THE RELATION BETWEEN TRANSVERSE AND RADIAL VELOCITY DISTRIBUTIONS FOR OBSERVATIONS OF AN ISOTROPIC VELOCITY FIELD

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ABSTRACT

We examine the case of a random isotropic velocity field, in which one of the velocity components (the “radial” component, with magnitude $v_z$) can be measured easily, while measurement of the velocity perpendicular to this component (the “transverse” component, with magnitude $v_T$) is more difficult and requires long-time monitoring. Particularly important examples are the motion of galaxies at cosmological distances and the interpretation of Gaia data on the proper motion of stars in globular clusters and dwarf galaxies. We address two questions: what is the probability distribution of $v_T$ for a given $v_z$, and for what choice of $v_z$ is the expected value of $v_T$ maximized? We show that, for a given $v_z$, the probability that $v_T$ exceeds some value $v_0$ is $p(v_T \geq v_0 | v_z) = p_z(\sqrt{v_0^2 + v_T^2})/p_z(v_z)$, where $p_z(v_z)$ is the probability distribution of $v_z$. The expected value of $v_T$ is maximized by choosing $v_z$ as large as possible whenever $\ln p_z(\sqrt{v_0^2})$ has a positive second derivative, and by taking $v_z$ as small as possible when this second derivative is negative.

Subject headings: (cosmology:) large-scale structure of the Universe, proper motions

1. INTRODUCTION

Measuring radial velocities in astronomy through Doppler shifts in spectral lines is easy. Measuring transverse velocities through shifts in angular position is more challenging. Redshift measurements allowed Vesto Slipher to determine the radial velocities of distant galaxies more than 100 years ago, while the transverse motion of galaxies at cosmological distances has never been measured.

However, the latter situation may soon change (Darling, Truebenbach, & Payne 2018). As noted previously by Sandage (1962) and Loeb (1998), precision redshift measurements taken over a significant time span would allow for a “real time” measurement of the evolution of the Hubble parameter; there have been attempts to measure this effect using H I 21 cm absorption line redshifts (Darling 2012). Similarly, precision astrometry might soon allow for the measurement of galactic proper motions (and hence, galaxy transverse velocities) in real time (Peebles et al. 2001; Nusser, Branchini, & Davis 2012; Quercellini 2012; Darling & Truebenbach 2018; Darling et al. 2018). The measurement of transverse velocities of distant galaxies using microlensing was considered by Grieger, Kayser, & Refsdal (1986) and Gould (1995), while Hamden, et al. (2010) explored the possibility of using perspective rotation in clusters. The possibility of making transverse velocity measurements with Gaia is discussed in detail by Nusser et al. (2012), while Darling et al. (2018) examine the potential for ngVLA to measure these transverse velocities.

This leads to an obvious question: given the opportunity to monitor a limited set of galaxies with known radial velocities $v_z$, which of these are most likely to have the largest transverse velocities $v_T$? (Here $v_z$ corresponds to the peculiar line-of-sight velocity, with the contribution from the Hubble expansion subtracted off.) This question is perhaps less relevant for an all-sky survey such as Gaia, but other instruments such as ngVLA might monitor a limited sample of distant galaxies.

Given a wide dispersion in the magnitudes of the total velocity, $v$, one might naturally assume that the galaxies with the largest radial velocities would also tend to have the largest transverse velocities. However, if $v$ is narrowly distributed around a single value, then the largest radial velocities would correspond to the smallest transverse velocities. This is easily seen for the case where $v$ is identically the same for all of the objects in question; in this case $\sqrt{v_T^2 + v_z^2}$ is maximized when $v_z = 0$.

Which of these two arguments is correct? Both of them are relevant. As we will see, the largest transverse velocities can correspond to either the largest or the smallest values of the radial velocity, $v_z$, depending on the distribution of $v_z$.

This paper addresses the following questions: given an isotropic random velocity field, along with a known distribution of radial velocities $v_z$, what is the corresponding distribution of transverse velocities, $v_T$, and for what choice of $v_z$ is the expected value of $v_T$ maximized? While this discussion is motivated within the context of galaxy velocities, these questions are quite general, and it seems likely that our results would be applicable to other areas of astronomy as well, such as star clusters. We address these questions mathematically in the next section, and briefly discuss our results in Section 3.

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1 The transverse velocities of the much closer LMC and SMC have both been measured using the Hubble Space Telescope (Kallivayalil et al. 2006; Kallivayalil, van der Maree, & Alcock 2006).
2. THE DISTRIBUTION OF TRANSVERSE VELOCITIES FOR A GIVEN RADIAL VELOCITY

Consider a set of sources with an isotropic velocity distribution. We begin by examining the relation between the distribution of the magnitude of the total velocity, \( v \), and the distribution of the magnitude of a single component, \( v_z \), where we will usually assume that it is the latter that can be observationally inferred. This is a well-known problem (Feller 1971, 29-33), and the results we derive here are not new. Note that the derivation is simplified if we take \( v_z \) to be the magnitude of a single component of the velocity, rather than the actual (positive or negative) value of the velocity component; this will be our convention throughout most of this paper. As a single component of an isotropic velocity field is symmetrically distributed about zero, it is trivial to go from the distribution of the magnitude of a single component to the distribution of that component itself.

Following Feller (1971), the relation between \( v \) and \( v_z \) is

\[
v_z = |v| \cos \theta,
\]

where \( \theta \) is the angle between the line of sight and the velocity vector of the moving object. Note that \( |\cos \theta| \) is uniformly distributed between 0 and 1, and it is independent of \( v \). Consequently, the right-hand side of equation (1) is the product of two independent random variables, the first with probability distribution function (PDF) \( p(v) \), and the second with a uniform distribution. Recall that for independent random variables \( x \) and \( y \), with PDFs \( p_x(x) \) and \( p_y(y) \), respectively, the PDF of the product \( z = xy \) is

\[
p(z) = \int p_x \left( \frac{z}{y} \right) p_y(y) \frac{dy}{|y|}.
\]

Then the PDF for \( v_z \), which we will denote \( p_z(v_z) \), is related to \( p(v) \) through

\[
p_z(v_z) = \int_{v_z}^{\infty} p(v) \frac{dv}{v}.
\]

Taking the derivative gives \( p'_z(v_z) = -p(v_z)/v_z \), where \( p'_z \) is the derivative of \( p_z \) with respect to its argument. But this functional equation is valid regardless of the independent variable used in the equation, so we can set this variable to be \( v \) instead of \( v_z \), which gives us \( p \) in terms of \( p_z \):

\[
p(v) = -v p'_z(v).
\]

For example, suppose that the distribution of radial velocities is a Gaussian, so that the distribution of the magnitude of the radial velocities is a one-sided Gaussian:

\[
p_z(v_z) = \sqrt{\frac{2}{\pi \sigma}} \exp\left(-\frac{v^2}{2\sigma^2}\right), \quad v_z \geq 0.
\]

Then equation (2) gives, for the distribution of the magnitude of the total velocity,

\[
p(v) = \sqrt{\frac{2}{\pi \sigma^3}} v^2 \exp\left(-\frac{v^2}{2\sigma^2}\right).
\]

Equations (5) and (6) correspond, of course, to the Maxwell distribution of velocities in a gas.

While \( p(v) \) can have essentially any functional form (subject to the condition that it be a probability distribution normalized to unity), this is not the case for \( p_z(v_z) \). For instance, equation (2) implies that \( p'_z(v_z) < 0 \): the distribution of \( v_z \) must have its maximum value at \( v_z = 0 \) and decrease monotonically with \( v_z \). An observed distribution of \( v_z \) that violated this condition would imply a deviation from an isotropic velocity field.

Equations (3) and (4) are restatements of previously-derived results, but now consider the question of interest to us: for a given observed value of \( v_z \), what is the distribution of \( v_T \), denoted \( p(v_T|v_z) \)? Consider first the cumulative distribution function (CDF), which is the probability that \( v_T \) is less than a given value \( v_0 \); this quantity is denoted \( p(v_T \leq v_0|v_z) \). It will turn out to be easier and more useful to work with the complementary CDF, which is \( p(v_T \geq v_0|v_z) \). Since \( v^2 = v_T^2 + v_z^2 \), we can express this CDF in terms of \( v_T \):

\[
p(v_T \geq v_0|v_z) = p(v \geq \sqrt{v_0^2 + v_z^2}).
\]

The right-hand side can be simplified if we have an expression for \( p(v|v_z) \), the PDF of \( v \) for a given fixed value of \( v_z \). Equation (4) implies that \( p(v|v_z) \) is just a uniform distribution between 0 and \( v \), given by \( 1/v \), so we can use Bayes theorem:

\[
p(v|v_z) = p(v_z|v) \frac{p(v)}{p_z(v_z)},
\]

\[
= \frac{1}{v} \frac{p(v)}{p_z(v_z)}.
\]

Integrating equation (5) over \( v \) while keeping \( v_z \) fixed gives us the CDF in equation (4), namely

\[
p(v_T \geq v_0|v_z) = \int_{v_0}^{\infty} \frac{1}{v} \frac{p(v)}{p_z(v_z)} dv.
\]

Using equation (5), we can perform the integral to give

\[
p(v_T \geq v_0|v_z) = \frac{p_z(\sqrt{v_0^2 + v_z^2})}{p_z(v_z)}.
\]

Equation (11) provides complete information about the distribution of \( v_T \) for a given \( v_z \), and it will be the main expression we work with. However, for completeness we will also derive the corresponding PDF. This is just the negative of the derivative of the right-hand side of equation (11) with respect to \( v_0 \), evaluated at \( v_0 = v_T \). We obtain

\[
p(v_T|v_z) = -\frac{v_T}{\sqrt{v_T^2 + v_z^2}} \frac{p'_z(\sqrt{v_T^2 + v_z^2})}{p_z(v_z)}.
\]

\[
= \frac{v_T}{v_T^2 + v_z^2} \frac{p(\sqrt{v_T^2 + v_z^2})}{p_z(v_z)},
\]

where we used equation (4) to go from equation (12) to equation (13). Any of equations (11)-(13) provides all of the information on the distribution of transverse velocities for an observed radial velocity, but equation (11) is the simplest and most informative of these.

As an example, consider again a Gaussian distribution of radial velocities, with the distribution of the magnitudes of these radial velocities given by equation (5).
Substituting this distribution into equation (11), we obtain
\[
p(v_T \geq v_0 | v_z) = \exp(-v_0^2/2\sigma^2). \tag{14}
\]
Thus, for the special case of a Gaussian distribution of radial velocities, the distribution of \(v_T\) is also a Gaussian, and it is independent of the value of \(v_z\).

Now we will address our second question of interest: for what choice of \(v_z\) is the expected value of \(v_T\) maximized? From equation (11), we see that there are two main possible cases. If \(p_z(\sqrt{v_0^2 + v_z^2}) = p_z(v_z)\) is an increasing function of \(v_z\), then the probability of observing a value of \(v_T\) greater than \(v_0\) always increases with \(v_z\) for any value of \(v_0\). In this case, choosing the largest value of \(v_z\) will maximize the expected value of \(v_T\). Conversely, if \(p_z(\sqrt{v_0^2 + v_z^2}) = p_z(v_z)\) is a decreasing function of \(v_z\), then the probability of observing \(v_T\) greater than \(v_0\) always decreases with \(v_z\) for any value of \(v_0\), and the expected value of \(v_T\) is maximized for the smallest observed value of \(v_z\). (The theoretical best value in this case is \(v_z = 0\).)

As an example, consider the Subbotin family of distributions, given by
\[
p_z(v_z) = \frac{\beta}{\Gamma(1/\beta)} \exp(-v_z^\beta) \tag{17}
\]
where \(\beta\) is a positive constant and \(\Gamma\) is the gamma function. The corresponding expressions for the magnitude of the total velocity \(v\) are (from equation (12))
\[
p(v) = \frac{\beta^2}{\Gamma(1/\beta)} v^\beta \exp(-v^\beta). \tag{18}
\]

For these distributions, \(f(v_z) = -v_z^{3/2}\) (plus an irrelevant constant), which is a convex function for \(\beta < 2\) and a concave function for \(\beta > 2\). Thus, for \(\beta < 2\), we maximize the expected value of \(v_T\) by choosing \(v_z\) as large as possible, while for \(\beta > 2\), we need to choose \(v_z\) as small as possible. The case \(\beta = 2\) corresponds to the Gaussian distribution, for which \(f'(v_z) = 0\), so \(p_z(\sqrt{v_0^2 + v_z^2}) = p_z(v_z)\) is constant and \(p(v_T \geq v_0 | v_z)\) is independent of \(v_z\), as have already noted.

We can express these conditions on \(p_z(v_z)\) more simply by writing the distribution in the form
\[
p_z(v_z) = \exp[f(v_z^2)], \tag{15}
\]
where the function \(f\) is defined by equation (15), or, alternatively, by
\[
f(v_z) = \ln p_z(\sqrt{v_z}). \tag{16}
\]

When \(p_z(v_z)\) is written in this way, our conditions for maximizing \(v_T\) become particularly transparent. The quantity \(p_z(\sqrt{v_0^2 + v_z^2})/p_z(v_z)\) is an increasing (decreasing) function of \(v_z\) when \(f(v_0^2 + v_z^2) - f(v_z^2)\) is an increasing (decreasing) function of \(v_z\). Now consider the conditions on \(f\) needed to make \(f(v_0^2 + v_z^2) - f(v_z^2)\) an increasing function of \(v_z\). This will be the case when the derivative of \(f\) with respect to \(v_z\) is positive, i.e., \(f'(v_0^2 + v_z^2) - f'(v_z^2) > 0\), where we have used the fact that \(v_z > 0\) by definition, and the prime denotes the derivative of \(f\) with respect to its argument. It is clear that this condition on \(f\) will be satisfied as long as \(f(v_z^2)\) is a convex function, i.e., having positive second derivative. Similarly, \(f(v_0^2 + v_z^2) - f(v_z^2)\) will be a decreasing function of \(v_z\) as long as \(f'(v_z^2)\) is a concave function, i.e., with negative second derivative. Thus, we maximize the expected value of \(v_T\) by choosing \(v_z\) large as possible whenever \(f''(v_z^2) > 0\) and by choosing \(v_z\) small as possible whenever \(f''(v_z^2) < 0\), with \(f\) defined by equations (15) or (16).

Probability distributions for which \(\ln p(\sqrt{x})\) is either always a concave function or always a convex function have been examined previously in the context of signal processing (Benveniste, Goursat, & Ruget 1980; Palmer, Kreutz-Delgado, & Makeig 2010). Palmer et al. introduced the terms strong sub-Gaussianity and strong super-Gaussianity to refer to distributions for which \(\ln p(\sqrt{x})\) is concave or convex, respectively. With this definition, \(v_T\) will be maximized by choosing the largest value of \(v_z\) when \(p_z(v_z)\) is strongly super-Gaussian, and by choosing the smallest value of \(v_z\) when \(p_z(v_z)\) is strongly sub-Gaussian.

We illustrate \(p_z(v_z)\) and \(p(v)\) for the Subbotin distribution in Figs. 1 and 2, respectively, for the cases \(\beta = 1\) (solid, black) and \(\beta = 4\) (dashed, red). The expected value of \(v_T\) is maximized for the largest observed value of \(v_z\) for \(\beta = 1\) and for the smallest observed value of \(v_z\) for \(\beta = 4\).

2 This family of distributions, extended from \(-\infty\) to \(\infty\), goes by a variety of other names, including the power exponential distribution, the exponential power distribution, and the generalized normal distribution.
(exponential), \( \beta = 2 \) (Gaussian) and \( \beta = 4 \). The form of these functions agrees with the intuitive argument outlined in Section 1. The \( \beta = 1 \) distribution for \( p(v) \) has a larger tail than the Gaussian at large \( v \) and corresponds to a case for which \( v_T \) is maximized at large \( v_z \). On the other hand, the \( \beta = 4 \) distribution for \( v \) is more sharply peaked at a single value of \( v \) than is the Gaussian, and it corresponds to the case where the expected value of \( v_T \) is maximized at small \( v_z \). Indeed, in the limit \( \beta \to \infty \), the distribution of \( v \) approaches a delta function in \( v \), corresponding to the case discussed in Section 1 where \( v \) is identically the same for all objects in the sample.

Most simple monotonically decreasing distributions for \( p_z(v_z) \) correspond to a form for \( p_z(\sqrt{v_0^2 + v_z^2})/p_z(v_z) \) that either increases or decreases monotonically, thereby maximizing the expected \( v_T \) for the largest or smallest values of \( v_z \), respectively. These include, for example, the one-sided Gaussian distribution, the uniform distribution, the exponential distribution, the Gamma distribution with shape parameter \( < 1 \), and the one-sided Cauchy distribution.

However, these are not the only possibilities. Consider, for example, these two distributions for \( v_z \):

\[
p_z(v_z) = 1.716 \exp(-v_z/v_z^4), \tag{19}
\]

and

\[
p_z(v_z) = 0.723 \exp\left(-\frac{v_z^4}{1 + v_z^2}\right), \tag{20}
\]

where the normalization constants are determined numerically. Each of these distributions is designed to mimic the behavior of the \( \beta = 1 \) and \( \beta = 4 \) Subbotin distributions in the appropriate limits: distribution (19) goes to \( \beta = 1 \) at small \( v_z \) and \( \beta = 4 \) at large \( v_z \), while distribution (20) does the opposite. These distributions correspond to total velocity distributions of the form

\[
p(v) = 1.716 \exp(-v/v^4),
\]

and

\[
p(v) = 0.723 \frac{v^4(1 + v^3)}{\exp\left(-\frac{v^4}{1 + v^3}\right)},
\]

respectively. While both of these distributions are exceptionally contrived, there is nothing pathological about their form, as can be seen in Figs. 3 and 4.

**Fig. 2.** As Fig. 1, for the distribution of the magnitude of the total velocity, \( v \) (equation 15) for \( \beta = 1 \) (blue, dotted), \( \beta = 2 \) (solid, black) and \( \beta = 4 \) (dashed, red). The expected value of \( v_T \) is maximized for the largest observed value of \( v_z \) for \( \beta = 1 \) and for the smallest observed value of \( v_z \) for \( \beta = 4 \).

**Fig. 3.** The distributions of the magnitude of a single component of the velocity, \( v_z \) given by equations (19) (solid, black) and (20) (blue, dashed).

Now consider the behavior of \( p_z(\sqrt{v_0^2 + v_z^2})/p_z(v_z) \) for these two distributions. For the distribution given by equation (19), our function \( f(v_z) \) defined by equations (15) or (16) is

\[
f(v_z) = -\frac{1}{2} - v_z^2,
\]

which has positive second derivative at small \( v_z \) and negative second derivative at large \( v_z \). Thus, the probability \( p(v_T \geq v_0 | v_z) = p_z(\sqrt{v_0^2 + v_z^2})/p_z(v_z) \) increases with \( v_z \) at small \( v_z \), reaches a maximum value, and then decreases with \( v_z \) at large \( v_z \). Furthermore, the value of \( v_z \) at which \( p(v_T \geq v_0 | v_z) \) attains its maximum value is itself a function of \( v_0 \). Thus, while one can determine a single optimal value of \( v_z \) for which the probability that
$v_T$ exceeds $v_0$ is maximized, this value of $v_z$ will now depend on $v_0$.

Conversely, for the distribution defined by equation (20), the function $f(v_z)$ is

$$f(v_z) = -\frac{v_z^2}{1 + v_z^{3/2}},$$

which has negative second derivative at small $v_z$ and positive second derivative at large $v_z$. Consequently, $p(v_T \geq v_0|v_z) = p_z(\sqrt{v_0^2 + v_z^2})/p_z(v_z)$ increases as we take either $v_z \to 0$ or $v_z \to \infty$. Thus, we can maximize $v_T$ for a given $v_z$ by choosing $v_z$ either as small as possible or as large as possible.

The appearance of the distributions in Fig. 1 suggests an alternate way to determine the optimal value of $v_z$ that maximizes the expected value of $v_T$. The Subbotin distribution with $\beta = 1$ has positive kurtosis, while $\beta = 4$ has negative kurtosis, where we define the kurtosis as

$$\kappa = \frac{\langle v_z^4 \rangle}{\langle v_z^2 \rangle^2} - 3,$$

so that $\kappa = 0$ for the Gaussian distribution. For our discussion of kurtosis (only) we will take $v_z$ to be the actual $z$ component of the velocity, rather than its magnitude, so that $\langle v_z \rangle = 0$. Then the Subbotin distribution (equation 17) extended to negative values of $v_z$ has kurtosis

$$\kappa = \frac{\Gamma(5/\beta)\Gamma(1/\beta)}{\Gamma(3/\beta)^2} - 3,$$

which is indeed positive for $\beta < 2$ and negative for $\beta > 2$.

It might appear that kurtosis can provide a simpler criterion for the value of $v_z$ that maximizes the expected $v_T$: positive kurtosis distributions for $p_z(v_z)$ (which have larger tails than a Gaussian) indicate that the largest value of $v_z$ should be chosen, while negative kurtosis distributions point toward the smallest value of $v_z$. However, this argument is only partially correct. Palmer et al. (2010) show that, in fact, all strong sub-Gaussian distributions have negative kurtosis, and all strong super-Gaussian distributions have positive kurtosis. However, the converse is not true. This is obvious, since the distributions in equations (19) and (20) have negative and positive values of $\kappa$, respectively, and yet the first maximizes $v_T$ at a fixed value of $v_z$, while the second maximizes $v_T$ at either large or small values of $v_z$. Thus, while kurtosis can provide a useful guide, the rigorously correct procedure to maximize the expected value of $v_T$ is to maximize the right-hand side of equation (11).

3. DISCUSSION AND CONCLUSIONS

We have derived an expression for the distribution of the transverse velocity, $v_T$, for a given fixed value of the radial velocity, $v_z$, valid for any isotropic velocity distribution (or indeed, for any isotropic vector field) in equation (11). Our results indicate that the expected value for $v_T$ can be maximized by choosing the largest possible value of $v_z$, if $v_z$ has a strongly super-Gaussian distribution, and for the smallest possible value of $v_z$ if the distribution is strongly sub-Gaussian, where these terms are defined in the previous section.

We now circle back to the question which originally motivated this investigation: what about the peculiar velocity field of galaxies? While current observations are beginning to probe this distribution (e.g., Tully et al. 2013; Springob et al. 2014; Tully, Courtois, & Sorce 2016), the data are still too noisy to provide a precise estimate of $p_z(v_z)$. The uncertainties in the measured peculiar velocities are typically of order the velocities themselves at cosmological distances (Watkins & Feldman 2015). However, this problem can be mitigated by binning the velocity data. Using the catalog of Tully et al. (2013), Sorce (2015) derived a bias-corrected distribution for $v_z$ which is consistent with a Gaussian distribution. This is precisely the unique distribution for which the value of $v_T$ is insensitive to the value of $v_z$. It is also consistent with the theoretical model of Sheth and Diaferio (2001), which predicts a form for $p_z(v_z)$ that looks Gaussian at small $v_z$. However, their model also predicts an exponential distribution for $p_z(v_z)$ at large $v_z$. For the exponential distribution, we expect that $v_T$ will be largest when $v_z$ is maximized. This suggests that if one were monitoring a limited set of galaxies over a long time span, efforts should be concentrated on those with the largest radial peculiar velocities.

Future data sets to which these results might be applied include measurements of radial peculiar velocities from distance calibrators such as SN Ia (Riess 1999) or gravitational wave sources (Chen, Fishbach, & Holz 2018) or from the kinetic Sunyaev-Zel’dovich effect (Akravi et al. 2018). The derivations presented here can also be used as a constraint on models of the peculiar velocity field of galaxies in the standard ΛCDM cosmology, such as those in Sheth & Diaferio (2001). In addition, our derivations can be applied to new astrometric data from the Gaia satellite on the proper motion of stars in globular clusters and dwarf galaxies (Helmi,
et al. 2018) in an attempt to constrain their mass distribution (Milone, et al. 2018) or rotation (Bianchini, et al. 2018) as well as the possible existence of an intermediate black hole at their center (e.g., Kiziltan, et al. 2017).

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