SYMBOLIC AND CANCELLATION-FREE FORMULAE FOR SCHUR ELEMENTS

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Abstract. In this paper we give a symbolical formula and a cancellation-free formula for the Schur elements associated to the simple modules of the degenerate cyclotomic Hecke algebras. As some direct applications, we show that the Schur elements are symmetric with respect to the natural symmetric group action and are integral coefficients polynomials and we give a different proof of Ariki-Mathas-Rui’s criterion on the semi-simplicity of degenerate cyclotomic Hecke algebras.

1. Introduction

Schur elements play a powerful role in the representation theory of symmetric algebras, see for example [CR, Chap. 9] and [GP, Chap. 7]. In the case of the degenerate cyclotomic Hecke algebras, Brundan and Kleshchev [BK1, Theorem A2] showed that these algebras are symmetric algebras for all parameters, which enable us to use the Schur elements to determine when Specht modules are projective irreducible and whether the algebra is semi-simple.

We recently obtain the explicit formula for Schur elements for the degenerate cyclotomic Hecke algebras in [Z] by computing the trace form on some nice idempotents of these algebras. This paper continues our study on the Schur elements for the degenerate cyclotomic Hecke algebras. Our motivations are that the Schur elements for the degenerate cyclotomic Hecke algebras should have the analogue properties as that of Schur elements for cyclotomic Hecke algebras, which is motivated by Brundan and Kleshchev’s fundamental work [BK2, Corolary 1.3].

Inspired by Geck, Iancu and Malle’s work [GIM] and Mathas’ work [M04], we give an L-symbolical formula for the Schur elements (Theorem 3.4), which enable us to show that Schur elements are “invariant” with respect to the natural symmetric group action. Using the generalized hook length of partitions, we obtain a cancellation-free formula for Schur elements following Chlouveraki and Jacon’s work [CJ]. As some direct applications, we show that Schur elements are integral coefficients polynomials in parameters and then give a different proof of Ariki-Mathas-Rui’s criterion on the semi-simplicity of the degenerate cyclotomic Hecke algebras.

The lay-out of this paper is as follows. In section 2 we introduce the necessary definitions and fix the notation. The L-symbolical formula for the Schur elements for the degenerate cyclotomic Hecke algebras are given in Section 3 and then a cancellation-free formula for Schur elements for the degenerate cyclotomic Hecke algebras are determined in Section 4. Finally, some direct application of our formulæ and remarks are given in Section 5.

2. Preliminaries

We recall the definition and some properties of degenerate cyclotomic Hecke algebras, some notations and facts about (multi)partitions, and the formula for Schur elements.

2.1. Let $m, n$ be positive integers. Recall from [ST] or [Co] that the complex reflection group $W_{m,n}$ of type $G(m, 1, n)$ is the finite group generated by elements $s_0, s_1, \ldots, s_{n-1}$ subject to the relations

\[
s_i^m = 1, \quad s_0 s_i s_0 = s_1 s_0 s_1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad i > 1
\]

In particular, the subgroup $\langle s_1, \ldots, s_{n-1} \rangle$ of $W_{m,n}$ is isomorphic to the symmetric group $S_n$ of order $n$ with simple transposition $s_i = (i, i+1)$ for $i = 1, \ldots, n-1$. It is well-known that $W_{m,n} \cong (\mathbb{Z}/m\mathbb{Z})^n \rtimes S_n$. Clearly, $W_{1,n}$ is the Weyl group of type $A_n$ and $W_{2,n}$ is the Weyl group of type $B_n$.

2.2. Definition. Let $R$ be a commutative ring and $Q = (q_1, \ldots, q_m) \in R^m$. The degenerate cyclotomic Hecke algebra is the unital associative $R$-algebra $H(Q) := H_{m,n}(Q)$ generated by $s_0, s_1, \ldots, s_{n-1}$ and subjected to relations

\[
\begin{align*}
(i) \quad & (s_0 - q_1) \ldots (s_0 - q_m) = 0, \\
(ii) \quad & s_0(s_1 s_0 s_1 + s_1) = (s_1 s_0 s_1 + s_1) s_0,
\end{align*}
\]

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We may and will identify $H$ of all $2.4.$ $= 0$ only if $2.5.$ $A$ $\lambda$ $\frac{2}{\ell}$. Then $W$ clearly, $\mathcal{H}_{i,n}(Q)$ is exactly the group algebra $RS_n$ and $\mathcal{H}(Q)$ is a (degenerate cyclotomic) deformation of the complex reflection group $W_{m,n}$. Kleshchev [K, Theorem 7.5.6] have proved that $\mathcal{H}$ is a free $R$-module with basis $\{x_1^i x_2^j \cdots x_n^w \mid 0 \leq i_1, \ldots, i_n < m, w \in S_n\}$. The following theorem says that $\mathcal{H}(Q)$ is a symmetric algebra for all parameters $q_1, \ldots, q_m$ in $R$.

2.3. Theorem ([BK1], Theorem A2). Let $\tau : \mathcal{H}(Q) \rightarrow R$ be the $R$-linear map determined by

$$\tau(x_1^{i_1} \cdots x_n^i w) := \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = m - 1 \text{ and } w = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\tau$ is a non-degenerate trace form on $\mathcal{H}(Q)$ all $m$-tuples $Q = (q_1, \ldots, q_m) \in R^m$.

2.4. Assumption. This paper is concerned with the Schur elements for degenerate cyclotomic Hecke algebras $\mathcal{H}(Q)$ with the parameters $Q = (q_1, \ldots, q_m) \in R^m$. The $m$-tuples of parameters $Q$ is separated over $R$ if

$$P_{\mathcal{H}}(Q) := n! \prod_{1 \leq i < j \leq m} |d_i - c_j| (d + q_i - q_j)$$

is invertible in $R$.

Ariki-Mathas-Rui [AMR, Theorem 6.11] have showed that when $R$ is a field $\mathcal{H}(Q)$ is semisimple if and only if $P_{\mathcal{H}}(Q) \neq 0$. This criterion will be recovered form our results later.

2.5. A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ is a decreasing sequence of non-negative integers. We define the length of $\lambda$ to be the smallest integer $\ell(\lambda)$ such that $\lambda_i = 0$ for all $i > \ell(\lambda)$. We write $|\lambda| := \sum_{i \geq 1} \lambda_i$ and we say that $\lambda$ is a partition of $|\lambda| \in \mathbb{N}$. The diagram of $\lambda$ is the set $[\lambda] := \{(i, j) \mid i \geq 1, 1 \leq j \leq \lambda_i\}$. The elements of $[\lambda]$ are called the nodes of $\lambda$ and a node $x = (i, j)$ is called removable if $[\lambda]\backslash\{(i, j)\}$ is still the diagram of a partition. Note that if $(i, j)$ is removable, then $j = \lambda_i$.

Recall that an $m$-multipartition of $n$ is an ordered $m$-tuple $\lambda = (\lambda^1 : \cdots : \lambda^m)$ of partitions $\lambda^i$ such that $n = \sum_{i=1}^{m} |\lambda^i|$. We define the length of $\lambda$ be $\ell(\lambda) = \max\{\ell(\lambda^s) \mid 1 \leq s \leq m\}$ and denote by $\mathcal{P}(m, n)$ the set of all $m$-multipartitions of $n$. The diagram of $\lambda$ is the set

$$[\lambda] := \{(i, j, c) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times m \mid 1 \leq j \leq \lambda_i\} \text{ where } m = \{1, \ldots, m\}.$$ 

Similarly the elements of $[\lambda]$ are called the nodes of $\lambda$; more generally, a node is any element of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times m$. We may and will identify $[\lambda]$ with the $m$-tuple of diagrams of the partitions $\lambda^i$, for $1 \leq c \leq m$.

Given a partition $\lambda$, let $\bar{\lambda} = (\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots)$ be the partition which is conjugate to $\lambda$; thus, $\bar{\lambda}_i$ is the number of nodes in column $i$ of the diagram of $\lambda$. Recall that the $(i, j)$-th hook in the diagram of $[\lambda]$ is the collection of nodes to the right of and below the node $(i, j)$, including the node $(i, j)$ itself. The $(i, j)$-th hook length $h^\lambda_{i,j} = \lambda_i - i + \bar{\lambda}_j - j + 1$ is the number of nodes in the $(i, j)$-th hook.

2.6. Remark. Assume that $\lambda$ is a partition. Then $\ell(\lambda) = \bar{\lambda}_1$ and $(i, j)$ is a removable node of $\lambda$ if and only if $j = \lambda_i$ and $i = \bar{\lambda}_j$.

We will need the following lemma, whose proof is an easy combinatorial exercise. For the convenience of reader especially for myself, we contain the proof.

2.7. Lemma. Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \mu_{k+1} = 0)$ be a partition and let $y$ be an indeterminate. Then for any integer $\ell$ such that $1 \leq \ell \leq \mu_1$, we have

$$\frac{1}{\mu_1 + y} \prod_{1 \leq i \leq \mu_1} \frac{\mu_i - i + 1 + y}{\mu_i - i + y} = \frac{1}{\ell - \mu_1 - 1 + y} \prod_{\ell \leq j \leq \mu_1} \frac{j - \mu_j - 1 + y}{j - \mu_j + y}.$$ 

Proof. Observe that $\mu_{\mu_1+i+s} = i$ for all $1 \leq s \leq \mu_i - \mu_{i+1} - 1 \leq \mu_i$ and that for any integer $1 \leq \ell \leq \mu_1$, there exists an integer $1 \leq t \leq k$ such that $\mu_{t+1} + 1 \leq \ell \leq \mu_t$. Therefore,

$$\frac{1}{\ell - \mu_1 - 1 + y} \prod_{\ell \leq j \leq \mu_1} \frac{j - \mu_j - 1 + y}{j - \mu_j + y} = \frac{1}{\mu_1 + y} \prod_{1 \leq i \leq \mu_1} \frac{\mu_i - i + 1 + y}{\mu_i - i + y} = \frac{1}{\mu_1 + y} \prod_{1 \leq i \leq \mu_1} \frac{\mu_i - i + 1 + y}{\mu_i - i + y}.$$ 


As a consequence, we complete the proof. □

Assume that $R$ is a field and that $Q$ is separated. Then $\{S^\lambda \mid \lambda \in \mathcal{P}(m, n)\}$ is a complete set of pairwise non-isomorphic irreducible $H(Q)$-modules and let $\chi^\lambda$ be the character of $S^\lambda$. Following Geck’s results on symmetrizing form (see [GP, Theorem 7.2.6]), we obtain the following definition for the Schur elements of $H(Q)$ associated to the irreducible representations of $H(Q)$.

2.8. **Definition.** Assume that $R$ is a field and that $Q$ is separated. The Schur elements of $H(Q)$ are the elements $s^\lambda(Q) \in R$ such that

$$\tau = \sum_{\lambda \in \mathcal{P}(m, n)} \frac{\chi^\lambda}{s^\lambda(Q)}.$$ 

The formula for the Schur elements for $H(Q)$ are given by the following theorem.

2.9. **Theorem ([Z], Theorem 7.9).** Let $\lambda = (\lambda^1; \ldots; \lambda^m)$ be an $m$-multipartition of $n$. Then

$$s^\lambda(Q) = \prod_{(i, j) \in [\lambda]} h^{\lambda_{ij}}_{ij} \prod_{1 \leq s < t \leq m} X_{st}^\lambda,$$

where, for $1 \leq s < t \leq m$,

$$X_{st}^\lambda = \prod_{(i, j) \in [\lambda]} \left( j - i + q_{t} - q_{s} \right) \prod_{(i, j) \in [\lambda]} \left( j - i - \lambda_{i}^{t} + q_{s} - q_{t} \right) \prod_{1 \leq k \leq \lambda_{i}^{s}} \frac{j - i - k + \lambda_{i}^{t} + 1 + q_{s} - q_{t}}{j - i - k + \lambda_{i}^{t} + q_{s} - q_{t}}.$$ 

3. **The L-symbolical formula for the Schur elements.**

In this section we are going to give the L-symbolical formula for Schur elements, which is inspired by Geck, Iancu and Malle’s work [GIM] and Mathas’ work [M04, Corollary 6.5]. Before to do this, we need the notation of L-symbol of (multi)partitions introduced by Malle [Ma].

3.1. **Definition.** Let $\lambda$ be a partition and fix an integer $L$ such that $L \geq \ell(\lambda)$. The L-beta numbers for $\lambda$ are the integers $\beta^{L}_{\lambda} = \lambda_{i} + L - i$ for $i = 1, \ldots, L$. For an m-multipartition $\lambda = (\lambda^{1}; \ldots; \lambda^{m})$, we call the $m \times L$ matrix $B_{\lambda}^{L} = (\beta^{L}_{\lambda})_{s, i}$ the L-symbol of $\lambda$.

From now on, we denote by $B_{\lambda}^{L}$ the set of the L-beta numbers for a partition $\lambda$. Observe that

$$B_{\lambda}^{L} = \{ \beta^{L}_{\lambda} = \lambda_{i} + L - i \mid 1 \leq i \leq \ell(\lambda) \} \cup \{ L - i - \ell(\lambda) + 1 \leq i \leq L \}$$

and that if we change $L$ to $L + 1$ then the beta set $B_{\lambda}^{L}$ is shifted to $B_{\lambda}^{L + 1} = \{ k + 1 \mid k \in B_{\lambda}^{L} \} \cup \{ 0 \}$. We say that a function of beta numbers is invariant under beta shifts if it is unchanged by such transformations; equivalently, the function is independent of $L$ provided that $L$ is large enough. For example, the formula for $s^\lambda(Q)$ is invariant under beta shifts since $s^\lambda(Q)$ does not depend on $L$.

The following lemma is the key to the L-symbolical formula for the Schur elements.

3.2. **Lemma.** Assume that $\lambda$, $\mu$ are partitions and that $x$ is an indeterminate $x$. Put

$$X_{\lambda \mu}(x) := \prod_{(i, j) \in [\mu]} (j - i - x) \prod_{(i, j) \in [\lambda]} \left( j - i - \mu_{1} + x \right) \prod_{1 \leq k \leq \mu_{1}} \frac{j - i + \mu_{k} - k + 1 + x}{j - i + \mu_{k} - k + x}$$

and

$$Y_{\lambda \mu}^{L}(x) := (-1)^{\binom{L}{2}}_{\lambda \mu} x^{L} \prod_{a \in B_{\lambda}^{L} 1 \leq r \leq a} \prod_{b \in B_{\mu}^{L} 1 \leq j \leq b} \frac{a - b + x}{j - x}$$

for a non-negative integer $L$.

Then $X_{\lambda \mu}(x) = Y_{\lambda \mu}^{L}(x)$ for any integer $L \geq \max \{ \ell(\lambda), \ell(\mu) \}$.

**Proof.** First, we show that $Y_{\lambda \mu}^{L}(x) = Y_{\lambda \mu}^{L + 1}(x)$ for all integers $L \geq \max \{ \ell(\lambda), \ell(\mu) \}$. In the case of $L + 1$, $B_{\lambda}^{L + 1} = \{ i + 1 \mid i \in B_{\lambda}^{L} \} \cup \{ 0 \}$ and $B_{\mu}^{L + 1} = \{ i + 1 \mid i \in B_{\mu}^{L} \} \cup \{ 0 \}$. Therefore, by definition,

$$Y_{\lambda \mu}^{L + 1}(x) = (-1)^{\binom{L+1}{2}}_{\lambda \mu} x^{L + 1} \prod_{a \in B_{\lambda}^{L + 1} 1 \leq r \leq a} \prod_{b \in B_{\mu}^{L + 1} 1 \leq j \leq b} \frac{a - b + x}{j - x}$$

for a non-negative integer $L$.

Then $X_{\lambda \mu}(x) = Y_{\lambda \mu}^{L}(x)$ for any integer $L \geq \max \{ \ell(\lambda), \ell(\mu) \}$.
\[
\lambda \in \mathbb{N}^k \quad \text{and that } \lambda \in \mathbb{N}^k. \]
Applying Lemma 2.7, we prove the lemma for partitions \( \lambda \) and \( \mu \) with \( |\mu| = r \). As a consequence, we prove the Lemma for all partitions \( \lambda \) and \( \mu \).

The following property of \( X_{\lambda \mu}(x) \) will be used later.

3.3. Corollary. Keep notations as Lemma 3.2. Then \( X_{\lambda \mu}(x) = X_{\mu \lambda}(-x) \). 

Proof. Applying Lemma 3.2, it suffices to show that \( Y_{\lambda \mu}^L(x) = Y_{\mu \lambda}^L(-x) \) for any integer \( L \geq \max\{\ell(\lambda), \ell(\mu)\} \). In fact, this follows directly by the definitions of \( Y_{\lambda \mu}^L(x) \) and of \( Y_{\mu \lambda}^L(-x) \) given in Lemma 3.2.

Now we can obtain the \( L \)-symbolical formula for Schur elements.

3.4. Theorem. Suppose that \( \lambda = (\lambda^1; \ldots; \lambda^m) \) is an \( m \)-multipartition of \( n \) with \( L \)-symbol \( B_L^\lambda = (\beta_i^s)_{s,i} \) such that \( L \geq \ell(\lambda) \). Then

\[
s_\lambda(Q) = (-1)^{\binom{m}{2}} \prod_{1 \leq s < t \leq m} (q_s - q_t)^L \prod_{1 \leq s \leq m} \prod_{\alpha \in B_L^\lambda \atop 1 \leq k \leq \alpha_s} (k + q_s - q_t) \prod_{1 \leq s \leq m} \prod_{1 \leq i < j \leq L} (\beta_i^s - \beta_j^s).
\]

Proof. Note that for an \( m \)-multipartition \( \lambda \) of \( n \) with \( L \)-symbol \( B_L^\lambda = (\beta_i^s)_{s,i} \) such that \( L \geq \ell(\lambda) \) we have the following well-known fact, see for example [M, Examples I.1(4)],

\[
\prod_{(i,j,s) \in [\lambda]} h_{i,j}^s = \prod_{s=1}^m \prod_{1 \leq i < j \leq L} (\beta_i^s - \beta_j^s).
\]
Now let
\[ \nu_{\lambda} = (-1)^{m}(z)\prod_{1 \leq s < t \leq m} (q_{s} - q_{t})^{L} \prod_{1 \leq s, t \leq m, \alpha_{s} \in B_{L}} \prod_{1 \leq k \leq \alpha_{s}} (k + q_{s} - q_{t}); \]
\[ \delta_{\lambda} = \prod_{1 \leq s < t \leq m} \prod_{(\alpha_{s}, \alpha_{t}) \in B_{L} \times B_{L}'} (\alpha_{s} - \alpha_{t} + q_{s} - q_{t}) \prod_{1 \leq s \leq m} \prod_{1 \leq i \leq L} (\beta_{i}^{s} - \beta_{j}^{s}). \]

Then
\[ \frac{\nu_{\lambda}}{\delta_{\lambda}} = \prod_{1 \leq s \leq m} \prod_{1 \leq i < j \leq L} \frac{\alpha_{s}!}{(\beta_{i}^{s} - \beta_{j}^{s})!} \prod_{1 \leq s, t \leq m} (-1)^{\binom{m}{2}} \left( q_{s} - q_{t} \right)^{L} \prod_{1 \leq s, t \leq m} \prod_{1 \leq k \leq \alpha_{s}} \left( k + q_{s} - q_{t} \right) \prod_{(\alpha_{s}, \alpha_{t}) \in B_{L} \times B_{L}'} (\alpha_{s} - \alpha_{t} + q_{s} - q_{t}). \]

As a consequence, following Theorem 2.9, the theorem follows immediately by applying Lemma 3.2 for \( \lambda = \lambda^{s}, \mu = \mu^{t} \) and \( x = q_{s} - q_{t} \). We complete the proof. \( \square \)

3.5. Remark. Geck, Iancu and Malle [GIM] use a clever specialization argument due to Orellana [O] to compute the Schur elements for the cyclotomic Hecke algebras using the Markov trace of the Hecke algebras \( H_{q}(S_{n}) \), which does not work for degenerate case because the trace form \( \tau \) on degenerate cyclotomic Hecke algebras is not a Markov trace. It is very interesting to know whether there is a “degenerate” version Markov trace for degenerate cyclotomic Hecke algebras satisfying the similar properties as that of usual Markov trace.

4. A CANCELLATION-FREE FORMULA FOR THE SCHUR ELEMENTS

In this section, we give a cancellation-free formula for the Schur elements, which is inspired by Chlouveraki and Jacon’s work [CJ]. We need the following notation. Let \( \lambda \) and \( \mu \) be partitions. If \( (i, j) \) is a node of \( [\lambda] \) and we define the generalized hook length of the node \( (i, j) \) with respect to \( (\lambda, \mu) \) to be the integer
\[ h_{i,j}^{\lambda,\mu} = \lambda_{i} - i + \lambda_{j} - j + 1. \]
Observe that if \( \lambda = \mu \) then \( h_{i,j}^{\lambda,\mu} = h_{i,j}^{\lambda}. \)

The following lemma is crucial to the cancellation-free formula for the Schur elements.

4.1. Lemma. Assume that \( \lambda, \mu \) are partitions and that \( x \) is an indeterminate \( x \). Let \( X_{\mu,\lambda}(x) \) be the one defined in Lemma 3.2 and define
\[ Z_{\mu,\lambda}(x) := \prod_{(i,j) \in [\lambda]} \left( h_{i,j}^{\lambda,\mu} + x \right) \prod_{(i,j) \in [\mu]} \left( h_{i,j}^{\mu,\lambda} - x \right). \]

Then \( X_{\mu,\lambda}(x) = Z_{\mu,\lambda}(x) \).

Proof. We will proceed our proof by induction on the number of nodes of \( \lambda \). We do not need to do the same for \( \mu \), because Corollary 3.3 implies that \( Z_{\mu,\lambda}(x) = X_{\mu,\lambda}(x) = X_{\lambda,\mu}(-x) = Z_{\mu,\lambda}(-x) \).

If \( \lambda = (0) \) then, by definition, we have
\[ X_{\mu,\lambda}(x) = \prod_{(i,j) \in [\mu]} (j - i - x) \]
\[ = \prod_{1 \leq i \leq \ell(\mu)} \prod_{1 \leq j \leq \mu_{i}} (t - i - x) \]
\[ = \prod_{1 \leq i \leq \ell(\mu)} \prod_{1 \leq j \leq \mu_{i}} (\mu_{i} - i - j + 1 - x) \]
\[ = \prod_{(i,j) \in [\mu]} \left( h_{i,j}^{\mu,\lambda} - x \right) \]
\[ = Z_{\mu,\lambda}(x), \]
where the third equality follows by setting \( t = \mu_{i} - i - j \) and \( 1 \leq t \leq \mu_{i}, \) which implies that \( 1 \leq j \leq \mu_{i} \).

Now assume that the assertion holds for all \( |\lambda| \leq k - 1 \). We want to show that it also holds when \( |\lambda| = k \geq 1 \). Note that in this case, there exists \( i \) such that \( (i, j) \) is a removable node of \( \lambda \) where \( j = \lambda_{i} \). Let
\( \nu \) be the partition obtained from \( \lambda \) by removing the removable node \((i, j)\). So \([\lambda] = [\nu] \cup \{(i, j)\}\) and we get that
\[
X_{\lambda}(x) = X_{\nu}(x)(j - i + x - 1) \prod_{1 \leq i \leq \lambda_1} \frac{\mu_i - i + j + x - 1}{\mu_i - i + j - i + x}.
\]

On the other hand, we have
\[
Z_{\lambda}(x) = Z_{\nu}(x) \prod_{(i, j) \in [\nu]} \frac{(h_{i,j}^\nu - x)}{(h_{i,j}^\nu - x)} \prod_{(i, j) \in [\mu]} \frac{(h_{i,j}^\mu - x)}{(h_{i,j}^\mu - x)}
= Z_{\nu}(x) \prod_{1 \leq i \leq \lambda_1} \frac{\mu_i - i + j + x - 1}{\mu_i - i + j - i + x} \prod_{1 \leq i \leq \lambda_1} \frac{\mu_i - i + j - x}{\mu_i - i + j - i + x}
= Z_{\nu}(x) \prod_{1 \leq i \leq \lambda_1} \frac{\mu_i - i + j - x}{\mu_i - i + j - i + x}
= Z_{\nu}(x) \prod_{1 \leq i \leq \lambda_1} \frac{\mu_i - i + j - x}{\mu_i - i + j - i + x},
\]

where the third equality follows by Remark 2.6 and the fourth one follows by using Lemma 2.7 for \( \ell = j \) and \( y = i - j - x \); because that both \( X_{\lambda}(x) \) and \( Z_{\lambda}(x) \) are invariant under beta-shifts, without loss of generality, we may assume that \( \mu_1 \) are large enough. Thus \( X_{\lambda}(x) = Z_{\lambda}(x) \) for partitions \( \lambda \) and \( \mu \) with \([\lambda] = k\). As a consequence, we complete the proof. \( \square \)

The following is the cancellation-free formula for Schur elements, which is symmetric.

4.2. Theorem. Let \( \lambda = (\lambda^1; \ldots; \lambda^m) \) be an \( m \)-multipartition of \( n \). Then
\[
s_{\lambda}(Q) = \prod_{1 \leq s \leq m} \prod_{(i, j) \in [\lambda^s]} \prod_{1 \leq t \leq m} (h_{i,j}^{\lambda^s} + q_s - q_t) = \prod_{1 \leq s \leq m} \prod_{(i, j) \in [\lambda^s]} \prod_{1 \leq t \leq m} (h_{i,j}^{\lambda^s} + q_s - q_t).
\]

Proof. Following Theorem 2.9, to prove the theorem, it is enough to show that, for all \( 1 \leq s < t \leq m \),
\[
X_{\lambda}^s = Z_{\lambda}^s(q_s - q_t),
\]
which follows directly by applying Lemma 4.1 for \( \lambda = \lambda^t \), \( \mu = \lambda^s \) and \( x = q_s - q_t \). As a consequence, the proof is completed. \( \square \)

5. Applications

In this section we give here several direct applications of the \( L \)-symbolic formula (Theorem 3.4) and the cancellation-free formula (Theorem 4.2) obtained in §§3 and 4.

5.1. Let \( S_m \) be the symmetric group of order \( m \) with simple transposes \( s_i = (i, i + 1) \) for \( i = 1, \ldots, m - 1 \). Note that there is an action of \( S_m \) on the set of \( m \)-multipartitions of \( n \) (by permuting components) and also on the rational functions in \( q_1, \ldots, q_m \) (by permuting parameters). As a direct application of Theorem 3.4, we obtain the following symmetry formula for Schur elements, which can be also obtained by observing that when \( \mathcal{H}(Q) \) is semi-simple the Specht modules are determined up to isomorphism by the action of Jucys-Murphy elements \( x_1, \ldots, x_n \) of \( \mathcal{H}(Q) \) and that the relation \( \prod_{i=1}^{m}(x_i - q) = 0 \) is invariant under the \( S_m \)-action (see also [Z, Remark 7.4]).

5.2. Corollary. Assume that \( R \) is a field and that \( \mathcal{H}(Q) \) is semi-simple. Then \( s_{\sigma(\lambda)}(Q) = \sigma(s_{\lambda}(Q)) \) for all \( m \)-multipartitions \( \lambda \) and all \( \sigma \in S_m \).

Proof. For \( i = 1, \ldots, m - 1 \), we let \( X_{s, i+1}(q_i, q_{i+1}) = X_{\lambda^{\lambda^{i+1}}}(q_i, q_{i+1}) \). Then, following Corollary 3.3,
\[
X_{s, i+1}(q_i, q_{i+1}) = X_{\lambda^{i+1}}(q_i, q_{i+1}) = X_{\lambda^{i+1}}(q_i, q_{i+1}) = X_{\lambda^{i+1}}(s_i(q_i, q_{i+1})).
\]
By the proof of Theorem 3.4, we have that \( s_{\lambda}(Q) = \prod_{(i, j) \in [\lambda]} h_{i,j}^{\lambda^s} \prod_{1 \leq s < t \leq m} X_{s, t}(q_s, q_t) \). As a consequence, we complete the proof. \( \square \)

As a direct application of the cancellation-free-formula, we can obtain the following fact on Schur elements, which can also be proved by a analogue argument as that of [GP, Proposition 7.3.9].

5.3. Corollary. Assume that \( R \) is a field and that \( \mathcal{H}(Q) \) is semi-simple. Then \( s_{\lambda}(Q) \in \mathbb{Z}[q_i] \) for all \( m \)-multipartitions \( \lambda \).
A second application of cancellation-free formula is that we can easily recover a well-known semi-simplicity criterion for the degenerate cyclotomic Hecke algebra due to Ariki-Mathas-Rui [AMR, Theorem 6.11]. To do this, let us assume that \( q_1, \ldots, q_m \) are indeterminates and \( R = \mathbb{Q}(q_1, \ldots, q_m) \). Then the resulting “generic” degenerate cyclotomic Hecke algebra \( \mathcal{H}(Q) \) is split semi-simple. Now assume that \( \theta: \mathbb{Z}[q_1, \ldots, q_m] \to \mathbb{K} \) is a specialization and let \( \mathbb{K}\mathcal{H}(Q) \) be the specialized algebra, where \( \mathbb{K} \) is any field. Note that for all \( \lambda \in \mathcal{P}(m, n) \), we have \( s_{\lambda}(Q) \in \mathbb{Z}[q_1, \ldots, q_m] \) according to Corollary 5.3. Then by [GP, Theorem 7.2.6], \( \mathbb{K}\mathcal{H}(Q) \) is (split) semi-simple if and only if, for all \( \lambda \in \mathcal{P}(m, n) \), we have \( \theta(s_{\lambda}(Q)) \neq 0 \). From this, we can deduce the following:

5.4. **Theorem** ([AMR], Theorem 6.11). Assume that \( \mathbb{K} \) is a field. The algebra \( \mathbb{K}\mathcal{H}(Q) \) is (split) semi-simple if and only if \( \theta(P_{\mathcal{H}(Q)}) \neq 0 \), where \( P_{\mathcal{H}(Q)} \) is defined in Assumption 2.4.

**Proof.** Assume first that \( \theta(P_{\mathcal{H}(Q)}) = 0 \). We distinguish three cases:

1. If \( \theta(n!) = 0 \), then we have \( \theta(h_{1,n-i+1}^{\nu, \eta}) = 0 \) for \( \eta = ((n); (0); \ldots; (0)) \in \mathcal{P}(m, n) \). Thus, for this \( m \)-multipartition, we have \( \theta(s_{\lambda}(Q)) = 0 \), which implies that \( \mathbb{K}\mathcal{H}(Q) \) is not semi-simple.

2. If there exist \( 1 \leq s < t \leq m \) and \( 0 \leq k < n \) such that \( \theta(k + q_s - q_t) = 0 \), then we have \( \theta(h_{1,n-k}^{\nu, \lambda'}) + q_s - q_t = 0 \) for \( \lambda \in \mathcal{P}(m, n) \) such that \( \lambda^s = (n), \lambda^t = (0) \). We have \( \theta(s_{\lambda}(Q)) = 0 \) and \( \mathbb{K}\mathcal{H}(Q) \) is not semi-simple.

3. If there exist \( 0 \leq s < t \leq m - 1 \) and \( 0 < k < n \) such that \( \theta(k + q_s - q_t) = 0 \), then we have \( \theta(h_{1,n+k}^{\nu, \lambda'} + q_s - q_t) = 0 \) for \( \lambda \in \mathcal{P}(m, n) \) such that \( \lambda^s = (0), \lambda^t = (n) \). Again, we have \( \theta(s_{\lambda}(Q)) = 0 \) and \( \mathbb{K}\mathcal{H}(Q) \) is not semi-simple.

Conversely, if \( \mathbb{K}\mathcal{H}(Q) \) is not semi-simple, then there exists \( \lambda \in \mathcal{P}(m, n) \) such that \( \theta(s_{\lambda}(Q)) = 0 \). As for all \( 1 \leq s, t \leq m \) and \( (i, j) \in [\lambda^s] \), we have \( -n < h_{i,j}^{\nu, \lambda'} < n \), we conclude that \( \theta(P_{\mathcal{H}(Q)}) = 0 \).

5.5. We end by giving some remarks for the study of Schur elements for degenerate cyclotomic Hecke algebras. Recall from [BMM, Definition 2.11] that the generic degree of \( W_{m,n} \) are certain “spetsial” specializations of the rational functions \( s_{\eta}(Q)/s_{\lambda}(Q) \) where \( \eta = ((n); (0); \ldots; (0)) \)—so \( s_{\eta}(Q) \) is the Schur elements of the rational elements, which are polynomial with rational coefficients. Moreover, for these specializations \( s_{\eta}(Q) \) is equal to the Poincaré of the coinvariant algebra of the reflection representation of \( W_{m,n} \). It would be very interesting to know whether there are the “spetsial” specializations for degenerate cyclotomic Hecke algebras.

Finally it is very possible that there are suitable specializations for degenerate cyclotomic Hecke algebras, which enable us to define the \( a \)-function and \( A \)-function attached to very irreducible characters of a degenerate cyclotomic Hecke algebra of complex reflection group \( W_{m,n} \) by involving the Schur elements, which are constant on the Rouquier block as Chlouveraki have done for cyclotomic Hecke algebras in [C].

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