\sum_{m=n} \ldots = (\frac{\eta_1^2}{\Delta_1} + \frac{\eta_2^2}{\Delta_2}) |e\rangle \langle e| + (\frac{\eta_1^2}{\Delta_1} n_0 + \frac{\eta_2^2}{\Delta_2} n_N) |g\rangle \langle g|
\]

(S31)
and
\[ \sum_{m \neq n} \ldots = \frac{g_1 \eta_1}{\Delta_1} e^{i\theta} a_0|e\rangle\langle g| + \frac{g_2 \eta_2}{\Delta_2} a_N|e\rangle\langle g| + \frac{g_1 \eta_2}{\Delta_1} e^{-i\theta} a_0^\dagger|g\rangle\langle e| + \frac{g_2 \eta_1}{\Delta_2} a_N^\dagger|g\rangle\langle e|, \] (S32)

respectively, where we have neglected the high-frequency terms containing \( \exp[\pm i(\Delta_2 - \Delta_1)t] \) in Eq. \( \text{(S32)} \). Moreover, we have also discarded the Stark shift terms of the state \( |f\rangle \) in Eq. \( \text{(S31)} \) since \( |f\rangle \) is decoupled from the other states in the effective Hamiltonian. It is clear from Eqs. \( \text{(S31)} \) and \( \text{(S32)} \) that such a Ladder-type scheme with two two-photon resonant transitions can also mimic a two-level giant atom with two coupling points in the synthetic frequency dimension. In particular, the two-photon processes induced Stark shifts [e.g., \( \eta_1^2/\Delta_1, \eta_2^2/\Delta_2 \) and \( g_1^2 n_0/\Delta_1, g_2^2 n_N/\Delta_2 \) in Eq. \( \text{(S31)} \)] can be removed if \( \Delta_1 = -\Delta_2, |\eta_1| = |\eta_2|, \) and \( |g_1| = |g_2| \), such that the giant-atom effects become more ideal. Finally, we would like to point out that the Ladder-type scheme can be extended to include more two-photon processes such that an effective giant atom with multiple atom-waveguide coupling points can be simulated.

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[1] L. Yuan, Y. Shi, and S. Fan, Photonic gauge potential in a system with a synthetic frequency dimension, Opt. Lett. 41, 741-744 (2016).
[2] L. Yuan and S. Fan, Bloch oscillation and unidirectional translation of frequency in a dynamically modulated ring resonator, Optica 3, 1014-1018 (2016).
[3] L. Yuan, M. Xiao, Q. Lin, and S. Fan, Synthetic space with arbitrary dimensions in a few rings undergoing dynamic modulation, Phys. Rev. B 97, 104105 (2018).
[4] L. Yuan, Q. Lin, M. Xiao, and S. Fan, Synthetic dimension in photonics, Optica 5, 1396-1405 (2018).
[5] D. F. Walls and G. J. Milburn, Quantum Optics (Springer-Verlag, Berlin, 1994).
[6] C. Qin, F. Zhou, Y. Peng, D. Souzas, X. Zhu, B. Wang, J. Dong, X. Zhang, A. Ali, and P. Lu, Spectrum Control through Discrete Frequency Diffraction in the Presence of Photonic Gauge Potentials, Phys. Rev. Lett. 120, 133901 (2018).
[7] C. Qin, B. Wang, Z. J. Wong, S. Longhi, and P. Lu, Discrete diffraction and Bloch oscillations in non-Hermitian frequency lattices induced by complex photonic gauge fields, Phys. Rev. B 101, 064303 (2020).
[8] N. R. A. Lee, M. Pechal, E. A. Wollack, P. Arrangoiz-Ariola, Z. Wang, and A. H. Safavi-Naeni, Propagation of microwave photons along a synthetic dimension, Phys. Rev. A 101, 053807 (2020).
[9] J. S. C. Hung, J. H. Busnaina, C. W. S. Chang, A. M. Vadiraj, I.Nsanzineza, E. Solano, H. Alaeian, E. Rico, and C. M. Wilson, Quantum Simulation of the Bosonic Creutz Ladder with a Parametric Cavity, Phys. Rev. Lett. 127, 100503 (2021).
[10] G. Andersson, S. W. Jolin, M. Scigliuzzo, R. Borgani, M. O. Tholén, J. C. R. Hernández, V. Shumeiko, D. B. Haviland, and P. Delsing, Squeezing and multimode entanglement of surface acoustic wave phonons, arXiv:2007.05826v2.
[11] F. Deppe, M. Marientoni, E. P. Menzel, A. Marx, S. Saito, K. Kakuyanagi, H. Tanaka, T. Meno, K. Semb, H. Takayanagi, E. Solano, and R. Gross, Two-photon probe of the Jaynes-Cummings model and controlled symmetry breaking in circuit QED, Nat. Phys. 4, 686-691 (2008).
[12] X. Gu, A. F. Kockum, A. Miranowicz, Y.-x. Liu, and F. Nori, Microwave photonics with superconducting quantum circuits, Phys. Rep. 718, 1-102 (2017).
[13] L. Guo, A. F. Kockum, F. Marquardt, and G. Johansson, Oscillating bound states for a giant atom, Phys. Rev. Res. 2, 043014 (2020).
[14] S. Longhi, Photonic simulation of giant atom decay, Opt. Lett. 45, 3017-3020 (2020).
[15] S. Longhi, Superradiance paradox in waveguide lattices, Opt. Lett. 45, 3297-3300 (2020).
[16] D. F. James and J. Jerke, Effective Hamiltonian theory and its applications in quantum information, Can. J. Phys. 85, 625-632 (2007).