Newtonian Atlas for Dust-Filled FRW Universe

N. S. Manton\footnote{email: N.S.Manton@damtp.cam.ac.uk}

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.

Abstract

The metric of an FRW universe filled with pressureless dust is shown to agree, close to any spacetime point, with a curved Newtonian-type metric where Einstein’s equations simplify to those of Newtonian gravity. The agreement is shown to quadratic order in the local coordinates, so the curvatures agree. This result is established by expressing both metrics in Riemann normal form. This approach gives a local Newtonian understanding of cosmology that avoids the paradoxes of global Newtonian cosmology.

Keywords: FRW Universe, Newtonian Atlas, Riemann Curvature, Normal Coordinates
1 Introduction

It is well known that the evolution equations of a Friedmann–Robertson–Walker (FRW) universe \[1\] are the same as those obtained in Newtonian cosmology \[2, 3, 4, 5\]. In particular, in the most Newtonian case, that of pressureless dust with no cosmological constant, the evolution can be interpreted as due to the Newtonian self-gravity of the dust. The FRW and Newtonian geometries are however globally different. The FRW universe has spacetime curvature, and the cross sections at fixed comoving time \(t\) are generically spatially curved. If the spatial sections have positive curvature, they are of finite size. The Newtonian universe is flat and infinite.

The Newtonian universe can consist of a bounded ball of matter, surrounded by an infinite empty region \[6\], and the matter may be discrete \[7\]. Alternatively it can consist of a uniform distribution of matter over infinite space. Both models have disturbing features \[8, 9\]. In the first case there is a definite centre, so a lack of homogeneity except near the centre. In the second case, one problem is that the gravitational potential \(\Phi\) of uniform matter is indeterminate if calculated by the inverse square law. Also the Hubble flow, where the matter velocity away from a point is proportional to the distance from the point, has the property that beyond a certain distance the velocity is greater than the speed of light. The first problem can be resolved by solving Poisson’s equation for the potential \(\Phi\). There is a spherically symmetric solution around a given point which grows quadratically with distance. This potential is also obtained if one imagines removing a ball of matter around the point, arguing that the potential inside is then constant, and then putting the ball back. The solution has the paradoxical property of again having a definite centre despite the homogeneity of the background matter density, and it leads to the acceleration of matter being towards this centre. The second problem is less severe if one regards the speed of light as irrelevant in Newtonian dynamics.

We will show that the Newtonian view is more accurate and devoid of paradox if one regards the FRW universe as covered by an atlas of overlapping patches. This develops the idea that Newtonian dynamics is effective in small regions of the universe, see e.g. \[10, 5\]. We will examine in detail the geometry of these patches and how neighbouring patches fit together. Our patches are of finite size but do not have a sharp physical cutoff. The cutoff will be purely mathematical, as in differential geometry, where one covers a manifold with an atlas of overlapping charts.

Around each point \(O\) of an FRW spacetime we establish that there is a Newtonian approximation up to quadratic order in the Riemann normal
coordinates centred at O. At this order one captures the Riemann curvature, so the Newtonian patches are not just tangent planes, but osculate the spacetime manifold.

By Newtonian patch, with a Newtonian-type metric, we mean the curved spacetime that is regarded as capturing the Newtonian limit of general relativity \[1, 11\]. It is a solution of Einstein’s equations in the weak field limit, close to flat spacetime, with uniform dust as source. The solution depends on the local Newtonian potential $\Phi$, and the geodesic motion of non-relativistic test particles as well as the dust itself experiences an acceleration proportional to the gradient of $\Phi$. It is sufficient to work with the metric whose source is dust at rest. The Hubble flow away from or towards the central point (both its velocity and acceleration) produces corrections at higher order.

Technically, we will show that by coordinate transformations around O, we can put the FRW metric into Riemann normal form up to quadratic order in the coordinates (with the coordinates vanishing at O). We will also put the Newtonian-type metric into Riemann normal form, and show that the FRW and Newtonian-type metrics are then the same. The Hubble flows also agree at linear order. The Riemann normal form is where the metric to leading order is Minkowskian, all the first derivatives of the metric vanish, so the Christoffel symbols vanish at O, and additionally the second derivatives of the metric are directly related to the Riemann curvature \[12\]. It occurs when the coordinates are geodesic normal coordinates to quadratic order. To find it, we will proceed by algebraic trial and error, and not actually construct geodesics emanating from O. This method is quite simple because of the isotropy of the metric around O.

It is interesting to keep track of the number of parameters that occur in the metrics, at quadratic order. The Robertson–Walker metric, expanded about O, has three continuous parameters, the Hubble parameter $H$, the deceleration $q$, and the spatial curvature $K = k/R_0^2$. $k$ here has its usual discrete values, 1, 0 or −1. However, the Riemann curvature has just two parameters, a spatial part and a time-space part, which are combinations of the three just mentioned. Then the Friedmann equations (the Einstein equations) relate both curvature parameters to the matter density $\rho$ at O. (The matter density in comoving coordinates is a Lorentz scalar.) The metric in Riemann normal form therefore depends on just one quantity, the density $\rho$.

Exactly the same final parameter count occurs in the Newtonian patches. The metric is determined by the potential $\Phi$, which in turn is determined by the density. At quadratic order there is no further information.

This is rather curious. It means that from the spacetime curvature alone, at one point, one cannot determine all of $K, q$ and $H$, but only two combina-
tions of these. However, the Hubble parameter $H$ is separately determined from the flow of matter away from $O$, and is measured using redshifts. In turn, the Hubble flow controls the rate of change of the density $\rho$ and the form of the energy-momentum tensor in the neighbourhood of $O$. In detail, the energy-momentum tensor depends on the coordinate system, so it is different in comoving coordinates and Riemann normal coordinates.

It is now fashionable to interpret the redshift of photons as due to the spatial expansion of the whole universe [1, 10]. In an FRW universe, and calculating over long time intervals, it is clearly convenient to use comoving coordinates, and the well known result that redshift $z$ depends only on the ratio of the scale factors at the times of emission and receipt of photons is mathematically elegant.

However, space doesn’t really expand, at least not locally. Near $O$, one just has a Newtonian patch of a Lorentzian curved spacetime, whose metric one can approximate using Riemann normal coordinates. Over short distances and times even the curvature can be ignored, so spacetime is approximately Minkowskian. In these coordinates there is a Hubble flow of light-emitting matter, and the redshift is a Doppler effect due to the recession of the emitters from $O$.

This remark is conceptual rather than practical. The more sophisticated point of view, that redshift is due to the expansion of space is helpful, but it relies on the global form and symmetries of the FRW metric and on the use of comoving coordinates. Locally, in normal coordinates, redshift is a Doppler effect. The use of Newtonian patches justifies this traditional understanding of redshift.

2 FRW metrics in Riemann normal form

Here we show, by suitable coordinate transformations, that an FRW metric, expanded around a spacetime point $O$, can be brought to Riemann normal form.

We recall that a Robertson–Walker metric is of the form

$$ds^2 = -dt^2 + R(t)^2 \frac{dx' \cdot dx'}{(1 + \frac{1}{2}kx' \cdot x')^2}$$  (1)

where $k$ is 1, 0 or $-1$. $t$ and $x'$ are comoving coordinates. The spatial metric here, the expression multiplied by $R(t)^2$, is one way of writing a metric of constant curvature $k$. Without loss of generality, we choose $O$ to be at $x' = 0$.
and (following a shift of $t$ if necessary) at $t = 0$. The standard way to write the Taylor expansion of the scale factor $R(t)$ around $t = 0$, up to quadratic order, is

\[
R(t) = R_0 \left( 1 + H t - \frac{1}{2} qH^2 t^2 \right).
\] (2)

$H$ is the Hubble parameter and $q$ the deceleration at $t = 0$. [If $H$ is zero, one needs to write $R(t) = R_0(1 - \frac{1}{2} Qt^2)$.] Defining $\mathbf{x} = R_0 \mathbf{x}'$, the metric becomes

\[
d s^2 = -dt^2 + \left( 1 + 2Ht + (1 - q)H^2 t^2 \right) \frac{d\mathbf{x} \cdot d\mathbf{x}}{\left( 1 + \frac{k}{4R_0^2} \mathbf{x} \cdot \mathbf{x} \right)^2},
\] (3)

and further expanding about $O$, i.e. about $\mathbf{x} = 0$, we find to quadratic order

\[
d s^2 = -dt^2 + \left( 1 + 2Ht + (1 - q)H^2 t^2 - \frac{1}{2} K \mathbf{x} \cdot \mathbf{x} \right) d\mathbf{x} \cdot d\mathbf{x}.
\] (4)

$K = k/R_0^2$ is the curvature of the spatial cross section at $t = 0$, with the scale factor taken into account, and it takes any real value. Our analysis is formal throughout. We do not draw attention to higher order terms, and in all expressions it is implied that there are higher order corrections that have been dropped.

From the Einstein equations, one obtains the Friedmann equations for the scale factor $R(t)$. In the simplest case that the matter is pressureless dust of density $\rho(t)$, and there is no cosmological constant, the Friedmann equations are

\[
\frac{\dot{R}}{R} = -\frac{4\pi G}{3}\rho
\] (5)

and

\[
\frac{\ddot{R}}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho,
\] (6)

together with the mass conservation constraint

\[
\rho R^3 = M = \text{constant}.
\] (7)

These are related, as by eliminating $\rho$ in favour of $M$ in (6), and differentiating, one obtains (5).

In terms of the Taylor coefficients of $R$, as in (2), eqs.(5) and (6) reduce at $t = 0$ to

\[
qH^2 = \frac{4\pi G}{3}\rho_0,
\] (8)

\[
H^2 + K = \frac{8\pi G}{3}\rho_0,
\] (9)
where $\rho_0 = \rho(0)$. It follows that $K = (2q - 1)H^2$.

The metric (4) is the one we shall now manipulate using coordinate transformations. It describes the FRW geometry up to quadratic order in both space and time around $O$. Standard calculations will correctly determine the Riemann curvature at $O$. However, by bringing the metric to Riemann normal form, the curvature will become manifest.

The leading terms in (4) have Minkowski form, but we need to remove the term linear in $t$. The desired coordinate change, involving quadratic terms in the coordinates (and no cubic terms yet), is

\[
\begin{align*}
  t &= \tilde{T} - \frac{1}{2} H \tilde{X} \cdot \tilde{X}, \\
  \mathbf{x} &= \tilde{X} - H \tilde{T} \tilde{X},
\end{align*}
\]

implying

\[
\begin{align*}
  dt &= d\tilde{T} - H \tilde{X} \cdot d\tilde{X}, \\
  d\mathbf{x} &= d\tilde{X} - H(\tilde{X} d\tilde{T} + \tilde{T} d\tilde{X}).
\end{align*}
\]

One finds, keeping all terms at quadratic order,

\[
\begin{align*}
  ds^2 &= -d\tilde{T}^2 + d\tilde{X} \cdot d\tilde{X} \\
  &\quad + H^2 \tilde{X} \cdot \tilde{X} d\tilde{T}^2 - 2H^2 \tilde{X} \cdot d\tilde{X} \tilde{T} d\tilde{T} - (2 + q)H^2 \tilde{T}^2 d\tilde{X} \cdot d\tilde{X} \\
  &\quad - H^2(\tilde{X} \cdot d\tilde{X})^2 - \left( H^2 + \frac{1}{2} K \right) \tilde{X} \cdot \tilde{X} d\tilde{X} \cdot d\tilde{X}.
\end{align*}
\]

The Minkowskian leading terms and absence of linear terms imply that the coordinates $\tilde{T}, \tilde{X}$ are normal coordinates to lowest order, and that the Christoffel symbols at $O$ vanish. To put the metric in Riemann normal form at quadratic order, we need a further coordinate transformation, involving cubic terms, so that the quadratic terms in the metric can be expressed as a quadratic form in the antisymmetrised quantities $X^\mu dX^\nu - X^\nu dX^\mu$, where $X^\mu$ runs over all four coordinates.

Spherical symmetry around $O$ makes it relatively easy to do this. We carry out the coordinate change

\[
\begin{align*}
  \tilde{T} &= T + \frac{1}{6} (3 + q) H^2 T \mathbf{X} \cdot \mathbf{X}, \\
  \tilde{X} &= \mathbf{X} + \frac{1}{3} (3 + q) H^2 T^2 \mathbf{X} + \frac{1}{6} \left( 2H^2 + \frac{1}{2} K \right) (\mathbf{X} \cdot \mathbf{X}) \mathbf{X},
\end{align*}
\]

and hence

\[
\begin{align*}
  d\tilde{T} &= dT + \frac{1}{6} (3 + q) H^2 (\mathbf{X} \cdot \mathbf{X} dT + 2T \mathbf{X} \cdot d\mathbf{X}),
\end{align*}
\]
\[
\begin{align*}
\tilde{d}\mathbf{X} &= d\mathbf{X} + \frac{1}{3}(3 + q)H^2(2T \mathbf{X} dT + T^2 d\mathbf{X}) \\
&\quad + \frac{1}{6} \left(2H^2 + \frac{1}{2}K\right)(2\mathbf{X}(\mathbf{X} \cdot d\mathbf{X}) + (\mathbf{X} \cdot \mathbf{X})d\mathbf{X}) .
\end{align*}
\] (14)

This preserves the spherical symmetry of (12). The coefficients \(\frac{1}{6}(3 + q)H^2\) etc. are not initially fixed, but by simple algebra one can show that with the values shown the metric becomes

\[
\begin{align*}
 ds^2 &= -dT^2 + d\mathbf{X} \cdot d\mathbf{X} \\
&\quad - \frac{1}{3} qH^2 (\mathbf{X} dT - T d\mathbf{X}) \cdot (\mathbf{X} dT - T d\mathbf{X}) \\
&\quad - \frac{1}{3} (H^2 + K)((\mathbf{X} \cdot \mathbf{X})d\mathbf{X} \cdot d\mathbf{X} - (\mathbf{X} \cdot d\mathbf{X})^2) .
\end{align*}
\] (15)

The spatial terms \((\mathbf{X} \cdot \mathbf{X})d\mathbf{X} \cdot d\mathbf{X} - (\mathbf{X} \cdot d\mathbf{X})^2\) can be rewritten as

\[
(X^1 dX^2 - X^2 dX^1)^2 + (X^2 dX^3 - X^3 dX^2)^2 + (X^3 dX^1 - X^1 dX^3)^2,
\] (16)

so the metric (15) has the Riemann normal form

\[
\begin{align*}
 ds^2 &= -dT^2 + d\mathbf{X} \cdot d\mathbf{X} \\
&\quad - \frac{1}{3} qH^2 \left((X^1 dT - T dX^1)^2 + (X^2 dT - T dX^2)^2 + (X^3 dT - T dX^3)^2\right) \\
&\quad - \frac{1}{3} (H^2 + K) \left((X^1 dX^2 - X^2 dX^1)^2 + (X^2 dX^3 - X^3 dX^2)^2 \right. \\
&\quad \left. + (X^3 dX^1 - X^1 dX^3)^2\right).
\end{align*}
\] (17)

(As usual, we identify \(X^0 = T\).) Spherical symmetry alone allows the term \((X^1 dT - T dX^1)(X^2 dX^3 - X^3 dX^2)\) and its cyclic permutations in the metric, but they are ruled out by inversion symmetry, and do not arise under the coordinate change.

For a general metric in Riemann normal form, the quadratic terms have the structure

\[
 ds^2_{\text{quad.}} = \sum_{\mu, \nu, \sigma, \tau} C_{\mu\nu\sigma\tau} (X^\mu dX^\nu - X^\nu dX^\mu)(X^\sigma dX^\tau - X^\tau dX^\sigma) ,
\] (18)

where the coefficients \(C_{\mu\nu\sigma\tau}\) have the same symmetry properties as the Riemann tensor, namely antisymmetry under exchange of \(\mu, \nu\) or of \(\sigma, \tau\), symmetry under exchange of \(\mu\nu\) with \(\sigma\tau\), and the cyclic symmetry \(C_{\mu\nu\sigma\tau} + C_{\mu\sigma\nu\tau} + C_{\mu\tau\nu\sigma} = 0\). (As explained in [12], this was the way Riemann initially understood curvature. Note also that our index conventions differ from those in...
By expanding out, and using the standard formula for the Riemann tensor \( R_{\mu\nu\sigma\tau} \), one finds \( R_{\mu\nu\sigma\tau} = 12 C_{\mu\nu\sigma\tau} \).

For the metric (17), the consequences of the Einstein equations can therefore be verified directly. The non-vanishing Riemann tensor components are

\[
R_{00} = -qH^2, \quad R_{ijij} = -(H^2 + K),
\]

and the further components implied by the symmetries of the Riemann tensor. The Ricci tensor is diagonal, with components

\[
R_{00} = -3(H^2 + K), \quad G_{ii} = -(2q - 1)H^2 + K.
\]

The energy-momentum tensor for dust is \( T_{\mu\nu} = \rho u_{\mu}u_{\nu} \), where \( u^\mu \) is its local 4-velocity. In the comoving frame \( u^\mu = (1, 0, 0, 0) \) so the only non-vanishing component is \( T^{tt} = \rho \). Our changes of coordinates do not change the 4-velocity at \( O \), so the only non-vanishing component at \( O \) (in Riemann normal coordinates) becomes \( T_{00} = \rho_0 \). The Einstein equations are therefore \( G_{00} = -8\pi G\rho_0 \) and \( G_{ii} = 0 \). These reproduce the equations (8) and (9).

Notice that eqs. (8) and (9) imply that the metric (17) simplifies further, to

\[
ds^2 = -dT^2 + dX \cdot dX
\]

\[
-\frac{4\pi G}{9} \rho_0 \left( (X^1dT - TdX^1)^2 + (X^2dT - TdX^2)^2 + (X^3dT - TdX^3)^2 \right)
\]

\[
-\frac{8\pi G}{9} \rho_0 \left( (X^1dX^2 - X^2dX^1)^2 + (X^2dX^3 - X^3dX^2)^2 + (X^3dX^1 - X^1dX^3)^2 \right).
\]

The curvature of a dust-filled FRW universe therefore has a universal structure multiplied by the (time-varying) density. The ratio of coefficients 2 : 1 is a constant feature.

From the curvature of spacetime, one can determine \( qH^2 \) and \( H^2 + K \), but not \( H \) and \( K \) separately. However, \( H \) is independently related to the flow of matter. Let us find the flow, calculating to linear order in the normal coordinates around \( O \). We start with the comoving velocity 4-vector \( u^\mu = (1, 0, 0, 0) \) and apply the coordinate change (10). We also apply the further coordinate change (13), although this has essentially no effect. In coordinates
\( X^\mu = (T, X), \)

\[
U^\mu = u^\nu \frac{\partial X^\mu}{\partial x^\nu} = \frac{\partial X^\mu}{\partial t} = (1, HX^1, HX^2, HX^3) \tag{22}
\]
at linear order. The 4-vector \( U^\mu \) represents the Hubble flow, with radial velocity proportional to spatial distance from \( O \).

At this order we can verify the conservation of the energy-momentum tensor. As \( \rho(t) R(t)^3 \) is conserved, we have to linear order \( \rho(t) = \rho_0(1 - 3HT) \), which implies that

\[
\rho(T) = \rho_0(1 - 3HT). \tag{23}
\]
The energy-momentum tensor near \( O \) is

\[
T^{\mu\nu} = \rho U^\mu U^\nu \tag{24}
\]
with components

\[
T^{00} = \rho_0(1 - 3HT),
\]
\[
T^{0i} = T^{i0} = \rho_0(1 - 3HT)HX^i = \rho_0 HX^i,
\]
\[
T^{ij} = \rho_0(1 - 3HT)HX^iX^j = 0, \tag{25}
\]
where the final expressions retain terms only up to linear order in the coordinates. As the Christoffel symbols now vanish at \( O \), the energy-momentum conservation law is that of Minkowski space at this order. We find

\[
\partial_0 T^{00} + \partial_i T^{i0} = -3\rho_0 H + 3\rho_0 H = 0,
\]
\[
\partial_0 T^{0j} + \partial_i T^{ij} = 0, \tag{26}
\]
so energy-momentum conservation is verified, but only at \( O \). In summary, we see that \( H \) is determined either from the Hubble flow of matter, or from the time derivative of \( \rho \).

It would be interesting to extend this analysis, working with \( T^{\mu\nu} \) at quadratic order and Christoffel symbols at linear order, to verify energy-momentum conservation at linear order.
3 Newtonian Spacetime Patches

Here we show that the Newtonian approximation to spacetime structure is valid in a spacetime patch around the point O. By Newtonian approximation we mean the weak field approximate solution of Einstein’s equations in the presence of static or slowly moving matter. Geodesic motion of a test particle in this curved Newtonian spacetime patch reproduces the test particle’s Newtonian motion in flat spacetime under the influence of the gravitational potential.

The Newtonian potential Φ is a solution of the Poisson equation

\[ \nabla^2 \Phi = 4\pi G \rho . \]  

(27)

For a spatially uniform matter density \( \rho \), which may depend on time, the solution we choose is

\[ \Phi(t, x) = \frac{2\pi G \rho}{3} x \cdot x . \]  

(28)

Here, \((t, x)\) are time and space coordinates that vanish at O. They are not the comoving coordinates that appear in eq.(4). The solution (28) is adapted to O as centre, and is only useful in a patch around O. Another solution would be used in the neighbourhood of another point.

Where Φ is small, and \( \rho \) slowly varying, the metric [11]

\[ ds^2 = -(1 + 2\Phi) dt^2 + (1 - 2\Phi) dx \cdot dx \]  

(29)

satisfies the linearised Einstein equations, provided Φ satisfies eq.(27). It is essential that \( 2|\Phi| \) is everywhere much less than 1, and this limits the region of validity of (29). The temporal part of the metric, \(-(1 + 2\Phi) dt^2\) is sufficient to reproduce the Newtonian equation of motion of non-relativistic test particles,

\[ \ddot{x} = -\nabla \Phi = -\frac{4\pi G}{3} \rho x . \]  

(30)

The acceleration towards the central point O is proportional to the distance from O. Eq.(30) also controls the acceleration of the background matter, the dust. If we write \( x = R(t)x' \), where \( x' \) is a comoving coordinate that is time-independent, then (30) becomes

\[ \ddot{R} = -\frac{4\pi G}{3} \rho . \]  

(31)

which is the Friedmann equation [3]. Moreover, conservation of matter requires that \( \rho R^3 \) is constant, as in (7).
If we had used the solution adapted to another centre, the acceleration would have been towards that centre. This apparent contradiction is resolved if we can show that the different Newtonian patches all give correct local descriptions of the FRW universe. We return to this point in Section 4.

We want to show next that, by a change of coordinates, the Newtonian-type metric (29), with \( \Phi = \frac{2\pi G}{3} \rho \mathbf{x} \cdot \mathbf{x} \),

\[
ds^2 = -dt^2 + d\mathbf{x} \cdot d\mathbf{x} - \frac{4\pi G}{3} \rho \mathbf{x} \cdot d\mathbf{x} dt^2 - \frac{4\pi G}{3} \rho \mathbf{x} \cdot d\mathbf{x} \cdot d\mathbf{x},
\]

(32)
can be reduced to the Riemann normal form of the FRW metric near the point O. As before, this coordinate change preserves spherical symmetry about O, so the formulae are very similar to those in Section 2.

Note that (32) differs from flat Minkowski space only at quadratic order, so only cubic coordinate changes are needed, with no quadratic terms. A further simplification is that we can treat the response of the metric to the time-varying density as instantaneous. In fact, we can effectively ignore the time-dependence of \( \rho \). We know that \( \rho(t) = \rho_0(1 - 3Ht) \), to linear order in \( t \), but the \( t \)-dependence only occurs in the metric at cubic order, beyond the order to which we are working. So we rewrite (32) as

\[
ds^2 = -dt^2 + d\mathbf{x} \cdot d\mathbf{x} - \frac{4\pi G}{3} \rho_0 \mathbf{x} \cdot d\mathbf{x} dt^2 - \frac{4\pi G}{3} \rho_0 \mathbf{x} \cdot d\mathbf{x} \cdot d\mathbf{x}.
\]

(33)

In (33) we carry out the change of coordinates

\[
t = T - \frac{4\pi G}{9} \rho_0 T \mathbf{X} \cdot \mathbf{X},
\]

\[
x = \mathbf{X} - \frac{2\pi G}{9} \rho_0 T^2 \mathbf{X} + \frac{2\pi G}{9} \rho_0 (\mathbf{X} \cdot \mathbf{X}) \mathbf{X},
\]

(34)

and hence

\[
dt = dT - \frac{4\pi G}{9} \rho_0 (\mathbf{X} \cdot \mathbf{X}) dT + 2T \mathbf{X} \cdot d\mathbf{X},
\]

\[
dx = d\mathbf{X} - \frac{2\pi G}{9} \rho_0 (2T \mathbf{X} \cdot dT + T^2 d\mathbf{X})
\]

\[
+ \frac{2\pi G}{9} \rho_0 (2\mathbf{X} \cdot d\mathbf{X} + (\mathbf{X} \cdot d\mathbf{X})).
\]

(35)

The resulting metric is exactly the same at quadratic order as (21), the Riemann form of the FRW metric near O. The spacetime curvature depends on just the instantaneous density \( \rho_0 \).

The coordinate change (34) is easily inverted by taking the cubic terms to the other side, and in them replacing \((T, \mathbf{X})\) by \((t, \mathbf{x})\). Then, applied to (21)
one would obtain the metric (33). Therefore, by composing the coordinate changes, one can obtain the Newtonian-type metric (33) starting from the FRW metric (4), at least up to quadratic order.

The Newtonian-type metric doesn’t determine the Hubble parameter \( H \). As we showed at the end of section 2, a radial Hubble flow with any value of \( H \) is consistent, provided that the size of the Newtonian patch is limited so that the velocities relative to \( O \) stay non-relativistic. As before, the value of \( H \) is correlated with the time-dependence of \( \rho \) in the coordinates we are using (either \( t \) or \( T \)). At linear order, \( \rho(T) = \rho_0(1 - 3HT) \).

### 4 Overlapping Newtonian Patches

We have shown that the metric and comoving dust in an FRW universe, in the neighbourhood of a spacetime point \( O \), can be described by a Newtonian-type metric and a radial Hubble flow. The spatial homogeneity of the FRW universe implies that this result must be true for any choice of \( O \). The local metric will be of identical form for patches centred at two distinct points with the same comoving time. In particular, for a pair of points with a small separation, whose Newtonian patches significantly overlap, the apparent inhomogeneity of the Newtonian-type metric should disappear after a coordinate transformation.

Recall the metric (33),

\[
    ds^2 = -dt^2 + dx \cdot dx - \kappa x \cdot dx dt^2 - \kappa x \cdot dx \cdot dx \, ,
\]

where we have written \( \kappa = \frac{4\pi G}{3} \rho_0 \). This is apparently inhomogeneous in \( x \), as the structure changes if we set \( x = y + \varepsilon \), where \( \varepsilon \) is a small constant vector, and expand around \( \varepsilon \). However, with a more subtle coordinate transformation, the metric expressed in terms of \( y \) has effectively the same form as the original metric in terms of \( x \). This is not an exact result, but true to first order in \( \varepsilon \) and up to a certain order in the expansion in \( y \).

Let us simplify the algebra by choosing \( \varepsilon = (\varepsilon, 0, 0) \). The required coordinate change is of the form

\[
    t = u - \varepsilon \kappa y_1 u \, , \\
    x_1 = y_1 + \varepsilon \frac{1}{2} \varepsilon \kappa (y_1^2 - y_2^2 - y_3^2 - u^2) \, , \\
    x_2 = y_2 + \varepsilon \kappa y_1 y_2 \, , \\
    x_3 = y_3 + \varepsilon \kappa y_1 y_3 \, ,
\]
which includes all types of quadratic term consistent with the vector nature of $\varepsilon$, and which reduces to the trivial coordinate change when $\varepsilon = 0$. The coefficients are such that after the coordinate change, if one keeps the terms up to first order in $\varepsilon$ and up to quadratic order in $u$ and $y$, then the metric becomes

$$ds^2 = -du^2 + dy \cdot dy - \kappa y \cdot y du^2 - \kappa y \cdot y dy \cdot dy,$$

which has the same form as (36).

Because the metric preserves its form to this order, one can say that the Newtonian patch around O is geometrically homogeneous, despite appearances. This gives a novel resolution of the apparent paradox in Newtonian cosmology that the physics picks out a choice of centre.

To further clarify why we can be satisfied with a result to this order, we show in the Appendix how a simpler metric, the round metric on a 2-sphere, exhibits its homogeneity when one compares the expansions of the metric around two closely separated points.

5 Conclusions

We have considered the standard, spatially homogeneous FRW universe filled with pressureless dust. We have shown that a patch of the universe, centred at any spacetime point O, may be brought by coordinate changes to Newtonian form, where the spacetime metric is expressed in terms of a local Newtonian gravitational potential $\Phi$. Our calculations have made use of the Riemann normal form of the metric, where the Riemann curvature tensor is directly related to the quadratic terms in the expansion of the metric around a chosen point. The transformations from FRW to Riemann form, and from Newtonian to Riemann form, have been given explicitly. Combining the first transformation with the formal inverse of the second is straightforward, and transforms the FRW metric to Newtonian form. The agreement is at quadratic order in the expansion of the metric around O. This means that the explicit metrics we have presented, (21) and (33), which are truncated at quadratic order, osculate the FRW spacetime.

This approach, which fully uses the ideas of general relativity, and involves fairly complicated nonlinear coordinate changes, avoids some of the paradoxes of a purely Newtonian cosmology. For example, there is no need to regard matter as occurring only in a large but finite ball. Our approach also avoids the paradox that in an infinite Newtonian cosmology one finds that
the acceleration of matter appears to be towards a fixed centre, violating the notion of spatial homogeneity.

We have shown how the Newtonian patches centred at nearby points are geometrically consistent with each other. The Newtonian atlas of patches is therefore consistent with homogeneity. Ultimately, this is possible because the local Newtonian potential $\Phi$ is not a geometrical invariant, but varies (by more than constant shifts) under coordinate transformations.

It would be interesting to extend the analysis here to include a cosmological constant, and to allow for small-scale density inhomogeneities.

## Appendix

Here we consider how the homogeneity of the round metric on the 2-sphere is manifested when one works with expansions around neighbouring points.

We use a stereographic coordinate $z = z_1 + iz_2$. The metric is

$$ds^2 = \frac{dzd\bar{z}}{(1 + z\bar{z})^2}$$

and is invariant under SU(2) Möbius transformations

$$z = \frac{\alpha w + \beta}{-\beta w + \alpha}$$

with $|\alpha|^2 + |\beta|^2 = 1$. The metric has the same form in terms of $w$ as it has in terms of $z$.

Now suppose that $\alpha = 1$ and $\beta = \varepsilon$, with $\varepsilon$ small and real, and work to first order in $\varepsilon$. The Möbius transformation has the form

$$z = w + \varepsilon + \varepsilon w^2,$$

so $dz = dw + 2\varepsilon w dw$. To first order in $\varepsilon$ and exactly in $w$ or $z$, one can check that the metric is formally invariant. Notice that the transformation (41) is a small translation with a quadratic correction. The patches centred at $z = 0$ and $w = 0$ have large overlap. Formally, there is no problem dropping terms of higher order in $\varepsilon$, but they are only small provided $\varepsilon |w| \ll 1$.

The next step is the most interesting, and gets closest to the issue of the effective homogeneity of the Newtonian-type metric. Consider the expansion of the metric (39) around the origin, to quadratic order in $|z|$,

$$ds^2 = (1 - 2z\bar{z})dzd\bar{z},$$

13
and carry out the coordinate change (41). One finds

$$ds^2 = (1 - 2w \bar{w} - 6\varepsilon(w\bar{w}^2 + w^2\bar{w}))dw d\bar{w}.$$ (43)

The form of (42) is reproduced, with $z$ replaced by $w$, but only if one drops the terms of first order in $\varepsilon$ and cubic in $|w|$. This is to be expected. The approximation (42) has ignored terms which are quartic in $|z|$, and from these one obtains, using (41), further terms that are first order in $\varepsilon$ and cubic in $|w|$, which cancel those in (43).

The Newtonian-type metric is analogous. This has terms with coefficients quadratic in $x$, but there are potentially quartic terms which have been dropped. One can only expect the metric to be effectively homogeneous under shifts of the origin if one drops terms that are simultaneously first order in the small shift parameter $\varepsilon$ and cubic in the coordinates $x$. This agrees with what we found in Section 4.

Also worth noting is that if one uses the real coordinates on the 2-sphere $z_1$ and $z_2$, and analogously $w_1$ and $w_2$, then the metric (42) is

$$ds^2 = (1 - 2z_1^2 - 2z_2^2)(dz_1^2 + dz_2^2),$$ (44)

and the coordinate change (41) is $z_1 = w_1 + \varepsilon + \varepsilon(w_2^2 - w_1^2)$ and $z_2 = w_2 + 2\varepsilon w_1 w_2$. The metric now has considerable similarity to (36), and the coordinate change to the second and third of (37).

Acknowledgements

I am grateful to Gary Gibbons and George Ellis for explaining their ideas about the Newtonian approach to cosmology, and to them and John Barrow for general advice on FRW spacetimes.

References

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, New York, Wiley, 1972.

[2] E.A. Milne, A Newtonian expanding universe, *Quart. J. Math.* 5, 64-72 (1934).

[3] W.H. McCrea and E.A. Milne, Newtonian universes and the curvature of space, *Quart. J. Math.* 5, 73-80 (1934).

[4] H. Bondi, *Cosmology (2nd ed.)*, Cambridge, Cambridge University Press, 1960.
[5] C. Callan, R.H. Dicke and P.J.E. Peebles, Cosmology and Newtonian mechanics, *Am. J. Phys.* 33, 105-108 (1965).

[6] W.H. McCrea, On the significance of Newtonian cosmology, *Astronomical J.* 60, 271-274 (1955).

[7] G.W. Gibbons and G.F.R. Ellis, Discrete Newtonian cosmology, arXiv:1308.1852 (2013).

[8] D. Layzer, On the significance of Newtonian cosmology, *Astronomical J.* 59, 268-270 (1954).

[9] J.D. Norton, The force of Newtonian cosmology: acceleration is relative, *Philosophy of Science* 62, 511-522 (1995).

[10] E.R. Harrison, *Cosmology: the Science of the Universe* (2nd ed.), Cambridge, Cambridge University Press, 2000.

[11] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, San Francisco, Pearson Addison Wesley, 2004.

[12] M. Spivack, *A Comprehensive Introduction to Differential Geometry*, Vol. 2, Berkeley, Publish or Perish, 1979.