Analytical Method for Geometric Nonlinear Problems Based on Offshore Derricks

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Abstract: The marine derrick sometimes operates under extreme weather conditions, especially wind; therefore, the buckling analysis of the components in the derrick is one of the critical contents of engineering safety research. This paper aimed to study the local stability of marine derrick and propose an analytical method for geometrically nonlinear problems. The rod in the derrick is simplified as a compression rod with simply supported ends, which is subjected to transverse uniform load. Considering the second-order effect, the differential equations were used to establish the deflection, rotation angle, and bending moment equations of the derrick rod under the lateral uniform load. This method was defined as a geometrically nonlinear analytical method. Moreover, the deflection deformation and stability of the derrick members were analyzed, and the practical calculation formula was obtained. The Ansys analysis results were compared with the calculation results in this paper.

Keywords: geometric nonlinear problem; derrick leg; deflection equation; axial force second-order effect; Ansys

1. Introduction

With an inevitable trend of oil exploitation in the ocean, recently the focus has shifted to the design and manufacturing of offshore platforms. The derrick, a large and complex metal steel frame structure, is an essential part of the offshore drilling platform. The instability and sudden destruction of components in the derrick may cause the entire derrick to collapse at any time, oil leakage, and huge pollution to the marine environment. Therefore, the study on the local stability of marine derricks is of great significance to the marine environment. Compared with the onshore drilling frame, offshore derrick is placed in a worse environment. In addition to load during work, it is also affected by sea wind, ocean current, and waves; therefore, the stress on the offshore derrick in the marine environment is also more complicated. In recent years, a large number of studies have been conducted on the bearing capacity [1] and stability [2,3] of the derrick. Song [2] proposed a structural stability optimization method through a nonlinear analysis of the whole derrick. Taking the influence of wind load into consideration, the stability of the derrick was analyzed by finite element software in [3]. Moreover, since the loss of stability of a single rod component is one of the main factors affecting the bearing capacity of the structure, in addition to the overall stability analysis of the derrick, checking the deflection of the components in the derrick is also necessary.

The elastic curve of the buckling deformation of the slender compression rod was first proposed by Euler, and subsequently, an increasing number of scholars have studied the force and moment at the rod end [4–6]. Analysis of the nonlinear buckling problem of continuous compression rods with three rigid supports was conducted by Panayotounakos [4],
who obtained a parametric solution to this problem. The closed-form solution of the differential equation of the cantilever column under the end co-planar load is given in [5], while in [6], the influence of transverse deformation was considered, and the exact parametric analytic solution of the same problem was established.

The analysis on the large deflection of the rod subjected transverse uniform load is provided in [7,8]. The buckling problem of the compression rod subjected to end force, bending moments, and uniform load, have been further studied in [9–12]. Through the nonlinear analysis, the authors of [10] obtained an analytical solution for the initial bending rod subjected to a concentrated co-planar load consisting of component force and uniform load. The research model for rods is more suitable for studying the structural components due to the existence of distributed axial force.

Through simplification of the offshore derrick rods, this article extends the previous geometric nonlinear analysis method and provides a new analysis method for the bending problem of the strut under lateral uniform load. In this paper, the differential equation for the bending deformation of the compression rod was derived and the general solution was proposed. The geometric parameters were introduced to the deflection equation, and the parameters were determined by the boundary conditions and the balance conditions of the force, and the solution of the differential equation with determined parameters was obtained. This method is used to calculate the deflection deformation of simply supported compression rod under the action of transverse uniform load and rod end bending moment, and the analytical solution is provided. The formulas for the deflection, rotation angle, and bending moment of the simplified model were provided by this method. The local stability of the derrick was checked by Ansys finite element software, and the position of the maximum displacement of the H-shaped cross-section member was obtained. The deflection deformation of the H-section rod obtained by this method is compared with that by ANSYS software (Ansys 18.2, Mechanical APDL Product Launcher 18.2, USA).

2. Simplified Calculation Model of Derrick Members

Derrick is a light truss as shown in Figure 1a, which is a rod system structure composed of many slender rods.

![Figure 1](image_url)

**Figure 1.** Derrick force diagram: (a) marine derrick; (b) diagram of the derrick leg; and (c) finite element model of the rod of derrick leg.
The loss of the stability of a single member of the structure is one of the reasons that affect the overall safety of the derrick, and hence the deflection of a single member of the derrick was checked. It is assumed that the internal force of the rods in the structure is dominated by the axial force. Therefore, the node can be simplified as a hinge point. The derrick leg bears the greatest force. Therefore, in this paper, the rods in the derrick leg were simplified into ideal hinged joints with constant cross-section compression rods, and the wind load in extreme weather was simplified as a uniform horizontal load. Figure 1b is the force diagram of the rods of the derrick leg. Figure 1a shows a partially enlarged view of the derrick thigh and marks the simplified member of the actual member of Figure 1b. The force analysis of the rods of the derrick leg was carried out by Ansys software. Figure 1c is the force diagram of the finite element model of the rods of the derrick leg.

3. The Analytical Method for the Bending Problem of the Compression Rod

3.1. Differential Equations and Solutions for Bending Compression

Firstly, the differential equations and solutions of the bent compression rod should be calculated. An equal-section compression rod subjected to transverse uniform load $q$ is detached from the elastic structure and called beam–column [13–15]. As shown in Figure 2, it is in a bending equilibrium state, and the axial pressure on the sections of the rod ends is $P$. Due to the effect of elastic restraint, shear forces $Q_0$, $Q_L$, and moment $M_0$, $M_L$ respectively, act on the sections of the rod ends. The rectangular coordinate system is established with one end of the rod as the origin $O$, the rod axis before deformation as the X-axis, and the lateral deflection direction of the rod (the same direction as $q$) as the positive direction of the Y-axis.

![Figure 2. The force diagram of the compression rod.](image-url)
The differential equation for the bending deformation of the rod is represented as

\[ EI \frac{d^2y}{dx^2} + P(y - y_0) = M_0 - Q_0x + \frac{qx^2}{2}. \]  

(1)

After sorting out, the above equation can be written as

\[ EI \frac{d^2y}{dx^2} + Py = M_0 - Q_0x + \frac{qx^2}{2} + Py_0. \]  

(2)

Making \( K^2 = P/EI \), the above equation can be written as

\[ \frac{d^2y}{dx^2} + k^2y = \frac{1}{EI} \left( M_0 - Q_0x + \frac{qx^2}{2} + Py_0 \right). \]  

(3)

The above equation is a second-order non-homogeneous differential equation with constant coefficients. The characteristic value of the homogeneous equation corresponding to the above equation is

\[ r^2 + k^2 = 0. \]  

(4)

The root of the characteristic equation is a conjugate complex number, which is

\[ r_{1,2} = \pm ki. \]  

(5)

The solution of the homogeneous equation corresponding to the differential Equation (1) is

\[ y = A_1 \cos kx + A_2 \sin kx, \]  

(6)

where \( A_1 \) and \( A_2 \) are undetermined integration constants. Since the right end of the non-homogeneous Equation (1) is a quadratic function of \( x \), taking into account the theory of differential equations [16,17], it can be determined that the power of the particular solution \( y^* \) of the non-homogeneous equation should be the same as that of the right end function, which is also a quadratic equation. The general form of \( y^* \) can be written as

\[ y^* = A_5x^2 + A_3x + A_4, \]  

(7)

where \( A_5, A_3, \) and \( A_4 \) are undetermined integration constants. Thus, the general solution of differential Equation (1) can be obtained as follows:

\[ y = A_1 \cos kx + A_2 \sin kx + A_5x^2 + A_3x + A_4 \]  

(8)

Calculated length coefficient \( \mu \) of the sine curve, inflection point position \( \zeta \), and undetermined coefficient \( A \) are introduced, and the above equation can take the form

\[ y = A \sin \left( \frac{x}{\mu L} - \frac{\zeta}{\mu} \right) \pi + A_5x^2 + A_3x + A_4. \]  

(9)

3.2. Decomposition of the Deflection Curve

According to Equation (9), the lateral deflection deformation of the rod can be obtained by the superposition of a sine curve and a quadratic parabola.

Figure 3 shows the decomposition principle of the deflection curve of the bending deformation of the rod [18]. Figure 3a is the force diagram of the rod. Figure 3b is a schematic diagram of the compression rod in the curve deformation equilibrium state. Figure 3c shows a sine curve decomposed by the deflection curve function, which takes the X-axis as the baseline, and the distance between the two inflection bending points is \( \mu L \).
The coordinate of the first inflection point in the positive direction of the X-axis is $\zeta L$. Only the pressure $P$ parallel to the X-axis and the bending moment $M_c$ act on any section of the curve. Figure 3d is a quadratic parabola decomposed by the deflection curve function. The shape of this quadratic curve must meet the demand that, under the combined action of transverse load $q$, rod end shear force $Q_0$, $Q_L$, and axial force $P$, no bending moment and shear force, but only axial force will be generated on any section perpendicular to the axis of the rod after bending deformation [19]. That is to say, the rod is in the curved equilibrium state of axial compression due to the influence of the above loads.

3.3. Determination of the Undetermined Coefficient

According to the analytical method, the process of solving differential equations under particular conditions is transformed into the process of determining the undetermined coefficient according to the equilibrium condition of the force, the boundary condition of the rod, and the deformation continuity condition [20,21].

First of all, the coefficient $A_5$ and $A_3$ of the quadratic parabola is determined. Because the sine curve is projected to the curve shown in Figure 3d, it can be called the initial curve $y_i$, which is

$$y_i = A_5x^2 + A_3 + A_4.$$  \hspace{1cm} (10)

The rotation equation of the quadratic curve is

$$\tan \theta_i = 2A_5x + A_3.$$  \hspace{1cm} (11)

At the rod end $x = 0$, the slope of the parabola should be equal to the tangent value of the angle between the line of resultant force action of the axial force $P$ and the horizontal reaction force $Q_0$ at the rod end and the X-axis, that is

$$A_3 = \frac{Q_0}{P},$$  \hspace{1cm} (12)

where $Q_0$ is the shear force of the rod at the origin end. When $Q_0$ is the same as the $q$ direction, namely, the direction of the Y-axis is the same, the positive value is taken. On the
contrary, \( Q_L \) takes a negative value when the shear force at rod L end has the same direction as the \( q \) and Y-axis.

In the same way, at the other end of the rod, \( x = L \) have

\[
2LA_5 + \frac{Q_0}{P} = \frac{Q_L}{P},
\]

(13)

where \( Q_L \) is the shear force of the rod at the origin. From the above equation, \( A_5 \) is

\[
A_5 = \frac{Q_L - Q_0}{2PL}.
\]

(14)

Since the algebraic sum of the force equals zero in the direction of the Y-axis, \( qL = Q_L - Q_0 \), put it into the above equation, \( A_5 \) can be written as

\[
A_5 = \frac{q}{2P}.
\]

(15)

After \( A_3 \) is determined, the initial curve \( y_i \) can be expressed as

\[
y_i = \frac{q}{2P}x^2 + \frac{Q_0}{P}x + A_4.
\]

(16)

Check now whether the bending moment on any cross-section of \( y_i \) subjected to \( P \) and \( Q_0 \) is zero. Figure 4a shows the relationship between \( P, Q_0 \), and initial curve \( y_i \). \( A_4 \) is the distance between the \( x = 0 \) section of the \( y_i \) curve and the origin of the rectangular coordinate system. When \( y_i < A_4 \), the bending moment of the cross-section of the rod is

\[
P(A_4 - y_i) + \frac{qy_i^2}{2} + Q_0x = P\left[A_4 - \left(\frac{q}{2P}x^2 + \frac{Q_0}{P}x + A_4\right)\right] + \frac{qy_i^2}{2} + Q_0x = 0.
\]

(17)

Figure 4. Schematic diagram of initial deflection curve of compression rod: (a) diagram of the relationship between the initial deflection curve and the load and (b) Diagram of shear force composition on the vertical section of initial deflection curve.
When \( y_i > A_4 \), the bending moment of the cross-section of the rod is

\[
\frac{q x^2}{2} - P(y_i - A_4) + Q_0 x = \frac{q x^2}{2} - P \left( \frac{q x^2}{2} + \frac{Q_0}{P} x + A_4 \right) - A_4 + Q_0 x = 0. \tag{18}
\]

From the calculation, it can be derived that the determined integral constant satisfies the demand that no bending moment generates under \( P, Q_0 \), and \( Q_L \) within the whole length of the rod. Moreover, the rod under \( P, Q_0 \), and \( Q_L \) will not produce a shear force on any section perpendicular to the rod axis (quadratic curve). The proof is as follows.

The rotation angle equation of the initial deflection curve \( y_i \) is given by

\[
tan \theta_i = tan \alpha = \frac{q x + Q_0}{P}, \tag{19}
\]

where \( \alpha \) is the included angle between the axis and the tangent line of any point on curve \( y_i \) which has a distance of \( x \) from the origin.

Shear \( V \) on the section perpendicular to the curved rod axis in \( y_i \) is composed of the projections of \( xq \), \( Q_0 \), and \( P \) on the section, as shown in Figure 4, and the shear value is

\[
V = xqcosa + Q_0cosa - Psina. \tag{20}
\]

The above equation can be written as

\[
\frac{V}{sina} = xqcota + Q_0cota - P = \frac{xq + Q_0}{tana} - P. \tag{21}
\]

Putting the expression of \( \theta_i \) into the above equation, we can obtain

\[
V = 0, \tag{22}
\]

confirmation is finished.

After confirming \( A_5, A_3 \) the deflection curve of the bending deformation of the rod can be written as

\[
y = Asin \left( \frac{x}{\mu L} - \frac{\zeta}{\mu} \right) \pi + \frac{q}{2P} x^2 + \frac{Q_0}{P} x + A_4, \tag{23}
\]

where \( A, \zeta, A_4 \) can be determined by the boundary condition and the force balance condition. From the differential relationship between deflection and rotation angle, the rotation angle equation of the rod can be obtained as

\[
tan \theta_x = A \frac{\pi}{\mu L} \cos \left( \frac{x}{\mu L} - \frac{\zeta}{\mu} \right) \pi + \frac{q}{P} x + \frac{Q_0}{P}. \tag{24}
\]

The bending moment on the section with a distance of \( x \) to the origin is

\[
M_x = -EI \frac{d^2 y}{dx^2} = EIA \left( \frac{\pi^2}{\mu L} \right) x^2 \sin \left( \frac{x}{\mu L} - \frac{\zeta}{\mu} \right) \pi - EI \frac{q}{P}. \tag{25}
\]

Equations (23)–(25) are the base equations of the analytical method for bending problems of the compression rod subjected to transverse uniform load.

4. Mechanics Principles of the Analytical Method

The solution of the pressure rod bending balance differential equation is decomposed into the form of a sine curve and multi-curve superposition. Through the analysis of the specific force balance state of the pressure rod, the solution of the pressure rod differential equation problem is transformed into the problem of determining the undetermined geometric parameters of the torsion curve equation on the basis of the boundary conditions.
and the equilibrium conditions of the pressure rod. Now, analyze the principles of mechanics for the analytical method proposed above. How the compression rod can be in the curve equilibrium state with the initial deflection of \( y_i \) is the main question. The answer is that, according to the differential relationship between the deflection curve function \( y_i \) and the bending moment, the bending moment \( M_i \) that enables the rod to reach the initial deflection can be determined as follows:

\[
M_i = -EI \frac{d^2y_i}{dx^2} = -EI \frac{q}{P}.
\]  

(26)

The above equation shows that the required initial bending deformation can be realized only when there is the bending moment \( M_i \) at both ends of the rod.

Now, the analytical method for the bending problem of the compression rod subjected to transverse uniform loads can be clearly explained. The basic principles of mechanics are the following:

1. Divide the load on the rod into two groups. The first group is composed of axial force \( P \), transverse uniform load \( q \), rod end shear forces \( Q_0 \) and \( Q_L \), and initial bending moment \( M_i \). The second group is composed of axial force \( P \) and bending moment \( M_0 - M_i, M_L - M_i \) at rod ends as shown in Figure 3;

2. Under loads of the first group, the deformation of the rod is completed as follows. Firstly, a bending moment \( M_i \) is imposed on the rod ends respectively, to obtain a bending equilibrium state with the initial deflection of \( y_i \). Next, add \( P, q, Q_0, \) and \( Q_L \) to the bending rod. As mentioned above, on the rod whose initial deformation is \( y_i \), \( P, q, Q_0, \) and \( Q_L \) only generate axial force instead of bending moment and shear force, and hence no new displacement will be generated. The deformation deflection curve of the rod is still the quadratic parabola \( y_i \);

3. The rod under loads of the second group is only affected by the axial force \( P \) and the bending moments at rod ends, which is in a typical “Euler” bending equilibrium state [6,22]. The deformed deflection curve of the rod is sinusoidal. It is worth noting that the bending moments at rod ends are \( M_0 - M_i \) and \( M_L - M_i \) because \( M_i \) was deducted from the original bending moments at rod ends \( M_0 \) and \( M_L \), respectively. Instead, \( M_i \) is added to the first group;

4. Due to axial force \( P \) is in both of the above two groups, the lateral displacement produced under loads of the two groups can be superposed. The superposition value is the final deformation value of the rod. This is the significance of mechanics of the solution for the bending equilibrium differential equations of the rod—Equation (23).

5. Application of Analytical Methods

5.1. In the Case That Bending Moments Exist at Rod Ends

The rod of the derrick leg is simplified as the simply ends-supported compression rod subjected to the transverse uniform load, as shown in Figure 5a. It can also be called the simply supported beam bearing axial pressure and transverse uniform loads. In order to make the derived equations have a more extensive range of application, the bending moments \( M_0 \) and \( M_L \) are applied to both ends of the rod, respectively, which is a statically determinate structure. First, the rod end force and the bending moment under transverse uniform loads and bending moments at rod ends should be obtained, as illustrated in Figure 5b,c, the superposition of which is the final bending moment diagram of the rod, as shown in Figure 5d. Figure 5a shows the simply ends-supported equal-section compression rod subjected to the transverse uniform load. Figure 5b shows the bending equilibrium deflection curve of the rod after deformation under pressure. Figure 6c,d is the sine curve and the quadratic parabola decomposed by the lateral deflection curve of the rod, respectively; drawing these schematic diagrams correctly is very helpful for analyzing and understanding the problem. It is noted that the forces acting on the rod are the resultant force \( qL \) of the transversely distributed load \( q \), the resultant force of the bending moment \( M_0 \), the shear force \( Q_0 \) and the axial force \( P \) at the rod end \( O \), the resultant force of \( M_L, Q_L \), etc.
and \( P \) at the rod end \( L \). The above forces should converge at one point. The above three resultant forces are shown in Figure 6. Since the action point of the resultant force at the rod, end \( O \) is obtained by simplifying \( M_0 \) and the axial force \( P \) translation \( \frac{e_0}{\epsilon} = M_0/P \) at the rod end \( O \), on the Y-axis of the rectangular coordinate system with the rod end \( O \) as the origin, the point whose distance from the origin is \( e_0 \) the action position of the action line of the resultant force at rod end \( O \). In the same way, translating \( \epsilon_L = M_L/P \) from rod end \( L \) can determine the sine curve decomposed by the line position of the resultant force of \( M_L, Q_L, \) and \( P \). It can be imaged to straighten the initial deflection curve \( y_i \) (in this case, consider \( y_i \) as the baseline of \( y \)), and then translate the sine curve obtained after straightening \( y_i \) with \( M_i \). Although the intercept \( A_4 \) between the initial deflection \( y_i \) and the Y-axis is determined by boundary conditions, the quadratic parabola \( y_i \) can still be drawn based on the action line of the resultant force at two ends of the rod, as shown in Figure 6d.

Figure 5. Schematic diagram of bending moment superposition of simply supported compression rods: (a) compression rod force diagram; (b) bending moment diagram under transverse uniform loads; (c) bending moment diagram under bending moments at rod ends; and (d) bending moment diagram of simply supported compression rods.

Figure 6. Decomposition diagram of the deflection curve of simply supported compression rods: (a) compression rod force diagram; (b) bending equilibrium deflection curve of rods; (c) sine curve decomposed by the deflection curve; and (d) quadratic parabola decomposed by the deflection curve.
Make the algebraic sum of moments zero on rod end L, we obtain
\[ M_L - M_0 + \frac{qL^2}{2} + Q_0 L = 0, \]  
which can be solved as follows:
\[ Q_0 = \frac{M_0 - M_L}{L} - \frac{qL}{2}. \]

Putting the above equation into Equation (23), the deflection curve equation is given by
\[ y = A sin \left( \frac{x}{\mu L} - \frac{\zeta}{\mu} \right) \pi + \frac{q}{2P} x^2 + \frac{M_0 - M_L}{PL} x - \frac{qL}{2P} x + A_4. \]

The bending moment equation of the rod takes the form
\[ M = EIA \left( \frac{\pi}{\mu L} \right)^2 \sin \left( \frac{x}{\mu L} - \frac{\zeta}{\mu} \right) \pi - EI \frac{q}{P}. \]

With the known condition that the bending moment at the rod end O is \( M_0 \), substituting \( x = 0 \) into the above equation, we can obtain
\[ M_0 = EIA \left( \frac{\pi}{\mu L} \right)^2 \sin \left( - \frac{\zeta}{\mu} \right) \pi - EI \frac{q}{P}. \]

After sorting out, the following equation can be obtained
\[ A \left( \frac{\pi}{\mu L} \right)^2 \sin \left( - \frac{\zeta}{\mu} \right) \pi = \frac{q}{P} + \frac{M_0}{EI}. \]

In the same way, with the condition that the bending moment at rod end L is \( M_L \), substituting \( x = L \) into Equation (25), the following equation can be obtained after sorting out:
\[ A \left( \frac{\pi}{\mu L} \right)^2 \sin \left( \frac{1}{\mu} - \frac{\zeta}{\mu} \right) \pi = \frac{q}{P} + \frac{M_L}{EI}. \]

Dividing the above two equations, one may obtain
\[ \frac{\sin \left( - \frac{\zeta}{\mu} \right) \pi}{\sin \left( \frac{1}{\mu} - \frac{\zeta}{\mu} \right) \pi} = \frac{q}{P} + \frac{M_0}{EI} \div \frac{q}{P} + \frac{M_L}{EI}. \]

where \( \mu \) can be determined by the equation
\[ \mu = \sqrt{\frac{\pi EI}{L^2}} / P, \]
removing the position coefficient \( \zeta \) of the inflection point from the above equation, substituting \( \zeta \) into Equation (32) or (33) to obtain \( A_4 \), and then using the boundary condition of \( y = 0 \) and \( x = 0 \) to determine \( A_4 \). After all the coefficients are determined, the lateral displacement of any cross-section of the compression rod can be obtained by Equation (23).

5.2. In the Case That Bending Moments at Rod Ends Are Equal

Figure 7 shows the condition of \( M_0 = M_L \). By Equation (28), we have
\[ Q_0 = -\frac{qL}{2}. \]
Figure 7. Schematic diagram of force superposition of a simply supported compression rod with equal bending moments at both ends under transverse uniform loads: (a) forces diagram of rod components; (b) deflection curve of the rod after deformation; (c) sine curve decomposed by the deflection curve; and (d) quadratic parabola decomposed by the deflection curve.

By substituting the above equation and \( x = 0 \) into Equation (25), we can obtain

\[
M_0 = EIA \left( \frac{\pi}{\mu L} \right)^2 \sin \left( -\frac{\zeta}{\mu} \right) \pi - EI \frac{q}{P}.
\]  

(36)

Similarly, we have

\[
M_L = EIA \left( \frac{\pi}{\mu L} \right)^2 \sin \left( \frac{1}{\mu} - \frac{\zeta}{\mu} \right) \pi - EI \frac{q}{P}.
\]  

(37)

Combining the above two equations, the result can be obtained as

\[
\sin \left( -\frac{\zeta}{\mu} \right) \pi - \sin \left( \frac{1}{\mu} - \frac{\zeta}{\mu} \right) \pi = 0.
\]  

(38)

The above equation can be represented in the following form:

\[
2 \cos \left( \frac{\zeta}{\mu} - \frac{1}{2\mu} \right) \pi \sin \frac{\pi}{2\mu} = 0
\]  

(39)

From the above equation, it can be solved as follows:

\[
\zeta = \frac{\mu + 1}{2}.
\]  

(40)

The above result can also be derived from the sine curve decomposed, as shown in Figure 7c. According to the symmetric relation, there is

\[
\mu L - \zeta L = \frac{\mu L - L}{2}.
\]  

(41)
The same result as in Equation (40) can also be obtained from the above equation. By substituting Equation (40) into Equation (36), the integration constant can be solved

\[
A = \frac{-M_0 - EI \frac{q}{P}}{P \cos \frac{\pi}{2P}}. \tag{42}
\]

From the boundary condition of \( y = 0 \) at \( x = 0 \), there is

\[
y_0 = A \sin \left( -\frac{\xi}{\mu} \right) \pi + A_4 = 0. \tag{43}
\]

It can be obtained that

\[
A_4 = \frac{-M_0 - EI \frac{q}{P}}{P}. \tag{44}
\]

Therefore, the deflection curve equation can be written as

\[
y = \frac{M_0 + EI \frac{q}{P}}{P} \left[ \cos \left( \frac{x \mu}{P} - \frac{1}{2P} \right) \pi \cos \frac{\pi}{2P} - 1 \right] + \frac{qL}{2P} x^2 - \frac{qL^2 x}{2P}. \tag{45}
\]

Next, the moment equilibrium conditions for any section are tested. By Equation (45), according to the differential relationship between deflection, rotation angle, and curvature [23], the rotation angle equation and the bending moment equation can be obtained as

\[
tan \theta = -\frac{M_0 + EI \frac{q}{P} \pi}{\mu} + \frac{q}{P} x - \frac{qL}{2P}, \tag{46}
\]

\[
M = \left( M_0 + EI \frac{q}{P} \right) \frac{\cos \left( \frac{x \mu}{P} - \frac{1}{2P} \right) \pi}{\cos \frac{\pi}{2P}} - EI \frac{q}{P}, \tag{47}
\]

respectively.

The bending moments at rod ends can be obtained as \( M_0 \) and \( M_L \) by substituting \( x = 0 \) and \( x = L \) into Equation (47), respectively. According to the equilibrium condition of the external moment, the bending moment of any section with a distance \( x \) from the origin can be obtained as

\[
M = M_0 + P \left\{ \frac{M_0 + EI \frac{q}{P}}{P} \left[ \cos \left( \frac{x \mu}{P} - \frac{1}{2P} \right) \pi \right] - 1 \right\} + \frac{qL}{2P} x^2 - \frac{qL^2 x}{2P} \tag{48}
\]

The checking calculation shows that the bending moment of the section is balanced with the external moment, and the deflection curve equation determined by the analytical method meets the requirements of deformation, boundary conditions, and the equilibrium conditions of the moment. Therefore, the calculation result is the correct solution for the differential equation of the deflection curve.

Substituting \( x = L/2 \) into Equation (45), the maximum lateral displacement in midspan can be obtained as

\[
y_{max} = \frac{y}{L} = \frac{M_0 + EI \frac{q}{P}}{P} \left[ \frac{1}{\cos \frac{\pi}{2P}} - 1 \right] - \frac{qL^2}{8P}. \tag{49}
\]
6. Numerical Analysis of Compression Rod

6.1. Simplified Model

When analyzing the local stability of the derrick, the main support members of the thigh of the derrick are analyzed. It is assumed that the internal force of the rods in the structure is dominated by the axial force. Therefore, the node can be simplified as a hinge point. In order to be more in line with engineering reality, the bending moments are respectively applied to both ends of the rod.

6.2. Parameters

After calculation and analysis, the maximum compressive stress appears at the derrick leg. The self-weight loading mode of the derrick is shown in Figure 8.

Figure 8. Self-weight load acting mode of the derrick.

The derrick thigh adopts a large section of H-shaped steel, which is Q345 steel. The cross-sectional size of the H-section is 400 mm × 300 mm × 12 mm × 20 mm. The elastic modulus of Q345 steel is $2.06 \times 10^5$ MPa, the Poisson’s ratio is 0.3, and the density is 7850 kg/m$^3$. The effective height of the tower derrick is 45 m, the upper part of the derrick is $2.4 \times 2.4$ m, and the lower part of the derrick is $9.144 \times 9.144$ m. The main supporting member of the derrick thigh is 3 m long.

6.3. Ansys Modeling

The Ansys modeling software (American APDL mechanical product launcher 18.2, Ansys 18.2) is employed to simulate the force of derrick members. Ansys software has a
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6.3. Ansys Modeling

The Ansys modeling software (American APDL mechanical product launcher 18.2, Ansys 18.2) is employed to simulate the force of derrick members. Ansys software has a rich element library, a powerful solver, and convenient post-processing functions. It can be used to perform nonlinear analysis of engineering structures to obtain more accurate results. The numerical simulation flow chart of Ansys software for the main force-bearing members of the derrick thigh is shown in Figure 9.

![Ansys numerical simulation flowchart](image)

Figure 9. Ansys numerical simulation flowchart.

The Cartesian coordinate system was used for modeling, and the length of the rod was the Z-axis. The cross section of the rod was H-shaped steel, and the height and width of the H-shaped steel cross section were X-axis and Y-axis, respectively. The model uses a three-dimensional elastic beam element beam188 element. Two points were selected to determine a line, and then the mesh was divided into 100 units. Constraints were imposed on both ends of the model, with UX, UY, UZ, and ROTZ at the lower end, and UX, UY, and ROTZ at the upper end, forming the boundary conditions of the simply supported constraints. The meshing and constraint drawing of H-section bars is shown in Figure 10.
A bending moment and a uniform load were applied in the X direction, and a vertical force was applied in the Z direction.

Table 1 shows the weight of the derrick components and the load values under normal working conditions. According to the force estimation, the maximum longitudinal force of the derrick thigh is 1961.03 KN, the transverse uniform load is wind load, and the value is 4.4 KN/m. In order to achieve conditions closer to the actual engineering situation, a simulation analysis was carried out on the same bending moment at the rod end.

Table 1. Basic parameters of the derrick.

| Project                        | Load Value | Unit   |
|--------------------------------|------------|--------|
| Structural weight              | 1200       | KN     |
| Self-weight of the second floor| 36.13      | KN     |
| Swimming system weight         | 300        | KN     |
| Crane weight                   | 107.212    | KN     |
| Maximum hook load              | 4500       | KN     |
| Maximum drill string weight    | 2200       | KN     |
| Maximum pulling force of Working rope | 793.288 | KN     |
| Standing root load             | 62.125     | KN     |
| Wind load                      | 4.4        | KN/m   |

7. Analysis of Calculation Results

7.1. Displacement

After calculation and analysis by Ansys software, the X-direction displacement diagram of the derrick under the load of the above working conditions was obtained, as shown in Figure 11. It can be observed from the figure that the maximum displacement occurring in the middle of the rod is consistent with the deflection curve derived from the above
formula; therefore, the derived formula is practical and reliable. The total deformation of the rod is shown in Figure 12. The maximum displacement value of the derrick thigh member is 1.82448 mm. Therefore, the use of large cross-section H-shaped steel as the main structure of the derrick thigh can meet the design and use requirements.

Figure 11. X-direction displacement diagram of the rod of the derrick leg.

Figure 12. The total displacement of the rod of the derrick leg.
7.2. Stress

Under the action of the transverse uniform load and the longitudinal load, the finite element simulation calculation of the main force-bearing rod of the derrick leg was carried out to obtain the stress cloud diagram. The von Mises stress cloud diagram is shown in Figure 13. The maximum stress is 226.637 MPa and the minimum stress is 0.121319 MPa. H-section steel is mainly stressed by the upper and lower flanges. The deformation of the middle part of the rod caused by the different bending moments at both ends is small, resulting in small stress in the rod.

![Von Mises stress diagram of derrick legs.](image)

Figure 13. Von Mises stress diagram of derrick legs.

7.3. Comparative Results

The data calculated by Ansys numerical simulation is shown in Table 2. The maximum displacement value calculated by the geometric nonlinear analysis method proposed in this paper is 1.748 mm and that obtained by Ansys is 1.824448 mm. The relative error of the results of the two methods is 4.19%, which verifies that the formula proposed in this article is practical and reliable.

Table 2. Ansys numerical simulation results.

| Simulation Chart Name         | Max Position          | Min Position          |
|-------------------------------|-----------------------|-----------------------|
| Total displacement map        | Span                  | 0.014614 mm           | Lower flange          |
| X-direction displacement map  | Span                  | 0 mm                  | Both ends             |
| Von Mises stress diagram      | Flanges at both ends  | 0.121319 MPa          | Mid-span and upper flange |

Based on the derrick member model above, after changing the value of the axial force and using Ansys to perform nonlinear analysis on the main force members of the derrick thigh, the results of the relationship between load and lateral displacements are shown in Figure 14. With the increase of the axial force, the deformation of the derrick member...
gradually increases, mainly because the geometric nonlinearity of the derrick member becomes more and more obvious with the increase of the axial force. It can be derived that the geometric nonlinear analytical method has practical significance for the safety calculation of actual engineering, which further proves the practicability of the formula in this paper.

Figure 14. Load–maximum lateral displacement relationship diagram.

8. Conclusions

The partial instability of the components in the derrick will cause the derrick to collapse, oil leakage, and marine environment pollution. Therefore, the local stability calculation of the derrick is one of the important contents of engineering safety research. Through calculation and analysis, the thigh of offshore derrick is the maximum axial force [24].

This article simplified the model of the member at the thigh of the derrick, the parameter solutions for the differential Equation (1) describing the nonlinear geometric problem of simply supported compression rods subjected to transverse uniform loads [25] is obtained. The deflection curve equation can be decomposed into a superposition of a parabola and a sine curve, and the deflection theory calculation formula Equation (9), considering the second-order effect of axial force is established. The method proposed in this paper is used to analyze the deflection and deformation of the members at the thigh of the derrick, and the results obtained are compared with the results of the Ansys nonlinear analysis. The comparison results verify the feasibility and practicability of the proposed method. The analytical method proposed in this paper provides a theoretical basis for the calculation of the local stability of the offshore derrick.

Author Contributions: Conceptualization: C.L., H.C., M.H., P.Q. and X.L.; writing—original draft preparation: C.L.; writing—review and editing: C.L., H.C., M.H., P.Q. and X.L.; supervision: H.C.; funding acquisition: P.Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China, grant number 41907239.

Acknowledgments: We kindly thank the National Natural Science Foundation of China.
Conflicts of Interest: The authors declare no conflict of interest.

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