Data-driven Stochastic Model Predictive Control

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Abstract

We propose a novel data-driven stochastic model predictive control (MPC) algorithm to control linear time-invariant systems with additive stochastic disturbances in the dynamics. The scheme centers around repeated predictions and computations of optimal control inputs based on a non-parametric representation of the space of all possible trajectories, using the fundamental lemma from behavioral systems theory. This representation is based on a single measured input-state-disturbance trajectory generated by persistently exciting inputs and does not require any further identification step. Based on stochastic MPC ideas, we enforce the satisfaction of state constraints with a pre-specified probability level, allowing for a systematic trade-off between control performance and constraint satisfaction. The proposed data-driven stochastic MPC algorithm enables efficient control where robust methods are too conservative, which we demonstrate in a simulation example.

Keywords: data-driven control, non-parametric representation, predictive control for linear systems, stochastic disturbances, uncertain systems

1. Introduction

Model predictive control (MPC) is an optimization-based method for the feedback control of dynamical systems that has been applied with great success (Kabzan et al., 2019; Bujarbaruah et al., 2020; Nubert et al., 2020). Within MPC literature, stochastic MPC (SMPC) has emerged as a framework to systematically incorporate probabilistic descriptions of uncertainties (Mesbah, 2016). Via chance constraints, requiring state or output constraints to be satisfied with a pre-specified probability level, SMPC allows for a systematic trade-off between control performance and constraint satisfaction. This is particularly important for MPC of uncertain systems when optimal performance requires operation in the vicinity of constraints in applications where rare or transient constraint violations are acceptable, such as in building climate control (Oldewurtel et al., 2012), electric grids (Jiang et al., 2019), battery systems (Kumar et al., 2018), process control (Jurado et al., 2015), or finance (Graf Plessen et al., 2019). For safety-critical applications, SMPC allows to provide probabilistic safety guarantees in the presence of uncertainty and the safe use of SMPC is enabled by employing failsafe or robust backup plans (Koller et al., 2018; Wabersich et al., 2021; Brüdigam et al., 2021a; Brüdigam et al., 2021b).

Since closed-loop performance and constraint satisfaction heavily depend on sufficiently accurate predictions, integrating data into MPC schemes promises to improve control where identification
is costly and models are uncertain. In learning-based MPC, most research focuses on safely improving a model of the system during closed-loop operation, by leveraging online measurements to decrease uncertainty, while establishing robust constraint satisfaction with an a priori approximate model with bounds on uncertainty (Hewing et al., 2020), akin to robust MPC (RMPC) in the model-based literature (Bemporad and Morari, 1999; Rawlings et al., 2017). An appealing alternative to improving a model, however, is to directly implement an MPC controller purely from data. Recently, purely data-driven frameworks based on Willems’ fundamental lemma (Willems et al., 2005) have received increasing attention. Informally, the lemma states that for all discrete-time linear time-invariant (LTI) systems, the time-shifted vectors of any input-output trajectory generated by a persistently exciting input signal span the vector space of all input-output trajectories of the system. As a consequence, the system can be represented with only a single measured trajectory and without any further model identification, enabling to solve control and analysis problems directly from data (Markovsky and Rapisarda, 2008; De Persis and Tesi, 2019; van Waarde et al., 2020; Berberich and Allgöwer, 2020). Markovsky and Dörfler (2021) provide a concise and comprehensive recent review.

By replacing the model with trajectory data, one effectively obtains a non-parametric representation of the subspace spanning the system behavior. This behavioral subspace can be directly searched by varying the coefficients of the linear combination of the basis (or library of trajectories, Markovsky and Dörfler, 2021), making the framework naturally well suited for finding optimal future input-output sequences within MPC (Yang and Li, 2015; Coulson et al., 2019; Berberich et al., 2021a). In the face of stochastic disturbances and noisy data, however, achieving (nearly) optimal performance and providing theoretical guarantees for the closed-loop behavior is difficult.

Even though data-driven MPC schemes have been successfully applied in challenging real-world problems (Elokda et al., 2021) and can perform remarkably well even for nonlinear systems (Elokda et al., 2021; Berberich et al., 2021b), the fundamental lemma itself only applies for deterministic LTI systems and exact measurement data. To deal with inconsistent data in practice, the equality within the fundamental lemma can be relaxed to an inequality, with a regularized slack variable (Coulson et al., 2019; Berberich et al., 2021a). However, if the data supposedly spanning the behavioral subspace is perturbed, e.g., by noise, the rank may likely increase, the behavioral subspace is erroneously enlarged, and all prediction guarantees are lost. Coulson et al. (2021) consider systems subject to stochastic disturbances with unknown probability distribution, resulting in a noisy data trajectory. They are able to give strong probabilistic guarantees on out-of-sample performance while enforcing safety via probabilistic constraint satisfaction for the worst-case probability distribution that would explain the data, albeit only the open-loop is considered and closed-loop guarantees such as recursive feasibility are left for future work. Berberich et al. (2021a) presented a data-driven MPC scheme for systems subject to bounded measurement noise with stability and robustness guarantees, providing the first theoretical analysis of closed-loop properties that result from a purely data-driven MPC scheme.

So far, however, works within data-driven MPC have either not considered additive stochastic disturbances in the dynamics, treated disturbances robustly, or have not fully exploited the performance potential of SMPC by not using possibly available knowledge about the probability distribution of the disturbance, which could also be approximated via, e.g., Gaussian processes (Umlauft et al., 2017).

In this work, we present a novel data-driven stochastic MPC algorithm for the performance-oriented control of unknown LTI systems in the presence of additive stochastic disturbances. The algorithm
requires only a single initially measured persistently exciting input-state-disturbance trajectory and leverages chance constraints to efficiently control against disturbances where robust methods are too conservative. In order to combine the deterministic fundamental lemma with stochastic dynamics, we split the state into a nominal part (affected by inputs) and an error part (affected by disturbances). Then, only the deterministic nominal state is used online for prediction and control, while offline probabilistic error predictions allow for the computation of new, tightened constraints from data, guaranteeing the pre-specified chance constraints. As a consequence, our approach allows for a systematic incorporation of probabilistic uncertainties without leaving the deterministic data-driven control framework and without introducing slack variables. If the system is unstable, the inflation of the error state during the prediction horizon will render the satisfaction of (chance) constraints infeasible. We keep the disturbance-induced error small by applying a pre-stabilizing data-driven state feedback, with gains derived from the initially measured trajectory (De Persis and Tesi, 2019).

This work is structured as follows. Section 2 introduces the notation, before the problem is formulated in Section 3. Preliminary results related to the data-driven framework and SMPC are introduced in Section 4, before the data-driven SMPC algorithm is presented in Section 5. Section 6 shows a simulation example and the work closes with a conclusion in Section 7.

2. Notation

We write $I_n$ for the $n \times n$ identity matrix and $0$ for any zero matrix or vector. We abbreviate the weighted 2-norm $\sqrt{z^T Z z}$ by $\|z\|_Z$, the integer sequence $\{a \ldots b\}$ by $\mathbb{N}_b^a$ and let $A \succ 0$ denote a pos. def. matrix. Data sequences of input or state vectors are shortened to $s_{[1,k]} = (s_1, \ldots, s_k)$, while $s_{[1,k]} = [s_1^T, \ldots, s_k^T]^T$ denotes the vector that results from stacking the data. For a given current state $x_k$, we write $x_{l|k}$ for the predicted state $l$ steps ahead. For any sequence of input- or state-vectors $s_{[0,N-1]}$ with length $N$, the corresponding Hankel matrix $H_L(s_{[0,N-1]})$ of order $L < N$ is defined as follows

$$H_L(s_{[0,N-1]}) = \begin{bmatrix} s_0 & s_1 & \cdots & s_{N-L} \\ s_1 & s_2 & \cdots & s_{N-L+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{L-1} & s_L & \cdots & s_{N-1} \end{bmatrix}. \tag{1}$$

3. Problem Formulation

We want to track a reference signal $r_k$ with the state $x_k$ or output $y_k$ of a discrete-time LTI system with unknown system matrices $A$, $B$, and $B_d$ and dynamics

$$x_{k+1} = Ax_k + Bu_k + B_d d_k, \quad \tag{2a}$$
$$y_k = Cx_k + Du_k, \quad \tag{2b}$$

where $x_k \in \mathbb{R}^n$ denotes the state, $u_k \in \mathbb{R}^m$ denotes the input, $y_k$ denotes the output and $d_k \in D \in \mathbb{R}^{m_d}$ denotes a random disturbance sampled from a known probability distribution, with possibly nonzero mean. It is assumed that the realizations of the disturbance $d_k$ are independent and identically distributed. The output equation (2b) is given in its general form here, since the preliminary results in the remainder of this section hold for all systems (2). However, we assume $C = I$ and
In all following sections, making the output $y_k$ equal to the state $x_k$. To solve the tracking control problem, we assume to only have access to one disturbed trajectory $\{x^d_k, u^d_k, d^d_k\}_{k=1}^N$, where $u^d_{[1,N]}$ and $d^d_{[1,N]}$ are assumed to be persistently exciting of order $L + 1$. This assumption is not very restrictive in practice, since $d_k$ is random and we can choose appropriate inputs $u_k$. In our problem setting, we assume that inputs are constrained by a set $U$ and that state constraints, given by a set $X$, must be satisfied up to a probability level specified by a risk parameter, based on the linear constraints

\[ u_k \in U = \{ u \in \mathbb{R}^m \mid G_u u \leq g_u \}, \quad (3a) \]
\[ x_k \in X = \{ x \in \mathbb{R}^n \mid G_x x \leq g_x \}. \quad (3b) \]

4. Preliminaries

In this chapter, we first introduce the preliminary results from data-driven system theory for LTI systems without disturbances, before briefly introducing stochastic model predictive control ideas to deal with stochastic disturbances.

4.1. Trajectory-based Representation of LTI Systems

To represent an LTI system with trajectory data, the data needs to have been generated by a persistently exciting input signal (Willems et al., 2005).

Definition 1 The input sequence $u_{[0,N-1]} \in \mathbb{R}^m$ of length $N$ is persistently exciting of order $L$ if the Hankel matrix $H_L(u_{[0,N-1]})$ has full rank $mL$.

The following result is called fundamental lemma (Willems et al., 2005), which we state in input-state-space (De Persis and Tesi, 2019).

Lemma 2 (Willems et al., 2005; De Persis and Tesi, 2019) Consider an LTI system such as (2) with $d_k = 0$. If the input sequence $u^d_{[0,N-1]}$ is persistently exciting of order $L + 1$, then any $(L + 1)$-long input-state sequence $\{u_{[k,k+L]}, x_{[k,k+L]}\}$ is a valid trajectory of the system if and only if there is an $\alpha \in \mathbb{R}^{N-L}$ such that

\[ \begin{bmatrix} u_{[k,k+L]} \\ x_{[k,k+L]} \end{bmatrix} = \begin{bmatrix} H_{L+1} \left( u^d_{[0,N-1]} \right) \\ H_{L+1} \left( x^d_{[0,N-1]} \right) \end{bmatrix} \alpha. \]

Lemma 2 describes a non-parametric system representation, with $\alpha$ acting as a decision variable to search through all possible trajectories. The following result allows for the construction of a stabilizing state-feedback from input-state data.

Theorem 3 (De Persis and Tesi, 2019) Consider an LTI system such as (2) with $d_k = 0$, let $U = [u^d_0 \cdots u^d_{N-1}]$ be a persistently exciting input sequence, and let $X = [x^d_0 \cdots x^d_{N-1}]$, $X_+ = [x^d_1 \cdots x^d_N]$ be the corresponding state data written into matrices shifted by one, with rank $[U^\top \ X^\top] = n + m$. For any matrix $W \in \mathbb{R}^{N \times n}$ satisfying

\[ XW > 0, \quad \begin{bmatrix} XW \\ W^\top X_+ \\ XW \end{bmatrix} > 0, \]

the state feedback $u_k = K x_k$, with $K = UW (XW)^{-1}$ stabilizes system (2).
4.2. Stochastic Model Predictive Control

To deal with the probabilistic additive disturbance $d_k$ in the system (2a), we employ ideas from SMPC (Lorenzen et al., 2016). In SMPC, typically, nominal state constraints $x_k \in X$ are replaced by chance constraints

$$\Pr (x_k \in X) \geq p, \quad k = 1 \ldots L, \quad (4)$$

with a risk parameter $p \in (0, 1]$ specifying the probability of constraint satisfaction. In order to allow for a tractable deterministic reformulation of the probabilistic chance constraint expression (4), the disturbance $d_k$ in (2a) is assumed to be a realization of an independent and identically distributed (iid) random variable, of which the probability distribution is assumed to be known.

While chance constraints are not suitable if constraint violation is catastrophic, chance constraints lead to less conservative control actions and allow for a deliberate trade-off between performance and constraint satisfaction with the risk parameter $p$. The control becomes more conservative as $p$ approaches 1; a choice $p = 1$ is equivalent to robust constraint satisfaction.

For tractability of the resulting optimal control problem in the predictive control scheme, the chance constraint (4) needs to be reformulated into a deterministic expression. This can be done by decomposing the state into a deterministic (nominal) and a stochastic part. The solver then only works with nominal predictions, which need to satisfy tighter constraints. This constraint tightening can be computed offline, for example as presented in Lorenzen et al. (2016). In Section 5.2, we elaborate on constraint tightening in the data-driven case.

5. Data-Driven Stochastic MPC

In this section, we develop the data-driven stochastic predictive control scheme for discrete-time LTI systems with additive stochastic disturbances. In order to be able to predict future states from past data, we first extend Lemma 2 to the case of disturbed systems.

**Lemma 4 (Extended Fundamental Lemma)** Consider the discrete time LTI system (2). If the input and disturbance sequences $u_{[0,N-1]}^d$ and $d_{[0,N-1]}^d$ are persistently exciting of order $L + 1$, then any $(L + 1)$-long input-disturbance-state trajectory of system (2) can be expressed as

$$\begin{bmatrix} u_{[k,k+L]}^d \\ d_{[k,k+L]}^d \\ x_{[k,k+L]} \end{bmatrix} = \begin{bmatrix} H_u \\ H_d \\ H_x \end{bmatrix} \alpha \quad (5)$$

with $\alpha \in \mathbb{R}^{N - L}$ and the Hankel matrices $H_u = H_{L+1} (u_{[0,N-1]}^d)$, $H_d = H_{L+1} (d_{[0,N-1]}^d)$, and $H_x = H_{L+1} (x_{[0,N-1]}^d)$, according to (1).

**Proof** Lemma 4 follows directly from Lemma 2 by setting the input to $\tilde{u}^d = [u^d \; d^d]^T$ and ordering the rows of the Hankel matrices accordingly.

Note that we need $N \geq (m + m_d + 1)(L + 1) + n - 1$ for given initial state $x_0$, input sequence $u_{[k,k+L]} \in \mathbb{R}^{m(L+1)}$, and disturbances $d_{[k,k+L]} \in \mathbb{R}^{m_d(L+1)}$, such that the system of equations (5) is not overdetermined.
5.1. Data-driven System Representation

Based on the formulation in Lemma 4, we split the prediction of state trajectories into the prediction of disturbance-free, nominal state trajectories \( z_{[0,L]} \), only influenced by inputs \( u_{[0,L]} \), and the prediction of state error trajectories \( e_{[0,L]} \), only influenced by disturbances \( d_{[0,L]} \), i.e.,

\[
x_k = z_k + e_k.
\]  

(6)

This allows us to only use deterministic nominal state predictions in the predictive control scheme, similarly to model-based tube-RMPC (Rawlings et al., 2017) and SMPC (Mesbah, 2016). The state error predictions \( e_k \) are used to tighten the nominal state constraints offline. The underlying dynamics for nominal state and error state are described by \( z_{l+1|k} = Az_{l|k} + Bu_{l|k} \) and \( e_{l+1|k} = Ae_{l|k} + B_d d_{l|k} \), respectively.

To counteract an inflation of the error state due to unstable dynamics in \( A \), the nominal state and state error dynamics are pre-stabilized by decomposing the input \( u_k \) into a stabilizing state feedback component, computed from data (see Theorem 3), and a new decision variable \( v_k \), yielding

\[
u_k = K x_k + v_k.
\]  

(7)

The resulting pre-stabilized system model is given by

\[
z_{l+1|k} = (A + BK) z_{l|k} + B v_{l|k}, \quad e_{l+1|k} = (A + BK) e_{l|k} + B_d d_{l|k}.
\]

These dynamics can be equivalently described by a data-driven system representation derived from Lemma 4: The nominal state trajectories \( z_{[k,k+L]} \) are generated by setting \( d_{[k,k+L]} = 0 \) in (5), assuming the disturbance-free case, and the error state \( e_{[k,k+L]} \) trajectories are generated by setting \( u_{[k,k+L]} = 0 \), assuming zero inputs. Thus, the data-driven representation for the prediction starting from \( x_k \) is split into two systems of equations

\[
\begin{bmatrix}
u_{[0,k,L|k]} \\ 0
\end{bmatrix}
= \begin{bmatrix} H_a - \bar{K}H_x \\ H_d \\ H_x
\end{bmatrix} \alpha_z,
\]  

(8a)

\[
\begin{bmatrix}
0 \\ d_{[0,k,L|k]} \\ e_{[0,k,L|k]}
\end{bmatrix}
= \begin{bmatrix} H_a - \bar{K}H_x \\ H_d \\ H_x
\end{bmatrix} \alpha_e,
\]  

(8b)

where, \( \alpha_z \) acts as decision variable for the online nominal state prediction (8a) in the optimal control problem, while \( \alpha_e \) acts as the decision variable for the offline error state prediction (8b) used for the probabilistic constraint tightening. \( K \) denotes a block-diagonal expansion of the feedback gain \( K \).

5.2. Probabilistic Constraint Tightening

In order to guarantee (chance) constraint satisfaction of the states \( x_k \), we subject the predicted nominal states \( z_k \) to tighter constraints (Mesbah, 2016). With the evolution of the state error depending on the disturbance in (8b), we can directly compute tightened constraints for an \( L \)-step prediction, similar to Lorenzen et al. (2016). The state constraints (3b) are required to hold with probability level \( p \in (0,1) \) for each predicted state \( x_{l|k} \) with \( l \in \mathbb{N}_1^L \), given the current state \( x_{0|k} = x_k \), i.e.,

\[
\Pr ( x_{l|k} \in X \mid x_{0|k} = x_k ) := \Pr ( G_x x_{l|k} \leq g_x \mid x_{0|k} = x_k ) \geq p \forall l \in \mathbb{N}_1^L,
\]  

(9)
For brevity, the conditional dependency \( x_{0|k} = x_k \) is omitted in the rest of this work. In terms of the state decomposition (6), the chance constraint (9) is expressed as
\[
\Pr \left( G_x z_{l|k} \leq g_x - G_x e_{l|k} \right) \geq p \quad \forall l \in \mathbb{N}_1^L.
\] (10)
By introducing a new parameter \( \tilde{\eta} \in \mathbb{R}^{r_x} \), we can split (10) into a deterministic constraint for the nominal predicted state \( z_{l|k} \) and a probabilistic constraint only depending on the predicted state error \( e_{l|k} \), i.e.,
\[
\exists \tilde{\eta} : \quad G_x z_{l|k} \leq \tilde{\eta}, \quad \Pr \left( \tilde{\eta} \leq g_x - G_x e_{l|k} \right) \geq p.
\]
Resulting from this expression, tightened state constraints for the nominal state prediction \( z_{l|k} \in \mathbb{Z}_l \) can be defined, which guarantee the satisfaction of the state constraint (9), where
\[
Z_l = \{ z \in \mathbb{R}^n \mid G_x z \leq \eta_l \}, \quad l \in \mathbb{N}_1^L,
\] (11)
with \( \eta_l \) given by solving
\[
\forall l \in \mathbb{N}_1^L : \quad \eta_l = \max_{\tilde{\eta}} \tilde{\eta} \quad \text{s.t.} \quad \Pr \left( \tilde{\eta} \leq g_x - G_x e_{l|k} \right) \geq p (12)
\]
offline, where the operator \( \max(\cdot) \) is applied element-wise to the multidimensional input vector.
Due to the state feedback component in (7), tightened input constraints also have to be found (Lorenzen et al., 2016). Analogously to (11) and (12), the stochastically tightened input constraint sets are defined as
\[
\forall l = \{ u \in \mathbb{R}^m \mid G_u u \leq \mu_l \}, \quad l \in \mathbb{N}_0^L,
\]
with \( \mu_l \) given by solving
\[
\forall l \in \mathbb{N}_0^{L-1} : \quad \mu_l = \max_{\tilde{\mu}} \tilde{\mu} \quad \text{s.t.} \quad \Pr \left( \tilde{\mu} \leq g_u - G_u Ke_{l|k} \right) \geq p. (13)
\]
For the actual control, the optimal input is recomputed and adapted to the actual disturbance realization, ensuring that applied inputs satisfy input constraints.
In MPC literature, it has been shown that an additional, suitable constraint on the terminal state, i.e., \( z_{L|k} \), can be beneficial to ensure recursive feasibility and stability (Rawlings et al., 2017). For a given (desired) terminal constraint set \( \mathbb{X}_f = \{ x \in \mathbb{R}^n \mid G_x f x \leq g_x, f \} \), the corresponding tightened terminal constraint set \( Z_f \) can be computed similarly to (11) by replacing \( G_x, g_x \) with \( G_{x,f}, g_{x,f} \), and setting \( l = L \). In order to solve the chance-constrained optimization problems (12) and (13), knowledge about the probability distribution of the state error predictions is necessary. For a given initial state error \( e_0 = 0 \) and disturbance sequence \( d_{[0|k, L|k]} \), we can retrieve the state error prediction \( e_{[0|k, L|k]} = H_x \alpha_\varepsilon \) by choosing \( \alpha_\varepsilon \) as
\[
\alpha_\varepsilon = \left[ H_u - \tilde{K} H_x \right]^\dagger \begin{bmatrix} 0 \\ H_d \\ [H_x]_{[1,n]} \end{bmatrix} \begin{bmatrix} \frac{d_{[0|k, L|k]}}{e_0} \\ 0 \end{bmatrix}.
\]
The probability distribution of the state error can then be derived by exploiting the dependence of the prediction on the probabilistic disturbance. Instead of solving the chance-constrained optimization problems (12) and (13) analytically, sampling-based approaches can also be applied (Lorenzen et al., 2016). Sampling techniques are an attractive alternative to analytical methods as they are independent of the underlying disturbance distribution, and state error probability distributions are not required to be determined explicitly. However, chance constraint satisfaction can then only be guaranteed with a predefined level of confidence \( \beta \).
5.3. Data-driven SMPC

Based on the results in the previous sections, we now state the optimal control problem, which is solved at every time-step $k$ with current state $x_k$:

$$
\begin{align*}
\text{minimize} & \quad \sum_{l=0}^{L-1} \left( \|z_{l,k} - r_k\|^2_Q + \|Kz_{l,k} + v_{l,k}\|^2_R \right) + \|z_{L,k}\|^2_P + \lambda_\alpha \|\alpha_k\|^2 \\
\text{s.t.} & \quad z_{0,k} = x_k, \\
& \quad \begin{bmatrix} v_{0,k} & L_k & z_{0,k} & L_k \end{bmatrix} = \begin{bmatrix} H_u - KH_x \\
H_d \\
H_x \end{bmatrix} \alpha_k, \\
& \quad z_{l,k} \in \mathbb{Z}l \quad \forall l \in \mathbb{N}^L, \\
& \quad v_{l,k} + Kz_{l,k} \in \mathbb{V}l \quad \forall l \in \mathbb{N}^{L-1}, \\
& \quad z_{L,k} \in \mathbb{Z}f,
\end{align*}
$$

with prediction horizon $L$, state reference $r_k$, state sequence $Z_k = [z_{0,k} \ldots z_{L,k}]$, input sequence $V_k = [v_{0,k} \ldots v_{L-1,k}]$, and the positive definite weighting matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, and $P \in \mathbb{R}^{n\times n}$ for states, inputs, and terminal state. For practical reasons, we introduce a parameter $\lambda_\alpha > 0$ to penalize large norms of the decision variable $\alpha_k$. This regularization increases the prediction accuracy in case of noisy data, since any noise inside the Hankel matrix (14c) is amplified by $\alpha_k$. The optimal control problem (14) is solved online at each sampling instant and the first element $v_{0,k}^*$ of the optimal control sequence $v_{0,k}^* = H_u \alpha_k^*$ is applied to the system, yielding the state feedback $u_k^* = v_{0,k}^* + Kx_k$.

In summary, the presented probabilistic data-driven predictive control algorithm is split into an offline and online phase: Offline, an input-state-disturbance trajectory with persistently exciting inputs is measured and the bounds for the tightened constraints are computed according to (11),(5.2). Online, the optimal control problem (14) is solved repeatedly to retrieve control inputs for system (2).

6. Simulation Example

We test our method on a double-mass-spring-damper system, shown in Figure 1, and show in this section that our proposed control algorithm is able to track a reference signal similarly to the corresponding SMPC strategy with access to a perfect model in the noise-free case and only marginally worse in the case of noisy data. Additionally, we include the corresponding model-based RMPC
strategy in the results to show how robustly dealing with disturbances can lead to very conservative control actions even in a simple simulation example. We consider a discrete-time version of the system sampled with $\Delta t = 0.1$ s, which takes the form

$$x_{k+1} = \begin{bmatrix} 0.36 & 0.64 & 0.07 & 0.02 \\ 0.42 & 0.58 & 0.02 & 0.07 \\ -9.34 & 9.34 & 0.23 & 0.58 \\ 5.88 & -5.88 & 0.39 & -0.39 \end{bmatrix} x_k + \begin{bmatrix} 0.29 \\ 0.03 \\ 4.90 \\ 1.07 \end{bmatrix} u_k + \begin{bmatrix} 0.03 \\ 0.20 \\ 1.07 \\ 3.48 \end{bmatrix} d_k. \tag{15}$$

The state vector $x = [\theta_1 \theta_2 \omega_1 \omega_2]$ consists of the respective angles and angular velocities of the masses. The input $u = M_u$ and the disturbance $d = M_d$ are the torques affecting mass 1 and mass 2, respectively. For the application of the data-driven stochastic MPC scheme, the system matrices are assumed to be unknown and only data in the form of a single measured trajectory $\{x^d(k), u^d(k), d^d(k)\}_{k=1}^N$ is available. The control objective is to track a step reference. The state is constrained by $-x_{\text{max}} \leq x \leq x_{\text{max}}$, with $x_{\text{max}} = [\pi \pi 2\pi \text{s}^{-1} 2\pi \text{s}^{-1}]^\top$, while the input is constrained by $|u_k| \leq 0.3 \text{Nm}$. The disturbance is sampled randomly from a truncated normal distribution with a mean of 0.1 Nm and a variance of 0.1 Nm, truncated at the bounds of the interval $[-0.2 \text{Nm}, 0.4 \text{Nm}]$. To retrieve the measured trajectory, we simulate the system in open-loop for 5 s, $N = 50$ time-steps, with inputs chosen randomly from the admissible set. We further corrupt this trajectory, as well as future online state measurements, by adding measurement noise $\epsilon_k$ sampled from a truncated zero-mean normal distribution with $\|\epsilon\|_\infty \leq \bar{\epsilon} := 0.1$ and $\bar{\epsilon}/3$ variance. Although the noise is not explicitly considered in the control design and raises theoretical issues, we demonstrate the applicability of our approach in a more realistic scenario.

For the pre-stabilization, the feedback gain $K = [0.11 \ -0.15 \ -0.05 \ -0.03]$ is computed according to Theorem 3 using data of a persistently exciting nominal state prediction of length $L$. For the probabilistic constraint tightening (see Section 5.2), we choose a risk parameter $p = 0.8$ and use a sampling based approach (Lorenzen et al., 2016) to solve the probabilistic optimization problems (12), (13), such that the chance constraint is fulfilled with confidence $\beta = 0.999$. For the optimal control problem, we use a quadratic cost function as defined in (14a), with weighting matrices $Q = \text{diag}(100, 100, 1, 1)$, $R = 1$, and $P = Q$. The norm of the decision variable $x$ is penalized with the weight $\lambda_x = 100$ and we choose a horizon length of $L = 10$ time-steps. Since the regularization of $x$ indirectly penalizes large control values, the same input weighting matrix $R$ does not lead to identical control behavior for data-driven and model-based SMPC. For a fair comparison, we increased the input cost weight $R$ in the model-based case until the control effort is...
Figure 3: Trajectories of 10 runs with noisy offline and online measurements. Exemplary inputs and disturbances are shown for the run with the brightly-colored trajectories.

7. Conclusion

We presented a novel data-driven predictive control strategy that enables performance-oriented control of constrained unknown LTI systems with stochastic disturbances. The strategy uses only past measured data in a non-parametric system representation and does not require any prior model identification. In our case study, the method performs similarly to classical model-based SMPC even if the data for the system representation is corrupted by measurement noise. On the one hand, our proposed approach extends the use of data-driven control to systems subject to known disturbances. On the other hand, the integration of the data-driven framework enables the application of SMPC to control domains where model identification is difficult, costly, or impossible.
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