MOTIVIC GAUSS-BONNET FORMULAS

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ABSTRACT. The apparatus of motivic stable homotopy theory provides a notion of Euler characteristic for smooth projective varieties, valued in the Grothendieck-Witt ring of the base field. Previous work of the first author and recent work of Déglise-Jin-Khan establishes a “Gauß-Bonnet formula” relating this Euler characteristic to pushforwards of Euler classes in motivic cohomology theories. In this paper, we apply this formula to SL-oriented motivic cohomology theories to obtain explicit characterizations of this Euler characteristic. The main new input is a unicity result for pushforward maps in SL-oriented theories, identifying these maps concretely in examples of interest.

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1. INTRODUCTION

Let k be a field of characteristic different from 2, and let X be a smooth projective k-scheme. Let SH(k) denote the motivic stable homotopy category over k; recall that this comes equipped with the structure of a symmetric monoidal category, whose tensor product we denote \( \wedge_k \) and whose unit object (the motivic sphere spectrum) we denote \( 1_k \).

Our starting point in this paper is the following fact (which shall be reviewed in \( \S 2 \)):

**Proposition 1.1.** The infinite suspension spectrum \( \Sigma^\infty_+ X \in \text{SH}(k) \) is dualizable. In particular, we can associate to X a natural Euler characteristic \( \chi(X/k) \in \text{End}_{\text{SH}(k)}(1_k) \), defined as the composition

\[
1_k \xrightarrow{\delta} \Sigma^\infty_+ X \wedge_k (\Sigma^\infty_+ X)^\vee \xrightarrow{\tau} (\Sigma^\infty_+ X)^\vee \wedge_k \Sigma^\infty_+ X \xrightarrow{\epsilon} 1_k,
\]

where the maps \( \delta \) and \( \epsilon \) are the “coevaluation” and “evaluation” that comprise the duality, and \( \tau \) is the symmetry isomorphism.

A theorem of Morel identifies \( \text{End}_{\text{SH}(k)}(1_k) \) with \( GW(k) \), the Grothendieck group of \( k \)-vector spaces equipped with a nondegenerate symmetric bilinear form, so we may think of the Euler characteristic \( \chi(X/k) \) as a class in \( GW(k) \). It is then natural to wonder whether there is an explicit interpretation of this Euler characteristic in terms of symmetric bilinear forms. An intuitive speculation is that the Euler characteristic...
should be given by the value of some cohomology theory on $X$, equipped with an “intersection pairing”.

One of the main results in this paper is to make precise and confirm this speculation. To state the result, we recall that classes in $GW(k)$ can be represented not just by nondegenerate symmetric bilinear forms on $k$-vector spaces, but also by nondegenerate symmetric bilinear forms on perfect complexes over $k$. With this in mind, we give the following explicit interpretation of the Euler characteristic $\chi(X/k)$:

**Notation 1.2.** Assume for simplicity that $X$ is of pure dimension $d$. We have the Hodge cohomology groups $H^i(X; \Omega^j_{X/k})$ for $0 \leq i, j \leq d$ and the canonical trace map

$$\text{Tr}: H^d(X; \Omega^d_{X/k}) \to k.$$ 

We define a perfect complex of $k$-vector spaces (with zero differential),

$$\text{Hdg}(X/k) := \bigoplus_{i,j=0}^d H^i(X, \Omega^j_{X/k})[j-i],$$

and the trace map defines a nondegenerate symmetric bilinear form on $\text{Hdg}(X/k)$ via the pairings

$$H^i(X, \Omega^j_{X/k}) \otimes_k H^{d-i}(X, \Omega^{d-j}_{X/k}) \to H^d(X, \Omega^d_{X/k}) \xrightarrow{\text{Tr}} k,$$

where the first map denotes the cup product (that this is indeed a nondegenerate symmetric bilinear form will be argued in [8, A]). We thus obtain a Grothendieck-Witt class $(\text{Hdg}(X/k), \text{Tr}) \in GW(k)$.

The following formula for $\chi(X/k)$ was proposed by J.P. Serre.

**Theorem 1.3.** We have $\chi(X/k) = (\text{Hdg}(X/k), \text{Tr}) \in GW(k)$.

In the case $k = \mathbb{R}$, a class in $GW(k)$ is determined by two $\mathbb{Z}$-valued invariants, rank and signature, and Theorem 1.3 reproves the following known result (see [1] Theorem 1 and [18, Theorem A]).

**Corollary 1.4.** Suppose $k = \mathbb{R}$ and $X$ is of even pure dimension $2n$. Then the symmetric bilinear form

$$H^n(X, \Omega^n_{X/\mathbb{R}}) \times H^n(X, \Omega^n_{X/\mathbb{R}}) \xrightarrow{\text{Tr}} H^{2n}(X, \Omega^{2n}_{X/\mathbb{R}})$$

has signature equal to $\chi^{\text{top}}(X(\mathbb{R}))$, the classical Euler characteristic of the real points of $X$ in the analytic topology. In particular, we have

$$|\chi^{\text{top}}(X(\mathbb{R}))| \leq \dim_{\mathbb{R}} H^n(X, \Omega^n_{X/\mathbb{R}}).$$

Let us now explain our methods for proving Theorem 1.3. The idea is to use the theory of Euler classes in motivic cohomology theories. More specifically, our focus is on cohomology theories represented by SL-oriented motivic ring spectra; recall that this refers to a commutative monoid object $E$ in $\text{SH}(k)$ equipped with a compatible system of Thom classes for oriented vector bundles (where an orientation is a specified trivialization of the determinant line bundle). The example of interest for proving Theorem 1.3 is Hermitian $K$-theory; other examples of interest include Chow-Witt theory, ordinary motivic cohomology, and algebraic K-theory (the last two are actually GL-oriented, meaning they have Thom classes for all vector bundles).

Given an SL-oriented motivic ring spectrum $E \in \text{SH}(k)$, one may define certain pushforward maps in twisted $E$-cohomology. Namely, for $Y, Z$ two smooth quasi-projective $k$-schemes, a proper morphism $f : Z \to Y$ of relative dimension $d \in \mathbb{Z}$, and a line bundle $L \to Y$, there is a pushforward map

$$f_* : E^{a,b}(Z; \omega_{Z/k} \otimes f^* L) \to E^{a-2d,a-b}(Y; \omega_{Y/k} \otimes L),$$

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where $\omega_{-/k}$ denotes the canonical bundle. This is defined abstractly via the six-functor formalism for motivic stable homotopy theory.

We note two key examples of these pushforwards, assuming our smooth projective variety $X$ is of pure dimension $d$ for simplicity:

- the structural morphism $\pi: X \to \text{Spec}(k)$ gives a pushforward map
  \[ \pi_*: \mathcal{E}^{2d,d}(X, \omega_{X/k}) \to \mathcal{E}^{0,0}(\text{Spec} k); \]

- given a vector bundle $p: V \to X$, the zero section $s: X \to V$ gives a pushforward map
  \[ s_*: \mathcal{E}^{0,0}(X) \to \mathcal{E}^{2d,d}(V; p^* \det^{-1}(V)). \]

The first should be thought of as a kind of integration map. The second allows us to define the Euler class of a vector bundle $V \to X$,
\[ e^\mathcal{E}(V) := s^* s_*(1) \in \mathcal{E}^{2d,d}(X; \det^{-1}(X)), \]
where $s$ again denotes the zero-section, and $1 \in \mathcal{E}^{0,0}(X)$ denotes the unit element.

We may now state the following “Gauß-Bonnet formula” equating the Euler characteristic with the “integral of the Euler class of the tangent bundle”, using the above notions; this result is a fairly immediate consequence of a result proven in [20]:

**Theorem 1.5** (Motivic Gauß-Bonnet). Let $\mathcal{E}$ be an SL-oriented motivic ring spectrum in $\text{SH}(k)$. Let \( u: 1_k \to \mathcal{E} \) denote the unit map, inducing the map \( u_*: \text{GW}(k) \cong 1_k^*(\text{Spec} k) \to \mathcal{E}^{0,0}(\text{Spec} k). \) Then we have
\[ u_*(\chi(X/k)) = \pi_*(e_\mathcal{E}(T_{X/k})) \in \mathcal{E}^{0,0}(\text{Spec} k). \]

A general motivic Gauß-Bonnet formula is also proven in [9], which implies the above formula by applying the unit map; our method is somewhat different from [9] in that we replace their general theory of Euler classes with the more special version for SL-oriented theories used here.

As stated above, we deduce Theorem 1.3 from Theorem 1.5 by considering the example of Hermitian K-theory, $\mathcal{E} = \text{KO}$. In this case, the map $u_*: \text{GW}(k) \to \text{KO}^{0,0}(\text{Spec} k)$ is an isomorphism. The deduction requires an explicit understanding of both the Euler class $e^\mathcal{E}(T_{X/k})$ and the pushforward $\pi_*$ in Hermitian K-theory; the former is fairly straightforward, but the latter requires new input.

What we do is identify the abstractly defined projective pushforward maps in Hermitian K-theory with the concrete ones defined in terms of pushforward of sheaves and Grothendieck-Serre duality. This comparison follows from a unicity result we prove for pushforward maps in an SL-oriented theory $\mathcal{E}$, characterizing them, under certain further hypotheses on $\mathcal{E}$, in terms of their behavior in the case of the inclusion of the zero-section of a vector bundle (which is governed by Thom isomorphisms). We leave the detailed statement of this theorem to the body of the paper, as it would take too long to spell out here.

1.1. Outline. In [2] we review the basic setup of motivic homotopy theory, as well as relevant aspects of the dualizability result Proposition 1.1. In [3] we review basic facts about SL-oriented motivic ring spectra. In [4] we describe the abstractly defined pushforwards in the twisted cohomology theory arising from an SL-oriented motivic ring spectrum. In [5] we prove the general Gauß-Bonnet formula for SL-oriented motivic ring spectra. In [6] we axiomatize the features of the twisted cohomology theory arising from an SL-oriented motivic ring spectrum. In [7] we use these axiomatics to prove our unicity/comparison theorem characterizing the pushforward maps in SL-oriented theories. And finally in [8] we apply the previous results in specific examples of SL-oriented theories to obtain various concrete consequences, in particular proving Theorem 1.3.
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2. Duality and Euler characteristics

In this section we review the strong dualizability of smooth projective schemes as objects of the stable motivic homotopy category, which supplies a notion of Euler characteristic for these schemes. We additionally recall a result from [20] that gives an alternative characterization of this Euler characteristic.

2.1. Preliminaries. We first recall the basic framework of stable motivic homotopy theory that will be employed throughout.

Notation 2.1. Throughout the paper, let $B$ denote a separated quasi-compact base scheme (further hypotheses to be made when necessary). Let $\text{Sch}_B$ denote the category of quasi-projective $B$-schemes, that is, $B$-schemes $X \rightarrow B$ that admit a closed immersion $i : X \hookrightarrow U$ over $B$, with $U$ an open subscheme of $\mathbb{P}^N_B$ for some $N$. Let $\text{Sch}^\text{pr}_B$ denote the subcategory of $\text{Sch}_B$ with the same objects as $\text{Sch}_B$ but with morphisms the proper morphisms. Let $\text{Sm}_B$ denote the full subcategory of $\text{Sch}_B$ with objects the smooth (quasi-projective) $B$-schemes. (The same notation will be used when working over schemes other than $B$.)

Notation 2.2. Given $X \in \text{Sch}_B$, we let $\text{SH}(X)$ denote the stable motivic homotopy category over $X$. We shall rely on the six-functor formalism for this construction as given in [3, 15]. In particular, for each morphism $f : Y \rightarrow X$ in $\text{Sch}_B$ one has the adjoint pairs of functors

$$
\text{SH}(X) \xrightarrow{f_*} \text{SH}(Y), \quad \text{SH}(Y) \xleftarrow{f^*} \text{SH}(X);
$$

natural isomorphisms $(gf)^* \cong f^*g^*$, $(gf)_! \cong f_!g_!$, $(gf)_* \cong g_*f_*$, $(gf)_! \cong g_!f_!$ for composable morphisms, with the usual associativity; a natural transformation $\eta^f_! : f_! \rightarrow f_*$, which is an isomorphism if $f$ is proper; and various base-change morphisms, which we will recall as needed. In addition, for $f$ smooth there is a further adjoint pair

$$
\text{SH}(Y) \xrightarrow{f_!} \text{SH}(X).
$$

There is also the symmetric monoidal structure on $\text{SH}(X)$; we denote the product by $\wedge_X$ and the unit by $1_X \in \text{SH}(X)$.

Remark 2.3. For $p : E \rightarrow X$ an affine space bundle in $\text{Sch}_B$, the composition $p_! \circ p^*$ is an autoequivalence of $\text{SH}(X)$ (this is a formulation of homotopy invariance).

Notation 2.4. For a morphism $f : Z \rightarrow X$ in $\text{Sch}_B$, we set $Z/X_{B,M} := f_!(1_Z) \in \text{SH}(X)$.

Remark 2.5. The assignment $Z \mapsto Z/X_{B,M}$ extends to a functor $(-)/X_{B,M} : (\text{Sch}_X^{\text{pr}})^{\text{op}} \rightarrow \text{SH}(X)$. This is described in a number of places, for example [21, §1]; we recall the construction for the reader’s convenience, referring to loc. cit. for details.
Let $g : Z \to Y$ be a proper morphism in $\text{Sch}_X$. Let $\pi_Y : Y \to X$ and $\pi_Z : Z \to X$ denote the structural morphisms. As mentioned in Notation 2.6, we have a natural isomorphism $\eta^*_p : g_! \cong g_*$. We may then define the map $g^* := g/X_{BM} : Y/X_{BM} \to Z/X_{BM}$ as the composition

$$\pi_Y(1_Y) \xrightarrow{u_\eta} \pi_Y ! g_* g^*(1_Y) \xrightarrow{(\eta^*_p)^{-1}} \pi_Y ! g ! g^*(1_Y) \cong \pi_Z(1_Z),$$

where $u_\eta : \text{id}_{\text{SH}(Y)} \to g_* : \text{id}$ is the unit of the adjunction. It follows directly from the definitions that $(gh)^* = h^* g^*$ for composable proper morphisms $g, h$. This establishes the claimed functoriality.

**Notation 2.6.** For $X \in \text{Sch}_B$, let $H_*(X)$ denote the pointed unstable motivic homotopy category over $X$. We have the infinite suspension functor $\Sigma^{\infty}_T : H_*(X) \to \text{SH}(X)$.

For $\pi_Y : Y \to X$ in $\text{Sm}_X$, we let $Y_+ \subseteq H_*(X)$ denote the pointed presheaf represented by $Y$. There is a canonical identification $\Sigma^{\infty}_T Y_+ \cong \pi_Y (1_Y) \in \text{SH}(X)$, and we write $Y/X$ to denote this object.

**Remark 2.7.** The assignment $Y \mapsto Y/X$ extends to a functor $(-)/X : \text{Sm}_X \to \text{SH}(X)$. This is evident from the description of $Y/X$ as $\Sigma^{\infty}_T Y_+$. That is, to a morphism $f : Z \to Y$ in $\text{Sm}_X$ this functor assigns the morphism $f_* := \Sigma^{\infty}_T f : Z/X \to Y/X$.

Alternatively, we may define a natural transformation $f_* : \pi_{Z+} \circ f^* \to \pi_{Y+}$ as the composition

$$\pi_{Z+} \circ f^* = \pi_{Y+} \circ f_+ \circ f^* \xrightarrow{\nu_f} \pi_{Z+},$$

where $\nu_f : f_+ f^* \to \text{id}_{\text{SH}(Y)}$ is the counit of the adjunction; then $f_* = \Sigma^{\infty}_T f : Z/X \to Y/X$ is recovered as $f_*(1_X)$.

**Notation 2.8.** Let $p : V \to X$ be a vector bundle over some $X \in \text{Sch}_B$, with zero-section $s : X \to V$. We have the endofunctors

$$\Sigma^{-V}, \Sigma^V : \text{SH}(X) \to \text{SH}(X)$$

defined by $\Sigma^{-V} := s^! p^*$ and $\Sigma^V := p_! \circ s_*$. These are in fact inverse autoequivalences.

These endofunctors can also be defined in terms of Thom spaces. Letting $0_V := s(X) \subseteq V$, the Thom space of the vector bundle is defined as

$$\text{Th}_X(V) := V/(V \setminus 0_V) \in H_*(X).$$

There is a canonical identification $\Sigma^V (1_X) \cong \Sigma^{\infty}_T \text{Th}_X(V)$. We shall often abuse notation by writing $\text{Th}_X(V)$ for $\Sigma^\infty_T \text{Th}_X(V)$, and in parallel we shall write $\text{Th}_X(-V)$ for $\Sigma^{-V}(1_X)$. With these notational conventions, there are canonical natural isomorphisms

$$\Sigma^V (-) \cong \text{Th}_X(V) \wedge_X (-), \quad \Sigma^{-V} (-) \cong \text{Th}_X(-V) \wedge_X (-).$$

Finally, if $\pi_X : X \to B$ moreover lies in $\text{Sm}_B$, we set

$$\text{Th}(V) := V/(V \setminus 0_V) \in H_*(B),$$

and we have a canonical isomorphism $\Sigma^{\infty}_T \text{Th}(V) \cong \pi_X (\text{Th}_X(V))$. We repeat the notational conventions above, often writing $\text{Th}(V)$ for $\Sigma^\infty_T \text{Th}(V)$ and $\text{Th}(-V)$ for $\pi_X (\text{Th}_X(-V))$.

**Remark 2.9.** Let $D_{\text{perf}(X)_{\text{iso}}}$ denote the subcategory of isomorphisms in the perfect derived category $D_{\text{perf}(X)}$. Then the assignment $V \mapsto \Sigma^V$ extends to a functor

$$\Sigma^V : D_{\text{perf}(X)_{\text{iso}}} \to \text{Aut}(\text{SH}(X)).$$

Moreover, a distinguished triangle $E' \to E \to E'' \to E'[1]$ in $D_{\text{perf}(X)}$ gives a natural isomorphism $\Sigma^{E'} \circ \Sigma^{E''} \cong \Sigma^E$. 

Remark 2.10. For \( f : Y \to X \) a smooth morphism in \( \text{Sch}_B \) with relative tangent bundle \( T_f \to Y \), there are canonical natural isomorphisms
\[
f_! \cong f'_! \circ \Sigma^{-T_f}, \quad f^! \cong \Sigma^{T_f} \circ f^*.
\]
See [15, Theorem 6.18(2)]. In addition, for \( V \to X \) a vector bundle there are canonical natural isomorphisms (see [15], the beginning of §5.2)
\[
f'_! \circ \Sigma^{\pm f_*V} \cong \Sigma^{\pm V} \circ f'_!, \quad f^* \circ \Sigma^{\pm V} \cong \Sigma^{\pm f_*V} \circ f^*
\]
the latter valid for an arbitrary morphism \( f : Y \to X \) in \( \text{Sm}_B \).

2.2. Duality for smooth projective schemes. We now recall how to construct the dual of a smooth projective scheme in the stable motivic homotopy category.

Lemma 2.11. Let \( X \in \text{Sch}_B \) and \( \pi_Y : Y \to X \) be an object of \( \text{Sm}_X \). View \( Y \times_X Y \) as a \( Y \)-scheme via the projection \( p_2 : Y \times_X Y \to Y \) onto the second factor. Then there are canonical isomorphisms
\[
\pi_Y(Y/X_{B,M.}) \cong p_{2!} \Sigma^{-p_1^*T_Y/X} (1_{Y \times_X Y}) \cong Y \times_X Y/Y_{B,M.}
\]
in \( \text{SH}(Y) \) and a canonical isomorphism
\[
\pi_{Y!}(Y \times_X Y/Y_{B,M.}) \cong Y/X_{B,M.} \wedge_X Y/X
\]
in \( \text{SH}(X) \).

Proof. Consider the commutative square
\[
\begin{array}{ccc}
Y \times_X Y & \xrightarrow{p_2} & Y \\
p_1 \downarrow & & \downarrow \pi_Y \\
Y & \xrightarrow{\pi_Y} & X
\end{array}
\]
This gives us the canonical isomorphism
\[
T_{Y \times_X Y/Y} \cong p_1^*T_Y/X.
\]
We have the identities and canonical isomorphisms
\[
\pi_Y^*(Y/X_{B,M.}) = \pi_Y^* \pi_Y!(1_Y) \cong \pi_Y^* \pi_Y \Sigma^{-T_Y/X} (1_Y) \quad \text{(base change)}
\]
\[
\cong p_{2!} p_1^* \Sigma^{-T_Y/X} (1_Y) \cong p_{2!} \Sigma^{-p_1^*T_Y/X} (1_{Y \times_X Y}),
\]
which gives us the first isomorphism in \( \text{SH}(Y) \). The second follows from
\[
p_{2!} \Sigma^{-p_1^*T_Y/X} (1_{Y \times_X Y}) \cong p_{2!} \Sigma^{-T_Y \times_X Y} (1_{Y \times_X Y}) \cong p_{2!} (1_{Y \times_X Y}) = Y \times_X Y/Y_{B,M.}.
\]
Finally, to give the isomorphism in \( \text{SH}(X) \), we have
\[
\pi_{Y!}(Y \times_X Y/Y_{B,M.}) \cong \pi_{Y!} (p_{2!} \Sigma^{-T_Y \times_X Y} (1_{Y \times_X Y})) \cong \pi_{Y!} p_{12} (\Sigma^{-T_Y \times_X Y} (1_{Y \times_X Y}))
\]
\[
\cong \pi_{Y!} p_{12} (\Sigma^{-p_1^*T_Y/X} (1_{Y \times_X Y})) \cong \pi_{Y!} \Sigma^{-T_Y/X} p_{12} (1_{Y \times_X Y}) \cong \pi_{Y!} p_{12} p_1^* (1_Y)
\]
\[
\cong \pi_Y^* \pi_Y^* (1_Y) \cong \pi_Y^* (1_Y \wedge_Y \pi_Y^*(Y/X)) \quad \text{(base change)}
\]
\[
\cong \pi_Y^* (1_Y) \wedge_X (Y/X) = Y/X_{B,M.} \wedge_X Y/X. \quad \text{(projection formula)}
\]
The base change isomorphisms follow from [15, Theorem 6.18(3)] and the projection formula is [15, Theorem 6.18(7)].
\[\square\]
Construction 2.12. Let $\pi_Y : Y \to X$ be an object of $\text{Sm}_X$ that is proper (i.e. a smooth projective scheme over $X$). We shall describe the construction of a duality between the objects $Y/X$ and $Y/X_{B,\text{M}}$ in $\text{SH}(X)$.

We first construct the coevaluation map $\delta_{Y/X} : 1_Y \to Y/X \land_X Y/X_{B,\text{M}}$. Applying the functoriality of $(-)/X_{B,\text{M}}$ from Remark 2.5 to the proper map $\pi_Y$ gives the map

$$\pi_Y^* : 1_Y = X/X_{B,\text{M}} \to Y/X_{B,\text{M}} = \pi_Y!(1_Y) \cong \pi_Y!(\Sigma^{T_Y/X}(1_Y))$$

in $\text{SH}(X)$, and the diagonal $\Delta_{Y/X} : Y \to Y \times_X Y$ induces via the functoriality of $(-)/Y$ discussed in Remark 2.7 the map

$$\Delta_{Y/X*} : 1_Y = Y/Y \to Y \times_Y Y/Y$$

in $\text{SH}(Y)$. We may put together these two maps to obtain the composition

$$1_Y \xrightarrow{\pi_Y^*} \pi_Y!(\Sigma^{T_Y/X}(1_Y)) \xrightarrow{\pi_Y!(\Sigma^{T_Y/X}(\Delta_{Y/X})))} \pi_Y!(\Sigma^{T_Y/X}(Y \times_Y Y/Y)),$$

where we consider $\pi_Y$ denotes the symmetry isomorphism, and $\pi_Y^* = \pi_Y!(1_Y)$.

Using the identification $\pi_Y!(\Sigma^{T_Y/X}(Y \times_Y Y/Y)) \cong Y/X \land_X Y/X_{B,\text{M}}$ from Lemma 2.11, this gives the desired map $\delta_{Y/X}$.

We now construct the evaluation map $\epsilon_{Y/X} : Y/X_{B,\text{M}} \land_X Y/X \to 1_X$ (which does not require properness of $\pi_Y$). Here we apply the functoriality of $(-)/Y$ to the proper map $\Delta_{Y/X}$ to obtain the map

$$\Delta_{Y/X} : Y \times_X Y/X_{B,\text{M}} \to Y/Y_{B,\text{M}} = 1_Y$$

in $\text{SH}(Y)$, and the functoriality of $(-)/Y$ to the map $\pi_Y$ to obtain the map

$$\pi_{Y*} : Y/X \to X/X = 1_X.$$

Putting these together yields the composition

$$\pi_{Y_2}(Y \times_X Y/Y_{B,\text{M}}) \xrightarrow{\pi_{Y/X}(\Delta')} \pi_{Y/X_2}(1_Y) = Y/X \xrightarrow{\pi_Y} 1_X.$$

Now using the identification $Y/X_{B,\text{M}} \land_X Y/X \cong \pi_{Y_2}(Y \times_X Y/Y_{B,\text{M}})$ from Lemma 2.11, we get the desired map $\epsilon_{Y/X}$.

It is shown in [15] Corollary 6.13 that the triple $(Y/X_{B,\text{M}}, \delta_{Y/X}, \epsilon_{Y/X})$ is the dual of $Y/X$ in $\text{SH}(X)$.

The above duality allows one to define an Euler characteristic for smooth projective schemes:

Definition 2.13. For $Y \to X$ in $\text{Sm}_X$ and proper, the Euler characteristic $\chi(Y/X) \in \text{End}_{\text{SH}(X)}(1_X)$ is the composition

$$1_X \xrightarrow{\delta_{Y/X}} Y/X \land_X Y/X_{B,\text{M}} \xrightarrow{\tau} Y/X_{B,\text{M}} \land_X Y/X \xrightarrow{\epsilon_{Y/X}} 1_X,$$

where $\tau$ denotes the symmetry isomorphism, and $\delta_{Y/X}$ and $\epsilon_{Y/X}$ are as constructed.

To finish this section, we give establish an alternative characterization of the Euler characteristic $\chi(Y/X)$ just defined. We proved this result (Lemma 2.15) below in the case $B = \text{Spec} k$ in [21]; the proof in this more general setting is exactly the same.

Construction 2.14. The diagonal $\Delta_X : X \to X \times_B X$ induces the map

$$\text{Th}(\Delta) := \pi_{X_1} \Sigma^{T_X/B}(\Delta) : \text{Th}(\neg T_{X/B}) \to \text{Th}(\neg p_1^* T_{X/B}) = \text{Th}(\neg T_{X/B}) \land_B X/B,$$

where we consider $X \times_B X$ as an $X$-scheme via $p_1$. Furthermore, the Morel-Voevodsky purity isomorphism gives the isomorphism in $H_0(B)$

$$X \times_B X/(X \times_B X \setminus \Delta_X(X) \to X \times_B X) \cong N_{\Delta_x}$$

which induces the isomorphism in $\text{SH}(B)$

$$\text{Th}(\neg p_1^* T_{X/B})/\text{Th}(\neg j^* p_1^* T_{X/B}) \cong \text{Th}(\neg T_{X/B} \oplus N_{\Delta_x}).$$

where $j : X \times_B X \setminus \Delta_X(X) \to X \times_B X$ is the inclusion.
We define the map $\beta_{X/B} : \text{Th}(-T_{X/B}) \to X/B$ as the composition

$$\text{Th}(-T_{X/B}) \xrightarrow{\text{Th}(\Delta_X)} \text{Th}(-p_1^*T_{X/B}) \xrightarrow{\pi} \text{Th}(-p_1^*T_{X/B})/\text{Th}(-j^*p_1^*T_{X/B}) \cong \text{Th}(-T_{X/B} \oplus N_{\Delta_X}) \cong \text{Th}(-T_{X/B} \oplus T_{X/B}) \cong \Sigma_0 \infty X_+$$

Here the isomorphism $N_{\Delta_X} \cong T_{X/B}$ is furnished by the composition

$$T_{X/B} \xrightarrow{p_1^*} \Delta_X^* p_2^* T_{X/B} \cong \Delta_X^* (p_1^* T_{X/B} \oplus p_2^* T_{X/B}) \cong \Delta_X^* T_{X \times_B X/B} \xrightarrow{\pi} N_{\Delta_X}$$

**Lemma 2.15.** For $X$ smooth and proper over $B$, $\chi(X/B)$ is equal to the composition

$$1_B \xrightarrow{\pi_X} \text{Th}(-T_{X/B}) \xrightarrow{\beta_{X/B}} X/B \xrightarrow{\pi_X} 1_B$$

**Proof.** Let $p_i : X \times_B X \to X$, $i = 1, 2$, be the projections. The map $\text{ev}_X : X/B^\vee \wedge_B X/B \to 1_B$ is the composition

$$X/B^\vee \wedge_B X/B = \text{Th}(-T_{X/k}) \wedge_B X/B$$

$$= \text{Th}(-p_1^*T_{X/k}) \xrightarrow{\pi_X} \text{Th}(-p_1^*T_{X/B})/\text{Th}(-j^*p_1^*T_{X/B}) \cong \text{Th}(-T_{X/B} \oplus N_{\Delta_X}) \cong \text{Th}(-T_{X/B} \oplus T_{X/B}) \cong X/B \xrightarrow{\pi_X} 1_B.$$ 

Thus $\pi_X \circ \beta_{X/B} = \text{ev}_X \circ \text{Th}(\Delta_X)$. Also, $\pi_X^* = \pi_X^*$ and $\pi_X^*$ is given by

$$1_B \xrightarrow{\delta_{X/B}} X/B \wedge X/B^\vee = \text{Th}(-p_2^*T_{X/B}) \xrightarrow{p_2} \text{Th}(-T_{X/B}).$$

It follows from the construction of the map $\delta_{X/B}$ described above that

$$\delta_{X/B} = \text{Th}(\Delta_X) \circ \pi_X^*.$$ 

This gives us the commutative diagram

\[
\begin{tikzcd}
X/B \wedge_B X/B^\vee \ar{rr}{\text{ev}_X} \ar{dd}{\delta_{X/B}} & & X/B^\vee \wedge_B X/B \ar{dd}{\pi_X^*} \\
\text{Th}(-p_1^*T_{X/B}) \ar{rr}{\pi_X} \ar{ur}{\text{Th}(\Delta_X)} & & \text{Th}(-p_1^*T_{X/B}) \ar{ur}{\delta_{X/B}} \\
\text{Th}(-T_{X/B}) \ar{rr}{\beta_{X/B}} & & X/B \ar{uu}{\pi_X^*}
\end{tikzcd}
\]

□

3. **SL-oriented theories**

Ananyevskiy discusses SL-oriented theories in [2] in the context of the motivic stable homotopy category $\text{SH}(k)$, where $k$ is a field. Essentially all of his constructions go through without change in the setting of a separated noetherian base-scheme $B$ of finite Krull dimension; the most one needs to do is replace a few of his arguments that rely on Jonanolou covers with some properties coming out of the six-functor formalism. We will recall and suitably extend Ananyevskiy’s treatment here without any claim of originality.

**Definition 3.1.** A *motivic commutative ring spectrum* in $\text{SH}(B)$ is a triple $(E, \mu, u)$ with $E$ in $\text{SH}(B)$, and $\mu : E \wedge_B E \to E$, $u : 1_B \to E$ morphisms in $\text{SH}(B)$, defining a symmetric monoid object in the symmetric monoidal category $\text{SH}(B)$. We usually drop the explicit mention of the multiplication $\mu$ and unit $u$ unless these are needed.
Notation 3.2. For a rank $r$ vector bundle $V \to X$, we have the determinant bundle $\det V \to X$ given as $\det V = \Lambda^r V$. For $V_1 \to X$, $V_2 \to X$ vector bundles, we have a canonical isomorphism

$$\alpha_{V_1, V_2} : \det(V_1 \oplus V_2) \to \det V_1 \otimes_{\mathcal{O}_X} \det V_2$$

classified by requiring, for a local basis of sections $s_1, \ldots, s_n$ of $V_1$ and $s_1', \ldots, s_m'$ of $V_2$, we have

$$\alpha_{V_1, V_2}((s_1^1, 0) \wedge \ldots \wedge (s_n^1, 0) \wedge (0, s_1^2) \wedge \ldots \wedge (0, s_m^2)) = (s_1^1 \wedge \ldots \wedge s_n^1) \otimes (s_1^2 \wedge \ldots \wedge s_m^2).$$

This extends to a canonical and natural isomorphism

$$\alpha_E : \det(V) \to \det V_1 \otimes_{\mathcal{O}_X} \det V_2$$

for each exact sequence of vector bundles on $X$.

\(\text{(E)}\)

$$0 \to V_1 \to V \to V_2 \to 0,$$

for example by choosing local splittings.

Definition 3.3. An SL-orientation on a motivic commutative ring spectrum $E$ in $\text{SH}(B)$ is given by an element $\text{th}_{V, \theta} \in \mathcal{E}^{2r, r}(\text{Th}(V))$ for each pair $(V, \theta)$ consisting of a rank $r$ vector bundle $V \to X$ with $X \in \text{Sm}_B$ and an isomorphism $\theta : \det V \to \mathcal{O}_X$ of invertible sheaves, satisfying the following axioms:

(i) **Functoriality:** For $f : Y \to X$ in $\text{Sm}_B$, we have the vector bundle $f^* V \to Y$ and isomorphism $f^* \theta : \det f^* V \cong f^* \det V \to \mathcal{O}_Y$. Then $f^*(\text{th}_{V, \theta}) = \text{th}_{f^* V, f^* \theta}$.

(ii) **Products:** Given vector bundles $V_1 \to X$, $V_2 \to X$ and isomorphisms $\theta_1 : \det V_1 \to \mathcal{O}_X$, $\theta_2 : \det V_2 \to \mathcal{O}_X$, we have the canonical isomorphism $\theta_1 \wedge \theta_2 : \det(V_1 \oplus V_2) \to \mathcal{O}_X$ defined as

$$\theta_1 \wedge \theta_2 = (\theta_1 \otimes \theta_2) \circ \alpha_{V_1, V_2}.$$

Then $\text{th}_{V_1 \oplus V_2, \theta_1 \wedge \theta_2} = \text{th}_{V_1, \theta_1} \cup \text{th}_{V_2, \theta_2}$.

(iii) **Normalization:** Let $V = \mathcal{O}_X$ with $\theta_V : \mathcal{O}_X \to \mathcal{O}_X$ the identity. Then $\text{Th}(V) = \Sigma_T X_+$ and $\text{th}_{V, \alpha} \in \mathcal{E}^{2, 1}(\text{Th}(V))$ is the image of the unit $u \in \mathcal{E}^{0, 0}(B)$ under the composition

$$\mathcal{E}^{0, 0}(B) \xrightarrow{\pi_X} \mathcal{E}^{0, 0}(X) \xrightarrow{\text{suspension}} \mathcal{E}^{2, 1}(\Sigma_T X_+).$$

An SL-oriented motivic spectrum is a pair $(E, \text{th}_-)$ with $E$ a motivic commutative ring spectrum and $\text{th}_-$ an SL-orientation on $E$.

Variant 3.4. A GL-orientation or simply orientation on a motivic commutative ring spectrum $E$ is an assignment $(V \to X) \mapsto \text{th}_V \in \mathcal{E}^{2r, r}(\text{Th}(V))$, where $V \to X$ is a rank $r$ vector bundle on $X \in \text{Sm}_B$, satisfying the evident modifications of the axioms (i)-(iii) in Definition 3.3, i.e. deleting the conditions on the determinant line bundle. The pair $(E, \text{th}_-)$ is then called a GL-oriented motivic spectrum or more simply, an oriented motivic spectrum.

Notation 3.5. For $E \in \text{SH}(B)$, $X \in \text{Sm}_B$ and $Z \subset X$ a closed subset, we have the $E$-cohomology with supports $\mathcal{E}_Z^a(X)$ defined as

$$\mathcal{E}_Z^a(X) := \mathcal{E}^a(X/(X \setminus Z)).$$

For instance, if $V \to X$ is a vector bundle, then $\mathcal{E}^{a, b}(\text{Th}(V)) = \mathcal{E}^a_0(V)$, where $0_V \subset V$ is the image of the 0-section. For $q : L \to X$ a line bundle on some $X \in \text{Sm}_B$, we define

$$\mathcal{E}^a,b(X; L) := \mathcal{E}^{a+2, b+1}(\text{Th}(L)) = \mathcal{E}^a_0(L).$$

and extend this to cohomology with supports in $Z \subset X$ by

$$\mathcal{E}_Z^{a,b}(X; L) := \mathcal{E}_Z^{a+2, b+1}(L).$$
Similarly, for a vector bundle \( q : V \to X \) and \( Z \subset X \) a closed subset, define \( \mathcal{E}_Z^{a,b}(\text{Th}(V)) \) as \( \mathcal{E}_Z^{a,b}(V) \).

**Lemma 3.6.** Let \( (\mathcal{E}, \text{th}(-)) \) be an SL-oriented motivic spectrum and let \( q : V \to X \) be a rank \( r \) vector bundle on \( X \) with isomorphism \( \theta : \det V \to \mathcal{O}_X \). Then sending \( x \in \mathcal{E}_Z^{a,b}(X) \) to \( q^*(x) \cup \theta_{V,\theta} \) defines an isomorphism
\[
\vartheta_{V,\theta} : \mathcal{E}_Z^{a,b}(X) \to \mathcal{E}_Z^{a+2r,b+r}(\text{Th}(V))
\]
natural in \((X,V,\theta)\).

**Proof.** The naturality of the maps \( \vartheta_{V,\theta} \) follows from the functoriality of the Thom classes (i).

By properties (i)-(iii) of the Thom class, it follows that for \( V = \bigoplus_{i=1}^n \mathcal{O}_X \), then \( \det V \to \mathcal{O}_X \) the canonical isomorphism \( \theta(e_1 \wedge \ldots \wedge e_n) = 1 \), the map \( \vartheta_{V,\theta} \) is the suspension isomorphism
\[
\mathcal{E}_Z^{a,b}(X) \cong \mathcal{E}_Z^{a+2r,b+r}(\Sigma_r X/\Sigma_r X \backslash Z) \cong \mathcal{E}_Z^{a+2r,b+r}(\text{Th}(V)).
\]
The naturality of the maps \( \vartheta_{V,\theta} \) allow one to use a Mayer-Vietoris sequence for a trivializing open cover of \( X \) for \( V \) to show that \( \vartheta_{V,\theta} \) is an isomorphism in general. \( \square \)

**Construction 3.7.** Let \( q : V \to X \) be a rank \( r \) vector bundle and let \( p : L \to X \) be the determinant bundle \( \det V \); let \( p' : L^{-1} \to X \) be the inverse of \( L \). Then we have canonical isomorphisms
\[
\alpha_{V,L^{-1}} : \det(V \oplus L^{-1}) \to \mathcal{O}_X, \quad \alpha_{L^{-1} \oplus L} : \det(L^{-1} \oplus L) \to \mathcal{O}_X.
\]

Let \( p_V : q^*(L^{-1} \oplus L) \to V, q_L \oplus p'_L : p^*(V \oplus L^{-1}) \to L \) be the pull-back bundles. For \((\mathcal{E}, \text{th}(-))\) an SL-oriented motivic spectrum, we thus have isomorphisms
\[
\vartheta_{q^*(L^{-1} \oplus L), q^* \alpha_{L^{-1} \oplus L}} : \mathcal{E}_0(V) \to \mathcal{E}_0(q^*(L^{-1} \oplus L))(p^*(V \oplus L^{-1}))
\]
and
\[
\vartheta_{p^*(V \oplus L^{-1}), p^* \alpha_{V \oplus L^{-1}}} : \mathcal{E}_0(L) \to \mathcal{E}_0(p^*(V \oplus L^{-1}))(p^*(V \oplus L^{-1}))
\]
However, as \( X \)-schemes \( q^*(L^{-1} \oplus L) \) and \( p^*(V \oplus L^{-1}) \) are both canonically isomorphic to \( V \oplus L^{-1} \oplus L \to X \), and via this isomorphism, the closed subsets \( q^{-1}(0_L \oplus 0_L) \cap p'_L^{-1}(0_V) \) and \( p^{-1}(0_V \oplus 0_L) \cap p'_L^{-1}(0_V) \) are both equal to \( 0_{V \oplus L^{-1} \oplus L} \). This gives us a canonical isomorphism
\[
\phi : \mathcal{E}_0(q^*(L^{-1} \oplus L))(p^*(V \oplus L^{-1}))(p^*(V \oplus L^{-1})) \to \mathcal{E}_0\left(\frac{L^{-1} \oplus L}{(0_{L^{-1}} \oplus 0_L) \cap p'_L^{-1}(0_V) \cap p^{-1}(0_V \oplus 0_L) \cap p'_L^{-1}(0_V)}\right)
\]
We set \( \mathcal{E}_b(X; L) := \mathcal{E}_a^{a+2r,b+r}(\text{Th}(L)) \) and define the Thom isomorphism
\[
\vartheta_V : \mathcal{E}_b(X; L) \to \mathcal{E}_b^{a+2r,b+r}(\text{Th}(V))
\]
as
\[
\vartheta_V := \vartheta_{r^{-1}(L^{-1} \oplus L), q^* \alpha_{L^{-1} \oplus L}} \circ \phi \circ \vartheta_{p^*(V \oplus L^{-1}), p^* \alpha_{V \oplus L^{-1}}}.
\]
We extend this construction to cohomology with supports in the evident manner.

**Notation 3.8.** The product structure on \( \mathcal{E} \)-cohomology induced by the multiplication on \( \mathcal{E} \) extends to a product structure on the “twisted” \( \mathcal{E} \)-cohomology
\[
\cup : \mathcal{E}_a(X; L) \otimes \mathcal{E}_c(X; M) \to \mathcal{E}_a^a(X; L \otimes M)
\]
as follows: We have the cup product
\[
\cup : \mathcal{E}_a^{a+2,b+1}(L) \otimes \mathcal{E}_c^{c+2,d+1}(M) \to \mathcal{E}_a^{a+c+4,b+4+d+2}(L \otimes M)
\]
and the canonical isomorphism \( \alpha_{L,M} : \det(L \oplus M) \to L \otimes M \) together with the Thom isomorphism \( \vartheta_{L \otimes M} \) gives the map
\[
\cup : \mathcal{E}^{a,b}(X; L) \otimes \mathcal{E}^{c,d}(X; M) \to \mathcal{E}^{a+c,b+d}(X; L \otimes M)
\]
as well as a version with supports.

**Definition 3.9.** Let \( q : V \to X \) be a rank \( r \) vector bundle, let \( p : q^* \det^{-1} V \to V \) be the pull-back of \( \det^{-1} V \to X \). The canonical Thom class \( \text{th}(V) \in \mathcal{E}^{2r,r}(\text{Th}(V); \det^{-1} V) \) is defined as follows: We have the canonical identification
\[
\mathcal{E}^{2r,r}(\text{Th}(V); \det^{-1} V) = \mathcal{E}^{0,r}_{V \otimes \det^{-1} V}(V \oplus \det^{-1} V)
\]
We have as well the isomorphism
\[
\alpha_{V, \det^{-1} V} : \det(V \oplus \det^{-1} V) \to \mathcal{O}_X
\]
giving us the Thom class \( \text{th}_{V \otimes \det^{-1} V, \alpha} \in \mathcal{E}^{2r+2, r+1}_{V \otimes \det^{-1} V}(V \oplus \det^{-1} V) \). We define \( \text{th}(V) \in \mathcal{E}^{2r,r}(\text{Th}(V); \det^{-1} V) \) to be element corresponding to
\[
\text{th}_{V \otimes \det^{-1} V, \alpha} \in \mathcal{E}^{2r+2, r+1}_{V \otimes \det^{-1} V}(q^* \det^{-1} V).
\]

In other words, for \( \pi_X : X \to B \) the structure morphism, the Thom class \( \text{th}(V) \) is an element of
\[
\mathcal{E}^{2r+2, r+1}(\pi_X^* (\Sigma^{\det^{-1}(V)}(1_X)))
\]
\[
= \text{Hom}_{\text{SH}(B)}(\pi_X^*(\Sigma^{\det^{-1}(V)}(1_X)), \mathcal{E})
\]
\[
= \text{Hom}_{\text{SH}(X)}(\Sigma^{\det^{-1}(V)}(1_X), S^{2r+2, r+1} \otimes \pi_X^* \mathcal{E})
\]
\[
= \text{Hom}_{\text{SH}(X)}(1_X, S^{2r+2, r+1} \otimes \Sigma^{-(\det^{-1}(V))} \pi_X^* \mathcal{E}).
\]

Via the multiplication on \( \mathcal{E} \), \( \text{th}(V) \) defines the map
\[
\times \text{th}(V) : \pi_X^* \mathcal{E} \to S^{2r+2, r+1} \otimes \Sigma^{-\det^{-1}(V)} \pi_X^* \mathcal{E}
\]
For each object \( x \in \text{SH}(X) \), we thus have the map
\[
\times \text{th}(V)_x : \text{Hom}_{\text{SH}(X)}(x, \pi_X^* \mathcal{E}) \to \text{Hom}_{\text{SH}(X)}(x, S^{2r+2, r+1} \otimes \Sigma^{-\det^{-1}(V)} \pi_X^* \mathcal{E})
\]
or applying the adjunction
\[
\times \text{th}(V)_x : \text{Hom}_{\text{SH}(B)}(\pi_X^* \mathcal{E}, x) \to \text{Hom}_{\text{SH}(B)}(\pi_X^* \Sigma^{\det^{-1}(V)} x, S^{2r+2, r+1} \otimes \mathcal{E})
\]

**Lemma 3.10.** The map \( \times \text{th}(V)_x \) is an isomorphism for all \( x \in \text{SH}(X) \).

**Proof.** Using the adjunction, we can just as well work in \( \text{SH}(X) \). The collection of objects \( x \) for which \( \times \text{th}(V)_x \) is an isomorphism is closed under arbitrary direct sums, hence is a localizing subcategory of \( \text{SH}(X) \). For \( x = 1_X \), we have already seen that \( \times \text{th}(V)_x \) is an isomorphism, and for \( p : Y \to X \) in \( \text{Sm}_X \), applying \( p^* \) and using the adjunction with \( p_! \) shows that \( \times \text{th}(V)_x \) is an isomorphism for \( x = Y/X \). Clearly \( \times \text{th}(V)_x \) being an isomorphism implies that \( \times \text{th}(V)_{S^{a,b} \otimes x} \) is an isomorphism for all \( a,b \); as \( \text{SH}(X) \) is generated as a localizing subcategory by the objects \( S^{a,b} Y/X \), this proves the lemma.

**Notation 3.11.** As a matter of notation, we define
\[
\mathcal{E}^{a,b}(\pi_X^* x, \text{Th}_X(V); L) := \text{Hom}_{\text{SH}(B)}(\pi_X^* \Sigma^{\det(L)} x, S^{a+2, b+1} \otimes \mathcal{E})
\]
\[
\mathcal{E}^{a,b}(x; L) := \text{Hom}_{\text{SH}(B)}(\pi_X^* \Sigma^L x, S^{a+2, b+1} \otimes \mathcal{E}),
\]
and the Thom isomorphism \( \times \text{th}(V)_x \) as
\[
\times \text{th}(V)_x : \mathcal{E}^{a,b}(x; L) \to \mathcal{E}^{a+2r, b+r}(\pi_X^* x, \text{Th}_X(V), L \otimes \det^{-1} V).
\]
One computes easily that the Thom isomorphism
\[ \partial_V : E^{a,b}(X; \det V) \to E^{a+2r,b+r}(\text{Th}(V)) \]
is given by \( \partial_V(x) = q^*(x) \cup \text{th}(V) \).

**Remark 3.12.** If the SL-orientation on \( E \) extends to a GL-orientation, then the analog of Lemma [5.5] holds: For \( V \to X \) a rank \( r \) vector bundle on \( X \in \text{Sm}_B \), sending \( x \in E^a_b(X) \) to \( q^*(x) \) defines an isomorphism
\[ \partial_V : E^{a,b}_q(X) \to E^{a+2r,b+r}_q(\text{Th}(V)) \]
natural in \( (X,V) \). In particular, the Thom classes for line bundles give isomorphisms
\[ \partial_L : E^{a,b}_q(X) \to E^{a,b}_q(X; L) \]
for each line bundle \( L \).

**Proposition 3.13.** Let \( E \) be an SL-oriented theory, \( L, M \to X \) line bundles on an \( X \in \text{Sm}_B \), \( Z \subset X \) a closed subset. There is a natural isomorphism
\[ \psi_{L,M} : E^{*,*}_Z(X; L) \to E^{*,*}_Z(X; L \otimes M^\otimes 2) \]

**Proof.** Let \( s : X \to L \otimes M \) be the zero-section. We have the Thom isomorphisms
\[ \begin{align*}
E^{*,*}_Z(X; L) &\cong E^{s+1,*,+2}_s(L \otimes M; M^{-1}), & E^{*,*}_Z(X; L \otimes M^\otimes 2) &\cong E^{s+1,*,+2}_s(L \otimes M; M),
\end{align*} \]
which reduces us to showing that there is a natural isomorphism
\[ \psi_L : E^{*,*}_Z(X; L) \to E^{*,*}_Z(X; L^{-1}) \]

For this, we follow the proof of [2, Lemma 2]. We have the morphism of \( X \)-schemes
\[ L \otimes L^{-1} = L \times_X L^{-1} \xrightarrow{\mu} X \times_B \mathbb{A}^1 \]
defined by \( \mu(x,y) = x \cdot y \), and let \( Y := \mu^{-1}(X \times 1) \). Letting \( L_0 = L \setminus L^{-1} \setminus L_0^{-1} \), \( Y \) is a closed subscheme of \( L \times_X L^{-1} \), projecting isomorphically to \( L_0 \) via \( p_1 \) and isomorphically to \( L^{-1} \) via \( p_2 \). Let \( p : L \to X \), \( q : L^{-1} \to X \) be the structure maps.

We have the commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{p_1} & L \\
| & & | \\
L_0 & \xleftarrow{p_1} & L \\
| & & | \\
L_0 & \xleftarrow{\tilde{p}_1} & L/L_0 \\
\end{array}
\]
with rows cofiber sequences. As the first two vertical maps are isomorphisms in \( H(B) \), the map \( \tilde{p}_1 \) induces an isomorphism
\[ \Sigma^\infty_T \tilde{p}_1 : \Sigma^\infty_T L \times_B L^{-1}/Y \to \Sigma^\infty_T \text{Th}(L) \]
in \( SH(B) \). Similarly, we have the isomorphism
\[ \Sigma^\infty_T \tilde{p}_2 : \Sigma^\infty_T L \times_B L^{-1}/Y \to \Sigma^\infty_T \text{Th}(L^{-1}) \]
in \( SH(B) \). Replacing \( X \) with \( U := X \setminus Z \), we have the isomorphisms
\[ \Sigma^\infty_T \tilde{p}_1 U : \Sigma^\infty_T L \times_B L^{-1} \times_X U/Y \times_X U \to \Sigma^\infty_T \text{Th}(L \times_X U) \]
and
\[ \Sigma^\infty_T \tilde{p}_2 U : \Sigma^\infty_T L \times_B L^{-1} \times_X U/Y \times_X U \to \Sigma^\infty_T \text{Th}(L^{-1} \times_X U) \]
This gives the diagram of isomorphisms (after applying $\Sigma^2$)

\[
\begin{array}{c}
\Sigma^{TX/B}(f^*) : X/B = \pi Xf(1_X) \to \pi Yf'\Sigma^{TX/B - T_Y/B}(1_Y)
\end{array}
\]

For $i : Y \to X$ a closed immersion, the short exact sequence

\[
0 \to T_Y/B \to f^*T_X/B \to N_f \to 0
\]

gives the isomorphism $\Sigma^{TX/B} - T_Y/B \cong \Sigma N_i$ and the map

\[
\Sigma^{TX/B}(i^*) : X/B \to \text{Th}(N_i).
\]

**Lemma 4.1.** (i) The map $\Sigma^{TX/B}(i^*) : X/B \to \text{Th}(N_i)$ is equal to the composition of the quotient map $X \to X/(X \setminus Y)$ followed by the Morel-Voevodsky purity isomorphism $X/(X \setminus Y) \cong \text{Th}(N_f)$.

(ii) Let $s : Y \to V$ be the zero-section of a vector bundle $p : V \to Y$. Then

\[
s^* : V/B_{B,M} \to Y/B_{B,M}
\]

is gotten by applying $\pi Y! \circ \Sigma^{-V}$ to the quotient map $V/Y \to \text{Th}_Y(V)$.

**Proof.** Since $i$ is a closed immersion $i_t = i_*$ and $\eta_t^* : i_t \to \eta$ is the identity. Thus $i^*$ is the composition

\[
\pi X!(1_X) \xrightarrow{\eta t} \pi Xi_iti^*(1_X) = \pi Xii_1iY = \pi Y!(1_Y).
\]

Letting $j : U := X \setminus Y \to X$ be the inclusion, the localization triangle

\[
jj^* \to \text{id}_X \to ji^*
\]

gives the isomorphism $i_iti^*(1_X) \cong X/(X \setminus Y) \cong \text{Th}(N_i)$, the latter isomorphism being the purity isomorphism. In addition, the identity $\pi Xii_1iY = \pi Y!(1_Y)$ gives after twisting by $\Sigma^{TX/B}$ the isomorphism

\[
\pi X\Sigma^{TX/B}i(1_Y) = \pi Xf\text{Th}(N_i).
\]

Writing $\pi X! = \pi Xf \circ \Sigma^{-TX/B}$ we see that $\Sigma^{TX/B}(i^*)$ is the composition

\[
X/B := \pi Xf(1_X) \xrightarrow{\eta t} \pi Xf_iiti^*(1_X) \cong \pi Xf\text{Th}(N_i)
\]

as claimed.

4. **Projective pushforward in twisted cohomology**

In this section we describe how one gets a pushforward map in twisted $\mathcal{E}$-cohomology for $\mathcal{E}$ an $\text{SL}$-oriented theory. We rely on the six-functor formalism. This is a bit different from the treatment of projective pushforward given by Ananyevskiy in [2]: in that treatment one relies on the factorization of an arbitrary projective morphism $Y \to X$ into a closed immersion $Y \to X \times \mathbb{P}^N$ followed by a projection $X \times \mathbb{P}^N \to X$. This factorization property will however reappear in our treatment when we discuss the unicity of the pushforward maps in section 6.

For $f : Y \to X$ a proper map in $\text{Sm}_B$ the map $f^*$ gives rise to

\[
\Sigma^{TX/B}(f^*) : X/B = \pi Xf(1_X) \to \pi Yf'\Sigma^{TX/B - T_Y/B}(1_Y)
\]

For $i : Y \to X$ a closed immersion, the short exact sequence

\[
0 \to T_Y/B \to f^*T_X/B \to N_f \to 0
\]

gives the isomorphism $\Sigma^{TX/B} - T_Y/B \cong \Sigma N_i$ and the map

\[
\Sigma^{TX/B}(i^*) : X/B \to \text{Th}(N_i).
\]
Statement (ii) follows from (i), noting that
\[
\Sigma^{-V} \text{Th}_Y(V) = \Sigma^{-V} \Sigma^V(1_Y) = 1_Y.
\]

**Notation 4.2.** For \( f : Y \to Z \) in \( \text{Sch}_B \) a smooth morphism, there is also natural transformation
\[
f_* : \pi_Y! \circ \Sigma^{T_{Y/Z}} \circ f^* \to \pi_Z!\]
In particular, for \( j : U \to X \) an open immersion, we have
\[
j_* : U/B_{B,M.} = \pi_U!(1_U) \to \pi_X!(1_X) = X/B_{B,M.}.
\]
For \( Z \subset X \) a closed subset with open complement \( U \), we define
\[
X_Z/B_{B,M.} := \pi_X!(\Sigma^\infty X/U).
\]
giving the canonical distinguished triangle in \( \text{SH}(B) \)
\[
U/B_{B,M.} \to X/B_{B,M.} \to X_Z/B_{B,M.} \to \Sigma_2^! U/B_{B,M.}
\]

**Lemma 4.3.** Let \( (\mathcal{E}, \text{th}_{(-)}) \) be an \( \text{SL} \)-oriented motivic spectrum in \( \text{SH}(B) \). Then for \( X \in \text{Sm}_B \) of dimension \( d \) over \( B \), \( Z \subset X \) a closed subset, \( p : L \to X \) a line bundle there is a canonical isomorphism
\[
\rho_{X,Z,L} : \mathcal{E}_Z^{a,b}(X; \omega_{X/B} \otimes L) \cong \mathcal{E}_{a-2d_X,b-d_X}(X_Z/B_{B,M.}; L)
\]

**Proof.** We have
\[
\mathcal{E}_Z^{a,b}(X; \omega_{X/B} \otimes L) = \text{Hom}_{\text{SH}(X)}(S^{a-4,-b-2} \land_X \Sigma^\omega_{X/B} \otimes L(X/(X\backslash Z)), \pi_X^* \mathcal{E})
\]
and
\[
\mathcal{E}_{a-2d_X,b-d_X}(X_Z/B_{B,M.}; L) = \text{Hom}_{\text{SH}(X)}(S^{2d_X-a-2,d_X-b-1} \land_X \Sigma^{-T_{X/B}} \otimes L(X/(X\backslash Z)), \pi_X^* \mathcal{E})
\]
Let \( x = S^{2d_X-a-2,d_X-b-1} \land_X \Sigma^{-T_{X/B}} \otimes L(X/(X\backslash Z)) \). Then since \( \omega_{X/B} = \text{det}^{-1} T_{X/B} \), we have
\[
\Sigma^{T_{X/B}} \otimes \text{det}^{-1} T_{X/B} \cong S^{2d_X-a-2,d_X-b-1} \land_X \Sigma^\omega_{X/B} \otimes L(X/(X\backslash Z))
\]
By Lemma 4.10 we have the isomorphism
\[
\mathcal{E}_{a-2d_X,b-d_X}(X_Z/B_{B,M.}; L) = \text{Hom}_{\text{SH}(X)}(x, \pi_X^* \mathcal{E})
\]
\[
\times \text{th}(T_{X/B})_* \cong \text{Hom}_{\text{SH}(X)}(\Sigma^{T_{X/B}} \otimes \text{det}^{-1} T_{X/B}, S^{2d_X+2,d_X+1} \land_X \pi_X^* \mathcal{E})
\]
\[
\cong \text{Hom}_{\text{SH}(X)}(S^{a-4,-b-2} \land_X \Sigma^\omega_{X/B} \otimes L(X/(X\backslash Z)), \pi_X^* \mathcal{E})
\]
\[
\delta(\omega_{X/B} \otimes L) \cong \mathcal{E}_Z^{a,b}(X; \omega_{X/B} \otimes L).
\]

Using the isomorphisms \( \rho_{X,Z,L} \), we make the following definition.

**Definition 4.4.** Let \( (\mathcal{E}, \text{th}_{(-)}) \) be an \( \text{SL} \)-oriented motivic spectrum in \( \text{SH}(B) \) let \( f : Y \to X \) be a proper morphism of relative dimension \( d \) in \( \text{Sm}_B, \) let \( L \to X \) be a line bundle and let \( Z \subset X \) be a closed subset set. Define
\[
f_* : \mathcal{E}_{f^{-1}(Z)}^{a,b}(Y, \omega_{Y/B} \otimes f^* L) \to \mathcal{E}_Z^{a-2d_X,b-d_X}(X, \omega_{X/B} \otimes L)
\]
to be the unique map making the diagram
\[
\begin{array}{ccc}
\mathcal{E}_{a-2d_Y-2,b-d_Y-1}^{a,b}(Y_{f^{-1}(Z)}/B_{B,M.}; f^* L) & (f^*)_* & \mathcal{E}_{a-2d_X-2,b-d_X-1}^{a,b}(X_Z/B_{B,M.}; L) \\
\rho_{Y,f^{-1}Z,f^*L} & & \rho_{X,Z,L} \\
\mathcal{E}_{f^{-1}(Z)}^{a,b}(Y, \omega_{Y/B} \otimes f^* L) & f_* & \mathcal{E}_Z^{a-2d_X,b-d_X}(X, \omega_{X/B} \otimes L)
\end{array}
\]
commute.

Let \( p : V \to Y \) be a rank \( r \) vector bundle on some \( Y \in \text{Sm}_B \), with 0-section \( s : Y \to V \). Letting \( L = \det V \), the exact sequence

\[
0 \to p^* V \to T_{V/B} \xrightarrow{dp} p^* T_{Y/B} \to 0
\]
gives the canonical isomorphism \( \omega_{V/B} \cong p^*(\det V \otimes \omega_{Y/B}) \), or \( p^* \det V \cong \omega_{V/B} \otimes \omega_{Y/B}^{-1} \).

Letting \( (E, \theta) \) be an SL-oriented theory, we have the pushforward map

\[
s_* : E_{a,b}^*(Y) \to E_{a+2r,b+r}^*(V, p^* \det V)
\]

**Lemma 4.5.** Let \( 1_Y^Y \in E_{0,0}^*(Y) \) be the unit \( \pi_Y^Y(u) \). Then \( s_*(1_Y^Y) \) is the image of \( \theta_Y \) in \( E_{2r,r}^*(V, p^* \det^{-1} V) \) under the “forget supports” map

\[
E_{0,v}^{2r,r}(V, p^* \det^{-1} V) \to E_{2r,r}^*(V, p^* \det^{-1} V).
\]

**Proof.** The exact sequence

\[
0 \to p^* V \to T_{V/B} \xrightarrow{dp} p^* T_{Y/B} \to 0
\]
gives us the isomorphism

\[
\omega_{V/B}^{-1} \otimes \det^{-1} V \cong p^* \omega_{Y/B}^{-1}.
\]

Keeping this in mind, we have the commutative diagram defining \( s_* \)

(4.1)

\[
\begin{array}{c}
E_{-2dY,-dY}^0(Y/B_{B,M}; \omega_{Y/B}^{-1}) \xrightarrow{(s_*)^*} E_{-2dY,-dY}^0(V/B_{B,M}; p^* \omega_{Y/B}^{-1})
\
\rho_{Y,Y,s^*p^*\omega_{Y/B}^{-1}} \downarrow \downarrow \rho_{V,v^*p^*\omega_{V/B}^{-1}}
\
E_Y^{0,0}(Y) \xrightarrow{s_*} E_{0,v}^{2r,r}(V, \det^{-1}_V)
\end{array}
\]

Here the lower \( s_* \) is the one we are considering and the upper \( s_* \) is the map with supports. Thus, we need to show that the upper \( s_* \) satisfies \( s_*(1_Y^Y) = \theta_Y \).

By Lemma 4.1(2), the map \( s^* : V_{0v}/B_{B,M} \to Y/B_{B,M} \) is gotten by applying \( \pi_Y \circ \Sigma^{-V} \) to identity map \( V_{0v}/Y \to \text{Th}_V(V) \); after twisting by \( \Sigma^{-T_{V/B}} \) this gives the map

\[
s^* : \Sigma^{-T_{V/B}}V_{0v}/Y \to \Sigma^{-T_{V/B}}(1_Y).
\]

The isomorphism \( \rho_{Y,Y,s^*p^*\omega_{Y/B}^{-1}} \) is the inverse of the Thom isomorphism

\[
(\pi_Y^Y \mathcal{E})^{-2dY,-dY}(\Sigma^{-T_{V/B}}(1_Y); \omega_{V/B}^{-1})
\]

\[
\xrightarrow{\theta(T_{V/B})} (\pi_Y^Y \mathcal{E})^{0,0}(\Sigma_{T_{V/B}} \Sigma^{-T_{V/B}}(1_Y)) = \mathcal{E}_{0,0}^0(Y)
\]

and the isomorphism \( \rho_{Y,0v^*p^*\omega_{Y/B}^{-1}} \) is similarly the inverse of the Thom isomorphism

\[
(\pi_Y^Y \mathcal{E})^{-2dY,-dY}(\Sigma^{-T_{V/B}}(V_{0v}/V), p^* \omega_{Y/B}^{-1})
\]

\[
\xrightarrow{\theta(T_{V/B})} (\pi_Y^Y \mathcal{E})^{2r,r}(\Sigma_{T_{V/B}} \Sigma^{-T_{V/B}}(V_{0v}/V), \omega_{V/B} \otimes p^* \omega_{Y/B}^{-1})
\]

\[
= \mathcal{E}_{2r,r}(\text{Th}_V(V), \det^{-1}_V).
\]
Using the isomorphism \( T_{V/B} \cong p^*(T_{Y/B} \oplus V) \), we can rewrite this as the composition
\[
(\pi_Y^* \mathcal{E})^{-2d_Y-d_Y} (p^* \Sigma^{-T_{V/B}}(\text{Th}_Y(V)), \omega_{V/B}^{-1})
\]
\[
\partial(T_{V/B} \oplus V) \to (\pi_Y^* \mathcal{E})^{2r,r} (\Sigma^{T_{V/B}+V} \Sigma^{-T_{V/B}}(\text{Th}_Y(V)), \omega_{V/B} \otimes p^* \omega_{V/B}^{-1})
\]
\[
= \mathcal{E}^{2r,r}(\text{Th}(V), \det_{V}^{-1}).
\]

The multiplicativity property (ii) of the Thom classes gives
\[
\partial(T_{Y/B} \oplus V) = \partial(V) \circ \partial(T_{Y/B})
\]
so putting all this together we see that
\[
s_* = \partial(T_{Y/B} \oplus V) \circ \partial(T_{Y/B})^{-1} = \partial(V). \quad \square
\]

**Remark 4.6.** If we have a GL-orientation on \( \mathcal{E} \), we have functorial pushforward maps
\[
f_* : \mathcal{E}^a_v(Y) \to \mathcal{E}^{a-2d, b-d}(X)
\]
for \( f : Y \to X \) a projective morphism in \( \text{Sm}_B \), of relative dimension \( d \), with \( W \subset Y \), \( Z \subset X \) closed subsets with \( f(W) \subset Z \). All the results of this section hold in the oriented context after deleting the twist by line bundles. This follows from Remark 3.12.

5. Motivic Gauss-Bonnet

**Definition 5.1.** Let \( p : V \to X \) be a rank \( r \) vector bundle on some \( X \in \text{Sm}_B \), and let \( \mathcal{E} \in \text{SH}(B) \) be an SL-oriented motivic ring spectrum. The *Euler class* \( e^\mathcal{E}(X) \in \mathcal{E}^{2r,r}(X, \det^{-1}V) \) is defined as
\[
e^\mathcal{E}(X) := s_* s_*(1^\mathcal{E}_X); \quad 1^\mathcal{E}_X \in \mathcal{E}^{0,0}(X) \ \text{the unit.}
\]

**Theorem 5.2** (Motivic Gauss-Bonnet). Let \( \mathcal{E} \in \text{SH}(B) \) be an SL-oriented motivic ring spectrum, \( \pi_X : X \to B \) a smooth and projective \( B \)-scheme, let \( u_\mathcal{E} : 1_B \to \mathcal{E} \) be the unit map. Then
\[
\pi_{X/B}* (e^\mathcal{E}(T_{X/B})) = u_\mathcal{E}*(\chi(X/B)) \in \mathcal{E}^{0,0}(B).
\]

**Proof.** We have the Thom isomorphism
\[
\partial_{-T_X/B} : \mathcal{E}^a_v(X; \omega_{X/N}) \to \mathcal{E}^{a-2d_X, b-d_X}(\text{Th}(-T_X/B)).
\]
By Lemma 2.13 it suffices to show that the map
\[
\beta_{X/B}^* : \mathcal{E}^{0,0}(X) \to \mathcal{E}^{0,0}(\text{Th}(-T_X/B))
\]
sends \( 1^\mathcal{E}_X \) to \( \partial_{-T_X/B}(e^\mathcal{E}(T_X/B)) \).

Following the definition of \( \beta_{X/B} \), we see that this map is gotten by applying \( \pi_{X/B}^* \circ \Sigma^{-T_{X/B}} \) to the following sequence (\( X \times_B X \) is an \( X \)-scheme via \( p_1 \))
\[
1_X \xrightarrow{\Delta_X} X \times_B X \xrightarrow{\pi_X} X \times_B X \Delta_X/X \cong \text{Th}_X(N_{\Delta_X}) \cong \text{Th}_X(T_{X/B})
\]
where the second to last map is the purity isomorphism and the last one is induced by the composition
\[
T_{X/B} \xrightarrow{i_X} T_{X/B} \oplus T_{X/B} \cong \Delta^*(p_1^* T_{X/B} \oplus p_2^* T_{X/B}) = \Delta^*(T_{X \times_B X/B}).
\]

Via this sequence, the inclusion \( \Delta_X \) gets first transformed to the 0-section of \( N_{\Delta} \) and then the zero-0 section of \( \text{Th}_X(T_{X/B}) \). But by Lemma 1.15, the pushforward map for the zero-section of a vector bundle is equal to the corresponding Thom isomorphism, in other words, the unit \( u_\mathcal{E} \in \mathcal{E}^{0,0}(X) \) gets mapped by \( \beta_{X/B} \) to its image under the Thom isomorphism, that is \( \text{th}(T_{X/B}) \in \mathcal{E}^{2d_X, d_X}(\text{Th}(T_{X/B}), \det^{-1}V) \), and thus
\[
e^\mathcal{E}_X(T_{X/B}) = \beta_{X/B}(u^\mathcal{E}_X).
\]

\[\square\]
6. SL-oriented cohomology theories

Our ultimate goal is to apply the Gauß-Bonnet theorem when projective pushforwards are defined on a representable cohomology theory in some concrete manner, not necessarily relying on the six-functor formalism. For this, we need a suitable axiomatic for such theories; we use a modification of the axioms of Panin-Smith [28, 29].

Definition 6.1. We let Sm-L/B denote the category of triples (X, Z, L) with X in Sm_B, Z ⊂ Z a closed subset and L → X a line bundle. A morphism (f, ̃f) : (X, Z, L) → (Y, W, M) is a morphism f : X → Y with Z ⊃ f^{-1}(W), together with an isomorphism of line bundles ̃f : L → f^*M. We let Sm-L^{pr}/B denote the category with the same objects as Sm-L/B, but with morphisms (f, ̃f) : (X, Z, L) → (Y, W, M) a proper morphism f : X → Y in Sm_B, with f(Z) ⊂ W, and ̃f : L → f^*M an isomorphism of line bundles.

Definition 6.2. An SL-oriented cohomology theory on Sm_B consists of the following data:

(D1) A functor H^{*, *} : Sm-L/B^{op} → Bi-gr Ab, (X, Z, L) ↦ H^{*, *}_Z(X; L); we often write f^* for H^{*, *}(f, ̃f).

(D2) A functor H_{*, *} : Sm-L^{pr}/B → GrAb, (X, Z, L) ↦ H^{Z, *}_X(X, L); we often write f_* for H_{*, *}(f, ̃f).

(D3) Natural isomorphisms, for X of dimension d_X

\[ H^{2d_X-n, d_X-m}_Z(X, \omega_X/B \otimes L) \xrightarrow{\alpha_{X,Z,L}} H^{Z, *}_{n,m}(X, L). \]

(D4) An element 1 ∈ H^0_0(B; O_B). For x := (X, Z, L), y := (Y, W, M) in Sm-L/B, a bi-graded cup product map

\[ \cup_{x,y} : H^{*}_Z(X, L) \otimes H^{*, *}_W(Y, M) → H^{*, *}_{Z \times W}(X \times_B Y, p_1^*L \otimes p_2^*M) \]

(D5) For Z ⊂ W closed subsets of an X ∈ Sm_B a bi-graded boundary map

\[ \delta^*_X,Z,W : H^{*, *}_{Z,W}(X \setminus W; j_W^*L) → H^{*, *-1}_W(X, L). \]

We write H^{*, *}(X, L) for H_X^{*, *}(X, L) and H^{*, *}_Z(X, L, O_X); we use the analogous notation for H_{*, *}. We write \cup for \cup_{x,y} and δ for δ^*_X,Z,W when the context makes the meaning clear.

For f : Y → X a proper map of relative dimension d in Sm_B, with Z ⊂ X, W ⊂ Y closed subsets with f(W) ⊂ Z and L → X a line bundle, combining D2 and D3 gives us pushforward maps

\[ f_* : H^*_W(Y, \omega_Y/B \otimes f^*L) → H^*_{Z, -2d, -d}(X, \omega_X/B \otimes L) \]

defined as the composition

\[ H^*_W(Y, \omega_Y/B \otimes f^*L) \xrightarrow{\alpha_{Y,W,f^*L}^{-1}} H^W_{2d_Y - n, d_Y - n}(Y, f^*L) \]

\[ f_*: H^Z_{2d_X - n, d_X - n}(X, L) \xrightarrow{\alpha_{X,Z,L}} H^*_{Z, -2d, -d}(X, \omega_X/B \otimes L). \]

These data satisfy are required to satisfy the following axioms:

(A1) H^{*, *} and H_{*, *} are additive: H^{*, *} transforms disjoint unions to products and H_{*, *} transforms disjoint unions to coproducts; in particular, H^{*, *}_Z(∅, L) = 0 and H^{Z, *}_{*, *}(∅, L) = 0.
(A2) Let

\[
\begin{array}{c}
Y' \xrightarrow{g'} Y \\
f' \downarrow \quad \downarrow f \\
X' \xrightarrow{g} X
\end{array}
\]

be a cartesian diagram in Sch, with \(X,Y,X',Y'\) in Sm (sometimes called a transverse cartesian diagram in Sm) and with \(f,f'\) proper of relative dimension \(d\). This gives us the isomorphism

\[
f'^*\omega_{Y'/X} \cong \omega_{Y'/Y}.
\]

Let \(Z \subset X\) be a closed subset, let \(W \subset Y\) be a closed subset with \(f(W) \subset Z\), let \(Z' = g^{-1}(Z)\), \(W' = g'^{-1}(W)\). Let \(L \rightarrow X\) be a line bundle on \(X\) and let \(L' = g'^*(L)\). Then the diagram

\[
\begin{array}{c}
H^*_W(Y', \omega_{Y'/B} \otimes \omega_{Y'/Y}^{-1} \otimes g^*L') \xrightarrow{g'^*} H^*_W(Y, \omega_{Y/B} \otimes f^*L) \\
\downarrow f' \quad \downarrow f \\
H^*_Z(X', \omega_{X'/B} \otimes \omega_{X'/X}^{-1} \otimes L') \xrightarrow{g^*} H^*_Z(X, \omega_{X/B} \otimes L)
\end{array}
\]

commutes.

(A3) For \(Z \subset W\) closed subsets of \(X \in Sm\), let \(U = X \setminus Z\) with inclusion \(j : U \rightarrow X\). For \(L \rightarrow X\) a line bundle, this gives us the morphisms \((id, id) : (X, W, L) \rightarrow (X, Z, L)\) and \((j, id) : (U, W \setminus Z, j^*L) \rightarrow (X, W, L)\). Then the sequence

\[
\cdots \xrightarrow{\delta_{Z,W,X}} H^*_Z(X, L) \rightarrow H^*_W(X, L)
\]

is exact. Moreover, the maps \(\delta_{Z,W,X}\) are natural with respect to the pullback maps \(g^*\) and the proper pushforward maps \(f_*\).

(A4) Let \(i : Y \rightarrow X\) be a closed immersion in Sm, let \(W \subset Y\) be a closed subset, \(L \rightarrow X\) a line bundle. Let \(Z = i(W)\), giving the morphism \((i, id) : (Y, W, i^*L) \rightarrow (X, Z, L)\) in Sm-L'. Then

\[
i_* : H^*_W(Y, i^*L) \rightarrow H^*_Z(X, L)
\]

is an isomorphism.

(A5) The cup products \(\cup\) of D4 are associative with unit 1. The maps \(f^*\) and \(f_*\) are compatible with cup products: \((f \times g)^*(\alpha \cup_{x,y} \beta) = f^*(\alpha) \cup_{x,y} g^*(\beta)\). Moreover, using the isomorphisms of D3, the cup products induce products \(\cup_{x,y}\) on \(H\) and one has \((f \times g)_*(\alpha \cup_{x,y} \beta) = f_*(\alpha) \cup_{x,y} g_*(\beta)\). Finally, the boundary maps \(\delta_{Z,W,X}\) are module morphism: retaining the notation of D4, for \(\alpha \in H^*_Z(W \setminus X; j^*_W L)\) and \(\beta \in H^*_T(Y, M)\), we have

\[
\delta_{X \times Y, Z \times T, W \times T}(\alpha \cup \beta) = \delta_{X,Z,W}(\alpha) \cup \beta.
\]

(A6) Let \(i : Y \rightarrow X\) be a closed immersion in Sm of codimension \(c\), \(\pi_Y : Y \rightarrow B\) the structure map. Let \(1^Y_B \in H^{0,0}(Y)\) be the element \(\pi_Y^*(1)\). Then \(\vartheta(i) := \alpha_{X,Y}(i_*(1^Y_B)) \in H^{p,c}_Y(X, det^{-1} N_i)\) is central, that is, for each \((U, T, M) \in Sm-L'/B\), and each \(\beta \in H^*_T(U, M)\), we have

\[
\tau^*(\beta \cup \vartheta(i)) = \vartheta(i) \cup \beta
\]

where \(\tau : X \times_B U \rightarrow U \times_B X\) is the symmetry isomorphism.
Let \( p \) be a morphism in \( \text{Sm-L}/B \). Suppose that the induced map \( f : Y/W/B \to X/Z/B \) is an isomorphism in \( \text{SH}(B) \). Then

\[
f^* : H^*_Z(Y) \to H^*_W(Y, f^*)
\]

is an isomorphism.

**Definition 6.3.** A twisted cohomology theory on \( \text{Sm}_B \) is given by the data \( D1, D4, D5 \) above, satisfying the parts of the axioms \( A1, A3-A7 \) that only involve \( H^{*,*} \).

Suppose in addition that one of the following conditions holds:

(i) the SL-orientation of \( E \) extends to a GL-orientation;

(ii) \( \eta \) acts invertibly on \( E \);

(iii) \( 2 \) acts invertibly on \( E \) and \( E^{-1,0}_+(U) = 0 \) for affine \( U \) in \( \text{Sm}_B \).

**Example 6.4.** The primary example of an SL-oriented cohomology theory on \( \text{Sm}_B \) is the one induced by an SL-oriented motivic spectrum \( \mathcal{E} \in \text{SH}(B) \):

\[
(X, Z, L) \mapsto \mathcal{E}_{X,Z}^*(X; L).
\]

One defines, for \( X \in \text{Sm}_B \) of dimension \( d_X \) over \( B \),

\[
\mathcal{E}_{m,n}^Z(X; L) := \mathcal{E}_{Z}^{2d_X - m, d_X - n}(X; \omega_{X/B} \otimes L);
\]

we extend the definition to arbitrary \( X \in \text{Sm}_B \) by taking the sum over the connected components of \( X \) and write this also as \( \mathcal{E}_{Z}^{2d_X - m, d_X - n}(X; \omega_{X/B} \otimes L) \) by considering \( d_X \) as a locally constant functor on \( X \).

The pushforward maps for a projective morphism of relative dimension \( d, f : Y \to X \), closed subsets \( W \subset Y, Z \subset X \) with \( f(W) \subset Z \) and line bundle \( L \to X \) are given by the pushforward

\[
f_* : \mathcal{E}_{Y,\omega}^{2d_Y - m, 2d_Y - n}(Y; \omega_Y/B \otimes f^* L) \to \mathcal{E}_{X,\omega}^{2d_X - m, 2d_X - n}(X; \omega_{X/B} \otimes L).
\]

7. **Comparison isomorphisms**

We recall the element \( \eta \in \text{Hom}_{\text{SH}(B)}(1_B, S^{-1,-1} \wedge 1_B) \) induced by the map of \( B \)-schemes \( \eta : \mathbb{A}^2 \{0\} \to \mathbb{P}^1, \eta(a, b) = (a : b) \). As every \( \mathcal{E} \in \text{SH}(B) \) is a module for \( 1_B \), we have the map \( x \eta : \mathcal{E} \to S^{-1,-1} \wedge \mathcal{E} \) for each \( x \in \text{SH}(B) \). We say that \( \mathcal{E} \) is an \( \eta \)-inverted spectrum if \( x \eta \) is an isomorphism in \( \text{SH}(B) \).

We consider the following situation: fix an SL-oriented motivic spectrum \( \mathcal{E} \in \text{SH}(B) \). This gives us the twisted cohomology theory \( \mathcal{E}^{*,*} \) underlying the oriented cohomology defined by \( \mathcal{E} \). Let \( (\mathcal{E}^{*,*}, \hat{\eta}_{*,*}) \) be an extension of \( \mathcal{E}^{*,*} \) to an oriented cohomology theory on \( \text{Sm}_B \), in other words, we define new pushforward maps

\[
\hat{f}_* : \mathcal{E}_{Y,\omega}^{*,*}(Y; \omega_Y/B \otimes f^* L) \to \mathcal{E}_{X,\omega}^{*,* - 2d, s - d}(X; \omega_{X/B} \otimes L)
\]

The main result of this section is a comparison theorem. Before stating the result we recall the decomposition of \( \text{SH}(B)[1/2] \) into plus and minus parts.

We have the involution \( \tau : 1_B \to 1_B \) induced by the symmetry isomorphism \( \tau : \mathbb{P}^1 \wedge \mathbb{P}^1 \to \mathbb{P}^1 \wedge \mathbb{P}^1 \). In \( \text{SH}(B)[1/2] \), this gives us the idempotents \((id + \tau)/2, (id - \tau)/2\), and so decomposes \( \text{SH}(B)[1/2] \) into \(+1\) and \(-1\) “eigenspaces” for \( \tau \):

\[
\text{SH}(B)[1/2] = \text{SH}(B)_+ \times \text{SH}(B)_-
\]

We decompose \( \mathcal{E} \in \text{SH}(B)[1/2] \) as \( \mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_- \).

**Theorem 7.1.** Suppose the pushforward maps

\[
f_* : \mathcal{E}_{W,\omega}^{*,*}(Y; \omega_Y/B \otimes f^* L) \to \mathcal{E}_{X,\omega}^{*,* - 2d, s - d}(X; \omega_{X/B} \otimes L)
\]

agree for \( W, Z, L, X = V \) a vector bundle over \( Y \) and \( f : Y \to V \) the zero-section. Suppose in addition that one of the following conditions holds:

(i) the SL-orientation of \( \mathcal{E} \) extends to a GL-orientation;

(ii) \( \eta \) acts invertibly on \( \mathcal{E} \);

(iii) \( 2 \) acts invertibly on \( \mathcal{E} \) and \( \mathcal{E}_{+}^{-1,0}(U) = 0 \) for affine \( U \) in \( \text{Sm}_B \).
Then \( f_* = \hat{f}_* \) for all \( X, Y, Z, W, L, f \) for which the push-forward is defined.

**Proof.** By the standard argument of deformation to the normal cone, it follows that \( f_* = \hat{f}_* \) for all \( f : Y \to X \) a closed immersion, \( Z, W, L \). As every proper map in \( \text{Sm}_B \) is projective, \( f \) admits a factorization \( f = p \circ i \), with \( i : Y \to X \times_B \mathbb{P}^N \) a closed immersion and \( p : X \times_B \mathbb{P}^N \to X \) the projection. By functoriality of the pushforward maps, it suffices to check that \( p_* = \hat{p}_* \).

In case (i), this follows from [29, Theorems 3.35, 3.36] and [28, Theorem 2.5]. Indeed, the cohomology theory associated to a GL-oriented motivic spectrum \( E \) satisfies the axioms of Panin-Smirnov, so we may apply the results cited.

In case (ii), we use Lemma 7.2 below. Indeed, if \( N \) is odd, we may apply the closed immersion \( X \times_B \mathbb{P}^N \to X \times_B \mathbb{P}^{N+1} \) as a hyperplane, so we reduce to the case \( N \) even, in which case both \( p_* \) and \( \hat{p}_* \) are inverse to the map \( i_* \), where \( i : X \to X \times_B \mathbb{P}^N \) is the section associated to the point \( (1 : 0 : \ldots : 0) \) of \( \mathbb{P}^N \).

In case (iii) we may work in the category \( \text{SH}(B)[1/2] \). We decompose \( E \in \text{SH}(B)[1/2] \) as \( E = E_+ \oplus E_- \) and similarly decompose the pushforward maps \( f_* \) and \( \hat{f}_* \). By Lemma 7.5, \( \eta \) acts invertibly on \( \text{SH}(B)^- \) and the projection of \( \eta \) to \( \text{SH}(B)^+ \) is zero. By Lemma 7.3 below, the SL-orientation of \( E \) induces an SL-orientation on the projection \( E^+_\ast \) that extends to a GL-orientation. By (i), this implies that \( f^+_\ast = \hat{f}^+_\ast \). By (ii), \( f^-_\ast = \hat{f}^-_\ast \), so \( f_\ast = \hat{f}_\ast \).

**Lemma 7.2 ([2, Theorem 1]).** Let \( E \in \text{SH}(B) \) be an SL-oriented motivic spectrum on which \( \eta \) acts invertibly. Let \( 0 \in \mathbb{P}^N(\mathbb{Z}) \) be the point \( (1 : 0 : \ldots : 0) \). For \( X \in \text{Sm}_B, L \to X \) a line bundle and \( Z \subset X \) a closed subset, the pushforward map

\[
i_\ast : E_Z^{\ast -2N, \ast -N}(X, \omega_X/B \otimes L) \to E_+^{\ast -1}(Z)(X \times_B \mathbb{P}^N, \omega_{\mathbb{P}^N/B} \otimes p^* L)
\]

is an isomorphism.

**Proof.** Using a Mayer-Vietoris sequence, we see that the statement is local on \( X \) for the Zariski topology, so we may assume that \( L = \mathcal{O}_X \). If we prove the statement for the pair \((X, X)\) and \((X \setminus Z, X \setminus Z)\) the local cohomology sequence gives the result for \((X, Z)\), thus we may assume that \( Z = X \), and we reduce to showing that

\[
i_\ast : E^{\ast -2N, \ast -N}(X, \omega_X/B) \to E^{\ast, \ast}(X \times_B \mathbb{P}^N, \omega_{\mathbb{P}^N/B})
\]

is an isomorphism.

This is [2, Theorem 4.6] in case \( B = \text{Spec} \, k \), \( k \) a field. The proof over a general base-scheme is exactly the same. \( \square \)

**Lemma 7.3.** Suppose that \( E \in \text{SH}(B) \) is SL oriented and that \( E^{-1, 0}(U) = 0 \) for all affine \( U \) in \( \text{Sm}_B \). Then the induced SL orientation on \( E_+ \in \text{SH}(B)_+ \) extends to a GL orientation.

**Proof.** Let \( u \in \Gamma(X, \mathcal{O}_X^\times) \) be a unit on some \( X \in \text{Sm}_B \). Then the map

\[
x \times u : X \times_B \mathbb{P}^1 \to X \times_B \mathbb{P}^1; \quad (x, [t_0 : t_1]) \mapsto (x, [ut_0 : t_1])
\]

induces the identity on \( S^{2, 1} \wedge \Sigma^\infty_+ X_+ \) in \( \text{SH}(B) \). Indeed, let \([u] : X/B \to X/B \wedge \mathbb{G}_m\) be the map induced by \( u : X \to \mathbb{G}_m \). The argument given by Morel [25, 6.3.4], that

\[
\Sigma^\infty_+ \times u = \text{id} + \eta[u]
\]

in case \( B = \text{Spec} \, k \), \( k \) a field, is perfectly valid over a general base-scheme: this only uses the fact that for \( X \) and \( Y \) pointed spaces over \( B \), one has

\[
\Sigma^\infty_+ X \times Y \cong \Sigma^\infty_+ X_+ \oplus \Sigma^\infty_+ Y \oplus \Sigma^\infty_+ X \wedge Y
\]

and that the map \( x \times u : S^1 \wedge \mathbb{G}_m \wedge X_+ \to S^1 \wedge \mathbb{G}_m \wedge X_+ \) is the \( S^1 \)-suspension of the composition

\[
S^1 \wedge \mathbb{G}_m \wedge X_+ \xrightarrow{\text{id} \wedge \mu \wedge \text{id}} S^1 \wedge (\mathbb{G}_m \times \mathbb{G}_m) \wedge X_+ \xrightarrow{\text{id} \wedge \mu \wedge \text{id}} S^1 \wedge \mathbb{G}_m \wedge X_+
\]
where \( \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \) is the multiplication. As \( \eta \) goes to zero in \( \text{SH}(B)_+ \), it follows that \( \Sigma^\infty \times u = \text{id} \) in \( \text{SH}(B)_+ \).

Now take \( g \in \Gamma(X, \text{GL}_n(O_X)) \), let \( u = \det g \), let \( m_u \in \Gamma(X, \text{GL}_n(O_X)) \) be the diagonal matrix with entries \( u, 1, \ldots, 1 \) and let \( h = m_u^{-1} \cdot g \in \Gamma(X, \text{SL}_n(O_X)) \). We have
\[
\text{Th}_X(O_X^n) = (\mathbb{P}^1)^n \times X_+
\]
Since \( E_+ \) is \( \text{SL} \)-oriented, the map \( \text{Th}(h) : \text{Th}_X(O_X^n) \to \text{Th}_X(O_X^n) \) induces the identity on \( E^{**} \) and thus
\[
\text{Th}(g)^* = \text{Th}(m_u)^* : E^{**}_+(\text{Th}_X(O_X^n)) \to E^{**}_+(\text{Th}_X(O_X^n))
\]
But as \( \text{Th}(m_u) = (x \circ u) \circ \text{id} \), our previous computation shows that \( \text{Th}(m_u)^* = \text{id} \).

Now let \( V \to X \) be a rank \( r \) vector bundle on some \( X \in \text{Sm}_B \), chose a trivializing affine open cover \( \mathcal{U} = \{ U_i \} \) of \( X \) and let \( \phi_i : V|_{U_i} \to U_i \times \mathbb{A}^r \) be a local framing. We have the suspension isomorphism
\[
\text{Th}(V_{U_i}) \cong \text{Th}(U_i \times \mathbb{A}^r) = \Sigma^r U_{i+}
\]
giving the isomorphism
\[
\theta_i : E^{a,b}(U_i) \to E^{2r+a,r+b}(V|_{U_i}).
\]
Since \( \text{GL}_r(O_U) \) acts trivially on \( E^{**}(\text{Th}(U_i \times \mathbb{A}^r)) \), the isomorphism \( \theta_i \) is independent of the choice of framing \( \phi_i \). In addition, the assumption \( \mathcal{E}^{-1,0}(U_i \cap U_j) = 0 \) implies
\[
\mathcal{E}^{2r-1,r}_{0V_{U_i \cap U_j}}(V|_{U_i \cap U_j}) = 0
\]
for all \( i, j \). By Mayer-Vietoris, the sections
\[
\theta_i(1_{U_i}) \in \mathcal{E}^{2r,r}_{V_{U_i}}(V|_{U_i})
\]
uniquely extend to an element
\[
\theta_V \in \mathcal{E}^{2r,r}_{V}(V)
\]
The independence of the \( \theta_i \) on the choice of framing and the uniqueness of the extension readily implies the functoriality of \( \theta_V \) and similarly implies the product formula \( \theta_V \circ \theta_W = \theta_{V \oplus W} = \theta_V \circ \theta_W \). By construction, \( \theta_V \) is the suspension of the unit over \( U_i \), another application of independence of the choice of framing and the uniqueness of the extension shows that this is the case over every open subset \( U \subset X \) for which \( V|_U \) is the trivial bundle. Finally, the independence and uniqueness shows that \( V \to \theta_V \) is an extension of the \( \text{SL} \) orientation on \( E_+ \) induced by that of \( E \).

**Lemma 7.4**. For \( u \in \Gamma(X, O_X^\infty) \) we have
\[
[u] \eta = \eta [u] : \Sigma^\infty X_+ \to \Sigma^\infty X_+ \otimes \mathbb{G}_m
\]
**Proof.** We use the decomposition
\[
\Sigma^\infty X_+ \otimes \mathbb{G}_m \otimes \mathbb{G}_m = \Sigma^\infty X_+ \otimes \mathbb{G}_m \oplus \Sigma^\infty X_+ \otimes \mathbb{G}_m \oplus \Sigma^\infty X_+ \otimes \mathbb{G}_m \otimes \mathbb{G}_m
\]
Via this, \( \eta \) is the map
\[
[s] \otimes [t] \mapsto [st] - [s] - [t]
\]
so \( \eta [u] \) sends \( [t] \) to \( [ut] - [u] - [t] \) and \( \text{id} \mathbb{G}_m \otimes \eta [u] \) sends \( [s] \otimes [t] \) to \( [s] \otimes [ut] - [s] \otimes [u] - [s] \otimes [t] \), so \( u \eta \) is given by
\[
[s] \otimes [t] \mapsto [st] - [s] - [t] \mapsto [u] \otimes [st] - [u] \otimes [s] - [u] \otimes [t].
\]
We have the automorphism \( \mathbb{G}_m^3 \) \( [u] \otimes [s] \otimes [t] \mapsto [s] \otimes [t] \otimes [u] \). As this is given by a linear map with determinant one, this is \( \mathbb{A}^1 \)-homotopic to the identity, so \( \text{id} \mathbb{G}_m \otimes \eta [u] \) is the map
\[
[s] \otimes [t] \mapsto [s] \otimes [t] \otimes [u] \mapsto [u] \otimes [s] \otimes [t] \mapsto [u] \otimes [st] - [u] \otimes [s] - [u] \otimes [t] = [u] \eta ([s] \otimes [t]).
\]
\( \square \)
Lemma 7.5. The projection η_- of η to SH(B)_- is an isomorphism and the projection η_+ of η to SH(B)_+ is zero.

Proof. Morel proves this in [25] §6 in the case of a field, but the proof works in general. In some detail, the map τ is the map on \( \mathbb{A}^2/(\mathbb{A}^2 \setminus \{0\}) \) induced by the linear map \((x, y) \mapsto (y, x)\). The matrix identity

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\cdot 
\begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}
\cdot 
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\cdot 
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

shows that the maps \((x, y) \mapsto (y, x)\) and \((x, y) \mapsto (-x, y)\) are \(\mathbb{A}^1\)-homotopic. By the arguments in Lemma 7.3 this latter map induces the map \(1 + \eta[-1] = 1 + [-1]\eta\) in SH(B), giving the identity

\[
(1 + \eta[-1])_\_ = (1 + [-1] \eta)_\_ = -\text{id} \mapsto \eta \cdot ([-1]/2) = (-1)/2 \cdot \eta = \text{id}_{\text{SH}(B)_-}
\]

For η_+, the projector to SH(B)_+ is given by the idempotent \((1/2)(\tau + 1) = (1/2)(2 + \eta[-1]), \) so \(\eta_+ = (1/2)\eta \cdot (2 + \eta[-1]). \)

Since the map \(\tau : \mathbb{P}^1 \wedge \mathbb{P}^1 \rightarrow \mathbb{P}^1 \wedge \mathbb{P}^1\) is 1 + \eta[-1] and \(\mathbb{P}^1 = \mathbb{A}^1 \wedge \mathbb{G}_m\), the symmetry \(e : \mathbb{G}_m \wedge \mathbb{G}_m \rightarrow \mathbb{G}_m \wedge \mathbb{G}_m\) is \(-1 \times \eta[-1]. \) From our formula for \(\eta([s] \wedge [t])\) we see that \(\eta e = \eta\) which gives \(\eta \cdot (2 + \eta[-1]) = 0. \)

8. Applications

In this section, we apply the motivic Gauss-Bonnet formula and the comparison results to make computations of the motivic Euler characteristic \(\chi(X/B)\) in different contexts.

8.1. Motivic cohomology and cohomology of the Milnor sheaves. We work over the base-scheme \(B = \text{Spec} k, \) with \(k\) a perfect field. In SH(k) we have the motivic cohomology spectrum \(HZ\) representing Voevodsky’s motivic cohomology (see e.g. [22] §6.2 for a construction valid in arbitrary characteristic). By [34], there is a natural isomorphism

\[
HZ^{a,b}(X) \cong \text{CH}^b(X, 2b - a)
\]

for \(X \in \text{Sm}_B, \) where \(\text{CH}^b(X, 2b - a)\) is Bloch’s higher Chow group [3].

\(HZ\) admits a localization sequence: for \(i : Z \rightarrow X\) a closed immersion of codimension \(d\) in \(\text{Sm}_k,\) there is a canonical isomorphism

\[
HZ_{\mathcal{Z}}^{a,b}(X) \cong HZ^{-a-2d,b-d}(Z)
\]

See for example [7]. In particular, for \(p : V \rightarrow X\) a rank \(r\) vector bundle over \(X \in \text{Sm}_k,\) we have the isomorphism

\[
HZ_{\mathcal{O}_V}(V) \cong HZ^{0,0}(X)
\]

which gives us Thom classes \(\partial_{HZ} \in HZ_{\mathcal{O}_V}(V)\) corresponding to the unit \(1_X^{HZ} \in HZ_{\mathcal{O}_V}(X). \) Thus \(HZ\) is a GL-oriented motivic spectrum.

Let \(X\) be a smooth projective \(k\)-scheme of dimension \(n\) over \(k.\) For a class \(x \in HZ^{2n,n}(X),\) the isomorphism \(HZ^{2n,n}(X) \cong \text{CH}^n(X, 0) = \text{CH}^n(X)\) allows one to represent \(x\) as the class of a 0-cycle \(\tilde{x} = \sum\ i \cdot p_i\), with the \(p_i\) closed points of \(X.\) One has the degree \(\text{deg}_k(p_i) := [k(p_i) : k]\) and extending by linearity gives the degree \(\text{deg}_k(\tilde{x}),\) which one shows passes to rational equivalence to define a degree map

\[
\text{deg}_k : HZ^{2n,n}(X) \rightarrow \text{CH}^0(\text{Spec} k) = \mathbb{Z}.
\]

As a GL-oriented theory, \(HZ\) has Chern classes for vector bundles: \(c_r(V) \in HZ^{2r,r}(X)\) for \(V \rightarrow X\) a vector bundle over some \(X \in \text{Sm}_k,\) \(r = 0, 1, \ldots,\)

Theorem 8.1. Let \(X \in \text{Sm}_k\) be projective of dimension \(d_X.\) Then

\[
u^{HZ}(\chi(X/k)) = \text{deg}_k(c_{d_X}(TX/k)).
\]
Proof. One has well-defined pushforward maps on $\text{CH}^*(\mathbb{A}^1, \ast)$ for projective morphisms (see e.g. []). Via the isomorphism $HZ^{a,b}(X) \cong \text{CH}^a(X, 2b - a)$ this gives push-forward maps $\tilde{f}_*$ on $HZ^{a,*}$ for $f : Y \to X$ a projective morphism in $\text{Sm}_B$, making $(X, Z) \to HZ^*_Z(X)$ a GL-oriented cohomology theory on $\text{Sm}_k$. In addition, for $\pi_X : X \to \text{Spec} k$ in $\text{Sm}_k$ projective of dimension $n$, the map $\pi_X^* : HZ_{2n,n}(X) \to \text{CH}^0(\text{Spec} k) = \mathbb{Z}$ is $\text{deg}_k$, and for $i : Y \to X$ a closed immersion, the map $i_*$ is given by the localization theorem, which readily implies that $i_* = i_*$. By our comparison theorem (really the theorem of Panin-Smirnov), it follows that $\tilde{f}_* = f_*$ for all projective $f$.

Finally, one has $c_{d_X} = s^* s_*(1_{X^\mathbb{A}^1}) = e^{HZ}(V)$ ([13, Corollary 6.3]), so applying the motivic Gauß-Bonnet theorem gives the statement. □

One can also obtain the same result by using the cohomology of the Milnor sheaves as bi-graded cohomology theory. The homotopy $t$-structure on $\text{SH}(k)$ has heart the abelian category of homotopy modules $\Pi_*(k)$; we let $H_0 : \text{SH}(k) \to \Pi_*(k)$ be the associated functor. The fact that $HZ^{a,n}(\text{Spec} F) = K^a_n(F)$ for $F$ a field says that $H_0 HZ$ is canonically isomorphic to the homotopy module $(K^a_n)_n$, which is in fact a cycle module in the sense of Rost. This gives us the isomorphism

$$H_0 HZ^{a,b}(X) \cong H^a(X, K^M_b)$$

The isomorphism (Bloch’s formula) $H^a(X, K^M_n) \cong \text{CH}^a(X)$ gives us as above Thom classes $0^{K^M_n}(V) \in H_0 HZ^{2r,r}(V)$, giving $H_0 HZ$ a GL-orientation. As for $HZ^{a,*}$ one has explicitly defined pushforward maps on $H^*(\mathbb{A}^1, K^M_n)$ which give $H_0 HZ^{a,*}$ the structure of a GL-oriented cohomology theory on $\text{Sm}_k$ and for which the pushforward map for the zero-section of a vector bundle is given by the Thom isomorphism. Since the pushforward on $H^a(X, K^M_n)$ agrees with the classical pushforward on $\text{CH}^a$, the same proof as for Theorem 8.1 gives

Theorem 8.2. Let $X \in \text{Sm}_k$ be projective of dimension $d_X$. Then

$$\text{deg}_k(c(d_X(T_{X/k}))) \in H^0(\text{Spec} k) = Z.$$

8.2. Algebraic $K$-theory. Let $B$ be a regular separated base-scheme of finite Krull dimension. Algebraic $K$-theory on $\text{Sm}_B$ is represented by the motivic commutative ring spectrum $KGL \in \text{SH}(B)$ (see []). Just as for $HZ$, the localization theorem:

$$KGL^{a,b}(X) \cong KGL^{a-2c,b-c}(Z)$$

for $i : Z \to X$ a closed immersion of codimension $d$ in $\text{Sm}_B$ gives Thom class $0^{KGL}(V) \in KGL^{2r,r}(V)$ for $V \to X$ a rank $r$ vector bundle over $X \in \text{Sm}_B$, and makes KGL a GL-oriented motivic spectrum.

Explicitly, KGL represents Quillen $K$-theory on $\text{Sm}_B$ via $KGL^{a,b} \cong K_{2b-a}$ and the Thom class for a rank $r$ vector bundle $p : V \to X$ is represented by the Koszul complex $\text{Kos}_V(p^* V^\vee, \text{can})$. Here can $: p^* V^\vee \to V$ is the dual of the tautological section $V \to p^* V$ and $\text{Kos}_V(p^* V^\vee, \text{can})$ is the complex with $\text{Kos}_V(p^* V^\vee, \text{can})^{-r} = \Lambda^p V^\vee$

with differential $\Lambda^r p^* V^\vee \to \Lambda^{r-1} p^* V^\vee$ given with respect to a local framing of $V^\vee$ by $d(e_i \wedge \ldots \wedge e_i) = \sum_{j=1}^r (-1)^{j-1} \text{can}(e_i) \cdot e_i \wedge \ldots \wedge e_i \wedge \ldots \wedge e_i$.

$\text{Kos}_V(p^* V^\vee, \text{can})$ is a locally free resolution of $s_*(\mathcal{O}_X)$, where $s : X \to V$ is the zero-section, and by the identification of $KGL^{2r,r}(V)$ with the Grothendieck group of the triangulated category of perfect complexes on $V$ with support contained in $0_V$, $\text{Kos}_V(p^* V^\vee, \text{can})$ gives rise to a class $[\text{Kos}_V(p^* V^\vee, \text{can})] \in KGL^{2r,r}(V)$, which maps to $1_X$ under the isomorphism $KGL^{2r,r}(V) \cong KGL^{0,0}(X)$ given by the localization theorem.
Just as for motivic cohomology, one has explicit pushforward maps in $K$-theory given by Quillen’s localization and devissage theorems identifying the $K$-theory with support $K^Z(X)$, for $Z \subset X$ a closed subscheme of $X \in \operatorname{Sm}_B$, with the $K$-theory of the abelian category of coherent sheaves $\operatorname{Coh}_Z$ on $Z$, denoted $G(Z)$. For a projective morphism $f : Y \to X$, one has $\hat{f}_* : G(Y) \to G(X)$ defined by using a suitable subcategory of $\operatorname{Coh}_Y$ on which $f_*$ is exact. On $K_0$, this recovers the usual formula

$$\hat{f}_*([\mathcal{F}]) = \sum_{j=0}^{\dim Y} (-1)^j [R^j f_*(\mathcal{F})]$$

for $\mathcal{F} \in \operatorname{Coh}_Y$. Via the isomorphisms $KGL^a_b(X) \cong G_{2a-b}(Z)$, this gives pushforward maps $\hat{f}_*$ for $KGL^a_b$, defining a GL-oriented cohomology theory on $\operatorname{Sm}_B$.

For the zero-section of a vector bundle, $s : X \to V$, $\hat{s}_*$ agrees with the pushforward $s_*$ using the Thom isomorphism/localization theorem, hence by our comparison theorem (really the theorem of Panin-Smirnov) we have $\hat{f}_* = f_*$ for all projective $f$.

**Theorem 8.3.** Let $\pi_X : X \to B$ be a smooth projective morphism with $B$ a regular separated scheme of finite Krull dimension. Then

$$u^\text{KGL}(\chi(X/B)) = \sum_{j=0}^{\dim_B X} \sum_{i=0}^{\dim_B X} (-1)^{j+i}[R^j \pi_X, \Omega^i_{X/B}] \in K_0(B) = KGL^{0,0}(B).$$

**Proof.** Let $p : T_{X/B} \to X$ be the relative tangent bundle and $s : X \to T_{X/B}$ the zero-section. We have

$$e^\text{KGL}(T_{X/B}) = s^*(\text{th}(T_{X/B})) = s^*(\text{Kos}_{T_{X/B}}(p^* T_{X/B}, \text{can}))$$

Since $T_{X/B} = \Omega_{X/B}$, and $s^*(\text{can})$ is the zero-map, it follows that, in $K_0(X)$,

$$s^*(\text{Kos}_{T_{X/B}}(p^* T_{X/B}, \text{can})) = \sum_{i=0}^{\dim_B X} (-1)^i[\Omega^i_{X/B}]$$

and thus

$$\pi_X(s^* e^\text{KGL}(T_{X/B})) = \sum_{j=0}^{\dim_B X} \sum_{i=0}^{\dim_B X} (-1)^{j+i}[R^j \pi_X, \Omega^i_{X/B}]$$

and we conclude by applying the motivic Gauß-Bonnet theorem. $\square$

### 8.3. Milnor-Witt cohomology and Chow-Witt groups

In this case, we work over a perfect base-field $k$. The Milnor-Witt sheaves $K^\text{MW}_{*}$ as constructed by Hopkins-Morel give rise to an $\text{SL}$-oriented theory as follows. Morel describes an isomorphism of $K^\text{MW}_0$ with the sheafification $\mathcal{GW}$ of the Grothendieck-Witt rings; the map of sheaves of abelian groups $\mathbb{G}_m \to \mathcal{GW}^\times$ sending a unit $u$ to the one-dimensional form $\langle u \rangle$ allows one to define a twisted version

$$K^\text{MW}_*(L) := K^\text{MW}_{*} \times_{\mathbb{G}_m} L^\times$$

as a Nisnevich sheaf on $X \in \operatorname{Sm}_k$ for $L \to X$ a line bundle. One may use the Rost-Schmid complex to compute $H^Z_2(X, K^\text{MW}_{*}(L))$ for $Z \subset X$ a closed subset, which gives a purity theorem: for $i : Z \to X$ a codimension $d$ closed immersion in $\operatorname{Sm}_k$ and $L \to X$ a line bundle, there is a canonical isomorphism

$$H^Z_2(X, K^\text{MW}_{*}(L)) \cong H^{*-d}(Z, K^\text{MW}_{*-d}(i^* L \otimes \det N_i)),$$

where $N_i \to Z$ is the normal bundle of $i$. Applying this to the zero-section of a rank $r$ vector bundle $p : V \to X$ gives the isomorphism

$$H^0_0(X, \mathcal{GW}) \cong H^r_0(V, K^\text{MW}_{r}(p^* \det^{-1} V))$$

in particular, an isomorphism $\phi : \det V \to \mathcal{O}_X$ gives the Thom class

$$\theta_{V, \phi} \in H^r_0(V, K^\text{MW}_r)$$
corresponding to the unit section \(1_X \in H^0(X, GW)\).

On the other hand, Morel’s computation of the 0th graded homotopy sheaf of the sphere spectrum [25] Theorem 6.4.1, Remark 6.4.2 gives the identity in \(\Pi_*(k)\)

\[
H_0(1_k) \cong (\mathcal{K}^\text{MW}_n)_{n \in \mathbb{Z}}
\]

which gives the natural isomorphism

\[
H_0(1_k)_{Z}^{s+b}(X) \cong H_0^Z(X, \mathcal{K}^\text{MW}_b)
\]

This is moreover compatible with twisting by a line bundle, on the \(H_0(1_k)\) side using the Thom space construction

\[
H_0(1_k)^{s+r}(X; L) := H_0(1_k)^{s+2,r+1}(L)
\]

and on the Milnor-Witt cohomology side using the twisted Milnor-Witt sheaves.

The Thom class \(\theta_{V, \phi} \in H^r_{\text{top}}(V, \mathcal{K}^\text{MW}_r)\) gives the Thom class

\[
\theta_{V, \phi} \in H_0(1_k)^{2r,r}(V)
\]

which make \(H_0(1_k)\) an SL-oriented theory (see e.g [20, §3.2]), and the resulting canonical Thom class

\[
\text{th}_V \in H_0(1_k)^{2r,r}(V; \det^{-1} V) = H^r_{\text{top}}(V, \mathcal{K}^\text{MW}_r(\det^{-1} V))
\]

agree with the image of \(1_X \in H^0(X, GW)\) under the Rost-Schmid isomorphism [25.1].

Moreover, under Morel’s isomorphism

\[
\text{End}_{\text{SH}(k)}(1_X) \cong GW(k)
\]

and the isomorphism

\[
H^0(\text{Spec } k, \mathcal{K}^\text{MW}_0) \cong GW(k)
\]

the unit map \(u_{H_0(1_k)} : 1_k \to H_0(1_k)\). induces the identity map on \(\pi_{0,0}\).

Let \(\pi_X : X \to \text{Spec } k\) be smooth and projective over \(k\) of dimension \(d\). Using the Rost-Schmid complex one has generators for \(H^d(X, \mathcal{K}^\text{MW}_d \omega_{X/k})\) as formal sums \(\tilde{x} = \sum \alpha_i \cdot p_i\), with \(\alpha_i \in k(p_i)\) and \(p_i \in X\) closed points. Since \(k\) is perfect, the finite extension \(k(p_i)/k\) is separable and one can define

\[
\widetilde{\deg}_k(\tilde{x}) := \sum \text{Tr}_{k(p_i)/k} \alpha_i \in GW(k)
\]

where \(\text{Tr}_{k(p_i)/k} : GW(k(p_i)) \to GW(k)\) is the Scharlau trace map. One shows that this descends to a map

\[
\widetilde{\deg}_k : H^{2d, d}_0(X; \omega_{X/k}) = H^d(X, \mathcal{K}^\text{MW}_d \omega_{X/k}) \to H^0_0(\text{Spec } k) = GW(k)
\]

The methods of this paper give a new proof of the result given in [20] Lemma 1.5

**Theorem 8.4.** Let \(k\) be a perfect field. For \(\pi_X : X \to \text{Spec } k\) smooth and projective over \(k\), we have

\[
\chi(X/k) = \widetilde{\deg}_k(e_{H_0(1_k)}(T_{X/k})).
\]

**Proof.** The proof is essentially the same as the other Gauß-Bonnet theorems we have discussed. Fasel [10] has defined pushforward maps

\[
\tilde{f} : H^*_W(X, \mathcal{K}^\text{MW}_b(\omega_{X/k} \otimes f^* L)) \to H^{a-d}_Z(Y, \mathcal{K}^\text{MW}_b(-d, L))
\]

for each projective morphism \(f : X \to Y\) in \(\text{Sm}_k\) of relative dimension \(d\) and line bundle \(L \to Y\), with \(Z \subset Y\), \(W \subset X\) closed subsets with \(f(W) \subset Z\). In the case of the structure map \(\pi_X : X \to \text{Spec } k\), the pushforward \(\pi_* : H^d(X, \mathcal{K}^\text{MW}_d \omega_{X/k}) \to H^0(\text{Spec } k, \mathcal{K}^\text{MW}_0) = GW(k)\) is the map \(\widetilde{\deg}_k\).

For \(s : X \to V\) the zero-section of a vector bundle, \(\tilde{s}_s\) is the Thom isomorphism \(s_*\). Thus, if we pass to the \(\eta\)-inverted theory, \(H_0(1_X)_{\eta} := H_0(1_k)[\eta^{-1}]\), our comparison theorem says that \(f_{\eta} \circ s_* = f_{\eta} \circ s_*\) for all projective morphisms \(f\) in \(\text{Sm}_k\). We have \(\mathcal{K}^\text{MW}_s[\eta^{-1}] \cong W\), the sheaf of Witt rings, and the map \(\mathcal{K}^\text{MW}_0 = GW \to \mathcal{K}^\text{MW}_0[\eta^{-1}] \cong..."
$W$ is the canonical map $q : \mathcal{G}W \rightarrow W$ realizing $W$ as the quotient of $\mathcal{G}W$ by the subgroup generated by the hyperbolic form. Applying our motivic Gauß-Bonnet theorem gives the identity

$$q(\chi(X/k)) = q(\deg_k(e^{H_0}(T_{X/k}))) \text{ in } W(k).$$

In addition, the map

$$(\text{rank}, q) : \mathcal{G}W \rightarrow \mathbb{Z} \times W$$

is injective. We can recover the rank by applying $H_0$ to the unit map $1_k : H \rightarrow H\mathbb{Z}$ and using Theorem 8.2, which completes the proof. □

8.4. Hermitian $K$-theory and Witt theory. The description of the “rank” of $\chi(X/B)$ given by Theorem 8.2.1 can be refined to give a formula for $\chi(X/k)$ itself in terms of Hodge cohomology by using hermitian $K$-theory. By work of Panin-Walter [30], Schlichting [31] and Schlichting-Tripathi [32], hermitian $K$-theory is represented by a motivic commutative ring spectrum $KO \in \mathcal{SH}(B)$ for $B$ a regular noetherian separated base-scheme of finite Krull dimension, assuming 2 is invertible on $B$. Panin-Walter give $KO$ an SL-orientation. There a direct connection with Schlichting’s Grothendieck-Witt groups [31] given by functorial isomorphisms

$$KO^{2r,r}(X; L) \cong \text{GW}(D^{\text{perf}}(X), L[r], \text{can})$$

where $L \rightarrow X$ is a line bundle and $\text{GW}(D^{\text{perf}}(X), L[r], \text{can})$ is the Grothendieck-Witt group of $L[r]$-valued quadratic forms on $D^{\text{perf}}(X)$.

**Definition 8.5.** Let $L \rightarrow X$ is a line bundle be a line bundle. An $L[n]$-valued quadratic form on $C \in D^{\text{perf}}(X)$ is a map

$$\phi : C \otimes^L C \rightarrow L[n]$$

in $D^{\text{perf}}(X)$ which is:

(i) non-degenerate: the induced map $C \rightarrow \mathcal{RHom}(C, L[n])$ is an isomorphism in $D^{\text{perf}}(X)$;

(ii) symmetric: $\phi \circ \tau = \phi$, where $\tau : C \otimes^L C \rightarrow C \otimes^L C$ is the commutativity isomorphism.

For a rank $r$ vector bundle $p : V \rightarrow X$, the Thom class $\theta^r_V \in KO^{2r,r}(V; p^* \text{ det}^{-1} V)$ is given by the Koszul complex $\text{Kos}(p^*V^\vee, s^\vee_{\text{can}})$, where the quadratic form

$$\phi : \text{Kos}(p^*V^\vee, s^\vee_{\text{can}}) \otimes \text{Kos}(p^*V^\vee, s^\vee_{\text{can}}) \rightarrow p^* \text{ det}^{-1} V[r] = \Lambda^r V^\vee[r]$$

is given by the usual exterior product

$$- \wedge - : \Lambda^i V^\vee \otimes \Lambda^{r-i} V^\vee \rightarrow \Lambda^r V^\vee.$$

Moreover, there are isomorphisms for $i < 0$

$$KO^{2r-i,r}(X; L) \cong W^{r-i}(D^{\text{perf}}(X), L[r], \text{can})$$

where $W^{r-i}(D^{\text{perf}}(X), L[r], \text{can})$ is Balmer’s triangulated Witt group. Ananyevskiy, in the case $B = \text{Spec} k$, with $2 \in k^\times$, shows that this isomorphism induces an isomorphism of $\eta$-inverted hermitian $K$-theory with Witt-theory

$$KO[\eta^{-1}]^{*,*} \cong W^*[\eta, \eta^{-1}]$$

where one gives $\eta$ bi-degree $(-1, -1)$ and an element $\alpha \eta^n$ with $\alpha \in W^m$ has bi-degree $(m-n, -n)$; the same proof works over arbitrary $B$ (with assumptions as at the beginning of this subsection).

For $f : Y \rightarrow X$ a proper map of relative dimension $d_f$ in $\text{Sm}_B$, we follow Calmes-Hornbostel in defining a push-forward map

$$\hat{f}_* : KO^{2r,r}(Y, \omega_{Y/B} \otimes f^* L) \rightarrow KO^{2r-2d_f,r-d_f}(X, \omega_{X/B} \otimes L)$$
by Grothendieck-Serre duality. In [S] this is worked out for the η-inverted theory KOη and for \( B = \text{Spec } k \), however, the same construction works for KO over the general base-scheme \( B \) and goes as follows: Given a quadratic form \( \phi : C \otimes L \rightarrow \omega_{Y/B} \otimes f^* L[r] \) we have the corresponding isomorphism

\[ \hat{\phi} : C \rightarrow \mathcal{R}\text{Hom}(C, \omega_{Y/k} \otimes f^* L[r]) = \mathcal{R}\text{Hom}(C, \omega_{Y/X} \otimes f^*(\omega_{X/k} \otimes L[r])) \]

Grothendieck-Serre duality gives the isomorphism

\[ Rf_*\mathcal{R}\text{Hom}(C, \omega_{Y/X} \otimes f^*(\omega_{X/k} \otimes L[r])) \xrightarrow{\psi} \mathcal{R}\text{Hom}(Rf_*C, \omega_{X/B} \otimes L[r - d_\eta]) \]

which gives the isomorphism

\[ \psi \circ \hat{\phi} : Rf_*C \rightarrow \mathcal{R}\text{Hom}(Rf_*C, \omega_{X/B} \otimes L[r - d_\eta]) \]

or

\[ Rf_!(\phi) : Rf_*C \otimes L \rightarrow \omega_{X/B} \otimes L[r - d_\eta] \]

which one shows is symmetric; explicitly, one has \( f_!(C, \phi) = (Rf_*C, Rf_!(\phi)) \).

If we apply this to \( V = X/B \), \( f = \pi_X : X \rightarrow B \) a smooth and proper \( B \)-scheme, we have the following formula for \( \hat{\pi}_X(e^\text{KO}(T_X/B)) \)

\[ \hat{\pi}_X(e^\text{KO}(T_X/B)) = (\oplus_{i,j=0}^{\dim X} \text{dim} R^i\pi_X, \text{dim} \Omega^j_{X/B}) [j - i], \text{Tr} \]

where \( \text{Tr} : (\oplus_{i,j=0}^{\dim X} R^i\pi_X, \text{dim} \Omega^j_{X/B}) [j - i] \otimes (\oplus_{i,j=0}^{\dim X} R^i\pi_X, \text{dim} \Omega^j_{X/B}) [j - i] \rightarrow O_B \)

is the quadratic form in \( D^b(B) \) given by the composition

\[ R^i\pi_X, \text{dim} \Omega^j_{X/B} \otimes R^{d_\pi X - i} \pi_X, \text{dim} \Omega^{d_\pi X - j}_{X/B} \rightarrow R^{d_\pi X} \pi_X, \text{dim} \Omega^{d_\pi X}_{X/B} \rightarrow \rightarrow \text{Tr}, O_B, \text{dim} X = \dim_\pi X \]

Indeed, for \( s : X \rightarrow T_X/B \) the 0-section, we have

\[ e^\text{KO}(T_X/B) = s^*(\text{Kos}(T_X/B), \phi) = (\oplus_{i,j=0}^{d_\pi X} \text{dim} \Omega^j_{X/B}) [j - i], s^*\phi \]

with \( s^*\phi \) given by the products

\[ \text{dim} \Omega^j_{X/B} [j] \otimes \text{dim} \Omega^{d_\pi X - j}_{X/B} [d_\pi X - j] \rightarrow \omega_{X/B} [d_\pi X] \]

and thus \( \hat{\pi}_X(e^\text{KO}(T_X/B)) \) is \( \oplus_{i,j=0}^{\dim X} R^i\pi_X, \text{dim} \Omega^j_{X/B} [j - i] \) with the quadratic form \( \text{Tr} \) as described above.

By passing to the \( \eta \)-inverted theory \( KO_\eta \), our comparison theorem gives

\[ q \circ \hat{\pi}_X = q \circ \pi_X \]

as maps \( KO^{2d_\pi d}(X, \omega_{X/B}) \rightarrow KO_\eta(B) \).

**Theorem 8.6.** Let \( X \) be a smooth projective \( B \)-scheme.

1. We have

\[ u^{\text{KO}_\eta}(\chi(X/B)) = (\oplus_{i,j=0}^{\dim X} R^i\pi_X, \text{dim} \Omega^j_{X/B}) [j - i], \text{Tr} \]

in \( KO^{0,0}_\eta(B) = W(D^b(B)) = W(B) \).

2. We have the forgetful map \( f : GW(B) \rightarrow K_0(B) \) forgetting the quadratic form. Suppose that the map

\[ (f, q) : GW(B) \rightarrow K_0(B) \times W(B) \]

is injective (this is the case if for example \( B \) is the spectrum of a local ring) then

\[ u^{\text{KO}_\eta}(\chi(X/B)) = (\oplus_{i,j=0}^{\dim X} H^i(X, \Omega^j_{X/B}) [j - i], \text{Tr}) \]

in \( GW(D^b(B)) = GW(B) \).
(3) Suppose $B$ is in $\text{Sm}_k$ for $k$ a perfect field. Then the image $\chi(X/B)$ of $\chi(X/B)$ in $\pi_{0,0}(1_B)(B) = H^0(B, \mathcal{G}W)$ is given by

$$\chi(X/B) = \left[\left(\bigoplus_{i,j=0}^{\dim X} R^i\pi_X^*\Omega^j_{X/B}\right)[j-i], \text{Tr} \right] \in H^0(B, \mathcal{G}W).$$

In particular, if $B = \text{Spec} k$, then

$$\chi(X/k) = \left(\bigoplus_{i,j=0}^{\dim X} H^i(X, \Omega^j_{X/k})[j-i], \text{Tr} \right) \in \text{GW}(k).$$

**Proof.** The first statement follows from our comparison theorem and the motivic Gauß-Bonnet theorem. (2) follows from (1) and the case of algebraic $K$-theory Theorem 8.3. Finally, (3) follows from (2): from Morel’s theorem identifying $1^{0,0}_k(\text{Spec} k)$ with $\text{GW}(k)$, it follows that the unit $u^\text{KO}$ induces an isomorphism

$$1^{0,0}_k(\text{Spec} k) \cong \text{KO}^{0,0}(\text{Spec} k) = \text{GW}(k).$$

**Corollary 8.7.** 1. For $X$ a smooth and projective $k$-scheme of odd dimension $2n - 1$, $\chi(X/k) = m \cdot H$, $H$ the hyperbolic form $x^2 - y^2$, with

$$m = \sum_{i+j<2n-1} \dim_k H^i(X, \Omega^j_{X/k}) + \sum_{0 \leq i < j \atop i+j = 2n-1} \dim_k H^i(X, \Omega^j_{X/k}).$$

2. For $X$ a smooth and projective $k$-scheme of even dimension $2n$, $\chi(X/k) = m \cdot H + Q$, with

$$m = \sum_{i+j<2n} \dim_k H^i(X, \Omega^j_{X/k}) + \sum_{0 \leq i < j \atop i+j = 2n} \dim_k H^i(X, \Omega^j_{X/k})$$

and $Q$ the symmetric bilinear form

$$H^n(X, \Omega^n_{X/k}) \times H^n(X, \Omega^n_{X/k}) \to H^{2n}(X, \Omega^1_{X/k}) \overset{\text{Tr}}{\to} k$$

**Proof.** In case $\dim X = 2n - 1$, the quadratic form $\text{Tr}$ reduces to perfect pairings

$$H^i(X, \Omega^j_{X/k}) \otimes H^{2n-1-i}(X, \Omega^{2n-1-j}_{X/k}) \to k$$

for $i+j<2n-1$, or $0 \leq i < j$ and $i+j = 2n-1$, which identifies $\text{Tr}$ with the canonical (hyperbolic) form on $V \oplus V^\vee$, where

$$V = \bigoplus_{i+j<2n-1} H^i(X, \Omega^j_{X/k}) \oplus \bigoplus_{0 \leq i < j \atop i+j = 2n-1} H^i(X, \Omega^j_{X/k}).$$

The argument in the even dimensional case is the same, except that one has the remaining factor coming from the pairing on $H^n(X, \Omega^n_{X/k}).$

The next result was obtained in 1974 independently by Abelson [11 Theorem 1] and Kharlamov [13 Theorem A] using an argument of Milnor’s relying on the Lefschetz fixed point theorem.

**Corollary 8.8.** Let $X$ be a smooth projective $k$-scheme of even dimension $2n$ and let $\sigma: k \to \mathbb{R}$ be an embedding. Then

$$|\chi^\text{top}(X(\mathbb{R}))| \leq \dim_k H^n(X, \Omega^n_{X/k}).$$

**Proof.** We know that $\chi^\text{top}(X(\mathbb{R}))$ is the signature of $\sigma_*(\chi(X/k))$ (see [20] Remarks 1.11)). The description of $\chi(X/k)$ given by Corollary 8.7 gives the desired inequality

$$|\text{sig}_\sigma(\chi(X/k))| \leq \dim_k H^n(X, \Omega^n_{X/k}).$$

\[\square\]
8.5. Descent for the motivic Euler characteristic. With the explicit formula for χ(X/k) given by Theorem 8.6 we may find χ(X/k) for forms X of an Xσ by the usual twisting construction; this works for all manners of descent but we confine ourselves to the case of Galois descent.

Let X0, X be smooth projective schemes of even dimension 2n over a perfect field k of characteristic different from 2, K ⊃ k a finite Galois extension with Galois group G. Let X := X ×k K, X0K := X0 ×k K and suppose we have an isomorphism ϕ : X ×k K → X0 ×k K. This gives us the cocycle (ψσ ∈ AutK(X ×K K)σ∈G, ψσ := ϕσ ◦ ϕ−1.

Letting

\[ b_0 : H^n(X_0, Ω^n_{X_0/k}) × H^n(X_0, Ω^n_{X_0/k}) → k, \]
\[ b : H^n(X, Ω^n_{X/k}) × H^n(X_0, Ω^n_{X_0/k}) → k \]

be the respective symmetric bilinear forms Tr(x ∪ y); the isomorphism ϕ induces an isometry

\[ φ^* : (H^n(X_0K, Ω^n_{X_0/K}), b_0) → (H^n(X_K, Ω^n_{X_K/K}), b_K) \]

and the associated cocycle (ψ−1 σ ∈ O(b0)(K))σ, from which one gets b back from b0 by twisting by this co-cycle. Explicitly, one recovers the k-vector space H^n(X, Ω^n_{X/K}) from H^n(X_0K, Ω^n_{X_0/K}) as the G-invariants for the map \( x → ψ^−1_σ(x^σ) \). Letting \( A \in GL(H^n(X_0, Ω^n_{X_0/k})) \) be a change of basis matrix comparing the k-forms H^n(X_0, Ω^n_{X_0/k}) ⊂ H^n(X_0K, Ω^n_{X_0/K}) and H^n(X, Ω^n_{X/K}) ⊂ H^n(X_0K, Ω^n_{X_0/K}), we recover b (up to k-isometry) as

\[ b(x, y) = b_0(Ax, Ay) =: b^A_0(x, y) \]

With this notation, Corollary 8.7 (2) gives

\[ χ(X_0/k) = [b_0 + m • H], \quad χ(X/k) = [b^A_0 + m • H] \]

in GW(k).

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