Lipschitz type inequalities for noncommutative perspectives of operator monotone functions in Hilbert spaces

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Abstract
Assume that \( f : [0, \infty) \to \mathbb{R} \) is a continuous function. We can define the perspective \( P_f(B, A) \) by setting
\[
P_f(B, A) := A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},
\]
where \( A, B > 0 \). We show in this paper among others that
\[
\|P_f(B, P) - P_f(A, P)\| \\leq \|P\|^2 \|B - A\| \left\{ \begin{array}{ll}
\frac{P_f(m_2, p) - P_f(m_1, p)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
f'(\frac{m}{p}) & \text{if } m_1 = m_2 = m
\end{array} \right.
\]
for all \( A \geq m_1 > 0, B \geq m_2 > 0 \) and \( P \geq p > 0 \). If \( f \) is operator monotone on \( [0, \infty) \), then for all \( C \geq n_1 > 0, D \geq n_2 > 0, Q > q > 0 \) we also have

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\[ \| \mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) \| \leq \frac{\| Q \|^2 \| D - C \|}{q^2} \left\{ \begin{array}{ll}
\frac{\mathcal{P}_f(q, n_2) - \mathcal{P}_f(q, n_1)}{n_2 - n_1} & \text{if } n_2 \neq n_1, \\
\left[ f \left( \frac{q}{n_2} \right) - f' \left( \frac{q}{n_2} \right) \right] & \text{if } n_2 = n_1 = n.
\end{array} \right. \]

Some applications for weighted operator geometric mean and relative operator entropy are also given.

**Keywords** Operator monotone functions · Noncommutative perspectives

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## 1 Introduction

Let \( B(H) \) be the Banach algebra of bounded linear operators on a complex Hilbert space \( H \). The absolute value of an operator \( A \) is the positive operator \( |A| \) defined as \( |A| := (A^*A)^{1/2} \).

It is known that [3] in the infinite-dimensional case the map \( f(A) := |A| \) is not Lipschitz continuous on \( B(H) \) with the usual operator norm, i.e. there is no constant \( L > 0 \) such that

\[ \| |A| - |B| \| \leq L \| A - B \| \]

for any \( A, B \in B(H) \).

However, as shown by Farforovskaya in [10, 11] and Kato in [18], the following inequality holds

\[ \| |A| - |B| \| \leq \frac{2}{\pi} \| A - B \| \left( 2 + \log \left( \frac{\| A \| + \| B \|}{\| A - B \|} \right) \right) \]

for any \( A, B \in B(H) \) with \( A \neq B \).

If the operator norm is replaced with Hilbert–Schmidt norm \( \| C \|_{HS} := (\text{tr}C^*C)^{1/2} \) of an operator \( C \), then the following inequality is true [1]

\[ \| |A| - |B| \|_{HS} \leq \sqrt{2} \| A - B \|_{HS} \]

for any \( A, B \in B(H) \).

The coefficient \( \sqrt{2} \) is best possible for a general \( A \) and \( B \). If \( A \) and \( B \) are restricted to be self-adjoint, then the best coefficient is 1.

It has been shown in [3] that, if \( A \) is an invertible operator, then for all operators \( B \) in a neighborhood of \( A \) we have
\[ \|A| - |B|\| \leq a_1\|A - B\| + a_2\|A - B\|^2 + O\left(\|A - B\|^3\right), \]  

(2)

where

\[ a_1 = \|A^{-1}\|\|A\| \text{ and } a_2 = \|A^{-1}\| + \|A^{-1}\|^3\|A\|^2. \]

In [2] the author also obtained the following *Lipschitz type inequality*

\[ \|f(A) - f(B)\| \leq f'(a)\|A - B\| \]  

(3)

where \( f \) is an *operator monotone function* on \((0, \infty)\) and \( A, B \geq aI_H > 0 \).

One of the central problems in perturbation theory is to find bounds for

\[ \|f(A) - f(B)\| \]

in terms of \( \|A - B\| \) for different classes of measurable functions \( f \) for which the function of operator can be defined. For some results on this topic, see [5, 12] and the references therein.

Consider a complex Hilbert space \((H, \langle \cdot, \cdot \rangle)\). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be *strictly positive* (denoted by \( T > 0 \)) if \( T \) is positive and invertible. A real valued continuous function \( f \) on \((0, \infty)\) is said to be operator monotone if \( f(A) \geq f(B) \) holds for any \( A \geq B > 0 \).

We have the following integral representation for the power function when \( t > 0, r \in (0, 1] \), see for instance [4, p. 145]

\[ t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \]  

(4)

Observe that for \( t > 0, t \neq 1 \), we have

\[ \int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1-t} \ln\left(\frac{u + t}{u + 1}\right) \]

for all \( u > 0 \).

By taking the limit over \( u \to \infty \) in this equality, we derive

\[ \frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)}, \]

which gives the representation for the logarithm

\[ \ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \]  

(5)

for all \( t > 0 \).

In 1934, Löwner [20] had given a definitive characterization of operator monotone functions as follows, see for instance [4, p. 144-145]:

\[ \frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)}, \]

which gives the representation for the logarithm

\[ \ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \]  

(5)

for all \( t > 0 \).
Theorem 1.1 A function $f : [0, \infty) \rightarrow \mathbb{R}$ is operator monotone if and only if it has the representation

$$f(t) = f(0) + bt + \int_0^\infty \frac{t \lambda}{1 + \lambda} \, dw(\lambda)$$

(6)

where $b \geq 0$ and a positive measure $w$ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} \, dw(\lambda) < \infty.$$  

We recall the important fact proved by Löwner and Heinz that states that the power function $f : [0, 1) \rightarrow \mathbb{R}$, $f(t) = t^a$ is an operator monotone function for any $a \in [0, 1]$. The function $\ln$ is also operator monotone on $(0, 1)$.

For other examples of operator monotone functions, see [15] and [16]. For Kwong matrices and operator monotone functions on $(0, 1)$, see [22].

Let $f$ be a continuous function defined on the interval $I$ of real numbers, $B$ a self-adjoint operator on the Hilbert space $H$ and $A$ a positive invertible operator on $H$. Assume that the spectrum $Sp(A^{-1/2}BA^{-1/2}) \subset \hat{I}$. Then by using the continuous functional calculus, we can define the perspective $\mathcal{P}_f(B, A)$ by setting

$$\mathcal{P}_f(B, A) := A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$  

If $A$ and $B$ are commutative, then

$$\mathcal{P}_f(B, A) = Af(BA^{-1})$$

provided $Sp(BA^{-1}) \subset \hat{I}$.

For any function $f : (0, \infty) \rightarrow \mathbb{R}$ the transpose $\tilde{f}$ of $f$ is defined by

$$\tilde{f}(x) = xf(x^{-1}), \quad x > 0.$$  

It is well known that (see for instance [24]), if $f : (0, \infty) \rightarrow \mathbb{R}$ is continuous on $(0, \infty)$, then for all $A, B > 0$,

$$\mathcal{P}_f(A, B) = \mathcal{P}_f(B, A).$$

If $f$ is non-negative and operator monotone on $(0, \infty)$, then $\tilde{f}$ is operator monotone on $(0, \infty)$, see [24].

The following inequality is of interest, see [24] and [19]:

Theorem 1.2 Assume that $f$ is non-negative and operator monotone on $(0, \infty)$. If $A \geq C > 0$ and $B \geq D > 0$, then

$$\mathcal{P}_f(A, B) \geq \mathcal{P}_f(C, D).$$
It is well known that (see [8] and [21] or [9]), if \( f \) is an operator convex function defined in the positive half-line, then the mapping

\[
(B, A) \mapsto \mathcal{P}_f(B, A)
\]

defined in pairs of positive invertible operators, is operator convex.

Motivated by the above results, we show in this paper among others that

\[
\| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) \|
\leq \frac{\|P\| \|B - A\|}{p^2} \left\{ \begin{array}{ll}
\frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
f'(\frac{m}{p}) & \text{if } m_1 = m_2 = m
\end{array} \right.
\]

for all \( A \geq m_1 > 0, B \geq m_2 > 0 \) and \( P \geq p > 0 \).

If \( f \) is operator monotone on \( [0, \infty) \), then for all \( C \geq n_1 > 0, D \geq n_2 > 0, Q > q > 0 \) we also have

\[
\| \mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) \|
\leq \frac{\|Q\| \|D - C\|}{q^2} \left\{ \begin{array}{ll}
\frac{\mathcal{P}_f(q, n_2) - \mathcal{P}_f(q, n_1)}{n_2 - n_1} & \text{if } n_2 \neq n_1, \\
\left[ f\left(\frac{q}{n}\right) - q f'\left(\frac{q}{n}\right)\right] & \text{if } n_2 = n_1 = n.
\end{array} \right.
\]

Some applications for weighted operator geometric mean and relative operator entropy are also given.

2 Some preliminary facts

We start to the following identity of interest [7]:

**Lemma 2.1** Assume that the function \( f : [0, \infty) \to \mathbb{R} \) is operator monotone and has the representation (6). Then for all \( U, V > 0 \),

\[
f(V) - f(U) = b(V - U)
\]

\[
+ \int_0^\infty \lambda^2 \left[ \int_0^1 ((1 - t)U + tV + \lambda)^{-1} \right.
\]

\[
\times (V - U)((1 - t)U + tV + \lambda)^{-1} dt \left. \right] d\lambda.
\]

**Proof** Since the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) and has the representation (6), then for \( U, V > 0 \) we have the representation
\begin{equation}
  f(V) - f(U) = b(V - U) + \int_0^\infty \lambda \left[ V(V + \lambda)^{-1} - U(U + \lambda)^{-1} \right] dw(\lambda). \quad (8)
\end{equation}

Observe that for $\lambda > 0$

\[ V(V + \lambda)^{-1} - U(U + \lambda)^{-1} = (V + \lambda - \lambda)(V + \lambda)^{-1} - (U + \lambda - \lambda)(U + \lambda)^{-1} = (V + \lambda)(V + \lambda)^{-1} - \lambda(V + \lambda)^{-1} - (U + \lambda)(U + \lambda)^{-1} + \lambda(U + \lambda)^{-1} = \lambda \left[ (U + \lambda)^{-1} - (V + \lambda)^{-1} \right]. \]

Therefore, (8) becomes, see also [16]

\begin{equation}
  f(V) - f(U) = b(V - U) + \int_0^\infty \lambda^2 \left[ (U + \lambda)^{-1} - (V + \lambda)^{-1} \right] dw(\lambda). \quad (9)
\end{equation}

Let $T, S > 0$. The function $f(t) = -t^{-1}$ is operator monotonic on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

\[ \nabla f_T(S) := \lim_{t \to 0} \frac{f(T + tS) - f(T)}{t} = T^{-1}ST^{-1} \quad (10) \]

for $T, S > 0$.

Consider the continuous function $f$ defined on an interval $I$ for which the corresponding operator function is Gâteaux differentiable and for $C, D$ self-adjoint operators with spectra in $I$ we consider the auxiliary function defined on $[0, 1]$ by

\[ f_{C,D}(t) = f((1 - t)C + tD), \quad t \in [0, 1]. \]

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C, D] := \{(1 - t)C + tD, \quad t \in [0, 1]\}$, then we have, by the properties of the Bochner integral, that

\begin{equation}
  f(D) - f(C) = f_{C,D}(1) - f_{C,D}(0) = \int_0^1 \frac{d}{dt} f_{C,D}(t) dt = \int_0^1 \nabla f_{(1-t)C+tD}(D - C) dt. \quad (11)
\end{equation}

This equality can also be seen as Taylor’s formula with integral reminder, see for instance [6, p. 112].

If we write this equality for the function $f(t) = -t^{-1}$ and $C, D > 0$, then we get the representation

\begin{equation}
  C^{-1} - D^{-1} = \int_0^1 ((1 - t)C + tD)^{-1}(D - C)((1 - t)C + tD)^{-1} dt. \quad (12)
\end{equation}

Now, if we replace in (12) $C = U + \lambda$ and $D = V + \lambda$ for $\lambda > 0$, then
\[(U + \lambda)^{-1} - (V + \lambda)^{-1} = \int_0^1 \left( (1-t)U + tV + \lambda \right)^{-1} \left( (1-t)U + tV + \lambda \right)^{-1} dt. \tag{13} \]

By the representation (9), we derive (7). \(\square\)

We have the following identity for the difference of perspectives in the first variable [7]:

**Theorem 2.2** Assume that the function \(f : [0, \infty) \to \mathbb{R}\) is operator monotone and has the representation (6). Then for all \(A, B, P > 0,\)

\[
P_f(B, P) - P_f(A, P) = b(B - A) + \int_0^\infty \lambda^2 \left[ \int_0^1 P((1-t)A + tB + \lambda P)^{-1}(B - A) \right. \\
\times \left. ((1-t)A + tB + \lambda P)^{-1} P dt \right] d\lambda. \tag{14} \]

**Proof** If we take \(V = P^{-1/2}BP^{-1/2}\) and \(U = P^{-1/2}AP^{-1/2}\) in (7), then we get

\[
f\left( P^{-1/2}BP^{-1/2} \right) - f\left( P^{-1/2}AP^{-1/2} \right) = b\left( P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2} \right) \\
\quad + \int_0^\infty \lambda^2 \left[ \int_0^1 \left( (1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda \right)^{-1} \right. \\
\quad \times \left. \left( P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2} \right) \right] \\
\quad \times \left( (1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda \right)^{-1} dt \right] d\lambda. \tag{15} \]

Observe that

\[
P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2} = P^{-1/2}(B - A)P^{-1/2}, \quad \text{and} \]

\[
(1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda = P^{-1/2}((1-t)A + tB + \lambda P)P^{-1/2}, \]

which gives

\[
\left( (1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda \right)^{-1} = P^{1/2}((1-t)A + tB + \lambda P)^{-1}P^{1/2} \]

and by (15),
\[ f\left( P^{-1/2}BP^{-1/2} \right) - f\left( P^{-1/2}AP^{-1/2} \right) = bP^{-1/2}(B - A)P^{-1/2} \]
\[ + \int_0^\infty \lambda^2 \left[ \int_0^1 P^{1/2}((1 - t)A + tB + \lambda P)^{-1} P^{1/2}P^{-1/2}(B - A)P^{-1/2} \right] \right] \, dw(\lambda) \]
\[ = bP^{-1/2}(B - A)P^{-1/2} \]
\[ + \int_0^\infty \lambda^2 \left[ \int_0^1 P^{1/2}((1 - t)A + tB + \lambda P)^{-1}(B - A) \right. \]
\[ - \left. ((1 - t)A + tB + \lambda P)^{-1} P^{1/2} \right] \, dw(\lambda). \]

If we multiply both sides of (16) by \( P^{1/2} \) we obtain the desired identity (14). □

**Lemma 2.3** Assume that the function \( f : [0, \infty) \to \mathbb{R} \) is operator monotone and has the representation (6). Then for all \( U, V > 0 \),
\[
\tilde{f}(V) - \tilde{f}(U) = f(0)(V - U) + \int_0^\infty \lambda \left( \int_0^1 (1 + \lambda[(1 - t)U + tV])^{-1} \right. \]
\[ \times (V - U)(1 + \lambda[(1 - t)U + tV])^{-1} \right) \, dw(\lambda). \]

**Proof** From (6) we have
\[
f(t) = f(0) + bt + t \int_0^\infty \frac{\lambda}{t + \lambda} \, dw(\lambda), \quad t > 0.
\]
If we put \( \frac{1}{t} \) instead of \( t \) we get
\[
f\left( \frac{1}{t} \right) = f(0) + \frac{1}{t} \int_0^\infty \frac{\lambda}{\frac{1}{t} + \lambda} \, dw(\lambda)
\]
\[ = f(0) + \frac{1}{t} \int_0^\infty \frac{t\lambda}{1 + t\lambda} \, dw(\lambda)
\]
and by multiplication with \( t > 0 \), we get
\[
\tilde{f}(t) = b + tf(0) + \int_0^\infty \frac{t\lambda}{1 + t\lambda} \, dw(\lambda)
\]
\[ = b + tf(0) + \int_0^\infty \left( 1 - \frac{1}{1 + t\lambda} \right) \, dw(\lambda).
\]
Therefore
\[ f(V) - f(U) = f(0)(V - U) + \int_0^\infty \left[ (1 + U\lambda)^{-1} - (1 + V\lambda)^{-1} \right] dw(\lambda). \quad (18) \]

From (12) we get
\[
(1 + U\lambda)^{-1} - (1 + V\lambda)^{-1} = \int_0^1 (1 - (1 + U\lambda) + t(1 + V\lambda) - (1 + U\lambda)) \times ((1 - t)(1 + U\lambda) + t(1 + V\lambda))^{-1} dt
\]
\[
= \int_0^1 x(1 + \lambda[(1 - t)U + tV])^{-1}(V - U)(1 + \lambda[(1 - t)U + tV])^{-1} dt.
\]
Therefore, by (18) we get
\[
\begin{align*}
\tilde{f}(V) - \tilde{f}(U) &= f(0)(V - U) + \int_0^\infty \left[ (1 + U\lambda)^{-1} - (1 + V\lambda)^{-1} \right] dw(\lambda) \\
&= f(0)(V - U) + \int_0^\infty \lambda \left( \int_0^1 (1 + \lambda[(1 - t)U + tV])^{-1} dt \right) dw(\lambda) \\
&\quad \times (V - U)(1 + \lambda[(1 - t)U + tV])^{-1} dt)
\end{align*}
\]
and the identity (17) is proved. \(\square\)

The dual identity is as follows [7]:

**Theorem 2.4** Assume that the function \( f : [0, \infty) \to \mathbb{R} \) is operator monotone and has the representation (6). Then for all \( C, \ D, \ Q > 0 \),
\[
\mathcal{P}_f(D, Q) - \mathcal{P}_f(C, Q) = f(0)(D - C) + \int_0^\infty \lambda \left( \int_0^1 Q[Q + \lambda[(1 - t)C + tD]]^{-1}(D - C) \right. \\
\left. \times [Q + \lambda[(1 - t)C + tD]]^{-1} Q dt \right) dw(\lambda).
\]

**Proof** If we take \( V = Q^{-1/2}DQ^{-1/2} \) and \( U = Q^{-1/2}CQ^{-1/2} \) in (17), then we get
If we multiply both sides by \( Q^{1/2} \) we get the desired result (21). □

**Corollary 2.5** Assume that the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is operator monotone and has the representation (6). Then for all \( C, D, Q > 0 \),

\[
\mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) = f(0)(D - C) + \int_0^\infty \lambda \left( \int_0^1 \frac{Q + \lambda[(1 - t)C + tD]}{(Q + \lambda[(1 - t)C + tD])^{-1}(D - C)} dt \right) dw(\lambda).
\]

(23)

We also have:

**Corollary 2.6** Assume that the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is operator monotone and has the representation (6). Then for all \( A, B, C, D > 0 \),
\[ \mathcal{P}_f(A, B) - \mathcal{P}_f(C, D) = b(A - C) + f(0)(B - D) \]
\[ + \int_0^\infty \lambda^2 \left[ \int_0^1 B((1 - t)C + tA + \lambda B)^{-1}(A - C) \right. \]
\[ \times ((1 - t)C + tA + \lambda B)^{-1}Bdt \left. dw(\lambda) \right] \]
\[ + \int_0^\infty \lambda \left( \int_0^1 C[C + \lambda[(1 - t)D + tB]]^{-1}(B - D) \right. \]
\[ \times [C + \lambda[(1 - t)D + tB]]^{-1}Cdt \left. dw(\lambda) \right]. \]

**Proof** Observe that
\[ \mathcal{P}_f(A, B) - \mathcal{P}_f(C, D) = \mathcal{P}_f(A, B) - \mathcal{P}_f(C, B) + \mathcal{P}_f(C, B) - \mathcal{P}_f(C, D). \] (25)

Since, by (14),
\[ \mathcal{P}_f(A, B) - \mathcal{P}_f(C, B) \]
\[ = b(A - C) + \int_0^\infty \lambda^2 \left[ \int_0^1 B((1 - t)C + tA + \lambda B)^{-1}(A - C) \right. \]
\[ \times ((1 - t)C + tA + \lambda B)^{-1}Bdt \left. dw(\lambda) \right] \]
\[ + \int_0^\infty \lambda \left( \int_0^1 C[C + \lambda[(1 - t)D + tB]]^{-1}(B - D) \right. \]
\[ \times [C + \lambda[(1 - t)D + tB]]^{-1}Cdt \left. dw(\lambda) \right], \]
and by (23),
\[ \mathcal{P}_f(C, B) - \mathcal{P}_f(C, D) \]
\[ = f(0)(B - D) + \int_0^\infty \lambda \left( \int_0^1 C[C + \lambda[(1 - t)D + tB]]^{-1}(B - D) \right. \]
\[ \times [C + \lambda[(1 - t)D + tB]]^{-1}Cdt \left. dw(\lambda) \right), \]

hence by (25)–(27) we obtain (24). \( \square \)

## 3 Lipschitz type inequalities

We have the following Lipschitz type inequality for the perspective in the first variable:

**Lemma 3.1** Assume that the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is operator monotone and has the representation (6). Then for all \( A \geq m_1 > 0, B \geq m_2 > 0 \) and \( P \geq p > 0 \),

\[
\| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) - b(B - A) \|
\leq \frac{\|P\|^2 \|B - A\|}{p^2} \left\{ \begin{array}{ll}
\left( \frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1} - b \right) & \text{if } m_1 \neq m_2, \\
\left( f' \left( \frac{m_1}{p} \right) - b \right) & \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

**Proof** Assume that \( m_1 \neq m_2 \). From (14), by taking the norm, we get that

\[
\| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) - b(B - A) \|
\leq \int_0^\infty \lambda^2 \left[ \int_0^1 \|P((1 - t)A + tB + \lambda P)^{-1}(B - A) \times ((1 - t)A + tB + \lambda P)^{-1}P\| \, dt \right] \, dw(\lambda)
\leq \|P\|^2 \|B - A\| \int_0^\infty \lambda^2 \left( \int_0^1 \|((1 - t)A + tB + \lambda P)^{-1}\|^2 \, dt \right) \, dw(\lambda)
\]

for \( A, B, P > 0 \).

We have

\[(1 - t)A + tB + \lambda P \geq (1 - t)m_1 + tm_2 + \lambda p,\]

which implies that

\[((1 - t)A + tB + \lambda P)^{-1} \leq ((1 - t)m_1 + tm_2 + \lambda p)^{-1}\]

for all \( t \in [0, 1] \) and \( \lambda \geq 0 \).

By taking the norm, we then get

\[\|((1 - t)A + tB + \lambda P)^{-1}\| \leq ((1 - t)m_1 + tm_2 + \lambda p)^{-1},\]

which implies that

\[\|((1 - t)A + tB + \lambda P)^{-1}\|^2 \leq ((1 - t)m_1 + tm_2 + \lambda p)^{-2},\]

for all \( t \in [0, 1] \) and \( \lambda \geq 0 \).

By (29) we derive

\[
\| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) - b(B - A) \|
\leq \|P\|^2 \|B - A\| \int_0^\infty \lambda^2 \left( \int_0^1 ((1 - t)m_1 + tm_2 + \lambda p)^{-2} \, dt \right) \, dw(\lambda).
\]

From the identity (14) for \( B = m_2, A = m_1 \) and \( P = p \) we get
\[ \mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p) = b(m_2 - m_1) + \int_0^\infty \lambda^2 \left( \int_0^1 p((1-t)m_1 + tm_2 + \lambda p)^{-1} (m_2 - m_1) \right) \right) \, dw(\lambda) \]
\[ = b(m_2 - m_1) + (m_2 - m_1)p^2 \int_0^\infty \lambda^2 \left( \int_0^1 ((1-t)m_1 + tm_2 + \lambda p)^{-2} dt \right) \, dw(\lambda), \]

which gives
\[ \int_0^\infty \lambda^2 \left( \int_0^1 ((1-t)m_1 + tm_2 + \lambda p)^{-2} dt \right) \, dw(\lambda) = \frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{(m_2 - m_1)p^2} - \frac{b}{p^2} \]

and the inequality in the first branch of (28) is proved.

Let \( m_1 = m_2 = m \). Let \( \epsilon > 0 \). Then \( B + \epsilon \geq m + \epsilon > 0 \). From the first branch of (28) we get
\[ \| \mathcal{P}_f(B + \epsilon, P) - \mathcal{P}_f(A, P) - b(B + \epsilon - A) \| \]
\[ \leq \|P\|^2 \|B + \epsilon - A\| \left[ \frac{\mathcal{P}_f(m + \epsilon, p) - \mathcal{P}_f(m, p)}{\epsilon p^2} - \frac{b}{p^2} \right]. \]

(31)

and by taking the limit over \( \epsilon \to 0^+ \), using the continuity and differentiability of \( f \),
\[ \| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) - b(B - A) \| \leq \|P\|^2 \|B - A\| \left( \frac{\partial \mathcal{P}_f(m, p)}{\partial x p^2} - \frac{b}{p^2} \right). \]

(32)

Since
\[ \mathcal{P}_f(x, y) := yf \left( \frac{x}{y} \right), \]

hence
\[ \frac{\partial \mathcal{P}_f(x, y)}{\partial x} := f' \left( \frac{x}{y} \right) \]

which give that
\[ \frac{\partial \mathcal{P}_f(m, p)}{\partial x} = f' \left( \frac{m}{p} \right) \]

and by (32) we deduce the second inequality in (28). \( \square \)

If the parameter \( b \geq 0 \) is not available, then we can state the following more practical bounds:
Theorem 3.2  Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator monotone. Then for all $A \geq m_1 > 0$, $B \geq m_2 > 0$ and $P \geq p > 0$,

$$\| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) \| \leq \frac{\|P\|^2\|B - A\|}{p^2} \begin{cases} \left( \frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1} \right) & \text{if } m_1 \neq m_2, \\ f'(\frac{m}{p}) & \text{if } m_1 = m_2 = m. \end{cases} \quad (33)$$

Proof  By the triangle inequality we get from (28) that

$$\| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) \| - b\|B - A\| \leq \| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) - b(B - A) \| \leq \begin{cases} \frac{\|P\|^2\|B - A\|}{p^2} \left[ \frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1} - b \right] & \text{if } m_1 \neq m_2, \\ \frac{\|P\|^2\|B - A\|}{p^2} \left[ f'(\frac{m}{p}) - b \right] & \text{if } m_1 = m_2 = m, \end{cases}$$

which implies that

$$\| \mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) \| \leq \begin{cases} \frac{\|P\|^2\|B - A\|}{p^2} \left[ \frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1} - b \right] + b\|B - A\| \\ \frac{\|P\|^2\|B - A\|}{p^2} \left[ f'(\frac{m}{p}) - b \right] + b\|B - A\| \\ \frac{\|P\|^2\|B - A\|}{p^2} \left[ \frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1} \right] + b\|B - A\| \left( 1 - \frac{\|P\|^2}{p^2} \right) \\ \frac{\|P\|^2\|B - A\|}{p^2} \left[ f'(\frac{m}{p}) \right] + b\|B - A\| \left( 1 - \frac{\|P\|^2}{p^2} \right). \end{cases} \quad (34)$$

Observe that $1 - \frac{\|P\|^2}{p^2} \leq 0$ and since $b \geq 0$, we get by (34) the desired result (33). □

Theorem 3.3  Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator monotone and has the representation (6). Then for all $C \geq n_1 > 0$, $D \geq n_2 > 0$, $Q > q > 0$,
\[
\begin{aligned}
&\| \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - f(0)(D-C) \| \\
&\leq \frac{\|Q\|^2\|D-C\|}{q^2} \left\{ \begin{array}{ll}
\frac{\mathcal{P}_f(q,n_2) - \mathcal{P}_f(q,n_1)}{n_2-n_1} - f(0) & \text{if } n_2 \neq n_1, \\
\frac{f\left(\frac{q}{n}\right) - f\left(\frac{q}{n}\right)}{q} & \text{if } n_2 = n_1 = n.
\end{array} \right.
\end{aligned}
\]

(35)

**Proof** From the representation (23) we get, by taking the norm, that
\[
\| \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - f(0)(D-C) \| \\
\leq \|Q\|^2\|D-C\| \int_0^\infty \lambda \left( \int_0^1 \| Q + \lambda[(1-t)C + tD] \|^{-2} \right) dt \, d\lambda.
\]

(36)

Since \( C \geq n_1 > 0, D \geq n_2 > 0, Q > q > 0, \)
\[
Q + \lambda[(1-t)C + tD] \geq q + \lambda[(1-t)n_1 + m_2],
\]

namely
\[
(Q + \lambda[(1-t)C + tD])^{-1} \leq (q + \lambda[(1-t)n_1 + m_2])^{-1},
\]

which implies that
\[
\| (Q + \lambda[(1-t)C + tD])^{-1} \| \leq (q + \lambda[(1-t)n_1 + m_2])^{-1}.
\]

Therefore
\[
\| (Q + \lambda[(1-t)C + tD])^{-1} \|^2 \leq (q + \lambda[(1-t)n_1 + m_2])^{-2}
\]

and by integration,
\[
\int_0^\infty \lambda \left( \int_0^1 \| Q + \lambda[(1-t)C + tD] \|^{-2} \right) dt \, d\lambda \\
\leq \int_0^\infty \lambda \left( \int_0^1 (q + \lambda[(1-t)n_1 + m_2])^{-2} dt \right) d\lambda.
\]

(37)

By utilising (36) and (37) we obtain
\[
\begin{aligned}
&\| \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - f(0)(D-C) \| \\
&\leq \|Q\|^2\|D-C\| \int_0^\infty \lambda \left( \int_0^1 (q + \lambda[(1-t)n_1 + m_2])^{-2} dt \right) d\lambda.
\end{aligned}
\]

(38)

If in the identity (23) we choose \( D = n_2, C = n_1 \) and \( Q = q \) then we get...
\[ \mathcal{P}_f(q, n_2) - \mathcal{P}_f(q, n_1) \]
\[ = f(0)(n_2 - n_1) + \int_0^\infty \lambda \left( \int_0^1 q[q + \lambda[(1 - t)n_1 + n_2]]^{-1}q dt \right) dw(\lambda) \]
\[ \times [q + \lambda[(1 - t)n_1 + n_2]]^{-1}q dt \right) dw(\lambda) \]
\[ = f(0)(n_2 - n_1) \]
\[ + q^2(n_2 - n_1) \int_0^\infty \lambda \left( \int_0^1 [q + \lambda[(1 - t)n_1 + n_2]]^{-2} dt \right) dw(\lambda). \] (39)

If \( n_2 \neq n_1 \), then by (39) we get
\[ \int_0^\infty \lambda \left( \int_0^1 [q + \lambda[(1 - t)n_1 + n_2]]^{-2} dt \right) dw(\lambda) \]
\[ = \frac{1}{q^2} \left[ \mathcal{P}_f(q, n_2) - \mathcal{P}_f(q, n_1) - f(0) \right]. \] (40)

By making use of (38) and (40) we derive the first branch in (35).

Let \( n_1 = n_2 = n \). Let \( \epsilon > 0 \). Then \( D + \epsilon \geq n + \epsilon > 0 \). From the first branch of (35) we get
\[ \| \mathcal{P}_f(Q, D + \epsilon) - \mathcal{P}_f(Q, C) - f(0)(D + \epsilon - C) \| \]
\[ \leq \| Q \|_2 \| D + \epsilon - C \| \left[ \mathcal{P}_f(q, n + \epsilon) - \mathcal{P}_f(q, n) \right] \]
\[ = \frac{1}{q^2} \left[ \mathcal{P}_f(q, n_2) - \mathcal{P}_f(q, n_1) - f(0) \right]. \]

and by taking the limit over \( \epsilon \to 0^+ \), using the continuity and differentiability of \( f \),
\[ \| \mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) - f(0)(D - C) \| \]
\[ \leq \| Q \|_2 \| D - C \| \left[ \mathcal{P}_f(q, n) - f(0) \right]. \] (41)

Since
\[ \mathcal{P}_f(x, y) := yf \left( \frac{x}{y} \right), \]

hence
\[ \frac{\partial \mathcal{P}_f(x, y)}{\partial y} := f \left( \frac{x}{y} \right) + yf' \left( \frac{x}{y} \right) \left( \frac{x}{y} \right)' = f \left( \frac{x}{y} \right) - \frac{yx}{y^2}f' \left( \frac{x}{y} \right) \]
\[ = f \left( \frac{x}{y} \right) - \frac{x}{y}f' \left( \frac{x}{y} \right), \]

which give that
\[ \frac{\partial \mathcal{P}_f(q, n)}{\partial y} = f\left(\frac{q}{n}\right) - \frac{q}{n} f'\left(\frac{q}{n}\right) , \]

and the second branch of (35) is also proved. \( \square \)

**Corollary 3.4** Assume that the function \( f : [0, \infty) \to \mathbb{R} \) is operator monotone. Then for all \( C \geq n_1 > 0, D \geq n_2 > 0, Q > q > 0, \)
\[ \| \mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) \| \leq \frac{\|Q\|^2\|D - C\|}{q^2} \left\{ \begin{array}{ll}
\frac{\mathcal{P}_f(q, n_2) - \mathcal{P}_f(q, n_1)}{n_2 - n_1} & \text{if } n_2 \neq n_1 , \\
\left[ f\left(\frac{q}{n}\right) - \frac{q}{n} f'\left(\frac{q}{n}\right) \right] & \text{if } n_2 = n_1 = n .
\end{array} \right. \] (42)

The proof is similar to the one provided in the proof of Theorem 3.2.

**Corollary 3.5** Assume that the function \( f : [0, \infty) \to \mathbb{R} \) is operator monotone and has the representation (6). Then for all \( A \geq m_1 > 0, B \geq m_2 > 0, C \geq n_1 > 0, D \geq n_2 > 0, \)
\[ \| \mathcal{P}_f(A, B) - \mathcal{P}_f(C, D) - b(A - C) - f(0)(B - D) \| \leq \frac{\|B\|^2\|A - C\|}{m_2} \left\{ \begin{array}{ll}
\frac{\mathcal{P}_f(m_1, m_2) - \mathcal{P}_f(n_1, m_2)}{m_1 - n_1} & , m_1 \neq n_1 \\
\left[ f\left(\frac{m_1}{m_2}\right) - b \right] , m_1 = n_1 \end{array} \right. \]
\[ + \frac{\|C\|^2\|B - D\|}{n_1} \left\{ \begin{array}{ll}
\frac{\mathcal{P}_f(n_1, m_2) - \mathcal{P}_f(n_1, n_2)}{m_2 - n_2} & - f(0) , m_2 \neq n_2 \\
\left[ f\left(\frac{n_1}{n_2}\right) - \frac{n_1}{n_2} f'\left(\frac{n_1}{n_2}\right) - f(0) \right] , m_2 = n_2 .
\end{array} \right. \] (43)

**Proof** From Theorems 3.2 and 3.3 we have
\[ \| \mathcal{P}_f(A, B) - \mathcal{P}_f(C, D) - b(A - C) - f(0)(B - D) \| \\
= \| \mathcal{P}_f(A, B) - \mathcal{P}_f(C, B) + \mathcal{P}_f(C, B) - \mathcal{P}_f(C, D) - b(A - C) - f(0)(B - D) \| \\
\leq \| \mathcal{P}_f(A, B) - \mathcal{P}_f(C, B) - b(A - C) \| \\
+ \| \mathcal{P}_f(C, B) - \mathcal{P}_f(C, D) - f(0)(B - D) \| \\
\leq \frac{\| B \|^2}{m_2} \| A - C \| \begin{cases} \\
\left[ \frac{\mathcal{P}_f(m_1, m_2) - \mathcal{P}_f(n_1, m_2)}{m_1 - n_1} - b \right], & m_1 \neq n_1 \\
\left[ f'(\frac{m_1}{m_2}) - b \right], & m_1 = n_1 \\
\end{cases} \\
+ \frac{\| C \|^2}{n_1} \| B - D \| \begin{cases} \\
\left[ \frac{\mathcal{P}_f(n_1, m_2) - \mathcal{P}_f(n_1, n_2)}{m_2 - n_2} - f(0) \right], & m_2 \neq n_2 \\
\left[ f\left(\frac{n_1}{n_2}\right) - \frac{n_1}{n_2} f'(\frac{n_1}{n_2}) - f(0) \right], & m_2 = n_2, \\
\end{cases} \\
\]

which proves (43). □

4 Some examples

If \( f_r : [0, \infty) \to [0, \infty), f_r(t) = t^r, r \in [0, 1], \) then

\[ \mathcal{P}^r_f(B, A) := A^{1/2} \left( A^{-1/2}BA^{-1/2} \right)^r A^{1/2} =: A^r_{\#}, B, \]

is the \textit{weighted operator geometric mean} of the positive invertible operators \( A \) and \( B \) with the weight \( r \).

Observe also that

\[ \mathcal{P}^r_f(x, y) = y^{1/2} \left( y^{1/2}xy^{-1/2} \right)^r y^{1/2} = x^r y^{1-r}, \ x, y > 0. \]

From (33) we get for the power function

\[ \| A^r_{\#} - A^m_{\#} \| \leq \frac{\| B \|^2 \| D - A \|}{p^{r+1}} \begin{cases} \\
\left( \frac{m_2 - m_1}{m_2 - m_1} \right)^r & \text{if } m_1 \neq m_2, \\
rm^{r-1} & \text{if } m_1 = m_2 = m \\
\end{cases} \] (44)

for all \( A \geq m_1 > 0, B \geq m_2 > 0 \) and \( P \geq p > 0 \).

From (42) we obtain

\[ \| D^r_{\#}Q - C^r_{\#}Q \| \leq \frac{\| Q \|^2 \| D - C \|}{q^{2-r}} \begin{cases} \\
\left( \frac{n_2^{1-r} - n_1^{1-r}}{n_2 - n_1} \right) & \text{if } n_2 \neq n_1, \\
\frac{(1 - r)}{n_{r}} & \text{if } n_2 = n_1 = n \\
\end{cases} \] (45)

for all \( C \geq n_1 > 0, D \geq n_2 > 0, Q > q > 0. \)
If we take the function $f = \ln$, then

$$P_{\ln}(B, A) := A^{1/2} \ln \left( A^{-1/2} BA^{-1/2} \right) A^{1/2} =: S(A|B),$$

is the relative operator entropy, for positive invertible operators $A$ and $B$.

Kamei and Fujii [13, 14] defined the relative operator entropy $S(A|B)$, for positive invertible operators $A$ and $B$, which is a relative version of the operator entropy considered by Nakamura-Umegaki [23].

Observe also that

$$P_{\ln}(x, y) := y^{1/2} \ln \left( y^{-1/2} xy^{-1/2} \right) y^{1/2} = y \ln \left( \frac{x}{y} \right), \quad x, y > 0.$$  

Let $\varepsilon > 0$. If we use the inequality (33) for the function $f_{\varepsilon}(t) = \ln(t + \varepsilon), \ t \in [0, \infty)$, then we obtain

$$\left\| P^{1/2} \ln \left( P^{-1/2} BP^{-1/2} + \varepsilon \right) P^{1/2} - P^{1/2} \ln \left( P^{-1/2} AP^{-1/2} + \varepsilon \right) P^{1/2} \right\|
\leq \frac{\|P\|^2 \|B - A\|}{p} \left\{ \begin{array}{ll}
\frac{\ln(m_2 + \varepsilon) - \ln(m_1 + \varepsilon)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{1}{m + \varepsilon} & \text{if } m_1 = m_2 = m,
\end{array} \right. \tag{46}$$

where $A \succeq m_1 > 0, B \succeq m_2 > 0$ and $P \succeq p > 0$.

If we take in $\varepsilon \to 0^+$ in (46), then we get

$$\|S(P|B) - S(P|A)\| \leq \frac{\|P\|^2 \|B - A\|}{p} \left\{ \begin{array}{ll}
\frac{\ln m_2 - \ln m_1}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{1}{m} & \text{if } m_1 = m_2 = m,
\end{array} \right. \tag{47}$$

where $A \succeq m_1 > 0, B \succeq m_2 > 0$ and $P \succeq p > 0$.

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