Study of new class of $q$-fractional derivative and its properties

M. Momenzadeh, S. Norouzpoor
Near East University
Lefkosa, TRNC, Mersin 10, Turkey
Email: mohammad.momenzadeh@neu.edu.tr
setareh.norouzpoor@neu.edu.tr

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Abstract

There are several approaches to the fractional differential operator. Generalized $q$-fractional difference operator was defined in the aid of $q$-iterated Cauchy integral and $q$-calculus techniques. We introduce Caputo type derivative related to this operator and some properties of this operator as boundness are investigated. Moreover, related $q$-difference equation is discussed and in the aid of concavity of operator, existence and uniqueness are studied.

1 Introduction

Since 1695, that a letter related to fractional derivative was written, fractional calculus has created. There are several types of fractional integral and derivative operators that arise from different aspects. Recent applications of fractional differential equations in explaining natural phenomena, motivate more and more scientists to work in this area. One approach to fractional integral operator is using Cauchy integral. Authors in [9] used $q$-calculus techniques to develop $q$-fractional difference equations. In this paper, we state some definitions and concepts of $q$-calculus. Then we discuss about the concave operator on Banach space and theorem of fixed point based on these conditions is written. Some properties of $q$-fractional derivative and integral operators are discussed next, boundness and condition on operators in case $\alpha \in (0, 1)$ are explained. General case can be reached easily by using induction and similar results hold. In the first section, let us introduce some familiar concepts of $q$-calculus. Most of these definitions and concepts are available in [2] and [4]. We use $[n]_q!$ as a $q$–analogue of any complex number. Naturally, we can define $[n]_q!$ as

$$[a]_q = \frac{1 - q^a}{1 - q} \quad (q \neq 1); \quad [0]_q! = 1; \quad [n]_q! = [n]_q! [n - 1]_q \quad n \in \mathbb{N}, \quad a \in \mathbb{C}.$$ 

The $q$-shifted factorial and $q$-polynomial coefficient are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N},$$

$$(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - q^j a), \quad |q| < 1, \quad a \in \mathbb{C}.$$ 

$$\left( \begin{array}{c} n \\ k \end{array} \right)_q = \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k},$$
Let for some $0 \leq \alpha < 1$, the function $|f(x)x^\alpha|$ is bounded on the interval $(0, A]$, then Jakson integral defines as \[2\]

$$
\int f(x)d_qx = (1-q)x\sum_{i=0}^{\infty} q^i f(q^i x)
$$

converges to a function $F(x)$ on $(0, A]$, which is a $q$–antiderivative of $f(x)$. Suppose $0 < a < b$, the definite $q$–integral is defined as

$$
\int_{a}^{b} f(x)d_qx = (1-q)b\sum_{i=0}^{\infty} q^i f(q^ib) - \int_{a}^{b} f(x)d_qx
$$

Let $C_q^n[a, b]$ denotes the space of all continues functions with continuous $q$-derivatives up to order $n - 1$ on the interval $[a, b]$. Associated norm function of $C_q^n[a, b]$ is defined by \[5\]

$$
\|f\| = \sum_{i=0}^{n-1} \max_{a \leq x \leq b} |(D_q^i f)(x)|, f \in C_q^n[a, b]
$$

Generalize $q$-exponent expression is defined \[9\]

$$(x - y)^{(\alpha)} = \prod_{k=0}^{\infty} \left( \frac{x - y (q^k)^{\alpha}}{x - y (q^k)^{\alpha + \alpha}} \right) = \frac{x^\alpha (\frac{x}{q}; q^\alpha)_{\infty}}{(q^{\alpha+1}; q^\alpha)_{\infty}}$$

where the normal definition can be expressed as \[8\] \[7\]

$$
(x - a)^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty} \left( \frac{1 - a q^k}{1 - a q^{k+\alpha}} \right) = \frac{(\frac{x}{q} - \lambda)^{\alpha}}{(q^{\alpha+1}; q^\alpha)_{\infty}}$$

$q$-derivative of this expression respect to $x$ and $y$ can be written respectively as

$$
xD_q\left[(x^p - y^p)^{(\alpha)}\right]_q = x^{p-1} [p\alpha]_q (x^p - y^p)^{(\alpha-1)}
$$

$$
xD_q\left[(x^p - y^p)^{(\alpha)}\right]_q = -y^{p-1} [p\alpha]_q (x^p - (yq)^p)^{(\alpha-1)}
$$

We can express the $q$-Gamma function by using this definition as \[3\]

$$
\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}
$$

The useful lemma was proved in \[9\], which express following integral for $\alpha$ and $\lambda > -1$

$$
\int_{a}^{b} t^{p-1}(x^p - (qt)^p)^{(\alpha-1)}(t^p - a^p)^{(\lambda)}_q \, dt = \frac{1}{[\lambda]_q} \left( \frac{\Gamma_{q^p}(\alpha)\Gamma_{q^p}(\lambda - 1)}{\Gamma_{q^p}(\alpha + \lambda - 1)} \right) \left( (x^p - a^p)^{(\alpha+\lambda)}_q \right)
$$

2
There are several approaches to fractional differential operators. One of demonstration methods of fractional differential equation is using the iterated Cauchy integrals. In [9], authors calculated

\[
\int_0^x (t_1)^{p-1} d_q t_1 \int_0^t_1 (t_2)^{p-1} d_q t_2 \ldots \int_0^{t_{n-1}} (t_n)^{p-1} f(t_n) dt_n = \frac{1}{\prod_{k=1}^{n-1} [kp]_q} \int_0^x w^{p-1} f(w) \prod_{k=0}^{n-1} (x^p - (wq)^p q^k) d_q w
\]

In the aid of this calculation, generalized \(q\)-fractional difference integral operator is defined by [9]

\[
J_{p,q}^\alpha (f(x)) = \frac{(1-q)^{\alpha-1}}{(1-q^p)^{(\alpha-1)}} \int_0^x w^{p-1} f(w) (w^p - (wq)^p q^{\alpha-1}) d_q w
\]

\[
e^{\frac{\alpha}{\Gamma_q(\alpha)}} \int_0^x w^{p-1} f(w) (w^p - (wq)^p q^{\alpha-1}) d_q w
\]

Proposition 2 Assume that all \(q\)-Jackson integral in following identification are convergent, then

\[
(\text{c}D_{a+}^\alpha)_{p,q} f(x) = (\text{D}_{a+}^\alpha)_{p,q} f(x) - (\text{D}_{a+}^\alpha)_{p,q} f(a) = (\text{D}_{a+}^\alpha)_{p,q} f(x) - \frac{[f(0)]}{\Gamma_q(1-\alpha)} (x^p - a^p q^{\alpha})
\]

In the next proposition we make the form of Aputo derivative easier:

**Definition 1** Let \(0 < a < b < \infty\), \(f : [a, b] \to R\) be a \(q\)-integrable function, and \(\alpha \in (0, 1)\) and \(p > 0\) two fixed reals. The Caputo type \(q\)-fractional derivatives of order \(\alpha\) are defined by

\[
(\text{c}D_{a+}^\alpha)_{p,q} f(t) = (\text{D}_{a+}^\alpha)_{p,q} f(t) - f(a)
\]

\[
= \frac{[f(0)]}{\Gamma_q(1-\alpha)} (x^p - a^p q^{\alpha})
\]
Proof. Use definition (1) and identification (4) and q-integral by part to reach
\[
(^c D_{a^+, p, q}^\alpha f)(x) = \frac{[p]_q^\alpha}{[\rho(1 - \alpha)]_q^\rho(1 - \alpha)} \int_a^x [D_q f(w)] (x^p - w^p)^{(1-\alpha)} \, dw
\]
Here, we used the property that is inspired by definition of Caputo type derivative. Now apply (3) to reach the relation.

**Corollary 3** The direct consequence of this proposition is the following identity
\[
(^c D_{a^+, p, q}^\alpha f)(x) = J_{a^+, p, q}^{1-\alpha} \left( w^{1-p} D_q (f) \right)(x)
\]
Which shows the invers property of Caputo unification q-fractional difference operator.

\[\blacksquare\]

2 Concave operator and Fixed point Theorem

In this section, we introduce some concepts of Banach space and operator defined on this space. Actually, a cone is defined in this space to make a partial order for defined functions and in the aid of this definition fixed point theorem is investigated. Let us start by defining Normal cone and concavity of operator

**Definition 4** Let \((E, \| \cdot \|)\) be a real Banach space, then nonempty closed convex set \(P \subset E\) is called a Cone, if the following conditions hold true:
1. For \(x \in P\) and \(r \geq 0\) as scalar, \(rx \in P\).
2. If \(x \in P\) and \(-x \in P\) then \(x\) should be zero element of \((E, \| \cdot \|)\) which we denote it by \(\theta\).

Generalized concave operator is an operator which satisfy \(A_2\) in following properties
\(A_1\) \(T : P \to P\) is increasing in \(P\).
\(A_2\) For \(x \in P\) and \(t \in (0, 1)\), there exist \(\varphi(t) \in (t, 1]\) with respect to \(t\) such that \(T(tx) \geq \varphi(t)T(x)\).

These conditions warranty the fixed point value for given operator. We recall this proposition as follow

**Proposition 5** Assume that the Cone \(P\) is normal, operator \(T : P \to P\) is satisfied \(A_1\) and \(A_2\). Moreover, \(x_0 \in P\) implies that \(T(h) + x_0 \in P_h\), then operator equation \(x = T(x) + x_0\) has a unique solution in \(P_h\). In addition, there exist \(u_0, v_0 \in P_h\) such that fixed point \(x^* \in \{u_0, v_0\}\) and for any starting point \(x_0 \in P\), constructing successively the sequence \(x_{n+1} = T(x_n)\) tends to the fixed point when \(n \to \infty\).

In this stage, we consider 3 conditions that verify concavity of given q-difference operator to reach fixed point theorem. For instance, assume following conditions:
- \(C_1\) Assume that \(f(t, u(t))\) be two variables continuous function from \([a, b] \times \mathbb{R}\) to real line. In addition, assume that \(f(t, u(t))\) be increasing function respect to the second variable, means \(f(t, u(t)) \leq f(t, v(t))\) where \(-\infty \leq u(t) \leq v(t) \leq \infty\).
- \(C_2\) Let \(0 < \lambda < 1\) be a constant and \(0 < y\), then there exist \(\varphi(\lambda) > \lambda\) such that
  \[
  f(t, \lambda x + (\lambda - 1)y) \geq \varphi(\lambda)f(t, x).
  \]
- \(C_3\) Last condition is positivity of \(f(t, 0)\), means \(f(t, 0) > 0\), specifically \(f(t, 0) \neq 0\) for all possible \(a \leq t \leq b\).

In [11], fixed point theorem based on these conditions were investigated and authors in [10] used this properties to discusse about positive solutions of nonlinear operator equations. In fact, there are several fixed point theorems with different conditions that leads to existence and uniqueness of q-difference equation. These conditions prepare the situations to approach uniqueness of solution as well. Moreover, succesive approximation iteration can be studied in terms of these conditions.
3 Some Properties of Integral and Derivative Operator

In this section, some properties of Caputo-type derivative and integral operator are studied. These properties lead to solving related $q$-fractional difference equation. First, we should state the boundedness of $q$–fractional integral operator.

**Proposition 6** Defined $q$-fractional integral operator $J_{a^+,p,q}^\alpha$ is linear and bounded from $C_q[a,b]$ to $C_q[a,b].$ Means

$$\|J_{a^+,p,q}^\alpha(f)(x)\| \leq A_{p,q,\alpha} \|f(x)\|$$

(15)

**Proof.** Assume that $f \in C_q[a,b],$ simple calculation in the aid of (3) shows that

$$\|J_{a^+,p,q}^\alpha(f)(x)\| \leq \|f(x)\| \left\| \left(\frac{[p]_q}{\Gamma_{p}(\alpha + 1)}(x^p - a^p)\right)^{\alpha}\right\|$$

Linearity is obvious.

**Remark 7** In the aid of Corollary (1), we can demonstrate Caputo-type derivative by integral operator, so we can see that for $f \in C_q^2[a,b],$ $(cD_{a^+,p,q}^\alpha f)(x)$ is bounded. We can see that

$$\left\| (cD_{a^+,p,q}^\alpha f)(x) \right\| \leq \left( \max_{a \leq w \leq x} \left| w^{1-p}D_q f(w) \right| \right) \frac{\left(\frac{[p]_q}{\Gamma_{p}(2 - \alpha)}(x^p - (aq^{-1})^p)^{(1-\alpha)}\right)}{\Gamma_{p}(\alpha + 1)}$$

Since $f \in C_q^2[a,b],$ we can see that $\|f\| = \max_{a \leq x \leq b} |(D_q f)(x)| + \max_{a \leq x \leq b} |(f)(x)| < \infty$ and operator is bounded.

Let us consider the inverse operator of integral operator, that we introduced it before. In the next proposition, we show how the Caputo type derivative can be operate on integral operator. Semi-group property of integral operator is investigated[9].

**Proposition 8** Let $0 < a < b < \infty$ and $f \in C_q^2[a,b]$ also assume that $\alpha \in (0,1)$ and $p > 0$ be two fixed reals such that derivative and integral operator are defined, then following identities are true:

$$cD_{a^+,p,q}^\alpha \left((J_{a^+,p,q}^\alpha f)(x)\right) = f(x)$$

(16)

$$J_{a^+,p,q}^\alpha \left((cD_{a^+,p,q}^\alpha f)(x)\right) = f(x) - f(a)$$

(17)

**Proof.** Proof of the first part is based on interchanging of integral and applying the definitions. This is proved at[22] for derivative operator and the procedure is similar for this operator. We prove the second identity by using corollary (3) and semi group property of integral operator, so

$$J_{a^+,p,q}^\alpha \left((cD_{a^+,p,q}^\alpha f)(x)\right) = J_{a^+,p,q}^{1-\alpha} \left(w^{1-p}D_q f(w)\right)(x) = J_{a^+,p,q}^1 \left(w^{1-p}D_q f\right)(x) = \int_0^x D_q f(w) dw = f(x) - f(a)$$


4 Cauchy problem of general q-fractional operator

In this section, Cauchy problem related to introduced Caputo type derivative is investigated. We construct solution of this difference equation in the aid of discussed properties in last section. Moreover we apply a fixed point theorem based on property of operator to guarantee existence and uniqueness of this difference equation. We start by direct solution of q-difference equation in following lemma:

Lemma 9 let \(0 < \zeta < 1\) and \(\alpha \in (0, 1)\), if \(f(t)\) is a continuous function on \((0, 1)\) and \(u \in C^2_q[a, b]\), then solution of the following boundary value problem

\[
(\frac{d}{dt})^\alpha_{a^+, p, q}u(t) = f(t, u(t)) \quad 0 < a < t < b < \infty \\
qu(a) = \zeta
\]

\(\text{can be written as}
\)

\[
u(t) = \zeta + \frac{([p]_q)^{1-\alpha}}{\Gamma_q(\alpha)} \int_a^t w^{p-1} f(w, u(w))(p^q - (w^q)^{(\alpha-1)}) dq w
\]

Proof. Apply integral operator in both sides of (18) and use (8) and (17) to reach the solution. ■

Following theorem shows the conditions on the operator to verify existence and uniqueness for given q-difference equation. Moreover, successive approximation is introduced and let us to consider numerical solution of this equation. Main theorem of this article determines solution and nature of this approximation.

Theorem 10 Assume that all conditions \(C_1 - C_3\) holds true. Then following q-fractional differential equation problem with given initial values has a unique solution. In addition, following sequence shows successive approximation approach for the solution.

\[
(\frac{d}{dt})^\alpha_{a^+, p, q}u(t) = f(t, u(t)) \quad 0 < a < t < b < \infty \\
qu(a) = \zeta
\]

successive approximation can be written as

\[
u_n(t) = \zeta + \int_a^b G_p(t, wq)f(w, u_{n-1}(w)) dq w
\]

Lemma 11 where we have

\[
G_p(t, w) = \begin{cases} 
\left([p]_q\right)^{1-\alpha} q^{p-1}\Gamma_q(\alpha) & \text{for } a \leq w \leq t \leq b \\
0 & \text{for } a \leq t \leq w \leq b
\end{cases}
\]

Proof. Here, we consider the Banach space as \((C[a, b], ||.||)\) which \(||.||\) is the supreme norm. The cone \(P\) is defined as \(\{ x(t) \in C[a, b] \mid x(t) \geq 0 \}\) which is the standard normal cone. According to proposition (4), solution function is continuous function on \([a, b]\) so it is easy to see that for positive \(p\) as parameter, \(w^{p-1}(p-w)^{(\alpha-1)}\) is positive and belongs to \(P\). Now let us define the operator \(T\) as follow

\[
T(v(t)) = \zeta + \int_a^b G_p(t, wq)f(w, v(w)) dq w
\]

Conditions on function \(f(w, v(w))\) in \(C_1 - C_3\) show that \(T\) is increasing operator. Conditions of fixed point theorem for \(T\) can be easily found and successive approximation resulted from the nature of constructing theorem. ■
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