Evacuation from a Finite 2D Square Grid Field by a Metamorphic Robotic System

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Abstract

We consider evacuation from a finite two-dimensional (2D) square grid field by a metamorphic robotic system (MRS). An MRS is composed of anonymous memoryless modules. Each module of an MRS executes an identical distributed algorithm and moves autonomously while keeping the connectivity of modules. Since the modules are memoryless, an MRS utilizes its shape to remember the progress of execution. The number of available shapes that an MRS can form depends on the number of modules, which is thus an important complexity measure for a behavior of an MRS. In this paper, we investigate the minimum number of modules required to solve the evacuation problem with several conditions. First, we consider a rectangular field surrounded by walls with at least one exit and show that two modules are necessary and sufficient for evacuation from any rectangular field if the modules are equipped with a global compass, which allows the modules to have a common sense of direction. Then, we focus on the case where modules do not have a global compass and show that four (resp. seven) modules are necessary and sufficient for restricted (resp. any) initial states of an MRS. We also show that two modules are sufficient in the special case where an MRS is on a wall in an initial configuration. Finally, we extend these results to another type of fields, that is, mazes.

1 Introduction

Modular robotic systems composed of a large number of modules, each of which autonomously moves, attract much attention both in robotics [16–18] and theoretical computer science [1, 10–13]. Since the modules collectively achieve a given task, the main focus has been put on distributed coordination. One of the most important distributed computing models is the metamorphic robotic system (MRS) [12, 13], that consists of a set of anonymous modules in a two-dimensional (2D) square grid. Each cell of the grid can accept at most one module at each time step, and each module can perform two types of movements, i.e., rotation and sliding. A rotation is a rotation by π/2 clockwise or counter-clockwise around another static module, and a sliding is a straight movement along a straight line of static modules. However, the modules must keep connectivity, which is defined by side-adjacency of cells occupied by the modules. The modules are anonymous, uniform, and oblivious; they are indistinguishable, execute a common algorithm, and are memoryless. The modules are synchronously activated in each time step, observe the positions of other modules, compute their movement, and perform the movement. The MRS changes its shape and moves...
toward a destination by local movements of modules. Existing literature \cite{1,10,13} has investigated the computational power of the MRS with a variety of problems and evaluation criteria.

Dumitrescu et al. considered reconfiguration, that requires the MRS to change its initial shape to a given target shape \cite{13}. They showed that any horizontally convex shape can be translated into a horizontal chain shape, i.e., a line. A shape of the MRS is horizontally convex if the set of modules in the same row is connected. Thus, any horizontally convex shape can be translated into another horizontally convex shape via the chain shape by the reversibility of local movements. They also consider decidability of related reconfiguration problems.

Dumitrescu et al. considered locomotion of the MRS to a given direction \cite{12}. They demonstrated how the MRS achieves fastest locomotion to a given vertical direction and a given diagonal direction. Chen et al. proposed a locomotion algorithm for modules that can observe cells within a constant distance \cite{1}.

Doi et al. proposed search by the MRS \cite{10,11}. The MRS is required to reach a target cell in an unknown finite 2D rectangular field from an arbitrary initial configuration. The authors focused on the number of modules necessary and sufficient for the MRS to achieve search because the shape of anonymous modules serves as memory of the MRS; modules can store, for example, a moving direction and progress of search. They also demonstrated the effect of the global compass; when the modules agree on the north, south, east, and west, we say they are equipped with the global compass. They showed that when the modules are equipped with the global compass, three modules are necessary and sufficient from an arbitrary initial shape. Thus, the MRS guarantees self-stabilizing search. When the modules are not equipped with the global compass, modules cannot perform any movement in some initial shapes due to their symmetry. For example, when the four modules form a square, none of them can move because symmetric modules may perform symmetric movements, and no module can serve as a static module for movement. They showed that when the modules are not equipped with the global compass but share a common handedness, five modules are necessary and sufficient from restricted initial shapes. They also showed that seven modules with the same ability are necessary and sufficient from arbitrary initial shapes. Table 1 summarizes above results.
Table 1: Summary of the related work and our contribution

| Problem       | Algorithm | Field shape | Global compass | Visibility | $n$  | Initial states                                      |
|---------------|-----------|-------------|----------------|------------|------|-----------------------------------------------------|
| Shape formation (line) | [13] | Any         | Yes            | Unlimited  | $\geq 2$ | Horizontally or vertically convex shapes            |
| Locomotion    | [12]      | Any         | Yes            | $7 \times 7$ | 2    | Horizontally & vertically convex shapes             |
|               | [1]       | Any         | Yes            | $7 \times 7$ | 4    | Restricted (4 forbidden states)                     |
| Search        | [11]      | Rectangular | No             | $11 \times 11$ | 7    | Any                                                 |
| Evacuation    | Sect. 3   | Rectangular | Yes            | $5 \times 5$ | 2    | Any                                                 |
|               | Sect. 4.1 | Rectangular | No             | $7 \times 7$ | 4    | Restricted (4 forbidden states)                     |
|               | Sect. 4.2 | Rectangular | No             | $11 \times 11$ | 7    | Any                                                 |
|               | Sect. 4.3 | Rectangular | No             | $5 \times 5$ | 2    | Starting on a wall                                  |
|               | Sect. 5   | Maze        | Yes/No/No      | $5/7/11 \times 5/7/11$ | 2/4/7 | The same as the cases of rectangular fields       |
Our contribution. In this paper, we first consider evacuation from an unknown finite 2D rectangular field by the MRS. A field is surrounded by walls, and there is at least one exit on the walls. The MRS is required to exit from the field without any a priori knowledge of the field, the position of the exit, or its initial position. To the best of our knowledge, this is the first time evacuation is considered for the MRS. We show that when the modules are equipped with a global compass, two modules are necessary and sufficient for evacuation from an arbitrary initial shape, and when the modules are not equipped with a global compass but share a common handedness, four modules are necessary and sufficient from restricted initial shapes. Then, we show that in the latter setting, seven modules are necessary and sufficient for evacuation from an arbitrary initial shape, and two modules are necessary and sufficient for evacuation from a configuration where the MRS is initially on a wall. This result separates the problem of moving to a wall from the problem of searching for an exit on the wall. Finally, we demonstrate that our evacuation algorithm can be extended to evacuation from an unknown maze composed of fields.

Related work. In distributed computing theory, a variety of theoretical distributed computing models for modular robotic systems have been proposed. The mobile robot system considers a set of anonymous mobile robots moving in the 2D Euclidean space [15]. The robots are anonymous, uniform, communication-less, and have no access to the global coordinate system. Each robot repeats a cycle, where it observes the positions of other robots, computes its next position, and moves to the next position. A robot is oblivious if the input to computation is the preceding observation, otherwise non-oblivious. Di Luna et al. showed that a constant number of oblivious mobile robots can simulate a single non-oblivious mobile robot by encoding the contents of the local memory of a non-oblivious robot to their geometric positions [14].

The programmable particle model consists of anonymous particles with constant size local memory in the infinite 2D triangular grid [6]. Each vertex of the triangular grid can accept at most one particle at each time step. Each particle can detect whether a neighboring vertex is occupied by another particle or not and communicate with particles at neighboring vertices. Each particle moves by repeating expansion and contraction: a particle is expanded when it occupies two neighboring vertices and contracted when it occupies a single vertex. Daymude et al. showed that programmable particles collectively form a global counter by forming a chain shape, and they can form a convex hull of an object of unknown size [5]. Di Luna et al. considered shape formation by programmable particles and showed that the programmable particles can simulate a Turing machine [8] and RAM [7].

Cooperative evacuation has been considered for mobile robots initially placed at the center of a disk with an exit on its boundary [2]. The goal is to minimize the time required for all the robots to exit. Czyzowicz et al. considered mobile robots equipped with two types of communication medium; wireless communication and local communication [2]. They presented evacuation algorithms and lower bounds of evacuation time for two and three robots. Czyzowicz et al. considered evacuation of three robots equipped with wireless communication when at least one of them is faulty [3], and evacuation of at most four robots when one of the robots is the “queen” and the goal is to guide the queen to the exit [4]. Evacuation has many applications such as exploration, rescue, map construction, and navigation in dangerous terrain, disaster area, and so on.

All these results are expected to serve as theoretical foundations for many related areas, such as robotics, navigation, autonomous vehicles, nano-manufacturing, molecular robotics, and understanding natural systems.

Organization. This paper is organized as follows: Section 2 defines our model and problems considered here. First, we consider the evacuation problem from a rectangular field. In Section 3, we discuss evacuation by an MRS composed of modules equipped with a global compass. Then, in Section 4, we consider the problem with modules not equipped with a global compass. Finally, we extend these results to another type of field, that is, a maze in Section 5. We conclude this paper in Section 6.
2 Preliminaries

2.1 Model

We consider the rectangular MRS introduced in [1, 10–13] and a 2D square grid where each square cell \( c_{i,j} \) is labeled by the underlying \( x\)-\( y \) coordinate system as shown in Fig. 1. Each cell \( c_{i,j} \) has eight adjacent cells: East (E) \( c_{i+1,j} \), NorthEast (NE) \( c_{i+1,j+1} \), North (N) \( c_{i,j+1} \), NorthWest (NW) \( c_{i-1,j+1} \), West (W) \( c_{i-1,j} \), SouthWest (SW) \( c_{i-1,j-1} \), South (S) \( c_{i,j-1} \), and SouthEast (SE) \( c_{i+1,j-1} \). In addition, we say that the four cells, N, S, E, and W, are side-adjacent to \( c_{i,j} \). An infinite sequence of cells with the same \( x \) (resp. \( y \)) coordinate is called a column (resp. a row). A field is a rectangular subgrid in the square 2D grid. There are the interior and the exterior in the field. The interior is surrounded by walls that separate it from the exterior. A wall is composed of cells in the same row or column, and a module cannot occupy a cell of a wall. We assume that the interior is sufficiently large in contrast with the number of modules. Without loss of generality, we assume that \( c_{0,0} \) (resp. \( c_{w-1,h-1} \)) is the southwesternmost cell (resp. the northeasternmost cell) of the interior. The interior has at least one exit to the exterior. The exit is on the wall and it consists of sequence of cells. The interior is only connected to the exterior by an exit. These cell labels are used just for description, and there is no way to distinguish cells. We consider various types of fields and detail them in Section 2.3.

An MRS, \( R \), consists of \( n \) (\( \geq 2 \)) anonymous modules, each of which occupies a distinct cell in the square grid at discrete time step \( t = 0, 1, 2, \cdots \). The configuration \( C_t \) of \( R \) at time \( t \) is the set of cells occupied by the modules at time \( t \). An execution is an evolution of configurations \( C_0, C_1, C_2, \cdots \). The evolution is generated by movements of modules. Let \( M_t \) be a set of modules that move at time \( t \). We call a set \( B_t \) of the modules that do not move at time \( t \) (that is, \( B_t = C_t \setminus M_t \)) a backbone. Modules have two types of movement, rotation and sliding, which are both guided by backbone modules as shown in Fig. 2. By a rotation, a module \( m \) moves either clockwise or counter-clockwise around a side-adjacent backbone module \( b \) by \( \pi/2 \) (Fig. 2(a)). By a 1-sliding, a module \( m \) moves to a side-adjacent cell along two backbone modules, \( b_1 \) and \( b_2 \) (Fig. 2(b)). In order to take this movement, the backbone module \( b_1 \) (resp. \( b_2 \)) must be side-adjacent to \( m \) (resp. \( b_1 \)). A \( k \)-sliding
\((k \geq 2, 3, \cdots)\) can be defined in the same way, and requires \((k + 1)\) backbone modules along the track of the sliding. For these movements, the cells that \(m\) passes through during a movement must not be occupied by any modules.

The connectivity of configuration \(C_t\) is represented by a connectivity graph \(G_t = (C_t, E_t)\). The edge set \(E_t\) contains an edge \((c, c')\) for \(c, c' \in C_t\) if and only if cells \(c\) and \(c'\) are side-adjacent. If a connectivity graph \(G_t\) induced by a configuration \(C_t\) is connected, we say a configuration \(C_t\) is connected. Any execution \(C_0, C_1, C_2, \cdots\) of an MRS must satisfy the following three conditions:

- **Connectivity**: for any time \(t\), \(C_t\) is connected.
- **Single backbone**: for any time \(t\), \(B_t\) is connected.
- **No interference**: for any time \(t\), the trajectories of any two moving modules never overlap.

Modules of an MRS are uniform; they are anonymous and execute an identical deterministic distributed algorithm. Modules are oblivious, that is, they have no memory. Modules behave synchronously; at each time step, each module observes the cells in its neighborhood and decides its movement. We say that a cell \(c_{i', j'}\) is a \(k\)-neighborhood of cell \(c_{i, j}\) if \(|i' - i| \leq k\) and \(|j' - j| \leq k\). We assume that a module can distinguish whether each cell in its \(k\)-neighborhood is occupied by a module or a wall and that \(k\) is constant with respect to the size of a field. A view of a module \(m\) is \((2k + 1) \times (2k + 1)\) square subgrid centered at the cell that \(m\) occupies. A distributed algorithm of neighborhood size \(k\) is defined by a total function that maps a view to the cell to which a module moves in this step. We assume that a module knows \(n\), and an algorithm is not universal; an algorithm designed for an MRS composed of exactly \(n\) modules.

If the modules are equipped with a global compass, they share common North, South, East, and West directions. Otherwise, they do not know directions and their observations may be inconsistent. However, we assume that the modules agree on the clockwise direction; they share a common handedness.

The state of an MRS \(R\) in \(C_t\) is the local shape of \(R\). We denote by \(S^n\) a state of \(R\) composed of \(n\) modules. If the modules are equipped with a global compass, a state of \(R\) contains global directions. Otherwise, it does not contain any direction because the modules cannot distinguish any rotation of their state in the lack of a global compass.

If the modules are equipped with a global compass, the execution of a given algorithm is uniquely determined by \(C_0\) because the modules can agree on a total ordering among themselves. On the other hand, if the modules are not equipped with a global compass, there exist multiple executions from \(C_0\) depending on the local compass of each module.

### 2.2 Problems

Here, we consider the following evacuation problem.

**Definition 1** (Evacuation). A metamorphic robotic system is required to evacuate from the interior through an exit starting from any initial position in the interior of a field and any initial shape.

We call the evacuation from the interior to the exterior through an exit on a field “the evacuation from the field”, for short.

As one of building blocks of the proposed evacuation algorithms, we use existing algorithms \cite{1, 11} that solve the following locomotion problem.

**Definition 2** (Locomotion). A metamorphic robotic system in an infinite 2D square grid is required to keep on moving in one direction.

Hence, to solve the locomotion problem, modules of an MRS must break their initial symmetry and agree on a moving direction.
In this section, we will describe the types of fields of the evacuation problem more precisely. First, we prove the following lemma from the basic moves of an MRS.

**Lemma 1.** An exit of a field must consist of at least two cells.

**Proof.** This lemma can be proven from the fact that every basic move shown in Fig. 2 requires two or more cells that are not occupied by walls, as indicated in Fig. 3. Note that modules of an MRS must keep their connectivity during a move; thus, each module is impossible to pass an exit one by one apart from the other modules.

For contradiction, we assume that there is an exit composed of one cell. To pass the exit by a rotation, there must be a cell with a backbone module next to the exit (Fig. 3(a)). However, this is impossible because the cell is occupied by the wall. Similarly, passing the exit by a \( k \)-sliding is also impossible because doing so also requires a backbone module in the wall (Fig. 3(b)). On the other hand, if the exit consists of two or more cells, the backbone module does not need to be located in the cell that is part of a wall; thus, a module can pass through the exit by a rotation or a \( k \)-sliding. \( \square \)

We consider the evacuation problem in the following types of fields. From Lemma 1, we assume that every exit consists of two or more side-adjacent cells.

**Definition 3.** A rectangular field \( F \) has a rectangular interior whose width and height are \( w \) and \( h \) cells, respectively (\( w, h \geq 2 \)). The interior of \( F \) is surrounded by four walls located at its North, South, East, and West, whose thickness is one cell. The walls have at least one exit.

**Definition 4.** A maze \( M \) is composed of rectangular fields. The walls of a rectangular field in \( M \) are shared by other rectangular fields in \( M \). An exit of a rectangular field in \( M \) connects to an exit of another rectangular field or the exterior. The maze \( M \) has at least one exit to the exterior.

Figures 4 and 5 illustrate the examples of rectangular fields and mazes, respectively.

We represent a side-adjacency graph of a set \( V_c \) of cells by \( G(V_c) = (V_c, E_c) \) where an edge \( e = \{c, c'\} \) is in \( E_c \) if and only if cells \( c \) and \( c' \) in \( V_c \) are side-adjacent. For a field \( F \), we denote the set of all the wall cells, the set of all exit cells, and the set of all cells in the interior by \( W_F \), \( X_F \), and \( I_F \), respectively.

To guarantee the solvability of evacuation from a maze, we assume that every maze \( M \) considered here satisfies the following conditions:
1. In the side-adjacency graph $G(I_M)$, every cell $c$ in $I_M$ has at least one path to a cell that is side-adjacent to an exit cell in $X_M$.

2. The side-adjacency graph $G(W_M \cup X_M)$ is connected.

Figure 6 shows examples that do not satisfy one of these conditions. Condition 1 guarantees the existence of an evacuation path for every cell in the interior. The maze in Fig. 6(a) does not satisfy Condition 1; thus, an MRS cannot evacuate from the interior of the maze because it cannot find any exit. Condition 2 ensures that an exit can be reached by trailing walls. To evacuate from the maze in Fig. 6(b), an MRS $R$ must place some kind of token on a cell to remember that $R$ has already searched the cell so that it does not search the same cell again. However, the MRS (and modules) assumed here does not have such function. Thus, we assume that Condition 2 is satisfied.

3. Evacuation from a rectangular field with a global compass

Here, we prove the following theorem to show that $n = 2$ is necessary and sufficient for evacuation from any rectangular field if modules are equipped with a global compass.

**Theorem 1.** Two modules equipped with a global compass are necessary and sufficient for a metamorphic robotic system to solve the evacuation problem in any given rectangular field from arbitrary initial states.

The necessary condition of Theorem 1 (i.e., an MRS composed of only one module cannot solve the evacuation problem) can be proven easily because any movement of a module requires at least one other backbone module. In the remainder of this section, we prove that $n = 2$ is sufficient by demonstrating an algorithm that solves the evacuation problem with two modules.

Movements of an MRS $R$ in the proposed algorithm is composed of the following two parts:

- **Reaching a wall:** $R$ moves in one direction from its initial position and its initial shape until reaching a wall.
Figure 7: An example evacuation path of an MRS from a rectangular field

Figure 8: Locomotion by two modules with a global compass

- **Searching for an exit**: $R$ moves along the walls to find an exit and evacuates from the interior through the exit found.

Figure 7 illustrates an example trail of an MRS performing these movements. The proposed algorithm uses the locomotion algorithm (Fig. 8) proposed in [1] for the reaching a wall part. In the algorithm, module $a$ rotates in a clockwise direction around backbone module $b$ if the westmost module $a$ is side-adjacent to module $b$. The MRS moves east by repeating these movements until a module of the MRS touches the wall cell.

The algorithm starts searching for an exit after reaching a wall. The MRS moves along walls until reaching a corner or an exit, as depicted in Fig. 9. While an MRS can move in an arbitrary direction (clockwise or counter-clockwise), we assume that an MRS moves so that the wall cells are always on its left-hand side (i.e., the clockwise direction). When an MRS arrives at a corner, it changes direction $\pi/2$ in a clockwise direction (Fig. 10). After turning, the MRS moves along walls again with the movements shown in Fig. 9.

After reaching a wall, an MRS eventually finds an exit of the interior of a rectangular field by repeating the movements shown in Figs. 9 and 10. In a rectangular field, only exits have convex shapes; thus, an MRS can easily distinguish them from other wall cells. The MRS evacuates from a rectangular field with the movements in Fig. 11 when it finds an exit.

We have shown that an MRS composed of two modules equipped with a global compass can evacuate from an arbitrary rectangular field. Therefore, we have proven Theorem 1. A module in this algorithm requires a visibility range of $5 \times 5$ (i.e., 2-neighborhood). This visibility range derives from the fact that a module decides its movement based on whether there is a wall cell side-adjacent to its neighbor module. For instance, if the cell is not a wall cell (e.g., the first state in Fig. 8), a module rotates $\pi/2$ around its neighbor module in a clockwise direction. Otherwise, the module must rotate in a counter-clockwise direction (e.g., the first state in Fig. 9).

With this evacuation algorithm, an MRS cannot stop after the evacuation and keeps moving along the outside of the walls. The following theorem shows this fact formally.

**Theorem 2.** Let $R$ be an MRS composed of two or more modules equipped with a global compass and having limited visibility. There is no algorithm that can solve the evacuation problem from any initial state and any initial position and can stop $R$ from moving after evacuation through an exit from any rectangular field.

**Proof.** For contradiction, we assume that there exists an algorithm $\mathcal{A}$ that can solve the evacuation problem from any initial state and any initial position and can stop an MRS from moving after evacuation. However, this would imply that an MRS equipped with a global compass can solve the evacuation problem from any initial state and any initial position and can stop an MRS from moving after evacuation through an exit from any rectangular field. This contradicts Theorem 1. Therefore, there is no such algorithm $\mathcal{A}$. 

1 Movements after changing direction may differ from those shown in the figure due to the different field size and locations of exits. The figure only shows the case where the width of the interior is larger than three cells (Fig. 11(a)) and the case where the width is two cells (Fig. 11(b)). We omit the other cases here, but the movements of the modules are mostly the same as Fig. 11.
Figure 9: Two modules with a global compass move along a wall

(a) A corner composed of three or more cells

(b) A corner composed of two cells

Figure 10: Two modules with a global compass turn around a corner

completing evacuation. Let $F$ be a rectangular field sufficiently large so that each module cannot see any corner from any exit (Fig. 12(a)). Let us consider evacuation from the rectangular field $F$. Without loss of generality, we assume that $F$ has only one exit $\{c_{a,-1}, c_{a+1,-1}\}$ in the south wall. Since $A$ is a deterministic algorithm, the terminated state of MRS $R$ is uniquely determined from the initial position and the initial shape of $R$. Let $S$ and $P$ be the terminated state and the terminated position of $R$ reached by $A$ from a given initial position and an initial shape.

Assume another rectangular field $F'$ whose size is the same as $F$ (Fig. 12(b)). Unlike $F$, $F'$ has only one exit $\{c_{a,h}, c_{a+1,h}\}$ on the north wall. Let $S'$ and $P'$ be the initial state and the initial position of $R$ in $F'$ in which each module of $R$ has an identical view to that of $S$ and $P$. Figure 12(c) and (d) depict the views of modules in $F$ and $F'$, respectively. Consider evacuation in $F'$ from the initial state $S'$ and the initial position $P'$. Since the views of the modules in $F'$ are identical to $F$, no module in $F'$ moves. Note that a module cannot distinguish between a cell in the exterior and one in the interior. Therefore, the algorithm $A$ cannot solve the evacuation problem in the rectangular field $F'$. This contradicts the assumption.

4 Evacuation from a rectangular field without a global compass

In this section, we consider evacuation by modules not equipped with a global compass. First, we consider evacuation from restricted initial states in section 4.1. Then, we discuss how we can remove this restriction on initial shapes in section 4.2. Finally, we show that an MRS composed of only two modules that are not equipped with a global compass can solve the evacuation problem by restricting the MRS’s initial position on a wall.
4.1 Starting from restricted initial states

We prove the following theorem to show that \( n = 4 \) is necessary and sufficiency for evacuation from any rectangular field if the initial shape of an MRS is restricted.

**Theorem 3.** Four modules not equipped with a global compass are necessary and sufficient for the metamorphic robotic system to solve the evacuation problem in any given rectangular field from the allowed initial states.

We prove the necessary part of Theorem 3 with the following lemma:

**Lemma 2.** Consider the metamorphic robotic system \( R \) composed of less than four modules that are not equipped with a global compass in a sufficiently large rectangular field. For any deterministic algorithm \( A \), there exist initial states in which \( R \) cannot evacuate from a rectangular field.

**Proof.** The proof mostly follows Lemma 5 in [11]. Here, we assume that each module cannot observe any wall in its initial state because walls are too far from its initial location.

For \( n = 2 \), as depicted in Fig. 13, there is the case where each module tries to rotate around the other module. This is impossible because there is no backbone module.

For \( n = 3 \), there are initial states from which an MRS cannot move. We can classify these initial states into “line-type” and “L-type” and briefly describe each.

In the line-type initial state (Fig. 14), the modules of both ends can move only by a rotation. If we assume that these modules rotate in the same direction (e.g., a clockwise direction) to avoid a collision, the subsequent state is also line-type. While its direction is different from the first
one, the only possible movement of the modules is a rotation. This movement results in the initial line-type state.

Let us consider the L-type initial states illustrated in Fig. 15. In these initial states, only the left and the bottom modules can move. However, if only one of them rotates around the middle module, the resulting shape is line-type (Figs. 15(a) and (b)); thus, an MRS cannot move as we saw before. If both modules rotate, the resulting state is L-type (Fig. 15(c)). In case of sliding, repeating four 1-slidings ends in the initial state, as depicted in Figs. 15(d) and (e).

As observed above, there exist initial states for $n < 4$ where an MRS cannot move. Since an MRS cannot find an exit from these initial states, it is impossible for it to evacuate from a rectangular field.

Then, we show an evacuation algorithm for an MRS composed of four modules not equipped with a global compass.

Lemma 3. Four modules not equipped with the global compass are sufficient for a metamorphic robotic system to solve the evacuation problem in any given rectangular field from all the initial states where evacuation is possible.

As shown in the proof of Lemma 4 in [11], for an MRS composed of four modules not equipped with a global compass, there are initial states from which an MRS cannot break the symmetry (Fig. 16). Therefore, we exclude these initial states from the proposed algorithm and consider only the remaining initial states (Fig. 17).

As in Section 3, we build the proposed algorithm with two parts: the “reaching a wall” part and the “searching for an exit” part. In the reaching a wall part, we use the locomotion algorithm for five modules proposed in [11] with a slight modification for four modules (Fig. 18). After an MRS reaches a wall, the algorithm switches to the searching for an exit part. The MRS changes its direction of movement as shown in Fig. 19 and moves along walls. When the MRS reaches a corner, it changes its direction by $\pi/2$ clockwise (Fig. 20). When the MRS finds an exit, it evacuates from the rectangular field (Fig. 21).

We have shown that an MRS composed of four modules that are not equipped with a global compass can evacuate from any rectangular field if the initial shape of an MRS is restricted; thus, Theorem 3 was proven.

Unlike in Section 3, this algorithm can stop an MRS from moving after evacuation. This fact does not contradict Theorem 2. This is because the algorithm can utilize states in Fig. 16 as terminal states after evacuation. In other words, no state in Fig. 16 is given to the algorithm as an initial state; thus, the algorithm does not need to consider evacuation from these states.

This algorithm requires a visibility range of $7 \times 7$ (i.e., 3-neighborhood) for the same reason as in Section 3.

4.2 Starting from arbitrary initial states

The limitation of the initial states in Fig. 16 is derived from the fact that an MRS cannot move to a wall from these initial states. The locomotion algorithm for seven or more modules proposed
Figure 15: L-type forbidden initial states of three modules without a global compass

Figure 16: Forbidden initial states of four modules without a global compass

in [11] and illustrated in Fig. 22 can resolve this limitation as stated in the following lemma:

**Lemma 4** (from [11]). *Seven modules not equipped with a global compass are necessary and sufficient for a metamorphic robotic system to perform locomotion from an arbitrary initial state.*

Namely, in the reaching a wall part, an MRS reaches a wall by the locomotion algorithm. The movements in the searching for an exit part are similar to those in Section 4.1. An MRS changes its direction of movement and moves along walls (Fig. 23). Then, it turns a corner (Fig. 24). Eventually, it finds an exit and evacuates from the rectangular field through the exit found (Fig. 25).

This locomotion algorithm occupies five cells in the same row in the second and third states in Fig. 22. In other words, the width and height of a rectangular field must be at least five cells each. Therefore, fields to which this evacuation algorithm is applicable are limited as follows:

**Theorem 4.** *Seven modules not equipped with a global compass are necessary and sufficient for a metamorphic robotic system to solve the evacuation problem from arbitrary initial states in any given rectangular field whose width and height are at least five cells each.*

This evacuation algorithm requires a visibility range of $11 \times 11$ (i.e., 5-neighborhood). This range is necessary to avoid collisions of modules when a module takes 5-slidings appeared in, for example, Fig. 23. Similarly to that in Section 3, this algorithm cannot stop an MRS from moving after evacuation.

### 4.3 Starting on a wall

Here, we consider evacuation when an MRS composed of modules not equipped with a global compass starts on a wall. Unlike in Sections 4.1 and 4.2, two modules are necessary and sufficient
Figure 17: Allowed initial states of four modules without a global compass

Figure 18: Locomotion by four modules without the global compass

for evacuation in this case.

The movements of the two modules obey those for two modules equipped with a global compass (Figs. 9–11). These movements are possible without a global compass. The reason is as follows. The modules must agree with their direction of movement among modules without a global compass. This can be achieved by utilizing the direction in which wall cells are observed in their views as follows. At each step, a module \( m \) observes neighborhood cells and obtains its view \( v \). If a module is side-adjacent to a wall cell in a direction \( d \) in view \( v \), \( m \) determines its direction of movement as \( d + \pi/2 \) in a clockwise direction. For example, if a module \( m \) observes based on its local compass that its neighbor module is side-adjacent to a wall cell in the north, \( m \) changes direction of movement to the east. This decision is based only on the handedness shared among the modules; thus, a global compass is not required.

However, this method has a limitation on field size. Let us consider the cases depicted in Fig. 20. In Fig. 20(a), modules \( a \) and \( b \) may have the same view in an initial state. If module \( a \) (resp. \( b \)) recognizes the right (resp. left) direction as the north, module \( a \) (resp. \( b \)) waits for \( b \) (resp. \( a \)) to rotate. Then, both modules do not move. As a result, the MRS cannot move. On the other hand, in the case shown in Fig. 20(b), the views of the two modules are different. Therefore, \( b \) becomes a backbone module and \( a \) rotates around \( b \). Then, the MRS can move along the walls in a clockwise direction. To avoid this case, the following theorem restricts the size of a rectangular field.

**Theorem 5.** Two modules not equipped with a global compass are necessary and sufficient for a metamorphic robotic system to solve the evacuation problem from a side of a wall in any given rectangular field whose width and height are at least three cells.

The visibility range and the termination possibility of this algorithm are the same as those of Section 3.

5 Evacuation from a maze

By extending the movements shown in Sections 3 and 4, an MRS can evacuate from more complex fields. Here, we consider evacuation from mazes. The method introduced here is applicable to various models, regardless of the existence of a global compass and the restrictions of an initial position. The number of modules and the visibility range required to solve the evacuation problem are the same as those in rectangular fields.

We defined a maze as a set of rectangular fields interconnected by exits; this is Definition 4. With this definition, we can apply evacuation algorithms for rectangular fields directly to mazes.

\(^2\)Only the evacuation algorithm for four modules that are not equipped with a global compass presented in
Figure 19: Four modules without a global compass move along walls

(a) A corner composed of three or more cells

(b) A corner composed of two cells

Figure 20: Four modules without a global compass turning a corner

We show an example evacuation path of an MRS composed of two modules equipped with a global compass in Fig. 27. First, an MRS reaches a wall from its initial position and shape by locomotion and finds an exit by moving along the walls. When an MRS finds an exit, it enters another rectangular field adjacent to the current field through the exit and finds an exit of the next field. By repeating these movements, an MRS eventually finds an exit to the exterior and evacuates from the maze. This approach simulates the well-known wall follower method and the correctness of our approach follows this method.

6 Conclusion

In this paper, we considered evacuation from a finite 2D square grid field by a metamorphic robotic system (MRS). We focused on the minimum number of modules required to solve the evacuation problem with several conditions. First, we considered a rectangular field surrounded by walls with at least one exit and showed that two modules are necessary and sufficient for any rectangular field if the modules are equipped with a global compass. Then, we focused on cases where modules do not have a global compass and showed that four (resp. seven) modules are necessary and sufficient for restricted (resp. any) initial states of an MRS. Additionally, we showed that two modules are sufficient if an MRS is on a wall in its initial position. Finally, we extended these results to another type of fields, that is, mazes.

There are various open problems for evacuation by an MRS. While we showed the proposed

Section 4.2 should be modified so that it does not stop an MRS after completing evacuation.

https://en.wikipedia.org/wiki/Maze_solving_algorithm#Wall_follower
evacuation algorithms in Sec. 3 can be extended for a maze, it is unknown that every evacuation algorithm for a rectangular field can be extended for a maze. It would be interesting to consider whether this is possible or not. It is also unclear that the assumptions we made for the proposed algorithms, e.g., the visibility range and common handedness, are optimal or not. Another open problem is evacuation for other types of fields such as convex fields. It also would be interesting to find an evacuation algorithm for two or more MRSs, while avoiding a collision between them.

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Figure 23: Seven modules without a global compass moving along walls

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Figure 24: Seven modules without a global compass turning a corner

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Figure 25: Evacuation through an exit by seven modules without a global compass
Figure 26: Field size limitation in the case of two modules without a global compass

(a) a $5 \times 2$ rectangular field  
(b) a $5 \times 3$ rectangular field

Figure 27: An example evacuation path of an MRS from a maze