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Large Deviations for Heavy-Tailed Factor Models

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Abstract

We study large deviation probabilities for a sum of dependent random variables from a heavy-tailed factor model, assuming that the components are regularly varying. Depending on the regions considered, the probabilities are determined by different parts of the model.

Key words: large deviations, heavy tails, regular variation, factor models

MSC: 60F10, 60G50

1 Introduction

This paper is devoted to the study of large deviations of sums of dependent random variables, where the dependence is generated through a factor model. Factor models are important in both financial theory and practice, because their form of structural dependence is both intuitive and tractable. From a theoretical point of view, different types of factor models give intuition to economic phenomena: the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT) are examples where factor structure is a fundamental property (see e.g. Cochrane (2001)). From an applied point of view, factor models are useful as approximations of other models and for dimension reduction. In many cases, reducing the number of dimensions of a model can make it tractable in practice.

Often, the random variables or vectors involved are assumed to be normally distributed, or at least light-tailed. A random variable $X$ is called light-tailed if its tail-distribution $P(X > \lambda)$ tends to zero faster than $e^{-c\lambda}$ for some $c > 0$.  

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Empirical studies of financial time series often conclude that data are heavy-tailed, i.e. the previous condition is not satisfied (see e.g. Cont (2001) for a review of some of these empirical findings). Consequently, light-tailed factor models may not be suited for describing the tail-properties of financial data. Therefore, it is of interest to incorporate the assumption of heavy tails into a factor model. As we will see, heavy-tailed factor models display qualitatively different behavior from standard light-tailed models.

We restrict ourselves to the class of regularly varying random variables and vectors. This class is fairly rich and includes popular distributions such as Pareto and student’s $t$. See e.g. Embrechts et al. (1997) and Resnick (2004) for treatments of the univariate and multivariate case, respectively.

A random variable $X$ is regularly varying if there exist $\alpha \geq 0$ and $p \in [0, 1]$ such that

$$\lim_{x \to \infty} \frac{P(X > tx)}{P(|X| > x)} = pt^{-\alpha} \quad \text{and} \quad \lim_{x \to \infty} \frac{P(X \leq -tx)}{P(|X| > x)} = (1 - p)t^{-\alpha},$$

for any $t > 0$. We refer to $p$ as the tail balance parameter. The definition can also be formulated in terms of sequences instead of a continuous parameter $x$. Clearly, regularly varying random variables are heavy-tailed according to the above definition.

Since we will allow for dependence between factors, we also need the corresponding class of random vectors. For random vectors, regular variation is defined through convergence of measures. Specifically, an $\mathbb{R}^d$-valued random vector $X$ is said to be regularly varying if there exist a sequence $a_n \to \infty$ and a measure $\mu$ on $\mathbb{R}^d$ such that

$$\lim_{n \to \infty} nP(a_n^{-1}X \in B) = \mu(B)$$

and $\mu(B) < \infty$ for every Borel set $B \subset \mathbb{R}^d$ satisfying $0 \notin \overline{B}$ and $\mu(\partial B) = 0$, where $\overline{B}$ and $\partial B$ denote the closure and boundary of $B$, respectively. The definition implies that the measure $\mu$ satisfies $\mu(tB) = t^{-\alpha}\mu(B)$ for some $\alpha > 0$ and every $t > 0$. We write $X \in \text{RV}(\alpha, \mu)$. See Hult and Lindskog (2006) for details about equivalent definitions of regular variation.

Using this class of distributions, we define a factor model for the column vector $\mathbf{R}_n$ by letting

$$\mathbf{R}_n = \mathbf{A}_n \mathbf{F}_d + \mathbf{\varepsilon}_n,$$

where $\mathbf{F}_d = (F_1, \ldots, F_d)^T$ is a regularly varying random vector with limit measure $\mu$ and tail index $\alpha_F$, $\varepsilon_i$ are i.i.d. regularly varying random variables with tail index $\alpha_\varepsilon$ and $E(\varepsilon_1) = 0$, and $\mathbf{A}_n$ denotes the matrix $(L_i)_{i=1}^n$ with rows $L_i = (L_{i1}, \ldots, L_{id})$ given by $n$ i.i.d. copies of the random vector $L = \ldots$
(L₁, . . . , Lₙ) satisfying E|L_j|^α F + δ < ∞ for j = 1, . . . , d and some δ > 0. The different parts of the factor model Λₙ, Fₙ and εₙ are assumed to be mutually independent. The components of Fₙ are referred to as factors, L_ij as factor loadings and ε_i as idiosyncratic components. The moment condition on L ensures that the factors account for large deviations of the product ΛₙFₙ, not the factor loadings.

A sum of the components of this model can be expressed as

\[ S_n = \sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \sum_{j=1}^{d} L_{ij} F_j + \sum_{i=1}^{n} \varepsilon_i, \]  
(3)

and the purpose of the rest of the paper is to analyze the large deviation behavior of this sum.

The tail probability P(S_n > λ) exhibits different asymptotic behavior depending on the relation between the tail indices of the independent sums Σ_i=1^n ε_i and Σ_i=1^n Σ_j=1^d L_ij F_j. Recall that (see e.g. Embrechts et al. (1997)) if two independent regularly varying random variables X and Y have different tail indices, 0 < α_X < α_Y, then

\[ P(X + Y > \lambda) \sim P(X > \lambda), \quad \text{as } \lambda \to \infty, \]

where \( a(x) \sim b(x) \) as \( x \to \infty \) denotes \( \lim_{x \to \infty} a(x)/b(x) = 1 \). This means that the random variable with heaviest tail, or smallest tail index, dominates the tail probability of the sum. On the other hand (see e.g. Embrechts et al. (1997)), if X₁, X₂, . . . are i.i.d. regularly varying random variables with tail balance parameter p, we have with n fixed,

\[ P(\sum_{i=1}^{n} X_i > \lambda) \sim npP(|X_1| > \lambda), \quad \text{as } \lambda \to \infty. \]  
(4)

In fact, relation (4) is still valid when \( n \to \infty \) if \( \lambda = \lambda_n \) increases sufficiently fast. Asymptotic probabilities of this kind are referred to as large deviation probabilities.

For an appropriate choice of \( \lambda_n \) we have

\[ P(\sum_{i=1}^{n} X_i > \lambda_n) \sim npP(|X_1| > \lambda_n), \quad \text{as } n \to \infty. \]  
(5)

We refer to A. Nagaev (1969a,b) and S. Nagaev (1979) for classical work on large deviations with heavy-tailed distributions, and Cline and Hsing (1991) and Mikosch and Nagaev (1998) for details about the choice of sequence \( \lambda_n \) under different distributional assumptions.

In this paper we consider regularly varying random variables with tail indices
larger than 1, for which relation (5) holds if $\lambda_n$ is such that $n/\lambda_n \to 0$ as $n \to \infty$.

For tail probabilities of the sum $S_n$ given by (3), we have two different situations. As $n \to \infty$ with $\lambda_n \sim n$, the tail behavior of $S_n$ is determined by the tail probability of the sum $\sum_{i=1}^{n} \sum_{j=1}^{d} L_{ij} F_j$, whereas, when $\lambda \to \infty$ with $n$ fixed, it is determined by the sum with the heaviest tail.

Motivated by this, we study the influence of the choice of $\lambda_n$ on the behavior of large deviation probabilities of the form $P(S_n > \lambda_n)$, when $n \to \infty$. In the main result of the paper, Theorem 1, we identify regions where either the factor part or the idiosyncratic part determine the large deviation behavior. We also identify conditions under which there exists a separating sequence $\lambda_n$ such that both sums contribute to the large deviation probability of $S_n$.

2 Main Result

In this section we investigate under which conditions the factors and the idiosyncratic components in (3) contribute to the large deviation probability $P(S_n > \lambda_n)$ as $n \to \infty$.

Consider the model given by (2) and the sum (3). Denoting $S_{n,j}^L = \sum_{i=1}^{n} L_{ij}$, we write the sum as

$$S_n = \sum_{j=1}^{d} S_{n,j}^L F_j + \sum_{i=1}^{n} \varepsilon_i. \quad (6)$$

By the law of large numbers,

$$\lim_{n \to \infty} \frac{S_{n,j}^L}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} L_{ij} = EL_j \text{ a.s.},$$

which suggests that

$$P(\sum_{j=1}^{d} S_{n,j}^L F_j > \lambda_n x) \sim P(\sum_{j=1}^{d} (EL_j) F_j > \frac{\lambda_n}{n} x), \quad \text{as } \frac{\lambda_n}{n} \to \infty.$$

This is verified using Lemmas 1 and 3 in Section 3 below. Indeed, under the integrability assumptions $E|L_j|^\alpha \mu < \infty$ on $L$, it follows by Lemma 1 below that $E|S_{n,j}^L|^\alpha \mu < \infty$ and that $E(\sup_k |S_{n,j}^L|^\alpha \mu < \infty$. Now, applying Lemma 3, we conclude that for fixed $x > 0$

$$\lim_{n \to \infty} \frac{P(\sum_{j=1}^{d} S_{n,j}^L F_j > \lambda_n x)}{P(|F_d| > \lambda_n/n)} = x^{-\alpha} \mu \left( (EL)^{-1}(1, \infty) \right).$$
where \( \{(EL)^{-1}(1, \infty)\} = \{z \in \mathbb{R}^d : (EL)z \in (1, \infty)\} \),

We now consider the tail-behavior of the sum \( S_n \). If \( F_d \) and \( \varepsilon_1 \) have the same tail indices, we expect \( F_d \) to dominate the extremal behavior, i.e. we expect the idiosyncratic components to become less relevant as \( n \) grows due to the law of large numbers. Thus, the variation of the sum is mainly due to variation of the factors. In the following theorem, which is the main result of the paper, we state the behavior of the tail probability of our sum under different assumptions.

**Theorem 1.** Let \( F_d = (F_1, \ldots, F_d)^T \) be a regularly varying random vector, \( F_d \in RV(\alpha_F, \mu) \), and let \( \varepsilon_i \) be a sequence of i.i.d. regularly varying random variables, \( \varepsilon_1 \in RV(\alpha_\varepsilon) \), with tail balance parameter \( p \). Consider the factor model given in (2) and the sum \( S_n \) in equation (6).

Let \( \gamma_n \gg \rho_n \) denote \( \lim_{n \to \infty} \gamma_n / \rho_n = \infty \).

(i) If \( \alpha_F \leq \alpha_\varepsilon \), then for any \( \lambda_n \gg n \),

\[
\lim_{n \to \infty} \frac{P(S_n > \lambda_n x)}{P(|F_d| > \lambda_n/n)} = x^{-\alpha_F} \mu \left((EL)^{-1}(1, \infty)\right).
\]

(ii) Assume that \( P(|F_d| > x) = L_{|F|}(x)x^{-\alpha_F} \) and \( P(|\varepsilon_1| > x) = L_{|\varepsilon|}(x)x^{-\alpha_\varepsilon} \), where \( \alpha_F > \alpha_\varepsilon > 1 \). Define \( \theta_F = (\alpha_F - 1)/(\alpha_F - \alpha_\varepsilon) \), \( \theta_\varepsilon = \theta_F - 1 \). If \( \alpha_F > \alpha_\varepsilon \), we have three different possibilities:

(a) If \( \lambda_n \gg n^{\theta_F} \), then

\[
\lim_{n \to \infty} \frac{P(S_n > \lambda_n x)}{nP(|\varepsilon| > \lambda_n)} = px^{-\alpha_\varepsilon}.
\]

(b) If \( n \ll \lambda_n \ll n^{\theta_F} \), then

\[
\lim_{n \to \infty} \frac{P(S_n > \lambda_n x)}{P(|F_d| > \lambda_n/n)} = x^{-\alpha_F} \mu \left((EL)^{-1}(1, \infty)\right).
\]

(c) If \( \lambda_n \sim n^{\theta_F} \), and

\[
\lim_{n \to \infty} \frac{L_{|\varepsilon|}(n^{\theta_F})}{L_{|F|}(n^{\theta_F})} = C \in [0, \infty],
\]

then for \( 0 \leq C < \infty \),

\[
\lim_{n \to \infty} \frac{P(S_n > \lambda_n x)}{P(|F_d| > \lambda_n/n)} = x^{-\alpha_F} \mu \left((EL)^{-1}(1, \infty)\right) + x^{-\alpha_\varepsilon} pC
\]

(8)
\[
\lim_{n \to \infty} \frac{P(S_n > \lambda_n x)}{nP(|\varepsilon| > \lambda_n)} = px^{-\alpha}.
\]

Remark 1. Theorem 1 (ii) (c) provides a choice for \(\lambda_n\) that, given the tail indices of \(F\) and \(\varepsilon\), yields the asymptotic behavior (8). Qualitatively, it also shows that for both parts to contribute to the large deviation behavior, the idiosyncratic part must have heavier tail than the factors.

Remark 2. Condition (7) can be difficult to verify. The slowly varying functions of the norms are often not known, and are not easy to calculate explicitly. Examples where condition (7) is satisfied include:

(a) \(L|F|(x) \to c_1, L|\varepsilon|(x) \to c_2\)
(b) \(L|F|(x) = a_1 \log x + b_1, L|\varepsilon|(x) = a_2 \log x + b_2\).

Example 1. As an illustration of the application of Theorem 1, we consider the case of independent Pareto-distributed factors and idiosyncratic components. Let \(\alpha_F = 5\) and \(\alpha_\varepsilon = 3\). Assume that \(d = 10\), i.e. \(F_{10} = (F_1, \ldots, F_{10})\). We have \(L|F| = L|\varepsilon| = 1\) so that \(C = 1\). With \(\lambda_n = n^{(5-1)/(5-3)} = n^2\) we obtain

\[
P(\sum_{i=1}^{n} R_i > \lambda_n x) = P(\sum_{j=1}^{10} S_{n,j}^{L} F_j + \sum_{i=1}^{n} \varepsilon_i > \lambda_n x)
\sim \sum_{j=1}^{10} P(\frac{S_{n,j}^{L} F_j}{n} > nx) + npP(\varepsilon_1 > n^2 x)
\sim n^{-5} \left( \sum_{j=1}^{10} (EL_j)^{-5} x^{-5} + px^{-3} \right).
\]

The above results rely on the regular variation of the components involved. In the case of light-tailed random variables, the decomposition in Theorem 1 is no longer valid. We illustrate this in the following corollary by assuming light-tailed factors.

Corollary 1. Let \(X > 0\) be a light-tailed random variable with tail distribution \(F_X(x) \sim e^{-g(x)}\), where \(g(x) = cx \to \infty\), as \(x \to \infty\) for some \(c > 0\). Let \(Y_i, i = 1, 2, \ldots\) be a sequence of i.i.d. regularly varying random variables with tail-index \(\alpha > 0\), \(Y_i \in RV(\alpha)\). Then, for any sequence \(\lambda_n\) such that \(\lambda_n/n \to \infty\),

\[
\lim_{n \to \infty} \frac{P(nX + \sum_{i=1}^{n} Y_i > \lambda_n)}{P(\sum_{i=1}^{n} Y_i > \lambda_n)} = 1.
\]
3 Proofs and Technical Results

Lemma 1. Let $X_i$, $i = 1, 2, \ldots$ be a sequence of i.i.d. random variables $E|X_1|^r < \infty$, $r > 1$. Then $E(\sup_k |\sum_{i=1}^k X_i|/k)^r < \infty$.

Proof. The result follows directly from the $L^p$ maximum inequality for martingales, see e.g. Durrett (1996).

To prove Lemma 3, we use the following multivariate version of Breiman’s lemma proved by Basrak et al. (2002).

Lemma 2 (Breiman’s lemma). Let $X$ be a $d \times 1$ regularly varying random vector, $X \in RV(\alpha, \mu)$ and let $A$ be a $k \times d$ random matrix, independent of $X$. If $0 < E|A|_{\infty}^{\alpha+\delta} < \infty$ for some $\delta > 0$, then

$$
\lim_{n \to \infty} \frac{P(AX \in a_n B)}{P(|X| > a_n)} = E(\mu(A^{-1}(B))).
$$

for any Borel set $B \subset \mathbb{R}^k$ satisfying $0 \notin B$ and $\mu(\partial B) = 0$.

Lemma 3. Let $X$ be a $d \times 1$ regularly varying random vector, $X \in RV(\alpha, \mu)$ and let $A_n \neq 0$ be a sequence of $1 \times d$ random vectors independent of $X$ such that $A_n \to A \neq 0$ a.s., as $n \to \infty$ and $E(\sup_n |A_n|_{\infty}^{\alpha+\delta}) < \infty$, where, $|A|_{\infty} = \sup_{|x| = 1} |Ax|$.

Then, for $0 < \lambda_n \uparrow \infty$ and $x > 0$, we have

$$
\lim_{n \to \infty} \frac{P(A_n X > \lambda_n x)}{P(|X| > \lambda_n)} = x^{-\alpha} \mu(A^{-1}(1, \infty)).
$$

Proof of Lemma 3. We split $X$ into positive and negative parts, $X = X^+ - X^-$, where $X^+ = (X_1^+, \ldots, X_d^+)$, $X^- = (X_1^-, \ldots, X_d^-)$. The infimum and supremum of the vector $A_k$ is interpreted component-wise, i.e. $\sup_{k > M} A_k = (\sup_{k > M} A_{k,1}, \ldots, \sup_{k > M} A_{k,d})$ and analogously for the infimum. Fix $M > 0$. For $n > M$ we have,

$$
P(A_n X > \lambda_n x) = P(\sup_{k > M} A_k (X^- - X^+) > \lambda_n x) \\
\leq P(\inf_{k > M} A_k X^+ - \sup_{k > M} A_k X^- > \lambda_n x) \\
= P((\sup_{k > M} A_k, \inf_{k > M} A_k)(X^+, -X^-)^T > \lambda_n x). \quad (9)
$$

The same argument also provides a lower bound,

$$
P(A_n X > \lambda_n x) \geq P((\inf_{k > M} A_k, \sup_{k > M} A_k)(X^+, -X^-)^T > \lambda_n x). \quad (10)
$$
The probability $P(A_nX > \lambda_n x)/P(|X| > \lambda_n)$ is thus bounded from above and below. To determine these bounds, we need to show regular variation of the vector $(X^+, -X^-)^T$.

Let $E_1 = \mathbb{R}^d \setminus \{0\}$ and $E_2 = \{z' \in \mathbb{R}^d \setminus \{0\} : z' = (z^+, -z^-)^T, z \in \mathbb{R}^d \setminus \{0\}\}$ and define the continuous transformation

$$T : E_1 \rightarrow E_2$$

$$x \rightarrow (x^+, -x^-)^T.$$

Any relatively compact set $K_2$ of $E_2$ is of the form

$$K_2 = \{z' = (z^+, -z^-) \in \mathbb{R}^d \setminus \{0\} : z \in \mathbb{R}^d \setminus \{0\}\},$$

bounded away from 0, i.e. $0 \notin \overline{K_2}$. Since $z' \neq 0 \Rightarrow z \neq 0$, it is obvious that the inverse images of these sets in $\mathbb{R}^d \setminus \{0\}$ are bounded away from 0 as well.

Hence, if $K_2$ is compact in $\mathbb{R}^d \setminus \{0\}$ then $K_1 = T^{-1}(K_2)$ is compact in $\mathbb{R}^d \setminus \{0\}$. Therefore, vague convergence of a sequence of measures $\mu_n$ on $E_1$ implies vague convergence of the induced measures $\hat{\mu}_n$ on $E_2$. Specifically, since $|T(x)| = |x|$ and $T(ax) = aT(x)$ for any $a > 0$,

$$P(T(X) \in \lambda_n B) = \frac{P(X \in T^{-1}(\lambda_n B))}{P(|X| > \lambda_n)} = \frac{P(X \in \lambda_n T^{-1}(B))}{P(|X| > \lambda_n)} \Rightarrow \mu(T^{-1}(B)).$$

Therefore, the vector $T(X) = (X^+, -X^-)^T$ is regularly varying.

Since $E(\sup_n |A_n|) < \infty$ it follows that $E(|\sup_{k>\infty} A_k|, \inf_{k>\infty} A_k)|_{\infty} < \infty$, so we can use Lemma 2 to determine the bounds (9) and (10). This yields

$$E\mu((\inf_{k>\infty} A_k, \sup_{k>\infty} A_k)^{-1}(1, \infty))x^{-\alpha} \leq \liminf_{n \rightarrow \infty} \frac{P(A_nX > \lambda_n x)}{P(|X| > \lambda_n)} \leq \limsup_{n \rightarrow \infty} \frac{P(A_nX > \lambda_n x)}{P(|X| > \lambda_n)} \leq E\mu((\sup_{k>\infty} A_k, \inf_{k>\infty} A_k)^{-1}(1, \infty))x^{-\alpha}. \quad (11)$$

Since $A_n \xrightarrow{a.s.} n \rightarrow \infty A$ we have $\inf_{k>\infty} A_k \xrightarrow{a.s.} M \rightarrow \infty A$ and $\sup_{k>\infty} A_k \xrightarrow{a.s.} M \rightarrow \infty A$. It remains to verify that we can evaluate these limits inside the expectations. We have

$$\mu((\inf_{k>\infty} A_k, \sup_{k>\infty} A_k)^{-1}(1, \infty)) \leq \mu((\sup_{k>\infty} A_k, \inf_{k>\infty} A_k)^{-1}(1, \infty)) \leq \mu((\sup_{k} A_k, \inf_{k} A_k)^{-1}(1, \infty)).$$
and

\[
E\mu((\sup_k A_k, \inf_k A_k)^{-1}(1, \infty)) = E\mu(z \in \mathbb{R}^d : (\sup_k A_k, \inf_k A_k)(z^+, -z^-)^T > 1) \\
\leq E\mu(z \in \mathbb{R}^d : (\sup_k |A_k|)1_{2d}(z^+, z^-)^T > 1) \\
= E(\sup_k |A_k|)^{\alpha}(z \in \mathbb{R}^d : 1_{2d}^T|z| > 1) < \infty,
\]

with \(z = (|z_1|, \ldots, |z_d|)\). Hence, by the dominated convergence theorem,

\[
limit_{M \to \infty} E\mu(z \in \mathbb{R}^d : (\sup_{k>M} A_k, \inf_{k>M} A_k)(z^+, -z^-)^T > 1) \\
= E\mu(z \in \mathbb{R}^d : (A, A)(z^+, -z^-)^T > 1) \\
= E\mu(z \in \mathbb{R}^d : Az > 1).
\]

A similar calculation applies to the lower bound in equation (11), with the same limit. Letting \(M \to \infty\) in that equation yields the conclusion.

Before proving Theorem 1, we state a partial result.

**Lemma 4.** Assume that \(X\) is a regularly varying \(d\)-dimensional random vector, \(X \in RV(\mu, \alpha X)\), and \(Y_i\) is a sequence of i.i.d. regularly varying random variables, \(Y_1 \in RV(\alpha Y)\), with tail balance parameter \(p\). Let \(A_n\) be a sequence of \(d\)-dimensional random vectors satisfying \(E(\sup_n |A_n|)^{\alpha_X + \delta} < \infty\), for some \(\delta > 0\), and \(A_n \xrightarrow{a.s.} \mathbb{A} \neq 0\). Furthermore assume that \(A_n, Y_i, \text{and } X\) are independent for all \(i\) and \(n\).

Consider the tail probabilities

\[
\mathcal{F}_{||X||}(x) = P(||X| > x), \\
\mathcal{F}_n(x) = P(nA_nX + \sum_{i=1}^n Y_i > x), \\
\mathcal{F}_1(x) = P(nA_nX > x), \\
\mathcal{F}_2(x) = P(\sum_{i=1}^n Y_i > x),
\]

where \(x > 0\). Assume that there exists a sequence \(\lambda_n \gg n\) such that

\[
\lim_{n \to \infty} \frac{\mathcal{F}_2(\lambda_n x)}{\mathcal{F}_{||X||}(\lambda_n/n)} = Qx^{-\alpha_Y},
\]

(12)
where \( Q \in [0, \infty] \). Then,
\[
\lim_{n \to \infty} \frac{F_1(\lambda_n)}{F(\lambda_n x)} = \frac{1}{x^{-\alpha x} + x^{-\alpha y} Q/\mu_{A^{-1}}}
\]  
(13)
and
\[
\lim_{n \to \infty} \frac{F_2(\lambda_n)}{F(\lambda_n x)} = \frac{1}{x^{-\alpha y} + x^{-\alpha x} \mu_{A^{-1}}/Q}
\]  
(14)
where, \( \mu_{A^{-1}} = \mu(A^{-1}(1, \infty)) \). If \( Q \) is zero or infinite, we interpret the right hand side of relations (13)-(14) as limits.

**Proof of Lemma 4.** We first note that if \( U \) and \( V \) are independent random variables, we have
\[
P(U + V > x) \geq P(U > (1 + \delta)x)P(|V| < \delta x) + P(|U| < \delta x)P(V > (1 + \delta)x).
\]
Therefore, setting \( U = nA_nX \) and \( V = \sum_{i=1}^{n} Y_i \), we get
\[
\mathcal{F}^*(x) \geq \mathcal{F}_1((1 + \delta)x)P(|\sum_{i=1}^{n} Y_i| < \delta x) + \mathcal{F}_2((1 + \delta)x)P(|nA_nX| < \delta x).
\]  
(15)
Furthermore, since for \( \delta \in (0, 1/2) \) we have
\[
\{U + V > x\} \subset \{U > (1 - \delta)x\} \cup \{V > (1 - \delta)x\} \cup \{U > \delta x, V > \delta x\},
\]
it follows that
\[
\mathcal{F}^*(x) \leq \mathcal{F}_1((1 - \delta)x) + \mathcal{F}_2((1 - \delta)x) + \mathcal{F}_1(\delta x)\mathcal{F}_2(\delta x).
\]  
(16)
Relation (13) is then obtained by dividing both sides in (15) and (16) by \( \mathcal{F}_1(\lambda_n) \), and inverting.

The lower bound of \( \mathcal{F}^*(x) \) consists of two parts. The first part is
\[
\lim_{n \to \infty} \frac{\mathcal{F}_1((1 + \delta)x\lambda_n)}{\mathcal{F}_1(\lambda_n)} P(|\sum_{i=1}^{n} Y_i| < \delta \lambda_n x) = \lim_{n \to \infty} \frac{\mathcal{F}_1((1 + \delta)x\lambda_n)}{\mathcal{F}_1(\lambda_n/n)} \frac{\mathcal{F}_1(\lambda_n/n)}{\mathcal{F}_1(\lambda_n)} P(|\sum_{i=1}^{n} Y_i| < \delta \lambda_n x) = x^{-\alpha x} (1 + \delta)^{-\alpha x},
\]
where we have used Lemma 3 and the fact that \( n/\lambda_n \to 0 \), as \( n \to \infty \), i.e. \( \lambda_n \) is in the large deviation region, which implies that
\[
\lim_{n \to \infty} P(|\sum_{i=1}^{n} Y_i| < \delta \lambda_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} P(|nA_nX| < \delta \lambda_n) = 1.
\]
The second part is
\[
\lim_{n \to \infty} \frac{F_2((1 + \delta)\lambda_n x)}{F_1(\lambda_n)} P(|nA_nX| < \delta\lambda_n x)
\]
\[
= \lim_{n \to \infty} \frac{F_2((1 + \delta)\lambda_n x)}{F(X|)(\lambda_n/n)} \frac{F(X|)(\lambda_n/n)}{F_1(\lambda_n)} P(|nA_nX| < \delta\lambda_n x)
\]
\[
= Q\left((1 + \delta)x\right)^{-\alpha_Y} (\mu(A^{-1}(1, \infty)))^{-1},
\]
using assumption (12) and Lemma 3.

The upper bound of \(F^*(x)\) is treated similarly, although it consists of three parts. The first part is treated using Lemma 3 as above. The second part is
\[
\lim_{n \to \infty} \frac{F_2((1 - \delta)\lambda_n x)}{F_1(\lambda_n)}
\]
\[
= \lim_{n \to \infty} \frac{F_2((1 - \delta)\lambda_n x)F(X|)(\lambda_n/n)}{F(X|)(\lambda_n/n)} \frac{F(X|)(\lambda_n/n)}{F_1(\lambda_n)}
\]
\[
= Q\left((1 - \delta)x\right)^{-\alpha_Y} (\mu(A^{-1}(1, \infty)))^{-1}.
\]

The third and last part is
\[
\lim_{n \to \infty} \frac{F_1(\delta\lambda_n z)}{F_1(\lambda_n z)} F_2(\delta\lambda_n) = 0.
\]

Hence, with \(\mu_A^{-1} = \mu(A^{-1}(1, \infty))\), it follows that
\[
\frac{1}{(1 - \delta)z^{-\alpha_X} + Q/\mu_A^{-1}(1 - \delta)x^{-\alpha_Y}} \leq \liminf_{n \to \infty} \frac{F_1(\lambda_n)}{F(\lambda_n x)}
\]
\[
\leq \limsup_{n \to \infty} \frac{F_1(\lambda_n)}{F(\lambda_n x)} \leq \frac{1}{(1 + \delta)z^{-\alpha_X} + Q/\mu_A^{-1}(1 + \delta)x^{-\alpha_Y}}.
\]

Letting \(\delta \to 0\) proves (13). Relation (14) is shown analogously.

Proof of Theorem 1. We only derive relation (8), the other relations are proved in a similar fashion. First, we compute \(Q\) in (12). This gives us the sequence \(\lambda_n\) via the tail indices. We then apply Lemma 4 to obtain the results.
We have, with \( F_2(\lambda_n x) = P(\sum_{i=1}^{n} \varepsilon_i > \lambda_n x) \),
\[
\frac{F_2(\lambda_n x)}{F_{|F|}(\frac{\lambda_n}{n})} = \frac{F_2(\lambda_n x)}{n F_2(\lambda_n)} \frac{n F_2(\lambda_n)}{F_{|F|}(\frac{\lambda_n}{n})} \frac{L_1[(\lambda_n n)^{\theta_1}]}{L_1[(\lambda_n n)^{\theta_2}]} n \lambda_n^{-\alpha} \frac{L_1[(\lambda_n n)^{\theta_3}]}{L_1[(\lambda_n n)^{\theta_4}]}.
\]

From (5) we get \( \lim_{n \rightarrow \infty} I_1 = px^{-\alpha} \) and, by assumption, \( \lim_{n \rightarrow \infty} I_2 = C \). For simplicity, we restrict ourselves to the case \( I_3 = 1 \). This condition gives the expression for \( \lambda_n \). We then have \( Q = pC \). Applying Lemma 4 we obtain, with \( \mu_{L^{-1}} = \mu(L^{-1}(1, \infty)) \),
\[
\lim_{n \rightarrow \infty} \frac{F_2(\lambda_n x)}{F_{|F|}(\frac{\lambda_n}{n})} = \lim_{n \rightarrow \infty} \frac{F'(\lambda_n x)}{F'(\frac{\lambda_n}{n})} \frac{F_1(\lambda_n)}{F_{|F|}(\frac{\lambda_n}{n})}
= (x^{-\alpha} + Qx^{-\alpha}/\mu_{L^{-1}}) \mu_{L^{-1}} = \mu_{L^{-1}}x^{-\alpha} + px^{-\alpha} C,
\]
and we arrive at relation (8).

\( \square \)

**Proof of Corollary 1.** Considering equation (12), we have
\[
\frac{F_2(\lambda_n)}{F_X(\lambda_n)} = \frac{P(\sum_{i=1}^{n} Y_i > \lambda_n)}{P(X > \lambda_n)} \sim e^{-g(\lambda_n/n)} n \lambda_n^{-\alpha},
\]
so that
\[
\log Q = \lim_{n \rightarrow \infty} g(\lambda_n/n) + \log n - \alpha \log \lambda_n = \infty.
\]
Hence, using equation (14) we obtain the result. \( \square \)

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