The asymptotics of the Touchard polynomials: a uniform approximation

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Abstract

The asymptotic expansion of the Touchard polynomials \( T_n(z) \) (also known as the exponential polynomials) for large \( n \) and complex values of the variable \( z \), where \( |z| \) may be finite or allowed to be large like \( O(n) \), has been recently considered in [4]. When \( z = -x \) is negative, it is found that there is a coalescence of two contributory saddle points when \( n/x = 1/e \). Here we determine the expansion when \( n \) and \( x \) satisfy this condition and also a uniform two-term approximation involving the Airy function in the neighbourhood of this value. Numerical results are given to illustrate the accuracy of the asymptotic approximations obtained.

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1. Introduction

The Touchard polynomials \( T_n(z) \), also known as exponential polynomials, are defined by

\[
T_n(z) = e^{-z} \sum_{k=0}^{\infty} \frac{k^n z^k}{k!} = e^{-z} \left( \frac{d^n}{dz^n} e^z \right) \tag{1.1}
\]

and were first introduced in a probabilistic context in 1939 by J. Touchard [5]. They have the generating function

\[
\exp \left[ z(e^t - 1) \right] = \sum_{n=0}^{\infty} T_n(z) \frac{t^n}{n!} \tag{1.2}
\]

and possess the alternative representation given by

\[
T_n(z) = \sum_{k=0}^{n} S(n,k) z^k, \tag{1.3}
\]

where \( S(n,k) \) is the Stirling number of the second kind [2, p. 624].

In [4] we considered the asymptotic expansion of \( T_n(z) \) for large \( n \) and complex values of the variable \( z \) by an application of the method of steepest descents applied to a contour integral.
representation. In this treatment |z| was finite or allowed to be large like O(n). It was found that there is an infinite number of saddle points of the integrand but that the precise number contributing to the expansion of \( T_n(z) \) depended on the values of \( n \) and |z|. When \( z = -x \) (\( x > 0 \)), which is the central issue in this note, we have the expansions [4, Theorem 2]

\[
T_{n-1}(-x) \sim \begin{cases} 
\Re \frac{\sqrt{2} \Gamma(n) e^{x+n/t_0}}{\sqrt{\pi (1 + t_0)}} \prod_{s=0}^{\infty} \frac{c_{2s}(t_0) \Gamma(s + \frac{1}{2})}{n^{s+\frac{1}{2}}} & (\mu > 1/e) \\
\frac{\Gamma(n) e^{x+n/t_0}}{\sqrt{2 \pi (1 + t_0)}} \prod_{s=0}^{\infty} \frac{c_{2s}(t_0) \Gamma(s + \frac{1}{2})}{n^{s+\frac{1}{2}}} & (0 < \mu < 1/e)
\end{cases}
\tag{1.4}
\]

as \( n \to \infty \), where \( t_0 \) is one of the conjugate pair of roots of \( t e^t = -n/x \) with smallest modulus in the first expression and the smaller (negative) root in the second expression. Explicit expressions for the coefficients \( c_{2s}(t_0) \) with \( s \leq 2 \) are given in [4]. In the case of the upper formula in (1.4), two conjugate saddles contribute to the expansion of \( T_{n-1}(-x) \) when \( 1/e < \mu < \mu_1 \), where \( \mu_1 \equiv 3.11179 \); when \( \mu \geq \mu_1 \), there are other conjugate pairs of contributory saddles but these are not included in the upper formula in (1.4) as they as subdominant as \( n \to \infty \).

When \( \mu := n/x = 1/e \), the two contributory saddle points coalesce to form a double saddle where the Poincaré-type expansions in (1.4) break down. In this note we obtain a uniform approximation for \( T_{n-1}(-x) \) involving the Airy function together with an expansion valid when \( \mu = 1/e \). Some numerical examples are given to illustrate the accuracy of the approximations obtained.

2. An integral representation

From (1.2) we obtain the integral representation

\[
T_n(z) = \frac{n!}{2\pi i} \int_{C} e^{ze^t} t_{n+1} dt,
\]

where the integration path is a closed circuit described in the positive sense surrounding the origin. We let \( z = -x \), where the variable \( x > 0 \) is either finite or large like \( O(n) \). Since \( |\exp(-xe^t)| \to 0 \) as \( \Re(t) \to +\infty \) when \( |\Im(t)| < \frac{1}{2} x \), it follows that the closed path above may be opened up into a loop\(^1\), which commences at +∞, encircles the origin and returns to +∞. Then, introducing the scaled Touchard polynomial \( \hat{T}_n(z) \) by

\[
\hat{T}_n(z) = \frac{1}{n!} T_n(z),
\]

we have

\[
\hat{T}_{n-1}(-x) = \frac{e^x}{2\pi i} \int_{C} \frac{(0^+)}{e^{\psi(t)}} dt,
\tag{2.1}
\]

where

\[
\psi(t) \equiv \psi(t; \mu) := -\frac{e^t}{\mu} - \log t, \quad \mu := \frac{n}{x}.
\tag{2.2}
\]

Saddle points of the integrand occur when \( \psi' \equiv 0 \); that is when

\[
te^t = -\mu,
\]

\(^1\)In [4], where \( z \) is a complex variable and \( n \geq 1 \), the closed path around the origin was opened up into a loop which commences at \(-\infty\), encircles the origin in the positive sense and returns to \(-\infty\).
for which there is an infinite number of (complex) roots. For a full discussion of the distribution of the saddle points see [4]. When $0 < \mu < 1/e$, there are two saddles on the negative real axis given by the negative values of the Lambert-W function; see [2, p. 111]. When $\mu = 1/e$, these two saddles coalesce to form a double saddle point at $t = -1$ and when $\mu > 1/e$ the saddles move off the real axis to form a complex conjugate pair.

In Fig. 1 we show examples of the steepest paths through the contributory saddles when (i) $0 < \mu < 1/e$, (ii) $1/e < \mu < \mu_1$ and (iii) $\mu = 1/e$, where $\mu_1$ is specified in Section 1. The $t$-plane has a branch cut along $[0, \infty)$. In case (i), the saddles $t_0$ and $t_1$ are situated on the negative real axis, with $t_0 \in (0, -1)$ and $t_1 \in (-1, -\infty)$ given by the negative branch of the Lambert-W function [2, p. 111]. The paths of steepest descent emanating from $t_0$ pass to $+\infty$ and the paths of steepest ascent from $t_1$ asymptotically approach the lines $\Im(t) = \pm \pi$ as $\Re(t) \to +\infty$. The integration path in (2.1) can then be deformed to pass over the steepest descent path emanating from $t_0$. In case (ii), the saddles $t_0$ and $t_1$ form a conjugate pair and the integration path is the path labelled $ABCD$ in Fig. 1(b). When $\mu \geq \mu_1$, however, there are additional conjugate pairs of saddles (dependent on the value of $\mu$) but these are subdominant as $n \to \infty$. In case (iii), the saddles $t_0$ and $t_1$ coalesce to form a double saddle at $t = -1$; the integration path then becomes the path $CSB$ in Fig. 1(c).

![Figure 1: Paths of steepest descent and ascent through the saddles](image)

**Figure 1:** Paths of steepest descent and ascent through the saddles when (a) $0 < \mu < 1/e$, (b) $1/e < \mu < \mu_1$ and (c) $\mu = 1/e$. The saddles are denoted by heavy dots; the arrows indicate the direction of integration taken along steepest descent paths. There is a branch cut along $[0, \infty)$.

### 3. The asymptotics of $\hat{T}_{n-1}(-x)$ for $\mu \simeq 1/e$

Both the expansions in (1.4) cease to be valid in the neighbourhood of the double saddle at $t = -1$. We now determine an expansion valid at the coalescence point when $\mu = 1/e$ and a uniform two-term approximation when $\mu \simeq 1/e$.

#### 3.1 The expansion of $\hat{T}_{n-1}(-x)$ when $\mu = 1/e$

When $\mu = 1/e$, the integration path can be deformed to coincide with the steepest descent path $C$ that enters $t = -1$ in the direction $\arg t = \pi/3$ and leaves to $t = -1$ in the direction $\arg t = -\pi/3$; see Fig. 1(c). If we put

$$-u = \psi(t) - \psi(-1) = \frac{\tau^3}{3!} + \frac{5\tau^4}{4!} + \frac{23\tau^5}{5!} + \frac{119\tau^6}{6!} + \frac{719\tau^7}{7!} + \cdots, \quad \tau = t + 1$$

we find upon inversion using Mathematica that

$$\tau(w) = (6w)^{1/3} - \frac{5w^{2/3}}{2 \cdot 6^{1/3}} + \frac{33w}{40} - \frac{1463w^{4/3}}{720 \cdot 6^{2/3}} + \frac{126827w^{5/3}}{151200 \cdot 6^{1/3}} - \frac{15451w^2}{44800} + \cdots.$$
where \( w = e^{-\pi i}u \) on the path \( SB \) and \( w = e^{\pi i}u \) on the path \( SC \) in Fig. 1(c). Then, upon differentiation of \( \tau(w) \), we have

\[
\frac{1}{2\pi i} \int e^{-nu} \frac{d\tau}{du} \, du = \frac{1}{2\pi i} \int_0^\infty e^{-nu} \left\{ \frac{d\tau(ue^{-\pi i})}{du} - \frac{d\tau(ue^{\pi i})}{du} \right\} \, du
\]

\[
= \frac{1}{\pi} \int_0^\infty e^{-nu} \left\{ -\frac{6^{1/3} \sin \frac{\pi}{3}}{3u^{2/3}} + \frac{5 \sin \frac{\pi}{3}}{3 \cdot 6^{1/3}u^{1/3}} + \frac{1463 \sin \frac{\pi}{3}}{540 \cdot 6^{2/3}u^{1/3}} + \cdots \right\} \, du
\]

\[
= -\frac{1}{3\pi} \left\{ \frac{\Gamma(\frac{1}{3}) \sin \frac{\pi}{3}}{\left( \frac{2 \pi}{3} \right)^{1/3}} - \frac{5\Gamma(\frac{4}{3}) \sin \frac{\pi}{3}}{6 \left( \frac{2 \pi}{3} \right)^{2/3}} - \frac{1463\Gamma(\frac{4}{3}) \sin \frac{\pi}{3}}{6480\left( \frac{2 \pi}{3} \right)^{4/3}} + \cdots \right\}.
\]

Hence we obtain the following result.

**Theorem 1.** Let \( \mu = n/x = 1/e \). Then as \( n \to \infty \), we have the expansion for the scaled Touchard polynomial

\[
\hat{T}_{n-1}(-x) \sim (-)^{n-1} \frac{e^{\pi i n^2}}{3\pi} \sum_{m=0}^{\infty} \left( - \right)^m B_m \frac{\Gamma(\frac{1}{3}m + \frac{1}{3})}{\left( \frac{2 \pi}{3} \right)^{(m+1)/3}} \sin \pi \left( \frac{2}{3}m + \frac{1}{3} \right),
\]

where

\[
B_0 = 1, \quad B_1 = \frac{5}{6}, \quad B_3 = \frac{1463}{6480}, \quad B_4 = \frac{126827}{1088640}, \quad B_6 = \frac{4732223}{167961600}. \quad \ldots
\]

We note the omission of the coefficients with index \( m = 2, 5, \ldots \); these terms do not contribute to the expansion on account of the vanishing of the sine factor.

**3.2 A uniform approximation for \( \hat{T}_{n-1}(-x) \) when \( \mu \sim 1/e \)**

Let \( \mu = 1/(e\xi) \) with \( \xi > 0 \); when \( \xi \geq 1 \) the saddles \( t_0 \) and \( t_1 \) are real, whereas when \( \xi < 1 \) the saddles form a conjugate pair. To obtain a uniform approximation valid for \( \xi \approx 1 \) we apply the standard cubic transformation [1]

\[
\psi(t) = \frac{1}{3} u^3 - \zeta u + \beta \quad (3.2)
\]

to the integrand in (2.1). The quantities \( \zeta \) and \( \beta \) depend on \( \xi \) and are determined by the requirement that the saddles \( t_0 \) and \( t_1 \) correspond to \( u = \zeta^{1/2} \) and \( u = -\zeta^{1/2} \), respectively; that is

\[
\beta = \frac{1}{2} \{ \psi(t_0) + \psi(t_1) \}, \quad \psi(t_j) = 1/t_j - \log t_j \quad (j = 0, 1) \quad (3.3)
\]

and

\[
\frac{\zeta^{3/2}}{2} = \frac{1}{2} \{ \psi(t_1) - \psi(t_0) \} \quad (\xi > 1), \quad \frac{\zeta^{3/2}}{2} = \frac{1}{2} \{ \psi(t_0) - \psi(t_1) \} \quad (\xi < 1). \quad (3.4)
\]

For \( x > 0 \), the quantity \( \zeta \geq 0 \) when \( \xi \geq 1 \) and \( \zeta < 0 \) when \( \xi < 1 \).

The integral for \( \hat{T}_{n-1}(-x) \) in (2.1) then becomes

\[
\hat{T}_{n-1}(-x) = \frac{e^{\pi i n^2}}{2\pi i} \int_{C'} e^{n(u^3 - \zeta u)} \, dt \, du,
\]

where \( C' \) is the image in the \( u \)-plane of the integration path. With the substitution

\[
g(u) := \frac{dt}{du} = A_0 + B_0 + (u^2 - \zeta)G(u),
\]
where

\[ A_0 = \frac{1}{2} \{ g(\zeta^{\frac{1}{2}}) + g(-\zeta^{\frac{1}{2}}) \}, \quad B_0 = \frac{1}{2\zeta^{\frac{1}{2}}} \{ g(\zeta^{\frac{1}{2}}) - g(-\zeta^{\frac{1}{2}}) \}, \]  

we then obtain [6, p. 369], [3, p. 67]

\[ \hat{T}_{n-1}(-x) = e^{x+\beta} \left\{ \frac{A_0}{n^{1/3}} U(n^{2/3}\zeta) - \frac{B_0}{n^{2/3}} U'(n^{2/3}\zeta) + \frac{n^{-1}}{2\pi i} \int_C e^{n(\frac{1}{2}u^3 - \zeta u)} G'(u) \, du \right\}, \]

where the prime denotes differentiation with respect to the argument concerned and the function \( U(z) \) is given by

\[ U(z) = \frac{1}{2\pi i} \int_C e^{\frac{1}{2}z^3 - z \tau} \, d\tau. \]

The path \( C' \) in the \( u \)-plane when \( \xi > 1 \) and \( \xi < 1 \) can be shown to be asymptotic to the rays \( \arg u = \pm \pi/3 \) traversed in the direction from the upper half-plane to the lower half-plane (we omit these details). From [2, Eq. (9.5.4)], the function \( U(z) \) is therefore given by the Airy function \( -\text{Ai}(z) \).

From [6, p. 367], we have

\[ g(\zeta^{\frac{1}{2}}) = \left( \frac{2\zeta^{\frac{1}{2}}}{\psi''(t_0)} \right)^{1/2}, \quad g(-\zeta^{\frac{1}{2}}) = \left( \frac{-2\zeta^{\frac{1}{2}}}{\psi''(t_1)} \right)^{1/2} \quad (\zeta \neq 0), \]

where \( \psi''(t_j) = (1 + t_j)/t_j^2 \) \( (j = 0, 1) \). Then, from (3.5), we find

\[ A_0 = \frac{\zeta^{\frac{1}{2}}}{\sqrt{2}} \left\{ \left( \frac{1}{\psi''(t_0)} \right)^{1/2} + \left( -\frac{1}{\psi''(t_1)} \right)^{1/2} \right\}, \quad B_0 = \frac{\zeta^{-\frac{1}{2}}}{\sqrt{2}} \left\{ \left( \frac{1}{\psi''(t_0)} \right)^{1/2} - \left( -\frac{1}{\psi''(t_1)} \right)^{1/2} \right\} \]

when \( \xi > 1 \), and

\[ A_0 = \sqrt{2}\xi^{\frac{1}{2}} \Re \left[ \left( \frac{i}{\psi''(t_1)} \right)^{1/2} \right], \quad B_0 = \sqrt{2}\xi^{-\frac{1}{2}} \Im \left[ \left( \frac{i}{\psi''(t_1)} \right)^{1/2} \right] \]

when \( \xi < 1 \). Hence, upon neglecting the third term in braces in (3.6) (which is \( o(n^{-1}) \)), we obtain the following result.

**Theorem 2.** Let \( \mu = n/x = 1/(e\xi) \), where \( \xi > 0 \). Then we have the uniform two-term approximation for the scaled Touchard polynomial

\[ \hat{T}_{n-1}(-x) \sim (-n^{-1}e^{x+\beta}) \left\{ \frac{A_0}{n^{1/3}} \text{Ai}(n^{2/3}\zeta) - \frac{B_0}{n^{2/3}} \text{Ai}'(n^{2/3}\zeta) \right\} \]

as \( n \to \infty \). The quantities \( \beta \) and \( \xi \) are defined in (3.3) and (3.4), where \( \xi \geq 0 \) for \( \xi \geq 1 \) and \( \xi < 0 \) when \( \xi < 1 \). The coefficients \( A_0 \) and \( B_0 \) are given in (3.7) and (3.8).

At coalescence when \( \xi = 1 \) \( (t = -1, u = 0) \) we have \( A_0 = g(0) = t'(0), B_0 = g'(0) = (t'(0))^2 + t''(0) \), where \( t(u) = dt/du \) and, by repeated differentiation of (3.2),

\[ t'(0) = \left( \frac{2}{\psi'''(-1)} \right)^{1/3}, \quad t''(0) = -\frac{\psi''(1)}{6\psi'''(-1)} \left( \frac{2}{\psi'''(-1)} \right)^{2/3}. \]

Since \( \psi'''(-1) = 1, \psi'''(1) = 5 \), we obtain \( A_0 = 2^{1/3}, B_0 = -\frac{5}{6} \cdot 2^{2/3} \). Use of the standard values \( \text{Ai}(0) = 3^{-2/3}/\Gamma(\frac{2}{3}), \text{Ai}'(0) = -3^{-1/3}/\Gamma(\frac{2}{3}) \), together with \( \psi(-1) = -1 - \pi i \) (so that \( \Re(\beta) = -1 \)), then shows that the approximation (3.9) at coalescence becomes

\[ \hat{T}_{n-1}(-x) \sim (-n^{-1}e^{x+\beta}) \left\{ \frac{\Gamma(\frac{1}{3})}{(\frac{3}{2})^{1/3}} \frac{\Gamma(\frac{2}{3})}{(\frac{3}{2})} \right\} \left( \mu = 1/e \right) \]

(3.10)
as \( n \to \infty \). This is seen to agree with the first two terms of the expansion in (3.1).

### 3.3 Numerical examples

In Table 1 we illustrate the accuracy of the expansion (3.1) by presenting values of the absolute relative error in \( \hat{T}_{n-1}(-x) \) for different values of \( n \) and truncation index \( m \) at the coalescence point \( \mu = 1/e \). The value of \( \hat{T}_{n-1}(-x) \) was computed from (1.3). Similarly, in Table 2, we show the absolute relative error in \( \hat{T}_{n-1}(-x) \) for different values of the coalescence parameter \( \xi \) using the two-term approximation in (3.9) and, when \( \xi = 1 \), using (3.10).

#### Table 1: Values of absolute relative error in the computation of \( \hat{T}_{n-1}(-x) \) using the asymptotic expansion (3.1) for different \( n \) and truncation index \( m \) when \( \mu = 1/e \).

| \( m \) | \( n = 50 \) | \( n = 80 \) | \( n = 121 \) |
|---|---|---|---|
| 0 | \( 2.514 \times 10^{-1} \) | \( 2.095 \times 10^{-1} \) | \( 1.788 \times 10^{-1} \) |
| 1 | \( 8.558 \times 10^{-3} \) | \( 5.390 \times 10^{-3} \) | \( 3.585 \times 10^{-3} \) |
| 3 | \( 2.744 \times 10^{-3} \) | \( 1.437 \times 10^{-3} \) | \( 8.144 \times 10^{-4} \) |
| 4 | \( 1.638 \times 10^{-4} \) | \( 6.490 \times 10^{-5} \) | \( 2.868 \times 10^{-5} \) |
| 6 | \( 6.184 \times 10^{-5} \) | \( 2.029 \times 10^{-5} \) | \( 7.616 \times 10^{-6} \) |

#### Table 2: Values of absolute relative error in the computation of \( \hat{T}_{n-1}(-x) \) using the uniform approximation (3.9) and (3.10) for different values of the coalescence parameter \( \xi \).

| \( \xi \) | \( n = 81 \) | \( n = 100 \) | \( \xi \) | \( n = 81 \) | \( n = 100 \) |
|---|---|---|---|---|---|
| 0.80 | \( 5.243 \times 10^{-3} \) | \( 8.179 \times 10^{-3} \) | 1.01 | \( 5.300 \times 10^{-3} \) | \( 4.301 \times 10^{-3} \) |
| 0.90 | \( 7.413 \times 10^{-3} \) | \( 3.322 \times 10^{-3} \) | 1.05 | \( 5.204 \times 10^{-3} \) | \( 4.222 \times 10^{-3} \) |
| 0.95 | \( 5.545 \times 10^{-3} \) | \( 4.540 \times 10^{-3} \) | 1.10 | \( 5.122 \times 10^{-3} \) | \( 4.153 \times 10^{-3} \) |
| 0.99 | \( 5.356 \times 10^{-3} \) | \( 4.355 \times 10^{-3} \) | 1.20 | \( 5.010 \times 10^{-3} \) | \( 4.060 \times 10^{-3} \) |
| 1.00 | \( 5.324 \times 10^{-3} \) | \( 4.326 \times 10^{-3} \) | 1.40 | \( 4.878 \times 10^{-3} \) | \( 3.951 \times 10^{-3} \) |

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