Distinguishing conceptions of multiplicative reasoning in Chinese elementary students: What sense correct and incorrect solutions might make to them?

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Abstract
We address the question: What may be the conceptual sources of Chinese elementary students’ correct and incorrect solutions to multiplicative problem situations? We focus on situations for eliciting evidence about their ability to add and subtract sets of composite units—mental structures that underlie conceiving of whole numbers as a single entity composed of smaller units (e.g., “3” is composed of three “1s”). A conception postulated to underlie this ability is termed Same-Unit Coordination (SUC). We attribute student errors to reasonable-to-them spontaneous use of a previously established, repetitively practiced way of operating on 1s (“Ones”) contained within composite units—a conception we term Totaling. We analyze qualitative data to illuminate this phenomenon and quantitative data to depict its scope. These analyses support our claim that student solutions, seen by an observer as correct or erroneous, can be explained as (a) reasonable from the students’ frame of reference and (b) possibly arising from instructional focus on mastering multiplication facts to find totals of 1s in equal-size sets.

Keywords
constructivism, elementary maths, multiplicative reasoning, units, whole numbers

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1. Introduction

Elaborating on Davydov’s (1992) definition of multiplication, Boulet (1998) demonstrated that, by defining multiplication as a transformation of units, researchers could understand the principal commonality in multiplying whole numbers, integers, rational numbers, and irrational numbers. Whereas repeated addition fits concrete models often used to introduce multiplication, the transformation of units explains what distinguishes multiplicative reasoning from additive reasoning (Simon et al., 2018; Tzur et al., 2013). Specifically, in additive reasoning, a student only needs to operate on one kind of units, whereas in multiplicative reasoning a student has to coordinate at least two types of units, unit of one (1s) and composite units (units larger than and composed of 1s) – a coordination the leads to a unit transformation. Thereafter, students come to a new mathematical world, in which they have to choose which quantities to use while solving problems. Researchers seem to agree that multiplicative reasoning is a leap from additive reasoning, providing a basis for meaningfully using not only multiplication and division but also fractional, proportional, and algebraic reasoning (Lamon, 2007; NCTM, 2000). Lacking such a basis seems a key hurdle in students’ progress toward the latter, more advanced concepts (Xin, 2008; Xin et al., 2008) and a possible cause for being and feeling ‘stuck’ in mathematics. Within multiplicative reasoning, Tzur et al. (2018) demonstrated further that a particular scheme, called Same-Unit Coordination (SUC), is a critical precursor for students’ advanced multiplicative reasoning in general and for reasoning in a place value, base ten number system in particular.

SUC entails being cognizant of sets of equal-size quantities as being composed of 1s while operating additively on the sets. When a task involves, say, 9 pagodas with 3 floors each (9P₃) and 4 pagodas with 3 floors each (4P₃), a correct answer for the sum of pagodas is 13P₃ and for the difference is 5P₃. The reason for the name of this scheme is that the child operates only on the composite units (sets) without losing track of the 1s that compose them. We term such tasks as SUC-sum and SUC-difference (SUC-diff), respectively.

Some students may incorrectly calculate the two sub-totals of 1s (27 and 12), concluding there are 15 more pagodas in one set and 39 pagodas in all. One may also ask children how many more bags are in a box containing 9 bags of black beans (3 kg each bag) than in a box with 4 bags of white beans (also 3 kg each bag). A correct response would be 5 bags – not 15 bags. Explaining reasoning that underlies such incorrect responses is important, as SUC is a conceptual requisite for conceptions that afford operating on different levels of units, such as Ones and Tens (1s and 10s) in a place value, base ten number system (Tzur et al., 2018) and later also like-denominator fractions (e.g., 2/13 + 4/13 = 6/13).

Our focus on SUC as a conceptual basis in multiplicative reasoning draws on decades-long research that focused on conceptual sources of students’ problem solving (e.g., SUC tasks) and number sense (Steffe & Cobb, 1988; Verschaffel et al., 2007). This work has also included Asian countries (Fuson & Kwon, 1992; Yang & Cobb, 1995), as Asian students show high test scores (Huang & Leung, 2006) while revealing inconsistencies regarding number sense, often depicted as ‘misconceptions’ (Kajander & Lovric, 2009; Lin et al., 2016). How can one reconcile, say, Taiwanese students’ top scores on Programme for International Student Assessment [PISA] (OECD, 2020) with Yang’s (2019) findings that about 70% of such students did not know how many digits the sum of two 3-digit numbers would have?

Our conception-focused stance leads us to infer conceptual sources of task-irrelevant solutions (i.e., finding totals of 1s), which otherwise may be labeled as lack of reasonableness and number sense (Johnson, 2014). We postulate commonalities and differences in two conceptions (detailed in the next section), termed Totaling and multiplicative Double Counting (mDC), that we expect students to construct prior to SUC. In our study of Chinese elementary students, the former conception seems reflective of practices and textbooks used for teaching multiplication, which echo the credo,
practice makes perfect (熟能生巧). The latter, identified in research on multiplicative reasoning with US students (Tzur et al., 2013), seemed applicable to our work with Chinese students.

Accordingly, our study focuses on explaining why the mDC conception may better support a shift to SUC reasoning, whereas Totaling is not likely to do so. To explore whether the tollaling error is popular among Chinese elementary students, we address the research questions:

1. Do and to what extent Chinese elementary students make totalling errors when solving SUC problems?
2. What may be the conceptual sources of Chinese elementary students’ correct and incorrect solutions to multiplicative problem situations?

2. Literature review and conceptual framework

Our research studies in China, including this one, examine the applicability of conceptual learning progressions identified in Western countries. This is consistent with a growing body of cross-cultural research on mathematics learning (Norton et al., 2018) and teaching (Cai et al., 2009; Huang et al., 2015) in western and east Asian countries. We focus on the extent to which findings about Western children’s construction of conceptions for reasoning with whole numbers may serve to analyze Chinese children’s conceptions. This focus reflects our Chinese team members’ work in the context of reform in China’s mathematics education, which stresses reasoning and problem solving (Ministry of Education [MoE], 2011). While some researchers in China attempt to characterize students’ “misconceptions”, the conceptual roots of students’ errors seems to be overlooked. Our focus is consistent with recent work by Norton and colleagues (Norton et al., 2018; Norton & Boyce, 2013), who used a constructivist approach to examine common cognitive cores of children’s conceptual learning across cultural contexts. China is a strategic research site because its reform reflects a stance on learning as an active process—a central premise of constructivism. Next, we briefly review the literature on multiplicative reasoning, then depict the conceptual framework and the cultural background of our study.

2.1 Multiplicative reasoning model

Drawing on a constructivist qualitative teaching experiment that studied students’ counting and numerical reasoning (Steffe & von Glasersfeld, 1985), Steffe (1992) pointed out that children’s counting types serve as important cognitive roots for the development of students’ number conceptions. At the core of his work is the notion of ‘number as a composite unit’ (a number, larger than 1, that is composed of several 1s or other units smaller than that number). With the improvement of the level of unit coordination, students’ conceptions of number develop. Based on Steffe’s (1992) number sequence theory, there are two learning trajectories (models) of students’ development in multiplicative reasoning with 1s and composite units.

The first model comprises 3-level multiplicative concepts proposed by Hackenberg (2007), who emphasized the levels of units that students coordinate and interiorize as they engage in multiplicative reasoning. Multiplicative Concept-1 (MC1) involves coordinating two levels of units in activity. Students who have interiorized two levels of units (MC2) know, ahead of acting, that the coordination can be made. Students who have interiorized three levels of units (MC3) can coordinate all levels of units in all situations, that is, 1s, units composed of 1s, and units composed of composite units (e.g., 4 is composed of six units of four 1s). Kosko and Singh (2018) and Kosko (2019) developed an assessment of elementary students’ multiplicative concepts according to Hackenberg’s model and found that 50% of students at 3rd grade have arrived at least to MC1, and around 50% of students in grade 4 and 5 arrived to MC3.
Different from Hackenberg’s study, Tzur et al. (2013), based on Steffe’s (1992) work, proposed a 6-scheme model of multiplicative reasoning. In their model, the focus is not only on the levels of units and units coordination but also on transitions between them. Their model includes multiplicative Double Counting (mDC, consistent with MC2, which will be explained later), Same Unit Coordination (SUC), Units Differentiation and Selection (UDS, consistent with MC3), Mixed-Unit Coordination (MUC, consistent with MC3), and two division schemes. SUC and UDS are two important transitional schemes between mDC and MUC. Their model indicated quantitative evidence for the roles of the vital stages, from mDC to MUC (Tzur et al., 2017, 2018; Zwanch & Wilkins, 2021). What seems needed is a better explanation of plausible conceptual sources of students’ errors while solving multiplicative-related tasks.

2.2 A constructivist conceptual framework

Piaget’s (1985) construct of assimilation underlies our explanation of mathematical knowing. Assimilation posits that a person’s problem solving is regulated by schemes (hereafter we use conceptions) they already established. Such conceptions comprise one’s goal in a problem situation, a mental activity the goal triggers, and a result anticipated to ensue from that activity (von Glasersfeld, 1995). We use conception to emphasize that, to the problem solver, a solution is reasonable.

A student’s solution may become unreasonable in their frame of reference when their assimilatory conceptions reveal a gap between the current and a desirable state. Such a gap leads to a perturbation, and possibly to a change in one’s goals and/or activities (von Glasersfeld, 1995). To an observer, if a change occurs it may illuminate aspects of the learner’s conceptions.

Steffe and others (Steffe, 1992; Steffe & Cobb, 1988) articulated numerical conceptions in terms of mental units and mental activities children use. A key distinction they made is between 1s (Ones) and composite units—numbers composed of smaller units, such as 1s (e.g., 12 is conceived as twelve 1s, or as four units of three 1s). Children initially operate on 1s to compose larger units. Later on, they establish and thus can operate on composite units as entities in and of themselves.

This study focuses on the two first conceptions in a conceptual progression in children’s multiplicative reasoning (Tzur et al., 2013). They postulated the first, Multiplicative Double Counting (mDC), as an initial form of such reasoning. It involves distributing items of one composite unit (e.g., 3 floors per pagoda) over items of another composite unit (e.g., 4 pagodas), and then figuring out the accrual of 1s in all of them (e.g., 1-pagoda-is-3-floors, ..., 4-pagodas-is-12-floors). MDC is multiplicative as it involves unit transformation (Simon et al., 2018). In our example, the coordination of composite units (pagodas) with unit rate (floor-per-pagoda) produced a total of 1s (floors). We alluded to the second conception, SUC, in the Introduction. SUC is an additive, unit-preserving operation (Schwartz, 1991) that takes as input not 1s but rather sets of composite units – without losing sight of the 1s in them.

In this study we inferred a third conception into which students committing the error pattern under study would assimilate SUC tasks. We term it Totaling Equal-Size Numbers (shortened as “Totaling”). Unlike in mDC, in Totaling a child focuses on calculating the number of 1s that result from multiplying two numbers (e.g., modifying 4 pagodas with 3 floors into 4*3 = 12). As Totaling is one outcome of our analysis, we articulate its features throughout the paper. Here, we point out two of its key features. First, a problem solver’s goal when using mDC or Totaling seems compatible (finding totals of 1s). The calculation could also be similar (e.g., retrieving 4*3 = 12). Second, the solver’s mental activities to obtain the total of 1s differ markedly, as in Totaling the solver disregards the composite units whereas in mDC they coordinates the number of composite units (one multiplicand) with the number of 1s in each (the other multiplicand).
2.3 Study context: Mathematics education reform in China

Aiming to study conceptual commonalities of Western and Eastern students, we acknowledge the context in which we conducted our study, namely, the mathematics education reform in China (MoE, 2011). This reform set out to overhaul authoritative practices of 'transmitting' mathematics to passive students, who practice teacher-directed methods and apply memorized facts accurately and quickly (Huang & Leung, 2006). Instead, goals for students' learning center on critical and creative thinking skills while solving non-routine problems to develop subject literacy. For grades 1–9, subject literacy means number and symbol sense (with ability to calculate), reasoning and modeling, spatial/geometric conceptions, and analyzing data. These new standards stressed that listening carefully, thinking actively, hands-on activities, self-exploration, collaboration, and communication are all important learning methods, and the teacher's role in teaching is the organizer, leader, and collaborator of learning (MoE, 2011). Yet, Zhang et al. (2018) found that most Chinese mathematics teachers have not transformed static curriculum into dynamic classroom instruction, and mainly follow the textbooks.

Our study focuses on three ways of numerical reasoning—the mDC, SUC, and Totaling conceptions—that seem to underlie how elementary Chinese students may reason when solving multiplicative situations. As Tzur et al. (2013) identified the first two conceptions in research with US students, we designed this study to determine their applicability to Chinese students. We postulated the third conception (Totaling), which was identified in this study for the first time, during the analysis leading to writing this paper.

Following Inhelder’s assertion that a scheme could be procedural (Karmiloff-Smith & Inhelder, 1974), we believed it was important to explain this way of solving SUC tasks conceptually and link it with instruction that may give rise to it. Considering a conception (e.g., Totaling) as a goal-directed activity, we illustrate this linkage by a few representative examples of problems Chinese students solve when learning multiplication.

Figure 1 shows problems in a widely used Chinese textbook (Liu et al., 2014). A student may assimilate all problems into a conception with a goal to find the total of 1s in a number of equal-size composite units (e.g., 12 × 3, 18 × 4). The first worked-out example presents two ways to find the cost.
of 3 floats (in 1s, not composite units): by repeatedly adding 12 (left side) or by considering three
groups of vertically organized bills of 10-Yuan + 2-Yuan. The arrows indicate how to first find sub-
totals of 1s (10 × 3, 2 × 3) and then add them (30 + 6).

Figure 2 shows how the two-step method of finding totals of 1s is linked to operating on abstract
symbols within a place value, base ten number system. One ‘multiplication chart’ for 3 × 12 (left)
shows sub-totals of 1s being re-grouped for composite units of 10 and for 1s separately (30 for 3
× 10; 6 for 3 × 2). Another multiplication chart (right) shows the same method for buying 4 balls
at 18 Yuan each. We note that the cost of each item is supposed to give a realistic sense of the sit-
uation. However, price tags chosen do not stand for single Chinese bills. Rather, these numbers were
chosen to foster operating separately on units of 10 (“Tens,” or “10s”) and of 1s while applying the
distributive property to their place value meanings (e.g., 12 × 3 = (10 + 2) × 3 = 10 × 3 + 2 × 3 = 30 + 6
= 36). These examples seem to direct students to focus on 1s and not on the composite units com-
prising those 1s (e.g., “12,” or “18,” or “10 + 8,” etc.). Thus, these examples indicate possible instruc-
tional sources of elementary Chinese students’ correct and incorrect solutions to SUC tasks. Our
study, however, aims to explain conceptual sources of those solutions.

3. Methods

This study arose as secondary analyses of findings from two earlier studies. One study focused on
Chinese students’ responses to mDC and SUC tasks. Another study, a series of constructivist teach-
ing experiments (Cobb & Steffe, 1983), focused on students’ construction of multiplicative concep-
tions. Here, we conjoin (a) quantitative data about errors students made when solving SUC tasks with
(b) an in-depth analysis of illustrative SUC errors made by eight students.

3.1 Sample

3.1.1 Quantitative. Following local procedures for human subjects and a protocol approved by the
second author’s US university, all students (N = 836) in second (n = 172), third (n = 160), fourth
(n = 173), fifth (n = 176), and sixth (n = 155) grades at one school in a mid-size city in northeast
China participated in this study (about 50–50 gender split). These students took a newly developed
and validated assessment that includes four mDC, three SUC-diff, and three SUC-sum items (see
Instruments below). The language of all participating students is Mandarin Chinese.

3.1.2 Qualitative. In our teaching experiment (Steffe et al., 2000), we analyzed data from work with
eight Chinese students. Crucially, our analysis across those students does not focus on any particular
person. Rather, we focused on a phenomenon for which those students’ work is a case—how elemen-
tary Chinese students may assimilate and solve mDC and SUC tasks. Thus, we searched in our data
for segments with compelling evidence to illustrate the conceptions we infer to underlie their work.

Figure 2. Linking multiplication as sub-totals of 1s to symbols reflecting base-10 organization.
We found such compelling data in the work of two exemplars, a 4th grader (Tobi) and a 5th grader (Ella; pseudonyms). We chose them based on conceptual analyses of correct and erroneous solutions of all 8 students. Our data show that, at different times in the teaching experiments, each of the 8 students transitioned from Totaling through mDC to SUC. For this paper, we looked for data segments that best illustrate each of those conceptions, which happened to be from the work with Tobi and Ella.

Ella’s work elucidates the phenomenon of incorrectly solving SUC tasks through assimilation into an mDC conception, which also seemed to enable a shift to task-relevant reasoning. During the first 10 teaching episodes, our team fostered her construction of number as a composite unit, using tasks adapted from the work (in the US) of teams led by the second author. In this paper we use data from the 11th episode, in which the team began working with Ella and her partner, Hilda, to promote and study the phenomenon of transition to mDC and/or SUC conceptions.

Tobi’s work (with Kate) elucidates the phenomenon of incorrectly solving SUC tasks through assimilation into a Totaling conception, which also seemed to hamper a shift to task-relevant reasoning. We use data from the first episode, in which the team engaged Tobi and Kate in solving SUC tasks to infer whether they had constructed this conception. We chose Tobi as he epitomizes two interrelated elements of assimilating the SUC task into a Totaling conception: (a) immediately and astoundingly calculating sub-totals of 1s as well as their sums/differences and (b) indicating no perturbation after seeing Kate’s correct solution. We stress that this snapshot of Tobi’s work is not his most advanced reasoning. During the teaching experiment we adapted situations that fostered his transition to SUC and beyond.

### 3.2 Instruments

For the quantitative analysis, we extracted data about participant solutions to six SUC tasks (3 SUC-diff and 3 SUC-sum) that are a component of a new instrument we developed and validated, in tandem, in the US and China. Construct validity of the entire instrument was established through two cycles of eight experts’ feedback to items we developed based on previous work (Johnson et al., 2018). In all, answer-generating items (no multiple-choice), a student is asked to show the work leading to their constructed answer. We then used three rounds of a back-translation process. A round began with two, native Chinese speaking team members translating the assessment from English. Then, a third, native Chinese speaking member, with a PhD in math education, translated the Chinese version back to English having no access to the English version. Finally, the second author compared the back-translated version against the original one.

We administered the final Chinese version to students in one school in a northeast city in China. Rasch analysis showed Eigenvalue of variance explained by the measure = 21.0, accounting for 40.4% of the observed variance (expected = 40.1%), Eigenvalues of unexplained contrasts were all less than 1.8, with none accounting for more than 3.4% of observed variance, satisfying criteria for unidimensionality (Raîche, 2005) and indicating that the SUC subtest is part of a single overarching dimension of multiplicative reasoning. Cronbach’s alpha values for inter-item reliability were high for the entire assessment (0.93), and acceptable for the 6-item SUC subset (0.74) and the 3-item SUC-sum subset (0.7). Table 1 shows the SUC-diff and SUC-sum items. We used the terms “in all” to decrease students’ use of a “key words” strategy while selecting an arithmetical operation.

### 3.3 Data collection and analysis

#### 3.3.1 Quantitative

Graduate research assistants administered the assessment during a regular mathematics lesson. Using a matrix, each class was randomly assigned one of four different versions,
each consisting of problems (randomly sequenced) to assess the 8 conceptions. Students were directed to show their work and write the answers in pen. We tested two null hypotheses: (a) the relative difficulty of SUC items is the same at each grade-level – tested for differential item functioning (DIF) by comparing mean item measure logits by grade level; and (b) the proportion of students making Totaling errors is the same across grade levels, using both Cramer’s V and Chi-Square ($\chi^2$) tests of association.

3.3.2 Qualitative. Our conceptual analysis focused on data from the work with Ella and Tobi, including video recording, still pictures of their work, and team members’ field notes. Within a teaching experiment, ongoing analysis sessions of data take place between episodes. In those, team members produced logs of main events and students’ written work in each episode, which were then compared/combined to guide future analysis.

After all teaching episodes were completed, we identified data segments in our initial logs that seemed valuable for the retrospective analysis, which proceeded in three back-and-forth cycles. One cycle involved 1–3 researchers examining selected video files and creating highly detailed logs. Another cycle involved small teams (2–4) in reviewing the annotated segments to identify and code critical events for the analysis of students’ conceptions. This cycle included rigorous, line-by-line interpretations of (transcribed) students’ language and actions. These sub-teams posed questions, made hypotheses grounded in the data, and searched for confirming or disconfirming evidence (Strauss & Corbin, 1994). A third cycle, involving small teams in co-analyzing across cases, led to identifying mDC and Totaling conceptions (with Ella and Tobi as exemplars).

4. Results

Our analysis supports a threefold claim about an error pattern in Chinese elementary students’ solutions to SUC tasks: It is widespread; it intensifies as students progress from 2nd to 3rd/4th grade; and it seems rooted in assimilating SUC tasks into one of two conceptions—mDC or Totaling. Students’ performance when solving tasks may appear similar, because a solver’s goal in both conceptions (mDC, Totaling) is compatible—finding the total of 1s in equal-size units. Yet, we distinguish these conceptions in terms of mental operations on units that this goal triggers. In Totaling, a student attends predominantly to 1s (sub-totals and global totals); in mDC, a student invokes a simultaneous coordination of both composite units and 1s, while anticipating and/or enacting the distribution of 1s into each of the given composite units (Tzur et al., 2013).

### Table 1. SUC-diff and SUC-sum tasks used in the validated measure (Omitting request to show work).

| SUC-diff                                                                 | SUC-sum                                                                 |
|-------------------------------------------------------------------------|-------------------------------------------------------------------------|
| 2.1 Store A has 23 shelves; on each shelf there are 10 boxes. Store A has 9 more shelves than Store B (each shelf with 10 boxes). In all, how many shelves, with 10 boxes each, does Store B have? | 2.3a RanRan put 4 bags in Box A, with 3 candies in each bag. Then, she put 9 bags in Box B, also with 3 candies in each bag. Together, Box A and Box B have ____ bags of 3 candies. |
| 2.2 Together, Ting and Yong have 17 boxes, with 5 moon cookies in each box. Ting has 11 of these boxes. Of the 17 boxes, how many boxes of 5 moon cookies does Yong have? | 2.4 School A has 18 rows of chairs, 7 chairs in each row. School B has 23 rows, also 7 chairs in each. In all, how many rows of 7 chairs do School A and School B have? |
| 2.3b RanRan put 4 bags in Box A, with 3 candies in each bag. Then, she put 9 bags in Box B, also with 3 candies in each bag. How many more bags of 3 candies each are there in Box B than in Box A? | 2.5 Yesterday, a store sold 15 boxes. Each box has 6 tomatoes. Today, they sold 7 boxes. Each box also has 6 tomatoes. In all, how many boxes (6 tomatoes each) did they sell in these two days? |
We begin this section with quantitative analysis to indicate the scope of the error pattern found in Chinese students. However, the main thrust of our study is not quantitative. Rather, we provide a model explaining the errors as rooted in conceptions we infer students use to make sense of and solve the SUC tasks. Our conceptual analysis also helps explain a grade-reversal (3rd/4th graders committing the totaling error more than 2nd graders).

4.1 Quantitative indicators of totaling errors

Table 2 presents grade-level, Rasch modeling of student ability estimates on the 3-item subscales of SUC-diff and SUC-sum. A DIF analysis led us to reject the null hypothesis that SUC items are equally difficult for students at different grade levels. We found systematic differences in the difficulty of each SUC item 2nd, 3rd, and 4th graders. Whereas differences between 3rd and 4th graders on all six SUC items were not substantive, they were statistically significant between 2nd graders and 3rd graders (DIF contrasts ranging between 0.95 and 1.65 logits, df between 152–160, t-values between 3.2 and 4.2, p-values between .02 and 0.0005). Significant differences were also found between 2nd and 4th graders (DIF contrasts of 0.91 and 1.44 logits, df between 150–161, t-values between 2.1 and 3.3, p-values between .03 and 0.001).

We further examined which of the two item types, SUC-diff or SUC-sum were more difficult for students. Table 2 shows a statistically significant difference in favor of students’ abilities on the former (mean logit score) at each grade level (2–5) and across all grades. Thus, we focused our qualitative analysis on totaling errors students made when solving SUC-sum problems.

While the estimated ability scores reflect the expected improvement with grade, we focused on the grade-reversal found in proportions of students committing the error of totaling (1s) when solving SUC-sum problems. The mean percentage of totaling errors on the SUC-sum subscale for 2nd graders (9.7%) is less than for 3rd graders (28.0%), 4th graders (18.5%), or 5th graders (13.0%). The difference in these proportions was significant (Cramer’s V = .228, p = .01). A chi-square test of association showed a significant difference between grades 2 and 3 ($\chi^2 = 27.4, p < .0005$) and between grades 2 and 4 ($\chi^2 = 8.2, p = .004$), but not for grade 5.

These results are intriguing: Why would solving SUC-sum problems be more difficult for 3rd and 4th graders and why would they commit more of the totaling error than 2nd graders? We contend these findings indicate a way of operating on composite units that prioritizes and is augmented by practicing multiplication to determine totals of 1s. The mastery of facts/procedures in 3rd and 4th grades seems to lead more students to assimilate SUC tasks into a Totaling conception. To explain the nature and conceptual source of this phenomenon, we turn to qualitative analysis of data illuminating how students understood and approached SUC-sum tasks.

4.2 Conceptual analysis of the typical, totaling error

Using assimilation into available conceptions as a lens, we present inferences that, as observers, we make to explain why students’ incorrect solutions would be reasonable to them. We infer that a
student who gives an incorrect answer (39, 132, or 287) assimilates the SUC-sum task into the goal-directed operations afforded by either a Totaling or mDC conception. Either conception involves recognizing the givens in a task as including an identical number of 1s that constitute each, equal-size composite unit (e.g., 3 floors in each pagoda) and a number of such composite units (e.g., 9 pagodas and 4 pagodas). Accordingly, the student further recognizes and interprets the givens as two sets of equal-size units, which brings forth a goal of determining the total of 1s. Such recognition provides a plausible cause for a shift in the student’s attention – from the task-relevant sum of composite units to the 1s given. Tobi and Ella’s exemplars illustrate this erroneous shift (manifested by all 8 students in our teaching experiment). First, we show traces of students’ written work; then we analyze data excerpts from Tobi (Totaling) and Ella (mDC).

4.2.1 Evidence from students’ written responses. Figure 3 presents four exemplars of written work that support our inference of their conception(s). The left-most example indicates a student has carried out the first step in her or his head, while leaving a trace of the second step (adding sub-totals). The next two examples show both steps of the process: determining the sub-totals of 1s and then the global total of 1s. The last example indicates the emphasis on writing the Chinese symbol for “unit” (个), with numbers indicating the term “unit” pertained to 1s.

The conceptual explanation above supports the grade-reversal finding. As shown in Background, multiplication is introduced in China with worked-out examples and student solutions by first calculating and then adding sub-totals of 1s. All four examples of erroneous solutions in Figure 3 indicate students likely assimilated this task into a Totaling conception indicative of instruction they received. We assert this based on resemblance of the students’ two-step calculation, with its clear focus on sub-totals of 1s, to the practiced calculation taught in schools. We expect that, from 2nd to 3rd and 4th grades, these experiences would increase assimilation of tasks involving equal-size units into a conception with a goal of figuring out the total of 1s, then using multiplication of sub-totals and adding those to a global total. With or without this “unit” symbol, the Totaling conception we infer to underlie the method leading to the typical error (e.g., 39) would make this erroneous answer a reasonable response to the student.

4.2.2 Evidence from cognitive-focused sessions. Students’ written traces pointed us to a plausible source of the error. We thus sought out data to distinguish the conceptions into which students assimilate the SUC-sum task. Tobi and Ella’s solutions provided the most compelling evidence of this distinction. As explained in Methods, Tobi illustrates a student whose Totaling conception did not support a shift to SUC whereas Ella’s mDC did support such a desirable shift.

Tobi: Assimilating both SUC tasks into Totaling conception. The researcher wrote and told Tobi and Kate that their team had built 8 pagodas with 7 floors each (8P7) and his team had built 13 such pagodas (13P7). He then posed the task, using vocal intonation to emphasize the unit at issue - pagodas (line 08:10). Figure 4 presents the SUC-sum task, with Tobi’s written work (lower-left) as seen at the completion of his solution.
Table 3 presents data from Tobi’s spontaneous, immediate actions before he and Kate shared and explained their answers (T stands for Tobi, K for Kate, RE for the English-speaking researcher, and RC for the Chinese-speaking researcher; numbers show video-recording running time).

Data in Table 3 support our inference about Tobi’s assimilation of the SUC-sum task into the Totaling conception. In spite of the researcher’s emphasis of the unit asked about (pagodas) in anticipation of the possible error, Tobi operated on 1s within the two given sets of units (8P7 and 13P7). We infer this was triggered by recognizing a situation with two sets of equal-size units (pagodas with 7 floors each), and thus setting the long-practiced goal of finding a global total of 1s. His available way of operating seemed to hinder assimilating the task into a conception (SUC) in which the goal, and mental activity to accomplish it, would take composite units—not 1s—as input. Until seeing Kate’s answer, his solution seemed reasonable to him.

Upon hearing Kate’s correct response (“21 Pagodas”), Tobi could make sense of her response. He seemed to have a perturbation that led to recognizing his error, and thus to changing his answer. Critically, Tobi’s perturbation did not yield a change in his assimilation of the follow-up, SUC-diff task (see Table 4). Our inference is supported by the following sequence of events: spontaneously solving one SUC-sum task incorrectly (Table 3), shifting to operation on composite units (after seeing Kate’s answer), and then reverting to his erroneous way for the next, SUC-diff task. It suggests not only the precedence taken by his Totaling conception but also a possible level of SUC that would not afford realizing the need to change his strategy.

The contrast between Tobi’s and Kate’s correct SUC reasoning and answers to both SUC tasks supports our inference about his assimilation of the SUC-diff task also into the Totaling conception. Prior to working on this task, he changed his answer to the SUC-sum task (to fit with Kate). Yet, this did not change his work on the SUC-diff task. Instead, Tobi again used sub-totals and swiftly found the difference in 1s between the two sets. His explanation of 91–56 indicates this response was reasonable for him. Indeed, Tobi could again adjust his answer while explaining where he went wrong (“I needed to divide 35 by 7, so the result is 5”), but his statement of the units in the difference still indicated his attention to the 1s (“Floors”).

Tobi’s incorrect solutions to both SUC tasks may be explained as a confusion of units. We agree, while stressing such a confusion was indicated neither in Kate’s nor in many student responses to the written assessment. An alternative explanation, which we always consider, could be that Tobi manifests many students’ implicit belief they should use all numbers given in a problem. We contend it

![Figure 4. Tobi’s work to find and add two sub-totals of 1s (56 [8*7]; 91 [13*7]; 56 + 91 = 147).](image)
does not pertain to Tobi’s case. His solution after seeing Kate’s answer indicated he understood the need to not only use given numbers but also what operations would be needed with those numbers to obtain a correct, task-relevant answer (e.g., divide 35 by 7). While operating only on the given

Table 3. Data excerpt from Tobi’s masterful summation of sub-totals (of 1s).

| Time  | Person: Utterances and Actions |
|-------|--------------------------------|
| 08:10 | RE: I am asking about **pagodas**. How many **pagodas** do you and I have together? |
| 08:22 | T: (Immediately begins to calculate sub-totals. In an algorithm format, he first writes 56 (apparently memorized \(7 \times 8\)) on the paper along with the “+” sign. He then pauses a bit, while holding the pen so it’s pointing to the 13 (Pagoda) symbol of RE’s team, seemingly engaged in calculating \(7 \times 13\) in his head.) |
| 08:38 | T: (Moves the pen back to the lower-left corner and writes the sub-total of “91” below “56”, immediately beginning to add both numbers. First, he writes “7” for the 1s place, as seen in Figure 4. After 3 more seconds he writes “14” in a smaller-font for the 100s and 10s places. Then, he looks at Kate’s correct answer of 21, seemingly realizing his answer was wrong. RE first let him finish his calculations.) |
| 08:42 | RC: But my question is, How many pagodas? |
| 08:49 | K & T (together): Twenty-one (21). |
| 09:15 | RE: Okay. Why 21? |
| 09:20 | K: We have 13 pagodas, and RE has 8 pagodas; and \(8 + 13\) is 21. |
| 09:35 | RE: how did you know 21? You just knew it, or you added? |
| 09:50 | K: I calculated in my head. |
| 09:52 | RC: How? What is the process? |
| 09:56 | K: Because \(3 + 8 = 11, 11 + 10 = 21\). |
| 10:14 | RE (to Tobi): How did you get 21? |
| 10:20 | T: I did this way; I gave 7 pagodas to it [13 pagodas, so] it is 20; then \(20 + 1\) is 21. |

Table 4. Data excerpt from Tobi’s masterful subtraction of sub-totals (of 1s).

| Time  | Person: Utterances and Actions |
|-------|--------------------------------|
| 10:45 | RE: Who has more **pagodas**, your team or my team? |
| 10:56 | T & K (together): Your team. |
| 10:57 | RE: How many more **pagodas** (emphasis in RE’s intonation) does my team have? |
| 11:25 | T (after barely 3 s of silently computing in his head he utters): Thirty-five (35). |
| 11:28 | RE (to T): Why 35? |
| 11:30 | T Because 13 pagodas times 7 is 91 floors; and, last time I calculated, the result [of our team] was 56 floors. So, I used 91-56 = 35. |
| 12:03 | RE (to T): You did all of these [calculations] in your head? 91-56, all in your head? (T nods, “Yes” and RE continues): How did you know this? |
| 12:27 | T (explains his calculation of 91-56, which follows a vertical algorithm): One (1) is not enough to subtract 6, so I borrowed 10 from 9; so \(11-6 = 5, 9-1 = 8\), and \(8-5 = 3\), so it is 35. |
| 13:33 | RE (to T, to elicit a perturbation): My team has 35 more **pagodas** than your team? |
| 13:52 | T (Nodding yes.) |
| 14:19 | RE (to both): We have different answers; Kate said 5 more pagodas and Tobi said 35 more pagodas. Can we have both, or do we need to decide which answer is correct? |
| 14:54 | T: I needed to divide 35 by 7, so the result is 5 [agreeing with Kate]. |
| 14:59 | RE: Why do you divide 35 by 7? |
| 15:01 | T: Because each pagoda has 7 floors; 35 are how many more floors [your team has] than us. So … the result is 5 floors. [Note: “Floors” was the unit T stated, wrongly.] |
| 15:50 | RE: So, Tobi, your answer is 35 more floors, [but] 5 more pagodas? |
| 16:20 | T (Nodding yes.) |
composite units and setting aside the 1s, as Kate did, is more consistent with the SUC conception, using all numbers to find the totals first and then dividing is mathematically correct. Because in Tobi’s solution to the SUC-sum task he spontaneously reverted to using Totaling, we looked for another, sound explanation.

Our analysis of Tobi’s Totaling conception articulates what could be a plausible conceptual source of such confusion—we expect a student to operate on sub-totals in order to find the total of 1s. In the Discussion section, we further address this predominant focus on 1s over a focus on the task-relevant composite units, including why we consider it as (a) conceptually less advanced than the mDC conception and (b) an inadequate foundation for constructing SUC. Next, we turn to Ella – our exemplar of a student who assimilates the SUC-sum task into an mDC conception.

**Ella: Assimilating SUC tasks into an mDC conception enables a shift to SUC.** Ella was a participant in a different teaching experiment than Tobi, led by the first author (Chinese native speaker). Figure 5 presents Ella’s work on the initial, SUC-sum task during the 11th episode: “Ella has 8 pagodas, 4 floors each. Hilda has 13 pagodas, also 4 floors each. How many pagodas (4 floors each) do Hilda and Ella have in all?” Ella drew a diagram with schematic lines (no hash-marks) to represent her 8 pagodas. She enumerated each inscription of the composite units at the bottom and the total of 1s at the top. (Note: She circled the number 32 and drew the three diagonal lines later, when explaining her solution to RC.) Then, she created a third space on the paper titled “T,” indicating that she intended to find the total of 1s. Next, she drew and enumerated lines just for the first 10 pagodas. Then, realizing this process means multiplying 13 × 4, she used the algorithm for 13 × 4 = 52. This led her to draw just one more line for the last of the next three pagodas, symbolized with 13 at the bottom and 52 at the top. Having found the two sub-totals of 1s (32, 52), and indicating no perturbation, she completed the calculation of the desired, global total of 1s by adding 52 + 32 = 84 vertically. Table 5 presents parts of Ella’s responses to RC’s probing of her work.

Our inference that Ella assimilated the SUC-sum task into an mDC conception is based on five aspects of her work. First, she seemed highly intent in producing each sub-total as a distribution of (four) 1s over the given number of composite units. Her numbered lines indicated an intention to keep track of all composite units. Second, she used diagonal lines to show each composite unit of four 1s is being distributed into each of the first few pagodas. Her explanation made this distribution of units explicit; it served as her justification of the multiples of 4 she wrote on the top of each line. Third, she circled the answer to her calculation in finding the first sub-total (32). This indicated it met her anticipated, reasonable result for this activity—she found the sub-total of 1s in the first set. Fourth, Ella could truncate the process of producing 13 lines by skipping from the 10th to the 13th pagoda, without losing track of the given number of composite units. It was this explicit conception of the 13 pagodas (4 floors each), symbolized with only 10 lines and eventually with one more line,
that she linked to the vertical algorithm (not multiplying $11 \times 4$). Her work to produce both sub-totals indicated to us assimilation into the mDC conception, which for her was also linked with the taught, vertical algorithm (adding those sub-totals, $32 + 52 = 84$).

The fifth aspect is, arguably, the most telling. On her own, Ella’s assimilation of the SUC-sum task led her to initially operate only on 1s. She could clearly identify 1s as the unit she operated on (“Then I have 32 floors and Helen has 52 floors…”). Yet, upon assimilating RC’s question (“Do you remember what was my question?”), Ella realized she replied with the wrong unit. Through interactions with Hilda, but no direction from RC, Ella eventually shifted her attention back to the composite units (“pagodas”) that were to be added ($8 + 13 = 21$). Ella’s shift back to composite units is encouraging; it suggests she could notice their error and reinterpret the task. Moreover, the fifth aspect we highlight focuses on her realization of the unit shift needed, because for her the initial focus on sub-totals of 1s no longer constituted an adequate response. Ella reconsidered her solution (84), thereafter seeing it as task-irrelevant and thus not reasonable.

To highlight the difference between the Totaling and mDC conceptions, we contrast our inferences about the conceptual source of unit shift by Ella with that of Tobi. Ella’s shift from 1s to composite units seemed supported by her mental activity of coordinating both types of units. Her initial focus was on the 1s within that coordination. However, being probed by RC without being told directly (“Altogether – What?”) could yield Ella’s realization of that focus on 1s being task-irrelevant, leading to the shift to the composite units she also used, and were thus available, in her mDC conception. Ella did not need to receive a more specific hint for the needed shift, such as seeing the correct answer (21 pagodas). In contrast, Tobi’s Totaling conception was not supportive of such a shift in units. He did recognize the need to change his answer after seeing Kate’s

| Time  | Person: Utterances and Actions |
|-------|--------------------------------|
| 32:30 | RC: So, how many pagodas do you have in all? |
|       | E: Eighty-four (84). [Here, Hilda first explained her solution.] |
| 36:25 | RC (to E): [Could you explain] how did you get 84 pagodas? |
| 36:31 | E: I first drew [my] 8 pagodas. Each pagoda has 4 floors. Then, I multiplied step-by-step. This (points to the first pagoda with diagonals) is $1 \times 4 = 4$. |
| 36:55 | RC: So, you used the [vertically drawn] lines to represent the pagodas? |
| 36:59 | E: (Nodding yes.) |
| 37:03 | RC: What is the meaning of the numbers above each [vertical] line? |
| 37:05 | E: These are the numbers of floors of the pagodas. (Here, as if continuing the $1 \times 4 = 4$ she draws the three diagonal lines from “4” above the first line to the bottom of the next three lines) And this is the second pagoda, $2 \times 4 = 8$, then $3 \times 4 = 12$. (A little later) Yeah, then $5 \times 4, 6 \times 4, 7 \times 4$, and $8 \times 4$; so, it is 32. Then I drew [H’s] 13 pagodas. |
| 37:51 | RC: But you did not finish all pagodas; correct? |
| 37:57 | E: Yeah. There are 10 pagodas. I should have drawn [all pagodas]; but I skipped the 11th and the 12th [a little later] then calculated $14 \times 13$; no, it is $4 \times 13$. … Then – I have 32 floors and Hilda has 52 floors, so I added the floors together, and got 84 floors. |
| 39:38 | RC (Checks if Ella notices the unit shift): Do you remember what was my question? |
| 39:42 | E: Altogether. The two of us. RC: (Narrowing the question): Altogether – what? |
| 39:50 | E: How many pagodas. (A little later) Oh, [we answered] how many floors altogether. |
| 40:20 | RC: So, did you answer my question? |
| 40:23 | E: (After H and E agreed they did not): Then I should use $8 + 10 = 18$ (indicating focus on pagodas but confused the total of 13P with the 10P she actually drew). |
| 40:27 | RC: What is the meaning of $8 + 10$? |
| 40:30 | E: (Now realizing her error with 10 pagodas): No. It is 13. (A little later, after more conversation about the units with H) Ah. 13 + 8. So, 21 pagodas altogether. |

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Table 5. Data excerpt from Ella’s explanation of her solution to the SUC-sum task.
answer. Yet, in the following SUC-diff task he again operated on 1s. Inferring assimilation into the Totaling conception explains why it made sense for him to continue operating on sub-totals of 1s and then add/subtract them. This conceptual difference, namely, availability of the composite units for further operations in mDC but not in the Totaling conception, underlies our stance that the former is more supportive of SUC than the latter.

To further support this stance, we end this section with data from Ella’s work on the SUC-diff task that followed her realization of unit shift. We note that this paper does not focus on Ella’s conceptual progress toward SUC. Yet, these data help to further contrast Ella with Tobi as exemplars—and explain why mDC provides a better foundation for constructing SUC. Whereas Tobi’s case manifests the Totaling conception as a plausible cause for reverting to operating on 1s, Ella manifests how mDC supports one’s focus on composite units.

Using the same numbers from the SUC-sum task (Ella = 8P₄, Hilda = 13P₄), Figure 6 presents Ella’s solution to the SUC-diff task that RC introduced to both students: “How many more pagodas does Hilda have than Ella?” First, Ella immediately and correctly wrote “P” for the unit being “pagodas” (left), followed by the subtraction algorithm to find the difference is 5 pagodas. Then, likely expecting RC would ask to explain it, she spontaneously drew and enumerated 13 lines as inscriptions of her pagodas (right) without writing the accrual of 1s. She paused for a few seconds, likely thinking how to show the subtraction of 8 pagodas to RC. Next, she wrote “-1” over the first line and “-2” over the second line. She paused again, erased the “2” and replaced it by “1” (minus sign was already there) and continued writing “-1” above each of the first 8 pagodas. Finally, she wrote “5 more units” (5个) as her answer. It took Ella 2 min to complete the entire task, which RC followed with a request to explain it (Table 5 above). She then explained to RC the “-1” symbols showed she was subtracting 8 pagodas from 13 pagodas.

Ella’s solutions of SUC-sum (incorrect, correct) and SUC-diff (all correct) tasks, seem to support our stance that mDC is more supportive of the SUC conception than the Totaling conception. Our four-fold focus is on the availability to the student of composite units for assimilating SUC tasks. First, Ella immediately triggered the subtraction of 13–5 while recognizing composite units as task-relevant. This is contrasted with her prior operations on sub-totals of 1s for the SUC-sum task, and with Tobi’s operations on sub-totals of 1s for both SUC tasks. Second, while using a somewhat similar drawing, Ella kept the task-relevant components of her previous diagram while omitting the task-irrelevant components (not writing the accrual of 1s). We infer this omission as rooted in the distinction she then made between 1s and composite units. Third, Ella did not draw two sets of composite units (hers and Hilda’s). Rather, she drew a single set of 13 lines to represent both given sets of composite units. We infer she could conceptualize the set of 8Ps as “nested” within her set of 13Ps.
using a single image for both sets (Steffé, 1992). Fourth, Ella could explicate the process of subtracting 8Ps as a repeated process of removing single pagodas she considered to constitute her set, until reaching the 8th line. We thus infer that, for Ella, each pagoda was conceived as a unit in and of itself—not mainly as a container of 1s (e.g., 4 floors).

These four aspects suggest Ella assimilated the task into an emerging SUC conception affording isolation of composite units (“pagodas”) in the given situation and thus setting her task-relevant goal of finding their difference. At first, to solve the SUC-sum task, her mDC conception seemed to underlie a conceptual focus on 1s, leading to an observable behavior to find sub-totals and then a global total of 1s (like Tobi). Yet, Ella’s mDC conception also included the needed ingredients to: (a) realize her task-irrelevant response, (b) shift to the task-relevant unit, and (c) maintain the task-relevant focus on composite units while explicitly distinguishing them from the 1s (“four floors”) that constituted each. In contrast, Tobi’s Totaling conception allowed fixing his recurring errors upon seeing Kate’s answers, but it did not seem to support a shift in his own focus on 1s and the two-step procedure of finding sub-totals and global totals of those 1s.

5. Discussion

We presented analyses of Chinese elementary students’ solutions to SUC tasks. The error we found is consistent with their US counterparts (Tzur et al., 2018). We discuss the implications of our study for theory building, for future research, and for practice that extends the previous work.

5.1 Implications for theory building

We see two implications of this study for theory building. First, it helps explain cognitive sources of procedural understandings that seem common to children across cultures. Second, it further illuminates why erroneous solutions are reasonable from a student’s frame of reference.

5.1.1 Explaining conceptual sources of procedural understanding. We presented novel analyses of conceptual sources of procedural ways students use to assimilate tasks involving both 1s and equal-size composite units. This underlies the contribution of a novel distinction, between mDC and Totaling, the latter illuminated by Tobi’s case. We explained how Totaling underlies responses reflective of the errors we found also on the written assessment and in Ella and Hilda’s initial responses to the SUC-sum task. Using such a conceptual analysis, we highlighted commonalities and differences between the Totaling and mDC conceptions, thus moving beyond categorizing the former as procedural understanding.

Totaling and mDC seem common to students in different cultures (China, USA). In both, a student recognizes two types of units that also constitute a SUC task (Tzur et al., 2013). Both conceptions lead to setting a goal that foregrounds the 1s, namely, finding the sum (or difference in) 1s in the given sets. Setting this goal by Chinese students seems heightened by teaching and textbooks that focus on a two-step procedure (Figures 1 and 2).

MDC differs from Totaling in the explicit operation on and thus greater availability of composite units. MDC is marked by the distribution of 1s into each of the composite units in a many-to-1 manner (Clark & Kamii, 1996; see Figure 5). Such a simultaneous attention to 1s and composite units (Steffé, 1992) can serve in isolating and operating only on the composite units. In this sense, we theorize that mDC is conceptually more advanced than focusing on totals of 1s, in that mDC includes a conceptual “ingredient” that affords attending to, distinguishing, and operating simultaneously on composite units and 1s that constitute them.

5.1.2 Foregrounding assimilatory conceptions (not ‘Mis’). Postulating two plausible conceptions into which students may assimilate SUC tasks manifests a stance of depicting such answers as

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conceptions, while not referring to erroneous solutions as ‘misconceptions’. Seen through the lens of assimilation, such a stance emphasizes aspects and distinctions (e.g., Totaling vs. mDC) of what students do know (an asset view). Misconception seems to focus on students’ mathematical deficits, as seen from the observer’s first-order model. That is, it does not distinguish what an observer would rightly consider mathematically unreasonable and wrong from what (and why) students committing particular errors would consider reasonable.

A related issue is that of taking correct performance as indicative of correct reasoning, and incorrect – for the opposite. In our study, Tobi’s initial incorrect solution to the SUC-sum task (147P) would indicate ‘misconception’, his later answer of 21P once he saw Kate’s answer would indicate a correct conception, and his incorrect solution just a few minutes later (35P instead of 5P) would, again, indicate a ‘misconception’. Similarly, Ella’s initial incorrect solution to the SUC-task (84P) indicates a misconception. But how should it be judged in comparison to her self-corrected solution of 21P? These examples illustrate the four (2 × 2) possibilities we theorize: task-relevant (adequate) conception with either correct or incorrect performance and task-irrelevant conception with those two possibilities. Our study demonstrates the importance of analyzing conceptual sources of task-irrelevant conceptions, by articulating how similar errors may be rooted in different conceptions (e.g., mDC vs. Totaling).

5.2 Implications for future research

We see two key implications of this study for future research. First, the similarity of conceptual sources inferred to underlie Chinese and US students’ solutions insinuate a common progression in reasoning. Demonstrating this common ways of reasoning for mDC and SUC, our study contributes to the body of research in other areas, such as fractions (Norton et al., 2018). Future research could extend this work to (a) other conceptions found in Chinese and US students (forthcoming) and (b) similar and/or other conceptions in students of other cultures.

Second, it seems important to extend research on inadequate conceptual understanding found particularly in Asian students (Yang, 2015, 2019). The grade-reversal in solving the SUC-sum task we found, with more 3rd and 4th graders committing a Totaling error than 2nd graders, provides an intriguing evidence for how procedural mastery may limit proper conceptualization. International studies, such as PISA (OECD, 2020), have continually ranked Chinese students at the top. Important as those accomplishments are, they might mask inadequate conceptual understanding, such as lack of attention to composite units revealed in our study.

5.3 Implications for practice

Arguably, an imperative implication for practice is to foster teachers’ recognition of the error we found, appreciation for its plausible conceptual sources, and ability to foster student construction of and operations on composite units. Previous work, which found similar errors in elementary students in the US (Tzur et al., 2018), suggested this would be equally important across social-cultural boundaries. In China, our study has been a first attempt to examine both the scope of this error and its conceptual sources. Our work with elementary Chinese teachers indicates that the majority of them are unaware of this error pattern, as well as its linkage to inadequate reasoning when solving SUC tasks and place-value, base-ten reasoning (Tzur et al., 2018).

Professional development (PD) to foster teachers’ recognition and appreciation of these errors, and ways to address them, revealed that student solutions to SUC tasks can benefit from focusing on the distinction between the mDC and Totaling conceptions that may give rise to it. Our work to create such PD in China is just starting, and capitalizes on a recent PD model used in the US. In such PD, teachers’ guided reflection on video records (e.g., Tobi and Ella) focuses on steering
away from correctness of answers and time to obtain them. Instead, teachers learn to notice and infer the extent to which a student’s conception of number as composite unit supports mDC and/or SUC reasoning, and research-based instructional methods to foster the shift from the former to the latter.

6. Concluding remarks

Our findings of intriguing and counter-intuitive units confusion indicate that SUC is anything but trivial. Inferring two different conceptions (mDC and Totaling) demonstrates a way forward in postulating conceptual sources of a prevalent error pattern. We hope this study indicates possible mismatches between adequate conceptualization and mastery, and thus illuminates challenges facing researchers and teachers when “practice-practice-practice” falls short of making perfect.

Contributorship

All authors contributed to the conceptualization and design of the study. Rui Ding and Ron Tzur conducted the research and drafted the manuscript. Bingqian Wei and Xuanzhu Jin joined in the data collection and data analysis. Alan Davis and Xianyan Jin contributed to the development of the tools. All authors read and approved the final manuscript.

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