A VANISHING EXT-BRANCHING THEOREM FOR 
(GL_{n+1}(F), GL_n(F))

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Abstract. We prove a conjecture of Dipendra Prasad on the Ext-branching problem from GL_{n+1}(F) to GL_n(F), where F is a p-adic field.

1. Introduction

Let $G_n$ be the general linear group over a p-adic field, and Alg($G_n$) the category of smooth representations. If $\pi \in$ Alg($G_n$), let Wh($\pi$) denote the space of Whittaker functionals on $\pi$. If $\pi$ is irreducible then Wh($\pi$) is one or zero dimensional. We say that $\pi$ is generic or degenerate, respectively. Let $\pi_1$ be an irreducible representation of $G_{n+1}$ and $\pi_2$ an irreducible representation of $G_n$. It is well known [ACRS] that $\dim \operatorname{Hom}_{G_n}(\pi_1, \pi_2) \leq 1$ and it is 1 if both representations are generic. On the other hand, Dipendra Prasad has proved in [Pr] that

$$\operatorname{EP}(\pi_1, \pi_2) := \sum (-1)^i \dim \operatorname{Ext}^i_{G_n}(\pi_1, \pi_2) = \dim \operatorname{Wh}(\pi_1) \cdot \dim \operatorname{Wh}(\pi_2).$$

Thus, if $\pi_1$ and $\pi_2$ are generic then $\operatorname{EP}(\pi_1, \pi_2) = 1$. Since $\dim \operatorname{Hom}_{G_n}(\pi_1, \pi_2) = 1$, Prasad has conjectured that $\operatorname{Ext}^i_{G_n}(\pi_1, \pi_2)$ vanish for $i > 0$ if both representations are generic. In this paper we prove this conjecture and more.

If $\pi_1$ is projective as $G_n$-module (for example if $\pi_1$ is a cuspidal representation) then the conjecture is true, moreover, without assuming that $\pi_2$ is generic. On the other hand, if $\pi_2$ is degenerate, then $\operatorname{EP}(\pi_1, \pi_2) = 0$. If $\pi_2$ is also a quotient of $\pi_1$ then we can conclude that $\operatorname{Ext}^i_{G_n}(\pi_1, \pi_2) \neq 0$ for some $i > 0$. Thus a necessary condition for $\pi_1$ to be $G_n$-projective is not to have degenerate quotients. In this paper we show that this is also a sufficient condition. The proof relies heavily on the Hecke algebra methods from our earlier paper [CS]. We show that this condition is satisfied if $\pi_1$ is an essentially square integrable representation. Therefore essentially square integrable representations of $G_{n+1}$ are projective $G_n$-modules. Moreover, any two essentially square integrable representations of $G_{n+1}$ are isomorphic as $G_n$-modules. This result generalizes, in some sense, the classical result of Bernstein and Zelevinsky which says that any two cuspidal representations of $G_{n+1}$ are isomorphic when restricted to the mirabolic subgroup $M_{n+1}$ of $G_{n+1}$.

Vanishing of higher extension spaces for generic representations is based on the theory of Bernstein-Zelevinsky derivatives [BZ1], [BZ2] with the following, additional ingredient. The theory of derivatives describes how a smooth representation of $G_{n+1}$ restricts to $M_{n+1}$. Instead of $M_{n+1}$ one can consider the transpose $M^+_{n+1}$ of $M_{n+1}$, and develop a theory of derivatives with respect to $M^+_{n+1}$. Thus we have two notions of derivatives: those with respect to $M_{n+1}$ are called right derivatives
and those with respect to $M^T_{n+1}$ are called left derivatives. Since $M^T_{n+1}$ is not conjugated to $M_{n+1}$ in $G_{n+1}$, the information provided by left and right derivatives taken together is stronger, and is essential to our combinatorial arguments. For $\pi \in \text{Alg}(G_n)$ the left $(i)\pi$ and the right $\pi^{(i)}$ derivatives and are related by the isomorphism $(\pi^\vee)^{(i)} \cong ((i)\pi)^\vee$ which we establish under finite length assumption on $\pi$. This is enough for our applications, but in the Appendix we remove this assumption for representations of depth 0. We also prove there the uniqueness of Whittaker functionals for depth 0 representations by a classical Gelfand pair argument (as opposed to the invariant distributions argument [IZY]).

2. Bernstein-Zelevinsky derivatives

In this section we study Bernstein-Zelevinsky derivatives, or simply derivatives, as functors from $\text{Alg}(G_n)$ to $\text{Alg}(G_{n-i})$. We state and prove a “second adjointness formula” for these functors, as well as an Ext version of the formula. Mirabolic group will appear in the next section.

2.1. Notation. Let $G_n = \text{GL}_n(F)$, where $F$ is $p$-adic field. Let $\nu(g) = |\det(g)|$ be the character of $G_n$, where $| \cdot |$ is the absolute value on $F$. Let $B_n$ be the Borel subgroup of $G_n$ consisting of upper triangular matrices and let $U_n$ be the unipotent radical of $B_n$. Let

$$R_{n-i} = \left\{ \begin{pmatrix} g & x \\ 0 & u \end{pmatrix} : g \in G_{n-i}, u \in U_i, x \in \text{Mat}_{n-i,i}(F) \right\}.$$ 

We have an obvious Levi decomposition $R_{n-i} = G_{n-i}E_{n-i}$, where $E_{n-i}$ is the unipotent radical of $R_{n-i}$. Moreover, $E_{n-i} = N_{n-i}U_i$ where $N_{n-i}$ is the unipotent radical of the maximal parabolic subgroup $P_{n-i}$ consisting of block upper triangular matrices and Levi factor $G_{n-i} \times G_i$. Fix a non-zero additive character $\psi$ of $F$. Let $\psi_i$ be the character of $E_{n-i}$ defined by

$$\psi_i(\begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix}) = \psi(u_{1,2} + \ldots + u_{i-1,i})$$

where $u_{1,2}, \ldots, u_{i-1,i}$ are the entries of $u$ above the diagonal. Let $\delta_{R_n}$ be the modular character of $R_{n-i}$. The modular character is trivial on the unipotent radical $E_{n-i}$, and it is equal to $\nu^i$ on the Levi factor $G_{n-i}$. Let $\pi$ be a smooth representation of $G_n$ on a vector space $V$. The right $i$-th Bernstein-Zelevinsky derivative of $\pi$ is a smooth representation $\pi^{(i)}$ of $G_{n-i}$ on the vector space $V^{(i)}$ defined by

$$V^{(i)} = V/\langle \pi(e)v - \psi_i(e)v : e \in E_{n-i}, v \in V \rangle.$$ 

The representation $\pi^{(i)}$ is the natural action of the Levi factor $G_{n-i}$ on $V^{(i)}$ twisted by $\delta_{R_{n-i}}^{-1/2}$, that is, Bernstein-Zelevinsky derivatives in this paper are normalized.

From the definition of derivatives, and the factorization $R_{n-i} = G_{n-i}E_{n-i}$, we have the following Frobenius reciprocity. For any smooth representation $\pi$ of $G_n$ and smooth representation $\sigma$ of $G_{n-i}$:

$$\text{Hom}_{G_n}(\pi, \text{Ind}_{R_{n-i}}^{G_n} (\sigma \otimes \psi_i)) \cong \text{Hom}_{G_{n-i}}(\pi^{(i)}, \sigma).$$

We called the derivative right because there is also a left derivative, which is taken with respect to the transpose of the groups used to define right derivatives.
More precisely, let $\theta_n(g) = (g^{-1})^T$ be the outer automorphism of $G_n$ where $g^T$ is the transpose of $g$ (over the usual diagonal here, but it may be over the opposite diagonal if more convenient). Then the left derivative of $\pi$ is defined by

$$(^i)\pi = \theta_n^{-1}(\theta_n(\pi)(^i)).$$

In other words, the underlying vector space for $^i\pi$ is

$$(^i)V = V/(\pi(e)v - \psi^\top_i(e)v : e \in E^-_{n-i}, v \in V),$$

where $\psi^\top_i$ is the character of $E^-_{n-i}$ defined by

$$\psi^\top_i (\begin{pmatrix} 1 & 0 \\ \frac{1}{u} & 0 \end{pmatrix}) = \psi(u_{2,1} + \ldots + u_{i,i-1}).$$

**Lemma 2.1.** Let $\pi$ be a finite length representation of $G_n$. Then $(\pi^\vee)(^i) \cong (^i)\pi^\vee$.

**Proof.** We have

$$(\pi^\vee)(^i) \cong ((\pi^\vee)_{N^\top_i})_{U^\top_i,\psi^\top_i} \cong ((\pi_{N^\top_i})^\vee)_{U^\top_i,\psi^\top_i}$$

where the second isomorphism is due to Casselman. Now note that $(U^\top_i,\psi^\top_i)$ is conjugated to the pair $(U_i,\psi_i)$ by the permutation matrix in $G_i$ of longest length. Thus, in order to finish, it suffices to show that for every $G_{n-i} \times G_i$-module $\sigma$ of finite length there is an isomorphism

$$(\sigma^\vee)_{U_i,\psi_i} \cong (\sigma_{U_i,\psi_i})^\vee.$$

of smooth $G_{n-i}$ modules. If $K_i$ is an open compact in $G_i$ and $\chi_i$ a character of $K_i$, then there is a natural isomorphism of smooth $G_{n-i}$-modules

$$(\sigma^\vee)_{K_i,\chi_i} \cong (\sigma_{K_i,\chi_i})^\vee.$$

Since $\sigma$ is of finite length, by a result of Rodier [Ro], there exists a pair $(K_i,\chi_i)$ such that the natural map $\sigma_{K_i,\chi_i} \rightarrow \sigma_{U_i,\psi_i}$ (and similarly one for $\sigma^\vee$) is an isomorphism. This completes the proof of lemma. \hfill $\square$

For $\pi$ of depth 0 we shall remove the finite length condition in the Appendix.

We now state two interesting consequences of Lemma 2.1. The first one is not needed in this work, however, we cannot resist not to state it.

**Lemma 2.2.** Let $\pi$ be an irreducible representation of $G_n$. If the irreducible subquotients of $(^i)\pi$ are multiplicity free, then $(^i)\pi$ is a direct sum of its irreducible subquotients.

**Proof.** The key observation is that, in view of Lemma 2.1 we have two ways to compute $(^i)\pi$:

$$(^i)\pi = \theta_n^{-1}(\theta_n(\pi)(^i)) = ((\pi^\vee)(^i))^\vee.$$ 

Since $\pi$ is irreducible, we have $\theta_n(\pi) \cong \pi^\vee$, and if we denote by $\sigma$ either of two isomorphic representations $\theta_n(\pi)(^i)$ and $(\pi^\vee)(^i)$, we see that on one hand $(^i)\pi$ is obtained from $\sigma$ by applying a co-variant functor $\theta$, and on the other hand by applying the contra-variant functor taking the contragradient. Since these two functors coincide on irreducible representations, corollary follows. \hfill $\square$

The second consequence is a “second adjointness” for Bernstein-Zelevinsky derivates:
Lemma 2.3. For any admissible representation \( \pi \) of \( G_n \) and smooth representation \( \sigma \) of \( G_{n-i} \),

\[
\text{Hom}_{G_n}(\text{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i), \pi) \cong \text{Hom}_{G_{n-i}}(\sigma, (i) \pi).
\]

**Proof.** The proof consists of four isomorphisms

\[
\begin{align*}
\text{Hom}_{G_n}(\text{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i), \pi) & \cong \text{Hom}_{G_n}((\pi^\vee, \text{Ind}^{G_n}_{R_{n-i}}((\sigma^\vee \otimes \psi_i))) \\
& \cong \text{Hom}_{G_{n-i}}((\pi^\vee)^{(i)}, \sigma^\vee) \\
& \cong \text{Hom}_{G_{n-i}}((i) \pi^\vee, \sigma^\vee) \\
& \cong \text{Hom}_{G_{n-i}}(\sigma, (i) \pi)
\end{align*}
\]

where the first and the last are dualizing isomorphisms that are valid since \( \pi \) is assumed admissible, the second is the Frobenius reciprocity, and the third is provided by Lemma 2.1. \( \square \)

Lemma 2.4. For any admissible representation \( \pi \) of \( G_n \) and smooth representation \( \sigma \) of \( G_{n-i} \),

\[
\text{Ext}_{G_n}^j(\text{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i), \pi) \cong \text{Ext}_{G_{n-i}}(\sigma, (i) \pi).
\]

**Proof.** In order to compute the right hand side we need to use a projective resolution of \( \sigma \). Using the induction in stages,

\[
\text{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i) \cong \text{Ind}^{G_n}_{R_{n-i}}(\sigma \boxtimes \text{ind}^{G_i}_{U_i}(\psi_i)).
\]

The Gelfand-Graev representation \( \text{ind}^{G_i}_{U_i}(\psi_i) \) is projective by [CS]. Thus, if \( \sigma \) is projective it follows that \( \text{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i) \) is projective, since the parabolic induction takes projective modules into projective modules. So we have shown that taking a projective resolution of \( \sigma \) also gives a projective resolution of \( \text{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i) \). Hence lemma follows from Lemma 2.3. \( \square \)

2.2. Zelevinsky segments. Here we follow [Ze]. Let \( \rho \) be a cuspidal representation of \( G_r \). For any \( a, b \in \mathbb{C} \) with \( b - a \in \mathbb{Z}_{\geq 0} \), we have a Zelevinsky segment let \( \Delta = [\nu^a \rho, \nu^{a+1} \rho, \ldots, \nu^b \rho] \). The absolute length of \( \Delta \) is defined to be \( r(b - a + 1) \), and the relative \( b - a + 1 \). We can truncate \( \Delta \) form each side to obtain two segments of absolute length \( r(b - a) \):

\[
\begin{align*}
\Delta^- &= [\nu^{a+1} \rho, \ldots, \nu^b \rho] \\
\Delta^- &= [\nu^a \rho, \ldots, \nu^{b-1} \rho].
\end{align*}
\]

Moreover, if we perform the truncation \( k \)-times, the resulting segments will be denoted by \( \langle k \rangle \Delta \) and \( \Delta^{(k)} \). The induced representation \( \nu^a \rho \times \nu^{a+1} \rho \times \ldots \times \nu^b \rho \) contains a unique irreducible submodule denoted by \( \langle \Delta \rangle \).

**Proposition 2.5.** Let \( i > 0 \) be an integer. The \( i \)-th left and right derivatives of \( \langle \Delta \rangle \) vanish unless \( i = r \) when

\[
\langle \Delta \rangle = \langle \Delta^- \rangle \text{ and } \langle \Delta \rangle^{(r)} = \langle \Delta^- \rangle.
\]
Recall that a representation \( \pi \) is Whittaker generic if the highest possible derivative is non-zero.

**Corollary 2.6.** Let \( \Delta_1, \ldots, \Delta_k \) be Zelevinsky segments. If \( \pi = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_k \rangle \) is degenerate, then at least one segment \( \Delta \) has the relative length greater than one. If some derivative of \( \pi \) has a generic subquotient, then \( \Delta \) has the relative length two, and \( -\Delta \) or \( \Delta^- \) contributes to the cuspidal support of the generic subquotient of the left or right, respectively, derivative of \( \pi \).

**Proof.** If \( k = 1 \), this follows from Proposition 2.5. In general we use that \( \pi(i) \) has a filtration whose subquotients are \( \langle \Delta_1 \rangle^{(i_1)} \times \ldots \times \langle \Delta_k \rangle^{(i_k)} \) where \( i_1 + \ldots + i_k = i \).

The induced representation \( \nu^a\rho \times \nu^{a+1}\rho \times \ldots \times \nu^b\rho \) also contains a unique irreducible quotient denoted by \( \text{St}(\Delta) \). This representation is an essentially square integrable representation i.e. its matrix coefficients are square integrable when restricted to the derived subgroup. Every essentially square integrable representation is isomorphic to \( \text{St}(\Delta) \) for some segment \( \Delta \).

**Proposition 2.7.** Let \( i > 0 \) be an integer. The \( i \)-th left and right derivatives of \( \text{St}(\Delta) \) vanish unless \( i = jr \) when

\[
^{(i)}\text{St}(\Delta) = \text{St}(\Delta^{(j)}) \quad \text{and} \quad \text{St}(\Delta)^{(i)} = \text{St}((^{(j)}\Delta)).
\]

### 3. Bernstein-Zelevinsky filtration

In this section we begin our study of the restriction problem from \( G_{n+1} \) to \( G_n \). Using the second adjointness formula, for both left and right derivatives, we prove that degenerate representations of \( G_n \) cannot be quotients of essentially square integrable representations of \( G_{n+1} \).

**3.1. Bernstein-Zelevinsky functors.** Let \( M_{n+1} \subseteq G_{n+1} \) be the mirabolic subgroup

\[
M_{n+1} = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} : g \in G_n, u \in \text{Mat}_{n,1}(F) \right\}.
\]

be the mirabolic subgroup of \( G_{n+1} \). We have an obvious Levi decomposition \( M_{n+1} = G_nE_n \). Abusing notation, let \( \psi \) be the character of \( E_n \) defined by \( \psi(u) = \psi(u_n) \) where \( u_n \) is the bottom entry of the column vector \( u \). Note that the stabilizer of \( \psi \) in \( G_n \) is \( M_n \). We have a pair of functors

\[
\Phi^- : \text{Alg}(M_{n+1}) \to \text{Alg}(M_n) \quad \text{and} \quad \Phi^+ : \text{Alg}(M_n) \to \text{Alg}(M_{n+1})
\]

defined by \( \Phi^- (\tau) = \tau_{E_n, \psi} \) and \( \Phi^+ (\tau) = \text{ind}^{M_{n+1}}_{M_nE_n} (\tau \boxtimes \psi) \). We also have a pair of functors

\[
\Psi^- : \text{Alg}(M_{n+1}) \to \text{Alg}(G_n) \quad \text{and} \quad \Psi^+ : \text{Alg}(G_n) \to \text{Alg}(M_{n+1})
\]

defined by \( \Psi^- (\tau) = \tau_{E_n} \) and \( \Psi^+ \) is simply the inflation. All functors are normalized as in [BZ2]. Any \( \tau \in \text{Alg}(M_{n+1}) \) has an \( M_{n+1} \)-filtration

\[
\tau_n \subseteq \ldots \subseteq \tau_0 = \tau
\]
where, \( \tau_i = (\Phi^+)^i(\Phi^-)^i(\tau) \), and
\[
\tau_i / \tau_{i+1} = (\Phi^+)^i\Psi^+\Psi^- (\Phi^-)^i(\tau).
\]
Observe that \( \Psi^- (\Phi^-)^i(\tau) = \tau^{(i+1)} \), is the \( i + 1 \)st derivative, and the subquotients of the filtration, considered as \( G_n \)-modules, are
\[
\tau_i / \tau_{i+1} \cong \text{ind}^{G_n}_{R_{n-i}} (\nu^{1/2} \cdot \tau^{(i+1)} \otimes \psi_i),
\]
where we have used notation from the previous section. In particular, \( \tau_n \) is a multiple of the Gelfand-Graev representation. We derive some consequences of this filtration that we shall need later.

**Lemma 3.1.** Let \( \tau \in \text{Alg}(M_{n+1}) \) such that its derivatives are all finitely generated. When \( \tau \) is considered a \( G_n \)-module, its Bernstein components are finitely generated.

**Proof.** Recall that \( P_{n-i} \supseteq R_{n-i} \) is the maximal parabolic subgroup of \( G_n \) with the Levi factor \( G_i \times G_{n-i} \). Using induction in stages, the \( i \)-th subquotient in the Bernstein-Zelevinsky filtration of \( \Pi \) can be written as
\[
\text{Ind}^{G_n}_{P_{n-i}} (\nu^{1/2} \cdot \tau^{(i+1)} \otimes \text{ind}^{G_i}_{U_i}(\psi_i)).
\]
By the assumption \( \tau^{(i+1)} \) is a finitely generated \( G_{n-i} \)-module and the Bernstein components of the Gelfand-Graev representation \( \text{ind}^{G_i}_{U_i}(\psi_i) \) are finitely generated [BH]. Lemma follows since parabolic induction sends finitely generated modules to finitely generated modules.

**Lemma 3.2.** Let \( \pi_1 \in \text{Alg}(G_{n+1}) \) and \( \pi_2 \) an admissible representation of \( G_n \). If \( \pi_2 \) is a quotient of \( \Pi \) then, for some \( i, j \geq 0 \),
\[
\text{Hom}_{G_{n-i}}(\nu^{1/2} \cdot \tau^{(i+1)}_1, \pi_2) \neq 0 \quad \text{and} \quad \text{Hom}_{G_{n-j}}(\nu^{-1/2} \cdot \tau^{(j+1)}_1, \pi_2) \neq 0.
\]
**Proof.** In order to prove the first isomorphism, we restrict \( \pi_1 \) to \( G_n \) by way of \( M_{n+1} \), and use the second adjointness formula. For the second we restrict to \( o \ G_n \) by way of \( M_{n+1}^+ \), i.e. we reverse the roles of left and right derivatives.

**3.2. Essentially square integrable representations.**

**Theorem 3.3.** Let \( \Delta = [\nu^{a} \rho, \ldots, \nu^{b} \rho] \) be a segment of absolute length \( n + 1 \), where \( \rho \) is a cuspidal representation of \( G_r \). Let \( \pi \) be an irreducible \( G_n \)-module. If \( \pi \) is a quotient of \( \text{St}(\Delta) \) then \( \pi \) is generic.

**Proof.** Let \( l = b - a + 1 \), in particular, \( n + 1 = lr \). Assume \( \pi \) is a degenerate representation of \( G_n \). Let \( m = \{\Delta_1, \ldots, \Delta_k\} \) be a multisegment such that \( \pi \) is a subquotient of \( \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_k \rangle \). If \( \pi \) is a quotient of \( \text{St}(\Delta) \) then, by Lemma 3.2, \( \pi \) contains \( \nu^{1/2} \cdot \text{St}(\Delta)^{(i+1)} \) as a generic submodule for some \( i \). Now we can apply Corollary 2.6: the relative length of each segment in \( m \) is 1 or 2 and one of them is \( [\nu^{-1/2} \rho, \nu^{c+1/2} \rho] \) where \( \nu^{c+1/2} \rho \) is contributes to the cuspidal support of \( \nu^{1/2} \cdot \text{St}(\Delta)^{(i+1)} \). It follows that \( \nu^{1/2} \cdot \text{St}(\Delta)^{(i+1)} \) is a generalized Steinberg representation corresponding to a segment ending in \( \nu^{b+1/2} \rho \), and containing \( \nu^{c+1/2} \rho \). Thus, for every \( d = c + 1, \ldots, b \), \( \nu^{d+1/2} \rho \) contributes to the cuspidal support of \( \pi \). This implies that for every \( d = c + 1, \ldots, b \) there is a segment in \( m \) ending in \( \nu^{d+1/2} \rho \).
Similarly, if we use the second identity in Lemma 3.2, then for every $d = a, \ldots, c - 1$ there exists a segment in $m$ beginning with $\nu^{d-1/2} \rho$. These segments are different than those ending in $\nu^{d+1/2} \rho$, where $d = c + 1, \ldots, b$, since all segments in $m$ have relative length at most 2. Together with $[\nu^{-1/2} \rho, \nu^{+1/2} \rho]$ we see that $m$ contains segments of total relative length $\geq l+1$ and absolute length $(l+1)r = (n+1)+r > n$. This is a contradiction. □

4. Hecke algebra methods

The main goal of this section is to prove projectivity of an essentially square integrable representation $\pi_1$ of $G_{n+1}$ when restricted to $G_n$. The proof uses Hecke algebras and identifies all Bernstein components of $\pi_1$ with sign-projective module of the Hecke algebra corresponding to the Bushnell-Kutzko type $[BK1]$, $[BK2]$, $[BK3]$. As a consequence, any two essentially square integrable representations of $G_{n+1}$ are isomorphic when restricted to $G_n$.

4.1. Hecke algebras. Let $\Delta = [\nu^a \rho, \ldots, \nu^b \rho]$ be a Zelevinsky segment. Let $m = b-a+1$. The Bernstein component of $\text{St}(\Delta)$ is equivalent to the category of representations of a Hecke algebra $\mathcal{H}_m$ arising from a simple Bushnell-Kutzko type $\tau_\Delta$, that is, if $\pi$ is a smooth representation in the Bernstein component, then $\text{Hom}(\tau_\Delta, \pi)$ is the corresponding $\mathcal{H}_m$-module. The algebra $\mathcal{H}_m$ is isomorphic to the Iwahori Hecke algebra of $GL_m(E)$, for some field $E$. Thus, as an abstract algebra, $\mathcal{H}_m$ is generated by $\theta_1, \ldots, \theta_m$, and $T_w$ ($w \in S_m$) satisfying the following relations:

1. $\theta_k \theta_l = \theta_l \theta_k$ for any $k, l = 1, \ldots, m$;
2. $T_{s_k} \theta_k - \theta_{k+1} T_{s_k} = (q-1) \theta_k$, where $q$ is a prime power depending on $\tau_\Delta$ and $s_k$ is the transposition of numbers $k$ and $k + 1$;
3. $T_{s_l} \theta_l = \theta_l T_{s_l}$ if $l \neq k, k + 1$.

Let $\mathcal{A}_m = \mathbb{C}[\theta_1^{\pm 1}, \ldots, \theta_m^{\pm 1}]$ and $\mathcal{H}_{S_m}$ be the span of $T_w$, $w \in S_m$. Then $\mathcal{H}_m \cong \mathcal{A}_m \otimes \mathcal{H}_{S_m}$. The finite dimensional algebra $\mathcal{H}_{S_m}$ has a one dimensional sign representation $\text{sgn}(T_w) = (-1)^{\ell(w)}$, where $\ell$ is the length function on $S_m$. An irreducible representation $\pi$ in the component is Whittaker generic if and only if $\text{Hom}(\tau_\Delta, \pi)$ contains the sign type $[CS]$.

Let $\Delta_1, \ldots, \Delta_r$ be segments such that for $i \neq j$, the cuspidal representations $\rho_i$ and $\rho_j$ are not unramified twists of each other. The Bernstein component of $\text{St}(\Delta_1 \times \cdots \times \Delta_r)$ is equivalent to the category of representations of a Hecke algebra $\mathcal{H}$ arising from a semi-simple Bushnell-Kutzko type $\tau$. We have $\mathcal{H} \cong \mathcal{H}_{m_1} \otimes \cdots \otimes \mathcal{H}_{m_r}$ and $\mathcal{H} \cong \mathcal{A} \otimes \mathcal{H}_S$ where $\mathcal{A} \cong \mathcal{A}_{m_1} \otimes \cdots \otimes \mathcal{A}_{m_r}$ and $\mathcal{H}_S \cong \mathcal{H}_{S_{m_1}} \otimes \cdots \otimes \mathcal{H}_{S_{m_r}}$. The subalgebra $\mathcal{A}$ is isomorphic to the ring of Laurent polynomials in $m = m_1 + \cdots + m_r$ variables, while $\mathcal{H}_S$ is spanned by $T_w$, $w \in S = S_{m_1} \times \cdots \times S_{m_r}$. An irreducible representation $\pi$ in the component can be written as $\pi_1 \times \cdots \times \pi_r$ where $\pi_i$ is in the component of $\text{St}(\Delta_i)$, thus it clear that $\pi$ is Whittaker generic if and only if $\text{Hom}(\tau, \pi)$ contains the sign type of $\mathcal{H}_S$.

4.2. Some projective modules. Let $\chi$ be a character of $\mathcal{A}$. The $\mathcal{H}$-module $\mathcal{H} \otimes A \chi$ is called the principal series representation of $\mathcal{H}$. A twisted Steinberg representation of $\mathcal{H}$ is any one-dimensional $\mathcal{H}$-module such that the restriction to $\mathcal{H}_S$ is the sign
type. For example, if \( \pi = \text{St}(\Delta_1) \times \cdots \times \text{St}(\Delta_r) \), then \( \text{Hom}(\tau, \pi) \) is a twisted Steinberg representation.

The following is Theorem 2.1 in [CS]. (It is stated there for \( \mathcal{H} \) arising from the singleton partition \((m)\) but the proof is applicable to a general partition \((m_1, \ldots, m_r)\)).

**Theorem 4.1.** Let \( \Pi \) be an \( \mathcal{H} \)-module. Assume that

1. \( \Pi \) is projective and finitely generated.
2. \( \dim \text{Hom}_{\mathcal{H}}(\Pi, \pi) \leq 1 \) for an irreducible principal series representation \( \pi \).
3. A twisted Steinberg representation is a quotient of \( \Pi \).

Then \( \Pi \cong \mathcal{H} \otimes_{\mathcal{H}_S} \text{sgn} \).

As in [CS], we have the following corollary.

**Corollary 4.2.** Let \( \Gamma \) be the summand of the Gelfand-Graev representation corresponding to the Bernstein component of \( \text{St}(\Delta_1) \times \cdots \times \text{St}(\Delta_r) \). Then we have an isomorphism \( \text{Hom}(\tau, \Gamma) \cong \mathcal{H} \otimes_{\mathcal{H}_S} \text{sgn} \) of \( \mathcal{H} \)-modules.

### 4.3 Projectivity for Hecke algebras

Let \( Z \) be the center of \( \mathcal{H} \). Recall that \( Z = \mathcal{A}^S \), in particular, \( \mathcal{H} \) is finitely generated \( Z \)-module. Let \( \mathcal{J} \) be a maximal ideal in \( Z \). Let \( \mathcal{H} \) denote the \( \mathcal{J} \)-adic completion of \( \mathcal{H} \). For every \( \mathcal{H} \)-module \( \Pi \), let \( \hat{\Pi} \) denote the \( \mathcal{J} \)-adic completion of \( \Pi \). If \( \Pi \) is finitely generated, then \( \hat{\Pi} \cong \hat{\mathcal{H}} \otimes_{\mathcal{H}} \Pi \).

**Theorem 4.3.** Let \( \Pi \) be a finitely generated \( \mathcal{H} \)-module and \( \mathcal{J} \) a maximal ideal in \( Z \). Let \( \pi \) be the unique irreducible \( \mathcal{H} \)-module annihilated by \( \mathcal{J} \) and containing the sign type. Assume that

1. \( \dim \text{Hom}_{\mathcal{H}}(\Pi, \pi) = 1 \)
2. \( \Pi \) has no other irreducible quotients annihilated by \( \mathcal{J} \).
3. \( \Pi \) contains a torsion free element for \( \mathcal{A} \).

Then \( \hat{\Pi} \cong \hat{\mathcal{H}} \otimes_{\mathcal{H}_S} \text{sgn} \).

**Proof.** In order to simplify notation, write \( \Sigma = \mathcal{H} \otimes_{\mathcal{H}_S} \text{sgn} \). Since \( \Pi \) is finitely generated, \( \hat{\Pi} / \mathcal{J}\hat{\Pi} \cong \Pi / \mathcal{J}\Pi \) is a finite dimensional \( \mathcal{H} \)-module, annihilated by \( \mathcal{J} \). By (2) it must be generated by the sign type subspace. Let \( r \) be the dimension of the sign type in \( \Pi / \mathcal{J}\Pi \). By Frobenius reciprocity, we have a surjection \( f : \Sigma^r \to \Pi / \mathcal{J}\Pi \) which descends to a surjection \( \hat{f} : (\Sigma / \mathcal{J}\Sigma)^r \to \hat{\Pi} / \mathcal{J}\hat{\Pi} \). Observe that \( \hat{f} \) is bijective on the sign type, since the sign type in \( \Sigma / \mathcal{J}\Sigma \) is one-dimensional. Since \( \pi \) is the unique irreducible quotient of \( \Sigma / \mathcal{J}\Sigma \) and \( \hat{f} \) is bijective on the sign type, it follows that \( \pi^r \) is a quotient of \( \Pi / \mathcal{J}\Pi \). This forces \( r = 1 \) by (1) and, by Nakayama lemma, we have a surjection \( \hat{f} : \hat{\Sigma} \to \hat{\Pi} \). Since \( \hat{\Sigma} \cong \hat{\mathcal{A}} \), as \( \hat{\mathcal{A}} \)-modules, (3) implies that the surjection is in fact an isomorphism. \( \square \)

**Corollary 4.4.** Let \( \Pi \) be a finitely generated \( \mathcal{H} \)-module and \( \mathcal{J} \) a maximal ideal in \( Z \). Assume that the conditions of Theorem 4.3 are satisfied. Then, for all \( \mathcal{H} \)-modules \( \sigma \) annihilated by \( \mathcal{J} \) and for all \( i > 0 \),

\[ \text{Ext}^i_{\mathcal{H}}(\Pi, \sigma) = 0. \]
Proof. Since $\sigma$ is annihilated by $J$, we have
\[
\operatorname{Ext}_H^i(\Pi, \sigma) \cong \operatorname{Ext}_\hat{H}^i(\hat{\Pi}, \sigma).
\]
The latter spaces are trivial by projectivity of $\hat{H} \otimes_H \text{sgn}$. $\square$

Corollary 4.5. Let $\Pi$ be a finitely generated $H$-module. Assume that the conditions of Theorem 4.3 are satisfied for every maximal ideal in $\mathcal{Z}$. Then $\Pi \cong H \otimes_H \text{sgn}$. Proof. Corollary 4.4 implies that $\operatorname{Ext}_H^i(\Pi, \sigma) = 0$ for all finite length modules. Since $\Pi$ is also finitely generated, it is projective by Theorem 8.1 in the appendix of [CS]. Now we can apply Theorem 4.1. $\square$

4.4. Projectivity for groups. Now we can apply the Hecke-module results to the restriction problem, one Bernstein component at the time. Let $\pi_1$ be an irreducible generic representation of $G_{n+1}$ and fix a Bushnell-Kutzko type $\tau$ for $G_n$. Let $\Pi = \text{Hom}(\tau, \pi_1)$ be the corresponding $H$-module. Note that the conditions (1) and (3) in Theorem 4.3 are satisfied for every maximal ideal $J$. Indeed, the condition (1) because all irreducible generic $G_n$-representations are quotients of $\pi_1$ with multiplicity one and (3) because $\pi_1$, restricted to $G_n$, contains the Gelfand-Graev representation, a free $A$-module. Theorem 4.3 implies the following local Ext vanishing result for groups.

Theorem 4.6. Let $\pi_1$ be an irreducible generic representation of $G_{n+1}$. Let $J$ be a maximal ideal of the Bernstein center of $G_n$. Assume that no degenerate irreducible representation of $G_n$ annihilated by $J$ is a quotient of $\pi_1$. Then $\operatorname{Ext}_{G_n}^i(\pi_1, \pi_2) = 0$, $i > 0$, for all irreducible representation $\pi_2$ of $G_n$ annihilated by $J$.

Finally since, by Theorem 3.3 essentially square integrable representation have no degenerate quotients, Corollary 4.5 implies:

Theorem 4.7. Let $\pi_1$ be an essentially square integrable representation of $G_{n+1}$. Then $\pi_1$ considered a $G_n$-module, is projective. Moreover, if $\pi_1'$ is another essentially square integrable representation of $G_{n+1}$ then $\pi_1$ and $\pi_1'$ are isomorphic as $G_n$-modules.

5. Vanishing of Ext’s

The purpose of this section is to prove:

Theorem 5.1. Let $\pi_1$ be an irreducible generic representation of $G_{n+1}$ and $\pi_2$ an irreducible generic representation of $G_n$. Then
\[
\operatorname{Ext}_{G_n}^i(\pi_1, \pi_2) = 0 \text{ if } i > 0 \text{ and } = \mathbb{C} \text{ if } i = 0.
\]

Let us explain the strategy of the proof. Fix $\pi_2$, and assume that $\pi_2$ is a subquotient of $\rho_1 \times \ldots \times \rho_k$ where $\rho_i$ are cuspidal representations. Let $m(\pi_1)$ be the integer that counts the number of cuspidal representations $\rho$ in the support of $\pi_1$ such that $\rho$ is an unramified twist of a $\rho_i$, for some $1 \leq i \leq k$. The proof is by induction on $m(\pi_1)$. The base case $m(\pi_1) = 0$ is easy. It is deduced from the Bernstein-Zelevinsky filtration of $G_{n+1}$ where the bottom piece is the Gelfand-Graev representation of
Assume now that $\pi_1 = \text{St}(m_1)$ and $\pi_2 = \text{St}(m_2)$ for a pair of generic multisegments i.e. no two segments are linked. Let $\Delta = [\nu_0 \rho, \ldots, \nu_b \rho]$ be a segment in $m_1$ such that $\rho$ contributes to $m(\pi_2)$. Assume that $\Delta$ is also a shortest such segment. Write $\pi_1 = \text{St}(\Delta) \times \pi$ where $\pi = \text{St}(m)$ and $m = m_1 \setminus \Delta$. Let $r$ be the integer such that $\rho \in \text{Alg}(G_r)$. Let $\rho' \in \text{Alg}(G_r)$ be another cuspidal representation such that no unramified twist of $\rho'$ appears in the cuspidal supports of $\pi_1$ and $\pi_2$. Now both $\rho' \times \text{St}(-\Delta) \times \pi$ and $\rho' \times \text{St}(\Delta^-) \times \pi \in \text{Alg}(G_{n+1})$ are irreducible and satisfy the induction assumption. We shall use this information to prove the theorem for $\pi_1$.

5.1. Transfer. Let $l = s + r$. Recall that $P_r$ is the maximal parabolic of $G_l$ with the Levi $G_s \times G_r$. Starting with $\sigma \in \text{Alg}(G_s)$ and $\tau \in \text{Alg}(M_r)$ one can manufacture two representation of $M_l$. The first one is obtained by the (normalized) induction from $P_r \cap M_l$ and, abusing notation, denoted by $\sigma \times \tau$. The second is obtained by the normalized induction from $P_r^\top \cap M_l$ but only after $\sigma$ is multiplied by $\nu^{-1/2}$, see [BZ2] page 457, where the definition uses a different subgroup, but conjugated in $M_l$. This representation is denoted by $\tau \times \sigma$.

Our interest in these representations comes from the following, 4.13 Proposition in [BZ2].

**Proposition 5.2.** Let $\rho \in \text{Alg}(G_r)$, $\sigma \in \text{Alg}(G_s)$ and $\tau \in \text{Alg}(M_r)$. Let $\rho|_M$ and $\sigma|_M$ denote restrictions to $M_r$ and $M_s$, respectively.

1. There exists an exact sequence in $\text{Alg}(M_l)$

$$0 \to (\rho|_M) \times \sigma \to \rho \times \sigma \to \rho \times (\sigma|_M) \to 0$$

2. If $\Omega$ is any of the four functors $\Phi^\pm$ and $\Psi^\pm$, then

$$\Omega(\sigma \times \tau) = \sigma \times \Omega(\tau).$$

3. $\Psi^-(\tau \times \sigma) = \Psi^-(\tau) \times \sigma$, and there exists an exact sequence in $\text{Alg}(M_{l-1})$

$$0 \to \Phi^-(\tau) \times \sigma \to \Phi^-(\tau \times \sigma) \to \Psi^-(\tau) \times (\sigma|_M) \to 0$$

**Proposition 5.3.** Let $\Delta = [\nu_0 \rho, \ldots, \nu_b \rho]$ be a segment where $\rho \in \text{Alg}(G_r)$. Let $\tau_r = (\Phi^+)^{-1}(1) \in \text{Alg}(M_r)$, the Gelfand-Graev module. Then $\text{St}(\Delta)|_M$ is isomorphic to $\tau_r \times \text{St}(-\Delta)$.

**Proof.** Recall that $\rho|M \cong \tau_r$ (this is true for every cuspidal representation). Note that $\text{St}(\Delta)$ is a quotient of $\nu^0 \rho \times \text{St}(-\Delta)$. By Proposition 5.2 (1), we have an exact sequence of mirabolic modules

$$0 \to \tau_r \times \text{St}(-\Delta) \to \nu^0 \rho \times \text{St}(-\Delta) \to \nu^0 \rho \times (\text{St}(-\Delta)|_M) \to 0$$

By Proposition 5.2 (2), any derivative of the quotient in the above sequence is equal to $\nu^0 \rho \times \text{St}^{(k)}(\Delta)$ with $k > 1$. Since $\nu^0 \rho$ and $^{(k)}\Delta$ are not linked, the corresponding subquotients in the Bernstein-Zelevinsky filtration are irreducible. Observe that they are non-isomorphic to the subquotients of the Bernstein-Zelevinsky filtration of $\text{St}(\Delta)$. Hence the projection from $\nu^0 \rho \times \text{St}(-\Delta)$ onto $\text{St}(\Delta)$, restricted to $\tau_r \times \text{St}(-\Delta)$ gives the desired isomorphism.

Now we arrive to a key result:
Corollary 5.4. Let \( \rho, \rho' \in \text{Alg}(G_r) \) be two irreducible cuspidal representations. Let \( \Delta = [\nu^0 \rho, \ldots, \nu^b \rho] \), and \( \pi \in \text{Alg}(G_s) \). Then we have an isomorphism of mirabolic modules

\[
\text{St}(\Delta)|_M \times \pi \cong \rho'|_M \times (\text{St}(-\Delta) \times \pi).
\]

Proof. By Proposition 5.3, we can substitute \( \text{St}(\Delta)|_M = \tau_r \times \text{St}(-\Delta) \). Furthermore, we have a natural isomorphism

\[
(\tau_r \times \text{St}(-\Delta)) \times \pi \cong \tau_r \times (\text{St}(-\Delta) \times \pi)
\]

given by the induction in stages in two different orders. We finish by observing that \( \tau_r = \rho'|_M \).

\(\square\)

Now we continue with the proof of vanishing for \( \pi_1 = \text{St}(\Delta) \times \pi \), notation as in the start of the section. By Proposition 5.2 (1) there is an exact sequence in \( \text{Alg}(M_{n+1}) \)

\[
0 \rightarrow (\text{St}(\Delta)|_M) \times \pi \rightarrow \text{St}(\Delta) \times \pi \rightarrow \text{St}(\Delta) \times (\pi|_M) \rightarrow 0.
\]

Likewise, there is an exact sequence in \( \text{Alg}(M_{n+1}) \)

\[
0 \rightarrow \rho'|_M \times (\text{St}(-\Delta) \times \pi) \rightarrow \rho' \times (\text{St}(-\Delta) \times \pi) \rightarrow \rho' \times (\text{St}(-\Delta) \times \pi)|_M \rightarrow 0.
\]

Note that the submodules in the two sequences are isomorphic by Corollary 5.4. Furthermore, by the choice of \( \rho' \),

\[
\text{Ext}^i_{G_n} (\rho' \times (\text{St}(-\Delta) \times \pi)|_M, \pi_2) = 0 \text{ if } i \geq 0.
\]

Now we can apply the induction assumption to \( \rho' \times \text{St}(-\Delta) \times \pi \) and conclude that

\[
\text{Ext}^i_{G_n} ((\text{St}(\Delta)|_M) \times \pi, \pi_2) = 0 \text{ if } i > 0 \text{ and } = C \text{ if } i = 0.
\]

Hence, in order to establish the conjecture for the pair \( (\pi_1, \pi_2) \), it suffices to show that

\[
\text{Ext}^i_{G_n} (\text{St}(\Delta) \times (\pi|_M), \pi_2) = 0 \text{ if } i \geq 0,
\]

and to do this it suffices to show vanishing for each subquotient in the Bernstein-Zelevinsky filtration of \( \text{St}(\Delta) \times (\pi|_M) \). By Proposition 5.2 part (2), the derivatives of \( \text{St}(\Delta) \times (\pi|_M) \) are computed on the second factor. Thus, combining with the second adjointness formula, it suffices to show that

- \( \text{Ext}^i_{G_n} (\nu^{1/2} \text{St}(\Delta) \times \pi^{(i+1)}, \pi_2) = 0 \) for \( i, j \geq 0 \).

Alternatively, by reversing the roles of left and right derivatives, it suffices to show that

- \( \text{Ext}^j_{G_n} (\nu^{-1/2} \text{St}(\Delta) \times \pi^{(i+1)}, \pi_2^{(i)}) = 0 \) for \( i, j \geq 0 \).

Hence it suffices to show that the Bernstein center spectra of \( \nu^{1/2} \text{St}(\Delta) \times \pi^{(i+1)} \) and of \( \pi^{(i)}_2 \) are different for all \( i \), or, they are different for \( \nu^{-1/2} \text{St}(\Delta) \times \pi^{(i+1)} \) and \( \pi^{(i)}_2 \) for all \( i \). The strategy is to show that, if both statements fail, then \( m_2 \) contains linked segments.
5.2. **Combinatorics.** Let $m = \{\Delta_1, \ldots, \Delta_k\}$ be a multisegment. Then $St(m)$ is generic but reducible if some segments are linked. However, if $\Delta_i$ and $\Delta_j$ are linked, then they can be replaced by $\Delta_i \cap \Delta_j$ and $\Delta_i \cup \Delta_j$. This process (called recombination henceforth) eventually leads to a generic segment such that the corresponding irreducible generic representation is the unique generic subquotient in $St(m)$. Important observation is that the points in the spectrum of the Bernstein center are uniquely represented by generic multisegments! The following is a key lemma.

**Lemma 5.5.** Let $m$ be a generic multisegment and $m'$ a multisegment obtained by truncating $m$ from the right. Then the generic segment corresponding to $m'$ by recombination is also obtained from $m$ by truncating from the right.

**Proof.** This proved by induction on the number of steps in the recombination process. If that number is 0 there is nothing to prove. Otherwise there is a pair of linked segments $\Delta'$ and $\Delta''$ in $m'$ such that the first step in the recombination is replacing $\Delta'$ and $\Delta''$ by $\Delta' \cap \Delta''$ and $\Delta' \cup \Delta''$. It is trivial to see that the resulting multisegment is also obtained by right truncation from $m$. The proof follows by induction. □

5.3. **Finishing the proof.** Let $l = b - a + 1$ be the relative length of $\Delta$. We note that $(i) \pi_2$ is glued from $St(m'_2)$ where $m'_2$ runs over all multisegments obtained from $m_2$ by truncating from the right $i$-times. By the previous lemma the Bernstein center spectrum of $(i) \pi_2$ is given by such generic multisegments. Likewise, $St(\Delta) \times \pi^{(i+1)}$ is glued from $St(\Delta) \times St(m')$ where $m'$ runs over all multisegments obtained from $m$ by truncating from the left $i+1$-times, and to determine the Bernstein center spectrum we need to consider only generic $m'$. However, $\{\Delta\} \cup m'$ needs not be generic. There could be segments in $m'$ linked to $\Delta$. Since $\Delta$ is not linked to any segment in $m$ and $m'$ is obtained from $m$ by left truncation, it follows that linking occurs over the right end point of $\Delta$. Let $\Delta_0$ be the longest segment in $m'$ linked to $\Delta$. It is easy to see that $\Delta \cup \Delta_0$ is a segment in the generic multisegment corresponding to $\{\Delta\} \cup m'$ by the recombination process. Note that $\Delta \cup \Delta_0$ starts with $\nu^a \rho$ and is of relative length at least $l$. Thus the Bernstein spectra of $\nu^{1/2}(St(\Delta) \times \pi^{(i+1)})$ and $(i) \pi_2$ can have a point in common only if $m_2$ contains a segment starting with $\nu^{a+1/2} \rho$ and of relative length at least $l$. Similarly, the Bernstein spectra of $\nu^{-1/2}(St(\Delta) \times \pi^{(i+1)})$ and $\pi_2^{(i)}$ can have a point in common only if $m_2$ contains a segment ending with $\nu^{b-1/2} \rho$ and of length at least $l$. In other words we have constructed a pair of linked segments in $m_2$, a contradiction. This completes the proof of the Ext-vanishing theorem.

6. **Appendix**

Let $O$ be the ring of integers in $F$, and $\varpi$ the uniformizer. Let $\psi$ be the character of $F$ of conductor $\varpi O$. Let $G = GL_n(F)$ and $U$ the group of unipotent upper triangular matrices. Let $\psi_U : U \to \mathbb{C}$ be the Whittaker character defined by

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n})$$
where $u_{i,j}$ denote the entries of the matrix $u$. Let $I \subseteq G$ be the pro-$p$ radical of the Iwahori group consisting of matrices with coefficients in $O$ such that the entries below the diagonal are divisible by $\varpi$ and on the diagonal are congruent to 1 modulo $\varpi$. Smooth representations $\pi$ of $G$ generated by $\pi^I$ form a direct summand of the category $\text{Alg}(G)$, see [BS]. These representations are also known as depth zero representations of $G$. The group $I$ has a decomposition

$$I = (I \cap U^T)(I \cap T)(I \cap U)$$

where $T$ is the torus of diagonal matrices. Let $\psi^I$ be the character of $I$, trivial on the first two factors and given by the restriction of $\psi_U$ on the last factor.

**Theorem 6.1.** Let $\pi$ be a smooth $G$-module generated by $\pi^I$. Then the natural map $\pi^I, \psi^I \to \pi_U, \psi_U$ is an isomorphism of vector spaces.

**Proof.** We shall prove this firstly for $V = \text{ind}_G^I(1)$. Let $N$ be the normalizer of $T$ in $G$. We shall exploit the decomposition $G = INI$ to compute $V^I, \psi^I$. Let $w \in N$, and $I_w = I \cap w^{-1}Iw$. The double coset $IwI$ supports $V^I, \psi^I$ if and only if the restriction of $\psi^I$ to $I_w$ is trivial. On the other hand, we shall exploit the decomposition $G = INU$ to compute $V_U, \psi_U$. As an $U$-module, $V$ decomposes as a direct sum of $V_w$ where $V_w$ is the submodule of functions supported on $IwU$. Note that $V_w = \text{ind}_{U_w}^U(1)$ where $U_w = U \cap w^{-1}Iw$. By Frobenius reciprocity, $(V_w)_{U, \psi_U} = \mathbb{C}$ if and only if the restriction of $\psi$ to $U_w$ is trivial. It is easy to see that the two conditions are identical, for a given $w$. Moreover, in that case $IwI = Iw(I \cap U)$, and the non-zero function in $V^I, \psi^I$ supported on $IwI$ belongs to $V_w$. It projects non-trivially to $(V_w)_{U, \psi_U} = \mathbb{C}$ and this completes the proof for $V = \text{ind}_G^I(1)$.

For general $\pi$, we have a surjection $\text{ind}_G^I(\pi^I) \to \pi$. Let $\pi_1$ be the kernel of this surjection. We have an exact sequence

$$\text{ind}_G^I(\pi_1^I) \to \text{ind}_G^I(\pi^I) \to \pi \to 0.$$

Observe that $\text{ind}_G^I(X) \cong \text{ind}_G^I(1) \otimes X$, for any vector space $X$ with the trivial action of $I$. Hence the theorem holds for the first two modules in the exact sequence, and hence for $\pi$ by an easy diagram chase. $\square$

Combining with $(\pi^I, \psi^I)^\vee \cong (\pi^\vee)^I, \bar{\psi}^I$ we have:

**Corollary 6.2.** Let $\pi$ be a smooth $G$-module generated by $\pi^I$. Then there exists an isomorphism of vector spaces

$$(\pi_{U, \psi_U})^\vee \cong (\pi^\vee)_{U, \bar{\psi}_U}.$$

**Corollary 6.3.** Lemma 2.1 is true for smooth representations of $G_n$ generated by $I$-fixed vectors.

We now give a simple(r) proof of uniqueness of Whittaker functionals for irreducible representations of $G$ generated by $I$-fixed vectors, i.e. of depth 0. We need the following.
Theorem 6.4. The Hecke algebra $\mathcal{H}_\psi$ of compactly supported $(I, \bar{\psi})$-biinvariant functions on $G$ is commutative.

Proof. This is a classical Gelfand pair argument. Let $g^\top$ denote the opposite diagonal transpose of $g \in G$. The involution $g \mapsto g^\top$ preserves the pair $(I, \bar{\psi})$ and hence define an (anti) involution of the Hecke algebra. Using the decomposition $G = INI$ we compute the support of the algebra and note that the involution preserves the double cosets in the support. Hence the (anti) involution is equal to the identity i.e. $\mathcal{H}_\psi$ is commutative. □

If $\pi$ is a smooth $G$-module, then $\pi^\top, \psi'$ is naturally a $\mathcal{H}_\psi$-module, and for irreducible $\pi$ it can be at most one-dimensional since $\mathcal{H}_\psi$ is commutative. Thus we have obtained:

Corollary 6.5. Let $\pi$ be a smooth, irreducible $G$-module, generated by $\pi^\top$. Then

$$\dim(\pi_{U, \psi_U}) \leq 1.$$ 

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