Initial Data for Black Holes and Black Strings in 5d.

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We explore time-symmetric hypersurfaces containing apparent horizons of black objects in a 5d spacetime with one coordinate compactified on a circle. We find a phase transition within the family of such hypersurfaces: the horizon has different topology for different parameters. The topology varies from $S^4$ to $S^2 \times S^2$. This phase transition is discontinuous – the topology of the horizon changes abruptly. We explore the behavior around the critical point and present a possible phase diagram.

Several black objects solutions exist in spacetimes with dimensionality greater then four containing compact dimensions. Among these solutions are black strings (BS) and black holes (BH) – two black objects with distinct horizon topology. However no general analytic solution is known for a black object in a compactified space-time in more then 4d spacetime.

Gregory and Laflamme (GL) [1,2] discovered that a uniform black string, a product of a Schwarzschild solution with a line, develops a dynamical instability if a compactification radius is ‘too large’. They postulated the existence of a new branch of non-uniform black string solutions. The endpoint of an unstable uniform string was expected to be a BH. However, Horowitz and Maeda [3,4] argued that the GL instability cannot cause a uniform BS to decay to a BH, at least in a finite affine time. This is because of the ‘no tear’ property of horizons. They conjectured that the endstate of such a decay would be rather a non-uniform BS. Non-uniform BS solutions connected to the GL point were constructed perturbatively by Gubser [5]. Wiseman [6] found non-uniform BS solutions using fully non-linear numerical calculations. However, Wiseman stresses that these solutions are most likely unstable.

We consider here the five dimensional spacetime with one compact dimension of length $2L$. We denote the coordinate along the compact dimension by $z$. A neutral black object has a characteristic dimensionfull parameter, its asymptotic 4d mass, $M$. It is convenient to describe the system by a single dimensionless parameter, $\zeta \equiv G_4 M/(2L)$, where the effective Newton constant is $G_4 = G_5/2L$. If $\zeta \ll 1$ the black object is expected to resemble a 5d Schwarzschild BH since local measurements cannot probe the compactness of the $z$ direction. The horizon of this BH is therefore expected to have an $S^4$ topology. At the opposite end, $\zeta \gg 1$, one expects that the horizon would extend over the compact dimensions wrapping it completely. The horizon’s topology of this object, $S^2 \times S^1$, suggests the name ‘BS’. An uniform BS is described by a 4d Schwarzschild solution times a circle. A non-uniform BS is characterized by a non-trivial $z$-dependence along the circle. An attempt, based on a thermodynamical reasoning, to construct the phase diagram that would include consistently different phases of solutions was recently undertaken by Kol [7].

In this letter we make a step toward the self-consistent study of black objects in compact 5d background. We solve numerically the initial-value problem for a moment of time-symmetry for spacetimes containing black objects. We determine the apparent horizon of the existing black objects in these solutions and we follow the transition from spherical to cylindrical topology in a family of solutions with a different dimensionless parameter $\zeta$. Although the configurations we find are not 5d static solutions, these solutions, give us some insight on the behavior of the black objects in a compactified background. A similar approach was previously applied [8] to explore the time-symmetric BH problem in the brane world scenario.

We consider a time-symmetric slice through the black object’s space-time. To generate the black object solution we consider a configuration with an artificial matter surrounding the origin. This matter gives rise to a non-flat spatial metric along the slice. We solve numerically for this metric. Then we solve for the apparent horizon of the black object. The method allows us to determine the spatial geometry of the black object. Since we solve along a time-symmetric slice, this spatial geometry constitutes the initial data for subsequent dynamical time evolution of this black object’s geometry. Limiting ourselves only to the determination of the initial data we find explicitly time-symmetric solutions with a distinct horizon topology and construct the phase diagram for a family of such solutions parametrized by $\zeta$. We show that there is a phase transition within this family. We identify the critical $\zeta_c$ and show explicitly that if $\zeta_c$ is reached from below a corresponding BH solutions becomes deformed and the horizon has a cigar-like shape. If $\zeta_c$ is approached from above, the corresponding BS solution becomes more and more non-uniform. However, this cannot be considered as evidence for existence of a non-uniform static BS solutions. This initial data may relax to a uniform string during its dynamical evolution. We find that the transition between both topologies is not smooth: the spherical-like horizon jumps suddenly to become a cylindrical-like horizon. Put differently, neither the BS pinches nor the BH has its south and north poles touching each other.

The method that we describe here enables us to study properties of black objects with different apparent horizon’s topology in a single consistent numerical scheme. Our method does not require a prescription of the topology of the horizon. The topology is not predetermined.
but rather it is a derived result of numerics. Usually, to obtain a static black object solution of the 5d Einstein equations one would had to specify the topology of the horizon. In other words there is no a single numerical scheme that can find black objects with distinct horizon topology.

Let us consider a time-symmetric, spacelike hypersurface \( \Sigma_t \) with a vanishing extrinsic curvature. The Hamiltonian constraint reads

\[
R^{(4)} = 16\pi G_5 T_{\mu\nu} t^\mu t^\nu ,
\]

where \( t^\mu \) is the unit normal to \( \Sigma_t \) and \( T_{\mu\nu} \) is the 5d stress-energy tensor. From now on we work in units where \( c = G_5 = 1 \). The momentum constraint is trivially satisfied provided that the matter is static and the extrinsic curvature of this slice vanishes.

We choose the metric on \( \Sigma_t \) to be conformally flat

\[
dt^2 = \psi^2 (dr^2 + dz^2 + r^2 d\Omega_2^2) .
\]

This ansatz has been adopted for simplicity. We didn’t expect that the static solution will be conformally flat. However, we expect that the trend we discuss is insensitive to this assumption. The method can be easily generalized to non-conformal choices. More complicated metrics will be considered elsewhere.

With this choice of the metric the constraint equation takes the form

\[
\nabla^2 \psi + \rho \psi^3 = 0 ,
\]

where \( \nabla^2 \equiv \frac{1}{r^2} \partial_r \left( r^2 \partial_r \right) + \partial_{zz} \). This choice of the metric the constraint equation (3) is well posed and has a unique solution. The rescaling of the matter density is unimportant as we are interested only in the external vacuum part. The solution for \( \psi \) is found using relaxation.

We obtain a sequence of momentary time-symmetric solutions by fixing the density of the artificial matter and its location as \( \rho = 10^6 \Theta(0.5 - r) \Theta(0.5 - z) \), and varying continuously the length of the compact asymptotic circle, \( 2L \). The figures below are obtained for this source. We checked that our results are not affected by the specific choice of the source. Taking the smooth distribution \( \tilde{\rho} = 10^6 \exp(-r^2/\sigma_r^2) \left[ \exp(-z^2/\sigma_z^2) + \exp(-(z - 2L)^2/\sigma_z^2) \right] \) with various \( \sigma_r \) and \( \sigma_z \) we found the same overall picture. The values of \( \zeta_c \) varied, as expected, depending on the source. The variations in \( \zeta_c \) are \( \zeta_c = .98 - 1.8 \). Moreover, taking \( \sigma_r \approx 0.5, \sigma_z \gg L \), i.e. practically cylindrical source, we were able to reproduce the uniform BS solution. The position of the horizon in this solution (see Eq. (5) and the subsequent discussion) was determined to within 0.1%. This provides an independent check on the overall accuracy of our calculation in addition to other standard tests of convergence, errors scaling etc.

To solve the equation for \( \psi \) we used grids, covering the domain \( 0 < z < L \), and \( 0 < r < R_{cut} \), with typical grid spacings of \( \Delta r \sim 0.02 \) and \( \Delta z \sim 0.01 \). The value of \( R_{cut} \), where the grid was cutoff and the asymptotic b.c. (iv) was implemented, has been taken as \( R_{cut} = 5, 10 \) and 20. We checked that the results are insensitive to variation of \( R_{cut} \), provided that \( R_{cut} > 5 \).

To envisage the spatial metric around the black objects in Fig.1 we plot the contours of \( \psi \) in two cases that correspond to a BH solution and a non-uniform BS solution. The matter is located near the origin and is encircled by the horizon in either case. The geometry outside the apparent horizon has an axisymmetric structure and it becomes asymptotically flat. The \( \psi \) contour lines are spherical near the origin and become cylindrical as \( r \) increases.

Once we obtain \( \psi \) we determine the existence of the apparent horizon. An apparent horizon is defined by a zero-expansion of the null rays generating the horizon [10]. For the time-symmetric hypersurface \( \Sigma_t \) this condition can be written as

\[
\nabla^{(4)}_\mu n^\mu = 0 , \tag{5}
\]

where \( n^\mu \) is the the unit normal to the apparent horizon.
The parameter condition of the non-uniformity of a BS \cite{5} is \( \lambda \). For a uniform string there are two horizons. The inner spherical apparent horizon, designated by the dotted thick curve, and the outer cylindrical horizon, designated by the solid curve.

To simplify the treatment, we distinguish between two different topologies for the horizon.

1. When the horizon has the topology \( S^2 \times S^1 \) we choose cylindrical coordinates \((r, z)\) and we solve for a curve \( r = h(z) \).

2. When the horizon has the topology of \( S^3 \) it is convenient to transform to spherical coordinates \( R, \chi \) defined by \( r = R \sin(\chi), z = R \cos(\chi) \). The horizon in the \( R, \chi \) plane is given by a curve \( R = h(\chi) \).

The unit normal to the curve that defines the horizon is

\[ n^\mu = (C, -Ch') \, . \tag{6} \]

The parameter \( C(r, z) \) could be read from the normalization condition \( n^\mu n_\mu = 1 \). In both cases we solve numerically equation (5) to obtain the position of the apparent horizon.

A useful qualitative parameter employed as a measure of the non-uniformity of a BS \cite{5} is \( \lambda \equiv 1/2(r_{\text{max}}/r_{\text{min}} - 1) \) where \( r_{\text{min}} \) and \( r_{\text{max}} \) are the minimal and the maximal 4d Schwarzschild radii of the apparent horizon. For a uniform string \( \lambda = 0 \). In the BH phase \( \lambda = \infty \). Therefore, for a BH we define another parameter \( \lambda' \equiv R_{\text{max}}/R_{\text{min}} - 1 \), where \( R_{\text{max}} \) and \( R_{\text{min}} \) are the 5d Schwarzschild radii of the horizon. This parameter gives an idea of deformation of the BH’s horizon.

We find that there are two topologically distinct apparent horizon solutions. At small \( \zeta \) the topology of the horizon is \( S^3 \) and the horizon is close to be exactly spherical. When \( \zeta \) increases we see that the horizon begins to deform, deviating from a spherical shape but still remaining topologically 3-sphere. At a certain value, \( \zeta_m \sim 1.78 \), a phase transition takes place – the topology of the apparent horizon changes form \( S^3 \) to \( S^2 \times S^1 \). In fact there are two apparent horizons in the BH phase. The outer horizon, has a cylindrical topology, while the inner one has a spherical topology. In Fig.2 we plot the sequence of solutions parametrized by \( \zeta \). In this figure there are finite values of \( \zeta \) when \( \lambda = 0 \) or \( \lambda' = 0 \). In fact for these \( \zeta \) the deformation of the horizon becomes so small that cannot be resolved by our numerics and we put the corresponding lamindas to zero.

Another interesting result is the measure of geometrical deformation of the horizons. In the BS and the BH phases this measure is supplied by \( \lambda \) and \( \lambda' \) respectively. The non-uniformity of the BS is displayed in the upper panel of Fig. 2. Near the critical point the most non-uniform string has \( \zeta \sim 0.22 \). The non-uniformity disappear as \( \zeta \) increases. For \( \zeta \geq 3.0 \) the BS becomes a uniform BS. The deformation of the horizon in the BH phase could be seen in the bottom panel of the same Figure. The maximal radius, \( R_{\text{max}} \), always occurs at the axis, \( r = 0 \). The BH becomes more and more oblate and stretched along the symmetry axis, as we approach the critical point. The most deformed BH has \( \lambda' \sim 0.15 \) just
before the transition.

The phase transition is discontinuous. At the critical value of $\zeta_c$ the spherical horizon jumps suddenly to become a cylindrical. To get insight on the behavior of the phase transition, in the BH phase we have computed the proper distance along the $r = 0$ axis from the BH horizon at the axis to $z = L$:

$$\ell = \int_{\psi(r=0)}^{L} \psi(r=0,z)dz . \quad (7)$$

This distance decreases as we increase $\zeta$. One could expect that as horizon grows and as $\ell \to 0$ the north pole of the BH would tend towards its south pole and they will touch. However, we find that as $\zeta \to \zeta_c$ $\ell$ reaches a finite value. We plot the behavior of $\ell$ as a function of $\zeta - \zeta_c$ in Fig. 3. One observes that $\ell$ tends to a positive constant just before the transition.

![FIG. 3. The proper distance from the BH to the reflection plane $z = L$ does not decrease to zero but tends to a finite value as we approach the transition point.](image)

The initial data that we have constructed here is analogous to the Misner initial data [11] for a family of momentary static two equal mass BHs in 4d GR. Misner choose a sequence of conformally flat metrics with the conformal factor parametrized by a certain parameter $\mu$ that is related to the mass of the BHs and their proper mutual separation. As $\mu$ varies the shape of the initial apparent horizons varies. If the BHs are close enough, that occurs for small $\mu$, a new apparent horizon suddenly appears, surrounding both BHs on the initial hypersurface. In other words, there is a critical $\mu_0$ that divides two distinct possibilities for the topology of the apparent horizon on the initial slice. Just by looking at this initial data sequence one has an indication that the event horizons for two BHs will merge and form a distorted BH during an actual evolution. The value of $\mu$ when this merge occurs, generally would not coincide with the theoretical, $\mu_0$. In fact, the numerical evolution [12,13] of Misner initial data shows that the qualitative picture obtained for the sequence of the initial data is correct. The actual critical value of $\mu$ does not coincide with $\mu_0$, they are, however, not that different from each other.

Here we have an infinite array of BHs that are approaching each other simultaneously. We have shown that there is a family of initial data parametrized by $\zeta$.

When the separate BHs in the array are getting closer they become distorted form the spherical shape. At a critical value $\zeta_c$ the separate horizons are engulfed suddenly by a single cylindrical-like horizon. The effective cylindrical horizon after the transition is non-uniform.

It is important to stress that the sudden jumps of the apparent horizon topology are different from the first order transition that can happen between different phases of static solutions, discussed in [5,7]. This is because the apparent horizon that we discuss here is not casual as it is defined only locally. The event horizon is a global concept and it is expected to be larger than the apparent horizon. Only for static solutions both horizons coincide. Since our solution is not static the jumps of the apparent horizon cannot exclude the possibility of a smooth transition between the BHs and the BSs event horizons, discussed in [14].

The concrete numerical value of $\zeta_c$ isn’t important as it is just a number characteristic to the specific initial data sequence. However we expect that the qualitative behavior would be similar in the static solutions as well. We believe that a dynamical evolution of our initial data would confirm this qualitative picture and will yield actual critical value, $\zeta_c$.

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