Exponential functions of Al-Tememe acceleration methods for improving the values of integrations numerically of second kind

1Ali Hassan Mohammed, 2 Shatha Hadier Theyab

1Dep. of Mathematics / Faculty of Education for Girls / University of Kufa / Iraq
prof.ali57hassan@gmail.com

2Dep. of Mathematics / Faculty of Education for Girls / University of Kufa / Iraq
Shathahaid193@gmail.com

Abstract-- The aims of study are to introduce acceleration methods that called acceleration methods of exponential functions, which we generally call Al-Tememe's acceleration methods of the second kind discovered by (Ali Hassan Mohammed). It is useful to improve the numerical results of integrals with continuous integrands in which the error is of the 4th order, and related to accuracy, the number of used, sub partial intervals and how fast to get results especially to accelerate the results got by Simpson's method. Also, it is possible to utilize it in improving the results of differential equations numerically of the main error of the forth order.

1.Introduction

There are numerical methods for calculating single integrals that are bounded in their integration intervals.

1. Trapezoidal Rule
2. Midpoint Rule
3. Simpson’s Rule

It is called Newton–Cotes formulas.

The study will introduce Simpson's method to find approximate values of single integrals of continuous integrands through using exponential acceleration methods, which come within Al-Tememe's acceleration series of the second kind. We will compare these methods with respect to accuracy and the speed of approaching these values to the real value (analytical) of those integrals.

Let's assume that integration is J:
\[ J = \int_{x_0}^{x_{2n}} f(x)dx \ldots \quad (1) \]
whereas \( f(x) \) is a continuous integral lies above X axis in the interval \([x_0, x_{2n}]\), and it what is required to find approximate value of J.

Generally, Newton–Cotes formula for integration (1) can be written in the following form:
\[ J = \int_{x_0}^{x_{2n}} f(x)dx = \bar{G}(h) + E_{\bar{G}}(h) + R_{\bar{G}} \quad (2) \]
whereas $G(h)$ represents (Lagrangian – Approximation) the value of integration value $J$, and $G$ refers to the type of the rule, $E_G(h)$ is the correction terms that can be added to $G(h)$ and $R_G$ is the remainder. The Simpson’s rule value $G(h)$ that is referred to by $S(h)$ is:

$$S(h) = \sum_{i=0}^{n} \left[ \frac{4}{3} f(a) + 4 f(a + h) + 2 f(a + 2h) + 4 f(a + 3h) + \cdots + 2 f(a + (2n - 2)h) + 4 f(a + (2n - 1)h) + f(b) \right]$$

and the general formula for $E_G(h)$ is the following:

$$E_G(h) = ES(E + \cdots)$$

2. Al-Tememe's acceleration of exponential functions of the second kind

We will present the acceleration methods that come within Al-Tememe's series of acceleration and we call exponential accelerations.

It is mentioned above that the error in Simpson's rule can be written as the following:

$$E = A_1 h^4 + A_2 h^6 + A_3 h^8 + \cdots \quad (3)$$

$$= h^4 (A_1 + A_2 h^2 + \cdots) \approx h^4 e^{h^2}$$

It is assumed that $S(h)$ is the approximate value of integration in Simpson's rule, so:

$$E = J - S(h) \approx h^4 e^{h^2}$$

If we assume that we calculate two values for $J$ calculated numerically based on Simpson's rule as $S_1(h_1)$ when $h = h_1$ and $S_2(h_2)$ when $h = h_2$:

$$J - S_1(h_1) \approx h_1^4 e^{h_1^2} \quad (4)$$

$$J - S_2(h_2) \approx h_2^4 e^{h_2^2} \quad (5)$$

From the equations (4) and (5), we get:

$$A_{h^2 e^{h^2}}^S \approx \frac{(h_1^4 e^{h_1^2}) S_1(h_1) - (h_2^4 e^{h_2^2}) S_2(h_2)}{h_1^4 e^{h_1^2} - h_2^4 e^{h_2^2}}$$

The formula (6) is called the first Al-Tememe's acceleration of exponential rule of the second kind that referred to by $A_{h^2 e^{h^2}}^S$.

Similarly, the second exponential acceleration rule of Al-Tememe can be written, since the error $E$ can be written as:

$$E = (A_1 h^4 + A_2 h^6 + A_3 h^8 + \cdots) = h^4 (A_1 - A_2 h^2 - A_3 h^4 - \cdots) \approx h^2 e^{-h^2},$$
\[ e^{-h^2} = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \cdots \]  \[ \text{[2]} \]

We call the formula (7) as Al-Tememe's second exponential acceleration rule of the second kind that referred to by \( A^s_{h^2 e^{-h^2}} \).

Similarly, we find the third exponential acceleration rule that we will call Al-Tememe's third exponential acceleration rule of the second kind, which is referred to by \( (A^s_{e^{h^2} e^{-h^2}}) \) and the fourth exponential acceleration rule of the second type referred to by \( (A^s_{e^{h^2} e^{-2h^2}}) \) respectively:

\[ E = \frac{E}{A^s_{h^2}} + \frac{1}{2!} h^2 + \frac{1}{4!} h^4 + \cdots \]

\[ \text{[3]} \]

3. Examples

We will review some integrals that have continuous integrands on the interval of integration using exponential acceleration methods of Al-Tememe to improve the results numerically:

3.1: \( l = \int f(x) \, dx \) and its analytical value is 0.26875540853367 and it is rounded for 14 decimal.

3.2: \( l = \int g(x) \, dx \) and its analytical value is 0.19106335261395 and it is rounded to 14 decimal.

3.3: \( l = \int h(x) \, dx \) and its analytical value is 0.30869426037404 and it is rounded to 14 decimal.

4. The results

The integrand of integration \( l = \int f(x) \, dx \) is continuous in the integration interval \([7,8]\), and the formula of correction terms of Simpson's rule as above mentioned (equation 3).

We put \( \text{EPS}=10^{-12} \) (represents the absolute error of the subsequent value- previous value). We get the results shown in table (1). We got correct values through accelerating \( A^s_{h^2 e^{h^2}} \) and \( A^s_{e^{h^2} e^{-h^2}} \) and \( A^s_{h^2 (e^{h^2} - 1)} \) to 12 decimal when \( n=10,12 \) while by using Simpson's method without acceleration.
was correct to 10 decimal when n=12. Also, we get the same accuracy by $A_{h^2}^{e^{-n}}$ when n=10. Moreover, we got correct acceleration values for $A_{e^{-n}}^{e^{-n^{-1}}}$ to 13 decimal when n=8. Also get the same accuracy for $A_{h^2(e^{-n^{-1}})}$ when n=8, 10.

The integrand of integration $l = \int_7^8 \frac{1}{x^3 + x^3} \, dx$ is continuous in the integration interval $[7, 8]$, and the formula of correction terms of Simpson's rule as above mentioned (equation 3).

We got the results shown in table (2). We got correct values through accelerating $A_{e^{h^2}}^{h^2}$ to 12 decimal when n=14 and for all other acceleration methods. But the value by Simpson's method without acceleration was correct to 9 decimal when n=14.

The integrand of integration $l = \int_9^{10} \frac{1}{\sqrt{1+y^2}} \, dx$ is continuous in the integration interval $[9, 10]$, and the formula of correction terms of Simpson's rule as above mentioned (equation 3).
We get the results shown in table (3). We get correct values through accelerating $A^v_{h^v e^v}$ to 12 decimal when n=12 also for all other acceleration methods. But the value by Simpson's method without acceleration was correct to 10 decimal when n=12.

5. Conclusion

We conclude from the mentioned tables that these acceleration methods have the same efficiency and give high accuracy of results in limited number of partial intervals with slight difference.

![Table](image-url)
Table no. (2) to calculate integration \( I = \int_{-1}^{1} \frac{1}{x^2+x^3} \, dx = 0.19106335261395 \) by Simpson’s rule with the exponential acceleration methods of Al-Tememe of second kind.

Table no. (3) to calculate integration \( I = \int_{0}^{1} \frac{1}{\sqrt{1+x^4}} \, dx = 0.30869426037404 \) by Simpson’s rule with the exponential acceleration methods of Al-Tememe of second kind.

| n  | \( A^h \) | \( A^h e^{-h^2} \) | \( A^h e^{(-h^2 - 1)} \) | \( A^h e^{h^2} \) | \( A^h e^{h^2 - 1} \) |
|----|------------|-------------------|-------------------|-------------------|-------------------|
| 2  | 0.30869431838189 | 0.30869425958506 | 0.30869426014971 | 0.30869426035923 | 0.30869426037212 |
| 4  | 0.3086942601772 | 0.30869426034922 | 0.30869426035578 | 0.30869426037362 | 0.30869426037392 |
| 6  | 0.30869426040523 | 0.30869426044074 | 0.30869426044573 | 0.30869426044857 | 0.30869426045180 |
| 8  | 0.30869426046745 | 0.30869426047349 | 0.30869426047451 | 0.30869426047451 | 0.30869426047451 |
| 10 | 0.3086942604909 | 0.30869426049292 | 0.30869426049317 | 0.30869426049317 | 0.30869426049317 |
| 12 | 0.30869426051349 | 0.30869426051550 | 0.30869426051575 | 0.30869426051575 | 0.30869426051575 |

6. References

[1] Fox L., “Romberg Integration for a Class of Singular Integrands”, Comput. J., 10, pp. 87-93, 1966

[2] D. Zwillinger, “Standard Mathematical Tables and Formulae”, 35th Edition, Boca Raton, London, New York Washington