Voting behavior in proportional elections from agent–based models

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Abstract

In the talk we reviewed universal aspects of voting behavior in proportional elections and universality breaking patterns, as established in the existing literature. Focus was made on the Brazilian elections, which are characterized by compulsory voting. We showed how agent–based models can qualitatively and/or quantitatively reproduce the observed empirical distributions. As an example, we discussed the multi–state voter model over a network based on interacting cliques and zealot candidates.

Keywords: social physics; voter model; complex networks.

1. Introduction

Recent years have witnessed an increasing interest from the community of Statistical Physics towards research topics typical of Social Sciences, although the intuition that the human society may be regarded as a physical system is at least as old as Physics itself. For instance, in the introduction of his most famous philosophical essay, Leviathan, published in 1651, T. Hobbes wrote “Nature (the art whereby God hath made and governs the world) is by the art of man, as in many other things, so in this also imitated, that it can make an artificial animal”. The human society was described by Hobbes like a mechanical monster, composed of interacting parts moved by men; as such, it must fulfill physical laws similar to those governing the interacting matter.

It is worthwhile noticing that Leviathan was published just after Galilei’s Dialogue (1632) and just before Newton’s Principia (1687). Yet, whereas the latter had to become milestones of modern science, Hobbes’ attempt to give Sociology a scientific foundation was condemned to remain a dream for a very long time. It is hoped nowadays that such an enterprise — if it ever has a chance to succeed — may rely on the joint frameworks of Statistical Mechanics and Network Science (cf. Castellano et al. (2009) for a thorough review of ideas and results).

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2. Universality (and breaking patterns) of voting behavior

An example of Leviathan at work is given in Fortunato and Castellano (2007), where it is shown that the excess of the number of votes received in proportional elections by candidates over the average number of votes per intra–party competitor has a universal distribution, invariant in time and independent of the country. If we denote by

- $N$ := # of votes received by a party in an electoral district,
- $Q$ := # of candidates proposed by a party in an electoral district,
- $v$ := # of votes received by a candidate in an electoral district,

then $v_0 = N/Q$ amounts to the number of received votes per candidate and $x = v/v_0$ represents the aforementioned excess of votes ($x \lesssim 1$ means that a candidate has performed better/worse than the average). The analysis of Fortunato and Castellano (2007) focuses in particular on the political elections held along the past fifty years in Italy, Poland and Finland (see Fig. 1, Left). In order to interpret the empirical evidence, the authors of Fortunato and Castellano (2007) propose an agent–based model based on word of mouth, where political opinions originate from the candidates and propagate down stochastic trees. With an appropriate choice of the underlying parameters, the model reproduces faithfully the observed voting behavior.

A recent extension of the same analysis to a larger pool of countries Chatterjee et al. (2013) confirms the universal trend in Denmark and Estonia, but proves it to be hopelessly broken in countries such as Brazil, Slovenia, Greece, etc. In each of these, the probability law of the intra–party excess of votes displays specific deviations from the universal curve, while persisting in time. Therefore, the discrepancies — as the authors of Chatterjee et al. (2013) claim — “are likely to be due to peculiar differences in the election rules”. An interesting case is indeed represented by Brazil, which has been first studied in Costa Filho et al. (1999). Here, the distribution of the excess of votes breaks the universality pattern for $x < 1$ (see Fig. 1, Right), where it displays an almost flat shape. Since the only peculiar difference characterizing Brazil is that it adopts a compulsory voting system, the Brazilian case can help shed light on the origin of the universality. Unfortunately, compulsory voting can be hardly encoded in the word of mouth model of Fortunato and Castellano (2007) without introducing major modifications, thus it makes sense to approach the Brazilian case differently.
3. The multi–state voter model on community–based networks

An agent–based model where voting is compulsory is the voter model (VM) proposed by Clifford and Sudbury (1973) and by Holley and Liggett (1975), in that here every agent expresses a preference at each time step of the stochastic dynamics. In order to adapt the VM to the Brazilian case, at least three modifications have to be considered:

- The original VM is a binary model, unfit to describe elections with more than two candidates. This difficulty is easily overcome by turning to a multi–state variant, considered with different purposes in Böhme and Gross (2012); Hubbell (2001); McKane et al. (2004); Pigolotti et al. (2005); Starnini et al. (2012);
- The stochastic dynamics of both the binary VM and its multi–state variant drives ultimately the system towards consensus, a uniform state where all agents share the same political preference. The simplest way to avoid this seems to be the use of committed agents (also known in the literature as zealots), that is to say agents who never change political preference, while taking part in the stochastic dynamics as opinion donors. Zealots have been originally introduced in Mobilia (2003) in the framework of the binary VM and have been subsequently used by several authors to investigate various aspects of opinion dynamics. It is legitimate to interpret political candidates as zealots;
- Opinion diffusion down stochastic trees, similar to the model of Fortunato and Castellano (2007) is not possible in the framework of the VM. Yet, trees can be regarded as communities. For this reason, we consider a community based voter network, i.e. a network partitioned into non–overlapping cliques (sub–networks where the agents are all pairwise connected), all having the same size. Each clique hosts one candidate and candidates belong all to the same party. Voters belonging to different cliques are then stochastically connected via Erdős–Rényi links with flat probability $0 < p < 1$. An example of such a network is given in Fig. 2.

In Palombi and Toti (2013) we studied a variant of the VM including all the above features. In particular, we showed that such a model is characterized by a well defined thermodynamic limit, obtained by keeping the average degree of each agent constant while increasing the network size. If $V$ denotes the set of agents, including $Q$ zealot candidates, and $V_0$ denotes the set of dynamic agents (those who can change political opinion in time), then for $x \in V_0$ the average degree of $x$ is given by

$$\mathbb{E}[\text{deg}(x)] = \delta_{\text{in}} + \delta_{\text{out}},$$

(1)
where

\[ \delta_{\text{in}} = \left( \frac{|V|}{Q} - 1 \right), \quad \delta_{\text{out}} = p(Q - 1) \left( \frac{|V|}{Q} - 1 \right), \quad (2) \]

represent the intra- and extra-clique average degrees. We keep both \( \delta_{\text{in}} \) and \( \delta_{\text{out}} \) constant as \( |V| \to \infty \), which is trivially achieved by imposing

\[ |V|/Q = \omega_1, \quad p(Q - 1) = \omega_2, \quad (3) \]

with \( \omega_1 \gg 1 \) and \( \omega_2 > 0 \) two independent constants. Each pair \((\omega_1, \omega_2)\) makes the model approach the thermodynamic limit along a specific trajectory.

In Fig. 3 we show the distribution \( F_{\text{VM}}(x) \) of the intra–party excess of votes at \((\omega_1, \omega_2) = (2000.0, 0.3)\) and \((\omega_1, \omega_2) = (2000.0, 6)\) for various network sizes, as obtained from Monte Carlo simulations. We see that \( F_{\text{VM}}(x) \) resembles the distribution observed in real Brazilian elections. It should be noticed that the dependence of the distribution upon \( |V| \) is minimal: \( F_{\text{VM}}(x) \) is very close to the thermodynamic limit at all network sizes, while finite size effects are only visible along the right tails. Moreover, the central part of the distribution follows a power law \( F_{\text{VM}}(x) \propto x^{-\alpha} \), with \( \alpha \) depending on the model parameters. In particular, \( \alpha \) increases with \( \omega_2 \).

4. Mean Field Theory approximation

Besides Monte Carlo simulations, Mean Field Theory (MFT) represents the simplest approach to studying birth–death models such as the VM. The system is described in this framework in terms of macroscopic states and its probability density follows from a Master Equation. If we denote by \( v_{(i)} \) the number of votes received by the \( k \)-th zealot from the \( i \)-th community, then we are interested in the time evolution of the vector \( v(t) = \{v_{(i)}(t)\}_{i,k=1}^{Q} \). In the thermodynamic limit, \( v(t) \) has an increasingly large number of components, each ranging between 0 and \( \omega_1 \) (besides contributions from candidates). For this reason, we introduce normalized variables

\[ \phi_{(i)}^{(k)} = \frac{Q v_{(i)}^{(k)}}{|V|} = \omega_1^{-1} v_{(i)}^{(k)}, \quad i,k = 1, \ldots, Q. \quad (4) \]

The time evolution of the probability density \( P(\phi, t) \) of the vector \( \phi = \{\phi_{(i)}^{(k)}\}_{i,k=1}^{Q} \) is described by a Fokker–Planck equation

\[ \partial_t P(\phi, t) = - \sum_{i=1}^{Q} \sum_{\ell \neq i}^{1,Q} \partial_{\phi_{(i)}^{(\ell)}} \left[ A_{(i)}^{(\ell)}(\phi) P(\phi, t) \right] + \frac{1}{2} \sum_{i,j=1}^{Q} \sum_{\ell,m \neq i}^{1,Q} \partial_{\phi_{(i)}^{(\ell)}} \partial_{\phi_{(j)}^{(m)}} \left[ B_{(\ell,m)}^{(i,j)}(\phi) P(\phi, t) \right], \quad (5) \]
where $\partial_t = \partial/\partial t$ and $\partial_i^{(i)} = \partial/\partial \phi_i^{(i)}$. Not surprisingly, the analytic structure of the Fokker–Planck equation reflects the specific topology of the network. The coefficient functions $A_{\ell}^{(i)}$ and $B_{m}^{(i)}$ of the equation are given by

$$
\tau A_{\ell}^{(i)} = -\left[1 + \omega_1 \omega_2 (1 - \omega_1^{-1})\right] \phi_t^{(i)} + \frac{\omega_1 \omega_2 (1 - \omega_1^{-1})}{Q - 1} \left\{ \sum_{k \neq \ell} \phi_t^{(k)} - \sum_{k \neq \ell} \phi_t^{(i)} \right\},
$$

$$
\tau B_{m}^{(i)} = -2(1 - \delta_{tm}) \phi_{m}^{(i)} \phi_t^{(i)} - (1 - \delta_{tm}) \frac{\omega_2}{(Q - 1)} \left\{ \phi_t^{(i)} \left(1 - \omega_1^{-1} + \sum_{k \neq m} \phi_t^{(k)} - \sum_{k \neq m} \phi_t^{(m)} \right) \right\} 
$$

$$
+ \phi_{m}^{(i)} \left(1 - \omega_1^{-1} + \sum_{k \neq m} \phi_t^{(k)} - \sum_{k \neq m} \phi_t^{(m)} \right) 
+ 2\delta_{tm} \phi_t^{(i)} \left[1 + \frac{\omega_2 (1 - \omega_1^{-1}) - \omega_1^{-1}}{2} - \phi_t^{(i)} \right]
$$

$$
+ \delta_{tm} \frac{\omega_2}{(Q - 1)} \left(1 - \omega_1^{-1} - 2\phi_t^{(i)} \right) \left[1 - \omega_1^{-1} + \sum_{k \neq \ell} \phi_t^{(k)} - \sum_{k \neq \ell} \phi_t^{(i)} \right], \quad \ell, m \neq i,
$$

with $\tau = \omega_1 (1 + \omega_2) (1 - \omega_1^{-1})$ denoting the autocorrelation time of the system. Unfortunately, no analytic solution of the Fokker–Planck equation is known so far. Nevertheless, numerical solutions of it show that the MFT description of the model is rather accurate for $x \gtrsim 1.0 \times 10^{-2}$ and a broad spectrum of model parameters.

5. Conclusions

In the talk, we showed that agent–based models allow to theoretically investigate complex phenomena with qualitative (and in some case quantitative) agreement. Agent–based models find application in a wide range of problems, such as opinion dynamics, epidemic spreading, ecological models, brain functionalities, etc. We focused on their role in political elections. We reviewed universal aspects of proportional voting systems and their breaking patterns. Specifically, we concentrated on the Brazilian case, which is characterized by compulsory voting and presented a variant of the voter model, whose vote distribution resembles that observed in real Brazilian elections.

References

Böhme, G.A., Gross, T., 2012. Fragmentation transitions in multistate voter models. Phys. Rev. E 85, 066117. doi:10.1103/PhysRevE.85.066117.

Castellano, C., Fortunato, S., Loreto, V., 2009. Statistical physics of social dynamics. Reviews of Modern Physics 81, 591–646. doi:10.1103/RevModPhys.81.591, arXiv:0710.3256.

Chatterjee, A., Mitrović, M., Fortunato, S., 2013. Universality in voting behavior: an empirical analysis. Scientific Reports 3. doi:10.1038/srep01049, arXiv:1212.2142.

Clifford, P., Sudbury, A., 1973. A model for spatial conflict. Biometrika 60, 581–588. doi:10.1093/biomet/60.3.581.

Costa Filho, R.N., Almeida, M.P., Andrade, J.S., Moreira, J.E., 1999. Scaling behavior in a proportional voting process. Phys. Rev. E 60, 1067–1068.

Fortunato, S., Castellano, C., 2007. Scaling and Universality in Proportional Elections. Phys. Rev. Lett. 99, 138701. doi:10.1103/PhysRevLett.99.138701, arXiv:arXiv:physics/0612140.

Holley, R., Liggett, T.M., 1975. Ergodic theorems for weakly interacting infinite systems and the voter model. The Annals of Probability 3, 643–663. doi:10.2307/2959329.

Hubbell, S.P., 2001. The Unified Neutral Theory of Biodiversity and Biogeography (MPB-32) (Monographs in Population Biology). Princeton University Press.

McKane, A.J., Alonso, D., Solé, R.V., 2004. Analytic solution of hubbell’s model of local community dynamics. Theoretical Population Biology 65, 67 – 73. doi:http://dx.doi.org/10.1016/j.tpb.2003.08.001.
Mobilia, M., 2003. Does a single zealot affect an infinite group of voters? Physical Review Letters 91, 028701. doi:10.1103/PhysRevLett.91.028701.

Palombi, F., Toti, S., 2013. Stochastic Dynamics of the Multi-State Voter Model over a Network based on Interacting Cliques and Zealot Candidates. ArXiv e-prints arXiv:1309.0396.

Pigolotti, S., Flammini, A., Marsili, M., Maritan, A., 2005. Species lifetime distribution for simple models of ecologies. Proc Natl Acad Sci U S A 102, 15747–51+. doi:10.1073/pnas.0502648102.

Starnini, M., Baronchelli, A., Pastor-Satorras, R., 2012. Ordering dynamics of the multi-state voter model. J. Stat. Mech. P10027. arXiv:1207.5810.