Integration of semidirect product Lie 2-algebras *

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Abstract

The semidirect product of a Lie algebra and a 2-term representation up to homotopy is a Lie 2-algebra. Such Lie 2-algebras include many examples arising from the Courant algebroid appearing in generalized complex geometry. In this paper, we integrate such a Lie 2-algebra to a strict Lie 2-group in the finite dimensional case.

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1 Introduction

The Courant bracket (skew-symmetric version) is a Lie-like bracket which satisfies a weakened version of the Jacobi identity. The standard Courant bracket on the Courant algebroid $TM \oplus T^*M$ is now widely used in Hitchin and Gualtieri’s program [14, 12] of generalized complex geometry. In this program, $TM \oplus T^*M$ is viewed as a generalized tangent bundle. With the help of the Courant bracket, generalized complex structures can unify both complex and symplectic structures. They are aimed at understanding mirror symmetry. In particular, we expect this object to help understand what controls the global symmetry in generalized complex geometry.

Much earlier than us, long before the program of generalized complex geometry took shape in mathematics, Kinyon and Weinstein set off to solve this problem in their pioneering work [15]. Their approach is to associate with the Courant bracket a Lie-Yamaguti algebra and use Kikkawa’s construction integrating Lie-Yamaguti algebras. They succeed in finding a canonical left loop structure with nonassociative multiplication, (which generalizes a group structure), associated to a finite dimensional version of Courant bracket. However, as the authors themselves point out, it is not satisfactory because it does not obey the following testing requirement,

Testing Requirement: the construction, when restricted to a subspace where the Courant bracket becomes a Lie bracket, needs to reproduce a group structure.

They also hint at an alternative approach, in which one views a Courant bracket as a part of an $L_\infty$-structure (see Definition 2.1), as in [18]. A Lie-Yamaguti algebra contains a bilinear and a trilinear operation, as this $L_\infty$-structure does. However, the significant difference is that the $L_\infty$-algebra is graded while the trilinear operation of Lie-Yamaguti algebras is not required to be completely skew-symmetric.

We now take this alternative approach benefiting from recent progresses on the problem of integrating $L_\infty$-algebras [11, 13, 20]. However, directly applying their method in our case gives a quite abstract space which we can not make explicit. In order to find an explicit formula, our first step [24] is to realize such an $L_\infty$-algebra associated to the Courant bracket as a semidirect product of a Lie algebra with a representation up to homotopy. Now we set off to integrate such a semidirect product type of Lie 2-algebra.

The Courant bracket is defined on the direct sum $X(M) \oplus \Omega^1(M)$ of vector fields and 1-forms on a manifold $M$ by

$$[[X + \xi, Y + \eta]] \triangleq [X, Y] + L_X \eta - L_Y \xi + \frac{1}{2} d(\xi(Y) - \eta(X)).$$

The factor of $\frac{1}{2}$ spoils the Jacobi identity. Summarizing the properties of the Courant bracket, Liu, Weinstein and Xu introduced the notion of Courant algebroid in [16]. Courant algebroids have many applications in the theory of Manin pairs and moment maps, generalized complex structures, $L_\infty$-algebras and symplectic supermanifolds, gerbes as well as BV algebras and topological field theories.

The Courant bracket lives on infinite dimensional spaces. This makes the integration procedure much more technical. There are two ways to find some finite dimensional toy-cases, which are interesting by themselves. One way is to restrict a Courant algebroid over a point. The integration is given in [24] via a Lie 2-algebra of string type. Another way is to linearize as Kinyon and Weinstein did in their paper: take $M$ to be the dual space $V^*$ of a vector space $V$, and consider the linear vector fields, that is, the linear maps $V^* \to V^*$, along with the values of 1-forms at 0.
The resulting bracket on the finite dimensional space $\mathfrak{gl}(V) \oplus V$ is given by
\[
\llbracket A + u, B + v \rrbracket = [A,B] + \frac{1}{2}(Av - Bu).
\]

The factor of $\frac{1}{2}$ spoils again the Jacobi identity. Furthermore, there is a canonical $V$-valued nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ which is given by
\[
\langle A + u, B + v \rangle = \frac{1}{2}(Av + Bu).
\]

The direct sum $\mathfrak{gl}(V) \oplus V$ with this bracket and this pairing is called an omni-Lie algebra by Weinstein [27]. Its Dirac structures characterize Lie algebra structures on $V$. Recently, several aspects of the theory of omni-Lie algebras have been studied in depth [9, 10, 15, 21, 25].

In this paper, we will restrict ourselves to the finite dimensional case as Kinyon and Weinstein do. Kinyon and Weinstein also hint at the possibility of solving the infinite dimensional issue of the Courant bracket by making the Courant bracket descend to a bracket on the vector bundle $TM \oplus T^*M$ which is finite dimensional. We postpone the discussion on this part to a further work [22].

Our method to integrate the semidirect product is simply to realize a 2-term representation up to homotopy $\mathcal{V}$ of a Lie algebra $\mathfrak{g}$ as an $L_\infty$-morphism from $\mathfrak{g}$ to the crossed module of Lie algebras $\text{End}(\mathcal{V})$ given by (29) (see Theorem 3.1 and Theorem 3.2). It is rather easy to integrate a strict morphism between strict Lie 2-algebras. But to integrate an $L_\infty$-morphisms between strict Lie 2-algebras is not that straightforward. We had to solve a set of PDE’s, which is a modification of the one that Lie solved to integrate Lie algebra morphisms. We had succeeded in solving them [23], but the model we obtained is infinite dimensional. However, thanks to Noohi’s timely paper on butterfly method of $L_\infty$-morphisms [8], where he realizes an $L_\infty$-morphism of strict Lie 2-algebras as a zig-zag of strict morphisms, we are able to avoid the above-mentioned technical difficulties.

First we form the semidirect product in the strict case both at the Lie group level and at the Lie algebra level (Theorem 5.6 and Theorem 5.4). Using the butterfly, we strictify an $L_\infty$-morphism between strict Lie 2-algebras. Then by using the strict morphism, we form a semidirect product Lie 2-algebra which is equivalent to the original one (Theorem 5.11). Thus we obtain a strict Lie 2-group integrating the semidirect product at the Lie algebra level (Corollary 5.12). Using the zig-zag provided by butterfly methods, we eventually solve our integration problem within the finite dimensional world.

Rephrasing Kinyon-Weinstein’s result in our higher language, we can associate a group-like object, which is a strict Lie 2-group, to an omni-Lie algebra, meeting the testing requirement that Kinyon and Weinstein ask. Moreover, given any such strict Lie 2-group coming from integration, we can differentiate it to obtain a Lie 2-algebra which contains the complete information of the omni-Lie algebra that we start with. Thus it is appropriate to say that this strict Lie 2-group is the integration of an omni-Lie algebra.

Since we realize the integration of a semidirect product Lie 2-algebra as a strict Lie 2-group, we also provide a strictification of the Lie 2-group we constructed in [24] integrating the string type Lie 2-algebra $\mathbb{R} \to \mathfrak{g} \oplus \mathfrak{g}^*$. Note that there is not yet a canonical way to strictify a Lie 2-group, so we are quite lucky to achieve this.

The paper is organized as follows. In Section 2 we briefly recall some notions and basic facts about $L_\infty$-algebras, representation up to homotopy of Lie algebras, crossed modules of Lie algebras (resp. Lie groups), Lie 2-groups, and butterflies. In Section 3 we construct the DGLA associated to a complex of vector spaces (Theorem 3.1). In particular, a 2-term representation up to homotopy
of a Lie algebra is equivalent to an $L_\infty$-morphism from this Lie algebra to the crossed module of Lie algebras $\text{End}(\mathcal{V})$, which is obtained by a truncation of the above-mentioned DGLA constructed in Theorem 3.1. In Section 4 we give the integration of the crossed module of Lie algebras $\text{End}(\mathcal{V})$, which turns out to be a crossed module of $\text{Lie}$ groups $\text{Aut}(\mathcal{V})$. In Section 5 we first construct a strict Lie 2-group associated to any strict morphism of crossed modules of Lie groups from $(H_1, H_0, t, \Phi)$ to $\text{Aut}(\mathcal{V})$ (Theorem 5.3). Then with the help of the butterfly, we give the integration of semidirect product Lie 2-algebras. In Section 6 we give the integration of omni-Lie algebras as an application.

**Notations:** $d$ is the differential in a complex of vector spaces, $\delta$ is the differential in the DGLA associated with a complex of vector spaces. $i$ is the inclusion map. $i_1$ and $i_2$ are the inclusion to the first factor and the second factor respectively. $\text{Id}$ is the identity map.

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## 2 Preliminaries

### 2.1 Lie 2-algebras

**Definition 2.1.** An $L_\infty$-algebra is a graded vector space $L = L_0 \oplus L_1 \oplus \cdots$ equipped with a system $\{l_k\}_{1 \leq k < \infty}$ of linear maps $l_k : \wedge^k L \rightarrow L$ of degree $\deg(l_k) = k - 2$, where the exterior powers are interpreted in the graded sense and the following relation with Koszul sign $'\text{Ksgn}'$ is satisfied for all $n \geq 0$:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} \text{sgn}(\sigma) \text{Ksgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0,$$

where the summation is taken over all $(i, n-i)$-unshuffles with $i \geq 1$.

For $n = 1$, we have

$$l_1^2 = 0, \quad l_1 : L_{i+1} \rightarrow L_i,$$

which means that $L$ is a complex, so we write $d = l_1$ as usual. For $n = 2$, we have

$$d l_2(X, Y) = l_2(dX, Y) + (-1)^p l_2(X, dY), \quad \forall X \in L_p, Y \in L_q,$$

which means that $d$ is a derivation with respect to $l_2$. We view $l_2$ as a bracket $[\cdot, \cdot]$. However, it is not a Lie bracket, the obstruction of Jacobi identity is controlled by $l_3$ and $l_3$ also satisfies higher coherence laws.

In particular, if the $k$-ary brackets are zero for all $k > 2$, we recover the usual notion of differential graded Lie algebras (DGLA). If $L$ is concentrated in degrees $< n$, we get the notion of $n$-term $L_\infty$-algebras. A 2-term $L_\infty$-algebra is also called a **Lie 2-algebra** in this paper. For details about Lie 2-algebras, see [3, 4, 17].

**Definition 2.2.** A 2-term representation up to homotopy of a Lie algebra $\mathfrak{g}$ consists of

1. A 2-term complex of vector spaces $\mathcal{V} : V_1 \xrightarrow{d} V_0$.

2. Two linear actions (not strict necessarily) $\mu_0$ and $\mu_1$ act on $V_0$ and $V_1$, which are compatible with $d$, i.e. for any $X \in \mathfrak{g}$, we have

$$d \circ \mu_1(X) = \mu_0(X) \circ d.$$
3. A 2-form $\nu \in \Omega^2(g; \text{End}(V_0, V_1))$ such that
\[
\begin{align*}
\mu_0[X_1, X_2] - [\mu_0(X_1), \mu_0(X_2)] &= d \circ \nu(X_1, X_2), \quad (3) \\
\mu_1[X_1, X_2] - [\mu_1(X_1), \mu_1(X_2)] &= \nu(X_1, X_2) \circ d, \quad (4)
\end{align*}
\]
as well as
\[
[\mu(X_1), \nu(X_2, X_3)] + \text{c.p.} = \nu([X_1, X_2], X_3) + \text{c.p.,} \quad (5)
\]
where c.p. means cyclic permutations.

We usually write $\mu = \mu_0 + \mu_1$ and denote a 2-term representation up to homotopy of a Lie algebra $g$ by $(\mathcal{V}, \mu, \nu)$.

Let $(\mathcal{V}, \mu, \nu)$ be a 2-term representation up to homotopy of $g$. Then we can form a new 2-term complex
\[
g \ltimes \mathcal{V} : V_1 \longrightarrow \longrightarrow g \oplus V_0.
\]
Define a graded bracket $[\cdot, \cdot] : \wedge^2(g \ltimes \mathcal{V}) \longrightarrow g \ltimes \mathcal{V}$ by setting
\[
\begin{align*}
\{ (X, \xi), (Y, \eta) \} &= ([X, Y], \mu_0(X)(\eta) - \mu_0(Y)(\xi)), \\
\{ (X, \xi), m \} &= \mu_1(X)(m), \\
\{ m, n \} &= 0,
\end{align*}
\]
for any $(X, \xi), (Y, \eta) \in g \oplus V_0$ and $m, n \in V_1$. One should note that $[\cdot, \cdot]$ is not a Lie bracket. Instead, we have
\[
[[([X, \xi], (Y, \eta)], (Z, \gamma)] + \text{c.p.} = d(\nu(X, Y)(\gamma)) + \text{c.p.}
\]
Define the 3-ary bracket $[\cdot, \cdot, \cdot] \colon \wedge^3(g \ltimes \mathcal{V}) \longrightarrow g \ltimes \mathcal{V}$ by setting:
\[
\{(X, \xi), (Y, \eta), (Z, \gamma) \} = -\nu(X, Y)(\gamma) + \text{c.p.}
\]
Proposition 2.3. \cite{[1]} If $(\mathcal{V}, \mu, \nu)$ is a 2-term representation up to homotopy of a Lie algebra $g$, then $(g \ltimes \mathcal{V}, [\cdot, \cdot], [\cdot, \cdot, \cdot])$ is a Lie 2-algebra.

2.2 2-term DGLAs and crossed modules of Lie algebras

Definition 2.4. A crossed module of Lie algebras is a quadruple $(h_1, h_0, dt, \phi)$, which we denote by $h$, where $h_1$ and $h_0$ are Lie algebras, $dt : h_1 \longrightarrow h_0$ is a Lie algebra morphism and $\phi : h_0 \longrightarrow \text{Der}(h_1)$ is an action of Lie algebra $h_0$ on Lie algebra $h_1$ as a derivation, such that
\[
dt(\phi_X(A)) = [X, dt(A)], \quad \phi_{dt(A)}(B) = [A, B].
\]

Example 2.5. For any Lie algebra $\mathfrak{t}$, the adjoint action $\text{ad}$ is a Lie algebra morphism from $\mathfrak{t}$ to $\text{Der}(\mathfrak{t})$. Then $(\mathfrak{t}, \text{Der}(\mathfrak{t}), \text{ad}, \text{Id})$ is a crossed module of Lie algebras.

Theorem 2.6. There is a one-to-one correspondence between 2-term DGLAs and crossed modules of Lie algebras.

In short, the formula for the correspondence can be given as follows: A 2-term DGLA $L_1 \longrightarrow L_0$ gives rise to a Lie algebra crossed module with $h_1 = L_1$ and $h_0 = L_0$, where the Lie brackets are given by:
\[
\begin{align*}
[A, B]_{h_1} &= [d(A), B], \quad \forall A, B \in L_1, \\
[X, X']_{h_0} &= [X, X'], \quad \forall X, X' \in L_0.
\end{align*}
\]
and \( dt = d, \phi : \mathfrak{h}_0 \to \text{Der}(\mathfrak{h}_1) \) is given by \( \phi_X(A) = [X, A] \). The DGLA structure gives the Jacobi identity for \([\cdot, \cdot]_{\mathfrak{h}}\) and \([\cdot, \cdot]_{\mathfrak{h}_0}\), and various other conditions for crossed modules.

Conversely, a crossed module \((\mathfrak{h}_1, \mathfrak{h}_0, dt, \phi)\) gives rise to a 2-term DGLA with \( d = dt, L_1 = \mathfrak{h}_1 \) and \( L_0 = \mathfrak{h}_0 \), where the brackets are given by:

\[
[A, B] \triangleq 0, \quad \forall A, B \in \mathfrak{h}_1, \\
[X, X'] \triangleq [X, X']_{\mathfrak{h}_0}, \quad \forall X, X' \in \mathfrak{h}_0, \\
[X, A] \triangleq \phi_X(A).
\]

### 2.3 Strict Lie 2-groups and crossed modules of Lie groups

A group is a monoid where every element has an inverse. A 2-group is a monoidal category where every object has a weak inverse and every morphism has an inverse. Denote the category of smooth manifolds and smooth maps by \( \text{Diff} \), a (semistrict) Lie 2-group is a 2-group in \( \text{DiffCat} \), every object has a weak inverse and every morphism has an inverse. Denote the category of smooth manifolds and smooth maps by \( \text{Diff} \), a (semistrict) Lie 2-group is a 2-group in \( \text{DiffCat} \), where \( \text{DiffCat} \) is the 2-category consisting of categories, functors, and natural transformations in \( \text{Diff} \). For more details, see [5]. Here we only give the definition of strict Lie 2-groups.

**Definition 2.7.** A strict Lie 2-group is a Lie groupoid \( C \) such that

1. The space of morphisms \( C_1 \) and the space of objects \( C_0 \) are Lie groups.
2. The source and the target \( s, t : C_1 \to C_0 \), the identity assigning function \( i : C_0 \to C_1 \) and the composition \( \circ : C_1 \times_{C_0} C_1 \to C_1 \) are all Lie group morphisms.

In the following we will denote the composition \( \circ \) in the Lie groupoid structure by \( \cdot \) : \( C_1 \times_{C_0} C_1 \to C_1 \) and call it the vertical multiplication. Denote the Lie 2-group multiplication by \( \cdot_{\mathfrak{h}} : C \times C \to C \) and call it the horizontal multiplication.

It is well known that strict Lie 2-groups can be described by crossed modules of Lie groups.

**Definition 2.8.** A crossed module of Lie groups is a quadruple \((H_1, H_0, t, \Phi)\), which we denote simply by \( \mathbb{H} \), where \( H_1 \) and \( H_0 \) are Lie groups, \( t : H_1 \to H_0 \) is a morphism, and \( \Phi : H_0 \times H_1 \to H_1 \) is an action of \( H_0 \) on \( H_1 \) preserving the Lie group structure of \( H_1 \) such that the Lie group morphism \( t \) is \( H_0 \)-equivariant:

\[
t\Phi(g)(h) = gt(h)g^{-1}, \quad \forall g \in H_0, h \in H_1,
\]

and \( t \) satisfies the so called Perffler identity:

\[
\Phi(t(h))(h') = hh'h^{-1}, \quad \forall h, h' \in H_1.
\]

**Theorem 2.9.** There is a one-to-one correspondence between crossed modules of Lie groups and strict Lie 2-groups.

Roughly speaking, given a crossed module \((H_1, H_0, t, \Phi)\) of Lie groups, there is a strict Lie 2-group for which \( C_0 = H_0 \) and \( C_1 = H_0 \rtimes H_1 \), the semidirect product of \( H_0 \) and \( H_1 \). In this strict Lie 2-group, the source and target maps \( s, t : C_1 \to C_0 \) are given by

\[
s(g, h) = g, \quad t(g, h) = t(h) \cdot g,
\]

the vertical multiplication \( \cdot_{\nu} \) is given by:

\[
(g', h') \cdot_{\nu} (g, h) = (g, h' \cdot h), \quad \text{where} \quad g' = t(h) \cdot g,
\]

the horizontal multiplication \( \cdot_\mathfrak{h} \) is given by

\[
(g, h) \cdot_\mathfrak{h} (g', h') = (g \cdot g', h \cdot \Phi g h').
\]
2.4 Morphisms of crossed modules of Lie algebras and butterflies

Definition 2.10. Let $L$ and $W$ be Lie 2-algebras. A Lie 2-algebra morphism $f : L \rightarrow W$ consists of:

- two linear maps $f_0 : L_0 \rightarrow W_0$ and $f_1 : L_1 \rightarrow W_1$ preserving the differential $d$,
- a skew-symmetric bilinear map $f_2 : \wedge^2 L_0 \rightarrow W_1$,

such that the following equalities hold for all $X, Y, Z \in L_0$, $A \in L_1$,

\[
\begin{align*}
    f_0[X, Y] - [f_0(X), f_0(Y)] &= df_2(X, Y), \\
    f_1[X, A] - [f_0(X), f_1(A)] &= f_2(X, dA), \\
    [f_0(X), f_2(Y, Z)] + c.p. + f_1(l_3(X, Y, Z)) &= f_2([X, Y], Z) + c.p. + l_3(f_0(X), f_0(Y), f_0(Z)).
\end{align*}
\]

In particular, if $L$ and $W$ are 2-term DGLAs, the last equality in the above definition turns out to be

\[
[f_0(X), f_2(Y, Z)] + c.p. = f_2([X, Y], Z) + c.p.. \tag{13}
\]

Since 2-term DGLAs are the same as crossed modules of Lie algebras, it is straightforward to obtain the definition of morphisms of crossed modules of Lie algebras. Let

\[
g = (g_1, g_0, dt, \phi), \quad h = (h_1, h_0, dt, \phi)
\]

be crossed modules of Lie algebras. Here we use the same notations $dt, \phi$. This will not lead to confusion since the correct interpretation will always be clear from the context.

Definition 2.11. A morphism $f : g \rightarrow h$ consists of:

- two linear maps $f_0 : g_0 \rightarrow h_0$ and $f_1 : g_1 \rightarrow h_1$ preserving the morphism $dt$,
- a skew-symmetric bilinear map $f_2 : \wedge^2 g_0 \rightarrow h_1$,

such that the following equalities hold for all $X, Y, Z \in g_0$, $A \in g_1$,

\[
\begin{align*}
    f_0[X, Y] - [f_0(X), f_0(Y)] &= df_2(X, Y), \\
    f_1[\phi X A] - \phi f_0(X)f_1(A) &= f_2(X, dt(A)), \\
    [f_0(X), f_2(Y, Z)] + c.p. &= f_2([X, Y], Z) + c.p..
\end{align*}
\]

The morphism $f$ is called a strict morphism if $f_2 = 0$.

Definition 2.12. Two Lie 2-algebras are said to be equivalent if there is a Lie 2-algebra morphism which induces an equivalence of the underlying 2-term complexes of vector spaces.

In particular, the equivalence of crossed modules of Lie algebras is defined to be the equivalence of the corresponding Lie 2-algebras.

The theory of “butterflies”, developed by Aldrovandi and Noohi in \cite{2, 8}, is a nice way to describe (nonstrict) morphisms of crossed modules of Lie algebras and of Lie groups.

\footnote{Since we view 2-term $L_\infty$-algebras as Lie 2-algebras, our Lie 2-algebra morphisms are exactly $L_\infty$-algebra morphisms.}
Definition 2.13. A butterfly from \( g \) to \( h \) is a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g}_1 & \xrightarrow{\kappa} & \mathfrak{h}_1 \\
\downarrow & \downarrow & \downarrow \\
\mathfrak{g}_0 & \xrightarrow{\sigma} & \mathfrak{h}_0 \\
\end{array}
\]

in which both the diagonal sequence are complexes of Lie algebras and the NE-SW sequence is short exact, such that for every \( A \in \mathfrak{g}_1, B \in \mathfrak{h}_1 \) and \( e \in \mathfrak{t} \), we have

\[
[e, \kappa(A)] = \kappa(\phi_{\sigma(e)}A), \quad [e, \iota(B)] = \iota(\phi_{\rho(e)}B).
\]

Definition 2.14. Let \( G \) and \( \mathbb{H} \) be two crossed modules of Lie groups. A butterfly \( E : G \longrightarrow \mathbb{H} \) is a commutative diagram

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\kappa} & H_1 \\
\downarrow & \downarrow & \downarrow \\
G_0 & \xrightarrow{\sigma} & H_0 \\
\end{array}
\]

in which both diagonal sequences are complexes of Lie groups, and the NE-SW sequence is short exact. Furthermore, for every \( x \in E, \alpha \in H_1, \beta \in G_1 \), the following equalities hold:

\[
\iota(\Phi_{\rho(x)}\alpha) = x\iota(\alpha)x^{-1}, \quad \kappa(\Phi_{\sigma(x)}\beta) = x\kappa(\beta)x^{-1}.
\]

The butterfly \( E \) is an equivalence between \( G \) and \( \mathbb{H} \) if and only if the NW-SE sequence is also short exact.

Remark 2.15. For people who understand crossed modules as Lie 2-groups, the butterfly \( E \) above is an H.S. morphism between the Lie groupoids \( G_1 \times G_0 \Rightarrow G_0 \) and \( H_1 \times H_0 \Rightarrow H_0 \). Moreover this H.S. morphism preserves the 2-group structure maps. This coincides with the notion of generalized morphisms between Lie 2-groups in [28]. The notion of equivalence coincides with the notion of Morita equivalence therein. In fact \( E \) is an equivalence if and only if \( E \) is a Morita bibundle of the underlying groupoids \( G_1 \times G_0 \Rightarrow G_0 \) and \( H_1 \times H_0 \Rightarrow H_0 \).

It is easy to see that

Corollary 2.16. A crossed module of Lie groups \( G = (G_1,G_0,t,\Phi) \) is equivalent to the Lie group \( G_0/G_1 \) viewed as a trivial crossed module \((1,G_0/G_1,1,1)\) if and only if the Lie group morphism \( t \) is injective.

In [2], Noohi has proved that any morphism between crossed modules of Lie algebras can be integrated to a butterfly of crossed modules of Lie groups. More precisely, for any morphism \( f : g \longrightarrow h \), where \( f = (f_0,f_1,f_2) \), he defines a bracket \([\cdot,\cdot]\) on \( \mathfrak{g}_0 \oplus \mathfrak{h}_1 \),

\[
[(X,A),(Y,B)] = ([X,Y],[A,B] + \phi_{f_0(x)}B - \phi_{f_0(y)}A - f_2(X,Y)),
\]

In [2]. Noohi has proved that any morphism between crossed modules of Lie algebras can be integrated to a butterfly of crossed modules of Lie groups. More precisely, for any morphism \( f : \mathfrak{g} \longrightarrow \mathfrak{h} \), where \( f = (f_0,f_1,f_2) \), he defines a bracket \([\cdot,\cdot]\) on \( \mathfrak{g}_0 \oplus \mathfrak{h}_1 \),

\[
[(X,A),(Y,B)] = ([X,Y],[A,B] + \phi_{f_0(x)}B - \phi_{f_0(y)}A - f_2(X,Y)),
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\[
[(X,A),(Y,B)] = ([X,Y],[A,B] + \phi_{f_0(x)}B - \phi_{f_0(y)}A - f_2(X,Y)),
\]
and maps $\kappa, i_2, \sigma, \rho$,

\[
\begin{align*}
\kappa : g_1 & \to g_0 \oplus h_1, \quad \kappa(A) = (-f_1(A), dt(A)), \\
i_2 : h_1 & \to g_0 \oplus h_1, \quad i_2(B) = (0, B), \\
\sigma : g_0 \oplus h_1 & \to g_0, \quad \sigma(X, B) = X, \\
\rho : g_0 \oplus h_1 & \to h_0, \quad \rho(X, B) = f_0(X) + dt(B).
\end{align*}
\]

Then he obtains a butterfly

\[
\begin{array}{c}
g_1 \\
\downarrow \arrow{\kappa} \downarrow \downarrow \downarrow \\
g_0 \oplus h_1 \\
\downarrow \arrow{\sigma} \downarrow \downarrow \downarrow \\
g_0 \\
\downarrow \downarrow \downarrow \arrow{i_2} \downarrow \arrow{\rho} \downarrow \downarrow \downarrow \\
h_1 \\
\end{array}
\]

(16)

This correspondence of butterflies and morphisms is one-to-one, and the morphism corresponding to a butterfly is an equivalence of the corresponding Lie 2-algebras associated to the crossed modules if and only if the NW-SE sequence is a short exact sequence (see [8, Remark 5.5]). One can integrate the butterfly (16) to a butterfly of crossed modules of Lie groups using the fact that a short exact sequence of Lie algebras integrates to the short exact sequence of the corresponding simply connected Lie groups. Then it is easy to see that an equivalence of crossed module of Lie algebras integrates to an equivalence of the corresponding crossed module of Lie groups. See Proposition 3.4 in [8] for more details.

3 2-term representations up to homotopy of a Lie algebra

By a complex of vector spaces, we mean a graded vector space $V_\bullet$ endowed with a degree minus one endomorphism $d$ satisfying $d^2 = 0$:

\[
(V_\bullet, d) : \cdots \to V_k \xrightarrow{d} V_{k-1} \to \cdots \to V_0.
\]

(17)

An element $u \in V_k$ is called a homogeneous element of degree $k$. From this graded vector space, we can form a new graded vector space $\text{End}(V_\bullet)$, the degree $k$ part $\text{End}^k(V_\bullet)$ of which consists of linear maps $T : V_\bullet \to V_\bullet$ which increase the degree by $k$. We denote $|T|$ the degree of $T$. We introduce an operator of degree minus one $\delta$ on $\text{End}(V_\bullet)$ by setting:

\[
\delta(T) = d \circ T - (-1)^k T \circ d, \quad \forall T \in \text{End}^k(V_\bullet).
\]

(18)

We have also the super-commutator bracket $[,]$ on $\text{End}(V_\bullet)$ given by the linear expansion of the formula for homogeneous elements:

\[
[T, S] = T \circ S - (-1)^{|T||S|} S \circ T.
\]

(19)

**Theorem 3.1.** With the above notations, $(\delta$ and $[,]$ are given by (18) and (19) respectively), $(\text{End}(V_\bullet), [,], \delta)$ is a DGLA.
The definition of $\delta$ we denote $\text{Ker}(\delta)$, which we denote by $\text{End}(\delta)$.

Furthermore, for any $T \in \text{End}^k(V)$ and $S \in \text{End}^l(V)$,

\[
\delta[T, S] = \delta(T) \circ S - (-1)^{kl} S \circ \delta(T) = d \circ T \circ S - (-1)^{k+1} T \circ S \circ d - (-1)^{kl} d \circ S \circ T + (-1)^{kl+1} S \circ T \circ d.
\]

On the other hand, we have

\[
\begin{align*}
[\delta(T), S] &= \delta(T) \circ S - (-1)^{kl} T \circ d \circ S - (-1)^{kl} S \circ d \circ T + (-1)^{kl} S \circ T \circ d,
[\delta(S), \delta(T)] &= T \circ \delta(S) - (-1)^{kl} \delta(S) \circ T = T \circ d \circ S - (-1)^{k+1} T \circ d \circ S - (-1)^{kl+1} S \circ T + (-1)^{kl+1} S \circ d \circ T.
\end{align*}
\]

Therefore, we have

\[
\delta[T, S] = [\delta(T), S] + (-1)^k [\delta(S), \delta(T)],
\]

which completes the proof.



**Proof.** It is straightforward to see that the bracket defined by (19) is a super Lie bracket. Next we need to prove that the operator $\delta$ defined by (18) satisfies $\delta^2 = 0$ and that $\delta$ is also a derivation with respect to the super Lie bracket $[\cdot, \cdot]$. Since $d^2 = 0$, for any $T \in \text{End}^k(V)$, we have

\[
\delta^2(T) = \delta(d \circ T - (-1)^k T \circ d) = d \circ d \circ T - (-1)^{k-1} d \circ T \circ d - (-1)^k d \circ T \circ d - T \circ d \circ d = 0.
\]

The corresponding crossed module of the 2-term DGLA (22) is as follows: the Lie algebra $\mathfrak{t}_1$ as a vector space is $\text{End}^1(V)$. Its Lie bracket is given by

\[
[A, B]_{\mathfrak{t}_1} = [\delta(A), B] = A \circ d \circ B - B \circ d \circ A.
\]
The Lie algebra $k_0$ is the Lie sub-algebra $End^d_0(V)$ of $End^0(V)$,

$$k_0 \triangleq End^d_0(V), \quad [X,X']_{k_0} = X \circ X' - X' \circ X. \quad (24)$$

Furthermore, the Lie algebra morphism $dt$ and the action $\phi$ are given by

$$\phi_X(A) = [X,A], \quad dt = \delta. \quad (25)$$

We denote this crossed module of Lie algebras also by $End(V)$, i.e.

$$End(V) = (t_1, t_0, dt, \phi), \quad (26)$$

where $t_1$ and $t_0$ are given by (23) and (24) respectively.

**Theorem 3.2.** The following two statements are equivalent:

a. $\langle V, \mu, \nu \rangle$ is a 2-term representation up to homotopy of a Lie algebra $g$.

b. $\langle \mu, \nu \rangle$ is a morphism from Lie algebra $g$ to the crossed module of Lie algebras $End(V)$, where the Lie algebra $g$ is viewed as the trivial crossed module $(0,g,0,0)$.

**Proof.** If $(V \xrightarrow{d} V_0, \mu, \nu)$ is a 2-term representation up to homotopy of $g$, then, since for any $X \in g$ we have

$$d \circ \mu(X) = \mu(X) \circ d,$$

we obtain that $\text{Im}(\mu) \subset \text{Ker}(\delta)$. Furthermore, it is straightforward to see that the conditions (3) and (4) are equivalent to (12) and (5) is equivalent to (13). \hfill \blacksquare

## 4 Integrating the crossed module of Lie algebras $End(V)$

The Lie algebra structure on $k_0$ given by (24) is very clear. The Lie group $K_0$ defined by

$$K_0 = \left\{ \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} : B_0 \in GL(V_0), B_1 \in GL(V_1) \text{ such that } B_0 \circ d = d \circ B_1 \right\} \quad (27)$$

is a Lie group whose Lie algebra is $k_0$.

The Lie algebra structure on $t_1$ given by (23), i.e. the Lie bracket $[\cdot, \cdot]_{t_1}$ is less clear. We consider the integration of the Lie algebra $t_1$ and the Lie algebra morphism $\delta$ by embedding $t_1$ into the Lie algebra $gl(V_0 \oplus V_1)$.

**Theorem 4.1.** The set $K_1$ given by

$$K_1 = \{M \in End(V_0, V_1) \text{ such that } \begin{pmatrix} I + d \circ M & 0 \\ 0 & M \circ d + I \end{pmatrix} \in GL(V_0 \oplus V_1) \}, \quad (28)$$

is a Lie group with multiplication

$$M_1 \cdot M_2 = M_1 + M_2 + M_1 \circ d \circ M_2, \quad (29)$$

identity $0$ and inverse

$$M^{-1} = -(I + M \circ d)^{-1} \circ M = -M \circ (I + d \circ M)^{-1}. \quad (30)$$
Its Lie algebra is \( \mathfrak{k}_1 \) given by \[ (\int \delta)(M) = \begin{pmatrix} I + d \circ M & 0 \\ 0 & M \circ d + I \end{pmatrix}, \] (30)
differentiates to the Lie algebra morphism \( \delta : \mathfrak{k}_1 \to \mathfrak{k}_0 \). The action \( \Phi \) of the Lie group \( K_0 \) on \( K_1 \) given by
\[
\Phi \left( \begin{array}{cc} B_0 & 0 \\ 0 & B_1 \end{array} \right) M = B_1 \circ M \circ B_0^{-1}, \quad \left( \begin{array}{cc} B_0 & 0 \\ 0 & B_1 \end{array} \right) \in K_0, \quad M \in K_1,
\] (31)
differentiates to the action \( \phi \) of Lie algebra \( \mathfrak{k}_0 \) on \( \mathfrak{k}_1 \) in (25). Furthermore,
\[
\text{Aut}(\mathcal{V}) \cong (K_1, K_0, \int \delta, \Phi)
\] (32)
is a crossed module of Lie groups whose differentiation is the crossed module of Lie algebras \( \text{End}(\mathcal{V}) \) given by (26).

**Proof.** It is not hard to see that the group structure of \( K_1 \) has the property that the injective map \( M \mapsto \begin{pmatrix} I + d \circ M & 0 \\ 0 & M \circ d + I \end{pmatrix} \) is a group morphism from \( K_1 \) to \( \text{GL}(\mathcal{V}_0 \oplus \mathcal{V}_1) \). It is obvious that 0 is the identity of the multiplication (29). It is also not hard to see that
\[
M \cdot (-I + M \circ d)^{-1} \circ M = M + (I + M \circ d)(-I + M \circ d)^{-1} \circ M = 0,
\]
\[
(-M \circ (I + d \circ M)^{-1}) \cdot M = (-M \circ (I + d \circ M)^{-1})(I + d \circ M) + M = 0.
\]
Furthermore, the equality
\[
M + M \circ d \circ M = M \circ (I + d \circ M) = (I + M \circ d) \circ M
\]
yields that
\[
-(I + M \circ d)^{-1} \circ M = -M \circ (I + d \circ M)^{-1}.
\]
We can view \( K_1 \) as a subgroup of \( \text{GL}(\mathcal{V}_0 \oplus \mathcal{V}_1) \).

Now we identify \( \mathfrak{t}_1 \) and \( K_1 \) to their images in the corresponding bigger matrix spaces. By exponentiating, we have
\[
\exp \left( \begin{pmatrix} d \circ A & 0 \\ A \circ d \end{pmatrix} \right) = \begin{pmatrix} I + d \circ M & 0 \\ 0 & M \circ d + I \end{pmatrix},
\]
where
\[
M = A + \frac{A \circ d \circ A}{2!} + \frac{A \circ d \circ A \circ d \circ A}{3!} + \cdots.
\]

Since the exponential map is an isomorphism near 0, by [26, Prop.3.18], \( K_1 \) is a Lie sub-group of \( \text{GL}(\mathcal{V}_0 \oplus \mathcal{V}_1) \) whose Lie algebra is \( \mathfrak{t}_1 \). It is not hard to see that the differentiation of \( \int \delta \) given by (30) is the \( \delta \) given by (18). Since the Lie algebra action is given by the commutator, it follows that the group action is given by the adjoint action, which turns out to be (31). In other words, (31) is constructed so that its differentiation is the \( \phi \) in (25).
Finally, we prove that \( \text{Aut}(V) = (K_1, K_0, \int \delta, \Phi) \) is a crossed module of Lie groups. First of all, we have
\[
(\int \delta) \Phi \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} M = (\int \delta)(B_1 \circ M \circ B_0^{-1}) = \begin{pmatrix} I + d \circ B_1 \circ M \circ B_0^{-1} \\ 0 \end{pmatrix} \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 \\ I + B_1 \circ M \circ B_0^{-1} \circ d \end{pmatrix} = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 \\ I + d \circ M \circ B_0^{-1} \circ d \end{pmatrix} = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 \\ I + d \circ M \circ B_0^{-1} \circ d \end{pmatrix} ^{-1}.
\]
Furthermore, we have
\[
\Phi(\int \delta)(M') = \Phi \begin{pmatrix} I + d \circ M & 0 \\ 0 & I + M \circ d \end{pmatrix} M' = (I + M \circ d) \circ M' \circ (I + d \circ M)^{-1}.
\]
On the other hand, using the facts that \( M^{-1} = -M \circ (I + d \circ M)^{-1} \) and \( I = (I + d \circ M) \circ (I + d \circ M)^{-1} \), we have
\[
M \cdot M' \cdot M^{-1} = (M + M' + M \circ d \circ M') \cdot (-M \circ (I + d \circ M)^{-1}) = M + M' + M \circ d \circ M' - M \circ (I + d \circ M)^{-1} - M \circ d \circ M \circ (I + d \circ M)^{-1}
\]
\[
= M' \circ M \circ (I + d \circ M)^{-1} - M \circ d \circ M \circ (I + d \circ M)^{-1} + M \circ d \circ M' \circ (I - d \circ M \circ (I + d \circ M)^{-1})
\]
\[
= M' \circ (I + d \circ M)^{-1} + M \circ d \circ M' \circ (I + d \circ M)^{-1}.
\]
Thus we have
\[
\Phi(\int \delta)(M') = M \cdot M' \cdot M^{-1},
\]
which implies that \( \int \delta \) satisfies the Pfeiffer identity. Therefore, \( \text{Aut}(V) = (K_1, K_0, \int \delta, \Phi) \) is a crossed module of Lie groups. \( \blacksquare \)

5 Integrating semidirect product Lie 2-algebras

5.1 Semidirect product Lie 2-groups–strict case

Let \( \mathbb{H} = (H_1, H_0, t, \Phi) \) and \( \mathbb{G} = (G_1, G_0, t, \Phi) \) be crossed modules of Lie groups, and let \( V = (V_1 \xrightarrow{d} V_0) \) be a 2-term complex of vector spaces. In the following, \( x, y, z \) are elements of \( H_0 \), \( a, b, c \) are elements of \( H_1 \), \( \xi, \eta, \gamma \) are elements of \( V_0 \), and \( m, n, p, q \) are elements of \( V_1 \). We omit the group multiplication \( \cdot \) in \( H_i \)'s, \( K_i \)'s, and \( G_i \)'s, and denote the composition of morphisms by \( \circ \).

**Definition 5.1.** A strict morphism from \( \mathbb{H} \) to \( \mathbb{G} \) is a pair \((F_1, F_0)\), where \( F_i : H_i \rightarrow G_i \) are morphisms of Lie groups, such that
\[
F_0 \circ t = t \circ F_1, \quad F_1(\Phi_x a) = \Phi_{F_0(x)} F_1(a).
\]
Let \((\Psi_1, \Psi_0)\) be a strict morphism of crossed modules of Lie groups from \(\mathbb{H}\) to \(\text{Aut}(V)\). By Definition 5.1 and Theorem 4.1, we have

\[
\Psi_1(\Phi x a) = \Psi_0(x) \circ \Psi_1(a) \circ (\Psi_0(x))^{-1},
\]

and

\[
\int \delta \Psi_1(a) = \begin{pmatrix}
I + d \circ \Psi_1(a) & 0 \\
\Psi_1(a) \circ d + I
\end{pmatrix} = \Psi_0(t(a)).
\]

More precisely, for any \(\xi \in V_0\) and \(m \in V_1\), we have

\[
\xi + d \circ \Psi_1(a)(\xi) = \Psi_0(t(a))(\xi),
\]

and

\[
m + \Psi_1(a) \circ dm = \Psi_0(t(a))(m).
\]

**Lemma 5.2.** For any \(a, b \in H_1\), we have

\[
\Psi_1(ab) = \Psi_1(a) + \Psi_0(t(a)) \circ \Psi_1(b).
\]

**Proof.** By (29) and the fact that \(\Psi_1\) is a morphism, we have

\[
\Psi_1(ab) = \Psi_1(a) \Psi_1(b) = \Psi_1(a) + \Psi_1(b) + \Psi_1(a) \circ d \circ \Psi_1(b) = \Psi_1(a) + (\int \delta \Psi_1(a) \circ \Psi_1(b) = \Psi_1(a) + \Psi_0(t(a)) \circ \Psi_1(b).
\]

**Lemma 5.3.** For any \(a \in H_1\), we have

\[
\Psi_0(t(a^{-1})) \circ \Psi_1(a) + \Psi_1(a^{-1}) = 0.
\]

**Proof.** By (29), we have

\[
0 = \Psi_1(a^{-1} a) = \Psi_1(a^{-1}) \Psi_1(a) = \Psi_1(a) + \Psi_1(a^{-1}) + \Psi_1(a^{-1}) \circ d \circ \Psi_1(a).
\]

Thus,

\[
\Psi_0(t(a^{-1})) \circ \Psi_1(a) + \Psi_1(a^{-1}) = (\int \delta \Psi_1(a^{-1})) \circ \Psi_1(a) - \Psi_1(a) - \Psi_1(a^{-1}) \circ d \circ \Psi_1(a) = (I + \Psi_1(a^{-1}) \circ d) \circ \Psi_1(a) - \Psi_1(a) - \Psi_1(a^{-1}) \circ d \circ \Psi_1(a) = 0.
\]

**Theorem 5.4.** Given a strict morphism \((\Psi_1, \Psi_0)\) of crossed modules of Lie groups from \((H_1, H_0, t, \Phi)\) to \(\text{Aut}(V)\) given by (32), there is a strict Lie 2-group

\[
\begin{array}{ccc}
H_0 \times H_1 \times V_0 \times V_1 & \xrightarrow{s} & t \\
\downarrow & & \\
H_0 \times V_0,
\end{array}
\]

(37)
in which the source and the target maps are given by
\[ s(x, a, \xi, m) = (x, \xi), \]
\[ t(x, a, \xi, m) = (t(a)x, \xi + dm), \]
the vertical multiplication \( \cdot_v \) is given by
\[ (y, b, \eta, n) \cdot_v (x, a, \xi, m) = (x, ba, \xi, m + n), \quad y = t(a)x, \quad \eta = \xi + dm, \]
the horizontal multiplication \( \cdot_h \) of morphisms is defined by
\[ (x, a, \xi, m) \cdot_h (y, b, \eta, n) = \left( xy, a\Phi_x b, \xi + \Psi_0(x)(\eta), m + \Psi_0(t(a)x)(n) + \Psi_1(a) \circ \Psi_0(x)(\eta) \right), \]
and the horizontal multiplication \( \cdot_h \) of objects is defined by
\[ (x, \xi) \cdot_h (y, \eta) = (xy, \xi + \Psi_0(x)(\eta)). \]
The identities of arrows and objects are \((1_{H_0}, 1_{H_1}, 0, 0)\) and \((1_{H_0}, 0)\) respectively. The inverse of \((x, a, \xi, m)\) with respect to \( \cdot_h \) is
\[ \left( x^{-1}, \Phi_{x^{-1}}, e^{-1}, -\Psi_0(x^{-1})(\xi), -\Psi_0((t(a)x)^{-1})(m) + \Psi_0((t(a)x)^{-1}) \circ \Psi_1(a)(\xi) \right). \]

We call \((\psi_1, \psi_0)\) a representation on \( \mathbb{V} \) of the crossed module \((H_1, H_0, t, \Phi)\), and the strict Lie 2-group \([37]\) the semidirect product of \((H_1, H_0, t, \Phi)\) with this representation.

**Proof.** Obviously, the horizontal multiplication respects the source map. By \([34]\) and \([35]\), the horizontal multiplication also respects the target map.

To see the horizontal multiplication is indeed a functor, we need to show that
\[ \left( t(a)x, b, \xi + dm, n \right) \cdot_h \left( t(c)y, d, \eta + dp, q \right) \cdot_v \left( x, a, \xi, m \right) \]
is equal to
\[ \left( t(a)x, b, \xi + dm, n \right) \cdot_h \left( t(c)y, d, \eta + dp, q \right) \cdot_v \left( x, a, \xi, m \right), \]
where \( x, y \in H_0, \ a, b, c, d \in H_1, \ \xi, \eta, \gamma \in V_0, \) and \( m, n, p, q \in V_1. \) By straightforward computations, the first expression is equal to
\[ \left( xy, ba\Phi_x dc, \xi + \Psi_0(x)(\eta), m + n + \Psi_0(t(ba)x)(p + q) + \Psi_1(ba) \circ \Psi_0(x)(\eta) \right). \quad (38) \]
The second expression is equal to
\[ \left( xy, (b\Phi_{t(a)x} d)(a\Phi_x c), \xi + \Psi_0(x)(\eta), \right) \]
\[ m + n + \Psi_0(t(a)x)(p) + \Psi_0(t(ba)x)(q) + \Psi_1(a) \circ \Psi_0(x)(\eta) + \Psi_1(b) \circ \Psi_0(t(a)x)(\eta + dp) \].

By the definition of crossed modules, it is not hard to see that
\[ (b\Phi_{t(a)x} d)(a\Phi_x c) = b\Phi_{t(a)}(\Phi_x d)a\Phi_x c = ba(\Phi_x d)a^{-1}a\Phi_x c = ba\Phi_x (dc). \]
Thus we only need to prove that
\[ \Psi_0(t(ba)x)(p) + \Psi_1(ba) \circ \Psi_0(x)(\eta) \]
\[ = \Psi_0(t(a)x)(p) + \Psi_1(a) \circ \Psi_0(x)(\eta) + \Psi_1(b) \circ \Psi_0(t(a)x)(\eta + dp). \]  
(39)

By (29), we have
\[ \Psi_1(ba) = \Psi_1(b) + \Psi_1(a) \circ d \circ \Psi_1(a). \]  
(40)

So it suffices to prove that
\[ \Psi_0(t(ba)x)(p) = \Psi_0(t(a)x)(p) + \Psi_1(b) \circ \Psi_0(t(a)x)(dp), \]
and \[ \Psi_1(b) \circ \Psi_0(t(a)x)(\eta) = \Psi_1(b) \circ \Psi_0(x)(\eta) + \Psi_1(b) \circ d \circ \Psi_1(a) \circ \Psi_0(x)(\eta), \]
which hold if and only if for any \( \xi \in V_0, m \in V_1 \), we have
\[ \{ \int \delta \}(\Psi_1(b))(m) = I + \Psi_1(b) \circ dm, \]
\[ \{ \int \delta \}(\Psi_1(a))(\xi) = I + d \circ \Psi_1(a)(\xi). \]

These are exactly (31) and (35). Thus the horizontal multiplication \( \cdot \) is indeed a functor.

To see that the horizontal multiplication \( \cdot \) is strictly associative, we only need to check it on the space of arrows, i.e. we need to verify that
\[ ((x, a, \xi, m) \cdot (y, b, \eta, n)) \cdot (z, c, \gamma, p) = (x, a, \xi, m) \cdot ((y, b, \eta, n) \cdot (z, c, \gamma, p)). \]  
(41)

By straightforward computations, we obtain that the left hand side is equal to
\[ \left( xy z, a(\Phi_x b)(\Phi_y c), \xi + \Psi_0(x)(\eta) + \Psi_0(xy)(\gamma), \ight. \]
\[ m + \Psi_0(t(a)x)(n) + \Psi_1(a) \circ \Psi_0(x)(\eta) + \Psi_0(t(a)bx)(p) + \Psi_1(a) \circ \Psi_0(xy)(\gamma) \}
\[ + \Psi_0(t(a)x) \circ \Psi_0(t(b)y)(p) + \Psi_0(t(a)x) \circ \Psi_1(b) \circ \Psi_0(y)(\gamma) + \Psi_1(a) \circ \Psi_0(x) \circ \Psi_0(y)(\gamma) \} \]

and the right hand side is equal to
\[ \left( xy z, a(\Phi_x b(\Phi_y c), \xi + \Psi_0(x)(\eta) + \Psi_0(x)(\gamma), m + \Psi_0(t(a)x)(n) + \Psi_1(a) \circ \Psi_0(x)(\eta) \ight. \]
\[ + \Psi_0(t(a)x) \circ \Psi_0(t(b)y)(p) + \Psi_0(t(a)x) \circ \Psi_1(b) \circ \Psi_0(y)(\gamma) + \Psi_1(a) \circ \Psi_0(x) \circ \Psi_0(y)(\gamma) \} \]

Since \( \Psi_0 \) is a morphism of Lie groups and \( \Phi \) acts as an automorphism, we only need to show that
\[ \Psi_1(a\Phi_x b) \circ \Psi_0(xy)(\gamma) = \Psi_0(t(a)x) \circ \Psi_1(b) \circ \Psi_0(y)(\gamma) + \Psi_1(a) \circ \Psi_0(x) \circ \Psi_0(y)(\gamma). \]

Since \( \Psi_1(a\Phi_x b) = \Psi_1(a) + \Psi_1(\Phi_x b) + \Psi_1(a) \circ d \circ \Psi_1(\Phi_x b), \) it is equivalent to
\[ \Psi_1(\Phi_x b) \circ \Psi_0(x) \circ \Psi_0(y)(\gamma) + \Psi_1(a) \circ d \circ \Psi_1(\Phi_x b) \circ \Psi_0(x) \circ \Psi_0(y)(\gamma) \]
\[ = \Psi_0(t(a)x) \circ \Psi_1(b) \circ \Psi_0(y)(\gamma), \]
which holds by (35).
Finally, we show that \((1_{H_0}, 1_{H_1}, 0, 0)\) and \((1_{H_0}, 0)\) are identities of arrows and objects respectively. It is straightforward to see that for any \((x, a, \xi, m)\), we have

\[
(x, a, \xi, m)(x^{-1}, \Phi_{x^{-1}} a^{-1}, -\Psi_0(x^{-1})(\xi), -\Psi_0((t(a)x)^{-1})(m) + \Psi_0((t(a)x)^{-1}) \circ \Psi_1(a)(\xi)) = (1_{H_0}, 1_{H_1}, 0, 0).
\]

On the other hand, we have

\[
(x^{-1}, \Phi_{x^{-1}} a^{-1}, -\Psi_0(x^{-1})(\xi), -\Psi_0((t(a)x)^{-1})(m) + \Psi_0((t(a)x)^{-1}) \circ \Psi_1(a)(\xi))(x, a, \xi, m) = (1_{H_0}, 1_{H_1}, 0, \Psi_0((t(a)x)^{-1}) \circ \Psi_1(a)(\xi) + \Psi_1(\Phi_{x^{-1}} a^{-1}) \circ \Psi_0(x^{-1})(\xi)).
\]

To see that \((x^{-1}, \Phi_{x^{-1}} a^{-1}, -\Psi_0(x^{-1})(\xi), -\Psi_0((t(a)x)^{-1})(m) + \Psi_0((t(a)x)^{-1}) \circ \Psi_1(a)(\xi))\) is the inverse of \((x, a, \xi, m)\), we need to prove that

\[
\Psi_0((t(a)^{-1})) \circ \Psi_1(a)(\xi) + \Psi_1(a^{-1})(\xi) = 0. \tag{42}
\]

This is exactly Lemma 5.3. The proof is finished. ■

By Theorem 2.9, we have

**Corollary 5.5.** Given a strict morphism \((\Psi_1, \Psi_0)\) of crossed modules of Lie groups from \((H_1, H_0, t, \Phi)\) to \(\text{Aut}(V)\) as in (23), there is a crossed module of Lie groups \(\left( H_1 \ltimes V_1, H_0 \ltimes V_0, t \times d, \Phi \right)\), where \(t \times d : H_1 \ltimes V_1 \to H_0 \ltimes V_0\) is given by

\[
(t \times d)(a, m) = (t(a), dm),
\]

the group structure on \(H_1 \ltimes V_1\) is given by

\[
(a, m)(b, n) = (ab, m + \Psi_0(t(a))(n)),
\]

the group structure on \(H_0 \ltimes V_0\) is the semidirect product, i.e.

\[
(x, \xi)(y, \eta) = (xy, \xi + \Psi_0(x)(\eta)),
\]

and the action of \(H_0 \ltimes V_0\) on \(H_1 \ltimes V_1\) is given by

\[
\Phi_{(x, \xi)}(a, m) = \left( \Phi_x a, \Psi_0(x)(m) - \Psi_0(x) \circ \Psi_1(a) \circ \Psi_0(x^{-1})(\xi) \right). \tag{43}
\]

**Proof.** It is not hard to see that the condition that \(\Phi\) acts as an automorphism, i.e.

\[
\Phi_{(x, \xi)}((a, n)(b, p)) = \Phi_{(x, \xi)}((a, n)) \Phi_{(x, \xi)}(b, p),
\]

is equivalent to the condition

\[
\Psi_1(ab) = \Psi_0(t(a)) \circ \Psi_1(b) + \Psi_1(a),
\]

which is proved in Lemma 5.2. Furthermore, it is not hard to see that the morphism \(t \times d\) is \(H_0 \ltimes V_0\)-equivariant. The Pfeiffer identity holds by (34), (35) and (33). ■

Take differentiation we also obtain the infinitesimal version of Corollary 5.5.
Theorem 5.6. Given a strict morphism \((\psi_1, \psi_0)\) of crossed modules of Lie algebras from \(\mathfrak{h}\) to \(\text{End}(V)\), there is a crossed module of Lie algebras

\[
(\mathfrak{h}_1 \ltimes V_1, \mathfrak{h}_0 \ltimes V_0, dt \times d, \phi),
\]

(44)

where the Lie algebra structure on \(\mathfrak{h}_1 \ltimes V_1\) is given by

\[
[(A,m), (B,n)] = (\left[ A, B \right], \psi_0(dt(A))(n) - \psi_0(dt(B))(m)),
\]

the Lie algebra structure on \(\mathfrak{h}_0 \ltimes V_0\) is the semidirect product

\[
[(X,\xi), (Y,\eta)] = (\left[ X, Y \right], \psi_0(X)(\eta) - \psi_0(Y)(\xi)),
\]

the Lie algebra morphism \(dt \times d\) is given by

\[
(dt \times d)(A,m) = (dt(A), dm),
\]

and the action \(\phi\) is given by

\[
\phi_{(x,\xi)}(A,m) = (\phi_A X, \psi_0(X)(m) - \psi_1(A)(\xi)).
\]

Let \((H_1, H_0, t, \Phi)\) be the simply connected integration of \((\mathfrak{h}_1, \mathfrak{h}_0, dt, \phi)\) and \((\Psi_1, \Psi_0)\) is the integration (see Remark 5.7) of \((\psi_1, \psi_0)\). Then the crossed module of Lie groups \((H_1 \ltimes V_1, H_0 \ltimes V_0, t \times d, \Phi)\) given in Corollary 5.5 is the simply connected integration of \((\mathfrak{h}_1 \ltimes V_1, \mathfrak{h}_0 \ltimes V_0, dt \times d, \phi)\).

We call \((\psi_1, \psi_0)\) a representation of \(\mathfrak{h}\) on \(V\), and the Lie 2-algebra corresponding to (44) the semidirect product of \(\mathfrak{h}\) with this representation.

Remark 5.7. Lie’s II and III theorems hold for crossed modules. Given a crossed module of Lie algebras \(\mathfrak{h}\), there is a unique crossed module of Lie groups \(\mathbb{H} = (H_1, H_0, t, \Phi)\) such that its differentiation is \(\mathfrak{h}\) and \(H_i\)'s are connected and simply connected. We call \(\mathbb{H}\) the simply connected integration of \(\mathfrak{h}\). Moreover, given any crossed module of Lie groups \(G\) whose differentiation is \(\mathfrak{g}\), a morphism of crossed module of Lie algebras \((\psi_1, \psi_0) : \mathfrak{h} \rightarrow \mathfrak{g}\) can be integrated to a morphism of crossed module of Lie groups \((\Psi_1, \Psi_0) : \mathbb{H} \rightarrow G\) where \(\mathbb{H}\) is the simply connected integration of \(\mathfrak{h}\). See [8] for more details about the integration of morphisms of crossed modules.

If in Corollary 5.5 we take \(\Psi_0, \Psi_1\) to be the identity map, then we obtain the crossed module of Lie groups

\[
(K_1 \ltimes V_1, K_0 \ltimes V_0, \int \delta \times d, \Phi),
\]

which plays the role of \(GL(V) \ltimes V\) in the classical case of a vector space \(V\) acted upon by \(GL(V)\), where \(K_1\) and \(K_0\) are given by (28) and (27). If in Theorem 5.6 we take \(\psi_0, \psi_1\) to be the identity map, then we also obtain a crossed module of Lie algebras

\[
(\mathfrak{k}_1 \ltimes V_1, \mathfrak{k}_0 \ltimes V_0, \delta \times d, \phi),
\]

which plays the role of \(\mathfrak{gl}(V) \ltimes V\) in the classical case of a vector space \(V\) acted upon by \(\mathfrak{gl}(V)\), where \(\mathfrak{k}_1\) and \(\mathfrak{k}_0\) are given by (23) and (21).
5.2 Nonstrict case

**Definition 5.8.** A strict Lie 2-group is called an integration of a Lie 2-algebra if its differentiation is a Lie 2-algebra equivalent to the given Lie 2-algebra.

**Remark 5.9.** When one studies the integration of the string Lie 2-algebras \([7, 13, 19]\), the models one finds are (only equivalent) not the same, as 2-groups. Equivalence of 2-groups corresponds to equivalence of Lie 2-algebras on the infinitesimal level. Thus, motivated by the examples arising from the integration of string Lie 2-algebras, we define our integration up to equivalence as above.

The final aim of this paper is to integrate the Lie 2-algebra which is the semidirect product of a Lie algebra \(g\) with a 2-term representation up to homotopy \(V\) (Proposition 2.3). By Theorem 3.2 a 2-term representation up to homotopy is equivalent to a nonstrict morphism \((\mu, \nu)\) from \(g\) to \(\text{End}(V)\). Moreover, in the last subsection we study the integration of the Lie 2-algebra which is the semidirect product of \(h\) with \(V\) via a strict morphism. Thus we give the integration of the Lie 2-algebra which is the semidirect product of a Lie algebra \(g\) with a 2-term representation up to homotopy \(V\) by strictifying the nonstrict morphism \((\mu, \nu)\) via the corresponding butterfly.

For any butterfly \(e\) from \(g\) to \(h\),

![Diagonal triangle]

we obtain a crossed module of Lie algebras

\[
(g_1 \times_{g_0} e, e, dt, \phi),
\]

where \(g_1 \times_{g_0} e\) is short for the fibre product \(g_1 \times_{dt, g_0, \sigma} e\). The Lie algebra structure on \(g_1 \times_{g_0} e\) is given by

\[
[(A_1, e_1), (A_2, e_2)] = ([A_1, A_2], [e_1, e_2]),
\]

the Lie algebra morphism \(dt : g_1 \times_{g_0} e \to e\) are defined by

\[
dt(A, e) = e, \quad \forall A \in g_1, \quad e \in e, \quad dt(A) = \sigma(e),
\]

and the action \(\phi\) of \(e\) on \(g_1 \times_{g_0} e\) is given by

\[
\phi_e(A, e_1) = (\phi_{\sigma(e)} A, [e, e_1]).
\]

Furthermore, there are two strict morphisms of crossed modules of Lie algebras, which are
(pr₁, σ) and (ψ₁, ψ₀),

\[
\begin{array}{c}
g_1 \times_{g_0} e \\
\psi_1 \\
g_1 \\
\kappa \\
dt \\
h_1 \\
\sigma \\
p = ψ₀ \\
g_0 \\
\rho = ψ₀
\end{array}
\]

where ψ₀ = ρ : e → h₀ and ψ₁ : g₁ ×ₕ₀ e → h₁ is defined by

\[
ψ₁(A, e) = e - κ(A), \quad ∀ A ∈ g₁, e ∈ e.
\]

The following conclusion is straightforward.

**Proposition 5.10.** Given a butterfly (42), the map (pr₁, σ) constructed in the diagram (47) is a strict morphism from the crossed module \((g_1 × g_0 e, e, dt, φ)\) to \(g\). The map (pr₁, σ) is also an equivalence of the underlying 2-term complexes. Thus, the corresponding Lie 2-algebras are equivalent.

At the end of this section, we focus on nonstrict morphisms \((µ, ν)\) from a Lie algebra \(g\), which is viewed as a trivial crossed module of Lie algebras, to the crossed module of Lie algebras \(\text{End}(V)\), which is given by (26). Obviously, the corresponding butterfly is \(ε = g ⊕ \mathfrak{t}₁\) given by (16). The crossed module of Lie algebras given by (46) turns out to be \((σ⁻¹(0), g ⊕ \mathfrak{t}₁, dt, φ)\). Moreover, it is straightforward to see that \(σ⁻¹(0)\) is exactly \(\mathfrak{t}₁\). Thus the corresponding crossed module of Lie algebras is

\[(\mathfrak{t}₁, g ⊕ \mathfrak{t}₁, i₂, \text{ad}),\]

where the Lie algebra structure on \(g ⊕ \mathfrak{t}₁\) is given by

\[
[(X, A), (Y, B)] = [(X, Y), [µ(X), B] + [A, µ(Y)] + [A, B]_{\mathfrak{t}₁} - ν(X, Y)],
\]

and the adjoint action \(\text{ad}\) is given by

\[
\text{ad}_{(X, A)} B = φ_{µ(X)} B + [A, B]_{\mathfrak{t}₁} = [µ(X), B] + A ∘ d(B) - B ∘ d(A).
\]

Furthermore, the Lie algebra morphism \(ψ₁ : \mathfrak{t}₁ → \mathfrak{t}₁\) given by (48) is exactly the identity map \(\text{Id}\) and the Lie algebra morphism \(ψ₀ = ρ : g ⊕ \mathfrak{t}₁ → \mathfrak{t}₀\) is given by

\[
ψ₀(X, A) = µ(X) + δ(A).
\]

The semidirect product of (49) with \((ψ₁, ψ₀)\) as in Theorem 5.6 is

\[(\mathfrak{t}₁ × V₁, (g ⊕ \mathfrak{t}₁) × V₀, i₂ × d, \tilde{φ}),\]

where the action \(\tilde{φ}\) is given by

\[
\tilde{φ}_{(X, A, ξ)}(B, m) = (\text{ad}_{(X, A)} B, (µ(X) + δ(A))(m) - Bξ).
\]
Theorem 5.11. With the above notations, given a (nonstrict) morphism of crossed modules \((\mu, \nu) : g \to \text{End}(V)\), we have a strict morphism \((\psi_1, \psi_0) = (\text{id}, \rho)\) from the crossed module defined by \([\mathfrak{M}]\) to \(\text{End}(V)\). The semidirect product Lie 2-algebra \([\mathfrak{M}]\) is equivalent to the Lie 2-algebra \(g \ltimes V\) given by Proposition \([\mathfrak{M}]\).

Proof. We only need to show that the Lie 2-algebras \((\mathfrak{t}_1 \ltimes V_1, (g \oplus \mathfrak{t}_1) \ltimes V_0, i_2 \times d, \tilde{\phi})\) and \(g \ltimes V\) are equivalent. Define \(f_0 : (g \oplus \mathfrak{t}_1) \ltimes V_0 \to g \ltimes V_0\) by
\[
f_0(X, A, \xi) = (X, \xi), \quad \forall (X, A) \in g \oplus \mathfrak{t}_1, \ \xi \in V_0,
\]
define \(f_1 : \mathfrak{t}_1 \ltimes V_1 \to V_1\) by
\[
f_1(A, m) = m, \quad \forall A \in \mathfrak{t}_1, \ m \in V_1.
\]
Obviously, \((f_0, f_1)\) respects the differential, i.e
\[
(0 + d) \circ f_1 = f_0 \circ (i_2 \times d).
\]
Define \(f_2 : \wedge^2((g \oplus \mathfrak{t}_1) \ltimes V_0) \to V_1\) by
\[
f_2((X, A, \xi), (Y, B, \eta)) = A\eta - B\xi.
\]
By straightforward computations, we have
\[
f_0((X, A, \xi), (Y, B, \eta)) - [f_0(X, A, \xi), f_0(Y, B, \eta)] = [\rho(X, A)(\eta) - \rho(Y, B)(\xi) - (\mu(X)(\eta) - \mu(Y)(\xi))
= d \circ A\eta - d \circ B\xi
= df_2((X, A, \xi), (Y, B, \eta)).
\]
On the other hand,
\[
f_1(\phi_{(X, A, \xi)})(B, m) = f_1(\text{ad}_{(X, A)}B, \rho(X, A)(m) - \psi_1(B)(\xi))
= (\mu(X) + \delta(A))(m) - B\xi
= \mu(X)(m) + A \circ dm - B\xi,
\]
\[
\phi_{f_0(X, A, \xi)}f_1(B, m) = \phi_{(X, \xi)}m = \mu(X)(m).
\]
Thus
\[
f_1(\phi_{(X, A, \xi)})(B, m) - \phi_{f_0(X, A, \xi)}f_1(B, m) = A \circ dm - B\xi
= f_2((X, A, \xi), (i_2 \times d)(B, m)).
\]
Finally, it is straightforward to obtain that
\[
f_2([([X, A, \xi], (Y, B, \eta), (Z, C, \gamma)] + c.p.
= [f_0(X, A, \xi), f_2((Y, B, \eta), (Z, C, \gamma))] + c.p.
= \mu(X)(B\gamma - C\eta) + \mu(Y)(C\xi - A\gamma) + \mu(Z)(A\eta - B\xi),
\]
which implies that \((f_0, f_1, f_2)\) is a Lie 2-algebra morphism.

Furthermore, it is obvious that \((f_0, f_1)\) also induces an equivalence of the underlying complexes of vector spaces. Thus the Lie 2-algebra \((\mathfrak{t}_1 \ltimes V_1, (g \oplus \mathfrak{t}_1) \ltimes V_0, i_2 \times d, \tilde{\phi})\) given by \([\mathfrak{M}]\) is equivalent to the Lie 2-algebra \(g \ltimes V\) given by Proposition \([\mathfrak{M}]\). 

By Theorem 5.10 and Theorem 5.11, we have
Corollary 5.12. The strict Lie 2-group corresponding to the simply connected integration of the crossed module of Lie algebras is an integration of the Lie 2-algebra $g \ltimes V$.

Remark 5.13. In Corollary 5.12, we construct a strict Lie 2-group integrating the Lie 2-algebra $g \ltimes V$. In [24, Theorem 4.4], the Lie 2-group constructed to integrate the string type Lie algebra $\mathbb{R} \to g \oplus g^*$ is not strict since the associator is not trivial. Thus this nonstrict Lie 2-group is equivalent to a strict Lie 2-group, which is the integration result given in Corollary 5.12.

A sub-Lie algebra of a Lie algebra is defined by an injective Lie algebra morphism. Similarly, we have:

Definition 5.14. Let $f : L \to W$ be a Lie 2-algebra morphism as in Definition 2.10. Then $(L, f)$ is called a sub-Lie-2-algebra of $W$ if $f_0$ and $f_1$ are injective. When the inclusion map $f$ is obvious, we also call that $L$ is a sub-Lie-2-algebra of $W$.

Remark 5.15. When $f_0$ and $f_1$ are injective, the linear category corresponding to $L$ is a subcategory of the one corresponding to $W$. In [6], the notion of Lie sub-2-algebra is used and it is also called a sub-Lie-2-algebra. We did not find its exact definition. It seems that [6] requires a Lie sub-2-algebra to be a subcomplex which is closed under the differential and the brackets. If this is the case, a Lie sub-2-algebra of a strict Lie 2-algebra must be strict. However, our definition of sub-Lie-2-algebra is much weaker as we will see below.

Consider the canonical inclusion map $(i_2, i_1 \times \text{Id})$ from $g \ltimes V$ to the strict Lie 2-algebra, which is given by

\[
i_2(m) = (0, m), \quad \forall \ m \in V_1,
\]
\[
(i_1 \times \text{Id})(X, \xi) = (X, 0, \xi), \quad \forall \ (X, \xi) \in g \oplus V_0.
\]

By straightforward computations, we have

\[
[(i_1 \times \text{Id})(X, \xi), (i_1 \times \text{Id})(Y, \eta)] = ([X, Y], -\nu(X, Y), \mu(X)(\eta) - \mu(Y)(\xi)),
\]
\[
(i_1 \times \text{Id})[(X, \xi), (Y, \eta)] = ([X, Y], 0, \mu(X)(\eta) - \mu(Y)(\xi)).
\]

Therefore,

\[
(i_1 \times \text{Id})[(X, \xi), (Y, \eta)] - [(i_1 \times \text{Id})(X, \xi), (i_1 \times \text{Id})(Y, \eta)] = (i_2 \times \text{Id})(\nu(X, Y), 0).
\]

Define $\tilde{\nu} : \wedge^2 (g \oplus V_0) \to \mathfrak{g} \times V_1$ by

\[
\tilde{\nu}((X, \xi), (Y, \eta)) = (\nu(X, Y), 0).
\]

The following proposition is straightforward.

Proposition 5.16. Given a nonstrict morphism $(\mu, \nu)$ from a Lie algebra $g$ to $\text{End}(V)$, the map $(i_2, i_1 \times \text{Id}, \tilde{\nu})$ is a Lie 2-algebra morphism from $g \ltimes V$ to the strict Lie 2-algebra. Consequently, the Lie 2-algebra $g \ltimes V$ is a sub-Lie-2-algebra of the strict Lie 2-algebra.
6 Integrating the omni-Lie algebra \( \mathfrak{gl}(V) \oplus V \)

The notion of omni-Lie algebra was introduced by A. Weinstein in [27] to characterize Lie algebra structures on a vector space \( V \). On the direct sum space \( \mathfrak{gl}(V) \oplus V \), the nondegenerate symmetric \( V \)-valued pairing \( \langle \cdot , \cdot \rangle \) is given by

\[
\langle (A, u), (B, v) \rangle = \frac{1}{2} (A v + B u),
\]

and the bracket operation \([\cdot , \cdot] \) is given by

\[
[[A, u], (B, v)] = ([A, B], \frac{1}{2} (Av - Bu)).
\]  
(51)

The quadruple \((\mathfrak{gl}(V) \oplus V, \langle \cdot , \cdot \rangle , [[\cdot , \cdot] , [\cdot , \cdot] ))\) is called the omni-Lie algebra associated to the vector space \( V \). A Dirac structure of the omni-Lie algebra \( \mathfrak{gl}(V) \oplus V \) is a maximal isotropic subspace on which the bracket \( [\cdot , \cdot] \) becomes a Lie bracket upon restriction. For any skew-symmetric bilinear operation \([\cdot, \cdot] : V \wedge V \to V, \) the induced linear map \( \text{ad} : V \to \mathfrak{gl}(V) \) is defined by

\[
\text{ad}_u(v) = [u, v], \quad \forall \, u, v \in V.
\]

The graph of the map \( \text{ad} \), which we denote by \( \mathfrak{g}_{\text{ad}} \subset \mathfrak{gl}(V) \oplus V \), is given by

\[
\mathfrak{g}_{\text{ad}} = \{(\text{ad}_u, u) \in \mathfrak{gl}(V) \oplus V | \forall \, u \in V\}.
\]  
(52)

Denote by \( i : \mathfrak{g}_{\text{ad}} \to \mathfrak{gl}(V) \oplus V \) the natural embedding map. Obviously, \( \mathfrak{g}_{\text{ad}} \) is a maximal isotropic subspace of \( \mathfrak{gl}(V) \oplus V \) since the bilinear operation \([\cdot , \cdot] \) is skew-symmetric. It is shown in [27] that \([\cdot, \cdot] \) is a Lie algebra structure on \( V \) if and only if \( \mathfrak{g}_{\text{ad}} \) is a Dirac structure. In this case, the map \( \text{ad} : V \to \mathfrak{gl}(V) \) is a Lie algebra morphism. Consequently, the map

\[
V \to \mathfrak{g}_{\text{ad}} : v \mapsto (\text{ad}_v, v),
\]  
(53)

is a Lie algebra isomorphism.

The factor of \( \frac{1}{2} \) in (51) spoils the Jacobi identity. More precisely, we have

\[
[[[A, u], (B, v)], (C, w)] + c.p. = \frac{1}{4} ([A, B]w + [B, C]u + [C, A]v)
\]
\[
\triangleq T((A, u), (B, v), (C, w)).
\]

Thus \([\cdot, \cdot] \) is not a Lie bracket. However, the Jacobiator is an exact term and we can extend the omni-Lie algebra \( \mathfrak{gl}(V) \oplus V \) to the Lie 2-algebra whose degree-0 part is \( \mathfrak{gl}(V) \oplus V \),

\[
\left\{ \begin{array}{c}
V \overset{0+1d}{\longrightarrow} \mathfrak{gl}(V) \oplus V, \\
l_2(e_1, e_2) = [e_1, e_2], & \text{for } e_1, e_2 \in \mathfrak{gl}(V) \oplus V, \\
l_2(e, f) = [e, df], & \text{for } e \in \mathfrak{gl}(V) \oplus V, f \in V, \\
l_3(e_1, e_2, e_3) = -T(e_1, e_2, e_3), & \text{for } e_1, e_2, e_3 \in \mathfrak{gl}(V) \oplus V.
\end{array} \right.
\]  
(54)

such that the Jacobiator is measured by a ternary bracket taking value in the degree-1 part \( V \). This Lie 2-algebra is the semidirect product of the Lie algebra \( \mathfrak{gl}(V) \) with the representation up to homotopy \((\mu, \nu)\) on the 2-term complex \( V \overset{1d}{\longrightarrow} V \),

\[
\mu(A)(u) = \frac{1}{2} Au, \quad \nu(A, B) = \frac{1}{4} [A, B].
\]
By Theorem 3.2, a 2-term representation up to homotopy of a Lie algebra is the same as a morphism from the Lie algebra to a certain crossed module of Lie algebras. In the example of the omni-Lie algebra $\mathfrak{gl}(V) \oplus V$, the morphism is from $\mathfrak{gl}(V)$ to the crossed module $(\mathfrak{gl}(V), \mathfrak{gl}(V), \text{Id}, \text{ad})$. The corresponding butterfly is as follows:

![Butterfly diagram]

The Lie algebra structure on $\mathfrak{gl}(V) \oplus \mathfrak{gl}(V)$ is given by

$$ [(A_1, A_2), (B_1, B_2)] = (\frac{1}{2}[(A_1, B_1) + [A_2, B_2] - \frac{1}{4}[A_1, B_1]), $$

and the maps $\kappa, i_2, \sigma, \rho$ are given by

$$ \kappa(0) = (0, 0), $$
$$ i_2(A) = (0, A), $$
$$ \sigma(A_1, A_2) = A_1, $$
$$ \rho(A_1, A_2) = \frac{1}{2}A_1 + A_2. $$

By Theorem 5.1 we obtain a strict morphism $(\psi_1, \psi_0) = (\text{Id}, \rho)$ from the crossed module of Lie algebras $(\mathfrak{gl}(V), \mathfrak{gl}(V) \oplus \mathfrak{gl}(V), i_2, \text{ad})$ to $(\mathfrak{gl}(V), \mathfrak{gl}(V), \text{Id}, \text{ad})$. Furthermore, by Theorem 5.11 the semidirect product

$$ (\mathfrak{gl}(V) \ltimes V, (\mathfrak{gl}(V) \oplus \mathfrak{gl}(V)) \ltimes V, i_2 \times \text{Id}, \tilde{\phi}) $$

is equivalent to the Lie 2-algebra (54) via the morphism $(f_0, f_1, f_2)$ given by

$$ f_0(A, B, u) = (A, u), $$
$$ f_1(A, u) = u, $$
$$ f_2((A, B, u), (A', B', v)) = Bv - B'u. $$

Here the action $\tilde{\phi}$ is given by

$$ \tilde{\phi}_{(A,B,u)}(C, v) = \left(\frac{1}{2}A + B, C\right), $$

and the Lie algebra $\mathfrak{gl}(V) \ltimes V$ is the semidirect product via the natural action of $\mathfrak{gl}(V)$ on $V$. The Lie algebra structure on $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(V)) \ltimes V$ is given by

$$ [(A_1, A_2, u), (B_1, B_2, v)] = (\frac{1}{2}A_1 + A_2, (A_1, B_1) - (\frac{1}{2}B_1 + B_2)(u)). $$

By Corollary 5.12 we obtain the following crossed module of Lie groups as the integration of the Lie 2-algebra (54)

$$ (\mathcal{G} \ltimes V, \mathcal{G} \ltimes V, (\int i_2) \times \text{Id}, \Phi), $$

(61)
where \( \mathcal{G} \) is the connected and simply connected Lie group of the Lie algebra \( \mathfrak{gl}(V) \), \( \mathfrak{S} \) is the connected and simply connected Lie group of the Lie algebra \( \mathfrak{gl}(V) \oplus \mathfrak{gl}(V) \) with the Lie bracket \( \mathfrak{S} \), \( \Phi \) is the integration of \( \tilde{\phi} \), and \( \int i_2 : \mathcal{G} \to \mathfrak{S} \) is the unique map integrating \( i_2 : \mathfrak{gl}(V) \to \mathfrak{gl}(V) \oplus \mathfrak{gl}(V) \).

We conclude by the following proposition.

**Proposition 6.1.** The strict Lie 2-group corresponding to \( (61) \) is an integration of the Lie 2-algebra \( \mathfrak{S} \) associated to an omni-Lie algebra \( \mathfrak{gl}(V) \oplus V \).

**Remark 6.2.** In general, \( GL(V) \) is neither connected nor simply connected. It has two connected components \( GL(V)_- \) and \( GL(V)_+ \) determined by the sign of the determinant. When \( \dim(V) = 1 \), \( \mathcal{G} = GL(V)_+ = \mathbb{R}^\times \). When \( \dim(V) = 2 \), \( \pi_1(GL(V)_+) = \mathbb{Z} \) and \( \mathcal{G} \) is a \( \mathbb{Z} \)-cover of \( GL(V)_+ \). When \( \dim(V) \geq 3 \), \( \pi_1(GL(V)_+) = \mathbb{Z}_2 \), and \( \mathcal{G} \) is a double cover of \( GL(V)_+ \).

**Remark 6.3.** The Lie 2-algebra \( \mathfrak{S} \) is simply equivalent to the Lie algebra \( \mathfrak{gl}(V) \). Thus by our definition, \( GL(V) \) is also an integration of \( \mathfrak{S} \). However the omni-Lie bracket \( \{ \cdot, \cdot \} \) does not appear in \( \mathfrak{gl}(V) \) anymore, while \( [\cdot, \cdot] \) is a part of the structure in the more complicated equivalent object \( \mathfrak{S} \) as shown in \( (61) \). Moreover by Proposition 5.16, the Lie 2-algebra \( \mathfrak{S} \) is a sub-Lie-2-algebra of \( \mathfrak{S} \). Thus we believe that \( \mathfrak{S} \) is a more meaningful integration of \( \mathfrak{S} \) than simply \( GL(V) \).

Now we study the testing requirement for our construction. By the definition of Dirac structures, we need to extend \( \mathfrak{S}_{\text{ad}} \) to a sub-Lie-2-algebra of \( \mathfrak{S} \). Since the preimage of \( \mathfrak{S}_{\text{ad}} \) under the differential of \( \mathfrak{S} \) is the center of the Lie algebra \( V \), the only possible ways to extend \( \mathfrak{S}_{\text{ad}} \) are described as follow: we take a sub-complex of \( \mathfrak{S} \) of which the degree-0 part is \( \mathfrak{S}_{\text{ad}} \) and the degree-1 part \( W \) is a subspace of the center of the Lie algebra \( V \),

\[
W \xrightarrow{0+i} \mathfrak{S}_{\text{ad}},
\]

where \( i : W \to V \) is the natural inclusion. The Lie 2-algebra structure on \( \mathfrak{S} \) is given by

\[
\begin{align*}
\ell_1((\text{ad}_u, u), (\text{ad}_v, v)) &= ([\text{ad}_{[u,v]}, [u,v]], \text{ad}_{[u,v]}), & \text{for} \ (\text{ad}_u, u), (\text{ad}_v, v) & \in \mathfrak{S}_{\text{ad}}, \\
\ell_2((\text{ad}_u, u), f) &= 0, & \text{for} \ (\text{ad}_u, u) & \in \mathfrak{S}_{\text{ad}}, f \in W, \\
\ell_2(f, g) &= 0, & f, g & \in W, \\
\ell_3 &= 0.
\end{align*}
\]

Obviously, it is a strict Lie 2-algebra. In fact, as a crossed module of Lie algebras, it is isomorphic to the crossed module \((W, V, i, \text{ad})\) via the map \( (53) \). Since \( i : W \to V \) is injective, \((W, V, i, \text{ad})\) is equivalent to the Lie algebra \( V/W \). It is not hard to see that

\[
(i, i) : (W \xrightarrow{0+i} \mathfrak{S}_{\text{ad}}) \longrightarrow (V \xrightarrow{0+i} \mathfrak{gl}(V) \oplus V)
\]

is a strict Lie 2-algebra morphism, i.e. \( i \) and \( i \) preserve the bracket and there is no “\( f_2 \)”-term. Therefore, the strict Lie 2-algebra \( W \xrightarrow{0+i} \mathfrak{S}_{\text{ad}} \) is a sub-Lie-2-algebra of the Lie 2-algebra \( \mathfrak{S} \) (see Definition 5.1.1). Our integration of \( \mathfrak{S} \) consists first of all in passing to an equivalent Lie
which is isomorphic to \( gl \)

Then we obtain a crossed module of Lie algebras: 

\[
\text{2-algebra which is strict, and then in integrating the latter Lie 2-algebra. Since the integration of equivalent Lie 2-algebras gives equivalent Lie 2-groups (see Section 2.4), the integration of our Lie 2-algebra \( GL \) is simply \( G(V)/W \), where \( G(V) \) is the simply connected Lie group of the Lie algebra \( V \). Thus taking \( W = 0 \), we recover exactly the group structure corresponding to \( V \).

Now we make this more explicit by tracing the equivalence ofLie 2-algebras and give the precise equivalent sub-Lie-2-algebra. We view the integration of the latter as a sub-Lie 2-group of the integration \( GL \). Finally we verify this sub-Lie 2-group is equivalent to a Lie group. This part of calculation is logically redundant, but we see more clearly some nonstrict phenomenon happening via pulling back a sub-Lie-2-algebra by an equivalence.

The preimage under the map \( (f_0, f_1) \) given by \( GL \) and \( GL \) of this sub-Lie-2-algebra \( W^{0+i} \) is the subcomplex \( gl(V) \times W^{i} \) (\( \text{ad}_V \oplus \text{gl}(V) \times V \), where the vector space \( \text{ad}_V \oplus \text{gl}(V) \times V \) is given by

\[
(\text{ad}_V \oplus \text{gl}(V)) \times V = \{(\text{ad}_{u}, A, u) \in (\text{gl}(V) \oplus \text{gl}(V)) \times V | \forall u \in V, A \in \text{gl}(V)\},
\]

which is isomorphic to \( \text{gl}(V) \times V \). The map \( i_2 \times i \) is given by

\[
(i_2 \times i)(A, c) = (0, A, c).
\]

It is not hard to see that the complex \( \text{gl}(V) \times W \) is not closed under the Lie brackets on \( \text{gl}(V) \times V \) and \( (\text{gl}(V) \oplus \text{gl}(V)) \times V \) because we have

\[
[(\text{ad}_{u}, A, u), (\text{ad}_{v}, B, v)] = (\text{ad}_{[u,v]}, \frac{1}{2}(\text{ad}_B[u] + [A, \text{ad}_u]) + [A, B] + \frac{1}{4}\text{ad}_{[u,v]}, [u, v] + Av - Bu).
\]

However this complex is the image of a strict Lie 2-algebra which we define as follows. Consider the complex \( \text{gl}(V) \times W \) \( (\text{ad}_V \oplus \text{gl}(V)) \times V \), where \( (\text{Id} \times i)(A, c) = (A, c) \), with the following Lie bracket operation

\[
[(A, c), (A', c')] = ([A, A'], 0), \quad \text{on} \ \text{gl}(V) \times W, \quad (63)
\]

\[
[(A, u), (B, v)] = \left(\frac{1}{2}([\text{ad}_u, B] + [A, \text{ad}_v]) + [A, B] + \frac{1}{4}\text{ad}_{[u,v]}, [u, v]\right), \quad \text{on} \ \text{gl}(V) \times V. \quad (64)
\]

Define the action \( \phi \) of the Lie algebra \( \text{gl}(V) \times V \) on the Lie algebra \( \text{gl}(V) \times W \) by

\[
\phi_{(A, u)}(B, c) = ([A, B] + \frac{1}{2}([\text{ad}_u, B], 0).
\]

Then we obtain a crossed module of Lie algebras:

\[
(\text{gl}(V) \times W, \text{gl}(V) \times V, \text{Id} \times i, \phi).
\]

Define \( \psi_0 : \text{gl}(V) \times V \rightarrow (\text{gl}(V) \oplus \text{gl}(V)) \times V \) by

\[
\psi_0(A, u) = (\text{ad}_u, A, u)
\]

and let \( \psi_1 : \text{gl}(V) \times W \rightarrow \text{gl}(V) \times V \) be the natural inclusion map. Furthermore, define \( \psi_2 : \wedge^2(\text{gl}(V) \times V) \rightarrow \text{gl}(V) \times V \) by

\[
\psi_2((A, u), (B, v)) = (0, Av - Bu).
\]
Then it is not hard to see that $(\psi_0, \psi_1, \psi_2)$ is a Lie 2-algebra morphism. To summarize, we have the following commutative diagram of Lie 2-algebras:

\[
\begin{array}{ccc}
gl(V) \times W & \xrightarrow{(\psi_i)} & gl(V) \ltimes V \\
pr_{W, \text{ad}}pr_V \times pr_V & & pr_V, \sigma \times \text{Id} \\
W & \xrightarrow{(i, \delta)} & V \rightarrow gl(V) \oplus V.
\end{array}
\]

(66)

It is not hard to see that the vertical arrows are equivalences of Lie 2-algebras. Thus the crossed module of Lie algebras (65) is the pull-back of the sub-Lie-2-algebra $W \rightarrow G_{ad}$ that we are interested in. The two Lie algebras $gl(V) \times W$ and $gl(V) \times V$ with the Lie brackets (63) and (64) are both extensions (the first a trivial one) of Lie algebras fitting in the following diagram of Lie algebras

\[
\begin{array}{c}
0 \rightarrow gl(V) \xrightarrow{i_1} gl(V) \times W \xrightarrow{pr_W} W \rightarrow 0 \\
\text{Id} \downarrow & & \text{Id} \times i \downarrow & & i \downarrow \\
0 \rightarrow gl(V) \xrightarrow{i_1} gl(V) \times V \xrightarrow{pr_V} V \rightarrow 0,
\end{array}
\]

where $i_1$ is the inclusion into the first factor. Thus this diagram integrates to a commutative diagram of simply connected Lie groups

\[
\begin{array}{c}
1 \rightarrow G \xrightarrow{i_1} G \times W \xrightarrow{pr_W} W \rightarrow 1 \\
\text{Id} \downarrow & & \int (\text{Id} \times i) \downarrow & & i \downarrow \\
1 \rightarrow G \xrightarrow{i} G(gl(V) \times V) \rightarrow G(V) \rightarrow 1,
\end{array}
\]

where $G(gl(V) \times V)$ is the simply connected Lie group integrating Lie algebra $gl(V) \times V$ with Lie bracket (64). The Lie group $G(gl(V) \times V)$ has no explicit form, however we know that $\int (\text{Id} \times i)$ is injective since $\text{Id} \times i$ is injective. Thus by Corollary 2.16 the integrated simply connected crossed module given by $G \times W$ and $G(gl(V) \times V)$ is equivalent as a Lie 2-group to the Lie group $G(gl(V) \times V)/G \times W \cong G(V)/W$.

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