Recovering the Lowest Layer of Deep Networks with High Threshold Activations

Surbhi Goel*1 and Rina Panigrahy†2

1University of Texas at Austin
2Google

Abstract

Giving provable guarantees for learning neural networks is a core challenge of machine learning theory. Most prior work gives parameter recovery guarantees for one hidden layer networks, however, the networks used in practice have multiple non-linear layers. In this work, we show how we can strengthen such results to deeper networks – we address the problem of uncovering the lowest layer in a deep neural network under the assumption that the lowest layer uses a high threshold before applying the activation, the upper network can be modeled as a well-behaved polynomial and the input distribution is Gaussian.

1 Introduction

Understanding the landscape of learning neural networks has been a major challenge in machine learning. Various works give parameter recovery guarantees for simple one-hidden-layer networks where the hidden layer applies a non-linear activation $u$ after transforming the input $x$ by a matrix $W$, and the upper layer is the weighted sum operator: thus $f(x) = \sum a_i u(w_i^T x)$ (Sedghi and Anandkumar (2014); Goel et al. (2016); Zhang et al. (2016); Li and Yuan (2017); Ge et al. (2017)). However, the networks used in practice have multiple non-linear layers and it is not clear how to extend these known techniques to deeper networks.

We first study a two layer network where the node in the second layer instead of a weighted-sum (as in a one layer network) is a multivariate polynomial $P$ in all its inputs, that is, the outputs of the first layer (note that even though we are talking about a two layer network, $P$ could represent the operation of all the layers above the first layer in a deeper network). More formally,

$$f_W(x) = P(u(w_1^T x), u(w_2^T x), \ldots, u(w_d^T x))$$

for $P(X_1, \ldots, X_d) = \sum_{r \in \mathbb{Z}_+^d} c_r \cdot \prod_j X_j^{r_j}$.

We assume that the input $x$ is generated from the standard Gaussian distribution and there is an underlying true network (parameterized by some unknown $W^*$) from which the labels are generated. We strengthen previous results for recovering weights of one hidden layer networks to this larger class of networks represented by a polynomial where $u$ is a “high threshold” (high bias term) before applying the activation: $u(a-t)$ instead of $u(a)$. The intuition for using high thresholds is as follows:

*surbhi@cs.utexas.edu. Work done while the author was at Google.
†rinap@google.com
a high threshold is looking for a high correlation of the input $a$ with a direction $w_i^*$. Thus even if the function $f$ is applying a complex transform, the identity of these high threshold directions may be preserved in the training data generated using $f$.

1.1 Our Contributions

Our main result shows that for $P$ being a multivariate polynomial, if the thresholds of the lowest layer are higher than $\Omega(\sqrt{\log d})$ where $d$ is the dimension, then the lowest layer is learnable under certain assumptions on the coefficients of the polynomial $P$. Furthermore even for lower thresholds, we give a polynomial sample complexity bound for recovering the weights which allows an exponential time algorithm. Our results also apply to the problem of union of halfspaces as it can be expressed by the OR function at the top node which can be written as a polynomial. Finally we note that if we have a deep network with depth larger than two, then assuming the layers two onwards can be written as a polynomial (or taylor series) we can learn the lowest layer under the conditions required by our assumptions.

Learning with linear terms in $P$. Suppose $P$ has a linear component then we show that increasing the threshold $t$ is equivalent to amplifying the coefficients of the linear part. Instead of dealing with the polynomial $P$ it turns out that we can roughly think of it as $P(\mu X_1, ..., \mu X_d)$ where $\mu$ decreases exponentially in $t$ ($\mu \approx e^{-t^2}$). As $\mu$ decreases it has the effect of diminishing the non-linear terms more strongly so that relatively the linear terms stand out. Taking advantage of this effect we manage to show that if $t$ exceeds a certain threshold the nonlinear terms drop in value enough so that the directions $w_i^*$ can be learned approximately by existing methods.

**Theorem 1** (Informal version of Claim 2, Theorem 12). If $t > c\sqrt{\log d}$ for large enough constant $c > 0$ and $P$ has linear terms with absolute value of coefficients at least $1/poly(d)$ and all coefficients at most $O(1)$, we can recover the weight vector $w_i^*$ within error $1/poly(d)$ in time $poly(d)$.

These approximations of $w_i^*$ obtained collectively can be further refined by looking at directions along which there is a high gradient in $f$; for monotone functions we show how in this way we can recover $w_i^*$ up to $\epsilon$-precision.

**Theorem 2.** (informal version of Theorem 5) Under the conditions of the previous theorem, for monotone $P$, there exists a procedure to refine the angle to precision $\epsilon$ in time $poly(1/\epsilon, d)$ starting from an estimate that is $1/poly(d)$ close.

The above mentioned theorems hold for $u$ being sign and ReLU. Theorem 1 also holds for $u$ being sigmoid with $t \geq c \log d$.

Learning intersection of halfspaces far from the origin. When $P$ is monotone and $u$ is the sign function, learning $W^*$ is equivalent to learning a union of half spaces. We learn the weights by learning sign of $P$ which is exactly the union of halfspaces $(w_i^*)^T x = t$. Thus our algorithm can also be viewed as a polynomial time algorithm for learning a union of large number of half spaces with high thresholds or alternately (viewing the complement) for learning intersection of halfspaces that are far from the origin – to our knowledge this is the first polynomial time algorithm for this problem under the given assumptions (see earlier work by Vempala (2010) for an exponential time algorithm). Refer to Section 2.3 for more details.

1 We suppress $W$ when it is clear from context.
Application to learning Deep Networks. If $P$ is used to model the operation performed by layers two and higher (as long as it has a taylor series expansion), we note that such linear components may easily be present in $P$: consider for example the simple case where $P(X) = u(u^T X - b)$ where $u$ is say the sigmoid or the logloss function. The taylor series of such functions has a linear component – note that since the linear term in the taylor expansion of $u(x)$ has coefficient $u'(0)$, for expansion of $u(x - b)$ it will be $u'(-b)$ which is $\Theta(e^{-b})$ in the case of sigmoid. In fact one may even have a tower (deep network) or such sigmoid/logloss layers and the linear components will still be present – unless they are made to cancel out precisely; however, the coefficients will drop exponentially in the depth of the networks and the threshold $b$.

Sample complexity with low thresholds and no explicit linear terms. Even if the threshold is not large or $P$ is not monotone, we show that $W^*$ can be learned with a polynomial sample complexity (although possibly exponential time complexity) by finding directions that maximize the gradient of $f$.

**Theorem 3** (informal version of Corollary 1). If $u$ is the sign function and $w_i^*$’s are orthogonal then in $\text{poly}(1/\epsilon,d)$ samples one can determine $W^*$ within precision $\epsilon$ if the coefficient of the linear terms in $P(\mu(X_1 + 1), \mu(X_2 + 1), \mu(X_3 + 1), \ldots)$ is least $1/\text{poly}(d)$.

Learning without explicit linear terms. We further provide evidence that $P$ may not even need to have the linear terms – under some restricted cases (section), we show how such linear terms may implicitly arise even though they may be entirely apparently absent. For instance consider the case when $P = \sum X_i X_j$ that does not have any linear terms. Under certain additional assumptions we show that one can recover $w_i^*$ as long as the polynomial $P(\mu(X_1 + 1), \mu(X_2 + 1), \mu(X_3 + 1), \ldots)$ (where $\mu$ is $e^{-t}$ has linear terms components larger than the coefficients of the other terms). Note that this transform when applied to $P$ automatically introduces linear terms. Note that as the threshold increases applying this transform on $P$ has the effect of “gathering” linear components from all the different monomials in $P$ and penalizing the higher degree monomials. We show that if $W^*$ is a sparse binary matrix then we can recover $W^*$ when activation $u(a) = e^{\rho a}$ under certain assumptions about the structure of $P$. When we assume the coefficients are positive then these results extend for binary low $l_1$- norm vectors without any threshold. Lastly, we show that for even activations ($\forall a, u(a) = u(-a)$) under orthogonal weights, we can recover the weights with no threshold.

Learning with high thresholds at deeper layers. We also point out how such high threshold layers could potentially facilitate learning at any depth, not just at the lowest layer. If there is any cut in the network that takes inputs $X_1, \ldots, X_d$ and if the upper layers operations can be modeled by a polynomial $P$, then assuming the inputs $X_i$ have some degree of independence we could use this to modularly learn the lower and upper parts of the network separately (Appendix E).

### 1.2 Related Work

Various works have attempted to understand the learnability of simple neural networks. Despite known hardness results (Livni et al. (2014); Goel et al. (2016); Brutzkus and Globerson (2017)), there has been an array of positive results under various distributional assumptions on the input and the underlying noise in the label. Most of these works have focused on analyzing one hidden layer neural networks. A line of research has focused on understanding the dynamics of gradient descent on these networks for recovering the underlying parameters under gaussian input.
distribution (Du et al. (2017b,a); Li and Yuan (2017); Zhong et al. (2017a); Zhang et al. (2017); Zhong et al. (2017b)). Another line of research borrows ideas from kernel methods and polynomial approximations to approximate the neural network by a linear function in a high dimensional space and subsequently learning the same (Zhang et al. (2015); Goel et al. (2016); Goel and Klivans (2017b,a)). Tensor decomposition methods (Anandkumar and Ge (2016); Janzamin et al. (2015)) have also been applied to learning these simple architectures.

The complexity of recovering arises from the highly non-convex nature of the underlying optimization problem. The main result we extend in this work is due to Ge et al. (2017). They show how to learn a single hidden layer neural network by designing a loss function that exhibits a "well-behaved" landscape for optimization avoiding this complexity. However, much like most other results, it is unclear how to extend to deeper networks. The only known result for networks with more than one hidden layer is due to Goel and Klivans (2017b). Combining kernel methods with isotonic regression, they provably learn networks with sigmoids in the first hidden layer and a single unit in the second hidden layer in polynomial time. We however model the above layer as a multivariate polynomial allowing for larger representation. Another work due to Arora et al. (2014) deals with learning a deep generative network when several random examples are generated in an unsupervised setting. By looking at correlations between input coordinates they are able to recover the network layer by layer. We use some of their ideas in section 4 when the weights of the lowest layer are sparse.

1.3 Notation

We denote vectors and matrices in bold face. $\|\cdot\|_p$ denotes the $l_p$-norm of a vector. $\|\cdot\|$ without subscript implies the $l_2$-norm. For matrices $\|\cdot\|$ denotes the spectral norm and $\|\cdot\|_F$ denotes the forbenius norm. $\mathcal{N}(0,\Sigma)$ denotes the multivariate gaussian distribution with mean 0 and covariance $\Sigma$. For a scalar $x$ we will use $\phi(x)$ to denote the p.d.f. of the univariate standard normal distribution with mean zero and variance 1. For a vector $x$ we will use $\phi(x)$ to denote the p.d.f. of the multivariate standard normal distribution with mean zero and variance 1 in each direction. $\Phi$ denotes the c.d.f. of the standard gaussian distribution. Also define $\Phi^c = 1 - \Phi$. Let $h_i$ denote the $i$th normalized Hermite polynomial (Wikipedia (2018)). For a function $f$, let $\hat{f}_i$ denote the $i$th coefficient in the hermite expansion of $f$, that is, $\hat{f}_i = \mathbb{E}_{g \sim \mathcal{N}(0,1)}[f(g) h_i(g)]$. For a given function $f$ computed by the neural network, we assume that the training samples $(x,y)$ are such that $x \in \mathbb{R}^n$ is distributed according to $\mathcal{N}(0,1)$ and label has no noise, that is, $y = f(x)$.

Note: Most proofs are deferred to the Appendix for clear presentation.

2 Recovery in the presence of Linear term

In this section we consider the case when $P$ has a positive linear component and we wish to recover the true parameters $W^\ast$. The algorithm has two-steps: 1) use existing learning algorithm (SGD on carefully designed loss (Ge et al. (2017))) to recover an approximate solution, 2) refine the approximate solution by performing local search (for monotone $P$). The intuition behind the first step is that high thresholds enable $P$ to in expectation be approximately close to a one-hidden-layer network which allows us to transfer algorithms for learning one-hidden-layer networks with approximate guarantees. Secondly, with the approximate solutions as starting points, we can evaluate the closeness of the estimate of each weight vector to the true weight vector using simple correlations. The intuition of this step is to correlate with a function that is large only in the direction of the true weight vectors. This equips us with a way to design a local search based algorithm to refine the estimate to small error.
We will first list our assumptions. For simplicity in this section we will work with \( P \) where the highest degree in any \( X_i \) is 1. The degree of the overall polynomial can still be \( d \). See Appendix B.7 for the extension to general \( P \). Also note that many of our arguments become much simpler if we assume that \( W^* \) is orthogonal thus it may be useful to the reader to bear that case in mind for the rest of the section.

**Assumptions 1** (Structure of polynomial). We assume that \( P \) is such that \( P(X_1, \ldots, X_k) = c_0 + \sum_{i \in [d]} c_i X_i + \sum_{S \subseteq [d]: |S| > 1} c_S \prod_{j \in S} X_j \) where \( c_i = \Theta(1)^2 \) for all \( i \in [d] \) and for all \( S \subseteq [d] \) such that \( |S| > 1 \), \( |c_S| \leq O(1) \). Also \( \kappa(W^*) = O(1) \).

Thus we have, \( f(x) = c_0 + \sum_{i \in [d]} c_i u((w_i^*)^T x) + \sum_{S \subseteq [d]: |S| > 1} c_S \prod_{j \in S} u((w_j^*)^T x) \). Denote \( f_{lin}(x) = c_0 + \sum_{i \in [d]} c_i u((w_i^*)^T x) \) to be the linear part of \( f \).

We will restrict the activation function to be a high threshold function with decaying gaussian tails.

**Assumptions 2.** Activation function \( u \) is a positive high threshold activation with threshold \( t \), that is, the bias term is \( t \). \( E_{g\sim\mathcal{N}(0,\sigma^2)}[u_t(g)] \leq \rho(t, \sigma) \) where \( \rho \) is a positive decreasing function of \( t \). Also, \( \vert \hat{u}_k \vert = t^{\Theta(1)}(\rho(t, 1) \) for \( k = 2, 4 \) where \( \hat{u}_k \) is the \( k \)th Hermite coefficient of \( u \).

Observe that for "high-threshold" ReLU, that is, \( u_t(a) = \max(0, a - t) \), \( E_{g\sim\mathcal{N}(0,\sigma^2)}[u_t(g)] \) is bounded by \( \approx e^{-\frac{t^2}{2\sigma^2}} \) (see Lemma 11) which decays exponentially with \( t \). Thus, for large enough \( t \), the gaussian weight of the tail is small. The following assumption dictates how large a threshold is needed.

**Assumptions 3** (Value of \( t \)). \( t \) is large enough such that \( \rho(t, \|W^*\|) \approx d^{-\eta} \) and \( \rho(t, 1) \approx d^{-p\eta} \) with for large enough constant \( \eta > 0 \) and \( p \in (0, 1] \).

For example, for high threshold ReLU, \( t = C \sqrt{\eta \log d} \) for large enough constant \( C > 0 \) suffices to get the above assumption (\( \kappa(W^*) \) is a constant).

### 2.1 Step 1: Approximate Recovery using Landscape Design

High-threshold activation are useful for learning as in expectation, they ensure that \( f \) is close to \( f_{lin} \) since the product terms have low expected value. This is made clear by the following lemmas.

The first lemma shows that the expected value of a product of activations is exponentially small in the number of terms in the product; again note that in the orthogonal case this is easy to observe as the the terms are independent and the expectation of the product becomes a product of expectations.

**Lemma 1.** For \( |S| > 1 \), under Assumption 2, we have,

\[
\mathbb{E} \left[ \prod_{j \in S} u_t((w_j^*)^T x) \right] \leq \rho(t, 1) (\kappa(W^*) \rho(t, \|W^*\|))^{\vert S \vert - 1}.
\]

So if \( \mu := \kappa(W^*) \rho(t, \|W^*\|) \), then \( \mathbb{E}[\prod_{j \in S} X_j(x)] \leq \rho(t, 1) \mu^{\vert S \vert - 1} \)

The second lemma bounds the closeness of \( f \) and \( f_{lin} \) in expectation.

---

\( ^2 \)We can handle \( \in [d^{-C}, d^{C}] \) for some constant \( C \) by changing the scaling on \( t \).

\( ^3 \)For similar bounds for sigmoid and sign refer to Appendix B.0.
Lemma 2. Let $\Delta(x) = f(x) - f_{\text{lin}}(x)$. Under Assumptions [1, 2] and [3] if $t$ is such that $d\rho(t, ||W^*||) \leq c$ for some small enough constant $c > 0$ we have,

$$E[|\Delta(x)|] \leq O \left( d^3 \rho(t, 1) \rho(t, ||W^*||) \right) = O \left( d^{-(1+p)\eta+3} \right) .$$

Note: We should point out that $f(x)$ and $f_{\text{lin}}(x)$ are very different point wise; they are just close in expectation under the distribution of $x$. In fact, if $d$ is some constant then even the difference in expectation is some small constant.

This closeness suggests that algorithms for recovering under the labels from $f_{\text{lin}}$ can be used to recover with labels from $f$ approximately.

Learning $f_{\text{lin}}$ using Landscape Design. Ge et al. (2017) proposed an algorithm for learning one-hidden-layer networks by designing a well behaved loss function based on correlations to recover the underlying weight vectors. We can reinterpret their result as applying a simple variant of PCA to recover the $w^*_i$. Again the main idea is very easy for the orthonormal case so we explain that first. An application of PCA can be thought of as finding principal directions as the local $k$th optimization evaluates to $max z$ to

$$\sum_{i}^{} |c_i(z^T w_i^*)^4 = \sum_{i}^{} c_i(z^T w_i^*)^4$$

which is equivalent to $max |u| = \sum_{i}^{} c_i u_i^4$ by substituting $u = W^* z$. Note that if all the $c_i$ are equal this is like maximizing sum of fourth powers on the sphere which is achieved when exactly one $u_i$ is $\pm 1$ which corresponds to $z$ being along a $w_i^*$

To handle non-orthogonal weights they showed that the local minima of the following modified optimization\footnote{Ge et al. (2017) do not mention $\hat{u}_2$ but it is necessary when the weight vectors are non-orthogonal for the correct reduction. Since for us, this value can be small, we mention the dependence.} corresponds to some transform of each of the $w_i^*$ – thus it can be used to recover $w_i^*$, one at a time.

$$\min z \ G_{\text{lin}}(z) := -\text{sgn}(\hat{u}_4) E[f_{\text{lin}}(x) H_4(z^T x)] + \lambda E[f_{\text{lin}}(x) H_2(z^T x)] - \hat{u}_2 z$$

where $H_2(z^T x) = ||z||^2 h_2 \left( \frac{z^T x}{||z||^2} \right) = \frac{(z^T x)^2}{\sqrt{2}} - \frac{||z||^2}{\sqrt{2}}$ and $H_4(z^T x) = ||z||^4 h_4 \left( \frac{z^T x}{||z||^2} \right) = \sqrt{6} (z^T x)^4 - \frac{||z||^2 (z^T x)^2}{2} + \frac{||z||^4}{4}$ (see Appendix \textit{A.1} for more details). Using properties of Hermite polynomials, we have $E[f_{\text{lin}}(x) H_2(z^T x)] = \hat{u}_2 \sum_{i}^{} c_i(z^T w_i^*)^2$ and similarly $E[f_{\text{lin}}(x) H_4(z^T x)] = \hat{u}_4 \sum_{i}^{} (z^T w_i^*)^4$.

Thus

$$G_{\text{lin}}(z) = -||\hat{u}_4|| \sum_{i}^{} c_i(z^T w_i^*)^4 + \lambda \hat{u}_2 \left( \sum_{i}^{} c_i(z^T w_i^*)^2 - 1 \right)^2 .$$

Let $T$ be a diagonal matrix with $T_{ii} = \sqrt{c_i}$ then using results from Ge et al., (2017), it can be shown that the approximate local minima of this problem are close to columns of $(TW^*)^{-1}$.

Definition 1 (($\epsilon, \tau$)-local minimum/maximum). $z$ is an $(\epsilon, \tau)$-local minimum of $F$ if $||\nabla F(z)|| \leq \epsilon$ and $\lambda_{\text{min}}(\nabla^2 F(z)) \leq \tau$.

Claim 1 (Ge et al. (2017)). An $(\epsilon, \tau)$-local minima of the Lagrangian formulation $z$ with $\epsilon \leq O \left( \frac{\sqrt{\tau^3}}{|\hat{u}_4|} \right)$ is such that for an index $i$ $|z^T w_i| = 1 \pm O(\epsilon/\lambda \hat{u}_2) \pm O(d \tau/|\hat{u}_4|)$ and $\forall j \neq i$, $|v^T w_j| = O(\sqrt{\tau/|\hat{u}_4|})$ where $w_i$ are columns of $(TW^*)^{-1}$.

Note that these are not exactly the directions $w_i^*$ that we need, one way to think about is that we can get the correct directions by estimating all columns and then inverting.
Moving from $f_{\text{fin}}$ to $f$. Consider the loss with $f$ instead of $f_{\text{fin}}$:

$$\min z : G(z) = -\text{sgn}(\hat{u}_4)\mathbb{E}[f(x)H_4(z^T x)] + \lambda(\mathbb{E}[f(x)H_2(z^T x)] - \hat{w}_2)^2$$

We previously showed that $f$ is close to $f_{\text{fin}}$ in expectation due to the high threshold property. This also implies that $G_{\text{fin}}$ and $G$ are close and so are the gradients and (eigenvalues of) Hessians of the same. This closeness implies that the landscape properties of one approximately transfers to the other function. More formally,

**Theorem 4.** Let $Z$ be an $(\epsilon, \tau)$-local minimum of function $A$. If $||\nabla (B - A)(Z)|| \leq \rho$ and $||\nabla^2 (B - A)(Z)|| \leq \gamma$ then $Z$ is an $(\epsilon + \rho, \tau + \gamma)$-local minimum of function $B$ and vice-versa.

We will now apply above lemma on our $G_{\text{fin}}(z)$ and $G(z)$.

**Claim 2.** For $\lambda = \Theta(|\hat{u}_4|/\hat{u}_2^2) \approx d^\eta$, an $(\epsilon, \tau)$-approximate local minima of $G$ (for small enough $\epsilon, \tau \leq d^{-2\eta}$) is an $O(\log d)d^{-(1+\eta)n+3}$-approximate local minima of $G_{\text{fin}}$. This implies $z$ is such that for an index $i$, $|z^T w_i| = 1+O(1)d^{-2/3\eta n+3}$ and $\forall j \neq i, |z^T w_j| = O(1)d^{-1/3\eta n+3/2}$ where $w_i$ are columns of $(TW^*)^{-1}$ (ignoring log $d$ factors).

**Note:** For ReLU, setting $t = \sqrt{C\log d}$ for large enough $C > 0$ we can get closeness $1/poly(d)$ to the columns of $(TW^*)^{-1}$. Refer Appendix B.6 for details for sigmoid. Ge et al. (2017) also provides an alternate optimization that when minimized simultaneously recovers the entire matrix $W^*$ instead of having to learn columns of $(TW^*)^{-1}$ separately. We show how our techniques can also be extended to the simultaneous optimization (see Appendix B.4) to recover $W^*$ by optimizing a single objective.

### 2.2 Step 2: Approximate to Arbitrarily Close for Monotone $P$

Assuming $P$ is monotone, we can show that the approximate solution from the previous analysis can be refined to arbitrarily closeness using a random search method by approximately finding the angle of our current estimate to the true direction.

The idea at a high level is to correlate the label with $\delta'(z^T x - t)$ where $\delta$ is the Dirac delta function. It turns out that the correlation is maximized when $z$ is equal to one of the $w_i^*$. Correlation with $\delta'(z^T x - t)$ equates to checking how fast the correlation of $f$ with $\delta(z^T x - t)$ is changing as you change $t$. To understand this look at the case when our activation $u$ is the sign function then note that correlation of $u_t(w^T x - t)$ with $\delta(w^T x - t)$ is very high as its correlation with $\delta(w^T x - t')$ is 0 when $t' < t$ and significant when $t' > t$. So as we change $t'$ slightly from $t - \epsilon$ to $t + \epsilon$ there is a sudden increase. If $z$ and $w$ differ then it can be shown that correlation of $u(w^T x - t)$ with $\delta'(z^T x - t)$ essentially depends on $\cot(\alpha)$ where $\alpha$ is the angle between $w$ and $z$ (for a quick intuition note that one can prove that $\mathbb{E}[u_t(w^T x)\delta'(z^T x)] = c\cot(\alpha)$. See Lemma 13 in Appendix).

In this section we will not correlate exactly with $\delta'(z^T x - t)$ but instead we will use this high level idea to estimate how fast the correlation with $\delta(z^T x - t')$ changes between two specific values as one changes $t'$, to get an estimate for $\cot(\alpha)$. Secondly since we can not perform a smooth optimization over $z$, we will do a local search by using a random perturbation and iteratively checking if the correlation has increased. We can assume that the polynomial $P$ doesn’t have a constant term $c_0$ as otherwise it can easily be determined and canceled out.

We will refine the weights one by one. WLOG, let us assume that $w_1^* = e_1$ and we have $z$ such that $z^T w_1^* = z_1 = \cos^{-1}(\alpha_1)$. Let $l(z, t, \epsilon)$ denote $\{x : z^T x \in [t - \epsilon, t]\}$ for $z \in S^{n-1}$.

\[^5\text{for example with REU activation, } f \text{ will be } c_0 \text{ most of the time as other terms in } P \text{ will never activate. So } c_0 \text{ can be set to say the median value of } f.\]
Theorem 5. Given a vector $z \in \mathbb{S}^{d-1}$ such that it is $1/poly(d)$-close to the underlying true vector $w_1^*$, that is $z^T w_1^* \leq 1/poly(d)$, running RefineEstimate for $O(T)$ iterations outputs a vector $z^* \in \mathbb{S}^{d-1}$ such that $z^*^T w_1^* \leq (1 - \frac{c}{d})^T \gamma$ for some constant $c > 0$. Thus after $O(d \log(1/\epsilon))$ iterations $\cos^{-1}((z^*)^T w_1^*) \leq \epsilon$.

We prove the correctness of the algorithm by first showing that EstimateTanAlpha gives a multiplicative approximation to $\tan(\alpha)$. The following lemma captures this property.

Lemma 3. EstimateTanAlpha$(z, \epsilon)$ for sufficiently small $\epsilon < \eta/t$ outputs $s$ such that $s = (1 \pm O(\eta)) \tan(\alpha_1) \pm O(\epsilon)$ where $\alpha_1$ is the angle between $z$ and $w_1^*$ for $\eta = de^{-\Omega(t^2)}$.

Proof. We first show that the given probability when computed with $\text{sgn}(x^T w_1^* - t)$ is a well defined function of the angle between the current estimate and the true parameter up to multiplicative error. Subsequently we show that the computed probability is close to the one we can estimate using $f(x)$ since the current estimate is close to one direction. The following two lemmas capture these properties.

Lemma 4. For $t, t'$ such that $t' \leq t/\cos(\alpha_1)$ and $\epsilon \leq 1/t'$, we have

$$\Pr[x^T w_1^* \geq t \text{ and } x \in l(z, t', \epsilon)] \leq \Phi^{-}(\frac{t - t^* \cos(\alpha_1)}{\sin(\alpha_1)}) \pm O(\epsilon)t'$$

where $t^* \in [t' - \epsilon, t']$ depends on $\epsilon$ and $t'$.

Lemma 5. For $t' \in [0, t/\cos(\alpha_1)]$, we have

$$\Pr[\text{sgn}(f(x)) \mid x \in l(z, t', \epsilon)] = \Pr[\text{sgn}(w_1^*)^T x - t] \mid x \in l(z, t, \epsilon)] + de^{-\Omega(t^2)}.$$

The following lemma bounds the range of $t_1$ and $t_2$. 

\begin{algorithm}[h]
\caption{RefineEstimate($\epsilon_1, \epsilon_2$)}
Compute $s := \text{EstimateTanAlpha}(z, \epsilon_1)$. Set estimate of angle between $z$ and $w_1^*$ to be $\alpha = \tan^{-1}(s)$.

Perturb current estimate $z$ by a vector along the $d-1$ dimensional hyperplane normal to $z$ such that $z' = z + 0.1 \sin(\alpha/d) \Delta z$ where $\Delta z \sim \mathcal{N}(0, I_{d-1})$.

Compute $s' := \text{EstimateTanAlpha}(z', \epsilon_1)$. Set estimate of angle between $z'$ and $w_1^*$ to be $\alpha' = \tan^{-1}(s')$.

if $\alpha' \leq 0.01\alpha/d$ then

Set $z \leftarrow z'$

end if

Repeat till $\alpha' \leq \epsilon_2$.

\end{algorithm}

\begin{algorithm}[h]
\caption{EstimateTanAlpha$(z, \epsilon)$}
Find $t_1$ and $t_2$ such that $\Pr[\text{sgn}(f(x)) \mid x \in l(z, t', \epsilon)]$ at $t_1$ is 0.4 and at $t_2$ is 0.6.

Return $\frac{t_2-t_1}{\Phi^{-}(0.6) - \Phi^{-}(0.4)}$.

\end{algorithm}
Lemma 6. We have $0 \leq t_1 \leq t_2 \leq \frac{t}{\cos(\alpha)}$.

Using the above, we can thus show that,

$$
\frac{t_2 - t_1}{\Phi^{-1}(0.6) - \Phi^{-1}(0.4)} = \left(1 - \frac{\eta_1 \pm O(\epsilon) t_1}{\Phi^{-1}(0.6) - \Phi^{-1}(0.4)}\right) \frac{\eta_2 \pm O(\epsilon) t_2}{\Phi^{-1}(0.6) - \Phi^{-1}(0.4)}
$$

where $\eta > \eta_1, \eta_2 > 0$ are the noise due to estimating using $f$ and $p_1 \in [0.6 - \eta_1 \pm O(\epsilon) t_1, 0.6]$ and $p_2 \in [0.4 - \eta_2 \pm O(\epsilon) t_2, 0.4]$ as long as $t_1, t_2 \in [0, t/\cos(\alpha)]$. As long as $\eta_2 + O(\epsilon) t_2 < 1$, we have,

$$
\frac{t_2 - t_1}{\Phi^{-1}(0.6) - \Phi^{-1}(0.4)} = (1 \pm O(\eta_1 + \eta_2 + \epsilon t_2)) \tan(\alpha_1) \pm O(\epsilon) = (1 + O(\eta)) \tan(\alpha_1) \pm O(\epsilon)
$$

Note that we can estimate the conditional probability using samples from the training data and by Chernoff bounds, we would require $\text{poly}(d/\epsilon)$ samples. As for finding $t_1$ and $t_2$, we can approximately do so by performing a binary search. This would incur an additional error which is in our control and can be handled in a standard way as the other estimate errors we show in the following analysis. For the ease of exposition, we assume that we get the exact values. Subsequently we show (proof in Appendix B.5) that with constant probability, a random perturbation reduces the angle by a factor of $(1 - 1/d)$ of the current estimate.

Lemma 7. By applying a random Gaussian perturbation along the $d - 1$ dimensional hyperplane normal to $z$ with the distribution $\mathcal{N}(0, \mathbf{1}_{d-1})$ scaled by $0.1 \sin(\alpha/d)$ and projecting back to the unit sphere, with constant probability, the angle $\alpha_1 (\leq \pi/2)$ with the true vector decreases by at least $0.004 \alpha/d$.

Since $\tan(\alpha)$ is a multiplicative approximation to $\tan(\alpha_1)$ and the angle is small (initially is small and our check ensures it stays small), we have $\tan(\alpha) = \Theta(\alpha)$ and $\tan(\alpha_1) = \Theta(\alpha_1)$ thus $\alpha = C\alpha_1 \pm O(\epsilon)$ for some constant $C > 0$. Therefore, the angle reduces by $\Omega(\alpha_1/d)$.

Thus if the angle decreases by a factor of $1 - Cd^{-1}$. Since our estimates are multiplicative, we can indeed verify the decrease in the value of $\tan$ for small enough $\eta$ (by assumption). Thus our check ensures that $\alpha_1$ goes down multiplicatively in every successful iteration sparing the additive loss of $\epsilon$ (this can be handled by choosing polynomially small $\epsilon$ in our estimate). Hence the algorithm will halt after $O(d \log(1/\epsilon))$ iterations and output an arbitrarily close solution.

2.3 Learning Intersection of Halfspaces far from the Origin

Here we apply the results from the previous sections to the problem of learning unions of high threshold halfspaces.

Theorem 6. Given non-noisy labels from a intersection of halfspaces that are at a distance $\Omega(\sqrt{\log d})$ from the origin and are each a constant angle apart, there is an algorithm to recover the underlying weights to $\epsilon$ closeness up to scaling in polynomial time.

Proof. Consider intersection of halfspaces far from the origin $g(x) = \bigwedge \text{sgn}(x^T w_i^* + t)$. The complement of the same can be expressed as $f(x) = \bigvee \text{sgn}(-x^T w_i^* - t)$. Observe that $\bigvee X_i$ is equivalent to $P(X_1, \ldots, X_d) = 1 - \prod (1 - X_i)$. Thus

$$
f(x) = \bigvee \text{sgn}(-x^T w_i^* - t) = 1 - \prod (1 - \text{sgn}(-x^T w_i^* - t)).
$$
Since \( P \) and \( \text{sgn} \) here satisfies our assumptions \[1, 2\] for \( t = \Omega(\sqrt{\log d}) \) (see Lemma \[11\]) we can apply Theorem \[12\] using the complement labels to recover the vectors \( w^*_i \) approximately. Subsequently, refining to arbitrarily close using Theorem \[5\] is possible due to the monotonicity of the union. Thus we can recover the vectors to arbitrary closeness in polynomial time.

Note that in the above proof we assumed that there was no noise in the samples. Bounded random noise can be handled using standard methods: the statistical quantities we estimate will be concentrated around the expected values with high probability.

3 Sample Complexity

In this section we will show how the ideas for the monotone setting work for non-monotone \( P \) even if it may not have any linear terms however we only manage to prove polynomial sample complexity for finding \( w^*_i \) instead of polynomial time complexity. The main idea as in the previous section is to correlate with \( \delta'(z^T x - t) \) and find \( \arg \max_{|z|_2 = 1} \mathbb{E}[f(x)\delta'(z^T x - t)] \). We will show that the correlation goes to infinity when \( z \) is one of \( w^*_i \) and bounded if it is far from all of them. From a practical standpoint we calculate \( \delta'(z^T x - s) \) by measuring correlation with \( \frac{1}{2\epsilon}(\delta(z^T x - s + \epsilon) - \delta(z^T x - s - \epsilon)) \). In the limit as \( \epsilon \to 0 \) this becomes \( \delta'(z^T x - s) \). \( \delta(z^T x - s) \) in turn is estimated using \( \frac{1}{\epsilon}(\text{sgn}(z^T x - s + \epsilon) - \text{sgn}(z^T x - s)) \), as in the previous section, for an even smaller \( \epsilon \); however, for ease of exposition, in this section, we will assume that correlations with \( \delta(z^T x - s) \) can be measured exactly.

Let us recall that \( f(x) = P(u((w^*_1)^T x), u((w^*_2)^T x), \ldots, u((w^*_d)^T x)) \). Let \( C_1(f, z, s) \) denote \( E[f(x)\delta'(z^T x - s)] \) and let \( C_2(f, z, s) \) denote \( E[f(x)\delta'(z^T x - s) - \delta(z^T x - s + \epsilon)] \).

If \( u = \text{sgn} \) then \( P \) has degree at most 1 in each \( X_i \). Let \( \frac{\partial P}{\partial x_i} \) denote the symbolic partial derivative of \( P \) with respect to \( X_i \); so, it drops monomials without \( X_i \) and factors off \( X_i \) from the remaining ones. Let us separate dependence on \( X_i \) in \( P \) as follows:

\[
P(X_1, \ldots, X_d) = X_i Q_i(X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_d) + R_1(X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_d)
\]

then \( \frac{\partial P}{\partial x_i} = Q_i \).

We will overload the polynomial \( P \) such that \( P[x] \) to denote the polynomial computed by substituting \( X_i = u((w^*_i)^T x) \) and similarly for \( Q \) and \( R \). Under this notation \( f(x) = P[x] \). We will also assume that \( ||P(X)|| \leq ||X||^{O(1)} = ||X||^{\epsilon_1} \) (say). By using simple correlations we will show:

**Theorem 7.** If \( u \) is the \( \text{sgn} \) function, \( P(X) \leq ||X||^{\epsilon_1} \) and for all \( i \), \( E[Q_i|x]|(w^*_i)^T x = t| \geq \epsilon_3 \) then using \( \text{poly}\left(\frac{d}{\epsilon_3\epsilon_2}\right) \) samples one can determine the \( w^*_i \)’s within error \( \epsilon_2 \).\[6\]

Note that if all the \( w^*_i \)'s are orthogonal then \( X_i \) are independent and \( E[Q_i|x]|(w^*_i)^T x = t] \) is just value of \( Q_i \) evaluated by setting \( X_i = 1 \) and setting all the the remaining \( X_j = \mu \) where \( \mu = E[X_j] \). This is same as \( 1/\mu \) times the coefficient of \( X_i \) in \( P(\mu(X_1 + 1), \ldots, \mu(X_d + 1)) \).

**Corollary 1.** If \( u \) is the \( \text{sgn} \) function and \( w^*_i \)'s are orthogonal then in sample complexity \( \text{poly}\left(\frac{d}{\epsilon_3\epsilon_2}\right) \) one can determine \( W^* \) within error \( \epsilon_2 \) in each entry, if the coefficient of the linear terms in \( P(\mu(X_1 + 1), \mu(X_2 + 1), \mu(X_3 + 1), \ldots) \) is larger than \( \epsilon_3 \mu \), where \( \mu = E[X_i] \).

The main point behind the proof of Theorem \[7\] is that the correlation is high when \( z \) is along one of \( w^*_i \) and negligible if it is not close to any of them.

\[6\]The theorem can be extended to ReLU by correlating with the second derivative \( \delta'' \) (see Appendix \[21\]).
Lemma 8. Assuming $P(X) < ||X||^{c_1}$. If $z = w_i^*$ then $C_2(f, z, t) = \phi(t)E\left[\frac{\partial P}{\partial x_i} \big| z^T x = t\right] + \epsilon d^{O(1)}$. Otherwise if all angles $\alpha_i$ between $z$ and $w_i^*$ are at least $\epsilon_2$ it is at most $\epsilon d^{O(1)}/\epsilon_2$.

We will use the notation $g(x)|_{x=s}$ to denote $g(x)$ evaluated at $x = s$. Thus Cauchy’s mean value theorem can be stated as $g(x + \epsilon) - g(x) = \epsilon g'(s)(s = s' \in [x, x + \epsilon])$. We will overload the notation a bit: $\phi(z^T x = s)$ will denote the probability density that $v z^T x = s$; so if $z$ is a unit vector this is just $\phi(s)$; $\phi(z_1^T x = s_1, z_2^T x = s_2)$ denotes the probability density that both $z_1^T x = s_1, z_2^T x = s_2$; so again if $z_1, z_2$ are orthonormal then this is just $\phi(s_1)\phi(s_2)$.

The following claim interprets correlation with $\delta(z^T x - s)$ as the expected value along the corresponding plane $z^T x = s$.

Claim 3. $E[f(x)\delta(z^T x - s)] = E[f(x)|z^T x = s]\phi(z^T x = s)$.

The following claim computes the correlation of $P$ with $\delta'(z^T x - s)$.

Claim 4. $E[P|x]\delta'(z^T x = s)) = \phi'(s)E[P|x]|z^T x = s + \sum_i |\cot(\alpha_i)|\phi(z^T x = s, (w_i^*)^T x = t))E\left[\frac{\partial P}{\partial x_i}|z^T x = s, (w_i^*)^T x = t\right]$.

We use this to show that the correlation is bounded if all the angles are lower bounded.

Claim 5. If $P(X) \leq ||X||^{c_1}$ and if $z$ has an angle of at least $\epsilon_2$ with all the $w_i^*$’s then $C_2(f, z, s) \leq \epsilon d^{O(1)}/\epsilon_2$.

Above claims can be used to prove main Lemma 8. Refer to the Appendix C for proofs.

Proof of Theorem 7. If we wish to determine $w_i^*$ within an angle of accuracy $\epsilon_2$ let us set $\epsilon$ to be $O(\epsilon_3\epsilon_2\phi(t)d^{-c})$. From Lemma 8 for some large enough $c$, this will ensure that if all $\alpha_i > \epsilon_2$ the correlation is $o(\phi(t)\epsilon_3)$. Otherwise it is $\phi(t)\epsilon_3(1 \pm o(1))$. Since $\phi(t) = poly(1/d)$, given poly($\frac{d}{\epsilon \epsilon_3}$) samples, we can test if a given direction is within accuracy $\epsilon_2$ of a $w_i^*$ or not.

4 Stronger Results under Structural Assumptions

Under additional structural assumptions on $W^*$ such as the weights being binary, that is, in $\{0, 1\}$, sparsity or certain restrictions on activation functions, we can give stronger recovery guarantees. Proofs have been deferred to Appendix D.

Theorem 8. For activation $u_t(a) = e^{\rho(a-t)}$. Let the weight vectors $w_i^*$ be 0, 1 vectors that select the coordinates of $x$. For each $i$, there are exactly $d$ indices $j$ such that $w_{ij} = 1$ and the coefficient of the linear terms in $P(\mu_1 X_1 + 1, \mu_2 X_2 + 1, \mu_3 X_3 + 1, \ldots)$ for $\mu = e^{-\rho t}$ is larger than the coefficient of all the product terms (constant factor gap) then we can learn the $W^*$.

In order to prove the above, we will construct a correlation graph over $x_1, \ldots, x_n$ and subsequently identify cliques in the graph to recover $w_i^*$’s.

With no threshold, recovery is still possible for disjoint, low $l_1$-norm vector. The proof uses simple correlations and shows that the optimization landscape for maximizing these correlations has local maxima in the $w_i^*$’s.

Theorem 9. For activation $u(a) = e^{a}$. If all $w_i^* \in \{0, 1\}^n$ are disjoint, then we can learn $w_i^*$ as long as $P$ has all positive coefficients and product terms have degree at most 1 in each variable.
For even activations, it is possible to recover the weight vectors even when the threshold is 0. The technique used is the PCA like optimization using hermite polynomials as in Section 2. Denote 

\[ C(S, \mu) = \sum_{S \subseteq S' \subseteq [n]} c_{S'} \mu^{\left| S' \right|}. \]

**Theorem 10.** If the activation is even and for every \( i, j \): 

\[ C(\{i\}, \hat{u}_0) + C(\{j\}, \hat{u}_0) > \frac{6\hat{u}_0^2}{\hat{u}_0^2} C(\{i, j\}, \hat{u}_0) \]

then there exists an algorithm that can recover the underlying weight vectors.

5 Conclusion

In this work we show how high threshold activations make it easier to recover the weights of a two layer neural network with multinomial activation. We show that for a large class of activations in the second layer, the weights can be learned with high precision. Our ideas can be used for learning the lowest layer of a deep neural network by viewing the network above as a multinomial. Furthermore, even if the threshold is low we can show that the sample complexity is polynomially bounded however time complexity may be exponential. An interesting open direction is to apply these methods to learn all layers of a deep network recursively. It would also be interesting to obtain stronger results if the high thresholds were only present at a higher layer and not the first layer.

References

Anima Anandkumar and Rong Ge. Efficient approaches for escaping higher order saddle points in non-convex optimization. *arXiv preprint arXiv:1602.05908*, 2016.

Sanjeev Arora, Aditya Bhaskara, Rong Ge, and Tengyu Ma. Provable bounds for learning some deep representations. In Eric P. Xing and Tony Jebara, editors, *Proceedings of the 31st International Conference on Machine Learning*, volume 32 of *Proceedings of Machine Learning Research*, pages 584–592, Beijing, China, 22–24 Jun 2014. PMLR. URL [http://proceedings.mlr.press/v32/arora14.html](http://proceedings.mlr.press/v32/arora14.html).

Alon Brutzkus and Amir Globerson. Globally optimal gradient descent for a convnet with gaussian inputs. *arXiv preprint arXiv:1702.07966*, 2017.

Simon S Du, Jason D Lee, and Yuandong Tian. When is a convolutional filter easy to learn? *arXiv preprint arXiv:1709.06129*, 2017a.

Simon S Du, Jason D Lee, Yuandong Tian, Barnabas Poczos, and Aarti Singh. Gradient descent learns one-hidden-layer cnn: Don’t be afraid of spurious local minima. *arXiv preprint arXiv:1712.00779*, 2017b.

Rong Ge, Jason D Lee, and Tengyu Ma. Learning one-hidden-layer neural networks with landscape design. *arXiv preprint arXiv:1711.00501*, 2017.

Surbhi Goel and Adam Klivans. Eigenvalue decay implies polynomial-time learnability for neural networks. *arXiv preprint arXiv:1708.03708*, 2017a.

Surbhi Goel and Adam Klivans. Learning depth-three neural networks in polynomial time. *arXiv preprint arXiv:1709.06010*, 2017b.

Surbhi Goel, Varun Kanade, Adam Klivans, and Justin Thaler. Reliably learning the ReLU in polynomial time. *arXiv preprint arXiv:1611.10258*, 2016.
Majid Janzamin, Hanie Sedghi, and Anima Anandkumar. Beating the perils of non-convexity: Guaranteed training of neural networks using tensor methods. *arXiv preprint arXiv:1506.08473*, 2015.

Yuanzhi Li and Yang Yuan. Convergence analysis of two-layer neural networks with ReLU activation. *arXiv preprint arXiv:1705.09886*, 2017.

Roi Livni, Shai Shalev-Shwartz, and Ohad Shamir. On the computational efficiency of training neural networks. In *Advances in Neural Information Processing Systems*, pages 855–863, 2014.

Jorge Nocedal and Stephen J Wright. Numerical optimization 2nd, 2006.

Hanie Sedghi and Anima Anandkumar. Provable methods for training neural networks with sparse connectivity. *arXiv preprint arXiv:1412.2693*, 2014.

Santosh S. Vempala. A random-sampling-based algorithm for learning intersections of halfspaces. *J. ACM*, 57(6):32:1–32:14, November 2010. ISSN 0004-5411. doi: 10.1145/1857914.1857916. URL [http://doi.acm.org/10.1145/1857914.1857916](http://doi.acm.org/10.1145/1857914.1857916).

Wikipedia. Hermite polynomials — Wikipedia, the free encyclopedia, 2018. URL [https://en.wikipedia.org/w/index.php?title=Hermite_polynomials&oldid=851173493](https://en.wikipedia.org/w/index.php?title=Hermite_polynomials&oldid=851173493). [Online; accessed 26-September-2018].

Qiuyi Zhang, Rina Panigrahy, Sushant Sachdeva, and Ali Rahimi. Electron-proton dynamics in deep learning. *arXiv preprint arXiv:1702.00458*, 2017.

Yuchen Zhang, Jason D Lee, Martin J Wainwright, and Michael I Jordan. Learning halfspaces and neural networks with random initialization. *arXiv preprint arXiv:1511.07948*, 2015.

Yuchen Zhang, Jason D Lee, and Michael I Jordan. l1-regularized neural networks are improperly learnable in polynomial time. In *International Conference on Machine Learning*, pages 993–1001, 2016.

Kai Zhong, Zhao Song, and Inderjit S Dhillon. Learning non-overlapping convolutional neural networks with multiple kernels. *arXiv preprint arXiv:1711.03440*, 2017a.

Kai Zhong, Zhao Song, Prateek Jain, Peter L Bartlett, and Inderjit S Dhillon. Recovery guarantees for one-hidden-layer neural networks. *arXiv preprint arXiv:1706.03175*, 2017b.
A Prerequisites

A.1 Hermite Polynomials

Hermite polynomials form a complete orthogonal basis for the gaussian distribution with unit variance. For more details refer to Wikipedia (2018). Let $h_i$ be the normalized hermite polynomials. They satisfy the following,

**Fact 0.** $E[h_n(x)] = 0$ for $n > 0$ and $E[h_0(x)] = 1$.

**Fact 1.** $E_{a \sim N(0, I)}[h_i(a)h_j(a)] = \delta_{ij}$ where $\delta_{ij} = 1$ iff $i = j$.

This can be extended to the following:

**Fact 2.** For $a, b$ with marginal distribution $N(0, 1)$ and correlation $\rho$, $E[h_i(a)h_j(b)] = \delta_{ij}\rho^j$.

Consider the following expansion of $u$ into the hermite basis ($h_i$),

$$u(a) = \sum_{i=0}^{\infty} \hat{u}_i h_i(a).$$

**Lemma 9.** For unit norm vectors $u, v$, $E[u(v^T x)h_j(w^T x)] = \hat{u}_j(v^T w)^j$.

**Proof.** Observe that $v^T x$ and $w^T x$ have marginal distribution $N(0, 1)$ and correlation $v^T w$. Thus using Fact 2,

$$E[u(v^T x)h_j(w^T x)] = \sum_{i=1}^{\infty} \hat{u}_i E[h_i(v^T x)h_j(w^T x)] = \sum_{i=1}^{\infty} \hat{u}_i \delta_{ij}(v^T w)^j = \hat{u}_j(v^T w)^j.$$  

\[\Box\]

For gaussians with mean 0 and variance $\sigma^2$ define weighted hermite polynomials $H_i^\sigma(a) = |\sigma|^i h_i(a/\sigma)$. Given input $v^T x$ for $x \sim N(0, I)$, we suppress the superscript $\sigma = ||v||$.

**Corollary 2.** For a non-zero vector $v$ (not necessarily unit norm) and a unit norm vector $w$, $E[H_i(v^T x)h_j(w^T x)] = \delta_{ij}(v^T w)^j$.

**Proof.** It follows as the proof of the previous lemma,

$$E[u(v^T x)h_j(w^T x)] = \sum_{i=1}^{\infty} \hat{u}_i E[h_i(v^T x)h_j(w^T x)] = \sum_{i=1}^{\infty} \hat{u}_i \delta_{ij}(v^T w)^j = \hat{u}_j(v^T w)^j.$$  

\[\Box\]

**Fact 3.** $h_n(x + y) = 2^{-\frac{n}{2}} \sum_{k=0}^{n} (\binom{n}{k}) h_{n-k}(x/\sqrt{2})h_k(y/\sqrt{2})$.

**Fact 4.** $h_n(\gamma x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma^n - 2k(\gamma^2 - 1)^k \binom{n}{2k} \frac{(2k)!}{k!} 2^{-k} h_{n-2k}(x)$.

**Fact 5.** $\alpha(n, m, \gamma) = E[h_m(x)h_n(\gamma x)] = \gamma^{n-2k} (\gamma^2 - 1)^k \binom{n}{2k} \frac{(2k)!}{k!} 2^{-k}$ for $k = \frac{n-m}{2}$ if $k \in \mathbb{Z}^+$ else 0.

A.2 Properties of Matrices

Consider matrix $A \in \mathbb{R}^{m \times m}$. Let $\sigma_i(A)$ to be the $i$th singular value of $A$ such that $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_m(A)$ and set $\kappa(A) = \sigma_1(A)/\sigma_m(A)$.

**Fact 6.** $|\det(A)| = \prod_{i=1}^{m} \sigma_i(A)$.

**Fact 7.** Let $B$ be a $(m - k) \times (m - k)$ principal submatrix of $A$, then $\kappa(B) \leq \kappa(A)$.
A.3 Activation Functions

Lemma 10. For \( u \) being a high threshold ReLU, that is, \( u_t(a) = \max(0, a - t) \) we have for \( t \geq C \) for large enough constant \( C > 0 \), \( \mathbb{E}_{g \sim N(0, \sigma^2)}[u_t(g)] \leq e^{-\frac{t^2}{2\sigma^2}} \). Also, \( \hat{u}_4, \hat{u}_2 = \Theta(1) e^{-\frac{t^2}{2\sigma^2}} \).

Proof. We have

\[
\mathbb{E}_{g \sim N(0, \sigma^2)}[u_t(g)] = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \max(0, g - t) e^{-\frac{g^2}{2\sigma^2}} dg
\]

\[
= \frac{1}{\sqrt{2\pi\sigma}} \int_{t}^{\infty} (g - t) e^{-\frac{g^2}{2\sigma^2}} dg
\]

\[
\leq \frac{1}{\sqrt{2\pi\sigma}} \int_{t}^{\infty} g e^{-\frac{g^2}{2\sigma^2}} dg
\]

\[
= \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{t^2}{2\sigma^2}}^{\infty} e^{-h} dh
\]

\[
= \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}.
\]

Also,

\[
\hat{u}_4 = \mathbb{E}_{g \sim N(0, 1)}[u_t(g) h_4(g)]
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max(0, g - t)(g^4 - 6g^2 + 3)e^{-\frac{g^2}{2}} dg
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{t}^{\infty} (g - t)(g^4 - 6g^2 + 3)e^{-\frac{g^2}{2}} dg
\]

\[
\geq \frac{1}{\sqrt{2\pi}} (t^4 - 6t^2) \frac{1}{t} e^{-\frac{t^2}{2} - 1 - \frac{1}{2\pi}}
\]

\[
\geq \Omega \left( t^3 e^{-\frac{t^2}{2}} \right).
\]

To upper bound,

\[
\hat{u}_4 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max(0, g - t)(g^4 - 6g^2 + 3)e^{-\frac{g^2}{2}} dg
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{t}^{\infty} (g - t)(g^4 - 6g^2 + 3)e^{-\frac{g^2}{2}} dg
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_{t}^{\infty} 2g^5 e^{-\frac{g^2}{2}} dg
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\frac{t^2}{2}}^{\infty} h^2 e^{-h} dh
\]

\[
= O \left( t^4 e^{-\frac{t^2}{2}} \right).
\]

Similar analysis holds for \( \hat{u}_2 \).

Observe that \( \text{sgn} \) can be bounded very similarly replacing \( g - t \) by 1 which can affect the bounds up to only a polynomial in \( t \) factor.
Lemma 11. For \( u \) being a high threshold \( \text{sgn} \), that is, \( u_t(a) = \text{sgn}(a-t) \) we have for \( t \geq C \) for large enough constant \( C > 0 \), \( \mathbb{E}_{g \sim N(0, \sigma^2)}[u_t(g)] \leq e^{-t^2/2\sigma^2} \). Also, \( \hat{u}_4, \hat{u}_2 = t \Theta(1) e^{-t/2} \).

For sigmoid, the dependence varies as follows:

Lemma 12. For \( u \) being a high threshold sigmoid, that is, \( u_t(a) = \frac{1}{1 + e^{-(a-t)}} \) we have for \( t \geq C \) for large enough constant \( C > 0 \), \( \mathbb{E}_{g \sim N(0, \sigma^2)}[u_t(g)] \leq e^{-t^2/2} + \sigma^2 \). Also, \( \hat{u}_4, \hat{u}_2 = \Theta(e^{-t/2}) \).

Proof. We have

\[
\mathbb{E}_{g \sim N(0, \sigma^2)}[u_t(g)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + e^{-(g-t)}} e^{-g^2/2\sigma^2} \, dg
\]

\[
= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{e^{-t} + e^{-g}} e^{-g^2/2\sigma^2} \, dg
\]

\[
\leq \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^g e^{-g^2/2\sigma^2} \, dg
\]

\[
= \frac{e^{-t} e^{\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(g^2 + \sigma^2)/2\sigma^2} \, dg
\]

\[
= e^{-t} e^{\sigma^2/2}
\]

Also,

\[
\hat{u}_4 = \mathbb{E}_{g \sim N(0, 1)}[u_t(g) h_4(g)]
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{e^{-(g-t)}} e^{-g^2/2} \, dg
\]

\[
= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t} e^{-(g-t)} (g^4 - 6g^2 + 3) e^{-g^2/2} \, dg
\]

\[
\geq \frac{e^{-t}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{e^{-t} + e^{-g}} (g^4 - 6g^2 + 3) e^{-g^2/2} \, dg
\]

\[
\geq \frac{e^{-t}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{2} (g^4 - 6g^2 + 3) e^{-g^2/2} \, dg
\]

\[
= \Omega(e^{-t}).
\]

\[
\square
\]

We can upper bound similarly and bound \( \hat{u}_2 \).

B Approximate Recovery with Linear Terms

B.1 Constrained Optimization View of Landscape Design

Let us consider the linear case with \( w^*_i \)'s are orthonormal. Consider the following maximization problem for even \( l \geq 4 \),

\[
\max_{z \in S^{n-1}} \text{sgn}(\hat{u}_l) \cdot \mathbb{E} \left[ f(x) \cdot H_l(z^T x) \right]
\]
where $h_l$ is the $l$th hermite polynomial. Then we have,

$$
\text{sgn}(\hat{u}_l) \cdot \mathbb{E} [f(x) \cdot h_l \left( z^T x \right)] = \text{sgn}(\hat{u}_l) \cdot \mathbb{E} \left[ \left( \sum_{i=1}^{k} c_i u_i((w_i^*)^T x) \right) \cdot h_l \left( z^T x \right) \right]
$$

$$
= \text{sgn}(\hat{u}_l) \cdot \sum_{i=1}^{k} c_i \mathbb{E} \left[ u_i((w_i^*)^T x) \cdot h_l \left( z^T x \right) \right]
$$

$$
= \hat{u}_l \sum_{i=1}^{k} c_i (w_i^*)^T z l.
$$

It is easy to see that for $z \in S^{n-1}$, the above is maximized at exactly one of the $w_i$'s (up to sign flip for even $l$) for $l \geq 3$ as long as $u_l \neq 0$. Thus, each $w_i$ is a local minima of the above problem.

Let $L(z) = -\sum_{i=1}^{k} c_i z_i^l$. For constraint $||z||^2 = 1$, we have the following optimality conditions (see Nocedal and Wright (2006) for more details).

First order:

$$
\nabla L(z) - \frac{z^T \nabla L(z)}{||z||^2} z = 0 \text{ and } ||z||^2 = 1.
$$

This applied to our function gives us that for $\lambda = -\sum_{i=1}^{k} c_i z_i^l$ ($\lambda < 0$),

$$
-l_i c_i z_i^l - 2\lambda z_i = 0
$$

The above implies that either $z_i = 0$ or $z_i^{l-2} = -\frac{\lambda}{c_i l}$ with $||z||_2 = 1$. For this to hold $z$ is such that for some set $S \subseteq [n]$, $|S| > 1$, only $i \in S$ have $z_i \neq 0$ and $\sum_{i \in S} z_i^2 = 1$. This implies that for all $i \in S$, $z_i^{l-2} = -\frac{\lambda}{c_i l}$.

Second order:

For all $w \neq 0$ such that $w^T z = 0, w^T (\nabla^2 L(z) - 2\lambda I) w \geq 0$.

For our function, we have:

$$
\nabla^2 L(z) = -(l-1) \text{diag}(c \cdot z)^{l-2}
$$

$$
\implies (\nabla^2 L(z))_{ij} = \begin{cases} 2(l-1)\lambda & \text{if } i = j \text{ and } i \in S \\ 0 & \text{otherwise.} \end{cases}
$$

The last follows from using the first order condition. For the second order condition to be satisfied we will show that $|S| = 1$. Suppose $|S| > 2$, then choosing $w$ such that $w_i = 0$ for $i \not\in S$ and such that $w^T z = 0$ (it is possible to choose such a value since $|S| > 2$), we get $w^T (\nabla^2 L(z) - 2\lambda I) w = 2(l-2)\lambda ||w||^2$ which is negative since $\lambda < 0$, thus these cannot be global minima. However, for $|S| = 1$, we cannot have such a $w$, since to satisfy $w^T z = 0$, we need $w_i = 0$ for all $i \in S$, this gives us $w^T (\nabla^2 L(z) - 2\lambda I) w = -2\lambda ||w||^2$ which is always positive. Thus $z = \pm e_i$ are the only local minima of this problem.

B.2 Important Results from Ge et al. (2017)

Lemma 13 (Ge et al. (2017)). If $z$ is an $(\epsilon, \tau)$-local minima of $F(z) = -\sum_i \alpha_i z_i^4 + \lambda (\sum_i z_i^2 - 1)^2$ for $\epsilon \leq \sqrt{\tau^3/\alpha_{\text{min}}}$ where $\alpha_{\text{min}} = \min_i \alpha_i$, then
• (Lemma 5.2) $|z|_{2nd} \leq \sqrt{\frac{1}{\alpha_{min}}}$ where $|z|_{2nd}$ denotes the magnitude of the second largest entry in terms of magnitude of $z$.

• (Derived from Proposition 5.7) $z_{max} = \pm 1 \pm O(d\tau/\alpha_{min}) \pm O(\epsilon/\lambda)$ where $|z|_{max}$ is the value of the largest entry in terms of magnitude of $z$.

B.3 Omitted Proofs for One-by-one Recovery

Proof of Lemma 5.4 Let $O \in \mathbb{R}^{d \times d}$ be the orthonormal basis (row-wise) of the subspace spanned by $w_i^*$ for all $i \in [d]$ generated using Gram-schmidt (with the procedure done in order with elements of $|S|$ first). Now let $O_S \in \mathbb{R}^{[S] \times d}$ be the matrix corresponding to the first $S$ rows and let $O_S^T \in \mathbb{R}^{(d-[S]) \times n}$ be that corresponding to the remaining rows. Note that $OW^*$ ($W^*$ also has the same ordering) is an upper triangular matrix under this construction.

\[
\mathbb{E} \left[ \prod_{j \in S} u_t((w_j^*)^T x) \right] = \frac{1}{(2\pi)^{n/2}} \int x \prod_{i \in S} u_t(x^T w_i^*) e^{-\frac{|x|^2}{2}} dx
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int x \prod_{i \in S} u_t((O_S w_i^*)^T O_S x) e^{-\frac{|O_S x|^2}{2}} dx
\]

\[
= \left( \frac{1}{(2\pi)^{n/2}} \int x' \prod_{i \in S} u_t((O_S w_i^*)^T x') e^{-\frac{|x'|^2}{2}} dx' \right) \left( \frac{1}{(2\pi)^{n/2}} \int x' \prod_{i \in S} e^{-\frac{|x'|^2}{2}} dx' \right)
\]

\[
= \frac{|\det(O_S W_S^*)|^{-1}}{(2\pi)^{|S|/2}} \int b \prod_{i \in S} u_t(b_i) e^{-\frac{|(O_S W_S^*)^T b|^2}{2}} db
\]

Now observe that $O_S W_S^*$ is also an upper triangular matrix since it is a principal sub-matrix of $OW^*$. Thus using Fact 6 and 7, we get the last equality. Also, the single non-zero entry row has non-zero entry being 1 ($|w_i^*| = 1$ for all $i$). This gives us that the inverse will also have the single non-zero entry row has non-zero entry being 1. WLOG assume index 1 corresponds to this row. Thus we can split this as following

\[
\mathbb{E} \left[ \prod_{j \in S} u_t((w_j^*)^T x) \right] \leq |\det(O_S W_S^*)|^{-1} \left( \frac{1}{\sqrt{2\pi}} \int b_1 u_t(b_1) e^{-\frac{b_1^2}{2}} db_1 \right) \left( \prod_{i \in S \setminus \{1\}} \frac{1}{\sqrt{2\pi}} \int b_i u_t(b_i) e^{-\frac{b_i^2}{2}} db_i \right)
\]

\[
\leq |\det(O_S W_S^*)|^{-1} \left( \frac{1}{\sqrt{2\pi}} \int b_1 u_t(b_1) e^{-\frac{b_1^2}{2}} db_1 \right) \left( \prod_{i \in S \setminus \{1\}} \frac{1}{\sqrt{2\pi}} \int b_i u_t(b_i) e^{-\frac{b_i^2}{2}} db_i \right)
\]

\[
\leq \rho(t, 1) (\kappa(W^*) \rho(t, \|W^*\|)|S|^{-1}
\]
Proof of Claim 1. Consider the SVD of matrix $M = UD^T$. Let $W = UD^{-1/2}$ and $y_i = \sqrt{c_i}W^Tw_i^*$ for all $i$. It is easy to see that $y_i$ are orthogonal. Let $F(z) = G(Wz)$:

$$F(z) = |\hat{u}_4| \sum_i c_i(z^TW^Tw_i^*)^4 - \lambda \hat{u}_2^2 \left( \sum_i c_i(z^TW^Tw_i^*)^2 - 1 \right)^2$$

$$= |\hat{u}_4| \sum_i \frac{1}{c_i} (z^Ty_i)^4 - \lambda \hat{u}_2^2 \left( \sum_i (z^Ty_i)^2 - 1 \right)^2.$$

Since $y_i$ are orthogonal, for means of analysis, we can assume that $y_i = e_i$, thus the formulation reduces to $\max_z |\hat{u}_4| \sum_i \frac{1}{c_i}(z_i)^4 - \lambda' (||z||^2 - 1)^2$ up to scaling of $\lambda' = \lambda \hat{u}_2^2$. Note that this is of the form in Lemma 13, hence using that we can show that the approximate local minimas of $F(z)$ are close to $y_i$ and thus the local maximas of $G(z)$ are close to $Wy_i = \sqrt{c_i}WW^Tw_i^* = \sqrt{c_i}M^{-1}w_i^*$ due to the linear transformation. This can alternately be viewed as the columns of $(TW^*)^{-1}$ since $TW^*M^{-1}(TW^*)^T = I$.

Proof of Theorem 4. Let $Z$ be an $(\epsilon, \tau)$-local minimum of $A$, then we have $||\nabla A(Z)|| \leq \epsilon$ and $\lambda_{\min}(\nabla^2 A(Z)) \geq -\tau$. Observe that

$$||\nabla B(Z)|| = ||\nabla (A + (B - A)(Z))|| \leq ||\nabla A(Z)|| + ||\nabla (B - A)(Z)|| \leq \epsilon + \rho.$$

Also observe that

$$\lambda_{\min}(\nabla^2 B(Z)) = \lambda_{\min}(\nabla^2 (A + (B - A))(Z))$$

$$\geq \lambda_{\min}(\nabla^2 A(Z)) + \lambda_{\min}(\nabla^2 (B - A)(Z))$$

$$\geq -\tau - ||\nabla^2 (B - A)(Z)|| \geq -\tau - \gamma$$

Here we use $|\lambda_{\min}(M)| \leq ||M||$ for any symmetric matrix. To prove this, we have $||M|| = \max_{x \in S^n} ||Mx||$. We have $x = \sum_i x_i v_i$ where $v_i$ are the eigenvectors. Thus we have $Mx = \sum_i x_i \lambda_i(M)v_i$ and $\sum x_i^2 = 1$. Which gives us that $||M|| = \sqrt{\sum x_i^2 \lambda_i^2(M)} \geq |\lambda_{\min}(M)|$.

Proof of Lemma 2. Expanding $f$, we have

$$E[|\Delta(x)|] = E \left[ \sum_{S \subseteq [d]: |S| > \rho} \sum_{j \in S} c_j \prod_{j \in S} u_t((w_j^*)^T x) \right]$$

using Lemma 1

$$\leq \sum_{S \subseteq [d]: |S| > \rho} |c_S| E \left[ \prod_{j \in S} u_t((w_j^*)^T x) \right]$$

$$\leq C \sum_{S \subseteq [d]: |S| > \rho} \rho(t, 1) \left( \frac{1}{\sigma_{\min}(W^*) \rho(t, ||W^*||)} \right)^{|S|-1}$$

$$= C \sum_{i=1}^d \binom{d}{i} \rho(t, 1) \left( \frac{1}{\sigma_{\min}(W^*) \rho(t, ||W^*||)} \right)^{i-1}$$

using $\binom{d}{i} \leq d^i$

$$\leq C \sum_{i=1}^d d \rho(t, 1) \left( \frac{d}{\sigma_{\min}(W^*) \rho(t, ||W^*||)} \right)^{i-1}$$

using assumption on $t$

$$\leq C d^2 \rho(t, 1) \left( \frac{d}{\sigma_{\min}(W^*) \rho(t, ||W^*||)} \right)$$
Lemma 14. For any function $L$ such that $||L(z, x)|| \leq C(z)||x||^{O(1)}$ where $C$ is a function that is not dependent on $x$, we have $||E[\Delta(x)L(x)]|| \leq C(z)d^{-(1+p)\eta+3}O(\log d)$.

Proof. We have

\[
\begin{align*}
||E[\Delta(x)L(x)]|| & \leq E[||\Delta(x)||L(x)||] \\
& \leq E[||\Delta(x)||C(z)||x||^{O(1)}] \\
& = C(z) \left( E[||\Delta(x)|| ||x||^{O(1)} | ||x|| \geq c]Pr[||x|| \geq c] \\
& + E[||\Delta(x)|| ||x||^{O(1)} | ||x|| < c]Pr[||x|| < c] \right) \\
& \leq C(z)E[||x||^{O(1)} ||x|| \geq c]Pr[||x|| \geq c] + cE[||\Delta(x)||] \\
& = C(z)(c^{O(1)} \varepsilon^{-2} + c^{O(1)}E[||\Delta(x)||]).
\end{align*}
\]

Now using Lemma 2 to bound $E[||\Delta(x)||]$, for $c = \Theta(\sqrt{\eta} \log d)$ we get the required result. \qed

Lemma 15. For $||z|| = O(1)$ and $\lambda = \Theta(\|\hat{u}_4\|/\|\hat{u}_2\|) \approx d^\eta$, $||\nabla G(z)|| \geq \Omega(1) d^{-\eta}$.

Proof. Let $K = \kappa(W^*)$ which by assumption is $\theta(1)$. We will argue that local minima of $G$ cannot have $z$ with large norm. First lets argue this for $G_{\text{lin}}(z)$. We know that $G_{\text{lin}}(z) = -\alpha \sum (z^T w_i^*)^4 + \lambda \beta^2((\sum (z^T w_i^*)^2) - 1)^2$ where $\alpha = \|\hat{u}_4\|$ and $\beta = \|\hat{u}_2\|$. We will argue that $z^T \nabla G_{\text{lin}}(z)$ is large if $z$ is large.

\[
z^T \nabla G_{\text{lin}}(z) = -4\alpha \sum (z^T w_i^*)^3(z^T w_i^*) + 2\lambda \beta^2 \left( \sum (z^T w_i^*)^2 - 1 \right) \left( \sum 2(z^T w_i^*)(z^T w_i^*) \right)
\]

Let $y = W^* z$ then $K ||z|| \geq ||y|| \geq ||z||/K$ since $K$ is the condition number of $W^*$. Then this implies

\[
z^T \nabla G_{\text{lin}}(z) = -4\alpha \sum y_i^4 + 4\lambda \beta^2(||y||^2 - 1)||y||^2
\]

\[
= 4||y||^2((-\alpha + \lambda \beta^2)||y||^2 + \lambda \beta^2)
\]

\[
\geq ||y||^4(-\alpha + \lambda \beta^2) \geq \Omega(1)d^{-\eta}||y||^4
\]

Since $||y|| \geq ||z||/K = O(1)$ by assumptions on $\lambda, z$ we have $z^T \nabla G_{\text{lin}}(z) \geq \Omega(\lambda \beta^2||y||^4) = \Omega(1)d^{-\eta}||z||^4$. This implies $||\nabla G_{\text{lin}}(z)|| = \Omega(1)d^{-\eta}||z||^3$.

Now we need to argue for $G$.

\[
G(z) - G_{\text{lin}}(z)
\]

\[
= -\text{sgn}(\hat{u}_4)E[(f_{\text{lin}}(x) + \Delta(x))H_4(z^T x)] + \lambda E[(f_{\text{lin}}(x) + \Delta(x))H_2(z^T x)] - \beta^2
\]

\[
+ \text{sgn}(\hat{u}_4)E[(f_{\text{lin}}(x))H_4(z^T x)] - \lambda E[(f_{\text{lin}}(x))H_2(z^T x)] - \beta^2
\]

\[
= -\text{sgn}(\hat{u}_4)\|z\|^4E[\Delta(x)h_4(z^T x/||z||)] + \lambda ||z||^4E[\Delta(x)h_2(z^T x/||z||)]
\]

\[
+ 2\lambda ||z||^4E[\Delta(x)h_2(z^T x/||z||)]E[f_{\text{lin}}(x)h_2(z^T x/||z||)] - 2\lambda \beta ||z||^2E[\Delta(x)h_2(z^T x/||z||)]
\]

\[
+ 2\lambda |z|^4E[\Delta(x)h_2(z^T x/||z||)]E[f_{\text{lin}}(x)h_2(z^T x/||z||)] - 2\lambda \beta |z|^2E[\Delta(x)h_2(z^T x/||z||)]
\]

20
Now $h_4(z^T x/||z||)$ doesn’t have a gradient in the direction of $z$ so $z^T \nabla h_4(z^T x/||z||) = 0$. Similarly $z^T \nabla h_2(z^T x/||z||) = 0$. So

$$z^T \nabla (G(z) - G_{\text{lin}}(z))$$

$$= -4\text{sgn}(\hat{u}_4)||z||^4 E[\Delta(x)h_4(z^T x/||z||)] + 4\lambda||z||^4 (E[\Delta(x)h_2(z^T x/||z||)])^2$$

$$+ 8\lambda||z||^4 E[\Delta(x)h_2(z^T x/||z||)] E[f_{\text{lin}}(x)h_2(z^T x/||z||)] - 4\lambda||z||^2 E[\Delta(x)h_2(z^T x/||z||)]$$

We know that $E[f_{\text{lin}}(x)h_2(z^T x/||z||)]$ has a factor of $\beta$ giving us using Lemma 13

$$|z^T \nabla (G(z) - G_{\text{lin}}(z))| \leq O(\log d)d^{-(1+p)\eta+3}||z||^4.$$ 

So $z^T \nabla G(z)$ is also $\Omega(||z||^4)$.

**Proof of Claim**. We have $G - G_{\text{lin}}$ as follows,

$$G(z) - G_{\text{lin}}(z)$$

$$= -\text{sgn}(\hat{u}_4)E[(f_{\text{lin}}(x) + \Delta(x))H_4(z^T x)] + \lambda(E[(f_{\text{lin}}(x) + \Delta(x))H_2(z^T x)] - \hat{u}_2)^2$$

$$+ \text{sgn}(\hat{u}_4)E[(f_{\text{lin}}(x))H_4(z^T x)] - \lambda(E[(f_{\text{lin}}(x))H_2(z^T x)] - \hat{u}_2)^2$$

$$= -\text{sgn}(\hat{u}_4)E[\Delta(x)H_4(z^T x)] + \lambda(E[\Delta(x)H_2(z^T x)])^2$$

$$+ 2\lambda E[\Delta(x)H_2(z^T x)] E[f_{\text{lin}}(x)H_2(z^T x) - \hat{u}_2]$$

Thus we have,

$$\nabla (G(z) - G_{\text{lin}}(z))$$

$$= -\text{sgn}(\hat{u}_4)E[\Delta(x)\nabla H_4(z^T x)] + 2\lambda E[\Delta(x)H_2(z^T x)] E[\Delta(x)\nabla H_2(z^T x)]$$

$$+ 2\lambda E[f_{\text{lin}}(x)H_2(z^T x)] E[f_{\text{lin}}(x)\nabla H_2(z^T x)]$$

Observe that $H_2$ and $H_4$ are degree 2 and 4 (respectively) polynomials thus norm of gradient and hessian of the same can be bounded by at most $O(||z||^4)$. Using Lemma 14 we can bound each term by roughly $O(\log d)d^{-(1+p)\eta+3}||z||^4$. Note that $\lambda$ being large does not hurt as it is scaled appropriately in each term. Subsequently, using Lemma 15 we can show that $||z||$ is bounded by a constant since $||G(z)|| \leq d^{2\eta}$. Similar analysis holds for the hessian too.

Now applying Theorem 4 gives us that $z$ is an $(O(\log d)d^{-(1+p)\eta+3}, O(\log d)d^{-(1+p)\eta+3})$-approximate local minima of $G_{\text{lin}}$. This implies that it is also an $(\epsilon' := C \log(d) d^{-(1+2p)\eta+3}, \tau' := C \log(d) d^{-(1+2p/3)\eta+3})$-approximate local minima of $G_{\text{lin}}$ for large enough $C > 0$ by increasing $\tau$. Observe that $\sqrt{\tau'/|\hat{u}_4|} = C^{3/2} log^{3/2}(d)d^{-(3/2+p)\eta+9/2}/d^{-\eta/2} = C^{3/2} log^{3/2}(d)d^{-(1+p)\eta+9/2} \geq \epsilon'$. Now using Claim 4 we get the required result.

**B.4 Simultaneous Recovery**

Ge et al. (2017) also showed simultaneous recovery by minimizing the following loss function $G_{\text{lin}}$ defined below has a well-behaved landscape.

$$G_{\text{lin}}(W) = E \left[ f_{\text{lin}}(x) \sum_{j,k \in [d], j \neq k} \psi(w_j, w_k, x) \right] - \gamma E \left[ f_{\text{lin}}(x) \sum_{j \in [d]} H_4(w_j^T x) \right]$$ (1)
Theorem 12. Let $c$ be a sufficiently small universal constant (e.g. $c = 0.01$ suffices), and suppose the activation function $u$ satisfies $u_4 \neq 0$. Assume $\gamma \leq c$, $\lambda \geq \Omega(|u_4|/u_2^2)$, and $W^*$ be the true weight matrix. The function $G_{\text{lin}}$ satisfies the following:

1. Any saddle point $W$ has a strictly negative curvature in the sense that $\lambda_{\text{min}}(\nabla^2 G_{\text{lin}}(W)) \geq -\tau_0$, where $\tau_0 = c \min\{|\hat{u}_4|/d, \lambda u_2^2\}$.
2. Suppose $W$ is an $(\epsilon, \tau_0)$-approximate local minimum, then $W$ can be written as $W^{-T} = PDW^* + \mathbb{E}$ where $D$ is a diagonal matrix with $D_{ii} \in \{\pm 1 \pm O(\gamma|\hat{u}_4|/\lambda u_2^2) \pm O(\epsilon/\lambda)\}$, $P$ is a permutation matrix, and the error term $||\mathbb{E}|| \leq O(ed/\hat{u}_4)$.

We show that this minimization is robust. Let us consider the corresponding function $G$ to $G_{\text{lin}}$ with the additional non-linear terms as follows:

$$
G(W) = \mathbb{E} \left[ f(x) \sum_{j,k \in [d], j \neq k} \psi(w_j, w_d, x) \right] - \gamma \mathbb{E} \left[ f(x) \sum_{j \in [d]} H_4(w_j, x) \right] 
+ \lambda \sum_{i} (\mathbb{E} [f(x)H_2(w_i, x)] - \hat{u}_2)^2
$$

Now we can show that $G$ and $G_{\text{lin}}$ are close as in the one-by-one case.

$$
R(W) := G(W) - G_{\text{lin}}(W) 
= \mathbb{E} [\Delta(x)A(W, x)] - \gamma \mathbb{E} [\Delta(x)B(W, x)] + \lambda \left( \mathbb{E} [f(x)C(W, x)]^2 - \mathbb{E} [f_{\text{lin}}(x)C(W, x)]^2 \right) 
= \mathbb{E} [\Delta(x)A(W, x)] - \gamma \mathbb{E} [\Delta(x)B(W, x)] + \lambda \mathbb{E} [\Delta(x)C(W, x)(f(x') + f_{\text{lin}}(x'))C(W, x')] 
= \mathbb{E} [\Delta(x)A(W, x)] - \gamma \mathbb{E} [\Delta(x)B(W, x)] + \lambda \mathbb{E} [\Delta(x)D(W, x)] 
= \mathbb{E} [\Delta(x)A(W, x) - \gamma B(W, x) + \lambda D(W, x)] 
= \mathbb{E} [\Delta(x)L(W, x)]
$$

where $A(W, x) = \sum_{j,k \in [d], j \neq k} \psi(w_j, w_d, x)$, $B(W, x) = \sum_{j \in [d]} H_4(w_j, x)$, $C(W, x) = \sum_i H_2(w_i, x)$, $D(W, x) = C(W, x)\mathbb{E}[(f(x') + f_{\text{lin}}(x'))C(W, x')]$ and $L(W, x) = A(W, x) - \gamma B(W, x) + \lambda D(W, x)$.

Using similar analysis as the one-by-one case, we can show the required closeness. It is easy to see that $||\nabla L||$ and $||\nabla^2 L||$ will be bounded above by a constant degree polynomial in $O(\log d)d^{-(1+p)\eta+3}\max ||w_i||^4$.

No row can have large weight as if any row is large, then looking at the gradient for that row, it reduces to the one-by-one case, and there it can not be larger than a constant. Thus we have the same closeness as in the one-by-one case.

Combining this with Theorem 11 and 12 we have the following theorem:

Theorem 12. Let $c$ be a sufficiently small universal constant (e.g. $c = 0.01$ suffices), and under Assumptions 1, 2 and 3. Assume $\gamma \leq c$, $\lambda = \Theta(d^p)$, and $W^*$ be the true weight matrix. The function $G$ satisfies the following:

1. Any saddle point $W$ has a strictly negative curvature in the sense that $\lambda_{\text{min}}(\nabla^2 G_{\text{lin}}(W)) \geq -\tau$ where $\tau_0 = O(\log d)d^{-\Omega(1)}$. 

22
2. Suppose $W$ is a $(d^{-\Omega(1)}, d^{-\Omega(1)})$-approximate local minimum, then $W$ can be written as $W^{-T} = PDW^* + E$ where $D$ is a diagonal matrix with $D_{ii} \in \{\pm 1 \pm O(\gamma) \pm d^{-\Omega(1)}\}$, $P$ is a permutation matrix, and the error term $||E|| \leq O(\log d) d^{-\Omega(1)}$.

Using standard optimization techniques we can find a local minima.

B.5 Approximate to Arbitrary Close

Lemma 16. If $u$ is the sign function then $E[u(w^T x)\delta'(z^T x)] = c|\cot(\alpha)|$ where $w, z$ are unit vectors and $\alpha$ is the angle between them and $c$ is some constant.

Proof. WLOG we can work the in the plane spanned by $z$ and $w$ and assume that $z$ is the vector $i$ along and $w = i \cos \alpha + j \sin \alpha$. Thus we can replace the vector $x$ by $ix + jy$ where $x, y$ are normally distributed scalars. Also note that $u' = \delta$ (Dirac delta function).

$$E[u(w^T x)\delta'(z^T x)] = E[u(x \cos \alpha + y \sin \alpha)\delta'(x)]$$

$$= \int_y \int_x u(x \cos \alpha + y \sin \alpha)\delta'(x)\phi(x)\phi(y)dxdy$$

Using the fact that $\int_x \delta'(x)h(x)dx = h'(0)$ this becomes

$$= \int_y \phi(y)[(\partial/\partial x)u(x \cos \alpha + y \sin \alpha)\phi(x)]_{x=0}dy$$

$$= \int_y \phi(y)[u'(x \cos \alpha + y \sin \alpha)\cos \alpha + \phi'(x)u(x \cos \alpha + y \sin \alpha)]_{x=0}dy$$

$$= \int_{y=-\infty}^{\infty} \phi(y)\phi(0)\delta(y \sin \alpha) \cos \alpha dy$$

Substituting $s = y \sin \alpha$ this becomes

$$= \int_{s=-\infty/\sin \alpha}^{\infty/\sin \alpha} \phi(s/\sin \alpha)\phi(0)\delta(s) \cos \alpha(1/\sin \alpha)ds$$

$$= \sgn(\sin \alpha) \cot(\alpha)\phi(0) \int_{s} \phi(s/\sin \alpha)\delta(s)ds$$

$$= |\cot(\alpha)|\phi(0)\phi(0)$$

$\square$

Proof of Lemma 16 Let us compute the probability of lying in the $e$-band for any $t$:

$$Pr[x \in l(z, t, \epsilon)] = Pr[t - \epsilon \leq z^T x \leq t] = \frac{1}{\sqrt{2\pi||z||}} \int_{t-\epsilon}^{t} e^{-\frac{g^2}{2||z||^2}} dg = \frac{\epsilon}{\sqrt{2\pi||z||}} e^{-\frac{\epsilon^2}{2||z||^2}}$$

where the last equality follows from the mean-value theorem for some $\bar{t} \in [t - \epsilon, t]$.

Next we compute the following:

$$Pr[x^T w^*_1 \geq t \text{ and } x \in l(z, t', \epsilon)]$$
\[
\begin{align*}
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{x} \text{sgn}(x - t) \mathbb{1}[x \in l(z, t', \epsilon)]e^{-\frac{|x|^2}{2}} dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{x_{1}=t}^{\infty} e^{-\frac{x_{1}^2}{2}} \left( \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{x_{-1}} \mathbb{1}[x_{-1} \in l(-1, t' - z_{1}x_{1}, \epsilon)]e^{-\frac{|x_{-1}|^2}{2}} dx_{-1} \right) dx_{1} \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{x_{1}=t}^{\infty} e^{-\frac{x_{1}^2}{2}} \Pr[x_{-1} \in l(-1, t - z_{1}x_{1}, \epsilon)] dx_{-1} \\
&= \frac{\epsilon}{2\pi \|z_{-1}\|} \int_{g'=t'-\epsilon}^{t'} \int_{x_{1}=t}^{\infty} e^{-\frac{g_{1}^2}{2\|z_{1}\|^2}} \Phi \left( \frac{\|z\|^2 - g_{1}z_{1}}{\|z_{-1}\| \|z\|} \right) dg \\
&= \frac{\epsilon}{\sqrt{2\pi}} e^{-\tfrac{t^2}{2}} \Phi \left( \frac{t - t^* \cos(\alpha)}{\sin(\alpha)} \right)
\end{align*}
\]

where the last equality follows from the mean-value theorem for some \( t^* \in [t' - \epsilon, t'] \). Combining, we get:

\[
\Pr[\mathbf{x}^T \mathbf{w}_i^* \geq t \text{ and } \mathbf{x} \in l(z, t', \epsilon) | \mathbf{x} \in l(z, t, \epsilon)] = e^{-\tfrac{t^2}{2}} \Phi \left( \frac{t - t^* \cos(\alpha)}{\sin(\alpha)} \right) = \Phi \left( \frac{t - t^* \cos(\alpha)}{\sin(\alpha)} \right) + O(\epsilon) t'
\]

for \( \epsilon \leq 1/t' \).

**Proof of Lemma 5.** Recall that \( P \) is monotone with positive linear term, thus for high threshold \( u \) (0 unless input exceeds \( t \) and positive after) we have \( \text{sgn}(f(\mathbf{x})) = \vee \text{sgn}(\mathbf{x}^T \mathbf{w}_i^* - t) \). This is because, for any \( i \), \( P \) applied to \( X_i > 0 \) and \( \forall j \neq i, X_j = 0 \) gives us \( c_i \) which is positive. Also, \( P(0) = 0 \). Thus, \( \text{sgn}(P) = 1 \) if any of the inputs are positive. Using this, we have,

\[
\Pr[\text{sgn}(f(\mathbf{x})) | \mathbf{x} \in l(z, t', \epsilon)] \geq \Pr[\text{sgn}(\mathbf{w}_i^*)^T \mathbf{x} - t | \mathbf{x} \in l(z, t', \epsilon)]
\]

Also,

\[
\Pr[\text{sgn}(f(\mathbf{x})) | \mathbf{x} \in l(z, t', \epsilon)] \leq \sum \Pr[\text{sgn}(\mathbf{x}^T \mathbf{w}_i^* - t) | \mathbf{x} \in l(z, t', \epsilon)]
\]

\[
= \Pr[\text{sgn}(\mathbf{w}_i^*)^T \mathbf{x} - t | \mathbf{x} \in l(z, t', \epsilon)] + \sum_{i \neq 1} \Pr[\text{sgn}(\mathbf{x}^T \mathbf{w}_i^* - t) | \mathbf{x} \in l(z, t', \epsilon)]
\]

\[
\leq \Pr[\text{sgn}(\mathbf{w}_1^*)^T \mathbf{x} - t | \mathbf{x} \in l(z, t, \epsilon)] + \eta
\]

where \( \sum_{i \neq 1} \Pr[\text{sgn}(\mathbf{x}^T \mathbf{w}_i^* - t) | \mathbf{x} \in l(z, t', \epsilon)] \leq \eta \). We will show that \( \eta \) is not large, if \( z \) is close to one of the vectors, it can not be close to the others thus \( \alpha_i \) will be large for all \( i \neq j \). Let us now formally bound \( \eta \),

\[
\sum_{i \neq 1} \Pr[\text{sgn}(\mathbf{x}^T \mathbf{w}_i^* - t) | \mathbf{x} \in l(z, t', \epsilon)] \leq \sum_{i \neq 1} \left( \Phi \left( \frac{t - t_i^* \cos(\alpha_i)}{\sin(\alpha_i)} \right) + O(\epsilon) t'_i \right)
\]
\[ \leq \sum_{i \neq 1} \left( \Phi^c \left( \frac{t - t^* \cos(\alpha_i)}{\sin(\alpha_i)} \right) + O(\epsilon) t' \right) \]

\[ \leq \sum_{i \neq 1} \left( \Phi^c \left( \frac{t - t' \cos(\alpha_i)}{\sin(\alpha_i)} \right) + O(\epsilon) t' \right) \]

\[ \leq \sum_{i \neq 1} \frac{1}{\sqrt{2\pi\gamma_i}} e^{-\frac{\gamma_i^2}{2}} + O(\epsilon) kt' \]

where \( \gamma_i = \frac{t - t' \cos(\alpha_i)}{\sin(\alpha_i)} \). The above follows since \( \gamma_i \geq 0 \) by assumption on \( t' \). Under the assumption, let \( \beta = \max_{i \neq 1} \cos(\alpha_i) \) we have

\[ \gamma_i \geq \frac{t \left(1 - \frac{\beta}{\cos(\alpha_i)} \right)}{\sqrt{1 - \beta^2}} = \Omega(t) \]

under our setting. Thus we have,

\[ \sum_{i \neq 1} \Pr[\text{sgn}(x^T \mathbf{w}^*_i - t) | x \in l(z, t', \epsilon)] \leq d e^{-\Omega(t^2)} + O(\epsilon) dt = de^{-\Omega(t^2)} \]

for small enough \( \epsilon \).

**Proof of Lemma 6.** Let us assume that \( \epsilon < c/t' \) for sufficiently small constant \( c \), then we have that

\[ \sum_{i \neq 1} \Pr[\text{sgn}(f(x)) \text{ and } x \in l(z, t_2, \epsilon) | x \in l(z, t_2, \epsilon)] \geq \Pr[x^T \mathbf{w}^*_i \geq t \text{ and } x \in l(z, t_2, \epsilon) | x \in l(z, t_2, \epsilon)] \]

\[ \geq \Phi^c \left( \frac{t - t^* \cos(\alpha_1)}{\sin(\alpha_1)} \right) - 0.1 \]

\[ \Rightarrow 0.7 \geq \Phi^c \left( \frac{t - t^* \cos(\alpha_1)}{\sin(\alpha_1)} \right) \]

\[ \Rightarrow (\Phi^c)^{-1}(0.7) \leq \frac{t - t^* \cos(\alpha_1)}{\sin(\alpha_1)} \]

\[ \Rightarrow t_2 \leq \frac{t - (\Phi^c)^{-1}(0.7) \sin(\alpha_1)}{\cos(\alpha_1)} + O(1) \leq \frac{t}{\cos(\alpha)} + O(1) \]

Similarly for \( t_1 \). Now we need to argue that \( t_1, t_2 \geq 0 \). Observe that

\[ \Pr[\text{sgn}(f(x)) \text{ and } x \in l(z, 0, \epsilon) | x \in l(z, 0, \epsilon)] \]

\[ \leq \sum \Pr[x^T \mathbf{w}^*_i \geq t \text{ and } x \in l(z, 0, \epsilon) | x \in l(z, 0, \epsilon)] \]

\[ = \sum \Phi^c \left( \frac{t - \epsilon \cos(\alpha_1)}{\sin(\alpha_1)} \right) + O(\epsilon^2) dt \leq d e^{-\Omega(t^2)} < 0.4 \]

Thus for sufficiently large \( t = \Omega(\sqrt{\log d}) \), this will be less than 0.4. Hence there will be some \( t_1, t_2 \geq 0 \) with probability evaluating to 0.4 since the probability is an almost increasing function of \( t \) up to small noise in the given range (see proof of Lemma 5).

**Proof of Lemma 7.** Let \( V \) be the plane spanned by \( \mathbf{w}^*_1 \) and \( z \) and let \( \mathbf{v}_1 = \mathbf{w}^*_1 \) and \( \mathbf{v}_2 \) be the basis of this space. Thus, we can write \( z = \cos(\alpha_1)\mathbf{v}_1 + \sin(\alpha_1)\mathbf{v}_2 \).
Let us apply a Gaussian perturbation $\Delta z$ along the tangential hyperplane normal to $z$. Say it has distribution $\epsilon N(0, I_{d-1})$ along any direction tangential to the vector $z$. Let $\epsilon_1$ be the component of $\Delta z$ on to $V$ and let $\epsilon_2$ be the component perpendicular to it. We can write the perturbation as $\Delta z = \epsilon_1(\sin(\alpha_1)v_1 - \cos(\alpha_1)v_2) + \epsilon_2 v_3$ where $v_3$ is orthogonal to both $v_1$ and $v_2$.

So the new angle $\alpha_1'$ of $z$ after the perturbation is given by

$$\cos(\alpha_1') = \frac{v_1'(z + \Delta z)}{||z + \Delta z||} = \frac{\cos(\alpha_1) + \epsilon_1 \sin(\alpha_1)}{\sqrt{1 + ||\Delta z||^2}}$$

Note that with constant probability $\epsilon_1 \geq \epsilon$ as $\Delta z$ is a Gaussian variable with standard deviation $\epsilon$. And with high probability $||\Delta z|| < \epsilon \sqrt{d}$. We will set $\epsilon = 0.1 \sin(\alpha)/d$ since $\alpha < \pi/2$, we have $0.5\alpha \leq \sin(\alpha) \leq \alpha$. Also $\sin(\alpha) \leq 2 \sin(\alpha_1)$ by the approximation guarantee. Thus with constant probability:

$$\cos(\alpha_1') \geq \frac{\cos(\alpha_1) + \epsilon \sin(\alpha_1)}{\sqrt{1 + \epsilon^2 d}}$$

$$\geq (\cos(\alpha_1) + \epsilon \sin(\alpha_1))(1 - 0.5\epsilon^2 d)$$

$$\geq \cos(\alpha_1) + \epsilon \sin(\alpha_1) - \epsilon^2 d$$

$$\geq \cos(\alpha_1) + 0.8\epsilon \sin(\alpha_1)).$$

Thus change in $\cos(\alpha_1)$ is given by $\Delta \cos(\alpha_1) \geq 0.8\epsilon \sin(\alpha_1)$. Now change in the angle $\alpha_1$ satisfies by the Mean Value Theorem:

$$\Delta \cos(\alpha_1) = \Delta \alpha_1 \left[ \frac{d}{dx} \cos(x) \right]_{x \in [\alpha_1, \alpha_1']}$$

$$\implies -\Delta \alpha_1 = \Delta \cos(\alpha_1) \left[ \frac{1}{\sin(x)} \right]_{x \in [\alpha_1, \alpha_1']}$$

$$\geq \frac{0.8\epsilon \sin(\alpha_1))}{\sin(\alpha_1)} = 0.8\epsilon = 0.04\alpha/d.$$

\[\square\]

### B.6 Sigmoid Activations

Observe that for sigmoid activation, Assumption 2 is satisfied for $\rho(t, \sigma) = e^{-t+\sigma^2/2}$. Thus to satisfy Assumption 3, we need $t = \Omega(\eta \log d)$.

Note that for such value of $t$, the probability of the threshold being crossed is small. To avoid this we further assume that $f$ is non-negative and we have access to an oracle that biases the samples towards larger values of $f$; that after $x$ is drawn from the Gaussian distribution, it retains the sample $(x, f(x))$ with probability proportional to $f(x)$ - so $P_f[x]$ in the new distribution. This enables us to compute correlations even if $E[x, \tilde{N}(0, 1)f(x)]$ is small. In particular by computing $E[h(x)]$ from this distribution, we are obtaining $E[f(x)h(x)]/E[f(x)]$ in the original distribution. Thus we can compute correlations that are scaled.

We get our approximate theorem:

**Theorem 13.** For $t = \Omega(\log d)$, columns of $(TW^*)^{-1}$ can be recovered within error $1/\text{poly}(d)$ using the algorithm in polynomial time.
B.7 Polynomials $P$ with higher degree in one variable

In the main section we assumed that the polynomial has degree at most 1 in each variable. Let us give a high level overview of how to extend this to the case where each variable is allowed a large degree. $P$ now has the following structure,

$$P(X_1, \ldots, X_d) = \sum_{r \in \mathbb{Z}^d_+} c_r \prod_{i=1}^d X_i^{r_i}$$

If $P$ has a higher degree in $X_i$ then Assumption 2 changes to a more complex (stronger) condition. Let $q_i(x) = \sum_{r \in \mathbb{Z}^d_+ : \|r\| = 0} c_r x^{r_i}$, that is $q_i$ is obtained by setting all $X_j$ for $j \neq i$ to 0.

Assumptions 4. \( \mathbb{E}_{g \sim \mathcal{N}(0, \sigma^2)}[|u_t(g)|^r] \leq \rho(t, \sigma) \) for all $r \in \mathbb{Z}_+^d$. \( \mathbb{E}[q_i(u_t(g))h_k(g)] = t^{O(1)} \rho(t, 1) \) for $k = 2, 4$. Lastly, for all $d \geq i > 1$, \( \sum_{r \in \mathbb{Z}^d_+ : \|r\| = 0} |c_r| \leq d^{O(i)} \).

The last assumption holds for the case when the degree is a constant and each coefficient is upper bounded by a constant. It can hold for decaying coefficients.

Let us collect the univariate terms $P_{\text{uni}}(X) = \sum_{i=1}^d q_i(X_i)$. Corresponding to the same we get $f_{\text{uni}}$. This will correspond to the $f_{\text{lin}}$ we had before. Note that the difference now is that instead of being the same activation for each weight vector, now we have different ones $q_i$ for each. Using $H_4$ correlation as before, now we get that:

$$\mathbb{E}[f_{\text{uni}}(x)H_4(z^T x)] = \sum_{i=1}^d \tilde{q}_i \circ u_t(z^T w_i^*)^4$$ and $$\mathbb{E}[f_{\text{uni}}(x)H_2(z^T x)] = \sum_{i=1}^d \tilde{q}_i \circ u_t(z^T w_i^*)^2$$

where $\tilde{q}_i \circ u_t$ are hermite coefficients for $q_i \circ u_t$. Now the assumption guarantees that these are positive which is what we had in the degree 1 case.

Second we need to show that even with higher degree, $\mathbb{E}[|f(x) - f_{\text{uni}}(x)|]$ is small. Observe that

**Lemma 17.** For $r$ such that $\|r\|_0 > 1$, under Assumption 4 we have,

$$\mathbb{E} \left[ \prod_{i=1}^d (u_t((w^*_i)^T x))^r_i \right] \leq \rho(t, 1) O(\rho(t, \|W^*\|))^{\|r\|_0 - 1}.$$ 

The proof essentially uses the same idea, except that now the dependence is not on $\|r\|_1$ but only the number of non-zero entries (number of different weight vectors). With this bound, we can now bound the deviation in expectation.

**Lemma 18.** Let $\Delta(x) = f(x) - f_{\text{uni}}(x)$. Under Assumptions 4 if $t$ is such that $\rho(t, \|W^*\|) \leq d^{-C}$ for large enough constant $C > 0$, we have, $\mathbb{E}[|\Delta(x)|] \leq d^{O(1)} \rho(t, 1) \rho(t, \|W^*\|)$.

**Proof.** We have,

$$\mathbb{E}[|\Delta(x)|] = \mathbb{E} \left[ \sum_{r \in \mathbb{Z}^d_+ : \|r\|_0 > 1} c_r \prod_{i=1}^d (u_t((w^*_j)^T x))^r_i \right]$$

\(^7\)For example, this would hold for any $u$ bounded in $[-1, 1]$ such as sigmoid or sign.
Let \( c_r \) denote a unit vector along \( z \) and \( \hat{z} \) be the component along \( z \). So \( x = x_0 \hat{z} + y z \perp \). Interpreting \( x \) as a function of \( x_0 \) and \( y \):

\[
C_1(f, z, s) = \int y \int_{x_0} f(x) \delta(x_0 - s) \phi(x_0) \phi(y) dx_0 dy
\]

where the second equality follows from \( \int f(x) \delta(x-a)f(x) = [f(x)]_x=a \).

\[ \square \]

\section{Sample Complexity}

\textbf{Proof of Lemma 3.} C1(f, z, s) = E[f(x)\delta(z^T x - s)] = \int x f(x)\delta(z^T x - s)\phi(x)dx Let \( x_0 \) be the component of \( x \) along \( z \) and \( y \) be the component along \( z \perp \). So \( x = x_0 \hat{z} + y z \perp \). Interpreting \( x \) as a function of \( x_0 \) and \( y \):

\[
C_1(f, z, s) = \int y \int_{x_0} f(x) \delta(x_0 - s) \phi(x_0) \phi(y) dx_0 dy
\]

Since \( \int f(x) \delta(x-a)f(x) = [f(x)]_x=a \) this implies:

\[ \square \]

Thus as before, if we choose \( t \) appropriately, we get the required results. Similar ideas can be used to extend to non-constant degree under stronger conditions on the coefficients.
\[
= \sum \int_y \left[ \phi(x_0) \frac{\partial P}{\partial X_i} \frac{\partial X_i}{\partial x_0} \right]_{x_0=s} \phi(y) dy + \phi'(s) \int_y P[x] \phi(y) dy
\]

Note that \( \frac{\partial X_i}{\partial x_0} u(x^T w_i^* - t) = u'(x^T w_i^* - t) \dot{z}^T w_i^* \). If \( u \) is the sign function then \( u'(x) = \delta(x) \). So focusing on one summand in the sum we get

\[
\int_y \left[ \phi(x_0) \frac{\partial P}{\partial X_i} \frac{\partial X_i}{\partial x_0} \right]_{x_0=s} \phi(y) dy
\]

\[
= \int_y [\phi(x_0) u'(x^T w_i^* - t)(\dot{z}^T w_i^*) \frac{\partial P}{\partial X_i}]_{x_0=s} \phi(y) dy
\]

\[
= \int_y (\dot{z}^T w_i^*)[\phi(x_0) \delta(\dot{z}^T w_i^* x_0 + (w_i^*)^T y - t) \frac{\partial P}{\partial X_i}]_{x_0=s} \phi(y) dy
\]

\[
= (\dot{z}^T w_i^*) \int_y \phi(s) \delta(s(w_i^*)^T \dot{z} + (w_i^*)^T y - t) \frac{\partial P}{\partial X_i} \phi(y) dy
\]

Again let \( y = y_0(w_i^*)' + z \) where \( z \) is perpendicular to \( w_i^* \) and \( z \). And \( (w_i^*)' \) is perpendicular component of \( w_i^* \) to \( z \). Interpreting \( x = t \dot{z} + y_0(w_i^*)' + z \) as a function of \( y_0, z \) we get:

\[
= (\dot{z}^T w_i^*) \int_z \phi(s) \delta(s(w_i^*)^T \dot{z} + (w_i^*)^T y_0 - t) \phi(y_0) \phi(z) \frac{\partial P}{\partial X_i} d y_0 d z
\]

Note that by substituting \( v = ax \) we get \( \int_{x=-\infty}^{x=\infty} f(x) \delta(ax-b) dx = \int_{x=-\infty/a}^{x=\infty/a} f(x) \delta(ax-b) dx = sgn(a) \frac{1}{a} f(\frac{b}{a}) \). So this becomes:

\[
= \frac{\dot{z}^T w_i^*}{|(w_i^*)^T (w_i^*)'|} \int_z \phi(s) \frac{\partial P}{\partial X_i} y_0=x_0=s, y_0=(w_i^*)^T w_i^* \phi(z) dz
\]

\[
= \frac{\dot{z}^T w_i^*}{|(w_i^*)^T (w_i^*)'|} \int_z \phi(y_0) \phi(x_0) \phi(z) \frac{\partial P}{\partial X_i} x_0=s, y_0=(w_i^*)^T w_i^* \phi(z) dz
\]

\[
= \frac{\dot{z}^T w_i^*}{|(w_i^*)^T (w_i^*)'|} \phi \left( y_0 = \frac{t-s \dot{z}^T w_i^*}{(w_i^*)^T (w_i^*)} \right) \phi(x_0 = t) \int_z [\phi(z) \frac{\partial P}{\partial X_i} x_0=t,y_0=(w_i^*)^T w_i^*] dz
\]

\[
= \frac{\dot{z}^T w_i^*}{|(w_i^*)^T (w_i^*)'|} \int_z [\phi(x) \frac{\partial P}{\partial X_i} x_0=t,y_0=(w_i^*)^T w_i^*] dx
\]

\[
= \frac{\dot{z}^T w_i^*}{|(w_i^*)^T (w_i^*)'|} \phi(z^T x = s, x^T w_i^* = t) E \left[ \frac{\partial P}{\partial X_i} | x^T z = s, x^T w_i^* = t \right].
\]

Let \( \alpha_i \) be the angle between \( z \) and \( w_i^* \). Then this is

\[
= | \cot(\alpha_i) | \phi(z^T x = t, (w_i^*)^T x = t) E \left[ \frac{\partial P}{\partial X_i} | x^T z = s, x^T w_i^* = t \right]
\]

29
Thus, overall correlation
\[
\sum_{i \in S} \cot(\alpha_i)|\phi(z^T x = s, x^T w_i^* = t)\mathbb{E}\left[\frac{\partial P}{\partial X_i} | x^T z = s, x^T w_i^* = t\right] + \phi'(s)\mathbb{E}[P(x) | x^T z = s]
\]

\[\square\]

Proof of Claim \footnote{Note that for small $\alpha$, $|\cot \alpha| = O(1/\alpha) \leq O(1/\epsilon_2)$. Since $P(X) \leq ||X||^{c_1}$, we have $f(x) \leq d^{c_1}$ (as with sgn function each $\text{sgn}(w_i^T x) \leq 1$) and all $Q_i[x], R_i[x]$ are at most $2d^{c_1}$. By Cauchy’s mean value theorem $C_2(f, z, t) = 2\epsilon[\mathbb{E}[f(x)\delta'(z^T x = s)]\in t \pm \epsilon].$ Note that $\cot(\alpha_i)\phi(z^T x = t, (w_i^*)^T x = t) = \cot(\alpha_i)\phi(t \tan(\alpha_i/2))$ which is a decreasing function of $\alpha_i$ in the range $[0, \pi]$ So if all $\alpha_i$ are upper bounded by $\epsilon_2$ then by above corollary,
\[
C_2(f, z, s) \leq 2\epsilon n \cot(\epsilon_2)\phi(z^T x = t, (w_i^*)^T x = t)(2d^{c_1}) + (2d^{c_1})
\]
\[
= 2\epsilon n \cot(\epsilon_2)\phi(t \tan(\epsilon_2/2))(2d^{c_1}) + (2d^{c_1})
\]
\[
\leq \frac{\epsilon}{\epsilon_2}d^{O(1)}.
\]

\[\square\]

Observe that the above proof does not really depend on $P$ and holds for for any polynomial of $u((w_i^*)^T x)$ as long as the polynomial is bounded and the $w_i^*$ are far off from $z$.

Proof Of Lemma \footnote{If $z = w_i^*$, then
\[
C_2(f, z, t) = \mathbb{E}[f(x)(\delta(z^T x = t - \epsilon) - \delta(z^T x = t + \epsilon))]
\]
\[
= \mathbb{E}[u((w_i^*)^T x)Q_i[x](\delta(z^T x = t - \epsilon) - \delta(z^T x = t + \epsilon))]
\]
\[
+ \mathbb{E}[R_i[x](\delta(z^T x = t - \epsilon) - \delta(z^T x = t + \epsilon))]
\]

Since $u((w_i^*)^T x) = 0$ for $z^T x = t - \epsilon$ and 1 for $z^T x = t + \epsilon$, and using the Cauchy mean value theorem for the second term this is
\[
= \mathbb{E}[Q_i[x]\delta(z^T x = t - \epsilon)] + 2\epsilon[\mathbb{E}[R_i[x]\delta'(z^T x - s_1)]_{s_1 \in t \pm \epsilon}
\]
\[
= \mathbb{E}[Q_i[x]\delta(z^T x = t)] + E[R_i[x](\delta(z^T x - t - \epsilon) - \delta(z^T x - t))]
\]
\[
= \phi(t)E[Q_i[x]|z^T x = t + \epsilon] + \epsilon[C_2(Q_i, z, s_2)]_{s_2 \in [t, t+\epsilon]} + 2\epsilon[C_2(R_i, z, s)]_{s \in t \pm \epsilon}
\]
\[
= \phi(t)E[Q_i[x]|z^T x = t] + \epsilon d^{O(1)}
\]
The last step follows from Claim \footnote{applied on $Q_i$ and $R_i$ as all the directions of $w_i^*$ are well separated from $z = w_i^*$ and $w_i^*$ is absent from both $Q_i$ and $R_i$. Also the corresponding $Q_i$ and $R_i$ are bounded.} applied on $Q_i$ and $R_i$ as all the directions of $w_i^*$ are well separated from $z = w_i^*$ and $w_i^*$ is absent from both $Q_i$ and $R_i$. Also the corresponding $Q_i$ and $R_i$ are bounded.}

\[\square\]

C.1 ReLU activation

If $u$ is the ReLU activation, the high level idea is to use correlation with the second derivative $\delta''$ of the Dirac delta function instead of $\delta'$. More precisely we will compute $C_3(f, z, s) = \mathbb{E}[f(\delta'(z^T x = s - \epsilon) - \delta'(z^T x - s + \epsilon))].$ Although we show the analysis only for the ReLU activation, the same idea works for any activation that has non-zero derivative at 0.

Note that now $u' = \text{sgn}$ and $u'' = \delta$. 

30
For ReLU activation, Lemma 8 gets replaced by the following Lemma. The rest of the argument is as for the sgn activation. We will need to assume that $P$ has constant degree and sum of absolute value of all coefficients is $\text{poly}(d)$

**Lemma 19.** Assuming polynomial $P$ has constant degree, and sum of the magnitude of all coefficients is at most $\text{poly}(d)$, if $z = w_i^*$ then $C_3(f, z, t) = \mathbb{E}[\phi(t) \frac{\partial}{\partial x_i} P + \phi(t) \sum_{j \neq i} \cos(\alpha_j) \text{sgn}(x^T w_i^* - t) \frac{\partial}{\partial w_j} P + \phi'(t) P |x^T w_i^* - t| + \epsilon d^{O(1)}]$. Otherwise if all angles $\alpha_i$ between $z$ and $w_i$ are at least $\epsilon_2$ it is at most $\epsilon d^{O(1)}/\epsilon_2$.

We will prove the above lemma in the rest of this section. First we will show that $z$ is far from any of the $w_i^*$’s then $\mathbb{E}[P.\delta''(z^T x - s)]$ is bounded.

**Lemma 20.** If the sum of the absolute value of the coefficients of $P$ is bounded by $\text{poly}(d)$, its degree is at most constant, $\alpha_i > \epsilon_2$ then $\mathbb{E}[P.\delta''(z^T x - s)]$ is $d^{O(1)}/\epsilon_2$.

**Proof.** Let $x_0$ be the component of $x$ along $z$ and $y$ be the component of $x$ in the space orthogonal to $z$ as before. We have $x = x_0 z + y$ and $\frac{\partial x}{\partial x_0} = z$. We will look at monomials $M_i$ in $P = \sum_i M_i$.

As before since $\int_x \delta''(x-a)f(x)dx = \left[ \frac{d^2 f}{dx^2} \right]_{x=a}$ we get

$$\mathbb{E}[f(x)\delta''(z^T x - s)] = \int_y f(x)\phi(x)\delta''(z^T x - s)dy$$

$$= \int_y \left[ \frac{\partial^2}{\partial x_0^2}(P|x|\phi(x)) \right]_{x_0=s} dy$$

$$= \sum_i \int_y \left[ \frac{\partial^2}{\partial x_0^2}(M_i|x|\phi(x_0)) \right]_{x_0=s} \phi(y)dy$$

Now consider a monomial $M = X_1^{i_1}...X_k^{i_k}$.

Take the symbolic second derivative $\frac{\partial^2}{\partial x_0^2}$ of $M|x|\phi(x_0)$ w.r.t $x_0$. This will produce a polynomial involving $X_i$’s, $\frac{\partial X_i}{\partial x_0}$, $\frac{\partial^2 X_i}{\partial x_0^2}$, $\phi$, $\phi'$, $\phi''$. Let us examine each of these terms.

$$\frac{\partial}{\partial x_0} X_i|x| = \frac{\partial}{\partial x_0} u(x^T w_i^* - t)$$

$$= \text{sgn}(x^T w_i^* - t) (w_i^*)^T \frac{\partial}{\partial x_0} x$$

$$= \text{sgn}(x^T w_i^* - t) ((w_i^*)^T z)$$

$$= \cos(\alpha_i) \text{sgn}(x^T w_i^* - t)$$

Thus $\frac{\partial}{\partial x_0} X_i|x|$ is a bounded function of $x$. We have

$$\frac{\partial^2}{\partial x_0^2} X_i(x) = \frac{\partial}{\partial x_0} \text{sgn}(x^T w_i^* - t)((w_i^*)^T z) = \cos^2(\alpha_i) \delta(x^T w_i^* - t).$$

Again as before

$$\int_y [\delta(x^T w_i^* - t)g(x)\phi(x_0)\phi(y)]_{x_0=s}dy$$

$$= (1/|\sin(\alpha_i)|)\mathbb{E}[g(x)|x_0 = s, x^T w_i^* = t] \phi(x_0 = s, x^T w_i^* = t)$$

Note that if the degree is bounded, since $|\sin(\alpha_i)|$ is at least $\epsilon_2$ expected value of each monomial obtained is bounded. So the total correlation is $\text{poly}(d)/\epsilon_2$. $\square$
Proof of Lemma 20. As in the case of sgn activation, if $z = w_i^*$,

$$
\mathbb{E}[P[x] \delta'(x^Tw_i^*-s)]
$$

$$
= \int_x P[x] \phi(x) \delta'(z^T x - s)dx
$$

$$
= \int_y \left[ \frac{\partial}{\partial x_0} (P[x] \phi(x)) \right]_{x_0=s} dy
$$

$$
= \int_y \left[ \sum_j \left( \frac{\partial P}{\partial x_j} \right)[x] \frac{\partial X_j}{\partial x_0} \phi(x) + P[x] \frac{\partial \phi(x)}{\partial x_0} \right]_{x_0=s} dy
$$

$$
= \int_y \phi(y) \left[ \text{sgn}(x_0 - t) \phi(s) \left( \frac{\partial P}{\partial x_i} \right)[x] + \sum_{j \neq i} \text{sgn}(x_i^Tw_j^* - t) \cos(\alpha_j) \phi(s) \left( \frac{\partial P}{\partial X_j} \right)[x] + P\phi'(x_0) \right]_{x_0=s} dy
$$

For $s = t + \epsilon$ this is

$$
= \left( \int_y \phi(y) \left( \phi(t) \left( \frac{\partial P}{\partial x_i} \right)[x] + \sum_{j \neq i} \cos(\alpha_j) \text{sgn}(x_i^Tw_j^* - t) \phi(t) \left( \frac{\partial P}{\partial X_j} \right)[x] + P\phi'(t) \right) dy \right) + \epsilon d^O(1)
$$

$$
= \mathbb{E} \left[ \phi(t) \left( \frac{\partial P}{\partial X_i} \right)[x] + \phi(t) \sum_{j \neq i} \cos(\alpha_j) \text{sgn}(x_i^Tw_j^* - t) \left( \frac{\partial P}{\partial X_j} \right)[x] + \phi'(t)P \left| x_i^Tw_j^* = t \right] \right] + \epsilon d^O(1)
$$

If $z$ is away from every $w_i^*$ by at least $\epsilon_2$ then again $\mathbb{E}[PC_3(f, z, t)] = \mathbb{E}[P\delta'(x_i^Tw_i^*-s)](s \in [t - \epsilon, t + \epsilon]) = \epsilon d^O(1)/\epsilon_2$.

D. Structural Restrictions Helps Learning

D.1 Proof of Theorem 8

To construct this correlation graph, we will run the following Algorithm 3. Denote $T_i := \{ j : w_{ij} =$
1. Let us compute \( \mathbb{E}[f(x)x_i x_j] \):

\[
\mathbb{E}[f(x)x_i x_j] = \sum_{S \subseteq [d]} c_S \mathbb{E} \left[ \prod_{p \in S} u(x^T w_p^*) x_i x_j \right] \\
= \sum_{S \subseteq [d]} c_S e^{-\rho|S|} \mathbb{E} \left[ e^{\rho \sum_{p \in S} x^T w_p^* x_i x_j} \right] \\
= \sum_{S \subseteq [d]} c_S e^{-\rho|S|} \mathbb{E} \left[ e^{\rho \sum_{q \in \cup_{p \in S} x^T w_p^* x_q x_j} x_i x_j} \right] \\
= \sum_{S \subseteq [d]} c_S e^{-\rho|S|} \mathbb{E} \left[ 1_{i, j \in \cup_{p \in S} T_p} \mathbb{E} [x_i e^{\rho x_i}] \mathbb{E} [x_j e^{\rho x_j}] \prod_{q \in \cup_{p \in S} T_p \setminus \{i, j\}} \mathbb{E} [e^{\rho x_q}] \right] \\
= \sum_{S \subseteq [d]} c_S e^{-\rho|S|} \mathbb{E} \left[ 1_{i, j \in \cup_{p \in S} T_p} \rho^2 e^{\rho^2/2} \mathbb{E} [x_j e^{\rho x_j}] \prod_{q \in \cup_{p \in S} T_p \setminus \{i, j\}} e^{\rho^2/2} \right] \\
= \sum_{S \subseteq [d]} c_S e^{-\rho|S|} \mathbb{E} \left[ 1_{i, j \in \cup_{p \in S} T_p} \rho^2 e^{\rho^2/2} \mathbb{E} [x_j e^{\rho x_j}] \prod_{q \in \cup_{p \in S} T_p \setminus \{i, j\}} e^{\rho^2/2} \right] \\
= \sum_{S \subseteq [d]} c_S e^{-\rho|S|} \mathbb{E} \left[ 1_{i, j \in \cup_{p \in S} T_p} \rho^2 e^{\rho^2/2} \mathbb{E} [x_j e^{\rho x_j}] \prod_{q \in \cup_{p \in S} T_p \setminus \{i, j\}} e^{\rho^2/2} \right]
\]

By assumption, for all \( p, T_p \) are disjoint. Now, if \( i, j \in T_r \) for some \( r \), we have

\[
\mathbb{E}[f(x)x_i x_j] = \rho^2 \sum_{S \subseteq [d], r \in S} c_S e^{-\rho|S|(1-\rho d)/2}
\]

Similarly, if \( i \in T_{r_1} \) and \( j \in T_{r_2} \) with \( r_1 \neq r_2 \), we have

\[
\mathbb{E}[f(x)x_i x_j] = \rho^2 \sum_{S \subseteq [d], r_1, r_2 \in S} c_S e^{-\rho|S|(1-\rho d)/2}
\]

It is easy to see that these correspond to coefficients of \( X_r \) and \( X_{r_1} X_{r_2} \) (respectively) in the following polynomial:

\[
Q(X_1, \ldots, X_n) = \rho^2 P(\mu(X_1 + 1), \ldots, \mu(X_n + 1))
\]

for \( \mu = e^{-\rho^2/2} \). For completeness we show that this is true. We have,

\[
Q(X_1, \ldots, X_n) = \rho^2 \sum_{S \subseteq [d]} c_S \prod_{j \in S} \mu(X_j + 1) \\
= \rho^2 \sum_{S \subseteq [d]} c_S \mu^{|S|} \prod_{j \in S} (X_j + 1)
\]

The coefficient of \( X_r \) in the above form is clearly \( \rho^2 \sum_{S \subseteq [d], r \in S} c_S \mu^{|S|} \) (corresponds to picking the 1 in each product term). Similarly coefficient of \( X_{r_1} X_{r_2} \) is \( \rho^2 \sum_{S \subseteq [d], r_1, r_2 \in S} c_S \mu^{|S|} \).

Now as in the assumptions, if we have a gap between these coefficients, then we can separate the large values from the small ones and form the graph of cliques. Each clique will correspond to the corresponding weight vector.
D.2 Proof of Theorem 9

Consider \( f(x) = \sum_{S} c_{S} \prod_{i \in S} e^{x^T w_i^*} \) where \( w_i^* = \sum_{j \in S_i} e_j \) for \( S_i \subseteq [n] \) such that for all \( i \neq j \), \( S_i \cap S_j = \emptyset \) and \( c_{S} \geq 0 \). Let us compute \( g(z) = e^{-\frac{1}{2} \|z\|^2} E \left[ f(x) e^{z^T x} \right] \) where \( z = \sum_{i} z_i e_i \) for some \( \alpha_i \).

\[
E \left[ f(x) e^{z^T x} \right] = E \left[ \left( \sum_{S} c_{S} \prod_{i \in S} e^{\sum_{j \in S_i} x_j} \right) e^{z^T x} \right] \\
= \sum_{S} c_{S} E \left( \prod_{i \in S} e^{\sum_{j \in S_i} z_j x_j} \right) \\
= \sum_{S} c_{S} \left( \prod_{p \in \cup_{i \in S_i} S} e^{(1+z_p)x_p} \right) \left( \prod_{q \in [n] \setminus \cup_{i \in S_i} S} e^{z_q x_q} \right) \\
= \sum_{S} c_{S} \left( \prod_{p \in \cup_{i \in S_i} S} \mathbb{E} \left[ e^{(1+z_p)x_p} \right] \right) \left( \prod_{q \in [n] \setminus \cup_{i \in S_i} S} \mathbb{E} \left[ e^{z_q x_q} \right] \right) \\
= \sum_{S} c_{S} \left( \prod_{p \in \cup_{i \in S_i} S} e^{(1+z_p)^2} \right) \left( \prod_{q \in [n] \setminus \cup_{i \in S_i} S} e^{z_q^2} \right) \\
= \sum_{S} c_{S} e^{\frac{\|z\|^2}{2}} \prod_{p \in \cup_{i \in S_i} S} e^{\frac{1}{2} z_p} \\
\implies g(z) = \sum_{S} c_{S} \prod_{p \in \cup_{i \in S_i} S} e^{\frac{1}{2} z_p}
\]

Consider the following optimization problem:

\[
\max_{z} \left( g(z) - \lambda \|z\|_1 - \gamma \|z\|_2^2 \right)
\]

subject to \( h(z) \)

for \( \lambda, \gamma > 0 \) to be fixed later.

We can assume that \( z_i \geq 0 \) for all \( i \) at a local maxima else we can move in the direction of \( e_i \) and this will not decrease \( g(z) \) since \( c_{S} \geq 0 \) for all \( S \) and will decrease \( \|z\|_1 \) and \( \|z\|_2 \) making \( h(z) \) larger. From now on, we assume this for any \( z \) at local maxima.

We will show that the local maximas of the above problem will have \( z \) such that most of the mass is equally divided among \( j \in S_i \) for some \( i \) and close to 0 everywhere else.

**Lemma 21.** There exists at most one \( i \) such that there exists \( j \in S_i \) with \( |z_j| \geq \beta \) for \( \gamma \) satisfying \( 4\gamma < \min_{i \neq j} \left[ e^{\beta} \sum_{S : i \in S, j \notin S_2 \cap \cup_{S \in S_2} c_{S} e^{\frac{\|z\|_2}{2}}} \right] \).

**Proof.** Let us prove by contradiction. Assume that there is a local maxima such that there are at least 2 indices say 1, 2 such that \( \exists j \in S_1, k \in S_2, |z_j|, |z_k| \geq \beta \). Now we will show that there exists a perturbation such that \( g(z) \) can be improved. Now consider the following perturbation, \( z + \epsilon s e_j - \epsilon s e_k \) for \( s \in \{\pm 1\} \). Observe that \( \|z\|_1 \) remains unchanged for \( \epsilon < \beta \) also \( \|z\|_2 \) changes by \( 2s^2 \epsilon^2 + 2(z_j - z_k) \epsilon \). We have

\[
E_a[h(z + \epsilon s e_j - \epsilon s e_k) - h(z)]
\]

34
= \sum_{S:1\in S,2\notin S} c_S \prod_{p\in U_{\in S} S_i} e^{\frac{1}{2} + z_p} \left( E_S [e^{\alpha}] - 1 \right) + \sum_{S:1\notin S,2\in S} c_S \prod_{p\in U_{\in S} S_i} e^{\frac{1}{2} + z_p} \left( E_S [e^{-\alpha}] - 1 \right) - \gamma E_S \left[2e^2 + 2(z_j - z_k)\sigma\right]

\geq \frac{e^2}{2} \left(-4\gamma + \sum_{S:1\in S,2\notin S} c_S \prod_{p\in U_{\in S} S_i} e^{\frac{1}{2} + z_p} + \sum_{S:1\notin S,2\in S} c_S \prod_{p\in U_{\in S} S_i} e^{\frac{1}{2} + z_p} \right)

The inequality follows since \( E [e^{\alpha}] = \frac{e^{e^2\sigma^2}}{2} \geq 1 + \frac{e^2}{2} \). Observe that

\sum_{S:1\in S,2\notin S} c_S \prod_{p\in U_{\in S} S_i} e^{\frac{1}{2} + z_p} \geq \sum_{S:1\in S,2\notin S} c_S e^{\frac{\left|\cup_{S_i|S_i|} S_i\right|}{2}} e^{\sum_{p\in U_{\in S} S_i} z_p} \geq \sum_{S:1\in S,2\notin S} c_S e^{\frac{\left|\cup_{S_i|S_i|} S_i\right|}{2}} e^{\sum_{p\in U_{\in S} S_i} z_p}

For chosen value of \( \gamma \), there will always be an improvement, hence can not be a local maxima. \( \square \)

**Lemma 22.** At the local maxima, for all \( i \in [n] \), \( z \) is such that for all \( j, k \in S_i \), \( z_j = z_k \) at local maxima.

**Proof.** We prove by contradiction. Suppose there exists \( j, k \) such that \( z_j < z_k \). Consider the following perturbation: \( z + \epsilon (z_k - z_j)(e_j - e_k) \) for \( 1 \leq \epsilon > 0 \). Observe that \( g(z) \) depends on only \( \sum_{i\in S_i} z_i \) and since that remains constant by this update \( g(z) \) does not change. Also note that \( ||z||_1 \) does not change. However \( ||z||_1^2 \) decreases by \( 2\epsilon (1 - \epsilon)(z_k - z_j)^2 \) implying that overall \( h(z) \) increases. Thus there is a direction of improvement and thus it can not be a local maxima. \( \square \)

**Lemma 23.** At the local maxima, \( ||z||_1 \geq \alpha \) for \( \lambda < \sum_S c_S e^{\frac{\left|\cup_{S_i|S_i|} S_i\right|}{2}} e^{\sum_{p\in U_{\in S} S_i} z_p} - \gamma(2\alpha + 1) \).

**Proof.** We prove by contradiction. Suppose \( ||z||_1 < \alpha \), consider the following perturbation, \( z + \epsilon 1 \). Then we have

\[
\begin{align*}
   h(z + \epsilon 1) - h(z) &= \sum_{S:1\in S,2\notin S} c_S e^{\frac{\left|\cup_{S_i|S_i|} S_i\right|}{2}} + \sum_{p\in U_{\in S} S_i} z_p (e^{\sum_{S_i|S_i|} S_i} - 1) - n\lambda - n\gamma \epsilon (2||z||_1 + \epsilon) \\
   &= \sum_{S:1\in S,2\notin S} c_S e^{\frac{\left|\cup_{S_i|S_i|} S_i\right|}{2}} ||\cup_{S_i|S_i|} S_i||e - n\lambda - n\gamma \epsilon (2\alpha + 1)
\end{align*}
\]

For given \( \lambda \) there is a direction of improvement giving a contradiction that this is the local maxima. Combining the above, we have that we can choose \( \lambda, \gamma = \text{poly}(n, 1/\epsilon, s) \) where \( s \) is a parameter that depends on structure of \( f \) such that at any local maxima there exists \( i \) such that for all \( j \in S_i \), \( z_j \geq 1 \) and for all \( k \notin \cup_{j\in S_i} \), \( z_k \leq \epsilon \).

**D.3 Proof of Theorem 10**

Let function \( f = \sum_{S\subseteq [n]} c_S \prod_{x\in S} u((w_i^*)^T x) \) for orthonormal \( w_i^* \). WLOG, assume \( w_i^* = e_i \). Let \( u(x) = \sum_{i=0}^{\infty} \hat{u}_i h_i(x) \) where \( h_i \) are hermite polynomials and \( u'(x) = u(x) - \hat{u}_0 = \sum_{i=1}^{\infty} \hat{u}_i h_i(x) \). This implies \( E[u'(x)] = 0 \). Observe that,

\[
   \prod_{i\in S} u(x_i) = \prod_{i\in S} (u'(x_i) + \hat{u}_0) = \sum_{k=0}^{\left|S\right|-k} \hat{u}_0^{\left|S\right|-k} \sum_{S'\subseteq S:|S'|=k} \prod_{i\in S'} u'(x_i).
\]
Let us consider correlation with $h_4(z^T x)$. This above can be further simplified by observing that when we correlate with $h_4$, $\prod_{i \in S'} u'(x_i) h_4(z^T x) = 0$ for $|S'| \geq 2$. Observe that $h_4(z^T x) = \sum_{d_1, \ldots, d_n \in [4]} c(d_1, \ldots, d_n) \prod h_{d_i}(x_i)$ for some coefficients $c$ which are functions of $z$. Thus when we correlate $\prod_{i \in S'} u'(x_i) h_4(z^T x)$ for $|S'| \geq 3$ then we can only get a non-zero term if we have at least $h_{2k}(x_i)$ with $k \geq 1$ for all $i \in S'$. This is not possible for $|S'| \geq 3$, hence, these terms are 0. Thus,

$$
E \left[ \prod_{i \in S} u(x_i) h_4(z^T x) \right] = \sum_{k=0}^{2} \sum_{S' \subseteq S : |S'| = k} \sum_{u'(x_i) h_4(z^T x)} E \left[ \prod_{i \in S'} u'(x_i) h_4(z^T x) \right].
$$

Lets compute these correlations.

$$
E \left[ u'(x_i) h_4(z^T x) \right] = E \left[ \sum_{p > 0} \hat{u}_p h_2 p (x_i) h_4(z_i x_i + z_{-i} x_{-i}) \right]
$$

$$
= \frac{1}{4} \sum_{p > 0} \hat{u}_2 p h_2 p (x_i) \sum_{k=0}^{4} \binom{4}{k} h_{4-k}(z_i x_i \sqrt{2}) h_k(z_{-i} x_{-i} \sqrt{2})
$$

$$
= \frac{1}{4} \sum_{p > 0} \hat{u}_2 p h_2 p (x_i) \left( h_4(z_i x_i \sqrt{2} + 6 h_2(z_i x_i \sqrt{2})(2||z_{-i}||^2 - 1) + 3(2||z_{-i}||^2 - 1)^2) \right)
$$

$$
= \frac{1}{4} \left( \hat{u}_4 \alpha(4, 4, z_i \sqrt{2}) + \hat{u}_2 \alpha(4, 2, z_i \sqrt{2}) + 6 \hat{u}_2 \alpha(2, 2, z_i \sqrt{2})(2||z_{-i}||^2 - 1) \right)
$$

$$
= \frac{1}{4} \left( 4 \hat{u}_4 z_i^4 + 12 \hat{u}_2 z_i^2 (2z_i^2 - 1) + 6 \hat{u}_2 (2z_i^2)(2||z_{-i}||^2 - 1) \right)
$$

$$
= \hat{u}_4 z_i^4 + 3 \hat{u}_2 z_i^2 (2||z||^2 - 1)
$$

$$
E \left[ u'(x_i) u'(x_j) h_4(z^T x) \right] = E \left[ \sum_{p, q > 0} \hat{u}_p \hat{u}_q h_2 p (x_i) h_2 q (x_j) h_4(z_i x_i + z_j x_j + z_{-i} x_{-i}) \right]
$$

$$
= \frac{1}{4} \sum_{p, q > 0} \hat{u}_2 \hat{u}_2 h_2 p (x_i) h_2 q (x_j) \sum_{k=0}^{4} \binom{4}{k} h_{4-k}((z_i x_i + z_j x_j) \sqrt{2}) h_k(z_{-i} x_{-i} \sqrt{2})
$$

$$
= \frac{1}{4} \sum_{p, q > 0} \hat{u}_2 \hat{u}_2 h_2 p (x_i) h_2 q (x_j) \left( h_4((z_i x_i + z_j x_j) \sqrt{2}) + 6 h_2((z_i x_i + z_j x_j) \sqrt{2})(2||z_{-i}||^2 - 1) + 3(2||z_{-i}||^2 - 1)^2 \right)
$$

$$
= \frac{1}{16} \sum_{p, q} \hat{u}_p \hat{u}_q \sum_{k=0}^{4} \binom{4}{k} E \left[ h_2 p (x_i) h_2 q (x_j) h_{4-k}(2z_i x_i) h_k(2z_j x_j) \right]
$$

$$
+ \frac{3}{4} (2||z_{-i}||^2 - 1) \sum_{p, q < 0} \hat{u}_2 \hat{u}_2 \sum_{k=0}^{2} \binom{2}{k} E \left[ h_2 p (x_i) h_2 q (x_j) h_{2-k}(2z_i x_i) h_k(2z_j x_j) \right]
$$

$$
= 36
$$
We will compute (1) and (2):

\[ 1 = \frac{1}{16} \sum_{k=0}^{4} \binom{4}{k} \sum_{p,q>0} \hat{u}_{2p} \hat{u}_{2q} \mathbb{E} \left[ h_{2p}(x_i) h_{4-k}(2z_i x_i) \right] \mathbb{E} \left[ h_{2q}(x_j) h_k(2z_j x_j) \right] \]

\[ = \frac{1}{16} \sum_{k=0}^{4} \binom{4}{k} \sum_{p,q>0} \hat{u}_{2p} \hat{u}_{2q} \alpha(4-k, 2p, 2z_i) \alpha(k, 2q, 2z_j) \]

\[ = \frac{3}{8} \hat{u}_{2}^2 \alpha(2, 2, 2z_i) \alpha(2, 2, 2z_j) \]

\[ = 6 \hat{u}_{2}^2 z_i^2 \]

Similarly,

\[ 2 = \frac{3}{4} \left( 2 \left\| z - y \right\| - 1 \right) \sum_{p,q>0} \hat{u}_{2p} \hat{u}_{2q} \sum_{k=0}^{2} \binom{2}{k} \mathbb{E} \left[ h_{2p}(x_i) h_{2q}(x_j) h_{2-k}(2z_i x_i) h_k(2z_j x_j) \right] \]

\[ = \frac{3}{4} \left( 2 \left\| z - y \right\| - 1 \right) \sum \binom{2}{k} \hat{u}_{2}^2 \alpha(2-k, 2, 2z_i) \alpha(k, 2, 2z_j) = 0. \]

Combining, we get

\[ \mathbb{E} \left[ u'(x_i) u'(x_j) h_4(z^T x) \right] = 6 \hat{u}^2 z_i^2 z_j^2. \]

Further, taking correlation with \( f \), we get:

\[ \mathbb{E} \left[ f(x) h_4(z^T x) \right] \]

\[ = \sum_{S \subseteq \{n\}} c_S \sum_{k=0}^{2} \hat{u}_0^{\left| S \right| - k} \sum_{S' \subseteq S: \left| S' \right| = k} \mathbb{E} \left[ \prod_{i \in S'} u'(x_i) h_4(z^T x) \right] \]

\[ = \sum_{S \subseteq \{n\}} c_S \hat{u}_0^{\left| S \right| - 2} \left( \hat{u}_0 \sum_{i \in S} (\hat{u}_4 z_i^4 + 3 \hat{u}_2 z_i^2 (2 \left\| z \right\| - 1)) + 6 \hat{u}_2^2 \sum_{j \neq k \in S} z_j^2 z_k^2 \right) + \text{constant} \]

\[ = \sum_{i=1}^{n} \alpha_i z_i^4 + (2 \left\| z \right\| - 1) \sum_{i=1}^{n} \beta_i z_i^2 + \sum_{1 \leq i \neq j \leq n} \gamma_{ij} z_j^2 z_k^2 + \text{constant} \]

where \( \alpha_i = \hat{u}_4 \sum_{S' \subseteq \{n\}} c_{S'} \hat{u}_0^{\left| S' \right| - 1}, \beta_i = 3 \hat{u}_2 \sum_{S' \subseteq \{n\}} c_{S'} \hat{u}_0^{\left| S' \right| - 1} \) and \( \gamma_{ij} = 6 \hat{u}_2^2 \sum_{S' \subseteq \{n\}} c_{S'} \hat{u}_0^{\left| S' \right| - 2} \).

If \( \gamma_{ij} < \alpha_i + \alpha_j \) for all \( i,j \) then the local maxima of the above are exactly \( e_i \). To show that this holds, we prove by contradiction. Suppose there is a maxima where \( z_i, z_j \neq 0 \). Then consider the following second order change \( z_i^2 \rightarrow z_i^2 + \varepsilon \) and \( z_j^2 \rightarrow z_j^2 - \varepsilon \) where \( \varepsilon \leq \min z_i^2, z_j^2 \) and \( s \) is 1 with probability 0.5 and -1 otherwise. Observe that the following change does not violate the constraint and in expectation affects the objective as follows:

\[ \Delta = \mathbb{E} \left[ \alpha_i (2 \varepsilon z_i^2 + \varepsilon^2) + \alpha_j (-2 \varepsilon z_j^2 + \varepsilon^2) + \beta_i \varepsilon - \beta_j \varepsilon + \gamma_{ij} (s \varepsilon z_j^2 - s \varepsilon z_i^2 - \varepsilon^2) \right] \]

\[ = (\alpha_i + \alpha_j - \gamma_{ij}) \varepsilon^2 > 0 \]

Thus there is a direction in which we can improve and hence it cannot be a maxima.
E Modular Learning by Divide and Conquer

Finally we point out how such high threshold layers could potentially facilitate the learning of deep functions $f$ at any depth, not just at the lowest layer. Note that essentially for Lemma 2 to hold, outputs $X_1, \ldots, X_d$ needn’t be present after first layer but they could be at any layer. If there is any cut in the network that outputs $X_1, \ldots, X_d$, and if the upper layer functions can be modeled by a polynomial $P$, then again assuming the inputs $X_i$ have some degree of independence one can get something similar to Lemma 2 that bounds the non-linear part of $P$. The main property we need is $E[\prod_{i \in S} X_i]/E[X_j] < \mu^{\lvert S \rvert - 1}$ for a small enough $\mu = 1/\text{poly}(d)$ which is essentially replaces Lemma 1. Thus high threshold layers can essentially reduce the complexity of learning deep networks by making them roughly similar to a network with lower depth. Thus such a cut essentially divides the network into two simpler parts that can be learned separately making it amenable to a divide and conquer approach. If there a robust algorithm to learn the lower part of the network that output $X_i$, then by training the function $f$ on that algorithm would recover the lower part of the network, having learned which one would be left with learning the remaining part $P$ separately.

**Remark 1.** If there is a layer of high threshold nodes at an intermediate depth $l$, $u$ is sign function, if outputs $X_i$ at depth $l$ satisfy the following type of independence property: $E[\prod_{i \in S} X_i]/E[X_j] < \mu^{\lvert S \rvert - 1}$ for a small enough $\mu = 1/\text{poly}(d)$, if there is a robust algorithm to learn $X_i$ from $\sum c_iX_i$ that can tolerate noise, then one can learn the nodes $X_i$, from a function $P(X_1, \ldots, X_d)$.