Asymptotic relatively more efficient test with auxiliary information:
the case of the $Z$-test and the chi-square test

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Abstract

The main goal of this article is to study how an auxiliary information can be used to improve the efficiency of two famous statistical tests: the $Z$-test and the chi-square test. Many definitions of auxiliary information can be found in the statistical literature. In this article, the notion of auxiliary information is discussed from a very general point of view and depends on the relevant test. These two statistical tests are modified so that this information is taken into account. It is shown in particular that the efficiency of these new tests is improved in the sense of Pitman’s ARE. Some statistical examples illustrate the use of this method.

1 Introduction

Main motivation

The main goal of this article is to present two new statistical tests which exploit a given auxiliary information. The new tests are based on modification of familiar statistical tests, the $Z$-test and the chi-square goodness-of-fit test and exploit a known auxiliary information in a way to get a more efficient test. These modifications are made so that, under the null hypothesis ($H_0$), the asymptotic behavior of the random variables involved by these test statistics does not change and, under ($H_1$), the probability of rejecting the null hypothesis is higher than that of the classical test. A description of the theoretical framework which allows for comparisons between two asymptotic statistical tests is provided below. To illustrate all results of this paper and to show how they can be used in a concrete way, these results will be applied with real data.

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Auxiliary information

Although auxiliary information has been discussed extensively throughout the statistical literature, a generally-accepted definition does not exist. Statistical methods, such as stratification, calibration or Raking-Ratio, assume that a priori auxiliary information is given by the probability of sets of one or more partitions, and is hence known to the statistician. The following example illustrates this definition: a statistician working on a population of human beings knows the real proportion $p_M$ of men and $p_F$ of women in this population and wants to exploit this information in order to improve the estimates based on the sample taken from this population. If the statistician has a sample available, the frequency $\hat{p}_M$ of men and $\hat{p}_F$ of women in this sample is very likely to deviate from the known proportions $p_M$ and $p_F$. The methods cited above therefore aim to correct this difference with the hope of improving the estimates or the efficiency of statistical tests based on this sample. For instance, to estimate the unknown proportion $p_S$ of sick people in the population, the raked estimator or Horvitz-Thompson estimator

$$\hat{p}_S = \frac{\hat{p}_{M\cap S}}{\hat{p}_M} p_M + \frac{\hat{p}_{F\cap S}}{\hat{p}_F} p_F,$$

can be considered, where $\hat{p}_{M\cap S}$ and $\hat{p}_{F\cap S}$ are respectively the proportion of sick men and sick women in the population. This estimator is more precise than the frequency $\hat{p}_S$ of sick people. Nevertheless, such a definition of the auxiliary information places constraints and is limited to only a handful of studies. The current paper instead defines auxiliary information as information that asymptotically reduces variance of the main estimator implied by the test statistic. No additional assumptions are made about the nature and source of the information the statistician has at his disposal. A formal definition will be given later for each
of the two tests and depends on that one. Auxiliary information has been covered extensively in the statistical literature to study the estimators improved with additional information. To the author’s knowledge, it has not been processed to study how to use the information to obtain more efficient tests. The advantage of improving the efficiency of these tests is important since it makes it possible to accept smaller samples for a fixed level and power. The presented results are general and are applicable to many areas like medicine, biostatistics, economics and industry.

Asymptotic comparison of two tests

The asymptotic relative efficiency (ARE) plays the key role in this paper to compare different tests. More precisely, it is the Pitman relative efficiency which would be used to compare the new tests which exploit an auxiliary information versus the classical test which does not take into account this information. For two statistical tests \( i = 1, 2 \) let \( n_i(\alpha, \beta, \theta) \) be the minimal sample sizes needed to test at a level \( \alpha \) and a power at least \( 1 - \beta \) a null hypothesis \( (H_0) : \theta = 0 \) against a sequence of composite hypotheses \( (H_1) : \theta = \theta_n \) with \( \theta_n \) a vanishing sequence. Notice that \( \theta \) is a parameter which can be a real or a real vector. The Pitman’s ARE of the first test with respect to the second test is defined, when this limit exists, as

\[
e_p = \lim_{\theta \to 0} \frac{n_2(\alpha, \beta, \theta)}{n_1(\alpha, \beta, \theta)},
\]

(1)

If this relative efficiency is smaller than 1, the second test needs a smaller size sample to attain the same level and power as the first test. In other words, the second test would be more efficient. As the same way, if this relative efficiency is larger than 1, the first test is then more effective.

Framework

Let \( X_1, \ldots, X_n, X \) be i.i.d. random variables defined on the same probability space \( (\Omega, T, \mathbb{P}) \) with same unknown distribution \( P = \mathbb{P}^X \) on some measurable space \( (\mathcal{X}, T') \). In order to get a probability space, the measurable space \( (\mathcal{X}, T') \) is endowed with \( P \). Let denote \( P(f) = \mathbb{E}[f(X)] \) and \( \mathbb{P}_n, \alpha_n \) respectively the empirical measure and process defined by

\[
\mathbb{P}_n(f) = \frac{1}{n} \sum_{i=1}^{n} f(X_i),
\]

\[
\alpha_n(f) = \sqrt{n}(\mathbb{P}_n(f) - P(f)),
\]

for all \( P \)-measurable functions \( f : \mathcal{X} \to \mathbb{R} \). By convenience, for \( A \in T', \mathbb{P}_n(A) = \mathbb{P}_n(1_A) \). For a \( P \)-measurable function \( f = (f_1, \ldots, f_m) : \mathcal{X} \to \mathbb{R}^m \) let denote

\[
P[f] = (P(f_1), \ldots, P(f_m)),
\]

\[
\mathbb{P}_n[f] = (\mathbb{P}_n(f_1), \ldots, \mathbb{P}_n(f_m)).
\]

The framework of this paper is non-parametric: no additional conditions are assumed on the law of \( X \). The results are therefore applicable to a wide range of fields.

Organization

The new statistical tests which exploit the auxiliary information are presented and justified below. The two following sections describe the methods and main results for the new tests with auxiliary information. Section 2 concerns the Z-test while Section 3 deals with the chi-square test. Section 4 provides examples for each of these improved tests as well as a non-exhaustive list of the literature surrounding the topic of auxiliary information.

2 Pitman’s ARE for the Z-test

Notation

In this section, suppose that the random variables \( X_i \) are real. The common expectation and the variance of all variables \( X_i \) are respectively denoted by \( \mathbb{E}[X] \) and \( \sigma^2 \). This section focuses on the statistical Z-test based on the null hypothesis \( (H_0) : \theta = 0 \) with \( \theta = \mathbb{E}[X] - \mu \) and the alternative \( (H_1) : \theta = h/\sqrt{n} \) for some \( h \in \mathbb{R} \). The classical statistic for this hypothesis is given by

\[
Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n},
\]

where \( \hat{\sigma}_n \) is a consistent estimator of the standard deviation \( \sigma \), \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \) is the empirical mean. Notice that if \( \sigma \) is a known value then \( \hat{\sigma}_n = \sigma \) can be taken. Under \( (H_0) \), the statistic \( Z_n \) converges weakly to the normal distribution \( \mathcal{N}(0, 1) \) while under \( (H_1) \) this statistic converges to \( \mathcal{N}(h/\sigma, 1) \). Then the statistical test based on the rejection region \( |Z_n| > \Phi(1 - \alpha/2) \) is an asymptotic confidence level \( \alpha \), where \( \Phi \) is the inverse of the standard normal cumulated distribution function.

Auxiliary information

In this context, an auxiliary information is an information which could be used to obtain an estimator of \( \mathbb{E}[X] \) with a lower variance than the natural empirical estimator \( \bar{X}_n \). To be in the most general framework, suppose
that a known statistic $\bar{X}_n^\Sigma$ exploits an auxiliary information in the sense that it satisfies the following weak convergence

$$\sqrt{n}(\bar{X}_n^\Sigma - \mathbb{E}[X]) \xrightarrow{n \to +\infty} \mathcal{N}(0, (\sigma^*)^2),$$

(2)

where $\sigma > \sigma^*$. In other words, $\bar{X}_n^\Sigma$ is an estimator which uses the auxiliary information. Some examples of this known statistic and their associated value $\sigma^*$ for this test are given in Section [3]. The new statistic based on the Z-test with the auxiliary information is defined by

$$Z_n^\Sigma = \frac{\sqrt{n}(\bar{X}_n^\Sigma - \mu)}{\sigma_n^*},$$

where $\sigma_n^*$ is a consistent estimator of $\sigma^*$. As the same way as $Z_n$, the new statistic $Z_n^\Sigma$ converges weakly to the normal distribution $\mathcal{N}(0, 1)$ under $(H_0)$ and $\mathcal{N}(h/\sigma^*, 1)$ under $(H_1)$.

Result

The following proposition suggests that the $Z$-test is improved with the exploitation of an auxiliary information. This result is trivial since it is a direct application of the ARE definition.

Proposition 1. The Pitman’s ARE $e_P$ of the classical test with respect to the new test which takes into account the auxiliary information satisfies $e_P = (\sigma^*/\sigma)^2 < 1$.

Proof. It is an application of Theorem 14.19 of [10] with $\theta = \mathbb{E}[X] - \mu$, $T_{n,1} = \bar{X}_n$, $M_1(\theta) = \theta, \sigma_1(\theta) = \sigma_n$ and $T_{n,2} = \bar{X}_n^\Sigma, M_2(\theta) = \theta, \sigma_2(\theta) = \sigma_n^*$ which both satisfy Van der Vaart’s condition (14.5). □

The interest of this proposition lies in its applications presented in Section [3].

3 Pitman’s ARE for the goodness-of-fit test

Notation

In this section no assumption is made on the random variables $X_i$, the distribution $P$ or the set $\mathcal{X}$. A parameter $M \in \mathbb{N} \setminus \{0,1\}$ and a partition $A = (A_1, \ldots, A_M) \subset \mathcal{T}'$ of $\mathcal{X}$ such that $P(A_i) \neq 0$ for all $i = 1, \ldots, M$ are fixed. Let denote $A^* = (A_1, \ldots, A_{M-1})$ and remind that $P[A^*]$ and $P_n[A^*]$ are the vectors respectively defined by

$$P[A^*] = (P(A_1), \ldots, P(A_{M-1})) \in \mathbb{R}^{M-1},$$

$$P_n[A^*] = (P_n(A_1), \ldots, P_n(A_{M-1})) \in \mathbb{R}^{M-1}.$$  (3)

The goal of this test is to check if $P[A^*] = P_0[A^*] = (P_0(A_1), \ldots, P_0(A_{M-1}))$ for some measure $P_0$. The null hypothesis is

$$(H_0) : \Theta = 0_{M-1},$$

where $0_{M-1} = (0, \ldots, 0) \in \mathbb{R}^{M-1}$ and

$$\Theta = P[A^*] - P_0[A^*] = (P(A_1) - P_0(A_1), \ldots, P(A_{M-1}) - P_0(A_{M-1})).$$

The simple sequence of alternative hypothesis considered for this test is

$$(H_1) : \Theta = h/\sqrt{n}$$

for some $h \in \mathbb{R}^{M-1}$. The chi-square test is based on the behavior of the random vector $\sqrt{n}(P_n[A^*] - P_0[A^*])$ which converges weakly under $(H_0)$ and $(H_1)$ respectively to the multivariate normal law $\mathcal{N}(0, \Sigma)$ and $\mathcal{N}(h, \Sigma)$ where

$$\Sigma = \text{Diag}(P_0[A^*]) - P_0[A^*] \cdot P_0[A^*].$$  (4)

According to Sherman–Morrison formula, $\Sigma$ is invertible and

$$\Sigma^{-1} = \text{Diag} \left( \frac{1}{P_0(A_1)}, \ldots, \frac{1}{P_0(A_{M-1})} \right) + \frac{1}{P_0(A_M)} \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}.$$

The statistic for the classic chi-square of goodness-of-fit test is given by

$$\chi_n^2 = n \sum_{i=1}^{M} \frac{(P_n(A_i) - P_0(A_i))^2}{P_0(A_i)}$$

$$= n \sum_{i=1}^{M-1} \frac{(P_n(A_i) - P_0(A_i))^2}{P_0(A_i)} + \frac{n \left( \sum_{i=1}^{M-1} P_n(A_i) - P_0(A_i) \right)^2}{P_0(A_M)}$$

$$= n(P_n[A^*] - P_0[A^*]) \cdot \Sigma^{-1} \cdot (P_n[A^*] - P_0[A^*])^t$$

$$= Z_n^t \cdot Z_n.$$

$Z_n = \sqrt{n}(P_n[A^*] - P_0[A^*]) \cdot \Sigma^{-1/2}.$

Under $(H_0)$, $Z_n$ converges weakly to $\mathcal{N}(0_{M-1}, \text{Id}_{M-1})$ and the statistic $\chi_n^2$ converges to $\chi^2(M-1)$, a chi-square distribution with $M-1$ degrees of freedom. This proof is inspired by the first proof proposed in [3]. Other proofs of the convergence of the chi-squared statistics under $(H_0)$ can be found in the last cited paper. Under $(H_1)$, $Z_n$ converges to $\mathcal{N}(\text{M}, \text{Id}_{M-1})$ with $\text{M} = h \cdot \Sigma^{-1/2}$ which leads to say that the statistic $\chi_n^2$ converges to a non-central chi-square distribution $\chi^2(M-1; \lambda)$ – see for example [12] – with $M-1$ degrees of freedom and a non-centrality parameter

$$\lambda = \text{M} \cdot \text{M}^t = h \cdot \Sigma^{-1} \cdot h^t.$$
Auxiliary information

Suppose that an auxiliary information is available and that our aim is to take into account this information to improve the chi-square test. Here, the auxiliary information is defined as an estimator \( \hat{P}^{(N)}_n [A^*] \) of \( P[A^*] \), given by \( \hat{P}_n[A^*] \), with lower variance than the natural and empirical estimator \( P_n[A^*] \). Formally, the random vector \( X_n^{\lambda} [A^*] = (P_n(A_1), \ldots, P_n(A_{M-1})) \in \mathbb{R}^{M-1} \) is supposed to satisfy

\[
\sqrt{n}(P_n[A^*] - P[A^*]) \overset{d}{\to} N(0, \Sigma^\lambda),
\]

where \( \Sigma^\lambda \) is a \( M \times M \) invertible covariance matrix such that \( \Sigma - \Sigma^\lambda \) is semi-definite positive. \( \cdotp \) \( \cdotp \) \( \cdotp \)

Condition 5 is what is called auxiliary information in this paper in the case of the chi-square test and this property will be essential for the next main result. Some examples of auxiliary information and matrices \( \Sigma^\lambda \) for this test which satisfy this hypothesis are given in Section 4. Consistent estimator \( \hat{\Sigma}^\lambda_n \) of \( \Sigma^\lambda \), like its empirical estimator, is considered. The chi-square statistic with auxiliary information is defined by

\[
\chi_n^\lambda = n(P_n[A^*] - P_0[A^*]) \cdot (\hat{\Sigma}^\lambda_n)^{-1} \cdot (P_n[A^*] - P_0[A^*])^t = Z_n \cdot (Z_n)^t,
\]

\[
Z_n = \sqrt{n}(P_n[A^*] - P_0[A^*]) \cdot (\hat{\Sigma}^\lambda_n)^{-1/2}.
\]

Define the chi-square statistics with auxiliary information by matricially multiplying \( \sqrt{n}(P_n[A^*] - P_0[A^*]) \) by \( (\hat{\Sigma}^\lambda_n)^{-1/2} \) for the definition of \( Z_n^\lambda \) is motivated by the fact that \( Z_n^\lambda \) and therefore \( \chi_n^\lambda \) follow the same law as \( Z_n \) and \( \chi_n^2 \) under \( (H_0) \). The statistical test based on the rejection region \( \chi_n^\lambda > t \), for some \( t > 0 \), has the same alpha risk than the classical chi-square test based on the decision \( \chi_n^2 > t \). Under \( (H_1) \), the random vector \( Z_n^\lambda \) converges weakly to \( \mathcal{N}(M^\lambda, \text{Id}_{M-1}) \) where \( M^\lambda = h \cdot (\Sigma^\lambda)^{-1/2} \) while \( \chi_n^{\lambda^2} \) converges to the non-central chi-square distribution \( \chi^2(M-1; \lambda^\lambda) \) with \( M-1 \) degrees of freedom and the non-centrality parameter

\[
\lambda^\lambda = M^\lambda \cdot (M^\lambda)^t = h \cdot (\Sigma^\lambda)^{-1} \cdot h^t.
\]

If condition (5) is satisfied then \( \lambda \leq \lambda^\lambda \). The next paragraph shows that the efficiency of the chi-square test is increased when the auxiliary information is used.

Result

The main result concerning the efficiency of the chi-square test with auxiliary information is given by the following proposition.

**Proposition 2.** The Pitman’s ARE of the classical chi-square test with respect to the new chi-square test which takes into account the auxiliary information is given by

\[
e_P = \frac{h \cdot \Sigma^{-1} \cdot h^t}{h \cdot (\Sigma^\lambda)^{-1} \cdot h^t}.
\]

This efficiency is bounded by

\[
e_P \leq \lambda_{\text{max}}((\Sigma^\lambda)^{-1} \Sigma) \leq 1,
\]

where \( \lambda_{\text{max}}(\cdot) \) is the largest eigenvalue of a matrix.

This proposition bounds to the Pitman’s ARE of the new test and suggests that this one is more efficient than the classical chi-square test.

**Proof.** The minimal sample size needed to attain the level \( \alpha \) and the power \( 1 - \beta \) for the classical chi-square test and the new chi-square test with auxiliary information are respectively denoted \( n_1(\alpha, \beta; \Theta) \) and \( n_2(\alpha, \beta; \Theta) \). The power of the tests without and with auxiliary information are respectively given by \( \pi_n, \pi_n^\lambda \) where

\[
\pi_n = \mathbb{P}(Z_n \cdot Z_n^\lambda > Q_{M-1}(\alpha)|H_1),
\]

\[
\pi_n^\lambda = \mathbb{P}(Z_n^\lambda \cdot (Z_n^\lambda)^t > Q_{M-1}(\alpha)|H_1),
\]

where \( Q_{M}(t) \) is the \( t \)-quantile of the \( \chi^2 \) distribution, that is

\[
Q_{M}(t) = \inf\{x : t \leq \mathbb{P}(X \leq x)\},
\]

for a chi-square variable \( X \sim \chi^2(M-1) \). These powers can be approximated thanks to the non-central approximation:

\[
\pi_n = 1 - F_{M-1}(Q_{M-1}(\alpha)) + o(1)
\]

\[
= Q_{(M-1)/2} \left( \sqrt{\lambda_\lambda}, \sqrt{Q_{M-1}(\alpha)} \right) + o(1),
\]

\[
\pi_n^\lambda = 1 - F_{M-1}^\lambda(Q_{M-1}(\alpha)) + o(1)
\]

\[
= Q_{(M-1)/2} \left( \sqrt{\lambda^\lambda}, \sqrt{Q_{M-1}(\alpha)} \right) + o(1),
\]

where \( F_{M-1}, F_{M-1}^\lambda \) are respectively the distribution functions of the non-central chi-square distribution \( \chi^2(M-1; \lambda_\lambda), \chi^2(M-1; \lambda^\lambda) \) with \( \lambda_\lambda = n\Theta \cdot \Sigma^{-1} \cdot \Theta^t, \lambda^\lambda = n\Theta \cdot (\Sigma^\lambda)^{-1} \cdot \Theta^t \), \( Q_{(M-1)/2} \) is the Marcus-Q-function and \( o(1) \) are sequences vanishing when \( n \to +\infty \). Sequence of powers \( \pi_n, \pi_n^\lambda \) satisfy \( \pi_n \to 1 - \beta \) and \( \pi_n^\lambda \to 1 - \beta \) if and only if \( \lambda_n \to G(1 - \beta) \) and \( \lambda^\lambda \to G(1 - \beta) \) when \( \Theta \to \Theta_{M-1} \) and \( G \) denoting the reciprocal of the application

\[
x \mapsto Q_{(M-1)/2} \left( \sqrt{x}, \sqrt{Q_{M-1}(\alpha)} \right).
\]
This statement implies that
\[
\lim_{\Theta \to 0} \frac{n_2(\alpha, \beta, \Theta)}{n_1(\alpha, \beta, \Theta)} \times \frac{\Theta \cdot (\Sigma^\top)^{-1} \cdot \Theta^t}{\Theta \cdot \Sigma^{-1} \cdot \Theta^t} = 1,
\]
and consequently,
\[
e_P = \lim_{\Theta \to 0} \frac{n_2(\alpha, \beta, \Theta)}{n_1(\alpha, \beta, \Theta)} \times \frac{\Theta \cdot (\Sigma^\top)^{-1} \cdot \Theta^t}{\Theta \cdot \Sigma^{-1} \cdot \Theta^t} = \frac{h \cdot (\Sigma^\top)^{-1} \cdot h^t}{h \cdot (\Sigma^\top)^{-1} \cdot h^t}.
\]
Since
\[
\frac{h \cdot (\Sigma^\top)^{-1} \cdot h^t}{h \cdot (\Sigma^\top)^{-1} \cdot h^t} = \frac{h \cdot (\Sigma^\top)^{-1/2} \Sigma (\Sigma^\top)^{-1/2} \cdot h^t}{h \cdot h^t}.
\]
then by Rayleigh-Ritz theorem,
\[
e_P \leq \max_{x \neq \theta} \frac{x \cdot ([\Sigma^\top]^{-1/2} \Sigma (\Sigma^\top)^{-1/2}) \cdot x^t}{x \cdot x^t} \leq \lambda_{\text{max}}((\Sigma^\top)^{-1/2} \Sigma (\Sigma^\top)^{-1/2}) = \lambda_{\text{max}}((\Sigma^\top)^{-1} \Sigma).
\]
Condition 5 implies that \(\lambda_{\text{max}}((\Sigma^\top)^{-1} \Sigma) \leq 1\). \(\square\)
Notice that if \(M = 2\) then \(e_P = \Sigma^\top / \Sigma\).

4  Statistical examples

Presentation

This section describes and justifies two methods, the Raking-Ratio method and the general auxiliary information, allowing to obtain an asymptotic reduction of variance and therefore an auxiliary information as defined in this paper. Other methods can also be used to obtain a variance reduction. For example, from a sufficient statistic \(S\) the Rao-Blackwell theorem allows from an estimator \(\hat{\theta}\) to construct a more precise estimator \(\mathbb{E}[\hat{\theta}|S]\) of \(\theta\). This method is not detailed but suggests to use as auxiliary information \(X^N\) = \(\mathbb{E}[(X_n|S)]\) for the Z-test and \(P_n[A^*] = P_n[A^*|S] = (P_n(A_1|S), \ldots, P_n(A_{M-1}|S))\) for the chi-square test with a sufficient statistic \(S\). Subsection 4.1 deals with the Raking-Ratio method, that is a method which takes into account an auxiliary information given by the probabilities of sets of given partitions. Subsection 4.2 deals with the general auxiliary information. Examples and numerical simulations are given in these two subsections.
the same weak convergence. The two following paragraphs apply these theoretical results in the case of the improved Z-test and chi-square test.

**Raking-Ratio for the Z-test**

**General case.** Result of Section 2 can be applied if an empirical estimator of \(E[X]\) satisfying condition 2 is known. The Raking-Ratio method gives a better estimator by exploiting iteratively the auxiliary information given by the knowledge of all \(P[A^{(N)}]\). In our case, the raked empirical mean is given by

\[
\overline{X}_n^{(N)} = \sum_{i=1}^{n} q_{n,i}^{(N)} X_i,
\]

where \(q_{n,i}^{(N)}\) is the weight of \(X_i\) for the \(N\)-th iteration of the Raking-Ratio method, that is, for all 1 \(\leq i \leq n\), \(q_{0,i}^{(N)} = 1/n\) and for \(N \in \mathbb{N}\),

\[
q_{n,i}^{(N+1)} = q_{n,i}^{(N)} \left( \sum_{j=1}^{M} P(A_j^{(N+1)}|A_j^{(N+1)})(X_j) \right).
\]

The factor in bracket can be interpreted as corrections which operates the auxiliary information. An example of calculation of the raked empirical mean is given at the appendix A of [1]. The asymptotic variance of \(\sqrt{n}X_n^{(N)}\) is denoted by \((\sigma^{(N)})^2\). Albertus and Berthet proved that condition 2 is satisfied by taking \(X^\triangledown = \overline{X}_n^{(N)}\) for some fixed \(N \in \mathbb{N}\), since they established that

\[
(\sigma^{(N)})^2 = \sigma^2 - \sum_{k=1}^{N} (\Phi_k^{(N)})^t \cdot C_k \cdot \Phi_k^{(N)},
\]

where \(C_k \in \mathcal{M}_{m_k m_k}\) and \(\Phi_k^{(N)} \in \mathcal{M}_{m_k 1}\) are the matrix and the vector defined respectively by

\[
C_k = \text{Diag}(P[A^{(k)}]) - P[A^{(k)}]^t \cdot P[A^{(k)}],
\]

\[
\Phi_k^{(N)} = \sum_{1 \leq L \leq N-k} (-1)^L P_{A^{(i_1)}|A^{(k)}} P_{A^{(i_2)}|A^{(k)}} \cdots P_{A^{(i_{L-1})}|A^{(k)}} \left( \begin{array}{c} E[X|A_i^{(1)}] \\ \vdots \\ E[X|A_i^{(L-1)}] \end{array} \right) + \left( \begin{array}{c} E[X|A_i^{(L)}] \\ \vdots \\ E[X|A_i^{(N-k)}] \end{array} \right),
\]

and \(P_{A^{(i)}|A^{(j)}} \in \mathcal{M}_{m_j m_i}\) are stochastic matrices defined for all \(i, j \in \mathbb{N}^*\) by

\[
(P_{A^{(i)}|A^{(j)}})_{k,l} = P[A_i^{(j)}|A_k^{(i)}],
\]

for all 1 \(\leq i \leq m_i\) and 1 \(\leq k \leq m_j\). Since \(C_k\) are covariance matrices, and in particular semi-definite positive matrices, then \(\sigma^{(N)} \leq \sigma\) for all \(N \in \mathbb{N}\). Notice that these last matrices depend only on the auxiliary information given by all \(P[A^{(N)}]\).

**Simple case.** In [1] the author gave, in a general way, some examples of possible and explicit values of \(\sigma^{(N)}\) for \(N = 1, 2\) when the simple case \(A^{(1)} = \{A, A^C\}, A^{(2)} = \{B, B^C\}\) is considered. In our case these values are

\[
(\sigma^{(1)})^2 = \sigma^2 - \frac{p_A^2}{p_{\overline{A}}} \Delta_A^2,
\]

\[
(\sigma^{(2)})^2 = \sigma^2 - \frac{p_B^2}{p_{\overline{B}}} \Delta_B^2 - K \Delta_A^2,
\]

where \(\Delta_A = E[X|A] - E[X]\) and \(\Delta_B = E[X|B] - E[X]\) and

\[
p_A = P(A), \quad p_{\overline{A}} = P(A^C),
\]

\[
p_B = P(B), \quad p_{\overline{B}} = P(B^C), \quad p_{A \cap B} = P(A \cap B),
\]

\[
K = p_{AP_{\overline{A}}} + p_{BP_{\overline{B}}}(p_{A \cap B} - p_{APB}).
\]

If \(A^{(2k-1)} = A^{(1)}\) and \(A^{(2k)} = A^{(2)}\) for \(k > 1\) then the value \(\sigma^{(\infty)}\), the standard deviation of \(\overline{X}_n^{(N)}\) when \(N\) goes to infinity, that is when the Raking-Ratio method converges, is given by the following formula – see (2.12) of [1]:

\[
(\sigma^{(\infty)})^2 = \sigma^2 - \frac{Kp_{APB}}{p_{APB}p_{\overline{A}}p_{\overline{B}}} - \frac{(p_{A} - p_{APB})^2}{p_{APB}},
\]

\[
K = p_A \Delta_A^2 + p_B \Delta_B^2 - p_{APB}(\Delta_A - \Delta_B)^2 - 2p_{APB} \Delta_A \Delta_B.
\]

When the events \(A\) and \(B\) are independent then

\[
(\sigma^{(\infty)})^2 = (\sigma^{(2)})^2 = (\sigma^{(1)})^2 - \frac{p_B^2}{p_{\overline{B}}} \Delta_B^2.
\]

In the independent case, since \(\sigma^{(\infty)} = \sigma^{(2)}\), the Raking-Ratio method could be stopped with \(N = 2\) steps.

**Numerical simulation.** The previous results are applied with \(X\) following the distribution given by Figure 1 and the following independent sets

\[
A = \{X \in [-0.5, 0] \cup [0.5, 1]\}, \quad B = \{X \leq 0\}
\]

which satisfy

\[
p_A = p_{\overline{A}} = p_B = p_{\overline{B}} = 0.5, \quad E[X|A] = 1/6, \quad E[X|B] = -0.5.
\]
With these sets, the empirical estimator with auxiliary information is given by
\[ X^{(N)}_n = \sum_{i=1}^{n} q^{(N)}_{n,i} X_i \]
where \( q^{(0)}_{n,i} = 1/n \) and for \( N \in \mathbb{N} \),
\[
q^{(2N+1)}_{n,i} = \frac{q^{(2N)}_{n,i}}{2} \times \frac{1_A(X_i)}{\sum_{k=1}^{n} q^{(2N)}_{n,k} 1_A(X_k)} + \frac{q^{(2N)}_{n,i}}{2} \times \frac{1_B(X_i)}{\sum_{k=1}^{n} q^{(2N)}_{n,k} 1_B(X_k)} + \frac{q^{(2(N+1))}}{2} \times \frac{1_{\overline{A}}(X_i)}{\sum_{k=1}^{n} q^{(2(N+1))}_{n,k} 1_{\overline{A}}(X_k)}.
\]

For example for \( N = 1 \),
\[
\overline{X}^{(1)}_{n} = \frac{1}{2} \left( \frac{\sum_{i=1}^{n} X_i 1_{X_i \in A}}{\sum_{i=1}^{n} 1_{X_i \in A}} + \frac{\sum_{i=1}^{n} X_i 1_{X_i \in \overline{A}}}{\sum_{i=1}^{n} 1_{X_i \in \overline{A}}} \right).
\]
The asymptotic variances of \( X^{(N)}_n \) for \( N = 1 \) and \( N = 2 \) or \( N = \infty \) are equal to
\[
(\sigma^{(1)})^2 = \sigma^2 - 1/36 = 19/72 \approx 0.264,
\]
\[
(\sigma^{(\infty)})^2 = (\sigma^{(2)})^2 = \sigma^2 - 1/4 = 1/24 \approx 0.042.
\]

Figure 2 represents the distribution of \( \sqrt{n} \overline{X}_n \) and \( \sqrt{n} X^{(N)}_n \) for \( N = 1, 2 \) which are close to \( \mathcal{N}(0, \sigma^2) \) and \( \mathcal{N}(0, (\sigma^{(N)})^2) \). The decrease in variance is particularly visible for \( N = 2 \) or \( N = \infty \).

According to Proposition 1 the Pitman’s ARE for \( N = 1 \) is equal to
\[
e^{(1)}_P = (\sigma^{(1)}/\sigma)^2 = 19/21 \approx 0.905.
\]

For \( N = 2 \) or \( N = \infty \),
\[
e^{(2)}_P = (\sigma^{(2)}/\sigma)^2 = 1/7 \approx 0.143
\]
\[
e^{(\infty)}_P = (\sigma^{(\infty)}/\sigma)^2 = e^{(2)}_P.
\]

Figure 3 represents the distribution of \( Z_n \) and \( \overline{Z}_n \) when \( \theta = 0.5/\sqrt{n} \) and \( n = 100 \). This figure illustrates that \( Z_n, \overline{Z}^{(1)}_n, \overline{Z}^{(2)}_n \) are asymptotically close to respectively \( \mathcal{N}(-\sqrt{6}/7, 1), \mathcal{N}(-\sqrt{18}/19, 1) \) and \( \mathcal{N}(\sqrt{6}/19, 1) \). The case \( N = 2 \) or \( N = \infty \) is the most interesting because what makes the tests with auxiliary information more effective is highlighted, that is to say that the expectation of \( Z^{\overline{N}}_n \) takes expected values greater than that of \( Z_n \).

Figure 3: Distribution of \( Z_n \) and \( Z^{\overline{N}}_n \) for \( N = 1, 2 \) and \( n = 100 \) under \( (H_1) \)

Raking-Ratio for the chi-square tests

General case. To use the result of Section 3 an estimator of \( P[\mathcal{A}^*] \) more efficient than the empirical estimator \( P_n[\mathcal{A}^*] \) in the sense given by condition 5 must be known to the statistician. The Raking-Ratio algorithm gives again a better estimator with the auxiliary information by using the knowledge of all \( P[\mathcal{A}^{(N)}] \) for \( N \geq 1 \). The raked estimator of \( P[\mathcal{A}^*] \) is
\[
P^{(N)}_n[\mathcal{A}^*] = \left( \frac{P^{(N)}_n(A_1), \ldots, P^{(N)}_n(A_{M-1})}{P^{(N)}_n(A)} \right),
\]
where \( P^{(N)}_n(A) \) is iteratively defined, for a measurable set \( A \), by \( P^{(0)}_n(A) = P_n(A) \) and for all \( N \in \mathbb{N} \),
\[
P^{(N+1)}_n(A) = \sum_{j=1}^{m_n+1} \frac{P^{(N)}_n(A \cap A^{(N+1)}_j)P(A^{(N+1)}_j)}{P^{(N)}_n(A^{(N+1)}_j)}.
\]
Results of Albertus and Berthet imply in particular that the process

$$a_n^*(A^*) = \sqrt{n} \left( \mathbb{P}^n(A^*) - P(A^*) \right),$$

converges to the singular multivariate normal distribution $N(0, \Sigma(N))$ with $\Sigma(N)$ is the covariance matrix defined by

$$\Sigma(N) = \Sigma - \sum_{k=1}^{N} (\Phi_k(N))^t \cdot C_k \cdot \Phi_k(N)$$  \hspace{1cm} (9)

where $\Sigma$ is defined by (4). $C_k \in \mathcal{M}_{mk, m_k}$ are the same covariance matrices defined above by (6) and vectors $\Phi_k(N) \in \mathcal{M}_{mk, M-1}$ are the matrices whose $i$th column is given for all by

$$(\Phi_k(N))_{i} = \begin{pmatrix} P(A_1|A_1^{(k)}) \\ \vdots \\ P(A_n|A_n^{(k)}) \end{pmatrix} + \sum_{1 \leq l \leq N-k} \sum_{k<l_1<l_2<\cdots<l_l \leq N} (-1)^l P_{\mathcal{A}^{(l)}|A^{(i)}} P_{\mathcal{A}^{(l)}|A^{(i)}} \begin{pmatrix} P(A_1|A_1^{(l)}) \\ \vdots \\ P(A_n|A_n^{(l)}) \end{pmatrix},$$

$$(\Phi_N(N))_{i} = \begin{pmatrix} P(A_1|A_1^{(N)}) \\ \vdots \\ P(A_n|A_n^{(N)}) \end{pmatrix}$$

for all $i = 1, \ldots, M - 1$, $N \geq 1$ and $1 \leq k < N$ and $P_{\mathcal{A}^{(l)}|A^{(i)}} \in \mathcal{M}_{m_k, m_k}$ are stochastic matrices defined by (7). The matrix $\Sigma(N)$ ensures condition (5) so the results of Proposition 2 can be applied with $\mathbb{P}^n(A^*) = \mathbb{P}^n(A^*)$ for any $N \geq 1$.

**Simple case.** The simple case with the following parameters is studied. The auxiliary information is given by the knowledge of $P[A^{(1)}]$ and $P[A^{(2)}]$ with $A^{(1)} = \{A_1, A_1^C\}$, $A^{(2)} = \{A_2, A_2^C\}$. The categories for the chi-square test are given by $M = 2$ and $\mathcal{A} = \{A, A^C\}$. Matrices $\Sigma^{(1)}$ and $\Sigma^{(2)}$ require calculating $\Phi_1^{(1)}, \Phi_1^{(2)}$ and $\Phi_2^{(2)}$ which are in this case equal to

$$\Phi_1^{(1)} = \begin{pmatrix} P(A|A_1) \\ P(A|A_1^C) \end{pmatrix},$$

$$\Phi_2^{(2)} = \begin{pmatrix} P(A_2|A_2) \\ P(A_2^C|A_2) \end{pmatrix},$$

$$\Phi_1^{(2)} = \begin{pmatrix} P(A_2|A_1) \\ P(A_2^C|A_1) \end{pmatrix} - \begin{pmatrix} P(A_2|A_1^C) \\ P(A_2^C|A_1^C) \end{pmatrix} \cdot \begin{pmatrix} P(A) \\ P(A) \end{pmatrix}$$

**Numerical simulation.** Results are applied with $\mathcal{A} = \{A, A^C\}, \mathcal{A}^{(1)} = \{A_1, A_1^C\}, \mathcal{A}^{(2)} = \{A_2, A_2^C\}$ with

- $A = \{X \leq 0.5\}$,
- $A_1 = \{X \in [-0.5, 0] \cup [0.5, 1]\}$,
- $A_2 = \{X \leq 0\}$.

In particular $P[A^*] = P(A) = 3/4$ and event $A$ is dependent from the events $A^{(1)}, A^{(2)}$. With these values, matrices $\Sigma$ and $\Sigma(N)$ are real values and

$$\Sigma = P(A)(1 - P(A)) = 3/16,$$

$$\Phi_1^{(1)} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix},$$

$$\Phi_1^{(2)} = \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix},$$

$$\Phi_2^{(2)} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

$$C_1 = C_2 = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

According to (9),

$$\Sigma^{(1)} = \Sigma - \Phi_1^{(1)} \cdot C_1 \cdot \Phi_1^{(1)} = 1/8,$$

$$\Sigma^{(2)} = \Sigma - \Phi_1^{(2)} \cdot C_1 \cdot \Phi_1^{(2)} - \Phi_2^{(2)} \cdot C_2 \cdot \Phi_2^{(2)} = 1/16.$$

Condition (5) is met as expected for $N = 1, 2$. According to Proposition 2 with $\mathbb{P}^n(A^*) = \mathbb{P}^n(A^*) = \mathbb{P}^n(A)$, the Pitman’s ARE for $N = 1$ is equal to

$$e_p^{(1)} = \Sigma^{(1)} / \Sigma = 2/3,$$

and for $N = 2$

$$e_p^{(2)} = \Sigma^{(2)} / \Sigma = 1/3.$$

Figure 4 illustrates the asymptotic distribution of $\sqrt{n}(\mathbb{P}_n(A) - P(A))$ and $\sqrt{n}(\mathbb{P}_n^{(N)}(A) - P(A))$ for $N = 1, 2$. The cases $\mathbb{P}_n(A) = \mathbb{P}_n^{(N)}(A)$ with $N = 1, 2$ are

$\sqrt{n}(\mathbb{P}_n(A) - P(A)) = N(0, 0.06)$

$\sqrt{n}(\mathbb{P}_n^{(N)}(A) - P(A)) = N(0, 0.12)$

$\sqrt{n}(\mathbb{P}_n^{(N)}(A) - P(A)) = N(0, 0.17)$.
knowledge of probabilities

Tarasenko [9] defined the auxiliary information as the
iliary information can be. For example, Dmitriev and
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Many definitions of the auxiliary information could be

4.2 General auxiliary information

Literature

Many definitions of the auxiliary information could be
found among the literature concerning this subject.
Some authors gave their own definition of the auxiliary
information and they established some results in
order to prove the efficiency of the use of auxiliary
information. Nevertheless, most part of these definitions
do not include some natural examples of what auxi-
liary information can be. For example, Dmitriev and
Tarasenko [9] defined the auxiliary information as the
knowledge of probabilities \( P(g_1), \ldots, P(g_m) \) for some
measurable functions \( g_1, \ldots, g_m \) or the knowledge of an
approximation of these probabilities. They determined
the projection of the empirical measure which minimize
the Kullback-Leibler divergence over the set of the prob-
ability measures satisfying the information. For another
example, in [17] [18] [19], Zhang defined the auxiliary
information as a known function which the expectation
cancels, that is a measurable real-valued function \( g \) such
that \( \mathbb{E}[g(X)] = 0 \) and he established some results when
an information of this kind is known. However, these
definitions do not scope the concept of general auxil-
ary information presented in the next paragraph. For
instance, the knowledge of the variance of \( X \) can be
not supported. As a matter of fact, \( \text{Var}(X) = \mathbb{E}[h(X)] \)
with \( h(X) = (X - \mathbb{E}[X])^2 \) implies that \( \mathbb{E}[g(X)] = 0 \) for
\( g(X) = h(X) - \text{Var}(X) \) and \( g \) can be not computable if
\( \mathbb{E}[X] \) is unknown.

General auxiliary information

A very general definition is given by Tarima and Pavlov –
see [15] – where the auxiliary information is defined
as an unbiased estimator which satisfies a CLT. Their
result is general since the auxiliary information can be
given by several sources of auxiliary information and can
even be uncertain, that is an estimate of the true auxil-
liary information under certain conditions of asymptotic
normality. To illustrate the results of Sections 2 and 3
the two following paragraphs present the case when the
conditional mean of interest variables is known. More
formally, suppose that \( X \) is real random variable such
that its conditional expectation conditional on an event
is known. In this paragraph, let work on real random
variables \( X_1, \ldots, X_n \) such that the expectation of \( X \)
conditional on the event that \( X \) belongs to some pre-
defined set \( C \in T' \) is known. Thereby the auxiliary
information is given by \( \mathbb{E}[X|C] = \mathbb{E}[X1_C]/P(C) \). This
kind of auxiliary information is not supported by the
definitions of auxiliary information given by Dmitriev
and Tarasenko or Zhang recalled below.

Conditional mean auxiliary information for the

Z-test

General case. In this paragraph, the way the auxiliary
information \( \mathbb{E}[X|C] \) can be exploited in the case of the
Z-test is presented. The natural empirical estimator of
the auxiliary information \( \mathbb{E}[X|C] \) is denoted by

\[
\tilde{P}_n(X|C) = \frac{P_n(X1_C)}{P_n(C)}.
\]

(10)

The aim of this paragraph is to take into account this
auxiliary information to suggest an estimator \( \tilde{X}_n \)
of \( \mathbb{E}[X] \) with a lower variance than the natural empirical
estimator \( \overline{X}_n \). With this new estimator, condition 2
would be satisfied. To make the parallel with the arti-
cle of Tarima and Pavlov, the same notation than their
article is adopted. Suppose in this paragraph that one
have one exact auxiliary information but an uncertain
auxiliary information can be considered, given for exam-
ple by an estimate based on another larger independent
sample. In our case there is \( I = 1 \) data source, \( J_1 = 1 \)
auxiliary information and

\[ \Theta = E[X], \quad \hat{\Theta} = \mathcal{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \]
\[ \bar{B} = B = E[X|C], \quad \bar{\mathcal{B}} = \mathbb{P}_n(X|C). \]

With these values,

\[ K_{11} = \text{Var}(\hat{\Theta}) = \sigma^2 \]
\[ K_{12} = \text{Cov}(\bar{B}, \hat{\Theta}) = \text{Cov}(\mathbb{P}_n(X|C), \mathcal{X}_n) \]
\[ K_{22}' = \text{Var}(\bar{B}) = \text{Var}(\mathbb{P}_n(X|C)), \]
\[ K_{22}'' = \text{Var}(\bar{B}) = 0, \]
\[ K_{22} = K_{22}'' + K_{22}' = K_{22}' = \text{Var}(\mathbb{P}_n(X|C)). \]

The elements \( K_{11}, K_{12}, K_{22} \) are unknown or could not be expressed simply since the estimator \( \mathbb{P}_n(X|C) \) is a quotient of the empirical measure. Therefore the first suggested estimator of Tarima and Pavlov

\[ \hat{\Theta}^0 = \hat{\Theta} - K_{12} K_{22}^{-1} (\bar{B} - \bar{B}) = \mathcal{X}_n - K_{12} K_{22}^{-1} (\mathbb{P}_n(X|C) - E[X|C]), \]

which exploits the auxiliary information \( E[X|C] \) is uncomputable. This case is common and it is for that reason that these authors suggested to replace these unknown values by consistent estimators of them. Values \( \hat{K}_{11}, \hat{K}_{12}, \hat{K}_{22} \) can be estimated respectively by the following values

\[ \hat{K}_{11} = \hat{\sigma}^2, \]
\[ \hat{K}_{12} = \frac{1}{n} (\mathbb{P}_n(X^2|C) - \mathcal{X}_n E[X|C]), \]
\[ \hat{K}_{22} = \frac{1}{n \mathcal{P}_n(C)} (\mathbb{P}_n(X^2|C) - E[X|C]), \]

where \( \hat{\sigma}^2 \) is a consistent estimator of \( \sigma^2 \). With these consistent estimators, Tarima and Pavlov suggest to use the statistic

\[ \hat{\Theta}^* = \hat{\Theta} - \hat{K}_{12} \hat{K}_{22}^{-1} (\bar{B} - \bar{B}) = \mathcal{X}_n - \hat{K}_{12} \hat{K}_{22}^{-1} (\mathbb{P}_n(X|C) - E[X|C]). \]

By taking \( a_n = \sqrt{n} \) the first Tarima and Pavlov conditions mentioned in Section 1.3 of their paper are respected. More precisely, \( \zeta_n = 0, \Sigma_{22}' = 0 \) and

\[ \xi_n = a_n (\hat{\Theta} - \Theta) \xrightarrow{n \to +\infty} \xi = N(0, \Sigma_{11}), \]
\[ \tau_n = a_n (\bar{B} - B) \xrightarrow{n \to +\infty} \tau = N(0, \Sigma_{22}'), \]
\[ nK_{22}' = \text{Var}(a_n(X|C)) \xrightarrow{n \to +\infty} \Sigma_{22}', \]

with

\[ a_n(X|C) = \sqrt{n} (\mathbb{P}_n(X|C) - E[X|C]), \]
\[ \Sigma_{11} = K_{11} = \sigma^2, \]
\[ \Sigma_{22}' = \frac{\text{Var}(X|C)}{P(C)}, \]

where \( \text{Var}(X|C) \) is the variance of \( X \) conditional on the event \( C \) defined by

\[ \text{Var}(X|C) = E[X^2|C] - E[X|C]^2. \]  

According to Proposition 1 of Tarima and Pavlov,

\[ a_n \left( \hat{\Theta} - \Theta \right) \xrightarrow{n \to +\infty} N \left( 0, (\sigma^\forall)^2 \right), \]

where

\[ (\sigma^\forall)^2 = \Sigma_{11} - \Sigma_{22}'^2 = \sigma^2 - \frac{\text{Cov}^2(X1_C, X)}{P(C) \text{Var}(X|C)}, \]
\[ \Sigma_{12} = \frac{\text{Cov}(\xi, \tau)}{P(C)}, \]
\[ \Sigma_{22} = \frac{\text{Var}(X|C)}{P(C)}. \]

Conditions of Proposition 2 of Tarima and Pavlov are respected since the following asymptotic behaviour is satisfied:

\[ a_n^2 (\hat{K}_{12} - K_{12}) \xrightarrow{n \to +\infty} 0, \]
\[ a_n^2 (\hat{K}_{22} - K_{22}) \xrightarrow{n \to +\infty} 0. \]

By taking \( X_\forall^n = \hat{\Theta}^* \), Proposition 2 of Tarima and Pavlov imply that

\[ \sqrt{n} (X_\forall^n - E[X]) \xrightarrow{n \to +\infty} \mathcal{N} \left( 0, (\sigma^\forall)^2 \right). \]

According to Proposition 1 of this article, the Pitman’s ARE \( e_P \) is equal to

\[ e_P = (\sigma^\forall)^2 = 1 - \frac{\text{Cov}^2(X1_C, X)}{\sigma^2 P(C) \text{Var}(X|C)}. \]

**Numerical simulation.** If the previous results are applied with \( C = \{|X| \leq 0.5\} \), that is the auxiliary information is given by the knowledge of the value \( E[X| |X| \leq 0.5] \) which is zero in the case of the law given by Figure 4 then

\[ P(C) = 1/2, \]
\[ \text{Cov}(X1_C, X) = 1/16, \]
\[ \text{Var}(X|C) = 1/8, \]

which imply that \( (\sigma^\forall)^2 = 11/48 \approx 0.229 \) and \( e_P = 11/14 \approx 0.786 \). Figure 5 represents
the distribution of $\sqrt{n}X_n^i$, $\sqrt{n}X_n^{\Sigma}$ which are respectively close to $N(0, \sigma^2)$ and $N(0, (\sigma^\Sigma)^2)$.

$I = 1$, $J_1 = 1$ and

$$\Theta = P[A^*], \quad \hat{\Theta} = P_n[A^*],$$

$$\bar{B} = B = E[X|C], \quad \hat{B} = P_n(X|C),$$

where $P_n(X|C)$ is the conditional expectation given by (10). The notation with $\Theta$ is taken from the original paper and should not be confused with that of Section 3.

With these values, one have

$$K_{11} = \text{Var}(\hat{\Theta}) = \text{Var}(P_n[A^*]),$$

$$K_{12} = \text{Cov}(\hat{B}, \hat{\Theta}) = \text{Cov}(P_n(X|C), P_n[A^*]) = (\text{Cov}(P_n(X|C), P_n(A_i)))_{1 \leq i \leq M-1},$$

$$K'_{22} = \text{Var}(\hat{B}) = \text{Var}(P_n(X|C)),$$

$$K''_{22} = \text{Var}(\bar{B}) = 0,$$

$$K_{22} = K'_{22} + K''_{22} = K''_{22} = \text{Var}(P_n(X|C)).$$

Notice that values $K'_{22}, K''_{22}$ and $K_{22}$ does not change from the previous paragraph since the auxiliary information is exact and represents the same information.

The elements $K_{11}, K_{12}, K_{22}$ are still unknown or could not be expressed simply since the estimator $P_n(X|C)$ is a quotient of the empirical measure. Thus, the first suggested estimator of Tarima and Pavlov

$$\hat{\Theta}^0 = P_n[A^*] - K_{12}^{-1} K_{11}^{-1} (P_n(X|C) - E[X|C]),$$

which exploits the auxiliary information $E[X|C]$ is impossible to evaluate. Values $K_{11}, K_{12}, K_{22}$ can be estimated by the following consistent estimators $\hat{K}_{11} = \Sigma_{1,n}$ and

$$\hat{K}_{22} = \frac{1}{n} (P_n(X^2|C) - E[X|C]^2)$$

$$\hat{K}_{12} = \frac{1}{n} (P_n(X1_{A_i}|C) - E[X|C]P_n(A_i))_{1 \leq i \leq M-1}.$$

With these estimators, Tarima and Pavlov suggest to use the statistic

$$\hat{\Theta}^* = \hat{\Theta} - \hat{K}_{12}^{-1} \hat{K}_{11}^{-1} (\hat{B} - \bar{B})$$

$$= P_n[A^*] - \hat{K}_{12}^{-1} (P_n(X|C) - E[X|C]).$$

By taking $a_n = \sqrt{n}$ the first Tarima and Pavlov conditions mentioned in Section 1.3 of their paper are respected. More precisely, $\zeta_{1n} = 0$, $\Sigma'_{22} = 0$ and

$$\xi_n = a_n(\hat{\Theta} - \Theta) \xrightarrow{n \to +\infty} \xi = N(0, \Sigma_{11}),$$

$$\tau_n = a_n(\hat{B} - B) \xrightarrow{n \to +\infty} \tau = N(0, \Sigma'_{22}),$$

$$nK''_{22} = \text{Var}(a_n(X|C)) \xrightarrow{n \to +\infty} \Sigma'_{22},$$

Conditional mean auxiliary information for the chi-square test

**General case.** Same notations of the previous paragraph are repeated. Suppose again that one exact auxiliary information is known but an uncertain auxiliary information can again also be considered. In our case,
with $\Sigma_{22} = \text{Var}(X|C)/P(C)$ where $\text{Var}(X|C)$ defined by (1) and $\Sigma_{11} = \Sigma$ given by (4). By Proposition 1 of Tarima and Pavlov,

$$a_n(\Theta^0 - \Theta) \xrightarrow{n \to +\infty} N(0_{M-1}, \Sigma^\nabla),$$

where

$$\Sigma^\nabla = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^t,$$

$$\Sigma_{12} = \text{Cov}(\xi, \tau) = \frac{1}{P(C)}(\text{Cov}(X1_C, 1_A))_{1 \leq i \leq M-1},$$

$$\Sigma_{22} = \Sigma_{22}^\nabla = \frac{\text{Var}(X|C)}{P(C)}.$$

Conditions of Proposition 2 of Tarima and Pavlov are respected since the following asymptotic behaviour is satisfied according to the distribution of large number:

$$a_n^2(\hat{K}_{12} - K_{12}) = a_n^2(\hat{K}_{22} - K_{22}) - a_n^2(K_{12} - \Sigma_{12}) \xrightarrow{n \to +\infty} 0_{M-1},$$

$$a_n^2(\hat{K}_{22} - K_{22}) = a_n^2(\hat{K}_{22} - \Sigma_{22}) - a_n^2(K_{22} - \Sigma_{22}) \xrightarrow{n \to +\infty} 0.$$

By taking $\Sigma^\nabla[A^*] = \hat{\Theta}^*$, Proposition 2 of Tarima and Pavlov implies that

$$\sqrt{n}(\mathbb{P}_n[A^*] - P[A^*]) \xrightarrow{n \to +\infty} N(0_{M-1}, \Sigma^\nabla).$$

Matrix $\Sigma^\nabla$ satisfies (5) then Proposition 2 of this paper can be applied to the chi-square test which exploits the auxiliary information given by the knowledge of $\mathbb{E}[X|C]$.

**Numerical simulation.** The previous results are applied with $X$ distributed as Figure 4 and these following values: $C = \{|X| \leq 0.5\}$, $M = 2$, $A = \{A, A^c\}$ where $A = \{|X| \leq 0\}$ satisfies $P(A) = 1/2$. With these values, the auxiliary information is given by

$$\mathbb{E}[X|C] = \mathbb{E}[X| |X| \leq 0.5] = 0,$$

that is the statistician knows the mean of the interest random variable when this last one is between 0.5 and 0.5. In this case,

$$\Sigma = P(A)(1 - P(A)) = 1/4,$$

$$\Sigma_{12} = \frac{\text{Cov}(X1_C, 1_A)}{P(C)} = -1/6,$$

$$\Sigma_{22} = \frac{\text{Var}(X|C)}{P(C)} = 1/4,$$

$$\Sigma^\nabla = \Sigma - \frac{\Sigma_{12}^2}{\Sigma_{22}} = 5/36.$$

By Proposition 2, the Pitman’s ARE $e_P$ is

$$e_P = \Sigma^\nabla/\Sigma = 5/9.$$

Figure 8 represents the distribution of $\sqrt{n}(\mathbb{P}_n(A) - P(A))$ and $\sqrt{n}(\mathbb{P}_n^\nabla(A) - P(A))$ for large value of $n.$

Figure 9 represents the distribution function of $\chi_n^2$ and $\chi_n^\nabla^2$, for the hypothesis $(H_1)$ with $h = 0.5$ and $n = 100$, which are respectively close to $\chi^2(1; 1)$ and $\chi^2(1; 9/5)$.

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References

[1] Albertus, M. (2020). Raking-ratio empirical process with auxiliary information learning. *ESAIM: Probability and Statistics*.

[2] Albertus, M. and Berthet, P. (2019). Auxiliary information: The raking-ratio empirical process. *Electronic Journal of Statistics*, 13(1):120–165.

[3] Benhamou, E. and Melot, V. (2018). Seven Proofs of the Pearson Chi-Squared Independence Test and its Graphical Interpretation. *SSRN Electronic Journal*.

[4] Binder, D. A. and Théberge, A. (1988). Estimating the Variance of Raking-Ratio Estimators. *The Canadian Journal of Statistics / La Revue Canadienne de Statistique*, 16:47.

[5] Brackstone, G. J. and Rao, J. N. K. (1979). An investigation of Raking-Ratio estimators. *Sankhya: The Indian Journal of Statistics*, 41:97–114.

[6] Cochran, W. G. (1952). The $\chi^2$ Test of Goodness of Fit. *The Annals of Mathematical Statistics*, 23(3):315–345.

[7] Deming, W. E. and Stephan, F. F. (1940). On a Least Squares Adjustment of a Sampled Frequency Table When the Expected Marginal Totals are Known. *The Annals of Mathematical Statistics*, 11(4):427–444.

[8] Deville, J.-C. (2002). La correction de la non-réponse par calage généralisé. *Journées de Méthodologie Statistique, Paris. INSEE*.

[9] Dmitriev, Y. G. and Tarasenko, P. F. (1992). The use of a priori information in the statistical processing of experimental data. *Russian Physics Journal*, 35(9):888–893.

[10] Ireland, C. T. and Kullback, S. (1968). Contingency tables with given marginals. *Biometrika*, 55(1):179–188.

[11] Konijn, H. S. (1981). Biases, variances and covariances of raking ratio estimators for marginal and cell totals and averages of observed characteristics. *Metrika*, 28(1):109–121.

[12] Patnaik, P. B. (1949). The Non-Central $\chi^2$-and F-Distribution and their Applications. Technical Report 1.

[13] Sinkhorn, R. (1964). A relationship between arbitrary positive matrices and doubly stochastic matrices. *The Annals of Mathematical Statistics*.

[14] Stephan, F. F. (1942). An Iterative Method of Adjusting Sample Frequency Tables When Expected Marginal Totals are Known. *The Annals of Mathematical Statistics*, 13(2):166–178.

[15] Tarima, S. and Pavlov, D. (2006). Using auxiliary information in statistical function estimation. *ESAIM - Probability and Statistics*, 10:11–23.

[16] Vaart, A. W. v. d. (1998). *Asymptotic Statistics*. Cambridge university press.

[17] Zhang, B. (1995). M-estimation and quantile estimation in the presence of auxiliary information. *Journal of Statistical Planning and Inference*, 44(1):77–94.

[18] Zhang, B. (1997a). Estimating a distribution function in the presence of auxiliary information. *Metrika*, 46(3):221–244.

[19] Zhang, B. (1997b). Quantile processes in the presence of auxiliary information. *Annals of the Institute of Statistical Mathematics*, 49(1):35–55.