Active Brownian particles with velocity-alignment and active fluctuations

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New Journal of Physics 14 (2012) 073033 (23pp)
Received 29 April 2012
Published 13 July 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/7/073033

Abstract. We consider a model of active Brownian particles (ABPs) with velocity alignment in two spatial dimensions with passive and active fluctuations. Here, active fluctuations refers to purely non-equilibrium stochastic forces correlated with the heading of an individual active particle. In the simplest case studied here, they are assumed to be independent stochastic forces parallel (speed noise) and perpendicular (angular noise) to the velocity of the particle. On the other hand, passive fluctuations are defined by a noise vector independent of the direction of motion of a particle, and may account, for example, for thermal fluctuations. We derive a macroscopic description of the ABP gas with velocity-alignment interaction. Here, we start from the individual-based description in terms of stochastic differential equations (Langevin equations) and derive equations of motion for the coarse-grained kinetic variables (density, velocity and temperature) via a moment expansion of the corresponding probability density function. We focus here on the different impact of active and passive fluctuations on onset of collective motion and show how active fluctuations in the active Brownian dynamics can change the phase-transition behaviour of the system. In particular, we show that active angular fluctuations lead to an earlier breakdown of collective motion and to the emergence of a new bistable regime in the mean-field case.

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1. Introduction

The fascinating self-organization phenomenon of collective motion in biology, such as exhibited by flocks of birds or schools of fish, has attracted the attention of scientists from very different backgrounds over the last few decades. Whereas biologists focus on the evolutionary advantages and the biological function of moving in groups [1–3], physicists address rather the question about universal laws governing the collective dynamics either from the theoretical point of view [4–10] or, as done recently, by analysing certain model systems experimentally [11–14]. In general, physicists are inclined towards the analysis of simple, but generic, models of collective motion such as the minimal model proposed by Vicsek et al [4]. The Vicsek model can be considered as a non-equilibrium extension of the XY model, where self-propelled particles (SPPs) move with a finite speed in continuous space and at discrete times align their velocities with the average velocity of other particles sensed in their local neighbourhood in the presence of noise. The Vicsek model exhibits a non-equilibrium phase transition leading to the onset of long-range order at a sufficiently low noise and high particle density. Detailed studies of the Vicsek model have led to intensive debate on the nature of the phase transition (continuous versus discontinuous) [7–9, 15, 16].

One of the main characteristics of the Vicsek model, as well as of many other individual-based models of collective motion, is the assumption of a constant speed of individuals. In general, this does not hold for the motion of real biological agents [17]. Furthermore, recent experimental realizations of swarming in non-living systems are based on rectification of the fluctuation due to asymmetric friction and therefore imply a non-negligible role for speed fluctuations [11, 12].

One possibility to introduce speed fluctuations into the dynamics of active particles is the addition of random increments from a well-defined probability distribution with zero mean to the speed of an individual in numerical simulations [18, 19]. This approach is useful in analysing the robustness of the collective dynamics with respect to noise. However, in this kind of model the speed dynamics is always independent of the interactions between individuals.
One more different approach is based on the formulation of stochastic equations of motion for the positions $r_i$ and velocities $v_i$ of $N$ interacting individuals ($i = 1, \ldots, N$):

\[
\frac{dr_i}{dt} = v_i, \quad (1.1a)
\]
\[
\frac{dv_i}{dt} = F_i + \eta_i. \quad (1.1b)
\]

Here, $m_i$ is the mass of the focal individual, $F_i$ is an effective force vector governing the deterministic dynamics, whereas $\eta_i$ models an effective stochastic force, which accounts for the randomness in the motion of the individual. In general, both terms can depend on the position and velocities of all other individuals. In addition, the stochastic force $\eta_i$ must, by definition, explicitly depend on time, which is not necessarily the case for the deterministic force $F_i$.

From a physical point of view, the velocity equation (1.1b) can be considered as a Langevin equation with actual physical forces on the right-hand side. However, the full description of all physical forces determining the motion of an individual will be extremely complex and not very practical in most cases. Thus, from a more general point of view, we can consider $F_i$ and $\eta_i$ to be effective forces responsible for accelerations of the focal individual in response to external stimuli and/or due to internal decision processes. The functional form of $F_i$ and $\eta_i$ can be set using reasonable modelling assumptions based on qualitative empirical observations [1, 10] or, if available, directly on experimental measurements of interactions between individuals [20–22].

This class of variable speed models not only overcomes the problems mentioned above, but also allows a straightforward mathematical analysis of the dynamical behaviour due to the usage of stochastic differential equations. We name active particles with self-propulsion, where both the magnitude and the direction of the velocity are (stochastic) degrees of freedom, as active Brownian particles (ABPs). This allows clearly distinguishing them from the so-called SPPs, which usually refer to active particles moving with constant speed [4, 23]. Originally, ABP referred to Brownian agents in far-from-equilibrium conditions with an internal energy depot, which can take up energy from the environment and convert it into kinetic energy of motion [24–27]. However, as non-equilibrium conditions and local energy conversion are characteristic features of all active particle systems, we suggest the above general definition of ABP.

Recently, two of the authors considered non-interacting ABPs with so-called passive and active fluctuations [28]. Passive fluctuations are assumed to be independent of the direction of motion, and may be motivated, for example, by a fluctuating environment. This could be, e.g., thermal fluctuations, but in general also other non-equilibrium processes. Active fluctuations, on the other hand, are assumed to be correlated with the current direction of motion of the active particle. In the simplest case, they consist of independent stochastic processes parallel and perpendicular to the instantaneous direction of motion. Such fluctuations may be associated with fluctuations of the actual propulsion mechanism and are, as per definition, a pure far-from-equilibrium phenomenon.

The main focus of this work is on the analysis of the impact of different fluctuation types on collective dynamics of interacting ABP. Here, we consider a generic, effective alignment interaction between individual particles, which represents a simple extension of the Vicsek model to the variable speed case. In particular, we will derive and analyse kinetic equations of ABP with velocity alignment and both passive and active fluctuations. Please note that already in the context of the Vicsek model, it was argued that different noise implementations may change the nature of the phase transition [8, 16, 29].
2. Active Brownian particles (ABPs) with velocity alignment

2.1. Microscopic description

We describe the ABP gas by a system of coupled stochastic differential equations. The dynamics of the $i$th point-like particle ($i = 1, \ldots, N$) with mass $m_i = m$ is determined by the following set of Langevin equations interpreted in the sense of Stratonovich:

$$\frac{d r_i}{d t} = v_i,$$

$$m_i \frac{d v_i}{d t} = -\gamma(v_i)v_i + \tilde{\mu}(u_{A_i} - v_i) + \eta_i(t).$$

The self-propulsion of active particles is described by a velocity-dependent friction coefficient $\gamma(v_i)$, which can take both negative and positive values. Here, we use a piecewise linear function, the so-called Schienbein–Gruler friction [25, 30]:

$$-\gamma(v_i)v_i = -\alpha \left(1 - \frac{v_0}{|v_i|}\right) v_i = \alpha(v_0 - |v_i|)e_{i,v}.$$  (2.2)

The interaction of particles is considered to be a linear velocity-alignment mechanism with characteristic time $\tau_{\tilde{\mu}} = m\tilde{\mu}^{-1}$. In (2.1b), the local mean velocity is denoted by

$$u_{A_i} = \frac{1}{N_{A_i}} \sum_{j=1}^{N_{A_i}} v_j.$$  (2.3)

The focal particle $i$ aligns its velocity with the average velocity of all particles $N_{A_i}$ within a metric interaction region $A_i$ around its current position. In general, $A_i$ can have different geometries; however, a common choice in two spatial dimensions is a disc-like interaction zone with radius $\epsilon$. The velocity-alignment strength is denoted by $\tilde{\mu}$. The summation in (2.3) includes the $i$th particle itself so that in the case of no neighbours ($N_{A_i} = 1$) the alignment force vanishes, consistently. If the local mean velocity (2.3) is equal to zero, the alignment interaction reduces to Stokes friction $\dot{v}_i \propto -\tilde{\mu} v_i$ decelerating the particles (‘social friction’).

In the limiting case of constant speeds $|v_i| \approx v_0$, the alignment interaction is related to the Kuramoto model [32] describing the synchronization phenomena of coupled oscillators. In this case, the polar angle defining the direction of motion corresponds to the phase of the oscillator [33, 34]. Peruani et al derived a mean-field theory for SPPs with constant speed and Kuramoto interaction [35].
Figure 1. Visualization of passive (a) and active (b) fluctuations. The dashed line shows the trajectory of the $i$th particle. Speed noise changes the speed of the particle only, whereas angular noise acts on the direction of motion. On the other hand, the components of the velocity vector are changed by passive fluctuations in the same way.

The above-mentioned velocity-alignment interaction has already been used to describe collective dynamics, e.g., by Czirók et al [36] to describe the formation of complex bacterial colonies as well as by Niwa [37] to characterize the dynamics of fish schools based on the Langevin equations.

In (2.1b), $\eta_i(t)$ denotes a stochastic force, introduced in [28], contributing three terms to the velocity dynamics. The vector $e_{i,\varphi}$ points perpendicular to the direction of motion and has the unit length one.

$$\eta_i(t) = \sqrt{2d_E} \left( \xi_{i,x}(t) \right) + \sqrt{2d_v} \xi_{i,v}(t) e_{i,v}(t) + \sqrt{2d_\varphi} \xi_{i,\varphi}(t) e_{i,\varphi}(t). \quad (2.4)$$

The first expression on the right-hand side denotes the so-called passive or external noise acting in the same way on both components of the velocity vector $v_i$ with intensity $d_E$. It might be associated, for example, with ordinary Brownian fluctuations. Thus, in principle, it can model essentially an equilibrium phenomenon linked to thermal fluctuations. In the present context, however, it may also be of non-equilibrium origin. We refer to this type of noise as passive, as it does not ‘carry’ any information on the propulsion of a particle.

We refer to the remaining two terms together as active or internal noise. It is motivated by the assumption that these internal fluctuations are directly linked to fluctuations of the driving ‘motor’ or internal decision processes of the Brownian agent, i.e. animals. These active fluctuations are subdivided into two parts: speed noise, acting along the direction of motion with intensity $d_v$, and pure angular fluctuations, changing the direction of motion of the particle with intensity $d_\varphi$ without affecting the particle’s speed. In the case of constant speed it is not possible to distinguish between active and passive fluctuations as the angle defining the direction of motion is the only degree of freedom. Figure 1 illustrates the different effects of active and passive fluctuations on the particle velocities.

The random processes $\xi_{i,v}(t)$ are assumed to be Gaussian white noise with zero mean and vanishing correlations: $\langle \xi_{i,v}(t) \rangle = 0$, $\langle \xi_{i,v}(t) \xi_{j,v}(t') \rangle = \delta_{ij} \delta_{v\nu} \delta (t-t') \quad (i, j \in \{1, \ldots, N\}$,
ν, µ ∈ {x, y, v, ϕ}). Due to the non-equilibrium nature of the self-propelled dynamics and active fluctuations, the fluctuation–dissipation theorem does not hold here in general.

2.2. Natural units and dimensionless parameters

First of all, we identify the relevant dimensionless parameters and characteristic length and time scales in (2.1):

\[ L = \left[ \frac{mv_0}{\alpha} \right], \quad T = \left[ \frac{m}{\alpha} \right]. \quad (2.5) \]

Apparently, the velocity scale is defined by \( v_0 \). In this section, physical values in SI units are indicated by a prime. In order to reduce the number of parameters, the following transformation into dimensionless quantities \( r, v \) and \( t \) is used:

\[ r' = \frac{mv_0}{\alpha} r, \quad t' = \frac{m}{\alpha} t, \quad v' = v_0 v. \quad (2.6) \]

The Gaussian processes \( \xi(t') \) are transformed in the same manner:

\[ \xi(t') = \sqrt{\frac{\alpha}{m}} \xi(t). \quad (2.7) \]

Using these transformations, one can rewrite the microscopic Langevin dynamics (2.1) of the \( i \)th particle in dimensionless quantities as follows:

\[ \frac{dr_i}{dt} = v_i, \quad (2.8a) \]

\[ \frac{dv_i}{dt} = e_{i,v} - v_i + \mu \left( u_{Ai} - v_i \right) + \sqrt{2D_E \left( \xi_{i,x} \right)} + \sqrt{2D_v \xi_{i,v}} e_{i,v} + \sqrt{2D_\phi \xi_{i,\phi}} e_{i,\phi}. \quad (2.8b) \]

In (2.8), the dimensionless velocity-alignment strength \( \mu \) and noise intensities \( D_v \) were introduced (\( v \in \{E, v, \phi\} \)):

\[ \mu = \frac{\tilde{\mu}}{\alpha}, \quad D_v = \frac{d_v}{mav_0^2}. \quad (2.9) \]

Hence, the dynamical behaviour is governed by these four dimensionless parameters.

3. Hydrodynamic theory of ABPs

A hydrodynamic description based on symmetry properties of flocks was first proposed by Toner and Tu (TT) [5, 6]. They proposed a general continuum model for collective motion without considering a particular microscopic dynamics and determined the scaling exponents for density and velocity fluctuations. Hydrodynamic equations for SPP with aligning collisions were derived by Bertin et al [38, 39] using a Boltzmann approach, which allows one to link microscopic model parameters to the parameters of a coarse-grained theory of TT type. Bertin et al assumed a small collective velocity, which implies the validity of the theory only close to the transition point. Recently, Ihle [40] derived a kinematic description of the Vicsek model accounting also for multiparticle interactions. Related hydrodynamic descriptions of self-propelled rods were also studied by Marchetti and co-workers [41, 42].
In order to predict the macroscopic behaviour of the ABP gas, a hydrodynamic description is derived directly from the microscopic Langevin dynamics in the following. We use a moment expansion of the corresponding probability density function (see also appendix A for further details). The approach used in this work goes back to Riethmüller et al [43], who studied gravity-driven granular flow. It was successfully used to analyse the mean-field dynamics of ABP with velocity alignment in one dimension [44] and in two dimensions with passive fluctuations only [45].

Let \( p(\mathbf{r}_j, v_j, t) \) \( d^2r_j \) \( d^2v_j \) be the probability to find the \( j \)th particle at time \( t \) in the velocity interval \( [v_j, v_j+dv_j] \) and within the spatial region \( [\mathbf{r}_j, \mathbf{r}_j+dr_j] \). The dynamics of the one-particle probability density function \( p(\mathbf{r}_j, v_j, t) \) can easily be derived from the dynamics of the joint probability density function \( P = P(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, v_1, v_2, \ldots, v_N, t) \) by integrating over all particle positions and velocities except the \( j \)th. Due to the fact that any \( j \) could be chosen, the particle index is omitted henceforward. The dynamics of the single-particle probability density function \( p(\mathbf{r}, v, t) \) reads

\[
\frac{\partial p}{\partial t} = -\nabla_r (w p) - \nabla_v \left[ \left( \frac{v}{|v|} - v \right) p + \mu (u_A - v) p - D_\psi \left( \frac{v}{|v|^2} \right) p \right] + \frac{\partial^2}{\partial v_x^2} \left( \frac{D_x v_x^2 + D_v v_y^2}{|v|^2} p \right) + \frac{\partial^2}{\partial v_y^2} \left( \frac{D_x v_x^2 + D_v v_y^2}{|v|^2} p \right) + \frac{\partial}{\partial v_x} \left[ \frac{2 v_x v_y}{|v|^2} (D_v - D_\psi) p \right] + \Delta_v (D_E p). \tag{3.1}
\]

Moments of the velocity components are defined by

\[
M^{(mn)} = \langle v_x^m v_y^n \rangle = \frac{1}{\rho(\mathbf{r}, t)} \int d^2v \ v_x^m v_y^n p(\mathbf{r}, v, t). \tag{3.2}
\]

The spatial density of particles corresponds directly to \( M^{(00)} \):

\[
\rho(\mathbf{r}, t) = \int d^2v \ p(\mathbf{r}, v, t). \tag{3.3}
\]

Multiplying (3.2) with the density and taking the temporal derivative yields

\[
\frac{\partial}{\partial t} (\rho M^{(mn)}) = \int d^2v \ v_x^m v_y^n \frac{\partial p}{\partial t}. \tag{3.4}
\]

We may rewrite the integral on the right-hand side of (3.4) by inserting the Fokker–Planck equation (3.1) and using the definition (3.2) to obtain a general equation of motion for the \( M^{(mn)} \) moment of the probability density function:

\[
\frac{\partial}{\partial t} (\rho \{v_x^m v_y^n\}) = -\frac{\partial}{\partial x} \left( \rho \{v_x^{m+1} v_y^n\} \right) - \frac{\partial}{\partial y} \left( \rho \{v_x^m v_y^{n+1}\} \right) + (m+n) \rho \left[ \left( \frac{v_x^m v_y^n}{|v|} \right) - \left( v_x^m v_y^n \right) - D_\psi \left( \frac{v_x^m v_y^n}{|v|^2} \right) \right] + \mu \rho \left[ m u_{A,x} \{v_x^{m-1} v_y^n\} + n u_{A,y} \{v_x^m v_y^{n-1}\} - (m+n) \{v_x^m v_y^n\} \right]
\]
Here, we use the approximation

\[ + m(m - 1) \rho \left[ D_v \left( \frac{v_x^{m-2} v_y^{n+2}}{|v|^2} \right) + D_v \left( \frac{v_x^m v_y^n}{|v|^2} \right) + D_E \left( \frac{v_x^{m-2} v_y^n}{|v|^2} \right) \right] \]

\[ + n(n - 1) \rho \left[ D_v \left( \frac{v_x^{m+2} v_y^{n-2}}{|v|^2} \right) + D_v \left( \frac{v_x^m v_y^n}{|v|^2} \right) + D_E \left( \frac{v_x^{m} v_y^{n-2}}{|v|^2} \right) \right] \]

\[ + 2mn \rho \left( \frac{v_x^m v_y^n}{|v|^2} \right) (D_v - D_v). \] (3.5)

Until now, no explicit approximations were made except the mean field assumption \((N_A \gg 1)\).

In order to derive hydrodynamic equations from (3.5), we rewrite the individual velocity \(v\) as a sum of a mean-field velocity \(u(r, t)\) at the position of the focal particle plus some deviation \(\delta v(r, t)\):

\[ v = u(r, t) + \delta v(r, t). \] (3.6)

We assume that the expectation value of \(\delta v_{k, r}^x(r, t)\) vanishes for odd \(k\):

\[ \langle \delta v_{k, r}^x \rangle = 0 \quad \text{for} \quad k = 1, 3, 5, \ldots \] (3.7)

This corresponds to the assumption of a symmetric \(p(r, v, t)\) with respect to \(u(r, t)\) in the velocity space. Furthermore, we assume that

\[ \langle (\delta v_x)^2 \delta v_y \rangle = \langle (\delta v_x)^2 \delta v_x \rangle = 0. \] (3.8)

In addition to the mean-field velocity, we introduce the covariance matrix as a measure of velocity fluctuations around the mean-field velocity:

\[ \hat{T} = \begin{pmatrix} \langle (\delta v_x)^2 \rangle & \langle \delta v_x \delta v_y \rangle \\ \langle \delta v_y \delta v_x \rangle & \langle (\delta v_y)^2 \rangle \end{pmatrix} = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix} = \hat{T}^T. \] (3.9)

In the following, \(\hat{T}\) denotes the ‘temperature tensor’.

In the lowest order \((m = n = 0)\), one obtains from (3.5) the continuity equation for the local density

\[ \frac{\partial \rho}{\partial t} + \nabla_r (\rho \cdot u) = 0, \] (3.10)

reflecting the conservation of the particle number \(N\).

To obtain evolution equations for the mean-field velocity and the temperature tensor, we need to evaluate the expectation values of fractional expressions of the form \(v_x^m v_y^n / |v|^k\) in (3.5). Here, we use the approximation

\[ \frac{v_x^m v_y^n}{|v|^k} \approx \frac{v_x^m v_y^n}{\sqrt{v_x^2 + v_y^2}} \approx \frac{v_x^m v_y^n}{V^k}, \] (3.11)

with \(V = V(u, \hat{T}) = \sqrt{|u|^2 + \text{tr}(\hat{T})}\). This approximation provides naturally a closure of the hierarchy of moment equations with \(m + n \leq 2\). The validity of the approximation decreases with increasing variance in the distribution of individual speeds (see appendix B). Thus, although we will derive the mean-field theory for the general case, we will focus in our
analysis on the case of small speed fluctuations (e.g. \( D_v \ll 1 \)), where the above approximation is expected to provide the best results.

From equation (3.4) for the first-order moments with \( m = 1, n = 0 \) and \( m = 0, n = 1 \) combined with (3.10), we can derive the evolution equation for the mean-field velocity field as

\[
\frac{\partial u}{\partial t} + (u \nabla) u = \left( \frac{1}{V} - 1 \right) u - \frac{D_v}{V^2} u + \mu (u_A - u) - \frac{1}{\rho} \left[ \nabla^T (\rho \hat{T}) \right]^T. \tag{3.12}
\]

The above equation (3.12) is a partial integro-differential equation since

\[
u_A(r, t) = A^{-1} \int_A d^2r' u(r', t). \tag{3.13}
\]

Csahők and Czirók [46] have suggested a hydrodynamic velocity equation for collective motion of bacteria with similar active propulsion and interaction terms as derived in equation (3.12). We should note that, in general, for a finite number of neighbours \( u_A \) will be a fluctuating quantity that is neglected here by assuming \( N_A \gg 1 \).

Finally, we can combine (3.10) and (3.12) together with the three second-order equations obtained from (3.5) for \( m + n = 2 \) \( \{(m, n) \in \{(2, 0), (0, 2), (1, 1)\} \)\). This yields the equation for the temperature tensor

\[
\frac{1}{2} \left[ \frac{\partial \hat{T}}{\partial t} + (u \nabla) \hat{T} \right] = \left( \frac{1}{V} - 1 \right) \hat{T} - \mu \hat{T} + D_E \hat{I} + \frac{D_v}{V^2} \hat{B}_v + \frac{D_\phi}{V^2} \hat{B}_\phi + \frac{1}{2} \left[ (\hat{T} \nabla) u^T + ((\hat{T} \nabla) u^2) \right]^T,
\]

with \( \hat{I} \) being the identity matrix and

\[
\hat{B}_v = (uu^T + \hat{T}) = \begin{pmatrix}
    u_x^2 + T_{xx} & u_x u_y + T_{xy} \\
    u_x u_y + T_{xy} & u_y^2 + T_{yy}
\end{pmatrix}, \tag{3.15a}
\]

\[
\hat{B}_\phi = \begin{pmatrix}
    u_y^2 + T_{yy} - T_{xx} & u_x u_y - 2T_{xy} \\
    u_x u_y - 2T_{xy} & u_x^2 + T_{xx} - T_{yy}
\end{pmatrix}. \tag{3.15b}
\]

From (3.12) and (3.14), one can already identify the general impact of the different fluctuation types: the active angular fluctuations \( D_\phi \) introduce an additional effective friction which affects both the velocity and temperature dynamics, whereas the active velocity fluctuations \( D_v \) and passive fluctuations \( D_E \) drive the temperature equation.

The kinetic theory contains the important limiting case of ordinary Brownian motion, which corresponds to \( \mu, D_\phi, D_v \) and \( v_0 \) equal to zero. In this special case, we recover the fluctuation–dissipation relation known from equilibrium statistical mechanics.

\section*{4. Spatially homogeneous, steady-state solutions}

Under the assumption of a spatially homogeneous system, we may drop all spatial derivatives in (3.10), (3.12) and (3.14). Furthermore, in the steady-state case, we may choose the coordinate system such that \( u = (|u|, 0) \) is parallel to the \( x \)-axis of the laboratory frame of reference. In this case, we assume that the average deviations parallel and perpendicular to the direction
of motion become independent \(\langle \delta v_x \delta v_y \rangle = \langle \delta v_x | \delta v_y \rangle = \langle \delta v_x \rangle \langle \delta v_y \rangle\), i.e. the temperature tensor assumes a diagonal form

\[
\hat{T} = \begin{pmatrix}
\langle (\delta v_x)^2 \rangle & \langle \delta v_x \rangle \langle \delta v_y \rangle \\
\langle \delta v_y \rangle \langle \delta v_x \rangle & \langle (\delta v_y)^2 \rangle
\end{pmatrix} = \begin{pmatrix} T_\parallel & 0 \\ 0 & T_\perp \end{pmatrix},
\]

(4.1)

where \(T_\parallel\) is the mean-squared velocity deviation parallel to \(u\), whereas \(T_\perp\) is the mean-squared deviation perpendicular to \(u\). We define \(T = \text{tr}(\hat{T})\) as the (scalar) kinetic temperature. The simplifying assumption of independent average fluctuations parallel and perpendicular to the average velocity is justified by the broken symmetry of the collective translational stationary state [6]. Static correlations in the macroscopic mean-field velocity fluctuations would imply collective rotational motion (see, e.g., [28]), which are not stable in our model.

In the spatially homogeneous and stationary case, we obtain the following set of homogeneous equations:

\[
\frac{\text{d} |u|}{\text{d} t} = 0 = \left(\langle |u|^2 + T \rangle^{-1/2} - 1 - D_v \langle |u|^2 + T \rangle^{-1}\right) |u|, \quad (4.2a)
\]

\[
\frac{1}{2} \frac{\text{d} T_\parallel}{\text{d} t} = 0 = \left(\langle |u|^2 + T \rangle^{-1/2} - 1 - \mu\right) T_\parallel + D_v \left(\frac{T_\parallel - T_\perp}{|u|^2 + T}\right) + D_v \left(\frac{|u|^2 + T}{|u|^2 + T}\right) + D_E, \quad (4.2b)
\]

\[
\frac{1}{2} \frac{\text{d} T_\perp}{\text{d} t} = 0 = \left(\langle |u|^2 + T \rangle^{-1/2} - 1 - \mu\right) T_\perp + D_v \left(\frac{|u|^2 + T - T_\parallel}{|u|^2 + T}\right) + D_v \left(\frac{T_\perp}{|u|^2 + T}\right) + D_E. \quad (4.2c)
\]

The corresponding steady-state solutions to (4.2a)–(4.2c) can in principle be found but are quite complex and hard to analyse. A simpler equation for the temperature can be found by adding up (4.2b) and (4.2c), which yields

\[
\frac{\text{d} |u|}{\text{d} t} = 0 = \left(\langle |u|^2 + T \rangle^{-1/2} - 1 - D_v \langle |u|^2 + T \rangle^{-1}\right) |u|, \quad (4.3a)
\]

\[
\frac{1}{2} \frac{\text{d} T}{\text{d} t} = 0 = \left(\langle |u|^2 + T \rangle^{-1/2} - 1 - \mu\right) T + D_v |u|^2 \langle |u|^2 + T \rangle^{-1} + D_v + 2D_E. \quad (4.3b)
\]

In the following, we discuss the time-independent stationary solutions of (4.3), which correspond to fixed points of the dynamical system. One pair of fixed points reads

\[
|u|_1 = 0, \quad (4.4a)
\]

\[
T_1^{(\pm)} = \frac{2D_E + D_v}{1 + \mu} + \frac{1}{2} (1 + \mu)^{-2} \left(1 \pm \sqrt{1 + 4 \left(2D_E + D_v\right) (1 + \mu)}\right). \quad (4.4b)
\]

However, only one of these fixed points is physically relevant. The solution \((|u|_1, T_1^{(-)})\) will be neglected because it predicts unphysical behaviour for vanishing noise intensities. In this case, both \(|u|_1\) and \(T_1^{(-)}\) are equal to zero. This corresponds to all particles being at rest, which is not possible for permanently self-driven particles considered here. The solution \((|u|_1, T_1^{(\ast)})\) with vanishing mean speed and finite fluctuations is called the disordered state. It is stable above a
critical noise intensity and below a critical coupling strength, respectively. The temperature in the disordered state is independent of the intensity of angular fluctuations $D_\phi$.

Two other solutions with nonzero mean speed are given by the following expressions:

\[ |u|_2^{(\pm)} = \sqrt{\frac{1}{2} \left( 1 \pm \sqrt{1 - 4D_\phi} \right) - D_\phi + \frac{D_v + 2DE}{\mu} }, \]

(4.5a)

\[ T_2 = \frac{D_\phi + D_v + 2DE}{\mu} . \]

(4.5b)

We refer to these fixed points as the ordered state. For vanishing noise intensities ($D_E = D_v = D_\phi = 0$), one solution reads $|u|_2^{(+)} = 1$ and $T_2 = 0$. It corresponds to all particles moving perfectly aligned in the same direction with a stationary speed of individual particles $|v_i| = 1$. In general, the solution $|u|_2^{(+)}$ is stable for sufficiently weak noise, or strong alignment, respectively. The second (ordered) solution $|u|_2^{(-)}$ is always unstable.

In (4.3), the intensities of passive noise $D_E$ and active speed noise $D_v$ enter the equation with the same sign. The theory predicts a qualitatively identical influence of passive fluctuations and speed noise on the mean-field behaviour of the system.

For weak interaction $\mu < 1$, the theory always predicts a continuous transition of the mean velocity from the ordered to the disordered state (figures 2(a) and (b)). In the case of strong interaction $\mu > 1$ and not too large passive fluctuations, a new regime emerges. In this regime, both the ordered and the disordered state are stable macroscopic configurations (figures 2(c) and (d)). The theory predicts that, depending on the initial condition, the system relaxes into one of these stable macroscopic states. Therefore, a discontinuous transition from collective motion to the disordered state for increasing angular fluctuations is expected. In figure 3, we show the phase diagram with respect to external noise $D_E$ and active angular fluctuations $D_\phi$ for $\mu = 10$ and $D_v = 0$. In the bistable regime, if the mean velocity $u_A = u$ is low, the velocity-alignment force reduces to Stokes friction (social friction) which dominates the individual particle dynamics and as a consequence leads to a stable disordered state. Surprisingly, the transition point $D_{\phi,c} = 0.25$ does not depend on the coupling strength $\mu$. From the Fokker–Planck equation (3.1) one can see that $D_\phi > 0$ leads to an effective drift against the collective direction of motion of the particles. Subsequently, the individual particle velocities decrease, and above $D_{\phi,c}$ no self-driven motion along the consensus direction motion is possible.

Please note that the mean speed in the limiting case of strong interaction $\mu \to \infty$ still depends on the intensity of the angular fluctuations.

\[ \lim_{\mu \to \infty} |u|_2^{(\pm)} = \sqrt{\frac{1}{2} \left( 1 \pm \sqrt{1 - 4D_\phi} \right) - D_\phi} . \]

(4.6)

The analytical results (4.4) and (4.5) are compared with numerical simulations of the Langevin dynamics (2.8). We used the stochastic Heun scheme for the numerical integration, which converges in quadratic mean to the solution of the Langevin equations interpreted in the sense of Stratonovich [47]. The simulations were performed with $N = 10^4$ particles considering a square spatial area of length $L$ and periodic boundary conditions. To ensure a spatially homogeneous system, global interaction was used ($A_i = L \times L$). In order to detect the bistable region, both the ordered and the disordered state were used as the initial condition. The results of the numerical simulations in comparison with analytical predictions of the mean-field theory are shown in figures 2 and 3.
In the absence of active speed fluctuations, the theory is mainly confirmed by the numerical simulations. This case was discussed in detail in [45]. If the intensity of the (active) speed noise is small in comparison with intensities of the active angular and passive fluctuations, the theoretical predictions are in good agreement with the numerical results.

Deviations from the theory are due to the approximations used, especially (3.11). This approximation implies that the dynamics of the local velocity field follows closely the individual dynamics or more specifically: the decrease of the mean speed is related to a decrease of the individual particle velocities with a narrow distribution of speeds around the average (local) population speed. Thus, the agreement between theory and simulations is better for a strong velocity alignment, whereas deviations are found in the case of large variance in the distribution of the particle speeds.

Figure 2. Comparison of the numerical Langevin simulations of (2.8) (points) and theoretical prediction (4.3) (lines) in the case of vanishing speed noise and passive noise ($D_E = D_v = 0$). The dashed lines indicate unstable states, whereas the solid lines represent stable configurations. The results of simulations initially started in the disordered state are indicated by (black) squares. Stars (red) represent the results of simulations performed using the ordered state as the initial condition. The simulations were performed with global coupling $A_i = L \times L$, periodic boundary conditions and $N = 10^4$ particles. Parameter: $\mu = 0.1$ (a), (b); $\mu = 10$ (c), (d); numerical time step $\Delta t = 1 \times 10^{-4}$. 

New Journal of Physics 14 (2012) 073033 (http://www.njp.org/)
According to the phase diagram in figure 3, it is possible to move from the bistable regime (C) into the only ordered regime (A) by increasing the external noise strength $D_E$. This implies the rather counter-intuitive effect of ‘generating’ order from disorder by increasing noise. If we consider the system to be initially in the disordered state in the bistable regime, an increase in $D_E$ will eventually lead to a destabilization of the current state and to a transition from disordered to ordered solution. The actual transition depends on the protocol $D_E(t)$ as shown in figure 4. However, if the noise increase is not too strong, the system will eventually relax to the ordered state of collective motion.

We point out that the bistable behaviour found differs from the previously reported bistabilities in collective motion (see, e.g., [44, 48]), which were either a consequence of a nonlinear friction function in one-dimensional dynamics [44] or caused by Morse-type interactions of active particles in three dimensions [48]. Noise in the cited cases did not create the coexistence of the two phases but merely induced switching between the two states with switching times which appeared to be exponentially distributed.
Figure 4. Switching of the globally coupled system from the disordered state to the ordered state with increasing noise intensity for step-like noise change (top) and a linear switching protocol (bottom). The system is initialized in the disordered state with a noise intensity of the passive fluctuations $D_E = 0.02$ corresponding to the bistable regime. At time $t = 100$, the noise intensity is increased either due to an instantaneous step-like switch or in finite time via a linear protocol to the final value $D_E = 0.2$, which corresponds to the ordered regime. In addition to the mean speed $|u| = u$, we show also the directional order parameter $\Phi = \left| \sum_{i=1}^{N} e_{i,v} \right| / N$ with $e_{i,v} = v_i / |v_i|$. Other parameters: $D_\phi = 0.2$, $D_v = 0$, $N = 10^4$ and $\mu = 10$.

5. Discussion and outlook

In this work, we focused on the impact of the different fluctuation types on the collective dynamics of ABPs with velocity alignment. We derived a hydrodynamic theory for the coarse-grained observables spatial density $\rho$, velocity-field $u$ and the temperature tensor $\hat{T}$ in the presence of both fluctuation types. However, approximations are necessary in order to obtain a closed set of equations, which may lead to deviations of theoretical predictions from the numerical simulations. For simplicity, we restricted ourselves to stationary, spatially homogeneous systems where analytical results can be obtained. This case corresponds either to global coupling (interaction length comparable to the system size) or to local dynamics of the system at length-scales where gradients in the macroscopic variables can be neglected. Our analysis shows that the introduction of active fluctuations has a striking influence on the mean-field behaviour of active particles with velocity-alignment and variable speed: in the case of
strong interaction $\mu > 1$, a new bistable regime emerges that cannot be found for passive noise only. The corresponding transition from the ordered to the disordered state is discontinuous for increasing angular noise, whereas it is always continuous in the mean field for a purely passive noise [45]. Our analytical results agree well with the simulation results for strong alignment $\mu \gg 1$ and low active speed fluctuations $D_v$, which is related to the approximation used for the expectation values of rational expressions appearing in the general moment equation and the corresponding implicit moment closure.

The onset of order in the spatially homogeneous system is related to the synchronization of coupled oscillators [33, 34]. If we identify the modulus of the velocity with the amplitude and the direction of motion with the phase of an oscillator, the active particle system with velocity-alignment corresponds to a system of coupled noisy oscillators with the same eigenfrequency in a co-rotating frame of reference. In the limit of small coupling $\mu \ll 1$, the critical noise strengths for the transition from order to disorder have to be small as well ($D_E, D_v, D_\phi \ll 1$). Thus, the fluctuation in the modulus of the velocity at the critical point will be negligible. The system reduces effectively to coupled identical phase oscillators, which is a special case of the well-studied Kuramoto model [32]. In this context, our results for strong alignment can be linked to amplitude death in a system of strongly coupled limit-cycle oscillators [49]. The effective social friction in the disordered state due to the velocity alignment slows down the individual particles and stabilizes the disordered regime.

We focused here on the case of global interactions; however, the hydrodynamic theory provides the basis for the stability analysis of the spatial inhomogeneous solution. It is known from the Vicsek model that density fluctuations occur in the case of local velocity-alignment interaction [7]. Recently, a number of studies analysed the coarse-grained dynamics of Vicsek-type models showing the instability of the ordered homogeneous state of collective motion close to the transition point [38–40, 50]. Similar density inhomogeneities can be observed for ABPs with local coupling and passive noise for sufficiently small interaction regions ($A_i \ll L \times L$) [45]. Although our results for the homogeneous system in the presence of active fluctuations cannot be directly generalized for the case of local interactions, they are nevertheless relevant for it, as they provide predictions on local dynamics in high-density regions. In the regime where only the ordered solution is stable (e.g. for purely passive fluctuations), regions of high density are correlated with regions of high order. Thus, they correspond to clusters or bands performing collective translational motion. This is also the case with the classical Vicsek model [7]. However, the situation changes drastically in the bistable regime. Here, high-density structures may be both ordered (mobile) or disordered and stationary.

In figure 5, we show an example of a simulation for local coupling. We assume that a focal particle interacts with its neighbours within a metric range $\epsilon = 1$ ($L = 100$). The parameters are chosen to correspond to the mean-field bistable regime ($\mu = 10, D_\phi = 0.2, D_v = 0.0, D_E = 0.01$). After a short initial transient, the system reaches a metastable state of multiple stationary clusters at $t \approx 1000$, with few small mobile clusters in between (figure 5(A)). At time $t \approx 11000$, a sufficiently large mobile cluster forms due to fluctuations. It is able to ‘break up’ the stationary high-density aggregates and we observe a sudden onset of large-scale collective motion (figure 5(A)). At time $t \approx 20000$, we observe again the spontaneous formation of a stationary (disordered) cluster due to the collision of clusters moving locally in different directions at $x \approx 70, y \approx 30$ (figure 5(C)). This cluster represents effectively an absorbing obstacle for the remaining mobile structures and leads eventually to a breakdown of collective motion (figure 5(D)). However, at the end of the simulation run at $t \approx 40000$ we
Figure 5. ABPs with local coupling (disc-like interaction zone $A_i$ with interaction radius $\epsilon = 1$, $L = 100$, $N = 10\,000$) in the bistable regime. (A–E) Spatial snapshots at $t = 10\,000$ (A), $t = 15\,500$ (B), $t = 20\,000$ (C), $t = 30\,000$ (D) and $t = 39\,900$ (E). Each particle is shown by its velocity vector, with the colour indicating the direction of motion according to (F). (G) Time evolution of the total mean velocity $U(t) = \left| \sum_{i=1}^{N} V_i \right|$ of the corresponding simulation run with the arrows indicating the snapshot times. The initial condition was a random distribution of particles and their directions of motion (disordered state). Other parameters: $D_{\phi} = 0.2$, $D_v = 0.0$, $D_E = 0.01$, $\mu = 10$; numerical time step $\Delta t = 5 \times 10^{-3}$.

observe again the onset of collective motion (figure 5(E)). Clearly, the bistable local dynamics predicted by the mean-field theory leads to complex spatial and temporal dynamics for local coupling. Please note that the time scales governing the large-scale dynamics are of the order of $\sim 10^4$. This is five orders of magnitude larger than the interaction time scale $\mu^{-1} = 0.1$ which governs the microscopic dynamics. This large spread in the relevant time scales makes this system difficult to analyse by standard numerical methods and may require novel computational techniques, such as for example the equation-free approach [51–53].

In the context of local coupling, we should mention recent studies on collective dynamics in different variants of the Vicsek model [54, 55] and in a run-and-tumble model [56]. In these models, the velocity of individuals was assumed to be directly a function of the local density of individuals [55, 56] or of the local order parameter [54]. These models show qualitatively similar phase-separation dynamics, resembling the behaviour observed here for local coupling. However, in our model we do not explicitly include a dependence of the individual speed on the mean-field velocity and density, but it emerges naturally due to the interplay of the velocity-alignment force and the variable speed of individuals.
Please note that the strong clustering observed in the bistable regime of our model with local coupling (figure 5) has to be distinguished from the so-called giant-number fluctuations predicted by the Toner and Tu theory [6, 23]. The term ‘giant-number fluctuations’ refers to huge density fluctuations in the ordered phase associated with convective flows of active matter induced by fluctuations in the order parameter. Such giant-number fluctuations are also present in our system in the ordered regime. In contrast, the large disordered clusters in the bistable regime are associated with coexistence of different phases in our model and a corresponding ‘absorption’ of particles from the ordered, moving phase into disordered (stationary) high-density clusters.

In conclusion, the ability of ABPs to adjust their speed in response to interactions with the environment is certainly more realistic, in comparison to the constant speed restriction in SPP models. However, this extension is far from being trivial. The mean-field approximation gives important insights into the emergent collective behaviour of the system and demonstrates clearly how the type of fluctuations can have a dramatic impact on the large-scale collective dynamics.

Acknowledgments

LSG thanks the DFG for the support via IRTG 1740 and RG gratefully acknowledges the support by the German National Academic Foundation (Studienstiftung des deutschen Volkes).

Appendix A. Derivation of the moment equation

The general evolution equation (3.5) for the moments \( M^{(mn)} \) is derived in this section. Therefore it is exploited that the dynamics of the ABP gas, which is determined by four stochastic differential equations (2.8), can equally be described by the corresponding Fokker–Planck equation, which describes the dynamics of the total probability density function \( P (r_1, \ldots, r_N, v_1, \ldots, v_N, t) \). The Stratonovich interpretation of the stochastic differential equation is used, which yields

\[
\frac{\partial P}{\partial t} = \sum_{i=1}^{N} \left\{ -\nabla_{r_i} (v_i, P) - \nabla_{v_i} \left[ \left( \frac{v_i}{|v_i|} - v_i + \mu (u_{A_i} - v_i) - D_\psi \left( \frac{v_i}{|v_i|^2} \right) \right) P \right] + \frac{\partial^2}{\partial (e_x, v_i)^2} \left[ \frac{D_v (e_x, v_i)^2 + D_\psi (e_x, v_i)^2}{|v_i|^2} P \right] + \frac{\partial^2}{\partial (e_y, v_i)^2} \left[ \frac{D_v (e_y, v_i)^2 + D_\psi (e_y, v_i)^2}{|v_i|^2} P \right] + \frac{\partial^2}{\partial (e_x, v_i) \partial (e_y, v_i)} \left[ 2 \frac{(e_x, v_i) (e_y, v_i)}{|v_i|^2} (D_v - D_\psi) P \right] + \Delta_{v_i} (D_E P) \right\}. \tag{A.1}
\]

The average velocity \( u_{A_i} \) around the \( i \)th particle is given by the integral

\[
u_{A_i} = \frac{1}{N} \sum_{j=1}^{N} \frac{\int_{A_i} d^2 r_1 \cdots d^2 r_N \int d^2 v_1 \cdots d^2 v_N v_j P (r_1, \ldots, r_N, v_1, \ldots, v_N, t)}{\int_{A_i} d^2 r_1 \cdots d^2 r_N \int d^2 v_1 \cdots d^2 v_N P (r_1, \ldots, r_N, v_1, \ldots, v_N, t)}, \tag{A.2}
\]

and hence (A.1) is a nonlinear Fokker–Planck equation. The interaction region of the \( i \)th particle is denoted by \( A_i \).
In the main text, the one-particle probability density function \( p(r, v, t) \) is introduced. Its dynamics can be easily derived from (A.1) by integrating over all particle positions and velocities except one, arbitrarily chosen particle as well as using the mean field approximation

\[
\mathbf{u}_A (r, t) \approx \int_A d^2r \int d^2v \frac{v p(r, v, t)}{\int_A d^2r \int d^2v p(r, v, t)}. \tag{A.3}
\]

This assumption certainly holds for high particle densities and leads to an effective single-particle description:

\[
\frac{\partial p}{\partial t} = -\nabla_r (v p) - \nabla_v \left[ \left( \frac{v}{|v|} - v \right) p + \mu (\mathbf{u}_A - v) p - D_\psi \left( \frac{v}{|v|} \right) p \right] + \frac{\partial^2}{\partial v_x^2} \left( \frac{D_x v_x^2 + D_\psi v_x^2}{|v|^2} p \right) + \frac{\partial^2}{\partial v_y^2} \left( \frac{D_y v_y^2 + D_\psi v_y^2}{|v|^2} p \right) + \frac{\partial^2}{\partial v_x \partial v_y} \left[ \frac{2 v_x v_y}{|v|^2} (D_v - D_\psi) p \right] + \Delta_\psi (D_E p). \tag{A.4}
\]

Multiplying the moments of the velocity components

\[
M^{(mn)} = \langle v_x^m v_y^n \rangle = \frac{1}{\rho(r, t)} \int d^2v v_x^m v_y^n p(r, v, t) \tag{A.5}
\]

and the spatial density

\[
M^{(00)} = \rho(r, t) = \int d^2v p(r, v, t) \tag{A.6}
\]

yields

\[
\frac{\partial}{\partial t} \left( \rho M^{(mn)} \right) = \int d^2v v_x^m v_y^n \frac{\partial p}{\partial t}. \tag{A.7}
\]

We may rewrite the integral on the right-hand side of (A.7) by inserting the Fokker–Planck equation (A.4)

\[
\frac{\partial}{\partial t} \left( \rho M^{(mn)} \right) = - \int d^2v v_x^m v_y^n \frac{\partial}{\partial x} (v_x p) - \int d^2v v_x^m v_y^n \frac{\partial}{\partial y} (v_y p) - \int d^2v v_x^m v_y^n \frac{\partial}{\partial v_x} \left( \left( |v|^{-1} - 1 \right) v_x p + \mu (\mathbf{u}_A - v_x) p - D_\psi \left( \frac{v_x}{|v|} \right) p \right) - \int d^2v v_x^m v_y^n \frac{\partial}{\partial v_y} \left( \left( |v|^{-1} - 1 \right) v_y p + \mu (\mathbf{u}_A - v_y) p - D_\psi \left( \frac{v_y}{|v|} \right) p \right) + \int d^2v v_x^m v_y^n \frac{\partial^2}{\partial v_x^2} \left( \frac{D_x v_x^2 + D_\psi v_x^2}{|v|^2} p \right) + \int d^2v v_x^m v_y^n \frac{\partial^2}{\partial v_y^2} \left( \frac{D_y v_y^2 + D_\psi v_y^2}{|v|^2} p \right) + \int d^2v v_x^m v_y^n \frac{\partial^2}{\partial v_x \partial v_y} \left[ \frac{2 v_x v_y}{|v|^2} (D_v - D_\psi) p \right] + \int d^2v v_x^m v_y^n \frac{\partial^2}{\partial v_x^2} (D_E p) + \int d^2v v_x^m v_y^n \frac{\partial^2}{\partial v_y^2} (D_E p). \tag{A.8}
\]
Integrals containing a derivative with respect to $v_x$ and $v_y$, respectively, are integrated by parts and rewritten using the moment definition (A.5). Furthermore, it is assumed that $p(r, v, t)$ decreases fast enough, so that

$$\lim_{v_x \to \infty} \left[ \lim_{v_y \to \infty} \left( v^m v^n p(r, v, t) \right) \right] = 0 \quad (A.9)$$

holds. In this way, partial differential equations for the moments (macroscopic variables) of the probability density function are derived directly from the microscopic Langevin description (2.1). The general evolution equation for the moment $M^{(mn)}$ reads

$$\frac{\partial}{\partial t} \left( \rho \langle v_x^m v_y^n \rangle \right) = -\frac{\partial}{\partial x} \left( \rho \langle v_x^{m+1} v_y^n \rangle \right) - \frac{\partial}{\partial y} \left( \rho \langle v_x^m v_y^{n+1} \rangle \right)$$

$$+ (m+n) \rho \left[ \frac{v_x^m v_y^n}{|v|^2} - \langle v_x^m v_y^n \rangle - D_v \left( \frac{v_x^m v_y^n}{|v|^2} \right) \right]$$

$$+ \mu \rho \left[ m u_{A,x} \langle v_x^{m-1} v_y^n \rangle + n u_{A,y} \langle v_x^m v_y^{n-1} \rangle - (m+n) \langle v_x^m v_y^n \rangle \right]$$

$$+ m(m-1) \rho \left[ D_v \left( \frac{v_x^{m+2} v_y^{n-2}}{|v|^2} \right) + D_v \left( \frac{v_x^m v_y^{n+2}}{|v|^2} \right) + D_{E} \langle v_x^m v_y^{n+2} \rangle \right]$$

$$+ n(n-1) \rho \left[ D_v \left( \frac{v_x^{m+2} v_y^{n-2}}{|v|^2} \right) + D_v \left( \frac{v_x^m v_y^{n+2}}{|v|^2} \right) + D_{E} \langle v_x^m v_y^{n+2} \rangle \right]$$

$$+ 2mn \rho \left( \frac{v_x^m v_y^n}{|v|^2} \right) \left( D_v - D_{E} \right). \quad (A.10)$$

It is shown in section 3 how a closed set of hydrodynamic equations up to the second order can be derived from (A.10).

**Appendix B. Moment closure approximation**

Here, we briefly discuss the moment approximation in (3.11)

$$\left\langle \frac{v_x^m v_y^n}{|v|^2} \right\rangle \approx \left\langle \frac{v_x^m v_y^n}{v_x^2 + v_y^2} \right\rangle \approx \frac{\langle v_x^m v_y^n \rangle}{\sqrt{v_x^2 + v_y^2}}$$

(B.1)

whereby it is sufficient to restrict the discussion to the relevant (low order) cases with $k \in \{1, 2\}$, $(m, n) \in \{m, n \in \mathbb{N} \mid 1 \leq m + n \leq 2\}$.

In order to investigate the validity of the above approximation, we assume a stationary one-particle probability density function for the speed $|v|$ and the angle $\varphi$, which may be decomposed according to

$$p(|v|, \varphi) = p_v(|v|) p_\varphi(\varphi). \quad (B.2)$$

Using this assumption we can rewrite (B.1) in terms of independent expectation values with respect to the speed and angle as

$$\left\langle \frac{v_x^m v_y^n}{|v|^2} \right\rangle \approx |v|^{m+n-k} \left\langle \cos^m \varphi \cdot \sin^n \varphi \right\rangle \varphi, \quad (B.3a)$$
It can be easily seen that the different terms (B.3a)–(B.3c) differ only with respect to the speed-dependent terms. Therefore, the validity of the approximation is foremost determined by the properties of the speed distribution.

By assuming a reasonable probability density of speeds, e.g. the stationary speed distribution for the Schienbein–Gruler function (see, e.g., [57], p 15), one can easily show that the deviations between the exact expectation value (B.3a) and the approximations (B.3b) and (B.3c) are of the order $\mathcal{O}(\sigma_v^2/v_i^2)$, with $\sigma_v^2$ being the variance of the speed distribution and $\tilde{v}$ being the first speed moment. This analysis indicates that the approximation (3.11) works best for small fluctuations of individual speeds around $\tilde{v}$. Please note that the mean speed $\bar{v}$ differs in general from the preferred speed of an individual $v_0$ in the Schienbein–Gruler friction, due to the velocity–alignment interaction. Thus, despite deriving the theory for the general case, we focused our discussion in the main text on the case $\bar{v} > 0$, in particular for strong alignment $\mu$ (or weak passive noise $D_E$), and ensures that the moment closure yields a good approximation. An alternative moment closure may be obtained by a series expansion of the corresponding (exact) expectation value in (3.11). This would lead to a ‘normal form’ of driving term in the velocity equation as in the Toner–Tu theory [5, 6]. However, this will lead to an infinite hierarchy of moment equations. In order to obtain a self-consistent set of coarse-grained equations, assumptions about higher moments have to be made (see, e.g., [44, 45]).

Appendix C. The stochastic Heun method

A system of stochastic differential equations with multiplicative Gaussian white noise ($\langle \xi_i(t) \rangle = 0$, $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij} \delta(t - t')$),

$$\dot{x} = f(x, t) + g(x, t) \xi(t)$$

(C.1)
can numerically be integrated applying the stochastic Heun method. This integration scheme converges in quadratic mean to the solution of the Langevin equations interpreted in the sense of Stratonovich [47] and reads as follows:

$$\tilde{x}_i(t + \Delta t) = x_i(t) + f_i(x(t), t) \Delta t + \sum_{j=1}^{m} g_{ij}(x(t), t) \sigma_j(t) \sqrt{\Delta t},$$

(C.2a)

$$x_i(t + \Delta t) = x_i(t) + \frac{1}{2} \left[ f_i(x(t), t) + f_i(\tilde{x}(t + \Delta t), t + \Delta t) \right] \Delta t$$

$$+ \frac{1}{2} \sum_{j=1}^{m} \left[ g_{ij}(x(t), t) + g_{ij}(\tilde{x}(t + \Delta t), t + \Delta t) \right] \sigma_j(t) \sqrt{\Delta t}.$$  

(C.2b)
For the integration, Gaussian random variables $\sigma_j$ with zero mean and variance one are used.

\[
\langle \sigma_i(t) \rangle = 0, \quad \text{(C.3a)}
\]

\[
\langle \sigma_i(t) \sigma_j(t) \rangle = \delta_{ij}. \quad \text{(C.3b)}
\]

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