Drift, diffusion, and third order derivatives in Fokker-Planck equations: one specific case

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I present a case where there is an exact re-interpretation for the third order derivative term in a Fokker-Planck equation, purely in terms of ordinary drift and diffusion.

I. INTRODUCTION

There are many situations in optics where we would like to treat system dynamics using those methods developed for Fokker-Planck equations, or their stochastic analogues. Some notable cases are for the quantum-optical parametric oscillator[1] and the cold-atom Gross-Pitaevski equation for Bose-Einstein (and other) condensates[2–4]. However, some (many) systems in fact give rise to partial differential equations containing extra terms, notably derivative terms of higher order than second. Fortunately, there are often grounds for considering such terms to negligible, so they are neglected (“truncated”) – thus giving rise to the so-called “truncated Wigner” phase space[1, 2] descriptions of quantum optical dynamics. Nevertheless, just because a term is small, that does not preclude it gradually accumulating and so providing a significant distortion. An ideal test bed for such comparisons is the second-order nonlinear interaction present in the parametric oscillator[1]. Such a FPE could be of the simple form

\[ \partial_t P(x; t) = \partial_x \left[ A(x) + \partial_x B(x) + \partial_x^2 C(x) \right] P(x; t). \]  (3)

However, in some contexts we can generate FPE-like equations that have additional third order derivative terms. An example is when using the Wigner representation to derive a FPE for the optical parametric oscillator[1]. For the purpose of this short note, they have some useful recurrence properties,

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \]  (5)
\[ \partial_x H_n(x) = 2nH_{n-1}(x), \]  (6)
\[ H_{n+1}(x) = xH_n(x) - \partial_x H_n(x). \]  (7)

When used appropriately, these recurrence properties enable us to convert a FPE containing third-order derivative terms into a form containing only second order ones[11].

II. FOKKER-PPLANCK EQUATIONS

Fokker-Planck equations (FPE’s) are usually of the form[3–10]

\[ \frac{\partial P(x; t)}{\partial t} = \left[ \frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] P(x; t) \]  (1)
\[ = \frac{\partial}{\partial x} \left[ A(x) + \partial_x B(x) \right] P(x; t). \]  (2)

This partial differential equation describes how the probability distribution function \( P \) evolves in space \( (x) \) as a function of time \( (t) \). The form given here is one dimensional, but of course multi-dimensional generalizations also exist. Here \( A(x) \) is a drift term causing probability to flow deterministically, and \( B(x) \) is a diffusion term causing probability to spread away from some give point.

III. HERMITE POLYNOMIALS

Hermite polynomials are a set of orthogonal polynomials, and, as such, can be used to describe any arbitrary function using the form

\[ P(x; t) = \sum_n a_n(t) H_n(x). \]  (4)

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IV. TRANSFORMING THE THIRD DERIVATIVES

Using eqn. (4) we can decompose the full FPE as follows

$$\sum_n a_n(t)H_n(x) = \partial_x [A(x) + \partial_x B(x) + \partial_x^2 C(x)]$$

$$+ \sum_n a_n(t)H_n(x) \quad (8)$$

Note that since there is no cross-coupling between the $H_n(x)$ components, and because the $H_n(x)$ are orthogonal, we can write

$$\partial_t a_n(t)H_n(x) = \partial_x [A(x) + \partial_x B(x) + \partial_x^2 C(x)] a_n(t)H_n(x) \quad (9)$$

A. A single $H_n$ component

I now assume $C(x) = C$ is a constant\(^1\), and consider the RHS derivative part of eqn. (9). Considering just one component $H_n$ of the Hermite decomposition of $P$, we have a RHS from (9) of

$$\partial_t a_n(t)H_n(x) = \partial_x [A(x) + \partial_x B(x) + \partial_x^2 C] H_n(x) \quad (10)$$

$$= \partial_x A(x)H_n(x) + \partial_x^2 B(x)H_n(x) + \partial_x^2 C(2xH_n - H_{n+1}) \quad (11)$$

$$= \partial_x A(x)H_n(x) + \partial_x^2 B(x)H_n(x) + \partial_x^2 2xH_n + \partial_x^2 C H_{n+1} \quad (12)$$

$$= \partial_x A(x)H_n(x) + \partial_x^2 B(x)H_n(x) + \partial_x^2 2xH_n + \partial_x^2 2(n+1)H_n \quad (13)$$

$$= \partial_x [A(x) + 2(n+1)C] H_n(x) + \partial_x^2 [B(x) + 2x] H_n(x) \quad (14)$$

$$= \partial_x [A'_n(x) + \partial_x B'_n(x)] H_n(x), \quad (15)$$

where

$$A'_n(x) = A(x) + 2(n+1)C \quad (16)$$

$$B'_n(x) = B(x) + 2xC. \quad (17)$$

So for a distribution function $P$ decomposed into Hermite polynomials $H_n$, a third-order derivative term with a constant prefactor $C$ has two effects:

1. The first, and presumably most important feature is that the drift term $A$ gains a constant positive contribution. We would typically expect such a directional property, since $\partial_x^2$ is clearly antisymmetric in nature. The strength of this effective drift is dependent on $n$, the polynomial order, so that fine $x$ structure in $P$ (requiring higher order contributions) drifts faster than coarse features;

2. The second, more minor effect is to add an antisymmetric adjustment to the diffusion.

The transformed counterpart to eqn. (9) is

$$\partial_t a_n(t)H_n(x) = \partial_x [A'_n(x) + \partial_x B'_n(x)] a_n(t)H_n(x). \quad (18)$$

This therefore is a nice specific example where we can re-cast a third order derivative term into the readily understood first order (drift) and second order (diffusion) terms. If implemented in some numerical scheme, it would require repetition of the following steps:

(a) the distribution $P(t)$ to be decomposed into $H_n$ with weights $a_n(t)$,

(b) each $H_n$ to evolve away from an exact Hermite polynomial under the influence of the drift and diffusion for some suitably small time interval $\delta t$,

(c) an evolved distribution $P(t + \delta t)$ to be calculated.

Whether or not this is useful in practise is left as an exercise for those dealing with such situations. However, even if such an implementation is not done, and the third-order derivatives are simple truncated as usual, the behaviour of $A'_n$ can be used to put constraints on the size of spatial features on distributions $P$ evolved using a truncated FPE. Thus it might at least be put to use as an intermittently applied test on the validity of a simulation of a truncated FPE model.

B. Rearranged $H_n$ component

We might attempt an alternative strategy that tries to avoid the requirement of decomposing the distribution function $P$. However, the one given below will not work, but for the sake of completeness I give it here.

Firstly, assume $C(x) = C$ is a constant and consider the RH derivative part of eqn. (9).

$$\partial_t a_n(t)H_n(x) = \partial_x A(x)H_n(x)a_n(t) + \partial_x^2 B(x)H_n(x)a_n(t) + \partial_x^2 C(2xH_n - H_{n+1})a_n(t) \quad (19)$$

$$= \partial_x A(x)H_n(x)a_n(t) + \partial_x^2 B(x)H_n(x)a_n(t) + \partial_x^2 C(2xH_n - H_{n+1})a_n(t) \quad (20)$$

$$= \partial_x A(x)H_n(x)a_n(t) + \partial_x^2 B(x)H_n(x)a_n(t) + \partial_x^2 C H_{n+1}a_n(t). \quad (21)$$

\(^1\) ??as it is for the truncated-Wigner dpo case?
Note the appearance of the $H_{n+1}$. Since we aim to reinstate the summation over all $H_n$, we might reassign this to $B_{n+1}$, so
\begin{equation}
A''_n (x) = A(x) \tag{22}
\end{equation}
\begin{equation}
B''_n (x) a_n (t) = B(x) a_n (t) + 2 x C a_n (t) - (1 - \delta_n) a_{n-1}(t). \tag{23}
\end{equation}

So we see no drift modification, but a diffusion reduction instead. However, the $H_n$ equations are still cross-coupled, as $B''_n a_n (t)$ has gained a dependence on $a_{n-1}$. In the case of an even $n$ the diffusion adjustment will depend on $n - 1$ which is a measure of the odd-ness of $P$ (as projected onto $H_{n-1}$); the converse is true for odd $n$. This has turned out to be just a re-representation of the effective-drift adjustment seen for $A'$ in the first method.

C. Rebuilding the FPE

Another (unsuccessful) attempt to make a useful application of the special case in Sec. IV A is to try to reinstate the summation over $n$, to convert eqn. [18] back into a true FPE. Thus
\begin{equation}
\partial \sum_{n} a_n (t) H_n (x) = \partial_x [A'_n (x) + \partial_x B'_n (x)] \sum_{n} a_n (t) H_n (x) \tag{24}
\end{equation}
\begin{equation}
\Rightarrow \partial_t P (x; t) = \partial_x [A'(x) + \partial_x B'(x)] P(x; t). \tag{25}
\end{equation}
Unfortunately this fails because $A'_n$ is dependent on $n$, and we cannot reach the desire $n$-independent form for $A$ seen in the target eqn. [23]. And even if I instead attempt to remove that $n$ dependence, using eqn. [5], i.e. $2 n H_n = 2 x H_{n+1} - H_{n+2}$, I end up cross-coupling the $H_n$ contributions instead, which is no better.

V. CONCLUSION

I have shown that in one specific case, the Hermite polynomial recurrence relations can be used to develop an exact relationship between a constant third order derivative term in an FPE, and the commonly understood diffusion and drift terms. Although there seems no clear path to use this as a basis to solve FPE’s with third order terms in general, it can still be useful in applying check to an ongoing numerical solution to a truncated (standard) FPE equation.

Note

This is a previously unpublished fragment of my PhD research, which I have dusted off and put here on the arXiv, in case someone finds it useful.

[1] P. Kinsler and P. D. Drummond, Phys. Rev. A 43, 6194 (1991), doi:10.1103/PhysRevLett.64.236
[2] A. Sinatra, C. Lobo, and Y. Castin, J. Phys. B 35, 3599 (2002), doi:10.1088/0953-4075/35/17/301
[3] C. W. Gardiner, J. R. Anglin, and T. I. A. Fudge, J. Phys. B 35, 1555 (2002), c, doi:10.1088/0953-4075/35/6/310
[4] N. Cherroret and T. Wellens, Phys. Rev. E 84, 021114 (2011), doi:10.1103/PhysRevE.84.021114
[5] P. D. Drummond and P. Kinsler, Phys. Rev. A 40, 4813 (1989), doi:10.1103/PhysRevA.40.4813
[6] P. Kinsler and P. D. Drummond, Phys. Rev. A 52, 783 (1995), doi:10.1103/PhysRevA.52.783
[7] P. Kinsler, Phys. Rev. A 53, 2000 (1996), doi:10.1103/PhysRevA.53.2000
[8] H. Risken, The Fokker-Planck Equation Methods of Solution and Applications, vol. 18 of Springer Series in Synergetics (Springer, Berlin Heidelberg, 1989), ISBN 978-3-540-61530-9, URL http://link.springer.com/book/10.1007/978-3-642-61544-3
[9] C. W. Gardiner, Handbook of Stochastic Methods, vol. 13 of Springer Series in Synergetics (Springer, 2004), 3rd ed., ISBN 978-3-540-20882-2.
[10] L. Arnold, Stochastic Differential Equations: Theory and Applications (Wiley, New York, 1974), ISBN 0-471-03359-6.
[11] MathWorld, Hermite polynomial, URL http://mathworld.wolfram.com/HermitePolynomial.html