Lattices freely generated by posets within a variety.

Part II: Finitely generated varieties

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1 Introduction

This article constitutes the second part of an essay dedicated to lattices freely generated by finite posets within a variety. The first part dealt with four "easy" cases, namely the variety of all semilattices, (general) lattices, distributive lattices, and Boolean lattices respectively. Special attention was paid to semilattices with a view to applications in Part II.

In the present Part II we are officially concerned with finitely generated (f.g.) varieties \( V \) of lattices, in the usual sense that all subdirectly irreducible members are finite and, up to isomorphism, there are only finitely many of them. We wrote "officially" because quite a few preliminaries, hopefully interesting in their own right, will have to be digested before we come to f.g. varieties in section 6 and 7.

The following problem is posed in section 2. Given finite lattices \( L_1, \ldots, L_t \), how much additional information about a subdirect product \( L \) thereof is needed in order to compute \( L \)? Specifically, let \( \pi_i : L \to L_i \) be the \( i \)-th projection map and \( \sigma_i : L_i \to L \) the corresponding smallest pre-image map. It turns out that the knowledge of the connection maps \( \alpha_{i,j} := \pi_i \circ \sigma_j \) from \( L_j \) to \( L_i \) \((1 \leq i, j \leq t)\) is sufficient, even if the \( \pi_i \)'s and \( \sigma_i \)'s themselves are unknown. Where one would get the connection maps from, will be seen in section 6.

When computing \( L \) from the connection maps it pays to replace the seemingly natural set \( J(L) \) of join irreducibles by the larger scaffolding \( G(L) \), which is defined as the union of the sets \( \sigma_i(L_i \setminus \{0\}) \). Following [3] we show in section 3 that the \( \lor \)-semilattice freely generated by the partial \( \lor \)-semilattice \((G(L), \lor)\) is isomorphic to \( L \setminus \{0\} \). The benefit is that free \( \lor \)-semilattices can be viewed as certain closure systems \( C \) which are amenable to the implication \( n \)-algorithm introduced in Part I (it is fully discussed in [12]). Namely, this algorithm is applicable whenever \( C \) is given by an implicational base \( \Sigma \).

The scaffolding \( G(L) \) contains the join core \( K_\lor(L) \) which in turn contains \( J(L) \). Section 4 investigates \( K_\lor(L) \) when \( L \) is a modular lattice. The view of \( K_\lor(L) \) as linear hypergraph (pioneered in [7]) generalizes the projective geometry view of complemented modular lattices.

Section 5 fine-tunes the implication \( n \)-algorithm to the situation where \( C \) is isomorphic to a modular lattice, and thus consists of all order ideals of a poset which simultaneously are closed with respect to some linear hypergraph.
In section 6 finitely generated varieties $\mathcal{V}$ enter the stage. Let $L$ be the lattice freely generated within $\mathcal{V}$ by some finite poset $(P, \leq)$. The calculation of $L$ is based on two essential ideas.

First, $L$ is a subdirect product with factors $L_i$ from among the finitely many subdirect irreducibles of $\mathcal{V}$. Crucially, since $L$ is free, the connection maps $\alpha_{i,j}$ between the $L_i$’s (section 2) can be calculated in miraculous ways and they yield $G(L)$.

Second, the fact that the partial semilattice $(G(L), \lor)$ freely generates $L$ as a semilattice, makes $L$ a closure system to which the $(A, B)$-algorithm applies.

Section 7 focuses on the variety $\mathcal{V}$ which is generated by the smallest modular nondistributive lattice $M_3$. For most of the 318 posets $P$ with $|P| = 6$ the cardinality of the $\mathcal{V}$-free lattice generated by $P$ was computed in [6].

2 Retrieving a subdirect product from its connection maps

Let $\phi : L \to L_0$ be an epimorphism of lattices such that each $y \in L_0$ has a smallest pre-image $\sigma(y) = \bigwedge\{x \in L : \phi(x) = y\}$. In particular that is the case in our situation where all lattices are finite. It is shown in [3] that $\sigma : L_0 \to L$ is an injective $\lor$-homomorphism.

Let $L \subseteq \prod_{1 \leq i \leq t} L_i$ be a subdirect product of lattices $L_1, \ldots, L_t$. Then all restricted projections $\pi_i : L \to L_i$ have smallest pre-images $\sigma_i : L_i \to L$ and

$$J(L) = \bigcup_{1 \leq i \leq t} \sigma_i(J(L_i)).$$

(1)

This is implicit in [3] and explicitly in [1, Thm 3.4]. Mutatis mutandis the same holds for meet irreducibles and biggest pre-images, but these will not concern us here. Observe that with $\sigma_j$ also $\alpha_{i,j} := \pi_i \circ \sigma_j : L_j \to L_i$ is a $\lor$-homomorphism. One readily checks that

$$\alpha_{i,i} = id, \quad \alpha_{i,j} \circ \alpha_{j,k} \leq \alpha_{i,k}$$

(2)

for all $1 \leq i, j, k \leq t$. Conversely, suppose one is given lattices $L_1, \ldots, L_t$ and any family $\alpha_{i,j} : L_j \to L_i$ of $\lor$-homomorphisms that satisfy (2). Then these connection maps $\alpha_{i,j}$ are induced by a suitable subdirect product as above. Namely, defining $\sigma_i' : L_i \to \prod_{1 \leq i \leq t} L_i$ by

$$\sigma_i'(y) := (\alpha_{j,i}(y))_{1 \leq j \leq t}$$

(3)

the following takes place.
Theorem 1 \[\text{Given a set of connection maps satisfying } \{\text{(1)}\}, \text{there is a unique subdirect product } L \subseteq \prod_{1 \leq i \leq t} L_i \text{ such that the maps } \sigma_i' \text{ in } \{\text{(3)}\} \text{ are the smallest pre-image maps of the projections } \pi_i : L \to L_i \ (1 \leq i \leq t).\]

According to Theorem \[\text{(1)}\] and \[\text{(11)}\], the subdirect product \(L\) can be calculated as the \(\lor\)-subsemilattice of \(\prod_{1 \leq i \leq t} L_i\) generated by \(\bigcup_{1 \leq i \leq t} \sigma_i'(\text{J}(L_i))\).

Example 1 Consider the two lattices \(L_1 = \{a, b, \cdots, n\}\) and \(L_2 = \{0, 2, 3, \cdots, 12, 1\}\) (so 1 is top) which are coupled by the \(\lor\)-homomorphisms \(\alpha_{1,2}\) and \(\alpha_{2,1}\) as indicated. For instance, as required in \[\text{(2)}\], we have
\[
(\alpha_{1,2} \circ \alpha_{2,1})(k) = \alpha_{1,2}(11) = c \leq k.
\]

![Figure 1:](image)

One verifies that:
\[
\begin{align*}
J(L_1) & = \{b, c, d, e, f, h, i\} \\
J(L_2) & = \{2, 3, 4, 5, 6, 8, 9, 11\}
\end{align*}
\]
\[
\begin{align*}
\sigma_1'(b) & = (b, 2) & \sigma_2'(2) & = (a, 2) \\
\sigma_1'(c) & = (c, 3) & \sigma_2'(3) & = (a, 3) \\
\sigma_1'(d) & = (d, 4) & \sigma_2'(4) & = (a, 4) \\
\sigma_1'(e) & = (e, 10) & \sigma_2'(5) & = (b, 5) \\
\sigma_1'(f) & = (f, 11) & \sigma_2'(6) & = (b, 6) \\
\sigma_1'(h) & = (h, 11) & \sigma_2'(8) & = (d, 8) \\
\sigma_1'(i) & = (i, 12) & \sigma_2'(9) & = (d, 9) \\
\sigma_2'(11) & = (c, 11)
\end{align*}
\]

It turns out that here taking suprema of \(\text{pairs}\) (as opposed to triplets, quadruplets, \cdots) of elements of \[\text{(1)}\] suffices to generate the subdirect product \(L \subseteq L_1 \times L_2\).
When generating \( L \) from \( J(L) \), some elements of \( L \), say \((g, 1)\), may be duplicated many times:

\[
(b, 5) \lor (d, 8) = (b, 5) \lor (d, 9) = (b, 6) \lor (d, 8) = (b, 6) \lor (d, 9) \\
= (c, 11) \lor (b, 5) = (c, 11) \lor (b, 6) = (c, 11) \lor (d, 8) = (c, 11) \lor (d, 9) = (g, 1).
\]

This is inefficient and a better way will be approached in section 3.

3 The algorithmic advantage of the scaffolding \( G(L) \) over \( J(L) \)

It is straightforward to design a general purpose algorithm for calculating the subalgebra generated by a given subset of a universal algebra. Using hashing techniques the inefficient regeneration of elements, as in Example \([1]\) can be partly cured. For the case where the universal algebra is a not too large lattice, this approach has been taken (among other methods) in \([1]\). Similarly the authors of \([10]\) proceed to compute the lattice \( L \) of all submodules of a module. We note that some of the theory developed in \([7]\) is rediscovered.

In contrast, our philosophy is the following. For \( a \in L \) put \( J(a) := \{ p \in J(L) \mid p \leq a \} \). We shall identify \( L \) with the isomorphic closure system \( \mathcal{C} = \{ J(x) \mid x \in L \} \) and seek some suitable implicational base \( \Sigma \) for \( \mathcal{C} \). The point is that with the implication \( n \)-algorithm of Part I, \( \mathcal{C} \) can be computed faster as the set \( \mathcal{C}(\Sigma) \) of all \( \Sigma \)-closed subsets of \( J(L) \).
One way to come up with such a $\Sigma$ is as follows. For any lattice $L$, let $R \subseteq L$ be such that it contains the join core $K_\vee(L)$. Recall from Part I, section 5, that an implicational base $\Sigma$ of $\mathcal{C} \cong L$ is then obtained by collecting all the implications $A \to J(\vee A)$ where $A$ ranges over those subsets of $J(L)$ for which $\bigvee A \in R$. In particular, $\Sigma$ contains all the implications $\{p\} \to J(p)$ with $p$ ranging over $J(L)$.

In this section we shall exhibit a convenient set $R = G(L)$ by merely exploiting that $L$ is subdirectly reducible. Thus let $L$ be a subdirect product of lattices $L_1, \cdots, L_t$ where, additionally to section 2, the $L_i$'s must be subdirectly irreducible.

Akin to (11) we define the scaffolding ("Gerüst" in [3]) of $L$ as

$$G(L) := \bigcup_{1 \leq i \leq t} \sigma_i(L_i \setminus \{\emptyset\}).$$

Despite appearances, $G(L)$ is not dependent on the particular subdirect decomposition of $L$. For modular $L$ this will be shown in section 4. As seen in Part I, as a subset of $L$ it automatically becomes a partial semilattice $(G(L), \vee)$. It turns out that $(G(L), \vee)$ freely generates $L$, that is,

$$F_\vee \left( G(L), \vee \right) \simeq L \setminus \{\emptyset\}.$$  \hspace{1cm} (6)

Here comes the proof of (10), which essentially is a translation of [3, 1.3]:

For all $x \in L$ the full ideal $\varepsilon(x) := \{a \in G(L) : a \leq x\}$ contains $J(x)$ because of (11) and (5). Therefore the map $\varepsilon : L \to F_\vee \left( G(L), \vee \right)$ satisfies

$$(\forall x, z \in L) \quad (x \leq z \iff \varepsilon(x) \subseteq \varepsilon(z)).$$

In order to show that $\varepsilon$ is onto and hence an isomorphism of semilattices, consider $\kappa_i(x) := \sigma_i(\phi_i(x))$. Then $\kappa_i : L \to L$ is a kernel operator, i.e. is anti-extensive, idempotent, and preserves suprema. Fix $A \in F_\vee (G(L), \vee)$ and any $a \in \varepsilon(\vee A) \supseteq A$. We need to show that $a \in A$. For some $i \in \{1, \cdots, t\}$, we have $a = \kappa_i(a) \leq \kappa_i(\vee A) = \vee \kappa_i(A)$. Thus $\vee \kappa_i(A) \in G(L)$, and $\kappa_i(A) \subseteq A$ since $A$ is hereditary. Therefore $\vee \kappa_i(A) \in A$ by definition of $\vee$-ideal. Because $A$ is hereditary, $a \in A$.

4 The join core of modular lattices

As in Part I the natural closure operator associated to a lattice $L$ maps $X \subseteq J(L)$ to $\overline{X} := J(\vee X)$. Recall that the set $E(L)$ of essential elements consists of those $x \in L$ which (alias $J(x)$) contain a proper quasiclosed generating set. Further the join core $K_\vee(L)$ is $E(L) \cup A(L)$ with $A(L) \subseteq J(L)$ the set of atoms of $L$. From (10) and Theorem 2 in Part I follows that the scaffolding comprises $K_\vee(L)$. In fact, according to [9, Thm. 7] one has

$$K_\vee(L) = \bigcup_{1 \leq i \leq t} \sigma_i(K_\vee(L_i))$$

(7)
where \( L \) is any subdirect product of subdirectly irreducible lattices \( L_i \) \((1 \leq i \leq t)\). The proof of (7) involves an application of Duquenne’s multi-purpose \( M_{3\times N_5}\)-lemma.

Here we show that for modular lattices \( K_\nu(L) \) is readily found. Along the way a proof of (7) in the modular case unfolds. In fact the union in (7) turns out to be disjoint. Lemma 1 and Theorem 2 below are based on [3, p.58]; we use the opportunity to mend some minor typos and expand some arguments.

**Lemma 1** ([3]): Let \( \phi : L \to L_0 \) be a lattice epimorphism with smallest pre-images \( \sigma : L_0 \to L \), and fix a nonzero element of type \( v = \sigma \phi(v) \). Let \( v/w \) be any prime quotient and let \( r/s \) be a prime (i.e. covering) quotient in \( L_0 \) which is projective to the prime quotient \( \phi v/\phi w \). Then \( r = \sigma r \) and \( s = \bigvee \{ u \leq r \mid \phi u \leq s \} \) yield a prime quotient \( r/s \) which is projective to \( v/w \).

**Proof:** Whereas \( r = \sigma r \), generally \( s \neq \sigma s \). Rather \( s \) is the greatest element below \( r \) that maps to \( s \). In particular \( r/s \) is a prime quotient (similarly for other quotients to come). One readily verifies that with \( v/w \) also \( \phi v/\phi w \) is a prime quotient. By assumption there are prime quotients \( v_i/w_i \) \((0 \leq i \leq n)\) in \( L_0 \) such that \( v_i/w_i \) is transposed to \( v_{i-1}/w_{i-1} \) \((1 \leq i \leq n)\) and such that \( v_0/w_0 = r/s \) and \( v_n/w_n = \phi v/\phi w \). For all \( 0 \leq i \leq n \) we put \( u_i = v_i - \sigma v_i \) and \( w_i := \bigvee \{ u \in L \mid u \leq u_i, \phi u \leq w_i \} \). Notice that \( v_n/w_n = v/w \) since \( w \) is obviously the largest element below \( v \) with \( \phi \)-image \( \leq w_n \). Ditto, \( v_0/w_0 = r/s \).

In order to see that \( u_i/w_i \) is transposed to \( u_{i-1}/w_{i-1} \) \((1 \leq i \leq n)\), assume w.l.o.g. that \( v_{i-1}/w_{i-1} \) transposes up to \( v_i/w_i \), that is, \( v_{i-1} \land w_i = w_{i-1} \) and \( v_{i-1} \lor w_i = v_i \). We conclude that \( \phi(v_{i-1}\land w_i) = \phi(v_{i-1})\land\phi(w_i) = v_{i-1}\land w_i = w_{i-1} \). Hence \( \bigvee_{i-1} \land w_i = v_{i-1} \) by definition of \( u_{i-1} \). The inequality \( > \) follows from \( w_{i-1} \leq u_{i-1} \) and (clearly) \( w_{i-1} \leq w_i \). Recalling that \( \sigma \) is a join homomorphism it follows from \( v_{i-1} \lor w_i = v_i \) that \( v_{i-1} \lor \sigma w_i = v_i \). Hence \( v_{i-1} \lor w_i = v_i \).

Call an element \( v \neq 0 \) of a lattice \( L \) **sub-irreducible** if all prime quotients \( v/w \) are mutually projective. For instance each join irreducible is sub-irreducible.

**Theorem 2** ([3]): Let \( L \) be a modular lattice. Then \( G(L) \) consists of all sub-irreducible elements. The lattices \( L_i \) \((1 \leq i \leq t)\) in (3) are unique up to isomorphism and the union in (3) is disjoint.

**Proof:** Let us fix any sub-irreducible \( x \in L \) and argue why we must have \( x \in \sigma_i(L_i \setminus \{0\}) \) for some \( i \). Assume to the contrary that \( x \notin \sigma_i(L_i \setminus \{0\}) \) for all \( i \). Then \( x \) is not the smallest element of any \( \theta_i \)-class (where \( \theta_i \) is the kernel of \( L \to L_i \)), and so for each \( i \) there is some lower cover \( y \prec x \) which is in the same \( \theta_i \)-class as \( x \). It cannot be that for all \( i \) all \( y \prec x \) are in the same \( \theta_i \)-class as \( x \) because then \( \bigwedge \{ \theta_i \mid 1 \leq i \leq t \} \neq 0 \), contradicting the fact that \( L \) is a subdirect product of the \( L_i \)'s. Hence there is an \( i \) and lower covers \( y_1, y_2 \) of \( x \) such that \( (x, y_1) \in \theta_i \) but \( (x, y_2) \notin \theta_i \). Yet this cannot be since \( x/y_1 \) by assumption is projective to \( x/y_2 \). It follows that \( \{ x \in L \mid x \text{ is sub-irreducible} \} \) is a subset of \( G(L) \). So far, modularity was not used.
Conversely, pick \( v \in G(L) \), say \( v = \sigma_i \phi(v) \). Let \( v/w \) be a prime quotient and let \( \phi v/\phi w \) be its image in \( L_i \). Since \( L_i \) is modular, it is simple, and so any fixed prime quotient \( r/s \) in \( L_i \) will be projective to \( \phi v/\phi w \). By Lemma 1 there is a prime quotient \( r/s \) that is projective to \( v/w \). Crucially, since \( r/s \) depends on \( r/s \) (and not on \( v/w \)), \( r/s \) is projective to any other prime quotient \( v/w' \) as well. Hence \( v \) is sub-irreducible.

It is well known that \( L \) being modular the projectivity classes of prime quotients correspond bijectively to the simple factors \( L_i (1 \leq i \leq t) \). This establishes the disjointness of the union in (5).

In order to succinctly describe the subset \( K_\psi(L) \) of \( G(L) \) in the modular case, call \( x \in L \) a line top if \( x \) has \( n \geq 3 \) lower covers \( x_i \) and their meet \( x \) is a lower cover of each \( x_i \). So far \( L \) could be any finite lattice. Since all prime quotients of the interval sublattice \([x, x]\) are clearly mutually projective, each line top \( x \) is in \( G(L) \). Actually nonclosed quasiclosed generating sets of \( J(x) \) are easy to find [13, p.156], and so even \( x \in E(L) \). For modular lattices \( L \) the line tops are the only reducible elements. This was first shown in [9, Thm.9], other proofs are mentioned in [13, p.157]. Hence it follows from

\[
K_\psi(L) = J(L) \cup \{ x \in L \mid x \text{ is a line top} \} \quad (8)
\]

for each finite modular lattice. As in (1) the line tops occurring in any of the simple factors \( L_i \) match the line tops of \( L \), and by Theorem 2 the union in (7) is disjoint. An at least 3-element subset \( l \subseteq J(L) \) maximal with the property that \( p \lor q = \lor l \) for all distinct \( p, q \in l \), is called a line of \( L \). It is shown in (9) that the line tops \( x \) of \( L \) are exactly the elements of type \( x = \lor l \) with \( l \) a line. One can have \( x = \lor l_1 = \lor l_2 \) for \( l_1 \neq l_2 \). Furthermore \( |l_1 \cap l_2| \leq 1 \) for all lines \( l_1 \neq l_2 \). A collection \( \Lambda \) of lines for which each line top \( x \) contains exactly one \( l \in \Lambda \) with \( \lor l = x \), is called a base of lines.

Example 2 Consider the lattice \( L \) in Example 1 which happens to be modular and which is a subdirect product of the simple lattices \( L_1, L_2 \) in Fig 1. The line tops of \( L_1 \) are \( g, k, n \). The only line for \( g \) is \( l = l(g) = \{ b, c, d \} \). The line top \( k \) houses the two lines \( \{ f, d, h \} \) and \( \{ f, d, h \} \). Let us pick, say, \( l(k) = \{ f, d, h \} \). Similarly, say \( l(n) = \{ e, h, i \} \). The resulting base of lines is \( \Lambda_1 = \{ l(g), l(k), l(n) \} \). In the same way, one possible base of lines for \( L_2 \) is \( \Lambda_2 = \{ l(7), l(10), l(12), l(1) \} \) (e.g. \( l(10) = \{ 4, 5, 6 \} \)). They are shown in Fig 3. The corresponding base of lines for \( L \) is \( \Lambda := \Lambda_1 \cup \Lambda_2 \), where \( \Lambda_i := \sigma_i(\Lambda'_i) \) is defined in the obvious way (see Fig 3). Generally the number of connected components of any base of lines of a modular lattice equals the number of its simple factors.

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*In the nonmodular case \( L_i \) is merely subdirectly irreducible. Finding \( r/s \) becomes more subtle and also weak projectivities must be dealt with [3, p.58].

† This crisp name has been recently introduced by C. Herrmann; in [1] and elsewhere line tops were called \( M_n \)-elements.
For instance, the pre-image of the line top 12 $\in L_2$, which can be evaluated as

$$\sigma_2(12) = \sigma_2(2 \lor 9) = \sigma_2(2) \lor \sigma_2(9) = (a, 2) \lor (d, 9) = (d, 12),$$

is a line top of $L$ and hence in $K_\vee(L)$. From Fig.2 one sees (after a while) that the element $(j, 10)$ is sub-irreducible, i.e. the quotient $(j, 10)/(e, 10)$ is projective to $(j, 10)/(g, 10)$. According to Theorem 2 this forces $(j, 10) \in G(L)$. Indeed, one checks that $(j, 10) = \sigma_1(j) \in \sigma_1(L_1 \setminus \{0\})$. Clearly $(j, 10) \notin K_\vee(L)$ since $(j, 10)$ is neither join-irreducible nor a line top. Notice that e.g. $(b, 7) \in L \setminus G(L)$.

5 Algorithmic details in the modular case

Let $L$ be any modular lattice with a base of lines $\Lambda$. As seen in section 4, the join core $R = K_\vee(L)$ consists of all the join irreducibles and the line tops $\overline{\Lambda}$ corresponding to the lines $l \in \Lambda$. Therefore (section 3) an implicational base $\Sigma$ of the closure system $\mathcal{C} = \{J(x) \mid x \in L\}$ isomorphic to $L$ is obtained by taking all implications $A \rightarrow J(\bigvee A)$ where $A$ is such that $\bigvee A \in R$. Here, besides $\{p\} \rightarrow J(p)$ ($p \in J(L)$), it suffices to take the implications $A \rightarrow J(\bigvee A)$ of type $l \rightarrow J(\overline{l})$ with $l \in \Lambda$. The purpose of section 5 is to exploit this special type of $\Sigma$ in order to speed up the implication $n$-algorithm presented in Part I. The section is quite technical and may be skipped without loss of continuity.

To fix ideas, let us return to the lattice $L$ of Example 2. Put $J_i := \sigma_i(J(L_i))$ for $i = 1, 2$. The family $J[\Lambda]$ of all $\Lambda$-closed subsets $Z \subseteq J$ consists exactly of the sets $Z = X \uplus Y$ where $X$ and $Y$ range over the accordingly defined families $J_1[\Lambda_1]$, respectively $J_2[\Lambda_2]$. Hence it makes sense to determine $J_1[\Lambda_1]$ and $J_2[\Lambda_2]$ apart, and afterwards worry to weave in all implications $p \rightarrow J(p)$. 
We identify subsets of $J_1 = \{(b, 2), \ldots, (e, 10)\}$ with their characteristic 0, 1-vectors but besides 0, 1 introduce other symbols $2, l, \delta, \varepsilon$ in order to get multi-valued rows that compactly encode certain families of subsets of $J_1$. For simplicity we write $b2$ for $(b, 2)$ and so forth. For starters, let $l_1 = \{b2, c3, d4\}$ be our first line from $\Lambda_1 = \{l_1, l_2, l_3\}$. The family $J_1[l_1] := \{X \subseteq J_1 : |X \cap l_1| \in 0, 1, 3\}$ of all $l_1$-closed subsets can be represented by the first (multivalued) row below in this (multivalued) context:

| b2 | c3 | d4 | f11 | h11 | i12 | e10 |
|----|----|----|-----|-----|-----|-----|
| l  | l  | l  | 2   | 2   | 2   | 2   |
| \varepsilon | \varepsilon | 0 | 2   | 2   | 2   | 2   |
| \delta | \delta | 1 | 2   | 2   | 2   | 2   |
| \varepsilon | \varepsilon | 0 | \varepsilon | \varepsilon | 2 | 2 |
| \delta | \delta | 1 | \delta | \delta | 2 | 2 |
| \varepsilon | \varepsilon | 0 | 2   | 0   | \varepsilon | \varepsilon |
| \delta | \delta | 1 | 0   | 2   | \varepsilon | \varepsilon |
| \delta | \delta | 1 | 1   | 1   | \delta | \delta |

Table 1:

When we identify subsets with their characteristic vectors, then $l l l$ in the first row is a shorthand for the family $\{000, 100, 010, 001, 111\}$ of all $l_1$-closed subsets of $\{b2, c3, d4\}$, and a symbol 2 at any position means that the corresponding element is free to be present or not. We need two more symbols.

Let $\varepsilon \varepsilon$ be a shorthand for $\{00, 01, 10\}$ ("at most one 1") and $\delta \delta$ a shorthand for $\{00, 11\}$ ("dichotomy": all 1’s or all 0’s). The second and third row encode the fact that $J_1[l_1] = \mathcal{F}_1 \cup \mathcal{F}_2$ where

\[
\mathcal{F}_1 := \{X \in J_1[l_1] : d4 \notin X\} = \{X \subseteq J_1 : |X \cap \{b2, c3\}| \in \{0, 1\}\}
\]

\[
\mathcal{F}_2 := \{X \in J_1[l_1] : d4 \in X\} = \{X \subseteq J_1 : |X \cap \{b2, c3\}| \in \{0, 2\}\}
\]

Consider the next line $l_2 = \{d4, f11, h11\}$ of $\Lambda_1$. The fact that we did split $\mathcal{F}$ with respect to $d4$, which is the intersection of $l_1$ and $l_2$, benefits the imposition of $l_2$. Thus the fourth and fifth row encode all $X \subseteq J_1$ which are $\{l_1, l_2\}$-closed. Splitting each row with respect to $h11$ (the intersection of $l_2$ and $l_3 = \{h11, i12, e10\}$) yields the last four rows. They encode the family $J_1[\Lambda_1]$ of all $\Lambda_1$-closed sets $X \subseteq J_1$.

As seen, since $\sigma$ is injective and join-preserving, the poset $(J_1, \leq)$ induced by $(L, \leq)$ is isomorphic to the poset $(J(L_1), \leq)$ induced by $(L_1, \leq)$. Ditto $(J_2, \leq) \simeq (J_2(L_2), \leq)$. However, $(J(L), \leq)$ features more comparabilities than the disjoint union of $(J_1, \leq)$ and $(J_2, \leq)$:
Our task to generate $L$ as the closure system $\mathcal{C} = \{J(x) \mid x \in L\}$ of all $\Lambda$-closed order ideals $X$ of $(J, \leq)$ amounts to determine those $X = Y \cup Z$ ($Y \in J_1[\Lambda_1], Z \in J_2[\Lambda_2]$) that happen to be order ideals of $(J, \leq)$. For $X = Y \cup Z$ to be an order ideal of $(J, \leq)$ it is necessary (but not sufficient) that $Y$ and $Z$ be order ideals of $(J_1, \leq)$ and $(J_2, \leq)$ respectively. In particular, since $d4 \leq i12$ in $(J_1, \leq)$ and since the fourth row from below in table I has 0 at position d4, the i12-component $\varepsilon_2$ can safely be switched to 0. Accordingly the other $\varepsilon_2$ turns to be 2. (If there had been 1 at position i12, then the row must have been deleted.) This yields $r_1$ below. Together with three similarly obtained rows we get a certain subset $J_1[\Lambda_1]$ of $J_1[\Lambda_1]$:
we impose all implications \( \{ p \} \rightarrow J(p) \) with the \((a, B)\)-algorithm from section 2 in Part I. Of course, instead of \( J(p) \) it suffices to take the smaller set of lower covers of \( p \). For singleton premises (as in \( \{ a \} \rightarrow B \)) the implication \( n \)-algorithm can be streamlined to the \((a, B)\)-algorithm discussed in [12].

Consider e.g.

\[
    r_1 s_5 = (\varepsilon, \varepsilon, 0, 2, 0, 0, 2, 1, 1, 1, 0, 0, 2, 0, 0)
\]

which we shall work from left to right. The implication \( \{ b2 \} \rightarrow \{ a2 \} \) holds already since each \( X \in r_1 s_5 \) has \( a2 \in X \). Similarly for \( \{ c3 \} \rightarrow \{ a3 \} \) and \( \{ d4 \} \rightarrow \{ a4 \} \). As to

\footnote{We can read \( d4 < d9 \) from Figure 3, the algorithm "knows" it from the given implication \( \{ d9 \} \rightarrow J(d9) \).}

The following table lists the order ideals in \( J_1 \) of \( A_1 \) and \( J_2 \) of \( A_2 \):

|   | \( b2 \) | \( c3 \) | \( d4 \) | \( f11 \) | \( h11 \) | \( i12 \) | \( e10 \) |
|---|---|---|---|---|---|---|---|
| \( r_1 \) | \( \varepsilon \) | \( \varepsilon \) | 0 | 2 | 0 | 0 | 2 |
| \( r_2 \) | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| \( r_3 \) | \( \delta \) | \( \delta \) | 1 | 0 | 0 | \( \varepsilon \) | \( \varepsilon \) |
| \( r_4 \) | 1 | 1 | 1 | 1 | 1 | \( \delta \) | \( \delta \) |

Table 2: \( J_1^*[A_1] \)

Computing \( J_2[A_2] \) along the same lines, and again filling in the "immediate" 0’s and 1’s forced by the poset \( (J, \leq) \), yields this subset \( J_2[A_2] \) of \( J_2[A_2] \) (where \( \delta\delta\delta \) is "000 or 111"):

|   | \( a2 \) | \( a3 \) | \( a4 \) | \( b6 \) | \( b5 \) | \( c11 \) | \( d9 \) | \( d8 \) |
|---|---|---|---|---|---|---|---|---|
| \( s_1 \) | \( \varepsilon \) | \( \varepsilon \) | 0 | 2 | 0 | 2 | 0 | 0 |
| \( s_2 \) | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| \( s_3 \) | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 |
| \( s_4 \) | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| \( s_5 \) | 1 | 1 | 1 | 0 | 0 | 2 | 0 | 0 |
| \( s_6 \) | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| \( s_7 \) | 1 | 1 | 1 | 1 | 1 | \( \delta \) | \( \delta \) | \( \delta \) |

Table 3: \( J_2^*[A_2] \)

Each member of \( \mathcal{C} \simeq L \), i.e. each \( \Lambda \)-closed order ideal \( X \) of \( (J, \leq) \), is of the form \( X = Y \cup Z \) for some \( i \in \{ 1, \ldots, 4 \} \), \( Y \in r_i \) and for some \( j \in \{ 1, \ldots, 7 \} \), \( Z \in s_j \). In order to get these \( X \)'s we first discard the "concatenated" rows \( r_i s_j \) which do not contain any order ideal of \( (J, \leq) \). For instance \( r_1 s_4 \) is of that kind: Because \[ d4 < d9 \], no order ideal \( X \) has 1 at \( d9 \) but 0 at \( d4 \). We say that 1 and 0 clash. As another example, suppose \( X \) was an order ideal contained in \( r_2 s_1 \). From \( c3, h11 \in X \) (see \( r_2 \)) and \( c3 \geq a3 \) and \( h11 \geq a2 \) follows \( a3, a2 \in X \). But this cannot be since the \( \varepsilon\varepsilon \) in \( s_1 \) forces \( |X \cap \{ a3, a2 \}| \leq 1 \). Hence also \( r_2 s_1 \) contains no order ideals.

In this way one finds that at most

\[
    r_1 s_1, r_1 s_2, r_1 s_3, r_1 s_5, r_1 s_7, r_2 s_5, r_3 s_3, r_3 s_4, r_3 s_5, r_3 s_6, r_3 s_7, r_4 s_5, r_4 s_7
\]

contain order ideals. In order to filter them from each of these 13 concatenated rows we impose all implications \( \{ p \} \rightarrow J(p) \) with the \((a, B)\)-algorithm from section 2 in Part I. Of course, instead of \( J(p) \) it suffices to take the smaller set of lower covers of \( p \). For singleton premises (as in \( \{ a \} \rightarrow B \)) the implication \( n \)-algorithm can be streamlined to the \((a, B)\)-algorithm discussed in [12].
\{f11\} \rightarrow \{c11\}, the corresponding components are both 2 and hence can be turned to 
a, b respectively. The next not yet holding implication is \{e10\} \rightarrow \{b5, b6, a3\}. Since say 
b6 \notin X for all \(X \in r_1s5\), we turn 2 to 0 at position e10. So far we have 
\[
\rho = (\varepsilon, \varepsilon, 0, a, 0, 0, 1, 1, 1, 0, 0, b, 0, 0).
\]

In order to impose the next not yet holding implication, i.e. \(\{c11\} \rightarrow \{a2, a4, c3\}\), we 
need to split \(\rho\) as follows:

\[
\rho_1 = (\varepsilon, \varepsilon, 0, 0, 0, 0, 0, 0, 0, \varepsilon, 0, 0, 0, 0, 0)
\]
\[
\rho_2 = (0, 1, 0, 2, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0)
\]

Notice that 0 in \(\rho_1\) turns \(a\) to 0 at position \(f11\). Further, 1 in \(\rho_2\) turns \(a\) to 2 at position 
\(f11\), and \(\varepsilon\) to 1 at position \(c3\). Hence the other \(\varepsilon\) becomes 0. The other concatenated 
rows in (9) are treated similarly, and the result is this:

\[
\begin{array}{cccccccccccccccc}
\hline
b2 & c3 & d4 & f11 & h11 & i12 & e10 & a2 & a3 & a4 & b6 & b5 & c11 & d9 & d8 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & \varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\delta & \delta & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
\delta & \delta & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & \delta & \delta & \delta \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

Table 4:

Exactly the 34 \(\Lambda\)-closed order ideals \(X = J(x) (x \in L)\) are encoded in this table. For 
instance, letting \(\varepsilon\varepsilon = 01\) in row \(\rho_1\) we get \(X = \{c3, a2, a3, a4\}\) which is \(J(c7)\) (see Fig.2).

Let us recap the described procedure. Steps (d),(e) and (f) convey an extension of the 
method that pays off for large \(t\).

**Summary:** Calculating a modular subdirect product from the connection 
maps.
(a) For each factor lattice $L_i$ find a base of lines $\Lambda'_i$ ($1 \leq i \leq t$).

(b) Calculate the connected components $(J_i, \Lambda_i)$ where $J_i := \sigma_i(J(L_i))$ and $\Lambda_i := \sigma_i(\Lambda'_i)$. Here $\sigma_i : L_i \rightarrow \prod_{j=1}^{t} L_j$ is calculated as $\sigma_i(x) := (\alpha_{j,i}(x) : 1 \leq j \leq t)$.

(c) Using the described $\delta, \varepsilon, l$-algorithm (more details in [12]) compute a context of each $J_i[\Lambda_i]$, i.e. compute a compact representation for the family of all $\Lambda_i$-closed subsets of $J_i$ ($1 \leq i \leq t$). Advantageous, but not strictly necessary are certain subfamilies $J^*_i[\Lambda_i] \subseteq J_i[\Lambda_i]$ (as in table 2, table 3) because they have shorter contexts, which reduces the size of a same graph $G$ in the next step.

(d) Let $J := \bigcup_{1 \leq i \leq t} J_i$. As subset of the known lattice $\prod_{1 \leq i \leq t} L_i$ the set $J$ becomes partially ordered. Consider the graph $G$ whose vertices are the rows occurring in the contexts $J^*_i[\Lambda_i]$ ($1 \leq i \leq t$). Let $k_i$ be the number of rows of $J^*_i[\Lambda_i]$. By definition these $t$ contexts constitute disjoint $k_i$-cliques of $G$. Moreover, two rows from distinct cliques are declared adjacent if they contain components 1 and 0 respectively that clash (with respect to the partial ordering $(J, \leq)$, as seen in the $t=2$ example).

(e) Calculate all $t$-element anticliques of $G$, for instance with the algorithm of [14].

(f) The rows $\rho$ concatenated from the $k_1k_2\cdots k_t$ transversals \{\(\rho_i \mid 1 \leq i \leq t\)\} of the $t$ contexts comprise precisely the $\Lambda$-closed subsets of $J$. Such a row $\rho$ is good in the sense of containing at least one order ideal $X$ of $(J, \leq)$ if and only if \{\(\rho_i \mid 1 \leq i \leq t\)\} is an anticlique of $G$; and the latter have been computed in (e). Using the $(a, B)$-algorithm to impose all implications \{\(p \rightarrow J(p) \mid p \in J\)\} on a good row $\rho$ filters the order ideals $X$ from it.

6 Application to lattices freely generated by posets within f.g. varieties

Recall that a variety $\mathcal{V}$ of lattices is finitely generated if it is generated by a single finite lattices. Equivalently, and more to the point for us, $\mathcal{V}$ has up to isomorphism only finitely many subdirectly irreducibles $S_1, \ldots, S_r$, and they are all finite. Thus every $L \in \mathcal{V}$ is a subdirect product of lattices $L_i$ ($1 \leq i \leq t$) where each $L_i$ is isomorphic to some $S_k$. Possibly $L_i \simeq L_j \simeq S_k$ for $i \neq j$.

Let $(P, \leq)$ be a finite poset. We wish to compute the lattice $F\mathcal{V}(P, \leq)$ freely generated by $(P, \leq)$ within $\mathcal{V}$ as defined in Part I. If we knew the precise structure of the connection maps $\alpha_{ij} : L_j \rightarrow L_i$, then we could construct the subdirect product $F\mathcal{V}(P, \leq)$ as in section 2 and 3.

This works out as follows. Restricting the projections $F\mathcal{V}(P, \leq) \rightarrow L_i$ to $P$ yields a $P$-labelling of $L_i$, i.e. a monotone map $\lambda$ from $P$ onto a generating set of $L_i$. Conversely,
by the universal mapping property, each $P$-labelling arises in this way. We will use the following poset as a standard poset as in Part I:

**Example 3** The finitely generated variety $\mathcal{V} = \mathcal{V}(5N_5)$ has $r = 3$ subdirectly irreducibles $S_1, S_2, S_3$ which (renamed) are these:

If $P$ is the poset of Figure 5, what are the $P$-labellings of these lattices? The twelve $P$-labellings $\lambda$ of $D_2$ are monotone maps and hence the sets $\lambda^{-1}(1)$ yield the nonempty filters of $(P, \leq)$. We encountered the twelve $P$-labellings of $D_2$ already in Part I but we computed the free distributive lattice $FD(P, \leq)$ by other means.

These are the seven $P$-labellings of $N_5$ (for readability $0, 1, 2, 3, 4$ are written only once):
Here are two $P$-labellings of $5N_5$:

![Diagrams of $\lambda_{13}$, $\lambda_{14}$, $\lambda_{15}$, $\lambda_{16}$, $\lambda_{17}$, $\lambda_{18}$, $\lambda_{19}$, $\lambda_{20}$, $\lambda_{20}'$]

Figure 7:

Figure 8:

They are equivalent in the sense that $\lambda_{20}' = \alpha \circ \lambda_{20}$ for some automorphism $\alpha$ of $5N_5$. One can show that up to automorphism $\lambda_{20}$ is the only $P$-labelling of $5N_5$. Therefore $FV(P, \leq)$ is a certain subdirect product of 20 lattices $L_i$, twelve of which are isomorphic to $D_2$, seven to $N_5$, and one to $5N_5$.

It turns out, crucially, that the connecting $\lor$-morphism $\alpha_{i,j} : L_j \to L_i$ is the biggest $\lor$-homomorphism that maps labels below corresponding labels [3, Satz 3.6]. For instance, between $L_{14}$ and $L_{17}$ we have
Notice that the $\lor$-homomorphism $\alpha = \alpha_{17,14}$ does not respect meets, $\alpha(c \land g) \neq \alpha(c) \land \alpha(g)$, but $\alpha_{14,17}$ happens to be a lattice homomorphism.

Let $F\mathcal{V}_1(P, \leq), F\mathcal{V}_2(P, \leq)$, and $F\mathcal{V}_3(P, \leq)$ be the factor lattices of $F\mathcal{V}(P, \leq)$ obtained by taking the subdirect products of $L_1, \cdots, L_{12}$ respectively $L_{13}, \cdots, L_{19}$, respectively $L_{20}$. These homogeneous components of $F\mathcal{V}(P, \leq)$ are $F\mathcal{V}_3(P, \leq) \simeq \mathbb{N}_5$ and $F\mathcal{V}_1(P, \leq) \simeq FD(P, \leq)$; as well as $F\mathcal{V}_2(P, \leq)$ which is depicted below together with its previously mentioned factor lattices $L_{14}$ and $L_{17}$:
Observe that say $\pi_{14} \circ \alpha_{17}$ coincides indeed with the initially given $\alpha_{14,17}$. For instance,

$$(\pi_{14} \circ \alpha_{17})(2) = \pi_{14}(f) = 1 = \alpha_{14,17}(2).$$

From the diagram above it is evident that the join irreducible $f \in \mathcal{FV}_2(P, \leq)$ belongs to both $\alpha_{14}(1)$ and $\alpha_{17}(2)$, which illustrates that the union in (1) needs not be disjoint for non-modular lattices $L$.

Having the connection maps $\alpha_{i,j}$ ($1 \leq i, j \leq 20$) at hand, the lattice $F := \mathcal{FV}(P, \leq)$ can be computed as discussed in section 2. Specifically, recall that the maps $\alpha_{i,j}$ yield the maps $\sigma'_i$ from $L_i$ to $L_1 \times \cdots \times L_{20}$ according to (3). The $\sigma'_i$ in turn yield the subsets $J(F)$ and $G(F)$ of $L_1 \times \cdots \times L_{20}$ according to (1) and (5). Now $F$ can be computed by running the $(A, B)$-algorithm on all implications $A \rightarrow J(\bigvee A)$ where $A \subseteq J(F)$ is such that $\bigvee A \in G(F)$.

The above method works to compute $\mathcal{FV}(P, \leq)$ for any finitely generated variety $\mathcal{V}$ of lattices. Our particular choice of $\mathcal{V} = \mathcal{V}(5, N_5)$ was motivated by some extra feature of this variety. Namely, recall that $FL(P, \leq)$ is the lattice freely generated by $\langle P, \leq \rangle$ within the (not f.e.) variety of all lattices. It turns out that when $FL(P, \leq)$ happens to be finite, it coincides with $\mathcal{FV}(P, \leq)$. Finiteness takes place if and only if $P$ has no subposet.

\footnote{The reader may check that also $f \in \alpha_{18}(2) \cap \alpha_{19}(2)$.}
isomorphic to $1 + 1 + 1$ or $2 + 2$ or $1 + 4$. Here, say $2 + 2$ denotes the disjoint union of two 2-element chains.

If partial semilattices $(P, \lor')$ rather than mere posets $(P, \leq)$ are at stake, everything "should" stay the same, except that there are usually less $(P, \lor')$-labellings $\lambda_j$. The latter by definition are not just monotone but also respect the declared suprema, i.e. $\lambda_j(a \lor' b) = \lambda_j(a) \lor \lambda_j(b)$. Detailed proofs are still pending.

6.1 A symmetry exploiting variation

Consider the natural epimorphism

$$f : TV(P, \leq) \to FD(P, \leq)$$

The idea is to calculate $TV(P, \leq)$ as the disjoint union of the interval sublattices $f^{-1}(x)$ with $x$ ranging over $x \in FD(P, \leq)$. Before going into further details, notice that this approach is appealing when $(P, \leq)$ has a large automorphism group $\Omega$ and hence decays into few and large $\Omega$-orbits $\Omega_i$. This is because $f^{-1}(x) \simeq f^{-1}(y)$ for all $x, y \in \Omega_i$, and thus only one $f^{-1}(x)$ per orbit needs to be computed.

As to the computation of $K := f^{-1}(x)$, let $TV(P, \leq)$ be a subdirect product of the subdirectly irreducible lattices $L_i$ and let $\phi_i : \prod_{1 \leq j \leq t} L_j \to L_i$ be the canonical projections, and $\sigma_i : L_i \to \prod_{1 \leq j \leq t} L_j$ the corresponding (known) smallest pre-image maps. Setting $K_i = \phi_i(K) \subseteq L_i$ it is clear that $K$ is the subdirect product of the lattices $K_i$ ($1 \leq i \leq t$).

When the sublattices $K_i$ of $L_i$ are known, one can calculate the scaffolding $G(K) = \bigcup_{1 \leq i \leq t} \sigma_i(K_i \setminus \{0\})$.

Here the maps $\sigma_i$ are still the same as for $L_i$. Using the $(A, B)$-algorithm or variations thereof the lattice $K$ can then be computed as the closure system of all $\lor$-ideals of the partial semilattice $(G(K), \lor)$.

But how is $K_i$ computed? Each $x$ in $FD(P, \leq)$ can be written, in many ways, as a lattice polynomial of elements of $P$. Considered within $TV(P, \leq)$ some of these lattice polynomials may yield distinct elements. Let $\text{DNF}(x)$ be the unique disjunctive normal form of $x$, and identify $\text{DNF}(x)$ with the corresponding element in $TV(P, \leq)$. Similarly define $\text{CNF}(x)$ in terms of the conjunctive normal form. To fix ideas, say $a, b, c, d \in P \subseteq FD(P, \leq)$ and the corresponding elements in $TV(P, \leq)$ are $a', b', c', d'$. If $x = ((a \lor b) \land c) \lor d$, then

$$\text{DNF}(x) = (a' \lor c') \lor (b' \lor c') \lor d' \leq ((a' \lor b') \land c') \lor d' \leq (a' \lor b' \lor d') \land (c' \lor d') = \text{CNF}(x).$$

Provided that $(P, \leq)$ is unordered (an antichain), it is shown in [1, Thm.3.3] that for all $x \in FD(P, \leq)$ one has

$$K = f^{-1}(x) = [\text{DNF}(x), \text{CNF}(x)]$$
Thus, for \( x \) as above we get
\[
K_i = [\phi_i(DNF(x)), \phi_i(CNF(x))] \quad \text{with e.g.}
\]
\[
\phi_i(DNF(x)) = (\phi_i(a') \lor \phi_i(b') \lor \phi_i(d')) \land (\phi_i(c') \lor \phi_i(d')).
\]
This is readily evaluated because \( \phi_i(a'), \ldots, \phi_i(d') \) are just some of the known labels of \( L_i \). When \((P, \leq)\) is not an antichain, Thm.3.3 in [1] needs to be adapted. Probably this is easy.

7 The smallest modular non-distributive variety

The smallest modular nondistributive variety is \( \mathcal{V} = \mathcal{V}(M_3) \), and it has \( D_2 \) and \( M_3 \) as subdirectly irreducibles. Since \( \mathcal{V} \) is finitely generated, the computation of \( FM_3(P, \leq) := F\mathcal{V}(P, \leq) \) works according to section 6. On the other hand, \( FM_3(P, \leq) \) is a modular lattice, and so the specialities of section 5 apply.

Specifically, in step (a) at the end of section 5 each \( \Lambda_i \) merely consists of one 3-element line. Step (b) involves the calculation of all \( P \)-labellings of \( D_2 \) and \( M_3 \), as well as the biggest \( \lor \)-morphisms \( \alpha_{j,i} \) that map labels below corresponding labels. The explicit programming of all of that was done with Mathematica. Steps (c) to (f) were condensed considerably, but for finitely generated modular varieties with more or bigger subdirectly irreducibles these steps would presumably pay off.

Our variety \( \mathcal{V}(M_3) \) enjoys an extra property akin to the variety \( \mathcal{V}(5N_5) \) in section 6. That is, whenever the free modular lattice \( FM(P, \leq) \) generated by the finite poset \( P \) happens to be finite, then \( FM(P, \leq) \) coincides with \( FM_3(P, \leq) \). Finiteness takes place if and only if \( P \) has no subposet isomorphic to \( 1+1+1+1 \) of \( 1+2+2 \). All of this is due to Wille 1973. We mention that an English version, and also a more explicit one with helpful drawings, of Wille’s German proof, features in [6].

For all \( 1 + 2 + 5 + 16 + 63 = 87 \) posets \( P \) with \(|P| \leq 5 \) the lattices \( FD(P, \leq) \) have been drawn (some in compressed form) in [11]. In [6] their cardinalities were recalculated and confirmed. In fact the cardinalities of \( FD(P, \leq) \) and \( FM_3(P, \leq) \) are calculated in [6] for almost all posets \( P \) with \(|P| \leq 6 \), and the numbers \( s \) and \( s + t \) of subdirectly irreducible factors of \( FD(P, \leq) \) respectively \( FM_3(P, \leq) \) are listed. Observe that \( FM_3(P, \leq) \) has length \( s + 2t \). As to "almost", the list lacks 14 out of 318 six element posets which due to their high symmetry blew up \( FM_3(P, \leq) \) too much. However, chances are good that implementing the symmetry exploiting ideas of 6.1 would finish the job. We mention that the cardinalities

\[
28, \quad 138, \quad 629, \quad 2784
\]

of \( 1+1+1 \) (the Dedekind lattice), \( 1+1+2 \) (the so called Takeuchi lattice), \( 1+1+3 \), and \( 1+1+4 \) match the explicit formula for \(|FM_3(1+1+n)|\) found in [8].

Here is the data for the first few 6-element posets.

| P      | 1326 | 296198143 | 26+45 | 936 | 160224000 | 24+39 |
|   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|
| 886 | 160228750 | 23+39 | 1058 | 6306868 | 26+37 |
| 670 | 434366 | 24+26 | 407 | 68915 | 22+20 |
| 590 | 2472286 | 23+22 | 354 | 64461 | 21+18 |
| 304 | 64461 | 21+18 | 490 | 213428 | 23+22 |
| 325 | 64004 | 20+18 | 298 | 63640 | 19+18 |
| 255 | 20984 | 20+15 | 218 | 20392 | 19+14 |
| 191 | 20184 | 18+14 | 209 | 20379 | 18+14 |
| 188 | 20181 | 17+14 | 170 | 19986 | 17+14 |
| 168 | 19984 | 16+14 | 9944 | 34+133 |
| 2024 | 2610806855 | 28+51 | 1195 | 179700889 | 26+43 |
| 596 | 153926 | 23+22 | 428 | 121130 | 22+22 |
| 1326 | 296198143 | 26+45 | 472 | 138454 | 22+22 |
| 318 | 63872 | 20+18 | 492 | 210044 | 22+23 |
| 325 | 63943 | 20+18 | 298 | 63640 | 19+18 |
| 670 | 434366 | 24+26 | 987 | 1007808 | 25+27 |
| 434 | 14616 | 22+15 | 243 | 3311 | 20+11 |
| 488 | 60962 | 22+18 | 273 | 32449 | 20+12 |
| 234 | 2895 | 19+9 | 184 | 2626 | 18+9 |
| 194 | 2665 | 18+9 | 174 | 2604 | 17+9 |
| 194 | 2665 | 18+9 | 243 | 3311 | 20+11 |
| 188 | 756 | 18+7 | 138 | 584 | 18+7 |
| 273 | 4936 | 20+12 | 154 | 649 | 18+7 |
| 127 | 415 | 17+5 | 100 | 361 | 16+5 |
| 167 | 1060 | 18+8 | 104 | 369 | 16+5 |
| 108 | 377 | 16+5 | 94 | 353 | 15+5 |
| 198 | 622 | 19+6 | 108 | 243 | 17+4 |
| 180 | 821 | 18+6 | 100 | 216 | 16+3 |
| 80 | 190 | 15+3 | 110 | 242 | 16+3 |
| 83 | 195 | 15+3 | 76 | 186 | 14+3 |
| 81 | 195 | 16+4 | 130 | 686 | 17+6 |
| 73 | 170 | 15+3 | 59 | 151 | 14+3 |
| 63 | 157 | 14+3 | 56 | 148 | 13+3 |
| 490 | 213428 | 22+23 | 167 | 1060 | 18+8 |
| 110 | 639 | 16+6 | 194 | 2784 | 18+10 |
Notice that the cardinality of $FM_3(P, \leq)$ is printed boldface whenever $FM(P, \leq)$ is infinite. As mentioned, otherwise the two cardinalities coincide. Thus e.g. $FM(\mathcal{W}) \cong FM_3(\mathcal{W})$ has 756 elements, 18 factors $D_2$, 7 factors $M_3$ and length 32.

Call a poset $P$ good if it does neither contain $1+1+1+1$ nor $1+2+2$ as subposet, and whence induces a finite lattice $FM(P, \leq)$. All 1101 good 7-elements posets $P$ and their cardinalities $|FM(P, \leq)|$ are listed in [6, 8.2] as well.

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