An Improved Separation of Regular Resolution from Pool Resolution and Clause Learning

Preliminary version. Comments appreciated.

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Abstract

We prove that the graph tautology principles of Alekhnovich, Johannsen, Pitassi and Urquhart have polynomial size pool resolution refutations that use only input lemmas as learned clauses and without degenerate resolution inferences. We also prove that these graph tautology principles can be refuted by polynomial size DPLL proofs with clause learning, even when restricted to greedy, unit-propagating DPLL search.

1 Introduction

The problem SAT of deciding the satisfiability of propositional CNF formulas is of great theoretical and practical interest. Even though it is NP-complete, industrial instances with hundreds of thousands variables are routinely solved by state of the art SAT solvers. Most of these solvers are based

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on the DPLL procedure extended with clause learning, restarts, variable selection heuristics, and other techniques.

The basic DPLL procedure without clause learning is equivalent to tree-like resolution. The addition of clause learning makes DPLL considerably stronger. In fact, clause learning together with unlimited restarts is capable of simulating general resolution proofs [12]. However, the exact power of DPLL with clause learning but without restarts is unknown. This question is interesting not only for theoretical reasons, but also because of the potential for better understanding the practical performance of various refinements of DPLL with clause learning.

Beame, Kautz, and Sabharwal [3] gave the first theoretical analysis of DPLL with clause learning. Among other things, they noted that clause learning with restarts simulates general resolution. Their construction required the DPLL algorithm to ignore some contradictions, but this situation was rectified by Pipatsrisawat and Darwiche [12] who showed that SAT solvers which do not ignore contradictions can also simulate resolution. These techniques were also applied to learning bounded width clauses by [2].

Beame et al. [3] also studied DPLL clause learning without restarts. Using a method of “proof trace extensions”, they were able to show that DPLL with clause learning and no restarts is strictly stronger than any “natural” proof system strictly weaker than resolution. Here, a natural proof system is one in which proofs do not increase in length when variables are restricted to constants. The class of natural proof systems is known to include common proof systems such as tree-like or regular proofs. The proof trace method involves introducing extraneous variables and clauses, which have the effect of giving the clause learning DPLL algorithm more freedom in choosing decision variables for branching.

There have been two approaches to formalizing DPLL with clause learning as a static proof system rather than as a proof search algorithm. The first is pool resolution with a degenerate resolution inference, due originally to Van Gelder [10] and studied further by Bacchus et al. [9]. Pool resolution requires proofs to have a depth-first regular traversal similarly to the search space of a DPLL algorithm. Degenerate resolution allows resolution inferences in which one or both of the hypotheses may be lacking occurrences of the resolution literal. Van Gelder argued that pool resolution with degenerate resolution inferences simulates a wide range of DPLL algorithms with clause learning. He also gave a proof, based on [1], that pool resolution with degenerate inferences is stronger than regular resolution, using extraneous variables similar to proof trace extensions.

The second approach is due to Buss-Hoffmann-Johannsen [7] who in-
roduced a “partially degenerate” resolution rule called w-resolution, and a
proof system regWRTI based on w-resolution and clause learning of “input
lemmas”. They proved that regWRTI exactly captures non-greedy DPLL
with clause learning. By “non-greedy” is meant that contradictions may
need to be ignored by the DPLL search.

Both [9] and [7] gave improved versions of the proof trace extension
method so that the extraneous variables depend only on the set of clauses
being refuted and not on resolution refutation of the clauses. The drawback
remains, however, that the proof trace extension method gives contrived sets
of clauses and contrived resolution refutations.

It remains open whether any of DPLL with clause learning, pool resolu-
tion (with or without degenerate inferences), or the regWRTI proof system
can polynomially simulate general resolution. One approach to answering
these questions is to try to separate pool resolution (say) from general res-
olution. So far, however, separation results are known only for the weaker
system of regular resolution, based on work of Alekhnovitch et al. [1], who
gave an exponential separation between regular resolution and general res-
olution. Alekhnovitch et al. [1] proved this separation for two families of
tautologies, variants of the graph tautologies GT' and the “Stone” pebbling
tautologies. Urquhart [15] subsequently gave a related separation. In the
present paper, we call the tautologies GT' the guarded graph tautologies,
and henceforth denote them GGT instead of GT'; their definition is given
in Section 2.

Thus, an obvious question is whether pool resolution (say) has poly-
nomial size proofs of the GGT tautologies or the Stone tautologies. The
main result of the present paper resolves the first question by showing that
pool resolution does indeed have polynomial size proofs of the graph tau-
tologies GGT. Our proofs apply to the original GGT principles, without
the use of extraneous variables in the style of proof trace extensions; our
refutations use only the traditional resolution rule and do not require de-
genenerate resolution inferences or w-resolution inferences. In addition, we
use only learning of input clauses; thus, our refutations are also regWRTI
proofs (and in fact regRTI proofs) in the terminology of [7]. As a corollary of
the characterization of regWRTI by [7], the GGT principles have polynomial
size refutations that can be found by a DPLL algorithm with clause learning
and without restarts (under the appropriate variable selection order).

1 Huang and Yu [10] also gave a separation of regular resolution and general resolution,
but only for a single set of clauses. Goerdt [8] gave a quasipolynomial separation of regular
resolution and general resolution.
It is still open if there are polynomial size pool resolution refutations for the Stone principles. However, it is plausible that our methods could extend to give such refutations. It seems more likely that our proof methods could extend to the pebbling tautologies used by [15], as the hardness of those tautologies is due to the addition of randomly chosen “guard” literals, similarly to the GGT tautologies. A much more ambitious project would be to show that pool resolution or regRTI can simulate general resolution, or that DPLL with clause learning and without restarts can simulate general resolution. It is far from clear that this is true, but, if so, our methods below may represent a step in that direction.

The outline of the paper is as follows. Section 2 begins with the definitions of resolution, degenerate resolution, and w-resolution, and then regular, tree, and pool resolution. After that, we define the graph tautologies GT_n and the guarded versions GGT_n, and state the main theorems about proofs of the GGT_n principles. Section 3 gives the proof of the theorems about pool resolution and regRTI proofs. Several ingredients are needed for the proof. The first idea is to try to follow the regular refutations of the graph tautology clauses GT_n as given by Stålmarck [14] and Bonet and Galesi [5]: however, these refutations cannot be used directly since the transitivity clauses of GT_n are “guarded” in the GGT_n clauses and this yields refutations which violate the regularity/pool property. So, the second idea is that the proof search process branches as needed to learn transitivity clauses. This generates additional clauses that must be proved: to handle these, we develop a notion of “bipartite partial order” and show that the refutations of [14, 5] can still be used in the presence of a bipartite partial order. The tricky part is to be sure that exactly the right set of clauses is derived by each subproof. Some straightforward bookkeeping shows that the resulting proof is polynomial size.

Section 4 discusses how to modify the refutations constructed for Section 3 so that they are “greedy” and “unit-propagating. These conditions means that proofs cannot ignore contradictions, nor contradictions that can be obtained by unit propagation. The greedy and unit-propagating conditions correspond well to actual implemented DPLL proof search algorithms, since they backtrack whenever a contradiction can be found by unit propagation. Section 4 concludes with an explicit description of a polynomial time DPLL clause learning algorithm for the GGT_n clauses.

Subsequent to the circulation of a preliminary version of the present paper, Buss and Johanssen [in preparation] have succeeded giving polynomial size regRTI proofs of the pebbling tautologies of [15].
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2 Preliminaries and main results

Propositional formulas are defined over a set of variables and the connectives \( \land, \lor \) and \( \neg \). We use the notation \( \overline{x} \) to express the negation \( \neg x \) of \( x \). A literal is either a variable \( x \) or a negated variable \( \overline{x} \). A clause \( C \) is a set of literals, interpreted as the disjunction of its members. The empty clause, \( \square \), has truth value \( False \). We shall only use formulas in conjunctive normal form, CNF; namely, a formula will be a set (conjunction) of clauses. We often use disjunction (\( \lor \)) and union (\( \cup \)) interchangeably.

**Definition** The various forms of resolution take two clauses \( A \) and \( B \) called the premises and a literal \( x \) called the resolution variable, and produce a new clause \( C \) called the resolvent.

\[
\begin{array}{c}
A & B \\
\hline
C
\end{array}
\]

In all cases below, it is required that \( \overline{\overline{x}} \notin A \) and \( x \notin B \). The different forms of resolution are:

*Resolution rule.* The hypotheses have the forms \( A := A' \lor x \) and \( B := B' \lor \overline{x} \). The resolvent \( C \) is \( A' \lor B' \).

*Degenerate resolution rule.* \( \square \square \) If \( x \in A \) and \( \overline{x} \in B \), we apply the resolution rule to obtain \( C \). If \( A \) contains \( x \), and \( B \) doesn’t contain \( \overline{x} \), then the resolvent \( C \) is \( A \lor B \). If \( A \) doesn’t contain \( x \), and \( B \) contains \( \overline{x} \), then the resolvent \( C \) is \( A \). If neither \( A \) nor \( B \) contains the literal \( x \) or \( \overline{x} \), then \( C \) is the lesser of \( A \) or \( B \) according to some tiebreaking ordering of clauses.

*w-resolution rule.* \( \square \) From \( A \) and \( B \) as above, we infer \( C := (A \setminus \{x\}) \lor (B \setminus \{\overline{x}\}) \). If the literal \( x \notin A \) (resp., \( \overline{x} \notin B \)), then it is called a phantom literal of \( A \) (resp., \( B \)).

**Definition** A resolution derivation, or proof, of a clause \( C \) from a CNF formula \( F \) is a sequence of clauses \( C_1, \ldots, C_s \) such that \( C = C_s \) and such that each clause from the sequence is either a clause from \( F \) or is the resolvent
of two previous clauses. If the derived clause, $C_s$, is the empty clause, this is called a resolution refutation of $F$. The more general systems of degenerate and w-resolution refutations are defined similarly.

We can represent a derivation as a directed acyclic graph (dag) on the vertices $C_1, \ldots, C_s$, where each clause from $F$ has out-degree 0, and all the other vertices from $C_1, \ldots, C_s$ have edges pointing to the two clauses from which they were derived. The empty clause has in-degree 0. We use the terms “proof” and “derivation” interchangeably.

Resolution is sound and complete in the refutational sense: a CNF formula $F$ has a refutation if and only if $F$ is unsatisfiable, that is, if and only if $\neg F$ is a tautology. Furthermore, if there is a derivation of a clause $C$ from $F$, then $C$ is a consequence of $F$: that is, for every truth assignment $\sigma$, if $\sigma$ satisfies $F$ then it satisfies $C$. Conversely, if $C$ is a consequence of $F$ then there is a derivation of some $C' \subseteq C$ from $F$.

A resolution refutation is regular provided that, along any path in the directed acyclic graph, each variable is resolved at most once. A resolution derivation of a clause $C$ is regular provided that, in addition, no variable appearing in $C$ is used as a resolution variable in the derivation. A refutation is tree-like if the underlying graph is a tree; that is, each occurrence of a clause occurring in the refutation is used at most once as a premise of an inference.

We next define a version of pool resolution, using the conventions of \cite{7} who called this “tree-like regular resolution with lemmas”. The idea is that clauses obtained previously in the proof can be used freely as learned lemmas. To be able to talk about clauses previously obtained, we need to define an ordering of clauses.

**Definition** Given a tree $T$, the postorder ordering $<_T$ of the nodes is defined as follows: if $u$ is a node of $T$, $v$ is a node in the subtree rooted at the left child of $u$, and $w$ is a node in the subtree rooted at the right child of $u$, then $v <_T w <_T u$.

**Definition** A pool resolution proof from a set of initial clauses $F$ is a resolution proof tree $T$ that fulfills the following conditions: (a) each leaf is labeled with either a clause of $F$ or a clause (called a “lemma”) that appears earlier in the tree in the $<_T$ ordering; (b) each internal node is labeled with a clause and a literal, and the clause is obtained by resolution from the clauses labeling the node’s children by resolving on the given literal; (c) the proof tree is regular; (d) the roof is labeled with the conclusion clause. If
the labeling of the root is the empty clause \( \Box \), the pool resolution proof is a pool refutation.

The notions of degenerate pool resolution proof and pool w-resolution proof are defined similarly, but allowing degenerate resolution or w-resolution inferences, respectively. Note that the two papers [16, 9] defined pool resolution to be the degenerate pool resolution system, so our notion of pool resolution is more restrictive than theirs. (Our definition is equivalent to the one in [6], however.)

A “lemma” in part (a) of the above definition is called an input lemma if it is derived by input subderivation, namely by a subderivation in which each inference has at least one hypothesis which is a member of \( F \) or is a lemma.

Next we define various graph tautologies, sometimes also called “ordering principles”. They will all use a size parameter \( n > 1 \), and variables \( x_{i,j} \) with \( i,j \in [n] \) and \( i \neq j \), where \([n] = \{0, 1, 2, \ldots, n-1\}\). A variable \( x_{i,j} \) will intuitively represent the condition that \( i \prec j \) with \( \prec \) intended to be a total, linear order. We will thus always adopt the simplifying convention that \( x_{i,j} \) and \( x_{j,i} \) are the identical literal. This identification makes no essential difference to the complexity of proofs of the tautologies, but it reduces the number of literals and clauses, and simplifies the definitions.

The following principle is based on the tautologies defined by Krishna-murthy [11]. These tautologies, or similar ones, have also been studied by [14, 5, 4, 13, 17].

**Definition** Let \( n > 1 \). Then GT\(_n\) is the following set of clauses involving the variables \( x_{i,j} \), for \( i,j \in [n] \) with \( i \neq j \).

\[
(a) \quad \bigvee_{j \neq i} x_{j,i}, \text{ for each value } i < n.
\]

\[
(g) \quad \text{The transitivity clauses } T_{i,j,k} := \overline{x}_{i,j} \lor \overline{x}_{j,k} \lor \overline{x}_{k,i} \text{ for all distinct } i,j,k \in [n].
\]

Note that the clauses \( T_{i,j,k}, T_{j,k,i} \) and \( T_{k,i,j} \) are identical. For this reason Van Gelder [16] uses the name "no triangles" (NT) for a similar principle.

The next definition is from [1], who used the notation GT\(_n\)'. They used particular functions \( r \) and \( s \) for their lower bound proof, but since our upper bound proof does not depend on the details of \( r \) and \( s \) we leave them unspecified. We require that \( r(i,j,k) \neq s(i,j,k) \) and that the set \( \{r(i,j,k), s(i,j,k)\} \nsubseteq \{i,j,k\} \). In addition, w.l.o.g., \( r(i,j,k) = r(j,k,i) = r(k,i,j) \), and similarly for \( s \).
Definition Let $n \geq 1$, and let $r(i, j, k)$ and $s(i, j, k)$ be functions mapping $[n]^3 \rightarrow [n]$ as above. The guarded graph tautology $\text{GGT}_n$ consists of the following clauses:

$(\alpha_\emptyset)$ The clauses $\bigvee_{j \neq i} x_{j,i}$, for each value $i < n$.

$(\gamma'_\emptyset)$ The guarded transitivity clauses $T_{i,j,k} \lor x_{r,s}$ and $T_{i,j,k} \lor \overline{x}_{r,s}$, for all distinct $i, j, k$ in $[n]$, where $r = r(i, j, k)$ and $s = s(i, j, k)$.

Our main result is:

**Theorem 1** The guarded graph tautology principles $\text{GGT}_n$ have polynomial size pool resolution refutations.

The proof of Theorem 1 will construct pool refutations in the form of regular tree-like refutations with lemmas. A key part of this is learning transitive closure clauses that are derived using resolution on the guarded transitivity clauses of $\text{GGT}_n$. A slightly modified construction, that uses a result from [7], gives instead tree-like regular resolution refutations with input lemmas. This will establish the following:

**Theorem 2** The guarded graph tautology principles $\text{GGT}_n$ have polynomial size, tree-like regular resolution refutations with input lemmas.

A consequence of Theorem 2 is that the $\text{GGT}_n$ clauses can be shown unsatisfiable by non-greedy polynomial size DPLL searches using clause learning. This follows via Theorem 5.6 of [7], since the refutations of $\text{GGT}_n$ are regRTI, and hence regWRTI, proofs in the sense of [7].

However, as discussed in Section 4, we can improve the constructions of Theorems 1 and 2 to show that the $\text{GGT}_n$ principles can be refuted also by greedy and unit-propagating polynomial size DPLL searches with clause learning.

### 3 Proof of main theorem

The following theorem is an important ingredient of our upper bound proof.

**Theorem 3** (Stålmarck [14]; Bonet-Galesi [5]; Van Gelder [17]) The sets $\text{GT}_n$ have regular resolution refutations $P_n$ of polynomial size $O(n^3)$.

We do not include a direct proof of Theorem 3 here, which can be found in [14, 5, 17]. The present paper uses the proofs $P_n$ as a “black box”; the only
property needed is that the $P_n$’s are regular and polynomial size. Lemma 4 below is a direct generalization to Theorem 3; in fact, when specialized to the case of $\pi = \emptyset$, it is identical to Theorem 3.

The refutations $P_n$ can be modified to give refutations of $\text{GGT}_n$ by first deriving each transitive clause $T_{i,j,k}$ from the two guarded transitivity clauses of $(\gamma'_\emptyset)$. This however destroys the regularity property, and in fact no polynomial size regular refutations exist for $\text{GGT}_n$.

As usual, a partial order on $[n]$ is an antisymmetric, transitive relation binary relation on $[n]$. We shall be mostly interested in “partial specifications” of partial orders: partial specifications are not required to be transitive.

**Definition** A partial specification, $\tau$, of a partial order is a set of ordered pairs $\tau \subseteq [n] \times [n]$ which are consistent with some (partial) order. The minimal partial order containing $\tau$ is the transitive closure of $\tau$. We write $i \prec_{\tau} j$ to denote $\langle i, j \rangle \in \tau$, and write $i \prec_{\tau}^* j$ to denote that $\langle i, j \rangle$ is in the transitive closure of $\tau$.

The $\tau$-minimal elements are the $i$’s such that $j \prec_{\tau} i$ does not hold for any $j$.

We will be primarily interested in particular kinds of partial orders, called “bipartite” partial orders, that can be associated with partial orders. A bipartite partial order is a partial order that does not have any chain of inequalities $x \prec y \prec z$.

**Definition** A bipartite partial order is a binary relation $\pi$ on $[n]$ such that the domain and range of $\pi$ do not intersect. The set of $\pi$-minimal elements is denoted $M_\pi$.

The righthand side of Figure 1 shows an example. The bipartiteness of $\pi$ arises from the fact that $M_\pi$ and $[n] \setminus M_\pi$ partition $[n]$ into two sets. Note that if $i \prec_{\pi} j$, then $i \in M_\pi$ and $j \notin M_\pi$. In addition, $M_\pi$ contains the isolated points of $\pi$.

**Definition** Let $\tau$ be a specification of a partial order. The bipartite partial order $\pi$ that is associated with $\tau$ is defined by letting $i \prec_{\pi} j$ hold for precisely those $i$ and $j$ such that $i$ is $\tau$-minimal and $i \prec_{\tau}^* j$.

It is easy to check that the $\pi$ associated with $\tau$ is in fact a bipartite partial order. The intuition is that $\pi$ retains only the information about whether $i \prec_{\tau}^* j$ for minimal elements $i$, and forgets the ordering that $\tau$ imposes on non-minimal elements. Figure 1 shows an example of how to obtain a bipartite partial order from a partial specification.

We define the graph tautology $\text{GT}_{\pi,n}$ relative to $\pi$ as follows.
Figure 1: Example of a partial specification of a partial order (left) and the associated bipartite partial order (right).

Definition Let \( \pi \) be a bipartite partial order on \([n]\). Then \( GT_{\pi,n} \) is the set of clauses containing:

\((\alpha)\) The clauses \( \bigvee_{j \neq i} x_{j,i} \), for each value \( i \in M_\pi \).

\((\beta)\) The transitivity clauses \( T_{i,j,k} := \overline{x}_{i,j} \lor \overline{x}_{j,k} \lor \overline{x}_{k,i} \) for all distinct \( i, j, k \) in \( M_\pi \). (Vertices \( i, j, k' \) in Figure 2 show an example.)

\((\gamma)\) The transitivity clauses \( T_{i,j,k} \) for all distinct \( i, j, k \) such that \( i, j \in M_\pi \) and \( i \not< \pi k \) and \( j \prec \pi k \). (As shown in Figure 2.)

The set \( GT_{\pi,n} \) is satisfiable if \( \pi \) is nonempty. As an example, there is the assignment that sets \( x_{j,i} \) true for some fixed \( j \not\in M_\pi \) and every \( i \in M_\pi \), and sets all other variables false. However, if \( \pi \) is applied as a restriction, then \( GT_{\pi,n} \) becomes unsatisfiable. That is to say, there is no assignment which satisfies \( GT_{\pi,n} \) and is consistent with \( \pi \). This fact is proved by the regular derivation \( P_\pi \) described in the next lemma.

Definition For \( \pi \) a bipartite partial order, the clause \( (\bigvee \pi) \) is defined by

\[
(\bigvee \pi) := \{ \overline{x}_{i,j} : i \prec \pi j \},
\]

Lemma 4 Let \( \pi \) be a bipartite partial order on \([n]\). Then there is a regular derivation \( P_\pi \) of \((\bigvee \pi)\) from the set \( GT_{\pi,n} \).

The only variables resolved on in \( P_\pi \) are the following: the variables \( x_{i,j} \) such that \( i, j \in M_\pi \), and the variables \( x_{i,k} \) such that \( k \not\in M_\pi \), \( i \in M_\pi \), and \( i \not< \pi k \).

Lemma 4 implies that if \( \pi \) is the bipartite partial order associated with a partial specification \( \tau \) of a partial order, then the derivation \( P_\pi \) does not resolve on any literal whose value is set by \( \tau \). This is proved by noting that if \( i \prec \tau j \), then \( j \not\in M_\pi \).
Figure 2: A bipartite partial order $\pi$ is pictured, with the ordered pairs of $\pi$ shown as directed edges. (For instance, $j \prec_\pi k$ holds.) The set $M_\pi$ is the set of minimal vertices. The nodes $i, j, k$ shown are an example of nodes used for a transitivity axiom $\overline{\pi}_{i,j} \lor \overline{\pi}_{j,k} \lor \overline{\pi}_{k,i}$ of type ($\gamma$). The nodes $i, j, k'$ are an example of the nodes for a transitivity axiom of type ($\beta$).

Note that if $\pi$ is empty, $M_\pi = [n]$ and there are no clauses of type ($\gamma$). In this case, GT$_{\pi,n}$ is identical to GT$_n$, and $P_\pi$ is the same as the refutation of GT$_n$ of Theorem [3].

**Proof** By renumbering the vertices, we can assume w.l.o.g. that $M_\pi = \{0, \ldots, m-1\}$. For each $k \geq m$, there is at least one value of $j$ such that $j \prec_\pi k$: let $J_k$ be an arbitrary such value $j$. Note $J_k < m$.

Fix $i \in M_\pi$; that is, $i < m$. Recall that the clause of type ($\alpha$) in GT$_{\pi,n}$ for $i$ is $\bigvee_{j \neq i} x_{j,i}$. We resolve this clause successively, for each $k \geq m$ such that $i \not<_{\pi} k$, against the clauses $\overline{\pi}_{i,J_k,k}$ of type ($\gamma$)

\[
\overline{\pi}_{i,J_k} \lor \overline{\pi}_{J_k,k} \lor \overline{\pi}_{k,i}
\]

using resolution variables $x_{k,i}$. (Note that $J_k \neq i$ since $i \not<_{\pi} k$.) This yields a clause $T'_{i,m}$:

\[
\bigvee_{k \geq m \atop i \not<_{\pi} k} \overline{\pi}_{i,J_k} \lor \bigvee_{k \geq m \atop i \not<_{\pi} k} \overline{\pi}_{J_k,k} \lor \bigvee_{k \geq m \atop i \not<_{\pi} k} x_{k,i} \lor \bigvee_{k < m \atop k \neq i} x_{k,i}.
\]

The first two disjuncts shown above for $T'_{i,m}$ come from the side literals of the clauses $\overline{\pi}_{i,J_k,k}$; the last two disjuncts come from the literals in $\bigvee_{j \neq i} x_{j,i}$ which were not resolved on. Since a literal $\overline{\pi}_{i,J_k}$ is the same literal as $x_{J_k,i}$ and since $J_k < m$, the literals in the first disjunct are also contained in the fourth disjunct. Thus, eliminating duplicate literals, $T'_{i,m}$ is equal to the clause

\[
\bigvee_{k \geq m \atop i \not<_{\pi} k} x_{J_k,k} \lor \bigvee_{k \geq m \atop i \not<_{\pi} k} x_{k,i} \lor \bigvee_{k < m \atop k \neq i} x_{k,i}.
\]

Repeating this process, we obtain derivations of the clauses $T'_{i,m}$ for all $i < m$. The final disjuncts of these clauses, $\bigvee_{i \not<_{\pi} k < m} x_{k,i}$, are the same as the
(α∅) clauses in GTₘ. Thus, the clauses T'ᵢₘ give all (α∅) clauses of GTₘ, but with literals ㏖ⱼₖ and ᵃⱼᵢ added in as side literals. Moreover, the clauses of type (β) in GTᵢₙ are exactly the transitivity clauses of GTₘ. All these clauses can be combined exactly as in the refutation of GTₘ described in Theorem 3 but carrying along extra side literals ㏖ⱼₖ and ᵃⱼᵢ, or equivalently carrying along literals ㏖ⱼₖ for ㏖ⱼₖ ≺ᵢₙ k, and ㏖ᵢₖ for i ≺ᵢₙ k. Since the refutation of GTₘ uses all of its transitivity clauses and since each ㏖ⱼₖ literal is also one of the ㏖ᵢₖ’s, this yields a resolution derivation Pₙ of the clause 

\{㏖ᵢₖ : i ≺ᵢₙ k\}.

This is the clause (∨ᵢₙ) as desired.

Finally, we observe that Pₙ is regular. To show this, note that the first parts of Pₙ deriving the clauses T'ᵢₘ are regular by construction, and they use resolution only on variables ᵃⱼᵢ with k ≥ m, i < m, and i ≠ πₖ. The remaining part of Pₙ is also regular by Theorem 3 and uses resolution only on variables ᵃᵢⱼ with i, j ≤ m. □

Proof of Theorem 1 We will show how to construct a series of “LR partial refutations”, denoted R₀, R₁, R₂, . . .; this process eventually terminates with a pool resolution refutation of GGTₙ. The terminology “LR partial” indicates that the refutation is being constructed in left-to-right order, with the left part of the refutation properly formed, but with many of the remaining leaves being labeled with bipartite partial orders instead of with valid learned clauses or initial clauses from GGTₙ. We first describe the construction of the pool refutation, and leave the size analysis to the end.

An LR partial refutation R is a tree with nodes labeled with clauses that form a correct pool resolution proof, except possibly at the leaves (the initial clauses). Furthermore, it must satisfy the following conditions.

a. R is a tree. The root is labeled with the empty clause. Each non-leaf node in R has a left child and right child; the clause labeling the node is derived by resolution from the clauses on its two children.

b. For each clause C occurring in R, the clause C⁺ and the set of ordered pairs τ(C) are defined by

C⁺ := \{㏖ᵢⱼ : ㏖ᵢⱼ occurs in some clause on the branch from the root node R to C\},

and τ(C) = \{(i, j) : ㏖ᵢⱼ ∈ C⁺\}. Note that C ⊆ C⁺ holds by definition. In many cases, τ(C) will be a partial specification of a partial order,
but this is not always true. For instance, if $C$ is a transitivity axiom, then $\tau(C)$ has a 3-cycle and is not consistent as a specification of a partial order.

d. Each finished leaf $L$ is labeled with either a clause from $\text{GGT}_n$ or a clause that occurs to the left of $L$ in the postorder traversal of $R$.

e. For an unfinished leaf labeled with clause $C$, the set $\tau(C)$ is a partial specification of a partial order. Furthermore, letting $\pi$ be the bipartite partial order associated with $\tau(C)$, the clause $C$ is equal to $(\bigvee \pi)$.

Property e. is particularly crucial and is novel to our construction. As shown below, each unfinished leaf, labeled with clause $C = (\bigvee \pi)$, will be replaced by a derivation $S$. The derivation $S$ often will be based on $P_{\pi}$, and thus might be expected to end with exactly the clause $C$; however, some of the resolution inferences needed for $P_{\pi}$ might be disallowed by the pool property. This can mean that $S$ will instead be a derivation of a clause $C'$ such that $C \subseteq C' \subseteq C^+$. The condition $C' \subseteq C^+$ is required because any literal $x \in C' \setminus C$ will be handled by modifying the refutation $R$ by propagating $x$ downward in $R$ until reaching a clause that already contains $x$. The condition $C' \subseteq C^+$ ensures that such a clause exists. The fact that $C' \supseteq C$ will mean that enough literals are present for the derivation to use only (non-degenerate) resolution inferences — by virtue of the fact that our constructions will pick $C$ so that it contains the literals that must be present for use as resolution literals. The extra literals in $C' \setminus C$ will be handled by propagating them down the proof to where they are resolved on.

The construction begins by letting $R_0$ be the “empty” refutation, containing just the empty clause. Of course, this clause is an unfinished leaf, and $\tau(\emptyset) = \emptyset$. Thus $R_0$ is a valid LR partial refutation.

For the induction step, $R_i$ has been constructed already. Let $C$ be the leftmost unfinished clause in $R_i$. $R_{i+1}$ will be formed by replacing $C$ by a refutation $S$ of some clause $C'$ such that $C \subseteq C' \subseteq C^+$.

We need to describe the (LR partial) refutation $S$. Let $\pi$ be the bipartite partial order associated with $\tau(C)$, and consider the derivation $P_{\pi}$ from Lemma[4]. Since $C = (\bigvee \pi)$ by condition e., the final line of $P_{\pi}$ is the clause $C$.

The intuition is that we would like to let $S$ be $P_{\pi}$. The first difficulty with this is that $P_{\pi}$ is dag-like, and the LR-refutation is intended to be tree-like, This difficulty, however, can be circumvented by just expanding $P_{\pi}$, which is regular, into a tree-like regular derivation with lemmas by the simple expedient of using a depth-first traversal of $P_{\pi}$. The second, and more serious, difficulty is that $P_{\pi}$ is a derivation from $\text{GT}_n$, not $\text{GGT}_n$. Namely,
the derivation $P_\pi$ uses the transitivity clauses of $\text{GT}_n$ instead of the guarded transitivity clauses of $\text{GGT}_n$. The transitivity clauses $T_{i,j,k} := \overline{x}_{i,j} \lor \overline{x}_{j,k} \lor \overline{x}_{k,i}$ in $P_\pi$ are handled one at a time as described below. We will use four separate constructions: in case (i), no change to $P_\pi$ is required; cases (ii) and (iii) require small changes; and in the fourth case, the subproof $P_\pi$ is abandoned in favor of “learning” the transitivity clause.

Before doing the four constructions, it is worth noting that Lemma 4 implies that no literal in $C^+$ is used as a resolution literal in $P_\pi$. To prove this, suppose $x_{i,j}$ is a resolution variable in $P_\pi$. Then, from Lemma 4 we have that at least one of $i$ and $j$ is $\pi$-minimal and that $i \not\prec \pi j$ and $j \not\prec \pi i$. Thus $i \not\prec r(C)$ and $j \not\prec r(C)i$, so $r(C)$ contains neither $x_{i,j}$ nor $\overline{x}_{t,j}$.

By the remark made after Lemma 4, no literal in $C^+$ is used as a resolution literal in $P_\pi$.

(i) If an initial transitivity clause of $P_\pi$ already appears earlier in $R_i$ (that is, to the left of $C$), then it is already learned, and can be used freely in $P_\pi$.

In the remaining cases (ii)-(iv), the transitivity clause $T_{i,j,k}$ is not yet learned. Let the guard variable for $T_{i,j,k}$ be $x_{r,s}$, so $r = r(i,j,k)$ and $s = s(i,j,k)$.

(ii) Suppose case (i) does not apply and that the guard variable $x_{r,s}$ or its negation $\overline{x}_{r,s}$ is a member of $C^+$. The guard variable thus is used as a resolution variable somewhere along the branch from the root to clause $C$. Then, as just argued above, Lemma 4 implies that $x_{r,s}$ is not resolved on in $P_\pi$. Therefore, we can add the literal $x_{r,s}$ or $\overline{x}_{r,s}$ (respectively) to the clause $T_{i,j,k}$ and to every clause on any path below $T_{i,j,k}$ until reaching a clause that already contains that literal. This replaces $T_{i,j,k}$ with one of the initial clauses $T_{i,j,k} \lor x_{r,s}$ or $T_{i,j,k} \lor \overline{x}_{r,s}$ of $\text{GGT}_n$. By construction, it preserves the validity of the resolution inferences of $R_i$ as well as the regularity property. Note this adds the literal $x_{r,s}$ or $\overline{x}_{r,s}$ to the final clause $C'$ of the modified $P_\pi$. This maintains the property that $C \subseteq C' \subseteq C^+$.

(iii) Suppose case (i) does not apply and that $x_{r,s}$ is not used as a resolution variable anywhere below $T_{i,j,k}$ in $P_\pi$ and is not a member of $C^+$. In this case, $P_\pi$ is modified so as to derive the clause $T_{i,j,k}$ from the two GGT$_n$ clauses $T_{i,j,k} \lor x_{r,s}$ and $T_{i,j,k} \lor \overline{x}_{r,s}$ by resolving on $x_{r,s}$. This maintains the regularity of the derivation. It also means that henceforth $T_{i,j,k}$ will be learned.

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If all of the transitivity clauses in $P_\pi$ can be handled by cases (i)-(iii), then we use $P_\pi$ to define $R_{i+1}$. Namely, let $P'_\pi$ be the derivation $P_\pi$ as modified by the applications of cases (ii) and (iii). The derivation $P'_\pi$ is regular and dag-like, so we can recast it as a tree-like derivation $S$ with lemmas, by using a depth-first traversal of $P'_\pi$. The size of $S$ is linear in the size of $P'_\pi$, since only input lemmas need to be repeated. The final line of $S$ is the clause $C'$, namely $C$ plus the literals introduced by case (ii). The derivation $R_{i+1}$ is formed from $R_i$ by replacing the clause $C$ with the derivation $S$ of $C'$, and then propagating each new literal $x \in C' \setminus C$ down towards the root of $R_i$, adding $x$ to each clause below $S$ until reaching a clause that already contains $x$. The derivation $S$ contains no unfinished leaf, so $R_{i+1}$ contains one fewer unfinished leaves than $R_i$.

On the other hand, if even one transitivity axiom $T_{i,j,k}$ in $P_\pi$ is not covered by the above three cases, then case (iv) must be used instead. This introduces a completely different construction to form $S$:

(iv) Let $T_{i,j,k}$ be any transitivity axiom in $P_\pi$ that is not covered by cases (i)-(iii). In this case, the guard variable $x_{r,s}$ is used as a resolution variable in $P_\pi$ somewhere below $T_{i,j,k}$; in general, this means we cannot use resolution on $x_{r,s}$ to derive $T_{i,j,k}$ while maintaining the desired pool property. Hence, $P_\pi$ is no longer used, and we instead will form $S$ with a short left-branching path that “learns” $T_{i,j,k}$. This will generate two or three new unfinished leaf nodes. Since unfinished leaf nodes in a LR partial derivation must be labeled with clauses from bipartite partial orders, it is also necessary to attach short derivations to these unfinished leaf nodes to make the unfinished leaf clauses of $S$ correspond correctly to bipartite partial orders. These unfinished leaf nodes are then kept in $R_{i+1}$ to be handled at later stages.

There are separate constructions depending on whether $T_{i,j,k}$ is a clause of type ($\beta$) or ($\gamma$); details are given below.

First suppose $T_{i,j,k}$ is of type ($\gamma$), and thus $\overline{x}_{j,k}$ appears in $C$. (Refer to Figure 2.) Let $x_{r,s}$ be the guard variable for the transitivity axiom $T_{i,j,k}$.

The derivation $S$ will have the form

\[
\overline{x}_{i,j}, \overline{x}_{j,k}, \overline{x}_{k,i}, x_{r,s} \quad S_1 \ldots \quad \overline{x}_{i,j}, \overline{x}_{j,k}, x_{r,s} \quad S_2 \ldots
\]

\[
\overline{x}_{i,j}, \overline{x}_{j,k}, \overline{x}_{k,i}, \overline{x}_{r,s}
\]

\[
\overline{x}_{i,j}, \overline{x}_{j,k}, \overline{x}_{k,i}, \overline{x}_{r,s}
\]

\[
\overline{x}_{i,j}, \overline{x}_{j,k}, \overline{x}_{k,i}, \overline{x}_{r,s}
\]
The notation $\pi_{-[jk]}$ denotes the disjunction of the negations of the literals in $\pi$ omitting the literal $\pi_{j,k}$. We write “$iR(j)$” to indicate literals $x_{i,\ell}$ such that $j \prec_{\pi} \ell$. (The “$R(j)$” means “range of $j$”.) Thus $\pi_{-[jk:iR(j)]}$ denotes the clause containing the negations of the literals in $\pi$, omitting $\pi_{j,k}$ and any literals $\pi_{i,\ell}$ such that $j \prec_{\pi} \ell$. The clause $\pi_{-[jk;jR(i)]}$ is defined similarly, and the notation extends to more complicated situations in the obvious way.

The upper leftmost inference of $S$ is a resolution inference on the variable $x_{r,s}$. Since $T_{i,j,k}$ is not covered by either case (i) or (ii), the variable $x_{r,s}$ does not appear in or below clause $C$ in $R_i$. Thus, this use of $x_{r,s}$ as a resolution variable does not violate regularity. Furthermore, since $T_{i,j,k}$ is of type $(\gamma)$, we have $i \not\prec_{\pi} j, j \not\prec_{\pi} i, i \not\prec_{\pi} k, k \not\prec_{\pi} i$. Thus the literals $x_{i,j}$ and $x_{i,k}$ do not appear in or below $C$, so they also can be resolved on without violating regularity.

Let $C_1$ and $C_2$ be the final clauses of $S_1$ and $S_2$, and let $C_1^-$ be the clause below $C_1$ and above $C$. The set $\tau(C_2)$ is obtained by adding $(j,i)$ to $\tau(C)$, and similarly $\tau(C_1^-)$ is $\tau(C)$ plus $(i,j)$. Since $T_{i,j,k}$ is type $(\gamma)$, we have $i, j \in M_\pi$. Therefore, since $\tau(C)$ is a partial specification of a partial order, $\tau(C_2)$ and $\tau(C_1^-)$ are also both partial specifications of partial orders. Let $\pi_2$ and $\pi_1$ be the bipartite orders associated with these two partial specifications (respectively). We will form the subproof $S_1$ so that it contains the clause $(\vee_{\pi_1})$ as its only unfinished clause. This will require adding inferences in $S_1$ which add and remove the appropriate literals. The first step of this type already occurs in going up from $C_1^-$ to $C_1$ since this has removed $\pi_{j,k}$ and added $\pi_{i,k}$, reflecting the fact that $j$ is not $\pi_1$-minimal and thus $x_{i,k} \in \pi_1$ but $x_{j,k} \notin \pi_1$. Similarly, we will form $S_2$ so that its only unfinished clause is $(\vee_{\pi_2})$.

We first describe the subproof $S_2$ of $S$. The situation is pictured in Figure 3, which shows an extract from Figure 2, the edges shown in part (a) of the figure correspond to the literals present in the final line $C_2$ of $S_2$. In particular, recall that the literals $\pi_{i,\ell}$ such that $j \prec_{\pi} \ell$ are omitted from the last line of $S_2$. (Correspondingly, the edge from $i$ to $\ell_1$ is omitted from Figure 3.) The last line $C_2$ of $S_2$ may not correspond to a bipartite partial order as it may not partition $[n]$ into minimal and non-minimal elements; thus, the last line of $S_2$ may not qualify to be an unfinished node of $R_{i+1}$.

(An example of this in Figure 3(a) is that $j \prec_{\tau(C_2)} i \prec_{\tau(C_2)} \ell_2$, corresponding to $\pi_{j,i}$ and $\pi_{i,\ell_2}$ being in the last line of $S_2$.) The bipartite partial order $\pi_2$ associated with $\tau(C_2)$ is equal to the bipartite partial order that agrees with $\pi$ except that each $i \prec_{\pi} \ell$ condition is replaced with the condition $j \prec_{\pi_2} \ell$. (This is represented in Figure 3(b) by the fact that the edge from $i$ to $\ell_2$ has been replaced by the edge from $j$ to $\ell_2$. Note that the vertex $i$ is
no longer a minimal element of \( \pi \); that is, \( i \notin M_{\pi_2} \). We wish to form \( S_2 \) to be a regular derivation of the clause \( \overline{\pi}_{j,i} \pi_{[jk;IR(j)]} \) from the clause \( \overline{\sqrt{\pi}_2} \).

The subproof of \( S_2 \) for replacing \( \overline{x}_{i,\ell_2} \) in \( \overline{\pi} \) with \( \overline{x}_{j,\ell_2} \) in \( \overline{\pi}_2 \) is as follows, letting \( \overline{\pi}^* \) be \( \overline{\pi}_{[jk;IR(j)];\ell_2} \).

\[
\begin{align*}
S'[\ldots] & \quad \text{rest of } S_2 \\
\overline{x}_{j,i}, \overline{x}_{i,\ell_2}, \overline{\pi}_{\ell_2, j} & \quad \overline{x}_{j,k}, \overline{x}_{j,\ell_2}, \overline{x}_{j,i}, \overline{\pi}^* \\
\overline{x}_{j,k}, \overline{x}_{i,\ell_2}, \overline{\pi}_{j,i} \quad \overline{\sqrt{\pi}_2} & \quad (1)
\end{align*}
\]

The part labeled “rest of \( S_2 \)” will handle similarly the other literals \( \ell \) such that \( i \prec_{\pi} \ell \) and \( j \not\prec_{\pi} \ell \). The final line of \( S'_2 \) is the transitivity axiom \( T_{j,i,\ell_2} \). This is a GT\(_{n} \) axiom, not a GGT\(_{n} \) axiom; however, it can be handled by the methods of cases (i)-(iii). Namely, if \( T_{j,i,\ell_2} \) has already been learned by appearing somewhere to the left in \( R_i \), then \( S'_2 \) is just this single clause. Otherwise, let the guard variable for \( T_{j,i,\ell_2} \) be \( x_{r',s'} \). If \( x_{r',s'} \) is used as a resolution variable below \( T_{j,i,\ell_2} \), then replace \( T_{j,i,\ell_2} \) with \( T_{j,i,\ell_2} \lor x_{r',s'} \) or \( T_{j,\ell_2} \lor \overline{x}_{r',s'} \), and propagate the \( x_{r',s'} \) or \( \overline{x}_{r',s'} \) to clauses down the branch leading to \( T_{j,i,\ell_2} \) until reaching a clause that already contains that literal. Finally, if \( x_{r',s'} \) has not been used as a resolution variable in \( R_i \) below \( C \), then let \( S'_2 \) consist of a resolution inference deriving (and learning) \( T_{j,i,\ell_2} \) from the clauses \( T_{j,i,\ell_2}, x_{r',s'} \) and \( T_{j,i,\ell_2}, \overline{x}_{r',s'} \).

To complete the construction of \( S_2 \), the inference (1) is repeated for each value of \( \ell \) such that \( i \prec_{\pi} \ell \) and \( j \not\prec_{\pi} \ell \). The result is that \( S_2 \) has one unfinished leaf clause, and it is labelled with the clause \( \overline{\sqrt{\pi}_2} \).

We next describe the subproof \( S_1 \) of \( S \). The situation is shown in Figure 4. As in the formation of \( S_2 \), the final clause \( C_1 \) in \( S_1 \) may need to be modified in order to correspond to the bipartite partial order \( \pi_1 \) which is associated with \( \tau(C_1) \). First, note that the literal \( \overline{x}_{j,k} \) is already replaced by \( \overline{x}_{i,k} \) in the final clause of \( S_1 \). The other change that is needed is that, for every \( \ell \) such that \( j \prec_{\pi} \ell \) and \( i \not\prec_{\pi} \ell \), we must replace \( \overline{x}_{j,\ell} \) with \( \overline{x}_{i,\ell} \) since we have \( j \not\prec_{\pi_1} \ell \) and \( i \prec_{\pi_1} \ell \). Vertex \( \ell_3 \) in Figure 4 is an example of a such a value \( \ell \). The ordering in the final clause of \( S_1 \) is shown in part (a), and the desired ordered pairs of \( \pi_1 \) are shown in part (b). Note that \( j \) is no longer

Figure 3: The partial orders for the fragment of \( S_2 \) shown in (1).
Figure 4: The partial orders for the fragment of $S_1$ shown in (2).

a minimal element in $\pi_1$.

The replacement of $x_{j,\ell}$ with $x_{i,\ell}$ is effected by the following inference, letting $\pi^*$ now be $\pi^*_{jk; jR(i); j\ell}$.

\[
\begin{align*}
S_1' & \vdash \ldots \ldots \\
\forall i, j, \ell : x_{i,\ell}, x_{j,\ell}, \pi^* & \vdash \ldots \ldots \\
\end{align*}
\]

The “rest of $S_1$” will handle similarly the other literals $\ell$ such that $j \prec_\pi \ell$ and $i \not\prec_\pi \ell$. Note that the final clause of $S_1'$ is the transitivity axiom $T_{i,j,\ell}$. The subproof $S_1'$ is formed in exactly the same way that $S_2'$ was formed above. Namely, depending on the status of the guard variable $x_{r,s}$ for $T_{i,j,\ell}$, one of the following is done: (i) the clause $T_{i,j,\ell}$ is already learned and can be used as is, or (ii) one of $x_{r,s}$ or $x_{r,s'}$ is added to the clause and propagated down the proof, or (iii) the clause $T_{i,j,\ell}$ is inferred using resolution on $x_{r,s}$ and becomes learned.

To complete the construction of $S_1$, the inference (2) is repeated for each value of $\ell$ such that $j \prec_\pi \ell$ and $i \not\prec_\pi \ell$. The result is that $S_1$ has one unfinished leaf clause, and it corresponds to the bipartite partial order $\pi_1$.

That completes the construction of the subproof $S$ for the subcase of (iv) where $T_{i,j,k}$ is of type ($\gamma$). Now suppose $T_{i,j,k}$ is of type ($\beta$). (For instance, the values $i, j, k'$ of Figure 2) In this case the derivation $S$ will have the form

\[
\begin{align*}
T_{i,j,k}, x_{r,s} & \vdash \ldots \ldots \\
T_{i,j,k}, x_{r,s} & \vdash \ldots \ldots \\
S_3 & \vdash \ldots \ldots \\
S_4 & \vdash \ldots \ldots \\
S_5 & \vdash \ldots \ldots \\
\end{align*}
\]

where $x_{r,s}$ is the guard variable for $T_{i,j,k}$. We write $[\pi_{jR(i); j\ell}]$ to mean the negations of literals in $\pi$ omitting any literal $\overline{x}_{j,\ell}$ such that both $i \prec_\pi \ell$ and
\( k \prec_\pi \ell \). Similarly, \( \overline{\pi}_{\prec \pi} \cdot [jR(i),kR(ij)] \) indicates the negations of literals in \( \pi \), omitting the literals \( \overline{x}_{j,\ell} \) such that \( i \prec_\pi \ell \) and the literals \( \overline{x}_{k,\ell} \) such that either \( i \prec_\pi \ell \) or \( j \prec_\pi \ell \).

Note that the resolution on \( x_{r,s} \) used to derive \( T_{i,j,k} \) does not violate regularity, since otherwise \( T_{i,j,k} \) would have been covered by case (ii). Likewise, the resolutions on \( x_{i,j}, x_{i,k} \) and \( x_{j,k} \) do not violate regularity since \( T_{i,j,k} \) is of type (\( \beta \)).

The subproof \( S_5 \) is formed exactly like the subproof \( S_2 \) above, with the exception that now the literal \( \overline{x}_{j,k} \) is not present. Thus we omit the description of \( S_5 \).

We next describe the construction of the subproof \( S_4 \). Let \( C_4 \) be the final clause of \( S_4 \); it is easy to check that \( \tau(C_4) \) is a partial specification of a partial order. As before, we must derive \( C_4 \) from the clause \( (\overline{\pi}_4) \) where \( \pi_4 \) is the bipartite partial order associated with the partial order \( \tau(C_4) \). A typical situation is shown in Figure 5. As pictured there, it is necessary to add the literals \( \overline{x}_{i,\ell} \) such that \( j \prec_\pi \ell \) and \( i \neq \pi \ell \), while removing \( \overline{x}_{j,\ell} \); examples of this are \( \ell \) equal to \( \ell_2 \) and \( \ell_3 \) in Figure 5. At the same time, we must add the literals \( \overline{x}_{k,\ell} \) such that \( j \prec_\pi \ell \) and \( k \neq \pi \ell \), while removing \( \overline{x}_{j,\ell} \); examples of this are \( \ell \) equal to \( \ell_1 \) and \( \ell_2 \) in the same figure.

For a vertex \( \ell_3 \) such that \( j \prec_\pi \ell_3 \) and \( k \prec_\pi \ell_3 \) but \( i \neq \pi \ell_3 \), this is done similarly to the inferences (1) and (2) but without the side literal \( \overline{x}_{j,k} \):

\[
\begin{array}{c}
S'_4 \vdash \cdots \vdash \overline{x}_{i,j}, \overline{x}_{i,\ell_3}, \overline{x}_{i,j}, \overline{x}_{i,\ell_3}, \overline{x}_{i,j}, \overline{x}_{i,\ell_3}, \pi^* \\
\overline{x}_{j,\ell_3}, \overline{x}_{k,j}, \overline{x}_{j,\ell_3}, \overline{x}_{i,j}, \overline{x}_{i,\ell_3}, \pi^*
\end{array}
\]

Here \( \pi^* \) is \( \overline{\pi}_{\prec \pi} \cdot [jR(i\cap k), j\ell_3j] \). The transitivity axiom \( T_{i,j,\ell_3} \) shown as the last line of \( S'_4 \) is handled exactly as before. This construction is repeated for all such \( \ell_3 \)’s.

The vertices \( \ell_1 \) such that \( j \prec_\pi \ell_1 \) and \( i \prec_\pi \ell_1 \) but \( k \neq \pi \ell_1 \) are handled in exactly the same way. (The side literals of \( \pi^* \) change each time to reflect the literals that have already been replaced.)

Finally, consider a vertex \( \ell_2 \) such that \( i \neq \pi \ell_2 \) and \( j \prec_\pi \ell_2 \) and \( k \neq \pi \ell_2 \). This is handled by the derivation

\[
\begin{array}{c}
S'_4 \vdash \cdots \vdash \overline{x}_{i,j}, \overline{x}_{i,\ell_2}, \overline{x}_{i,\ell_2}, \pi^* \\
\overline{x}_{i,j}, \overline{x}_{i,\ell_2}, \overline{x}_{i,\ell_2}, \pi^*
\end{array}
\]
As before, the set $\pi^*$ of side literals is changed to reflect the literals that have already been added and removed as $S_4$ is being created. The subproofs $S''_4$ and $S''''_4$ of the transitivity axioms $T_{i,j,\ell_2}$ and $T_{k,j,\ell_2}$ are handled exactly as before, depending on the status of their guard variables.

Finally, we describe how to form the subproof $S_3$. For this, we must form the bipartite partial order $\pi_3$ which is associated with the partial order $\tau(C_3)$, where $C_3$ is the final clause of $S_3$. To obtain $\pi_3$, we need to add the literals $x_{i,\ell}$ such that $i \not\preceq \pi \ell$ and such that either $j \preceq \pi \ell$ or $k \preceq \pi \ell$, while removing any literals $x_{j,\ell}$ and $x_{k,\ell}$. This is done by exactly the same construction used above in (3). The literals in $\pi^* - [jR(i);kR(i,\ell)]$ are exactly the literals needed to carry this out. The construction is quite similar to the above constructions, and we omit any further description.

That completes the description of how to construct the LR partial refutations $R_i$. The process stops once some $R_i$ has no unfinished clauses. We claim that the process stops after polynomially many stages.

To prove this, recall that $R_{i+1}$ is formed by handling the leftmost unfinished clause using one of cases (i)-(iv). In the first three cases, the unfinished clause is replaced by a derivation based on $P_{\pi}$ for some bipartite order $\pi$. Since $P_{\pi}$ has size $O(n^3)$, this means that the number of clauses in $R_{i+1}$ is at most the number of clauses in $R_i$ plus $O(n^3)$. Also, by construction, $R_{i+1}$ has one fewer unfinished clauses than $R_i$. In case (iv) however, $R_{i+1}$ is formed by adding up to $O(n)$ many clauses to $R_i$ plus adding either two or three new unfinished leaf clauses. In addition, case (iv) always causes at least one transitivity axiom $T_{i,j,k}$ to be learned. Therefore, case (iv) can occur at most $2 \binom{n}{3} = O(n^3)$ times. Consequently at most $3 \cdot 2 \binom{n}{3} = O(n^3)$ many unfinished clauses are added throughout the entire process. It follows that the process stops with $R_i$ having no unfinished clauses for some $i \leq 6 \binom{n}{3} = O(n^3)$. Therefore there is a pool refutation of $\text{GGT}_n$ with $O(n^6)$ lines.

By inspection, each clause in the refutation contains $O(n^2)$ literals. This is because the largest clauses are those corresponding to (small modifications...
of bipartite partial orders, and because bipartite partial orders can contain at most $O(n^2)$ many ordered pairs. Furthermore, the refutations $P_n$ for the graph tautology $\text{GT}_n$ contain only clauses of size $O(n^2)$.

Q.E.D. Theorem 1

Theorem 2 is proved with nearly the same construction. In fact, the only change needed for the proof is the construction of $S$ from $P'_\pi$. Recall that in the proof of Theorem 1 the pool derivation $S$ was formed by using a depth-first traversal of $P$. This is not sufficient for Theorem 2 since now the derivation $S$ must use only input lemmas. Instead, we use Theorem 3.3 of [7], which states that a (regular) dag-like resolution derivation can be transformed into a (regular) tree-like derivation with input lemmas. Forming $S$ in this way from $P'_\pi$ suffices for the proof of Theorem 2: the lemmas of $S$ are either transitive closure axioms derived earlier in $R_i$ or are derived by input subproofs earlier in the post-ordering of $S$. Since the transitive closure axioms that appeared earlier in $R_i$ are derived by resolving two $\text{GGT}_n$ axioms, the lemmas used in $S$ are all input lemmas.

The transformation of Theorem 3.3 of [7] may multiply the size of the derivation by the depth of the original derivation. Since it is possible to form the proofs $P_\pi$ with depth $O(n)$, the overall size of the pool resolution refutations with input lemmas is $O(n^7)$. This completes the proof of Theorem 2.

4 Greedy, unit-propagating DPLL with clause learning

This section discusses how the refutations in Theorems 1 and 2 can be modified so as to ensure that the refutations are greedy and unit-propagating.

**Definition** Let $R$ be a tree-like regular $w$-resolution refutation with input lemmas. For $C$ a clause in $R$, let $C^+$ be the set of literals which occur as literals or phantom literals in clauses on the path from $C$ to the root of $R$. (Recall that “phantom literals” are literals used for $w$-resolution that are not actually present in the clauses.) Also, let $\Gamma(C)$ be the set of clauses of $\Gamma$ plus every clause $D <_R C$ in $R$ that has been derived by an input subproof and thus is available as a learned clause to aid in the derivation of $C$.

The refutation $R$ is **greedy and unit-propagating** provided that, for each clause $C$ of $R$, if there is an input derivation from $\Gamma(C)$ of some clause $C' \subseteq C^+$ which does not resolve on any literal in $C^+$, then $C$ is derived in $R$ by such a derivation.
Note that, as proved in [3], the condition that there is an input derivation from $\Gamma(C)$ of some $C' \subseteq C^+$ which does not resolve on $C^+$ literals is equivalent to the condition that if all literals of $C^+$ are set false then unit propagation yields a contradiction from $\Gamma(C)$. (In [3], these are called “trivial” proofs.) This justifies the terminology “unit-propagating”.

The definition of “greedy and unit-propagating” is actually a bit more restrictive than necessary, since DPLL algorithms may actually learn multiple clauses at once, and this can mean that $C$ is not derived from a single input proof but rather from a combination of several input proofs as described in the proof of Theorem 5.1 in [7].

**Theorem 5** The guarded graph tautology principles $\text{GGT}_n$ have greedy, unit-propagating, polynomial size, tree-like, regular $w$-resolution refutations with input lemmas.

**Proof** We indicate how to modify the proofs of Theorems [1] and [2]. We again build tree-like LR partial refutations satisfying the same properties a.-e. as before, except now $w$-resolution inferences are permitted. Instead of being formed in distinct stages $R_0, R_1, R_2, \ldots$, the $w$-resolution refutation $R$ is constructed by one continuing process. This construction incorporates all of transformations (i)-(iv) and also incorporates the construction of Theorem 3.3 of [7].

At each point in the construction, we will be scanning the so-far constructed partial $w$-resolution refutation $R$ in preorder, namely in depth-first, left-to-right order. That is to say, the construction recursively processes a node in the proof tree, then its left subtree, and then its right subtree. During most steps of the preorder scan, the partial refutation $R$ is modified by changing the part that comes subsequently in the preorder, but the construction may also add and remove literals from clauses below the current clause $C$. When the preorder scan reaches a clause $C$ that has an input derivation $R'$ from $\Gamma(C)$ of some $C' \subseteq C$ that does not resolve on $C^+$, then some such $R'$ is inserted into $R$ at that point. When the preorder scan reaches an unfinished leaf $C = C_0$, then a (possibly exponentially large) derivation $P^*_\pi$ is added as its derivation. The construction continues processing $R$ by scanning $P^*_\pi$ in preorder, with the end result that either (1) $P^*_\pi$ is successfully processed and reduced to only polynomial size or (2) the preorder scan of $P^*_\pi$ reaches a transitivity clause $T_{i,j,k}$ of the type that triggered case (iv) of Theorem [1]. In the latter case, the preorder scan backs up to the root clause $C_0$ of $P^*_\pi$, replaces $P^*_\pi$ with the derivation $S$ constructed in case (iv) of Theorem [1] and restarts the preorder scan at clause $C_0$. 

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We describe the actions of the preorder scan in more detail. Initially, 
R is the “empty” derivation, with the empty clause as its only (unfinished) 
clause. A clause C encountered during the preorder scan of R is handled by 
one of the following.

(i') Suppose that some \( C' \subseteq C^+ \) can be derived by an input derivation 
from \( \Gamma(C) \) that does not resolve on any literals of \( C^+ \). Fix any 
such \( C' \subseteq C^+ \), and replace the subderivation in \( R \) of the clause C 
with such a derivation of \( C' \) from \( \Gamma(C) \). Any extra literals in \( C' \setminus C \) 
are in \( C^+ \) and are propagated down until reaching a clause where they 
already appear, or occur as a phantom literal. There may also be literals 
in \( C \setminus C' \); these literals are removed as necessary from clauses 
below \( C' \) in \( R \) to maintain the property of \( R \) containing correct w-
resolution inferences. Note that this can convert resolution inferences 
into w-resolution inferences.

The clause \( C' \) is now a learned clause. Note that this case includes 
transitivity clauses \( C = C' = T_{i,j,k} \) that satisfy the conditions of cases 
(i)-(iii) of Theorem 1

(ii') If case (i') does not apply, and \( C \) is not a leaf node, then \( R \) is unchanged 
at this point and the depth-first traversal proceeds to the next clause.

(iii') If \( C \) is an unfinished clause of the form \((\lor \pi)\), let \( P_\pi \) be as before. 
Recall that no literal in \( C^+ \) is resolved on in \( P_\pi \). Unwind the proof \( P_\pi \) 
into a tree-like regular refutation \( P^*_\pi \) that is possibly exponentially big, 
and attach \( P^*_\pi \) to \( R \) as a proof of \( C \). Mark the position of \( C \) by setting 
\( C_0 = C \) in case it is necessary to later backtrack to \( C \). Then continue 
the preorder scan by traversing into \( P^*_\pi \).

(iv') Otherwise, \( C \) is an initial clause of the form \( T_{i,j,k} \) and since case (i') 
does not apply, one of \( T_{i,j,k} \)'s guard literals \( x \), namely \( x = x_{r,s} \) or 
\( x = \overline{x}_{r,s} \), is in \( C^+ \). If \( C \) is not inside the most recently added \( P^*_\pi \) or if 
\( x \in C_0^+ \), then replace \( T_{i,j,k} \) with \( T_{i,j,k} \lor x \), and propagate the literal \( x \) 
downward in the refutation until reaching a clause where it appears as 
a literal or a phantom literal. Otherwise, the preorder scan backtracks 
to the root clause \( C_0 \) of \( P^*_\pi \), and replaces \( P^*_\pi \) with the partial resolution 
refutation \( S \) formed in case (iv) of Theorem 1.

It is clear that this process eventually halts with a valid greedy, unit-
propagating, tree-like w-resolution refutation. We claim that it also yields 
a polynomial size refutation. Consider what happens when a derivation \( P^*_\pi \)
is inserted. If case \((iv')\) is triggered, then the proof \(S\) is inserted in place of \(P^*\), so the size of \(P^*\) does not matter. If case \((iv')\) is not triggered, then, as in the proof of Theorem 3.3 of [7], the preorder scan of \(P^*\) modifies (the possibly exponentially large) \(P^*\) to have polynomial size. Indeed, as argued in [7], any clause \(C\) in \(P^*\) will occur at most \(d_C\) times in the modified version of \(P^*\) where \(d_C\) is the depth of the derivation of \(C\) in the original \(P^*\). This is because, \(C\) will have been learned by an input derivation once it has appeared no more than \(d_C\) times in the modified derivation \(P^*\). This is proved by induction on \(d_C\).

Consider the situation where \(S\) has just been inserted in place of \(P^*\) in case \((iii')\). The transitivity clause \(T_{i,j,k}\) is not yet learned at this point, since otherwise case \((i')\) would have applied. We claim, however, that \(T_{i,j,k}\) is learned as \(S\) is traversed. To prove this, since \(T_{i,j,k}\) is manifestly derived by an input derivation and since its guard literals \(x_{r,s}\) and \(\overline{x}_{r,s}\) do not appear in \(C_0^+\), it is enough to show that the clause \(T_{i,j,k}\) is reached in the preorder traversal scan of \(S\). This, however, is an immediate consequence of the fact that \(T_{i,j,k}\) was reached in the preorder scan of \(P^*\) and triggered case \((iv')\), since if case \((i')\) applies to \(T_{i,j,k}\) or to any clause below \(T_{i,j,k}\) in the preorder scan of \(S\), then it certainly also applies \(T_{i,j,k}\) or some clause below \(T_{i,j,k}\) in the preorder scan of \(P^*\).

The size of the final refutation \(R\) is bounded the same way as in the proof of Theorem 2, and this completes the proof of Theorem 5.

\[\square\]

**Theorem 6** There are DPLL search procedures with clause learning which are greedy, unit-propagating, but do not use restarts, that refute the \(\text{GGT}_n\) clauses in polynomial time.

We give a sketch of the proof. The construction for the proof of Theorem 5 requires only that the clauses \(T_{i,j,k}\) are learned whenever possible, and does not depend on whether any other clauses are learned. This means that the following algorithm for DPLL search with clause learning will always succeed in finding a refutation of the \(\text{GGT}_n\) clauses: At each point, there is a partial assignment \(\tau\). The search algorithm must do one of the following:

1. If unit propagation yields a contradiction, then learn a clause \(T_{i,j,k}\) if possible, and backtrack.

2. Otherwise, if there are any literals in the transitive closure of the bipartite partial order associated with \(\tau\) which are not assigned a value, branch on one of these literals to set its value. (One of the true or false assignments yields an immediate conflict, and may allow learning a clause \(T_{i,j,k}\).)
(3) Otherwise, determine whether there is a clause $T_{i,j,k}$ which is used in the proof $P_\pi$ whose guard literals are resolved on in $P_\pi$. (See Lemma [4]) If not, do a DPLL traversal of $P_\pi$, eventually backtracking from the assignment $\tau$.

(4) Otherwise, the clause $T_{i,j,k}$ blocks $P_\pi$ from being traversed in polynomial time. Branch on its variables in the order given in the proof of Theorem [1]. From this, learn the clause $T_{i,j,k}$.

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