OPTIMAL POWER FOR AN ELLIPTIC EQUATION RELATED TO SOME CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

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Abstract. In this paper we analyze the following elliptic problem related to some Caffarelli-Kohn-Nirenberg inequalities:

$$-\text{div}(|x|^{-2\gamma} \nabla u) - \lambda \frac{u}{|x|^{2(\gamma+1)}} = |\nabla u|^p |x|^{-\gamma p} + cf, \quad u > 0 \text{ in } \Omega, \quad u|_{\partial \Omega} \equiv 0,$$

where \( \Omega \subset \mathbb{R}^N \) is a domain such that \( 0 \in \Omega, \ N \geq 3, \) and \( c, \lambda, \gamma, p \) are positive constants verifying \( 0 < \lambda \leq \Lambda_{N, \gamma} = \left( \frac{N-2(\gamma+1)}{2} \right)^2, \ -\infty < \gamma < \frac{N-2}{2} \) and \( p > 0. \) Our study concerns to existence of solutions to the former problem. More precisely, first we determine a critical threshold for the power \( p, \) in the sense that, beyond this value it does not exist any positive supersolution to our problem, not even in a very weak sense. In addition, we show existence of solutions for all the values \( p > 0 \) below this threshold, with the restriction \( \gamma > \frac{-N(1-p)+2}{2}, \) whenever the righthand side verifies \( f(x) \leq |x|^{-2(\gamma+1)} \) if \( \gamma > -1. \) When \( \frac{-N(1-p)+2}{2} < \gamma \leq -1 \) it suffices that \( f \in L^2/|x|^p(\Omega). \) The existence of solutions for \( 0 < p < 1 \) and \( \gamma \leq \frac{-N(1-p)+2}{2} \) is an open question.

1. Introduction and main results. We devote these notes to study the problem

$$\begin{cases}
-\text{div}(|x|^{-2\gamma} \nabla u) - \lambda \frac{u}{|x|^{2(\gamma+1)}} = |\nabla u|^p |x|^{-\gamma p} + cf, & \text{in } \Omega, \\
\rule{0cm}{0.2cm} u > 0 & \text{in } \Omega, \\
\rule{0cm}{0.2cm} u \equiv 0, & \text{on } \partial \Omega,
\end{cases}$$

where \( \Omega \subset \mathbb{R}^N, \ N \geq 3, \) is a domain that contains the origin. We assume that \( f \) is a nonnegative function under certain hypothesis that will be specified later and \( c, \lambda, \alpha, \gamma \) are positive constants such that \( 0 < \lambda \leq \Lambda_{N, \gamma} = \left( \frac{N-2(\gamma+1)}{2} \right)^2, \ -\infty < \gamma < \frac{N-2}{2} \) and \( p > 0. \) Our analysis is motivated in part by the work [3], where the authors consider equations of type

$$-\text{div}(|x|^{-2\gamma} \nabla u) = g(u, x),$$

with \( -\infty < \gamma < \frac{N-2}{2}. \) The function \( g \) satisfies certain conditions that vary along the paper, depending on the subject to analyze. In fact, the authors treat here

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a more general operator of $p$-laplacian type. Notice that if we take $g(\nabla u, u, x) = \lambda \frac{|u|^r}{|x|^\beta} + |\nabla u|^p |x|^{-\gamma} + cf$ our problem is related to (2), it has an additional gradient term that is not considered in [3].

A suitable frame to study existence of solutions to this kind of equations is the following energy setting. The weighed Sobolev Space $\mathcal{D}^{1,2} \Omega$ is defined as the completion of $C^\infty(\Omega)$ with respect to the norm

$$||u||_{2,\gamma} = \left( \int_\Omega \left( |u|^2 + |\nabla u|^2 \right) |x|^{-2\gamma} dx \right)^{\frac{1}{2}}.$$ (3)

Analogously, $\mathcal{D}^{1,2}_{0,\gamma} \Omega$ is the completion of $C^\infty_0 (\Omega)$ with respect to the norm above.

For these spaces there exists a critical exponent for the Sobolev inequality. It is a consequence of the well known Caffarelli-Kohn-Nirenberg inequality, see [13].

**Theorem 1.1.** Let $u \in \mathcal{D}^{1,2}_{0,\gamma} \Omega$, then there exists a positive constant $C = C(\gamma, \Omega)$ such that

$$\left( \int_\Omega |u|^{2^* \gamma} |x|^{-2\gamma} dx \right)^\frac{1}{2^*} \leq C \left( \int_\Omega |\nabla u|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}},$$

being

$$2^* = \begin{cases} \frac{2N}{N-2}, & \gamma \geq 0 \\ \frac{2(N-\gamma)}{N-2(\gamma+1)}, & \gamma < 0. \end{cases} (4)$$

The next result is an extension of the Hardy-Sobolev inequality to the weighted Sobolev space $\mathcal{D}^{1,2}_{0,\gamma} \Omega$:

**Theorem 1.2.** Let $N \geq 3$ and $-\infty < \gamma < \frac{N-2}{2}$. For all $u \in \mathcal{D}^{1,2}_{0,\gamma} \Omega$ it holds that

$$\Lambda_{N,\gamma} \int_\Omega |u|^2 |x|^{-2(\gamma+1)} dx \leq \int_\Omega |\nabla u|^2 |x|^{-2\gamma} dx,$$

with $\Lambda_{N,\gamma} = \left( \frac{N-2(\gamma+1)}{2} \right)^2$, optimal and not achieved.

Indeed different kind of estimates on the remaining terms in the previous inequality have been shown. In this direction, the following result is suitable for the problem in consideration, see [1] and references therein for further literature on this topic.

**Theorem 1.3.** Let $N \geq 3$ and $-\infty < \gamma < \frac{N-2}{2}$. Then

$$\int_\Omega |\nabla u|^2 |x|^{-2\gamma} dx - \Lambda_{N,\gamma} \int_\Omega \frac{|u|^2}{|x|^{2\gamma+1}} \geq C(r, \beta, \gamma, |\Omega|) \left( \int_\Omega |\nabla u|^r |x|^{-\beta \gamma} \right)^{\frac{1}{r}},$$

being $1 < r < 2$, and $r \leq \beta < \infty$ if $\gamma \leq 0$, while if $\gamma > 0$ then $1 \leq \beta < 2 + \delta (N, r, \gamma)$, for some positive constant $\delta$.

For $N \geq 3$ and $-\infty < \gamma < \frac{N-2}{2}$, the weight $|x|^{-2\gamma}$ is admissible to ensure that the classical Harnack inequality holds for solutions to (2) with $g = 0$, see for instance [15]. Moreover, inspired in a preceding work [10], the authors show in [3] the next weak Harnack inequality.

**Theorem 1.4.** Let $u \in \mathcal{D}^{1,2}_{loc} \Omega$ be a nonnegative weak supersolution to (2) with $g = 0$ on $B_r \subset \subset \Omega$. Then

$$\left( \frac{1}{\mu(B_r)} \int_{B_r} u^\kappa |x|^{-2\gamma} dx \right) \leq C \inf_{B_r} u,$$
where $\mu(A) = \int_A |x|^{-2\gamma}$ for $A \subset \mathbb{R}^N$, $C = C(r, N, \kappa)$ and $0 < \kappa < \frac{N}{2}$, being $2^*_\gamma$ the critical Sobolev exponent defined in (4).

In the sequel, we will also consider weighted Sobolev spaces with more general powers, namely, $\mathcal{D}^{1,q}_0(\Omega)$ with analogous definition. The space $W^{1,r}(\Omega; |x|^\beta dx)$, for some $\beta \in \mathbb{R}$, denotes the completion of $C^\infty(\Omega)$ with the norm

$$\|u\|_{W^{1,r}(\Omega; |x|^\beta dx)} = \left( \int_\Omega \left( |u|^r + |\nabla u|^r \right) |x|^\beta dx \right)^{\frac{1}{r}}.$$

Turning again our attention to equation (2) with $g(u, x) = u^{\alpha} |x|^2(\gamma + 1)$ and $\alpha > 1$, it is shown that there exists no positive entropy solution. Moreover, the authors prove that the problem exhibits complete blow-up. Conversely, there exists a nontrivial supersolution whenever the second term is $\lambda u^{\alpha} |x|^2(\gamma + 1)$, with $\alpha \leq 1$ and $0 < \lambda \leq \Lambda_{N, \gamma}$.

Inspired in this work, we wish to find out how adding a gradient term in the equation affects the existence of solutions to these problems. As a preceding study we refer to [5], where the problem under consideration is

$$\Delta u = \lambda \frac{u}{|x|^2} + |\nabla u|^p + cf,$$

with $0 < \lambda \leq \Lambda_N = \left( \frac{N-2}{2} \right)^2$, $c > 0$ and $f \geq 0$. The authors find the critical exponent, in the sense that, beyond this value there do not exist positive supersolutions, not even very weak supersolutions, concept that we will precise later on. In addition, they show that the nonexistence is due to global blow-up of the approximations to the problem. Our model extends these results corresponding to the case $\gamma = 0$. The lack of spatial symmetry of our operator if $\gamma \neq 0$ leads to develop new techniques to achieve some results related with comparison.

Another foregoing work in this direction is [11], where the optimality of the power $p$ is determined to have very weak solutions to

$$\Delta u = \lambda \frac{u}{|x|^2} + u^p + cf.$$

Furthermore, regarding existence and optimal summability of solutions to the elliptic equation

$$\text{div} (M(x)\nabla u) = \theta \frac{u}{|x|^2} + f,$$

we refer to [9]. See also [18] for the analysis of regularity of this kind of operators.

We organize this work as follows. In Section 2 we precise what we understand by very weak super and subsolutions to our problem and determine the candidate to be the critical exponent for the nonexistence result. We also give some preliminary results, concerning the integrability of our solutions. Section 3 is devoted to establish a comparison principle, that has interest itself since it is an alternative proof of the comparison result shown in [6]. Moreover, the concern of our proof resides in its application to more general operators. In Section 4 we show that above the critical exponent we have nonexistence of solutions. This nonexistence result is the strongest we can expect, since we prove that it is not possible to have any very weak positive supersolution, namely the class of solutions in the distributional sense. Furthermore, we see that in the range of values for which there is not existence of solutions, the approximating problems exhibit complete blow-up. This is the purpose of Section 5. Finally, in Section 6 we conclude this paper by
showing that below this critical exponent, we have existence of positive very weak supersolutions.

2. Nonexistence results if $p \geq p_+(\lambda)$. The aim of this work is to characterize the nonexistence of very weak positive solutions to \((1)\) in terms of $p$. We specify this notion of solutions. Note that it is the more general setting for which the equation has meaning in a distributional sense.

**Definition 2.1.** We say that $u \in L^1_{\text{loc}}(\Omega)$ is a very weak supersolution to problem \((1)\) if $u \in L^1_{\text{loc}}(\Omega; |x|^{-2(\gamma+1)}dx)$, $|\nabla u| \in L^p_{\text{loc}}(\Omega; |x|^{-\gamma p}dx)$ and it holds that

$$
\int_{\Omega} -\text{div}(|x|^{-2\gamma} \nabla \phi) u \, dx \geq \int_{\Omega} \left( |\nabla u|^p |x|^{-\gamma p} + \frac{u}{|x|^{2(\gamma+1)}} + cf \right) \phi \, dx,
$$

for every $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$. The definition of a subsolution is the analogous reversing the inequalities. If $u$ is a very weak super and subsolution we say that $u$ is a very weak solution.

It is well known that the problem \((1)\) has not positive very weak supersolutions if $\lambda > \Lambda_{N,\gamma} = \left( \frac{N-2(\gamma+1)}{2} \right)^2$, as a consequence of the optimality of the constant $\Lambda_{N,\gamma}$ in the Hardy inequality, see for instance [2]. Therefore, we will assume that $0 < \lambda \leq \Lambda_{N,\gamma}$ in the sequel.

To search for the optimal exponent we consider a radial solution to \((1)\) with $f \equiv 0$, namely $u(x) = A |x|^{-\alpha}$. It is easy to check that $a = \frac{2-p(\gamma+1)}{p-1}$ and $A^p-1a^p = -a^2 + (N-2(\gamma+1))a - \lambda = q(a)$. The left hand side is positive, then $q(a)$ must be positive. Thus $a \in (a^-, a^+)$ where $a^+$, $a^-$ are the two roots of $q$ given by

$$
a_+ = \frac{N-2(\gamma+1)}{2} \pm \sqrt{\Lambda_{N,\gamma} - \lambda}. \tag{5}
$$

The fact that $a^- < a < a^+$ is equivalent to $p_-(\lambda) < p < p_+(\lambda)$, that is

$$
p_-(\lambda) = \frac{a^+ + 2(1 + \gamma)}{a^+ + \gamma + 1}, \quad p_+(\lambda) = \frac{a^- + 2(1 + \gamma)}{a^- + \gamma + 1}.
$$

However, we observe that if $\gamma < -1$ the above exponents will be no longer well defined for all values of $\lambda$. This fact is new with respect to the problem treated in [5], (case $\gamma = 0$). Indeed, the optimal power should be written as $p_+(\lambda, \gamma)$, though we omit the dependence on $\gamma$ to simplify the notation. We will see that if $p \geq 1 = p^+(\lambda)$ and $\gamma \leq -1$, there do not exist very weak supersolutions to \((1)\). Hence the candidate to be the optimal power is

$$
p_+(\lambda) = \begin{cases} 
\frac{a^- + 2(1 + \gamma)}{a^- + \gamma + 1}, & \text{if } -1 \leq \gamma \leq \frac{N-2}{2}, \\
1, & \text{if } \gamma < -1.
\end{cases} \tag{6}
$$

**Remark 1.** Notice that whenever $\gamma \geq -1$ the function $p_+(\lambda)$ is nonincreasing in $\lambda$. If $\lambda \to 0$, then $p_+(\lambda) \to 2$, whereas $p_+(\lambda) = \frac{2(\gamma+1)+N}{N}$ if $\lambda = \Lambda_{\gamma,N}$ and $\gamma \geq -1$. Therefore, $p_+(\lambda) \in \left[ \min \left\{ \frac{2(\gamma+1)+N}{N}, 1 \right\}, 2 \right]$.

We give some preliminaries concerning to the regularity and the admissible weights for the solutions to the operator in divergence form. The proofs of these results are analogous to the Lemmas 2.2 and 2.3 in [5] and we omit them.
Lemma 2.2. Let \( u \in L^1_{\text{loc}}(\Omega; |x|^{-2(\gamma+1)}dx) \) be a nonnegative nontrivial function verifying
\[
-\text{div}(|x|^{-2\gamma}\nabla u) - \lambda \frac{u}{|x|^{2(\gamma+1)}} \geq 0,
\]
in a distributional sense. Then there exists a positive constant \( c \) and a small ball \( B_R(0) \subset \Omega \), such that \( u(x) \geq c|x|^{-\alpha} \) in \( B_R(0) \), with \( \alpha \) given by (5).

Lemma 2.3. Consider the equation in \( \Omega \)
\[
-\text{div}(|x|^{-2\gamma}\nabla \omega) - \beta \frac{\omega}{|x|^{2(\gamma+1)}} = g,
\]
with \( g \in L^1_{\text{loc}}(\Omega) \) nonnegative and \( \beta \leq \Lambda_{N,\gamma} \). If equation (7) has a very weak supersolution, then \( |x|^{-\alpha} g \in L^1_{\text{loc}}(\Omega) \), with \( \alpha \) defined in (5).

Remark 2. If we consider equation (1) with \( \lambda < \Lambda_{N,\gamma} \) we can split \( \beta = \lambda + (\beta - \lambda) \) for some \( \lambda < \beta \leq \Lambda_{N,\gamma} \). Then, applying this result we get
\[
u \in L^1_{\text{loc}}(\Omega; |x|^{-2(\gamma+1)-\alpha}dx) \text{ and } |\nabla \nu|^p \in L^1_{\text{loc}}(\Omega; |x|^{-\alpha}dx).
\]

Our nonexistence proof is strongly based on the contradiction of the Hardy-Sobolev inequality in Theorem 1.2, if we admit the existence of a very weak supersolution. This inequality holds only for functions vanishing on the boundary. However, in the case \( p = p_+ = N \) and \( \lambda = \Lambda_{N,\gamma} \) we will need to use an analogous argument for functions that do not vanish on the boundary. To overcome this fact we introduce a version of the well known Picone’s inequality for measures (see also [21] or its version for the \( p \)-laplacian [7]).

Lemma 2.4. Let \( \phi \) be a positive function verifying that \( -\text{div}(|x|^{-q\gamma}|\nabla \phi|^{q-2}\nabla \phi) \) is a positive Radon measure if \( 1 < q < N \) and \( -\infty < \beta < \frac{N-q}{q} \). Then, for all \( v \in D^1_{Q}(\Omega) \) it holds that
\[
\int_{\Omega} |\nabla v|^q |x|^{-q\beta}dx \geq \int_{\Omega} \frac{|v|^q}{\phi^{q-1}} \left( -\text{div}(|x|^{-q\beta}|\nabla \phi|^{q-2}\nabla \phi) \right) dx
+
\int_{\partial \Omega} \frac{|v|^q}{\phi^{q-1}} (|\nabla \phi|^{q-2}\nabla \phi, \nu) |x|^{-q\beta}d\nu.
\]

For the proof of this lemma we refer to [3, 4]. Now we show

Lemma 2.5. For any \( v \in D^1_{Q}(\Omega) \) we have that
\[
\int_{\Omega} v^q |x|^{-q(\beta+1)}dx \leq c\|v\|_{\beta,q}^q
\]
with the norm \( \| \cdot \|_{\beta,q} \) defined in (3), being \( \beta \) and \( q \) as in Lemma 2.4.

Proof. We have to show that
\[
\inf_{v \neq 0} \frac{\int_{\Omega} v^q |x|^{-q(\beta+1)}dx + \int_{\Omega} |\nabla v|^q |x|^{-q\beta}dx}{\int_{\Omega} v^q |x|^{-q(\beta+1)}dx} > c > 0.
\]

One can easily check that \( w(x) = |x|^{-\frac{N-q(\beta+1)}{q}} \in L^{q-1}(\Omega; |x|^{-q(\beta+1)}) \) and it verifies
\[-\text{div}(|x|^{-q\beta}|\nabla w|^{q-2}\nabla w) = \Lambda_{N,q,\beta} w^{q-1} |x|^{-q(\beta+1)},\]
with $Λ_{N,q,β} = \left(\frac{N-q(β+1)}{q}\right)^q$.

Then, for any $v \in D_β^{1,q}(Ω)$ by the previous lemma we get
\[
\int_Ω |∇v|^q |x|^{-qβ} dx ≥ -\int_Ω \frac{v^q}{w^{q-1}} div \left(|∇w|^{q-2}∇w |x|^{-qβ}\right) dx
+ \int_Ω \frac{v^q}{w^{q-1}} (|∇w|^{q-2}∇w, ν)|x|^{-qβ} dν
= Λ_{N,q,β} \int_Ω \frac{v^q}{|x|^{(β+1)}} dx - c_\delta(q, β, Ω) \int_{∂Ω} v^q |x|^{-qβ} dν.
\]

On the other hand the Sobolev trace embedding yields that
\[
\int_{∂Ω} v^q |x|^{-qβ} dν ≤ C(q) \|v\|_{q, β, q}^q.
\]

If we substitute this inequality above, we obtain (9) and conclude the proof. □

3. A comparison principle. The following comparison principle will be the key to prove both, existence of solutions for the corresponding values of the parameter $p$ and, in case of nonexistence of solutions, to show the global blow-up for the approximating problems.

We note that this is the analogous comparison result due to N. Alaa and M. Pierre (see [6]), for our operator. However, such an extension is not straight forward, since in [6] the proof is based on the isoperimetric inequality. To skip this difficulty we follow the ideas in [19], which are strongly based in the Schauder fixed point. We have devoted this section to see this proof in detail.

Throughout this section, we define the exponent
\[
q = \begin{cases} 
N - 1, & γ ≥ 0 \\
\frac{N - 2γ}{N - 2}, & γ < 0,
\end{cases}
\]
which is related to the integrability of the gradient of the solutions to the following problem. More precisely

**Lemma 3.1.** Let $-∞ ≤ γ ≤ \frac{N-2}{2}$, $N ≥ 3$ and $w$ be a solution to
\[
\begin{cases} 
- div(|x|^{-2γ}∇w) = g & \text{in } Ω, \\
w = 0 & \text{on } ∂Ω,
\end{cases}
\]
If $g ∈ L_{loc}^1(Ω)$ then $∇w ∈ L^{\tilde{q}}(Ω; |x|^{-2\tilde{γ}} dx)$, for any $\tilde{q} < q$, with $q$ given in (10).

For the proof of this Lemma we refer to Lemma 2.6 in [3].

We also recall the extension of the Sobolev inequality for this setting, see Theorem 1.1, with critical Sobolev exponent
\[
2^*_γ = \begin{cases} 
\frac{2N}{N - 2}, & γ ≥ 0 \\
\frac{2(N - 2γ)}{N - 2(γ + 1)}, & γ < 0.
\end{cases}
\]

We have all the preliminaries to show the main result of this section.
Proposition 1. Let \( w \in L^1_{\text{loc}}(\Omega) \) be such that \( \nabla w \in L^{\tilde{q}}(\Omega; |x|^{-2\gamma} dx) \), for any \( \tilde{q} < q \) with \( q \) given in (10), satisfying
\[
\begin{align*}
-\text{div}(|x|^{-2\gamma} \nabla w) &\leq \langle h, \nabla w \rangle \quad \Omega, \\
w &\equiv 0 \quad \partial \Omega,
\end{align*}
\]
where \( h : \mathbb{R}^N \rightarrow \mathbb{R}^N \) verifies
\[
\int_{\Omega} |h|^{\tilde{q}} |x|^{2\gamma \frac{q'}{q}} < +\infty.
\]
Then \( w \leq 0 \).

We begin with this existence result, for which we establish the following notation. We denote by \( H^* \) the dual space of \( \mathcal{D}^{1,2}_{0,\gamma}(\Omega) \) with the norm
\[
\| \phi \|_{H^*} := \sup_{\phi \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega), \phi \neq 0} \frac{\int_{\Omega} \phi \varphi dx}{\left( \int_{\Omega} |\nabla \varphi|^2 |x|^{-2\gamma} dx \right)^{\frac{q'}{2}}}.
\]

Lemma 3.2. There exists a unique solution \( u \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega), u \geq 0 \) in \( \Omega \) to the problem
\[
\begin{align*}
-\text{div}(|x|^{-2\gamma} \nabla u) &= c \langle h, \nabla u \rangle + f \quad x \in \Omega, \\
u &= 0 \quad x \in \partial \Omega,
\end{align*}
\]
where \( h : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is such that \( |h| \in L^{q'}(\Omega; |x|^{2\gamma \frac{q'}{q}} dx) \) with \( q \) given in (10) and \( f \in H^* \). In particular we have
\[
\| u \|_{\mathcal{D}^{1,2}_{0,\gamma}(\Omega)} \leq C\| f \|_{H^*}.
\]

Proof. Let us define the operator
\[
T : \mathcal{D}^{1,2}_{0,\gamma}(\Omega) \rightarrow \mathcal{D}^{1,2}_{0,\gamma}(\Omega)
\]
\[
u \mapsto T(\nu) = v,
\]
such that \( v \) verifies \(-\text{div}(|x|^{-2\gamma} \nabla v) = c \langle h, \nabla u \rangle + f \). The continuity of \( T \) easily follows. We show that \( T \) is compact. Note that the term \( c \langle h, \nabla \nu \rangle \) is in \( H^* \), since for any \( \varphi \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega) \) it holds that
\[
\int_{\Omega} c \langle h, \nabla \nu \rangle \varphi dx 
\]
\[
\leq \left( \int_{\Omega} c |h|^q |x|^{2\gamma \frac{q'}{q}} dx \right)^{\frac{1}{q}} \left( \int_{\Omega} c |\nabla \nu|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} c |\varphi|^{2\gamma} |x|^{-2\gamma} dx \right)^{\frac{1}{2}} 
\]
\[
\leq C \left( \int_{\Omega} c |\nabla \varphi|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}}.
\]
We have applied the Sobolev inequality in Lemma 1.1 for the exponent (11). The fact that also \( f \in H^* \) yields that the sequence \( \nu_n = T(\nu_n) \) is uniformly bounded in \( \mathcal{D}^{1,2}_{0,\gamma}(\Omega) \). By Rellich Theorem we get the existence of some \( v \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega) \) such that for a subsequence, still labeled with \( n \), it holds that
\[
\nu_n \rightarrow v \quad \text{strongly in } L^2(\Omega; |x|^{-2\gamma} dx) \quad \text{and weakly in } \mathcal{D}^{1,2}_{0,\gamma}(\Omega).
\]
To show the strong convergence in $D_{0,\gamma}^{1,2}(\Omega)$, we take $v_n - v$ as test function in (14) and get
\[
\int_{\Omega} |\nabla (v_n - v)|^2 |x|^{-2\gamma} \, dx = \int_{\Omega} c(h, \nabla u_n)(v_n - v) \, dx + \int_{\Omega} f(v_n - v) \, dx \\
- \int_{\Omega} \nabla v \nabla (v_n - v) |x|^{-2\gamma} \, dx = \int_{\Omega} c(h, \nabla v)(v_n - v) \, dx + o(1),
\]
due to (16) and $f \in H^r$. With the remaining term we argue as follows:
\[
\int_{\Omega} c(h, \nabla v)(v_n - v) \, dx \leq c \int_{\Omega} |h||\nabla v||T_m(v_n - v)| \, dx \\
+ c \int_{\Omega_{m,n}} |h||\nabla v||v_n - v| \, dx = I_1 + I_2,
\]
being $T_m(s) = \max\{-m, \min(s, m)\}$ and $\Omega_{m,n} = \Omega \cap \{v_n - v > m\}$. Observe that
\[
I_1 \leq \left( \int_{\Omega_{m,n}} c|h|^q|x|^{2\gamma\frac{1}{n}} \, dx \right)^{\frac{1}{p'}} \left( \int_{\Omega_{m,n}} c|\nabla v|^2|x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \\
\times \left( \int_{\Omega_{m,n}} c|T_m(v_n - v)|^{2\gamma\frac{1}{n}} \, dx \right)^\frac{1}{p'}.
\]
The strong convergence $v_n \to v$ in $L^2(\Omega; |x|^{-2\gamma} \, dx)$ implies that $T_m(v_n - v) \to 0$ strongly in $L^r(\Omega; |x|^{-2\gamma} \, dx)$ for any $r \geq 1$. In particular for $r = 2\gamma$, thus $I_1 = o(1)$.

Regarding the second term, it holds that
\[
I_2 \leq \left( \int_{\Omega_{m,n}} c|v_n - v|^2 \, dx \right)^\frac{1}{p'} \left( \int_{\Omega_{m,n}} c|\nabla v|^2|x|^{-2\gamma} \, dx \right)^\frac{1}{2} \\
\times \left( \int_{\Omega_{m,n}} c|v_n - v|^{2\gamma\frac{1}{n}} \, dx \right)^\frac{1}{p'} \\
\leq C \left( \int_{\Omega_{m,n}} c|h|^q|x|^{2\gamma\frac{1}{n}} \, dx \right)^{\frac{1}{p'}} \left( \int_{\Omega_{m,n}} c|\nabla(v_n - v)|^2|x|^{-2\gamma} \, dx \right)^\frac{1}{2} \\
\leq C \left( \int_{\Omega_{m,n}} c|h|^q|x|^{2\gamma\frac{1}{n}} \, dx \right)^{\frac{1}{p'}}.
\]
Summing up everything we have shown that
\[
\int_{\Omega} |\nabla (v_n - v)|^2 |x|^{-2\gamma} \, dx \leq C \sup_n \left( \int_{\Omega_{m,n}} c|h|^q|x|^{2\gamma\frac{1}{n}} \, dx \right)^{\frac{1}{p'}}.
\]
We verified that $v_n - v$ is uniformly bounded in $L^2(\Omega; |x|^{-2\gamma} \, dx)$, thus $\mu(\{|v_n - v| > m\}) \to 0$ as $n \to \infty$. Recall that in $D_{h,\gamma}^{1,2}(\Omega)$ the measure of any set $A \subset \mathbb{R}^N$ is given by $\mu(A) = \int_{A} |x|^{-2\gamma} \, dx$. Taking successively limits as $m \to \infty$ and as $n \to \infty$ we get
\[
\lim_{n \to \infty} \int_{\Omega} |\nabla (v_n - v)|^2 |x|^{-2\gamma} \, dx = 0.
\]
Hence $v_n$ is strongly convergent in $D_{0,\gamma}^{1,2}(\Omega)$ which leads to the compactness of $T$.

We show that there exists $M > 0$, such that $\|u\|_{D_{h,\gamma}^{1,2}(\Omega)} \leq M$, for every $u \in D_{0,\gamma}^{1,2}(\Omega)$ and every $\sigma \in [0,1]$ verifying $u = \sigma T(u)$. Arguing by contradiction, we
assume that for every $n$ there exist $u_n \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega)$ and $\sigma_n \in [0,1]$ such that $u_n = \sigma_n T(u_n)$ and $\|u_n\|_{\mathcal{D}^{1,2}_{0,\gamma}(\Omega)} \geq n$. We consider $w_n = \frac{u_n}{\|u_n\|_{\mathcal{D}^{1,2}_{0,\gamma}(\Omega)}}$ that satisfies

$$-\text{div}(|x|^{-2\gamma}\nabla w_n) = \sigma_n c(h, \nabla w_n) + \sigma_n \frac{f}{\|u_n\|_{\mathcal{D}^{1,2}_{0,\gamma}(\Omega)}}.$$ 

Since $\|u_n\|_{\mathcal{D}^{1,2}_{0,\gamma}(\Omega)} \to \infty$ applying Kato inequality, see [16, 12], this identity implies that for $n$ large

$$-\text{div}(|x|^{-2\gamma}\nabla w_n^+) \leq C\|\nabla w_n^+\|.$$ 

Then, by Lemma 3.3 below it follows that $w^+ \equiv 0$, which contradicts the fact that $\|w_n\|_{\mathcal{D}^{1,2}_{0,\gamma}(\Omega)} = 1$. We have proved that $T$ has a unique fixed point, which is the desired solution.

Finally we show the last estimate of the proposition. Set $v_1 = \frac{w}{\|w\|_{H^\gamma}}$. Then, repeating the same computation as in the previous step, we reach that

$$\|v_1\|_{\mathcal{D}^{1,2}_{0,\gamma}(\Omega)} \leq C(h),$$

and hence (15) follows. \hfill \Box

**Lemma 3.3.** Let $w \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega)$ satisfy

$$-\text{div}(|x|^{-2\gamma}\nabla w) \leq (h, \nabla w) \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega,$$

with $h : \mathbb{R}^N \to \mathbb{R}^N$ and $|h| \in L^q(\Omega; |x|^{2\gamma_q'}) dx$ and $q$ defined in (10). Then $w \leq 0$.

**Proof.** Taking $w_k = (w - k)^+$ as test function, we obtain

$$\int_{\Omega_k} |x|^{-2\gamma}|\nabla w_k|^2 dx \leq \int_{\Omega_k} |h| |\nabla w_k| w_k dx,$$

where $\Omega_k = \{x \in \Omega : w(x) > k, |\nabla w| > 0\}$. Now, applying Hölder and Sobolev inequalities to the righthand side we get

$$\int_{\Omega_k} |\nabla w_k|^2 |x|^{-2\gamma} dx \leq \left( \int_{\Omega_k} |\nabla w_k|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} \left( \int_{\Omega_k} |w_k|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} \left( \int_{\Omega_k} |h|^q |x|^{2\gamma_q'} dx \right)^{\frac{1}{q'}} \leq CS \left( \int_{\Omega_k} |\nabla w_k|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} \left( \int_{\Omega_k} |h|^q |x|^{2\gamma_q'} dx \right)^{\frac{1}{q'}}.$$

Let us argue by contradiction assuming that sup $w > 0$. Denote by $M = \sup w$ or $M = \infty$. In case $M = \infty$, we have that

$$\lim_{k \to M} \mu(\Omega_k) = 0,$$

since $w \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega)$. If $M < \infty$, again thanks to the fact that $w \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega)$ we have that $\nabla w = 0$ a.e. in $\{x \in \Omega : w = M\}$, by Stampacchia’s Theorem. Therefore (18) also holds, which implies that $\|h\|_{L^q(\Omega_k; |x|^{2\gamma_q'} dx)} \to 0$. Then, there exists $k_0 < M$ such that $\|h\|_{L^q(\Omega_k; |x|^{2\gamma_q'} dx)} < \frac{1}{25M}$ for $k \geq k_0$. Substituting this fact in (17), it yields that $w \leq k_0$ a.e. in $\Omega$. Hence sup $w \leq k_0 < M$, a contradiction. \hfill \Box

Notice that the solutions to (12) do not necessarily belong to the energy space $\mathcal{D}^{1,2}_{0,\gamma}(\Omega)$, thus we must look for a test function within $L^\infty(\Omega)$. This is the reason for the next result.
Lemma 3.4. There exists a unique solution $\phi \in \mathcal{D}_{0,\gamma}^{1,2}(\Omega)$ to problem
\begin{equation}
-\text{div}(|x|^{-2\gamma}\nabla \phi) = -\text{div}(h \phi) + |x|^{-2\gamma}, \quad \phi|_{\partial \Omega} = 0, \quad (19)
\end{equation}
with $h : \mathbb{R}^N \to \mathbb{R}^N$ a function verifying $|h| \in L^q(\Omega; |x|^{2\gamma+q} \, dx)$ and $\text{div}(h) \in L^{\frac{2}{q}}(\Omega; |x|^{\gamma(q-2)} \, dx)$, with $q$ given in (10). Moreover, $\phi \in L^\infty(\Omega)$ and it is non-negative.

Proof. We use again Schauder fixed point theorem and define the operator
\[ T : \mathcal{D}_{0,\gamma}^{1,2}(\Omega) \to \mathcal{D}_{0,\gamma}^{1,2}(\Omega) \]
\[ \phi \mapsto T(\phi) = v \]
such that $-\text{div}(|x|^{-2\gamma}\nabla v) = -\text{div}(h \phi) + |x|^{-2\gamma}, \quad v|_{\partial \Omega} \equiv 0.$

The continuity and compactness follow straightforward as in Lemma 3.2. Finally, let $\phi$ and $\sigma \in [0,1]$ verify that $\sigma T(\phi) = \phi$, that is
\begin{equation}
-\text{div}(|x|^{-2\gamma}\nabla \phi) = -\sigma \text{div}(h \phi) + \sigma |x|^{-2\gamma}. \quad (20)
\end{equation}

To see that there exists a constant $M > 0$ such that $\|\phi\|_{\mathcal{D}_{0,\gamma}^{1,2}(\Omega)} \leq M$, we consider $v \in \mathcal{D}_{0,\gamma}^{1,2}(\Omega)$ a nonnegative function satisfying
\begin{equation}
-\text{div}(|x|^{-2\gamma}\nabla v) = \sigma h \nabla v + |\phi|^{2\gamma-2}\phi |x|^{-2\gamma}, \quad v|_{\partial \Omega} \equiv 0. \quad (21)
\end{equation}

The existence of such a solution is guaranteed by Lemma 3.2 if we take $f = |\phi|^{2\gamma-2}\phi |x|^{-2\gamma}$. Indeed, multiplying by any nontrivial $\varphi \in \mathcal{D}_{0,\gamma}^{1,2}(\Omega)$, we get
\[
\int_{\Omega} |\phi|^{2\gamma-2}\phi \varphi |x|^{-2\gamma} \, dx \leq \left( \int_{\Omega} \varphi^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\phi|^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{2\gamma-1}{2\gamma}} \leq CS \left( \int_{\Omega} |\nabla \varphi|^{2} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\phi|^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{2\gamma-1}{2\gamma}}.
\]

In particular,
\[
\|\phi\|^{2\gamma-2}\phi |x|^{-2\gamma} \|_{H^*} = \sup_{\varphi \in \mathcal{D}_{0,\gamma}^{1,2}(\Omega), \varphi \neq 0} \frac{\int_{\Omega} |\phi|^{2\gamma-2}\phi \varphi |x|^{-2\gamma} \, dx}{\left( \int_{\Omega} |\nabla \varphi|^{2} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}}} \leq CS \left( \int_{\Omega} |\phi|^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{2\gamma-1}{2\gamma}},
\]
namely, $|\phi|^{2\gamma-2}\phi |x|^{-2\gamma} \in H^*$. Multiplying by $v$ and $\phi$ equations (20) and (21) respectively, and subtracting we get
\[
\int_{\Omega} |\phi|^{2\gamma} |x|^{-2\gamma} \, dx = \sigma \int_{\Omega} v |x|^{-2\gamma} \, dx \leq C \left( \int_{\Omega} v^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \leq C \left( \int_{\Omega} |\nabla v|^{2} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \leq C \|\phi\|^{2\gamma-2}\phi |x|^{-2\gamma} \|_{H^*} \leq C \left( \int_{\Omega} |\phi|^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{2\gamma-1}{2\gamma}}.
\]
where we have also used (15). Hence,
\[
\left( \int_{\Omega} \phi^{2\gamma} |x|^{-2\gamma} dx \right)^{\frac{1}{2\gamma}} \leq C \tag{22}
\]
To get a bound for the gradient, we multiply (20) by $\phi$. Integrating by parts, we obtain
\[
\int_{\Omega} |\nabla \phi|^2 |x|^{-2\gamma} dx = \sigma \int_{\Omega} h \phi \nabla \phi dx + \sigma \int_{\Omega} \phi |x|^{-2\gamma} dx \leq C \left( \int_{\Omega} \phi^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} + \frac{1}{\sigma} \left( \int_{\Omega} |\nabla \phi|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |h|' |x|^{-\gamma} \frac{q}{q'} dx \right)^{\frac{1}{2}}.
\]
Therefore,
\[
\left( \int_{\Omega} |\nabla \phi|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} \phi^{2\gamma} |x|^{-2\gamma} dx \right)^{\frac{1}{2\gamma}} \left( \int_{\Omega} |h|' |x|^{-\gamma} \frac{q}{q'} dx \right)^{\frac{1}{2}} + C.
\]
Taking into account (13) and (22) this implies that $\|\phi\|_{D^{1,2}_0(\Omega)} \leq C$. By the fixed point Schauder theorem, we have shown that there exists $\phi \in D^{1,2}_{0,\gamma}(\Omega)$ a unique solution to (19).

The positivity of $\phi$ follows from a comparison principle for problem (19), analogous to Lemma 3.3 and the fact that zero is a subsolution to (19).

More precisely, if we denote as $g = w_1 - w_2$, the difference of a sub and a supersolution to (19), respectively, it verifies
\[
-\text{div}(|x|^{-2\gamma} \nabla g^+) \leq -\text{div}(h g^+),
\]
where we have applied also Kato’s inequality. We multiply again by $g_k = (g^+ - k)^+$ to obtain (17) which shows that $g^+ \leq 0$. Then $\phi \geq 0$, since 0 is a subsolution to (19).

To conclude the proof we verify that $\phi \in L^\infty(\Omega)$. To this end, we argue as in [3] and consider in (19) the following test function, $v = \text{sign}(\phi)(|\phi| - k)^+$. We find that
\[
\int_{A(k)} |x|^{-2\gamma} |\nabla \phi|^2 dx = \int_{A(k)} \langle h, \nabla \phi \rangle \phi dx + \int_{A(k)} v|x|^{-2\gamma} dx, \tag{23}
\]
where $A(k) = \{x \in \Omega : |\phi(x)| > k\}$. Note that $\nabla \phi = \nabla v$ in $A(k)$ and also that $v = 0$ in $\Omega \setminus A(k)$. We observe as well that $\mu(A(k)) \to 0$ as $k \to \infty$, otherwise we would contradict the fact that $\phi \in D^{1,2}_{0,\gamma}(\Omega)$. Our purpose is to show that indeed, there exists $k_0$ such that $\mu(A(k)) = 0$ for any $k \geq k_0$. We treat first the integral on the right hand side,
\[
\int_{A(k)} |x|^{-2\gamma} v dx \leq \left( \int_{A(k)} v^{2\gamma} |x|^{-2\gamma} dx \right)^{\frac{1}{2\gamma}} \mu(A(k))^{1-\frac{1}{2\gamma}} \leq CS \left( \int_{A(k)} |\nabla \phi|^2 |x|^{-2\gamma} dx \right)^{\frac{1}{2}} \mu(A(k))^{1-\frac{1}{2\gamma}}, \tag{24}
\]
where we have used the Sobolev inequality and the identity $\nabla v = \nabla \phi$ in $A(k)$. We deal now with the second term on the left. We take the following Hölder exponents:
\( \frac{1}{2} + \frac{1}{q^*} + \frac{1}{q} + \frac{1}{\beta} = 1 \), where \( q' > q' \) with \( q \) given in (10), to see that

\[
\int_{A(k)} (h, \nabla \phi) \, dx \leq \left( \int_{A(k)} |h|^q |x|^{2\gamma} \, dx \right)^{\frac{1}{q'}} \left( \int_{A(k)} \phi^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \times \left( \int_{A(k)} |\nabla \phi|^2 |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \mu(A(k))^{\frac{1}{2}}
\]

\[
\leq C \left( \int_{A(k)} |\nabla \phi|^2 |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \mu(A(k))^{\frac{1}{2}}.
\]

Substituting this estimate and (24) into (23), and applying once more Sobolev inequality, it follows

\[
\left( \int_{A(k)} v^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \leq CS \left( \int_{A(k)} |\nabla \phi|^2 |x|^{-2\gamma} \, dx \right)^{\frac{1}{2}} \leq C \mu(A(k))^{1 - \frac{1}{2\gamma} + C \mu(A(k))^{\frac{1}{2}}}.\]

Note that \( \frac{1}{2} < 1 - \frac{1}{2\gamma} \), hence

\[
\left( \int_{A(k)} v^{2\gamma} |x|^{-2\gamma} \, dx \right)^{\frac{1}{2\gamma}} \leq C \mu(A(k))^{\frac{1}{2}}. \tag{25}
\]

Taking into account that \( A(m) \subset A(k) \) if \( 0 < k < m \), it is easy to deduce that

\[
\mu(A(m))^{\frac{1}{2\gamma}} (m - k) \leq \left( \int_{A(m)} v^{2\gamma} |x|^{2\gamma} \, dx \right)^{\frac{1}{2\gamma}} \leq \left( \int_{A(k)} v^{2\gamma} |x|^{2\gamma} \, dx \right)^{\frac{1}{2\gamma}}.
\]

Plugging this inequality into (25) yields

\[
\mu(A(m)) \leq \frac{C}{(m - k)^{2\gamma}} \mu(A(k))^{\frac{2\gamma}{m}}
\]

with \( \frac{2\gamma}{m} > 1 \). Then, the classical Stampacchia Theorem implies that there exists \( k_0 \) such that \( \mu(A(k)) = 0 \) for any \( k \geq k_0 \) as we wished to prove. \( \square \)

**Proof of Proposition 1.** We take as test function \( \phi \in L^\infty(\Omega) \), \( \phi \geq 0 \) verifying (19), to find that

\[
\int_{\Omega} -\text{div}(|x|^{-2\gamma} \nabla \phi) \, dx \leq \int_{\Omega} (h, \nabla w) \phi \, dx = - \int_{\Omega} \text{div}(h \phi) w \, dx
\]

\[
= \int_{\Omega} -\text{div}(|x|^{-2\gamma} \nabla \phi) w \, dx - \int_{\Omega} w |x|^{-2\gamma} \, dx.
\]

Integrating by parts in the first of the integrals and canceling terms we arrive to

\[
0 \geq \int_{\Omega} w |x|^{-2\gamma} \, dx.
\]

Thus \( w \leq 0 \) and the proposition holds. \( \square \)
4. **Nonexistence results if** \( p \geq p_+(\lambda) \). We state the main result of this section.

**Theorem 4.1.** Assume that \( f \geq 0 \). If \( p \geq p_+(\lambda) \), where the optimal exponent \( p_+(\lambda) \) is defined in (6), then the equation (1) has not any positive very weak supersolution.

In case \( f \equiv 0 \), then the unique nonnegative very weak supersolution is \( u \equiv 0 \).

**Proof.** We separate into cases the proof of this result. In all of them we argue by contradiction. We assume that there exists a solution and finally we contradict Hardy inequality.

**Non existence result for** \( \gamma > -1 \) **and** \( p_+(\lambda) > 1 \).

**Case** \( p > p_+(\lambda) \): Suppose that there exists a very weak supersolution to (1). Then, it holds that

\[-\text{div}(|x|^{-2\gamma} \nabla u) - \lambda \frac{u}{|x|^{2(\gamma + 1)}} > 0,\]

and by Lemma 2.2 there exists \( c > 0 \) and \( B_r(0) \) such that \( u(x) \geq c|x|^{-a} \) in \( B_r(0) \). Let \( \phi \in C_0^\infty(B_r(0)) \). Taking \( |\phi|^{p'} \) as test function in (1), since \( f \geq 0 \) we get

\[
\int_{B_r(0)} \lambda \frac{u}{|x|^{2(\gamma + 1)}} |\phi|^{p'} \leq p' \int_{B_r(0)} |\phi|^{p'-2} \phi \nabla u \nabla \phi |x|^{-2\gamma} - \int_{B_r(0)} |\nabla u|^{p} |\phi|^{p'} |x|^{-p\gamma}
\]

\[
\leq (\eta - 1) \int_{B_r(0)} |\nabla u|^{p} |\phi|^{p'} |x|^{-p\gamma} + C(p) \int_{B_r(0)} |\nabla \phi|^{p'} |x|^{-p'\gamma}.
\]

Choosing \( \eta \) small and noting that \( u(x) \geq c|x|^{-a} \) in \( B_r(0) \), it follows

\[
\int_{B_r(0)} \lambda |\phi|^{p'} |x|^{-2(\gamma + 1) - a} \leq C(p) \int_{B_r(0)} |\nabla \phi|^{p'} |x|^{-p'\gamma},
\]

(26)

contradicting Hardy inequality if \( p > p_+(\lambda) \).

**Case** \( p = p_+(\lambda) \) **and** \( \lambda < \Lambda_{N,\gamma} \): As before, we assume that there exists a very weak supersolution to problem (1). Again by Lemma 2.2, there exists a positive constant, \( c_0 \) such that

\[
u(x) \geq c_0 |x|^{-a}, \quad \text{in } B_{\eta}(0).
\]

(27)

Furthermore, since \( \lambda < \Lambda_{N,\gamma} \), applying Lemma 2.3 we get the estimates given in (8). Without loss of generality, we fix some \( \eta \leq e^{-1} \).

We define \( \omega(x) = |x|^{-a} \left( \log \left( \frac{1}{|x|} \right) \right)^\beta \), with \( \beta > 0 \) small enough, chosen later conveniently. Recall that \( \lambda < \Lambda_{N,\gamma} \), \( p_+(\lambda) < 2 \), hence \( \omega \in D^{1,2}_{\gamma}(B_{\eta}(0)) \cap W^{1,p_+(\lambda)}(B_{\eta}(0); |x|^{-2\gamma} dx) \). One can compute that

\[
-\text{div}(|x|^{-2\gamma} \nabla \omega) - \lambda \frac{\omega}{|x|^{2(\gamma + 1)}} = \frac{\beta}{|x|^{2(\gamma + 1) + a}} \left( (N - 2(\gamma + 1 + a^-)) \left( \log \left( \frac{1}{|x|} \right) \right)^{\beta - 1} + (1 - \beta) \left( \log \left( \frac{1}{|x|} \right) \right)^{\beta - 2} \right),
\]

and

\[
|\nabla \omega|^{p_+(\lambda)} = |x|^{-(a^- + 1)p_+(\lambda)} \left( a^- \left( \log \left( \frac{1}{|x|} \right) \right)^\beta + \beta \left( \log \left( \frac{1}{|x|} \right) \right)^{\beta - 1} \right)^{p_+(\lambda)}.
\]

Therefore, we deduce that

\[
-\text{div}(|x|^{-2\gamma} \nabla \omega) - \lambda \frac{\omega}{|x|^{2(\gamma + 1)}} \leq \beta^2 h(x) |\nabla \omega|^{p_+(\lambda)} |x|^{-p_+(\lambda)\gamma},
\]
Proof of the claim. We can take $v$ we claim that
\[
\text{Moreover, by (29) it holds that}
\]
\[
\text{It is not difficult to check that}
\]
\[
\text{(28) becomes into}
\]
On the other hand, $u_1$ satisfies
\[
- \text{div}(|x|^{-2\gamma} \nabla u_1) - \lambda \frac{u_1}{|x|^{2(\gamma+1)}} \geq c_1 1 - p_+^+(\lambda) |\nabla u_1|^{p_+^+(\lambda)} |x|^{-p_+^+ \gamma} (30) \]

Therefore, in a similar way we obtain an estimate analogous to (26),
\[
\lambda \int_{B_\eta(0)} u_1 |\phi|^{p_+^+(\lambda)} |x|^{-2(\gamma+1)} \leq C(p) \int_{B_\eta(0)} |\nabla \phi|^{p_+^+(\lambda)} |x|^{-p_+^+ \gamma}.
\]

But then
\[
C(p) \int_{B_\eta(0)} |\nabla \phi|^{p_+^+(\lambda)} |x|^{-p_+^+ \gamma} \geq \lambda \int_{B_\eta(0)} \omega |\phi|^{p_+^+(\lambda)} |x|^{-2(\gamma+1)}
\]
\[
= \lambda \int_{B_\eta(0)} \log \left( \frac{1}{|x|} \right)^\beta |\phi|^{p_+^+(\lambda)} |x|^{-2(\gamma+1) + a^-},
\]
which contradicts the optimality of the constant $\Lambda_{N,\gamma}$ in Hardy inequality.

**Proof of the claim.** We can take $\beta$ small enough such that $c_1 1 - p_+^+(\lambda) \geq \|h\|_\infty \beta \frac{1}{2}$, and (28) becomes into
\[
- \text{div}(|x|^{-2\gamma} \nabla u_1) - \lambda \frac{u_1}{|x|^{2(\gamma+1)}} \geq \beta \frac{1}{2} \|h\|_\infty |\nabla u_1|^{p_+^+(\lambda)} |x|^{-p_+^+ \gamma}.
\]

The regularity of $\omega$ and the fact that $u_1$ verifies (8), assure that $v \in W^{1,p_+^+(\lambda)}(B_\eta(0))$ and
\[
\int_{B_\eta(0)} \frac{v}{|x|^{2(\gamma+1)+a^-}} \, dx < \infty, \quad \int_{B_\eta(0)} |\nabla v|^{p_+^+(\lambda)} |x|^{-p_+^+ \gamma - a^-} \, dx < \infty. \tag{29}
\]

It is not difficult to check that $v$ satisfies
\[
- \text{div}(|x|^{-2\gamma} \nabla v) - \lambda \frac{v}{|x|^{2(\gamma+1)}} \leq \frac{1}{2} p_+^+(\lambda) h(x)|\nabla \omega|^{p_+^+(\lambda)-2} |x|^{-p_+^+ \gamma} (\nabla \omega, \nabla v) = m(x) \nabla v \quad \text{in } B_\eta(0),
\]

where $m(x) \sim -\beta \frac{1}{2} p_+^+(\lambda) \frac{x}{|x|^{2(\gamma+1)}}$. Notice that $m$ has not the required regularity to apply the comparison principle stated in Proposition 1. We proceed then using Kato’s inequality:
\[
- \text{div}(|x|^{-2\gamma} \nabla v^+) - \lambda \frac{v^+}{|x|^{2(\gamma+1)}} + \beta \frac{1}{2} p_+^+(\lambda) \left( \frac{x}{|x|^{2(\gamma+1)}} , \nabla v^+ \right) \leq 0. \tag{30}
\]

Moreover, by (29) it holds that
\[
\int_{B_\eta(0)} |\nabla v^+|^{p_+^+(\lambda)} |x|^{-p_+^+(\lambda) \gamma - a^-} \, dx < \infty. \tag{31}
\]
Since $N > (\gamma + 1)p + a^-$, applying Caffarelli-Kohn-Nirenberg inequalities (see [13]), it gives

$$\int_{B_\eta(0)} \frac{(v^+)^p}{|x|p+(\gamma+1)+a^-} dx < \infty. \quad (32)$$

Let $2\theta = \beta \frac{1}{2} p_+(\lambda)$. For $\beta$ small $|x|^{-2\theta}$ is an admissible weight for Caffarelli-Kohn-Nirenberg inequalities. Then, equation (30) becomes into

$$- \text{div}(|x|^{-2(\gamma+\theta)} \nabla v^+) - \lambda \frac{v^+}{|x|^{2(\gamma+\theta+1)}} \leq 0. \quad (33)$$

We consider now some nonnegative function $\varphi$ verifying the following problem

$$\begin{cases}
-\text{div}(|x|^{-2(\gamma+\theta)} \nabla \varphi) - \lambda \frac{\varphi}{|x|^{2(\gamma+\theta+1)}} = \frac{1}{|x|^{2(\gamma+\theta+1)}}, \\
\varphi|_{\partial B_\eta(0)} \equiv 0.
\end{cases}$$

The idea is to take $\varphi$ as test function in (33). Integrating by parts we would get that

$$\int_{B_\eta(0)} \frac{v^+}{|x|^{2(\gamma+\theta+1)}} dx \leq 0,$$

as we wanted to prove. An easy computation shows that

$$\varphi(x) = c \left(\frac{1}{|x|^\eta} - \frac{1}{\eta + \frac{1}{2} a}\right), \quad \tilde{a} = (\Lambda_{N,\gamma+\theta})^{1/2} - \sqrt{\Lambda_{N,\gamma+\theta} - \lambda},$$

$$\Lambda_{N,\gamma+\theta} = \left(\frac{N - 2(\gamma + 1 + \theta)}{2}\right)^2.$$ 

We recall that $\lambda < \Lambda_{N,\gamma}$. Thus taking $\beta$ small we can assure that $\tilde{a}$ is a real number.

Since $\varphi$ lacks of the regularity to be used as test function in (33), we take instead the approximation $\varphi_\eta \in C^\infty(B_\eta(0))$: 

$$\varphi_\eta(x) = c \left(\frac{1}{(|x| + \frac{1}{2} \tilde{a})^\eta} - \frac{1}{(\eta + \frac{1}{2} \tilde{a})^\eta}\right).$$

It is not difficult to see that we just need the following two conditions to perform the integration by parts using $\varphi_\eta(x)$,

$$\int_{B_\eta(0)} \frac{v^+}{|x|^{2(\gamma+\theta+1)} + \tilde{a}} dx < \infty \quad \text{and} \quad \int_{B_\eta(0)} \frac{|
abla v^+|}{|x|^{2(\gamma+\theta)+\tilde{a}+1}} dx < \infty. \quad (34)$$

To fulfill the first condition we observe that, as a function of $\theta$, $\tilde{a}$ verifies $\tilde{a}(0) = a^-$ and $\tilde{a}(\theta) > 0$ for $\theta > 0$. Therefore, $\tilde{a}(\theta) > a^-$ for $\theta > 0$, small. We claim that

$$\int_{B_\eta(0)} \frac{v^+}{|x|^\sigma} dx < \infty, \quad (35)$$

for some $\sigma > 2(\gamma + 1) + a^-$. This shows (34) for $\theta$ sufficiently small. We proceed now with the proof of the claim. Note that

$$\int_{B_\eta(0)} \frac{v^+}{|x|^\sigma} dx$$

$$\leq \left(\int_{B_\eta(0)} \frac{(v^+)^p}{|x|^{p+(\gamma+1)+a^-}} dx\right)^{\frac{1}{p^+}} \left(\int_{B_\eta(0)} |x|^{-p_+(\lambda)} \left(\frac{\sigma - p_+(\lambda)(\gamma+1)+a^-}{p_+(\lambda)}\right) dx\right)^{\frac{1}{p_+(\lambda)}}.$$
The first integral is bounded by (32) and to show that the second integral is also bounded we define
\[ g(\sigma) = p_+'(\lambda) \left( \sigma - \frac{p_+(\lambda)(\gamma + 1) + a^-}{p_+(\lambda)} \right). \]

It is not difficult to check that
\[ g(2(\gamma + 1) + a^-) = 2(\gamma + 1 + a^-) < N, \]

since \( \lambda < \Lambda_N \) and (35) is proved.

To show (34) we note that
\[
\int_{B_{\eta}(0)} \frac{|\nabla u^+|}{|x|^{2(\gamma + \theta) + \frac{a^-}{2} + 1}} \, dx \\
\leq \left( \int_{B_{\eta}(0)} \frac{|\nabla u^+|^{p_+}(\lambda)}{|x|^{\gamma p_+(\lambda) + a^-}} \, dx \right)^{\frac{1}{p_+}} \left( \int_{B_{\eta}(0)} |x|^{-g(2(\theta + \gamma + 1) + \delta)} \right)^{\frac{1}{g}} < \infty,
\]

for \( \theta \) sufficiently small, since (31) and (36) hold. \( \square \)

**Case** \( p = p_+(\lambda) \) and \( \lambda = \Lambda_N, \gamma > 1 \): If \( u \) solves (1) by Lemmas 2.2 and 2.3 we get that \( u \) verifies (27). Furthermore,
\[
\int_{B_{\eta}(0)} \frac{|\nabla u|^{p_+}(\lambda)}{|x|^{\gamma p_+(\lambda) + a^-}} \, dx < \infty,
\]

where \( p_+(\lambda) = \frac{2(\gamma + 1) + N}{N} \) and \( a^- = \frac{N-2(\gamma+1)}{2} \). Take \( \phi \in C^\infty(B_{\eta}(0)) \) such that \( \phi \geq 0 \) and \( \phi = 1 \) in \( B_{\eta_1}(0) \subset B_{\eta}(0) \). We have that
\[
\int_{B_{\eta}(0)} \frac{|\nabla(\phi u)|^{p_+}(\lambda)}{|x|^{\gamma p_+(\lambda) + a^-}} \, dx < \infty.
\]

Since \( a^- + p_+(\lambda)(\gamma + 1) < N \) whenever \( \gamma > -1 \), \( |x|^{-p_+(\lambda)(\gamma + 1) - a^-} \) is an admissible weight in Caffarelli-Kohn-Nirenberg inequalities. Thus,
\[
c_1 \int_{B_{\eta}(0)} \frac{(\phi u)^{p_+}(\lambda)}{|x|^{\gamma p_+(\lambda) + a^-}} \, dx \leq \int_{B_{\eta}(0)} \frac{|\nabla(\phi u)|^{p_+}(\lambda)}{|x|^{\gamma p_+(\lambda) + a^-}} \, dx < \infty,
\]

and then,
\[
\int_{B_{\eta_1}(0)} \frac{u^{p_+}(\lambda)}{|x|^{\gamma p_+(\lambda) + a^-}} \, dx < \infty.
\]

Therefore, if we denote by \( \beta = \gamma + \frac{a^-}{p_+(\lambda)} \) we have that \( u \in D^{1,p_+(\lambda)}(B_{\eta_1}(0)) \). Since \( \beta = \gamma + \frac{a^-}{p_+(\lambda)} < \frac{N-p_-(\lambda)}{p_+(\lambda)} \) is an admissible weight in Caffarelli-Kohn-Nirenberg inequalities. We can apply then Lemma 2.5 to obtain
\[
\int_{B_{\eta_1}(0)} \frac{u^{p_+(\lambda)}}{|x|^{p_+(\lambda)(\gamma + 1) + a^-}} \, dx \leq c_1 \|u\|^{p_+(\lambda)}_{\beta,p_+(\lambda)} < \infty.
\]

But taking into account Lemma 2.2 this implies that
\[
\int_{B_{\eta_1}(0)} |x|^{-N} \, dx < \infty,
\]

which is a contradiction.

**Non existence result for** \( \gamma \leq -1 \) and \( p_+(\lambda) = 1 \).

**Case** \( p > p_+(\lambda) = 1 \): The proof can be performed exactly as in the case \( \gamma > -1 \).
Case $p = p_+(\lambda) = 1$: Again Lemmas 2.2 and 2.3 ensure that (27) and
\[
\int_{B_\eta(0)} \frac{\nabla u}{|x|^{\gamma + a}} \, dx < \infty
\] (38)
hold in some $B_\eta(0)$. Consider $\phi \in C^\infty(B_\eta(0))$ such that $\phi \geq 0$ and $\phi = 1$ in $B_{\eta_1} \subset B_\eta(0)$, so that
\[
\int_{B_\eta(0)} \frac{|\nabla (\phi u)|}{|x|^{\gamma + a}} \, dx < \infty.
\]
Note that, writing $a^- = \delta \frac{N-2(\gamma+1)}{2}$, with $\delta \in (0,1]$, we have $N - \gamma - 1 - a^- = \frac{(1-\delta)(N-2(\gamma+1)) + N}{2} > 0$, hence $\gamma + 1 + a^-$ is an admissible weight to apply Caffarelli-Kohn-Nirenberg inequalities. Therefore,
\[
c_1 \int_{B_\eta(0)} \frac{(\phi u)}{|x|^{\gamma + 1 + a}} \, dx \leq \int_{B_\eta(0)} \frac{|\nabla (\phi u)|}{|x|^{\gamma + a - 1}} \, dx < \infty,
\]
hence,
\[
\int_{B_{\eta_1}(0)} \frac{u}{|x|^{\gamma + 1 - a}} \, dx < \infty.
\]
It follows that $u \in D^{1,1}_{\gamma + a -}(B_{\eta_1}(0))$ and since $\gamma + a^- < N - 1$ we can use Lemma 2.5 to get that
\[
\int_{B_{\eta_1}(0)} \frac{u}{|x|^{\gamma + 1 + a}} \, dx \leq c\|u\|_{1,\gamma + a^-} < \infty. \quad (39)
\]
We also note that, in particular, $u$ verifies.

\[-\text{div}(|x|^{-2\gamma} \nabla u) \geq \lambda u |x|^{-2(\gamma + 1)}.
\]
On the other hand, a simple calculation shows that, for any $b > 0$ the function
\[w_n = \frac{1}{(|x| + \frac{1}{n})^b} - \frac{1}{(|\eta_1 + \frac{1}{n})^b},\]
satisfies
\[-\text{div}(|x|^{-2\gamma} \nabla w_n) = |x|^{-2\gamma - 1} \left( \frac{b(N - 1 - 2\gamma)}{(|x| + \frac{1}{n})^{b+1}} - \frac{|x|b(b+1)}{(|x| + \frac{1}{n})^{b+2}} \right),
\]
and hence,
\[-\text{div}(|x|^{-2\gamma} \nabla w_n) \leq \lambda w_n |x|^{-2(\gamma + 1)}.
\]
If we denote by $g = w_n - u$ and apply Kato’s inequality, we obtain that
\[-\text{div}(|x|^{-2\gamma} \nabla g^+) \leq \lambda g^+ |x|^{-2(\gamma + 1)}
\]
in $B_{\eta_1}(0)$. We show that $g^+ = 0$ arguing as in (33). We consider
\[
\varphi = \frac{1}{(|x| + \frac{1}{n})^{a^-}} - \frac{1}{(|\eta_1 + \frac{1}{n})^{a^-}},
\]
which is admissible as test function if
\[
\int_{B_{\eta_1}(0)} \varphi g^+ |x|^{-2(\gamma + 1)} \, dx \leq \int_{B_{\eta_1}(0)} g^+ |x|^{-2(\gamma + 1) - a^-} \, dx \leq \int_{B_{\eta_1}(0)} g^+ |x|^{-\gamma - 1 - a^-} \, dx < \infty,
\]
and
\[ \int_{B_{r_n}(0)} |\nabla \varphi| |\nabla g^+| |x|^{-2\gamma} dx \leq c \int_{B_{r_n}(0)} |\nabla g^+| |x|^{-2\gamma-1-a} dx \leq c \int_{B_{r_n}(0)} |\nabla g^+| |x|^{-\gamma-a} dx < +\infty, \]
by (39) and the fact that \( \gamma \leq -1 \).
As before, we can deduce that \( g^+ = 0 \) and so \( u \geq w_n \). But then, (39) gives that for any \( n \)
\[ \int_{B_{r_n}} w_n |x|^{-1-\gamma-a} \leq c \|u\|_{1,\gamma+a}, \]
a contradiction if we take \( b > N - (\gamma + 1 + a^-) = (1-\delta)(N-2(\gamma+1)+N) > 0 \).

5. Blow-up if \( p \geq p_+(\lambda) \). In this section we will show that if \( p \geq p_+(\lambda) \), problem (40) arises complete blow-up as \( n \to \infty \), as consequence of the nonexistence of very weak supersolutions to (1). To achieve this aim, we study the following approximating problems
\[
\begin{cases}
-\text{div}(|x|^{-2\gamma} \nabla u_n) = \lambda u_n a_n(x) + |\nabla u_n|^p b_n(x) + cf, & \text{in } \Omega, \\
u_n > 0, & \text{in } \Omega, \\
u_n \equiv 0, & \text{on } \partial\Omega,
\end{cases}
\]
being
\[
a_n(x) = \frac{1}{|x|^{2(\gamma+1)} + \frac{1}{n}}, \quad b_n(x) = \begin{cases}
\frac{1}{|x|^p + \frac{1}{n}}, & \text{if } \gamma \geq 0, \\
\frac{1}{|x|^{2\gamma} (|x|^{p-2} + \frac{1}{n})}, & \text{if } \gamma < 0,
\end{cases}
\]
and \( f \geq 0, f \neq 0 \).
For existence of solutions to those problems see [3].

The main result of this section is enclosed in the following theorem.

**Theorem 5.1.** Let \( p \geq p_+(\lambda) \). If \( u_n \in W^{1,p}_0(\Omega) \) is a solution to problem (40), then \( u_n(x_0) \to \infty \), \( \forall x_0 \in \Omega \).

Before performing the proof we need some preliminaries. The first one consists of a property satisfied by solutions to the divergence operator, see [3, 10].

**Lemma 5.2.** Let \( u \) be the unique positive solution to problem:
\[
\begin{cases}
-\text{div}(|x|^{-2\gamma} \nabla u) = g, & \text{in } \Omega, \\
u \geq 0, & \text{in } \Omega, \\
u \equiv 0, & \text{on } \partial\Omega,
\end{cases}
\]
where \( g \) is a positive function such that \( g|x|^{2\gamma} \in L^\infty(\Omega) \). Then, for any ball \( B_r \subset \Omega \) such that \( B_{2r} \subset \Omega \), there exists a positive constant \( c = c(r, N, \gamma) \) such that
\[
\frac{u(x)}{\text{dist}(x, \partial\Omega)} \geq c \int_{B_{2r}} g(y)\text{dist}(y, \partial\Omega) dy, \quad \text{for every } x \in \Omega.
\]

**Remark 3.** Note that, by this result for any compact set \( K \subset \subset \Omega \), there exists a constant \( c = c(K, N, \gamma) \) such that
\[
\frac{u(x)}{\text{dist}(x, \partial\Omega)} \geq c \int_K g(y)\text{dist}(y, \partial\Omega) dy, \quad \text{for every } x \in \Omega,
\]
since there exists a finite number of balls $B_r$, with $i = 1, \ldots, \ell$, such that $K \subset \cup_i B_{2r_i}$ and $B_{4r_i} \subset \Omega$.

Finally, thanks to the comparison principle shown in Lemma 1, we get the following result, whose proof is similar to Lemma 3.4 in [5].

**Lemma 5.3.** Let $\omega_1, \omega_2 \in D_{0, \gamma}^{1, 2}(\Omega) \cap L^\infty(\Omega)$ be positive super and subsolutions, respectively, to the following equation:

$$-\text{div}(|x|^{-2\gamma} \nabla v) = \lambda v a_n(x) + \frac{|\nabla v|^p}{1 + s|\nabla v|^p} b_n(x) + g, \quad \text{in } \Omega,$$

with $a_n, b_n$ given in (40), $s > 0$ and $g \in L^\beta(\Omega)$, with $\beta > N/2$. Then $\omega_1 \geq \omega_2$ in $\Omega$.

With this comparison result we can argue as in [8] with an iteration procedure, to show the existence of the minimal solution $v_j \in D_{0, \gamma}^{1, 2}(\Omega) \cap L^\infty(\Omega)$ to the problem

$$\begin{cases}
-\text{div}(|x|^{-2\gamma} \nabla v_j) = \lambda v_j a_n(x) + \frac{|\nabla v_j|^p}{1 + \frac{s}{j}|\nabla v_j|^p} b_n(x) + \alpha f, & \text{in } \Omega, \\
v_j = 0, & \text{on } \partial \Omega.
\end{cases} \quad (42)$$

By Lemma 5.3 we can prove that $v_j \leq v_{j+1}$ and $v_j \leq u_n$ for every $j$. We define

$$\omega_n = \lim_{j \to \infty} v_j \leq u_n.$$

We deduce that $\omega_n \leq \omega_{n+1}$, since $a_n, b_n$ are nondecreasing sequences.

**Proof of Theorem 5.1.** We perform this proof arguing by contradiction. We assume that there exists $x_0 \in \Omega$ such that $u_n(x_0) \leq C$ for all $n$. We get a contradiction after several steps.

**Step 1.** First of all, we prove that $|\nabla T_k(u_n)|$ is uniformly bounded (for $k$ fixed) in $L^2_{\text{loc}}(\Omega; |x|^{-2\gamma} dx)$. In fact, if we take $T_k(u_n) \phi$ as test function in (40), being $\phi \geq 0$ such that $\phi \in C_0^\infty(K)$, with $K$ some compact set $K \subset \subset \Omega$, integrating by parts we get

$$\int_{\Omega} |x|^{-2\gamma} |\nabla T_k(u_n)|^2 \phi \, dx + \int_{\Omega} |x|^{-2\gamma} \nabla T_k(u_n) T_k(u_n) \nabla \phi \, dx = \int_{\Omega} (\lambda u_n a_n(x) + |\nabla u_n|^p b_n(x) + cf) T_k(u_n) \phi \, dx \leq C k. \quad (43)$$

Here we used Lemma 5.2 at $x_0 \in \text{supp}(\phi) = K$ and the assumption $u(x_0) \leq C$. Note that the second integral can be estimated

$$\int_{\Omega} |x|^{-2\gamma} \nabla T_k(u_n) T_k(u_n) \nabla \phi \, dx \leq \varepsilon \int_{\Omega} |\nabla T_k(u_n)|^2 |\nabla \phi| |x|^{-2\gamma} dx + C(\varepsilon) \int_{\Omega} |T_k(u_n)|^2 |\nabla \phi| |x|^{-2\gamma} dx \leq \varepsilon \int_{\Omega} \nabla |T_k(u_n)|^2 |\nabla \phi| |x|^{-2\gamma} dx + C.$$

Since the last integral is bounded, together with (43) it gives that $|\nabla T_k(u_n)|$ is bounded in $L^2_{\text{loc}}(\Omega; |x|^{-2\gamma} dx)$, as we wanted to show.

**Step 2.** $T_k(v_j) \to T_k(\omega_n)$ strongly in $D_{0, \gamma}^{1, 2}(\Omega)$. In particular, this ensures that $\nabla v_j \to \nabla \omega_n$ a. e. in $\Omega$. The following identity

$$\int_{\Omega} |\nabla T_k(v_j)|^2 |x|^{-2\gamma} \, dx = - \int_{\Omega} \text{div}(|x|^{-2\gamma} \nabla T_k(v_j)) T_k(v_j) \, dx,$$
guarantees that $-\text{div}(|x|^{-2\gamma} \nabla T_k(v_j)) \geq 0$. Therefore, we deduce

$$-\int_\Omega \text{div}(|x|^{-2\gamma} \nabla T_k(v_j))T_k(v_j) \, dx \leq -\int_\Omega \text{div}(|x|^{-2\gamma} \nabla T_k(v_j))T_k(u_n) \, dx$$

$$\leq \left( \int_\Omega |\nabla T_k(v_j)|^2 |x|^{-2\gamma} \, dx \right)^\frac{1}{2} \left( \int_\Omega |\nabla T_k(u_n)|^2 |x|^{-2\gamma} \, dx \right)^\frac{1}{2}.$$ 

Thanks to the first step we can conclude that $T_k(v_j)$ is uniformly bounded in $\mathcal{D}^{1,2}_0(\Omega)$ and $T_k(v_j) \rightharpoonup T_k(\omega_n)$ weakly in $\mathcal{D}^{1,2}_0(\Omega)$. Using the semicontinuity of the norm

$$\int_\Omega |\nabla T_k(\omega_n)|^2 |x|^{-2\gamma} \, dx \leq \liminf_{j \to \infty} \int_\Omega |\nabla T_k(v_j)|^2 |x|^{-2\gamma} \, dx$$

$$\leq \limsup_{j \to \infty} \int_\Omega |\nabla T_k(v_j)|^2 |x|^{-2\gamma} \, dx.$$ 

Moreover, taking into account again that $-\text{div}(|x|^{-2\gamma} \nabla T_k(v_j)) \geq 0$, we get that

$$\int_\Omega |\nabla T_k(v_j)|^2 |x|^{-2\gamma} \, dx = -\int_\Omega \text{div}(|x|^{-2\gamma} \nabla T_k(v_j))T_k(v_j) \, dx$$

$$\leq -\int_\Omega \text{div}(|x|^{-2\gamma} \nabla T_k(v_j))T_k(\omega_n) \, dx$$

$$\leq \left( \int_\Omega |\nabla T_k(v_j)|^2 |x|^{-2\gamma} \, dx \right)^\frac{1}{2} \left( \int_\Omega |\nabla T_k(\omega_n)|^2 |x|^{-2\gamma} \, dx \right)^\frac{1}{2}.$$ 

It yields

$$\limsup_{j \to \infty} \int_\Omega |\nabla T_k(v_j)|^2 |x|^{-2\gamma} \, dx \leq \int_\Omega |\nabla T_k(\omega_n)|^2 |x|^{-2\gamma} \, dx,$$

and the strong convergence is shown.

**Step 3.** $T_k(\omega_n)$ is uniformly bounded (with respect to $n$) in $(\mathcal{D}^{1,2}_0)_{\text{loc}}(\Omega)$. We take $T_k(v_j)\phi$ as test function in (42), for some nonnegative function $\phi$ as in the first step. Let us denote $g_k(x) = \left( \lambda v_j a_n(x) + \frac{|\nabla v_j|^p}{1+\frac{\gamma}{p}}b_n(x) + \alpha f \right) T_k(v_j) \phi$. Using Lemma 5.2 for problem (42) and the fact that $v_j(x_0) \leq u_n(x_0) \leq C$ we deduce that $g_k$ is uniformly bounded (with respect to $n$ and $j$) in $L^{1}\text{loc}(\Omega)$. Proceeding as in (43), we can prove then that $|\nabla T_k(v_j)|$ is uniformly bounded (with respect to $j$ and $n$) in $L^q\text{loc}(\Omega; |x|^{-2\gamma} \, dx)$. By the strong convergence we conclude that $|\nabla T_k(\omega_n)|$ is uniformly bounded (with respect to $n$) in $(\mathcal{D}^{1,2}_0)_{\text{loc}}(\Omega)$.

**Step 4.** $|\nabla \omega_n|$ is uniformly bounded in $L^{\tilde{q}}_{\text{loc}}(\Omega; |x|^{-2\gamma} \, dx)$, with $\tilde{q} < q$ given in (10). In the previous step we deduced the boundedness in $L^1_{\text{loc}}(\Omega)$ of the second member of equation (42). Then, by Lemma 3.1 we obtain that $|\nabla v_j|$ is uniformly bounded in $L^q_{\text{loc}}(\Omega; |x|^{-2\gamma} \, dx)$, so it is $|\nabla w_n|$, by the a.e. convergence of the gradients.

**Step 5.** $\omega_n$ is a supersolution of problem (40), that is, $\omega_n$ verifies

$$-\text{div}(|x|^{-2\gamma} \nabla \omega_n) \geq \lambda \omega_n a_n(x) + |\nabla \omega_n|^p b_n(x) + cf, \quad \text{ in } \mathcal{D}'(\Omega). \quad (44)$$
Thanks to the a.e. convergence of the gradients, we can apply Fatou’s lemma to obtain
\[ C \geq u_n(x_0) \]
\[ \geq C'(\Omega) \text{dist}(x_0, \partial \Omega) \int_K (\lambda v_j a_n(x) + |\nabla v_j|^p b_n(x) + \alpha f) \text{dist}(x, \partial \Omega) \, dx \]
\[ \geq C'(\Omega) \text{dist}(x_0, \partial \Omega) \int_K (\lambda w_n a_n(x) + |\nabla w_n|^p b_n(x) + \alpha f) \text{dist}(x, \partial \Omega) \, dx. \] 

(45)

But then, we can take some nonnegative test function, \( \phi \in C_0^\infty(\Omega) \), in (42) and using once more Fatou’s Lemma together with the convergence of the gradients obtained in Step 4, to end with the proof of (44).

**Step 6.** We pass to the limit in (44) and find a contradiction. By the monotonicity of \( \omega_n \) we can conclude that
\[ a_n(x) \omega_n \rightharpoonup \frac{\omega}{|x|^{2\gamma}} \text{ in } L^1_{\text{loc}}(\Omega). \]
By (45) we have as well
\[ \int_K b_n(x)|\nabla \omega_n|^p \text{dist}(x, \partial \Omega) \, dx < C \text{ uniformly with respect to } n \text{ for any } K \subset \subset \Omega. \]
Now we apply Lemma 5.4 below to state that
\[ \nabla T_k(\omega_n) \to \nabla T_k(\omega) \text{ strongly in } (L^2_{\text{loc}}(\Omega; |x|^{-2\gamma} dx))^N. \]
Taking \( T_k(\omega_n) \phi \) as test function in (44), performing the corresponding integrations by parts and passing to the limit as \( n \to \infty \) by Fatou Lemma we arrive to
\[ -\text{div}(|x|^{-2\gamma} \nabla \omega) - \lambda \frac{\omega}{|x|^{2(\gamma+1)}} \geq |\nabla \omega|^p |x|^{-\gamma p} + cf, \]
a contraction with Theorem 5.1. \( \square \)

To finish the proof of the previous result we show the following lemma.

**Lemma 5.4.** Let \( \{w_n\}_{n \in \mathbb{N}} \) be a sequence such that \(-\text{div}(|x|^{-2\gamma} \nabla w_n) \geq 0 \) in \( D'(\Omega) \). Moreover, suppose that \( T_k(w_n) \) is uniformly bounded in \( (D^1_0)^2_{\text{loc}}(\Omega) \) for \( k \) fixed. In addition, assume that \( \nabla w_n \) is uniformly bounded in \( L^q_{\text{loc}}(\Omega; |x|^{-2\gamma} dx) \), for some \( \tilde{q} < q \) given in (10). Furthermore, suppose that \( w_n \leq w \), where \( w \) is the weak limit \( w_n \rightharpoonup w \) in \( W^{1,\tilde{q}}_{\text{loc}}(\Omega; |x|^{-2\gamma} dx) \). Then, \( \nabla T_k(w_n) \to \nabla T_k(w) \) strongly in \( (L^2_{\text{loc}}(\Omega; |x|^{-2\gamma} dx))^N \).

**Proof.** Since \( w_n \leq w \), for any bounded regular domain \( K \subset \subset \Omega \) it holds that
\[ \|\nabla T_k(w)\|_{L^2(\Omega; |x|^{-2\gamma})} \leq \|\nabla T_k(w_n)\|_{L^2(\Omega; |x|^{-2\gamma})}, \]
and for any nonnegative function \( \phi \in C_0^\infty(\Omega) \) we have
\[ \int_{\Omega} -\text{div}(|x|^{-2\gamma} \nabla w_n)(T_k(w_n)\phi) \, dx \leq \int_{\Omega} -\text{div}(|x|^{-2\gamma} \nabla w_n)(T_k(w)\phi) \, dx. \]
(47)
We observe that,
\[ \int_{\Omega} -\text{div}(|x|^{-2\gamma} \nabla w_n)(T_k(w_n)\phi) \, dx = \int_{\Omega} \phi |\nabla T_k(w_n)|^2 |x|^{-2\gamma} \, dx \]
\[ + \int_{\Omega} T_k(w_n) \nabla \phi \nabla w_n |x|^{-2\gamma} \, dx, \]
(48)
while the other term in (47) can be estimated as follows

\[ \int_{\Omega} -\operatorname{div}(\phi)^{2\gamma} \nabla w_n(T_k(\omega)\phi) \, dx = \int_{\Omega} \phi \nabla T_k(w_n) \nabla T_k(w) \phi^{2\gamma} \, dx + \int_{\Omega} T_k(w) \nabla \phi \nabla w_n \phi^{2\gamma} \, dx + \frac{1}{2} \int_{\Omega} \phi \nabla T_k(w_n) \phi^{2\gamma} \, dx. \]

This estimate together with (47) and (48) yield

\[ \frac{1}{2} \int_{\Omega} \phi \nabla T_k(w_n) \phi^{2\gamma} \, dx - \frac{1}{2} \int_{\Omega} \phi \nabla T_k(w) \phi^{2\gamma} \, dx \leq \int_{\Omega} (T_k(w) - T_k(w_n)) \nabla \phi \nabla w_n \phi^{2\gamma} \, dx \]

\[ \leq \left( \int_{\Omega} \left| T_k(w) - T_k(w_n) \right| \phi \nabla \phi \phi^{2\gamma} \, dx \right)^{\frac{1}{\gamma}} \left( \int_{\Omega} \left| \nabla \phi \right| \left| \nabla w_n \right| \phi^{2\gamma} \, dx \right)^{\frac{1}{\gamma}}. \]

Taking into account that \( w_n \rightarrow w \) in \( L^{\frac{\gamma}{\gamma-2}}(\Omega; |x|^{-2\gamma} \, dx) \) implies that \( T_k(w_n) \rightarrow T_k(w) \) uniformly in \( L^{\frac{\gamma}{\gamma-2}}(\Omega; |x|^{-2\gamma} \, dx) \) for any \( 1 \leq r \) and passing to the limit, we get

\[ \limsup_{n \to \infty} \frac{1}{2} \int_{\Omega} \phi \left( \left| \nabla T_k(w_n) \right|^{2} - \left| \nabla T_k(w) \right|^{2} \right) \phi^{2\gamma} \, dx \leq 0, \quad (49) \]

for any nonnegative \( \phi \in C^\infty_0(\Omega) \). If we denote by \( \varpi_n = \phi T_k(w_n) \) and \( \varpi = \phi T_k(w) \) we have that \( \varpi_n \rightharpoonup \varpi \) weakly in \( (D^{1,2})_{\text{loc}}(\Omega) \) and \( \varpi_n \rightarrow \varpi \) strongly in \( L^{2}(\Omega; |x|^{-2\gamma} \, dx) \). From (49) it follows that

\[ \limsup_{n \to \infty} \left( \left\| \nabla \varpi_n \right\|_{L^{2}(\Omega; |x|^{-2\gamma} \, dx)} - \left\| \nabla \varpi \right\|_{L^{2}(\Omega; |x|^{-2\gamma} \, dx)} \right) \leq 0. \]

But on the other hand, the previous limit is nonnegative by (46), thus it goes to zero concluding the proof. \( \square \)

6. **Existence of solutions** if \( p < p_+(\lambda) \). We proceed now to show existence of solutions to problem (1) when \( p < p_+(\lambda) \). We observe that, we have shown the nonexistence result if \( \gamma \leq -1 \). Hence through this section we will consider \( \gamma > -1 \) and \( p < p_+(\lambda) \leq 2 \). We divide the proof in two cases: \( \lambda < \Lambda_{N,\gamma} \) and \( \lambda = \Lambda_{N,\gamma} \).

We include here a result concerning weighted Sobolev Spaces (see for instance [17]), that will be used in our proofs.

**Theorem 6.1.** Let \( 1 \leq r < p \) and \( \alpha, \beta \in \mathbb{R} \). Whenever \((\alpha+1)r < (\beta+1)q\), it holds that \( L^q(\Omega; |x|^\alpha \, dx) \cap D_{\Omega,\gamma}^{1,p}(\Omega) \rightarrow L^r(\Omega; |x|^\beta \, dx) \).

**Theorem 6.2.** Assume that, \( 1 < p < p_+^{\lambda}(\lambda) \) and \( \lambda < \Lambda_{N,\gamma} \). For certain \( c_0 > 0 \) fixed, if \( c < c_0 \) and \( f(x) \leq |x|^{-2(\gamma+1)} \) then, problem (1) has a very weak solution.

**Proof.** We first note that in this case \( \gamma > -1 \). We perform the proof in several steps.

**Step 1.** We begin by constructing \( \omega_1 \) a supersolution to (1) in \( \Omega = B_1(0) \), such that \( \omega_1 \in L^1(B_1(0); |x|^{-2(\gamma+1)} \, dx) \cap D_{0,\gamma}^{1,p}(B_1(0)) \).

i) Case \( p^-_{\lambda} < p < p_+^{\lambda} \). Consider \( \omega_1 = A(|x|^{-\beta} - 1) \). It is easy to check that, taking \( \beta = \frac{p-2}{p-1} (\gamma + 1) > 0 \) and \( 0 < A^{-1} \beta^p < \beta (N - 2(\gamma + 1) - \beta) - \lambda \),
\( \omega_1 \) verifies
\[
-\text{div}(|x|^{-2\gamma}\nabla \omega_1) \geq \lambda \frac{\omega_1}{|x|^{2(\gamma+1)}} + |\nabla \omega_1|^p|x|^{-\gamma p} + \frac{\lambda A}{x^{2(\gamma+1)}}
\geq \lambda \frac{\omega_1}{|x|^{2(\gamma+1)}} + |\nabla \omega_1|^p|x|^{-\gamma p} + cf,
\]
(50)
for \( c \) small. Furthermore, since \( p^- (\lambda) \in \left( \frac{N-2(\gamma+1)}{N}, \frac{N+2(\gamma+1)}{N} \right) \), then, \( p > \frac{N}{N-2-\gamma} \).
This fact and \( \gamma > -1 \) allow us to see easily that \( \omega_1 \in L^1(\Omega; |x|^{-2(\gamma+1)}dx) \cap \mathcal{D}^1_{0;\gamma}(B_1(0)) \).

ii) Case \( 1 < p \leq p^- (\lambda) \). We take \( w_1 = A(|x|^{-\beta}-1) \) being \( \beta \in (a^-, a^+) \) sufficiently close to \( a^- \) to ensure that \( \omega_1 \in L^1(\Omega; |x|^{-2(\gamma+1)}dx) \cap \mathcal{D}^1_{0;\gamma}(B_1(0)) \) and verifying \( (\beta + 1 + \gamma)p < \beta + 2(\gamma + 1) \). As before, we take \( 0 < A^{p^-1} \beta < \beta (N - 2(\gamma + 1) - \beta) - \lambda \) and we see that \( w_1 \) satisfies (50).

**Step 2.** For \( \Omega = B_R(0) \) with \( R > 1 \) we just observe that defining \( \omega(x) = \omega_1(x/R) \), then \( \omega \) verifies also (50).

**Step 3.** We notice that there exists \( \delta > 0 \) such that
\[
\omega^{1+\delta}|x|^{-2(\gamma+1)} \in L^1(\Omega), \quad \omega^{(2-\rho)+\beta}|x|^{-\gamma p} \in L^1(\Omega).
\]
(51)
It suffices to take \( 0 < \delta < \min \left\{ \frac{N-2(\gamma+1)}{\beta}, \frac{N}{\beta} - \frac{p}{2-p}, 1 \right\} \). Note that in case \( p^- (\lambda) < p < p^+ (\lambda) \), both quantities are positive, since \( p > \frac{N}{N-2-\gamma} \). If \( 1 < p \leq p^- (\lambda) \) this minimum is positive because \( \beta \sim a^- \) and \( p \leq \frac{N+2(\gamma+1)}{N} \).

**Step 4.** As supersolution to (1) in a general domain \( \Omega \), we take \( \bar{\omega} = s(\omega - v) \), being \( s > 0 \) a constant to choose conveniently later on, \( \omega \) a supersolution in \( B_R(0) \supset \Omega \) defined in Step 2 and \( v \) being a solution to
\[
\begin{aligned}
-\text{div}(|x|^{-2\gamma} \nabla v) &= 0, \quad \text{in } \Omega, \\
v &= \omega, \quad \text{on } \partial \Omega.
\end{aligned}
\]
Note that \( v \in \mathcal{D}^1_{0;\gamma}(\Omega) \), see [3]. Therefore, thanks to Hardy inequality it implies that \( v \in L^2_{\text{loc}}(\Omega; |x|^{-2(\gamma+1)}dx) \), and since \( (1-2\gamma)p < 2(1-p\gamma) \) by Theorem 6.1 we have that \( v \in \mathcal{D}^1_{0;\gamma}(\Omega) \). Finally, we see that \( \bar{\omega} \) is a supersolution to (1). Indeed, using the following inequality
\[
|\alpha + \beta|^p \leq (1 + \kappa)^{p-1} |\alpha|^p + \left( 1 + \frac{1}{\kappa} \right)^{p-1} |\beta|^p,
\]
we obtain that
\[
-\text{div}(|x|^{-2\gamma} \nabla \bar{\omega}) - \lambda \frac{\bar{\omega}}{|x|^{2(\gamma+1)}} = s \left( -\text{div}(|x|^{-2\gamma} \nabla \omega) - \lambda \frac{\omega}{|x|^{2(\gamma+1)}} \right)
+ s\lambda \frac{v}{|x|^{2(\gamma+1)}} \geq s|\nabla \omega|^p|x|^{-\gamma p} + s\lambda \frac{v}{|x|^{2(\gamma+1)}}
\geq s|x|^{-\gamma p} \left( \frac{1}{1+\kappa} \right)^{p-1} |\nabla \bar{\omega}|^p s^{-p} - \left( \frac{1 + \frac{1}{\kappa}}{1+\kappa} \right)^{p-1} |\nabla v|^p
+ s\lambda \frac{v}{|x|^{2(\gamma+1)}}
\geq \left( \frac{1 + 1}{1+\kappa} \right)^{p-1} |\nabla v|^p
+ s\lambda \frac{v}{|x|^{2(\gamma+1)}}.
\]
where we have taken \( s = \frac{1}{1 + \kappa} \). Notice that \( 2(\gamma + 1) > \gamma p \), hence there exists a positive constant \( c_0 \) such that for any \( \phi \in C^\infty_0(\Omega) \)
\[
\int_\Omega \left( \frac{\lambda}{1 + \kappa} \frac{v}{|x|^{2(\gamma + 1)}} - \frac{1}{\kappa p - 1} \frac{1}{|x|^{\gamma p}} \right) \phi \, dx \geq \int_\Omega c_0 |x|^{-2(\gamma + 1)} \phi \, dx.
\]
Consequently, taking \( c \) small and \( f(x) \leq |x|^{-2(\gamma + 1)} \) it holds
\[
-\text{div}(|x|^{-2\gamma} \nabla \omega) - \lambda \frac{\tilde{\omega}}{|x|^{2(\gamma + 1)}} \geq |\nabla \omega|^p |x|^{-\gamma p} + cf,
\]
in the distributional sense.

**Step 5:** Let us consider again the approximation problems
\[
\begin{align*}
-\text{div}(|x|^{-2\gamma} \nabla u_n) &= \lambda u_n a_n(x) + \frac{|\nabla u_n|^p}{1 + \frac{1}{n}|\nabla u_n|^p} b_n(x) + cf, & \text{in } \Omega, \\
u_n &> 0, & \text{in } \Omega, \\
u_n &\equiv 0, & \text{on } \partial \Omega,
\end{align*}
\]
being \( a_n(x) \) and \( b_n(x) \) defined in \((41)\). Note that \( u_n \in D^{1,2}_{0,\gamma}(\Omega) \cap L^\infty(\Omega) \). Applying then the comparison principle in Section 3, we have that \( u_n \leq u_{n+1} \leq \tilde{\omega} \) for all \( n \).

Let us denote by \( \bar{u} = \lim_{n \to \infty} u_n \leq \tilde{\omega} \).

**Step 6:** We find a priori estimates for problem \((52)\). Using \( \phi_n = (1 + u_n)^\delta - 1 \) as test function in \((52)\), with \( \delta \) as in \((51)\), we obtain that
\[
\delta \int_\Omega \frac{|\nabla u_n|^2}{(1 + u_n)^{1-\delta}} |x|^{-2\gamma} \, dx \\
\leq \int_\Omega \left( \lambda a_n(x) u_n + \frac{|\nabla u_n|^p}{1 + \frac{1}{n}|\nabla u_n|^p} b_n(x) + f \right) ((1 + u_n)\delta - 1) \, dx.
\]
Thus,
\[
\delta \int_\Omega \frac{|\nabla u_n|^2}{(1 + u_n)^{1-\delta}} |x|^{-2\gamma} \, dx + \lambda \int_\Omega u_n |x|^{-2(\gamma + 1)} \, dx + \int_\Omega \frac{|\nabla u_n|^p}{1 + \frac{1}{n}|\nabla u_n|^p} b_n(x) \, dx
\]
\[
\leq \int_\Omega \lambda \tilde{\omega} (1 + \tilde{\omega})^\delta |x|^{-2(\gamma + 1)} \, dx + \varepsilon \int_\Omega \frac{|\nabla u_n|^2}{(1 + u_n)^{1-\delta}} |x|^{-2\gamma} \, dx
\]
\[
+ C(\varepsilon) \int_\Omega \tilde{\omega} (\frac{2-p}{2})^{\delta+p} |x|^{-2\gamma} \, dx + \int_\Omega f (1 + \tilde{\omega})^{\delta} \, dx \leq C + \varepsilon \int_\Omega \frac{|\nabla u_n|^2}{(1 + u_n)^{1-\delta}} |x|^{-2\gamma} \, dx.
\]
These estimates, Fatou’s Lemma and step 3 yield
\[
\frac{1}{k} \int_\Omega |\nabla T_k u_n|^2 |x|^{-2\gamma} \, dx \leq C, \quad \int_\Omega u_n |x|^{-2(\gamma + 1)} \, dx \leq C,
\]
\[
\int_\Omega |\nabla u_n|^p |x|^{-p\gamma} \, dx \leq C.
\]
In particular, \( T_k u_n \to T_k \bar{u} \) in \( D^{1,2}_{0,\gamma}(\Omega) \).

**Step 7:** We show that \( \nabla T_k u_n \to \nabla T_k \bar{u} \) strongly in \( D^{1,2}_{0,\gamma}(\Omega) \). We take \( \phi_n = \phi(T_k (u_n) - T_k (\bar{u})) \) as test function in \((52)\), being \( \phi(s) = e^{\frac{\gamma}{2} s^2} \), to get
\[
\int_\Omega \nabla u_n \nabla (T_k u_n - T_k \bar{u}) \phi_n |x|^{-2\gamma} \, dx
\]
\[
\leq \int_\Omega \left( \lambda u_n |x|^{-2(\gamma + 1)} + |\nabla u_n|^p |x|^{-p\gamma} + f \right) \phi_n \, dx.
\]
We estimate the left hand side in the previous expression as follows

\[
\int_\Omega \nabla u_n \nabla (T_k u_n - T_k \bar{u}) \phi'_n |x|^{-2\gamma} dx
\]

\[
= \int_\Omega \left( \nabla T_k u_n + \nabla G_k u_n \right) \nabla (T_k u_n - T_k \bar{u}) \phi'_n |x|^{-2\gamma} dx
\]

\[
= \int_\Omega |\nabla (T_k u_n - T_k \bar{u})|^2 |x|^{-2\gamma} dx + \int_\Omega \nabla T_k \bar{u} \nabla (T_k u_n - T_k \bar{u}) \phi'_n |x|^{-2\gamma} dx
\]

\[
+ \int_\Omega \nabla G_k u_n \nabla (T_k u_n - T_k \bar{u}) \phi'_n |x|^{-2\gamma} dx,
\]

being \( G_k(s) = s - T_k(s) \). Notice that \( G_k u_n \nabla (T_k u_n - T_k \bar{u}) = 0 \), thus the last integral vanishes. On the other hand, it holds that

\[
\int_\Omega \nabla T_k \bar{u} \nabla (T_k u_n - T_k \bar{u}) \phi'_n |x|^{-2\gamma} dx \leq \frac{1}{2} \int_\Omega |\nabla T_k \bar{u}||\nabla (T_k u_n - T_k \bar{u})||x|^{-2\gamma} dx
\]

\[
+ \frac{1}{2} \int_\Omega |\nabla T_k \bar{u}|^2 |\nabla (T_k u_n - T_k \bar{u})||x|^{-2\gamma} dx \to 0,
\]

since \( T_k(u_n) \to T_k \bar{u} \) strongly in \( L^2_{\text{loc}}(\Omega; |x|^{-2\gamma} dx) \), \( T_k(u_n) \to T_k \bar{u} \) in \( D^{1,2}_{0,\gamma}(\Omega) \) and \( T_k \bar{u} \in D^{1,2}_{0,\gamma}(\Omega) \).

We deal now with the right hand side member in (53). Note that

\[
\int_\Omega \lambda \left( u_n |x|^{-2(\gamma + 1)} + f \right) \phi_n dx \to 0,
\]

(55)
due to the uniform convergence \( \phi_n \to 0 \) in \( L^1_{\text{loc}}(\Omega; |x|^{-2(\gamma + 1)} dx) \cap L^1_{\text{loc}}(\Omega) \) and the uniform bound \( \|u_n\|_{L^1_{\text{loc}}(\Omega; |x|^{-2(\gamma + 1)} dx)} \leq C \). On the other hand, thanks to the monotonicity of the sequence \( \{u_n\} \), we know that \( \phi_n \chi_{\{u_n \geq k\}} = 0 \), hence

\[
\int_\Omega |\nabla u_n|^p |x|^{-p\gamma} \phi_n dx = \int_{\{u_n < k\}} |\nabla u_n|^p |x|^{-p\gamma} \phi_n dx
\]

\[
= \int_\Omega |\nabla T_k u_n|^p |x|^{-p\gamma} \phi_n dx \leq \int_\Omega \left( |\nabla T_k u_n|^2 |x|^{-2\gamma} + C(p) \right) \phi_n dx
\]

(56)

\[
= \int_\Omega |\nabla T_k u_n - \nabla T_k \bar{u}|^2 |x|^{-2\gamma} \phi_n dx - \int_\Omega |\nabla T_k \bar{u}|^2 |x|^{-2\gamma} \phi_n dx
\]

\[
+ 2 \int_\Omega |\nabla T_k u_n||\nabla T_k \bar{u}| |x|^{-2\gamma} \phi_n dx + \int_\Omega C(p) \phi_n dx.
\]

The fact that \( T_k \bar{u} \in D^{1,2}_{0,\gamma}(\Omega) \) and \( \phi_n \to 0 \) in \( L^1_{\text{loc}}(\Omega; |x|^{-2\gamma} dx) \) imply that

\[
\int_\Omega |\nabla T_k \bar{u}|^2 |x|^{-2\gamma} \phi_n dx \to 0, \quad \text{and} \quad \int_\Omega C(p) \phi_n dx \to 0.
\]

Finally, since we have verified that \( \|\nabla T_k u_n\|_{L^2_{\text{loc}}(\Omega; |x|^{-2\gamma} dx)} \leq C, T_k \bar{u} \in D^{1,2}_{0,\gamma}(\Omega) \) and \( \phi_n \to 0 \) in \( L^2_{\text{loc}}(\Omega; |x|^{-2\gamma} dx) \) it follows that

\[
\int_\Omega |\nabla T_k u_n||\nabla T_k \bar{u}| |x|^{-2\gamma} \phi_n dx \to 0.
\]

We have shown that the last three integrals in (56) go to zero. Hence summing up and substituting (54) and (55) into (53), we deduce that

\[
\int_\Omega |\nabla T_k u_n - \nabla T_k \bar{u}|^2 |x|^{-2\gamma} \phi'_n dx = \int_\Omega |\nabla T_k u_n - \nabla T_k \bar{u}|^2 |x|^{-2\gamma} \phi_n dx + o(1).
\]
We note that the existence in the case 

Then, applying Hardy inequality to the first term on the righthand side, we deduce 

Using Hölder inequality, it follows that 

\begin{equation}
\limsup_{k \to \infty} \int_{\{u_n > k\}} |\nabla u_n|^p |x|^{-p\gamma} dx 
\leq \limsup_{k \to \infty} \int_{\{u_n > k\}} |\nabla G_k u_n|^p |x|^{-p\gamma} (1 + u_n - T_k u_n)^s dx = 0.
\end{equation}

By the regularity of \( \omega \), the first integral on the righthand side goes to zero as \( k \to \infty \). Furthermore, 

\begin{equation}
q_{\gamma} := \frac{2N}{N(1 - p\gamma) + 2}.
\end{equation}

Observe that since \( f \leq |x|^{-2(\gamma + 1)} \) and \( N \geq 2(\gamma + 1) \), the last integral is bounded. Since \( p(\lambda) > 1 \), then \( \gamma > -1 \) and the second integral on the left hand side is also bounded. Finally we just notice that 

\begin{equation}
\int_{\Omega} |u_n|^{\frac{2N}{N - 2\gamma}} |x|^{- \frac{2N}{N - 2\gamma} \gamma} dx \leq \left( \int_{\Omega} |x|^{-2\gamma} |\nabla u_n|^2 dx \right)^{\frac{N - 2}{N - 2\gamma}}.
\end{equation}
Since $p < p^+(\lambda)$, being the radial function $\lambda > \Lambda_{N, \gamma}$. To ensure the existence of a minimal solution to these problems we just point out Theorem 6.4. Assume that $0 < p \leq 1 = p^+(\lambda)$, $-1 \geq \gamma > -\frac{N(1-p)+2}{2}$ and $\lambda < \Lambda_{N, \gamma}$. If $f(x) \in L^{2/p}(\Omega)$, problem (1) has a very weak solution.

Proof. Proceeding as before we verify (57). We use the same Hölder exponents to deal with the second integral on the lefthand side, while for the last term we take $\frac{1}{q} - \frac{N}{2N} + \frac{p}{2} = 1$, being $q = \frac{2N}{N(1-p)+2}$ as before. It leads to
\[
\int_{\Omega} f u_n \, dx \leq \varepsilon \int_{\Omega} |u_n|^{\frac{2N}{N-p}} |x|^{-\frac{2N}{N-p}} \, dx + C_1(\varepsilon) \int_{\Omega} |x|^{\gamma} \, dx + C_2(\varepsilon) \int_{\Omega} f^2 \, dx.
\]
To ensure that the second integral on the right hand side is also bounded we have to assume that $\gamma > -\frac{N(1-p)+2}{2}$. Arranging terms as in the previous case we show that $u_n$ is uniformly bounded in $D_{0, \gamma}^{1, 2}(\Omega)$. We obtain the desired solution passing to the limit in (52).

Theorem 6.5. Assume that $\lambda = \Lambda_{N, \gamma}$ and hence $p < p^+(\lambda) = \frac{N+2(\gamma+1)}{N}$. For certain $c_0 > 0$ fixed, if $\varepsilon < c_0$ and $f(x) \leq c |x|^{-2(\gamma+1)}$ with $\gamma > -1$ then problem (1) has a very weak solution. If $\gamma \leq -1$ and $f \in L^{2/p}(\Omega)$ there exists a weak solution to problem (1).

Proof. We restrict ourselves to the case $\Omega = B_r(0)$, since the result for a general domain follows as in Step 4 in the previous result. We consider the approximating problems given in (52), for $\lambda = \Lambda_{N, \gamma}$ and $\Omega = B_r(0)$. Namely,
\[
\begin{cases}
-\text{div}(|x|^{-2\gamma} \nabla u_n) = \Lambda_{\gamma, N} u_n a_n(x) + \frac{\nabla u_n |p}{1 + \frac{1}{2} |\nabla u_n|^p} b_n(x) + cf, & \text{in } B_r(0), \\
u_n > 0, & \text{in } B_r(0), \\
u_n \equiv 0, & \text{on } \partial B_r(0).
\end{cases}
\]
To ensure the existence of a minimal solution to these problems we just point out that $\lambda_1(a_n) < \Lambda_{N, \gamma}$, where $\lambda_1(a_n)$ denotes the first eigenvalue for the divergence operator with weight $a_n(x)$, and then we can argue as before.

Now we construct a supersolution to (1) in $B_r(0)$. For certain $R > r > 0$, consider the radial function
\[
\omega(x) = \left( \frac{|x|}{R} \right)^{-\frac{N-2(\gamma+1)}{2}} \left( \log \left( \frac{R}{|x|} \right) \right)^{\frac{1}{2}} - \left( \frac{r}{R} \right)^{-\frac{N-2(\gamma+1)}{2}} \left( \log \left( \frac{R}{r} \right) \right)^{\frac{1}{2}}.
\]
It is tedious but simple to show that
\[
-\text{div}(|x|^{-2\gamma} \nabla \omega) - \Lambda_{N, \gamma} \frac{\omega}{|x|^{2(\gamma+1)}} = \frac{1}{4} \frac{\omega}{|x|^{2(\gamma+1)}} \left( \log \left( \frac{R}{|x|} \right) \right)^{-2} + \frac{CA_{N, \gamma}}{|x|^{2(\gamma+1)}},
\]
being $C = \left( \frac{R}{r} \right)^{-\frac{N-2(\gamma+1)}{2}} \left( \log \left( \frac{R}{r} \right) \right)^{\frac{1}{2}}$.

On the other hand, \[
|\nabla \omega|^p |x|^{-p}
= R^{\frac{(N-2(\gamma+1))p}{2}} \left( \log \left( \frac{R}{|x|} \right) \right)^{\frac{p}{2}} \left( \frac{N-2(\gamma+1)}{2} + \frac{1}{2} \left( \log \left( \frac{R}{|x|} \right) \right)^{-1} \right)^p.
\]
Since $p < \frac{N+2(\gamma+1)}{N}$ we can find a suitable constant such that $c \omega$ is a supersolution to (1) in $B_r(0)$, with $\lambda = \Lambda_{N, \gamma}$ and $\omega = 0$ on $\partial B_r(0)$. Moreover, $\omega \in D_{0, \gamma}^{1, 2}(B_r(0))$.\]
for every $q < 2$. Applying the comparison principle in Section 3, we deduce that $u_n \leq u_{n+1} \leq \omega$. Thus $u_n$ converges pointwise to $u \leq \omega$, $u \in L^q(B_r(0); |x|^{-2\gamma})$, for all $q < 2^*_\gamma$.

Let us denote by $H_\gamma(B_r(0))$ the completion of $C_0^\infty(B_r(0))$ with respect to the norm
\[
\|\phi\|_{H_\gamma(B_r(0))}^2 = \int_{B_r(0)} |\nabla\phi|^2 |x|^{-2\gamma} \, dx - A_{\gamma,N} \int_{B_r(0)} |\phi|^2 |x|^{-2(\gamma+1)}.
\]

$H_\gamma(B_r(0))$ is a Hilbert space and verifies $D_{0,\gamma}^{1,2}(\Omega) \subset H_\gamma(B_r(0)) \subset D_{0,\gamma}^{1,2}(B_r(0))$, for all $q < 2$. Let us show that $\{u_n\}$ is uniformly bounded in $H_\gamma(B_r(0))$. Indeed, using $u_n$ as test function in (58) and taking into account that $u_n \leq \omega$, we get that
\[
\|u_n\|_{H_\gamma(B_r(0))}^2 = \int_{B_r(0)} |\nabla u_n|^p |x|^{-\gamma p} \omega \, dx + c_0 \int_{B_r(0)} \omega f \, dx.
\]
We observe that $\omega \in L^1(\Omega; |x|^{-2(\gamma+1)} \, dx)$, hence the last integral is bounded if $f(x) \leq c|x|^{-2(\gamma+1)}$. If $\gamma \leq -1$ it suffices taking $f \in L^{2/p}(\Omega)$ to ensure the boundedness of this term. To estimate the second integral, we apply Hölder, Young and the improved Hardy-Sobolev inequalities (see [1]), to deduce that
\[
\int_{B_r(0)} |\nabla u_n|^p |x|^{-\gamma p} \omega \, dx = \int_{B_r(0)} |\nabla u_n|^p |x|^{-\gamma p} \left( \log \left( \frac{R}{|x|} \right) \right)^p \left( \log \left( \frac{R}{|x|} \right) \right)^{-p} \omega \, dx
\]
\[
\leq \varepsilon \int_{B_r(0)} |\nabla u_n|^2 \left( \log \left( \frac{R}{|x|} \right) \right)^{-2} |x|^{-2\gamma} \, dx + C(\varepsilon) \int_{B_r(0)} \omega^{\frac{2p}{2\gamma}} \left( \log \left( \frac{R}{|x|} \right) \right)^{\frac{2p}{2\gamma}} \, dx
\]
\[
\leq \varepsilon \|u_n\|_{H_\gamma(B_r(0))}^2 + C(\varepsilon) \int_{B_r(0)} \omega^{\frac{2p}{2\gamma}} \left( \log \left( \frac{R}{|x|} \right) \right)^{\frac{2p}{2\gamma}} \, dx.
\]
It is easy to see that the last integral is bounded since $p < \frac{N+2(\gamma+1)}{N}$. Therefore, taking $\varepsilon$ small we have shown that $\|u_n\|_{H_\gamma(B_r(0))} \leq C$ and thus $u_n \rightharpoonup u$ in $H_\gamma(B_r(0))$. Furthermore,
\[
\|u\|_{H_\gamma(B_r(0))}^2 \leq \|u_n\|_{H_\gamma(B_r(0))}^2.
\]
To show that indeed, $u_n \rightharpoonup u$ strongly in $H_\gamma(B_r(0))$, we consider the functional
\[
F_n : H_\gamma(B_r(0)) \to \mathbb{R}
\]
\[
F_n = -\text{div}(|x|^{-2\gamma} \nabla u_n) - \Lambda_{\gamma,N} a_n(x) u_n.
\]
It is not difficult to check that $F_n \in H_\gamma^*(B_r(0))$, the dual space of $H_\gamma(B_r(0))$. In fact, $\{F_n\}$ is uniformly bounded in $H_\gamma^*(B_r(0))$. For any $\phi \in H_\gamma(B_r(0))$, we have that
\[
|\langle F_n, \phi \rangle| \leq \int_{B_r(0)} |\nabla u_n|^p |x|^{-\gamma p} \phi \, dx + c \int_{B_r(0)} f \phi \, dx.
\]
If $\gamma > -1$, using that $f \leq c_0 |x|^{-2(\gamma+1)}$, we can see that the last integral is bounded by $C\|\phi\|_{H_\gamma(B_r(0))}$. Whereas if $\gamma \leq -1$ we just observe that $\phi \in D_{0,\gamma}^{1,q}(B_r(0))$, for any $q < 2$. In particular for $q = \frac{p}{2\gamma}$. Therefore, since $f \in L^{p/2}(\Omega)$ the last integral is also bounded in this case. To estimate the second integral we argue similarly as
before to find that
\[
\int_{B_r(0)} |\nabla u_n|^p |x|^{-\gamma p} \phi \leq \left( \int_{B_r(0)} |\nabla u_n|^2 \left( \log \frac{R}{|x|} \right)^{-2} |x|^{-2\gamma} \right)^\frac{p}{2} \\
\times \left( \int_{B_r(0)} \phi \frac{2\gamma}{p} \left( \log \frac{R}{|x|} \right) \frac{2p}{2p - 2\gamma} \right) \leq C \left( \int_{B_r(0)} \phi \frac{2\gamma}{p} \left( \log \frac{R}{|x|} \right) \right) \frac{2p}{2p - 2\gamma}.
\]

If \( \gamma \leq 0 \) we observe that \( p < \frac{N+2(\gamma+1)}{N} \) yields \( \frac{2p}{2p - 2\gamma} < 2_\gamma^* = \frac{2(N-2\gamma)}{N-2(\gamma+1)}. \) Furthermore, since \( \Lambda_{\gamma,N} \) is not achieved, we have
\[
\left( \int_{B_r(0)} \phi \frac{2\gamma}{p} \left( \log \frac{R}{|x|} \right) \frac{2p}{2p - 2\gamma} \right) \leq C \left( \int_{B_r(0)} \phi \frac{2\gamma}{p} \left( \log \frac{R}{|x|} \right) \right) \frac{2p}{2p - 2\gamma} \cdot
\]

being \( \eta = (2_\gamma^*)' = \frac{2(N-2\gamma)}{N-2}, \) for \( r \) sufficiently small. On the other hand, if \( \gamma > 0 \) recalling that \( 2_\gamma^* = \frac{2N}{2N-2} \) we have
\[
\left( \int_{B_r(0)} \phi \frac{2\gamma}{p} \left( \log \frac{R}{|x|} \right) \frac{2p}{2p - 2\gamma} \right) \leq o(r) \left( \int_{B_r(0)} \phi \frac{2\gamma}{p} \left( \log \frac{R}{|x|} \right) \right) \frac{2p}{2p - 2\gamma} \leq c \|\phi\|_{H_\gamma(B_r(0))},
\]

with \( \eta' = (2_\gamma^*)' = \frac{2N}{2N-2}. \) This means that \( \langle F_n, \phi \rangle \leq c \|\phi\|_{H_\gamma(B_r(0))}, \) namely \( F_n \) is uniformly bounded in \( H_\gamma(B_r(0)) \). In consequence, \( F_n \rightharpoonup F \) in the weak-star topology of \( H_\gamma^*(B_r(0)). \) But then observe that for any \( \phi \in C_0^\infty(\Omega) \), thanks to the weak convergence \( u_n \rightharpoonup u \) in \( H_\gamma(B_r(0)) \), it holds that
\[
\langle F_n, \phi \rangle = \int_{B_r(0)} \left( \|
abla u_n\| \|
abla \phi\| |x|^{-2\gamma} - \Lambda_{\gamma,N} u_n a_n(x) \phi \right) dx \\
\rightarrow \int_{B_r(0)} \left( \|
abla u\| \|
abla \phi\| |x|^{-2\gamma} - \Lambda_{\gamma,N} u |x|^{-2(\gamma+1)} \phi \right) dx.
\]

Therefore, \( F = -\text{div}(|x|^{-2\gamma} \nabla u) - \Lambda_{\gamma,N} u |x|^{-2(\gamma+1)} \in H_\gamma^*(B_r(0)) \) and by density,
\[
\langle F_n, u \rangle \rightarrow \langle F, u \rangle = \|u\|_{H_\gamma(B_r(0))}.
\]
Moreover, since $u_n \leq u$ and $F_n = -\text{div}(|x|^{-2\gamma}\nabla u_n) - \Lambda_{\gamma,N}a_n(x)u_n \geq 0$, we deduce

\[
\|u_n\|_{H^s(B_r(0))} \leq \int_{B_r(0)} -\left(\text{div}(|x|^{-2\gamma}\nabla u_n) + \Lambda_{\gamma,N}a_n(x)u_n\right) u_n \, dx = \langle F_n, u_n \rangle
\]

\[
\leq \int_{B_r(0)} -\left(\text{div}(|x|^{-2\gamma}\nabla u_n) + \Lambda_{\gamma,N}a_n(x)u_n\right) u \, dx = \langle F_n, u \rangle.
\]

This estimate together with (60) give us that $\|u_n\|_{H^s(B_r(0))} \leq \|u\|_{H^s(B_r(0))}$. Then, (59) implies the strong convergence $u_n \to u$ in $H^s(B_r(0))$. Now, it is easy to check that $u$ is supersolution to (1) in $B_r(0)$, passing to the limit in (58) thanks to the strong convergence, and applying Fatou’s Lemma.

\[\square\]

**Open problem:** Note that the question about existence of solutions in the range of parameters $0 < p < 1$ and $-\infty < \gamma \leq -\frac{N(1-p)^2}{2}$ remains open.

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