The Limit Cycles of Liénard Equations in the Strongly Nonlinear Regime

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Abstract

Liénard systems of the form $\ddot{x} + \epsilon f(x)\dot{x} + x = 0$, with $f(x)$ an even function, are studied in the strongly nonlinear regime ($\epsilon \to \infty$). A method for obtaining the number, amplitude and loci of the limit cycles of these equations is derived. The accuracy of this method is checked in several examples. Lins-Melo-Pugh conjecture for the polynomial case is true in this regime.

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1 Introduction

Many systems in nature display self-sustained oscillations: there is an internal balance between amplification and dissipation, and they do not require an external periodic forcing to oscillate. For instance, the beating of a heart, some chemical reactions, self-excited vibrations in bridges and airplane wings, Bénard-von Karman vortex street in the wake of a cylinder, etc. These phenomena can be modelled by the stable limit cycles found in specific nonlinear autonomous dynamical systems. These are called 'self-oscillators' [1].

Limit cycles are isolated closed trajectories in phase space (an inherently nonlinear phenomenon) [2]. They describe the periodic motions of the system. A very well known example having one limit cycle is the van der Pol equation: \( \ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \), where \( \dot{x}(t) = dx(t)/dt \). It displays a wide range of behavior, from weakly nonlinear to strongly nonlinear relaxation oscillations when the parameter \( \epsilon \) is modified, making it a good model for many practical situations [3, 4, 5]. The existence, uniqueness and non-algebraicity of its limit cycle has been shown for the whole range of the parameter \( \epsilon \) that controls the nonlinearity [3, 4, 5].

Liénard equation,

\[
\ddot{x} + \epsilon f(x)\dot{x} + x = 0, \tag{1}
\]

with \( \epsilon \) a real parameter, is a generalization of the van der Pol self-oscillator. There are no general results about the existence, number,
amplitude and loci of the limit cycles of this system \[9, 10\]. A nonperturbative method for obtaining information about the number of limit cycles and their location in phase space when \(f(x)\) is an even polynomial is presented in [11]. When \(f(x)\) is a polynomial of degree \(N = 2n + 1\) or \(2n\), Lins, Melo and Pugh conjectured (LMP-conjecture) that \(n\) is the maximum number of limit cycles allowed [12]. This conjecture is true if \(f(x)\) is of degree 2, if \(f(x)\) is of degree 3 or if \(f(x)\) is even and of degree 4. Different results about the necessary conditions that certain families of \(f(x)\) must satisfy to have \(n\) limit cycles have been given in [13] and references therein.

In this work, we are interested in the strongly nonlinear regime \((\epsilon \to \infty)\) of Liénard equation when the viscous term \(f(x)\) is a continuous even function, otherwise arbitrary. We give a method to find the limit cycles in this regime, and we claim that LMP-conjecture is true in that limit.

2 Liénard Equation

We start by considering the modified form of Liénard equation (1) after the change of variables \(\dot{x}(t) = y(x)\) and \(\ddot{x}(t) = y(x)y'(x)\) (where \(y'(x) = dy/dx\)):

\[
yy' + \epsilon f(x)y + x = 0.
\] (2)

The variables are now the coordinates \((x, \dot{x}) = (x, y)\) on the plane. Our interest in the shape of limit cycles in phase space \((x, y)\) justifies the
elimination of the time variable.

2.1 Symmetries

A limit cycle \( C_l \) of equation (2) is a closed orbit around the origin \((0, 0)\), the only fixed point of the system. This curve \( C_l \equiv (x, y_\pm(x)) \) cuts the \( x \)-axis in two points: \((-a_1, 0)\) and \((a_2, 0)\) with \(a_1, a_2 > 0\). Write \((x, y_+(x))\), with \(y_+(x) > 0\) and \(-a_1 < x < a_2\), the positive \( y \)-branch of the limit cycle and \((x, y_-(x))\), with \(y_-(x) < 0\) and \(-a_1 < x < a_2\), the negative one. The inversion symmetry \((x, y) \leftrightarrow (-x, -y)\) is verified by equation (2) because \(f(x)\) is an even function. Since the flow lines do not intersect themselves, a limit cycle and its transformed by this symmetry must be the same curve, then \(y_+(x) = -y_-(x)\) and \(a_1 = a_2 = a\). Therefore, in the following, we will restrict ourselves to the positive branches of limit cycles, \((x, y_+(x))\), with \(-a \leq x \leq a\). The amplitude of oscillation will be the number \(a\). For a given \(\epsilon\) this amplitude \(a\) identifies the limit cycle (Fig. 1). Thus, the number of limit cycles of system (2) is equal to the number of different possible amplitudes \(a\).

Let us remark also the parameter inversion symmetry \((\epsilon, x, y) \leftrightarrow (-\epsilon, x, -y)\) of equation (2): if \(C_l \equiv (x, y_\pm(x))\) is a limit cycle for a given \(\epsilon\) then \(\overline{C}_l \equiv (x, \overline{y}_\pm(x)) = (x, -y_\mp(x))\) is a limit cycle for \(-\epsilon\). Moreover if \(C_l\) is stable (or unstable) then \(\overline{C}_l\) is unstable (or stable, respectively). This is a consequence of the fact that each limit cycle encloses the origin \((0, 0)\) and this point changes its stability when \(\epsilon\) changes of sign. In the
regime $\epsilon \to \infty$ the two eigenvalues of the linear part of system (2) in the origin are $\lambda_1 \sim -\epsilon^{-1} f(0)^{-1}$ and $\lambda_2 \sim -\epsilon f(0)$. In order to have a well defined stability, we impose the condition $f(0) \neq 0$. This symmetry is used in our computer-simulations to find the unstable limit cycles. For a given $\epsilon$ the unstable limit cycles are the stable ones for $-\epsilon$ after reflection on the $x$-axis. Summarizing, in order to clasify all the limit cycles for a given even function $f(x)$, it is enough to find the positive $y$-branch $y_+(x)$ of the limit cycles of equation (2) when $\epsilon \to +\infty$.

2.2 Scaling

An easier understanding of the behavior of equation (2) in the strongly nonlinear regime is obtained by performing the change of variable $y = \epsilon z$. With this change Eq. (2) reads:

$$zz' + f(x)z = -\epsilon^{-2}x$$

(3)

Different scalings can be considered:

On one hand, if we consider $x$ and $f(x)$ of order 1 in the oscillation region $-a \leq x \leq a$ of a limit cycle $z(x)$, close to the extreme points (where $z$ is of order less than $\epsilon^{-2}$), the limit cycle obeys the equation:

$$zz' + \epsilon^{-2}x = yy' + x = 0.$$  

(4)

Integrating this expression we obtain: $x^2 + y^2 = a^2$, where $a$ is the amplitude of the limit cycle. If $x = a + \delta x$ and $y = \delta y$, then close to the
extreme points \((\pm a, 0)\) we have: 
\[2a\delta x + \delta y^2 = 0,\]
with \(\delta y \ll \epsilon^{-2} \).

On the other hand, if \(z\) is of order bigger than \(\epsilon^{-2}\) Eq. (3) is reduced to:

\[z(z' + f(x)) = 0. \tag{5}\]

It is clear that all relevant information about the limit cycles of Eq. (2) when \(\epsilon \to \infty\) is contained in Eq. (5).

3 Limit Cycle Solutions

Local solutions of Eq. (5) are:

\[z_1(x) = 0, \quad \text{or} \quad z_2(x) = -F(x) + C,\]

where \(F(x) = \int_0^x f(t) dt\) is an odd function and \(C\) is a constant. The positive \(y\)-branch \(y_+\) of a limit cycle solution of Eq. (1) and of amplitude \(a\) will be a solution \(z_1(x)\) of Eq. (5) verifying (i) \(z_1(-a) = z_1(a) = 0\) and (ii) \(z_1(0) > 0\). The solution \(z_1(x) = 0\) for \(-a \leq x \leq a\) does not represent a limit cycle because it does not verify condition (ii). The solution \(z_2(x) = -F(x) + C\) for \(-a \leq x \leq a\) is not a limit cycle either because it does not verify condition (i). Therefore the positive \(y\)-branch of a limit cycle (solution of Eq. (5)) must be a piecewise function built with \(z_1(x) = 0\) and \(z_2(x) = -F(x) + C\) (with \(C > 0\) by condition (ii)) as integrant blocks.
3.1 Two-Piecewise Solutions

We can start trying two-piecewise functions called *two-piecewise limit cycles*. Write \( x_i \) the *gluing point* of the limit cycle \( z_l(x) \) where \( z_1(x) \) and \( z_2(x) \) are glued: \( z_l(x) \) and \( z'_l(x) \) must be continous in \( x_i \) because the velocity \( \dot{x}(t) = \epsilon^{-1}z_l(x) \) and the acceleration \( \ddot{x}(t) = \epsilon^{-2}z_l(x)z'_l(x) \) are continuos in the oscillation. Continuity in \( z_l(x) \) means \( C = F(x_i) > 0 \). Continuity in \( z'_l(x) \) means \( f(x_i) = 0 \). The point \(-x_i\) is also a zero of \( f(x) \) (in the following we will suppose \( f(x) \) has a finite number of zeroes) and it is the *gluing point* of the negative \( y \)-branch \( y_- \) of the limit cycle. Thus each pair of zeroes \( \pm x_i \) of \( f(x) \) can generate at most one limit cycle. If \( f(x) \) is a polynomial of degree \( 2n \), there will be at most \( n \) (two-piecewise) limit cycles.

The stability of a limit cycle \( y(x) \) solution of Eq. (2), with \(-a < x < a\), is determined by the sign of the integral:

\[
\sigma \equiv -\int_{-a}^{a} \frac{\epsilon f(x)}{y(x)} dx = -\int_{-a}^{a} \frac{f(x)}{z(x)} dx.
\]

If \( \sigma < 0 \) the limit cycle is stable and if \( \sigma > 0 \) it is unstable. In the two-piecewise case this quantity \( \sigma \) is controlled essentially by the region where \( z(x) = 0 \), more exactly by the sign of \( f(x) \) close to the gluing point \( x_i \):

\[
\text{sign}[\sigma] = -\text{sign} \left[ \int_{x \sim x_i} \frac{f(x)}{z(x)} dx \right]_{z(x)=0} = -\text{sign} [f(x \sim x_i)]_{z(x)=0}.
\]

In the next section it is shown that if \( x_i \) is a gluing point, \( F(x_i) \) must be a maximum of \( F(x) \). Therefore, for \( \epsilon > 0 \), if \( x_i \equiv s_i < 0 \) the limit cycle
whose positive $y$-branch is $y_+(x) = \varepsilon z_+(x)$ is stable because, at the side where $z(x) = 0$, $f(x \sim s_i) > 0$. If $x_i \equiv u_i > 0$ it is unstable because, at the side where $z(x) = 0$, $f(x \sim u_i) < 0$.

Thus the expression of the positive $y$-branch $y_+(x) = \varepsilon z_+(x)$ (solution of Eq. (5)), with $-a_i^s < x < a_i^s$, of a stable (two-piecewise) limit cycle is (Fig. 2(a)):

\[
z_i^s(x) = \begin{cases} 
0 & \text{if } -a_i^s < x < s_i \\
-F(x) + F(s_i) & \text{if } s_i < x < a_i^s 
\end{cases} 
\]

(6)

and the form for an unstable (two-piecewise) limit cycle, $y_+(x) = \varepsilon z_i^u(x)$, with $-a_i^u < x < a_i^u$ is (Fig. 2(b)):

\[
z_i^u(x) = \begin{cases} 
-F(x) + F(u_i) & \text{if } -a_i^u < x < u_i \\
0 & \text{if } u_i < x < a_i^u 
\end{cases} 
\]

(7)

where $a_i^{s,u} > 0$ represent the amplitude of each limit cycle, and $s_i$ and $u_i$ are the gluing points of the two pieces of each cycle, $z_i^s$ and $z_i^u$, respectively. Recall that $s_i < 0$ for the stable cycle and $u_i > 0$ for the unstable one. In both cases $F(s_i) > 0$, $F(u_i) > 0$ and $f(s_i) = f(u_i) = 0$.

### 3.2 Number of Two-Piecewise Solutions

**STABLE CYCLES:** We study in more detail the stable limit cycles given by Eq. (6). The coordinate $y_+(x) = \varepsilon z_i^s(x)$ of the limit cycle vanishes in the extreme points $\pm a_i^s$ and then $F(s_i) = F(a_i^s)$. As $z_i^s(x)$ is positive for $s_i < x < a_i^s$ the amplitude $a_i^s$ is defined by:

\[a_i^s = \min \{x > s_i, F(x) = F(s_i)\}.\]
Also the property $a^s_i > |s_i|$ is fulfilled.

Let us invert the problem. If $s^* < 0$ verifies $f(s^*) = 0$ and $F(s^*) > 0$, is it possible to build a stable (two-piecewise) limit cycle, as given by Eq. (6), with $s^*$ as gluing point?. First we define $a^*$ by the rule:

$$a^* = \min \{x > s^*, F(x) = F(s^*)\}.$$

Geometrically $a^*$ represents the $x$-coordinate of the first crossing point between the straight $z = F(s^*)$ and the curve $F(x)$ (Fig. 3(a)). If $a^* < |s^*|$ it is not possible to build the limit cycle and we can eliminate this $s^*$ as a possible gluing point. If $a^* > |s^*|$ the point $s^*$ is a gluing point candidate. We rename all the pairs $(s^*, a^*)$ verifying this last property as $(\bar{s}_i, \bar{a}^s_i)$ and collect them into the set:

$$\mathcal{A}^s \equiv \{(\bar{s}_i, \bar{a}^s_i), f(\bar{s}_i) = 0, F(\bar{s}_i) > 0, \bar{s}_{i+1} < \bar{s}_i < 0, \bar{a}^s_i > |\bar{s}_i|\} \quad (8)$$

By construction $\bar{a}^s_{i+1} > \bar{a}^s_i$.

There are two different situations when two successive pairs, $(\bar{s}_i, \bar{a}^s_i)$ and $(\bar{s}_{i+1}, \bar{a}^s_{i+1})$, are ordered:

(a) $-\bar{a}^s_{i+1} < \bar{s}_{i+1} < -\bar{a}^s_i < \bar{s}_i$. In this case it is possible to build a two-piecewise limit cycle with the pair $(\bar{s}_i, \bar{a}^s_i)$ as indicated by Eq. (6). This pair is picked out and renamed once more as $(s_i, a^s_i)$.

(b) $-\bar{a}^s_{i+1} < -\bar{a}^s_i < \bar{s}_{i+1} < \bar{s}_i$. Now the construction of a limit cycle derived from the pair $(\bar{s}_i, \bar{a}^s_i)$ is not possible. If a initial condition (with $z = 0$) in the interval $[-\bar{a}^s_i, \bar{s}_{i+1}]$ is given, the system will jump from the
point \( \bar{s}_{i+1} > \bar{a}_i^s \), then the curve does not close at \( \bar{a}_i^s \) and there is no a limit cycle with amplitude \( \bar{a}_i^s \). This pair is rejected.

If there is only one pair \((\bar{s}_1, \bar{a}_1^s)\), we consider it satisfies (a).

All the existing (two-piecewise) stable limit cycles can be found comparing the pairs \( i \) and \( i + 1 \) under rules (a)-(b) and iterating this process. All the pairs selected by condition (a) (and renamed as \((s_i, a_i^s)\)) are collected into the set:

\[
A^s \equiv \{(s_i, a_i^s) \in \bar{A}^s, (s_i, a_i^s) \text{ verifies (a)}\}.
\] (9)

The algorithm above proposed shows also that \( F(x) \) must be a local maximum at each gluing point \( s_i \in A^s \).

The number, \( l_s = \text{card}(A^s) \), of pairs \((s_i, a_i^s)\) is the number of stable (two-piecewise) limit cycles of system (5).

**UNSTABLE CYCLES**: The same process can be repeated for the unstable cycles by considering the points \( u^* > 0 \), where \( f(u^*) = 0 \) and \( F(u^*) > 0 \) and finding their partners \( a^* \) defined as (Fig. 3(b)):

\[
a^* = \max \{x < u^*, F(x) = F(u^*)\}.
\]

Moreover the gluing point candidates must verify \(|a^*| > u^*\). After collecting the pairs \((u^*, a^*)\) fulfilling this last condition, we have, as in (8), the set:

\[
\bar{A}^u \equiv \{(\bar{u}_i, \bar{a}_i^u), f(\bar{u}_i) = 0, F(\bar{u}_i) > 0, \bar{u}_{i+1} > \bar{u}_i > 0, |\bar{a}_i^u| > \bar{u}_i\}.
\] (10)
A similar algorithm as indicated above can be applied in this case with the following modified rules:

(a') $\bar{u}_i < -\bar{a}_i^u < \bar{u}_{i+1} < -\bar{a}_{i+1}^u$. In this case there exists an unstable two-piecewise limit cycle resulting from the pair $(\bar{u}_i, \bar{a}_i^u)$ and given by Eq. (7). This pair is picked out and renamed $(u_i, a_i^u)$.

(b') $\bar{u}_i < \bar{u}_{i+1} < -\bar{a}_i^u < -\bar{a}_{i+1}^u$. The pair $(\bar{u}_i, \bar{a}_i^u)$ does not produce a such limit cycle and is rejected.

If there is only one pair $(\bar{u}_1, \bar{a}_1^u)$ we consider it satisfies (a').

We iterate the process given by rules (a’)-(b’). All the pairs selected by condition (a’) are collected into the set:

$$\mathcal{A}^u \equiv \{(u_i, a_i^u)\} = \{((\bar{u}_i, \bar{a}_i^u) \in \mathcal{A}^u, (\bar{u}_i, \bar{a}_i^u) \text{ verifies (a')}\}. \quad (11)$$

The algorithm above proposed shows also that $F(x)$ must be a local maximum at each gluing point $u_i \in A^u$.

The number, $l_u = \text{card}(\mathcal{A}^u)$, of pairs $(u_i, a_i^u)$ is the number of unstable (two-piecewise) limit cycles of system (5). Obviously, $l_s - 1 \leq l_u \leq l_s + 1$.

We claim that the total number $l$ of limit cycles of Eq. (1) in the strongly nonlinear regime is $l = l_s + l_u$, where $l_s$ and $l_u$ are the number of stable and unstable limit cycles of Eq. (1), respectively. The amplitudes of these limit cycles are given by the numbers $a_i^s$ and $a_i^u$, respectively.

We remark also that each pair of zeroes $\pm x_i$ of $f(x)$ produces at most one limit cycle. If $f(x)$ is a polynomial of degree $2n$ there will be at most $n$ limit cycles. Therefore, LMP-conjecture is true in the strongly nonlinear regime.
3.3 Shape of the Limit Cycles

The method introduced above allows us to find the number \( l \) of limit cycles and their amplitudes. However, two-piecewise solutions (Eqs. (6)-(7)) are only a first approach to the problem of finding the exact loci of the limit cycles of equation (1) in the \( \epsilon \to \infty \) regime. A better approximation to the shape of the limit cycles is presented in this section.

**STABLE CYCLES:** A stable two-piecewise solution \( z^*_i(x) \) of Eq. (5), identified by the pair \((s_i, a^*_i)\), is built of two blocks glued at \( s_i \): 
\[
z_{1,i}^*(x) = 0 \quad \text{and} \quad z_{2,i}^*(x) = -F(x) + F(s_i) \quad \text{see Eq.(6)}.
\]
The role of the point \( s_i \) is to allow the limit cycle to jump from the left side where \( z_{1,i}^*(x) = 0 \) to the maximal amplitude \( a^*_i \) through \( z_{2,i}^*(x) = z_{2,i}(x) \). Depending of \( f(x) \), it is also possible to find some points inside the interval \( -a^*_i < x < s_i \) and with similar jumping properties than \( s_i \). If this is the case, the exact limit cycle in the interval \( [-a^*_i, s_i] \) is not \( z^*_i(x) = z^*_{1,i}(x) = 0 \), and the piece \( z^*_{1,i}(x) \) in this interval must be corrected in the following way (Fig. 4):

(a) Define \( \{ s_{i,1}, \ldots, s_{i,n_i} \} \), the largest set of points where \( F(x) \) is a local maximum and verifying:

\[
f(s_{i,1}) = \cdots = f(s_{i,n_i}) = 0,
\]
\[
-a^*_i < s_{i,1} < s_{i,2} < \cdots < s_{i,n_i} < s_i,
\]
\[ 0 < F(s_{i,1}) < F(s_{i,2}) < \cdots < F(s_{i,n_i}) < F(s_i). \]

The values \( \{s_{i,1}, \ldots, s_{i,n_i}\} \) will be called the \textit{jumping} points of the cycle \((s_i, a_i^s)\).

(b) For each pair \((i, m)\), \(m = 1, \ldots, n_i\), we define \(b_{i,m} = \min\{x > s_{i,m}, F(x) = F(s_{i,m})\}\). It follows that \(b_{i,m} < s_{i,m+1}\) for every \(m\) because \(F(s_{i,m}) < F(s_{i,m+1})\). Thus we have the ordered set:

\[-a_i < s_{i,1} < b_{i,1} < s_{i,2} < b_{i,2} < \cdots < s_{i,n_i} < b_{i,n_i} < s_i.\]

(c) The correct expression \(z_i^s(x)\) of the stable limit cycle \((a_i^s, s_i)\) solution of Eq. (5) is:

\[
z_i^s(x) = \begin{cases} 
z_{i,1}^s(x) & \text{if } -a_i^s < x < s_i, \\
-F(x) + F(s_i) & \text{if } s_i < x < a_i^s,
\end{cases}
\]  

where

\[
z_{i,1}^s(x) = \begin{cases} 
0 & \text{if } -a_i^s < x < s_{i,1} \\
-F(x) + F(s_{i,1}) & \text{if } s_{i,1} < x < b_{i,1} \\
0 & \text{if } b_{i,1} < x < s_{i,2} \\
-F(x) + F(s_{i,2}) & \text{if } s_{i,2} < x < b_{i,2} \\
0 & \text{if } b_{i,2} < x < s_{i,3} \\
\vdots & \vdots \\
-F(x) + F(s_{i,n_i}) & \text{if } s_{i,n_i} < x < b_{i,n_i} \\
0 & \text{if } b_{i,n_i} < x < s_i.
\end{cases}
\]

Note that the dynamics in the jumping points \(s_{i,m}\) has the correct scale to be governed by Eq. (5). Then the continuity in \(z(x)\) and \(z'(x)\) can be imposed in these points in the gluing process. Nevertheless, a more subtle study precises this gluing process in the points \(b_{i,m}\) because other scales not present in Eq. (5) take place in the dynamics. Thus
the continuity of $z'(x)$ (the acceleration) is lost in the points $b_{i,m}$. This
fact is not a problem in order to consider the expression (12) as a fine
approximation (up to order $\epsilon^{-2}$) to the shape of the stable limit cycle,
$y_+(x) = \epsilon z_1^s(x)$, solution of Eq. (1) in the strongly nonlinear regime.

**UNSTABLE CYCLES:** Similarly to the previous case, the piece $z_{1,i}^u(x)$
in the unstable two-piecewise solution (7) must be corrected. If the limit
cycle is identified by the pair $(u_i, a_i^u)$ the correction in the interval $[u_i, a_i^u]$ is
given by the following steps (Fig. 4):

(a) Define $\{u_{i,1}, \ldots, u_{i,p_i}\}$, the largest set of points where $F(x)$ is a local
maximum and verifying:

\[
\begin{align*}
f(u_{i,1}) = \cdots = f(u_{i,p_i}) &= 0, \\
u_i < u_{i,1} < u_{i,2} < \cdots < u_{i,p_i} < a_i^u, \\
F(u_i) > F(u_{i,1}) > F(u_{i,2}) > \cdots > F(u_{i,p_i}) > 0.
\end{align*}
\]

The values $\{u_{i,1}, \ldots, u_{i,p_i}\}$ are the *jumping* points of the cycle $(u_i, a_i^u)$.

(b) For each pair $(i, m)$, $m = 1, \ldots, p_i$, we define $d_{i,m} = \max\{x < u_{i,m}: F(x) = F(u_{i,m})\}$. It follows that $d_{i,m} > u_{i,m-1}$ because $F(u_{i,m}) < F(u_{i,m-1})$. Thus we have the ordered set:

\[
u_i < d_{i,1} < u_{i,1} < d_{i,2} < u_{i,2} < \cdots < d_{i,p_i} < u_{i,p_i} < a_i^u.
\]

(c) The correct expression $z_{1,i}^u(x)$ of the unstable limit cycle $(a_i^u, u_i)$ solution of Eq. (5) is:

\[
z_{1,i}^u(x) = \begin{cases} 
-F(x) + F(u_i) & \text{if } -a_i^u < x < u_i \\
-\epsilon z_{1,i}^s(x) & \text{if } u_i < x < a_i^u.
\end{cases}
\]
where

\[
z_{1,i}^u(x) = \begin{cases} 
0 & \text{if } u_i < x < d_{i,1} \\
-F(x) + F(u_{i,1}) & \text{if } d_{i,1} < x < u_{i,1} \\
0 & \text{if } u_{i,1} < x < d_{i,2} \\
-F(x) + F(u_{i,2}) & \text{if } d_{i,2} < x < u_{i,2} \\
0 & \text{if } u_{i,2} < x < d_{i,3} \\
\vdots & \vdots & \vdots \\
-F(x) + F(u_{i,p_i}) & \text{if } d_{i,p_i} < x < u_{i,p_i} \\
0 & \text{if } u_{i,p_i} < x < a_i^u.
\end{cases}
\]

Similar comments to those of the stable case apply here. Therefore the expression (13) is a fine approximation (up to order \( \epsilon^{-2} \)) to the shape of the unstable limit cycle, \( y_+(x) = \epsilon z_{1,i}^u(x) \), solution of Eq. (1) in the strongly nonlinear regime.

### 4 Examples

We illustrate in this section the method introduced in Sections 2-3 for finding the number, amplitude and loci of the limit cycles of equation (1) by means of some examples.

(1) \( f(x) = 5x^4 - 3x^2 - 1 \), then \( F(x) = x^5 - x^3 - x \). The only real solutions of \( f(x) = 0 \) are \( x = \pm0.9157 \). The only local maximum verifying \( F(x) > 0 \) is \( s^* = -0.9157 \). The only value \( a^* \) verifying \( F(a^*) = F(s^*) \) is \( a^* = 1.3837 \). Moreover, it verifies \( a^* > |s^*| \), then we can rename this pair as \((s^*, a^*) = (\bar{s}_1, \bar{a}_1)\) and because there is only one pair, \((\bar{s}_1, \bar{a}_1^u) = (s_1, a_1^u)\).
The limit cycle when $\epsilon \to \infty$ is then ($F(s_1) = 1.0397$):

$$z_l(x) = \begin{cases} 
0 & \text{if } -1.3857 < x < -0.9157 \\
-x^5 + x^3 + x + 1.0397 & \text{if } -0.9157 < x < 1.3857 
\end{cases}$$

The amplitudes $a_{exp}$ for different values of the parameter $\epsilon$ can be calculated by numerical integration of Eq. (1). For instance:

| $\epsilon$ | $a_{exp}$ |
|------------|-----------|
| 1          | 1.4099    |
| 10         | 1.3975    |
| 100        | 1.3874    |

Let us remark the agreement between our analytical approach and the behavior of the system.

(2) $f(x) = 5x^4 - 3x^2 + 0.1$, then $F(x) = x^5 - x^3 + 0.1x$. The solutions of $f(x) = 0$ are $x_1 = \pm 0.7514$ and $x_2 = \pm 0.1882$. The only local maxima verifying $F(x) > 0$ are $s^* = -0.9157$ and $u^* = 0.1882$. In the case of $s^*$ the only value $a^*$ verifying $F(a^*) = F(s^*)$ is $a^* = 1.0045$. Moreover, it verifies $a^* > |s^*|$, then we can rename this pair as $(s^*, a^*) = (\bar{s}_1, \bar{a}_s) = (s_1, a_1^*)$. The stable limit cycle, when $\epsilon \to \infty$, is then ($F(s_1) = 0.1096$):

$$z_l(x) = \begin{cases} 
0 & \text{if } -1.0045 < x < -0.7514 \\
-x^5 + x^3 - 0.1x + 0.1096 & \text{if } -0.7514 < x < 1.0045 
\end{cases}$$

The amplitudes $a_{exp}$ for some values of the parameter $\epsilon$ are:

| $\epsilon$ | $a_{exp}$ |
|------------|-----------|
| 1          | 1.0234    |
| 10         | 1.0164    |
| 100        | 1.0096    |
In the case of $u^*$ the closest point $a^* < 0$ to the origin verifying $F(a^*) = F(u^*)$ is $a^* = -0.3945$. Moreover, it verifies $|a^*| > u^*$, then we can rename this pair as $(u^*, a^*) = (\bar{u}_1, \bar{a}_1^u) = (s_1, a_1^u)$. The unstable limit cycle when $\epsilon \to \infty$ is then $F(u_1) = 0.0124$:

$$z_l(x) = \begin{cases} -x^5 + x^3 - 0.1x + 0.0124 & \text{if } -0.3945 < x < 0.1882 \\ 0 & \text{if } 0.1882 < x < 0.3945 \end{cases}$$

The experimental amplitudes $a_{exp}$ for different values of the parameter $\epsilon$ are:

| $\epsilon$ | $a_{exp}$ |
|---|---|
| 1 | 0.3909 |
| 10 | 0.3927 |
| 100 | 0.3943 |

(3) $f(x) = -(x^2 - 0.09)(x^2 - 0.49)(x^2 - 0.81)$, then $F(x) = -0.1428x^7 + 0.278x^5 - 0.1713x^3 + 0.3572x$. The solutions of the equation $f(x) = 0$ are: $x = \pm 0.3$, $x = \pm 0.7$ and $x = \pm 0.9$. The only local maxima are $u_\alpha^* = 0.9$ and $u_\beta^* = 0.3$. The value $a_\alpha^* = 0.5443$ verifies $F(a_\alpha^*) = F(u_\alpha^*)$ and $|a_\alpha^*| < u_\alpha^* = 0.9$ then this pair can not build a limit cycle. On the other hand, the value $a_\beta^* = 1.0485$ is the closest point to the origin verifying $F(a_\beta^*) = F(u_\beta^*)$. Moreover, it verifies $|a_\beta^*| > u_\beta^*$. Then, we can rename this pair $(u_\beta^*, a_\beta^*) = (\bar{u}_1, \bar{a}_1^u) = (u_1, a_1^u)$. As $u_1^* < |a_1^u|$ the value $u_1^*$ is the jumping point $u_{1,1} = 0.9$ of the cycle $(u_1, a_1^u)$. The only (unstable) limit
cycle of this system is therefore \( F(u_1) = 0.0067 \) and \( F(u_{1,1}) = 0.0031 \):

\[
z_l(x) = \begin{cases} 
0.1428x^7 - 0.278x^5 + 0.1713x^3 - 0.3572x + 0.0067 & \text{if } -1.0485 < x < 0.3 \\
0 & \text{if } 0.3 < x < 0.5433 \\
0.1428x^7 - 0.278x^5 + 0.1713x^3 - 0.3572x + 0.0031 & \text{if } 0.5433 < x < 0.9 \\
0 & \text{if } 0.9 < x < 1.0485
\end{cases}
\]

The experimental amplitudes \( a_{exp} \) obtained by direct integration are:

| \( \epsilon \) | \( a_{exp} \) |
|----------------|-------------|
| 500            | 1.0497      |
| 1000           | 1.0493      |
| 5000           | 1.0489      |

\section{5 Conclusions}

Periodic self-oscillations can arise in nonlinear systems. These are represented by isolated closed curves in phase space that we call limit cycles. The knowledge of the number, amplitude and loci of these solutions in a general nonlinear system is an unsolved problem.

In this work, we have studied the Liénard equation in the strongly nonlinear regime. An effective algorithm for obtaining its limit cycles solutions (number, amplitude and loci) has been proposed. There exists an strong agreement between our analytical approach and the numerical integration of the system. Moreover, we claim that Lins-Melo-Pugh conjecture is true in this regime when the nonlinear viscous term \( f(x) \) is an even function.
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Figure Captions

**Figure 1**: A typical limit cycle \( C_l \equiv (x, y_{\pm}(x)) \) of Eq. (1). The oscillatory dynamics \( x(t) \) given by this solution verifies \(-a \leq x \leq a\), where \( a \) is the amplitude of oscillation. Let us observe the inversion symmetry \((x, y) \leftrightarrow (-x, -y)\) of this curve.

**Figure 2**: Diagrams (a)-(b) represent two-piecewise limit cycle solutions of Eq. (5), a stable and an unstable, respectively. Its amplitudes are \( a^s_i \) and \( a^u_i \). The functions \( z_1(x) \) and \( z_2(x) = -F(x) + C \) are glued at the points \( \pm s_i \) in the stable case (a) and at the point \( \pm u_i \) in the unstable case (b).

**Figure 3**: Diagrams (a)-(b) show algorithms for obtaining the limit cycles, stable and unstable, respectively. (a) The pair \((s^*, a^*)\) is rejected because \( a^* < |s^*| \). \( \mathcal{A}^s = \{(\tilde{s}_i, \tilde{a}_i), i = 1, 2, 3\} \). \( \mathcal{A}^s = \{(s_3, a_3^s)\} \). (b) The pair \((u^*, a^*)\) is rejected because \( |a^*| < u^* \). \( \mathcal{A}^u = \{(\tilde{u}_i, \tilde{a}_i), i = 1, 2, 3\} \). \( \mathcal{A}^u = \{(u_1, a_1^u), i = 2, 3\} \).

**Figure 4**: Shape of the limit cycles. (a) There are four limit cycles: two stables collected into \( \mathcal{A}^s = \{(s_i, a_i^s), i = 1, 2\} \), and two unstables into \( \mathcal{A}^u = \{(u_i, a_i^u), i = 1, 2\} \). (b) The unstable cycle \((u_1, a_1^u)\) has the jumping point \( u_{1,1} \). The stable one \((s_2, a_2^s)\) has the jumping point \( s_{2,1} \). Let us remark the repeating shape of the four limit cycles.