Canonical Theory of 2D Gravity Coupled to Conformal Matter

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Abstract

A canonical quantization for two dimensional gravity models, including a dilaton gravity model, is performed in a way suitable for the light-cone gauge. We extend the theory developed by Abdalla et al. [1] and obtain the correlation functions by using the screening charges of the symmetry algebra. It turns out that the role of the cosmological constant term in the Liouville theory is played by the screening charge of the symmetry algebra and the resulting theory looks like a chiral part of the Liouville theory or a non-critical open string theory. Moreover, we show that the dilaton gravity theory has a symmetry realized by the centrally extended Poincaré algebra instead of the $SL(2, \mathbb{R})$ Kac-Moody algebra which is a symmetry of an ordinary two dimensional gravity theory.
1 Introduction

Since Polyakov and his collaborators solved the two dimensional quantum gravity theory in the continuum approach[2], the $SL(2, \mathbb{R})$ symmetry of the two dimensional quantum gravity theory has played an important role. Although this symmetry is transparent in the light-cone gauge, remarkable progress has been made in the conformal gauge based on the results of papers [3]. For example, correlation functions have been calculated by the path integral formalism in the conformal gauge [4]. However, it would be better to know what happens in the canonical formalism in the light-cone gauge. In this paper we will investigate a canonical formalism of the two dimensional gravity theory including the dilaton gravity theory in a gauge independent way which has been developed by Abdalla et.al. [1]. We will show that in the case of the dilaton gravity theory [5], the residual symmetry is represented by the Kac-Moody version of the centrally extended Poincaré algebra instead of the $SL(2, \mathbb{R}) \otimes U(1)$ Kac-Moody algebra. The action we will investigate is

$$S = -\int_\Sigma d^2x \sqrt{-\det(g)} \left( \frac{\alpha}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \beta \phi R[g] + \Lambda_0 + \frac{1}{2} \sum_{i=1}^N g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i \right), \quad (1)$$

where $\phi$ and $\phi^i (i = 1, \ldots, N)$ are scalar fields, $g$ is the metric and $R[g]$ is the scalar curvature constructed from $g$. $\Sigma$ is the space-time on which these fields are defined. For simplicity, we assume the space-time $\Sigma$ is a cylinder with the coordinate system $(x^0, x^1)$, where $x^0$ is a time coordinate which takes any real value, while $x^1$ is a spatial one with values lying between $-\pi$ and $\pi$. The action (1) contains three parameters, $\alpha$, $\beta$ and $\Lambda_0$, where $\Lambda_0$ is a cosmological constant. We assume $\beta \neq 0$ in this paper. The case $\alpha = 0$ corresponds to the two dimensional dilaton gravity theory, because the CGHS action [3]

$$S_{CGHS} = \int d^2x \sqrt{-\det(g)} \left[ e^{-2\Phi} \left( R[g] + 4 f^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \Lambda_0 \right) - \frac{1}{2} \sum_{i=1}^N f^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i \right] \quad (2)$$

turns out to be a special case of (1) with $\alpha = 0$ and $\beta = 1$ if we define $\phi = -e^{-2\Phi}$ and $g_{\mu\nu} = e^{-2\Phi} f_{\mu\nu}$. On the other hand, the case $\alpha \neq 0$ corresponds to the usual two dimensional gravity theory coupled to scalar fields, one of which has a background charge. So, we will call the theory (1) with $\alpha = 0$ the dilaton gravity theory and the one with $\alpha \neq 0$ the $SL(2, \mathbb{R})$ gravity theory. The reason why we have used the term “$SL(2, \mathbb{R})$” will become clear in the following sections. After we quantize the theory, we will calculate correlation functions for “tachyons”$^1$. It turns out that the screening charges of the symmetry algebra play the role of the cosmological constant terms in the conformal gauge when one computes correlation functions.

$^1$ But this method is suitable for the light-cone gauge.

$^2$ As will be explained, “tachyons” in this paper may not correspond to tachyons in the conformal gauge theory.
2 Classical Theory

The action (1) can be written as
\[
S = \int \Sigma d^2x (-\text{det}(g))^{-\frac{1}{2}} \left[ \frac{\alpha}{2} \left( g_{11} (\partial_\phi)^2 + g_{00} (\partial_1 \phi)^2 - 2g_{01} \partial_0 \phi \partial_1 \phi \right) + \Lambda_0 \text{det}(g) \right] + \beta \left\{ \partial_0 \phi \partial_0 g_{11} + \partial_1 \phi \partial_1 g_{00} - 2\partial_0 \phi \partial_1 g_{01} + \frac{g_{01}}{g_{11}} (\partial_0 \phi \partial_1 g_{11} - \partial_1 \phi \partial_0 g_{11}) \right\} + \frac{1}{2} \sum_{i=1}^{N} \left\{ g_{11} (\partial_0 \phi^i)^2 + g_{00} (\partial_1 \phi^i)^2 - 2g_{01} \partial_0 \phi^i \partial_1 \phi^i \right\} + \text{(surface term)}. \tag{3}
\]

If we assume \( x^0 \) is the “time”, the Lagrangian does not contain “velocities” of the \( g_{00} \) and \( g_{01} \) components. Therefore they are Lagrange multipliers. This fact means that the canonical momenta for \( g_{00} \) and \( g_{01} \) become zero, reflecting the gauge invariance of the theory. After the gauge is fixed, the Hamiltonian becomes as follows:
\[
H = \oint dx^1 \left( \sqrt{-\text{det}(g)} \frac{G_0 + g_{01}}{g_{11}} G_1 \right) + \text{(surface term)}, \tag{4}
\]

where
\[
G_0 = \frac{\alpha}{2} (\partial_\phi)^2 - \frac{\alpha}{2\beta^2} \left( g_{11} p^{11} \right)^2 + \frac{1}{\beta} g_{11} p^{11} \pi + \beta \frac{\partial_1 g_{11}}{g_{11}} \partial_\phi - 2\beta \partial_1^2 \pi + g_{11} \Lambda_0
\]
\[
+ \frac{1}{2} \sum_{i=1}^{N} \left\{ (\pi_i)^2 + (\partial_1 \phi^i)^2 \right\}, \tag{5}
\]
\[
G_1 = \pi \partial_1 \phi - p^{11} \partial_1 g_{11} - 2g_{11} \partial_1 p^{11} + \sum_{i=1}^{N} \pi_i \partial_1 \phi^i. \tag{6}
\]

It is easy to see that \( G_0 \) and \( G_1 \) are secondary constraints and there are no more constraints. Note that since the Hamiltonian is a linear combination of constraints, all physical observables do not evolve as time goes by. The two secondary constraints obtained above generate diffeomorphism transformations. In fact, \( T_\pm = \frac{1}{2} (G_0 \pm G_1) \) obey Virasoro algebras
\[
\left\{ T_\pm(x^1), T_\pm(x^{1'}) \right\}_{P.B.} = \pm \left( T_\pm(x^1) + T_\pm(x^{1'}) \right) \partial_1 \delta(x^1 - x^{1'}). \tag{7}
\]

The equation (7) also means that \( G_0 \) and \( G_1 \) are first class constraints.

We introduce new phase space variables which are very useful when we quantize the theory. The change of variables is divided in two steps. First, let us define
\[
\chi \equiv 4\beta^2 g_{11}, \quad \omega \equiv \frac{1}{4\beta^2} \left( p^{11} - \beta \frac{\partial_1 \phi}{g_{11}} \right), \quad P_0^0 \equiv \pi + \beta \frac{\partial_1 g_{11}}{g_{11}}, \quad P_1^0 \equiv \partial_1 \phi. \tag{8}
\]
\[ P_i^\pm \equiv \frac{1}{\sqrt{2}} \left( \pi_i \pm \partial_i \phi^i \right) \quad (i = 1, \ldots, N). \] (9)

Secondly, following [1], we define the following fields,

\[
\begin{align*}
J^+ & \equiv \frac{1}{2g_{11}} (G_1 - G_0) + \frac{\Lambda_0}{2} = \alpha \chi \omega^2 - 2\beta \omega \left( P_S^0 - \alpha P_S^1 \right) - 4\beta^2 \partial_i \omega - \frac{2\beta^2}{\chi} \sum_{i=1}^{N} (P_i^-)^2, \quad (10) \\
J^0 & \equiv \chi \omega + \gamma \left( P_S^0 + \alpha P_S^1 \right) + 2\beta P_S^1, \quad J^- \equiv \chi, \quad P_S \equiv P_S^0 + \alpha P_S^1, \quad (11)
\end{align*}
\]

where we have introduced a free parameter \( \gamma \) for later use. The most remarkable thing about the above definition is that it resembles the Wakimoto construction for the \( SL(2, \mathbb{R}) \) algebra [3] except for the last term of \( J^+ \). But it turns out that this difference does not matter. In fact, the Poisson brackets of these variables are

\[
\begin{align*}
\{ J^0(x^1), J^\pm(x^{1'}) \}_{P.B.} & = \pm J^\pm(x^{1'}) \delta(x^1 - x^{1'}), \quad (12) \\
\{ J^+(x^1), J^-(x^{1'}) \}_{P.B.} & = -2 \left( \alpha J^0(x^1) - (\beta + \alpha \gamma) P_S(x^1) \right) \delta(x^1 - x^{1'}) \\
& \quad + 4\beta^2 \partial_i \delta(x^1 - x^{1'}), \quad (13) \\
\{ J^0(x^1), J^0(x^{1'}) \}_{P.B.} & = 2\gamma(2\beta + \alpha \gamma) \partial_i \delta(x^1 - x^{1'}), \quad (14) \\
\{ J^0(x^1), P_S(x^{1'}) \}_{P.B.} & = 2(\beta + \alpha \gamma) \partial_i \delta(x^1 - x^{1'}), \quad (15) \\
\{ P_S(x^1), P_S(x^{1'}) \}_{P.B.} & = 2\alpha \partial_i \delta(x^1 - x^{1'}), \quad (16)
\end{align*}
\]

and the other brackets are zero. In the case \( \alpha \neq 0 \), we can put \( \gamma = \frac{\beta}{\alpha} \) and (12) – (13) become the commutation relations for the \( SL(2, \mathbb{R}) \otimes U(1) \) Kac-Moody algebra. On the other hand, if \( \alpha = 0 \), then those brackets form the Kac-Moody version of the centrally extended Poincaré algebra [1]. In this case the simplest choice for \( \gamma \) is \( \gamma = 0 \). We call these current-like new phase space variables \( SL(2, \mathbb{R}) \) variables for brevity. In the light-cone gauge, which is defined by the gauge fixing condition \( g_{00} = 0 \) and \( g_{01} = -1 \), it turns out that the \( SL(2, \mathbb{R}) \) variables are the generators of the residual symmetry [1].

As for the Poisson brackets of \( SL(2, \mathbb{R}) \) variables and matter ones, all brackets are zero except for

\[
\{ J^+(x^1), P_i^-(x^{1'}) \}_{P.B.} = 4\beta^2 \frac{P_i^-(x^{1'})}{J^-(x^1)} \partial_i \delta(x^1 - x^{1'}) \quad (i = 1, \ldots, N). \quad (17)
\]

This fact causes difficulties when we quantize the theory because the inverse of the operator \( J^- \) must be taken into account. However, if we put the non-zero modes of \( P_i^- \) to be zero, the Dirac bracket for \( J^+ \) and \( P_i^- \) vanishes. In this sense we will deal with “chiral” matter in this paper.
Before embarking on the quantization of the theory, we must construct the BRST charge. To do that we will employ the so-called BFV methods ([8] is a good review for this theory). According to them the BRST charge is constructed from the first class constraints and their structure constants. Since we obtained two first class constraints, $G_0$ and $G_1$, it is natural to use them. However, we can also choose linear combinations of them which have forms easy to deal with, namely,

$$J^+ - \frac{\Lambda_0}{2} = \frac{1}{2g_{11}} (G_1 - G_0) \approx 0, \quad T_G + T_M^+ = G_1 \approx 0,$$

(18)

where

$$T_G = \frac{1}{4\beta^2} \left( J^- J^+ - \alpha \left( J^0 \right)^2 \right) - \partial J^0 + \frac{\beta + \alpha \gamma}{2\beta^2} J^0 P_S - \frac{\gamma(2\beta + \alpha \gamma)}{4\beta^2} (P_S)^2 + \gamma \partial_1 P_S, \quad T_M^+ = \frac{1}{2} \sum_{i=1}^N (P_i^+)^2. \quad (19)$$

$T_G$ is nothing but the Sugawara form for the $SL(2, \mathbb{R}) \otimes U(1)$ Kac-Moody algebra (in the case $\alpha \neq 0$) or the Kac-Moody version of the centrally extended Poincaré algebra (in the case $\alpha = 0$) with twisted terms, namely derivative terms for $J^0$ and $P_S$. The BRST charge constructed from (18) by BFV method is

$$Q = \oint dx^1 \left\{ c^+ \left( J^+ - \frac{\Lambda_0}{2} \right) + c \left( T_G + T_M^+ \right) + cc^+ \partial_1 b^+ - bc \partial_1 c \right\}, \quad (21)$$

where $b$, $c$, $b^+$ and $c^+$ are fermionic ghost fields.

3 Quantization

In this section we will quantize the system respecting the diffeomorphism symmetry of the classical theory. Since the “space” is a circle, we can decompose the fields into Fourier components,

$$J^{\pm,0}(x^1) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} J^{\pm,0}_n e^{-\sqrt{-1}nx^1}, \quad P_S(x^1) = \frac{\zeta}{2\pi} \sum_{n \in \mathbb{Z}} a_n^e e^{-\sqrt{-1}nx^1},$$

(22)

$$P_i^+(x^1) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} a_i^i e^{-\sqrt{-1}nx^1} \quad (i = 1, \ldots, N),$$

(23)

$$b(x^1) = \frac{\sqrt{-1}}{2\pi} \sum_{n \in \mathbb{Z}} b_n e^{-\sqrt{-1}nx^1}, \quad c(x^1) = \sum_{n \in \mathbb{Z}} c_n e^{-\sqrt{-1}nx^1}, \quad (24)$$

where
Consequently, the non-trivial commutation relations become

\begin{align}
  b^+(x^1) &= \frac{\sqrt{-1}}{2\pi} \sum_{n \in \mathbb{Z}} b_n^+ e^{-\sqrt{-1}nx^1}, \
  c^+(x^1) &= \sum_{n \in \mathbb{Z}} c_n^+ e^{-\sqrt{-1}nx^1}, \quad (25)
\end{align}

where we have introduced an undetermined normalization constant \( \zeta \) preserved also in the quantum theory except for the possible renormalization of \( \kappa \), with \( \kappa = \frac{\sqrt{-1}}{2\pi} \). The Jacobi identities hold. We will study the case \( \alpha \neq 0 \) separately.

\( J_m^\pm, a_m^\pm = a_{-m}^\pm, b_m^\pm = b_{-m}, c_m^\pm = c_{-m}, b_0^+ = b_0^-, c_0^+ = c_0^- \) are the Virasoro generators for the matter sector with the central charge \( c_M = N \). The normal ordering symbol \( (\cdot) \) means that non-negative modes are to the right of negative ones with respect to its arguments. Since real variables in the classical theory become Hermitian operators in the quantum theory, the Hermiticity relations turn out to be that \( J_m^{\pm, \dagger} = J_{-m}^{\pm, \dagger}, a_m^{\dagger} = a_{-m}, b_m^\dagger = b_{-m}, c_m^\dagger = c_{-m}, b_0^+ = b_0^-, c_0^+ = c_0^- \) and \( L_m^\dagger = L_{-m} \), while the Hermiticity relations for \( a_m^S \) depend upon \( \zeta \). The conventional quantization procedure requires that Poisson brackets must be replaced by commutators multiplied by \( -\sqrt{-1} \). We apply this procedure for \( P_i^+, b, c, b^+ \) and \( e^+ \).

\( [a_m^i, a_n^j] = m\delta^{ij}\delta_{m+n,0} \quad (i, j = 1, \ldots, N) \quad (28) \)

\( \{b_m, c_n\} = \{c_m, b_n\} = \{b_m^+, c_n^+\} = \{c_m^+, b_n^+\} = \delta_{m+n,0}. \quad (29) \)

As for the Poisson bracket algebra \((22) - (16)\), we assume that these algebraic relations are preserved also in the quantum theory except for the possible renormalization of \( c \)-number “anomaly” terms, namely,

\begin{align}
  [J_0^m, J_n^\pm] &= \pm \sqrt{-1} J_{m+n}^\pm, \
  [J_0^m, J_n^\pm] &= -2\sqrt{-1} (\alpha J_0^{m+n} - (\beta + \gamma) \zeta a_{m+n}^S) - 2(\kappa \alpha - \kappa' (\beta + \gamma)) m\delta_{m+n,0}, \quad (31)
\end{align}

\begin{align}
  [J_0^m, J_0^n] &= \kappa m\delta_{m+n,0}, \quad [J_0^m, a_0^S] = \frac{\kappa'}{\zeta} m\delta_{m+n,0}, \quad [a_0^S, a_0^S] = \frac{4\pi \alpha (\beta + \gamma)}{\zeta^2} m\delta_{m+n,0} \quad (32)
\end{align}

with

\( 4\pi \alpha (\beta + \gamma) = \alpha \kappa', \quad (33) \)

where \( \kappa \) and \( \kappa' \) are unknown parameters. The equation (33) is one of the conditions to make Jacobi identities hold. We will study the case \( \alpha \neq 0 \) and the case \( \alpha = 0 \) separately.

In the case where \( \alpha \neq 0 \), if we put \( \gamma = -\frac{\beta}{\alpha} \) and \( \zeta = \sqrt{4\pi \alpha} \), then \( \kappa' = 0 \) from (33). Consequently, the non-trivial commutation relations become

\begin{align}
  [J_0^m, J_n^\pm] &= \pm \sqrt{-1} J_{m+n}^\pm, \quad [J_0^m, J_n^\pm] = -2\sqrt{-1} J_0^{m+n} + km\delta_{m+n,0}, \quad (34)
\end{align}

\begin{align}
  [J_0^m, J_0^n] &= \frac{k}{2} m\delta_{m+n,0}, \quad [a_m^S, a_n^S] = m\delta_{m+n,0}. \quad (35)
\end{align}
where $k = -2\kappa$ and $J_m^-$ is replaced by $\alpha J_m^-$. This algebra is the $SL(2, \mathbb{R}) \otimes U(1)$ Kac-Moody algebra. Next, we will construct $T_G$ from $SL(2, \mathbb{R})$ variables as in (19). Recall that classical $T_G$ is the generator of the spatial diffeomorphism in the gravity sector. Since we respect the diffeomorphism symmetry, Poisson brackets for $T_G$ and the $SL(2, \mathbb{R})$ variables are replaced by commutators with the same structure constants taking into account some possible $c$-number renormalizations. We know that the Fourier components of the required $T_G$ become

$$L^G_m = \frac{1}{k + 2} \sum_{n \in \mathbb{Z}} \left( \frac{1}{2} : J_{m-n}^- J_n^+ : + \frac{1}{2} : J_{m-n}^+ J_n^- : - : J_{m-n}^0 J_n^0 : \right) + \sqrt{-1} m J_m^0$$

$$+ \frac{1}{2} \sum_{n \in \mathbb{Z}} : a_{m-n}^S a_n^S : + \sqrt{-1} m Q S a_m^S + \left( \frac{Q_S^2}{2} - \frac{k}{4} \right) \delta_{m,0},$$

(36)

where $Q_S = \frac{\beta}{\sqrt{c}}$. It is a well-known fact that the generators for the $SL(2, \mathbb{R})$ Kac-Moody algebra can be expressed by free fields $[3]$ such that

$$J_m^+ = \beta_m,$$

$$J_m^0 = - \sqrt{-1} \sum_{n \in \mathbb{Z}} : \beta_{m-n} \gamma_n : + \sqrt{-1} \xi a_m^C - \frac{\sqrt{-1}}{2} \delta_{m,0},$$

(37)

$$J_m^- = - \sum_{n \in \mathbb{Z}} : \beta_{m-n}^{-1} \gamma_n \gamma_1 : + 2\xi \sum_{n \in \mathbb{Z}} : \gamma_{m-n} a_n^C : - (1 + km) \gamma_m,$$

(38)

where $\xi^2 = 1 + \frac{k}{2}$ and the canonical commutation relations for the $\beta$, $\gamma$ and $a^C$ fields are $[\beta_m, \gamma_n] = \delta_{m+n,0}$ and $[a_m^C, a_n^C] = m \delta_{m+n,0}$. The normal ordering for $\beta$ and $\gamma$ zero modes is defined such that $: \beta_0 \gamma_0 : = \gamma_0 \beta_0$ and the Hermiticity relations become $\beta_m^\dagger = \beta_m$, $\gamma_m^\dagger = - \gamma_m$ and $a_m^{C\dagger} = - a_m^C$ ($a_m^{-C\dagger} = a_m^{-C}$) if $k + 2 > 0$ ($k + 2 < 0$).

In the case where $\alpha = 0$, if we put $\zeta = \kappa'$ and redefine $J_m^0 \rightarrow J_m^0 + \frac{\kappa'}{2\kappa} a_m^S$, then (30) – (32) become

$$\begin{align*}
[J_m^0, J_n^\pm] &= \pm \sqrt{-1} J_{m+n}^\pm, \\
[J_m^+, J_n^-] &= \sqrt{-1} a_{m+n}^S + m \delta_{m+n,0}, \\
[J_m^0, a_n^S] &= m \delta_{m+n,0},
\end{align*}$$

(39)

(40)

where we have replaced $J_m$ by $2\beta\kappa' J_m^-$. This algebra is the Kac-Moody version of the centrally extended Poincaré algebra. Next, we must determine quantum $T_G$ expressed by $SL(2, \mathbb{R})$ variables. In the case of the $SL(2, \mathbb{R})$ gravity, $T_G$ is nothing but the Sugawara form for the $SL(2, \mathbb{R})$ Kac-Moody algebra with twisted term. In general, for a Kac-Moody algebra associated with a semi-simple Lie algebra, the Sugawara form can be constructed if one knows the Killing form of the Lie algebra. In the present case, the associated Lie algebra is the centrally extended Poincaré algebra which is not semi-simple and its Killing form is degenerate. However, fortunately there is a bilinear form on the centrally extended Poincaré
algebra which has the same property as the Killing form and is not degenerate. Using this bilinear form we can construct $T_G$ whose Fourier components are

$$L^G_m = \sum_{n \in \mathbb{Z}} \left( \frac{1}{2} : J^-_{m-n} J^+_{n} : + \frac{1}{2} : J^+_{m-n} J^-_{n} : + : J^0_{m-n} a^+_n : \right) + \sqrt{-1} m J^0_m + \frac{1}{2} \sum_{n \in \mathbb{Z}} : a^S_{m-n} a^+_n : + \sqrt{-1} m Q_s a^S_m + Q_s \delta_{m,0},$$

(41)

where $Q_s$ is a real free parameter. There exists a free field representation for $SL(2,R)$ variables also in the case of the dilaton gravity theory, namely,

$$J^+_{m} = \beta_m, \quad J^0_{m} = -\sqrt{-1} \sum_{n \in \mathbb{Z}} : \beta_{m-n} \gamma_n : + a^C_m - \frac{1}{2} a^S_m - \sqrt{-1} \frac{1}{2} \delta_{m,0},$$

(42)

$$J^-_{m} = \sqrt{-1} \sum_{n \in \mathbb{Z}} : \gamma_{m-n} a^S_n : - m \gamma_m,$$

(43)

with canonical commutation relations, $[\beta_m, \gamma_n] = \delta_{m+n,0}$ and $[a^C_m, a^S_n] = m \delta_{m+n,0}$.

$L^G_m$ expressed by free fields has the same form in both the $SL(2,R)$ gravity theory and the dilaton gravity theory, namely,

$$L^G_m = \sum_{n \in \mathbb{Z}} (m + n) : \gamma_{m-n} \beta_m :$$

$$+ \frac{1}{2} \sum_{n \in \mathbb{Z}} : a^D_{m-n} a^D_n : + \sqrt{-1} (m + 1) Q_D a^D_m + \frac{1}{2} \sum_{n \in \mathbb{Z}} : a^L_{m-n} a^L_n : + (m + 1) Q_L a^L_m,$$

(44)

where

$$Q_D \equiv \begin{cases} Q_S & (SL(2,R) \text{ Gravity}) \\ \frac{1}{\sqrt{2}} (Q_S + 1) & (\text{Dilaton Gravity}) \end{cases},$$

(45)

$$Q_L \equiv \begin{cases} \frac{1}{2} \xi & (SL(2,R) \text{ Gravity}) \\ \frac{1}{\sqrt{2}} (Q_S - 1) & (\text{Dilaton Gravity}) \end{cases},$$

(46)

$$a^D_m \equiv \begin{cases} a^S_m - \sqrt{-1} Q_D \delta_{m,0} & (SL(2,R) \text{ Gravity}) \\ \frac{1}{\sqrt{2}} (a^S_m + a^C_m) - \sqrt{-1} Q_D \delta_{m,0} & (\text{Dilaton Gravity}) \end{cases},$$

(47)

$$a^L_m \equiv \begin{cases} a^C_m - Q_L \delta_{m,0} & (SL(2,R) \text{ Gravity}) \\ \frac{1}{\sqrt{2}} (a^S_m - a^C_m) - Q_L \delta_{m,0} & (\text{Dilaton Gravity}) \end{cases}. $$

(48)

The condition $Q_D - Q_L = \sqrt{2}$ distinguishes the dilaton gravity theory.

The quantum BRST charge $Q$ and the total Virasoro generator $L_m$ are defined as follows:

$$Q = \sum_{n \in \mathbb{Z}} : e^+_n (\beta_n - A \delta_{n,0}) : + \sum_{n \in \mathbb{Z}} : e^-_n (L^G_n + L_n^M + L_n^{gh+}) :$$

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\[-\frac{1}{2} \sum_{m,n \in \mathbb{Z}} (m - n) : c_m c_n b_{m+n} : -c_0, \quad (49)\]

\[L_m \equiv \{Q, b_m\} = L^G_m + L^M_m + L^{gh+}_m + L^{gh}_m, \quad (50)\]

where

\[L^{gh+}_m = \sum_{n \in \mathbb{Z}} (m + n) : c_m b^+_n : , \quad L^{gh}_m = \sum_{n \in \mathbb{Z}} (m - n) : b_m c_n : -\delta_{m,0} \quad (51)\]

are the Virasoro generators for \(b^+ c^+\) and \(bc\) ghosts respectively with the central charges \(c_{gh+} = -2\) and \(c_{gh} = -26\). The normal ordering for \(b^+ c^+\) and \(bc\) ghost zero modes is defined such that :

\[b^+_0 c^+_0 : = -c^+_0 b^+_0 \quad \text{and} \quad b^+_0 c_0 : = -c_0 b^+_0. \quad \text{In the above definition for the quantum BRST charge (49), we replaced} \quad \pi \Lambda_0 \text{by} \quad \Lambda \text{taking into account a possible renormalization for the cosmological constant. It is easy to check that the BRST charge defined by (49) is nilpotent if and only if the total central charge vanishes, namely,} \]

\[Q_D^2 - Q_L^2 = 2 - \frac{N}{12}. \quad (52)\]

4 Correlation Functions

An observable is an Hermitian operator whose commutator with the BRST charge vanishes. Here, we will give some examples of observables. First, we introduce a vertex operator which is defined such that

\[V(\vec{p})(x^1) \equiv e^{\sqrt{-1} \vec{p} \cdot (\vec{\phi}(x^1) + \vec{\rho}Qx^1)} = e^{\sqrt{-1} \vec{p} \cdot \vec{\phi}^-(x^1)} e^{\sqrt{-1} \vec{p} \cdot \vec{\phi}^+(x^1)} e^{\sqrt{-1} \vec{p} \cdot (\vec{q}^+(\vec{a}_0 + \vec{\rho}Q)x^1)}, \quad (53)\]

where \(\vec{\rho}_Q = \left(p^D_Q, p^L_Q, p^1_Q, \ldots, p^N_Q\right) \equiv \left(Q_D, \sqrt{-1}Q_D, 0, \ldots, 0\right)\) and the components of \(\vec{\phi}(x^1)\) are

\[\phi^I(x^1) \equiv q^I + a^I_0 x^1 + \sqrt{-1} \sum_{n \neq 0} \frac{a^I_n}{n} e^{-\sqrt{-1}nx^1} \quad (I = D, L, 1, \ldots, N). \quad (54)\]

In equation (53) we have defined that \(\phi^+_I(x^1)(\phi^-_I(x^1))\) is the positive(negative) frequency mode part of \(\phi^I(x^1)\), and \(q^I_d, q^I_L\) and \(q^I_0\) are canonical conjugate operators for \(a^I_d, a^I_L\) and \(a^I_0\) respectively, which satisfy \(\left[q^I_0, a^I_0\right] = \sqrt{-1}\delta^I_0\). Note that \(V(\vec{p})(x^1)\) is Hermitian when the exponents of the normal ordered exponential functions are Hermitian. Therefore \(V(\vec{p})(x^1)\) is a candidate for an observable. To see the condition that \(V(\vec{p})(x^1)\), or something made from
V(\vec{p})(x^1)$, becomes an observable, we must calculate the commutator of $V(\vec{p})(x^1)$ with the BRST charge, which becomes

$$[Q, V(\vec{p})(x^1)] = -\sqrt{-1} \partial_1 W(\vec{p})(x^1) - \sqrt{-1} (h(\vec{p}) - 1) \partial_1 c(x^1) V(\vec{p})(x^1),$$

(55)

where $W(\vec{p})(x^1) \equiv c(x^1) V(\vec{p})(x^1)$ and $h(\vec{p}) \equiv \frac{1}{2} \vec{p} \cdot \vec{p} + \vec{p} \cdot \vec{p}_Q$ is the conformal weight of $V(\vec{p})(x^1)$ with respect to the total Virasoro algebra. Therefore if $h(\vec{p}) = 1$, the integral of $V(\vec{p})(x^1)$ with respect to $x^1$ is an observable if $V(\vec{p})(x^1)$ takes the same values both at the beginning and the end point of the integral contour. We will call $V(\vec{p})(x^1)$ with the conformal weight 1 a tachyon vertex operator\footnote{As we shall see in the following, $V(\vec{p})(x^1)$ is not a tachyon vertex operator in the Liouville theory, although it looks like a chiral component of one. But we will use this term for simplicity.}. Under the condition $h(\vec{p}) = 1$, $W(\vec{p})(x^1)$ also becomes an observable. The following operator is also an observable.

$$\psi(x^1) = \beta(x^1) V(\vec{p}_\psi)(x^1),$$

(56)

where $\beta(x^1) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \beta_n e^{-\sqrt{-1} n x^1}$ and

$$\vec{p}_\psi = (p^L_\psi, p^D_\psi, p^1_\psi, \ldots, p^N_\psi) \equiv \left\{ \begin{array}{ll}
\left( -\frac{1}{2}, 0, 0, \ldots, 0 \right) & (SL(2, \mathbb{R}) \text{ Gravity })
\left( -\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, 0, \ldots, 0 \right) & (\text{Dilaton Gravity })
\end{array} \right..$$

(57)

As is easily checked, the commutators of $\psi(x^1)$ and the generators of the $SL(2, \mathbb{R})$ Kac-Moody algebra or the Kac-Moody version of the centrally extended Poincaré algebra become zero or total derivatives. Therefore $\psi(x^1)$ can be considered as a screening current of these algebras\footnote{\textsuperscript{3} As we shall see in the following, $V(\vec{p})(x^1)$ is not a tachyon vertex operator in the Liouville theory, although it looks like a chiral component of one. But we will use this term for simplicity.} and the integral of $\psi(x^1)$ is an observable. Note that the forms of screening currents are similar to the integrands of the cosmological constant terms in both the Liouville theory\footnote{\textsuperscript{3} As we shall see in the following, $V(\vec{p})(x^1)$ is not a tachyon vertex operator in the Liouville theory, although it looks like a chiral component of one. But we will use this term for simplicity.} and the dilaton gravity theory\footnote{\textsuperscript{3} As we shall see in the following, $V(\vec{p})(x^1)$ is not a tachyon vertex operator in the Liouville theory, although it looks like a chiral component of one. But we will use this term for simplicity.}. In fact, as will be shown in the following, the integrals of these screening currents play the role of cosmological constant terms in correlation functions\footnote{\textsuperscript{3} As we shall see in the following, $V(\vec{p})(x^1)$ is not a tachyon vertex operator in the Liouville theory, although it looks like a chiral component of one. But we will use this term for simplicity.}.

The $r + 3$ point correlation function for tachyons are defined as follows:

$$Z(\vec{p}_a, \vec{p}_b, \vec{p}_c, \vec{p}_1, \ldots, \vec{p}_r; s) = \langle 0; SL(2, \mathbb{C}) | b^+_0 \delta(\beta_0 - \Lambda) W(\vec{p}_a)(x^1_a) \times W(\vec{p}_b)(x^1_b) W(\vec{p}_c)(x^1_c) \prod_{j=1}^{r-1} \int_{C_j} dx^1_j V(\vec{p}_j)(x^1_j) \prod_{k=1}^{s} \int_{S_k} dx^1_k \psi(x^1_k) \rangle | 0; SL(2, \mathbb{C}) \rangle,$$

(58)

where $| 0; SL(2, \mathbb{C}) \rangle$ is called the $SL(2, \mathbb{C})$ vacuum defined by the following conditions,

$$a^I_m | 0; SL(2, \mathbb{C}) \rangle = 0 \quad (I = D, L, 1, \ldots, N) \quad (m \geq 0),$$

(59)
\[ \beta_m \mid 0; SL(2, C) \rangle = b_m^+ \mid 0; SL(2, C) \rangle = 0 \quad (m \geq 1), \]
\[ \gamma_m \mid 0; SL(2, C) \rangle = c_m^+ \mid 0; SL(2, C) \rangle = 0 \quad (m \geq 0), \]
\[ b_m \mid 0; SL(2, C) \rangle = 0 \quad (m \geq -1), \quad c_m \mid 0; SL(2, C) \rangle = 0 \quad (m \geq 2). \]

The name of the \( SL(2, C) \) vacuum comes from the fact that this vacuum vanishes upon operation of the Virasoro generators \( L_{-1}, L_0 \) and \( L_1 \) which are the generators for M"obius transformations in conformal field theories. The reason why we bring up the \( SL(2, C) \) vacuum is that it is a physical state; namely, is annihilated by the BRST charge. Note that if \( h(\bar{p}) = 1, \mid \bar{p} \rangle = \oint dx_1 W(\bar{p})(x^1)b_0^+ \delta(\beta_0 - \Lambda) \mid 0; SL(2, C) \rangle \) is also a physical state which we will call a Fock vacuum. Since \( W(\bar{p})(x^1) \) is independent of its position in a correlation function of observables, \( \langle 0; SL(2, C) \mid b_0^+ \delta(\beta_0 - \Lambda)W(\bar{p}_a)(x^1) \rangle \) in (18) can be considered as the bra corresponding to the Fock vacuum \( \langle \bar{p}_a \rangle \). We inserted \( s \) screening charges and three \( W(\bar{p})(x^1) \)'s in equation (18). If the number of the inserted \( W(\bar{p})(x^1) \)'s is not three, the correlation function vanishes identically because of the contribution from bc ghost.\(^4\)

The correlation function defined above may become singular because the operator product \( V(\bar{p}_1)(x^1)V(\bar{p}_2)(x^2) \) may be singular at \( x^1 = x^2 \). Therefore we must regularize it. We employ the analytic continuation for the regularization scheme; in practice, we analytically continue the coordinates of the operators \( z \equiv e^{\sqrt{-1}\pi x^i} \) from the unit circle to the whole complex plane such that the absolute values of the \( z \)'s become smaller from left to right; for example, \( |z_j| > |z_k| \) for \( j < k \), where \( z_j = e^{\sqrt{-1}\pi x^j} \). Under the regularization scheme stated above and the mass shell conditions, \( h(\bar{p}_a) = h(\bar{p}_b) = h(\bar{p}_c) = h(\bar{p}_1) = \cdots = h(\bar{p}_r) = h(\bar{p}_c) = 1 \), we obtain the following form for the correlation function,

\[ Z(\bar{p}_a, \bar{p}_b, \bar{p}_c, \bar{p}_1, \ldots, \bar{p}_r; s) = \langle 0; SL(2, C) \mid c_{0c-1} \mid \bar{v} \rangle \cdot A(\bar{p}_a, \bar{p}_b, \bar{p}_c, \bar{p}_1, \ldots, \bar{p}_r; s), \]

where \( \bar{v} = \bar{p}_a + \bar{p}_b + \bar{p}_c + \sum_{j=1}^{r} \bar{p}_j + s\bar{\psi} \). Note that \( \langle 0; SL(2, C) \mid c_{0c-1} \mid \bar{v} \rangle = \delta (\bar{v} + 2\bar{p}_Q) \times \langle 0; SL(2, C) \mid c_{0c-1} \mid -2\bar{p}_Q \rangle \), where the delta function ensures the momentum conservation and \( \langle 0; SL(2, C) \mid c_{0c-1} \mid -2\bar{p}_Q \rangle \) is considered as the vacuum to vacuum amplitude, by which the correlation function must be divided. The amplitude \( A(\bar{p}_a, \bar{p}_b, \bar{p}_c, \bar{p}_1, \ldots, \bar{p}_r; s) \) in equation (18) can be written in the following form,

\[ A(\bar{p}_a, \bar{p}_b, \bar{p}_c, \bar{p}_1, \ldots, \bar{p}_r; s) = \left( \frac{\Lambda}{2\pi} \right)^s \left( -\sqrt{-1} \right)^{r+s} \prod_{j=1}^{r} dz_j \prod_{k=1}^{s} dw_k \]
\[ \times \left[ \prod_{j=1}^{r} z_j \bar{p}_a \bar{p}_b (z_j - 1) \bar{p}_c \bar{p}_j \right] \left[ \prod_{k=1}^{s} w_k \bar{p}_a \bar{p}_b (w_k - 1) \bar{p}_c \bar{p}_k \right] \]
\[ \times \left[ \prod_{i<j}^{r} (z_i - z_j) \bar{p}_i \bar{p}_j \right] \left[ \prod_{j<k}^{s} (w_j - w_k) \bar{p}_k \bar{p}_j \right] \left[ \prod_{j=1}^{r} \prod_{k=1}^{s} (z_j - w_k) \bar{p}_k \bar{p}_j \right]. \]

\(^4\) In the path integral language, these three \( c \) ghosts are needed to cancel the Grassmann integrals of \( c \) ghost zero modes which correspond to conformal Killing vectors.
The paths of integrations are divided into three groups. The paths which belong to the first group start from 1 and end at \( \infty \). The paths which belong to the second group start from 0 and end at 1. The paths which belong to the third group start from \( \infty \) and end at 0. \( C_1, \ldots, C_r \) and \( S_1, \ldots, S_l \) belong to the first group while \( C_{r+1}, \ldots, C_{r+r+2} \) and \( S_{1+m}, \ldots, S_{1+m+n} \) belong to the second group and \( C_{r+r+1}, \ldots, C_{r+r+2+r_3} \) and \( S_{1+m+1}, \ldots, S_{1+m+n} \) belong to the third group, where \( r_1 + r_2 + r_3 = r \) and \( l + m + n = s \). Moreover, in each of the three groups, the \( S_k \)'s lie above the \( C_j \)'s and \( C_j(S_j) \) lies above \( C_i(S_i) \) if \( i < j \).

When \( r = 0 \), we can perform the integral of (64) using the formula of [12]. The result is as follows:

\[
A(\vec{p}_a; \vec{p}_b; \vec{p}_c; s) = \left( \frac{\Lambda}{2 \pi \sqrt{-1}} \right)^s e^{\sqrt{-1} \pi \theta} J_{l,m,n}(\alpha; \beta; \rho),
\]

(65)

where \( e^{\sqrt{-1} \pi \theta} \) is an irrelevant phase factor and \( J_{l,m,n}(\alpha; \beta; \rho) \) in (65) is defined as follows:

\[
J_{l,m,n}(\alpha; \beta; \rho) = \prod_{k=1}^{s} \frac{\Gamma(1 - \rho)}{\Gamma(1 - k \rho)} \times \prod_{k=1}^{l} \frac{1}{\Gamma(-\alpha - (s - k) \rho)} \prod_{k=1}^{m} \frac{1}{\Gamma(-\gamma - (s - k) \rho)} \prod_{k=1}^{n} \frac{1}{\Gamma(-\beta - (s - k) \rho)} \times \prod_{k=1}^{m+n} \Gamma(1 + \alpha + (k - 1) \rho) \prod_{k=1}^{n+l} \Gamma(1 + \gamma + (k - 1) \rho) \prod_{k=1}^{l+m} \Gamma(1 + \beta + (k - 1) \rho),
\]

(66)

where \( \alpha = \vec{p}_\psi \cdot \vec{p}_a \), \( \beta = \vec{p}_\psi \cdot \vec{p}_b \), \( \rho = \frac{1}{2} \vec{p}_\psi \cdot \vec{p}_c \) and \( \gamma = -2 - \alpha - \beta - 2(\gamma - 1) \rho = \vec{p}_\psi \cdot \vec{p}_c \).

As an immediate example of a correlation function, we will calculate the three point function for the dilaton gravity theory. In this case, owing to the fact \( \rho = \frac{1}{2} \vec{p}_\psi \cdot \vec{p}_c = 0 \), the amplitude becomes much simpler;

\[
A(\vec{p}_a; \vec{p}_b; \vec{p}_c; s) = e^{\sqrt{-1} \pi \theta} \left( \frac{\sqrt{-1} \Lambda}{2 \Gamma(-\vec{p}_a \cdot \vec{p}_\psi) \Gamma(-\vec{p}_b \cdot \vec{p}_\psi) \Gamma(-\vec{p}_c \cdot \vec{p}_\psi)} \right)^s \times \frac{1}{(\sin (\pi \vec{p}_a \cdot \vec{p}_\psi))^l (\sin (\pi \vec{p}_b \cdot \vec{p}_\psi))^m (\sin (\pi \vec{p}_c \cdot \vec{p}_\psi))^n}.
\]

(67)

Compared with the results of [11], our amplitude seems to be a chiral part of theirs.

In the rest of this section, we will study correlation functions without matter (i.e., \( N = 0 \)). For this purpose, we will introduce the notion of chirality. The solutions of the mass shell condition \( h(\vec{p}) = 1 \) have simple forms in this case; namely, \( p^\pm = \pm \sqrt{-1} (p^D + \sqrt{-1} Q_D) - Q_L \).

We will say that the momentum with a plus(minus) sign has positive(negative) chirality and write it as \( \vec{p}^+ (\vec{p}^-) \). And the chiralities of a set of momenta \( (\vec{p}_a, \vec{p}_b, \vec{p}_c, \vec{p}_1, \ldots, \vec{p}_r) \) is denoted by, for example, \((-++,+-,+,+,+,+)\) if the chirality of the momenta \( \vec{p}_a, \vec{p}_c, \vec{p}_1, \ldots, \vec{p}_r \) are positive and that of \( \vec{p}_a \) is negative. Since we already know the general three point function for the dilaton gravity theory, we will investigate three point functions for the \( SL(2, \mathbb{R}) \) gravity
theory without matter. It is easy to show that the amplitudes with chirality (+, +, +) and (−, −, −) vanish. Therefore we will study the amplitudes with chirality (+, +, −) and (−, −, +). The amplitudes with other chiralities are obtained from these amplitudes owing to the symmetry of the amplitudes. When the chirality is (+, +, −), \( \alpha + \beta = -1 + (2 - s)\rho \) and \( \gamma = -1 - s\rho \). Therefore \( J_{l,m,n}(\alpha, \beta; \rho) \) becomes as follows:

\[
J_{l,m,n}(\alpha, \beta; \rho) = \frac{-\pi (\Gamma (-\rho))^s}{\Gamma \left(1 + \frac{1}{2}\vec{p}_a \cdot \vec{p}_a\right) \Gamma \left(1 + \frac{1}{2}\vec{p}_b \cdot \vec{p}_b\right) \Gamma \left(1 + \frac{1}{2}\vec{p}_c \cdot \vec{p}_c\right)} \prod_{k=1}^{m} \frac{1}{\sin \left[\pi \left(\frac{1}{2}\vec{p}_a \cdot \vec{p}_a + (k + n)\rho\right)\right]} \frac{1}{\sin \left[\pi \left(\frac{1}{2}\vec{p}_a \cdot \vec{p}_a + n\rho\right)\right]} \prod_{k=1}^{m} \sin \left[\frac{\pi}{2}\vec{p}_a \cdot \vec{p}_a \right].
\]

(68)

When the chirality is (−, −, +), \( \alpha + \beta = -2 + \rho(1 - s) \) and \( \gamma = (1 - s)\rho \). It turns out that, if \( m \neq 0 \), the amplitude vanishes. Therefore we assume \( m = 0 \).

\[
J_{l,0,n}(\alpha, \beta; \rho) = \frac{(-1)^n (\Gamma (-\rho))^s}{\Gamma \left(1 + \frac{1}{2}\vec{p}_c \cdot \vec{p}_c\right)} \prod_{k=1}^{s} \frac{1}{k - 1 - \frac{1}{2}\vec{p}_a \cdot \vec{p}_a}.
\]

(69)

These amplitudes seem to be those of open string theories [13].

5 Conclusions

In this paper we investigated a canonical formalism for the two dimensional quantum gravity theory, formally, in a gauge independent way. Our method is useful at least in the light-cone gauge. The resulting theory seems to be a chiral part of the Liouville theory, i.e. the conformal gauge theory, or a non-critical open string theory reflecting the asymmetry property of the light-cone gauge. However, to compare with the results of the conformal gauge theory, it is necessary to investigate the correspondence between the two; for example, we must know much about the correspondence between the “tachyon” vertex operators in our theory and the tachyon vertex operators in the conformal gauge theory. The second problem is that we discarded half of the dynamical degrees of freedom for the matter sector. The theory including all the degree of freedom must be studied. In the presence of the degree of freedom, which we discarded in this paper, the constraints we took in this paper are not convenient. We tried to use some other constraints, but they do not seem to be suitable for quantization.

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