Stability analysis of boundary layer flow and heat transfer over a permeable exponentially shrinking sheet in the presence of thermal radiation and partial slip

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Abstract. In this paper, the steady boundary layer fluid flow with heat transfer over an exponentially shrinking sheet with thermal radiation, partial slip and suction is studied. The similarity transformation was applied to the governing partial differential equations to transform into a set of ordinary differential equations which are then solved numerically using bvp4c function in Matlab. It is reveal that dual solutions exist in our observations. A stability analysis is performed to determine which solution is linearly stable and physically realizable.

1. Introduction

Studies of viscous fluid flow and heat transfer in boundary layer caused by shrinking sheet have been considered by several authors recently. Wang [1] is the first studied about shrinking sheet. Bhattacharyya and Vajravelu [2] proposed the study stagnation-point flow towards an exponentially shrinking sheet. Next, Saleh et al. [3] studied steady stagnation flow for the mixed convection towards a vertical shrinking sheet.

Many researchers have discovered dual solutions in their computations such as Merrill et al. [4], Rosca and Pop [5], Naramgari and Sulochana [6] and Mishra and Singh [7]. Hence, to determine which solution is stable and vice versa, a stability analysis is performed. According to Merkin [8], the solutions with positive eigenvalues indicate stability and the solutions with negative eigenvalue imply otherwise. Since then, many researchers employed the stability analysis to determine which solution is stable. Ishak [9] and Yasin et al. [10] investigated shrinking sheet problem and found that lower branch is not stable while upper branch is opposite manner. In this present study, the aim is to analyze the stability analysis on boundary layer of fluid flow with heat transfer over an exponentially shrinking sheet.

2. Mathematical formulation

Let us consider a steady incompressible, viscous and two-dimensional convection boundary layer flow and heat transfer over a permeable exponentially shrinking sheet. Under these conditions, the equations of governing problem are as follows:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(1)
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}
\]
\[
\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y}
\]

Here \( u \) and \( v \) are defined as the components of velocity along \( x \) and \( y \) axes, \( T \) represents the temperature of fluid, \( \alpha \) for the thermal diffusivity, \( \nu \) represents the kinematic viscosity, \( \rho \) is the density and \( C_p \) is the specific heat of the fluid at constant pressure. The appropriate boundary conditions for the problem are given by.

\[
v = v_0(x), \quad u = U + Nv \frac{\partial u}{\partial y}, \quad T = T_\infty + D \frac{\partial T}{\partial y} \quad \text{at} \quad y = 0
\]

\[u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty\]

where \( U = -U_0 \exp(x/L), \quad T_\infty = T_0 \exp(x/2L) \) and \( v_0(x) = V_0 \exp(x/2L) \). \( U \) is the shrinking velocity, \( T_\infty \) is the variable temperature at the sheet, \( L \) is the length, \( U_0 \) is the velocity, \( T_0 \) is the temperature and \( V_0 \) is the mass flux velocity with \( V_0 < 0 \) for suction and \( V_0 > 0 \) for injection. Here, \( N = N_i \exp(-x/2L) \) and \( D = D_i \exp(-x/2L) \) where \( N \) and \( D \) are the velocity and thermal slip factor respectively while \( N_i \) and \( D_i \) are initial value for velocity and thermal slip factor, respectively.

Using Rosseland approximation for the radiation flux \( q_r \), we can write

\[
q_r = -\frac{4\sigma^*}{\kappa^*} \frac{\partial T^4}{\partial y}
\]

where \( \kappa^* \) and \( \sigma^* \) are the mean absorption coefficient and the Stefan-Boltzmann constant while \( T^4 \) as below:

\[
T^4 \equiv 4T_\infty^3 T - 3T_\infty^4
\]

We now introduce the stream function \( \psi \), which is defined as \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \). Then, we assume the similarity variables defined as

\[
\psi = \sqrt{2\nu L U_0} f(\eta) \exp(x/2L), \quad T = T_0 \exp(x/2L) \theta(\eta), \quad \eta = y \sqrt{\frac{U_0}{2\nu L}} \exp(x/2L).
\]

In terms of these variables, we have

\[
u = U_0 \exp(x/L) f'(\eta), \quad v = -\sqrt{\frac{U_0}{2L}} \exp(x/2L) [f(\eta) + \eta f'(\eta)]
\]

where primes indicate differentiation with respect to \( \eta \). Using equations (5) and (7), equations (2) and (3) become

\[
f'''' + f f'' - 2f^2 = 0
\]

\[
\frac{1}{Pr} \frac{\theta''}{1 + \frac{4}{3} R} + \frac{\theta'}{f} + \theta f' - f' \theta = 0
\]
where $\Pr = \frac{V}{\alpha}$ is the Prandtl number and $R = \frac{4\sigma T_0^3}{k_w k}$ is the parameter for radiation. The transformed boundary conditions are:

$$f(0) = S, \quad f'(0) = -1 + \lambda f''(0), \quad \theta(0) = 1 + \delta \theta'(0)$$

$$f'(-\eta) \to 0, \quad \theta'(\eta) \to 0 \quad \text{as} \quad \eta \to \infty$$

(11)

where $\lambda = N \sqrt{U_0/2L} (\eta > 0)$, $\delta = D \sqrt{U_0/2vL} (\eta > 0)$ and $S = -V \sqrt{U_0/2vL}$. $\lambda$ represent parameter for velocity slip, $\delta$ represent parameter for thermal slip and $S$ is the parameter for the constant mass transfer with $S > 0$ and $S < 0$ are for suction and injection.

The physical quantities that are used in this problem are the local Nusselt number, $Nu_x$, and skin friction coefficient, $C_f$, which are given as follows

$$Nu_x = \frac{L}{T_0 \exp(x/2L)} \left( \frac{\partial T}{\partial y} + q_c \right)_{y=0}, \quad C_f = \frac{\mu}{\rho [U_0 \exp(x/L)]^2} \left( \frac{\partial u}{\partial y} \right)_{y=0}.$$  

(12)

Equations (5) and (7) are substitute into equation (12), we obtain

$$(2/Re_x)^2 Nu_x = -(1 + R) \theta'(0), \quad (2Re_x)^2 C_f = f''(0).$$

(13)

Here $Re_x$ represent the local Reynolds number which is define as $Re_x = (U_0 L/\nu) \exp(x/L)$.

3. Stability analysis

In this paper, we found that there are dual solutions. Following Weidman et al. [11], a variable $\tau$ has to be introduced. We consider unsteady case for equations (2) and (3), which are can be replace by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

(14)

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_c}{\partial y}$$

(15)

where $t$ indicates the time. Then, we introduce new similarity variables as below:

$$\psi = \sqrt{2\nu L U_0} \eta \exp(x/2L), \quad T = T_0 \exp(x/2L) \theta(\eta, \tau),$$

$$\eta = \sqrt{U_0 \exp(x/2L), \quad \tau = \frac{U_0 t}{2L}} \exp(x/L)$$

(16)

We substitute equation (16) into equations (14) and (15) and obtain equations as follows:

$$\frac{\partial^3 \psi}{\partial \eta^3} + f \frac{\partial^3 f}{\partial \eta^3} - 2 \left( \frac{\partial f}{\partial \eta} \right)^2 - \frac{\partial^3 f}{\partial \eta^3 \partial \tau} = 0$$

(17)

$$1 \frac{\partial^3 \theta}{\partial \eta^3} \left( \frac{\partial^3 f}{\partial \eta^3} \right)^2 + f \frac{\partial^3 \theta}{\partial \eta^3 \partial \tau} - \frac{\partial^3 f}{\partial \eta^3 \partial \tau} = 0$$

(18)

alongside boundary conditions as follows:
The stability of dual solutions is determined by adopting the analysis suggested by Weidman et al. [11]:

\[
f(\eta, \tau) = f_0(\eta) + e^{-\gamma \tau} F(\eta, \tau), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{-\gamma \tau} G(\eta, \tau),
\]

where \( \gamma \) denotes an unknown eigenvalue while \( F(\eta, \tau) \) and \( G(\eta, \tau) \) are small relative to \( f_0(\eta) \) and \( \theta_0(\eta) \). Next, we substitute equation (20) into equations (17) and (18). As a result, we get the following equations

\[
\frac{\partial^2 F}{\partial \eta^2} + f_0 \frac{\partial^2 F}{\partial \eta^2} + f_0 F - (4f_0' - \gamma) \frac{\partial F}{\partial \eta} - \frac{\partial^2 F}{\partial \eta \partial \tau} = 0, \tag{21}
\]

\[
\frac{1}{Pr} \left( 1 + \frac{4}{3} R \right) \frac{\partial^2 G}{\partial \eta^2} + f_0 \frac{\partial G}{\partial \eta} + \theta_0 F - f_0' G - \theta_0 \frac{\partial F}{\partial \eta} + \gamma G - \frac{\partial G}{\partial \tau} = 0 \tag{22}
\]

and the boundary conditions take the following form:

\[
F(0, \tau) = 0, \quad \frac{\partial F}{\partial \eta}(0, \tau) = \lambda \frac{\partial^2 F}{\partial \eta^2}(0, \tau), \quad G(0, \tau) = \delta \frac{\partial G}{\partial \eta}(0, \tau)
\]

\[
\frac{\partial F}{\partial \eta}(\eta, \tau) \rightarrow 0, \quad G(\eta, \tau) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty \tag{23}
\]

After that, we set \( \tau = 0 \), \( F = F_0(\eta) \) and \( G = G_0(\eta) \) to obtain the following:

\[
F_0'' + f_0 F_0'' + f_0' F_0 - (4f_0' - \gamma) F_0' = 0, \tag{24}
\]

\[
\frac{1}{Pr} \left( 1 + \frac{4}{3} R \right) G_0'' + f_0 G_0' + \theta_0' F_0 - f_0' G_0 - \theta_0 F_0' + \gamma G_0 = 0 \tag{25}
\]

along with the boundary conditions:

\[
F_0(0) = 0, \quad F_0'(0) = \lambda F_0''(0), \quad G_0(0) = \delta G_0'(0)
\]

\[
F_0'(\eta) \rightarrow 0, \quad G_0(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty \tag{26}
\]

The stability of the problem can be tested via the smallest eigenvalue \( \gamma \). Therefore, the condition \( F_0'(\eta) \rightarrow 0 \) as \( \eta \rightarrow \infty \) has been put at rest as suggested by Harris et al. [12] and for fixed value of eigenvalue, \( \gamma \).

4. Results and discussion

We apply similarity transformation method to deduce the partial differential equations governing the fluid flow to a set of nonlinear nonsimilar equations. Then, the nonlinear system of equations will be solved using bvp4c function in Matlab, the governing ordinary differential equations (9) and (10) alongside boundary conditions (11) were solved numerically. In this paper, we used the values of \( S \) in the region \( 2.18 \leq S \leq 2.24 \) while the values of \( Pr = 0.7, \quad R = 0.1, \quad \lambda = 0.1 \) and \( \delta = 0.1 \) are fixed.

To support the numerical results obtained, we compare our results with Sharma et al. [13]. Table 1 shows the values of the \( \theta'(0) \) for various values of \( Pr \). The table shows that our numerical results are in good agreement with the previous works. Hence, we are assured that our results are correct.
1 represents the variation of $f''(0)$ with $S$ while Figure 2 illustrates the variation of $-\theta'(0)$ with $S$.

From Figure 1 and 2, we can clearly see that there are dual solutions within a certain range. Further, velocity and temperature profiles in Figures 3 and 4 are drawn to prove the existence of the second solution. From these figures, it is clearly seen that the thickness of the boundary layer presents thinner for the first solution compared to the second solution.

**Table 1.** Value of $\theta'(0)$ for several values of Prandtl number with $S = R = \lambda = \delta = 0$ & $f'(0) = 1$.

| Pr   | Sharma et al. [13] | Present study |
|------|--------------------|---------------|
| 1    | -0.954789          | -0.955216     |
| 2    | -1.471461          | -1.471361     |
| 3    | -1.869073          | -1.868998     |
| 5    | -2.500125          | -2.500074     |
| 10   | -3.660350          | -3.660326     |

**Figure 1.** Variation of $f''(0)$ with $S$.

**Figure 2.** Variation of $-\theta'(0)$ with $S$.

**Figure 3.** Velocity profile for several values of $S$.

**Figure 4.** Temperature profile for several values of $S$. 
Since the numerical computation admits dual solutions, we perform a stability analysis in order to identify which solution is stable. The stability of the flow can be tested by looking at the polarity of the smallest eigenvalue $\gamma$ itself. The solution is unstable if the value of the smallest eigenvalue $\gamma$ is negative while it is stable if the smallest eigenvalue $\gamma$ vice versa. Table 2 displays the smallest eigenvalues $\gamma$ for some values of $S$. From Table 2, we found that smallest eigenvalue $\gamma$ for the first solution (upper branch) is positive and for the second solution (lower branch) is negative. Hence, we can conclude that it is physically significant and stable for the first solution while it is physically insignificant and unstable for second solution. Therefore, our further discussion is about first solution only. Based on Figure 1, it’s shown that the skin friction coefficient increases as $S$ increase. The local Nusselt number also increases as $S$ increase as display in Figure 2.

| $S$  | Upper Branch | Lower Branch |
|------|--------------|--------------|
| 2.18 | 0.3015       | -0.2985      |
| 2.20 | 0.3952       | -0.3903      |
| 2.22 | 0.4758       | -0.4671      |
| 2.24 | 0.5427       | -0.5316      |

5. Conclusion
In this study, we observed that dual solutions exist. Hence, we carried out the stability analysis to identify which solution is stable and unstable. After the stability analysis, we seen that the first solution obtained are stable while it is otherwise for the second solution and can be ignored.

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