Quantum phase space functions and relations of entropic localisation measures

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Abstract. The concept of quantum phase space offers a view on quantum mechanics, which is different from the standard Hilbert space approach, but which more closely resembles the classical phase space. Due to the properties of quantum mechanics there are several equivalent quantum phase space descriptions, and one cannot always prefer one or another as they all have certain merits and drawbacks. For example, the Husimi-Kano $Q$ function is a probability distribution and thus gives rise to entropic quantities, namely the Rényi-Wehrl entropies, of which several properties are known. The Wigner function, on the other hand, has an easier physical explanation, but may take negative values. In this article, we investigate entropic measures of localisation for a state in quantum phase space by using the Beckner-Brascamp-Lieb inequality to relate different phase-space functions.

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1. Introduction

In classical physics, the concept of a phase space is well-known and widely applied, e.g. in Lagrangian and Hamiltonian mechanics. A (pure) state of such a system may be described by a point in phase space, and the dynamics is given by a trajectory in this space. Statistical ensembles of classical states may accordingly be described by a probability distribution on the phase space, and the dynamics of points induces dynamics of such probabilities. On the other hand, the standard approach to quantum mechanics uses the concept of a Hilbert space with vectors in and operators on the Hilbert space. But there also exists the phase-space approach, which is arguably more similar to classical physics and exhibits different phenomena of quantum mechanics. Mathematically, the quantum phase space formalism is an equivalent formulation of quantum mechanics, which adapts classical phase space to quantum mechanics. In this article we present a method to relate different entropic quantities on quantum phase space to each other and to Hilbert space concepts such as purity by using tools from functional analysis.

The rest of this article is organised as follows: In section 2 we introduce the basics of quantum phase space distributions and discuss entropies and entropy-related measures and their interpretation. In section 3 we derive relations of these quantities and give some examples. We finally conclude our discussion in section 4 and offer an outlook on future work. In the appendix we summarise concepts and results from quantum mechanics and mathematics, which are used in the text.

2. Quantum phase space distributions and entropic measures of localisation

Let us consider a single particle with position $x$ and momentum $p$, whose classical phase space is $\mathbb{R} \times \mathbb{R}$. In the quantum case we take the same mathematical space, but have to fulfil e.g. the Heisenberg uncertainty relation, so that a pure state cannot be a point (delta distribution). We can introduce quantum phase space in the following way: we consider a state (density matrix) $\rho$ and combine $x$ and $p$ into $\alpha \in \mathbb{C}$ (cf. Appendix A); provided the expression is well-defined, we define the $s$-ordered phase space functions, $s \in \mathbb{C}$, by (cf. e.g. [1] and also [2] with different sign convention for $s$)

$$W^{(s)}(\alpha) = \frac{1}{\pi^2} \int_{\xi \in \mathbb{C}} \text{Tr} \left( \rho e^{\xi a^\dagger - \xi^* a} \right) e^{-\langle \xi \alpha^* - \xi^* \alpha \rangle} e^{+\frac{\xi^2|\xi|^2}{2}} d^2 \xi. \tag{1}$$

With $\hat{D}_s(\xi) := e^{\xi a^\dagger - \xi^* a + \frac{\xi^2|\xi|^2}{2}}$, the “expectation value” function $\xi \mapsto \text{Tr} \rho \hat{D}_s(\xi)$ is known as the $s$-ordered characteristic function, and $W^{(s)}$ is its symplectic Fourier transform. The three most prominent phase-space functions are the GLAUBER-SUDARSHAN $P$ function ($s = +1$), the WIGNER function $W$ ($s = 0$) and the HUSIMI-KANO $Q$ function ($s = -1$). While $P$ need not even be a true function, $W$ and $Q$ are real-valued functions.

‡ Another common definition of the WIGNER function is $W_\rho(x, p) := \frac{1}{2\pi \hbar} \int_{\xi \in \mathbb{R}} \langle x + \frac{\xi}{2}\vert \rho \vert x - \frac{\xi}{2} \rangle e^{-\frac{i\xi p}{\hbar}} d\xi$ [1, p. 68], which is easier to interpret, but has the disadvantage that it has a dimension of inverse action; by relating phase-space domains with $dx dp = 2\hbar d^2\alpha$, we find $W_\rho(\alpha) = 2\hbar \cdot W_\rho(x, p)$. 

[1]
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As all phase space distributions contain the full information on the density matrix, they can be transformed into each other. In particular, if \( \text{Re}(s) < \text{Re}(t) \), there holds \[ W(s) = \frac{2}{\pi(t-s)} \int_{\beta \in \mathbb{C}} W(t)(\beta) \exp \left[ -2 \frac{|\alpha - \beta|^2}{t-s} \right] \ d^2\beta; \] this can be written as a two-dimensional convolution, \( W(s) = \frac{2}{\pi(t-s)} W(t) * f_{\frac{2}{t-s}} \) with a Gaussian \( f_a : \mathbb{C} \to \mathbb{R}^+ \), \( f_a(z) := e^{-a|z|^2} \) and may be seen as a continuous averaging or smoothing of \( W(t) \).

We shall now introduce some measures of localisation of a quantum state, which give rise to entropic quantities.

2.1. The Rényi-Wehrl entropies

The Husimi-Kano \( Q \) distribution can be written in the form \( Q_\rho(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle \), which explicitly shows that it is non-negative and normalised, so that it can directly be used to define entropies; for simplicity we shall always use the natural logarithm.

**Definition 1 (Rényi-Wehrl entropies) .**

For a density operator \( \rho \) on \( \mathcal{H} = L^2(\mathbb{R}) \), the Rényi-Wehrl entropy of order \( q \in \mathbb{R}^+ \setminus \{1\} \) is defined by

\[
R_q(\rho) := \frac{1}{1-q} \ln \int_{\alpha \in \mathbb{C}} Q_\rho(\alpha)^q \ d^2\alpha,
\]

and for \( q \in \{0, 1, \infty\} \) we take the appropriate limit. In particular, we get the (standard) Wehrl entropy \( W(\rho) := R_1(\rho) = -\int_{\alpha \in \mathbb{C}} Q_\rho(\alpha) \ln Q_\rho(\alpha) \ d^2\alpha \).

A Rényi-Wehrl entropy is thus the Rényi entropy of the \( Q \) function or the entropy with respect to the frame of coherent states (cf. Appendix A). To illustrate this, we calculate Rényi-Wehrl entropies for a general squeezed state \( \rho_{\alpha, \xi} = |\alpha, \xi\rangle \langle \alpha, \xi| \): we have \( R_q(\rho_{\alpha, \xi}) = \frac{\ln q}{q-1} + \ln \pi + \ln \cosh |\xi| \), and this splits into three parts: (i) a Rényi part (with appropriate limits), (ii) a constant part \( \ln \pi \) and (iii) a squeezing part, which is related to the maximum absolute overlap \( \sqrt{\text{sech}|\xi|} \) of coherent states and states squeezed by \( \hat{S}(\xi) \).

\[ \quad \text{§ Originally, the Wehrl entropy was defined by } \tilde{W}(\rho) := -\frac{1}{2} \int_{\alpha \in \mathbb{C}} \tilde{Q}_\rho(\alpha) \ln \tilde{Q}_\rho(\alpha) \ d^2\alpha = W(\rho) - \ln \pi \text{ using the non-normalised Husimi function } \tilde{Q}_\rho(\alpha) := \langle \alpha | \rho | \alpha \rangle \lim_{q \to 1} R_q(\rho) \text{. Note, however, that the limit } q \to 1 \text{ in our definition is only sensible due to the normalisation of } Q_\rho. \]

\[ \quad \text{§ Note that this is an instance of the Wehrl-Lieb inequality for } W(\rho) \text{, which states that } W(\rho) \geq 1 + \ln \pi \text{ with equality, if and only if } \rho \text{ is a coherent state.} \]
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2.2. The Süßmann measure

Another measure of localisation of a quantum state in phase space is Süßmann’s uncertainty area with dimension of an action, which is defined by

$$\delta_\rho := \frac{2\hbar}{\int W_\rho(\alpha)^2 d^2\alpha},$$

from which the Rényi-Süßmann entropy $S_\delta := \ln[(2\pi\hbar)^{-1}\delta_\rho]$ can be constructed [9]. Using $\text{Tr}(\rho_1\rho_2) = \pi \int_{\alpha \in C} W_{\rho_1}(\alpha)W_{\rho_2}(\alpha) d^2\alpha$ [1, p. 71] we see that the Süßmann measure is directly related to the purity $\text{Tr}(\rho^2)$ or the linear entropy $1-\text{Tr}(\rho^2)$ of the state: there holds $\delta_\rho \geq 2\pi\hbar$ with equality, if and only if $\rho$ is pure.

3. Relations of phase space functions

We shall now present a relation of phase-space quantities for different phase-space functions $W^{(s)}$, in particular the WIGNER and the HUSIMI functions. Using the concept of $p$-norms (cf. Appendix B, Appendix B), we can rewrite the Rényi-WEHRL entropies as

$$R_q(\rho) = \frac{q}{1-q} \ln \|Q_\rho\|_q, \quad q > 1,$$

and the Rényi-Süßmann entropy as

$$S_\delta = \ln \left( \pi \int W_\rho(\alpha)^2 d^2\alpha \right)^{-1} = \ln \left( \frac{1}{\pi} \|W_\rho\|_2^{-2} \right) = -\ln \pi - 2 \ln \|W_\rho\|_2. \quad (5)$$

Solving for the norm of the phase-space function these equations read

$$\|Q_\rho\|_q = \exp \left( \frac{1-q}{q} R_q(\rho) \right) \quad \text{and} \quad \|W_\rho\|_2 = \exp \left[ -\frac{1}{2}(S_\delta + \ln \pi) \right]. \quad (6)$$

Now, we can use $Q = \frac{2}{\pi}(W * f_2)$ and plug this into the BECKNER-BRASCAMP-LIEB inequality (Theorem 11) for $p = 2$. Then, for $1 + \frac{1}{r} = \frac{1}{2} + \frac{1}{q}$ or $q = \frac{2r}{r+2}$, it follows

$$e^{\frac{1-q}{q} R_r(\rho)} = \|Q_\rho\|_r \leq \left( \frac{C_q}{C_r} \right)^2 \frac{2}{\pi} \|W_\rho\|_2 \|f_2\|_q \quad (7)$$

$$e^{\frac{1}{2}R_2(\rho)} \leq e^{-\frac{1}{2}(S_\delta + \ln \pi)} \cdot \frac{\pi}{2q} \left( \frac{\pi}{2q} \right)^{\frac{1}{q}} \quad (8)$$

We can now consider special cases of this inequality and start by mentioning that $r \geq 2$ is necessary for $q \geq 1$. For the case of $r = 2$, the Rényi-WEHRL collision entropy, we find $q = 1$, and the expression reduces to

$$e^{-\frac{1}{2}R_2(\rho)} \leq e^{-\frac{1}{2}(S_\delta + \ln \pi)} \quad (9)$$

The other main example is $r \to \infty$, from which there follows $q = 2$, so that eq. (7) then reads

$$e^{-R_\infty(\rho)} \leq \pi^{-1/2} \cdot e^{-\frac{1}{2}(S_\delta + \ln \pi)} \quad (10)$$

In this case, we can explicitly see that equality holds for coherent states.

1 Alternatively, $\delta_\rho = (\int W_\rho(x, p)^2 \, dx \, dp)^{-1}$ and $\text{Tr}(\rho_1\rho_2) = 2\pi\hbar \int_{x, p \in \mathbb{R}} W_{\rho_2}(x, p)W_{\rho_2}(x, p) \, dx \, dp$. 


3.1. Generalisations

We do not need to restrict to the case of the Sütßmann measure only, but can consider all quantities $\|W_\rho\|_p$ for $p \in [1; \infty]$. Note that none of these quantities change under Gaussian operations such as displacement and squeezing or with time in free evolution or in a harmonic potential.

For example, in the case that $p = 1$, the normalisation of the Wigner function, $\int_{\alpha \in \mathbb{C}} W_\rho(\alpha) \, d^2\alpha = 1$, implies $\|W_\rho\|_1 \geq 1$ with equality, if and only if the Wigner function is non-negative. As the Wigner function is often considered to be a “non-classical” probability distribution, we may e.g. define $C_\rho := \ln \|W_\rho\|_1$ as the “non-classicality” of a quantum state. With $p = 1$ and thus $r = q$, our inequality reads

$$\|Q_\rho\|_q \leq \frac{2}{\pi} \cdot \|W_\rho\|_1 \cdot \|f_2\|_q \quad \text{with} \quad \|f_2\|_q = \left(\frac{\pi}{2q}\right)^{1/q}. \quad (11)$$

In the case of $q \in \{1, 2, \infty\}$ the the right-hand side factor becomes $\pi/2, \sqrt{\pi}/2$ and 1, respectively.

Another example is $p = \infty$, which yields the maximum of the Wigner function on phase space. In this case, we have $1 + \frac{1}{q} = \frac{1}{q}$ or $q = \frac{r+1}{r}$, so that only $r = \infty$ and $q = 1$ remains possible, and the inequality then reads $\|Q_\rho\|_\infty \leq \|W_\rho\|_\infty$.

3.2. Perspectives

It might be possible to prove the Hudson-Piquet theorem [1], which states that a pure state has a non-negative Wigner function, if and only if it is Gaussian, in a functional-analytic way (the standard proofs use methods from the theory of functions [10, 11]): By construction, both the Wigner and the Husimi function are normalised in the sense that $\int_{\alpha \in \mathbb{C}} W(\alpha) \, d^2\alpha = \int_{\alpha \in \mathbb{C}} Q(\alpha) \, d^2\alpha = 1$. As the latter one is non-negative everywhere, there also holds $\|Q\|_1 = 1$, but this is not true for the former. There holds $\|W\|_1 \geq 1$ with equality, if and only if $W$ is non-negative, i.e. $W \geq 0$. Using $Q = \frac{2}{\pi}(W * f_2)$, from Theorem [11] there follows $1 = \|Q\|_1 \leq \frac{2}{\pi} \|W\|_1 \cdot \|f_2\|_1 = \|W\|_1$, and we have to check the conditions for equality. The purity of the state can be reformulated in norms by $\|W_\rho\|_2 = \pi^{-1/2}$. However, the conditions of equality from Theorem [11] do not apply here.

4. Conclusion

In this work, we have found relations between entropic quantities in quantum phase space, in particular, for Wigner and Husimi functions, by using the Beckner-Brascamp-Lieb inequality, which points out specific properties of Gaussian functions in a natural way. Some of these quantities, e.g. the Sütßmann measure, are directly related to Hilbert-space properties. The relations which we found for Wigner and

As an example, there roughly holds $\|W_{(m)}(m)\|_1 \approx \sqrt{m+1}$ for Fock states, so that $C_\rho \approx \frac{m+1}{2}$. 

* As an example, there roughly holds $\|W_{(m)}(m)\|_1 \approx \sqrt{m+1}$ for Fock states, so that $C_\rho \approx \frac{m+1}{2}$. 


HUSIMI distributions can, in principle, be generalised to other s-ordered phase-space functions, and one may hope to find more relations for different distributions.

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Appendix A. Quantum mechanics: Harmonic oscillators and Gaussians

A single spinless, pointlike particle in a one-dimensional harmonic oscillator is described by the separable, infinite-dimensional complex system Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. The position and momentum operators, $\hat{x}$ and $\hat{p}$, fulfill the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar\mathbb{1}_\mathcal{H}$; in position representation, they read $\hat{x} = x$ and $\hat{p} = -i\hbar\nabla$, and the Hamiltonian operator is $\hat{H} = \frac{p^2}{2m} + \hat{V} = -\frac{\hbar^2}{2m}\Delta + V(x)$ with potential $V(x) := \frac{1}{2}m\omega x^2$.

Classical position and momentum are combined into a dimensionless parameter $\alpha = \sqrt{\frac{2m}{\hbar}}x + \frac{i}{\sqrt{2m\omega}}p \in \mathbb{C}$ with differentials $dx\,dp = 2\hbar\,d\alpha\,d\text{Im}(\alpha) = 2\hbar\,d^2\alpha$. By “canonical quantisation”—replacing $(x, p)$ by $(\hat{x}, \hat{p})$—$\alpha$ turns into an operator $\hat{a}$ with $[\hat{x}, \hat{p}] = i\hbar \Leftrightarrow [\hat{a}, \hat{a}^\dagger] = 1$. On $\mathcal{H}$ with the orthonormal FOCK (or photon-number) basis $\{ |n\rangle | n \in \mathbb{N}_0 \}$, $\hat{a}^\dagger$ and $\hat{a}$ are the unbounded creation and annihilation operators with spectrum $\emptyset$ and $\mathbb{C}$, respectively, where $\hat{a}^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle$ and $\hat{a} |n\rangle = \sqrt{n} |n - 1\rangle$, except for $\hat{a}|0\rangle = 0$. The number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ with $\hat{n}|n\rangle = n|n\rangle$ has spectrum $\mathbb{N}_0$, and $\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2}) = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$, related to the classical overall energy $E_{\text{tot}} = \hbar\omega^2\alpha^2$.

Two systems of unitary operators, the displacement operators $\hat{D}(\alpha) := e^{\hat{a}^\dagger - \alpha^* \hat{a}}$ for $\alpha \in \mathbb{C}$ and the squeezing operators $\hat{S}(\xi) := e^{-\frac{i}{2}(\hat{a}^\dagger \hat{a} - \xi^2)}$ for $\xi = r e^{i\phi} \in \mathbb{C}$, define coherent states $|\alpha\rangle := \hat{D}(\alpha) |0\rangle$ (note that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$) and squeezed vacuum states $|\xi\rangle := \hat{S}(\xi) |0\rangle$, respectively. The set $\{ |\alpha\rangle | \alpha \in \mathbb{C} \}$ does not form an orthonormal basis, but somewhat similar, by $\int_{|\alpha| \leq \tau} |\alpha\rangle \langle \alpha| \, d^2\alpha = \pi \mathbb{1}$ (pointwise), an overcomplete tight continuous frame with frame bound $\pi$ [12] and, more generally, a positive operator-valued measure (POVM) [13] [14]. The two-mode and ideal squeezed (coherent) states $|\alpha, \xi\rangle$ are defined by $\hat{S}(\xi)\hat{D}(\alpha) |0\rangle$ or $\hat{D}(\alpha)\hat{S}(\xi) |0\rangle$, respectively [15] p. 1042, and are related by $\hat{S}(\xi)\hat{D}(\alpha) = \hat{D}(\beta)\hat{S}(\xi)$ with $\beta = \alpha \cosh r + \alpha^* e^{i\phi} \sinh r$ [16] p. 18 with different signs. Due to $\hat{D}(\alpha + \beta) = \hat{D}(\alpha)\hat{D}(\beta) e^{\alpha^* \beta^*} = \hat{D}(\alpha)\hat{D}(\beta) e^{i(\text{Re}\alpha \text{Im}\beta - \text{Im}\alpha \text{Re}\beta)}$ the map $\alpha \mapsto \hat{D}(\alpha)$ is a unitary projective representation of the abelian group $\langle \mathbb{C}, + \rangle \cong (\mathbb{R}^2, +)$; although $\hat{S}(\xi)^{-1} = \hat{S}(-\xi)$, this is not true for the squeezing operators due to $[\hat{a}^2, \hat{a}^\dagger^2] = 2\{\hat{a}, \hat{a}^\dagger\} = 4(\hat{n} + \frac{1}{2}) \neq \text{const}$.

Appendix B. Functional analysis: Norms, convolutions and some integrals

In order to derive relations between entropic measures, we use some well-known mathematical concepts from functional analysis, which we will review here. The most important concept in our calculations is the $p$-norms (cf. e.g. [17] [18]).
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Definition 2 (p-norms).
For a measure space \((X, \mu)\) — in particular, for \(X = \mathbb{R}^n\) with the Lebesgue measure — and a measurable function \(f : X \to \mathbb{C}\), the p-norm \(\| \cdot \|_p : X \to \mathbb{R}_0^+\) is (provided the expression is finite) defined by \(\|f\|_p := \left(\int_{x \in X} |f(x)|^p \, d\mu(x)\right)^{1/p}\), if \(p \in [1; \infty)\), and \(\|f\|_\infty := \text{ess sup}_{x \in X} |f(x)| := \inf \{a \in \mathbb{R} | \mu\{x \in X | |f(x)| > a\} = 0\}\).

Note that \(\|f\|_p = \|g\|_p \cdot \|h\|_p\) for \(f(x, y) = g(x) \cdot h(y)\). The convolution of functions \(f, g : \mathbb{R}^n \to \mathbb{C}\) is given by \((f \ast g)(x) := \int_{y \in \mathbb{R}^n} f(x-y)g(y) \, dy\). For a one-dimensional Gaussian function \(f_a(x) := e^{-ax^2}\) with \(a > 0\) we find \((f_a \ast f_b)(x) := \sqrt{\frac{2}{a+b}} \int_{y \in \mathbb{R}} xu \, du\) with \(\sqrt{\frac{2}{a+b}}\) and variance \(\sigma^2 = 2\) (standard deviation \(\sigma = \sqrt{\sigma^2}\)).

We can now discuss a strengthened YOUNG’s inequality ([11, p. 169], [20, p. 168f.]): The conjugate of \(p \in [1; \infty]\) is \(p' = \frac{p}{p-1}\) (with \(\frac{1}{p} := \infty\) and vice versa), so that \(\frac{1}{p} + \frac{1}{q} = 1\) and \(1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}\) is equivalent to \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q}\). We set \(C_p := \sqrt{p^{1/p}/p'^{1/p'}}\) with \(C_1 := C_\infty := 1\); note that \(C_p C_p' = 1\) and, in particular, \(C_2 = 1\).

Theorem 1 (Beckner-Brascamp-Lieb inequality).
For \(p, q, r \in [1; \infty]\) such that \(1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}\) and functions \(f, g : \mathbb{R}^n \to \mathbb{C}\), there holds \(\|f \ast g\|_r \leq (C_p C_q C_r^{-1})^p \|f\|_p \|g\|_q\), with equality, for \(n = 1\) and \(p, q \in (1; \infty)\), if and only if \(f(x) = A e^{-\gamma_p'(x-\alpha)^2+ibx}\) and \(g(x) = B e^{-\gamma_q'(x-\beta)^2+ibx}\) for some \(A, B \in \mathbb{C}, \alpha, \beta, \delta \in \mathbb{R}\) and \(\gamma \in \mathbb{R}^+\).

We finally remind the reader that all integrals over \(\mathbb{C}\) correspond to integrals over \(\mathbb{R}^2\). For \(a, b, c \in \mathbb{C}\) with \(\text{Re}(a) > 0\), there holds \(\int_{x \in \mathbb{R}} e^{-ax^2+bx+c} \, dx = \sqrt{\frac{2}{a}} e^{\frac{b^2}{4a}}\) where the square root takes positive real part, and \(\int_{x=0}^{\infty} x^{m-1} e^{-ax^2} \, dx = \frac{1}{2a^m} \Gamma\left(\frac{m+1}{2}\right)\) for \(m \in \mathbb{C}\) with \(\text{Re}(m) > 0\). In particular, for \(f_a(x) := e^{-ax^2}\), there holds \(\|f_a\|_p = \left(\frac{\pi a}{2p}ight)^{n/2p}\).

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