INVERSE PROBLEM OF DETERMINING AN ORDER OF THE RIEemann-LIouville TIME-FRACTIONAL DERIVATIVE

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Abstract. The inverse problem of determining the order of the fractional Riemann-Liouville derivative with respect to time in the subdiffusion equation with an arbitrary positive self-adjoint operator having a discrete spectrum is considered. Using the classical Fourier method it is proved, that the value of the norm $|u(t)|$ of the solution at a fixed time instance recovers uniquely the order of derivative. A list of examples is discussed, including a linear system of fractional differential equations, differential models with involution, fractional Sturm-Liouville operators, and many others.

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1. Main result

It has long been known that to model subdiffusion (anomalous or slow diffusion) processes, it is necessary to use differential equations of fractional order $\rho \in (0, 1)$. But in this case, unlike differential equations of integer order, an order of the fractional derivative $\rho$ is often unknown and difficult to be directly measured. The determination of this parameter is called the inverse problem of determining the order of the fractional derivative. These inverse problems are not only theoretically interesting, but also necessary for finding solutions to initial-boundary value problems and studying properties of solutions. The paper [1] by Li, Liu, Yamamoto surveys works on such inverse problems.

In the present paper, we are concerned with inversion for order in the subdiffusion equation with the Riemann-Liouville time-fractional derivative.

Let $H$ be a separable Hilbert space with the scalar product $(\cdot, \cdot)$ and the norm $|| \cdot ||$ and $A: H \rightarrow H$ be an arbitrary positive selfadjoint operator in $H$. Suppose that $A$ has a complete in $H$ system of orthonormal eigenfunctions $\{\upsilon_k\}$ and a countable set of nonnegative eigenvalues $\lambda_k$. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e. $0 < \lambda_1 \leq \lambda_2 \cdots$.

Using the definitions of a strong integral and a strong derivative, fractional analogues of integrals and derivatives can be determined for vector-valued functions (or simply functions) $h: \mathbb{R}_+ \rightarrow H$, while the well-known formulae and properties are preserved (see, for example, [2]). Recall that the fractional integration of order $\rho < 0$ of the function $h(t)$ defined on $[0, \infty)$ has the form

\[ \partial_t^{-\rho} h(t) = \frac{1}{\Gamma(-\rho)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\rho+1}} d\xi, \quad t > 0, \]

provided the right-hand side exists. Here $\Gamma(\rho)$ is Euler’s gamma function. Using this definition one can define the Riemann-Liouville fractional derivative of order $\rho$, $0 < \rho < 1$, as

\[ \partial_t^\rho h(t) = \frac{d}{dt} \partial_t^{-1} h(t). \]

If in this definition we interchange the differentiation and fractional integration, then we get the definition of a regularized derivative, that is, the definition of a fractional derivative in the sense of Caputo:

\[ D_t^\rho h(t) = \partial_t^{-1} \frac{d}{dt} h(t). \]
Note that if \( \rho = 1 \), then fractional derivative coincides with the ordinary classical derivative of the first order:

\[
\partial_t h(t) = \frac{d}{dt} h(t).
\]

Let \( \rho \in (0, 1) \) be a fixed number and let \( C([a, b]; H) \) stand for a set of continuous functions \( u(t) \) of \( t \in (a, b) \) with values in \( H \). Consider the Cauchy type problem:

\[
\begin{align*}
    \partial_t^\rho u(t) + Au(t) &= 0, \quad 0 < t \leq T; \\
    \lim_{t \to 0} \partial_t^\rho u(t) &= \varphi,
\end{align*}
\]

where \( \varphi \) is a given vector in \( H \). If \( \rho = 1 \), then the initial condition has the form \( u(0) = \varphi \). This problem is called a forward problem.

**Definition 1.1.** A function \( u(t) \) with the properties \( \partial_t^\rho u(t), Au(t) \in C([0, T]; H) \) and satisfying conditions (1.1) is called the solution of the forward problem (1.1).

Let us denote by \( E_{\rho, \mu}(t) \) the Mittag-Leffler function of the form

\[
E_{\rho, \mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}.
\]

We first prove the existence and uniqueness of a solution of problem (1.1).

**Theorem 1.2.** For any \( \varphi \in H \) problem (1.1) has a unique solution and this solution has the form

\[
u(t) = \sum_{k=1}^{\infty} E_{\rho, \mu}(-\lambda_k t^\rho)(\varphi, v_k) v_k.
\]

The problem (1.1) for various operators \( A \) has been considered by a number of authors. Let us mention only some of these works. The case of one spatial variable \( x \in \mathbb{R} \) and subdiffusion equation with \( Au = u_{xx} \) considered, for example, in the book of A.A. Kilbas et al. [3] and monograph of A. V. Pskhu [4] and references in these works. The paper Gorenflo, Luchko and Yamamoto [5] is devoted to the study of subdiffusion equations in Sobolev spaces. In the paper by Kubica and Yamamoto [6], initial-boundary value problems for equations with time-dependent coefficients are considered. In the multidimensional case \( x \in R^N \), instead of the differential expression \( u_{xx} \), authors of the papers [3], [7]-[9] considered the Laplace operator and Umarov [10] considered pseudodifferential operators with constant coefficients in the whole space \( \mathbb{R}^N \).

A result similar to the above, in the case when the fractional part of the equation (1) is the Caputo derivative, was obtained by M. Ruzhansky et al. [11]. In the case when \( A \) is an arbitrary elliptic differential operator, this theorem was proved in [12].

Obviously solution (1.2) depends on \( \rho \in (0, 1) \). Now let us consider the order of fractional derivative \( \rho \) as a unknown parameter and consider an inverse problem: can we identify uniquely this parameter \( \rho \), if we have as a additional information the norm

\[
W(t_0, \rho) = ||u(t_0)||^2 = d_0
\]

at a fixed time instant \( t_0 > 0 \)?

Problem (1.1) with extra condition (1.3) is called the inverse problem.

To solve this inverse problem we fix a number \( \rho_0 \in (0, 1) \) and consider the problem for \( \rho \in [\rho_0, 1) \).

**Definition 1.3.** A pair \( \{u(t), \rho\} \) of the solution \( u(t) \) to the forward problem and the parameter \( \rho \in [\rho_0, 1] \) satisfying the additional condition (1.3) is called the solution of the inverse problem.

**Lemma 1.4.** Given \( \rho_0 \) from interval \( 0 < \rho_0 < 1 \), there exists a number \( T_0 = T_0(\rho_0, \lambda_1) \), such that for all \( t_0 \geq T_0 \) and for arbitrary \( \varphi \in H \) function \( W(t_0, \rho) \) decreases monotonically with respect to \( \rho \) in \( [\rho_0, 1] \).

The main result of the paper is the following:

**Theorem 1.5.** Let \( \varphi \in H \) and \( t_0 \geq T_0 \). Then the inverse problem has a unique solution \( \{u(t), \rho\} \) if and only if

\[
W(t_0, 1) \leq d_0 \leq W(t_0, \rho_0).
\]
Theorem 1.3 gives a positive answer to the problem posed in review article by Z. Li et al. [11] (p. 440) in the Conclusions and Open Problems section: Is it possible to identify uniquely the order of fractional derivatives if an additional information about the solution is specified at a fixed time instant as "the observation data"?

Note that our result shows, that it is possible to restore the order of the fractional derivative by using the value of $W(t, \rho)$ at a fixed time instant $t_0$ as "the observation data".

The inverse problem of determining the order of time fractional derivative in subdiffusion equations has been studied by a number of authors (see a survey paper [1] and references therein, [13]-[23]). It is necessary to note that in all these publications the following relation equations has been studied by a number of authors (see a survey paper [1] and references at a monitoring point condition (1.3) guarantees both uniqueness and the existence of a solution.

However, as Theorem 2 states, unlike (1.4), order of fractional derivatives if an additional information about the solution is specified at a fixed time instant as "the observation data". Examples $W(t, \rho)$ by using the value of $h(t)$, $0 < t < T$,

at a monitoring point $x_0 \in \Omega$. But this condition, as a rule, can guarantee only the uniqueness of the solution of the inverse problem (see [13]-[16]). However, as Theorem 2 states, unlike (1.4), condition (1.3) guarantees both uniqueness and the existence of a solution.

Hatano et al. [17] considered the equation $\partial_t^\rho u = \triangle u$ with the Dirichlet boundary condition and the initial function $\varphi(x)$ (see also [18]). They proved the following property of the parameter $\rho$: if $\varphi \in C_0^\infty(\Omega)$ and $\triangle \varphi(x_0) \neq 0$, then

$$
\rho = \lim_{t \to 0} \left| t \partial_t u(x_0, t)[u(x_0, t) - \varphi(x_0)]^{-1} \right|.
$$

For the best of our knowledge, only in the paper [19] by J. Janno the existence problem is considered. Giving an extra boundary condition $Bu(., t) = h(t).0 < t < T$ the author succeeded to prove the existence theorem for determining the order $\rho$, $0 < \rho < 1$, of the Caputo derivative and the kernel of the integral operator in the equation.

We also note the following recent papers. In the paper by Z. Li and Z. Zhang [20] the authors proved the uniqueness in determining both orders of fractional time derivatives and spatial derivatives in diffusion equations. The proof relies on the eigenfunction expansion and the asymptotics of the Mittag-Leffler function. The authors of [22] discuss similar issues discussed in the present paper. As an additional information for inverse problem they have considered the value of projection of the solution onto the first eigenfunction at a fixed time instance. Note, that results of paper [22] are applicable only in case, when the first eigenvalue of the corresponding elliptic operator is equal to zero. We also mention the paper [23], in which a result similar to Theorem 2 was proved for the subdiffusion equation with the Caputo derivative.

In conclusion, we give the following remarks:

1) As the operator $A$, one can take any equations of mathematical physics considered in Section 6 of the article by M. Ruzhansky et al. [11], including the classical Sturm-Liouville problem, differential models with involution, fractional Sturm-Liouville operators, harmonic and anharmonic oscillators, Landau Hamiltonians, fractional Laplacians, harmonic and anharmonic operators on the Heisenberg group.

2) Further, let us take $\mathbb{R}^N$ as a Hilbert space $H$ and $N$-dimensional symmetric quadratic matrix $A = \{a_{i,j}\}$ with constant elements $a_{i,j}$ as operator $A$. In this case, the problem (1.1) coincides with the Cauchy problem for a linear system of fractional differential equations.

3) You can also consider various options for the function $W(t, \rho)$. Examples $W(t, \rho) = ||Au(t)||^2$, $W(t, \rho) = (u, \varphi)$.

2. Forward problem

In the present section we prove Theorem 1.2.

To prove the existence of the forward problem’s solution we remind the following estimate of the Mittag-Leffler function with a negative argument (see, for example, [24], p. 136)

$$
|E_{\rho,\mu}(-t)| \leq \frac{C}{1 + t}, \quad t > 0.
$$

(2.1)
In accordance with Definition 1.1 we will first show that for function one has \( Au(t) \in C((0, T]; H) \). To do this, consider the sum

\[
S_j(t) = \sum_{k=1}^{j} t^{\rho-1} E_{\rho, \rho}(-\lambda k t^\rho) (\varphi, v_k).
\]

Then

\[
AS_j(t) = \sum_{k=1}^{j} \lambda k t^{\rho-1} E_{\rho}(-\lambda k t^\rho) (\varphi, v_k).
\]

Due to the Parseval equality we may write

\[
||AS_j(t)||^2 = \sum_{k=1}^{j} |\lambda k t^{\rho-1} E_{\rho}(-\lambda k t^\rho) (\varphi, v_k)|^2 \leq C t^{-2} ||\varphi||^2.
\]

Here we used estimate (2.1) and the inequality \( \lambda t^\rho (1 + \lambda t^\rho)^{-1} < 1 \).

Hence, we obtain \( Au(t) \in C((0, T]; H) \).

Further, from equation (1.1) one has \( \partial_t^\rho S_j(t) = -AS_j(t) \). Therefore, from above reasoning, we finally have \( \partial_t^\rho u(t) \in C((0, T]; H) \).

It is not hard to verify the fulfillment of equation (1.1) (see, for example, [25], p. 173 and [26]) and the initial condition therein.

Now we prove the uniqueness of the forward problem’s solution.

Suppose that problem (1.1) has two solutions \( u_1(t) \) and \( u_2(t) \). Our aim is to prove that \( u(t) = u_1(t) - u_2(t) \equiv 0 \). Since the problem is linear, then we have the following homogenous problem for \( u(t) \):

\[
(2.2) \quad \partial_t^\rho u(t) + Au(t) = 0, \quad t > 0;
\]

\[
(2.3) \quad \lim_{t \to 0} \partial_t^{\rho-1} u(t) = 0.
\]

Set

\[ w_k(t) = (u(t), v_k). \]

It follows from (2.2) that for any \( k \in \mathbb{N} \)

\[
\partial_t^\rho w_k(t) = (\partial_t^\rho u(t), v_k) = -(Au(t), v_k) = -u(t), Av_k = -\lambda_k w_k(t).
\]

Therefore, we have the following Cauchy problem for \( w_k(t) \) (see (2.3)):

\[
\partial_t^\rho w_k(t) + \lambda_k w_k(t) = 0, \quad t > 0; \quad \lim_{t \to 0} \partial_t^{\rho-1} w_k(t) = 0.
\]

This problem has the unique solution (see, for example, [24], p. 173 and [26]). Therefore, \( w_k(t) = 0 \) for \( t > 0 \) and for all \( k \geq 1 \). Then by the Parseval equation we obtain \( u(t) = 0 \) for all \( t > 0 \). Hence uniqueness of the solution is proved.

Thus the proof of Theorem 1.2 is complete.

3. INVERSE PROBLEM

**Lemma 3.1.** Given \( \rho_0 \) from the interval \( 0 < \rho_0 < 1 \), there exists a number \( T_0 = T_0(\rho_0, \lambda_1) \), such that for all \( t_0 \geq T_0 \) and \( \lambda \geq \lambda_1 \) functions \( e_{\rho} = t_0^{\rho} E_{\rho, \rho}(-\lambda t_0^\rho) \) are positive and they decrease monotonically with respect to \( \rho \in [\rho_0, 1] \).

**Proof.** Let us denote by \( \delta(1; \beta) \) a contour oriented by non-decreasing arg \( \zeta \) consisting of the following parts: the ray \( \arg \zeta = -\beta, |\zeta| \geq 1 \), the arc \( -\beta \leq \arg \zeta \leq \beta, |\zeta| = 1 \), and the ray \( \arg \zeta = \beta, |\zeta| \geq 1 \). If \( 0 < \beta < \pi \), then the contour \( \delta(1; \beta) \) divides the complex \( \zeta \)-plane into two unbounded parts, namely \( G^{(-1};1; \beta) \) to the left of \( \delta(1; \beta) \) by orientation, and \( G^{(+1};1; \beta) \) to the right of it. The contour \( \delta(1; \beta) \) is called the Hankel path.

Let \( \beta = \frac{1}{2} \pi, \rho \in [\rho_0, 1) \). Then by the definition of this contour \( \delta(1; \beta) \), we arrive at (see (1.1), formula (2.29), p. 135, note \( -\lambda t_0^\rho \in G^{(-1};1; \beta) \))

\[
(3.1) \quad t_0^{\rho-1} E_{\rho, \rho}(-\lambda t_0^\rho) = -\frac{1}{\lambda^{2} t_0^{\rho+1} \Gamma(-\rho)} + \frac{\rho}{2 \pi i \lambda^{2} t_0^{\rho+1}} \int_{\delta(1; \beta)} \frac{\zeta^{1/\rho} \zeta^{1/2} \zeta^{-1}}{\zeta + \lambda t_0^\rho} d\zeta = f_1(\rho) + f_2(\rho).
\]
To prove the lemma, it suffices to show that the derivative \( \frac{d}{d\rho} c_\lambda(\rho) \) is negative for all \( \rho \in [\rho_0, 1) \), since the positivity of \( c_\lambda(1) = e^{-\lambda t} > 0 \).

It is not hard to estimate the derivative \( f'_1(\rho) \). Indeed, let \( \Psi(\rho) \) be the logarithmic derivative of the gamma function \( \Gamma(\rho) \) (for the definition and properties of \( \Psi \) see [20]). Then \( \Gamma'(\rho) = \Gamma(\rho)\Psi(\rho) \), and therefore,

\[
f'_1(\rho) = \frac{\ln t_0 - \Psi(-\rho)}{\lambda^2 t_0^{\rho+1} \Gamma(-\rho)}.\]

Since \[
\frac{1}{\Gamma(-\rho)} = -\frac{\rho}{\Gamma(1-\rho)} - \frac{\rho(1-\rho)}{\Gamma(2-\rho)} \Psi(-\rho) = -\frac{1}{\Gamma(2-\rho)} + \frac{1}{\rho} = -\frac{1}{\rho^\rho - \Gamma(1-\rho)} = -\frac{1}{\rho - 1},
\]
the function \( f'_1(\rho) \) can be represented as follows

\[
f'_1(\rho) = \frac{1}{\lambda^2 t_0^{\rho+1}} \frac{\rho(1-\rho)\Psi(2-\rho) - \ln t_0 + 1 - 2\rho}{\Gamma(2-\rho)} = -\frac{f_{11}(\rho)}{\lambda^2 t_0^{\rho+1} \Gamma(2-\rho)}.
\]

If \( \gamma \approx 0.57722 \) is the Euler-Mascheroni constant, then \( \Psi(2-\rho) < 1 - \gamma \). Therefore,

\[
f_{11}(\rho) > \rho(1-\rho)\ln t_0 - (1-\gamma) + 2\rho - 1 - 2\rho - 1 = 0.
\]

For \( t_0 = e^{1-\gamma}e^{2/\rho} \) one has \( \rho(1-\rho)\ln t_0 - (1-\gamma) + 2\rho - 1 = 1 \). Hence, \( f_{11}(\rho) \geq 1 \), provided \( t_0 \geq T_0 \) and

\[
T_0 = e^{1-\gamma}e^{2/\rho_0}.
\]

Thus, by virtue of (3.2), for all such \( t_0 \) we arrive at

\[
f'_1(\rho) \leq -\frac{1}{\lambda^2 t_0^{\rho+1}}.
\]

To estimate the derivative \( f'_2(\rho) \), we denote the integrand in (3.1) by \( F(\zeta, \rho) \):

\[
F(\zeta, \rho) = \frac{1}{2\pi i \rho \lambda^2 t_0^{\rho+1}} \frac{e^{1/\rho} \zeta^{1/\rho+1}}{\zeta + \lambda t_0^\rho}.
\]

Note, that the domain of integration \( \delta(1; \beta) \) also depends on \( \rho \). To take this circumstance into account when differentiating the function \( f'_2(\rho) \), we rewrite the integral (3.1) in the form:

\[
f_2(\rho) = f_{2+}(\rho) + f_{2-}(\rho) + f_{21}(\rho),
\]

where

\[
f_{2+}(\rho) = e^{i\beta} \int_1^\infty F(s e^{i\beta}; \rho) ds,
\]

\[
f_{21}(\rho) = i \int_0^{i\beta} F(e^{i\rho}; \rho) e^{i\beta} dy = i i \int_{-1}^1 F(e^{i\beta s}; \rho) e^{i\beta s} ds.
\]

Let us consider the function \( f_{2+}(\rho) \). Since \( \beta = \frac{3\pi}{4} \rho \) and \( \zeta = s e^{i\beta} \), then

\[
e^{1/\rho} = e^{i\pi/4}(1-s)^{1/2}.
\]

The derivative of the function \( f_{2+}(\rho) \) has the form

\[
f'_2(\rho) = \int_1^\infty e^{i\pi/4} \left[ (1-s)^{1/2} s^{1/\rho+1} e^{2\alpha \rho} \left( \frac{1}{\rho} \frac{1}{\rho} (1-i) s^{1/2} - 1 \right) \ln s + 2t_0 - \frac{1}{\rho} \ln t_0 - \frac{t_0}{\rho} e^{i\alpha \rho} + \lambda t_0^\rho \ln t_0 \right] ds.
\]

where \( I = e^{i\alpha}(2\pi i \rho \lambda^2 t_0^{\rho+1}) \) and \( a = \frac{3\pi}{4} \). By virtue of the inequality \( |se^{i\alpha \rho} + \lambda t_0^\rho| \geq \lambda t_0^\rho \) we arrive at

\[
|f'_2(\rho)| \leq C \int_1^\infty e^{i\pi/4} \left[ \int_1^\infty \frac{1}{\rho} \frac{1}{\rho} (1-i) s^{1/2} - 1 \right] ds + \ln t_0 ds.
\]
Lemma 3.2. Let $0 < \rho \leq 1$ and $m \in \mathbb{N}$. Then

$$I(\rho) = \frac{1}{\rho} \int_1^\infty e^{-\frac{1}{\rho} s^{-\rho + \frac{m}{\rho} + 1}} ds \leq C_m.$$

Proof. Set $r = s^{\frac{1}{\rho}}$. Then

$$s = r^{\rho}, \quad ds = \rho r^{\rho - 1} dr.$$

Therefore,

$$I(\rho) = \int_1^\infty e^{-\frac{1}{\rho} r^{\rho (m - 1 + 2\rho)}} dr \leq \int_1^\infty e^{-\frac{1}{\rho} r^{\rho + 1}} dr = C_m.$$

Lemma 3.2 is proved.

Application of this lemma gives (note, $\frac{1}{\rho} t_0 s < s^{\frac{1}{\rho}}$, provided $s \geq 1$)

$$|f_2(\rho)| \leq \frac{C}{\lambda^2 t_0^{2\rho + 1}} \left( \frac{C_0}{\rho} + C t_0 \right) \leq \frac{C}{\lambda^2 t_0^{2\rho + 1}} \left( \frac{1}{\rho} + \ln t_0 \right).$$

Function $f_2(\rho)$ has exactly the same estimate.

Now consider the function $f_{21}(\rho)$. For its derivative we have

$$f_{21}'(\rho) = \frac{a}{2\pi i \lambda^2 t_0^{\rho + 1}} \int_{-1}^1 e^{i a s} e^{ia \rho s} \left[ t_0 s - \ln t_0 - \frac{ia e^{ia \rho s} + \lambda t_0^\rho}{e^{ia \rho s} + \lambda t_0^\rho} \right] ds.$$

Therefore,

$$|f_{21}'(\rho)| \leq C \frac{\ln t_0}{\lambda^2 t_0^{2\rho + 1}}.$$

Taking into account the estimate (3.4) and the estimates of $f_{22}'$ and $f_{21}'$, we have

$$\frac{d}{d\rho} e(\rho) < -\frac{1}{\lambda^2 t_0^{2\rho + 1}} + \frac{C}{\lambda^2 t_0^{2\rho + 1}} \left( \frac{1}{\rho} + \ln t_0 \right).$$

In other words, this derivative is negative if

$$t_0 > \frac{C}{\lambda_0} \left( \frac{1}{\rho_0} + \ln t_0 \right).$$

Hence, there exists a number $T_0 > T_0(\lambda_0, \rho_0)$ (see also (3.3)) such, that for all $t_0 \geq T_0$

$$\frac{d}{d\rho} [t_0^{\rho - 1} E_{\rho, \rho}(-\lambda_0 t_0^\rho)] < 0, \quad \lambda \geq \lambda_0, \quad \rho \in [\rho_0, 1].$$

Lemma 3.1 is proved.

Since

$$W(t, \rho) = ||u(t)||^2 = \sum_{k=1}^\infty |(\varphi, v_k)|^2 |t_{\rho - 1} E_{\rho, \rho}(-\lambda_k t_{\rho})|^2,$$

then Lemma 1.4 follows immediately from Lemma 3.1. Theorem 1.5 is an easy consequence of these two lemmas.
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