Differential Inclusion Problems with Convolution and Discontinuous Nonlinearities

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Abstract. The paper investigates a new type of differential inclusion problem driven by a weighted \((p,q)\)-Laplacian and subject to Dirichlet boundary condition. The problem fully depends on the solution and its gradient. The main novelty is that the problem exhibits simultaneously a nonlocal term involving convolution with the solution and a multivalued term describing discontinuous nonlinearities for the solution. Results stating existence, uniqueness and dependence on parameters are established.

1. Introduction. The aim of this paper is to study the quasilinear differential inclusion problem

\[
\begin{cases}
-\Delta_p u - \mu(x) \Delta_q u \in f(x, \rho * u, \nabla(\rho * u)) + [g(u), g(u)] & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

on a bounded domain \(\Omega \subset \mathbb{R}^N\), for \(N \geq 2\), with the boundary \(\partial \Omega\). In order to simplify the presentation, we assume from the beginning that \(p < N\). The case \(p \geq N\) is actually simpler and can be handled along the same lines. As can be seen from the statement, (1) is a nonstandard problem, in particular it is nonlocal, for which the meaning of the data are described below.

Here \(\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)\) and \(\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)\), with \(1 < q < p < +\infty\), \(p' = \frac{p}{p-1}\) and \(q' = \frac{q}{q-1}\), are the \(p\)-Laplacian and the \(q\)-Laplacian, respectively. The leading operator in problem (1) is the (negative) weighted \((p,q)\)-Laplacian \(-\Delta_p - \mu(x)\Delta_q\) with the nonnegative weight \(\mu \in L^\infty(\Omega)\). An important case is for \(\mu = 0\) with the (negative) \(p\)-Laplacian \(-\Delta_p\) as driving operator. Another

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important case is for \( \mu = 1 \), where the driving operator is the (negative) \((p,q)\)-Laplacian \(-\Delta_p - \Delta_q\).

The right-hand side of (1) is described by a Carathéodory function \( f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) (i.e., \( f(\cdot, s, \xi) \) is measurable on \( \Omega \) for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \) and \( f(x, \cdot, \cdot) \) is continuous for a.e. \( x \in \Omega \)). In (1), \( f(x, \rho * u, \nabla(\rho * u)) \) is a convection term, i.e., it depends on the solution \( u \) and its gradient \( \nabla u \). In fact, it is much more than a convection term because it involves a function \( \rho \in L^1(\mathbb{R}^N) \) (that can be seen as a parameter) through the convolution \( \rho * u \) with the solution \( u \in W^{1,p}_0(\Omega) \). Due to the convolution, in order to have a meaningful formulation it is convenient to consider the Sobolev space \( W^{1,p}_0(\Omega) \) embedded in \( W^{1,p}(\mathbb{R}^N) \) given by the extension with 0 outside \( \Omega \), so \( E(u) = \hat{u} \). Then for \( \rho \in L^1(\mathbb{R}^N) \) and \( u \in W^{1,p}_0(\Omega) \subset W^{1,p}(\mathbb{R}^N) \) the convolution \( \rho * u \) is defined by

\[
\rho * u(x) = \int_{\mathbb{R}^N} \rho(x-y)u(y)dy \quad \text{for a.e. } x \in \mathbb{R}^N.
\]

Notice that it holds

\[
\text{supp } \rho * u \subset \overline{\Omega} + \text{supp } \rho.
\]

The gradient \( \nabla(\rho * u) \) appearing in (1) makes sense taking into account that

\[
\rho * u = \rho * \hat{u} \in W^{1,p}(\mathbb{R}^N).
\]

In (1) we also have a function \( g \in L^1_{\text{loc}}(\mathbb{R}) \) for which we set

\[
g(s) = \lim_{\delta \to 0} \text{essinf}_{|\tau - s| < \delta} g(\tau), \quad \forall s \in \mathbb{R}
\]

and

\[
\overline{g}(s) = \lim_{\delta \to 0} \text{esssup}_{|\tau - s| < \delta} g(\tau), \quad \forall s \in \mathbb{R}.
\]

If the function \( g \) is continuous, then the interval \([g(u(x)), \overline{g}(u(x))]\) collapses to the singleton \( g(u(x)) \). Consequently, in this case (1) reduces to the quasilinear Dirichlet equation

\[
\begin{cases}
-\Delta_p u - \mu(x) \Delta_q u = f(x, \rho * u, \nabla(\rho * u)) + g(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

(4)

involving convection and convolution.

The multivalued term \([g(u), \overline{g}(u)]\) in (1) is actually the generalized gradient of a locally Lipschitz function as will be shown explicitly in the next section. This fact qualifies problem (1) as a hemivariational inequality, which is of a special type due to the convection term \( f(x, \rho * u, \nabla(\rho * u)) \) exhibiting composition with convection. Besides their substantial mathematical interest, the hemivariational inequalities represented a major progress by passing from convex nonsmooth potentials to nonconvex nonsmooth potentials in order to model complicated phenomena with various contact laws in mechanics and engineering. For many results and applications in this direction we refer to [6, 9, 10, 12, 13].

It is worth mentioning that problems (1) and (4) do not have variational (smooth or nonsmooth) structure, so the variational methods are not applicable. This causes considerable technical difficulties in dealing with such problems. Generally, the variational structure is lost when there is a convection term since the presence of the gradient \( \nabla u \) in the right-hand side prevents the construction of potential. Here
the situation is even more difficult because of the composition with the convolution $\rho \ast u$, which is a nonlocal operator. If $f = 0$, problem (1) becomes a nonsmooth variational problem with discontinuous nonlinearities (see [4, 9]), while (4) is a quasilinear elliptic equation that can be treated by using the smooth critical point theory. If $g = 0$, problems (1) and (4) reduce to those considered in [8] whose results are extended in the present work.

The main novelty of the paper is that we are able to deal simultaneously with various top difficulties: weighted $(p,q)$-Laplacian, convection, convolution, discontinuous nonlinearity and multivalued term. Under verifiable conditions we prove existence of solutions. Strengthening the hypotheses, we provide a uniqueness result, too. Moreover, we also investigate the dependence of the solution set to problems (1) and (4) with respect to $\mu \in L^\infty(\Omega)$ and $\rho \in L^1(\mathbb{R}^N)$ considered as parameters. Results on dependence regarding $\mu \in \mathbb{R}$ for problems without convolution, multivalued terms and discontinuous nonlinearities are given in [1] (see also [7]).

The rest of the paper is organized as follows. Section 2 is devoted to the mathematical background needed in the sequel. Section 3 contains our existence result. Section 4 presents our uniqueness result. Section 5 focuses on the dependence with respect to parameters.

2. Mathematical background. This section provides the necessary mathematical background for our results on problem (1), in particular (4).

We start by briefly reviewing the multivalued pseudomonotone operators. More details can be found in [3, 11, 14]. Let $X$ be a reflexive Banach space with the norm $\| \cdot \|$, its dual $X^*$ and the duality pairing $\langle \cdot, \cdot \rangle$ between $X$ and $X^*$. The norm convergence in $X$ and $X^*$ is denoted by $\to$, while the weak convergence is denoted by $\rightharpoonup$. A multivalued map $A : X \to 2^{X^*}$ is called bounded if it maps bounded sets into bounded sets. It is said to be coercive if there is a function $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(t) \to +\infty$ as $t \to +\infty$ such that

$$\langle \xi, u - u_0 \rangle \geq c(\|u\|)\|u\|$$

for all $\xi \in A(u)$ and some $u_0 \in X$. A multivalued map $A : X \to 2^{X^*}$ is called pseudomonotone if

(i) for each $v \in X$, the set $Av \subset X^*$ is nonempty, bounded, closed and convex;

(ii) $A$ is upper semicontinuous from each finite dimensional subspace of $X$ to $X^*$ endowed with the weak topology;

(iii) for any sequences $(u_n) \subset X$ and $(u_n^*) \subset X^*$ satisfying $u_n \rightharpoonup u$ in $X$, $u_n^* \in Au_n$ for all $n$ and $\limsup_{n \to \infty} \langle u_n^*, u - u_n \rangle \leq 0$,

and for each $v \in X$ there exists $u^*(v) \in Au$ such that

$$\langle u^*(v), u - v \rangle \leq \liminf_{n \to \infty} \langle u_n^*, u_n - v \rangle.$$

We recall the main theorem for pseudomonotone operators (see, e.g., [3, Theorem 2.125]).

**Theorem 2.1.** Let $X$ be a reflexive Banach space, let $A : X \to 2^{X^*}$ be a pseudomonotone, bounded and coercive operator, and let $\eta \in X^*$. Then there exists at least one $u \in X$ with $\eta \in Au$. 

Next we outline some basic elements of nonsmooth analysis related to locally Lipschitz functions. An extensive study of this topic can be found in \([5, 4, 9, 13]\). A function \(\Phi : X \to \mathbb{R}\) on a Banach space \(X\) is called locally Lipschitz if for every \(u \in X\) there is a neighborhood \(U\) of \(u\) in \(X\) and a constant \(L_u > 0\) such that

\[|\Phi(v) - \Phi(w)| \leq L_u \|v - w\|, \quad \forall \, v, w \in U.\]

The generalized directional derivative of a locally Lipschitz function \(\Phi : X \to \mathbb{R}\) at \(u \in X\) in the direction \(v \in X\) is defined as

\[\Phi^0(u; v) := \limsup_{w \to u, \, t \to 0^+} \frac{1}{t} (\Phi(w + tv) - \Phi(w))\]

and the generalized gradient of \(\Phi\) at \(u \in X\) is the subset of the dual space \(X^*\) given by

\[\partial \Phi(u) := \{u^* \in X^* : \langle u^*, v \rangle \leq \Phi^0(u; v), \quad \forall \, v \in X\}.\]

It is useful to point out that a continuous and convex function \(\Phi : X \to \mathbb{R}\) is locally Lipschitz and its generalized gradient \(\partial \Phi : X \to 2^{X^*}\) coincides with the subdifferential of \(\Phi\) in the sense of convex analysis. As another important example, if \(\Phi : X \to \mathbb{R}\) is a continuously differentiable function, the generalized gradient of \(\Phi\) is just the differential \(D\Phi\) of \(\Phi\).

The preceding notions of subdifferentiability theory for locally Lipschitz functions are needed to handle the multivalued term \([g(u), \overline{g}(u)]\) in problem (1) in the way as explained in the following. Given \(g : \mathbb{R} \to \mathbb{R}\) satisfying \(g \in L^1_{loc}(\mathbb{R})\), we introduce

\[G(s) = \int_0^s g(t) \, dt \quad \text{for all } s \in \mathbb{R}.\]

The function \(G : \mathbb{R} \to \mathbb{R}\) is locally Lipschitz and one can show that the generalized gradient \(\partial G(s)\) of \(G\) at any \(s \in \mathbb{R}\) is the compact interval in \(\mathbb{R}\) determined by

\[\partial G(s) = [g(s), \overline{g}(s)],\]

where \(g(s)\) and \(\overline{g}(s)\) are precisely the functions in (2) and (3), respectively (see, e.g., [5, Example 2.2.5]).

Now we describe the functional setting for problems (1) and (4). We suppose that \(1 < q < p < +\infty\) and consider the Sobolev spaces \(W_{0}^{1,p}(\Omega)\) and \(W_{0}^{1,q}(\Omega)\) endowed with the norms \(\|u\| := \|
abla u\|_p\) and \(\|
abla u\|_q\), respectively, where \(\| \cdot \|_r\) stands for the usual \(L^r\)-norm. The duals of the spaces \(W_{0}^{1,p}(\Omega)\) and \(W_{0}^{1,q}(\Omega)\) are \(W^{-1,p'}(\Omega)\) and \(W^{-1,q'}(\Omega)\), respectively. As usual, we denote by \(p^*\) the Sobolev critical exponent, that is \(p^* = Np/(N - p)\) (recall that we assume \(p < N\)). As \(p \in (1, N)\), Rellich-Kondrachov theorem asserts that \(W_{0}^{1,p}(\Omega)\) is compactly embedded into \(L^q(\Omega)\) if \(1 \leq \theta < p^*\) and continuously embedded for \(\theta = p^*\). Thus for every \(r \in [1, p^*]\) there exists a positive constant \(S_r\) such that

\[\|u\|_r \leq S_r \|u\|, \quad \forall \, u \in W_{0}^{1,p}(\Omega).\]

The (negative) \(p\)-Laplacian \(-\Delta_p : W_{0}^{1,p}(\Omega) \to W^{-1,p'}(\Omega)\) is defined by

\[\langle -\Delta_p u, \varphi \rangle = \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx\]

for all \(u, \varphi \in W_{0}^{1,p}(\Omega)\).

Among many properties of this nonlinear operator, we mention that \(-\Delta_p\) is strictly monotone and continuous, so pseudomonotone. If \(p = 2\) it is just the ordinary
The (negative) Laplacian operator \(-\Delta\). Similarly, we have the definition of the (negative) \(q\)-Laplacian \(-\Delta_q\) \(W^{1,q}_0(\Omega) \to W^{-1,q'}(\Omega)\). By virtue of the embedding \(W^{1,p}_0(\Omega) \hookrightarrow W^{1,q}_0(\Omega)\), there exists a constant \(k > 0\) such that

\[
\|\nabla u\|_q \leq k \|\nabla u\|_p, \quad \forall \ u \in W^{1,p}_0(\Omega).
\]

(8)

Then the differential operator \(-\Delta_p - \mu(x)\Delta_q\) driving inclusion (1) and equation (4) is well defined on \(W^{1,p}_0(\Omega)\) since \(\mu \in L^\infty(\Omega)\). Moreover, taking into account that the sum of pseudomonotone operators is pseudomonotone, the nonlinear operator \(-\Delta_p - \mu(x)\Delta_q\) is pseudomonotone, actually maximal monotone because it was supposed that \(\mu(x) \geq 0\) for a.e. \(x \in \Omega\).

We add a few things about the convolution \(\rho \ast u\), where \(\rho \in L^1(\mathbb{R}^N)\) and \(u \in W^{1,p}_0(\Omega) \subset W^{1,p}(\mathbb{R}^N)\). The weak partial derivatives of the convolution \(\rho \ast u\) of \(\rho \in L^1(\mathbb{R}^N)\) and \(u \in W^{1,p}_0(\Omega)\) have the expressions

\[
\frac{\partial}{\partial x_i}(\rho \ast u) = \rho \ast \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N), \quad \forall \ i = 1, \ldots, N.
\]

(9)

The Tonelli’s and Fubini’s theorems, in conjunction with Hölder’s inequality, lead to the estimates

\[
\|\rho \ast u\|_{L^r(\mathbb{R}^N)} \leq \|ho\|_{L^1(\mathbb{R}^N)} \|u\|_r
\]

for every \(r \in [1, p^*]\) and

\[
\left\| \rho \ast \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} \leq \|ho\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}, \quad \forall \ i = 1, \ldots, N
\]

(10)

(11)

(see [2]). It is clear that (9)–(11) entail that the linear mapping \(u \in W^{1,p}_0(\Omega) \mapsto \rho \ast u \in W^{1,p}(\mathbb{R}^N)\) is continuous. A consequence of (11) concerns the gradient \(\nabla (\rho \ast u)\) of the convolution \(\rho \ast u\). Namely, by the convexity of the function \(t \mapsto t^p\) on \((0, +\infty)\), (9) and (11), we can derive

\[
\left\| \nabla (\rho \ast u) \right\|_{L^p(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |\nabla (\rho \ast u)|^p dx
\]

\[
= \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N} (\rho \ast \frac{\partial u}{\partial x_i})^2 \right)^{\frac{p}{2}} dx
\]

\[
\leq \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N} \left| \rho \ast \frac{\partial u}{\partial x_i} \right|^p \right) dx
\]

\[
\leq N^{p-1} \sum_{i=1}^{N} \left\| \rho \ast \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}^p
\]

\[
\leq N^{p-1} \|ho\|_{L^1(\mathbb{R}^N)} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}^p
\]

\[
\leq N^p \|ho\|_{L^1(\mathbb{R}^N)} \|u\|_{L^p(\mathbb{R}^N)}^p.
\]
By a (weak) solution to problem (1) with \( g \in L_{\text{loc}}^1(\mathbb{R}) \) we mean any \( u \in W^{1,p}_0(\Omega) \) for which it holds \( \bar{g}(u), \bar{g}(u) \in L^p(\Omega) \) and
\[
\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} \mu(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx \\
- \int_{\Omega} f(x, \rho * u, \nabla (\rho * u)) \varphi \, dx \geq \int_{\Omega} \min \{g(u(x))\varphi(x), \bar{g}(u(x))\varphi(x)\} \, dx \quad \text{for all} \quad \varphi \in W^{1,p}_0(\Omega).
\] (13)

By replacing \( \varphi \in W^{1,p}_0(\Omega) \) with \( -\varphi \) it is seen that (13) is equivalent to
\[
\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} \mu(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx \\
- \int_{\Omega} f(x, \rho * u, \nabla (\rho * u)) \varphi \, dx \leq \int_{\Omega} \max \{g(u(x))\varphi(x), \bar{g}(u(x))\varphi(x)\} \, dx \quad \text{for all} \quad \varphi \in W^{1,p}_0(\Omega).
\] (14)

As can be readily seen from (13) (or (14)), (2) and (3), we emphasize that for the Dirichlet equation (4) the usual notion of weak solution is retrieved. Indeed, if \( g : \mathbb{R} \to \mathbb{R} \) is continuous, then the interval \([\bar{g}(u), \bar{g}(u)]\) reduces to the singleton \( g(u) \), thus \( u \in W^{1,p}_0(\Omega) \) is a (weak) solution to equation (4) provided \( g(u) \in L^p(\Omega) \) and
\[
\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} \mu(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x) \, dx \\
- \int_{\Omega} f(x, \rho * u, \nabla (\rho * u)) \varphi \, dx = \int_{\Omega} g(u(x))\varphi(x) \, dx \quad \text{for all} \quad \varphi \in W^{1,p}_0(\Omega).
\] (15)

3. Existence of solutions. In this section we focus on the existence of solutions to problems (1) and (4). The hypotheses that we assume are the following:

\((H_f)\) There are constants \( a_1, a_2 \geq 0 \), \( \alpha, \beta \in [0, p - 1) \), \( r \in [1, p^*) \), and a function \( \omega \in L^r(\Omega) \) such that
\[
|f(x, s, \xi)| \leq \omega(x) + a_1|s|^{\alpha} + a_2|\xi|^\beta
\]
for a.e. \( x \in \Omega \), all \( s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \).

\((H_g)\) The function \( g : \mathbb{R} \to \mathbb{R} \) is measurable and there exist constants \( c > 0 \) and \( \sigma \in (1, p) \) such that
\[
|g(t)| \leq c(1 + |t|^\sigma - 1) \quad \text{for a.e.} \quad t \in \mathbb{R}.
\]
Condition \((H_f)\) has been used in [8].

Our existence result on problems (1) and (4) is as follows.

**Theorem 3.1.** Assume that conditions \((H_f)\) and \((H_g)\) hold. Then problem (1) admits at least one solution. In particular, if the function \( g \) is continuous, then a solution to problem (4) exists.
Proof. By (8), the operator $-\Delta_p - \mu \Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous (see, e.g., [3, Lemma 2.11]). This operator is bounded, too.

Recall the function $G : \mathbb{R} \to \mathbb{R}$ in (5) corresponding to $g : \mathbb{R} \to \mathbb{R}$ in the right-hand side of (1). Assumption $(H_g)$ implies that $G : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.

Then the functional $\Phi : L^\sigma(\Omega) \to \mathbb{R}$ given by

$$\Phi(v) = \int_\Omega G(v(x)) \, dx \quad \text{for all } v \in L^\sigma(\Omega) \quad (16)$$

is Lipschitz continuous on the bounded subsets of $L^\sigma(\Omega)$. Hence we can see it as a locally Lipschitz function on $W_0^{1,p}(\Omega)$ (note that $\sigma < p$) and its generalized gradient can be regarded as a multivalued map $\partial \Phi : W_0^{1,p}(\Omega) \to 2^{L^{\sigma'}(\Omega)}$ with $\sigma' = \sigma/(\sigma - 1)$. This multivalued operator is bounded thanks to the fact that $\Phi$ is Lipschitz continuous on the bounded subsets of $L^\sigma(\Omega)$.

Due to the presence of convolution, it is convenient to identify $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ with the function $\tilde{f} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ obtained by extending $f(x,s,\xi)$ by 0 outside $\Omega$. Then the growth condition in assumption $(H_f)$ ensures that the Nemytskii operator $N_f : W_1^{1,p}(\mathbb{R}^N) \to L^{(p')'}(\mathbb{R}^N) \subset W^{-1,p'}(\mathbb{R}^N)$ associated with the function $\tilde{f}(x,s,\xi)$, that is

$$N_f(u) = f(x,u,\nabla u),$$

is well defined, continuous and bounded. Using the inclusion map $E : W_0^{1,p}(\Omega) \to W_1^{1,p}(\mathbb{R}^N)$ obtained by taking the zero extension outside $\Omega$ (see Section 1) and its adjoint map $E^* : W^{-1,p'}(\mathbb{R}^N) \to W^{-1,p'}(\Omega)$, we introduce the multivalued operator $A : W_0^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ as

$$Au = -\Delta_p u - \mu \Delta_q u - E^* N_f(\rho \ast E u) - \partial \Phi(u), \quad \forall \ u \in W_0^{1,p}(\Omega). \quad (17)$$

The preceding comments clarify that the multivalued operator $A : W_0^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ is bounded.

We claim that the multivalued operator $A : W_0^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ in (17) is pseudomonotone. To this end, let the sequences $(u_n) \subset W_0^{1,p}(\Omega)$ and $(u'_n) \subset W^{-1,p'}(\Omega)$ satisfy $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u'_n \in Au_n$ for all $n$ and

$$\limsup_{n \to \infty} \langle u'_n, u_n - u \rangle \leq 0. \quad (18)$$

From (17) we deduce that

$$u'_n = -\Delta_p u_n - \mu \Delta_q u_n - E^* N_f(\rho \ast E u_n) - \xi_n, \quad (19)$$

with

$$\xi_n \in \partial \Phi(u_n). \quad (20)$$

Hypothesis $(H_f)$, Hölder’s inequality, (10), (12) and (7) allow us to obtain the estimate

$$\left| \int_\Omega f(x,\rho \ast u_n, \nabla(\rho \ast u_n))(u_n - u) \, dx \right| \leq \|\omega\|_r \|u_n - u\|_r + a_1 \|\rho \ast u_n\|_{L^{p^*}([\mathbb{R}^N])} \|u_n - u\|_{\frac{p^*}{p - p^*}}$$

$$+ a_2 \left\| \nabla(\rho \ast u_n) \right\|_{L^p([\mathbb{R}^N])} \|u_n - u\|_{\frac{p}{p^*}} \leq \|\omega\|_r \|u_n - u\|_r + a_1 \|\rho\|_{L^1(\mathbb{R}^N)} S_{p^*} \|u_n\|^{p^*} \|u_n - u\|_{\frac{p^*}{p - p^*}}$$

$$+ a_2 N^{p^*} \|\rho\|_{L^1(\mathbb{R}^N)} \|u_n\|^{p^*} \|u_n - u\|_{\frac{p^*}{p - p^*}}.$$
The sequential strongly-weakly closedness of the graph of the generalized gradient $E^*$ implies the strong convergence $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$ and the weak convergence $\xi_n \rightarrow \xi$ in $\partial G$. Consequently, the foregoing estimate yields

$$\lim_{n \rightarrow \infty} \langle E^* N_f(\rho \ast E u), u_n - u \rangle = 0. \quad (21)$$

Applying Aubin-Clarke theorem (see [5, Theorem 2.7.5]), (16) and (6), we infer from (20) that $\xi_n \in L^{\sigma'}(\Omega) \subset W^{-1,p'}(\Omega)$ and

$$\xi_n(x) \in \partial G(u_n(x)) = [g(u_n(x)), g(u_n(x))] \quad \text{for a.e. } x \in \Omega. \quad (22)$$

Then by (22), Hölder’s inequality and hypothesis $(H_f)$ we find the estimate

$$\|\langle \xi_n, u_n - u \rangle \| = \left| \int_{\Omega} \xi_n(x) (u_n(x) - u(x)) \, dx \right| \leq \int_{\Omega} |\xi_n(x)| \| u_n(x) - u(x) \| \, dx \leq \left( \int_{\Omega} |\xi_n(x)|^{\sigma'} \, dx \right)^{\frac{1}{\sigma'}} \left( \int_{\Omega} \| u_n(x) - u(x) \|^{p-\sigma} \, dx \right)^{\frac{1}{p-\sigma}} \leq c \left( \int_{\Omega} (1 + |u_n(x)|^{\sigma-1})^{\sigma'} \, dx \right)^{\frac{1}{\sigma'}} \| u_n - u \|_\sigma.$$

In view of Rellich-Kondrachov theorem, the weak convergence $u_n \rightarrow u$ in $W^{1,p}(\Omega)$ implies the strong convergence $u_n \rightarrow u$ in $L^\sigma(\Omega)$. Consequently, the foregoing estimate yields

$$\lim_{n \rightarrow \infty} \langle \xi_n, u_n - u \rangle = 0. \quad (23)$$

At this point we gather (18), (19), (21) and (23) resulting in

$$\limsup_{n \rightarrow \infty} \langle -\Delta_p u_n - \mu(x) \Delta_q u_n, u_n - u \rangle \leq 0. \quad (24)$$

Observe that the monotonicity of $-\Delta_q : W_0^{1,q}(\Omega) \rightarrow W^{-1,q'}(\Omega)$ in conjunction with $\mu(x) \geq 0$ for a.e. $x \in \Omega$ entails

$$\langle -\Delta_p u_n, u_n - u \rangle = \langle -\Delta_p u_n - \mu(x) \Delta_q u_n, u_n - u \rangle + \langle \mu(x) \Delta_q u_n - \Delta_q u_n, u_n - u \rangle + \langle \mu(x) \Delta_q u, u_n - u \rangle \leq \langle -\Delta_p u_n - \mu(x) \Delta_q u_n, u_n - u \rangle + \langle \mu(x) \Delta_q u, u_n - u \rangle.$$

Then (24) and $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$ give

$$\limsup_{n \rightarrow \infty} \langle -\Delta_p u_n, u_n - u \rangle \leq 0. \quad (25)$$

On the basis of $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$ and of (25) we can apply the $(S_+)$-property of the operator $-\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ (see, e.g., [3, Chapter 2]) getting the strong convergence $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$.

Assume in addition that $u_n^* \rightarrow u^*$ in $W^{-1,p'}(\Omega)$. Exploiting the continuity of the operators $-\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$, $-\Delta_q : W_0^{1,q}(\Omega) \rightarrow W^{-1,q'}(\Omega)$ and $E^* N_f(\rho \ast E u) : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$, we are able to deduce from (19) that

$$\xi_n \rightarrow \xi = -\Delta_p u - \mu \Delta_q u - E^* N_f(\rho \ast E u) - u^*.$$

The sequential strongly-weakly closedness of the graph of the generalized gradient $\partial \Phi : W_0^{1,p}(\Omega) \rightarrow 2^{L^{p'}(\Omega)} \subset 2^{W^{-1,p'}(\Omega)}$ (see, e.g., [5, Proposition 2.1.5]) can be
and the fact that $\mu$ follows that $A$ (17) is generalized pseudomonotone. Since as shown before it is also bounded, it follows that $A$ is pseudomonotone (see [3, Proposition 2.123]).

In order to prove that $A : W^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ in (17) is coercive, pick arbitrary $v \in W^{1,p}(\Omega)$ and $\xi \in \partial \Phi(v)$, with $\Phi$ in (16). By assumption $(H_f)$, $(22)$, $(2)$, $(3)$ and $(7)$, we infer that

$$\langle \xi, v \rangle = \int_{\Omega} \xi(x) v(x) \, dx \leq \int_{\Omega} |\xi(x)||v(x)| \, dx$$

$$\leq \int_{\Omega} c(1 + |v(x)|^{p-1})|v(x)| \, dx$$

$$\leq cS \|v\|^p + cS_1\|v\|.$$ 

From assumption $(H_f)$, Hölder’s inequality, $(10)$, $(12)$ and $(7)$ it is available the estimate

$$\int_{\Omega} f(x, \rho * v, \nabla(\rho * v)) v \, dx$$

$$\leq \|\omega\|_{L^p(\mathbb{R}^n)} \|v\|_r + a_1\|\rho * v\|^p_{L^p(\mathbb{R}^n)} \|v\|_p^{p-1} + a_2\|\nabla(\rho * v)\|_{L^p(\mathbb{R}^n)}^p \|v\|_r$$

$$\leq \|\omega\|_{L^p(\mathbb{R}^n)} \|v\|_r + a_1\|\rho\|^p_{L^p(\mathbb{R}^n)} S_{\rho, p, x} \|v\|_r \|v\|_r$$

$$+ a_2N^{\beta} \|\rho\|_{L^1(\mathbb{R}^n)} S_{\rho, p, x} \|v\|_r \|v\|_r \|v\|_r.$$ 

Through the above estimates, the monotonicity of $-\Delta_q : W^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ and the fact that $\mu(x) \geq 0$ for a.e. $x \in \Omega$, we arrive at

$$\langle -\Delta_p v - \mu(x)\Delta_q v - E^*N_f(\rho * Ev) - \xi, v \rangle$$

$$\geq \|v\|^p - \|\omega\|_{L^p(\mathbb{R}^n)} \|v\|_r - a_1\|\rho\|^p_{L^p(\mathbb{R}^n)} S_{\rho, p, x} \|v\|_r \|v\|_r$$

$$- a_2N^{\beta} \|\rho\|_{L^1(\mathbb{R}^n)} S_{\rho, p, x} \|v\|_r \|v\|_r \|v\|_r - cS \|v\|^\sigma - cS_1\|v\|.$$ 

Taking into account (17) and since $1, \alpha + 1, \beta + 1, \sigma < p$, we can conclude that the multivalued operator $A : W^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ is coercive.

Summarizing, we have shown that all the conditions required to apply Theorem 2.1 are satisfied in the case of the multivalued operator $A : W^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ in (17). Accordingly, by applying Theorem 2.1 we are entitled to assert the existence of $u \in W^{1,p}(\Omega)$ solving the equation $Au = 0$, which means exactly that $u$ is a solution of problem (1), that is (13) holds true. In the case where the function $g$ is continuous, $u$ becomes a solution to (4), that is, (15) is valid. This completes the proof.

4. A uniqueness result. Our objective in this section is to provide sufficient conditions to have a unique solution to problems (1) and (4). Basically, this occurs under additional assumptions of Lipschitz-like conditions.

$$\lim_{n \to \infty} \langle u_n^*, u_n \rangle = \langle u^*, u \rangle.$$
Theorem 4.1. (a) Assume that conditions \((H_f)\), \((H_g)\) and \(1 < q < 2 = p\) are satisfied and there exist constants \(b_1, c_1, d_1 > 0\) such that
\[
|f(x, s_1, \xi_1) - f(x, s_2, \xi_2)| \leq c_1|s_1 - s_2| + d_1|\xi_1 - \xi_2| \tag{26}
\]
for a.e. \(x \in \Omega\), all \(s_1, s_2 \in \mathbb{R}\), \(\xi_1, \xi_2 \in \mathbb{R}^N\) and
\[
\max\{g(s_1)(s_1 - s_2), \varphi(s_1)(s_1 - s_2)\} - \min\{g(s_2)(s_1 - s_2), \varphi(s_2)(s_1 - s_2)\} \leq b_1(s_1 - s_2)^2, \forall s_1, s_2 \in \mathbb{R}. \tag{27}
\]
If \(\rho \in L^1(\mathbb{R}^N)\) is such that
\[
(b_1 + c_1\|\rho\|_{L^1(\mathbb{R}^N)})S_2^2 + d_1\|\rho\|_{L^1(\mathbb{R}^N)}S_2 < 1, \tag{28}
\]
then for every \(\mu \in L^\infty(\Omega)\) problem (1) has exactly one solution.

(b) Assume that \(\mu \geq 0\) is a constant and conditions \((H_f)\), \((H_g)\), (26), (27) and \(q = 2 < p < +\infty\) are satisfied. If \(\mu > 0\) is sufficiently large, precisely if
\[
\mu > (b_1 + c_1\|\rho\|_{L^1(\mathbb{R}^N)})S_2^2 + d_1\|\rho\|_{L^1(\mathbb{R}^N)}S_2, \tag{29}
\]
then problem (1) has exactly one solution.

Proof. (a) The existence of a solution follows from Theorem 3.1. If \(u_1, u_2 \in W^{1,p}_0(\Omega)\) are solutions to problem (1) with \(1 < q < 2 = p\), by testing (13) and (14) with \(\varphi = u_1 - u_2\) and using hypotheses (26) and (27), it holds
\[
\|\nabla(u_1 - u_2)\|^2 \leq (-\Delta u_1 + \Delta u_2, u_1 - u_2) + \mu(x)(-\Delta u_1 + \Delta u_2, u_1 - u_2)
\]
\[
\leq \int_{\Omega} (f(x, \rho * u_1, \nabla(\rho * u_1)) - f(x, \rho * u_2, \nabla(\rho * u_2)))|u_1 - u_2| \, dx
\]
\[
+ \int_{\Omega} \max\{g(u_1(x))(u_1(x) - u_2(x)), \varphi(u_2(x))(u_1(x) - u_2(x))\} \, dx
\]
\[
- \int_{\Omega} \min\{g(u_2(x))(u_1(x) - u_2(x)), \varphi(u_2(x))(u_1(x) - u_2(x))\} \, dx
\]
\[
\leq \int_{\Omega} (c_1|\rho * u_1 - \rho * u_2| + d_1|\nabla(\rho * u_1 - \rho * u_2)|)|u_1 - u_2| \, dx
\]
\[
+ \int_{\Omega} b_1(u_1(x) - u_2(x))^2 \, dx.
\]
Then we get from Hölder’s inequality and (10) that
\[
\|u_1 - u_2\|^2 \leq (b_1 + c_1\|\rho\|_{L^1(\mathbb{R}^N)})\|u_1 - u_2\|^2
\]
\[
+ d_1\|\nabla(\rho * (u_1 - u_2))\|_{L^2(\mathbb{R}^N)}\|u_1 - u_2\|^2. \tag{30}
\]
Through (9) and (10) we see that
\[
\|\nabla(\rho * (u_1 - u_2))\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla(\rho * (u_1 - u_2))|^2 \, dx
\]
\[
= \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} \left(\rho * \frac{\partial}{\partial x_i}(u_1 - u_2)\right)^2\right) \, dx
\]
\[
\leq \|\rho\|^2_{L^1(\mathbb{R}^N)}\|u_1 - u_2\|^2. \tag{31}
\]
We find from (30), (31) and (7) that
\[
(1 - b_1S_2^2 - c_1\|\rho\|_{L^1(\mathbb{R}^N)}S_2^2 - d_1\|\rho\|_{L^1(\mathbb{R}^N)}S_2\|u_1 - u_2\|^2 \leq 0.
\]
Thanks to assumption (28), we conclude that \(u_1 = u_2\).
(b) The existence of a solution is guaranteed by Theorem 3.1. Let \( u_1, u_2 \in W_0^{1,p}(\Omega) \) be solutions to problem (1) with \( q = 2 < p < +\infty \). Proceeding as in part (a), we can rely on (13), (14), (26) and (27) to see that

\[
\mu \| u_1 - u_2 \|^2 \leq \langle -\Delta_p u_1 + \Delta_p u_2, u_1 - u_2 \rangle + \mu \langle -\Delta u_1 + \Delta u_2, u_1 - u_2 \rangle
\]

\[
\leq \int_\Omega (c_1 |\rho * u_1 - \rho * u_2| + d_1 |\nabla (\rho * u_1 - \rho * u_2)|) |u_1 - u_2| \, dx
\]

\[
+ \int_\Omega b_1 (u_1(x) - u_2(x))^2 \, dx.
\]

Then arguing as in part (a) we are led to

\[
(\mu - b_1 S_2^2 - c_1 \| \rho \|_{L^1(\mathbb{R}^N)} S_2^2 - d_1 \| \rho \|_{L^1(\mathbb{R}^N)} S_2) \| u_1 - u_2 \|^2 \leq 0.
\]

Now it suffices to address hypothesis (29) for obtaining \( u_1 = u_2 \). The proof is thus complete. \( \square \)

In the case of problem (4) with \( g \) continuous, Theorem 4.1 takes the form.

**Corollary 1.** (a) Assume that the function \( g : \mathbb{R} \to \mathbb{R} \) is continuous, \( 1 < q < 2 = p \) and conditions \((H_f), (H_g), (26)\) and

\[
(g(s_1) - g(s_2))(s_1 - s_2) \leq b_1 (s_1 - s_2)^2, \quad \forall \ s_1, s_2 \in \mathbb{R}, \tag{32}
\]

with some positive constant \( b_1 \), are satisfied. If \( \rho \in L^1(\mathbb{R}^N) \) fulfills (28), then for every \( \mu \in L_+^\infty(\Omega) \) equation (1) has exactly one solution.

(b) Assume that \( \mu \geq 0 \) is a constant and conditions \((H_f), (H_g), (26), (32)\) and \( q = 2 < p < +\infty \) are verified. If \( \mu \) satisfies (29), then equation (4) has exactly one solution.

**Proof.** The corollary follows readily from Theorem 4.1 noticing that, under the hypothesis of continuity for the function \( g \), condition (27) reads as (32), which is apparent from (2) and (3). \( \square \)

**Remark 1.** If the function \( g : \mathbb{R} \to \mathbb{R} \) is nonincreasing or Lipschitz continuous, then condition (32) is fulfilled.

5. Dependence on parameters. In this section we investigate the dependence of the solutions to problems (1) and (4) with respect to the parameters \( \mu \in L_+^\infty(\Omega) \) and \( \rho \in L^1(\mathbb{R}^N) \).

The following statement deals with the dependence on the parameter \( \rho \in L^1(\mathbb{R}^N) \) in problem (1).

**Theorem 5.1.** Assume that conditions \((H_f), (H_g)\) are satisfied. If \( \rho_n \to \rho \) in \( L^1(\mathbb{R}^N) \) and \( u_n \in W_0^{1,p}(\Omega) \) is a solution to the inclusion

\[
\begin{align*}
-\Delta_p u_n - \mu \Delta q u_n & \in f(x, \rho_n * u_n, \nabla (\rho_n * u_n)) + [g(u_n), \overline{g}(u_n)] \quad \text{in } \Omega, \\
u_n & = 0 \quad \text{on } \partial \Omega, \tag{33}
\end{align*}
\]

then there is a subsequence of \( (u_n) \) still denoted \( (u_n) \) such that \( u_n \to u \) in \( W_0^{1,p}(\Omega) \) for some solution \( u \in W_0^{1,p}(\Omega) \) to differential inclusion (1). In particular, if \( g \) is
continuous, for each sequence $\rho_n \to \rho$ in $L^1(\mathbb{R}^N)$ and $u_n \in W^{1,p}_0(\Omega)$ solution to the equation
\[
\begin{cases}
-\Delta_p u_n - \mu \Delta_q u_n = f(x, \rho_n * u_n, \nabla(\rho_n * u_n)) + g(u_n) & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]
there is a subsequence of $(u_n)$ still denoted $(u_n)$ such that $u_n \to u$ in $W^{1,p}_0(\Omega)$ for some solution $u \in W^{1,p}_0(\Omega)$ of equation (4).

Proof. A solution $u_n$ to problem (33) exists according to Theorem 3.1. By hypothesis $(H_f)$, Hölder’s inequality, (10), (12) and (7) we infer that
\[
\left| \int_{\Omega} f(x, \rho_n * v, \nabla(\rho_n * v)) v \, dx \right| 
\leq \| \omega \|_{r'} \| v \|_r + a_1 \| \rho_n * v \|_{L^p(\mathbb{R}^N)} \| v \|_{\frac{p^*}{p}} 
+ a_2 \| \nabla(\rho_n * v) \|_{L^p(\mathbb{R}^N)} \| v \|_{\frac{p^*}{p}} 
\leq \| \omega \|_{r'} \| v \|_r + a_1 \| \rho_n \|_{L^1(\mathbb{R}^N)} S_{p^*}^\alpha \| v \|_\alpha \| v \|_{\frac{p^*}{p}} 
+ a_2 N^\beta \| \rho_n \|_{L^1(\mathbb{R}^N)} \| v \|_\beta \| v \|_{\frac{p^*}{p}} 
\text{for all } v \in W^{1,p}_0(\Omega), \text{ all } n.
\]
Due to the boundedness of the sequence $(\rho_n)$ in $L^1(\mathbb{R}^N)$ and since $\alpha, \beta < p - 1$, there exist constants $C_1 > 0$ and $\delta_1 \in (0,1)$ independent of $n$ such that
\[
\int_{\Omega} f(x, \rho_n * v, \nabla(\rho_n * v)) v \, dx \leq \delta_1 \| v \|^p + C_1 \text{ for all } v \in W^{1,p}_0(\Omega), \text{ all } n. \tag{34}
\]
On the other hand, by assumption $(H_g)$, (22), (2), (3) and (7) we derive
\[
|\langle \xi_n, u_n \rangle| = \left| \int_{\Omega} \xi_n(x) u_n(x) \, dx \right| 
\leq \int_{\Omega} |\xi(x)||u_n(x)| \, dx
\leq \int_{\Omega} c(1 + |u_n(x)|^{\sigma-1})|u_n(x)| \, dx
\leq cS_{\sigma}^\alpha \| u_n \|_{\sigma} + cS_{1} \| u_n \|_{1}
\text{for every } \xi_n \in L^{\sigma'}(\Omega) \subset W^{-1,\sigma'}(\Omega) \text{ as in } (22). \text{ Since } 1 < \sigma < p, \text{ there exist constants } C_2 > 0 \text{ and } \delta_2 \in (0, 1 - \delta_1) \text{ independent of } n \text{ such that}
\]
\[
|\langle \xi_n, u_n \rangle| \leq \delta_2 \| u_n \|^p + C_2 \text{ for all } n. \tag{35}
\]
Let us use $u_n$ as test function in (33). Then (34) and (35) imply that the sequence $(u_n)$ is bounded in $W^{1,p}_0(\Omega)$, so up to a subsequence $u_n \to u$ in $W^{1,p}_0(\Omega)$ for some $u \in W^{1,p}_0(\Omega)$. Along a reasoning similar to the one developed in the proof of Theorem 3.1, this time essentially based on the boundedness of the sequence $(\rho_n)$ in $L^1(\mathbb{R}^N)$, we are able to reach (24). From now on we can continue as in the proof of Theorem 3.1 achieving the strong convergence $u_n \to u$ in $W^{1,p}_0(\Omega)$. Here the validity of the $(S_\lambda)$-property of the operator $-\Delta_p - \mu \Delta_q$ on $W^{1,p}_0(\Omega)$ as checked in the mentioned proof is crucial. We now note that (10) and (11) yield the strong convergence $\rho_n * u_n \to \rho * u$ in $W^{1,p}(\mathbb{R}^N)$. Moreover, passing to a relabeled subsequence we may suppose that $u_n(x) \to u(x), \rho_n * u_n(x) \to \rho * u(x)$ and $\rho_n * \nabla u_n(x) \to \rho * \nabla u(x)$ for a.e. $x \in \Omega$. 


Recall that \( u_n \) solves (33) which by means of (6) reads as
\[
\int_{\Omega} |\nabla u_n(x)|^{p-2} \nabla u_n(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} \mu(x)|\nabla u_n(x)|^{p-2} \nabla u_n(x) \cdot \nabla \varphi(x) \, dx
- \int_{\Omega} f(x, \rho_n * u_n, \nabla (\rho * u_n)) \varphi \, dx
\geq \int_{\Omega} \min\{g(u_n(x)), \varphi(x), \bar{g}(u_n(x)), \varphi(x)\} \, dx
= \int_{\Omega} \min[\partial G(u_n(x)), \varphi(x)] \, dx \text{ for all } \varphi \in W_0^{1,p}(\Omega).
\]

On the basis of the continuity of the operators \(-\Delta_p - \mu(x)\Delta_q\) and \(N_f\) in conjunction with the strong convergence \(u_n \to u\) in \(W_0^{1,p}(\Omega)\) and the strong convergence \(\rho_n * u_n \to \rho * u\) in \(W^{1,p}(\mathbb{R}^N)\), besides the graph closedness of the generalized gradient \(\partial G\) of the locally Lipschitz function \(G\) in (5), we can pass to the limit in the above inequality as \(n \to +\infty\) obtaining through Fatou’s lemma that \(u\) is a solution of (1). The proof is thus complete.

We turn to the dependence with respect to the parameter \(\mu \in L^\infty(\Omega)\). For simplicity, in the statement below \(\rho \in L^1(\mathbb{R}^N)\) is kept fixed.

**Theorem 5.2.** Assume that conditions \((H_f)\) and \((H_g)\) are satisfied.

(a) For any sequence \(\mu_n \to 0\) in \(L^\infty(\Omega)\) as \(n \to \infty\), with \(\mu_n \geq 0\) for a.e. \(x \in \Omega\), there exists a relabeled subsequence of solutions \((u_n)\) to the differential inclusions
\[
\begin{cases}
- \Delta_p u_n - \mu_n(x)\Delta_q u_n \in f(x, \rho * u_n, \nabla (\rho * u_n)) + [g(u_n), \bar{g}(u_n)] & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega
\end{cases}
\tag{36}
\]
such that \(u_n \to u\) in \(W_0^{1,p}(\Omega)\) as \(n \to \infty\), with \(u \in W_0^{1,p}(\Omega)\) solution to
\[
\begin{cases}
- \Delta_p u \in f(x, \rho * u, \nabla (\rho * u)) + [g(u), \bar{g}(u)] & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\tag{37}
\]

In particular, if \(g\) is continuous, for any sequence \(\mu_n \to 0\) in \(L^\infty(\Omega)\), with \(\mu_n \geq 0\) for a.e. \(x \in \Omega\), there exists a relabeled subsequence of solutions \((u_n)\) with
\[
\begin{cases}
- \Delta_p u_n - \mu_n(x)\Delta_q u_n = f(x, \rho * u_n, \nabla (\rho * u_n)) + g(u_n) & \text{in } \Omega \\
u_n = 0 & \text{on } \partial \Omega
\end{cases}
\tag{38}
\]
such that \(u_n \to u\) in \(W_0^{1,p}(\Omega)\), with \(u \in W_0^{1,p}(\Omega)\) solution to the equation
\[
\begin{cases}
- \Delta_p u = f(x, \rho * u, \nabla (\rho * u)) + g(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\tag{39}
\]

(b) Let \((\mu_n)\) be a sequence of positive real numbers with \(\mu_n \to +\infty\) as \(n \to \infty\). If \(u_n \in W_0^{1,p}(\Omega)\) is a solution of inclusion (36) with \(\mu_n(x) \equiv \mu_n\), then it holds \(u_n \to 0\) in \(W_0^{1,q}(\Omega)\) as \(n \to \infty\). In particular, this holds true if \(g\) is continuous and \(u_n \in W_0^{1,p}(\Omega)\) is a solution of equation (38) with \(\mu_n(x) \equiv \mu_n\).

**Proof.** (a) The existence of a solution \(u_n\) to differential inclusion (36) is ensured by Theorem 3.1. Arguing as in the proof of Theorem 5.1 we establish two key estimates: there exist constants \(C_1 > 0\) and \(\delta_1 \in (0, 1)\) independent of \(n\) such that
\[
\int_{\Omega} f(x, \rho * v, \nabla (\rho * v)) v \, dx \leq \delta_1 \|v\|^p + C_1 \quad \text{for all } v \in W_0^{1,p}(\Omega), \text{ all } n
\tag{39}
\]

\[
\frac{1}{2} \int_{\Omega} |\nabla u_n(x)|^{p-2} \nabla u_n(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} \mu(x)|\nabla u_n(x)|^{p-2} \nabla u_n(x) \cdot \nabla \varphi(x) \, dx
- \int_{\Omega} f(x, \rho_n * u_n, \nabla (\rho * u_n)) \varphi \, dx
\geq \int_{\Omega} \min\{g(u_n(x)), \varphi(x), \bar{g}(u_n(x)), \varphi(x)\} \, dx
= \int_{\Omega} \min[\partial G(u_n(x)), \varphi(x)] \, dx \text{ for all } \varphi \in W_0^{1,p}(\Omega).
\]
and there exist constants $C_2 > 0$ and $\delta_2 \in (0, 1 - \delta_1)$ independent of $n$ such that (35) holds for every $\xi_n \in L^{r'}(\Omega) \subset W^{-1,p'}(\Omega)$ satisfying (22). Testing problem (36) with the solution $u_n$, by means of estimates (39) and (35) it is straightforward to confirm the sequence $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, thereby along a relabeled subsequence $u_n \to u$ in $W_0^{1,p}(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$. Following the same pattern as in the proof of Theorem 2.1, through (36), Rellich-Kondrachov compact embedding theorem and estimates (39) and (35) we are able to show that

$$\lim_{n \to +\infty} \langle -\Delta_p u_n, u_n - u \rangle = 0,$$

thus $u_n \to u$ in $W_0^{1,p}(\Omega)$ because the operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ satisfies the $(S_+)$-property. The strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$ enables us to pass to the limit as $n \to \infty$ in the definition of solution $u_n$ to (36), that is

$$\int_{\Omega} |\nabla u_n(x)|^{p-2}\nabla u_n(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} \mu_n(x)|\nabla u_n(x)|^{p-2}\nabla u_n(x) \cdot \nabla \varphi(x) \, dx$$

$$- \int_{\Omega} f(x, \rho \ast u_n, \nabla(\rho \ast u_n))\varphi \, dx$$

$$\geq \int_{\Omega} \min\{g(u_n(x))\varphi(x), \overline{g}(u_n(x))\varphi(x)\} \, dx$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Bearing in mind that $\mu_n \to 0$ in $L^\infty(\Omega)$, this renders that $u$ resolves (37).

(b) Let a sequence $\mu_n \to +\infty$ in $\mathbb{R}$ as $n \to \infty$ and let $u_n \in W_0^{1,p}(\Omega)$ be a solution of inclusion (36) with $\mu_n(x) \equiv \mu_n$ for each $n$. By (14) we can write

$$\|u_n\|^p \leq \int_{\Omega} |\nabla u_n(x)|^p \, dx + \mu_n \int_{\Omega} |\nabla u_n(x)|^q \, dx$$

$$\leq \int_{\Omega} f(x, \rho \ast u_n, \nabla(\rho \ast u_n)) \, dx$$

$$+ \int_{\Omega} \max\{g(u_n(x))u_n, \overline{g}(u_n(x))u_n\} \, dx.$$

On account of estimates (39) and (35) with any $\xi_n \in L^{r'}(\Omega) \subset W^{-1,p'}(\Omega)$ satisfying (22), we see that the sequence $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, so up to a relabeled subsequence we have $u_n \to u$ in $W_0^{1,p}(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$.

Returning to (36), we note that $u_n$ is a solution to

$$\begin{cases}
-\frac{1}{\mu_n}\Delta_p u_n - \Delta_q u_n & \frac{1}{\mu_n}f(x, \rho \ast u_n, \nabla(\rho \ast u_n)) + \frac{1}{\mu_n}[g(u_n), \overline{g}(u_n)] \quad \text{in} \quad \Omega \\
\mu_n = 0 & \text{on} \quad \partial\Omega
\end{cases} \quad \text{(40)}$$

Acting with $u_n - u$ in (40) and taking into account that $\mu_n \to +\infty$ whereas the sequences $(u_n)$ and $(\Delta_p u_n)$ remain bounded in $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively, it turns out

$$\lim_{n \to +\infty} \langle -\Delta_q u_n, u_n - u \rangle = 0.$$

This together with the weak convergence $u_n \to u$ in $W_0^{1,q}(\Omega)$ permits to invoke the $(S_+)$-property of the operator $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ for obtaining the strong convergence $u_n \to u$ in $W_0^{1,q}(\Omega)$. Based on the previous analysis, we can
pass to the limit in (40) leading to $\Delta_q u = 0$, which yields $u = 0$. The proof is thus complete.

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