The Integral Representation of Solutions
to a Boundary Value Problem on the Half-Line
for a System of Linear ODEs
with Singularity of First Kind

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Abstract

We consider a problem of finding vanishing at infinity $C^1([0, \infty))$-solutions to non-homogeneous system of linear ODEs which has the pole of first order at $x = 0$. The resonant case where the corresponding homogeneous problem has nontrivial solutions is of main interest. Under the conditions that the homogeneous system is exponentially dichotomic on $[1, \infty)$ and the residue of system’s operator at $x = 0$ does not have eigenvalues with real part 1, we construct the so-called generalized Green function. We also establish conditions under which the main non-homogeneous problem can be reduced to the Noetherian one with nonzero index.

1 Introduction

In the space $\mathbb{R}^n$ endowed with a scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ the following linear singular system is considered:

$$y' = \left( \frac{A}{x} + B(x) \right) y + \frac{a}{x} + f(x).$$

\(\text{1} \quad (*)\)
Here $A \in \text{Hom}(\mathbb{R}^n)$ is a linear operator, $a \in \mathbb{R}^n$ is a constant vector, $B(\cdot) : [0, \infty) \mapsto \text{Hom}(\mathbb{R}^n)$ and $f(\cdot) : [0, \infty) \mapsto \mathbb{R}^n$ are continuous bounded mappings for which there exists a constant $M > 0$ such that $\|B(x)\| \leq M$ and $\|f(x)\| \leq M$ for all $x \in [0, \infty)$.
(The norm of a linear operator in $\mathbb{R}^n$ is considered to be concordant with the norm in $\mathbb{R}^n$.)

We seek a solution $y(x)$ of the system (1) which satisfies the following conditions:

$$y(\cdot) \in C^1([0, \infty) \mapsto \mathbb{R}^n), \quad y(+\infty) = 0. \quad (2)$$

The stated problem belongs to the class of singular ones on account of both having a singularity at the point $x = 0$ and unboundedness of the interval where the independent variable is defined. The problems of such a kind often arise when constructing and investigating solutions of various equations of mathematical physics. Majority of papers devoted to study of such problems deal with second and higher order equations (see e.g. [1]–[12]). Despite the fact that corresponding bibliography amounts to several hundreds of titles, we failed to find a ready-made procedure for establishing existence conditions and integral representation of solutions to the problem (1)–(2).

The necessity of such representation naturally arises when solving the problem about perturbations of solutions to singular non-linear boundary value problems on the semi-axis [13, 14].

While considering the above problem, we did not exclude the so-called resonance case when the corresponding homogeneous problem has non-trivial solutions. In this connection results of papers [15]–[22] should be mentioned, which are devoted to the problem of existence of solutions to linear non-homogeneous systems bounded on the entire axis, in particular, extension of Fredholm and Noether theory over such systems. It should be noted that in papers [23]–[25] the authors find quite general sufficient conditions for boundary value problems on a finite interval with non-integrable singularities to have the Fredholm property with index zero.

The present paper is organized as follows. Section 2 contains an auxiliary result about the structure of a fundamental operator of a linear homogeneous system with continuous (however non-analytic) coefficients on the interval $(0, x_0)$ and singular point of the first kind at $x = 0$. In section 3 we describe additional conditions imposed on the linear homogeneous system and classify its solutions in accordance with their asymptotical behavior when $x \to +0$ and $x \to +\infty$. In section 4 the existence criterion for the solution to a boundary value problem with homogeneous boundary conditions is established and the Green function for this problem is constructed. Finally, in section 5 the main result is stated — the theorem about existence and integral representation of solutions to the problem (1)–(2).
2 The structure of the fundamental operator of linear system near the singular point of first kind

Consider the linear homogeneous system associated with (1):
\[
y' = \left( \frac{A}{x} + B(x) \right) y.
\] (3)

In the analytical theory of differential equations the structure of the fundamental operator of the system (3) is completely investigated under the assumption that the mapping \( B(\cdot) \) is holomorphic in the neighborhood of the singular point \( x = 0 \) (see e.g. [26]). In the case where \( B(\cdot) \) is continuous only, the following proposition which is a simple modification of the result stated in [16, p. 275] holds true.

**Proposition 1.** There exist numbers \( x_0 \in (0, \infty) \), \( K > 0 \), and \( r > 0 \) such that the fundamental operator of the system (3) admits the representation in the form
\[
Y(x) = (E + U(x)) x^A, \quad x \in (0, x_0],
\] (4)
where \( E \in \text{Hom}(\mathbb{R}^n) \) is a unit operator and the mapping \( U(\cdot) \in C^1((0, \infty) \mapsto \text{Hom}(\mathbb{R}^n)) \) satisfies the estimate
\[
\|U(x)\| \leq Kx |\ln x|^r, \quad x \in (0, x_0].
\]

**Proof.** The mapping \( Y(\cdot) : (0, x_0] \mapsto \text{Hom}(\mathbb{R}^n) \) defined by (4) is a fundamental operator of the system (3) if \( U(x) \) satisfies the equation
\[
U' = \frac{1}{x} (AU - UA) + B(x) (E + U), \quad x \in (0, x_0].
\]
After the substitution \( x = e^{-t} \) we obtain the following equation for the operator \( V(t) := U(e^{-t}) \):
\[
\dot{V} = VA - AV - e^{-t} B(e^{-t}) (E + V).
\] (5)
Thus we are to find the solution to this equation which satisfies the inequality
\[
\|V(t)\| \leq Kt^r e^{-t}, \quad t \in [t_0, \infty)
\]
for a certain value of \( t_0 > 0 \).

The equation (5) can be identified in \( \mathbb{R}^{n^2} \) with the system of the form
\[
\dot{v} = Av + e^{-t} (H(t)v + h(t))
\] (6)
where \( A \in \text{Hom}(\mathbb{R}^{n^2}) \) is a constant operator and the mappings \( H(\cdot) \in C([t_0, \infty) \mapsto \text{Hom}(\mathbb{R}^{n^2})) \) and \( h(\cdot) \in C([t_0, \infty) \mapsto \mathbb{R}^{n^2}) \) satisfy the inequalities \( \|H(t)\| \leq M \), \( \|h(t)\| \leq M \) for any \( t \in [t_0, \infty) \).

Now the required result can be obtained as an obvious consequence of Lemma 1 and Lemma 2 stated below. \[\square\]
Lemma 1. Let \( A \in \text{Hom}(\mathbb{R}^N) \). Then there exists a mapping \( G_A(\cdot) \in C^\infty(\mathbb{R} \to \text{Hom}(\mathbb{R}^N)) \) such that for any function \( f(t) \in C([t_0, \infty) \to \mathbb{R}^N) \) satisfying the estimate
\[
\|f(t)\| \leq M_f e^{-t}, \ t \in [t_0, \infty),
\]
with some constant \( M_f > 0 \) the system
\[
\dot{y} = Ay + f(t)
\]
possesses a bounded on the semi-axis \([t_0, \infty)\) solution of the form
\[
y(t) = \int_{t_0}^{\infty} G_A(t-s)f(s)ds.
\]
This solution satisfies the inequality
\[
\|y(t)\| \leq C_A M_f e^{-t} (1 + (t-t_0)^r)
\]
where \( C_A \) is a positive constant depending on \( A \) only and \( r \) is the maximal dimension of Jordan blocks corresponding to eigenvalues with the real part equal to \(-1\) in the normal form matrix of the operator \( A \).

If, in addition, \( f(t) = o(e^{-t}), \ t \to \infty \), then the solution \( y(t) \) has the property \( y(t) = o(e^{-t^r}), \ t \to \infty \).

Proof. We give the proof of the first part of the Proposition for the case where \( r \geq 1 \). Note that there exist three projectors \( P_i : \mathbb{R}^N \to \mathbb{R}^N, i = 1,3 \), such that \( P_i P_k = 0, i \neq k, P_1 + P_2 + P_3 = E \), and for some constants \( K_A > 0, \gamma_1 > -1, \gamma_2 < -1 \) the following inequalities hold true:
\[
\|e^{A\tau} P_1\| \leq K_A e^{\gamma_1 \tau}, \tau \leq 0,
\]
\[
\|e^{A\tau} P_2\| \leq K_A (1 + \tau^{r-1}) e^{-\gamma_1 \tau}, \tau \geq 0,
\]
\[
\|e^{A\tau} P_3\| \leq K_A e^{\gamma_2 \tau}, \tau \geq 0.
\]

Now we define a function \( G_A(\tau) \) as follows:
\[
G_A(\tau) = \begin{cases} 
-e^{A\tau} P_1, & \tau \leq 0, \\
e^{A\tau} (P_2 + P_3), & \tau > 0.
\end{cases}
\]

The function
\[
y(t) := \int_{t_0}^{\infty} G_A(t-s)f(s)ds
\]
\[
\equiv \int_{t_0}^{t} e^{A(t-s)} P_2 f(s)ds + \int_{t_0}^{t} e^{A(t-s)} P_3 f(s)ds - \int_{t_0}^{\infty} e^{A(t-s)} P_1 f(s)ds
\]

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is well defined and there exists a constant $C_A > 0$ dependent on the operator $A$ only such that

$$\| y(t) \| \leq K_A M_f \left( \int_{t_0}^{t} (1 + (t - s)^r - 1) e^{-(t-s)} e^{-s} ds + \int_{t_0}^{\infty} e^{\gamma_2(t-s)} e^{-s} ds + \int_{t}^{\infty} e^{\gamma_1(t-s)} e^{-s} ds \right)$$

$$\leq K_A M_f e^{-t} \left( (t - t_0) + \frac{(t - t_0)^r}{r} + \frac{1 - e^{-(|\gamma_2| - 1)(t-t_0)}}{|\gamma_2| - 1} + \frac{1}{\gamma_1 + 1} \right)$$

$$\leq C_A M_f e^{-t} \left( 1 + (t - t_0)^r \right).$$

Therefore, for $y(t)$ the inequality (7) holds true. One can easily make sure by the direct check that this function is in fact the solution to the system (8).

Now let $f(t) = o(e^{-t})$, $t \to \infty$. Then for an arbitrary $\epsilon > 0$ one can choose $T(\epsilon) > t_0$ in such a way that $\| f(t) \| \leq \epsilon e^{-t}$ for $t \geq T(\epsilon)$. Represent the solution $y(t)$ in the form

$$y(t) = \int_{t_0}^{T(\epsilon)} G_A(t - s) f(s) ds + \int_{t}^{\infty} G_A(t - s) f(s) ds.$$

In accordance with what has been proved above, the norm of the second addend does not exceed $C_A \epsilon e^{-t} (1 + (t - T(\epsilon))^r)$ for any $t \geq T(\epsilon)$. For the first addend, when $t \geq T(\epsilon)$ we have:

$$\int_{t_0}^{T(\epsilon)} G_A(t - s) f(s) ds = \int_{t_0}^{T(\epsilon)} e^{A(t-s)} P_2 f(s) ds + \int_{t_0}^{T(\epsilon)} e^{A(t-s)} P_3 f(s) ds.$$

If $r = 0$, then $P_2 = 0$ and

$$\left\| \int_{t_0}^{T(\epsilon)} G_A(t - s) f(s) ds \right\| = O(e^{\gamma_2 t}) = o(e^{-t}), \quad t \to \infty.$$

If $r > 0$, then

$$\left\| \int_{t_0}^{T(\epsilon)} G_A(t - s) f(s) ds \right\| = O(e^{-t}) = o(e^{-t} r), \quad t \to \infty.$$
Lemma 2. Assume that $H(\cdot) \in C([t_0, \infty) \to \text{Hom}(\mathbb{R}^N))$, $h(\cdot) \in C([t_0, \infty) \to \mathbb{R}^N))$, and that there exist constants $M > 0$, $m > 0$ such that $\|H(t)\| \leq M$, $\|h(t)\| \leq m$ for any $t \geq t_0$. Let $C_A$ and $r$ be the numbers defined in Lemma 1. If the inequalities

$$t_0 > r, \quad q := 2C_A M t_0^r e^{-t_0} < 1$$

hold true, then the system (7) has a solution $v(t)$ such that

$$\|v(t)\| \leq 2C_A m t_0^r e^{-t}, \quad t \geq t_0.$$

If, in addition, $h(t) \to 0$, $t \to \infty$, then $v(t) = o(t^r e^{-t})$, $t \to \infty$.

Proof. In view of the Lemma 1, we are going to find the solution to the system (6) satisfying the integral equation

$$v(t) = \int_{t_0}^{\infty} G_A(t-s)e^{-s}(H(s)v(s) + h(s))ds.$$  \hspace{1cm} (10)

Denote

$$\mathcal{G}[v(\cdot)](t) := \int_{t_0}^{\infty} G_A(t-s)e^{-s}(H(s)v(s) + h(s))ds$$

and define the space of functions

$$\mathcal{M}_{t_0,C} := \{v(t) \in C([t_0, \infty) \to \mathbb{R}^N) : \|v(t)\| \leq Ct^r e^{-t}, t \geq t_0\}.$$ 

Let us show that if (9) holds true, then it is possible to choose the constant $C > 0$ in such a way that $\mathcal{G} : \mathcal{M}_{t_0,C} \mapsto \mathcal{M}_{t_0,C}$ and this mapping is a contraction in the uniform metric.

The Lemma 1 implies that

$$\|\mathcal{G}[v(\cdot)](t)\| \leq C_A (M \sup_{t \geq t_0} (Ct^r e^{-t}) + m) e^{-t} (1 + (t - t_0)^r)$$

$$\leq 2C_A (MCt_0^r e^{-t_0} + m) t^r e^{-t}, \quad t_0 > r,$$

for any function $v(t) \in \mathcal{M}_{t_0,C}$. Besides, when $t_0 > r$, for any $v(t)$, $u(t) \in \mathcal{M}_{t_0,C}$ we obtain:

$$\|\mathcal{G}[v(\cdot) - u(\cdot)](t)\| \leq C_A M e^{-t} (1 + (t - t_0)^r) \sup_{t \geq t_0} \|v(t) - u(t)\|$$

$$\leq 2C_A M t_0^r e^{-t_0} \sup_{t \geq t_0} \|v(t) - u(t)\| = q \sup_{t \geq t_0} \|v(t) - u(t)\|,$$
Since \( q < 1 \), it is clear that \( G \) is a contraction mapping on \( M_{t_0, C} \) once the following inequality holds true:

\[
2C_A(MCt_0^q e^{-t_0} + m) \leq C.
\]

Hence, by setting

\[
C := \frac{2C_A m}{1 - q}
\]

we guarantee the existence of a unique solution \( v(t) \in M_{t_0, C} \) to the equation (10).

Now, suppose, in addition, that \( h(t) \to 0, \ t \to \infty \). Since the solution \( v(t) \) can be represented in the form

\[
v(t) = \int_{t_0}^{\infty} G_A(t - s)f(s)ds
\]

where \( f(t) = e^{-t}(H(t)v(t) + h(t)) = o(e^{-t}), \ t \to \infty \), then in accordance with the Lemma 1 we obtain: \( v(t) = o(t^\alpha e^{-t}), \ t \to \infty \).

3 Additional conditions for the linear homogeneous system and their corollaries

Hereafter we assume that for the linear homogeneous system (3) conditions A, B described below hold true. These conditions concern local properties of the system in neighborhoods of the points \( x = 0 \) and \( x = +\infty \).

A: the characteristic polynomial of the operator \( A \) has no roots with real part equal to 1;

B: the system (3) is exponentially dichotomic on the semi-axis \([x_0, \infty)\) for some (and therefore, for any) positive \( x_0 \).

Let \( y(x, y_0) \) be a solution to the system (3) satisfying the initial condition \( y(x_0, y_0) = y_0 \). For the sake of generality we assume that the characteristic polynomial of the operator \( A \) has roots with real parts both less and greater than 1 and the system (3) has both bounded and unbounded solutions on the half-line \([x_0, \infty)\).

Under the conditions A and B there exist subspaces \( V_+ \) and \( U_- \) with the following properties:

1. There exists \( \alpha > 0 \) such that for any subspace \( V_- \) which is a direct supplement of \( V_+ \) to \( \mathbb{R}^n \) one can choose a constant \( c_0 > 0 \) in such a way that

\[
\|y(x, y_0)\| \leq c_0 \left( \frac{x}{s} \right)^{1+\alpha} \|y(s, y_0)\|, \quad 0 < x \leq s \leq x_0, \quad \text{if} \ y_0 \in V_+; \quad (11)
\]

\[
\|y(x, y_0)\| \leq c_0 \left( \frac{x}{s} \right)^{1-\alpha} \|y(s, y_0)\|, \quad 0 < s \leq x \leq x_0, \quad \text{if} \ y_0 \in V_-.
\]

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(This property results from the Proposition 1 and the condition A.)

2. There exists a constant \( \gamma > 0 \) such that for any subspace \( U_+ \) which is a direct supplement of \( U_- \) to \( \mathbb{R}^n \) one can choose a constant \( c_* > 0 \) in such a way that

\[
\| y(x, y_0) \| \leq c_* e^{-\gamma(x-s)} \| y(s, y_0) \|, \quad x_0 \leq s \leq x, \quad \text{if } y_0 \in U_-; \quad \text{and} \quad \| y(x, y_0) \| \leq c_* e^{\gamma(x-s)} \| y(s, y_0) \|, \quad x_0 \leq x \leq s, \quad \text{if } y_0 \in U_+. \quad (13)
\]

(See remark 3.4 in [16, p. 235].)

If the subspace \( \ker A \) is non-trivial, then there exists a subspace \( V_0^- \) isomorphic to the subspace \( \ker A \) and having the next property:

3. For any \( y_* \in V_0^- \) there exists a unique vector \( \zeta \in \ker A \) such that

\[
y(x, y_*) = (E + \Theta(x)) \zeta, \quad x \to +0,
\]

where \( \Theta(\cdot) \in C^1([0, x_0] \to \text{Hom}(\mathbb{R}^n)) \) and \( \Theta(x) = x(E-A)^{-1}B(0) + o(x), \quad x \to +0. \) At the same time, \( V_0^- \cap V_+ = \{0\} \) and the subspace \( V_+ \oplus V_0^- \) coincides with the subspace of initial values (for \( x = x_0 \)) of continuously differentiable on \([0, \infty)\) solutions to the system (13). (See the corollary from the Proposition 2 which is stated in section 5.)

Now the space \( \mathbb{R}^n \) can be represented as the direct sum of six subspaces \( L_1, \ldots, L_6 \) defined in the following way:

1) \( L_1 := U_- \cap V_+; \)

2) \( L_2 \) is a direct supplement of the subspace \( L_1 \) to \( U_- \cap (V_+ \oplus V_0^-) \), so that

\[
L_1 \oplus L_2 = U_- \cap (V_+ \oplus V_0^-);
\]

3) \( L_3 \) is a direct supplement of the subspace \( U_- \cap (V_+ \oplus V_0^-) \) to \( U_- \), so that

\[
L_1 \oplus L_2 \oplus L_3 = U_-;
\]

4) \( L_4 \) is a direct supplement of the subspace \( L_1 = U_- \cap V_+ \) to \( V_+ \), so that

\[
V_+ = L_1 \oplus L_4;
\]

5) \( L_5 \) is a direct supplement of the subspace \( (U_- \cap (V_+ \oplus V_0^-)) \oplus L_4 \) to \( V_+ \oplus V_0^- \), so that

\[
L_1 \oplus L_2 \oplus L_4 \oplus L_5 = V_+ \oplus V_0^-;
\]

and taking into account the equalities \( (L_1 \oplus L_4) \cap V_0^- = \{0\} \) and \( \dim L_2 + \dim L_5 = \dim V_0^- \) we choose \( L_5 \subset V_0^-; \)

6) \( L_6 \) is a direct supplement of the subspace \( L_1 \oplus \cdots \oplus L_5 = U_- \oplus L_4 \oplus L_5 \) to \( \mathbb{R}^n. \)
If the two subspaces $U_+$ and $V_-$, which are direct supplements of the subspaces $U_-$ and $V_+$ respectively, are defined by the equalities

\[ U_+ := L_4 \oplus L_5 \oplus L_6, \quad V_- := L_2 \oplus L_3 \oplus L_5 \oplus L_6, \]

then the above assumptions allow us to distinguish six types of solutions to the system (3). Namely: if $y_0 \in L_1$, then the solution $y(x, y_0)$ satisfies the inequalities (11) and (13); the solution for which $y_0 \in L_2$ fulfills the inequality (13), and there exists unique $y_* \in V_0$ such that

\[ \|y(x, y_0) - y(x, y_*)\| = o(x), \quad x \to 0; \]

the solution for which $y_0 \in L_3$ satisfies the inequalities (12) and (13), besides, for this solution the derivative $y'(+0; y_0)$ does not exist; for the solution with $y_0 \in L_4$ the inequalities (11) and (13) hold true; the solution having initial value from $L_5$ fulfills the inequality (14), and there is a unique $\zeta \in \ker A$ for which (15) is valid; finally, if $y_0 \in L_6$, then the solution $y(x, y_0)$ satisfies inequalities (12) and (14), and for such a solution the derivative $y'(+0; y_0)$ does not exist.

Let $E = P_1 + \cdots + P_6$ be the decomposition of the unit operator into the sum of mutually disjunctive projectors generated by the decomposition $\mathbb{R}^n = L_1 \oplus \cdots \oplus L_6$. Define the following operators:

\[ Q_+ := P_1 + P_4, \quad Q_- := P_2 + P_3 + P_5 + P_6, \]
\[ P_- := P_1 + P_2 + P_3, \quad P_+ := P_4 + P_5 + P_6. \]

It is clear that the projectors $Q_+, \ Q_-$ correspond to the decomposition $\mathbb{R}^n = V_+ \oplus V_-$, while $P_-, \ P_+$ correspond to the decomposition $\mathbb{R}^n = U_- \oplus U_+$, and there exist constants $C_0 > 0$ and $C_\gamma > 0$ such that for the normalized at the point $x_0$ evolution operator $Y(x; x_0)$ of the system (3) the following estimates are valid:

\[ \|Y(x; x_0)Q_+ Y^{-1}(s; x_0)\| \leq C_0 \left(\frac{x}{s}\right)^{1+\alpha}, \quad 0 < x \leq s \leq x_0, \tag{16} \]
\[ \|Y(x; x_0)Q_- Y^{-1}(s; x_0)\| \leq C_0 \left(\frac{x}{s}\right)^{1-\alpha}, \quad 0 < s \leq x \leq x_0, \tag{17} \]

and

\[ \|Y(x; x_0)P_- Y^{-1}(s; x_0)\| \leq C_\gamma e^{-\gamma(x-s)}, \quad x_0 \leq s \leq x, \tag{18} \]
\[ \|Y(x; x_0)P_+ Y^{-1}(s; x_0)\| \leq C_\gamma e^{-\gamma(s-x)}, \quad x_0 \leq x \leq s. \tag{19} \]
4 Generalized Green function for the boundary value problem with homogeneous boundary conditions

Consider the boundary value problem of the form

\[ y' = \left( \frac{A}{x} + B(x) \right) y + g(x), \quad (20) \]

\[ y(\cdot) \in C^1([0, \infty) \mapsto \mathbb{R}^n), \quad y(+0) = 0, \quad y(+\infty) = 0, \quad (21) \]

in the case of function \( g(\cdot) \in C([0, \infty) \mapsto \mathbb{R}^n) \) vanishing at infinity: \( g(x) \to 0 \) when \( x \to +\infty \). Let \( m := \sup_{x \in [0, \infty)} \| g(x) \| \).

First, we prove that any element of \( \ker A \) can be brought into correspondence with at least one solution which is continuously differentiable on \([0, \infty)\).

**Proposition 2.** Under the condition \( A \), for any \( \zeta \in \ker A \) there exists a solution to the system (20) of the form

\[ y_{\zeta}(x) = \zeta + \zeta_1 x + o(x), \quad x \to +0, \quad (22) \]

where \( \zeta_1 := (E - A)^{-1}(B(0)\zeta + g(0)) \). Conversely, every continuously differentiable on \([0, \infty)\) solution to the system (20) can be represented in the form (22).

**Proof.** The change of dependent variable \( y = \zeta + \zeta_1 x + z \) in (20) leads to the system

\[ z' = \left( \frac{A}{x} + B(x) \right) z + \tilde{g}(x) \]

where \( \tilde{g}(x) = (B(x) - B(0))\zeta + g(x) - g(0) + xB(x)\zeta_1 = o(1), \ x \to +0. \) After the substitution \( x = e^{-t} \) we obtain the system

\[ \dot{z} = -(A + e^{-t}B(e^{-t}))z - e^{-t}\tilde{g}(e^{-t}). \quad (23) \]

The value \( t_0 > 0 \) can be chosen sufficiently large, so that the conditions of Lemma 2 hold true for this system. In accordance with this Lemma and taking into account that the characteristic polynomial of the operator \(-A\) has no roots with the real part equal to \(-1\), there exists the solution \( \tilde{z}(t) \) to the system (23) satisfying the equality

\[ \tilde{z}(t) = -\int_{t_0}^{\infty} G_{-A}(t-s)e^{-s}(B(e^{-s})\tilde{z}(s) + \tilde{g}(e^{-s})) \, ds \]

and, besides, having the property \( \tilde{z}(t) = o(e^{-t}), \ t \to \infty. \) But in such a case the function \( z(x) := \tilde{z}(-\ln x) = o(x), \ x \to 0, \) generates the required solution \( y(x) = \zeta + \zeta_1 x + z(x) \) of the system (20). The second part of the Proposition is obvious.
Corollary 1. There exists a mapping $\Theta(\cdot) \in C^1([0,x_0] \mapsto \text{Hom}(\mathbb{R}^n))$ of the form $\Theta(x) = x(E - A)^{-1}B(0) + o(x)$, $x \rightarrow +0$, such that for any $\zeta \in \ker A$ the function $y_\zeta(x) = (E + \Theta(x))\zeta$ is a solution to the homogeneous system \((\ref{23})\) corresponding to the vector $\zeta$.

Proposition 3. The family of functions defined as

$$
\bar{y}_v(x) = Y(x;x_0)v + \int_0^x Y(x;x_0)Q_-Y^{-1}(s;x_0)g(s)ds
$$

$$
+ \int_{x_0}^x Y(x;x_0)Q_+Y^{-1}(s;x_0)g(s)ds,
$$

where $v$ is an arbitrary vector from $\mathbb{V} + \mathbb{V}^0$, determines all solutions to the system \((\ref{24})\) of the class $C^1([0,\infty) \mapsto \mathbb{R}^n)$. Each of such solutions satisfies the condition $\bar{y}_v(+0) = 0$ iff $v \in \mathbb{L}_1 \oplus \mathbb{L}_4 = \mathbb{V}_+$.  

Proof. In view of the estimates \((\ref{16}), (\ref{17})\), for any $x \in [0,x_0)$ the integrals in the formula \((\ref{24})\) satisfy the inequalities

$$
\left\| \int_0^x Y(x;x_0)Q_-Y^{-1}(s;x_0)g(s)ds \right\| \leq mC_0x^{1-\alpha} \int_0^x s^{\alpha-1}ds = mC_0 \frac{x}{\alpha};
$$

$$
\left\| \int_{x_0}^x Y(x;x_0)Q_+Y^{-1}(s;x_0)g(s)ds \right\| \leq mC_0x^{1+\alpha} \int_x^0 s^{-1-\alpha}ds \leq mC_0 \frac{x^{1+\alpha}(x^{-\alpha} - x_0^{-\alpha})}{\alpha} \leq mC_0 \frac{x}{\alpha}.
$$

By means of direct check, one can easily verify that each function of the set \((\ref{24})\) is a solution to the system \((\ref{24})\). From the definition of $\mathbb{V}_+ \oplus \mathbb{V}^0$, properties of the spaces $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_4, \mathbb{L}_5$ (see p. 9) it follows that for any $v \in \mathbb{V}_+ \oplus \mathbb{V}^0$ there exists a limit $\lim_{x \to +0} Y(x;x_0)v = : \zeta(v) \in \ker A$, and $Y(x;x_0)v = \zeta(v) + O(x)$, $x \to 0$. Therefore, $\bar{y}_v(x) = \zeta(v) + O(x)$, $x \to 0$, and the difference $\bar{y}_v(x) - y_\zeta(x)$, where $y_\zeta(x)$ is the solution from the Proposition 2 is a solution to the system \((\ref{3})\). Moreover, $\|\bar{y}_v(x) - y_\zeta(x)\| = O(x)$, $x \to 0$. This implies that $\|\bar{y}_v(x) - y_\zeta(x)\| = o(x)$, $x \to 0$, and thus, $\bar{y}_v(x_0) - y_\zeta(x_0) \in \mathbb{L}_1 \oplus \mathbb{L}_4$. Taking into account the Proposition 3 we can conclude that $\bar{y}_v(x) \in C^1([0,x_0] \mapsto \mathbb{R}^n)$.

Since each non-trivial solution to the system \((\ref{3})\) with the initial condition $y_0 \in \mathbb{L}_2 \oplus \mathbb{L}_5$ has a non-zero limit when $x \to +0$, the equality $\bar{y}_v(+0) = 0$ is equivalent to $v \in \mathbb{L}_1 \oplus \mathbb{L}_4$. \qed
It is well known (see e.g. [16]) that all solutions to the system (20) which are bounded on the semi-axis \([x_0, \infty)\) form a family

\[
\hat{y}_u(x) = Y(x; x_0)u + \int_{x_0}^{x} Y(x; x_0)P_\gamma Y^{-1}(s; x_0)g(s)\,ds
- \int_{x}^{\infty} Y(x; x_0)P_\gamma Y^{-1}(s; x_0)g(s)\,ds
\]

where \(u\) is an arbitrary vector from \(U_-\).

It is also known that the following proposition holds true:

**Proposition 4.** If \(g(x) \to 0, x \to \infty\), then \(\hat{y}_u(x) \to 0, x \to \infty\).

**Proof.** For the sake of completeness let us sketch the proof.

For an arbitrary \(\epsilon > 0\) let choose the value \(x(\epsilon) > x_0\) in such a way that \(\|g(x)\| < \epsilon\) for any \(x > x(\epsilon)\). Then for \(x > x(\epsilon)\) we have

\[
\hat{y}_u(x) = Y(x; x_0)u + \int_{x_0}^{x(\epsilon)} Y(x; x_0)P_\gamma Y^{-1}(s; x_0)g(s)\,ds
+ \int_{x(\epsilon)}^{x} Y(x; x_0)P_\gamma Y^{-1}(s; x_0)g(s)\,ds + \int_{x}^{\infty} Y(x; x_0)P_\gamma Y^{-1}(s; x_0)g(s)\,ds.
\]

The first addend in this expression tends to zero when \(x \to \infty\), norm of each of the last two addends does not exceed \(\epsilon K/\gamma\), and for the second addend it holds

\[
\left\| \int_{x_0}^{x(\epsilon)} Y(x; x_0)P_\gamma Y^{-1}(s; x_0)g(s)\,ds \right\| = O(e^{-\gamma x}), \quad x \to \infty.
\]

\(\square\)

Now, to find all solutions to the system (20) which satisfy the conditions (21) we bind parameters \(v \in L_1 \oplus L_4\) and \(u \in L_1 \oplus L_2 \oplus L_3\) by means of the equality \(\tilde{y}_v(x_0) = \hat{y}_u(x_0)\) which can be rewritten in the form

\[
P_- u - Q_+ v = \int_{x_0}^{x} Q_- Y^{-1}(s; x_0)g(s)\,ds + \int_{x_0}^{\infty} P_+ Y^{-1}(s; x_0)g(s)\,ds
\]

or, equivalently,

\[
(P_1 + P_2 + P_3)u - (P_1 + P_4)v
= \int_{0}^{x_0} (P_2 + P_3 + P_5 + P_0)Y^{-1}(s; x_0)g(s)\,ds
+ \int_{x_0}^{\infty} (P_4 + P_5 + P_6)Y^{-1}(s; x_0)g(s)\,ds.
\]
From this it follows that

\[ P_1 u = P_1 v, \quad P_2 u = \int_{x_0}^{x} P_2 Y^{-1}(s; x_0)g(s) \, ds, \]
\[ P_3 u = \int_{x_0}^{x} P_3 Y^{-1}(s; x_0)g(s) \, ds, \quad P_4 v = -\int_{x_0}^{\infty} P_4 Y^{-1}(s; x_0)g(s) \, ds, \]

and the function \( g(x) \) must satisfy the additional condition

\[ \int_{x_0}^{\infty} (P_3 + P_4) Y^{-1}(s; x_0)g(s) \, ds = 0. \]  \hfill (25)

Therefore, if the condition (25) holds true, the solutions to the problem (20)–(21) can be given by the formula

\[
y = Y(x; x_0) \left( P_1 v + \int_{x_0}^{x} P_1 Y^{-1}(s; x_0)g(s) \, ds \right.
\]
\[ + \int_{x_0}^{x} (P_2 + P_3) Y^{-1}(s; x_0)g(s) \, ds
\]
\[ \left. - \int_{x}^{\infty} (P_4 + P_5 + P_6) Y^{-1}(s; x_0)g(s) \, ds \right). \]  \hfill (26)

This formula can also be rewritten in the following way:

\[
y = Y(x; x_0) \left( P_1 v - \int_{x_0}^{\infty} P_4 Y^{-1}(s; x_0)g(s) \, ds \right.
\]
\[ + \int_{x_0}^{x} Q_+ Y^{-1}(s; x_0)g(s) \, ds + \int_{x}^{\infty} Q_- Y^{-1}(s; x_0)g(s) \, ds \right). \]

Having defined the sets

\[ D := \{(x, s) : 0 < x < s < x_0\} \cup \{(x, s) : x_0 \leq s \leq x\}, \]
\[ D_+ := \{(x, s) : 0 < s \leq x\}, \quad D_- := \{(x, s) : 0 < x < s\} \]

and the functions

\[ G_1(x, s) := \begin{cases} Y(x; x_0)P_1 Y^{-1}(s; x_0), & (x, s) \in D \cap D_+, \\ -Y(x; x_0)P_1 Y^{-1}(s; x_0), & (x, s) \in D \cap D_-, \\ 0, & (x, s) \in (D_+ \cup D_-) \setminus D, \end{cases} \]
\[ G_2(x, s) := \begin{cases} Y(x; x_0)(P_2 + P_3) Y^{-1}(s; x_0), & (x, s) \in D_+, \\ -Y(x; x_0)(P_4 + P_5 + P_6) Y^{-1}(s; x_0), & (x, s) \in D_-, \end{cases} \]
\[ G(x, s) := G_1(x, s) + G_2(x, s), \]

and taking into account the formula (26) we get the following result.
Proposition 5. There exists a solution to the boundary value problem (20)–(21) iff the condition (25) holds true, and in this case all solutions to the problem are defined by the formula

\[ y = Y(x; x_0)v + \int_0^\infty G(x, s)g(s)\,ds, \quad \forall v \in L_1. \]

Now we are going to interpret the condition (25) in terms of solutions to the adjoint (with respect to the scalar product \(\langle \cdot, \cdot \rangle\)) homogeneous system

\[ \eta' = -\left(\frac{A^*}{x} + B^*(x)\right)\eta. \]  

(27)

Let \(\eta(x, \eta_0)\) denote the solution to this system satisfying the initial condition \(\eta(x_0, \eta_0) = \eta_0\). In what follows, without loss of generality we assume that the scalar product in \(\mathbb{R}^n\) is determined in such a way that \(P^*_j = P_j, \ j = 1, \ldots, 6\).

Let \(L_1([0, \infty) \mapsto \mathbb{R}^n)\) be the space of functions \(f(\cdot) : [0, \infty) \mapsto \mathbb{R}^n\) for which \(\int_0^\infty \|f(x)\|\,dx < \infty\).

Proposition 6. The solution \(\eta(x, \eta_0)\) belongs to \(L_1([0, \infty) \mapsto \mathbb{R}^n)\) iff \(\eta_0 \in L_5 \oplus L_6\).

Proof. As is well known, \([Y^{-1}(x; x_0)]^*\) is a fundamental operator of the adjoint system normalized at the point \(x_0\), and

\[ \langle \eta(x, \eta_0), y(x, y_0) \rangle \equiv \langle \eta_0, y_0 \rangle. \]

Let \(y_0 := (P_1 + \cdots + P_4)\eta_0 \neq 0\). If, in addition, we suppose that \(Q_+y_0 \neq 0\), then in view of (16)

\[ \|Q_+y_0\|^2 \leq \|y(x, Q_+y_0)\|\|\eta(x, \eta_0)\| \leq c_0(x/x_0)^{1+\alpha}\|Q_+y_0\|\|\eta(x, \eta_0)\| \]

for all \(x \in (0, x_0)\), and thus \(\|\eta(x, \eta_0)\| \geq \|Q_+y_0\|(x/x_0)^{-1-\alpha}/c_0\) when \(x \in (0, x_0)\). This implies that \(\eta(x, \eta_0) \notin L_1([0, \infty) \mapsto \mathbb{R}^n)\).

If \(Q_+y_0 = 0\), then \(y_0 = (P_2 + P_3)y_0 = \mathcal{P}_-y_0 \neq 0\). Hence, in view of (18)

\[ \|\mathcal{P}_-y_0\|^2 \leq \|y(x, \mathcal{P}_-y_0)\|\|\eta(x, \eta_0)\| \leq c_*e^{-\gamma(x-x_0)}\|\mathcal{P}_-y_0\|\|\eta(x, \eta_0)\| \]

for all \(x \geq x_0\). This also implies that \(\eta(x, \eta_0) \notin L_1([0, \infty) \mapsto \mathbb{R}^n)\).

On the other hand, if \(y_0 = 0\), then \(\eta_0 = (P_3 + P_5)\eta_0\). Now from the inequalities (17) and (19) it follows that

\[ \|\eta(x, \eta_0)\| \leq \left\| [Y^{-1}(x; x_0)]^*(P_5 + P_6)\right\|\|\eta_0\| \leq C_0 \left(\frac{x_0}{x}\right)^{1-\alpha}\|\eta_0\| \]

when \(x \in (0, x_0)\), and

\[ \|\eta(x, \eta_0)\| \leq \left\| [Y^{-1}(x; x_0)]^*(P_5 + P_6)\right\|\|\eta_0\| \leq C_*e^{-\gamma(x-x_0)}\|\eta_0\| \]

when \(x \geq x_0\). Hence, \(\eta(x, \eta_0)\) belongs to \(L_1([0, \infty) \mapsto \mathbb{R}^n)\). \(\square\)
The above proof has the following corollary:

**Proposition 7.** The condition (25) holds true iff the function $g(x)$ is orthogonal (in sense of the scalar product $(\cdot, \cdot)_{L^2} := \int_0^\infty (\cdot, \cdot) \, dx$) to each solution of the adjoint system (27) belonging to the space $L^1([0, \infty) \mapsto \mathbb{R}^n)$.

Now, let us show that the problem (20)–(21) has a generalized Green function $\mathcal{G}(x, s)$ defined by the following properties:

1. For any $s > 0$ and $x \in [0, \infty) \setminus \{s\}$, it holds
   \[
   \mathcal{L} \mathcal{G}(x, s) = -F(x; x_0) \Pi^{-1}(s; x_0)
   \]
   where $\mathcal{L} := \frac{d}{dx} - \left(\frac{A}{x} + B(x)\right)$, $\Pi := P_5 + P_6$, and $F(\cdot, x_0) \in C([0, \infty) \mapsto \text{Hom}(\mathbb{R}^n))$ is a bounded mapping with the "biorthonormality" property with respect to the space of solutions of the adjoint system which belong to $L^1([0, \infty) \mapsto \mathbb{R}^n)$:
   \[
   \int_0^\infty \Pi^{-1}(x; x_0) F(x; x_0) \, dx = \Pi.
   \]
   E.g., we may set
   \[
   F(x; x_0) := \frac{\kappa^{1+\beta} x^\beta}{\Gamma(1+\beta)e^{\kappa x}} Y(x; x_0) \Pi
   \]
   where $\kappa$ is an arbitrary number greater than $\gamma$ and $\beta > 0$ is an arbitrary number with the property that all real parts of eigenvalues of the matrix $A$ exceed $-\beta$. Obviously, $F(+0; x_0) = F(+\infty; x_0) = 0$.

2. For any $x > 0$ the unit jump property is valid: $\mathcal{G}(x + 0, x) - \mathcal{G}(x - 0, x) = E$.

3. The condition of orthogonality to the space of solutions to the corresponding homogeneous boundary value problem is fulfilled:
   \[
   \int_0^\infty P_1 Y^*(x; x_0) \mathcal{G}(x, s) \, dx = 0.
   \]

4. For any $s > 0$ the boundary conditions $\mathcal{G}(+0, s) = \mathcal{G}(+\infty, s) = 0$ are satisfied.

5. For any $g(\cdot) \in C([0, \infty) \mapsto \mathbb{R}^n)$ satisfying (25), the boundedness condition holds true:
   \[
   \sup_{x \in [0, \infty)} \int_0^\infty \|\mathcal{G}(x, s) g(s)\| \, ds < \infty.
   \]

Observe that the operator differential equation $\mathcal{L} Y = -F(x; x_0)$ has a particular solution
\[
Y = N(x; x_0) := -\int_0^\infty G(x, t) F(t; x_0) \, dt
\]
which can be represented in the form
\[
N(x; x_0) = Y(x; x_0) \left( \Pi - \int_0^x \frac{\kappa^{1+\beta}t^\beta}{\Gamma(1+\beta)} \, dt \Pi \right).
\]
(Note that generally \( N(x; x_0) \) is unbounded on \((0, x_0)\), but it vanishes at infinity.)

It is easily seen that the conditions 1–4 hold true for the operator
\[
\mathcal{G}(x, s) := G(x, s) + Y(x; x_0)P_1 M(s; x_0) + N(x; x_0)\Pi Y^{-1}(s; x_0)
\]
(28)

once we set
\[
M(s; x_0) := - \left[ \int_0^\infty P_1 Y^*(x; x_0) Y(x; x_0) P_1 \, dx \right]_{L_1}^{-1} \cdot \int_0^\infty \! \! P_1 Y^*(x; x_0) \left( G(x, s) + N(x; x_0)\Pi Y^{-1}(s; x_0) \right) \, dx.
\]

To show that the condition 5 is fulfilled it remains only to verify that \( M(s; x_0) \) is absolutely integrable on \([0, \infty)\). This property can be easily obtained from the next six estimates for the function
\[
J(s; x_0) := \| P_1 Y^*(x; x_0)(G(x, s) + N(x; x_0)\Pi Y^{-1}(s; x_0)) \|
\]
which are based on inequalities (16)–(19).

1) Let \( x < s < x_0 \). In this case \( G(x, s) = - Y(x; x_0)(P_1 + P_4 + \Pi)Y^{-1}(s; x_0) \), and therefore there exits a constant \( C_1(x_0) > 0 \) such that
\[
J(s; x_0) \leq \| Y(x; x_0)P_1\| (\|Y(x; x_0)(P_1 + P_4)Y^{-1}(s; x_0)\| + (\kappa x)^{1+\beta}\|Y(x; x_0)\Pi Y^{-1}(s; x_0)\|) \\
\leq C_1(x_0)x^{1+\alpha}(x/s)^{1+\alpha} + x^\alpha \leq C_1(x_0)x^{1+\alpha}(1 + x^\alpha).
\]

2) Let \( s \leq x < x_0 \). Now \( G(x, s) = Y(x; x_0)(P_2 + P_3)Y^{-1}(s; x_0) \), and there exists a constant \( C_2(x_0) > 0 \) such that
\[
J(s; x_0) \leq \| Y(x; x_0)P_1\| (\|Y(x; x_0)(P_2 + P_3)Y^{-1}(s; x_0)\| + (\kappa x)^{1+\beta}\|Y(x; x_0)\Pi Y^{-1}(s; x_0)\|) \leq C_2(x_0)x^2s^\alpha.
\]

3) Let \( s < x_0 \leq x \). Now \( G(x, s) = Y(x; x_0)(P_2 + P_3)Y^{-1}(s; x_0) \), hence,
\[
J(s; x_0) \leq \| Y(x; x_0)P_1\| (\|Y(x; x_0)(P_2 + P_3)Y^{-1}(s; x_0)\| + \|N(x; x_0)\Pi Y^{-1}(s; x_0)\|) \\
\leq C_0C_s e^{-\gamma(x-x_0)} \left( \frac{x_0}{s} \right)^{1-\alpha} (C_s + \sup_{x=[x_0, \infty)} \|N(x; x_0)\|) \\
\leq C_3(x_0)e^{-\gamma s^\alpha s^{-1}}
\]
for some constant $C_3(x_0) > 0$.

4) Let $x < x_0 \leq s$. Since $G(x, s) = -Y(x; x_0)(P_4 + \Pi)Y^{-1}(s; x_0)$, then there exists a constant $C_4(x_0) > 0$ such that

$$J(s; x_0) \leq \|Y(x; x_0)P_1\|\|\|Y(x; x_0)P_1Y^{-1}(s; x_0)\|\| + (\kappa x)^{1+\beta}\|Y(x; x_0)\|Y^{-1}(s; x_0)\| \leq C_4(x_0)x^{1+\alpha}e^{-\gamma s}.$$

5) Let $x_0 \leq x < s$. In this case we also have $G(x, s) = -Y(x; x_0)(P_4 + \Pi)Y^{-1}(s; x_0)$. Hence,

$$J(s; x_0) \leq \|Y(x; x_0)P_1\|\|\|Y(x; x_0)P_1Y^{-1}(s; x_0)\|\| + \|Y(x; x_0)\|Y^{-1}(s; x_0)\| \leq 2C_5^2e^{-\gamma(x-x_0)}e^{-\gamma(s-x)} = C_5(x_0)e^{-\gamma s}$$

where $C_5(x_0) := 2C_5^2e^{\gamma x_0}$.

6) Finally, let $x_0 \leq s \leq x$. Now $G(x, s) = Y(x; x_0)(P_1 + P_2 + P_3)Y^{-1}(s; x_0)$, and there exists a constant $C_6(x_0) > 0$ such that

$$J(s; x_0) \leq \|Y(x; x_0)P_1\|\|\|Y(x; x_0)(P_1 + P_2 + P_3)Y^{-1}(s; x_0)\|\| + \|N(x; x_0)\|Y^{-1}(s; x_0)\| \leq C_6(x_0)\left(e^{-\gamma(2x-s)} + e^{-\gamma(x+s)}\right).$$

The above arguments prove the following theorem.

**Theorem 1.** There exists a solution to the boundary value problem (20)–(21) iff the function $g(x)$ is orthogonal (in terms of the scalar product $\langle \cdot, \cdot \rangle_{L_2} := \int_{\mathbb{R}}^\infty \langle \cdot, \cdot \rangle_{L_2} \, dx$) to all solutions to the adjoint system (27) which belong to $L_1([0, \infty) \mapsto \mathbb{R}^n)$. If the orthogonality condition holds true, then the problem (20)–(21) has the family of solutions which can be represented as the sum of two mutually orthogonal components

$$y = Y(x; x_0)v + \int_0^\infty \mathcal{G}(x, s)g(s) \, ds$$

where $v \in L_1$ is an arbitrary vector and $\mathcal{G}(x, s)$ is the generalized Green function defined by (28).

## 5 The main theorem

Let us turn back to the main problem of finding solutions to the system (1) which possess the properties (2). It is clear that a continuously differentiable on $[0, \infty)$ solution $g(x)$ to the system (1), provided that it exists, must satisfy the equality $Ay(\|0\| + a = 0$. Thus we require the following condition to hold true:

C: $a \in \text{im} A$.  

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The orthogonal decomposition $\mathbb{R}^n = \text{im} A^* \oplus \ker A$ together with the condition $C$ imply existence of the unique $\eta \in \text{im} A^*$ for which $A\eta + a = 0$.

Hence, it is naturally to formulate main boundary value problem in the following way: find all $\zeta \in \ker A$ for which the boundary value problem for the system (1) with the boundary conditions

$$y(+0) = \eta + \zeta, \quad y(\infty) = 0$$

is solvable in the class $C^1([0, \infty) \mapsto \mathbb{R}^n)$ and construct an integral representation of corresponding solutions. This problem is resolved by the following theorem.

**Theorem 2.** Let the system (1) satisfy the conditions $A - C$ and $f(x) \to 0$ when $x \to +\infty$. Then the main boundary value problem is solvable iff

$$\int_0^{\infty} P_5 Y^{-1}(x; x_0) \left[ f(x) + B(x)\eta \right] dx = 0. \quad (29)$$

Provided that (29) holds true, the main boundary value problem has the family of solutions defined by the formulae

$$y = Y(x; x_0)(v_1 + v_2) + \eta + \int_0^{\infty} G(x, s)(f(s) + B(s)\eta) \, ds, \quad (30)$$

$$\zeta = (E + U(x_0))^{-1} v_2 + (E + \Theta(x_0))^{-1} w \quad (31)$$

where $v_1 \in \mathbb{L}_1$, $v_2 \in \mathbb{L}_2$ are arbitrary vectors and

$$w := - \int_0^{\infty} P_5 Y^{-1}(x; x_0) \left[ f(x) + B(x)\eta \right] dx. \quad (32)$$

There exist positive constants $K_1(\alpha, \gamma, x_0)$, $K_2(\alpha, \gamma, x_0)$ such that

$$\int_0^{\infty} \| G(x, s)(f(s) + B(s)\eta) \| \, ds \leq K_1(\alpha, \gamma, x_0) \sup_{x \in [0, \infty)} \| f(s) + B(s)\eta \|,$$

$$\| w \| \leq K_2(\alpha, \gamma, x_0) \sup_{x \in [0, \infty)} \| f(s) + B(s)\eta \|.$$ 

**Proof.** We seek the solution to the problem (1)–(2) in the form

$$y = \eta + \varphi(x) + Y(x; x_0)v + e^{-\kappa x} Y(x; x_0)w + y_0(x) \quad (33)$$

where $\kappa > \gamma$ is an arbitrary number, $v \in \mathbb{L}_1 \oplus \mathbb{L}_2$ is an arbitrary constant vector, $w \in \mathbb{L}_5$ is a constant vector which is to be determined, $\varphi(\cdot) \in C^1([0, \infty) \mapsto \mathbb{R}^n)$ is an arbitrary function with the properties

$$\varphi(0) = 0, \quad \lim_{x \to \infty} \varphi(x) = -\eta, \quad \lim_{x \to \infty} \varphi'(x) = 0,$$
and $y_0(x)$ is a solution to the problem (20)–(21) with

$$g(x) := f(x) + B(x)\eta - \mathcal{L}\varphi(x) + \kappa e^{-\kappa x}Y(x; x_0)w \quad \text{when } x > 0.$$  

Observe that there exists $\lim_{x \to +0} g(x)$. In virtue of the Theorem 11, existence of the solution $y_0(x)$ is guaranteed by the orthogonality conditions which can be given in the form

$$\int_0^\infty \langle [Y^{-1}(s; x_0)]^* P_j b, f(s) + B(s)\eta - \mathcal{L}\varphi(s) \rangle ds + w = 0,$$

$$\int_0^\infty \langle [Y^{-1}(s; x_0)]^* P_6 b, f(s) + B(s)\eta - \mathcal{L}\varphi(s) \rangle ds = 0, \quad \forall b \in \mathbb{R}^n.$$  

Since $\langle [Y^{-1}(s; x_0)]^* P_j b, \varphi(s) \rangle |_{s=0}^\infty = 0$, $j = 5, 6$, and $\mathcal{L}[Y^{-1}(s; x_0)]^* = 0$, these conditions are equivalent to (32), (29). The orthogonality conditions also imply that

$$y_0(x) := \int_0^\infty \mathcal{G}(x, s) g(s) \, ds$$

$$= \int_0^\infty G(x, s) g(s) \, ds + Y(x; x_0)P_1 \int_0^\infty M(s; x_0) g(s) \, ds.$$  

The second addend is inessential owing to the presence of an arbitrary vector $v \in \mathbb{L}_1 \oplus \mathbb{L}_2$ in the formula (33).  

Next, it is not hard to show that

$$\int_0^\infty G(x, s)\mathcal{G}\varphi(s) \, ds = \varphi(x) - Y(x; x_0)P_1\varphi(x_0)$$

and

$$\int_0^\infty G(x, s)\kappa e^{-\kappa s} Y(s; x_0)w \, ds = -e^{-\kappa x} Y(x; x_0)w.$$  

Taking into account these equalities, one can rewrite the formula (33) in the form (30).  

Finally, in view of (11), (15), (33), and the equality $y_0(+0) = 0$ we obtain (31).

Now, observe that from the definition of $\mathbb{L}_5$ it follows that the constant $C_7(x_0) := \max_{x \in [0, x_0]} \|Y(x; x_0)P_5\|$ is well defined. Let $\bar{g}(s) := f(s) + B(s)\eta$. Making use of (29) and estimates similar to those which were obtained for the function $J(s; x; x_0)$ in previous section, we have:
1) if $0 \leq x \leq x_0$, then
\[
\int_0^\infty \|G(x, s)\hat{g}(s)\| \, ds \leq \sup_{s \in [0, \infty)} \|\hat{g}(s)\| \left( \int_0^x \|Y(x; x_0)(P_2 + P_3)Y^{-1}(s; x_0)\| \, ds \right. \\
+ \left. \int_x^{x_0} \|Y(x; x_0)(P_1 + P_4 + P_5)Y^{-1}(s; x_0)\| \, ds \right. \\
+ \left. \int_{x_0}^\infty \|Y(x; x_0)(P_4 + P_5)Y^{-1}(s; x_0)\| \, ds \right)
\]
\[
\leq \sup_{s \in [0, \infty)} \|\hat{g}(s)\| \left( C_0 \int_0^x \frac{x}{s} \, ds + C_0 \int_x^{x_0} \left( \frac{x}{s} \right)^{1-\alpha} \, ds + C_7(x_0)(x_0/s)^{1-\alpha} \right) \, ds \\
+ C_\ast \int_{x_0}^\infty \left( C_0 \cdot \left( \frac{x}{x_0} \right)^{1+\alpha} + C_7(x_0) \right) e^{-\gamma(s-x_0)} \, ds \right.
\]
\[
\leq K_1(\alpha, \gamma, x_0) \sup_{s \in [0, \infty)} \|\hat{g}(s)\|.
\]
2) if $0 < x_0 < x$, then
\[
\int_0^\infty \|G(x, s)\hat{g}(s)\| \, ds \leq \sup_{s \in [0, \infty)} \|\hat{g}(s)\| \left( \int_0^x \|Y(x; x_0)(P_2 + P_3)Y^{-1}(s; x_0)\| \, ds \right. \\
+ \left. \int_x^{x_0} \|Y(x; x_0)(P_1 + P_2 + P_3)Y^{-1}(s; x_0)\| \, ds \right. \\
+ \left. \int_{x_0}^\infty \|Y(x; x_0)(P_4 + P_5)Y^{-1}(s; x_0)\| \, ds \right)
\]
\[
\leq \sup_{s \in [0, \infty)} \|\hat{g}(s)\| \left( C_0 C_\ast \int_0^{x_0} \left( \frac{x}{s} \right)^{1-\alpha} e^{-\gamma(x-x_0)} \, ds + C_\ast \int_{x_0}^x e^{-\gamma(x-s)} \, ds + \\
+ C_\ast \int_x^\infty e^{-\gamma(s-x)} \, ds \right) \leq K_1(\alpha, \gamma, x_0) \sup_{s \in [0, \infty)} \|\hat{g}(s)\|.
\]

Finally, the inequality for $\|w\|$ is easily obtained with help of the estimates from the proof of Proposition 6.

**Conclusions.** The results obtained can be interpreted in terms of linear equations in Banach spaces in the following way. Let $Y$ be the Banach space of continuous mappings $y(\cdot) : [0, \infty) \to \mathbb{R}^n$ such that $\lim_{x \to +\infty} y(x) = 0$, and $X \subset Y$ be the Banach space of mappings satisfying $y(0) = 0$ (these spaces are endowed with usual supremum norm). Consider the closed linear operator $\mathcal{L} : X \to Y$ defined on the dense domain $D(\mathcal{L}) = \{y(\cdot) \in X \cap C^1([0, \infty) \to \mathbb{R}^n) : \lim_{x \to +\infty} y'(x) = 0\}$ by $\mathcal{L}y(x) = y'(x) - Ay(x)/x - B(x)y(x)$. From Proposition 6 it follows that the range $R(\mathcal{L})$ is a closed subspace of Banach space $Y$. Hence, the operator $\mathcal{L}$ is normally solvable, moreover, it is both $n$-normal with $n(\mathcal{L}) = \dim \ker \mathcal{L} = \dim L_1$ and $d$-normal with $d(\mathcal{L}) = \text{codim} R(\mathcal{L}) = \dim (L_5 + L_6)$. This means that we have established conditions under which the operator $\mathcal{L}$ is a Noetherian one with index $n(\mathcal{L}) - d(\mathcal{L})$. 20
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