On quasicomplete \( k \)-surfaces in 3-dimensional space-forms

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Abstract
In the study of immersed surfaces of constant positive extrinsic curvature in space-forms, it is natural to substitute completeness for a weaker property, which we here call quasicompleteness. We determine the global geometry of such surfaces under the hypotheses of quasicompleteness. In particular, we show that, for \( k > \text{Max}(0, -c) \), the only quasicomplete immersed surfaces of constant extrinsic curvature equal to \( k \) in the 3-dimensional space-form of constant sectional curvature equal to \( c \) are the geodesic spheres. Together with earlier work of the author, this completes the classification of quasicomplete immersed surfaces of constant positive extrinsic curvature in 3-dimensional space-forms.

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1 | INTRODUCTION

When working with immersed surfaces in space-forms, it is natural to impose the condition of completeness. However, when studying surfaces of constant positive extrinsic curvature, various phenomena (see, e.g., [9]) indicate that we should instead use the weaker condition of quasicompleteness, which we define as follows. Let \( (S, e) \) be an immersed surface in some 3-dimensional space-form \( X \), and let \( I_e, II_e \) and \( III_e \) denote its three fundamental forms. We say that \( (S, e) \) is quasicomplete whenever the Riemannian metric \( I_e + III_e \) is complete. Note that an analogous concept is used in [3–5] and [8] to study what the authors there call flat fronts.

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It is useful to know what quasicompleteness still allows us to say concerning the global geometry of the first fundamental forms of immersed surfaces of constant extrinsic curvature, and it is the purpose of this paper to address this problem. Given \( c \in \mathbb{R} \) and \( m \in \mathbb{N} \), let \( X^m_c \) denote the \( m \)-dimensional space-form of constant sectional curvature equal to \( c \). Choose \( k > 0 \), and let \((S, e)\) be a quasicomplete immersed surface in \( X^3_c \) of constant extrinsic curvature equal to \( k \). By Gauss’ Theorem, \((S, I_e)\) has constant curvature equal to \((c + k)\), and thus carries an \( X^2_{c+k} \)-structure in the sense of Weyl.

We construct the completion of \( S \) as follows. We define the pseudodistance \( d \) over the set of Cauchy sequences in \( S \) by

\[
d((x_m)_{m \in \mathbb{N}}, (y_m)_{m \in \mathbb{N}}) := \lim_{m \to \infty} d(x_m, y_m). \tag{1}
\]

We identify two Cauchy sequences whenever the pseudodistance between them vanishes. We then define the completion \( \bar{S} \) to be the space of Cauchy sequences furnished with the metric that \( d \) defines. As every point of \( S \) trivially identifies with the constant sequence at that point, \( S \) naturally embeds as a dense, open subset of \( \bar{S} \), and we denote \( \partial S := \bar{S} \setminus S \).

**Theorem 1.1.** For all \( c \in \mathbb{R} \) and \( k > 0 \), the \( X^2_{c+k} \)-structure of \( S \) extends uniquely to an \( X^2_{c+k} \)-structure of \( \bar{S} \) with geodesic boundary. Furthermore, \( e \) extends to a continuous function \( \bar{e} : \bar{S} \to X^3_c \) that sends components of \( \partial S \) locally isometrically into geodesics in \( X^3_c \).

**Remark 1.1.** We draw the reader’s attention to the intriguing analogy between Theorem 1.1 and [1, Corollary E].

In terms of the global geometry of \((S, I_e)\), this has the following useful consequence.

**Corollary 1.2.** Any two points of \((S, I_e)\) are joined by a length-minimising geodesic. Furthermore, when \( S \) is simply-connected and \( c + k \leq 0 \), this geodesic is unique.

A nice application of Theorem 1.1 is the following extension of a classical result of Liebmann.

**Theorem 1.3.** When \( k > \max(0, -c) \), the only quasicomplete immersed surfaces in \( X^3_c \) of constant extrinsic curvature equal to \( k \) are the geodesic spheres.

**Remark 1.2.** The classical version of this result, which holds for complete immersed surfaces, is proven in [11, Theorem III.5.2] and [2, Theorem V.1.5].

In [10, Theorem 2.8.3], we extended the theorem of Volkov–Vladimirova and Sasaki to the quasicomplete case, proving that the only quasicomplete immersed surfaces in \( \mathbb{H}^3 \) of constant extrinsic curvature equal to 1 are the horospheres and the level sets of distance functions to complete geodesics. Likewise, in [9, Theorems 1.3.1 and 1.3.2], we classified quasicomplete immersed surfaces in \( \mathbb{H}^3 \) of constant extrinsic curvature equal to \( 0 < k < 1 \). Together with Theorem 1.3, this yields a complete classification of quasicomplete immersed surfaces of constant positive extrinsic curvature in 3-dimensional space forms. Significantly, the corresponding classification of complete surfaces is unfinished, precisely because of the difficulties arising from the case of surfaces
of constant extrinsic curvature equal to $0 < k < 1$ in $\mathbb{H}^3$. We view this as further evidence of the naturality of the quasicompleteness condition.

2 \quad LABOURIE SPACE

Throughout this paper, $k$ will be a positive real number. For all such $k$, we define a $k$-surface in $X^3_c$ to be a quasicomplete, immersed surface $(S, e)$ of constant extrinsic curvature equal to $k$. Let $S^1 X^3_c$ denote the unit sphere bundle over $X^3_c$, and let $\pi: S^1 X^3_c \to X^3_c$ denote the canonical projection. Given a $k$-surface $(S, e)$, we denote by $\hat{e} : S \to S^1 X^3_c$ its outward-pointing, unit, normal vector field and we call $(S, \hat{e})$ its Gauss lift. We define a Gauss lift in $S^1 X^3_c$ to be any immersed surface obtained from some $k$-surface in this manner. Note that the first fundamental form of $\hat{e}$ with respect to the Sasaki metric over $S^1 X^3_c$ is related to the fundamental forms of $e$ by

$$\hat{I}_e = I_e + III_e = I_e(\cdot, \cdot) + I_e(A\cdot, A\cdot),$$

where $A$ here denotes the shape operator of $e$. It follows by definition of quasicompleteness that every Gauss lift is complete.

Theorem 1.1 will follow in a straightforward manner from properties of the space of Gauss lifts that we now describe. We first recall the concept of Cheeger–Gromov convergence, which serves to define its topology. We define a marked surface in $S^1 X^3_c$ to be a triple $(S, f, p)$, where $(S, f)$ is a complete immersed surface, and $p \in S$. In what follows, we will often identify marked surfaces that only differ from one another by reparametrisations which respect the marking. Let $(S_m, f_m, p_m)_{m \in \mathbb{N}}$ be a sequence of marked surfaces in $S^1 X^3_c$. We say that this sequence converges towards the marked surface $(S_\infty, f_\infty, p_\infty)$ in the Cheeger–Gromov sense whenever there exists a sequence $(\Phi_m)_{m \in \mathbb{N}}$ of functions such that

1. for all $m$, $\Phi_m : S_\infty \to S_m$ and $\Phi_m(p_\infty) = p_m$; and
2. for every relatively compact open subset $V_p$ of $S_\infty$, there exists $M$ such that
3. $(f_m \circ \Phi_m)_{m \geq M}$ converges to $f_\infty$ in the $C^1_{\text{loc}}$ sense over $V_p$.

We call $(\Phi_m)_{m \in \mathbb{N}}$ a sequence of convergence maps of $(S_m, f_m, p_m)_{m \in \mathbb{N}}$ with respect to $(S_\infty, f_\infty, p_\infty)$.

We define a marked Gauss lift in $S^1 X^3_c$ to be a triple $(S, \hat{e}, p)$ where $(S, \hat{e})$ is a Gauss lift and $p \in S$. We define a marked tube in $S^1 X^3_c$ to be a triple $(S, \hat{e}, p)$ where $\hat{e} : S \to S^1 X^3_c$ is a covering map of the unit normal bundle over some complete geodesic in $X^3_c$. Let $\text{MGL}_{k,c}$ and $\text{MT}_c$ denote, respectively, the sets of reparametrisation equivalence classes of marked Gauss lifts and marked tubes in $S^1 X^3_c$. We denote

$$\text{MGL}_{k,c} := \text{MGL}_{k,c} \cup \text{MT}_c,$$

and we furnish this set with the quotient of the Cheeger–Gromov topology. Isom$(X^3_c)$ acts continuously on $\text{MGL}_{k,c}$ by post-composition, and we define Labourie space by

$$\mathcal{L}_{k,c} := \text{MGL}_{k,c} / \text{Isom}(X^3_c).$$

In [6], Labourie proves the following result.
**Theorem 2.1** (Labourie (1997)). \( \mathcal{L}_{k,c} \) is compact.

**Remark 2.1.** The reader may consult our recent review [10] for an alternative proof of Theorem 2.1.

**Remark 2.2.** In [7], Labourie shows that \( \overline{\text{MGL}}_{k,c} \) is naturally laminated by surfaces, where each leaf \((S, \hat{e})\) is simply the level set of the forgetful map \( f : (S, \hat{e}, p) \mapsto (S, \hat{e}) \). This perspective will be helpful in understanding the sublaminations \( F_\pm \) that will be introduced below.

Define \( d : \text{MGL}_{k,c} \to ]0, \infty] \) by

\[
d(S, \hat{e}, p) := d(p, \partial S),
\]

where distance is taken with respect to the metric induced by \((\pi \circ \hat{e})\). Note that this function is infinite if and only if \((S, \pi \circ \hat{e})\) is complete. We extend \( d \) to a function defined over \( \overline{\text{MGL}}_{k,c} \) by setting it equal to zero over \( \text{MT}_c \). This function is trivially invariant under the action of \( \text{Isom}(X_c^3) \), and thus descends to a function over \( \mathcal{L}_{k,c} \).

**Lemma 2.2.** \( d \) is lower semi-continuous, that is, for all \( x \in \mathcal{L}_{k,c} \),

\[
\liminf_{y \to x} d(y) \geq d(x).
\]

**Remark 2.3.** Note that equidistant surfaces to some fixed geodesic \( \Gamma \) in \( \mathbb{H}^3 \) have constant sectional curvature equal to 1. For any marked Gauss lift \((S, \hat{e}, p)\) of any such surface, \( d(S, \hat{e}, p) = \infty \). However, upon letting the radius tend to zero, we obtain sequences converging towards marked tubes. This shows that \( d \) is not continuous over \( \mathcal{L}_{-1,1} \). Using rotationally symmetric surfaces about geodesics in \( \mathbb{H}^3 \) (see, e.g., [12, chapter 7.\( F \)]), we see that \( d \) is likewise not continuous over \( \mathcal{L}_{-1, k} \) for \( 0 < k < 1 \).

**Proof.** Let \((S_m, \hat{e}_m, p_m)_{m \in \mathbb{N}} \) be a sequence in \( \overline{\text{MGL}}_{k,c} \) converging towards the limit \((S_\infty, \hat{e}_\infty, p_\infty)\) and let \((\Phi_m)_{m \in \mathbb{N}} \) be a corresponding sequence of convergence maps. We may suppose that \((S_\infty, \hat{e}_\infty, p_\infty)\) is not a tube, for otherwise the result holds trivially. In particular, \( r_\infty := d(S_\infty, \hat{e}_\infty, p_\infty) > 0 \). For all \( r > 0 \), let \( B_r \) denote the open ball of radius \( r \) about \( p_\infty \) in \( S_\infty \) with respect to the metric induced by \((\pi \circ \hat{e}_\infty)\). Choose \( \varepsilon > 0 \) and note that \( B := B_{r_\infty - \varepsilon} \) is relatively compact.

Let \( M \) be such that, for all \( m \geq M \), the restriction of \( \Phi_m \) to \( B \) is a diffeomorphism onto its image. For all \( m \geq M \), and for all \( r \), let \( B_{r,m} \) denote the open ball of radius \( r \) about \( p_\infty \) with respect to the metric induced by \((\pi \circ \hat{e}_m \circ \Phi_m)\). For sufficiently large \( m \), the distance of \( \partial B_{r_\infty - 2\varepsilon} \) from \( p_\infty \) with respect to this metric is greater than \((r_\infty - 3\varepsilon)\), so that

\[
\overline{B}_{r_\infty - 3\varepsilon, m} \subseteq B_{r_\infty - 2\varepsilon}.
\]

In particular, for all such \( m \), this closed ball is compact, and so

\[
d(S_m, \hat{e}_m, p_m) \geq r_\infty - 3\varepsilon.
\]

It follows that

\[
\liminf_{m \to \infty} r_m \geq r_\infty - 3\varepsilon.
\]
As $\varepsilon > 0$ is arbitrary, the result follows.

Define $n : \text{MGL}_{k,c} \to [\sqrt{k}, \infty]$ by

$$n(S, \hat{e}, p) := \|A(p)\|,$$

where $A$ here denotes the shape operator of the immersion $(\pi \circ \hat{e})$, and $\| \cdot \|$ denotes its operator norm. We extend this function to one defined over the whole of $\text{MGL}_{k,c}$ by setting it equal to infinity over $\text{MT}_c$. This function is trivially continuous, and, as it is invariant under the action of $\text{Isom}(X^3_c)$, it descends to a continuous function over $\mathcal{L}_{k,c}$.

**Lemma 2.3.** For all $B > 0$, there exists $\delta := \delta_1(B, c, k)$ such that, for all $x \in \mathcal{L}_{k,c}$, if $d(x) < \delta$, then $n(x) > B$.

**Proof.** As $n$ is continuous, $n^{-1}([0, B])$ is closed, and therefore compact. As $d$ is lower semi-continuous, it attains its minimum $\delta$ over this set. As this set contains no tubes, $\delta > 0$. It follows that if $d(x) < \delta$, then $x \notin n^{-1}([0, B])$, so that $n(x) > B$, as desired.

We define the set $U_{k,c}$ of umbilic points of $\text{MGL}_{k,c}$ by

$$U_{k,c} = n^{-1}(\{\sqrt{k}\}).$$

(8)

Note that no marked tube is umbilic, whilst a marked Gauss lift $(S, \hat{e}, p)$ is umbilic whenever the immersion $(\pi \circ \hat{e})$ is umbilic at $p$.

We introduce sublaminations $F_{\pm}$ of $\text{MGL}_{k,c} \setminus U_{k,c}$ by curves as follows. Let $g$ and $\hat{g}$ denote, respectively, the metric of $X^3_c$ and the Sasaki metric of $S_1X^3_c$. Given $x := (S, \hat{e}, p) \in \text{MGL}_{k,c} \setminus U_{k,c}$, let $M_x$ denote the matrix of $\hat{e}^* \pi^* g$ with respect to $\hat{e}^* \hat{g}$. As $x$ is non-umbilic, by (2), this matrix has distinct eigenvalues. Let $L_{x,-}, L_{x,+} \subseteq T_p S$ denote the respective eigenspaces of its lesser and greater eigenvalues. $L_{\pm}$ define smooth distributions over the set of non-umbilic points in $S$, and the desired laminations are obtained upon integrating these distributions in $S$.

By definition, the restrictions of $F_{\pm}$ to each leaf $(S, \hat{e})$ of $\text{MGL}_{k,c}$ define smooth, transverse foliations of the complement of the set of umbilic points of this surface. More precisely, when $(S, \hat{e})$ is a marked Gauss lift, the leaves of $F_-$ are the lines of greater curvature of $(\pi \circ \hat{e})$, whilst the leaves of $F_+$ are the lines of lesser curvature of this immersion. Likewise, when $(S, \hat{e})$ is a tube, the leaves of $F_-$ are its transverse curves, whilst the leaves of $F_+$ are its longitudinal curves.

Define $d_+ : \text{MGL}_{k,c} \setminus U_{k,c} \to ]0, \infty]$ such that, for all $(S, \hat{e}, p) \in \text{MGL}_{k,c} \setminus U_{k,c}, d_+(S, \hat{e}, p)$ is the distance along $F_+$ to $U_{k,c} \cup \partial S$. We extend $d_+$ to a function over $\text{MGL}_{k,c}$ by setting it equal to zero over $U_{k,c}$ and infinity over $\text{MT}_c$. This function is trivially invariant under the action of $\text{Isom}(X^3_c)$, and thus descends to a function over $\mathcal{L}_{k,c}$.

**Lemma 2.4.** $d_+$ is lower semi-continuous, that is, for all $x \in \mathcal{L}_{k,c}$,

$$\liminf_{y \to x} d_+(y) \geq d_+(x).$$

(9)
Proof. Let \((S_m, \hat{e}_m, p_m)_{m \in \mathbb{N}}\) be a sequence in \(\overline{\text{MGL}}_{k,c}\) converging to \((S_\infty, \hat{e}_\infty, p_\infty)\), and let \((\Phi_m)_{m \in \mathbb{N}}\) be a corresponding sequence of convergence maps. Without loss of generality, we may suppose that \((S_\infty, \hat{e}_\infty, p_\infty)\) is non-umbilic, for otherwise (9) holds trivially. For all \(m \in \mathbb{N} \cup \{\infty\}\), denote \(d_{+,:m} := d_+(S_m, \hat{e}_m, p_m)\) and let \(F_{m,+}\) denote the leaf of \(F_+\) passing through \(p_m\). Choose \(\varepsilon > 0\), and let \(F_{\infty,+}^\varepsilon\) denote the closed segment in \(F_{\infty,+}\) of length \(2(d_{\infty,+} - \varepsilon)\) centred on \(p_\infty\). Let \(V_p\) be a relatively compact neighbourhood of this segment of \(S_\infty\). Let \(M\) be such that, for all \(m \geq M\), the restriction of \(\Phi_m\) to \(V_p\) is a diffeomorphism onto its image. Trivially, \((\Phi_m^{-1}(F_{m,+}) \cap V_p)_{m \geq M}\) converges in the \(C^\infty_{\text{loc}}\) sense to \(F_{\infty,+} \cap V_p\). In particular, for sufficiently large \(m\), \(F_{m,+}\) also contains a segment of length \(2(d_{\infty,+} - 2\varepsilon)\) centred on \(p_m\), so that

\[
\liminf_{m \to \infty} d_{m,+} \geq d_{\infty,+} - 2\varepsilon.
\]

As \(\varepsilon > 0\) is arbitrary, it follows that

\[
\liminf_{m \to \infty} d_+(S_m, \hat{e}_m, p_m) \geq d_+(S_\infty, \hat{e}_\infty, p_\infty),
\]

as desired. \(\square\)

**Lemma 2.5.** For all \(B > 0\), there exists \(\delta := \delta_2(B, c, k)\) such that, for all \(x \in \mathcal{L}_{k,c}\), if \(d(x) < \delta\), then \(d_+(x) > B\).

Proof. As \(d_+\) is lower semi-continuous, \(d^{-1}_+([0, B])\) is closed and therefore compact. As \(d\) is lower semi-continuous, it attains a minimum \(\delta\) over this set. As this set contains no tubes, \(\delta > 0\). It follows that if \(d(x) < \delta\), then \(x \notin d^{-1}_+([0, B])\), so that \(d_+(x) > B\), as desired. \(\square\)

We define the function \(\kappa_+ : \overline{\text{MGL}}_{k,c} \to [-\infty, \infty]\) as follows. For \(x := (S, \hat{e}, p) \in \overline{\text{MGL}}_{k,c} \setminus U_{k,c}\), consider the restriction of \((\pi \circ \hat{e})\) to the leaf of \(F_+\) passing through \(p\). We define \(\kappa_+(S, \hat{e}, p)\) to be the geodesic curvature in \(X^3_c\) of this curve at \(p\). Note, in particular, that when \((S, \hat{e}, p)\) is not a marked tube, this quantity also provides an upper bound for the geodesic curvature in \((S, I_\varepsilon)\) of this curve at this point. We extend \(\kappa_+\) to a function over the whole of \(\overline{\text{MGL}}_{k,c}\) by setting it equal to \(-\infty\) over \(U_{k,c}\).

**Lemma 2.6.** \(\kappa_+\) is continuous away from the set of umbilic points.

Proof. Let \((S_m, \hat{e}_m, p_m)_{m \in \mathbb{N}}\) be a sequence in \(\overline{\text{MGL}}_{k,c}\) converging to \((S_\infty, \hat{e}_\infty, p_\infty)\) and let \((\Phi_m)_{m \in \mathbb{N}}\) be a corresponding sequence of convergence maps. It suffices to consider the case where \((S_\infty, \hat{e}_\infty, p_\infty)\) is a marked tube and \((S_m, \hat{e}_m, p_m)\) is not a marked tube for any finite \(m\). Let \(\gamma_\infty : ]-2\varepsilon, 2\varepsilon[ \to S_\infty\) be a unit-speed parametrisation of the leaf of \(F_+\) passing through \(p_\infty\) such that \(\gamma_\infty(0) = p_\infty\). By definition \((\pi \circ \hat{e}_\infty \circ \gamma_\infty)\) is a unit-speed parametrised geodesic segment in \(X^3_c\). Let \(V_p\) be a relatively compact open subset of \(S_\infty\) containing \(\gamma_\infty([-2\varepsilon, 2\varepsilon])\). Let \(M_1 > 0\) be such that, for all \(m \geq M_1\), \(d_+(S_m, \hat{e}_m, p_m) > \varepsilon\). For all such \(m\), let \(\gamma_m : ]-\varepsilon, \varepsilon[ \to S_m\) be a unit-speed parametrisation of the leaf of \(F_+\) passing through \(p_m\) such that \(\gamma_m(0) = p_m\). Let \(M_2 \geq M_1\) be such that, for all \(m \geq M_2\), the restriction of \(\Phi_m\) to \(V_p\) is a diffeomorphism onto its image and \(\gamma_m([-\varepsilon, \varepsilon]) \subseteq \Phi_m(V_p)\). Trivially, \((\Phi_m^{-1} \circ \gamma_m)_{m \geq M_2}\) converges in the \(C^\infty_{\text{loc}}\) sense to \(\gamma_\infty\). Consequently, \(\lim_{m \to \infty} (\pi \circ \hat{e}_m \circ \gamma_m) = \lim_{m \to \infty} (\pi \circ \hat{e}_m \circ \Phi_m \circ (\Phi_m^{-1} \circ \gamma_m) = \pi \circ \hat{e}_\infty \circ \gamma_\infty\),
where convergence is here in the $C^\infty_{\text{loc}}$ sense. In particular,

$$\lim_{m \to \infty} \kappa_+(S_m, \hat{e}_m, p_m) = 0,$$

and the result follows.

**Lemma 2.7.** For all $\varepsilon > 0$, there exists $\delta := \delta_3(B, c, k)$ such that, for all $x \in \mathcal{L}_{k, c}$, if $d(x) < \delta$, then $\kappa_+(x) < \varepsilon$.

**Proof.** By Lemma 2.3, there exists $\delta_1 > 0$ such that, if $d(x) \leq \delta_1$, then $n(x) > \sqrt{k}$, and $x$ is therefore not umbilic. As $\kappa_+$ is continuous, $d^{-1}([0, \delta_1]) \cap \kappa_+^{-1}([\varepsilon, \infty[)$ is closed and thus compact. As $d$ is lower semi-continuous, it attains a minimum $\delta_2$ over this set. As this set contains no tubes, $\delta_2 > 0$. It follows that if $d(x) < \delta_2$, then $x \not\in \kappa_+^{-1}([\varepsilon, \infty[)$, so that $\kappa_+(x) < \varepsilon$, as desired. $\square$

For all $\delta > 0$, we define the subset $\mathcal{L}^\delta_{k, c}$ of $\mathcal{L}_{k, c}$ by

$$\mathcal{L}^\delta_{k, c} := d^{-1}([0, \delta]).$$

(10)

By Lemma 2.5, for all $B > 0$, upon choosing $\delta$ sufficiently small, we may suppose that $d_+ \geq B$ over this set. We then define $d^B(S, \hat{e}, p)$ to be the supremum of $d$ over the closed interval in $F_+$ of length $2B$ centred on $p$. We likewise define $\kappa^B_+(S, \hat{e}, p)$ to be the supremum of $\kappa_+$ over the same closed interval. Proceeding as before, we obtain the following two results.

**Lemma 2.8.** $d^B$ is lower semi-continuous. In particular, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in \mathcal{L}^\delta_{k, c}$, $d^B(x) < \varepsilon$.

**Lemma 2.9.** $\kappa^B_+$ is continuous. In particular, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in \mathcal{L}^\delta_{k, c}$, $\kappa^B_+(x) < \varepsilon$.

### 3 | CONVEXITY

We now construct charts of $\overline{S}$ in $X^2_{c+k}$. It will suffice to consider the case where $S$ is simply connected. Denote

$$r_0 := \begin{cases} \frac{\pi}{2\sqrt{c+k}} & \text{if } c + k > 0, \\ \infty & \text{otherwise.} \end{cases}$$

(11)

Note that, when $(c + k) > 0$, $r_0$ is the radius of a hemisphere in $X^2_{c+k}$.

Let $(S, e, p)$ be a simply connected marked $k$-surface, and let $g$ denote the metric induced over $S$ by $e$. Recall that $g$ has constant curvature equal to $(c + k)$. Let $\text{Exp}_p : T_p S \to S$ denote the exponential map of $S$ at $p$, and let $\text{Dom}_p$ denote its domain. Recall that $\text{Dom}_p$ is open and star-shaped about 0. Define $U_p \subseteq T_p S$ by

$$U_p := B_{r_0}(0) \cap \text{Dom}_p.$$
Let $\phi : S \to X^2_{c+k}$ be a local isometry, and denote $q := \phi(p)$. Denote $\hat{\phi} := \hat{\phi}_p := \phi \circ \text{Exp}_p$. Note that this function restricts to a diffeomorphism from $U_p$ onto its image $V_p := \hat{\phi}(U_p)$. Let $\hat{\psi} := \hat{\psi}_p$ denote its inverse. Define $\psi := \psi_p : V_p \to S$ by

$$\psi := \text{Exp}_p \circ \hat{\psi},$$

and note that $\psi(q) = p$, and $\phi \circ \psi = \text{Id}$.

Recall that a subset of $X^2_{c+k}$ is said to be convex whenever it contains every length-minimising geodesic between any two of its elements.

**Lemma 3.1.** $V_p$ is convex.

**Proof.** Choose $\alpha, \beta \in U_p$, and denote $a := \hat{\phi}(\alpha)$ and $b := \hat{\phi}(\beta)$. We show that the unique geodesic segment in $X^2_{c+k}$ joining $a$ and $b$ is contained in $V_p$. We suppose that $\alpha$ and $\beta$ are non-colinear, as the colinear case is trivial. Consider the closed geodesic triangle $T$ in $X^2_{c+k}$ with vertices $a$, $b$ and $q$, as in Figure 1. Let $A$, $B$ and $Q$ denote, respectively, the edges lying opposite these three points. We also denote by $A$ and $B$ the respective extensions of $A$ and $B$ to complete geodesics in $X^2_{c+k}$.

Let $\Omega$ denote the unique sector of $X^2_{c+k}$ bounded by $A$ and $B$ and containing $T$. Let $(\hat{q}_t)_{t \in [0,1]}$ be a parametrisation of $Q$, and, for all $t \in [0,1]$, let $\Gamma_t$ denote the unique geodesic in $X^2_{c+k}$ joining $q$ and $\hat{q}_t$.

For all $s \in [0,1]$, denote $a_s := \hat{\phi}(s\alpha)$ and $b_s := \hat{\phi}(s\beta)$, let $Q_s$ denote the minimising geodesic segment joining $a_s$ and $b_s$, and let $T_s$ denote the geodesic triangle with vertices $a_s$, $b_s$ and $q$. Denote

$$I := \{ s \in [0,1] \mid Q_s \not\subset V_p \}.$$

It trivially suffices to show that $I$ is empty. Suppose the contrary. Note that $I$ is closed and therefore contains a least element $s_0 > 0$, say. Let $q'$ be a point of $Q_{s_0} \setminus V_p$. Let $\gamma : [0,1] \to T$ denote the unique geodesic joining $q$ and $q'$. Note that

$$\lim_{r \to 1} (d \circ \psi \circ \gamma)(r) = 0.$$
For all $r$ sufficiently close to 1, let $F_{r,+}$ denote the leaf of $F_+$ passing through $(\psi \circ \gamma)(r)$ and let $L_r$ denote its image under $\phi$. By (14) and Lemmas 2.5 and 2.9, $L_r$ converges to a complete geodesic $L_1$, say, in $X_{c+k}^2$ as $r$ tends to 1.

We first claim that $L_1 \cap T = Q_{S_0'}$. Indeed, otherwise $L_1' := L_1 \cap \text{Int}(T_{S_0})$ would be non-trivial. However, by (14) and Lemma 2.8, $(d \circ \gamma)$ would vanish over this set, which is absurd, and the assertion follows. In particular, there exists $\epsilon > 0$ such that, for all $r \in [1 - \epsilon, 1]$, and for all $t \in [0, 1]$, $L_r$ meets $\Gamma_t$ transversally.

We now claim that, for all such $r$, $L_r$ does not leave $T_{S_0}$ through $Q_{S_0}$. To this end, we show that $(L_r \cap \Omega)_{r \in [1 - \epsilon, 1]}$ foliates some open subset of $\Omega$.

Finally, as $L_1 \cap T = Q_{S_0'}$, $Q_{S_0}$ lies on the boundary of this open subset so that, for all $r$ sufficiently close to 1, $L_r$ indeed does not leave $T_{S_0}$ through $Q_{S_0}$.

It follows by the preceding discussion that, for all $r$ sufficiently close to 1, the connected component of $L_r \cap \Omega$ contains $\gamma(r)$ lies in $T_{S_0} \setminus Q_{S_0}$. In particular, for all such $r$ and for all $t \in [0, 1]$, the path from $q$ to $q_{r,t}$ obtained by moving first along $\gamma$ to $\gamma(r)$, and then along $L_r$ to $q_{r,t}$ homotopes in $T_{S_0}$ to a segment of $\Gamma_t$. As this holds, in particular, when $t = 0$, we obtain

$$d(\text{Exp}_p(s_0 \alpha)) = \lim_{r \to 1} (d \circ \gamma)(q_{r,0}) \leq \liminf_{r \to 1} (d B \circ \gamma)(r) = 0.$$ 

This is absurd, and the result follows.

**Lemma 3.2.** $\partial V_p \cap B_{r_0}(q)$ is a union of disjoint geodesics.

**Proof.** Choose $\alpha \in \partial U_p \cap B_{r_0}(0)$ and define $\gamma : [0, 1] \to X_{c+k}^2$ by $\gamma(t) := \hat{\phi}(t \alpha)$. Note that

$$\lim_{t \to 1} (d \circ \gamma)(t) = 0. \quad (15)$$

For all $t$ sufficiently close to 1, let $L_t$ denote the image under $\phi$ of the leaf of $F_+$ passing through $(\psi \circ \gamma)(t)$. For all such $t$, as $d$ is non-vanishing over this leaf, $L_t$ can only intersect $\partial V_p$ along $\partial B_{r_0}(0)$. However, by (15) and Lemmas 2.5 and 2.9, $(L_t)_{t \in [1 - \epsilon, 1]}$ converges to a complete geodesic $L_1$, say, in $X_{c+k}^2$. It follows that $V_p$ contains a geodesic passing through $\hat{\phi}(\alpha)$, and the boundary component containing $\alpha$ is thus a geodesic. As $\alpha \in \partial U_p \cap B_{r_0}(0)$ is arbitrary, this completes the proof.

We now prove our main results.

**Proof of Theorem 1.1.** It trivially suffices to prove the result when $S$ is simply-connected. Choose $p \in S$, and note that $\psi$ extends to a homeomorphism from $\overline{V}_p \cap B_{r_0}(q)$ into an open subset of $\overline{S}$. As $p \in S$ is arbitrary, this yields an extension of the $X_{c+k}$-structure of $S$ to an $X_{c+k}$-structure over $\overline{S}$ with geodesic boundary. This extension is trivially unique. As $\epsilon$ is a local isometry with respect to the metric $d$ defined in (1), it trivially extends to a continuous function $\hat{\epsilon} : \overline{S} \to X_c^3$.

It remains only to show that $\hat{\epsilon}$ sends components of $\partial S$ locally isometrically to geodesics in $X_c^3$. Choose $p \in S$ such that $\partial U_p \cap B_{r_0}(0) \neq \emptyset$, let $\alpha$ be a point of this set, and define $\gamma : [0, 1] \to X_{c+k}^2$ by $\gamma(t) := \hat{\phi}(t \alpha)$. For all $t$ sufficiently close to 1, let $L_t$ denote the image under $\phi$ of the leaf of $F_+$ passing through $(\psi \circ \gamma)(t)$. By Lemmas 2.5 and 2.9, $L_t$ converges to a complete geodesic in
that we know lies along $\partial V$. For all $t$ sufficiently close to 1, let $\delta_t : \gamma(t) \mapsto X_{c+k}^2$. From the remarks of Lemma 2.9, $(\pi \circ \hat{e} \circ \psi \circ \delta_t)_{t \in [1-\epsilon, \epsilon]}$ converges to a unit-speed parametrisation of a geodesic segment in $X_c^3$. It follows that $\hat{e}$ indeed maps $\partial S$ locally isometrically to geodesic segments in $X_c^3$, and this completes the proof.

**Proof of Corollary 1.2.** As $\partial S$ is a disjoint union of complete geodesics, no length minimising curve in $\overline{S}$ between two points of $S$ can meet the boundary, and the first assertion follows. Suppose now that $c + k \leq 0$, so that, in particular, $r_0$ is infinite. Choose $p \in S$, and note that $(d \circ \psi_p)$ extends to a continuous function over $V_p$ that is positive over $V_p$ and vanishes along the boundary. As $\psi_p$ is a local isometry, it therefore has the path-lifting property, and is thus a covering map of $S$. As $V_p$ is convex, it is simply connected, and is therefore the universal cover of $S$, and the result follows.

**Proof of Theorem 1.3.** We may suppose that $S$ is simply connected. We first show that $S$ is complete. Indeed, suppose the contrary. We claim that $S$ is isometric to a hemisphere in $X_{c+k}^2$. Indeed, let $\Sigma$ denote the surface obtained by doubling $S$ along its boundary, let $\overline{\Sigma}$ denote its universal cover, and let $\hat{\pi} : \overline{\Sigma} \to \Sigma$ denote the canonical projection. Note that $\pi^{-1}(\partial S)$ is a union of *disjoint* complete geodesics in $\overline{\Sigma}$. As $\overline{\Sigma}$ is complete, it is isometric to $X_{c+k}^2$, from which it follows that $\pi^{-1}(\partial S)$ consists of a single complete geodesic $\Gamma$. As $S$ identifies with one component of the complement of $\Gamma$, we conclude that $S$ is indeed isometric to a hemisphere in $X_{c+k}^2$, as asserted. Finally, let $\hat{e} : \overline{S} \to X_c^3$ denote the continuous extension of $e$. By Theorem 1.1, $\hat{e}$ maps $\partial S$ locally isometrically into a geodesic in $X_c^3$. This is absurd, as $X_c^3$ contains no complete geodesic of length $2\pi/\sqrt{c+k}$, and it follows that $S$ is complete, as asserted.

The result now follows from the classical case where $S$ is complete. We sketch the proof for the reader’s convenience. Note first that $S$ is isometric to the sphere $X_{c+k}^2$. Let $I_e$ and $II_e$ denote the first and second fundamental forms of $e$, and note that both define Riemannian metrics over $S$. Let $\phi$ denote the Hopf differential, that is, the $(2,0)$-component, of $I_e$ with respect to $II_e$. As $e$ has constant extrinsic curvature, $\phi$ is holomorphic and therefore vanishes, as $S$ carries no non-trivial holomorphic quadratic differential. $II_e$ is thus a constant multiple of $I_e$, and we conclude by the fundamental theorem of surface theory that $(S,e)$ is a geodesic sphere, as desired.

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