Group Theoretical Approach to the Coherent and the Squeeze States of a Time-Dependent Harmonic Oscillator with a Singular Term

Jung Kon Kim and Sang Pyo Kim

Department of Physics, Kunsan National University, Kunsan 573-360, Korea

Abstract

For a time-dependent harmonic oscillator with an inverse squared singular term, we find the generalized invariant using the Lie algebra of $SU(2)$ and construct the number-type eigenstates and the coherent states using the spectrum-generating Lie algebra of $SU(1,1)$. We obtain the evolution operator in both of the Lie algebras. The number-type eigenstates and the coherent states are constructed group-theoretically for both the time-independent and the time-dependent harmonic oscillators with the singular term. It is shown that the squeeze operator transforms unitarily the time-dependent basis of the spectrum-generating Lie algebra of $SU(1,1)$ for the generalized invariant, and thereby evolves the initial vacuum into a final coherent vacuum.
I. INTRODUCTION

The adiabatic method has been one of the most general and frequently used approximate methods for time-dependent quantum systems [1]. The task to find the exact quantum states of these systems, however, is mathematically difficult in general. The exact quantum states are known only for a few systems. The solutions for the physically interesting, nonlinear, time-dependent classical harmonic oscillators even dated back to Ermakov [2], and they are now referred to as the Ermakov system.

There are several methods to find the exact quantum states for such time-dependent quantum systems. The first method which was developed several decades ago by Lewis and Riesenfeld [3] introduces an interesting quantum-mechanically conserved quantity, now known as either the Lewis-Riesenfeld invariant or the generalized invariant, for a time-dependent quantum harmonic oscillator and finds the exact quantum states in terms of the eigenstates of the invariant up to some time-dependent phase factors. The Ermakov system also has a generalized invariant called the Lewis-Ray-Reid invariant [4]. The second method which was developed by Wei and Norman [5] finds directly the evolution operator in the disentangled exponentials for a time-dependent quantum system that has some Lie algebraic structure. The path integral [6] is still another method.

As one of the exactly solvable time-dependent quantum systems, harmonic oscillators have been studied intensively and have had a wide application for a long period of time in various branches of physics from quantum optics to gravitational wave detection. Among the various methods introduced to find the exact quantum states of time-dependent harmonic oscillators, the most frequently used methods are the generalized invariant method and the evolution operator method that are based on the Lie algebras of $SU(2)$ and $SU(1, 1)$. First, the generalized invariant method was originally introduced for a time-dependent quantum harmonic oscillator with the aid of an auxiliary equation [3]. There are further elaborations and diverse applications of this method [7]. The second method is the evolution operator method in disentangled exponentials [5] as further elaborated in Ref. [8]. Using one variant
of the generalized invariant method one of the authors (S.P.K) has recently found a connection between the classical and the quantum harmonic oscillators, and has obtained explicitly a class of exactly solved time-dependent quantum harmonic oscillators \[9\]. Quite independently of these two methods, Popov and Perelomov have found the exact quantum states of a time-dependent quantum harmonic oscillator in the Gaussian form using the classical integrals of motion \[10\].

In this paper, we shall extend the group-theoretical approach to a time-dependent harmonic oscillator with an inverse squared singular term (hereafter, anharmonic oscillator will refer to harmonic oscillator with an inverse squared singular term). The time-dependent quantum anharmonic oscillator has already been analyzed as an uncoupled Ermakov system \[4\] and was independently treated in Ref. \[11\]. In particular, the time-independent anharmonic oscillator can be regarded as a reduced system of the Calogero model in the center of mass system \[12\]. The main purpose of this paper is to uncover the underlying group structure for the time-dependent anharmonic oscillator by introducing the two relevant Lie algebras of $SU(2)$ and $SU(1,1)$, and to construct the exact quantum states group-theoretically. It is shown that the Lie algebra of $SU(2)$ is useful in finding the generalized invariant, whereas the spectrum-generating Lie algebra of $SU(1,1)$ yields the number-type eigenstates and the coherent states. It is observed that in the Lie algebra of $SU(2)$ the generalized invariants for both the time-dependent harmonic oscillator and the time-dependent anharmonic oscillator have the same invariant equation, the only modification being in the representation of the Lie algebra, whose solutions can be read from the classical integrals of motion for the time-dependent harmonic oscillator \[3\]. We also obtain the evolution operator in both of the Lie algebras, which is again expressed by the classical integrals as expected. Because the exact quantum states are determined by the eigenstates of the generalized invariant up to some time-dependent phase factors, we introduce the spectrum-generating Lie algebra of $SU(1,1)$ for the generalized invariant and construct the number-type eigenstates and the coherent states. Finally, it is found that the squeeze operator is simply the Hermitian conjugate of the evolution operator, transforms unitarily the basis of the Lie algebra of
SU(1, 1) for the generalized invariant, and thereby evolves the initial vacuum into a coherent vacuum.

The organization of this paper is as follows: In Sec. II, we introduce the two Lie algebras of SU(2) and SU(1, 1) for a time-independent anharmonic oscillator. In Sec. III, we find the generalized invariant and the evolution operator for a time-dependent anharmonic oscillator. In Sec. IV, we construct the number-type eigenstates and the coherent states for both the time-independent and the time-dependent anharmonic oscillators. In Sec. V, we show that the squeeze operator transforms unitarily the basis of the Lie algebra of SU(1, 1) and is the Hermitian conjugate of the evolution operator.

II. SU(2) AND SU(1, 1) GROUPS OF THE ANHARMONIC OSCILLATOR

A time-independent quantum anharmonic oscillator,

\[ \hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega_0^2 \hat{q}^2}{2} + \frac{c}{\hat{q}^2} \]  \hspace{1cm} (2.1)

where \( c \) is a constant, has the Lie algebra of SU(2),

\[ \left[ \frac{i}{2} \hat{L}_0, \hat{L}_\pm \right] = \pm \hat{L}_\pm, \left[ \hat{L}_+, \hat{L}_- \right] = 2 \left( \frac{i}{2} \hat{L}_0 \right), \]  \hspace{1cm} (2.2)

with the following choice of the Hermitian basis:

\[ \hat{L}_- = \frac{\hat{p}^2}{2} + \frac{c}{\hat{q}^2}, \hat{L}_0 = \hat{p} \hat{q} + \frac{\hat{q} \hat{p}}{2}, \hat{L}_+ = \frac{\hat{q}^2}{2}. \]  \hspace{1cm} (2.3)

One may introduce the standard basis for the Lie algebra of SU(1, 1) as

\[ \hat{K}_0 = \frac{1}{2} \left( \hat{L}_- + \hat{L}_+ \right), \hat{K}_\pm = \frac{1}{2} \left( \hat{L}_+ - \hat{L}_- \mp i \hat{L}_0 \right), \]  \hspace{1cm} (2.4)

whose group structure is

\[ \left[ \hat{K}_0, \hat{K}_\pm \right] = \pm \hat{K}_\pm, \left[ \hat{K}_+, \hat{K}_- \right] = -2 \hat{K}_0. \]  \hspace{1cm} (2.5)

In particular, one can represent the standard basis of the SU(1, 1) algebra as
in terms of the creation and the annihilation operators of a harmonic oscillator. It should be noted that the anharmonic oscillator is one of the two perturbations of the harmonic oscillator which have finite-dimensional Lie algebras (the other being that with a linear force term). All other extended harmonic oscillators with perturbation terms except for those with the linear force term and the inverse squared singular term have infinite-dimensional Lie algebra, for which the Weyl ordered basis in Ref. [13] may be used. The bases in Eqs. (2.3) and (2.4) are chosen from among the bases of such Lie algebras for a later use. One can also use the following basis:

\[
\hat{\Gamma}_0 = \frac{1}{2} \hat{L}_0, \quad \hat{\Gamma}_\pm = -2\hat{L}_{\pm} \pm \frac{1}{8} \hat{L}_+ ,
\]

as a spectrum-generating algebra of \(SU(1,1)\) [14].

**III. TIME-DEPENDENT QUANTUM ANHARMONIC OSCILLATOR**

We now turn to a time-dependent quantum anharmonic oscillator of the form of Eq. (2.1), but with a variable frequency squared:

\[
\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2(t)\hat{q}^2}{2} + \frac{c}{\hat{q}^2}.
\]

The classical equation of motion is the equation for the anharmonic oscillator:

\[
\ddot{q} + \omega^2(t)q - \frac{2c}{q^3} = 0.
\]

The time-dependent anharmonic oscillator was first analyzed by Camiz et al. [11] and independently in Ref. [6]. The Lewis-Ray-Reid invariant was used for the time-dependent anharmonic oscillator as an uncoupled Ermakov system [4]. The exact eigenfunctions in Gaussian form up to some time-dependent factors are similar in many respects to those for the Caldirola-Kanai oscillator [15]. In this paper we shall, however, uncover the underlying group structure introduced in Sec. II and elaborate further on the coherent and squeezed...
states of the time-dependent anharmonic oscillator. Two methods, the generalized invariant method and the evolution operator method, will be employed below for the time-dependent anharmonic oscillator of Eq. 3.1.

A. Generalized Invariant Method

The generalized invariant method finds quantum-mechanical invariants for a time-dependent quantum system and obtains the exact quantum states as the eigenstates of the invariant. In the Heisenberg picture, the quantum-mechanical invariants obey the equation (in units of $\hbar = 1$)

$$\frac{d}{dt} \hat{I}(t) = \frac{\partial}{\partial t} \hat{I}(t) - i \left[ \hat{I}(t), \hat{H}(t) \right] = 0.$$  \hspace{1cm} (3.3)

Since the invariant equation is linear, the quantum-mechanical invariants form a linear space of operators. In particular, when the Hamiltonian has Lie algebras, the quantum-mechanical invariants belong to the same Lie algebras. In the case of a time-dependent harmonic oscillator, the Hamiltonian has the Lie algebras of $SU(2)$ and $SU(1, 1)$, therefore the generalized invariant is a quantum-mechanical invariant that has the same Lie algebras of $SU(2)$ and $SU(1, 1)$. The generalized invariant is determined uniquely up to a one-parameter coefficient, so one gets the same generalized invariant, regardless of the basis used, as long as the basis spans the same Lie algebra as the Hamiltonian. In our case of the anharmonic oscillator, we shall get the same generalized invariant irrespective of whether the basis in Eq. (2.3) or the basis in Eq. (2.4) is used.

We use the basis in Eq. (2.3) rather than the basis in Eq. (2.4) in analogy to the harmonic oscillator case, and we find a generalized invariant of the form

$$\hat{I}(t) = \sum_{k=0,\pm} g_k(t) \hat{L}_k.$$  \hspace{1cm} (3.4)

The invariant equation can be written as a vector equation:
\[
\frac{d}{dt} \begin{pmatrix} g_-(t) \\ g_0(t) \\ g_+(t) \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ \omega^2(t) & 0 & -1 \\ 0 & 2\omega^2(t) & 0 \end{pmatrix} \begin{pmatrix} g_-(t) \\ g_0(t) \\ g_+(t) \end{pmatrix}.
\]

(3.5)

It is worthy to compare Eq. (3.5) with Eq. (11) of Ref. [9(a)] and Eq. (3.5) of Ref. [9(c)] for the unperturbed time-dependent quantum harmonic oscillator

\[
\hat{H}_0(t) = \frac{\hat{p}^2}{2} + \frac{\omega^2(t)\hat{q}^2}{2}.
\]

(3.6)

The only differences between these two cases are the elements \(\hat{L}_- = \hat{p}^2/2 + c/\hat{q}^2\) of \(SU(2)\) for the anharmonic oscillator and \(\hat{L}_- = \hat{p}^2/2\) (i.e., \(c = 0\)) for the harmonic oscillator. In either case, both of the Hamiltonians have the same Lie algebraic structure

\[
\hat{H}(t) = \hat{L}_- + \omega^2(t)\hat{L}_+.
\]

(3.7)

Therefore, one may expect the same invariant equation. However, it is only through a group-theoretical approach that one can manifestly explain the reason for the same invariant equation (Eq. (3.5)) or the same auxiliary equation [3]

\[
\ddot{\rho}(t) + \omega^2(t)\rho(t) = \frac{1}{\rho^2(t)}
\]

(3.8)

by using the following substitution as was done in Refs. [9(a)] and [9(c)];

\[
g_-(t) = \rho^2(t) \quad g_0(t) = -\rho(t)\dot{\rho}(t) \quad g_+(t) = \ddot{\rho}(t) + \frac{1}{\rho^2(t)}.
\]

(3.9)

From the connection [9(a), (c)] between the classical and quantum harmonic oscillators, the generalized invariant is given straightforwardly by

\[
\begin{pmatrix} g_-(t) \\ g_0(t) \\ g_+(t) \end{pmatrix} = \begin{pmatrix} Q_0^2 & -2Q_1Q_0 & Q_1^2 \\ -P_1Q_0 & P_0Q_0 + P_1Q_1 & -P_0Q_1 \\ P_1^2 & -2P_1P_0 & P_0^2 \end{pmatrix} \begin{pmatrix} g_-(t_0) \\ g_0(t_0) \\ g_+(t_0) \end{pmatrix}
\]

(3.10)

where

\[
\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} P_0(t, t_0) & P_1(t, t_0) \\ Q_1(t, t_0) & Q_0(t, t_0) \end{pmatrix} \begin{pmatrix} p(t_0) \\ q(t_0) \end{pmatrix}
\]

(3.11)

represents the classical integrals of motion for the harmonic oscillator in Eq. (3.6).
The evolution operator method evaluates directly the evolution operator for a time-dependent quantum system with a Lie algebraic structure. There are the two typical methods to express the evolution operator. One method is the global exponential operator of Magnus [16]. The other is the product of disentangled exponential operator introduced by Wei and Norman [5], which has recently been applied to time-dependent harmonic oscillators [8]. We shall follow the technique developed in Ref. [8].

The evolution operator for the quantum anharmonic oscillator in Eq. (3.1) obeys the evolution equation

$$i \frac{\partial}{\partial t} \hat{U}(t, t_0) = \hat{H}(t) \hat{U}(t, t_0) \quad (3.12)$$

with the initial value $\hat{U}(t_0, t_0) = 1$. The evolution operator is unitary for a Hermitian Hamiltonian. We search for the disentangled evolution operator of the form

$$\hat{U}(t, t_0) = \exp \left( il_+(t) \hat{L}_+ \right) \exp \left( l_0(t) \frac{i}{2} \hat{L}_0 \right) \exp \left( il_-(t) \hat{L}_- \right) \quad (3.13)$$

in the basis of Eq. (2.3) and of the form

$$\hat{U}(t, t_0) = \exp \left( ik_+(t) \hat{K}_+ \right) \exp \left( k_0(t) \hat{K}_0 \right) \exp \left( ik_-(t) \hat{K}_- \right) \quad (3.14)$$

in the basis of Eq. (2.4). For the evolution operator in Eq. (3.13), we have a set of coupled differential equations:

$$\dot{l}_+(t) + l_+^2(t) + \omega^2(t) = 0$$

$$\dot{l}_0(t) + 2il_+(t) = 0$$

$$\dot{l}_-(t) + i \exp l_0(t) = 0 \quad (3.15)$$

with the initial values $l_0(t_0) = l_\pm(t_0) = 0$. One solution is found to be

$$l_+(t) = \frac{\dot{x}(t)}{x(t)} \quad (3.16)$$
where

$$\ddot{x}(t) + \omega^2(t)x(t) = 0. \quad (3.17)$$

$x(t)$ satisfies the classical equation of motion for the harmonic oscillator in Eq. (3.6), as expected [8]. The other solutions are obtained by substituting Eq. (3.16) into Eq. (3.15) and integrating.

For the evolution operator in Eq. (3.14), we have the following set of differential equations:

$$\dot{k}_+ (t) + \frac{1}{2} \left( \omega^2(t) - 1 \right) (k_+^2 - 1) - i \left( \omega^2(t) + 1 \right) k_+ (t) = 0,$$

$$\dot{k}_0 (t) + \left( \omega^2(t) - 1 \right) k_0 (t) - i \left( \omega^2(t) + 1 \right) = 0,$$

$$\dot{k}_- (t) - \frac{1}{2} \left( \omega^2(t) - 1 \right) \exp k_0 (t) = 0 \quad (3.18)$$

with the initial values $k_0 (t_0) = k_\pm (t_0) = 0$. One solution is found to be

$$k_+ (t) = \frac{2}{\omega^2(t) - 1} \frac{\dot{y}(t)}{y(t)}, \quad (3.19)$$

where

$$\ddot{y}(t) + \left\{ \frac{2\omega(t)\dot{\omega}(t)}{\omega^2(t) - 1} - i \left( \omega^2(t) + 1 \right) \right\} \dot{y}(t) - \frac{(\omega^2(t) - 1)^2}{4} y(t) = 0. \quad (3.20)$$

Again, the other solutions are obtained by substituting Eq. (3.19) into Eq. (3.18) and integrating.

### IV. SQUEEZE OPERATOR AND COHERENT STATES

#### A. Time-Independent Harmonic Oscillator

We review briefly the spectrum-generating algebra of $SU(1, 1)$ for the time-independent anharmonic oscillator of Eq. (2.1). In order to find the exact eigenstates, we rescale the basis of Eq. (2.4):
\[ \hat{K}_0 = \frac{1}{2} \left( \frac{\omega_0 \hat{q}^2 + \hat{\mathbf{p}}^2 / \omega_0}{2} + \frac{c}{\omega_0 \hat{q}^2} \right), \]
\[ \hat{K}_\pm = \frac{1}{2} \left( \frac{\omega_0 \hat{q}^2 - \hat{\mathbf{p}}^2 / \omega_0}{2} - \frac{c}{\omega_0 \hat{q}^2} \mp i \frac{\hat{\mathbf{q}} \hat{\mathbf{p}} + \hat{\mathbf{p}} \hat{\mathbf{q}}}{2} \right). \] (4.1)

The time-independent anharmonic oscillator can now be rewritten simply as

\[ \hat{H} = 2\omega_0 \hat{K}_0. \] (4.2)

The basis in Eq. (4.1) still has the same group structure as that of Eq. (2.5) and forms the spectrum-generating algebra of \( SU(1, 1) \) for the time-independent anharmonic oscillator. It is Eq. (4.2) together with Eq. (2.5) that determines the eigenvalues and group-theoretically generates the spectrum of eigenstates of the anharmonic oscillator.

The Casimir operator for the basis of Eq. (4.1) is

\[ \hat{C} = \frac{1}{2} \hat{K}_0^2 - \frac{1}{4} \left( \hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+ \right). \] (4.3)

Using the representation in Eq. (4.1), the Casimir operator becomes

\[ \hat{C} = k_0(k_0 - 1) \hat{\varepsilon} = -\left( \frac{3 - 8c}{16} \right) \hat{\varepsilon}, \] (4.4)

where \( \hat{\varepsilon} \) is the identity element, so we have two unitary irreducible representations \( D^{\dagger}(k_0) \) that have positive discrete series [17] with Bargmann indices [18]

\[ k_0 = \frac{1}{2} \left( 1 \pm \sqrt{\frac{1}{4} + 2c} \right), \] (4.5)

and consist of the eigenstates of \( \hat{K}_0 \),

\[ \hat{K}_0 |n, k_0\rangle = (n + k_0) |n, k_0\rangle, \] (4.6)

where \( n = 0, 1, 2, \cdots \). The positivity of the Bargmann index is ensured by the condition \( c > -1/8 \) that prevents the wave functions from falling into the center of motion [19]. In the pure harmonic oscillator representation \( (c = 0) \), the representation with the Bargmann index \( k_0 = 1/4 \) corresponds to even photon-number states and \( k_0 = 3/4 \) to odd photon-number states [20]. The operators \( \hat{K}_+ \) and \( \hat{K}_- \) acting on \( |n, k_0\rangle \) increase and decrease the
eigenvalues by one unit; thus, they behave as the raising and the lowering operators of the number-type eigenstates. After some algebra, the number-type states are found to be

\[ |n, k_0\rangle = \left( \frac{\Gamma(2k_0)}{n!\Gamma(n + 2k_0)} \right)^{1/2} (\hat{K}_+)^n |0, k_0\rangle. \quad (4.7) \]

Now, we introduce two definitions of the coherent state. First, following Perelomov [20],

the coherent state defined as an eigenstate of \( \hat{K}_- \),

\[ \hat{K}_- |z, k_0\rangle = z |z, k_0\rangle, \quad (4.8) \]

is

\[ |z, k_0\rangle = \left( 1 - |z|^2 \right)^{k_0} \sum_{n=0}^{\infty} \left( \frac{\Gamma(2k_0)}{n!\Gamma(n + 2k_0)} \right)^{1/2} z^n |n, k_0\rangle. \quad (4.9) \]

Second, the coherent state is defined inequivalently as

\[ |z, k_0\rangle = \hat{S}(\zeta) |0, k_0\rangle, \quad (4.10) \]

where

\[ \zeta = (\tanh^{-1}(zz^*)^{1/2} \frac{z}{z^*})^{1/2}, \quad (4.11) \]

and

\[ \hat{S}(\zeta) = \exp \left( \zeta \hat{K}_+ - \zeta^* \hat{K}_- \right). \quad (4.12) \]

The operator in Eq. (4.12) is the squeeze operator [21]. The coherent state in Eq. (4.10) is obviously not an eigenstate of \( \hat{K}_- \). Using the disentangling technique, one can express the squeeze operator in the form [22]

\[ \hat{S}(\zeta) = \exp \left( z\hat{K}_+ \right) \exp \left( \log(1 - |z|)^2 \hat{K}_0 \right) \exp \left( -z^* \hat{K}_- \right), \quad (4.13) \]

and thereby the coherent states as

\[ |z, k_0\rangle = \left( 1 - |z|^2 \right)^{k_0} \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + 2k_0)}{n!\Gamma(2k_0)} \right)^{1/2} z^n |n, k_0\rangle. \quad (4.14) \]
It should be noted that the coherent state in Eq. (4.8) defined as an eigenstate of the lowering operator is not equivalent to that in Eq. (4.14) defined by the squeeze operator. This is in strong contrast with the harmonic oscillator case for which the corresponding group is the Heisenberg group and the two representations are equivalent [20]. The coherent states of $SU(1,1)$ are discussed with an emphasis on the harmonic oscillator representation in Ref. [24].

In coordinate representation, the orthonormal eigenfunction and energy eigenvalue are found in terms of the Laguerre polynomials [25] as

$$
\Phi_n(q) = \left( \frac{\omega_0^{1/2} n!}{\Gamma(n + 2k_0)^3} \right)^{1/2} \exp \left( -\omega_0 q^2 / 2 \right) (\omega_0 q^2)^{k_0 - 1/4} L_n^{(2k_0 - 1)}(\omega_0 q^2),
$$

$$
E_n = 2\omega_0(n + k_0). \tag{4.15}
$$

Equation (4.15) agrees with the group-theoretical result of Eq. (4.6), as expected.

B. Time-Dependent Anharmonic Oscillator

The exact quantum states of the time-dependent harmonic oscillator of Eq. (3.6) are determined from the number states of its generalized invariant up to some time-dependent phase factors. Similarly, the exact quantum states of the time-dependent anharmonic oscillator can be found from the number-type eigenstates of the generalized invariant in Eq. (3.4) up to some time-dependent phase factors. This is one of the great advantages of the generalized invariant method.

In order to find the eigenstates, first let us introduce a canonical transformation

$$
\hat{q}_c = \hat{q}, \quad \hat{p}_c = \hat{p} + g_0(t) \hat{q}, \tag{4.16}
$$

and rewrite the generalized invariant as

$$
\tilde{I}(t) = \frac{g_-(t)\hat{p}_c^2}{2} + \frac{\omega_0^2 \hat{q}_c^2}{2g_-(t)} + \frac{cg_-(t)}{\hat{q}_c^2} \tag{4.17}
$$

where
\[ \omega_0^2 = g_+(t)g_-(t) - g_0^2(t) \] (4.18)

is a constant of motion. The constancy is a consequence of the invariance of Eq. (3.3) [3].

In analogy with Eq. (4.1) we now introduce a time-dependent basis of \( SU(1,1) \) as

\[
\begin{align*}
\hat{K}_{c,0}(t) &= \frac{1}{2} \left( \frac{\omega_0 \hat{q}_c^2 / g_-(t) + g_+(t) \hat{p}_c^2 / \omega_0}{2} + \frac{cg_-(t)}{\omega_0 \hat{q}_c^2} \right), \\
\hat{K}_{c,\pm}(t) &= \frac{1}{2} \left( \frac{\omega_0 \hat{q}_c^2 / g_-(t) - g_+(t) \hat{p}_c^2 / \omega_0}{2} - \frac{cg_-(t)}{\omega_0 \hat{q}_c^2} \mp i \frac{\hat{p}_c \hat{q}_c + \hat{q}_c \hat{p}_c}{2} \right). \quad (4.19)
\end{align*}
\]

Of course, it holds that \( \hat{K}_{c,+}^* = \hat{K}_{c,-} \). The time-dependent anharmonic oscillator can still be rewritten as

\[ \hat{I}(t) = 2\omega_0 \hat{K}_{c,0}(t). \] (4.20)

We obtain the Casimir operator just by replacing the basis of Eq. (4.1) with the basis of Eq. (4.19) in Eq. (4.3). The Bargmann index, however, does not change and is given by the same value as in Eq. (4.3). The only modification is the number-type eigenstate

\[ \hat{K}_{c,0}(t) |n, k_0, t\rangle = (n + k_0) |n, k_0, t\rangle. \] (4.21)

One of the properties of the generalized invariant is that the eigenvalues are always time-independent, whereas the eigenstates are time-dependent. By applying the raising operators we obtain the number-type eigenstate

\[ |n, k_0, t\rangle = \left( \frac{\Gamma(2k_0)}{n! \Gamma(n + 2k_0)} \right)^{1/2} \hat{K}_{c,+}^n |0, k_0, t\rangle \] (4.22)

and the coherent state

\[ |z, k_0, t\rangle = (1 - |z|^2)^{k_0} \sum_{n=0}^{\infty} \left( \frac{\Gamma(2k_0)}{n! \Gamma(n + 2k_0)} \right)^{1/2} |n, k_0, t\rangle. \] (4.23)

After a bit of algebra gymnastics, we find that

\[ \frac{\partial}{\partial t} \hat{K}_{c,t}(t) = -\frac{1}{2g_-(t)} \frac{dg_-(t)}{dt} \hat{K}_0(t) - i \frac{g_-(t)}{2\omega_0} \frac{dg_0(t)}{dt} \left( \hat{K}_{c,+}(t) - \hat{K}_{c,-}(t) \right) \] (4.24)

and
\[
\langle n, k_0, t \mid \frac{\partial}{\partial t} \mid n, k_0, t \rangle = -i \frac{g_- (t)}{2\omega_0} \frac{d}{dt} \left( \frac{g_0 (t)}{g_- (t)} \right) (n + k_0).
\]

(4.25)

The expectation value of the Hamiltonian of the anharmonic oscillator in Eq. (3.1) is also given by

\[
\langle n, k_0, t \mid \hat{H}(t) \mid n, k_0, t \rangle = \frac{\omega_0^2 + g_0^2 (t) + \omega^2 (t) g_-^2 (t)}{\omega_0 g_- (t)} (n + k_0).
\]

(4.26)

Equations (4.25) and (4.26) for the anharmonic oscillator are the same as Eq. (36) in Ref.[9.(b)] and Eqs. (4.6) and (4.8) in Ref.[9(c)] for the harmonic oscillator of Eq. (3.6), except for a modification by \((n + k_0)\). Equation (4.24), however, differs from Eq. (4.5) in Ref.[9(c)] because in our case the relevant algebra that was used to construct the number-type eigenstate was that of \(SU(1,1)\) compared to the Heisenberg group of the creation and the annihilation operators for a harmonic oscillator. It is a well-known result of the generalized invariant that the exact quantum states of the anharmonic oscillator are

\[
|\psi, n, k_0, t\rangle = \exp \left( -i \int (h(t) - i \epsilon (t)) \right) (n + k_0) \mid n, k_0, t \rangle
\]

(4.27)

where

\[
h(t) = \frac{\omega_0^2 + g_0^2 + \omega^2 (t) g_-^2 (t)}{\omega_0 g_- (t)}, \quad \epsilon (t) = -i \frac{g_- (t)}{2\omega_0} \frac{d}{dt} \left( \frac{g_0 (t)}{g_- (t)} \right).
\]

(4.28)

By replacing \(\omega_0 q^2/2\) with \(\omega_0 q^2/g_- (t)\) in Eq. (4.13), one obtains straightforwardly the orthonormal eigenfunctions

\[
\Phi_n (q, t) = \left( \frac{\omega^{1/2} n!}{\Gamma (n + 2 k_0)^3} \right)^{1/2} \exp \left( -\omega_0 q^2 / 2 g_- (t) \right) \left( \frac{\omega_0 g^2}{g_- (t)} \right)^{k_0 - 1/4} L_n^{(2k_0 - 1)} \left( \frac{\omega_0 q^2}{g_- (t)} \right).
\]

(4.29)

Furthermore, with the generalized canonical transformation

\[
\hat{\tilde{q}} = \frac{\hat{q}}{\rho}, \quad \hat{\tilde{p}} = \rho \hat{p} - \dot{\rho} \hat{q},
\]

(4.30)

one can transform the generalized invariant of Eq. (3.4) into a time-independent one:

\[
\hat{I}(t) = \frac{1}{2} \left( \hat{\tilde{p}}^2 + \hat{\tilde{q}}^2 \right) + \frac{c}{\hat{\tilde{q}}^2}.
\]

(4.31)
V. EVOLUTION OPERATOR METHOD

In Sec. III, we have seen that the time-dependent anharmonic oscillator of Eq. (3.1) preserves the Lie algebra of $SU(1, 1)$ with the time-dependent basis of Eq. (4.19) during its evolution, and that enables us to find the exact quantum states group-theoretically. In this section, we shall investigate systematically the evolution of the basis.

We assume that

$$\hat{H} = \frac{\hat{p}^2}{2} + \omega_0^2 \hat{q}^2 + \frac{c}{\hat{q}^2}, \quad t \leq t_0,$$

$$\hat{H}(t) = \frac{\hat{p}^2}{2} + \frac{\omega^2(t)\hat{q}^2}{2} + \frac{c}{\hat{q}^2}, \quad t \geq t_0,$$

(5.1)

for an initial time $t_0$. Then, it is easy to see that

$$\hat{K}_{c,0}(t) = u_0(t)\hat{K}_0 + u_+(t)\hat{K}_+ + u_-(t)\hat{K}_-$$

(5.2)

with

$$u_0(t) = \frac{1}{2} \left( g_-(t) + \frac{1}{g_-(t)} + \frac{g_0^2(t)}{\omega_0^2 g_-(t)} \right),$$

$$u_\pm(t) = \frac{1}{4} \left( -g_-(t) + \frac{1}{g_-(t)} \pm 2i \frac{g_0(t)}{\omega_0} + \frac{g_0^2(t)}{\omega_0^2 g_-(t)} \right),$$

(5.3)

and

$$\hat{K}_{c,+}(t) = \nu_0(t)\hat{K}_0 + \nu_+(t)\hat{K}_+ + \nu_-(t)\hat{K}_-$$

(5.4)

where

$$\nu_0(t) = \frac{1}{2} \left( -g_-(t) + \frac{1}{g_-(t)} - i \frac{g_0(t)}{\omega_0 g_-(t)} - \frac{g_0^2(t)}{\omega_0^2 g_-(t)} \right),$$

$$\nu_\pm(t) = \frac{1}{2} \left( g_-(t) \pm \frac{1}{g_-(t)} \mp i \frac{g_0(t)}{2 \omega_0 g_-(t)} \mp i \frac{g_0(t)}{\omega_0} \mp \frac{g_0^2(t)}{2 \omega_0^2 g_-(t)} \right),$$

(5.5)

and $\hat{K}_{c,-}(t) = \hat{K}_{c,+}^*(t)$. Before the initial time, the generalized invariant in Eq. (3.4) is simply the Hamiltonian of Eq. (2.1) itself, $g_\pm(t_0) = 1$, and $g_0(t_0) = 0$; so $u_0(t_0) = \nu_+(t_0) = 1$, and all the other terms vanish, as expected.
Since \( u(t) = u^*_t(t) \), one may write \( \hat{K}_{c,0}(t) \) as

\[
\hat{K}_{c,0}(t) = \hat{S}^\dagger(\xi)\hat{K}_0\hat{S}(\xi)
\]

(5.6)

where the squeeze operator

\[
\hat{S}(\xi) = \exp (\xi\hat{K}_+ - \xi^*\hat{K}_-)
\]

(5.7)

has the squeeze parameter

\[
(\xi\xi^*)^{1/2} = \frac{1}{2}\tanh^{-1}\left(\frac{2(u_+u^*_+)^{1/2}}{u_0}\right), \quad \frac{\xi^*}{\xi} = \frac{u^*_+}{u_+}.
\]

(5.8)

It follows then that the vacuum state at an arbitrary later time is the coherent state of the initial vacuum state

\[
|0, k_0, t_0\rangle = \hat{S}^\dagger(\xi)|0, k_0, t_0\rangle
\]

(5.9)

and that the number-type eigenstate is again the coherent state of the same initial number-type eigenstate

\[
|n, k_0, t\rangle = \hat{S}^\dagger(\xi)|n, k_0, t_0\rangle.
\]

(5.10)

In the case of the time-dependent harmonic oscillator, it was pointed out in Refs. [8] and [26] that the exact quantum states are the squeeze states of the initially prepared quantum states. This also holds for the time-dependent anharmonic oscillator.

In order to see how an initially prepared quantum state evolves into a quantum state an arbitrary later time, one can also use the evolution operator obtained in Sec. III. In spite of the facts that both of the evolution operators, Eqs. (3.13) and (3.14), are the same from the uniqueness of the evolution operator for the evolution equation (Eq. (3.12)) and that it is easier to solve Eq. (3.15) than Eq. (3.18), the evolution operator of Eq. (3.14) is more useful because the basis of the spectrum-generating algebra of the anharmonic oscillator is Eq. (4.1). The initial state prepared as one of the number-type eigenstate in Eq. (4.6),

\[
\hat{H}|n, k_0, t_0\rangle = 2\omega_0(n + k_0)|n, k_0, t_0\rangle,
\]

(5.11)
evolves at an arbitrary later time according to
\[ |n, k_0, t⟩ = \hat{U}(t, t_0) |n, k_0, t_0⟩ \quad (5.12) \]
into the same number-type eigenstate with the same energy eigenvalue \(2\omega_0(n + k_0)\) of the 
generalized invariant \(\hat{I}(t)\). From the evolution of the Hamiltonian in the Heisenberg picture, 
it follows that the basis transforms unitarily as
\[ \hat{K}_{c,0}(t) = \hat{U}(t, t_0) \hat{K}_{0} \hat{U}^†(t, t_0). \quad (5.13) \]
One can see that the squeeze operator is the Hermitian conjugate of the evolution operator:
\[ \hat{S}(\xi) = \hat{U}^†(t, t_0). \quad (5.14) \]
The generalized invariant method gives exactly the same result as the evolution operator method.

For example, we consider a class of time-dependent harmonic oscillators
\[ \hat{H}_0(t) = \frac{\hat{p}^2}{2} + \frac{\omega_0^2 t^\alpha \hat{q}^2}{2}, \quad (5.15) \]
which have the one-parameter-dependent generalized invariant [9(c)]
\[
\begin{align*}
g_-(t) &= \frac{\pi}{2} \left( \frac{2\omega_0}{2 + \alpha} \right)^{\alpha\nu} c_1 z^{2\nu} \left[ J^2_\nu(z) + N^2_\nu(z) \right], \\
g_0(t) &= -\frac{\pi}{2} \frac{\omega_0}{c_1} \left[ (\nu J_\nu(z) + z J'_\nu(z)) J_\nu(z) + (\nu N_\nu(z) + z N'_\nu(z)) N_\nu(z) \right], \\
g_+(t) &= \frac{\pi}{2} \left( \frac{2}{2 + \alpha} \right)^{-\alpha\nu} c_1 z^{-2\nu} \left[ (\nu J_\nu(z) + z J'_\nu(z))^2 + (\nu N_\nu(z) + z N'_\nu(z))^2 \right],
\end{align*}
\] (5.16)
where \(\nu = 1/(2 + \alpha)\), \(z = 2\omega_0 t^{(2+\alpha)/2}/(2 + \alpha)\), and \(c_1\) is a constant. Then, we are able to find the 
generalized invariant for the time-dependent anharmonic oscillator,
\[ H(t) = \frac{\hat{p}^2}{2} + \frac{\omega_0^2 t^\alpha \hat{q}^2}{2} + \frac{c}{q^2}, \quad (5.17) \]
by substituting Eq. (5.16) into Eq. (3.4) with the basis of Eq. (2.3) and by rewriting in the basis of Eq. (2.4).
VI. DISCUSSION

The quantum harmonic oscillator with an inverse squared singular term is a well-known problem that was exactly solved. In this problem, the singular term is related to the repulsive potential of the angular momentum in the radial motion of a three-dimensional isotropic harmonic oscillator, up to some power of the radius [14,19]. It is also related to the relative motion of the two-body Calogero model [12], a prototype of exactly solvable models. In spite of its familiarity in physics, the majority of approaches so far has been analytic rather than group theoretical. Even the group theoretical approach to this problem has not gone beyond the spectrum-generating algebra of $SU(1,1)$ and the spectrum of energy eigenvalues [14], in contrast with the harmonic oscillator problem which has been intensively studied group-theoretically based on the Heisenberg group and $SU(1,1)$.

In this paper, we considered the time-dependent harmonic oscillator with the inverse squared singular term. Even though the exact eigenfunctions of the time-dependent anharmonic oscillator were analytically found in Ref. [11], we uncovered the underlying groups $SU(2)$ and $SU(1,1)$, found the generalized invariant, and constructed the number-type eigenstates and coherent states group-theoretically, which are the main results of this paper. Based on the Lie algebra of $SU(2)$, it was observed that the generalized invariant in Eq. (3.4) for the time-dependent anharmonic oscillator in Eq. (3.1) had the same invariant equation (Eq. (3.6)) as was determined by the classical integrals of motion for the time-dependent harmonic oscillator of Eq. (3.6) and was determined by the classical integrals of motion for the time-dependent harmonic oscillator of Eq. (3.6) as did the evolution operator in Eq. (3.13) [8]. We showed that the generalized invariant preserved the same spectrum-generating Lie algebra of $SU(1,1)$ by introducing the time-dependent basis in Eq. (4.19) and had the time-dependent number-type eigenstates of Eq. (4.22) and the coherent states of Eq. (4.23). The exact quantum states were given by Eq. (4.27), whose time-dependent phase factors had the same form (Eq. (4.28)), except for the Bargmann index, as those for the harmonic oscillator. It was shown that the squeeze operator, (Eq. (5.7)) transformed unitarily the time-dependent basis according to Eq. (5.6). Therefore, the number-type eigenstate transformed as in Eq.
by the squeeze operator, and in particular the vacuum at an arbitrary later time was a coherent state of the initial vacuum. The eigenstate evolved as a squeeze state of the initial number-type state. Finally, it was shown that the squeeze operator was the Hermitian conjugate of the evolution operator. It was manifest that the task to find the generalized invariant was relatively easier than that to evaluate the evolution operator directly. As an example, the harmonic oscillator of Eq. (5.17) was worked out explicitly.

Further application of the generalized invariant method to time-dependent extended harmonic oscillators with perturbation terms beyond the linear force term and the inverse squared singular term may be tremendously difficult, but promising.

ACKNOWLEDGMENTS

This work was supported in part by a Non Directed Research Fund, Korea Research Foundation, 1994 and by the Basic Research Institute Program of the Korea Ministry of Education under Contract No. 94-2427.
REFERENCES

[1] A. Messiah, *Quantum Mechanics II* (Wiley, New York, 1962).

[2] V. P. Ermakov, Univ. Izv. Kiev 20 Ser III 9,1 (1880).

[3] H. R. Lewis, Jr., J. Math. Phys. 9, 1976(1968); H. R. Lewis, Jr., and W. B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).

[4] J. R. Ray and J. L. Reid, Phys. Lett. A 71, 317 (1979); J. L. Reid and J. R. Ray, J. Math. Phys. 21, 1583 (1980); J. R. Ray, Lett. A 78, 4 (1980); J. G. Hartley and J. R. Ray, Phys. Rev. A 24, 2873 (1981).

[5] J. Wei and E. Norman, J. Math. Phys. 4, 575 (1963); S. V. Prants, J. Phys. A: Math. Gen. 19, 3457 (1986); F. Wolf and H. J. Korsch, Phys. Rev. A 37, 1934 (1988); C. M. Cheng and P. C. W. Fung, J. Phys. A: Math. Gen. 21, 4115 (1988); F. Salmistraro and R. Rosso, J. Math. Phys. 34, 3964 (1993).

[6] D. C. Khandekar and S. V. Lawande, J. Math. Phys. 16, 384 91975); D. C. Khandekar and S. V. Lawande, J. Math. Phys. 20, 1870 (1979).

[7] N. J. Günther and P. G. L. Leach, J. Math. Phys. 18, 572 (1977); J. G. Hartley and J. R. Ray, Phys. Rev. D 25, 382 (1982); F. M. Fernandez, J. Math. Phys. 28, 2908 (1987); Phys. Rev. A 40, 41 (1989); G. Profilo and G. Soliani, Phys. Rev. A 44, 2057 (1991); M. Hirayama, Prog. Theor. Phys. 86, 341 (1991); J. B. Xu, T. Z. Qian, and X. C. Gao, Phys. Rev. A 44, 1485 (1991); X. C. Gao, J. B. Xu, and T. Z. Qian, Phys. Rev. A 44, 7016 (1991).

[8] C. F. Lo, Phys. Rev. A 43, 404 (1991); *ibid* 45, 5262 (1992).

[9] (a) K. H. Cho and S. P. Kim, J. Phys. A: Math. Gen. 27, 1387 (1994); (b) S. P. Kim and S-W. Kim, Phys. Rev. D 49, R1679 (1994); (c) S. P. Kim, J. Phys. A: Math. Gen. 27, 3927 (1994).
[10] V. S. Povov and A. M. Perelomov, Sov. Phys. JETP 29, 738 (1969); ibid 30, 910 (1970).

[11] P. Camiz, a. Gerardi, C. Marchioro, E. Presutti, and E. Scacciatelli, J. Math. Phys. 12, 2040 (1971); V. V. Dodonov, V. I. Man’ko, and D. E. Nikonov, Phys. Lett. A 162, 359 (1992).

[12] F. Calogero, J. Math. Phys. 12, 419 (1971).

[13] C. M. Bender and G. V. Dunne, Phys. Rev. D 40, 2739 (1989); ibid 40, 3504 (1989).

[14] B. G. Wybourne, Classical Groups for Physicists (wiley, New York, 1974).

[15] P. Caldirola, Nuovo Cimento 18, 393 (1941); E. Kanai, Prog. Theor. Phys. 3, 440 (1948).

[16] W. Magnus, Commun. Pure Appl. Math. 7, 649 (1954).

[17] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) 287 A, 532 (1965).

[18] V. Bargmann, Ann. Math. 48, 568 (1947).

[19] L. D. Landau and E. M. Lifshitz. Quantum Mechanics (Pergamon, New York, 1958).

[20] C. C. Gerry and S. Silverman, J. Math. Phys. 23, 1995 (1982).

[21] A. M. Perelomov, Commun. Math. Phys. 26, 222 (1972).

[22] H. P. Yuen, Phys. Rev. A 13, 2226 (1976); J. N. Hollenhorst, Phys. Rev. D 19, 1669 (1979).

[23] D. R. Traux, Phys. Rev. D 31, 1988 (1985).

[24] C. C. Gerry, J. B. Togeas, and S. Silverman, Phys. Rev. D 28, 1939 (1983); C. C. Gerry, Phys. Rev. D 31, 2721 (1985); G. Dattoli, P. Di Lazzaro, and A. Torre, Phys. Rev. A 35, 1582 (1987); C. C. Gerry, Phys. Rev. A 39, 3204 (1989); Y. Brihaye, S. Gilber, P. Kosin’ski, and P. Mas’lanka, J. Phys. A: Math. Gen. 23, 1985 (1990).

[25] P. M. Morse and H. Feshbach, Methods of Theoretical Physics I (McGraw-Hill, New
York, 1953).

[26] I. A. Pedrosa, Phys. Rev. D 36, 1279 (1987).