Asymptotic Expansion of the Variance of Random Zeros on Complex Manifolds

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Abstract
Linear statistics of random zero sets are integrals of smooth differential forms over the zero set and as such are smooth analogues of the volume of the random zero set inside a fixed domain. We derive an asymptotic expansion for the variance of linear statistics of the zero divisors of random holomorphic sections of powers of a positive line bundle on a compact Kähler manifold. This expansion extends the leading-order asymptotics (in the codimension one case) given by Shiffman–Zelditch in 2010.

Keywords Random zeros · Kaehler manifold · Holomorphic section · Bergman kernel · Asymptotic expansion · Positive line bundle

1 Introduction
The distribution of zeros of random polynomials has a long history going back to Bloch and Polya [9], Kac [23], Littlewood and Offord [25], Maslova [28], and others, who studied the zeros of random real polynomials with i.i.d. coefficients. Zeros of random polynomials of one complex variable were studied by Hammersley [19] in 1956 and in the 1990s by Hannay [20] and others [13,18,30] and in the 2000s in [10,33,36]. In higher dimensions, zeros of $m$ simultaneous random polynomials in $\mathbb{C}^m$ were investigated by Edelman and Kostlan [17] in 1995 and more recently in [1,11,16,41].

Polynomials on $\mathbb{C}^m$ of degree $k$ can be identified with holomorphic sections of the $k$-th power of the hyperplane section bundle on $\mathbb{C} \mathbb{P}^m$, and thus a natural generalization is to holomorphic sections of powers of a positive holomorphic line bundle on a compact Kähler manifold. The zeros of random sections of such line bundles form the

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subject of a series of papers [7,8,31,34,35] based on the asymptotic expansion of the Bergman kernel for powers of a line bundle introduced by Catlin [14] and Zelditch [39] (extending the leading-order asymptotics of Tian [37]) and further developed in [3,4,15,26,27,38,40]. Additional references for zeros of random polynomials and random holomorphic sections can be found in [2].

This article concerns the fluctuations of the zero sets of random holomorphic sections. We study the “linear statistics” of random zero sets; these are the integrals of a smooth test form over the zero sets of random holomorphic sections. Linear statistics can be regarded as smooth approximations of “discontinuous statistics” such as the volume (or cardinality in the point case) of a random zero set inside a fixed domain, as studied in [34]. Linear statistics were studied by Forrester and Honner [18], Sodin and Tsirelson [36], and Nazarov and Sodin [29] for functions of one complex variable, in [35] for holomorphic sections on compact Kähler manifolds, and by Bayraktar [1] for polynomials in several complex variables. We shall apply the methods of [35] to give a sharp asymptotic expansion (Theorem 1.1) for the case of divisors (codimension 1), and we also apply the results of [26] to compute the sub-leading term of this expansion.

To describe our framework, we begin with a positively curved Hermitian holomorphic line bundle \((L, h) \to M\) over a compact complex manifold \(M\) of dimension \(m\). Recall that the curvature form \(\Theta_h \in D^{1,1}(M)\) of \(L\) is given locally by \(\Theta_h = -\partial \bar{\partial} \log h(e_L, \bar{e}_L)\), where \(e_L\) is a nonvanishing local holomorphic section of \(L\). We then give \(M\) the Kähler form \(\omega = \frac{i}{2} \Theta_h\), and we give the spaces \(H^0(M, L^k)\) of holomorphic sections the Hermitian inner products induced by the Kähler form \(\omega\) and the Hermitian metrics \(h^k\):

\[
\langle s_1, s_2 \rangle = \int_M h^k(s_1, \bar{s}_2) \frac{1}{m!} \omega^m, \quad s_1, s_2 \in H^0(M, L^k),
\]

(1)

For a section \(s^k \in H^0(M, L^k)\) (not identically zero), we let \(Z_{s^k} \in D^{1,1}(M)\) denote the current of integration along the zero set (divisor) of \(s^k\):

\[
(Z_{s^k}, \psi) = \int_{\{s^k = 0\}} \psi, \quad \psi \in D^{m-1,m-1}(M),
\]

taking into account multiplicities. Here, \(D^{p,q}(M)\) denotes the space of smooth \((p, q)\)-forms on the (compact) manifold \(M\). In particular, if \(\dim M = 1\), then \(Z_{s^k} = \sum_j n_j \delta_{a_j}\), where \(s^k\) has zeros of multiplicity \(n_j\) at the point \(a_j\). (In our probabilistic setting, all multiplicities equal one almost surely.)

The following asymptotic expansion for the variance of linear statistics of random zeros is the main result of this article:

**Theorem 1.1** Let \((L, h)\) be a positive Hermitian holomorphic line bundle over a compact Kähler manifold \((M, \omega)\) of dimension \(m\), where \(\omega = \frac{i}{2} \Theta_h\). We give the spaces \(H^0(M, L^k)\) the standard Gaussian probability measures (see (6)) corresponding to the inner products (1).

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Let \( \psi \in \mathcal{D}_{\mathbb{R}}^{m-1,m-1}(M) \) be a real \( C^\infty \) \((m - 1, m - 1)\)-form on \( M \). Then for random sections \( s^k \in H^0(M, L^k) \), the variances \( \text{Var}(Z_{s^k}, \psi) \) have an asymptotic expansion

\[
\text{Var}(Z_{s^k}, \psi) \sim A_0 k^{-m} + A_1 k^{-m-1} + \ldots + A_j k^{-m-j} + \ldots.
\]

(2)

The leading and sub-leading coefficients are given by

\[
A_0 = \pi^{m-2} \zeta(m+2) \frac{\|\partial \bar{\partial} \psi\|_2^2}{4},
\]

(3)

\[
A_1 = -\pi^{m-2} \zeta(m+3) \left\{ \frac{1}{8} \int_M \rho |\partial \bar{\partial} \psi|^2 \Omega_M + \frac{1}{4} \|\partial^* \partial \bar{\partial} \psi\|_2^2 \right\},
\]

(4)

where \( \rho \) is the scalar curvature and \( \Omega_M = \frac{1}{m!} \omega^m \) is the volume form on \( M \).

In fact, all the coefficients \( A_j \) can be given in terms of \( \partial \bar{\partial} \psi \), curvature invariants of \( M \), and their derivatives; see Theorem 5.2.

Theorem 1.1 sharpens the result in [35], where it was shown that

\[
\text{Var}(Z_{s^k}, \psi) = A_0 k^{-m} + O(k^{-m-\frac{1}{2}+\varepsilon}).
\]

Similar results for the variance in higher codimension were also given in [35].

In particular, for \( \dim M = 1 \) we have:

Corollary 1.2 Let \( M \) be a compact Riemann surface and let \((L, h) \to (M, \omega)\) be as in Theorem 1.1. Then for \( \psi \in C^\infty_{\mathbb{R}}(M) \),

\[
\text{Var}(Z_{s^k}, \psi) = \frac{\zeta(3)}{16\pi} \|\Delta \psi\|^2 k^{-1} - \frac{\pi^3}{2880} \left\{ \int_M \rho |\Delta \psi|^2 \omega + \|d \Delta \psi\|_2^2 \right\} k^{-2} + O(k^{-3}).
\]

(5)

The leading term of (5) was obtained by Sodin and Tsirelson [36] for the cases where \( M \) is \( \mathbb{CP}^1 \), the hyperbolic disk, and \( \mathbb{C}^1 \), where it was shown that \( \text{Var}(Z_{s^k}, \psi) = \frac{\zeta(3)}{16\pi} \|\Delta \psi\|^2 k^{-1} + o(k^{-1}) \).

2 Background

We summarize here the notation and results that are used in this paper. We recall that the standard Gaussian measure \( \gamma_V \) on an \( n \)-dimensional complex vector-space \( V \) with a Hermitian inner product is given by

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\[ \gamma_V(v) = \prod_{j=1}^{n} \left( \frac{i}{2\pi} e^{-|c_j|^2} \, dc_j \land d\bar{c}_j \right) \]
\[ = \frac{1}{\pi^n} e^{-\sum_{j=1}^{n} |c_j|^2} \prod_{j=1}^{n} \frac{i}{2} dc_j \land d\bar{c}_j, \quad v = \sum_{j=1}^{n} c_j v_j \in V, \quad (6) \]

where \( \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( V \). (The Gaussian measure \( \gamma_V \) is independent of the choice of orthonormal basis.) We let \( \mathbf{E} \) denote the expected value of a (real or vector-space valued) random variable, and we let \( \text{Var} \) denote the variance of a (real) random variable.

Throughout this paper, \( (L, h) \rightarrow M \) denotes a positive Hermitian holomorphic line bundle over an \( m \)-dimensional compact Kähler manifold \( M \) with Kähler form \( \omega = \frac{i}{2} \Theta_h \). We give the spaces \( H^0(M, L^k) \) of holomorphic sections of \( L^k = L^\otimes k \) the inner products given by (1) together with their corresponding standard Gaussian probability measures.

Let \( \{S^k_1, S^k_2, \ldots, S^k_{n_k}\} \) be a basis for \( H^0(M, L^k) \), \( k \geq 1 \). The reproducing kernel \( K_k(z, w) \) of the orthogonal projection \( K_k : \mathcal{L}^2(M, L^k) \rightarrow H^0(M, L^k) \) is given by the Bergman kernel

\[ K_k(z, w) = \sum_{j=1}^{n_k} S^k_j(z) \otimes S^k_j(w) = \mathbf{E} \left( S^k(z) \otimes S^k(w) \right) \in L^k_z \otimes \overline{L^k_w}. \quad (7) \]

The Bergman projection \( K_k \) lifts to an orthogonal projection \( \Pi_k : \mathcal{L}^2(X) \rightarrow \mathcal{H}^2_k(X) \), where \( X = \{ \lambda \in L^* : \|\lambda\| = 1 \} \) is the circle bundle of the dual line bundle \( L^* \), and \( \mathcal{H}^2_k(X) = \{ f \in \mathcal{H}^2(X) : f(e^{i\theta} x) = e^{ik\theta} f(x) \} \), where \( \mathcal{H}^2(X) \) is the Hardy space of CR-holomorphic \( \mathcal{L}^2 \) functions on \( X \) (see [32,34,39]). The reproducing kernel for \( \Pi_k \) is the Szegő kernel \( \Pi_k ((z, \theta_1), (w, \theta_2)) = e^{i(\theta_1 - \theta_2)} \Pi_k(z, w) \) with norm given by

\[ |\Pi_k(z, w)| = \|K_k(z, w)\|_h^k(z) \otimes \overline{h^k(w)} . \]

It was shown in [31] using the Poincaré–Lelong formula,

\[ Z_{sk} = \frac{i}{2\pi} \partial \bar{\partial} \log \|s^k\|_h^k + \frac{k}{\pi} \omega, \quad (8) \]

that the expected value of the zero current of a Gaussian random section \( s^k \in H^0(M, L^k) \) is given by

\[ \mathbf{E}Z_{sk} = \frac{i}{2\pi} \partial \bar{\partial} \log \Pi_k(z, z) + \frac{k}{\pi} \omega . \quad (9) \]

On the other hand, the variance of \( Z_{sk} \) depends on the normalized Szegő kernel introduced in [34]:

\[ P_k(z, w) = \frac{|\Pi_k(z, w)|}{\Pi_k(z, z)^{\frac{1}{2}} \Pi_k(w, w)^{\frac{1}{2}}} . \quad (10) \]
We note that $P_k(z, z) = 1$ and $0 \leq P_k(z, w) \leq 1$ by Cauchy–Schwarz. The following variance formula from [34] involves the normalized kernel $P_k(z, w)$ together with the function

$$G(t) := -\frac{1}{4\pi^2} \int_0^t \frac{\log(1 - s)}{s} \, ds = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{t^n}{n^2}, \quad -1 \leq t \leq 1. \quad (11)$$

**Theorem 2.1** [34, Theorem 3.1] Let $(L, h) \to (M, \omega)$ be as in Theorem 1.1. Let $Q_k : M \times M \to [0, +\infty)$ be the function given by

$$Q_k(z, w) = G\left(P_k(z, w)^2\right). \quad (12)$$

Then the variance of the linear statistics $(Z_{sk}, \psi)$ of a random section $s_k \in H^0(M, L^k)$ is given by

$$\text{Var}(Z_{sk}, \psi) = \int_{M \times M} Q_k(z, w) i \bar{\partial} \partial \psi(z) \wedge i \bar{\partial} \partial \psi(w), \quad (13)$$

for all real $(m - 1, m - 1)$-forms $\psi$ on $M$ with $C^2$ coefficients.

**Remark** The function $\tilde{G}(t)$ in [34] equals $G(t^2)$ here. Although Theorem 2.1 involves $G(t)$ only for $0 \leq t \leq 1$, we allow $t$ to take negative values in the proof of Theorem 1.1.

To obtain the asymptotic formula of Theorem 1.1, we shall apply the asymptotics of the Bergman–Szegő kernel to formula (13). To do this, we must overcome the fact that $G(t)$ is not differentiable at $t = P_k(z, z) = 1$.

As in [34,35], we shall apply the off-diagonal asymptotics of $\Pi_k(z, w)$ and $P_k(z, w)$ from [32,34]. We recall that the Szegő kernel $\Pi_k$ decays rapidly away from the diagonal as $k$ increases:

**Proposition 2.2** [34, Proposition 2.6] Let $(L, h) \to (M, \omega)$ be as in Theorem 1.1, and let $p > 0$. we have

$$P_k(z, w) = O(k^{-p}) \quad \text{uniformly for } \text{dist}(z, w) \geq \left(2p + 1 + \log k \right)^{1/2}. \quad (12)$$

**Corollary 2.3** [34, Lemma 3.4] Under the hypotheses of Proposition 2.2, we have

$$Q_k(z, w) = O(k^{-p}) \quad \text{uniformly for } \text{dist}(z, w) \geq \left(p + 1 + \log k \right)^{1/2}. \quad (13)$$

**Proof** The bound follows immediately from (12), Proposition 2.2, and the fact that $G(t) = O(t)$ for $0 \leq t \leq 1$. \hfill$\square$

**Remark** The normalized Szegő kernel satisfies the sharper decay rate away from the diagonal:

$$P_k(z, w) = O\left(e^{-c\sqrt{k} \text{dist}(z, w)}\right).$$
(See, for example, [6, Theorem 2.5] and [24].) However, Proposition 2.2 suffices for our purposes.

Corollary 2.3 allows us to replace the integral in (13) with an integral over a small shrinking neighborhood of the diagonal, as we shall demonstrate below. We then shall apply the “near-diagonal” asymptotic expansion of the Szegő kernel from [26,32].

To give the asymptotic expansion in a neighborhood $U$ of a point $z_0 \in M$, let $\Phi_{z_0} : U \to U' \subset \mathbb{C}^m$ be a local coordinate chart with $\Phi_{z_0}(z_0) = 0$ and we write, by abuse of notation,

$$z_0 + u \equiv \Phi_{z_0}^{-1}(u) \in U,$$

for $u \in U'$.

**Theorem 2.4** [32, Theorem 3.1] Let $(L, h) \to (M, \omega)$ be as in Theorem 1.1, and let $z_0 \in M$ and $\varepsilon, b \in \mathbb{R}^+$. Then using normal coordinates at $z_0$,

$$\Pi_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 + \frac{v}{\sqrt{k}} \right) = \frac{k^m}{\pi^m} e^{\varepsilon u \cdot \bar{v} - \frac{1}{2}(|u|^2 + |v|^2)} \left[ 1 + \sum_{r=1}^{n} k^{-r} p_r(u, v) + k^{-(n+1)/2} E_{kn}(u, v) \right],$$

where the $p_r$ are polynomials in $(u, \bar{u}, v, \bar{v})$ (depending on $z_0$ and the choice of coordinates) of degree $\leq 5r$ and of the same parity as $r$, and

$$\|D^j E_{kn}(u, v)\| \leq C_{jnek} k^\varepsilon \text{ for } |u| + |v| < b\sqrt{\log k}, \quad (14)$$

for $j, n \geq 0$. Furthermore, if one chooses smoothly varying coordinate charts $\Phi_z$ for $z$ varying in a closed domain $V$, then the constants $C_{jnek}$ can be chosen independently of $z_0 \in V$.

Here $\|D^j F(u, v)\|$ denotes the sum of the norms of the derivatives of $F$ of order $j$ with respect to $u, \bar{u}, v, \bar{v}$. The estimate (14) is equivalent to equation (96) in [32], where the result was shown to hold for almost-complex symplectic manifolds. (A short proof of Theorem 2.4 is given in [34, Sect. 5]; see [5,21] for alternative approaches. For real-analytic metrics, the expansion holds for $|u| + |v| < \varepsilon k^{1/4}$; see [22].)

We assume henceforth that $M$ is covered by a finite collection of open sets $V$ such that there are smoothly varying coordinate charts $\Phi_z$ for $z$ in a closed domain $V$. This will guarantee uniformity of remainder estimates for $z_0 \in M$.

Although the terms of the expansion in Theorem 2.4 depend on the choice of coordinates, if one uses specific coordinates introduced by Bochner [12], then the terms of the expansion are well described in terms of curvature invariants, as was given in [26]. In particular, choosing a nonvanishing holomorphic section $e_L$ of $L$ over a neighborhood of $z_0$, we let $\varphi = -\log h(e_L, \bar{e}_L)$ so that $\varphi$ is the Kähler potential:

$$\omega = \frac{i}{2} \partial \bar{\partial} \varphi = \frac{i}{2} \sum_{p, q=1}^{m} g_{p\bar{q}} dz^p \wedge d\bar{z}^q, \quad g_{p\bar{q}} = \frac{\partial^2 \varphi}{\partial z^p \partial \bar{z}^q}. \quad (15)$$
We recall that the components of the curvature tensor are given by

$$R_{p\bar{q}j\bar{l}} = -\frac{\partial^2 g_{p\bar{q}}}{\partial z^j \partial \bar{z}^l} + \sum_{r,s=1}^m g^{r\bar{s}} \frac{\partial g_{ps}}{\partial z^j} \frac{\partial g_{r\bar{q}}}{\partial \bar{z}^l}. \quad (16)$$

The scalar curvature $\rho$ is given by

$$\rho = \sum_{p,q,j,l} g^{p\bar{q}} g^{j\bar{l}} R_{p\bar{q}j\bar{l}}. \quad (17)$$

We write

$$R_{z_0}(u, \bar{v}, u, \bar{v}) = \sum_{p,q,j,l} R_{p\bar{q}j\bar{l}}(z_0) u^p \bar{v}^q u^j \bar{v}^l,$$

for $u = (u^1, \ldots, u^m), v = (v^1, \ldots, v^m)$.

We shall use the following result from [26] to obtain the expressions for $A_0$ and $A_1$ in (3)–(4).

**Theorem 2.5** [26, Theorem 2.3] Assuming the hypotheses of Theorem 2.4, suppose that the local holomorphic section $e_L$ and local coordinates at $z_0$ are chosen such that the Kähler potential $\varphi = -\log h(e_L, e_L)$ is of the form

$$\varphi(z_0 + z) = \frac{1}{2} \frac{\partial^4 \varphi}{\partial z^p \partial z^j \partial \bar{z}^q \partial \bar{z}^l}(z_0^0) z^p z^j \bar{z}^q \bar{z}^l + O(|z|^5). \quad (18)$$

Then

$$p_1(u, v) = 0, \quad (19)$$

$$p_2(u, v) = \frac{1}{2} \rho(z_0) + \frac{1}{8} R_{z_0}(u, \bar{u}, u, \bar{u}) + \frac{1}{8} R_{z_0}(v, \bar{v}, v, \bar{v}) - \frac{1}{4} R_{z_0}(u, \bar{v}, u, \bar{v}). \quad (20)$$

Note that (18) implies that the coordinates are normal coordinates at $z_0$.

### 3 The Leading and Sub-leading Terms

We shall prove in this section the following weaker form of Theorem 1.1:

**Proposition 3.1** Assuming the hypotheses of Theorem 1.1, we have for all $\varepsilon > 0$,

$$\text{Var}(Z_{x_\varepsilon}, \psi) = \frac{\pi^{m-2}}{k^m} \left[ \frac{\zeta(m + 2)}{4} \| \partial \bar{\partial} \psi \|_2^2 \right] - \zeta(m + 3) \left[ \frac{1}{8} \left( \frac{1}{k} \| \partial \bar{\partial} \psi \|_2^2 + \frac{1}{4} \| \partial^* \bar{\partial} \psi \|_2^2 \right)^{k-1} + O(k^{-3/2+\varepsilon}) \right].$$
To prove Proposition 3.1, we shall derive off-diagonal asymptotics of $Q_k(z, w)$ (Lemma 3.4). To obtain these asymptotics, we first apply Theorem 2.4:

**Lemma 3.2** Under the hypotheses of Theorem 2.4, there are constants $c_{n\varepsilon b}$ independent of $z_0$ such that

$$P_k(z_0 + \frac{u}{\sqrt{k}}, z_0)^2 = e^{-|u|^2} \left( 1 + \sum_{r=1}^{n} k^{-r/2} a_r(u) + k^{-(n+1)/2} \tilde{E}_{kn}(u) \right),$$

where the $a_r(u)$ are polynomials in $(u, \bar{u})$ of degree $\leq 5r$, of the same parity as $r$, and with $a_r(0) = 0$, $d a_r(0) = 0$, and

$$|\tilde{E}_{kn}(u)| \leq |u|^2 c_{n\varepsilon b} k^\varepsilon$$

for $|u| < b\sqrt{\log k}$.

**Proof** The result with the weaker remainder estimate

$$P_k(z_0 + \frac{u}{\sqrt{k}}, z_0)^2 = e^{-|u|^2} \left( 1 + \sum_{r=1}^{n} k^{-r/2} a_r(u) + k^{-(n+1)/2} \tilde{E}_{kn}(u) \right),$$

is an immediate consequence of Theorem 2.4 with $j = 0$. To obtain the sharper estimate, let

$$\lambda_k(u) := P_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right)^2.$$

We first show that $a_n(0) = \tilde{E}_{kn}(0) = 0$. Let $n \geq 1$ and suppose by induction that $a_j(0) = 0$ for $j \leq n - 1$. We then have by the weaker estimate (21),

$$1 = \lambda_k(0) = 1 + k^{-n/2} a_n(0) + k^{-(n+1)/2} \tilde{E}_{kn}(0),$$

and hence $\tilde{E}_{kn}(0) = -a_n(0)k^{1/2}$. But $\tilde{E}_{kn}(0) = O(k^\varepsilon)$; therefore, $a_n(0) = 0$ and hence $\tilde{E}_{kn}(0) = 0$.

Since $\lambda_k(u) \leq \lambda_k(0) = 1$ by Cauchy–Schwarz, we have $d \lambda_k(0) = 0$. To show that $d a_n(0) = 0$, we suppose by induction that $d a_j(0) = 0$ for $j \leq n - 1$. Then

$$0 = d \lambda_k(0) = k^{-n/2} d a_n(0) + k^{-(n+1)/2} d \tilde{E}_{kn}(0),$$

and it follows by the same argument as above that $d a_n(0) = d \tilde{E}_{kn}(0) = 0$. (Thus the polynomial $a_n$ has no constant or linear terms.)
Since $d\tilde{E}_{kn}(0) = 0$, we have by Theorem 2.4

$$\left| \frac{\partial}{\partial u^j} \tilde{E}_{kn}(u) \right| \leq \int_0^1 \left| \sum_{l=1}^m u^l \frac{\partial^2 \tilde{E}_{kn}(tu) + \bar{u}^l \frac{\partial^2 \tilde{E}_{kn}(tu)}}{\partial u^l \partial u^j} \right| \, dt \leq C_{2neb} \, |u| \, k^\varepsilon,$$

for $|u| < b \sqrt{\log k}$. Thus

$$|\tilde{E}_{kn}(u)| \leq \int_0^1 2 \left| \sum_{j=1}^m u^j \frac{\partial \tilde{E}_{kn}(tu)}{\partial u^j} \right| \, dt \leq 2\sqrt{m} \, C_{2neb} \, |u|^2 \, k^\varepsilon,$$

for $|u| < b \sqrt{\log k}$.

\hfill \Box

**Theorem 3.3** Under the hypotheses of Theorem 2.5, there are constants $c_{neb}$ independent of $z_0$ such that

$$P_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right)^2 = e^{-|u|^2} \left[ 1 + \frac{1}{4} R_{z_0}(u, \bar{u}, u, \bar{u}) k^{-1} + \sum_{r=3}^{n} k^{-r/2} a_r(u) + k^{-(n+1)/2} \tilde{E}_{kn}(u) \right],$$

where the $a_r$ are polynomials in $(u, \bar{u})$ of degree $\leq 5r$ of the same parity as $r$ and with $a_r(0) = 0$, $da_r(0) = 0$, and

$$|\tilde{E}_{kn}(u)| \leq |u|^2 \, c_{neb} k^\varepsilon \text{ for } |u| < b \sqrt{\log k},$$

for $n \geq 3$.

**Proof** By Lemma 3.2 and (19), we have

$$P_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right)^2 = \frac{\Pi_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right) \Pi_k \left( z_0, z_0 + \frac{u}{\sqrt{k}} \right)}{\Pi_k \left( z_0, z_0 \right) \Pi_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 + \frac{u}{\sqrt{k}} \right)} = e^{-|u|^2} \left\{ 1 + k^{-1} \left[ p_2(u, 0) + p_2(0, u) - p_2(0, 0) - p_2(u, u) \right] + \cdots \right\},$$

and therefore $a_1(u) = 0$. By (20),

$$p_2(u, 0) = p_2(0, u) = \frac{1}{2} \rho(z_0) + \frac{1}{4} R_{z_0}(u, \bar{u}, u, \bar{u}), \quad p_2(0, 0) = p_2(u, u) = \frac{1}{2} \rho(z_0).$$

Therefore, $a_2(u) = \frac{1}{4} R_{z_0}(u, \bar{u}, u, \bar{u})$. \hfill \Box
Henceforth in this paper, we assume that the Kähler potential \( \varphi \) satisfies (18).

**Lemma 3.4** For \( |u| < b \sqrt{\log k} \) and \( \varepsilon > 0 \), we have

\[
Q_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right) = G \left( e^{-|u|^2} \left[ 1 + \frac{1}{4} R_{z_0} (u, \tilde{u}, u, \tilde{u}) k^{-1} \right] \right) + O \left( |u| k^{-3/2 + \varepsilon} \right). \tag{22}
\]

**Proof** By Theorem 3.3 (with \( n = 2 \) and (12)), there is a constant \( C \) (depending only on \( b \) and \( \varepsilon \)) such that for \( |u| < b \sqrt{\log k} \), \( k \gg 0 \),

\[
Q_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right) = G \left( e^{-|u|^2} \left[ 1 + \frac{1}{4} R_{z_0} (u, \tilde{u}, u, \tilde{u}) k^{-1} \right] \right) + G' (t_k (u)) \cdot O \left( |u| k^{-3/2 + \varepsilon} \right),
\]

\[
t_k (u) = e^{-|u|^2} \left[ 1 + \frac{1}{4} R_{z_0} (u, \tilde{u}, u, \tilde{u}) k^{-1} + C_k (u) |u|^2 k^{-3/2 + \varepsilon} \right], \quad |C_k (u)| \leq C. \tag{23}
\]

Note that we can choose \( k_0 \) such that for \( 0 < |u| < b \sqrt{\log k} \) and \( k \geq k_0 \), we have

\[
0 < 1 + \frac{1}{4} R_{z_0} (u, \tilde{u}, u, \tilde{u}) k^{-1} \pm C |u|^2 k^{-3/2 + \varepsilon} \leq e^{\frac{1}{2} |u|^2} \tag{25}
\]

and thus \( 0 < t_k (u) \leq e^{-\frac{1}{2} |u|^2} \) so that \( G' (t_k (u)) \) and \( G (e^{-|u|^2} \left[ 1 + \frac{1}{4} R_{z_0} (u, \tilde{u}, u, \tilde{u}) k^{-1} \right]) \) are well defined for \( k \geq k_0 \). To complete the proof, we must show that

\[
|u| G' (t_k (u)) = O \left( \sqrt{\log k} \right), \quad \text{for} \quad 0 < |u| < b \sqrt{\log k}, \quad k \geq k_0. \tag{26}
\]

Let \( s = e^{-\frac{1}{2} |u|^2} \). By (24)–(25), for \( k \geq k_0 \) we have

\[
|u| G' (t_k (u)) \leq |u| G' \left( e^{-\frac{1}{2} |u|^2} \right) = -\frac{1}{4 \pi^2} \sqrt{-2 \log s} \frac{\log (1 - s)}{s}.
\]

We then have

\[
\lim_{u \to 0} \sup |u| G' (t_k (u)) \leq \lim_{s \to 1^-} \frac{-1}{4 \pi^2} \sqrt{-2 \log s} \frac{\log (1 - s)}{s} = 0, \quad \text{uniformly for} \quad k \geq k_0.
\]

Thus, \( |u| G' (t_k (u)) \) is bounded for \( |u| \leq 1, \ k \geq k_0 \). On the other hand, \( G' (e^{-\frac{1}{2} |u|^2}) \) is bounded for \( |u| \geq 1 \), yielding (26). Lemma 3.4 then follows from (23). \( \Box \)

We begin the proof of Proposition 3.1 following the method of [35]: By Theorem 2.1, we have

\[
\text{Var}(Z^h, \psi) = \int_M \mathcal{I}_k i \bar{\partial} \partial \psi, \tag{27}
\]

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where
\[ \mathcal{I}_k(z_0) = \int_M Q_k(z, z_0) i \partial \bar{\partial} \psi(z), \quad \forall \ z_0 \in M. \] (28)

We write
\[ i \partial \bar{\partial} \psi = f \Omega_M, \quad f \in C_\infty^\infty (M). \] (29)
so that
\[ \mathcal{I}_k(z_0) = \int_M Q_k(z, z_0) f(z) \Omega_M(z). \] (30)

We now choose local coordinates and local frame satisfying Eq. (18) as described, for example, in [12] or [26, Lemma 2.7]. By Corollary 2.3, we can approximate \( \mathcal{I}_k(z_0) \) by integrating (30) over a small ball about \( z_0 \):
\[ \mathcal{I}_k(z_0) = \int_{|u| \leq b \sqrt{\log k}} Q_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right) f \left( z_0 + \frac{u}{\sqrt{k}} \right) \Omega_M \left( z_0 + \frac{u}{\sqrt{k}} \right) \]
\[ + O \left( \frac{1}{k^{m+2}} \right), \]
(31)
where we let \( b = \sqrt{m+3} \). By (16) and (18),
\[ R_{jlpq}(z_0) = -\frac{\partial^4 \varphi}{\partial z^j \partial z^l \partial z^p \partial z^q} (z_0), \]
(32)
and thus
\[ \omega \left( z_0 + \frac{u}{\sqrt{k}} \right) = \frac{i}{2} \partial \bar{\partial} \varphi \left( z_0 + \frac{u}{\sqrt{k}} \right) \]
\[ = \frac{i}{2} \sum_{j=1}^{m} \frac{1}{k} du_j \wedge d\bar{u}_j - \frac{i}{2} \sum_{j,l,p,q} \left( \frac{1}{k^2} R_{jlpq}(z_0) u^p \bar{u}^q + O \left( \frac{|u|^3}{k^{5/2}} \right) \right) du^j \wedge d\bar{u}^l. \]
(33)
Hence
\[ \Omega_M \left( z_0 + \frac{u}{\sqrt{k}} \right) \]
\[ = \frac{1}{m!} \omega^m = \left[ 1 - \left( \sum_{j,p,q} R_{jlpq}(z_0) u^p \bar{u}^q \right) \right] k^{-1} + O \left( k^{-3/2 + \varepsilon} \right) \]
\[ \frac{1}{k^m} v_u, \]
(34)
for \( |u| \leq b \sqrt{\log k} \), where
\[ v_u = \left( \frac{i}{2} du^1 \wedge d\bar{u}^1 \right) \wedge \cdots \wedge \left( \frac{i}{2} du^m \wedge d\bar{u}^m \right) \]
(35)
denotes the Euclidean volume form. We then have by (31), (34), and Lemma 3.4,

\[ \mathcal{I}_k(z_0) = \frac{1}{k^m} \int_{|u| \leq b \sqrt{\log k}} \left[ G \left( e^{-\frac{1}{2}|u|^2} \left[ 1 + \frac{1}{4} \sum_{j,l,p,q} R_{jlpq}(z_0)u^j \bar{u}^l u^p \bar{u}^q k^{-1} \right] \right) \right. \]

\[ + O \left( k^{-3/2+\varepsilon} \right) \} \times \left\{ f(z_0) + 2 \text{Re} \sum_{j} f_j(z_0)u^j k^{-1/2} \right. \]

\[ + \sum_{j,l} \left[ \text{Re} f_{jl}(z_0)u^j u^l + f_{jl}(z_0)u^j \bar{u}^l \right] k^{-1} + O(k^{-3/2+\varepsilon}) \right\} \times \left[ 1 - \left( \sum_{j,p,q} R_{jlpq}(z_0)u^p \bar{u}^q \right) k^{-1} + O(k^{-3/2}) \right] \nu_u + O(k^{-m-2}) , \]

(36)

where we write

\[ f_j = \frac{\partial f}{\partial \bar{z}^j} , \quad f_{jl} = \frac{\partial^2 f}{\partial \bar{z}^j \partial \bar{z}^l} , \quad f_{jl} = \frac{\partial^2 f}{\partial \bar{z}^j \partial \bar{z}^l} \cdot \]

Since the integral of the \( O(k^{-3/2+\varepsilon}) \) terms in (36) over the \( (b \sqrt{\log k}) \)-ball is \( O(k^{-3/2+\varepsilon'}) \), we have

\[ \mathcal{I}_k(z_0) = \frac{1}{k^m} \int_{|u| \leq b \sqrt{\log k}} \left[ G \left( e^{-\frac{1}{2}|u|^2} \left[ 1 + \frac{1}{4} \sum_{j,l,p,q} R_{jlpq}(z_0)u^j \bar{u}^l u^p \bar{u}^q k^{-1} \right] \right) \right. \]

\[ \left. \times \left\{ f(z_0) + 2 \text{Re} \sum_{j} f_j(z_0)u^j k^{-1/2} + \sum_{j,l} \left[ \text{Re} f_{jl}(z_0)u^j u^l + f_{jl}(z_0)u^j \bar{u}^l \right] k^{-1} \right\} \times \left[ 1 - \sum_{j,p,q} R_{jlpq}(z_0)u^p \bar{u}^q k^{-1} \right] \nu_u + O(k^{-m-3/2+\varepsilon'}) \right. \]

(37)

Since \( G(t) = O(t) \) for \( 0 \leq t \leq 1 \), it follows from (25) that the integrand in (37) is \( O \left( e^{-\frac{1}{2}|u|^2} (1 + |u|^8) \right) \) for \( k \geq k_0, u \in \mathbb{C}^m \). Since

\[ \int_{|u| \geq b \sqrt{\log k}} e^{-\frac{1}{2}|u|^2} (1 + |u|^r) \nu_u = O(k^{-b^2/2+\varepsilon}) , \quad r \geq 0 , \]

(38)

we can replace the domain of integration \( \{|u| \leq b \sqrt{\log k} \} \) with \( \mathbb{C}^m \) in (37) (under our assumption \( b^2 = m + 3 \).

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We say that a function $F$ on $\mathbb{C}^m$ has \textit{polynomial growth} if $|F(u)| = O(1 + |u|^r)$ for some $r \in \mathbb{R}_+$. We shall use the following estimate in the proof of Proposition 3.1:

\textbf{Lemma 3.5} Let $F \in \mathcal{C}^0(\mathbb{C}^m)$ such that $F$ has polynomial growth, and let $\alpha(u)$ be a polynomial in $(u, \bar{u})$ with $\alpha(0) = 0$, $d\alpha(0) = 0$. Then

$$
\int_{\mathbb{C}^m} G\left(e^{-|u|^2}[1 + \alpha(u)k^{-1}]\right) F(u) \, v_u
\leq \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2}[1 + n\alpha(u)k^{-1}]F(u) \, v_u + O(k^{-3/2}).
$$

(39)

for $k \gg 0$.

\textbf{Proof} Since $\alpha(0) = 0$ and $d\alpha(0) = 0$, we can choose $k_1$ such that

$$
1 + |\alpha(u)|k^{-1} \leq e^{\frac{|u|^2}{2}}, \quad \text{for} \ k \geq k_1, \ u \in \mathbb{C}^m.
$$

(40)

Since $|G(t)| \leq G(|t|) = O(|t|)$ for $|t| \leq 1$, for $k \geq k_1$ we have by (40)

$$
\left| \int_{\mathbb{C}^m} G\left(e^{-|u|^2}[1 + \alpha(u)k^{-1}]\right) F(u) \, v_u \right| \leq \int_{\mathbb{C}^m} G\left(e^{-|u|^2}[1 + |\alpha(u)|k^{-1}]\right) |F(u)| \, v_u
\leq \int_{\mathbb{C}^m} G\left(e^{-|u|^2/2}\right) |F(u)| \, v_u
\leq C \int_{\mathbb{C}^m} e^{-|u|^2/2} |F(u)| \, v_u < \infty.
$$

(41)

Thus the left side of (39) is well defined and finite, for $k \geq k_1$. Furthermore, for $p \geq 0$, $q \geq 1$,

$$
\sum_{n=1}^{\infty} \int_{\mathbb{C}^m} \frac{1}{n^q} e^{-n|u|^2} |u|^p \, v_u = \sum_{n=1}^{\infty} \frac{1}{n^{q+m+p/2}} \int_{\mathbb{C}^m} e^{-n|u|^2} |u|^p \, v_u < \infty
$$

(42)

and therefore

$$
\left| \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2}[1 + n\alpha(u)k^{-1}]F(u) \, v_u \right|
\leq \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2}[1 + n|\alpha(u)||]F(u) \, v_u < \infty,
$$

(43)

for $k \geq 1$. Hence the right side of (39) is also well defined.
By (11),
\[
\int_{\mathbb{C}^m} G \left( e^{-|u|^2} [1 + \alpha(u)k^{-1}] \right) F(u) \nu_u \\
= \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} [1 + \alpha(u)k^{-1}]^n F(u) \nu_u. \tag{44}
\]

Let
\[
E := \int_{\mathbb{C}^m} G \left( e^{-|u|^2} [1 + \alpha(u)k^{-1}] \right) F(u) \nu_u \\
- \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} [1 + n\alpha(u)k^{-1}] F(u) \nu_u \\
= \frac{1}{4\pi^2} \sum_{n=2}^{\infty} \int_{\mathbb{C}^m} \frac{e^{-n|u|^2}}{n^2} \sum_{j=2}^{n} \binom{n}{j} \alpha(u)^j k^{-j} |F(u)| \nu_u.
\]

Then
\[
|E| \leq \frac{1}{4\pi^2} k^{-3/2} \sum_{n=2}^{\infty} \frac{e^{-n|u|^2}}{n^2} \sum_{j=2}^{n} \binom{n}{j} |\alpha(u)|^j k^{-j+3/2} |F(u)| \nu_u.
\]

Since \(- j + 3/2 \leq - j/4\) for \(j \geq 2\), we have for \(k \geq k_1^4\),
\[
|E| \leq \frac{1}{4\pi^2} k^{-3/2} \sum_{n=2}^{\infty} \int_{\mathbb{C}^m} \frac{e^{-n|u|^2}}{n^2} \sum_{j=2}^{n} \binom{n}{j} \left( |\alpha(u)|k^{-1/4} \right)^j |F(u)| \nu_u \\
\leq \frac{1}{4\pi^2} k^{-3/2} \sum_{n=1}^{\infty} \int_{\mathbb{C}^m} \frac{e^{-n|u|^2}}{n^2} \left( 1 + |\alpha(u)|k^{-1/4} \right)^n |F(u)| \nu_u \\
= \frac{1}{4\pi^2} k^{-3/2} \int_{\mathbb{C}^m} G \left( e^{-|u|^2} \left( 1 + |\alpha(u)|k^{-1/4} \right) \right) |F(u)| \nu_u.
\]

By (41),
\[
\int_{\mathbb{C}^m} G \left( e^{-|u|^2} \left( 1 + |\alpha(u)|k^{-1/4} \right) \right) |F(u)| \nu_u < \infty
\]
for \(k \geq k_1^4\), and hence \(|E| = O(k^{-3/2})\). \(\Box\)

We now continue the proof of Proposition 3.1. Applying Lemma 3.5 with
\[
\alpha(u) = \frac{1}{4} \sum_{j,l,p,q} R_{jlpq}(z_0) u^j \bar{u}^l u^p \bar{u}^q,
\]
\(\Box\) Springer
Eqs. (37)–(38) yield

\[
\mathcal{I}_k(z_0) = \frac{1}{4\pi^2 k_m} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \left[ 1 + \frac{n}{4} \sum_{j,l,p,q} R_{jlpq}(z_0) u^j \bar{u}^l u^p \bar{u}^q k^{-1} \right] \\
\times \left[ f(z_0) + 2\text{Re} \sum_{j} f_j(z_0) u^j k^{-1/2} + \sum_{j,l} \left( \text{Re} f_{jl}(z_0) u^j u^l + f_{jl}(z_0) u^j \bar{u}^l \right) k^{-1} \right] \\
\times \left[ 1 - \sum_{j,p,q} R_{jlpq}(z_0) u^p \bar{u}^q k^{-1} \right] v_u + O \left( \frac{1}{k_m + 3/2 - \varepsilon} \right). \tag{45}
\]

Then after gathering terms, we have

\[
\mathcal{I}_k(z_0) = \frac{1}{4\pi^2 k_m} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \left\{ f(z_0) + 2\text{Re} \sum_{j} f_j(z_0) u^j k^{-1/2} \\
+ \left( \frac{n}{4} \sum_{j,l,p,q} R_{jlpq}(z_0) u^j \bar{u}^l u^p \bar{u}^q - \sum_{j,p} R_{jlp\bar{p}}(z_0) u^p \bar{u}^q \right) f(z_0) k^{-1} \right. \\
+ \sum_{j,l} \left( \text{Re} f_{jl}(z_0) u^j u^l + f_{jl}(z_0) u^j \bar{u}^l \right) k^{-1} \left. \right\} v_u + O \left( \frac{1}{k_m + 3/2 - \varepsilon} \right). \tag{46}
\]

By making a change of variable \( u^j \mapsto e^{i\theta} u^j \) in (46) for a fixed index \( j \) and noting that the volume form is invariant under this transformation, one sees that terms where \( u^j \) is not paired with \( \bar{u}^j \) have vanishing integrals. So we obtain

\[
\mathcal{I}_k(z_0) = \frac{1}{4\pi^2 k_m} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \left\{ f(z_0) \\
+ \left( \frac{n}{4} \sum_{j,l,p,q} R_{jlpq}(z_0) u^j \bar{u}^l u^p \bar{u}^q - \sum_{j,p} R_{jlp\bar{p}}(z_0) u^p \bar{u}^q \right) f(z_0) \\
+ \sum_{p} f_{p\bar{p}}(z_0) |u^p|^2 \right\} v_u + O \left( \frac{1}{k_m + 3/2 - \varepsilon} \right). \tag{47}
\]
With the change of variables \( u = \frac{1}{\sqrt{n}} v \), we have

\[
I_k(z_0) = \frac{\xi(m + 2)}{4\pi^2} \left( \int_{\mathbb{C}^m} e^{-|v|^2} v v_n f(z_0) k^{-m} \right) + \frac{\xi(m + 3)}{4\pi^2} \left( \int_{\mathbb{C}^m} e^{-|v|^2} \left[ \frac{1}{4} \sum_{j,l,p,q} R_{jlpq}(z_0) v^j v^l v^p v^q - \sum_{j,p} R_{jpp}(z_0) |v^p|^2 \right] f(z_0) \right) + \sum_{p} f_p \tilde{p}(z_0) |v^p|^2 \right) v v_n k^{-m-1} + O \left( k^{-m-3/2+\varepsilon} \right).
\]

By the Wick formula and (32),

\[
\frac{1}{\pi^m} \int_{\mathbb{C}^m} \left( \sum_{j,l,p,q} R_{jlpq}(z_0) v^j v^l v^p v^q \right) e^{-|v|^2} v v_n = \sum_{j,l,p,q} R_{jlpq}(z_0) (\delta_{j,l} \delta_{p,q} + \delta_{j,p} \delta_{l,q}) = \sum_{j,p} \left[ R_{jpp}(z_0) + R_{jlp}(z_0) \right] = 2 \sum_{j,p} R_{jlp}(z_0) = 2 \rho(z_0),
\]

where we recall that \( \rho(z_0) \) denotes the scalar curvature at \( z_0 \). Thus

\[
I_k(z_0) = \frac{\pi^{m-2}}{4} \frac{\xi(m + 2)}{\xi(m + 2)} f(z_0) k^{-m} + \frac{\pi^{m-2}}{4} \frac{\xi(m + 3)}{\xi(m + 3)} \left[ -\frac{1}{2} \rho(z_0) f(z_0) + \sum_{p} f_p \tilde{p}(z_0) \right] k^{-m-1} + O \left( k^{-m-3/2+\varepsilon} \right).
\]

(49)

Corollary 2.3 and Theorem 2.4 guarantee that the remainder estimate \( O(k^{-m-3/2+\varepsilon}) \) in (49) is uniform over \( z_0 \in M \).

Note that \( \sum_{p} f_p \tilde{p}(z_0) = -\tilde{\partial}^{*} \tilde{\partial} f(z_0) \) since the \( z^p \) are normal coordinates at \( z_0 \). Hence by (27) and (49), we have

\[
\text{Var}(Z, \psi) = \frac{\pi^{m-2}}{4} \frac{\xi(m + 2)}{\xi(m + 2)} \left( \int_M f^2 \Omega_M \right) k^{-m} - \frac{\pi^{m-2}}{4} \frac{\xi(m + 3)}{\xi(m + 3)} \int_M \left[ -\frac{1}{2} \rho f^2 + f \tilde{\partial}^* \tilde{\partial} f \right] \Omega_M k^{-m-1} + O(k^{-m-3/2+\varepsilon}).
\]

(50)

Since \( f = \imath \partial \tilde{\partial} \psi \), we have

\[
f^2 = |\partial \tilde{\partial} \psi|^2, \quad \int_M f^2 \Omega_M = \int_M |\partial \tilde{\partial} \psi|^2 \Omega_M = \| \partial \tilde{\partial} \psi \|^2_2.
\]

(51)
Furthermore,

$$\int_M f \bar{\partial}^* \partial f \Omega_M = (\bar{\partial}^* \partial f, f) = \| \bar{\partial} f \|^2 = \| \bar{\partial}^* (\bar{\partial} \partial \psi) \|^2 = \| \bar{\partial} \partial \psi \|^2. \quad (52)$$

The formula of Proposition 3.1 follows from (50)–(52).

4 Proof of Theorem 1.1.

To complete the proof of Theorem 1.1, it suffices to show that

$$\text{Var}(Z_{sk}, \psi) = A_0 k^{-m} + A_1 k^{-m-1} + \cdots + A_p k^{-m-p} + O(k^{-m-p-1}), \quad (53)$$

for \( p \geq 2 \). We shall verify (53) by generalizing Lemmas 3.4–3.5. First we have

Lemma 4.1 Let \( p \geq 2, \varepsilon > 0 \). For \( |u| < b \sqrt{\log k} \), we have

$$Q_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 \right) = G \left( e^{-|u|^2} \left[ 1 + \sum_{j=2}^p a_j(u) k^{-j/2} \right] \right) + O \left( |u| k^{-(p+1)/2+\varepsilon} \right), \quad (54)$$

where the \( a_j(u) \) are as in Theorem 3.3 and \( a_2(u) = \frac{1}{4} R_{z_0}(u, \bar{u}, u, \bar{u}) \).

Proof Repeat the proof of Lemma 3.4 with \( \frac{1}{4} R_{z_0}(u, \bar{u}, u, \bar{u}) k^{-1} \) replaced by \( \sum_{j=2}^p a_j(u) k^{-j/2} \) and with \( k^{-3/2+\varepsilon} \) replaced by \( k^{-(p+1)/2+\varepsilon} \). \( \square \)

To state the generalization of Lemma 3.5, we use the following notation: For integers \( j \geq 2 \), we let

$$P(j) = \{ (r_2, \ldots, r_p) \in \mathbb{N}^{p-1} : \sum_{\lambda=2}^p \lambda r_{\lambda} = j \},$$

where \( \mathbb{N} \) denotes the non-negative integers. For \( r = (r_2, \ldots, r_p) \in \mathbb{N}^{p-1} \) and \( n \in \mathbb{Z}_+ \), we let

$$\binom{n}{r} = \frac{n(n-1) \cdots (n-|r|+1)}{r_2! \cdots r_p!} = \begin{cases} \frac{n!}{r_2! \cdots r_p! (n-|r|)!} & \text{for } n \geq |r| \\ 0 & \text{for } n \in \{0, 1, \ldots, |r| - 1\} \end{cases} \quad (55)$$

denote the multinomial coefficients, where \( |r| = r_2 + \cdots + r_p \). We note the polynomial identity

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\[
\left(1 + \sum_{j=2}^{p} C_j x^j\right)^n = 1 + \sum_{j=2}^{np} B_{pj}(C_2, \ldots, C_p; n)x^j,
\]

\[
B_{pj}(C_2, \ldots, C_p; n) = \sum_{r \in p(j)} \left(\frac{n}{r}\right) \prod_{\lambda=2}^{C_p} C_{\lambda}^r.
\] (56)

The following generalizes Lemma 3.5:

**Lemma 4.2** Let \( F \in C^0(\mathbb{C}^m) \) such that \( F \) has polynomial growth. Then for \( p \geq 2 \),

\[
\int_{\mathbb{C}^m} G \left( e^{-|u|^2} \left[ 1 + \sum_{j=2}^{p} a_j(u)k^{-j/2} \right] \right) F(u) \nu_u
\]

\[
\leq \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \left[ 1 + \sum_{j=2}^{p} b_{pj}(u, n)k^{-j/2} \right] F(u) \nu_u + O(k^{-p/2-1/4}).
\] (57)

where

\[
b_{pj}(u, n) = B_{pj} (a_2(u), \ldots, a_p(u); n).\] (58)

**Proof** Fix \( p \geq 2 \). Since the polynomials \( a_j \) have no constant or linear terms, we can choose \( k_0 \) such that

\[
1 + \sum_{j=2}^{p} |a_j(u)|k^{-j/2} \leq e^{|u|^2/2}, \text{ for } k \geq k_0, \; u \in \mathbb{C}^m.
\]

Since \( |G(t)| \leq G(|t|) = O(|t|) \) for \( |t| \leq 1 \), we then have

\[
\int_{\mathbb{C}^m} \left| G \left( e^{-|u|^2} \left[ 1 + \sum_{j=2}^{p} a_j(u)k^{-j/2} \right] \right) F(u) \right| \nu_u
\]

\[
\leq \int_{\mathbb{C}^m} G \left( e^{-|u|^2} \left[ 1 + \sum_{j=2}^{p} |a_j(u)|k^{-j/2} \right] \right) |F(u)| \nu_u
\]

\[
\leq \int_{\mathbb{C}^m} G \left( e^{-|u|^2/2} \right) |F(u)| \nu_u < \infty
\] (59)

and thus the left side of (57) is well defined and finite, for \( k \geq k_0 \).
By (11) and (56),

\[
G\left(e^{-|u|^2}\left[1 + \sum_{j=2}^{p} a_j(u)k^{-j/2}\right]\right) = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n|u|^2} \left[1 + \sum_{j=2}^{p} a_j(u)k^{-j/2}\right]^n
\]

\[
= \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n|u|^2} \left[1 + \sum_{j=2}^{np} b_{pj}(u, n)k^{-j/2}\right],
\]

(60)

for \(k \geq k_0\), where \(b_{pj}(u, n)\) is given by (58).

Furthermore, since \(\binom{n}{r} \geq 0\) for \(n \in \mathbb{Z}_+\), we have

\[
|b_{pj}(u, n)| \leq \sum_{r \in \mathcal{P}(j)} \binom{n}{r} \prod_{\lambda=2}^{p} |a_{\lambda}(u)|^{r_{\lambda}} = B_{pj}(|a_2(u)|, \ldots, |a_p(u)|; n), \text{ for } n \in \mathbb{Z}_+,
\]

and thus

\[
1 + \sum_{j=2}^{np} |b_{pj}(u, n)|k^{-j/2} \leq 1 + \sum_{j=2}^{np} B_{pj}(|a_2(u)|, \ldots, |a_p(u)|; n)k^{-j/2}
\]

\[
= \left[1 + \sum_{j=2}^{p} |a_j(u)|k^{-j/2}\right]^n.
\]

Hence by (59),

\[
\sum_{n=1}^{\infty} \int_{\mathbb{C}^m} e^{-n|u|^2} \left[1 + \sum_{j=2}^{np} |b_{pj}(u, n)|k^{-j/2}\right] |F(u)| v_u
\]

\[
\leq \sum_{n=1}^{\infty} \int_{\mathbb{C}^m} e^{-n|u|^2} \left[1 + \sum_{j=2}^{np} |a_j(u)|k^{-j/2}\right]^n |F(u)| v_u
\]

\[
= 4\pi^2 \int_{\mathbb{C}^m} G\left(e^{-|u|^2}\left[1 + \sum_{j=2}^{p} |a_j(u)|k^{-j/2}\right]\right) |F(u)| v_u < \infty, \quad (61)
\]

for \(k \geq k_0\). In particular, the right side of (57) is also well defined and finite.
As before, let
\[
E := \int_{\mathbb{C}^m} G \left( e^{-|u|^2} \left[ 1 + \sum_{j=2}^{p} a_j(u) k^{-j/2} \right] \right) F(u) v_u
\]
\[= \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \left[ 1 + \sum_{j=2}^{p} b_{pj}(u, n) k^{-j/2} \right] F(u) v_u.
\]

Therefore
\[
|E| \leq \frac{1}{4\pi^2} k^{-p/2-1/4} \sum_{n=2}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \sum_{j=p+1}^{np} |b_{pj}(u, n)| k^{(p-j)/2+1/4} |F(u)| v_u.
\]

Since
\[
p - j + \frac{1}{2} \leq \frac{-j}{2p+2} \quad \text{for } j \geq p + 1,
\]
\[
|E| \leq \frac{1}{4\pi^2} k^{-p/2-1/4} \sum_{n=2}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \sum_{j=p+1}^{np} |b_{pj}(u, n)| k^{-j/(4p+4)} |F(u)| v_u.
\]

By (61),
\[
\sum_{n=2}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-n|u|^2} \sum_{j=p+1}^{np} |b_{pj}(u, n)| k^{-j/(4p+4)} |F(u)| v_u < \infty
\]
for \(k \geq k_0^{2p+2}\), and thus \(|E| = O(k^{-p/2-1/4})\).

**Continuation of the Proof of Theorem 1.1** Fix \(p \geq 2\) and let \(b = \sqrt{m + p + 1}\). Recalling (29), we have an expansion of the form
\[
i \bar{\partial} \partial u \left( z_0 + \frac{u}{\sqrt{k}} \right) = f \left( z_0 + \frac{u}{\sqrt{k}} \right) \Omega_M \left( z_0 + \frac{u}{\sqrt{k}} \right)
\]
\[= \left[ f(z_0) + 2\text{Re} \sum_{q=1}^{m} f_q(z_0) u^q k^{-1/2} + \sum_{j=2}^{p} \beta_j(u) k^{-j/2} + O(k^{-p/2-1/2+\epsilon}) \right] \frac{1}{k^m v_u}.
\]

for \(|u| \leq b \sqrt{\log k}\), where \(\beta_j(u)\) is a homogeneous polynomial in \((u, \bar{u})\) of degree \(j\).
Repeating the derivation of (37) using (31), (62), and Lemma 4.1, we obtain

\[
I_k(z_0) = \frac{1}{k^m} \int_{\mathbb{C}^m} G \left( e^{-|u|^2} \left[ 1 + \sum_{j=2}^{p} a_j(u) k^{-j/2} \right] \right) \times \left[ f(z_0) + 2\text{Re} \left( \sum_{q=1}^{m} f_q(z_0) u^q k^{-1/2} + \sum_{j=2}^{p} \beta_j(u) k^{-j/2} \right) \right] u + O(k^{-m-p/2-1/4}).
\]  

(63)

Here we set \( \varepsilon = 1/4 \) and again used (38). By (63) and Lemma 4.2, we then have

\[
I_k(z_0) = \frac{1}{4\pi^2 k^m} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-|u|^2} \left[ 1 + \sum_{j=2}^{p} b_{pj}(u, n) k^{-j/2} \right] \times \left[ f(z_0) + 2\text{Re} \left( \sum_{q=1}^{m} f_q(z_0) u^q k^{-1/2} + \sum_{j=2}^{p} \beta_j(u) k^{-j/2} \right) \right] u + O \left( \frac{1}{k^{m+p/2+1/4}} \right).
\]  

(64)

Furthermore by Theorem 3.3 and (58),

\[
b_{pj}(u, n) = \sum_{r \in \mathcal{P}(j)} \binom{n}{r} \prod_{\lambda=2}^{j/2} a_{\lambda}(u)^{r_{\lambda}} = \sum_{l=0}^{[j/2]} b_{pj}(u)n^l \quad (r = (r_2, \ldots, r_p)),
\]  

(65)

where \( b_{pj}(u) \) is a polynomial in \( (u, \bar{u}) \) of degree \( \leq 5j \) and of the same parity as \( j \). (The highest power of \( n \) in \( b_{pj} \) is \( [j/2] \) since \( j = 2r_2 + \cdots + pr_p \geq 2|r| = 2\text{deg}_{\mathbb{Q}[n]} \binom{n}{r} \) for \( r \in \mathcal{P}(j) \). However, this bound is not needed in the proof.)

Since the polynomials \( a_{\lambda}(u) \) contain only terms of degree 2 or higher in \( (u, \bar{u}) \), it follows that \( \prod_{\lambda=2}^{j/2} a_{\lambda}(u)^{r_{\lambda}} \) contains only terms of degree \( \geq 2|r| = 2\text{deg}_{\mathbb{Q}[n]} \binom{n}{r} \). It then follows from (65) that \( b_{pj}(u) \) is of the form

\[
b_{pj}(u) = \sum_{q=2l}^{5j} b_{pj}(u),
\]

where \( b_{pj}(u) \) is homogeneous of degree \( q \) in \( (u, \bar{u}) \), and \( \sum' \) denotes the sum with \( q \equiv j(2) \). Thus (64) can be written in the form

\[
I_k(z_0) = \frac{1}{4\pi^2 k^m} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbb{C}^m} e^{-|u|^2} \sum_{j=0}^{2p} \sum_{l=0}^{[j/2]} \sum_{q=2l}^{5j} \sum_{q=2l}^{5j} S_{pj}(u)n^l k^{-j/2} u + O \left( \frac{1}{k^{m+p/2+1/4}} \right).
\]  

(66)
where \( S_{pjlq}(u) \) is a homogeneous polynomial in \((u, \bar{u})\) of degree \(q\). (In particular, \(S_{p000}(u) = f(z_0)\).)

As before, we make the change of variables \(u = \frac{1}{\sqrt{n}}v\) so that (66) becomes

\[
\mathcal{I}_k(z_0) = \frac{1}{4\pi^2 km} \sum_{n=1}^{\infty} \int_{\mathbb{C}^m} \sum_{j=0}^{2p} \sum_{l=0}^{|j/2|} \sum_{q=2l}^{5j} e^{-|v|^2} S_{pjlq}(v) \frac{k^{-j/2}}{n^{2+m+q/2-l}} v_\nu + O\left(\frac{1}{k^{m+p/2+1/4}}\right)
\]

\[
= \frac{1}{4\pi^2 km} \sum_{j=0}^{2p} k^{-j/2} \sum_{l=0}^{|j/2|} \sum_{q=2l}^{5j} \xi(2 + m + q/2 - l) \int_{\mathbb{C}^m} e^{-|v|^2} S_{pjlq}(v) v_\nu
\]

\[+ O\left(\frac{1}{k^{m+p/2+1/4}}\right). \tag{67}\]

Since the terms of the sum in (67) with \(q\) odd (in which \(S_{pjlq}(v)\) has odd degree) vanish, and since \(q\) and \(j\) have the same parity in the sum, (67) reduces to a sum over \(q\) and \(j\) even. I.e., substituting \(j = 2\tilde{j}, q = 2\tilde{q}\), we have

\[
\mathcal{I}_k(z_0) = \frac{1}{4\pi^2 km} \sum_{j=0}^{p} k^{-j} \sum_{l=0}^{\tilde{j}} \sum_{\tilde{q}=l}^{5\tilde{j}} \xi(2 + m + \tilde{q} - l) \int_{\mathbb{C}^m} e^{-|v|^2} S_{p(2\tilde{j})l(2\tilde{q})}(v) v_\nu
\]

\[+ O\left(\frac{1}{k^{m+p/2+1/4}}\right). \tag{68}\]

Finally, by (27) and (68) with \(p\) replaced by \(2p+2\), we obtain the asymptotic expansion (53) of Theorem 1.1. Proposition 3.1 provides the values of the leading and sub-leading terms of the expansion. \(\square\)

### 5 Remarks

By further refining the local coordinate condition (18), one can obtain information about additional terms of the asymptotic expansion (53). I.e., for \(n \geq 4\), one can choose holomorphic Bochner coordinates (also called K-coordinates) of order \(n\) at \(z_0\), which satisfy:

\[
\psi(z_0 + z) = |z|^2 + \sum_{|J| \geq 2, |K| \geq 2 \atop |J| + |K| \leq n} a_{JK} z^J \bar{z}^K + O(|z|^{n+1}). \tag{69}\]

See [12, 26]. Equation (18) describes Bochner coordinates of order 4. It was shown in [26, Theorem 2.8] that with Bochner coordinates of order \(r + 2\), the polynomials \(p_r(u, v)\) of Theorem 2.4 are curvature invariants (and are of degree \(\leq 2r\) instead of \(5r\)). It then follows by tracing through the proof of Theorem 1.1 in Sect. 4 that the coefficients \(A_j\) in Theorem 1.1 are integrals involving curvature invariants and \(\partial \bar{\partial} \psi\). To state this precisely, we introduce the following definition:
Definition 5.1 Let $M$ be a Riemannian manifold and suppose that $f \in C^\infty(M)$. An $f$-curvature invariant of $M$ is a scalar field (smooth function) on $M$ that is a contraction of tensor products of $f$ and its derivatives and the curvature tensor of $M$ and its derivatives.

The proof of Theorem 1.1 yields the following result:

Theorem 5.2 Let $(\mathcal{L}, h) \rightarrow (M, \omega)$ be as in Theorem 1.1. Suppose that $\psi \in \mathcal{D}^{m-1,m-1}(M)$ and let $i\partial \bar{\partial} \psi = f \Omega M$, where $\Omega M$ is the volume form on $M$. Then the coefficients $A_j$ of the asymptotic expansion (2) of $\text{Var}(Z_{sk}, \psi)$ are of the form $A_j = \int_M A_j \Omega M$, where $A_j$ is a linear combination of $f$-curvature invariants of $M$.

Formulas for the polynomials $p_3$ and $p_4$ of Theorem 2.4 were also given in [26] using Bochner coordinates, and these can be used together with (64) and (68) to obtain a formula for $A_2$. (The integral in (68) can be evaluated using the Wick formula, and $5\tilde{j}$ can be replaced by $2\tilde{j}$.)

In addition to the linear statistics studied here, the following “number statistics” are also of interest: For a domain $U \subset M$ with smooth boundary, we let $N_k^U$ denote the number of simultaneous zeros in $U$ of $m$ independent Gaussian holomorphic sections of $L^k \rightarrow M$. It was shown in [34] that the variance of $N_k^U$ has the asymptotics

$$\text{Var} \left( N_k^U \right) = \nu_{nm} \text{Vol}(\partial U) \left( k^{m-1/2} + O(k^{m-1/2} + \varepsilon) \right),$$

where $\nu_{nm}$ is a universal constant (given explicitly in [34]). In particular, for dimension $m = 1$, we have

$$\text{Var} \left( N_k^U \right) = \frac{\zeta(3/2)}{8\pi^{3/2}} \text{Length}(\partial U) \left( k^{1/2} + O(k^{\varepsilon}) \right).$$

The analogy with linear statistics leads to the conjecture that $\text{Var}(N_k^U)$ has an asymptotic expansion. In dimension 1, an asymptotic expansion should follow by the methods of this paper. The higher dimensional case requires a more complicated analysis.

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