TITCHMARSH-WEYL THEORY FOR VECTOR-VALUED DISCRETE SCHRÖDINGER OPERATORS

KESHAV RAJ ACHARYA

Department of Mathematics, Embry–Riddle Aeronautical University
Daytona Beach, FL 32114-3900, U.S.A.
acharyak@erau.edu

Abstract: We develop the Titchmarsh-Weyl theory for vector-valued discrete Schrödinger operators and show that the Weyl $m$ functions associated with these operators map complex upper half plane to the Siegel upper half space. We also discuss about the Weyl disk and Weyl circle corresponding to these operators.

Key Words: Discrete Schrödinger operator, Titchmarsh-Weyl $m$-function.

AMS (MOS) Subject Classification: 39A70, 47A05, 34B20.

1. Introduction

The goal of this paper is to extend the Titchmarsh-Weyl theory for vector valued discrete Schrödinger operators. We consider a discrete Schrödinger equation in $d-$ dimensional space of the form

(1.1) \quad y(n + 1) + y(n - 1) + B(n)y(n) = zy(n), \quad z \in \mathbb{C}

where $y(n) = [y_1(n) \ y_2(n) \ldots \ y_d(n)]^t$ ( $t$ stands for a transpose), is a vector valued sequence in $l^2(I, \mathbb{C}^d)$. Usually $I = \mathbb{Z}$ or $I = \mathbb{N}$. Here $l^2(I, \mathbb{C}^d)$ is a Hilbert space of square summable vector valued sequences with the inner product

$$\langle u, v \rangle = \sum_{n \in I} u(n)^* v(n),$$

where “*” stands for conjugate transpose and $B(n)$ is a symmetric $d \times d$ matrix. We denote the space of all $d \times d$ complex matrices by $\mathbb{C}^{d \times d}$. The equation (1.1) can be generalized to a $d-$dimensional Jacobi equation of the form

(1.2) \quad A(n)y(n + 1) + A(n - 1)y(n - 1) + B(n)y(n) = zy(n), \quad z \in \mathbb{C}

with $A(n), B(n)$ are sequences of $d \times d$ matrices. If $I = \mathbb{N}$ The equation (1.2) can be written in the form:

$$\begin{pmatrix}
B(1) & A(1) & 0 \\
A(1) & B(2) & A(2) \\
0 & A(2) & B(3) \\
\vdots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
y(1) \\
y(2) \\
\vdots
\end{pmatrix}
= z
\begin{pmatrix}
y(1) \\
y(2) \\
\vdots
\end{pmatrix}.$$
The matrix

\[
J = \begin{pmatrix}
B(1) & A(1) & 0 \\
A(1) & B(2) & A(2) \\
0 & A(2) & B(3) \\
& \ddots & \ddots \\
& & & & \ddots \\
& & & & & \ddots & \ddots \\
\end{pmatrix}
\]

is called a block Jacobi matrix. Some studies about the block Jacobi matrix can be found in the paper [10]. Equation (1.1) is a particular case of Jacobi equation with \( A(n) \equiv 1 \).

The equation (1.1) induces a discrete Schrödinger operator \( J \) on \( l^2(I, \mathbb{C}^d) \) as

\[
J y(n) = y(n+1) + y(n-1) + B(n)y(n).
\]

It can be easily observed that if \( B(n) \) is a Hermitian matrix, \( B(n)^* = B(n) \), then \( J \) is a self-adjoint operator on \( l^2(\mathbb{N}, \mathbb{C}^d) \). Then, the spectrum of \( J \) is a set of real numbers: \( \sigma(J) \subset \mathbb{R} \).

To get a solution of the equation (1.1), we may fix any two vectors \( c, d \in \mathbb{C}^d \) to two consecutive sites, that is, we fix the values \( u(k) = c, u(k+1) = d \) and evolve according to (1.1). In particular, we fix \( u(0) \) and \( u(1) \) then any \( u(n) \) is obtained by solving the difference equation (1.1) using transformation matrices:

\[
T(m; z) = \begin{pmatrix}
I & B(m) \\
-I & 0
\end{pmatrix}
\]

where \( I \) is an \( d \times d \) identity matrix. Let

\[
A(n; z) = T(n; z) \times \cdots \times T(1, z) \times I.
\]

Then, \( u \) solves (1.1) for every \( n \) if and only if

\[
\begin{pmatrix}
u(n+1) \\
u(n)
\end{pmatrix} = A(n; z) \begin{pmatrix}
u(1) \\
u(0)
\end{pmatrix}
\]

and evolve according to (1.1). In particular, we fix \( u(0) \) and \( u(1) \) then any \( u(n) \) is obtained by solving the difference equation (1.1) using transformation matrices:

(1.3)

\[
T(m; z) = \begin{pmatrix}
zI & -B(m) \\
I & 0
\end{pmatrix}
\]

where \( I \) is an \( d \times d \) identity matrix. Let

(1.4)

\[
A(n; z) = T(n; z) \times \cdots \times T(1, z) \times I.
\]

Then, \( u \) solves (1.1) for every \( n \) if and only if

(1.5)

\[
\begin{pmatrix}
u(n+1) \\
u(n)
\end{pmatrix} = A(n, m; z) \begin{pmatrix}
u(m+1) \\
u(m)
\end{pmatrix}
\]

For every pair of vectors \( c, d \in \mathbb{C}^d \), there exists a solution of (1.1), therefore, the space of solutions of (1.1) is a \( 2d \)-dimensional vector space. In [1], it is shown that are exactly \( d \) linearly independent solutions of (1.1) that are in \( l^2(\mathbb{N}, \mathbb{C}^d) \).

It is now convenient to fix a basis of the solution space of (1.1). An easier way is to prescribe a pair of initial conditions. For \( z \in \mathbb{C} \), let

(1.6)

\[
U(n, z) = (u_1(n), u_2(n), \ldots, u_d(n)), \quad V(n, z) = (v_1(n), v_2(n), \ldots, v_d(n))
\]

where \( u_i(n) = [u_{1,i}(n) \ u_{2,i}(n) \ \cdots \ u_{d,i}(n)]^t \quad v_i(n) = [v_{1,i}(n) \ v_{2,i}(n) \ \cdots \ v_{d,i}(n)]^t \) are solutions of (1.1). Thus, both of the sets \( U(n, z) \) and \( V(n, z) \) consists of \( d \) linearly independent solutions of (\( \tau - z \))\( u(n) = 0 \), where \( \tau \) is the expression on the left side of equation (1.1). For our convenience, we call these sets as matrix valued solutions of (1.1). We further suppose that these solutions satisfy the following initial conditions

(1.7)

\[
U(0, z) = -I, \quad V(0, z) = 0, \quad U(1, z) = 0, \quad V(1, z) = I.
\]

By iterating the difference equation, we see that for fixed \( n \in \mathbb{N} \), \( U(n, z) \), \( V(n, z) \) are polynomial of degree \( n - 2 \) over \( \mathbb{C}^{d \times d} \). So \( \overline{U(n, z)} = U(n, \overline{z}) \) and \( \overline{V(n, z)} = V(n, \overline{z}) \).

We generalize the equation (1.5) for the matrix valued solutions \( U(n, z), V(n, z) \) as

\[
\begin{pmatrix}
U(n+1, z) \\
U(n, z)
\end{pmatrix} = A(n; z) \begin{pmatrix}
U(1, z) \\
U(0, z)
\end{pmatrix}
\]

\[
= A(n; z) \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\]

\[
= A(n; z) \mathbb{J},
\]

where \( \mathbb{J} = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix} \)
Lemma 1.1. Suppose \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), and \( W(z) = \begin{pmatrix} U(n + 1, z) & V(n + 1, z) \\ U(n, z) & V(n, z) \end{pmatrix} \) then
\[
W^t W = W W^t = I
\]

Proof. Notice that \( T(n; z)^t T(n; z) = T(n; z) T(n; z)^t = I \) for any \( n \) so that
\[
A(n; z)^t A(n; z) = A(n; z) A(n; z)^t = I. \quad \text{Then}
\]
\[
W^t W = (A(n; z))^t A(n; z)
= A^t A(n; z) A(n; z) J
= A^t J J
= I
\]
Exactly the same way we can see: \( W W^t = I \). \( \square \)

Definition 1.2. The Wronskian of any two sequences \( f(n, z), g(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d) \) is defined by
\[
W_n(f, g) = [f^*(n + 1, \bar{z}) g(n, z) - f^*(n, \bar{z}) g(n + 1, z)].
\]

This definition incorporate with the definition in one dimensional space and in the continuous case. In [1], it is shown that for fixed \( z \in \mathbb{C} \), if \( f(n, z), g(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d) \) are any two solutions of \( (1.1) \) then \( W_n(f, g) \) is independent of \( n \). Moreover, the Wronskian \( W_n \) is linear in both arguments.

For \( f(n, z), g(n, z) \in l^2(\mathbb{N}_0, \mathbb{C}^d) \) the Green’s identity corresponding to equation \( (1.1) \) is given by
\[
\sum_{j=0}^{n} \left( f^*(\tau g) - (\tau f)^* g \right)(j) = W_0(f, g) - W_n(f, g).
\]

We extend the definition of Wronskian and the Green’s identity for the matrix valued solutions \( U(n, z), V(n, z) \), each contains \( d \) linearly independent solutions of \( (1.1) \) for fixed \( z \in \mathbb{C} \).
\[
W_n(U, V) = [U^*(n + 1, z) V(n, z) - U^*(n, z) V(n + 1, z)].
\]

It is shown in [1] that the Wronskian \( W_n(U, V) \) is a matrix independent of \( n \in \mathbb{N} \). We extend the Green’s Identity for these matrix valued solutions.
\[
(1.8) \quad \sum_{j=0}^{N} \left( F(j, z)^*(\tau G(j, z)) - (\tau F(j, z))^* G(j, z) \right) = W_0(\tilde{F}, G) - W_N(\tilde{F}, G).
\]

Again, the proof of the Green’s identity can be found in [1].

2. Titchmarsh-Weyl m function

The theory of Titchmarsh-Weyl \( m \) functions is very important tool in the spectral theory of Jacobi and Schrödinger operators. In order to study the asymptotic behavior of solutions of Jacobi and Schrödinger equations, one need to study these \( m \) functions. Moreover, the absolutely continuous, singular continuous and essential spectrum of the operators associated with these equations are well explained in terms of \( m \) functions. These \( m \) functions were first introduced in 1910 by H. Weyl in [16] for Sturn-Liouville differential equations. It was further studied by E. C. Titchmarsh in [15] and established the connection between the analyticity of the solution and the spectrum of the operator of Sturn-Liouville differential equations. For further history of \( m \) function, one can see [6]. The theory of \( m \) functions in one dimensional space has been widely studied, some of which can be found in the papers [2] [3] [8] [11] [13] [14].

The Titchmarsh-Weyl \( m \) function for the vector-valued discrete Schrödinger operators associated to the equation \( (1.1) \) is defined in terms of solutions as follows.

Definition 2.1. Let \( z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\} \). The Titchmarsh-Weyl \( m \) function is defined as the unique complex matrix \( M(z) \in \mathbb{C}^{d \times d} \) such that
\[
(2.1) \quad F(n, z) = U(n, z) + V(n, z) M(z)
\]
where \( U(n, z), V(n, z) \) are matrix valued solutions consisting of \( d \) linearly independent solutions with initial values \( (1.7) \) and the matrix valued solution \( F(n, z) \) is a set of \( d \) linearly independent solutions of \( (1.1) \) that are in \( l^2(\mathbb{N}, \mathbb{C}^d) \).
This definition, is in fact well defined. As we mentioned above that there are only \( d \) linearly independent solutions in \( l^2(\mathbb{N}_0, \mathbb{C}^d) \), if there is another \( M(z) \) satisfying the above conditions then the solutions from both \( U(n, z) \) and \( V(n, z) \) will be in \( l^2(\mathbb{N}_0, \mathbb{C}^d) \). The solution \( V(n, z) \) is such that \( V(0, z) = 0 \) which implies that \( V(n, z) \) is the set of eigen-functions for the self-adjoint operator \( J \). This contradicts that the spectrum of \( J \) is a set of real numbers.

**Theorem 2.2.** [1] Let \( z \in \mathbb{C}^+ \). If \( (\tau - z)F = 0 \) and \( F \) is a \( d \times d \) matrix valued solution whose \( d \) columns are linearly independent solutions of (1.1) that are in \( l^2(\mathbb{N}, \mathbb{C}^d) \). Then

\[
(2.2) \quad M(z) = -F(1, z)F(0, z)^{-1}.
\]

Moreover,

\[
(2.3) \quad M(z) = (m_{ij}(z))_{d \times d} \in \mathbb{C}^{d \times d}, \quad m_{ij}(z) = \langle \delta_j, (J - z)^{-1}\delta_i \rangle.
\]

**Proof.** If the matrix valued solution \( F \) is given by (2.1) then \( F(0, z) = -I \) and \( F(1, z) = M(z) \). So (2.2) holds. Suppose \( G(n, z) \) is any \( d \times d \) matrix valued solution then it is a constant (matrix) multiple of the solution set \( F(n, z) \) from (2.1) because (2.1) is a set of \( d \) linearly independent solutions. That is,

\[
G(n, z) = F(n, z)C
\]

where \( C \) is a \( d \times d \) scalar invertible matrix.

\[
F(n, z) = G(n, z)C^{-1}
\]

so that

\[
-G(1, z)G(0, z)^{-1} = -F(1, z)CC^{-1}F(0, z)^{-1} = -F(1, z)F(0, z)^{-1} = M(z).
\]

Let \( F(n, z) \) as in (2.2) and let

\[
g_i = (J - z)^{-1}\delta_i
\]

where \( \delta_i(n) \in l^2(\mathbb{N}, \mathbb{C}^d) \) such that the values of \( \delta_i(n) = 0 \) if \( i \neq 0 \) and \( \delta_i(i) = [1, 0, \ldots, 0]^t \). Then

\[
(J - z)g_i = \delta_i. \quad \text{So} \quad (\tau - z)g_i(n) = 0 \text{ for } n \geq 2.
\]

Moreover, \( g_i \in l^2 \) for all \( i = 1, 2, \ldots, d \). Let

\[
G(n, z) = [g_1, g_2, \ldots, g_d] .
\]

Then \( G(n, z) = F(n, z)C, \quad C \in \mathbb{C}^{d \times d} \). By comparing values at

\[
n = 1, \quad G(1, z) = [g_1(1), g_2(1), \ldots, g_d(1)].
\]

Here

\[
g_1(1) = (J - z)^{-1}\delta_1(1)
\]

and

\[
g_1 = [g_{11}, g_{21}, \ldots, g_{d1}]^t, \quad g_{11} = \langle \delta_i, g_1 \rangle, \quad i = 1, 2, \ldots, d.
\]

Then

\[
M(z) = G(1, z)C^{-1}
\]

and

\[
M(z) = (m_{ij}(z)) = (\langle \delta_j, (J - z)^{-1}\delta_i \rangle)C^{-1}.
\]

To find the value of \( C \), we compare values at \( n = 2 \).

First \( (J - z)G(1, z) = (\delta_1, \delta_2, \ldots, \delta_d) \) so

\[
(J - z)G(1, z) = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix} = I
\]

It follows that

\[
G(2, z) + B(1)G(1, z) - zG(1, z) = I
\]

\[
G(2, z) = (z - B(1))G(1, z) + I \quad \text{.........(i)}
\]

Also,

\[
F(2, z) = (z - B(1))F(1, z) - F(0, z)C
\]
$G(2, z) = (z - B(1))G(1, z) - G(0, z)$

Comparing (i) and (ii), we get $-F(0, z)C = I$ and so $I.C = I \implies C = I$. Hence (2.3) holds. That is

$$M(z) = (m_{ij}(z)) = ((\delta_j, (J - z)^{-1}\delta_i)).$$

(2.4)

This result allows us to connect the $m$ function with a matrix valued Borel measure using functional calculus for these resolvent operators $(\delta_j, (J - z)^{-1}\delta_i)$, where $\delta_i(n) \in l^2(\mathbb{N}, \mathbb{C}^d)$ such that the values of $\delta_i(n) = 0$ if $i \neq 0$ and $\delta_i(i) = [1, 0, \ldots 0]^t$

By functional calculus,

$$m_{ij}(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu_{ij}$$

where $\mu_{ij}$ is a spectral measure for the vectors $\delta_j$ and $\delta_i$. Therefore,

$$M(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu, \mu = (\mu_{ij})_{d \times d}$$

and

$$M(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu = \left( \int_{\mathbb{R}} \frac{1}{t - z} d\mu_{ij} \right)_{d \times d}$$

The matrix valued measure $\mu$ is a spectral measure of the $d-$dimensional discrete Schrödinger operator $J$.

For each $i, j$ the entries $m_{ij}(z)$ maps complex upper half plane to itself. For if $z \in \mathbb{C}^+$, $\text{Im } m_{ij}(z) = \frac{1}{2\pi} (m_{ij}(z) - m_{ij}(\bar{z})) = \int_{\mathbb{R}} \frac{\mu_{ij}}{|t - z|^2} > 0$

Suppose $\overline{M(z)}$ denotes the complex conjugate of $M(z)$ obtained by taking the complex conjugate of each entry of $M(z)$. Then by integral representation of $m_{ij}(z)$, we have $m_{ij}(z) = m_{ij}(\bar{z})$

so that $\overline{M(z)} = M(\bar{z})$.

Also, $M(z) = (m_{ij}(z)) = ((\delta_j, (J - z)^{-1}\delta_i))$ so that

$$m_{ij}(z) = (\delta_j, (J - z)^{-1}\delta_i) = \langle (J - z)(J - z)^{-1}\delta_j, (J - z)^{-1}\delta_i \rangle = \langle (J - \bar{z})^{-1}\delta_j, (J - \bar{z})^{-1}\delta_i \rangle$$

Since $J$ is self adjoint, $(J - \bar{z})^* = (J - z)$

$$m_{ij}(z) = \langle (J - \bar{z})^{-1}\delta_j, \delta_i \rangle = \overline{\langle \delta_i, (J - \bar{z})^{-1}\delta_j \rangle} = m_{ji}(\bar{z}) = m_{ji}(z)$$

for all $i, j$. Hence $M(z)^t = M(z)$. Thus we proved the following proposition.

**Proposition 2.3.** $M(z)^* = M(\bar{z})$,

The imaginary part of $M(z)$ is $\text{Im } M(z) = \frac{1}{2\pi} (M(z) - M(z)^*)$ and it is clear from the above observation that $\text{Im } M(z) > 0$.

Let $S$ be a subspace of $\mathbb{C}^{d \times d}$, consisting of all symmetric matrices with positive definite imaginary part. That is,

$$S = \{ M \in \mathbb{C}^{d \times d} : \frac{1}{2\pi} (M - M^*) > 0 \}$$

The space $S$ is called a Seigel upper half space.

From above discussion we proved

**Theorem 2.4.** For $z \in \mathbb{C}^+$, the map $z \mapsto M(z)$ maps complex upper half plane $\mathbb{C}^+$ to Seigel upper half space $S$. 

Lemma 3.2. For a matrix valued solution \( E \) of (1.1) on a compact interval \([0, N]\). Suppose \( U(n, z), V(n, z) \) are the matrix valued solutions of (1.1) with initial values (1.7). For \( z \in \mathbb{C}^+ \), define a matrix valued solution \( F(n, z) = U(n, z) + V(n, z)M^\beta_N(z) \) satisfying a boundary condition

\[
\beta_2 F(N, z) + \beta_1 F(N + 1, z) = 0 \quad \text{where} \\
\beta = [\beta_1, \beta_2] \in \mathbb{R}^{d \times 2d}, \beta_1, \beta_2 \in \mathbb{R}^{d \times d}, \beta^T \beta = I, \ \beta J \beta^T = 0.
\]

The unique coefficient \( M^\beta_N(z) \) is called the Weyl \( m \) function on the interval \([0, N]\).

On solving we see that,

\[
M^\beta_N(z) = -(\beta_2 V(N, z) + \beta_1 V(N + 1, z))^{-1}(\beta_2 U(N, z) + \beta_1 U(N + 1, z)).
\]

Note that \((\beta_2 V(N, z) + \beta_1 V(N + 1, z))\) is invertible. Since \( z, N, \beta \) varies, \( M^\beta_N(z) \) becomes a function of these arguments, and since \( U, V \) are matrix polynomials with entries meromorphic functions of \( z \).

Lemma 3.1. The Weyl \( m \) function \( M^\beta_N(z) \) on \([0, N]\) is symmetric.

Proof. Let \( U(z) = \begin{pmatrix} (U(N + 1) / U(N) \end{pmatrix} = A(N; z) \begin{pmatrix} (U(1) / U(0) \end{pmatrix} = A(N; z) \begin{pmatrix} 0 \ -I \end{pmatrix} \) and \( V(z) = \begin{pmatrix} (V(N + 1) / V(N) \end{pmatrix} = A(N; z) \begin{pmatrix} (V(1) / V(0) \end{pmatrix} = A(N; z) \begin{pmatrix} I \ -0 \end{pmatrix} \)

Using equation (3.2), the Weyl \( m \) function can be written as \( M^\beta_N(z) = -(\beta \overline{V}(z))^{-1}(\beta \overline{U}(z)) \).

Suppose \( E = \beta \overline{U}(z) \) and \( F = \beta \overline{V}(z) \) so that \( M^\beta_N(z) = -F^{-1}E \). Now,

\[
M^\beta_N(z)^T - M^\beta_N(z) = F^{-1}E - (F^{-1}E)^T \\
= F^{-1}(FE^T - EF^T)F^{-T} \\
= F^{-1}[\beta \overline{V}(\beta \overline{U})^T - \beta \overline{U}(\beta \overline{V})^T]F^{-T} \\
= F^{-1}\beta[\overline{V}U^T - \overline{U}V^T]\beta^T F^{-T} \\
= F^{-1}\beta \begin{pmatrix} A(N; z) \ & \ 0 && 0 \ -I \end{pmatrix} \begin{pmatrix} A(N; z) \ & \ 0 && 0 \ -I \end{pmatrix} \beta^T F^{-T} \\
= -F^{-1}\beta \begin{pmatrix} A(N; z) \ & \ 0 && 0 \ -I \end{pmatrix} \beta^T F^{-T} \\
= -F^{-1}\beta J \beta^T F^{-T} \\
= 0
\]

\( \square \)

Lemma 3.2. For a matrix valued solution \( F(n, z) = U(n, z) + M^\beta_N(z)V(n, z) \) of (1.1) we have \( W_N(\overline{F}, F) = 2i \ \text{Im} \ M - 2i \ \text{Im} \ \sum_{j=0}^{N} F(j, z)^*F(j, z) \).

Proof. We use the Greens identity (1.8) with \( G = F \).

\[
\sum_{j=0}^{N} \left( F(j, z)^*(\tau F(j, z)) - (\tau F(j, z))^*F(j, z) \right) = W_0(\overline{F}, F) - W_N(\overline{F}, F) \\
(z - \overline{z}) \sum_{j=0}^{N} F(j, z)^*F(j, z) = W_0(\overline{F}, F) - W_N(\overline{F}, F)
\]
Theorem 3.4. The map $M_N^\beta(z)$ are respectively called the Weyl disk and Weyl circle.

Definition 3.3. (3.3)

For $F(n, z) = U(n, z) + M_N^\beta(z)V(n, z)$, using the linearity of the Wronskian we get

\[
\begin{align*}
\sum_{j=0}^{N} \left( F(j, z)^*(\tau F(j, z)) - (\tau F(j, z))^* F(j, z) \right) &= W_0(\bar{F}, F) - W_N(\bar{F}, F) \\
(z - \bar{z}) \sum_{j=0}^{N} F(j, z)^* F(j, z) &= W_0(\bar{F}, F) - W_N(\bar{F}, F) \\
&= W_0(U + VM, U + VM) - W_N(\bar{F}, F) \\
&= W_0(U, U) + W_0(U, VM) + W_0(VM, U) + W_0(VM, VM) - W_N(\bar{F}, F).
\end{align*}
\]

Then we have

\[
\begin{align*}
&\text{Here } W_0(U, U) = W_0(VM, VM) = 0, W_0(U, VM) = -M, W_0(\bar{U}, VM) = M \\
&(z - \bar{z}) \sum_{j=0}^{N} F(j, z)^* F(j, z) = M - \bar{M} - W_N(\bar{F}, F) \\
&2i \operatorname{Im} z \sum_{j=0}^{N} F(j, z)^* F(j, z) = 2i \operatorname{Im} M - W_N(\bar{F}, F) \\
&W_N(\bar{F}, F) = 2i \operatorname{Im} M - 2i \operatorname{Im} z \sum_{j=0}^{N} F(j, z)^* F(j, z) \\
&\square
\end{align*}
\]

The condition on $\beta$ in the boundary condition (3.1) implies that $\beta_1$ and $\beta_2$ are invertible. Equation (3.2) is written as

\[
M_N^\beta(z) = - (\beta_2 V(N, z) + \beta_1 V(N + 1, z))^{-1} (\beta_2 U(N, z) + \beta_1 U(N + 1, z))
\]

\[
= - (\beta_1^{-1} \beta_2 V(N, z) + V(N + 1, z))^{-1} (\beta_1^{-1} \beta_2 U(N, z) + U(N + 1, z))
\]

\[
= - (\gamma V(N, z) + V(N + 1, z))^{-1} (\gamma U(N, z) + U(N + 1, z)), \quad \gamma = \beta_1^{-1} \beta_2 \in \mathbb{R}^{d \times d}.
\]

Again solving for $\gamma$ we have,

\[
\gamma = -F(N + 1, z)F(N, z)^{-1}.
\]

Observe that $\Im \gamma \leq 0$

Let $W(N, z, M) = \begin{pmatrix} U(N + 1, z) & V(N + 1, z) \\ U(N, z) & V(N, z) \end{pmatrix} [I \ M]$. Define a matrix function

\[
E(M, N) = -iW(N, z, M)^* JW(N, z, M)
\]

Observe that

\[
E(M, N) = -i[F(N + 1, z)^*, F(N, z)^*]J \begin{pmatrix} F(N + 1, z) \\ F(N, z) \end{pmatrix}
\]

\[
= -iW_N(\bar{F}, F)
\]

\[
(3.3) = -2 \operatorname{Im} M + 2 \operatorname{Im} z \sum_{j=0}^{N} F(j, z)^* F(j, z).
\]

Definition 3.3. Let $z \in \mathbb{C}^+$. The set

\[
\mathcal{D}(N, z) = \{ M \in C^{d \times d} | E(M, N) \leq 0 \}
\]

and $C(N, z) = \{ M \in C^{d \times d} | E(M, N) = 0 \}$

are respectively called the Weyl disk and Weyl circle.

Clearly, $C_N(z) = \{ M_N^\beta(z) : \beta \in \mathbb{R}^{d \times d}, \text{satisfying (3.1)} \}$. Thus for any complex symmetric matrix $M \in \mathbb{C}^{d \times d}$

\[
M \in C(N, z) \iff \operatorname{Im}(-F(N + 1, z)F(N, z)^{-1}) = 0
\]

Theorem 3.4. The map $z \mapsto M_N^\beta(z)$ maps complex upper half plane to Seigel half space.
That is

\[ 2 \text{Im} M = 2 \text{Im} z \sum_{j=0}^{N} F(j, z)^* F(j, z) = 0. \]

That is

\[ \frac{\text{Im} M}{\text{Im} z} = \sum_{j=0}^{N} F(j, z)^* F(j, z) > 0. \]

This implies that \( \text{Im} M \) is positive definite. \( \square \)

**Lemma 3.5** (Nesting property of Weyl disks). Let \( z \in \mathbb{C}^+ \). Then

\[ \mathcal{D}(N + 1, z) \subset \mathcal{D}(N, z), \quad N \in \mathbb{N} \]

**Proof.** Let \( M \in \mathcal{D}(N + 1, z) \). From (3.3) we have

\[ E(M, N) = -2 \text{Im} M + 2 \text{Im} z \sum_{j=0}^{N} F(j, z)^* F(j, z) \]

\[ \leq -2 \text{Im} M + 2 \text{Im} z \sum_{j=0}^{N+1} F(j, z)^* F(j, z) \]

\[ = E(M, N + 1) \leq 0. \]

This shows that \( M \in \mathcal{D}(N, z) \). Hence the result.

From above we have,

\[ E(M, N) = -i[I, M^*] J \begin{pmatrix} U(N + 1, z)^* & U(N, z)^* \\ V(N + 1, z)^* & V(N, z)^* \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix} \]

\[ = -i[I, M^*] \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix} \]

\[ = -i[W_N(\bar{U}, U) + W_N(\bar{U}, V) + M^* W_N(\bar{V}, U) + M^* W_N(\bar{V}, V) M] \]

Using \( W_N(\bar{V}, V)^* = -W_N(\bar{V}, V) \) and \( W_N(\bar{U}, V)^* = -W_N(\bar{U}, V) \), \( E(M, N) \) can be written as

\[ E(M, N) = -i \left\{ [M - W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)]^* W_N(\bar{V}, V) [M - W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^*] \right. \]

\[ + W_N(\bar{U}, U) + W_N(\bar{U}, V) W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^* \}

**Lemma 3.6.** For \( z \in \mathbb{C}^+ \), \( N \in \mathbb{N} \),

\[ \mathcal{W} = W^* JW \]

Notice that \( W^* JW = \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix} \). From lemma [1.1] we see that

\[ W^* JW = J. \]

Then

\[ \mathcal{W}^4 JW = (W^* JW)^4 J(W^* JW) \]

\[ = W^4 J W^4 W^* JW \]

\[ = -W^4 JW \]

\[ = J. \]

On the other hand,

\[ \mathcal{W}^4 JW = \begin{pmatrix} W_N(\bar{U}, U)^4 & W_N(\bar{U}, V)^4 \\ W_N(\bar{V}, U)^4 & W_N(\bar{V}, V)^4 \end{pmatrix} J \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix} \]

\[ = \begin{pmatrix} W_N(\bar{U}, U)^* & W_N(\bar{U}, V)^* \\ W_N(\bar{V}, U)^* & W_N(\bar{V}, V)^* \end{pmatrix} J \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix} \]
By direct computation we see that
\begin{align}
\tag{3.5}
-W_N(V, \bar{V})^*W_N(U, \bar{V}) + W_N(U, \bar{V})*W_N(V, U) &= -I \\
\tag{3.6}
-W_N(V, \bar{V})^*W_N(U, \bar{V}) + W_N(U, \bar{V})*W_N(V, U) &= 0.
\end{align}

From equation (3.6) we have
\[ W_N(\bar{U}, V)^* = W_N(\bar{V}, V)^*W_N(U, \bar{V})W_N(V, \bar{V})^{-1} = -W_N(\bar{V}, V)W_N(U, \bar{V})W_N(V, \bar{V})^{-1}. \]

Using this we get
\[ W_N(\bar{U}, V)W_N(\bar{V}, V)^{-1}W_N(\bar{U}, V)^* + W_N(\bar{U}, U) = W_N(\bar{U}, V)W_N(\bar{V}, V)^{-1}[-W_N(\bar{V}, V)W_N(U, \bar{V})W_N(V, \bar{V})^{-1}] + W_N(\bar{U}, U) = -W_N(\bar{U}, V)W_N(U, \bar{V})W_N(V, \bar{V})^{-1} + W_N(\bar{U}, U). \]

Also from equation (3.6) we get,
\[ W_N(U, \bar{V})^*W_N(\bar{U}, U) = -I + W_N(V, \bar{V})^*W_N(\bar{U}, U) \]
\[ -W_N(U, \bar{V})^*W_N(\bar{U}, U)^* = -I + W_N(V, \bar{V})^*W_N(\bar{U}, U) \]
\[ (W_N(\bar{U}, V)W_N(U, \bar{V}))^* = I - W_N(V, \bar{V})^*W_N(\bar{U}, U) \]
\[ W_N(\bar{U}, V)W_N(U, \bar{V}) = I - W_N(\bar{U}, U)^*W_N(V, \bar{V}) \]
\[ W_N(\bar{U}, V)W_N(U, \bar{V}) = I + W_N(\bar{U}, U)W_N(V, \bar{V}) \]

Then,
\[ W_N(\bar{U}, V)W_N(\bar{V}, V)^{-1}W_N(\bar{U}, V)^* + W_N(\bar{U}, U) = -(I + W_N(\bar{U}, U)W_N(V, \bar{V})W_N(V, \bar{V})^{-1} + W_N(\bar{U}, U) = -W_N(V, \bar{V})^{-1}. \]

Using lemma 3.4 and equation 3.2 we can express $E(M, N)$ in the form
\[ E(M, N) = -i\left\{ [M - W_N(\bar{V}, V)^{-1}W_N(\bar{U}, V)^*]W_N(\bar{V}, V) \right. \]
\[ \left. [M - W_N(\bar{V}, V)^{-1}W_N(\bar{U}, V)^* - W_N(V, \bar{V})^{-1}] \right\}. \]

Thus it can be expressed as
\[ E(M, N) = -[(M - C_N(z))^*R(N, z)^{-2}(M - C_N(z)) - R(N, \bar{z})^2] \]
where $C_N(z) = W_N(\bar{V}, V)^{-1}W_N(\bar{U}, V)^*$ and $R(N, z) = (iW_N(\bar{V}, V))^{-1/2}$. So the equation of Weyl circle can be written as
\[ (M - C_N(z))^*R(N, z)^{-2}(M - C_N(z)) = R(N, \bar{z})^2 \]

**Theorem 3.7.** For all $z \in \mathbb{C}^+$, $\lim_{N \to \infty} R(N, z)$ exists and $\lim_{N \to \infty} R(N, z) \geq 0$.

**Proof.** By Green’s identity we have
\[ 2\text{Im} z \sum_{j=0}^{N} V(j, z) V(j, \bar{z}) = iW_N(\bar{V}, V) = R(N, z)^{-2} > 0. \]

Also $R(N, z)^{-2}$ is nondecreasing. Thus, $R(N, z)$ is non increasing and so $\lim_{N \to \infty} R(N, z)$ exists.\qed

**Theorem 3.8.** For all $z \in \mathbb{C}^+$, $\lim_{N \to \infty} C_N(z)$ exists.

**Proof.** For any $M \in C(N, Z)$ we have
\[ (M - C_N(z))^*R(N, z)^{-2}(M - C_N(z)) = R(N, \bar{z})^2 \]
which follows that
\[ \left( R(N, z)^{-1}(M - C_N(z))R(N, \bar{z})^{-1} \right)^* \left( R(N, z)^{-1}(M - C_N(z))R(N, \bar{z})^{-1} \right) = I \]
Suppose \( U = \left( R(N, z)^{-1}(M - C_N(z)) R(N, \bar{z})^{-1} \right) \) so that \( U^* U = I \) that is \( U \) is unitary. Also,
\[
M = C_N(z) + R(N, z) U R(N, \bar{z})
\]

Suppose \( M \in C_{N+1}(z) \subset C_N(z) \) then we have
\[
M = C_{N+1}(z) + R(N + 1, z) U_{N+1} R(N + 1, \bar{z}) \quad \text{and} \quad M = C_N(z) + R(N, z) U_N R(N, \bar{z})
\]
Equating and taking operator norm on both sides we get
\[
\|C_{N+1}(z) - C_N(z)\| = \|R(N + 1, z) U_{N+1} R(N + 1, \bar{z}) - R(N, z) U_N R(N, \bar{z})\|
\]
\[
\leq \|R(N + 1, z) U_{N+1} R(N + 1, \bar{z}) - R(N, z) U_N R(N + 1, \bar{z})\| + \|R(N, z) U_N R(N + 1, \bar{z}) - R(N, z) U_N R(N, \bar{z})\|
\]
\[
\leq \|R(N + 1, z) - R(N, z)\| \|U_{N+1}\| \|U_N\| \|R(N + 1, \bar{z}) - R(N, \bar{z})\|
\]

This shows that \( C_N(z) \) is a Cauchy sequence, hence converges. \( \square \)

Let \( C_0(z) = \lim_{N \to \infty} C_N(z) \) and \( R_0(z) = \lim_{N \to \infty} R(N, z) \).
Define \( D_0(z) = \{ M \in \mathbb{C}^{d \times d} : (M - C_0(z))^* R_0(z)^{-2} (M - C_0(z)) \leq R_0(z)^2 \} \) then
\[
D_0(z) = \cap_{N \geq 1} D(N, z).
\]

**Theorem 3.9.** Let \( z \in \mathbb{C}^+ \) and \( M \in \mathbb{C}^{d \times d} \). Then for \( F(N, z) = U(N, z) + V(N, z) M \) we have

(1) \( M \) is inside \( D_0(z) \) if and only if
\[
\sum_{N=1}^{\infty} F(N, z)^* F(N, z) \leq \frac{\text{Im} M}{\text{Im} z}
\]

(2) \( M \) is on the boundary of \( D_0(z) \) if and only if
\[
\sum_{N=1}^{\infty} F(N, z)^* F(N, z) = \frac{\text{Im} M}{\text{Im} z}
\]

**Proof.** Let \( M \in D_0(z) \). Then \( M \in D(N, z) \) for all \( N \). So from \( \ref{eq:3.3} \) we have
\[
E(M, N) = -2 \text{Im} M + 2 \text{Im} z \sum_{j=0}^{N} F(j, z)^* F(j, z) \leq 0
\]
which follows that
\[
\sum_{j=0}^{N} F(j, z)^* F(j, z) \leq \frac{\text{Im} M}{\text{Im} z}.
\]
Taking limit as \( N \to \infty \) we get
\[
\sum_{N=1}^{\infty} F(N, z)^* F(N, z) \leq \frac{\text{Im} M}{\text{Im} z}.
\]

Conversely, for any \( N \) we have,
\[
\sum_{j=1}^{N} F(j, z)^* F(j, z) \leq \sum_{j=1}^{\infty} F(j, z)^* F(j, z) \leq \frac{\text{Im} M}{\text{Im} z}.
\]
So \( E(M, N) \leq 0 \) for all \( N \) and hence \( M \in D_0(z) \). Similar explanation also proves (2). \( \square \)

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References

[1] K. R. Acharya, A note on multidimensional discrete Schrödinger operators, \textit{The Nepal Math. Sc. Report} Vol 34, No.1, 2016, 1-10.

[2] J. Behrndt, J. Rohleder, Titchmarsh-Weyl Theory for Schrödinger operators on unbounded domain, arXiv: 12085224v2.

[3] S. L. Clark and F. Gesztesy, WeylTitchmarsh M-function asymptotics for matrixvalued Schrödinger operators, \textit{Proc. London Math. Soc.} (3) 82 (2001), no. 3, 701-724.

[4] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, Schrödinger Operators: With Applications to Quantum Mechanics and Global Geometry, Springer, 2008.

[5] D. Damanik, A. Pushnitski, B. Simon, The analytic theory of matrix orthogonal polynomials, \textit{Surv. Approx. Theory}, 4: 1-85, 2008.

[6] W. N. Everitt, A personal history of the m-coefficient, \textit{J. Comput. Appl. Math.} 171(2004), no. 1-2, 185-197.

[7] J. S. Geronimo, Scattering theory and matrix orthogonal polynomials on the real line, \textit{Circuits Systems Signal Process.}, 1(3-4): 472-495, 1982.

[8] F. Gesztesy, E. Rsekanovskii, On matrix-valued Herglotz functions. \textit{Math. Machr.}, 218: 61-138, 2000.

[9] F. Gesztesy, B. Simon, G. Teschl, Zeros of the Wronskian and renormalized oscillation theory, \textit{Amer. J. Math.} 118 (1996), 571 - 594.

[10] R. Kozhan, Equivalence classes of block Jacobi matrices. \textit{Proc. Amer. Math. Soc.}, (139) 799-805, 2011.

[11] C. Remling, The absolutely continuous spectrum of Jacobi Matrices, \textit{Annals of Math.}, 174, 125-171, 2011.

[12] C. Remling, The absolutely continuous spectrum of one-dimensional Schrödinger operators, \textit{Math. Phys. Anal. Geom.}, 10(4), 359–373, 2007.

[13] B. Simon, \textit{m}-functions and the absolutely continuous spectrum of one-dimensional almost periodic Schrödinger operators, \textit{Differential equation (Birmingham, Ala., 1983)}, 519, North-Holland Math. Stud. 92, North-Holland, Amsterdam, 1984.

[14] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, Mathematical Monographs and Surveys, Vol.72, American Mathematical Society, Providence, 2000.

[15] E.C. Titchmarsh, \textit{Eigenfunction Expansions Associated with Second-order Differential Equations}, Part I, Second Edition, Clarendon Press, Oxford, 1962.

[16] H. Weyl, \textit{Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklung willkürlicher Funktionen}, Math. Ann., 68 (1910), no. 2, 220-269.