ON NON-EQUILIBRIUM PHYSICS AND STRING THEORY

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In this article we review the relation between string theory and non-equilibrium physics based on our previously published work. First we explain why a theory of quantum gravity and non-equilibrium statistical physics should be related in the first place. Then we present the necessary background from the recent research in non-equilibrium physics. The review discusses the relationship of string theory and aging phenomena, as well as the connection between AdS/CFT correspondence and the Jarzynski identity. We also discuss the emergent symmetries in fully developed turbulence and the corresponding non-equilibrium stationary states. Finally we outline a larger picture regarding the relationship between non-perturbative quantum gravity and non-equilibrium statistical physics of critical phenomena. This relationship can be understood as a natural generalization of the well-known Wilsonian relation between local quantum field theory and equilibrium statistical physics of critical phenomena. According to this picture the AdS/CFT duality is just an example of a more general connection between non-perturbative quantum gravity and non-equilibrium physics. In the appendix of this review we discuss a new kind of complementarity between thermodynamics and statistical physics which should be important in the context of black hole complementarity.

1. Outline of the Review

Quantum gravity and non-equilibrium physics present two outstanding problems of contemporary physics. In this article we review the surprising relation between string theory, viewed as a consistent theory of quantum gravity and matter, and non-equilibrium physics. We follow our publications, Refs. 1, 2, 3, 4, 5. (See also Refs. 6, 7.) In section 2 we explain why a theory of quantum gravity and non-equilibrium statistical physics should be related in the first place. Then in section 3 we present the necessary background from the recent research in non-equilibrium physics. In section 4 we review the relationship of string theory and aging phenomena as presented in Refs. 1, 2, 3. In section 5 we review the connection between the AdS/CFT correspondence and the Jarzynski identity as presented in Ref. 4. Next, in section 6 we review the proposal presented in Ref. 5 regarding the emergent symmetries in fully developed turbulence and the corresponding non-equilibrium stationary states. Finally, in section 7 we conclude our discussion by presenting a much larger picture regarding the relationship between non-perturbative quantum gravity and non-equilibrium statistical physics. According to this picture, the AdS/CFT duality is just an example of a more general connection between quantum gravity and non-equilibrium physics. In the appendix of the review we discuss a new kind of complementarity between thermodynamics and statistical physics which should be important in the context of black hole complementarity. Our aim in this review is to emphasize the natural relationship between non-perturbative quantum gravity, in the context of string theory, and non-equilibrium statistical physics of critical phenomena. We expect that this new relation between quantum gravity and non-equilibrium physics will be as illuminating and insightful as the well-known Wilsonian paradigm.

2. From Quantum Gravity to Non-equilibrium physics

We start our discussion by a brief review of the canonical thinking about gravitational thermodynamics and the reason for the deep connection between quantum gravity and non-equilibrium physics. (For the comprehensive treatment of this subtle issue see Ref. 8.) The basic intuition of black hole thermodynamics (derived from Einstein’s equations) can be turned on its head to claim that the Einstein’s equations can be understood thermodynamically. The first law is naively written as

\[ dE = TdS \] (2.1)
where according to the second law \( dS \geq 0 \). Here \( T = \frac{\hbar a}{2\pi c} \) is the Hawking-Unruh temperature and \( a \) is the local acceleration, whereas the entropy is given by the holographic Bekenstein-Hawking formula
\[
S = \frac{Ac^3}{4G_N\hbar}.
\]

Notice that the gravitational entropy is not extensive, i.e. it does not scale with the volume of the system, but with the boundary of the volume, i.e. the area \( A \). (For the subtlety of the concept of entropy in the non-inertial frames see Ref. 16.) The claim is that Einstein’s gravitational equations follow from these two laws of gravitational thermodynamics. The factors of \( \hbar \) cancel between the temperature and entropy (temperature being perturbative in \( \hbar \) and entropy being non-perturbative in \( \hbar \)) in Einstein’s gravitational equations. Also the usual \( 8\pi G_N \) factor from Einstein’s equations is now represented as \( 8\pi G_N = 2\pi(4G_N) \), where the \( 2\pi \) factor (from the formula for \( T \)) is fixed by the relation between the imaginary (or Euclidean) time \( t_E \) in the exponential measure of the Euclidean path integral \( \exp(-\frac{S_E}{\hbar}) \) (\( S_E \) being the Euclidean action) and the Boltzmann-Gibbs equilibrium exponential measure \( \exp(-\beta H) \), \( H \) being the Hamiltonian, i.e. \( \beta\hbar = t_E \), the Euclidean time also being periodic with the period determined by the acceleration of the particle \( 2\pi c/a \).

Thus according to this intuition the Einstein equations essentially are the expression of the first law combined with the second law \( dS \geq 0 \) which is implied by the focusing theorem (i.e. the attractive nature of gravity).

Our first basic observation is that this reasoning implies an underlying non-equilibrium description! The temperature is only locally defined and it is given by the local acceleration, or the local gravitational field, as implied by the equivalence principle. Thus gravitational thermodynamics should be essentially of the non-equilibrium character and thus quantum gravity, especially its locally holographic formulation, should be essentially mapped onto non-equilibrium physics. (Note that some prescient statements about the relation between non-equilibrium physics (especially the fluctuation-dissipation theorem and Onsager’s relations) and black hole thermodynamics can be found in Refs. 19, 20.)

3. Systems far from equilibrium

Next we turn to a brief review of the current research of systems far from equilibrium. Understanding systems far from equilibrium is one of the most challenging problems in contemporary physics. It is also one of the most important ones, due to the omnipresence of non-equilibrium processes in nature. In virtually every discipline one observes an increasing focus on processes that take place far from equilibrium. Notwithstanding this strong interest in non-equilibrium phenomena, we still lack a unifying theoretical framework for the description of non-equilibrium interacting many-body systems. Still, some remarkable recent progress has been achieved through the study of specific model systems with generic properties, using the large toolbox of numerical and analytical methods provided by statistical physics.

This progress has been achieved in three different areas. Firstly, the study of relaxation processes taking place in systems suddenly brought out of equilibrium has yielded an increased understanding of physical aging in non-frustrated systems, especially in cases where the system is characterized by a single, algebraically growing length scale. Secondly, the discovery of various work and fluctuation theorems has provided very general statements valid for large classes of systems. Thirdly, the in-depth investigation of steady-state properties, especially in one-dimensional transport models, has yielded a better understanding of the differences between equilibrium and non-equilibrium steady states.

In the following we briefly discuss some of these recent results, as this will allow us to put into a broader context the research summarized in this review.

At first look a unifying description of processes far from equilibrium seems to be hopeless, due to the apparently erratic and history-dependent properties in that regime. It is therefore very remarkable that the out-of-equilibrium properties of many systems with slow dynamics can be organized in terms of a simple scaling picture. Glasses are one example of such systems. In many cases they are made by rapidly cooling...
(quenching) a molten liquid to below some characteristic temperature threshold. If this cooling happens sufficiently fast, normal crystallization no longer takes place and the material remains in some non-equilibrium state. Studying the mechanical properties of polymeric glasses quenched to low temperatures, Struick\textsuperscript{31} observed that the time-dependent creep curves of the mechanical response can all be mapped onto a single master curve. This master curve turned out to be the same for very different materials, pointing to a universal behavior far from equilibrium.

These experiments, which were the first systematic studies of aging in out-of-equilibrium systems, revealed the following important features of physical aging: (i) slow (i.e. non-exponential) dynamics, (ii) breaking of time-translation invariance, and (iii) dynamical scaling\textsuperscript{24}

In recent years remarkable progress has been achieved in the study of the out-of-equilibrium dynamical behavior, and especially of aging, in cases where the system is characterized by a single dynamical length \( L \) that grows as a power-law of time \( t: L(t) \sim t^{1/z} \), where \( z \) is the dynamical exponent. This simple growth law is encountered in many systems, ranging from ferromagnets undergoing phase-ordering\textsuperscript{32} to reaction-diffusion systems.\textsuperscript{33} The theoretical description of aging phenomena in these systems starts from the observation that dynamical correlation and response functions transform in a specific way under the dynamical scale transformation \( t \rightarrow b^z t, \vec{r} \rightarrow b^z \vec{r} \), where \( b \) is a scale factor. Of special interest are the two-time correlation and response functions, \( C(t, s, \vec{r}) \) and \( R(t, s, \vec{r}) \), that are formally defined through the equations (here we use the language of magnetic systems, but this can of course be generalized easily) \( C(t, s, \vec{r}) = \langle \phi(t, \vec{r}) \phi(s, \vec{0}) \rangle \) and \( R(t, s, \vec{r}) = \delta(\phi(t, \vec{r})/\partial h(s, \vec{0})|_{h=0}. \) Here \( \phi(t, \vec{r}) \) is the space-time-dependent order-parameter, \( h(s, \vec{r}) \) is the conjugate magnetic field, and \( s \) and \( t > s \) are two different times called waiting and observation time. In the special case \( s = 0 \), these quantities yield the autocorrelation function \( C(t, s) \) and the autoresponse function \( R(t, s) \). Due to dynamical scaling, these quantities display the following simple scaling form in the dynamical scaling or aging regime\textsuperscript{34,35} where all the times involved are large compared to any microscopic time scale: \( C(t, s) = s^{-b} f_C(t/s) \) and \( R(t, s) = s^{-1-a} f_R(t/s) \), where \( f_C \) and \( f_R \) are dynamical scaling functions, whereas \( a \) and \( b \) are non-equilibrium exponents. The dynamical scaling functions of two-time quantities and the associated non-equilibrium exponents are often found to be universal and to depend only on some global features of the system under investigation.

The presence of dynamical scaling yields some hope that symmetry principles could be used successfully for the description of this type of non-equilibrium processes, along lines similar to those that yielded the successful use of conformal invariance in equilibrium critical phenomena.

The theory of local scale invariance\textsuperscript{25,29,37} presents a novel approach for the analytical treatment of stochastic systems. Starting from the stochastic Langevin equation for some quantity of interest (as for example the order parameter in magnetic systems), it proposes to split this equation into a deterministic part and a noise part. A crucial observation is thereby that the deterministic part admits much richer space-time symmetries than mere dynamical scale invariance. Exploiting these non-trivial symmetries, all averages within the initial noisy theory can be reduced exactly to averages within the deterministic, noiseless theory. This is a very general result as it depends only on the (generalized) Galilean invariance of the deterministic part.

Interestingly, the space-time symmetries of the noiseless theory strongly constrain the possible forms of \( n \)-point functions. Especially for two-point functions, as the two-time response and correlation functions, the scaling functions are completely fixed due to these symmetries. This allows the model-independent determination of universal scaling functions of response and correlation functions in aging systems. The scaling functions obtained from this approach have been found to be correct in exactly solvable models and to describe faithfully over many time decades the numerically determined scaling functions of more complex systems.\textsuperscript{26}

A second research thrust focuses on generic fluctuation properties of systems driven out of a steady state.\textsuperscript{28,29,38,39} The progress in this field is closely related to the discovery of various fluctuation and work theorems that provide generic statements applicable to large classes of systems. One well known example is
the Crooks relation where a system in equilibrium is driven out of equilibrium following a specific protocol. Repeating this process many times allows to compute the probability distribution $P_F(W_d)$ of the dissipative work $W_d$. Using the time-reversed process one can also determine the probability distribution for this process, $P_R$. Comparing these two distributions yields the simple relation

$$P_F(\beta W_d)/P_R(-\beta W_d) = e^{\beta W_d},$$

(3.1)

where $\beta = 1/(k_B T)$. For the special case of equilibrium initial and final states one can also relate the free energy difference to the following average over all processes leading from one state to the other, $\langle e^{-\beta W} \rangle = e^{-\beta \Delta G}$, with $W$ being the work done on the system, and $G$ the free energy. This is the celebrated Jarzynski work theorem.

The Jarzynski and Crooks relations have been generalized to many different situations, and detailed and integral fluctuation theorems have been proposed for a range of quantities, from driving entropy production\textsuperscript{39, 40} to adiabatic and non-adiabatic trajectory entropies.\textsuperscript{41}

Finally, many efforts have also been devoted to gain a better understanding and characterize more completely non-equilibrium steady states. A well known example for a non-equilibrium steady state is of course the steady state realized in fully developed turbulence.\textsuperscript{42} Even though turbulence is far from being well understood, remarkable recent progress has been achieved in the characterization of the steady states in simpler model systems. Thus for one-dimensional transport models, which to a large extent can be solved analytically, a very detailed understanding of non-equilibrium steady states has been achieved.\textsuperscript{26}

One of the hallmark of non-equilibrium steady states is entropy production. While entropy production is a system-dependent quantity, some remarkable universal properties have been discovered. Thus, the probability distribution of the total entropy production satisfies a detailed fluctuation theorem in large classes of systems (see, e.g., Refs. \textsuperscript{27, 43, 44, 45, 46}, and a kink appears in its large deviation function (and in that of related currents) at zero entropy production.\textsuperscript{30, 47–49}

In the following three sections we turn to the unexpected relation between string theory and these research thrusts: the aging phenomena, the Jarzynski identity and the steady states in fully developed turbulence.

### 4. Aging Phenomena and String Theory

In this section we review the connection between aging phenomena and string theory. The presentation is based on our previous work.\textsuperscript{1–3} Let us start by stating in more detail some of the known specifics about the dynamical scaling and scale invariance in dynamical systems with aging.\textsuperscript{44} The general set-up for the study of aging behavior is as follows: One considers a coarse-grained order parameter $\phi(t, \vec{r})$, conjugate to a generalized field $h(t, \vec{r})$, which is usually assumed to be fully disordered at $t = 0$, i.e. $\langle \phi(0, \vec{r}) \rangle = 0$. For a magnetic system, the order parameter is of course the magnetization whereas the conjugate field is an external magnetic field. In the following we consider the case where the order parameter is not conserved by the dynamics (model A dynamics). As already pointed out in the previous section, one studies the scaling behavior of two-time correlation functions (here and in the following we assume that spatial translation invariance holds)

$$C(t, s) = \langle \phi(t, \vec{r})\phi(s, \vec{r}) \rangle \sim s^{-b} f_C(t/s)$$

(4.1)

as well as of two-time response functions

$$R(t, s) = \frac{\delta \langle \phi(t, \vec{r}) \rangle}{\delta h(t, \vec{r})} \sim s^{-1-a} f_R(t/s)$$

(4.2)

where $a$ and $b$ are non-equilibrium exponents whereas $f_C$ and $f_R$ are scaling functions. This scaling behavior is expected in the aging regime, defined by both $t$ and $s$ as well as $t - s$ being much larger than the characteristic microscopic time scale. Note also that this scaling assumes a single characteristic length scale $L$ which scales with time $t$ as

$$L(t) \sim t^{1/z}$$

(4.3)
where $z$ is the dynamical exponent. For large values of the argument one expects

$$f_C(x) \sim x^{-\lambda_C/z}, \quad f_R(x) \sim x^{-\lambda_R/z}$$

with new non-equilibrium exponents $\lambda_C$ and $\lambda_R$. The aging behavior just summarized is called simple or full aging and has been observed in many exactly solvable models, in numerical simulations of more complex models as well as in actual experiments.

Given the success of conformal invariance in equilibrium critical phenomena, it is natural to ask whether scaling functions and the values of non-equilibrium exponents might be deduced from symmetry principles by invoking generalized dynamical scaling with a space-time dependent scale factor $b = b(t, \vec{r})$. This program has been reviewed in Ref. 37 and here we concentrate on the specific case when $L(t) \sim t^{1/2}$, i.e. $z = 2$. This is a very important case as it encompasses systems undergoing phase-ordering with non-conserved dynamics.\[37\]

The generalization for $z \neq 2$ has been discussed in Ref. 37.

The theoretical description of aging systems with a dynamical exponent $z = 2$ starts from a stochastic Langevin equation which for a non-conserved order parameter reads:

$$2M \partial_t \phi = \nabla^2 \phi - \frac{\delta V[\phi]}{\delta \phi} + \eta$$

where $V$ is a Ginzburg-Landau potential and $\eta$ is a Gaussian white noise that arises due to the contact with a heat bath.\[50\] The theory of local scale invariance then permits to show\[37, 51\] that under rather general conditions all averages (i.e., correlation and response functions) of the noisy theory (4.5) can be reduced exactly to averages of the corresponding deterministic, noiseless theory.

For the $z = 2$ case the relevant symmetry structure is the Schrödinger group.\[52–54\] It is well known that the free diffusion equation

$$2M \partial_t \phi = \nabla^2 \phi$$

is invariant under the Schrödinger group (the free diffusion equation being essentially equivalent to the free Schrödinger equation). The Schrödinger group is defined through the space-time transformations

$$t \to t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \vec{r} \to \vec{r}' = \frac{R \vec{r} + \vec{v}t + \vec{a}}{\gamma t + \delta}$$

where $\alpha, \beta, \gamma, \delta$ and $\vec{v}, \vec{a}$ are real parameters and $\alpha \delta - \beta \gamma = 1$, whereas $R$ denotes a rotation matrix in $d$ spatial dimensions.

The Schrödinger group is an action on space and time coordinates extending the usual Galilean symmetries to include anisotropic scaling $\vec{x} \to \lambda \vec{x}, t \to \lambda^2 t$. Throughout this section, we will only discuss the non-relativistic case, $z = 2$. These actions close on a larger group generated by temporal translations $H$, spatial translations $P_i$, Galilean boosts $K_i$, rotations $M_{ij}$, dilatations $D$ and the special ‘conformal’ transformation $C$.

The commutation relations satisfied by these generators, aside from the obvious commutators of rotation and translations and those that vanish, read\[32\]

$$[D, K_i] = K_i, \quad [D, P_i] = -P_i, \quad [P_i, K_j] = \delta_{ij} N$$
$$[D, C] = 2C, \quad [C, P_i] = -K_i, \quad [H, C] = D$$
$$[H, K_i] = P_i, \quad [D, H] = -2H$$

We note that $\{H, D, C\}$ generate an $SL(2, \mathbb{R})$ subalgebra, with $\{P_i, K_i\}$ forming an $SL(2, \mathbb{R})$ doublet, and the generator $N$ is central. We will refer to the full algebra as $Schr_d$, where $d$ is the spatial dimension. This algebra made its first appearance long ago, for example, as the invariance group of the Schrödinger equation with zero potential.
If one considers field theories with this symmetry, one finds, in complete analogy with the conformal field theory bootstrap,\textsuperscript{53,54} that correlation functions are of a restricted form.\textsuperscript{57} For example, for operators that are scalars under rotations, the two-point function is essentially given by
\[
\langle O_1(t_1, \vec{x}_1)O_2(t_2, \vec{x}_2) \rangle = \delta_{\Delta_1, \Delta_2} \delta_{n_1+n_2, 0} (t_1-t_2)^{-\Delta_1} \exp \left( \frac{i n_2 (\vec{x}_1 - \vec{x}_2)^2}{2 (t_1-t_2)} \right).
\]
(4.11)

In the holographic context, this result was obtained using real-time methods\textsuperscript{55,61,62} in Ref. \textsuperscript{58} and using Euclidean methods in Refs. \textsuperscript{59,60,63} Similarly, higher point functions are of a constrained form as well.

In holography, one makes use of a space-time possessing $Schr_d$ as its algebra of isometries. Such a space-time may be taken to have metric\textsuperscript{61,62}
\[
ds^2 = \frac{L^2}{z^2} \left( dz^2 - \frac{\beta^2}{z^2} dt^2 - 2 dtd\xi + d\vec{x}^2 \right),
\]
(4.12)
where $\vec{x} = (x_1, \ldots, x_d)$ are the spatial coordinates of the dual field theory and $z$ is the holographic direction (with the “boundary” located at $z = 0$). The parameters $\beta$ and $L$ are length scales, and all the coordinates have units of length.

The space-time (4.12) has Killing vectors
\[
M_{ij} = x_j \partial_i - x_i \partial_j
\]
(4.13)
\[
P_i = \partial_i, \quad H = \partial_t, \quad N = \partial_\xi
\]
(4.14)
\[
C = z t \partial_z + t^2 \partial_t + \frac{1}{2} \left( z^2 + \vec{x}^2 \right) \partial_\xi + tx^i \partial_i
\]
(4.15)
\[
D = z \partial_z + 2t \partial_t + x^i \partial_i
\]
(4.16)
\[
K_i = t \partial_i + x^i \partial_\xi
\]
(4.17)
providing a representation of the $Schr_d$ algebra (4.8) acting on bulk (scalar) fields. Since $N$ is central, such fields can be taken to be equivariant with respect to $N$, i.e., their $\xi$-dependence can be taken to be of the form $e^{in\xi}$ for fixed $n \in \mathbb{R}$. In gauge/gravity duality, the asymptotic ($z \to 0$) values of fields propagating in this geometry act as sources for operators in the dual field theory. The $Schr_d$ algebra acts on those operators in a way that can be deduced from the field asymptotics.\textsuperscript{55,61} We thus get another distinct representation of $Schr_d$ that acts on (scalar) operators of the dual field theory
\[
M_{ij} = x_j \partial_i - x_i \partial_j
\]
(4.18)
\[
P_i = \partial_i, \quad H = \partial_t, \quad N = \partial_\xi
\]
(4.19)
\[
D = 2t \partial_t + x \partial_x + \Delta \partial_\xi
\]
(4.20)
\[
C = t^2 \partial_t + tx \partial_x + \frac{x^2}{2} \partial_\xi + \Delta t \partial_\xi
\]
(4.21)
\[
K_i = t \partial_i + x_i \partial_\xi
\]
(4.22)
where $\Delta$ is the scaling dimension; for equivariant fields, $\partial_\xi$ evaluates to $in$.

4.1. The Aging Algebra and Correlation Functions

The aging algebra, which we will denote as $Age_d$, is obtained by discarding the time translation generator $H$ from the $Schr_d$ algebra. Indeed, the form of the algebra is such that it makes sense to do so. This is the simplest possible notion of time-dependent dynamics, and it is considered as a rather special form of non-equilibrium physics. We consider the problem of constructing an appropriate space-time geometry possessing $Age_d$ as its isometry algebra, and then compute some simple correlation functions, following Ref. \textsuperscript{2}.

$^d\xi$ is often taken to be compact, so that the spectrum of operators in the dual theory is discrete. This is not without problems in the bulk, as $\partial_\xi$ is null in the geometry (4.12).
Since $H$ has been discarded, such correlation functions are not generically time-translation invariant. To see what sort of time-dependence to expect, let us consider a construction that often appears in the literature. Consider a diffusive system with (white) noise $\eta$ and a time-dependent potential $v(t)$ governed by a wave-function $\phi$ satisfying

$$2M\partial_t\phi = \nabla^2 \phi - \frac{\delta V}{\delta \phi} - v(t)\phi + \eta.$$  \hspace{1cm} (4.23)

As pointed out in Ref. 25, correlation functions in this system can be studied by examining deterministic dynamics governed by the aging group. Note that the ‘gauge transformation’

$$\phi \rightarrow \phi \exp\left(-\frac{1}{2M} \int^t_\tau v(\tau) d\tau \right),$$  \hspace{1cm} (4.24)

removes the time dependent potential term from equation (4.23). This means that this sort of time-dependence can be mapped to a system governed by the Schrödinger group. A simple physical model for this type of time dependence is the out of equilibrium decay of a system towards equilibrium, following some sort of quench. In the special case

$$v(t) \sim 2MKt^{-1},$$  \hspace{1cm} (4.25)

one finds that the wavefunctions are related via a scaling function to the wavefunctions of the Schrödinger problem

$$\phi_{Age} = t^K \phi_{Schr}.$$  \hspace{1cm} (4.26)

Thus, the local scale-invariance of aging systems is largely determined by studying first the Schrödinger fields. In particular, the correlators of operators in an $Age_d$-invariant theory can be expressed in terms of the correlators of operators in a $Schr_d$-invariant theory. Schematically, for the two-point function of a scalar operator $O$, the result we want to reproduce is

$$\langle O(t_1)O(t_2) \rangle_{Age} \sim \left(\frac{t_1}{t_2}\right)^\# \langle O(t_1)O(t_2) \rangle_{Schr},$$  \hspace{1cm} (4.27)

where $\#$ is a constant which characterizes the breaking of time-translation invariance. We will present the details of this relationship below.

Above, we gave two representations of $Schr_d$, one acting on (scalar) operators of a field theory, and one acting on the holographic bulk (scalar) fields. We now consider removing $H$ from the algebra, and ask how the representation of the remaining generators might be modified. We will consider first the representation on operators, as that is what appears in the literature. Let us assume that the time and spatial coordinates $t$ and $\vec{x}$ and the non-relativistic mass (equivalently the coordinate $\xi$) retain their standard meaning. Consequently, we take the representation of $M_{ij}, P_i, N, K_i$ and $D$ to be unchanged from that of $Schr_d$. However, it is possible that the form of the generator $C$ could be modified consistent with the $Age_d$ algebra. Suppose then that we write $C_A = C + \delta C$, where $C_A$ is the representation of $C$ in the aging algebra. In order for $C_A$ to satisfy the commutators of the $Age_d$ algebra (which are unchanged from $Schr_d$) we need

$$[P_i, \delta C] = 0, \quad [N, \delta C] = 0, \quad [K_i, \delta C] = 0, \quad [D, \delta C] = 2\delta C.$$  \hspace{1cm} (4.28)

The first three commutators are easily seen to imply that $\delta C$ can only be of the form $\delta C = g_1(t)\vec{\xi} + g_2(t)\partial_\xi$. The fourth commutator then fixes $g_i(t)$ ($i = 1, 2$):

$$[D, \delta C] = 2\delta C \Rightarrow t\partial_t g_i(t) = g_i(t) \Rightarrow g_i(t) = K_i t,$$  \hspace{1cm} (4.29)

\[The diffusion equation should be regarded as a Wick rotated version of a Schrödinger-like equation (or equivalently, a Schrödinger-like equation is obtained by considering $M$ to be imaginary). In following sections, we will always work in Lorentzian signature. One does not expect that such Wick rotations are innocuous in general.\]
and we then conclude\[^5\] that the most general form is
\[
C_A = t^2 \partial_t + tx \partial_x + \frac{x^2}{2} \partial_x + (\Delta + K_1)t \partial_x + K_2 t \partial_{\xi},
\]
where $K_1$ and $K_2$ are constants. Note that in the undeformed case $K_1 = K_2 = 0$, and that when evaluated on equivariant fields, $K_2$ is accompanied by the eigenvalue of $\partial_{\xi}$ which is imaginary. Thus, $\delta C = K t \partial_x$, where $K = K_1 + i \alpha K_2$ is naturally a complex number. The physical significance of the real and imaginary parts will be discussed more fully below, but for now we note that we expect $K$ to make an appearance in time non-translation invariant features of correlation functions.

$K$ has been described in the literature\[^5\] as a ‘quantum number’ labeling representations. As pointed out in Ref.\[^2\] we believe that it is better to think of this parameter as a property of the algebra instead; indeed, later we will see $K$ emerge in a holographic setup as a parameter appearing in the metric, rather than being associated with any particular field. We further note that if we were to demand $[\partial_t, C_A] = D$, we would find that $K = 0$, thus recovering the full Schrödinger algebra in its standard representation.

### 4.2. A Geometric Realization of the Aging Group

Next, following Ref.\[^2\] we will explore how we might implement $Age_d$ as an isometry algebra. There are a variety of possible constructions that present themselves. First, we might consider the breaking of time translation invariance to be associated with the introduction of a ‘temporal defect’. In relativistic gauge/gravity duality, there is a way to introduce spatial defects\[^6\] by placing a D-brane along an $AdS_{d-1}$ slice of $AdS_d$.

Such a brane intersects the boundary along a co-dimension one curve, which can be coordinatized as $z = 0$. The choice of slicing preserves as much symmetry as possible: clearly $P_x$ is broken along with $K_x$ and $M_{xj}$ leaving unbroken $SO(d - 1, 2)$. Because of Lorentz invariance, presumably such a construction works for temporal defects in the relativistic case (see Ref.\[^6\] for related work on time-dependent holography). Such a construction would involve placing an S-brane along a suitable slice.

Similarly, spatial defects\[^7\] can be introduced in non-relativistic holography in much the same way. One can imagine placing a D-brane along a $Schr_{d-1}$ slice of the $Schr_d$ space-time. Here, $P_z$, $K_x$ and $M_{xj}$ would be broken. It is fairly obvious though that in this case, a temporal defect cannot be constructed in this way. The basic reason is that time is much different than space in a non-relativistic theory, and at the algebraic level, we seek to break $H$ only. We will see signs of this sort of difficulty below.

The key feature of such metrics is that $Age_d$ invariance allows the metric components to depend on the invariant combination $T = \beta t / z^2$. This is scale invariant (and dimensionless) but clearly transforms under time translations. A fairly general Ansatz for the metric is then of the form:\[^2\]
\[
ds^2 = \frac{L^2}{z^2} \left[ f_{zz}(T) \, dz^2 + \frac{2\beta}{z} f_{zt}(T) \, dzdt - \frac{\beta^2}{z^2} f_{tt}(T) \, dt^2 + f(T) \left( -2 dt d\xi + d\vec{x}^2 \right) \right].
\]
As a regularity requirement, the functions $f_{zz}$ and $f$ are chosen such that $\det g = -L^8 f_{zz} f^3 / z^8$ is non-zero everywhere other than $z = \infty$. It is clear that the Ansatz\[^4, 31\] already admits $\{ M_{i j}, P_i, K_i, D, N \}$ as isometries and the generators are in the same form as in the Schrödinger space-time,\[^11, 13\]. If the metric functions $f_{zz}(T)$ and $f(T)$ are independent, one concludes that there are no further isometries.

Although in this construction the full $Schr_d$ algebra is present locally, we clearly need to be careful near $t = 0$. If we interpret the blowing up of the components of $H$ as an indication that we lose $H$ as a generator, we obtain precisely what we want in the dual field theory. It should be emphasized again however that this is a coordinate singularity; for example, the norm of the vector field $H$ is well-behaved everywhere in the metric
\[
ds^2 = \frac{L^2}{z^2} \left[ dz^2 + \frac{2\alpha \beta}{z} dzdt - \frac{\beta^2}{z^2} \left( 1 + \frac{\alpha}{T} \right) dt^2 - 2 dt d\xi + d\vec{x}^2 \right].
\]
\[^4\]Here we refer to the generators of the $AdS_d$ isometry, $SO(d, 2)$. 
\[\]
We note also that this metric breaks a discrete symmetry enjoyed by the Schrödinger metric, namely ‘CT’, \( t \to -t, \xi \to -\xi \). The following change of coordinates would take this metric back to the standard Schrödinger form,

\[
\xi' = \xi + \frac{\alpha \beta}{2} \ln T = \xi + \frac{\alpha \beta}{2} \ln \left( \frac{\beta t}{z^2} \right).
\] (4.33)

The multi-valuedness of the logarithm in the complex \( t \) plane will play an important role in the physical interpretation of the point \( t = 0 \) and in the calculation of correlation functions that we present below.

### 4.3. Correlation Functions

Now we review the correlations functions in the background of the aging geometry as presented in Ref. [2]. Consider a scalar field on the geometry \( (4.32) \). We look for solutions to the scalar wave equation for the above aging geometry that are of the form

\[
\phi(z, t, \xi, x) = e^{Q(T)} e^{i n \xi} \phi_S(z, t, x; n) = e^{Q(T)} e^{i n \xi} \int \frac{d^d p}{(2\pi)^d} \frac{d\omega}{2\pi} e^{i \vec{p} \cdot \vec{x} - i \omega t} \phi_S(z, \omega, p; n),
\] (4.34)

where \( Q(T) \) is some function and \( \phi_S \) denotes a solution of the wave equation on the standard Schrödinger background \( (4.12) \) (which is translationally invariant). If such a form exists, then the appearance of time translation non-invariance in the scalar is entirely in the scale-invariant prefactor \( e^{Q(T)} \). We note that given the fact that the form of the dilatation operator is the same for both \( Age_d \) and \( Schr_\xi \), the corresponding fields have the same conformal dimension, even though there is \( z \)-dependence in the prefactor, precisely because \( T \) is dilatation invariant. We note also that the Ansatz for the scalar fields is reminiscent of the expected form \( (4.26) \).

Indeed, for the geometry \( (4.32) \), the scalar Laplacian is such that solutions are of the form \( (4.34) \), with

\[
Q(T) = \frac{\beta}{2} \int_0^T dT' \left[ f_1(T') - 1 \right].
\] (4.35)

For the Schrödinger field, in the asymptotic region we have

\[
\phi_S(z \to 0, t, \vec{x}; n) \sim z^{-\Delta} \phi_S^{(0)}(t, \vec{x}; n) + ...
\] (4.36)

and thus the aging field behaves as

\[
\phi(z \to 0, t, \vec{x}; n) \sim z^{-\Delta} \left( \frac{\beta t}{z^2} \right)^{-in\beta/2} \phi_S^{(0)}(t, \vec{x}; n) + ...
\] (4.37)

As we mentioned, although there is an unusual factor of \( z \) present here, because the generator \( D \) is unmodified, the scaling dimension of \( \phi \) is the same as \( \phi_S \). As we will see below, the extra factor of \( z \) should be thought of as a ‘wavefunction renormalization’ factor that should be absorbed into the definition of the dual operator — it is included in the source for \( Age \) fields.

Now, one might assume since the prefactor is common to both the source and the vacuum expectation value, that it cancels out in the evaluation of the Green function. We will show below that this is naive — it is related to depending too much on Euclidean methods. When one carefully examines the real-time correlation functions, one finds that the prefactor makes its presence felt. Note also that the prefactor is generally complex. If we confine ourselves to \( \alpha \in \mathbb{R} \), which one normally would do in real geometry, then formally the prefactor is a phase. However, \( Q(T) \) is generally a complex function, and in our case, it is a multi-valued function about \( t = 0 \). It is then not too much of a stretch to go all the way to a complexified geometry, in the sense of taking \( \alpha \in \mathbb{C} \). As we have discussed above, the real and imaginary parts of \( \alpha \) (or \( K \) in the language of Section \ref{sec:4.1}) have a direct physical interpretation, and we will encounter precisely that in the \( Age_d \)-invariant correlation functions that we study below.
The Schrödinger theory is renormalizable\cite{55} the on-shell action can be made finite by the inclusion of a number of Schrödinger invariant counterterms. One can show that the boundary renormalization of the Age theory follows that of Schrödinger closely. Indeed, the bulk action

\[ S = -\frac{1}{2} \int d^{d+1}x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \phi + m_0^2 / L^2 |\phi|^2 \right) \]  

reduces on-shell to the boundary term

\[ S_{os} = \frac{1}{2} \int d^{d+1}x \sqrt{\gamma} \bar{\phi} n \cdot \partial \phi \]  

Here we take the ‘boundary’ to be a constant $z$ slice, with normal $n = \frac{1}{L} (z \partial z + \alpha \beta \partial k)$ when we use $f_T(T) = 1 + \alpha / T$. Since the on-shell solutions are of the form $\phi = e^{Q(T)} \phi_{Sch}$, we get

\[ S_{os} = \frac{1}{2L} \int d^{d+1}x \sqrt{\gamma} |e^{Q(T)}| \bar{\phi} \partial z \phi_{Sch} \]  

with $\gamma$ the induced spatial metric. This is precisely the same form as in the Schrödinger case, with the inclusion of the prefactors appropriate to Age fields. We see again that one must be careful in this case in interpreting powers of $z$ — since the Age fields are of dimension $\Delta$, the scale invariant $z$-dependent prefactors must be associated with the normalization of operators. They are not to be canceled by the addition of counterterms.

As in Ref.\cite{2} we will focus on the calculation of the two-point functions of scalar operators dual to $\phi$. As we stated above, because of the time-dependence of the metric, it is dangerous to attempt to employ Euclidean continuation, and thus, as in Ref.\cite{2} we will consider the real time correlators very carefully using the Skenderis-Van Rees method\cite{56,57} following\cite{58} closely.

### 4.3.1. Review of Schrödinger calculations

We now outline the results found in Ref.\cite{58} for the case of the Schrödinger geometry. Generally, correlators are computed by constructing solutions along segments of a contour in the complex time plane, where the choice of contour determines the nature of the correlator considered. In the case of a time ordered correlator, the contour is as shown in Fig. 1.

![Contour in the complex t-plane corresponding to the time-ordered correlator.](image)
A vertical segment corresponds to Euclidean signature, while a horizontal segment corresponds to Lorentzian signature. One computes the solutions in the presence of a δ-function source on the real axis along each segment, and matches them at junctions.

4.3.2. Schrödinger Solutions: Lorentzian

In Lorentzian signature, we use the notation $q = \sqrt{q^2} = \sqrt{k^2 - 2\omega n}$. Solutions of the scalar wave equation on the Schrödinger space-time are given in terms of Bessel functions. For $q^2 < 0$, both the $K_\nu$ and $I_\nu$ solutions are regular everywhere, while for $q^2 > 0$, $I_\nu$ diverges for large $z$ and is discarded. Noting that $q$ has a branch point at $\omega = \sqrt{k^2}/2n$, we facilitate integration along real $\omega$ by properly deforming $q$ to $q_\epsilon = -\sqrt{2\omega n + k^2 - i\epsilon}$. With these comments, we arrive at the general solution in Lorentzian signature

$$
\phi_S(z, t, \vec{x}; \xi) = e^{in\xi} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega t + i\vec{k} \cdot \vec{x}} z^{\frac{d}{2} + 1} \left( A_s(\omega, \vec{k}) K_\nu(q_\epsilon z) + \theta(-q^2) B_s(\omega, \vec{k}) I_\nu(|q| z) \right).
$$

(4.41)

4.3.3. Schrödinger Solutions: Euclidean

Next, we consider a similar analysis in Euclidean signature. To do so, we Wick rotate the Schrödinger metric to (more precisely, along $M_0$ and $M_2$ respectively, we write $t = \pm (T + i\tau)$)

$$
ds^2 = \frac{L^2}{z^2} \left[ dz^2 + \frac{\beta^2}{z^2} dt^2 - 2i dr d\xi + d\vec{x}^2 \right].
$$

(4.42)

Although this metric is complex, it is possible to trace carefully through the analysis. The general solution of the Schrödinger problem is

$$
\phi_S(z, \tau, \vec{x}; \xi) = e^{in\xi} \int \frac{d\omega_E}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega_E \tau + i\vec{k} \cdot \vec{x}} z^{\frac{d}{2} + 1} A(\omega_E, \vec{k}) K_\nu(q_E z),
$$

(4.43)

where now $q_E = \sqrt{q_E^2} = \sqrt{k^2 - i2\omega E n}$. Note that in this case, the branch point is at imaginary $\omega_E$, and so no $i\epsilon$ insertion is necessary. In writing this, we have assumed that $\tau \in (-\infty, \infty)$ and thus $\phi_S$ has no normalizable mode. One has to be careful with this. For example, if $\tau \in [0, \infty)$, we write $\omega_E = -i\omega$ for $\phi$ and $\omega_E = i\omega$ for $\bar{\phi}$ and the following mode is allowable

$$
\phi_S \sim e^{in\xi} e^{-\omega t + i\vec{k} \cdot \vec{x}} z^{\frac{d}{2} + 1} I_\nu(q_E z)
$$

(4.44)

as long as $\omega > 0$ and $-2\omega n + \sqrt{k^2/2n} < 0$, or equivalently $\omega > \sqrt{k^2/2n}$. For $\tau \in (-\infty, 0]$, no such mode is present.

4.3.4. Correlators (Schrödinger)

The correlators are computed by recognizing the asymptotics as sources for corresponding operators

$$
e^{iS_{\phi\psi}^{(0)}} |\phi^{(0)}_S, \phi^{(0)}_\bar{\phi} \rangle = \langle e^{i \int \bar{\phi} \partial^\dagger \phi^{(0)}_S + \bar{\phi}^{(0)}_\bar{\phi} \partial \phi^{(0)}_S} \rangle
$$

(4.45)

where the fields have asymptotic expansions

$$
\phi_S = e^{in\xi} \left\{ z^{\Delta -} (\phi(0) + z^2 \phi(2) + o(z^4)) + z^{\Delta +} (\bar{\psi}(0) + z^2 \bar{\psi}(2) + o(z^4)) \right\}
$$

(4.46)

with $\Delta_{\pm} = 1 + d/2 \pm \nu$ and

$$
\phi_{(2m)} = \frac{1}{2m(2\Delta_+ - (d + 2) - 2m)} S^{2m-2} \phi_{(2m-2)},
$$

(4.47)

where $S = 2m \partial_t + \nabla^2$ is the Schrödinger operator. In Ref. [35] the bulk to boundary propagator was derived

$$
K_{(n)}(t, \vec{x}, z; t', \vec{y}) = \frac{2z^{1+d/2}}{\Gamma(\nu)} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t-t') + i\vec{k} \cdot \vec{x}} \left( \frac{q_E}{2} \right)^\nu K_\nu(q_E z),
$$

(4.48)
which determines the time ordered correlator

\[ \langle T \left( \hat{O}_{(\epsilon)}(\vec{x}, t) \hat{O}_{(\epsilon)}^\dagger(\vec{x}', t') \right) \rangle_{\text{Schr.}} = \frac{1}{\pi^{d/2} T(\nu)} \left( \frac{n}{2i} \right)^{\Delta_+ - 1} \frac{\theta(t - t')}{(t - t')^{\Delta_+}} e^{i \nu \frac{(\vec{x} - \vec{x}')^2 + i \epsilon}{2(t - t')}} \]  

(4.49)

The derivation of the final result involves a somewhat difficult contour integral.

### 4.4. Aging Correlators

Now, we turn our attention to the computation of correlators holographically in the Age geometry. Because of the physical interpretation, that of a quench at \( t = 0 \), we focus on correlators of operators inserted at times \( t > 0 \). We have argued that the point \( t = 0 \) is much like a horizon, and so one should in fact confine oneself to the \( t > 0 \) patch. As we have also discussed, the Age solutions have a discontinuity across \( t = 0 \) given by \( \phi(-\epsilon, \vec{x}) = e^{i \pi n \beta / 2} \phi(+\epsilon, \vec{x}) \). To calculate the Age correlators, we propose that one should use the same Keldysh contour as above, fixing the solutions to the Age problem formally by requiring the continuity of the associated Schrödinger solutions. This prescription uniquely determines the two-point correlation function, and as we shall see, gives the expected form (4.27). We proceed to construct solutions on the segments \( M_0, M_1 \) and \( M_2 \).

#### 4.4.1. Age Solutions: Lorentzian

With the same setup as for the Schrödinger problem above, we arrive at the general solution in Lorentzian signature

\[ \phi(t, \xi, \vec{x}, z) = e^{Q(T)} e^{i \kappa \xi} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i \omega t + i \vec{k} \cdot \vec{x} + \frac{\beta^2}{2} f_{tt}(T_E) d\tau^2 + 2i \tau d \xi \pm 2i \frac{\beta T_E}{z} (f_{tt}(T_E) - 1) d \tau d z + \hat{d} \vec{x}^2} \]  

(4.50)

#### 4.4.2. Age Solutions: Euclidean

Next, we consider a similar analysis in Euclidean signature. To do so, we take complex time \( t = \pm (-T + i \tau) \) (this is appropriate to \( M_0 \) and \( M_2 \) respectively) and thus replace the Schrödinger metric by

\[ ds^2 = \frac{L^2}{z^2} \left[ dz^2 + \frac{\beta^2}{z^2} f_{tt}(T_E) d\tau^2 + 2i d\tau d \xi \pm 2i \frac{\beta T_E}{z} (f_{tt}(T_E) - 1) d \tau d z + \hat{d} \vec{x}^2 \right] \]  

(4.51)

where \( T_E = \pm \beta (-T + i \tau) / z^2 \). The general solution of the Age problem is

\[ \phi(\tau, \xi, \vec{x}, z) = e^{Q(T_E)} e^{i \kappa \xi} \int \frac{d\omega_E}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i \omega_E \tau + i \vec{k} \cdot \vec{x} + \frac{\beta^2}{2} f_{tt}(T_E) d\tau^2 + 2i \tau d \xi \pm 2i \frac{\beta T_E}{z} (f_{tt}(T_E) - 1) d \tau d z + \hat{d} \vec{x}^2} \]  

(4.52)

where now \( q_E = \sqrt{q_E^2} = \sqrt{\hat{k}^2 - i 2 \omega_E n} \).

#### 4.4.3. Time-ordered Correlator

As in the Schrödinger case, the solution along the \( M_0 \) component is zero. Thus we have to require \( \phi_1(t_1 = -T, \vec{x}, z) = 0 \). We then also conclude that there is no normalizable solution on \( M_1 \). Thus, we should have a unique solution.

We place a \( \delta \)-function source at \( \vec{x} = 0, t_1 = \hat{t}_1 > 0 \) on \( M_1 \). That is, we want

\[ \phi_1(t_1, \vec{x}, z) \bigg|_{z=0} = z^{\Delta - 1} e^{i \kappa \xi} \delta(t_1 - \hat{t}_1) \delta(\vec{x}) \]  

(4.53)

This requires the field to be of the form

\[ \phi_1(t_1, \vec{x}, z) = \frac{2}{\Gamma(\nu)} e^{Q(T_1) - Q(T_1)} e^{i \kappa \xi} z^{1 + d/2 - in\alpha/\beta} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i \omega(t_1 - \hat{t}_1) + i \vec{k} \cdot \vec{x}} \left( \frac{q_E}{2} \right)^{\nu} K_\nu(q_E z). \]  

(4.54)

On \( M_2 \) we then have (where we match at \( t = T \))
The most general realization of these generators presented in Ref. 3 is

\[ \phi_2(\tau_2, \vec{x}, z) = \frac{2\pi i}{\Gamma(\nu)} e^{Q(-i\beta(\tau_2+IT)/z^2)} e^{in\xi z^{1+d/2-ina\beta}} \times \int \frac{dw}{2\pi} \frac{d^dk}{(2\pi)^d} e^{-\omega(\tau_2+IT-\hat{t}_1)+ik\vec{x}\theta(-q^2)} \left( \frac{|q|}{2} \right)^\nu J_\nu(|q|z). \] (4.55)

For \( t_1 > \hat{t}_1 \), the correlator \( K(t_1, \hat{t}_1) \) is essentially \( \phi_1 \) itself. Given the choice of \( Q(T) \), we have

\[ K_{age}(t_1, \hat{t}_1; \vec{x}) = \left( \frac{t_1}{\hat{t}_1} \right)^{\alpha \beta / 2} K_{Schr}(t_1, \hat{t}_1; \vec{x}). \] (4.56)

This result is the expected one — it displays the time-translation non-invariant scaling form, with exponent given by \( K = i\alpha\beta/2 \). We see that for real \( \alpha \), this time dependence is a phase\(^8\). It is only for \( \alpha \) complex that the correlator corresponds to a relaxation process. For generic \( \alpha \), the correlator ‘spirals in’ towards the Schrödinger correlator at late times. Note that for this physical interpretation, one expects that \( \text{Im}\ \alpha > 0 \). This is related holographically to normalizability of the solutions. Indeed if one traces the solution back to \( t = -T \), for \( \text{Im}\ \alpha < 0 \), the prefactor of the solution blows up as \( T \to \infty \), but is innocuous for \( \alpha \) in the upper half plane.

### 4.5. Comments on the 3-Point functions ang aging geometry

The above realization of aging does not capture the most general aging dynamics and that what has been described above is just a particular realization of the Schrödinger dynamics! This was discussed in Ref. 3. In what follows we clearly distinguish between this special case and the most general aging dynamics.

As in Ref. 3 for simplicity, let us consider a 1+1-dimensional theory with coordinates \( t, r \). We use \( \xi \) to denote the Fourier variable conjugate to the mass \( M \) of a certain primary operator. In the notation of Refs. 25 and 68, the Schrödinger and Age algebras are respectively spanned by the generators \{\( X_{-1}, X_0, X_1, M_0, Y_{\frac{1}{2}}, Y_{-\frac{1}{2}} \)\} and \{\( X_0, X_1, M_0, Y_{\frac{1}{2}}, Y_{-\frac{1}{2}} \)\}, which obey the following commutation relations

\[ [X_n, X_{n'}] = (n - n')X_{n+n'}, \quad [X_n, Y_m] = (\frac{4}{\pi} - m)Y_{n+m} \]

\[ [X_n, M_{n'}] = -n'M_{n+n'}, \quad [Y_m, M_{m'}] = (m - m')M_{m+m'}. \] (4.57)

The most general realization of these generators presented in Ref. 3 is

\[ X_{-1} = -\partial_t + \frac{g(t) - \gamma}{t} + \frac{h(t) - \delta}{t} i\partial_\xi, \quad X_0 = -t\partial_t - \frac{1}{2}r\partial_r - \frac{1}{2} + g(t) + h(t)i\partial_\xi, \]

\[ X_1 = -t^2\partial_t - tr\partial_r - \Delta t + \frac{i}{2}r^2\partial_\xi + t(g(t) + \gamma) + t(h(t) + \delta)i\partial_\xi, \]

\[ Y_{\frac{1}{2}} = -\partial_r, Y_{-\frac{1}{2}} = t\partial_t + ir\partial_\xi, M_0 = i\partial_\xi \equiv -M, \] (4.58)

where \( g(t), h(t) \) are arbitrary time-dependent functions and \( \gamma, \delta \) are arbitrary constants. In arriving at (4.58) we have kept the form of the generators \( M_0 \) and of the spatial translation \( Y_{\frac{1}{2}} \) and generalized Galilean-invariance \( Y_{\frac{1}{2}} \) unchanged. This general realization is central for the new results presented in what follows.

Next we solve the partial differential constraints imposed on the 3-point functions (namely that they are invariance under the Age generators). The conclusion is that the most general scalar 3-point function is

\[ G_A(\{t_1, r_1, \xi_1\}) = \left( \prod_{i=1}^{3} t_1^{r_i} \right) \exp \left( \sum_{i=1}^{3} t_i \int \frac{g(\tau)}{\tau} \right) \times \left( t_3 - t_1 \right)^{-\frac{1}{2} \Delta_{31,2} + \gamma_{31,2}} \left( t_1 - t_2 \right)^{-\frac{1}{2} \Delta_{32,1} + \gamma_{32,1}} \times \left( t_2 - t_1 \right)^{\frac{1}{2} \Delta_{21,3} + \gamma_{21,3}} \Theta \left( u_1, u_2, u_3, \frac{t_3(t_2 - t_1)}{t_2(t_3 - t_1)} \right). \] (4.59)

\(^8\)In the aging literature, many specific systems have been studied numerically for which no such phase is present. To be capable of seeing such a phase, one must at least have a complex order parameter. As an example, it is possible that such behavior could be seen in \( px + ipy \) superconductors.
and Schrödinger 3-point functions lies in the dependence of the former on yet to be found. As noted in Ref. 3 this is further evidenced by the fact that the 3-point correlators implied previous part of this section, is equally restrictive, and thus, the most general holographic Age background is one of the consequences of this observation is that the holographic realization of aging reviewed in the previous part of this section, is equally restrictive, and thus, the most general holographic Age background is unconstrained function of:

\[ u_1 = -2i(\xi_2 - \xi_1) + \frac{(r_2 - r_1)^2}{t_2 - t_1} + 2 \int_{t_1}^{t_2} dt \frac{h(\tau)}{\tau} + 2(\delta_2 - \delta_1) \ln(t_2 - t_1) \]

+ \frac{2(\delta_1 + \delta_2) \ln \frac{t_3 - t_2}{t_3 - t_1} - 2\delta_2 \ln t_2 + 2\delta_1 \ln t_1}{t_1}

\[ u_2 = -2i(\xi_3 - \xi_1) + \frac{(r_3 - r_1)^2}{t_3 - t_1} + 2 \int_{t_1}^{t_3} dt \frac{h(\tau)}{\tau} + 2(\delta_3 - \delta_1) \ln(t_3 - t_1) \]

+ \frac{2(\delta_1 + \delta_3) \ln \frac{t_3 - t_2}{t_2 - t_1} - 2\delta_3 \ln t_3 + 2\delta_1 \ln t_1}{t_1}

\[ u_3 = -2i(\xi_3 - \xi_2) + \frac{(r_3 - r_2)^2}{t_3 - t_2} + 2 \int_{t_2}^{t_3} dt \frac{h(\tau)}{\tau} + 2(\delta_3 - \delta_2) \ln(t_3 - t_2) \]

+ \frac{2(\delta_2 + \delta_3) \ln \frac{t_3 - t_1}{t_2 - t_1} - 2\delta_3 \ln t_3 + 2\delta_2 \ln t_2}{t_1} \quad (4.60) \]

For the Schrödinger 3-point correlator we find a similar expression, but without the dependence on the additional variable \( \frac{\tau_2(t_2 - t_1)}{t_2(t_3 - t_1)} \):

\[ G_S(\{t, r, \xi\}) = \left( \prod_{i=1}^{3} t_i \right) \exp \left( 3 \sum_{i=1}^{3} \int_{t_i}^{t_{i+1}} dt \frac{g(\tau)}{\tau} \right) \times (t_3 - t_1)^{-\frac{1}{2} \Delta_{31.2} + \Delta_{31.2}} \]

\[ \times (t_2 - t_1)^{-\frac{1}{2} \Delta_{32.1} + \Delta_{32.1}} \theta(u_1, u_2, u_3) \quad (4.61) \]

Note that despite the presence of the time-dependent prefactors this correlator is time-translation invariant. In fact, it is easy to check that a redefinition of the primary fields of the Schrödinger algebra, effected by factoring out appropriate time-dependent functions, gives the correlators of the type (4.61). However, this redefinition does not change the fact that X \( \rightarrow \) \( G_S(\{t, r, \xi\}) = 0 \). The fundamental difference between Age and Schrödinger 3-point functions lies in the dependence of the former on \( \frac{\tau_2(t_2 - t_1)}{t_2(t_3 - t_1)} \).

At this stage we pause to note that the analysis reviewed in the previous part of this section regarding the form of the Age correlators was too restrictive.

Since the time-dependent potential is introduced by a simple redefinition of the fields, \( \varphi_S(t, \vec{x}) = \exp(i \int d\tau v(\tau))/\varphi_A(t, \vec{x}) \), the relevant symmetry group is still the full Schrödinger and not the Age group. One of the consequences of this observation is that the holographic realization of aging reviewed in the previous part of this section, is equally restrictive, and thus, the most general holographic Age background is yet to be found. As noted in Ref. 3 this is further evidenced by the fact that the 3-point correlators implied by the aging metric found in Ref. 2 are dressed Schrödinger correlators (i.e. they are “fake” Age correlators), whereas the ones in (4.39) are not.

Finally, following Ref. 3 we comment on the holographic realization of the Age algebra in terms of metric isometries of a 1+3-dimensional space. The holographic dual space is parametrized by \( x^{\mu} \) coordinates: \( t, r, \xi \) and the holographic coordinate \( z \). The main observation is that once one identifies the Killing vectors \( K = K^\mu \frac{\partial}{\partial x^\mu} \) obeying the Age algebra, one can reverse engineer the metric by solving the Killing vector equations for the components of the metric, i.e. \( (g_{\mu\nu} \partial_\mu + g_{\nu\rho} \partial_\nu)K^\rho + K^\rho \partial_\rho g_{\mu\nu} = 0 \). It is natural to assume that \( Y_{\frac{1}{2}} \) and \( M_0 \) are bulk Killing vectors. If one makes the additional assumption that \( Y_{\frac{1}{2}} \), given by (4.58), is a bulk Killing vector then the problem becomes quite tractable. The bulk forms of the Killing vectors \( X_0 \) and \( X_1 \) are \( X_0 = -t \partial_t + \frac{\xi}{\gamma} \partial_t - \frac{1}{2} r \partial_r + X_3 \partial_\xi \) and \( X_1 = -t^2 \partial_t + X_3 \partial_t - tr \partial_r + X_1 \partial_\xi \), where \( X_3 \frac{\partial}{\partial x_3} \).
\[ X_1^z = \frac{\partial_i(t^2 g_{rr})}{\partial z g_{rr}} \]
\[ X_0^\xi = \frac{i}{2t^2 g_{rr}} \left( -\partial_z (g_{rr} S) - t \partial_t g_{rr} \partial_z S + t \partial_t S \partial_z g_{rr} + 2t T \partial_z g_{rr} + 2t C_1 \partial_z g_{rr} \right) \]
\[ X_1^\xi = \frac{i}{2t^2 g_{rr}} \left( -\partial_z (g_{rr} S) - 2g_{rr} \partial_z S - S \partial_z g_{rr} - t \partial_t g_{rr} \partial_z S + t \partial_t S \partial_z g_{rr} + 2t T \partial_z g_{rr} \right). \] (4.62)

Here \( g_{rr} = g_{rr}(t, z), S = S(t, z), T = T(t) \) and \( C_1 \) is an arbitrary constant. Solving the Killing vector equations corresponding to \( Y_{-1} \) and \( M_0 \) leads to a metric which is \( \xi, r \)-independent. Furthermore, solving the \( Y_2 \) Killing equations brings the metric to a form which coincides with the initial ansatz of Ref. [2]
\[ ds^2 = g_{tt}(t, z) dt^2 + g_{rr}(t, z) dr^2 + g_{zz}(t, z) dz^2 - 2i g_{rr}(t, z) dtdz + 2g_{zz}(t, z) ddz. \] The other components of the metric are determined by the remaining Killing equations: \( g_{zz} = \frac{C(t^2 g_{rr})^2}{g_{rr}}, g_{zt} = \frac{g_{rr} \partial_z S}{2t} + \frac{C_2 \partial_z g_{rr} \partial_z (t^2 g_{rr})}{t^2 g_{rr}^2} + C_1 \partial_z g_{rr} \)
\[ g_{tt} = C_3 g_{rr}^2 + \frac{C_2 (\partial_t (t^2 g_{rr}))^2}{t^4 g_{rr}^2} + \frac{2C_1 \partial_t (t^2 g_{rr})}{t^2} + \frac{g_{rr}(2T - S + \partial_t S)}{t^2}, \] (4.63)

where \( C_2, C_3 \) are additional integration constants. We stress that this metric is the most general solution of the reverse-engineering procedure, given the confines of the initial assumption that \( Y_{-1} \) becomes a bulk Killing vector while remaining unchanged. However, we cannot claim that we have identified the holographic dual of a general theory possessing the full symmetry of the Age algebra. The reason for this is that we are able to identify one more Killing vector of the metric compatible with the following bulk extension of \( X_{-1} \)
\[ -X_{-1} = \partial_t - \frac{\partial_t g_{rr}}{\partial z g_{rr}} \partial_z + i \left( -\frac{2C_2}{t^2 g_{rr}} + \frac{S}{2t^2} \frac{\partial_z S + 2T + 4C_1}{2t} + \frac{\partial_t g_{rr} \partial_z S}{2t^2 g_{rr}} \right) \partial_z. \] (4.64)

Thus the isometries of the above metric generate the full Schrödinger algebra, as in Ref. [2]. Naturally the correlators computed from this metric using holography exhibit the kind of "fake" aging discussed earlier, and are constrained by the full Schrödinger algebra. For the time being we can only trace this feature to the assumption made regarding the bulk realization of \( Y_{-1} \). (This assumption was also made in Ref. [2]) Relaxing this condition makes the problem of identifying the holographic metric of Aging much more complicated. This is still an important open problem.

5. The Jarzynski Identity and the AdS/CFT Duality

In this section, following Ref. [4], we review a deep analogy between the Jarzynski identity[28,69] one of the most remarkable results in the recent history of non-equilibrium statistical physics, and the AdS/CFT duality[20,74] one of the most influential developments in the recent history of quantum field theory and string theory. The Jarzynski identity has been tested in many experimental situations in non-equilibrium systems[28,69] and it has been also theoretically generalized[39,10,28]. On the other hand, the AdS/CFT duality has been used in fields as diverse as quantum gravity, quantum chromodynamics, nuclear physics, and condensed matter physics[21,32,14,24].

The Jarzynski identity[28,69] gives the exact relation between the thermodynamic free energy differences \( \Delta G \) and the irreversible work \( W \)
\[ \langle \exp (-\beta W) \rangle = \exp (-\beta \Delta G) \] (5.1)
where \( \beta^{-1} = k_B T \) with \( k_B \) denoting the Boltzmann constant and \( T \) the temperature. The average \( \langle ... \rangle \) is over all trajectories that take the system from an initial to the final equilibrium states. Note that this exact equality extends the well known inequality between work and change in free energy, \( W \geq \Delta G \), which follows from the second law of thermodynamics. The relation \( W \geq \Delta G \) is implied by the Jarzynski identity and Jensen’s inequality \( \langle e^A \rangle \geq e^{\langle A \rangle} \).
In the AdS/CFT correspondence, one computes the on-shell bulk action $S_{\text{bulk}}$ and relates it to the appropriate boundary correlators. The conjecture\textsuperscript{70–72} is then that the generating functional of the vacuum correlators of the operator $O$ for a $d$ dimensional conformal field theory (CFT) is given by the partition function $Z(\phi)$ in (Anti-de-Sitter) $\text{AdS}_{d+1}$ space

$$\langle \exp(\int JO) \rangle = Z(\phi) \rightarrow \exp[ -S_{\text{bulk}}(g, \phi, ...)] ,$$

where in the semiclassical limit the partition function $Z = \exp(-S_{\text{bulk}})$. Here $g$ denotes the metric of the $\text{AdS}_{d+1}$ space, and the boundary values of the bulk field $\phi$ are given by the sources $J$ of the boundary CFT. Essentially here one reinterprets the RG flow of the boundary non-gravitational theory in terms of bulk gravitational equations of motion, and then rewrites the generating functional of vacuum correlators of the boundary theory in terms of a semi-classical wave function of the bulk “universe” with specific boundary conditions. Note that we have written here a semi-classical expression for the correspondence, which is what is essentially used in many tests of this remarkable conjecture.\textsuperscript{61, 62, 79–82}

Obviously, there exists a naive formal similarity between the expressions (5.1) and (5.2), given the fact that $\int JO$ formally corresponds to generalized “work”. What was argued in Ref.\textsuperscript{4} is that this naive similarity is actually deeper and points to a profound analogy between the two relations. Given the fact that (5.1) is exact (under certain assumptions) and (5.2) is still regarded as conjectural, but extremely profound and technically powerful, this analogy might point a way for a formal “proof” of (5.2). Also, this analogy points to some novel views on the RG flows of quantum field theories as well as to natural generalizations of the AdS/CFT dictionary.

The precise definition of $W$ and $G$ is as follows.\textsuperscript{83} Given the canonical Liouville equation whose stationary solution is the Boltzmann distribution $e^{-\beta H(\vec{x}, t)}$, one can define the average over an ensemble of trajectories starting from the equilibrium Boltzmann distribution at $t = 0$ and evolving according to the Liouville equation. Each trajectory is weighted with the Boltzmann factor of the external work $W_t$ done on the system:

$$W_t = \int_0^t \frac{\partial H}{\partial t'}(\vec{x}', t') dt'.$$

By remembering that the exponent of the free energy difference is given by definition as

$$e^{-\beta \Delta G} = \int \frac{d\vec{x} e^{-\beta H(\vec{x}, t)}}{d\vec{y} e^{-\beta H(\vec{y}, 0)}}$$

and by using the results of Ref.\textsuperscript{83} one is lead to the Jarzynski identity

$$\exp (-\beta \Delta G) = \int \frac{d\vec{x} e^{-\beta H(\vec{x}, t)}}{d\vec{y} e^{-\beta H(\vec{y}, 0)}} = \langle \exp ( -\beta W_t) \rangle .$$

In Ref.\textsuperscript{4} this procedure was applied to the RG trajectories, by using the well-known formal dictionary between the Hamiltonian $H$ of a dynamical system in phase space and the action $S$ of a Euclidean quantum field theory\textsuperscript{84, 85}

$$\beta H(\vec{x}, t) \rightarrow S(\phi, \Lambda) (5.6)$$

where $\Lambda$ denotes the cut-off at which the action of the quantum field theory is evaluated according to the dynamical RG equation.\textsuperscript{84, 85} The RG evolution parameter (“RG time”) is given by the fact that the operation of rescaling formally corresponds to the “temporal” evolution

$$\Lambda \frac{\partial}{\partial \Lambda} \rightarrow \frac{\partial}{\partial \tau} .$$

In Ref.\textsuperscript{4} a Jarzynski-like identity was derived in the context of the renormalization group by following the proof\textsuperscript{83} of the Jarzynski identity for the case of the Liouville dynamics. Note that the renormalization group dynamics is not the Liouville dynamics, because of its fundamental irreversible nature and yet the logic
applied to the context of the Liouville dynamics can be used in the context of the renormalization group in order to arrive at the statement of the Jarzynski-like identity! The important point here is that in the context of the Wilsonian renormalization group one ultimately gets a stochastic like equation which is then solved by averaging over the renormalization group trajectories for the appropriate expressions involving the “free energy” and “work”. This then leads to a new Jarzynski-like identity involving averages over ensembles of RG (and not dynamical) trajectories! Here one should emphasize that both the RG “free energy” and “work” introduced below are defined with respect to the renormalization group formalism and are fully covariant.

More precisely, each RG trajectory is weighted with the appropriate “Boltzmann factor” of the "external work"
\[
W_\tau = \int_0^\tau \frac{\partial S}{\partial \tau'}(\phi_{\tau'}, \tau')d\tau'.
\]
(5.8)

Also, the “free energy” difference is given by definition as
\[
e^{-\Delta G} = \frac{\int D\phi e^{-S(\phi, \tau)}}{\int D\psi e^{-S(\psi, \tau_0)}}
\]
(5.9)

where \(\tau_0\) corresponds to the initial cutoff \(\Lambda_0\). We are thus lead to the RG form of the Jarzynski identity
\[
\exp (-\Delta G) = \frac{\int D\phi e^{-S(\phi, \tau)}}{\int D\psi e^{-S(\psi, \tau_0)}} = \langle \exp (-W_\tau) \rangle.
\]
(5.10)

This equations has not been considered in the literature before, even though it is of an exact form, presumably because the physical construction that leads to the relevant linear stochastic equation which implies this exact equality, is not really motivated without thinking about the original Jarzynski equality.

Next, we equate the work \(W_\tau\) with the work done by the external source. This can be understood very simply by invoking the conjugate relation between the sources and fields with respect to the covariant action. The relation of this type defines generalized forces (in this cases, sources) and thus
\[
W_\tau \equiv -\int J \phi
\]
(5.11)
can be understood as a generalized work (where the integral is over space). We think that this substitution is natural given the covariant nature of the RG Jarzynski identity, and the fact that in the case of vacuum averages, which we have argued replace the average over the RG trajectories, the only “covariant work” is done by sourcing the vacuum. We do not see any other natural candidate for such “covariant work”. We also identify the initial and the final conformal fixed point and apply the above proposal for the Jarzynski-like identity in the renormalizaton group context and then we are led to an AdS/CFT-like relation
\[
\langle \exp(\int J \phi) \rangle = \exp(-\Delta G).
\]
(5.12)

Note, that we have treated \(\phi\) as the fundamental field. The same reasoning can be applied to any general operator \(O\) in the above Euclidean quantum field theory. Now, we would like to appeal to the extra dimension \(\tau\) to argue that this formula can be rewritten as the actual AdS/CFT relation provided:

1) We assume a geometrization of assumed conformal invariance in the \(\tau\) direction, so that the metric in the \(\tau\) direction has the isometries of the conformal group associated with the assumed initial and final conformal fixed points. This leads us to asymptotically AdS metrics \(ds^2 = dr^2 + e^{2A} ds^2_{CFT}\), where \(\tau = Ar\) in the flat coordinate system (\(A\) determining the size of the bulk space) and where \(ds^2_{CFT}\) is the natural flat metric of the boundary CFT.

2) We assume a map between the choice of RG scheme to the choice of coordinates in the \(\tau\) extra dimensional space, thus effectively inducing gravitational interactions in this AdS space. This is reasonable from what we know about perturbative string theory and its relation to the Wilsonian RG\(^ {86–88}\) as well as from what we know about holographic RG in the context of AdS/CFT\(^ {89–96}\). Nevertheless, this might be harder to justify than the first assumption.
As another general caveat we note that the field theories for which we expect holographic duals are gauge theories for which we do not have a nice Wilsonian RG because a cutoff corresponding to a physical length scale typically breaks gauge invariance, and a cutoff for the gauge theory that could be geometrized is not known at present. Also, most field theories do not have a semi-classical gravity dual and thus AdS/CFT should work only for a limited number of quantum field theories. Most probably, the theories for which this duality works can be obtained from the open string sector, in which case AdS/CFT is really an open/closed string duality of a very specific kind (the gravity dual coming from the closed string sector). A nice discussion of this point is given by I. Heemskerk and J. Polchinski in the first reference of Refs. 97, 98, 99, 100, 101.

Finally, we recall that gravity is a very special interaction whose energy is given in terms of boundary data, or symbolically

\[ \Delta G = \Delta S_{\text{bulk}} \]  (5.13)

and thus the RG Jarzynski identity, with above assumptions, becomes the canonical AdS/CFT formula. Note that the semiclassical limit has to come in here, if the expression for the change of the free energy defined in the context of the RG Jarzynski identity is used so that the relative partition function is expanded in some appropriate WKB limit. That WKB expression for the relative partition function will necessarily involve an exponent of some effective action, which could be interpreted as an on-shell “bulk” action. Of course, the reason for the fundamental appearance of gravity is obscure in this heuristic argument. Presumably the true origin of gravity in the AdS/CFT duality should be sought in the open/closed string duality.

Next, we can try to apply the generalizations of Jarzynski’s identity in order to generalize AdS/CFT-like dualities. In that case we do not need to assume conformally invariant fixed points. On the side of non-equilibrium physics, this would correspond to the situation where one has a probability distribution of a steady state (ss) with some parameter \( \alpha \), \( \rho_{ss}(x; \alpha) \), with the corresponding (negative) “entropy” (in the sense of Boltzmann’s definition)

\[ \Phi(x; \alpha) = -\log \rho_{ss}(x; \alpha) \]  (5.14)

Given the general properties of probability distributions one can assert the following mathematical identity (for a discrete time evolution, labeled by \( i = 1, 2, ..., N \))

\[ \langle \prod_{i=0}^{N-1} \frac{\rho_{ss}(x_{i+1}; \alpha_{i+1})}{\rho_{ss}(x_{i+1}; \alpha_i)} \rangle = 1 \]  (5.15)

that implies in the limit \( N \to \infty \) the generalized Jarzynski identity

\[ \langle \exp(-\int_0^t dt' \frac{d\alpha}{dt'} \frac{\partial \Phi(x; \alpha)}{\partial \alpha}) \rangle = 1. \]  (5.16)

The usual Jarzynski identity follows when \( \Phi = -\beta(G - W) \). Given our dictionary between time and the logarithm of the cut-off \( \Lambda (t \to \tau) \) we can obviously translate this general Jarzynski formula into a general AdS/CFT-like formula,

\[ \langle \exp(-\int_0^\tau d\tau' \frac{d\alpha}{d\tau'} \frac{\partial \tilde{\Phi}(x; \alpha)}{\partial \alpha}) \rangle = 1 \]  (5.17)

which, curiously, involves the gradient of “entropy” \( \frac{\partial \tilde{\Phi}}{\partial \alpha} \). (In the usual AdS/CFT case \( \tilde{\Phi} = -(S_{\text{bulk}} + \int J \Omega) \).) This gradient of “entropy” corresponds to some kind of “entropic force”, a concept that has recently been invoked in the context of the holographic treatment of gravity. Thus, it is quite plausible that the concept of entropic force does play a very precise, albeit hidden, role in the AdS/CFT-like dualities. Such a generalized AdS/CFT formula should be useful in illuminating the puzzling duals of cosmological backgrounds or pure (non-conformal) Yang-Mills theory, or various condensed matter systems.
6. Turbulence, Emergent Gauge Symmetries and Strings

In this section we review the proposals\textsuperscript{[2]} for universal steady state distributions for fully developed turbulent flows in two and three dimensions (2d and 3d) building on the previous work which aims to relate modern quantum field theory, string theory and fluid dynamics.\textsuperscript{[105–111]} The basic dynamical equation for turbulent flow is the non-linear Navier-Stokes equation (we use the sum convention throughout the section)

\[ \rho(\partial_t \vec{v}_i + \vec{v}_j \partial_j \vec{v}_i) = -\partial_i p + \nu \partial_j^2 \vec{v}_i, \]  

(6.1)

with the incompressibility condition \( \partial_i \vec{v}_i = 0 \), where \( \vec{v}_i \) is a component of the velocity field of the flow, \( p \) is pressure, and \( \rho \) is the fluid density.\textsuperscript{[105]} In this section we are interested in fully developed turbulence, or turbulence in the limit of infinite Reynolds number \( R \). As \( R = L v / \nu \) goes to infinity, with \( v = \sqrt{\nu \vec{v}_i} \), whereas \( L \) is a characteristic scale and \( \nu \) is the kinematic viscosity, we effectively have the limit of vanishing viscosity \( \nu \to 0 \). In this regime, all the various possible symmetries are restored in a statistical sense, calling for a probabilistic description of what is in essence a deterministic system (strongly dependent on the boundary conditions). Therefore, in computing correlators of the velocity field, we should use the statistical and quantum field theoretic descriptions of turbulence.\textsuperscript{[105–111]}

In particular we are interested in the form of a steady state turbulent distribution \( P(\vec{v}) \) which should imply the various famous scaling laws (the Komogorov scaling in 3d\textsuperscript{[112]} and both Kolmogorov and Kraichnan scaling laws in 2d\textsuperscript{[122]})\textsuperscript{[112]}. The two-point correlators determined by these scalings should be computable from \( P(\vec{v}) \) as follows

\[ \langle \delta v_i(l) \delta v_j(0) \rangle = \frac{\int D\vec{v} \delta v_i(l) \delta v_j(0) P(\vec{v})}{\int D\vec{v} P(\vec{v})}. \]

(6.2)

where \( \delta v_i(l) \) is the \( i \)-th component of the difference vector \( \vec{v}(\vec{x} + \vec{l}) - \vec{v}(\vec{x}) \), with \( l = |\vec{l}| \) being the distance between the fluid elements located at positions \( \vec{x} \) and \( \vec{x} + \vec{l} \). The fundamental question is: what is \( P(\vec{v}) \)? In Ref.\textsuperscript{[3]} it was argued that the answer to this question is:

\[ P(\vec{v}) = \exp[-S_K(\vec{v})], \]

(6.3)

where we assume that the expression for the \( S_K(\vec{v}) \) is local and universal (see below). Both of these assumptions might be challenged: we can a priori expect non-local factors in \( S_K(\vec{v}) \). Also, the relative locations of the Kolmogorov and Kraichnan distributions in 2d turbulence energy spectra depend on the (location of the) forcing, which challenges universality. We also note that the notion of universality might be challenged by an a priori dependence on the boundary conditions. We will simply assume that universal distributions exist in the rest of this section, in spite of these caveats. Finally, we note that if universality applies to turbulence, it is expected at short distance\textsuperscript{[109–111]} as opposed to the usual long distance universality associated with quantum field theory and critical phenomena from equilibrium physics.

The central message is that \( S_K \) is what we call the Kolmogorov distribution in 3d, determined by the effective action for a 3d gauge theory based on volume preserving diffeomorphisms. Alternatively, \( S_K \) is what we call the Kraichnan distribution in 2d, determined by the effective action for a 2d gauge theory based on area preserving diffeomorphisms. In both cases the diffeomorphisms act in velocity space as emergent symmetries. In both cases we crucially use the Galilean symmetry in the space of velocities, so that the effective respective actions involve only derivatives in \( v_i \). We pause briefly to remark that in order to fully characterize systems far from equilibrium knowledge of the probability currents is needed in addition to the steady state probability distributions.\textsuperscript{[114–115]} However, a discussion of these currents is outside of the scope of the present review.

Even though the puzzle of fully developed turbulence is essentially a strongly coupled problem, we motivate our discussion (and ultimately, our proposals) by some rather elementary observations at weak coupling. We start with 2d and then move on to 3d. In 2d the Lagrangian description of the fluid is generated...
by the following Lagrangian\cite{116} (in the notation of Ref. \cite{116})

\[
L_2 = \int d^2y \rho_0 \left[ \frac{m}{2} \dot{x}^2 - V(\rho_0 |\partial y/\partial x|) \right].
\]  

(6.4)

where \(\rho_0\) is the constant density in the co-moving coordinates and the real space density is \(\rho_0 |\partial y/\partial x|\), where \(|\partial y/\partial x|\) denotes the Jacobian connecting the co-moving coordinates \(y\) and the continuum fields \(x_i(y,t)\). The Lagrangian \(L_2\) is invariant under area preserving diffeomorphisms in 2d\cite{116}:

\[
y'_i = y_i + f_i(y) ; \quad \delta x_a = \frac{\partial x_a}{\partial y_i} f_i(y),
\]

(6.5)

where \(\epsilon_{ij}\) being the Levi-Civita symbol in 2d

\[
f_i = \epsilon_{ij} \frac{\partial \Lambda(y)}{\partial y_j},
\]

(6.6)

and where \(\Lambda(y)\) is an arbitrary function, which generates these “gauge” transformations. This equation in turn leads to the 2d Poisson bracket action:

\[
\delta x_a = \epsilon_{ij} \frac{\partial \Lambda(y)}{\partial y_j} \equiv \{\Lambda, x_a\}.
\]

(6.7)

Suppose we consider small motions of the 2d fluid\cite{116}

\[
x_i = y_i + e \epsilon_{ij} A_j,
\]

(6.8)

where \(e\) is the small coupling (and where we have absorbed the factor of \(\rho_0\) compared to Ref. \cite{116}). To linear order the area preserving diffeomorphisms become the usual gauge transformations

\[
\delta A_i = \partial_i \Lambda,
\]

(6.9)

and the quadratic Lagrangian that is invariant under this transformation is just the usual Maxwell Lagrangian

\[
L_2 \sim \frac{1}{2e^2} \int d^2y [A_i^2 - (\nabla \times A)^2].
\]

(6.10)

The inclusion of non-linear terms can be done by extending the linear gauge transformations to their non-Abelian completion. This leads to a non-Abelian gauge theory, with the following map between the full area preserving diffeomorphism group and the non-Abelian transformations generated by a commutator of two matrices

\[
\{\Lambda, x\} \rightarrow [\lambda, X].
\]

(6.11)

Here we have used the standard\cite{122,124} mapping between the Poisson brackets and commutators of infinite square matrices \(\lambda\) and \(X\). The corresponding gauge theory action reads as

\[
S_A = \frac{1}{2e^2} \int d^2y [A_i^2 - (\partial_i A_j - \partial_j A_i + [A_i, A_j])^2].
\]

(6.12)

We will return to this gauge theory action in what follows. In 3d we can extend the presentation of Ref. \cite{116} by considering

\[
L_3 = \int d^3y \rho_0 \left[ \frac{m}{2} \dot{x}^2 - V(\rho_0 |\partial y/\partial x|) \right].
\]

(6.13)

The Lagrangian \(L_3\) is invariant under volume preserving diffeomorphisms

\[
\delta x_a = \frac{\partial x_a}{\partial y_i} \epsilon_{ijk} \frac{\partial \Lambda_1(y)}{\partial y_j} \frac{\partial \Lambda_2(y)}{\partial y_k}.
\]

(6.14)

where \(\Lambda_1\) and \(\Lambda_2\) are the generators of the volume preserving gauge transformations. This is equivalent to the following Nambu bracket\cite{122,124}

\[
\delta x_a \equiv \{\Lambda_1, \Lambda_2, x_a\},
\]

(6.15)
where, by definition

\[ \{ A, B, C \} \equiv \epsilon_{abc} \partial_a A \partial_b B \partial_c C. \]  

(6.16)

Here \( A, B, C \) are three functions of three spatial coordinates \( x, y, z \). This classical bracket seems to be naturally generalized to a triple algebraic structure.\[ \{ A_i, A_j, A_k \} \equiv \epsilon_{abc} A_a A_b A_c. \]  

(6.17)

Suppose we consider small motions of the 3d fluid in analogy with the 2d case:

\[ x_i = y_i + f \epsilon_{ijk} B_{jk}, \]  

(6.18)

where \( B_{jk} = -B_{kj} \) and \( f \) denotes a small coupling. The “quantization” of the volume preserving diffeomorphisms is a more involved problem.\[ B_{jk} \] Nevertheless, even in this case we expect a mapping between the full volume preserving diffeomorphism group and the 3-bracket

\[ \{ \lambda_1, \lambda_2, x \} \rightarrow [\lambda_1, \lambda_2, X], \]  

(6.19)

where \( \lambda_1 \) and \( \lambda_2 \) are the appropriate matrix realizations of \( \lambda_1 \) and \( \lambda_2 \). The 3d action invariant under the linear part of this transformation is

\[ S_B = \frac{1}{2f^2} \int d^3y [\dot{B}_{ij}^2 - (\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij})^2]. \]  

(6.20)

Note that the explicit linear volume preserving transformations are

\[ \delta B_{ij} = \partial_i \lambda_1 \partial_j \lambda_2 - \partial_i \lambda_2 \partial_j \lambda_1, \]  

(6.21)

where \( B_{ij} \) is dual to a 3-vector in three dimensions, \( a_i = \frac{1}{2} \epsilon_{ijk} B_{jk} \).

The above linear analyses seem removed from such a strongly coupled problem as fully developed turbulence.\[ \{ F, v \} \] Still the linear analysis is useful, because in the stationary case we can formally replace \( x_i \rightarrow v_i \) and talk about velocity Lagrangians. That the corresponding Lagrangians should be given in terms of the derivatives of velocity is fixed by the Galilean symmetry, \( v_i \rightarrow v_i + u_i \), for a fixed velocity with component \( u_i \). Moreover, if \( v \sim l^\alpha \), from \( \delta l = \{ F, l \} \) it follows that we have an emergent symmetry in the space of velocities \( \delta v = \{ F, v \} \). Given this emergent symmetry in 2d we have the following natural theory that is consistent with area preserving diffeomorphisms involving \( v_i \):

\[ S_K^{(2)} = \frac{1}{2g^2} \int d^2y [\dot{v}_j (\partial_i v_j - \partial_j v_i + t_0 \{ v_i, v_j \})]^2, \]  

(6.22)

where \( g^2 \) and \( t_0 \) denote the appropriate dimensionful coupling constants and \( i, j = 1, 2 \) (the factor \( \frac{1}{2g^2} \) is canonical). For \( t_0 = 1 \) this theory is invariant under

\[ \delta v_i = \{ F, v_i \}, \]  

(6.23)

where \( F \) generates area preserving gauge transformations in \( v_i \) space. (This is an emergent gauge symmetry in the space of velocities, which should be only the feature of the scaling regime that characterizes fully developed turbulence.) We want to argue that this is the stationary distribution we have been looking for in 2d. First, we attempt to justify this guess on more general grounds: Obviously we have the Galilean invariance in the inertial range \( v_i \rightarrow v_i + u_i \) (where \( u_i \) is constant) because \( S_K^{(2)} \) is a functional of the derivatives of \( v_i \). Second, it is reasonable to expect that \( S_K^{(2)} \) is governed by the conserved quantities. (Recall the case of equilibrium statistical mechanics, i.e. the Boltzmann-Gibbs distribution, where \( S \) is simply the energy, an additive conserved quantity.) In our situation we have vorticity (\( \omega = \nabla \times \vec{v} \)) squared and velocity squared, as the natural conserved quantities.\[ \{ \omega, \omega \} \] Therefore, if we start with the vorticity squared (\( \omega^2 \)) term we see that this is really the quadratic part of our guess for \( S_K^{(2)} \). This term is invariant under the linear gauge transformations in the space of velocities. However, by going to real space, we may invoke the full non-linear group of area preserving coordinate transformations generated by the Jacobian (the Poisson bracket) in 2d
space. Then by concentrating on the steady state regime we may claim the same symmetry in the \(v\) space which would lead us to the above proposal.

This action should be compared to the 2d Yang-Mills theory action \(S_A\) given in equation \(\text{(6.12)}\). We can use the covariant derivative \(\partial_i \rightarrow D_i \equiv \partial_i + A_i\), to rewrite the usual 2d Yang-Mills theory Lagrangian as \([D_i, D_j]^2 \equiv (\partial_i A_j - \partial_j A_i + [A_i, A_j])^2\). This procedure can be immediately generalized to 3d so that we have a theory consistent with volume preserving diffeomorphisms in the space of \(v_i\). By using the covariant derivative \(\partial_i \rightarrow D_i^v \equiv \partial_i + v_i\), we can immediately rewrite the above guess for the 2d action

\[
S_K^{(2)} = \frac{1}{2g^2} \int d^3 y ([D_i^v, D_j^v]^2),
\]

(6.24)

and then extrapolate our 2d proposal to the natural proposal for the Kolmogorov distribution in 3d

\[
S_K^{(3)} = \frac{1}{2g^2} \int d^3 y ([D_i^v, D_j^v, D_k^v]^2).
\]

(6.25)

Here \(\tilde{g}^2\) denotes the appropriate dimensionful coupling constant. The crucial non-linear part which replaces \(\int d^2 y (\{v_i, v_j\}^2)\) is

\[
S_K^{(3)} \sim \frac{1}{2g^2} \int d^3 y (\{v_i, v_j, v_k\}^2).
\]

(6.26)

This action is fixed now by the volume preserving diffeomorphisms in the 3d velocity space

\[
\delta v_i = \{F_1, F_2, v_i\},
\]

(6.27)

where \(F_1\) and \(F_2\) generate the volume preserving gauge transformations. The argument which leads to this proposal is just the repetition of the argument we have presented for the 2d Kraichnan distribution.

These are thus our explicit proposals for the turbulent distributions in 2d and 3d: they are encoded in the expressions for \(S_K^{(2)}\) and \(S_K^{(3)}\). Do these educated guesses give the correct results? We concentrate on the case of 2d turbulence. Let us remember that for the Kraichnan scaling in 2d we want to derive \(\langle \delta v_i(l) \delta v_j(0) \rangle \sim l^{2} \delta_{ij}\) (for \(t_0 = 1\), so that the natural dimension factor \(l/t_0 \rightarrow l\) and for the Kolmogorov scaling law \(\langle \delta v_i(l) \delta v_j(0) \rangle \sim l^{3/2} \delta_{ij}\). (Again using the natural dimensions). In order to discuss the validity of our proposal, we turn to the natural loop variables introduced by Migdal\[132,134\]. The natural loop (closed string) variable, the Migdal loop\[132,134\]

\[
W_M(C) \equiv \langle \exp(-\frac{1}{\nu} \int_C v_i dx_i) \rangle,
\]

(6.28)

(where \(C\) is a contour and the viscosity \(\nu\) plays the role of an effective \(\frac{1}{105-107, 132-134}\)) allows us to rewrite the Naiver-Stokes equations\[132,134\] as an effective Schrodinger equation

\[
i \nu \partial_t W_M(C) = H_C W_M(C),
\]

(6.29)

with the appropriate loop equation Hamiltonian \(H_C\[132,134\] (The loop equation for turbulence is a direct analog of the well-known loop equation for the non-Abelian gauge theory, also proposed by Migdal and collaborators.) In 3d Migdal observed a self-consistent scaling solution of this equation in the \(\nu \rightarrow 0\) limit (a WKB limit in this problem) \(W_M(C) \sim \exp(-\frac{A_0}{4A_0})\), which precisely corresponds to the Kolmogorov scaling. In 2d, the Kraichnan scaling leads to the area law\[105,107\] for the Migdal loop \(W_M(C) \sim \exp(-A_i/A_0)\). In the above gauge theory of 2d velocities this area law is very natural, because of the fact that in the corresponding 2d Yang-Mills theory, the Wilson loop (the natural analog of the Migdal loop)

\[
W(C) \equiv \langle \exp(-\int_C A_i dx_i) \rangle,
\]

(6.30)

obeys the same area law\[130\] \(W(C) \sim \exp(-A_i/A_0)\). Given the precise structural mapping between the 2d Kraichnan theory (defined by \(S_K^{(2)}\)) and the 2d gauge theory, this area law scaling for the loop variables is obeyed, and the Kraichnan scaling thus follows from our proposed steady state distribution. This has been checked in unpublished numerical experiments\[130\].
7. Conclusion: Quantum Gravity vs. Non-equilibrium Physics

In this concluding section of the review we want to argue for a more general relation of quantum gravity and non-equilibrium physics, as presented in Refs. \[137, 138\]. First we summarize some facts known in the theory of dynamical systems. \[139, 140\] Start with a non-linear Hamiltonian system with slight dissipation (see below).

1) Non-linearities generate positive dynamical Lyapunov exponents which ultimately lead to chaotic dynamics (the negative Lyapunov exponents are irrelevant on the long time scales). The chaotic dynamics manifests itself in the emergence of the attractor. There exist natural measures on this attractor, the famous Bowen-Ruelle-Sinai measure (BRS). \[139, 140\]

The relevant equations here are as follows. \[139, 140\] One starts from

\[
\frac{dp_a}{dt} = -\kappa \frac{\partial F}{\partial p_a}, \quad \frac{dq_a}{dt} = p_a. \quad (7.1)
\]

The dissipative term, described by the dissipative functional \(F\), for example

\[
F \sim \sum_{ij} a_{ij} p^i p^j \quad (7.2)
\]

is generated by looking at an infinite system because essentially an infinite reservoir can be understood as a thermostat. Then one integrates out the reservoir degrees of freedom to produce a finite non-Hamiltonian (dissipative) system. Because the dynamics is non-Hamiltonian, the volume of phase space is not preserved! The entropy production is, at the end, crucially related to the volume construction of the phase space volume. Note that it is important here that there is hyperbolic dynamics (i.e. chaos) in the effective dynamical equations

\[
\frac{dx}{dt} = g(x) \quad (7.3)
\]

so that the time evolution \(x(t) = f^t x(0)\) does have positive Lyapunov exponents \(x(t) \sim e^{lt}\), with \(l > 0\). This leads to chaotic time evolution and the stretching of some directions of a unit phase space volume. Finally, there are measures that are invariant under time evolution. The BRS measure \(\rho\) may be singular (it is defined on the attractor which is usually a fractal space) but the time evolution averages are indeed averages over this measure, so that

\[
\lim_{T \to \infty} \frac{1}{T + t} \int_{-T}^{t} d\tau O(f^{T+\tau} x) = \int \rho(dy)O(y) \quad (7.4)
\]

This is the crucial relation which calls for a holographic interpretation in the gravitational context. \[137\] Notice that the dynamics is that of a \(d+1\) dimensional system (in this case the extra dimension \(t\) is indeed a time parameter, but it could be some other evolution parameter such as the radial slicing of \(AdS\) space), that is related to an ensemble description of a \(d\) dimensional system, with a specific (BRS) measure.

2) A crucial aspect of this theory is entropy production and volume contraction (note that dissipation is crucial for the appearance of the attractor and the new measures). The introduction of the new measures is tantamount to the breaking of time-reversal symmetry - which is ultimately the consequence of dissipation. The central equation for the entropy \(S\) in terms of the invariant measure \(\rho\) is

\[
S(\rho) = \int \rho(dx)(-\nabla_x g) \quad (7.5)
\]

where the divergence is computed with respect to the phase space volume element. For the BRS measure one can show that \(S(\rho) \geq 0\). \[139, 140\] Once again, in view of the relation between quantum gravity and non-equilibrium physics argued in this review, it is tempting to interpret this entropy production precisely as the gravitational entropy in the gravitational context.

3) Given the natural dynamically generated measure on the attractor, in the near-to-equilibrium limit of the response functions for this general dynamics one gets an averaged, coarse grained description as required
by the ergodic theorem: i.e. an averaged ensemble description with respect to the above measure. The linear response formulae read as follows.\[ 139, 140 \] Modify the evolution equation by adding small perturbations
\[
\frac{dx}{dt} = g(x) + X_t(x)
\] (7.6)
to compute, via a perturbation in the BRS measure, \( \rho \rightarrow \rho + \delta \rho \)
\[
\int \delta t \rho(\text{dy}) O(y) = \int_{-\infty}^{t} \rho(\text{dy}) X_t(y) \nabla_y [O(f^{t-y})].
\] (7.7)
Note that this should be compared to the usual holographic dictionary for the computations of various correlations functions.

The obvious application of this set-up is in the context of the AdS/CFT correspondence by reinterpreting the evolution parameter as the AdS radial “time”
\[
ds^2 = dr^2 + g_{ij} dx^i dx^j
\] (7.8)
and by writing the AdS/CFT dictionary in terms of the above non-equilibrium result, with \( \rho \) being the Euclidean path integral (i.e. the Gibbs measure). Note that in this case one would consider the boundary of AdS as the attractor for the bulk quantum gravitational evolution viewed as a dynamical system out of equilibrium.

Similarly, this picture could provide a different view on the issue of cosmological holography. For the case of time dependent backgrounds the evolution parameter is time-like as in the context of dynamical systems out of equilibrium
\[
ds^2 = -dt^2 + h_{ij} dx^i dx^j.
\] (7.9)
For a de Sitter (dS)-like asymptotics the attractor (the holographic screen) does not have to be at all regular. For example, the \( I^+ \) of eternal inflation looks like a fractal. On that fractal-like attractor, we can envision placing a holographic dual, given the more general BRS measure\[ 139, 140 \] (and not the Gibbs measure). One obvious use of this picture is the context of dS holography where it has been conjectured that a non-standard CFT\[ 141 \] is dual to an asymptotic dS space, however with a non-standard (non-Gibbsian), BRS measure on the future fractal-like boundary.

Finally, we note that given the fact that the Schrödinger equation can be viewed as a geodesic equation\[ 137, 138 \] and given the fact the the master equation for non-equilibrium physics, such as the Liouville equation, is of the same nature as the Euclidean Schrödinger equation, it is natural to 1) derive the Schrödinger and the master equations from some underlying geometric Einstein equations\[ 137, 138 \] and 2) identify the non-equilibrium equations of this kind with the non-perturbative formulation of quantum gravity. This has been discussed in Refs.\[ 137, 138, 142, 143, 144, 145, 146, 137, 148, 149 \] According to this proposal the non-perturbative string theory viewed as non-perturbative theory of quantum gravity and matter is essentially a new generalized quantum theory, which is related to general non-equilibrium statistical physics, the same way the canonical quantum theory is related to the equilibrium statistical physics. (Note that ’tHooft has also emphasized the relation between dissipative non-equilibrium dynamics, holography and quantum gravity in Ref.\[ 150 \])

Of course, this more general proposal goes beyond what has been reviewed in this article. Nevertheless, we hope that the material presented in this review clearly points to a strong relation between quantum gravity, string theory and non-equilibrium statistical physics that should prove fruitful in finding out new results on both sides of this exciting connection.

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Appendix A. A New Kind of Complementarity

Finally we make a comment about the not so well known complementarity between thermodynamics and statistical physics first emphasized by Bohr and worked out in a bit more detail by Rosenfeld.

Let us examine the energy-time principle

$$\delta E \delta t \sim \hbar$$

and take the well known relation between the Euclidean time ($\tau = i t$) and the temperature ($\tau = \hbar / (k T)$). This relation follows from the formal relation between the Euclidean path integral and the Boltzmann-Gibbs equilibrium partition function (and it is the basis for the simplest derivation of the Unruh-Hawking formula for the uniformly accelerated observer.) By plugging the relation $\tau = \beta \hbar \equiv \hbar / (k T)$ into the Euclidean energy-time uncertainty $\delta E \delta \tau \sim \hbar$ we get (if we properly Wick-rotate) the Bohr-Rosenfeld relation

$$\delta E \delta T \sim k T^2.$$  

(A.2)

Now, this is curious because (as observed first by Niels Bohr and then by Leon Rosenfeld) if one were to compute the fluctuations of energy and temperature a la Boltzmann-Einstein, one would obtain exactly the same formula! We are not aware that Bohr and Rosenfeld made any connection to the energy-time relation. They simply observed that from the fluctuation of the internal energy $E$ and temperature $T$,

$$(\delta E)^2 \sim k T^2 C_V, \quad (\delta T)^2 \sim k T^2 / C_V.$$  

(A.3)

The statistics would enter in the expression for the specific heat $C_V$ at constant volume $V$, but upon multiplication the $C_V$ factor cancels. So the final formula is independent of statistics. The complementarity between the thermodynamic and the statistical descriptions of macroscopic systems as emphasized by Bohr and by Rosenfeld can be understood as follows. In order to assign a definite temperature to the system it is necessary to allow the system to exchange energy with a heat bath, and thus, it is impossible to assign to its energy any definite value. Similarly, in order to keep its energy constant, one must isolate the system, and thus, one cannot assign a definite temperature to it.

First, we observe that this relation between the fluctuations seems to be the same as the Euclidean energy time uncertainty plus the relation between the Euclidean (imaginary) time and temperature. Second, this relation between Euclidean time and temperature gives us the relation between temperature and acceleration (the Unruh-Hawking formula) which then implies

$$\delta E \delta a \sim \hbar a^2$$  

(A.4)

(\text{where we use the Unruh-Hawking relation } T \sim \hbar a). \text{ In the context of string theory}

$$\alpha' \delta E \sim \delta \alpha'$$

(A.5)

and thus one gets, using natural units, that

$$\delta x \delta a \sim \alpha' a^2$$  

(A.6)

which points to a complementarity between the position and surface gravity (acceleration).
Given the current discussion on the black hole complementarity, this not-so-well-known complementarity between thermodynamics and statistical physics, can be immediately extended to the gravitational case, where the main observation would be that the black hole thermodynamics is complementary to the statistical description of black holes given by quantum gravity (string theory).

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$$2M \partial_t \phi = \nabla^2 \phi - \frac{\delta V[\phi]}{\delta \phi} - v(t)\phi + \eta$$

where $v(t)$ is a time-dependent potential. Obviously, this equation can be reduced to the equation through a gauge transformation.

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