Voros Coefficients for the Hypergeometric Differential Equations and Eynard-Orantin’s Topological Recursion

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June 12, 2018

Abstract

We develop the theory of quantization of spectral curves via the topological recursion. We formulate a quantization scheme to spectral curves which is not necessarily admissible in the sense of Bouchard-Eynard 2017. Our main result establishes a relation between the Voros coefficient for quantum curves and the free energies are formulated for spectral curves associated with a confluent family of Gauss hypergeometric differential equations. As an application, we find explicit formulas of free energies for those spectral curves in terms of the Bernoulli numbers.

Contents

1 Introduction 2
2 Voros coefficients and Eynard-Orantin’s topological recursion 4
  2.1 Exact WKB analysis and Voros coefficients . . . . . . . . . . . . . . . . . . . . . 5
  2.2 Voros coefficient . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
  2.3 Eynard-Orantin’s topological recursion . . . . . . . . . . . . . . . . . . . . . . 7
  2.4 Free energy through the topological recursion . . . . . . . . . . . . . . . . . . 8
  2.5 Variational formulas for the correlation functions . . . . . . . . . . . . . . . . 11
3 Quantization of spectral curves 14
  3.1 An index of the defining polynomials . . . . . . . . . . . . . . . . . . . . . . . . 15
  3.2 Quantum curves . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16
  3.3 Preparations for the proof . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
  3.4 Proof of Theorem 3.6 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

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2010 Mathematics Subject Classification. Primary:34M60; Secondary:81T45

Keywords: Exact WKB analysis; Voros coefficients; Topological recursion; Quantum curves; Free energy; Gauss hypergeometric equation.
1 Introduction

The Voros coefficient for a Schrödinger-type linear ordinary differential equation is defined as a contour integral of the logarithmic derivative of WKB solutions. Its importance in the study of the global behavior of solutions of differential equations has been already recognized by the earlier work of Voros ([V]). For example, the Voros coefficient is an important ingredient for describing the Stokes phenomena and the monodromy group of a (1-dimensional) Schrödinger equation. Moreover, the concrete form of the Voros coefficient enables us to analyze the parametric Stokes phenomena explicitly. The concrete form of the Voros coefficient is now known for the Weber equation, the Whittaker equation, the Kummer equation and the Gauss hypergeometric differential equation ([SS, T08, KoT, ATT, AT]). The Voros coefficient also plays an essentially important role in a relationship between exact WKB analysis and cluster algebras ([IN]).

On the other hand, the topological recursion introduced by B. Eynard and N. Orantin ([EO1]) is a generalization of the loop equations that the correlation functions of the matrix model satisfy. For a Riemann surface $\Sigma$ and meromorphic functions $x$ and $y$ on $\Sigma$, it produces an infinite tower of meromorphic differentials $W_{g,n}(z_1, \ldots, z_n)$ on $\Sigma$. A triplet $(\Sigma, x, y)$ is called a spectral curve and $W_{g,n}(z_1, \ldots, z_n)$ is called a correlation function. Moreover, for a spectral curve, we can define free energies (also called symplectic invariants) $F_g$. Topological recursion attracts the interests of both mathematicians and physicists since the quantities produced by its framework are expected to encode information of various geometric or enumerative invariants. It is also known that the topological recursion is closely related to integrable systems. For more details see, e.g., the review paper [EO3].

A surprising connection between WKB theory and topological recursion was discovered recently. The quantization scheme connects WKB solutions with the topological recursion
More precisely, it was found that WKB solutions can be constructed by correlation functions for the spectral curve, which corresponds to the semi-classical limit of the differential equation, when the spectral curve satisfies the “admissibility” condition in the sense of \[\text{BE2} \text{ Definition } 2.7\].

Then the following question naturally arises: What quantity corresponds to the Voros coefficient in the topological recursion? In this paper, we answer this question for a family of the Gauss hypergeometric differential equations. That is, we show that the Voros coefficients are expressed as the difference values of the generating function of the free energies for the spectral curve corresponding to the differential equations in the case of a family of the Gauss hypergeometric differential equations. This is our first main result.

Let us describe the result for the differential equation

\[
\left\{ \hbar^2 \frac{d^2}{dx^2} - \left( \frac{\lambda_\infty^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_0^2}{x^2(x-1)^2} + \hbar \frac{\lambda_0}{x^2(x-1)} - \hbar^2 \frac{1}{4(x-1)^2} \right) \right\} \psi = 0
\]

which is equivalent to the Gauss hypergeometric differential equation via a certain gauge transformation, and the corresponding spectral curve

\[
y^2 - \frac{\lambda_\infty^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_0^2}{x^2(1-x)^2} = 0
\]

obtained as the classical limit of (1.1). Here \(\lambda_j\) is a complex constant which is related to the characteristic exponent at a regular singular point \(j\) of (1.1) \(j = 0, 1, \infty\). We will impose a generic assumption on these constants. Since (1.1) has three singular points at \(x = 0, 1, \infty\) on \(\mathbb{P}^1\), three Voros coefficients \(V^{(j)}(\lambda_0, \lambda_1, \lambda_\infty; \hbar)\) are associated to (1.1) depending on the singular point \(j = 0, 1, \infty\) (see \(\S 4.8\) for precise definition). Having this in mind, we state the main result for the Gauss hypergeometric differential equation (1.1) as follows.

**Theorem 1.1** (cf. Theorem 4.39). Let \(V^{(0)}(\lambda_0, \lambda_1, \lambda_\infty; \hbar)\) be the Voros coefficient of the Gauss hypergeometric differential equation (1.1) defined for the singular point \(x = 0\). We also let \(F(\lambda_0, \lambda_1, \lambda_\infty) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\lambda_0, \lambda_1, \lambda_\infty)\) be the generating function of the free energies associated with the spectral curve (1.2). Then, we have

\[
V^{(0)}(\lambda_0, \lambda_1, \lambda_\infty; \hbar) = F(\lambda_0, \lambda_1, \lambda_\infty; \hbar) - F(\lambda_0 - \hbar, \lambda_1, \lambda_\infty; \hbar) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_0} + \frac{1}{2} \frac{\partial^2 F_0}{\partial \lambda_0^2}.
\]

Similar formulae also hold for the Voros coefficients defined for the other singularities \(x = 1\) and \(\infty\).

It should be noted that, although several Voros coefficients are associated with a given differential equation (having several singular points) as in the case of Gauss hypergeometric differential equation, they are described in terms of the generating function of the free energies which is canonically (or uniquely) associated with a given spectral curve. These results imply that the free energy is more essential quantity that controls the Voros coefficient. To obtain the main results, we also study quantizations without assuming the admissibility condition in
the case when the degree of a polynomial $P(x, y)$ with respect to $y$ is two. The latter result is a partial extension of that of [BE2]. We need this generalization because the spectral curve (1.2) arising from Gauss hypergeometric differential equation is not admissible.

As applications of the main results, we get three-term difference equations which the generating function of the free energies satisfies. By solving them, we obtain concrete forms of the free energies and those of the Voros coefficients as well. For example, the explicit form of the free energy $F_g$ of the Gauss curve (1.2) for $g \geq 2$ is given by

$$F_g = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{(\lambda_0 + \lambda_1 + \lambda_\infty)^{2g-2}} + \frac{1}{(\lambda_0 - \lambda_1 + \lambda_\infty)^{2g-2}} \right\}. \quad (1.4)$$

Throughout the paper, $B_m$ is the $m$-th Bernoulli number (see (C.1) for the definition). The free energies of other examples which will be considered in this paper are also expressed in terms of the Bernoulli numbers. Consequently, the Voros coefficients of the quantum curves are written in terms of the Bernoulli polynomials. See §4 for details.

The paper is organized as follows: In §2 we recall some basic facts about the WKB analysis and Eynard-Orantin’s topological recursion. In §3 we study quantization. Our main result in this section is Theorem 3.6 which gives quantizations without assuming the admissibility condition in the case when the degree of a polynomial $P(x, y)$ with respect to $y$ is two. In §4 we state our main theorem. In each subsection we pick up the spectral curve corresponding to the semi-classical limit of a (confluent) hypergeometric differential equation and explain how the Voros coefficients are expressed in terms of the generating function of the free energies. Moreover, as an application of the main results we give concrete forms of the free energies and those of the Voros coefficients.

Acknowledgement

We are grateful to Takashi Aoki, Akishi Ikeda, Takahiro Kawai, Toshinori Takahashi, Yoshitsugu Takei and Mika Tanda for helpful discussions and communications. This work is supported, in part, by JSPS KAKENHI Grand Numbers 16K17613, 16H06337, 16K05177, 17H06127.

2 Voros coefficients and Eynard-Orantin’s topological recursion

In this section we briefly recall some basics about the exact WKB analysis, and Eynard-Orantin’s theory. See [KT] and [EO1] (or [EO3]) respectively for the details of them.
2.1 Exact WKB analysis and Voros coefficients

The differential equation which we will discuss in this paper is the second order ordinary differential equation with a small parameter $\hbar \neq 0$ of the form

\begin{equation}
\left\{ \hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \hbar \frac{d}{dx} + r(x, \hbar) \right\} \psi = 0,
\end{equation}

where $x \in \mathbb{C}$,

\begin{equation}
q(x, \hbar) = q_0(x) + \hbar q_1(x), \quad r(x, \hbar) = r_0(x) + \hbar r_1(x) + \hbar^2 r_2(x),
\end{equation}

with rational functions $q_j(x)$ and $r_j(x)$ ($j = 0, 1, 2$). We consider (2.1) as a differential equations in the complex projective line $\mathbb{P}^1$ in a usual way. A WKB type solution of (2.1) is a (formal) solution of (2.1) of the form

\begin{equation}
\psi(x, \hbar) = \exp \left[ \frac{1}{\hbar} f_{-1}(x) + f_0(x) + \hbar f_1(x) + \cdots \right].
\end{equation}

A typical way of constructing WKB type solutions is the following: The logarithmic derivative $S(x, \hbar)$ of solutions of (2.1) satisfies the Riccati equation

\begin{equation}
\hbar^2 \left( \frac{d}{dx} S(x, \hbar) + S(x, \hbar)^2 \right) + \hbar q(x, \hbar) S(x, \hbar) + r(x, \hbar) = 0.
\end{equation}

Eq. (2.4) admits a solution of the form

\begin{equation}
S(x, \hbar) := \hbar^{-1} S_{-1}(x) + S_0(x) + \hbar S_1(x) + \cdots = \sum_{n=-1}^{\infty} \hbar^n S_n(x).
\end{equation}

In fact, by substituting (2.5) into (2.4), and equating the both-sides like powers with respect to $\hbar$, we obtain

\begin{equation}
S_{-1}^2 + q_0(x) S_{-1} + r_0(x) = 0,
\end{equation}

\begin{equation}
\left( 2S_{-1} + q_0(x) \right) S_0 + q_1(x) S_{-1} + r_1(x) + \frac{dS_{-1}}{dx} = 0,
\end{equation}

\begin{equation}
\left( 2S_{-1} + q_0(x) \right) S_1 + S_0^2 + q_1(x) S_0 + r_2(x) + \frac{dS_0}{dx} = 0,
\end{equation}

and

\begin{equation}
\left( 2S_{-1} + q_0(x) \right) S_{n+1} + \sum_{j=0}^{n} S_{n-j} S_j + q_1(x) S_n + \frac{dS_n}{dx} = 0 \quad (n \geq 1).
\end{equation}

Eq. (2.6) has two solutions, and once we fix one of them, we can determine $S_n$ for $n \geq 0$ uniquely and recursively by (2.7) – (2.9). Thus we obtain two WKB type solutions of the form

\begin{equation}
\psi(x, \hbar) := \exp \left( \int_{x}^{x} S(x, \hbar) dx \right).
\end{equation}
Let \( \text{Disc}(x) \) be the discriminant of \((2.6)\), i.e.,
\[
\text{Disc}(x) := \left\{ q_0(x) \right\}^2 - 4r_0(x).
\]
Because
\[
\text{Disc}(x) = \left\{ y_+(x) - y_-(x) \right\}^2
\]
holds, where \( y_\pm(x) \) are two solutions of \( y^2 + q_0(x)y + r_0(x) = 0 \) (i.e., \((2.6)\)), \( y_+(x) \) and \( y_-(x) \) merge to each other at zeros of \( \text{Disc}(x) \). A point \( a \in \mathbb{C} \) is called a turning point of \((2.1)\) if it satisfies \( \text{Disc}(a) = 0 \). If it is a simple zero, we say it is a simple turning point. Near a simple turning point \( a \), two solutions \( y_\pm(x) \) behave like
\[
y_\pm(x) = c_0 \pm c_{1/2}(x - a)^{1/2} + c_1(x - a) \pm c_{3/2}(x - a)^{3/2} + \cdots \quad (c_{1/2} \neq 0),
\]
and hence
\[
y_+(x) - y_-(x) = 2(x - a)^{1/2} \left\{ c_{1/2} + c_{3/2}(x - a) + \cdots \right\} \quad (c_{1/2} \neq 0).
\]
A simple pole \( a \in \mathbb{C} \) of \( \text{Disc}(x) \) is called a simple-pole type turning point. Near a simple-pole type turning point, we have
\[
y_\pm(x) = \pm c_{-1/2}(x - a)^{-1/2} + c_0 \pm c_{1/2}(x - a)^{1/2} + c_1(x - a) + \cdots \quad (c_{-1/2} \neq 0),
\]
and hence,
\[
y_+(x) - y_-(x) = 2(x - a)^{-1/2} \left\{ c_{-1/2} + c_{1/2}(x - a) + \cdots \right\} \quad (c_{-1/2} \neq 0).
\]
We also call \( \infty \in \mathbb{P}^1 \) is a turning point (resp., simple-pole type turning point) if \( X = 0 \) is a zero (resp., simple-pole) of \( \text{Disc}(1/X)/X^4 \) (i.e., if \( X = 0 \) is a turning point (resp., simple-pole type turning point) of the differential equation obtained by the coordinate change \( X = 1/x \) from \((2.1)\)).

Turning points and Stokes curves, which is defined by
\[
\text{Im} \int_a^x \sqrt{\text{Disc}(x)} \, dx = 0,
\]
where \( a \) is a turning point or a simple-pole type turning point, play a central role in exact WKB analysis. See §4 where turning points and Stokes curves are illustrated for various differential equations. They are used to describe the region where Borel summability of WKB type solutions holds, and to give a connection formulas among the Borel sum of WKB type solutions. Although we do not discuss any such analytic properties of WKB type solutions in this paper, turning points and Stokes curves are still useful to “visualize” the map \( x(z) \) of the spectral curve (see §2.3) in our study.

It is more convenient for exact WKB analysis if the second order linear differential equation in question is represented in the so-called SL-form, i.e., the second order linear differential
equation with no first order term is more convenient (the word “SL” comes from the fact that the monodromy matrices of the equation of SL-form belong to SL(2, \(\mathbb{C}\))). A gauge transformation
\begin{equation}
\varphi := \exp \left( \frac{1}{2} \int x q(x, \hbar) dx \right) \psi
\end{equation}
of the unknown function \(\psi\) makes (2.1) into the SL-form:
\begin{equation}
\left\{ \hbar^2 \frac{d^2}{dx^2} - Q(x, \hbar) \right\} \varphi = 0,
\end{equation}
where
\begin{equation}
Q(x, \hbar) := \frac{1}{4} q(x, \hbar)^2 - r(x, \hbar) + \frac{1}{2} \hbar \frac{\partial}{\partial x} q(x, \hbar)
\end{equation}
\begin{equation}
= \frac{1}{4} q_0(x)^2 - r_0(x) + \frac{\hbar}{2} \left( q_0(x) q_1(x) - 2 r_1(x) + \frac{dq_0}{dx} \right) + \frac{\hbar^2}{4} \left( q_1(x)^2 + 2 \frac{dq_1}{dx} - 4 r_2(x) \right).
\end{equation}
Note that the leading term of the potential \(Q(x, \hbar)\) is \(\text{Disc}(x)/4\), where \(\text{Disc}(x)\) is given in (2.11).

\subsection{Voros coefficient}

A Voros coefficient is defined as a properly regularized integral of \(S(x, \hbar)\) from a singular point of (2.1) to itself. Here, by a singular point of (2.1) we mean a pole of \(\text{Disc}(x)\) of order greater than or equal to two. It depends, however, on the situation we consider how we regularize such an integral. Fortunately, all of the examples discussed in §4 have the property that \(S_n(x)\) with \(n \geq 1\) is integrable at any singular point of (2.1). In this situation we can define Voros coefficients by
\begin{equation}
V(\hbar) := \int_{\gamma_{b_1, b_2}} \left( S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \right) dx = \sum_{n=1}^{\infty} \hbar^n \int_{\gamma_{b_1, b_2}} S_n(x) dx,
\end{equation}
where \(\gamma_{b_1, b_2}\) is a path from a singular point \(b_1\) to a singular point \(b_2\). Note that Voros coefficients only depend on the class \([\gamma_{b_1, b_2}]\) of paths in the relative homology group
\[H_1(\mathbb{P}^1 \setminus \{\text{Turning points}\}, \{\text{Singular points}\}; \mathbb{Z}).\]
Such an integration contour (or a relative homology class) can be understood as a lift of path on \(x\)-plane onto the Riemann surface of \(S_{-1}(x)\) (i.e., two sheeted covering of \(x\)-plane) after drawing branch cuts and distinguishing the first/second sheet of the Riemann surface. In §4 we will show several examples of such contours, and compute the Voros coefficients associated with them. Our main examples are members of a confluent family of the Gauss hypergeometric equations.
### 2.3 Eynard-Orantin’s topological recursion

A starting point of Eynard-Orantin’s theory ([EO1]) is a spectral curve. Because we will not discuss the general case in this paper, we restrict ourselves to the case when a spectral curve is given by a triplet \((\mathbb{P}^1, x(z), y(z))\), where \(x(z)\) and \(y(z)\) are non-constant rational functions on \(\mathbb{P}^1\). Let \(R\) be the set of ramification points of \(x(z)\), i.e., \(R\) consists of zeros of \(dx(z)\) of any order and poles of \(x(z)\) whose orders are greater than or equal to two (here we consider \(x\) as a branched covering map from \(\mathbb{P}^1\) to itself).

We further assume that

- (A1) A function field \(\mathbb{C}(x(z), y(z))\) coincides with \(\mathbb{C}(z)\).
- (A2) For any ramification point \(r\) of \(x(z)\):
  1. If \(r\) is a zero of \(dx(z)\), and if \(y(z)\) is holomorphic near \(r\), then \(y'(r) \neq 0\).
  2. If \(r\) is a pole of \(x(z)\), and if \(Y(z) = -x(z)^2 y(z)\) is holomorphic near \(r\), then \(Y''(r) \neq 0\).
- (A3) All of the ramification points of \(x(z)\) are simple, i.e., the ramification index of each ramification point is two.
- (A4) We assume branch points are all distinct, where a branch point is defined as the image of a ramification point by \(x(z)\).

Because of the assumption (A1), we can find an irreducible polynomial \(P(x, y) \in \mathbb{C}[x, y]\) for which \(P(x(z), y(z)) = 0\) holds at all \(z\) except for poles of \(x(z)\) and \(y(z)\). Hence \((x(z), y(z))\) becomes a regular meromorphic parametrization in the sense of [SWP] §4 of the plane curve \(C := \{(x, y) \in \mathbb{C} | P(x, y) = 0\}\). We also call this curve \(C\) a spectral curve if there is no fear of confusions.

In the assumption (A2), note that the transformation \((x, y) \mapsto (X, Y) := (1/x, -x^2y)\) satisfies \(ydx = YdX\) (hence it is symplectic), and that \(r\) is a ramification point of \(X(z)\). This assumption ensures that Eynard-Orantin’s correlation function \(W_{g,n}(z_1, z_2, \cdots, z_n)\) becomes symmetric in their variables. (See Theorem 2.4 below.) In this assumption we also allow the case when \(y(z)\) has a pole at a ramification point (cf. [DN]).

The assumption (A3) is equivalent to saying that any zero of \(dx(z)\) is simple, and that the order of any pole of \(x(z)\) is less than or equal to two. From this assumption, for each \(r \in R\), there exists a neighborhood \(U\) of \(r\) such that \(x^{-1}(x(z)) \cap U\) consists of only two points for \(z \in U \setminus \{r\}\). Let \(x^{-1}(x(z)) \cap U = \{z, \overline{z}\}\). Then, we define a map \(\overline{\cdot} : U \setminus \{r\} \to \mathbb{P}^1\), called a conjugate map, by \(z \mapsto \overline{z}\). This conjugate map is characterized by the following three conditions:

\[(2.22) \quad \text{(a) } \overline{z} \neq z \text{ if } z \neq r, \quad \text{(b) } x(\overline{z}) = x(z), \quad \text{(c) } \lim_{z \to r} \overline{z} = z.\]

The conjugate map can be extended to \(U\) as a holomorphic map: if \(r \in \mathbb{C}\) is a zero of \(dx(z)\), and \(x(z) = x_0 + x_2(z-r)^2 + x_3(z-r)^3 + \cdots\) with a nonzero \(x_2\), then

\[(2.23) \quad \overline{z} = r - (z-r) - \frac{x_3}{x_2} (z-r)^2 + \cdots\]
near \( r \). If \( r \in \mathbb{C} \) is a pole of \( x(z) \), and \( x(z) = (z - r)^{-2}\{x_0 + x_1(z - r) + \cdots\} \) with a nonzero \( x_0 \), then
\[
(2.24) \quad \bar{z} = r - (z - r) + \frac{x_1}{x_0}(z - r)^2 + \cdots.
\]

**Remark 2.1.** The conjugate map is only defined near a ramification point. In the subsequent sections of this paper, however, we only study the case when the degree of \( P(x, y) \) with respect to \( y \) is two (cf. (AQ1) in §3.2). In this case the degree of \( x(z) \) as a rational function is two, or, in other words, \( x : \mathbb{P}^1 \to \mathbb{P}^1 \) is a 2-sheeted branched covering, and the conjugate map becomes a globally defined rational map from \( \mathbb{P}^1 \) to itself.

**Definition 2.2** ([EO1, Definition 4.2]). Eynard-Orantin’s correlation function \( W_{g,n}(z_1, \cdots, z_n) \) for \( g \geq 0 \) and \( n \geq 1 \) is defined as a multidifferential\(^1\) on \( (\mathbb{P}^1)^n \) using the recurrence relation (called Eynard-Orantin’s topological recursion)
\[
(2.25) \quad W_{g,n+1}(z_0, z_1, \cdots, z_n) := \sum_{r \in R} \text{Res}_{z=r} K_r(z_0, z) \left[ W_{g-1,n+2}(z, \bar{z}, z_1, \cdots, z_n) \right. \\
\left. + \sum_{I_1 \cup I_2 = \{1, 2, \cdots, n\}} W_{g_1, |I_1|+1}(z, z_{I_1}) W_{g_2, |I_2|+1}(\bar{z}, z_{I_2}) \right]
\]
for \( 2g + n \geq 2 \) with initial conditions
\[
(2.26) \quad W_{0,1}(z_0) := y(z_0) dx(z_0), \quad W_{0,2}(z_0, z_1) = B(z_0, z_1) := \frac{dz_0 dz_1}{(z_0 - z_1)^2}.
\]
Here we set \( W_{g,n} \equiv 0 \) for a negative \( g \),
\[
(2.27) \quad K_r(z_0, z) := \frac{1}{2(y(z) - y(z'))} \int_{z' = z}^{z = z} B(z_0, \zeta)
\]
is a recursion kernel defined near a ramification point \( r \in R \), \( \cup \) denotes the disjoint union, and the prime ‘ on the summation symbol in \( (2.25) \) means that we exclude terms for \((g_1, I_1) = (0, \emptyset)\) and \((g_2, I_2) = (0, \emptyset)\) (so that \( W_{0,1} \) does not appear) in the sum. We have also used the multi-index notation: for \( I = \{i_1, \cdots, i_m\} \subset \{1, 2, \cdots, n\} \) with \( i_1 < i_2 < \cdots < i_m \), \( z_I := (z_{i_1}, \cdots, z_{i_m}) \).

Some remarks are necessary here:

**Remark 2.3.** (i) Since a meromorphic differential appears in the denominator of \( (2.27) \), it might be better if we made our discussion in the \( K \)-group of the meromorphic differentials. However, we satisfy ourselves with a somewhat formal treatment as above, since it cancels out in \( (2.25) \), and the final result becomes a multidifferential.

\(^1\)We borrow this terminology from DN. We summarize in §A.5 our notations on multidifferentials.
(ii) Eq. (2.25) becomes a recurrence relation with respect to $2g + n$. In fact, a correlation function $W_{g',n'}$ appearing in the right-hand side of (2.25) satisfies $2g' + n' < 2g + (n + 1)$. More is true: correlation functions in the right-hand side of (2.25) are $\{W_{g',n'} \mid 0 \leq g' \leq g, 1 \leq n' \leq n, (g',n') \neq (0,1),(g,n)\}$ and $W_{g-1,n+1}$. Thus, theoretically or in a actual computation, we can also proceed in the total degree lexicographical ordering, i.e.,

$$
(g, n) = \begin{cases} (1,1) & g+n=2 \\ (0,3), (1,2), (2,1) & g+n=3 \\ (0,4), (1,3), (2,2), (3,1) & g+n=4 \\ & \ldots 
\end{cases}
$$

(iii) $B(z_0,\zeta)$ is replaced by the canonical (or fundamental) bilinear differential of the second kind when a genus of the spectral curve is not necessarily zero. This $B(z_0,\zeta)$ is also called simply the Bergman kernel in the literature.

(iv) Eynard and Orantin defined $R$ as a set of zeros of $dx(z)$ in [EO1], and they call an element of $R$ a branch point. As far as we know, it is [BE2] which pointed out that poles of $x(z)$ of order two or more also play the same role as zeros of $dx(z)$ in the topological recursion. In this paper we use this modified version. See [EO1] for the topological recursion associated to general spectral curve. See also [BHLMR], where higher multiple zero of $dx(z)$ is studied.

**Theorem 2.4.** Under the assumptions (A1), (A2) and (A3), we obtain

(i) $W_{g,n}(z_1,\ldots,z_n)$ is a symmetric meromorphic multidifferential.

(ii) Only singular points of $W_{g,n}(z_1,\ldots,z_n)$ for $2g + n > 2$ (i.e., $(g,n) \neq (0,1),(0,2)$) with respect to each variable are ramification points. They are poles with no residue. If we further assume that the degree of $x(z)$ is two, then

(iii) The following relation holds for $2g + n \geq 2$:

$$
W_{g,n}(z_1,\ldots,z_n) + W_{g,n}(\overline{z}_1,\ldots,\overline{z}_n) = \delta_{g,0} \delta_{n,2} \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}.
$$

See [EO1, Theorem 4.6], [EO1, Theorem 4.2] and [EO1, Theorem 4.4] respectively for the proof. (Although the original paper [EO1] does not include higher order poles of $x(z)$ to the set of ramification points, the proof can be given in the same way.)

It turns out that, ramification points given as higher order poles of $x(z)$ sometimes do not contribute to the topological recursion; namely, the correlation functions $W_{g,n}(z_1,\ldots,z_n)$ may be holomorphic at such ramification points for each variable $z_i$ ($i = 1,\ldots,n$) except for $(g,n) = (0,1)$, and the residue at those points in (2.25) may become zero. This happens for all examples which will be considered in [4]. Therefore we introduce the following notion.

**Definition 2.5.** A ramification point $r$ is said to be ineffective if the correlation functions $W_{g,n}(z_1,\ldots,z_n)$ for $(g,n) \neq (0,1)$ are holomorphic at $z_i = r$ for each $i = 1,\ldots,n$. A ramification point which is not ineffective is called effective. The set of effective ramification points is denoted by $R^*$ ($\subset R$).
The following properties of ineffective ramification points are important in this paper.

**Proposition 2.6.** Under the assumptions (A1), (A2) and (A3), we obtain the following:

(i) Let \( r \) be a ramification point. Then, \( r \) is an ineffective ramification point if and only if 
\[(y(z) - y(\overline{z}))dx(z)\] 
has a pole at \( r \).

(ii) If \( r \in R \) is an ineffective ramification point, then the residue at \( r \) in \((2.25)\) is zero.

The proof will be given in §B (cf. Propositions B.1 and B.4). The property (i) above implies that an ineffective ramification point often appears as a double pole of \( x(z) \).

These properties (Theorem 2.4 and Proposition 2.6) of the correlation functions are fundamental, and we often use them without any reference.

**Remark 2.7.** Correlation functions also satisfy \((2.25)\), where \( K_r(z_0, z) \) is now replaced by
\[(2.29) \quad K_{D,r}(z_0, z) := \frac{1}{(y(z) - y(\overline{z}))dx(z)} \int_{\zeta \in D(z)} B(z_0, \zeta)\]
with any divisor \( D(z) = [z] - \sum_{j=1}^{m} \nu_j [\beta_j] \) with \( \beta_j \in \mathbb{P}^1 \setminus R^* \) and \( \nu_j \in \mathbb{C} \) satisfying \( \sum_{j=1}^{m} \nu_j = 1 \). (See §A for the definition of the integral with a divisor.) This follows from the facts that
\[K_{D,r}(z_0, z) - K_r(z_0, z) = \frac{1}{(y(z) - y(\overline{z}))dx(z)} \left( \frac{1}{2(z_0 - z)} + \frac{1}{2(z_0 - \overline{z})} - \sum_{j=1}^{m} \frac{\nu_j}{z_0 - \beta_j} \right) dz_0\]
is anti-invariant under the conjugate map \( z \mapsto \overline{z} \), and \( \text{Res}_{z=r} \omega(z) = 0 \) holds for any meromorphic differential \( \omega(z) \) defined near a ramification point \( r \) which satisfies \( \omega(\overline{z}) = -\omega(z) \). In a computation, it is useful to choose \( D(z) = [z] - [\infty] \) if \( \infty \notin R^* \).

### 2.4 Free energy through the topological recursion

Free energy \( F_g \) \((g \geq 0)\) is a complex number defined for the spectral curve, and one of the most important objects in Eynard-Orantin’s theory. It is also called a symplectic invariant since it is “almost” invariant under symplectic transformations of spectral curves (see §EO4 for the details).

**Definition 2.8** ([EO1 Definition 4.3]). For \( g \geq 2 \), the \( g \)-th free energy \( F_g \) is defined by
\[(2.30) \quad F_g := \frac{1}{2 - 2g} \sum_{r \in R} \text{Res}_{z=r} \Phi(z)W_{g,1}(z) \quad (g \geq 2),\]
where \( \Phi(z) \) is a primitive of \( y(z)dx(z) \). The free energies \( F_0 \) and \( F_1 \) for \( g = 0 \) and 1 are also defined, but by a different manner (see §4.2.2 and §4.2.3 for the definition).

Note that the right-hand side of \((2.30)\) does not depend on the choice of the primitive because \( W_{g,1} \) has no residue at each ramification point (see Theorem 2.4(ii)).
In applications (and in our article), the generating series

\[(2.31) \quad F := \sum_{g=0}^{\infty} \hbar^{2g-2} F_g\]

de\(F_g\)’s is crucially important. We also call the generating series (2.31) the free energy of the spectral curve.

### 2.5 Variational formulas for the correlation functions

In §4 we will consider a family (or a deformation) of spectral curves parametrized by a complex parameter. For our purpose, we briefly recall the variational formulas obtained by [EO1, §5] which describe the differentiation of the correlation functions \(W_{g,n}\) and the free energies \(F_g\) with respect to the parameter.

Suppose that we have given a family \((P_1, x_\epsilon(z), y_\epsilon(z))\) of spectral curves parametrized by a complex parameter \(\epsilon\) which lies on a certain domain \(U \subset \mathbb{C}\) such that

- \(x_\epsilon(z), y_\epsilon(z)\) depend holomorphically on \(\epsilon \in U\). Furthermore, the ramification points of \(x_\epsilon(z)\) and the conjugate map \(z \mapsto \overline{z}\) also depend holomorphically on \(\epsilon \in U\).
- \(x_\epsilon(z), y_\epsilon(z)\) satisfy the assumptions (A1) – (A4) for any \(\epsilon \in U\).
- The cardinality of the set \(R_\epsilon\) of ramification points of \(x_\epsilon(z)\) is constant on \(\epsilon \in U\) (i.e. ramification points of \(x_\epsilon(z)\) are distinct for any \(\epsilon \in U\)).

Then, the correlation functions \(W_{g,n}(z_1, \ldots, z_n; \epsilon)\) and the \(g\)-th free energy \(F_g(\epsilon)\) defined from the spectral curve \((P_1, x_\epsilon(z), y_\epsilon(z))\) are holomorphic in \(\epsilon \in U\) (as long as \(z_i \not\in R_\epsilon\) for any \(i = 1, \ldots, n\)).

In order to formulate a variational formula for correlation functions, we need to introduce the notion of “differentiation with fixed \(x\)”. For a meromorphic differential \(\omega(z; \epsilon)\) on \(P_1\), which depends on \(\epsilon\) holomorphically, define

\[(2.32) \quad \delta_\epsilon \omega(z; \epsilon) := \left( \frac{\partial}{\partial \epsilon} \omega(z_\epsilon(x); \epsilon) \right)_{x=x_\epsilon(z)} (z \not\in R_\epsilon),\]

where \(z_\epsilon(x)\) is (any branch of) the inverse function of \(x = x_\epsilon(z)\) which is defined away from branchpoints (i.e. points in \(x_\epsilon(R_\epsilon)\)). In [EO1] the notation \(\delta_\Omega \omega(z; \epsilon)\) is used for \(\delta_\epsilon \omega(z; \epsilon)\) defined above. Such differentiation \(\delta_\epsilon\) can be generalized to multidifferentials in an obvious way. Then, under these assumptions, the variational formula is formulated as follows.

**Theorem 2.9** ([EO1 Theorem 5.1]). In addition to the above conditions, for any \(\epsilon \in U\), we further assume that

- If \(r_\epsilon \in R_\epsilon\) is a zero of \(dx_\epsilon(z)\), then the functions \(\partial x_\epsilon/\partial \epsilon\) and \(\partial y_\epsilon/\partial \epsilon\) are holomorphic (as functions of \(z\)) at \(r_\epsilon\), and \(dy_\epsilon(z)\) does not vanish (as a differential of \(z\)) at \(r_\epsilon\).
• If \( r_\varepsilon \in R_\varepsilon \) is a pole of \( x_\varepsilon(z) \) with an order greater than or equal to two, then
\[
\frac{\Omega_\varepsilon(z) B(z_1, z) B(z_2, z)}{dy_\varepsilon(z) dx_\varepsilon(z)}
\]
is holomorphic (as a differential in \( z \)) at \( r(\varepsilon) \), where

\[
(2.33) \quad \Omega_\varepsilon(z) := \frac{\partial y_\varepsilon}{\partial \varepsilon}(z) dx(z) - \frac{\partial x_\varepsilon}{\partial \varepsilon}(z) dy(z).
\]

• There exist a path \( \gamma \) in \( \mathbb{P}^1 \) passing through no ramification point and a function \( \Lambda_\varepsilon(z) \) holomorphic in a neighborhood of \( \gamma \) for which the following holds.

\[
(2.34) \quad \Omega_\varepsilon(z) = \int_{\zeta \in \gamma} \Lambda_\varepsilon(\zeta) B(z, \zeta).
\]

Then, \( W_{g,n}(z_1, \ldots, z_n; \varepsilon) \) and \( F_g(\varepsilon) \) defined from the spectral curve \( (\mathbb{P}^1, x_\varepsilon(z), y_\varepsilon(z)) \) satisfy the following relations:

(i) For \( 2g - 2 + n \geq 0 \),
\[
(2.35) \quad \delta_\varepsilon W_{g,n}(z_1, \ldots, z_n; \varepsilon) = \int_{\zeta \in \gamma} \Lambda_\varepsilon(\zeta) W_{g,n+1}(z_1, \ldots, z_n, \zeta; \varepsilon)
\]
holds on \( \varepsilon \in U \) as long as each of \( x_1, \ldots, x_n \) satisfies \( x_i \notin x_\varepsilon(R_\varepsilon) \).

(ii) For \( g \geq 1 \),
\[
(2.36) \quad \frac{\partial F_g}{\partial \varepsilon}(\varepsilon) = \int_{\gamma} \Lambda_\varepsilon(z) W_{g,1}(z; \varepsilon)
\]
holds on \( \varepsilon \in U \).

See [EO1, §5.1] (based on the Rauch’s variation formula; see [KK1] for example) for the proof. We note that, since we modify the definition of topological recursion by adding higher order poles of \( x(z) \) as ramification point, we also need to require the second condition in the above claim.

In examples discussed in [H] we will use the variational formula in a situation that \( \Lambda_\varepsilon(z) = \Lambda \) is a constant function (which is also independent of the parameter \( \varepsilon \)). In that case, applying the above formulas iteratively, we have

\[
(2.37) \quad \frac{\partial^n F_g}{\partial \varepsilon^n}(\varepsilon) = \Lambda^n \int_{\zeta_1 \in \gamma} \cdots \int_{\zeta_n \in \gamma} W_{g,n}(\zeta_1, \ldots, \zeta_n; \varepsilon).
\]
3 Quantization of spectral curves

With correlation functions \( W_{g,n}(z_1, \ldots, z_n) \) of a spectral curve \((\mathbb{P}^1, x(z), y(z))\) we associate a formal series defined by

\[
\varphi(z; \nu, \hbar) = \exp \left[ \frac{1}{\hbar} \int z W_{0,1}(\zeta) + \frac{1}{2!} \int_{\zeta_1 \in D(z; \nu)} \int_{\zeta_2 \in D(z; \nu)} \left( \frac{dx(\zeta_1) dx(\zeta_2)}{(x(\zeta_1) - x(\zeta_2))^2} \right) \right]
+ \sum_{m=1}^{\infty} \hbar^m \left\{ \sum_{2g+n-2=m, g \geq 0, n \geq 1} \frac{1}{n!} \int_{\zeta_1 \in D(z; \nu)} \cdots \int_{\zeta_n \in D(z; \nu)} W_{g,n}(\zeta_1, \ldots, \zeta_n) \right\},
\]

where

\[
D(z; \nu) = [z] - \sum_{\beta \in B} \nu_{\beta}
\]

is a divisor on \( \mathbb{P}^1 \) with a finite set \( B \subset \mathbb{P}^1 \setminus \mathbb{R}^* \) and \( \nu = (\nu_{\beta})_{\beta \in B} \) which is a tuple of complex numbers satisfying \( \sum_{\beta \in B} \nu_{\beta} = 1 \) (see [A] for the integral with a divisor). Note that the quantization by using the divisor with the parameter \( \nu \) was first introduced by [BE2]. This \( \varphi(z; \nu, \hbar) \) has the same form as WKB type solutions (2.3), and it is known that \( \psi(x; \nu, \hbar) = \varphi(z(x); \nu, \hbar) \) for some (or some class of) spectral curves with some specified divisors \( D(z; \nu) \) satisfies a linear ordinary differential equation, where \( z(x) \) is a inverse function of \( x(z) \). Such a linear ordinary differential equation is called a quantum curve of a spectral curve in question.

A typical example of a quantization is the Airy curve \( x(z) = z^2, y(z) = z \) \((z \in \mathbb{P}^1)\) (so that \( P(x(z), y(z)) = 0 \) with \( P(x, y) = y^2 - x \)). In this case we choose \( D = [z] - [\infty] \), and the corresponding quantum curve is the Airy equation \( \{(x \Phi/dx)^2 - x\} \psi = 0 \) ([Z], see also [EO1], [EO3]). Another example is \( x(z) = z^2 \) and \( y(z) = 1/z \) \((z \in \mathbb{P}^1)\). In this case \( P(x(z), y(z)) = 0 \) with \( P(x, y) = xy^2 - 1 \), and the corresponding quantum curve is \( \{(x \Phi/dx)^2 - 1\} \psi = 0 \) ([DN]).

A systematic study of a quantization is done in [BE2], and explicit forms of quantum curves are given under the assumption called admissibility. The SL-form of the Gauss hypergeometric equation which we will discuss in §4.8 however, does not satisfy the admissibility condition. In this section we study quantizations without the admissibility condition, while we restrict ourselves to the case when the degree of a polynomial \( P(x, y) \) with respect to \( y \) is two.

To be more precise, we assume the following conditions in addition to (A1) – (A4):

(AQ1) The rational functions \( (x(z), y(z)) \) satisfy \( P(x(z), y(z)) = 0 \) with an irreducible polynomial \( P(x, y) = p_0(x)y^2 + p_1(x)y + p_2(x) \in \mathbb{C}[x, y] \), where \( p_0(x) \) is a nonzero polynomial.

(AQ2) The differential \( (y(z) - y(\bar{z})) dx(z) \) does not vanish except for at ramification points.

By the assumption (AQ1), the conjugate map is now a rational map defined globally (cf. Remark [2,1]). Hence the assumption (AQ2) makes sense. Properties which follow immediately from these assumptions are
Proposition 3.1. If assumptions of (A1) – (A4), (AQ1) and (AQ2) hold, then $y(z)$ and $y(\overline{z})$ are two (different) solutions of $P(x(z), y) = 0$.

Proof. Because $x(z)$ is a non-constant rational function, $R \neq \mathbb{P}^1$ follows, and hence $g(z)$ does not identically coincide with $y(\overline{z})$ by (AQ2). We also obtain

$$0 = P(x(\overline{z}), y(\overline{z})) = P(x(z), y(\overline{z})).$$

□

Remark 3.2. The assumption (AQ2) ensures that two solutions $y(z)$ and $y(\overline{z})$ of $P(x(z), y) = 0$ merge to each other only at ramification points. Because we assume that any ramification point is simple by (A3), $y$ as a function of $x$ has an expansion of the form (2.13) or (2.15) at $x(r) = a$ with a ramification point $r$ if $r$ is not a pole of $y(z)$.

An example of a spectral curve which does not satisfy (AQ2) is

$$P(x, y) = y^2 - x^3 - x^2, \quad x(z) = -1 + z^2, \quad y(z) = z(z^2 - 1).$$

Because $R = \{0, \infty\}$ and $\overline{z} = -z$, we have $y(1) = y(\overline{1}) = 0$, whereas $z = 1$ is not a ramification point of $x(z)$. Thus (AQ2) is violated. At $(x(z), y(z))|_{z=1} = (0, 0)$, a curve defined by $\{(x, y) \mid P(x, y) = 0\}$ has a normal crossing. Hence the point $x(1) = 0$ should become a double turning point of its quantum curve (if it exists). As illustrated by this example, (AQ2) excludes multiple turning points. See [IS], in which the cases are discussed where quantum curves have double turning points.

3.1 An index of the defining polynomials

To state assumptions for the divisor $D(z; \mathcal{L})$ in (3.1), we introduce an index at $x_0 \in \mathbb{P}^1$ by

$$\rho(x_0; P) := \begin{cases} \text{ord}_{x_0} Q_0(x) & (x_0 \neq \infty), \\ \text{ord}_0 Q_0^{(\infty)}(x) & (x_0 = \infty) \end{cases}$$

for a polynomial $P(x, y) = p_0(x)y^2 + p_1(x)y + p_2(x) \in \mathbb{C}[x, y]$, where

$$Q_0(x) := \frac{1}{4} \left( \frac{p_1(x)}{p_0(x)} \right)^2 - \frac{p_2(x)}{p_0(x)} \quad \text{and} \quad Q_0^{(\infty)}(x) := \frac{1}{x^4}Q_0(1/x).$$

Here $\text{ord}_{x_0} g(x)$ for a rational function $g(x)$ is defined as an integer $p$ for which

$$g(x) = (x - x_0)^p(c_0 + c_1(x - x_0) + \cdots), \quad c_0 \neq 0$$

holds as the Laurent expansion of $g(x)$ at $x_0$.

Remark 3.3. (i) This index $\rho(x_0; P)$ is related to the type of singularities of the quantum curve. We have expected that the leading term with respect to $\hbar$ of the quantum curve is given by the second order differential operator $\mathcal{L} := (p_0(x))^{-1}P(x, \hbar(d/dx))$, where we use the normal ordering with respect to $x$ and $\hbar(d/dx)$. Then $Q_0(x)$ is the leading term with respect to $\hbar$ of the potential of the SL-form of $\mathcal{L}$ (cf. (2.19)). It is well-known that
(i) $x_0 \in \mathbb{P}^1$ is a regular singular point of the SL-form of $\mathcal{L}$ iff $\rho(x_0; P) = -1, -2$.

(ii) $x_0 \in \mathbb{P}^1$ is an irregular singular point of the SL-form of $\mathcal{L}$ iff $\rho(x_0; P) \leq -3$.

Note also that $x_0 \in \mathbb{P}^1$ is not necessarily a regular singular point of $\mathcal{L}$ even if the condition (i) above holds.

(ii) Our index $\rho(x_0; P)$ remains invariant under a symplectic transformation of the form $(x, y) \mapsto (X, Y) = (x, y + g(x))$ for some function $g(x)$, or, a gauge transformation $\mathcal{L} \mapsto e^{-\int g(x)dx} \circ \mathcal{L} \circ e^{\int g(x)dx}$.

**Proposition 3.4.** We assume (A1) – (A4), (AQ1) and (AQ2). Then, for $\alpha \in \mathbb{P}^1$,

$$
\text{ord}_\alpha \left[ (y(z) - y(\overline{z})) \frac{dx}{dz}(z) \right] = \begin{cases} 
\frac{1}{2} \rho(x(\alpha); P) & (\alpha \notin R), \\
\rho(x(\alpha); P) + 1 & (\alpha \in R).
\end{cases}
$$

**Proof.** Because $y(z)$ and $y(\overline{z})$ are two solutions of $P(x(z), y) = 0$, we have

$$
\{ y(z) - y(\overline{z}) \}^2 = \left\{ \left( \frac{p_1(x)}{p_0(x)} \right)^2 - 4 \frac{p_2(x)}{p_0(x)} \right\} \bigg|_{x = x(z)} = 4Q_0(x(z)).
$$

Hence if $\alpha$ is a regular point of $x(z)$, we have

$$
\text{ord}_\alpha [y(z) - y(\overline{z})] = \frac{1}{2} \text{ord}_{x(\alpha)} Q_0(x) = \frac{1}{2} \rho(x(\alpha); P).
$$

If $\alpha$ is a pole of $x(z)$, we utilize a transformation

$$
X(z) = \frac{1}{x(z)}, \quad Y(z) = -x(z)^2 y(z).
$$

By this transformation, we have

$$
(y(z) - y(\overline{z})) dx(z) = (Y(z) - Y(\overline{z})) dX(z).
$$

Furthermore $Y(z)$ and $Y(\overline{z})$ are two solutions of $P(\infty)(X, Y) = 0$, where

$$
P(\infty)(X, Y) = P(1/X, -X^2) = X^4 p_0(1/X) Y^2 - X^2 p_1(1/X) Y + p_2(1/X).
$$

Hence

$$
\{ Y(z) - Y(\overline{z}) \}^2 = \frac{1}{X^4} \left\{ \left( \frac{p_1(1/X)}{p_0(1/X)} \right)^2 - 4 \frac{p_2(1/X)}{p_0(1/X)} \right\} \bigg|_{X = X(z)} = 4Q_0(\infty)(X(z)).
$$

Thus

$$
\text{ord}_\alpha [Y(z) - Y(\overline{z})] = \frac{1}{2} \text{ord}_0 Q_0(\infty)(x) = \frac{1}{2} \rho(x(\alpha); P).
$$

Now we divide the proof into four cases.
(a) $\alpha \not\in R$ and $\alpha$ is a regular point of $x(z)$: In this case $\mathrm{ord}_\alpha [dx/dz] = 0$ follows, and hence
\begin{equation}
(3.16) \quad \mathrm{ord}_\alpha \left[ (y(z) - y(\overline{z})) \frac{dx}{dz}(z) \right] = \mathrm{ord}_\alpha \left[ y(z) - y(\overline{z}) \right] = \frac{1}{2} \rho(x(\alpha); P).
\end{equation}

(b) $\alpha \not\in R$ and $\alpha$ is a simple pole of $x(z)$: Since $\alpha$ is a regular point of $X(z)$ with $X(\alpha) = 0$ and $X'(\alpha) \neq 0$, we obtain
\begin{align*}
(3.17) \quad \mathrm{ord}_\alpha \left[ (y(z) - y(\overline{z})) \frac{dx}{dz}(z) \right] &= \mathrm{ord}_\alpha \left[ (Y(z) - Y(\overline{z})) \frac{dX}{dz}(z) \right] \\
&= \mathrm{ord}_\alpha \left[ Y(z) - Y(\overline{z}) \right] \\
&= \frac{1}{2} \rho(x(\alpha); P).
\end{align*}

(c) $\alpha \in R$ and $\alpha$ is a simple zero of $dx(z)$: In this case
\begin{equation}
(3.18) \quad \mathrm{ord}_\alpha \left[ x(z) - x(\alpha) \right] = 2 \quad \text{and} \quad \mathrm{ord}_\alpha \left[ \frac{dx}{dz}(z) \right] = 1
\end{equation}
hold. Hence
\begin{align*}
\mathrm{ord}_\alpha \left[ (y(z) - y(\overline{z})) \frac{dx}{dz}(z) \right] &= \frac{1}{2} \rho(x(\alpha); P) \times (\text{ord}_\alpha [x(z) - x(\alpha)]) + 1 \\
&= \rho(x(\alpha); P) + 1.
\end{align*}

(d) $\alpha \in R$ and $\alpha$ is a double pole of $x(z)$: Since
\begin{equation}
(3.19) \quad \mathrm{ord}_\alpha \left[ X(z) - X(\alpha) \right] = 2 \quad \text{and} \quad \mathrm{ord}_\alpha \left[ \frac{dX}{dz}(z) \right] = 1,
\end{equation}
we have
\begin{align*}
(3.20) \quad \mathrm{ord}_\alpha \left[ (y(z) - y(\overline{z})) \frac{dx}{dz}(z) \right] &= \mathrm{ord}_\alpha \left[ (Y(z) - Y(\overline{z})) \frac{dX}{dz}(z) \right] \\
&= \frac{1}{2} \rho(x(\alpha); P) \times (\text{ord}_\alpha [X(z) - X(\alpha)]) + 1 \\
&= \rho(x(\alpha); P) + 1.
\end{align*}

\[\square\]

Remark 3.5. It follows from Proposition 3.4 that, if $r \in R$ satisfies $\rho(r; P) \leq -2$, then $r$ is an ineffective ramification point (cf. Proposition 2.6).
3.2 Quantum curves

In order to give our theorem in a clear form, we introduce

\begin{align*}
\text{Sing}(P) := \{ b \in \mathbb{P}^1 \mid \rho(b; P) \leq -2 \}, \\
\text{Sing}_2(P) := \{ b \in \mathbb{P}^1 \mid \rho(b; P) = -2 \}
\end{align*}

for a polynomial \( P(x, y) = p_0(x)y^2 + p_1(x)y + p_2(x) \in \mathbb{C}[x, y] \). We also use the following symbols:

\begin{align*}
\Delta(z) &:= y(z) - y(\overline{z}), \\
C_\beta &:= \text{Res}_{z=\beta} \Delta(z)dx(z).
\end{align*}

**Theorem 3.6.** We assume (A1) – (A4), (AQ1) and (AQ2). Let

\begin{equation}
D(z; \nu) := [z] - \sum_{\beta \in B} \nu_\beta [\beta]
\end{equation}

be a divisor on \( \mathbb{P}^1 \), where \( B := x^{-1}(\text{Sing}(P)) \), and \( \nu = (\nu_\beta)_{\beta \in B} \) is a tuple of complex numbers satisfying

\begin{equation}
\sum_{\beta \in B} \nu_\beta = 1.
\end{equation}

Then \( \psi(x; \nu, \hbar) := \varphi(z(x); \nu, \hbar) \), where \( z(x) \) denotes an inverse function of \( x(z) \) and \( \varphi(z, \hbar) \) is defined by \((3.1)\) with the integration divisor \((3.25)\), is a WKB type formal solution of \( \hat{P}\psi = 0 \), where \( \hat{P} \) is defined by

\begin{equation}
\hat{P} := \hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \frac{d}{dx} + r(x, \hbar)
\end{equation}

with

\begin{align*}
q(x, \hbar) &= q_0(x) + \hbar q_1(x), \\
r(x, \hbar) &= r_0(x) + \hbar r_1(x) + \hbar^2 r_2(x)
\end{align*}

whose leading terms are respectively given by

\begin{align*}
q_0(x) &= \frac{p_1(x)}{p_0(x)}, \\
r_0(x) &= \frac{p_2(x)}{p_0(x)}
\end{align*}

and their lower order terms are determined by

\begin{align*}
x'(z)q_1(x(z)) &= -\frac{\Delta'(z)}{\Delta(z)} + \frac{2}{z - \overline{z}} - \sum_{\beta \in B} \nu_\beta + \nu_\overline{\beta}, \\
x'(z)r_1(x(z)) &= \frac{1}{2} x'(z) \left| \frac{\partial q_0}{\partial x} \right|_{x=x(z)} + \frac{1}{2} x'(z)q_0(x(z))q_1(x(z)) + \frac{1}{2} \Delta(z) \sum_{\beta \in B} \frac{\nu_\beta - \nu_\overline{\beta}}{z - \beta}, \\
x'(z)r_2(x(z)) &= \Delta(z) \sum_{\beta \in B_1} \frac{\nu_\beta \nu_\overline{\beta}}{C_\beta} \frac{1}{z - \beta}.
\end{align*}

Here we set \( B_1 := x^{-1}(\text{Sing}_2(P)) \).
We call the equation $\hat{P}\psi = 0$ given by Theorem 3.6 a quantum curve of the spectral curve.

Remark 3.7. (i) The set $B$ may contain the infinity. In Theorem 3.6 we use the convention that if $\beta \in B$ is the infinity, $1/(z - \beta)$ in $q_1$, $r_1$ and $r_2$ is zero. In what follows, we use this convention.

(ii) It follows from Proposition 3.4 that

\begin{align}
B &= \text{the set of poles of } \Delta(z)dx(z), \\
B_1 &= \text{the set of simple poles of } \Delta(z)dx(z).
\end{align}

Hence, $C_\beta \neq 0$ for $\beta \in B_1$. The set $B$ may contain ramification points (see Example 3.8 (i)). It follows, however, from the definition of $\text{Sing}(P)$ and Remark 3.5 (v) that ramification points contained in $B$ are ineffective. Note also that, since $\text{Sing}(P)$ and $\text{Sing}_2(P)$ are defined in terms of order of $Q_0(x)$ (or $Q_0^\infty(x)$), they are closed by the conjugation map; that is, if $\beta \in B$ (resp. $\beta \in B_1$), then $\beta \in B$ (resp. $\beta \in B_1$).

(iii) It is not obvious that $q_1(x)$, $r_1(x)$ and $r_2(x)$ defined by (3.30) – (3.32) are rational functions in $x$. But for the examples in §4 $q_1(x)$, $r_1(x)$ and $r_2(x)$ become rational functions in $x$.

(iv) Because the degree of $x(z)$ is two by our assumption (AQ1), we have two inverse functions of $x(z)$. By Theorem 3.6 we can construct two WKB type formal solutions from these two inverse functions, and they form a basis of the solution space of $\hat{P}\psi = 0$. In other words, we first fix an inverse function $z(x)$ of $x(z)$. Next we set $\varphi^\dagger(z) := \varphi(z)$. Then linearly independent WKB type formal solutions of $\hat{P}\psi = 0$ are given by $\varphi(z(x); \hbar)$ and $\varphi^\dagger(z(x); \hbar)$.

Example 3.8. (i) The Airy curve ($[Z]$): As a first example, let us consider a curve defined by

\begin{align}
\{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\}
\end{align}

with $P(x, y) = y^2 - x$. Its rational parametrization is given by

\begin{align}
x(z) = z^2, \quad y(z) = z.
\end{align}

For this spectral curve, we have $R = \{0, \infty\}$ and $\tau = -z$. Since

\begin{align}
Q_0(x) = x, \quad Q_\infty(x) = \frac{1}{x^3},
\end{align}

we have

\begin{align}
\text{Sing}(P) = \{\infty\}, \quad B = \{\infty\}, \quad B_1 = \emptyset.
\end{align}

Therefore we choose $D(z) = [z] - [\infty]$. It is easy to see $q_0(x) = 0$, $r_0(x) = -x$. Since

\begin{align}
\Delta(z) = 2y(z) = 2z,
\end{align}

we have

\begin{align}
x'(x)q_1(x(z)) = -\frac{1}{z} + \frac{2}{z - (-z)} - \frac{2\nu_\infty}{z - \infty} = 0.
\end{align}
We also have
\[(3.40)\]
\[r_1(z) = r_2(x) = 0.\]
Hence the quantum curve of the Airy curve is given by \(\hat{P}\psi = 0\) with
\[(3.41)\]
\[\hat{P} = \hbar^2 \frac{d^2}{dx^2} - x.\]

(ii) As a second example, we consider a curve \(\{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\}\) with
\[(3.42)\]
\[P(x, y) = (x^2 - 1)y^2 - \alpha^2 \quad (\alpha \neq 0).\]
A rational parametrization of this curve is given by
\[(3.43)\]
\[x(z) = \frac{1}{2} \left( z + \frac{1}{z} \right), \quad y(z) = \frac{2\alpha}{z^2 - 1}.\]
For this parametrization, we have \(R = \{1, -1\}\) and \(\overline{z} = 1/z\). Because
\[(3.44)\]
\[Q_0(x) = \frac{\alpha^2}{x^2 - 1}, \quad Q_0(x) = \frac{\alpha^2}{x^2(1 - x^2)},\]
we find
\[(3.45)\]
\[\text{Sing}(P) = \{\infty\}, \quad B = B_1 = \{0, \infty\},\]
and we choose \(D = [z] - \nu_0[0] - \nu_\infty[\infty]\) with \(\nu_0 + \nu_\infty = 1\). Since \(\Delta = y(z) - y(\overline{z}) = 2y(z)\), we have
\[(3.46)\]
\[x'(z)q_1(x(z)) = -\frac{y'}{y} + \frac{2}{z - 1/z} - \frac{\nu_0 + \nu_\infty}{z - 0} - \frac{\nu_\infty + \nu_0}{z - \infty} = \frac{2(z^2 + 1)}{z(z^2 - 1)}.\]
Therefore
\[(3.47)\]
\[q_1(x) = \frac{4}{(z - 1/z)^2} = \frac{2x}{x^2 - 1}.\]
By a similar computation, we also have
\[(3.48)\]
\[r_1(x) = \frac{\alpha(\nu_0 - \nu_\infty)}{x^2 - 1}.\]
Since
\[(3.49)\]
\[\Delta(z)dx(z) = \frac{2\alpha}{z} dz,\]
we obtain \(C_0 = -C_\infty = 2\alpha\). Hence
\[(3.50)\]
\[x'(z)r_2(x(z)) = 2y \left[ \frac{\nu_0 \nu_\infty}{C_0} \frac{1}{z - 0} + \frac{\nu_\infty \nu_0}{C_\infty} \frac{1}{z - \infty} \right] = 2y \times \frac{\nu_0 \nu_\infty}{C_0} \frac{1}{z} = \frac{2\nu_0 \nu_\infty}{z^2 - 1}.\]
Thus
\[(3.51)\]
\[r_2(x) = \frac{4\nu_0 \nu_\infty}{(z - 1/z)^2} = \frac{\nu_0 \nu_\infty}{x^2 - 1}.\]
Hence we conclude that the quantum curve is given by \(\hat{P}\psi = 0\) with
\[(3.52)\]
\[\hat{P} = \hbar^2 \frac{d^2}{dx^2} + \frac{2x}{x^2 - 1} \hbar^2 \frac{d}{dx} - \frac{\alpha^2}{x^2 - 1} + \hbar \alpha(\nu_0 - \nu_\infty) \frac{1}{x^2 - 1} + \hbar^2 \frac{\nu_0 \nu_\infty}{x^2 - 1}.\]
This equation is further investigated in §4.
3.3 Preparations for the proof

Our proof of Theorem 3.6 is based on “diagonal specialization” which allows us to relate the topological recursion (2.25) to the recursive relations (2.6)–(2.9) satisfied by the coefficients of WKB solution ([DM]; see also [BE2, IS]). In this subsection we prepare several statements for the proof.

In the remaining part of this section, we denote $D(z; \nu)$ by $D(z)$ for simplicity. We also assume the assumptions of Theorem 3.6.

Let us define

$$
\hat{T}(z, h) = \frac{1}{h} T_{-1}(z) + \hat{T}_0(z) + h \hat{T}_1(z) + \cdots := \frac{d}{dz} \log \varphi(z; \nu, h),
$$

that is,

$$
\hat{T}(z, h) dz = \frac{1}{h} W_{0,1}(z) + \int_{\zeta \in D(z)} \left( W_{0,2}(z, \zeta) - \frac{dx(z) dx(\zeta)}{(x(z) - x(\zeta))^2} \right) dz + \sum_{m=1}^{\infty} h^m \left\{ \sum_{g \geq 0, n \geq 1} \frac{1}{(n-1)!} \int_{\zeta_1 \in D(z)} \cdots \int_{\zeta_{n-1} \in D(z)} \frac{1}{2g+n-2=m} G_{g,n}(z, \zeta_1, \cdots, \zeta_{n-1}) \right\}.
$$

Then, in order to prove Theorem 3.6 it is enough to show that $\hat{T}(z, h)$ satisfies the Riccati equation associated with $\hat{P}$ after a change of variable $x = x(z)$. Comparing the coefficients, we have

$$
\hat{T}_m(z) dz = \begin{cases} W_{0,1}(z) & (m = -1), \\
\int_{\zeta \in D(z)} \left\{ W_{0,2}(z, \zeta) - \frac{dx(z) dx(\zeta)}{(x(z) - x(\zeta))^2} \right\} & (m = 0), \\
\sum_{2g+n-2=m \atop g \geq 0, n \geq 1} \frac{1}{(n-1)!} G_{g,n}(z, \cdots, z) & (m \geq 1),
\end{cases}
$$

where

$$
G_{g,n}(z_0, z_1, \cdots, z_{n-1}) := \begin{cases} W_{g,1}(z_0) & (n = 1), \\
\int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_{n-1} \in D(z_{n-1})} W_{g,n}(z_0, \zeta_1, \cdots, \zeta_{n-1}) & (n \geq 2)
\end{cases}
$$

for $2g+n \geq 2$. Note that $G_{g,n}(z_0, z_1, \cdots, z_{n-1})$ is a meromorphic differential (1-form) in the first variable $z_0$, and is a meromorphic functions (0-form) in the remaining variables $z_1, \cdots, z_{n-1}$.

We will derive a recurrence relation satisfied by $\hat{T}_m$ by using several properties (especially the topological recursion (2.25)) of $W_{g,n}$.

For the later convenience, we set

$$
G_{g,n} = W_{g,n} = 0
$$

for either $g < 0$ or $n \leq 0$. 

Proposition 3.9.

(i) \( G_{g,n}(z_0, z_1, \ldots, z_{n-1}) \) for \( 2g + n \geq 3 \) is holomorphic in each variable in \( \mathbb{P}^1 \setminus (R^* \cup B) \). All singular points of them are poles.

(ii) For \((g,n) = (0,2)\),

\[
G_{0,2}(z_0, z_1) = \left( \frac{1}{z_0 - z_1} - \sum_{\beta \in B} \frac{\nu_\beta}{z_0 - \beta} \right) dz_0.
\]

Proof. Because only singularities of \( W_{g,n} \) for \( 2g + n \geq 3 \) are poles whose residues vanish, we obtain (i). By a straightforward computation using \( W_{0,2}(z_0, z_1) = B(z_0, z_1) = dz_0 dz_1 / (z_0 - z_1)^2 \), we can prove (ii).

Proposition 3.10.

(i) \( G_{0,2}(\overline{z}_0, z_1) = \left( \frac{1}{\overline{z}_0 - z_1} - \sum_{\beta \in B} \frac{\nu_\beta}{\overline{z}_0 - \beta} \right) dz_0. \)

(ii) For \( 2g + n \geq 3 \),

\[
G_{g,n}(\overline{z}_0, z_1, \ldots, z_{n-1}) = -G_{g,n}(z_0, z_1, \ldots, z_{n-1}).
\]

Proof. By Theorem 2.4 (iii), we have

\[
W_{0,2}(z_0, z_1) + W_{0,2}(\overline{z}_0, z_1) = \frac{dx(z_0)dx(z_1)}{(x(z_0) - x(z_1))^2}.
\]

Because \( W_{0,2} \) is symmetric in its variables, we also have

\[
W_{0,2}(z_0, z_1) + W_{0,2}(z_0, \overline{z}_1) = \frac{dx(z_0)dx(z_1)}{(x(z_0) - x(z_1))^2}.
\]

From these two relations, we have

\[
W_{0,2}(z_0, \overline{z}_1) = W_{0,2}(\overline{z}_0, z_1).
\]

Thus

\[
G_{0,2}(\overline{z}_0, z_1) = \int_{\zeta_1 \in D(z_1)} W_{0,2}(\overline{z}_0, \zeta_1) = \int_{\zeta_1 \in D(z_1)} W_{0,2}(z_0, \overline{\zeta}_1)
= \sum_{\beta \in B} \nu_\beta \left( \int_{\beta} \frac{d\overline{\zeta}_1}{(\overline{z}_0 - \overline{\zeta}_1)^2} \right) dz_0 = \sum_{\beta \in B} \nu_\beta \left( \int_{\beta} \frac{d\overline{\zeta}_1}{(\overline{z}_0 - \overline{\zeta}_1)^2} \right) dz_0
= \sum_{\beta \in B} \nu_\beta \left( \frac{1}{z_0 - z_1} - \frac{1}{z_0 - \beta} \right) dz_0 = \left( \frac{1}{z_0 - z_1} - \sum_{\beta \in B} \frac{\nu_\beta}{z_0 - \beta} \right) dz_0.
\]

The relation (ii) is a direct consequence of Theorem 2.4 (iii).
We now state two propositions which give recurrence relations of \( \{G_{g,n}\} \). These relations play a key role in the proof of Theorem 3.6.

**Proposition 3.11.** We have

\[
G_{1,1}(z_0) = \frac{B(z_0, \overline{z_0})}{\Delta(z_0)d\overline{z}(z_0)}
\]

and

\[
G_{0,3}(z_0, z_1, z_2) = G_{0,2}(z_0, z_1) \left\{ \frac{G_{0,2}(z_0, z_2)}{\Delta(z_0)d\overline{z}(z_0)} - \frac{G_{0,2}(\overline{z_0}, z_2)}{\Delta(z_1)d\overline{z}(z_1)} \right\} + \frac{G_{0,2}(z_0, \overline{z_0})G_{0,2}(\overline{z_1}, z_2)}{\Delta(z_1)d\overline{z}(z_1)}
\]

\[
+ G_{0,2}(z_0, z_2) \left\{ \frac{G_{0,2}(z_0, z_1)}{\Delta(z_0)d\overline{z}(z_0)} - \frac{G_{0,2}(\overline{z_2}, z_1)}{\Delta(z_2)d\overline{z}(z_2)} \right\} + \frac{G_{0,2}(z_0, \overline{z_2})G_{0,2}(\overline{z_2}, \overline{z_1})}{\Delta(z_2)d\overline{z}(z_2)}
\]

\[- \sum_{\beta \in B_i} \frac{\nu_{\beta}r_{\beta}}{C_{\beta}} \{G_{0,2}(z_0, \beta) - G_{0,2}(z_0, \overline{\beta})\}.
\]

**Proposition 3.12.** For \( 2g + n \geq 3 \) we have

\[
G_{g,n+1}(z_0, \zeta_i) = \frac{1}{\Delta(z_0)d\overline{z}(z_0)} \int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_n \in D(z_n)} W_{g-1,n+2}(z_0, \overline{z_0}, \zeta_i)
\]

\[
+ \sum_{j=1}^{n} G_{0,2}(z_0, z_j) \left\{ \frac{G_{g,n}(z_0, z_{I \setminus \{j\}})}{\Delta(z_0)d\overline{z}(z_0)} - \frac{G_{g,n}(\overline{z_j}, z_{I \setminus \{j\}})}{\Delta(z_j)d\overline{z}(z_j)} \right\}
\]

\[
+ \sum_{j=1}^{n} \left\{ \frac{G_{0,2}(z_0, \overline{z_j})G_{g,n}(z_0, z_{I \setminus \{j\}})}{\Delta(z_0)d\overline{z}(z_0)} + \frac{G_{0,2}(z_0, \overline{z_j})G_{g,n}(\overline{z_j}, z_{I \setminus \{j\}})}{\Delta(z_j)d\overline{z}(z_j)} \right\}
\]

\[
+ \frac{1}{\Delta(z_0)d\overline{z}(z_0)} \sum_{\substack{I_1 + I_2 = I, \\
2g_1 + |I_1| \geq 2, \\
2g_2 + |I_2| \geq 2}} G_{g_1,|I_1|+1}(z_0, z_{I_1})G_{g_2,|I_2|+1}(\overline{z_0}, z_{I_2}),
\]

where \( I = \{1, 2, \cdots, n\} \).

**Proof of Proposition 3.11.** It follows from the topological recursion (2.25) for \((g, n) = (1, 0)\) that

\[
G_{1,1}(z_0) = \sum_{r \in R^*} \text{Res}_{\xi = r} \left[ \frac{G_{0,2}(z_0, \xi)}{\Delta(\xi)d\overline{\xi}(\xi)} W_{0,2}(\xi, \overline{\xi}) \right]
\]

holds (cf. (2.29) and Proposition 2.6 (ii)). Here we remind the readers that the set \( R^* \) consists of the effective ramification points in the sense of Definition 2.5. Because, under the assumption (AQ2), the singular points of the integrand are contained in \( R^* \cup \{z_0\} \), the residue theorem gives

\[
G_{1,1}(z_0) = - \text{Res}_{\xi = z_0} \left[ \frac{G_{0,2}(z_0, \xi)}{\Delta(\xi)d\overline{\xi}(\xi)} W_{0,2}(\xi, \overline{\xi}) \right] = \frac{W_{0,2}(z_0, \overline{z_0})}{\Delta(z_0)d\overline{z}(z_0)}.
\]
Here we have used (3.58) to compute the residue. Similarly, it follows from the topological recursion (2.25) for \((g, n) = (0, 2)\) that
\[
G_{0,3}(z, 0, z_2) = \sum_{r \in R^*} \operatorname{Res}_{\xi=r} [g_{0,3}(z, 0, z_2)]
\]
where
\[
g_{0,3}(\xi; 0, z_1, z_2) = K_D(\xi) (z_0, \xi) \left\{ G_{0,2}(\xi, z_1)G_{0,2}(\xi, z_2) + G_{0,2}(\xi, z_2)G_{0,2}(\xi, z_1) \right\}
= G_{0,2}(\xi, 0) \Delta(\xi) dx(\xi) \left\{ G_{0,2}(\xi, z_1)G_{0,2}(\xi, z_2) + G_{0,2}(\xi, z_2)G_{0,2}(\xi, z_1) \right\}.
\]
Since \(g_{0,3}(\xi; 0, z_1, z_2)\) is holomorphic in \(\xi\) except for points in \(R^* \cup B \cup \{ z_0, z_1, z_1, z_2, z_2 \}\), the residue theorem gives
\[
G_{0,3}(z_0, z_1, z_2) = - \sum_{r \in \{ z_0, z_1, z_1, z_2, z_2 \}} \operatorname{Res}_{\xi=r} [g_{0,3}(\xi; 0, z_1, z_2)] - \sum_{\beta \in B} \operatorname{Res}_{\xi=\beta} [g_{0,3}(\xi; 0, z_1, z_2)].
\]
By Proposition 3.10 we find that \(g_{0,3}(\xi; 0, z_1, z_2)\) has simple poles at \(z_0, z_1, z_1, z_2, z_2\), and its residues are given respectively by
\[
\operatorname{Res}_{\xi=z_0} [g_{0,3}(\xi; 0, z_1, z_2)] = \frac{1}{\Delta(0) dx(0)} \left\{ G_{0,2}(z_0, z_1)G_{0,2}(z_0, z_2) + G_{0,2}(z_0, z_2)G_{0,2}(z_0, z_1) \right\},
\]
\[
\operatorname{Res}_{\xi=z_1} [g_{0,3}(\xi; 0, z_1, z_2)] + \operatorname{Res}_{\xi=z_2} [g_{0,3}(\xi; 0, z_1, z_2)] = \frac{G_{0,2}(z_0, z_1)G_{0,2}(z_1, z_2)}{\Delta(z_1) dx(z_1)} + \frac{G_{0,2}(z_0, z_1)G_{0,2}(z_1, z_2)}{\Delta(z_1) dx(z_1)},
\]
\[
\operatorname{Res}_{\xi=z_2} [g_{0,3}(\xi; 0, z_1, z_2)] + \operatorname{Res}_{\xi=z_2} [g_{0,3}(\xi; 0, z_1, z_2)] = \frac{G_{0,2}(z_0, z_2)G_{0,2}(z_2, z_1)}{\Delta(z_2) dx(z_2)} + \frac{G_{0,2}(z_0, z_2)G_{0,2}(z_2, z_1)}{\Delta(z_2) dx(z_2)}.
\]
Next we compute the residue at \(\beta \in B\). By Proposition 3.9 (ii) and Proposition 3.10 (i), we have the Laurent expansion as
\[
G_{0,2}(\xi, z_1)G_{0,2}(\xi, z_2) = \frac{\nu_3 \nu_7}{(\xi - \beta)^2} (1 + O(\xi - \beta))(d\xi)^2
\]
near \(\xi = \beta\) (note that \(\bar{\beta} \in B\) since \(B\) is closed by the conjugation map; see Remark 3.7 (ii)). Hence, if \(\Delta(\xi) dx(\xi)\) has a double or higher order pole at \(\xi = \beta\), then
\[
\frac{G_{0,2}(\xi, \xi)G_{0,2}(\xi, z_1)G_{0,2}(\xi, z_2)}{\Delta(\xi) dx(\xi)}
\]
is holomorphic at \(\xi = \beta\) and its residue is zero. If \(\Delta(\xi) dx(\xi)\) has a simple pole at \(\beta\), i.e., if \(\beta \in B_1\) (cf. Remark 3.7 (ii)), then
\[
\frac{G_{0,2}(\xi, \xi)G_{0,2}(\xi, z_1)G_{0,2}(\xi, z_2)}{\Delta(\xi) dx(\xi)} = \frac{\nu_3 \nu_7}{C_\beta} G_{0,2}(\beta) \left( \frac{1}{\xi - \beta} + O(1) \right) d\xi
\]
when $\xi \to \beta$. Hence we obtain

$$(3.73) \quad \text{Res}_{\xi=\beta} \left[ \frac{G_{0,2}(z,\xi)G_{0,2}(\xi,z)}{\Delta(\xi)dx(\xi)} \right] = \begin{cases} 0 & (\beta \notin B_1), \\ \frac{\nu_{\beta,\xi}}{C_\beta} G_{0,2}(z,\beta) & (\beta \in B_1). \end{cases}$$

Then, the desired equality follows from

$$(3.74) \quad \sum_{\beta \in B_1} \frac{\nu_{\beta,\xi}}{C_\beta} G_{0,2}(z,\beta) = \sum_{\beta \in B_1} \frac{\nu_{\beta,\xi}}{C_\beta} G_{0,2}(z,\beta)$$

(3.75) $C_\beta = \text{Res}_{z=\beta} \Delta(z)dx(z) = \text{Res}_{z=\beta} \Delta(\bar{z})dx(\bar{z}) = -\text{Res}_{z=\beta} \Delta(z)dx(z) = -C_\beta$ $(\beta \in B)$.

This completes the proof. \qed

**Proof of Proposition 3.12.** The topological recursion gives

$$(3.76) \quad G_{g,n+1}(z_0,z_I) = \sum_{r \in R^*} \text{Res}_{\xi=r} \left[ f_1(\xi; z_0, z_I) + f_2(\xi; z_0, z_I) + f_3(\xi; z_0, z_I) \right].$$

Here

$$(3.77) \quad f_1(\xi; z_0, z_I) = \frac{G_{0,2}(z_0,\xi)}{\Delta(\xi)dx(\xi)} \int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_n \in D(z_n)} W_{g-1,n+2}(\xi,\overline{\zeta},\zeta_I),$$

$$(3.78) \quad f_2(\xi; z_0, z_I) = \frac{G_{0,2}(z_0,\xi)}{\Delta(\xi)dx(\xi)} \sum_{j=1}^n \left\{ G_{0,2}(\xi, z_j)G_{g,n}(\overline{\xi}, z_{I\setminus\{j\}}) + G_{0,2}(\overline{\xi}, z_j)G_{g,n}(\xi, z_{I\setminus\{j\}}) \right\},$$

$$(3.79) \quad f_3(\xi; z_0, z_I) = \frac{G_{0,2}(z_0,\xi)}{\Delta(\xi)dx(\xi)} \sum_{g_1+g_2=g, I_1\cup I_2=I} G_{g_1,|I_1|+1}(\xi, z_{I_1})G_{g_2,|I_2|+1}(\overline{\xi}, z_{I_2}),$$

where $\sum''$ in $f_3(\xi; z_0, z_I)$ means that we take the sum for

$$(3.80) \quad 2g_1 + |I_1| \geq 2 \quad \text{and} \quad 2g_2 + |I_2| \geq 2.$$

Because $f_1(\xi; z_0, z_I)$ and $f_3(\xi; z_0, z_I)$ are holomorphic except for $R^* \cup \{z_0\}$, the residue theorem gives

$$(3.81) \quad \sum_{r \in R^*} \text{Res}_{\xi=r} \left[ f_1(\xi; z_0, z_I) + f_3(\xi; z_0, z_I) \right]$$

$$= \frac{1}{\Delta(z_0)dx(z_0)} \int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_n \in D(z_n)} W_{g-1,n+2}(z_0, \overline{z_0}, \zeta_I) + \frac{1}{\Delta(z_0)dx(z_0)} \sum_{g_1+g_2=g, I_1\cup I_2=I} G_{g_1,|I_1|+1}(z_0, z_{I_1})G_{g_2,|I_2|+1}(\overline{z_0}, z_{I_2}).$$
Although $G_{0,2}(\xi, z_j)$ has a simple pole at $\xi = \beta \in B$, $f_2(\xi; z_0, z_1)$ is holomorphic at $\xi = \beta$ because $\Delta(\xi)dx(\xi)$ has a pole at $\beta$ (cf. Remark 3.12 (ii)). It is also holomorphic near $\xi = \beta$ with $\beta \in B$. Hence the singular points of $f_2(\xi; z_0, z_1)$ are contained in $R^s \cup \{z_0\} \cup \{z_j, \varpi_j\}_{j=1}^n$. Then, by the residue theorem we obtain

\begin{equation}
\sum_{r \in R^s} \text{Res} f_2(\xi; z_0, z_1)
= \frac{1}{\Delta(z_0)dx(z_0)} \sum_{j=1}^n \left\{ G_{0,2}(z_0, z_j)G_{g,n}(\varpi_0, z_{I\setminus \{j\}}) + G_{0,2}(\varpi_0, z_j)G_{g,n}(z_0, z_{I\setminus \{j\}}) \right\}
- \frac{1}{\Delta(z_j)dx(z_j)} \sum_{j=1}^n G_{0,2}(z_0, z_j)G_{g,n}(\varpi_j, z_{I\setminus \{j\}}) + \sum_{j=1}^n \frac{G_{0,2}(z_0, \varpi_j)}{\Delta(z_j)dx(z_j)} G_{g,n}(\varpi_j, z_{I\setminus \{j\}}).
\end{equation}

Here the first term comes from the residue at $z_0$, the second term from the residue at $z_j$, and the third term from the residue at $\varpi_j$. In computing the residue at $\varpi_j$, we have used the relation $\Delta(\varpi_j)dx(\varpi_j) = -\Delta(z_j)dx(z_j).$

Let us consider the diagonal specialization of $W_{g,n}$. For the purpose, we define

\begin{equation}
H_{g,n}(z) := \frac{1}{(n-1)!} G_{g,n}(z_0, z_1, \ldots, z_{n-1}) \bigg|_{z_0 = z_1 = \cdots = z_{n-1} = z}.
\end{equation}

This is a meromorphic differential in $z$.

**Proposition 3.13.**

\begin{align}
H_{1,1}(z) &= \frac{W_{0,2}(z, \varpi)}{\Delta(z)dx(z)}, \\
H_{0,3}(z) &= \frac{\partial}{\partial z_0} \left( \frac{G_{0,2}(\varpi, z)}{\Delta(z)x'(z_0)} \right) \bigg|_{z_0 = z} + \frac{G_{0,2}(z, \varpi)G_{0,2}(\varpi, z)}{\Delta(z)dx(z)} \\
&- \sum_{\beta \in B_1} \frac{\nu_{\beta}u_{\varpi}}{2C_{\beta}} (G_{0,2}(z, \beta) - G_{0,2}(\varpi, \beta)), \\
H_{g,n+1}(z) &= -\frac{1}{\Delta(z)x'(z)} \left[ \frac{\partial}{\partial z} H_{g-1,n+2}(z) - \frac{\partial}{\partial z_0} \left( \frac{G_{g-1,n+2}(z_0, z, \ldots, z)}{(n+1)!} \right) \bigg|_{z_0 = z} \right] \\
&- \frac{d}{dz} \left( \frac{1}{\Delta(z)x'(z)} \right) H_{g,n}(z) - \frac{1}{\Delta(z)x'(z)} \frac{\partial}{\partial z_0} \left( \frac{G_{g,n}(z_0, z, \ldots, z)}{(n-1)!} \right) \bigg|_{z_0 = z} \\
&- \frac{G_{0,2}(z, \varpi) - G_{0,2}(\varpi, z)}{\Delta(z)dx(z)} H_{g,n}(z) \\
&- \frac{1}{\Delta(z)dx(z)} \sum_{g_1 + g_2 = g, n_1 + n_2 = n, 2g_1 + n_1 \geq 2, 2g_2 + n_2 \geq 2} H_{g_1,n_1+1}(z) H_{g_2,n_2+1}(z) \quad (2g + n \geq 3).
\end{align}

Here and in what follows, we use the following convention: For a meromorphic differential $f(z)dz$, its $z$-derivative means

\begin{equation}
\frac{\partial}{\partial z} (f(z)dz) := \frac{\partial f}{\partial z}(z)dz.
\end{equation}
Proof. Because \( G_{0,2}(z_j, z_k) \) is singular along \( z_j = z_k \), we need a careful treatment for the terms which contain it. For example, the first term in the right-hand side of (3.65) becomes

\[
(3.87) \\
\lim_{z_0, z_1, z_2 \to z} \left[ G_{0,2}(z_0, z_1) \frac{G_{0,2}(\overline{z_0}, z_2)}{\Delta(z_0) dx(z_0)} - \frac{G_{0,2}(z_1, z_2)}{\Delta(z_1) dx(z_1)} \right] = \lim_{z_0, z_1, z_2 \to z} \left[ \left( \frac{dz_0}{z_0 - z_1} \right) + (\text{holomorphic along } z_0 = z_1) \right] \times \left( \frac{G_{0,2}(\overline{z_0}, z_2)}{\Delta(z_0) dx(z_0)} - \frac{G_{0,2}(z_1, z_2)}{\Delta(z_1) dx(z_1)} \right) = \frac{\partial}{\partial z_0} \left( \frac{G_{0,2}(z_0, z)}{\Delta(z_0) x'(z_0)} \right) \bigg|_{z_0 = z}
\]

after the diagonal specialization. In the same manner, we can compute the third term in the right-hand side of (3.65), and obtain the same result. This proves (3.85). We use the same computation to obtain (3.86) together with Theorem 2.4 (iii) and the relation

\[
(3.88) \\
\frac{1}{dx(z)} \int_{\zeta_1 \in D(z)} \ldots \int_{\zeta_n \in D(z)} \frac{W_{g-1,n+2}(z, \zeta_1, \ldots, \zeta_n)}{n!} \\
= - \frac{1}{dx(z)} \int_{\zeta_1 \in D(z)} \ldots \int_{\zeta_n \in D(z)} \frac{W_{g-1,n+2}(z, z, \zeta_j)}{n!} \\
= \frac{1}{x'(z)} \left\{ - \frac{\partial}{\partial z} H_{g-1,n+2}(z) + \frac{\partial}{\partial z_0} \left( \frac{G_{g-1,n+2}(z_0, z, \cdot, \cdot, \cdot)}{(n + 1)!} \right) \right\} \bigg|_{z_0 = z}
\]

Proposition 3.14.

(3.89) \( \hat{T}_0(z)dz = -G_{0,2}(z, z) \),

(3.90) \( \hat{T}_1(z) = -\frac{1}{\Delta(z)x'(z)} \int \frac{d\hat{T}_0(z)}{dz} - \frac{\partial}{\partial z} \left( \frac{1}{\Delta(z)x'(z)} \right) \hat{T}_0(z) \\
- \frac{G_{0,2}(z, z) + G_{0,2}(z, \overline{z})}{\Delta(z) dx(z)} \hat{T}_0(z) - \frac{1}{\Delta(z)x'(z)} \hat{T}_0(z)^2 \\
- \sum_{\beta \in B_1} \frac{\nu_\beta |\zeta|}{2C_\beta} G_{0,2}(z, \beta) - G_{0,2}(z, \overline{\beta}) \\
\frac{\partial}{\partial z} \left( \frac{1}{\Delta(z)x'(z)} \right) \hat{T}_1(z) \bigg|_{j=0} \sum_{j=0}^m \hat{T}_{m-j}(z) \hat{T}_j(z) \quad (m \geq 1).\)

Proof. It follows from the definition (3.55) of \( \hat{T}_m(z) \) and Theorem 2.4 (iii) that

\[
\hat{T}_0(z)dz = \int_{\zeta \in D(z)} \left( W_{0,2}(z, \zeta) - \frac{dx(z)dx(\zeta)}{(x(z) - x(\zeta))^2} \right) = - \int_{\zeta \in D(z)} W_{0,2}(z, \zeta) = -G_{0,2}(z, z).
\]

Thus we obtain (3.89).
We may write $\hat{T}_m$ for $m \geq 1$ as

\begin{equation}
(3.92) \quad \hat{T}_m(z)dz = \sum_{2g+n-2=m, \atop g \geq 0, n \geq 1} H_{g,n}(z).
\end{equation}

Then, by using Proposition 3.13 we find

\begin{equation}
(3.93) \quad \hat{T}_1(z)dz = H_{0,3}(z) + H_{1,1}(z)
= \frac{\partial}{\partial z} \left( \frac{G_{0,2}(\bar{z}_0, z)}{\Delta(z_0)x'(z_0)} \right) \bigg|_{z_0 = z}
+ \frac{G_{0,2}(z, \bar{z})G_{0,2}(\bar{z}, z)}{\Delta(z)dx(z)}
- \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\beta}}{2C_\beta} \left( G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta}) \right)
+ \frac{W_{0,2}(z, \bar{z})}{\Delta(z)dx(z)}.
\end{equation}

Because

\begin{equation}
W_{0,2}(z, \bar{z}) = W_{0,2}(\bar{z}, z) = \frac{\partial}{\partial z_1} \left( G_{0,2}(\bar{z}, z_1) \right) \bigg|_{z_1 = z}
\end{equation}
holds, we conclude that

\begin{equation}
(3.94) \quad \hat{T}_1(z)dz = \frac{\partial}{\partial z} \left( \frac{1}{\Delta(z)x'(z)} \right) G_{0,2}(\bar{z}, z)
+ \frac{1}{\Delta(z)x'(z)} \frac{\partial}{\partial z} \left( G_{0,2}(\bar{z}, z) \right)
+ \frac{G_{0,2}(z, \bar{z})G_{0,2}(\bar{z}, z)}{\Delta(z)dx(z)}
- \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\beta}}{2C_\beta} \left( G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta}) \right)
\end{equation}

Then, the equality (3.90) follows from this equality together with (3.89).
Similarly, for $m \geq 1$, (3.55) gives

$$
(3.95) \quad \hat{T}_{m+1}(z)dz = \sum_{2g+n-2=m+1, \quad g \geq 0, n \geq 1} H_{g,n}(z) = \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} H_{g,n+1}(z)
$$

$$
= -\frac{1}{\Delta(z)x'(z)} \frac{\partial}{\partial z_0} \left( \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} H_{g-1,n+2}(z_0) \right)
$$

$$
- \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} \frac{G_{g-1,n+2}(z_0, z, \cdots, z)}{(n+1)!} + \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} \frac{G_{g,n}(z_0, z, \cdots, z)}{(n-1)!}
$$

$$
- \frac{d}{dz} \left( \frac{1}{\Delta(z)x'(z)} \right) \hat{T}_m(z)dz + \frac{G_{0,2}(z, z) - G_{0,2}(z, z)}{\Delta(z)x(z)} \hat{T}_m(z)dz
$$

$$
- \frac{1}{\Delta(z)x(z)} \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} \sum_{g_1+g_2=g} \sum_{n_1+n_2=n, \quad 2g_1+n_1 \geq 2, \quad 2g_2+n_2 \geq 2} H_{g_1,n_1+1}(z)H_{g_2,n_2+1}(z)
$$

From this expression we obtain (3.91), because we can compute as

$$
(3.96) \quad \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} H_{g-1,n+2}(z_0) + \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} \left( \frac{G_{g,n}(z_0, z, \cdots, z)}{(n-1)!} - \frac{G_{g-1,n+2}(z_0, z, \cdots, z)}{(n+1)!} \right)
$$

$$
= \left\{ \begin{array}{ll}
\sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} H_{g,n}(z_0) & \text{if } m \text{ is even} \\
\sum_{2g+n-2=m, \quad g \geq 0, n \geq 2} H_{g,n}(z_0) + G_{m^2+1}(z_0) & \text{if } m \text{ is odd}
\end{array} \right.
$$

$$
= \hat{T}_m(z_0)dz_0,
$$

and

$$
(3.97) \quad \frac{1}{dx(z)} \sum_{2g+n-2=m, \quad g \geq 0, n \geq 0} \sum_{g_1+g_2=g, \quad n_1+n_2=n, \quad 2g_1+n_1 \geq 2, \quad 2g_2+n_2 \geq 2} H_{g_1,n_1+1}(z)H_{g_2,n_2+1}(z) = \frac{1}{x'(z)} \sum_{m_1+m_2=m, \quad m_1,m_2 \geq 1} \hat{T}_{m_1}(z)\hat{T}_{m_2}(z)dz
$$

$$
= \frac{1}{x'(z)} \left( \sum_{j=0}^{m} \hat{T}_{m-j}(z)\hat{T}_j(z) - 2\hat{T}_0(z)\hat{T}_m(z) \right)dz
$$

$$
= \frac{1}{x'(z)} \sum_{j=0}^{m} \hat{T}_{m-j}(z)\hat{T}_j(z)dz + \frac{2G_{0,2}(z, z)}{x'(z)} \hat{T}_m(z),
$$

where we set $m_j = 2g_j + (n_j + 1) - 2$ for $j = 1, 2$ to rewrite summation.
3.4 Proof of Theorem 3.6

Now we give a proof of Theorem 3.6. We will compare the recursive relations (3.89)–(3.91) satisfied by the functions \( \hat{T}_m \) for \( m \geq -1 \) to those (2.7)–(2.9) satisfied by the expansion coefficients of the WKB solutions (which will be denoted by \( T_m \) for \( m \geq -1 \)) after the coordinate change \( x = x(z) \).

Transformation of the Riccati equation (2.4) to \( z \)-variable. Let \( \psi(x, h) \) be the WKB solution of \( \hat{P}\psi = 0 \), where \( \hat{P} \) is given by the formula (3.27) with the coefficients specified by (3.28)–(3.32). Let \( S(x, h) \) be a logarithmic derivative of \( \psi(x, h) \), and \( T(z, h) \) is that of \( \psi(x(z), h) \); namely,

\[
S(x(z), h) = \left. \frac{d}{dx} \log \psi(x, h) \right|_{x=x(z)} = \frac{1}{x'(z)} \frac{d}{dz} \log \psi(x(z), h) = \frac{1}{x'(z)} T(z, h).
\]

Since \( S(x, h) \) satisfies the Riccati equation (2.4) associated with (3.27), \( T(z, h) \) satisfies

\[
\hbar^2 \left\{ \frac{d}{dz} T + T^2 \right\} + \hbar \left\{ U(z, h) - \hbar \frac{x''(z)}{x'(z)} \right\} T + V(z, h) = 0,
\]

where ‘ designates derivative with respect to \( z \), and we set

\[
U(z, h) = U_0(z) + \hbar U_1(z) := x'(z) q(x(z), h),
\]

\[
V(z, h) = V_0(z) + \hbar V_1(z) + \hbar^2 V_2(z) := (x'(z))^2 r(x(z), h).
\]

Hence

\[
T(z, h) = \frac{1}{\hbar} T_{-1}(z) + T_0(z) + \hbar T_1(z) + \cdots
\]

satisfies (3.98) if and only if the following recurrence relations hold:

\[
T_{-1}^2 + U_0(z) T_{-1} + V_0(z) = 0,
\]

\[
\{ 2 T_{-1}(z) + U_0(z) \} T_{m+1} + \left\{ U_1(z) - \frac{x''(z)}{x'(z)} \right\} T_m + \sum_{j=0}^m T_{m-j} T_j + \frac{d}{dz} T_m + V_{m+2}(z) = 0 \quad (m \geq -1),
\]

where we set \( V_m(z) = 0 \) for \( m \geq 3 \). These equations determine the coefficients \( \{ T_m \} \) of \( T(z) \) uniquely.

Our task for the proof of Theorem 3.6 is to show that

\[
T_m(z) = \hat{T}_m(z) \quad (m \geq -1).
\]

Equivalently, we must prove that \( \{ \hat{T}_m \} \) satisfies the recurrence relations (3.102) and (3.103). The rest of this subsection is devoted to a proof of (3.104).
Proof of $T_{-1}(z) = \hat{T}_{-1}(z)$. Since
\begin{equation}
U_0(z) = x'(z)q_0(x(z)), \quad V_0(z) = x'(z)^2 r_0(x(z)),
\end{equation}
and $(x(z), y(z))$ satisfies $y(z)^2 + q_0(x(z))y(z) + r_0(x(z)) = 0$, we find
\begin{equation}
\hat{T}_{-1}(z) = \frac{W_{0,1}(z)}{dz} = y(z)x'(z)
\end{equation}
satisfies (3.102). Thus we have verified (3.104) for $m = -1$.

Proof of $T_0(z) = \hat{T}_0(z)$. Since $y(z)$ and $y(\overline{z})$ are two solutions of $P(x(z), y) = 0$,
\begin{equation}
y(z) + y(\overline{z}) = -q_0(x(z)).
\end{equation}
Therefore,
\begin{equation}
2T_{-1}(z) + U_0(z) = x'(z)\{2y(z) + q_0(x(z))\} = x'(z)\{y(z) - y(\overline{z})\} = x'(z)\Delta(z)
\end{equation}
holds. Then, (3.103) for $m = -1$ implies that the desired relation $T_0(z) = \hat{T}_0(z)$ is equivalent to
\begin{equation}
x'(z)\Delta(z)\hat{T}_0 + \left\{U_1(z) - \frac{x''(z)}{x'(z)}\right\}\hat{T}_{-1} + \frac{d}{dz}\hat{T}_{-1} + V_1(z) = 0.
\end{equation}
In order to verify (3.109), we first prove the following.

Lemma 3.15. The function $U_1(z)$ is expressed as
\begin{equation}
U_1(z) = -\frac{\Delta'(z)}{\Delta(z)} + \frac{G_{0,2}(\overline{z}, z) + G_{0,2}(z, \overline{z})}{dz}.
\end{equation}

Proof of Lemma 3.15. It follows from Proposition 3.9 (ii) and Proposition 3.10 (i) that
\begin{equation}
\frac{G_{0,2}(\overline{z}, z) + G_{0,2}(z, \overline{z})}{dz} = \frac{2}{z - \overline{z}} - \sum_{\beta \in B} \nu_{\beta} \left(\frac{1}{z - \beta} + \frac{1}{z - \overline{\beta}}\right) = \frac{2}{z - \overline{z}} - \sum_{\beta \in B} \nu_{\beta} + \nu_{\overline{\beta}}.
\end{equation}
Here we have used the fact that $B$ is closed under the conjugate map (cf. Remark 3.7 (ii)) in the last equality. This proves the relation (3.110).

Let us prove
\begin{equation}
V_1(z) = -x'(z)\Delta(z)\hat{T}_0 - \left\{U_1(z) - \frac{x''(z)}{x'(z)}\right\}\hat{T}_{-1} - \frac{d}{dz}\hat{T}_{-1},
\end{equation}
\[31\]
which is equivalent to (3.109). By using Proposition 3.14 and Lemma 3.15, the right-hand side becomes

$$\begin{align*}
(3.113) & \quad -x'(z)\Delta(z)\hat{T}_0 - \left\{ U_1(z) - \frac{x''}{x'} \right\} \hat{T}_{-1} - \frac{d}{dz}\hat{T}_{-1} \\
& = x'(z)\Delta(z)\frac{G_{0,2}(\beta, z)}{dz} - \left\{ -\frac{\Delta'(z)}{\Delta(z)} + \frac{G_{0,2}(\beta, z) + G_{0,2}(z, \beta)}{dz} - \frac{x''(z)}{x'(z)} \right\} x'(z)y(z) \\
& \quad - \frac{d}{dz}(x'(z)y(z)) \\
& = x'(z)\Delta(z)\frac{G_{0,2}(\beta, z)}{dz} + \frac{\Delta'(z)}{\Delta(z)}x'(z)y(z) \\
& \quad - G_{0,2}(\beta, z) + G_{0,2}(z, \beta) x'(z)y(z) - x'(z)y'(z).
\end{align*}$$

Substituting

$$y(z) = \frac{y(z) - y(\beta)}{2} + \frac{y(z) + y(\beta)}{2} = \frac{1}{2}\Delta(z) - \frac{1}{2}q_0(x(z))$$

into the last expression, we find

$$\begin{align*}
(3.114) & \quad \text{(RHS of (3.112))} \\
& = \frac{1}{2}x'(z)\Delta(z)\frac{G_{0,2}(\beta, z) - G_{0,2}(z, \beta)}{dz} - \frac{1}{2}x'(z)\frac{\Delta'(z)}{\Delta(z)}q_0(x(z)) \\
& \quad + \frac{1}{2}x'(z)^2 \frac{dq_0}{dx} \bigg|_{x=x(z)} + \frac{1}{2}x'(z)q_0(x(z)) \frac{G_{0,2}(\beta, z) + G_{0,2}(z, \beta)}{dz}.
\end{align*}$$

Then, thanks to the equalities (3.110) and

$$\begin{align*}
(3.115) & \quad \frac{G_{0,2}(\beta, z) - G_{0,2}(z, \beta)}{dz} = -\sum_{\beta \in B} \frac{\nu_\beta}{z - \beta} + \sum_{\beta \in B} \frac{\nu_\beta}{z - \beta} = \sum_{\beta \in B} \frac{\nu_\beta - \nu_\beta'}{z - \beta},
\end{align*}$$

we conclude that the right-hand side of (3.112) is equal to $V_1(z)$. Thus we have verified (3.109), and hence, (3.104) holds for $m = 0$.

**Proof of $T_1(z) = \hat{T}_1(z)$**. It follows from Proposition 3.14 and Lemma 3.15 that $\hat{T}_1(z)$ satisfies

$$\begin{align*}
(3.116) & \quad x'(z)\Delta(z)\hat{T}_1(z) + \left\{ U_1(z) - \frac{x''(z)}{x'(z)} \right\} \hat{T}_0(z) + \hat{T}_0(z)^2 + \frac{d\hat{T}_0}{dz}(z) \\
& \quad + x'(z)\Delta(z) \sum_{\beta \in B_1} \frac{\nu_\beta \nu_\beta'}{2C_\beta} G_{0,2}(\beta, \beta) - G_{0,2}(z, \beta) = 0.
\end{align*}$$

In view of (3.103) for $m = 0$, the desired relation $T_1(z) = \hat{T}_1(z)$ is equivalent to the following claim:

**Lemma 3.16.** The function $V_2(z)$ is expressed as

$$\begin{align*}
(3.117) & \quad V_2(z) = x'(z)\Delta(z) \sum_{\beta \in B_1} \frac{\nu_\beta \nu_\beta'}{2C_\beta} G_{0,2}(\beta, \beta) - G_{0,2}(z, \beta)
\end{align*}$$

32
Proof of Lemma 3.16. It follows from Proposition 3.9 (ii) and the relation (3.75) that

\[
\sum_{\beta \in B_1} \frac{\nu_1 \nu_2 \Gamma G_{0,2}(z, \beta) - G_{0,2}(z, \beta)}{\beta} \frac{1}{2C_\beta} dz = \sum_{\beta \in B_1} \frac{\nu_1 \nu_2 \Gamma 1}{2C_\beta} \frac{1}{z - \beta} \frac{1}{2C_\beta} dz - \sum_{\beta \in B_1} \frac{\nu_1 \nu_2 \Gamma 1}{2C_\beta} \frac{1}{z - \beta} \frac{1}{2C_\beta} dz
\]

In the last equality, we have also used the fact that the set \( B_1 \) is closed under the conjugate map (cf. Remark 3.7 (ii)). This proves (3.117).

Thus we have verified that (3.104) holds for \( m = 1 \).

Proof of \( T_m(z) = \hat{T}_m(z) \) for \( m \geq 2 \). Proposition 3.14 and Lemma 3.15 show that

\[
x'(z) \Delta(z) \hat{T}_{m+1}(z) + \left\{ U_1(z) - \frac{x''(z)}{x'(z)} \right\} \hat{T}_0(z) + \sum_{j=0}^m \hat{T}_{m-j}(z) \hat{T}_j(z) + \frac{d \hat{T}_m}{dz} = 0 \quad (m \geq 1)
\]

holds. This is the same as the equation (3.103) for \( m \geq 1 \) satisfied by \( T_{m+1}(z) \). Thus we have proved the desired relation (3.104) for all \( m \geq 2 \).

This completes the proof of Theorem 3.6.

4 Relations between Voros coefficients and free energies via the topological recursion

4.1 Family of the Gauss hypergeometric equation

In this section we study Voros coefficients of the family of the Gauss hypergeometric equation and the free energies of their corresponding spectral curves. One of the distinguishable features of this family is that all member of it are linked by a confluence process. The traditional confluence diagram is shown in Fig. 4.1 and it is (basically) obtained by a confluence of singular points. This is also related to particular solutions of Painlevé equations.

![Figure 4.1: Traditional confluence diagram of the Gauss hypergeometric equation.](image)

In the WKB theoretic study, or in the semi-classical setting, however, it is more suitable to include a coalescence of a turning point and a singular point in the usual confluence process. From this viewpoint, we show in Fig. 4.2 the confluence diagram of defining polynomials.
Figure 4.2: Semi-classical version of the confluence diagram of a family of the Gauss hypergeometric equations. A straight line (resp., a wiggly line) in the figure denotes the confluence of singular points (resp., the coalescence of a turning point and a singular point). Here a simple pole of the potential is also regarded as a turning point.

\[ P(x, y) = y^2 - Q_{cl}(x, \lambda) \] Concrete forms of \( Q_{cl}(x, \lambda) \) is given in Table 4.1. In this diagram the number in a parenthesis denotes an order of a pole of a quadratic differential \( Q_{cl}(x, \lambda)(dx)^2 \) (we can also consider that we glue together two coordinate patches \( \mathbb{C}_x \times \mathbb{C}_y \) and \( \mathbb{C}_X \times \mathbb{C}_Y \) by a symplectic transformation \( X = 1/x, Y = -x^2y \), and the number before parenthesis denotes a number of poles of this order. For example, a symbol \( 2(1) + (-2) + (-4) \) means that \( Q_{cl}(dx)^2 \) has two simple zeros, one pole of order two, and one pole of order four. Because we can fix two poles at 0 and \( \infty \) without loss of generality, this symbol is regarded as representing

\[ Q_{cl} = \frac{(x - a_1)(x - a_2)}{x^2} \quad (a_1 \neq a_2, a_1, a_2 \neq 0), \]

or the Kummer curve (note that, because

\[ q(x)(dx)^2 = \frac{1}{(X=1/x)}q(1/X)(dX)^2 = \frac{(1 - a_1X)(1 - a_2X)}{X^4}(dX)^2 \]

holds, \( x = \infty \) is a pole of order four).

This diagram has two ends, and the corresponding differential equations play an important role in exact WKB analysis: The Airy equation is used as a canonical equation to derive the connection formula of WKB solutions of a simple turning point ([AKT2]), and the equation corresponding to \( y^2 - 1/x \) is used as that of a simple-pole type turning point ([Ko1], [Ko2]).
Each polynomial of the second line of the diagram from the ends (i.e., Weber, Whittaker, Bessel, and Legendre) admits one parameter $\lambda$. Interestingly, differential equations having these polynomials as principal symbols are also used as canonical equations to study so-called fixed singular points of the Borel transform of WKB solutions, and hence they are important to study parametric Stokes phenomenons (Weber: [V], [SS], [T08], [AKT2], Whittaker: [KoT], [KKKoT], Bessel: [AIT], Legendre: [Ko4], [KKKoT]). To analyze the Borel transform of WKB solutions of these equations, and to analyze parametric Stokes phenomenons, explicit expressions of Voros coefficients play a central role. Voros coefficients of the Kummer equation and the Gauss hypergeometric equation were also studied (see, e.g., [AT], [Ta] and [ATT]). Concerning on the computation of explicit forms of Voros coefficients, all of the studies listed above except for [V] and [SS] used a technique developed in [T08]. Its basic idea is to derive a difference equation of Voros coefficients from contiguity relations of solutions of hypergeometric equations. We also use his idea to determine the difference values of Voros coefficients.

\[ Q_{cl}(x, \Delta) = 0. \]

| Polynomial | Equation | Condition |
|------------|----------|-----------|
| Gauss ($\text{§}4.8$) | \[ \lambda_{\infty}^2 x^2 - \{\lambda_{\infty}^2 + \lambda_0^2 - \lambda_1^2\} x + \lambda_0^2 \] | $\lambda_0, \lambda_1, \lambda_{\infty} \neq 0$, $\lambda_{\infty} \neq \lambda_0 \pm \lambda_1$, $\lambda_{\infty} \neq -(\lambda_0 \pm \lambda_1)$. |
| Degenerate Gauss ($\text{§}4.7$) | \[ \frac{\lambda_{\infty}^2 x + \lambda_1^2 - \lambda_{\infty}^2}{x(x-1)^2} \] | $\lambda_1, \lambda_{\infty} \neq 0$, $\lambda_{\infty} \neq \pm \lambda_1$. |
| Kummer ($\text{§}4.6$) | \[ \frac{x^2 + 4\lambda_{\infty} x + 4\lambda_0^2}{4x^2} \] | $\lambda_0 \neq 0$, $\lambda_{\infty} \neq \pm \lambda_0$. |
| Legendre ($\text{§}4.5$) | \[ \frac{\lambda^2}{x^2 - 1} \] | $\lambda \neq 0$. |
| Bessel ($\text{§}4.4$) | \[ \frac{x + \lambda^2}{4x^2} \] | $\lambda \neq 0$. |
| Whittaker ($\text{§}4.3$) | \[ \frac{x - 4\lambda}{4x} \] | $\lambda \neq 0$. |
| Weber ($\text{§}4.2$) | \[ \frac{1}{4}x^2 - \lambda \] | $\lambda \neq 0$. |

Table 4.1: Spectral curves $y^2 - Q_{cl}(x, \Delta) = 0$.  

\[ y^2 - \frac{(x + \lambda^2)}{(4x^2)} \] as the defining polynomial of the Bessel curve, where one may think $y^2 - \frac{(x^2 + \lambda^2)}{(4x^2)} = 0$, the traditional one, as the Bessel curve. The diagram explains its reason: the former is represented by (1) + (−2) + (−3) as indicated in the confluence diagram, while the latter is represented by 2(1) + (−2) + (−4), the same type of the Kummer curve. Thus the Bessel curve of the traditional one is a special case of the Kummer curve.

\[ \frac{\lambda_{\infty}^2 x^2}{x^2 - 1} \]

[^2]: We choose $y^2 - \frac{(x^2 + \lambda^2)}{(4x^2)}$ as the defining polynomial of the Bessel curve, where one may think $y^2 - \frac{(x^2 + \lambda^2)}{(4x^2)} = 0$, the traditional one, as the Bessel curve. The diagram explains its reason: the former is represented by (1) + (−2) + (−3) as indicated in the confluence diagram, while the latter is represented by 2(1) + (−2) + (−4), the same type of the Kummer curve. Thus the Bessel curve of the traditional one is a special case of the Kummer curve.
Our arguments to analyze quantum curves are as follows: we first express Voros coefficients of the quantum curve as the difference values of the generating function of free energies of the classical curve by using WKB solutions (3.1). We then derive a difference equation which the generating function of free energies satisfies. Here we use some difference values of Voros coefficients which is obtained from the contiguity relations of the quantum curve. Finally we solve this difference equation to obtain the explicit expression of the free energy, and then determine Voros coefficients.

We have summarize our results in Table 4.3 for free energies, and in Table 4.2 for the SL-form of quantum curves. All of the potentials in Table 4.2 have the form
\[ Q_{cl}(x, \hat{\lambda}) + \hbar^2 R(x) \quad (\hat{\lambda} = \lambda + \hbar \nu / 2), \]
where \( R(x) \) has a double pole at a regular singular point of the corresponding quantum curve, and the quadratic differential \( R(x)(dx)^2 \) behaves as
\[ R(x)(dx)^2 \sim -\frac{1}{4(x - p)^2}(dx)^2 \]

| Type        | Potential                                                                 |
|-------------|---------------------------------------------------------------------------|
| Gauss (§4.8)| \( \hat{\lambda}^2 x^2 - \{ \hat{\lambda}^2_\infty + \hat{\lambda}^2 - \hat{\lambda}^2_1 \} x + \hat{\lambda}^2_0 \) - \( \hbar^2 \frac{x^2 - x + 1}{4x^2(x - 1)^2} \) |
| Degenerate Gauss (§4.7) | \( \hat{\lambda}^2 x + \hat{\lambda}^2_1 - \hat{\lambda}^2_\infty \) \( \frac{x}{x(x - 1)^2} \) - \( \hbar^2 \frac{x^2 - x + 1}{4x^2(x - 1)^2} \) |
| Kummer (§4.6) | \( \frac{x^2 + 4\hat{\lambda}^2_\infty x + 4\hat{\lambda}^2_0}{4x^2} - \frac{\hbar^2}{4x^2} \) |
| Legendre (§4.5) | \( \frac{\hat{\lambda}^2}{x^2 - 1} - \frac{\hbar^2}{4}(x^2 - 1)^2 \) |
| Bessel (§4.4) | \( \frac{x + \hat{\lambda}^2}{4x^2} - \frac{\hbar^2}{4x^2} \) |
| Whittaker (§4.3) | \( \frac{x - 4\hat{\lambda}}{4x} - \frac{\hbar^2}{4x^2} \) |
| Weber (§4.2) | \( \frac{1}{4}x^2 - \hat{\lambda} \) |

Table 4.2: SL-forms \( \{(\hbar d/dx)^2 - Q(x, \hat{\lambda}; \nu; \hbar)\} \psi = 0 \) of quantum curves of spectral curves in Table 4.1. In this table \( \hat{\lambda} = \lambda - \hbar \nu / 2, \hat{\lambda}_j = \lambda_j - \hbar \nu_j / 2 \) \( (j = 0, 1, \infty) \). For all cases, the leading term of the potential is given by \( Q_{cl}(x, \hat{\lambda}) \). Some of the quantum curves in this table have slightly different lower order terms with respect to \( \hbar \) from differential equations in the literature.
Table 4.3: Free energies for the Spectral curves $y^2 - Q_{cl}(x, \lambda) = 0$ in Table 4.1. In this table $B_{2g}$ denotes the $2g$-th Bernoulli number (see (C.1) for the definition).

when $x$ approaches to a double pole $p$. Hence

- the principal part of the potential of the quantum curve with respect to $\hbar$ is the same with that of the corresponding classical curve,

- the first term of the potential is obtained from that of the classical curve by replacing $\lambda$ with $\hat{\lambda}$.

Near a double pole of the principal part of the potential, the subleading term $R(x)$ plays an important role (cf. Remark 4.15).

Note that each quantum curve of the Gauss, the Kummer, and the degenerate Gauss curves has two or more Voros coefficients. Our study shows that all of the Voros coefficients are expressed only by the free energy of each curve, i.e., only by one power series. Thus we can expect that the free energy is more fundamental object in exact WKB analysis than the Voros
coefficients. We hope its theoretical role will be clarified by future studies. As a by-product of our study, our study also provides a way to find out the difference equations of the Free energies, which determine their explicit forms. (Some of the explicit forms of them are already known: [HZ IM SW] etc.)

In ending this subsection, we briefly recall contiguity relations of a family of the Gauss hypergeometric equation. A recent result of Oshima (see [O] and references therein) shows that his framework is efficient in a study of linear differential equations of rational coefficients theoretically and in practice. As is discussed in [IKo], it would give a computation method of Voros coefficients in a unified and algorithmic manner. Although it is enough to use the traditional contiguity relations because we only discuss second order hypergeometric equations in this paper, here we give them along Oshima’s theory for future works.

In our study we need contiguity relations of the following three operators:

\[(4.5) \quad P_{\text{Hermite}}(\mu) := h^2 RAd(\partial_x^{-\mu/h}) RAd(e^{x^2/(2h)}) \partial_x\]
\[(4.6) \quad P_{\text{Kummer}}(\kappa, \mu) := h^2 RAd(\partial_x^{-\mu/h}) RAd(x^{\kappa/h} e^{-x/h}) \partial_x\]
\[(4.7) \quad \left(\begin{array}{c}
\begin{array}{c}
a = -\mu + h, \ c = -\kappa - \mu + h \\
\leftrightarrow \kappa = a - c, \ \mu = -a + h
\end{array}
\end{array}\right)\]
\[(4.8) \quad P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) := RAd(\partial^{-\mu/h}) RAd(x^{\kappa_1/h} (1 - x)^{\kappa_2/h}) \partial_x\]
\[(4.9) \quad \left(\begin{array}{c}
\begin{array}{c}
a = -(\mu + \kappa_1 + \kappa_2), \ b = -\mu + h, \ c = -\mu - \kappa_1 + h \\
\leftrightarrow \kappa_1 = b - c, \ \kappa_2 = -a + c - h, \ \mu = -b + h
\end{array}
\end{array}\right)\]

Here RAd denotes the reduced adjoint operator ([O §2.2]). From these expressions, we obtain contiguity relations of them (cf. [O §5, §4]).

**Theorem 4.1.**

(i) \[\partial_x \circ P_{\text{Hermite}}(\mu) = P_{\text{Hermite}}(\mu - h) \circ \partial_x.\]
(ii) \[\partial_x \circ P_{\text{Kummer}}(\kappa, \mu) = P_{\text{Kummer}}(\kappa, \mu - h) \circ \partial_x, \quad (x + (h - \mu)(h\partial_x)^{-1}) \circ P_{\text{Kummer}}(\kappa, \mu) = P_{\text{Kummer}}(\kappa + h, \mu) \circ (x - (\mu(h\partial_x)^{-1})].\]
(iii) \[\partial_x \circ P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) = P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu - h) \circ \partial_x, \quad (x + (h - \mu)(h\partial_x)^{-1}) \circ P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) = P_{\text{Gauss}}(\kappa_1 + h, \kappa_2, \mu) \circ (x - \mu(h\partial_x)^{-1}), \quad (x - 1 + (h - \mu)(h\partial_x)^{-1}) \circ P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) = P_{\text{Gauss}}(\kappa_1, \kappa_2 + h, \mu) \circ (x - 1 - \mu(h\partial_x)^{-1}).\]

In order to adjust parameters of them to those of our quantum curves we use
Proposition 4.2. The SL-forms of the operators given above are

\[ \text{Ad} \left( e^{-x^2/(4\hbar)} \right) P_{\text{Hermite}} = (\hbar \partial_x)^2 - \left( \frac{1}{4} x^2 - \mu + \frac{1}{2} \hbar \right), \]

\[ \frac{1}{x} \text{Ad} \left( x^{c/(2\hbar)} e^{-x/(2\hbar)} \right) P_{\text{Kummer}} = (\hbar \partial_x)^2 - \frac{x^2 + (4a - 2c)x + c^2 - 2hc}{4x^2}, \]

and

\[ \frac{1}{x(1-x)} \text{Ad} \left( x^{c/(2\hbar)} (1-x)^{(a+b-c+h)/(2\hbar)} \right) P_{\text{Gauss}} = (\hbar \partial_x)^2 - \frac{(a-b)^2 - h^2}{4x^2(x-1)^2} x^2 + (4ab - 2ac - 2bc + 2ch)x + c^2 - 2hc. \]

Here Ad denotes the usual adjoint operator.

4.2 Weber equation

As the beginning of our study of quantum curves, we discuss the Weber curve defined by

\[ P(x, y) := y^2 - \frac{x^2}{4} + \lambda = 0 \quad (\lambda \neq 0). \]

This is a model case of our study of quantum curves. Some of the results in this subsection are studied in [Yu]. We parametrize this curve as

\[ \begin{cases} 
  x = x(z) = \sqrt{\lambda} \left( z + \frac{1}{z} \right), \\
  y = y(z) = \frac{\sqrt{\lambda}}{2} \left( z - \frac{1}{z} \right). 
\end{cases} \]

Since

\[ dx = \sqrt{\lambda} \frac{z^2 - 1}{z^2} dz, \]

ramification points are given by \( R = \{ +1, -1 \} \) (\( = R^* \)), and the conjugation map becomes \( z \to 1/z \). A straightforward computation gives

\[ y(\bar{z}) = -y(z), \quad \Delta(z) = 2y(z), \quad y(z)dx(z) = \frac{\lambda(z^2 - 1)^2}{2z^3} dz, \]

and, by using them, we also have

\[ W_{0,3}(z_1, z_2, z_3) = \frac{1}{2\lambda} \left\{ \frac{1}{(z_1 + 1)^2(z_2 + 1)^2(z_3 + 1)^2} - \frac{1}{(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2} \right\} dz_1 dz_2 dz_3, \]

\[ W_{1,1}(z) = -\frac{z^3}{\lambda(z^2 - 1)^4} dz, \quad W_{2,1}(z) = -\frac{21(z^{11} + 3z^9 + z^7)}{(z^2 - 1)^{10}\lambda^3} dz \]

39
from the topological recursion \((2.25)\), and also
\[
F_0(\lambda) = -\frac{3}{4}\lambda^2 + \frac{1}{2}\lambda^2 \log \lambda, \quad F_1(\lambda) = -\frac{1}{12} \log \lambda, \quad F_2(\lambda) = -\frac{1}{240}\lambda^2.
\]

Correlation functions and free energies have homogeneity properties with respect to \(\lambda\):

**Proposition 4.3.**

\[
W_{g,n}(z_1, \cdots, z_n) = \frac{1}{\lambda^{2g-2+n}} \times \left( W_{g,n}(z_1, \cdots, z_n) \bigg|_{\lambda=1} \right) \quad (2g + n \geq 3),
\]

\[
F_g(\lambda) = \frac{1}{\lambda^{2g-2}} \times F_g(1) \quad (g \geq 2).
\]

Because \(B = \{0, \infty\}\) and \(B_1 = \emptyset\), we can choose
\[
D(z; \nu) := [z] - \nu[0] - \nu[\infty] \quad (\nu_0 + \nu_\infty = 1, \nu = (\nu_0, \nu_\infty))
\]
as a divisor for the quantization. The quantum curve for \((4.13)\) constructed by Theorem 3.6 becomes
\[
\left\{ \hbar^2 \frac{d^2}{dx^2} - \left( \frac{x^2}{4} - \hat{\lambda}\right) \right\} \psi = 0,
\]
where
\[
\hat{\lambda} := \lambda - \frac{\hbar \nu}{2} \quad \text{and} \quad \nu := \nu_\infty - \nu_0.
\]

| \(x\) | \(a_1\) | \(a_2\) | \(\infty\) |
|---|---|---|---|
| \(z\) | 1 | -1 | 0, \(\infty\) |

**Table 4.4:** Correspondence of points for the Weber curve

There exist two simple turning points at \(a_1 = 2\sqrt{\lambda}\) and \(a_2 = -2\sqrt{\lambda}\). See Fig. 4.3 for Stokes curves emanating from them, and Table 4.4 for the correspondence of special points in the \(x\)-plane and the \(z\)-plane by the ramified covering \(x = x(z)\). The turning point \(a_1\) in the \(x\)-plane corresponds to the ramification point \(z = 1\). Because the map \(z \mapsto x = x(z)\) is double covering at \(z = 1\), there emanate 6 Stokes curves from the point \(z = 1\). Same is true for \(x = a_2\) and \(z = -1\). Note that \(z = 1\) and \(z = -1\) are turning points of the equation obtained from \((4.23)\) by the coordinate transformation \(x = x(z)\).

A WKB solution of \((4.23)\) is given by
\[
\psi(x, \lambda, \nu; \hbar) := \exp \left( \int^x S(x, \lambda, \nu; \hbar) dx \right),
\]
Figure 4.3: (a) Stokes curves of (4.23) with \( \lambda > 0 \) and the path \( \gamma \) on the \( x \)-plane. A wiggly line designates a branch cut to define \( S_{-1}(x, \lambda) \). The path \( \gamma \) will be used to define the Voros coefficient. (b) The inverse image of the Stokes curves and \( \gamma \) by \( x = x(z) \). Note that \( x(1) = a_1 \), \( x(-1) = a_2 \) and \( x(0) = x(\infty) = \infty \).

where \( S(x, \lambda, \nu; \hbar) = \sum_{n=-1}^{\infty} \hbar^n S_n(x, \lambda, \nu) \) is a solution of the Riccati equation associated with (4.23). First few coefficients are given by

\[
S_{-1}(x, \lambda) = \sqrt{\frac{x^2}{4} - \lambda}, \quad S_0(x, \lambda, \nu) = -\frac{x}{2(x^2 - 4\lambda)} + \frac{\nu}{2\sqrt{x^2 - 4\lambda}}, \quad \cdots.
\]

Here a branch of \( S_{-1}(x, \lambda) \) remains undetermined. As is explained in §2, once we fix a branch of \( S_{-1}(x, \lambda) \), all of \( S_n(x, \lambda, \nu) \) for \( n \geq 0 \) are determined uniquely.

By induction, we can show

\textbf{Proposition 4.4.} For \( n = 0, 1, 2, \cdots \), we have

\[
\begin{align*}
(i) & \quad S_n(x, \lambda, \nu) = O(x^{-2n-1}) \quad (x \to \infty). \\
(ii) & \quad S_n(\sqrt{\lambda}x, \lambda, \nu) = \lambda^{-n-1/2}S_n(x, 1, \nu).
\end{align*}
\]

Hence \( S_n(x, \lambda, \nu) \) for \( n \geq 1 \) are integrable from the infinity, and the Voros coefficient

\[
V(\lambda, \nu; \hbar) := \int_{\gamma} \left( S(x, \hbar) - \hbar^{-1}S_{-1}(x) - S_0(x) \right) dx = \sum_{n=1}^{\infty} \hbar^n \int_{\gamma} S_n(x, \lambda, \nu) dx
\]

is a well-defined (formal) power series with respect to \( \hbar \). Here \( \gamma \) is a path from the infinity to itself which crosses a branch cut once with \( a_1 \) on the right-hand side. For example, \( \gamma(t) = te^{i(\arg \lambda + \pi)/2} \) \( (t \in \mathbb{R}) \) satisfies the condition which is shown in Fig. 4.3. We also choose a branch of \( S_{-1}(x, \lambda) \) so that

\[
S_{-1}(x, \lambda) \sim \frac{x}{2}
\]
as \( x \to \infty \) along \( \gamma \). Note that \( S_{-1}(x, \lambda) \) behaves like

\[
S_{-1}(x, \lambda) \sim -\frac{x}{2}
\]

near the initial point of \( \gamma \) because \( \gamma \) crosses a cut once. In other words, the initial point and the end point of \( \gamma \) are on a different sheet of the Riemann surface of \( S_{-1}(x, \lambda) \). In Fig. 4.3, a solid (resp. dotted) portion of \( \gamma \) means that it is in a first (resp. second) sheet.

Let

\[
F(\lambda; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\lambda)
\]

be the free energy of the Weber curve (note that it does not depend on \( \nu \)). We now prove

**Theorem 4.5.** (i) The Voros coefficient of the quantum Weber curve (4.23) and the free energy (4.30) of the Weber curve are related as follows.

\[
V(\lambda, \nu; \hbar) = F\left(\lambda + \frac{\hbar}{2}; \hbar\right) - F\left(\lambda - \frac{\hbar}{2}; \hbar\right) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda} + \frac{\nu}{2} \frac{\partial^2 F_0}{\partial \lambda^2}.
\]

(Recall that \( \hat{\lambda} = \lambda - (\hbar \nu)/2 \) and \( \nu = \nu_{\infty} - \nu_0 \).)

(ii) The free energy (4.30) satisfies the following difference equation.

\[
F(\lambda + \hbar; \hbar) - 2F(\lambda; \hbar) + F(\lambda - \hbar; \hbar) = \frac{\partial^2 F_0}{\partial \lambda^2} \left( = \log \frac{1}{\lambda}\right).
\]

To prove Theorem 4.5 (i), we need the following two identities.

**Lemma 4.6.**

\[
\frac{\partial^n}{\partial \lambda^n} F_g = \int_{\zeta_1=0}^{\zeta_1=\infty} \cdots \int_{\zeta_n=0}^{\zeta_n=\infty} W_{g,n}(\zeta_1, \ldots, \zeta_n) \quad (2g + n \geq 3).
\]

**Lemma 4.7.**

\[
V(\lambda, \nu + 2; \hbar) - V(\lambda, \nu; \hbar) = -\log \left(1 - \frac{\nu + 1}{2\lambda}\right).
\]

**Proof of Lemma 4.6.** Because

\[
\Omega(z) = \frac{\partial y(z)}{\partial \lambda} \cdot dx(z) - \frac{\partial x(z)}{\partial \lambda} \cdot dy(z) = -\frac{dz}{z} = \int_{\zeta=0}^{\zeta=\infty} B(z, \zeta)
\]

holds, Theorem 2.9 (and (2.37)) gives (4.33). \(\Box\)

**Proof of Lemma 4.7.** The following proof of this lemma uses the same idea with that in [T08], while the resulting equation (4.34) has a simpler form because we only consider the difference of Voros coefficients with respect to \( \nu \) (not \( \lambda \)).
By comparing (4.23) and (4.10), we find that
\[ \tilde{\psi}(x, \nu) := e^{-x^2/(2\hbar)}\psi(x, \nu), \]
where \( \psi(x, \nu) \) is a WKB solution of (4.23) defined by (4.25) (because we have interested in \( x \) and \( \nu \) dependence only in this proof, we abbreviate in this way), is a WKB solution for the operator \( P_{\text{Hermite}} \) (cf. (4.5)) with
\[ \mu = \lambda - \hbar \nu + \frac{1}{2}. \]
It then follows from Theorem 4.1 (i) that
\[ \partial_x \tilde{\psi}(x, \nu) = (\text{const.}) \times \tilde{\psi}(x, \nu + 2), \]
where (const.) means that this does not depend on \( x \) (it may depend on \( \nu \)). By taking the logarithmic derivatives of both sides, we obtain
\[ S(x, \nu + 2) - S(x, \nu) = \frac{d}{dx} \log \left( \frac{x}{2\hbar} + S(x, \nu) \right). \]
By integration, we get
\[ \int_{\tilde{x}}^{x} \left\{ S_0(x, \nu + 2) - S_0(x, \nu) \right\} dx + V^{x, \tilde{x}}(\nu + 2) - V^{x, \tilde{x}}(\nu) = \log \left( \frac{x}{2\hbar} + S(x, \nu) \right) - \log \left( \frac{\tilde{x}}{2\hbar} + S(\tilde{x}, \nu) \right), \]
where
\[ V^{x, \tilde{x}}(\nu) = \int_{\tilde{x}}^{x} \left( S(x, \nu) - \hbar^{-1}S_{-1}(x) - S_0(x, \nu) \right) dx, \]
and \( x \) (resp., \( \tilde{x} \)) is a point on \( \gamma \) which will be taken a limit as \( x \) tends to the end point (resp., the initial point) of \( \gamma \). It follows from (4.28) and Proposition 4.4 (i) that
\[ \frac{x}{2\hbar} + S(x, \nu) = \frac{x}{\hbar} \left( 1 + O(|x|^{-2}) \right) \]
as \( x \) tends to the initial point of \( \gamma \). Hence we have
\[ \log \left( \frac{x}{2\hbar} + S(x, \nu) \right) = \log \frac{x}{\hbar} + O(|x|^{-2}). \]
In a similar manner, it follows from
\[ S_{-1}(\tilde{x}) \sim -\frac{\tilde{x}}{2} + \frac{\lambda}{\tilde{x}} + O(|\tilde{x}|^{-2}), \quad S_0(\tilde{x}) \sim -\frac{\nu + 1}{2\tilde{x}} + O(|\tilde{x}|^{-2}) \]
(care is needed for the branch, cf. (1.29)) and Proposition 4.4 (i) that
\[ \log \left( \frac{\tilde{x}}{2\hbar} + S(\tilde{x}, \nu) \right) = \log \left( \lambda - \frac{\hbar}{2}(\nu + 1) \right) - \log(\hbar \tilde{x}) + O(|\tilde{x}|^{-1}) \]
as \( \dot{x} \) tends to the end point of \( \gamma \). Hence

\[
\int_{\dot{x}}^{x} \{ S_0(x, \nu + 2) - S_0(x, \nu) \} dx + V^{x, \dot{x}}(\nu + 2) - V^{x, \dot{x}}(\nu) = \log(x\dot{x}) - \log \left( \lambda - \frac{\hbar \nu + 1}{2} \right) + O(|x|^{-2}) + O(|\dot{x}|^{-1}).
\]

Because \( S_0(x, \nu + 2) \) and \( S_0(x, \nu) \) do not depend on \( \hbar \), and \( V^{x, \dot{x}}(\nu + 2) \) and \( V^{x, \dot{x}}(\nu) \) do not contain constant term with respect to \( \hbar \), we finally obtain

\[
\int_{\dot{x}}^{x} \{ S_0(x, \nu + 2) - S_0(x, \nu) \} dx = \log(x\dot{x}) - \log \lambda + O(|x|^{-2}) + O(|\dot{x}|^{-1}),
\]

\[
V^{x, \dot{x}}(\nu + 2) - V^{x, \dot{x}}(\nu) = -\log \left( 1 - \frac{\hbar \nu + 1}{2\lambda} \right) + O(|x|^{-2}) + O(|\dot{x}|^{-1}).
\]

This proves Lemma 4.7.

\[\Box\]

**Proof of Theorem 4.5.** (i) By Theorem 3.6, the Voros coefficient can be rewritten as

\[
V(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \hbar^m \int_0^{\infty} \left( S(x, z, \lambda, \nu; \hbar) - \hbar^{-1} S_{-1}(x, z, \lambda) - S_0(x, \lambda, \nu) \right) \frac{dx}{dz} dz
\]

\[
= \sum_{m=1}^{\infty} \hbar^m \int_0^{\infty} \left\{ \sum_{2g+n-2=m, \ n \geq 0, \ n \geq 1} \frac{1}{n!} \left( \int_{\zeta_1 \in D(z; \nu)} \cdots \int_{\zeta_n \in D(z; \nu)} W_{g,n}(\zeta_1, \ldots, \zeta_n) \right) \right\} dz
\]

\[
= \sum_{m=1}^{\infty} \hbar^m \sum_{2g+n-2=m, \ g \geq 0, \ n \geq 1} \frac{1}{n!} \left( \int_{\zeta_1 \in D(\infty; \nu)} \cdots \int_{\zeta_n \in D(\infty; \nu)} W_{g,n}(\zeta_1, \ldots, \zeta_n) \right) - \int_{\zeta_1 \in D(0; \nu)} \cdots \int_{\zeta_n \in D(0; \nu)} W_{g,n}(\zeta_1, \ldots, \zeta_n).
\]

Because

\[
D(\infty; \nu) = \nu_0([\infty] - [0]) \quad \text{and} \quad D(0; \nu) = -\nu_\infty([\infty] - [0]),
\]

we have

\[
V(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \hbar^m \sum_{2g+n-2=m, \ g \geq 0, \ n \geq 1} \frac{\nu_0^n - (-\nu_\infty)^n}{n!} \int_0^{\infty} \cdots \int_0^{\infty} W_{g,n}(\zeta_1, \ldots, \zeta_n).
\]

\[\text{Note that } (4.47) \text{ also follows from the indefinite integral}
\]

\[
\int_{\dot{x}}^{x} \{ S_0(x, \nu + 2) - S_0(x, \nu) \} dx = \int_{\sqrt{x^2 - 4\lambda}}^{x} \frac{dx}{\sqrt{x^2 - 4\lambda}} = \log \left( x + \sqrt{x^2 - 4\lambda} \right).
\]

44
Now we use Lemma 4.6:

\[ V(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \hbar^m \sum_{g=0, n \geq 1} \frac{\nu_0^n - (-\nu_\infty)^n}{n!} \frac{\partial^n F_g}{\partial \lambda^n} \]

\[ = \sum_{n=1}^{\infty} \frac{\nu_0^n - (-\nu_\infty)^n}{n!} \hbar^n \frac{\partial^n F(\lambda, \hbar)}{\partial \lambda^n} - \frac{\nu_0 - (-\nu_\infty)}{\hbar} \frac{\partial F_0}{\partial \lambda} - \frac{\nu_0^2 - (-\nu_\infty)^2}{2!} \frac{\partial^2 F_0}{\partial \lambda^2} \]

Since \( \nu_0 = (1 - \nu)/2 \) and \( \nu_\infty = (1 + \nu)/2 \), we obtain (i).

(ii) By Lemma 4.7, we have

\[ V(\lambda, \nu; \hbar)|_{\nu=1} = V(\lambda, \nu; \hbar)|_{\nu=-1}. \]

It follows from (i) that

\[ V(\lambda, \nu; \hbar)|_{\nu=1} = F(\lambda; \hbar) - F(\lambda - \nu_\infty; \hbar) - \frac{\nu_0 + \nu_\infty}{\hbar} \frac{\partial F_0}{\partial \lambda} - \frac{\nu_0^2 - \nu_\infty^2}{2} \frac{\partial^2 F_0}{\partial \lambda^2}. \]

By substituting these two relations into (4.53), we obtain (ii).

**Theorem 4.8.** The free energy of the Weber curve (4.13) is given explicitly as follows:

\[ F_g(\lambda) = \frac{B_{2g}}{2g(2g-2)} \lambda^{2g-2} \quad (g \geq 2). \]

**Proof.** From Proposition C.3 (i), we find that

\[ F := (e^{\hbar \lambda/2} - e^{\hbar \lambda/2})^{-2} \log \lambda = (\hbar \partial_\lambda)^{-2} \log \lambda + \sum_{g=1}^{\infty} F_g(\lambda) \hbar^{2g-2} \]

is a solution of the difference equation in Theorem 4.5 (ii). Let \( \hat{F} = \sum_{g=0}^{\infty} \hat{F}_g \hbar^{2g-2} \) be other solution of it. Then, by Proposition C.5, each coefficient of \( F - \hat{F} \) with respect to \( \hbar \) must be a linear function in \( \lambda \). Thus it follows from Proposition 4.3 that \( F_g - \hat{F}_g \) should be zero for \( g \geq 2 \). 

**Corollary 4.9.** The Voros coefficient of the quantum Weber curve (4.22) is given explicitly as follows:

\[ V(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \frac{B_{m+1}(\nu + 1)/2}{m(m + 1)} \left( \frac{\hbar}{\lambda} \right)^m. \]

Here \( B_m(t) \) is the Bernoulli polynomial (see (C.3) for the definition).
Proof. By Theorem 4.5, we have
\[
V(\lambda, \nu; \hbar) = \left( e^{-\nu - 1} \hbar \partial \lambda / 2 - e^{-\nu - 1} \hbar \partial \lambda / 2 \right) F - \frac{\partial F_0}{\partial \lambda} \hbar^{-1} + \frac{\nu}{2} \frac{\partial^2 F_0}{\partial \lambda^2}
\]
\[
= e^{-\nu \hbar \partial \lambda / 2} \left( e^{\hbar \partial \lambda / 2} - e^{-\hbar \partial \lambda / 2} \right) F - \frac{\partial F_0}{\partial \lambda} \hbar^{-1} + \frac{\nu}{2} \frac{\partial^2 F_0}{\partial \lambda^2}.
\]
Here we use (4.57) and then Proposition C.2 (i) to obtain
\[
e^{-\nu \hbar \partial \lambda / 2} \left( e^{\hbar \partial \lambda / 2} - e^{-\hbar \partial \lambda / 2} \right) F
\]
\[
= e^{-\nu \hbar \partial \lambda / 2} \left( e^{\hbar \partial \lambda / 2} - e^{-\hbar \partial \lambda / 2} \right)^{-1} \log \lambda
\]
\[
= (\hbar \partial \lambda)^{-1} \log \lambda - \sum_{n=0}^{\infty} \frac{B_{n+1}(-\nu - 1/2)}{(n+1)!} (\hbar \partial \lambda)^n \log \lambda.
\]
We finally use (C.6) and \(B_1(X) = X - 1/2\), we have
\[
e^{-\nu \hbar \partial \lambda / 2} \left( e^{\hbar \partial \lambda / 2} - e^{-\hbar \partial \lambda / 2} \right) F
\]
\[
= (\hbar \partial \lambda)^{-1} \log \lambda - \sum_{n=0}^{\infty} \frac{B_{n+1}((\nu + 1)/2)}{(n+1)!} (-\hbar \partial \lambda)^n \log \lambda
\]
\[
= (\hbar \partial \lambda)^{-1} \log \lambda - \frac{\nu}{2} \log \lambda + \sum_{n=1}^{\infty} \frac{B_{n+1}((\nu + 1)/2)}{n(n+1)} \left( \frac{\hbar}{\lambda} \right)^n.
\]
This proves Corollary 4.9 (Because we use an inverse of a difference operator, there remain undetermined arbitrariness. They are, however, shown to be zero by a similar argument as that given in the end of the proof of Corollary 4.8 by noting the homogeneity property
\[
V_m(\lambda, \nu) := \int_\gamma S_m(x, \lambda, \nu) dx = \lambda^m V_m(1, \nu),
\]
which follows from Proposition 4.3 (ii.).)

\[\square\]

4.3 Whittaker equation

In this subsection, we study the Whittaker curve defined by
\[
P(x, y) := xy^2 - \frac{1}{4} x + \lambda = 0 \quad (\lambda \neq 0).
\]
Because the argument for this curve can be done parallel to that for the Weber curve (4.2), some proofs below are given rather sketchy.

We parametrize this curve by
\[
\begin{cases}
x(z) = -\frac{4\lambda}{z^2 - 1} = 2\lambda \left( \frac{1}{z + 1} - \frac{1}{z - 1} \right), \\
y(z) = \frac{1}{2} z.
\end{cases}
\]

46
Because

\[(4.65) \quad dx = \frac{8\lambda z}{(z^2 - 1)^2} dz\]

holds, we find that ramification points are given by \( R = \{0, \infty\} (= R^*) \), and the conjugate map is given by \( \bar{z} = -z \). We also have

\[(4.66) \quad y(\bar{z}) = -y(z), \quad \Delta(z) = 2y(z), \quad y(z)dx(z) = \frac{4\lambda z^2}{(z^2 - 1)^2} dz,\]

and

\[(4.67) \quad W_{1,1}(z) = \frac{(z^2 - 1)^2}{32\lambda z^4} dz, \quad W_{0,3}(z_1, z_2, z_3) = -\frac{dz_1 dz_2 dz_3}{4\lambda z_1^2 z_2^2 z_3^2},\]

\[(4.68) \quad W_{1,2}(z_1, z_2) = \frac{5z_1^4 + 3z_1^2 z_2^2 + 5z_2^4 - 12z_1^2 z_2^4 - 12z_1^4 z_2^2 + 10z_1^4 z_2^4 + z_1^6 z_2^6}{128\lambda^2 z_1^6 z_2^6} dz_1 dz_2,\]

\[(4.69) \quad W_{2,1}(z) = \frac{-9z_1^{12} + 22z_1^{10} - 103z_1^8 + 372z_1^6 - 583z_1^4 + 406z_1^2 - 105}{8192\lambda^3 z_1^{10}} dz,\]

\[(4.70) \quad F_0 = \frac{3}{2} \lambda^2 + \lambda^2 \log(4\lambda), \quad F_1 = \frac{1}{6} \log \lambda, \quad F_2 = -\frac{1}{120\lambda^2}.\]

We can verify that the correlation functions and free energies have certain homogeneity properties as well as the Weber case (cf. Proposition 4.3).

To quantize the Whittaker curve, we use a divisor

\[(4.71) \quad D(z; \nu) := [z] - \nu_+ [1] - \nu_- [-1] \quad (\nu = (\nu_+, \nu_-)),\]

where the coefficients satisfy \( \nu_+ + \nu_- = 1 \). The resulting quantum curve is given by (cf. Theorem 3.6)

\[(4.72) \quad \left\{ \hat{\hbar}^2 \frac{d^2}{dx^2} + \frac{\hbar^2}{x} \frac{d}{dx} - \frac{x - 4\hat{\lambda}}{4x} \right\} \psi = 0\]

where

\[(4.73) \quad \hat{\lambda} := \lambda - \frac{\hbar \nu}{2} \quad \text{and} \quad \nu := \nu_+ - \nu_- .\]

The SL-form of this equation is

\[(4.74) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} - \frac{x - 4\lambda}{4x} + \frac{\hbar^2}{4x^2} \right\} \varphi = 0.\]

Here we give a remark that, although the principal part of the equation discussed in [KoT] has the same form with (4.74), the lower degree terms of (4.74) with respect to \( \hbar \) are different.

There exist one turning point at \( x = a := 4\lambda \) and one simple-pole type turning point at \( x = 0 \) for the quantum curve (4.72) or (4.74). Fig. 4.4 shows its turning points and Stokes curves, and Table 4.5 gives the correspondence of special points. Contrary to the case of the
Table 4.5: Correspondence of points for the Whittaker curve

| $x$ | $a$ | 0 | $\infty$ |
|-----|-----|---|----------|
| $z$ | 0   | $\infty$ | $1, -1$ |

Figure 4.4: (a) Stokes curves of (4.72) with $\lambda = -1/4$ on the $x$-plane. A wiggly line designates a branch cut to define $S_{-1}(x) = \sqrt{\frac{1}{4} - \frac{\lambda}{x}}$. (b) The inverse image of the Stokes curves and $\gamma$ by $x = x(z)$. This figure is drawn in the $w$-plane, where $w = 1/(z + i)$, in order to include the point $z = \infty$ in the figure.

Weber curve, (4.72) has a simple-pole type turning point. It corresponds to a ramification point $z = \infty$, from which two curves are emanate. For the equation obtained from (4.72) by the coordinate transform $x = x(z)$, $z = 0$ is a turning point, while $z = \infty$ is neither a turning point nor a simple-pole type turning point: This point is a “ghost” point (cf. [Ko3], where a “ghost” point is called a “new” turning point). Thus Stokes phenomena of (Borel summed) WKB solutions also occurs when we cross curves emanating from $z = \infty$.

Now we use $\lambda$ as a deformation parameter for the spectral curve. Then, since

\[
\Omega(z) = \frac{\partial y}{\partial \lambda}(z) \cdot dx(z) - \frac{\partial x}{\partial \lambda}(z) \cdot dy(z) = \frac{2}{z^2 - 1} dz = \int_{\zeta = -1}^{\zeta = 1} B(z, \zeta),
\]

Theorem 2.9 gives the following variation formula.

Lemma 4.10. For $2g + n \geq 3$, we have

\[
\frac{\partial^n}{\partial \lambda^n} F_g = \int_{\zeta_1 = -1}^{\zeta_1 = 1} \cdots \int_{\zeta_n = -1}^{\zeta_n = 1} W_{g,n}(\zeta_1, \ldots, \zeta_n).
\]

Now we study the Voros coefficient

\[
V(\lambda, \nu; \hbar) := \int_{\gamma} (S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x)) dx
\]
of the quantum Whittaker curve (1.72). Here we choose the integration path \( \gamma \) in \( x \)-plane as indicated in Fig.4.4 (a); that is, \( \gamma \) starts from infinity (on the second sheet of the Riemann surface of \( S_{-1}(x) \)) and ends at infinity (on the first sheet of the Riemann surface) after crossing the branch cut. We will choose the branch of \( S_{-1}(x) \) which behaves as

\[
S_{-1}(x) = \sqrt{\frac{1}{4} - \frac{\lambda}{x} + \frac{1}{2}}
\]

when \( x \) tends to infinity on the first sheet (i.e. the endpoint of \( \gamma \) which corresponds to \( z = +1 \) in Fig.4.4 (b)). Then, as a counterpart of Theorem 4.5 for the Weber case, we have

**Theorem 4.11.** (i) The Voros coefficient of the quantum Whittaker curve (1.72) and the free energy \( F(\lambda; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_{g}(\lambda) \) of the Whittaker curve (4.63) are related as follows.

\[
V(\lambda, \nu; \hbar) = F\left(\hat{\lambda} + \frac{\hbar}{2}, \hat{\nu}\right) - F\left(\hat{\lambda} - \frac{\hbar}{2}, \hat{\nu}\right) - \frac{1}{\hbar} \frac{\partial F_{0}}{\partial \lambda} + \nu \frac{\partial^{2} F_{0}}{2 \partial \lambda^{2}}.
\]

(ii) The free energy of the Whittaker curve satisfies the following difference equation.

\[
F(\lambda + \hbar; \hbar) - 2F(\lambda; \hbar) + F(\lambda - \hbar; \hbar) = \frac{\partial^{2} F_{0}}{\partial \lambda^{2}} = 2 \log(4\lambda).
\]

**Proof.** The claims (i) and (ii) can be proved by exactly the same argument used in the proof of Theorem 4.5 for the Weber curve. For example, the equality (4.79) can be derived as follows.

\[
V(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \hbar^{m} \int_{-1}^{1} \frac{d}{dz} \left\{ \sum_{g=0, n \geq 1}^{2g+n-2=m} \frac{1}{n!} \int_{\zeta_{1} \in D(z; \nu)} \cdots \int_{\zeta_{n} \in D(z; \nu)} W_{g,n}(\zeta_{1}, \ldots, \zeta_{n}) \right\} dz
\]

\[
= \sum_{m=1}^{\infty} \hbar^{m} \left\{ \sum_{g=0, n \geq 1}^{2g+n-2=m} \frac{1}{n!} \int_{\zeta_{1} \in D(1; \nu)} \cdots \int_{\zeta_{n} \in D(1; \nu)} W_{g,n}(\zeta_{1}, \ldots, \zeta_{n}) - \sum_{g=0, \nu_{\pm} \geq 1}^{2g+n-2=m} \frac{1}{n!} \int_{\zeta_{1} \in D(-1; \nu)} \cdots \int_{\zeta_{n} \in D(-1; \nu)} W_{g,n}(\zeta_{1}, \ldots, \zeta_{n}) \right\}
\]

\[
= \sum_{m=1}^{\infty} \hbar^{m} \sum_{g=0, n \geq 1}^{2g+n-2=m} \frac{\nu_{-}^{n} - (-\nu_{+})^{n}}{n!} \frac{\partial^{n} F_{g}}{\partial \lambda^{n}}
\]

\[
= F(\lambda + \hbar \nu_{-}; \hbar) - F(\lambda - \hbar \nu_{+}; \hbar) - \frac{\nu_{-} + \nu_{+}}{\hbar} \frac{\partial F_{0}}{\partial \lambda} - \frac{\nu_{-}^{2} - \nu_{+}^{2}}{2} \frac{\partial^{2} F_{0}}{\partial \lambda^{2}}.
\]

(Note that \( \nu_{\pm} = (1 \pm \nu)/2 \).) Here we have used Lemma 4.10. The second equality (4.80) follows from (4.79) and the difference equation

\[
V(\lambda, \nu + 2; \hbar) - V(\lambda, \nu; \hbar) = -2 \log \left(1 - \frac{\nu + 1}{2 \lambda} \right)
\]

satisfied by the Voros coefficient (cf. Lemma 4.7). We omit the computational details. \qed
We can solve the difference equation (4.80) by a similar method used for the Weber equation in §4.2. Consequently, we obtain the following explicit formula for the free energy and the Voros coefficient for the quantum Whittaker curve (we omit the details again).

**Theorem 4.12.** The free energy of the Whittaker curve (4.63) is expressed as follows:

\[
F_g = \frac{B_{2g}}{g(2g-2)} \frac{1}{\lambda^{2g-2}} \quad (g \geq 2).
\]

**Corollary 4.13.** The Voros coefficient of the quantum Whittaker curve (4.72) is given explicitly as follows:

\[
V(\lambda, \nu; \hbar) = 2 \sum_{m=1}^{\infty} \frac{B_{m+1}(\nu + 1)/2}{m(m+1)} \left( \frac{\hbar}{\lambda} \right)^m.
\]

We conclude this subsection by giving one remark. We can also choose the rational parametrization of the Whittaker curve (4.63) in a different way:

\[
\begin{align*}
x(z) &= \lambda \frac{(z+1)^2}{z}, \\
y(z) &= \frac{(z-1)}{2(z+1)}. 
\end{align*}
\]

To this parametrization,

\[
\begin{align*}
dx(z) &= \lambda \frac{z^2 - 1}{z} dz, \\
y(z)dx(z) &= \frac{\lambda (z-1)^2}{2z^2} dz,
\end{align*}
\]
and $R = \{1, -1\}$. Correlation functions are
\begin{align}
W_{1,1}(z) &= -\frac{z^2}{\lambda(z - 1)^4(z + 1)^2}dz,
\end{align}
\begin{align}
W_{0,3}(z_1, z_2, z_3) &= -\frac{2dz_1dz_2dz_3}{\lambda(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2},
\end{align}
\begin{align}
W_{1,2}(z_1, z_2) &= \frac{\lambda^2(z_1 - 1)^6(z_1 + 1)^2(z_2 - 1)^6(z_2 + 1)^2}{1}
\times \left\{3z_1^6z_2^4 + 2z_1^5z_2^5 + 3z_1^4z_2^6
\right.
\left.
+ 4z_1^6z_2^3 - 8z_1^5z_2^4 - 8z_1^4z_2^5 + 4z_1^3z_2^6
\right.
\left.
+ 3z_1^6z_2^2 - 8z_1^5z_2^3 + 2z_1^4z_2^4 - 8z_1^3z_2^5 + 3z_1^2z_2^6
\right.
\left.
- 8z_1^5z_2^2 - 4z_1^4z_2^3 - 4z_1^3z_2^4 - 8z_1^2z_2^5
\right.
\left.
+ 2z_1^5z_2 + 2z_1^4z_2^2 + 56z_1^3z_2^3 + 2z_1^2z_2^4 + 2z_1z_2^5
\right.
\left.
- 8z_1^4z_2 - 4z_1^3z_2^2 - 4z_1^2z_2^3 - 8z_1^2z_2^4
\right.
\left.
+ 3z_1^4 - 8z_1^3z_2 + 2z_1^2z_2^2 - 8z_1z_2^3 + 3z_1^3
\right.
\left.
+ 4z_1^3 - 8z_1^2z_2 - 8z_1z_2^2 + 4z_2^3
\right.
\left.
+ 3z_1^2 + 2z_1z_2 + z_2^2\right\}dz_1dz_2,
\end{align}
\begin{align}
W_{2,1}(z) &= -\frac{8z^8 + 24z^7 + 41z^6 + 24z^5 + 8z^4}{\lambda^3(z - 1)^{(10)(z + 1)^4}}dz,
\end{align}
and so on (free energies coincide with those for the previous parametrization).

To quantize the Whittaker curve with this parametrization, we use a divisor
\begin{align}
D = [z] - \nu_0[0] - \nu_\infty[\infty] \quad (\nu_0 + \nu_\infty = 1),
\end{align}
and the result is
\begin{align}
\left\{\hbar^2 \frac{d^2}{dx^2} + \hbar^2 \frac{d}{x} \frac{d}{dx} - \frac{x - 4\lambda}{4x}\right\} \psi = 0 \quad \text{with} \quad \lambda := \frac{\hbar\nu}{2}, \quad \nu := \nu_\infty - \nu_0,
\end{align}
which has the same form with (4.72) for the previous parametrization (4.64) of the Whittaker curve (4.63), although we have used a different parametrization (4.85).

### 4.4 Bessel equation

Let us consider the Bessel curve defined by
\begin{align}
P(x, y) = y^2 - \frac{x + 4\lambda^2}{4x^2} = 0 \quad (\lambda \neq 0).
\end{align}
A rational parameterization of this curve is given by
\begin{align}
\left\{\begin{array}{l}
x = x(z) = 4\lambda^2(z^2 - 1),
\end{array}\right.
\left\{\begin{array}{l}
y = y(z) = \frac{z}{4\lambda(z^2 - 1)}.
\end{array}\right.
\end{align}
For this curve,

\[ dx = 8 \lambda^2 z dz \]

and \( R = \{0, \infty\} \). Note that \( \infty \in R \) is ineffective, and hence \( R^* = \{0\} \); cf. Proposition B.1. The conjugate map is given by \( \tau = -z \) for both ramification points. We also note that \( z = \pm 1 \) and \( \infty \) are poles of \( \Delta(z) = (y(z) - y(\tau)) dz(z) \) (i.e., \( B = \{+1, -1, \infty\} \)).

Let \( W_{g,n} \) and \( F_g \) be the correlation functions and free energies computed from the Bessel curve (4.95), respectively. Few of them are computed as

\[
\begin{align*}
W_{0,3}(z_1, z_2, z_3) &= \frac{dz_1 dz_2 dz_3}{2 \lambda z_1^2 z_2^2 z_3^2}, \quad W_{1,1}(z) = -\frac{z^2 - 1}{16 \lambda z^4} dz, \\
W_{1,2}(z_1, z_2) &= \frac{z_1^4 z_2^4 - 6 z_1^2 z_2^2 - 6 z_1^2 z_2^4 + 5 z_1^4 + 3 z_1^2 z_2^2 + 5 z_2^4}{32 \lambda^2 z_1^6 z_2^6} dz_1 dz_2, \\
W_{2,1}(z) &= -\frac{9 z^6 - 107 z^4 + 203 z^2 - 105}{1024 \lambda^3 z_{10}} dz, \quad F_0 = 3 \lambda^2 + \lambda^2 \log \left(-\frac{1}{16 \lambda^2}\right), \quad F_1 = -\frac{1}{24} \log \left(-\frac{1}{16 \lambda^2}\right), \quad F_2 = \frac{1}{960 \lambda^2}, \quad F_3 = -\frac{1}{16128 \lambda^2}.
\end{align*}
\]

Let us take a divisor

\[ D(z; \nu) = [z] - \nu_+ [1] - \nu_- [-1] - \nu_\infty [\infty] \quad (\nu = (\nu_+, \nu_-, \nu_\infty)) \]

with the parameters \( \nu_+, \nu_\infty \) satisfying \( \nu_+ + \nu_- + \nu_\infty = 1 \). Then, the quantization of the Bessel curve via Theorem 3.6 is given by

\[ \left\{ \hbar^2 \frac{d^2}{dx^2} + \hbar^2 \nu_\infty \frac{d}{dx} - \frac{x + 4 \lambda^2}{4 x^2} + \hbar \lambda (\nu_+ - \nu_-) \frac{x^2}{x^2} + \hbar^2 \frac{\nu_+ \nu_-}{x^2} \right\} \psi = 0. \]

This equation has a regular singular point at \( x = 0 \) and an irregular singular point at \( x = \infty \). We can verify that the equation (4.101) is equivalent to the Bessel equation

\[ \left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} - \left( 1 + \frac{\alpha^2}{u^2} \right) \right) \tilde{\psi} = 0 \]

via the change of variables \( u = \hbar^{-1} x^{1/2}, \alpha = \hbar^{-1} 2 \lambda - (\nu_+ - \nu_-) \) and the gauge transform \( \psi = x^{(\nu_+ + \nu_-)/2} \tilde{\psi} \). The SL-form of (4.101) is given by

\[ \left\{ \hbar^2 \frac{d^2}{dx^2} - Q(x, \hbar) \right\} \varphi = 0, \quad Q(x, \hbar) := \frac{x + 4 \lambda^2}{4 x^2} - \hbar^2 \frac{1}{4 x^2}, \]

where

\[ \hat{\lambda} := \lambda - \frac{\hbar \nu}{2} \quad \text{and} \quad \nu := \nu_+ - \nu_- \]

Note also that the special case \( \nu = 0 \) of the equation (4.103) has already constructed as a quantum curve in [FIMS, §4.1]; this example appears as the quantum differential equation for the equivariant \( \mathbb{CP}^1 \).

The free energies for the Bessel curve (4.95) satisfy the following:
Lemma 4.14. For $2g + n \geq 3$, we have

\begin{equation}
\frac{\partial^n}{\partial \lambda^n} F_g = \int_{\zeta_1 = -1}^{\zeta_1 = 1} \cdots \int_{\zeta_n = -1}^{\zeta_n = 1} W_{g,n}(\zeta_1, \ldots, \zeta_n) .
\end{equation}

The above variational formula follows from

\[ \Omega(z) = \frac{\partial y(z)}{\partial \lambda} dx(z) - \frac{\partial x(z)}{\partial \lambda} dy(z) = \frac{2dz}{z^2 - 1} = \int_{\zeta = -1}^{\zeta = 1} W_{0,2}(z, \zeta) \]

and Theorem 2.9.

Let $S(x, \hbar) = (d \log \psi(x, \hbar))/dx$ be the solution of the Riccati equation associated with

\begin{equation}
(4.101)
\end{equation}

The first few terms are given as

\begin{equation}
S_{-1}(x) = \frac{\sqrt{x + 4\lambda^2}}{2x}, \quad S_0(x) = \frac{(\nu_+ + \nu_- - \nu_\infty)x + 8\lambda^2(\nu_+ + \nu_-) - 4\lambda(\nu_+ - \nu_-)\sqrt{x + 4\lambda^2}}{4x(x + 4\lambda^2)}.
\end{equation}

The points $z = \pm 1$ correspond to the singular point $x = 0$ of these functions, where

\begin{equation}
S_{-1}(x) \sim \pm \frac{\lambda}{x}
\end{equation}

holds when $z \to \pm 1$, respectively.

Remark 4.15. It is easily checked that each $S_m(x)$ has at most simple pole at $x = 0$, which is a double pole of the leading term of $Q(x, \hbar)$. However, we can prove that $S_m(x)$ with $m \geq 1$ are in fact holomorphic at $x = 0$. We emphasize that this property follows form the fact that the behavior of the subleading term, which is denoted by $R(x)$ in the expression \[(4.13), of Q(x, \hbar)\ near x = 0 has the specific form: R(x) \sim -1/(4x^2) as x \to \infty.\ This allows us to define the Voros coefficient for a path from $x = 0$ as shown in Fig. 4.5 below. Similar pole-cancellation is always observed for other examples discussed in §4 when $Q(x, \hbar)$ has a double pole which is a regular singular point of the quantum curve. (Note also that the holomorphicity at the double pole is obvious from the view point of topological recursion since the correlation functions are holomorphic there.)

\begin{tabular}{|c|c|c|c|}
\hline
$x$ & $a$ & $0$ & $\infty$ \\
\hline
$z$ & $0$ & $1$ & $1$ & $\infty$ \\
\hline
\end{tabular}

Table 4.6: Correspondence of points for the Bessel curve

Let $\tilde{\gamma}_0$ (resp., $\tilde{\gamma}_{\infty}$) be a path on $z$-plane from $-1$ to $+1$ (resp., from $\infty$ to $\infty$) depicted in Fig. 4.5 (b). We denote by $\gamma_0$ (resp., $\gamma_{\infty}$) its image by $x(z)$ as indicated in Fig. 4.5 (a). The Voros coefficients of \[(4.101)\ for the paths $\gamma_0$ and $\gamma_{\infty}$ are defined by

\begin{equation}
(4.108)
V^{(j)}(\lambda, \nu; \hbar) := \int_{\gamma_j} \left( S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \right) dx \quad (j = 0, \infty).
\end{equation}

Actually, the Voros coefficients depend only on the difference $\nu = \nu_+ - \nu_-$ of the coefficients in the divisor $D(z; \nu)$ because the SL-form \[(4.103)\ of the quantum curve depends solely on $\nu$.\
Figure 4.5: (a) Stokes curves of (4.101) with $\lambda = 1$ (thick lines), and the paths $\gamma_0$ and $\gamma_\infty$ on $x$-plane. The point $x = a = -4\lambda^2$ is a turning point, and the wiggly line designates the branch cut to fix the branch of $S_{-1}(x)$. (b) (Pull back by $x(z)$ of) Stokes curves (thick lines) and the paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_\infty$ on $z$-plane. Under the map $x = x(z)$, $x = a$ and $x = 0$ correspond respectively to $z = 0$ and $z = \pm 1$.

**Lemma 4.16.** The Voros coefficient $V^{(\infty)}(\lambda, \nu; \hbar)$ for the path $\gamma_\infty$ is trivial:

$$V^{(\infty)}(\lambda, \nu; \hbar) = 0.$$  

**Proof.** In view of Fig.4.5 (a), the path $\gamma_\infty$ is homotopically equivalent to the residue cycle around $x = 0$. On the other hand, Theorem 3.6 implies that the integrant of (4.108) is holomorphic except for branchpoints, because the correlation function $W_{g,n}$ is holomorphic there if $2g - 2 + n \geq 1$ (cf. Remark 4.15). Hence, in particular, the integrant of (4.108) has no residue at $x = 0$. Thus we obtain (4.109).}

The other Voros coefficient $V^{(0)}$ is non-trivial as we will see next.

**Theorem 4.17.** (i) The Voros coefficient $V^{(0)}(\lambda, \nu; \hbar)$ of the quantum Bessel curve (4.101) associated with the path $\gamma_0$ in Fig.4.5 (a) and the free energy $F(\lambda; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\lambda)$ of the Bessel curve (4.94) are related as follows.

$$V^{(0)}(\lambda, \nu; \hbar) = F\left(\hat{\lambda} + \frac{\hbar}{2}; \hbar\right) - F\left(\hat{\lambda} - \frac{\hbar}{2}; \hbar\right) - \frac{1}{2} \frac{\partial F_0}{\partial \lambda} + \frac{\nu}{2} \frac{\partial^2 F_0}{\partial \lambda^2}.$$  

(Recall that $\hat{\lambda} = \lambda - (\hbar \nu)/2$ and $\nu = \nu_+ - \nu_-$.)

(ii) The free energy of the Bessel curve (4.94) satisfies the following difference equation.

$$F(\lambda + \hbar; \hbar) - 2F(\lambda; \hbar) + F(\lambda - \hbar; \hbar) = - \log \left(256\lambda^2 \left(\lambda^2 - \frac{\hbar^2}{16}\right)\right).$$
Proof. As well as Theorem 4.5 for the Weber case, we have

\begin{equation}
V^{(0)}(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \hbar^m \left\{ \sum_{\substack{g \geq 0, n \geq 1 \\text{and} \\text{all even}}} \frac{1}{n!} \int_{\zeta_1 \in D(1; \nu)} \cdots \int_{\zeta_n \in D(1; \nu)} W_{g,n}(\zeta_1, \ldots, \zeta_n) \right\} \frac{dz}{dz} \left(\frac{1}{\lambda} - \nu \right) \frac{2\nu}{\lambda} \frac{\partial^2 F_g}{\partial \lambda^2}.
\end{equation}

Here we have used Lemma 4.14 and the following relations for a meromorphic 1-form \( \omega(z) \) satisfying \( \omega(\bar{z}) = -\omega(z) \):

\begin{equation}
\int_{\zeta \in D(\beta, \nu)} \omega(\zeta) = \begin{cases} 
\frac{1 - \nu}{2} & \text{if } \beta = 1 \\
\frac{1 + \nu}{2} & \text{if } \beta = -1.
\end{cases}
\end{equation}

Thus we obtain (4.110) after substituting \( \nu_{\pm} = (1 \pm \nu)/2 \). The second claim (4.111) follows from (4.110) and the relation

\begin{equation}
V^{(0)}(\lambda, \nu + 2; \hbar) - V^{(0)}(\lambda, \nu; \hbar) = \log \left(\frac{\hat{\lambda}(\hat{\lambda} - \hbar)}{\lambda^2}\right)
\end{equation}

satisfied by the Voros coefficients (cf. Lemma 4.7). We omit the details.

As a corollary, we obtain an explicit formula for the free energy.

Theorem 4.18 ([IM, Theorem 4.3]). The free energy for the Bessel curve (4.94) is given by

\begin{equation}
F_g = - \frac{B_{2g}}{2g(2g - 2)} \frac{1}{(2\lambda)^{2g-2}} \quad (g \geq 2).
\end{equation}
We omit the proof of above claim since it is done similarly to that of Theorem 4.8. We could also find an explicit expression of the Voros coefficient (which is a generalization of a result obtained in [AIT, Theorem 2.1] corresponding to the special case $\nu = 0$).

**Corollary 4.19.** The Voros coefficient of the quantum Bessel curve (4.101) associated with the path $\gamma_0$ in Fig. 4.3 (a) has the following expression:

$$V^{(0)}(\lambda, \nu; \hbar) = -\sum_{m=1}^{\infty} \frac{B_{m+1}(\nu) + B_{m+1}(\nu + 1)}{m(m + 1)} \left(\frac{\hbar}{\lambda}\right)^m.$$  

4.5 Legendre equation

This subsection is devoted to study the Legendre curve defined by

$$P(x, y) := (x^2 - 1)y^2 - \lambda^2 = 0 \quad (\lambda \neq 0).$$

A rational parametrization of this curve is

$$\left\{ \begin{array}{l}
x(z) = \frac{1}{2} \left( \frac{z + 1}{z} \right), \\
y(z) = \frac{2\lambda z}{z^2 - 1}.
\end{array} \right.$$  

As in the Weber curve, ramification points are given by $R = R^* = \{1, -1\}$ (both of ramifications are poles of $y(z)$, which correspond to simple-pole type turning points of the quantum curve), and the conjugate map is $\bar{z} = 1/z$. We also have

$$y(\bar{z}) = -y(z), \quad y(z)dx(z) = \frac{\lambda}{z} dz.$$  

Examples of correlation functions and free energies are:

$$W_{1,1}(z) = \frac{z}{2\lambda(z^2 - 1)^2} dz, \quad W_{0,3}(z_1, z_2, z_3) = 0,$$

$$W_{1,2}(z_1, z_2) = \frac{z_1^2 z_2^2 + z_1^2 z_2^2 + 4z_1z_2 + 1}{4\lambda^2(z_1^2 - 1)^2(z_2^2 - 1)^2} dz_1 dz_2, \quad W_{2,1}(z) = -\frac{z^5 + 7z^3 + z}{8\lambda^3(z^2 - 1)^4} dz,$$

$$F_0 = -\frac{1}{2} \lambda^2 \log 2, \quad F_1 = -\frac{1}{4} \log \lambda, \quad F_2 = -\frac{1}{64\lambda^2}, \quad F_3 = -\frac{1}{256\lambda^4}.$$  

As explained in Example 3.8 (ii), we choose a divisor as

$$D(z; \nu) = [z] - \nu_0[0] - \nu_\infty[\infty] \quad (\nu_0 + \nu_\infty = 1),$$  

and the quantum curve is given by

$$\left\{ \begin{array}{l}
\hbar^2 \frac{d^2 \psi}{dx^2} + \frac{2x}{x^2 - 1} \hbar^2 \frac{d \psi}{dx} - \frac{\lambda^2}{x^2 - 1} + \frac{\lambda \hbar(\nu_0 - \nu_\infty)}{x^2 - 1} + \frac{\hbar^2 \nu_0 \nu_\infty}{x^2 - 1} \end{array} \right\} \psi = 0.$$  

56
The SL-form of \((4.124)\) is

\[
(4.125) \quad \left\{ \frac{\hbar^2}{2} \frac{d^2}{dx^2} - \left( \frac{\hat{\lambda}^2}{x^2 - 1} - \hbar^2 \frac{x^2 + 3}{4(x^2 - 1)^2} \right) \right\} \varphi = 0,
\]

where

\[
(4.126) \quad \hat{\lambda} := \lambda - \frac{\hbar \nu}{2} \quad \text{and} \quad \nu := \nu_0 - \nu_{\infty}.
\]

This equation \((4.124)\) has two simple-pole type turning points at \(x = 1\) and \(x = -1\). These two points together with Stokes curves emanating from them are shown in Fig.4.6 (a), where we choose \(\lambda = 1\) in the figure. Fig.4.6 (b) depicts the inverse image of Stokes curves in Fig.4.6 (a) by \(x = x(z)\).

\[
\begin{array}{c|ccc}
  x & 1 & -1 & \infty \\
  z & 1 & -1 & 0, \infty \\
\end{array}
\]

Table 4.7: Correspondence of points for the Legendre curve

Figure 4.6: (a) Stokes curves of \((4.124)\) with \(\lambda = 1\), and the path \(\gamma\) on the \(x\)-plane. (b) Stokes curves and the path \(\tilde{\gamma}\) on \(z\)-plane.

We use \(\lambda\) as a deformation parameter. Since

\[
(4.127) \quad \Omega(z) = \frac{\partial y(z)}{\partial \lambda} \cdot dx(z) - \frac{\partial x(z)}{\partial \lambda} \cdot dy(z) = \frac{dz}{z} = \int_{\zeta=0}^{\zeta=\infty} B(z, \zeta)
\]

holds, Theorem 2.9 implies the following.

**Lemma 4.20.** For \(2g + n \geq 3\), we have

\[
(4.128) \quad \frac{\partial^n}{\partial \lambda^n} F_g = \int_{\zeta_1=\infty}^{\zeta_1=0} \cdots \int_{\zeta_n=\infty}^{\zeta_n=0} W_{g,n}(\zeta_1, \cdots, \zeta_n).
\]
Now we study the Voros coefficient of (4.124). Let \( S(x, \hbar) = (d \log \psi(x, \hbar))/dx \) be the solution of the Riccati equation associated with (4.124). The first few terms are given as

\[
S_{-1}(x) = \frac{\lambda}{\sqrt{x^2 - 1}}, \quad S_0(x) = -\frac{x + (\nu_+ - \nu_-) \sqrt{x^2 - 1}}{2(x^2 - 1)}.
\]

Let us define a curve on \( x \)-plane by

\[
\gamma = \gamma(t) = it \quad (-\infty < t < \infty)
\]

as indicated in Fig. 4.6 (a), and define the Voros coefficient by

\[
V(\lambda, \nu; \hbar) := \int_\gamma (S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x)) dx.
\]

Here we fix the branch of \( S_{-1}(x) \) so that

\[
S_{-1}(x) = \frac{\lambda}{\sqrt{x^2 - 1}} \sim \frac{\lambda}{x}
\]

holds when \( x \) approaches to infinity along the path \( \gamma \). Note also that (4.131) depends only on the difference \( \nu = \nu_0 - \nu_\infty \) of the coefficients in the divisor \( D(z; \nu) \) because the SL-form (4.125) of the quantum curve depends solely on \( \nu \). Then, by the exactly same argument with that for the Weber curve, we obtain the following.

**Theorem 4.21.** (i) The Voros coefficient \( V(\lambda, \nu; \hbar) \) of the quantum Legendre curve (4.124) associated with the path \( \gamma \) in Fig. 4.6 (a) and the free energy \( F(\lambda; \hbar) = \sum_{g=0}^\infty \hbar^{2g-2} F_g(\lambda) \) of the Legendre curve (4.117) are related as follows.

\[
V(\lambda, \nu; \hbar) = F\left(\tilde{\lambda} - \frac{1}{2} \hbar; \hbar\right) - F\left(\tilde{\lambda} + \frac{1}{2} \hbar; \hbar\right) + \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda} \frac{\nu}{2} \frac{\partial^2 F_0}{\partial \lambda^2}.
\]

(Recall that \( \tilde{\lambda} = \lambda - (h\nu)/2 \) and \( \nu = \nu_0 - \nu_\infty \).

(ii) The free energy of the Legendre curve (4.117) satisfies the following difference equation.

\[
F(\lambda + \hbar; \hbar) - 2F(\lambda; \hbar) + F(\lambda - \hbar; \hbar) = -\log 2 + \log \left(\frac{\lambda^2 - \hbar^2}{\lambda^2 - \hbar^2/4}\right).
\]
Proof. As well as Therem 4.5 for the Weber case, we have

\[
\lambda, \nu ; \hbar \right) = \sum_{m=1}^{\infty} \hbar^m \int_0^\infty \frac{d}{dz} \left\{ \sum_{g \geq 0, n \geq 1} \frac{1}{n!} \int_{\zeta_1 \in D(z; \nu)} \cdots \int_{\zeta_n \in D(z; \nu)} W_{g,n}(\zeta_1, \ldots, \zeta_n) \right\} dz
\]

Here we have used Lemma 4.20. This proves the desired equality (4.133) since \( \nu_0 = (1 + \nu)/2 \) and \( \nu_\infty = (1 - \nu)/2 \). The other claim (4.134) follows from the relation

(4.136) \( V(\lambda, \nu + 2; \hbar) - V(\lambda, \nu; \hbar) = \log \left( \frac{(2\lambda - \nu \hbar + \hbar)(2\lambda - \nu \hbar - 3\hbar)}{(2\lambda - \nu \hbar)(2\lambda - \nu \hbar - 2\hbar)} \right) \)

satisfied by the Voros coefficient (cf. Lemma 4.7). We omit the details.

Then, the formula summarized in §C allows us to solve (4.134) and obtain an explicit expression of the free energy of the Legendre curve.

Theorem 4.22. The free energy for the Legendre curve (4.117) is given by

(4.137) \( F_g = \frac{(4 - 4^{1-g}) B_{2g}}{2g(2g - 2)} \frac{1}{\lambda^{2g-2}} \) (\( g \geq 2 \)).

The proof of the above claim is given similarly to that of Theorem 4.8, and we do not repeat it here. We could also find an explicit expression of the Voros coefficient (which recovers a result obtained in [Ko4]).

Corollary 4.23. The Voros coefficient of the quantum Legendre curve (4.124) associated with the path \( \gamma \) in Fig. 4.6 (a) has the following expression:

(4.138) \( V(\lambda, \nu; \hbar) = - \sum_{m=1}^{\infty} \frac{B_{m+1}(-\nu/2) + B_{m+1}(1 - \nu/2) - 2B_{m+1}(1 - \nu/2)}{m(m + 1)} \left( -\frac{\hbar}{\lambda} \right)^m. \)
### 4.6 Kummer equation

Let us consider the Kummer curve defined by

\begin{equation}
P(x, y) = 4x^2y^2 - (x^2 + 4\lambda_\infty x + 4\lambda_0^2) = 0 \quad (\lambda_0 \neq 0, \lambda_0 \pm \lambda_\infty \neq 0).
\end{equation}

A rational parameterization of this curve is

\begin{equation}
\begin{aligned}
x &= x(z) = \sqrt{\lambda_\infty^2 - \lambda_0^2} \left( z + \frac{1}{z} \right) - 2\lambda_\infty = \frac{\sqrt{\lambda_\infty^2 - \lambda_0^2}}{z} (z - \beta_{0+})(z - \beta_{0-}), \\
y &= y(z) = \frac{z^2 - 1}{2(z - \beta_{0+})(z - \beta_{0-})},
\end{aligned}
\end{equation}

where \( \tau = 1/z \) and \( \beta_{0\pm} = (\lambda_\infty \pm \lambda_0)/\sqrt{\lambda_\infty^2 - \lambda_0^2} \). Let \( W_{g,n} \) and \( F_g \) be the correlation functions and the free energies computed from (4.140) respectively. Few of them are computed as

\begin{align}
W_{0,3}(z_1, z_2, z_3) &= -\frac{dz_1 dz_2 dz_3}{2\sqrt{\lambda_\infty^2 - \lambda_0^2}} \left\{ \frac{\beta_{0+} + \beta_{0-} + 2}{(z_1 + 1)^2(z_2 + 1)^2(z_3 + 1)^2} - \frac{\beta_{0+} + \beta_{0-} - 2}{(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2} \right\}, \\
W_{1,1}(z) &= -\frac{z^2(z - \beta_{0+})(z - \beta_{0-})}{\sqrt{\lambda_\infty^2 - \lambda_0^2}(z^2 - 1)^4} dz, \\
F_0(\lambda_0, \lambda_\infty) &= \frac{(\lambda_\infty + \lambda_0)^2}{2} \log(\lambda_\infty + \lambda_0) + \frac{(\lambda_\infty - \lambda_0)^2}{2} \log(\lambda_\infty - \lambda_0) - 2\lambda_0^2 \log(2\lambda_0) - \frac{3}{2}(\lambda_\infty^2 - \lambda_0^2), \\
F_1(\lambda_0, \lambda_\infty) &= -\frac{1}{12} \log\left(\frac{\lambda_\infty^2 - \lambda_0^2}{\lambda_0}\right), \quad F_2(\lambda_0, \lambda_\infty) = \frac{\lambda_\infty^4 - 10\lambda_\infty^2 \lambda_0^2 - 7\lambda_0^4}{960\lambda_0^2(\lambda_\infty^2 - \lambda_0^2)^2}.
\end{align}

We can easily verify that the correlation functions and free energies have certain homogeneity properties (as a counterpart of Proposition 4.33 for the Weber case).

**Proposition 4.24.** For \( 2g + n \geq 3 \), we have the following relations:

\[
W_{g,n}(z_1, \ldots, z_n)\big|_{(\lambda_0, \lambda_\infty)=(r\lambda_0, r\lambda_\infty)} = r^{2-2g-n}W_{g,n}(z_1, \ldots, z_n),
\]

\[
F_g(r\lambda_0, r\lambda_\infty) = r^{2-2g}F_g(\lambda_0, \lambda_\infty),
\]

where \( r \) is an arbitrary nonzero complex number.

Firstly, we describe the quantization of (4.139). For a divisor

\begin{equation}
D(z; \nu) = [z] - \nu_{\infty+}[0] - \nu_{\infty-}[\infty] - \nu_{0+}[\beta_{0+}] - \nu_{0-}[\beta_{0-}],
\end{equation}

60
where the parameters $\nu_{\infty \pm}, \nu_{0 \pm}$ satisfy $\nu_{\infty +} + \nu_{\infty -} + \nu_{0 +} + \nu_{0 -} = 1$. Then, using Theorem 3.6 we find that (3.1) is a WKB solution of the following quantization of the Kummer curve

$$\left\{ \hbar^2 \frac{d^2}{dx^2} + q_1(x)\hbar^2 \frac{d}{dx} - \left( \frac{x^2 + 4\lambda_{\infty}x + 4\lambda_0^2}{4x^2} - \hbar r_1(x) - \hbar^2 r_2(x) \right) \right\} \psi = 0,$$

where

$$q_1(x) = \frac{\nu_{\infty +} + \nu_{\infty -}}{x}, \quad r_1(x) = \frac{\nu_{\infty +} - \nu_{\infty -}}{2x} + \frac{(\nu_{0 +} - \nu_{0 -})\lambda_0}{x^2}, \quad r_2(x) = \frac{\nu_{0 +} \nu_{0 -}}{x^2}.$$

The SL-form of (4.146) is

$$\left\{ \hbar^2 \frac{d^2}{dx^2} - Q(x, \hbar) \right\} \varphi = 0, \quad Q(x, \hbar) = \frac{x^2 + 4\hat{\lambda}_{\infty}x + 4\hat{\lambda}_0^2}{4x^2} - \frac{\hbar^2}{4x^2}.$$

Here

$$\hat{\lambda}_j := \lambda_j - \frac{\hbar \nu_j}{2} \quad \text{with} \quad \nu_j := \nu_{j +} - \nu_{j -} \quad (j = 0, \infty).$$

These equations are equivalent to the Kummer differential equation (i.e., the confluent hypergeometric equation) via a certain gauge transformation. Note also that the quantum curve (4.146) is closely related to the quantum differential equation associated with the degree 1 hypersurface in $\mathbb{C}P^1$. A realization of the equation as a quantum curve has already done in [FIMS, §4.1].

Figure 4.7: (a) Stokes curves of (4.146) with $(\lambda_0, \lambda_{\infty}) = (4/5, 1)$, and the paths $\gamma_0$ and $\gamma_{\infty}$ on the $x$-plane. The wiggly line designates the branch cut to fix the branch of $S_{-1}(x)$. (b) Stokes curves and the paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_{\infty}$ on $z$-plane.
Let us consider the Voros coefficients for the quantum Kummer curve \(4.146\)

\[
V^{(j)}(\lambda; \nu; \hbar) = \int_{\gamma_j} \left( S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \right) dx \quad (j = 0, \infty),
\]

where \(\lambda = (\lambda_0, \lambda_\infty)\), \(\nu = (\nu_0, \nu_\infty)\) and the integration contours \(\gamma_0, \gamma_\infty\) are given in Fig.4.7 (a). We fix the branch of \(S_{-1}(x)\) so that

\[
S_{-1}(x) = \sqrt{x^2 + 4\lambda_\infty x + 4\lambda_0^2} \sim \lambda_0 x
\]

holds when \(x\) tends to 0 on the first sheet (i.e. the endpoint of \(\gamma_0\) which corresponds to \(z = \beta_{0+}\) in Fig.4.7 (b)). Note that the branch behaves as \(S_{-1}(x) \sim +1/2\) when \(x\) approaches to \(\infty\) on the first sheet (i.e. the endpoint of \(\gamma_\infty\) which corresponds to \(z = \infty\) in Fig.4.7 (b)).

On the other hand, let \(F_g(\lambda) = F_g(\lambda_0, \lambda_\infty)\) be the free energies for \(4.139\) and

\[
F(\lambda; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\lambda)
\]

be their generating function (and also called the free energy). Then, we obtain

**Theorem 4.25.** (i) The Voros coefficients for the quantum Kummer curve \(4.146\) and the free energy of the Kummer curve \(4.139\) are related as follows.

\[
V^{(0)}(\lambda; \nu; \hbar) = F\left(\lambda_0 + \frac{1}{2} \hbar, \lambda_\infty; \hbar\right) - F\left(\lambda_0 - \frac{1}{2} \hbar, \lambda_\infty; \hbar\right) - \frac{1}{2} \frac{\partial F_0}{\partial \lambda_0} + \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0},
\]

\[
V^{(\infty)}(\lambda; \nu; \hbar) = F\left(\lambda_0, \lambda_\infty - \frac{1}{2} \hbar; \hbar\right) - F\left(\lambda_0, \lambda_\infty + \frac{1}{2} \hbar; \hbar\right) + \frac{1}{2} \frac{\partial F_0}{\partial \lambda_\infty} - \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_\infty}.
\]

(ii) The free energy of the Kummer curve \(4.139\) satisfies the following difference equations.

\[
F(\lambda_0 + \hbar, \lambda_\infty; \hbar) - 2F(\lambda_0, \lambda_\infty; \hbar) + F(\lambda_0 - \hbar, \lambda_\infty; \hbar) = \frac{\partial^2 F_0}{\partial \lambda_0^2} - \log (2\lambda_0 + \hbar) + 2 \log (2\lambda_0) - \log (2\lambda_0 - \hbar),
\]

\[
F(\lambda_0, \lambda_\infty + \hbar; \hbar) - 2F(\lambda_0, \lambda_\infty; \hbar) + F(\lambda_0, \lambda_\infty - \hbar; \hbar) = \frac{\partial^2 F_0}{\partial \lambda_\infty^2}.
\]

The proof of the above claims is done by a similar manner used in that of Theorem 4.5 for the Weber case. However, since the Kummer curve contains two deformation parameters \(\lambda_0\) and \(\lambda_\infty\), we need a slight modification.
Proof of Theorem 4.25 (i). For the proof, we will use the following facts.

Lemma 4.26. The following relations hold for $2g + n \geq 3$:

\[
\frac{\partial^n}{\partial \lambda_0^n} F_g = \int_{\zeta_1 = \beta_{0+}}^{\zeta_1 = \beta_{0-}} \cdots \int_{\zeta_n = \beta_{0+}}^{\zeta_n = \beta_{0-}} W_{g,n}(\zeta_1, \ldots, \zeta_n),
\]

\[
\frac{\partial^n}{\partial \lambda_\infty^n} F_g = \int_{\zeta_1 = \infty}^{\zeta_1 = \gamma} \cdots \int_{\zeta_n = \infty}^{\zeta_n = \gamma} W_{g,n}(\zeta_1, \ldots, \zeta_n).
\]

Lemma 4.27. Let $\omega(z)$ be a meromorphic differential whose poles are only on the set of ramification points of $x(z)$ in (4.140). Suppose also that $\omega(z)$ is anti-invariant under the conjugation map; that is, $\omega(\bar{z}) = -\omega(z)$ holds. Then we have

\[
\int_\infty^{\beta_{0+}} \omega(z) = -\int_0^{\beta_{0-}} \omega(z) = \frac{1}{2} \int_0^\infty \omega(z) + \frac{1}{2} \int_{\beta_{0-}}^{\beta_{0+}} \omega(z),
\]

\[
\int_\infty^{\beta_{0-}} \omega(z) = -\int_0^{\beta_{0+}} \omega(z) = \frac{1}{2} \int_\infty^0 \omega(z) - \frac{1}{2} \int_{\beta_{0-}}^{\beta_{0+}} \omega(z).
\]

Lemma 4.26 is a consequence of the variation formula (Theorem 2.9). Lemma 4.27 is proved by a straightforward computation.

Now let us derive (4.153). Firstly, for $1 \leq k \leq n$, we introduce the notation

\[
\left\{ \int_{\gamma_1} \right\}^k \left\{ \int_{\gamma_2} \right\}^{n-k} \omega(\zeta_1, \ldots, \zeta_n) = \int_{\zeta_1 \in \gamma_1} \cdots \int_{\zeta_k \in \gamma_1} \int_{\zeta_{k+1} \in \gamma_2} \cdots \int_{\zeta_n \in \gamma_2} \omega(\zeta_1, \ldots, \zeta_n),
\]

where $\omega(\zeta_1, \ldots, \zeta_n)$ is a symmetric meromorphic multidifferential and $\gamma_1$ and $\gamma_2$ are paths. Since the integration contour $\gamma_0$ corresponds to a path from $\beta_{0-}$ to $\beta_{0+}$ on $z$-plane (cf. Fig. 4.7 (b)), we have

\[
V^{(0)}(\lambda, \nu; h) = \sum_{m=1}^{\infty} h^m \int_{\beta_{0-}}^{\beta_{0+}} \left( S(x(z), h) - h^{-1} S_{-1}(x(z)) - S_0(x(z)) \right) \frac{dx}{dz} dz
\]

\[
= \sum_{m=1}^{\infty} h^m \int_{\beta_{0-}}^{\beta_{0+}} \left\{ \sum_{g \geq 0, n \geq 1} \frac{1}{n!} \int_{D(z; \nu)} \cdots \int_{D(z; \nu)} W_{g,n}(\zeta_1, \ldots, \zeta_n) \right\} dz
\]

\[
= \sum_{m=1}^{\infty} h^m \sum_{2g+n-2=m, g \geq 0, n \geq 1} \frac{1}{n!} \left\{ \nu_{\infty+} \int_0^{\beta_{0+}} + \nu_{\infty-} \int_{\beta_{0-}}^{\beta_{0+}} \right\} W_{g,n}(\zeta_1, \ldots, \zeta_n).
\]
Using Lemma 4.27 we get

\[
V^{(0)}(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

It follows from Lemma 4.26 that we obtain

\[
V^{(0)}(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^m \nu_0 n^k}{2 m^k (n - k)!} \frac{\partial^n F}{\partial \lambda^k \partial \lambda_\infty^{n-k}}
\]

Thus we obtain the desired equation (4.153). We omit the derivation of (4.154) since it can be obtained similarly.

Proof of Theorem 4.25 (ii). With the aid of contiguity relation (Theorem 4.1), we obtain the following relations (which is the counterpart of Lemma 4.7 for the Weber case).

Lemma 4.28.

\[
(4.162) \quad V^{(0)}(\lambda, \nu_0, \nu_\infty; \hbar) - V^{(0)}(\lambda, \nu_0 - 1, \nu_\infty - 1; \hbar) = - \log \left( \frac{\lambda_0^2 (2 \lambda_0 + 2 \lambda_\infty + \hbar)}{(\lambda_0 + \lambda_\infty) \lambda_0 (2 \lambda_0 + \hbar)} \right)
\]

\[
(4.163) \quad V^{(0)}(\lambda, \nu_0, \nu_\infty; \hbar) - V^{(0)}(\lambda, \nu_0, \nu_\infty - 2; \hbar) = - \log \left( \frac{(\lambda_\infty - \lambda_0) (2 \lambda_0 + 2 \lambda_\infty + \hbar)}{(\lambda_0 + \lambda_\infty) (2 \lambda_\infty - 2 \lambda_0 + \hbar)} \right)
\]

\[
(4.164) \quad V^{(\infty)}(\lambda, \nu_0, \nu_\infty; \hbar) - V^{(\infty)}(\lambda, \nu_0, \nu_\infty - 2; \hbar)
\]

\[
= - \log \left( \frac{4 (\lambda_\infty^2 - \lambda_0^2)}{(2 \lambda_\infty + 2 \lambda_0 + \hbar)(2 \lambda_\infty - 2 \lambda_0 + \hbar)} \right)
\]

64
In particular, the relations in Lemma 4.28 are specialized to

\[ \tag{4.165} V^{(0)}(\lambda, \nu; \hbar) \bigg|_{\mathcal{J} = (0, 0)} - V^{(0)}(\lambda, \nu; \hbar) \bigg|_{\mathcal{J} = (1, 0)} = \log \left( \frac{4\lambda_0^2}{(2\lambda_0 + \hbar)(2\lambda_0 - \hbar)} \right), \]

\[ \tag{4.166} V^{(\infty)}(\lambda, \nu; \hbar) \bigg|_{\mathcal{J} = (0, -1)} - V^{(\infty)}(\lambda, \nu; \hbar) \bigg|_{\mathcal{J} = (0, 1)} = 0. \]

Then, combining Theorem 4.25 (i) and the equalities (4.165) and (4.166), we can show the desired equalities (4.155) and (4.156). □

Theorem 4.25 suffices to obtain an explicit expression of the free energies of the Kummer curve.

Theorem 4.29. The free energy of the Kummer curve (4.139) is explicitly given by

\[ \tag{4.167} F_g(\lambda) = \frac{B_{2g}}{2g(2g - 2)} \left\{ \frac{1}{(\lambda_\infty + \lambda_0)^{2g - 2}} + \frac{1}{(\lambda_\infty - \lambda_0)^{2g - 2}} - \frac{1}{(2\lambda_0)^{2g - 2}} \right\} \quad (g \geq 2). \]

Proof. The difference equations (4.156) and (4.155) can be written by using a shift operator, which is an exponential of the differential operator \( \partial \lambda_j = \frac{\partial}{\partial \lambda_j} \) (\( j = 0, \infty \)), in the following forms:

\[ \tag{4.168} X_0 F(\lambda; \hbar) := e^{-\hbar\lambda_0} \left( e^{\hbar\lambda_0} - 1 \right)^2 F(\lambda; \hbar) = \partial F_0(\lambda) - \log (2\lambda_0 + \hbar) + 2 \log (2\lambda_0) - \log (2\lambda_0 - \hbar), \]

\[ \tag{4.169} X_\infty F(\lambda; \hbar) := e^{-\hbar\lambda_\infty} \left( e^{\hbar\lambda_\infty} - 1 \right)^2 F(\lambda; \hbar) = \partial F_0(\lambda). \]

Using (4.169) and Proposition C.1, we obtain

\[ \tag{4.170} F(\lambda; \hbar) = \hbar^{-2} F_0(\lambda) - \sum_{g=1}^{\infty} \frac{2g - 1}{(2g)!} B_{2g} h^{2g - 2} \frac{\partial^2 F_0(\lambda)}{\partial \lambda_\infty^{2g}}(\lambda) + \tilde{F}_0(\lambda_0; \hbar) + \tilde{F}_1(\lambda_0; \hbar) \lambda_\infty. \]

Here \( \tilde{F}_0(\lambda_0; \hbar) \) and \( \tilde{F}_1(\lambda_0; \hbar) \) are formal power series in \( \hbar \) whose coefficients depend only on \( \lambda_0 \) (that is, it is independent of \( \lambda_\infty \)). Then, we find

\[ \tag{4.171} X_0 F(\lambda; \hbar) = \hbar^{-2} X_0 F_0(\lambda) - h^{-2} X_\infty F_0(\lambda) + X_\infty F(\lambda; \hbar) + X_0(\tilde{F}_0(\lambda_0; \hbar) + \tilde{F}_1(\lambda_0; \hbar) \lambda_\infty), \]

where we use (4.170) and the following lemma.

Lemma 4.30. For \( g \geq 1 \),

\[ \tag{4.172} X_0 \frac{\partial^2 F_0}{\partial \lambda_\infty^{2g}}(\lambda) = X_\infty \frac{\partial^2 F_0}{\partial \lambda_\infty^{2g}}(\lambda). \]

Lemma 4.30 follows from (4.169), (4.168) and
Lemma 4.31.

\(\frac{\partial^2 F_0}{\partial \lambda_0^2} = \log (\lambda_\infty + \lambda_0) + \log (\lambda_\infty - \lambda_0) - 4 \log (2\lambda_0),\)
\(\frac{\partial^2 F_0}{\partial \lambda_\infty^2} = \log (\lambda_\infty + \lambda_0) + \log (\lambda_\infty - \lambda_0).\)

Lemma 4.31 is obtained by a straightforward computation. Then, using (4.169), (4.168), (4.171) and (4.173), we get

\[
X_0(\tilde{F}_0(\lambda_0; \hbar) + \tilde{F}_1(\lambda_0; \hbar)\lambda_\infty)
= h^{-2}(X_\infty - X_0)F_0(\Delta) - \log (2\lambda_0 + \hbar) - 2 \log (2\lambda_0) - \log (2\lambda_0 - \hbar)
\]
\[
= h^{-2}X_0(2\lambda_0^2 \log (2\lambda_0)) - 6 - \log (2\lambda_0 + \hbar) - 2 \log (2\lambda_0) - \log (2\lambda_0 - \hbar).
\]

Further calculation shows

\[
\tilde{F}_0(\lambda_0; \hbar) + \tilde{F}_1(\lambda_0; \hbar)\lambda_\infty
= 2h^{-2}\lambda_0^2 \log (2\lambda_0) + (\hbar \partial \lambda_0)^{-2}\{-6 - 4 \log (2\lambda_0)\}
+ \frac{1}{12} \log \lambda_0 - \sum_{m=1}^{\infty} \frac{B_{2m+2}}{2m(2m+2)} \left( \frac{\hbar}{2\lambda_0} \right)^{2m}
= \frac{1}{12} \log \lambda_0 - \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} \left( \frac{\hbar}{2\lambda_0} \right)^{2g-2},
\]

where we use Proposition C.3. From (4.170) and (4.175), we finally get

\[
F(\Delta; \hbar) = h^{-2}F_0(\Delta) - \frac{1}{12}\{ \log (\lambda_\infty + \lambda_0) + \log (\lambda_\infty - \lambda_0) - \log \lambda_0\}
- \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} \hbar^{2g-2} \left\{ \frac{1}{(\lambda_\infty + \lambda_0)^{2g-2}} + \frac{1}{(\lambda_\infty - \lambda_0)^{2g-2}} - \frac{1}{(2\lambda_0)^{2g-2}} \right\}.
\]

Therefore, the uniqueness of the formal solution (Proposition C.5) implies the desired formula (4.176). This completes the proof of Theorem 4.37 \(\square\)

Using Theorem 4.25 (i) and Theorem 4.37, we find that

**Corollary 4.32** (\cite{18}, Theorem 3.2). The Voros coefficients for the quantum Kummer curve (4.146) associated with the integration contours \(\gamma_0\) and \(\gamma_\infty\) in Fig. 4.7 (a) are explicitly written as follows:

\[
V^{(0)}(\Delta; \nu; \hbar) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \hbar^m}{m(m+1)} \left\{ \frac{B_{m+1}((1 - \nu_0 - \nu_\infty)/2)}{(\lambda_0 + \lambda_\infty)^m} + \frac{B_{m+1}((1 - \nu_0 + \nu_\infty)/2)}{(\lambda_0 - \lambda_\infty)^m} - \frac{B_{m+1}(1 - \nu_0) + B_{m+1}(1 - \nu_0)}{(2\lambda_0)^m} \right\},
\]
\[
V^{(\infty)}(\Delta; \nu; \hbar) = \sum_{m=1}^{\infty} \frac{(-1)^m \hbar^m}{m(m+1)} \left\{ \frac{B_{m+1}((1 - \nu_0 - \nu_\infty)/2)}{(\lambda_0 + \lambda_\infty)^m} - \frac{B_{m+1}((1 - \nu_0 + \nu_\infty)/2)}{(\lambda_0 - \lambda_\infty)^m} \right\}.
\]
The concrete forms of the Voros coefficients for the quantum Kummer curve \((4.146)\) are known by [13]. This corollary gives another proof of the results from the view point of the topological recursion.

**Proof.** From \((4.153)\) and \((4.168)\), we get

\[
V^{(0)}(\zeta, \nu, \hbar) = \left( e^{(1-\nu_0)\hbar \partial_{\lambda_0}/2} - e^{-(1+\nu_0)\hbar \partial_{\lambda_0}/2} \right) F(\lambda_0, \dot{\lambda}_\infty; \hbar) - \hbar^{-1} \frac{\partial F_0}{\partial \lambda_0} + \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0} = \sum l \frac{\partial}{\partial \lambda_0} \frac{\partial F_0}{\partial \lambda_0} - \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0}.
\]

Then, using the following lemma, we complete the proof of \((4.177)\). Similarly, we obtain \((4.178)\).

**Lemma 4.33.** The following relations hold.

\[
(4.179) \quad e^{(1-\nu_0)\hbar \partial_{\lambda_0}/2} \frac{\partial^2 F_0}{\partial \lambda_0^2}(\lambda_0, \dot{\lambda}_\infty) = \sum l \frac{\partial}{\partial \lambda_0} \frac{\partial F_0}{\partial \lambda_0} - \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0},
\]

\[
(4.180) \quad e^{(1-\nu_0)\hbar \partial_{\lambda_0}/2} \frac{\partial^2 F_0}{\partial \lambda_0^2}(\lambda_0, \dot{\lambda}_\infty) = \sum l \frac{\partial}{\partial \lambda_0} \frac{\partial F_0}{\partial \lambda_0} - \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0},
\]

**Proof.** Let us derive \((4.179)\). By Proposition [C.2 (i), we have

\[
\begin{aligned}
e^{(1-\nu_0)\hbar \partial_{\lambda_0}/2} & \frac{\partial^2 F_0}{\partial \lambda_0^2}(\lambda_0, \dot{\lambda}_\infty) = \left( e^{(1-\nu_0)\hbar \partial_{\lambda_0}/2} - e^{-(1+\nu_0)\hbar \partial_{\lambda_0}/2} \right) \left( \sum l \frac{\partial}{\partial \lambda_0} \frac{\partial F_0}{\partial \lambda_0} \right) \\
 & = \sum l \frac{\partial}{\partial \lambda_0} \frac{\partial F_0}{\partial \lambda_0} - \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0}.
\end{aligned}
\]

Then, using \(B_0(X) = 1\) and \(B_1(X) = X - 1/2\), we get

\[
\begin{aligned}
e^{(1-\nu_0)\hbar \partial_{\lambda_0}/2} & \frac{\partial^2 F_0}{\partial \lambda_0^2}(\lambda_0, \dot{\lambda}_\infty) = \sum l \frac{\partial}{\partial \lambda_0} \frac{\partial F_0}{\partial \lambda_0} - \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0}.
\end{aligned}
\]
Here, we find

\[
\frac{\partial^{k+l+1} F_0}{\partial \lambda_0^{k+1} \partial \lambda_\infty} = \frac{(-1)^{k+l}(k+l-2)!}{(\lambda_\infty + \lambda_0)^{k+l-1}} - \frac{(-1)^l(k+l-2)!}{(\lambda_\infty - \lambda_0)^{k+l-1}} \quad (l \geq 2),
\]

\[
\frac{\partial^{k+2} F_0}{\partial \lambda_0^{k+1} \partial \lambda_\infty} = \frac{(-1)^{k-1}(k-1)!}{(\lambda_\infty + \lambda_0)^{k}} + \frac{(k-1)!}{(\lambda_\infty - \lambda_0)^{k}} \quad (k \geq 1),
\]

\[
\frac{\partial^{k+1} F_0}{\partial \lambda_0^{k+1}} = \frac{(-1)^{k}(k-2)!}{(\lambda_\infty + \lambda_0)^{k-2}} - \frac{(k-2)!}{(\lambda_\infty - \lambda_0)^{k-2}} \frac{2(-1)^{k}(k-2)!}{\lambda_0^{k-2}} \quad (k \geq 2)
\]

by Lemma 4.31. Then, we have

\[
\frac{e^{(1-\nu_0)\partial \lambda_0/2}}{e^{\partial \lambda_0} - 1} \frac{\partial^2 F_0}{\partial \lambda_0^2} (\lambda_0, \lambda_\infty) = \sum_{m=1}^{\infty} \left\{ \sum_{k=0}^{m+1} \frac{(-\nu_\infty)m+1-k B_k((1-\nu_0)/2)}{2^{m+1-k}k!(m+1-k)!} \right\} \frac{(-1)^{m+1}(m-1)!}{(\lambda_\infty + \lambda_0)^m} \hbar^m
\]

\[
- \sum_{m=1}^{\infty} \left\{ \sum_{k=0}^{m+1} \nu_\infty^{m+1-k} B_k((1-\nu_0)/2) \right\} \frac{\nu_\infty}{2^{m+1-k}k!(m+1-k)!} \frac{(m-1)!}{(\lambda_\infty - \lambda_0)^m} \hbar^m
\]

\[
+ \frac{\partial F_0}{\partial \lambda_0} \hbar^{-1} - \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} - \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_0}.
\]

We finally use (C.7), we obtain (4.179). Next, let us start to prove (4.180). Since

\[
\frac{e^{(1-\nu_0)\partial \lambda_0/2}}{e^{\partial \lambda_0} - 1} \{ \log (2\lambda_0 + \hbar) - 2 \log (2\lambda_0) + \log (2\lambda_0 - \hbar) \}
\]

\[
= \frac{\nu_\infty}{e^{\partial \lambda_0} - 1} \{ e^{-\partial \lambda_0/2} \{ e^{(1-\nu_0)\partial \lambda_0/2}(e^{\partial \lambda_0} - 2e^{\partial \lambda_0/2} + 1) \} \log \lambda_0
\]

\[
= \left\{ \frac{e^{(2-\nu_0)\partial \lambda_0/2}}{e^{\partial \lambda_0} - 1} - 2 \frac{e^{(1-\nu_0)\partial \lambda_0/2}}{e^{\partial \lambda_0} - 1} e^{\nu_0\partial \lambda_0/2} + e^{-\nu_0\partial \lambda_0/2} \right\} \log \lambda_0
\]

holds, we obtain (4.180) by Proposition C.2.

\[\square\]

### 4.7 Degenerate Gauss equation

In this subsection we consider a spectral curve defined by

\[
P(x, y) = x(x - 1)^2y^2 - \lambda_\infty^2 x - \lambda_1^2 + \lambda_\infty^2 = 0.
\]

Here we impose the following generic conditions for parameters \( \Delta = (\lambda_\infty, \lambda_1) \):

\[
\lambda_\infty \neq 0, \quad \lambda_1 \neq 0, \quad \text{and} \quad \lambda_\infty \pm \lambda_1 \neq 0.
\]

This curve is parametrized by

\[
\begin{align*}
x(z) &= \frac{\lambda_1^2 - \lambda_\infty^2}{z^2 - \lambda_\infty^2}, \\
y(z) &= -\frac{z(z^2 - \lambda_\infty^2)}{z^2 - \lambda_1^2}.
\end{align*}
\]

68
For this curve, we have

\[ \overline{z} = -z, \quad dx(z) = \frac{2(\lambda_1^2 - \lambda_\infty^2)z}{(z^2 - \lambda_\infty^2)^2} \, dz = \frac{2(\lambda_1^2 - \lambda_\infty^2)\zeta}{(1 - \lambda_\infty^2\zeta^2)^2} \, d\zeta. \]

Thus ramification points are given by \( R = \{0, \infty\} \). First few correlation functions and free energies are given explicitly as follows:

\[ W_{0,3}(z_1, z_2, z_3) = -\frac{\lambda_\infty^2\lambda_1^2}{2(\lambda_1^2 - \lambda_\infty^2)} \frac{dz_1dz_2dz_3}{z_1^2z_2^2z_3^2}, \]

\[ W_{1,1}(z_1) = \frac{1}{16(\lambda_\infty^2 - \lambda_1^2)} \frac{(z_1^2 - \lambda_\infty^2)(z_1^2 - \lambda_1^2)}{z_1^4} \, dz_1, \]

\[ F_0(\lambda) = \lambda_\infty^2 \log \left( \frac{\lambda_\infty^2 - \lambda_1^2}{4\lambda_\infty^2} \right) + 2\lambda_\infty\lambda_1 \log \left( \frac{\lambda_\infty + \lambda_1}{\lambda_\infty - \lambda_1} \right) + \lambda_1^2 \log \left( \frac{\lambda_\infty^2 - \lambda_1^2}{4\lambda_1^2} \right), \]

\[ F_1(\lambda) = \frac{1}{12} \log(\lambda_\infty\lambda_1) - \frac{1}{6} \log(\lambda_\infty^2 - \lambda_1^2), \]

\[ F_2(\lambda) = \frac{\lambda_\infty^6 - 17\lambda_\infty^4\lambda_1^2 + 17\lambda_\infty^2\lambda_1^4 + \lambda_1^6}{960\lambda_\infty^2\lambda_1^2(\lambda_\infty^2 - \lambda_1^2)^2}. \]

For the quantization, we take a divisor as

\[ D(z; \nu) = [z] - \nu_1 + [\lambda_1] - \nu_1 - [-\lambda_1] - \nu_\infty + [\lambda_\infty] - \nu_\infty - [-\lambda_\infty] \quad (\nu_1 + \nu_1 - + \nu_\infty + + \nu_\infty - = 1), \]

and the quantum curve constructed by Theorem 3.6 becomes

\[ \left\{ \hbar^2 \frac{d^2}{dx^2} + \hbar^2 \left( \frac{1}{x} + \frac{\nu_\infty + + \nu_\infty -}{x - 1} \right) \frac{d}{dx} - \frac{\lambda_\infty^2x + \lambda_1^2 - \lambda_\infty^2}{x(x - 1)^2} \right. \]

\[ - \hbar \left( \frac{\lambda_\infty(\nu_\infty + - \nu_\infty -)}{x(x - 1)} - \frac{\lambda_1(\nu_1 + - \nu_1 -)}{x(x - 1)^2} \right) \]

\[ + \hbar^2 \left( \frac{\nu_\infty + + \nu_\infty -}{x(x - 1)} - \frac{\nu_1 + + \nu_1 -}{x(x - 1)^2} \right) \left\} \psi = 0. \]

The SL-form of the quantum curve is

\[ \left\{ \hbar^2 \frac{d^2}{dx^2} - Q(x, \hbar) \right\} \varphi = 0, \quad Q(x, \hbar) = \frac{\hat{\lambda}_\infty^2}{x(x - 1)} + \frac{\hat{\lambda}_1^2}{x(x - 1)^2} - \frac{\hbar^2}{4} \left( \frac{1}{x^2} + \frac{1}{x(x - 1)^2} \right), \]

where

\[ \hat{\lambda}_j := \lambda_j - \frac{\hbar\nu_j}{2} \quad \text{with} \quad \nu_j := \nu_{j +} - \nu_{j -} \quad (j = 1, \infty). \]

The quantum curve (4.194) has a simple turning point at \( x = a = (\lambda_\infty^2 - \lambda_1^2)/\lambda_\infty^2 \), and a simple-pole type turning point at \( x = 0 \). Its Stokes geometry is shown in the left of Fig. 4.8.

The right figure of Fig. 4.8 shows its inverse image by \( x = x(z) \). To include \( z = \infty \) in the figure, we further transform it by \( w = 1/(z + i) \), and the result is shown in Fig. 4.8.
Table 4.8: Correspondence of points for the degenerate Gauss curve

|   |   |   |   |
|---|---|---|---|
| $x$ | $a$ | 0 | 1 |
| $z$ | 0 | $\infty$ | $\lambda_1, -\lambda_1$ | $\lambda_\infty, -\lambda_\infty$ |

Figure 4.8: (a) Stokes curves of (4.194) with $(\lambda_\infty, \lambda_1) = (1, 2)$, and the paths $\gamma_\infty$, $\gamma_1$ on the $x$-plane. (b) Stokes curves and the paths $\tilde{\gamma}_\infty$ and $\tilde{\gamma}_1$ on $w$-plane, where the coordinate $w$ is defined by $w = 1/(z + i)$ from the coordinate $z$ in (4.186). Here we employ the additional change of coordinate from $z$ to $w$ in order to include the point $z = \infty$ in the figure; this point corresponds to the simple pole $x = 0$ from which Stokes curves emanate.

As deformation parameters, we choose

$$\lambda_\infty = -\text{Res}_{z = \lambda_\infty} y(z) dx(z) \quad \text{and} \quad \lambda_1 = \text{Res}_{z = \lambda_1} y(z) dx(z).$$

Since

$$\Omega_\infty(z) := \frac{\partial y(z)}{\partial \lambda_\infty} dx(z) - \frac{\partial x(z)}{\partial \lambda_\infty} dy(z) = -\frac{2\lambda_\infty}{z^2 - \lambda_\infty^2} dz = \int_{\zeta = -\lambda_\infty}^{\zeta = \lambda_\infty} B(z, \zeta),$$

$$\Omega_1(z) := \frac{\partial y(z)}{\partial \lambda_1} dx(z) - \frac{\partial x(z)}{\partial \lambda_1} dy(z) = \frac{2\lambda_1}{z^2 - \lambda_1^2} dz = \int_{\zeta = -\lambda_1}^{\zeta = \lambda_1} B(z, \zeta)$$

hold, the variation formula (Theorem 2.9) shows

**Lemma 4.34.** For $2g + n \geq 3$, we have

$$\frac{\partial^n}{\partial \lambda_\infty^n} F_g = \int_{\zeta_1 = -\lambda_\infty}^{\zeta_1 = \lambda_\infty} \cdots \int_{\zeta_n = -\lambda_\infty}^{\zeta_n = \lambda_\infty} W_{g,n}(\zeta_1, \cdots, \zeta_n),$$

$$\frac{\partial^n}{\partial \lambda_1^n} F_g = \int_{\zeta_1 = -\lambda_1}^{\zeta_1 = \lambda_1} \cdots \int_{\zeta_n = -\lambda_1}^{\zeta_n = \lambda_1} W_{g,n}(\zeta_1, \cdots, \zeta_n).$$

Let $S(x, \hbar)$ be the logarithmic derivative of the WKB solution of the quantum degenerate Gauss curve (4.194), and let us introduce the Voros coefficients by

$$V^{(j)}(\Delta, \nu; \hbar) = \int_{\gamma_j} \left( S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \right) dx \quad (j = \infty, 1),$$
Lemma 4.36. \(\Lambda = (\lambda_\infty, \lambda_1), \nu = (\nu_\infty, \nu_1)\) and the defining contours \(\gamma_\infty, \gamma_1\) are indicated in Fig. 4.8 (a). Here we fix the branch of \(S_{-1}(x)\) such that

\[
S_{-1}(x) = \sqrt{\frac{\lambda_\infty^2 x - (\lambda_\infty^2 - \lambda_1^2)}{x(x - 1)^2}} \sim -\frac{\lambda_\infty}{x}
\]

holds when \(x\) tends to \(\infty\) on the first sheet (i.e. the initial point of (4.208)). \(\lambda\) approaches to 1 on the first sheet (i.e. the initial point of (4.207)).

\[
\lambda_{\infty} \approx \frac{2 \log(2)}{\lambda_{\infty}}
\]

which corresponds to \(z = -\lambda_\infty\) in Fig. 4.8 (b). Note that the branch behaves as \(S_{-1}(x) \sim -\lambda_1/(x - 1)\) when \(x\) approaches to 1 on the first sheet (i.e. the initial point of \(\gamma_1\) which corresponds to \(z = -\lambda_1\) in Fig. 4.8 (b)).

On the other hand, let \(F_g(\Lambda) = F_g(\lambda_\infty, \lambda_1)\) be the free energies for (4.184) and

\[
F(\Lambda; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\Lambda)
\]

be their generating function (and also called the free energy). Then, we obtain

**Theorem 4.35.** (i) The Voros coefficients of the quantum degenerate Gauss curve (4.194) and the free energy of the degenerate Gauss curve (4.184) are related as follows.

\[
V^{(\infty)}(\Lambda, \nu; \hbar) = F(\hat{\lambda}_\infty +\frac{1}{2} \hbar, \hat{\lambda}_1) - F(\hat{\lambda}_\infty - \frac{1}{2} \hbar, \hat{\lambda}_1) - \frac{1}{h} \frac{\partial F_0}{\partial \lambda_\infty} + \frac{1}{2} \left( \nu_1 \frac{\partial}{\partial \lambda_1} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_\infty},
\]

\[
V^{(1)}(\Lambda, \nu; \hbar) = F(\hat{\lambda}_\infty, \hat{\lambda}_1 + \frac{1}{2} \hbar) - F(\hat{\lambda}_\infty, \hat{\lambda}_1 - \frac{1}{2} \hbar) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_1} + \frac{1}{2} \left( \nu_1 \frac{\partial}{\partial \lambda_1} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_1}.
\]

(ii) The free energy (4.184) satisfies the following difference equations.

\[
F(\lambda_\infty + \hbar, \lambda_1; \hbar) - 2 F(\lambda_\infty, \lambda_1; \hbar) + F(\lambda_\infty - \hbar, \lambda_1; \hbar) = 2 \log (\lambda_\infty^2 - \lambda_1^2) - 2 \log (\lambda_\infty + \frac{1}{2} \hbar) - 2 \log \lambda_\infty - 4 \log 2,
\]

\[
F(\lambda_\infty, \lambda_1 + \hbar; \hbar) - 2 F(\lambda_\infty, \lambda_1; \hbar) + F(\lambda_\infty, \lambda_1 - \hbar; \hbar) = 2 \log (\lambda_\infty^2 - \lambda_1^2) - 2 \log (\lambda_1 + \frac{1}{2} \hbar) - 2 \log \lambda_1 - 4 \log 2.
\]

To prove the above statement, we use Lemma 4.34 and the following lemma (cf. Lemma 4.26 and Lemma 4.28 for the Kummer case):

**Lemma 4.36.**

\[
V^{(\infty)}(\Lambda, \nu; \hbar)|_{\nu = (0, 1)} - V^{(\infty)}(\Lambda, \nu; \hbar)|_{\nu = (1, 0)} = \log \left( 1 - \frac{\hbar^2}{4 \lambda_\infty^2} \right),
\]

\[
V^{(1)}(\Lambda, \nu; \hbar)|_{\nu = (0, 1)} - V^{(1)}(\Lambda, \nu; \hbar)|_{\nu = (0, 1)} = \log \left( 1 - \frac{\hbar^2}{4 \lambda_1^2} \right).
\]

These claims are consequences of the variation formula (Theorem 2.9) and the contiguity relation (Theorem 4.11). Then, the proof of Theorem 4.35 is similar to that of Theorem 4.25 for the Kummer case.

In parallel with Theorem 4.37 and Corollary 4.32 for the Kummer case, we obtain
Theorem 4.37. The free energy of the degenerate Gauss curve (4.184) is explicitly given by
\[ F_g(\Delta) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{2}{(\lambda_\infty + \lambda_1)^{2g-2}} + \frac{2}{(\lambda_\infty - \lambda_1)^{2g-2}} - \frac{1}{(2\lambda_\infty)^{2g-2}} - \frac{1}{(2\lambda_1)^{2g-2}} \right\} \quad (g \geq 2). \] (4.209)

Corollary 4.38. The Voros coefficients for the quantum degenerate Gauss curve (4.194) are explicitly given by
\[ V^{(\infty)}(\Delta, \nu; \hbar) = \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \left\{ \frac{2B_{m+1}(\nu_\infty + \nu_1 + 1)/2}{(\lambda_\infty + \lambda_1)^m} + \frac{2B_{m+1}(\nu_\infty - \nu_1 + 1)/2}{(\lambda_\infty - \lambda_1)^m} \right\} \] (4.210)
\[ V^{(1)}(\Delta, \nu; \hbar) = V^{(\infty)}(\lambda_1, \lambda_\infty, \nu_1, \nu_\infty; \hbar). \] (4.211)

We can prove these equalities by a similar calculation performed in the case of Kummer equation (cf. Theorem 4.37 and Corollary 4.32). Thus we omit the proof.

4.8 Gauss hypergeometric equation

Our final example is the Gauss curve defined by
\[ P(x, y) = x^2(1-x)^2y^2 - \{ \lambda_\infty^2x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_0^2 \} = 0. \] (4.212)

Here we impose the following generic conditions for parameters \( \Delta = (\lambda_0, \lambda_1, \lambda_\infty) \):
\[ \lambda_0, \lambda_1, \lambda_\infty \neq 0 \quad \text{and} \quad \Lambda = (\lambda_\infty - \lambda_0 + \lambda_1)(\lambda_\infty + \lambda_0 - \lambda_1)(\lambda_\infty + \lambda_0 + \lambda_1)(\lambda_\infty - \lambda_0 - \lambda_1) \neq 0. \] (4.213)

A rational parameterization of this curve is given by
\[ \begin{cases} x = x(z) = \frac{\sqrt{\Lambda}}{4\lambda_\infty^2} \left( z + \frac{1}{z} \right) + \frac{\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2}{2\lambda_\infty^2} = \frac{\sqrt{\Lambda}}{4\lambda_\infty^2} \left( z - \beta_{0+} \right) \left( z - \beta_{0-} \right) \frac{z}{z}, \\ y = y(z) = \frac{\sqrt{\Lambda}(z - \beta_{0+})}{4\lambda_\infty^3z(z^2 - 1)} = \frac{\sqrt{\Lambda}(z - \beta_{0+})(z - \beta_{0-})(z - \beta_{1+})(z - \beta_{1-})}{z}, \end{cases} \] (4.214)
where \( \overline{z} = 1/z \) and
\[ \beta_{0\pm} = -\frac{(\lambda_\infty \pm \lambda_0)^2 - \lambda_1^2}{\sqrt{\Lambda}}, \quad \beta_{1\pm} = \frac{(\lambda_\infty \pm \lambda_1)^2 - \lambda_0^2}{\sqrt{\Lambda}}. \] (4.215)
First few terms of the correlation functions and free energies are computed as

\begin{equation}
W_{0,3}(z_1, z_2, z_3) = \left\{ \frac{(\beta_{0+} + \beta_{0-} + 2)(\beta_{1+} + \beta_{1-} + 2)}{(z_1 + 1)^2(z_2 + 1)^2(z_3 + 1)^2} - \frac{(\beta_{0+} + \beta_{0-} - 2)(\beta_{1+} + \beta_{1-} - 2)}{(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2} \right\} \times \frac{dz_1 dz_2 dz_3}{4\lambda_\infty},
\end{equation}

\begin{equation}
W_{1,1}(z) = -\frac{z(z - \beta_{0+})(z - \beta_{0-})(z - \beta_{1+})(z - \beta_{1-})}{2\lambda_\infty(z^2 - 1)^4} dz,
\end{equation}

\begin{equation}
F_0 = \frac{(\lambda_0 + \lambda_1 + \lambda_\infty)^2}{2} \log (\lambda_0 + \lambda_1 + \lambda_\infty) + \frac{(\lambda_0 - \lambda_1 + \lambda_\infty)^2}{2} \log (\lambda_0 - \lambda_1 + \lambda_\infty)
+ \frac{(\lambda_0 + \lambda_1 - \lambda_\infty)^2}{2} \log (\lambda_0 + \lambda_1 - \lambda_\infty) + \frac{(\lambda_0 - \lambda_1 - \lambda_\infty)^2}{2} \log (\lambda_0 - \lambda_1 - \lambda_\infty)
- 2\lambda_0^2 \log (2\lambda_0) - 2\lambda_1^2 \log (2\lambda_1) - 2\lambda_\infty^2 \log (2\lambda_\infty),
\end{equation}

\begin{equation}
F_1 = -\frac{1}{12} \log \left( \frac{\Lambda}{16\lambda_\infty \lambda_0 \lambda_1} \right),
\end{equation}

\begin{equation}
F_2 = \frac{1}{960\lambda_\infty^2 \lambda_0^2 \lambda_1^2 \Lambda^2} \left\{ (\lambda_0^2 + \lambda_1^2)\lambda_\infty^8 - (4\lambda_0^4 + 23\lambda_0^2 \lambda_1^2 + 4\lambda_1^4)\lambda_\infty^6
+ 2(\lambda_0^2 + \lambda_1^2)(3\lambda_0^4 + 8\lambda_0^2 \lambda_1^2 + 3\lambda_1^4)\lambda_\infty^4 - 2(2\lambda_0^8 - 11\lambda_0^6 \lambda_1^2 + 74\lambda_0^4 \lambda_1^4 - 11\lambda_0^2 \lambda_1^6 + 2\lambda_1^8)\lambda_\infty^2
+ (\lambda_0^2 - \lambda_1^2)^2(\lambda_0^2 - \lambda_1^2)(\lambda_0^4 - 22\lambda_0^2 \lambda_1^2 + \lambda_1^4)\lambda_\infty^2
+ \lambda_0^2 \lambda_1^2(\lambda_0^2 - \lambda_1^2)^2 \right\}.
\end{equation}

For the quantization, we use a divisor

\begin{equation}
D(z; \nu) = [z - \nu_{\infty_+}[0] - \nu_{\infty_-}[\infty] - \nu_{0+}[\beta_{0+}] - \nu_{0-}[\beta_{0-}] - \nu_{1+}[\beta_{1+}] - \nu_{1-}[\beta_{1-}],
\end{equation}

where the parameters $\nu_{\infty_\pm}, \nu_{0\pm}$ and $\nu_{1\pm}$ satisfy $\nu_{\infty_+} + \nu_{\infty_-} + \nu_{0+} + \nu_{0-} + \nu_{1+} + \nu_{1-} = 1$. Then Theorem \textbf{3.6} gives the quantum curve of the Gauss curve:

\begin{equation}
\left( h^2 \frac{d^2}{dx^2} + q_1(x) h^2 \frac{d}{dx} - \left( r_0(x) + h r_1(x) + h^2 r_2(x) \right) \right) \psi = 0,
\end{equation}

where

\begin{align*}
q_1(x) &= \frac{1 - \nu_{0+} - \nu_{0-}}{x} + \frac{1 - \nu_{1+} - \nu_{1-}}{x - 1}, \\
r_0(x) &= \frac{\lambda_\infty^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2) x + \lambda_0^2}{x^2(1 - x)^2}, \\
r_1(x) &= -\frac{(\nu_{\infty_+} - \nu_{\infty_-}) \lambda_\infty}{x(x - 1)} - \frac{(\nu_{0+} - \nu_{0-}) \lambda_0}{x^2(x - 1)} - \frac{(\nu_{1+} - \nu_{1-}) \lambda_1}{x(x - 1)^2}, \\
r_2(x) &= -\frac{\nu_{\infty_+} \nu_{\infty_-}}{x(x - 1)} + \frac{\nu_{0+} \nu_{0-}}{x^2(x - 1)} - \frac{\nu_{1+} \nu_{1-}}{x(x - 1)^2}.
\end{align*}

The SL-form of (4.222) is

\begin{equation}
\left\{ h^2 \frac{d^2}{dx^2} - Q(x, h) \right\} \psi = 0
\end{equation}

73
with
\[
Q(x, \hbar) = \frac{\lambda^2_{\infty} x^2 - (\lambda^2_{0} + \lambda^2_{1}) x + \lambda^2_{0}}{x^2(x-1)^2} - \hbar^2 \frac{x^2 - x + 1}{4x^2(x-1)^2}.
\]

Here
\[
\dot{\lambda}_j := \lambda_j - \hbar \nu_j/2 \quad \text{with} \quad \nu_j := \nu_{j+} - \nu_{j-} \quad (j = 0, 1, \infty).
\]

The equation (1.1) which appeared in §1 is a specialization of (4.223) (i.e. the parameters are chosen as $(\nu_0, \nu_1, \nu_{\infty}) = (1, 0, 0)$ in (1.1)).

There are two simple turning points at $a_1$ and $a_2$, which correspond to $z = +1$ and $-1$, respectively, by $x = x(z)$. Fig. 4.9 depicts the Stokes curves emanating from them, and see Table 4.9 for the correspondence of special points in $x$-plane and $z$-plane.

| $x$ | $a_1$ | $a_2$ | 0 | 1 | $\infty$ |
|-----|-------|-------|---|---|---------|
| $z$ | 1     | -1    | $\beta_{0+}, \beta_{0-}$ | $\beta_{1+}, \beta_{1-}$ | 0, $\infty$ |

Table 4.9: Correspondence of points for the Gauss curve

Figure 4.9: (a) Stokes curves of (4.222) with $(\lambda_0, \lambda_1, \lambda_{\infty}) = (3/4, 2/3, 1/2)$ (thick lines), and the paths $\gamma_0$, $\gamma_1$ and $\gamma_{\infty}$ on $x$-plane. (b) (Pull back by $x(z)$ of) Stokes curves (thick lines) and the paths $\tilde{\gamma}_0$, $\tilde{\gamma}_1$ and $\tilde{\gamma}_{\infty}$ on $z$-plane.

Let $S(x, \hbar)$ be the logarithmic derivative of the WKB solution of the quantum Gauss curve (4.222), and let us introduce the Voros coefficients by
\[
V^{(j)}(\Delta, \gamma; \hbar) = \int_{\gamma_{j}} \left( S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \right) dx \quad (j = 0, 1, \infty),
\]
where $\Delta = (\lambda_0, \lambda_1, \lambda_\infty)$, $\xi = (\nu_0, \nu_1, \nu_\infty)$ and $\gamma_0, \gamma_1, \gamma_\infty$ are paths from a singular point $0, 1, \infty$ to itself (on different sheet) given in Fig. 4.9 (a). We fix the branch of $S_{-1}(x)$ such that

$$S_{-1}(x) = \sqrt{\frac{\lambda_\infty^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2) x + \lambda_0^2}{x^2 (x - 1)^2}} \sim +\frac{\lambda_0}{x}$$

holds when $x$ tends to 0 on the first sheet (i.e. the endpoint of $\gamma_0$ which corresponds to $z = \beta_{0+}$ in Fig. 4.9 (b)). Note that the branch we have chosen here behaves as $S_{-1}(x) \sim +\lambda_1/(x - 1)$ when $x$ approaches to 1 on the first sheet (i.e. the endpoint of $\gamma_1$ which corresponds to $z = \beta_{1+}$ in Fig. 4.9(b)) and $S_{-1}(x) \sim +\lambda_\infty/x$ when $x$ approaches to $\infty$ on the first sheet (i.e. the initial point of $\gamma_\infty$ which corresponds to $z = \infty$ in Fig. 4.9(b)).

On the other hand, let $F_g(\Delta) = F_g(\lambda_0, \lambda_1, \lambda_\infty)$ be the free energies for (4.212) and

$$F(\Delta; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\Delta)$$

be the free energy for the Gauss curve (4.212). Then we have the following statements.

**Theorem 4.39.**

(i) The Voros coefficients of the quantum Gauss curve (4.222) and the free energy (4.228) of the Gauss curve are related as follows.

$$V^{(0)}(\Delta, \xi; \hbar) = F(\hat{\lambda}_0 + \frac{1}{2} \hbar, \hat{\lambda}_1, \hat{\lambda}_\infty; \hbar) - F(\hat{\lambda}_0 - \frac{1}{2} \hbar, \hat{\lambda}_1, \hat{\lambda}_\infty; \hbar) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_0} + \frac{1}{2} \left( \nu_0 \frac{\partial}{\partial \lambda_0} + \nu_1 \frac{\partial}{\partial \lambda_1} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial}{\partial \lambda_0} F_0,$$

(ii) The free energy (4.228) satisfies the following difference equations.

$$F(\lambda_0 + \hbar, \lambda_1, \lambda_\infty; \hbar) - 2F(\lambda_0, \lambda_1, \lambda_\infty; \hbar) + F(\lambda_0 - \hbar, \lambda_1, \lambda_\infty; \hbar) = \frac{\partial^2 F_0}{\partial \lambda_0^2} + \log \frac{4\lambda_0^2}{4\lambda_0^2 - \hbar^2},$$

$$F(\lambda_0, \lambda_1 + \hbar, \lambda_\infty; \hbar) - 2F(\lambda_0, \lambda_1, \lambda_\infty; \hbar) + F(\lambda_0, \lambda_1 - \hbar, \lambda_\infty; \hbar) = \frac{\partial^2 F_0}{\partial \lambda_1^2} + \log \frac{4\lambda_1^2}{4\lambda_1^2 - \hbar^2},$$

$$F(\lambda_0, \lambda_1, \lambda_\infty + \hbar; \hbar) - 2F(\lambda_0, \lambda_1, \lambda_\infty; \hbar) + F(\lambda_0, \lambda_1, \lambda_\infty - \hbar; \hbar) = \frac{\partial^2 F_0}{\partial \lambda_\infty^2} + \log \frac{4\lambda_\infty^2}{4\lambda_\infty^2 - \hbar^2}.$$
We omit the proof of Theorem 4.39 because it can be done by a similar argument employed in the proof of Theorem 4.25 for the Kummer case, together with the following lemmas (cf. Lemmas 4.26 and 4.28 for the Kummer case):

**Lemma 4.40.** The following relations hold for \(2g + n \geq 3\):

\[
\frac{\partial^n}{\partial \lambda_0^n} F_g = \int_{\zeta_1 = \beta_0+}^{\zeta_1 = \beta_0-} \cdots \int_{\zeta_n = \beta_0+}^{\zeta_n = \beta_0-} W_{g,n}(\zeta_1, \ldots, \zeta_n),
\]

\[
\frac{\partial^n}{\partial \lambda_1^n} F_g = \int_{\zeta_1 = \beta_1+}^{\zeta_1 = \beta_1-} \cdots \int_{\zeta_n = \beta_1+}^{\zeta_n = \beta_1-} W_{g,n}(\zeta_1, \ldots, \zeta_n),
\]

\[
\frac{\partial^n}{\partial \lambda_\infty^n} F_g = \int_{\zeta_1 = \infty}^{\zeta_1 = \infty} \cdots \int_{\zeta_n = \infty}^{\zeta_n = \infty} W_{g,n}(\zeta_1, \ldots, \zeta_n).
\]

**Lemma 4.41.**

\[
V^{(0)}(\Delta, \nu_0, \nu_1, \nu_\infty; h) - V^{(0)}(\Delta, \nu_0 - 1, \nu_1 - 1, \nu_\infty; h) = \log \left[ \frac{\lambda_0^2 (2\lambda_\infty + 2\lambda_1 + 2h)(2\lambda_\infty - 2\lambda_0 - 2\lambda_1 - h)}{2\lambda_0(2\lambda_0 + h)(\lambda_\infty + \lambda_0 + \lambda_1)(\lambda_\infty - \lambda_0 - \lambda_1)} \right].
\]

\[
V^{(0)}(\Delta, \nu_0, \nu_1, \nu_\infty; h) - V^{(0)}(\Delta, \nu_0 - 1, \nu_\infty + 1; h) = \log \left[ \frac{(\lambda_\infty + \lambda_0 - \lambda_1)(\lambda_\infty - \lambda_0 - \lambda_1 - \frac{1}{2}h)}{(\lambda_\infty - \lambda_0 - \lambda_1)(\lambda_\infty + \lambda_0 - \lambda_1 - \frac{1}{2}h)} \right],
\]

\[
V^{(0)}(\Delta, \nu_0, \nu_1, \nu_\infty; h) - V^{(0)}(\Delta, \nu_0 - 1, \nu_1, \nu_\infty + 1; h) = \log \left[ \frac{\lambda_0^2 (2\lambda_\infty - 2\lambda_0 + 2\lambda_1 + h)(2\lambda_\infty - 2\lambda_0 - 2\lambda_1 - h)}{2\lambda_0(2\lambda_0 + h)(\lambda_\infty - \lambda_0 + \lambda_1)(\lambda_\infty - \lambda_0 - \lambda_1)} \right].
\]

These claims are consequences of the variation formula (Theorem 4.39) and the contiguity relation (Theorem 4.39 (iii)).

Finally, we obtain the following explicit expressions of the free energy and the Voros coefficients in terms of Bernoulli numbers from the difference equations in Theorem 4.39 (ii).

**Corollary 4.42.** For \(g \geq 2\), we have

\[
F_g = \frac{B_{2g}}{2g(2g - 2)} \left\{ \frac{1}{(\lambda_0 + \lambda_1 + \lambda_\infty)^{2g-2}} + \frac{1}{(\lambda_0 - \lambda_1 + \lambda_\infty)^{2g-2}} \right. \\
+ \frac{1}{(\lambda_0 + \lambda_1 - \lambda_\infty)^{2g-2}} + \frac{1}{(\lambda_0 - \lambda_1 - \lambda_\infty)^{2g-2}} - \frac{1}{(2\lambda_0)^{2g-2}} - \frac{1}{(2\lambda_1)^{2g-2}} - \frac{1}{(2\lambda_\infty)^{2g-2}} \right\}.
\]
Corollary 4.43 (cf. [ATT2], Theorem 1.1). The Voros coefficients for the quantum Gauss curve (4.222) are explicitly given by

\[
V^{(0)}(\Lambda, \nu; \hbar) = V^{(0)}(\lambda_0, \lambda_1, \lambda_\infty, \nu_0, \nu_1, \nu_\infty; \hbar) \\
= \sum_{m=1}^{\infty} \frac{B_m}{m(m+1)} \left\{ \frac{B_{m+1}(\nu_0 + \nu_1 + \nu_\infty + 1/2)}{(\lambda_0 + \lambda_1 + \lambda_\infty)^m} + \frac{B_{m+1}(\nu_0 - \nu_1 + \nu_\infty + 1/2)}{(\lambda_0 - \lambda_1 + \lambda_\infty)^m} \\
+ \frac{B_{m+1}(\nu_0 - \nu_1 - \nu_\infty + 1/2)}{(\lambda_0 - \lambda_1 - \lambda_\infty)^m} \right\},
\]

(4.242)

\[
V^{(1)}(\Lambda, \nu; \hbar) = V^{(0)}(\lambda_1, \lambda_0, \lambda_\infty, \nu_1, \nu_\infty; \hbar),
\]

(4.243)

\[
V^{(\infty)}(\Lambda, \nu; \hbar) = V^{(0)}(\lambda_\infty, \lambda_1, \lambda_0, \nu_\infty, \nu_1, \nu_\infty; \hbar).
\]

(4.244)

We can prove these equalities by a similar calculation performed in the case of Kummer equation (cf. Theorem 4.37 and Corollary 4.32 for the Kummer case). Thus we omit the proof.

A Meromorphic multidifferentials

The correlation function \( W_{g,n}(z_1, z_2, \ldots, z_n) \) is a meromorphic multidifferential, i.e., a meromorphic section of the line bundle \( \pi_1^*(T^*\mathbb{P}^1) \otimes \pi_2^*(T^*\mathbb{P}^1) \otimes \cdots \otimes \pi_n^*(T^*\mathbb{P}^1) \) on \( (\mathbb{P}^1)^n \), where \( \pi_j : (\mathbb{P}^1)^n \to \mathbb{P}^1 \) denotes the \( j \)-th projection ([DN]). Thus a multidifferential is a meromorphic differential in \( \mathbb{P}^1 \) for each variable. If all of the residues with respect to each variable vanish, then we call it a multidifferential of the second kind. We summarize here some notations on multidifferential which we use in this paper.

**Local coordinate representation.** In a local coordinate, we express a meromorphic multidifferential \( \Omega \) on \( (\mathbb{P}^1)^n \) as

\[
\Omega = \Omega(z_1, z_2, \cdots, z_n) = \omega(z_1, z_2, \cdots, z_n) dz_1 dz_2 \cdots dz_n
\]

(A.1)

(we omit the tensor product \( \otimes \) in its expression). If \( \Omega \) is symmetric under the permutation of variables, i.e.,

\[
\Omega(z_1, \cdots, z_j, \cdots, z_k, \cdots, z_n) = \Omega(z_1, \cdots, z_k, \cdots, z_j, \cdots, z_n)
\]

(A.2)

for any \( j, k \in \{1, 2, \cdots, n\} \), \( \Omega \) is said to be a symmetric multidifferential.

**Integration.** Its integral with respect to \( j \)-th variable is denoted by

\[
\int_{z_j=a}^{z_j=b} \Omega(z_1, z_2, \cdots, z_n) := \left( \int_a^b \omega(z_1, z_2, \cdots, z_n) dz_j \right) dz_1 \cdots dz_{j-1} dz_{j+1} \cdots dz_n
\]

(A.3)

or

\[
\int_{z_j \in \gamma} \Omega(z_1, z_2, \cdots, z_n) := \left( \int_\gamma \omega(z_1, z_2, \cdots, z_n) dz_j \right) dz_1 \cdots dz_{j-1} dz_{j+1} \cdots dz_n
\]

(A.4)
for an integration path $\gamma$ in $\mathbb{P}^1$. If $\Omega$ is symmetric under the permutation of the variables, we write a multiple integral with a same integration contour $\gamma$ like

\[
\int_{\gamma} \cdots \int_{\gamma} \Omega(z_1, \cdots, z_n) := \int_{\gamma} \cdots \int_{\gamma} dz_1 \cdots dz_n \omega(z_1, \cdots, z_n).
\]

Further we sometimes drop off the word “$n$-th” if it is clear from the context.

**Integration with a divisor.** Let $\omega$ be a meromorphic differential on some domain $U$ in $\mathbb{P}^1$ of the second kind. Following [BE2], for a divisor $D(z; \nu) = [z] - \sum_{k=1}^{m} \nu_k [p_k]$, where $z, p_1, p_2, \cdots, p_m \in \mathbb{P}^1 \setminus \{\text{poles of } \omega\}$, and $\nu = (\nu_k)_{k=1}^{m}$ being a tuple of complex numbers satisfying $\sum_{k=1}^{m} \nu_k = 1$, we define the integral of $\omega$ with $D(z; \nu)$ by

\[
\int_{D(z; \nu)} \omega := \sum_{k=1}^{m} \nu_k \int_{p_k}^{z} \omega.
\]

Because $\omega$ is of the second kind, this integral does not depend on the choice of paths in $U$ from $p_j$ to $z$ $(j = 1, 2, \cdots, m)$. For a multidifferential $\Omega = \Omega(z_1, z_2, \cdots, z_n)$ of the second kind, we define

\[
\int_{D(z; \nu)} \cdots \int_{D(z; \nu)} \Omega(z_1, \cdots, z_j, \cdots, z_n) := \sum_{k=1}^{m} \nu_k \int_{p_k}^{z_j=\hat{z}} \cdots \int_{p_k}^{z_m=\hat{z}} \Omega(z_1, \cdots, z_j, \cdots, z_n).
\]

As in (A.5), we write the multiple integral as

\[
\int_{D(z; \nu)} \cdots \int_{D(z; \nu)} \Omega(z_1, \cdots, z_n) := \int_{z_1 \in D(z; \nu)} \cdots \int_{z_n \in D(z; \nu)} dz_1 \cdots dz_n \omega(z_1, \cdots, z_n)
\]

for a symmetric meromorphic multidifferential of the second kind.

**Pull-back.** Finally, for a holomorphic map $\phi$ from some domain in $\mathbb{P}^1$ to $\mathbb{P}^1$, we write the pullback of $\Omega$ by $\phi$ with respect to the $j$-th variable as

\[
\Omega(z_1, \cdots, \phi(z_j), \cdots, z_n) := \omega(z_1, \cdots, \phi(z_j), \cdots, z_n)dz_1 \cdots d\phi(z_j) \cdots dz_n.
\]

We frequently use this expression mainly when $\phi$ is conjugate map defined near a ramification point.

**B Ineffectiveness of ramification points**

Following [BE2], we define $R$ as a set of ramification points of $x(z)$, not as a set of zeros of $dx(z)$ as in [EO1]. However, this modification does not cause difference when the ramification points are ineffective (in the sense of Definition 2.5). Here we give a criterion for the ineffectiveness of ramification points (cf. Proposition 2.6).
Proof of Lemma B.2. Taking a square root with an appropriate branch, we obtain

(b) The differential \( (y(z) - y(\overline{z}))dx(z) \) has a pole at \( r \).

Proof. First let us give a remark on the pole order of correlation functions at a ramification point satisfying the above condition.

Lemma B.2. If \( (y(z) - y(\overline{z}))dx(z) \) has a pole at a ramification point \( r \), then the pole order of \( (y(z) - y(\overline{z}))dx(z) \) at \( z = r \) is greater than or equal to two.

Proof of Lemma B.2. It is enough to prove that \( (y(z) - y(\overline{z}))dx(z) \) never has a simple pole at ramification point. In other words, it suffices to prove that, there is no \( r \in R \) satisfying \( \rho(x(r); P) = -2 \) (cf. Proposition B.1).

Suppose for contradiction that a point \( r \in R \) satisfies \( \rho(x(r); P) = -2 \).

We also assume that \( x(r) \neq \infty \) for simplicity. (The case \( x(r) = \infty \) can be treated by a similar way.) Then, the function \( Q_0(x) \) defined by (3.10) has an expression

\[
Q_0(x) = \frac{c_0}{(x - x(r))^2(1 + f(x))},
\]

where \( c_0 \) is a nonzero constant, and \( f(x) \) is a rational function of \( x \) which vanishes at \( x = x(r) \). Taking a square root with an appropriate branch, we obtain

\[
y(z) - y(\overline{z}) = \frac{2\sqrt{c_0}}{x(z) - x(r)}(1 + f(x(z)))^{1/2}.
\]

Since the left hand side is anti-invariant under the involution \( z \mapsto \overline{z} \), the above equality and the relation \( x(z) = x(\overline{z}) \) imply

\[
\frac{1}{x(z) - x(r)} = -\frac{1}{x(z) - x(r)}
\]

which leads a contradiction. This proves that no \( r \in R \) satisfying \( \rho(x(r); P) = -2 \).

Remark B.3. By a similar argument presented in the proof of Lemma B.2, we can also show that, there is no \( r \in R \) satisfying \( \rho(x(r); P) = -2m \) for some \( m \geq 1 \).

Now let us prove Proposition B.1.

Let us assume (b), that is, \( (y(z) - y(\overline{z}))dx(z) \) has a pole at \( r \), and look at the behavior of \( W_{0,3}(z_0, z_1, z_2) \) and \( W_{1,1}(z_0) \) when \( z_0 \) approaches to \( r \). By deforming the residue contour around \( r \), we can decompose the contribution of residue at \( z = r \) to \( W_{0,3}(z_0, z_1, z_2) \) as

\[
(B.1) \quad \text{Res}_{z=r} K_r(z_0, z)(B(z, z_1)B(\overline{z}, z_2) + B(\overline{z}, z_1)B(z, z_2))
\]

\[
= \frac{1}{2\pi i} \left( \oint_{z \in C_r, z_0 \neq r} - \oint_{z \in C_{r_0}} - \oint_{z \in C_{r_2}} \right) K_r(z_0, z)(B(z, z_1)B(\overline{z}, z_2) + B(\overline{z}, z_1)B(z, z_2)).
\]
Here $C_{r,z_0}$ is a contour satisfying
\[ \{ \text{the domain bounded by } C_{r,z_0,\overline{z_0}} \} \cap R \cap \{ z_0, \overline{z_0}, \cdots, z_n, \overline{z_n} \} = \{ r, z_0, \overline{z_0} \}, \]
and the other contours $C_{z_0}$ and $C_{\overline{z_0}}$ are defined similarly. Then, the first integral in the right hand-side of (B.1) is holomorphic at $z_0 = r$, while the other two integrals are evaluated as
\[
(B.2) \quad - \frac{1}{2\pi i} \left( \oint_{z \in C_{z_0}} + \oint_{z \in C_{\overline{z_0}}} \right) K_r(z_0, z) \left( B(z, z_1)B(\overline{z}, \overline{z}_2) + B(\overline{z}, z_1)B(z, \overline{z}_2) \right)
= \frac{1}{(y(z_0) - y(\overline{z}_0))dx(z_0)} \left( B(z_0, z_1)B(\overline{z}_0, \overline{z}_2) + B(\overline{z}_0, z_1)B(z_0, \overline{z}_2) \right).
\]
Therefore $W_{0,3}(z_0, z_1, z_2)$ is holomorphic at $z_0 = r$ (and hence, it is holomorphic at $z_i = r$ for $i = 1, 2$ as well) under the assumption on the pole property of $(y(z) - y(\overline{z}))dx(z)$ at $r$.

On the other hand, the behavior of $W_{1,1}(z_0)$ when $z_0$ approaches to $r$ is described in a similar manner:
\[
(B.3) \quad \left. \text{Res} \right|_{z=r} K_r(z_0, z)B(z, \overline{z}) = \frac{1}{2\pi i} \oint_{z \in C_{r,z_0,\overline{z_0}}} K_r(z_0, z)B(z, \overline{z}) + \frac{1}{(y(z_0) - y(\overline{z}_0))dx(z_0)}B(z_0, \overline{z}_0).
\]
Then we can conclude that $W_{1,1}(z_0)$ is holomorphic at $z_0 = r$ because $(y(z_0) - y(\overline{z}_0))dx(z_0)$ has a double or higher order pole at $r$ (cf. Lemma [B.2]) while $B(z_0, \overline{z}_0)$ has a double pole there.

Using the induction on $2g - 2 + n$, we can conclude that the correlation functions $W_{g,n}$ are holomorphic at $r$ with respect to each variable $z_i$, as follows. For general $(g, n)$, the contribution of the residue at $z = r$ to $W_{g,n+1}(z_0, z_1, \cdots, z_n)$ is given by the following form
\[
(B.4) \quad \left. \text{Res} \right|_{z=r} K_r(z_0, z)F_{g,n}(z, \overline{z}, z_1, \cdots, z_n)
= \frac{1}{2\pi i} \left( \oint_{z \in C_{r,z_0,\overline{z_0}}} - \oint_{z \in C_{z_0}} - \oint_{z \in C_{\overline{z_0}}} \right) K_r(z_0, z)F_{g,n}(z, \overline{z}, z_1, \cdots, z_n).
\]
Although we omit an explicit expression of $F_{g,n}(z, \overline{z}, z_1, \cdots, z_n)$ (which can be read off from (2.23)), we know it has at most double pole at $z = r$ under the induction hypothesis. Then, by the similar argument for $W_{1,1}(z_0)$ presented above, we can verify that the right hand-side of (B.4) is holomorphic at $z_0 = r$. Thus we have verified that $r$ is an ineffective ramification point.

Conversely, let us assume (a). Then, it follows from the definition of ineffectiveness that the correlation functions $W_{0,3}(z_0, z_1, z_2)$ must be holomorphic at $z_0 = r$. Then, in view of (B.2), we can conclude that the differential $(y(z_0) - y(\overline{z}_0))dx(z_0)$ must have a pole at $z_0 = r$ (otherwise the right-hand side of (B.2) never becomes holomorphic at $z_0 = r$).

Thus we have proved the equivalence between the conditions (a) and (b). This also completes the proof of Proposition 2.6 (i).

The remaining task for a proof of Proposition 2.6 is to show
Proposition B.4. If \( r \) is an ineffective ramification point, then the residue at \( r \) in \((2.25)\) becomes zero.

Proof. Since \( \int_{z_0}^z B(z_0, z) \) is holomorphic and vanishes at \( z = r \), we can verify that \( K_r(z_0, z) \) has a double (or more higher order) zero at \( z = r \) if \( r \) is ineffective (cf. Proposition B.1). Therefore, \( K_r(z_0, z)F_{g,n}(z, z_1, \cdots, z_n) \) in \((B.4)\) is holomorphic and has no residue at \( z = r \). This completes the proof.

C Bernoulli numbers and difference equations

Here we summarize several useful formulas of Bernoulli numbers which are applied to solve difference equations satisfied by free energies and Voros symbols in \(\S4\).

The Bernoulli number \( \{B_n\}_{n \geq 0} \) can be defined through the generating function as

\[
\frac{w}{e^w - 1} = \sum_{n=0}^{\infty} B_n \frac{w^n}{n!}.
\]

From this definition we find

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_{2k+1} = 0 \text{ for } k \geq 1.
\]

The Bernoulli polynomials, which we denote by \( \{B_n(X)\}_{n \geq 0} \), can also be defined through the generating function:

\[
\frac{we^{Xw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(X) \frac{w^n}{n!}.
\]

Other useful expressions are also known, e.g.,

\[
B_n(X) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} X^k = \frac{\partial^X}{e^\partial X - 1} X^n.
\]

Important relations for the Bernoulli polynomials are

\[
B_n(1 - X) = (-1)^n B_n(X),
\]

\[
(-1)^n B_n(-X) = B_n(X) + nX^{n-1},
\]

\[
B_n(X + Y) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B_k(X) Y^{n-k}.
\]

Special values of the Bernoulli polynomials are

\[
B_n(0) = B_n,
\]

\[
B_n\left(\frac{1}{2}\right) = \left(\frac{1}{2^{n-1}} - 1\right) B_n.
\]

We use these Bernoulli numbers and Bernoulli polynomials to give particular solutions of some difference equations. Its basic idea is due to [AT].
Proposition C.1.

(i) \[\frac{1}{e^w - 1} = \frac{1}{w} + \sum_{n=0}^{\infty} \frac{B_{n+1}}{n+1} \frac{w^n}{n!}.\]

(ii) \[\frac{e^w}{(e^w - 1)^2} = \frac{1}{(e^{w/2} - e^{-w/2})^2} = \frac{1}{w^2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{w^n}{n!}.\]

(iii) \[\frac{1}{(e^w - 1)^2} = \frac{1}{w^2} - \frac{1}{w} - \sum_{n=0}^{\infty} \left( \frac{B_{n+1}}{n+1} + \frac{B_{n+2}}{n+2} \right) \frac{w^n}{n!}.\]

Proof. The statement (i) follows from (C.1). By differentiating (i) with respect to \(w\), we get (ii). The statement (iii) follows from (i), (ii) and (C.10)

(C.10) \[\frac{1}{(e^w - 1)^2} = \frac{1 - e^w + e^w}{(e^w - 1)^2} = -\frac{1}{e^w - 1} + \frac{e^w}{(e^w - 1)^2}.\]

Proposition C.2.

(i) \[\frac{e^x w}{e^w - 1} = \frac{1}{w} + \sum_{n=0}^{\infty} \frac{B_{n+1}(X)}{n+1} \frac{x^n}{n!}.\]

(ii) \[\frac{e^{(1+x)} w}{(e^w - 1)^2} = \frac{1}{w^2} + \frac{X}{w} + \sum_{n=0}^{\infty} \left\{ X \frac{B_{n+1}(X)}{n+1} - \frac{B_{n+2}(X)}{n+2} \right\} \frac{x^n}{n!}.\]

(iii) \[\frac{e^{(1-x)} w}{(e^w - 1)^2} = \frac{1}{w^2} - \frac{X}{w} + \sum_{n=0}^{\infty} \left\{ X \frac{B_{n+1}(X)}{n+1} - \frac{B_{n+2}(X)}{n+2} \right\} \frac{(-x)^n}{n!}.\]

Proof. The first statement is a direct consequence of (C.3). After differentiating (i) with respect to \(w\), we have

(C.11) \[\frac{X e^x w}{e^w - 1} - \frac{e^{(x+1)} w}{(e^w - 1)^2} = -\frac{1}{w^2} + \sum_{n=0}^{\infty} \frac{B_{n+2}(X)}{n+2} \frac{x^n}{n!}.\]

Then we substitute (i) into the first term of the right-hand side to obtain (ii). To obtain (iii), we replace \(X\) by \(-X\) in (ii), and apply (C.6).

Proposition C.3. Particular solutions of a difference equation

\[F(\lambda + \hbar) - 2F(\lambda) + F(\lambda - \hbar) = G(\lambda)\]

for (i) \(G(\lambda) = \log \lambda\), (ii) \(G(\lambda) = \log(\lambda - \hbar)\), and (iii) \(G(\lambda) = \log(\lambda - \hbar/2)\), are respectively
given by

(i) \( F(\lambda) = (h\partial_\lambda)^{-2} \log \lambda + \sum_{n=1}^{\infty} \frac{B_{n+2}}{n(n+2)} \left( -\frac{\hbar}{\lambda} \right)^n \)

\( = (h\partial_\lambda)^{-2} \log \lambda + \sum_{m=1}^{\infty} \frac{B_{2m+2}}{2m(2m+2)} \left( \frac{\hbar}{\lambda} \right)^{2m} . \)

(ii) \( F(\lambda) = (h\partial_\lambda)^{-2} \log \lambda - (h\partial_\lambda)^{-1} \log \lambda + \frac{3}{4} \log \lambda + \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{B_{n+1}}{n+1} + \frac{B_{n+2}}{n+2} \right\} \left( -\frac{\hbar}{\lambda} \right)^n . \)

(iii) \( F(\lambda) = (h\partial_\lambda)^{-2} \log \lambda - \frac{1}{2} (h\partial_\lambda)^{-1} \log \lambda - \frac{1}{24} \log \lambda \\
- \sum_{n=1}^{\infty} \left\{ \frac{B_{n+1}(1/2)}{2n(n+1)} - \frac{B_{n+2}(1/2)}{n(n+2)} \right\} \left( \frac{\hbar}{\lambda} \right)^n . \)

Proof. (i) By using an infinite order differential operator \( e^{h\partial_\lambda} \), the equation becomes

\[ e^{-h\partial_\lambda} (e^{h\partial_\lambda} - 1)^2 F = \log \lambda. \]

It follows from

\[ e^{-w} (e^w - 2)^2 \left\{ \frac{1}{w^2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{w^n}{n!} \right\} = 1 \]

(cf. Proposition C.1 (ii)) that

\[ e^{-h\partial_\lambda} (e^{h\partial_\lambda} - 1)^2 \left\{ (h\partial_\lambda)^{-2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{(h\partial_\lambda)^n}{n!} \right\} = \text{id}. \]

Hence we find that

\[ \left\{ (h\partial_\lambda)^{-2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{(h\partial_\lambda)^n}{n!} \right\} \log \lambda \]

gives a solution of \( \text{[C.12]} \). This proves (i).

(ii) It is enough to find a solution of

\[ F(\lambda + 2\hbar) - 2F(\lambda + \hbar) + F(\lambda) = \log \lambda. \]

This can be expressed as

\[ (e^{h\partial_\lambda} - 1)^2 F = \log \lambda. \]

Thus Proposition C.1 (iii) gives a particular solution in this case.
In this case the difference equation becomes

\[(C.18) \quad F(\lambda + \frac{3}{2}h) - 2F(\lambda + \frac{1}{2}h) + F(\lambda - \frac{1}{2}h) = \log \lambda,\]

or

\[(C.19) \quad e^{-h\partial \lambda/2} (e^{h\partial \lambda} - 1)^2 F = \log \lambda.\]

To obtain a solution of this equation, we use Proposition C.2 (iii) with \(X = 1/2\) and \((C.9)\).

\[\square\]

**Corollary C.4.** A particular solution of a difference equation

\[(C.20) \quad F(\lambda + h) - 2F(\lambda) + F(\lambda - h) = \log(\lambda^2 - h^2/4)\]

is given by

\[(C.21) \quad F(\lambda) = 2(h\partial \lambda)^{-2} \log \lambda - \frac{1}{12} \log \lambda + 2 \sum_{m=1}^{\infty} \frac{B_{2m+2}(1/2)}{2m(2m+2)} \left( \frac{h}{\lambda} \right)^{2m}.\]

**Proof.** Because

\[(C.22) \quad \log(\lambda^2 - h^2/4) = \log(\lambda - h/2) + \log(\lambda + h/2),\]

a solution of \((C.20)\) is given by \(F(\lambda) = F_1(\lambda) + F_2(\lambda)\), where \(F_1(\lambda)\) and \(F_1(\lambda)\) are respectively solutions of

\[(C.23) \quad F_1(\lambda + h) - 2F_1(\lambda) + F_1(\lambda - h) = \log(\lambda - h/2),\]

\[(C.24) \quad F_2(\lambda + h) - 2F_2(\lambda) + F_2(\lambda - h) = \log(\lambda + h/2).\]

A solution of the first equation is given by Proposition C.3 (iii). A solution of the second is also given by Proposition C.3 (iii), where we replace \(h\) in Proposition C.3 (iii) by \(-h\). Hence

\[(C.25) \quad F(\lambda) = 2(h\partial \lambda)^{-2} \log \lambda - \frac{1}{12} \log \lambda + 2 \sum_{n \geq 1, \text{even}} \left\{ \frac{B_{n+1}(1/2)}{2n(n+1)} - \frac{B_{n+2}(1/2)}{n(n+2)} \right\} \left( \frac{h}{\lambda} \right)^{n}\]

is a solution of \((C.20)\). Because \(B_{2k+1}(1/2) = 0\) for \(k \geq 1\), we obtain \((C.21)\). \[\square\]

In the last, we study solutions of homogeneous difference equations to discuss a uniqueness of solutions.

**Proposition C.5.** A formal power series solution \(F(\lambda) = \sum_{n \geq 0} F_n(\lambda)h^n\) of a difference equation

\[(C.26) \quad F(\lambda + h) - F(\lambda) = 0\]

should satisfy \(\partial \lambda F_n = 0 \ (n \geq 0)\). Here we regard \(h\) as a small parameter and \(F_n(\lambda + h)\) for \(n \geq 0\) is defined by the Taylor expansion:

\[(C.27) \quad F_n(\lambda + h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} \frac{d^k}{d\lambda^k} F_n(\lambda).\]

84
Proof. From (C.27), we have

\[ F(\lambda + \hbar) - F(\lambda) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{1}{(n-k)!} d^{n-k} F_k. \]

Hence \( F(\lambda + \hbar) - F(\lambda) = 0 \) implies

\[ \frac{d}{d\lambda} F_0 = 0, \]

\( \frac{d}{d\lambda} F_1 + \frac{1}{2!} \frac{d^2}{d\lambda^2} F_0 = 0, \)

\( \frac{d}{d\lambda} F_2 + \frac{1}{2!} \frac{d^2}{d\lambda^2} F_1 + \frac{1}{3!} \frac{d^3}{d\lambda^3} F_0 = 0, \)

and so on. Hence, by induction, we can show \( \partial_\lambda F_n = 0 \) for \( n \geq 0. \)

\[ \square \]

**Corollary C.6.** A formal power series solution \( F(\lambda) = \sum_{n \geq 0} F_n(\lambda) h^n \) of a difference equation

\[ F(\lambda + \hbar) - 2F(\lambda) + F(\lambda - \hbar) = 0 \]

should satisfy \( \partial_\lambda^2 F_n = 0 \) \( (n \geq 0) \).

Proof. By using Proposition C.5 twice, we get

\[ \frac{d^2}{d\lambda^2} F_0 = 0, \]

\( \frac{d^2}{d\lambda^2} F_1 + \frac{1}{2!} \frac{d^3}{d\lambda^3} F_0 = 0, \)

\( \frac{d^2}{d\lambda^2} F_2 + \frac{1}{2!} \frac{d^3}{d\lambda^3} F_1 + \frac{1}{3!} \frac{d^4}{d\lambda^4} F_0 = 0, \)

and so on. These relations prove corollary.

\[ \square \]

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