The Spatial SIS model of epidemic spreading on networks

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Abstract. In this paper, we define a Spatial Susceptible-Infectious-Susceptible model for discussing how disease spreads on spatial driven network. The spreading rate is affected by spatial distance of links and infective medium. Numerical studies exhibit that infective medium is bad for disease spreading. With the increasing of $\beta$, spreading behavior needs longer time to reach to the steady state, but reaches to smaller steady infected ratio. In addition, also by numerical study, when spreading rate is small, epidemic behavior is affected by some special nodes.

Keywords: S-SIS model, spatial driven network, epidemic spreading

1. Introduction
The behavior of epidemic spreading has been described by several models[1-3]. In 1975, Bailey introduced a model, which was known as the susceptible-infectious-susceptible (SIS) model[1]. In this model, individuals could be divided into two classes: susceptible (S) individuals who don’t have the disease yet but can catch it if they contact to someone who does, and infective (I) individuals who have the disease and can pass it on. In SIS model, infective individuals recover to susceptible individuals automatically after a few steps.

The SIS model cannot be solved exactly on a network, but Pastor-Satorras and Vespignani used mean-field theory to solve SIS model on scale-free network[4,5]. In these works, each of the links has equal effective spreading rate under mean-field approximation. The epidemic behavior is completely determined by degree distribution. The SIS model can also be extended to describe the real epidemic spreading behavior to predict the infection density of disease propagation and to reduce the duration of epidemic outbreak[6,7].

However, many real systems are embedded in space[8-11]. In such real systems, spatial distances between individuals play important role not only in structure but also in dynamical process. Several spatial network models have been proposed for studying the spatial characters and the spatial dependence of dynamical process[12-15]. Scientists have revealed two important phenomena. The first one is on average the majority of our social relationships are in our neighborhood[15]. The second one is the disease has higher probability to spread on short spatial distance than on long spatial distance. Therefore, spatial properties are of great significance to study the epidemic spreading.

In this paper, we study the epidemic spreading process on growing spatial driven network[13]. We focus on how network spatial structure and spreading rate influence spreading behavior. The
spreading rate is not a constant, but it is inversely proportional to the power function of spatial distance for each link. Monte Carlo method is used in the simulation of dynamics.

2. Spatial Susceptible-Infectious-Susceptible Model

In SIS model[1], each node in the network represents an individual, and each link represents a connection. The disease spreads from one node to another through a link. These nodes have two states: susceptible and infected. At each time step, each susceptible node is infected with rate $\nu$ if it links with one or more infected nodes. At the same time, infected nodes are cured and become susceptible again with rate $\delta$. Thus, the effective spreading rate $\lambda = \nu/\delta$. Simplified, we always set $\delta = 1$. According to the mean-field approximation, in the traditional SIS model, each link has the same spreading rate $\lambda$. But in real systems, the disease always spreads through human contact, air and other infective media. Most of these factors are affected by spatial distance. Thus, we give the definition of Spatial Susceptible-Infectious-Susceptible (S-SIS) Model:

We assume that $\lambda_0$ denotes the infective rate of the disease itself. The cure rate $\delta = 1$ for simplified. Inspired from previous works[16-18], we define the effective spreading rate $\lambda_{sj}$ from infective node $j$ to its susceptible neighbor $s$ is inversely proportional to their spatial distance as follows:

$$\lambda_{sj} = \lambda_0 \left( \frac{\min_{k \in \Omega_s} d_{kj}}{d_{sj}} \right)^\beta,$$

where $\Omega_s$ is the set of all susceptible neighbors of node $j$. The distance between infected node and its nearest susceptible neighbor is used to rescale $\lambda_{sj}$ ($0 < \lambda_{sj} \leq \lambda_0$). Thus, the spreading rate from infective node to its nearest susceptible neighbor is $\lambda_0$. For other susceptible neighbors, the spreading rate is inversely proportional to their spatial distance with power $\beta$. We assume $\beta$ is the parameter of infective medium. For example, malaria spreads by mosquitoes. Many empirical studies and theoretical analysis on the pattern of animal foraging and migration have uncovered that the mobility travel step size for animal follows a power-law[19-21]. What’s more, the step size of human traveling follows a power-law as well. For smaller $\beta$, the medium is easier to spread disease; for larger $\beta$ it is more difficult.

In S-SIS model, the time-discrete dynamics is defined by synchronously updating the states of all individuals with the following rules: If node $s$ is infected at time $t-1$, it is susceptible at time $t$. Otherwise, if node $s$ is susceptible at time $t-1$ and it connects to one or more infected nodes at the same time, then node $s$ is infected at time $t$ with probability $\lambda_s$. The value of $\lambda_s$ depends on all infective neighbors and can be written as

$$\lambda_s = 1 - \prod_{k \in \varphi_s} (1 - \lambda_{sk})$$

$\varphi_s$ is the set of all infective neighbors of node $s$. In order to discuss how network spatial structure and infective medium influence the epidemic behavior, we set $\lambda_0 = 1$.

3. Numerical Results

3.1. Spatial driven network

The individuals are connected by spatial driven networks[13]. The network is fixed on the plan, each of the nodes has a two-dimensional coordinate $(x, y)$ ($0 \leq x < 1, 0 \leq y < 1$). The networks are constructed by adding nodes and links in the following way: In initial state ($t = m_0$), we generate $m_0 + 1$ all-to-all connected nodes. At every time step, a new node $n$ is randomly located in $(x_n, y_n)$ on the plan. Node $n$ connects with $m$ previous nodes, which are selected by the probability $\pi_i = \frac{d_{ni}}{\sum_j d_{nj}}$ ($0 \leq \alpha$). $d_{ni}$ is the spatial distance between new node $n$ and
previous node \( i \). Close the iteration loop by generating the next new node and so forth, until the network size reaches the desired value \( N = 10000 \). The degree distribution grows exponentially and is determined by \( m \) as follows:

\[
p(k) = \frac{1}{m+1} \left( \frac{m}{m+1} \right)^{k-m}, (k \geq m).
\]

In this paper, we set \( m_0 = 2 \) and \( m = 2 \), which means that spatial networks with different \( \alpha \) have the same degree distribution. The spatial distance distribution follows \( p(d) \sim d^{-(\alpha-1)} \).

3.2. S-SIS model on spatial driven networks

![Figure 1](image.png)

Figure 1. The infected nodes ratio \( \rho(t) \) as a function of time \( t \) with \( \alpha = 4 \) and \( \beta = 4 \). Different symbols represent different simulations.

The disease spreads in spatial driven networks following the S-SIS model. For each \( \beta \) in S-SIS model, we do 20 simulations on the same spatial driven network. In every simulation, we start by selecting one node randomly and assume it is infected. Figure 1 shows that, when \( \alpha = 4 \) and \( \beta = 4 \), the steady infected nodes ratio \( \rho_s \) is different for different simulations. \( \rho_s \) is the average of infected nodes ratio \( \rho(t) \) from \( t = 9000 \) to \( t = 10000 \). The only difference for these 5 simulations in figure 1 is the first infected node. In this case, the initial state of infected node affects the spreading behavior. But in traditional SIS models, \( \rho_s \) has nothing to do with the first infected node.

We also discuss this phenomenon with \( \alpha = 1, 2, 3, 4 \) and \( \beta = 1, 2, 3, 4 \) in figure 2. According to the simulation results, we divide the spreading processes into two classes. For the first class, \( \rho_s \) is a fixed value for different simulations with the same \( \alpha \) and \( \beta \). In this case, the value of \( \rho_s \) is determined by \( \alpha \) and \( \beta \). The first class marks with red zone in figure 2. For the second class, \( \rho_s \) has different values for different simulations. In this case, the value of \( \rho_s \) is determined by the first infected node besides \( \alpha \) and \( \beta \), as figure 1 shows. The second class marks with gray zone in figure 2. The numbers on red zone represent the value of \( \rho_s \) with corresponding \( \alpha \) and
When $\alpha = 3$ and $\beta = 2$, the steady infected nodes ratio is close to $\rho_s \approx 0.42037$, for different simulations.

**Figure 2.** States of the spreading process with $\alpha = 1, 2, 3, 4$ and $\beta = 1, 2, 3, 4$. The numbers on red zone represent the value of $\rho_s$ with corresponding $\alpha$ and $\beta$.

$\beta$. e.g. When $\alpha = 3$ and $\beta = 2$, the steady infected nodes ratio is close to $\rho_s \approx 0.42037$, for different simulations.

**Figure 3.** The infected nodes ratio $\rho(t)$ as a function of time $t$ in two special cases, (a) $\beta = 1$ and (b) $\beta = 2$ with $\alpha = 1, 2, 3, 4$, respectively.

Figure 3 shows the evolution of infected nodes ratio $\rho(t)$ as a function of time $t$ for different spatial driven networks and different $\beta$. According to eq. (3), spatial driven networks with different $\alpha$ have the same degree distribution. But in figure 3(a) and figure 3(b) the spreading behaviors are different. In S-SIS model the degree distribution is no more the main element for epidemic dynamics. We have discussed this property in Ref. [16]. Comparing figure 3(a) and figure 3(b), we reveal that the curves for $\alpha = 3, \beta = 2$ and $\alpha = 4, \beta = 2$ have platform areas when $\rho(t)$ grows as a function of time $t$. In figure 3(b) insets, circles mark this phenomenon. $\beta = 2$ means the medium is difficult to spread disease in epidemic spreading, $\alpha = 3$ and $\alpha = 4$ means short spatial distance network. Both epidemic dynamics and network spatial structure go against disease spreading. But when disease spreads to some special nodes, disease erupts again.
In figure 4, we plot $\rho(t)$ as a function of time $t$ for different $\beta = 1, 2, 3, 4$ on spatial driven network with $\alpha = 1$. In figure 4 insert, we only plot three curves with $\beta = 1, 2, 3$ when $\alpha = 2$. Because in figure 2, we have discussed that, when $\alpha = 2$ and $\beta = 4$, the value of $\rho_s$ is determined by the first infected node. We do not mention this case in figure 4 insert. In figure 4 and figure 4 insert, we clearly notice a great influence of infective medium on effective spreading time of disease and the steady infected nodes ratio $\rho_s$. The effective spreading time is the number of steps to reach to the steady state. With the increasing of $\beta$, the disease needs longer effective spreading time to reach to the steady state, $\rho_s$ gets smaller value. According to eq. (2), for larger $\beta$, $\lambda_{sj}$ gets a smaller value. Thus, these infective media have smaller probability to transmit disease, which cause smaller $\rho_s$ and longer effective spreading time.

4. Summery
In summary, this paper presents Spatial Susceptible-Infectious-Susceptible model to discuss epidemic dynamics on spatial driven network. In this model, infective rate of a node is determined by its spatial connections and infective medium. For larger $\alpha$ and $\beta$, spreading steady state $\rho_s$ is determined by the first infected node. In others, $\rho_s$ is independent of the first infected node. With the increasing of $\beta$, epidemic behavior needs longer time to reach the steady state, but the value of $\rho_s$ is smaller. Larger $\beta$ makes infective medium go against epidemic spreading. In the case of $\beta = 2$ and $\alpha = 3, 4$, when disease spreads to some special nodes, disease erupts again. It will be another interesting work, finding properties of special nodes in S-SIS model, in the future.

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